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CONTENTS

Alireza Amini-Harandi, Mahdi Goli, and Hamid Reza Hajisharifi
A fixed point theorem for generalized nonexpansive mappings 1117

Enes Ata
On the new fractional operators generating modified gamma and beta functions 1127

Merve Avcı Ardıç, Havva Kavurmacı Önalan, Ahmet Ocak Akdemir, and Anh Tuan Nguyen
New approaches for m-convex functions via fractional integral operators with strong kernels 1145

Gülhan Ayar and Mehmet Akif Akyol
New curvature tensors along Riemannian submersions 1161

Abdullah Aydın
Statistically unbounded p-convergence in lattice-normed Riesz spaces . . 1185

Hamid Beddani, Moustafa Beddani, and Zoubir Dahmani
An existence etudy for a tripled system with $p-$Laplacian involving $\varphi-$Caputo derivatives 1197

István Blahota
Approximation by a special de la Vallée Poussin type matrix transform mean of Walsh-Fourier series 1213

Oleksandr Boichuk, Dmytro Bihun, and Oleksandr Pokutnyi
ν,μ dichotomy and bounded solutions of differential equations 1223

Ghazala Gulshan, Hüseyin Budak, Rashida Hussain, Asad Sadiq, and Pınar Kösem
On new version of Hermite-Jensen-Mercer inequalities for newly defined quantum integral 1235

Miskolc Mathematical Notes

VOLUME 24 (2023), NUMBER 3

CONTENTS

Hüseyin Budak and Hasan Kara
On quantum Hermite-Jensen-Mercer inequalities 1247

Huanyin Chen and Marjan Sheibani
Block representations for the g-Drazin inverse in Banach algebras 1259

Dragos-Patru Covei
Existence theorems for equations and systems in R^N with k_i-Hessian operator 1273

S. Deniz, F. Özger, Z. Ö. Özger, S. A. Mohiuddine, and M. T. Ersoy
Numerical solution of fractional Volterra integral equations based on rational Chebyshev approximation 1287

Zhibin Du and Carlos M. da Fonseca
The determinants of certain double banded $(0,1)$ Toeplitz matrices 1307

Hajar EL Mir, Badr Nejjar, and Lahcen Oukhtite
On traces of permuting n-derivations on prime ideals 1317

Hajar El Mir, Abdellah Mamouni, and Lahcen Oukhtite
Additive mappings satisfying certain algebraic equations in prime rings . 1329

Hiranmoy Garai, Pratikshan Mondal, and Lakshmi Kanta Dey
On Common fixed point results via implicit relations 1337

Ayazul Hasan
Characterizations of generalized submodules of QTAG-modules 1351

Ahmed Mohamed Hassan and Samy Affan
New oscillation criteria for second-order delay dynamic equations with a sub-linear negative neutral term on time scales 1361

Miskolc Mathematical Notes

VOLUME 24 (2023), NUMBER 3

CONTENTS

Fatih Hezenci and Huseyin Budak
New generalizations of some important inequalities for Sarıkaya fractional integrals 1377

İmdat İşcan
Construction of a new class of functions with their some properties and certain inequalities: n-fractional polynomial convex functions 1389

Dilek Karakaya, Yasin Yazlik, and Merve Kara
On a solvable system of difference equations of sixth-order 1405

Wende Li, Jianlong Chen, Yukun Zhou, and Yuanyuan Ke
Pseudo core invertibility and DMP invertibility in two semigroups of a ring with involution 1427

Naim Braha, Valdete Loku, and Ram Mohapatra
Tauberian theorems for Nörlund-Cesáro mean via statistically convergence in m-Normed spaces 1439

Bilel Mefteh
On the existence of solutions for an infinite system of integral equations . 1453

Celil Nebiyev and Hilal Başak Özdemir
$\oplus$-g-Radical supplemented modules 1469

Anju Panwar, Pinki Lamba, Vladimir Rakočević, and Dhananjay Gopal
New fixed point results of some enriched contractions in CAT(0) spaces . 1477

Ramasamy Saravanakumar and Hemen Dutta
State estimation of memristor-based stochastic neural networks with mixed variable delays 1495

Miskolc Mathematical Notes

VOLUME 24 (2023), NUMBER 3

CONTENTS

Arpan Sardar and Uday Chand De
Conformal vector fields on $N(k)$-paracontact and para-Kenmotsu manifolds 1515

Shagun Sharma and Sumit Chandok
Convergence, optimal points and applications 1527

Jaraldpushparaj Simon, Gnanaprakasam Britto Antony Xavier, Sina Etemad, and Shahram Rezapour
Mittag-Leffler function for real index and its application in solving difference and differential equations 1541

Thanin Sitthiwirattham, Muhammad Aamir Ali, Asghar Ali, and Hüseyin Budak
A new q-Hermite-Hadamard's inequality and estimates for midpoint type inequalities for convex functions 1555

Sanjiv Subba and Tikaram Subedi
NJ-semicommutative Rings 1569

Adeolu Taiwo, Timilehin Opeyemi Alakoya, Lateef Olakunle O. Jolaoso, and Oluwatosin Temitope Mewomo
Viscosity S-iteration algorithm for finding common fixed point of nonexpansive mappings and application to nonexpansive semigroups . . . 1581

Abbas Kareem Wanas and Páll-Szabó Ágnes Orsolya
Applications of Horadam polynomials on a new family of bi-prestarlike functions 1605

Milan Lj. Zlatanović, Miloš Z. Petrović, and Miroslav D. Maksimović
Curvature properties of projective semi-symmetric linear connections . . 1615

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A FIXED POINT THEOREM FOR GENERALIZED NONEXPANSIVE MAPPINGS

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Abstract. In this paper, we obtain a generalization of a fixed point theorem given by Popescu [O. Popescu, Comput. Math. Appl., vol. 62, no. 10, pp. 3912–3919, 2011]. An example is also given to support our main result.

2010 *Mathematics Subject Classification:* 47H04; 47H10; 54H25

Keywords: fixed point, approximate fixed point sequence, metric space, generalized nonexpansive mapping

1. Introduction and Preliminares

Let (X,d) be a metric space and let $T\colon X \to X$ be a mapping. We say that $x \in X$ is a fixed point for T, if $Tx = x$. The sequence $(x_n) \subseteq X$ is said to be an approximate fixed point sequence (a.f.p.s., for short) for T, if $d(x_n, Tx_n) \to 0$.

In 2011, Popescu [6] proved the following result.

Theorem 1. *Let (X,d) be a complete metric space and let $T\colon X \to X$ be a mapping satisfying*

$$\begin{aligned}&\frac{1}{2}d(x,Tx) \le d(x,y) \;\Rightarrow\\ &d(Tx,Ty) \le ad(x,y) + b(d(x,Tx)+d(y,Ty)) + c(d(x,Ty)+d(y,Tx)),\end{aligned}$$

for all $x,y \in X$, with $a \ge 0$, $b > 0$, $c > 0$ and $a+2b+2c = 1$. Then, T has a unique fixed point.

Definition 1. Let (X,d) be a metric space, $\lambda \in [0,1)$ and let $a,b,c \in \mathbb{R}$ with $a+2b+2c = 1$. We say that a mapping $T\colon X \to X$ is a (λ,a,b,c)-generalized nonexpansive mapping if for all $x,y \in X$,

$$\begin{aligned}&\lambda d(x,Tx) \le d(x,y) \;\Rightarrow\\ &d(Tx,Ty) \le ad(x,y) + b(d(x,Tx)+d(y,Ty)) + c(d(x,Ty)+d(y,Tx)).\end{aligned}$$

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In the following example, we see that many previously known classes of generalized nonexpansive mappings are actually (λ, a, b, c)-generalized nonexpansive, for some $\lambda \in [0,1)$ and $a, b, c \in \mathbb{R}$.

Example 1.

(i) Every nonexpansive mapping is $(0,1,0,0)$-generalized nonexpansive.
(ii) Every mapping which satisfies condition (C) [2,4] is $(\frac{1}{2},1,0,0)$-generalized nonexpansive.
(iii) Every Kannan nonexpansive mapping [5] is $(0,0,0,1)$-generalized nonexpansive.
(iv) Every Reich nonexpansive mapping [5] is $(0,a,\alpha,0)$-generalized nonexpansive, where $a \geq 0$ and $\alpha \geq 0$.
(v) Every generalized α-nonexpansive mapping [5] is $(\frac{1}{2},1-2\alpha,0,\alpha)$-generalized nonexpansive, where $\alpha \in [0,1)$.
(vi) Every Reich-Suzuki nonexpansive mapping [5] is $(\frac{1}{2},1-2\alpha,\alpha,0)$-generalized nonexpansive, where $\alpha \in [0,1)$.
(vii) Every Gobel- Kirk-Shimi generalized nonexpansive mapping [1] is $(0,a,b,c)$-generalized nonexpansive, where $a \geq 0$, $b > 0$ and $c > 0$.
(viii) The class of $(0,a,b,0)$-generalized nonexpansive mappings, where $a > 0$ and $b > 0$, was defined and studied by Greguš [3].
(ix) The class of $(\frac{1}{2},a,b,c)$-generalized nonexpansive mappings, where $a \geq 0$, $b > 0$ and $c > 0$, was defined and studied by Popescu [6].

In connection with Theorem 1, the following problem arises:

Problem 1. *Determine all $(\lambda, a, b, c) \in [0,1) \times \mathbb{R}^3$ with $a + 2b + 2c = 1$, such that every (λ, a, b, c)-generalized nonexpansive mapping $T : X \to X$ has a fixed point, where (X, d) is a complete metric space.*

Fixed point theory for generalized nonexpansive mappings, that recall in example 1, have been studied by some authors [1–6]. In this paper, we give a partial answer to the problem 1, which is also a generalization of Theorem 1.

2. Main result

García-Falset et al. [2], studied a class of mappings satisfying the following condition.

Definition 2. Let C be a nonempty subset of a metric space X. For $\mu \geq 1$, we say that a mapping $T : C \to X$ satisfies condition (E_μ) on C if for each $x, y \in C$,

$$d(x,Ty) \leq \mu d(x,Tx) + d(x,y).$$

We say that T satisfies condition (E) on C whenever T satisfies (E_μ) on C for some $\mu \geq 1$.

Proposition 1. *Let $T\colon X \to X$ be $(\frac{1}{2}, a, b, c)$-generalized nonexpansive mapping with $b \geq 0$, $c \geq 0$ and $b + c < 1$. Then, T satisfies condition (E) with $\mu = \frac{3+b+c}{1-b-c}$.*

Proof. Since $\frac{1}{2}d(x,Tx) \leq d(x,Tx)$, for each $x \in X$, then

$$\begin{aligned} d(Tx,T^2x) &\leq ad(x,Tx) + b(d(x,Tx) + d(Tx,T^2x)) + cd(x,T^2x) \\ &\leq ad(x,Tx) + b(d(x,Tx) + d(Tx,T^2x)) + c(d(x,Tx) + d(Tx,T^2x)). \end{aligned}$$

Using the assumptions $b + c < 1$ and $a + 2b + 2c = 1$, from the above we conclude that (note that the above inequality holds for any (λ, a, b, c)-generalized nonexpansive mapping with $\lambda \in [0,1)$),

$$d(Tx,T^2x) \leq d(x,Tx) \quad \text{for each } x \in X. \tag{2.1}$$

Now, we prove that for each $x, y \in X$,

$$\text{either} \quad \frac{1}{2}d(x,Tx) \leq d(x,y) \quad \text{or} \quad \frac{1}{2}d(Tx,T^2x) \leq d(Tx,y). \tag{2.2}$$

On the contrary, we assume that there exist $x, y \in X$ such that

$$\frac{1}{2}d(x,Tx) > d(x,y) \quad \text{and} \quad \frac{1}{2}d(Tx,T^2x) > d(Tx,y).$$

Hence by (2.1), we have

$$\begin{aligned} d(x,Tx) &\leq d(x,y) + d(y,Tx) \\ &< \frac{1}{2}d(x,Tx) + \frac{1}{2}d(Tx,T^2x) \\ &\leq \frac{1}{2}d(x,Tx) + \frac{1}{2}d(x,Tx) = d(x,Tx), \end{aligned}$$

a contradiction. So, (2.2) holds for each $x, y \in X$.
Now, from (2.2) we get that for each $x, y \in X$, either

$$d(Tx,Ty) \leq ad(x,y) + b(d(x,Tx) + d(y,Ty)) + c(d(x,Ty) + d(y,Tx)), \tag{2.3}$$

or

$$d(T^2x,Ty) \leq ad(Tx,y) + b(d(Tx,T^2x) + d(y,Ty)) + c(d(Tx,Ty) + d(y,T^2x)). \tag{2.4}$$

Let $x, y \in X$. We first assume that (2.3) holds. Then

$$\begin{aligned} d(x,Ty) &\leq d(x,Tx) + d(Tx,Ty) \\ &\leq d(x,Tx) + ad(x,y) \\ &\quad + b(d(x,Tx) + d(y,Ty)) + c(d(x,Ty) + d(y,Tx)) \\ &\leq (1+b+c)d(x,Tx) + (b+c)d(x,Ty) + (1-b-c)d(x,y). \end{aligned}$$

Since $b+c<1$, we obtain

$$\begin{aligned} d(x,Ty) &\leq (\frac{1+b+c}{1-b-c})d(x,Tx)+d(x,y) \\ &\leq (\frac{3+b+c}{1-b-c})d(x,Tx)+d(x,y). \end{aligned}$$

Now assume that (2.4) holds. Then by (2.1), we have

$$\begin{aligned} d(x,Ty) &\leq d(x,Tx)+d(Tx,T^2x)+d(T^2x,Ty) \\ &\leq 2d(x,Tx)+ad(Tx,y) \\ &\quad +b(d(Tx,T^2x)+d(y,Ty))+c(d(Tx,Ty)+d(y,T^2x)) \\ &\leq 2d(x,Tx)+ad(Tx,y) \\ &\quad +b(d(Tx,T^2x)+d(y,Tx)+d(x,Tx)+d(x,Ty)) \\ &\quad +c(d(x,Tx)+d(x,Ty)+d(Tx,y)+d(Tx,T^2x)) \\ &\leq 2d(x,Tx)+ad(Tx,y) \\ &\quad +b(d(y,Tx)+2d(x,Tx)+d(x,Ty)) \\ &\quad +c(d(x,Ty)+d(Tx,y)+2d(x,Tx)). \end{aligned}$$

Since $b+c<1$, we obtain

$$\begin{aligned} (1-b-c)d(x,Ty) &\leq (2+2b+2c)d(x,Tx)+(1-b-c)d(Tx,y) \\ &\leq (2+2b+2c)d(x,Tx)+(1-b-c)(d(x,Tx)+d(x,y)). \end{aligned}$$

Hence

$$d(x,Ty) \leq (\frac{3+b+c}{1-b-c})d(x,Tx)+d(x,y).$$

□

Now, we are ready to state our first main result.

Theorem 2. *Let (X,d) be a complete metric space and let $T\colon X\to X$ be a (λ,a,b,c)-generalized nonexpansive mapping with $\lambda\in[0,1)$, $b>0$, $c>0$ and $b+c<1$. Then, T has an a.f.p.s.*

Proof. Let $m=\inf_{x\in X} d(x,Tx)$. On the contrary, assume that $m>0$. Let (x_n) be a sequence in X such that

$$m=\lim_{n\to\infty} d(x_n,Tx_n). \tag{2.5}$$

By the diagonal argument, we may assume all the following limits exist:

$$\lim_{n\to\infty} d(T^i x_n, T^j x_n), \quad \forall\, i,j\in\mathbb{N}\cup\{0\}. \tag{2.6}$$

Replacing x by Tx_n and y by T^2x_n in Definition 1, for each $n\in\mathbb{N}$ we obtain

$$d(T^2x_n,T^3x_n)\leq ad(Tx_n,T^2x_n)+bd(Tx_n,T^2x_n)$$

$$+bd(T^2x_n,T^3x_n)+cd(Tx_n,T^3x_n).$$

Thus

$$\begin{aligned} m &= \lim_{n\to\infty} d(T^2x_n,T^3x_n) \\ &\le a\lim_{n\to\infty} d(Tx_n,T^2x_n)+b\lim_{n\to\infty} d(Tx_n,T^2x_n) \\ &\quad +b\lim_{n\to\infty} d(T^2x_n,T^3x_n)+c\lim_{n\to\infty} d(Tx_n,T^3x_n). \end{aligned} \tag{2.7}$$

From (2.1) and (2.5), we obtain

$$m=\lim_{n\to\infty} d(T^kx_n,T^{k+1}x_n), \quad \text{for each } k\in\mathbb{N}. \tag{2.8}$$

Hence by using (2.8) and (2.7), we get

$$m\le \frac{1}{2}\lim_{n\to\infty} d(Tx_n,T^3x_n),$$

and so (note the conditions (2.5) and (2.6) are also satisfied by (Tx_n))

$$m\le \frac{1}{2}\lim_{n\to\infty} d(T^2x_n,T^4x_n).$$

Now we prove by induction that

$$m\le \frac{1}{k}\lim_{n\to\infty} d(Tx_n,T^{k+1}x_n) \quad \text{for each } k\ge 2. \tag{2.9}$$

Assume that (2.9) holds for some $k\ge 2$, then

$$m\le \frac{1}{k}\lim_{n\to\infty} d(T^2x_n,T^{k+2}x_n).$$

Therefore from (2.9) and for sufficiently large n, we have

$$\begin{aligned} \lambda d(Tx_n,T^2x_n) &\le \lim_{n\to\infty} d(Tx_n,T^2x_n)=m \\ &\le \frac{1}{k}\lim_{n\to\infty} d(Tx_n,T^{k+1}x_n) \\ &\le d(Tx_n,T^{k+1}x_n), \end{aligned}$$

and so, we may assume that

$$\lambda d(Tx_n,T^2x_n)\le d(Tx_n,T^{k+1}x_n) \ \text{ for each } n\in\mathbb{N}.$$

Replacing x by Tx_n and y by $T^{k+1}x_n$ in Definition (1), for each $n\in\mathbb{N}$ we obtain

$$\begin{aligned} d(T^2x_n,T^{k+2}x_n) &\le ad(Tx_n,T^{k+1}x_n) \\ &\quad +bd(Tx_n,T^2x_n)+bd(T^{k+1}x_n,T^{k+2}x_n) \\ &\quad +cd(Tx_n,T^{k+2}x_n)+cd(T^2x_n,T^{k+1}x_n). \end{aligned}$$

Now we claim that

$$a\lim_{n\to\infty} d(Tx_n,T^{k+1}x_n)\le akm. \tag{2.10}$$

In the case $a < 0$, the inequality follows from (2.9). In the case $a \geq 0$, (2.10) follows from the inequality

$$d(Tx_n, T^{k+1}x_n) \leq d(Tx_n, T^2x_n) + d(T^2x_n, T^3x_n) + \ldots + d(T^kx_n, T^{k+1}x_n).$$

Thus

$$\begin{aligned} km &\leq \lim_{n\to\infty} d(T^2x_n, T^{k+2}x_n) \\ &\leq a \lim_{n\to\infty} d(Tx_n, T^{k+1}x_n) \\ &\quad + b \lim_{n\to\infty} d(Tx_n, T^2x_n) + b \lim_{n\to\infty} d(T^{k+1}x_n, T^{k+2}x_n) \\ &\quad + c \lim_{n\to\infty} d(Tx_n, T^{k+2}x_n) + c \lim_{n\to\infty} d(T^2x_n, T^{k+1}x_n) \\ &\leq akm + 2bm + c \lim_{n\to\infty} d(Tx_n, T^{k+2}x_n) + c(k-1)m, \end{aligned}$$

and so

$$\begin{aligned} m &\leq \frac{c}{(c+2b)(k+1) - 4b} \lim_{n\to\infty} d(Tx_n, T^{k+2}x_n) \\ &\leq \frac{1}{k+1} \lim_{n\to\infty} d(Tx_n, T^{k+2}x_n), \end{aligned}$$

and so (2.9) holds.
Let

$$M = \max\left\{ \left(\frac{a+b+c}{b}\right) m, \left(\frac{2(b+c)}{1-c}\right) m, \left(\frac{b+c}{b}\right) m \right\}.$$

We show that for each $k \geq 2$,

$$\lim_{n\to\infty} d(Tx_n, T^{k+1}x_n) \leq M. \tag{2.11}$$

To prove the claim, note that by (2.1) and (2.9) and for each $k \geq 2$, there exists $N \in \mathbb{N}$ such that for $n \geq N$,

$$\begin{aligned} \lambda d(x_n, Tx_n) \leq \lim_{n\to\infty} d(x_n, Tx_n) = m &\leq \frac{1}{k} \lim_{n\to\infty} d(Tx_n, T^{k+1}x_n) \\ &\leq \frac{1}{k} \lim_{n\to\infty} d(x_n, T^kx_n) \leq d(x_n, T^kx_n). \end{aligned}$$

Thus for sufficiently large n, we obtain

$$\begin{aligned} d(Tx_n, T^{k+1}x_n) \leq{}& ad(x_n, T^kx_n) + b(d(x_n, Tx_n) + d(T^kx_n, T^{k+1}x_n)) \\ &+ c(d(x_n, T^{k+1}x_n) + d(Tx_n, T^kx_n)). \end{aligned} \tag{2.12}$$

Assume first that $a \geq 0$. Then by (2.12),

$$\begin{aligned} d(Tx_n, T^{k+1}x_n) \leq{}& a(d(x_n, Tx_n) + d(Tx_n, T^{k+1}x_n) + d(T^kx_n, T^{k+1}x_n)) \\ &+ b(d(x_n, Tx_n) + d(T^kx_n, T^{k+1}x_n)) \\ &+ c(d(x_n, Tx_n) + d(Tx_n, T^{k+1}x_n)) \end{aligned}$$

$$+c(d(Tx_n,T^{k+1}x_n)+d(T^kx_n+T^{k+1}x_n)),$$

and so by letting $n\to\infty$, we get

$$(1-a-2c)\lim_{n\to\infty}d(Tx_n,T^{k+1}x_n)\le 2(a+b+c)m.$$

Hence for each $k\ge 2$

$$\lim_{n\to\infty}d(Tx_n,T^{k+1}x_n)\le\left(\frac{a+b+c}{b}\right)m\le M.$$

If $a<0$ and $a+c\le 0$, then by (2.12),

$$\begin{aligned}d(Tx_n,T^{k+1}x_n)&\le ad(x_n,T^kx_n)+b(d(x_n,Tx_n)+d(T^kx_n,T^{k+1}x_n))\\&\quad+c(d(x_n,Tx_n)+d(Tx_n,T^{k+1}x_n))+c(d(x_n,Tx_n)+d(x_n+T^kx_n)).\end{aligned}$$

Thus by letting $n\to\infty$, we obtain

$$(1-c)\lim_{n\to\infty}d(Tx_n,T^{k+1}x_n)\le(a+c)\lim_{n\to\infty}d(x_n,T^kx_n)+2(b+c)m,$$

and so (note that $c<1$ and $a+c\le 0$)

$$\lim_{n\to\infty}d(Tx_n,T^{k+1}x_n)\le\frac{2(b+c)}{1-c}m\le M.$$

Now if $a<0$ and $a+c>0$, then thanks to (2.12),

$$\begin{aligned}d(Tx_n,T^{k+1}x_n)&\le ad(x_n,T^kx_n)+b(d(x_n,Tx_n)+d(T^kx_n,T^{k+1}x_n))\\&\quad+(a+c)d(x_n,T^{k+1}x_n)-ad(x_n,T^{k+1}x_n)+cd(Tx_n,T^kx_n)\\&\le(-a)(d(x_n,T^{k+1}x_n)-d(x_n,T^kx_n))\\&\quad+b(d(x_n,Tx_n)+d(T^kx_n,T^{k+1}x_n))\\&\quad+(a+c)(d(x_n,Tx_n)+d(Tx_n,T^{k+1}x_n))\\&\quad+c(d(Tx_n,T^{k+1}x_n)+d(T^kx_n,T^{k+1}x_n))\\&\le(-a)d(T^kx_n,T^{k+1}x_n)\\&\quad+b(d(x_n,Tx_n)+d(T^kx_n,T^{k+1}x_n))\\&\quad+(a+c)(d(x_n,Tx_n)+d(Tx_n,T^{k+1}x_n))\\&\quad+c(d(Tx_n,T^{k+1}x_n)+d(T^kx_n,T^{k+1}x_n)).\end{aligned}$$

Thus by letting $n\to\infty$,

$$(1-a-2c)\lim_{n\to\infty}d(Tx_n,T^{k+1}x_n)\le 2(b+c)m.$$

Hence

$$\lim_{n\to\infty}d(Tx_n,T^{k+1}x_n)\le(\frac{b+c}{b})m\le M.$$

Therefore (2.11) holds. Now from (2.9) and (2.11), we obtain

$$m \leq \frac{1}{k} \lim_{n\to\infty} d(Tx_n, T^{k+1}x_n) \leq \frac{1}{k}M, \quad \text{for each } k \geq 2,$$

and so by letting $k \to \infty$, we get $m = 0$, a contradiction. So, T has an a.f.p.s. □

Now we state our second main result.

Theorem 3. *Let (X,d) be complete metric space and let $T : X \to X$ be $(\frac{1}{2}, a, b, c)$-generalized nonexpansive mapping with $b > 0$, $c > 0$ and $b + c < 1$. Then, T has a unique fixed point.*

Proof. For each $n \in \mathbb{N}$ we set $K_n := \{x \in X : d(x,Tx) \leq \frac{1}{n}\}$. By Theorem 2, $\inf_{x\in X} d(x,Tx) = 0$, therefore for each $n \in \mathbb{N}$, $K_n \neq \varnothing$. We show that diam $(K_n) \to 0$ as $n \to \infty$, where for each $n \in \mathbb{N}$, diam $(K_n) := \sup_{x,y\in K_n} d(x,y)$. Let $n \in \mathbb{N}$ and $x, y \in K_n$. We have either $d(x,y) \leq \frac{1}{n}$ or $d(x,y) > \frac{1}{n}$. If $d(x,y) > \frac{1}{n}$, then $\frac{1}{2}d(x,Tx) \leq \frac{1}{n} < d(x,y)$. Hence

$$d(Tx,Ty) \leq ad(x,y) + b(d(x,Tx) + d(y,Ty)) + c(d(x,Ty) + d(y,Tx)).$$

Then

$$\begin{aligned}
d(x,y) &\leq d(x,Tx) + d(Tx,Ty) + d(y,Ty) \leq \frac{2}{n} + d(Tx,Ty) \\
&\leq \frac{2}{n} + ad(x,y) + b(d(x,Tx) + d(y,Ty)) + c(d(x,Ty) + d(y,Tx)) \\
&\leq \frac{2}{n} + \frac{2b}{n} + ad(x,y) + c(d(x,y) + d(y,Ty) + d(x,y) + d(x,Tx)) \\
&\leq \frac{2+2b+2c}{n} + (a+2c)d(x,y) \\
&\leq \frac{2+2b+2c}{n} + (1-2b)d(x,y).
\end{aligned}$$

So

$$d(x,y) \leq \frac{1+b+c}{nb}.$$

Then for each $x, y \in K_n$, we have

$$d(x,y) \leq \max\left\{\frac{1}{n}, \frac{1+b+c}{nb}\right\}.$$

Thus diam $(K_n) \to 0$ when $n \to \infty$. Since for each $n \in \mathbb{N}$, diam $(\overline{K_n}) = diam(K_n)$ and (K_n) is a decreasing sequence of nonempty sets, so $(\overline{K_n})$ is a decreasing sequence of nonempty closed sets such that diam $(\overline{K_n}) \to 0$ as $n \to \infty$. Therefore $\bigcap_{n\in\mathbb{N}} \overline{K_n}$ is singleton, say, $\bigcap_{n\in\mathbb{N}} \overline{K_n} = \{x_0\}$. Then there exists a sequence $(x_n) \subseteq X$ with $x_n \in K_n$ for each $n \in \mathbb{N}$, such that $x_n \to x_0$. Now by Proposition 1 there exists $\mu \geq 1$, such that for each $n \in \mathbb{N}$,

$$d(x_n, Tx_0) \leq \mu d(x_n, Tx_n) + d(x_n, x_0).$$

Hence as $n \to \infty$ we obtain $d(x_0, Tx_0) = 0$, that is, x_0 is a fixed point for T.
To prove the uniqueness, suppose that $x, y \in X$ are two fixed point of T. Since $\frac{1}{2}d(x,Tx) = 0 \leq d(x,y)$ then (note that $b > 0$)

$$\begin{aligned} d(x,y) = d(Tx,Ty) &\leq ad(x,y) + b(d(x,Tx) + d(y,Ty)) + c(d(x,Ty) + d(y,Tx)) \\ &\leq ad(x,y) + c(d(x,y) + d(y,Ty) + d(x,y) + d(x,Tx)) \\ &= (a+2c)d(x,y) \leq (1-2b)d(x,y), \end{aligned}$$

and so $x = y$. □

In the following example, we define a metric space (X,d) and a mapping $T\colon X \to X$ which satisfies the assumptions of Theorem 3, but it is easy to see that T is not in any classes of generalized nonexpansive mappings which are listed in Example 1. This example also shows that Theorem 3 is a real generalization of Theorem 2.1 of [6].

Example 2. Let (X,d) be a metric space such that $X := \{1,2,3,4,5\}$ and $d(1,2) = d(4,5) = d(3,4) = 1$, $d(1,5) = d(2,3) = d(1,3) = d(2,5) = 1.7$, $d(1,4) = d(2,4) = 2.2$ and $d(3,5) = 1.8$. Let $T\colon X \to X$ define by $T1 = 5$, $T2 = 3$ and $T3 = T4 = T5 = 4$. It is straightforward to show that T is $(\frac{1}{2}, -0.2, 0.3, 0.3)$-generalized nonexpansive mapping, and so by Theorem 3 has a unique fixed point. But we show that T does not satisfy the assumptions of Theorem 2.1 of [6]. On the contrary, assume that T is $(\frac{1}{2}, a, b, c)$-generalized nonexpansive mapping with $a \geq 0$, $b > 0$ and $c > 0$. Then, for $x = 1$ and $y = 2$ we have $\frac{1}{2}d(1,T1) \leq d(1,2)$. Then (note that $a \geq 0$)

$$\begin{aligned} 1.8 = d(5,3) &= d(T1,T2) \\ &\leq ad(1,2) + b(d(1,T1) + d(2,T2)) + c(d(1,T2) + d(2,T1)) \\ &= a + (2b+2c)1.7 \\ &\leq (a+2b+2c)1.7 = 1.7, \end{aligned}$$

a contradiction.

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ON THE NEW FRACTIONAL OPERATORS GENERATING MODIFIED GAMMA AND BETA FUNCTIONS

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Abstract. In this paper, we introduce three new fractional operators, MRL fractional integral, MRL fractional derivative and MC fractional derivative operators, which including generalized M-series in their kernels and give some of their fundamental properties. Then we apply Laplace, Mellin and beta integral transformations to the new fractional operators and obtain conclusions involving classical gamma and beta and modified gamma and beta functions. As examples, we also obtain similar conclusions by applying new fractional operators to the functions z^{λ} and $(1-az)^{-\lambda}$. Furthermore, we present the relationships of the new fractional operators with other fractional operators in the literature. Finally, we compare the behavior of the function z^2 in classical Riemann-Liouville and Caputo fractional operators and MRL and MC fractional operators for orders $\varepsilon = 0.2, 0.4, 0.6, 0.8$.

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1. Introduction

It is known how to take the integer integral and derivative of a function such as the first, second, third, fourth. Is it possible to take the fractional integral and derivative of the same function from orders 0.2, 0.4, 0.6, 0.8 etc? First, in the correspondence between Leibniz and L'Hospital in 1695, "Can integer derivatives be extended to fractional derivatives?" [11, 13] the question arose and caught the attention of many famous mathematicians (Riemann, Grünwald, Letnikov, Liouville, Caputo, Euler, Abel, Fourier, Kobel, Erdelyi, Hadamard, Riesz and Laplace).

The fractional integral and derivative has been one of the interesting topics in mathematics for a long time, and this interest is increasing day by day. In the literature, there are many studies by researchers on fractional operators (see for example [1–4, 7–10, 12, 14–19, 22, 23] and reference therein).

In this paper, we define three new fractional operators with generalized M-series in their kernels. Then we apply Laplace, Mellin and beta integral transformations to the new fractional operators and give some examples. Furthermore, we present

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the relationships of the new fractional operators with other fractional operators in the literature. Finally, we examine the behavior of the function z^2 in classical and new fractional operators, see Figure 1 and Figure 2.

2. Preliminaries

In this section, we give relevant material which will be used throughout the paper. The gamma function [5] for $\Re(x) > 0$ is defined by

$$\Gamma(x) = \int_0^\infty \Delta^{x-1} \exp(-\Delta) d\Delta.$$

The beta function [5] for $\Re(x) > 0$ and $\Re(y) > 0$ is given by

$$B(x,y) = \int_0^1 \Delta^{x-1}(1-\Delta)^{y-1} d\Delta.$$

The Pochhammer symbol [5] for $\Re(\lambda) > -n$, $n \in \mathbb{N}_0$ and $\lambda \neq 0, -1, -2, \ldots$ is defined by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}, \quad (\lambda)_0 \equiv 1.$$

The generalized M-series [20] for $\Re(\alpha) > 0$, $\xi_1, \ldots, \xi_p, \mu_1, \ldots, \mu_q \neq 0, -1, -2, \ldots$ is given by

$${}_p^\alpha M_q^\beta(\xi_1, \ldots, \xi_p; \mu_1, \ldots, \mu_q; z) = \sum_{n=0}^\infty \frac{(\xi_1)_n \ldots (\xi_p)_n}{(\mu_1)_n \ldots (\mu_q)_n} \frac{z^n}{\Gamma(\alpha n + \beta)}.$$

The modified gamma function [6] for $\Re(\alpha) > 0$, $\Re(\rho) > 0$, $\Re(x) > 0$, $\xi_1, \ldots, \xi_p$, $\mu_1, \ldots, \mu_q \neq 0, -1, -2, \ldots$ is given by

$$\begin{aligned} {}^M\Gamma_{p,q}^{(\alpha,\beta)}(x;\rho) &= {}^M\Gamma_{p,q}^{(\alpha,\beta)}(\xi_1, \ldots, \xi_p; \mu_1, \ldots, \mu_q; x; \rho) \\ &= \int_0^\infty \Delta^{x-1}\, {}_p^\alpha M_q^\beta\left(\xi_1, \ldots, \xi_p; \mu_1, \ldots, \mu_q; -\Delta - \frac{\rho}{\Delta}\right) d\Delta. \end{aligned}$$

The modified beta function [6] for $\Re(\alpha) > 0$, $\Re(\rho) > 0$, $\Re(x) > 0$, $\Re(y) > 0$, $\xi_1, \ldots, \xi_p$, $\mu_1, \ldots, \mu_q \neq 0, -1, -2, \ldots$ is defined by

$$\begin{aligned} {}^M B_{p,q}^{(\alpha,\beta)}(x,y;\rho) &= {}^M B_{p,q}^{(\alpha,\beta)}(\xi_1, \ldots, \xi_p; \mu_1, \ldots, \mu_q; x, y; \rho) \\ &= \int_0^1 \Delta^{x-1}(1-\Delta)^{y-1}\, {}_p^\alpha M_q^\beta\left(\xi_1, \ldots, \xi_p; \mu_1, \ldots, \mu_q; \frac{-\rho}{\Delta(1-\Delta)}\right) d\Delta. \end{aligned}$$

The Mellin transform [11] for $\rho \in \mathbb{C}$ is defined by

$$\mathfrak{M}\{f(w);\rho\} = \int_0^\infty w^{\rho-1} f(w) dw.$$

The Laplace transform [11] for $\Re(s) > 0$ is given by

$$\mathfrak{L}\{f(w);s\} = \int_0^\infty \exp(-sw) f(w) dw.$$

The beta transform [21] for $\Re(x) > 0$ and $\Re(y) > 0$ is defined by

$$\mathfrak{B}\{f(w);x,y\} = \int_0^1 w^{x-1}(1-w)^{y-1} f(w)dw.$$

The Riemann-Liouville fractional integral [13] for $\Re(\varepsilon) > 0$ is given by

$$I_z^{\varepsilon}\{f(z)\} = \frac{1}{\Gamma(\varepsilon)} \int_0^z (z-\Delta)^{\varepsilon-1} f(\Delta)d\Delta. \tag{2.1}$$

The Riemann-Liouville fractional derivative [13] for $\Re(\varepsilon) > 0$, $m-1 < \Re(\varepsilon) < m$, $m \in \mathbb{N}$ is defined by

$$D_z^{\varepsilon}\{f(z)\} = \frac{1}{\Gamma(m-\varepsilon)} \frac{d^m}{dz^m} \int_0^z (z-\Delta)^{m-\varepsilon-1} f(\Delta)d\Delta. \tag{2.2}$$

The Caputo fractional derivative [13] for $\Re(\varepsilon) > 0$, $m-1 < \Re(\varepsilon) < m$, $m \in \mathbb{N}$ is given by

$${}^{c}D_z^{\varepsilon}\{f(z)\} = \frac{1}{\Gamma(m-\varepsilon)} \int_0^z (z-\Delta)^{m-\varepsilon-1} f^{(m)}(\Delta)d\Delta. \tag{2.3}$$

3. New fractional operators

In this section, we introduce three new fractional operators in the classical Riemann-Liouville fractional integral, Riemann-Liouville fractional derivative and Caputo fractional derivative operators sensitivity and give some of their basic properties. Also, we call new Riemann-Liouville fractional integral, Riemann-Liouville fractional derivative and Caputo fractional derivative operators as MRL fractional integral, MRL fractional derivative and MC fractional derivative operators, respectively.

Definition 3.1. New Riemann-Liouville fractional integral operator for $\Re(\alpha) > 0$, $\Re(\varepsilon) > 0$, $\xi_1, \ldots, \xi_p, \mu_1, \ldots, \mu_q \neq 0, -1, -2, \ldots$ is defined as:

$$\begin{aligned}\widehat{I}_z^{\varepsilon}\{f(z);\alpha,\beta;w\} &= \widehat{I}_z^{\varepsilon}\left\{f(z);\xi_1,\ldots,\xi_p;\mu_1,\ldots,\mu_q;\alpha,\beta;w\right\} \\ &:= \frac{1}{\Gamma(\varepsilon)} \int_0^z (z-\Delta)^{\varepsilon-1} f(\Delta)\, {}_p^{\alpha}M_q^{\beta}\left(\xi_1,\ldots,\xi_p;\mu_1,\ldots,\mu_q;\frac{-wz^2}{\Delta(z-\Delta)}\right) d\Delta.\end{aligned}$$

Definition 3.2. New Riemann-Liouville fractional derivative operator for $\Re(\alpha) > 0$, $\Re(\varepsilon) > 0$, $m-1 < \Re(\varepsilon) < m$, $m \in \mathbb{N}$, $\xi_1, \ldots, \xi_p, \mu_1, \ldots, \mu_q \neq 0, -1, -2, \ldots$ is defined as:

$$\begin{aligned}&\widehat{D}_z^{\varepsilon}\{f(z);\alpha,\beta;w\} = \widehat{D}_z^{\varepsilon}\left\{f(z);\xi_1,\ldots,\xi_p;\mu_1,\ldots,\mu_q;\alpha,\beta;w\right\} \\ &:= \frac{1}{\Gamma(m-\varepsilon)} \frac{d^m}{dz^m} \int_0^z (z-\Delta)^{m-\varepsilon-1} f(\Delta)\, {}_p^{\alpha}M_q^{\beta}\left(\xi_1,\ldots,\xi_p;\mu_1,\ldots,\mu_q;\frac{-wz^2}{\Delta(z-\Delta)}\right) d\Delta.\end{aligned}$$

Definition 3.3. New Caputo fractional derivative operator for $\Re(\alpha)>0$, $\Re(\varepsilon)>0$, $m-1<\Re(\varepsilon)<m$, $m\in\mathbb{N}$, $\xi_1,\ldots,\xi_p,\mu_1,\ldots,\mu_q\neq 0,-1,-2,\ldots$ is defined as:

$$
\begin{aligned}
&{}^{c}\widehat{D}_z^{\varepsilon}\left\{f(z);\alpha,\beta;w\right\}={}^{c}\widehat{D}_z^{\varepsilon}\left\{f(z);\xi_1,\ldots,\xi_p;\mu_1,\ldots,\mu_q;\alpha,\beta;w\right\}\\
&:=\frac{1}{\Gamma(m-\varepsilon)}\int_0^z(z-\Delta)^{m-\varepsilon-1}f^{(m)}(\Delta)\,{}_p^{\alpha}M_q^{\beta}\left(\xi_1,\ldots,\xi_p;\mu_1,\ldots,\mu_q;\frac{-wz^2}{\Delta(z-\Delta)}\right)d\Delta.
\end{aligned}
$$

Remark 3.1. If we put $w=0$ and $\beta=1$ to the new fractional operators, we get the classical fractional operators (2.1), (2.2) and (2.3).

Theorem 3.1. *The MRL fractional integral for* $\lambda_1,\lambda_2\in\mathbb{R}$, $\Re(\alpha)>0$, $\Re(\varepsilon)>0$, $\xi_1,\ldots,\xi_p,\mu_1,\ldots,\mu_q\neq 0,-1,-2,\ldots$ *is linear:*

$$
\widehat{I}_z^{\varepsilon}\left\{\lambda_1 f(z)+\lambda_2 g(z);\alpha,\beta;w\right\}=\lambda_1\widehat{I}_z^{\varepsilon}\left\{f(z);\alpha,\beta;w\right\}+\lambda_2\widehat{I}_z^{\varepsilon}\left\{g(z);\alpha,\beta;w\right\}.
$$

Proof. By using MRL fractional integral operator, we have

$$
\begin{aligned}
&\widehat{I}_z^{\varepsilon}\left\{\lambda_1 f(z)+\lambda_2 g(z);\alpha,\beta;w\right\}\\
&=\frac{\lambda_1}{\Gamma(\varepsilon)}\int_0^z(z-\Delta)^{\varepsilon-1}f(\Delta)\,{}_p^{\alpha}M_q^{\beta}\left(\xi_1,\ldots,\xi_p;\mu_1,\ldots,\mu_q;\frac{-wz^2}{\Delta(z-\Delta)}\right)d\Delta\\
&\quad+\frac{\lambda_2}{\Gamma(\varepsilon)}\int_0^z(z-\Delta)^{\varepsilon-1}g(\Delta)\,{}_p^{\alpha}M_q^{\beta}\left(\xi_1,\ldots,\xi_p;\mu_1,\ldots,\mu_q;\frac{-wz^2}{\Delta(z-\Delta)}\right)d\Delta\\
&=\lambda_1\widehat{I}_z^{\varepsilon}\left\{f(z);\alpha,\beta;w\right\}+\lambda_2\widehat{I}_z^{\varepsilon}\left\{g(z);\alpha,\beta;w\right\}.
\end{aligned}
$$

□

Theorem 3.2. *The MRL fractional derivative for* $\lambda_1,\lambda_2\in\mathbb{R}$, $\Re(\alpha)>0$, $\Re(\varepsilon)>0$, $m-1<\Re(\varepsilon)<m$, $m\in\mathbb{N}$, $\xi_1,\ldots,\xi_p,\mu_1,\ldots,\mu_q\neq 0,-1,-2,\ldots$ *is linear:*

$$
\widehat{D}_z^{\varepsilon}\left\{\lambda_1 f(z)+\lambda_2 g(z);\alpha,\beta;w\right\}=\lambda_1\widehat{D}_z^{\varepsilon}\left\{f(z);\alpha,\beta;w\right\}+\lambda_2\widehat{D}_z^{\varepsilon}\left\{g(z);\alpha,\beta;w\right\}.
$$

Proof. By using MRL fractional derivative operator, we have

$$
\begin{aligned}
&\widehat{D}_z^{\varepsilon}\left\{\lambda_1 f(z)+\lambda_2 g(z);\alpha,\beta;w\right\}\\
&=\frac{\lambda_1}{\Gamma(m-\varepsilon)}\frac{d^m}{dz^m}\int_0^z(z-\Delta)^{m-\varepsilon-1}f(\Delta)\,{}_p^{\alpha}M_q^{\beta}\left(\xi_1,\ldots,\xi_p;\mu_1,\ldots,\mu_q;\frac{-wz^2}{\Delta(z-\Delta)}\right)d\Delta\\
&\quad+\frac{\lambda_2}{\Gamma(m-\varepsilon)}\frac{d^m}{dz^m}\int_0^z(z-\Delta)^{m-\varepsilon-1}g(\Delta)\,{}_p^{\alpha}M_q^{\beta}\left(\xi_1,\ldots,\xi_p;\mu_1,\ldots,\mu_q;\frac{-wz^2}{\Delta(z-\Delta)}\right)d\Delta\\
&=\lambda_1\widehat{D}_z^{\varepsilon}\left\{f(z);\alpha,\beta;w\right\}+\lambda_2\widehat{D}_z^{\varepsilon}\left\{g(z);\alpha,\beta;w\right\}.
\end{aligned}
$$

□

Theorem 3.3. *The MC fractional derivative for* $\lambda_1,\lambda_2\in\mathbb{R}$, $\Re(\alpha)>0$, $\Re(\varepsilon)>0$, $m-1<\Re(\varepsilon)<m$, $m\in\mathbb{N}$, $\xi_1,\ldots,\xi_p,\mu_1,\ldots,\mu_q\neq 0,-1,-2,\ldots$ *is linear:*

$$
{}^{c}\widehat{D}_z^{\varepsilon}\left\{\lambda_1 f(z)+\lambda_2 g(z);\alpha,\beta;w\right\}=\lambda_1{}^{c}\widehat{D}_z^{\varepsilon}\left\{f(z);\alpha,\beta;w\right\}+\lambda_2{}^{c}\widehat{D}_z^{\varepsilon}\left\{g(z);\alpha,\beta;w\right\}.
$$

Proof. By using MC fractional derivative operator, we have

$$
{}^{c}\widehat{D}_z^{\varepsilon}\left\{\lambda_1 f(z)+\lambda_2 g(z);\alpha,\beta;w\right\}
$$

$$= \frac{\lambda_1}{\Gamma(m-\varepsilon)} \int_0^z (z-\Delta)^{m-\varepsilon-1} f^{(m)}(\Delta)\, {}_p^{\alpha}M_q^{\beta}\left(\xi_1,\ldots,\xi_p;\mu_1,\ldots,\mu_q;\frac{-wz^2}{\Delta(z-\Delta)}\right) d\Delta$$
$$+ \frac{\lambda_2}{\Gamma(m-\varepsilon)} \int_0^z (z-\Delta)^{m-\varepsilon-1} g^{(m)}(\Delta)\, {}_p^{\alpha}M_q^{\beta}\left(\xi_1,\ldots,\xi_p;\mu_1,\ldots,\mu_q;\frac{-wz^2}{\Delta(z-\Delta)}\right) d\Delta$$
$$= \lambda_1 {}^c\widehat{D}_z^{\varepsilon}\{f(z);\alpha,\beta;w\} + \lambda_2 {}^c\widehat{D}_z^{\varepsilon}\{g(z);\alpha,\beta;w\}. \qquad \square$$

Theorem 3.4. *Let $f(z) = \sum_{n=0}^{\infty} a^n z^n$ analytic function for $|z| < r$. Then, the MRL fractional integral for $\Re(\alpha) > 0$, $\Re(\varepsilon) > 0$, $\xi_1,\ldots,\xi_p,\mu_1,\ldots,\mu_q \neq 0,-1,-2,\ldots$ is provided as:*

$$\widehat{I}_z^{\varepsilon}\{f(z);\alpha,\beta;w\} = \sum_{n=0}^{\infty} a^n\, \widehat{I}_z^{\varepsilon}\{z^n;\alpha,\beta;w\}.$$

Proof. By using MRL fractional integral operator, we have

$$\widehat{I}_z^{\varepsilon}\{f(z);\alpha,\beta;w\} = \sum_{n=0}^{\infty} \frac{a^n}{\Gamma(\varepsilon)} \int_0^z (z-\Delta)^{\varepsilon-1} \Delta^n\, {}_p^{\alpha}M_q^{\beta}\left(\xi_1,\ldots,\xi_p;\mu_1,\ldots,\mu_q;\frac{-wz^2}{\Delta(z-\Delta)}\right) d\Delta$$
$$= \sum_{n=0}^{\infty} a^n\, \widehat{I}_z^{\varepsilon}\{z^n;\alpha,\beta;w\}. \qquad \square$$

Theorem 3.5. *Let $f(z) = \sum_{n=0}^{\infty} a^n z^n$ analytic function for $|z| < r$. Then, the MRL fractional derivative for $\Re(\alpha) > 0$, $\Re(\varepsilon) > 0$, $m-1 < \Re(\varepsilon) < m$, $m \in \mathbb{N}$, $\xi_1,\ldots,\xi_p$, $\mu_1,\ldots,\mu_q \neq 0,-1,-2,\ldots$ is provided as:*

$$\widehat{D}_z^{\varepsilon}\{f(z);\alpha,\beta;w\} = \sum_{n=0}^{\infty} a^n\, \widehat{D}_z^{\varepsilon}\{z^n;\alpha,\beta;w\}.$$

Proof. By using MRL fractional derivative operator, we have

$$\widehat{D}_z^{\varepsilon}\{f(z);\alpha,\beta;w\} = \sum_{n=0}^{\infty} \frac{a^n}{\Gamma(m-\varepsilon)} \frac{d^m}{dz^m} \int_0^z (z-\Delta)^{m-\varepsilon-1} \Delta^n$$
$$\times\, {}_p^{\alpha}M_q^{\beta}\left(\xi_1,\ldots,\xi_p;\mu_1,\ldots,\mu_q;\frac{-wz^2}{\Delta(z-\Delta)}\right) d\Delta$$
$$= \sum_{n=0}^{\infty} a^n\, \widehat{D}_z^{\varepsilon}\{z^n;\alpha,\beta;w\}. \qquad \square$$

Theorem 3.6. *Let $f(z) = \sum_{n=0}^{\infty} a^n z^n$ analytic function for $|z| < r$. Then, the MC fractional derivative for $\Re(\alpha) > 0$, $\Re(\varepsilon) > 0$, $m-1 < \Re(\varepsilon) < m$, $m \in \mathbb{N}$, $\xi_1,\ldots,\xi_p$, $\mu_1,\ldots,\mu_q \neq 0,-1,-2,\ldots$ is provided as:*

$${}^c\widehat{D}_z^{\varepsilon}\{f(z);\alpha,\beta;w\} = \sum_{n=0}^{\infty} a^n\, {}^c\widehat{D}_z^{\varepsilon}\{z^n;\alpha,\beta;w\}.$$

Proof. By using MC fractional derivative operator, we have

$$\begin{aligned}{}^{c}\widehat{D}_{z}^{\varepsilon}\left\{f(z);\alpha,\beta;w\right\} &= \sum_{n=0}^{\infty}\frac{a^{n}}{\Gamma(m-\varepsilon)}\int_{0}^{z}(z-\Delta)^{m-\varepsilon-1} \\ &\quad\times {}_{p}^{\alpha}M_{q}^{\beta}\left(\xi_{1},\ldots,\xi_{p};\mu_{1},\ldots,\mu_{q};\frac{-wz^{2}}{\Delta(z-\Delta)}\right)\frac{d^{m}}{d\Delta^{m}}\left(\Delta^{n}\right)d\Delta \\ &= \sum_{n=0}^{\infty}a^{n}\,{}^{c}\widehat{D}_{z}^{\varepsilon}\left\{z^{n};\alpha,\beta;w\right\}. \end{aligned}$$

□

4. Application of integral transforms

In this section, we apply Laplace, Mellin and beta transforms to the MRL fractional integral, MRL fractional derivative and MC fractional derivative operators.

4.1. *Application of Laplace transform*

Theorem 4.1. *The Laplace transform of MRL fractional integral for $\Re(\alpha)>0$, $\Re(s)>0$, $\Re(\varepsilon)>0$, $\xi_{1},\ldots,\xi_{p},\mu_{1},\ldots,\mu_{q}\neq 0,-1,-2,\ldots$ is obtained as:*

$$\mathfrak{L}\left\{\widehat{I}_{z}^{\varepsilon}\left\{f(z);\alpha,\beta;w\right\};s\right\}=\frac{1}{s}\,\widehat{I}_{z}^{\varepsilon}\left\{f(z);\xi_{1},\ldots,\xi_{p},1;\mu_{1},\ldots,\mu_{q};\alpha,\beta;\frac{1}{s}\right\}.$$

Proof. By using MRL fractional integral operator and Laplace transform, we have

$$\begin{aligned}&\mathfrak{L}\left\{\widehat{I}_{z}^{\varepsilon}\left\{f(z);\alpha,\beta;w\right\};s\right\} \\ &=\frac{1}{s}\frac{1}{\Gamma(\varepsilon)}\int_{0}^{z}(z-\Delta)^{\varepsilon-1}f(\Delta)\,{}_{p+1}^{\alpha}M_{q}^{\beta}\left(\xi_{1},\ldots,\xi_{p},1;\mu_{1},\ldots,\mu_{q};\frac{-\left(s^{-1}\right)z^{2}}{\Delta(z-\Delta)}\right)d\Delta \\ &=\frac{1}{s}\,\widehat{I}_{z}^{\varepsilon}\left\{f(z);\xi_{1},\ldots,\xi_{p},1;\mu_{1},\ldots,\mu_{q};\alpha,\beta;\frac{1}{s}\right\}. \end{aligned}$$

□

Corollary 4.1. *The Laplace transform of the MRL fractional integral operator yields the following result in relation to the modified beta function for $f(z)=z^{\lambda}$:*

$$\mathfrak{L}\left\{\widehat{I}_{z}^{\varepsilon}\left\{z^{\lambda};\alpha,\beta;w\right\};s\right\}=\frac{1}{s}\frac{z^{\varepsilon+\lambda}}{\Gamma(\varepsilon)}\,{}^{M}B_{p+1,q}^{(\alpha,\beta)}\left(\xi_{1},\ldots,\xi_{p},1;\mu_{1},\ldots,\mu_{q};\lambda+1,\varepsilon;\frac{1}{s}\right).$$

Theorem 4.2. *The Laplace transform of MRL fractional derivative for $\Re(\alpha)>0$, $\Re(s)>0$, $\Re(\varepsilon)>0$, $m-1<\Re(\varepsilon)<m$, $m\in\mathbb{N}$, $\xi_{1},\ldots,\xi_{p},\mu_{1},\ldots,\mu_{q}\neq 0,-1,-2,\ldots$ is obtained as:*

$$\mathfrak{L}\left\{\widehat{D}_{z}^{\varepsilon}\left\{f(z);\alpha,\beta;w\right\};s\right\}=\frac{1}{s}\widehat{D}_{z}^{\varepsilon}\left\{f(z);\xi_{1},\ldots,\xi_{p},1;\mu_{1},\ldots,\mu_{q};\alpha,\beta;\frac{1}{s}\right\}.$$

Proof. By using the MRL fractional derivative operator and the Laplace transform, the desired result can be easily achieved with similar calculations. □

Corollary 4.2. *The Laplace transform of the MRL fractional derivative operator yields the following result in relation to the modified beta function for* $f(z) = z^{\lambda}$:

$$\mathfrak{L}\left\{\widehat{D}_z^{\varepsilon}\left\{z^{\lambda};\alpha,\beta;w\right\};s\right\} = \frac{1}{s}\frac{\Gamma(m+\lambda+1-\varepsilon)z^{\lambda-\varepsilon}}{\Gamma(m-\varepsilon)\Gamma(\lambda+1-\varepsilon)} \times {}^{M}B_{p+1,q}^{(\alpha,\beta)}\left(\xi_1,\ldots,\xi_p,1;\mu_1,\ldots,\mu_q;\lambda+1,m-\varepsilon;\frac{1}{s}\right).$$

Theorem 4.3. *The Laplace transform of MC fractional derivative for* $\Re(\alpha) > 0$, $\Re(s) > 0$, $\Re(\varepsilon) > 0$, $m-1 < \Re(\varepsilon) < m$, $m \in \mathbb{N}$, $\xi_1,\ldots,\xi_p,\mu_1,\ldots,\mu_q \neq 0,-1,-2,\ldots$ *is obtained as:*

$$\mathfrak{L}\left\{{}^{c}\widehat{D}_z^{\varepsilon}\left\{f(z);\alpha,\beta;w\right\};s\right\} = \frac{1}{s}{}^{c}\widehat{D}_z^{\varepsilon}\left\{f(z);\xi_1,\ldots,\xi_p,1;\mu_1,\ldots,\mu_q;\alpha,\beta;\frac{1}{s}\right\}.$$

Proof. By using the MC fractional derivative operator and the Laplace transform, the desired result can be easily achieved with similar calculations. □

Corollary 4.3. *The Laplace transform of the MC fractional derivative operator yields the following result in relation to the modified beta function for* $f(z) = z^{\lambda}$:

$$\mathfrak{L}\left\{{}^{c}\widehat{D}_z^{\varepsilon}\left\{z^{\lambda};\alpha,\beta;w\right\};s\right\} = \frac{1}{s}\frac{\Gamma(\lambda+1)z^{\lambda-\varepsilon}}{\Gamma(m-\varepsilon)\Gamma(\lambda+1-m)} \times {}^{M}B_{p+1,q}^{(\alpha,\beta)}\left(\xi_1,\ldots,\xi_p,1;\mu_1,\ldots,\mu_q;\lambda+1-m,m-\varepsilon;\frac{1}{s}\right).$$

4.2. *Application of Mellin transform*

Theorem 4.4. *The Mellin transform of MRL fractional integral for* $\Re(\alpha) > 0$, $\Re(\rho) > 0$, $\Re(\varepsilon) > 0$, $\Re(\varepsilon+\rho) > 0$, $\xi_1,\ldots,\xi_p,\mu_1,\ldots,\mu_q \neq 0,-1,-2,\ldots$ *is obtained as:*

$$\mathfrak{M}\left\{\widehat{I}_z^{\varepsilon}\left\{f(z);\alpha,\beta;w\right\};\rho\right\} = \frac{\Gamma(\varepsilon+\rho)}{\Gamma(\varepsilon)}\frac{{}^{M}\Gamma_{p,q}^{(\alpha,\beta)}(\rho;0)}{z^{2\rho}}\, I_z^{\varepsilon+\rho}\left\{z^{\rho}f(z)\right\}.$$

Proof. By using MRL fractional integral operator and Mellin transform, we have

$$\mathfrak{M}\left\{\widehat{I}_z^{\varepsilon}\left\{f(z);\alpha,\beta;w\right\};\rho\right\} = \frac{1}{\Gamma(\varepsilon)}\int_0^z (z-\Delta)^{\varepsilon-1}f(\Delta) \times \int_0^{\infty} w^{\rho-1}\,{}_{p}^{\alpha}M_q^{\beta}\left(\xi_1,\ldots,\xi_p;\mu_1,\ldots,\mu_q;\frac{-wz^2}{\Delta(z-\Delta)}\right)dwd\Delta.$$

By putting $k = \frac{wz^2}{\Delta(z-\Delta)}$ and multiplied by $\frac{\Gamma(\varepsilon+\rho)}{\Gamma(\varepsilon+\rho)}$, we have

$$\mathfrak{M}\left\{\widehat{I}_z^{\varepsilon}\left\{f(z);\alpha,\beta;w\right\};\rho\right\} = \frac{\Gamma(\varepsilon+\rho)}{\Gamma(\varepsilon)}\frac{z^{-2\rho}}{\Gamma(\varepsilon+\rho)}\int_0^z (z-\Delta)^{\varepsilon+\rho-1}\Delta^{\rho}f(\Delta) \times \int_0^{\infty} k^{\rho-1}\,{}_{p}^{\alpha}M_q^{\beta}\left(\xi_1,\ldots,\xi_p;\mu_1,\ldots,\mu_q;-k\right)dkd\Delta$$

$$= \frac{\Gamma(\varepsilon+\rho)}{\Gamma(\varepsilon)} \frac{{}^{M}\Gamma_{p,q}^{(\alpha,\beta)}(\rho;0)}{z^{2\rho}} \, I_z^{\varepsilon+\rho} \left\{ z^{\rho} f(z) \right\}. \qquad \square$$

Corollary 4.4. *The Mellin transform of the MRL fractional integral operator yields the following result in relation to the modified gamma function for $f(z) = z^{\lambda}$:*

$$\mathfrak{M}\left\{ \widehat{I}_z^{\varepsilon} \left\{ z^{\lambda}; \alpha, \beta; w \right\}; \rho \right\} = \frac{\Gamma(\varepsilon+\rho)\Gamma(\lambda+\rho+1) z^{\lambda+\varepsilon}}{\Gamma(\varepsilon)\Gamma(\lambda+\varepsilon+2\rho+1)} \, {}^{M}\Gamma_{p,q}^{(\alpha,\beta)}(\rho;0).$$

Theorem 4.5. *The Mellin transform of MRL fractional derivative for $\Re(\alpha) > 0$, $\Re(\rho) > 0$, $\Re(\varepsilon) > 0$, $m-1 < \Re(\varepsilon) < m$, $\Re(m-\varepsilon) > 0$, $\Re(m-\varepsilon+\rho) > 0$, $m \in \mathbb{N}$, $\xi_1, \ldots, \xi_p$, $\mu_1, \ldots, \mu_q \neq 0, -1, -2, \ldots$ is obtained as:*

$$\mathfrak{M}\left\{ \widehat{D}_z^{\varepsilon} \left\{ f(z); \alpha, \beta; w \right\}; \rho \right\} = \frac{{}^{M}\Gamma_{p,q}^{(\alpha,\beta)}(\rho;0)\Gamma(m-\varepsilon+\rho)}{\Gamma(m-\varepsilon)} \frac{d^m}{dz^m} \left(\frac{1}{z^{2\rho}} \, I_z^{m-\varepsilon+\rho} \left\{ z^{\rho} f(z) \right\} \right).$$

Proof. By using MRL fractional derivative operator and Mellin transform, we have

$$\begin{aligned} \mathfrak{M}\left\{ \widehat{D}_z^{\varepsilon} \left\{ f(z); \alpha, \beta; w \right\}; \rho \right\} &= \frac{1}{\Gamma(m-\varepsilon)} \frac{d^m}{dz^m} \int_0^z (z-\Delta)^{m-\varepsilon-1} f(\Delta) \\ &\quad \times \int_0^{\infty} w^{\rho-1} \, {}_p^{\alpha}M_q^{\beta} \left(\xi_1, \ldots, \xi_p; \mu_1, \ldots, \mu_q; \frac{-wz^2}{\Delta(z-\Delta)} \right) dw d\Delta. \end{aligned}$$

By putting $k = \frac{wz^2}{\Delta(z-\Delta)}$ and multiplied by $\frac{\Gamma(m-\varepsilon+\rho)}{\Gamma(m-\varepsilon+\rho)}$, we have

$$\begin{aligned} &\mathfrak{M}\left\{ \widehat{D}_z^{\varepsilon} \left\{ f(z); \alpha, \beta; w \right\}; \rho \right\} \\ &\quad = \frac{\Gamma(m-\varepsilon+\rho)}{\Gamma(m-\varepsilon)} \frac{d^m}{dz^m} \frac{z^{-2\rho}}{\Gamma(m-\varepsilon+\rho)} \int_0^z (z-\Delta)^{m-\varepsilon+\rho-1} \Delta^{\rho} f(\Delta) \\ &\qquad \times \int_0^{\infty} k^{\rho-1} \, {}_p^{\alpha}M_q^{\beta} \left(\xi_1, \ldots, \xi_p; \mu_1, \ldots, \mu_q; -k \right) dk d\Delta \\ &\quad = \frac{{}^{M}\Gamma_{p,q}^{(\alpha,\beta)}(\rho;0)\Gamma(m-\varepsilon+\rho)}{\Gamma(m-\varepsilon)} \frac{d^m}{dz^m} \left(\frac{1}{z^{2\rho}} \, I_z^{m-\varepsilon+\rho} \left\{ z^{\rho} f(z) \right\} \right). \qquad \square \end{aligned}$$

Corollary 4.5. *The Mellin transform of the MRL fractional derivative operator yields the following result in relation to the modified gamma function for $f(z) = z^{\lambda}$:*

$$\begin{aligned} \mathfrak{M}\left\{ \widehat{D}_z^{\varepsilon} \left\{ z^{\lambda}; \alpha, \beta; w \right\}; \rho \right\} &= \frac{\Gamma(m+\rho-\varepsilon)\Gamma(m+\lambda+1-\varepsilon)\Gamma(\lambda+\rho+1) z^{\lambda-\varepsilon}}{\Gamma(m-\varepsilon)\Gamma(\lambda+1-\varepsilon)\Gamma(m+\lambda+2\rho+1-\varepsilon)} \\ &\quad \times {}^{M}\Gamma_{p,q}^{(\alpha,\beta)}(\rho;0). \end{aligned}$$

Theorem 4.6. *The Mellin transform of MC fractional derivative for $\Re(\alpha) > 0$, $\Re(\rho) > 0$, $\Re(\varepsilon) > 0$, $m-1 < \Re(\varepsilon) < m$, $\Re(m-\varepsilon) > 0$, $\Re(m-\varepsilon+\rho) > 0$, $m \in \mathbb{N}$,*

$\xi_1,\ldots,\xi_p$, $\mu_1,\ldots,\mu_q \neq 0,-1,-2,\ldots$ *is obtained as:*

$$\mathfrak{M}\left\{{}^c\widehat{D}_z^{\varepsilon}\left\{f(z);\alpha,\beta;w\right\};\rho\right\} = \frac{{}^M\Gamma_{p,q}^{(\alpha,\beta)}(\rho;0)\Gamma(m-\varepsilon+\rho)}{\Gamma(m-\varepsilon)z^{2\rho}}\, I_z^{m-\varepsilon+\rho}\left\{z^{\rho}f^{(m)}(z)\right\}.$$

Proof. By using MC fractional derivative operator and Mellin transform, we have

$$\mathfrak{M}\left\{{}^c\widehat{D}_z^{\varepsilon}\left\{f(z);\alpha,\beta;w\right\};\rho\right\} = \frac{1}{\Gamma(m-\varepsilon)}\int_0^z (z-\Delta)^{m-\varepsilon-1}f^{(m)}(\Delta)$$
$$\times\int_0^{\infty} w^{\rho-1}\,{}_p^{\alpha}M_q^{\beta}\left(\xi_1,\ldots,\xi_p;\mu_1,\ldots,\mu_q;\frac{-wz^2}{\Delta(z-\Delta)}\right)dwd\Delta.$$

By putting $k=\frac{wz^2}{\Delta(z-\Delta)}$ and multiplied by $\frac{\Gamma(m-\varepsilon+\rho)}{\Gamma(m-\varepsilon+\rho)}$, we have

$$\begin{aligned}
&\mathfrak{M}\left\{{}^c\widehat{D}_z^{\varepsilon}\left\{f(z);\alpha,\beta;w\right\};\rho\right\}\\
&= \frac{{}^M\Gamma_{p,q}^{(\alpha,\beta)}(\rho;0)\Gamma(m-\varepsilon+\rho)}{\Gamma(m-\varepsilon)z^{2\rho}}\frac{1}{\Gamma(m-\varepsilon+\rho)}\int_0^z (z-\Delta)^{m-\varepsilon+\rho-1}\Delta^{\rho}f^{(m)}(\Delta)d\Delta\\
&= \frac{{}^M\Gamma_{p,q}^{(\alpha,\beta)}(\rho;0)\Gamma(m-\varepsilon+\rho)}{\Gamma(m-\varepsilon)z^{2\rho}}\, I_z^{m-\varepsilon+\rho}\left\{z^{\rho}f^{(m)}(z)\right\}.
\end{aligned}$$ □

Corollary 4.6. *The Mellin transform of the MC fractional derivative operator yields the following result in relation to the modified gamma function for* $f(z)=z^{\lambda}$:

$$\mathfrak{M}\left\{{}^c\widehat{D}_z^{\varepsilon}\left\{z^{\lambda};\alpha,\beta;w\right\};\rho\right\} = \frac{\Gamma(\lambda+1)\Gamma(m+\rho-\varepsilon)\Gamma(\lambda+\rho+1-m)z^{\lambda-\varepsilon}}{\Gamma(m-\varepsilon)\Gamma(\lambda+1-m)\Gamma(\lambda+2\rho+1-\varepsilon)}$$
$$\times\,{}^M\Gamma_{p,q}^{(\alpha,\beta)}(\rho;0).$$

4.3. *Application of beta transform*

Theorem 4.7. *The beta transform of MRL fractional integral for* $\Re(\alpha)>0$, $\Re(x)>0$, $\Re(y)>0$, $\Re(\varepsilon)>0$, $\xi_1,\ldots,\xi_p,x,\mu_1,\ldots,\mu_q,x+y\neq 0,-1,-2,\ldots$ *is obtained as:*

$$\mathfrak{B}\left\{\widehat{I}_z^{\varepsilon}\left\{f(z);\alpha,\beta;w\right\};x,y\right\} = B(x,y)\,\widehat{I}_z^{\varepsilon}\left\{f(z);\xi_1,\ldots,\xi_p,x;\mu_1,\ldots,\mu_q,x+y;\alpha,\beta;1\right\}.$$

Proof. By using MRL fractional integral operator and beta transform, we have

$$\begin{aligned}
&\mathfrak{B}\left\{\widehat{I}_z^{\varepsilon}\left\{f(z);\alpha,\beta;w\right\};x,y\right\}\\
&= \frac{B(x,y)}{\Gamma(\varepsilon)}\int_0^z (z-\Delta)^{\varepsilon-1}f(\Delta)\,{}_{p+1}^{\alpha}M_{q+1}^{\beta}\left(\xi_1,\ldots,\xi_p,x;\mu_1,\ldots,\mu_q,x+y;\frac{-z^2}{\Delta(z-\Delta)}\right)d\Delta\\
&= B(x,y)\,\widehat{I}_z^{\varepsilon}\left\{f(z);\xi_1,\ldots,\xi_p,x;\mu_1,\ldots,\mu_q,x+y;\alpha,\beta;1\right\}.
\end{aligned}$$ □

Corollary 4.7. *The beta transform of the MRL fractional integral operator yields the following result in relation to the modified beta function for* $f(z)=z^{\lambda}$:

$$\mathfrak{B}\left\{\widehat{I}_z^{\varepsilon}\left\{z^{\lambda};\alpha,\beta;w\right\};x,y\right\}=\frac{B(x,y)}{\Gamma(\varepsilon)}z^{\varepsilon+\lambda}$$
$$\times\,{}^{M}B_{p+1,q+1}^{(\alpha,\beta)}\left(\xi_1,\ldots,\xi_p,x;\mu_1,\ldots,\mu_q,x+y;\lambda+1,\varepsilon;1\right).$$

Theorem 4.8. *The beta transform of MRL fractional derivative for* $\Re(\alpha)>0$, $\Re(x)>0$, $\Re(y)>0$, $\Re(\varepsilon)>0$, $m-1<\Re(\varepsilon)<m$, $m\in\mathbb{N}$, $\xi_1,\ldots,\xi_p,x$, $\mu_1,\ldots,\mu_q$, $x+y\neq 0,-1,-2,\ldots$ *is obtained as:*

$$\mathfrak{B}\left\{\widehat{D}_z^{\varepsilon}\left\{f(z);\alpha,\beta;w\right\};x,y\right\}=B(x,y)\,\widehat{D}_z^{\varepsilon}\left\{f(z);\xi_1,\ldots,\xi_p,x;\mu_1,\ldots,\mu_q,x+y;\alpha,\beta;1\right\}.$$

Proof. By using the MRL fractional derivative operator and the beta transform, the desired result can be easily achieved with similar calculations. □

Corollary 4.8. *The beta transform of the MRL fractional derivative operator yields the following result in relation to the modified beta function for* $f(z)=z^{\lambda}$:

$$\mathfrak{B}\left\{\widehat{D}_z^{\varepsilon}\left\{z^{\lambda};\alpha,\beta;w\right\};x,y\right\}=\frac{B(x,y)\Gamma(m+\lambda+1-\varepsilon)z^{\lambda-\varepsilon}}{\Gamma(m-\varepsilon)\Gamma(\lambda+1-\varepsilon)}$$
$$\times\,{}^{M}B_{p+1,q+1}^{(\alpha,\beta)}\left(\xi_1,\ldots,\xi_p,x;\mu_1,\ldots,\mu_q,x+y;\lambda+1,m-\varepsilon;1\right).$$

Theorem 4.9. *The beta transform of MC fractional derivative for* $\Re(\alpha)>0$, $\Re(x)>0$, $\Re(y)>0$, $\Re(\varepsilon)>0$, $m-1<\Re(\varepsilon)<m$, $m\in\mathbb{N}$, $\xi_1,\ldots,\xi_p,x$, $\mu_1,\ldots,\mu_q$, $x+y\neq 0,-1,-2,\ldots$ *is obtained as:*

$$\mathfrak{B}\left\{{}^{c}\widehat{D}_z^{\varepsilon}\left\{f(z);\alpha,\beta;w\right\};x,y\right\}=B(x,y)\,{}^{c}\widehat{D}_z^{\varepsilon}\left\{f(z);\xi_1,\ldots,\xi_p,x;\mu_1,\ldots,\mu_q,x+y;\alpha,\beta;1\right\}.$$

Proof. By using the MC fractional derivative operator and the beta transform, the desired result can be easily achieved with similar calculations. □

Corollary 4.9. *The beta transform of the MC fractional derivative operator yields the following result in relation to the modified beta function for* $f(z)=z^{\lambda}$:

$$\mathfrak{B}\left\{{}^{c}\widehat{D}_z^{\varepsilon}\left\{z^{\lambda};\alpha,\beta;w\right\};x,y\right\}=\frac{B(x,y)\Gamma(\lambda+1)z^{\lambda-\varepsilon}}{\Gamma(m-\varepsilon)\Gamma(\lambda+1-m)}$$
$$\times\,{}^{M}B_{p+1,q+1}^{(\alpha,\beta)}\left(\xi_1,\ldots,\xi_p,x;\mu_1,\ldots,\mu_q,x+y;\lambda+1-m,m-\varepsilon;1\right).$$

5. Examples

In this section, we compute the MRL fractional integrals, MRL fractional derivatives and MC fractional derivatives of functions z^{λ} and $(1-az)^{-\lambda}$ and obtain results that generate the classical gamma and beta and modified gamma and beta functions.

Example 5.1. The MRL fractional integral of $f(z) = z^{\lambda}$ for $\Re(\alpha) > 0$, $\Re(w) > 0$, $\Re(\lambda+1) > 0$, $\Re(\varepsilon) > 0$, $\xi_1, \ldots, \xi_p, \mu_1, \ldots, \mu_q \neq 0, -1, -2, \ldots$ is found as:

$$\widehat{I}_z^{\varepsilon}\left\{z^{\lambda}; \alpha, \beta; w\right\} = \frac{z^{\varepsilon+\lambda}}{\Gamma(\varepsilon)}\, {}^{M}B_{p,q}^{(\alpha,\beta)}(\lambda+1, \varepsilon; w).$$

By using MRL fractional integral operator, we have

$$\begin{aligned}
\widehat{I}_z^{\varepsilon}\left\{z^{\lambda}; \alpha, \beta; w\right\} &= \frac{1}{\Gamma(\varepsilon)} \int_0^z (z-\Delta)^{\varepsilon-1} \Delta^{\lambda}\, {}_p^{\alpha}M_q^{\beta}\left(\xi_1, \ldots, \xi_p; \mu_1, \ldots, \mu_q; \frac{-wz^2}{\Delta(z-\Delta)}\right) d\Delta \\
&= \frac{z^{\varepsilon+\lambda}}{\Gamma(\varepsilon)} \int_0^1 u^{\lambda}(1-u)^{\varepsilon-1}\, {}_p^{\alpha}M_q^{\beta}\left(\xi_1, \ldots, \xi_p; \mu_1, \ldots, \mu_q; \frac{-w}{u(1-u)}\right) du \\
&= \frac{z^{\varepsilon+\lambda}}{\Gamma(\varepsilon)}\, {}^{M}B_{p,q}^{(\alpha,\beta)}(\lambda+1, \varepsilon; w).
\end{aligned}$$

Example 5.2. The MRL fractional derivative of $f(z) = z^{\lambda}$ for $\Re(\alpha) > 0$, $\Re(w) > 0$, $\Re(\lambda+1) > 0$, $\Re(\varepsilon) > 0$, $m-1 < \Re(\varepsilon) < m$, $\Re(m-\varepsilon) > 0$, $\Re(\lambda+1-\varepsilon) > 0$, $m \in \mathbb{N}$, $\xi_1, \ldots, \xi_p$, $\mu_1, \ldots, \mu_q \neq 0, -1, -2, \ldots$ is found as:

$$\widehat{D}_z^{\varepsilon}\left\{z^{\lambda}; \alpha, \beta; w\right\} = z^{\lambda-\varepsilon} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\varepsilon)} \frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(\lambda+1, m-\varepsilon; w)}{B(\lambda+1, m-\varepsilon)}.$$

By using MRL fractional derivative operator, we have

$$\begin{aligned}
&\widehat{D}_z^{\varepsilon}\left\{z^{\lambda}; \alpha, \beta; w\right\} \\
&= \frac{1}{\Gamma(m-\varepsilon)} \frac{d^m}{dz^m} \int_0^z (z-\Delta)^{m-\varepsilon-1} \Delta^{\lambda}\, {}_p^{\alpha}M_q^{\beta}\left(\xi_1, \ldots, \xi_p; \mu_1, \ldots, \mu_q; \frac{-wz^2}{\Delta(z-\Delta)}\right) d\Delta \\
&= \frac{z^{\lambda-\varepsilon}}{\Gamma(m-\varepsilon)} \frac{\Gamma(m+\lambda+1-\varepsilon)}{\Gamma(\lambda+1-\varepsilon)} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1)}\, {}^{M}B_{p,q}^{(\alpha,\beta)}(\lambda+1, m-\varepsilon; w) \\
&= z^{\lambda-\varepsilon} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\varepsilon)} \frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(\lambda+1, m-\varepsilon; w)}{B(\lambda+1, m-\varepsilon)}.
\end{aligned}$$

Example 5.3. The MC fractional derivative of $f(z) = z^{\lambda}$ for $\Re(\alpha) > 0$, $\Re(w) > 0$, $\Re(\lambda+1) > 0$, $\Re(\varepsilon) > 0$, $m-1 < \Re(\varepsilon) < m$, $\Re(m-\varepsilon) > 0$, $\Re(\lambda+1-\varepsilon) > 0$, $\Re(\lambda+1-m) > 0$, $m \in \mathbb{N}$, $\xi_1, \ldots, \xi_p$, $\mu_1, \ldots, \mu_q \neq 0, -1, -2, \ldots$ is found as:

$${}^{c}\widehat{D}_z^{\varepsilon}\left\{z^{\lambda}; \alpha, \beta; w\right\} = z^{\lambda-\varepsilon} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\varepsilon)} \frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(\lambda+1-m, m-\varepsilon; w)}{B(\lambda+1-m, m-\varepsilon)}.$$

By using MC fractional derivative operator, we have

$$
\begin{aligned}
&{}^{c}\widehat{D}_{z}^{\varepsilon}\left\{z^{\lambda};\alpha,\beta;w\right\}\\
&=\frac{1}{\Gamma(m-\varepsilon)}\int_{0}^{z}(z-\Delta)^{m-\varepsilon-1}\,{}_{p}^{\alpha}M_{q}^{\beta}\left(\xi_{1},\ldots,\xi_{p};\mu_{1},\ldots,\mu_{q};\frac{-wz^{2}}{\Delta(z-\Delta)}\right)\frac{d^{m}}{d\Delta^{m}}\left(\Delta^{\lambda}\right)d\Delta\\
&=z^{\lambda-\varepsilon}\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-m)\Gamma(m-\varepsilon)}\frac{\Gamma(\lambda+1-\varepsilon)}{\Gamma(\lambda+1-\varepsilon)}\,{}^{M}B_{p,q}^{(\alpha,\beta)}(\lambda+1-m,m-\varepsilon;w)\\
&=z^{\lambda-\varepsilon}\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\varepsilon)}\frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(\lambda+1-m,m-\varepsilon;w)}{B(\lambda+1-m,m-\varepsilon)}.
\end{aligned}
$$

Example 5.4. The MRL fractional integral of $f(z)=(1-az)^{-\lambda}$ for $\Re(\alpha)>0$, $\Re(w)>0$, $\Re(\varepsilon)>0$, $\xi_{1},\ldots,\xi_{p},\mu_{1},\ldots,\mu_{q},\lambda\neq 0,-1,-2,\ldots$ is found as:

$$
\widehat{I}_{z}^{\varepsilon}\left\{(1-az)^{-\lambda};\alpha,\beta;w\right\}=\frac{z^{\varepsilon}}{\Gamma(\varepsilon)}\sum_{n=0}^{\infty}\frac{(\lambda)_{n}(az)^{n}}{n!}\,{}^{M}B_{p,q}^{(\alpha,\beta)}(n+1,\varepsilon;w).
$$

By using MRL fractional integral operator, we have

$$
\begin{aligned}
&\widehat{I}_{z}^{\varepsilon}\left\{(1-az)^{-\lambda};\alpha,\beta;w\right\}\\
&=\frac{1}{\Gamma(\varepsilon)}\int_{0}^{z}(z-\Delta)^{\varepsilon-1}(1-a\Delta)^{-\lambda}\,{}_{p}^{\alpha}M_{q}^{\beta}\left(\xi_{1},\ldots,\xi_{p};\mu_{1},\ldots,\mu_{q};\frac{-wz^{2}}{\Delta(z-\Delta)}\right)d\Delta\\
&=\frac{z^{\varepsilon}}{\Gamma(\varepsilon)}\sum_{n=0}^{\infty}\frac{(\lambda)_{n}(az)^{n}}{n!}\int_{0}^{1}u^{n}(1-u)^{\varepsilon-1}\,{}_{p}^{\alpha}M_{q}^{\beta}\left(\xi_{1},\ldots,\xi_{p};\mu_{1},\ldots,\mu_{q};\frac{-w}{u(1-u)}\right)du\\
&=\frac{z^{\varepsilon}}{\Gamma(\varepsilon)}\sum_{n=0}^{\infty}\frac{(\lambda)_{n}(az)^{n}}{n!}\,{}^{M}B_{p,q}^{(\alpha,\beta)}(n+1,\varepsilon;w).
\end{aligned}
$$

Example 5.5. The MRL fractional derivative of $f(z)=(1-az)^{-\lambda}$ for $\Re(\alpha)>0$, $\Re(w)>0$, $\Re(\varepsilon)>0$, $m-1<\Re(\varepsilon)<m$, $\Re(m-\varepsilon)>0$, $\Re(n+1-\varepsilon)>0$, $m\in\mathbb{N}$, $n\in\mathbb{N}_{0}$, $\xi_{1},\ldots,\xi_{p}$, $\mu_{1},\ldots,\mu_{q},\lambda\neq 0,-1,-2,\ldots$ is found as:

$$
\widehat{D}_{z}^{\varepsilon}\left\{(1-az)^{-\lambda};\alpha,\beta;w\right\}=z^{-\varepsilon}\sum_{n=0}^{\infty}\frac{(\lambda)_{n}(az)^{n}}{n!}\frac{\Gamma(n+1)}{\Gamma(n+1-\varepsilon)}\frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(n+1,m-\varepsilon;w)}{B(n+1,m-\varepsilon)}.
$$

By using MRL fractional derivative operator, we have

$$
\begin{aligned}
&\widehat{D}_{z}^{\varepsilon}\left\{(1-az)^{-\lambda};\alpha,\beta;w\right\}\\
&=z^{-\varepsilon}\sum_{n=0}^{\infty}\frac{(\lambda)_{n}(az)^{n}}{n!}\frac{\Gamma(m+n+1-\varepsilon)}{\Gamma(n+1-\varepsilon)\Gamma(m-\varepsilon)}\frac{\Gamma(n+1)}{\Gamma(n+1)}\,{}^{M}B_{p,q}^{(\alpha,\beta)}(n+1,m-\varepsilon;w)\\
&=z^{-\varepsilon}\sum_{n=0}^{\infty}\frac{(\lambda)_{n}(az)^{n}}{n!}\frac{\Gamma(n+1)}{\Gamma(n+1-\varepsilon)}\frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(n+1,m-\varepsilon;w)}{B(n+1,m-\varepsilon)}.
\end{aligned}
$$

Example 5.6. The MC fractional derivative of $f(z) = (1-az)^{-\lambda}$ for $\Re(\alpha) > 0$, $\Re(w) > 0$, $\Re(\varepsilon) > 0$, $m-1 < \Re(\varepsilon) < m$, $\Re(m-\varepsilon) > 0$, $\Re(n+1-\varepsilon) > 0$, $n+1-m > 0$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $\xi_1, \ldots, \xi_p, \mu_1, \ldots, \mu_q, \lambda \neq 0, -1, -2, \ldots$ is found as:

$$
\begin{aligned}
&{}^{c}\widehat{D}_z^{\varepsilon}\left\{(1-az)^{-\lambda}; \alpha, \beta; w\right\} \\
&\quad = z^{-\varepsilon} \sum_{n=0}^{\infty} \frac{(\lambda)_n (az)^n}{n!} \frac{\Gamma(n+1)}{\Gamma(n+1-\varepsilon)} \frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(n+1-m, m-\varepsilon; w)}{B(n+1-m, m-\varepsilon)}.
\end{aligned}
$$

By using MC fractional derivative operator, we have

$$
\begin{aligned}
&{}^{c}\widehat{D}_z^{\varepsilon}\left\{(1-az)^{-\lambda}; \alpha, \beta; w\right\} \\
&= z^{-\varepsilon} \sum_{n=0}^{\infty} \frac{(\lambda)_n (az)^n}{n!} \frac{\Gamma(n+1)}{\Gamma(n+1-m)\Gamma(m-\varepsilon)} \frac{\Gamma(n+1-\varepsilon)}{\Gamma(n+1-\varepsilon)} \, {}^{M}B_{p,q}^{(\alpha,\beta)}(n+1-m, m-\varepsilon; w) \\
&= z^{-\varepsilon} \sum_{n=0}^{\infty} \frac{(\lambda)_n (az)^n}{n!} \frac{\Gamma(n+1)}{\Gamma(n+1-\varepsilon)} \frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(n+1-m, m-\varepsilon; w)}{B(n+1-m, m-\varepsilon)}.
\end{aligned}
$$

6. Conclusions

In this paper, MRL fractional integral, MRL fractional derivative and MC fractional derivative operators, which including generalized M-series at their kernels, are defined. Since the generalized M-series has a more general form then most of the special functions, many fractional operators becomes the special cases of the fractional operators (MRL and MC fractional operators) introduced here.

The definitions of some popular fractional operators that have been defined recently and their relationships with the new fractional operators (MRL and MC fractional operators) are given below.

Özarslan and Özergin [17]:

$$
D_z^{\varepsilon,w}\{f(z)\} = \begin{cases} \frac{1}{\Gamma(-\varepsilon)} \int_0^z (z-\Delta)^{-\varepsilon-1} f(\Delta) \exp\left(\frac{-wz^2}{\Delta(z-\Delta)}\right) d\Delta, \\ \frac{d^m}{dz^m} D_z^{\varepsilon-m,w}\{f(z)\} \end{cases}
$$

and relationships

$$
\widehat{D}_z^{\varepsilon}\{f(z); 1; 1; 1, 1; w\} = \frac{d^m}{dz^m} D_z^{\varepsilon-m,w}\{f(z)\}.
$$

Srivastava et al. [22]:

$$
D^{\varepsilon,w}_{z,\left(\{K_l\}_{l\in\mathbf{N_0}}\right)}\{f(z)\} = \begin{cases} \frac{1}{\Gamma(-\varepsilon)} \int_0^z (z-\Delta)^{-\varepsilon-1} f(\Delta) \Theta\left(\{K_l\}_{l\in\mathbf{N_0}}; \frac{-wz^2}{\Delta(z-\Delta)}\right) d\Delta, \\ \frac{d^m}{dz^m} D^{\varepsilon-m,w}_{z,\left(\{K_l\}_{l\in\mathbf{N_0}}\right)}\{f(z)\} \end{cases}
$$

and relationships

$$\widehat{D}_z^{\varepsilon}\left\{f(z);\xi;\mu;1,1;w\right\}=\frac{d^m}{dz^m}D_{z,\left(\{K_l\}_{l\in\mathbf{N_0}}\right)}^{\varepsilon-m,w}\left\{f(z)\right\}.$$

Parmar [18]:

$$D_z^{\varepsilon,w;k}\left\{f(z)\right\}=\begin{cases}\frac{1}{\Gamma(-\varepsilon)}\int\limits_0^z(z-\Delta)^{-\varepsilon-1}f(\Delta)\exp\left(\frac{-wz^{2k}}{\Delta^k(z-\Delta)^k}\right)d\Delta,\\ \frac{d^m}{dz^m}D_z^{\varepsilon-m,w;k}\left\{f(z)\right\}\end{cases}$$

and relationships

$$\widehat{D}_z^{\varepsilon}\left\{f(z);1;1;1,1;w\right\}=\frac{d^m}{dz^m}D_z^{\varepsilon-m,w;1}\left\{f(z)\right\}.$$

Agarwal et al. [3]:

$$D_z^{\varepsilon,w;\kappa,\eta}\left\{f(z)\right\}=\begin{cases}\frac{1}{\Gamma(-\varepsilon)}\int\limits_0^z(z-\Delta)^{-\varepsilon-1}f(\Delta)\,{}_1F_1\left(\xi;\mu;\frac{-wz^{\kappa+\eta}}{\Delta^{\kappa}(z-\Delta)^{\eta}}\right)d\Delta,\\ \frac{d^m}{dz^m}D_z^{\varepsilon-m,w;\kappa,\eta}\left\{f(z)\right\}\end{cases}$$

and relationships

$$\widehat{D}_z^{\varepsilon}\left\{f(z);\xi;\mu;1,1;w\right\}=\frac{d^m}{dz^m}D_z^{\varepsilon-m,w;1,1}\left\{f(z)\right\}.$$

Kıymaz et al. [15]:

$$D_z^{\varepsilon,w}\left\{f(z)\right\}=\frac{1}{\Gamma(m-\varepsilon)}\int_0^z(z-\Delta)^{m-\varepsilon-1}f^{(m)}(\Delta)\exp\left(\frac{-wz^2}{\Delta(z-\Delta)}\right)d\Delta$$

and relationships

$${}^c\widehat{D}_z^{\varepsilon}\left\{f(z);1;1;1,1;w\right\}=D_z^{\varepsilon,w}\left\{f(z)\right\}.$$

Agarwal et al. [2]:

$$D_z^{\varepsilon,w;k}\left\{f(z)\right\}=\frac{1}{\Gamma(m-\varepsilon)}\int_0^z(z-\Delta)^{m-\varepsilon-1}f^{(m)}(\Delta)\exp\left(\frac{-wz^{2k}}{\Delta^k(z-\Delta)^k}\right)d\Delta$$

and relationships

$${}^c\widehat{D}_z^{\varepsilon}\left\{f(z);1;1;1,1;w\right\}=D_z^{\varepsilon,w;1}\left\{f(z)\right\}.$$

Baleanu et al. [8]:

$$D_z^{\varepsilon}\left\{f(z);p,q\right\}=\begin{cases}\frac{1}{\Gamma(-\varepsilon)}\int\limits_0^z(z-\Delta)^{-\varepsilon-1}f(\Delta)\exp\left(\frac{-pz}{\Delta}-\frac{qz}{z-\Delta}\right)d\Delta,\\ \frac{d^m}{dz^m}D_z^{\varepsilon-m}\left\{f(z);p,q\right\}\end{cases}$$

and relationships

$$\widehat{D}_z^{\varepsilon}\left\{f(z);1;1;1,1;w\right\}=\frac{d^m}{dz^m}D_z^{\varepsilon-m}\left\{f(z);w,w\right\}.$$

Kıymaz et al. [14]:

$$D_z^{\varepsilon}\{f(z);p,q\} = \frac{1}{\Gamma(m-\varepsilon)}\int_0^z (z-\Delta)^{m-\varepsilon-1} f^{(m)}(\Delta)\exp\left(\frac{-pz}{\Delta}-\frac{qz}{z-\Delta}\right)d\Delta$$

and relationships

$${}^{c}\widehat{D}_z^{\varepsilon}\{f(z);1;1;1,1;w\} = D_z^{\varepsilon}\{f(z);w,w\}.$$

Çetinkaya et al. [10]:

$$D^{\varepsilon;(\xi,\mu;\kappa,\eta;p,q)}\{f(z)\} = \begin{cases} \frac{1}{\Gamma(-\varepsilon)}\int_0^z (z-\Delta)^{-\varepsilon-1} f(\Delta)\,{}_1F_1\left(\xi;\mu;-p\left(\frac{z}{\Delta}\right)^{\kappa}-q\left(\frac{z}{z-\Delta}\right)^{\eta}\right)d\Delta, \\ \frac{d^m}{dz^m}D^{\varepsilon-m;(\xi,\mu;\kappa,\eta;p,q)}\{f(z)\}, \end{cases}$$

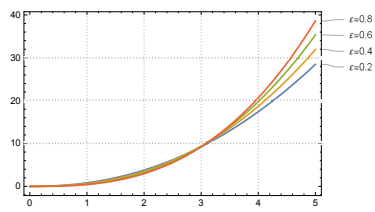
$$\mathbf{D}^{\varepsilon;(\xi,\mu;\kappa,\eta;p,q)}\{f(z)\} = D^{\varepsilon-m;(\xi,\mu;\kappa,\eta;p,q)}\left\{f^{(m)}(z)\right\}$$

and relationships

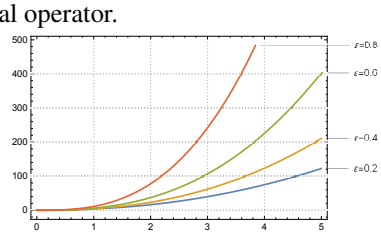
$$\widehat{D}_z^{\varepsilon}\{f(z);\xi;\mu;1,1;w\} = \frac{d^m}{dz^m}D^{\varepsilon-m;(\xi,\mu;1,1;w,w)}\{f(z)\},$$

$${}^{c}\widehat{D}_z^{\varepsilon}\{f(z);\xi;\mu;1,1;w\} = \mathbf{D}^{\varepsilon;(\xi,\mu;1,1;w,w)}\{f(z)\}.$$

Finally, we present the behavior of the function z^2 in both the classical fractional operators and the new fractional operators, see Figure 1 and Figure 2.

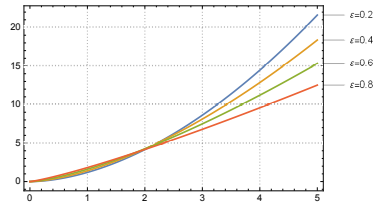


(A) Behavior of the function z^2 in the classical Riemann-Liouville fractional integral operator.

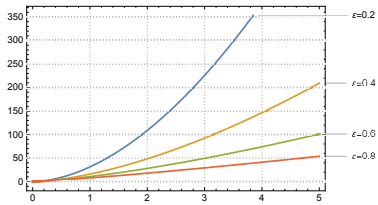


(B) Behavior of the function z^2 in MRL fractional integral operator for values $p = q = \xi_1 = \mu_1 = 1$, $\alpha = \beta = 1$, $w = 2$ and generalized M-series index $n = 0, 1$.

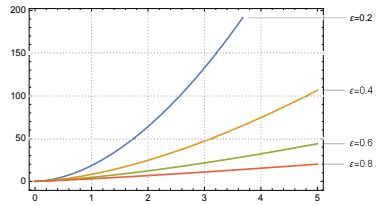
FIGURE 1. Comparison of classical Riemann-Liouville fractional integral operator with MRL fractional integral operator.



(A) Behavior of the function z^2 in the classical Riemann-Liouville and Caputo fractional derivative operators.



(B) Behavior of the function z^2 in MRL fractional derivative operator for values $p = q = \xi_1 = \mu_1 = 1$, $\alpha = \beta = m = 1$, $w = 2$ and generalized M-series index $n = 0, 1$.



(C) Behavior of the function z^2 in MC fractional derivative operator for values $p = q = \xi_1 = \mu_1 = 1$, $\alpha = \beta = m = 1$, $w = 2$ and generalized M-series index $n = 0, 1$.

FIGURE 2. Comparison of classical Riemann-Liouville and Caputo fractional derivative operators with MRL and MC fractional derivative operators.

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NEW APPROACHES FOR m-CONVEX FUNCTIONS VIA FRACTIONAL INTEGRAL OPERATORS WITH STRONG KERNELS

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Abstract. We have established this paper on $m-$convex functions, which can be expressed as a general form of the convex function concept. First of all, some inequalities of Hadamard type are proved with fairly simple conditions. Next, an integral identity containing Atangana-Baleanu fractional integral operators is obtained to prove new inequalities for differentiable m-convex functions. Using this identity, various properties of $m-$convex functions and classical inequalities, some new integral inequalities have been proved.

2010 *Mathematics Subject Classification:* 26A33, 26A51, 26D10

Keywords: m-convex functions, Atangana-Baleanu integral operators, normalization function, Euler Gamma function

1. Introduction

Although the definition of convex functions has an aesthetic form due to its algebraic and geometric structure, it has been an important place in mathematical analysis with its applications in many fields. Several modified versions and variants of convex functions have been established, which have become the center of consideration of researchers in many branches of applied sciences, especially in areas such as statistics, numerical analysis, convex programming and approximation theory.

The definition of $m-$convex function, which is one of these general forms, is given as follows.

Definition 1. [30] The function $\Upsilon : [0,b] \to \mathbb{R}$ is said to be m-convex $m \in [0,1]$, if for every $x_1, x_2 \in [0,b]$ and $\tau \in [0,1]$, we have

$$\Upsilon(\tau x_1 + m(1-\tau)x_2) \leq \tau\Upsilon(x_1) + m(1-\tau)\Upsilon(x_2).$$

Let us recall what important results were obtained in some of the basic studies available in the literature for m-convex functions. In the study presented in [12], firstly some basic properties of m-convex functions are given; then, results including Hermite-Hadamard inequality for m-convex functions are obtained by Dragomir. In

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[29], Set *et al.* gave place to two basic theorems for m-convex functions involving fractional integrals. While the first of these basic results is a Hermite-Hadamard type result obtained via fractional integrals for m-convex functions; the second theorem pays attention to an inequality obtained including fractional integrals for a new function defined with the help of an m-convex function. In [22], Mehreen and Anwar generalized the inequality obtained by Set *et al.* for a new function defined by m-convex function via Katugampola fractional integrals. In [21], addition to the results including Hermite-Hadamard inequality for m-convex and (α,m)-convex functions, Klaričić Bakula *et al.* have also obtained inequalities for the product of m-convex functions and the product of (α,m) -convex functions. In [8], Chen generalized the results on the product of two m-convex functions and the product of two (α,m)-convex functions of Klaričić Bakula *et al*. with the help of Riemann-Liouville fractional integrals. Different from the above definition, in [25] Pavić and Avcı Ardıç obtained interesting results for m-convex functions by using the intervals of real numbers that contain zero. For other results obtained on m-convex functions, see the references in [6], [14] and [24].

Mathematics is a language that explains nature and events and is as old and ancient as the history of humanity. The adventure, which started with simple calculations based on classical analysis, continued towards more advanced problems with the increasing needs of humanity. New orientations in mathematics were needed because it was insufficient to explain real world problems, physical phenomena and the infrastructure that would form the basis of engineering sciences with classical analysis. At this point, the orientation of mathematicians has been towards fractional analysis. Because it is known that fractional analysis produces more effective results for the solutions of differential equation systems and differs from classical analysis in terms of memory effect.

The development in fractional analysis has gained momentum with the definition of fractional derivative operators and associated integral operators. New operators are changing with the structural differences in their core structures and their effectiveness in the application areas, and each new fractional operator has brought a new approach method to the field. Especially the operators defined by strong kernels with non-local and non-singular properties have been preferred by mathematicians. To provide more information related to new fractional operators, integral inequalities and applications, see the papers [2–4, 7, 9–11, 13, 15–20, 23] and [26–29].

Now, recall the Atangana-Baleanu fractional integral operators associated with the fractional derivative described with a kernel structure containing the Mittag-Leffler function.

Definition 2. [5] The fractional integral associate to the new fractional derivative with non-local kernel of a function $\Upsilon \in H^1(x_1,x_2)$ as defined:

$$ {}^{AB}_{x_1}I^{\alpha}_{t}\{\Upsilon(t)\} = \frac{1-\alpha}{B(\alpha)}\Upsilon(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{x_1}^{t}\Upsilon(u)(t-u)^{\alpha-1}du $$

where $x_2 > x_1, \alpha \in [0,1]$, $B(\alpha)$ is normalization function.

In [1], the authors have given the right hand side of integral operator as following;

$$\left({}^{AB}I^{\alpha}_{x_2}\right)\{\Upsilon(t)\} = \frac{1-\alpha}{B(\alpha)}\Upsilon(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_t^{x_2}\Upsilon(u)(u-t)^{\alpha-1}du.$$

Here, $\Gamma(\alpha)$ is the Gamma function. Since the normalization function $B(\alpha)$ is positive, it immediately follows that the fractional $AB-$integral of a positive function is positive. It should be noted that, when the order $\alpha \longrightarrow 1$, we recover the classical integral. Also, the initial function is recovered whenever the fractional order $\alpha \longrightarrow 0$.

The paper is organized as follows: In section 2, some new integral inequalities have been derived via Atangana-Baleanu fractional integrals for $m-$convex functions. In section 3, a new integral identity containing Atangana-Baleanu fractional integrals as well as generalizations of some integral inequalities available in the literature for $m-$convex functions has been proved. Besides, some new Hadamard type integral inequalities based on this integral identity are given.

2. INEQUALITIES FOR m-CONVEX FUNCTIONS WITHOUT USING THEIR DERIVATIVES

In this section, we will first give the inequalities that we have obtained without using the derivative of a function that is $m-$convex. Next, we have established some new inequalities for the product of two $m-$convex functions. Of course, let's state once again that we make use of the Atangana-Baleanu integral operator while obtaining these results.

Theorem 1. *Let $0 \le x_1 < \frac{x_2}{m}$, $\xi, m \in (0,1]$ and $\Upsilon : [0,\infty) \longrightarrow \mathbb{R}$ be an m-convex function. If $\Upsilon \in L[x_1, \frac{x_2}{m}]$, we have the following*

$$\frac{1}{(x_2-x_1)^{\xi}}\left({}^{AB}_{x_1}I^{\xi}_{x_2}\{\Upsilon(x_2)\}\right) \le \frac{\xi}{B(\xi)\Gamma(\xi)}\left[\frac{\Upsilon(x_1)}{\xi+1} + \frac{m\Upsilon\left(\frac{x_2}{m}\right)}{\xi(\xi+1)}\right] + \frac{1-\xi}{(x_2-x_1)^{\xi}B(\xi)}\Upsilon(x_2) \tag{2.1}$$

and

$$\frac{1}{(x_2-x_1)^{\xi}}\left({}^{AB}I^{\xi}_{x_2}\{\Upsilon(x_1)\}\right) \le \frac{\xi}{B(\xi)\Gamma(\xi)}\left[\frac{\Upsilon(x_2)}{\xi+1} + \frac{m\Upsilon\left(\frac{x_1}{m}\right)}{\xi(\xi+1)}\right] + \frac{1-\xi}{(x_2-x_1)^{\xi}B(\xi)}\Upsilon(x_1). \tag{2.2}$$

Proof. Let us consider the definition of the m-convex function, we can write for all $\tau \in [0,1]$

$$\Upsilon(\tau x_1 + (1-\tau)x_2) \le \tau\Upsilon(x_1) + m(1-\tau)\Upsilon\left(\frac{x_2}{m}\right) \tag{2.3}$$

and

$$\Upsilon(\tau x_2 + (1-\tau)x_1) \le \tau\Upsilon(x_2) + m(1-\tau)\Upsilon\left(\frac{x_1}{m}\right). \tag{2.4}$$

If we multiply both sides of (2.3) by $\tau^{\xi-1}$, and integrate the resulting inequality over $[0,1]$ with respect to τ, we get

$$\int_0^1 \tau^{\xi-1}\Upsilon(\tau x_1+(1-\tau)x_2)\,d\tau \tag{2.5}$$
$$\leq \int_0^1 \tau^{\xi-1}\left(\tau\Upsilon(x_1)+m(1-\tau)\Upsilon\left(\frac{x_2}{m}\right)\right)d\tau = \frac{\Upsilon(x_1)}{\xi+1}+\frac{m\Upsilon\left(\frac{x_2}{m}\right)}{\xi(\xi+1)}.$$

If appropriate variable changing is made for the left hand side of the above inequality, we obtain

$$\int_0^1 \tau^{\xi-1}\Upsilon(\tau x_1+(1-\tau)x_2)\,d\tau = \frac{1}{(x_2-x_1)^{\xi}}\int_{x_1}^{x_2}(x_2-y)^{\xi-1}\Upsilon(y)\,dy. \tag{2.6}$$

Substituting the equation in (2.6) for the inequality in (2.5), then multiplying the two sides of the inequality by $\frac{\xi}{B(\xi)\Gamma(\xi)}$ and adding by $\frac{1-\xi}{(x_2-x_1)^{\xi}B(\xi)}\Upsilon(x_2)$ we get the inequality in (2.1).

Similar steps are followed to obtain the inequality in (2.2). First, we multiply both sides of (2.4) by $\tau^{\xi-1}$, and integrate the resulting inequality over $[0,1]$. After these operations, if we multiply the two sides of the last inequality by $\frac{\xi}{B(\xi)\Gamma(\xi)}$ and add by $\frac{1-\xi}{(x_2-x_1)^{\xi}B(\xi)}\Upsilon(x_1)$, we obtain the inequality in (2.2). □

Remark 1. In inequalities (2.1) and (2.2), if we choose $\xi=1$ and evaluate the obtained inequalities together, we achieve Theorem 2 of Dragomir's paper as in [12].

Theorem 2. *Let $0\leq x_1<x_2$, $\xi,m\in(0,1]$ and $\Upsilon:[0,\infty)\longrightarrow\mathbb{R}$ be an m-convex function. If $\Upsilon\in L[mx_1,x_2]$, we have the following inequality*

$$\frac{1}{(mx_2-x_1)^{\xi}}\left[{}^{AB}_{x_1}I^{\xi}_{mx_2}\{\Upsilon(mx_2)\}+{}^{AB}I^{\xi}_{mx_2}\{\Upsilon(x_1)\}\right] \tag{2.7}$$
$$+\frac{1}{(x_2-mx_1)^{\xi}}\left[{}^{AB}I^{\xi}_{x_2}\{\Upsilon(mx_1)\}+{}^{AB}_{mx_1}I^{\xi}_{x_2}\{\Upsilon(x_2)\}\right]$$
$$\leq\frac{m+1}{B(\xi)\Gamma(\xi)}(\Upsilon(x_1)+\Upsilon(x_2))+\frac{1-\xi}{(mx_2-x_1)^{\xi}B(\xi)}(\Upsilon(x_1)+\Upsilon(mx_2))$$
$$+\frac{1-\xi}{(x_2-mx_1)^{\xi}B(\xi)}(\Upsilon(mx_1)+\Upsilon(x_2)).$$

Proof. For the proof, we will basically make use of the $m-$convex function definition and the definition of Atangana-Baleanu fractional integral oprator. If we use the definition of m-convex function, we can write for all $\tau\in[0,1]$ and $x_1,x_2\in[0,\infty)$

$$\Upsilon(\tau x_1+m(1-\tau)x_2)\leq\tau\Upsilon(x_1)+m(1-\tau)\Upsilon(x_2),$$
$$\Upsilon((1-\tau)x_1+m\tau x_2)\leq(1-\tau)\Upsilon(x_1)+m\tau\Upsilon(x_2),$$
$$\Upsilon(\tau x_2+m(1-\tau)x_1)\leq\tau\Upsilon(x_2)+m(1-\tau)\Upsilon(x_1)$$

and

$$\Upsilon\left((1-\tau)x_2+m\tau x_1\right)\leq(1-\tau)\Upsilon\left(x_2\right)+m\tau\Upsilon\left(x_1\right).$$

If we add these four inequalities above, multiply both sides of the new inequality by $\tau^{\xi-1}$ and integrate the last inequality over $[0,1]$ with respect to τ, we obtain

$$\begin{aligned}\int_0^1\tau^{\xi-1}\Upsilon\left(\tau x_1+m(1-\tau)x_2\right)d\tau+\int_0^1\tau^{\xi-1}\Upsilon\left((1-\tau)x_1+m\tau x_2\right)d\tau \\ +\int_0^1\tau^{\xi-1}\Upsilon\left(\tau x_2+m(1-\tau)x_1\right)d\tau+\int_0^1\tau^{\xi-1}\Upsilon\left((1-\tau)x_2+m\tau x_1\right)d\tau \\ \leq\frac{(m+1)}{\xi}\left(\Upsilon(x_1)+\Upsilon(x_2)\right).\end{aligned}\tag{2.8}$$

Here, following equalities hold:

$$\int_0^1\tau^{\xi-1}\Upsilon\left(\tau x_1+m(1-\tau)x_2\right)d\tau=\frac{1}{(mx_2-x_1)^{\xi}}\int_{x_1}^{mx_2}(mx_2-y)^{\xi-1}\Upsilon(y)dy,$$

$$\int_0^1\tau^{\xi-1}\Upsilon\left((1-\tau)x_1+m\tau x_2\right)d\tau=\frac{1}{(mx_2-x_1)^{\xi}}\int_{x_1}^{mx_2}(y-x_1)^{\xi-1}\Upsilon(y)dy,$$

$$\int_0^1\tau^{\xi-1}\Upsilon\left(\tau x_2+m(1-\tau)x_1\right)d\tau=\frac{1}{(x_2-mx_1)^{\xi}}\int_{mx_1}^{x_2}(y-mx_1)^{\xi-1}\Upsilon(y)dy$$

and

$$\int_0^1\tau^{\xi-1}\Upsilon\left((1-\tau)x_2+m\tau x_1\right)d\tau=\frac{1}{(x_2-mx_1)^{\xi}}\int_{mx_1}^{x_2}(x_2-y)^{\xi-1}\Upsilon(y)dy.$$

Here, if we take these equations into account in (2.8) and firstly multiply both sides of the inequality by $\frac{\xi}{B(\xi)\Gamma(\xi)}$ and secondly add the terms $\frac{1-\xi}{(mx_2-x_1)^{\xi}B(\xi)}\left(\Upsilon(x_1)+\Upsilon(mx_2)\right)$, $\frac{1-\xi}{(x_2-mx_1)^{\xi}B(\xi)}\left(\Upsilon(mx_1)+\Upsilon(x_2)\right)$ to both sides of the resulting inequality, we obtain the inequality in (2.7). □

Remark 2. In (2.7), if we choose $\xi=1$, we achieve Theorem 5 in [12].

Theorem 3. *Assume that the assumptions given in the Theorem 1 are valid. Then, we have the following inequalities:*

$$\begin{aligned}&\frac{\Upsilon\left(\frac{x_1+x_2}{2}\right)}{B(\xi)\Gamma(\xi)}+\frac{1-\xi}{2(x_2-x_1)^{\xi}B(\xi)}\Upsilon\left(x_2\right)+\frac{m^{\xi+1}\left(1-\xi\right)}{2(x_2-x_1)^{\xi}B(\xi)}\Upsilon\left(\frac{x_1}{m}\right)\\ &\leq\frac{1}{2(x_2-x_1)^{\xi}}\left[{}^{AB}_{x_1}I^{\xi}_{x_2}\left\{\Upsilon(x_2)\right\}+m^{(\xi+1)AB}I^{\xi}_{\frac{x_2}{m}}\left\{\Upsilon\left(\frac{x_1}{m}\right)\right\}\right]\\ &\leq\frac{\xi}{2B(\xi)\Gamma(\xi)}\left[\frac{\Upsilon\left(x_1\right)+m^2\Upsilon\left(\frac{x_2}{m^2}\right)}{\xi+1}+\frac{m\left(\Upsilon\left(\frac{x_2}{m}\right)+\Upsilon\left(\frac{x_1}{m}\right)\right)}{\xi\left(\xi+1\right)}\right]\end{aligned}\tag{2.9}$$

$$+\frac{1-\xi}{2(x_2-x_1)^{\xi}B(\xi)}\Upsilon(x_2)+\frac{m^{\xi+1}(1-\xi)\Upsilon\left(\frac{x_1}{m}\right)}{2(x_2-x_1)^{\xi}B(\xi)}.$$

Proof. Since Υ is m-convex function, we can write $w_1=\tau x_1+(1-\tau)x_2$ and $w_2=(1-\tau)x_1+\tau x_2$ in the inequality

$$\Upsilon\left(\frac{w_1+w_2}{2}\right)\leq\frac{1}{2}\left(\Upsilon(w_1)+m\Upsilon\left(\frac{w_2}{m}\right)\right)$$

where all w_1 and w_2 belongs to $[0,\infty)$, we get

$$\Upsilon\left(\frac{x_1+x_2}{2}\right)\leq\frac{1}{2}\left[\Upsilon(\tau x_1+(1-\tau)x_2)+m\Upsilon\left(\frac{(1-\tau)x_1+\tau x_2}{m}\right)\right]. \tag{2.10}$$

If we multiply both sides of (2.10) by $\tau^{\xi-1}$, and integrate the resulting inequality over $[0,1]$ with respect to τ, we get

$$\begin{aligned}&\frac{1}{\xi}\Upsilon\left(\frac{x_1+x_2}{2}\right)\\&\leq\frac{1}{2(x_2-x_1)^{\xi}}\left[\int_{x_1}^{x_2}(x_2-y)^{\xi-1}\Upsilon(y)dy+m^{\xi+1}\int_{\frac{x_1}{m}}^{\frac{x_2}{m}}\left(y-\frac{x_1}{m}\right)^{\xi-1}\Upsilon(y)dy\right].\end{aligned} \tag{2.11}$$

If we multiply both sides of the inequality in (2.11) by $\frac{\xi}{B(\xi)\Gamma(\xi)}$ and add the terms $\frac{1-\xi}{2(x_2-x_1)^{\xi}B(\xi)}\Upsilon(x_2)$ and $\frac{m^{\xi+1}(1-\xi)}{2(x_2-x_1)^{\xi}B(\xi)}\Upsilon(\frac{x_1}{m})$ to both sides of the resulting inequality, we obtain the first inequality in (2.9).

We will follow a similar method to prove the second inequality in (2.9). Since Υ is m-convex function, we can write

$$\begin{aligned}&\frac{1}{2}\left[\Upsilon(\tau x_1+(1-\tau)x_2)+m\Upsilon\left(\frac{(1-\tau)x_1+\tau x_2}{m}\right)\right]\\&\leq\frac{1}{2}\left[\tau\Upsilon(x_1)+m(1-\tau)\Upsilon\left(\frac{x_2}{m}\right)+m\left((1-\tau)\Upsilon\left(\frac{x_1}{m}\right)+m\tau\Upsilon\left(\frac{x_2}{m^2}\right)\right)\right].\end{aligned}$$

After multiplying both sides of the above inequality by $\tau^{\xi-1}$, and integrating the resulting inequality over $[0,1]$ with respect to τ, if we multiply by $\frac{\xi}{B(\xi)\Gamma(\xi)}$ both sides of the last inequality and add the terms $\frac{1-\xi}{2(x_2-x_1)^{\xi}B(\xi)}\Upsilon(x_2)$ and $\frac{m^{\xi+1}(1-\xi)}{2(x_2-x_1)^{\xi}B(\xi)}\Upsilon(\frac{x_1}{m})$, we deduce the second inequality in (2.9). □

Remark 3. In (2.9), if we choose $\xi=1$, we achieve a result similar to the result in Theorem 4 in [12].

Theorem 4. *Under the assumptions of Theorem 1, let us define the mapping* $\Psi(w_1,w_2)_{(\tau)}:[0,1]\to\mathbb{R}$,

$$\Psi(w_1,w_2)_{(\tau)}=\frac{1}{2}\left[\Upsilon(\tau w_1+m(1-\tau)w_2)+\Upsilon((1-\tau)w_1+m\tau w_2)\right].$$

We have for all $\tau \in [0,1]$

$$\frac{\xi}{B(\xi)\Gamma(\xi)(x_2-x_1)^{\xi}}\int_{x_1}^{x_2}(x_2-y)^{\xi-1}\Psi\left(y,\frac{x_1+x_2}{2}\right)_{\left(\frac{x_2-y}{x_2-x_1}\right)}dy \tag{2.12}$$
$$\leq \frac{1}{2(x_2-x_1)^{\xi}}\left({}^{AB}_{x_1}I^{\xi}_{x_2}\{\Upsilon(x_2)\}\right)+\frac{m}{2B(\xi)\Gamma(\xi)}\Upsilon\left(\frac{x_1+x_2}{2}\right)-\frac{1-\xi}{2(x_2-x_1)^{\xi}B(\xi)}\Upsilon(x_2).$$

Proof. Considering that Υ is m-convex function, we can write

$$\begin{aligned}\Psi(w_1,w_2)_{(\tau)} &\leq \frac{1}{2}\left[\tau\Upsilon(w_1)+m(1-\tau)\Upsilon(w_2)+(1-\tau)\Upsilon(w_1)+m\tau\Upsilon(w_2)\right]\\ &= \frac{1}{2}\left[\Upsilon(w_1)+m\Upsilon(w_2)\right]\end{aligned}$$

and

$$\Psi\left(w_1,\frac{x_1+x_2}{2}\right)_{(\tau)} \leq \frac{1}{2}\left[\Upsilon(w_1)+m\Upsilon\left(\frac{x_1+x_2}{2}\right)\right]. \tag{2.13}$$

If we write $w_1=\tau x_1+(1-\tau)x_2$ in the equation (2.13), and after that if we multiply the both sides of resulting inequality with $\tau^{\xi-1}$ and integrate the last inequality over $[0,1]$, we have

$$\frac{1}{(x_2-x_1)^{\xi}}\int_{x_1}^{x_2}(x_2-y)^{\xi-1}\Psi\left(y,\frac{x_1+x_2}{2}\right)_{\left(\frac{x_2-y}{x_2-x_1}\right)}dy \tag{2.14}$$
$$\leq \frac{1}{2(x_2-x_1)^{\xi}}\int_{x_1}^{x_2}(x_2-y)^{\xi-1}\Upsilon(y)dy+\frac{m}{2\xi}\Upsilon\left(\frac{x_1+x_2}{2}\right).$$

To complete the proof, we must multiply the inequality in (2.14) by $\frac{\xi}{B(\xi)\Gamma(\xi)}$ and we must add $\frac{1-\xi}{2(x_2-x_1)^{\xi}B(\xi)}\Upsilon(x_2)$. So we obtain the requested result. □

Remark 4. In (2.12), if we choose $\xi=1$, we achieve the $\alpha=1$ special case of the inequality in Theorem 6 given by Set *et al.* in [29].

Theorem 5. *Let* $0\leq x_1<x_2$, *and* $\Upsilon,\Omega:[0,\infty)\to[0,\infty)$ *be functions such that* $\Upsilon\Omega\in L[x_1,x_2]$. *If* Υ *and* Ω *are* m_1*-convex and* m_2*-convex on* $[x_1,x_2]$ *respectively with* $m_1,m_2\in(0,1]$ *we obtain*

$$\frac{1}{(x_2-x_1)^{\xi}}\left({}^{AB}_{x_1}I^{\xi}_{x_2}\{(\Upsilon\Omega)(x_2)\}\right)\leq\frac{\xi}{B(\xi)\Gamma(\xi)}\left[\frac{\Upsilon(x_1)\Omega(x_1)}{\xi+2}\right. \tag{2.15}$$
$$\left.+\left(m_2\Upsilon(x_1)\Omega\left(\frac{x_2}{m_2}\right)+m_1\Omega(x_1)\Upsilon\left(\frac{x_2}{m_1}\right)\right)\frac{1}{(\xi+1)(\xi+2)}\right]$$
$$+\frac{2m_1m_2}{B(\xi)\Gamma(\xi)(\xi+1)(\xi+2)}\Upsilon\left(\frac{x_2}{m_1}\right)\Omega\left(\frac{x_2}{m_2}\right)+\frac{1-\xi}{(x_2-x_1)^{\xi}B(\xi)}(\Upsilon\Omega)(x_2)$$

and

$$\frac{1}{(x_2-x_1)^{\xi}}\left({}^{AB}I_{x_2}^{\xi}\{(\Upsilon\Omega)(x_1)\}\right) \leq \frac{\xi}{B(\xi)\Gamma(\xi)}\left[\frac{\Upsilon(x_2)\Omega(x_2)}{\xi+2}\right. \tag{2.16}$$
$$\left.+\left(m_2\Upsilon(x_2)\Omega\left(\frac{x_1}{m_2}\right)+m_1\Omega(x_2)\Upsilon\left(\frac{x_1}{m_1}\right)\right)\frac{1}{(\xi+1)(\xi+2)}\right]$$
$$+\frac{2m_1m_2}{B(\xi)\Gamma(\xi)(\xi+1)(\xi+2)}\Upsilon\left(\frac{x_1}{m_1}\right)\Omega\left(\frac{x_1}{m_2}\right)+\frac{1-\xi}{(x_2-x_1)^{\xi}B(\xi)}(\Upsilon\Omega)(x_1)$$

where $\xi \in (0,1]$.

Proof. We will start by proving the inequality in the equation (2.15). In the hypothesis of the Theorem 5, it is given that the functions Υ and Ω are m_1-convex and m_2-convex on $[x_1,x_2]$ respectively. So, we can write

$$\Upsilon(\tau x_1+(1-\tau)x_2) \leq \tau\Upsilon(x_1)+m_1(1-\tau)\Upsilon\left(\frac{x_2}{m_1}\right) \tag{2.17}$$

and

$$\Omega(\tau x_1+(1-\tau)x_2) \leq \tau\Omega(x_1)+m_2(1-\tau)\Omega\left(\frac{x_2}{m_2}\right). \tag{2.18}$$

If we multiply (2.17) and (2.18), we obtain

$$\Upsilon(\tau x_1+(1-\tau)x_2)\Omega(\tau x_1+(1-\tau)x_2) \leq \tau^2\Upsilon(x_1)\Omega(x_1) \tag{2.19}$$
$$+\tau(1-\tau)m_2\Upsilon(x_1)\Omega\left(\frac{x_2}{m_2}\right)+\tau(1-\tau)m_1\Omega(x_1)\Upsilon\left(\frac{x_2}{m_1}\right)$$
$$+(1-\tau)^2m_1m_2\Upsilon\left(\frac{x_2}{m_1}\right)\Omega\left(\frac{x_2}{m_2}\right).$$

After multiplying both sides of (2.19) by $\tau^{\xi-1}$, and integrating the resulting inequality over $[0,1]$ with respect to τ, if we multiply both sides of the last inequality by $\frac{\xi}{B(\xi)\Gamma(\xi)}$ and add the term $\frac{1-\xi}{(x_2-x_1)^{\xi}B(\xi)}(\Upsilon\Omega)(x_2)$ we complete the proof of the inequality in the equation (2.15).

Secondly, to prove the inequality in the equation (2.16), one can reach the result by using the method similar to the proof of the first inequality for the product of the following inequalities:

$$\Upsilon(\tau x_2+(1-\tau)x_1) \leq \tau\Upsilon(x_2)+m_1(1-\tau)\Upsilon\left(\frac{x_1}{m_1}\right)$$

and

$$\Omega(\tau x_2+(1-\tau)x_1) \leq \tau\Omega(x_2)+m_2(1-\tau)\Omega\left(\frac{x_1}{m_2}\right).$$

Based on the above estimates, we can easily obtain the desired results. □

Corollary 1. *If we choose $m_1 = m_2 = 1$ in Theorem 5, we have the following inequalities*

$$\frac{1}{(x_2-x_1)^{\xi}}\left({}^{AB}_{x_1}I^{\xi}_{x_2}\{(\Upsilon\Omega)(x_2)\}\right) \leq \frac{\xi}{B(\xi)\Gamma(\xi)}\left[\frac{\Upsilon(x_1)\Omega(x_1)}{\xi+2}\right.$$
$$\left.+(\Upsilon(x_1)\Omega(x_2)+\Omega(x_1)\Upsilon(x_2))\frac{1}{(\xi+1)(\xi+2)}\right]$$
$$+\frac{2}{B(\xi)\Gamma(\xi)(\xi+1)(\xi+2)}\Upsilon(x_2)\Omega(x_2)+\frac{1-\xi}{(x_2-x_1)^{\xi}B(\xi)}(\Upsilon\Omega)(x_2)$$

and

$$\frac{1}{(x_2-x_1)^{\xi}}\left({}^{AB}I^{\xi}_{x_2}\{(\Upsilon\Omega)(x_1)\}\right) \leq \frac{\xi}{B(\xi)\Gamma(\xi)}\left[\frac{\Upsilon(x_2)\Omega(x_2)}{\xi+2}\right.$$
$$\left.+(\Upsilon(x_2)\Omega(x_1)+\Omega(x_2)\Upsilon(x_1))\frac{1}{(\xi+1)(\xi+2)}\right]$$
$$+\frac{2}{B(\xi)\Gamma(\xi)(\xi+1)(\xi+2)}\Upsilon(x_1)\Omega(x_1)+\frac{1-\xi}{(x_2-x_1)^{\xi}B(\xi)}(\Upsilon\Omega)(x_1)$$

where Υ and Ω are convex functions on $[0,\infty)$.

Remark 5. In inequalities (2.15) and (2.16), if we choose $\xi = 1$ and evaluate the obtained inequalities together, we achieve Theorem 2.4 in [21].

3. Hermite-Hadamard type inequalities for m-convex functions via Atangana-Baleanu integral operators

The lemma containing the Atangana-Baleanu integral operator below incorporates the left side of the Hermite-Hadamard inequality.

Lemma 1. *Let $x_1 < x_2$, $x_1, x_2 \in J^{\circ}$ and $\Upsilon : J \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function on J°. If $\Upsilon' \in L[x_1, mx_2]$, identity for Atangana-Baleanu integral operators in equation (3.1) is valid for all $\tau, \xi \in [0,1]$ and $m \in (0,1]$*

$$\begin{aligned}
&{}^{AB}_{\frac{x_1+mx_2}{2}}I^{\xi}_{mx_2}\{\Upsilon(mx_2)\} + {}^{AB}I^{\xi}_{\frac{x_1+mx_2}{2}}\{\Upsilon(x_1)\} \\
&\quad - \frac{(mx_2-x_1)^{\xi}}{2^{\xi-1}B(\xi)\Gamma(\xi)}\Upsilon\left(\frac{x_1+mx_2}{2}\right) - \frac{1-\xi}{B(\xi)}[\Upsilon(x_1)+\Upsilon(mx_2)] \\
&= \frac{(mx_2-x_1)^{\xi+1}}{B(\xi)\Gamma(\xi)}\left\{\int_0^{\frac{1}{2}}\tau^{\xi}\Upsilon'(\tau x_1+m(1-\tau)x_2)d\tau\right. \\
&\quad \left. - \int_{\frac{1}{2}}^{1}(1-\tau)^{\xi}\Upsilon'(\tau x_1+m(1-\tau)x_2)d\tau\right\}.
\end{aligned} \tag{3.1}$$

Proof. By making use of integration by parts, we can write

$$\frac{(mx_2-x_1)^{\xi+1}}{B(\xi)\Gamma(\xi)}\int_0^{\frac{1}{2}}\tau^{\xi}\Upsilon'(\tau x_1+m(1-\tau)x_2)d\tau \tag{3.2}$$
$$=\frac{(mx_2-x_1)^{\xi+1}}{B(\xi)\Gamma(\xi)}\left[\tau^{\xi}\frac{\Upsilon(\tau x_1+m(1-\tau)x_2)}{x_1-mx_2}\Bigg|_0^{\frac{1}{2}}-\int_0^{\frac{1}{2}}\xi\tau^{\xi-1}\frac{\Upsilon(\tau x_1+m(1-\tau)x_2)}{x_1-mx_2}d\tau\right]$$
$$=-\frac{(mx_2-x_1)^{\xi}}{2^{\xi}B(\xi)\Gamma(\xi)}\Upsilon\left(\frac{x_1+mx_2}{2}\right)+\frac{\xi}{B(\xi)\Gamma(\xi)}\int_{\frac{x_1+mx_2}{2}}^{mx_2}(mx_2-y)^{\xi-1}\Upsilon(y)\,dy$$

and

$$-\frac{(mx_2-x_1)^{\xi+1}}{B(\xi)\Gamma(\xi)}\int_{\frac{1}{2}}^{1}(1-\tau)^{\xi}\Upsilon'(\tau x_1+m(1-\tau)x_2)d\tau \tag{3.3}$$
$$=-\frac{(mx_2-x_1)^{\xi+1}}{B(\xi)\Gamma(\xi)}\left[(1-\tau)^{\xi}\frac{\Upsilon(\tau x_1+m(1-\tau)x_2)}{x_1-mx_2}\Bigg|_{\frac{1}{2}}^{1}\right.$$
$$\left.+\int_{\frac{1}{2}}^{1}\xi(1-\tau)^{\xi-1}\frac{\Upsilon(\tau x_1+m(1-\tau)x_2)}{x_1-mx_2}d\tau\right]$$
$$=-\frac{(mx_2-x_1)^{\xi}}{2^{\xi}B(\xi)\Gamma(\xi)}\Upsilon\left(\frac{x_1+mx_2}{2}\right)+\frac{\xi}{B(\xi)\Gamma(\xi)}\int_{x_1}^{\frac{x_1+mx_2}{2}}(y-x_1)^{\xi-1}\Upsilon(y)\,dy.$$

If we add (3.2) and (3.3), and then by adding $\frac{1-\xi}{B(\xi)}\Upsilon(x_1)+\frac{1-\xi}{B(\xi)}\Upsilon(mx_2)$ to two sides of resulting equality, we complete the proof of Lemma 1. □

Theorem 6. *Let $x_1<x_2$, $x_1,x_2\in J^{\circ}$ and $\Upsilon:J\subset[0,\infty)\longrightarrow\mathbb{R}$ be a differentiable function on J° and $\Upsilon'\in L[x_1,mx_2]$. If $|\Upsilon'|$ is an m-convex function on $[x_1,x_2]$, following inequality is achieved*

$$\Bigg|{}^{AB}_{\frac{x_1+mx_2}{2}}I^{\xi}_{mx_2}\{\Upsilon(mx_2)\}+{}^{AB}I^{\xi}_{\frac{x_1+mx_2}{2}}\{\Upsilon(x_1)\}$$
$$-\frac{(mx_2-x_1)^{\xi}}{2^{\xi-1}B(\xi)\Gamma(\xi)}\Upsilon\left(\frac{x_1+mx_2}{2}\right)-\frac{1-\xi}{B(\xi)}[\Upsilon(x_1)+\Upsilon(mx_2)]\Bigg|$$
$$\leq\frac{(mx_2-x_1)^{\xi+1}}{B(\xi)\Gamma(\xi)}\left(\frac{1}{2^{\xi+1}(\xi+1)}\right)\left(|\Upsilon'(x_1)|+m|\Upsilon'(x_2)|\right)$$

for all $\xi\in[0,1]$ and $m\in(0,1]$.

Proof. By using the equality in (3.1), property of modulus, we have

$$\Bigg|{}^{AB}_{\frac{x_1+mx_2}{2}}I^{\xi}_{mx_2}\{\Upsilon(mx_2)\}+{}^{AB}I^{\xi}_{\frac{x_1+mx_2}{2}}\{\Upsilon(x_1)\}$$

$$
\begin{aligned}
&\left. -\frac{(mx_2-x_1)^{\xi}}{2^{\xi-1}B(\xi)\Gamma(\xi)}\Upsilon\left(\frac{x_1+mx_2}{2}\right)-\frac{1-\xi}{B(\xi)}\left[\Upsilon(x_1)+\Upsilon(mx_2)\right]\right| \\
&\leq \frac{(mx_2-x_1)^{\xi+1}}{B(\xi)\Gamma(\xi)}\left\{\int_0^{\frac{1}{2}}\tau^{\xi}\left|\Upsilon'(\tau x_1+m(1-\tau)x_2)\right|d\tau\right. \\
&\quad \left.+\int_{\frac{1}{2}}^{1}(1-\tau)^{\xi}\left|\Upsilon'(\tau x_1+m(1-\tau)x_2)\right|d\tau\right\}.
\end{aligned}
$$

By applying m-convexity of $|\Upsilon'|$, we obtain

$$
\begin{aligned}
&\left|{}^{AB}_{\frac{x_1+mx_2}{2}}I^{\xi}_{mx_2}\{\Upsilon(mx_2)\}+{}^{AB}I^{\xi}_{\frac{x_1+mx_2}{2}}\{\Upsilon(x_1)\}\right. \\
&\qquad \left. -\frac{(mx_2-x_1)^{\xi}}{2^{\xi-1}B(\xi)\Gamma(\xi)}\Upsilon\left(\frac{x_1+mx_2}{2}\right)-\frac{1-\xi}{B(\xi)}\left[\Upsilon(x_1)+\Upsilon(mx_2)\right]\right| \\
&\leq \frac{(mx_2-x_1)^{\xi+1}}{B(\xi)\Gamma(\xi)}\left\{\int_0^{\frac{1}{2}}\tau^{\xi}\left[\tau\left|\Upsilon'(x_1)\right|+m(1-\tau)\left|\Upsilon'(x_2)\right|\right]d\tau\right. \\
&\quad \left.+\int_{\frac{1}{2}}^{1}(1-\tau)^{\xi}\left[\tau\left|\Upsilon'(x_1)\right|+m(1-\tau)\left|\Upsilon'(x_2)\right|\right]d\tau\right\}.
\end{aligned}
$$

We complete the proof by making the necessary calculations in above. □

Theorem 7. *Let $x_1<x_2$, $x_1,x_2\in J^{\circ}$ and $\Upsilon: J\subset[0,\infty)\longrightarrow\mathbb{R}$ be a differentiable function on J° and $\Upsilon'\in L[x_1,mx_2]$. If $|\Upsilon'|^q$ is an m-convex function on $[x_1,x_2]$, following inequality is achieved for all $\xi\in[0,1]$ and $m\in(0,1]$*

$$
\begin{aligned}
&\left|{}^{AB}_{\frac{x_1+mx_2}{2}}I^{\xi}_{mx_2}\{\Upsilon(mx_2)\}+{}^{AB}I^{\xi}_{\frac{x_1+mx_2}{2}}\{\Upsilon(x_1)\}\right. \\
&\qquad \left. -\frac{(mx_2-x_1)^{\xi}}{2^{\xi-1}B(\xi)\Gamma(\xi)}\Upsilon\left(\frac{x_1+mx_2}{2}\right)-\frac{1-\xi}{B(\xi)}\left[\Upsilon(x_1)+\Upsilon(mx_2)\right]\right| \\
&\leq \frac{(mx_2-x_1)^{\xi+1}}{B(\xi)\Gamma(\xi)}\left(\frac{1}{2^{\xi+1}(\xi+1)}\right)^{\frac{1}{p}} \\
&\quad \times\left\{\left(\frac{1}{2^{\xi+2}(\xi+2)}\left|\Upsilon'(x_1)\right|^q+\frac{m(\xi+3)}{2^{\xi+2}(\xi+1)(\xi+2)}\left|\Upsilon'(x_2)\right|^q\right)^{\frac{1}{q}}\right. \\
&\quad \left.+\left(\frac{(\xi+3)}{2^{\xi+2}(\xi+1)(\xi+2)}\left|\Upsilon'(x_1)\right|^q+\frac{m}{2^{\xi+2}(\xi+2)}\left|\Upsilon'(x_2)\right|^q\right)^{\frac{1}{q}}\right\}
\end{aligned}
$$

where $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. If we use Hölder inequality, we have

$$\begin{aligned}
&\Bigg|{}^{AB}_{\frac{x_1+mx_2}{2}}I^{\xi}_{mx_2}\{\Upsilon(mx_2)\}+{}^{AB}I^{\xi}_{\frac{x_1+mx_2}{2}}\{\Upsilon(x_1)\} \\
&\qquad -\frac{(mx_2-x_1)^{\xi}}{2^{\xi-1}B(\xi)\Gamma(\xi)}\Upsilon\left(\frac{x_1+mx_2}{2}\right)-\frac{1-\xi}{B(\xi)}\left[\Upsilon(x_1)+\Upsilon(mx_2)\right]\Bigg| \\
&\leq \frac{(mx_2-x_1)^{\xi+1}}{B(\xi)\Gamma(\xi)}\left\{\left(\int_0^{\frac{1}{2}}\tau^{\xi}d\tau\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\tau^{\xi}\left|\Upsilon'(\tau x_1+m(1-\tau)x_2)\right|^q d\tau\right)^{\frac{1}{q}}\right. \\
&\qquad \left.+\left(\int_{\frac{1}{2}}^{1}(1-\tau)^{\xi}d\tau\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}(1-\tau)^{\xi}\left|\Upsilon'(\tau x_1+m(1-\tau)x_2)\right|^q d\tau\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

By using m-convexity of $|\Upsilon'|^q$, we get

$$\begin{aligned}
&\Bigg|{}^{AB}_{\frac{x_1+mx_2}{2}}I^{\xi}_{mx_2}\{\Upsilon(mx_2)\}+{}^{AB}I^{\xi}_{\frac{x_1+mx_2}{2}}\{\Upsilon(x_1)\} \\
&\qquad -\frac{(mx_2-x_1)^{\xi}}{2^{\xi-1}B(\xi)\Gamma(\xi)}\Upsilon\left(\frac{x_1+mx_2}{2}\right)-\frac{1-\xi}{B(\xi)}\left[\Upsilon(x_1)+\Upsilon(mx_2)\right]\Bigg| \\
&\leq \frac{(mx_2-x_1)^{\xi+1}}{B(\xi)\Gamma(\xi)}\left\{\left(\int_0^{\frac{1}{2}}\tau^{\xi}d\tau\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\tau^{\xi}\left[\tau\left|\Upsilon'(x_1)\right|^q+m(1-\tau)\left|\Upsilon'(x_2)\right|^q\right]d\tau\right)^{\frac{1}{q}}\right. \\
&\qquad \left.+\left(\int_{\frac{1}{2}}^{1}(1-\tau)^{\xi}d\tau\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}(1-\tau)^{\xi}\left[\tau\left|\Upsilon'(x_1)\right|^q+m(1-\tau)\left|\Upsilon'(x_2)\right|^q\right]d\tau\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

If we calculate the integrals above, we have the desired result. □

Theorem 8. *Suppose that the assumptions given in the Theorem 7 are valid. Then, we have the following inequality:*

$$\begin{aligned}
&\Bigg|{}^{AB}_{\frac{x_1+mx_2}{2}}I^{\xi}_{mx_2}\{\Upsilon(mx_2)\}+{}^{AB}I^{\xi}_{\frac{x_1+mx_2}{2}}\{\Upsilon(x_1)\} \\
&\qquad -\frac{(mx_2-x_1)^{\xi}}{2^{\xi-1}B(\xi)\Gamma(\xi)}\Upsilon\left(\frac{x_1+mx_2}{2}\right)-\frac{1-\xi}{B(\xi)}\left[\Upsilon(x_1)+\Upsilon(mx_2)\right]\Bigg| \\
&\leq \frac{(mx_2-x_1)^{\xi+1}}{B(\xi)\Gamma(\xi)}\left(\frac{1}{2^{\xi p+1}(\xi p+1)}\right)^{\frac{1}{p}} \\
&\qquad \times\left[\left(\frac{|\Upsilon'(x_1)|^q+3m|\Upsilon'(x_2)|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{3|\Upsilon'(x_1)|^q+m|\Upsilon'(x_2)|^q}{8}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Proof. By Hölder's inequality in a different variant, we can write

$$\Bigg|{}^{AB}_{\frac{x_1+mx_2}{2}}I^{\xi}_{mx_2}\{\Upsilon(mx_2)\}+{}^{AB}I^{\xi}_{\frac{x_1+mx_2}{2}}\{\Upsilon(x_1)\}$$
$$-\frac{(mx_2-x_1)^{\xi}}{2^{\xi-1}B(\xi)\Gamma(\xi)}\Upsilon\left(\frac{x_1+mx_2}{2}\right)-\frac{1-\xi}{B(\xi)}[\Upsilon(x_1)+\Upsilon(mx_2)]\Bigg|$$
$$\leq\frac{(mx_2-x_1)^{\xi+1}}{B(\xi)\Gamma(\xi)}\left\{\left(\int_0^{\frac{1}{2}}\tau^{\xi p}d\tau\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|\Upsilon'(\tau x_1+m(1-\tau)x_2)\right|^q d\tau\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^1(1-\tau)^{\xi p}d\tau\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|\Upsilon'(\tau x_1+m(1-\tau)x_2)\right|^q d\tau\right)^{\frac{1}{q}}\right\}$$

By taking into account the m-convexity of $|\Upsilon'|^q$, we have

$$\Bigg|{}^{AB}_{\frac{x_1+mx_2}{2}}I^{\xi}_{mx_2}\{\Upsilon(mx_2)\}+{}^{AB}I^{\xi}_{\frac{x_1+mx_2}{2}}\{\Upsilon(x_1)\}$$
$$-\frac{(mx_2-x_1)^{\xi}}{2^{\xi-1}B(\xi)\Gamma(\xi)}\Upsilon\left(\frac{x_1+mx_2}{2}\right)-\frac{1-\xi}{B(\xi)}[\Upsilon(x_1)+\Upsilon(mx_2)]\Bigg|$$
$$\leq\frac{(mx_2-x_1)^{\xi+1}}{B(\xi)\Gamma(\xi)}\left\{\left(\int_0^{\frac{1}{2}}\tau^{\xi p}d\tau\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left[\tau\left|\Upsilon'(x_1)\right|^q+m(1-\tau)\left|\Upsilon'(x_2)\right|^q\right]d\tau\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^1(1-\tau)^{\xi p}d\tau\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left[\tau\left|\Upsilon'(x_1)\right|^q+m(1-\tau)\left|\Upsilon'(x_2)\right|^q\right]d\tau\right)^{\frac{1}{q}}\right\}.$$

We complete the proof by calculating the integrals. □

Theorem 9. *Assume that the assumptions given in the Theorem 7 are valid. Then, we have the following inequality:*

$$\Bigg|{}^{AB}_{\frac{x_1+mx_2}{2}}I^{\xi}_{mx_2}\{\Upsilon(mx_2)\}+{}^{AB}I^{\xi}_{\frac{x_1+mx_2}{2}}\{\Upsilon(x_1)\}$$
$$-\frac{(mx_2-x_1)^{\xi}}{2^{\xi-1}B(\xi)\Gamma(\xi)}\Upsilon\left(\frac{x_1+mx_2}{2}\right)-\frac{1-\xi}{B(\xi)}[\Upsilon(x_1)+\Upsilon(mx_2)]\Bigg|$$
$$\leq\frac{(mx_2-x_1)^{\xi+1}}{B(\xi)\Gamma(\xi)}\left(\frac{q-1}{2^{\xi\left(\frac{q-p}{q-1}\right)+1}(\xi(q-p)+q-1)}\right)^{\frac{1}{p}}$$
$$\times\left\{\left(\frac{1}{2^{\xi p+2}(\xi p+2)}\left|\Upsilon'(x_1)\right|^q+\frac{m(\xi p+3)}{2^{\xi p+2}(\xi p+1)(\xi p+2)}\left|\Upsilon'(x_2)\right|^q\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{(\xi p+3)}{2^{\xi p+2}(\xi p+1)(\xi p+2)}\left|\Upsilon'(x_1)\right|^q+\frac{m}{2^{\xi p+2}(\xi p+2)}\left|\Upsilon'(x_2)\right|^q\right)^{\frac{1}{q}}\Bigg\}$$

where $q \geq p > 1$.

Proof. Applying by different way of Hölder inequality, we have

$$\Bigg|{}^{AB}_{\frac{x_1+mx_2}{2}}I^{\xi}_{mx_2}\{\Upsilon(mx_2)\}+{}^{AB}I^{\xi}_{\frac{x_1+mx_2}{2}}\{\Upsilon(x_1)\}$$
$$-\frac{(mx_2-x_1)^{\xi}}{2^{\xi-1}B(\xi)\Gamma(\xi)}\Upsilon\left(\frac{x_1+mx_2}{2}\right)-\frac{1-\xi}{B(\xi)}\left[\Upsilon(x_1)+\Upsilon(mx_2)\right]\Bigg|$$
$$\leq\frac{(mx_2-x_1)^{\xi+1}}{B(\xi)\Gamma(\xi)}\Bigg\{\left(\int_0^{\frac{1}{2}}\tau^{\xi\left(\frac{q-p}{q-1}\right)}d\tau\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\tau^{\xi p}\left|\Upsilon'(\tau x_1+m(1-\tau)x_2)\right|^q d\tau\right)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{1}{2}}^{1}(1-\tau)^{\xi\left(\frac{q-p}{q-1}\right)}d\tau\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}(1-\tau)^{\xi p}\left|\Upsilon'(\tau x_1+m(1-\tau)x_2)\right|^q d\tau\right)^{\frac{1}{q}}\Bigg\}.$$

If we use m-convexity of $|\Upsilon'|^q$ above and calculate the integrals, we complete the proof. □

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NEW CURVATURE TENSORS ALONG RIEMANNIAN SUBMERSIONS

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Abstract. In 1966, B. O'Neill [The fundamental equations of a submersion, Michigan Math. J., Volume 13, Issue 4 (1966), 459-469.] defined some fundamental equations and curvature relations between the total space, the base space and the fibres on a submersion. In the present paper, we define new curvature tensors on Riemannian submersions such as Weyl projective curvature tensor, concircular curvature tensor, conharmonic curvature tensor, conformal curvature tensor and $M-$projective curvature tensor, respectively. Finally, we obtain some results in case of the total space of Riemannian submersions has umbilical fibres for any curvature tensors mentioned by the above.

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Keywords: Riemannian submersion, Weyl projective curvature tensor, M-projective curvature tensor, concircular curvature tensor, conformal curvature tensor, conharmonic curvature tensor

1. Introduction and Preliminaries

In differential geometry, an important tool to define the curvature of $n-$ dimensional spaces (such as Riemannian manifolds) is the Riemannian curvature tensor. The tensor has played an important role both general relativty and gravity. In this manner, Mishra in [13] defined some new curvature tensors on Riemannian manifolds such as concircular curvature tensor, conharmonic curvature tensor, conformal curvature tensor, respectively. Taking into account the paper of Mishra, Pokhariyal and Mishra defined the Wely projective curvature tensor on Riemannian manifolds [17]. Afterwards, Ojha defined $M-$ Projective curvature tensor [15].

Riemannian submersion appears to have been studied and its differential geometry was first defined by O'Neill 1966 and Gray 1967. We note that Riemannian submersions have been studied widely not only in mathematics, but also in theoretical pyhsics because of their applications in the Yang-Mills theory, Kaluza Klein theory, super gravity, relativity and superstring theories (see [3, 4, 9, 10, 14, 18]). Most of the studies related to Riemannian submersion can be found in the books ([5, 6]). In 1966, B. O'Neill has defined a paper related to some fundamental equations of a

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submersions. In that paper, he has given some curvature relations on Riemannian submersions.

In this study, in addition to the curvature relations previously defined on Riemannian submersion, we investigate new curvature tensors on a Riemannian submersion and the curvature properties of these tensors. In the present paper, in the first part of our study, the basic definitions and theorems that we will use throughout the paper are given. In sections 2-6 include the Weyl projective curvature tensor, concircular curvature tensor, conharmonic curvature tensor, conformal curvature tensor and M-projective curvature tensor relations for a Riemannian submersion respectively. Also various results are obtained by examining the conditions for having total umbilical fibers.

Now, we will give the basic definitions and theorems without proofs that we will use throughout the paper.

Definition 1. Let (M,g) and (N,g_N) be Riemannian manifolds, where-$dim(M) > dim(N)$. A surjective mapping $\pi : (M,g) \to (N,g_N)$ is called a *Riemannian submersion* [16] if:

(S1) The rank of π equals $dim(N)$.
In this case, for each $q \in N$, $\pi^{-1}(q) = \pi_q^{-1}$ is a k-dimensional submanifold of M and called a *fiber*, where $k = dim(M) - dim(N)$. A vector field on M is called *vertical* (resp. *horizontal*) if it is always tangent (resp. orthogonal) to fibers. A vector field X on M is called *basic* if X is horizontal and π-related to a vector field X_* on N, i.e. , $\pi_*(X_p) = X_{*\pi(p)}$ for all $p \in M$, where π_* is derivative or differential map of π. We will denote by $\mathcal{V}$ and $\mathcal{H}$ the projections on the vertical distribution $ker\pi_*$, and the horizontal distribution $ker\pi_*^{\perp}$, respectively. As usual, the manifold (M,g) is called *total manifold* and the manifold (N,g_N) is called *base manifold* of the submersion $\pi : (M,g) \to (N,g_N)$.

(S2) π_* preserves the lengths of the horizontal vectors.

These conditions are equivalent to say that the derivative map π_* of π, restricted to $ker\pi_*^{\perp}$, is a linear isometry.

If X and Y are the basic vector fields, π-related to X_N, Y_N, we have the following facts:

(1) $g(X,Y) = g_N(X_N,Y_N) \circ \pi$,
(2) $h[X,Y]$ is the basic vectr field π-related to $[X_N,Y_N]$,
(3) $h(\nabla_X Y)$ is the basic vector field π-related to $\nabla^N{}_{X_N} Y_N$,

for any vertical vector field $\mathcal{V}$, $[X,Y]$ is the vertical.

The geometry of Riemannian submersions is characterized by O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$, defined as follows:

The geometry of Riemannian submersions is characterized by O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$, defined as follows:

$$\mathcal{T}_E F = \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F, \tag{1.1}$$

$$\mathcal{A}_E F = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F \tag{1.2}$$

for any vector fields E and F on M, where ∇ is the Levi-Civita connection, $\mathcal{V}$ and $\mathcal{H}$ are orthogonal projections on vertical and horizontal spaces, respectively.

Now, we are going to give an example for a Riemannian submersion as follows:

Example 1. Let $\phi : \left(\mathbb{R}^4, g_{\mathbb{R}^4}\right) \longrightarrow \left(\mathbb{R}^2, g_{\mathbb{R}^2}\right)$ be a submersion defined by

$$\phi\left(u_1, u_2, u_3, u_4\right) = \left(\frac{1}{\sqrt{2}}(u_1 - u_2), \sqrt{u_3^2 + u_4^2}\right).$$

Then, the Jacobian matrix of ϕ is:

$$\phi_* = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{u_3}{W} & \frac{u_4}{W} \end{bmatrix}$$

where $W = \sqrt{u_3^2 + u_4^2}$. The rank of the map equal to 2. It means that the map is a submersion. A straight computations yields

$$\ker\phi_* = \operatorname{span}\left\{V_1 = \frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2},\ \ V_2 = u_4\frac{\partial}{\partial u_3} - u_3\frac{\partial}{\partial u_4}\right\}$$

and

$$(\ker\phi_*)^{\perp} = \operatorname{span}\left\{X_1 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial u_1} - \frac{\partial}{\partial u_2}\right),\ \ X_2 = \frac{u_3}{W}\frac{\partial}{\partial u_3} + \frac{u_4}{W}\frac{\partial}{\partial u_4}\right\}.$$

Also by direct computations yields

$$\phi_*\left(X_1\right) = \partial v_1 \text{ and } \phi_*\left(X_2\right) = \partial v_2$$

Thus, it is easy to see that

$$g_{\mathbb{R}^2}\left(\phi_*\left(X_i\right), \phi_*\left(X_i\right)\right) = g_{\mathbb{R}^4}\left(X_i, X_i\right),\ \ i = 1, 2$$

Hence ϕ is a Riemannian submersion.

Definition 2. Let (M, g) and (G, g') Riemannian manifolds of $n-$ dimensional and the horizontal distribution of (M, g) is $\mathcal{H}$. Let's show $(1,3)$-order curvature tensor field on $X^h(M)$ with R^*. For any $X, Y, Z \in \chi^h(M)$ and $p \in M$

$$R^G_{\pi(p)}(\pi_{*p}X_p, \pi_{*p}Y_p, \pi_{*p}Z_p). \tag{1.3}$$

We now recall the following curvature relations for a Riemannian submersion from [6] and [16].

Theorem 1. *(M,g) and (G,g') Riemannian manifolds,*

$$\pi : (M,g) \to (G,g')$$

a Riemannian submersion and R^M, R^G and $\hat{R}$ be Riemannian curvature tensors of M,G and $(\pi^{-1}(x),\hat{g}_x)$ fibre respectively. In this case, there are the following equations for any $U,V,W,F \in \chi^v(M)$ and $X,Y,Z,H \in \chi^h(M)$

$$\begin{aligned} g(R^M(X,Y)Z,H) &= g(R^G(X,Y)Z,H) + 2g(A_XY,A_ZH) \\ &\quad - g(A_YZ,A_XH) + g(A_XZ,A_YH), \end{aligned} \tag{1.4}$$

$$\begin{aligned} g(R^M(X,Y)Z,V) &= -g((\nabla_Z A)_X Y,V) - g(A_XY,T_VZ) \\ &\quad + g(A_YZ,T_VX) - g(A_XZ,T_VY), \end{aligned} \tag{1.5}$$

$$\begin{aligned} g(R^M(X,Y)V,W) &= g((\nabla_V A)_X Y,W) - g((\nabla_W A)_X Y,V) \\ &\quad + g(A_XVA_YW) + g(A_XW,A_YV) \\ &\quad - g(T_VX,T_WY) + g(T_WX,T_VY), \end{aligned} \tag{1.6}$$

$$\begin{aligned} g(R^M(X,V)Y,W) &= g((\nabla_X T)_V W,Y) - g((\nabla_V A)_X Y,W) \\ &\quad - g(T_VX,T_WY) + g(A_XV,A_YW), \end{aligned} \tag{1.7}$$

$$g(R^M(U,V)W,X) = g((\nabla_U T)_V W,X) - g((\nabla_V T)_U W,X) \tag{1.8}$$

and

$$g(R^M(U,V)W,F) = g(\hat{R}(U,V)W,F) + g(T_UW,T_VF) - g(T_VW,T_UF). \tag{1.9}$$

Definition 3. [6] Let (M,g) be a Riemannian manifold and a local orthonormal frame of the vertical distribution ν is $\{U_j\}_{1\le j\le r}$. Then N, the horizontal vector field on (M,g) is locally defined by

$$N = \sum_{j=1}^{r} T_{U_j}U_j.$$

Proposition 1. *Let (M,g) and (G,g') Riemannian manifolds,*

$$\pi : (M,g) \to (G,g')$$

a Riemannian submersion and $\{X_i,U_j\}$ be a π-compatible frame.

In this case, for any $U,V \in \chi^v(M)$ and $X,Y \in \chi^h(M)$, the Ricci tensor S^M holds the following equations [6]:

$$\begin{aligned} (i) \qquad S^M(U,V) &= \hat{S}(U,V) - g(N,T_UV) \\ &\quad + \sum_i \{g((\nabla_{X_i}T)_U V,X_i) + g(A_{X_i}U,A_{X_i}V)\}, \end{aligned} \tag{1.10}$$

$$\begin{aligned} (ii) \qquad S^M(X,Y) &= S^G(X',Y')\circ\pi + \frac{1}{2}\{g(\nabla_X N,Y) + g(\nabla_Y N,X)\} \\ &\quad - 2\sum_i g(A_XX_i,A_YX_i) - \sum_j g(T_{U_j}X,T_{U_j}Y), \end{aligned} \tag{1.11}$$

$$(iii) \qquad S^M(U,X) = g(\nabla_U N, X) - \sum_j g(\nabla_{U_j} T)_{U_j} U, X) \tag{1.12}$$
$$+ \sum_i \{g((\nabla_{X_i} A)_{X_i} X, U) - 2g(A_X X_i, T_U X_i)\}.$$

Proposition 2. *[6] Let's take the scalar curvatures of $(M,g),(G,g')$ Riemannian manifolds and $x \in G, \pi^{-1}(x)$ fibre with r^M, r^G and $\hat{r}$, respectively. In a*

$$\pi : (M,g) \to (G,g')$$

Riemannian submersion, (M,g) depends on the scalar curvature of the Riemannian manifold r^G and the scalar curve of any lift $\hat{r}$. In this case

$$r^M = \hat{r} + r^G \circ \pi - ||N||^2 - ||A||^2 - ||T||^2 + 2\sum_i g(\nabla_{X_i} N, X_i). \tag{1.13}$$

2. Weyl projective curvature tensor along a Riemannian submersion

In this section, we examine the Weyl projective curvature tensor relations between the total space, the base space and fibres on a Riemannian submersion. We also give a corollary in case of the Riemannian submersion has totally umbilical fibres case.

Definition 4. [13] Let take an n-dimensional differentiable manifold M^n with differentiability class C^∞. In the $n-$dimensional space V_n, the tensor

$$P^*(X,Y)Z = R^M(X,Y)Z - \frac{1}{n-1}\{S^M(Y,Z)X - S^M(X,Z)Y\}.$$

is called Weyl projective curvature tensor, where Ricci tensor of total space denoted by S^M.

Now, we have the following main theorem.

Theorem 2. *Let, (M,g) and (G,g') Riemannian manifolds,*

$$\pi : (M,g) \to (G,g')$$

a Riemannian submersion and R^M, R^G and $\hat{R}$ be Riemannian curvature tensors, S^M, S^G and $\hat{S}$ be Ricci tensors of M, G and the fibre respectively. Then for any $U,V,W,F \in \chi^v(M)$ and $X,Y,Z,H \in \chi^h(M)$, we have the following relations for Weyl projective curvature tensor:

$$\begin{aligned} g(P^*(X,Y)Z,H) = {} & g(R^G(X,Y)Z,H) + 2g(A_X Y, A_Z H) \\ & - g(A_Y Z, A_X H) + g(A_X Z, A_Y H) \\ & - \frac{1}{n-1}\Bigg\{ g(X,H)\Bigg[S^G(Y',Z') \circ \pi + \frac{1}{2}(g(\nabla_Y N, Z) + g(\nabla_Z N, Y)) \\ & - 2\sum_i g(A_Y X_i, A_Z X_i) - \sum_j g(T_{U_j} Y, T_{U_j} Z)\Bigg] \end{aligned}$$

$$
\begin{aligned}
&\quad - g(Y,H)\Big[S^G(X',Z')\circ\pi + \frac{1}{2}\left(g(\nabla_X N,Z)+g(\nabla_Z N,X)\right)\\
&\quad - 2\sum_i g(A_X X_i, A_Z X_i) - \sum_j g(T_{U_j}X, T_{U_j}Z)\Big]\Big\},\\
g(P^*(X,Y)Z,V) &= -g((\nabla_Z A)_X Y,V) - g(A_X Y, T_V Z)\\
&\quad + g(A_Y Z, T_V X) - g(A_X Z, T_V Y),\\
g(P^*(X,Y)V,W) &= g((\nabla_V A)_X Y,W) - g((\nabla_W A)_X Y,V) + g(A_X V, A_Y W)\\
&\quad - g(A_X W, A_Y V) - g(T_V X, T_W Y) + g(T_W X, T_V Y),\\
g(P^*(X,V)Y,W) &= g((\nabla_X T)_V W,Y) + g((\nabla_V A)_X Y,W)\\
&\quad - g(T_V X, T_W Y) + g(A_X V, A_Y W)\\
&\quad + \frac{1}{n-1}\Big\{g(V,W)\Big[S^G(X',Y')\circ\pi + \frac{1}{2}\{g(\nabla_X N,Y)\\
&\quad + g(\nabla_Y N,X)\} - 2\sum_i g(A_X X_i, A_Y X_i) - \sum_j g(T_{U_j}X, T_{U_j}Y)\Big]\Big\},\\
g(P^*(U,V)W,X) &= g((\nabla_U T)_V W,X) - g((\nabla_V T)_U W,X)
\end{aligned}
$$

and

$$
\begin{aligned}
g(P^*(U,V)W,F) &= g(\hat{R}(U,V)W,F) + g(T_U W, T_V F)\\
&\quad - g(T_V W, T_U F) - \frac{1}{n-1}\Big\{g(F,U)\Big[\hat{S}(V,W) - g(N,T_V W)\\
&\qquad + \sum_i \left(g((\nabla_{X_i}T)_V W, X_i) + g(A_{X_i}V, A_{X_i}W)\right)\Big]\\
&\quad - g(F,V)\Big[\hat{S}(U,W) - g(N,T_U W)\\
&\qquad + \sum_i \left(g((\nabla_{X_i}T)_U W, X_i) + g(A_{X_i}U, A_{X_i}W)\right)\Big]\Big\}.
\end{aligned}
$$

Proof. We only give the proof of the 1st equation of this theorem. The following equations are obtained inner production with H to P^* and using (1.4) and (1.11) equations.

$$
\begin{aligned}
g(R^M(X,Y)Z,H) &= g(R^G(X,Y)Z,H) + 2g(A_X Y, A_Z H) - g(A_Y Z, A_X H)\\
&\quad + g(A_X Z, A_Y H),
\end{aligned}
$$

$$S^M(Y,Z) = S^G(Y',Z')\circ\pi + \frac{1}{2}\{g(\nabla_Y N,Z)+g(\nabla_Z N,Y)\} \\ -2\sum_i g(A_Y Y_i, A_Z Y_i) - \sum_j (T_{U_j}Y, T_{U_j}Z)$$

and

$$S^M(X,Z) = S^G(Y',Z')\circ\pi + \frac{1}{2}\{g(\nabla_X N,Z)+g(\nabla_Z N,X)\} \\ -2\sum_i g(A_X X_i, A_Z X_i) - \sum_j (T_{U_j}X, T_{U_j}Z).$$

When these equations are substituted in P^*, the given result is obtained. Other equations are similarly proved by using Theorem 1 and Proposition 1. □

Corollary 1. *Let $\pi:(M,g)\to(G,g')$ be a Riemannian submersion, where (M,g) and (G,g') Riemannian manifolds. If the Riemannian submersion has total umbilical fibres that is $N=0$ and then the Weyl projective curvature tensor is given by*

$$g(P^*(X,Y)Z,H) = g(R^G(X,Y)Z,H) + 2g(A_X Y, A_Z H) - g(A_Y Z, A_X H) + g(A_X Z, A_Y H) \\ -\frac{1}{n-1}\Bigg\{ g(X,H)\Bigg[S^G(Y',Z')\circ\pi - 2\sum_i g(A_Y X_i, A_Z X_i) - \sum_j g(T_{U_j}Y, T_{U_j}Z)\Bigg] \\ - g(Y,H)\Bigg[S^G(X',Z')\circ\pi - 2\sum_i g(A_X X_i, A_Z X_i) - \sum_j g(T_{U_j}X, T_{U_j}Z)\Bigg]\Bigg\},$$

and

$$g(P^*(U,V)W,F) = g(\hat{R}(U,V)W,F) + g(T_U W, T_V F) - g(T_V W, T_U F) \\ -\frac{1}{n-1}\Bigg\{ g(F,U)\Bigg[\hat{S}(V,W) + \sum_i (g((\nabla_{X_i}T)_V W, X_i) + g(A_{X_i}V, A_{X_i}W))\Bigg] \\ - g(F,V)\Bigg[\hat{S}(U,W) + \sum_i (g((\nabla_{X_i}T)_U W, X_i) + g(A_{X_i}U, A_{X_i}W))\Bigg]\Bigg\}.$$

for any $U,V,W,F\in\chi^v(M)$ and $X,Y,Z,H\in\chi^h(M)$,

3. Concircular curvature tensor along a Riemannian submersion

In this section, curvature relations of concircular curvature tensor in a Riemannian submersion are examined and showing that the Riemannian submersion with concircular curvature tensor has no the totally umbilical fibres.

Definition 5. In the $n-$dimensional space V_n, the tensor

$$C^*(X,Y,Z,H) = R^M(X,Y,Z,H) - \frac{r^M}{n(n-1)}[g(X,H)g(Y,Z) - g(Y,H)g(X,Z)],$$

is called concircular curvature tensor, where scalar tensor denoted by r^M [13].

Now, we have the following main theorem.

Theorem 3. *Let, (M,g) and (G,g') Riemannian manifolds,*

$$\pi : (M,g) \to (G,g')$$

a Riemannian submersion and R^M, R^G and $\hat{R}$ be Riemannian curvature tensors, r^M, r^G and $\hat{r}$ be scalar curvature tensors of M, G and the fibre respectively. Then for any $U,V,W,F \in \chi^v(M)$ and $X,Y,Z,H \in \chi^h(M)$, we have the following relations

$$\begin{aligned}
g(C^*(X,Y)Z,H) &= g(R^G(X,Y)Z,H) + 2g(A_XY,A_ZH) \\
&\quad - g(A_YZ,A_XH) + g(A_XZ,A_YH) \\
&\quad - \frac{r^M}{n(n-1)}\left\{g(Y,Z)g(X,H) - g(X,Z)g(Y,H)\right\}, \\
g(C^*(X,Y)Z,V) &= -g((\nabla_Z A)_X Y,V) - g(A_XT,T_VZ) \\
&\quad + g(A_YZ,T_VX) - g(A_XZ,T_VY), \\
g(C^*(X,Y)V,W) &= g((\nabla_V A)_X Y,W) - g((\nabla_W A)_X Y,V) + g(A_XV,A_YW) \\
&\quad - g(A_XW,A_YV) - g(T_VX,T_WY) + g(T_WX,T_VY), \\
g(C^*(X,V)Y,W) &= g((\nabla_X T)_V W,Y) + g((\nabla_V A)_X Y,W) - g(T_VX,T_WY) \\
&\quad + g(A_XY,A_YW) - \frac{r^M}{n(n-1)}\{-g(X,Y)g(V,W)\}, \\
g(C^*(U,V)W,X) &= g((\nabla_U T)_V W,X) - g((\nabla_V T)_U W,X)
\end{aligned}$$

and

$$\begin{aligned}
g(C^*(U,V)W,F) &= g(\hat{R}(U,V)W,F) + g(T_UW,T_VF) - g(T_VW,T_UF) \\
&\quad - \frac{r^M}{n(n-1)}\left\{g(V,W)g(U,F) - g(U,W)g(V,F)\right\}
\end{aligned}$$

where

$$r^M = \hat{r} + r^G \circ \pi - ||A||^2 - ||T||^2.$$

Proof. Let's prove the 2nd equation of this theorem. Taking inner product C^* with V then we have

$$g(C^*(X,Y)Z,V) = g(R(X,Y)Z,V) - \frac{r^M}{n(n-1)}\{g(Y,Z)g(X,V) - g(X,Z)\}.$$

Then using equation (1.5), we get

$$\begin{aligned}
g(C^*(X,Y)Z,V) &= -g((\nabla_Z A)_X Y,V) - g(A_XT,T_VZ) \\
&\quad + g(A_YZ,T_VX) - g(A_XZ,T_VY).
\end{aligned}$$

which completes the proof of the second equation. Other equations are similarly proved by using Theorem 1, Proposition 1 and Proposition 2. □

Corollary 2. *Let $\pi : (M,g) \to (G,g')$ be a Riemannian submersion, where (M,g) and (G,g') Riemannian manifolds. Then the concircular curvature tensor of Riemannian submersion has no total umbilical fibres.*

4. Conharmonic curvature tensor along a Riemannian submersion

In this section, curvature relations of conharmonic curvature tensor in a Riemannian submersion are examined.

Definition 6. In the $n-$dimensional space V_n, the tensor

$$L^*(X,Y,Z,H) = R^M(X,Y,Z,H) - \frac{1}{n-2}[g(Y,Z)Ric(X,H) - g(X,Z)Ric(Y,H) + g(Y,H)Ric(Y,Z) - g(Y,H)Ric(X,Z)]$$

is called conharmonic curvature tensor, where Ricci tensor denoted by Ric [13].

In a similar way, we have the following main theorem.

Theorem 4. *Let, (M,g) and (G,g') Riemannian manifolds,*

$$\pi : (M,g) \to (G,g')$$

a Riemannian submersion and R^M, R^G and $\hat{R}$ be Riemannian curvature tensors, S^M, S^G and $\hat{S}$ be Ricci tensors of M, G and the fibre respectively. Then for any $U,V,W,F \in \chi^v(M)$ and $X,Y,Z,H \in \chi^h(M)$, we have the following relations

$$\begin{aligned} g(L^*(X,Y)Z,H) = {} & g(R^G(X,Y)Z,H) + 2g(A_XY,A_ZH) - g(A_YZ,A_XH) \\ & + g(A_XZ,A_YH) - \frac{1}{(n-2)}\Bigg\{ g(Y,Z)\Bigg[S^G(X',H')\circ\pi \\ & + \frac{1}{2}(g(\nabla_XN,H) + g(\nabla_HN,X)) - 2\sum_i g(A_XX_i,A_HX_i) \\ & - \sum_j g(T_{U_j}X,T_{U_j}H)\Bigg] - g(X,Z)\Bigg[S^G(Y',H')\circ\pi \\ & + \frac{1}{2}(g(\nabla_YN,H) + g(\nabla_HN,Y)) - 2\sum_i g(A_YX_i,A_HX_i) \\ & - \sum_j g(T_{U_j}Y,T_{U_j}H)\Bigg] + g(X,H)\Bigg[S^G(Y',Z')\circ\pi \\ & + \frac{1}{2}(g(\nabla_YN,Z) + g(\nabla_ZN,Y)) \end{aligned}$$

$$
\begin{aligned}
&-2\sum_i g(A_Y X_i, A_Z X_i) - \sum_j g(T_{U_j} Y, T_{U_j} Z)\Big] \\
&- g(Y,H)\Big[S^G(X',Z')\circ\pi + \frac{1}{2}\left(g(\nabla_X N, Z) + g(\nabla_Z N, X)\right) \\
&-2\sum_i g(A_X X_i, A_Z X_i) - \sum_j g(T_{U_j} X, T_{U_j} Z)\Big]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
g(L^*(X,Y)Z,V) = &-g((\nabla_Z A)_X Y, V) - g(A_X T, T_V Z) + g(A_Y Z, T_V X) \\
&- g(A_X Z, T_V Y) - \frac{1}{(n-2)}\Big\{g(Y,Z)\Big[g(\nabla_X N, V) \\
&- \sum_j g((\nabla_{U_j} T)_{U_j} X, V) + \sum_i (g((\nabla_{X_i} A)_{X_i} X, V) \\
&-2g(A_V X_i, T_X X_i))\Big] - g(X,Z)\Big[g(\nabla_Y N, V) \\
&- \sum_j g((\nabla_{U_j} T)_{U_j} Y, V) + \sum_i (g((\nabla_{X_i} A)_{X_i} Y, V) \\
&-2g(A_V X_i, T_Y X_i))\Big]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
g(L^*(X,Y)V,W) = &\, g((\nabla_V A)_X Y, W) - g((\nabla_W A)_X Y, V) + g(A_X V, A_Y W) \\
&- g(A_X W, A_Y V) - g(T_V X, T_W Y) + g(T_W X, T_V Y),
\end{aligned}
$$

$$
\begin{aligned}
g(L^*(X,V)Y,W) = &\, g((\nabla_X T)_V W, Y) + g((\nabla_V A)_X Y, W) - g(T_V X, T_W Y) \\
&+ g(A_X Y, A_Y W) - \frac{1}{(n-2)}\Big\{-g(V,W)\Big[S^G(X',Y')\circ\pi \\
&+ \frac{1}{2}\left(g(\nabla_X N, Y) + g(\nabla_Y N, X)\right) - 2\sum_i g(A_X X_i, A_Y X_i) \\
&- \sum_j g(T_{U_j} X, T_{U_j} Y)\Big] - g(X,Y)\Big[\hat{S}(V,W) - g(N, T_V W) \\
&+ \sum_i (g((\nabla_{X_i} T)_V W, X_i) + g(A_{X_i} V, A_{X_i} W))\Big]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
g(L^*(U,V)W,X) = &\, g((\nabla_U T)_V W, X) - g((\nabla_V T)_U W, X) \\
&- \frac{1}{(n-2)}\Big\{g(V,W)\Big[g(\nabla_U N, X) - \sum_j g(\nabla_{U_j} T)_{U_j} U, X) \\
&+ \sum_i \{g((\nabla_{X_i} A)_{X_i} X, U) - 2g(A_X X_i, T_U X_i)\}\Big]
\end{aligned}
$$

$$- g(U,W)\Big[g(\nabla_V N,X) - \sum_j g(\nabla_{U_j}T)_{U_j}V,X)$$
$$+ \sum_i \{g((\nabla_{X_i}A)_{X_i}X,V) - 2g(A_X X_i, T_V X_i)\}\Big]\Big\}$$

and

$$\begin{aligned} g(L^*(U,V)W,F) = {} & g(\hat{R}(U,V)W,F) + g(T_U W, T_V F) - g(T_V W, T_U F) \\ & - \frac{1}{(n-2)}\Big\{ g(V,W)\Big[\hat{S}(U,F) - g(N,T_U F) \\ & + \sum_i (g((\nabla_{X_i}T)_U F, X_i) + g(A_{X_i}U, A_{X_i}F))\Big] \\ & - g(U,W)\Big[\hat{S}(V,F) - g(N,T_V F) \\ & + \sum_i (g((\nabla_{X_i}T)_V F, X_i) + g(A_{X_i}V, A_{X_i}F))\Big] \\ & + g(F,U)\Big[\hat{S}(V,W) - g(N,T_V W) \\ & + \sum_i (g((\nabla_{X_i}T)_V W, X_i) + g(A_{X_i}V, A_{X_i}W))\Big] \\ & - g(F,V)\Big[\hat{S}(U,W) - g(N,T_U W) \\ & + \sum_i (g((\nabla_{X_i}T)_U W, X_i) + g(A_{X_i}U, A_{X_i}W))\Big]\Big\}. \end{aligned}$$

Proof. Let's prove the 3th equation of this theorem. The following equations are obtained inner production with W to L^* and by using equation (1.6)

$$\begin{aligned} g(L^*(X,Y)V,W) = {} & g(R^M(X,Y)V,W) - \frac{1}{n-2}\{g(X,W)S(Y,V) \\ & - g(Y,W)S(X,V) + g(Y,V)S(X,W) - g(X,V)S(Y,W)\}. \end{aligned}$$

One can easily obtain the other equations by using Theorem 1 and Proposition 1. □

Corollary 3. *Let $\pi : (M,g) \to (G,g')$ be a Riemannian submersion, where (M,g) and (G,g') Riemannian manifolds. If the Riemannian submersion has total umbilical fibres that is $N = 0$ and then the conharmonic curvature tensor is given by*

$$\begin{aligned} g(L^*(X,Y)Z,H) = {} & g(R^G(X,Y)Z,H) + 2g(A_X Y, A_Z H) - g(A_Y Z, A_X H) \\ & + g(A_X Z, A_Y H) - \frac{1}{(n-2)}\Big\{ g(Y,Z)\Big[S^G(X',H')\circ\pi \\ & - 2\sum_i g(A_X X_i, A_H X_i) - \sum_j g(T_{U_j}X, T_{U_j}H)\Big] \end{aligned}$$

$$
\begin{aligned}
&- g(X,Z)\Big[S^G(Y',H')\circ\pi - 2\sum_i g(A_Y X_i, A_H X_i)\\
&- \sum_j g(T_{U_j}Y, T_{U_j}H)\Big] + g(X,H)\Big[S^G(Y',Z')\circ\pi\\
&- 2\sum_i g(A_Y X_i, A_Z X_i) - \sum_j g(T_{U_j}Y, T_{U_j}Z)\Big]\\
&- g(Y,H)\Big[S^G(X',Z')\circ\pi - 2\sum_i g(A_X X_i, A_Z X_i)\\
&- \sum_j g(T_{U_j}X, T_{U_j}Z)\Big]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
g(L^*(X,Y)Z,V) = &-g((\nabla_Z A)_X Y, V) - g(A_X T, T_V Z) + g(A_Y Z, T_V X)\\
&- g(A_X Z, T_V Y) - \frac{1}{(n-2)}\Big\{g(Y,Z)\Big[-\sum_j g((\nabla_{U_j}T)_{U_j}X, V)\\
&+ \sum_i \big(g((\nabla_{X_i}A)_{X_i}X, V) - 2g(A_V X_i, T_X X_i)\big)\Big]\\
&- g(X,Z)\Big[-\sum_j g((\nabla_{U_j}T)_{U_j}Y, V)\\
&+ \sum_i \big(g((\nabla_{X_i}A)_{X_i}Y, V) - 2g(A_V X_i, T_Y X_i)\big)\Big]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
g(L^*(X,V)Y,W) = &\, g((\nabla_X T)_V W, Y) + g((\nabla_V A)_X Y, W) - g(T_V X, T_W Y)\\
&+ g(A_X Y, A_Y W) - \frac{1}{(n-2)}\Big\{-g(V,W)\Big[S^G(X',Y')\circ\pi\\
&- 2\sum_i g(A_X X_i, A_Y X_i) - \sum_j g(T_{U_j}X, T_{U_j}Y)\Big]\\
&- g(X,Y)\Big[\hat{S}(V,W) + \sum_i \big(g((\nabla_{X_i}T)_V W, X_i) + g(A_{X_i}V, A_{X_i}W)\big)\Big]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
g(L^*(U,V)W,X) = &\, g((\nabla_U T)_V W, X) - g((\nabla_V T)_U W, X)\\
&- \frac{1}{(n-2)}\Big\{g(V,W)\Big[\sum_j g(\nabla_{U_j}T)_{U_j}U, X)\\
&+ \sum_i \{g((\nabla_{X_i}A)_{X_i}X, U) - 2g(A_X X_i, T_U X_i)\}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&- g(U,W)\Big[\sum_j g(\nabla_{U_j}T)_{U_j}V,X) \\
&+ \sum_i \{g((\nabla_{X_i}A)_{X_i}X,V) - 2g(A_XX_i,T_VX_i)\}\Big]\Big\}
\end{aligned}
$$

and

$$
\begin{aligned}
g(L^*(U,V)W,F) = {} & g(\hat{R}(U,V)W,F) + g(T_UW,T_VF) \\
& - g(T_VW,T_UF) - \frac{1}{(n-2)}\Big\{g(V,W)\Big[\hat{S}(U,F) \\
& + \sum_i (g((\nabla_{X_i}T)_UF,X_i) + g(A_{X_i}U,A_{X_i}F))\Big] \\
& - g(U,W)\Big[\hat{S}(V,F) + \sum_i (g((\nabla_{X_i}T)_VF,X_i) + \\
& g(A_{X_i}V,A_{X_i}F))\Big] + g(F,U)\Big[\hat{S}(U,V) \\
& + \sum_i (g((\nabla_{X_i}T)_UV,X_i) + g(A_{X_i}U,A_{X_i}V))\Big] \\
& - g(F,V)\Big[\hat{S}(U,W) + \sum_i (g((\nabla_{X_i}T)_UW,X_i) + g(A_{X_i}U,A_{X_i}W))\Big]\Big\}.
\end{aligned}
$$

5. Conformal curvature tensor along a Riemannian submersion

In this section, we find some curvature relations of conformal curvature tensor in a Riemannian submersion and give a corollary in case of the Riemannian submersion has totally umbilical fibres.

Definition 7. In the $n-$dimensional space V_n, the tensor

$$
\begin{aligned}
V^*(X,Y,Z,H) = {} & R^M(X,Y,Z,H) - \frac{1}{n-2}[g(X,H)Ric(Y,Z) - g(Y,H)Ric(X,Z) \\
& + g(Y,Z)Ric(X,H) - g(X,Z)Ric(Y,H)] \\
& + \frac{r^M}{(n-1)(n-2)}[g(X,H)g(Y,Z) - g(Y,H)g(X,Z)],
\end{aligned}
$$

is called conformal curvature tensor, where Ricci tensor and scalar tensor denoted by Ric and r^M respectively [13].

Theorem 5. *Let, (M,g) and (G,g') Riemannian manifolds,*

$$\pi : (M,g) \to (G,g')$$

a Riemannian submersion and R^M, R^G and $\hat{R}$ be Riemannian curvature tensors, S^M, S^G and $\hat{S}$ be Ricci tensors and r^M, r^G and $\hat{r}$ be scalar curvature tensors of M, G and

the fibre respectively. Then for any $U,V,W,F \in \chi^v(M)$ and $X,Y,Z,H \in \chi^h(M)$, we have the following relations

$$
\begin{aligned}
g(V^*(X,Y)Z,H) = {} & g(R^G(X,Y)Z,H) + 2g(A_XY,A_ZH) - g(A_YZ,A_XH) \\
& + g(A_XZ,A_YH) - \frac{1}{(n-2)}\Bigg\{ g(X,H)\Bigg[S^G(Y',Z')\circ\pi \\
& + \frac{1}{2}\left(g(\nabla_YN,Z) + g(\nabla_ZN,Y)\right) - 2\sum_i g(A_YX_i,A_ZX_i) \\
& - \sum_j g(T_{U_j}Y,T_{U_j}Z)\Bigg] - g(Y,H)\Bigg[S^G(X',Z')\circ\pi \\
& + \frac{1}{2}\left(g(\nabla_XN,Z) + g(\nabla_ZN,X)\right) - 2\sum_i g(A_XX_i,A_ZX_i) \\
& - \sum_j g(T_{U_j}X,T_{U_j}Z)\Bigg] + g(Y,Z)\Bigg[S^G(X',H')\circ\pi \\
& + \frac{1}{2}\left(g(\nabla_XN,H) + g(\nabla_HN,X)\right) \\
& - 2\sum_i g(A_XX_i,A_HX_i) - \sum_j g(T_{U_j}X,T_{U_j}H)\Bigg] \\
& - g(X,Z)\Bigg[S^G(Y',H')\circ\pi + \frac{1}{2}\left(g(\nabla_YN,H) + g(\nabla_HN,Y)\right) \\
& - 2\sum_i g(A_YX_i,A_HX_i) - \sum_j g(T_{U_j}Y,T_{U_j}H)\Bigg]\Bigg\} \\
& + \frac{r^M}{(n-1)(n-2)}\{g(Y,Z)g(X,H) - g(X,Z)g(Y,H)\}, \\
g(V^*(X,Y)Z,V) = {} & -g((\nabla_ZA)_XY,V) - g(A_XT,T_VZ) + g(A_YZ,T_VX) \\
& - g(A_XZ,T_VY) - \frac{1}{(n-2)}\Bigg\{ g(Y,Z)\Bigg[g(\nabla_XN,V) \\
& - \sum_j g((\nabla_{U_j}T)_{U_j}X,V) + \sum_i \left(g((\nabla_{X_i}A)_{X_i}V,X)\right. \\
& -2g(A_VX_i,T_XX_i))\Bigg] - g(X,Z)\Bigg[g(\nabla_YN,V) \\
& - \sum_j g((\nabla_{U_j}T)_{U_j}Y,V) + \sum_i \left(g((\nabla_{X_i}A)_{X_i}V,Y)\right.
\end{aligned}
$$

$$
\begin{aligned}
&\qquad -2g(A_V X_i, T_Y X_i))\Big]\Big\},\\
g(V^*(X,Y)V,W) &= g((\nabla_V A)_X Y, W) - g((\nabla_W A)_X Y, V) + g(A_X V, A_Y W)\\
&\quad - g(A_X W, A_Y V) - g(T_V X, T_W Y) + g(T_W X, T_V Y),\\
g(V^*(X,V)Y,W) &= g((\nabla_X T)_V W, Y) + g((\nabla_V A)_X Y, W) - g(T_V X, T_W Y)\\
&\quad + g(A_X Y, A_Y W) - \frac{1}{(n-2)}\Big\{ - g(V,W)\Big[S^G(X',Y')\circ\pi\\
&\quad + \frac{1}{2}\left(g(\nabla_X N, Y) + g(\nabla_Y N, X)\right) - 2\sum_i g(A_X X_i, A_Y X_i)\\
&\quad - \sum_j g(T_{U_j} X, T_{U_j} Y)\Big] - g(X,Y)\Big[\hat{S}(V,W) - g(N, T_V W)\\
&\quad + \sum_i \left(g((\nabla_{X_i} T)_V W, X_i) + g(A_{X_i} V, A_{X_i} W)\right)\Big]\Big\}\\
&\quad + \frac{r^M}{(n-1)(n-2)}\{g(X,Y)g(V,W)\},\\
g(V^*(U,V)W,X) &= g((\nabla_U T)_V W, X) - g((\nabla_V T)_U W, X)\\
&\quad - \frac{1}{(n-2)}\Big\{ g(V,W)\Big[g(\nabla_U N, X) - \sum_j g(\nabla_{U_j} T)_{U_j} U, X)\\
&\quad + \sum_i \{g((\nabla_{X_i} A)_{X_i} X, U) - 2g(A_X X_i, T_U X_i)\}\Big]\\
&\quad - g(U,W)\Big[g(\nabla_V N, X) - \sum_j g(\nabla_{U_j} T)_{U_j} V, X)\\
&\quad + \sum_i \{g((\nabla_{X_i} A)_{X_i} X, V) - 2g(A_X X_i, T_V X_i)\}\Big]\Big\}
\end{aligned}
$$

and

$$
\begin{aligned}
g(V^*(U,V)W,F) &= g(\hat{R}(U,V)W,F) + g(T_U W, T_V F) - g(T_V W, T_U F)\\
&\quad - \frac{1}{(n-2)}\Big\{ g(F,U)\Big[\hat{S}(V,W) - g(N, T_V W)\\
&\quad + \sum_i \left(g(\nabla_{X_i} T)_V W, X_i + g(A_{X_i} V, A_{X_i} W)\right)\Big]\\
&\quad - g(F,V)\Big[\hat{S}(U,W) - g(N, T_U W) + \sum_i \left(g((\nabla_{X_i} T)_U W, X_i)\right.\\
&\quad \left. + g(A_{X_i} U, A_{X_i} W)\right)\Big] + g(V,W)\Big[\hat{S}(U,F) - g(N, T_U F)
\end{aligned}
$$

$$+\sum_i (g((\nabla_{X_i}T)_U F, X_i) + g(A_{X_i}U, A_{X_i}F))\Big]$$
$$-g(U,W)\Big[\hat{S}(V,F) - g(N, T_V F) + \sum_i (g((\nabla_{X_i}T)_V F, X_i)$$
$$+ g(A_{X_i}V, A_{X_i}F))\Big]\Big\}$$
$$+\frac{r^M}{(n-1)(n-2)}\{g(V,W)g(U,F) - g(U,W)g(V,F)\}$$

where

$$r^M = \hat{r} + r^G \circ \pi - ||N||^2 - ||A||^2 - ||T||^2 + 2\sum_i g(\nabla_{X_i}N, X_i).$$

Proof. Let's prove the 4th equation of this theorem. The following equations are obtained inner production with W to V^*

$$g(V^*(X,V)Y,W) = g(R^M(X,V)Y,W) - \frac{1}{n-2}\{g(X,W)S^M(V,Y)$$
$$-g(V,W)S^M(X,Y) + g(V,Y)S(X,W) - g(X,Y)S^M(V,W)\}$$
$$+\frac{r^M}{(n-1)(n-2)}\{g(X,W)g(Y,V) - g(X,V)g(Y,W)\}.$$

Then using equations (1.7)-(1.11), we have the desired result. From the Theorem 1, Proposition 1 and Proposition 2 the above equations are obtained. □

Corollary 4. *Let $\pi : (M,g) \to (G,g')$ be a Riemannian submersion, where (M,g) and (G,g') Riemannian manifolds. If the Riemannian submersion has total umbilical fibres that is $N = 0$ and then the conformal curvature tensor is given by*

$$g(V^*(X,Y)Z,H) = g(R^G(X,Y)Z,H) + 2g(A_XY, A_ZH) - g(A_YZ, A_XH)$$
$$+g(A_XZ, A_YH) - \frac{1}{(n-2)}\Big\{g(X,H)\Big[S^G(Y',Z')\circ\pi$$
$$-2\sum_i g(A_YX_i, A_ZX_i) - \sum_j g(T_{U_j}Y, T_{U_j}Z)\Big] - g(Y,H)\Big[S^G(X',Z')$$
$$\circ\pi - 2\sum_i g(A_XX_i, A_ZX_i) - \sum_j g(T_{U_j}X, T_{U_j}Z)\Big]$$
$$+g(Y,Z)\Big[S^G(X',H')\circ\pi - 2\sum_i g(A_XX_i, A_HX_i)$$
$$-\sum_j g(T_{U_j}X, T_{U_j}H)\Big] - g(X,Z)\Big[S^G(Y',H')\circ\pi$$

$$
\begin{aligned}
&\quad -2\sum_i g(A_Y X_i, A_H X_i) - \sum_j g(T_{U_j} Y, T_{U_j} H)\Big]\Big\} \\
&\quad + \frac{r^M}{(n-1)(n-2)}\{g(Y,Z)g(X,H) - g(X,Z)g(Y,H)\}, \\
g(V^*(X,Y)Z,V) &= -g((\nabla_Z A)_X Y, V) - g(A_X T, T_V Z) + g(A_Y Z, T_V X) - g(A_X Z, T_V Y) \\
&\quad - \frac{1}{(n-2)}\Big\{ g(Y,Z)\Big[-\sum_j g((\nabla_{U_j} T)_{U_j} X, V) \\
&\quad + \sum_i (g((\nabla_{X_i} A)_{X_i} X, V) - 2g(A_V X_i, T_X X_i))\Big] \\
&\quad - g(X,Z)\Big[-\sum_j g((\nabla_{U_j} T)_{U_j} Y, V) \\
&\quad + \sum_i (g((\nabla_{X_i} A)_{X_i} Y, V) - 2g(A_V X_i, T_Y X_i))\Big]\Big\}, \\
g(V^*(X,V)Y,W) &= g((\nabla_X T)_V W, Y) + g((\nabla_V A)_X Y, W) - g(T_V X, T_W Y) \\
&\quad + g(A_X Y, A_Y W) - \frac{1}{(n-2)}\Big\{ -g(V,W)\Big[S^G(X',Y') \circ \pi \\
&\quad - 2\sum_i g(A_X X_i, A_Y X_i) - \sum_j g(T_{U_j} X, T_{U_j} Y)\Big] - g(X,Y) \\
&\quad \times \Big[\hat{S}(V,W) + \sum_i (g((\nabla_{X_i} T)_V W, X_i) + g(A_{X_i} V, A_{X_i} W))\Big]\Big\}, \\
g(V^*(U,V)W,X) &= g((\nabla_U T)_V W, X) - g((\nabla_V T)_U W, X) \\
&\quad - \frac{1}{(n-2)}\Big\{ g(V,W)\Big[\sum_j g(\nabla_{U_j} T)_{U_j} U, X) \\
&\quad + \sum_i \{g((\nabla_{X_i} A)_{X_i} X, U) - 2g(A_X X_i, T_U X_i)\}\Big] \\
&\quad - g(U,W)\Big[\sum_j g(\nabla_{U_j} T)_{U_j} V, X) \\
&\quad + \sum_i \{g((\nabla_{X_i} A)_{X_i} X, V) - 2g(A_X X_i, T_V X_i)\}\Big]\Big\}
\end{aligned}
$$

and

$$
g(V^*(U,V)W,F) = g(\hat{R}(U,V)W,F) + g(T_U W, T_V F) - g(T_V W, T_U F) - \frac{1}{(n-2)}
$$

$$\times\Big\{g(F,U)\Big[\hat{S}(V,W)+\sum_i(g(\nabla_{X_i}T)_VW,X_i+g(A_{X_i}V,A_{X_i}W))\Big]$$
$$-g(F,V)\Big[\hat{S}(U,W)+\sum_i(g((\nabla_{X_i}T)_UW,X_i)+g(A_{X_i}U,A_{X_i}W))\Big]$$
$$+g(V,W)\Big[\hat{S}(U,F)+\sum_i(g((\nabla_{X_i}T)_UF,X_i)+g(A_{X_i}U,A_{X_i}F))\Big]$$
$$-g(U,W)\Big[\hat{S}(V,F)+\sum_i(g((\nabla_{X_i}T)_VF,X_i)+g(A_{X_i}V,A_{X_i}F))\Big]\Big\}$$
$$+\frac{r^M}{(n-1)(n-2)}\{g(V,W)g(U,F)-g(U,W)g(V,F)\}$$

where

$$r^M=\hat{r}+r^G\circ\pi-||A||^2-||T||^2.$$

Finally, we investigate the $M-$projective curvature tenson on a Riemannian submersion and give a corollary in case of the totall umbilical fibres.

6. M-PROJECTIVE CURVATURE TENSOR ALONG A RIEMANNIAN SUBMERSION

In this section, curvature relations of M-projective curvature tensor in a Riemannian submersion are examined and obtain a corollary using the curvature tensor.

Definition 8. Let take an n-dimensional differentiable manifold $M^n$with differentiability class C^∞. In 1971 on a n -dimensional Riemannian manifold, ones [17] defined a tensor field W^* as

$$W^*(X,Y)Z=R^M(X,Y)Z-\frac{1}{2(n-1)}[S^M(Y,Z)X$$
$$-S^M(X,Z)Y+g(Y,Z)QX-g(X,Z)QY]$$

tensor W^* as M-projective curvature tensor.

In addition, on an $n-$dimensional Riemannian manifold M^n the Ricci operator Q is defined by

$$S^M(X,Y)=g(QX,Y).$$

Theorem 6. *Let, (M,g) and (G,g') Riemannian manifolds,*

$$\pi:(M,g)\to(G,g')$$

a Riemannian submersion and R^M, R^G and $\hat{R}$ be Riemannian curvature tensors, S^M, S^G and $\hat{S}$ be Ricci tensors of M,G and the fibre respectively. Then for any $U,V,W,F\in\chi^v(M)$ and $X,Y,Z,H\in\chi^h(M)$, we have the following relations for M-projective curvature tensor:

$$g(W^*(X,Y)Z,H)=g(R^G(X,Y)Z,H)+2g(A_XY,A_ZH)-g(A_YZ,A_XH)$$

$$
\begin{aligned}
&+g(A_XZ,A_YH)-\frac{1}{2(n-1)}\Bigg\{g(X,H)\Bigg[S^G(Y',Z')\circ\pi\\
&+\frac{1}{2}\left(g(\nabla_YN,Z)+g(\nabla_ZN,Y)\right)-2\sum_i g(A_YX_i,A_ZX_i)\\
&-\sum_j g(T_{U_j}Y,T_{U_j}Z)\Bigg]-g(Y,H)\Bigg[S^G(X',Z')\circ\pi+\frac{1}{2}\left(g(\nabla_XN,Z)\right.\\
&\left.+g(\nabla_ZN,X)\right)-2\sum_i g(A_XX_i,A_ZX_i)-\sum_j g(T_{U_j}X,T_{U_j}Z)\Bigg]\\
&+g(Y,Z)\Bigg[S^G(X',H')\circ\pi+\frac{1}{2}\left(g(\nabla_XN,H)+g(\nabla_HN,X)\right)\\
&-2\sum_i g(A_XX_i,A_HX_i)-\sum_j g(T_{U_j}X,T_{U_j}H)\Bigg]\\
&-g(X,Z)\Bigg[S^G(Y',H')\circ\pi+\frac{1}{2}\left(g(\nabla_YN,H)+g(\nabla_HN,Y)\right)\\
&-2\sum_i g(A_YX_i,A_HX_i)-\sum_j g(T_{U_j}Y,T_{U_j}H)\Bigg]\Bigg\},\\
g(W^*(X,Y)Z,V)&=-g((\nabla_ZA)_XY,V)-g(A_XY,T_VZ)+g(A_YZ,T_VX)-g(A_XZ,T_VY)\\
&-\frac{1}{2(n-1)}\Bigg\{g(Y,Z)\Bigg[g(\nabla_XN,V)-\sum_j g((\nabla_{U_j}T)_{U_j}X,V)\\
&+\sum_i\left(g((\nabla_{X_i}A)_{X_i}X,V)-2g(A_VX_i,T_XX_i)\right)\Bigg]\\
&-g(X,Z)\Bigg[g(\nabla_YN,V)-\sum_j g((\nabla_{U_j}T)_{U_j}Y,V)\\
&+\sum_i\left(g((\nabla_{X_i}A)_{X_i}Y,V)-2g(A_VX_i,T_YX_i)\right)\Bigg]\Bigg\},\\
g(W^*(X,Y)V,W)&=g((\nabla_VA)_XY,W)-g((\nabla_WA)_XY,V)+g(A_XV,A_YW)\\
&-g(A_XW,A_YV)-g(T_VX,T_WY)+g(T_WX,T_VY),\\
g(W^*(X,V)Y,W)&=g((\nabla_XT)_VW,Y)+g((\nabla_VA)_XY,W)-g(T_VX,T_WY)\\
&+g(A_XY,A_YW)-\frac{1}{2(n-1)}\Bigg\{-g(V,W)\Bigg[S^G(X',Y')\circ\pi\\
&+\frac{1}{2}\left(g(\nabla_XN,Y)+g(\nabla_YN,X)\right)
\end{aligned}
$$

$$
\begin{aligned}
&-2\sum_i g(A_X X_i, A_Y X_i) - \sum_j g(T_{U_j}X, T_{U_j}Y)\Big] - g(X,Y)\Big[\hat{S}(V,W) \\
&- g(N, T_V W) + \sum_i (g((\nabla_{X_i}T)_V W, X_i) + g(A_{X_i}V, A_{X_i}W))\Big]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
g(W^*(U,V)W,X) = {} & g((\nabla_U T)_V W, X) - g((\nabla_V T)_U W, X) \\
& - \frac{1}{2(n-1)}\Big\{g(V,W)\Big[g(\nabla_U N, X) - \sum_j g(\nabla_{U_j}T)_{U_j}U, X) \\
& + \sum_i \{g((\nabla_{X_i}A)_{X_i}X, U) - 2g(A_X X_i, T_U X_i)\}\Big] \\
& - g(U,W)\Big[g(\nabla_V N, X) - \sum_j g(\nabla_{U_j}T)_{U_j}V, X) \\
& + \sum_i \{g((\nabla_{X_i}A)_{X_i}X, V) - 2g(A_X X_i, T_V X_i)\}\Big]\Big\}
\end{aligned}
$$

and

$$
\begin{aligned}
g(W^*(U,V)W,F) = {} & g(\hat{R}(U,V)W,F) + g(T_U W, T_V F) - g(T_V W, T_U F) \\
& - \frac{1}{2(n-1)}\Big\{g(F,U)\Big[\hat{S}(V,W) - g(N, T_V W) \\
& + \sum_i (g(\nabla_{X_i}T)_V W, X_i + g(A_{X_i}V, A_{X_i}W))\Big] \\
& - g(F,V)\Big[\hat{S}(U,W) - g(N, T_U W) \\
& \quad + \sum_i g((\nabla_{X_i}T)_U W, X_i) + g(A_{X_i}U, A_{X_i}W)\Big] \\
& + g(V,W)\Big[\hat{S}(U,F) - g(N, T_U F) \\
& \quad + \sum_i (g((\nabla_{X_i}T)_U F, X_i) + g(A_{X_i}U, A_{X_i}F))\Big] \\
& - g(U,W)\Big[\hat{S}(V,F) - g(N, T_V F) \\
& \quad + \sum_i (g((\nabla_{X_i}T)_V F, X_i) + g(A_{X_i}V, A_{X_i}F))\Big]\Big\}.
\end{aligned}
$$

Proof. Let's prove the 6th equation of this theorem. The following equations are obtained inner production with F to W^* and using (1.9) and (1.10) equations.

$$
g(W^*(U,V)W,F) = g(R^M(U,V)W,F) - \frac{1}{2(n-1)}\Big\{g(F,U)S^M(U,V)
$$

$$-g(F,V)S^M(U,W)+g(V,W)S^M(U,F)-g(U,W)S^M(V,F)\Big\},$$

$$g(R^M(U,V)W,F)=g(\hat{R}(U,V)W,F)+g(T_UW,T_VF)-g(T_VW,T_UF)$$

and

$$S^M(U,V)=\hat{S}(U,V)-g(N,T_UV)+\sum_i\{g((\nabla_{X_i}T)_UV,X_i)+g(A_{X_i}U,A_{X_i}V)\}.$$

When these equations are substituted in W^*, the given result is obtained. Other equations are similarly proved by using Theorem 1 and Proposition 1. □

Corollary 5. *Let $\pi:(M,g)\to(G,g')$ be a Riemannian submersion, where (M,g) and (G,g') Riemannian manifolds. If the Riemannian submersion has total umbilical fibres that is $N=0$ and then the M-projective curvature tensor is given by*

$$\begin{aligned}
g(W^*(X,Y)Z,H)=&\,g(R^G(X,Y)Z,H)+2g(A_XY,A_ZH)-g(A_YZ,A_XH)\\
&+g(A_XZ,A_YH)-\frac{1}{2(n-1)}\Big\{g(X,H)\Big[S^G(Y',Z')\circ\pi\\
&-2\sum_i g(A_YX_i,A_ZX_i)-\sum_j g(T_{U_j}Y,T_{U_j}Z)\Big]\\
&-g(Y,H)\Big[S^G(X',Z')\circ\pi-2\sum_i g(A_XX_i,A_ZX_i)\\
&\quad-\sum_j g(T_{U_j}X,T_{U_j}Z)\Big]\\
&+g(Y,Z)\Big[S^G(X',H')\circ\pi-2\sum_i g(A_XX_i,A_HX_i)\\
&\quad-\sum_j g(T_{U_j}X,T_{U_j}H)\Big]\\
&-g(X,Z)\Big[S^G(Y',H')\circ\pi-2\sum_i g(A_YX_i,A_HX_i)\\
&\quad-\sum_j g(T_{U_j}Y,T_{U_j}H)\Big]\Big\},\\
g(W^*(X,Y)Z,V)=&-g((\nabla_ZA)_XY,V)-g(A_XY,T_VZ)+g(A_YZ,T_VX)\\
&-g(A_XZ,T_VY)-\frac{1}{2(n-1)}\Big\{g(Y,Z)\Big[-\sum_j g((\nabla_{U_j}T)_{U_j}X,V)\\
&+\sum_i(g((\nabla_{X_i}A)_{X_i}X,V)-2g(A_VX_i,T_XX_i))\Big]
\end{aligned}$$

$$
\begin{aligned}
&- g(X,Z)\Big[-\sum_j g((\nabla_{U_j}T)_{U_j}Y,V) \\
&+\sum_i (g((\nabla_{X_i}A)_{X_i}Y,V)-2g(A_VX_i,T_YX_i))\Big]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
g(W^*(X,V)Y,W) = {} & g((\nabla_XT)_VW,Y)+g((\nabla_VA)_XY,W)-g(T_VX,T_WY) \\
&+g(A_XY,A_YW)-\frac{1}{2(n-1)}\Big\{-g(V,W)\Big[S^G(X',Y')\circ\pi \\
&-2\sum_i g(A_XX_i,A_YX_i)-\sum_j g(T_{U_j}X,T_{U_j}Y)\Big] \\
&-g(X,Y)\Big[\hat{S}(V,W)+\sum_i (g((\nabla_{X_i}T)_VW,X_i)+g(A_{X_i}V,A_{X_i}W))\Big]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
g(W^*(U,V)W,X) = {} & g((\nabla_UT)_VW,X)-g((\nabla_VT)_UW,X) \\
&-\frac{1}{2(n-1)}\Big\{g(V,W)\Big[\sum_j g(\nabla_{U_j}T)_{U_j}U,X) \\
&+\sum_i\{g((\nabla_{X_i}A)_{X_i}X,U)-2g(A_XX_i,T_UX_i)\}\Big] \\
&-g(U,W)\Big[\sum_j g(\nabla_{U_j}T)_{U_j}V,X) \\
&+\sum_i\{g((\nabla_{X_i}A)_{X_i}X,V)-2g(A_XX_i,T_VX_i)\}\Big]\Big\}
\end{aligned}
$$

and

$$
\begin{aligned}
g(W^*(U,V)W,F) = {} & g(\hat{R}(U,V)W,F)+g(T_UW,T_VF)-g(T_VW,T_UF)-\frac{1}{2(n-1)} \\
&\times\Big\{g(F,U)\Big[\hat{S}(V,W)+\sum_i (g(\nabla_{X_i}T)_VW,X_i+g(A_{X_i}V,A_{X_i}W))\Big] \\
&-g(F,V)\Big[\hat{S}(U,W)+\sum_i g((\nabla_{X_i}T)_UW,X_i)+g(A_{X_i}U,A_{X_i}W)\Big] \\
&+g(V,W)\Big[\hat{S}(U,F)+\sum_i (g((\nabla_{X_i}T)_UF,X_i)+g(A_{X_i}U,A_{X_i}F))\Big] \\
&-g(U,W)\Big[\hat{S}(V,F)+\sum_i (g((\nabla_{X_i}T)_VF,X_i)+g(A_{X_i}V,A_{X_i}F))\Big]\Big\}.
\end{aligned}
$$

7. Conclusion

The authors investigate new curvature tensors along Riemannian submersions and obtain some results by using totally umbilical fibres. Therefore, it will be worth examining new curvature tensors along Riemannian submersions. Based on this study, pseudo-projective curvature tensor and quasi-conformal curvature tensor for a Riemannian submersion have been studied in [2]. Again other features such as flatness, symmetry conditions, and a variety of specific conditions on these curvature tensors can be investigated. A sequence of inequalities for Riemannian submersions and various applications between Riemannian submersions and Riemannian manifolds can also be established using some special functions [7, 8].

Riemannian submersions have applications in theoretical pyhsics, too. Other example is in robotic theory, for the modeling and control of certain types of redundant robotic chains [1] (see: https://ieeexplore.ieee.org/document/1284418). Moreover, tensor analysis performed in this work has potential applications in dynamics of rigid bodies, electricity and magnetism, as well as in special theory of relativity, due to the fact that the tensors new considered are of prime interest in the fields of research [11, 12].

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STATISTICALLY UNBOUNDED p-CONVERGENCE IN LATTICE-NORMED RIESZ SPACES

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Abstract. The statistically unbounded p-convergence is an abstraction of the statistical order, unbounded order, and p-convergences. We investigate the concept of the statistically unbounded convergence on lattice-normed Riesz spaces with respect to statistical p-decreasing sequences. Also, we get some relations between this concept and the other kinds of statistical convergences on Riesz spaces.

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1. Introduction

A Riesz space is an ordered vector space, which was introduced by Riesz in [21]. Most of the spaces encountered in the analysis are Riesz spaces. Moreover, Riesz spaces have many applications in measure theory, operator theory, optimization, problems of Banach spaces, measure theory, and applications in economics (cf. [1–3, 19, 24]).

The *order convergence* is one of the fundamental concepts in the study of Riesz spaces, which is not topological in general (cf. [15]). However, even without a topological structure, several kinds of continuous operators can be defined. Another fundamental concept in the theory of Riesz spaces is *unbounded order convergence*, which was firstly introduced in [20] under the name individual convergence. A lot of work has been done since then (cf. [7, 9, 14, 23]). The unbounded order convergence will be the basic tool of this paper. Lattice-valued norms on vector lattices provide natural and efficient tools in the theory of vector lattices. It is enough to mention the theory of lattice-normed vector lattices (cf. [10, 17, 18]).

The theory of the *statistical convergence* is an active area of research, which is a generalization of the ordinary convergence of a real sequence, and the idea of the statistical convergence was firstly introduced by Zygmund [16]. After then, Fast [12] and Steinhaus [22] independently improved that idea. Several applications and generalizations of the statistical convergence of sequences have been investigated by

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several authors (cf. [4–6, 8, 11, 13]. Most of the works on the theory of the statistical convergence are handled with respect to a given topology. But, Ercan introduced a concept of the statistical convergence in Riesz spaces without topology in [11]. Then Şençimen and Pehlivan extended this concept to Riesz spaces with respect to the order convergence; see [8]. Recently, Aydın et al., have investigated some studies about the statistical convergence on Riesz spaces, Riesz algebras, and locally solid Riesz spaces; see [4–6]. In the present paper, we aim to introduce the concept of the statistically unbounded p-convergence on lattice-normed spaces and illustrate the usefulness of lattice-valued norms for the investigation of different types of statistical convergence in Riesz spaces.

The structure of the paper is as follows. In Section 2, we give several notions related to lattice-normed spaces and statistical convergence. In Section 3, we introduce the concept of the statistically unbounded convergence in lattice-normed spaces. In the last section, we show some main results.

2. Preliminaries

We begin the section with some basic concepts related to the theory of Riesz space and refer to [1, 2, 17–19, 24] for more details.

Definition 1. A real-valued vector space E with a partial order relation "$\leq$" on E (i.e. it is an antisymmetric, reflexive and transitive relation) is called an *ordered vector space* whenever, for every $x, y \in E$, we have

(a) $x \leq y$ implies $x+z \leq y+z$ for all $z \in E$,
(b) $x \leq y$ implies $\lambda x \leq \lambda y$ for every $0 \leq \lambda \in \mathbb{R}$.

An ordered vector space E is called a *Riesz space* or a *vector lattice* if, for any two vectors $x, y \in E$, the infimum and the supremum

$$x \wedge y = \inf\{x, y\} \text{ and } x \vee y = \sup\{x, y\}$$

exist in E, respectively. A Riesz space is called *Dedekind complete* if every nonempty bounded above subset has a supremum (or, equivalently, whenever every nonempty bounded below subset has an infimum). For an element x in a Riesz spaces E, *the positive part*, *the negative part*, and *module* of x, respectively

$$x^+ := x \vee 0, \quad x^- := (-x) \vee 0 \ \textit{and} \ |x| := x \vee (-x).$$

In the present paper, the vertical bar $|\cdot|$ of elements of the Riesz spaces will stand for the module of the given elements. A Riesz space E is called *Archimedean* whenever $\frac{1}{n}x \downarrow 0$ holds in E for each $x \in E_+$. Unless otherwise stated, we assume that all vector lattices are real and Archimedean.

A sequence (x_n) in a Riesz space E is said to be *increasing* whenever $x_1 \leq x_2 \leq \cdots$, and *decreasing* if $x_1 \geq x_2 \geq \cdots$. Then we denote them by $x_n \uparrow$ and $x_n \downarrow$ respectively. Moreover, if $x_n \uparrow$ and $\sup x_n = x$ then we write $x_n \uparrow x$. Similarly, if $x_n \downarrow$ and $\inf x_n = x$

then we write $x_n \downarrow x$. Then we call that (x_n) is increasing or decreasing as *monotonic*. Moreover, we remind that a sequence (x_n) in a Riesz space X is called

- *order convergent* to $x \in X$ ($x_n \xrightarrow{\text{o}} x$, for short) whenever there exists a sequence $(q_n) \downarrow 0$ in X such that $|x_n - x| \leq q_n$ for all $n \in \mathbb{N}$,
- *unbounded order convergent* (or *uo*-convergent, for short) to $x \in X$ if $|x_n - x| \wedge u \xrightarrow{\text{o}} 0$ for every $u \in X_+$, in this case, we write $x_n \xrightarrow{\text{uo}} x$.

It is clear that order convergence implies *uo*-convergence because for any $u \geq 0$ $|a-b| \wedge u \leq |a-b|$. But, the converse need not be true. For example, consider the sequence (e_n) of the standard unit vectors in c_0. Then $e_n \xrightarrow{\text{uo}} 0$, but does not converge in order because (e_n) is not order bounded in c_0. In L_p-spaces, where $1 \leq p < \infty$, the *uo*-convergence of sequences is equivalent to almost everywhere convergence (cf. [14]). Also, if $E := \ell_p$, $1 \leq p < \infty$, c_0 or c then *uo*-convergence is equivalent to the coordinate-wise convergence (cf. [9]).

Definition 2. Let X be a vector space and E be a Riesz space. Then $p : X \to E_+$ is called an *E-valued vector norm* whenever it satisfies the following conditions:

(1) $p(x) = 0 \Leftrightarrow x = 0$;
(2) $p(\lambda x) = |\lambda| p(x)$ for all $\lambda \in \mathbb{R}$;
(3) $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$.

Then the triple (X, p, E) is called a *lattice-normed space*, abbreviated as *LNS*. If, in addition, X is a Riesz space and the vector norm p is monotone (i.e., $|x| \leq |y| \Rightarrow p(x) \leq p(y)$ holds for all $x, y \in X$) then the triple (X, p, E) is called a *lattice-normed vector lattice* or a *lattice-normed Riesz space*. We abbreviate it as *LNRS*. While dealing with *LNRS*s, we shall keep in mind also the following examples.

Example 1. Let X be a normed space with a norm $\|\cdot\|$. Then $(X, \|\cdot\|, \mathbb{R})$ is an *LNS* .

Example 2. Let X be a Riesz space. Then $(X, |\cdot|, X)$ is an *LNRS* .

We abbreviate the convergence $p(x_n - x) \xrightarrow{\text{o}} 0$ as $x_n \xrightarrow{\text{p}} x$ and say in this case that (x_n) *p-converges* to x in an *LNS* (X, p, E). Moreover, an *LNRS* (X, p, E) is called *op-continuous* if $x_n \xrightarrow{\text{o}} 0$ implies $p(x_n) \xrightarrow{\text{o}} 0$. In an *LNS* (X, p, E) a subset A of X is called *p-bounded* if there exists $e \in E$ such that $p(a) \leq e$ for all $a \in A$. A vector $e \in X$ is called a *p-unit* if, for any $x \in X_+$, $p(x - ne \wedge x) \xrightarrow{o} 0$. We refer the reader for more information on *LNS*s to [7, 10, 17, 18].

Now, we recall some basic properties of the concepts related to the statistical convergence. Consider a set K of positive integers. Then the *natural density* of K is defined by

$$\delta(K) := \lim_{n \to \infty} \frac{1}{n} \left| \{k \leq n : k \in K\} \right|,$$

where the vertical bar of sets will stand for the cardinality of the given sets. We refer the reader to an exposition on the natural density of sets to [12, 13]. In the same way, a sequence $x = (x_k)$ is called *statistical convergent* to L provided that

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0$$

for each $\varepsilon > 0$. Then it is written by $S - \lim x_k = L$. We take the following notions from [6, 8]. Let (x_n) be a sequence in a Riesz space E. Then (x_n) is called

- *statistically order decreasing* to 0 if there exists a set $K = \{n_1 < n_2 < \cdots\} \subseteq \mathbb{N}$ with $\delta(K) = 1$ such that (x_{k_n}) is decreasing and $\inf_{k_n \in K} (x_{k_n}) = 0$, and so, it is abbreviated as $x_n \downarrow^{\text{st}} 0$,
- *statistically order convergent* to $x \in E$ if there exists a sequence $q_n \downarrow^{\text{st}} 0$ with a set $K = \{n_1 < n_2 < \cdots\} \subseteq \mathbb{N}$ such that $\delta(K) = 1$ and $|x_{k_n} - x| \leq q_{k_n}$ for every $k_n \in K$, and so, we write $x_n \xrightarrow{\text{st-o}} x$,
- *statistically unbounded order convergent* to $x \in E$ if, for every $u \in E_+$, there exists a sequence $q_n \downarrow^{st} 0$ and a subset K of the natural numbers such that $\delta(K) = 1$ and
$$|x_{k_n} - x| \wedge u \leq q_{k_n}$$
for all $k_n \in K$, and so, we abbreviate it as $x_n \xrightarrow{\text{st-uo}} x$.

Moreover, a sequence in an *LNS* (X, p, E) is said to be

- *statistically p-decreasing* to 0 if there exists a set $K = \{n_1 < n_2 < \cdots\} \subseteq \mathbb{N}$ such that $\delta(K) = 1$ and $p(x_{k_n}) \downarrow 0$ on K, and so, we abbreviate it as $x_n \downarrow^{\text{st}_\text{p}} 0$,
- *statistically p-convergence* to x if there exists a sequence $q_n \downarrow^{\text{st}_\text{p}} 0$ in X with a set $K = \{n_1 < n_2 < \cdots\} \subseteq \mathbb{N}$ such that $\delta(K) = 1$ and $p(x_{k_n} - x) \leq q_{k_n}$ for all $k_n \in K$, and so, we write $x_n \xrightarrow{\text{st}_\text{p}} x$.

3. Statistically unbounded p-convergence

We introduce the statistically unbounded convergence on *LNRS*s in this section. Recall that a sequence (x_n) in an *LNRS* (X, p, E) is said to be *unbounded p-convergent* to $x \in X$ (shortly, $x_n \xrightarrow{up} x$), if $p(|x_n - x| \wedge u) \xrightarrow{\text{o}} 0$ for all $u \in X_+$; see [7, Def.6]. The following definition is motivated by the notion of up-convergence.

Definition 3. Let (X, p, E) be an *LNRS* and (x_n) be a sequence in X. Then (x_n) is said to be *statistically unbounded p-convergence* to x if, for any positive element $u \in X_+$, there exists a sequence $q_n \downarrow^{\text{st}_\text{p}} 0$ in X with an index set $K = \{n_1 < n_2 < \cdots\} \subseteq \mathbb{N}$ such that $\delta(K) = 1$ and

$$p(|x_{k_n} - x| \wedge u) \leq q_{k_n}$$

for all $k_n \in K$. Then we write $x_n \xrightarrow{\text{st-up}} x$.

It is clear that a sequence $x_n \xrightarrow{\text{st-up}} x$ holds in an *LNRS* (X,p,E) if and only if $(x_n - x) \xrightarrow{\text{st-up}} 0$ if and only if $p(|x_n - x| \wedge u) \xrightarrow{\text{st-o}} 0$ in E for all $u \in X_+$.

Example 3. Let X^* be the algebraic dual of a Riesz space X and Y be a sublattice of X^* such that $\langle X,Y\rangle$ is a dual system. Then define an *LNRS* $(X,p,\mathbb{R}^Y)$ with a lattice norm $p : X \to \mathbb{R}^Y$ denoted by $p(x)[f] := |f|(|x|)$. Take a sequence (x_n) in X. Thus, it is clear that $|x_n| \wedge u \xrightarrow{|\sigma|(X,Y)} 0$ for every $u \in X_+$, and so, we have $x_n \xrightarrow{\text{st-up}} 0$.

The next observations follow directly from the basic definitions and results, and so, we omit their proofs.

Remark 1.

(i) It follows from the inequality $|a-b| \wedge u \leq |a-b|$ in Riesz spaces and by the monotonicity of p in *LNRS*s that statistical p-convergence implies *st-up*-convergence.
(ii) Statistically unbounded p-convergence coincides with the notion of statistical p-convergence for order bounded sequences.
(iii) Consider an *LNRS* $(L_p(\mu), \|\cdot\|, \mathbb{R})$ for $1 \leq p < \infty$, where μ is a finite measure. Then the convergence in measure implies *st-up*-convergence of sequences in $L_p(\mu)$; see [23].
(iv) Take the *LNRS*s $(L_p, |\cdot|, L_p)$ for $1 \leq p < \infty$. Then the abstraction of a.e.-convergence in L_p-spaces implies *st-up*-convergence of sequences in L_p; see [14].
(v) Consider an *LNRS* $(E, |\cdot|, E)$ for an arbitrary Riesz space E. Then statistically unbounded order convergence and *st-up*-convergence agree.
(vi) Every *up*-convergent sequence is statistically unbounded p-convergent to its *up*-limit.
(vii) A statistical p-decreasing sequence is *st-up*-convergent.
(viii) Take an *LNRS* (X,p,E), where E is ℓ_p $(1 \leq p < \infty)$, c_0 or c. Then the coordinate-wise convergence of a sequence implies *st-up*-convergence.
(ix) An order convergent sequence in *op*-continuous *LNRS*s is *st-up*-convergent to its order limit.

The converse of the properties of Remark 1 need not be true in general. To see some of them, we give the following examples.

Example 4. Consider the *LNRS* $(c_0, |\cdot|, c_0)$, where c_0 is the set of all convergent to zero real sequences. Take the sequence (e_n) of the standard unit vectors in c_0. Then (e_n) is statistically unbounded p-convergent to 0. However, it is not p-convergent because it is not p-bounded in c_0.

Example 5. Let's consider the *LNRS* $(\mathbb{R}^2, |\cdot|, \mathbb{R}^2)$, where Euclidean space $\mathbb{R}^2$ with the coordinate-wise ordering. Take a sequence (x_n) in $\mathbb{R}^2$ denoted by

$$x_n := \begin{cases} (0,n), & k = n^3 \\ (1, \frac{1}{n}), & k \neq n^3 \end{cases}$$

for all n, where $k \in \mathbb{N}$. Thus, (x_n) is statistically unbounded p-convergent to $(1,0) \in \mathbb{R}^2$. But, it does not up-convergent.

Example 6. Let's take the *LNRS* c of all convergent real sequences. Consider a sequence (x_n) in *LNRS* $(c, |\cdot|, c)$ denoted by $x_n := (x_k^n) = (x_1^n, x_2^n, \cdots, x_k^n, \cdots) \in E$ such that

$$x_k^n := \begin{cases} 1, & k = n^2 \\ \frac{1}{k+1}, & k \neq n^2 \end{cases}$$

for all $n, k \in \mathbb{N}$. Then it is clear that $x_n \xrightarrow{\text{st-up}} 0$. Observe that the whole sequence (x_n) is not monotonic.

It is clear from Example 5 that a statistical p-monotone decreasing sequence need not be monotone convergent in general. It is well known that a subsequence of order convergent sequence is order convergent to its order limit. However, this need not be true for *st-up*-convergence. To illustrate this, we consider the following example.

Example 7. Take the Riesz space $\mathbb{R}$ and a sequence (x_n) in $\mathbb{R}$ defined by

$$x_n := \begin{cases} (-1)^n(2n+1), & k = n^5 \\ \frac{1}{2n+1}, & k \neq n^5 \end{cases}$$

for all n, where $k \in \mathbb{N}$. Fix $u \in \mathbb{R}_+$. Thus, for the sequence $q_n := \frac{1}{n+1} \downarrow 0$, we have $|x_n| \wedge u \leq q_n$ for all $n \in \mathbb{N}$ such that $n \neq k^5$ for all $k \in \mathbb{N}$. Hence, we obtain $x_n \xrightarrow{\text{st-up}} 0$. But, by choosing a subsequence (x_{m_n}) of (x_n) such that $m_n = k^5$ for some $k \in \mathbb{N}$, it is clear that (x_{m_n}) is not *st-up*-convergent.

By the following sense, we see that the lattice operations in an *LNRS* are statistically unbounded p-continuous.

Theorem 1. *If $x_n \xrightarrow{\text{st-up}} x$ and $y_n \xrightarrow{\text{st-up}} y$ satisfy then $x_n \vee y_n \xrightarrow{\text{st-up}} x \vee y$ in LNRSs.*

Proof. Let $x_n \xrightarrow{\text{st-up}} x$ and $y_n \xrightarrow{\text{st-up}} y$ be in an *LNRS* (X, p, E). Take an arbitrary $u \in E_+$. Then there exist sequences $q_n \downarrow^{\text{st}_\text{p}} 0$ and $t_n \downarrow^{\text{st}_\text{p}} 0$ with $\delta(K) = \delta(M) = 1$ such that $p|(x_{k_n} - x| \wedge u) \leq q_{k_n}$ and $p(|x_{m_n} - y| \wedge u) \leq t_{m_n}$ for each $k_n \in K$ and $m_n \in M$. By using the inequality $|a \vee b - a \vee c| \leq |b - c|$ (cf. [2, Thm.1.9(2)]) in Riesz spaces, we have

$$\begin{aligned} p(|x_n \vee y_n - x \vee y| \wedge u) &\leq p(|x_n \vee y_n - x_n \vee y| \wedge u) + p(|x_n \vee y - x \vee y| \wedge u) \\ &\leq p(|y_n - y| \wedge u) + p(|x_n - x| \wedge u) \end{aligned}$$

for every $n \in \mathbb{N}$. So, it follows that

$$p(|x_{j_n} \vee y_{j_n} - x \vee y| \wedge u) \leq p(|y_{j_n} - y| \wedge u) + p(|x_{j_n} - x| \wedge u) \leq q_{j_n} + t_{j_n}$$

for each $j_n \in J$, where $J := K \cap J$. Therefore, we have $x_n \vee y_n \xrightarrow{\text{st-up}} x \vee y$ by $(q_{j_n} + t_{j_n}) \downarrow 0$. □

Theorem 2. *Let (X, p, E) be an LNRS, $x_n \xrightarrow{\text{st-up}} x$ and $y_n \xrightarrow{\text{st-up}} y$. Then the following statements hold;*

- (i) *if $x_n \xrightarrow{\text{st-up}} x$ and $x_n \xrightarrow{\text{st-up}} z$ then $x = y$;*
- (ii) *$\lambda x_n + \beta y_n \xrightarrow{\text{st-up}} \lambda x + \beta y$ for all $\lambda, \beta \in \mathbb{R}$;*
- (iii) *$|x_n| \xrightarrow{\text{st-up}} |x|$;*
- (iv) *$x_n^+ \xrightarrow{\text{st-up}} x^+$;*
- (v) *if $x_n \geq y_n$ for all $n \in \mathbb{N}$ then we have $x \geq y$.*

Proof. The properties (iii) and (iv) are results of Theorem 1, and so, we omit their proof. Moreover, (v) can be obtained by applying (iv), and (ii) has a straightforward proof.

To prove (i), consider that a sequence (x_n) in an *LNRS* (X, p, E) satisfies $x_n \xrightarrow{\text{st-up}} x$ and $x_n \xrightarrow{\text{st-up}} y$. Fix a positive element $u \in X_+$. Then there exist sequences $q_n \downarrow^{\text{st}_\text{p}} 0$ and $t_n \downarrow^{\text{st}_\text{p}} 0$ with index sets $K, M \subseteq \mathbb{N}$ such that $\delta(K) = \delta(M) = 1$ and

$$p(|x_{k_n} - x| \wedge u) \leq q_{k_n} \quad \text{and} \quad p(|x_{m_n} - y| \wedge u) \leq t_{m_n}$$

hold for all $k_n \in K$ and $m_n \in M$. Take $J := K \cap M$, and so, we have $\delta(J) = 1$. It follows $p(|x_{j_n} - x| \wedge u) \leq q_{j_n}$ and $p(|x_{j_n} - y| \wedge u) \leq r_{j_n}$ for each $j_n \in J$. Thus, we observe

$$0 \leq p(|x - y| \wedge u) \leq p(|x_{m_n} - x| \wedge u) + p(|x_{m_n} - y| \wedge u) \leq q_{j_n} + r_{j_n}$$

for every $j_n \in J$. Thus, $(q_{j_n} + r_{j_n}) \downarrow 0$ on J, and so, we obtain $|x - y| \wedge u = 0$. Since u is arbitrary, we get $x = y$. □

By applying Theorem 2(v), we give the following result.

Corollary 1. *Let (X, p, E) be an LNRS and (x_n), (y_n) and (z_n) be sequences in X such that $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$. Then $x_n \xrightarrow{\text{st}_\text{p}} x$ and $z_n \xrightarrow{\text{st}_\text{p}} x$ implies $y_n \xrightarrow{\text{st}_\text{p}} x$.*

In the following theorem, we give a relation between *st-up-* and order convergence.

Theorem 3. *Any monotone st-up-convergent sequence in an LNRS is order convergent to its st-up-limit.*

Proof. Suppose that $x_n \xrightarrow{\text{st-up}} x$ and $x_n \downarrow$ hold in an *LNRS* (X, p, E). Then, for a fixed arbitrary m, we have $x_m - x_n \in X_+$ for each $n \geq m$. Thus, by using Theorem 2, we get $x_m - x_n \xrightarrow{\text{st-up}} x_m - x \in X_+$. So, it is clear that x is a lower bound of (x_n) because m is arbitrary. Choose another lower bound w of (x_n), i.e., $x_n \geq w$ for all n.

Then, again by applying Theorem 2, we obtain $x_n - x \xrightarrow{\text{st-up}} x - w \in X_+$. Hence, we have $w \leq x$, i.e., we get $x_n \downarrow x$. □

It follows from Remark 1(i) that statistical p-convergence implies *st-up*-convergence. For the converse, we give the following result.

Theorem 4. *Let (x_n) be a monotone and $x_n \xrightarrow{\text{st-up}} x$ sequence in an LNRS (X,p,E). Then $x_n \xrightarrow{\text{st}_\text{p}} x$.*

Proof. Assume that $0 \leq x_n \uparrow$ holds in X by without loss of generality. It follows from Theorem 3 that $x_n \xrightarrow{\text{o}} x$ for some $x \in X$. Thus, we get $0 \leq x - x_n \leq x$ for all $n \in \mathbb{N}$. Fix $u \in X_+$. Then we have $p((x-x_n)\wedge u) \xrightarrow{\text{st-o}} 0$ because of $x_n \xrightarrow{\text{st-up}} x$. If, in particular, we choose u as $x \in X_+$ then we get $p(x-x_n) = p((x_n - x)\wedge x) \xrightarrow{\text{st-o}} 0$, i.e., we have $x_n \xrightarrow{\text{st}_\text{p}} x$. □

4. Main results

Recall that if (x_n) is a sequence satisfying the property P for all $n \in \mathbb{N}$ except a set of natural density zero then we say that (x_n) satisfies the property P for almost all n, and it is abbreviated by a.a.n.; see [13].

Theorem 5. *A sequence $x_n \xrightarrow{\text{st-up}} x$ holds in LNRSs if and only if there is another sequence (y_n) such that $x_n = y_n$ for a.a.n and $y_n \xrightarrow{\text{st-up}} x$.*

Proof. Assume that there is a sequence (y_n) in an *LNRS* (X,p,E) such that $x_n = y_n$ for a.a.n and $y_n \xrightarrow{\text{st-up}} x$. Take an arbitrary $u \in X_+$. Thus, there exists a sequence $q_n \downarrow^{\text{st}_\text{p}} 0$ with a set $\delta(K) = 1$ such that $p(|y_{k_n} - x| \wedge u) \leq q_{k_n}$ for all $k_n \in K$. It follows from $x_n = y_n$ for a.a.n that the index sets $\{k_n \in K : p(|x_{k_n} - x| \wedge u) \leq q_{k_n}\}$ and $\{k_n \in K : p(|y_{k_n} - x| \wedge u) \leq q_{k_n}\}$ are a.a.n. Therefore, there is a subset J of K such that $\delta(J) = 1$ and $p(|x_{j_n} - x| \wedge u) \leq q_{j_n}$ for each $j_n \in J$. So, we have $x_n \xrightarrow{\text{st-up}} x$. □

In the general case, Example 7 shows that a subsequence of *st-up*-convergent sequence need not be *st-up*-convergent. But, in the following work, we give a positive result for this.

Theorem 6. *Let (X,p,E) be an LNRS and (x_n) be a sequence in X. If $x_n \xrightarrow{\text{st-up}} x$ and $0 \leq x_n \downarrow$ then every subsequence (x_{k_n}) with an index set $\delta(\{k_1, k_2, \cdots, k_i, \cdots\}) = 1$ is st-up-convergent to x.*

Proof. Assume that $x_n \downarrow$ and $x_n \xrightarrow{\text{st-up}} x$ in an *LNRS* (X,p,E). Then, for any $u \in X_+$, there exists a sequence $q_n \downarrow^{\text{st}_\text{p}} 0$ with a set $\delta(K) = 1$ such that $p(|x_{k_n} - x| \wedge u) \leq q_{k_n}$ for all $k_n \in K$. Thus, it can be seen that $x_{k_n} \xrightarrow{\text{st-up}} x$ holds.

On the other hand, it follows from Theorem 4 that $x_n \xrightarrow{\text{st}_\text{p}} x$, i.e., $p(x_n - x) \xrightarrow{\text{st-o}} 0$. Thus, we have $p(x_n - x) \downarrow^{\text{st}} 0$ because of the monotonicity of p and $(x_n - x) \downarrow$. Then, for an arbitrary $M = \{m_1, \cdots, m_n, \cdots\} \subseteq \mathbb{N}$ such that $\delta(M) = 1$ and $M \neq K$, we have $p(x_{m_n} - x) \downarrow 0$ by applying [8, Thm.3]. Hence, $p(x_{m_n} - x) \downarrow^{\text{st}} 0$, and so, $p(x_{m_n} - x) \xrightarrow{\text{st-o}} 0$, i.e., $(x_{m_n} - x) \xrightarrow{\text{st}_\text{p}} 0$. Therefore, we get $x_{m_n} \xrightarrow{\text{st-up}} x$. □

Recall that a Dedekind complete Riesz space X^δ is said to be a Dedekind completion of a Riesz space X whenever X is Riesz isomorphic to a majorizing order dense Riesz subspace of X^δ. It is well known that every Archimedean Riesz space has a Dedekind completion (cf. [2, Thm.2.24]). A sequence $x_n \xrightarrow{\text{o}} 0$ in a Riesz space X if and only if $x_n \xrightarrow{\text{o}} 0$ in X^δ (cf. [14, Cor. 2.9]).

Theorem 7. *Let (X, p, E) be an LNRS and e be a p-unit in X. Then we have $x_n \xrightarrow{\text{st-up}} 0$ in X if and only if $p(|x_n| \wedge e) \xrightarrow{\text{st-o}} 0$ in E.*

Proof. Suppose that $x_n \xrightarrow{\text{st-up}} 0$ holds in X. Then, by taking $u = e \in E_+$, we get $p(|x_n| \wedge e) \xrightarrow{\text{st-o}} 0$ in E.

Now, assume that $p(|x_n| \wedge e) \xrightarrow{\text{st-o}} 0$ in E. Then there exists a sequence $q_n \downarrow^{\text{st}_\text{p}} 0$ with a set $\delta(K) = 1$ such that $p(|x_{k_n}| \wedge e) \leq q_{k_n}$ for all $k_n \in K$, i.e., we have $p(|x_{k_n}| \wedge e) \xrightarrow{\text{o}} 0$ on K. Take any $u \in X_+$. Then we observe that

$$\begin{aligned} p(|x_{k_n}| \wedge u) &\leq p(|x_{k_n}| \wedge (u - u \wedge ne)) + p(|x_{k_n}| \wedge (u \wedge ne)) \\ &\leq p(u - u \wedge ne) + np(|x_{k_n}| \wedge e) \end{aligned}$$

holds in E^δ for each $k_n \in K$ and for every $n \in \mathbb{N}$. So, we have

$$\limsup_{k_n} p(|x_{k_n}| \wedge u) \leq p(u - u \wedge ne) + n \limsup_{k_n} p(|x_{k_n}| \wedge e)$$

holds in E^δ for all $n \in \mathbb{N}$. Since $p(|x_{k_n}| \wedge e) \xrightarrow{\text{o}} 0$ in E, we have $p(|x_{k_n}| \wedge e) \xrightarrow{\text{o}} 0$ in E^δ. Thus, we see $\limsup_{k_n} p(|x_{k_n}| \wedge e) = 0$ in E^δ. Thus

$$\limsup_{k_n} p(|x_{k_n}| \wedge u) \leq p(u - u \wedge ne)$$

holds in E^δ for all $n \in \mathbb{N}$. Since e is a p-unit, we have $\limsup_{k_n} p(|x_{k_n}| \wedge u) = 0$ in E^δ or $p(|x_{k_n}| \wedge u) \xrightarrow{\text{o}} 0$ in E^δ. It follows that $p(|x_{k_n}| \wedge u) \xrightarrow{\text{o}} 0$ in E. Hence, $x_n \xrightarrow{\text{st-up}} 0$. □

Theorem 8. *Let (X, p, E) be an LNRS. Then define a Riesz norm $p^\delta : X^\delta \to E^\delta$ by $p^\delta(z) = \sup_{0 \leq x \leq |z|} p(x)$ for every $z \in X^\delta$. Then we have $x_n \xrightarrow{\text{st-up}} x$ in (X, p, E) if and only if $x_n \xrightarrow{\text{st-up}} x$ in $(X^\delta, p^\delta, E^\delta)$.*

Proof. It follows from [7, Prop.4.1] that $(X^\delta, p^\delta, E^\delta)$ is an *LNRS*.

Take a sequence $x_n \xrightarrow{\text{st-up}} 0$ in (X, p, E) and fix an arbitrary positive element z in X^δ_+. Then, for any positive element $u \in X_+$, there exists a sequence $q_n \downarrow^{\text{st}_\text{p}} 0$ in E with a set $\delta(K) = 1$ such that $p(|x_{k_n} - x| \wedge u) \leq q_{k_n}$ for every $k_n \in K$. On the other hand, since X is majorizing in its Dedekind completion X^δ, there exists $u \in X$ such that $0 < z \leq u$. By using $|x_{k_n} - x| \wedge z \leq |x_{k_n} - x| \wedge u$, we obtain

$$p^\delta(|x_{k_n} - x| \wedge z) \leq p(|x_{k_n} - x| \wedge u) \leq q_{k_n}$$

for each $k_n \in K$. Thus, $x_n \xrightarrow{\text{st-up}} x$ in $(X^\delta, p^\delta, E^\delta)$ because $q_{k_n} \downarrow 0$ in X implies $q_{k_n} \downarrow 0$ in X^δ; [14, Cor.2.9].

For the converse, suppose that $x_n \xrightarrow{\text{st-up}} 0$ in $(X^\delta, p^\delta, E^\delta)$ and u be a positive element in X_+. Then there exists a sequence $q_n \downarrow^{\text{st}_\text{p}} 0$ in E^δ with a set $\delta(K) = 1$ such that $p^\delta(|x_{k_n} - x| \wedge u) \leq q_{k_n}$ for every $k_n \in K$. Again by applying [14, Cor.2.9], $q_{k_n} \downarrow 0$ in X^δ implies $q_{k_n} \downarrow 0$ in X. So, we have

$$p(|x_{k_n} - x| \wedge u) = p^\delta(|x_{k_n} - x| \wedge u) \leq q_{k_n}$$

for every $k_n \in K$ because of (x_n) in X. It follows that $x_n \xrightarrow{\text{st-up}} x$ in (X, p, E). □

Proposition 1. *A statistical order convergent sequence in op-continuous LNRSs is st-up-convergent.*

Proof. Let $x_n \xrightarrow{\text{st-o}} x$ be in an *op*-continuous *LNRS* (X, p, E). Then there exists a sequence $s_n \downarrow^{\text{st}} 0$ in X with an index set I such that $\delta(I) = 1$ and $|x_{i_n} - x| \leq s_{i_n}$ for each $i_n \in I$. Hence, we get $p(x_{i_n} - x) \leq p(s_{i_n})$, and so, $p(|x_{i_n} - x| \wedge u) \leq p(s_{i_n})$ for every $i_n \in I$ and each $u \in X_+$. By using $s_{i_n} \downarrow 0$ on I in X, it follows from *op*-continuity of (X, p, E) that $p(s_{i_n}) \downarrow 0$ on I in E. Therefore, we get the desired result, $x_n \xrightarrow{\text{st-up}} 0$. □

Recall that a subset A of a Riesz space E is called solid if, for each $x \in A$ and $y \in E$, $|y| \leq |x|$ implies $y \in A$. Also, a solid vector subspace of a Riesz space is referred to as an ideal. Moreover, an order closed ideal is called a band (cf. [1, 3]).

Proposition 2. *Let B be a band in an LNRS (X, p, E). If $b_n \xrightarrow{\text{st-up}} x$ is a sequence in B then $x \in B$.*

Proof. Assume that $b_n \xrightarrow{\text{st-up}} x$ satisfies for a sequence (b_n) in B and some $x \in X$. By applying Theorem 1, we have $|b_n| \wedge |z| \xrightarrow{\text{st-up}} |x| \wedge |z|$ for any $z \in B^d := \{z \in X : |z| \wedge |b| = 0 \text{ for all } b \in B\}$. Also, $|b_n| \wedge |z| = 0$ for all n because (b_n) is a sequence in B. Therefore, we obtain $|x| \wedge |z| = 0$. So, $x \in B^{dd}$. As a result, it follows from [2, Thm.1.39] that we have $B = B^{dd}$, and so, we obtain $x \in B$. □

Proposition 3. *Let B be a projection band in an LNRS (X, p, E) and p_B be the corresponding band projection of B. Then $x_n \xrightarrow{\text{st-up}} x$ implies $P_B(x_n) \xrightarrow{\text{st-up}} P_B(x)$.*

Proof. Let $x_n \xrightarrow{\text{st-up}} x$ be in an *LNRS* (X,p,E). Then, for any $u \in X_+$, there exists a sequence $q_n \downarrow^{\text{st}_\text{p}} 0$ with a set $\delta(K) = 1$ such that $p(|x_{k_n} - x| \wedge u) \leq q_{k_n}$ for all $k_n \in K$. Also, it is well known that a band projection p_B is a lattice homomorphism and it satisfies the inequality $0 \leq P_B \leq I$. Hence, we have

$$p(|P_B(x_{k_n}) - P_B(x)| \wedge u) = p((P_B|x_{k_n} - x|) \wedge u) \leq p(|x_{k_n} - x| \wedge u) \leq q_{k_n}$$

for every $k_n \in K$. Thus, we have $P_B(x_n) \xrightarrow{\text{st-up}} P_B(x)$. □

Theorem 9. *Let (X,p,E) be an LNRS and Y be a sublattice of X. If a sequence (y_n) in Y is st-up-convergent to zero in Y then it is st-up-convergent to zero in X for each of the following cases:*

(i) *Y is majorizing in X;*
(ii) *Y is p-dense in X;*
(iii) *Y is a projection band in X.*

Proof. Assume that (y_n) is a sequence in Y such that $y_n \xrightarrow{\text{st-up}} 0$ in Y and u be a positive element in X_+.

(i) Since Y is majorizing in X, there exists $y \in Y$ such that $u \leq y$. It follows from

$$0 \leq p(|y_n| \wedge u) \leq p(|y_n| \wedge y) \xrightarrow{\text{st-o}} 0,$$

that $p(|y_n| \wedge u) \xrightarrow{\text{st-o}} 0$, i.e., $y_n \xrightarrow{\text{st-up}} 0$ in X.

(ii) Take any $0 \neq w \in p(X)$. Thus, there is $y \in Y$ with $p(u-y) \leq w$. So, it follows that

$$p(|y_n| \wedge u) \leq p(|y_n| \wedge |u-y|) + p(|y_n| \wedge |y|) \leq w + p(|y_n| \wedge |y|).$$

Then $p(|y_n| \wedge u) \xrightarrow{\text{st-o}} 0$ because $0 \neq w \in p(X)$ is arbitrary and $p(|y_n| \wedge |y|) \xrightarrow{\text{st-o}} 0$. So, we get $y_n \xrightarrow{\text{st-up}} 0$ in X.

(iii) It is clear that $Y = Y^{\perp\perp}$ implies $X = Y \oplus Y^{\perp}$. Thus, $u = u_1 + u_2$ with $u_1 \in Y$ and $u_2 \in Y^{\perp}$. It follows from $y_n \wedge u_2 = 0$ and [2, Lem.1.4] that we have

$$p(|y_n| \wedge u) = p(|y_n| \wedge (u_1 + u_2)) \leq p(|y_n| \wedge u_1) \xrightarrow{\text{st-o}} 0.$$

As a result, we get $y_n \xrightarrow{\text{st-up}} 0$ in X. □

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AN EXISTENCE ETUDY FOR A TRIPLED SYSTEM WITH $p-$LAPLACIAN INVOLVING $\varphi-$CAPUTO DERIVATIVES

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Abstract. In this paper, we study the existence and uniqueness of solutions for a tripled system of fractional differential equations with nonlocal integro multi point boundary conditions by using the $p-$Laplacian operator and the $\varphi-$Caputo derivatives. The presented results are obtained by the two fixed point theorems of Banach and Krasnoselskii. An illustrative example is presented at the end to show the applicability of the obtained results. To the best of our knowledge, this is the first time where such problem is considered.

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1. Introduction

The fractional calculus has many significant roles in various scientific fields of research, see for instance [10, 20]. As applied results,the fractional order differential equations have attired attention of several scientists in different fields of research [8, 19]. However, most of the published works have been achieved by using the fractional derivatives of type Riemann-Liouville, Hadamard, Katugampola, Atangana-Baleanu, Grunwald Letnikov and Caputo. The fractional derivatives of functions with respect to some other functions [14] are different from the others since their kernels appear in terms of other functions (called φ). Recently, some fractional differential results have been considered in [4–6, 11].

In most of the present articles, Schauder, Krasnoselskii, Darbo, or Monch theories have been used to prove existence of solutions of nonlinear fractional differential equations with some restrictive conditions [3, 18]. Some authors have worked on the solutions for fractional problems with $p-$Laplacian operators. We cite, for example [1,2,12,16] where it has been studied nonlinear fractional equation with $p-$Laplacian operator for the solutions.

Here, we will mention some other research works for the reader. We begin by A. Devi, A. Kumar, D. Baleanu and A. Khan [2] where they worked on the stability results, for the following nonlinear FDEs involving Caputo derivatives of distinct

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orders and ψ_p Laplacian operator:

$$\begin{cases} {}^c\mathcal{D}^{r_1}\psi_p\left[{}^c\mathcal{D}^{r_2}\left(u(t)-\sum_{i=1}^m v_i(t)\right)\right]=-w(t,u(t)),t\in(0,1] \\ \psi_p\left[{}^c\mathcal{D}^{r_2}\left(u(t)-\sum_{i=1}^m v_i(t)\right)\right]\big|_{t=0}=0, \\ u(0)=\sum_{i=1}^m v_i(0), \\ u'(1)=\sum_{i=1}^m v_i'(1), \\ u^j(0)=\sum_{i=1}^m v_i^j(0),\ \text{for } j=2,3,...,n-1, \end{cases}$$

where $0<r_1\leq 1, n-1<r_2\leq n, n\geq 4$, and v_i, w are continuous functions. ${}^c\mathcal{D}^{r_1}$ and ${}^c\mathcal{D}^{r_2}$ denotes the derivative of fractional order r_1 and r_2 in Caputo's sense, respectively, and $\psi_p(z)=|z|^{p-2}z$ denotes the $p-$Laplacian operator and satisfies $\frac{1}{p}+\frac{1}{q}=1, (\psi_p)^{-1}=\psi_q$.

We can also cite the paper of A. Mahdjouba et al. [17] where they have investigated the existence and multiplicity of positive solutions of the following problem:

$$\begin{cases} \left(\psi_p\left[\mathcal{D}_{0^+}^r\left(u(t)\right)\right]\right)'+a_1(t)f\left(u(\theta_1(t)),v(\theta_2(t))\right)=0, 0<t<1, \\ \left(\psi_{\overline{p}}\left[\mathcal{D}_{0^+}^r\left(v(t)\right)\right]\right)'+a_2(t)f\left(u(\theta_1(t)),v(\theta_2(t))\right)=0, 0<t<1, \\ \mathcal{D}_{0^+}^r u(0)=u(0)=u'(0),\quad \mathcal{D}_{0^+}^m u(1)=\gamma\,\mathcal{D}_{0^+}^m u(\eta), \\ \mathcal{D}_{0^+}^r v(0)=v(0)=v'(0),\quad \mathcal{D}_{0^+}^m v(1)=\gamma\,\mathcal{D}_{0^+}^m v(\eta), \end{cases}$$

where $\eta\in(0,1), \gamma\in\left(0,\frac{1}{\eta^{r-m-1}}\right)$ $\mathcal{D}_{0^+}^r, \mathcal{D}_{0^+}^m$, are the standard Riemann–Liouville fractional derivatives with $r\in(2,3), m\in(1,2)$ such that $r\geq m+1$, $p-$Laplacian operator is defined as $\psi_p(z)=z|z|^{p-2}, p>1$, and the functions $f,g\in C\left(\mathbb{R}^2,\mathbb{R}\right)$.

Then, S. Etemad with his coauthors [9] have been concerned with the existence study for the following tripled impulsive fractional problem

$$\begin{cases} {}^c\mathcal{D}_{0^+}^{\kappa_m}x_m(t)=f_m(t,x(t)), m=1,2,3,\ \text{ and }\ t\in J' \\ x_m(a)=\Phi_m x, x_m'(a)=\Theta_m x, \\ \Delta x_m|_{t=t_k}=I_{m,k}(x(t_k)), \Delta x_m'|_{t=t_k}=\overline{I}_{m,k}(x(t_k)), \end{cases}$$

where $J=[a,b], J'=J-\{t_1,t_2,...,t_p\}, a=t_0<t_1<\ldots<t_p<t_{p+1}=b$, ${}^c\mathcal{D}_{0^+}^{\kappa_m}$, $m=1,2,3$, are the Caputo fractional derivatives such that $\kappa_m\in(1,2], f_m: J\times\mathbb{R}^3\to\mathbb{R}$, $x(t)=(x_1(t),x_2(t),x_3(t)), I_{m,k}, \overline{I}_{m,k}:\mathbb{R}^3\to\mathbb{R}$, $k=1,2,...,p$, are given functions, Φ_m,Θ_m are given operators, $\Delta x_m|_{t=t_k}=x(t_k^+)-x(t_k^-), \Delta x_m'|_{t=t_k}=x'(t_k^+)-x'(t_k^-)$, and

$$x(t_k^+)=\lim_{h\to 0^+}x_m(t_k+h), x(t_k^-)=\lim_{h\to 0^-}x_m(t_k+h).$$

In the present research work, we study the existence and uniqueness of solutions for the following problem:

$$\begin{cases} \mathcal{D}_{0^+}^{r_{1m};\varphi}\psi_p\left[\mathcal{D}_{0^+}^{r_{2m};\varphi}\left(u_m(t)-I_{0^+}^{\sigma;\varphi}G_m(t,u_1(t),u_2(t),u_3(t))\right)\right] \\ =H_m(t,u_1(t),u_2(t),u_3(t)), m=1,2,3,\ \text{ and }\ t\in J=(0,1] \\ \psi_p\left[\mathcal{D}_{0^+}^{r_2;\varphi}\left(u_m(t)-I_{0^+}^{\sigma;\varphi}G_m(t,u_1(t),u_2(t),u_3(t))\right)\right]\big|_{t=0}=0, \\ u_m(0)=0,\ \ u_m(1)=\sum_{i=1}^n\lambda_{im}u_m(\zeta_{im}),\ \zeta_{im}\in(0,1] \\ \varphi(1)-\varphi(0)=K>0. \end{cases} \tag{1.1}$$

Here, we take $\mathcal{D}_{0^+}^{r_{im};\varphi}, i=\overline{1,2}, m=\overline{1,2,3}$ as the $\varphi-$Caputo fractional derivatives of orders r_{im}, $0<r_{1m}<1<r_{2m}<2$, and $I_{0^+}^{\sigma;\varphi}, 0<\sigma<r_{2m}$, the fractional integral of order σ, $\lambda_{im}\in\mathbb{R}_+^*$, and $\varphi: J\to\mathbb{R}$ is an increasing function such that $\varphi'(t)\neq 0$, and $\psi_p(z)=|z|^{p-2}z$ denotes the $p-$Laplacian operator and satisfies $\frac{1}{p}+\frac{1}{q}=1, (\psi_p)^{-1}=\psi_q$. For all $t\in J$, $G_m, H_m: J\times\mathbb{R}^3\to\mathbb{R}$ is a given functions satisfying some assumptions that will be specified later.

2. $\varphi-$Caputo Derivatives

In this section, we introduce some notations and definitions of $\varphi-$Caputo approach, for details, see [6, 14, 19].

Let $\varphi: J\to\mathbb{R}$ be an increasing function with $\varphi'(t)\neq 0$, for all $t\in J$.

And throughout the paper, let $\mathcal{C}=C(J,\mathbb{R})$ denotes the Banach space of all continuous mappings from $[0,1]$ to $\mathbb{R}$ endowed with the norm $\|u\|=\sup\limits_{t\in[0,1]} u(t)$. It is clear that the space $\mathcal{C}\times\mathcal{C}\times\mathcal{C}$ endowed with the norm $\|(u_1,u_2,u_3)\|=\|u_1\|+\|u_2\|+\|u_3\|$ is a Banach space. We pose for all $r>0$, and $t\in[0,1], (t>s)$

$$\varphi_r(t,s)=\frac{\varphi'(s)(\varphi(t)-\varphi(s))^{r-1}}{\Gamma(r)}.$$

Definition 1. For $\alpha>0$, the left-sided $\varphi-$Riemann Liouville fractional integral of order α for an integrable function $u: J\to\mathbb{R}$ with respect to another function $\varphi: J\to\mathbb{R}$ that is an increasing differentiable function such that $\varphi'(t)\neq 0$, for all $t\in J$ is defined as follows

$$I_{a^+}^{\alpha;\varphi}u(t)=\int_a^t \varphi_\alpha(t,s)\,u(s)ds, \tag{2.1}$$

where Γ is the gamma function. Note that equation (2.1) is reduced to the Riemann Liouville and Hadamard fractional integrals when $\varphi(t)=t$ and $\varphi(t)=\ln t$, respectively.

Definition 2. Let $n\in\mathbb{N}$ and let $\varphi, u\in C^n(J)$ be two functions such that φ is increasing and $\varphi'(t)\neq 0$, for all $t\in J$. The left-sided $\varphi-$Riemann Liouville fractional derivative of a function u of order α is defined by

$$\begin{aligned}\mathcal{D}_{a^+}^{\alpha;\varphi}u(t)&=\left(\frac{1}{\varphi'(t)}\frac{d}{dt}\right)^n I_{a^+}^{n-\alpha;\varphi}u(t)\\&=\left(\frac{1}{\varphi'(t)}\frac{d}{dt}\right)^n\int_a^t \varphi_{n-\alpha}(t,s)\,u(s)ds,\end{aligned}$$

where $n=[\alpha]+1$.

Definition 3. Let $n \in \mathbb{N}$ and let $\varphi, u \in C^n(J)$ be two functions such that φ is increasing and $\varphi'(t) \neq 0$, for all $t \in J$. The left-sided $\varphi-$Caputo fractional derivative of a function u of order α is defined by

$$ {}^{c}\mathcal{D}_{a^+}^{\alpha;\varphi} u(t) = I_{a^+}^{n-\alpha;\varphi} \left(\frac{1}{\varphi'(t)} \frac{d}{dt} \right)^n u(t), $$

where $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}, n = \alpha$ for $\alpha \in \mathbb{N}$.

To simplify notation, we will use the symbol

$$ u_{\varphi}^{[n]}(t) = \left(\frac{1}{\varphi'(t)} \frac{d}{dt} \right)^n u(t). $$

From the definition, it is clear that

$$ {}^{c}\mathcal{D}_{a^+}^{\alpha;\varphi} u(t) = \begin{cases} \int_a^t \varphi_{n-\alpha}(t,s) u_{\varphi}^{[n]}(s) ds, & \text{if } \alpha \notin \mathbb{N}, \\ u_{\varphi}^{[n]}(t) & \text{if } \alpha \in \mathbb{N}. \end{cases} \tag{2.2} $$

This generalization (2.2) yields to the Caputo fractional derivative operator when $\varphi(t) = t$. Moreover, for $\varphi(t) = \ln t$, it gives the Caputo Hadamard fractional derivative.

2.1. *Auxiliary Lemma*

Lemma 1. *Let $\alpha, \beta > 0$, and $u \in L^1(J)$. Then*

$$ I_{a^+}^{\alpha;\varphi} I_{a^+}^{\beta;\varphi} u(t) = I_{a^+}^{\alpha+\beta;\varphi} u(t), \quad a.e.\ t \in J. $$

In particular, if $u \in C(J)$, then $I_{a^+}^{\alpha;\varphi} I_{a^+}^{\beta;\varphi} u(t) = I_{a^+}^{\alpha+\beta;\varphi} u(t)$, $t \in J$.

Next, we recall the property describing the composition rules for fractional φ-integrals and φ-derivatives.

Lemma 2. *Let $\alpha > 0$. The following holds:*
If $u \in C([a,b])$, then

$$ {}^{c}\mathcal{D}_{a^+}^{\alpha;\varphi} I_{a^+}^{\alpha;\varphi} u(t) = u(t), t \in [a,b]. $$

If $u \in C^n(J)$, $n-1 < \alpha < n$, then

$$ I_{a^+}^{\alpha;\varphi} {}^{c}\mathcal{D}_{a^+}^{\alpha;\varphi} u(t) = u(t) - \sum_{k=0}^{n-1} \frac{u_{\varphi}^{[k]}(a)}{k!} [\varphi(t) - \varphi(a)]^k, $$

for all $t \in [a,b]$. In particular, if $0 < \alpha < 1$, we have

$$ I_{a^+}^{\alpha;\varphi} {}^{c}\mathcal{D}_{a^+}^{\alpha;\varphi} u(t) = u(t) - u(a). $$

Lemma 3. *Let $t > a$, $\alpha \geq 0$ and $\beta > 0$. Then*

- $I_{a^+}^{\alpha;\varphi} [\varphi(t) - \varphi(a)]^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} [\varphi(t) - \varphi(a)]^{\beta+\alpha-1}$,

- ${}^{c}\mathcal{D}_{a^+}^{\alpha;\varphi}\left[\varphi(t)-\varphi(a)\right]^{\beta-1}=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left[\varphi(t)-\varphi(a)\right]^{\beta-\alpha-1}$,
- ${}^{c}\mathcal{D}_{a^+}^{\alpha;\varphi}\left[\varphi(t)-\varphi(a)\right]^{k}=0$, *for all* $k\in\{0,...,n-1\},n\in\mathbb{N}$.

Lemma 4. *Let* $\alpha>0,n\in\mathbb{N}$ *such that* $n-1<q\leq n$. *Then*

- ${}^{c}\mathcal{D}_{a^+}^{q;\varphi}I_{a^+}^{\alpha;\varphi}u(t)={}^{c}\mathcal{D}_{a^+}^{q-\alpha;\varphi}u(t)$; *if* $q>\alpha$.
- ${}^{c}\mathcal{D}_{a^+}^{q;\varphi}I_{a^+}^{\alpha;\varphi}u(t)u(t)=I_{a^+}^{\alpha-q;\varphi}u(t)$; *if* $\alpha>q$.

Lemma 5. *Given a function* $u\in C^{n}[a,b]$ *and* $0<q<1$, *we have*

$$\left|I_{a^+}^{q;\varphi}u(t_2)-I_{a^+}^{q;\varphi}u(t_1)\right|\leq\frac{2\|u\|}{\Gamma(q+1)}\left(\varphi(t_2)-\varphi(t_1)\right)^{q}.$$

Finally, we recall the fixed point theorems that will be used to prove the main results. (We have $\mathcal{C}$ a Banach space in each theorem).

Lemma 6 (Banach fixed point theorem, [7])**.** *Let* U *be a closed set in* $\mathcal{C}$ *and* $\mathcal{T}:U\to U$ *satisfies*

$$|\mathcal{T}u-\mathcal{T}v|\leq\alpha|u-v|,\ \textit{for some}\ \alpha\in(0,1),\textit{and for}\ u,v\in U.$$

Then $\mathcal{T}$ *admits one fixed point in* U.

Lemma 7 (Krasnoselskii fixed point theorem, [15])**.** *Let* M *be a closed, bounded, convex and nonempty subset of a Banach space* U. *Let* A,B *be operators such that*

(i) $Ax+By\in M$, *where* $x,y\in M$,
(ii) A *is compact and continuous,*
(iii) B *is a contraction mapping.*

Then there exists $z\in M$ *such that* $z=Az+Bz$.

Lemma 8 ([13])**.** *For the* $p-$*Laplacian operator* ψ_p, *the following conditions hold true:*

(1) *If* $|\delta_1|,|\delta_2|\geq\rho>0$, $1<p\leq2,\delta_1\delta_2>0$, *then*

$$|\psi_p(\delta_1)-\psi_p(\delta_2)|\leq(p-1)\rho^{p-2}|\delta_1-\delta_2|.$$

(2) *If* $p>2$, $|\delta_1|,|\delta_2|\leq\rho_*>0$, *then*

$$|\psi_p(\delta_1)-\psi_p(\delta_2)|\leq(p-1)\rho_*^{p-2}|\delta_1-\delta_2|.$$

Lemma 9 ([10])**.** *For nonnegative* $a_i,i=1,...,k$,

$$\left(\sum_{i=1}^{k}a_i\right)^{q}\leq k^{q-1}\left(\sum_{i=1}^{k}a_i^{q}\right),q\geq1.$$

Now, we pass to prove the following result.

Lemma 10. *For a given $h_m, g_m \in L^1(J,,\mathbb{R}^3)$, the unique solution of the linear fractional initial value problem*

$$\begin{cases} \mathcal{D}_{0^+}^{r_{1m};\varphi}\psi_p\left[\mathcal{D}_{0^+}^{r_{2m};\varphi}\left(u_m(t)-I_{0^+}^{\sigma;\varphi}g_m(t)\right)\right]=h_m(t), \\ m=1,2,3, \ \ and \ \ t\in J=(0,1] \\ \psi_p\left[\mathcal{D}_{0^+}^{r_2;\varphi}\left(u_m(t)-I_{0^+}^{\sigma;\varphi}g_m(t)\right)\right]\big|_{t=0}=0, \\ u_m(0)=0, \ \ u_m(1)=\sum_{i=1}^{n}\lambda_{im}u_m\left(\zeta_{im}\right), \ \zeta_{im}\in(0,1] \\ \varphi(1)-\varphi(0)=K>0. \end{cases}$$

is given by

$$\begin{aligned} u_m(t)=&\int_0^t\varphi_{r_{2m}}(t,s)\,\psi_q\left[\int_0^s\varphi_{r_{1m}}(s,e)\,h_m(e)de\right]ds+\int_0^t\varphi_\sigma(t,s)\,g_m(s)ds \\ &-(\varphi(t)-\varphi(0))\int_0^1\varphi_{r_{2m}}(1,s)\,\psi_q\left[\int_0^s\varphi_{r_{1m}}(s,e)\,h_m(e)de\right]ds \\ &+(\varphi(t)-\varphi(0))\left(\sum_{i=1}^{n}\frac{\lambda_{im}}{K}u_m(\zeta_i)-\frac{g_m(0)}{K}\right). \end{aligned} \tag{2.3}$$

Proof. For $0<r_{1m}<1<r_{2m}<2$, Lemma 2 yields

$$\psi_p\left[\mathcal{D}_{0^+}^{r_{2m};\varphi}\left(u_m(t)-I_{0^+}^{\sigma;\varphi}g_m(t)\right)\right]=I_{0^+}^{r_{1m};\varphi}h_m(t)+c_{1m}$$

by conditions $\psi_p\left[\mathcal{D}_{0^+}^{r_{2m};\varphi}(u_m(t)-g_m(t)\right]\big|_{t=0}=0$ we get $c_{1m}=0$. Then

$$\left[\mathcal{D}_{0^+}^{r_{2m};\varphi}\left(u_m(t)-I_{0^+}^{\sigma;\varphi}g_m(t)\right)\right]=\psi_q\left[I_{0^+}^{r_{1m};\varphi}h_m(t)\right]$$

so

$$u_m(t)=I_{0^+}^{r_{2m};\varphi}\left[\psi_q\left[I_{0^+}^{r_{1m};\varphi}h_m(t)\right]\right]+I_{0^+}^{\sigma;\varphi}g_m(t)+c_{2m}\left(\varphi(t)-\varphi(0)\right),$$

by conditions $u_m(0)=0$ and $u_m(1)=\sum_{i=1}^{n}\lambda_{im}u_m\left(\zeta_i\right)$, we get

$$c_{2m}=\sum_{i=1}^{n}\frac{\lambda_{im}}{K}u_m\left(\zeta_i\right)-\frac{g_m(0)}{K}-I_{0^+}^{r_{2m};\varphi}\left[\psi_q\left[I_{0^+}^{r_{1m};\varphi}h_m(t)\right]\right]\big|_{t=1}. \tag{2.4}$$

□

3. Main Results

Taking into account Lemma 10, we define an operator $\mathcal{T}:\mathcal{C}\times\mathcal{C}\times\mathcal{C}\to\mathcal{C}\times\mathcal{C}\times\mathcal{C}$

$$\mathcal{T}\left(u_1,u_2,u_3\right)(t)=\left(\mathcal{T}_1\left(u_1,u_2,u_3\right)(t),\mathcal{T}_2\left(u_1,u_2,u_3\right)(t),\mathcal{T}_3\left(u_1,u_2,u_3\right)(t)\right), \tag{3.1}$$

where (for $m=\overline{1,3}$)

$$\begin{aligned} \mathcal{T}_m\left(u_1,u_2,u_3\right)(t)=&\int_0^t\varphi_{r_{2m}}(t,s)\,\psi_q\left[\int_0^s\varphi_{r_{1m}}(s,e)\,H_m(e,u_1(e),u_2(e),u_3(e))de\right]ds \\ &+\int_0^t\varphi_\sigma(t,s)\,G_m(s,u_1(s),u_2(s),u_3(s))ds \end{aligned}$$

$$-\left(\varphi(t)-\varphi(0)\right)\int_0^1 \varphi_{r_{2m}}(1,s)\,\psi_q\left[\int_0^s \varphi_{r_{1m}}(s,e)\,H_m(e,u_1(e),u_2(e),u_3(e))de\right]ds$$
$$+\left(\varphi(t)-\varphi(0)\right)\left(\sum_{i=1}^{n}\frac{\lambda_{im}}{K}u_m(\zeta_i)-\frac{G_m(0,0,0,0)}{K}\right), \tag{3.2}$$

and

$$\varphi_{r_{2m}}(t,s)=\frac{\varphi'(s)\left(\varphi(t)-\varphi(s)\right)^{r_{2m}-1}}{\Gamma(r_{2m})},\quad \varphi_{r_{1m}}(t,s)=\frac{\varphi'(s)\left(\varphi(t)-\varphi(s)\right)^{r_{1m}-1}}{\Gamma(r_{1m})},$$
$$\varphi_{\sigma}(t,s)=\frac{\varphi'(s)\left(\varphi(t)-\varphi(s)\right)^{\sigma-1}}{\Gamma(\sigma)}.$$

For the sake of convenience, we use the following notations (for $m=\overline{1,3}$) :

$$\mathcal{K}_{1m}=\frac{2^{q-2}(K+1)K^{r_{2m}}}{\Gamma(1+r_{2m})}\left(\frac{k_mK^{r_{1m}}}{\Gamma(1+r_{1m})}\right)^{q-1},$$
$$\mathcal{K}_{2m}=\left(\frac{K^{\sigma}l_m}{\Gamma(1+\sigma)}+\sum_{i=1}^{n}|\lambda_{im}|\right),$$
$$\mathcal{K}_{3m}=\frac{2^{q-2}(K+1)K^{r_{2m}}}{\Gamma(1+r_{2m})}\left(\frac{\mathcal{N}K^{r_{1m}}}{\Gamma(1+r_{1m})}\right)^{q-1}+\frac{K^{\sigma}\mathcal{M}}{\Gamma(1+\sigma)}+\mathcal{M},$$
$$\mathcal{K}_{4m}=\left(\frac{K^{r_{1m}}(\rho\mathcal{A}_m+\mathcal{N})}{\Gamma(1+r_{1m})}\right)^{q-2},$$
$$\mathcal{K}_{5m}=\frac{(q-1)(1+K)K^{r_{2m}+r_{1m}}\mathcal{A}_m\mathcal{K}_{4m}}{\Gamma(1+r_{2m})\Gamma(1+r_{1m})}+\frac{K^{\sigma}\mathcal{B}_m}{\Gamma(1+\sigma)}+\sum_{i=1}^{n}|\lambda_{im}|.$$

3.1. *An Existence and Uniqueness Result*

Here, by using the Banach contraction mapping principle, we prove an existence and uniqueness result.

Theorem 1. *Let $H_m,G_m:[0,1]\times\mathbb{R}^3\to\mathbb{R}$ two continuous functions which satisfy the condition*

($\mathbb{A}_1$) there exist positive real constants $\mathcal{A}_m,\mathcal{B}_m$ such that, for all $t\in[0,1]$ and $u_i,v_i\in\mathbb{R}, i,m=\overline{1,3}$, we have

$$|H_m(t,u_1,u_2,u_3)-H_m(e,v_1,v_2,v_3)|\le\mathcal{A}_m\left(|u_1-v_1|+|u_2-v_2|+|u_3-v_3|\right),$$
$$|G_m(t,u_1,u_2,u_3)-G_m(e,v_1,v_2,v_3)|\le\mathcal{B}_m\left(|u_1-v_1|+|u_2-v_2|+|u_3-v_3|\right).$$

Then, system (1.1) admits a unique solution on $[0,1]$ provided that

$$3\left(\mathcal{K}_{21}+\mathcal{K}_{22}+\mathcal{K}_{23}\right)<1,\ \textit{and}\ \mathcal{K}_{51}+\mathcal{K}_{52}+\mathcal{K}_{53}<1 \tag{3.3}$$

is valid.

Proof. We transform system (1.1) into a fixed point problem, $(u,v,w)(z) = \mathcal{T}(u,v,w)(z)$, where the operator $\mathcal{T}$ is defined as in (3.1). Applying the Banach contraction mapping principle (Lemma 6), we show that the operator $\mathcal{T}$ has a unique fixed point, which is the unique solution of system (1.1).

Let $\sup_{t\in[0,1]} H_m(t,0,0,0) = \mathcal{N} < \infty$, and $\sup_{t\in[0,1]} G_m(t,0,0,0) = \mathcal{M} < \infty$. Next, we set $\mathbb{U}\rho = \{(u_1,u_2,u_3) \in \mathcal{C}\times\mathcal{C}\times\mathcal{C}, \|(u_1,u_2,u_3)\| \le \rho\}$,in which

$$\rho \ge \max\left\{ \sqrt[2-q]{3\mathcal{K}_{1m}}, 3\mathcal{K}_{3m}, m = \overline{1,3} \right\}.$$

Observe that $\mathbb{U}\rho$ is a bounded, closed, and convex subset of $\mathcal{C}$. First, we show that $\mathcal{T}\mathbb{U}\rho \subset \mathbb{U}\rho$.

For any $(u,v,w) \in \mathbb{U}\rho$, $t \in [0,1]$, using the condition $(\mathbb{A}_1)$, we have

$$\begin{aligned}|H_m(t,u,v,w)| &\le |H_m(t,u,v,w) - H_m(e,0,0,0)| + |H_m(e,0,0,0)| \\ &\le k_m(|u|+|v|+|w|) + N \le \rho\mathcal{A}_m + \mathcal{N}\end{aligned}$$

and

$$|G_m(t,u,v,w)| \le \rho\mathcal{B}_m + \mathcal{M}.$$

Then, we obtain

$$\begin{aligned}|\mathcal{T}_m(u_1,u_2,u_3)(t)| &\le \left|\int_0^t \varphi_{r_{2m}}(t,s)\psi_q\left[\int_0^s \varphi_{r_{1m}}(s,e)H_m(e,u_1(e),u_2(e),u_3(e))de\right]ds\right| \\ &+ \left|\int_0^t \varphi_\sigma(t,s)G_m(s,u_1(s),u_2(s),u_3(s))ds\right| \\ &+ |(\varphi(t)-\varphi(0))|\left|\int_0^1 \varphi_{r_{2m}}(1,s)\psi_q\left[\int_0^s \varphi_{r_{1m}}(s,e)H_m(e,u_1(e),u_2(e),u_3(e))de\right]ds\right| \\ &+ |(\varphi(t)-\varphi(0))|\left(\sum_{i=1}^n \frac{|\lambda_{im}|}{K}|u_m(\zeta_i)| + \frac{|G_m(0,0,0,0)|}{K}\right),\end{aligned}$$

by Lemma 5 we get

$$\begin{aligned}|\mathcal{T}_m(u_1,u_2,u_3)(t)| &\le \frac{(K+1)K^{r_{2m}}}{\Gamma(1+r_{2m})}\left|\psi_q\left[\int_0^s \varphi_{r_{1m}}(s,e)H_m(e,u_1(e),u_2(e),u_3(e))de\right]\right| \\ &+ \frac{K^\sigma(\rho\mathcal{B}_m+\mathcal{M})}{\Gamma(1+\sigma)} + \left(\sum_{i=1}^n \rho|\lambda_{im}| + \mathcal{M}\right),\end{aligned}$$

and by $\psi_q(z) = |z|^{q-2}z$, we have

$$\begin{aligned}|\mathcal{T}_m(u_1,u_2,u_3)(t)| &\le \frac{(K+1)K^{r_{2m}}}{\Gamma(1+r_{2m})}\left|\int_0^s \varphi_{r_{1m}}(s,e)H_m(e,u_1(e),u_2(e),u_3(e))de\right|^{q-1} \\ &+ \frac{K^\sigma(\rho\mathcal{B}_m+\mathcal{M})}{\Gamma(1+\sigma)} + \left(\sum_{i=1}^n \rho|\lambda_{im}| + \mathcal{M}\right)\end{aligned}$$

$$\begin{aligned}
&\leq \left(\frac{(\rho\mathcal{A}_m+\mathcal{N})K^{r_{1m}}}{\Gamma(1+r_{1m})}\right)^{q-1}\frac{(K+1)K^{r_{2m}}}{\Gamma(1+r_{2m})}\\
&\quad+\frac{K^{\sigma}(\rho\mathcal{B}_m+\mathcal{M})}{\Gamma(1+\sigma)}+\left(\sum_{i=1}^{n}\rho|\lambda_{im}|+\mathcal{M}\right)\\
&\leq \frac{(K+1)K^{r_{2m}}}{\Gamma(1+r_{2m})}\left(\frac{K^{r_{1m}}}{\Gamma(1+r_{1m})}\right)^{q-1}(\rho\mathcal{A}_m+\mathcal{N})^{q-1}\\
&\quad+\left(\frac{K^{\sigma}\mathcal{B}_m}{\Gamma(1+\sigma)}+\sum_{i=1}^{n}|\lambda_{im}|\right)\rho+\frac{K^{\sigma}\mathcal{M}}{\Gamma(1+\sigma)}+\mathcal{M}.
\end{aligned}$$

Thanks to Lemma 9 , for all $m=\overline{1,3}$ we get

$$\begin{aligned}
|\mathcal{T}_m(u_1,u_2,u_3)(t)| &\leq \frac{2^{q-2}(K+1)K^{r_{2m}}}{\Gamma(1+r_{2m})}\left(\frac{K^{r_{1m}}}{\Gamma(1+r_{1m})}\right)^{q-1}\left(\mathcal{A}_m^{q-1}\rho^{q-1}+\mathcal{N}^{q-1}\right)\\
&\quad+\left(\frac{K^{\sigma}\mathcal{B}_m}{\Gamma(1+\sigma)}+\sum_{i=1}^{n}|\lambda_{im}|\right)\rho+\frac{K^{\sigma}\mathcal{M}}{\Gamma(1+\sigma)}+\mathcal{M}\\
&\leq \frac{2^{q-2}(K+1)K^{r_{2m}}}{\Gamma(1+r_{2m})}\left(\frac{\mathcal{A}_mK^{r_{1m}}}{\Gamma(1+r_{1m})}\right)^{q-1}\rho^{q-1}\\
&\quad+\left(\frac{K^{\sigma}\mathcal{B}_m}{\Gamma(1+\sigma)}\sum_{i=1}^{n}|\lambda_{im}|\right)\rho+\frac{2^{q-2}(K+1)K^{r_{2m}}}{\Gamma(1+r_{2m})}\left(\frac{\mathcal{N}K^{r_{1m}}}{\Gamma(1+r_{1m})}\right)^{q-1}\\
&\quad+\frac{K^{\sigma}\mathcal{M}}{\Gamma(1+\sigma)}+\mathcal{M}\\
&\leq \mathcal{K}_{1m}\rho^{q-1}+\mathcal{K}_{2m}\rho+\mathcal{K}_{3m}\leq\frac{\rho}{3}.
\end{aligned}$$

Hence,

$$\|\mathcal{T}(u_1,u_2,u_3)\|\leq\sum_{m=1}^{3}\left(\mathcal{K}_{1m}\rho^{q-1}+\mathcal{K}_{2m}\rho+\mathcal{K}_{3m}\right)\leq\rho, \tag{3.4}$$

which gives us $\mathcal{T}\mathbb{U}\sigma\subset\mathbb{U}\sigma$.

Next, we show that $\mathcal{T}:\mathcal{C}\times\mathcal{C}\times\mathcal{C}\to\mathcal{C}\times\mathcal{C}\times\mathcal{C}$ is a contraction.

Using condition $(\mathbb{A}_1)$, for any $(u_1,u_2,u_3),(v_1,v_2,v_3)\in\mathcal{C}\times\mathcal{C}\times\mathcal{C}$ and for each $t\in[0,1]$, we have

$$\begin{aligned}
&|\mathcal{T}_m(u_1,u_2,u_3)-\mathcal{T}_m(v_1,v_2,v_3)|\\
&\leq\left|\int_0^t\varphi_{r_{2m}}(t,s)\psi_q\left(\int_0^s\varphi_{r_{1m}}(s,e)H_m(e,u_1(e),u_2(e),u_3(e))de\right)ds\right.\\
&\quad\left.-\int_0^t\varphi_{r_{2m}}(t,s)\psi_q\left(\int_0^s\varphi_{r_{1m}}(s,e)H_m(e,v_1(e),v_2(e),v_3(e))de\right)ds\right|
\end{aligned}$$

$$
\begin{aligned}
&+\left|\int_0^t \varphi_\sigma(t,s)\left(G_m(s,u_1(s),u_2(s),u_3(s))-G_m(s,v_1(s),v_2(s),v_3(s))\right)ds\right| \\
&+K\left|\int_0^1 \varphi_{r_{2m}}(1,s)\,\psi_q\left(\int_0^s \varphi_{r_{1m}}(s,e)\,H_m(e,u_1(e),u_2(e),u_3(e))de\right)ds\right. \\
&\left.-\int_0^1 \varphi_{r_{2m}}(1,s)\,\psi_q\left(\int_0^s \varphi_{r_{1m}}(s,e)\,H_m(e,v_1(e),v_2(e),v_3(e))de\right)ds\right| \\
&+\sum_{i=1}^n |\lambda_{im}|\,|u_m(\zeta_i)-v_m(\zeta_i)|,
\end{aligned}
$$

by Lemma 5 and Lemma 8, we get

$$
\begin{aligned}
&|\mathcal{T}_m(u_1,u_2,u_3)-\mathcal{T}_m(v_1,v_2,v_3)| \\
&\leq \frac{(1+K)K^{r_{2m}}}{\Gamma(1+r_{2m})}\left|\psi_q\left(\int_0^s \varphi_{r_{1m}}(s,e)\,H_m(e,u_1(e),u_2(e),u_3(e))\,de\right)\right. \\
&\quad\left.-\psi_q\left(\int_0^s \varphi_{r_{1m}}(s,e)\,H_m(e,v_1(e),v_2(e),v_3(e))\,de\right)\right| \\
&\quad+\left(\frac{K^\sigma \mathcal{B}_m}{\Gamma(1+\sigma)}+\sum_{i=1}^n |\lambda_{im}|\right)(|u_1-v_1|+|u_2-v_2|+|u_3-v_3|) \\
&\leq \frac{(1+K)K^{r_{2m}}(q-1)\,\mathcal{K}_{4m}}{\Gamma(1+r_{2m})}\left|\int_0^s \varphi_{r_{1m}}(s,e)\,H_m(e,u_1(e),u_2(e),u_3(e))\,de\right. \\
&\quad\left.-\int_0^s \varphi_{r_{1m}}(s,e)\,H_m(e,v_1(e),v_2(e),v_3(e))\,de\right| \\
&\leq \left[\frac{(q-1)\,\mathcal{A}_m(1+K)K^{r_{2m}+r_{1m}}\mathcal{K}_{4m}}{\Gamma(1+r_{2m})\Gamma(1+r_{1m})}+\frac{K^\sigma \mathcal{B}_m}{\Gamma(1+\sigma)}+\sum_{i=1}^n |\lambda_{im}|\right] \\
&\quad\cdot(|u_1-v_1|+|u_2-v_2|+|u_3-v_3|) \\
&\leq \mathcal{K}_{5m}(|u_1-v_1|+|u_2-v_2|+|u_3-v_3|).
\end{aligned}
$$

Hence,

$$
|\mathcal{T}(u_1,u_2,u_3)-\mathcal{T}_m(v_1,v_2,v_3)| \leq (\mathcal{K}_{51}+\mathcal{K}_{51}+\mathcal{K}_{53})(|u_1-v_1|+|u_2-v_2|+|u_3-v_3|).
$$

Since $\mathcal{K}_{51}+\mathcal{K}_{51}+\mathcal{K}_{53}<1$, by (3.3), the operator $\mathcal{T}$ is a contraction. Therefore, using the Banach contraction mapping principle (Lemma 6), the operator $\mathcal{T}$ has a unique fixed point. Hence, system (1.1) has a unique solution on $[0,1]$. The proof is completed. □

3.2. *An Existence Result*

Now we apply Krasnoselskii fixed point theorem (Lemma 7) to prove our second existence result. So, consider the following operator

$$\begin{aligned}\mathcal{T}(u_1,u_2,u_3))(t) &= \sum_{m=1}^{3}(\mathcal{T}_m(u_1,u_2,u_3))(t)\\ &= \sum_{m=1}^{3}(\mathcal{P}_{1m}(u_1,u_2,u_3)(t)+\sum_{m=1}^{3}(\mathcal{P}_{2m}(u_1,u_2,u_3)(t)\\ &= \mathcal{P}_1(u_1,u_2,u_3)(t)+\mathcal{P}_2(u_1,u_2,u_3)(t),\end{aligned}$$

where

$$\mathcal{P}_1(u_1,u_2,u_3)(t) = \sum_{m=1}^{3}(\mathcal{P}_{1m}(u_1,u_2,u_3)(t),$$

$$\mathcal{P}_2(u_1,u_2,u_3)(t) = \sum_{m=1}^{3}(\mathcal{P}_{2m}(u_1,u_2,u_3)(t),$$

and

$$\begin{aligned}(\mathcal{P}_{1m}(u_1,u_2,u_3)(t) &= \int_0^t \varphi_{r_{2m}}(t,s)\,\psi_q\left[\int_0^s \varphi_{r_{1m}}(s,e)\,H_m(e,u_1(e),u_2(e),u_3(e))de\right]ds\\ &-(\varphi(t)-\varphi(0))\int_0^1 \varphi_{r_{2m}}(1,s)\,\psi_q\left[\int_0^s \varphi_{r_{1m}}(s,e)\,H_m(e,u_1(e),u_2(e),u_3(e))de\right]ds\end{aligned}$$

and

$$\begin{aligned}(\mathcal{P}_{2m}(u_1,u_2,u_3)(t) &= \int_0^t \varphi_\sigma(t,s)\,G_m(s,u_1(s),u_2(s),u_3(s))ds\\ &+(\varphi(t)-\varphi(0))\left(\sum_{i=1}^{n}\frac{\lambda_{im}}{K}u_m(\zeta_i)-\frac{G_m(0,u_1(0),u_2(0),u_3(0))}{K}\right).\end{aligned}$$

Theorem 2. *Let $H_m, G_m : [0,1]\times\mathbb{R}^3 \to \mathbb{R}$ be continuous functions which satisfy condition ($\mathbb{A}_1$) in Theorem 1.*

In addition, we assume that there exist two positive constants $\Upsilon_{1m}, \Upsilon_{2m}$ such that, for all $t\in[0,1]$ and $u_i, v_i \in \mathbb{R}, i,m=\overline{1,3}$, we have

$$\begin{aligned}|H_m(t,u_1,u_2,u_3)| &\leq \Upsilon_{1m},\\ |G_m(t,u_1,u_2,u_3)| &\leq \Upsilon_{2m}.\end{aligned}$$

Moreover, assume that

$$\sum_{i=1}^{n}|\lambda_{im}| \leq \frac{1}{3},\ \textit{and}\ \left(\sum_{m=1}^{3}\frac{(q-1)\,\mathcal{A}_m\,(1+K)\,K^{r_{2m}+r_{1m}}\,\mathcal{K}_{4m}}{\Gamma(1+r_{2m})\,\Gamma(1+r_{1m})}\right)<1.$$

Then, problem (1.1) admits at least one solution on $[0,1]$.

Proof. The proof will be given in several steps. Let

$$\mathbb{U}_\delta = \{(u_1,u_2,u_3) \in \mathcal{C}\times\mathcal{C}\times\mathcal{C}, \|(u_1,u_2,u_3)\| \le \delta\},$$

in which

$$\delta \ge 3\max_{m=\overline{1,3}}\left[\frac{\left(\frac{\Upsilon_{1m}K^{r_{1m}}}{\Gamma(1+r_{1m})}\right)^{q-1}\frac{(K+1)K^{r_{2m}}}{\Gamma(1+r_{2m})}+\frac{K^\sigma\Upsilon_{2m}}{\Gamma(1+\sigma)}+\mathcal{M}}{\frac{1}{3}-\sum_{i=1}^n|\lambda_{im}|}\right].$$

Step 1. We prove that

$$|(\mathcal{T}(u_1,u_2,u_3))(t)| \le \delta.$$

Let $(u_1,u_2,u_3),(v_1,v_2,v_3),(w_1,w_2,w_3) \in \mathbb{U}_\delta$. As in the proof of Theorem 1, we have

$$\begin{aligned}&|(\mathcal{P}_{1m}(u_1,u_2,u_3)(t)+(\mathcal{P}_{2m}(u_1,u_2,u_3)(t)|\\ &\le\left(\frac{\Upsilon_{1m}K^{r_{1m}}}{\Gamma(1+r_{1m})}\right)^{q-1}\frac{(K+1)K^{r_{2m}}}{\Gamma(1+r_{2m})}+\frac{K^\sigma\Upsilon_{2m}}{\Gamma(1+\sigma)}+\mathcal{M}+\left(\sum_{i=1}^n|\lambda_{im}|\right)\delta\le\frac{\delta}{3}.\end{aligned}$$

Hence

$$\begin{aligned}|(\mathcal{T}(u_1,u_2,u_3))(t)| &= |(\mathcal{T}_1(u_1,u_2,u_3))(t)+(\mathcal{T}_2(u_1,u_2,u_3))(t)+(\mathcal{T}_3(u_1,u_2,u_3))(t)|\\ &\le |(\mathcal{T}_1(u_1,u_2,u_3))(t)|+|(\mathcal{T}_2(u_1,u_2,u_3))(t)|+|(\mathcal{T}_3(u_1,u_2,u_3))(t)|\\ &\le \sum_{m=1}^3|\mathcal{P}_{1m}(u_1,u_2,u_3)(t)+\mathcal{P}_{2m}(u_1,u_2,u_3)(t)|\le\delta.\end{aligned}$$

Accordingly, $\sum_{m=1}^3\mathcal{P}_{1m}(u_1,u_2,u_3)(t)+\mathcal{P}_{2m}(u_1,u_2,u_3)(t)\in\mathbb{U}_\delta$ and the condition (i) of Lemma 7 is satisfied.

Step 2. $\mathcal{P}_1$ is a contraction.

Let (u_1,u_2,u_3), $(v_1,v_2,v_3)\in\mathbb{U}_\delta$. We have the following estimate

$$\begin{aligned}&|\mathcal{P}_{1m}(u_1,u_2,u_3)(t)-\mathcal{P}_{1m}(v_1,v_2,v_3)(t)||\\ &\le\left|\int_0^t\varphi_{r_{2m}}(t,s)\psi_q\left(\int_0^s\varphi_{r_{1m}}(s,e)H_m(e,u_1(e),u_2(e),u_3(e))de\right)ds\right.\\ &\quad\left.-\int_0^t\varphi_{r_{2m}}(t,s)\psi_q\left(\int_0^s\varphi_{r_{1m}}(s,e)H_m(e,v_1(e),v_2(e),v_3(e))de\right)ds\right|\\ &\quad+K\left|\int_0^1\varphi_{r_{2m}}(1,s)\psi_q\left(\int_0^s\varphi_{r_{1m}}(s,e)H_m(e,u_1(e),u_2(e),u_3(e))de\right)ds\right.\\ &\quad\left.-\int_0^1\varphi_{r_{2m}}(1,s)\psi_q\left(\int_0^s\varphi_{r_{1m}}(s,e)H_m(e,v_1(e),v_2(e),v_3(e)de)\right)ds\right|\\ &\le\frac{(1+K)K^{r_{2m}}}{\Gamma(1+r_{2m})}\left|\psi_q\left(\int_0^s\varphi_{r_{1m}}(s,e)H_m(e,u_1(e),u_2(e),u_3(e))de\right)\right.\end{aligned}$$

$$-\psi_q\left(\int_0^s \varphi_{r_{1m}}(s,e)\,H_m(e,v_1(e),v_2(e),v_3(e))de\right)$$
$$\leq \frac{(q-1)\,\mathcal{A}_m\,(1+K)\,K^{r_{2m}+r_{1m}}\,\mathcal{K}_{4m}}{\Gamma(1+r_{2m})\,\Gamma(1+r_{1m})}\left(|u_1-v_1|+|u_2-v_2|+|u_3-v_3|\right).$$

So

$$|\mathcal{P}_1(u_1,u_2,u_3)(t)-\mathcal{P}_1(v_1,v_2,v_3)(t)||$$
$$\leq\left(\sum_{m=1}^{3}\frac{(q-1)\,\mathcal{A}_m\,(1+K)\,K^{r_{2m}+r_{1m}}\,\mathcal{K}_{4m}}{\Gamma(1+r_{2m})\,\Gamma(1+r_{1m})}\right)\left(|u_1-v_1|+|u_2-v_2|+|u_3-v_3|\right).$$

Since $\left(\sum_{m=1}^{3}\frac{(q-1)\mathcal{A}_m(1+K)K^{r_{2m}+r_{1m}}\mathcal{K}_{4m}}{\Gamma(1+r_{2m})\Gamma(1+r_{1m})}\right)<1$, the operator $\mathcal{P}_1$ is a contraction.

Step 3. $\mathcal{P}_2$ is compact and continuous.

Since H_m, G_m are a continuous functions, this implies that the operator $\mathcal{P}_2$ is continuous on $\mathbb{U}_\delta$. Moreover, $\mathcal{P}_2(u_1,u,u_3)$ is uniformly bounded by (3.4). Next, we show equicontinuity. Let $(u_1,u,u_3)\in\mathbb{U}_\delta$, we have

$$|(\mathcal{P}_{2m}(u_1,u_2,u_3)(t)||\leq\left|\int_0^t \varphi_\sigma(t,s)\,G_m(s,u_1(s),u_2(s),u_3(s))ds\right|$$
$$+K\left(\sum_{i=1}^{n}\frac{|\lambda_{im}|}{K}\,|u_m(\zeta_i)|+\frac{|G_m(0,u_1(0),u_2(0),u_3(0))|}{K}\right)$$
$$\leq\left(\frac{\Upsilon_{2m}K^\sigma}{\Gamma(1+\sigma)}+\sum_{i=1}^{n}|\lambda_{im}|+\Upsilon_{2m}\right)\delta.$$

So

$$|(\mathcal{P}_2(u_1,u_2,u_3)(t)|\leq\sum_{m=1}^{3}\left(\frac{\Upsilon_{2m}K^\sigma}{\Gamma(1+\sigma)}+\sum_{i=1}^{n}|\lambda_{im}|+\Upsilon_{2m}\right)\delta. \tag{3.5}$$

Moreover, $\mathcal{P}_2(u_1,u,u_3)$ is uniformly bounded by (3.5). Next, we show equicontinuity and $t_1,t_2\in[0,1]$ such that , $t_1<t_2$ we have

$$|(\mathcal{P}_{2m}(u_1,u_2,u_3)(t_2)-(\mathcal{P}_{2m}(u_1,u_2,u_3)(t_1)|$$
$$\leq\left|\int_0^{t_2}\varphi_\sigma(t_2,s)\,G_m(s,u_1(s),u_2(s),u_3(s))ds-\int_0^{t_1}\varphi_\sigma(t_1,s)\,G_m(s,u_1(s),u_2(s),u_3(s))ds\right|$$
$$\leq\frac{\Upsilon_{2m}}{\Gamma(1+\sigma)}\left(\varphi(t_2)-\varphi(t_1)\right)^\sigma.$$

So

$$|(\mathcal{P}_2(u_1,u_2,u_3)(t_2)-(\mathcal{P}_2(u_1,u_2,u_3)(t_1)|$$
$$\leq\left|\int_0^{t_2}\varphi_\sigma(t_2,s)\,G_m(s,u_1(s),u_2(s),u_3(s))ds-\int_0^{t_1}\varphi_\sigma(t_1,s)\,G_m(s,u_1(s),u_2(s),u_3(s))ds\right|$$

$$\leq \sum_{m=1}^{3}\left(\frac{\Upsilon_{2m}}{\Gamma(1+\sigma)}\right)(\varphi(t_2)-\varphi(t_1))^{\sigma}.$$

Consequently,

$$|(\mathcal{P}_2(u_1,u_2,u_3)(t_2)-(\mathcal{P}_2(u_1,u_2,u_3)(t_1)|\to 0, \quad \text{as } t_1\to t_2.$$

This shows that $\mathcal{P}_2\mathbb{U}_\delta$ is equicontinuous. Hence, by Arzelià-Ascoli theorem $\mathcal{P}_2$ is completely continuous on $\mathbb{U}_\delta$. As a consequence of Krasnoselskii's fixed point theorem, we conclude that it has a fixed point which is a solution of (1.1). The proof of Theorem 2 is thus completely achieved. □

4. An Illustrative Example

Consider the following nonlinear equation for all $t\in(0,1]$:

$$\begin{cases} {}^{c}\mathcal{D}_{0^+}^{\frac{1}{2};t^2}\psi_p\left[{}^{c}\mathcal{D}_{0^+}^{\frac{1}{4};t^2}\left(u(t)-I_{0^+}^{\sigma;\varphi}\left(\frac{e^t+1}{1+(u(t))^2}\right)\right)\right]=\frac{1}{1+t^2}\left(\frac{u(t)}{1+(u(t))^2}\right),\\ {}^{c}\mathcal{D}_{0^+}^{\frac{1}{4};t^2}\psi_p\left[{}^{c}\mathcal{D}_{0^+}^{\frac{1}{2};t^2}\left(v(t)-I_{0^+}^{\sigma;\varphi}\left(\frac{t^2+1}{1+(v(t))^2}\right)\right)\right]=\frac{t}{1+e^t}\left(\frac{v(t)}{1+(v(t))^2}\right),\\ {}^{c}\mathcal{D}_{0^+}^{\frac{3}{4};t^2}\psi_p\left[{}^{c}\mathcal{D}_{0^+}^{\frac{3}{4};t^2}\left(w(t)-I_{0^+}^{\sigma;\varphi}\left(\frac{e^t}{1+(w(t))^2}\right)\right)\right]=\frac{e^t}{1+t^2}\left(\frac{w(t)}{1+(w(t))^2}\right),\\ \psi_p\left[{}^{c}\mathcal{D}_{0^+}^{r_2;\varphi}\left(u(t)-I_{0^+}^{\sigma;\varphi}\left(\frac{e^t+1}{1+(u(t))^2}\right)\right)\right]\Big|_{t=0}=0,\\ \psi_p\left[{}^{c}\mathcal{D}_{0^+}^{r_2;\varphi}\left(v(t)-I_{0^+}^{\sigma;\varphi}\left(\frac{t^2+1}{1+(v(t))^2}\right)\right)\right]\Big|_{t=0}=0,\\ \psi_p\left[{}^{c}\mathcal{D}_{0^+}^{r_2;\varphi}\left(w(t)-I_{0^+}^{\sigma;\varphi}\left(\frac{e^t}{1+(w(t))^2}\right)\right)\right]\Big|_{t=0}=0,\\ u(0)=v(0)=w(0)=0,\\ u(1)=\sum_{i=1}^{n}\frac{1}{7(i!)}u(\zeta_i),v(1)=\sum_{i=1}^{n}\frac{1}{9(i!)}v(\zeta_i),w(1)=\sum_{i=1}^{n}\frac{1}{11(i!)}w(\zeta_i),\ \zeta_i\in(0,1] \end{cases} \tag{4.1}$$

and

$$\begin{aligned} &K=1,\\ &\Upsilon_{11}=\mathcal{A}_1=\frac{1}{2},\Upsilon_{12}=\mathcal{A}_2=\frac{1}{4},\Upsilon_{13}=\mathcal{A}_3=\frac{e}{2},\\ &\Upsilon_{21}=\mathcal{B}_1=\frac{1+e}{2},\Upsilon_{22}=\mathcal{B}_2=1,\Upsilon_{23}=\mathcal{B}_3=\frac{e}{2}. \end{aligned}$$

Thus, the assumptions $(\mathbb{A}_1)$ are satisfied and Theorem 1-2 implies that (4.1) has a unique solution on $[0,1]$.

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APPROXIMATION BY A SPECIAL DE LA VALLÉE POUSSIN TYPE MATRIX TRANSFORM MEAN OF WALSH-FOURIER SERIES

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Abstract. In this paper, we consider norm convergence for a special matrix-based de la Vallée Poussin-like mean of Fourier series for the Walsh system. We estimate the difference between the named mean above and the corresponding function in norm, and the upper estimation is given by the modulus of continuity of the function.

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Keywords: character system, Fourier series, Walsh-Paley system, rate of approximation, modulus of continuity, matrix transform

1. Definitions and notations

We follow the standard notions of dyadic analysis introduced by F. Schipp, W. R. Wade, P. Simon, and J. Pál [17] and others.

Let $\mathbb{P}$ be the set of positive natural numbers and $\mathbb{N} := \mathbb{P} \cup \{0\}$. Let denote by $\mathbb{Z}_2$ the discrete cyclic group of order 2, the group operation is the modulo 2 addition. Let be every subset open. The normalized Haar measure μ on $\mathbb{Z}_2$ is given in the way that $\mu(\{0\}) := \mu(\{1\}) := 1/2$. $G := \underset{k=0}{\overset{\infty}{\times}} \mathbb{Z}_2$, G is called the Walsh group. The elements of Walsh group G are sequences of numbers 0 and 1, that is $x = (x_0, x_1, \dots, x_k, \dots)$ with $x_k \in \{0,1\}$ $(k \in \mathbb{N})$.

The group operation on G is the coordinate-wise addition (denoted by $+$), the normalized Haar measure μ is the product measure and the topology is the product topology. Dyadic intervals are defined in the usual way

$$I_0(x) := G,\ I_n(x) := \{y \in G : y = (x_0, \dots, x_{n-1}, y_n, y_{n+1}, \dots)\}$$

for $x \in G, n \in \mathbb{P}$. They form a base for the neighbourhoods of G. Let $0 := (0 : i \in \mathbb{N}) \in G$ denote the null element of G and $I_n := I_n(0)$ for $n \in \mathbb{N}$.

Let $L_p(G)$ denote the usual Lebesgue spaces on G (with the corresponding norm $\|.\|_p$), where $1 \le p < \infty$.

For the sake of brevity in notation, we agree to write $L_\infty(G)$ instead of $C(G)$ and set $\|f\|_\infty := \sup\{|f(x)| : x \in G\}$. Of course, it is clear that the space $L_\infty(G)$ is not the

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same as the space of continuous functions, i.e. it is a proper subspace of it. But since in the case of continuous functions the supremum norm and the $L_\infty(G)$ norm are the same, for convenience we hope the reader will be able to tolerate this simplification in notation.

Next, we define the modulus of continuity in $L_p(G), 1 \le p \le \infty$, of a function $f \in L_p(G)$ by

$$\omega_p(f,\delta) := \sup_{|t|<\delta} \|f(.+t) - f(.)\|_p, \quad \delta > 0,$$

with the notation

$$|x| := \sum_{i=0}^{\infty} \frac{x_i}{2^{i+1}} \quad \text{for all } x \in G.$$

The Lipschitz classes in $L_p(G)$ (for each $\alpha > 0$) are defined as

$$\mathrm{Lip}(\alpha, p, G) := \{f \in L_p(G) : \omega_p(f,\delta) = O(\delta^\alpha) \text{ as } \delta \to 0\}.$$

We introduce some concepts of Walsh-Fourier analysis. The Rademacher functions are defined as

$$r_k(x) := (-1)^{x_k} \ (x \in G, k \in \mathbb{N}).$$

The Walsh-Paley functions are the product functions of the Rademacher functions. Namely, each natural number n can be uniquely expressed in the number system based 2, in the form

$$n = \sum_{k=0}^{\infty} n_k 2^k, \ n_k \in \{0,1\} \ (k \in \mathbb{N}),$$

where only a finite number of n_k's different from zero. Let the order of $n \in \mathbb{P}$ be denoted by $|n| := \max\{j \in \mathbb{N} : n_j \neq 0\}$. Walsh-Paley functions are $w_0 := 1$ and for $n \in \mathbb{P}$

$$w_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) = (-1)^{\sum_{k=0}^{|n|} n_k x_k}.$$

Let $\mathcal{P}_n$ be the collection of Walsh polynomials of order less than n, that is, functions of the form

$$P(x) = \sum_{k=0}^{n-1} a_k w_k(x),$$

where $n \in \mathbb{P}$ and $\{a_k\}$ is a sequence of complex numbers.

It is known [10] that the system $(w_n, n \in \mathbb{N})$ is the character system of $(G,+)$. The nth Fourier-coefficient, the nth partial sum of the Fourier series and the nth Dirichlet kernel is defined by

$$\hat{f}(n) := \int_G f w_n d\mu, \ S_n(f) := \sum_{k=0}^{n-1} \hat{f}(k) w_k, \ D_n := \sum_{k=0}^{n-1} w_k, \ D_0 := 0.$$

Fejér kernels are defined as the arithmetical means of Dirichlet kernels, that is,

$$K_n := \frac{1}{n}\sum_{k=1}^{n} D_k.$$

Let $T := (t_{i,j})_{i,j=1}^{\infty}$ be a doubly infinite matrix of numbers. It is always supposed that matrix T is upper triangular.

Let us define the (m,n)th matrix transform de La Vallée Poussin mean determined by the matrix T as

$$\sigma_{m,n}^{T}(f) := \sum_{k=m}^{n} t_{k,n} S_k(f),$$

where $m,n \in \mathbb{P}$ and $m \leq n$.

The (m,n)th matrix transform de La Vallée Poussin kernel is defined as

$$K_{m,n}^{T} := \sum_{k=m}^{n} t_{k,n} D_k.$$

It is very easy to verify that

$$\sigma_{m,n}^{T}(f;x) = \int_G f(u) K_{m,n}^{T}(u+x)\, d\mu(u).$$

We introduce the notation $\Delta t_{k,n} := t_{k,n} - t_{k+1,n}$, where $k \in \{1,\dots,n\}$ and $t_{n+1,n} := 0$.

2. Historical overview

Matrix transform means are common generalizations of several well-known summation methods. It follows by simple consideration that the Nörlund means, the Fejér (or the $(C,1)$) and the (C,α) means are special cases of the matrix transform summation method introduced above.

Our paper is motivated by the work of Móricz, Siddiqi [14] on the Walsh-Nörlund summation method and the result of Móricz and Rhoades [13] on the Walsh weighted mean method. As special cases, Móricz and Siddiqi obtained the earlier results given by Yano [23], Jastrebova [11] and Skvortsov [19] on the rate of the approximation by Cesàro means. The approximation properties of the Walsh-Cesàro means of negative order were studied by Goginava [9], the Vilenkin case was investigated by Shavardenidze [18] and Tepnadze [20]. A common generalization of these two results of Móricz and Siddiqi [14] and Móricz and Rhoades [13] was given by Nagy and the author [2].

In 2008, Fridli, Manchanda and Siddiqi generalized the result of Móricz and Siddiqi for homogeneous Banach spaces and dyadic Hardy spaces [8]. Recently, the author, Baramidze, Memić, Nagy, Persson, Tephnadze and Wall presented some results with respect to this topic [1],[3], [5],[12]. See [7, 22], as well. For the two-dimensional results see [4, 15, 16].

It is important to note that in the paper of Chripkó [6] some methods and results with respect to Jacobi-Fourier series gave us some ideas and used in this paper.

3. Auxiliary results

To prove Theorem 1 we need the following Lemmas.

Lemma 1 (Paley's Lemma [17], p. 7.)**.** *For $n \in \mathbb{N}$*

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n, \\ 0, & \text{if } x \notin I_n. \end{cases}$$

Lemma 2 ([17], p. 34.)**.** *For $j, n \in \mathbb{N},\ j < 2^n$ we have*

$$D_{2^n+j} = D_{2^n} + r_n D_j.$$

Lemma 3 (Yano's Lemma [24])**.** *The norm of the Fejér kernel is bounded uniformly. That is, for all $n \in \mathbb{P}$*

$$\|K_n\|_1 \leq 2.$$

In 2018, Toledo improved this result.

Lemma 4. *[21]*

$$\sup_{n \in \mathbb{P}} \|K_n\|_1 = \frac{17}{15}.$$

Lemma 5. *[14] Let $n \in \mathbb{P},\ g \in \mathcal{P}_{2^n},\ f \in L_p(G)$ $(1 \leq p \leq \infty)$. Then*

$$\left\| \int_G r_n(t) g(t) (f(\cdot + t) - f(\cdot)) d\mu(t) \right\|_p \leq \frac{1}{2} \|g\|_1 \omega_p \left(f, 2^{-n}\right)$$

holds.

In the next lemma, we give a decomposition of the kernels $K^T_{2^n, 2^{n+1}-1}$.

Lemma 6. *Let n be a positive integer, then we have*

$$K^T_{2^n, 2^{n+1}-1} = \sum_{k=0}^{2^n-1} t_{2^n+k, 2^{n+1}-1} D_{2^n} + r_n \sum_{k=1}^{2^n-2} \Delta t_{2^n+k, 2^{n+1}-1} k K_k$$
$$+ r_n t_{2^{n+1}-1, 2^{n+1}-1} (2^n - 1) K_{2^n-1} =: \sum_{j=1}^{3} K_{j,n}.$$

Proof. We write

$$K^T_{2^n, 2^{n+1}-1} = \sum_{l=2^n}^{2^{n+1}-1} t_{l, 2^{n+1}-1} D_l.$$

Now, we apply Lemma 2. We get

$$\sum_{l=2^n}^{2^{n+1}-1} t_{l, 2^{n+1}-1} D_l = \sum_{k=0}^{2^n-1} t_{2^n+k, 2^{n+1}-1} D_{2^n+k}$$

$$= \sum_{k=0}^{2^n-1} t_{2^n+k,2^{n+1}-1} D_{2^n} + r_n \sum_{k=1}^{2^n-1} t_{2^n+k,2^{n+1}-1} D_k.$$

Using Abel-transform

$$\sum_{k=1}^{2^n-1} t_{2^n+k,2^{n+1}-1} D_k = \sum_{k=1}^{2^n-2} \Delta t_{2^n+k,2^{n+1}-1} k K_k + t_{2^{n+1}-1,2^{n+1}-1}(2^n-1)K_{2^n-1}.$$

Summarizing these it completes the proof of Lemma 6. □

4. The rate of the approximation

Theorem 1. *Let $f \in L_p(G)$ $(1 \le p \le \infty)$. For every $n \in \mathbb{P}$, $\{t_{k,2^{n+1}-1} : 2^n \le k \le 2^{n+1}-1\}$ be a finite sequence of non-negative numbers such that*

$$\sum_{k=2^n}^{2^{n+1}-1} t_{k,2^{n+1}-1} = 1 \tag{4.1}$$

be satisfied.

a) If the finite sequence $\{t_{k,2^{n+1}-1} : 2^n \le k \le 2^{n+1}-1\}$ is non-decreasing for a fixed n and

$$t_{2^{n+1}-1,2^{n+1}-1} = O\left(\frac{1}{2^{n+1}-1}\right), \tag{4.2}$$

or

b) if the finite sequence $\{t_{k,2^{n+1}-1} : 2^n \le k \le 2^{n+1}-1\}$ is non-increasing for a fixed n, then

$$\left\|\sigma^T_{2^n,2^{n+1}-1}(f) - f\right\|_p \le c\omega_p\left(f, 2^{-n}\right)$$

holds.

Proof of Theorem 1. The proof is carried out in cases where $1 \le p < \infty$, while the proof of case $p = \infty$ is similar. Recall that by the case $p = \infty$ we mean that we are considering the space of continuous functions.

During our proofs c denotes a positive constant, which may vary at different appearances.

We use condition (4.1), the usual Minkowski inequality and Lemma 6

$$\begin{aligned}
\left\|\sigma^T_{2^n,2^{n+1}-1}(f) - f\right\|_p &= \left(\int_G |\sigma^T_{2^n,2^{n+1}-1}(f;x) - f(x)|^p d\mu(x)\right)^{\frac{1}{p}} \\
&= \left(\int_G \left|\int_G K^T_{2^n,2^{n+1}-1}(u)F(x,u)d\mu(u)\right|^p d\mu(x)\right)^{\frac{1}{p}} \\
&\le \sum_{j=1}^{3} \left(\int_G \left|\int_G K_{j,n}(u)F(x,u)d\mu(u)\right|^p d\mu(x)\right)^{\frac{1}{p}} =: \sum_{j=1}^{3} I_{j,n}
\end{aligned}$$

with notation $F(x,u) := f(x+u) - f(x)$.

Using generalized Minkowski inequality, Lemma 1 and condition (4.1) for the expressions $I_{1,n}$, we obtain

$$\begin{aligned} I_{1,n} &\leq \sum_{k=0}^{2^n-1} t_{2^n+k,2^{n+1}-1} \int_G D_{2^n}(u) \left(\int_G |F(x,u)|^p \, d\mu(x) \right)^{\frac{1}{p}} d\mu(u) \\ &\leq \sum_{k=2^n}^{2^{n+1}-1} t_{k,2^{n+1}-1} \omega_p \left(f, 2^{-n}\right) = \omega_p \left(f, 2^{-n}\right). \end{aligned}$$

Now, applying Lemma 4 and Lemma 5 we get

$$\begin{aligned} I_{2,n} &\leq \sum_{k=1}^{2^n-2} \left|\Delta t_{2^n+k,2^{n+1}-1}\right| k \\ &\quad \times \left(\int_G \left| \int_G r_n(u) K_k(u) F(x,u) d\mu(u) \right|^p d\mu(x) \right)^{\frac{1}{p}} \\ &\leq \sum_{k=1}^{2^n-2} \left|\Delta t_{2^n+k,2^{n+1}-1}\right| k \frac{1}{2} \|K_k\|_1 \omega_p \left(f, 2^{-n}\right) \leq \sum_{k=1}^{2^n-2} \left|\Delta t_{2^n+k,2^{n+1}-1}\right| k \omega_p \left(f, 2^{-n}\right). \end{aligned}$$

We write in case a)

$$\begin{aligned} \sum_{k=1}^{2^n-2} |\Delta t_{2^n+k,2^{n+1}-1}| k &= \sum_{k=1}^{2^n-2} (t_{2^n+k+1,2^{n+1}-1} - t_{2^n+k,2^{n+1}-1}) k \\ &= (2^n-2) t_{2^{n+1}-1,2^{n+1}-1} - \sum_{k=1}^{2^n-2} t_{2^n+k,2^{n+1}-1} \\ &\leq (2^{n+1}-1) t_{2^{n+1}-1,2^{n+1}-1} \end{aligned}$$

and using condition (4.2)

$$I_{2,n} \leq (2^{n+1}-1) t_{2^{n+1}-1,2^{n+1}-1} \omega_p \left(f, 2^{-n}\right) \leq c \omega_p \left(f, 2^{-n}\right).$$

We estimate the expression $I_{3,n}$ in case a). Lemma 4, Lemma 5 and condition (4.2) yield

$$\begin{aligned} I_{3,n} &\leq (2^n-1) t_{2^{n+1}-1,2^{n+1}-1} \\ &\quad \times \left(\int_G \left| \int_G r_n(u) K_{2^n-1}(u) F(x,u) d\mu(u) \right|^p d\mu(x) \right)^{\frac{1}{p}} \\ &\leq (2^{n+1}-1) t_{2^{n+1}-1,2^{n+1}-1} \frac{1}{2} \|K_{2^n-1}\|_1 \omega_p \left(f, 2^{-n}\right) \\ &\leq (2^{n+1}-1) t_{2^{n+1}-1,2^{n+1}-1} \omega_p \left(f, 2^{-n}\right) \leq c \omega_p \left(f, 2^{-n}\right). \end{aligned}$$

In case b) we estimate $I_{2,n}+I_{3,n}$. In this situation

$$\sum_{k=1}^{2^n-2}\left|\Delta t_{2^n+k,2^{n+1}-1}\right|k=\sum_{k=1}^{2^n-2}t_{2^n+k,2^{n+1}-1}-(2^n-2)t_{2^{n+1}-1,2^{n+1}-1},$$

so Lemma 4, Lemma 5 and condition (4.1) imply

$$\begin{aligned}
I_{2,n}+I_{3,n} &\leq \sum_{k=1}^{2^n-2}\left|\Delta t_{2^n+k,2^{n+1}-1}\right|k \\
&\quad\times\left(\int_G\left|\int_G r_n(u)K_k(u)F(x,u)d\mu(u)\right|^p d\mu(x)\right)^{\frac{1}{p}} \\
&\quad+(2^n-1)t_{2^{n+1}-1,2^{n+1}-1}\times \\
&\quad\times\left(\int_G\left|\int_G r_n(u)K_{2^n-1}(u)F(x,u)d\mu(u)\right|^p d\mu(x)\right)^{\frac{1}{p}} \\
&\leq\left(\sum_{k=1}^{2^n-2}t_{2^n+k,2^{n+1}-1}+(2^n-1)t_{2^{n+1}-1,2^{n+1}-1}-(2^n-2)t_{2^{n+1}-1,2^{n+1}-1}\right) \\
&\quad\times\frac{1}{2}\cdot\frac{17}{15}\omega_p\left(f,2^{-n}\right)=\sum_{k=1}^{2^n-1}t_{2^n+k,2^{n+1}-1}\frac{17}{30}\omega_p\left(f,2^{-n}\right) \\
&\leq\frac{17}{30}\omega_p\left(f,2^{-n}\right).
\end{aligned}$$

This completes the proof of our Theorem 1. □

Remark 1. We mention, that assuming (4.1) is natural, because many well-known means satisfy it and this equality is a part of regularity conditions [25, page 74.].

Corollary 1. *Let us suppose that the conditions in Theorem 1 are satisfied. If $f\in Lip(\alpha,p,G)$, then*

$$\|\sigma_{2^n,2^{n+1}-1}(f)-f\|_p=O\left(2^{-n\alpha}\right).$$

Remark 2. In case b) we can formulate the statement of Theorem 1 in following form

$$\left\|\sigma^T_{2^n,2^{n+1}-1}(f)-f\right\|_p\leq\frac{47}{30}\omega_p\left(f,2^{-n}\right).$$

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ν,μ DICHOTOMY AND BOUNDED SOLUTIONS OF DIFFERENTIAL EQUATIONS

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Abstract. For a μ,ν dichotomic systems generalization of Palmer's lemma was proved. Necessary and sufficient conditions of the existence of bounded on the whole axis solutions and quasi-isolutions that minimize the residual norm were obtained. Index of the corresponding operator was found.

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1. Introduction

It is well known Palmer lemma in finite-dimensional space (see [20], [21]), which connects the notion of an exponential dichotomy on the axes of the corresponding equation and the noetherianity of the corresponding differential operator. Namely, such a result, proved in the work of Palmer K. [20] in 1984 is hold.

Theorem 1 ([20]). *If the linear system*

$$x'(t) = A(t)x(t)$$

admits an exponential dichotomy on both semi axes $[0;+\infty)$ *and* $(-\infty;0]$*, then the operator*

$$L : BC^1(-\infty,+\infty) \to BC(-\infty,+\infty)$$

which defines by the following rule

$$(Lx)(t) = x'(t) - A(t)x(t)$$

is Noetherian.

Later [21], in 1988 he proved converse assertion.

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Theorem 2 ([21]). *Suppose that $A(t)$ - $n \times n$ matrix valued function, bounded and continuous on the interval J, where $J = (-\infty, +\infty)$, $[0, +\infty)$ or $(-\infty, 0]$. Suppose that the operator*

$$L : BC^1(J) \to BC(J),$$

defined below, is semi-Fredholm. Then if J semi axe, then homogeneous system admits an exponential dichotomy on J and if $J = (-\infty, +\infty)$ admits an exponential dichotomy on both semi axes $[0; +\infty)$ and $(-\infty, 0]$.

For difference equations in Banach space such assertion contains in [15] (see also [1]). Fredholm differential operators with unbounded operators were considered in [17].

Recently, various different kinds of nonuniform dichotomy are proposed (see [4], [7], [14], [18], [19], [22]), nonuniform polynomial dichotomy [3], [6], [8], ρ-nonuniform exponential dichotomy [5], nonuniform (μ, ν)-dichotomy (see [2], [9], [10], [12], [13], [25]).

In the present article we are going to prove Palmer lemma for the equation that admits the so called μ, ν-dichotomy in the space endowed with the Frechet topology (the topology of the corresponding space is generated by the system of semi-norms). It should be noted that exponential dichotomy for the differential equations in the Frechet spaces was developed in the paper [11].

In the paper [16] the author give an example of a semigroup of bounded operators $\{e^{m\Delta} : m \in \mathbb{N}\}$ in the Frechet space that has dichotomy but not in a Banach space, where Δ is the Laplace operator in unbounded domain.

On the other hand H.O. Walther [23], [24], recently, investigated the delay equation in the space $C((-\infty, 0]; \mathbb{R}^n)$ which is the Frechet space. The Frechet space $C((-\infty, 0]; \mathbb{R}^n)$ has the advantage that it contains all histories $x_t = x(t + \cdot), t \in \mathbb{R}$ of every solution of the differential equation $x'(t) = f(x_t)$ in contrast to a Banach space.

The exponential dichotomy for operators on the distribution space also requires the use of Frechet spaces instead of Banach spaces.

2. Statement of the problem

In the space $\mathbb{R}^n$ we consider a system of differential equation of the following form:

$$x'(t) + A(t)x(t) = f(t), \tag{2.1}$$

where vector-function $f \in L^m_\infty(\mathbb{R}^n)$, which means it has bounded semi-norms $||f||_m$,

$$L^m_\infty(\mathbb{R}^n) = \left\{ f : \mathbb{R} \to \mathbb{R}^n : ||f||_m = \sup_{t \in [-m;m]} ||f(t)|| < +\infty, m \in \mathbb{N} \right\},$$

matrix-valued function $A(t) \in \mathcal{L}(\mathbb{R}^n)$ which is strongly continuous and $||A||_m < +\infty$ for any m:

$$||A||_m = \sup_{t \in [-m,m]} ||A(t)||$$

and the homogeneous system

$$x'(t)+A(t)x(t)=0 \tag{2.2}$$

allows μ,ν dichotomy on the semi-axes $\mathbb{R}^s_+=[s;+\infty)$ and $\mathbb{R}^s_-=(-\infty;s]$ with matrix projector-valued functions $P_+(t),t\geq s$, $P_-(t),t\leq s$. We denote by $X(t,\tau)$ fundamental matrix (2.2) normalized at $t=\tau$. Let us recall the corresponding definition of ν,μ dichotomy [7].

Definition 1. System (2.2) allows μ,ν dichotomy on the interval J, if projector-valued function $P(t)=P^2(t),t\in J$ exist, such that

i) $X(t,s)P(s)=P(t)X(t,s)$; constants $\alpha,\beta,d>0$ and $\varepsilon\geq 0$ exist, such that

ii) $||X(t,s)P(s)||\leq d\left(\frac{\mu(t)}{\mu(s)}\right)^{-\alpha}\nu(|s|)^{\varepsilon},t\geq s,t,s\in J$

and

iii) $||X(t,s)Q(s)||\leq d\left(\frac{\mu(s)}{\mu(t)}\right)^{-\beta}\nu(|s|)^{\varepsilon},s\geq t,t,s\in J$ where $Q(t)=I-P(t)$.

3. Main results

Consider the case when $f\in L^m_\infty(\mathbb{R})$ and dichotomy from the point $s=0$ on the semi-axes $\mathbb{R}_+$, $\mathbb{R}_-$. In fact, we can consider a family of equations

$$x'_\varepsilon(t)+A(t)x_\varepsilon(t)=f_\varepsilon(t) \tag{3.1}$$

instead of the equation (2.1), where $f_\varepsilon(t)=e^{-\varepsilon|t|b}f(t)$. It is easy to see that

$$||f-f_\varepsilon||_m=\sup_{t\in[-m;m]}|(1-e^{-\varepsilon|t|b})f(t)|\leq ||f||_{L^m_\infty}(1-e^{-\varepsilon mb})\to 0,\varepsilon\to 0.$$

Thus, the sequence f_ε converges to function f in the corresponding Frechet space. Using projectors that correspond for the dichotomy on the semi-axes $P_+(t),t\geq 0$ and $P_-(t),t\leq 0$ we introduce such matrix

$$D=P_+(0)-(I-P_-(0))$$

and projectors $P_{N(D)}$ and $P_{N(D^*)}$ on the kernel and cokernel of the matrix D, respectively. First, let us explore the question concerning Lyapunov's terms of the homogeneous equation.

Further, for simplicity, we would consider the case when $\mu(t)=e^{at},\nu(t)=e^{bt}$. Note that the set of bounded solutions of the homogeneous equation on the semi-axes looks as follows

$$x(t,\xi_1,\xi_2)=\begin{cases} X(t,0)P_+(0)\xi_1, & t\geq 0\\ X(t,0)(I-P_-(0))\xi_2, & t\leq 0.\end{cases}$$

Indeed

$$||X(t,0)P_+(0)\xi_1||\leq d_1e^{-\alpha_1 at}||\xi_1||,t\geq 0.$$

Then

$$\sup_{t\in[0,m]} ||X(t,0)P_+(0)\xi_1|| \leq d_1||\xi_1||, t \geq 0.$$

Similarly

$$||X(t,0)(I-P_-(0))\xi_2|| \leq d_2 e^{\beta_2 at}||\xi_2||, t \leq 0.$$

$$\sup_{t\in[-m;0]} ||X(t,0)(I-P_-(0))\xi_2|| \leq d_2||\xi_2||, t \leq 0.$$

In order for this expression to define bounded on the entire axis solutions, it is necessary to unite them at zero. From the condition

$$x(0+,\xi_1,\xi_2) = x(0-,\xi_1,\xi_2)$$

we obtain such matrix equation

$$P_+(0)\xi_1 = (I-P_-(0))\xi_2.$$

It is easy to prove, that the set of solutions of such system coincides with the set of solutions of the equation

$$P_+(0)\xi = (I-P_-(0))\xi$$

which can be rewritten as follows

$$D\xi = 0.$$

It is known [1] that the set of solutions of such equation can be presented as

$$\xi = P_{N(D)}c, \ \ \forall c \in \mathbb{R}^n.$$

The dimensionality of projector $P_{N(D)}$ determines the linearly independent number of solutions of such system. If $r = dimP_{N(D)}$, then we can rewrite the set of solutions in the following form

$$\xi = P_{N(D)_r}c_r, \ \ \forall c_r \in \mathbb{R}^r \ \ (r = dimP_{N(D)}).$$

Note also that the definition of projector implies that

$$P_+(0)P_{N(D)} = (I-P_-(0))P_{N(D)}.$$

Then the set of bounded solutions of a homogeneous system can be represented in the following form

$$x(t,c_r) = X(t,0)P_+(0)P_{N(D)_r}c_r$$

or

$$x(t,c_r) = X(t,0)(I-P_-(0))P_{N(D)_r}c_r.$$

If we consider such limit as

$$\lim_{t\to+\infty} \frac{ln||x(t,c_r)||}{t} \leq \lim_{t\to+\infty} \frac{ln(d_1 e^{-\alpha_1 at}||P_{N(D)_r}c_r||)}{t} = -\alpha_1 a.$$

Similarly

$$\lim_{t\to-\infty} \frac{ln||x(t,c_r)||}{t} \leq \lim_{t\to\infty} \frac{ln(d_2 e^{\beta_2 at}||P_{N(D)_r}c_r||)}{t} = \beta_2 a.$$

Theorem 3. *Under the conditions μ,ν dichotomy, solutions of the system (2.1) for the right-hand side of $f \in L^m_\infty(\mathbb{R})$ exist if and only if the following solvability condition fulfills*

$$P_{N(D^*)}\left\{\int_0^\infty X(0,\tau)(I-P_+(\tau))f(\tau)d\tau+\int_{-\infty}^0 X(0,\tau)P_-(\tau)f(\tau)d\tau\right\}=0. \tag{3.2}$$

Under condition (3.2) the set of bounded solutions has the following form

$$x(t,c_r)=X(t,0)P_+(0)P_{N(D)_r}c_r+(G[f])(t,0),$$

where $(G[f])(t,0)$ is generalized Green's operator:

$$(G[f])(t,0)=\begin{cases} -\int_t^{+\infty}X(t,\tau)(I-P_+(\tau))f(\tau)d\tau \\ \qquad +\int_0^t X(t,\tau)P_+(\tau)f(\tau)d\tau, & t\geq 0 \\ \int_{-\infty}^t X(t,\tau)P_-(\tau)f(\tau)d\tau \\ \qquad -\int_t^0 X(t,\tau)(I-P_-(\tau))f(\tau)d\tau, & t\leq 0. \end{cases} \tag{3.3}$$

Moreover, if $f\in L^m_\infty(\mathbb{R})$, then $x,x'\in L^m_\infty(\mathbb{R})$.

Proof. For simplicity, we take $\mu(t)=e^{at}$, $\nu(t)=e^{bt}$, $a\geq b$. Bounded on semi-axes $\mathbb{R}^+$ and $\mathbb{R}^-$ solutions of the inhomogeneous family of equations (3.1), have the following form:

$$x(t,\xi)=\begin{cases} X(t,0)P_+(0)\xi-\int_t^{+\infty}X(t,\tau)(I-P_+(\tau))f(\tau)d\tau \\ \qquad +\int_0^t X(t,\tau)P_+(\tau)f(\tau)d\tau, & t\geq 0 \\ X(t,0)(I-P_-(0))\xi+\int_{-\infty}^t X(t,\tau)P_-(\tau)f(\tau)d\tau \\ \qquad -\int_t^0 X(t,\tau)(I-P_-(\tau))f(\tau)d\tau, & t\leq 0 \end{cases} \tag{3.4}$$

Indeed:

$$||X(t,0)P_+(0)\xi||\leq d_1e^{-\alpha_1 at}||\xi||, t\geq 0,$$

$$\begin{aligned}\int_0^t ||X(t,\tau)P_+(\tau)f(\tau)||d\tau &\leq d_1\int_0^t e^{-\alpha_1 a(t-\tau)+\varepsilon b\tau}||f||_{L^m_\infty}d\tau \\ &\leq \frac{d_1}{\alpha_1 a+\varepsilon b}(e^{\varepsilon bt}-e^{-\alpha_1 at})||f||_{L^m_\infty}.\end{aligned}$$

Similarly

$$\begin{aligned}\left|\left|\int_t^{+\infty}X(t,\tau)(I-P_+(\tau))f(\tau)d\tau\right|\right| &\leq \int_t^{+\infty}||X(t,\tau)(I-P_+(\tau))f(\tau)||d\tau \\ &\leq d_1\int_t^{+\infty}e^{-\beta_1 a(\tau-t)+\varepsilon b\tau}||f||_{L^m_\infty}d\tau \\ &\leq \frac{d_1e^{\varepsilon bt}}{\beta_1 a-\varepsilon b}||f||_{L^m_\infty}.\end{aligned}$$

Thus, adding the obtained inequalities, we have

$$||x(t,\xi)||\leq d_1e^{-\alpha_1 at}||\xi||$$

$$+\left(d_1 e^{\varepsilon bt}\left(\frac{1}{\alpha_1 a+\varepsilon b}+\frac{1}{\beta_1 a-\varepsilon b}\right)-\frac{d_1}{\alpha_1 a+\varepsilon b}e^{-\alpha_1 at}\right)||f||_{L_\infty^m}.$$

Finally we obtain

$$\sup_{t\in[0;m]}||x(t,\xi)||\le d_1||\xi|| \tag{3.5}$$
$$+\left(d_1 e^{\varepsilon bm}\left(\frac{1}{\alpha_1 a+\varepsilon b}+\frac{1}{\beta_1 a-\varepsilon b}\right)-\frac{d_1}{\alpha_1 a+\varepsilon b}e^{-\alpha_1 am}\right)||f||_{L_\infty^m}.$$

□

It is also easy to obtain appropriate estimates for negative real numbers and the derivative of the solutions. The boundedness of other integrals is checked in the same way. Based on that

$$\begin{aligned}
&\frac{\partial(-\int_t^{+\infty}X(t,\tau)(I-P_+(\tau))f(\tau)d\tau+\int_0^t X(t,\tau)P_+(\tau)f(\tau)d\tau)}{\partial t}\\
&=X(t,t)(I-P_+(t))f(t)-A(t)\int_t^{+\infty}X(t,\tau)(I-P_+(\tau))f(\tau)d\tau\\
&\quad+X(t,t)P_+(t)f(t)+A(t)\int_0^t X(t,\tau)P_+(\tau)f(\tau)d\tau\\
&=f(t)+A(t)\left\{-\int_t^{+\infty}X(t,\tau)(I-P_+(\tau))f(\tau)d\tau+\int_0^t X(t,\tau)P_+(\tau)f(\tau)d\tau\right\}
\end{aligned}$$

which proves that expression (3.4) defines all bounded solutions of the system (2.1) on the semi-axes.

In order for the system (3.4) to define bounded solutions on the whole axis, it is necessary and sufficient that the next condition is fulfilled

$$x(0+,\xi)=x(0-,\xi).$$

This condition is equivalent to the solvability of the matrix equation

$$\begin{aligned}
P_+(0)\xi-\int_0^{+\infty}X(0,\tau)(I-P_+(\tau))f(\tau)d\tau \\
=(I-P_-(0))\xi+\int_{-\infty}^0 X(0,\tau)P_-(\tau)f(\tau)d\tau.
\end{aligned} \tag{3.6}$$

If ξ is the solution of the equation (3.6), then substituting it into (3.4) leads us to the bounded on the whole axis solution of the system (2.1). Actually, based on the fact that matrix

$$D=P_+(0)-I+P_-(0)$$

always has a pseudo-inverse by Moore-Penrose, the set of the bounded on the whole axis solutions of the system (2.1) could be represented in the following form

$$x(t,\xi)=\begin{cases} X(t,0)P_+(0)\xi-\int_t^{+\infty}X(t,\tau)(I-P_+(\tau))f(\tau)d\tau \\ \qquad +\int_0^t X(t,\tau)P_+(\tau)f(\tau)d\tau, & t\geq 0 \\ X(t,0)(I-P_-(0))\xi+\int_{-\infty}^t X(t,\tau)P_-(\tau)f(\tau)d\tau- \\ \qquad -\int_t^0 X(t,\tau)(I-P_-(\tau))f(\tau)d\tau, & t\leq 0 \end{cases} \tag{3.7}$$

Let $g=\int_{-\infty}^0 X(0,\tau)P_-(\tau)f(\tau)d\tau+\int_0^{+\infty}X(0,\tau)(I-P_+(\tau))f(\tau)d\tau$.

Condition $P_{N(D^*)}g=0$ [1] is necessary and sufficient for the solvability of the equation $D\xi=g$. Then we obtain that the condition for the existence of bounded on the whole axis solutions of the system (2.1) is equivalent to the solvability of the following matrix equation:

$$D\xi=\int_0^{\infty}X(0,\tau)(I-P_+(\tau))f(\tau)d\tau+\int_{-\infty}^0 X(0,\tau)P_-(\tau)f(\tau)d\tau. \tag{3.8}$$

Since matrix D has pseudo-inverse by Moore-Penrose matrix, then equation (3.8) has solutions if and only if

$$P_{N(D^*)}\left\{\int_0^{\infty}X(0,\tau)(I-P_+(\tau))f(\tau)d\tau+\int_{-\infty}^0 X(0,\tau)P_-(\tau)f(\tau)d\tau\right\}=0.$$

Under this condition, equation (3.8) has a set of solutions

$$\xi=D^+\left(\int_0^{\infty}X(0,\tau)(I-P_+(\tau))f(\tau)d\tau+\int_{-\infty}^0 X(0,\tau)P_-(\tau)f(\tau)d\tau\right)+P_{N(D)}c, \tag{3.9}$$

where c is an arbitrary vector of the corresponding dimensionality. Substituting the obtained solutions into (3.4) we obtain a general view of the solutions bounded on the entire axis in this form

$$x(t,c_r)=X(t,0)P_+(0)P_{N(D)_r}c_r+(G[f])(t,0),$$

where $(G[f])(t,0)$ is generalized Green's operator:

$$(G[f])(t,0)=\begin{cases} -\int_t^{+\infty}X(t,\tau)(I-P_+(\tau))f(\tau)d\tau \\ \qquad +\int_0^t X(t,\tau)P_+(\tau)f(\tau)d\tau, & t\geq 0 \\ \int_{-\infty}^t X(t,\tau)P_-(\tau)f(\tau)d\tau \\ \qquad -\int_t^0 X(t,\tau)(I-P_-(\tau))f(\tau)d\tau, & t\leq 0. \end{cases} \tag{3.10}$$

Green's operator has the following property at 0 relative to the jump:

$$\begin{aligned} &(G[f])(0+0)-(G[f])(0-0) \\ &\qquad =-\int_0^{\infty}X(0,\tau)(I-P_+(\tau))f(\tau)d\tau+P_+(0)D^-g \\ &\qquad\quad -\int_{-\infty}^0 X(0,\tau)P_-(\tau)f(\tau)d\tau-(I-P_-(0))D^-g \\ &\qquad =-g+P_+(0)D^-g-D^-g+P_-(0)D^-g \end{aligned}$$

$$= (P_+(0) - I + P_-(0))D^- g - g = DD^- g - g$$
$$= -(I - DD^-)g = -P_{N(D^*)}g = -\int_{-\infty}^{+\infty} H(t)f(t)dt = 0.$$

The corresponding solution is bounded. Thus, we obtain the statement of the theorem.

Remark 1. If the condition (3.2) isn't hold then expression (3.9) defines the set of quasisolutions of the system (3.8) (elements ξ from the set (3.9) give the minimum of the norm $||D\xi - g||$).

Corollary 1. *For any $f \in L_\infty^m(\mathbb{R}^n)$ there are constants d, $N_1(m) > 0$ such that the following estimates are satisfied*

$$||x||_{L_\infty^m} \le d||P_{N(D)}c|| + N_1(m)||f||_{L_\infty^m},$$
$$||G[f]||_{L_\infty^m} \le N_1(m)||f||_{L_\infty^m}.$$

For any $f_\varepsilon(t) = e^{-\varepsilon|t|b}f(t) \in L_\infty^m(\mathbb{R}^n)$ there are constants d, $N_2(m) > 0$ such that the following estimates are satisfied

$$||x_\varepsilon||_{L_\infty^m} \le d||P_{N(D)}c|| + N_2(m)||f||_{L_\infty^m}$$

or

$$||x_\varepsilon||_{L_\infty^m} \le d||P_{N(D)}c|| + N_2(m)e^{\varepsilon mb}||f_\varepsilon||_{L_\infty^m}$$
$$||G[f_\varepsilon]||_{L_\infty^m} \le N_2(m)||f||_{L_\infty^m}$$

or

$$||G[f_\varepsilon]||_{L_\infty^m} \le N_2(m)e^{\varepsilon mb}||f_\varepsilon||_{L_\infty^m}$$

Proof. Together with the inequality (3.5) we can obtain the following inequalities with the right hand side $f(t)$:

$$\sup_{t\in[0;m]} ||x(t,\xi)|| \le d_1||\xi|| + \left(d_1 e^{\varepsilon bm}\left(\frac{1}{\alpha_1 a + \varepsilon b} + \frac{1}{\beta_1 a - \varepsilon b}\right) - \frac{d_1}{\alpha_1 a + \varepsilon b}e^{-\alpha_1 am}\right)||f||_{L_\infty^m},$$
$$\sup_{t\in[-m;0]} ||x(t,\xi)|| \le d_2||\xi|| + \left(d_2 e^{\varepsilon bm}\left(\frac{1}{\alpha_2 a - \varepsilon b} + \frac{1}{\beta_2 a + \varepsilon b}\right) - \frac{d_2}{\beta_2 a + \varepsilon b}e^{-\beta_2 am}\right)||f||_{L_\infty^m}.$$

If $d = max\{d_1, d_2\}$,

$$K_1(m) = \max\left\{d_1 e^{\varepsilon bm}\left(\frac{1}{\alpha_1 a + \varepsilon b} + \frac{1}{\beta_1 a - \varepsilon b}\right), d_2 e^{\varepsilon bm}\left(\frac{1}{\alpha_2 a - \varepsilon b} + \frac{1}{\beta_2 a + \varepsilon b}\right)\right\},$$
$$L_1(m) = \min\left\{\frac{d_1 e^{-\alpha_1 am}}{\alpha_1 a + \varepsilon b}, \frac{d_2 e^{-\beta_2 am}}{\beta_2 a + \varepsilon b}\right\},$$

and $N_1(m) = K_1(m) - L_1(m)$, then

$$||x||_{L_\infty^m} \le d||P_{N(D)}c|| + N_1(m)||f||_{L_\infty^m}$$

For $f_\varepsilon(t) = e^{-\varepsilon b|t|} f(t)$ we have

$$\sup_{t\in[0;m]} ||x(t,\xi)|| \le d_1||\xi|| + \left(d_1\left(\frac{1}{\alpha_1 a}+\frac{1}{\beta_1 a}\right) - \frac{d_1}{\beta_1 a}e^{-\alpha_1 am}\right)||f||_{L^m_\infty}$$

$$\sup_{t\in[-m;0]} ||x(t,\xi)|| \le d_2||\xi|| + \left(d_2\left(\frac{1}{\alpha_2 a}+\frac{1}{\beta_2 a}\right) - \frac{d_2}{\beta_2 a}e^{-\beta_2 am}\right)||f||_{L^m_\infty}$$

If

$$K_2 = \max\left\{d_1\left(\frac{1}{\alpha_1 a}+\frac{1}{\beta_1 a}\right), d_2\left(\frac{1}{\alpha_2 a}+\frac{1}{\beta_2 a}\right)\right\}$$

$$L_2(m) = \min\left\{\frac{d_1}{\beta_1 a}e^{-\alpha_1 am}, \frac{d_2}{\beta_2 a}e^{-\beta_2 am}\right\}$$

and $N_2(m) = K_2 - L_2(m)$, then

$$||x_\varepsilon||_{L^m_\infty} \le d||P_{N(D)}c|| + N_2(m)||f||_{L^m_\infty}$$

or

$$||x_\varepsilon||_{L^m_\infty} \le d||P_{N(D)}c|| + N_2(m)e^{\varepsilon mb}||f_\varepsilon||_{L^m_\infty}$$

□

Corollary 2. *Operator*

$$\mathcal{L} := \frac{d}{dt} + A(t).$$

under conditions μ,ν dichotomy is Noetherian with index $r-d$, as the operator from the space

$$H^m_\infty(\mathbb{R}^n) = \{g \in L^m_\infty(\mathbb{R}^n),\ \textit{such that } g' \in L^m_\infty(\mathbb{R}^n)\}$$

into space $L^m_\infty(\mathbb{R}^n)$. Thus $\mathcal{L} : H^m_\infty(\mathbb{R}^n) \to L^m_\infty(\mathbb{R}^n)$.

Proof. Indeed, it follows from the proof of the previous theorem that the operator

$$\mathcal{L} := \frac{d}{dt} + A(t)$$

is Noetherian. The following numbers determine its index. Let us denote the number of linear independent solvability conditions

$$+P_{N(D^*)}\left\{\int_0^\infty T(0,\tau)(I-P_+(\tau))f_\varepsilon(\tau)d\tau + \int_{-\infty}^0 T(0,\tau)P_-(\tau)f_\varepsilon(\tau)d\tau\right\} = 0,$$

as d, and the number of linearly independent bounded solutions of the homogeneous equation of the form $P_{N(D)}c$ as r. Then the index of the operator $\mathcal{L}$ is determined as

$$ind\ \mathcal{L} = r - d.$$

□

Remark 2. Let us consider the case when this system is considered only on semi-axis. In this case, the system has a set of bounded solutions in the following form:

$$x(t,\xi_1) = T(t,0)P_+(0)\xi_1 - \int_t^{+\infty} T(t,\tau)(I-P_+(\tau))f_\varepsilon(\tau)d\tau$$
$$+\int_0^t T(t,\tau)P_+(\tau)f_\varepsilon(\tau)d\tau, \qquad t \geq 0$$

for the arbitrary heterogeneity and vector ξ_1. The boundedness of the solution from this set is proved in the same way as Theorem 1.

Example 1. Consider an example of a two-dimensional system

$$\frac{dx_1(t)}{dt} = th\, tx_1(t) + f_{1\varepsilon}(t),$$
$$\frac{dx_2(t)}{dt} = -th\, tx_2(t) + f_{2\varepsilon}(t).$$

Let us show that this system admits a uniform exponential dichotomy on the axes. In this case, we show that

$$X(t,\tau) = diag\left\{\frac{e^t+e^{-t}}{e^\tau+e^{-\tau}}, \frac{e^\tau+e^{-\tau}}{e^t+e^{-t}}\right\}.$$

Projectors

$$P_+(\tau) = diag\{0,1\}, P_-(\tau) = diag\{1,0\},$$
$$f(t) = (f_1(t), f_2(t))^T, f_\varepsilon(t) = (f_{1\varepsilon}(t), f_{2\varepsilon}(t))^T = e^{-\varepsilon|t|b}(f_1(t), f_2(t))^T$$

and $||f_\varepsilon||_\tau = e^{-\varepsilon|\tau|b}||f||_{L_\infty(\mathbb{R})}$. It is easy to show that the system allows uniform exponential dichotomy on the axes under such conditions. So let us write down the necessary and sufficient condition of solvability of such system. In this case, the matrix $D = P_+(0) - (I - P_-(0)) = 0$ and respectively $P_{N(D)} = P_{N(D^*)} = I$. The solvability condition takes the following form

$$\int_{-\infty}^{+\infty} \frac{f_{1\varepsilon}(\tau)}{e^\tau+e^{-\tau}}d\tau = 0$$

or

$$\int_{-\infty}^{+\infty} \frac{e^{-\varepsilon|\tau|b}f_1(\tau)}{e^\tau+e^{-\tau}}d\tau = 0$$

The set of bounded solutions looks like this

$$x(t,c) = \begin{pmatrix} 0 \\ \frac{2}{e^t+e^{-t}}c_2 \end{pmatrix} + (G[f_\varepsilon])(t,0), \forall c_2 \in \mathbb{R}$$

where

$$(G[f_\varepsilon])(t,0) = \begin{pmatrix} -\int_t^{+\infty} \frac{e^t+e^{-t}}{e^\tau+e^{-\tau}} f_{1\varepsilon}(\tau)d\tau, \tau \leq t \\ \int_0^t \frac{e^\tau+e^{-\tau}}{e^t+e^{-t}} f_{2\varepsilon}(\tau)d\tau, \tau \geq t \end{pmatrix}, \qquad t \geq 0$$

$$(G[f_\varepsilon])(t,0) = \begin{pmatrix} \int_{-\infty}^{t} \frac{e^t+e^{-t}}{e^\tau+e^{-\tau}} f_{1\varepsilon}(\tau)d\tau, \tau \le t \\ -\int_{t}^{0} \frac{e^\tau+e^{-\tau}}{e^t+e^{-t}} f_{2\varepsilon}(\tau)d\tau, \tau \ge t \end{pmatrix}, \qquad t \le 0.$$

or in the form

$$x(t,c) = \begin{pmatrix} 0 \\ \frac{2}{e^t+e^{-t}} c_2 \end{pmatrix} + (G[f])(t,0), \forall c_2 \in \mathbb{R}$$

where

$$(G[f])(t,0) = \begin{pmatrix} -\int_{t}^{+\infty} \frac{e^t+e^{-t}}{e^\tau+e^{-\tau}} e^{-\varepsilon|\tau|b} f_1(\tau)d\tau, \tau \le t \\ \int_{0}^{t} \frac{e^\tau+e^{-\tau}}{e^t+e^{-t}} e^{-\varepsilon|\tau|b} f_2(\tau)d\tau, \tau \ge t \end{pmatrix}, \qquad t \ge 0$$

$$(G[f])(t,0) = \begin{pmatrix} \int_{-\infty}^{t} \frac{e^t+e^{-t}}{e^\tau+e^{-\tau}} e^{-\varepsilon|\tau|b} f_1(\tau)d\tau, \tau \le t \\ -\int_{t}^{0} \frac{e^\tau+e^{-\tau}}{e^t+e^{-t}} e^{-\varepsilon|\tau|b} f_2(\tau)d\tau, \tau \ge t \end{pmatrix}, \qquad t \le 0.$$

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ON NEW VERSION OF HERMITE-JENSEN-MERCER INEQUALITIES FOR NEWLY DEFINED QUANTUM INTEGRAL

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Abstract. In current study, we establish some new quantum Hermite Hadamard-Jensen-Mercer type integral inequalities by way of recently newly defined integral.Then we investigate the connections between our results and those in earlier works.Furthermore, we present some examples which satisfied our main outcomes. We expect that this innovative way opens many avenues for interested researchers will reconcile discovering further quantum approximations of Hermite-Hadamard type variants for other classes of convex functions, and, additionally, to discover uses in the aforementioned scientific disciplines.

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1. Quantum calculus

The investigation of quantum calculus is equivalent to the usual infinitesimal calculus without the concept of limits or the investigation of calculus without limits (quantum is from the Latin word ”quantus” and literally it means how much, in Swedish ”Kvant”). It has two main branches, q-calculus and h-calculus. And both of them were worked out by P. Cheung and V. Kac [14] in the early twentieth century. In the same era F.H. Jackson started working on quantum calculus or q-calculus, but Euler and Jacobi had already figured out this type of calculus. A number of studies have been recently conducted on the field of q-analysis, beginning from Euler, due to it vast applications in mathematical modelling and the existence of its connection in the framework between physics and mathematics. Tariboon and Ntouyas [22] proposed the quantum calculus concepts on finite intervals and obtained several q-analogues of classical mathematical objects. This inspired other researchers and, as a consequence, various novel results concerning quantum analogues of classical mathematical results have already been inaugurated in the literature. In [4], Alp et al. acquired some bonds for left-hand side of q-Hermite-Hadamard inequalities and

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quantum calculations by way of convex and quasi-convex functions for midpoint type inequalities. For more details, see [14, 18, 20–22] and the references cited therein.

In different mathematical fields, Quantum calculus has a broad area of application in number theory, combinatorics, orthogonal polynomials, simple hyper-geometric functions, and other sciences, quantum theory, physics and relativity theory. It has been discovered that quantum calculus is a sub field of the more general mathematical field of time scales calculus. New developments have recently been made in the research and methodology of dynamic derivatives on time scales. The research offers a unification and application of traditional differential and difference equations. Moreover, it is a alliance of the discrete theory with the continuous theory from a theoretical perspective. Time scales provide a unified framework for studying dynamic equations on both discrete and continuous domains. In studying quantum calculus, we are concerned with a specific time scale, called the q-time scale, defined as follows: $T := q^{N_0} := qt : t \in N_0$, see [2–5, 9–12] and the references cited therein.

On the other hand, the well known discrete Jensen inequality states that: let $0 < x_1 \leq x_2 \leq \ldots \leq x_n$ and let $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ non negative weights such that $\Sigma_{i=1}^{n} \mu_i = 1$. If F is a convex function on the interval $[a,b]$, then

$$F\left(\sum_{i=1}^{n} \mu_i x_i\right) \leq \sum_{i=1}^{n} \mu_i F(x_i),$$

where for all $x_i \in [a,b]$ and $\mu_i \in [0,1]$, $(i = \overline{1,n})$ [8]. The Hermite–Hadamard inequality states that, if a mapping $F : I \subseteq R \to R$ is a convex on I with $a, b \in I$ and $a < b$, then

$$F\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b F(x)dx \leq \frac{F(a)+F(b)}{2}.$$

The double inequality holds in the reversed direction if F is concave [7].

Theorem 1 ([17, Theoem 1.2])**.** *If F is convex function on $I = [a,b]$, then*

$$F\left(a+b-\sum_{i=1}^{n} \mu_i x_i\right) \leq F(a)+F(b)-\sum_{i=1}^{n} \mu_i F(x_i), \tag{1.1}$$

for each $x_i \in [a,b]$ and $\mu_i \in [0,1]$, $(i = \overline{1,n})$ with $\sum_{i=1}^{n} \mu = 1$.

Inequality (1.1) is known as the Jensen–Mercer inequality. By using inequality (1.1), Kian and Moslehian prove the Hermite-Jensen-Mercer inequality in [16]. Recently several papers have devoted to generalization of Hermite-Jensen-Mercer inequality. For more recent and related results connected with Jensen-Mercer inequality and Hermite-Jensen–Mercer inequality, see [1, 6, 19, 23, 24].

In this research, we establish the Hermite-Jensen-Mercer inequalities for T_q integrals.We also present some examples to illustrate our results.

2. Preliminaries of q-Calculus and Some Inequalities

In this section we present some required definitions and related inequalities about q-calculus. Also, here and further we use the following notation(see [14]):

$$[n]_q = \frac{1-q^n}{1-q} = 1+q+q^2+...+q^{n-1}, \quad q \in (0,1).$$

In [13], Jackson gave the q-Jackson integral from 0 to b for $0<q<1$ as follows:

$$\int_0^b F(x) \ d_q x \ = (1-q)\, b \sum_{n=0}^{\infty} q^n F(bq^n)$$

provided the sum converge absolutely.
Jackson in [13] gave the q-Jackson integral in a generic interval $[a,b]$ as:

$$\int_a^b F(x) \ d_q x \ = \int_0^b F(x) \ d_q x \ - \int_0^a F(x) \ d_q x \ .$$

Definition 1 ([21, Definition 3.3])**.** Let $F:[a,b]\to R$ be a continuous function. Then, the q_a-definite integral on $[a,b]$ is defined as

$$\int_a^b F(x)_a d_q x = (1-q)(b-a)\sum_{n=0}^{\infty} q^n F(q^n b+(1-q^n)a)$$

$$= (b-a)\int_0^1 F((1-t)a+tb)\, d_q t.$$

Many of the most renowned inequalities in the classical analysis such as Hölder's inequality, Hermite-Hadamard's inequality and Ostrowski's inequality, Cauchy-Bunyakovsky-Schwarz, Gruss, Gruss-Cebysev and other integral inequalities have been proven and applied for q-calculus.
In [4], Alp et al. proved the following q_a-Hermite-Hadamard inequalities for convex functions in the setting of quantum calculus:

Theorem 2 ([4, Theoem 6])**.** *Let $F:[a,b]\to R$ be a convex differentiable function on $[a,b]$ and $0<q<1$. Then q-Hermite-Hadamard inequalities are as follows:*

$$F\left(\frac{qa+b}{1+q}\right) \le \frac{1}{b-a}\int_a^b F(x) \ {}_a d_q x \ \le \frac{qF(a)+F(b)}{1+q}. \tag{2.1}$$

In [18] and [4], authors established some bounds for left and right hand sides of the inequality (2.1).
On the other hand, Bermudo et al. gave the following new definition and related Hermite-Hadamard type inequalities:

Definition 2 ([5, Definition 6])**.** Let $F:[a,b]\to R$ be a continuous function. Then, the q^b-definite integral on $[a,b]$ is defined as

$$\int_a^b F(x)\ {}^b d_q x = (1-q)(b-a)\sum_{n=0}^{\infty} q^n F(q^n a+(1-q^n)b)$$
$$= (b-a)\int_0^1 F(ta+(1-t)b)\ d_q t.$$

Theorem 3 ([5, Theorem 11])**.** *Let $F:[a,b]\to R$ be a convex function on $[a,b]$ and $0<q<1$. Then, q-Hermite-Hadamard inequalities are as follows:*

$$F\left(\frac{a+qb}{1+q}\right)\le \frac{1}{b-a}\int_a^b F(x)\ {}^b d_q x \le \frac{F(a)+qF(b)}{1+q}.$$

From Theorem 1 and Theorem 3, one can the following inequalities:

Corollary 1 ([5, Corollary 14])**.** *For any convex function $F:[a,b]\to R$ and $0<q<1$, we have*

$$F\left(\frac{qa+b}{1+q}\right)+F\left(\frac{a+qb}{1+q}\right)\le \frac{1}{b-a}\left\{\int_a^b F(x)\ {}_a d_q x + \int_a^b F(x)\ {}^b d_q x\right\}$$
$$\le F(a)+F(b)$$

and

$$F\left(\frac{a+b}{2}\right)\le \frac{1}{2(b-a)}\left\{\int_a^b F(x)\ {}_a d_q x + \int_a^b F(x)\ {}^b d_q x\right\}\le \frac{F(a)+F(b)}{2}.$$

2.1. *T_q-Integrals and Related Inequalities*

In this subsection, we present definitions and properties given by using trapezoids.

Alp and Sarikaya, by using the area of trapezoids, introduced the following generalized quantum integral which is called ${}_aT_q$-integral in [2].

Definition 3 ([2, Definition 3.1])**.** Let $F:[a,b]\to R$ is continuous function. for $x\in[a,b]$

$$\int_a^b F(s)\ {}_a d_q^T s = \frac{(1-q)(b-a)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n F(q^n b+(1-q^n)a)-F(b)\right]$$

$$= (b-a)\int_0^1 F(tb+(1-t)a)\,{}_0d_q^T t$$

where $0<q<1$.

Theorem 4 ([2, Theorem 4.1])**.** *Let $F:[a,b]\to R$ be a convex continuous function on $[a,b]$ and $0<q<1$. Then we have*

$$F\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b F(x)\ \ {}_ad_q^T x\ \ \le \frac{F(a)+F(b)}{2}$$

In [15], Kara et al. introduced the following generalized quantum integral which is called ${}^bT_q$-integral.

Definition 4 ([15, Definition 6])**.** Let $F:[a,b]\to R$ is continuous function. for $x\in[a,b]$,

$$\int_a^b F(s)\ \ {}^bd_q^T s\ \ = \frac{(1-q)(b-a)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n F(q^n a+(1-q^n)b)-f(a)\right]$$

$$= (b-a)\int_0^1 F(ta+(1-t)b)\,{}^1d_q^T t$$

where $0<q<1$.

Theorem 5 ([15, Theorem 11])**.** *Let $F:[a,b]\to R$ be a convex continuous function on $[a,b]$ and $0<q<1$. Then we have*

$$F\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b F(x)\ \ {}^bd_q^T x\ \ \le \frac{F(a)+F(b)}{2}.$$

3. Generalized Hermite-Jensen-Mercer inequalities for Quantum Integrals

In this section, we establish some new Quantum Hermite-Jensen-Mercer type inequalities by way of T_q integrals.

Theorem 6. *Let $F:[a,b]\to R$ be a convex function on $[a,b]$ and $0<q<1$. Then we have the following quantum Jensen-Mercer inequalities*

$$\begin{aligned} F\left(a+b-\frac{x+y}{2}\right) &\le F(a)+F(b)-\frac{1}{2(y-x)}\left[\int_x^y F(w)\,{}_xd_q^T w+\int_x^y F(w)\,{}^yd_q^T w\right] \\ &\le F(a)+F(b)-F\left(\frac{x+y}{2}\right) \end{aligned} \tag{3.1}$$

for all $x,y\in[a,b]$ with $x\le y$.

Proof. It follows from the Jensen-Mercer inequality, we have

$$F\left(a+b-\frac{u+v}{2}\right) \leq F(a)+F(b)-\frac{1}{2}[F(u)+F(v)]$$

for $u,v \in [a,b]$. Considering $u = tx+(1-t)y$ and $v = ty+(1-t)x$, where $t \in [0,1]$, we get

$$F\left(a+b-\frac{x+y}{2}\right) \leq F(a)+F(b)-\frac{1}{2}\left[F(tx+(1-t)y)+F(ty+(1-t)x)\right].$$

By ${}_aT_q$-integrating with respect to t over $[0,1]$ for $q \in (0,1)$, we have

$$\begin{aligned} &F\left(a+b-\frac{x+y}{2}\right) \\ &\leq F(a)+F(b)-\frac{1}{2}\left[\int_0^1 F(tx+(1-t)y)d_q^T t+\int_0^1 F(ty+(1-t)x)d_q^T t\right]. \end{aligned} \tag{3.2}$$

From Definition 3, and Theorem 4, we have

$$\int_0^1 F(tx+(1-t)y)\,d_q^T t = \frac{1}{y-x}\int_x^y F(w)\,{}_x d_q^T w \geq F\left(\frac{x+y}{2}\right) \tag{3.3}$$

and similarly by Definition 4 and Theorem 5, we have

$$\int_0^1 F(tx+(1-t)y)\,d_q^T t = \frac{1}{y-x}\int_x^y F(w)\,{}^y d_q^T w \geq F\left(\frac{x+y}{2}\right) \tag{3.4}$$

Using (3.3) and (3.4) in (3.2), we get the desired result (3.1). The proof is completed. □

Remark 1. If we take the limit $q \to 1$ in Theorem 6, then we have the following inequalities

$$\begin{aligned} F\left(a+b-\frac{x+y}{2}\right) &\leq F(a)+F(b)-\frac{1}{(y-x)}\int_x^y F(w)\,dw \\ &\leq F(a)+F(b)-F\left(\frac{x+y}{2}\right) \end{aligned}$$

which are given by Kian and Moslehian in [16, Theorem 2.1].

Corollary 2. *If we take $x = a$ and $y = b$ in (3.1) then we get*

$$\begin{aligned} F\left(\frac{a+b}{2}\right) &\leq F(a)+F(b)-\frac{1}{2(b-a)}\left[\int_a^b F(w)\,{}_a d_q^T w+\int_a^b F(w)\,{}^b d_q^T w\right] \\ &\leq F(a)+F(b)-F\left(\frac{a+b}{2}\right). \end{aligned}$$

Theorem 7. *Let $F:[a,b]\to R$ be a convex function on $[a,b]$ and $0<q<1$. Then we have the following quantum Jensen-Mercer inequalities*

$$F\left(a+b-\frac{x+y}{2}\right)\leq\frac{1}{y-x}\int_{a+b-y}^{a+b-x}F(w)\,{}_{(a+b-y)}d_q^T w \tag{3.5}$$
$$\leq F(a)+F(b)-\frac{F(x)+F(y)}{2}$$

for all $x,y\in[a,b]$ with $x\leq y$.

Proof. From Definition 3 and Theorem 4, we have

$$\int_0^1 F\left(a+b-(tx+(1-t)y)\right)_0 d_q^T t \tag{3.6}$$
$$=\int_0^1 F\left(t(a+b-x)+(1-t)(a+b-y)\right)_0 d_q^T t$$
$$=\frac{1}{y-x}\int_{a+b-y}^{a+b-x}F(w)\,{}_{(a+b-y)}d_q^T w$$
$$\geq F\left(a+b-\frac{x+y}{2}\right)$$

which gives proof of first inequality in (3.5). By Jensen-Mercer inequality, we obtain

$$\frac{1}{y-x}\int_{a+b-y}^{a+b-x}F(w)\,{}_{(a+b-y)}d_q^T w \tag{3.7}$$
$$=\int_0^1 F\left(a+b-(tx+(1-t)y)\right)_0 d_q^T t$$
$$\leq F(a)+F(b)-F(x)\int_0^1 t\,{}_0d_q^T t-F(y)\int_0^1(1-t)\,{}_0d_q^T t$$
$$=F(a)+F(b)-\frac{F(x)+F(y)}{2}.$$

Combining the inequalities (3.6) and (3.7), we get the required result. □

Remark 2. If we take the limit $q\to 1$ in Theorem 7, then then we have the following inequalities

$$F\left(a+b-\frac{x+y}{2}\right)\leq\frac{1}{y-x}\int_{a+b-y}^{a+b-x}F(w)\,dw \tag{3.8}$$
$$\leq F(a)+F(b)-\frac{F(x)+F(y)}{2}$$

which are proven by Kian and Moslehian in [16, Theorem 2.1].

Remark 3. If we choose $x=a$ and $y=b$ in Theorem 7, then Theorem 7 reduces to Theorem 4.

Theorem 8. *Let $F:[a,b]\to R$ be a convex function on $[a,b]$ and $0<q<1$. Then we have the following quantum Jensen-Mercer inequalities*

$$F\left(a+b-\frac{x+y}{2}\right)\leq\frac{1}{y-x}\int_{a+b-y}^{a+b-x}F(w)^{(a+b-x)}d_q^Tw \tag{3.9}$$
$$\leq F(a)+F(b)-\frac{F(x)+F(y)}{2}$$

for all $x,y\in[a,b]$ with $x\leq y$.

Proof. From Definition 4 and Theorem 5, we have

$$\int_0^1 F\left(a+b-(ty+(1-t)x)\right)_0 d_q^Tt \tag{3.10}$$
$$=\int_0^1 F\left(t(a+b-y)-(1-t)(a+b-x)\right)_0 d_q^Tt$$
$$=\frac{1}{y-x}\int_{a+b-y}^{a+b-x}F(w)^{(a+b-x)}d_q^Tw$$
$$\geq F\left(a+b-\frac{x+y}{2}\right)$$

which proves the first inequality in (3.9). By Jensen-Mercer inequality, we have

$$\frac{1}{y-x}\int_{a+b-y}^{a+b-x}F(w)^{(a+b-x)}d_q^Tw \tag{3.11}$$
$$=\int_0^1 F\left(a+b-(ty+(1-t)x)\right)_0 d_q^Tt$$
$$\leq F(a)+F(b)-F(y)\int_0^1 t\,{}_0d_q^Tt-F(x)\int_0^1(1-t)\,{}_0d_q^Tt$$
$$=F(a)+F(b)-\frac{F(x)+F(y)}{2}.$$

Combining the inequalities (3.10) and (3.11), we get required result. □

Remark 4. By taking the limit $q\to 1$ in Theorem 8, then inequalities (3.9) reduce to inequalities (3.8).

Remark 5. If we assign $x=a$ and $y=b$ in Theorem 8, then Theorem 8 reduces to Theorem 5.

4. Examples

In this section,we present some examples which illustrates the validity of our new results.

Example 1. Define the the convex function $F(x) = x^2$ on $[0,2]$. By considering $q = \frac{1}{2}$, $x = 0$, $y = 1$ in Theorem 6, we have

$$F\left(a+b-\frac{x+y}{2}\right) = F\left(\frac{3}{2}\right) = \frac{9}{4},$$

$$F(a)+F(b)-\frac{1}{2(y-x)}\left[\int_x^y F(w)\,{}_xd_q^T w + \int_x^y F(w)\,{}^yd_q^T w\right]$$
$$= 4-\frac{1}{2}\left[\int_0^1 w^2\,{}_0d_q^T w + \int_0^1 w^2\,{}^1d_q^T w\right] = 4-\left[\frac{5}{28}+\frac{5}{28}\right] = \frac{51}{14}$$

and

$$F(a)+F(b)-F\left(\frac{x+y}{2}\right) = 4-\frac{1}{4} = \frac{15}{4}$$

Thus, the inequalities (3.1) are verified.

Example 2. Let consider the convex function $F(x) = x^2$ on $[-\frac{3}{2},2]$. If $q = \frac{1}{2}$, $x = -\frac{1}{2}$, $y = \frac{1}{2}$ in (3.5), then we have

$$F\left(a+b-\frac{x+y}{2}\right) = F\left(\frac{1}{2}\right) = \frac{1}{4},$$

$$\frac{1}{y-x}\int_{a+b-y}^{a+b-x} F(w)\,{}_{(a+b-y)}d_q^T w = \int_0^1 w^2\,{}_0d_{\frac{1}{2}}^T w = \frac{5}{14},$$

and

$$F(a)+F(b)-\frac{F(x)+F(y)}{2} = \frac{25}{4}-\frac{1}{4} = 6.$$

Thus, the inequalities (3.5) are verified.

Example 3. If we apply the Theorem 8 to the convex function $F(x) = x^2$ on $[0,2]$ for $q = \frac{1}{2}$, $x = 0$, $y = 1$, we have

$$F\left(a+b-\frac{x+y}{2}\right) = F\left(\frac{3}{2}\right) = \frac{9}{4},$$

$$\frac{1}{y-x}\int_{a+b-y}^{a+b-x} F(w)\,{}^{(a+b-x)}d_q^T w = \int_1^2 w^2\,{}^2d_{\frac{1}{2}}^T w = \frac{33}{14},$$

and

$$F(a)+F(b)-\frac{F(x)+F(y)}{2} = 4-\frac{1}{2} = \frac{7}{2}.$$

Thus, the inequalities (3.9) are verified.

5. Conclusion

In the present work,we proved some new Quantum Hermite-Jensen-Mercer inequalities by using the innovative idea of T_q integrals.Some newly established inequalities could be turned into previous results of literature.It is a very new and interesting inequalities for which researcher can obtain similar inequalities for other kind of convexities in their new work.

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ON QUANTUM HERMITE-JENSEN-MERCER INEQUALITIES

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Abstract. A. M. Mercer prove a new version of well-known Jensen inequality which is called Jensen-Mercer inequality [16]. By using Jensen-Mercer inequality, Kian and Moslehian establish a new variant of Hermite-Hadamard inequality which is called Hermite-Jensen-Mercer inequality [15]. In this paper, we extend the Hermite-Jensen-Mercer inequality for quantum integrals by utilizing Jensen-Mercer inequality. Then we investigate the connections between our results and those in earlier works. Moreover we give some examples to illustrate the main results in this paper. This is the first paper focusing on Hermite-Jensen-Mercer inequalities for quantum integrals.

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1. Introduction

In quantum mathematical research, it is an unlimited analysis of calculus and is known as q - calculus. We get original mathematical formulas for q-calculus as q reaches 1^-. The beginning of the analysis of q-calculus was initiated by Euler (1707-1783). The results mentioned above lead to a thorough investigation into the q-calculus of the twentieth century. The concept of q-calculus is used in many areas in mathematics and physics such as numerical theory, orthogonal polynomials, integration, basic hypergeometric functions, mechanics, and quantum and relativity theory. For more information about q -calculus, one can refer to [5, 11, 13, 14] and references cited therein.

On the other hand, the celebrated discrete Jensen inequality states that: let $0 < \varkappa_1 \leq \varkappa_2 \leq \ldots \leq \varkappa_n$ and let $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ nonnegative weights such that $\Sigma_{i=1}^n \mu = 1$. If ψ is a convex function on the interval $[\kappa_1, \kappa_2]$, then

$$\psi\left(\sum_{i=1}^n \mu_i \varkappa_i\right) \leq \sum_{i=1}^n \mu_i \psi(\varkappa_i),$$

where for all $\varkappa_i \in [\kappa_1, \kappa_2]$ and $\mu_i \in [0,1]$, $(i = \overline{1,n})$ [10]. The Hermite–Hadamard inequality states that, if a mapping $\psi\colon I \subseteq R \to R$ is a convex on I with $\kappa_1, \kappa_2 \in I$ and

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$\kappa_1 < \kappa_2$, then

$$\psi\left(\frac{\kappa_1+\kappa_2}{2}\right) \leq \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\psi(\varkappa)d\varkappa \leq \frac{\psi(\kappa_1)+\psi(\kappa_2)}{2}.$$

The double inequality holds in the reversed direction if ψ is concave [9].

Theorem 1 ([16, Theoem 1.2])**.** *If ψ is convex function on $I=[\kappa_1,\kappa_2]$, then*

$$\psi\left(\kappa_1+\kappa_2-\sum_{i=1}^{n}\mu_i\varkappa_i\right) \leq \psi(\kappa_1)+\psi(\kappa_2)-\sum_{i=1}^{n}\mu_i\psi(\varkappa_i), \tag{1.1}$$

for each $\varkappa_i \in [\kappa_1,\kappa_2]$ and $\mu_i \in [0,1]$, $(i=\overline{1,n})$ with $\sum_{i=1}^{n}\mu=1$.

Inequality (1.1) is known as the Jensen-Mercer inequality. By using inequality (1.1), Kian and Moslehian prove the Hermite-Jensen-Mercer inequality in [15]. Recently several papers have devoted to generalization of Hermite-Jensen-Mercer inequality. For more recent and related results connected with Jensen-Mercer inequality and Hermite-Jensen–Mercer inequality, see [1,6–8,19].

In this study, we extend the Hermite-Jensen-Mercer for quantum integrals. We also present some examples to illustrate our results.

2. Preliminaries of q-calculus and some inequalities

In this section we present some required definitions and related inequalities about q-calculus. Also, here and further we use the following notation(see [14]):

$$[n]_q = \frac{1-q^n}{1-q} = 1+q+q^2+\ldots+q^{n-1}, \quad q\in(0,1).$$

In [13], Jackson gave the q-Jackson integral from 0 to κ_2 for $0<q<1$ as follows:

$$\int_0^{\kappa_2}\psi(\varkappa)\ d_q\varkappa = (1-q)\kappa_2\sum_{n=0}^{\infty}q^n\psi(\kappa_2 q^n)$$

provided the sum converge absolutely.

Jackson in [13] gave the q-Jackson integral in a generic interval $[\kappa_1,\kappa_2]$ as:

$$\int_{\kappa_1}^{\kappa_2}\psi(\varkappa)\ d_q\varkappa = \int_0^{\kappa_2}\psi(\varkappa)\ d_q\varkappa - \int_0^{\kappa_1}\psi(\varkappa)\ d_q\varkappa\ .$$

Definition 1 ([21, Definition 3.3])**.** Let $\psi\colon [\kappa_1,\kappa_2]\to\mathbb{R}$ be a continuous function. Then, the q_{κ_1}-definite integral on $[\kappa_1,\kappa_2]$ is defined as

$$\int_{\kappa_1}^{\kappa_2}\psi(\varkappa)\ {}_{\kappa_1}d_q\varkappa = (1-q)(\kappa_2-\kappa_1)\sum_{n=0}^{\infty}q^n\psi(q^n\kappa_2+(1-q^n)\kappa_1)$$

$$= (\kappa_2 - \kappa_1) \int_0^1 \psi((1-t)\kappa_1 + t\kappa_2) \ d_q t \ .$$

Many of the most well-known inequalities in the classical analysis such as Hölder's inequality, Hermite-Hadamard's inequality and Ostrowski's inequality, Cauchy-Bunyakovsky-Schwarz, Gruss, Gruss-Cebysev and other integral inequalities have been proven and applied for q-calculus . For more q-calculus results please refer [2–5, 12, 17, 18, 20–22].

In [4], Alp et al. proved the following q_{κ_1}-Hermite-Hadamard inequalities for convex functions in the setting of quantum calculus:

Theorem 2 ([4, Theoem 6])**.** *Let $\psi\colon [\kappa_1, \kappa_2] \to \mathbb{R}$ be a convex differentiable function on $[\kappa_1, \kappa_2]$ and $0 < q < 1$. Then q-Hermite-Hadamard inequalities are as follows:*

$$\psi\left(\frac{q\kappa_1 + \kappa_2}{1+q}\right) \leq \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \psi(\varkappa) \ {}_{\kappa_1}d_q\varkappa \ \leq \frac{q\psi(\kappa_1) + \psi(\kappa_2)}{1+q}. \tag{2.1}$$

In [18] and [4], authors established some bounds for left and right hand sides of the inequality (2.1).

On the other hand, Bermudo et al. gave the following new definition and related Hermite-Hadamard type inequalities:

Definition 2 ([5, Definition 6])**.** Let $\psi\colon [\kappa_1, \kappa_2] \to \mathbb{R}$ be a continuous function. Then, the q^{κ_2}-definite integral on $[\kappa_1, \kappa_2]$ is defined as

$$\int_{\kappa_1}^{\kappa_2} \psi(\varkappa) \ {}^{\kappa_2}d_q\varkappa \ = (1-q)(\kappa_2 - \kappa_1) \sum_{n=0}^{\infty} q^n \psi(q^n\kappa_1 + (1-q^n)\kappa_2)$$

$$= (\kappa_2 - \kappa_1) \int_0^1 \psi(t\kappa_1 + (1-t)\kappa_2) \ d_q t \ .$$

Theorem 3 ([5, Theorem 11])**.** *Let $\psi\colon [\kappa_1, \kappa_2] \to \mathbb{R}$ be a convex function on $[\kappa_1, \kappa_2]$ and $0 < q < 1$. Then, q-Hermite-Hadamard inequalities are as follows:*

$$\psi\left(\frac{\kappa_1 + q\kappa_2}{1+q}\right) \leq \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \psi(\varkappa) \ {}^{\kappa_2}d_q\varkappa \ \leq \frac{\psi(\kappa_1) + q\psi(\kappa_2)}{1+q}.$$

From Theorem 2 and Theorem 3, one can get the following inequalities:

Corollary 1 ([5, Corollary 14])**.** *For any convex function $\psi\colon [\kappa_1, \kappa_2] \to \mathbb{R}$ and $0 < q < 1$, we have*

$$\psi\left(\frac{q\kappa_1 + \kappa_2}{1+q}\right) + \psi\left(\frac{\kappa_1 + q\kappa_2}{1+q}\right) \tag{2.2}$$

$$\leq \frac{1}{\kappa_2-\kappa_1}\left\{\int_{\kappa_1}^{\kappa_2}\psi(\varkappa)\ {}_{\kappa_1}d_q\varkappa\ +\int_{\kappa_1}^{\kappa_2}\psi(\varkappa)\ {}^{\kappa_2}d_q\varkappa\ \right\}$$
$$\leq \psi(\kappa_1)+\psi(\kappa_2)$$

and

$$\psi\left(\frac{\kappa_1+\kappa_2}{2}\right)\leq \frac{1}{2(\kappa_2-\kappa_1)}\left\{\int_{\kappa_1}^{\kappa_2}\psi(\varkappa)\ {}_{\kappa_1}d_q\varkappa\ +\int_{\kappa_1}^{\kappa_2}\psi(\varkappa)\ {}^{\kappa_2}d_q\varkappa\ \right\} \tag{2.3}$$
$$\leq \frac{\psi(\kappa_1)+\psi(\kappa_2)}{2}.$$

In the next section, we'll get integral inequalities that are our main results.

3. Quantum Hermite-Jensen-Mercer inequalities

In this section, we prove some Quantum Hermite-Jensen-Mercer type inequalities for quantum integrals.

Theorem 4. *Let* $\psi\colon [\kappa_1,\kappa_2]\to\mathbb{R}$ *be a convex function on* $[\kappa_1,\kappa_2]$ *and* $0<q<1$. *Then we have the following quantum Jensen-Mercer inequalities*

$$\psi\left(\kappa_1+\kappa_2-\frac{\varkappa+\gamma}{2}\right) \tag{3.1}$$
$$\leq \psi(\kappa_1)+\psi(\kappa_2)-\frac{1}{2}\left[\frac{1}{\gamma-\varkappa}\int_{\varkappa}^{\gamma}\psi(\upsilon)\ {}^{\gamma}d_q\upsilon+\frac{1}{\gamma-\varkappa}\int_{\varkappa}^{\gamma}\psi(\upsilon)\ {}_{\varkappa}d_q\upsilon\right]$$
$$\leq \psi(\kappa_1)+\psi(\kappa_2)-\frac{1}{2}\left[\psi\left(\frac{\varkappa+q\gamma}{[2]_q}\right)+\psi\left(\frac{q\varkappa+\gamma}{[2]_q}\right)\right]$$
$$\leq \psi(\kappa_1)+\psi(\kappa_2)-\psi\left(\frac{\varkappa+\gamma}{2}\right)$$

for all $\varkappa,\gamma\in[\kappa_1,\kappa_2]$ *with* $\varkappa<\gamma$.

Proof. It follows from the Jensen-Mercer inequality,

$$\psi\left(\kappa_1+\kappa_2-\frac{\upsilon+\nu}{2}\right)\leq \psi(\kappa_1)+\psi(\kappa_2)-\frac{1}{2}\left[\psi(\upsilon)+\psi(\nu)\right]$$

for $\upsilon,\nu\in[\kappa_1,\kappa_2]$. By choosing $\upsilon=t\varkappa+(1-t)\gamma$ and $\nu=t\gamma+(1-t)\varkappa$ for $t\in[0,1]$, we obtain

$$\psi\left(\kappa_1+\kappa_2-\frac{\upsilon+\nu}{2}\right)\leq \psi(\kappa_1)+\psi(\kappa_2) \tag{3.2}$$
$$-\frac{1}{2}\left[\psi(t\varkappa+(1-t)\gamma)+\psi(t\gamma+(1-t)\varkappa)\right].$$

By taking q-integral of both sides of (3.2), we have

$$\psi\left(\kappa_1+\kappa_2-\frac{\upsilon+\nu}{2}\right)\leq\psi(\kappa_1)+\psi(\kappa_2) \tag{3.3}$$
$$-\frac{1}{2}\left[\int_0^1\psi(t\varkappa+(1-t)\gamma)\,d_qt+\int_0^1\psi(t\gamma+(1-t)\varkappa)\,d_qt\right].$$

From Definition 1 and Theorem 2, we get

$$\int_0^1\psi(t\gamma+(1-t)\varkappa)\,d_qt=\frac{1}{\gamma-\varkappa}\int_\varkappa^\gamma\psi(\upsilon)\;{}_\varkappa d_q\upsilon\geq\psi\left(\frac{q\varkappa+\gamma}{[2]_q}\right) \tag{3.4}$$

and similarly by Definition 2 and Theorem 3, we have

$$\int_0^1\psi(t\varkappa+(1-t)\gamma)\,d_qt=\frac{1}{\gamma-\varkappa}\int_\varkappa^\gamma\psi(\upsilon)\;{}^\gamma d_q\upsilon\geq\psi\left(\frac{\varkappa+q\gamma}{[2]_q}\right). \tag{3.5}$$

By substituting the inequalities (3.4) and (3.5) in (3.3), we obtain

$$\begin{aligned}&\psi\left(\kappa_1+\kappa_2-\frac{\varkappa+\gamma}{2}\right)\\&\leq\psi(\kappa_1)+\psi(\kappa_2)-\frac{1}{2}\left[\frac{1}{\gamma-\varkappa}\int_\varkappa^\gamma\psi(\upsilon)\;{}^\gamma d_q\upsilon+\frac{1}{\gamma-\varkappa}\int_\varkappa^\gamma\psi(\upsilon)\;{}_\varkappa d_q\upsilon\right]\\&\leq\psi(\kappa_1)+\psi(\kappa_2)-\frac{1}{2}\left[\psi\left(\frac{\varkappa+q\gamma}{[2]_q}\right)+\psi\left(\frac{q\varkappa+\gamma}{[2]_q}\right)\right]\end{aligned}$$

which gives the proof of first and second inequalities in (3.1).

Since ψ is a convex function we have

$$\psi\left(\frac{\varkappa+\gamma}{2}\right)=\psi\left(\frac{1}{2}\frac{\varkappa+q\gamma}{[2]_q}+\frac{1}{2}\frac{q\varkappa+\gamma}{[2]_q}\right)\leq\frac{1}{2}\left[\psi\left(\frac{\varkappa+q\gamma}{[2]_q}\right)+\psi\left(\frac{q\varkappa+\gamma}{[2]_q}\right)\right].$$

This proves the last inequality in (3.1). □

Remark 1. If we take the limit $q\to1^-$ in Theorem 4, then inequalities (3.1) reduce to inequalities (2.1) of Theorem 2.1 in [15].

Corollary 2. *If we choose $\varkappa=\kappa_1$ and $\gamma=\kappa_2$ in Theorem 4, we have*

$$\psi\left(\frac{\kappa_1+\kappa_2}{2}\right)\leq\psi(\kappa_1)+\psi(\kappa_2)-\frac{1}{2(\kappa_2-\kappa_1)}\left[\int_{\kappa_1}^{\kappa_2}\psi(\upsilon)\;{}^{\kappa_2}d_q\upsilon+\int_{\kappa_1}^{\kappa_2}\psi(\upsilon)\;{}_{\kappa_1}d_q\upsilon\right]$$

$$\leq \psi(\kappa_1)+\psi(\kappa_2)-\frac{1}{2}\left[\psi\left(\frac{\kappa_1+q\kappa_2}{[2]_q}\right)+\psi\left(\frac{q\kappa_1+\kappa_2}{[2]_q}\right)\right]$$
$$\leq \psi(\kappa_1)+\psi(\kappa_2)-\psi\left(\frac{\kappa_1+\kappa_2}{2}\right).$$

This inequalities give the first inequalities in (2.2)and (2.3).

Theorem 5. *Let $\psi\colon [\kappa_1,\kappa_2]\to\mathbb{R}$ be a convex function on $[\kappa_1,\kappa_2]$ and $0<q<1$. Then we have the following quantum Jensen-Mercer inequalities*

$$\psi\left(\kappa_1+\kappa_2-\frac{\varkappa+q\gamma}{[2]_q}\right)\leq\frac{1}{\gamma-\varkappa}\int_{\kappa_1+\kappa_2-\gamma}^{\kappa_1+\kappa_2-\varkappa}\psi(\upsilon)\,{}_{(\kappa_1+\kappa_2-\gamma)}d_q\upsilon \tag{3.6}$$
$$\leq \psi(\kappa_1)+\psi(\kappa_2)-\frac{\psi(\varkappa)+q\psi(\gamma)}{[2]_q}$$

for all $\varkappa,\gamma\in[\kappa_1,\kappa_2]$ with $\varkappa<\gamma$, where $0<q<1$.

Proof. From Definition 1 and Theorem 2, we get

$$\int_0^1\psi(\kappa_1+\kappa_2-(t\varkappa+(1-t)\gamma))\,d_qt$$
$$=\int_0^1\psi(t(\kappa_1+\kappa_2-\varkappa)+(1-t)(\kappa_1+\kappa_2-\gamma))\,d_qt$$
$$=\frac{1}{\gamma-\varkappa}\int_{\kappa_1+\kappa_2-\gamma}^{\kappa_1+\kappa_2-\varkappa}\psi(\upsilon)\,{}_{(\kappa_1+\kappa_2-\gamma)}d_q\upsilon$$
$$\geq\psi\left(\frac{q(\kappa_1+\kappa_2-\gamma)+\kappa_1+\kappa_2-\varkappa}{[2]_q}\right)$$
$$=\psi\left(\kappa_1+\kappa_2-\frac{\varkappa+q\gamma}{[2]_q}\right)$$

which gives prof of first inequality in (3.6). By Jensen-Mercer inequality, we obtain

$$\frac{1}{\gamma-\varkappa}\int_{\kappa_1+\kappa_2-\gamma}^{\kappa_1+\kappa_2-\varkappa}\psi(\upsilon)\,{}_{(\kappa_1+\kappa_2-\gamma)}d_q\upsilon=\int_0^1\psi(\kappa_1+\kappa_2-(t\varkappa+(1-t)\gamma))\,d_qt$$
$$\leq\int_0^1\psi(\kappa_1)+\psi(\kappa_2)-[t\psi(\varkappa)+(1-t)\psi(\gamma)]\,d_qt$$

$$= \psi(\kappa_1) + \psi(\kappa_2) - \left[\frac{\psi(\varkappa) + q\psi(\gamma)}{[2]_q}\right].$$

This completes the proof. □

Remark 2. If we take the limit $q \to 1^-$ in Theorem 5, then inequalities (3.6) reduce to inequalities (2.2) of Theorem 2.1 in [15]..

Remark 3. If we choose $\varkappa = \kappa_1$ and $\gamma = \kappa_2$ in Theorem 5, then Theorem 5 reduces to Theorem 2.

Theorem 6. *Let* $\psi\colon [\kappa_1, \kappa_2] \to \mathbb{R}$ *be a convex function on* $[\kappa_1, \kappa_2]$ *and* $0 < q < 1$. *Then we have the following quantum Jensen-Mercer inequalities*

$$\psi\left(\kappa_1 + \kappa_2 - \frac{q\varkappa + \gamma}{[2]_q}\right) \leq \frac{1}{\gamma - \varkappa} \int_{\kappa_1+\kappa_2-\gamma}^{\kappa_1+\kappa_2-\varkappa} \psi(\upsilon)\ {}^{(\kappa_1+\kappa_2-\varkappa)}d_q\upsilon \tag{3.7}$$

$$\leq \psi(\kappa_1) + \psi(\kappa_2) - \frac{q\psi(\varkappa) + \psi(\gamma)}{[2]_q}$$

for all $\varkappa, \gamma \in [\kappa_1, \kappa_2]$ *with* $\varkappa < \gamma$, *where* $0 < q < 1$.

Proof. From Definition 2 and Theorem 3, we get

$$\int_0^1 \psi(\kappa_1 + \kappa_2 - (t\gamma + (1-t)\varkappa))\, d_q t$$

$$= \int_0^1 \psi(t(\kappa_1 + \kappa_2 - \gamma) + (1-t)(\kappa_1 + \kappa_2 - \varkappa))\, d_q t$$

$$= \frac{1}{\gamma - \varkappa} \int_{\kappa_1+\kappa_2-\gamma}^{\kappa_1+\kappa_2-\varkappa} \psi(\upsilon)\ {}^{(\kappa_1+\kappa_2-\varkappa)}d_q\upsilon$$

$$\geq \psi\left(\frac{q(\kappa_1 + \kappa_2 - \varkappa) + \kappa_1 + \kappa_2 - \gamma}{[2]_q}\right)$$

$$= \psi\left(\kappa_1 + \kappa_2 - \frac{\gamma + q\varkappa}{[2]_q}\right)$$

which proves the first inequality in (3.7). By Jensen-Mercer Inequality, we have

$$\int_0^1 \psi(\kappa_1 + \kappa_2 - (t\gamma + (1-t)\varkappa))\, d_q t \leq \int_0^1 \psi(\kappa_1) + \psi(\kappa_2) - [t\psi(\gamma) + (1-t)\psi(\varkappa)]\, d_q t$$

$$= \psi(\kappa_1) + \psi(\kappa_2) - \left[\frac{\psi(\gamma) + q\psi(\varkappa)}{[2]_q}\right]$$

which completes the proof. □

Remark 4. If we take the limit $q \to 1^-$ in Theorem 6, then inequalities (3.7) reduce to inequalities (2.2) of Theorem 2.1 in [15].

Remark 5. If we choose $\varkappa = \kappa_1$ and $\gamma = \kappa_2$ in Theorem 6, then Theorem 6 reduces to Theorem 3.

4. Examples

We provide some examples of our main theorems in this section.

Example 1. We define a convex function $\psi\colon [\kappa_1, \kappa_2] = [-1, 2] \to \mathbb{R}$ by $\psi(x) = x^2$ for $\varkappa = 0$, $\gamma = 1$ and $q = \frac{1}{2}$. Then we have

$$\psi\left(\kappa_1 + \kappa_2 - \frac{\varkappa + \gamma}{2}\right) = \psi\left(\frac{1}{2}\right) = \frac{1}{4}, \tag{4.1}$$

and

$$\psi(\kappa_1) + \psi(\kappa_2) = 5,$$

$$\frac{1}{\gamma - \varkappa}\int_{\varkappa}^{\gamma} \psi(\upsilon)\, {}^{\gamma}d_q\upsilon = \int_0^1 \psi(\upsilon)\, {}^{1}d_{\frac{1}{2}}\upsilon = \frac{5}{21}$$

$$\frac{1}{\gamma - \varkappa}\int_{\varkappa}^{\gamma} \psi(\upsilon)\, {}_{\varkappa}d_q\upsilon = \int_0^1 \psi(\upsilon)\, {}_{0}d_{\frac{1}{2}}\upsilon = \frac{4}{7}.$$

Thus,

$$\psi(\kappa_1) + \psi(\kappa_2) - \frac{1}{2}\left[\frac{1}{\gamma - \varkappa}\int_{\varkappa}^{\gamma} \psi(\upsilon)\, {}^{\gamma}d_q\upsilon + \frac{1}{\gamma - \varkappa}\int_{\varkappa}^{\gamma} \psi(\upsilon)\, {}_{\varkappa}d_q\upsilon\right] = \frac{193}{42}. \tag{4.2}$$

Now, we observe that

$$\psi\left(\frac{\varkappa + q\gamma}{[2]_q}\right) = \psi\left(\frac{1}{3}\right) = \frac{1}{9} \qquad \psi\left(\frac{q\varkappa + \gamma}{[2]_q}\right) = \psi\left(\frac{2}{3}\right) = \frac{4}{9}.$$

Thus,

$$\psi(\kappa_1) + \psi(\kappa_2) - \frac{1}{2}\left[\psi\left(\frac{\varkappa + q\gamma}{[2]_q}\right) + \psi\left(\frac{q\varkappa + \gamma}{[2]_q}\right)\right] = \frac{85}{18}. \tag{4.3}$$

Again, we observe that

$$\psi(\kappa_1) + \psi(\kappa_2) - \psi\left(\frac{\varkappa + \gamma}{2}\right) = \frac{19}{4} \tag{4.4}$$

Finally, we conclude from the equalities (4.1–4.4) that the inequality (3.1) is valid and

$$\frac{1}{4} < \frac{193}{42} < \frac{85}{18} < \frac{19}{4}.$$

Example 2. We also consider the convex function $\psi\colon [-1,2] \to \mathbb{R}$ by $\psi(x) = x^2$ with $\varkappa = 0$, $\gamma = 1$ and $q = \frac{1}{2}$. Then we get

$$\psi\left(\kappa_1 + \kappa_2 - \frac{\varkappa + q\gamma}{[2]_q}\right) = \frac{4}{9}, \tag{4.5}$$

$$\frac{1}{\gamma - \varkappa} \int_{\kappa_1+\kappa_2-\gamma}^{\kappa_1+\kappa_2-\varkappa} \psi(\upsilon)\ {}_{(\kappa_1+\kappa_2-\gamma)}d_q\upsilon = \int_0^1 \psi(\upsilon)\ {}_0d_{\frac{1}{2}}\upsilon = \frac{4}{7} \tag{4.6}$$

and

$$\psi(\kappa_1) + \psi(\kappa_2) - \frac{\psi(\varkappa) + q\psi(\gamma)}{[2]_q} = \frac{14}{3}. \tag{4.7}$$

By the equalities (4.5–4.7), the inequality (3.6) is valid and

$$\frac{4}{9} < \frac{4}{7} < \frac{14}{3}.$$

Example 3. Let's consider again the convex function $\psi\colon [-1,2] \to \mathbb{R}$, $\psi(x) = x^2$ with $\varkappa = 0$, $\gamma = 1$ and $q = \frac{1}{2}$. Then we have

$$\psi\left(\kappa_1 + \kappa_2 - \frac{\varkappa + q\gamma}{[2]_q}\right) = \frac{1}{9}, \tag{4.8}$$

$$\frac{1}{\gamma - \varkappa} \int_{\kappa_1+\kappa_2-\gamma}^{\kappa_1+\kappa_2-\varkappa} \psi(\upsilon)\ {}^{(\kappa_1+\kappa_2-\varkappa)}d_q\upsilon = \int_0^1 \psi(\upsilon)\ {}^1d_{\frac{1}{2}}\upsilon = \frac{5}{21} \tag{4.9}$$

and

$$\psi(\kappa_1) + \psi(\kappa_2) - \frac{\psi(\varkappa) + q\psi(\gamma)}{[2]_q} = \frac{13}{3}. \tag{4.10}$$

From the equalities (4.8–4.10), the inequality (3.7) is valid and

$$\frac{1}{9} < \frac{5}{21} < \frac{13}{3}.$$

5. Conclusion

In this research paper, we generalize Hermite-Jensen-Mercer inequality for quantum integrals. This is the first study obtaining Hermite-Jensen-Mercer inequalities for quantum integrals. In the future works, authors can try to generalize our results by utilizing different kinds of convex function classes.

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BLOCK REPRESENTATIONS FOR THE g-DRAZIN INVERSE IN BANACH ALGEBRAS

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Abstract. We present new formulas for the g-Drazin inverse of the sum in a Banach algebra. The block representations for the g-Drazin inverse of a 2×2 block operator matrix are thereby established. These extend the known results obtained in [3, 7].

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1. Introduction

Let $\mathcal{A}$ be a complex Banach algebra with an identity 1. The commutant of $a\in\mathcal{A}$ is defined by $\mathrm{comm}(a)=\{x\in\mathcal{A}\mid xa=ax\}$. Let $\mathcal{A}^{\mathrm{qnil}}=\{a\in\mathcal{A}\ \mid\ 1-\mathrm{comm}(a)a\subseteq\mathcal{A}^{-1}\}$. As is well known, we have $a\in\mathcal{A}^{\mathrm{qnil}}\Leftrightarrow\lim\limits_{n\to\infty}\|a^n\|^{\frac{1}{n}}=0$. We say that $a\in\mathcal{A}$ has g-Drazin inverse (i.e. generalized Drazin inverse) if there exists $b\in\mathrm{comm}(a)$ such that $b=bab, a-a^2b\in\mathcal{A}^{\mathrm{qnil}}$. Such b is unique, if it exists, and it is denoted by a^d. When we have in the preceding $a-a^2b$ is nilpotent, then b is the Drazin inverse a^D of a. The Drazin and g-Drazin inverse are useful in matrix and operator theory. It has been applied in many fields such as ordinary differential equations, statistics and probability, Marcov chain, etc (see [2]).

The g-Drazin inverse of the sum of two elements in a Banach algebra has been studied by many authors, e.g. [3, 6, 8, 10, 11] and [12]. In [9, Theorem 2.1], Yang and Liu gave the representation of the Drazin inverse of two complex matrices P and Q such that $PQP=0, PQ^2=0$, which recovered the case $PQ=0$ studied by Hartwig et al. In [7], Ljubisavljevic and Cvetkovic-Ilic derived an expression of $(P+Q)^D$ under a weaker condition $PQP^2=0, PQPQ=0, PQ^2P=0$ and $PQ^3=0$. As the computational and applied requirements, the explicit representation of the g-Drazin inverse is expected under certain weaker conditions. This inspires us to derive the explicit formulas of $(a+b)^d$ in terms of a,b and their g-Drazin inverses in

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Banach algebras. In Section 2, we give the explicit representations for $(a+b)^d$ under the conditions $aba^2=0, abab=0, ab^2a=0$ and $ab^3=0$ in a Banach algebra $\mathcal{A}$. Furthermore, the representation of $(a+b)^d$ under the conditions $a^2ba=0, abab=0, ab^2a=0$ and $ab^3=0$ is presented. The known results are thereby extended to wider cases (see [7, Corollary 2.3] and [9, Theorem 2.1]).

Let $\mathcal{L}(X)$ denote the set of all bounded linear operators on a Banach space X. Let

$$M=\begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad (*)$$

where $A\in\mathcal{L}(X), D\in\mathcal{L}(Y)$ have g-Drain inverses and X,Y are complex Banach spaces. Then M is a bounded linear operator on $X\oplus Y$. In Section 3, we apply our results and give the representations for the g-Drazin inverse of some operator matrix M whose Shur complement is zero, i.e. $D=CA^dB$. As Drazin and g-Drazin inverses coincide with each other for complex matrices, our results also extend [1, Theorem 4.1 and Theorem 4.2] with alternative formulas for the Drazin inverse of 2×2 block complex matrices.

2. ADDITIVE RESULTS

In [3, Theorem 2.2 and Theorem 2.4], the authors investigated the g-Drazin invertibility of $a+b$ under the preceding conditions. The purpose of this section is to further study such problems and establish the explicit formulas of the g-Drazin inverse of the sum. If $p=p^2\in\mathcal{A}$ is an idempotent, We can represent $a\in\mathcal{A}$ as $a=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_p$, where $a_{11}=pap, a_{12}=pa(1-p), a_{21}=(1-p)ap$ and $a_{22}=(1-p)a(1-p)$. We use a^π to stand for the spectral idempotent $1-aa^d$ of $a\in\mathcal{A}^d$. We begin with

Lemma 1. *Let*

$$x=\begin{pmatrix} a & 0 \\ c & b \end{pmatrix}_p \text{ or } \begin{pmatrix} b & c \\ 0 & a \end{pmatrix}_p$$

Then

$$x^d=\begin{pmatrix} a^d & 0 \\ z & b^d \end{pmatrix}_p \text{ or } \begin{pmatrix} b^d & z \\ 0 & a^d \end{pmatrix}_p,$$

where

$$z=\sum_{i=0}^{\infty}(b^d)^{i+2}ca^ia^\pi+\sum_{i=0}^{\infty}b^ib^\pi c(a^d)^{i+2}-b^dca^d.$$

Proof. See [6, Theorem 2.1]. □

Lemma 2. *Let $a,b\in\mathcal{A}^d$. If $ab=0$, then $a+b\in\mathcal{A}^d$. In this case*

$$(a+b)^d=\sum_{i=0}^{\infty}b^ib^\pi(a^d)^{i+1}+\sum_{i=0}^{\infty}(b^d)^{i+1}a^ia^\pi.$$

Proof. See [4, Lemma 2.2]. □

We are now ready to extend [3, Theorem 2.2] and prove:

Theorem 1. *Let $a,b \in \mathcal{A}^d$. If $aba^2 = 0, abab = 0, ab^2a = 0$ and $ab^3 = 0$, then*

$$\begin{aligned}(a+b)^d = {} & a^d + (a^d)^3ba + (a^d)^2b + (a^d)^3b^2 \\ & + b(a^d)^2 + b(a^d)^4ba + b(a^d)^3b + b(a^d)^4b^2 + a(b^d)^9aba + (b^d)^8aba \\ & + a(b^d)^9a^2b + a(b^d)^8ab + a(b^d)^9ab^2 + (b^d)^8a^2b + (b^d)^7ab + (b^d)^8ab^2 \\ & + (ab+b^2)z(a + a^dba + aa^db + a^db^2) \\ & + \big(a(b^d)^4 + (b^d)^3\big)z(a^2ba + a^3b + a^2b^2),\end{aligned}$$

where

$$\begin{aligned} z = {} & -b^d(a^d)^3 + b^\pi(a^d)^4 + bb^\pi(a^d)^5 + (b^d)^4a^\pi \\ & - (b^d)^5a^d - (b^d)^4(a^d)^2 + \sum_{n=1}^{\infty} u_n, \\ u_n = {} & b^{3n-1}b^\pi(a^d)^{3n+3} + b^{3n}b^\pi(a^d)^{3n+4} + b^{3n+1}b^\pi(a^d)^{3n+5} \\ & + (b^d)^{3n+4}a^{3n}a^\pi + (b^d)^{3n+3}a^{3n-1}a^\pi + (b^d)^{3n+2}a^{3n-2}a^\pi. \end{aligned}$$

Proof. Set

$$M = \begin{pmatrix} a^3 + a^2b + aba + ab^2 & a^3b + a^2b^2 + abab + ab^3 \\ a^2 + ab + ba + b^2 & a^2b + ab^2 + bab + b^3 \end{pmatrix}.$$

Then $M = G + F$, where

$$G = \begin{pmatrix} a^2b + aba + ab^2 & a^3b + a^2b^2 \\ 0 & a^2b + ab^2 + bab \end{pmatrix}, \quad F = \begin{pmatrix} a^3 & 0 \\ a^2 + ab + ba + b^2 & b^3 \end{pmatrix}.$$

Clearly $F = A + B$, where

$$A = \begin{pmatrix} 0 & 0 \\ ab & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} a^3 & 0 \\ a^2 + ba + b^2 & b^3 \end{pmatrix}.$$

Moreover we have $B = H + K$, where

$$H = \begin{pmatrix} a^3 & 0 \\ a^2 + ba & 0 \end{pmatrix} \text{ and } K = \begin{pmatrix} 0 & 0 \\ b^2 & b^3 \end{pmatrix}.$$

In view of [3, Theorem 2.2], we have

$$(a+b)^d = (a+b, ab+b^2)M^d \begin{pmatrix} a \\ 1 \end{pmatrix},$$

where $M^d = F^d + (F^d)^2G$ and

$$F^d = B^d + (B^d)^2A,$$

$$B^d = (I-KK^d)\big[\sum_{n=0}^{\infty} K^n(H^d)^n\big]H^d + K^d\big[\sum_{n=0}^{\infty}(K^d)^nH^n\big](I-HH^d),$$
$$H^d = \begin{pmatrix} (a^d)^3 & 0 \\ (a^d)^4+b(a^d)^5 & 0 \end{pmatrix},$$
$$K^d = \begin{pmatrix} 0 & 0 \\ (b^d)^4 & (b^d)^3 \end{pmatrix}.$$

We compute that

$$I-HH^d = \begin{pmatrix} a^\pi & 0 \\ -(a+b)(a^d)^2 & 1 \end{pmatrix};$$
$$(H-H^2H^d)^n = \begin{pmatrix} a^{3n}a^\pi & 0 \\ (a+b)a^{3n-2}a^\pi & 0 \end{pmatrix};$$
$$I-KK^d = \begin{pmatrix} 1 & 0 \\ -b^d & b^\pi \end{pmatrix},$$
$$(K-K^2K^d)^n = \begin{pmatrix} 0 & 0 \\ b^{3n-1}b^\pi & b^{3n}b^\pi \end{pmatrix} \quad n\in\mathbb{N}.$$

Moreover we have

$$(H^d)^{n+1} = \begin{pmatrix} (a^d)^{3n+3} & 0 \\ (a^d)^{3n+4}+b(a^d)^{3n+5} & 0 \end{pmatrix}, \quad (K^d)^{n+1} = \begin{pmatrix} 0 & 0 \\ (b^d)^{3n+4} & (b^d)^{3n+3} \end{pmatrix}.$$

Therefore we have

$$\begin{aligned} B^d &= (I-KK^d)H^d + \sum_{n=1}^{\infty}(K-K^2K^d)^n(H^d)^{n+1} \\ &\quad + K^d(I-HH^d) + \sum_{n=1}^{\infty}(K^d)^{n+1}(H-H^2H^d)^n \\ &= \begin{pmatrix} (a^d)^3 & 0 \\ z & (b^d)^5 \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} z &= -b^d(a^d)^3 + b^\pi(a^d)^4 + bb^\pi(a^d)^5 + (b^d)^4a^\pi \\ &\quad - (b^d)^5a^d - (b^d)^4(a^d)^2 + \sum_{n=1}^{\infty} u_n, \\ u_n &= b^{3n-1}b^\pi(a^d)^{3n+3} + b^{3n}b^\pi(a^d)^{3n+4} + b^{3n+1}b^\pi(a^d)^{3n+5} \\ &\quad + (b^d)^{3n+4}a^{3n}a^\pi + (b^d)^{3n+3}a^{3n-1}a^\pi + (b^d)^{3n+2}a^{3n-2}a^\pi. \end{aligned}$$

Then we have

$$F^d = B^d(I+B^dA)$$

$$= \begin{pmatrix} (a^d)^3 & 0 \\ z & (b^d)^5 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ (b^d)^5 ab & 1 \end{pmatrix} = \begin{pmatrix} (a^d)^3 & 0 \\ z+(b^d)^{10}ab & (b^d)^5 \end{pmatrix}.$$

Moreover we have

$$M^d = F^d(I+F^dG) = \begin{pmatrix} (a^d)^3 & 0 \\ z+(b^d)^{10}ab & (b^d)^5 \end{pmatrix}$$
$$\begin{pmatrix} 1+a^db+(a^d)^2ba+(a^d)^2b^2 & aa^db+a^db^2 \\ z(a^2b+aba+ab^2) & z(a^3b+a^2b^2)+(b^d)^5(a^2b+bab+ab^2) \end{pmatrix}.$$

Therefore $(a+b)^d = (a+b, ab+b^2)M^d \begin{pmatrix} a \\ 1 \end{pmatrix} = vw$, where

$$\begin{aligned} v &= (a+b, ab+b^2) \begin{pmatrix} (a^d)^3 & 0 \\ z+(b^d)^{10}ab & (b^d)^5 \end{pmatrix} \\ &= \big((a^d)^2+b(a^d)^3+(a+b)bz+(a+b)(b^d)^9ab, (a+b)(b^d)^4\big), \\ w &= \begin{pmatrix} 1+a^db+(a^d)^2ba+(a^d)^2b^2 & aa^db+a^db^2 \\ z(a^2b+aba+ab^2) & z(a^3b+a^2b^2)+(b^d)^5(a^2b+bab+ab^2) \end{pmatrix} \\ &\quad \begin{pmatrix} a \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} a+a^dba+aa^db+a^db^2 \\ z(a^2ba+a^3b+a^2b^2)+(b^d)^5(a^2b+bab+ab^2) \end{pmatrix}. \end{aligned}$$

By the direct computation, we complete the proof. □

Bu and Zhang considered the Drazin inverses of block complex matrices under the conditions $PQP^2 = 0, QPQP = 0$ and $Q^2 = 0$ (see [1, Remark 3.1]). As an immediate consequence of Theorem 1, we now derive

Corollary 1. *Let $a, b \in \mathcal{A}^d$. If $aba^2 = 0, abab = 0$ and $b^2 = 0$, then*

$$(a+b)^d = a^d + (a^d)^2b + (a^d)^3ba + b(a^d)^2 + b(a^d)^3b + b(a^d)^4ba.$$

Proof. Using the notations in Theorem 1, we have $u_n = 0, z = (a^d)^4 + b(a^d)^5$. In view of Theorem 1, $a+b$ has g-Drazin inverse and

$$\begin{aligned} (a+b)^d &= a^d + (a^d)^3ba + (a^d)^2b + b(a^d)^2 + b(a^d)^4ba + b(a^d)^3b \\ &\quad + abz(a+a^dba+aa^db) \\ &= a^d + (a^d)^2b + (a^d)^3ba + b(a^d)^2 + b(a^d)^3b + b(a^d)^4ba, \end{aligned}$$

as asserted. □

Using the previous corollary, we now extend [3, Theorem 2.4].

Theorem 2. *Let $a,b\in\mathcal{A}^d$. If $a^2ba=0, abab=0, ab^2a=0$ and $ab^3=0$, then*

$$\begin{aligned}(a+b)^d=&\left[(a+b)^2b\Gamma+(a+b)\Lambda\right]\left[\Gamma+\Delta(a+b)a\right]\\&+\left[(a+b)^2b\Delta+(a+b)\Xi\right]\left[\Lambda+\Xi(a+b)a\right],\end{aligned}$$

where

$$\begin{aligned}\Gamma&=\alpha^d+z_2\beta\alpha+\alpha^dz_2\beta\alpha^2+z_1\beta^d\alpha^2,\\\Delta&=z_1+\alpha^dz_2\beta\alpha+z_1\beta^d\alpha,\\\Lambda&=\beta^d\alpha+(\beta^d)^2\alpha^2+\beta\alpha^d+\beta\alpha^dz_1\beta\alpha+\beta\alpha z_2\beta^d\beta\alpha\\&\quad+\beta\alpha^dz_2\beta\alpha^2+\beta\alpha z_2\beta^d\alpha^2,\\\Xi&=\beta^d+(\beta^d)^2\alpha+\beta\alpha z_2+\beta\alpha^dz_2\beta\alpha+\beta\alpha z_2\beta^d\alpha,\\\alpha&=b^2+ab,\beta=ba+a^2,\\\alpha^d&=(b^d)^2+(b^d)^4ab,\beta^d=(a^d)^2+b(a^d)^3+bab(a^d)^5,\\z_1&=\sum_{i=0}^{\infty}(\alpha^d)^{i+2}\beta^i\beta^\pi+\sum_{i=0}^{\infty}\alpha^i\alpha^\pi(\beta^d)^{i+2}-\alpha^d\beta^d,\\z_2&=\alpha^dz_1+z_1\beta^d.\end{aligned}$$

Proof. Let $\alpha=b^2+ab$ and $\beta=ba+a^2$. Since $ab^3=0$, it follows by Lemma 2 that α has g-Drazin inverse and $\alpha^d=(b^d)^2+(b^d)^4ab$. Since $a^2ba=0$, similarly, we have $\beta^d=(a^d)^2+b(a^d)^3+bab(a^d)^5$. Set

$$M=\begin{pmatrix}b^2+ab&1\\(a+b)a(a+b)b&ba+a^2\end{pmatrix}.$$

Then $M=A+B$, where

$$A=\begin{pmatrix}\alpha&1\\0&\beta\end{pmatrix}\text{ and }B=\begin{pmatrix}0&0\\\beta\alpha&0\end{pmatrix}.$$

In view of Lemma 1, $A^d=\begin{pmatrix}\alpha^d&z_1\\0&\beta^d\end{pmatrix}$, where

$$z_1=\sum_{i=0}^{\infty}(\alpha^d)^{i+2}\beta^i\beta^\pi+\sum_{i=0}^{\infty}\alpha^i\alpha^\pi(\beta^d)^{i+2}-\alpha^d\beta^d.$$

Let $z_2=\alpha^dz_1+z_1\beta^d$. Then

$$(A^d)^2=\begin{pmatrix}(\alpha^d)^2&z_2\\0&(\beta^d)^2\end{pmatrix}.$$

Clearly $B^2=0$, and so $B^d=0$. We compute that

$$(A^d)^2B=\begin{pmatrix}z_2\beta\alpha&0\\\beta^d\alpha&0\end{pmatrix},$$

$$(A^d)^3BA = \begin{pmatrix} \alpha^d z_2\beta\alpha^2 + z_1\beta^d\alpha^2 & \alpha^d z_2\beta\alpha + z_1\beta^d\alpha \\ (\beta^d)^2\alpha^2 & (\beta^d)^2\alpha \end{pmatrix},$$

$$B(A^d)^2 = \begin{pmatrix} 0 & 0 \\ \beta\alpha^d & \beta\alpha z_2 \end{pmatrix},$$

$$B(A^d)^3B = \begin{pmatrix} 0 & 0 \\ \beta\alpha^d z_1\beta\alpha + \beta\alpha z_2\beta^d\beta\alpha & 0 \end{pmatrix},$$

$$B(A^d)^4BA = \begin{pmatrix} 0 & 0 \\ \beta\alpha^d z_2\beta\alpha^2 + \beta\alpha z_2\beta^d\alpha^2 & \beta\alpha^d z_2\beta\alpha + \beta\alpha z_2\beta^d\alpha \end{pmatrix}.$$

We easily check that $ABAB = 0, ABA^2 = 0$ and $B^2 = 0$. In view of Corollary 1, we have

$$M^d = A^d + (A^d)^2B + (A^d)^3BA + B(A^d)^2 + B(A^d)^3B + B(A^d)^4BA = \begin{pmatrix} \Gamma & \Delta \\ \Lambda & \Xi \end{pmatrix},$$

where

$$\begin{aligned} \Gamma &= \alpha^d + z_2\beta\alpha + \alpha^d z_2\beta\alpha^2 + z_1\beta^d\alpha^2, \\ \Delta &= z_1 + \alpha^d z_2\beta\alpha + z_1\beta^d\alpha, \\ \Lambda &= \beta^d\alpha + (\beta^d)^2\alpha^2 + \beta\alpha^d + \beta\alpha^d z_1\beta\alpha + \beta\alpha z_2\beta^d\beta\alpha \\ &\quad + \beta\alpha^d z_2\beta\alpha^2 + \beta\alpha z_2\beta^d\alpha^2, \\ \Xi &= \beta^d + (\beta^d)^2\alpha + \beta\alpha z_2 + \beta\alpha^d z_2\beta\alpha + \beta\alpha z_2\beta^d\alpha, \end{aligned}$$

Let $N = \left(\begin{pmatrix} 1 \\ a \end{pmatrix}(b,1)\right)^2$. Then $N = \begin{pmatrix} b^2+ab & a+b \\ ab^2+a^2b & a^2+ab \end{pmatrix}$. As in the proof of [3, Theorem 2.4], we have

$$M = \begin{pmatrix} 1 & 0 \\ 0 & a+b \end{pmatrix}\begin{pmatrix} (a+b)b & 1 \\ a(a+b)b & a \end{pmatrix}, \quad N = \begin{pmatrix} (a+b)b & 1 \\ a(a+b)b & a \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & a+b \end{pmatrix}.$$

By using Cline's formula (see [5, Theorem 2.9]), N has g-Drazin inverse and

$$N^d = \begin{pmatrix} (a+b)b & 1 \\ a(a+b)b & a \end{pmatrix}[M^d]^2\begin{pmatrix} 1 & 0 \\ 0 & a+b \end{pmatrix}.$$

In view of [5, Theorem 2.7], $Q := \begin{pmatrix} 1 \\ a \end{pmatrix}(b,1)$ has g-Drazin inverse and $(Q^d)^2 = (Q^2)^d = N^d$. By using Cline's formula again,

$$\begin{aligned} (a+b)^d &= (b,1)N^d\begin{pmatrix} 1 \\ a \end{pmatrix} \\ &= (b,1)\begin{pmatrix} (a+b)b & 1 \\ a(a+b)b & a \end{pmatrix}[M^d]^2\begin{pmatrix} 1 & 0 \\ 0 & a+b \end{pmatrix}\begin{pmatrix} 1 \\ a \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
&= ((a+b)^2b, a+b)[M^d]^2 \begin{pmatrix} 1 \\ a^2+ba \end{pmatrix} \\
&= \big(((a+b)^2b, a+b)M^d\big)\big(M^d \begin{pmatrix} 1 \\ a^2+ba \end{pmatrix}\big) \\
&= \big[(a+b)^2b\Gamma + (a+b)\Lambda\big]\big[\Gamma + \Delta(a+b)a\big] \\
&\quad + \big[(a+b)^2b\Delta + (a+b)\Xi\big]\big[\Lambda + \Xi(a+b)a\big].
\end{aligned}
$$

This completes the proof. □

Corollary 2. *Let $a, b \in \mathcal{A}^d$. If $a^2ba = 0, abab = 0$ and $b^2 = 0$, then*

$$
\begin{aligned}
(a+b)^d = &\big[(a+b)ab\Gamma + (a+b)\Lambda\big]\big[\Gamma + \Delta(a+b)a\big] \\
&+ \big[(a+b)ab\Delta + (a+b)\Xi\big]\big[\Lambda + \Xi(a+b)a\big],
\end{aligned}
$$

where

$$
\begin{aligned}
&\Gamma = z_2\beta\alpha + z_1\beta^d\alpha^2, \quad \Delta = z_1 + z_1\beta^d\alpha, \\
&\Lambda = \beta^d\alpha + (\beta^d)^2\alpha^2 + \beta\alpha z_2\beta^d\beta\alpha + \beta\alpha z_2\beta^d\alpha^2, \\
&\Xi = \beta^d + (\beta^d)^2\alpha + \beta\alpha z_2 + \beta\alpha z_2\beta^d\alpha, \\
&\alpha = ab, \quad \beta = ba + a^2, \quad \alpha^d = 0, \quad \beta^d = (a^d)^2 + b(a^d)^3 + bab(a^d)^5, \\
&z_1 = (\beta^d)^2 + ab(a^d)^6, \quad z_2 = (\beta^d)^3 + ab(a^d)^8.
\end{aligned}
$$

Proof. This is obvious by Theorem 2. □

For complex matrices, Drain and g-Drazin inverses coincide with each other. Thus the preceding theorems give alternative formulas for the Drazin inverse of the sum of two block complex matrices provided in [7, Corollary 2.]. The following example illustrates that Theorem 2 is not a trivial generalization of [9, Theorem 2.1].

Example 1. Let

$$
a = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, b = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in M_4(\mathbb{C}).
$$

Then $a, b \in M_4(\mathbb{C})^d$. It is clear that $a^2ba = 0, abab = 0, ab^2a = 0$ and $ab^3 = 0$, but $aba = ab^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0$.

3. Block operator matrices

In [1], Bu and Zhang considered a class of block complex matrices with zero generalized Schur complement. The goal of this section is to provide explicit representations for the g-Drazin inverse of the operator matrix M given by $(*)$. We now derive.

Theorem 3. *Let $A \in \mathcal{L}(X)$ have g-Drazin inverse, $D \in \mathcal{L}(Y)$ and M be given by $(*)$. Let $W = AA^d + A^dBCA^d$. If AW has g-Drazin inverse, $AA^{\pi}BCA = 0, CA^{\pi}BCA = 0, AA^{\pi}BCB = 0, CA^{\pi}\ BCB = 0$ and $D = CA^dB$, then M has g-Drazin inverse. In this case*

$$M^d = P^d + (P^d)^2Q + (P^d)^3QP + Q(P^d)^2 + Q(P^d)^3Q + Q(P^d)^4QP,$$

where $P = \begin{pmatrix} A & AA^dB \\ C & CA^dB \end{pmatrix}$ *and* $Q = \begin{pmatrix} 0 & A^{\pi}B \\ 0 & 0 \end{pmatrix}$ *and*

$$P^d = \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} [(AW)^d]^2 \begin{pmatrix} A, AA^dB \end{pmatrix} + \sum_{i=1}^{\infty} \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} [(AW)^d]^{i+2} \begin{pmatrix} A^{i+1}A^{\pi} + AA^dBCA^{i-1}A^{\pi}, 0 \end{pmatrix}.$$

Proof. One easily checks that

$$M = \begin{pmatrix} A & B \\ C & CA^dB \end{pmatrix} = P + Q,$$

where

$$P = \begin{pmatrix} A & AA^dB \\ C & CA^dB \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & A^{\pi}B \\ 0 & 0 \end{pmatrix}.$$

By hypothesis, we have $PQPQ = 0, PQP^2 = 0, Q^2 = 0$. By virtue of Corollary 1, we have $M^d = P^d + (P^d)^2Q + (P^d)^3QP + Q(P^d)^2 + Q(P^d)^3Q + Q(P^d)^4QP$. Moreover we have

$$P = P_1 + P_2, P_1 = \begin{pmatrix} A^2A^d & AA^dB \\ CAA^d & CA^dB \end{pmatrix}, \quad P_2 = \begin{pmatrix} AA^{\pi} & 0 \\ CA^{\pi} & 0 \end{pmatrix}$$

and $P_2P_1 = 0$. By virtue of Lemma 1, P_2 has g-Drazin inverse. Obviously, we have

$$P_1 = \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} \begin{pmatrix} A, AA^dB \end{pmatrix}.$$

By hypothesis, we see that

$$\begin{pmatrix} A, AA^dB \end{pmatrix} \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} = AW$$

has g-Drazin inverse. By using Cline's formula, we see that

$$P_1^d = \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} [(AW)^d]^2 \begin{pmatrix} A, AA^dB \end{pmatrix}.$$

For any $i \in \mathbb{N}$, we compute that

$$(P_1^d)^i = \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} [(AW)^d]^{i+1} \begin{pmatrix} A, AA^dB \end{pmatrix} \text{ and } P_2^i = \begin{pmatrix} A^iA^\pi & 0 \\ CA^{i-1}A^\pi & 0 \end{pmatrix}.$$

According to Lemma 2, P has g-Drazin inverse and

$$\begin{aligned} P^d &= \sum_{i=0}^{\infty} P_1^i P_1^\pi (P_2^d)^{i+1} + \sum_{i=0}^{\infty} (P_1^d)^{i+1} P_2^i P_2^\pi \\ &= \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} [(AW)^d]^2 \begin{pmatrix} A, AA^dB \end{pmatrix} \\ &\quad + \sum_{i=1}^{\infty} \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} [(AW)^d]^{i+2} \begin{pmatrix} A^{i+1}A^\pi + AA^dBCA^{i-1}A^\pi, 0 \end{pmatrix}. \end{aligned}$$

This completes the proof. □

Corollary 3. *Let $A \in \mathcal{L}(X)$ have g-Drazin inverse, $D \in \mathcal{L}(Y)$ and M be given by $(*)$. Let $W = AA^d + A^dBCA^d$. If AW has g-Drazin inverse, $A^\pi BC = 0$ and $D = CA^dB$, then M has g-Drazin inverse. In this case*

$$M^d = P^d + (P^d)^2 Q,$$

where

$$P = \begin{pmatrix} A & AA^dB \\ C & CA^dB \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & A^\pi B \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{aligned} P^d &= \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} [(AW)^d]^2 \begin{pmatrix} A, AA^dB \end{pmatrix} \\ &\quad + \sum_{i=1}^{\infty} \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} [(AW)^d]^{i+2} \begin{pmatrix} A^{i+1}A^\pi + BCA^{i-1}A^\pi, 0 \end{pmatrix}. \end{aligned}$$

Proof. Since $QP = 0$, we obtain the result by Theorem 3. □

The following example illustrates that Theorem 3 is a nontrivial generalization of [9, Theorem 3.3].

Example 2. Let A, B, C and D be the linear operators on $\mathbb{C}^2$ given by 2×2 matrices over $\mathbb{C}$.

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \text{ and } D = 0.$$

Set $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Then M can be seen as the linear operator acting on $\mathbb{C}^2 \times \mathbb{C}^2$. We check that

$$AA^\pi BCA = 0, CA^\pi BCA = 0, AA^\pi BCB = 0, CA^\pi BCB = 0$$

and $D = CA^dB$, while $CA^\pi BC = \begin{pmatrix} 0 & 2 \\ 0 & -2 \end{pmatrix} \neq 0$.

Finally, we split the block operator matrix M in an other way and extend [1, Theorem 4.2] as follows.

Theorem 4. *Let $A \in \mathcal{L}(X)$ have g-Drazin inverse, $D \in \mathcal{L}(Y)$ and M be given by $(*)$. Let $W = AA^d + A^dBCA^d$. If AW has g-Drazin inverse, $BCA^\pi A^2 = 0$, $BCA^\pi AB = 0$, $BCA^\pi BCA^d = 0$ and $D = CA^dB$, then M has g-Drazin inverse. In this case*

$$M^d = P^d + (P^d)^2Q + (P^d)^3QP + Q(P^d)^2 + Q(P^d)^3Q + Q(P^d)^4QP,$$

where

$$P = \begin{pmatrix} A & B \\ CAA^d & CA^dB \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 0 \\ CA^\pi & 0 \end{pmatrix}$$

and

$$P^d = \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} [(AW)^d]^2 \begin{pmatrix} A, AA^dB \end{pmatrix} + \sum_{i=1}^{\infty} \begin{pmatrix} A^{i-1}A^\pi BCA^d \\ 0 \end{pmatrix} [(AW)^d]^{i+2} \begin{pmatrix} A, AA^dB \end{pmatrix}.$$

Proof. Clearly we have

$$M = \begin{pmatrix} A & B \\ C & CA^dB \end{pmatrix} = P + Q,$$

where

$$P = \begin{pmatrix} A & B \\ CAA^d & CA^dB \end{pmatrix} \text{ and } Q = \begin{pmatrix} 0 & 0 \\ CA^\pi & 0 \end{pmatrix}.$$

Then we check that $PQPQ = 0, PQP^2 = 0, Q^2 = 0$. In view of Corollary 1, we have

$$M^d = P^d + (P^d)^2Q + (P^d)^3QP + Q(P^d)^2 + Q(P^d)^3Q + Q(P^d)^4QP.$$

Moreover we have

$$P = P_1 + P_2, P_1 = \begin{pmatrix} A^2A^d & AA^dB \\ CAA^d & CA^dB \end{pmatrix}, \quad P_2 = \begin{pmatrix} AA^\pi & A^\pi B \\ 0 & 0 \end{pmatrix}$$

and $P_1P_2 = 0$. In light of Lemma 2, P_2 has g-Drazin inverse. As in the proof of Theorem 3, we have

$$P_1^d = \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} [(AW)^d]^2 \begin{pmatrix} A, AA^dB \end{pmatrix}.$$

For any $i \in \mathbb{N}$, we compute that

$$(P_1^d)^i = \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} [(AW)^d]^{i+1} \begin{pmatrix} A, AA^dB \end{pmatrix} \text{ and } P_2^i = \begin{pmatrix} A^iA^\pi & A^{i-1}A^\pi B \\ 0 & 0 \end{pmatrix}.$$

According to Lemma 2, P has g-Drazin inverse and

$$\begin{aligned} P^d &= \sum_{i=0}^{\infty} P_2^i P_2^{\pi} (P_1^d)^{i+1} + \sum_{i=0}^{\infty} (P_2^d)^{i+1} P_1^i P_1^{\pi} \\ &= \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} [(AW)^d]^2 \begin{pmatrix} A, AA^d B \end{pmatrix} \\ &\quad + \sum_{i=1}^{\infty} \begin{pmatrix} A^{i-1} A^{\pi} BCA^d \\ 0 \end{pmatrix} [(AW)^d]^{i+2} \begin{pmatrix} A, AA^d B \end{pmatrix}, \end{aligned}$$

and we are through. □

As an immediate consequence of Theorem 4, we have

Corollary 4. *Let $A \in \mathcal{L}(X)$ have g-Drazin inverse, $D \in \mathcal{L}(Y)$ and M be given by $(*)$. Let $W = AA^d + A^d BCA^d$. If AW has g-Drazin inverse, $BCA^{\pi} = 0$ and $D = CA^d B$, then M has g-Drazin inverse. In this case,*

$$M^d = P^d + Q(P^d)^2,$$

where

$$\begin{aligned} Q &= \begin{pmatrix} 0 & 0 \\ CA^{\pi} & 0 \end{pmatrix}, \\ P^d &= \begin{pmatrix} AA^d \\ CA^d \end{pmatrix} [(AW)^d]^2 \begin{pmatrix} A, AA^d B \end{pmatrix} \\ &\quad + \sum_{i=1}^{\infty} \begin{pmatrix} A^{i-1} A^{\pi} BCA^d \\ 0 \end{pmatrix} [(AW)^d]^{i+2} \begin{pmatrix} A, AA^d B \end{pmatrix}. \end{aligned}$$

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EXISTENCE THEOREMS FOR EQUATIONS AND SYSTEMS IN R^N WITH k_i-HESSIAN OPERATOR

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Abstract. We establish the existence of positive radial entire solutions for nonlinear equations and systems. Our main results obtained with the use of the Schauder-Tychonov fixed point theorem will complete the works of Kusano-Swanson and Holanda.

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1. Introduction

We consider the existence of entire radial solutions for the following nonlinear equation

$$S_{k_1}\left(\Lambda\left(D^2u_1\right)\right)-\alpha_1 S_{k_2}\left(\Lambda\left(D^2u_1\right)\right)=\lambda_1 f\left(|x|,u_1,|\nabla u_1|\right),\quad x\in\mathbb{R}^N, \tag{1.1}$$

and the nonlinear system

$$\begin{cases} S_{k_1}\left(\Lambda\left(D^2u_1\right)\right)-\alpha_1 S_{k_2}\left(\Lambda\left(D^2u_1\right)\right)=\lambda_1 f_1\left(|x|,u_1,u_2,|\nabla u_1|,|\nabla u_2|\right), & x\in\mathbb{R}^N,\\ S_{k_1}\left(\Lambda\left(D^2u_2\right)\right)-\alpha_2 S_{k_2}\left(\Lambda\left(D^2u_2\right)\right)=\lambda_2 f_2\left(|x|,u_1,u_2,|\nabla u_1|,|\nabla u_2|\right), & x\in\mathbb{R}^N, \end{cases} \tag{1.2}$$

where $N\geq 3$, $\alpha_i\in(0,\infty)$, $\lambda_i\in\mathbb{R}$, $k_i\in\{1,2,\ldots,N\}$ with $k_1>2k_2$, D^2u_i is the Hessian matrix of a C^2 function u_i defined over $\mathbb{R}^N$, $\Lambda\left(D^2u_i\right)=\left(\Lambda_1^i,\ldots,\Lambda_N^i\right)$ is the vector of eigenvalues of D^2u_i, $S_{k_i}\left(\Lambda\left(D^2u_i\right)\right)$ is the k_i- Hessian operator defined by

$$S_{k_i}\left(\Lambda\left(D^2u_i\right)\right)=\sum_{1\leq j_1<\ldots<j_{k_i}\leq N}\Lambda_{j_1}^i\cdot\ldots\cdot\Lambda_{j_{k_i}}^i,\quad i=1,2,$$

and the nonlinearities f, f_1 and f_2 satisfy some of the conditions that will be specified later.

Historically, when $\alpha_1\in(0,\infty)$, $k_1=N$ and $k_2=1$, (1.1) is referred in differential geometry as the equation for prescribed generalized Gaussian curvature, as pointed in the paper written with a long time ago by Kusano and Swanson [5] (for further details, see Pogorelov [6, Chap. 10-13]). In this situation, with $k_1=N$ and $k_2=1$,

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from mathematical point of view the authors [5] carried out a systematic study of (1.1) and gave sufficient conditions on f to admit solutions in $\mathbb{R}^N$.

Recently, in the case $\alpha_1 = \alpha_2 = 0$, the question of existence of solutions for the more general equations and systems of the form (1.1) and (1.2) was studied by Holanda [2]. So, the case $\alpha_1, \alpha_2 \in (0,\infty)$ is a whole new ball game. Observing the results obtained until now and practical character of the problems of the type (1.1)/(1.2), there are several reasons to restate the research of these more general equations and systems.

In this work, we shall consider directly the equation (1.1), the system (1.2) and we intend to obtain the same results as observed by Kusano and Swanson [5], in the particular case mentioned. We also point that, our results can be adapted to a more larger classes of problems that was studied by Holanda [2].

The outline of the paper is as follows. Section 2 present the hypotheses on the nonlinearities and our main results. In Section 3 we give some auxiliary results. Our main Theorems will be proved in Section 4.

2. The main results

2.1. *The scalar equation (1.1)*

We work with the following classes of functions f:

(C1) $|f(t,x,z)|$ is monotone nondecreasing function with respect to x and z for fixed values of other variables. Meaning, $x \to |f(t,x,z)|$ and $z \to |f(t,x,z)|$ are monotone nondecreasing, for fixed (t,z) and fixed (t,x) respectively;

(C2) $F(m) = \sup_{t\in[0,\infty)} \left|f\left(t, m\left(1+t^2\right), 2mt\right)\right| < \infty$ for all $m > 0$;

(C3) $\lim_{m\to\infty} \frac{F(m)}{m^{k_1}} = 0$;

(C4)
$$H(c) = \sup_{t\in[0,\infty)} \left|f\left(t, c + 2\alpha_{N;k_1,k_2} t^2, 4\alpha_{N;k_1,k_2} t\right)\right| < \infty$$
for all $c \in \mathbb{R}$, where $\alpha^1_{N;k_1,k_2}$ is defined by
$$\alpha^1_{N;k_1,k_2} = \left(\alpha_1 k_1 C_{N-1}^{k_2-1} / k_2 C_{N-1}^{k_1-1}\right)^{1/(k_1-k_2)}, \quad C_{N-1}^{k_2-1} = (N-1)!/(k_2-1)!(N-k_2)!;$$

(C5) $\lim_{c\to-\infty} H(c) = 0$.

The main results of the paper for the equation (1.1) are the following theorems.

Theorem 1. *Suppose that $f\colon [0,\infty)\times(0,\infty)\times[0,\infty) \to \mathbb{R}$ is a continuous function that satisfies the assumptions (C1) and (C2). Then, there exists $\lambda_0^1 > 0$ such that for all $\lambda_1 \in \left[-\lambda_0^1, \lambda_0^1\right]$ the equation (1.1) has an infinitude of positive radial entire solutions $u_1(x) = u_1(|x|)$ such that*
$$0 < \lim_{|x|\to\infty} \inf \frac{u_1(|x|)}{|x|^2} \quad \textit{and} \quad \lim_{|x|\to\infty} \sup \frac{u_1(|x|)}{|x|^2} < \infty. \tag{2.1}$$

Theorem 2. *Assume that $f\colon [0,\infty)\times(0,\infty)\times[0,\infty)\to[0,\infty)$ is a continuous function that satisfies the assumptions (C1), (C2) and (C3). Then, for all $\lambda_1\in[0,\infty)$ the equation (1.1) has an infinitude of positive radial entire solutions $u_1(x)=u_1(|x|)$ satisfying (2.1).*

Theorem 3. *If $f\colon [0,\infty)\times\mathbb{R}\times[0,\infty)\to\mathbb{R}$ is a continuous function that satisfies the assumptions (C1), (C4) and (C5) then, for all $\lambda_1\in\mathbb{R}$ the equation (1.1) has an infinitude of radial entire solutions $u_1(x)=u_1(|x|)$ which are positive in a neighborhood of infinity and satisfy (2.1).*

2.2. *The system of equations (1.2)*

Let $i=1,2$ and consider the following classes of functions f_1 and f_2:

(CS1) $|f_i(t,s,l,q,w)|$ is monotone nondecreasing function with respect to t, s, l, q and w for fixed values of other variables;

(CS2) $F_i(m_1,m_2)=\sup_{t\in[0,\infty)}\left|f_i\left(t,m_1\left(1+t^2\right),m_2\left(1+t^2\right),2m_1t,2m_2t\right)\right|<\infty$ for all $m_1,m_2>0$;

(CS3)
$$\lim_{m_1\to\infty}\frac{F_1(m_1,m_2)}{(m_1)^{k_1}}=0\quad\text{for all } m_2>0$$
and
$$\lim_{m_2\to\infty}\frac{F_2(m_1,m_2)}{(m_2)^{k_1}}=0\quad\text{for all } m_1>0;$$

(CS4)
$$H_i(c_1,c_2)=\sup_{t\in[0,\infty)}\left|f_i\left(t,c_1+2\alpha^1_{N;k_1,k_2}t^2,c_2+2\alpha^2_{N;k_1,k_2}t^2,4\alpha^1_{N;k_1,k_2}t,4\alpha^2_{N;k_1,k_2}t\right)\right|<\infty$$
for all $c_1,c_2\in\mathbb{R}$, where $\alpha^i_{N;k_1,k_2}$ is defined by
$$\alpha^i_{N;k_1,k_2}=\left(\alpha_ik_1C^{k_2-1}_{N-1}/k_2C^{k_1-1}_{N-1}\right)^{1/(k_1-k_2)},\quad C^{k_2-1}_{N-1}=(N-1)!/(k_2-1)!(N-k_2)!;$$

(CS5) $\lim_{c_1\to-\infty}H_1(c_1,c_2)=0$ for all $c_2\in\mathbb{R}$ and $\lim_{c_2\to-\infty}H_2(c_1,c_2)=0$ for all $c_1\in\mathbb{R}$.

Regarding the system (1.2), Our main results are the following theorems.

Theorem 4. *Let*
$$D=[0,\infty)\times(0,\infty)\times(0,\infty)\times[0,\infty)\times[0,\infty).$$
Suppose that f_1, $f_2\colon D\to\mathbb{R}$ are continuous functions that satisfy the assumptions (CS1) and (CS2). Then, there exists $\lambda^i_0>0$ such that for all $\lambda_i\in\left[-\lambda^i_0,\lambda^i_0\right]$ the system (1.2) has an infinitude of positive radial entire solutions $(u_1(x),u_2(x))=(u_1(|x|),u_2(|x|))$ such that
$$0<\liminf_{|x|\to\infty}\frac{u_i(|x|)}{|x|^2}\quad\textit{and}\quad\limsup_{|x|\to\infty}\frac{u_i(|x|)}{|x|^2}<\infty,\quad i=1,2.\tag{2.2}$$

Theorem 5. *Let*

$$D=[0,\infty)\times(0,\infty)\times(0,\infty)\times[0,\infty)\times[0,\infty).$$

Assume that f_1, $f_2\colon D\to[0,\infty)$ are continuous functions that satisfy the assumptions (CS1), (CS2) and (CS3). Then, for all $\lambda_i\in[0,\infty)$ the system (1.2) has an infinitude of positive radial entire solutions $(u_1(x),u_2(x))=(u_1(|x|),u_2(|x|))$ satisfying (2.2).

Theorem 6. *Let*

$$D=[0,\infty)\times\mathbb{R}\times\mathbb{R}\times[0,\infty)\times[0,\infty).$$

Suppose that f_1, $f_2\colon D\to\mathbb{R}$ are continuous functions that satisfy the assumptions (CS1), (CS4) and (CS5). Then, for all $\lambda_i\in\mathbb{R}$ the system (1.2) has an infinitude of radial entire solutions $(u_1(x),u_2(x))=(u_1(|x|),u_2(|x|))$ which are positive in a neighborhood of infinity and satisfy (2.2).

Comparing the results in the paper of [5] with our Theorems 1-3, we consider that our work is a consistent generalization of [5] from the mathematical point view and on the other hand Theorems 4-6 are excellent sources of inspiration for the futures works in treating a more general classes of equations and systems that was considered by Holanda [2].

3. Preliminary considerations

To prove our theorems, we make to introduce the following useful result.

Lemma 1. *Setting*

$$\varphi(t)=t^{k_1}-t^{k_2}\quad \text{for } t\in\mathbb{R},\ t_0=(k_2/k_1)^{1/(k_1-k_2)},$$

the following hold:

1. $\varphi(t_0)=\frac{k_2-k_1}{k_1}\left(\frac{k_2}{k_1}\right)^{k_2/(k_1-k_2)}<0$, $\varphi(1)=0$ *and* $\varphi(\infty):=\lim_{t\to\infty}\varphi(t)=\infty$;
2. $\varphi\colon[t_0,\infty)\to[\varphi(t_0),\infty)$ *is strictly increasing for all* $t>t_0$ *and in fact has a uniquely defined inverse function* $\phi\colon[\varphi(t_0),\infty)\to[t_0,\infty)$ *with* $\phi(0)=1$;
3. $\phi\colon[\varphi(t_0),\infty)\to[t_0,\infty)$ *is analytic and concave. In particular,* $\phi(t)\geq 1$ *for all* $t\geq 0$, $\phi(\infty):=\lim_{t\to\infty}\phi(t)=\infty$ *and for* $t>\varphi(t_0)$ *we have that* ϕ *is strictly increasing and it hold*
$$\phi'(t)=\frac{1}{k_1(\phi(t))^{k_1-1}-k_2(\phi(t))^{k_2-1}}>0$$
and
$$\phi''(t)=-\frac{k_1(k_1-1)(\phi(t))^{k_1-2}-k_2(k_2-1)(\phi(t))^{k_2-2}}{\left[k_1(\phi(t))^{k_1-1}-k_2(\phi(t))^{k_2-1}\right]^3}<0;$$
4. $\phi(s\xi)\leq\xi^{1/k_1}\phi(s)$ *for all* $s\geq 0$ *and* $\xi\geq 1$.

Proof of Lemma 1.

1. By a direct calculation

$$\begin{aligned}\varphi(t_0) &= \left(\frac{k_2}{k_1}\right)^{\frac{k_1}{k_1-k_2}} - \left(\frac{k_2}{k_1}\right)^{\frac{k_2}{k_1-k_2}} = \left(\frac{k_2}{k_1}\right)^{\frac{k_1-k_2+k_2}{k_1-k_2}} - \left(\frac{k_2}{k_1}\right)^{\frac{k_2}{k_1-k_2}} \\ &= \left(\frac{k_2}{k_1}\right)^{\frac{k_2}{k_1-k_2}} \left[\left(\frac{k_2}{k_1}-1\right)\right] = \frac{k_2-k_1}{k_1}\left(\frac{k_2}{k_1}\right)^{k_2/(k_1-k_2)},\end{aligned}$$

and from $k_1 > k_2$ we obtain $\varphi(t_0) < 0$. Clearly $\varphi(1) = 0$ and $\varphi(\infty) := \lim_{t\to\infty}\varphi(t) = \infty$.

2. Since φ is differentiable, we have

$$\varphi'(t) = k_1 t^{k_1-1} - k_2 t^{k_2-1} = t^{k_2-1}\left(k_1 t^{k_1-k_2} - k_2\right) \geq 0 \quad \text{for} \quad t \geq t_0$$

and so φ is strictly increasing on (t_0,∞) with the range $[\varphi(t_0),\infty)$. Whereas φ is bijective with the inverse function satisfying $\phi(0) = 1$.

3. It is important to note that the last part of the result follow from standard inversion theorem, from where

$$\phi'(t) = \frac{1}{(\varphi'\circ\phi)(t)}.$$

On the other hand

$$\frac{k_2}{k_1} \geq \frac{k_2(k_2-1)}{k_1(k_1-1)} \text{ and } \phi(t) > 0, \quad \text{for } t > \varphi(t_0),$$

implies $\phi''(t) < 0$ for all $t > \varphi(t_0)$.

4. By the basic fact $\sigma^{k_1} \geq \sigma$ for all $\sigma \geq 1$, we have that for each $t \geq 1$ and $\sigma \geq 1$

$$\begin{aligned}\varphi(t\sigma) &= (t\sigma)^{k_1} - (t\sigma)^{k_2} = t^{k_1}\sigma^{k_1} - t^{k_2}\sigma^{k_2} \\ &= \sigma^{k_1}\left(t^{k_1} - t^{k_2}\sigma^{k_2-k_1}\right) \geq \sigma^{k_1}\left(t^{k_1} - t^{k_2}\right) = \sigma^{k_1}\varphi(t)\end{aligned}$$

where, we have used $\sigma^{k_2-k_1} \leq 1$. It follows that

$$t\sigma = \phi(\varphi(t\sigma)) \geq \phi\left(\sigma^{k_1}\varphi(t)\right).$$

Let $s = \varphi(t)$ and $\xi = \sigma^{k_1}$, i.e., $t = \Phi(s)$ and $\xi^{1/k_1} = \sigma$, one can see that **4.** holds.

□

To end the section, for the readers' convenience, we recall the radial form of the k-Hessian operator, see for example [1, 3, 4].

Lemma 2. *Assume $u \in C^2[0,R)$ is radially symmetric with $u'(0)=0$. Then, the function u defined by $u(x)=u(r)$ where $r=|x|<R$ is $C^2(B_R)$, and*

$$\Lambda\left(D^2u(r)\right)=\begin{cases}\left(u''(r),\frac{u'(r)}{r},\ldots,\frac{u'(r)}{r}\right) & \text{for } r\in(0,R),\\ \left(u''(0),u''(0),\ldots,u''(0)\right) & \text{for } r=0;\end{cases}$$

$$S_k\left(\Lambda\left(D^2u(r)\right)\right)=\begin{cases}C_{N-1}^{k-1}u''(r)\left(\frac{u'(r)}{r}\right)^{k-1}+C_{N-1}^{k-1}\frac{N-k}{k}\left(\frac{u'(r)}{r}\right)^{k} & \text{for } r\in(0,R),\\ C_N^k\left(u''(0)\right)^k & \text{for } r=0;\end{cases}$$

where the prime denotes differentiation with respect to r.

4. Proofs of the main results

Proofs of Theorems 1-3. Setting $r=|x|$ we prove the existence of a radial solution $u_1(r)\in C^2$ to the problem (1.1). Observe that we can rewrite (1.1) as follows:

$$C_{N-1}^{k_1-1}\left[\frac{r^{N-k_1}}{k_1}\left(u_1'(r)\right)^{k_1}\right]'-\alpha_1C_{N-1}^{k_2-1}\left[\frac{r^{N-k_2}}{k_2}\left(u_1'(r)\right)^{k_2}\right]'=\lambda_1r^{N-1}f\left(r,u_1(r),\left|u_1'(r)\right|\right). \quad (4.1)$$

Then, the radial solution of (4.1) is a solution u, of the ordinary differential equation (4.1) with the initial conditions

$$u_1(0)=c_0 \quad \text{and} \quad u_1'(0)=0. \quad (4.2)$$

For $r>0$ it follows that

$$\frac{C_{N-1}^{k_1-1}r^{N-k_1}}{k_1}\left(u_1'(r)\right)^{k_1}-\frac{\alpha_1C_{N-1}^{k_2-1}r^{N-k_2}}{k_2}\left(u_1'(r)\right)^{k_2}=\int_0^r\lambda_1s^{N-1}f\left(s,u_1(s),\left|u_1'(s)\right|\right)ds,$$

or, equivalently

$$\left(\frac{u_1'(r)}{\alpha_{N;k_1,k_2}^1r}\right)^{k_1}-\left(\frac{u_1'(r)}{\alpha_{N;k_1,k_2}^1r}\right)^{k_2}=\int_0^r\frac{\lambda_1k_1r^{-N}s^{N-1}f\left(s,u_1(s),\left|u_1'(s)\right|\right)}{C_{N-1}^{k_1-1}\left(\alpha_{N;k_1,k_2}^1\right)^{k_1}}ds,\quad r>0. \quad (4.3)$$

From the definition of ϕ in Lemma 1, we see that (4.3) is equivalent to

$$\frac{u_1'(r)}{\alpha_{N;k_1,k_2}^1r}=\phi\left(\frac{\lambda_1r^{-N}}{C_{N-1}^{k_1-1}\left(\alpha_{N;k_1,k_2}^1\right)^{k_1}}\int_0^rs^{N-1}f\left(s,u_1(s),\left|u_1'(s)\right|\right)ds\right),\quad r>0. \quad (4.4)$$

Since $\lim_{r\to 0+} u_1'(r) = 0 = u_1'(0) = 0$ via L'Hôpital's rule and (4.2), the equation (4.4) can be extended by continuity

$$u'(r) = \alpha^1_{N;k_1,k_2} r\phi\left(\frac{\lambda_1 k_1 r^{-N}}{C_{N-1}^{k_1-1}\left(\alpha^1_{N;k_1,k_2}\right)^{k_1}}\int_0^r s^{N-1} f\left(s, u_1(s), \left|u_1'(s)\right|\right) ds\right), \quad r \geq 0,$$

for any C^1-function u. Then, (4.1) with the initial conditions (4.2) can be equivalently written as an integral equation

$$\begin{cases} u_1(r) = c_0 + \alpha^1_{N;k_1,k_2} \\ \qquad \displaystyle\int_0^r t\phi\left(\frac{\lambda_1 k_1 t^{-N}}{C_{N-1}^{k_1-1}\left(\alpha^1_{N;k_1,k_2}\right)^{k_1}}\int_0^t s^{N-1} f\left(s, u_1(s), \left|u_1'(s)\right|\right) ds\right) dt, \quad r \geq 0 \\ u_1(0) = c_0 \text{ and } u_1'(0) = 0. \end{cases} \tag{4.5}$$

To establish the existence of a solution to this problem (4.5), we use the Schauder-Tychonov fixed point theorem and hence $u_1(x) := u_1(r)$ is a radial entire solution of (1.1). □

Next, we are ready to prove our main results.

Proof of the Theorem 1. Our conditions permit to choose $\lambda_0^1 > 0$ such that

$$\frac{\lambda_0^1 k_1}{N C_{N-1}^{k_1-1}} \frac{F\left(2\alpha^1_{N;k_1,k_2}\right)}{\left(\alpha^1_{N;k_1,k_2}\right)^{k_1}} \leq \frac{1}{k_1^{k_2}}.$$

Denote by C^1 the Fréchet space of all C^1- functions in $[0,\infty)$, with the topology of uniform convergence of functions and their first derivatives on compact subintervals of $[0,\infty)$. Let $c_0 \in \left(0, 2\alpha^1_{N;k_1,k_2}\right)$ arbitrarily fixed and we consider the closed convex set

$$K = \left\{u_1 \in C^1 \,\middle|\, c_0 \leq u_1(r) \leq c_0 + 2\alpha^1_{N;k_1,k_2} r^2,\ 0 \leq u_1'(r) \leq 4\alpha^1_{N;k_1,k_2} r,\ r \geq 0\right\}. \tag{4.6}$$

We define the mapping $T: K \to C^1$ by

$$(Tu_1)(r) = c_0 + \alpha^1_{N;k_1,k_2}\int_0^r t\phi(w(t))\, dt, \quad r \geq 0,\ u \in K, \tag{4.7}$$

where

$$w(t) = \frac{\lambda_1 k_1 t^{-N}}{C_{N-1}^{k_1-1}\left(\alpha^1_{N;k_1,k_2}\right)^{k_1}}\int_0^t s^{N-1} f\left(s, u_1(s), \left|u_1'(s)\right|\right) ds, \quad \lambda_1 \in \left[-\lambda_0^1, \lambda_0^1\right].$$

From the above settings, the assumptions (C1) and (C2), if $u_1 \in K$, then for all $r \geq 0$ and $\lambda_1 \in \left[-\lambda_0^1, \lambda_0^1\right]$ we have

$$\begin{aligned}
|w(t)| &\leq \frac{|\lambda_1| k_1 t^{-N}}{C_{N-1}^{k_1-1}\left(\alpha_{N;k_1,k_2}^1\right)^{k_1}} \left|f\left(t, 2\alpha_{N;k_1,k_2}^1\left(1+t^2\right), 4\alpha_{N;k_1,k_2}^1 t\right)\right| \int_0^t s^{N-1} ds \\
&= \frac{|\lambda_1| k_1}{N C_{N-1}^{k_1-1}\left(\alpha_{N;k_1,k_2}^1\right)^{k_1}} \left|f\left(t, 2\alpha_{N;k_1,k_2}^1\left(1+t^2\right), 4\alpha_{N;k_1,k_2}^1 t\right)\right| \\
&\leq \frac{\lambda_0^1 k_1}{N C_{N-1}^{k_1-1}} \frac{F\left(2\alpha_{N;k_1,k_2}^1\right)}{\left(\alpha_{N;k_1,k_2}^1\right)^{k_1}} \leq \frac{1}{k_1^{k_2}} < \frac{k_1-k_2}{k_1}\left(\frac{k_2}{k_1}\right)^{k_2/(k_1-k_2)},
\end{aligned}$$

showing that T is well-defined on K.

Next, we observe that $\varphi(2) = 2^{k_1} - 2^{k_2} = 2^{k_2}\left(2^{k_1-k_2} - 1\right) \geq 1$, from where

$$2 = \phi(\varphi(2)) \geq \phi(1).$$

Also, with the use of Lemma 1, if $u \in K$ we have

$$\begin{aligned}
\alpha_{N;k_1,k_2}^1 \phi(w(t)) &\leq \alpha_{N;k_1,k_2}^1 \phi\left(\frac{1}{k_1^{k_2}}\right) \\
&\leq \alpha_{N;k_1,k_2}^1 \phi\left(\left(\frac{1}{k_1}+1\right)^{k_2}\right) \\
&\leq \alpha_{N;k_1,k_2}^1 \phi\left(\left(\frac{1}{k_1}+1\right)^{k_2}\right) \\
&\leq \alpha_{N;k_1,k_2}^1 \left(\frac{1}{k_1}+1\right)\phi(1) \\
&\leq 2\alpha_{N;k_1,k_2}^1 \left(\frac{1}{k_1}+1\right) \leq 4\alpha_{N;k_1,k_2}^1, \quad t \geq 0,
\end{aligned}$$

and hence

$$c_0 \leq (Tu_1)(r) \leq c_0 + 2\alpha_{N;k_1,k_2}^1 r^2, \quad r \geq 0.$$

Furthermore

$$0 \leq (Tu_1)'(r) \leq \alpha_{N;k_1,k_2}^1 r \phi(w(r)) \leq 4\alpha_{N;k_1,k_2}^1 r, \quad r \geq 0, \tag{4.8}$$

implying that T maps K into itself.

Next, we prove the continuity of T in C^1- topology. To do this, let $\{u_1^n(r)\}_{n\geq 0}$ be a sequence in K converging to $u_1 \in K$ in this topology, and define

$$w_n(t) = \frac{\lambda k_1 t^{-N}}{C_{N-1}^{k_1-1}\left(\alpha_{N;k_1,k_2}^1\right)^{k_1}} \int_0^t s^{N-1} f\left(s, u_1^n(s), \left|(u_1^n)'(s)\right|\right) ds,\ t \geq 0,\ \lambda_1 \in \left[-\lambda_0^1, \lambda_0^1\right].$$

Using (4.8) we have

$$|w_n(t) - w(t)| \leq \lambda_0^1 \left(\alpha_{N;k_1,k_2}^1\right)^{k_1} \sup_{0\leq r\leq t} \left|f\left(r, u_1^n(r), \left|(u_1^n)'(r)\right|\right) - f\left(r, u_1(r), \left|u_1'(r)\right|\right)\right|$$

and

$$\left|(Tu_1^n)'(t) - (Tu_1^n)'(t)\right| = \alpha_{N;k_1,k_2}^1 t \left|\phi(w_n(t)) - \phi(w(t))\right|.$$

The continuity of ϕ implies that $(Tu_1^n)'(t) \to (Tu_1)'(t)$ as $n \to \infty$ and the convergence is uniformly on every compact subinterval of $[0,\infty)$. Likewise, from (4.7) $(Tu_1^n)(t) \to (Tu_1)(t)$ as $n \to \infty$ uniformly on such subintervals, implying the continuity of T in C^1.

To prove that $T(K)$ has compact closure in C^1 via Arzela-Ascoli's theorem, we note that $(Tu_1)(r) \in C^2([0,\infty))$ for all $u_1 \in K$ and

$$\begin{aligned}
(Tu_1)''(r) &= \left((Tu_1)'(r)\right)' = \alpha_{N;k_1,k_2}\left(r\phi(w(r))\right)' \\
&= \alpha_{N;k_1,k_2}^1 \phi(w(r)) + \alpha_{N;k_1,k_2}^1 r\phi'(w(r))\, w'(r) \\
&= \alpha_{N;k_1,k_2}^1 \phi(w(r)) + \lambda_1 \frac{k_1\left(\alpha_{N;k_1,k_2}^1\right)^{1-k_1}}{C_{N-1}^{k_1-1}} \phi'(w(r)) \left[f\left(r, u_1(r), \left|u_1'(r)\right|\right)\right. \\
&\quad \left. - Nr^{-N}\int_0^r s^{N-1} f\left(s, u_1(s), \left|u_1'(s)\right|\right) ds\right]
\end{aligned}$$

for all $r \geq 0$. Then, Lemma 1 imply the uniform bound

$$\left|(Tu_1)''(r)\right| \leq \alpha_{N;k_1,k_2}^1 \phi\left(\frac{1}{k_1^{k_2}}\right) + 2\lambda_0^1 k_1 \left(\alpha_{N;k_1,k_2}^1\right)^{1-k_1} \phi'\left(-\frac{1}{k_1^{k_2}}\right), \quad r \geq 0.$$

Thus

$$\sup\left|(Tu_1)''(r)\right| < \infty \quad \text{for any } r \in [a,b] \subset [0,\infty).$$

By the Mean Value Theorem

$$\left|(Tu_1)'(r_2) - (Tu_1)'(r_1)\right| \leq \sup_{r\in[a,b]} \left|(Tu_1)''(r)\right| |r_2 - r_1|,$$

with

$$r_1, r_2 \in [a,b] \subset [0,\infty),$$

we obtain that $(TK)' = \{(Tu_1)' | u_1 \in K\}$ is locally equicontinuous on any compact of $[0,\infty)$. Similarly, TK is locally equicontinuous on compact sets from $[0,\infty)$, and

the local uniform boundedness of TK and $(TK)'$ is easily verified. Therefore, from Ascoli's Theorem, it follows that TK is relatively compact in the C^1-topology.

Finally, we can then apply the Schauder-Tychonov fixed point theorem to conclude that there exists an element $u_1 \in K$ such that $Tu_1 = u_1$. We have proved that u_1 satisfies (4.5), yielding a positive entire solution $u_1(x) = u_1(|x|)$ of equation (1.1) in $\mathbb{R}^N$.

To finish the proof of Theorem 1, the fact that $u_1(r)$ satisfies (2.1) follows from the inequalities

$$c_0 + \alpha^1_{N;k_1,k_2}(k_2/k_1)^{1/(k_1-k_2)}\frac{r^2}{2} \le u_1(r) \le c_0 + 2\alpha^1_{N;k_1,k_2} r^2, \quad r \ge 0, \tag{4.9}$$

where the left inequality in (4.9) is a consequence of the fact

$$\phi(t) > (k_2/k_1)^{1/(k_1-k_2)} \quad \text{for } t > \frac{k_2-k_1}{k_1}\left(\frac{k_2}{k_1}\right)^{k_2/(k_1-k_2)},$$

and the right inequality in (4.9) is obvious from (4.6). On the other hand since any $c_0 \in \left(0, 2\alpha^1_{N;k_1,k_2}\right)$ will serve as an initial value $u_1(0) = c_0$ in (4.5), there exists an infinitude of positive radial entire solutions of equation (1.1). □

Proof of Theorems 2 and 3. The proofs of Theorems 2 and 3 are virtually the same as that for Theorem 1, and the details will be omitted. For any further comments regarding the details, see the paper of Kusano and Swanson [5]. □

Proofs of Theorems 4-6. Setting $r = |x|$ we prove the existence of a radial solution $(u_1, u_2) \in C^2 \times C^2$ to the problem (1.2). Denote

$$G_1(|x|, u_1, u_2) = f_1(|x|, u_1, u_2, |\nabla u_1|, |\nabla u_2|)$$

and

$$G_2(|x|, u_1, u_2) = f_2(|x|, u_1, u_2, |\nabla u_1|, |\nabla u_2|).$$

We observe that we can rewrite (1.2) as follows:

$$\begin{cases} C_{N-1}^{k_1-1}\left[\frac{r^{N-k_1}}{k_1}\left(u_1'(r)\right)^{k_1}\right]' - \alpha_1 C_{N-1}^{k_2-1}\left[\frac{r^{N-k_2}}{k_2}\left(u_1'(r)\right)^{k_2}\right]' = \\ \lambda_1 r^{N-1} G_1(r, u_1(r), u_2(r)), \\ C_{N-1}^{k_1-1}\left[\frac{r^{N-k_1}}{k_1}\left(u_2'(r)\right)^{k_1}\right]' - \alpha_2 C_{N-1}^{k_2-1}\left[\frac{r^{N-k_2}}{k_2}\left(u_2'(r)\right)^{k_2}\right]' = \\ \lambda_2 r^{N-1} G_2(r, u_1(r), u_2(r)). \end{cases} \tag{4.10}$$

Then, the radial solution of (4.10) is a solution u of the ordinary differential system (4.10) with the initial conditions

$$(u_1(0), u_2(0)) = (c_1, c_2) \text{ and } \left(u_1'(0), u_2'(0)\right) = (0,0). \tag{4.11}$$

It follows that

$$\begin{cases} u_1'(r) = \alpha^1_{N;k_1,k_2} r\phi\left(\frac{\lambda_1 r^{-N}}{C_{N-1}^{k_1-1}\left(\alpha^1_{N;k_1,k_2}\right)^{k_1}} \int_0^r s^{N-1} G_1(s,u_1(s),u_2(s))\,ds\right), & r \geq 0, \\ u_2'(r) = \alpha^2_{N;k_1,k_2} r\phi\left(\frac{\lambda_2 r^{-N}}{C_{N-1}^{k_1-1}\left(\alpha^2_{N;k_1,k_2}\right)^{k_1}} \int_0^r s^{N-1} G_2(s,u_1(s),u_2(s))\,ds\right), & r \geq 0. \end{cases}$$

for any C^1-function (u_1,u_2). Then, for $r \geq 0$, (4.10) with the initial conditions (4.11) can be equivalently written as an integral system of equations

$$\begin{cases} u_1(r) = c_1 + \alpha^1_{N;k_1,k_2} \int_0^r t\phi\left(\frac{\lambda_1 k_1 t^{-N}}{C_{N-1}^{k_1-1}\left(\alpha^1_{N;k_1,k_2}\right)^{k_1}} \int_0^t s^{N-1} G_1(s,u_1(s),u_2(s))\,ds\right) dt, \\ u_2(r) = c_2 + \alpha^2_{N;k_1,k_2} \int_0^r t\phi\left(\frac{\lambda_2 k_1 t^{-N}}{C_{N-1}^{k_1-1}\left(\alpha^2_{N;k_1,k_2}\right)^{k_1}} \int_0^t s^{N-1} G_2(s,u_1(s),u_2(s))\,ds\right) dt, \\ (u_1(0),u_2(0)) = (c_1,c_2) \text{ and } (u_1'(0),u_2'(0)) = (0,0). \end{cases} \tag{4.12}$$

To establish the existence of a solution to this problem (4.12), we use the Schauder-Tychonov fixed point theorem and hence $(u_1(x),u_2(x)) := (u_1(r),u_2(r))$ is a radial entire solution of (1.2). □

Next, we are ready to prove our main results.

Proof of the Theorem 4. Let $i = 1,2$. We choose $\lambda_0^i > 0$, such that

$$\frac{\lambda_0^i k_1}{N C_{N-1}^{k_1-1}\left(\alpha^i_{N;k_1,k_2}\right)^{k_1}} F_i\left(2\alpha^1_{N;k_1,k_2}, 2\alpha^2_{N;k_1,k_2}\right) \leq \frac{1}{k_1^{k_2}}.$$

For simplicity, denote by C^1 the Fréchet space of all C^1- functions in $[0,\infty)$, with the topology of uniform convergence of functions and their first derivatives on compact subintervals of $[0,\infty)$.

For a fixed choice of c_i in $\left(0, 2\alpha^i_{N;k_1,k_2}\right)$ and λ_i a small positive parameter, the solutions of (1.2) are fixed point of the compact operator $(T_1,T_2) : K_1 \times K_2 \to C^1 \times C^1$ defined by

$$(T_i u_i)(r) = c_i + \alpha^i_{N;k_1,k_2} \int_0^r t\phi(w_i(t))\,dt, \quad r \geq 0,\ u \in K, \tag{4.13}$$

where

$$w_i(t) = \frac{\lambda_i k_1 t^{-N}}{C_{N-1}^{k_1-1}\left(\alpha^i_{N;k_1,k_2}\right)^{k_1}} \int_0^t s^{N-1} G_i(s,u_1(s),u_2(s))\,ds, \quad \lambda_i \in \left[-\lambda_0^i, \lambda_0^i\right],$$

on the closed convex set

$$K_i = \left\{u \in C^1 \,\middle|\, c_i \leq u_i(r) \leq c_i + 2\alpha^i_{N;k_1,k_2} r^2,\ 0 \leq u_i'(r) \leq 4\alpha^i_{N;k_1,k_2} r,\ r \geq 0\right\}. \tag{4.14}$$

Similar to the proof of Theorem 1, if $(u_1,u_2)\in K_1\times K_2$, for all $r\geq 0$ and $\lambda_i\in\left[-\lambda_0^i,\lambda_0^i\right]$ we have

$$|w_i(t)|\leq\frac{\lambda_0^i k_1}{NC_{N-1}^{k_1-1}}\frac{F_i\left(2\alpha_{N;k_1,k_2}^1,2\alpha_{N;k_1,k_2}^2\right)}{\left(\alpha_{N;k_1,k_2}^i\right)^{k_1}}\leq\frac{1}{k_1^{k_2}}<\frac{k_1-k_2}{k_1}\left(\frac{k_2}{k_1}\right)^{k_2/(k_1-k_2)}.$$

From the above analysis, (T_1,T_2) is well-defined on $K_1\times K_2$. Also, if $u_i\in K_i$ we have

$$\alpha_{N;k_1,k_2}^i\phi(w_i(t))\leq 4\alpha_{N;k_1,k_2}^i,\quad t\geq 0,$$

and hence

$$c_i\leq(T_iu_i)(r)\leq c_i+2\alpha_{N;k_1,k_2}^i r^2,\quad r\geq 0.$$

Therefore

$$0\leq(T_iu_i)'(r)\leq\alpha_{N;k_1,k_2}^i r\phi(w_i(r))\leq 4\alpha_{N;k_1,k_2}^i r,\quad r\geq 0,\tag{4.15}$$

implying that (T_1,T_2) maps $K_1\times K_2$ into itself.

Next, we prove the continuity of (T_1,T_2) in $C^1\times C^1$- topology. To do this, let $\{(u_1^n(r),u_2^n(r))\}_{n\geq 0}$ be a sequence in $K_1\times K_2$ converging to $(u_1,u_2)\in K_1\times K_2$ in this topology, and define

$$w_i^n(t)=\frac{\lambda_i k_1 t^{-N}}{C_{N-1}^{k_1-1}\left(\alpha_{N;k_1,k_2}^i\right)^{k_1}}\int_0^t s^{N-1}G_i(s,u_1^n(s),u_2^n(s))\,ds,\quad t\geq 0,\ \lambda_i\in[-\lambda_0,\lambda_0].$$

Using (4.15) we have

$$|w_i^n(t)-w_i(t)|\leq\lambda_0^i\alpha_{N;k_1,k_2}^{k_1}\sup_{0\leq r\leq t}|G_i(r,u_1^n(r),u_2^n(r))-G_i(r,u_1(r),u_2(r))|$$

and

$$\left|(T_iu_i^n)'(t)-(T_iu_i)'(t)\right|=\alpha_{N;k_1,k_2}^i t\left|\phi(w_i^n(t))-\phi(w_i(t))\right|.$$

Due to the continuity of ϕ we get the convergences $(T_iu_i^n)'(t)\to(T_iu_i)'(t)$ as $n\to\infty$ uniformly on every compact subinterval of $[0,\infty)$. Likewise, from (4.13) $(T_iu_i^n)(t)\to(T_iu_i)(t)$ as $n\to\infty$ uniformly on such subintervals. So (T_1,T_2) is continuous in $C^1\times C^2$-topology.

We prove that T_iK_i has compact closure in C^1. We note that $(T_iu_i)(r)\in C^2([0,\infty))$ for all $u_i\in K_i$ and

$$\begin{aligned}(T_iu_i)''(r)=\alpha_{N;k_1,k_2}^i\phi(w_i(r))+\lambda_i\frac{k_1\left(\alpha_{N;k_1,k_2}^i\right)^{1-k_1}}{C_{N-1}^{k_1-1}}\phi'(w_i(r))\,[G_i(r,u_1(r),u_2(r))\\ -Nr^{-N}\int_0^r s^{N-1}G_i(s,u_1(s),u_2(s))\,ds],\end{aligned}$$

for all $r \geq 0$. Lemma 1 imply the uniform bound

$$\left|(T_i u_i)''(r)\right| \leq \alpha^i_{N;k_1,k_2}\phi\left(\frac{1}{k_1^{k_2}}\right) + 2\lambda_0 k_1 \left(\alpha^i_{N;k_1,k_2}\right)^{1-k_1}\phi'\left(-\frac{1}{k_1^{k_2}}\right), \quad r \geq 0,$$

and so $(T_i K_i)' = \{(T_i u_i)' \,|\, u_i \in K_i\}$ is locally equicontinuous in $[0,\infty)$. Similarly $T_i K_i$ is locally equicontinuous, and the local uniform boundedness of $T_i K_i$ and $(T_i K_i)'$ is easily verified. Therefore, from Ascoli's Theorem, it follows that $T_i K_i$ and $(T_i K_i)'$ are relatively compact in any compact interval of $[0,\infty)$. Consequently, by a diagonal sequential process, we conclude that $T_i K_i$ is relatively compact in the C^1-topology. Finally, we apply the Schauder-Tychonov fixed point theorem to conclude that there exists an element $(u_1,u_2) \in K_1 \times K_2$ such that $(T_1 u_1, T_2 u_2) = (u_1,u_2)$.

Then (u_1,u_2) satisfies (4.12), yielding a positive entire solution

$$(u_1(x), u_2(x)) := (u_1(|x|), u_2(|x|)),$$

to the original system (1.2).

The fact that $u_i(r)$ satisfies (2.2) follows from the inequalities

$$c_i + \alpha^i_{N;k_1,k_2}(k_2/k_1)^{1/(k_1-k_2)}\frac{r^2}{2} \leq u_i(r) \leq c_i + 2\alpha^i_{N;k_1,k_2} r^2, \quad r \geq 0, \tag{4.16}$$

where the left inequality in (4.16) is a consequence of the fact

$$\phi(t) > (k_2/k_1)^{1/(k_1-k_2)} \quad \text{for} \quad t > \frac{k_2-k_1}{k_1}\left(\frac{k_2}{k_1}\right)^{k_2/(k_1-k_2)},$$

and the right inequality in (4.16) is obvious from (4.14). On the other hand since any

$$(c_1,c_2) \in \left(0, 2\alpha^1_{N;k_1,k_2}\right) \times \left(0, 2\alpha^2_{N;k_1,k_2}\right),$$

will serve as an initial value $(u_1(0), u_2(0)) = (c_1,c_2)$ in (4.12), there exists an infinitude of positive radial entire solutions of system (1.2) and, thus, Theorem 4 is proved. □

Proof of Theorems 5 and 6. Details of the proofs of Theorems 5 and 6 are omitted, since are virtually the same as that for Theorem 4, see also the paper of Kusano and Swanson [5] for any further comments, resulting a fixed point (u_1,u_2) of the mapping (T_1,T_2) defined by (4.13) in the set (4.14). □

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NUMERICAL SOLUTION OF FRACTIONAL VOLTERRA INTEGRAL EQUATIONS BASED ON RATIONAL CHEBYSHEV APPROXIMATION

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Abstract. We aim to give the numerical method for solving the fractional Volterra integral equations of first and second kinds. We here use the techniques based upon rational Chebyshev functions and Riemann-Liouville fractional integrals. Some illustrative experiments with a view of estimating error and graphics are given in order to show the validity and applicability of the technique. Our experiments show that the new technique has high accuracy and is very efficient when compare to the other approaches existing in literature.

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1. Introduction

The equation in which the unknown function $u(x)$ appears within an integral sign is called an integral equation. The linear integral equations

$$\lambda \int_a^b K(x,\rho)u(\rho)d\rho = f(x) \tag{1.1}$$

and

$$u(x) - \lambda \int_a^b K(x,\rho)u(\rho)d\rho = f(x) \tag{1.2}$$

of first and second kinds, respectively, play a particularly important role in the study of pure and applied mathematics. Here, the kernel $K(x,\rho)$ and a function $f(x)$ are given, and the function $u(x)$ is to be determined. Note that, in (1.1) and (1.2), the interval of integration (a,b) can be extended to infinity in one or both directions.

An Italian mathematician Volterra introduced an integral equation and later called his integral equation "Volterra integral equation", briefly, VIE. His work is summarized in his book [33] and it is one of the widely-studied integral equation in several

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branches of mathematics. A Volterra integral equation is one in which one of the integral's limits is variable. For example, the VIE corresponding to (1.2) is given by

$$u(x) - \lambda \int_a^x K(x,\rho)u(\rho)d\rho = f(x). \tag{1.3}$$

Several numerical and theoretical methods have been demonstrated by several researchers to solve Volterra integral equations such as (i) Adomian decomposition method [6], (ii) Taylor-series expansion method [17], (iii) Sinc-collocation method [22], (iv) quadrature method with variable step [24], (v) collocation method [16]. Moreover, Sezer et al. have developed powerful techniques for different kinds of VIE [3, 34]. Recently, a numerical solution was obtained based on Bernstein approximation method to solve VIE of first and second kind as well as their singular form [15].

Fractional operators have been used by many researchers to describe some complex phenomena. Many differential equations have been generalized and revisited with the aid of new fractional definitions [1, 2, 5, 8, 9, 23, 30, 32]. Correlatively, the theory of fractional integral equations (FIE) can be counted as a generalization of the classical idea of integral equations wherein calculus of integrals of any arbitrary real or complex order [25]. In the last three decades, this theory has got attention from several researchers because of its widespread application in engineering and science.

Consider $I = [a,b]$ $(-\infty < a < b < \infty)$ is an interval on the real line $\mathbb{R}$. Recall as in [25] that the left and right Riemann-Liouville fractional integrals ${}^{\alpha}\mathcal{F}^x_{a_+}u$ and ${}^{\alpha}\mathcal{F}^x_{b_-}u$, respectively, of order α, $0 < \alpha < 1$, for an absolutely continuous function $u : I \to \mathbb{R}$ are defined by

$${}^{\alpha}\mathcal{F}^x_{a_+}u(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1}u(s)ds \qquad (x > a)$$

and

$${}^{\alpha}\mathcal{F}^x_{b_-}u(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-s)^{\alpha-1}u(s)ds \qquad (x < b),$$

where $\Gamma(\alpha)$ is a Gamma function given by

$$\Gamma(\alpha) = \int_0^\infty s^{\alpha-1}e^{-s}ds.$$

In this study, we apply the rational Chebyshev functions to solve the following fractional Volterra integral equations (FVIE) of first and second kinds as follows:

$$\upsilon(x) = {}^{\alpha}\mathcal{F}^x_{a_+}(\kappa(x)u(x)) \tag{1.4}$$

and

$$u(x) = \upsilon(x) + \beta(x)\left[{}^{\alpha}\mathcal{F}^x_{a_+}(\kappa(x)u(x))\right], \tag{1.5}$$

where $\alpha \in (0,1)$ and $\kappa, \beta, u : [a,b] \to \mathbb{R}$ are absolutely continuous functions. Most recently, Usta [31] has obtained the numerical method for solving the fractional Volterra integral equations (1.4) and (1.5) which is based on Bernstein approximation

method. We now show that more efficient and accurate results are obtained by using the rational Chebyshev functions even for the lower degrees of iterations.

1.1. *Rational Chebyshev functions*

Orthogonal polynomials are very important and serve to approximate functions. Classical orthogonal polynomials were developed by P. L. Chebyshev in the late 19th century. Significance of the orthogonal polynomials can be seen especially in the solution of systems of linear equations and in the least-squares approximations.

With respect to the weighted function

$$w(x) = \frac{1}{\sqrt{1-x^2}},$$

the Chebyshev polynomials are orthogonal in $[-1,1]$ which can be formulated by the recurrence formula:

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1} \quad (n \geq 1).$$

Guo et al. [10] demonstrated the rational Chebyshev functions, in short, we will write RC functions, by

$$R_n(x) = T_n\left(\frac{x-1}{x+1}\right). \tag{1.6}$$

Consequently, one can write

$$R_0(x) = 1, \quad R_1(x) = \frac{x-1}{x+1}, \quad R_{n+1}(x) = 2\left(\frac{x-1}{x+1}\right)R_n(x) - R_{n-1}(x) \quad (n \geq 1). \tag{1.7}$$

We remark that RC functions are orthogonal with respect to the weight function

$$w(x) = \frac{1}{\sqrt{x}(x+1)}$$

in the interval $[0,\infty)$. In this case, the orthogonal property is

$$\int_0^\infty R_n(x)R_m(x)w(x)dx = \frac{s_m\pi}{2}\delta_{mn}; \; s_m = \begin{cases} 2 & if \;\; m=0 \\ 1 & if \;\; m \neq 1 \end{cases},$$

where δ_{mn} is the well-known Kronecher function. Parand and Razzaghi [21] used the technique based on RC functions with a view of solving higher-order ordinary differential equations. For more information about RC functions, we refer to the articles [7, 21].

2. Numerical schemes

We apply rational Chebyshev approximation method to fractional Volterra integral equations of first and second kinds in order to establish numerical schemes for numerical related solutions.

2.1. *Numerical scheme for FVIE of first kind*

In order to solve the first kind FVIE (1.4) numerically, we approximate the unknown function as

$$u(x) \simeq u_n(x) = \sum_{k=0}^{n} c_k R_k(x). \tag{2.1}$$

Substituting (2.1) in

$$\upsilon(x) = {}^{\alpha}\mathcal{F}_{a_+}^{x}(\kappa(x)u(x)) \tag{2.2}$$

we obtain

$$\upsilon(x) = \left[{}^{\alpha}\mathcal{F}_{a_+}^{x}\left(\kappa(x)\sum_{k=0}^{n} c_k R_k(x)\right)\right].$$

Let ε be any arbitrary small number. Replacing x by $x_j = j/n + \varepsilon$ and reorganizing the above equation, we achieve that

$$\sum_{k=0}^{n} c_k \left[{}^{\alpha}\mathcal{F}_{a_+}^{x_j}(\kappa(x_j)R_k(x_j))\right] = \upsilon(x_j),$$

which yields $[\mathbf{R}][\mathbf{C}] = [\mathbf{V}]$, where

$$[R] = \left[{}^{\alpha}\mathcal{F}_{a_+}^{x_j}(\kappa(x_j)R_k)\right]_{(n+1)\times(n+1)}, \quad j,k = 0,1,\cdots,n \tag{2.3}$$

and

$$[V] = \begin{bmatrix} v(x_0) \\ v(x_1) \\ \vdots \\ \vdots \\ v(x_n) \end{bmatrix}_{(n+1)\times 1}, \quad [\mathbf{C}] = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ \vdots \\ c_n \end{bmatrix}_{(n+1)\times 1}.$$

2.2. *Numerical scheme for FVIE of second kind*

In order to solve the second kind FVIE (1.5) numerically, we substitute (2.1) in

$$u(x) = \upsilon(x) + \beta(x)\left[{}^{\alpha}\mathcal{F}_{a_+}^{x}(\kappa(x)u(x))\right]. \tag{2.4}$$

Thus, we have the following equation

$$\sum_{k=0}^{n} c_k R_k(x) = \upsilon(x) + \beta(x){}^{\alpha}\mathcal{F}_{a_+}^{x}\left(\kappa(x)\sum_{k=0}^{n} c_k R_k(x)\right). \tag{2.5}$$

We have following relation if we rearrange the above equation.

$$\sum_{k=0}^{n} c_k \left\{R_k(x) - \beta(x){}^{\alpha}\mathcal{F}_{a_+}^{x}(\kappa(x)R_k(x))\right\} = v(x). \tag{2.6}$$

Let ε be any arbitrary small number. Replacing x by $x_j = j/n+\varepsilon$, we obtain

$$\sum_{k=0}^{n} c_k \left\{ R_k(x_j) - \beta(x_j)\, {}^{\alpha}\mathcal{F}_{a_+}^{x_j} \left(\kappa(x_j) R_k(x_j)\right) \right\} = v(x_j), \tag{2.7}$$

which yields $[\mathbf{R}][\mathbf{C}] = [\mathbf{V}]$, where $[\mathbf{R}]$ is an $(n+1)\times(n+1)$ matrix and given by

$$[\mathbf{R}] = \left[R_k(x_j) - \beta(x_j)\, {}^{\alpha}\mathcal{F}_{a_+}^{x_j} \left(\kappa(x_j) R_k(x_j)\right) \right]_{(n+1)\times(n+1)}, \quad j,k = 0,1,\cdots,n,$$

and $[\mathbf{C}]$ and $[\mathbf{V}]$ are $((n+1)\times 1)$ vectors given in (2.4). The system (2.2) is stable if the matrix $[\mathbf{R}]$ is invertible. Hence we can have the solution matrix $[\mathbf{C}]$ by the help of $[\mathbf{C}] = \left[\mathbf{R}^{-1}\right][\mathbf{V}]$. Finally, the approximate solution of 2nd FVIEs can be computed by substituting the matrix $[\mathbf{C}]$ in equation (1.5).

3. Numerical results and analysis

Since the outcomes of these experiments are computed in different references, they can be compared to those obtained using other computational technique. We provide the following notations for analysing error of the implemented procedure:

$$e_n(x) = |u(x) - u_n(x)| \text{ and } \|e_n\|_\infty = \max\left\{ e_n(x_j), j = 0,1,2,\cdots,n \right\},$$

where $u(x)$ and $u_n(x)$ are exact solution and approximate solution, respectively, of the test problems and x_j is the uniform grid on $[0,1]$. Also the root mean square error can calculated by

$$RmsE = \sqrt{\frac{\sum_{j=1}^{n} \left(u(x_j) - u_n(x_j)\right)^2}{n}}.$$

Experiment 1. Let us consider the Abel integral equation (see [12, 35]) as follows:

$$e^x - 1 = \int_0^x \frac{g(t)}{(x-t)^{1/2}} dt \tag{3.1}$$

with the exact solution

$$g(x) = \frac{e^x}{\sqrt{\pi}} erf(\sqrt{x}); \;\; erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2} ds.$$

Equation (3.1) can be considered as FVIE of first kind as follows:

$$e^x - 1 = \Gamma(1/2) \left[{}^{1/2}\mathcal{F}_{0_+}^{x} u(x) \right], \quad x \in [0,1].$$

Therefore, we have

$$\upsilon(x) = e^x - 1, \; \kappa(x) = \Gamma(1/2), \; a_+ = 0_+ \text{ and } \alpha = \frac{1}{2}.$$

For $n = 3$, we have the following matrices:

$$\mathbf{R} = \begin{bmatrix} 0.2 & -0.197354 & 0.189502 & -0.176691 \\ 1.17189 & -0.749508 & -0.163557 & 0.819708 \\ 1.6452 & -0.672325 & -0.935528 & 1.10479 \\ 2.00998 & -0.486655 & -1.49239 & 0.767512 \end{bmatrix}, \mathbf{V} = \begin{bmatrix} 0.0100502 \\ 0.409639 \\ 0.967309 \\ 1.7456 \end{bmatrix}$$

and therefore the matrix $\mathbf{C}$ can be computed as

$$\mathbf{R}^{-1} \cdot \mathbf{V} = \mathbf{C} = \begin{bmatrix} 2.86872 \\ 4.02452 \\ 1.5845 \\ 0.394493 \end{bmatrix}.$$

Thus, third order approximation will be

$$u_3(x) = \frac{8.87223x^3 - 1.20921x^2 + 2.57654x + 0.0342116}{(x+1)^3}$$

Similarly, for $n = 4$, one can get the following matrices:

$$\mathbf{R} = \begin{bmatrix} 0.2 & -0.197354 & 0.189502 & -0.176691 & 0.159327 \\ 1.0198 & -0.726484 & 0.0440107 & 0.576421 & -0.74932 \\ 1.42829 & -0.733834 & -0.572921 & 1.07927 & -0.372143 \\ 1.74356 & -0.631576 & -1.09533 & 1.0569 & 0.381453 \\ 2.00998 & -0.486655 & -1.49239 & 0.767512 & 0.974483 \end{bmatrix}, \mathbf{V} = \begin{bmatrix} 0.0100502 \\ 0.29693 \\ 0.665291 \\ 1.13828 \\ 1.7456 \end{bmatrix},$$

and the desired matrix can be calculated as:

$$\mathbf{R}^{-1} \cdot \mathbf{V} = \mathbf{C} = \begin{bmatrix} 3.76836 \\ 5.59245 \\ 2.59438 \\ 0.849141 \\ 0.115932 \end{bmatrix}.$$

According to these results, the fourth order approximation becomes

$$u_4(x) = \frac{12.9203x^4 + 0.746768x^3 + 4.78162x^2 + 2.15289x + 0.0370701}{(x+1)^4}.$$

This problem has been considered in [12, 35]. According to the Tables 1, 2 and 3, our new results are slightly better than those obtained by the methods used in [12, 35]. Figure 1 validates the theoretical results given in the problem. Figure 1 proves that new solutions are valid also for larger region.

TABLE 1. Estimated and exact solution of Experiment 1 [12].

x	$n=1$	$n=10$	$n=100$	Exact value
0.1	0.2154319668	0.2152921762	0.2152904646	0.2152905021
0.2	0.3267280013	0.3258941876	0.3258841023	0.3258840762
0.3	0.4300194238	0.4275954299	0.4275658716	0.4275656575

TABLE 2. Approximated and exact values for Experiment 1 [35].

x	$n=10$	$n=20$	$n=30$	Exact value
0.1	0.2152897706	0.2152904948	0.2152905016	0.2152905021
0.2	0.3258833448	0.325884069	0.3258840758	0.3258840763
0.3	0.427564926	0.4275656502	0.427565657	0.4275656576
0.4	0.5293323416	0.5293330658	0.5293330726	0.5293330731
0.5	0.6350311405	0.6350318647	0.6350318715	0.635031872
0.6	0.7470394417	0.7470401659	0.7470401727	0.7470401732
0.7	0.8671868544	0.8671875786	0.8671875854	0.867187586
0.8	0.9970886453	0.9970893695	0.9970893763	0.9970893769
0.9	1.138297847	1.138298572	1.138298578	1.138298579
1.	1.292387361	1.292388086	1.292388092	1.292388093

TABLE 3. Absolute errors of the present method for Experiment 1.

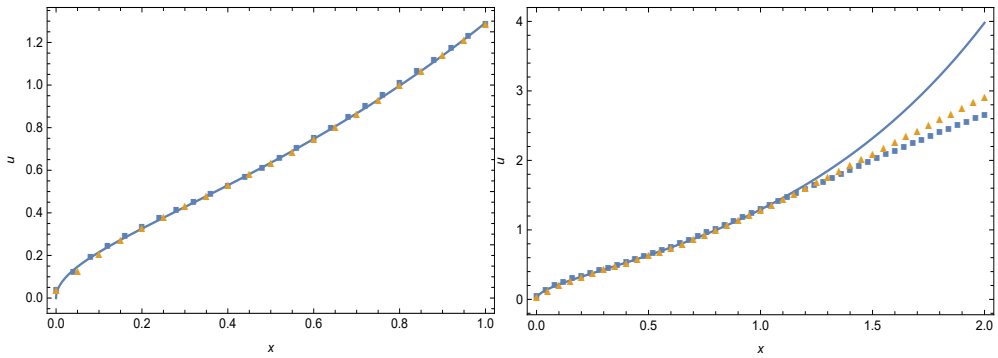
x	$n=10$	$n=20$	$n=30$
0.1	1.1714×10^{-5}	9.0681×10^{-7}	6.0623×10^{-9}
0.2	1.392×10^{-5}	9.2251×10^{-7}	6.2612×10^{-9}
0.3	1.6727×10^{-5}	9.5101×10^{-7}	6.6218×10^{-9}
0.4	2.0269×10^{-5}	9.9437×10^{-7}	7.1813×10^{-9}
0.5	2.4709×10^{-5}	1.0547×10^{-6}	7.9944×10^{-9}
0.6	3.0246×10^{-5}	1.1342×10^{-6}	9.1416×10^{-9}
0.7	3.7123×10^{-5}	1.2353×10^{-6}	1.0744×10^{-8}
0.8	4.5636×10^{-5}	1.3604×10^{-6}	1.2988×10^{-8}
0.9	5.6147×10^{-5}	1.512×10^{-6}	1.618×10^{-8}
1.	6.9096×10^{-5}	1.6929×10^{-6}	2.0842×10^{-8}

Experiment 2. Consider the FVIE of second kind as

$$u(x)=\sqrt{\pi}(1+x)^{-3/2}-0.02\frac{x^3}{1+x}+0.01x^{5/2}\left[{}^{1/2}\mathcal{F}^{x}_{0_+}u(x)\right],\quad x\in[0,1]$$

with the exact solution $u(x)=\sqrt{\pi}(1+x)^{-3/2}$. In this experiment, we have

$$\upsilon(x)=\sqrt{\pi}(1+x)^{-3/2}-0.02\frac{x^3}{1+x},\ \beta(x)=0.01x^{5/2},\ \kappa(x)=1,\ a_+=0_+ \text{ and } \alpha=\frac{1}{2}.$$



(A) Approximations for $n = 3$ (■) and $n = 4$ (▲) with the exact solution (−)

(B) Approximations for $n = 3$ (■) and $n = 4$ (▲) with the exact solution (−) for larger domain

FIGURE 1. Approximations in $x \in [0,1]$ and in $x \in [0,2]$.

For $n = 4$, we have the following matrices:

$$\mathbf{R} = \begin{bmatrix} 1. & -0.980198 & 0.921576 & -0.826457 & 0.698606 \\ 0.999802 & -0.58716 & -0.310162 & 0.951497 & -0.807464 \\ 0.998503 & -0.323734 & -0.788795 & 0.835695 & 0.24668 \\ 0.995047 & -0.134569 & -0.959698 & 0.395946 & 0.852922 \\ 0.988374 & 0.00778994 & -0.991319 & -0.0193642 & 0.994166 \end{bmatrix}, \mathbf{V} = \begin{bmatrix} 1.7462 \\ 1.25292 \\ 0.953476 \\ 0.754124 \\ 0.611735 \end{bmatrix},$$

and therefore the matrix $\mathbf{C}$ can be computed as

$$\mathbf{R}^{-1} \cdot \mathbf{V} = \mathbf{C} = \begin{bmatrix} 0.74613 \\ -0.913927 \\ 0.120294 \\ 0.00870883 \\ 0.000820654 \end{bmatrix}.$$

By proceeding as mentioned in Section 2, the fourth order RC approximation to this problem is obtained as:

$$u_4(x) = \frac{1.77246 + 4.43015x + 3.33129x^2 + 0.530591x^3 - 0.0379734x^4}{(1+x)^4}.$$

Similarly, for $n = 6$, one can get the following matrices:

$$\mathbf{R} = \begin{bmatrix} 1. & -0.980198 & 0.921576 & -0.826457 & 0.698606 & -0.543088 & 0.366061 \\ 0.999938 & -0.699667 & -0.0208101 & 0.728796 & -0.999093 & 0.669365 & 0.0623669 \\ 0.999543 & -0.488542 & -0.522019 & 0.998938 & -0.454573 & -0.554589 & 0.996835 \\ 0.998503 & -0.323734 & -0.788795 & 0.835695 & 0.24668 & -0.996071 & 0.399885 \\ 0.996504 & -0.191414 & -0.923635 & 0.547495 & 0.713269 & -0.823165 & -0.395785 \\ 0.993232 & -0.0828303 & -0.98098 & 0.248905 & 0.940395 & -0.409556 & -0.871125 \\ 0.988374 & 0.00778994 & -0.991319 & -0.0193642 & 0.994166 & 0.0284576 & -0.994792 \end{bmatrix},$$

$$\mathbf{V}=\begin{bmatrix}1.7462\\1.38856\\1.13781\\0.953476\\0.812708\\0.701714\\0.611735\end{bmatrix}.\ \text{Hence}\ \mathbf{R}^{-1}\cdot\mathbf{V}=\mathbf{C}=\begin{bmatrix}0.749586\\-0.907714\\0.124798\\0.0112702\\0.00190131\\0.000307628\\0.000032497\end{bmatrix}.$$

By using this matrix and RC functions, one can find the sixth order approximation as:

$$u_6(x)=\frac{1.77245+7.97598x+13.9589x^2+11.6271x^3+4.37473x^4+0.416709x^5-0.0198182x^6}{(1+x)^6}.$$

Even for $n=6$, we have $\|e_6\|_\infty=\max\{e_6(x_i),i=0,1,2,3\}=3.28592\times10^{-10}$, where the method in [18] gets

$$\|e_{24}\|_\infty=\max\{e_{24}(x_i),i=0,1,2,\cdots,24\}=2.507237\times10^{-6}$$

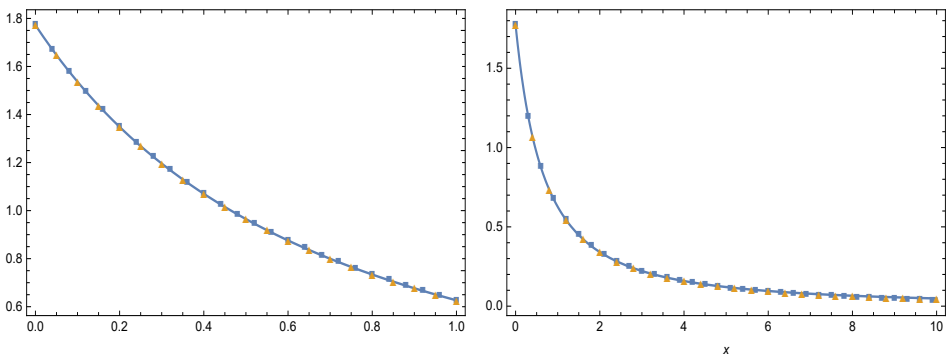
with ten iterations. Table 4 shows the errors obtained by proposed technique. One can deduce that, this method is more accurate than other techniques for solving Experiment 1. Figures are also given to show the efficiency of the technique. Our results are valid also for larger domain as it can be seen from Figure 2. All figures and tables demonstrate that the approximate solutions have high accuracy even for smaller n.

TABLE 4. Comparison of maximum errors $\|e_n\|_\infty$ obtained by present method and the results obtained in [18,31] for Experiment 1.

	Present method	Method in [31]	Method in [18] with 10 iterations
$n=12$	1.81437×10^{-13}	1.046918×10^{-11}	9.966408×10^{-5}
$n=18$	1.10668×10^{-15}	1.221245×10^{-15}	4.446827×10^{-5}
$n=24$	5.05891×10^{-18}	3.774758×10^{-14}	2.507237×10^{-6}

TABLE 5. Absolute and relative errors of approximations for Experiment 1.

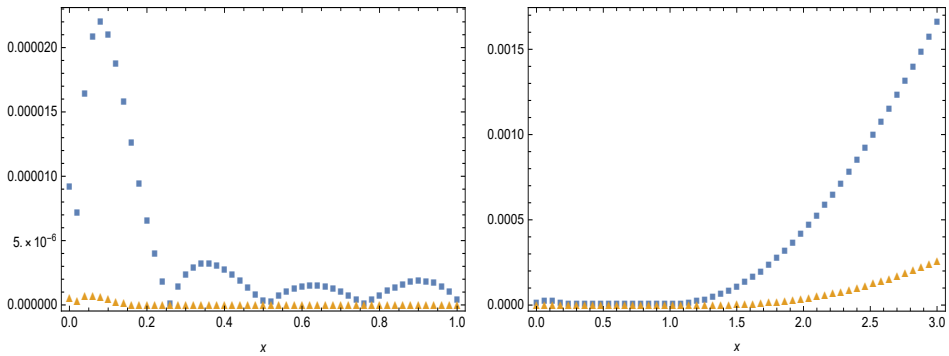
	Absolute errors		Relative Errors	
x	$n=3$	$n=6$	$n=3$	$n=6$
0.1	5.00546×10^{-3}	7.82261×10^{-6}	3.25805×10^{-3}	5.09173×10^{-6}
0.2	4.54269×10^{-3}	3.37788×10^{-5}	3.36907×10^{-3}	2.50519×10^{-5}
0.3	3.7144×10^{-4}	8.17478×10^{-6}	3.1062×10^{-4}	6.83623×10^{-6}
0.4	3.14638×10^{-3}	2.891×10^{-5}	2.94055×10^{-3}	2.70187×10^{-5}
0.5	4.21141×10^{-3}	9.99757×10^{-9}	4.36505×10^{-3}	1.03623×10^{-8}
0.6	2.72061×10^{-3}	2.50915×10^{-5}	3.1065×10^{-3}	2.86505×10^{-5}
0.7	2.72982×10^{-4}	6.16972×10^{-6}	3.41375×10^{-4}	7.7155×10^{-6}
0.8	2.90836×10^{-3}	2.1963×10^{-5}	3.96261×10^{-3}	2.99244×10^{-5}
0.9	2.74717×10^{-3}	4.34802×10^{-6}	4.0592×10^{-3}	6.42461×10^{-6}
1	3.06959×10^{-3}	2.00571×10^{-5}	4.89835×10^{-3}	3.20065×10^{-5}



(A) Approximations for $n=4$ (■) and $n=6$ (▲) with the exact solution (−)

(B) Approximations for $n=4$ (■) and $n=6$ (▲) with the exact solution (−) for larger domain

FIGURE 2. Approximations in $x\in[0,1]$ and in $x\in[0,3]$.



(A) Absolute errors for $n = 4$ (■) and $n = 6$ (▲)

(B) Absolute errors for $n = 4$ (■) and $n = 6$ (▲) for larger domain

FIGURE 3. Absolute errors in $x \in [0,1]$ and in $x \in [0,3]$.

Experiment 3. Consider the FVIE of second kind as

$$u(x) = \Gamma\left(\frac{2}{3}\right)x - \frac{1}{40}x^{8/3} + \frac{1}{27}\left[{}^{2/3}\mathcal{F}_{0_+}^{x}\, xu(x)\right], \quad x \in [0,1]$$

with the exact solution $u(x) = \Gamma\left(\frac{2}{3}\right)x$. In this experiment, we have

$$\upsilon(x) = \Gamma\left(\frac{2}{3}\right)x - \frac{1}{40}x^{8/3},\ \beta(x) = \frac{1}{27},\ \kappa(x) = x,\ a_+ = 0_+ \text{ and } \alpha = \frac{2}{3}.$$

For $n = 4$, we have the following matrices:

$$\mathbf{R} = \begin{bmatrix} 0.999989 & -0.980187 & 0.921566 & -0.826447 & 0.698597 \\ 0.997393 & -0.585534 & -0.309982 & 0.949724 & -0.80536 \\ 0.991986 & -0.320835 & -0.785028 & 0.829785 & 0.248151 \\ 0.98442 & -0.131823 & -0.950695 & 0.388677 & 0.84844 \\ 0.974972 & 0.00900327 & -0.977827 & -0.0241485 & 0.983044 \end{bmatrix}, \mathbf{V} = \begin{bmatrix} 0.0135411 \\ 0.351382 \\ 0.686449 \\ 1.0171 \\ 1.34199 \end{bmatrix}$$

and therefore the matrix $\mathbf{C}$ can be computed as

$$\mathbf{R}^{-1} \cdot \mathbf{V} = \mathbf{C} = \begin{bmatrix} 2.8232 \\ 3.99973 \\ 1.54097 \\ 0.435018 \\ 0.0719605 \end{bmatrix}.$$

By proceeding as mentioned in Section 2, the fourth order RC approximation to this problem is obtained as:

$$u_4(x) = \frac{8.87087x^4 + 5.02323x^3 + 6.56676x^2 + 1.20484x + 0.00138086}{(x+1)^4}$$

Similarly, for $n = 6$, one can get the following matrices:

$$\mathbf{R} = \begin{bmatrix} 0.999989 & -0.980187 & 0.921566 & -0.826447 & 0.698597 & -0.54308 & 0.366055 \\ 0.998631 & -0.698665 & -0.0210517 & 0.728197 & -0.997982 & 0.6683 & 0.0628871 \\ 0.995856 & -0.48636 & -0.520985 & 0.995716 & -0.451944 & -0.554561 & 0.994521 \\ 0.991986 & -0.320835 & -0.785028 & 0.829785 & 0.248151 & -0.992085 & 0.395664 \\ 0.987162 & -0.188449 & -0.916403 & 0.540234 & 0.711094 & -0.815699 & -0.397755 \\ 0.981469 & -0.0804581 & -0.970295 & 0.242074 & 0.933545 & -0.401143 & -0.868101 \\ 0.974972 & 0.00900327 & -0.977827 & -0.0241485 & 0.983044 & 0.0350695 & -0.986688 \end{bmatrix}$$

$$\mathbf{V} = \begin{bmatrix} 0.0135411 \\ 0.238982 \\ 0.463469 \\ 0.686449 \\ 0.907464 \\ 1.1261 \\ 1.34199 \end{bmatrix} \text{ and } \mathbf{R}^{-1} \cdot \mathbf{V} = \mathbf{C} = \begin{bmatrix} 3.59335 \\ 5.3876 \\ 2.55307 \\ 1.01763 \\ 0.323744 \\ 0.0746041 \\ 0.00990615 \end{bmatrix}.$$

Using the above information, the sixth order RC approximation to fractional Volterra integral equation (FVIE) can be computed as:

$$u_6(x) = \frac{12.9599x^6 + 13.4391x^5 + 32.0296x^4 + 18.4223x^3 + 8.48603x^2 + 1.32651x + 0.000241776}{(x+1)^6}.$$

Even for $n = 6$, we have $\|e_6\|_\infty = \max\{e_6(x_i), i = 0,1,2,3\} = 1.40454 \times 10^{-7}$, where the methods in [31] and [18] have found

$$\|e_{24}\|_\infty = \max\{e_{24}(x_i), i = 0,1,2,\cdots,24\} = 1.233508 \times 10^{-7}$$

and

$$\|e_{24}\|_\infty = \max\{e_{24}(x_i), i = 0,1,2,\cdots,24\} = 6.045722 \times 10^{-6}$$

with 24 iterations, respectively. Table 6 shows the errors obtained by proposed technique. One can deduce that this method is more accurate than other techniques for solving Experiment 2. Figures are also given to show the efficiency of the technique. Our results are valid also for larger domain as can be seen from Figure 4. All figures prove that the proposed method is very powerful even for smaller n.

TABLE 6. Comparison of maximum errors $\|e_n\|_\infty$ obtained by present method and the results obtained in [18,31] for Experiment 2.

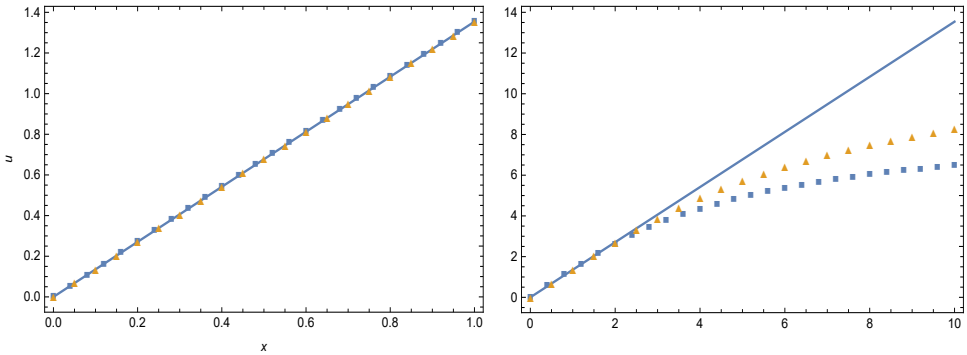
	Present method	Method in [31]	Method in [18] with 10 iterations
$n = 12$	9.089011×10^{-11}	1.714861×10^{-8}	2.492919×10^{-5}
$n = 18$	7.066315×10^{-11}	9.354121×10^{-8}	1.304662×10^{-5}
$n = 24$	7.770991×10^{-13}	1.233508×10^{-7}	6.045722×10^{-6}

Experiment 4. For the last experiment, we assume the FVIE of second kind as follows:

$$u(x) = 2\sqrt{\frac{x}{\pi}} - \left[{}^{1/2}\mathcal{F}_{0_+}^{x} u(x)\right], \quad x \in [0,1]$$

TABLE 7. Absolute and relative errors of approximations for Experiment 1.

	Absolute errors		Relative Errors	
x	$n=4$	$n=6$	$n=4$	$n=6$
0.1	3.288×10^{-3}	2.0572×10^{-4}	2.4282×10^{-2}	1.5192×10^{-3}
0.2	1.0523×10^{-3}	2.2107×10^{-5}	3.8857×10^{-3}	8.1628×10^{-5}
0.3	3.7546×10^{-4}	1.3241×10^{-5}	9.2423×10^{-4}	3.2594×10^{-5}
0.4	4.6486×10^{-4}	7.3716×10^{-6}	8.5823×10^{-4}	1.361×10^{-5}
0.5	4.1574×10^{-5}	9.823×10^{-7}	6.1403×10^{-5}	1.4508×10^{-6}
0.6	2.4577×10^{-4}	3.5124×10^{-6}	3.0249×10^{-4}	4.3231×10^{-6}
0.7	1.7586×10^{-4}	1.4662×10^{-6}	1.8552×10^{-4}	1.5468×10^{-6}
0.8	1.3202×10^{-4}	2.971×10^{-6}	1.2187×10^{-4}	2.7426×10^{-6}
0.9	3.5444×10^{-4}	5.5372×10^{-6}	2.9084×10^{-4}	4.5435×10^{-6}
1	7.4223×10^{-5}	2.8036×10^{-6}	5.4813×10^{-5}	2.0704×10^{-6}



(A) Approximations for $N=4$ (■) and $N=6$ (▲) with the exact solution (−)

(B) Approximations for $N=4$ (■) and $N=6$ (▲) with the exact solution (−) for larger domain.

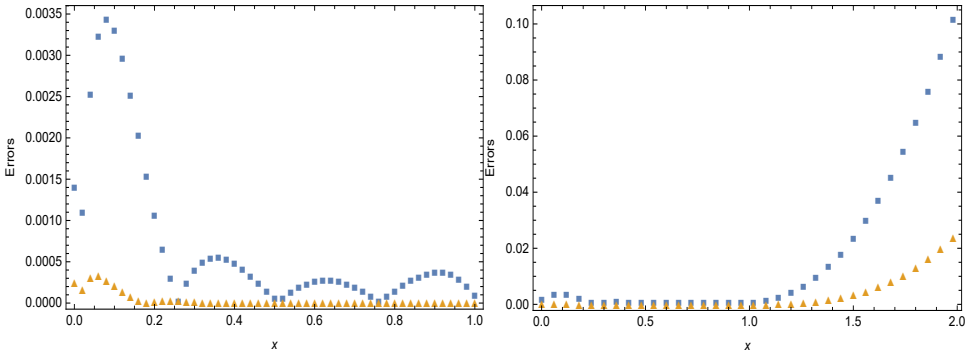
FIGURE 4. Approximations in $x\in[0,1]$ and in $x\in[0,10]$.

with the exact solution

$$u(x)=1-e^{x}\operatorname{erfc}(\sqrt{x});\operatorname{erfc}(z)=1-\frac{2}{\sqrt{\pi}}\int_0^z e^{-s^2}ds.$$

In this experiment, we have

$$\upsilon(x)=2\sqrt{\frac{x}{\pi}},\ \beta(x)=-1,\ \kappa(x)=1,\ a_+=0_+ \text{ and } \alpha=\frac{1}{2}.$$



(A) Absolute errors for $n = 4$ (■) and $n = 6$ (▲)

(B) Absolute errors for $n = 4$ (■) and $n = 6$ (▲)

FIGURE 5. Absolute errors

For $n = 4$, we have the following matrices:

$$\mathbf{R} = \begin{bmatrix} 1.11284 & -1.09154 & 1.02849 & -0.926144 & 0.788496 \\ 1.57536 & -0.997176 & -0.285323 & 1.27682 & -1.23037 \\ 1.80582 & -0.738525 & -1.11263 & 1.44574 & 0.0363306 \\ 1.9837 & -0.492692 & -1.58078 & 0.995238 & 1.06922 \\ 2.13401 & -0.26959 & -1.84194 & 0.418097 & 1.5496 \end{bmatrix}, \mathbf{V} = \begin{bmatrix} 0.112838 \\ 0.575363 \\ 0.805824 \\ 0.983698 \\ 1.13401 \end{bmatrix}$$

and the desired matrix can be calculated as:

$$\mathbf{R}^{-1} \cdot \mathbf{V} = \mathbf{C} = \begin{bmatrix} 0.283994 \\ -0.0961519 \\ -0.355155 \\ -0.118124 \\ -0.0663045 \end{bmatrix}.$$

Thus, we can get the following approximation

$$u_4(x) = \frac{-0.351742x^4 + 5.87456x^3 + 0.614194x^2 + 2.95169x + 0.07681}{(x+1)^4}.$$

By doing the similar calculations for $n = 6$, we have

$$\mathbf{R} = \begin{bmatrix} 1.11284 & -1.09154 & 1.02849 & -0.926144 & 0.788496 & -0.620922 & 0.429959 \\ 1.47428 & -1.076 & 0.109234 & 0.874858 & -1.32292 & 0.999436 & -0.118471 \\ 1.66117 & -0.911698 & -0.61436 & 1.46173 & -0.868575 & -0.508846 & 1.28017 \\ 1.80582 & -0.738525 & -1.11263 & 1.44574 & 0.0363306 & -1.31907 & 0.761709 \\ 1.9282 & -0.572162 & -1.45344 & 1.17315 & 0.789894 & -1.32601 & -0.278787 \\ 2.03623 & -0.415809 & -1.68568 & 0.805609 & 1.28484 & -0.90136 & -1.0784 \\ 2.13401 & -0.26959 & -1.84194 & 0.418097 & 1.5496 & -0.324764 & -1.46415 \end{bmatrix},$$

$$\mathbf{V}=\begin{bmatrix}0.112838\\0.474277\\0.66117\\0.805824\\0.928202\\1.03623\\1.13401\end{bmatrix}.\text{ Hence }\mathbf{C}=\begin{bmatrix}-0.285357\\-1.15588\\-1.19127\\-0.674898\\-0.368269\\-0.119576\\-0.0349047\end{bmatrix}.$$

Substituting the above matrix as coefficients of appropriate RC functions, we get

$$u_6(x)=\frac{-3.83015x^6+21.2857x^5-14.1179x^4+28.9658x^3+0.45665x^2+3.81248x+0.0705518}{(x+1)^6}.$$

Even for $n=6$, we have $\|e_6\|_\infty=\max\{e_6(x_i), i=0,1,2,3\}=1.09464\times10^{-3}$, where the methods in [31] and [18] have found

$$\|e_{12}\|_\infty=\max\{e_{12}(x_i), i=0,1,2,\cdots,24\}=1.543002\times10^{-3}$$

and $\|e_{18}\|_\infty=\max\{e_{18}(x_i), i=0,1,2,\cdots,18\}=6.999936\times10^{-3}$ respectively. Table 8 shows the errors obtained by proposed technique. One can deduce that, this method is more accurate than other techniques for solving Experiment 4. Figures are also given to show the efficiency of the technique. Our results are valid also for larger domain as can be seen from Figure 6. All figures prove that the proposed method gives effective results even for smaller n.

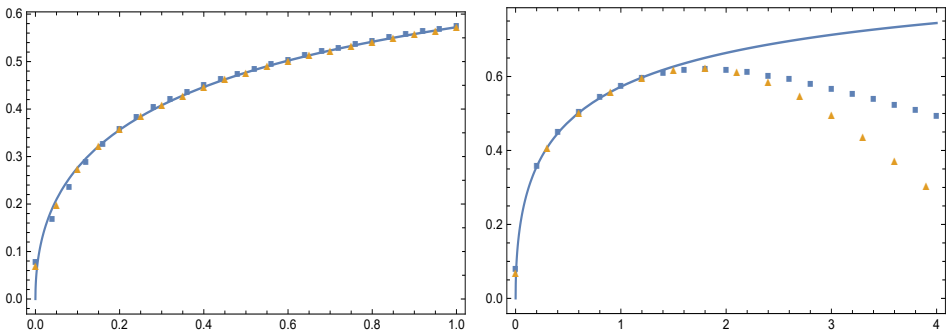
TABLE 8. Comparison of maximum errors $\|e_n\|_\infty$ obtained by present method and the results obtained in [18,31] for Experiment 4.

	Present method	Method in [31]	Method in [18] with 10 iterations
$n=12$	2.121911×10^{-4}	1.543002×10^{-3}	1.007444×10^{-2}
$n=18$	4.099874×10^{-5}	7.725839×10^{-4}	6.999936×10^{-3}
$n=24$	2.106335×10^{-7}	4.577828×10^{-4}	2.381096×10^{-3}

Figures and tables are given to show the efficiency of the technique.

TABLE 9. Absolute and relative errors of approximations for Experiment 4.

	Absolute errors		Relative Errors	
x	For $n=4$	For $n=6$	For $n=4$	For $n=6$
0.1	0.0141718	0.00314341	0.0512688	0.0113718
0.2	0.000237314	0.0011302	0.000666215	0.00317283
0.3	0.00284334	0.000579788	0.00696929	0.00142111
0.4	0.00204502	0.000284363	0.00458121	0.000637024
0.5	0.000892315	0.000272262	0.0018713	0.000570966
0.6	0.000382037	0.000244394	0.000761067	0.000486864
0.7	0.000459034	0.000181115	0.000877195	0.000346103
0.8	0.000745512	0.000152942	0.00137611	0.000282308
0.9	0.00084146	0.000168345	0.00150805	0.000301704
1.	0.00042765	0.000133346	0.000747096	0.000232953



(A) Approximations for $n=4$ (■) and $n=6$ (▲) with the exact solution (−)

(B) Approximations for $n=4$ (■) and $n=6$ (▲) with the exact solution (−) for larger domain

FIGURE 6. Approximations in $x \in [0,1]$ and in $x \in [0,4]$.

4. CONCLUSION

In our present discussion, we have presented the numerical method to solve the fractional Volterra integral equations ($x > a, 0 < \alpha < 1$) of first and second kinds as

$$\upsilon(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\kappa(x)u(s)}{(x-s)^{1-\alpha}} ds \quad \text{and} \quad u(x) = \upsilon(x) + \frac{\beta(x)}{\Gamma(\alpha)} \int_a^x \frac{\kappa(x)u(s)}{(x-s)^{1-\alpha}} ds.$$

Our technique is based on approximating unknown function with rational Chebyshev functions. We have provided numerical experiments in order to demonstrate the significance of our numerical schemes. Moreover, we have compared our illustrative experiments with the existing approximate methods in the literature and seen that our method has less error and hence the results based on our new technique has high accuracy and efficiency. For further studies, one can also investigate (i) some substantially general forms of the Riemann-Liouville fractional integrals [27, 28]; (ii) collocation methods based upon various orthogonal polynomials [11, 14, 26, 29]; (iii) Volterra and related integro-differential equations [4]. Our next aim is to combine rational Chebyshev functions with approximation theory [13, 19, 20] to solve fractional integral equation.

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THE DETERMINANTS OF CERTAIN DOUBLE BANDED $(0,1)$ TOEPLITZ MATRICES

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Abstract. We prove a conjecture proposed recently in Linear Algebra and its Applications about an exact formula for certain double banded $(0,1)$ Toeplitz matrices. Moreover, we extend the result to a more general setting.

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1. Introduction

For any integers $n > t \geqslant 2$, let $\Delta(n,t)$ be the determinant of the $n \times n$ matrix whose (i,j)-entry is 1, if

$$j - i \in \{-2,-1,0,1,t\},$$

and 0, otherwise. In [7], Shitov proves that $\Delta(n,n-k)$ is of period 4, for $n \geqslant 2k$. The procedure is quite ingenious and, as an immediate consequence, two conjectures stated in [2] by Anđelić and da Fonseca on $\Delta(n,n-1)$ and $\Delta(n,n-2)$ are proved. For alternative approaches, the reader is referred to [1, 5, 6]. A more general double banded $(0,1)$ Toeplitz matrix has been considered by the authors in [4], where an explicit formula to the period of the determinants of those matrices is determined.

Based on numerical experiments, Shitov [7] proposes several conjectures towards two directions. One direction is the following:

Conjecture 1. *For any given integer $c \geqslant 0$, the sequence $\Delta(2k+c,k+c)$ admits an exact formula.*

As mentioned in [7, Statements 9 and 10], two straightforward instances of this conjecture are $\Delta(8k,4k) = k^2+1$ and $\Delta(8k+2,4k+2) = k^2$.

The aim of this note is to prove the above conjecture. Indeed, we will obtain a more general result than what Shitov conjectured. A preparatory case will be considered

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in the next section. In Section 3, we prove Conjecture 1. In the last section, the conjecture is extended to a more general setting.

2. The preparatory case

We start this section with the definition of a special double banded $(0,1)$ Toeplitz matrix.

Definition 1. For nonnegative integers n,c,s, let $A_s\left(n,\frac{n+c}{2}\right)$ be the $n\times n$ matrix whose (i,j)-entry is equal to 1, if

$$j-i\in\left\{-s,-s+1,-s+2,-s+3,\frac{n+c}{2}\right\},$$

and 0, otherwise. In particular, we set $A_1(n)$ for the $n\times n$ matrix whose (i,j)-entry is equal to 1, if $j-i=-1,0,1,2$, and 0, otherwise.

Remark 1. Notice that we require

$$n\geqslant\max\{c+2,10-c-2s\},$$

i.e., $n\geqslant c+2$ and $n\geqslant 10-c-2s$. The reason why $n\geqslant c+2$ is because we want to guarantee that the upper band (with entries 1) satisfying $j-i=\frac{n+c}{2}$ contains at least one 1, i.e., $\frac{n-c}{2}\geqslant 1$. On the other hand, we consider $n\geqslant 10-c-2s$ because the matrices should have two disjoint bands (following the type of matrices in Conjecture 1).

We denote $\Delta_s(n,\frac{n+c}{2})=\det A_s\left(n,\frac{n+c}{2}\right)$ and $\Delta_1(n)=\det A_1(n)$. It is worth mentioning that $\Delta_2(2k+c,k+c)$ coincides with the notation $\Delta(2k+c,k+c)$ that Shitov used in Conjecture 1.

Our first goal is to establish a formula for $\Delta_1\left(n,\frac{n+c}{2}\right)$.

Theorem 1. *Assume that* $n\geqslant\max\{c+2,8-c\}$.

(i) *Suppose that c is even.*

- *Assume that* $c\equiv 0\pmod 4$. *Then*

$$\Delta_1\left(n,\frac{n+c}{2}\right)=\begin{cases}1 & \text{if } n\equiv c\pmod 8,\\ -\frac{n-c+6}{8} & \text{if } n\equiv c+2\pmod 8,\\ \frac{n-c+8}{4} & \text{if } n\equiv c+4\pmod 8,\\ -\frac{n-c+2}{8} & \text{if } n\equiv c+6\pmod 8.\end{cases}$$

- *Assume that* $c\equiv 2\pmod 4$. *Then*

$$\Delta_1\left(n,\frac{n+c}{2}\right)=\begin{cases}0 & \text{if } n\equiv c\pmod 8,\\ -\frac{n-c-2}{8} & \text{if } n\equiv c+2\pmod 8,\\ \frac{n-c+4}{4} & \text{if } n\equiv c+4\pmod 8,\\ -\frac{n-c-6}{8} & \text{if } n\equiv c+6\pmod 8.\end{cases}$$

(ii) *Suppose that c is odd.*

- *Assume that $c \equiv 1 \pmod 4$. Then*

$$\Delta_1\left(n, \frac{n+c}{2}\right) = \begin{cases} 1 & \text{if } n \equiv c \pmod 8, \\ \frac{n-c+6}{8} & \text{if } n \equiv c+2 \pmod 8, \\ -\frac{n-c}{4} & \text{if } n \equiv c+4 \pmod 8, \\ \frac{n-c+2}{8} & \text{if } n \equiv c+6 \pmod 8. \end{cases}$$

- *Assume that $c \equiv 3 \pmod 4$. Then*

$$\Delta_1\left(n, \frac{n+c}{2}\right) = \begin{cases} 0 & \text{if } n \equiv c \pmod 8, \\ \frac{n-c+14}{8} & \text{if } n \equiv c+2 \pmod 8, \\ -\frac{n-c+4}{4} & \text{if } n \equiv c+4 \pmod 8, \\ \frac{n-c+10}{8} & \text{if } n \equiv c+6 \pmod 8. \end{cases}$$

The proof of Theorem 1 is based on the following lemmas. They reveal the recurrence relations and initial conditions of $\Delta_1\left(n, \frac{n+c}{2}\right)$.

Notice that, as a special case of [4, Lemma 3.2], the recurrence relations of $\Delta_1\left(n, \frac{n+c}{2}\right)$ follow immediately.

Lemma 1 ([4, Lemma 3.2])**.** *For any integers n, c satisfying $n \geqslant \max\{c+2, 8-c\}$ and $c \geqslant 0$, we have*

$$\Delta_1\left(n, \frac{n+c}{2}\right) = \Delta_1\left(n-4, \frac{n+c}{2}\right) + \begin{cases} 0 & \text{if } n \equiv c \pmod 8, \\ (-1)^{c+1} & \text{if } n \equiv c+2, c+6 \pmod 8, \\ 2(-1)^c & \text{if } n \equiv c+4 \pmod 8. \end{cases}$$

Next, we determine the initial conditions of $\Delta_1\left(n, \frac{n+c}{2}\right)$.

Lemma 2. *When $c \geqslant 3$,*

$$\Delta_1\left(n, \frac{n+c}{2}\right) = \begin{cases} \Delta_1(c-2) + (-1)^{c+1} & \text{if } n = c+2, \\ \Delta_1(c) + 2(-1)^c & \text{if } n = c+4, \\ \Delta_1(c+2) + (-1)^{c+1} & \text{if } n = c+6, \\ \Delta_1(c+4) & \text{if } n = c+8. \end{cases}$$

Proof. The proofs of the four cases are similar, so we only show the first case, i.e., when $n = c+2$. From Lemma 1, we have

$$\Delta_1\left(n, \frac{n+c}{2}\right) = \Delta_1\left(n-4, \frac{n+c}{2}\right) + (-1)^{c+1}.$$

Since $n = c+2$, the above equation is in fact

$$\Delta_1(c+2, c+1) = \Delta_1(c-2, c+1) + (-1)^{c+1}.$$

The result follows now by observing that $A_1(c-2,c+1) = A_1(c-2)$. $\square$

The values of $\Delta_1(n)$ are known from [4, Theorem 3.3].

Lemma 3 ([4, Theorem 3.3])**.** *When $n \geqslant 1$,*

$$\Delta_1(n) = \begin{cases} (-1)^n & \text{if } n \equiv 0 \pmod 4, \\ (-1)^{n+1} & \text{if } n \equiv 1 \pmod 4, \\ 0 & \text{if } n \equiv 2,3 \pmod 4. \end{cases}$$

From Remark 1, we require that $n \geqslant \max\{c+2, 8-c\}$. But in order to make the recurrence relations shown in Lemma 1 applicable, we slightly adjust it into $n \geqslant \max\{c+2, 10-c\}$. Keeping this requirement in mind, we calculate $\Delta_1\left(n, \frac{n+c}{2}\right)$ with small values of n, for each c, which can be regarded as the initial conditions of the sequence $\Delta_1\left(n, \frac{n+c}{2}\right)$.

Lemma 4.

(i) *When $c = 0$,*

$$\Delta_1\left(n, \frac{n+c}{2}\right) = \begin{cases} 1 & \text{if } n = 8, \\ -2 & \text{if } n = 10, 14, \\ 5 & \text{if } n = 12. \end{cases}$$

(ii) *When $c = 1$,*

$$\Delta_1\left(n, \frac{n+c}{2}\right) = \begin{cases} 1 & \text{if } n = 7, 9, \\ 2 & \text{if } n = 11, \\ -3 & \text{if } n = 13. \end{cases}$$

(iii) *When $c = 2$,*

$$\Delta_1\left(n, \frac{n+c}{2}\right) = \begin{cases} 2 & \text{if } n = 6, \\ 0 & \text{if } n = 8, 10, \\ -1 & \text{if } n = 12. \end{cases}$$

(iv) *Assume that $c \geqslant 3$ is odd.*

- *If $c \equiv 1 \pmod 4$, then*

$$\Delta_1\left(n, \frac{n+c}{2}\right) = \begin{cases} 1 & \text{if } n = c+2, c+6, c+8, \\ -1 & \text{if } n = c+4. \end{cases}$$

- *If $c \equiv 3 \pmod 4$, then*

$$\Delta_1\left(n, \frac{n+c}{2}\right) = \begin{cases} 2 & \text{if } n = c+2, c+6, \\ -2 & \text{if } n = c+4, \\ 0 & \text{if } n = c+8. \end{cases}$$

(v) *Assume that $c \geqslant 4$ is even.*

– *If $c \equiv 0 \pmod 4$, then*

$$\Delta_1\left(n, \frac{n+c}{2}\right) = \begin{cases} -1 & \text{if } n = c+2, c+6, \\ 3 & \text{if } n = c+4, \\ 1 & \text{if } n = c+8. \end{cases}$$

– *If $c \equiv 2 \pmod 4$, then*

$$\Delta_1\left(n, \frac{n+c}{2}\right) = \begin{cases} 0 & \text{if } n = c+2, c+6, c+8, \\ 2 & \text{if } n = c+4. \end{cases}$$

Proof. The items (i)-(iii) can be obtained directly. When $c \geqslant 3$, (iv) and (v) follow from Lemmas 2 and 3. □

Finally, when $n \geqslant \max\{c+2, 8-c\}$, we can obtain $\Delta_1\left(n, \frac{n+c}{2}\right)$ as it is shown in Theorem 1, by combining the recurrence relations of Lemma 1 and the initial conditions found in Lemma 4.

3. A CONJECTURE OF SHITOV

In this section, we present the explicit expressions for $\Delta_2\left(n, \frac{n+c}{2}\right)$, which solve Conjecture 1. We remark that, with $n = 2k+c$,

$$\Delta_2\left(n, \frac{n+c}{2}\right) = \Delta_2(2k+c, k+c) = \Delta(2k+c, k+c),$$

according to the notation of Shitov.

Theorem 2. *Assume that $n \geqslant \max\{c+2, 6-c\}$ and $c \geqslant 0$.*

(i) *Suppose that c is even.*

- *Assume that $c \equiv 0 \pmod 4$. Then*

$$\Delta_2\left(n, \frac{n+c}{2}\right) = \begin{cases} \left(\frac{n-c}{8}\right)^2 + 1 & \text{if } n \equiv c \pmod 8, \\ -\left(\frac{n-c+6}{8}\right)^2 & \text{if } n \equiv c+2 \pmod 8, \\ \left(\frac{n-c-4}{8}\right)^2 & \text{if } n \equiv c+4 \pmod 8, \\ \frac{(3n-3c-2)(n-c+2)}{64} & \text{if } n \equiv c+6 \pmod 8. \end{cases}$$

- *Assume that $c \equiv 2 \pmod 4$. Then*

$$\Delta_2\left(n, \frac{n+c}{2}\right) = \begin{cases} \left(\frac{n-c}{8}\right)^2 & \text{if } n \equiv c \pmod 8, \\ -\frac{(n-c)^2+12(n-c)-92}{64} & \text{if } n \equiv c+2 \pmod 8, \\ \frac{(n-c+20)(n-c+4)}{64} & \text{if } n \equiv c+4 \pmod 8, \\ \frac{3(n-c)^2+36(n-c)+124}{64} & \text{if } n \equiv c+6 \pmod 8. \end{cases}$$

(ii) *Suppose that c is odd.*

- *Assume that* $c \equiv 1 \pmod 4$. *Then*

$$\Delta_2\left(n, \frac{n+c}{2}\right) = \begin{cases} \left(\frac{n-c+8}{8}\right)^2 & \text{if } n \equiv c \pmod 8, \\ -\left(\frac{n-c-2}{8}\right)^2 & \text{if } n \equiv c+2 \pmod 8, \\ \frac{(n-c)^2+8(n-c)+80}{64} & \text{if } n \equiv c+4 \pmod 8, \\ \frac{(3n-3c+14)(n-c+2)}{64} & \text{if } n \equiv c+6 \pmod 8. \end{cases}$$

- *Assume that* $c \equiv 3 \pmod 4$. *Then*

$$\Delta_2\left(n, \frac{n+c}{2}\right) = \begin{cases} \frac{(n-c)(n-c-16)}{64} & \text{if } n \equiv c \pmod 8, \\ -\frac{(n-c)^2-4(n-c)-124}{64} & \text{if } n \equiv c+2 \pmod 8, \\ \left(\frac{n-c+4}{8}\right)^2 & \text{if } n \equiv c+4 \pmod 8, \\ \frac{3(n-c)^2-12(n-c)+28}{64} & \text{if } n \equiv c+6 \pmod 8. \end{cases}$$

In order to provide the extension from $s = 1$ (Theorem 1) to $s = 2$ (Theorem 2), we recall the classical Dodgson's determinant-evaluation rule [3].

Lemma 5 ([3]). *For any $n \times n$ matrix A, $n \geqslant 2$, we have*

$$\det A \det A_2 = \det A_{11} \det A_{nn} - \det A_{1n} \det A_{n1},$$

where A_{ij} is the submatrix obtained from A by deleting the ith row and jth column, and A_2 is the principal submatrix of A induced by $\{2, \ldots, n-1\}$.

Applying Dodgson's determinant-evaluation rule to $A_1\left(n+1, \frac{n+c+2}{2}\right)$, we obtain

$$\begin{aligned} &\Delta_1\left(n+1, \frac{n+c+2}{2}\right)\Delta_1\left(n-1, \frac{n+c+2}{2}\right) \\ &= \left(\Delta_1\left(n, \frac{n+c+2}{2}\right)\right)^2 - \Delta_0\left(n, \frac{n+c+4}{2}\right)\Delta_2\left(n, \frac{n+c}{2}\right). \end{aligned} \tag{3.1}$$

Observe that $A_0\left(n, \frac{n+c+4}{2}\right)$ is an upper triangular matrix whose main diagonal entries are all equal to 1. This means that $\Delta_0\left(n, \frac{n+c+4}{2}\right) = 1$. An equivalent form of (3.1) can be obtained immediately as follows:

$$\begin{aligned} \Delta_2\left(n, \frac{n+c}{2}\right) &= \left(\Delta_1\left(n, \frac{n+c+2}{2}\right)\right)^2 \\ &\quad - \Delta_1\left(n+1, \frac{n+c+2}{2}\right)\Delta_1\left(n-1, \frac{n+c+2}{2}\right). \end{aligned} \tag{3.2}$$

The three determinants $\Delta_1(*,*)$ on the right-hand side of (3.2) are known from Theorem 1. It is worth indicating that here we need the precondition that $n \geqslant c+6$, or equivalently, $n \geqslant \max\{c+6, 6-c\}$, which guarantees the using of Theorem 1 to the three determinants $\Delta_1(*,*)$. Then the expression of $\Delta_2\left(n, \frac{n+c}{2}\right)$ can be obtained after some straightforward calculations, proving Theorem 2.

As to the remaining two cases $n = c+2$ and $n = c+4$, they just correspond to the two conjectures proposed in [2], which were confirmed in [6] (see also [1, 4, 5, 7]. Following the notations in the paper, they claim that

$$\Delta_2(c+2, c+1) = \begin{cases} -1 & \text{if } c \equiv 0 \pmod 4, \\ 0 & \text{if } c \equiv 1 \pmod 4, \\ 1 & \text{if } c \equiv 2 \pmod 4, \\ 2 & \text{if } c \equiv 3 \pmod 4, \end{cases}$$

and

$$\Delta_2(c+4, c+2) = \begin{cases} 0 & \text{if } c \equiv 0 \pmod 4, \\ 2 & \text{if } c \equiv 1 \pmod 4, \\ 3 & \text{if } c \equiv 2 \pmod 4, \\ 1 & \text{if } c \equiv 3 \pmod 4, \end{cases}$$

which agree with Theorem 2 as well.

Therefore, setting $n = 2k+c$, Theorem 2 presents the exact formula for $\Delta_2(2k+c, k+c)$, for any fixed integers $k \geqslant 1$ and $k+c \geqslant 3$ (other integers k, c are against the requirement $n \geqslant \max\{c+2, 6-c\}$ mentioned in Remark 1), confirming the conjecture of Shitov.

As examples, we can extend Statements 9 and 10 in [7], claiming that $\Delta_2(8k, 4k) = k^2+1$ and $\Delta_2(8k+2, 4k+2) = k^2$. In fact, from Theorem 2, we have:

Proposition 1. *For all integers $k \geqslant 1$ and $c \geqslant 0$,*

$$\Delta_2(8k+c, 4k+c) = \begin{cases} k^2+1 & \textit{if } c \equiv 0 \pmod 4, \\ (k+1)^2 & \textit{if } c \equiv 1 \pmod 4, \\ k^2 & \textit{if } c \equiv 2 \pmod 4, \\ k(k-2) & \textit{if } c \equiv 3 \pmod 4. \end{cases}$$

4. An extension

It is worth mentioning that the conjecture of Shitov is not only true for $\Delta(2k+c, k+c)$, but also valid for the determinants of a much larger family of matrices. First we introduce this family of matrices, which is a generalization of the matrix $A_s\left(n, \frac{n+c}{2}\right)$ defined in Definition 1.

Definition 2. For nonnegative integers n, r, s, t, let $A_{s,t}(n, r)$ be the $n \times n$ matrix whose (i, j)-entry is equal to 1, if

$$j - i \in \{-s, -s+1, -s+2, -s+3, r, r+1, \ldots, r+t-1\},$$

and 0, otherwise.

Remark 2. In particular, when $t = 1$ and $r = \frac{n+c}{2}$, $A_{s,t}(n, r)$ would be reduced to the matrix $A_s\left(n, \frac{n+c}{2}\right)$ investigated in previous sections.

Set $\Delta_{s,t}(n,r) = \det A_{s,t}(n,r)$. Now we recall several formulae obtained in our previous publication [4, Theorems 3.3 and 3.4], which lead to an explicit expression of $\Delta_{1,t}(n,r)$, in terms of the determinants of form $\Delta_1(*)$.

Assume that $n \equiv p \pmod 4$ and $n-r-\ell \equiv q_\ell \pmod 4$, for any nonnegative integer ℓ. When $n > r$, set

$$\nu = \max\{s \in \mathbb{Z} : n-r-4s > 0\},$$

where $\mathbb{Z}$ represents the set of the integers. Clearly $n-r-4\nu = q_0$, if $q_0 > 0$, and $n-r = 4(\nu+1)$, if $q_0 = 0$.

Theorem 3 ([4, Theorems 3.3 and 3.4])**.**

(*i*) *Assume that* $r = 0,1,2,3$. *For* $n \geqslant 1$, *we have*

$$\Delta_{1,t}(n,r) = \begin{cases} (-1)^n & \text{if } n \equiv 0 \pmod{r+t+1}, \\ (-1)^{n+1} & \text{if } n \equiv 1 \pmod{r+t+1}, \\ 0 & \text{if } n \equiv 2,3,\ldots,r+t \pmod{r+t+1}. \end{cases}$$

(*ii*) *Assume that* $4 \leqslant r \leqslant n-1$. *If* $r \geqslant \frac{n-1}{2}$, *we have*

$$\Delta_{1,t}(n,r) = (-1)^{n-p}\Delta_1(p) + (-1)^n \sum_{j=0}^{\nu} \big((-1)^{q_{\min\{t,n-r-4j\}+1}}\Delta_1(q_{\min\{t,n-r-4j\}+1})$$
$$+ (-1)^{q_1+1}\Delta_1(q_1)\big).$$

(*iii*) *Assume that* $r \geqslant n$. *For* $n \geqslant 1$, *we have*

$$\Delta_{1,t}(n,r) = \begin{cases} (-1)^n & \text{if } n \equiv 0 \pmod 4, \\ (-1)^{n+1} & \text{if } n \equiv 1 \pmod 4, \\ 0 & \text{if } n \equiv 2,3 \pmod 4. \end{cases}$$

Remark 3. We divided the result into three parts. This is due to the fact that $A_{1,t}(n,r)$ has one band when $r = 0,1,2,3$ or $r \geqslant n$, and two (disjoint) bands when $4 \leqslant r \leqslant n-1$.

Recall that the formulae about determinants of form $\Delta_1(*)$ are known (see Lemma 3). Thus we can get the following corollary immediately.

Corollary 1. *The sequence* $\Delta_{1,t}(n,r)$ *admits an exact formula under the conditions mentioned in Theorem 3.*

As in the previous section, based on the formulae of $\Delta_{1,t}(n,r)$ (Theorem 3), we can obtain a formula of $\Delta_{2,t}(n,r)$:

$$\Delta_{2,t}(n,r) = \big(\Delta_{1,\min\{t,n-r-1\}}(n,r+1)\big)^2$$
$$- \Delta_{1,t}(n+1,r+1)\Delta_{1,\min\{t,n-r-2\}}(n-1,r+1),$$

with the help of Lemma 5 (Dodgson's determinant-evaluation rule).

Corollary 2. *The sequence $\Delta_{2,t}(n,r)$ admits an exact formula, when $r = 0,1,2$ with $n \geqslant 2$, or $3 \leqslant r \leqslant n-1$ and $r \geqslant \frac{n-2}{2}$, or $r \geqslant n$ with $n \geqslant 2$.*

The above corollary extends the conjecture of Shitov to a much larger family of matrices.

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ON TRACES OF PERMUTING n-DERIVATIONS ON PRIME IDEALS

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Abstract. In this article we investigate some properties of permuting n-derivations acting on a prime ideal. More precisely, let $n \geq 2$ be a fixed positive integer, P be a prime ideal of a ring R such that R/P is $(n+1)!$-torsion free. If there exists a permuting n-derivation $\Delta\colon R^n \longrightarrow R$ such that the trace δ of Δ satisfies $\overline{[[\delta(x),x],x]} \in Z(R/P)$ for all $x \in R$, then $\Delta(R^n) \subseteq P$ or R/P is commutative.

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1. Introduction

Throughout this article, R will represent an associative ring with center $Z(R)$. Recall that an ideal P of R is said to be prime if $P \neq R$ and for all $x, y \in R$, $xRy \subseteq P$ implies that $x \in P$ or $y \in P$. Therefore, R is called a prime ring if and only if (0) is the only minimal prime ideal of R. R is n-torsion free if whenever $nx = 0$, with $x \in R$ implies $x = 0$. For any $x, y \in R$, the symbol $[x, y]$ will denote the commutator $xy - yx$; while the symbol $x \circ y$ will stand for the anti-commutator $xy + yx$. A mapping $f\colon R \longrightarrow R$ is said to be centralizing on a subset S of R if $[f(s), s] \in Z(R)$ for all $s \in S$. In particular, if $[f(s), s] = 0$ for all $s \in S$, then f is commuting on S. A map $d\colon R \longrightarrow R$ is a derivation of a ring R if d is additive and satisfies $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$.

Suppose n is a fixed positive integer and $R^n = R \times R \times \cdots \times R$, a map $\Delta\colon R^n \longrightarrow R$ is n-additive if it satisfies

$$\Delta(x_1, x_2, \ldots, x_i + x_i', \ldots, x_n) = \Delta(x_1, x_2, \ldots, x_i, \ldots, x_n) + \Delta(x_1, x_2, \ldots, x_i', \ldots, x_n)$$

for all $x_i, x_i' \in R$, $i = 1, 2, \ldots, n$. A map $\Delta\colon R^n \longrightarrow R$ is said to be permuting if $\Delta(x_1, x_2, x_3, \ldots, x_n) = \Delta(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}, \ldots, x_{\pi(n)})$ for all $x_i \in R$ and for every permutation $\pi \in S_n$; where S_n is the symmetric group on n symbols $\{1, 2, 3, \ldots, n\}$. A map $\delta\colon R \longrightarrow R$ is called the trace of Δ if $\delta(x) = \Delta(x, x, x, \ldots, x)$ for all $x \in R$. It is obvious to verify that if $\Delta\colon R^n \longrightarrow R$ is a permuting and n-additive map, then the

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trace δ of Δ satisfies the relation

$$\delta(x+y) = \delta(x) + \delta(y) + \sum_{i=1}^{n-1} \binom{n}{i} \Delta(x,x,\ldots,x,y,y,\ldots,y)$$

where x appears $(n-i)$-times and y appears i-times.

Park [4] introduced the notion of permuting n-derivation as follows: a permuting map $\Delta\colon R^n \longrightarrow R$ is said to be a permuting n-derivation if Δ is n-additive and

$$\Delta(x_1,x_2,\ldots,x_i x_i^{'},\ldots,x_n) = x_i\Delta(x_1,x_2,\ldots,x_i^{'},\ldots,x_n) + \Delta(x_1,x_2,\ldots,x_i,\ldots,x_n)x_i^{'}$$

for all $x_1,x_2,\ldots,x_i,x_i^{'},\ldots,x_n \in R$. Clearly, a 1-derivation is a usual derivation and a 2-derivation is a symmetric bi-derivation. However, in the case of $n=3$ we get the concept of symmetric tri-derivation.

Many results in the literature indicate how the global structure of a ring R is often tightly connected to the behavior of additive maps defined on R. A well known result due to Posner [1] states that a prime ring R which admits a nonzero centralizing derivation is commutative. Since then, several authors have done a great deal of work concerning commutativity of prime and semi-prime rings admitting different kinds of maps which are centralizing on some appropriate subsets of R ([6, 7] and [5] for a further references). More recently several authors have studied various identities involving trace of permuting n-derivations and have obtained interesting theorems. Indeed, motivated by the results due to Posner [1], Vukman obtained some results concerning the trace of symmetric bi-derivation in prime ring (see [2] and [3]). Ashraf [8] proved similar results for semi-prime ring. In [11], Jung and Park obtained the similar results to Posner's and Vukman's ones for permuting 3-derivations on prime and semiprime rings. In the year 2009, Park [4] introduced the concept of symmetric permuting n-derivation and obtained some results related to the commuting traces of permuting n-derivations in rings. Further, Ashraf and Jamal [10] obtained commutativity of rings admitting n-derivations whose traces satisfy certain polynomial conditions. In 2018, the authors in [9] generalize the result of Park by considering more general identities rather than centralizing mappings. More precisely, they proved that if R is $(n+1)!$-torsion free semi-prime ring admitting a permuting n-derivation Δ such that the trace δ of Δ satisfies $[[\delta(x),x],x] \in Z(R)$ for all $x \in R$. Then δ is commuting on R.

The present paper is motivated by the previous results and we here continue this line of investigation by considering a more general concept rather than commuting and centralizing traces of permuting n-derivations. In fact, we define the concepts of S-commuting maps and S-centralizing maps as follows:

Definition 1. Let R be a ring, S a subset of R and $f\colon R \longrightarrow R$ a map. Then

1. f is called S-commuting if $[f(x),x] \in S$ for all $x \in R$;
2. f is called S-centralizing if $[[f(x),x],y] \in S$ for all $x,y \in R$.

1.1. *Preliminary results*

The following fact is essential for the proofs of our results.

Lemma 1 ([4], Lemma 2.4)**.** *Let n be a fixed positive integer and let R be a $n!-$torsion free ring. Suppose that $y_1, y_2, \dots, y_n \in R$ satisfy $\lambda y_1 + \lambda^2 y_2 + \cdots + \lambda^n y_n = 0$ (or $\in Z(R)$) for $\lambda = 1, 2, \dots, n$. Then $y_i = 0$ (or $y_i \in Z(R)$) for all i.*

In ([4], Theorem 2.3) the author proved that if $n \geq 2$ is a fixed positive integer and R is a non-commutative $n!$-torsion free prime ring provided with a permuting n-derivation $\Delta\colon R^n \longrightarrow R$ such that the trace δ of Δ is commuting on R, then $\Delta = 0$. In the following lemma we prove the same result by considering a more general situation.

Lemma 2. *Let $n \geq 2$ be a fixed positive integer, P be a prime ideal of a ring R such that R/P is noncommutative $n!-$torsion free. If there exists a permuting n-derivation $\Delta\colon R^n \longrightarrow R$ such that the trace δ of Δ is P-commuting, then we have $\Delta(R^n) \subseteq P$.*

Proof. We are given that

$$\overline{[\delta(x), x]} = \overline{0} \quad \text{for all } x \in R. \tag{1.1}$$

An easier computation shows that the trace δ of Δ satisfies the relation

$$\delta(x+y) = \delta(x) + \delta(y) + \sum_{k=1}^{n-1} \binom{n}{k} h_k(x,y) \quad \text{for all } x, y \in R, \tag{1.2}$$

where $h_k(x,y) = \Delta(x, x, x, \dots, x, y, y, \dots, y)$; y appears k times and x appears n-k times. Let λ $(1 \leq \lambda \leq n)$ be any integer. By replacing x by $x + \lambda y$ in (1.1) and using (1.1), we obtain

$$\begin{aligned} &\lambda\Big(\overline{[\delta(x), y]} + {}^nC_1\overline{[h_1(x,y), x]}\Big) + \lambda^2\Big({}^nC_1\overline{[h_1(x,y), y]} + {}^nC_2\overline{[h_2(x,y), x]}\Big) + \dots \\ &\quad + \lambda^n\Big(\overline{[\delta(y), x]} + {}^nC_{n-1}\overline{[h_{n-1}(x,y), y]}\Big) = \overline{0} \end{aligned} \tag{1.3}$$

for all $x, y \in R$. From Lemma 1 and equation (1.3) we conclude that

$$\overline{[\delta(x), y]} + n\overline{[h_1(x,y), x]} = \overline{0} \quad \text{for all } x, y \in R. \tag{1.4}$$

Writing xy instead of y in (1.4), one can see that

$$\overline{x}\Big(\overline{[\delta(x), y]} + n\overline{[h_1(x,y), x]}\Big) + n\overline{\delta(x)[y, x]} = \overline{0} \quad \text{for all } x, y \in R. \tag{1.5}$$

Invoking (1.4), equation (1.5) implies that

$$n\overline{\delta(x)[y, x]} = \overline{0} \quad \text{for all } x, y \in R.$$

Since n divides $n!$, we find that R/P is n-torsion free and thus

$$\overline{\delta(x)[y, x]} = \overline{0} \quad \text{for all } x, y \in R. \tag{1.6}$$

Substituting ry for y in (1.6), we get

$$\overline{\delta(x)}(R/P)\overline{[y,x]} = \overline{0} \quad \text{for all } x,y \in R. \tag{1.7}$$

Since P is a prime ideal, then (1.7) implies $[y,x] \in P$ or $\delta(x) \in P$ for all $x,y \in R$. It follows that $\delta(x) \in P$ for all $x \in R$ such that $\bar{x} \notin Z(R/P)$.

Now, let $x \in R$ such that $\bar{x} \in Z(R/P)$ and let $y \in R$ with $\bar{y} \notin Z(R/P)$ $(\overline{\delta(y)} = 0)$. Then $\bar{y} + \lambda\bar{x} \notin Z(R/P)$. Thus we obtain

$$\overline{\delta(y+\lambda x)} = \overline{\delta(y)} + \overline{\delta(x)}\lambda^n + \sum_{k=1}^{n-1} {}^nC_k \overline{h_k(y,x)}\lambda^k = \overline{0}.$$

Therefore

$$\sum_{k=1}^{n-1} {}^nC_k \overline{h_k(y,x)}\lambda^k + \overline{\delta(x)}\lambda^n = \overline{0}. \tag{1.8}$$

Applying Lemma 1, equation (1.8) yields $\delta(x) \in P$ for all $x \in R$ $(\bar{x} \in Z(R/P))$. Hence, we conclude that

$$\delta(x) \in P \quad \text{for all } x \in R. \tag{1.9}$$

Let μ $(1 \leq \mu \leq n-1)$ be any integer. By (1.9) we have

$$\overline{\delta(\mu x + x_n)} = \overline{0} \quad \text{for all } x, x_n \in R. \tag{1.10}$$

For each $k = 1,2,\ldots,n$, let

$$P_k(x) = \Delta(x,x,\ldots,x,x_{k+1},x_{k+2},\ldots,x_n),$$

where $x, x_i \in R, i = k+1, k+2, \ldots, n$.

By viewing of (1.9), equation (1.10) yields

$$\sum_{k=1}^{n-1} {}^nC_k \overline{P_k(x)}\mu^k = \overline{0} \quad \text{for all } x \in R. \tag{1.11}$$

Thus Lemma 1 and (1.11) imply

$$n\overline{P_{n-1}(x)} = \overline{0} \quad \text{for all } x \in R. \tag{1.12}$$

Since R/P is n-torsion free ring, then equation (1.12) forces

$$\overline{P_{n-1}(x)} = \overline{0} \quad \text{for all } x \in R. \tag{1.13}$$

Let ν $(1 \leq \nu \leq n-2)$ be any integer. By (1.13) the relation

$$\overline{P_{n-1}(\nu x + x_{n-1})} = \nu^{n-1}\overline{P_{n-1}(x)} + \overline{P_{n-1}(x_{n-1})} + \sum_{k=1}^{n-2} {}^nC_k \overline{P_k(x)}\nu^k = \overline{0}$$

holds for all $x, x_{n-1} \in R$, and hence one can see that

$$\sum_{k=1}^{n-2} {}^nC_k \overline{P_k(x)}\nu^k = \overline{0} \quad \text{for all } x \in R. \tag{1.14}$$

Using Lemma 1 and (1.14), we get

$$^{n}C_{n-2}\overline{P_{n-2}(x)} = \overline{0} \quad \text{for all } x \in R,$$

in such a way that

$$\overline{P_{n-2}(x)} = \overline{0} \quad \text{for all } x \in R.$$

Now if we continue with the same method as above, we finally obtain

$$^{n}C_{1}\overline{P_{1}(x)} = \overline{0} \quad \text{for all } x \in R,$$

which leads to

$$\overline{P_{1}(x)} = \overline{0} \quad \text{for all } x \in R.$$

Thus we obtain

$$\Delta(x_1, x_2, \ldots, x_n) \in P$$

for all $x_i \in R$. This completes the proof of the lemma. □

2. Main results

In [[9], Theorem 1] it is proved that if $n \geq 2$ is a fixed positive integer and R is a $(n+1)!$-torsion free semiprime ring provided with a permuting n-derivation $\Delta\colon R^n \longrightarrow R$ such that the trace δ of Δ satisfies $[[\delta(x),x],x] = 0$ for all $x \in R$, then δ is commuting. Motivated by this result we continue our investigation of permuting n-derivations with the next theorem which extends the result of [[9], Theorem 1], ([4], Theorem 2.5) to the case of any semiprime ideals rather than the zero ideal. The following result shows that if the trace δ of a permuting n-derivation Δ is P-centralizing then it is P-commuting. In fact, we prove rather a more general result:

Theorem 1. *Let $n \geq 2$ be a fixed positive integer, P be a semiprime ideal of a ring R such that R/P is $(n+1)!$-torsion free. If R admits a permuting n-derivation Δ such that the trace δ of Δ satisfies $[[\delta(x),x],x] \in P$ for all $x \in R$, then δ is P-commuting.*

Proof. We are given that

$$\big[[\delta(x),x],x\big] \in P \quad \text{for all } x \in R. \tag{2.1}$$

Consider a positive integer k, $1 \leq k \leq n+1$. Replacing x by $x+ky$ in (2.1), we obtain

$$kQ_1(x,y) + k^2Q_2(x,y) + \cdots + k^{n+1}Q_{n+1}(x,y) \in P \quad \text{for all } x,y \in R, \tag{2.2}$$

where $Q_i(x,y)$ denotes the sum of the terms in which y appears i times. Using (2.2) together with Lemma 1 we find that

$$\big[[\delta(x),x],y\big] + \big[[\delta(x),y],x\big] + n\big[[\Delta(x,x,x,\ldots,y),x],x\big] \in P \tag{2.3}$$

for all $x,y \in R$. Replacing y by xy in (2.3), we get

$$\begin{aligned} x\big[[\delta(x),x],y\big] + \big[[\delta(x),x],x\big]y + x\big[[\delta(x),y],x\big] + \big[[\delta(x),x],x\big]y + [\delta(x),x][y,x] \\ + nx\big[[\Delta(x,x,x,\ldots,y),x],x\big] + n\big[[\delta(x),x],x\big]y + n[\delta(x),x][y,x] \\ n[\delta(x),x][y,x] + n\delta(x)[[y,x],x] \in P. \end{aligned} \tag{2.4}$$

Using (2.4) and (2.3) we find that

$$(2n+1)[\delta(x),x][y,x]+n\delta(x)\big[[y,x],x\big]\in P. \tag{2.5}$$

Similarly, replacing y by yx in (2.3), one can get

$$(2n+1)[y,x][\delta(x),x]+n\big[[y,x],x\big]\delta(x)\in P. \tag{2.6}$$

Substituting yz for y in (2.5), we have

$$(2n+1)\big([\delta(x),x]y[z,x]+[\delta(x),x][y,x]z\big)+n\delta(x)y\big[[z,x],x\big]+$$
$$n\delta(x)[y,x][z,x]+n\delta(x)[y,x][z,x]+n\delta(x)\big[[y,x],x\big]z\in P.$$

Using equation 2.5, we obtain

$$(2n+1)[\delta(x),x]y[z,x]+n\delta(x)y[[z,x],x]+2n\delta(x)[y,x][z,x]\in P.$$

Replacing y by $\delta(x)$ in the above relation we find that

$$(2n+1)[\delta(x),x]\delta(x)[z,x]+n\delta(x)^2[[z,x],x]+2n\delta(x)[\delta(x),x][z,x]\in P. \tag{2.7}$$

From (2.5) we have

$$(2n+1)\delta(x)[\delta(x),x][y,x]+n\delta(x)^2[[y,x],x]\in P.$$

Now using this relation in (2.7) we get

$$(2n+1)[\delta(x),x]\delta(x)[z,x]-(2n+1)\delta(x)[\delta(x),x][z,x]+2n\delta(x)[\delta(x),x][z,x]\in P.$$

Therefore

$$\Big((2n+1)[\delta(x),x]\delta(x)-\delta(x)[\delta(x),x]\Big)[z,x]\in P. \tag{2.8}$$

Puting $A=(2n+1)[\delta(x),x]\delta(x)-\delta(x)[\delta(x),x]$ and replacing z by yz in (2.8) we get

$$Ay[z,x]\in P \quad \text{for all } x,y,z\in R. \tag{2.9}$$

Taking $z=\delta(x)$ in (2.9) and multiplying by $(2n+1)\delta(x)$ we obtain

$$Ay(2n+1)[\delta(x),x]\delta(x)\in P \quad \text{for all } x,y\in R. \tag{2.10}$$

Substituting $y\delta(x)$ for y and $\delta(x)$ for z in (2.9) we find that

$$Ay\delta(x)[\delta(x),x]\in P \quad \text{for all } x,y\in R. \tag{2.11}$$

Substracting (2.11) from (2.10) one can see that

$$(2n+1)[\delta(x),x]\delta(x)-\delta(x)[\delta(x),x]\in P \quad \text{for all } x\in R. \tag{2.12}$$

Similarly, (2.6) gives

$$(2n+1)\delta(x)[\delta(x),x]-[\delta(x),x]\delta(x)\in P \quad \text{for all } x\in R. \tag{2.13}$$

Combining (2.12) and (2.13), we obtain

$$2n\Big(\delta(x)[\delta(x),x]+[\delta(x),x]\delta(x)\Big)\in P \quad \text{for all } x\in R.$$

Since $2n$ divides $(n+1)!$, then R is $2n$-torsion free and hence for all $x \in R$,

$$\delta(x)[\delta(x),x]+[\delta(x),x]\delta(x) \in P \quad \text{for all } x \in R. \tag{2.14}$$

Using (2.14) together with (2.13) one can see that $(2n+2)[\delta(x),x]\delta(x) \in P$ for all $x \in R$. Since $2(n+1)$ divides $(n+1)!$, then R is $2(n+1)$-torsion free and hence

$$[\delta(x),x]\delta(x) \in P \quad \text{for all } x \in R. \tag{2.15}$$

Similarly, from (2.13) and (2.14) we get

$$\delta(x)[\delta(x),x] \in P \quad \text{for all } x \in R. \tag{2.16}$$

Substituting $x+ky$ for x in equation (2.16), where $1 \leq k \leq 2n$, and implementing Lemma 1 we arrive at

$$\delta(x)[\delta(x),y]+n\delta(x)[\Delta(x,x,\dots,y),x]+n\Delta(x,x,\dots,y)[\delta(x),x] \in P. \tag{2.17}$$

Replacing y by yx in the last equation, one can see that

$$\delta(x)[\delta(x),yx]+n\delta(x)[y\delta(x)+\Delta(x,x,\dots,y)x,x]+ \\ n\big(y\delta(x)+\Delta(x,x,\dots,y)x\big)[\delta(x),x] \in P,$$

and thus

$$\delta(x)y[\delta(x),x]+\delta(x)[\delta(x),y]x+n\delta(x)y[\delta(x),x]+n\delta(x)[y,x]\delta(x)+ \\ n\delta(x)[\Delta(x,x,\dots,y),x]x+ny\delta(x)[\delta(x),x]+n\Delta(x,x,\dots,y)x[\delta(x),x] \in P. \tag{2.18}$$

Using (2.17) together with (2.18), we get

$$(n+1)\delta(x)y[\delta(x),x]+n\delta(x)[y,x]\delta(x)-n\Delta(x,x,\dots,y)[\delta(x),x]x+ \\ n\Delta(x,x,\dots,y)x[\delta(x),x] \in P,$$

hence

$$(n+1)\delta(x)y[\delta(x),x]+n\delta(x)[y,x]\delta(x)-n\Delta(x,x,\dots,y)[[\delta(x),x],x] \in P.$$

Accordingly,

$$(n+1)\delta(x)y[\delta(x),x]+n\delta(x)[y,x]\delta(x) \in P \quad \text{for all } x,y \in R. \tag{2.19}$$

Substituting xy for y in (2.19)

$$(n+1)\delta(x)xy[\delta(x),x]+n\delta(x)x[y,x]\delta(x) \in P \quad \text{for all } x,y \in R. \tag{2.20}$$

Left multiplying (2.19) by x, we obtain

$$(n+1)x\delta(x)y[\delta(x),x]+nx\delta(x)[y,x]\delta(x) \in P \quad \text{for all } x,y \in R. \tag{2.21}$$

Combining (2.20) and (2.21), we get

$$(n+1)[\delta(x),x]y[\delta(x),x]+n[\delta(x),x][y,x]\delta(x) \in P \quad \text{for all } x,y \in R. \tag{2.22}$$

Substituting yz for y in (2.6), we have

$$(2n+1)[yz,x][\delta(x),x]+n\big[[yz,x],x\big]\delta(x) \in P,$$

and thus

$$(2n+1)y[z,x][\delta(x),x]+(2n+1)[y,x]z[\delta(x),x]+ny\big[[z,x],x\big]\delta(x)+$$
$$n[y,x][z,x]\delta(x)+n[y,x][z,x]\delta(x)+n\big[[y,x],x\big]z\delta(x)\in P.$$

Using (2.6) one can see that

$$(2n+1)[y,x]z[\delta(x),x]+2n[y,x][z,x]\delta(x)+n\big[[y,x],x\big]z\delta(x)\in P.$$

Replacing y by $\delta(x)$ in the above relation we get

$$(2n+1)[\delta(x),x]z[\delta(x),x]+2n[\delta(x),x][z,x]\delta(x)\in P \quad \text{for all } x,z\in R. \tag{2.23}$$

Combining (2.22) and (2.23) we obtain

$$[\delta(x),x]z[\delta(x),x]\in P \quad \text{for all } x,z\in R.$$

Since P is a semi-prime ideal, we get $[\delta(x),x]\in P$ for all $x\in R$. □

In ([4], Theorem 2.5) it is proved that if $n\geq 2$ is a fixed positive integer and R is a $n!$-torsion free semiprime ring provided with a permuting n-derivation $\Delta\colon R^n\longrightarrow R$ such that the trace δ of Δ is centralizing on R, then δ is commuting on R. In the following corollary which is an immediate result from Theorem 1 we extends the result of Park to the case of any semiprime ideals rather than the zero ideal.

Corollary 1. *Let $n\geq 2$ be a fixed positive integer, P be a semiprime ideal of a ring R such that R/P is $(n+1)!$-torsion free. If there exists a permuting n-derivation $\Delta\colon R^n\longrightarrow R$ such that the trace δ of Δ is P-centralizing then δ is P-commuting.*

The following corollary extends the main result in ([4], Theorem 2.6), which is an analogue of Posner's Theorem for permuting n-derivations.

Corollary 2. *Let $n\geq 2$ be a fixed positive integer, P be a prime ideal of a ring R such that R/P is $n!$-torsion free. If there exists a permuting n-derivation $\Delta\colon R^n\longrightarrow R$ such that the trace δ of Δ is P-centralizing, then $\Delta(R^n)\subseteq P$ or R/P is commutative.*

Proof. Suppose that R/P is noncommutative, then it follows from Corollary 1 that δ is P-commuting. Hence Lemma 2 gives $\Delta(R^n)\subseteq P$. This guarantees the conclusion of the corollary. □

The fundamental result of [4] is immediate from Corollary 2 as follows

Corollary 3. *Let $n\geq 2$ be a fixed positive integer, R be a $n!-$torsion free prime ring. If there exists a permuting n-derivation $\Delta\colon R^n\longrightarrow R$ such that the trace δ of Δ is centralizing on R, then $\Delta=0$ or R is commutative.*

We continue with the next theorem for symmetric n-derivations which present a more general situation than P-centralizing and the identity $\big[[\delta(x),x],x\big]\in P$ for all $x\in R$. In fact we suggest to extend the result of [[9], Theorem 2] proving that if $n\geq 2$ is a fixed positive integer and R is a $(n+1)!$-torsion free semiprime ring

admitting a permuting n-derivation $\Delta\colon R^n \longrightarrow R$ such that the trace δ of Δ satisfies $[[\delta(x),x],x] \in Z(R)$ for all $x \in R$, then δ is commuting on R.

Theorem 2. *Let $n \geq 2$ be a fixed positive integer, P be a semiprime ideal of a ring R such that R/P is $(n+1)!$-torsion free. If R admits a permuting n-derivation Δ such that the trace δ of Δ satisfies $\overline{\big[[\delta(x),x],x\big]} \in Z(R/P)$ for all $x \in R$, then δ is P-commuting.*

Proof. Assume that

$$\Big[\big[[\delta(x),x],x\big],r\Big] \in P \quad \text{for all } \ r,x \in R. \tag{2.24}$$

Consider a positive integer $k, 1 \leq k \leq n+1$. Replacing x by $x+ky$ in (2.24), we obtain

$$kQ_1(x,y)+k^2Q_2(x,y)+\cdots+k^{n+1}Q_{n+1}(x,y) \in P \quad \text{for all } \ x,y \in R \tag{2.25}$$

where $Q_i(x,y)$ denotes the sum of the terms in which y appears i times. Using (2.25) together with Lemma 1 we have

$$\Big[\big[[\delta(x),x],y\big],r\Big]+\Big[\big[[\delta(x),y],x\big],r\Big]+n\Big[\big[[\Delta(x,x,\ldots,y),x],x\big],r\Big] \in P. \tag{2.26}$$

Replacing y by xy in (2.26), we get

$$\Big[(n+2)\big[[\delta(x),x],x\big]y+x\Big(\big[[\delta(x),x],y\big]+\big[[\delta(x),y],x\big]+ n\big[[\Delta(x,x,\ldots,y),x],x\big]\Big)+n\delta(x)\big[[y,x],x\big]+(2n+1)[\delta(x),x][y,x],r\Big] \in P. \tag{2.27}$$

Combining (2.26) with (2.27), we find that

$$(3n+3)\big[[\delta(x),x],x\big][y,x]+(3n+1)[\delta(x),x]\big[[y,x],x\big]+ n\delta(x)\Big[\big[[y,x],x\big],x\Big] \in P. \tag{2.28}$$

Substituting $\delta(x)$ for y in (2.28) to get

$$(6n+4)\big[[\delta(x),x],x\big][\delta(x),x] \in P.$$

Commuting with x gives

$$(6n+4)\big[[\delta(x),x],x\big]^2 \in P. \tag{2.29}$$

Replacing y by $[\delta(x),x]$ in (2.28), we arrive at

$$(3n+3)\big[[\delta(x),x],x\big]^2 \in P. \tag{2.30}$$

Now combining (2.29) and (2.30) to get

$$2\big[[\delta(x),x],x\big]R\big[[\delta(x),x],x\big] \in P.$$

Since R is $(n+1)!$-torsion free and also the center of a semi-prime ring is free from nilpotent elements, we have $\big[[\delta(x),x],x\big] \in P$. Using the result of Theorem 1 we conclude that δ is P-commuting. $\square$

Combining Theorem 2 with Lemma 2, we can prove the following corollary which generalize the main result of ([9], Corollary 1).

Corollary 4. *Let $n \geq 2$ be a fixed positive integer, P be a prime ideal of a ring R such that R/P is $(n+1)!$-torsion free. If R admits a permuting n-derivation Δ such that the trace δ of Δ satisfies $\overline{\big[[\delta(x),x],x\big]} \in Z(R/P)$ for all $x \in R$, then $\Delta(R^n) \subseteq P$ or R/P is commutative.*

The following example proves that the `semiprimeness` hypothesis in Theorems 1 and 2 is necessary.

Example 1. Let us consider $R = \left\{ \begin{pmatrix} 0 & x_1 & x_2 & x_3 \\ 0 & 0 & y_1 & y_2 \\ 0 & 0 & 0 & z_1 \\ 0 & 0 & 0 & 0 \end{pmatrix} | x_1,x_2,x_3,y_1,y_2,z_1 \in \mathbb{Z} \right\}$ provided with the derivation defined by $d_A(X) = [A,X]$ with $A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

It is straightforward to check that for $X = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ we have $[d_A(X),X] = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0_R$ proving that d_A is not commuting. Moreover, $[[d_A(X),X],Y] = 0$ for all $X,Y \in R$ and thus d_A is centralizing on R. Hence d_A satisfies the condition of Theorem 1 and 2 but d_A in not commuting.

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ADDITIVE MAPPINGS SATISFYING CERTAIN ALGEBRAIC EQUATIONS IN PRIME RINGS

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Abstract. In this paper we give a classification of endomorphisms and additive mappings of a prime ring satisfying certain algebraic identities. Moreover, we provide an example proving that the primeness hypothesis imposed in our theorems is not superfluous.

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1. Introduction

Throughout this article, R will represent an associative ring with center $Z(R)$. For any $x,y \in R$, the symbol $[x,y]$ will denote the commutator $xy-yx$; while the symbol $x \circ y$ will stand for the anti-commutator $xy+yx$. Recall that a ring R is said to be prime if for any $a,b \in R$, $aRb = \{0\}$ implies $a=0$ or $b=0$. R is 2-torsion free if whenever $2x=0$, with $x \in R$, implies $x=0$.

Recently many authors have obtained commutativity of prime and semiprime rings admitting suitably constrained additive mappings, as automorphisms, derivations, skew derivations and generalized derivations acting on appropriate subsets of the rings (for example, see [4], [5], [6], [8], [7], [3], [9][10]). In [2] it is proved that a prime ring R must be commutative if it admits derivation d satisfying any one of the properties $d(xy)-xy \in Z(R)$, $d(xy)+xy \in Z(R)$, $d(xy)-yx \in Z(R)$, $d(xy)+yx \in Z(R)$, $d(x)d(y)-xy \in Z(R)$, $d(x)d(y)+xy \in Z(R)$, for all $x,y \in R$. Further some authors have studied the situations replacing d with a generalized derivation F. More precisely, they proved that the prime ring R must be commutative if R is equipped with a generalized derivation F associated with a nonzero derivation d satisfying any one of the following conditions:

(i) $F(xy)-xy \in Z(R)$ for all $x,y \in I$,
(ii) $F(xy)+xy \in Z(R)$ for all $x,y \in I$,
(iii) $F(xy)-yx \in Z(R)$ for all $x,y \in I$,
(iv) $F(xy)+yx \in Z(R)$ for all $x,y \in I$,
(v) $F(x)F(y)-xy \in Z(R)$ for all $x,y \in I$,

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(vi) $F(x)F(y)+xy \in Z(R)$ for all $x,y \in I$, where I is a nonzero two sided ideal of R.

Inspired by these results, Ali et al. [1] have studied the situations replacing generalized derivation F with a multiplicative generalized derivation. Moreover, The authors in [3] considered the situation when the derivation d is replaced by an endomorphism g satisfying any one of the following conditions:

(i) $g(xy)-xy \in Z(R)$,
(ii) $g(xy)+xy \in Z(R)$,
(iii) $g(xy)-yx \in Z(R) \in Z(R)$, (iv) $g(xy)+yx \in Z(R)$, for all $x,y \in R$.

The present paper is motivated by the previous results and we here continue this line of investigation by examining a more general situations of algebraic identities involving an endomorphism g and an additive mappings ϕ satisfying any one of the following properties :

$$g(xy)-\phi(x)y \in Z(R) \quad \text{for all } x,y \in R$$
$$g(xy)-\phi(y)x \in Z(R) \quad \text{for all } x,y \in R$$

More precisely, we will discuss existence of such mappings and gives classifications of these mappings in case of prime ring.

1.1. *Preliminary considerations*

Lemma 1. *Let R be a prime ring, g an endomorphism of R and ϕ an additive mapping of R. If $g(x)g(y)-\phi(x)y=0$ for all $x,y \in R$, then one of the following assertions holds :*

(1) $g=\phi=I_R$;
(2) $\phi=0$ *and* $g(R^2)=0$.

Proof. Assume that

$$g(x)g(y)-\phi(x)y=0 \quad \text{for all } x,y \in R. \tag{1.1}$$

Substituting yu for y in equation (1.1) we find that

$$g(x)g(y)g(u)-\phi(x)yu=0 \quad \text{for all } u,x,y \in R.$$

Multiplying equation (1.1) by $g(u)$ and comparing with the last relation, one can see that

$$\phi(x)y(g(u)-u)=0 \quad \text{for all } u,x,y \in R$$

and therefore

$$\phi(x)R(g(u)-u)=0 \quad \text{for all } u,x \in R.$$

Since R is prime, then either $\phi=0$ or $g=I_d$.

If $\phi=0$, then equation (1.1) reduces to

$$g(x)g(y)=0 \quad \text{for all } x,y \in R$$

and thus $g(R^2)=0$.

If $g = I_d$, then the hypothesis becomes

$$(\phi(x) - x)y = 0 \quad \text{for all } x, y \in R$$

which, because of the primeness of R, implies that $\phi = I_d$. □

Lemma 2. *Let R be a prime ring, g an endomorphism of R and ϕ an additive mapping of R. If $g(x)g(y) - \phi(y)x = 0$ for all $x, y \in R$, then one of the following assertions holds :*

(1) $\phi = 0$ *and* $g(R^2) = 0$*;*
(2) $\phi = g = 0$.
(3) $\phi = g = I_R$ *and R is a commutative integral domain.*

Proof. Assume that

$$g(x)g(y) - \phi(y)x = 0 \quad \text{for all } x, y \in R. \tag{1.2}$$

Replacing x by xr in equation (1.2) we obtain

$$g(x)g(r)g(y) - \phi(y)xr = 0 \quad \text{for all } r, x, y \in R.$$

Using equation (1.2) one can see that

$$g(x)\Big(g(r)g(y) - g(y)r\Big) = 0 \quad \text{for all } r, x, y \in R.$$

Once again, using (1.2) we find that

$$g(x)\Big(\phi(y) - g(y)\Big)r = 0 \quad \text{for all } r, x, y \in R.$$

Accordingly,

$$g(x)\Big(\phi(y) - g(y)\Big) = 0 \quad \text{for all } x, y \in R.$$

Substituting xr for x and using the hypothesis, we obviously get

$$\phi(r)x\Big(\phi(y) - g(y)\Big) = 0 \quad \text{for all } r, x, y \in R$$

and thus

$$\phi(r)R\Big(\phi(y) - g(y)\Big) = 0 \quad \text{for all } r, y \in R.$$

Since R is prime, then either $\phi = 0$ in which case the relation (1.2) implies that $g(x)g(y) = 0$ and thus $g(R^2) = 0$ or $g = \phi$. In the last case the hypothesis forces

$$g(x)g(y) - g(y)x = 0 \quad \text{for all } x, y \in R. \tag{1.3}$$

Writing xr instead of x we find that

$$g(x)g(r)g(y) - g(y)xr = 0 \quad \text{for all } r, x, y \in R. \tag{1.4}$$

Invoking (1.3), equation (1.4) yields

$$g(r)xg(y) - g(y)xr = 0 \quad \text{for all } r, x, y \in R. \tag{1.5}$$

Substituting rt for r in equation (1.5), it is obvious to see that

$$g(r)g(t)xg(y) - g(y)xrt = 0 \quad \text{for all } r,t,x,y \in R. \tag{1.6}$$

By view of (1.3), the last equation becomes

$$g(t)rxg(y) - g(y)xrt = 0 \quad \text{for all } r,t,x,y \in R. \tag{1.7}$$

Taking $r = rt$ in the last relation and using it we are forced to conclude that

$$g(t)r[t,xg(y)] = 0 \quad \text{for all } r,t,x,y \in R \tag{1.8}$$

which implies that $g(t) = 0$ or $[t,xg(y)] = 0$ for all $t,x,y \in R$.

Now if $[t,xg(y)] = 0$ for all $t,x,y \in R$ then replacing x by xr we obtain

$$[t,x]Rg(y) = 0 \quad \text{for all } t,x,y \in R$$

and thus $[t,x] = 0$ or $g = 0$. Hence $g(t) = 0$ or $[t,x] = 0$ for all $t,x \in R$; using Brauer's trick we conclude that $g = 0$ or R is a commutative integral domain. □

Theorem 1. *Let R be a noncommutative prime ring, g an endomorphism of R and ϕ an additive mapping of R. The following assertions are equivalent:*

(1) $g(x)g(y) - \phi(x)y \in Z(R)$ *for all* $x,y \in R$;
(2) $g = \phi = I_R$ *or* $\big(\phi = 0$ *and* $g(R^2) \subseteq Z(R)\big)$.

Proof. For the nontrivial implication, assume that

$$g(x)g(y) - \phi(x)y \in Z(R) \quad \text{for all } x,y \in R. \tag{1.9}$$

Since $[g(x)g(y),g(z)] = 0$ for all $z \in Z(R)$, then equation (1.9) implies that

$$[\phi(x)y,g(z)] = 0 \quad \text{for all } x,y \in R. \tag{1.10}$$

Substituting yt for y, one can see that

$$\phi(x)y[t,g(z)] = 0 \quad \text{for all } t,x,y \in R \tag{1.11}$$

in such a way that

$$\phi(x)R[t,g(z)] = 0 \quad \text{for all } t,x \in R.$$

By view of the primeness of R we conclude that $\phi = 0$ or $g(z) \in Z(R)$ for all $z \in Z(R)$.

If $\phi = 0$ then the hypothesis becomes

$$g(x)g(y) \in Z(R) \quad \text{for all } x,y \in R$$

which means that $g(R^2) \subseteq Z(R)$.

If $g(Z(R)) \subseteq Z(R)$; replacing y by yz in equation (1.9), where $z \in Z(R)$, we get

$$[g(x)g(y),r]g(z) - [\phi(x)y,r]z = 0 \quad \text{for all } r,x,y \in R. \tag{1.12}$$

Using (1.9), equation (1.12) yields

$$[\phi(x)y,r](g(z) - z) = 0 \quad \text{for all } r,x,y \in R. \tag{1.13}$$

Writing tr instead of r in the last equation, we obtain

$$[\phi(x)y,r]R(g(z) - z) = 0 \quad \text{for all } r,x,y \in R \tag{1.14}$$

which implies that $[\phi(x)y,r]=0$ or $g(z)=z$.

If $[\phi(x)y,r]=0$ for $r,x,y\in R$, then substituting yt for y we find that

$$\phi(x)y[t,r]=0 \quad \text{for all } r,t,x,y\in R.$$

Hence $\phi=0$, because of the noncommutativity of R, which leads to $g(R^2)\subseteq Z(R)$.

If $g(z)=z$ for all $z\in Z(R)$, then considering (1.9) with $x=y=z$, we have

$$(\phi(z)-z)z\in Z(R) \quad \text{for all } z\in Z(R)$$

which implies that $\phi(z)\in Z(R)$. Replacing x by z in equation (1.9), where $z\in Z(R)$, we obtain

$$z[g(y),r]-\phi(z)[y,r]=0 \quad \text{for all } r,y\in R. \tag{1.15}$$

Substitution yx for y and using (1.15) we arrive at

$$\Big((g(y)-y)[x,r]+[y,r](g(x)-x)\Big)\phi(z)=0 \quad \text{for all } r,x,y\in R \tag{1.16}$$

in such a way that $\phi(z)=0$ in which case the hypothesis becomes $zg(y)\in Z(R)$ for all $y\in R$ and $z\in Z(R)$ and thus $g(R)\subseteq Z(R)$ or

$$(g(y)-y)[x,r]+[y,r](g(x)-x)=0 \quad \text{for all } r,x,y\in R.$$

Replacing y by ry we obviously get

$$(g(r)-r)g(y)[x,r]=0 \quad \text{for all } r,x,y\in R. \tag{1.17}$$

Substituting rt for r one can see that

$$(g(r)-r)g(y)R[x,r]=0 \quad \text{for all } r,x,y\in R \tag{1.18}$$

and thus

$$g(r)g(y)-rg(y)=0 \quad \text{for all } r,y\in R. \tag{1.19}$$

Writing yt instead of y and using (1.19), we obtain

$$(g(r)-r)yg(t)=0 \quad \text{for all } r,t,y\in R. \tag{1.20}$$

Hence $g=I_R$ because $g\neq 0$ and thus equation (1.9) reduces to

$$(\phi(x)-x)y\in Z(R) \quad \text{for all } x,y\in R. \tag{1.21}$$

Taking yt instead of y we have

$$(\phi(x)-x)yt\in Z(R) \quad \text{for all } t,x,y\in R \tag{1.22}$$

thereby obtaining

$$(\phi(x)-x)y=0 \quad \text{for all } x,y\in R \tag{1.23}$$

and thus $\phi=I_R$. □

Corollary 1. *Let R be a noncommutative prime ring, g an epimorphism of R and ϕ an additive mapping of R. The following assertions are equivalent:*

(1) $g(x)g(y)-\phi(x)y\in Z(R)$ *for all* $x,y\in R$;
(2) $g=\phi=I_R$.

Corollary 2. *[[3], Lemma 2] Let R be a prime ring and g a nonidentity endomorphism of R. If $g(x)g(y)-xy \in Z(R)$ for all $x,y \in R$, then R is a commutative integral domain.*

Theorem 2. *Let R be a noncommutative prime ring, g an endomorphism of R and ϕ an additive mapping of R. The following assertions are equivalent:*

(1) $g(x)g(y)-\phi(y)x \in Z(R)$ *for all* $x,y \in R$;
(2) $g=\phi=0$ *or* $\big(\phi=0$ *and* $g(R^2) \subseteq Z(R)\big)$.

Proof. For the nontrivial implication, suppose that

$$g(x)g(y)-\phi(y)x \in Z(R) \quad \text{for all } x,y \in R. \tag{1.24}$$

Since $[g(x)g(y),g(z)]=0$ for all $z \in Z(R)$, then equation (1.24) yields

$$[\phi(y)x,g(z)]=0 \quad \text{for all } x,y \in R. \tag{1.25}$$

Substituting xt for x, one can see that

$$\phi(y)x[t,g(z)]=0 \quad \text{for all } t,x,y \in R \tag{1.26}$$

in such a way that

$$\phi(y)R[t,g(z)]=0 \quad \text{for all } t,y \in R.$$

In light of primeness, we conclude that $\phi=0$ in which case the hypothesis becomes $g(x)g(y) \in Z(R)$ for all $x,y \in R$ and thus $g(R^2) \subseteq Z(R)$ or $g(z) \in Z(R)$ for all $z \in Z(R)$. In the last case; replacing x and y by z in equation (1.24), where $z \in Z(R)$, we conclude that

$$\phi(z) \in Z(R) \text{ for all } z \in Z(R).$$

Writing z instead of y and xy instead of x in equation (1.24), we get

$$g(x)g(y)g(z)-xy\phi(z) \in Z(R) \quad \text{for all } x,y \in R. \tag{1.27}$$

Multiplying equation (1.24) by $g(z)$ and using equation (1.27) one can see that

$$\phi(y)xg(z)-xy\phi(z) \in Z(R) \quad \text{for all } x,y \in R. \tag{1.28}$$

Replacing x by xr we obtain

$$\phi(y)xrg(z)-xry\phi(z) \in Z(R) \quad \text{for all } r,x,y \in R$$

and thus

$$\phi(y)xg(z)r-xy\phi(z)r+xyr\phi(z)-xry\phi(z) \in Z(R) \quad \text{for all } r,x,y \in R.$$

Taking $A(x,y,z)=\phi(y)xg(z)-xy\phi(z)$ we get

$$A(x,y,z)r+x[y,r]\phi(z) \in Z(R) \quad \text{for all } r,x,y \in R.$$

Commuting the last equation with u and using equation (1.28), we find that

$$A(x,y,z)[r,u]+[x[y,r],u]\phi(z)=0 \quad \text{for all } r,u,x,y \in R. \tag{1.29}$$

Writing ur instead of u and using equation (1.29), we may write

$$[x[y,r],r]\phi(z) = 0 \quad \text{for all } \ r,x,y \in R. \tag{1.30}$$

So that $[x[y,r],r] = 0$ which leads to a contradiction because R is noncommutative. Hence $\phi(z) = 0$; taking $y = z$ in equation (1.24), where $z \in Z(R)$, we have

$$g(x)g(z) \in Z(R) \ \text{ for all } \ x \in R.$$

Substituting xy for x, it follows that

$$g(x)g(z)g(y) \in Z(R) \ \text{ for all } \ x,y \in R$$

which leads to $g(R) \subseteq Z(R)$ or $g = 0$ in which case the hypothesis forces $\phi = 0$ or $g(z) = 0$ for all $z \in Z(R)$. In the last case replacing x by xz in (1.24), we obtain

$$\phi(y)xz \in Z(R) \ \text{ for all } \ x,y \in R$$

and thus for $x = xr$ one can see that

$$\phi(y)xrz \in Z(R) \ \text{ for all } \ r,x,y \in R.$$

Hence $\phi(y)x = 0$ and therefore $\phi = 0$ which leads to $g(R^2) \subseteq Z(R)$. □

Remark 1. Proceeding on the same lines with necessary variations, one can prove that the same conclusion remains valid for the identity $g(x)g(y) + \phi(x)y \in Z(R)$ for all $x,y \in R$.

Corollary 3. *Let R be a prime ring, g an epimorphism of R and ϕ an additive mapping of R. If $g(x)g(y) - \phi(y)x \in Z(R)$ for all $x,y \in R$, then R is a commutative integral domain.*

Corollary 4. *[[3], Lemma 3] Let R be a prime ring and g an endomorphism of R. If $g(x)g(y) - yx \in Z(R)$ for all $x,y \in R$, then R is a commutative integral domain.*

The following example proves that the `primeness` hypothesis in Theorem 1 and 2 is necessary.

Example 1. Let us consider $R = \mathbb{R}[X] \times M_n(\mathbb{R})$. It is straightforward to check that R is a noncommutative semi-prime ring. Moreover, let $g : R \longrightarrow R$ be the endomorphism defined on R by $g(P,M) = (P(0),0)$ and ϕ the additive mapping defined by $\phi(P,M) = (P^{'},0)$. Hence g and ϕ satisfy the conditions of Theorem 1 and 2 but $\phi \neq 0$ and $\phi \neq I_R$.

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ON COMMON FIXED POINT RESULTS VIA IMPLICIT RELATIONS

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Abstract. There are certain contractive conditions (and contractions) available in the literature that ensure the existence of common fixed points of a couple and a family of mappings. However, to verify the validity of these conditions, each result must be checked separately. Thus, it becomes legitimate to obtain some contraction or contractive conditions that can bypass the computational difficulties of checking contraction and contractive conditions via individual results and ensure the existence of common fixed points simultaneously. In this article, by the virtue of implicit relations, we acquire some contraction and contractive conditions, christened A-contraction and $\mathcal{A}$-contractive conditions, which serve the desired purpose. The utility of the established conditions is exhibited through some typical fixed point results and concrete examples.

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1. Introduction

Throughout the last century, *metric fixed point theory* plays an important role in *nonlinear analysis* due to its simplicity and applicability in different fields. The main aim of this theory is to derive some adequate conditions on a mapping so that we can get the guaranty of existence of a (sometimes unique) fixed point of the mapping. The first one among these adequate conditions is the *contraction* condition which was taken into consideration by Banach [2] in 1922. After this, the contraction condition of Banach have been extended in a variety of ways and as a consequence, a lot of adequate conditions for existence of fixed points of a mapping have been established. Among these adequate conditions, the contraction conditions of Kannan [8], Chatterjea [4], Reich [12], Ćirić [5] are remarkable. In order to verify the validity of these contraction conditions, we need to prove and remember separate results for each contraction conditions. So one may naturally ask whether any contraction condition can

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be derived which can accommodate all of the above mentioned contractions. Fortunately, the answer is *positive* due to the introduction of A-contraction condition by Akram et al. [1]. The main result is given below:

Theorem 1 (cf. [1, p. 29, Theorem 5])**.** *Let a self map T on a complete metric space (X,d) satisfies the condition:*

$$d(Tx,Ty) \leq g(d(x,y),d(x,Tx),d(y,Ty))$$

for all $x,y \in X$ and some $g \in A$, where A is the collection of all functions $g\colon \mathbb{R}_+^3 \to \mathbb{R}_+$ ($\mathbb{R}_+$ is the set of all non-negative real numbers) satisfying

(i) *g is continuous on the set $\mathbb{R}_+^3$ (with respect to the Euclidean metric on $\mathbb{R}^3$);*
(ii) *$u \leq kv$ for some $k \in [0,1)$ whenever $u \leq g(u,v,v)$ or $u \leq g(v,u,v)$ or $u \leq g(v,v,u)$ for all $u,v \in \mathbb{R}_+$.*

Then, T has a unique fixed point.

The mapping satisfying the contraction condition of this result is called an A-contraction mapping. It is to be noted that the contraction conditions of Banach, Kannan, Ćirić are particular cases of A-contraction for different g (see [1] for more details).

On the other hand, in *metric fixed point theory*, among the other type of adequate conditions for giving guaranty of existence of fixed point of a mapping, *contractive* condition is a remarkable one. This notion was initiated by Edelstein in 1962, see [6]. Afterwards, this contractive condition have been extended in many ways. Among all such extensions, $\mathcal{A}$-contractive and $\mathcal{A}'$-contractive conditions due to Garai et al. [7] are one of the most generalized ones, since these two contain a handful number of contractive conditions as particular cases.

Apart from all these, we know that in order to solve different kind of problems, the notion of fixed point has given arise some other notions. Such as in order to solve a pair or a family of equations, the notion of common fixed point arise; in order to find the approximate solution of an equation, the notion of best proximity point arise etc. In the literature, there are many contraction as well as contractive conditions which deal with the existence of common fixed points of a couple of mappings. But in order to verify the validity of these conditions, we need to prove separate results for each separate conditions. So it is now very natural to think about some contraction and contractive conditions which can accommodate a large number of contraction and contractive conditions. With the aim of thinking in this way, we find that if the notion of A-contraction and $\mathcal{A}$-contractive conditions can be established in case of two or more mappings, then our aim will be fulfilled. As a result, in this article, we first formulate the notions of A-contraction and $\mathcal{A}$-contractive conditions in case of a couple of mappings and also for a family of mappings. After this, we examine the validity of existence of common fixed points of mappings satisfying such contraction and contractive conditions. During this verification process, we show that completeness of the underlying metric space can give the guaranty of existence of common

fixed point in case of A-contraction condition but can't give in case of $\mathcal{A}$-contractive condition and also we show that compactness of the underlying metric space can give such guaranty in case of $\mathcal{A}$-contractive condition.

Before going to our main findings, we now recall the definition of orbit of two mappings and the corresponding definition of orbital continuity.

If X is a non-empty set; $T,S\colon X \to X$ are two mappings and $x_0 \in X$, then the orbit of (T,S), $O_{T,S}(x_0)$ is the set

$$O_{T,S}(x_0) = \{x_0, Tx_0, STx_0, TSTx_0, STSTx_0, \dots\}.$$

Moreover if X is a metric space, then the pair (T,S) of mappings is said to be orbitally continuous in pair if for any $x_0 \in X$ and for any sequence $\{x_n\} \in O_{T,S}(x_0)$, $x_n \to y \in X$ implies $Tx_n \to Ty$ and $Sx_n \to Sy$ as $n \to \infty$.

2. A-CONTRACTION AND $\mathcal{A}$-CONTRACTIVE MAPPINGS

In this section, we give the formal definitions of A-contraction and $\mathcal{A}$-contractive conditions in case of a couple and a family of mappings. Before this, we consider two collection of mappings.

Let A be the collection of all functions $g\colon \mathbb{R}^3_+ \to \mathbb{R}$ that satisfies the following conditions:

- (A_1) g is continuous on $\mathbb{R}^3_+$;
- (A_2) there exists $\gamma \in [0,1)$ such that if $u \le g(u,v,v)$ or $u \le g(v,u,v)$ or $u \le g(v,v,u)$, then $u \le \gamma v$;
- (A_3) $g(u,v,w) \le u+v+w$ for all $u,v,w \in \mathbb{R}_+$.

Let $\mathcal{A}$ be the collection of all functions $g\colon \mathbb{R}^3_+ \to \mathbb{R}$ that satisfies the following conditions:

- (A_1) g is continuous on $\mathbb{R}^3_+$;
- (A_2) if $v > 0$ and $u < g(u,v,v)$ or $u < g(v,u,v)$ or $u < g(v,v,u)$, then $u < v$;
- (A_3) $g(u,v,w) \le u+v+w$ for all $u,v,w \in \mathbb{R}_+$.

For examples of particular mappings 'g' belonging to the above two collections, the readers are referred to see [1, 7, 10, 11].

Now, we introduce the notion of A-contraction in case of a couple and a family of mappings.

Definition 1. Let (X,d) be a metric space and let $T,S\colon X \to X$ be two mappings. Then, the pair of mappings (T,S) is said to be A-contraction in pair if there exists $g \in A$ such that

$$d(Tx,Sy) \le g(d(x,y), d(x,Tx), d(y,Sy)) \quad \text{for all } x,y \in X.$$

Definition 2. Let (X,d) be a metric space and let $S_i\colon X \to X$, $i = 1,2,\dots,m$, be m number of mappings. Then, such collection of mappings is said to be A-contraction

in pair if there exists $g_i \in A$, $i = 1,2,\ldots,m$, such that

$$d(S_i x, S_{i+1} y) \leq g_i(d(x,y), d(x,S_i x), d(y,S_{i+1} y)) \quad \text{for all } x,y \in X,$$

where we assume that $S_{m+1} = S_1$.

Next, we introduce the notion of $\mathcal{A}$-contractive condition in case of a couple and a family of mappings.

Definition 3. Let (X,d) be a metric space and let $T,S\colon X \to X$ be two mappings. Then, the pair of mappings (T,S) is said to be $\mathcal{A}$-contractive in pair if there exists $g \in \mathcal{A}$ such that

$$d(Tx,Sy) < g(d(x,y), d(x,Tx), d(y,Sy)) \quad \text{for all } x,y \in X \text{ with } x \neq y.$$

Definition 4. Let (X,d) be a metric space and let $S_i\colon X \to X$, $i = 1,2,\ldots,m$, be m number of mappings. Then, such collection of mappings is said to be $\mathcal{A}$-contractive in pair if there exists $g_i \in \mathcal{A}$, $i = 1,2,\ldots,m$, such that

$$d(S_i x, S_{i+1} y) < g_i(d(x,y), d(x,S_i x), d(y,S_{i+1} y)) \quad \text{for all } x,y \in X \text{ with } x \neq y,$$

where we assume that $S_{m+1} = S_1$.

3. Main results

In the beginning of this section, we prove the following two common fixed point results involving the A-contraction condition.

Theorem 2. *Let (X,d) be a complete metric space and $T,S\colon X \to X$ be two mappings such that (T,S) is A-contraction in pair. If (T,S) is orbitally continuous in pair, then T and S have a unique common fixed point.*

Proof. Let $x_0 \in X$ be arbitrary. We consider the sequence $\{x_n\}$ in X by setting $x_n = Tx_{n-1}$ if n is odd and $x_n = Sx_{n-1}$ if n is even. We show that $\{x_n\}$ is Cauchy. If n is even, then we have

$$\begin{aligned} & d(Tx_n, Sx_{n-1}) \leq g(d(x_n,x_{n-1}), d(x_n,Tx_n), d(x_{n-1},Sx_{n-1})) \\ \Longrightarrow\ & d(x_{n+1},x_n) \leq g(d(x_n,x_{n-1}), d(x_n,x_{n+1}), d(x_{n-1},x_n)) \\ \Longrightarrow\ & d(x_{n+1},x_n) \leq \gamma d(x_n,x_{n-1}) \ \text{ where } \gamma \in [0,1). \end{aligned}$$

Again if n is odd, then we can similarly show that

$$d(x_{n+1},x_n) \leq \gamma d(x_n,x_{n-1}).$$

Thus the infinite series $\sum_{n=0}^{\infty} d(x_n,x_{n+1})$ is convergent and hence the sequence $\{x_n\}$ is Cauchy. Since (X,d) is complete, there exists $\alpha \in X$ such that $\lim_{n\to\infty} x_n = \alpha$. Consequently the two subsequences $\{x_{2n}\}$ and $\{x_{2n-1}\}$ converge to α also. By the orbital continuity of (T,S), it follows that $\{x_{2n+1}\}$ converge to $T\alpha$ and $\{x_{2n}\}$ converge to $S\alpha$. So we have $\alpha = T\alpha = S\alpha$, i.e., α is a common fixed point of T and S.

Next, we show the uniqueness of the common fixed point. Let α and α_1 be two common fixed points T and S. Then, we have

$$\begin{aligned} & d(T\alpha, S\alpha_1) \leq g(d(\alpha,\alpha_1), d(\alpha, T\alpha), d(\alpha_1, S\alpha_1)) \\ \Longrightarrow\ & d(\alpha,\alpha_1) \leq g(d(\alpha,\alpha_1), 0, 0) \\ \Longrightarrow\ & d(\alpha,\alpha_1) \leq 0, \end{aligned}$$

which implies that $\alpha = \alpha_1$. So α is the unique common fixed point of T and S. □

Theorem 3. *Let (X,d) be a complete metric space and $S_i\colon X \to X$, $i = 1,2,\ldots,m$, be m number of mappings such that the collection of mappings is A-contraction in pair. If the collection of mappings is orbitally continuous in pair, then the collection has a unique common fixed point.*

Proof. From Theorem 2, it follows that the the pair of mappings S_i, S_{i+1} have a unique common fixed point. We denote this common fixed point of S_i and S_{i+1} by α_i. Then, from the definition of A-contraction condition, we have

$$\begin{aligned} & d(S_1\alpha_1, S_2\alpha_2) \leq g(d(\alpha_1,\alpha_2), d(\alpha_1, S_1\alpha_1), d(\alpha_2, S_2\alpha_2)) \\ \Longrightarrow\ & d(\alpha_1,\alpha_2) \leq g(d(\alpha_1,\alpha_2), 0, 0) \ \Longrightarrow\ d(\alpha_1,\alpha_2) \leq 0 \ \Longrightarrow\ \alpha_1 = \alpha_2. \end{aligned}$$

Similarly we can show that $\alpha_2 = \alpha_3$, $\alpha_3 = \alpha_4$, $\ldots$, $\alpha_{m-1} = \alpha_m$. So α_1 is the unique common fixed point of the collection of mappings S_i, $i = 1,2,\ldots,m$. □

By choosing a particular g in Theorem 2, we have the following remark:

Remark 1. If we choose $g(u,v,w) = \alpha u$, where $0 \leq \alpha < 1$; $g(u,v,w) = \alpha(v+w)$, where $0 \leq \alpha < \frac{1}{2}$; $g(u,v,w) = \alpha_1 u + \alpha_2 v + \alpha_3 w$, where $0 \leq \alpha_1, \alpha_2, \alpha_3 < 1$ and $\alpha_1 + \alpha_2 + \alpha_3 < 1$; $g(u,v,w) = \alpha \max\{v,w\}$, where $0 \leq \alpha < 1$; $g(u,v,w) = \alpha\sqrt{vw}$, where $0 \leq \alpha < 1$ in Theorem 2, then we can obtain the the common fixed point results of corresponding contraction conditions of Banach [2], Kannan [8], Reich [12], Bianchini [3] and Khan [9] respectively as consequences.

Next, we show by an example that unlikely A-contraction mappings, $\mathcal{A}$-contractive mappings don't posses common fixed points if the underlying metric space is complete.

Example 1. Let us take $X = \{(x,y) : x > 0, y > 0\}$ and define a function $d\colon X \times X \to \mathbb{R}$ by

$$d((x_1,y_1),(x_2,y_2)) = \begin{cases} 0, & \text{if } (x_1,y_1) = (x_2,y_2) \\ 1 + \left|\frac{1}{x_1} - \frac{1}{x_2}\right| + \left|\frac{1}{y_1} - \frac{1}{y_2}\right|, & \text{if } (x_1,y_1) \neq (x_2,y_2). \end{cases}$$

Then, (X,d) is a complete metric space but (X,d) is not compact. Next, we define two mappings $T,S\colon X \to X$ by $T(x,y) = (2x,4y)$ and $S(x,y) = (4x,2y)$ for all $(x,y) \in X$. Let us choose $g \in \mathcal{A}$ defined by $g(u,v,w) = \frac{1}{3}u + \frac{1}{3}v + \frac{1}{3}w$.

Let $(x_1,y_1),(x_2,y_2)\in X$ be arbitrary with $(x_1,y_1)\neq(x_2,y_2)$. Then,

$$d(T(x_1,y_1),S(x_2,y_2))=d((2x_1,4y_1),(4x_2,2y_2))=1+\left|\frac{1}{2x_1}-\frac{1}{4x_2}\right|+\left|\frac{1}{4y_1}-\frac{1}{2y_2}\right|$$

and

$$\begin{aligned}
&g(d((x_1,y_1),(x_2,y_2)),d((x_1,y_1),T(x_1,y_1)),d((x_2,y_2),S(x_2,y_2)))\\
&=g(d((x_1,y_1),(x_2,y_2)),d((x_1,y_1),(2x_1,4y_1)),d((x_2,y_2),(4x_2,2y_2)))\\
&=\frac{1}{3}\left\{1+\left|\frac{1}{x_1}-\frac{1}{x_2}\right|+\left|\frac{1}{y_1}-\frac{1}{y_2}\right|+1+\left|\frac{1}{x_1}-\frac{1}{2x_1}\right|+\left|\frac{1}{y_1}-\frac{1}{4y_1}\right|\right.\\
&\qquad\left.+1+\left|\frac{1}{x_2}-\frac{1}{4x_2}\right|+\left|\frac{1}{y_2}-\frac{1}{2y_2}\right|\right\}\\
&=1+\frac{1}{3}\left|\frac{1}{x_1}-\frac{1}{x_2}\right|+\frac{1}{3}\left|\frac{1}{y_1}-\frac{1}{y_2}\right|+\frac{1}{6x_1}+\frac{1}{4y_1}+\frac{1}{4x_2}+\frac{1}{6y_2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
d(T(x_1,y_1),S(x_2,y_2))<g(&d((x_1,y_1),(x_2,y_2)),d((x_1,y_1),T(x_1,y_1)),d((x_2,y_2),\\
&S(x_2,y_2))).
\end{aligned}$$

Thus (T,S) is $\mathcal{A}$-contractive in pair, but T and S have no common fixed point.

Next, we show that instead of completeness if the underlying space is compact, then $\mathcal{A}$-contractive mappings do posses common fixed points.

Theorem 4. *Let (X,d) be a compact metric space. Let $T,S\colon X\to X$ be two mappings such that (T,S) is $\mathcal{A}$-contractive in pair, commutative and also orbitally continuous in pair. Further assume that card* $(\mathrm{Fix}(T))$*, card* $(\mathrm{Fix}(S))\leq 1$*. Then, T and S have a unique common fixed point.*

Proof. Let $x_0\in X$ be arbitrary. We consider the sequence $\{x_n\}$ in X by setting $x_n=Tx_{n-1}$ if n is odd and $x_n=Sx_{n-1}$ if n is even. Since (X,d) is compact, $\{x_n\}$ has a convergent subsequence, say, $\{x_{n_k}\}$. Let $\lim_{k\to\infty}x_{n_k}=\alpha$.

Next, we consider a sequence of real numbers $\{s_n\}$, defined by $s_n=d(x_n,x_{n+1})$ for all $n\in\mathbb{N}$. We prove that $s_n\to 0$ as $n\to\infty$. First we suppose that $x_n=x_{n+1}$ for some n. Without loss of generality, we assume that n is even. Then, by using the commutativity of T and S, we can show that $x_n,x_{n+1},x_{n+2},\dots$ all are fixed points of T. This together with the fact that card $(\mathrm{Fix}(T))\leq 1$ proves that $x_n=x_{n+1}=x_{n+2}=\dots$. Then, clearly $s_n\to 0$. So we now assume that $x_n\neq x_{n+1}$ for all n. If n is odd, then we have

$$\begin{aligned}
&d(Tx_{n-1},Sx_n)<g(d(x_{n-1},x_n),d(x_{n-1},Tx_{n-1}),d(x_n,Sx_n))\\
\implies\ &d(x_n,x_{n+1})<g(d(x_{n-1},x_n),d(x_{n-1},x_n),d(x_n,x_{n+1}))
\end{aligned}$$

$$\implies s_n < g(s_{n-1}, s_{n-1}, s_n) \implies s_n < s_{n-1}.$$

Again if n is even, then we can similarly show that $s_n < s_{n-1}$. Thus $\{s_n\}$ is a strictly decreasing sequence of non-negative real numbers. Hence $s_n \to b$ for some $b \in \mathbb{R}_+$. Therefore,

$$b = \lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{k\to\infty} d(x_{n_k}, x_{n_k+1}).$$

Now, we have three cases to consider.

Case I: Let all but finitely many n_k in the subsequence $\{x_{n_k}\}$ be even. Then, we have

$$b = \lim_{k\to\infty} d(x_{n_k}, x_{n_k+1}) = \lim_{k\to\infty} d(x_{n_k}, Tx_{n_k}) = d(\alpha, T\alpha).$$

Again

$$b = \lim_{k\to\infty} d(x_{n_k+1}, x_{n_k+2}) = \lim_{k\to\infty} d(Tx_{n_k}, STx_{n_k}) = d(T\alpha, ST\alpha).$$

If $b \neq 0$, then $\alpha \neq T\alpha$. Therefore,

$$\begin{aligned} & d(T\alpha, ST\alpha) < g(d(\alpha, T\alpha), d(\alpha, T\alpha), d(T\alpha, ST\alpha)) \\ \implies & d(T\alpha, ST\alpha) < d(\alpha, T\alpha) \implies b < b, \end{aligned}$$

which is a contradiction. So $b = 0$. Then, $\alpha = T\alpha$ and $T\alpha = ST\alpha \implies \alpha = S\alpha$. Therefore, α is a common fixed point of T and S.

Case II: Let all but finitely many n_k in the subsequence $\{x_{n_k}\}$ be odd. In this case, we can show that $b = d(\alpha, S\alpha) = (S\alpha, TS\alpha)$. Then, as in Case I, we can show that α is a common fixed point of T and S.

Case III: Let infinitely many n_k in the subsequence $\{x_{n_k}\}$ be even and infinitely many n_k be odd. Then, we can extract a subsequence $\{x_{n_{k_r}}\}$ from $\{x_{n_k}\}$ such that all n_{k_r} are even. Then we can show, as in Case I, that α is a common fixed point of T and S.

Next, we show the uniqueness of the common fixed point. Let $\alpha_1 (\neq \alpha)$ be another common fixed point T and S. Then, we have

$$\begin{aligned} & d(T\alpha, S\alpha_1) < g(d(\alpha, \alpha_1), d(\alpha, T\alpha), d(\alpha_1, S\alpha_1)) \\ \implies & d(\alpha, \alpha_1) < g(d(\alpha, \alpha_1), 0, 0) \leq d(\alpha, \alpha_1), \end{aligned}$$

which is a contradiction. So α is the unique common fixed point of T and S. □

Theorem 5. *Let (X, d) be a compact metric space. Let $S_i \colon X \to X$, $i = 1, 2, \ldots, m$, be m number of mappings such that the collection of mappings is $\mathcal{A}$-contractive in pair, orbitally continuous in pair and S_i, S_{i+1} are commutative for $i = 1, 2, \ldots, m$. Further assume that card $(\mathrm{Fix}(S_i)) \leq 1$ for $i = 1, 2, \ldots, m$. Then, the collection of mappings has a unique common fixed point.*

Proof. From Theorem 4, it follows that the the pair of mappings S_i, S_{i+1} have a unique common fixed point. We denote this common fixed point of S_i and S_{i+1} by α_i. If $\alpha_1 \neq \alpha_2$, then from the definition of $\mathcal{A}$-contractiveness, we have

$$\begin{aligned} & d(S_1\alpha_1, S_2\alpha_2) < g(d(\alpha_1,\alpha_2), d(\alpha_1, S_1\alpha_1), d(\alpha_2, S_2\alpha_2)) \\ \implies & d(\alpha_1,\alpha_2) < g(d(\alpha_1,\alpha_2),0,0) \leq d(\alpha_1,\alpha_2) \\ \implies & d(\alpha_1,\alpha_2) < d(\alpha_1,\alpha_2), \end{aligned}$$

which is a contradiction. So we must have $\alpha_1 = \alpha_2$. Similarly we can show that $\alpha_2 = \alpha_3$, $\alpha_3 = \alpha_4$, $\ldots$, $\alpha_{m-1} = \alpha_m$. So α_1 is the unique common fixed point of the collection of mappings S_i, $i = 1, 2, \ldots, m$. □

Finally, we show that if we add some mild additional condition on $\mathcal{A}$-contractive mappings, then such mappings also do posses common fixed points in complete metric spaces.

Theorem 6. *Let (X,d) be a complete metric space. Let $T,S\colon X \to X$ be two mappings such that (T,S) is $\mathcal{A}$-contractive in pair, commutative and also orbitally continuous in pair. Further assume that card* $(\mathrm{Fix}(T))$, *card* $(\mathrm{Fix}(S)) \leq 1$. *If for any $\varepsilon > 0$ and for any $x_0 \in X$, there exists a positive $\delta = \delta(\varepsilon, x_0)$ such that*

$$g(d(x,y), d(x,Tx), d(y,Sy)) < \varepsilon + \delta \implies d(Tx,Sy) \leq \frac{\varepsilon}{4} \quad \textit{for all } x,y \in O_{T,S}(x_0),$$

then T and S have a unique common fixed point.

Proof. Let $x_0 \in X$ be arbitrary. We consider the sequences $\{x_n\}$ in X and $\{s_n\}$ in $\mathbb{R}_+$ which we have considered in Theorem 4. Then, $\{s_n\}$ is strictly decreasing and hence $\{s_n\}$ is convergent to some $b \in \mathbb{R}_+$. We now show that $b = 0$. If possible let $b > 0$. Then, there exists $\delta > 0$ such that

$$g(d(x,y), d(x,Tx), d(y,Sy)) < 4b + \delta \implies d(Tx,Sy) \leq b \ \text{ for all } \ x,y \in O_{T,S}(x_0).$$

Since $\{s_n\}$ converges to b, for the above $\delta > 0$, there exists $n \in \mathbb{N}$ such that

$$s_n < b + \frac{\delta}{4}.$$

Without loss of generality, we assume that n is odd. Therefore,

$$\begin{aligned} & g(d(x_{n+1},x_n), d(x_{n+1},x_{n+2}), d(x_n,x_{n+1})) \leq d(x_{n+1},x_n) + d(x_{n+1},x_{n+2}) + d(x_n,x_{n+1}) \\ & = s_n + s_{n+1} + s_n < 3s_n < 3\left(b + \frac{\delta}{4}\right) < 4b + \delta. \end{aligned}$$

This implies that

$$d(Tx_{n+1}, Sx_n) \leq b \implies d(x_{n+2}, x_{n+1}) \leq b \implies s_{n+1} \leq b, \quad \text{a contradiction.}$$

Therefore $b = 0$, i.e., $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. Next, we show that $\{x_n\}$ is Cauchy. For this, let $\varepsilon > 0$ be arbitrary. Then, there exists $\delta > 0$ such that

$$g(d(x,y), d(x,Tx), d(y,Sy)) < \varepsilon + \delta \implies d(Tx,Sy) \leq \frac{\varepsilon}{4} \quad \text{for all } x,y \in O_{T,S}(x_0).$$

Without loss of generality, we show that $\delta \leq \varepsilon$. Since $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$, we get $N_1 \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) \leq \frac{\delta}{16} \quad \text{for all } n \geq N_1. \tag{3.1}$$

Let $n \geq N_1 + 1$ be arbitrary. Then, we show by induction that

$$d(x_n, x_{n+k}) \leq \varepsilon \quad \text{for all } k \in \mathbb{N}. \tag{3.2}$$

The statement for $k = 1$, follows from (3.1). Let (3.2) be true for $k = 1, 2, \ldots, m$. Our next aim is to show that $d(x_n, x_{n+m+1}) \leq \varepsilon$. To show this, we first assume that n and $n+m+1$ both are even. Then, we have

$$d(x_{n-2}, x_{n+m}) \leq d(x_{n-2}, x_{n-1}) + d(x_{n-1}, x_n) + d(x_n, x_{n+m}) \leq \frac{\delta}{8} + \varepsilon.$$

Therefore,

$$\begin{aligned} &g(d(x_{n-2}, x_{n+m}), d(x_{n-2}, Tx_{n-2}), d(x_{n+m}, Sx_{n+m})) \\ &\leq d(x_{n-2}, x_{n+m}) + d(x_{n-2}, x_{n-1}) + d(x_{n+m}, x_{n+m+1}) \\ &\leq \frac{\delta}{8} + \varepsilon + \frac{\delta}{16} + \frac{\delta}{16} < \varepsilon + \delta. \end{aligned}$$

Therefore,

$$d(x_{n-1}, x_{n+m+1}) \leq \frac{\varepsilon}{4}$$

and so

$$d(x_n, x_{n+m+1}) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n+m+1}) \leq \frac{\delta}{16} + \frac{\varepsilon}{4} \leq \varepsilon.$$

When both n and $n+m+1$ are odd or only one of n and $n+m+1$ is even, then by proceeding as in the above way, we can show that $d(x_n, x_{n+m+1}) \leq \varepsilon$. Thus by Principle of Mathematical induction, it follows that

$$d(x_n, x_{n+k}) \leq \varepsilon \text{ for all } n \geq N_1 + 1 \text{ and } k = 1, 2, \ldots.$$

Thus $\{x_n\}$ is a Cauchy sequence in X and hence there exists $\alpha \in X$ such that $x_n \to \alpha$ as $n \to \infty$. Then, by proceeding to the similar way of Theorem 4, we can show that α is the unique common fixed point of T and S. □

Theorem 7. *Let (X,d) be a complete metric space. Let $S_i \colon X \to X$, $i = 1, 2, \ldots, m$, be m number of mappings such that the collection of mapping is $\mathcal{A}$-contractive in pair, orbitally continuous in pair and S_i, S_{i+1} are commutative for $i = 1, 2, \ldots, m$.*

Further assume that card $(\mathrm{Fix}(S_i)) \leq 1$ *for* $i = 1,2,\ldots,m$. *If for any* $\varepsilon > 0$ *and for any* $x_0 \in X$, *there exists a positive* $\delta = \delta(\varepsilon, x_0)$ *such that*

$$g(d(x,y), d(x,S_ix), d(y,S_{i+1}y)) < \varepsilon + \delta \implies d(S_ix, S_{i+1}y) \leq \frac{\varepsilon}{4}$$

for all $x,y \in O_{T,S}(x_0)$, *then the collection of mappings has a unique common fixed point.*

Proof. Follows from the proofs of Theorem 6 and Theorem 5. □

Again by picking out particular g in Theorem 4, we have the following remark.

Remark 2. If we choose $g(u,v,w) = u$; $g(u,v,w) = \frac{1}{2}(v+w)$; $g(u,v,w) = \alpha_1 u + \alpha_2 v + \alpha_3 w$, where $0 \leq \alpha_1, \alpha_2, \alpha_3 < 1$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$; $g(u,v,w) = \max\{v,w\}$; $g(u,v,w) = \sqrt{vw}$ in Theorem 4, then we can obtain the the common fixed point results of corresponding contractive conditions of Banach [2], Kannan [8], Reich [12], Bianchini [3] and Khan [9] respectively as consequences.

We finish this section by demonstrating three supporting examples.

Example 2. Let us consider the set $\mathbb{R}^2$ equipped with the metric

$$d((x_1,y_1),(x_2,y_2)) = |x_1 - y_1| + |x_2 - y_2| \quad \text{for all } (x_1,y_1),(x_2,y_2) \in \mathbb{R}^2.$$

Let $X = \{(x,y) \in \mathbb{R}^2 : 3 \leq x \leq y\}$. Then, (X,d) is a complete metric space. Let us define $T,S\colon X \to X$ by $T(x,y) = (3,3)$ and $S(x,y) = \left(\frac{3+x}{2}, 3\right)$. Then, (T,S) is orbitally continuous in pair. We choose $g \in A$ defined by $g(u,v,w) = \frac{1}{3}v + \frac{1}{3}w$.

Let $(x_1,y_1),(x_2,y_2) \in X$ be arbitrary. Then,

$$d(T(x_1,y_1), S(x_2,y_2)) = d\left((3,3), \left(\frac{3+x_2}{2}, 3\right)\right) = \frac{1}{2}(x_2 - 3)$$

and

$$\begin{aligned}
&\frac{1}{3}d((x_1,y_1), T(x_1,y_1)) + \frac{1}{3}d((x_2,y_2), S(x_2,y_2)) \\
&= \frac{1}{3}d((x_1,y_1),(3,3)) + \frac{1}{3}d\left((x_2,y_2), \left(\frac{3+x_2}{2}, 3\right)\right) \\
&= \frac{1}{3}\left\{x_1 - 3 + y_1 - 3 + \frac{1}{2}(x_2 - 3) + y_2 - 3\right\}.
\end{aligned}$$

Now

$$\begin{aligned}
&\frac{1}{2}(x_2 - 3) - \frac{1}{3}\left\{x_1 - 3 + y_1 - 3 + \frac{1}{2}(x_2 - 3) + y_2 - 3\right\} \\
&= \frac{1}{6}\left\{3(x_2 - 3) - 2\left(x_1 + y_1 - 6 + \frac{x_2}{2} - \frac{9}{2} + y_2\right)\right\} \\
&= \frac{1}{6}\{3x_2 - 9 - 2x_1 - 2y_1 + 12 - x_2 + 9 - 2y_2\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{6}\{2x_2 - 2y_2 - 2x_1 - 2y_1 + 12\} \\
&\leq \frac{1}{6}(2x_2 - 2y_2), \text{ since } x_1, y_1 \geq 3 \\
&\leq 0, \text{ since } x_2 \leq y_2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{1}{2}(x_2 - 3) \leq \frac{1}{3}\left\{x_1 - 3 + y_1 - 3 + \frac{1}{2}(x_2 - 3) + y_2 - 3\right\} \\
\implies\ & d(T(x_1,y_1), S(x_2,y_2)) \leq \frac{1}{3}d((x_1,y_1), T(x_1,y_1)) + \frac{1}{3}d((x_2,y_2), S(x_2,y_2)) \\
\implies\ & d(T(x_1,y_1), S(x_2,y_2)) \leq g(d((x_1,y_1),(x_2,y_2)), d((x_1,y_1), T(x_1,y_1)), \\
& \qquad d((x_2,y_2), S(x_2,y_2))).
\end{aligned}
$$

Thus (T,S) is A-contraction in pair and so by Theorem 2, T and S have a unique common fixed point in X and $(3,3)$ is the unique common fixed point of T and S.

Example 3. Let us take $X = \{(x,y) : 0 \leq x \leq 2, 0 \leq y \leq 1\}$ and define a function $d\colon X \times X \to \mathbb{R}$ by $d((x_1,y_1),(x_2,y_2)) = |x_1 - y_1| + |x_2 - y_2|$ for all $(x_1,y_1),(x_2,y_2) \in X$. Then, (X,d) is a compact metric space. Next, we define two mappings $T,S\colon X \to X$ by

$$
T(x,y) = \left(\frac{x}{3}, 0\right) \text{ and } S(x,y) = \left(0, \frac{y}{4}\right)
$$

for all $(x,y) \in X$. Also we choose $g \in \mathcal{A}$ defined by $g(u,v,w) = \frac{1}{2}v + \frac{1}{2}w$.

Let $(x_1,y_1),(x_2,y_2) \in X$ be arbitrary with $(x_1,y_1) \neq (x_2,y_2)$. Then,

$$
d(T(x_1,y_1), S(x_2,y_2)) = d\left(\left(\frac{x_1}{3}, 0\right), \left(0, \frac{y_2}{4}\right)\right) = \frac{x_1}{3} + \frac{y_2}{4}
$$

and

$$
\begin{aligned}
& g(d((x_1,y_1),(x_2,y_2)), d((x_1,y_1), T(x_1,y_1)), d((x_2,y_2), S(x_2,y_2))) \\
&= \frac{1}{2}d\left((x_1,y_1), \left(\frac{x_1}{3}, 0\right)\right) + \frac{1}{2}d\left((x_2,y_2), \left(0, \frac{y_2}{4}\right)\right) \\
&= \frac{1}{2}\left\{\left|x_1 - \frac{x_1}{3}\right| + y_1 + x_2 + \left|y_2 - \frac{y_2}{4}\right|\right\} = \frac{x_1}{3} + \frac{3y_2}{8} + \frac{1}{2}(x_2 + y_1).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
d(T(x_1,y_1), S(x_2,y_2)) < g(&d((x_1,y_1),(x_2,y_2)), d((x_1,y_1), T(x_1,y_1)), d((x_2,y_2), \\
&S(x_2,y_2))).
\end{aligned}
$$

Hence (T,S) is $\mathcal{A}$-contractive in pair and so by Theorem 4, T and S have a unique common fixed point in X. Indeed $(0,0)$ is the unique common fixed point of T and S.

Example 4. Let us consider the set $C\left[0,\frac{1}{12}\right]$ of all real valued continuous functions of real numbers defined on $\left[0,\frac{1}{12}\right]$ and consider the sup metric d on $C\left[0,\frac{1}{12}\right]$. Then, $\left(C\left[0,\frac{1}{12}\right],d\right)$ is a complete metric space. Also we take $g\in\mathcal{A}$ defined by $g(u,v,w)=\frac{1}{2}u+\frac{1}{2}\max\{v,w\}$. After this, we consider two self-mappings T and S on X defined by

$$T(x(t))=\frac{t}{2}x(t) \text{ and } S(x(t))=tx(t)$$

for all $x\in C\left[0,\frac{1}{12}\right]$. Then, for $x,y\in C\left[0,\frac{1}{12}\right]$ and $t\in\left[0,\frac{1}{12}\right]$, we have

$$\begin{aligned}
|T(x(t))-S(y(t))| &= \left|\frac{t}{2}x(t)-ty(t)\right| \leq \left|\frac{t}{2}x(t)-\frac{t}{2}y(t)\right|+\left|\frac{t}{2}y(t)\right| \\
&= \frac{t}{2}|x(t)-y(t)|+\frac{t}{2}|y(t)| \leq \frac{t}{2}|x(t)-y(t)|+\frac{1}{2}(1-t)|y(t)| \\
&= \frac{t}{2}|x(t)-y(t)|+\frac{1}{2}|y(t)-ty(t)| \\
&= \frac{t}{2}|x(t)-y(t)|+\frac{1}{2}|y(t)-Sy(t)| \\
&\leq \frac{t}{2}d(x,y)+\frac{1}{2}d(y,Sy) \\
&\leq \frac{1}{4}d(x,y)+\frac{1}{2}\max\{d(x,Tx),d(y,Sy)\} \\
\implies d(Tx,Sy) &\leq \frac{1}{4}d(x,y)+\frac{1}{2}\max\{d(x,Tx),d(y,Sy)\} \\
&< \frac{1}{2}d(x,y)+\frac{1}{2}\max\{d(x,Tx),d(y,Sy)\} \\
\implies d(Tx,Sy) &< g(d(x,y),d(x,Tx),d(y,Sy)).
\end{aligned}$$

Thus (T,S) is $\mathcal{A}$-contractive in pair.

Now, let $x\in C\left[0,\frac{1}{12}\right]$ and $\varepsilon>0$ be arbitrary. Then, for any $y\in O_{T,S}(x)$, y will look like $y(t)=t\alpha(t)x(t)$ for some α, where α is a function of t. We choose $\delta=\frac{\varepsilon}{2}$. Then, for $y\in O_{T,S}(x)$, we have

$$\begin{aligned}
g(d(x,y),d(x,Tx),d(y,Sy)) &< \varepsilon+\frac{\varepsilon}{2} \\
\implies \frac{1}{2}d(x,y) \leq g(d(x,y),d(x,Tx),d(y,Sy)) &< \frac{3\varepsilon}{2} \\
\implies (1-t\alpha(t))|x(t)| &< 3\varepsilon.
\end{aligned}$$

Therefore,

$$d(Tx,Sy)=d\left(\frac{t}{2}x(t),t^2\alpha(t)x(t)\right)=\sup_{t\in\left[0,\frac{1}{12}\right]}\left(\frac{t}{2}-t^2\alpha(t)\right)|x(t)|$$

$$\leq \frac{1}{12} \sup_{t\in\left[0,\frac{1}{12}\right]} \left(\frac{1}{2} - t\alpha(t)\right) |x(t)| \leq \frac{1}{12} \sup_{t\in\left[0,\frac{1}{12}\right]} (1 - t\alpha(t)) |x(t)|$$
$$\leq \frac{3\varepsilon}{12} = \frac{\varepsilon}{4}.$$

Thus all the conditions of Theorem 6 hold good. So by the same theorem, T and S have a unique common fixed point in X. Indeed $x \in C\left[0, \frac{1}{12}\right]$ defined by $x(t) = 0$ for all t, is the unique common fixed point of T and S.

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CHARACTERIZATIONS OF GENERALIZED SUBMODULES OF QTAG-MODULES

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Abstract. We characterize the α-large submodules of α-modules in terms of certain sequences of ordinals, and give some of their interesting properties. Also, we deal with α-large submodules of the closure of an unbounded direct sum of uniserial modules, and the α-large submodules of the smallest α-pure fully invariant submodules of the closure containing a given element.

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1. Introduction and Fundamentals

Let R be any ring with unity. A uniserial module M is a module over a ring R, whose submodules are totally ordered by inclusion. This means simply that for any two submodules N_1 and N_2 of M, either $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$. A module M is called a serial module if it is a direct sum of uniserial modules. An element $x \in M$ is uniform, if xR is a non-zero uniform (hence uniserial) module and for any R-module M with a unique decomposition series, $d(M)$ denotes its decomposition length.

In 1976 Singh [16] introduced a class of modules called *TAG*-modules, defined by satisfying two properties relating to uniserial modules.

(I) Every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules.

(II) Given any two uniserial submodules U and V of a homomorphic image of M, for any submodule W of U, any non-zero homomorphism $f : W \to V$ can be extended to a homomorphism $g : U \to V$, provided the composition length $d(U/W) \leq d(V/f(W))$.

It was shown that the theory of these modules very closely paralleled the theory of torsion abelian groups; for this reason they were referred to as *TAG*-modules. In 1987 Singh showed that the second property, with minimal additional hypotheses, can be deduced from the first and studied the modules satisfying only the first property and called them *QTAG*-modules. The study of *QTAG*-modules and their structure began with work of Singh in [17]. Since then, many authors have written about the

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structure of $QTAG$-modules. Not surprisingly, many of the developments parallel the earlier development of the structure of torsion abelian groups. Our main goal here is to study α-modules, a class of the $QTAG$-modules and further advance the study of the structure of $QTAG$-modules and the parallels with torsion abelian groups. The present paper is a natural extension of work already done in this field and certainly contributes to the overall knowledge of the structure of $QTAG$-modules.

All rings below are assumed to be associative and with nonzero identity element; all modules are assumed to be unital $QTAG$-modules. For a uniform element $x \in M$, $e(x) = d(xR)$ and $H_M(x) = \sup\left\{ d\left(\frac{yR}{xR}\right) \mid y \in M,\ x \in yR \text{ and } y \text{ uniform} \right\}$ are the exponent and height of x in M, respectively. $H_k(M)$ denotes the submodule of M generated by the elements of height at least k and $H^k(M)$ is the submodule of M generated by the elements of exponents at most k. Let us denote by M^1, the submodule of M, containing elements of infinite height. The module M is h-divisible [9] if $M = M^1 = \bigcap_{k=0}^{\infty} H_k(M)$ and it is h-reduced if it does not contain any h-divisible submodule. In other words, it is free from the elements of infinite height. The module M is said to be bounded [16], if there exists an integer k such that $H_M(x) \leq k$ for every uniform element $x \in M$. A submodule N of M is h-pure [8] in M if $N \cap H_k(M) = H_k(N)$, for every integer $k \geq 0$. A submodule $B \subseteq M$ is a basic submodule [9] of M, if B is h-pure in M, $B = \oplus B_i$, where each B_i is the direct sum of uniserial modules of length i and M/B is h-divisible. A submodule F of M is said to be fully invariant [1] if each endomorphism of M sends F into itself. A fully invariant submodule $L \subseteq M$ is large [2], if $L + B = M$, for every basic submodule B in M.

For a module M and an ordinal α, $H_\alpha(M)$ is defined as $H_\alpha(M) = \bigcap_{\beta<\alpha} H_\beta(M)$. For an ordinal α, a submodule N of M is said to be α-pure [14], if $H_\beta(M) \cap N = H_\beta(N)$ for all $\beta \leq \alpha$. A submodule $N \subset M$ is nice [11] in M, if $H_\alpha(M/N) = (H_\alpha(M) + N)/N$ for all ordinals α, i.e. every coset of M modulo N may be represented by an element of the same height. The sum of all simple submodules of M is called the socle of M, denoted by $Soc(M)$. The cardinality of the minimal generating set of M is denoted by $g(M)$. For all ordinals α, $f_M(\alpha)$ is the αth-Ulm invariant of M and it is equal to $g\big(Soc(H_\alpha(M))/Soc(H_{\alpha+1}(M))\big)$. An ordinal α is said to be confinal with ω, if α is the limit of a countable ascending sequence of ordinals.

Imitating [12], for any uniform element $x \in M$, there exist uniform elements x_1, $x_2, \ldots$ such that $xR \supseteq x_1R \supseteq x_2R \supseteq \ldots$ and $d\left(\frac{x_iR}{x_{i+1}R}\right) = 1$. Now the Ulm-sequence of x is defined as $U_M(x) = (H_M(x),\ H_M(x_1),\ H_M(x_2), \ldots)$. These sequences are partially ordered because $U_M(x) \leq U_M(y)$ if $H_M(x_i) \leq H_M(y_i)$ for every i.

An h-reduced module M is totally projective [10] if it has a collection $\mathcal{N}$ of nice submodules such that (i) $0 \in \mathcal{N}$ (ii) if $\{N_i\}_{i\in I}$ is any subset of $\mathcal{N}$, then $\sum_{i\in I} N_i \in \mathcal{N}$ (iii) given any $N \in \mathcal{N}$ and any countable subset X of M, there exists $K \in \mathcal{N}$ containing $N \cup X$, such that K/N is countably generated.

It is interesting to note that almost all the results which hold for *TAG*-modules are also valid for *QTAG*-modules [14]. Many results, stated in the present paper, are clearly motivated from the paper [3]. Most of our notations and terminology will be standard being in agreement with [4] and [5].

2. α-MODULES

These α-modules were originally defined and carefully explored in [13]. Recall that a *QTAG*-module M is an α-module, where α is a limit ordinal, if $M/H_\beta(M)$ is totally projective for every ordinal $\beta < \alpha$. But there are some other closely related concepts which have been of interest: recall that a submodule $B \subseteq M$ is an α-basic submodule of an α-module M if B is totally projective of length at most α, B is α-pure submodule of M, and M/B is h-divisible; while a fully invariant submodule L of the α-module M is α-large if $M = B + L$, for all α-basic submodules B of M. A few recurring relationships between them, and certain related assertions in this direction were established in [6]. The same type of study was continued in [7] and a number of results have been obtained in terms of α-basic submodules and α-large submodules. Some new achievements in this theme for other important sorts of α-modules were established in [15]. The motivation for writing the present article is to promote in this direction some new concepts and to explore some structural consequences in the light of generalized submodules.

We start with the following subsection.

2.1. α-*large submodules*

The study of α-large submodules and its fascinating properties makes the theory of α-modules more interesting. We summarize only for information a few standard properties of this notion in the following: If L is an α-large submodule of M and $\beta <$ length of $L/H_\alpha(M)$, then $H_\beta(L)$ is α-large submodule of M. Likewise, an α-large submodule $L \subseteq M$ is totally projective, if M is totally projective.

Mimicking [12], for a sequence $n(L) = (n_0, n_1, n_2, \ldots)$ of non-negative, non-decreasing integers we may consider $L = \{x : x \in M, U_M(x) \geq n(L)\}$ as the submodule of M. This submodule is a large submodule of M. Since for any endomorphism f of M, $H_M(x) \leq H_M(f(x))$, L is fully invariant. Therefore with every large submodule L of M we may associate a sequence $n(L)$. In fact for every large submodule there is a sequence and, for every sequence, there is a large submodule.

If n is replaced by an arbitrary ordinal less than or equal to σ, then $n(L)$ may be extended to $\sigma(L)$, and all the definitions and results which hold for $n(L)$ may be

extended for $\sigma(L)$. In $\sigma(L)$, for any large submodule L of M, the sequence of L as $\sigma(L)$ denoted by M^{σ}. It is fairly easy to see that M^{σ} is a fully invariant submodule of M.

We first give the following strengthening concept.

Definition 1. Let $\sigma = \{\sigma_n\}_{n\in\mathbb{N}}$ be an increasing sequence of ordinals and symbol ∞; that is, for each n, either σ_n is an ordinal or $\sigma_n < \infty$ and $\infty < \infty$. With each such sequence σ we associate the fully invariant submodule M^{σ} of the *QTAG*-module M as

$$M^{\sigma} = \{x \in M : x \in H_{\sigma_n - n}(M), \forall\, n \in \mathbb{N}\}$$

Now we prove the following lemma.

Lemma 1. *Let α be an ordinal cofinal with ω and M a QTAG-module with a fully invariant submodule F such that $H_\alpha(M) = H_\alpha(F)$ and let $a \in Soc^n(M)$ for some natural n. Then there exists $b \in F$ such that $U_M(a-b) = U_M(a)$ and $(a-b)R \cap H_\alpha(M) = 0$.*

Proof. Let $a \in Soc^n(M)$ be any uniform element of exponent n and let $U_M(a) = (\sigma_0, \sigma_1 \dots)$. Now $\sigma_{n+i} = \infty$ $i = 0, 1, \dots$. If all $\sigma_j = \infty$, $j = 0, 1, \dots$, then $a \in H_\alpha(M)$ and since $H_\alpha(M) = H_\alpha(F)$, we get $a = b$. Similarly if $\sigma_j \neq \infty$, $0 \leq j \leq n$, then $aR \cap H_\alpha(M) = 0$, and we get $b = 0$. Thus the lemma is true for the trivial cases. Now suppose that $\sigma_k = \infty$ with $0 < k < n$, and k is the smallest index for which $\sigma_k = \infty$. Let $a' \in H_\alpha(M)$, $c \in F$ such that $a' = c'$ where $d\left(\frac{aR}{a'R}\right) = k$, $d\left(\frac{cR}{c'R}\right) = m+1$ and $m > \sigma_{k-1}$. Then $b = c'$ where $d\left(\frac{cR}{bR}\right) = m-k+1$. In this case $H_M(b') \geq m-k+i+1 > \sigma_i$ where $d\left(\frac{bR}{b'R}\right) = i$ and $0 \leq i \leq k$. This shows that $H_M(H_i((a-b)R)) = H_M(a') = \sigma_i$ where $d\left(\frac{aR}{a'R}\right) = i$ and $0 \leq i \leq k$. Therefore, $H_M(H_i((a-b)R)) = \infty$, for $i \geq k$, and we are done. □

We continue in this way by the following.

Lemma 2. *Let L be an α-large submodule of an α-module M, where α is cofinal with ω. Then $Soc^n(L)$ is a large submodule of $Soc^n(M)$ for some natural n.*

Proof. Let $a \in Soc^n(L)$ and let ψ be an endomorphism of $Soc^n(M)$. Then $H_\alpha(M) = H_\alpha(L)$ since $H_n(L)$ is α-large for every n and $H_\alpha(M)$ is contained in every α-large submodule of M. Thus by Lemma 1, one can easily deduce that $(a-b)R \cap H_\alpha(M) = 0 = (\psi(a)-c)R$, and $U_M(a-b) = U_M(a) \leq U_M(\psi(a)) = U_M(\psi(a)-c)$ for all $b, c \in L$. We are now in the hypothesis of [7, Lemma 2.3], there exists an endomorphism ϕ of M such that $\phi(a-b) = \psi(a)-c$. Therefore, $a-b \in L$ and L is fully invariant in M. Hence $\psi(a)-c \in L$, and it follows that $\psi(a) \in L \cap Soc^n(M) = Soc^n(L)$. The proof is over. □

And so, we prepare to prove the following.

Proposition 1. *Let L be an α-large submodule of an α-module M, where α is cofinal with ω. Then $L = M^{\sigma}$ where $\sigma = (\sigma_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence of ordinals.*

Proof. Clearly, $Soc^n(L)$ is a large submodule of $Soc^n(M)$ by Lemma 2. Since owing to the criterion from [7], there exists a strictly increasing sequence of ordinals such that $Soc^n(L) = Soc^n(M^{\sigma})$. Now $L + Soc^n(M) = M$ and $L/Soc^n(L)$ is h-divisible. Therefore, $U_M(x) \geq \sigma$, for every $x \in L$, and $L \subset M^{\sigma}$. But $Soc^n(M^{\sigma}) = M^{\sigma} \cap Soc^n(L)$. Furthermore, $M/Soc^n(L) = L/Soc^n(L) \oplus Soc^n(M)/Soc^n(L)$, therefore $M^{\sigma}/Soc^n(L) = L/Soc^n(L)$, and $L = M^{\sigma}$. □

The next corollary is a valuable consequence of the above proposition.

Corollary 1. *Let M be an α-module, where α is cofinal with ω. Then M^{σ} is an α-large submodule of M if and only if $M = M^{\sigma} + Soc^n(M)$, where $\sigma = (\sigma_n)_{n\in\mathbb{N}}$ and $\sigma_n < \infty$ for each $n \in \mathbb{N}$.*

We also need the following simple but however useful

Definition 2. Let $\mu = \{\mu_n\}_{n\in\mathbb{N}}$ be an increasing sequence of ordinals and symbol ∞. We say that μ is larger than σ almost everywhere if there exists a non-negative integer n such that $\mu_i \geq \sigma_i$, for all $i > n$. With each such sequence σ we associate the fully invariant submodule $\dot{M}^{\sigma}$ of the $QTAG$-module M containing $Soc^n(M) + M^{\sigma}$ as

$$\dot{M}^{\sigma} = \left\{ x \in M : x' \in M^{\sigma},\ d\left(\frac{xR}{x'R}\right) = n \text{ for some } n \in \mathbb{N} \right\}$$

We are now in a position to proceed by proving the following theorem.

Theorem 1. *Let L be a submodule of an α-module M, where α is cofinal with ω. Then L is an α-large submodule of M if and only if $L = M^{\sigma}$ and $M = \dot{M}^{\sigma}$ for some strictly increasing sequence of ordinals.*

Proof. Let L be an α-large submodule of M. By Proposition 1, there exists a sequence σ such that $L = M^{\sigma}$. But, by virtue of Corollary 1, $\dot{M}^{\sigma} \supset Soc^n(M) + M^{\sigma} = M$, so that with Definition 2 at hand we are done.

Conversely, suppose $L = M^{\sigma}$ and $M = \dot{M}^{\sigma}$. Let x be any uniform element in M, then $U_M(x) \geq \sigma$, for some increasing sequence μ of ordinals with $\mu_i \geq \sigma_i$, for all $i > n$ and for each $n \in \mathbb{N}$. Therefore, $H_M(x') \geq \sigma_i$ where $d\left(\frac{xR}{x'R}\right) = i$. Now we choose $y \in M$ such that $x' = y'$ where $d\left(\frac{xR}{x'R}\right) = n+1$, $d\left(\frac{yR}{y'R}\right) = r$, and $r \geq \sigma_{n+1}$. Therefore there exists an element $z \in M$ such that $z = y'$ where $d\left(\frac{yR}{y'R}\right) = r - n - 1$.

Of course we claim that $z \in M^{\sigma}$. Indeed, $z' = x'$ where $d\left(\frac{xR}{x'R}\right) = d\left(\frac{zR}{z'R}\right) = i$ for $i > n$. Thus, $H_M(z') = H_M(x') \geq \sigma_i$ where $d\left(\frac{xR}{x'R}\right) = d\left(\frac{zR}{z'R}\right) = i$. But $H_M(z') \geq r - n - 1 + i \geq \sigma_{n+1} - (n+1) + i$ where $d\left(\frac{zR}{z'R}\right) = i$ for $0 \leq i \leq n$. Now, σ being a strictly increasing sequence of ordinals, we have $\sigma_{j+1} \geq \sigma_j + 1$, $j = 0, 1, \ldots$. Thus, by continuing the same process, we obtain $\sigma_{n+1} \geq \sigma_i + n - i + 1$ for $i \leq j \leq n$. Hence $H_M(z') \geq \sigma_i$ where $d\left(\frac{zR}{z'R}\right) = i$. It follows that $z \in M^{\sigma}$, $x - z \in Soc^n(M)$, and $x = z + (x - z) \in M^{\sigma} + Soc^n(M)$. This shows that $M = M^{\sigma} + Soc^n(M)$. Thus, by what we have just seen above, in view of the Corollary 1, $L = M^{\sigma}$ is an α-large in M, as desired. The proof of the theorem is completed. □

As immediate consequence, we yield the following.

Corollary 2. *If N is an α-pure submodule of an α-module M and L is an α-large submodule of M, then $N \cap L$ is an α-large submodule of N, where α is cofinal with ω.*

The next proposition shed some light about the relationships between generalized submodules.

Proposition 2. *Let B be an α-basic submodule of an α-module M and let L be an α-large submodule of M, where α is cofinal with ω. Then $B \cap L$ is an α-basic submodule of M and M/L is a direct sum of uniserial modules.*

Proof. Choose $B \cap L = B \cap Soc^n(L)$, for some n. Since $Soc^n(L)$ is large in $Soc^n(M)$, $B \cap L$ is an α-basic submodule of $Soc^n(L)$. But $Soc^n(L)$ is h-pure in L and $L/Soc^n(L)$ is h-divisible. Therefore $B \cap L$ is an α-basic submodule of L. Now $M/L \cong B/(B \cap L)$ and by Corollary 2, $B \cap L$ is fully invariant in B. Henceforth, it follows that $B/(B \cap L)$ is a direct sum of uniserial modules. □

We thus deduce the following statement.

Proposition 3. *Let L_1 be an α-large submodule of an α-module M and let L_2 be an α-large submodule of L_1, where α is cofinal with ω. Then L_2 is an α-large submodule of M.*

2.2. *α-pure fully invariant submodules.*

The notion of a fully invariant submodule of a module is, of course, a classical notion in the theory of *QTAG*-modules. There are numerous observations of fully invariant submodules more enlightening than the definition: If F is a fully invariant submodule of an α-module M of length α, and if α is an ordinal cofinal with ω, then M/F is also an α-module. In addition, if F is an unbounded of length μ, then M/F is a totally projective module, and F is a μ-module. Moreover, if F is an unbounded,

fully invariant submodule of an α-module M, and if α is an ordinal cofinal with ω, then F is totally projective module only if M is totally projective module.

However, we now have the following.

Definition 3. Let $\sigma = \{\sigma_n\}_{n\in\mathbb{N}}$ be a strictly increasing sequence of ordinals. We say that σ has a gap at n if $\sigma_n + 1 < \sigma_{n+1}$ and finitely many gaps if there exists n such that $\sigma_n + i < \sigma_{n+i}$ for every $i = 1, 2, \ldots$. If no such n exists we say that σ has infinitely many gaps. Now σ satisfies the gap condition for a *QTAG*-module M if the Ulm-invariant of M corresponding to σ_n is non-zero.

Now, we proceed by proving

Lemma 3. *Let α be an ordinal cofinal with ω and let $\sigma = \{\sigma_n\}_{n\in\mathbb{N}}$ be a sequence of ordinals. If M is an α-module and σ has finitely many gaps, then $M^\sigma \supset H_t(M)$ for some $t \geq 0$.*

Proof. By hypothesis, there exists n such that $\sigma_n + i = \sigma_{n+i}$, $i = 1, 2, \ldots$. Let $t = \sigma_n$ and suppose $x \in H_t(M)$. Then $x = y'$ where $d\left(\frac{yR}{y'R}\right) = t$ for some $y \in M$. Thus, $H_M(x') \geq \sigma_n + i = \sigma_{n+i} \geq \sigma_i$ where $d\left(\frac{xR}{x'R}\right) = i$, for every $i = 0, 1, \ldots$. Therefore $x \in M^\sigma$ and $H_t(M) \subset M^\sigma$, □

Definition 4. For a sequence σ and an element x of a *QTAG*-module M, we let

$$\Gamma_x^\sigma = \left\{ i : H_M(x') < \sigma_i,\ d\left(\frac{xR}{x'R}\right) = i \right\}.$$

The careful reader will observe that Theorem 1 can be restated in terms of Γ_x^σ; namely

Remark 1. For an α-module M and a sequence σ, M^σ is α-large in M if and only if Γ_x^σ is finite for every $x \in M$.

So, we come to the following lemma.

Lemma 4. *Let M be an α-module, where α is cofinal with ω. If the closure $\overline{M}$ of M is an unbounded direct sum of uniserial modules and suppose $\sigma = \{\sigma_n\}_{n\in\mathbb{N}}$ is a strictly increasing sequence of ordinals. Then there exists $x \in \overline{M}$ such that Γ_x^v is infinite.*

Proof. Write $M = \bigoplus_{i=1}^{\infty} M_i$ where M_i is a direct sum of uniserial modules of length i. Let t_1 be an integer such that $\sigma_{t_1} + 1 < \sigma_{t_1+1}$ and choose $r_1 \geq \sigma_{t_1} + 2$ such that $M_{r_1} \neq 0$. Let $x_{r_1} \in M_{r_1}$ be any uniform element such that $H_{M_{r_1}}(x_{r_1}) = \sigma_{t_1} - t_1$. Now we repeat this operation for an integer t_2, we get $t_2 \geq e(x_{r_1})$, $\sigma_{t_2} + 1 < \sigma_{t_2+1}$ and $r_2 \geq \sigma_{t_2} + 2$ with $M_{r_2} \neq 0$ and choose $x_{r_2} \in M_{r_2}$ be any uniform element such that $H_{M_{r_2}}(x_{r_2}) = \sigma_{t_2} - t_2$.

Thus, by continuing the same process, we obtain a sequence x_{r_i} of elements such that $H_{M_{r_i}}(x_{r_i}) < H_{M_{r_{(i+1)}}}(x_{r_{(i+1)}}) < \dots$ and $t_1 + 2 \leq e(x_{r_1}) \leq t_2 < t_2 + 2 \leq e(x_{r_2}) \leq \dots$. Now let $y = x_j \in \prod_{j=1}^{\infty} M_j$ where $x_j = 0$ unless $j = ti$ for some i, in which case $x_j = x_t i$. Since $H_{M_t}(x_t)$ increases as t increases, and it follows that $y \in \overline{M}$. Furthermore, Γ_x^v is infinite. Consequently, $H_M(y') = H_M(x'_{r_i})$ such that $d\left(\frac{x_{r_i}R}{x'_{r_i}R}\right) = d\left(\frac{yR}{y'R}\right) = ti$ and $H_M(y') = \sigma_{ti} + 1 < \sigma_{ti+1}$ such that $d\left(\frac{yR}{y'R}\right) = ti + 1$ for all $i = 1, 2, \dots$. This ends the proof. □

Summarizing the above corresponding assertions, we immediately deduce

Proposition 4. *$\overline{M}^{\sigma}$ is α-large in $\overline{M}$ if and only if σ has finitely many gaps.*

Definition 5. Let M be a *QTAG*-module and $\lambda = \{\lambda_n\}_{n\in\mathbb{N}}$ is an increasing sequence of ordinals and symbol ∞. With each such sequence λ we associate the α-pure fully invariant submodule $\overline{M}^{(x)}$ of the closure $\overline{M}$ of M containing x as

$$\overline{M}^{(x)} = \left\{ y \in \overline{M} : y' \in H_{\lambda_n}(M),\ d\left(\frac{yR}{y'R}\right) = n, \text{ for some } n \geq 0 \right\}$$

So, we shall now prove the following proposition.

Proposition 5. *Let α be an ordinal cofinal with ω and let $\sigma = \{\sigma_n\}_{n\in\mathbb{N}}$ be a strictly increasing sequence of ordinals with infinitely many gaps. If M is an α-module, then there exists an α-pure fully invariant submodule F of the closure $\overline{M}$ of M such that F^{σ} is α-large in F.*

Proof. Let $\sigma = \{\sigma_n\}_{n\in\mathbb{N}}$ be a given sequence. Now we construct the another sequence $\lambda = \{\lambda_n\}_{n\in\mathbb{N}}$ form σ such that $\lambda_n = \sigma_{2n}$, for every $n = 0, 1, \dots$. The sequence λ has a gap at each $i = 0, 1, \dots$. Choose $M = \oplus M_n$ such that $M_{\lambda_n+1} \neq 0$ for every n. Then there exists $x \in \overline{M}$ such that $\lambda = H_{\overline{M}}(x)$.

We set $F = \overline{M}^{(x)}$. It is not difficult to show that Γ_y^{σ} is finite for every $y \in F$. Since $y' \in H_{\overline{M}}(x)$, where $d\left(\frac{yR}{y'R}\right) = t$, there exists an endomorphism of $\overline{M}$ which maps x onto y', and hence $H_F(y') \geq H_{\overline{M}}(x') = \sigma_{2i}$ where $d\left(\frac{yR}{y'R}\right) = t + i$ and $d\left(\frac{xR}{x'R}\right) = i$. But $H_F(y') = \lambda_{t+i}$, where $d\left(\frac{yR}{y'R}\right) = t + i$. Therefore $\lambda_{t+i} \geq \sigma_{2i} \geq \sigma_{t+i}$ for all $i \geq t$ and Γ_y^{σ} is finite. Thus, by Remark 1, F^{σ} is α-large in F. □

The following necessary and sufficient condition is of some interest.

Lemma 5. *Let $F = \overline{M}^{(x)}$ be an α-pure fully invariant submodule of the α-module M, where α is cofinal with ω, and let $\sigma = \{\sigma_n\}_{n\in\mathbb{N}}$ be a strictly increasing sequence of ordinals with infinitely many gaps satisfying the gap condition with respect to the closure $\overline{M}$ of M. Then F^σ is α-large in F if and only if for every $k \geq 0$, there exists $j_k \geq 0$, such that $\lambda_i \geq \sigma_{i+k}$, whenever $\lambda = \{\lambda_n\}_{n\in\mathbb{N}}$ and $i \geq j_k$.*

Proof. "Necessity". We know that Γ_x^σ is finite for every $x \in F$. Thus $t = 0$ there exists j_0 such that $i > j_0$ implies $\lambda_i \geq \sigma_i$. Now $H_t(F^\sigma)$ is also α-large in F. Define $\gamma = (\gamma_i)$ where $\gamma_i = \sigma_{i+k}$. We claim that $H_k(F^\sigma) = F^\gamma$. Clearly $H_k(F^\sigma) \subset F^\gamma$. Indeed, $F^\sigma = F \cap \overline{M}^\sigma$ since F is α-pure in $\overline{M}$.

After this, since σ satisfies the gap condition, there exists $a \in \overline{M}$ such that $\overline{M}^\sigma = \{\phi(a) : \phi \in \mathrm{End}\overline{M}\}$ and $H_{\overline{M}}(a) = \sigma$. Thus $H_{\overline{M}}(a') = \gamma$ such that $d\left(\dfrac{aR}{a'R}\right) = k$, and $\overline{M}^\gamma = H_k(\overline{M}^\sigma)$.

Observe that

$$F^\gamma = F \cap \overline{M}^\gamma = F \cap H_k(\overline{M}^\sigma) \subset H_k(F) \cap H_k(\overline{M}^\sigma) \subset H_k(F \cap \overline{M}^\sigma) = H_k(F^\sigma).$$

This substantiates our claim.

"Sufficiency". We need only show that Γ_x^σ is finite for every $x \in F$. Now, there exists k such that $x' = \overline{M}^\lambda$ where $d\left(\dfrac{xR}{x'R}\right) = k$. Therefore, $H_F(x') \geq \lambda_i$ such that $d\left(\dfrac{xR}{x'R}\right) = i+k$, and for $i \geq j_k$ such that $\lambda_i \geq \sigma_{i+k}$. Hence Γ_x^σ is finite for every $x \in F$ and it follows that F^σ is α-large in F. □

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NEW OSCILLATION CRITERIA FOR SECOND-ORDER DELAY DYNAMIC EQUATIONS WITH A SUB-LINEAR NEGATIVE NEUTRAL TERM ON TIME SCALES

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Abstract. In this paper, some sufficient conditions for the oscillation of all solutions of second order dynamic equations with a negative sub-linear neutral term are established. The obtained results provide a unified platform that adequately covers both discrete and continuous equations. Furthermore, it covers a wide range of equations by utilizing different time scales. Illustrative examples are provided.

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1. Introduction

The main focus of this paper is to provide new oscillation criteria for the second-order half-linear dynamic equation of the form

$$\left[r(t)\left(z^{\Delta}(t)\right)^{\gamma}\right]^{\Delta}+q(t)f(x(\delta(t)))=0, \tag{1.1}$$

where $z(t):=x(t)-p(t)x^{\alpha}(\tau(t))$. Under the following assumptions

(H1) $\alpha,\gamma\in\mathbb{Q}^{+}_{\text{odd}}$, where $\mathbb{Q}^{+}_{\text{odd}}:=\{a/b: a,b\in\mathbb{Z}^{+}\text{are odd }\}$, $\alpha\in(0,1]$;

(H2) $r\in C_{rd}([t_0,\infty)_{\mathbb{T}},(0,\infty))$, $p,q\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, $0\le p(t)\le p_0<1$, $q(t)\ge 0$ and $q(t)$ is not identically zero for large t;

(H3) $\tau,\delta\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{T}),\delta^{\Delta}\ge 0$, $\tau(t)\le t$, $\delta(t)\le t$, $\lim_{t\to\infty}\tau(t)=\lim_{t\to\infty}\delta(t)=\infty$ and $h(t)=\tau^{-1}(\delta(t))$;

(H4) $f\in C(\mathbb{R},\mathbb{R})$, $uf(u)>0$ for all $u\neq 0$, and there exists a positive constant k such that $f(u)/u^{\beta}\ge k$, β is a ratio of odd positive integers where $\beta\le\gamma$.

Furthermore, for sufficiently large t_1, we assume

$$R(v,u)=\int_u^v\frac{1}{r^{1/\gamma}(s)}\Delta s,\qquad v\ge u\ge t_0.$$

and assume that

$$R(t_0,t)\to\infty\qquad\text{as } t\to\infty. \tag{1.2}$$

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Also, we define

$$Q(t) = \int_t^\infty kq(s)\Delta s, \qquad t \geq t_0.$$

By a solution of (1.1), we mean a function $x \in C_{rd}[T_x,\infty)_{\mathbb{T}}$, $T_x \in [t_0,\infty)_{\mathbb{T}}$ which has the property $r(z^\Delta)^\alpha \in C^1_{rd}[T_x,\infty)_{\mathbb{T}}$ and satisfies (1.1) on $[T_x,\infty)_{\mathbb{T}}$. We consider only those solutions x of (1.1) which satisfy $\sup|x(t)| : t \in [T_x,\infty)_{\mathbb{T}} > 0$ for all $T \in [T_x,\infty)_{\mathbb{T}}$. A solution of (1.1) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is termed nonoscillatory.

Differential, difference equations, and dynamic equations on time scales have an enormous potential for applications in biology, engineering, economics, physics, neural networks, social sciences, etc. In particular, half-linear equations have numerous applications in the analyses of p-Laplace equations, Emden-Fowler equations, non-Newtonian fluid theory, porous medium problems, chemotaxis models, and so forth; see, e.g., [6, 11, 19, 25]. We also refer the reader to the papers [1, 2, 4–11, 13–18, 20–24, 28–34] for the oscillation and asymptotic behavior of different classes of half-linear equations.

In previous years, many papers studied the oscillatory behavior for different classes of dynamic equations on time scale. Many studies have been devoted to the oscillatory behavior of solutions to different classes of equations with nonnegative neutral coefficients; see, e.g., [2, 4, 31, 32] and the references cited therein. However, for equations with nonpositive neutral coefficients, there are relatively fewer results in the literature; see [5, 7, 17, 18, 28–30, 34]. for the oscillation and asymptotic behavior of different classes of half-linear equations.

For instance, Zhang *et al.*[33] investigated oscillatory behavior of solutions to a class of second-order nonlinear neutral delay dynamic equations with nonpositive neutral coefficients of the form

$$\left[r(t)\left(z^\Delta(t)\right)^\gamma\right]^\Delta + q(t)f(x(\delta(t))) = 0, \qquad t \in [t_0,\infty)_{\mathbb{T}}$$

where $\gamma \geq 1$ is a ratio of odd integers and $z(t) = x(t) - p(t)x(\tau(t))$ with $\int_{t_0}^\infty r^{-\frac{1}{\alpha}}(s)\Delta s = \infty$, and presented new oscillation criteria for

$$\left[r(t)\left(z'(t)\right)^\alpha\right]' + q(t)f(x(\delta(t))) = 0, \qquad t \geq t_0$$

under the assumption

$$\int_{t_0}^\infty r^{-\frac{1}{\alpha}}(s)\mathrm{d}s < \infty.$$

For $\mathbb{T} = \mathbb{Z}$, in [14] Grace and Graef presented some new oscillation criteria for second order nonlinear difference equations with a nonlinear nonpositive neutral term of the form

$$\Delta\left(a(t)\left(\Delta\left(x(t) - p(t)x^\alpha(t-k)\right)\right)^\gamma\right) + q(t)x^\beta(t+1-m) = 0$$

where α, β and γ are ratios of positive odd integers with $\gamma \geq \beta$ and $0 < \alpha \leq 1$.

In [26], Lin studied $\Delta\left(x_n - p_n x_{n-\tau}^{\alpha}\right) + q_n x_{n-\sigma}^{\beta} = 0,\ n \geq n_0$ where α and β are quotients of odd positive integers with $0 < \alpha < 1$

More precisely, to the best of our knowledge, no paper on the oscillation of second-order dynamic equations on a time scale appears on (1.1). Our aim is not only present some oscillation criteria for solutions of equation (1.1) but also present sufficient conditions which ensure that all solutions of (1.1) are oscillatory.

2. Auxiliary results

Lemma 1 ([9, Theorem 1.93])**.** *Assume that $v\colon \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\check{\mathbb{T}} := v(\mathbb{T})$ is a time scale. Let $y\colon \check{\mathbb{T}} \to \mathbb{R}$. If $v^{\Delta}(t)$ and $y^{\check{\Delta}}(v(t))$ exist for $t \in \mathbb{T}^k$, then*

$$(y(v(t)))^{\Delta} = y^{\check{\Delta}}(v(t))v^{\Delta}(t).$$

Lemma 2. *Let conditions (H1)-(H4) and (1.2) hold. Assume that $x(t)$ is a positive solution of (1.1). Then we have the following two cases:*

(I) $z(t) > 0,\ z^{\Delta}(t) > 0,\ (r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} \leq 0,$

(II) $z(t) < 0,\ z^{\Delta}(t) > 0,\ (r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} \leq 0,$

for $t \in [t_1, \infty)_{\mathbb{T}}$, where $t_1 \in [t_0, \infty)_{\mathbb{T}}$, is sufficiently large.

Proof. Suppose that there exists a $t_1 \geq t_0$ such that $x(t) > 0, x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \geq t_1$. From (1.1) it follows that

$$[r(t)(z^{\Delta}(t))^{\gamma}]^{\Delta} \leq -kq(t)x^{\beta}(\delta(t)) < 0.$$

Hence, $[r(t)(z^{\Delta}(t))^{\gamma}]$ is nonincreasing and of one sign. That is, there exists $t_2 \geq t_1$ such that $z^{\Delta}(t) > 0$ or $z^{\Delta}(t) < 0$ for $t \geq t_2$. We claim that $z^{\Delta}(t) > 0$ for $t \geq t_2$. For this, we assume that $z^{\Delta}(t) < 0$ for $t \geq t_2$. Then,

$$r(t)(z^{\Delta}(t))^{\gamma} \leq -C < 0, \qquad \text{for } t \geq t_2,$$

where $C = -r(t_2)(z^{\Delta}(t_2))^{\gamma}$. Thus, we conclude that

$$z(t) \leq z(t_2) - c^{1/\gamma} \int_{t_2}^{t} r^{-1/\gamma}(s)\Delta s.$$

By virtue of condition (1.2), $\lim_{t\to\infty} z(t) = -\infty$. Now, we consider two cases.

Case 1: If x is unbounded, then there exists a sequence $\{t_k\}$ such that $\lim_{k\to\infty} t_k = \infty$ and $\lim_{k\to\infty} x(t_k) = \infty$, where $x(t_k) = \max\{x(s); t_0 \leq s \leq t_k\}$, since

$$\lim_{t\to\infty} \tau(t) = \infty, \qquad \tau(t_k) > t_0$$

for all sufficiently large k. By $\tau(t) \leq t$

$$x(\tau(t_k)) = \max\{x(s); t_0 \leq s \leq \tau(t_k)\} \leq \max\{x(s); t_0 \leq s \leq t_k\} = x(t_k).$$

Therefore, for all large k,

$$z(t_k) = x(t_k) - p(t_k)x^{\alpha}(\tau(t_k)) \geq x(t_k) - p(t_k)x^{\alpha}(t_k)$$

$$\geq \left(1-\frac{p(t_k)}{x^{1-\alpha}(t_k)}\right)x(t_k)>0$$

which contradicts the fact that $\lim_{t\to\infty} z(t)=-\infty$.

Case 2: If $x(t)$ is bounded, then $z(t)$ is also bounded, which contradicts $\lim_{t\to\infty} z(t)=-\infty$. This completes the proof.

□

3. Main results

For simplicity, we consider

$$K(t)=\begin{cases}1, & \gamma=\beta;\\ MR^{\frac{\beta-\gamma}{\gamma}}(t,t_2); & \gamma>\beta,\end{cases}\qquad \text{for some } M>0.$$

$$Q(t)=\int_t^\infty q(s)\Delta s \ \text{ and } \psi(t)=r^{1/\gamma}(t)R(T,t),\qquad \text{for } T\geq t_0.$$

Theorem 1. *Assume that conditions (H1)-(H4) and (1.2) hold. If there exists a positive nondecreasing continuously differentiable function $\varphi(t)$, such that*

$$\limsup_{t\to\infty}\Big[\varphi(t)Q(t) + \int_{t_2}^t\left(kq(s)\varphi(s)-\frac{\gamma^\gamma}{\beta^\gamma(\gamma+1)^{\gamma+1}}\frac{r(\delta(s))(\varphi^\Delta(s))^{\gamma+1}}{(\delta^\Delta(s))^\gamma\varphi^\gamma(s)K^\gamma(\delta(\sigma(s)))}\right)\Delta s\Big]=\infty, \tag{3.1}$$

$$\limsup_{t\to\infty}\left[k\int_{h(t)}^t q(s)R^{\beta/\alpha}(h(s),h(t))\Delta s\right]>1,\qquad \textit{for } \beta=\alpha\gamma, \tag{3.2}$$

and

$$\limsup_{t\to\infty}\int_{h(t)}^t q(s)R^{\beta/\alpha}(h(s),h(t))\Delta s=\infty,\qquad \textit{for } \beta<\alpha\gamma, \tag{3.3}$$

then every solution of (1.1) is oscillatory.

Proof. Assume that x is a nonoscillatory solution of (1.1) such that $x(t)>0$, $x(\tau(t))>0$ and $x(\delta(t))>0$, for $t\in[t_1,\infty)_{\mathbb{T}}$. From Lemma 2, $z(t)$ satisfies either (I) or (II) for $t\in[t_1,\infty)_{\mathbb{T}}$.

Case 1 : Suppose that $z(t)$ satisfies Lemma 2 (I). From the definition of $z(t)$,

$$x(t)=z(t)+p(t)x^\alpha(\tau(t))\geq z(t),$$

in view of (1.1), we get

$$[r(t)(z^\Delta(t))^\alpha]^\Delta\leq -kq(t)z^\beta(\delta(t))<0. \tag{3.4}$$

Integrating (3.4) from t to u, letting $u\to\infty$, and using the increasing fact of $z(t)$, we get

$$r(t)(z^\Delta(t))^\gamma\geq z^\beta(\delta(t))\int_t^\infty kq(s)\Delta s=:Q(t)z^\beta(\delta(t)). \tag{3.5}$$

Define

$$\omega(t) = \varphi(t)\frac{r(t)(z^{\Delta}(t))^{\gamma}}{z^{\beta}(\delta(t))}.$$

It is clear that $\omega(t) > 0$ and

$$\begin{aligned}\omega^{\Delta}(t) &= [r(t)(z^{\Delta}(t))^{\gamma}]^{\Delta}\frac{\varphi(t)}{z^{\beta}(\delta(t))} + r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\gamma}\left(\frac{\varphi(t)}{z^{\beta}(\delta(t))}\right)^{\Delta}\\ &\le -kq(t)\varphi(t) + \varphi^{\Delta}(t)\frac{r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\gamma}}{z^{\beta}(\delta(\sigma(t)))}\\ &\quad - \beta\varphi(t)\frac{r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\gamma}z^{\Delta}(\delta(t))\delta^{\Delta}(t)}{z^{\beta+1}(\delta(t))}.\end{aligned}$$

Since $z^{\Delta}(t) > 0$, $\delta^{\Delta}(t) \ge 0$ and from the definition of $\omega(t)$, we obtain

$$\begin{aligned}\omega^{\Delta}(t) &\le -kq(t)\varphi(t) + \frac{\varphi^{\Delta}(t)}{\varphi(\sigma(t))}\omega(\sigma(t))\\ &\quad - \beta\delta^{\Delta}(t)\varphi(t)\frac{r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\gamma}}{z^{\beta}(\delta(\sigma(t)))}\frac{z^{\Delta}(\delta(t))}{z(\delta(\sigma(t)))}\\ &\le -kq(t)\varphi(t) + \frac{\varphi^{\Delta}(t)}{\varphi(\sigma(t))}\omega(\sigma(t))\\ &\quad - \beta\delta^{\Delta}(t)\frac{\varphi(t)z^{\Delta}(\delta(t))}{\varphi(\sigma(t))z(\delta(\sigma(t)))}\omega(\sigma(t)).\end{aligned} \tag{3.6}$$

From the definition of $\omega(t)$ and since $[r(t)(z^{\Delta})^{\gamma}]$ is nonincreasing, then we have

$$z^{\Delta}(\delta(t)) \ge \frac{z^{\frac{\beta}{\gamma}}(\delta(\sigma(t)))}{r^{\frac{1}{\gamma}}(\delta(t))\varphi^{\frac{1}{\gamma}}(\sigma(t))}\omega^{\frac{1}{\gamma}}(\sigma(t)).$$

This with (3.6) leads to

$$\begin{aligned}\omega^{\Delta}(t) &\le -kq(t)\varphi(t) + \frac{\varphi^{\Delta}(t)}{\varphi(\sigma(t))}\omega(\sigma(t))\\ &\quad - \beta\delta^{\Delta}(t)\frac{\varphi(t)}{\varphi^{\frac{\gamma+1}{\gamma}}(\sigma(t))r^{\frac{1}{\gamma}}(\delta(t))}\omega^{\frac{\gamma+1}{\gamma}}(\sigma(t))z^{\frac{\beta-\gamma}{\gamma}}(\delta(\sigma(t))).\end{aligned} \tag{3.7}$$

It is clear that $z^{\frac{\beta-\gamma}{\gamma}}(\delta(\sigma(t))) = 1$ for $\beta = \gamma$ and (3.7) takes the form

$$\begin{aligned}\omega^{\Delta}(t) &\le -kq(t)\varphi(t) + \frac{\varphi^{\Delta}(t)}{\varphi(\sigma(t))}\omega(\sigma(t))\\ &\quad - \beta\delta^{\Delta}(t)\frac{\varphi(t)}{\varphi^{\frac{\gamma+1}{\gamma}}(\sigma(t))r^{\frac{1}{\gamma}}(\delta(t))}\omega^{\frac{\gamma+1}{\gamma}}(\sigma(t)).\end{aligned} \tag{3.8}$$

On the other hand for $\beta < \gamma$. Since $[r(t)(z^{\Delta}(t))^{\gamma}]^{\Delta} \leq 0$, then there exists a constant $C > 0$ such that $r(t)(z^{\Delta}(t))^{\gamma} \leq r(t_2)(z^{\Delta}(t_2))^{\gamma} < C$ for some $t \geq t_2$, which leads to

$$z^{\Delta}(t) \leq C^{\frac{1}{\gamma}} r^{\frac{-1}{\gamma}}(t). \tag{3.9}$$

Integrating (3.9) from t to t_2, we get

$$\begin{aligned} z(t) &\leq z(t_2) + C^{\frac{1}{\gamma}} R(t_2,t) \\ &\leq C^{1/\gamma} R(t_2,t). \end{aligned}$$

This leads to,

$$z^{\frac{\beta-\gamma}{\gamma}}(\delta(\sigma(t))) > M R^{\frac{\beta-\gamma}{\gamma}}(\delta(\sigma(t)),t_2), \quad \beta < \gamma \quad \text{for some } M > 0. \tag{3.10}$$

Combining (3.8) and (3.10), we get

$$\begin{aligned} \omega^{\Delta}(t) &\leq -kq(t)\varphi(t) + \frac{\varphi^{\Delta}(t)}{\varphi(\sigma(t))}\omega(\sigma(t)) \\ &\quad - \beta M \delta^{\Delta}(t)\varphi(t) \frac{R^{\frac{\beta-\gamma}{\gamma}}(\delta(\sigma(t)),t_2)}{\varphi^{\frac{\gamma+1}{\gamma}}(\sigma(t)) r^{\frac{1}{\gamma}}(\delta(t))} \omega^{\frac{\gamma+1}{\gamma}}(\sigma(t)) \\ &\leq -kq(t)\varphi(t) + \frac{\varphi^{\Delta}(t)}{\varphi(\sigma(t))}\omega(\sigma(t)) \\ &\quad - \beta \delta^{\Delta}(t)\varphi(t) \frac{K(\delta(\sigma(t)))}{\varphi^{\frac{\gamma+1}{\gamma}}(\sigma(t)) r^{\frac{1}{\gamma}}(\delta(t))} \omega^{\frac{\gamma+1}{\gamma}}(\sigma(t)). \end{aligned} \tag{3.11}$$

Applying the inequality

$$B\omega - A\omega^{\frac{\gamma+1}{\gamma}} \leq \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^{\gamma}},$$

with $B = \frac{\varphi^{\Delta}(t)}{\varphi(\sigma(t))}$, $A = \beta\delta^{\Delta}(t)\varphi(t) \dfrac{K(\delta(\sigma(t)))}{\varphi^{\frac{\gamma+1}{\gamma}}(\sigma(t)) r^{\frac{1}{\gamma}}(\delta(t))}$, we get

$$\omega^{\Delta}(t) \leq -kq(t)\varphi(t) + \frac{\gamma^{\gamma}\beta^{-\gamma}}{(\gamma+1)^{\gamma+1}} \frac{r(\delta(t))(\varphi^{\Delta}(t))^{\gamma+1}}{(\delta^{\Delta}(t))^{\gamma}\varphi^{\gamma}(t)K^{\gamma}(\delta(\sigma(t)))}. \tag{3.12}$$

Integrating (3.12) from t_2 to t

$$\begin{aligned} \omega(t) \leq \omega(t_2) - \int_{t_2}^{t} \Bigg(& kq(t)\varphi(s) \\ & - \frac{\gamma^{\gamma}}{\beta^{\alpha}(\gamma+1)^{\gamma+1}} \frac{r(\delta(s))(\varphi^{\Delta}(s))^{\gamma+1}}{(\delta^{\Delta}(s))^{\gamma}\varphi^{\gamma}(s)K^{\gamma}(\delta(\sigma(s)))} \Bigg) \Delta s. \end{aligned} \tag{3.13}$$

From the definition of $\omega(t)$ and (3.5) we have

$$\omega(t) \geq \varphi(t)\frac{Q(t)z^{\beta}(\delta(t))}{z^{\beta}(\delta(t))} \geq \varphi(t)Q(t). \tag{3.14}$$

From (3.13) and taking (3.14) into account

$$\varphi(t)Q(t) + \int_{t_2}^{t}\left(kq(s)\varphi(s) - \frac{\alpha^{\alpha}}{\beta^{\alpha}(\alpha+1)^{\alpha+1}}\frac{r(\delta(s))(\varphi^{\Delta}(s))^{\alpha+1}}{(\delta^{\Delta}(s))^{\alpha}\varphi^{\alpha}(s)K^{\alpha}(\delta(\sigma(s)))}\right)\Delta s \leq \omega(t_2),$$

which contradicts (3.1).

Case 2 : Suppose that $z(t)$ satisfies Lemma 2 (II). Putting $y = -z > 0$, then $y^{\Delta} < 0$, and (1.1) takes the form

$$[r(t)(y^{\Delta}(t))^{\gamma}]^{\Delta} \geq kq(t)x^{\beta}(\delta(t)) \geq 0. \tag{3.15}$$

Since

$$\begin{aligned} y(t) = -z(t) &= p(t)x^{\alpha}(\tau(t)) - x(t) \\ &\leq p(t)x^{\alpha}(\tau(t)), \end{aligned}$$

by virtue of $0 \leq p(t) \leq p_0 < 1$, we have

$$y^{1/\alpha}(h(t)) \leq x(\delta(t)). \tag{3.16}$$

Now, inequalities (3.15) and (3.16) lead to

$$[r(t)(y^{\Delta}(t))^{\gamma}]^{\Delta} \geq kq(t)y^{\beta/\alpha}(h(t)). \tag{3.17}$$

Also for $t_2 \leq u \leq v$, we can write

$$y(u) - y(v) = \int_{u}^{v}\frac{1}{r^{1/\gamma}(s)}(-r(s)(y^{\Delta}(s))^{\gamma})^{1/\gamma}\Delta s,$$

$$y(u) \geq R(v,u)(-r(v)(y^{\Delta}(v))^{\gamma})^{1/\gamma}.$$

Setting $u = h(t)$ and $v = h(s)$, we get

$$y(h(s)) \geq R(h(s),h(t))(-r(h(t))(y^{\Delta}(h(t)))^{\gamma})^{1/\gamma}. \tag{3.18}$$

Integrating (3.17) from $h(t)$ to t, in view of (3.18), we get

$$-r(h(t))(y^{\Delta}(h(t)))^{\gamma} \geq k[-r(h(t))(y^{\Delta}(h(t)))^{\gamma}]^{\beta/\alpha\gamma}\int_{h(t)}^{t} q(s)R^{\beta/\alpha}(h(s),h(t))\Delta s,$$

which leads to

$$[Y(t)]^{1-\frac{\beta}{\alpha\gamma}} \geq k\int_{h(t)}^{t} q(s)R^{\beta/\alpha}(h(s),h(t))\Delta s,$$

where $Y(t) = -r(h(t))(y^{\Delta}(h(t)))^{\gamma}$. Therefore, we have

$$1 \geq k\int_{h(t)}^{t} q(s)R^{\beta/\alpha}(h(s),h(t))\Delta s, \qquad \text{for } \beta = \alpha\gamma,$$

which contradicts (3.2). Also, for $\beta < \gamma$, by (3.15) together with the fact that $Y^{\Delta}(t) \leq 0$ and $Y(t)$ is bounded, we get a contradiction with (3.3). This completes the proof.

□

Remark 1. Note that Theorem 1 holds when $Q(t) < \infty$ and the addition term $\varphi(t)Q(t)$ in condition (3.1) may improve some of the well-known results in the literature.

Corollary 1. *Let conditions (H1)-(H4) be satisfied and* (1.2) *hold. If* $Q(t) < \infty$, *then condition* (3.1) *replaced by*

$$\limsup_{t\to\infty} \int_{t_2}^{t} \left(kq(s)\varphi(s) - \frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}} \frac{r(\delta(s))(\varphi^{\Delta}(s))^{\gamma+1}}{(\delta^{\Delta}(s))^{\gamma}\varphi^{\gamma}(s)K^{\gamma}(\delta(\sigma(s)))} \right) \Delta s = \infty, \quad (3.19)$$

and conclusion of Theorem 1 remains intact.

Corollary 2. *Let conditions (H1)-(H4) be satisfied and* (1.2) *hold. With* $\varphi^{\Delta}(t) \leq 0$, *then condition* (3.1) *replaced by*

$$\limsup_{t\to\infty} \left[\varphi(t)Q(t) + \int_{t_2}^{t} kq(s)\varphi(s)\Delta s \right] = \infty,$$

and conclusion of Theorem 1 remains intact.

To add variety, we present a different approach of the condition (3.1) in the following.

Theorem 2. *Assume that conditions (H1)-(H4) and* (1.2) *hold. If*

$$\limsup_{t\to\infty} \left[\varphi(t)Q(t) + \int_{t_2}^{t} \left(kq(s)\varphi(s) - \frac{1}{4\beta} \frac{r^{1/\gamma}(\delta(s))(\varphi^{\Delta}(s))^2}{\delta^{\Delta}(s)Q^{\frac{1-\gamma}{\gamma}}(\sigma(s))\varphi(s)K(\delta(\sigma(s)))} \right) \Delta s \right] = \infty, \quad (3.20)$$

and (3.2) *or* (3.3), *then every solution of* (1.1) *is oscillatory.*

Proof. Assume that $x(t)$ is a nonoscillatory solution of (1.1) such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\delta(t)) > 0$, for $t \in [t_1, \infty)_{\mathbb{T}}$. From Lemma 2, $z(t)$ satisfies either (I) or (II) for $t \in [t_1, \infty)_{\mathbb{T}}$.

Case 1: First, we suppose that $z(t)$ satisfies (I). Since $\omega(t) = \varphi(t)\dfrac{r(t)(z^{\Delta}(t))^{\gamma}}{z^{\beta}(\delta(t))}$, then by (3.14), we conclude that

$$\left(\frac{\omega(\sigma(t))}{\varphi(\sigma(t))} \right)^{\frac{1-\gamma}{\gamma}} \geq Q^{\frac{1-\gamma}{\gamma}}(\sigma(t)). \quad (3.21)$$

Now, inequalities (3.11) and (3.21) imply

$$\omega^{\Delta}(t) \leq -kq(t)\varphi(t) + \frac{\varphi^{\Delta}(t)}{\varphi(\sigma(t))}\omega(\sigma(t)) - \beta\delta^{\Delta}(t)Q^{\frac{1-\gamma}{\gamma}}(\sigma(t))\frac{\varphi(t)K(\delta(\sigma(t)))}{\varphi^2(\sigma(t))r^{\frac{1}{\gamma}}(\delta(t))}\omega^2(\sigma(t)).$$

Apply the inequality

$$B\omega - A\omega^{\frac{\gamma+1}{\alpha}} \leq \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{B^{\gamma+1}}{A^{\gamma}}, \qquad \text{where } \gamma = 1,$$

with $B = \frac{\varphi^{\Delta}(t)}{\varphi(\sigma(t))}$, $A = \beta\delta^{\Delta}(t)Q^{\frac{1-\gamma}{\gamma}}(\sigma(t))\frac{\varphi(t)K(\delta(\sigma(t)))}{\varphi^2(\sigma(t))r^{\frac{1}{\gamma}}(\delta(t))}$, we get

$$\omega^{\Delta}(t) \leq -kq(t)\varphi(t) + \frac{1}{4\beta}\frac{r^{1/\gamma}(\delta(t))(\varphi^{\Delta}(t))^2}{\delta^{\Delta}(t)Q^{\frac{1-\gamma}{\gamma}}(\sigma(t))\varphi(t)K(\delta(\sigma(t)))}. \tag{3.22}$$

Integrating (3.22) from t_2 to t

$$\omega(t) \leq \omega(t_2) - \int_{t_2}^{t}\left(kq(s)\varphi(s) - \frac{1}{4\beta}\frac{r^{1/\gamma}(\delta(s))(\varphi^{\Delta}(s))^2}{\delta^{\Delta}(s)Q^{\frac{1-\gamma}{\gamma}}(\sigma(s))\varphi(s)K(\delta(\sigma(s)))}\right)\Delta s, \tag{3.23}$$

In view of (3.14), inequality (3.23) takes the form

$$k\varphi(t)Q(t) + \int_{t_2}^{t}\left(kq(s)\varphi(s) - \frac{1}{4\beta}\frac{r^{1/\gamma}(\delta(s))(\varphi^{\Delta}(s))^2}{\delta^{\Delta}(s)Q^{\frac{1-\gamma}{\gamma}}(\sigma(s))\varphi(s)K(\delta(\sigma(s)))}\right)\Delta s \leq \omega(t_2),$$

which contradicts (3.20).

Case 2: Suppose that $z(t)$ satisfies Lemma 2 (II). The proof can be performed a similar manner as in the proof of Theorem1. This completes the proof.

□

4. Applications and Examples

This section introduces some special cases for Eq. (1.1). For the non-neutral equation, i.e., Eq. (1.1) with $p(t) \equiv 0$, and $q(t)$ is either positive or negative for all large t, Eq. (1.1) is reduced to the equation

$$\left[r(t)\left((x(t))^{\Delta}\right)^{\gamma}\right]^{\Delta} \pm q(t)f(x(\delta(t))) = 0. \tag{E$\pm$}$$

From Theorem1 we conclude the following results.

Corollary 3. *Assume that conditions (H1)-(H4) and* (1.2) *hold. If there exists a positive function* $\phi(t)$ *with* $\phi^{\Delta}(t) \geq 0$ *such that condition* (3.1) *holds, then Eq. (E+) is oscillatory.*

Proof. The proof is omitted because it is included in the proof of Theorem 1-Case (1). □

It's noted that, results in [3] and the references cited therein are related to Corollary 3 for $\mathbb{T} = \mathbb{R}$. Also the results can be extended to the difference equations when $\mathbb{T} = \mathbb{Z}$, see [12, 27].

Corollary 4. *Assume that conditions (H1)-(H4) and* (1.2) *hold. If there exists a positive function* $\phi(t)$ *with* $\phi^{\Delta}(t) \geq 0$ *such that condition* (3.2) *or* (3.3) *holds, then Eq. (E-) is oscillatory.*

Proof. The proof is omitted because it is included in the proof of Theorem 1-Case (2). □

In the following, we investigate another special case

Theorem 3. *Let* $\alpha = 1$, *conditions (H1)-(H4), and* (1.2) *hold. Assume that condition* (3.2) *and*

$$\limsup_{t\to\infty} R^{\beta}(t_0, \delta(t)) > 1, \qquad \text{when } \beta = \gamma, \tag{4.1}$$

hold and condition (3.3) *and*

$$\limsup_{t\to\infty} R^{\beta}(t_0, \delta(t)) > 0, \qquad \text{when } \beta < \gamma, \tag{4.2}$$

hold. Then (1.1) *is oscillatory.*

Proof. Assume that (1.1) has a nonoscillatory solution $x(t)$. Without loss of generality, we assume that there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\delta(t))$ for $t \in\in [t_1, \infty)_{\mathbb{T}}$. According to Lemma 2, we have two possible cases for $z(t)$. Assume that $z(t)$ satisfies Lemma 2 (I). It follows that

$$\begin{aligned} z(t) &= z(t_2) + \int_{t_2}^{t} \frac{(r(s)(z^{\Delta}(s))^{\gamma})^{1/\gamma}}{a^{1/\gamma}(s)} \Delta s \\ &\geq (r(t)(z^{\Delta}(t))^{\gamma})^{1/\gamma} \int_{t_2}^{t} a^{-1/\gamma}(s) \Delta s \\ &=: \psi(t) z^{\Delta}(t). \end{aligned} \tag{4.3}$$

In view of (3.5) and (4.3), and using the decreasing fact of $r(t)z^{\Delta}(t)$, we find

$$\begin{aligned} w(t) &=: r(t)z^{\Delta}(t) \\ &\geq Q(t)\psi^{\beta}(\delta(t))(z^{\Delta}(\delta(t)))^{\beta} \\ &= Q(t)\psi^{\beta}(\delta(t)) \left(r^{-\beta/\gamma}(\delta(t))\right) \left(r(\delta(t))(z^{\Delta}(\delta(t)))^{\gamma}\right)^{\beta/\gamma} \\ &\geq Q(t)\psi^{\beta}(\delta(t)) \left(r^{-\beta/\gamma}(\delta(t))\right) \left(r(t)(z^{\Delta}(t))^{\gamma}\right)^{\beta/\gamma} \\ &= Q(t)\psi^{\beta}(\delta(t)) \left(r^{-\beta/\gamma}(\delta(t))\right) w^{\beta/\gamma}(t), \end{aligned}$$

hence

$$\begin{aligned} w^{1-\beta/\gamma}(t) &\geq Q(t)\psi^{\beta}(\tau(t))\left(r^{-\beta/\gamma}(\delta(t))\right) \\ &= Q(t)\left(\int_{t_2}^{\delta(t)} r^{-1/\gamma}(s)\Delta s\right)^{\beta} \\ &= R^{\beta}(t_2,\delta(t))Q(t). \end{aligned} \tag{4.4}$$

Taking lim sup of both sides of (4.4) as $t \to \infty$. For $\beta = \gamma$, we get a contradiction to (4.1) and (4.2) when $\beta < \gamma$. If $z(t)$ satisfies Lemma 2 (II), then the proof is similar to that of Theorem 1. □

Remark 2. Note that the obtained results in Theorem 3 are an improvement of the results in [18, 34] which guarantee that every solution of (1.1) is oscillatory.

Example 1. Let $\mathbb{T} = \mathbb{R}$. Consider the second order differential equation

$$\left(t^2\left(\left(x(t) - \frac{1}{2}x(t/3)\right)'\right)^3\right) + \frac{\lambda}{t^2}x^3(t/2) = 0, \qquad t \geq 1. \tag{4.5}$$

Here $\alpha = 1$, $\beta = 3$, $\gamma = 3$, $\lambda > 0$ is a constant $k = \lambda$, $r(t) = t^2$, $p(t) = 1/2$, $q(t) = 1/t^2$, $\tau(t) = t/3$, $\delta(t) = t/2$ and $h(t) = 3t/2$. It is clear that

$$\begin{aligned} R(t_0,t) &= \int_1^t s^{-2/3}ds \\ &= 3(\sqrt[3]{t} - 1), \qquad R(t_0,t) \to \infty \text{ as } t \to \infty. \end{aligned}$$

Applying Theorem 3, we have

$$\limsup_{t\to\infty}\left[k\int_{h(t)}^t q(s)R^{\beta}(h(s),h(t))\Delta s\right] = \limsup_{t\to\infty}\left[\lambda\int_{3t/2}^t \frac{81(\sqrt[3]{s}-\sqrt[3]{t})^3}{2s^2}ds\right] > 1$$

and

$$\limsup_{t\to\infty} R^{\beta}(t_0,\delta(t)) = \limsup_{t\to\infty}\left(3(\sqrt[3]{3t/2} - 1)\right) > 1.$$

For suitable λ and large t, every solution of Eq. (4.5) is oscillatory.

Remark 3. Theorem 3.1 of [18] can be applied to (4.5) which yields that every solution of equation (4.5) is oscillatory when $\lambda > \frac{2}{27}$ or $\lim_{t\to\infty} x(t) = 0$.

Example 2. Let $\mathbb{T} = \mathbb{R}$. Consider the second order differential equation

$$\left(\left(x(t) - \frac{1}{2}x(\sqrt{t})\right)^3\right)'' + \frac{m}{t^{5/4}}x(t/2) = 0, \qquad t \geq 1. \tag{4.6}$$

Here $\alpha = 1$, $\beta = 1$, $\gamma = 3$, $k = m > 0$ where m is a constant, $r(t) = 1$, $p(t) = 1/2$, $q(t) = 1/t^{5/4}$, $\tau(t) = \sqrt{t}$, $\delta(t) = t^{5/4}$ and $h(t) = \sqrt{t}$. Taking $\varphi(t) = t$, by using Theorem 1, we have

$$\begin{aligned}
&\limsup_{t\to\infty}\left[4mt^{3/4} + \int_1^t \left(\frac{m}{\sqrt[4]{s}} - \frac{9}{16\left(\frac{1}{4}s^{1/4}\right)^3\left(\sqrt[4]{s}-1\right)^{-2}}\right)ds\right] \\
&= \limsup_{t\to\infty}\left[4mt^{3/4} + \frac{4}{3}\left(m\left(t^{3/4}-1\right) - 36\left(\sqrt[4]{t}-1\right)^3\right)\right].
\end{aligned}$$

Hence, (4.6) oscillates for $m > 9$.
According to (3.2), we have

$$\limsup_{t\to\infty}\left[k\int_{h(t)}^t q(s)R^{\beta}(h(s),h(t))ds\right] = \limsup_{t\to\infty}\left[2m\int_{\sqrt{t}}^t \frac{\sqrt{t}-\sqrt{s}}{s^{5/4}}ds\right] \to \infty.$$

Therefore, every solution of (4.5) is oscillatory when $m > 9$.

Remark 4. It should be noted that, for $\mathbb{T} = \mathbb{R}$ and $\alpha = 1$, Theorem 1 improves the conditions of Theorem 2.1 of [13] and guarantees that every solution of (1.1) is oscillatory unlike in [13, 33, 34] .

Example 3. Let $\mathbb{T} = \mathbb{Z}$. Consider the second order difference equation

$$\Delta\left(\Delta\left(x(t) - \frac{1}{2}x^{1/3}(t-3)\right)^3\right) + 8(x-7) = 0, \qquad t \geq 1. \tag{4.7}$$

Here $\alpha = 1/3$, $\beta = 1$, $\gamma = 3$, $k = 1$, $r(t) = 1$, $p(t) = 1/2$, $q(t) = 8$, $\tau(t) = t-3$, $\delta(t) = t-7$ and $h(t) = t-4$. Taking $\varphi(t) = t$, by using Corollary 1, we have

$$\begin{aligned}
&\limsup_{t\to\infty}\left[\sum_{s=1}^t \left(kq(s)\varphi(s) - \frac{\gamma^{\gamma}}{\beta^{\gamma}(\gamma+1)^{\gamma+1}}\frac{r(\delta(s))(\Delta\varphi(s))^{\gamma+1}}{(\Delta\delta(s))^{\gamma}\varphi^{\gamma}(s)K^{\gamma}(\delta(\sigma(s)))}\right)\right] \\
&= \limsup_{t\to\infty}\left[\sum_{s=1}^t \left(8s - \frac{9}{16s^3(s-6)^{-2}}\right)\right] \to \infty.
\end{aligned}$$

Also, condition (3.2) implies that

$$\limsup_{t\to\infty}\left[k\sum_{s=h(t)}^t q(s)R^{\beta/\alpha}(h(t),h(s))ds\right] \to \infty,$$

which yields that (4.7) satisfies conditions (3.19) and (3.2). Therefore, every solution of (4.7) is oscillatory.

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NEW GENERALIZATIONS OF SOME IMPORTANT INEQUALITIES FOR SARıKAYA FRACTIONAL INTEGRALS

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Abstract. In this research paper, we investigate some new identifies for Sarıkaya fractional integrals which introduced by Sarıkaya and Ertuğral in [20]. The fractional integral operators also have been applied to Hermite-Hadamard type integral inequalities to provide their generalized properties. Furthermore, as special cases of our main results, we present several known inequalities such as Simpson, Bullen, trapezoid for convex functions.

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Keywords: simpson type inequalities, Sarıkaya fractional integrals, convex functions

1. Introduction

The convex functions for inequalities are introduced firstly by C. Hermite and J. Hadamard. Let $\digamma : I \to \mathbb{R}$ be a convex function on the interval I of real numbers and $\sigma, \rho \in I$ with $\sigma < \rho$. Then, the following double inequality

$$\digamma\left(\frac{\sigma+\rho}{2}\right) \leq \frac{1}{\rho-\sigma}\int_{\sigma}^{\rho} \digamma(\varkappa)d\varkappa \leq \frac{\digamma(\sigma)+\digamma(\rho)}{2} \tag{1.1}$$

is valid for all convex functions, which is known in the literature as the Hermite-Hadamard inequality. If $\digamma$ is concave, then both inequalities in (1.1) hold to the reverse direction. With the help of the inequality (1.1), many researchers have considered the Hermite-Hadamard inequality and related inequalities such as trapezoid, midpoint, and Simpson's inequality.

Over the years, considerable number of studies have been focused on obtaining trapezoid and midpoint type inequalities which give bounds for the right-hand side and left-hand side of the inequality (1.1), respectively. Dragomir and Agarwal first introduced trapezoid inequalities for convex functions in [5]. Kirmacı first established midpoint inequalities for convex functions in the paper [14]. In addition to these, Hwang and Tseng prove some new Hermite-Hadamard type inequalities for fractional integrals and introduce several applications for the beta function in [11] and some new Fejer-type inequalities for convex functions are proved in the paper

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[23]. Moreover, Hwang et al. establish some Hermite-Hadamard type, Bullen-type, and Simpson-type inequalities for fractional integrals and some applications for the beta function are also given in the paper [10]. Some fractional trapezoid and midpoint type inequalities for convex functions are proved in [22] and [12], respectively. Some generalized midpoint type inequalities for Riemann-Liouville fractional integrals are given in [1] and [4].

The results of Simpson-type for convex functions have been studied extensively by many mathematicians. To be more precise, some inequalities of Simpson's type for s-convex functions are investigated by using differentiable functions [15]. By using the differentiable convex function, the new variants of Simpson's type inequalities are proved in the papers [16,21]. For further information about Simpson type inequalities for various convex classes, we refer the reader to Refs. [2,3,6,7,9,13,17–19] and the references therein.

Now, we introduce some definitions and notations which are used frequently in throughout main section.

Definition 1. [20] Let us consider that a function $\varphi : [0,\infty) \to [0,\infty)$ satisfies the condition $\int_0^1 \frac{\varphi(\tau)}{\tau} d\tau < \infty$. Then, the following left-sided and right-sided Sarıkaya fractional integral operators are described as

$$ {}_{\sigma+}I_{\varphi}F(\varkappa) = \int_{\sigma}^{\varkappa} \frac{\varphi(\varkappa-\tau)}{\varkappa-\tau} F(\tau)d\tau, \quad \varkappa > \sigma \tag{1.2} $$

and

$$ {}_{\rho-}I_{\varphi}F(\varkappa) = \int_{\varkappa}^{\rho} \frac{\varphi(\tau-\varkappa)}{\tau-\varkappa} F(\tau)d\tau, \quad \varkappa < \rho, \tag{1.3} $$

respectively.

The most significant feature of Sarıkaya fractional integrals is that they generalize some important types of fractional integrals such as Riemann-Liouville fractional integral, k-Riemann-Liouville fractional integral, Hadamard fractional integrals, conformable fractional integral, Katugampola fractional integrals etc. These important special cases of the integral operators (1.2) and (1.3) are given as follows:

i. Let us consider $\varphi(\tau) = \tau$. Then, the operators (1.2) and (1.3) reduce to the Riemann integral.
ii. If we assign $\varphi(\tau) = \frac{\tau^{\alpha}}{\Gamma(\alpha)}$ and $\alpha > 0$, then the operators (1.2) and (1.3) reduce to the Riemann-Liouville fractional integrals $J^{\alpha}_{\sigma+}F(\varkappa)$ and $J^{\alpha}_{\rho-}F(\varkappa)$, respectively. Here, Γ is Gamma function defined by the integral formula

$$ \Gamma(\varkappa) := \int_0^{\infty} \tau^{\varkappa-1} e^{-\tau} d\tau, \ \varkappa \in \mathbb{R}^+. $$

iii. Suppose $\varphi(\tau)=\frac{1}{k\Gamma_k(\alpha)}\tau^{\frac{\alpha}{k}}$ and $\alpha,k>0$. The operators (1.2) and (1.3) reduce to the k-Riemann-Liouville fractional integrals $J^{\alpha}_{\sigma+,k}F(\varkappa)$ and $J^{\alpha}_{\rho-,k}F(\varkappa)$, respectively. Here, Γ_k is k-Gamma function defined by

$$\Gamma_k(\alpha)=\int_0^{\infty}\tau^{\alpha-1}e^{-\frac{\tau^k}{k}}d\tau, \qquad \mathcal{R}(\alpha)>0$$

and

$$\Gamma_k(\alpha)=k^{\frac{\alpha}{k}-1}\Gamma\left(\frac{\alpha}{k}\right), \qquad \mathcal{R}(\alpha)>0;k>0.$$

2. Some identities for Sarikaya fractional integrals

Throughout this paper for brevity, we define

$$\Delta(\varkappa)=\int_{\sigma}^{\varkappa}\frac{\varphi(\tau-\sigma)}{\tau-\sigma}d\tau \qquad \text{and} \qquad \Lambda(\varkappa)=\int_{\varkappa}^{\rho}\frac{\varphi(\rho-\tau)}{\rho-\tau}d\tau.$$

Lemma 1. *Let $F:[\sigma,\rho]\to\mathbb{R}$ be an absolutely continuous mapping (σ,ρ) such that $F'\in L_1([\sigma,\rho])$. Then, the following equality*

$$\frac{1}{2\Lambda(\sigma)}\left[{}_{\sigma+}I_{\varphi}F(\rho)+{}_{\rho-}I_{\varphi}F(\sigma)\right]-\frac{F(c)+F(d)}{2}=\frac{1}{2\Lambda(\sigma)}\int_{\sigma}^{\rho}\Psi(c,d;\varkappa)F'(\varkappa)d\varkappa$$

is valid. Here,

$$\Psi(c,d;\varkappa)=\begin{cases}\Lambda(\varkappa)-\Delta(\varkappa)-\Lambda(\sigma), & \varkappa\in[\sigma,c],\\ \Lambda(\varkappa)-\Delta(\varkappa), & \varkappa\in(c,d),\\ \Lambda(\varkappa)-\Delta(\varkappa)+\Lambda(\sigma), & \varkappa\in[d,\rho].\end{cases}$$

Proof. By using integration by parts, we obtain

$$\begin{aligned}\int_{\sigma}^{\rho}\Psi(c,d;\varkappa)F'(\varkappa)d\varkappa&=\int_{\sigma}^{\rho}[\Lambda(\varkappa)-\Delta(\varkappa)]F'(\varkappa)d\varkappa\\&\quad-\Lambda(\sigma)\int_{\sigma}^{c}F'(\varkappa)d\varkappa+\Lambda(\sigma)\int_{d}^{\rho}F'(\varkappa)d\varkappa\\&=[\Lambda(\varkappa)-\Delta(\varkappa)]F(\varkappa)|_{\sigma}^{\rho}-\int_{\sigma}^{\rho}\left[-\frac{\varphi(\rho-\varkappa)}{\rho-\varkappa}-\frac{\varphi(\varkappa-\sigma)}{\varkappa-\sigma}\right]F(\varkappa)d\varkappa\\&\quad-\Lambda(\sigma)[F(c)-F(\sigma)]+\Lambda(\sigma)[F(\rho)-F(d)]\\&=-\Delta(\rho)F(\rho)-\Lambda(\sigma)F(\sigma)+\left[{}_{\sigma+}I_{\varphi}F(\rho)+{}_{\rho-}I_{\varphi}F(\sigma)\right]\\&\quad-\Lambda(\sigma)[F(c)-F(\sigma)]+\Lambda(\sigma)[F(\rho)-F(d)]\\&=\left[{}_{\sigma+}I_{\varphi}F(\rho)+{}_{\rho-}I_{\varphi}F(\sigma)\right]-\Lambda(\sigma)[F(c)+F(d)].\end{aligned}$$

This ends the proof of Lemma 1. □

Corollary 1. *If we choose $\varphi(\tau)=\tau$ in Lemma 1, then the following equality holds:*

$$\frac{1}{(\rho-\sigma)}\int_{\sigma}^{\rho}F(\tau)\,d\tau-\frac{F(c)+F(d)}{2}=\frac{1}{2(\rho-\sigma)}\left[2\int_{\sigma}^{c}(\sigma-\varkappa)F'(\varkappa)\,d\varkappa\right.$$
$$\left.+\int_{c}^{d}(\sigma+\rho-2\varkappa)F'(\varkappa)\,d\varkappa+2\int_{d}^{\rho}(\rho-\varkappa)F'(\varkappa)\,d\varkappa\right].$$

Corollary 2. *Let us consider $\varphi(\tau)=\frac{\tau^{\alpha}}{\Gamma(\alpha)}$ in Lemma 1. Then, the following equality*

$$\frac{\Gamma(\alpha+1)}{2(\rho-\sigma)^{\alpha}}\left[J_{\sigma+}^{\alpha}F(\rho)+J_{\rho-}^{\alpha}F(\sigma)\right]-\frac{F(c)+F(d)}{2}=\frac{1}{2(\rho-\sigma)^{\alpha}}\int_{\sigma}^{\rho}\Psi_1(\alpha,\varkappa)F'(\varkappa)\,d\varkappa$$

is valid. Here,

$$\Psi_1(\alpha,\varkappa)=\begin{cases}(\rho-\varkappa)^{\alpha}-(\varkappa-\sigma)^{\alpha}-(\rho-\sigma)^{\alpha}, & \varkappa\in[\sigma,c],\\ (\rho-\varkappa)^{\alpha}-(\varkappa-\sigma)^{\alpha}, & \varkappa\in(c,d),\\ (\rho-\varkappa)^{\alpha}-(\varkappa-\sigma)^{\alpha}+(\rho-\sigma)^{\alpha}, & \varkappa\in[d,\rho].\end{cases}$$

Corollary 3. *If we assign $\varphi(\tau)=\frac{1}{k\Gamma_k(\alpha)}\tau^{\frac{\alpha}{k}}$ in Lemma 1, then the following inequality holds:*

$$\frac{\Gamma_k(\alpha+k)}{2(\rho-\sigma)^{\frac{\alpha}{k}}}\left[J_{\sigma+,k}^{\alpha}F(\rho)+J_{\rho-,k}^{\alpha}F(\sigma)\right]-\frac{F(c)+F(d)}{2}$$
$$=\frac{1}{2(\rho-\sigma)^{\frac{\alpha}{k}}}\int_{\sigma}^{\rho}\Psi_2(\alpha,k;\varkappa)F'(\varkappa)\,d\varkappa,$$

where

$$\Psi_2(\alpha,k;\varkappa)=\begin{cases}(\rho-\varkappa)^{\frac{\alpha}{k}}-(\varkappa-\sigma)^{\frac{\alpha}{k}}-(\rho-\sigma)^{\frac{\alpha}{k}}, & \varkappa\in[\sigma,c],\\ (\rho-\varkappa)^{\frac{\alpha}{k}}-(\varkappa-\sigma)^{\frac{\alpha}{k}}, & \varkappa\in(c,d),\\ (\rho-\varkappa)^{\frac{\alpha}{k}}-(\varkappa-\sigma)^{\frac{\alpha}{k}}+(\rho-\sigma)^{\frac{\alpha}{k}}, & \varkappa\in[d,\rho].\end{cases}$$

Lemma 2. *Suppose $F:[\sigma,\rho]\to\mathbb{R}$ is an absolutely continuous mapping (σ,ρ) such that $F'\in L_1([\sigma,\rho])$. Then, the following equality*

$$\frac{1}{2\Lambda(\sigma)}\left[{}_{\sigma+}I_{\varphi}F(\rho)+{}_{\rho-}I_{\varphi}F(\sigma)\right]-\Omega(\alpha,\beta)F\left(\frac{\sigma+\rho}{2}\right)$$
$$-(1-\Omega(\alpha,\beta))\frac{F(\sigma)+F(\rho)}{2}=\frac{1}{2\Lambda(\sigma)}\int_{\sigma}^{\rho}\Phi(\varkappa)F'(\varkappa)\,d\varkappa$$

is valid. Here, $\Omega(\alpha,\beta)=(1-\beta)^{\alpha}-\beta^{\alpha}$ *with* $\alpha>0$, $0\leq\beta\leq\frac{1}{2}$ *and*

$$\Phi(\varkappa)=\begin{cases}\Lambda(\varkappa)-\Delta(\varkappa)-\Omega(\alpha,\beta)\Lambda(\sigma), & \varkappa\in\left[\sigma,\frac{\sigma+\rho}{2}\right),\\ \Lambda(\varkappa)-\Delta(\varkappa)+\Omega(\alpha,\beta)\Lambda(\sigma), & \varkappa\in\left[\frac{\sigma+\rho}{2},\rho\right).\end{cases}$$

Proof. With the help of the integration by parts, we get

$$\begin{aligned}
&\int_{\sigma}^{\rho}\Phi(\varkappa)F'(\varkappa)d\varkappa\\
&\quad=\int_{\sigma}^{\rho}[\Lambda(\varkappa)-\Delta(\varkappa)]F'(\varkappa)d\varkappa\\
&\qquad-\left((1-\beta)^{\alpha}-\beta^{\alpha}\right)\Lambda(\sigma)\int_{\sigma}^{\frac{\sigma+\rho}{2}}F'(\varkappa)d\varkappa+\left((1-\beta)^{\alpha}-\beta^{\alpha}\right)\Lambda(\sigma)\int_{\frac{\sigma+\rho}{2}}^{\rho}F'(\varkappa)d\varkappa\\
&\quad=[\Lambda(\varkappa)-\Delta(\varkappa)]F(\varkappa)|_{\sigma}^{\rho}-\int_{\sigma}^{\rho}\left[-\frac{\varphi(\rho-\varkappa)}{\rho-\varkappa}-\frac{\varphi(\varkappa-\sigma)}{\varkappa-\sigma}\right]F(\varkappa)d\varkappa\\
&\qquad-\left((1-\beta)^{\alpha}-\beta^{\alpha}\right)\Lambda(\sigma)\left[F\left(\frac{\sigma+\rho}{2}\right)-F(\sigma)\right]\\
&\qquad+\left((1-\beta)^{\alpha}-\beta^{\alpha}\right)\Lambda(\sigma)\left[F(\rho)-F\left(\frac{\sigma+\rho}{2}\right)\right]\\
&\quad=-\Delta(\rho)F(\rho)-\Lambda(\sigma)F(\sigma)+\left[{}_{\sigma+}I_{\varphi}F(\rho)+{}_{\rho-}I_{\varphi}F(\sigma)\right]\\
&\qquad+\left((1-\beta)^{\alpha}-\beta^{\alpha}\right)\Lambda(\sigma)\left[F(\sigma)-2F\left(\frac{\sigma+\rho}{2}\right)+F(\rho)\right]\\
&\quad=\left[{}_{\sigma+}I_{\varphi}F(\rho)+{}_{\rho-}I_{\varphi}F(\sigma)\right]-2\Omega(\alpha,\beta)\Lambda(\sigma)F\left(\frac{\sigma+\rho}{2}\right)\\
&\qquad-(1-\Omega(\alpha,\beta))\Lambda(\sigma)(F(\sigma)+F(\rho)).
\end{aligned}$$

This ends the proof of Lemma 2. □

Corollary 4. *If we choose* $\varphi(\tau)=\tau$ *in Lemma 2, then the following equality holds:*

$$\begin{aligned}
&\frac{1}{(\rho-\sigma)}\int_{\sigma}^{\rho}F(\tau)d\tau-\Omega(\alpha,\beta)F\left(\frac{\sigma+\rho}{2}\right)-(1-\Omega(\alpha,\beta))\frac{F(\sigma)+F(\rho)}{2}\\
&\quad=\frac{1}{2(\rho-\sigma)}\int_{\sigma}^{\rho}\Phi_1(\varkappa)F'(\varkappa)d\varkappa.
\end{aligned}$$

Here,

$$\Phi_1(\varkappa)=\begin{cases}\sigma+\rho-2\varkappa-\Omega(\alpha,\beta)(\rho-\sigma), & \varkappa\in\left[\sigma,\frac{\sigma+\rho}{2}\right),\\ \sigma+\rho-2\varkappa+\Omega(\alpha,\beta)(\rho-\sigma), & \varkappa\in\left[\frac{\sigma+\rho}{2},\rho\right).\end{cases}$$

Corollary 5. *Let us consider* $\varphi(\tau)=\frac{\tau^{\alpha}}{\Gamma(\alpha)}$ *in Lemma 2. Then, we obtain*

$$\frac{\Gamma(\alpha+1)}{2(\rho-\sigma)^{\alpha}}\left[J_{\sigma+}^{\alpha}F(\rho)+J_{\rho-}^{\alpha}F(\sigma)\right]-\Omega(\alpha,\beta)F\left(\frac{\sigma+\rho}{2}\right)$$
$$-(1-\Omega(\alpha,\beta))\frac{F(\sigma)+F(\rho)}{2}=\frac{1}{2(\rho-\sigma)^{\alpha}}\int_{\sigma}^{\rho}\Phi_2(\alpha,\varkappa)F'(\varkappa)\,d\varkappa,$$

where

$$\Phi_2(\alpha,\varkappa)=\begin{cases}(\rho-\varkappa)^{\alpha}-(\varkappa-\sigma)^{\alpha}-\Omega(\alpha,\beta)(\rho-\sigma)^{\alpha}, & \varkappa\in\left[\sigma,\frac{\sigma+\rho}{2}\right),\\ \Lambda(\varkappa)-\Delta(\varkappa)+\Omega(\alpha,\beta)\Lambda(\sigma), & \varkappa\in\left[\frac{\sigma+\rho}{2},\rho\right).\end{cases}$$

3. New inequalities for Sarikaya fractional integrals

In this Section we present some new inequalities for convex function by using Sarıkaya fractional integrals.

Theorem 1. *Let us consider that the assumptions of Lemma 1 are valid. Let us also consider that the mapping* $|F'|$ *is convex on* $[\sigma,\rho]$*. Then, we have the following inequality*

$$\left|\frac{1}{2\Lambda(\sigma)}\left[{}_{\sigma+}I_{\varphi}F(\rho)+{}_{\rho-}I_{\varphi}F(\sigma)\right]-\frac{F(c)+F(d)}{2}\right| \tag{3.1}$$
$$\leq\frac{1}{\Lambda(\sigma)(\rho-\sigma)}\left[\frac{\omega_1(c,d)|F'(\sigma)|+\omega_2(c,d)|F'(\rho)|}{2}\right],$$

where ω_1 *and* ω_2 *are defined by*

$$\omega_1(c,d)=\int_{\sigma}^{\rho}|\Psi(c,d;\varkappa)|(\rho-\varkappa)\,d\varkappa \text{ and } \omega_2(c,d)=\int_{\sigma}^{\rho}|\Psi(c,d;\varkappa)|(\varkappa-\sigma)\,d\varkappa.$$

Proof. By taking modulus in Lemma 1, we get

$$\left|\frac{1}{2\Lambda(\sigma)}\left[{}_{\sigma+}I_{\varphi}F(\rho)+{}_{\rho-}I_{\varphi}F(\sigma)\right]-\frac{F(c)+F(d)}{2}\right|$$
$$\leq\frac{1}{2\Lambda(\sigma)}\int_{\sigma}^{\rho}|\Psi(c,d;\varkappa)|\left|F'(\varkappa)\right|d\varkappa.$$

By using convexity of $|F'|$, we obtain

$$\left|\frac{1}{2\Lambda(\sigma)}\left[{}_{\sigma+}I_{\varphi}F(\rho)+{}_{\rho-}I_{\varphi}F(\sigma)\right]-\frac{F(c)+F(d)}{2}\right|$$

$$\leq \frac{1}{2\Lambda(\sigma)} \int_{\sigma}^{\rho} |\Psi(c,d;\varkappa)| \left[\frac{\rho-\varkappa}{\rho-\sigma} |F'(\sigma)| + \frac{\varkappa-\sigma}{\rho-\sigma} |F'(\rho)| \right] d\varkappa$$

$$= \frac{1}{2\Lambda(\sigma)(\rho-\sigma)} \left[|F'(\sigma)| \int_{\sigma}^{\rho} |\Psi(c,d;\varkappa)| (\rho-\varkappa) d\varkappa \right.$$

$$\left. + |F'(\rho)| \int_{\sigma}^{\rho} |\Psi(c,d;\varkappa)| (\varkappa-\sigma) d\varkappa \right].$$

This finishes the proof of Theorem 1. □

Corollary 6. *In Theorem 1, suppose* $c = (1-\beta)\sigma + \beta\rho$ *and* $d = \beta\sigma + (1-\beta)\rho$ *with* $0 \leq \beta \leq \frac{1}{2}$. *Then, the inequality* (3.1) *reduces to the inequality*

$$\left| \frac{1}{2\Lambda(\sigma)} \left[{}_{\sigma+}I_{\varphi}F(\rho) + {}_{\rho-}I_{\varphi}F(\sigma) \right] - \frac{F((1-\beta)\sigma+\beta\rho) + F(\beta\sigma+(1-\beta)\rho)}{2} \right|$$

$$\leq \frac{1}{\Lambda(\sigma)(\rho-\sigma)} \left[\frac{\omega_1((1-\beta)\sigma+\beta\rho, \beta\sigma+(1-\beta)\rho)|F'(\sigma)|}{2} \right.$$

$$\left. + \frac{\omega_2((1-\beta)\sigma+\beta\rho, \beta\sigma+(1-\beta)\rho)|F'(\rho)|}{2} \right],$$

where ω_1 *and* ω_2 *are defined as in Theorem 1.*

Corollary 7. *In Corollary 6, let us consider* $\beta = 0$. *Then, we have*

$$\left| \frac{1}{2\Lambda(\sigma)} \left[{}_{\sigma+}I_{\varphi}F(\rho) + {}_{\rho-}I_{\varphi}F(\sigma) \right] - \frac{F(\sigma)+F(\rho)}{2} \right|$$

$$\leq \frac{1}{\Lambda(\sigma)(\rho-\sigma)} \left[\frac{\omega_1(\sigma,\rho)|F'(\sigma)| + \omega_2(\sigma,\rho)|F'(\rho)|}{2} \right],$$

where

$$\begin{cases} \omega_1(\sigma,\rho) = \int_{\sigma}^{\rho} |\Psi(\sigma,\rho;\varkappa)| (\rho-\varkappa) d\varkappa = \int_{\sigma}^{\rho} |\Lambda(\varkappa) - \Delta(\varkappa)| (\rho-\varkappa) d\varkappa, \\ \omega_2(\sigma,\rho) = \int_{\sigma}^{\rho} |\Psi(\sigma,\rho;\varkappa)| (\varkappa-\sigma) d\varkappa = \int_{\sigma}^{\rho} |\Lambda(\varkappa) - \Delta(\varkappa)| (\varkappa-\sigma) d\varkappa. \end{cases}$$

Remark 1. If we choose $\varphi(\tau) = \tau$ in Corollary 7, then Corollary 7 reduces to [5, Theorem 2.2].

Remark 2. Let us consider $\varphi(\tau) = \frac{\tau^{\alpha}}{\Gamma(\alpha)}$ in Corollary 7. Then, Corollary 7 reduces to [22, Theorem 3].

Corollary 8. *Let us note* $\varphi(\tau) = \frac{1}{k\Gamma_k(\alpha)}\tau^{\frac{\alpha}{k}}$ *in Corollary 7. Corollary 7 reduces to [8, Theorem 2.4].*

Corollary 9. *In Corollary 6, let* $\beta=\frac{1}{2}$*. Then, we obtain*

$$\left|\frac{1}{2\Lambda(\sigma)}\left[{}_{\sigma+}I_{\varphi}F(\rho)+{}_{\rho-}I_{\varphi}F(\sigma)\right]-F\left(\frac{\sigma+\rho}{2}\right)\right|$$
$$\leq\frac{1}{\Lambda(\sigma)(\rho-\sigma)}\left[\frac{\omega_1\left(\frac{\sigma+\rho}{2},\frac{\sigma+\rho}{2}\right)|F'(\sigma)|+\omega_2\left(\frac{\sigma+\rho}{2},\frac{\sigma+\rho}{2}\right)|F'(\rho)|}{2}\right].$$

Remark 3. If we assign $\varphi(\tau)=\tau$ in Corollary 9, then Corollary 9 reduces to [14, Theorem 2.2].

Remark 4. Consider $\varphi(\tau)=\frac{\tau^{\alpha}}{\Gamma(\alpha)}$ in Corollary 9. Then, Corollary 9 reduces to [10, Theorem 2.1].

Corollary 10. *For* $\varphi(\tau)=\frac{1}{k\Gamma_k(\alpha)}\tau^{\frac{\alpha}{k}}$ *in Corollary 9, we have*

$$\left|\frac{\Gamma_k(\alpha+k)}{2(\rho-\sigma)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\sigma+,k}F(\rho)+J^{\alpha}_{\rho-,k}F(\sigma)\right]-F\left(\frac{\sigma+\rho}{2}\right)\right| \tag{3.2}$$
$$\leq\frac{(\rho-\sigma)}{4\left(\frac{\alpha+k}{k}\right)}\left(\frac{\alpha}{k}-1+\frac{1}{2^{\left(\frac{\alpha-k}{k}\right)}}\right)\left(|F'(\sigma)|+|F'(\rho)|\right).$$

Corollary 11. *In Corollary 6, let us consider* $\beta=\frac{1}{4}$*. Then, we get*

$$\left|\frac{1}{2\Lambda(\sigma)}\left[{}_{\sigma+}I_{\varphi}F(\rho)+{}_{\rho-}I_{\varphi}F(\sigma)\right]-\frac{1}{2}\left[F\left(\frac{3\sigma+\rho}{4}\right)+F\left(\frac{\sigma+3\rho}{4}\right)\right]\right|$$
$$\leq\frac{1}{\Lambda(\sigma)(\rho-\sigma)}\left[\frac{\omega_1\left(\frac{3\sigma+\rho}{4},\frac{\sigma+3\rho}{4}\right)|F'(\sigma)|+\omega_2\left(\frac{3\sigma+\rho}{4},\frac{\sigma+3\rho}{4}\right)|F'(\rho)|}{2}\right].$$

Corollary 12. *If we choose* $\varphi(\tau)=\tau$ *in Corollary 11, then we have*

$$\left|\frac{1}{(\rho-\sigma)}\int_{\sigma}^{\rho}F(\varkappa)\,d\varkappa-\frac{1}{2}\left[F\left(\frac{3\sigma+\rho}{4}\right)+F\left(\frac{\sigma+3\rho}{4}\right)\right]\right|$$
$$\leq\frac{(\rho-\sigma)}{16}\left(|F'(\sigma)|+|F'(\rho)|\right).$$

Remark 5. Let us consider $\varphi(\tau)=\frac{\tau^{\alpha}}{\Gamma(\alpha)}$ in Corollary 11. Then, Corollary 11 reduces to [11, Theorem G].

Corollary 13. *Suppose* $\varphi(\tau)=\frac{1}{k\Gamma_k(\alpha)}\tau^{\frac{\alpha}{k}}$ *in Corollary 11. Then, we have*

$$\left|\frac{\Gamma_k(\alpha+k)}{2(\rho-\sigma)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\sigma+,k}F(\rho)+J^{\alpha}_{\rho-,k}F(\sigma)\right]-\frac{1}{2}\left[F\left(\frac{3\sigma+\rho}{4}\right)+F\left(\frac{\sigma+3\rho}{4}\right)\right]\right|$$

$$\leq (\rho-\sigma)\left[\frac{1}{8}+\frac{3^{\left(\frac{\alpha+k}{k}\right)}-2^{\left(\frac{\alpha+k}{k}\right)}+1}{4^{\left(\frac{\alpha+k}{k}\right)}\left(\frac{\alpha+k}{k}\right)}-\frac{1}{2\left(\frac{\alpha+k}{k}\right)}\right]\left(\left|F'(\sigma)\right|+\left|F'(\rho)\right|\right).$$

Theorem 2. *Let us consider that the assumptions of Lemma 2 are valid. Let us also consider that the mapping $|F'|$ is convex on $[\sigma,\rho]$. Then, we get the following inequality*

$$\begin{aligned}&\left|\frac{1}{2\Lambda(\sigma)}\left[{}_{\sigma+}I_{\varphi}F(\rho)+{}_{\rho-}I_{\varphi}F(\sigma)\right]-\Omega(\alpha,\beta)F\left(\frac{\sigma+\rho}{2}\right)\right.\\&\qquad\left.-(1-\Omega(\alpha,\beta))\frac{F(\sigma)+F(\rho)}{2}\right|\\&\leq\frac{1}{\Lambda(\sigma)(\rho-\sigma)}\left[\frac{\Upsilon_1(\alpha,\beta;\varkappa)\left|F'(\sigma)\right|+\Upsilon_2(\alpha,\beta;\varkappa)\left|F'(\rho)\right|}{2}\right],\end{aligned}$$

where $\Upsilon_1(\alpha,\beta;\varkappa)$ and $\Upsilon_2(\alpha,\beta;\varkappa)$ are defined by

$$\begin{cases}\Upsilon_1(\alpha,\beta;\varkappa)=\int\limits_{\sigma}^{\rho}|\Phi(\varkappa)|(\rho-\varkappa)\,d\varkappa,\\ \Upsilon_2(\alpha,\beta;\varkappa)=\int\limits_{\sigma}^{\rho}|\Phi(\varkappa)|(\varkappa-\sigma)\,d\varkappa.\end{cases}$$

Proof. By taking modulus in Lemma 2, we obtain

$$\begin{aligned}&\left|\frac{1}{2\Lambda(\sigma)}\left[{}_{\sigma+}I_{\varphi}F(\rho)+{}_{\rho-}I_{\varphi}F(\sigma)\right]-\Omega(\alpha,\beta)F\left(\frac{\sigma+\rho}{2}\right)\right.\\&\qquad\left.-(1-\Omega(\alpha,\beta))\frac{F(\sigma)+F(\rho)}{2}\right|\\&\leq\frac{1}{2\Lambda(\sigma)}\int\limits_{\sigma}^{\rho}|\Phi(\varkappa)|\left|F'(\varkappa)\right|d\varkappa.\end{aligned}$$

By using convexity of $|F'|$, we have

$$\begin{aligned}&\left|\frac{1}{2\Lambda(\sigma)}\left[{}_{\sigma+}I_{\varphi}F(\rho)+{}_{\rho-}I_{\varphi}F(\sigma)\right]-\Omega(\alpha,\beta)F\left(\frac{\sigma+\rho}{2}\right)\right.\\&\qquad\left.-(1-\Omega(\alpha,\beta))\frac{F(\sigma)+F(\rho)}{2}\right|\\&\leq\frac{1}{2\Lambda(\sigma)}\int\limits_{\sigma}^{\rho}|\Phi(\varkappa)|\left[\frac{\rho-\varkappa}{\rho-\sigma}\left|F'(\sigma)\right|+\frac{\varkappa-\sigma}{\rho-\sigma}\left|F'(\rho)\right|\right]d\varkappa\\&=\frac{1}{2\Lambda(\sigma)(\rho-\sigma)}\left[\left|F'(\sigma)\right|\int\limits_{\sigma}^{\rho}|\Phi(\varkappa)|(\rho-\varkappa)\,d\varkappa\right.\end{aligned}$$

$$+\left|F'(\rho)\right|\int_{\sigma}^{\rho}\left|\Phi(\varkappa)\right|(\varkappa-\sigma)\,d\varkappa\Bigg].$$

This completes the proof of Theorem 2. □

Corollary 14. *In Theorem 2, let us note that* $\beta=0$. *Then, we have*

$$\left|\frac{1}{2\Lambda(\sigma)}\left[{}_{\sigma+}I_{\varphi}F(\rho)+{}_{\rho-}I_{\varphi}F(\sigma)\right]-F\left(\frac{\sigma+\rho}{2}\right)\right|$$
$$\leq\frac{1}{\Lambda(\sigma)(\rho-\sigma)}\left[\frac{\Upsilon_1(\alpha,0;\varkappa)\left|F'(\sigma)\right|+\Upsilon_2(\alpha,0;\varkappa)\left|F'(\rho)\right|}{2}\right].$$

Remark 6. Let us consider that $\varphi(\tau)=\tau$ in Corollary 14, Then, Corollary 14 reduces to [14, Theorem 2.2].

Remark 7. If we select $\varphi(\tau)=\frac{\tau^{\alpha}}{\Gamma(\alpha)}$ in Corollary 14, then Corollary 14 reduces to [10, Theorem 2.1].

Remark 8. Let us now note that $\varphi(\tau)=\frac{1}{k\Gamma_k(\alpha)}\tau^{\frac{\alpha}{k}}$ in Corollary 14. Then, Corollary 14 reduces to inequality (3.2).

Corollary 15. *In Theorem 2, let us choose* $\beta=\frac{1}{2}$. *Then, we obtain*

$$\left|\frac{1}{2\Lambda(\sigma)}\left[{}_{\sigma+}I_{\varphi}F(\rho)+{}_{\rho-}I_{\varphi}F(\sigma)\right]-\frac{F(\sigma)+F(\rho)}{2}\right|$$
$$\leq\frac{1}{\Lambda(\sigma)(\rho-\sigma)}\left[\frac{\Upsilon_1\left(\alpha,\frac{1}{2};\varkappa\right)\left|F'(\sigma)\right|+\Upsilon_2\left(\alpha,\frac{1}{2};\varkappa\right)\left|F'(\rho)\right|}{2}\right].$$

Remark 9. For $\varphi(\tau)=\tau$ in Corollary 15, Corollary 15 reduces to [5, Theorem 2.2].

Remark 10. If we take $\varphi(\tau)=\frac{\tau^{\alpha}}{\Gamma(\alpha)}$ in Corollary 15, then Corollary 15 reduces to [22, Theorem 3].

Corollary 16. *Let us consider that* $\varphi(\tau)=\frac{1}{k\Gamma_k(\alpha)}\tau^{\frac{\alpha}{k}}$ *in Corollary 15, Then, Corollary 15 reduces to [8, Theorem 2.4].*

4. Conclusion

In this paper, we establish some new refinements of Hermite-Hadamard inequalities for convex functions by using special choices of parameters. Furthermore, we prove that our results generalize the inequalities obtained by Hwang and Tseng [11]. In the future work of the authors, generalization or improvement of our results can be examined by using different kind of convex function classes or other type fractional integral operators.

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CONSTRUCTION OF A NEW CLASS OF FUNCTIONS WITH THEIR SOME PROPERTIES AND CERTAIN INEQUALITIES: n-FRACTIONAL POLYNOMIAL CONVEX FUNCTIONS

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Abstract. In this paper, we introduce a new function class called n-fractional polynomial convex function and their some algebric properties. We obtain some refinements of the right-hand side of Hermite-Hadamard inequality for the class of functions whose derivatives in absolutely value at certain powers are n-fractional polynomial convex. Also, we compare the results obtained with both Hölder, Hölder-İşcan inequalities and power-mean, improved-power-mean integral inequalities and show that the result obtained with Hölder-İşcan and improved power-mean inequalities give better approach than the others. Some applications to special means of real numbers are also given.

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1. Preliminaries

Due to its robustness, convex functions and convex sets have been generalized and extended in many mathematics branches, in particular, many inequalities can be found in the literature via convexity theory. To the best of our knowledge, the Hermite–Hadamard inequality is a well-known, paramount and extensively useful inequality in the applied literature of mathematical inequalities. Let $\varphi\colon I \to \mathbb{R}$ be a convex function. Then the following inequalities hold

$$\varphi\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u}\int_u^v \varphi(x)dx \leq \frac{\varphi(u)+\varphi(v)}{2} \tag{1.1}$$

for all $u, v \in I$ with $u < v$. Both inequalities hold in the reversed direction if the function φ is concave. This double inequality is well known as the Hermite-Hadamard inequality [5]. Note that some of the classical inequalities for means can be derived from Hermite-Hadamard integral inequalities for appropriate particular selections of the mapping φ. Convexity theory provides powerful principles and techniques to

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study a wide class of problems in both pure and applied mathematics. See articles [1,3,4,6,8,9,11] and the references therein.

Definition 1 ([12, Definition 4])**.** Let $h\colon J \to \mathbb{R}$ be a non-negative function, $h \neq 0$. We say that $\varphi\colon I \to \mathbb{R}$ is an h-convex function, or that φ belongs to the class $SX(h,I)$, if φ is non-negative and for all $u,v \in I$, $\alpha \in (0,1)$ we have

$$\varphi(\alpha u + (1-\alpha)v) \leq h(\alpha)\varphi(u) + h(1-\alpha)\varphi(v).$$

If this inequality is reversed, then φ is said to be h-concave, i.e. $\varphi \in SV(h,I)$. It is clear that, if we choose $h(\alpha) = \alpha$ and $h(\alpha) = 1$, then the h-convexity reduces to convexity and definition of P-function, respectively.

Readers can look at [2] for studies on h-convexity.

In [7], İşcan gave a refinement of the Hölder integral inequality as follows:

Theorem 1 (Hölder-İşcan integral inequality [7, Theorem 2.1])**.** *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If φ and ψ are real functions defined on interval $[u,v]$ and if $|\varphi|^p$, $|\psi|^q$ are integrable functions on $[u,v]$ then*

$$\int_u^v |\varphi(x)\psi(x)|\,dx \leq \frac{1}{v-u}\left\{\left(\int_u^v (v-x)\,|\varphi(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_u^v (v-x)\,|\psi(x)|^q\,dx\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_u^v (x-u)\,|\varphi(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_u^v (x-u)\,|\psi(x)|^q\,dx\right)^{\frac{1}{q}}\right\}$$

An refinement of power-mean integral inequality as a different version of the Hölder-İşcan integral inequality can be given as follows:

Theorem 2 (Improved power-mean integral inequality [10])**.** *Let $q \geq 1$. If φ and ψ are real functions defined on interval $[u,v]$ and if $|\varphi|$, $|\varphi|\,|\psi|^q$ are integrable functions on $[u,v]$ then*

$$\int_u^v |\varphi(x)\psi(x)|\,dx$$
$$\leq \frac{1}{v-u}\left\{\left(\int_u^v (v-x)\,|\varphi(x)|\,dx\right)^{1-\frac{1}{q}}\left(\int_u^v (v-x)\,|\varphi(x)|\,|\psi(x)|^q\,dx\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_u^v (x-u)\,|\varphi(x)|\,dx\right)^{1-\frac{1}{q}}\left(\int_u^v (x-u)\,|\varphi(x)|\,|\psi(x)|^q\,dx\right)^{\frac{1}{q}}\right\}$$

The main purpose of this paper is to introduce the concept of n-fractional polynomial convex functions and establish some results connected with the right-hand side of new inequalities similar to the Hermite-hadamard inequality for these classes of functions. Some applications to special means of positive real numbers are also given.

2. THE DEFINITION OF N-FRACTIONAL POLYNOMIAL CONVEX FUNCTIONS

In this section, we introduce a new concept, which is called n-fractional polynomial convexity and we give by setting some algebraic properties for the n-fractional polynomial convex functions, as follows:

Definition 2. Let $n \in \mathbb{N}$. A non-negative function $\varphi\colon I \subset \mathbb{R} \to \mathbb{R}$ is called n-fractional polynomial convex function if the inequality

$$\varphi(tu+(1-t)v) \leq \frac{1}{n}\sum_{i=1}^{n} t^{1/i}\varphi(u) + \frac{1}{n}\sum_{i=1}^{n}(1-t)^{1/i}\varphi(v) \tag{2.1}$$

holds for every $u, v \in I$ and $t \in [0,1]$.

We will denote by $FPC(I)$ the class of all fractional n-fractional polynomial convex functions on interval I.

We note that, every fractional n-fractional polynomial convex function is a h-convex function with the function $h(t) = \frac{1}{n}\sum_{i=1}^{n} t^{1/i}$. Therefore, if $\varphi, \psi \in FPC(I)$, then

(i) $\varphi + \psi \in FPC(I)$ *and for* $c \in \mathbb{R}$ *(*$c \geq 0$*)* $c\varphi \in FPC(I)$. (see [12, Proposition 9]).

(ii) if φ and g be a similarly ordered functions on I , then $\varphi\psi \in FPC(I)$. (See [12, Proposition 10]).

Also, if $\varphi\colon I \to J$ is a convex and $\psi \in FPC(J)$ and nondecreasing, then $\psi \circ \varphi \in FPC(I)$ (see [12, Theorem 15]).

Remark 1. We especially note that; if we take $n = 1$ in the inequality (2.1), then the 1-polynomial convexity reduces to the clasical convexity.

With the help of the following remark, we can give all non-negative convex functions as an example of an n-fractional polynomial convex functions.

Remark 2. Every nonnegative convex function is also a n-fractional polynomial convex function. Indeed, since

$$t \leq t^{1/2} \leq t^{1/3} \leq \ldots \leq t^{1/n}$$

for all $t \in [0,1]$ and $n \in \mathbb{N}$. We can write

$$t \leq \frac{1}{n}\sum_{i=1}^{n} t^{1/i} \text{ and } 1-t \leq \frac{1}{n}\sum_{i=1}^{n}(1-t)^{1/i}$$

for all $t \in [0,1]$ and $n \in \mathbb{N}$. and thus, if φ is an nonnegative convex function on an interval $I \subseteq \mathbb{R}$, then we have

$$\varphi(tu+(1-t)v) \leq t\varphi(u) + (1-t)\varphi(v) \leq \frac{1}{n}\sum_{i=1}^{n} t^{1/i}\varphi(u) + \frac{1}{n}\sum_{i=1}^{n}(1-t)^{1/i}\varphi(v)$$

for all $u, v \in I$, $t \in [0,1]$ and $n \in \mathbb{N}$.

Theorem 3. *Let* $\varphi\colon [u,v]\to\mathbb{R}$ *be an n-fractional polynomial convex function, then* φ *is bounded on* $[u,v]$.

Proof. Let $M=\varphi(u)+\varphi(v)$ and $x\in[u,v]$ be an arbitrary point. There exists an $t\in[0,1]$ such that $x=tu+(1-t)v$. Also, since

$$\frac{1}{n}\sum_{i=1}^{n}t^{1/i}\leq 1 \text{ and } \frac{1}{n}\sum_{i=1}^{n}(1-t)^{1/i}\leq 1$$

for all $t\in[0,1]$, we can write

$$\varphi(x)\leq\frac{1}{n}\sum_{i=1}^{n}t^{1/i}\varphi(u)+\frac{1}{n}\sum_{i=1}^{n}(1-t)^{1/i}\varphi(v)\leq M.$$

It is also bounded from below as we see by writing an arbitrary point $x\in[u,v]$ in the form $(u+v)/2+t$ or $(u+v)/2-t, t\in[0,(v-u)/2]$.We can accept $x=(u+v)/2+t$ since it will not loss the generality. Then

$$\varphi\left(\frac{u+v}{2}\right)\leq\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{2}\right)^{1/i}\left[\varphi(x)+\varphi\left(\frac{u+v}{2}-t\right)\right]\leq\left[\varphi(x)+\varphi\left(\frac{u+v}{2}-t\right)\right]$$

or

$$\varphi(x)\geq\varphi\left(\frac{u+v}{2}\right)-\varphi\left(\frac{u+v}{2}-t\right).$$

Using M as the upper bound, $-\varphi\left(\frac{u+v}{2}-t\right)\geq -M$, so

$$\varphi(x)\geq\varphi\left(\frac{u+v}{2}\right)-M=m.$$

Thus, the proof is completed. □

Theorem 4. *Let* $\varphi_\alpha\colon [u,v]\to\mathbb{R}$ *be an arbitrary family of n-fractional polynomial convex functions and let* $\varphi(m)=\sup_\alpha\varphi_\alpha(m)$. *If* $J=\{m\in[u,v]:\varphi(m)<\infty\}$ *is nonempty, then J is an interval and* φ *is a n-fractional polynomial convex function on J.*

Proof. Let $t\in[0,1]$ and $m,r\in J$ be arbitrary. Then

$$\begin{aligned}\varphi(tm+(1-t)r)&=\sup_\alpha\varphi_\alpha(tm+(1-t)r)\\&\leq\sup_\alpha\left[\frac{1}{n}\sum_{i=1}^{n}t^{1/i}\varphi_\alpha(m)+\frac{1}{n}\sum_{i=1}^{n}(1-t)^{1/i}\varphi_\alpha(r)\right]\\&\leq\frac{1}{n}\sum_{i=1}^{n}t^{1/i}\sup_\alpha\varphi_\alpha(m)+\frac{1}{n}\sum_{i=1}^{n}(1-t)^{1/i}\sup_\alpha\varphi_\alpha(r)\\&=\frac{1}{n}\sum_{i=1}^{n}t^{1/i}\varphi(m)+\frac{1}{n}\sum_{i=1}^{n}(1-t)^{1/i}\varphi(r)<\infty.\end{aligned}$$

This shows simultaneously that J is an interval, since it contains every point between any two of its points, and that φ is a n-fractional polynomial convex function. This completes the proof of theorem. □

3. Hermite-Hadamard inequality for n-fractional polynomial convex functions

The goal of this paper is to establish some inequalities of Hermite-Hadamard type for n-fractional polynomial convex functions. In this section, we will denote by $L[u,v]$ the space of (Lebesgue) integrable functions on $[u,v]$.

Theorem 5. *Let $\varphi : [u,v] \to \mathbb{R}$ be a n-fractional polynomial convex function. If $u < v$ and $\varphi \in L[u,v]$, then the following Hermite-Hadaamrd type inequalities hold:*

$$\frac{n}{2\sum_{i=1}^{n}\left(\frac{1}{2}\right)^{1/i}}\varphi\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u}\int_u^v \varphi(x)dx \leq \left(\frac{\varphi(u)+\varphi(v)}{n}\right)\sum_{k=1}^{n}\frac{k}{k+1}. \tag{3.1}$$

Proof. From the propery of the n-fractional polynomial convex function of φ, we get

$$\begin{aligned}
\varphi\left(\frac{u+v}{2}\right) &= \varphi\left(\frac{(tu+(1-t)v)+[(1-t)u+tv]}{2}\right) \\
&= \varphi\left(\frac{1}{2}(tu+(1-t)v)+\frac{1}{2}[(1-t)u+tv]\right) \\
&\leq \frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{1/k}\varphi(tu+(1-t)v)+\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{1/k}\varphi((1-t)u+tv) \\
&= \frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{1/k}[\varphi(tu+(1-t)v)+\varphi((1-t)u+tv)].
\end{aligned}$$

By taking integral in the last inequality with respect to $t \in [0,1]$, we deduce that

$$\begin{aligned}
\varphi\left(\frac{u+v}{2}\right) &\leq \frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{1/k}\left[\int_0^1 \varphi(tu+(1-t)v)\,dt+\int_0^1 \varphi((1-t)u+tv)\,dt\right] \\
&= \frac{2}{(v-u)n}\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{1/k}\int_u^v \varphi(x)dx.
\end{aligned}$$

By using the property of the n-fractional polynomial convex function φ, if the variable is changed as $x = tu+(1-t)v$, then

$$\begin{aligned}
\frac{1}{v-u}\int_u^v \varphi(x)du &= \int_0^1 \varphi(tu+(1-t)v)\,dt \\
&\leq \int_0^1 \left[\frac{1}{n}\sum_{k=1}^{n} t^{1/k}\varphi(u)+\frac{1}{n}\sum_{k=1}^{n}(1-t)^{1/k}\varphi(v)\right]dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\varphi(u)}{n}\int_0^1 \sum_{k=1}^{n} t^{1/k}dt + \frac{\varphi(v)}{n}\int_0^1 \sum_{k=1}^{n} (1-t)^{1/k}dt \\
&= \frac{\varphi(u)}{n+1}\sum_{i=1}^{n}\int_0^1 t^{1/k}dt + \frac{\varphi(v)}{n}\sum_{k=1}^{n}\int_0^1 (1-t)^{1/k}dt \\
&= \frac{\varphi(u)}{n}\sum_{k=1}^{n}\frac{k}{k+1} + \frac{\varphi(v)}{n}\sum_{k=1}^{n}\frac{k}{k+1} \\
&= \left[\frac{\varphi(u)+\varphi(v)}{n}\right]\sum_{k=1}^{n}\frac{k}{k+1}
\end{aligned}
$$

where

$$\int_0^1 t^{1/k}dt = \int_0^1 (1-t)^{1/k} = \frac{k}{k+1}.$$

This completes the proof of theorem. □

Remark 3. In case of $n = 1$, the inequality (3.1) coincides with the the inequality (1.1)

4. Trapezoid type inequalities for n-fractional polynomial convex functions

The main purpose of this section is to establish new estimates that refine Hermite-Hadamard inequality for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is n-fractional polynomial convex function. Dragomir and Agarwal [3] used the following lemma:

Lemma 1. *Let $\varphi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $u, v \in I^\circ$ with $u < v$. If $\varphi' \in L[u,v]$.The following identity holds:*

$$\frac{\varphi(u)+\varphi(v)}{2} - \frac{1}{v-u}\int_u^v \varphi(x)dx = \frac{v-u}{2}\int_0^1 (1-2t)\varphi'(tu+(1-t)v)\,dt.$$

Theorem 6. *Let $\varphi\colon I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $u, v \in I^\circ$ with $u < v$ and assume that $\varphi' \in L[u,v]$. If $|\varphi'|$ is n-fractional polynomial convex function on interval $[u,v]$, then the following inequality holds*

$$
\begin{aligned}
&\left|\frac{\varphi(u)+\varphi(v)}{2} - \frac{1}{v-u}\int_u^v \varphi(x)dx\right| \\
&\leq \frac{v-u}{n}\sum_{k=1}^{n}\left[\frac{k(k+2^{1/k})}{2^{1/k}(k+1)(2k+1)}\right] A\left(\left|\varphi'(u)\right|, \left|\varphi'(v)\right|\right),
\end{aligned}
\tag{4.1}
$$

where $A(.,.)$ is the arithmetic mean.

Proof. Using Lemma 1 and the inequality

$$\left|\varphi'(tu+(1-t)v)\right| \leq \frac{1}{n}\sum_{k=1}^{n} t^{1/k}\left|\varphi'(u)\right| + \frac{1}{n}\sum_{k=1}^{n}(1-t)^{1/k}\left|\varphi'(v)\right|,$$

we get

$$\begin{aligned}
&\left|\frac{\varphi(u)+\varphi(v)}{2} - \frac{1}{v-u}\int_u^v \varphi(x)dx\right| \\
&\leq \left|\frac{v-u}{2}\int_0^1 (1-2t)\varphi'(tu+(1-t)v)\,dt\right| \\
&\leq \frac{v-u}{2}\int_0^1 |1-2t|\left(\frac{1}{n}\sum_{k=1}^{n} t^{1/k}\left|\varphi'(u)\right| + \frac{1}{n}\sum_{k=1}^{n}(1-t)^{1/k}\left|\varphi'(v)\right|\right)dt \\
&\leq \frac{v-u}{2n}\left(\left|\varphi'(u)\right|\int_0^1 |1-2t|\sum_{k=1}^{n} t^{1/k}dt + \left|\varphi'(v)\right|\int_0^1 |1-2t|\sum_{k=1}^{n}(1-t)^{1/k}dt\right) \\
&= \frac{v-u}{2n}\left(\left|\varphi'(u)\right|\sum_{k=1}^{n}\int_0^1 |1-2t|t^{1/k}dt + \left|\varphi'(v)\right|\sum_{k=1}^{n}\int_0^1 |1-2t|(1-t)^{1/k}dt\right) \\
&= \frac{v-u}{2n}\left(\left|\varphi'(u)\right|\sum_{k=1}^{n}\left[\frac{k(k+2^{1/k})}{2^{1/k}(k+1)(2k+1)}\right] + \left|\varphi'(v)\right|\sum_{k=1}^{n}\left[\frac{k(k+2^{1/k})}{2^{1/k}(k+1)(2k+1)}\right]\right) \\
&= \frac{v-u}{n}\sum_{k=1}^{n}\left[\frac{k(k+2^{1/k})}{2^{1/k}(k+1)(2k+1)}\right] A\left(\left|\varphi'(u)\right|,\left|\varphi'(v)\right|\right)
\end{aligned}$$

where

$$\int_0^1 |1-2t|t^{1/k}dt = \int_0^1 |1-2t|(1-t)^{1/k} = \frac{k(k+2^{1/k})}{2^{1/k}(k+1)(2k+1)}.$$

This completes the proof of theorem. □

Corollary 1. *If we take $n=1$ in the inequality (4.1), we get the following inequality:*

$$\left|\frac{\varphi(u)+\varphi(v)}{2} - \frac{1}{v-u}\int_u^v \varphi(x)dx\right| \leq \frac{v-u}{4}A\left(\left|\varphi'(u)\right|,\left|\varphi'(v)\right|\right).$$

This inequality coincides with the inequality in [3].

Theorem 7. *Let $\varphi\colon I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on I°, $u,v\in I^\circ$ with $u<v$, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$ and assume that $\varphi'\in L[u,v]$. If $|\varphi'|^q$ is an n-fractional polynomial convex function on interval $[u,v]$, then the following inequality holds*

$$\left|\frac{\varphi(u)+\varphi(v)}{2} - \frac{1}{v-u}\int_u^v \varphi(x)dx\right| \tag{4.2}$$

$$\leq \frac{v-u}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{n}\sum_{k=1}^{n}\frac{k}{k+1}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(\left|\varphi'(u)\right|^{q},\left|\varphi'(v)\right|^{q}\right),$$

where $A(.,.)$ is the arithmetic mean.

Proof. Using Lemma 1, Hölder's integral inequality and the following inequality

$$\left|\varphi'(tu+(1-t)v)\right|^{q} \leq \frac{1}{n}\sum_{k=1}^{n} t^{1/k}\left|\varphi'(u)\right|^{q} + \frac{1}{n}\sum_{k=1}^{n}(1-t)^{1/k}\left|\varphi'(v)\right|^{q}$$

which is the n-fractional polynomial convex function of $|\varphi'|^{q}$, we get

$$\begin{aligned}
&\left|\frac{\varphi(u)+\varphi(v)}{2} - \frac{1}{v-u}\int_{u}^{v}\varphi(x)dx\right| \\
&\leq \frac{v-u}{2}\left(\int_{0}^{1}|1-2t|^{p}dt\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left|\varphi'(tu+(1-t)v)\right|^{q}dt\right)^{\frac{1}{q}} \\
&\leq \frac{v-u}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{|\varphi'(u)|^{q}}{n}\sum_{k=1}^{n}\int_{0}^{1}t^{1/k}dt + \frac{|\varphi'(v)|^{q}}{n}\sum_{k=1}^{n}\int_{0}^{1}(1-t)^{1/k}dt\right)^{\frac{1}{q}} \\
&= \frac{v-u}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{n}\sum_{k=1}^{n}\frac{k}{k+1}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(\left|\varphi'(u)\right|^{q},\left|\varphi'(v)\right|^{q}\right),
\end{aligned}$$

where

$$\int_{0}^{1}|1-2t|^{p}dt = \frac{1}{p+1}.$$

This completes the proof of theorem. □

Corollary 2. *If we take $n=1$ in the inequality (4.2), we get the following inequality:*

$$\left|\frac{\varphi(u)+\varphi(v)}{2} - \frac{1}{v-u}\int_{u}^{v}\varphi(x)dx\right| \leq \frac{v-u}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} A^{\frac{1}{q}}\left(\left|\varphi'(u)\right|^{q},\left|\varphi'(v)\right|^{q}\right).$$

This inequality coincides with the inequality in [3].

Theorem 8. *Let $\varphi\colon I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on I°, $u,v\in I^{\circ}$ with $u<v$, $q\geq 1$and assume that $\varphi'\in L[u,v]$. If $|\varphi'|^{q}$ is an n-fractional polynomial convex function on the interval $[u,v]$, then the following inequality holds*

$$\begin{aligned}
&\left|\frac{\varphi(u)+\varphi(v)}{2} - \frac{1}{v-u}\int_{u}^{v}\varphi(x)dx\right| \qquad (4.3)\\
&\leq \frac{v-u}{2}\left(\frac{1}{2}\right)^{1-\frac{2}{q}}\left(\frac{1}{n}\sum_{k=1}^{n}\frac{k(k+2^{1/k})}{2^{1/k}(k+1)(2k+1)}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(\left|\varphi'(u)\right|^{q},\left|\varphi'(v)\right|^{q}\right),
\end{aligned}$$

where $A(.,.)$ is the arithmetic mean.

Proof. Assume first that $q>1$. From Lemma 1, power mean integral inequality and the property of the n-fractional polynomial convex function of $|\varphi'|^q$, we obtain

$$\begin{aligned}
&\left|\frac{\varphi(u)+\varphi(v)}{2}-\frac{1}{v-u}\int_u^v \varphi(x)dx\right| \\
&\leq \frac{v-u}{2}\left(\int_0^1 |1-2t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2t|\left|\varphi'(tu+(1-t)v)\right|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{v-u}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2t|\left[\frac{1}{n}\sum_{k=1}^{n} t^{1/k}\left|\varphi'(u)\right|^q\right.\right. \\
&\qquad\qquad \left.\left.+\frac{1}{n}\sum_{k=1}^{n}(1-t)^{1/k}\left|\varphi'(v)\right|^q dt\right]\right)^{\frac{1}{q}} \\
&= \frac{v-u}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\frac{|\varphi'(u)|^q}{n}\sum_{k=1}^{n}\int_0^1 |1-2t|\,t^{1/k}dt\right. \\
&\qquad\qquad \left.+\frac{|\varphi'(v)|^q}{n}\sum_{k=1}^{n}\int_0^1 |1-2t|\,(1-t)^{1/k}dt\right]^{\frac{1}{q}} \\
&= \frac{v-u}{2}\left(\frac{1}{2}\right)^{1-\frac{2}{q}}\left(\frac{1}{n}\sum_{k=1}^{n}\frac{k(k+2^{1/k})}{2^{1/k}(k+1)(2k+1)}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(\left|\varphi'(u)\right|^q,\left|\varphi'(v)\right|^q\right).
\end{aligned}$$

For $q=1$ we use the estimates from the proof of Theorem 6, which also follow step by step the above estimates. This completes the proof of theorem. □

Corollary 3. *Under the assumption of Theorem 8 with $q=1$, we get the conclusion of Theorem 6.*

Corollary 4. *If we take $n=1$ in the inequality (4.3), we get the following inequality:*

$$\left|\frac{\varphi(u)+\varphi(v)}{2}-\frac{1}{v-u}\int_u^v \varphi(x)dx\right| \leq \frac{v-u}{4}A^{\frac{1}{q}}\left(\left|\varphi'(u)\right|^q,\left|\varphi'(v)\right|^q\right).$$

If we take $q=1$ in the above inequality, then obtained inequality coincides with the inequality in [3].

Now, we will prove the Theorem 7 by using Hölder-İşcan integral inequality. Then we will show that the result we have obtained in this theorem gives a better approach than that obtained in the Theorem 7.

Theorem 9. *Let $\varphi\colon I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on I°, $u,v\in I^\circ$ with $u<v$, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$ and assume that $\varphi'\in L[u,v]$. If $|\varphi'|^q$ is an n-fractional*

polynomial convex function on interval $[u,v]$*, then the following inequality holds*

$$\begin{aligned}&\left|\frac{\varphi(u)+\varphi(v)}{2}-\frac{1}{v-u}\int_u^v\varphi(x)dx\right| \\ &\leq \frac{v-u}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\varphi'(u)|^q}{n}K_1(n)+\frac{|\varphi'(v)|^q}{n}K_2(n)\right)^{\frac{1}{q}} \\ &\quad+\frac{v-u}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\varphi'(u)|^q}{n}K_2(n)+\frac{|\varphi'(v)|^q}{n}K_1(n)\right)^{\frac{1}{q}},\end{aligned} \tag{4.4}$$

where

$$K_1(n)=\sum_{k=1}^{n}\frac{k^2}{(k+1)(2k+1)},\qquad K_2(n)=\sum_{k=1}^{n}\frac{k}{2k+1}.$$

Proof. Using Lemma 1, Hölder-İşcan integral inequality and the following inequality

$$\left|\varphi'(tu+(1-t)v)\right|^q\leq\frac{1}{n}\sum_{k=1}^{n}t^{1/k}\left|\varphi'(u)\right|^q+\frac{1}{n}\sum_{k=1}^{n}(1-t)^{1/k}\left|\varphi'(v)\right|^q$$

which is the n-fractional polynomial convex function of $|\varphi'|^q$, we get

$$\begin{aligned}&\left|\frac{\varphi(u)+\varphi(v)}{2}-\frac{1}{v-u}\int_u^v\varphi(x)dx\right| \\ &\leq\frac{v-u}{2}\left(\int_0^1(1-t)|1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1(1-t)\left|\varphi'(tu+(1-t)v)\right|^q dt\right)^{\frac{1}{q}} \\ &\quad+\frac{v-u}{2}\left(\int_0^1 t|1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left|\varphi'(tu+(1-t)v)\right|^q dt\right)^{\frac{1}{q}} \\ &\leq\frac{v-u}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\varphi'(u)|^q}{n}\sum_{k=1}^{n}\int_0^1(1-t)t^{1/k}dt\right. \\ &\qquad\qquad\left.+\frac{|\varphi'(v)|^q}{n}\sum_{k=1}^{n}\int_0^1(1-t)(1-t)^{1/k}dt\right)^{\frac{1}{q}} \\ &\leq\frac{v-u}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\varphi'(u)|^q}{n}\sum_{k=1}^{n}\int_0^1 t.t^{1/k}dt\right. \\ &\qquad\qquad\left.+\frac{|\varphi'(v)|^q}{n}\sum_{k=1}^{n}\int_0^1 t(1-t)^{1/k}dt\right)^{\frac{1}{q}} \\ &=\frac{v-u}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\varphi'(u)|^q}{n}K_1(n)+\frac{|\varphi'(v)|^q}{n}K_2(n)\right)^{\frac{1}{q}}\end{aligned}$$

$$+\frac{v-u}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\varphi'(u)|^q}{n}K_2(n)+\frac{|\varphi'(v)|^q}{n}K_1(n)\right)^{\frac{1}{q}}$$

where

$$\int_0^1(1-t)|1-2t|^p\,dt=\int_0^1 t\,|1-2t|^p\,dt=\frac{1}{2(p+1)},$$

$$\int_0^1(1-t)t^{1/k}dt=\int_0^1 t(1-t)^{1/k}dt=\frac{k^2}{(k+1)(2k+1)},$$

$$\int_0^1(1-t)(1-t)^{1/k}dt=\int_0^1 t.t^{1/k}dt=\frac{k}{2k+1}.$$

This completes the proof of theorem. □

Corollary 5. *If we take $n=1$ in the inequality (4.4), we get the following inequality:*

$$\left|\frac{\varphi(u)+\varphi(v)}{2}-\frac{1}{v-u}\int_u^v\varphi(x)dx\right|$$
$$\leq\frac{v-u}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{|\varphi'(u)|^q+2|\varphi'(v)|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{2|\varphi'(u)|^q+|\varphi'(v)|^q}{6}\right)^{\frac{1}{q}}\right].$$

Remark 4. The inequality (4.4) gives better results than the inequality (4.2). Indeed, using the inequality $x^r+y^r\leq 2^{1-r}(x+y)^r$, $x,y\in[0,\infty),0<r\leq 1$, by sample calculation we get

$$\frac{v-u}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\varphi'(u)|^q}{n}K_1(n)+\frac{|\varphi'(v)|^q}{n}K_2(n)\right)^{\frac{1}{q}}$$
$$+\frac{v-u}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\varphi'(u)|^q}{n}K_2(n)+\frac{|\varphi'(v)|^q}{n}K_1(n)\right)^{\frac{1}{q}}$$
$$\leq\frac{v-u}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}2^{1-1/q}\left(\frac{K_1(n)+K_2(n)}{n}\right)^{\frac{1}{q}}\left(|\varphi'(u)|^q+|\varphi'(v)|^q\right)^{\frac{1}{q}}$$
$$=\frac{v-u}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{n}\sum_{k=1}^{n}\frac{k}{k+1}\right)^{\frac{1}{q}}A^{\frac{1}{q}}\left(|\varphi'(u)|^q,|\varphi'(v)|^q\right)$$

which is the required.

Theorem 10. *Let $\varphi\colon I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on I°, $u,v\in I^\circ$ with $u<v$, $q\geq 1$ and assume that $\varphi'\in L[u,v]$. If $|\varphi'|^q$ is an n-fractional polynomial convex function on the interval $[u,v]$, then the following inequality holds for $t\in[0,1]$.*

$$\left|\frac{\varphi(u)+\varphi(v)}{2}-\frac{1}{v-u}\int_u^v\varphi(x)dx\right| \tag{4.5}$$

$$\leq \frac{v-u}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|\varphi'(u)|^q}{n}\sum_{k=1}^{n}M_1(k)+\frac{|\varphi'(v)|^q}{n}\sum_{k=1}^{n}M_2(k)\right)^{\frac{1}{q}}$$
$$+\frac{v-u}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|\varphi'(u)|^q}{n}\sum_{k=1}^{n}M_2(k)+\frac{|\varphi'(v)|^q}{n}\sum_{k=1}^{n}M_1(k)\right)^{\frac{1}{q}},$$

where

$$M_1(k)=\frac{k^2\left[\left(\frac{1}{2}\right)^{1+1/k}(5k+1)+1-k\right]}{(k+1)(2k+1)(3k+1)},\qquad M_2(k)=\frac{k\left[\left(\frac{1}{2}\right)^{1+1/k}k+1+k\right]}{(2k+1)(3k+1)}.$$

Proof. Assume first that $q>1$. From Lemma 1, improved power-mean integral inequality and the property of the n-fractional polynomial convex function of $|\varphi'|^q$, we obtain

$$\left|\frac{\varphi(u)+\varphi(v)}{2}-\frac{1}{v-u}\int_u^v\varphi(x)dx\right|$$
$$\leq \frac{v-u}{2}\left(\int_0^1(1-t)|1-2t|dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)|1-2t|\left|\varphi'(tu+(1-t)v)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{v-u}{2}\left(\int_0^1 t|1-2t|dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t|1-2t|\left|\varphi'(tu+(1-t)v)\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq \frac{v-u}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{|\varphi'(u)|^q}{n}\sum_{k=1}^{n}\int_0^1(1-t)|1-2t|t^{1/k}dt\right.$$
$$\left.+\frac{|\varphi'(v)|^q}{n}\sum_{k=1}^{n}\int_0^1(1-t)|1-2t|(1-t)^{1/k}dt\right)^{\frac{1}{q}}$$
$$+\frac{v-u}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{|\varphi'(u)|^q}{n}\sum_{k=1}^{n}\int_0^1 t|1-2t|t^{1/k}dt\right.$$
$$\left.+\frac{|\varphi'(v)|^q}{n}\sum_{k=1}^{n}\int_0^1 t|1-2t|(1-t)^{1/k}dt\right)^{\frac{1}{q}}$$
$$=\frac{v-u}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|\varphi'(u)|^q}{n}\sum_{k=1}^{n}M_1(k)+\frac{|\varphi'(v)|^q}{n}\sum_{k=1}^{n}M_2(k)\right)^{\frac{1}{q}}$$
$$+\frac{v-u}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|\varphi'(u)|^q}{n}\sum_{k=1}^{n}M_2(k)+\frac{|\varphi'(v)|^q}{n}\sum_{k=1}^{n}M_1(k)\right)^{\frac{1}{q}}$$

where

$$\int_0^1 (1-t)\,|1-2t|\,dt = \int_0^1 t\,|1-2t|\,dt = \frac{1}{4},$$

$$\begin{aligned}
M_1(k) &= \int_0^1 (1-t)\,|1-2t|\,t^{1/k} = \int_0^1 t\,|1-2t|\,(1-t)^{1/k}dt \\
&= \frac{k^2\left[\left(\frac{1}{2}\right)^{1+1/k}(5k+1)+1-k\right]}{(k+1)\,(2k+1)(3k+1)}, \\
M_2(k) &= \int_0^1 t\,|1-2t|\,t^{1/k}dt = \int_0^1 (1-t)\,|1-2t|\,(1-t)^{1/k}dt \\
&= \frac{k\left[\left(\frac{1}{2}\right)^{1+1/k}k+1+k\right]}{(2k+1)(3k+1)}.
\end{aligned}$$

For $q=1$ we use the estimates from the proof of Theorem 6, which also follow step by step the above estimates. This completes the proof of theorem. □

Corollary 6. *If we take $n=1$ in the inequality (4.5), we get the following inequality:*

$$\begin{aligned}
&\left|\frac{\varphi(u)+\varphi(v)}{2} - \frac{1}{v-u}\int_u^v \varphi(x)dx\right| \\
&\leq \frac{v-u}{8}\left[\left(\frac{|\varphi'(u)|^q}{4}+\frac{3\,|\varphi'(v)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{3\,|\varphi'(u)|^q}{4}+\frac{|\varphi'(v)|^q}{4}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Remark 5. The inequality (4.5) gives better result than the inequality (4.3). Indeed, If we use the inequality $x^r+y^r \leq 2^{1-r}(x+y)^r$, $x,y\in[0,\infty)$, $0<r\leq 1$, we get

$$\begin{aligned}
&\frac{v-u}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|\varphi'(u)|^q}{n}\sum_{s=1}^{n}M_1(k)+\frac{|\varphi'(v)|^q}{n}\sum_{s=1}^{n}K_2(s)\right)^{\frac{1}{q}} \\
&+\frac{v-u}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|\varphi'(u)|^q}{n}\sum_{s=1}^{n}M_2(k)+\frac{|\varphi'(v)|^q}{n}\sum_{s=1}^{n}K_1(s)\right)^{\frac{1}{q}} \\
&\leq \frac{v-u}{2}\left(\frac{1}{2}\right)^{1-\frac{2}{q}}\left(\frac{1}{n}\sum_{s=1}^{n}\left[M_1(k)+M_2(k)\right]\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(\left|\varphi'(u)\right|^q,\left|\varphi'(v)\right|^q\right),
\end{aligned}$$

where

$$M_1(k)+M_2(k) = \frac{k(k+2^{1/k})}{2^{1/k}(k+1)(2k+1)}$$

which completes the proof of remark.

5. Applications for special means

Throughout this section, for shortness, the following notations will be used for special means of two nonnegative numbers u,v with $v>u$:

1. The arithmetic mean
$$A:=A(u,v)=\frac{u+v}{2}.$$
2. The geometric mean
$$G:=G(u,v)=\sqrt{uv},\quad u,v\geq 0.$$
3. The harmonic mean
$$H:=H(u,v)=\frac{2uv}{u+v},\quad u,v>0.$$
4. The logarithmic mean
$$L:=L(u,v)=\begin{cases}\frac{v-u}{\ln v-\ln u}, & u\neq v;\\ u, & u=v;\end{cases}\quad u,v>0.$$
5. The p-logaritmic mean
$$L_p:=L_p(u,v)=\begin{cases}\left(\frac{v^{p+1}-u^{p+1}}{(p+1)(v-u)}\right)^{\frac{1}{p}}, & u\neq v, p\in\mathbb{R}\setminus\{-1,0\};\\ u, & u=v;\end{cases}\quad u,v>0.$$
6. The identric mean
$$I:=I(u,v)=\frac{1}{e}\left(\frac{v^v}{u^u}\right)^{\frac{1}{v-u}},\quad u,v>0.$$

These means are often used in numerical approximation and in other areas. However, the following simple relationships are known in the literature:
$$H\leq G\leq L\leq I\leq A.$$
It is also known that L_p is monotonically increasing over $p\in\mathbb{R}$, denoting $L_0=I$ and $L_{-1}=L$.

Proposition 1. *Let $u,v\in[0,\infty)$ with $u<v$ and $n\in(-\infty,0)\cup[1,\infty)\setminus\{-1\}$. Then, the following inequalities are obtained:*
$$\frac{n}{2\sum_{i=1}^{n}\left(\frac{1}{2}\right)^{1/i}}A^n(u,v)\leq L_n^n(u,v)\leq A(u^n,v^n)\frac{2}{n}\sum_{k=1}^{n}\frac{k}{k+1}.$$

Proof. The assertion follows from the inequalities (3.1) for the function
$$\varphi(x)=x^n,\ \ x\in[0,\infty).$$

□

Proposition 2. *Let $u,v \in (0,\infty)$ with $u < v$. Then, the following inequalities are obtained:*

$$\frac{n}{2\sum_{i=1}^{n}\left(\frac{1}{2}\right)^{1/i}} \leq L^{-1}(u,v) \leq \frac{2}{n}H^{-1}(u,v)\sum_{k=1}^{n}\frac{k}{k+1}.$$

Proof. The assertion follows from the inequalities (3.1) for the function

$$\varphi(x) = x^{-1}, \quad x \in (0,\infty).$$

□

Proposition 3. *Let $u,v \in (0,1]$ with $u < v$. Then, the following inequalities are obtained:*

$$\frac{2\ln G(u,v)}{n}\sum_{k=1}^{n}\frac{k}{k+1} \leq \ln I \leq \frac{n}{2\sum_{i=1}^{n}\left(\frac{1}{2}\right)^{1/i}}\ln A(u,v).$$

Proof. The assertion follows from the inequalities (3.1) for the function

$$\varphi(x) = -\ln x, \quad x \in (0,1].$$

□

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ON A SOLVABLE SYSTEM OF DIFFERENCE EQUATIONS OF SIXTH-ORDER

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Abstract. In this paper, we study the following two-dimesional system of difference equations

$$x_n = \frac{x_{n-4}y_{n-5}x_{n-6}}{y_{n-1}x_{n-2}\left(a+by_{n-3}x_{n-4}y_{n-5}x_{n-6}\right)}, \quad y_n = \frac{y_{n-4}x_{n-5}y_{n-6}}{x_{n-1}y_{n-2}\left(c+dx_{n-3}y_{n-4}x_{n-5}y_{n-6}\right)}, \; n \in \mathbb{N}_0,$$

where the parameters a,b,c,d and the initial values x_{-i}, y_{-i}, $i \in \{1,2,3,4,5,6\}$, are real numbers. We show that some subclasses of nonlinear two-dimensional system of difference equations are solvable in closed form. We also describe the forbidden set of solutions of the system of difference equations. Some numerical examples are given to demonstrate the theoretical results.

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Keywords: system of difference equations, closed-form, periodic solution, forbidden set

1. Introduction and preliminaries

Difference equations and systems of difference equations usually state the natural models of discrete processes, containing non-linear difference equations such as rational difference equations, Riccati difference equations, exponential difference equations. They play a key role in numerous applications in applied sciences such as Genetics, Biomathematics, Population Dynamics, Bioengineering, Biology and other sciences. So, there has been a lot of interest in difference equations which can be solved in explicit form or closed form (see, [1–4, 13, 14, 17, 19, 20, 23, 24, 36, 40–42, 44]) as well as in difference equations systems (see, e.g. [6–9, 11, 16, 21, 25–28, 30–35, 37–39, 45, 46, 48–50]). For example, the difference equations

$$x_{n+1} = \frac{x_{n-3}x_{n-4}}{x_n\left(\pm 1 \pm x_{n-1}x_{n-2}x_{n-3}x_{n-4}\right)}, \qquad n \in \mathbb{N}_0, \tag{1.1}$$

where the initial conditions are arbitrary real numbers, were studied in [12]. In addition, they investigated the behavior of the solutions of equations in (1.1).

The authors of [43] studied the periodicity and the solutions of the following systems of difference equations

$$x_{n+1} = \frac{y_n x_{n-1}}{\pm x_{n-1} \pm y_n}, \quad y_{n+1} = \frac{x_n y_{n-1}}{\pm y_{n-1} \pm x_n}, \quad n \in \mathbb{N}_0,$$

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where the initial conditions are non-zero real numbers.

The existence, uniqueness and attractivity of prime period two solutions of the following difference equation

$$x_{n+1} = a + bx_{n-1}e^{-x_n}, \qquad n \in \mathbb{N}_0,$$

where a, b are positive constants and the initial values x_{-1}, x_0 are positive numbers, was studied in [15].

Recently, Yazlik and Gungor solved the following non-linear difference equation

$$x_n = \frac{x_{n-4}x_{n-5}x_{n-6}}{x_{n-1}x_{n-2}\left(a + bx_{n-3}x_{n-4}x_{n-5}x_{n-6}\right)}, \qquad n \in \mathbb{N}_0, \tag{1.2}$$

in closed form. Also, the asymptotic behavior of well-defined solution of equation (1.2) was obtained in [47].

In [22], Ibrahim gave the solutions of the rational difference equation

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}\left(a + bx_n x_{n-2}\right)}, \qquad n \in \mathbb{N}_0, \tag{1.3}$$

where initial values x_0, x_{-1}, x_{-2} are non-negative real numbers with $bx_0x_{-2} \neq -a$ and $x_{-1} \neq 0$. In addition, he investigated some properties for difference equation (1.3) such as the local stability and the boundedness for the solutions.

Alzubaidi et al. [5] presented the solutions of the following recursive sequences

$$x_{n+1} = \frac{x_{n-2}x_{n-3}}{x_n\left(\pm 1 \pm x_{n-2}x_{n-3}\right)}, \qquad n \in \mathbb{N}_0, \tag{1.4}$$

where the initial conditions x_{-3}, x_{-2}, x_{-1} and x_0 are arbitrary real numbers. Also, they studied some dynamic behavior of equations in (1.4).

El-Dessoky et al. dealt with the existence of solutions and the periodicity character of the following systems of rational difference equations

$$x_{n+1} = \frac{x_n y_{n-3}}{y_{n-2}\left(\pm 1 \pm x_n y_{n-3}\right)}, \quad y_{n+1} = \frac{y_n x_{n-3}}{x_{n-2}\left(\pm 1 \pm y_n x_{n-3}\right)}, \quad n \in \mathbb{N}_0,$$

with initial conditions are non-zero real numbers in [10].

In [29], Kara et al. showed that the following system of difference equations

$$x_n = \frac{x_{n-2}y_{n-3}}{y_{n-1}\left(a_n + b_n x_{n-2}y_{n-3}\right)}, \quad y_n = \frac{y_{n-2}x_{n-3}}{x_{n-1}\left(\alpha_n + \beta_n y_{n-2}x_{n-3}\right)}, \quad n \in \mathbb{N}_0, \tag{1.5}$$

where the sequences $\forall n \in \mathbb{N}_0$, (a_n), (b_n), (α_n), (β_n) and the initial values x_{-j}, y_{-j}, $j \in \{1,2,3\}$ are non-zero real numbers, could be solved in the closed form. Further, they investigated the asymptotic behavior and periodicity of solutions of system (1.5) for the case when all the sequences (a_n), (b_n), (α_n), (β_n) are constant.

Also, Halim et al. [18] gave a representation formula for the general solutions to the following two-dimensional system of difference equations

$$x_{n+1} = \frac{y_{n-1}x_{n-2}}{y_n\left(a + by_{n-1}x_{n-2}\right)}, \quad y_{n+1} = \frac{x_{n-1}y_{n-2}}{x_n\left(a + bx_{n-1}y_{n-2}\right)}, \quad n \in \mathbb{N}_0,$$

where parameters a, b and initial values x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0 are real numbers. In addition, they gave some theoretical explanations related to the representation.

Motivated by the above papers we will expand the difference equation (1.2) to following system of difference equations

$$\begin{aligned} x_n &= \frac{x_{n-4}y_{n-5}x_{n-6}}{y_{n-1}x_{n-2}\left(a+by_{n-3}x_{n-4}y_{n-5}x_{n-6}\right)}, \\ y_n &= \frac{y_{n-4}x_{n-5}y_{n-6}}{x_{n-1}y_{n-2}\left(c+dx_{n-3}y_{n-4}x_{n-5}y_{n-6}\right)}, \end{aligned} \tag{1.6}$$

for $n \in \mathbb{N}_0$, where the parameters a,b and the initial values x_{-i}, $i \in \{1,2,3,4,5,6\}$, are real numbers. Our aim here is to show that system (1.6) is solvable in closed form by using the method of transformation. The forbidden set of initial values for solutions of system (1.6) is also described. Some numerical examples are given to demonstrate the theoretical results.

Definition 1. A solution $(x_n,y_n)_{n\geq -6}$ of system (1.6) is called *eventually periodic* with period p if there exist $n_0 \geq -6$ such that $x_{n+p} = x_n$ and $y_{n+p} = y_n$ for all $n \geq n_0$. If $n_0 = -6$, then the solution $(x_n,y_n)_{n\geq -6}$ of system (1.6) is said to be periodic with period p.

2. Solutions of the system (1.6) in closed form

In this section is studied solvability of system (1.6) and our main results are proved, also some applicateions are given. We will deal only with well-defined solutions to system of difference equations. Hence, we assume that

$$x_n \neq 0, \quad y_n \neq 0, \quad n \geq -6,$$

and

$$a+by_{n-3}x_{n-4}y_{n-5}x_{n-6} \neq 0, \quad c+dx_{n-3}y_{n-4}x_{n-5}y_{n-6} \neq 0, \quad n \in \mathbb{N}_0.$$

It is clear that if $x_{-j} = 0$ or $y_{-j} = 0$, for some $j \in \{1,2\}$, then x_0 or y_0 is not defined. Similarly, if $x_{-3} = 0$ or $y_{-3} = 0$, then $x_1 = 0$ or $y_1 = 0$ and y_2 or x_2 is not defined, respectively, while if $x_{-l} = 0$ or $y_{-l} = 0$, for some $l \in \{4,5,6\}$, then $x_0 = 0$ or $y_0 = 0$ so that x_1 or y_1 is not defined. On the other hand, we suppose that $x_{n_0} = 0$ for some $n_0 \in \mathbb{N}_0$ and $x_n \neq 0$, for every $n < n_0$, then from system (1.6), it follows that $x_{n_0-4} = 0$ or $y_{n_0-5} = 0$, which is impossible. Thus, the set

$$\left\{\vec{\mathbb{S}} : x_{-i} = 0 \text{ or } y_{-i} = 0,\ i \in \{1,2,3,4,5,6\}\right\}$$

is a subset of forbidden set of solutions of the initial values for system (1.6). Hence, for well-defined solutions of system (1.6), $(x_n,y_n)_{n\geq -6}$, we have that

$$x_n y_n \neq 0, \quad n \geq -6, \tag{2.1}$$

if and only if $x_{-j}y_{-j} \neq 0$, $j \in \{1,2,3,4,5,6\}$. After this, we assume that $(x_n,y_n)_{n\geq -6}$ is a solution of system (1.6) with holding the condition (2.1). Now, we will investigate

the solutions in 10 different cases depending on whether the parameters are zero or nonzero.

Case 1: Let $a = c = 0$, $bd \neq 0$. In this case, system (1.6) is equivalent to the system

$$x_n = \frac{1}{by_{n-1}x_{n-2}y_{n-3}}, \quad y_n = \frac{1}{dx_{n-1}y_{n-2}x_{n-3}}, \quad n \in \mathbb{N}_0. \tag{2.2}$$

Multiplying the first equation in system (2.2) by y_{n-1}, for all $n \in \mathbb{N}_0$ and the second by x_{n-1}, for all $n \in \mathbb{N}_0$, it follows that

$$x_n y_{n-1} = \frac{1}{bx_{n-2}y_{n-3}}, \quad y_n x_{n-1} = \frac{1}{dy_{n-2}x_{n-3}}, \quad n \in \mathbb{N}_0.$$

Now, we may use the change of variables

$$k_n = x_n y_{n-1}, \quad \widehat{k}_n = y_n x_{n-1}, \quad n \geq -2,$$

and transform (2.2) into the following equations

$$k_n = \frac{1}{bk_{n-2}} = k_{n-4}, \quad \widehat{k}_n = \frac{1}{d\widehat{k}_{n-2}} = \widehat{k}_{n-4}, \quad n \geq 2,$$

which means that $(k_n)_{n\geq -2}$ and $\left(\widehat{k}_n\right)_{n\geq -2}$ are four-periodic, that is,

$$k_{4n+i} = k_i, \quad \widehat{k}_{4n+i} = \widehat{k}_i,$$

where $n \in \mathbb{N}_0$, $i \in \{-2,-1,0,1\}$, from which along with the substitutions $k_n = x_n y_{n-1}$, $\widehat{k}_n = y_n x_{n-1}$, it follows that

$$x_{4n+i} = \frac{d}{b} x_{4(n-1)+i}, \quad y_{4n+i} = \frac{b}{d} y_{4(n-1)+i},$$

for every $n \in \mathbb{N}_0$, $i \in \{-2,-1,0,1\}$, and we get

$$x_{4n+i} = \left(\frac{d}{b}\right)^n x_i, \quad y_{4n+i} = \left(\frac{b}{d}\right)^n y_i, \tag{2.3}$$

for every $n \in \mathbb{N}_0$, $i \in \{-2,-1,0,1\}$.

Case 2: Let $b = d = 0$, $ac \neq 0$. In this case, system (1.6) is written as in the form

$$x_n = \frac{x_{n-4}y_{n-5}x_{n-6}}{ay_{n-1}x_{n-2}}, \quad y_n = \frac{y_{n-4}x_{n-5}y_{n-6}}{cx_{n-1}y_{n-2}}, \quad n \in \mathbb{N}_0. \tag{2.4}$$

Multiplying the first equation in system (2.4) by $y_{n-1}x_{n-2}$, for all $n \in \mathbb{N}_0$ and the second by $x_{n-1}y_{n-2}$, for all $n \in \mathbb{N}_0$, it follows that

$$x_n y_{n-1} x_{n-2} = \frac{x_{n-4}y_{n-5}x_{n-6}}{a}, \quad y_n x_{n-1} y_{n-2} = \frac{y_{n-4}x_{n-5}y_{n-6}}{c}, \quad n \in \mathbb{N}_0. \tag{2.5}$$

By change the change of variables

$$w_n = x_n y_{n-1} x_{n-2}, \quad \widehat{w}_n = y_n x_{n-1} y_{n-2}, \quad n \geq -4,$$

system (2.5) becomes

$$w_n = \frac{1}{a} w_{n-4}, \quad \widehat{w}_n = \frac{1}{c} \widehat{w}_{n-4}, \quad n \in \mathbb{N}_0. \tag{2.6}$$

From (2.6), we see that the sequences $(w_{4m+i})_{m\geq -1}$ and $(\widehat{w}_{4m+i})_{m\geq -1}$, for $i \in \{0,1,2,3\}$, are the solutions of the homogeneous linear first-order difference equation with constant coefficients, respectively,

$$s_m = \frac{1}{a} s_{m-1}, \quad \widehat{s}_n = \frac{1}{c} \widehat{s}_{n-1}, \quad m \in \mathbb{N}_0.$$

From which it follows that

$$s_m = \left(\frac{1}{a}\right)^{m+1} s_{-1}, \quad \widehat{s}_n = \left(\frac{1}{c}\right)^{m+1} \widehat{s}_{-1}, \quad m \in \mathbb{N}_0,$$

and consequently we have

$$w_{4m+i} = \left(\frac{1}{a}\right)^{m+1} w_{i-4}, \quad \widehat{w}_{4m+i} = \left(\frac{1}{c}\right)^{m+1} \widehat{w}_{i-4}, \quad m \in \mathbb{N}_0, \tag{2.7}$$

for $i \in \{0,1,2,3\}$. From (2.6), we easily get

$$x_n = \frac{w_n \widehat{w}_{n-3}}{\widehat{w}_{n-1} w_{n-4}} x_{n-6}, \quad y_n = \frac{\widehat{w}_n w_{n-3}}{w_{n-1} \widehat{w}_{n-4}} y_{n-6}, \quad n \in \mathbb{N}_0,$$

from which along with the solutions in (2.7), we obtain the general solutions of system (2.4)

$$x_{12m+4r+s} = x_{4r+s-12} \prod_{j=0}^{m} \left(\frac{w_{4(3j+r)+s} \widehat{w}_{4(3j+r)+s-3}}{\widehat{w}_{4(3j+r)+s-1} w_{4(3j+r-1)+s}} \frac{w_{4(3j+r-1)+s-2} \widehat{w}_{4(3j+r-2)+s-1}}{\widehat{w}_{4(3j+r-1)+s-3} w_{4(3j+r-2)+s-2}} \right),$$

and

$$y_{12m+4r+s} = y_{4r+s-12} \prod_{j=0}^{m} \left(\frac{\widehat{w}_{4(3j+r)+s} w_{4(3j+r)+s-3}}{w_{4(3j+r)+s-1} \widehat{w}_{4(3j+r-1)+s}}, \frac{\widehat{w}_{4(3j+r-1)+s-2} w_{4(3j+r-2)+s-1}}{w_{4(3j+r-1)+s-3} \widehat{w}_{4(3j+r-2)+s-2}} \right),$$

for every $m \in \mathbb{N}_0$, $r \in \{1,2,3\}$ and $s \in \{2,3,4,5\}$, from which along with (2.7) it follows that

$$x_{12m+4r+s} = x_{4r+s-12} \left(\frac{c}{a^2}\right)^{m+1}, \quad y_{12m+4r+s} = y_{4r+s-12} \left(\frac{a}{c^2}\right)^{m+1}, \tag{2.8}$$

for every $m \in \mathbb{N}_0$, $r \in \{1,2,3\}$ and $s \in \{2,3,4,5\}$.

Case 3: Let $a=d=0$, $bc\neq 0$. In this case, system (1.6) is expressed as

$$x_n=\frac{1}{by_{n-1}x_{n-2}y_{n-3}},\quad y_n=\frac{y_{n-4}x_{n-5}y_{n-6}}{cx_{n-1}y_{n-2}},\quad n\in\mathbb{N}_0. \tag{2.9}$$

Multiplying the first equation in system (2.9) by $y_{n-1}x_{n-2}y_{n-3}$, for all $n\in\mathbb{N}_0$ and the second by $x_{n-1}y_{n-2}x_{n-3}$, for all $n\in\mathbb{N}_0$, it follows that

$$x_ny_{n-1}x_{n-2}y_{n-3}=\frac{1}{b},\quad y_nx_{n-1}y_{n-2}x_{n-3}=\frac{x_{n-3}y_{n-4}x_{n-5}y_{n-6}}{c},\quad n\in\mathbb{N}_0. \tag{2.10}$$

By using the change of variables

$$r_n=x_ny_{n-1}x_{n-2}y_{n-3},\quad \widehat{r}_n=y_nx_{n-1}y_{n-2}x_{n-3},\quad n\in\mathbb{N}_0, \tag{2.11}$$

system (2.10) becomes

$$\begin{cases} r_n=\frac{1}{b}, & n\in\mathbb{N}_0,\\ \widehat{r}_n=\frac{r_{n-3}}{c}=\frac{1}{bc}, & n\geq 3.\end{cases} \tag{2.12}$$

From (2.11), we easily obtain

$$x_n=\frac{r_n}{y_{n-1}x_{n-2}y_{n-3}}=\frac{r_n}{\widehat{r}_{n-1}}x_{n-4},\quad y_n=\frac{\widehat{r}_n}{x_{n-1}y_{n-2}x_{n-3}}=\frac{\widehat{r}_n}{r_{n-1}}y_{n-4},\quad n\geq 1,$$

from which along with (2.12), we get the general solutions of system (2.9)

$$x_{4n+i}=c^nx_i,\quad y_{4n+i}=\left(\frac{1}{c}\right)^n y_i, \tag{2.13}$$

for every $n\in\mathbb{N}_0$ and $i\in\{0,1,2,3\}$.

Case 4: Let $b=c=0$, $ad\neq 0$. In this case, we obtain the system

$$x_n=\frac{x_{n-4}y_{n-5}x_{n-6}}{ay_{n-1}x_{n-2}},\quad y_n=\frac{1}{dx_{n-1}y_{n-2}x_{n-3}},\quad n\in\mathbb{N}_0. \tag{2.14}$$

By interchanging variables x_n, y_n and d instead of b and a instead of c, system (2.9) is transformed into (2.14). So, by interchanging x_{-j} and y_{-j}, for $j\in\{1,2,3,4,5,6\}$, the formula in (2.13) is transformed into the formula

$$x_{4n+i}=\left(\frac{1}{a}\right)^n x_i,\quad y_{4n+i}=a^ny_i, \tag{2.15}$$

for every $n\in\mathbb{N}_0$ and $i\in\{0,1,2,3\}$.

Case 5: Let $a=0$, $bcd\neq 0$. In this case, system (1.6) becomes

$$x_n=\frac{1}{by_{n-1}x_{n-2}y_{n-3}},\quad y_n=\frac{y_{n-4}x_{n-5}y_{n-6}}{x_{n-1}y_{n-2}\left(c+dx_{n-3}y_{n-4}x_{n-5}y_{n-6}\right)},\quad n\in\mathbb{N}_0. \tag{2.16}$$

Multiplying the first equation in system (2.16) by $y_{n-1}x_{n-2}y_{n-3}$, for all $n\in\mathbb{N}_0$ and the second by $x_{n-1}y_{n-2}x_{n-3}$, for all $n\in\mathbb{N}_0$, it follows that

$$x_ny_{n-1}x_{n-2}y_{n-3}=\frac{1}{b},\quad n\in\mathbb{N}_0,$$

$$y_n x_{n-1} y_{n-2} x_{n-3} = \frac{x_{n-3} y_{n-4} x_{n-5} y_{n-6}}{c + d x_{n-3} y_{n-4} x_{n-5} y_{n-6}}, \quad n \in \mathbb{N}_0. \tag{2.17}$$

By using the first equation of (2.17) in the second one, we get that

$$x_n y_{n-1} x_{n-2} y_{n-3} = \frac{1}{b}, \quad n \in \mathbb{N}_0,$$
$$y_n x_{n-1} y_{n-2} x_{n-3} = \frac{1}{bc+d}, \quad n \in \mathbb{N}_0,$$

from which it follows that

$$x_{4n+i} = \left(\frac{bc+d}{b}\right)^n x_i, \quad y_{4n+i_1} = \left(\frac{b}{bc+d}\right)^n y_{i_1}, \tag{2.18}$$

where $n \in \mathbb{N}$, $i \in \{0,1,2,3\}$ and $i_1 \in \{-1,0,1,2\}$.

Case 6: Let $b = 0$, $acd \neq 0$. In this case, system (1.6) is expressed as

$$x_n = \frac{x_{n-4} y_{n-5} x_{n-6}}{a y_{n-1} x_{n-2}}, \quad y_n = \frac{y_{n-4} x_{n-5} y_{n-6}}{x_{n-1} y_{n-2} \left(c + d x_{n-3} y_{n-4} x_{n-5} y_{n-6}\right)}, \quad n \in \mathbb{N}_0. \tag{2.19}$$

Multiplying the first equation in system (2.19) by $y_{n-1} x_{n-2} y_{n-3}$, for all $n \in \mathbb{N}_0$ and the second by $x_{n-1} y_{n-2} x_{n-3}$, for all $n \in \mathbb{N}_0$, it follows that

$$x_n y_{n-1} x_{n-2} y_{n-3} = \frac{1}{a} y_{n-3} x_{n-4} y_{n-5} x_{n-6}, \quad n \in \mathbb{N}_0,$$
$$y_n x_{n-1} y_{n-2} x_{n-3} = \frac{x_{n-3} y_{n-4} x_{n-5} y_{n-6}}{c + d x_{n-3} y_{n-4} x_{n-5} y_{n-6}}, \quad n \in \mathbb{N}_0. \tag{2.20}$$

By using the first equation of (2.20) in the second one, we get that

$$y_n x_{n-1} y_{n-2} x_{n-3} = \frac{y_{n-6} x_{n-7} y_{n-8} x_{n-9}}{ac + d y_{n-6} x_{n-7} y_{n-8} x_{n-9}}, \quad n \geq 3. \tag{2.21}$$

By employing the substitution $y_n x_{n-1} y_{n-2} x_{n-3} = \frac{1}{s_n}$, $n \geq -3$, to (2.21), we obtain the linear six-order equation

$$s_n = ac s_{n-6} + d, \qquad n \geq 3, \tag{2.22}$$

from which it easily follows that

$$s_{6n+i_2} = \frac{d + (ac)^n \left((1-ac) s_{i_2} - d\right)}{1-ac}, \quad n \in \mathbb{N}_0, \tag{2.23}$$

if $ac \neq 1$, and

$$s_{6n+i_2} = s_{i_2} + dn, \qquad n \in \mathbb{N}_0, \tag{2.24}$$

if $ac = 1$, where $i_2 \in \{-3,-2,-1,0,1,2\}$. The equalities in (2.23) and (2.24) are formulas for the general solution of (2.22). From the substitution $y_n x_{n-1} y_{n-2} x_{n-3} = \frac{1}{s_n}$, for $n \geq -3$, and $x_n y_{n-1} x_{n-2} y_{n-3} = \frac{1}{a s_{n-3}}$, for $n \in \mathbb{N}_0$, we get

$$x_n = \frac{s_{n-1} s_{n-5} s_{n-9}}{a^3 s_{n-3} s_{n-7} s_{n-11}} x_{n-12}, \quad n \geq 8,$$

$$y_n = \frac{a^3 s_{n-12}}{s_n} y_{n-12}, \quad n \geq 9,$$

from which it follows that

$$x_{12m+6r_1+i_1} = x_{6r_1+i_1-12} \frac{1}{a^{3m+3}} \prod_{j=0}^{m} \frac{s_{12j+6r_1+i_1-1} s_{12j+6r_1+i_1-5} s_{12j+6r_1+i_1-9}}{s_{12j+6r_1+i_1-3} s_{12j+6r_1+i_1-7} s_{12j+6r_1+i_1-11}},$$

$$\begin{aligned} y_{12m+6r_2+i_2} &= y_{6r_2+i_2-12} a^{3m+3} \prod_{j=0}^{m} \frac{s_{6(2j+r_2-2)+i_2}}{s_{6(2j+r_2)+i_2}} \\ &= y_{6r_2+i_2-12} a^{3m+3} \frac{s_{6(r_2-2)+i_2}}{s_{6(2m+r_2)+i_2}}, \end{aligned}$$

where

$$m \in \mathbb{N}_0, \quad r_1 \in \{1,2\}, \quad r_2 \in \{2,3\}, \quad i_1 \in \{2,3,4,5,6,7\} \quad \text{and}$$
$$i_2 \in \{-3,-2,-1,0,1,2\}.$$

From this, along with (2.23) and (2.24), we get

$$\begin{aligned} x_{12m+6r_1+i_1} &= x_{6r_1+i_1-12} \frac{1}{a^{3m+3}} \\ &\times \prod_{j=0}^{m} \frac{d+(ac)^{\left(2j+r_1+1+\lfloor \frac{i_1-4}{6} \rfloor\right)} \left((1-ac) s_{i_1-7-6\lfloor \frac{i_1-4}{6} \rfloor} - d\right)}{d+(ac)^{\left(2j+r_1+1+\lfloor \frac{i_1-6}{6} \rfloor\right)} \left((1-ac) s_{i_1-9-6\lfloor \frac{i_1-6}{6} \rfloor} - d\right)} \\ &\times \frac{d+(ac)^{\left(2j+r_1+\lfloor \frac{i_1-2}{6} \rfloor\right)} \left((1-ac) s_{i_1-5-6\lfloor \frac{i_1-2}{6} \rfloor} - d\right)}{d+(ac)^{\left(2j+r_1+\lfloor \frac{i_1-4}{6} \rfloor\right)} \left((1-ac) s_{i_1-7-6\lfloor \frac{i_1-4}{6} \rfloor} - d\right)} \\ &\times \frac{d+(ac)^{\left(2j+r_1+\lfloor \frac{i_1-6}{6} \rfloor\right)} \left((1-ac) s_{i_1-9-6\lfloor \frac{i_1-6}{6} \rfloor} - d\right)}{d+(ac)^{\left(2j+r_1-1+\lfloor \frac{i_1-2}{6} \rfloor\right)} \left((1-ac) s_{i_1-5-6\lfloor \frac{i_1-2}{6} \rfloor} - d\right)}, \end{aligned} \tag{2.25}$$

$$\begin{aligned} y_{12m+6r_2+i_2} &= y_{6r_2+i_2-12} a^{3m+3} \prod_{j=0}^{m} \frac{d+(ac)^{(2j+r_2-2)} \left((1-ac) s_{i_2} - d\right)}{d+(ac)^{(2j+r_2)} \left((1-ac) s_{i_2} - d\right)} \\ &= y_{6r_2+i_2-12} a^{3m+3} \frac{d+(ac)^{(r_2-2)} \left((1-ac) s_{i_2} - d\right)}{d+(ac)^{(2m+r_2)} \left((1-ac) s_{i_2} - d\right)}, \end{aligned} \tag{2.26}$$

if $ac \neq 1$, and

$$x_{12m+6r_1+i_1} = x_{6r_1+i_1-12} \frac{1}{a^{3m+3}} \tag{2.27}$$

$$\times \prod_{j=0}^{m} \left(\frac{s_{i_1-7-6\lfloor \frac{i_1-4}{6} \rfloor} + d\left(2j+r_1+1+\lfloor \frac{i_1-4}{6} \rfloor\right)}{s_{i_1-9-6\lfloor \frac{i_1-6}{6} \rfloor} + d\left(2j+r_1+1+\lfloor \frac{i_1-6}{6} \rfloor\right)} \right)$$
$$\times \left(\frac{s_{i_1-5-6\lfloor \frac{i_1-2}{6} \rfloor} + d\left(2j+r_1+\lfloor \frac{i_1-2}{6} \rfloor\right)}{s_{i_1-7-6\lfloor \frac{i_1-4}{6} \rfloor} + d\left(2j+r_1+\lfloor \frac{i_1-4}{6} \rfloor\right)} \right)$$
$$\times \left(\frac{s_{i_1-9-6\lfloor \frac{i_1-6}{6} \rfloor} + d\left(2j+r_1+\lfloor \frac{i_1-6}{6} \rfloor\right)}{s_{i_1-5-6\lfloor \frac{i_1-2}{6} \rfloor} + d\left(2j+r_1-1+\lfloor \frac{i_1-2}{6} \rfloor\right)} \right),$$

$$y_{12m+6r_2+i_2} = y_{6r_2+i_2-12} a^{3m+3} \frac{s_{i_2} + d(r_2-2)}{s_{i_2} + d(2m+r_2)}, \tag{2.28}$$

if $ac = 1$, where $m \in \mathbb{N}_0$, $r_1 \in \{1,2\}$, $r_2 \in \{2,3\}$, $i_1 \in \{2,3,4,5,6,7\}$ and $i_2 \in \{-3,-2,-1,0,1,2\}$.

Case 7: Let $c = 0$, $abd \neq 0$. In this case, system (1.6) becomes

$$x_n = \frac{x_{n-4}y_{n-5}x_{n-6}}{y_{n-1}x_{n-2}\left(a + by_{n-3}x_{n-4}y_{n-5}x_{n-6}\right)}, \quad y_n = \frac{1}{dx_{n-1}y_{n-2}x_{n-3}}, \quad n \in \mathbb{N}_0. \tag{2.29}$$

By interchanging variables x_n, y_n and d instead of b, a instead of c and b instead of d, system (2.16) is transformed into (2.29). So, by interchanging x_{-j} and y_{-j}, for $j \in \{1,2,3,4,5,6\}$, the formula in (2.18) is transformed into the formula

$$x_{4n+i_1} = \left(\frac{d}{ad+b}\right)^n x_{i_1}, \quad y_{4n+i} = \left(\frac{ad+b}{d}\right)^n y_i, \tag{2.30}$$

where $n \in \mathbb{N}$, $i \in \{0,1,2,3\}$ and $i_1 \in \{-1,0,1,2\}$.

Case 8: Let $d = 0$, $abc \neq 0$. In this case, system (1.6) is equivalent to the system

$$x_n = \frac{x_{n-4}y_{n-5}x_{n-6}}{y_{n-1}x_{n-2}\left(a + by_{n-3}x_{n-4}y_{n-5}x_{n-6}\right)}, \quad y_n = \frac{y_{n-4}x_{n-5}y_{n-6}}{cx_{n-1}y_{n-2}}, \quad n \in \mathbb{N}_0. \tag{2.31}$$

Similarly, by interchanging variables x_n, y_n and b instead of d, c instead of a and a instead of c, system (2.19) is transformed into (2.31). So, by interchanging x_{-j} and y_{-j}, for $j \in \{1,2,3,4,5,6\}$, the formula in (2.25)-(2.28) is transformed into the formula

$$\begin{aligned} x_{12m+6r_2+i_2} &= x_{6r_2+i_2-12} c^{3m+3} \prod_{j=0}^{m} \frac{b + (ac)^{(2j+r_2-2)}\left((1-ac)s_{i_2} - b\right)}{b + (ac)^{(2j+r_2)}\left((1-ac)s_{i_2} - b\right)} \\ &= x_{6r_2+i_2-12} c^{3m+3} \frac{b + (ac)^{(r_2-2)}\left((1-ac)s_{i_2} - b\right)}{b + (ac)^{(2m+r_2)}\left((1-ac)s_{i_2} - b\right)}, \end{aligned} \tag{2.32}$$

$$y_{12m+6r_1+i_1} = y_{6r_1+i_1-12} \frac{1}{c^{3m+3}} \tag{2.33}$$

$$\times\prod_{j=0}^{m}\frac{b+(ac)^{\left(2j+r_1+1+\lfloor\frac{i_1-4}{6}\rfloor\right)}\left((1-ac)s_{i_1-7-6\lfloor\frac{i_1-4}{6}\rfloor}-b\right)}{b+(ac)^{\left(2j+r_1+1+\lfloor\frac{i_1-6}{6}\rfloor\right)}\left((1-ac)s_{i_1-9-6\lfloor\frac{i_1-6}{6}\rfloor}-b\right)}$$
$$\times\frac{b+(ac)^{\left(2j+r_1+\lfloor\frac{i_1-2}{6}\rfloor\right)}\left((1-ac)s_{i_1-5-6\lfloor\frac{i_1-2}{6}\rfloor}-b\right)}{b+(ac)^{\left(2j+r_1+\lfloor\frac{i_1-4}{6}\rfloor\right)}\left((1-ac)s_{i_1-7-6\lfloor\frac{i_1-4}{6}\rfloor}-b\right)}$$
$$\times\frac{b+(ac)^{\left(2j+r_1+\lfloor\frac{i_1-6}{6}\rfloor\right)}\left((1-ac)s_{i_1-9-6\lfloor\frac{i_1-6}{6}\rfloor}-b\right)}{b+(ac)^{\left(2j+r_1-1+\lfloor\frac{i_1-2}{6}\rfloor\right)}\left((1-ac)s_{i_1-5-6\lfloor\frac{i_1-2}{6}\rfloor}-b\right)},$$

if $ac\neq 1$, and

$$x_{12m+6r_2+i_2}=x_{6r_2+i_2-12}c^{3m+3}\frac{s_{i_2}+b(r_2-2)}{s_{i_2}+b(2m+r_2)}, \tag{2.34}$$

$$\begin{aligned}y_{12m+6r_1+i_1}=&y_{6r_1+i_1-12}\frac{1}{c^{3m+3}}\prod_{j=0}^{m}\left(\frac{s_{i_1-7-6\lfloor\frac{i_1-4}{6}\rfloor}+b\left(2j+r_1+1+\lfloor\frac{i_1-4}{6}\rfloor\right)}{s_{i_1-9-6\lfloor\frac{i_1-6}{6}\rfloor}+b\left(2j+r_1+1+\lfloor\frac{i_1-6}{6}\rfloor\right)}\right)\\&\times\left(\frac{s_{i_1-5-6\lfloor\frac{i_1-2}{6}\rfloor}+b\left(2j+r_1+\lfloor\frac{i_1-2}{6}\rfloor\right)}{s_{i_1-7-6\lfloor\frac{i_1-4}{6}\rfloor}+b\left(2j+r_1+\lfloor\frac{i_1-4}{6}\rfloor\right)}\right)\\&\times\left(\frac{s_{i_1-9-6\lfloor\frac{i_1-6}{6}\rfloor}+b\left(2j+r_1+\lfloor\frac{i_1-6}{6}\rfloor\right)}{s_{i_1-5-6\lfloor\frac{i_1-2}{6}\rfloor}+b\left(2j+r_1-1+\lfloor\frac{i_1-2}{6}\rfloor\right)}\right),\end{aligned} \tag{2.35}$$

if $ac=1$, where $m\in\mathbb{N}_0$, $r_1\in\{1,2\}$, $r_2\in\{2,3\}$, $i_1\in\{2,3,4,5,6,7\}$ and $i_2\in\{-3,-2,-1,0,1,2\}$.

Case 9: Let $abcd\neq 0$. Employing the change of variables

$$u_n=\frac{1}{x_ny_{n-1}x_{n-2}y_{n-3}},\quad v_n=\frac{1}{y_nx_{n-1}y_{n-2}x_{n-3}},\quad n\geq -3, \tag{2.36}$$

system (1.6) is transformed into the following system of linear difference equations

$$u_n=av_{n-3}+b,\quad v_n=cu_{n-3}+d,\quad n\in\mathbb{N}_0, \tag{2.37}$$

from which it follows that

$$u_n=acu_{n-6}+b+ad, \tag{2.38}$$
$$v_n=acv_{n-6}+d+bc,\quad n\geq 3, \tag{2.39}$$

which are nonhomogeneous linear sixth-order difference equations with constant coefficients. If we apply the decomposition of indexes $n \to 6m+j$, for some $m \in \mathbb{N}_0$ and $j \in \{3,4,5,6,7,8\}$, to (2.38) and (2.39), then they become

$$u_{6m+j} = acu_{6(m-1)+j} + b + ad, \tag{2.40}$$

$$v_{6m+j} = acv_{6(m-1)+j} + d + bc, \tag{2.41}$$

which are first-order 6-equations. Let $u_{6m+j} = u_m^{(j)} = z_m$ and $v_{6m+j} = v_m^{(j)} = \widehat{z}_m$, for $m \geq -1$ and $j \in \{3,4,5,6,7,8\}$. Then Eqs. (2.40) and (2.41) can be written in the form

$$z_m = acz_{m-1} + b + ad, \tag{2.42}$$

$$\widehat{z}_m = ac\widehat{z}_{m-1} + d + bc, \quad m \in \mathbb{N}_0, \tag{2.43}$$

which are nonhomogeneous linear first-order difference equations with constant coefficients. Contrary to the usual, here, Eqs. (2.42) and (2.43) can be solved by using transformation $z_m = w_m + C$ and $\widehat{z}_m = \widehat{w}_m + \widehat{C}$, for chosen suitable values of C and $\widehat{C}$, to reduce them to homogeneous linear first-order difference equations with constant coefficients. That is, by employing these transformations to Eqs. (2.42) and (2.43), they become

$$w_m = acw_{m-1} + C(ac-1) + b + ad,$$

$$\widehat{w}_m = ac\widehat{w}_{m-1} + \widehat{C}(ac-1) + d + bc, \quad m \in \mathbb{N}_0,$$

which for $C = \frac{b+ad}{1-ac}$ and $\widehat{C} = \frac{d+bc}{1-ac}$, if $ac \neq 1$, reduce to the next equations

$$w_m = acw_{m-1},$$

$$\widehat{w}_m = ac\widehat{w}_{m-1}, \quad m \in \mathbb{N}_0,$$

whose general solutions are

$$w_m = w_{-1}(ac)^{m+1},$$

$$\widehat{w}_m = \widehat{w}_{-1}(ac)^{m+1}, \quad m \geq -1.$$

From this and by using the relation $z_m = w_m + \frac{b+ad}{1-ac}$ and $\widehat{z}_m = \widehat{w}_m + \frac{d+bc}{1-ac}$, we obtain

$$z_m = \left(z_{-1} + \frac{b+ad}{ac-1}\right)(ac)^{m+1} + \frac{b+ad}{1-ac}, \tag{2.44}$$

$$\widehat{z}_m = \left(\widehat{z}_{-1} + \frac{d+bc}{ac-1}\right)(ac)^{m+1} + \frac{d+bc}{1-ac}, \quad m \geq -1, \tag{2.45}$$

when $ac \neq 1$. If $ac = 1$, then Eqs. (2.42) and (2.43) becomes

$$z_m = z_{m-1} + b + ad,$$

$$\widehat{z}_m = \widehat{z}_{m-1} + d + bc, \quad m \geq -1,$$

from which they immediately follow that

$$z_m = z_{-1} + (m+1)(b+ad), \tag{2.46}$$

$$\widehat{z}_m = \widehat{z}_{-1} + (m+1)(d+bc), \quad m \geq -1. \tag{2.47}$$

From (2.44), (2.45), (2.46) and (2.47) and the substitutions $u_{6m+j} = u_m^{(j)} = z_m$ and $v_{6m+j} = v_m^{(j)} = \widehat{z}_m$, we can write the solutions of (2.40) and (2.41) as in the following

$$u_{6m+j} = \left(u_{j-6} + \frac{b+ad}{ac-1}\right)(ac)^{m+1} + \frac{b+ad}{1-ac},$$

$$v_{6m+j} = \left(v_{j-6} + \frac{d+bc}{ac-1}\right)(ac)^{m+1} + \frac{d+bc}{1-ac},$$

$m \geq -1$, $j \in \{3,4,5,6,7,8\}$, when $ac \neq 1$, or

$$u_{6m+j} = u_{j-6} + (m+1)(b+ad),$$

$$v_{6m+j} = v_{j-6} + (m+1)(d+bc).$$

$m \geq -1$, $j \in \{3,4,5,6,7,8\}$, when $ac = 1$. Now note that from (2.36) we have that

$$\begin{aligned} x_m &= \frac{1}{u_m y_{m-1} x_{m-2} y_{m-3}} = \frac{v_{m-1}}{u_m} x_{m-4} \\ &= \frac{v_{m-1} v_{m-5}}{u_m u_{m-4}} x_{m-8} = \frac{v_{m-1} v_{m-5} v_{m-9}}{u_m u_{m-4} u_{m-8}} x_{m-12}, \end{aligned}$$

$$\begin{aligned} y_m &= \frac{1}{v_m x_{m-1} y_{m-2} x_{m-3}} = \frac{u_{m-1}}{v_m} y_{m-4} \\ &= \frac{u_{m-1} u_{m-5}}{v_m v_{m-4}} y_{m-8} = \frac{u_{m-1} u_{m-5} u_{m-9}}{v_m v_{m-4} v_{m-8}} y_{m-12}, \end{aligned}$$

for $m \geq 6$, from which it follows that

$$\begin{aligned} x_{12m+6r+i} = x_{6r+i-12} \prod_{s=0}^{m} & \frac{v_{6\left(2s+r+\lfloor \frac{i-4}{6} \rfloor\right)+i-1-6\lfloor \frac{i-4}{6} \rfloor}}{u_{6\left(2s+r+\lfloor \frac{i-3}{6} \rfloor\right)+i-6\lfloor \frac{i-3}{6} \rfloor}} \frac{v_{6\left(2s+r+\lfloor \frac{i-8}{6} \rfloor\right)+i-5-6\lfloor \frac{i-8}{6} \rfloor}}{u_{6\left(2s+r+\lfloor \frac{i-7}{6} \rfloor\right)+i-4-6\lfloor \frac{i-7}{6} \rfloor}} \\ & \times \frac{v_{6\left(2s+r+\lfloor \frac{i-12}{6} \rfloor\right)+i-9-6\lfloor \frac{i-12}{6} \rfloor}}{u_{6\left(2s+r+\lfloor \frac{i-11}{6} \rfloor\right)+i-8-6\lfloor \frac{i-11}{6} \rfloor}} \end{aligned}$$

and

$$\begin{aligned} y_{12m+6r+i} = y_{6r+i-12} \prod_{s=0}^{m} & \frac{u_{6\left(2s+r+\lfloor \frac{i-4}{6} \rfloor\right)+i-1-6\lfloor \frac{i-4}{6} \rfloor}}{v_{6\left(2s+r+\lfloor \frac{i-3}{6} \rfloor\right)+i-6\lfloor \frac{i-3}{6} \rfloor}} \frac{u_{6\left(2s+r+\lfloor \frac{i-8}{6} \rfloor\right)+i-5-6\lfloor \frac{i-8}{6} \rfloor}}{v_{6\left(2s+r+\lfloor \frac{i-7}{6} \rfloor\right)+i-4-6\lfloor \frac{i-7}{6} \rfloor}} \\ & \times \frac{u_{6\left(2s+r+\lfloor \frac{i-12}{6} \rfloor\right)+i-9-6\lfloor \frac{i-12}{6} \rfloor}}{v_{6\left(2s+r+\lfloor \frac{i-11}{6} \rfloor\right)+i-8-6\lfloor \frac{i-11}{6} \rfloor}}, \end{aligned}$$

for every $m \in \mathbb{N}_0$, $r \in \{1,2\}$ and $i \in \{0,1,2,3,4,5\}$.

Case $ac \neq 1$**:** In this case, we obtain that

$$\begin{aligned} x_{12m+6r+i} &= x_{6r+i-12} \\ &\times \prod_{s=0}^{m} \frac{d+bc+\left((1-ac)\,v_{-7+i-6\lfloor\frac{i-4}{6}\rfloor}-(d+bc)\right)(ac)^{2s+r+1+\lfloor\frac{i-4}{6}\rfloor}}{b+ad+\left((1-ac)\,u_{-6+i-6\lfloor\frac{i-3}{6}\rfloor}-(b+ad)\right)(ac)^{2s+r+1+\lfloor\frac{i-3}{6}\rfloor}} \\ &\times \frac{d+bc+\left((1-ac)\,v_{-11+i-6\lfloor\frac{i-8}{6}\rfloor}-(d+bc)\right)(ac)^{2s+r+1+\lfloor\frac{i-8}{6}\rfloor}}{b+ad+\left((1-ac)\,u_{-10+i-6\lfloor\frac{i-7}{6}\rfloor}-(b+ad)\right)(ac)^{2s+r+1+\lfloor\frac{i-7}{6}\rfloor}} \\ &\times \frac{d+bc+\left((1-ac)\,v_{-15+i-6\lfloor\frac{i-12}{6}\rfloor}-(d+bc)\right)(ac)^{2s+r+1+\lfloor\frac{i-12}{6}\rfloor}}{b+ad+\left((1-ac)\,u_{-14+i-6\lfloor\frac{i-11}{6}\rfloor}-(b+ad)\right)(ac)^{2s+r+1+\lfloor\frac{i-11}{6}\rfloor}} \end{aligned} \tag{2.48}$$

and

$$\begin{aligned} y_{12m+6r+i} &= y_{6r+i-12} \\ &\times \prod_{s=0}^{m} \frac{b+ad+\left((1-ac)\,u_{-7+i-6\lfloor\frac{i-4}{6}\rfloor}-(b+ad)\right)(ac)^{2s+r+1+\lfloor\frac{i-4}{6}\rfloor}}{d+bc+\left((1-ac)\,v_{-6+i-6\lfloor\frac{i-3}{6}\rfloor}-(d+bc)\right)(ac)^{2s+r+1+\lfloor\frac{i-3}{6}\rfloor}} \\ &\times \frac{b+ad+\left((1-ac)\,u_{-11+i-6\lfloor\frac{i-8}{6}\rfloor}-(b+ad)\right)(ac)^{2s+r+1+\lfloor\frac{i-8}{6}\rfloor}}{d+bc+\left((1-ac)\,v_{-10+i-6\lfloor\frac{i-7}{6}\rfloor}-(d+bc)\right)(ac)^{2s+r+1+\lfloor\frac{i-7}{6}\rfloor}} \\ &\times \frac{b+ad+\left((1-ac)\,u_{-15+i-6\lfloor\frac{i-12}{6}\rfloor}-(b+ad)\right)(ac)^{2s+r+1+\lfloor\frac{i-12}{6}\rfloor}}{d+bc+\left((1-ac)\,v_{-14+i-6\lfloor\frac{i-11}{6}\rfloor}-(d+bc)\right)(ac)^{2s+r+1+\lfloor\frac{i-11}{6}\rfloor}}, \end{aligned} \tag{2.49}$$

for every $m \in \mathbb{N}_0$, $r \in \{1,2\}$ and $i \in \{0,1,2,3,4,5\}$.

Case $ac = 1$**:** In this case, we get that

$$\begin{aligned} x_{12m+6r+i} = x_{6r+i-12} &\prod_{s=0}^{m} \frac{v_{-7+i-6\lfloor\frac{i-4}{6}\rfloor}+\left(2s+r+1+\lfloor\frac{i-4}{6}\rfloor\right)(d+bc)}{u_{-6+i-6\lfloor\frac{i-3}{6}\rfloor}+\left(2s+r+1+\lfloor\frac{i-3}{6}\rfloor\right)(b+ad)} \\ &\times \frac{v_{-11+i-6\lfloor\frac{i-8}{6}\rfloor}+\left(2s+r+1+\lfloor\frac{i-8}{6}\rfloor\right)(d+bc)}{u_{-10+i-6\lfloor\frac{i-7}{6}\rfloor}+\left(2s+r+1+\lfloor\frac{i-7}{6}\rfloor\right)(b+ad)} \\ &\times \frac{v_{-15+i-6\lfloor\frac{i-12}{6}\rfloor}+\left(2s+r+1+\lfloor\frac{i-12}{6}\rfloor\right)(d+bc)}{u_{-14+i-6\lfloor\frac{i-11}{6}\rfloor}+\left(2s+r+1+\lfloor\frac{i-11}{6}\rfloor\right)(b+ad)} \end{aligned} \tag{2.50}$$

and

$$y_{12m+6r+i} = y_{6r+i-12} \prod_{s=0}^{m} \frac{u_{-7+i-6\lfloor \frac{i-4}{6} \rfloor} + \left(2s+r+1+\lfloor \frac{i-4}{6} \rfloor\right)(b+ad)}{v_{-6+i-6\lfloor \frac{i-3}{6} \rfloor} + \left(2s+r+1+\lfloor \frac{i-3}{6} \rfloor\right)(d+bc)}$$
$$\times \frac{u_{-11+i-6\lfloor \frac{i-8}{6} \rfloor} + \left(2s+r+1+\lfloor \frac{i-8}{6} \rfloor\right)(b+ad)}{v_{-10+i-6\lfloor \frac{i-7}{6} \rfloor} + \left(2s+r+1+\lfloor \frac{i-7}{6} \rfloor\right)(d+bc)} \tag{2.51}$$
$$\times \frac{u_{-15+i-6\lfloor \frac{i-12}{6} \rfloor} + \left(2s+r+1+\lfloor \frac{i-12}{6} \rfloor\right)(b+ad)}{v_{-14+i-6\lfloor \frac{i-11}{6} \rfloor} + \left(2s+r+1+\lfloor \frac{i-11}{6} \rfloor\right)(d+bc)},$$

for every $m \in \mathbb{N}_0$, $r \in \{1,2\}$ and $i \in \{0,1,2,3,4,5\}$.

From the above considerations, we see that the following result holds.

Theorem 1. *Assume that a,b,c,d and the initial values x_{-i}, y_{-i}, $i \in \{1,2,3,4,5,6\}$, are real numbers. Then, the next statements hold.*

a) *If $a = c = 0$ and $bd \neq 0$, then the general solutions of system (1.6) is given by formulas in (2.3).*
b) *If $b = d = 0$ and $ac \neq 0$, then the general solutions of system (1.6) is given by formulas in (2.8).*
c) *If $a = d = 0$ and $bc \neq 0$, then the general solutions of system (1.6) is given by formulas in (2.13).*
d) *If $b = c = 0$ and $ad \neq 0$, then the general solutions of system (1.6) is given by formulas in (2.15).*
e) *If $a = 0$ and $bcd \neq 0$, then the general solutions of system (1.6) is given by formulas in (2.18).*
f) *If $b = 0$, $acd \neq 0$ and $ac \neq 1$, then the general solutions of system (1.6) is given by formulas in (2.25)-(2.26).*
g) *If $b = 0$, $acd \neq 0$ and $ac = 1$, then the general solutions of system (1.6) is given by formulas in (2.27)-(2.28).*
h) *If $c = 0$ and $abd \neq 0$, then the general solutions of system (1.6) is given by formulas in (2.30).*
i) *If $d = 0$, $abc \neq 0$ and $ac \neq 1$, then the general solutions of system (1.6) is given by formulas in (2.32)-(2.33).*
j) *If $d = 0$, $abc \neq 0$ and $ac = 1$, then the general solutions of system (1.6) is given by formulas in (2.34)-(2.35).*
k) *If $abcd \neq 0$ and $ac \neq 1$, then the general solutions of system (1.6) is given by formulas in (2.48) and (2.49).*
l) *If $abcd \neq 0$ and $ac = 1$, then the general solutions of system (1.6) is given by formulas in (2.50) and (2.51).*

By the following theorem, we characterize the forbidden set of the initial values for system (1.6).

Theorem 2. *The forbidden set of the initial values for system* (1.6) *is the union of two sets*

$$\left\{\vec{\mathbb{S}} : x_{-i} = 0 \text{ or } y_{-i} = 0,\ i \in \{1,2,3,4,5,6\}\right\}$$

and

$$\bigcup_{m\in\mathbb{N}_0}\bigcup_{j=3}^{8}\left\{\vec{\mathbb{S}} : \frac{1}{x_{j-3}y_{j-4}x_{j-5}y_{j-6}} = (f\circ g)^{-m}\left(-\frac{d}{c}\right),\ \text{or}\right.$$
$$\left.\frac{1}{y_{j-3}x_{j-4}y_{j-5}x_{j-6}} = (g\circ f)^{-m}\left(-\frac{b}{a}\right)\right\},$$

where $\vec{\mathbb{S}} = (x_{-6},x_{-5},x_{-4},x_{-3},x_{-2},x_{-1},y_{-6},y_{-5},y_{-4},y_{-3},y_{-2},y_{-1})$.

Proof. We have already obtained that the set

$$\left\{\vec{\mathbb{S}} : x_{-i} = 0 \text{ or } y_{-i} = 0,\ i \in \{1,2,3,4,5,6\}\right\}$$

belongs to the forbidden set of the initial values for system (1.6). If $x_{-i} \neq 0 \neq y_{-i}$, $i \in \{1,2,3,4,5,6\}$, (i.e., $x_n y_n \neq 0,\ \ n \geq -6$), then such a solution $(x_n,y_n)_{n\geq -6}$ is not defined if and only if

$$a + by_{n-3}x_{n-4}y_{n-5}x_{n-6} = 0, \quad c + dx_{n-3}y_{n-4}x_{n-5}y_{n-6} = 0,$$

for some $n \in \mathbb{N}_0$, which correspond to the statements $y_{n-3}x_{n-4}y_{n-5}x_{n-6} = -\frac{a}{b}$ and $x_{n-3}y_{n-4}x_{n-5}y_{n-6} = -\frac{c}{d}$, for $n \in \mathbb{N}_0$, respectively. Hence, from (2.36), we get

$$u_{n-3} = -\frac{d}{c} \quad \text{and} \quad v_{n-3} = -\frac{b}{a}, \tag{2.52}$$

for $n \in \mathbb{N}_0$. Now, we consider system (2.37) and the functions

$$f(t) = at + b, \quad g(t) = ct + d$$

which correspond to the equations of (2.37). Thus, we can describe the solutions of (2.38) and (2.39) as follows:

$$u_{6(m-1)+j} = (f\circ g)^m (u_{j-6}),$$
$$v_{6(m-1)+j} = (g\circ f)^m (v_{j-6}),$$

for $m \in \mathbb{N}_0$ and $j \in \{3,4,5,6,7,8\}$. By using (2.52), we get

$$u_{j-6} = (f\circ g)^{-m}\left(-\frac{d}{c}\right), \tag{2.53}$$

$$v_{j-6} = (g\circ f)^{-m}\left(-\frac{b}{a}\right), \tag{2.54}$$

for $m \geq -1$ and $j \in \{3,4,5,6,7,8\}$, where $f^{-1}(t) = \frac{t-b}{a}$ and $g^{-1}(t) = \frac{t-d}{c}$. This means that if one of the conditions in (2.53) and (2.54) holds, then $6(m-1)$-th or

$6m$-th iteration in (1.6) can not be calculated. Consequently, desired result follows from (2.36).

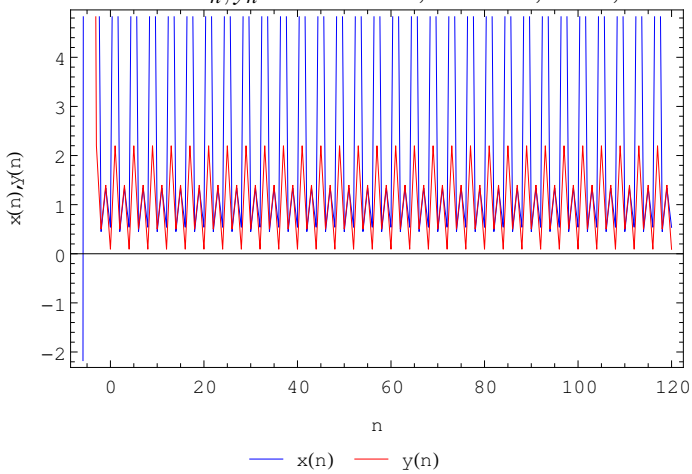
□

3. Numerical Examples

To support our theoretical results, we present numerical examples for the solutions of system (1.6) regard to the different values of a, b, c and d.

Example 1. Consider the system (1.6) with the initial values $x_{-6} = -15.71$, $x_{-5} = 87.76$, $x_{-4} = 48.97$, $x_{-3} = 12.23$, $x_{-2} = 0.45$, $x_{-1} = 1.34$, $y_{-6} = 5.2$, $y_{-5} = 8.6$, $y_{-4} = 22.7$, $y_{-3} = 2.2$, $y_{-2} = 0.5$, $y_{-1} = 1.4$ and the parameters, $a = 0$, $b = 1.34$, $c = 0$, $d = 1.34$ the solutions are represented as in the Figure 1.

FIGURE 1. Plots of x_n, y_n in case $a = 0$, $b = 1.34$, $c = 0$, $d = 1.34$



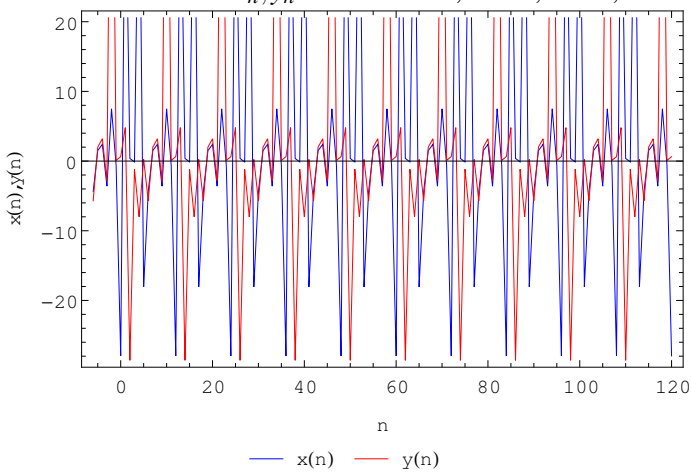
In this case, equations in (2.3) are satisfied. Hence, the solutions of system (1.6) have periodic solutions with period four.

Example 2. Consider the system (1.6) with the initial values $x_{-6} = -4.37$, $x_{-5} = 1.5$, $x_{-4} = 2.4$, $x_{-3} = -3.6$, $x_{-2} = 7.5$, $x_{-1} = -0.6$, $y_{-6} = -5.7$, $y_{-5} = 2$, $y_{-4} = 3.2$, $y_{-3} = -2.8$, $y_{-2} = 71.5$, $y_{-1} = 0.1$ and the parameters, $a = 1$, $b = 0$, $c = 1$, $d = 0$ the solutions are represented as in the Figure 2.

In this case, equations in (2.8) are satisfied. Hence, the solutions of system (1.6) have periodic solutions with period twelve.

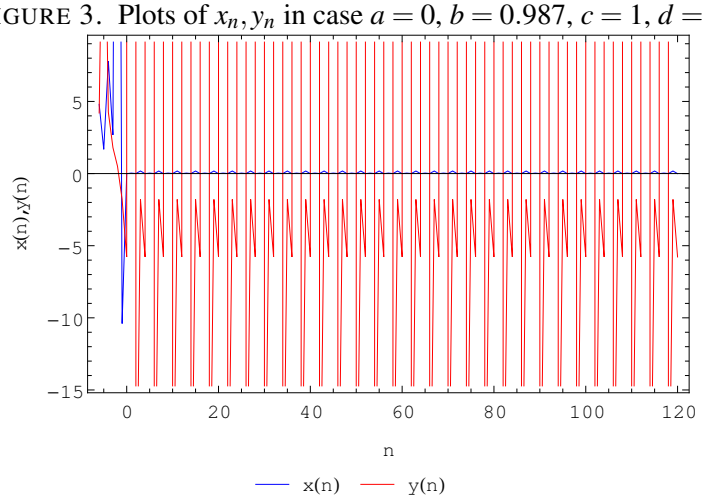
Example 3. Consider the system (1.6) with the initial values $x_{-6} = 4.8$, $x_{-5} = 1.68$, $x_{-4} = 7.8$, $x_{-3} = 2.68$, $x_{-2} = 70.8$, $x_{-1} = -10.4$, $y_{-6} = 4.2$, $y_{-5} = 25.6$, $y_{-4} = 4.26$,

FIGURE 2. Plots of x_n, y_n in case $a = 1$, $b = 0$, $c = 1$, $d = 0$



$y_{-3} = 1.8$, $y_{-2} = 0.5$, $y_{-1} = -1.79$ and the parameters, $a = 0$, $b = 0.987$, $c = 1$, $d = 0$ the solutions are represented as in the Figure 3.

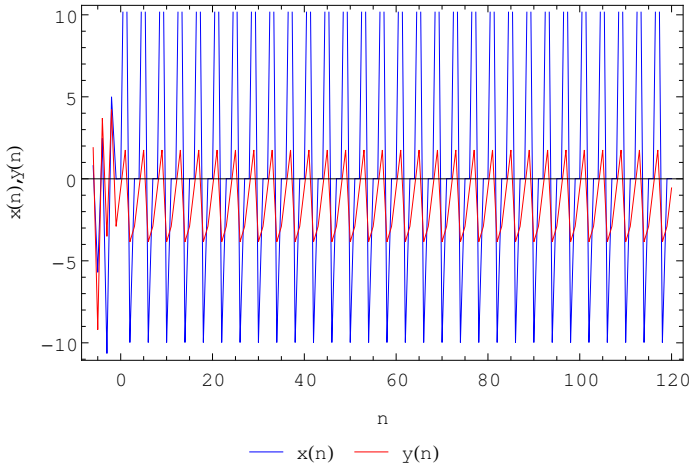
FIGURE 3. Plots of x_n, y_n in case $a = 0$, $b = 0.987$, $c = 1$, $d = 0$



In this case, equations in (2.13) are satisfied. Hence, the solutions of system (1.6) have periodic solutions with period four.

Example 4. Consider the system (1.6) with the initial values $x_{-6} = 0.8$, $x_{-5} = -5.7$, $x_{-4} = 2.47$, $x_{-3} = -10.65$, $x_{-2} = 5$, $x_{-1} = -0.01$, $y_{-6} = 1.9$, $y_{-5} = -9.2$, $y_{-4} = 3.7$, $y_{-3} = -3.51$, $y_{-2} = 4.22$, $y_{-1} = -2.9$ and the parameters, $a = 0$, $b = 2$, $c = 3$, $d = -4$ the solutions are represented as in the Figure 4.

FIGURE 4. Plots of x_n, y_n in case $a = 0$, $b = 2$, $c = 3$, $d = -4$



In this case, equations in (2.18) are satisfied. Hence, the solutions of system (1.6) have periodic solutions with period four.

4. CONCLUSION

In this study, we obtain solutions of the following system of difference equations

$$x_n = \frac{x_{n-4}y_{n-5}x_{n-6}}{y_{n-1}x_{n-2}\left(a + by_{n-3}x_{n-4}y_{n-5}x_{n-6}\right)}, \quad y_n = \frac{y_{n-4}x_{n-5}y_{n-6}}{x_{n-1}y_{n-2}\left(c + dx_{n-3}y_{n-4}x_{n-5}y_{n-6}\right)},$$

for $n \in \mathbb{N}_0$, where the parameters a, b and the initial values x_{-i}, $i \in \{1,2,3,4,5,6\}$, are real numbers. In addition, we show that some solvable subclasses of the class of nonlinear two-dimensional system of difference equations are solvable in closed form. We also describe the forbidden set of solutions of the system of difference equations. Some numerical examples are given to demonstrate the theoretical results.

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PSEUDO CORE INVERTIBILITY AND DMP INVERTIBILITY IN TWO SEMIGROUPS OF A RING WITH INVOLUTION

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Abstract. In 2004, Patrício and Puystjens characterized the relation between Drazin invertible elements (resp., Moore-Penrose invertible elements) of two semigroups pRp and $pRp+1-p$ of a ring R for some idempotent (resp., projection) $p \in R$. In this paper, we consider the relevant result for pseudo core invertible elements of such two semigroups of a ring for some projection, which is then applied to characterize the relation between pseudo core invertible elements of the matrix semigroup $AA^{\dagger}R^{m\times m}AA^{\dagger}+I_m-AA^{\dagger}$ and the matrix semigroup $A^{\dagger}AR^{n\times n}A^{\dagger}A+I_n-A^{\dagger}A$, where $A \in R^{m\times n}$ with $A^{\dagger}$ existing. Also, similar equivalence involving DMP invertible elements is investigated.

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Keywords: pseudo core invertibility, DMP invertibility, semigroups, matrices over rings

1. Introduction

Moore-Penrose inverses [22] and Drazin inverses [5] are well-known classical generalized inverses, which play important roles in many fields. Some characterizations of Moore-Penrose invertibility and Drazin invertibility are given, such as [13, 19–21, 23]. Let R be a ring with identity 1 and $R^{m\times n}$ the set of $m\times n$ matrices over R. In [21], Patrício and Puystjens first characterized the relation between Drazin invertible elements of two semigroups pRp and $pRp+1-p$ of R when p is an idempotent. Specifically, for any $x \in R$, they proved that $pxp+1-p$ is Drazin invertible in R if and only if pxp is Drazin invertible in pRp, in which case, $(pxp)^d = p(pxp+1-p)^d p \in pRp$ and $(pxp+1-p)^d = (pxp)^d+1-p \in pRp+1-p$. Similar equivalence appeared in the characterization of Moore-Penrose invertibility when R is a ring with an involution and p is a projection. Using the previous result, they related the Drazin invertible elements between the semigroup $AA^{-}R^{m\times m}AA^{-}+I_m-AA^{-}$ and the semigroup $A^{=}AR^{n\times n}A^{=}A+I_n-A^{=}A$, where A^{-} and $A^{=}$ are inner inverses of A, and also related the Moore-Penrose invertible elements between the semigroup $AA^{\dagger}R^{m\times m}AA^{\dagger}+I_m-AA^{\dagger}$ and the semigroup $A^{\dagger}AR^{n\times n}A^{\dagger}A+I_n-A^{\dagger}A$ when $A^{\dagger}$ exists.

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In addition to Drazin inverses and Moore-Penrose inverses, some new generalized inverses, such as core inverses, pseudo core inverses and DMP inverses, have also been introduced and deeply studied in recent years. The concept of the core inverse of a complex matrix was defined by Baksalary and Trenkler [1] in 2010, which was generalized to a ring with involution by Rakić et al. [24] later. As two extensions of the core inverse, in 2014, Manjunatha Prasad and Mohana [17], Malik and Thome [16] introduced, respectively, namely the core-EP inverse and DMP inverse which both exist for arbitrary square complex matrices. Later, Gao and Chen [7] defined the pseudo core inverse by three equations in a ring with involution, extending the core-EP inverse of a complex matrix. Additionally, Gao et al. [8] discussed existence criteria and formulae of the pseudo core inverse of a companion matrix over a ring with involution. Zhou and Chen [27] showed some characterizations of pseudo core inverses and the relation with other generalized inverses. For more details of pseudo core inverses and DMP inverses, for example, see [18, 28]. Noted that some generalized inverse matrices can provide a method to define pre-orders or partial orders and to analyze binary relations, for example, see [3,9,11,12,15]. Except that, many scholars have studied projections related to generalized inverses and their generalizations, such as [4, 14, 25].

Motivated by the above discussion, it is natural to ask whether similar equivalences hold for pseudo core invertibility and DMP invertibility. The article is organized as follows. In Section 2, we present some necessary definitions mentioned above and lemmas. In Section 3, we characterize the relation between pseudo core invertible elements (resp., DMP invertible elements) of two semigroups pRp and $pRp+1-p$ of R, where R is equipped with an involution and p is a projection. As an application, in Section 4, the relation between pseudo core invertible elements (resp., DMP invertible elements) of the semigroup $AA^{\dagger}R^{m\times m}AA^{\dagger}+I_m-AA^{\dagger}$ and the semigroup $A^{\dagger}AR^{n\times n}A^{\dagger}A+I_n-A^{\dagger}A$ is characterized, assuming that $A\in R^{m\times n}$ with $A^{\dagger}$ existing.

2. Preliminaries

Let R be a ring with involution $*$ and have identity 1, where an involution $*: a\mapsto a^*$ is an anti-isomorphism satisfying $(a^*)^*=a$, $(a+b)^*=a^*+b^*$ and $(ab)^*=b^*a^*$ for all $a,b\in R$.

Definition 1 ([10, 22])**.** An element $a\in R$ is said to be Moore-Penrose invertible if there exists $x\in R$ such that

$$(1)\ axa=a,\quad (2)\ xax=x,\quad (3)\ (ax)^*=ax,\quad (4)\ (xa)^*=xa.$$

Such x is called the Moore-Penrose inverse of a and is unique if it exists, denoted by $a^{\dagger}$. If $axa=a$, then a is called regular and x is called an inner inverse of a. We use a^{-} to denote an inner inverse of a. If the equations (1) and (3) hold, then x is called a $\{1,3\}$-inverse of a. We use $a^{(1,3)}$ to denote a $\{1,3\}$-inverse of a. The set of all $\{1,3\}$-invertible elements in R will be denoted by $R^{\{1,3\}}$.

Definition 2 ([5])**.** Let $a \in R$. Then a is said to be Drazin invertible if there exist $x \in R$ and a positive integer k such that

$$xa^{k+1} = a^k,\ ax^2 = x,\ ax = xa.$$

Such x is unique if it exists and called the Drazin inverse of a, denoted by a^d. If k is the smallest positive integer such that the above equations hold, then k is called the Drazin index of a and denoted by $\mathrm{ind}(a)$. When $\mathrm{ind}(a) = 1$, the Drazin inverse of a is called the group inverse of a, denoted by $a^{\#}$.

Definition 3 ([7])**.** Let $a \in R$. If there exist $x \in R$ and a positive integer k such that

$$xa^{k+1} = a^k,\ ax^2 = x,\ (ax)^* = ax,$$

then x is called the pseudo core inverse of a. It is unique and denoted by $a^{\textcircled{D}}$ when the pseudo core inverse of a exists. If a is pseudo core invertible, then it must be Drazin invertible. The smallest positive integer k satisfying the above equations is called the pseudo core index of a, which coincides with its Drazin index, and still denoted by $\mathrm{ind}(a)$. When $\mathrm{ind}(a) = 1$, the pseudo core inverse of a reduces to its core inverse $a^{\textcircled{\#}}$.

Lemma 1 ([26])**.** *Let $a \in R$. Then $x = a^{\textcircled{\#}}$ is the core inverse of a if and only if there exists $x \in R$ such that*

$$xa^2 = a,\ ax^2 = x,\ (ax)^* = ax.$$

Let $a \in R$. Gao and Chen [7] proved that a is pseudo core invertible if and only if a^n is core invertible for some positive integer n. Here we give the further result.

Lemma 2. *Let $a \in R$. Then a is pseudo core invertible with $\mathrm{ind}(a) = n$ if and only if n is the smallest positive integer such that a^n is core invertible. In this case, $a^{\textcircled{D}} = a^{n-1}(a^n)^{\textcircled{\#}}$ and $(a^n)^{\textcircled{\#}} = (a^{\textcircled{D}})^n$.*

Proof. Suppose that n is the smallest positive integer such that a^n is core invertible. Then we obtain

$$(a^n)^{\textcircled{\#}}(a^n)^2 = a^n,\ \ a^n\left((a^n)^{\textcircled{\#}}\right)^2 = (a^n)^{\textcircled{\#}}, \left(a^n(a^n)^{\textcircled{\#}}\right)^* = a^n(a^n)^{\textcircled{\#}}.$$

According to the proof of [7, Theorem 2.5], we get that $a^{\textcircled{D}} = a^{n-1}(a^n)^{\textcircled{\#}}$ with $\mathrm{ind}(a) \le n$. Assume $\mathrm{ind}(a) = k < n$, then it is easy to obtain that a^k is core invertible, a contradiction. Hence, $\mathrm{ind}(a) = n$.

Conversely, following the proof of [7, Theorem 2.5], we know that a^n is core invertible with $(a^n)^{\textcircled{\#}} = (a^{\textcircled{D}})^n$. We suppose that there exists a positive integer $k < n$ such that a^k is core invertible. Then according to the sufficiency of the proof, it follows that $k \ge \mathrm{ind}(a) = n$, a contradiction. Therefore, n is the smallest positive integer such that a^n is core invertible. □

Let $R^\dagger$ and R^d denote the sets of all Moore-Penrose invertible elements and Drazin invertible elements in R, respectively.

Definition 4 ([16])**.** Let $a \in R^d \cap R^\dagger$. The DMP inverse of a, denoted by $a^{d,\dagger}$, is the unique solution $x \in R$ of the system

$$xax = x,\ xa = a^d a,\ a^k x = a^k a^\dagger,$$

where $k = \mathrm{ind}(a)$. In fact, it can be proved that $a^{d,\dagger} = a^d a a^\dagger$.

3. Pseudo core invertibility and DMP invertibility in a corner ring

The following lemmas will be useful in the sequel.

Lemma 3 ([2]). *Let $a, b \in R$ with $ab = ba$ and $a^*b = ba^*$. If $a \in R^{\textcircled{\#}}$, then $a^{\textcircled{\#}}b = ba^{\textcircled{\#}}$.*

Lemma 4 ([26]). *Let $a, b \in R$ be core invertible. If $ab = ba = 0$ and $a^*b = 0$, then $a+b$ is core invertible, and $(a+b)^{\textcircled{\#}} = a^{\textcircled{\#}} + b^{\textcircled{\#}}$.*

Lemma 5 ([7]). *Let $a, b \in R$ be pseudo core invertible. If $ab = ba = 0$ and $a^*b = 0$, then $a+b$ is pseudo core invertible, and $(a+b)^{\textcircled{D}} = a^{\textcircled{D}} + b^{\textcircled{D}}$.*

Recall that an element $p \in R$ is a projection if $p^2 = p = p^*$. Let $p \in R$ be a projection. Then $pRp + 1 - p = \{pxp + 1 - p : x \in R\}$ is a (multiplicative) semigroup. The subrings of the form pRp are called corner rings. It should be remarked that for pseudo core inverses (resp., core inverses) and DMP inverses we still keep the usual notation as $(pxp)^{\textcircled{D}}$ (resp., $(pxp)^{\textcircled{\#}}$) and $(pxp)^{d,\dagger}$ both belong to pRp if they exist in R. Next, we illustrate the relation between pseudo core invertibility (resp., core invertibility) of the corresponding elements in two semigroups pRp and $pRp + 1 - p$ of R. The next two theorems will play an important role in the forthcoming section.

Theorem 1. *Let $p \in R$ be a projection and $x \in R$. Then the following statements hold.*

(1) *$pxp + 1 - p$ is core invertible in R if and only if pxp is core invertible in pRp. In this case,*

$$(pxp)^{\textcircled{\#}} = p(pxp + 1 - p)^{\textcircled{\#}} p \in pRp,$$

and

$$(pxp + 1 - p)^{\textcircled{\#}} = (pxp)^{\textcircled{\#}} + 1 - p \in pRp + 1 - p.$$

(2) *$pxp + 1 - p$ is pseudo core invertible with $\mathrm{ind}(pxp + 1 - p) = k$ in R if and only if pxp is pseudo core invertible with $\mathrm{ind}(pxp) = k$ in pRp. In this case,*

$$(pxp)^{\textcircled{D}} = p(pxp + 1 - p)^{\textcircled{D}} p \in pRp,$$

and

$$(pxp + 1 - p)^{\textcircled{D}} = (pxp)^{\textcircled{D}} + 1 - p \in pRp + 1 - p.$$

Proof. (1) Suppose that $pxp + 1 - p$ is core invertible. Then

$$\big((pxp + 1 - p)(pxp + 1 - p)^{\textcircled{\#}}\big)^* = (pxp + 1 - p)(pxp + 1 - p)^{\textcircled{\#}}.$$

Multiplying on both sides by p, we have

$$\big((pxp)p(pxp + 1 - p)^{\textcircled{\#}} p\big)^* = (pxp)p(pxp + 1 - p)^{\textcircled{\#}} p.$$

Moreover,

$$(pxp+1-p)^{\textcircled{\#}}(pxp+1-p)^2 = pxp+1-p.$$

Multiplying the both sides by p, it follows that $p(pxp+1-p)^{\textcircled{\#}}p(pxp)^2 = pxp$. Also,

$$(pxp+1-p)\left((pxp+1-p)^{\textcircled{\#}}\right)^2 = (pxp+1-p)^{\textcircled{\#}}.$$

Multiplying on both sides by p, we get

$$(pxp)p\left((pxp+1-p)^{\textcircled{\#}}\right)^2 p = p(pxp+1-p)^{\textcircled{\#}}p.$$

Since

$$(pxp+1-p)(1-p) = 1-p = (1-p)(pxp+1-p),$$
$$(pxp+1-p)^*(1-p) = 1-p = (1-p)(pxp+1-p)^*,$$

by Lemma 3, we have

$$(pxp+1-p)^{\textcircled{\#}}(1-p) = (1-p)(pxp+1-p)^{\textcircled{\#}},$$

that is

$$(pxp+1-p)^{\textcircled{\#}}p = p(pxp+1-p)^{\textcircled{\#}}.$$

Hence,

$$(pxp)\left(p(pxp+1-p)^{\textcircled{\#}}p\right)^2 = p(pxp+1-p)^{\textcircled{\#}}p.$$

According to Lemma 1, $(pxp)^{\textcircled{\#}} = p(pxp+1-p)^{\textcircled{\#}}p \in pRp$.

Conversely, if $(pxp)^{\textcircled{\#}}$ is the core inverse of pxp in $pRp \subseteq R$, then by Lemma 4,

$$\begin{aligned}(pxp+1-p)^{\textcircled{\#}} &= (pxp)^{\textcircled{\#}} + (1-p)^{\textcircled{\#}}\\ &= (pxp)^{\textcircled{\#}} + 1 - p\end{aligned}$$

since $(pxp)(1-p) = 0 = (1-p)(pxp)$ and $(pxp)^*(1-p) = 0$.

(2) According to Lemma 2, if $pxp+1-p$ is pseudo core invertible with $\mathrm{ind}(pxp+1-p) = k$, then k is the smallest positive integer such that $(pxp+1-p)^k = (pxp)^k + 1 - p = p\left(x(px)^{k-1}\right)p + 1 - p$ is core invertible, and therefore $p\left(x(px)^{k-1}\right)p = (pxp)^k$ is core invertible. We remark that k is also the smallest positive integer such that $(pxp)^k$ is core invertible according to (1). In fact, we suppose that there exists a positive integer $m < k$ such that $(pxp)^m$ is core invertible. It follows that $p\left(x(px)^{m-1}\right)p = (pxp)^m$ is core invertible, then $(pxp+1-p)^m = (pxp)^m + 1 - p = p\left(x(px)^{m-1}\right)p + 1 - p$ is core invertible according to (1), and therefore $m \geq \mathrm{ind}(pxp+1-p) = k$, a contradiction. Hence, pxp is pseudo core invertible with $\mathrm{ind}(pxp) = k$. For the expression of $(pxp)^{\textcircled{D}}$, by Lemma 2 we can obtain that

$$(pxp)^{\textcircled{D}} = (pxp)^{k-1}\left((pxp)^k\right)^{\textcircled{\#}} = (pxp)^{k-1}p\left((pxp)^k\right)^{\textcircled{\#}}$$

$$
\begin{aligned}
&= \left((pxp)^{k-1}+1-p\right)p\left((pxp)^{k}+1-p\right)^{\textcircled{\#}}p \\
&= p(pxp+1-p)^{k-1}\left((pxp+1-p)^{k}\right)^{\textcircled{\#}}p \\
&= p(pxp+1-p)^{\textcircled{D}}p.
\end{aligned}
$$

Conversely, if $(pxp)^{\textcircled{D}}$ is the pseudo core inverse of pxp in $pRp \subseteq R$ and $\mathrm{ind}(pxp) = k$, then by Lemma 5 it follows that $pxp+1-p$ is pseudo core invertible since $(pxp)(1-p) = 0 = (1-p)(pxp)$ and $(pxp)^*(1-p) = 0$, and

$$
\begin{aligned}
(pxp+1-p)^{\textcircled{D}} &= (pxp)^{\textcircled{D}} + (1-p)^{\textcircled{D}} \\
&= (pxp)^{\textcircled{D}} + 1 - p.
\end{aligned}
$$

Moreover, it can be derived from the necessity of the proof that $\mathrm{ind}(pxp+1-p) = k$. □

Similar equivalence involving DMP invertibility is given in the following result.

Theorem 2. *Let $p \in R$ be a projection and $x \in R$. Then $pxp+1-p$ is DMP invertible in R if and only if pxp is DMP invertible in pRp. In this case,*

$$(pxp)^{d,\dagger} = p(pxp+1-p)^{d,\dagger}p \in pRp,$$

and

$$(pxp+1-p)^{d,\dagger} = (pxp)^{d,\dagger} + 1 - p \in pRp + 1 - p.$$

Proof. According to [21, Theorem 1] and Definition 4, it is easy to obtain the sufficiency and necessity of the theorem. For the expressions of $(pxp)^{d,\dagger}$ and $(pxp+1-p)^{d,\dagger}$, since $(pxp+1-p)^d \in pRp+1-p$, $(pxp)^d \in pRp$ and $(pxp)^\dagger \in pRp$, it follows that

$$
\begin{aligned}
(pxp)^{d,\dagger} &= (pxp)^d(pxp)(pxp)^\dagger \\
&= p(pxp+1-p)^d p(pxp)p(pxp+1-p)^\dagger p \\
&= p(pxp+1-p)^d(pxp+1-p)(pxp+1-p)^\dagger p \\
&\quad - p(pxp+1-p)^d(1-p)(pxp+1-p)^\dagger p \\
&= p(pxp+1-p)^{d,\dagger}p,
\end{aligned}
$$

and

$$
\begin{aligned}
(pxp+1-p)^{d,\dagger} &= (pxp+1-p)^d(pxp+1-p)(pxp+1-p)^\dagger \\
&= ((pxp)^d+1-p)(pxp+1-p)((pxp)^\dagger+1-p) \\
&= (pxp)^d(pxp)(pxp)^\dagger + 1 - p \\
&= (pxp)^{d,\dagger} + 1 - p.
\end{aligned}
$$

□

4. Pseudo core invertibility and DMP invertibility in two matrix semigroups

Let $R^{m\times n}$ denote the set of $m\times n$ matrices over R. And $A^*\in R^{n\times m}$ is defined as (a^*_{ji}) for $A=(a_{ij})\in R^{m\times n}$. Let $P\in R^{m\times m}$ be a projection. In Section 3, we relate pseudo core invertible (resp., DMP invertible) elements between the semigroup $PR^{m\times m}P+I_m-P$ and the corner ring $PR^{m\times m}P$. If $A\in R^{m\times n}$ is Moore-Penrose invertible, then $AA^\dagger$ and $A^\dagger A$ are two projections. As an application, in this section, we will relate pseudo core invertible elements between the semigroup $AA^\dagger R^{m\times m}AA^\dagger+I_m-AA^\dagger$ and the semigroup $A^\dagger AR^{n\times n}A^\dagger A+I_n-A^\dagger A$ using Theorem 1. Also, the DMP invertibility case is investigated.

Let R be a ring with involution ι and S a ring with involution τ. Recall from [21] that $\varphi: R\to S$ is a (ι,τ)-invariant homomorphism if φ is a ring homomorphism and $\varphi(x^\iota)=(\varphi(x))^\tau$ for all $x\in R$. If ι and τ coincide, then it is written as ι-invariant for short. Let $A\in R^{m\times n}$ such that $A^\dagger$ exists. Suppose that $\phi_A: AA^\dagger R^{m\times m}AA^\dagger\to A^\dagger AR^{n\times n}A^\dagger A$ is defined by $\phi_A(AA^\dagger XAA^\dagger)=A^\dagger XA$. Then they called that A is $*$-invariant if ϕ_A is $*$-invariant. Furthermore, they showed that ϕ_A is $*$-invariant if and only if $A^\dagger YA=A^*Y(A^\dagger)^*$ for all $Y\in R^{m\times m}$.

Let $A\in R^{m\times n}$ such that $A^\dagger$ exists and $B\in R^{m\times m}$. Denote $\Gamma=AA^\dagger BAA^\dagger+I_m-AA^\dagger$ and $\Omega=A^\dagger BA+I_n-A^\dagger A$. In [21], Patrício and Puystjens proved that Γ is Moore-Penrose invertible if and only if Ω is Moore-Penrose invertible, under the hypothesis that A is $*$-invariant. Furthermore, they gave examples to explain that the $*$-invariance of A is sufficient but not necessary for this equivalence. Analogously, we will give equivalences for pseudo core inverses and DMP inverses providing that A is $*$-invariant, respectively. Moreover, the examples in [21] can also show that $*$-invariance of A is sufficient but not necessary for the pseudo core inverse case. For completeness of the conclusion, we employ examples again as follows.

Let $a\in R$. We first recall that a is $\{1,3\}$-invertible if and only if $a\in Ra^*a$ [10]. Also, an element a is core invertible if and only if $a\in R^\#\cap R^{\{1,3\}}$ [26].

Let $R=\mathbb{C}^{2\times 2}$ be the set of 2×2 matrices over the complex field $\mathbb{C}$.

Example 1. Take $A=\begin{bmatrix}1&0\\1&1\end{bmatrix}$, $B=\begin{bmatrix}1&0\\i&0\end{bmatrix}\in\mathbb{C}^{2\times 2}$ and transposition as the involution. Then $A^\dagger=\begin{bmatrix}1&0\\-1&1\end{bmatrix}$. By calculation,

$$\Gamma=\begin{bmatrix}1&0\\i&0\end{bmatrix},\ \Omega=\begin{bmatrix}1&0\\-1+i&0\end{bmatrix},$$

and we get that Γ and Ω are two idempotents, and hence are group invertible. In addition, it is easy to check that $\Omega\in R^{\{1,3\}}$ however $\Gamma\notin R^{\{1,3\}}$. Therefore, Ω is core invertible, but Γ is not core invertible.

The example above is to say that the equivalence that Γ is core invertible if and only if Ω is core invertible does not hold. Therefore, we consider to give a sufficient

condition that A is $*$-invariant for this equivalence concerning core invertibility analogously. Furthermore, the example as follow is to show that the $*$-invariance of A is not necessary for the equivalence that Γ is pseudo core invertible if and only if Ω is pseudo core invertible.

Example 2. Consider $A = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \in \mathbb{C}^{2\times 2}$ and let $*$ be the involution defined as the transposed conjugate. Then $A^\dagger = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Clearly Γ and Ω are pseudo core invertible since every complex matrix has a pseudo core inverse. We next prove that ϕ_A is not $*$-invariant. In fact, if we take $X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathbb{C}^{2\times 2}$, then $A^\dagger XA = \begin{bmatrix} 0 & \frac{2}{3} \\ 0 & 0 \end{bmatrix}$, $AXA^\dagger = \begin{bmatrix} 0 & \frac{3}{2} \\ 0 & 0 \end{bmatrix}$. Hence, $A^\dagger XA = A^* X (A^\dagger)^*$ does not hold.

Theorem 3. *Let $A \in R^{m\times n}$ be Moore-Penrose invertible and $B \in R^{m\times m}$. Consider the following conditions:*

(1) $\Gamma = AA^\dagger BAA^\dagger + I_m - AA^\dagger$ *is pseudo core invertible with* $\operatorname{ind}(\Gamma) = k$ *(core invertible if $k = 1$).*

(2) $\Omega = A^\dagger BA + I_n - A^\dagger A$ *is pseudo core invertible with* $\operatorname{ind}(\Omega) = k$ *(core invertible if $k = 1$).*

If A is $*$-invariant then (1)$\Leftrightarrow$(2), in which case

$$\Gamma^{\textcircled{D}} = A\Omega^{\textcircled{D}} A^\dagger + I_m - AA^\dagger$$

and

$$\Omega^{\textcircled{D}} = A^\dagger \Gamma^{\textcircled{D}} A + I_n - A^\dagger A.$$

Proof. Let us first consider the case $k = 1$, i.e., the core invertibility case.

Suppose that Γ is core invertible. Then by Theorem 1, we have that $AA^\dagger BAA^\dagger$ has a core inverse $\Gamma_0^{\textcircled{\#}}$ in $AA^\dagger R^{m\times m} AA^\dagger$. As $\Gamma_0^{\textcircled{\#}}(AA^\dagger BAA^\dagger)^2 = AA^\dagger BAA^\dagger$ then we have

$$A^\dagger \Gamma_0^{\textcircled{\#}} A (A^\dagger BA)^2 = A^\dagger BA.$$

Moreover, $AA^\dagger BAA^\dagger (\Gamma_0^{\textcircled{\#}})^2 = \Gamma_0^{\textcircled{\#}}$. Since

$$AA^\dagger (AA^\dagger BAA^\dagger) = (AA^\dagger BAA^\dagger) AA^\dagger,$$
$$AA^\dagger (AA^\dagger BAA^\dagger)^* = (AA^\dagger BAA^\dagger)^* AA^\dagger,$$

it follows that $AA^\dagger \Gamma_0^{\textcircled{\#}} = \Gamma_0^{\textcircled{\#}} AA^\dagger$ by Lemma 3, and therefore

$$A^\dagger BA (A^\dagger \Gamma_0^{\textcircled{\#}} A)^2 = A^\dagger \Gamma_0^{\textcircled{\#}} A.$$

Since ϕ_A is $*$-invariant, for all $Y \in R^{m\times m}$, we have

$$A^\dagger YA = A^* Y (A^\dagger)^*.$$

Also, $(AA^\dagger BAA^\dagger \Gamma_0^{\textcircled{\#}})^* = AA^\dagger BAA^\dagger \Gamma_0^{\textcircled{\#}}$. Multiplying on left side by A^*, it follows that

$$\begin{aligned}(AA^\dagger BAA^\dagger \Gamma_0^{\textcircled{\#}} A)^* &= A^* AA^\dagger BAA^\dagger \Gamma_0^{\textcircled{\#}} \\ &= A^* B(A^\dagger)^* A^* \Gamma_0^{\textcircled{\#}} \\ &= A^\dagger BAA^* \Gamma_0^{\textcircled{\#}},\end{aligned}$$

then multiplying on right side by $(A^\dagger)^*$, we get

$$\begin{aligned}(A^\dagger BAA^\dagger \Gamma_0^{\textcircled{\#}} A)^* &= A^\dagger BAA^* \Gamma_0^{\textcircled{\#}} (A^\dagger)^* \\ &= A^\dagger BAA^\dagger \Gamma_0^{\textcircled{\#}} A.\end{aligned}$$

Therefore, by Lemma 1, $A^\dagger \Gamma_0^{\textcircled{\#}} A$ is the core inverse of $A^\dagger A\Omega A^\dagger A$ in $A^\dagger AR^{n\times n}A^\dagger A$. Then by Theorem 1, it follows that $\Omega^{\textcircled{\#}} = A^\dagger \Gamma_0^{\textcircled{\#}} A + I_n - A^\dagger A$. Since $\Gamma_0^{\textcircled{\#}} = AA^\dagger \Gamma^{\textcircled{\#}} AA^\dagger$, we get

$$\Omega^{\textcircled{\#}} = A^\dagger \Gamma^{\textcircled{\#}} A + I_n - A^\dagger A.$$

The converse is analogous. If $\Omega^{\textcircled{\#}}$ exists, then $\Omega_0^{\textcircled{\#}} = A^\dagger A\Omega^{\textcircled{\#}} A^\dagger A$ is the core inverse of $A^\dagger BA = A^\dagger A\Omega A^\dagger A$ in the ring $A^\dagger AR^{n\times n}A^\dagger A$. Analogous to the necessity of the proof, we can check that $A\Omega_0^{\textcircled{\#}} A^\dagger$ is the core inverse of $AA^\dagger \Gamma AA^\dagger$ in $AA^\dagger R^{m\times m}AA^\dagger$. Therefore,

$$\begin{aligned}\Gamma^{\textcircled{\#}} &= A\Omega_0^{\textcircled{\#}} A^\dagger + I_m - AA^\dagger \\ &= A\Omega^{\textcircled{\#}} A^\dagger + I_m - AA^\dagger.\end{aligned}$$

For the general case, suppose $\Gamma^{\textcircled{D}}$ exists with $\operatorname{ind}(\Gamma) = k$. Then by Lemma 2,

$$(\Gamma^k)^{\textcircled{\#}} = \left(AA^\dagger (BAA^\dagger)^k AA^\dagger + I_m - AA^\dagger\right)^{\textcircled{\#}}$$

exists. Using the first part of the proof and keeping in mind that B is arbitrary, we can obtain that $\Omega^k = A^\dagger (BAA^\dagger)^k A + I_n - A^\dagger A$ is core invertible. Thus, $\Omega^{\textcircled{D}}$ exists with $\operatorname{ind}(\Omega) \le k$. Moreover,

$$\begin{aligned}\Omega^{\textcircled{D}} &= \Omega^{k-1}(\Omega^k)^{\textcircled{\#}} \\ &= \Omega^{k-1}\left(A^\dagger (BAA^\dagger)^k A + I_n - A^\dagger A\right)^{\textcircled{\#}} \\ &= \Omega^{k-1}\left(A^\dagger (\Gamma^k)^{\textcircled{\#}} A + I_n - A^\dagger A\right) \\ &= \left(A^\dagger (BAA^\dagger)^{k-1} A + I_n - A^\dagger A\right)\left(A^\dagger (\Gamma^k)^{\textcircled{\#}} A + I_n - A^\dagger A\right) \\ &= A^\dagger (BAA^\dagger)^{k-1} AA^\dagger (\Gamma^k)^{\textcircled{\#}} A + I_n - A^\dagger A \\ &= A^\dagger \Gamma^{k-1}(\Gamma^k)^{\textcircled{\#}} A + I_n - A^\dagger A \\ &= A^\dagger \Gamma^{\textcircled{D}} A + I_n - A^\dagger A.\end{aligned}$$

The converse is analogous and $\mathrm{ind}(\Gamma) \leq k$. For the expression of $\Gamma^{\textcircled{D}}$, we have

$$\begin{aligned}
\Gamma^{\textcircled{D}} &= \Gamma^{k-1}(\Gamma^k)^{\textcircled{\#}} \\
&= \Gamma^{k-1}\left(AA^\dagger(BAA^\dagger)^k AA^\dagger + I_m - AA^\dagger\right)^{\textcircled{\#}} \\
&= \Gamma^{k-1}\left(A(\Omega^k)^{\textcircled{\#}}A^\dagger + I_m - AA^\dagger\right) \\
&= \left(AA^\dagger(BAA^\dagger)^{k-1}AA^\dagger + I_m - AA^\dagger\right)\left(A(\Omega^k)^{\textcircled{\#}}A^\dagger + I_m - AA^\dagger\right) \\
&= AA^\dagger(BAA^\dagger)^{k-1}A(\Omega^k)^{\textcircled{\#}}A^\dagger + I_m - AA^\dagger \\
&= A\Omega^{k-1}(\Omega^k)^{\textcircled{\#}}A^\dagger + I_m - AA^\dagger \\
&= A\Omega^{\textcircled{D}}A^\dagger + I_m - AA^\dagger.
\end{aligned}$$

□

Finally, we will show that a similar equivalence holds for DMP inverses.

Lemma 6 ([6]). *Let $a, b \in R$ with $ab = ba$. If $a \in R^d$, then $a^d b = ba^d$.*

Theorem 4. *Let $A \in R^{m\times n}$ be Moore-Penrose invertible and $B \in R^{m\times m}$. Consider the following conditions:*

(1) $\Gamma = AA^\dagger BAA^\dagger + I_m - AA^\dagger$ *is DMP invertible.*
(2) $\Omega = A^\dagger BA + I_n - A^\dagger A$ *is DMP invertible.*

If A is $*$-invariant then (1)⇔(2), in which case

$$\Gamma^{d,\dagger} = A\Omega^{d,\dagger}A^\dagger + I_m - AA^\dagger$$

and

$$\Omega^{d,\dagger} = A^\dagger\Gamma^{d,\dagger}A + I_n - A^\dagger A.$$

Proof. According to [21, Propositions 5 and 6] and Definition 4, the sufficiency and necessity of the theorem are obtained. Since $A^\dagger A\Omega = \Omega A^\dagger A$ and $AA^\dagger\Gamma = \Gamma AA^\dagger$, it follows that $A^\dagger A\Omega^d = \Omega^d A^\dagger A$ and $AA^\dagger\Gamma^d = \Gamma^d AA^\dagger$ by Lemma 6. For the expressions of $\Gamma^{d,\dagger}$ and $\Omega^{d,\dagger}$, we can obtain

$$\begin{aligned}
\Gamma^{d,\dagger} &= \Gamma^d\Gamma\Gamma^\dagger \\
&= (A\Omega^d A^\dagger + I_m - AA^\dagger)(AA^\dagger BAA^\dagger + I_m - AA^\dagger)(A\Omega^\dagger A^\dagger + I_m - AA^\dagger) \\
&= A\Omega^d A^\dagger BA\Omega^\dagger A^\dagger + I_m - AA^\dagger \\
&= A\Omega^d A^\dagger A\Omega\Omega^\dagger A^\dagger + I_m - AA^\dagger \\
&= A\Omega^d\Omega\Omega^\dagger A^\dagger + I_m - AA^\dagger \\
&= A\Omega^{d,\dagger}A^\dagger + I_m - AA^\dagger,
\end{aligned}$$

and

$$\begin{aligned}\Omega^{d,\dagger} &= \Omega^d \Omega \Omega^\dagger \\ &= (A^\dagger \Gamma^d A + I_n - A^\dagger A)(A^\dagger B A + I_n - A^\dagger A)(A^\dagger \Gamma^\dagger A + I_n - A^\dagger A) \\ &= A^\dagger \Gamma^d A A^\dagger B A A^\dagger \Gamma^\dagger A + I_n - A^\dagger A = A^\dagger \Gamma^d A A^\dagger \Gamma \Gamma^\dagger A + I_n - A^\dagger A \\ &= A^\dagger \Gamma^d \Gamma \Gamma^\dagger A + I_n - A^\dagger A = A^\dagger \Gamma^{d,\dagger} A + I_n - A^\dagger A.\end{aligned}$$

□

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TAUBERIAN THEOREMS FOR NÖRLUND-CESÁRO MEAN VIA STATISTICALLY CONVERGENCE IN m-NORMED SPACES

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Abstract. In this paper, we have proved certain kinds of Tauberian theorems for Nörlund-Cesáro summability methods in m-normed space X in statistically sense. We give necessary and sufficient Tauberian condition for this method of summability, in statistically sense. Also, we prove that Tauberian theorems under these summability methods are valid under oscillating slow condition of Schmidt-type and Hardy-type conditions.

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1. Introduction

The notion of statistical convergence was introduced by Fast [10] and Schoenberg [23] independently. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on it was further investigated from various points of view. For example, statistical convergence has been investigated in summability theory, topological groups, topological spaces, function spaces, locally convex spaces, measure theory, fuzzy mathematics and many other areas.

Definition 1. A sequence (x_n) is statistically convergent to L, if for every $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \frac{|\{k \le n\colon |x_k - L| \ge \varepsilon\}|}{n} = 0,$$

where $|A|$, stands for cardinality of the set.

Let X be a real vector space of dimension $d \ge m$ (Here d can be infinite). A mapping $||\cdot,\dots,\cdot||\colon X^m \to \mathbb{R}$ satisfying the following four properties

(1) $||x_1,x_2,\dots,x_m|| = 0$ if and only if $x_1,x_2,\dots,x_m$ are linearly dependent;
(2) $||x_1,x_2,\dots,x_m||$ is invariant under permutation for every $x_1,x_2,\dots,x_m \in X$;
(3) $||\alpha x_1,x_2,\dots,x_m|| = |\alpha|||x_1,x_2,\dots,x_m||$ for all $x_1,x_2,\dots,x_m \in X$ and $\alpha \in \mathbb{R}$;

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(4) $||x,x_2,\dots,x_{m-1},y+z|| \leq ||x,x_2,\dots,x_{m-1},y|| + ||x,x_2,\dots,x_{m-1},z||$
for all $x_1,x_2,\dots,x_{m-1},y,z \in X$,

is called a m-norm on X, and the pair $(X,||\cdot,\dots,\cdot||)$ is called an m-normed space. A trivial example of an m-normed space is $X = R^m$ equipped with the following m-norm (Gunawan [11]):

$$||x_1,x_2,\dots,x_m||_E = \text{abs}\left(\begin{vmatrix} x_{11} & \dots & x_{1m} \\ \vdots & \ddots & \vdots \\ x_{m1} & \dots & x_{mm} \end{vmatrix}\right)$$

where $x_i = (x_{i1},\dots,x_{im}) \in \mathbb{R}^m$ for each $i = 1,\dots,m$. (The subscript E is for Euclidean.)

A sequence (x_k) in an m-normed space $(X,||\cdot,\dots,\dots||)$ is said to be convergent to $c \in X$ with respect to the m-norm if for each $\varepsilon > 0$ there exists an positive integer m_0 such that

$$||x_k - c, z_1, z_2, \dots, z_{m-1}|| < \varepsilon,$$

for all $k \geq m_0$ and for every $z_1,z_2,\dots,z_{m-1} \in X$.

Definition 2. A sequence (x_k) in an m-normed space $(X,||\cdot,\dots,\dots||)$ is said to be statistically-convergent to some $c \in X$ with respect to the m-norm, if for each $\varepsilon > 0$

$$\lim_{k\to\infty} \frac{1}{k} |\{n \leq k\colon ||x_n - c, z_1, z_2, \dots, z_{m-1}|| \geq \varepsilon\}| = 0,$$

for each $z_1,z_2,\dots,z_{m-1} \in X$.

This kind of the convergence we denote by $st^m - (x_k)$ and with $st^m - S(X)$ we will denote the set of all statistically convergent sequences in m-normed space X.

In what follows we give the concept of the summability method known as the generalized Nörlund summability method (N,p,q) (see Borwein [1], Stadtmüller and Tali [30]). In the recent years the Nörlund summability methods, are studied intensively, for more information see([15, 24, 25, 27, 28]). Given two non-negative sequences (p_n) and (q_n), the convolution $(p \star q)$ is defined by

$$R_n\colon = (p \star q)_n = \sum_{k=0}^{n} p_k q_{n-k} = \sum_{k=0}^{n} p_{n-k} q_k.$$

In this paper, we suppose $R_n \to \infty$ as $n \to \infty$. With $(C;1)-$ we will denote the Cesáro summability method. Let (x_n) be a sequence. When $(p \star q)_n \neq 0$ for all $n \in \mathbb{N}$, the generalized Nörlund-Cesáro transform of the sequence (x_n) is the sequence $N^n_{p,q}C^1_n$ obtained by putting

$$N^n_{p,q}C^1_n = \frac{1}{(p \star q)_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} x_v.$$

We say that the sequence (x_n) is generalized Nörlund-Cesáro summable to L determined by the sequences (p_n) and (q_n) or briefly summable $N^n_{p,q}C^1_n$ to L if

$$\lim_{n\to\infty} N^n_{p,q}C^1_n = L. \tag{1.1}$$

Definition 3. $N^n_{p,q}C^1_n$, is said to be statistically-convergent to some $c \in X$ with respect to the m-norm if for each $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{(p\star q)_n}\left|\left\{k \le (p\star q)_n\colon \left\|\frac{1}{(p\star q)_k}\sum_{i=0}^{k} p_i q_{k-i}\frac{1}{i+1}\sum_{v=0}^{i} x_v - L, z_1, z_2, \ldots, z_{m-1}\right\| \ge \varepsilon\right\}\right| = 0,$$

for each $z_1, z_2, \ldots, z_{m-1} \in X$.

This convergence we will denote by $st^m - lim_n N^n_{p,q}C^1_n = L$. Suppose throughout the paper we assume that the sequences (q_n) and (p_n) are satisfying the following conditions:

$$q_n \ge 1, \qquad \sum_{k=0}^{n} p_k \sim n, \qquad n \in \mathbb{N},$$
$$q_{\lambda_n - k} \le 2q_{n-k}, \qquad k = 0,1,2,3,\ldots, \qquad n; \lambda > 1,$$
$$q_{n-k} \le 2q_{\lambda_n - k}, \qquad k = 0,1,2,3,\ldots,\lambda_n; \qquad 0 < \lambda < 1,$$

where $\lambda_n = [\lambda \cdot n]$, $a_n \sim b_n$, means that there are constants C, C_1 such that $a_n \le Cb_n \le C_1 a_n$.

If

$$\lim_{n\to\infty} x_n = L \tag{1.2}$$

implies (1.1), then the method $N^n_{p,q}C^1_n$ is called to be regular. However, the converse is not always true.

Notice that (1.1) may imply (1.2) under a certain condition, which is called a Tauberian condition. For further results of Tauberian type theorems, a reader is referred to the following references: Braha [2–5,7], Canak, Braha and Totur [8], Canak, Erikli, Sezer and Yarasgil [9], Kiesel [16], Kiesel, Stadtmüller [17], Loku, Braha, Et, Tato [18], Loku, Braha [19]. Very recently, Savas, Sezer [21] and Braha, Loku [6], have studied the Tauberian Theorems in the 2-normed spaces. It is also worth mentioning that the papers Srivastava, Jena, Paikray and Mishra [29], Srivastava, Jena, Paikray [26], Jena, Paikray, Mishra [13], Jena, Pairay, Parida and Dutta [14], and Parida, Paikray, and Jena [20] are relevant in the context of our research.

In this paper, we present necessary and sufficient conditions under which the existence of the limit $st^m - \lim_{n\to\infty} x_n = L$ follows from that of $st^m - \lim_{n\to\infty} N^n_{p,q}C^1_n = L$,

in m-normed spaces. These conditions are one-sided or two-sided if (x_n) is a sequence of real or complex numbers, respectively. In what follows we will mention some concepts and definitions which we need in the sequel.

Theorem 1. *If (x_k) converges statistically to $L \in X$, in m-normed space X, such that $\|x_k - L, z_1, z_2, \ldots, z_{m-1}\| \leq M$, for every $k \in \mathbb{N}$, $z_1, z_2, \ldots, z_{m-1} \in X$. Then the sequence $(N^k_{p,q} C^1_k)$ is statistically convergent to L in m-normed space X. Conversely is not true.*

Proof. Let us suppose that $st - \lim_k x_k = L$, in m-normed space X. For every $\varepsilon > 0$, we get

$$\lim_{k\to\infty} \frac{|\{n \leq k\colon \|x_n - L, z_1, z_2, \ldots, z_{m-1}\| \geq \varepsilon\}|}{k} = 0,$$

for each $z_1, z_2, \ldots, z_{m-1} \in X$. Let us denote by

$$B_\varepsilon = \{k \in \mathbb{N}\colon \|x_k - L, z_1, z_2, \ldots, z_{m-1}\| \geq \varepsilon\}$$

and

$$\overline{B_\varepsilon} = \{k \in \mathbb{N}\colon |x_k - L, z_1, z_2, \ldots, z_{m-1}\| < \varepsilon\}.$$

Then

$$\begin{aligned}
&\left\| \frac{1}{R_k} \sum_{i=0}^{k} p_i q_{k-i} \frac{1}{i+1} \sum_{v=0}^{i} x_v - L, z_1, z_2, \ldots, z_{m-1} \right\| \\
&= \left\| \frac{1}{R_k} \sum_{i=0}^{k} p_i q_{k-i} \frac{1}{i+1} \sum_{v=0}^{i} (x_v - L), z_1, z_2, \ldots, z_{m-1} \right\| \\
&\leq \frac{1}{R_k} \sum_{\substack{i=0 \\ i\in B_\varepsilon}}^{k} p_i q_{k-i} \frac{1}{i+1} \sum_{v=0}^{i} \|x_v - L, z_1, z_2, \ldots, z_{m-1}\| \\
&\quad + \frac{1}{R_k} \sum_{\substack{i=0 \\ i\in \overline{B_\varepsilon}}}^{k} p_i q_{k-i} \frac{1}{i+1} \sum_{v=0}^{i} \|x_v - L, z_1, z_2, \ldots, z_{m-1}\| \\
&\leq \frac{M}{R_k} \sum_{\substack{i=0 \\ i\in B_\varepsilon}}^{k} p_i q_{k-i} + \frac{\varepsilon}{R_k} \sum_{\substack{i=0 \\ i\in \overline{B_\varepsilon}}}^{k} p_i q_{k-i} \\
&\leq \frac{M|B_\varepsilon|}{R_k} \cdot \max_{0\leq i\leq k} \{p_i q_{k-i}\} + \varepsilon \leq \frac{M|B_\varepsilon|}{k} \cdot \max_{0\leq i\leq k} \{p_i q_{k-i}\} + \varepsilon \to 0 + \varepsilon, \quad \text{as} \quad k \to \infty.
\end{aligned}$$

To show that converse is not true we will use into consideration this

Proposition 1 (Gunawan [11])**.** *A sequence in a standard m-normed space is convergent in the m-norm if and only if it is convergent in the standard $(m-1)$-norm and,*

by induction, in the standard $(m-r)$-norm for all $r = 1,\dots,m-1$. In particular, a sequence in a standard m-normed space is convergent in the m-norm if and only if it is convergent in the usual norm $||\cdot||_S = \langle\cdot,\cdot\rangle^{\frac{1}{2}}$.

Based on this fact we will give example which proves that converse does not holds, in the 2-norm.

Example 1. Let us consider that $X = \mathbb{R}^3$ and $||x,y|| = \max\{|x_1y_2 - x_2y_1|, |x_1y_3 - x_3y_1|, |x_2y_3 - y_2x_3|\}$, where $x = (x_1,x_2,x_3), y = (y_1,y_2,y_3)$. Define the sequence $(x_n) \in X$ as follows:

$$x_n = \left(2+(-1)^n, \frac{2}{3}+\frac{(-1)^n}{3}, \frac{2}{3}+\frac{2(-1)^n}{3}\right),$$

and $y = (y_1,y_2,y_3) \in X$.
The sequence $(N^n_{p,q}C^1_n)$ is

$$\begin{aligned} N^n_{p,q}C^1_n = \Bigg(&2+\frac{1}{2(n+1)}\left[\ln 2+\frac{(-1)^n}{2}\left(\Psi\left(\frac{n}{2}+\frac{3}{2}\right)-\Psi\left(\frac{n}{2}+1\right)\right)+\ln n+C\right],\\ &\frac{2}{3}+\frac{1}{6(n+1)}\left[\ln 2+\frac{(-1)^n}{2}\left(\Psi\left(\frac{n}{2}+\frac{3}{2}\right)-\Psi\left(\frac{n}{2}+1\right)\right)+\ln n+C\right],\\ &\frac{2}{3}+\frac{1}{3(n+1)}\left[\ln 2+\frac{(-1)^n}{2}\left(\Psi\left(\frac{n}{2}+\frac{3}{2}\right)-\Psi\left(\frac{n}{2}+1\right)\right)+\ln n+C\right]\Bigg), \end{aligned}$$

where $\Psi(z) = \frac{d}{dz}\{\log\Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)}$, C-Lebesgue constant and it is valid for all $y = (y_1,y_2,y_3) \in \mathbb{R}^3$. Let us denote by $L = \left(2, \frac{2}{3}, \frac{2}{3}\right)$, then

$$\begin{aligned} &\lim_{n\to\infty} ||N^n_{p,q}C^1_n - L, y||\\ &= \lim_{n\to\infty}\max\Bigg\{\Bigg|\frac{1}{2(n+1)}\left[\ln 2+\frac{(-1)^n}{2}\left(\Psi\left(\frac{n}{2}+\frac{3}{2}\right)-\Psi\left(\frac{n}{2}+1\right)\right)+\ln n+C\right]y_2\\ &\qquad -\frac{1}{6(n+1)}\left[\ln 2+\frac{(-1)^n}{2}\left(\Psi\left(\frac{n}{2}+\frac{3}{2}\right)-\Psi\left(\frac{n}{2}+1\right)\right)+\ln n+C\right]y_1\Bigg|,\\ &\qquad \Bigg|\frac{1}{2(n+1)}\left[\ln 2+\frac{(-1)^n}{2}\left(\Psi\left(\frac{n}{2}+\frac{3}{2}\right)-\Psi\left(\frac{n}{2}+1\right)\right)+\ln n+C\right]y_3\\ &\qquad -\frac{1}{3(n+1)}\left[\ln 2+\frac{(-1)^n}{2}\left(\Psi\left(\frac{n}{2}+\frac{3}{2}\right)-\Psi\left(\frac{n}{2}+1\right)\right)+\ln n+C\right]y_1\Bigg|,\\ &\qquad \Bigg|\frac{1}{6(n+1)}\left[\ln 2+\frac{(-1)^n}{2}\left(\Psi\left(\frac{n}{2}+\frac{3}{2}\right)-\Psi\left(\frac{n}{2}+1\right)\right)+\ln n+C\right]y_3 \end{aligned}$$

$$-\frac{1}{3(n+1)}\left[\ln 2+\frac{(-1)^n}{2}\left(\Psi\left(\frac{n}{2}+\frac{3}{2}\right)-\Psi\left(\frac{n}{2}+1\right)\right)+\ln n+C\right]y_2\Bigg|\Bigg\}$$
$$=0.$$

So (x_n) is $N^n_{p,q}C^1_n$-summable to $\left(2,\frac{2}{3},\frac{2}{3}\right)$ in 2-normed space X. Now we will prove that (x_n) does not converges to $\left(2,\frac{2}{3},\frac{2}{3}\right)$ in 2-normed space X. Let $y=(0,1,1)\in\mathbb{R}^3$ then

$$\begin{aligned}\lim_{n\to\infty}||x_n-L,y|| &= \lim_{n\to\infty}\left\|\left((-1)^n,\frac{(-1)^n}{3},\frac{2(-1)^n}{3}\right),(y_1,y_2,y_3)\right\| \\ &= \lim_{n\to\infty}\max\left\{\left|(-1)^n\cdot y_2-\frac{(-1)^n\cdot y_1}{3}\right|,\left|(-1)^n\cdot y_3-\frac{2(-1)^n y_1}{3}\right|,\right. \\ &\quad \left.\left|\frac{(-1)^n\cdot y_3}{3}-\frac{2(-1)^n\cdot y_2}{3}\right|\right\}\neq 0,\end{aligned}$$

sequence (x_n) is not convergence.

□

2. Tauberian theorem, for $st^m-N^n_{p,q}C^1_n$-summability method

In this section, we will show the Tauberian Theorem for $st^m-N^n_{p,q}C^1_n$-summability method.

Theorem 2. *Let (p_n) and (q_n) be two sequences of real numbers defined as above and*

$$st^m-\liminf_n\frac{R_{\lambda_n}}{R_n}>1,\quad \lambda>1 \tag{2.1}$$

where $\lambda_n=[\lambda n]$, denotes the integral parts of the $[\lambda n]$ for every $n\in\mathbb{N}$, and let $(N^n_{p,q}C^1_n)$ be a sequence of real numbers such that $\lim_n N^n_{p,q}C^1_n=L$, in m-normed space X. Then (x_n) is convergent to the same number L in m-normed space X, if and only if the following two conditions holds:

$$\begin{aligned}&\inf_{\lambda>1}\limsup_n\frac{1}{R_{\lambda_n}-R_n}\\ &\left|\left\{k\le(p\star q)_n:\ \left\|\frac{1}{(p\star q)_k}\sum_{i=k+1}^{\lambda_k}p_iq_{\lambda_k-i}\frac{1}{i+1}\sum_{v=0}^{i}x_v-x_k,z_1,z_2,\ldots,z_{m-1}\right\|\le-\varepsilon\right\}\right|\\ &=0,\end{aligned} \tag{2.2}$$

and

$$\inf_{0<\lambda<1} \limsup_{n} \frac{1}{R_n - R_{\lambda_n}} \left| \left\{ k \leq (p\star q)_n : \left\| \frac{1}{(p\star q)_k} \sum_{i=\lambda_k+1}^{k} p_i q_{k-i} \frac{1}{i+1} \sum_{v=0}^{i} x_k - x_v, z_1, z_2, \ldots, z_{m-1} \right\| \leq \varepsilon \right\} \right| = 0, \tag{2.3}$$

for any $z_1, z_2, z_3, \ldots, z_{m-1} \in X$, *and* $\varepsilon > 0$.

Remark 1. The symmetric counterparts of conditions (2.2) and (2.3) are the following

$$\sup_{\lambda>1} \liminf_{n} \frac{1}{R_{\lambda_n} - R_n} \left| \left\{ k \leq (p\star q)_n : \left\| \frac{1}{(p\star q)_k} \sum_{i=k+1}^{\lambda_k} p_i q_{\lambda_k-i} \frac{1}{i+1} \sum_{v=0}^{i} x_v - x_k, z_1, z_2, \ldots, z_{m-1} \right\| \leq -\varepsilon \right\} \right| = 0,$$

and

$$\sup_{0<\lambda<1} \liminf_{n} \frac{1}{R_n - R_{\lambda_n}} \left| \left\{ k \leq (p\star q)_n : \left\| \frac{1}{(p\star q)_k} \sum_{i=\lambda_k+1}^{k} p_i q_{k-i} \frac{1}{i+1} \sum_{v=0}^{i} x_k - x_v, z_1, z_2, \ldots, z_{m-1} \right\| \leq \varepsilon \right\} \right| = 0,$$

for any $z_1, z_2, z_3, \ldots, z_{m-1} \in X$, and $\varepsilon > 0$.

Remark 2. The sequence $(x_n) \in X$ is slowly oscillating (see Schmidt [22]) in m-normed space if:

$$\inf_{\lambda>1} \lim_{n\to\infty} \sup \max_{n\leq k\leq \lambda_n} ||x_k - x_n, z_1, z_2, z_3, \ldots, z_{m-1}|| = 0,$$

for all $z_1, z_2, z_3, \ldots, z_{m-1} \in X$, or equivalently

$$\inf_{0<\lambda<1} \lim_{n\to\infty} \sup \max_{\lambda_n\leq k\leq n} ||x_n - x_k, z_1, z_2, z_3, \ldots, z_{m-1}|| = 0,$$

for all $z_1, z_2, z_3, \ldots, z_{m-1} \in X$.
If we denote by $\Delta x_n = x_n - x_{n-1}$, then we can obtain the following form of the above conditions

$$\inf_{\lambda>1} \lim_{n\to\infty} \sup \max_{n\leq k\leq \lambda_n} \left\| -\sum_{i=k+1}^{n} \Delta x_i, z_1, z_2, z_3, \ldots, z_{m-1} \right\| = 0$$

and

$$\inf_{0<\lambda<1} \lim_{n\to\infty} \sup \max_{\lambda_n \le k \le n} \left\| \sum_{i=k+1}^{n} \Delta x_i, z_1, z_2, z_3, \ldots, z_{m-1} \right\| = 0,$$

for all $z_1, z_2, z_3, \ldots, z_{m-1} \in X$.

In what follows we will show some auxiliary lemmas which are needful in the sequel.

Lemma 1. *For the sequences of real numbers (p_n) and (q_n), condition given by relation (2.1) is equivalent to this one:*

$$st^m - \liminf_n \frac{R_n}{R_{\lambda_n}} > 1, \quad 0 < \lambda < 1.$$

Proof. It is similar to the Lemma 2.4 in Braha [5], and we omit it. □

Lemma 2. *Let (p_n) and (q_n) be the sequences defined as above and relation (2.1) is satisfied. If $x = (x_n)$ is $st^m - N^n_{p,q}C^1_n$-convergent to L, in m-normed space X. Then for every $\lambda > 0$,*

$$st^m - \lim_n \left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{k=n+1}^{\lambda_n} p_k q_{\lambda_n - k} \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \ldots, z_{m-1} \right\| = 0, \tag{2.4}$$

for $\lambda > 1$ and

$$st^m - \lim_n \left\| \frac{1}{R_n - R_{\lambda_n}} \sum_{k=\lambda_n+1}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \ldots, z_{m-1} \right\| = 0, \tag{2.5}$$

for $0 < \lambda < 1$, for every $z_1, z_2, z_3, \ldots, z_{m-1} \in X$.

Proof. Let us suppose that $\lambda > 1$. After some calculations we obtain:

$$\begin{aligned}
&\left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{k=n+1}^{\lambda_n} p_k q_{\lambda_n - k} \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \ldots, z_{m-1} \right\| \\
&\le \left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{k=0}^{\lambda_n} p_k q_{\lambda_n - k} \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \ldots, z_{m-1} \right\| \\
&\quad + \left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{k=0}^{n} p_k q_{\lambda_n - k} \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \ldots, z_{m-1} \right\| \\
&= \left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{k=0}^{\lambda_n} p_k q_{\lambda_n - k} \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \ldots, z_{m-1} \right\| \\
&\quad + \left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{k=0}^{n} p_k (q_{n-k} + q_{\lambda_n - k} - q_{n-k}) \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \ldots, z_{m-1} \right\|
\end{aligned}$$

$$
\begin{aligned}
&= \left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{k=0}^{\lambda_n} p_k q_{\lambda_n - k} \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \ldots, z_{m-1} \right\| \\
&\quad + \left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \ldots, z_{m-1} \right\| \\
&\quad + \left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{k=0}^{n} p_k (q_{\lambda_n - k} - q_{n-k}) \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \ldots, z_{m-1} \right\|. \qquad (2.6)
\end{aligned}
$$

We know that

$$
st^m - \limsup_n \frac{R_n}{R_{\lambda_n} - R_n} = st^m - \left(\liminf_n \frac{R_{\lambda_n}}{R_n} - 1 \right)^{-1} < \infty. \qquad (2.7)
$$

Now from relations (2.6) and (2.7), we get relation (2.4).

Case where $0 < \lambda < 1$. In this case we have:

$$
\begin{aligned}
&\left\| \frac{1}{R_n - R_{\lambda_n}} \sum_{k=\lambda_n+1}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \ldots, z_{m-1} \right\| \\
&\le \left\| \frac{1}{R_n - R_{\lambda_n}} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \ldots, z_{m-1} \right\| \\
&\quad + \left\| \frac{1}{R_n - R_{\lambda_n}} \sum_{k=0}^{\lambda_n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \ldots, z_{m-1} \right\| \\
&= \left\| \frac{1}{R_n - R_{\lambda_n}} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \ldots, z_{m-1} \right\| \\
&\quad + \left\| \frac{1}{R_n - R_{\lambda_n}} \sum_{k=0}^{\lambda_n} p_k (q_{\lambda_n - k} + q_{n-k} - q_{\lambda_n - k}) \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \ldots, z_{m-1} \right\| \\
&\le \left\| \frac{1}{R_n - R_{\lambda_n}} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \ldots, z_{m-1} \right\| \\
&\quad + \left\| \frac{1}{R_n - R_{\lambda_n}} \sum_{k=0}^{\lambda_n} p_k q_{\lambda_n - k} \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \ldots, z_{m-1} \right\| \\
&\quad + \left\| \frac{1}{R_n - R_{\lambda_n}} \sum_{k=0}^{\lambda_n} p_k (q_{n-k} - q_{\lambda_n - k}) \frac{1}{k+1} \sum_{v=0}^{k} x_v - L, z_1, z_2, z_3, \ldots, z_{m-1} \right\|. \qquad (2.8)
\end{aligned}
$$

Knowing that

$$
\limsup_n \frac{R_{\lambda_n}}{R_n - R_{\lambda_n}} = \left(\liminf_n \frac{R_n}{R_{\lambda_n}} - 1 \right)^{-1} < \infty. \qquad (2.9)
$$

Now from relations (2.8) and (2.9), we get relation (2.5). □

In what follows we will prove the Theorem 2.

Proof of Theorem 2. Necessity. Let us suppose that $st^m - \lim_n x_n = L$, and $st^m - \lim_n N^n_{p,q} C^1_n = L$, in m-normed space X. For every $\lambda > 1$ following Lemma 2 we get relation (2.2) and in case where $0 < \lambda < 1$, again applying Lemma 2 we obtain relation (2.3).

Sufficient: Assume that $st^m - \lim_n N^n_{p,q} C^1_n = L$, in m-normed space X and conditions (2.1), (2.2) and (2.3) are satisfied. In what follows we will prove that $st^m - \lim_n x_n = L$, in m-normed space X. Or equivalently, $st^m - \lim_n (N^n_{p,q} C^1_n - x_n) = 0$, in m-normed space X.

Given any $\varepsilon > 0$, by relation (2.2) we can choose some $\lambda_1 > 1$ such that

$$\liminf_n \left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{i=n+1}^{\lambda_{n_1}} p_i q_{\lambda_{n_1}-i} \frac{1}{i+1} \sum_{v=0}^{i} x_v - x_n, z_1, z_2, \ldots, z_{m-1} \right\| \geq -\varepsilon,$$

where $\lambda_{n_1} = \lambda_1 \cdot n$. From above relation and Lemma 2, we get:

$$\left\| \liminf_n \left(\frac{1}{R_{\lambda_n} - R_n} \sum_{i=n+1}^{\lambda_{n_1}} p_i q_{\lambda_{n_1}-i} \frac{1}{i+1} \sum_{v=0}^{i} x_v - x_n, z_1, z_2, \ldots, z_{m-1} \right) \right\| \geq -\varepsilon,$$

respectively

$$\left\| L - \limsup_n x_n, z_1, z_2, \ldots, z_{m-1} \right\| \geq -\varepsilon \;\Rightarrow\; \liminf_n \|L - x_n, z_1, z_2, \ldots, z_{m-1}\| \geq -\varepsilon. \tag{2.10}$$

From relation (2.3), there exists a $0 < \lambda_2 < 1$, such that

$$\limsup_n \left\| \frac{1}{R_{\lambda_n} - R_n} \sum_{i=n+1}^{\lambda_{n_2}} p_i q_{\lambda_{n_1}-i} \frac{1}{i+1} \sum_{v=0}^{i} x_n - x_v, z_1, z_2, \ldots, z_{m-1} \right\| \leq \varepsilon,$$

where $\lambda_{n_2} = \lambda_2 \cdot n$. From above relation and Lemma 2, we get:

$$\left\| \limsup_n \left(\frac{1}{R_{\lambda_n} - R_n} \sum_{i=n+1}^{\lambda_{n_1}} p_i q_{\lambda_{n_1}-i} \frac{1}{i+1} \sum_{v=0}^{i} x_n - x_v, z_1, z_2, \ldots, z_{m-1} \right) \right\| \leq \varepsilon,$$

respectively

$$\left\| \limsup_n x_n - L, z_1, z_2, \ldots, z_{m-1} \right\| \geq \varepsilon \;\Rightarrow\; \limsup_n \|x_n - L, z_1, z_2, \ldots, z_{m-1}\| \leq \varepsilon. \tag{2.11}$$

From equations (2.10) and (2.11), we get

$$-\varepsilon \leq \liminf_n \|x_n - L, z_1, z_2, \ldots, z_{m-1}\| \leq \limsup_n \|x_n - L, z_1, z_2, \ldots, z_{m-1}\| \leq \varepsilon.$$

Hence, we have obtain that

$$\lim_n \|x_n - L, z_1, z_2, \ldots, z_{m-1}\| = 0.$$

□

In what follows we will show that under conditions that (x_n) is slowly oscillation sequence, and $st^m - N^n_{p,q}C^1_n$ summability, it converges st^m to the same limit.

Theorem 3. *Let $(x_n) \in X$ be $st^m - N^{\lambda_n}_{p,q}C^1_n$-summable to L. If (x_n) slowly oscillating in m-normed space X, then (x_n) converges st^m- to L.*

Proof. First case, $\lambda > 1$.

Let us suppose that $st^m - N^{\lambda_n}_{p,q}C^1_n$ converges to L in m-normed space X. To prove that $st^m - (x_n) \to L$ in m-normed space X, it is enough to prove that

$$st^m - \lim_n ||N^{\lambda_n}_{p,q}C^1_n - x_n, z_1, z_2, \ldots, z_{m-1}|| = 0,$$

for every $z_1, z_2, \ldots, z_{m-1} \in X$. Let us start from

$$\begin{aligned}
\left\|N^n_{p,q}C^1_n - x_n, z_1, z_2, \ldots, z_{m-1}\right\| &= \left\| \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} x_v - x_n, z_1, z_2, \ldots, z_{m-1} \right\| \\
&= \left\| \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} (x_v - x_n), z_1, z_2, \ldots, z_{m-1} \right\| \\
&= \left\| \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} \sum_{j=v+1}^{n} \Delta x_j, z_1, z_2, \ldots, z_{m-1} \right\| \\
&\leq \max_{0 \leq v \leq n} \left\| \sum_{j=v+1}^{n} \Delta x_j, z_1, z_2, \ldots, z_{m-1} \right\|.
\end{aligned}$$

Taking limit superior in both sides of the above relation and then infimum, we get:

$$\inf_{\lambda > 1} \lim_{n \to \infty} \sup \left\|N^n_{p,q}C^1_n - x_n, z_1, z_2, \ldots, z_{m-1}\right\| = 0.$$

Hence, it is proved that (x_n) converges to L in m-normed space X. And from it follows it st^m- convergence.

Second case, $0 < \lambda < 1$. This case is similar to the previous one and for this reason we omit it. □

Next result show that if (x_n) satisfies Hardy[12] conditions, and is $st^m - N^n_{p,q}C^1_n$ summable, then (x_n) converges st^m- to same limit.

Theorem 4. *Let $(x_n) \in X$ be $st^m - N^n_{p,q}C^1_n$-summable to L. If (x_n) satisfies the following relation:*

$$n\Delta x_n = 0(1),$$

then (x_n) st^m- convergent to L.

Proof. It is enough to prove that

$$st^m - \lim_n ||N_{p,q}^n C_n^1 - x_n, z_1, z_2, \ldots, z_{m-1}|| = 0$$

for every $z_1, z_2, \ldots, z_{m-1} \in X$. Let us consider that $\lambda > 1$. From

$$n\Delta x_n = 0(1),$$

it follows that for every $\varepsilon > 0$, there exists a n_0 such that for every $n > n_0$ we have

$$|n\Delta x_n| < \varepsilon.$$

Let us start from

$$\begin{aligned}
\left\|N_{p,q}^n C_n^1 - x_n, y\right\| &= \left\| \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} (x_v - x_n), z_1, z_2, \ldots, z_{m-1} \right\| \\
&= \left\| \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{k+1} \sum_{v=0}^{k} \sum_{j=v+1}^{n} \Delta x_j, z_1, z_2, \ldots, z_{m-1} \right\| \\
&\leq \max_{0 \leq k \leq n} \left\| \sum_{j=k+1}^{n} \Delta x_j, z_1, z_2, \ldots, z_{m-1} \right\|.
\end{aligned}$$

From above relations, we get:

$$\left\|N_{p,q}^n C_n^1 - x_n, z_1, z_2, \ldots, z_{m-1}\right\| \leq \max_{0 \leq k \leq n} \left\| \sum_{j=k+1}^{n} \Delta x_j, z_1, z_2, \ldots, z_{m-1} \right\| \leq \varepsilon. \tag{2.12}$$

Hence, it is proved that (x_n) converges st^m to L in m-normed space X.

The second case where $0 < \lambda < 1$, can be proved in similar way. □

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ON THE EXISTENCE OF SOLUTIONS FOR AN INFINITE SYSTEM OF INTEGRAL EQUATIONS

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Abstract. The paper is devoted to prove the existence of solutions for an infinite system of nonlinear integral equations. This system is investigated in the WC-Banach algebra $C(I,c_0)$, the space of all continuous functions acting from an interval I into the sequence space c_0. Making use of the measure of weak noncompactness and the weak topology, we establish some fixed point theorems for the sum and the product of nonlinear weakly sequentially continuous operators acting on WC-Banach algebra involving w-contractive operators.

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1. Introduction

In this paper we are mainly concerned with the existence results for the following system of nonlinear integral equations [1, 11]

$$x_n(t) = a_n(x_n(t)) \int_0^1 b(t,s) f_n(s, x_n(s), x_{n+1}(s), \dots) ds + c_n(x_n(t)), \tag{1.1}$$

where $n = 1,2,\dots$ and $t \in I = [0,1]$. The infinite system of equations (1.1) will be investigated in the Banach space $C_0 = C(I,c_0)$ consisting of all functions acting from the interval I into the sequence space c_0, which are continuous on I. Obviously, the norm in the space C_0 has the form

$$\|x\|_{C_0} = \|(x_1, x_2, \dots)\|_{C_0} = \sup_{t\in I}\{\sup[|x_n(t)| : n = 1,2,\dots]\}.$$

Note that the previous system (1.1) may be written in the form

$$Ax.Bx + Cx = x, \tag{1.2}$$

where A, B, and C are three nonlinear operators defined on subsets of a Banach algebra.

The aim of the paper is to present new results in concern with the existence of fixed points of operators dealing with Banach algebras. More precisely, we will consider operators having the form (1.2), where A, B, and C are defined on subsets of a

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given Banach algebra and satisfy some conditions expressed in terms of a generalized Lipschitz continuity or the contractivity with respect to the De Blasi measure of weak noncompactness. This study is carried out in the frame work of the so-called WC-Banach algebra i.e., a Banach algebra in which the product of two weakly compact subsets is weakly compact. This concept has been introduced quite recently in [3] and it turned out to be very fruitful when investigating some problems of operators theory under weak topology [9].

Notice that it was shown in [10] that if the operator A involved in (1.2) is weakly compact and satisfy condition $(\mathcal{H})$ (this condition allows us to distinguish some classes of operators which transform each weakly convergent sequence in a Banach space X into a sequence containing a subsequence being strongly convergent in X) and B and C are w-contractive then the equation (1.2) has at least one solution (see [10, Theorem 3.2]). However, the condition (ii) in Theorem 3.2 in [10] is too restrictive and can be relaxed just by supposing the operator A is w-contractive. This result is developed in Theorem 4 (see section 3).

The remaining of the paper is organized as follows. After some preliminaries in section 2, we establish in section 3 some fixed point theorems for the sum and the product of nonlinear weakly sequentially continuous operators acting on WC-Banach algebra involving w-contractive operators. Finally, we provide in section 4 an example indicating that our results are still valid in the field of nonlinear integral equations.

2. Preliminaries

Throughout the paper we denote by $\mathbb{R}$ the set of real numbers. The symbol $\mathbb{N}$ stands for the set of natural numbers. By the symbol X we will denote a Banach space endowed with the norm $\|.\|$. For any $r > 0$, B_r denotes the closed ball in X centered at 0_X with radius r and $\mathcal{D}(A)$ denotes the domain of an operator A. Also Ω_X is the collection of all nonempty bounded subsets of X and $\mathcal{K}^w$ is the subfamily of Ω_X consisting of all weakly compact subsets of X. Now, $\rightharpoonup$ denotes the weak convergence and $\to$ denotes the strong convergence in X, respectively.

Further, let us recall the concept of the De Blasi measure of weak noncompactness [6] being the function $\omega\colon \Omega_X \to [0,+\infty)$, defined in the following way

$$\omega(M) = \inf\{r > 0 : \text{ there exits } K \in \mathcal{K}^w \text{ such that } M \subseteq K + B_r\},$$

for all $M \in \Omega_X$. For convenience we recall some basic properties of $\omega(.)$ needed below [6].

Lemma 1. *Let M_1, M_2 be two elements of Ω_X. Then, the following conditions are satisfied:*

(1) $M_1 \subseteq M_2$ implies $\omega(M_1) \leq \omega(M_2)$.
(2) $\omega(M_1) = 0$ if, and only if, M_1 is relatively weakly compact.
(3) $\omega(\overline{M_1}^w) = \omega(M_1)$, where $\overline{M_1}^w$ is the weak closure of the subset M_1.

(4) $\omega(M_1 \cup M_2) = \max\{\omega(M_1), \omega(M_2)\}$.
(5) $\omega(\lambda M_1) = |\lambda|\omega(M_1)$ *for all* $\lambda \in \mathbb{R}$.
(6) $\omega(\mathrm{co}(M_1)) = \omega(M_1)$, *where* $\mathrm{co}(M_1)$ *denotes the convex hull of* M_1.
(7) $\omega(M_1 + M_2) \leq \omega(M_1) + \omega(M_2)$.
(8) *if* $(M_n)_{n\geq 1}$ *is a decreasing sequence of nonempty bounded and weakly closed subsets of* X *with* $\lim_{n\to\infty} \omega(M_n) = 0$, *then* $M_\infty := \cap_{n=1}^{\infty} M_n$ *is nonempty and* $\omega(M_\infty) = 0$.

Definition 1 ([9, Definition 1.3.3])**.** An operator $A\colon \mathcal{D}(A) \subseteq X \to X$ is said to be weakly sequentially continuous on $\mathcal{D}(A)$ if for every sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(A)$, $x_n \rightharpoonup x$ implies $Ax_n \rightharpoonup Ax$.

Definition 2 ([9, Definition 1.4.3])**.** An operator $A\colon \mathcal{D}(A) \subseteq X \to X$ is said to be ω-contractive (or ω-α-contraction) if it maps bounded sets into bounded sets, and there exists some $\alpha \in [0,1)$ such that $\omega(A(S)) \leq \alpha\omega(S)$ for all bounded subsets $S \subseteq \mathcal{D}(A)$.

An operator $A\colon \mathcal{D}(A) \subseteq X \to X$ is said to be ω-condensing if it maps bounded sets into bounded sets, and $\omega(A(S)) < \omega(S)$ for all bounded sets $S \subseteq \mathcal{D}(A)$ with $\omega(S) > 0$.

Example 1 ([9, Corollary 2.3.2]). If $A\colon \mathcal{D}(A) \subseteq X \to X$ is Lipschitzian with a Lipschitz constant $\alpha \in [0,1)$ and is weakly sequentially continuous on X, then A is ω-α-contraction.

3. Fixed Point Theorems in WC-Banach Algebras

First, let us recall other definitions which will be used in our study.

Definition 3 ([3, Definition 2.2])**.** Let X be a Banach algebra. We say that X is a WC-Banach algebra, if the product $K^w.K'^w$ of arbitrary weakly compact subsets K^w, K'^w of X is weakly compact.

It is interesting to work with these WC-Banach algebras and find important characterizations under the weak topology setting. Some results in this direction were established in [3, 4]. We provide the following definition as a way to highlight the most important of these results [4].

Definition 4 ([4, Definition 3.1])**.** We will say that the Banach algebra X satisfies condition $(\mathcal{P})$ if for every two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in X such that $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$ for some $x, y \in X$, we have $x_n.y_n \rightharpoonup x.y$.

Recently, J. Banas and L. Olszowy [2] have shown the equivalence between WC-Banach algebra and a Banach algebra satisfying $(\mathcal{P})$.

Theorem 1 ([2, Theorem 2.9])**.** *A Banach algebra X satisfies condition $(\mathcal{P})$ if and only if X is the WC-Banach algebra.*

Example 2. Clearly, every finite dimensional Banach algebra is a WC-Banach algebra. If X is a WC-Banach algebra, then the set $\mathcal{C}(\mathcal{K}, X)$ (here $\mathcal{K}$ is a compact

Hausdorff space) of all continuous functions from $\mathcal{K}$ to X is also a WC-Banach algebra. The proof is based on Dobrakov's Theorem [7].

Theorem 2 ([7, Theorem 9])**.** *Let $\mathcal{K}$ be a compact Hausdorff space and X be a Banach space. Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathcal{C}(\mathcal{K},X)$, and $f \in \mathcal{C}(\mathcal{K},X)$. Then, $(f_n)_{n\in\mathbb{N}}$ is weakly convergent to f if and only if $(f_n(t))_{n\in\mathbb{N}}$ is weakly convergent to $f(t)$ for each $t \in \mathcal{K}$.*

Example 3 ([1, Example 2.9]). Now, let us consider the classical Banach sequence space c_0 consisting of all real (or complex) sequence converging to zero and equipped with the standard supremum (maximum) norm. In other words

$$c_0 = \left\{x = (x_n)_{n\in\mathbb{N}}\ ;\ \lim_{n\to\infty} x_n = 0\right\} \text{ and } \|x\| = \sup\{|x_n| :\ n \in \mathbb{N}\}.$$

We define the product of two elements $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}}$ in c_0 in the classical way:

$$x \cdot y = (x_n)_{n\in\mathbb{N}} \cdot (y_n)_{n\in\mathbb{N}} = (x_n y_n)_{n\in\mathbb{N}}.$$

Observe that for $x, y \in c_0$ we have

$$\begin{aligned}\|x \cdot y\| &= \sup\{|x_n y_n| :\ n \in \mathbb{N}\}\\ &\le \|x\| \sup\{|y_n| :\ n \in \mathbb{N}\}\\ &= \|x\|\|y\|.\end{aligned}$$

Thus c_0 is a Banach algebra (normalized).

We show that c_0 is the WC-Banach algebra. To this end let us recall [8] that in the Banach space c_0 the sequence $(x_k)_{k\in\mathbb{N}} = ((x_n^k)_{n\in\mathbb{N}})_{k\in\mathbb{N}}$ (denoted $((x_k^n))$ in the sequel), where $x_k = (x_n^k)_{n\in\mathbb{N}} \in c_0$ for any $k = 1,2,\ldots$, is convergent to an element $x = (x_n)_{n\in\mathbb{N}} \in c_0$ if and only if the sequence $(x_k)_{k\in\mathbb{N}}$ is bounded in c_0 and $\lim_{k\to\infty} x_n^k = x_n$ for any $n = 1,2,\ldots$. In other words, the sequence $(x_k)_{k\in\mathbb{N}} = ((x_k^n))$ is weakly convergent to $x = (x_n)_{n\in\mathbb{N}} \in c_0$ if only if the sequence $(x_k)_{k\in\mathbb{N}}$ is bounded in c_0 and is coordinatewise convergent to $x = (x_n)_{n\in\mathbb{N}}$.

Further, let us assume that W and W' are weakly compact subsets of the space c_0. Consider the product $W \cdot W'$. Let us take an arbitrary sequence $(z_k)_{k\in\mathbb{N}} \subset W \cdot W'$, $z_k = (z_n^k)_{n\in\mathbb{N}}$ for any $k = 1,2,\ldots$. This means that there exist two sequences $(x_k)_{k\in\mathbb{N}} \subset W$, $(y_k)_{k\in\mathbb{N}} \subset W'$ such that $z_k = x_k \cdot y_k$ for any $k = 1,2,\ldots$. Since the sets W and W' are weakly compact, without loss of generality we can assume that $x_k \rightharpoonup x = (x_n)_{n\in\mathbb{N}} \in W$ and $y_k \rightharpoonup y = (y_n)_{n\in\mathbb{N}} \in W'$ as $k \to \infty$. If we denote $x_k = (x_n^k)_{n\in\mathbb{N}}$, $y_k = (y_n^k)_{n\in\mathbb{N}}$ for each $k = 1,2,\ldots$, then in view of the above quoted characterization of the weak convergence in c_0 we deduce that $\lim_{k\to\infty} x_n^k = x_n$ for any $n = 1,2,\ldots$. This implies that $\lim_{k\to\infty} z_n^k = \lim_{k\to\infty} x_n^k y_n^k = x_n y_n$ for $n = 1,2,\ldots$. Obviously, the sequence $(z_k)_{k\in\mathbb{N}} = (x_k)_{k\in\mathbb{N}} \cdot (y_k)_{k\in\mathbb{N}}$ is bounded in c_0.

Thus we showed that the sequence $(z_k)_{k\in\mathbb{N}}$ is weakly convergent to the element $z = x\cdot y \in W\cdot W'$. This allows us to infer that the set $W\cdot W'$ is weakly compact in the space c_0.

To present the main fixed point results of this section we need the following theorems.

Theorem 3 ([5, Theorem 3.2])**.** *Let S be a nonempty, bounded, closed, and convex subset of a Banach space X and let $A\colon S\to S$ be a weakly sequentially continuous mapping. If A is ω-condensing, then it has, at least, a fixed point in S.*

Lemma 2 ([3, Lemma 2.4])**.** *Let M_1 and M_2 be two bounded subsets of a WC-Banach algebra X. Then, we have the following inequality*

$$\omega(M_1.M_2) \le \|M_2\|\omega(M_1) + \|M_1\|\omega(M_2) + \omega(M_1)\omega(M_2).$$

Now, we are ready to state and prove the main result of this section.

Theorem 4. *Let S be a nonempty, bounded, closed, and convex subset of a WC-Banach algebra X and let A, $C\colon X\to X$ and $B\colon S\to X$ be three weakly sequentially continuous operators, satisfying the following conditions:*

(i) A is regular and $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(S)$.
(ii) A, B, and C are w-contractive with constants α, β, and γ, respectively.
(iii) For each $y\in S$ the following implication holds

$$x = Ax.By + Cx \Rightarrow x\in S.$$

(iv) The following inequality is satisfied

$$\beta\Delta + \alpha\beta w(S) + \delta\alpha + \gamma < 1,$$

where, $\Delta = \|A(S)\|$ and $\delta = \|B(S)\|$.

Under the above assumptions the operator equation $x = Ax.Bx + Cx$ has at least one solution in the set S.

Proof. From assumption (i), it follows that for each $y\in S$, there is a unique $x_y\in X$ such that

$$\left(\frac{I-C}{A}\right)x_y = By,$$

or equivalently

$$Ax_y.By + Cx_y = x_y.$$

Since hypothesis (iii) holds, then $x_y\in S$. Hence, the operator $T = \left(\frac{I-C}{A}\right)^{-1}B\colon S\to S$ is well defined. At first, let us notice that after converting of the equality $T = \left(\frac{I-C}{A}\right)^{-1}B$, we obtain

$$T = AT.B + CT. \tag{3.1}$$

Let us now prove that T is ω-condensing. Let M be a subset of S with $\omega(M) > 0$ and taking into account the properties of the De Blasi measure of weak noncompactness, we get

$$\omega(TM) \leq \omega(A(TM).BM) + \omega(C(TM)).$$

The properties of ω in Lemma 2 and assumption (ii) on A, B and C yield

$$\omega(TM) \leq \delta\alpha\omega(TM) + \Delta\beta\omega(M) + \alpha\beta\omega(TM)\omega(M) + \gamma\omega(TM). \tag{3.2}$$

Note that from the inequality (3.2) and the fact that $\omega(TM)\omega(M) \leq \omega(S)\omega(M)$, one can deduce that

$$\omega(TM) \leq k_1\omega(M) \quad \text{where } k_1 = \frac{\Delta\beta + \alpha\beta\omega(S)}{1 - \delta\alpha - \gamma},$$

and from the inequality (3.2) and the fact that $\omega(TM)\omega(M) \leq \omega(TM)\omega(S)$, we can deduce that

$$\omega(TM) \leq k_2\omega(M) \quad \text{where } k_2 = \frac{\Delta\beta}{1 - \alpha\beta\omega(S) - \delta\alpha - \gamma}.$$

Then, we have

$$\omega(TM) \leq \min\{k_1, k_2\}\omega(M).$$

Using assumption (v), we have

$$\min\{k_1, k_2\} < 1,$$

and this prove that, the operator T is ω-condensing. Now, we prove that T is weakly sequentially continuous. For this purpose, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in S which converges weakly to some $x \in S$. In view of the equality (3.1) we obtain:

$$\omega(\{Tx_n : n \in \mathbb{N}\}) \leq \omega(\{A(Tx_n)Bx_n : n \in \mathbb{N}\}) + \omega(\{C(Tx_n) : n \in \mathbb{N}\}).$$

In the above estimate we have used that the set $\{Bx_n : n \in \mathbb{N}\}$ is relatively weakly compact (since $\{x_n : n \in \mathbb{N}\}$ is relatively weakly compact). Further, observe that according to assumption (ii) the operators A and C are w-contractive with constants α and γ, respectively. So, we obtain

$$\begin{aligned}\omega(\{Tx_n : n \in \mathbb{N}\}) &\leq \delta\alpha\omega(\{Tx_n : n \in \mathbb{N}\}) + \gamma\omega(\{Tx_n : n \in \mathbb{N}\}) \\ &\leq (\delta\alpha + \gamma)\omega(\{Tx_n : n \in \mathbb{N}\}) \\ &< \omega(\{Tx_n : n \in \mathbb{N}\}).\end{aligned}$$

Observe that conducting the same reasoning as above, on the basis of assumption (iv) we deduce that $\omega(\{Tx_n : n \in \mathbb{N}\}) = 0$. Thus the set $\{Tx_n : n \in \mathbb{N}\}$ is relatively weakly compact. Consequently, there exists a subsequence $(x_{n_i})_{i\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $Tx_{n_i} \rightharpoonup y$.

Going back to equality (3.1), to the weak sequential continuity of A, B and C and in view of condition $(\mathcal{P})$ we deduce that $Tx_{n_i} = A(Tx_{n_i}).Bx_{n_i} + C(Tx_{n_i}) \rightharpoonup Ay.Bx + Cy$. Then, $y = Ay.Bx + Cy$ so, $y = Tx$. Consequently, $Tx_{n_i} \rightharpoonup Tx$.

Now, we claim that $Tx_n \rightharpoonup Tx$. Suppose the contrary, then there exists a subsequence $(x_{n_i})_{i\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ and a weak neighborhood V^w of Tx, such that $Tx_{n_i} \notin V^w$ for all $i \in \mathbb{N}$. Since $(x_{n_i})_{i\in\mathbb{N}}$ converges weakly to x then arguing as before, we may extract a subsequence $(x_{n_{i_j}})_{j\in\mathbb{N}}$ of $(x_{n_i})_{i\in\mathbb{N}}$, such that $Tx_{n_{i_j}} \rightharpoonup Tx$, which is absurd, since $Tx_{n_{i_j}} \notin V^w$, for all $j \in \mathbb{N}$. As a result, T is weakly sequentially continuous.

Next, taking into account the above proved properties of the operator T and utilizing Theorem 3 we infer that the operator T has at least one fixed point x in the set S i.e., there exists $x \in S$ such that $x = Tx$. Hence, we obtain

$$x = \left(\frac{I-C}{A}\right)^{-1} Bx,$$

and consequently

$$\left(\frac{I-C}{A}\right) x = Bx.$$

From the above equality we get

$$x - Cx = Ax.Bx.$$

Finally, we have

$$x = Ax.Bx + Cx.$$

This completes the proof of our theorem. □

The above theorem yields the following corollaries.

Corollary 1 ([10])**.** *Let S be a nonempty, bounded, closed, and convex subset of a WC-Banach algebra X and let A, $C\colon X \to X$ and $B\colon S \to X$ be three weakly sequentially continuous operators, satisfying the following conditions:*

(i) A is regular and $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(S)$.
(ii) $A(S)$,$B(S)$, and $C(S)$ are relatively weakly compact.
(iii) For each $y \in S$ the following implication holds

$$x = Ax.By + Cx \Rightarrow x \in S.$$

Under the above assumptions the operator equation $x = Ax.Bx + Cx$ has at least one solution in the set S.

Now, we formulate the final result of this section.

Theorem 5. *Let S be a nonempty, bounded, closed, and convex subset of a WC-Banach algebra X and let A, $C\colon X \to X$ and $B\colon S \to X$ be three weakly sequentially continuous operators, satisfying the following conditions:*

(i) A and C are Lipschitzian with the Lipschitz constants α and γ, respectively, where $\alpha, \gamma \in [0,1)$.
(ii) A is regular.
(iii) $B(S)$ is relatively weakly compact.

(iv) For each $y \in S$ the following implication holds

$$x = Ax.By + Cx \Rightarrow x \in S.$$

(v) The following inequality is satisfied

$$\delta\alpha + \gamma < 1,$$

where, $\Delta = \|A(S)\|$ and $\delta = \|B(S)\|$.

Under the above assumptions the operator equation $x = Ax.Bx + Cx$ has at least one solution in the set S.

Proof. Let y be fixed in S and let's define the mapping

$$\begin{cases} \varphi_y \colon X \longrightarrow X, \\ \quad x \longrightarrow \varphi_y(x) = Ax.By + Cx. \end{cases}$$

Let $x_1, x_2 \in X$. The use of assumption (i) leads to

$$\begin{aligned} \|\varphi_y(x_1) - \varphi_y(x_2)\| &\leq \|Ax_1.B_y - Ax_2.B_y\| + \|Cx_1 - Cx_2\| \\ &\leq \|Ax_1 - Ax_2\|\|By\| + \|Cx_1 - Cx_2\| \\ &\leq (\delta\alpha + \gamma)\|x_1 - x_2\|. \end{aligned}$$

Now, an application of Banach's fixed point theorem [9] shows the existence of a unique point $x_y \in X$, such that

$$\varphi_y(x_y) = x_y.$$

Hence, the operator $T := \left(\frac{I-C}{A}\right)^{-1} B \colon S \to X$ is well defined. Moreover, the use of assumption (iv) allows us to have $T(S) \subset S$. Using arguments similar to those used in the proof of Theorem 4, we can deduce that the operator T is weakly sequentially continuous. By using Theorem 3, it is sufficient to check that T is ω-condensing. In order to achieve this, let M be a subset of S with $\omega(M) > 0$.

Using equality (3.1), we have

$$\omega(TM) \leq \omega(A(TM).BM) + \omega(C(TM)).$$

Making use of Lemmas 1, 2, and Example 1, together with the assumptions on A, B, and C enables us to have

$$\begin{aligned} \omega(T(M)) &\leq \omega(A(T(M))B(M)) + \omega(C(T(M))) \\ &\leq \delta\alpha\omega(T(M)) + \gamma\omega(T(M)) \\ &\leq (\delta\alpha + \gamma)\omega(T(M)). \end{aligned}$$

From the assumption (v), we have

$$\omega(T(M)) < \omega(T(M)).$$

Hence, $T(M)$ is relatively weakly compact, and in particular, T is ω-condensing. $\square$

4. Application to infinite systems of integral equations

This section is dedicated to show the applicability of the theory developed in the previous section to prove a result on the existence of solutions of the following infinite system of nonlinear integral equations [1,2]

$$x_n(t) = a_n(x_n(t)) \int_0^1 b(t,s) f_n(s, x_n(s), x_{n+1}(s), \dots) ds + c_n(x_n(t)), \tag{4.1}$$

where $n = 1,2,\dots$ and $t \in I = [0,1]$. The infinite system of equations (4.1) will be investigated in the Banach space $C_0 = C(I, c_0)$ consisting of all functions acting from the interval I into the sequence space c_0, which are continuous on I. Obviously, the norm in the space C_0 has the form

$$\|x\|_{C_0} = \|(x_1, x_2, \dots)\|_{C_0} = \sup_{t \in I}\{\sup[|x_n(t)| : \ n = 1,2,\dots]\}.$$

Moreover, let us recall that by $C(I)$ we will denote the space $C(I,\mathbb{R})$ equipped with the norm $\|x\|_{C(I)} = \sup\{|x(t)| : \ t \in I\}$. Let us mention that C_0 forms a WC-Banach algebra (cf. Example 2). Now, we formulate a few assumptions under which we will investigate the infinite system (4.1).

(i) For any $n \geq 1$, the function $a_n \colon \mathbb{R} \to (0,\infty)$ is Lipschitzian with the constant $\alpha \in [0,1)$.

(ii) For any sequence $(x_n)_{n\geq 1}$ with $x_n \to 0$ as $n \to \infty$ we have that $a_n x_n \to 0$ as $n \to \infty$.

(iii) The function $c_n \colon \mathbb{R} \to \mathbb{R}$ is Lipschitzian with the Lipschitz constant $\gamma \in [0,1)$ and $c_n(0) = 0$ for $n = 1,2,\dots$.

(iv) There exist two constants M_1 and $M_2 > 0$ such that $a_n(x) \leq M_1$ and $|c_n(x)| \leq M_2$, for any $x \in \mathbb{R}$ and for $n = 1,2,\dots$.

(v) The function f_n acts from the set $I \times \mathbb{R}^\infty$ into $\mathbb{R}$ for any $n = 1,2,\dots$. Moreover, we assume that there exist two sequences $(k_n)_{n\geq 1}$, $(l_n)_{n\geq 1}$ with positive terms such that $k_n \to 0$ as $n \to \infty$, $(l_n)_{n\geq 1}$ is bounded and the following inequality is satisfied

$$|f_n(t, x_n, x_{n+1}, \dots)| \leq k_n + l_n \sup\{|x_i| : \ i \geq n\}$$

for any $t \in I$, $x = (x_n)_{n\geq 1} \in c_0$ and for $n = 1,2,\dots$.

(vi) The family of functions $\{f_n\}_{\geq 1}$ is uniformly equicontinuous on the set $I \times c_0$. This means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $n \geq 1$ and $t \in I$, and for all $x = (x_n)_{n\geq 1}$, $y = (y_n)_{n\geq 1} \in c_0$ with $\|x - y\|_{c_0} \leq \delta$ we have that

$$|f_n(t, x_n, x_{n+1}, \dots) - f_n(t, y_n, y_{n+1}, \dots)| \leq \varepsilon.$$

(vii) The function $b(t,s) = b \colon \ I \times I \to \mathbb{R}$ is continuous in t uniformly with respect to the variable $s \in I$ and is integrable with respect to s for any $t \in I$.

Remark 1. Observe that from assumptions (i) and (iii) it follows that the functions a_n and c_n $(n = 1,2,\dots)$ are continuous on $\mathbb{R}$.

Remark 2. From the above formulated assumption we deduce that $K < \infty$ and $L < \infty$, where the constants K and L are defined by the equalities:

$$K = \sup\{k_n : \ n = 1,2,\dots\}, \qquad L = \sup\{l_n : \ n = 1,2,\dots\}.$$

Remark 3. In view of assumption (vii) the function

$$\overline{b}(t) = \int_0^1 b(t,s)ds$$

is well defined on I. Observe that this function is continuous on the interval I. In fact, for a fixed $\varepsilon > 0$ and for arbitrary $t_1, t_2 \in I$ such that $|t_2 - t_1| \leq \varepsilon$, we have

$$\begin{aligned} |\overline{b}(t_2) - \overline{b}(t_1)| &\leq \int_0^1 |b(t_2,s) - b(t_1,s)|ds \\ &\leq \int_0^1 \mu_1(b,\varepsilon)ds = \mu_1(b,\varepsilon), \end{aligned}$$

where the function $\mu_1(b,\varepsilon)$ denotes the modulus of continuity of the function $t \to b(t,s)$ defined by the formula

$$\mu_1(b,\varepsilon) = \sup\{|b(t_2,s) - b(t_1,s)| : \ t_1,t_2,s \in I, \ |t_2 - t_1| \leq \varepsilon\}.$$

Obviously, in view of assumption (vii) we have that $\mu_1(b,\varepsilon) \to 0$ as $\varepsilon \to 0$. This shows that the function $\overline{b} = \overline{b}(t)$ defined above is continuous on I. Taking into account the above statement we can define the finite constant $\overline{B}$ by putting

$$\overline{B} = \sup\left\{ \int_0^1 |b(t,s)|ds : \ t \in I \right\}.$$

Now, we formulate our further assumptions. To this end, let us put

$$M = \max\{M_1, M_2\}.$$

(viii) The following inequality holds

$$M\overline{B}L < 1.$$

where the constants L and $\overline{B}$ were defined earlier.

For further purposes we define the number r_0 by putting

$$r_0 = \frac{MK\overline{B} + M}{1 - ML\overline{B}}. \tag{4.2}$$

Finally, we assume that

(ix) $\alpha\left(\dfrac{K\overline{B} + L\overline{B}M}{1 - ML\overline{B}}\right) + \gamma < 1.$

Before formulating our main result we provide an auxiliary lemma which will be useful in our investigations.

Lemma 3 ([1, Lemma 4.6]). *Let the function $x(t) = (x_1(t), x_2(t), \dots) = (x_n(t))_{n\geq 1}$ be an element of the space $C_0 = C(I, c_0)$. Then*

$$\lim_{n\to\infty} \|x_n\|_{C(I)} = 0.$$

Theorem 6. *Under assumptions $(i)-(ix)$ the infinite system of integral equations (4.1) has at least one solution $x(t) = (x_n(t))_{n\geq 1}$ in the space $C_0 = C(I, c_0)$.*

Proof. Let us define the subset S of $C_0 = C(I, c_0)$ by

$$S = B_{r_0} := \{x \in C_0 : \|x\| \leq r_0\},$$

where r_0 is a number described by equality (4.2). Obviously, S is a nonempty, closed, convex, and bounded subset of $C_0 = C(I, c_0)$. To make lecture of the infinite system of equations let us consider three operators A, B, and C defined on C_0 as follow

$$\begin{cases} (Ax)(t) = (a_n(x_n(t)))_{n\geq 1} = (a_1(x_1(t)), a_2(x_2(t)), \dots), \\ (Bx)(t) = \left(\int\limits_0^1 b(t,s) f_n(s, x_n(s), x_{n+1}(s), \dots) ds \right)_{n\geq 1}, \quad . \\ (Cx)(t) = (c_n(x_n(t)))_{n\geq 1} = (c_1(x_1(t)), c_2(x_2(t)), \dots), \end{cases}$$

for $t \in I = [0,1]$. We show that these operators satisfy the assumptions of Theorem 5. We start with investigations concerning the operator A. At first, let us observe that assumption (i) guarantees that A is regular. This means that there is satisfied assumption (ii) of Theorem 5.

Next let us fix arbitrary $x = (x_n)_{n\geq 1}$ and $y = (y_n)_{n\geq 1} \in S$. If we take an arbitrary $t \in I$, then we get

$$\begin{aligned} \|A(x)(t) - A(y)(t)\|_{c_0} &= \|(a_1(x_1)(t) - a_1(y_1)(t), a_2(x_2)(t) - a_2(y_2)(t), \dots)\|_{c_0} \\ &\leq \sup\{|a_n(x_n)(t) - a_n(y_n)(t)| : \ n = 1, 2, \dots\} \\ &\leq \alpha \|x(t) - y(t)\|_{c_0}. \end{aligned}$$

From the last inequality and taking the supremum over t, we obtain

$$\|Ax - Ay\|_{C_0} \leq \alpha \|x - y\|_{C_0}.$$

This shows that the operator A is Lipschitzian with a Lipschitzian constant α.

Now, we will verify that A transforms the space C_0 into itself. Indeed, let us take a function $x(t) = (x_n(t))_{n\geq 1} \in C_0$. This means, that for each fixed $t \in I$ we have that $x_n(t) \to 0$ when $n \to \infty$. Hence, in view of assumption (ii) we get that $a_n(x_n(t)) \to 0$ as $n \to \infty$. This proves our claim.

Further on, we show that the operator A is weakly sequentially continuous. Let us take a sequence $(x_n)_{n\geq 1} \subset C_0$ which is weakly convergent to a function $x \in C_0$. This means by [7] that if we denote

$$x_n(t) = (x_1^n(t), x_2^n(t), x_3^n(t), \dots)$$

for $n = 1,2,\dots$ and for an arbitrary $t \in I$ and if we denote $x(t) = (x_1(t), x_2(t), x_3(t), \dots)$, then we have $x_1^n(t) \to x_1(t), \dots, x_k^n(t) \to x_k(t), \dots$ for any $t \in I$, if $n \to \infty$. Now, let us consider the sequence $(Ax_n)_{n\geq 1}$ i.e.,

$$\begin{aligned}(Ax_n)_{n\geq 1} &= (A(x_1^n, x_2^n, x_3^n, \dots))_{n\geq 1} = (a_1(x_1^n), a_2(x_2^n), a_3(x_3^n), \dots)_{n\geq 1} \\ &= (a_k(x_k^n))_{n\geq 1}.\end{aligned}$$

Then, for an arbitrarily fixed $t \in I$ we obtain

$$\begin{aligned}((Ax_n)(t))_{n\geq 1} &= (a_1(x_1^n(t)), a_2(x_2^n(t)), a_3(x_3^n(t)), \dots)_{n\geq 1} \\ &= (a_k(x_k^n(t)))_{n\geq 1}.\end{aligned}$$

Since, according to our assumptions we have that $x_k^n(t) \to x_k(t)$ as $n \to \infty$, $(k = 1,2,\dots)$, this implies that $a_k(x_k^n(t)) \to a_k(x_k(t))$, $(k = 1,2,\dots)$, which is a simple consequence of the continuity of each function a_k, $(k = 1,2,\dots)$, on the set $\mathbb{R}$ (cf. Remark 1). Since $(Ax_n)_{n\geq 1}$ is bounded by M_1, then we can again apply Dobrakov's theorem (see Theorem 2), A is weakly sequentially continuous.

In a similar way we can show that the operator C is well-defined, Lipschitzian with the Lipschitz constants γ, and weakly sequentially continuous.

In what follows we will consider the operator B. To this end let us take the set S. At he beginning we show that B transforms the set S into the space C_0. Thus, take an arbitrary function $x(t) = (x_n(t))_{n\geq 1} \in S$. Fix arbitrarily $n \geq 1$ and a number $t \in I$. Then, keeping in mind assumptions (v) and (vii), we obtain:

$$\begin{aligned}|(Bx)(t)| &\leq \int_0^1 |b(t,s)||f_n(s, x_n(s), x_{n+1}(s), \dots)|ds \\ &\leq \int_0^1 |b(t,s)|\{k_n + l_n \sup[|x_i(s)| : \ i \geq n]\}ds \\ &\leq k_n \int_0^1 |b(t,s)|ds + l_n \int_0^1 |b(t,s)| \sup[|x_i(s)| : \ i \geq n]ds \\ &\leq k_n \overline{B} + l_n \sup_{i\geq n}\{\sup[|x_i(t)| : \ t \in I]\}\overline{B},\end{aligned}$$

where $\overline{B}$ was defined in Remark 3.

Consequently, in view of assumption (v) and Remark 2, we get

$$|(Bx)(t)| \leq k_n \overline{B} + L\overline{B} \sup_{i\geq n}\{\sup[|x_i(t)| : \ t \in I]\},$$

for any $n \geq 1$ and for each $t \in I$. The above estimate implies the following one

$$|(Bx)(t)| \leq k_n \overline{B} + L\overline{B} \sup_{i\geq n} \|x_i\|_{C(I)}. \tag{4.3}$$

Now, taking into account that $k_n \to 0$ as $n \to \infty$ and keeping in mind Lemma 3 we conclude from estimate (4.3) that the operator B transforms the set S into the C_0.

Moreover, from estimate (4.3) we infer the following inequality

$$\|Bx\|_{C_0} \leq K\overline{B} + L\overline{B}\|x\|_{C_0}.$$

Consequently (since $x \in S$), we obtain

$$\|Bx\|_{C_0} \leq K\overline{B} + L\overline{B}r_0. \tag{4.4}$$

Further on, we show that the operator B is weakly sequentially continuous on the set S. To do it, let $(x_n)_{n\geq 1}$ be any sequence in S weakly converging to a function $x \in S$. Then, by using Dobrakov's theorem [7], we get for all $t \in I$, $x_k^n(t) \to x_k(t)$, if $n \to \infty$, $(k = 1, 2, \dots)$. Now, fix $\varepsilon > 0$. This means that for any $t \in I$, we have the following inequalities:

$$\begin{aligned}
\|Bx_n(t) - Bx(t)\|_{c_0} &= \max_{k\geq 1}\left\{\left|\int_0^1 b(t,s)(f_k(s,x_k^n(s),x_{k+1}^n(s),\dots)\right.\right. \\
&\qquad \left.\left. - f_k(s,x_k(s),x_{k+1}(s),\dots))ds\right|\right\} \\
&\leq \max_{k\geq 1}\left\{\int_0^1 |b(t,s)||f_k(s,x_k^n(s),x_{k+1}^n(s),\dots)\right. \\
&\qquad \left. - f_k(s,x_k(s),x_{k+1}(s),\dots)|ds\right\}.
\end{aligned}$$

Let $\delta > 0$. Since $x_k^n(t) \to x_k(t)$, then $\forall k \geq 1$ there exists $n_0 \geq 1$ such that for all $n \geq n_0$ and for all $t \in I$,

$$|x_k^n(t) - x_k(t)| \leq \delta, \qquad \forall\, n \geq n_0.$$

Hence, in view of assumption (vi), we obtain for any $t \in I$

$$\begin{aligned}
\|Bx_n(t) - Bx(t)\|_{c_0} &\leq \int_0^1 |b(t,s)|\varepsilon ds \\
&\leq \overline{B}\varepsilon, \qquad \forall\, n \geq n_0.
\end{aligned}$$

This shows that $Bx_n(t) \to Bx(t)$ in c_0, in particular, $Bx_n(t) \rightharpoonup Bx(t)$ in c_0. Since $(Bx_n)_{n\geq 1}$ is bounded by $K\overline{B} + L\overline{B}r_0$, then we can again apply Dobrakov's theorem [7] to obtain $Bx_n \rightharpoonup Bx$. We conclude that the operator B is weakly sequentially continuous.

Now, we are going to prove that for all $t \in I$, the set $\{Bx(t) : \ x \in S\}$ is relatively compact in c_0. At first let $t_1, t_2 \in I$, $n \geq 1$, and for a function $x \in S$, on the basis of the estimate from Remark 3, we have:

$$|(Bx)(t_2) - (Bx)(t_1)| \leq \int_0^1 |b(t_2,s) - b(t_1,s)||f_n(s,x_n(s),x_{n+1}(s),\dots)|ds$$

$$\leq \int_0^1 \mu_1(b,|t_2-t_1|)(K+Lr_0)ds$$
$$\leq (K+Lr_0)\mu_1(b,|t_2-t_1|).$$

This implies that the family of functions $\{Bx(t):\ x\in S\}$ is equicontinuous on the space c_0. Further, let $(Bx_k(t))_{k\geq 1}$ be a sequence of $B(S)(t)$. From estimate (4.4), we have

$$\sup_{n\geq 1}|Bx_n^k(t)| \leq \|Bx_k\|_{C_0} \leq K\overline{B}+L\overline{B}r_0.$$

for all $t\in I$ and for all $k\geq 1$. Then, the sequence $(Bx_n^k(t))_{k\geq 1}$ is equibounded in $\mathbb{R}$, then using Bolzano-Weirstrass's Theorem, there is a renamed subsequence such that

$$\lim_{k\to\infty} Bx_n^k(t) = g_n(t), \quad \text{for all } n\geq 1 \text{ and } t\in I.$$

Then, this means that the following condition is satisfied:

For all $t\in I$ and $\varepsilon>0$, there exists $k_0\geq 1$ such that for all $k\geq k_0$, we have

$$|Bx_n^k(t)-g_n(t)| \leq \frac{\varepsilon}{2}, \quad \text{for all } n\geq 1.$$

Taking into account that $\lim_{n\to\infty} Bx_n^{k_0}(t)=0$ then, for all $t\in I$, there exists $n_0\geq 1$ such that for all $n\geq n_0$, we have

$$|Bx_n^{k_0}(t)| \leq \frac{\varepsilon}{2}.$$

Now, for $t\in I$ and $n\geq n_0$, we have

$$|g_n(t)| \leq |g_n(t)-Bx_n^{k_0}(t)|+|Bx_n^{k_0}(t)| \leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

It follows that $(g_n(t))_{n\geq 1}\in c_0$. Then, $B(S)(t)$ is relatively compact in c_0. From the above established facts and Arzelá-Ascoli criterion [8] for the relative compactness, we deduce that the set $B(S)$ is relatively compact in C_0, in particular, $B(S)$ is relatively weakly compact in C_0. This means that there is satisfied assumption (iii) of Theorem 5.

In our next step we show that there is satisfied assumption (iv) of Theorem 5. To this end let us fix arbitrarily $y\in S$. Next, assume that an element $x\in C_0$ satisfies the equality

$$x = Ax.By+Cx.$$

This yields

$$\|x\|_{C_0} \leq \|Ax\|_{C_0}\|By\|_{C_0}+\|Cx\|_{C_0}.$$

Further, by hypothesis (iv) we obtain

$$\|x\|_{C_0} \leq M_1\|By\|_{C_0}+M_2$$
$$\leq M(K\overline{B}+L\overline{B}r_0+1)$$

Hence, in view of assumption $(viii)$ we obtain $\|x\|_{C_0} \leq r_0$. Finally, let us notice in view of equality (4.3) and hypothesis (ix) that

$$\begin{aligned}\alpha\|B(S)\|_{C_0} + \gamma &\leq \alpha\left(K\overline{B} + L\overline{B}\frac{MK\overline{B}+M}{1-ML\overline{B}}\right) + \gamma \\ &\leq \alpha\left(\frac{K\overline{B}+L\overline{B}M}{1-ML\overline{B}}\right) + \gamma < 1.\end{aligned}$$

Hence, the hypothesis (v) of Theorem 5 is satisfied which achieves the proof. □

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$\oplus$-g-RADICAL SUPPLEMENTED MODULES

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Abstract. In this work $\oplus$-g-radical supplemented modules are defined and some properties of these modules are investigated. It is proved that the finite direct sum of $\oplus$-g-radical supplemented modules is also $\oplus$-g-radical supplemented. Let M be a $\oplus$-g-radical supplemented R-module. If $\mathrm{Rad}_g(M) \ll_g M$, then M is $\oplus$-g-supplemented. If $\mathrm{Rad}_g(M) \ll M$, then M is $\oplus$-supplemented.

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1. Introduction

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let R be a ring and M be an R-module. We will denote a submodule N of M by $N \leq M$. Let M be an R-module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a *small submodule* of M and denoted by $N \ll M$. Let M be an R-module and $N \leq M$. If there exists a submodule K of M such that $M = N + K$ and $N \cap K = 0$, then N is called a *direct summand* of M and it is denoted by $M = N \oplus K$. For any R-module M, we have $M = M \oplus 0$. The intersection of all maximal submodules of M is called the *radical* of M and denoted by $\mathrm{Rad}(M)$. If M have no maximal submodules, then it is defined $\mathrm{Rad}(M) = M$. M is said to be *semilocal* if $M/\mathrm{Rad}(M)$ is semisimple. A submodule N of an R-module M is called an *essential submodule* of M and denoted by $N \trianglelefteq M$ in case $K \cap N \neq 0$ for every submodule $K \neq 0$, or equivalently, $K = 0$ for every $K \leq M$ with $N \cap K = 0$. Let M be an R-module and K be a submodule of M. K is called a *generalized small* (or briefly, *g-small*) submodule of M if for every essential submodule T of M with the property $M = K + T$ implies that $T = M$, then we write $K \ll_g M$ (in [13], it is called an *e-small submodule* of M and denoted by $K \ll_e M$). It is clear that every small submodule is a generalized small submodule but the converse is not true in general. Let M be an R-module. M is called a *hollow module* if every proper submodule of M is small in M. M is called a *generalized hollow* (or briefly, *g-hollow*) *module* if every proper submodule of M is g-small in M. Here it is clear

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that every hollow module is generalized hollow. The converse of this statement is not always true. M is called a *local module* if M has the largest submodule, i.e. a proper submodule which contains all other proper submodules. M is called a *generalized local* (briefly, *g-local*) if M has a large proper essential submodule which contain all proper essential submodules of M or M have no proper essential submodules. Let U and V be submodules of M. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M. M is said to be *supplemented* if every submodule of M has a supplement in M. If every submodule of M has a supplement that is a direct summand in M, then M is called a $\oplus$-*supplemented module*. Let M be an R-module and $U, V \leq M$. If $M = U + V$ and $M = U + T$ with $T \trianglelefteq V$ implies that $T = V$, or equivalently, $M = U + V$ and $U \cap V \ll_g V$, then V is called a *g-supplement* of U in M. M is said to be *g-supplemented* if every submodule of M has a g-supplement in M. M is said to be $\oplus$-*g-supplemented* if every submodule of M has a g-supplement that is a direct summand in M (see [10]). A module M is said to have the *Summand Sum Property* (*SSP*) if the sum of two direct summands of M is again a direct summand of M (see [12, Exercise 39.17(3)]). We say that a module M has $(D3)$ property if $M_1 \cap M_2$ is a direct summand of M for every direct summands M_1 and M_2 of M with $M = M_1 + M_2$ (see [8]). Let M be an R-module and $U, V \leq M$. If $M = U + V$ and $U \cap V \leq \mathrm{Rad}(V)$, then V is called a *generalized* (*radical*) *supplement* (briefly, *Rad-supplement*) of U in M. M is said to be *generalized* (*radical*) *supplemented* (briefly, *Rad-supplemented*) if every submodule of M has a Rad-supplement in M. M is said to be *generalized* (*radical*) $\oplus$-*supplemented* (briefly, *Rad-*$\oplus$*-supplemented*) if every submodule of M has a Rad-supplement that is a direct summand in M. The intersection of all essential maximal submodules of an R-module M is called the *generalized radical* of M and denoted by $\mathrm{Rad}_g(M)$ (in [13], it is denoted by $\mathrm{Rad}_e(M)$). If M have no essential maximal submodules, then we denote $\mathrm{Rad}_g(M) = M$. An R-module M is said to be *g-semilocal* if $M/\mathrm{Rad}_g(M)$ is semisimple (see [9]). Let M be an R-module and $U, V \leq M$. If $M = U + V$ and $U \cap V \leq \mathrm{Rad}_g(V)$, then V is called a *generalized radical supplement* (or briefly, *g-radical supplement*) of U in M. M is said to be *generalized radical supplemented* (briefly, *g-radical supplemented*) if every submodule of M has a g-radical supplement in M.

More informations about supplemented modules are in [2, 8, 12]. More results about $\oplus$-supplemented modules are in [4, 5, 8]. More details about generalized (radical) supplemented modules are in [11]. More details about generalized (radical) $\oplus$-supplemented modules are in [1, 3]. More informations about g-supplemented modules are in [6]. More informations about g-radical supplemented modules are in [7].

Now we will give some important properties of the generalized radical of any module.

Lemma 1. *Let M be an R-module. The following conditions hold.*

(1) $\mathrm{Rad}_g(M) = \sum_{L \ll_g M} L$.
(2) *$Rm \ll_g M$ for every $m \in \mathrm{Rad}_g(M)$.*
(3) *If $N \leq M$, then $\mathrm{Rad}_g(N) \leq \mathrm{Rad}_g(M)$.*
(4) *If $K, L \leq M$, then $\mathrm{Rad}_g(K) + \mathrm{Rad}_g(L) \leq \mathrm{Rad}_g(K+L)$.*
(5) *Let N be an R-module and $f\colon M \to N$ be an R-module homomorphism. Then $f(\mathrm{Rad}_g(M)) \leq \mathrm{Rad}_g(N)$.*
(6) *If $K, L \leq M$, then $\frac{\mathrm{Rad}_g(K)+L}{L} \leq \mathrm{Rad}_g\left(\frac{K+L}{L}\right)$.*
(7) *If $M = \bigoplus_{i \in I} M_i$, then $\mathrm{Rad}_g(M) = \bigoplus_{i \in I} \mathrm{Rad}_g(M_i)$.*

Proof. See [7, Lemma 2, Lemma 3 and Lemma 4]. □

2. $\oplus$-g-RADICAL SUPPLEMENTED MODULES

Definition 1. Let M be an R-module. If every submodule of M has a g-radical supplement that is a direct summand of M, then M is called a $\oplus$-generalized radical supplemented (briefly, $\oplus$-g-radical supplemented) module.

Clearly we can see that every $\oplus$-g-supplemented module is $\oplus$-g-radical supplemented, but the converse is not true in general (see Example 1 and Example 2). We also clearly can see that every $\oplus$-supplemented and every g-hollow modules are $\oplus$-g-radical supplemented. Let M be $\oplus$-g-radical supplemented R-module. Then M is g-radical supplemented and by [7, Theorem 1] M is g-semilocal.

Lemma 2. *Let M be an R-module and $M = M_1 \oplus M_2$. If M_1 and M_2 are $\oplus$-g-radical supplemented, then M is also $\oplus$-g-radical supplemented.*

Proof. Let U be any submodule of M. Since M_2 is $\oplus$-g-radical supplemented, $(M_1+U) \cap M_2$ has a g-radical supplement X that is a direct summand of M_2. Since X is a g-radical supplement of $(M_1+U) \cap M_2$ in M_2, $M_2 = (M_1+U) \cap M_2 + X$ and $(M_1+U) \cap X = (M_1+U) \cap M_2 \cap X \leq \mathrm{Rad}_g(X)$. By $M_2 = (M_1+U) \cap M_2 + X$, $M = M_1 \oplus M_2 = M_1 + (M_1+U) \cap M_2 + X = M_1 + U + X$. Since M_1 is $\oplus$-g-radical supplemented, $(U+X) \cap M_1$ has a g-radical supplement Y that is a direct summand of M_1. Since Y is a g-radical supplement of $(U+X) \cap M_1$ in M_1, $M_1 = (U+X) \cap M_1 + Y$ and $(U+X) \cap Y = (U+X) \cap M_1 \cap Y \leq \mathrm{Rad}_g(Y)$. By $M_1 = (U+X) \cap M_1 + Y$, $M = M_1 + U + X = (U+X) \cap M_1 + Y + U + X = U + X + Y$. Since $(M_1+U) \cap X \leq \mathrm{Rad}_g(X)$ and $(U+X) \cap Y \leq \mathrm{Rad}_g(Y)$, by Lemma 1, $U \cap (X+Y) \leq (U+Y) \cap X + (U+X) \cap Y \leq (M_1+U) \cap X + (U+X) \cap Y \leq \mathrm{Rad}_g(X) + \mathrm{Rad}_g(Y) \leq \mathrm{Rad}_g(X+Y)$. Hence $X+Y$ is a g-radical supplement of U in M. Since X is a direct summand of M_2 and Y is a direct summand of M_1, $X+Y$ is a direct summand of $M = M_1 \oplus M_2$. Hence M is $\oplus$-g-radical supplemented. □

Corollary 1. *Let M be an R-module and $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$. If M_i is $\oplus$-g-radical supplemented for every $i = 1, 2, \ldots, n$, then M is also $\oplus$-g-radical supplemented.*

Proof. Clear from Lemma 2. □

Proposition 1. *Let M be a $\oplus$-g-radical supplemented R-module. If $\mathrm{Rad}_g(M) \ll_g M$, then M is $\oplus$-g-supplemented.*

Proof. Let $U \leq M$. Since M is $\oplus$-g-radical supplemented, U has a g-radical supplement V that is a direct summand in M. Here $M = U + V$ and $U \cap V \leq \mathrm{Rad}_g(V)$. By Lemma 1, $\mathrm{Rad}_g(V) \leq \mathrm{Rad}_g(M) \ll_g M$. Since V is a direct summand of M, we can see that $\mathrm{Rad}_g(V) \ll_g V$. Hence $U \cap V \ll_g V$ and V is a g-supplement of U in M. Therefore, M is $\oplus$-g-supplemented. □

Proposition 2. *Let M be a $\oplus$-g-radical supplemented R-module. If $\mathrm{Rad}_g(M) \ll M$, then M is $\oplus$-supplemented.*

Proof. Similar to proof of Proposition 1. □

Lemma 3. *Let $M = M_1 \oplus M_2$ and $X, Y \leq M_2$. Then Y is a g-radical supplement of X in M_2 if and only if Y is a g-radical supplement of $M_1 + X$ in M.*

Proof.

$\Longrightarrow$: Since Y is a g-radical supplement of X in M_2, $M_2 = X + Y$ and $X \cap Y \leq \mathrm{Rad}_g(Y)$. Then $M = M_1 + M_2 = M_1 + X + Y$ and by Modular Law, $(M_1 + X) \cap Y = (M_1 + X) \cap M_2 \cap Y = (M_1 \cap M_2 + X) \cap Y = (0 + X) \cap Y = X \cap Y \leq \mathrm{Rad}_g(Y)$. Hence Y is a g-radical supplement of $M_1 + X$ in M.

$\Longleftarrow$: Since Y is a g-radical supplement of $M_1 + X$ in M, $M = M_1 + X + Y$ and $(M_1 + X) \cap Y \leq \mathrm{Rad}_g(Y)$. Then by Modular Law, $M_2 = M_2 \cap M = M_2 \cap (M_1 + X + Y) = M_1 \cap M_2 + X + Y = 0 + X + Y = X + Y$ and $X \cap Y \leq (M_1 + X) \cap Y \leq \mathrm{Rad}_g(Y)$. Hence Y is a g-radical supplement of X in M_2.

□

Proposition 3. *Let M be a $\oplus$-g-radical supplemented module. If every g-radical supplement submodule in M is a direct summand of M, then every direct summand of M is $\oplus$-g-radical supplemented.*

Proof. Let N be a direct summand of M and $M = N \oplus K$ with $K \leq M$. Since M is g-radical supplemented, by [7, Lemma 9], M/K is g-radical supplemented. By $\frac{M}{K} = \frac{N \oplus K}{K} \cong \frac{N}{N \cap K} = \frac{N}{0} \cong N$, N is also g-radical supplemented. Let $X \leq N$ and Y be a g-radical supplement of X in N. Since $M = N \oplus K$, by Lemma 3, Y is a g-radical supplement of $K + X$ in M. Since every g-radical supplement submodule in M is a direct summand of M, Y is a direct summand of M. By $Y \leq N$, Y is also a direct summand of N. Hence N is $\oplus$-g-radical supplemented. □

Lemma 4. *Let M be a $\oplus$-g-radical supplemented R-module and $K \leq M$. If $\frac{X+K}{K}$ is a direct summand of $\frac{M}{K}$ for every direct summand X of M, then $\frac{M}{K}$ is $\oplus$-g-radical supplemented.*

Proof. Let U/K be any submodule of M/K. Since M is $\oplus$-g-radical supplemented, U has a g-radical supplement X in M that is a direct summand in M. Since X is a g-radical supplement of U in M and $K \leq U$, by [7, Lemma 8], $\frac{X+K}{K}$ is a g-radical supplement of U/K in M/K. Since X is a direct summand of M, by hypothesis, $\frac{X+K}{K}$ is a direct summand of M/K. Hence M/K is $\oplus$-g-radical supplemented. □

Lemma 5. *Let M be a distributive and $\oplus$-g-radical supplemented R-module. Then every factor module of M is $\oplus$-g-radical supplemented.*

Proof. Let $K \leq M$ and X be a direct summand of M. Since X is a direct summand of M, there exists $Y \leq M$ such that $M = X \oplus Y$. Since $M = X \oplus Y$, $\frac{M}{K} = \frac{X+K}{K} + \frac{Y+K}{K}$. Since M is distributive, $(X+K) \cap (Y+K) = K$ and $\frac{X+K}{K} \cap \frac{Y+K}{K} = \frac{(X+K)\cap(Y+K)}{K} = \frac{K}{K} = 0$. Hence $\frac{M}{K} = \frac{X+K}{K} \oplus \frac{Y+K}{K}$ and by Lemma 4, M/K is $\oplus$-g-radical supplemented. □

Corollary 2. *Let M be a distributive and $\oplus$-g-radical supplemented R-module. Then every homomorphic image of M is $\oplus$-g-radical supplemented.*

Proof. Clear from Lemma 5. □

Lemma 6. *Let M be an R-module and K be a direct summand of M. Then* $\mathrm{Rad}_g(K) = K \cap \mathrm{Rad}_g(M)$.

Proof. Since K is a direct summand of M, there exists $T \leq M$ such that $M = K \oplus T$. By Lemma 1, $\mathrm{Rad}_g(M) = \mathrm{Rad}_g(K) \oplus \mathrm{Rad}_g(T)$. Then by Modular Law, $K \cap \mathrm{Rad}_g(M) = K \cap (\mathrm{Rad}_g(K) \oplus \mathrm{Rad}_g(T)) = \mathrm{Rad}_g(K) \oplus K \cap \mathrm{Rad}_g(T) = \mathrm{Rad}_g(K) + 0 = \mathrm{Rad}_g(K)$, as desired. □

Lemma 7. *Let M be a $\oplus$-g-radical supplemented R-module with $(D3)$ property. Then every direct summand of M is $\oplus$-g-radical supplemented.*

Proof. Let K be any direct summand of M. Then there exists $T \leq M$ such that $M = K \oplus T$. Let $U \leq K$. Since M is $\oplus$-g-radical supplemented, U has a g-radical supplement X that is a direct summand in M. Here $M = U + X$ and $U \cap X \leq \mathrm{Rad}_g(X)$. Since $U \leq K$, $M = U + X = K + X$ and since M has $(D3)$ property, $K \cap X$ is a direct summand of M. Then there exists $Y \leq M$ such that $M = (K \cap X) \oplus Y$. Here $K = (K \cap X) \oplus (K \cap Y)$. Since $M = U + X$ and $U \leq K$, by Modular Law, $K = U + (K \cap X)$. Since $U \cap X \leq \mathrm{Rad}_g(X) \leq \mathrm{Rad}_g(M)$, $U \cap K \cap X \leq K \cap X \cap \mathrm{Rad}_g(M)$. Since $K \cap X$ is a direct summand of M, by Lemma 6, $U \cap K \cap X \leq K \cap X \cap \mathrm{Rad}_g(M) = \mathrm{Rad}_g(K \cap X)$. Hence $K \cap X$ is a g-radical supplement of U that is a direct summand in K. Therefore, K is $\oplus$-g-radical supplemented. □

Corollary 3. *Let M be a $\oplus$-g-radical supplemented R-module with $(D3)$ property. Then M/X is $\oplus$-g-radical supplemented for every direct summand X of M.*

Proof. Let X be any direct summand of M. Then there exists $Y \leq M$ such that $M = X \oplus Y$. By Lemma 7, Y is $\oplus$-g-radical supplemented. Then by $\frac{M}{X} = \frac{X+Y}{X} \cong \frac{Y}{X \cap Y} = \frac{Y}{0} \cong Y$, M/X is also $\oplus$-g-radical supplemented. □

Corollary 4. *Let M be a $\oplus$-g-radical supplemented R-module with $(D3)$ property and $f\colon M \to N$ be an R-module epimorphism with N is an R-module and* $\mathrm{Ker}(f)$ *is a direct summand of M. Then N is $\oplus$-g-radical supplemented.*

Proof. Clear from Corollary 3, since $M/\mathrm{Ker}(f) \cong \mathrm{Im}(f) = N$. □

Lemma 8. *Let M be a $\oplus$-g-radical supplemented R-module, $K \leq M$ and $K = (K \cap M_1) \oplus (K \cap M_2)$ for every $M_1, M_2 \leq M$ with $M = M_1 \oplus M_2$. Then M/K is $\oplus$-g-radical supplemented.*

Proof. Let $U/K \leq M/K$. Since M is $\oplus$-g-radical supplemented, U has a g-radical supplement V that is a direct summand in M. Here there exists $X \leq M$ such that $M = V \oplus X$. By hypothesis, $K = (K \cap V) \oplus (K \cap X)$. Since V is a g-radical supplement of U in M and $K \leq U$, by [7, Lemma 8], $\frac{V+K}{K}$ is a g-radical supplement of U/K in M/K. Since $M = V \oplus X$, $\frac{M}{K} = \frac{V+K}{K} + \frac{X+K}{K}$. Here $\frac{V+K}{K} \cap \frac{X+K}{K} = \frac{(V+K)\cap(X+K)}{K} = \frac{(V+K)\cap X+K}{K} = \frac{(V+K\cap V+K\cap X)\cap X+K}{K} = \frac{(V+K\cap X)\cap X+K}{K} = \frac{V\cap X+K\cap X+K}{K} = \frac{0+K}{K} = \frac{K}{K} = 0$. Hence $\frac{M}{K} = \frac{V+K}{K} \oplus \frac{X+K}{K}$. Thus M/K is $\oplus$-g-radical supplemented. □

Corollary 5. *Let M be a $\oplus$-g-radical supplemented R-module, $f\colon M \to N$ be an R-module epimorphism with N be an R-module and* $\mathrm{Ker}(f) = (\mathrm{Ker}(f) \cap M_1) \oplus (\mathrm{Ker}(f) \cap M_2)$ *for every $M_1, M_2 \leq M$ with $M = M_1 \oplus M_2$. Then N is $\oplus$-g-radical supplemented.*

Proof. Clear from Lemma 8, since $M/\mathrm{Ker}(f) \cong \mathrm{Im}(f) = N$. □

Lemma 9. *Let M be a $\oplus$-g-radical supplemented R-module with SSP property. Then M/K is $\oplus$-g-radical supplemented for every direct summand K of M.*

Proof. Let K be any direct summand of M and $U/K \leq M/K$. Since M is $\oplus$-g-radical supplemented, U has a g-radical supplement V in M such that V is a direct summand of M. By [7, Lemma 8], $\frac{V+K}{K}$ is a g-radical supplement of U/K in M/K. Since K and V are direct summands of M and M has *SSP* property, $K+V$ is also a direct summand of M. Hence there exists $T \leq M$ such that $M = (K+V) \oplus T$. Since $M = (K+V) \oplus T$, $\frac{M}{K} = \frac{K+V+T}{K} = \frac{V+K}{K} + \frac{T+K}{K}$. Since $(V+K) \cap T = 0$, $\frac{V+K}{K} \cap \frac{T+K}{K} = \frac{(V+K)\cap(T+K)}{K} = \frac{(V+K)\cap T+K}{K} = \frac{0+K}{K} = 0$. Hence $\frac{M}{K} = \frac{V+K}{K} \oplus \frac{T+K}{K}$ and M/K is $\oplus$-g-radical supplemented. □

Corollary 6. *Let M be a $\oplus$-g-radical supplemented R-module with SSP property. Then every direct summand of M is $\oplus$-g-radical supplemented.*

Proof. Let T be any direct summand of M. Then there exists a submodule K of M such that $M = T \oplus K$. By Lemma 9, M/K is $\oplus$-g-radical supplemented. Since $\frac{M}{K} = \frac{T+K}{K} \simeq \frac{T}{T\cap K} = \frac{T}{0} \cong T$, T is also $\oplus$-g-radical supplemented. □

Remark 1. Let M be an R-module which has only four proper submodules 0, A, B, C with $C \leq A$, $C \leq B$, $A \nleq B$ and $B \nleq A$. Then M is g-radical supplemented but not $\oplus$-g-radical supplemented.

Example 1. Consider the $\mathbb{Z}$-module $\mathbb{Q}$. Since $\mathrm{Rad}_g(\mathbb{Q}) = \mathrm{Rad}(\mathbb{Q}) = \mathbb{Q}$, ${}_{\mathbb{Z}}\mathbb{Q}$ is $\oplus$-g-radical supplemented. But, since ${}_{\mathbb{Z}}\mathbb{Q}$ is not supplemented and every nonzero submodule of ${}_{\mathbb{Z}}\mathbb{Q}$ is essential in ${}_{\mathbb{Z}}\mathbb{Q}$, ${}_{\mathbb{Z}}\mathbb{Q}$ is not g-supplemented and hence ${}_{\mathbb{Z}}\mathbb{Q}$ is not $\oplus$-g-supplemented.

Example 2. Consider the $\mathbb{Z}$-module $\mathbb{Q} \oplus \mathbb{Z}_{p^2}$ for a prime p. It is easy to check that $\mathrm{Rad}_g(\mathbb{Z}_{p^2}) \neq \mathbb{Z}_{p^2}$. By Lemma 1, $\mathrm{Rad}_g\left(\mathbb{Q} \oplus \mathbb{Z}_{p^2}\right) = \mathrm{Rad}_g(\mathbb{Q}) \oplus \mathrm{Rad}_g(\mathbb{Z}_{p^2}) \neq \mathbb{Q} \oplus \mathbb{Z}_{p^2}$. Since $\mathbb{Q}$ and $\mathbb{Z}_{p^2}$ are $\oplus$-g-radical supplemented, by Lemma 2, $\mathbb{Q} \oplus \mathbb{Z}_{p^2}$ is $\oplus$-g-radical supplemented. But $\mathbb{Q} \oplus \mathbb{Z}_{p^2}$ is not $\oplus$-g-supplemented.

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NEW FIXED POINT RESULTS OF SOME ENRICHED CONTRACTIONS IN CAT(0) SPACES

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Abstract. Enriched contraction, a class that contains the Picard -Banach contractions and some nonexpansive mappings, are generalized from Banach space framework into a nonlinear context, namely, geodesic metric spaces of nonpositive curvature. We establish general implications that extend the well-known results for enriched mappings formed on Banach spaces. We also look at the results for fixed points involving the contractions i.e., the limit shadowing property and well-posedness as well as some instances to back up our findings.

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1. Introduction

In the analysis of the solutions to nonlinear functional systems, fixed point theory provides useful tools. The desired solution of a functional equation is written as the fixed point of an appropriate operator i.e.,

$$a = Sa,$$

where S is a self map defined on a nonempty set E.

A variety of contractions have been proposed by a number of mathematicians and employed them in a variety of frameworks to produce fixed points, common fixed points, and coincidence points for a map. Enriched contraction is one such contraction, which was very recently introduced by Berinde and Pacurar [2]. First, let's review the notion of enriched contraction.

"Let S be a self map on a normed linear space E. S is said to be an enriched contraction if there exist $b^* \in [0,\infty)$ and $\theta \in [0, b^*+1)$ such that

$$\|b^*(a-b) + Sa - Sb\| \leq \theta\|a-b\|$$

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for all $a,b \in E$. In this situation the map S is also known as a (b^*,θ)-enriched contraction." Picard-Banach contractions along with several nonexpansive maps, are included in the enriched contractions category.

Many of the contractions used to solve fixed point problems have gradually been generalized from linear spaces to differentiable manifolds. We continue along these lines by introducing symmetric contractions in the context of CAT(0) spaces. In addition, Mondal et al. [9] published a work presenting two different kinds of enriched contractions in Banach spaces.

The goal of this paper is to introduce enriched contractions into nonpositive curvature metric spaces. Fixed point results of enriched contractions are next investigated, and strong convergence theorems for the Kransnoselskii iteration which is adapted to approximate fixed points, are established. Examples are also given to demonstrate the generality of our new findings.

2. Preliminaries

Considering a self map S on a convex subset C of a linear space E, then for any $\lambda \in (0,1)$, the so-called averaged mapping (a term coined in [1]) S_λ given by

$$S_\lambda a = \lambda a + (1-\lambda)Sa, \qquad \forall a \in C.$$

Lemma 1. *[4] Let (E, d) be a CAT(0) space. Then*

$$d((1-\lambda)a \oplus \lambda b, c) \leq (1-\lambda)d(a,c) + \lambda d(b,c)$$

for all $a,b,c \in E$ and $\lambda \in [0,1]$.

Proposition 1. *[4] Let (E, d) be a CAT(0) space. For each $\lambda \in [0,1]$ there is a unique point $c \in [a,b]$ such that*

$$d(a,c) = \lambda d(a,b), \qquad d(b,c) = (1-\lambda)d(a,b),$$

for all $a,b \in E$.

The following lemma is a partial extension from Banach space to CAT(0) space setting of a conclusion presented in Corollary to Theorem 5 in [5] (it can be also found in [10]).

Lemma 2. *Assume that S is a self map on CAT(0) space E. Define the map $S_\lambda a : E \to E$ by*

$$S_\lambda a = \lambda a \oplus (1-\lambda)Sa, \qquad a \in E$$

Then, for any $\lambda \in [0,1)$,

$$Fix(S) = Fix(S_\lambda).$$

Proof. $S_\lambda = S$ when $\lambda = 0$, and the claim is simple. Consider that $\lambda \in (0,1)$ and that $a \in Fix(S)$. This suggests $a = Sa$ and as a result

$$d(a, S_\lambda a) = d(a, \lambda a \oplus (1-\lambda)Sa) \leq \lambda d(a,a) + (1-\lambda)d(a,Sa) = 0.$$

i.e., $d(a, S_\lambda a) = 0 \Rightarrow a \in Fix(S_\lambda)$.

Conversely, assume $a \in Fix(S_\lambda)$. This implies that $d(a, S_\lambda a) = 0$, so,

$$d(a, \lambda a \oplus (1-\lambda)Sa) = 0.$$

By Proposition 1,

$$d(a, Sa) = \lambda d(a, Sa)$$
$$d(a, Sa)(1-\lambda) = 0$$

given that $\lambda \neq 1$, this means that $d(a, Sa) = 0$. □

3. Enriched contractions in CAT(0) space

Definition 1. Let (E, d) be a CAT(0) space. If there exist $c^* \in [0,1)$ and $\lambda \in [0,1)$, then a self mapping S defined on E is an enriched contraction such that

$$d(\lambda a \oplus (1-\lambda)Sa, \lambda b \oplus (1-\lambda)Sb) \leq c^* d(a,b), \forall a, b \in E.$$

(1) S is also known as a (λ, c^*)-enriched contraction because it specifies the parameters c^* and λ. "A $(0, c^*)$-enriched contraction is a usual Banach contraction which can be easily seen."

(2) Let $E = [0,1] \times [0,1]$. Define the radial distance d_r between $a, b \in E$ to be the usual distance if they are on the same ray emanating from origin; otherwise take

$$d_r(a,b) = d(a,0) + d(0,b).$$

(Here d denotes the usual Euclidean distance and 0 denotes the origin.) Note that d_r is a metric on E and is known as a radial metric on E ([8]). Let S be a self map on E defined as $Sa = -a$, for all $a \in E$. Then S is an enriched contraction, not a contraction.

However, if S is a contraction, then there would exist $c^* \in [0,1)$ by the definition of contraction, such that

$$d(Sa, Sb) = d(-a, -b) = \|b - a\| = d(a,b) \leq c^*.d(a,b), \forall a, b \in [0,1] \times [0,1].$$

This leads to the contradiction $1 \leq c^* < 1$ for any $a \neq b$. Take $a, b \in E$ and $\lambda = \frac{1}{2}$. If a and b are on the same ray emanating from the origin, the enriched contraction condition is identical to

$$d\left(\frac{1}{2}a + \frac{1}{2}Sa, \frac{1}{2}b + \frac{1}{2}Sb\right) \leq c^* d(a,b).$$

Now we show that the above inequality holds for all $a, b \in E$.

$$d\left(\frac{1}{2}a + \frac{1}{2}Sa, \frac{1}{2}b + \frac{1}{2}Sb\right) = \frac{1}{2}\|a + Ta - b - Tb\| = \frac{1}{2}\|a - b + Ta - Tb\|$$
$$= \frac{1}{2}\|a - b - a + b\| = 0 \leq c^* d(a,b),$$

which is true. Now, if a and b are not on the same ray emanating from the origin, then enriched contraction condition becomes:

$$d_r\left(\frac{1}{2}a+\frac{1}{2}Sa,\frac{1}{2}b+\frac{1}{2}Sb\right)\leq c^*d_r(a,b),$$

which can equivalently be written :

$$\begin{aligned} d\left(\frac{1}{2}a+\frac{1}{2}Sa,0\right)+d\left(0,\frac{1}{2}b+\frac{1}{2}Sb\right) &\leq c^*d(a,0)+d(0,b)\\ \frac{1}{2}(|a+Sa|+|b+Sb|) &\leq c^*|a|+|b|\\ \Rightarrow \qquad \frac{1}{2}(|a+(-a)|+|b+(-b)|)=0 &\leq c^*|a|+|b|, \end{aligned}$$

which shows that S is an enriched contraction.

Theorem 1. *A self map S defined on CAT(0) space E is a (λ,c^*)-enriched contraction. Then,*

(a): *$Fix(S)=\{p\}$, for some $p\in E$.*

(b): *The sequence $\{a_n\}$ obtained from the Krasnoselskii iterative method [3]*

$$a_{n+1}=\lambda a_n\oplus(1-\lambda)Ta_n, n\geq 0, \tag{3.1}$$

converges to p, for any $a_0\in E$.

(c): *Also, the following holds*

$$d(a_{n+i-1},p)\leq\frac{(c^*)^i}{1-(c^*)}\cdot d(a_n,a_{n-1}) \qquad n=1,2,\ldots;i=1,2,\ldots$$

Proof. The mapping $S_\lambda:E\to E$ described by $S_\lambda a=(1-\lambda)a\oplus\lambda Sa$ satisfies enriched contractive condition, so

$$d(S_\lambda a,S_\lambda b)\leq c^*\cdot d(a,b), \tag{3.2}$$

for all $a,b\in E$, i.e., S_λ is a contraction. We also notice that the Picard iteration linked with S_λ is the Krasnoselskii iterative process $\{a_n\}$ linked with S and described by (3.2), i.e.,

$$a_{n+1}=S_\lambda a_n, n\geq 0. \tag{3.3}$$

Now, we take $a=a_n$ and $b=a_{n-1}$ in (3.2) to get

$$d(a_{n+1},a_n)\leq c^*\cdot d(a_n,a_{n-1}), n\geq 1, \tag{3.4}$$

which inductively implies

$$d(a_{n+1},a_n)\leq (c^*)^n\cdot d(a_1,a_0), n\geq 1, \tag{3.5}$$

We can derive from $c^*\in[0,1)$ that

$$\lim_{n\to\infty}d(a_{n+1},a_n)=0, \tag{3.6}$$

i.e., $\{a_n\}$ is asymptotically regular. for all n and $k > 0$, we have the following using the triangle inequality and (3.2),

$$\begin{aligned} d(a_{n+k+1}, a_{n+1}) &\leq c^* \cdot d(a_{n+k}, a_n) \\ &\leq c^*[d(a_{n+k}, a_{n+k+1}) + d(a_{n+k+1}, a_{n+1}) + d(a_{n+1}, a_n)] \\ &\leq \frac{c^*}{1-c^*}[d(a_{n+k}, a_{n+k+1}) + d(a_{n+1}, a_n)]. \end{aligned}$$

As a result of (3.6), $\lim_{n\to\infty} d(a_{n+k+1}, a_{n+1}) = 0$, uniformly w.r.t. k, demonstrating that $\{a_n\}$ is Cauchy sequence. Hence, $\{a_n\}$ is convergent and let us represent

$$p = \lim_{n\to\infty} a_n. \tag{3.7}$$

By taking $n \to \infty$ in (3.3) and employing the continuity of S_λ (which follows by the fact that S_λ is contraction), we obtain immediately

$$p = S_\lambda p,$$

i.e., $p \in Fix(S_\lambda)$.

Next, we show that p is the only fixed point of S_λ. Suppose that $q \neq p$ is yet another fixed point of S_λ. After that, by (3.2),

$$0 < d(p,q) = d(S_\lambda p, S_\lambda q) \leq c \cdot d(p,q) < d(p,q),$$

This is a contradiction. As a result, $Fix(S_\lambda) = \{p\}$ and, as a result of Lemma 2, $p \in Fix(S)$, which demonstrates (a).

(3.7) proves the conclusion (b).

To prove (c), first by (3.4) and (3.5), The following estimates are obtained commonly,

$$d(a_{n+m}, a_n) \leq (c^*)^n \cdot \frac{(1-(c^*)^m)}{1-c^*} \cdot d(a_1, a_0), \qquad n \geq 1, m \geq 1. \tag{3.8}$$

$$d(a_{n+m}, a_n) \leq \frac{c^*}{1-c^*} \cdot d(a_n, a_{n-1}), \qquad n \geq 1, m \geq 1. \tag{3.9}$$

By taking $m \to \infty$ in (3.8) and (3.9), we obtain

$$d(a_n, p) \leq \frac{(c^*)^n}{1-c^*} \cdot d(a_1, a_0), n \geq 1 \tag{3.10}$$

and

$$d(a_n, p) \leq \frac{c^*}{1-c^*} \cdot d(a_n, a_{n-1}), n \geq 1, \tag{3.11}$$

respectively. Finally, (3.10) and (3.11) can be combined to obtain the unifying error estimate in conclusion (c). □

The following example support the above theorem:

Example 1. Let $E = [0,1]$ and consider the usual metric d on E. Define a function $S : E \to E$ by

$$Sa = 1 - a$$

for all $a \in [0,1]$. Here, S is an enriched contraction. We note that Picard iteration $\{x_n\}$ defined by $a_{n+1} = Sa_n$ does not converge, for any initial value a_0 except the unique fixed point $\frac{1}{2}$ of S while Krasnoselskij iteration $\{x_n\}$ defined by (3.1) converges for any $\lambda \in (0,1)$ and any initial value $a_0 \in [0,1]$. We also observe that Krasnoselskij iteration converges faster for λ closer to but different from the unique fixed point $\frac{1}{2}$. The fact is illustrated by some numerical experiments performed for the starting value $a_0 = 1$ and for four different values of the control parameter $\lambda, \lambda \in \{0.1, 0.25, 0.3, 0.49\}$ which are presented in Table 1. Also graphical representation is also given in Figure 1

TABLE 1. **Results of the numerical experiments for** $Ta = 1 - a$

$\lambda = 0.1$	$\lambda = 0.25$	$\lambda = 0.3$	$\lambda = 0.49$
1	1	1	1
0.1	0.25	0.3	0.49
0.82	0.625	0.58	0.5002
0.244	0.4375	0.468	0.499996
0.7048	0.53125	0.5128	0.50000008
0.33616	0.484375	0.49488	0.499999998
0.631072	0.5078125	0.502048	0.5
0.3951424	0.49609375	0.4991808	0.5
0.58388608	0.501953125	0.50032768	0.5
0.432891136	0.499023438	0.499868928	0.5
0.553687091	0.500488281	0.500052429	0.5
0.457050327	0.499755859	0.499979028	0.5
0.534359738	0.50012207	0.500008389	0.5
0.472512209	0.499938965	0.499996645	0.5
0.521990233	0.500030518	0.500001342	0.5
0.482407814	0.499984741	0.499999463	0.5
0.514073749	0.500007629	0.500000215	0.5
0.488741001	0.499996185	0.499999914	0.5
0.509007199	0.500001907	0.500000034	0.5
0.492794241	0.499999046	0.499999986	0.5
0.505764608	0.500000477	0.500000005	0.5
0.495388314	0.499999762	0.499999998	0.5
0.503689349	0.500000119	0.500000001	0.5
0.497048521	0.49999994	0.5	0.5

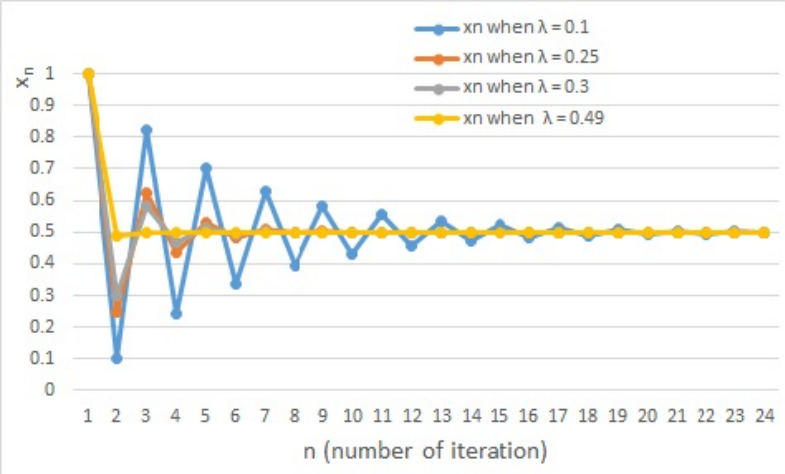

FIGURE 1. Convergence of x_n to the unique fixed point $\frac{1}{2}$ for different values of λ.

Remark 1. The contraction map principle in the context of a CAT(0) space is obtained by Theorem 1 if S is a $(0,c^*)$-contraction. The Krasnoselskii type iterative method simplifies to the Picard iteration in this case.

"The local variant of Banach contraction mapping principle (see, [6]), which involves an open ball B in a complete metric space (E,d) and a nonself contraction map of B into E with the key property of not displacing the center of the ball too far, is important in actual applications." The following theorem is an analogue of this conclusion in the case CAT(0) spaces.

Theorem 2. *Consider (E,d) is a complete CAT(0) space, $a_0 \in E, r > 0, B = B(a_0,r) := \{a \in E : d(a,a_0) < r\}$ and also, $S : B \to E$ is a (λ,c^*)-enriched contraction and*

$$d(Sa_0,a_0) < \frac{1-c^*}{1-\lambda} \cdot r,$$

then S has a fixed point.

Proof. We can select $\varepsilon < r$ in such a way that

$$d(Sa_0,a_0) < \frac{1-c^*}{1-\lambda} \cdot \varepsilon < \frac{1-c^*}{1-\lambda} \cdot r. \tag{3.12}$$

Since S is a (λ,c^*)-enriched contraction, there exists $c^* \in [0,1)$ such that

$$d(S_\lambda a, S_\lambda b) \le c^* d(a,b), \qquad \text{for all } a,b \in B,$$

for any $\lambda \in (0,1)$, where we denote as before $S_\lambda a = (1-\lambda)a \oplus \lambda Sa$.

By Proposition 1, we have

$$d(S_\lambda a_0, a_0) = d((1-\lambda)a_0 \oplus \lambda S a_0, a_0) = (1-\lambda)d(a_0, Sa_0)$$

and therefore (3.12) implies that $d(S_\lambda a_0, a_0) \leq (1-c^*)\varepsilon$.

We now prove that the closed ball $\overline{B_\varepsilon} := \{a \in E : d(a, a_0) \leq \varepsilon\}$ is invariant with respect to S_λ. Indeed, for any $a \in \overline{B_\varepsilon}$, we have

$$d(S_\lambda a, a_0) \leq d(S_\lambda a, S_\lambda a_0) + d(S_\lambda a_0, a_0) \leq c^* d(a, a_0) + (1-c^*)\varepsilon \leq \varepsilon.$$

Since $\overline{B_\varepsilon}$ is complete, the conclusion follows by Theorem 1. □

The following example demonstrates that while some mappings S are not contractions, a certain iterate of them is.

Example 2. Let $E = \mathbb{R}$ and $S : E \to E$ be given by

$$Sa = \begin{cases} 0, & \text{if} \quad a \in (-\infty, 3] \\ \frac{-1}{2} & \text{if} \quad a \in (3, \infty). \end{cases}$$

Then S is not a contraction as it is discontinuous at $a = 3$. But S^2 is a contraction as $S^2 a = 0$, if $a \in (-\infty, \infty)$.

We cannot use Theorem 1 in these instances, but the following result could be beneficial in applications.

Theorem 3. *Let (E, d) be a complete CAT(0) space and let V be a self mapping defined on E with the property that there exists a positive integer N such that V^N is a (λ, c^*)-enriched contraction. Then,*

(a): $Fix(V) = \{p\}$
(b): *The sequence $\{a_n\}$ obtained from the iterative process*

$$a_{n+1} = \lambda a_n \oplus (1-\lambda) S^N a_n, n \geq 0,$$

converges to p, for any $a_0 \in E$.

Proof. We apply Theorem 1 (a) for the mapping $S = V^N$ and obtain that $Fix(V^N) = \{p\}$. We also have

$$V^N(V(p)) = V^{N+1}(p) = V(V^N(p)) = V(p),$$

which shows that $V(p)$ is a fixed point of V^N. However, V^N has a unique fixed point, p, hence $V(p) = p$ and so $p \in Fix(V)$. The remaining part of the proof follows by Theorem 1 □

4. Enriched $\mathcal{A}$-contraction and enriched $\mathcal{A}'$-contraction in CAT(0) spaces

Mondal et al. [9] proposed two new forms of enriched contractions, and found fixed points of mappings meeting such contractions using the fixed point property of the average operator of a map . We will introduce enriched $\mathcal{A}$-contraction and enriched $\mathcal{A}'$-contraction in the context of CAT(0) space in this section, as follows.

Definition 2. "Let (E,d) be a CAT(0) space. Let $\mathcal{A}$ be the collection of all mappings $f:\mathbb{R}_+^3\to\mathbb{R}$ satisfying the following conditions:

($\mathcal{A}_1$): f is continuous;
($\mathcal{A}_2$): if $r\le f(s,r,s)$ or $r\le f(r,s,s)$, then there exists $k\in[0,1)$ such that $r\le ks$;
($\mathcal{A}_3$): for $\lambda>0$ and for all $r,s,t\in\mathbb{R}_+, \lambda f(r,s,t)\le f(\lambda r,\lambda s,\lambda t)$.

Let $S:E\to E$ be a mapping such that there exist $f\in\mathcal{A}$ with

$$d(S_\lambda a,S_\lambda b)\le \lambda f((b^*+1)d(a,b),d(a,Sa),d(b,Sb)) \tag{4.1}$$

for all $a,b\in E$ with $a\neq b$ and $b^*\in[0,\infty)$. Then S is said to be an enriched $\mathcal{A}$-contraction."

Example 3. Let $E=[0,1]\times[0,1]$. Define the radial distance d_r between $a,b\in E$ to be the usual distance if they are on the straight ray initiating from origin; otherwise take

$$d_r(a,b)=d(a,0)+d(0,b).$$

(Here d denotes the usual Euclidean distance and 0 denotes the origin.) Let S be a self map on E defined as $Sa=-2a$, for all $a\in E$. S is not an enriched contraction but is an enriched $\mathcal{A}$-contraction.

Indeed, if S would be an enriched contraction, then there would exist $b^*\in[0,+\infty)$ and $\theta\in[0,b^*+1)$ such that $\forall\, a,b\in E$,

$$\begin{aligned} d(\lambda a\oplus(1-\lambda)Sa,\lambda b\oplus(1-\lambda)Sb)&\le\theta d(a,b)\\ \|b^*(a-b)+Sa-Sb\|&\le\theta\|(a-b)\|.\end{aligned}$$

Take $\lambda=0$ and then $\theta\in[0,1)$,

$$\begin{aligned} d(Sa,Sb)&\le\theta d(a,b)\\ \|Sa-Sb\|&\le\theta\|(a-b)\|\\ \|-2a+2b\|&\le\theta\|(a-b)\|\\ 2\|a-b\|&\le\theta\|(a-b)\|,\end{aligned}$$

which yields to the contradiction $2\le 1$.

S is an enriched $\mathcal{A}$-contraction for $b^*=\frac{5}{4}$, $\lambda=\frac{1}{2}$ and $f(r,s,t)=\frac{s+t}{3}$. Assume $a,b\in E$ and a,b are on the straight ray initiating from origin, then the contractive

condition becomes;

$$d\left(\frac{1}{2}a+\frac{1}{2}Sa,\frac{1}{2}b+\frac{1}{2}Sb\right)\leq\frac{1}{2}f\left((b^*+1)d(a,b),d(a,Sa),d(b,Sb)\right),$$

which can be written as

$$\begin{aligned}\frac{1}{2}\|a-2a+b-2b\|&\leq\frac{1}{2}f\left((b^*+1)\|a-b\|,\|a-Sa\|,\|b-Sb\|\right)\\ \frac{1}{2}\|-a-b\|&\leq\frac{1}{2}\times\frac{1}{3}\left[\|a-Sa\|+\|b-Sb\|\right]\\ \frac{1}{2}\|a+b\|&\leq\frac{1}{6}\left[\|a+2a\|+\|b+2b\|\right]\\ \frac{1}{2}\|a+b\|&\leq\frac{3}{6}\left[\|a\|+\|b\|\right]=\frac{1}{2}\left[\|a\|+\|b\|\right],\end{aligned}$$

which is true.

Now, if a and b are not on the straight ray initiating from the origin, then the contractive condition becomes;

$$d_r\left(\frac{1}{2}a+\frac{1}{2}Sa,\frac{1}{2}b+\frac{1}{2}Sb\right)\leq\frac{1}{2}f\left((b^*+1)d_r(a,b),d_r(a,Sa),d_r(b,Sb)\right).$$

It gives that

$$\begin{aligned}d\left(\frac{1}{2}a+\frac{1}{2}Sa,0\right)d\left(0,\frac{1}{2}b+\frac{1}{2}Sb\right)&\leq\frac{1}{2}f\left((b^*+1)d(a,0)+d(0,b),d(a,0)\right.\\ &\qquad\left.+d(0,Sa),d(b,0)+d(0,Sb)\right)\\ \left\|\frac{1}{2}a+\frac{1}{2}Ta\right\|+\left\|\frac{1}{2}b+\frac{1}{2}Sb\right\|&\leq\frac{1}{6}[d(a,0)+d(0,Ta)+d(b,0)+d(0,Sb)]\\ \frac{1}{2}\|a-2a\|+\frac{1}{2}\|b-2b\|&\leq\frac{1}{6}[\|a\|+\|2a\|+\|b\|+\|2b\|]\\ \frac{1}{2}[\|a\|+\|b\|]&\leq\frac{3}{6}[\|a\|+\|b\|]=\frac{1}{2}[\|a\|+\|b\|],\end{aligned}$$

that is true. This shows that S is an enriched $\mathcal{A}$-contraction.

Theorem 4. *Let (E,d) be a complete CAT(0) space and S be an enriched $\mathcal{A}$-contraction. Then S has a unique fixed point in E and there exists $\lambda\in(0,1]$ such that the sequence $\{a_n\}$ defined by $a_{n+1}=\lambda a_n\oplus(1-\lambda)Sa_n$, $n\geq 0$ converges to that fixed point , for any $a_0\in E$.*

Proof. Let $b^*>0$. Set $\lambda=\frac{1}{b^*+1}>0$. Then

$$\begin{aligned}d(S_\lambda a,S_\lambda b)&\leq\lambda f(\frac{1}{\lambda}d(a,b),d(a,Sa),d(b,Sb))\\ &\leq f(d(a,b),\lambda d(a,Sa),\lambda d(b,Sb))\\ d(S_\lambda a,S_\lambda b)&\leq f(d(a,b),d(a,S_\lambda a),d(b,S_\lambda b))\end{aligned}\tag{4.2}$$

Let $a_0 \in E$ be arbitrary. Define $a_n = S_\lambda^n a_0$ for all $n \geq 1$. Let us put $a = a_n$ and $y = a_{n-1}$ in (4.2). Then

$$d(a_{n+1}, a_n) \leq f(d(a_n, a_{n-1}), d(a_n, a_{n+1}), d(a_{n-1}, a_n))$$

which implies that

$$d(a_{n+1}, a_n) \leq k.d(a_n, a_{n-1})$$

for some $k \in [0,1]$. From this we get

$$d(a_{n+1}, a_n) \leq k^n.d(a_1, a_0).$$

Now for all $m, n \geq 1$, we have

$$\begin{aligned} d(a_{n+m}, a_n) &\leq d(a_{n+m}, a_{n+m-1}) + d(a_{n+m-1}, a_{n+m-2}) + \ldots + d(a_{n+1}, a_n) \\ &\leq (k^{n+m-1} + k^{n+m-2} + \ldots + k^n) d(a_1, a_0) \\ &= k^n \left(\frac{1-k^m}{1-k} \right) d(a_1, a_0) \end{aligned}$$

This implies that $d(a_{n+m}, a_n) \to 0$ as $m, n \to \infty$. Hence $\{a_n\}$ is a Cauchy sequence in E and hence there exists an element $p \in E$ such that $a_n \to p$ as $n \to \infty$.

Now

$$\begin{aligned} d(S_\lambda p, a_{n+1}) &= d(S_\lambda p, S_\lambda a_n) \\ &\leq f(d(p, a_n), d(p, Sp), d(a_n, a_{n+1})) \\ &= f(d(p, a_n), d(p, S_\lambda p), d(a_n, a_{n+1})) \end{aligned}$$

Taking limit as $n \to \infty$, we get

$$d(S_\lambda p, p) \leq f(d(p,p), d(p, S_\lambda p), d(p,p)) \leq k \cdot d(p,p) = 0.$$

Hence, $S_\lambda p = p$ and consequently $Sp = p$. Therefore, p is a fixed point of S.

Let p^* be another fixed point of S and consequently, a fixed point of S_λ. Then

$$\begin{aligned} d(p, p^*) = d(S_\lambda p, S_\lambda p^*) &\leq f(d(p, p^*), d(p, Sp), d(p^*, Sp^*)) \\ &= f(d(p, p^*), d(p,p), d(p^*, p^*)) = f(d(p, p^*), 0, 0). \end{aligned}$$

This gives that $d(p, p^*) \leq k.0 = 0$ for some $k \in [0,1)$. Hence $p = p^*$.

If $b = 0$, then (4.1) reduces to

$$d(S_\lambda a, S_\lambda b) \leq \lambda f(d(a,b), d(a, Sa), d(b, Sb))$$

for all $a, b \in E$ with $a \neq b$ which can easily be solved. □

Definition 3. Let (E, d) be a CAT(0) space. Let $\mathcal{A}'$ be the collection of all mappings $f : \mathbb{R}_+^3 \to \mathbb{R}$ satisfying the following conditions:

- **($\mathcal{A}_1'$):** f is continuous;
- **($\mathcal{A}_2'$):** if $r \leq f(r,s,s)$ or $r \leq f(s,s,r)$, then there exists $k \in [0,1)$ such that $r \leq ks$;
- **($\mathcal{A}_3'$):** for $\lambda > 0$ and for all $r, s, t \in \mathbb{R}_+, \lambda f(r,s,t) \leq f(\lambda r, \lambda s, \lambda t)$;

($\mathcal{A}_4'$): if $t \leq t_1$, then $f(r,s,t) \leq f(r,s,t_1)$;
($\mathcal{A}_5'$): if $r \leq f(s,0,r+s)$, then $r \leq ks$ for some $k \in [0,1)$;
($\mathcal{A}_6'$): if $r \leq f(r,r,r)$ then $r = 0$.

Let $T : E \to E$ be a mapping such that there exist $f \in \mathcal{A}'$ with

$$d(S_\lambda a, S_\lambda b) \leq \lambda f((b^*+1)d(a,b), (b^*+1)d(a,Sb), (b^*+1)d(b,Sa)) \tag{4.3}$$

for all $a,b \in E$ with $a \neq b$ and $b^* \in [0,\infty)$. Then S is said to be an enriched $\mathcal{A}'$-contraction.

Example 4. Let $E = \mathbb{R}$ be endowed with the usual norm and $S : E \to E$ be defined by $Sa = 2 - a$, for all $a \in E$. S is not an enriched contraction but is an enriched $\mathcal{A}'$-contraction.

Indeed, if S would be an enriched contraction, then there would exist $b^* \in [0,+\infty)$ and $\theta \in [0, b^*+1)$ such that $\forall\, a,b \in \mathbb{R}$,

$$\|b^*(a-b) + Sa - Sb\| \leq \theta\|(a-b)\|$$

Take $b^* = 0$ and then $\theta \in [0,1)$,

$$\|a-b\| \leq \theta\|a-b\|,$$

which, for $a = 0$ and $b = 1$ yields the contradiction $1 \leq \theta < 1$. S is an enriched $\mathcal{A}'$-contraction for $b^* = 1$ and $f(r,s,t) = \frac{s+t}{6}$ (see [9]).

Theorem 5. *Let (E,d) be a complete CAT(0) space and S be an enriched $\mathcal{A}'$-contraction. Then S has a unique fixed point in E and there exists $\lambda \in (0,1]$ such that the sequence $\{a_n\}$ defined by $a_{n+1} = \lambda a_n \oplus (1-\lambda)Sa_n$, $n \geq 0$ converges to that fixed point , for any $a_0 \in E$.*

Proof. Let $b^* > 0$. Set $\lambda = \frac{1}{b^*+1}$ so that $0 < \lambda < 1$. Now note that

$$\begin{aligned} d(S_\lambda a, S_\lambda b) &\leq \lambda f(\frac{1}{\lambda}d(a,b), \frac{1}{\lambda}d(a,Sb), \frac{1}{\lambda}d(b,Sa)) \\ &\leq f(d(a,b), d(a,Sb), d(b,Sa)) \\ d(S_\lambda a, S_\lambda b) &\leq f(d(a,b), d(a,S_\lambda b), d(b,S_\lambda a)) \end{aligned} \tag{4.4}$$

for all $a,b \in E$ with $a \neq b$.

Let $a_0 \in E$ be arbitrary. Define $a_n = S_\lambda^n a_0$ for all $n \geq 1$. Let us put $a = a_n$ and $b = a_{n-1}$ in (4.4). Then

$$\begin{aligned} d(a_{n+1}, a_n) &\leq f(d(a_n, a_{n-1}), d(a_n, a_n), d(a_{n-1}, a_{n+1})) \\ &\leq f(d(a_n, a_{n-1}), d(a_n, a_n), d(a_{n-1}, a_n), d(a_n, a_{n+1})) \end{aligned}$$

which implies that

$$d(a_{n+1}, a_n) \leq k.d(a_n, a_{n-1})$$

for some $k \in [0,1]$. From this we get

$$d(x_{n+1}, x_n) \leq k^n.d(x_1, x_0).$$

Now for all $m,n \geq 1$, we have

$$\begin{aligned} d(a_{n+m},a_n) &\leq d(a_{n+m},a_{n+m-1}) + d(a_{n+m-1},a_{n+m-2}) + \ldots + d(a_{n+1},a_n) \\ &\leq (k^{n+m-1} + k^{n+m-2} + \ldots + k^n) d(a_1,a_0) \\ &= k^n \left(\frac{1-k^m}{1-k} \right) d(a_1,a_0) \end{aligned}$$

So, $d(a_{n+m},a_n) \to 0$ as $m,n \to \infty$. Hence $\{a_n\}$ is a Cauchy sequence in E and hence there exists an element $p \in E$ such that $a_n \to p$ as $n \to \infty$.

Now

$$d(S_\lambda p, a_{n+1}) = d(S_\lambda p, S_\lambda a_n) \leq f(d(p,a_n), d(p,a_{n+1}), d(a_n,Sp))$$

Taking limit as $n \to \infty$, we get

$$d(S_\lambda p, p) \leq f(d(p,p), d(p,p), d(p,S_\lambda p)) \leq k \cdot d(p,p) = 0.$$

Hence, $S_\lambda p = p$ and consequently $Sp = p$. Therefore, p is a fixed point of S.

Let p^* be another fixed point of S and hence, a fixed point of S_λ. Then

$$\begin{aligned} d(p,p^*) &= d(S_\lambda p, S_\lambda p^*) \\ &\leq f(d(p,p^*), d(p,Sp^*), d(p^*,Sp)) \\ &= f(d(p,p^*), d(p,p^*), d(p,p^*)) \end{aligned}$$

This gives that $d(p,p^*) = 0$. Hence $p = p^*$.

If $b = 0$, then (4.3) reduces to

$$d(S_\lambda a, S_\lambda b) \leq \lambda f(d(a,b), d(a,Sa), d(b,Sb))$$

for all $a,b \in E$ with $a \neq b$ which can easily be solved. □

Example 5. Let $E = \mathbb{R}$ be a CAT(0) space equipped with usual norm. Let S be a self map on E defined as $Sa = 4 - a$ for all $a \in E$.

We choose $f \in \mathcal{A}$, defined by $f(r,s,t) = \frac{1}{4} max\{s,t\}$ for $r,s,t \in \mathbb{R}_+$ and $\lambda = \frac{1}{2}, b = 1$. Then

$$d(S_\lambda a, S_\lambda b) = |S_\lambda a - S_\lambda b| = 0$$

and

$$\begin{aligned} \lambda f((b^*+1)d(a,b), d(a,Sa), d(b,Sb)) &= \frac{1}{2} \cdot \frac{1}{4} max\{d(a,Sa),(b,Sb)\} \\ &= \frac{1}{8} max\{|2a-4|, |2b-4|\} \end{aligned}$$

Therefore,

$$d(S_\lambda a, S_\lambda b) \leq \lambda f((b^*+1)d(a,b), d(a,Sa), d(b,Sb))$$

for all $a,b \in E$. Hence, S is an enriched $\mathcal{A}$-contraction on E. So, by Theorem 4, S has a unique fixed point. It is to be noticed that $a = 2$ is the unique fixed point of S.

Example 6. Let $E = \mathbb{R}$ be a CAT(0) space equipped with usual norm. Let S be a self map on E defined as $Sa = 1 - a$ for all $a \in E$.

Let us consider the mapping $f \in \mathcal{A}'$, defined by $f(r,s,t) = \frac{s+t}{2}$ for $r,s,t \in \mathbb{R}_+$ and $\lambda = \frac{1}{2}, b = 1$. Then

$$d(S_\lambda a, S_\lambda b) = |S_\lambda a - S_\lambda b| = 0$$

and

$$\begin{aligned}\lambda f((b^*+1)d(a,b),(b+1)d(a,Sb),(b+1)d(b,Sa)) &= \frac{2(d(a,Sb)+d(b,Sa))}{4}\\ &= \frac{|a-Sb|+|b-Sa|}{2}\\ &= \frac{|a-1+b|+|a+b-1|}{2}\end{aligned}$$

Hence,

$$d(S_\lambda a, S_\lambda b) \leq \lambda f((b^*+1)d(a,b),(b^*+1)d(a,Sb),(b^*+1)d(b,Sa))$$

for all $a,b \in E$. Therefore, S is an enriched $\mathcal{A}'$-contraction on E. So, by Theorem 5, S has a unique fixed point. It is to be noticed that $a = \frac{1}{2}$ is the unique fixed point of S.

Now, we study well-posedness and limit shadowing property of fixed point problem for both types of contractions.

Definition 4. Let S be a self map defined on a CAT(0) space (E,d). Then the fixed point problem concerning S is known as well-posed if the followings hold:

(i): S has a unique fixed point $p \in E$;

(ii): for any sequence $\{a_n\}$ in E with $\lim_{n\to\infty} d(a_n, Sa_n) = 0$, we have

$$\lim_{n\to\infty} d(a_n, p) = 0.$$

Theorem 6. *Let (E,d) be a complete CAT(0) space and S be an enriched $\mathcal{A}$-contraction (resp. enriched $\mathcal{A}'$-contraction). Then the fixed point problem is well posed.*

Proof. Theorem 1(resp. in Theorem 2) ensures that S possesses a unique fixed point p, say.

$$\begin{aligned}d(a_n, S_\lambda a_n) &= d(a_n, \lambda a_n + (1-\lambda)Sa_n)\\ &\leq (1-\lambda)d(a_n, Sa_n).\end{aligned}$$

for all $a \in E$. Let $\{a_n\}$ be a sequence in E such that $\lim_{n\to\infty} d(a_n, Sa_n) = 0$. Then $\lim_{n\to\infty} d(a_n, S_\lambda a_n) = 0$.

Then

$$\begin{aligned}d(a_n, p) &\leq d(a_n, S_\lambda a_n) + d(S_\lambda a_n, p)\\ &= d(a_n, S_\lambda a_n) + d(S_\lambda a_n, S_\lambda p).\end{aligned}$$

Now

$$d(S_\lambda a_n, S_\lambda p) \leq f(d(a_n,p), d(a_n, S_\lambda a_n), d(p,Sp)).$$

Therefore, we have

$$d(a_n,p) \leq d(a_n, S_\lambda a_n) + f(d(a_n,p), d(a_n,S_\lambda a_n), d(p,Sp)).$$

Taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} d(a_n,p) \leq f(\lim_{n\to\infty} d(a_n,p), 0, 0).$$

So, there exist $k \in [0,1)$ such that

$$\lim_{n\to\infty} d(a_n,p) \leq k \cdot 0 \qquad \Rightarrow \qquad \lim_{n\to\infty} d(a_n,p) = 0.$$

Hence the result follows. □

Definition 5. Let S be a self map defined on a CAT(0) space (E,d). Then the fixed point problem involving S is said to posses limit shadowing property in E if for any sequence $\{a_n\}$ in E such that $\lim_{n\to\infty} d(a_n, Sa_n) = 0$, we have $p \in E$ with $\lim_{n\to\infty} d(S^n p, a_n) = 0$.

Theorem 7. *Let (E,d) be a complete CAT(0) space and S be an enriched $\mathcal{A}$-contraction (resp. enriched $\mathcal{A}'$-contraction). Then the fixed point problem involving S possesses limit shadowing property.*

Proof. We have already shown in Theorem 4 (resp. in Theorem 5) that $\lim_{n\to\infty} d(a_n,p) = 0$, where p is the unique fixed point of S. Then for any $n \in \mathbb{N}, S^n p = p$ and therefore

$$\lim_{n\to\infty} d(a_n, S^n p) = 0.$$

i.e.,

$$\lim_{n\to\infty} d(S^n p, a_n) = 0.$$

Hence the fixed point problems has limit shadowing property in both the cases. □

5. Conclusion

(1) The existence and approximation of fixed points for the class of enriched contractions in the context of CAT(0) space are introduced and investigated in this article.
(2) We show that in a complete CAT(0) space, any enriched contraction has a unique fixed point that can be approximated using a Kransnoselskii type iterative procedure
(3) For enriched contractions in CAT(0) spaces, we derive a local fixed point conclusion (Theorem 2) along with an asymptotic fixed point consequence (Theorem 3). These results expand the results in [2] from Banach spaces to complete CAT(0) spaces in a significant way.

(4) Some convergence results are obtained for enriched $\mathcal{A}$-contraction and enriched $\mathcal{A}'$-contraction in the context of CAT(0) space and numerical examples are also provided to illustrate the findings of our paper. These outcomes are generalization of the results in [9] from Banach spaces to complete CAT(0) spaces.

6. Future Work

(1) Other important conclusions on the solution of the fixed point problem in CAT(0) spaces or in Banach spaces, metric spaces and generalized metric spaces could be derived using the approach considered in this study along with the ideas given in [7].
(2) In the case of CAT(k) spaces for $k > 0$, an approach identical to that described in this paper can be applied.

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STATE ESTIMATION OF MEMRISTOR-BASED STOCHASTIC NEURAL NETWORKS WITH MIXED VARIABLE DELAYS

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Abstract. This paper studies the state estimation problem for memristor-based stochastic neural networks (MSNNs) with mixed variable delays. A new Lyapunov-Krasovskii functional (LKF) with quadruple integral terms is incorporated. Then, asymptotic stability conditions are established for the error system using a linear matrix inequality technique. The estimator gain can be obtained by solving the linear matrix inequalities. Numerical simulations are given to demonstrate the effectiveness and superiority of the new scheme.

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Keywords: distributed variable delay, memristor-based stochastic neural networks, quadruple integral, state estimation

1. Introduction

The memristor-based neural networks (MNNs) made of hybrid complementary metal-oxide semiconductors have a vast range of applications in bioinspired engineering [5, 7, 8]. One can get a good range of useful applications in new classes of artificial neural systems [8], neural learning circuits [15], associative memories [5] by investigating MNNs. Recently, many research works have been published on MNNs [4, 12]. In the electronic circuit implementation of the neural networks (NNs), time delay plays a vital role due to the finite switching speed of amplifiers [3, 16]. The existence of time delays may cause divergence and instability in system performance [16]. A particular class of time delay exists in engineering systems, called time-varying interval delays, which generally exists in networked control systems [23]. In the previous decade, distributed delays were found in the realistic NN model, and significant research attention was brought to the stability of NNs with discrete and distributed time-varying delays [1, 24]. Thus, analyzing the dynamics of NNs with mixed variable delays has attracted more research attention [4, 12].

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In practice, synaptic transmission is a silent process brought on by the cause of random probabilities in the actual neuron system. The methods in [1, 20] show that some stochastic inputs can destabilize or degrade NNs. So, it is essential to study stochastic perturbations, and their effects on the NNs [18, 22]. In addition, the state estimation problem for stochastic neural networks (SNNs) is a more challenging task than for the deterministic NNs. Thus, it is of great importance to study state estimation problems for NNs with stochasticities [6, 19].

The state estimation problem for NNs has brought up a lot of research interest in the current decade (see [9, 10]). There are some difficulties in estimating the neurons in NNs due to the complicated composition of NNs. Therefore, it is challenging and essential to study the state estimation problem of NNs. In recent years, the state estimation of NNs was proposed in [2, 11, 14, 17]. In [18], synchronization of MSNNs with mixed delays has been proposed. Stability criteria for MSNNs with mixed delays were considered in [14]. State estimation problem for delayed generalized NNs was studied in [16]. Exponential stability of uncertain SNNs with mixed delays was established in [6]. Stability criteria for Markovian jump static SNNs with time delays are considered in [19]. To the author's best knowledge, state estimation of MSNNs with mixed variable delays has not been completely studied yet. Hence, our primary intention is to study the state estimation for delayed MSNNs.

The main contribution of this paper are as follows:

(1) Most of the existing results in the literature consider the state estimation problems for only deterministic cases. Therefore, the state estimation problem for MSNNs with mixed variable delays is considered in this paper. A novel approach is proposed to solve the considered problem.
(2) In this article, a stochastic quadruple integral LKF is proposed to deal with the state estimation problem for MSNNs with mixed time-varying delays.
(3) The quadruple integral term of activation function such as

$$\int_{-\vartheta_1}^{0}\int_{\rho}^{0}\int_{\chi}^{0}\int_{t+\kappa}^{t} e(s)dsd\kappa d\chi d\rho \quad \text{and} \quad \int_{-\vartheta_2}^{-\vartheta_1}\int_{\rho}^{0}\int_{\chi}^{0}\int_{t+\kappa}^{t} e(s)dsd\kappa d\chi d\rho$$

are utilized in estimating the time derivative of the LKF.

From the above comments, the main focus is on the state estimation problem for MSNNs with mixed variable delays. By establishing a new quadruple stochastic integral LKF and using the linear matrix inequalities (LMIs) technique, a sufficient condition for the feasibility of the state estimation problem has been presented, which assures the asymptotic stability of the estimation error system. At last, an example is provided to reveal the usefulness and superiority of our approach.

Notation 1. $\mathfrak{R}^n$ denotes the n-dimensional Euclidean space, and $\mathfrak{R}^{n\times n}$ denotes the set of all $n\times n$ real matrices. Y^T denotes the matrix transpose of Y. $X>0$ denotes that X is a real symmetric positive definite matrix. I denotes the identity matrix. $*$

denotes the term that is induced by symmetry. $\mathrm{diag}\{a,b,\ldots,z\}$ denotes the block-diagonal matrix with $a,b,\ldots,z$ in the diagonal entries.

2. Problem statement and Preliminaries

As in [5], the memductance of a memristor is given as $\varpi(\phi) = \frac{d\hat{q}(\phi)}{d\phi}$, where $d\hat{q}(\phi)$ denotes the charge passing through the memristor and ϕ denotes the flux. Since q and ϕ, respectively, represent the time integral of the current $i(t)$ and voltage $\nu(t)$ applied to the memristor; namely $q(t) = \int_{-\infty}^{t} i(\tau)d\tau$, and $\phi(t) = \int_{-\infty}^{t} \nu(\tau)d\tau$ the memductance ϖ is the function of voltage v and $\varpi(v) = \frac{d\hat{q}(\phi)}{d\phi} = \frac{dq/dt}{d\phi/dt} = \frac{i(t)}{\nu(t)}$.

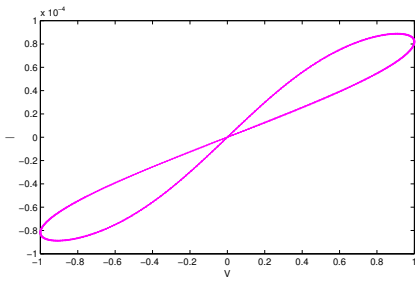

FIGURE 1. Typical current voltage characteristic of memristor with a sinusoidal current source [8].

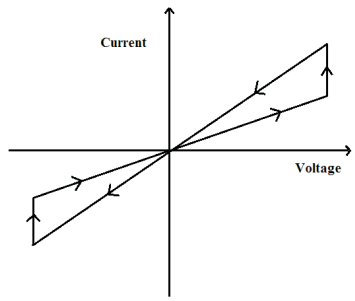

FIGURE 2. Characteristic of the Piece-wise linear model of memristor [8].

The mathematical model of the memductance is presented to accurately describe the memory function and the hysteresis properties of the memory, according to the current-voltage characteristics of the memory in Figure 1:

$$\varpi(\nu(t)) = \begin{cases} \varpi'(\nu(t)), & \nu(s)\downarrow, s\in(t-\rho,t]; \\ \varpi''(\nu(t)), & \nu(s)\uparrow, s\in(t-\rho,t]; \\ \lim_{s\to t^-}\varpi(\nu(s)), & \nu(s) \text{ unchage}, s\in(t-\rho,t], \end{cases} \tag{2.1}$$

where $\uparrow$ means increase, $\downarrow$ means decrease, ρ is a sufficiently small positive constant, $\lim_{s\to t^-}$ is either equal to $\varpi'(\nu(t))$ or $\varpi''(\nu(t))$, and it means that the memductance keeps the voltage value. Hence, the memductance function may be discontinuous. As mentioned in [5], the memristor must display two adequate equilibrium levels, R_0 and R_1, where $R_0 \gg R1$, which makes it easy to switch between high resistance levels. Fast resistance as low as possible, or vice versa when consuming as little energy as possible. Therefore, with this property a memductance of the memristor can be defined as (2.1), where the constants defined by $\varpi'(\nu(t)) = \varpi'$ and $\varpi''(\nu(t)) =$

ϖ''. Figure 2 shows the current-voltage characteristics of this memristor. The use of memristors to change the resistance of the circuit sensor in the connection of the NN called the MNN. Now, consider the MSNN model based on the methods in [8]:

$$\begin{aligned} dx(t) = [&-Ax(t) + W_0(x)f(x(t)) + W_1(x)f(x(t-\vartheta(t))) \\ &+ W_2(x)\int_{t-d(t)}^{t} f(x(s))ds]dt + [Ex(t) + E_1x(t-\vartheta(t)) \\ &+ H_0(x)f(x(t)) + H_1(x)f(x(t-\vartheta(t))) \\ &+ H_2(x)\int_{t-d(t)}^{t} f(x(s))ds]dv(t), \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} x(t) &= [x_1(t), x_2(t), \ldots, x_n(t)]^T \text{ is the state vector;} \\ A &= \operatorname{diag}\{a_1(t), a_2(t), \ldots, a_n(t)\}, \\ W_0(x) &= [w_{0ij}(f_j(x_j(t)) - x_i(t))]_{n\times n}, \\ W_1(x) &= [w_{1ij}(f_j(x_j(t-\vartheta(t))) - x_i(t))]_{n\times n}, \\ W_2(x) &= \left[w_{2ij}\left(\int_{t-d(t)}^{t} f_j(x_j(s))ds - x_i(t)\right)\right]_{n\times n}, \\ H_0(x) &= [h_{0ij}(f_j(x_j(t)) - x_i(t))]_{n\times n}, \\ H_1(x) &= [h_{1ij}(f_j(x_j(t-\vartheta(t))) - x_i(t))]_{n\times n}, \\ H_2(x) &= \left[h_{2ij}(\int_{t-d(t)}^{t} f_j(x_j(s))ds) - x_i(t))\right]_{n\times n} \end{aligned}$$

are the feedback connection weight matrices. $\vartheta(t)$, and $d(t)$ are respectively representing the time-varying and time-varying distributed delays.

Assumption 1. *There exist scalars $\vartheta_1 > 0$, $\vartheta_2 > 0$, $\mu \geq 0$ and $d > 0$ such that $0 \leq \vartheta_1 \leq \vartheta(t) \leq \vartheta_2$, $\dot{\vartheta}(t) \leq \mu$ and $0 \leq d(t) \leq d$, respectively.*

Assumption 2. *Each neuron activation function $f_i(t)$ $(i = 1, 2, \ldots, n)$ is continuous and bounded, and satisfies the Lipschitz condition: $|f(x_1) - f(x_2)| \leq F|(x_1 - x_2)|$, with $F \in \Re^{n\times n}$ and $x_1, x_2 \in \Re$, $x_1 \neq x_2$ are constants. The network measurements are assumed to satisfy:*

$$y(t) = Cx(t) + l(t, x(t)),$$

where $y(t) \in \Re^n$ is the network measurement, C is a known constant matrix with appropriate dimension, $l\colon \Re \times \Re^n \to \Re^m$ is the neuron-dependent nonlinear disturbances on the network outputs, and satisfies: $|l(t,x_1) - l(t,x_2)| \leq |L(x_1 - x_2)|$, with $L \in \Re^{n\times n}$. Consider the full-order state estimator ([21]):

$$d\hat{x}(t) = [-A\hat{x}(t) + W_0(x)f(\hat{x}(t)) + W_1(x)f(\hat{x}(t-\vartheta(t)))$$

$$
\begin{aligned}
&+W_2(x)\int_{t-d(t)}^{t} f(\hat{x}(s))ds + K[y(t)-C\hat{x}(t)-l(t,\hat{x}(t))]]dt + [E\hat{x}(t) \\
&+E_1\hat{x}(t-\vartheta(t)) + H_0(x)f(\hat{x}(t)) + H_1(x)f(\hat{x}(t-\vartheta(t))) \\
&+H_2(x)\int_{t-d(t)}^{t} f(\hat{x}(s))ds]dv(t),
\end{aligned} \tag{2.3}
$$

where $\hat{x}(t) \in \mathfrak{R}^n$ denotes the estimated state, and $K \in \mathfrak{R}^{n\times m}$ is the estimator gain matrix to be calculated. Define the error state $e(t) = x(t) - \hat{x}(t)$. From (2.2) and (2.3), one can obtain the following error system:

$$
\begin{aligned}
de(t) = [&-Ae(t) + W_0(x)\varphi(t) + W_1(x)\varphi(t-\vartheta(t)) + W_2(x)\Upsilon_1 \\
&- KC\psi(t)]dt + [Ee(t) + E_1e(t-\vartheta(t)) + H_0(x)\varphi(t) \\
&+ H_1(x)\varphi(t-\vartheta(t)) + H_2(x)\Upsilon_1]dv(t),
\end{aligned} \tag{2.4}
$$

where $\Upsilon_1 = \int_{t-d(t)}^{t}\varphi(s)ds$, $e(t) = [e_1(t), e_2(t), \ldots, e_n(t)]^T \in \mathfrak{R}^n$ is the state vector of the transformed system and $\varphi(t) = f(x(t)) - f(\widehat{x}(t))$ with $\varphi(t) = [\varphi_1(t), \ldots, \varphi_n(t)]^T$.

Definition 1. For the error-state system (2.4) and every $\varsigma \in L^2_{\mathcal{F}_0}([-\vartheta_2, 0]; \mathcal{R}^n)$ where ϑ_2 is the upper bound of the time delay, the trivial solution is globally asymptotically stable if it is locally stable in the sense of Lyapunov and globally attractive.

3. Main Results

Theorem 1. *For given positive scalars $\vartheta_1, \vartheta_2, \mu, d$ and the matrix K, the error-state system (2.4) of MSNN (2.2) and (2.3) is globally asymptotically mean-square stable if there exist matrices $\mathbb{P}_{5\times5} > 0$, $R_j > 0$, $(j = 1,2,3)$, $S_p > 0$, $(p = 1,2,3,4,5)$, $T_q > 0$, $Z_q > 0$, $q = (1,2,3,4,5,6)$ and any appropriate dimensional matrices $\mathcal{P}_s$, $\mathcal{Q}_s$, $\mathcal{R}_s$, $\mathcal{S}_s$, $\mathcal{T}_s$, $\mathcal{U}_s$, $\mathcal{V}_s$, $(s = 1,2)$, such that the following LMI hold for $i = 1,2,3$:*

$$
\widehat{\Pi}_i = \begin{bmatrix} \Pi_1^i & \Pi_2^i \\ * & \Pi_3^i \end{bmatrix} < 0, \tag{3.1}
$$

where

$$
\Pi_1^i = \Pi_{r\times s}, \quad (i = 1,2,3,\ r,s\ = 1,2,\ldots,12),
$$

$$
\mathbb{P} = \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} & P_{15} \\ * & P_{22} & P_{23} & P_{24} & P_{25} \\ * & * & P_{33} & P_{34} & P_{35} \\ * & * & * & P_{44} & P_{45} \\ * & * & * & * & P_{55} \end{bmatrix}, \quad R_j = \begin{bmatrix} R_{11}^j & R_{12}^j \\ * & R_{22}^j \end{bmatrix}, \quad (j = 1,2,3)
$$

$$
\begin{aligned}
\Pi_{1,1} = &-P_{11}A - A^TP_{11} + P_{12} + P_{12}^T + R_{11}^1 + \vartheta_1S_1 + \vartheta_{21}S_2 + \mathcal{P}_1 + \mathcal{P}_1^T + \vartheta_1\mathcal{S}_1 \\
&+ \vartheta_1\mathcal{S}_1^T + \vartheta_{21}\mathcal{T}_1 + \vartheta_{21}\mathcal{T}_1^T + \frac{\vartheta_1^2}{2}\mathcal{U}_1 + \frac{\vartheta_1^2}{2}\mathcal{U}_1^T + \frac{\vartheta_2^2-\vartheta_1^2}{2}\mathcal{V}_1 + \frac{\vartheta_2^2-\vartheta_1^2}{2}\mathcal{V}_1^T \\
&+ \varepsilon_1F^TF + \varepsilon_3L^TL,
\end{aligned}
$$

$$\Pi_{1,2} = \mathcal{P}_2^T - \mathcal{Q}_1 + \mathcal{R}_1 + \vartheta_1 \mathcal{S}_2^T + \vartheta_{21} \mathcal{T}_2^T + \frac{\vartheta_1^2}{2} \mathcal{U}_2^T + \frac{\vartheta_2^2 - \vartheta_1^2}{2} \mathcal{V}_2^T,$$

$$\Pi_{1,3} = -P_{12} + P_{13} - \mathcal{P}_1 + \mathcal{Q}_1, \quad \Pi_{1,4} = -\mathcal{R}_1 - P_{13}, \quad \Pi_{1,5} = P_{11} W_0 + P_{14} + R_{12}^1,$$

$$\Pi_{1,6} = P_{11} W_1, \quad \Pi_{1,7} = P_{15} - P_{14}, \quad \Pi_{1,8} = -P_{15}, \quad \Pi_{1,9} = -P_{11} KC,$$

$$\Pi_{1,10} = P_{11} W_2, \quad \Pi_{1,11} = -A^T P_{14} + P_{24}, \quad \Pi_{1,12} = -A^T P_{15} + P_{25},$$

$$\Pi_{2,2} = -(1-\mu) R_{11}^2 - \mathcal{Q}_2 - \mathcal{Q}_2^T + \mathcal{R}_2 + \mathcal{R}_2^T + \varepsilon_2 F^T F, \quad \Pi_{2,3} = -\mathcal{P}_2 + \mathcal{Q}_2,$$

$$\Pi_{2,4} = -\mathcal{R}_2, \quad \Pi_{2,6} = -(1-\mu) R_{12}^2, \quad \Pi_{3,3} = -R_{11}^1 + R_{11}^2 + R_{11}^3,$$

$$\Pi_{3,7} = -R_{12}^1 + R_{12}^2 + R_{12}^3, \quad \Pi_{3,11} = -P_{24} + P_{34}, \quad \Pi_{3,12} = -P_{25} + P_{35},$$

$$\Pi_{4,4} = -R_{11}^3, \quad \Pi_{4,8} = -R_{12}^3, \quad \Pi_{4,11} = -P_{34}, \quad \Pi_{4,12} = -P_{35},$$

$$\Pi_{5,5} = R_{22}^1 + d^2 S_3 + \vartheta_1^2 S_4 + \vartheta_{21}^2 S_5 - \varepsilon_1, \quad \Pi_{5,11} = W_0^T P_{14} + P_{44},$$

$$\Pi_{5,12} = W_0^T P_{15} + P_{45}, \quad \Pi_{6,6} = -(1-\mu) R_{22}^2 - \varepsilon_2, \quad \Pi_{6,11} = W_1^T P_{14},$$

$$\Pi_{6,12} = W_1^T P_{15}, \quad \Pi_{7,7} = -R_{22}^1 + R_{22}^2 + R_{22}^3, \quad \Pi_{7,11} = -P_{44} + P_{45}^T,$$

$$\Pi_{7,12} = -P_{45} + P_{55}, \quad \Pi_{8,8} = -R_{22}^3, \quad \Pi_{8,11} = -P_{45}^T,$$

$$\Pi_{8,12} = -P_{55}, \quad \Pi_{9,9} = -\varepsilon_3, \quad \Pi_{9,11} = -C^T K^T P_{14},$$

$$\Pi_{9,12} = -C^T K^T P_{15}, \quad \Pi_{10,10} = -S_3, \quad \Pi_{10,11} = W_2^T P_{14},$$

$$\Pi_{10,12} = W_2^T P_{15}, \quad \Pi_{11,11} = -S_4, \quad \Pi_{12,12} = -S_5, \quad \vartheta_{21} = \vartheta_2 - \vartheta_1,$$

$$\Pi_2^1 = \Big[\Omega_1 \ \ \Big(-\vartheta_1 \widetilde{\mathcal{P}} - \frac{\vartheta_1^2}{2} \widetilde{\mathcal{S}} - \frac{\vartheta_1^3}{6} \widetilde{\mathcal{U}} \Big) \ \ \widehat{E}^T P_{11} \ \ \vartheta_1 \widehat{A}^T T_1 \ \ \frac{\vartheta_1^2}{2} \widehat{A}^T T_3 \ \ \frac{\vartheta_1^3}{6} \widehat{A}^T T_5$$
$$\vartheta_1 \widehat{E}^T Z_1 \ \ \frac{\vartheta_1^2}{2} \widehat{E}^T Z_3 \ \ \frac{\vartheta_1^3}{6} \widehat{E}^T Z_5 \ \ \widetilde{\mathcal{P}} \ \ \vartheta_1 \widetilde{\mathcal{S}} \ \ \frac{\vartheta_1^2}{2} \widetilde{\mathcal{U}} \Big],$$

$$\Pi_2^2 = \Big[\Omega_2 \ \ \Big(-\vartheta_{21} \widetilde{\mathcal{Q}} - \frac{\vartheta_2^2 - \vartheta_1^2}{2} \widetilde{\mathcal{T}} - \frac{\vartheta_2^3 - \vartheta_1^3}{6} \widetilde{\mathcal{V}} \Big) \ \ \widehat{E}^T P_{11} \ \ \vartheta_{21} \widehat{A}^T T_2$$
$$\frac{\vartheta_2^2 - \vartheta_1^2}{2} \widehat{A}^T T_4 \ \ \frac{\vartheta_2^3 - \vartheta_1^3}{6} \widehat{A}^T T_6 \ \ \vartheta_{21} \widehat{E}^T Z_2 \ \ \frac{\vartheta_2^2 - \vartheta_1^2}{2} \widehat{E}^T Z_4 \ \ \frac{\vartheta_2^3 - \vartheta_1^3}{6} \widehat{E}^T Z_6$$
$$\widetilde{\mathcal{Q}} \ \ \vartheta_{21} \widetilde{\mathcal{T}} \ \ \frac{\vartheta_2^2 - \vartheta_1^2}{2} \widetilde{\mathcal{V}} \Big],$$

$$\Pi_2^3 = \Big[\Omega_2 \ \ \Big(-\vartheta_{21} \widetilde{\mathcal{R}} - \frac{\vartheta_2^2 - \vartheta_1^2}{2} \widetilde{\mathcal{T}} - \frac{\vartheta_2^3 - \vartheta_1^3}{6} \widetilde{\mathcal{V}} \Big) \ \ \widehat{E}^T P_{11} \ \ \vartheta_{21} \widehat{A}^T T_2$$
$$\frac{\vartheta_2^2 - \vartheta_1^2}{2} \widehat{A}^T T_4 \ \ \frac{\vartheta_2^3 - \vartheta_1^3}{6} \widehat{A}^T T_6 \ \ \vartheta_{21} \widehat{E}^T Z_2 \ \ \frac{\vartheta_2^2 - \vartheta_1^2}{2} \widehat{E}^T Z_4 \ \ \frac{\vartheta_2^3 - \vartheta_1^3}{6} \widehat{E}^T Z_6$$
$$\widetilde{\mathcal{R}} \vartheta_{21} \widetilde{\mathcal{T}} \ \ \frac{\vartheta_2^2 - \vartheta_1^2}{2} \widetilde{\mathcal{V}} \Big],$$

$$\Pi_3^1 = \Big[-\vartheta_1 S_1 \;\; \Big(-\vartheta_1 T_1 - \frac{\vartheta_1^2}{2} T_3 - \frac{\vartheta_1^3}{6} T_5\Big) \;\; -P_{11} \;\; -\vartheta_1 T_1 \;\; -\frac{\vartheta_1^2}{2} T_3 \;\; -\frac{\vartheta_1^3}{6} T_5$$
$$-\vartheta_1 Z_1 \;\; -\frac{\vartheta_1^2}{2} Z_3 \;\; -\frac{\vartheta_1^3}{6} Z_5 \;\; -Z_1 \;\; -\vartheta_1 Z_3 \;\; -\frac{\vartheta_1^2}{2} Z_5 \Big],$$

$$\Pi_3^2 = \Big[-\vartheta_{21} S_2 \;\; \Big(-\vartheta_1 T_2 - \frac{\vartheta_2^2 - \vartheta_1^2}{2} T_4 - \frac{\vartheta_2^3 - \vartheta_1^3}{6} T_6\Big) \;\; -P_{11} \;\; -\vartheta_{21} T_2$$
$$-\frac{\vartheta_2^2 - \vartheta_1^2}{2} T_4 \;\; -\frac{\vartheta_2^3 - \vartheta_1^3}{6} T_6 \;\; -\vartheta_{21} Z_2 \;\; -\frac{\vartheta_2^2 - \vartheta_1^2}{2} Z_4 \;\; -\frac{\vartheta_2^3 - \vartheta_1^3}{6} Z_6$$
$$-Z_2 \;\; -\vartheta_{21} Z_4 \;\; -\frac{\vartheta_2^2 - \vartheta_1^2}{2} Z_6 \Big],$$

$$\Pi_3^3 = \Big[-\vartheta_{21} S_2 \;\; \Big(-\vartheta_{21} T_2 - \frac{\vartheta_2^2 - \vartheta_1^2}{2} T_4 - \frac{\vartheta_2^3 - \vartheta_1^3}{6} T_6\Big) \;\; -P_{11} \;\; -\vartheta_{21} T_2$$
$$-\frac{\vartheta_2^2 - \vartheta_1^2}{2} T_4 \;\; -\frac{\vartheta_2^3 - \vartheta_1^3}{6} T_6 \;\; -\vartheta_{21} Z_2 \;\; -\frac{\vartheta_2^2 - \vartheta_1^2}{2} Z_4 \;\; -\frac{\vartheta_2^3 - \vartheta_1^3}{6} Z_6$$
$$-Z_2 \;\; -\vartheta_{21} Z_4 \;\; -\frac{\vartheta_2^2 - \vartheta_1^2}{2} Z_6 \Big],$$

$$\Omega_1 = \Big[\Big(\vartheta_1 P_{22} - \vartheta_1 A^T P_{12} - \vartheta_1 \mathcal{S}_1 - \frac{\vartheta_1^2}{2} \mathcal{U}_1\Big) \;\; \Big(-\vartheta_1 \mathcal{S}_2 - \frac{\vartheta_1^2}{2} \mathcal{U}_2\Big)$$
$$\big(-\vartheta_1 P_{22} + \vartheta_1 P_{23}^T\big) \;\; -\vartheta_1 P_{23}^T \;\; \big(\vartheta_1 W_0^T P_{12} + \vartheta_1 P_{24}^T\big) \;\; \vartheta_1 W_1^T P_{12}$$
$$\big(-\vartheta_1 P_{24}^T + \vartheta_1 P_{25}^T\big) \;\; -\vartheta_1 P_{25}^T \;\; -\vartheta_1 C^T K^T P_{12} \;\; \vartheta_1 W_2^T P_{12} \;\; 0 \;\; 0 \Big],$$

$$\Omega_2 = \Big[\Big(\vartheta_{21} P_{23} - \vartheta_{21} A^T P_{13} - \vartheta_{21} \mathcal{T}_1 - \frac{\vartheta_2^2 - \vartheta_1^2}{2} \mathcal{V}_1\Big) \;\; \Big(-\vartheta_{21} \mathcal{T}_2 - \frac{\vartheta_2^2 - \vartheta_1^2}{2} \mathcal{V}_2\Big)$$
$$\big(-\vartheta_{21} P_{23} + \vartheta_{21} P_{33}\big) \;\; -\vartheta_{21} P_{33} \;\; \big(\vartheta_{21} W_0^T P_{13} + \vartheta_{21} P_{34}^T\big)$$
$$\vartheta_{21} W_1^T P_{13} \;\; \big(-\vartheta_{21} P_{34}^T + \vartheta_{21} P_{35}^T\big) \;\; -\vartheta_{21} P_{35}^T \;\; -\vartheta_{21} C^T K^T P_{13}$$
$$\vartheta_{21} W_2^T P_{13} \;\; 0 \;\; 0 \Big],$$

$$\widehat{E} = [E \;\; E_1 \;\; 0 \;\; 0 \;\; H_0 \;\; H_1 \;\; 0 \;\; 0 \;\; 0 \;\; H_2 \;\; 0 \;\; 0],$$
$$\widehat{A} = [A \;\; 0 \;\; 0 \;\; 0 \;\; W_0 \;\; W_1 \;\; 0 \;\; 0 \;\; -KC \;\; W_2 \;\; 0 \;\; 0],$$
$$\widetilde{\mathcal{P}}^T = \big[\mathcal{P}_1^T \;\; \mathcal{P}_2^T \;\; 0_{1\times 10}\big], \quad \widetilde{\mathcal{Q}}^T = \big[\mathcal{Q}_1^T \;\; \mathcal{Q}_2^T \;\; 0_{1\times 10}\big],$$
$$\widetilde{\mathcal{R}}^T = \big[\mathcal{R}_1^T \;\; \mathcal{R}_2^T \;\; 0_{1\times 10}\big], \quad \widetilde{\mathcal{S}}^T = \big[\mathcal{S}_1^T \;\; \mathcal{S}_2^T \;\; 0_{1\times 10}\big], \quad \widetilde{\mathcal{T}}^T = \big[\mathcal{T}_1^T \;\; \mathcal{T}_2^T \;\; 0_{1\times 10}\big],$$
$$\widetilde{\mathcal{U}}^T = \big[\mathcal{U}_1^T \;\; \mathcal{U}_2^T \;\; 0_{1\times 10}\big], \quad \widetilde{\mathcal{V}}^T = \big[\mathcal{V}_1^T \;\; \mathcal{V}_2^T \;\; 0_{1\times 10}\big].$$

Proof. Choose the LKF:

$$V(e_t) = \sum_{p=1}^{5} V_p(e_t), \tag{3.2}$$

where

$$\begin{aligned}
V_1(e_t) &= \eta_1^T(t)\mathbb{P}\eta_1(t),\\
V_2(e_t) &= \int_{t-\vartheta_1}^{t} \eta_2^T(s)R_1\eta_2(s)ds + \int_{t-\vartheta(t)}^{t-\vartheta_1} \eta_2^T(s)R_2\eta_2(s)ds + \int_{t-\vartheta_2}^{t-\vartheta_1} \eta_2^T(s)R_3\eta_2(s)ds,\\
V_3(e_t) &= \Upsilon_2 e^T(s)S_1e(s)dsd\rho + \Upsilon_3 e^T(s)S_2e(s)dsd\rho + d\int_{-d}^{0}\int_{t+\rho}^{t} \varphi^T(s)S_3\varphi(s)dsd\rho\\
&\quad + \vartheta_1\Upsilon_2\varphi^T(s)S_4\varphi(s)dsd\rho + \vartheta_{21}\Upsilon_3\varphi^T(s)S_5\varphi(s)dsd\rho,\\
V_4(e_t) &= \Upsilon_2\mathfrak{z}^T(s)T_1\mathfrak{z}(s)dsd\rho + \Upsilon_3\mathfrak{z}^T(s)T_2\mathfrak{z}(s)dsd\rho + \Upsilon_4\mathfrak{z}^T(s)T_3\mathfrak{z}(s)dsd\chi d\rho\\
&\quad + \Upsilon_5\mathfrak{z}^T(s)T_4\mathfrak{z}(s)dsd\chi d\rho + \Upsilon_6\mathfrak{z}^T(s)T_5\mathfrak{z}(s)dsd\kappa d\chi d\rho\\
&\quad + \Upsilon_7\mathfrak{z}^T(s)T_6\mathfrak{z}(s)dsd\kappa d\chi d\rho,\\
V_5(e_t) &= \Upsilon_2\mathfrak{y}^T(s)Z_1\mathfrak{y}(s)dsd\rho + \Upsilon_3\mathfrak{y}^T(s)Z_2\mathfrak{y}(s)dsd\rho + \Upsilon_4\mathfrak{y}^T(s)Z_3\mathfrak{y}(s)dsd\chi d\rho\\
&\quad + \Upsilon_5\mathfrak{y}^T(s)Z_4\mathfrak{y}(s)dsd\chi d\rho + \Upsilon_6\mathfrak{y}^T(s)Z_5\mathfrak{y}(s)dsd\kappa d\chi d\rho\\
&\quad + \Upsilon_7\mathfrak{y}^T(s)Z_6\mathfrak{y}(s)dsd\kappa d\chi d\rho,
\end{aligned}$$

and

$$\begin{aligned}
\eta_1(t) &= \left[e^T(t) \quad \int_{t-\vartheta_1}^{t} e^T(s)ds \quad \int_{t-\vartheta_2}^{t-\vartheta_1} e^T(s)ds \quad \int_{t-\vartheta_1}^{t} \varphi^T(s)ds \quad \int_{t-\vartheta_2}^{t-\vartheta_1} \varphi^T(s)ds\right]^T,\\
\eta_2(t) &= \left[e^T(s) \quad \varphi^T(s)\right]^T,\\
\mathfrak{z}(s) &= -Ae(t) + W_0\varphi(t) + W_1\varphi(t-\vartheta(t)) + W_2\Upsilon_1 - KC\psi(t),\\
\mathfrak{y}(s) &= Ee(t) + E_1e(t-\vartheta(t)) + H_0\varphi(t) + H_1\varphi(t-\vartheta(t)) + H_2\Upsilon_1,
\end{aligned}$$

$$\begin{aligned}
\Upsilon_2 &= \int_{-\vartheta_1}^{0}\int_{t+\rho}^{t}, & \Upsilon_3 &= \int_{-\vartheta_2}^{-\vartheta_1}\int_{t+\rho}^{t}, & \Upsilon_4 &= \int_{-\vartheta_1}^{0}\int_{\rho}^{0}\int_{t+\chi}^{t},\\
\Upsilon_5 &= \int_{-\vartheta_2}^{-\vartheta_1}\int_{\rho}^{0}\int_{t+\chi}^{t}, & \Upsilon_6 &= \int_{-\vartheta_1}^{0}\int_{\rho}^{0}\int_{\chi}^{0}\int_{t+\kappa}^{t}, & \Upsilon_7 &= \int_{-\vartheta_2}^{-\vartheta_1}\int_{\rho}^{0}\int_{\chi}^{0}\int_{t+\kappa}^{t}.
\end{aligned}$$

Using Ito's formula ([13]), the generator $\mathcal{L}V$ for the evolution of $V(x_t)$ in (3.2) is therefore given by

$$dV(e_t) = \mathcal{L}V(x_t)dt + 2[e^T(t)P_{11}\mathfrak{y}(t) + \int_{t-\vartheta_1}^{t} e^T(s)dsP_{12}^T\mathfrak{y}(t) + \int_{t-\vartheta_2}^{t-\vartheta_1} e^T(s)dsP_{13}^T\mathfrak{y}(t)$$
$$+ \int_{t-\vartheta_1}^{t} \varphi^T(s)dsP_{14}^T\mathfrak{y}(t) + \int_{t-\vartheta_2}^{t-\vartheta_1} \varphi^T(s)dsP_{15}^T\mathfrak{y}(t)]dv(t), \tag{3.3}$$

where

$$\mathcal{L}V(e_t) = \sum_{p=1}^{5} \mathcal{L}V_p(x_t),$$
$$\mathcal{L}V_1(e_t) = 2\left[e^T(t) \quad \int_{t-\vartheta_1}^{t} e^T(s)ds \quad \int_{t-\vartheta_2}^{t-\vartheta_1} e^T(s)ds \quad \int_{t-\vartheta_1}^{t} \varphi^T(s)ds \quad \int_{t-\vartheta_2}^{t-\vartheta_1} \varphi^T(s)ds\right]$$
$$\mathbb{P} \times \left[\mathfrak{z}(t)^T \quad e(t)^T - e(t-\vartheta_1)^T \quad e(t-\vartheta_1)^T - e(t-\vartheta_2)^T\right.$$
$$\left.\varphi(t)^T - \varphi(t-\vartheta_1)^T \quad \varphi(t-\vartheta_1)^T - \varphi(t-\vartheta_2)^T\right]^T + \mathfrak{y}^T(t)P_{11}\mathfrak{y}(t),$$
$$\mathcal{L}V_2(e_t) \le \eta_2^T(t)R_1\eta_2(t) - \eta_2^T(t-\vartheta_1)[-R_1+R_2+R_3]\eta_2(t-\vartheta_1)$$
$$-(1-\mu)\eta_2^T(t-\vartheta(t))R_2\eta_2(t-\vartheta(t)) - \eta_2^T(t-\vartheta_2)R_3\eta_2(t-\vartheta_2),$$
$$\mathcal{L}V_3(e_t) = e^T(t)[\vartheta_1 S_1 + \vartheta_{21}S_2]e(t) - \int_{t-\vartheta_1}^{t} e^T(s)S_1e(s)ds - \int_{t-\vartheta_2}^{t-\vartheta_1} e^T(s)S_2e(s)ds$$
$$+\varphi^T(t)[d^2S_3 + \vartheta_1^2S_4 + \vartheta_{21}^2S_5]\varphi(t) - d\int_{t-d}^{t} \varphi^T(s)S_3\varphi(s)ds$$
$$-\vartheta_1\int_{t-\vartheta_1}^{t} \varphi^T(s)S_4\varphi(s)ds - \vartheta_{21}\int_{t-\vartheta_2}^{t-\vartheta_1} \varphi^T(s)S_5\varphi(s)ds,$$
$$\mathcal{L}V_4(e_t) = \mathfrak{z}^T(t)[\vartheta_1T_1 + \vartheta_{21}T_2 + \left(\frac{\vartheta_1^2}{2}\right)T_3 + \left(\frac{\vartheta_2^2-\vartheta_1^2}{2}\right)T_4 + \left(\frac{\vartheta_1^3}{6}\right)T_5$$
$$+\left(\frac{\vartheta_2^3-\vartheta_1^3}{6}\right)T_6]\mathfrak{z}(t) - \int_{t-\vartheta_1}^{t} \mathfrak{z}^T(s)T_1\mathfrak{z}(s)ds - \int_{t-\vartheta_2}^{t-\vartheta_1} \mathfrak{z}^T(s)T_2\mathfrak{z}(s)ds$$
$$-\Upsilon_2\mathfrak{z}^T(s)T_3\mathfrak{z}(s)dsd\rho - \Upsilon_3\mathfrak{z}^T(s)T_4\mathfrak{z}(s)dsd\rho$$
$$-\Upsilon_4\mathfrak{z}^T(s)T_5\mathfrak{z}(s)dsd\chi d\rho - \Upsilon_5\mathfrak{z}^T(s)T_6\mathfrak{z}(s)dsd\chi d\rho,$$
$$\mathcal{L}V_5(e_t) = \mathfrak{y}^T(t)[\vartheta_1Z_1 + \vartheta_{21}Z_2 + \left(\frac{\vartheta_1^2}{2}\right)Z_3 + \left(\frac{\vartheta_2^2-\vartheta_1^2}{2}\right)Z_4 + \left(\frac{\vartheta_1^3}{6}\right)Z_5$$
$$+\left(\frac{\vartheta_2^3-\vartheta_1^3}{6}\right)Z_6]\mathfrak{y}(t) - \int_{t-\vartheta_1}^{t} \mathfrak{y}^T(s)Z_1\mathfrak{y}(s)ds - \int_{t-\vartheta_2}^{t-\vartheta_1} \mathfrak{y}^T(s)Z_2\mathfrak{y}(s)ds$$
$$-\Upsilon_2\mathfrak{y}^T(s)Z_3\mathfrak{y}(s)dsd\rho - \Upsilon_3\mathfrak{y}^T(s)Z_4\mathfrak{y}(s)dsd\rho$$
$$-\Upsilon_4\mathfrak{y}^T(s)Z_5\mathfrak{y}(s)dsd\chi d\rho - \Upsilon_5\mathfrak{y}^T(s)Z_6\mathfrak{y}(s)dsd\chi d\rho.$$

By using Jensen's inequality implies

$$-d\int_{t-d}^{t}\varphi^T(s)S_3\varphi(s)ds \le -\Upsilon_1^T S_3\Upsilon_1,$$
$$-\vartheta_1\int_{t-\vartheta_1}^{t}\varphi^T(s)S_4\varphi(s)ds \le -\Upsilon_8^T S_4\Upsilon_8,$$
$$-\vartheta_{21}\int_{t-\vartheta_2}^{t-\vartheta_1}\varphi^T(s)S_5\varphi(s)ds \le -\Upsilon_9^T S_5\Upsilon_9,$$

where $\Upsilon_8 = \int_{t-\vartheta_1}^{t}\varphi(s)ds$, $\Upsilon_9 = \int_{t-\vartheta_2}^{t-\vartheta_1}\varphi(s)ds$. To derive the stability criteria, zero equations with any matrices $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$, $\mathcal{U}$ and $\mathcal{V}$ of appropriate dimensions, to be chosen as follows,

$$0 = 2\eta_3^T(t)\mathcal{P}[e(t)-e(t-\vartheta_1)-\int_{t-\vartheta_1}^{t}\mathfrak{z}(s)ds-\int_{t-\vartheta_1}^{t}\mathfrak{y}(s)dv(s)],$$
$$0 = 2\eta_3^T(t)\mathcal{Q}[e(t-\vartheta_1)-e(t-\vartheta(t))-\int_{t-\vartheta(t)}^{t-\vartheta_1}\mathfrak{z}(s)ds-\int_{t-\vartheta(t)}^{t-\vartheta_1}\mathfrak{y}(s)dv(s)],$$
$$0 = 2\eta_3^T(t)\mathcal{R}[e(t-\vartheta(t))-e(t-\vartheta_2)-\int_{t-\vartheta_2}^{t-\vartheta(t)}\mathfrak{z}(s)ds-\int_{t-\vartheta_2}^{t-\vartheta(t)}\mathfrak{y}(s)dv(s)],$$
$$0 = 2\eta_3^T(t)\mathcal{S}[\vartheta_1 e(t)-\int_{t-\vartheta_1}^{t}e(s)ds-\Upsilon_2\mathfrak{z}(s)dsd\rho-\Upsilon_2\mathfrak{y}(s)dv(s)d\rho],$$
$$0 = 2\eta_3^T(t)\mathcal{T}[\vartheta_{21}e(t)-\int_{t-\vartheta_2}^{t-\vartheta_1}e(s)ds-\Upsilon_3\mathfrak{z}(s)dsd\rho-\Upsilon_3\mathfrak{y}(s)dv(s)d\rho],$$
$$0 = 2\eta_3^T(t)\mathcal{U}[\left(\frac{\vartheta_1^2}{2}\right)e(t)-\Upsilon_2 e(s)dsd\rho-\Upsilon_4\mathfrak{z}(s)dsd\chi d\rho-\Upsilon_4\mathfrak{y}(s)dv(s)d\chi d\rho],$$
$$0 = 2\eta_3^T(t)\mathcal{V}[\left(\frac{\vartheta_2^2-\vartheta_1^2}{2}\right)e(t)-\Upsilon_3 e(s)dsd\rho-\Upsilon_5\mathfrak{z}(s)dsd\chi d\rho-\Upsilon_5\mathfrak{y}(s)dv(s)d\chi d\rho],$$

where

$$\eta_3^T(t) = \left[e^T(t)\ \ e^T(t-\vartheta(t))\right],\qquad \mathcal{P} = \left[\mathcal{P}_1^T\ \ \mathcal{P}_2^T\right]^T,\qquad \mathcal{Q} = \left[\mathcal{Q}_1^T\ \ \mathcal{Q}_2^T\right]^T,$$
$$\mathcal{R} = \left[\mathcal{R}_1^T\ \ \mathcal{R}_2^T\right]^T,\qquad \mathcal{S} = \left[\mathcal{S}_1^T\ \ \mathcal{S}_2^T\right]^T,\qquad \mathcal{T} = \left[\mathcal{T}_1^T\ \ \mathcal{T}_2^T\right]^T,$$
$$\mathcal{U} = \left[\mathcal{U}_1^T\ \ \mathcal{U}_2^T\right]^T,\qquad \mathcal{V} = \left[\mathcal{V}_1^T\ \ \mathcal{V}_2^T\right]^T.$$

It is clear that

$$\begin{aligned}
&-2\eta_3^T(t)\mathcal{P}\int_{t-\vartheta_1}^{t}\mathfrak{y}(s)dv(s)\\
&\le \eta_3^T(t)\mathcal{P}Z_1^{-1}\mathcal{P}^T\eta_3(t)+[\int_{t-\vartheta_1}^{t}\mathfrak{y}(s)dv(s)]^T Z_1[\int_{t-\vartheta_1}^{t}\mathfrak{y}(s)dv(s)],\\
&\quad -2\eta_3^T(t)\mathcal{Q}\int_{t-\vartheta(t)}^{t-\vartheta_1}\mathfrak{y}(s)dv(s)
\end{aligned}$$

$$
\begin{aligned}
&\leq \eta_3^T(t)\mathcal{Q}Z_2^{-1}\mathcal{Q}^T\eta_3(t)+[\int_{t-\vartheta(t)}^{t-\vartheta_1}\mathfrak{y}(s)dv(s)]^TZ_2[\int_{t-\vartheta(t)}^{t-\vartheta_1}\mathfrak{y}(s)dv(s)],\\
&\quad -2\eta_3^T(t)\mathcal{R}\int_{t-\vartheta_2}^{t-\vartheta(t)}\mathfrak{y}(s)dv(s)\\
&\leq \eta_3^T(t)\mathcal{R}Z_2^{-1}\mathcal{R}^T\eta_3(t)+[\int_{t-\vartheta_2}^{t-\vartheta(t)}\mathfrak{y}(s)dv(s)]^TZ_2[\int_{t-\vartheta_2}^{t-\vartheta(t)}\mathfrak{y}(s)dv(s)],\\
&\quad -2\eta_3^T(t)\mathcal{S}\Upsilon_2\mathfrak{y}(s)dv(s)d\rho\leq \vartheta_1\eta_3^T(t)\mathcal{S}Z_3^{-1}\mathcal{S}^T\eta_3(t)+\int_{-\vartheta_1}^{0}\Upsilon_{10}^TZ_3\Upsilon_{10}d\rho,\\
&\quad -2\eta_3^T(t)\mathcal{T}\Upsilon_3\mathfrak{y}(s)dv(s)d\rho\\
&\leq \vartheta_{21}\eta_3^T(t)\mathcal{T}Z_4^{-1}\mathcal{T}^T\eta_3(t)+\int_{-\vartheta_1}^{0}[\int_{-\vartheta_2}^{-\vartheta_1}\mathfrak{y}(s)dv(s)]^TZ_4[\int_{-\vartheta_2}^{-\vartheta_1}\mathfrak{y}(s)dv(s)]d\rho,\\
&\quad -2\eta_3^T(t)\mathcal{U}\Upsilon_4\mathfrak{y}(s)dv(s)d\chi d\rho\\
&\leq \left(\frac{\vartheta_1^2}{2}\right)\eta_3^T(t)\mathcal{U}Z_5^{-1}\mathcal{U}^T\eta_3(t)+\int_{-\vartheta_1}^{0}\int_{\rho}^{0}\Upsilon_{10}^TZ_5\Upsilon_{10}d\chi d\rho,\\
&\quad -2\eta_3^T(t)\mathcal{V}\Upsilon_5\mathfrak{y}(s)dv(s)d\chi d\rho\\
&\leq \left(\frac{\vartheta_2^2-\vartheta_1^2}{2}\right)\eta_3^T(t)\mathcal{V}Z_6^{-1}\mathcal{V}^T\eta_3(t)+\int_{-\vartheta_2}^{-\vartheta_1}\int_{\rho}^{0}\Upsilon_{10}^TZ_6\Upsilon_{10}d\chi d\rho.
\end{aligned}
$$

where $\Upsilon_{10}=\int_{t+\rho}^{t}\mathfrak{y}(s)dv(s)$. From inequalities in Assumption (A2) and in (2.3), one can have

$$
\begin{aligned}
&0\leq \varepsilon_1[e^T(t)F^TFe(t)-\varphi^T(t)\varphi(t)],\\
&0\leq \varepsilon_2[e^T(t-\vartheta(t))F^TFe(t-\vartheta(t))-\varphi^T(t-\vartheta(t))\varphi(t-\vartheta(t))],\\
&0\leq \varepsilon_3[e^T(t)L^TLe(t)-\psi^T(t)\psi(t)].
\end{aligned}\tag{3.4}
$$

Define

$$
\begin{aligned}
\xi^T(t)=\big[&e^T(t)\ \ e^T(t-\vartheta(t))\ \ e^T(t-\vartheta_1)\ \ e^T(t-\vartheta_2)\ \ \varphi^T(t)\\
&\varphi^T(t-\vartheta(t))\ \ \varphi^T(t-\vartheta_1)\ \ \varphi^T(t-\vartheta_2)\ \ \psi^T(t)\ \ \Upsilon_1^T\ \ \Upsilon_8^T\ \ \Upsilon_9^T\big].
\end{aligned}
$$

Combining (3.3) to (3.4) and applying Schur complement and the $It\widehat{o}$ isometry ([13]), one can deduce that

$$
\begin{aligned}
\mathcal{L}V(x_t)\leq{}&\frac{6}{\vartheta_1^4}\int_{-\vartheta_1}^{0}\int_{\rho}^{0}\int_{t+\chi}^{t}\int_{-\vartheta_1}^{t}\xi^T(s)\widehat{\Pi}_1\xi(s)dsd\kappa d\chi d\rho+\left(\frac{6}{\vartheta_{21}(\vartheta_2^3-\vartheta_1^3)}\right)\\
&\times\int_{-\vartheta_2}^{-\vartheta_1}\int_{\rho}^{0}\int_{t+\chi}^{t}\int_{t-\vartheta(t)}^{t-\vartheta_1}\xi^T(s)\widehat{\Pi}_2\xi(s)dsd\sigma d\chi d\rho+\left(\frac{6}{\vartheta_{21}(\vartheta_2^3-\vartheta_1^3)}\right)\\
&\times\int_{-\vartheta_2}^{-\vartheta_1}\int_{\rho}^{0}\int_{t+\chi}^{t}\int_{t-\vartheta_2}^{t-\vartheta(t)}\xi^T(s)\widehat{\Pi}_3\xi(s)dsd\sigma d\chi d\rho.
\end{aligned}\tag{3.5}
$$

From (3.5) and Schur's complement Lemma, it can be seen that the error system (2.4) is asymptotically stable. □

Theorem 2. *For given positive scalars $\vartheta_1, \vartheta_2, \mu, d$, the error-state system (2.4) of MSNN (2.2) and (2.3) is asymptotically stable if there exist matrices $\mathbb{P} > 0$, $R_j > 0$, $(j = 1,2,3)$, $S_p > 0$, $(p = 1,2,3,4,5)$, $T_q > 0$, $Z_q > 0$, $q = (1,2,3,4,5,6)$, $G > 0$, and any appropriate dimensional matrices $\mathcal{P}_s$, $\mathcal{Q}_s$, $\mathcal{R}_s$, $\mathcal{S}_s$, $\mathcal{T}_s$, $\mathcal{U}_s$, $\mathcal{V}_s$, $(s = 1,2)$, such that the following LMI hold for $i = 1,2,3$:*

$$\widehat{\Xi}_i = \begin{bmatrix} \Xi_1 & \Xi_2^i \\ * & \Xi_3^i \end{bmatrix} < 0, \tag{3.6}$$

where

$$\begin{aligned}
\Xi_1 &= \Xi_{r\times s} \quad \text{where } i = 1,2,3,\ r,s = 1,2,\ldots,12,\\
\Xi_{1,1} &= -P_{11}A - A_{11}^T + \alpha_2 P_{11} + R_{11}^1 + \vartheta_1 S_1 + \vartheta_{21} S_2 + \mathcal{P}_1 + \mathcal{P}_1^T + \vartheta_1 \mathcal{S}_1 + \vartheta_1 \mathcal{S}_1^T\\
&\quad + \vartheta_{21}\mathcal{T}_1 + \vartheta_{21}\mathcal{T}_1^T + \frac{\vartheta_1^2}{2}\mathcal{U}_1 + \frac{\vartheta_1^2}{2}\mathcal{U}_1^T + \frac{\vartheta_2^2 - \vartheta_1^2}{2}\mathcal{V}_1 + \frac{\vartheta_2^2 - \vartheta_1^2}{2}\mathcal{V}_1^T\\
&\quad + \varepsilon_1 F^T F + \varepsilon_3 L^T L,\\
\Xi_{1,2} &= \Pi_{1,2}, \quad \Xi_{1,3} = \alpha_3 P_{11} - \alpha_2 P_{11} - \mathcal{P}_1 + \mathcal{Q}_1, \quad \Xi_{1,4} = -\mathcal{R}_1 - \alpha_3 P_{11},\\
\Xi_{1,5} &= P_{11}W_0 + \alpha_4 P_{11} + R_{12}^1, \quad \Xi_{1,6} = P_{11}W_1, \quad \Xi_{1,7} = \alpha_5 P_{11} - \alpha_4 P_{11},\\
\Xi_{1,8} &= -\alpha_5 P_{11}, \quad \Xi_{1,9} = -GC, \quad \Xi_{1,10} = P_{11}W_2, \quad \Xi_{1,11} = -\alpha_4 A^T P_{11} + P_{24},\\
\Xi_{1,12} &= -\alpha_5 A^T P_{11} + P_{25}, \quad \Xi_{2,2} = \Pi_{2,2}, \quad \Xi_{2,3} = \Pi_{2,3}, \quad \Xi_{2,4} = \Pi_{2,4},\\
\Xi_{2,6} &= \Pi_{2,6}, \quad \Xi_{3,3} = \Pi_{3,3}, \quad \Xi_{3,7} = \Pi_{3,7}, \quad \Xi_{3,11} = \Pi_{3,11}, \quad \Xi_{3,12} = \Pi_{3,12},\\
\Xi_{4,4} &= -R_{11}^3, \quad \Xi_{4,8} = -R_{12}^3, \quad \Xi_{4,11} = -P_{34}, \quad \Xi_{4,12} = -P_{35},\\
\Xi_{5,5} &= R_{22}^1 + d^2 S_3 + \vartheta_1^2 S_4 + \vartheta_{21}^2 S_5 - \varepsilon_1, \quad \Xi_{5,11} = \alpha_4 W_0^T P_{11} + P_{44},\\
\Xi_{5,12} &= \alpha_5 W_0^T P_{11} + P_{45}, \quad \Xi_{6,6} = -(1-\mu)R_{22}^2 - \varepsilon_2, \quad \Xi_{6,11} = \alpha_4 W_1^T P_{11},\\
\Xi_{6,12} &= \alpha_5 W_1^T P_{11}, \quad \Xi_{7,7} = -R_{22}^1 + R_{22}^2 + R_{22}^3, \quad \Xi_{7,11} = -P_{44} + P_{45}^T,\\
\Xi_{7,12} &= -P_{45} + P_{55}, \quad \Xi_{8,8} = -R_{22}^3, \quad \Xi_{8,11} = -P_{45}^T, \quad \Xi_{8,12} = -P_{55},\\
\Xi_{9,9} &= -\varepsilon_3, \quad \Xi_{9,11} = -\alpha_4 C^T G^T, \quad \Xi_{9,12} = -\alpha_5 C^T G^T, \quad \Xi_{10,10} = -S_3,\\
\Xi_{10,11} &= \alpha_4 W_2^T P_{11}, \quad \Xi_{10,12} = \alpha_5 W_2^T P_{11}, \quad \Xi_{11,11} = -S_4, \quad \Xi_{12,12} = -S_5,\\
\Xi_2^1 &= \Big[\Sigma_1 \ \ \Big(-\vartheta_1 \widetilde{\mathcal{P}} - \frac{\vartheta_1^2}{2}\widetilde{\mathcal{S}} - \frac{\vartheta_1^3}{6}\widetilde{\mathcal{U}}\Big) \ \ \widehat{E}^T P_{11} \ \ \vartheta_1 \widehat{A}^T P_{11} \ \ \frac{\vartheta_1^2}{2}\widehat{A}^T P_{11} \ \ \vartheta_1 \widehat{E}^T P_{11}\\
&\qquad \frac{\vartheta_1^3}{6}\widehat{A}^T P_{11} \ \ \frac{\vartheta_1^2}{2}\widehat{E}^T P_{11} \ \ \frac{\vartheta_1^3}{6}\widehat{E}^T P_{11} \ \ \widetilde{\mathcal{P}} \ \ \vartheta_1 \widetilde{\mathcal{S}} \ \ \frac{\vartheta_1^2}{2}\widetilde{\mathcal{U}}\Big],\\
\Xi_2^2 &= \Big[\Sigma_2 \ \ \Big(-\vartheta_{21}\widetilde{\mathcal{Q}} - \frac{\vartheta_2^2 - \vartheta_1^2}{2}\widetilde{\mathcal{T}} - \frac{\vartheta_2^3 - \vartheta_1^3}{6}\widetilde{\mathcal{V}}\Big) \ \ \widehat{E}^T P_{11} \ \ \vartheta_{21}\widehat{A}^T P_{11}
\end{aligned}$$

$$\frac{\vartheta_2^2-\vartheta_1^2}{2}\widehat{A}^T P_{11} \quad \frac{\vartheta_2^3-\vartheta_1^3}{6}\widehat{A}^T P_{11} \;\; \vartheta_{21}\widehat{E}^T P_{11} \;\; \frac{\vartheta_2^2-\vartheta_1^2}{2}\widehat{E}^T P_{11}$$
$$\frac{\vartheta_2^3-\vartheta_1^3}{6}\widehat{E}^T P_{11} \quad \widetilde{\mathcal{Q}} \quad \vartheta_{21}\widetilde{\mathcal{T}} \quad \frac{\vartheta_2^2-\vartheta_1^2}{2}\widetilde{\mathcal{V}}\Bigg],$$

$$\Xi_2^3=\Bigg[\Sigma_2 \;\; \left(-\vartheta_{21}\widetilde{\mathcal{R}}-\frac{\vartheta_2^2-\vartheta_1^2}{2}\widetilde{\mathcal{T}}-\frac{\vartheta_2^3-\vartheta_1^3}{6}\widetilde{\mathcal{V}}\right) \;\; \widehat{E}^T P_{11} \;\; \vartheta_{21}\widehat{A}^T P_{11}$$
$$\frac{\vartheta_2^2-\vartheta_1^2}{2}\widehat{A}^T P_{11} \;\; \frac{\vartheta_2^3-\vartheta_1^3}{6}\widehat{A}^T P_{11} \;\; \vartheta_{21}\widehat{E}^T P_{11} \;\; \frac{\vartheta_2^2-\vartheta_1^2}{2}\widehat{E}^T P_{11} \;\; \frac{\vartheta_2^3-\vartheta_1^3}{6}\widehat{E}^T P_{11}$$
$$\widetilde{\mathcal{R}} \quad \frac{\vartheta_2^2-\vartheta_1^2}{2}\widehat{A}^T P_{11} \;\; \frac{\vartheta_2^2-\vartheta_1^2}{2}\widetilde{\mathcal{V}}\Bigg],$$

$$\Xi_3^1=\Bigg[-\vartheta_1 S_1 \;\; \left(-\vartheta_1 T_1-\frac{\vartheta_1^2}{2}T_3-\frac{\vartheta_1^3}{6}T_5\right) \;\; -P_{11} \;\; \vartheta_1(-2P_{11}+T_1)$$
$$\frac{\vartheta_1^2}{2}(-2P_{11}+T_3) \;\; \frac{\vartheta_1^3}{6}(-2P_{11}+T_5) \;\; \vartheta_1(-2P_{11}+Z_1) \;\; \frac{\vartheta_1^2}{2}(-2P_{11}+Z_3)$$
$$\frac{\vartheta_1^3}{6}(-2P_{11}+Z_5) \;\; -Z_1 \;\; -\vartheta_1 Z_3 \;\; -\frac{\vartheta_1^2}{2}Z_5\Bigg],$$

$$\Xi_3^2=\Bigg[-\vartheta_{21} S_2 \;\; \left(-\vartheta_{21} T_2-\frac{\vartheta_2^2-\vartheta_1^2}{2}T_4-\frac{\vartheta_2^3-\vartheta_1^3}{6}T_6\right) \;\; -P_{11} \;\; \vartheta_{21}(-2P_{11}+T_2)$$
$$\frac{\vartheta_2^2-\vartheta_1^2}{2}(-2P_{11}+T_4) \;\; \frac{\vartheta_2^3-\vartheta_1^3}{6}(-2P_{11}+T_6) \;\; \vartheta_{21}(-2P_{11}+Z_2)$$
$$\frac{\vartheta_2^2-\vartheta_1^2}{2}(-2P_{11}+Z_4) \;\; \frac{\vartheta_2^3-\vartheta_1^3}{6}(-2P_{11}+Z_6)$$
$$-Z_2 \;\; -\vartheta_{21} Z_4 \;\; -\frac{\vartheta_2^2-\vartheta_1^2}{2}Z_6\Bigg],$$

$$\Xi_3^3=\Bigg[-\vartheta_{21} S_2 \;\; \left(-\vartheta_{21} T_2-\frac{\vartheta_2^2-\vartheta_1^2}{2}T_4-\frac{\vartheta_2^3-\vartheta_1^3}{6}T_6\right) \;\; -P_{11} \;\; \vartheta_{21}(-2P_{11}+T_2)$$
$$\frac{\vartheta_2^2-\vartheta_1^2}{2}(-2P_{11}+T_4) \;\; \frac{\vartheta_2^3-\vartheta_1^3}{6}(-2P_{11}+T_6) \;\; \vartheta_{21}(-2P_{11}+Z_2)$$
$$\frac{\vartheta_2^2-\vartheta_1^2}{2}(-2P_{11}+Z_4) \;\; \frac{\vartheta_2^3-\vartheta_1^3}{6}(-2P_{11}+Z_6)$$
$$-Z_2 \;\; -\vartheta_{21} Z_4 \;\; -\frac{\vartheta_2^2-\vartheta_1^2}{2}Z_6\Bigg],$$

$$\Sigma_1=\Bigg[\left(\vartheta_1 P_{22}-\alpha_2\vartheta_1 A^T P_{11}-\vartheta_1 \mathcal{S}_1-\frac{\vartheta_1^2}{2}\mathcal{U}_1\right)$$
$$\left(-\vartheta_1 P_{22}+\vartheta_1 P_{23}^T\right) \;\; -\vartheta_1 P_{23}^T \;\; \left(\alpha_2\vartheta_1 W_0^T P_{11}+\vartheta_1 P_{24}^T\right) \;\; \alpha_2\vartheta_1 W_1^T P_{11}$$

$$\left(-\vartheta_1 P_{24}^T + \vartheta_1 P_{25}^T\right) \quad -\vartheta_1 P_{25}^T \quad -\alpha_2 \vartheta_1 C^T G^T \quad \alpha_2 \vartheta_1 W_2^T P_{11} \quad 0 \quad 0\Big],$$

$$\Sigma_2 = \Bigg[\left(\vartheta_{21} P_{23} - \alpha_3 \vartheta_{21} A^T P_{11} - \vartheta_{21} \mathcal{T}_1 - \frac{\vartheta_2^2 - \vartheta_1^2}{2} \mathcal{V}_1\right)$$
$$\left(-\vartheta_{21} \mathcal{T}_2 - \frac{\vartheta_2^2 - \vartheta_1^2}{2} \mathcal{V}_2\right) \quad \left(\vartheta_{21} P_{33} - \vartheta_{21} P_{23}\right) \quad -\vartheta_{21} P_{33}$$
$$\left(\alpha_3 \vartheta_{21} W_0^T P_{11} + \vartheta_{21} P_{34}^T\right) \quad \alpha_3 \vartheta_{21} W_1^T P_{11} \quad \left(\vartheta_{21} P_{35}^T - \vartheta_{21} P_{34}^T\right) \quad -\vartheta_{21} P_{35}^T$$
$$-\alpha_3 \vartheta_{21} C^T G^T \quad \alpha_3 \vartheta_{21} W_2^T P_{11} \quad 0 \quad 0\Big],$$

$$\widehat{E} = [E \;\; E_1 \;\; 0 \;\; 0 \;\; H_0 \;\; H_1 \;\; 0 \;\; 0 \;\; 0 \;\; H_2 \;\; 0 \;\; 0],$$

$$\widehat{A} = \left[A \;\; 0 \;\; 0 \;\; 0 \;\; W_0 \;\; W_1 \;\; 0 \;\; 0 \;\; -P_{11}^{-1} GC \;\; W_2 \;\; 0 \;\; 0\right].$$

Furthermore, the gain matrix K of the state estimator of (2.4) can be designed as $K = P_{11}^{-1} G$.

Proof. Defining $P_{11} K = G$, $P_{1y} = \alpha_y P_{11}$, $(y = 2,3,4,5)$. Pre and post multiplying Π_1 in (3.1) by diag$\{\overbrace{I,\ldots,I}^{15 \text{ times}}, P_{11}T_1^{-1}, P_{11}T_3^{-1}, P_{11}T_5^{-1}, P_{11}Z_1^{-1}, P_{11}Z_3^{-1}, P_{11}Z_5^{-1}, I, I, I\}$ and diag$\{\overbrace{I,\ldots,I}^{15 \text{ times}}, T_1^{-1}P_{11}, T_3^{-1}P_{11}, T_5^{-1}P_{11}, Z_1^{-1}P_{11}, Z_3^{-1}P_{11}, Z_5^{-1}P_{11}, I, I, I\}$ respectively. As well as pre and post multiplying Π_2, Π_3 in (3.1) by diag$\{\overbrace{I,\ldots,I}^{15 \text{ times}}, P_{11}T_2^{-1}, P_{11}T_4^{-1}, P_{11}T_6^{-1}, P_{11}Z_2^{-1}, P_{11}Z_4^{-1}, P_{11}Z_6^{-1}, I, I, I\}$ and diag$\{\overbrace{I,\ldots,I}^{15 \text{ times}}, T_2^{-1}P_{11}, T_4^{-1}P_{11}, T_6^{-1}P_{11}, Z_2^{-1}P_{11}, Z_4^{-1}P_{11}, Z_6^{-1}P_{11}, I, I, I\}$ respectively. Then, through the inequalities (see [1])

$$(-2P_{11} + T_q) \geq -P_{11}T_q^{-1}P_{11}, \; (-2P_{11} + Z_q) \quad \geq -P_{11}Z_q^{-1}P_{11}, \; (q = 1,2,\ldots,6),$$

Ξ_i, (i=1,2,3) in (3.6) can be obtained. From Theorem 1, the error system (2.4) is asymptotically stable. □

4. Examples

A numerical example is given in this section. For convenience, denote $f_{ij}(t) = f_j(x_j(t)) - x_i(t)$, $f_{ij}(t - \vartheta(t)) = f_j(x_j(t - \vartheta(t))) - x_i(t)$ and $f_{ij}(t - d(t)) = \int_{t-d(t)}^{t} f_j(x_j(s))ds) - x_i(t))$.

Example 1. Consider a two-dimensional MSNNs (2.2):

$$w_{11}^0(t) = \begin{cases} -0.20, & f_{11}(t) \downarrow, s \in (t-\rho, t]; \\ 0.17, & f_{11}(t) \uparrow, s \in (t-\rho, t]; \\ \lim_{s \to t^-} w_{11}^0(s), & f_{11}(s) \text{ unchage } s \in (t-\rho, t], \end{cases}$$

$$w_{12}^{0}(t)=\begin{cases}0.15, & f_{12}(t)\ \downarrow, s\in(t-\rho,t];\\ -0.19, & f_{12}(t)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} w_{12}^{0}(s), & f_{12}(s)\text{ unchage } s\in(t-\rho,t],\end{cases}$$

$$w_{21}^{0}(t)=\begin{cases}0.30, & f_{21}(t)\ \downarrow, s\in(t-\rho,t];\\ 0.12, & f_{21}(t)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} w_{21}^{0}(s), & f_{21}(s)\text{ unchage } s\in(t-\rho,t],\end{cases}$$

$$w_{22}^{0}(t)=\begin{cases}0.10, & f_{22}(t)\ \downarrow, s\in(t-\rho,t];\\ -0.45, & f_{22}(t)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} w_{22}^{0}(s), & f_{22}(s)\text{ unchage } s\in(t-\rho,t],\end{cases}$$

$$w_{11}^{1}(t)=\begin{cases}0.15, & f_{11}(t-0.7)\ \downarrow, s\in(t-\rho,t];\\ 0.56, & f_{11}(t-0.7)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} w_{11}^{1}(s), & f_{11}(t-0.7)\text{ unchage } s\in(t-\rho,t],\end{cases}$$

$$w_{12}^{1}(t)=\begin{cases}0.35, & f_{12}(t-0.7)\ \downarrow, s\in(t-\rho,t];\\ -0.26, & f_{12}(t-0.7)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} w_{12}^{1}(s), & f_{12}(s-0.7)\text{ unchage } s\in(t-\rho,t],\end{cases}$$

$$w_{21}^{1}(t)=\begin{cases}-0.14, & f_{21}(t-0.7)\ \downarrow, s\in(t-\rho,t];\\ 0.35, & f_{21}(t-0.7)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} w_{21}^{1}(s), & f_{21}(s-0.7)\text{ unchage } s\in(t-\rho,t],\end{cases}$$

$$w_{22}^{1}(t)=\begin{cases}0.25, & f_{22}(t-0.7)\ \downarrow, s\in(t-\rho,t];\\ 0.05, & f_{22}(t-0.7)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} w_{22}^{1}(s), & f_{22}(s-0.7)\text{ unchage } s\in(t-\rho,t],\end{cases}$$

$$w_{11}^{2}(t)=\begin{cases}0.20, & f_{11}(t-0.2)\ \downarrow, s\in(t-\rho,t];\\ -0.13, & f_{11}(t-0.7)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} w_{11}^{2}(s), & f_{11}(s-0.7)\text{ unchage } s\in(t-\rho,t],\end{cases}$$

$$w_{12}^{2}(t)=\begin{cases}-0.01, & f_{12}(t-0.2)\ \downarrow, s\in(t-\rho,t];\\ -0.52, & f_{12}(t-0.2)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} w_{12}^{2}(s), & f_{12}(s-0.2)\text{ unchage } s\in(t-\rho,t],\end{cases}$$

$$w_{21}^{2}(t))=\begin{cases}0.30, & f_{21}(t-0.2)\ \downarrow, s\in(t-\rho,t];\\ 0.05, & f_{21}(t-0.2)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} w_{21}^{2}(s), & f_{21}(s-0.2)\text{ unchage } s\in(t-\rho,t],\end{cases}$$

$$w_{22}^{2}(t)=\begin{cases}0.40, & f_{22}(t-0.2)\ \downarrow, s\in(t-\rho,t];\\ 0.14, & f_{22}(t-0.2)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} w_{22}^{2}(s), & f_{22}(t-0.2)\text{ unchage } s\in(t-\rho,t],\end{cases}$$

$$h_{11}^{0}(t)=\begin{cases}0.12, & f_{11}(t)\ \downarrow, s\in(t-\rho,t];\\ -0.20, & f_{11}(t)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} h_{11}^{0}(s), & f_{11}(s)\text{ unchage } s\in(t-\rho,t],\end{cases}$$

$$h_{12}^{0}(t)=\begin{cases}0.02, & f_{12}(t)\ \downarrow, s\in(t-\rho,t];\\ -0.21, & f_{12}(t)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} h_{12}^{0}(s), & f_{12}(s)\text{ unchage } s\in(t-\rho,t],\end{cases}$$

$$h_{21}^{0}(t)=\begin{cases}0.30, & f_{21}(t)\ \downarrow, s\in(t-\rho,t];\\ 0.25, & f_{21}(t)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} h_{21}^{0}(s), & f_{21}(s)\text{ unchage } s\in(t-\rho,t],\end{cases}$$

$$h_{22}^{0}(t)=\begin{cases}-0.09, & f_{22}(t)\ \downarrow, s\in(t-\rho,t];\\ 0.12, & f_{22}(t)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} h_{22}^{0}(s), & f_{22}(s)\text{ unchage } s\in(t-\rho,t],\end{cases}$$

$$h_{11}^{1}(t)=\begin{cases}-0.11, & f_{11}(t-0.7)\ \downarrow, s\in(t-\rho,t];\\ -0.03, & f_{11}(t-0.7)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} h_{11}^{1}(s), & f_{11}(s-0.7)\text{ unchage } s\in(t-\rho,t],\end{cases}$$

$$h_{12}^{1}(t)=\begin{cases}0.20, & f_{12}(t-0.7)\ \downarrow, s\in(t-\rho,t];\\ 0.10, & f_{12}(t-0.7)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} h_{12}^{1}(s), & f_{12}(s-0.7)\text{ unchage } s\in(t-\rho,t],\end{cases}$$

$$h_{21}^{1}(t)=\begin{cases}-0.15, & f_{21}(t-0.7)\ \downarrow, s\in(t-\rho,t];\\ -0.10, & f_{21}(t-0.7)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} h_{21}^{1}(s), & f_{21}(s-0.7)\text{ unchage } s\in(t-\rho,t],\end{cases}$$

$$h_{22}^{1}(t)=\begin{cases}0.20, & f_{22}(t-0.7)\ \downarrow, s\in(t-\rho,t];\\ -0.05, & f_{22}(t-0.7)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} h_{22}^{1}(s), & f_{22}(s-0.7)\text{ unchage } s\in(t-\rho,t],\end{cases}$$

$$h_{11}^{2}(t)=\begin{cases}0.22, & f_{11}(t-0.2)\ \downarrow, s\in(t-\rho,t];\\ -0.19, & f_{11}(t-0.2)\ \uparrow, s\in(t-\rho,t];\\ \lim_{s\to t^-} h_{11}^{2}(s), & f_{11}(s-0.2)\text{ unchage } s\in(t-\rho,t],\end{cases}$$

$$h_{12}^2(t) = \begin{cases} -0.10, & f_{12}(t-0.2) \downarrow, s \in (t-\rho, t]; \\ 0.73, & f_{12}(t-0.2) \uparrow, s \in (t-\rho, t]; \\ \lim_{s\to t^-} h_{12}^2(s), & f_{12}(s-0.2) \text{ unchage } s \in (t-\rho, t], \end{cases}$$

$$h_{21}^2(t) = \begin{cases} 0.10, & f_{21}(t-0.2) \downarrow, s \in (t-\rho, t]; \\ -0.26, & f_{21}(t-0.2) \uparrow, s \in (t-\rho, t]; \\ \lim_{s\to t^-} h_{21}^2(s), & f_{21}(s-0.2) \text{ unchage } s \in (t-\rho, t], \end{cases}$$

$$h_{22}^2(t) = \begin{cases} 0.50, & f_{22}(t-0.2) \downarrow, s \in (t-\rho, t]; \\ -0.14, & f_{22}(t-0.2) \uparrow, s \in (t-\rho, t]; \\ \lim_{s\to t^-} h_{22}^2(s), & f_{22}(s-0.2) \text{ unchage } s \in (t-\rho, t], \end{cases}$$

with the activation function $f_j(x_j(\cdot)), j = 1,2$. and $L = 0.5I$, $F = I$, $C = 0.5I$, $a_1 = 2, a_2 = 1$, $be_1^0 = 0.2, e_2^0 = 0.3$, $e_{11}^1 = 0.1, e_{12}^1 = -0.1, e_{21}^1 = -0.05, e_{22}^1 = 0.05$, then, by solving the LMI in Theorem 2 with $d = 0.2, \vartheta_1 = 0.2, \vartheta_2 = 0.7$, $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0.1$, and $\mu = 0.5$ the estimator gain matrix is obtained as $K = [0.2283, \ -0.0035; 0.0577, \ 0.0808]$, which verifies the feasibility of Theorem 2.

5. Conclusion

In this article, the state estimation problem for MSNNs with mixed variable delays via a new quadruple stochastic integral LKF has been investigated. A novel state estimation analysis have been considered in terms of LMIs. A desired state estimator gain matrix is obtained by solving LMIs. An example is provided to explain the usefulness of the proposed approach. In the future, it is possible to extend this results to the delayed bidirectional associative memory stochastic NNs.

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CONFORMAL VECTOR FIELDS ON $N(k)$-PARACONTACT AND PARA-KENMOTSU MANIFOLDS

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Abstract. In this article, we initiate the study of conformal vector fields in paracontact geometry. First, we establish that if the Reeb vector field ζ is a conformal vector field on $N(k)$-paracontact metric manifold, then the manifold becomes a para-Sasakian manifold. Next, we show that if the conformal vector field $\mathbf{V}$ is pointwise collinear with the Reeb vector field ζ, then the manifold recovers a para-Sasakian manifold as well as $\mathbf{V}$ is a constant multiple of ζ. Furthermore, we prove that if a 3-dimensional $N(k)$-paracontact metric manifold admits a Killing vector field $\mathbf{V}$, then either it is a space of constant sectional curvature k or, $\mathbf{V}$ is an infinitesimal strict paracontact transformation. Besides these, we also investigate conformal vector fields on para-Kenmotsu manifolds.

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1. Introduction

Some intrinsic properties of contact metric manifolds can be explained beautifully by the existence of conformal vector fields (briefly, CVF) on a contact metric manifold. For example, it is well circulated that [18] if an n-dimensional Riemannian manifold admits a maximal, i.e., $\frac{(n+1)(n+2)}{2}$-parameter group of conformal motions, then it is conformally flat. It is well known that a Sasakian manifold which is conformally flat has a constant curvature 1 [12]. Again, it is to be noted that a complete and connected Sasakian manifold of dimension greater than three is isometric to sphere if it admits a conformal motion. This result was determined by Okumura in 1962 [11]. Later study of contact manifolds admitting conformal motions was extended to $N(k)$-contact metric manifolds by Sharma [15]. Sharma also introduced the notion of holomorphic planar CVF in Hermitian manifolds [14]. Planar CVF have also been studied in [7, 14].

A vector field $\mathbf{V}$ on $\mathbf{M}^{2n+1}$ satisfying the equation

$$\pounds_{\mathbf{V}} g = 2\rho g, \tag{1.1}$$

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ρ being a smooth function and $\pounds$ is the Lie-derivative, is called a CVF. It is said to be non-trivial, if $\mathbf{V}$ is not Killing. If ρ vanishes, then the CVF $\mathbf{V}$ is named Killing. $\mathbf{V}$ is called homothetic, if ρ is constant. CVF have been studied by many authors such as [4, 5, 9, 10, 15, 16] and many others. The CVF $\mathbf{V}$ satisfies the followings [18]

$$\begin{aligned}
(\pounds_{\mathbf{V}}\nabla)(X_1,Y_1) &= (X_1\rho)Y_1-(Y_1\rho)X_1-g(X_1,Y_1)D\rho,\\
(\pounds_{\mathbf{V}}R)(X_1,Y_1)Z_1 &= g(\nabla_{X_1}D\rho,Z_1)Y_1-g(\nabla_{Y_1}D\rho,Z_1)X_1\\
&\quad+g(X_1,Z_1)\nabla_{Y_1}D\rho-g(Y_1,Z_1)\nabla_{X_1}D\rho,\\
(\pounds_{\mathbf{V}}S)(X_1,Y_1) &= -(2n-1)g(\nabla_{X_1}D\rho,Y_1)-(\triangle\rho)g(X_1,Y_1),\\
\pounds_{\mathbf{V}}r &= -4n\triangle\rho-2r\rho
\end{aligned}$$

for all vector fields X_1,Y_1,Z_1 on $\mathbf{M}^{2n+1}$, where $D\rho$ and $\triangle\rho = divD\rho$ respectively denote the gradient and Laplacian of ρ. Here, R being the curvature tensor, S indicates the Ricci tensor and r denotes the scalar curvature.

Definition 1. A vector field $\mathbf{V}$ satisfying the relation

$$\pounds_{\mathbf{V}}\eta = \sigma\eta,$$

σ being a scalar function, is named an infinitesimal paracontact transformation. It is called an infinitesimal strict paracontact transformation, if σ vanishes identically.

Sharma and Vrancken [16] generalized the theorem of Tanno [17] and proved:

Theorem 1. *If the CVF on $\mathbf{M}$ is an infinitesimal contact transformation, then it is an infinitesimal automorphism of $\mathbf{M}$.*

In this paper we observe that the above result of Theorem 1 is exactly the same in paracontact geometry and proof is the same as in [16].

Now, we recollect the subsequent outcome of Okumura [11], which reveals that the existence of a non-Killing vector field places a significant condition on a Sasakian manifold of dimension > 3.

Theorem 2. *If $\mathbf{M}^{2n+1}$, $n > 1$ be a Sasakian manifold with a non-Killing CVF $\mathbf{V}$, then $\mathbf{V}$ is special concircular. $\mathbf{M}^{2n+1}$ is isometric to a unit sphere if it is connected and complete as well.*

Again, to generalize the foregoing result Sharma and Blair [15] proved the following:

Theorem 3. *On a $(k,0)$-contact metric manifold $\mathbf{M}^{2n+1}$, let $\mathbf{V}$ be a non-Killing CVF. $\mathbf{M}^{2n+1}$ is Sasakian and $\mathbf{V}$ is concircular for $n > 1$, hence if $\mathbf{M}^{2n+1}$ is complete, it is isometric to a unit sphere. $\mathbf{M}^{2n+1}$ is either Sasakian or flat for $n = 1$.*

To generalized the above result, Sharma and Vrancken [16] established the following theorem:

Theorem 4. *Let $\mathbf{M}^{2n+1}$ be a (k,μ)-contact metric manifold which admits a non-Killing CVF $\mathbf{V}$, for $n > 1$,*

(i) *$\mathbf{M}^{2n+1}$ is Sasakian and $\mathbf{V}$ is concircular, in which case if $\mathbf{M}^{2n+1}$ is complete then it is isometric to a unit sphere, or*

(ii) *$\mu = 1, k = -n-1$. In addition, $\mathbf{M}^{2n+1}$ is isometric to the unit sphere S^{2n+1} if it is compact.*

Very recently, De, Suh and Chaubey [4] studied CVF on almost co-Kähler manifolds.

Inspired by the foregoing studies we are interested to investigate CVF on paracontact geometry.

The present article is laid out as: In Section 2, we recollect some facts about paracontact manifolds, in particular $N(k)$-paracontact metric manifolds. In Section 3, we classify CVF on $N(k)$-paracontact metric manifolds. Finally, we investigate CVF on para-Kenmotsu manifolds.

2. Paracontact manifolds

An almost paracontact structure on an $(2n+1)$-dimensional manifold $\mathbf{M}$ consists of a (1,1)-tensor field ϕ, a vector field ζ and a one-form η obeying the subsequent conditions:

$$\phi^2 X_1 = X_1 - \eta(X_1)\zeta, \qquad \eta(\zeta) = 1. \tag{2.1}$$

The manifold $\mathbf{M}^{2n+1}$ with an almost paracontact structure is called an almost paracontact manifold. From the foregoing definition we recovers that $\phi\zeta = 0$, $\eta \circ \phi = 0$ and rank of ϕ is $2n$. If the Nijenhuis tensor vanishes identically, then the almost paracontact manifold is named normal. $\mathbf{M}^{2n+1}$ is called an almost paracontact metric manifold if there exists a semi-Riemannian metric g such that

$$g(\phi X_1, \phi Y_1) = -g(X_1, Y_1) + \eta(X_1)\eta(Y_1)$$

for all $X_1, Y_1 \in \chi(\mathbf{M})$.

In $\mathbf{M}^{2n+1}$ the fundamental 2-form is defined by $\Phi(X_1, Y_1) = g(X_1, \phi Y_1)$. If $d\eta(X_1, Y_1) = g(X_1, \phi Y_1)$, then $\mathbf{M}^{2n+1}, \phi, \zeta, \eta, g)$ is named paracontact metric manifold.

In $\mathbf{M}^{2n+1}$, one can introduce a symmetric, trace-free (1,1)-tensor $h = \frac{1}{2}\pounds_\zeta \phi$ fulfilling [2, 19]

$$\begin{aligned} \phi h + h\phi = 0, \qquad h\zeta = 0, \\ \nabla_{X_1}\zeta = -\phi X_1 + \phi h X_1, \end{aligned} \tag{2.2}$$

for all $X_1 \in \chi(\mathbf{M})$. It is noted that the condition $h = 0$ means that ζ is a Killing vector field and then (ϕ, ζ, η, g) is called K-paracontact structure. $\mathbf{M}^{2n+1}$ is named a para-Sasakian manifold if and only if [19]

$$(\nabla_{X_1}\phi)Y_1 = -g(X_1, Y_1)\zeta + \eta(Y_1)X_1$$

holds, for any X_1, Y_1. A para-Sasakian manifold satisfies the relation

$$R(X_1,Y_1)\zeta = -[\eta(Y_1)X_1 - \eta(X_1)Y_1] \tag{2.3}$$

for any X_1, Y_1, but in contrast to the contact metric geometry, the equation (2.3) does not mean that the paracontact manifold is para-Sasakian manifold. It is to be noted that every para-Sasakian manifold is K-paracontact, but the converse is not always true, it is true only for three dimensional case [1].

A $\mathbf{M}^{2n+1}$ is named a (k,μ)-paracontact manifold if the curvature tensor R obeys [2]

$$R(X_1,Y_1)\zeta = k[\eta(Y_1)X_1 - \eta(X_1)Y_1] + \mu[\eta(Y_1)hX_1 - \eta(X_1)hY_1]$$

for all vector fields $X_1, Y_1 \in \chi(\mathbf{M})$ and k, μ are real constants.

In particular, a (k,μ)-paracontact manifold turns into a $N(k)$-paracontact manifold for $\mu = 0$ and hence, the curvature tensor obeys

$$R(X_1,Y_1)\zeta = k[\eta(Y_1)X_1 - \eta(X_1)Y_1].$$

In a $N(k)$-paracontact metric manifold $(\mathbf{M}^3, \phi, \zeta, \eta, g)$, the subsequent results hold [3, 13]:

$$\begin{aligned} QX_1 &= (\frac{r}{2} - k)X_1 + (3k - \frac{r}{2})\eta(X_1)\zeta, \\ S(X_1,Y_1) &= (\frac{r}{2} - k)g(X_1,Y_1) + (3k - \frac{r}{2})\eta(X_1)\eta(Y_1), \\ S(X_1,\zeta) &= 2k\eta(X_1). \end{aligned} \tag{2.4}$$

Lemma 1 ([8], Theorem 3.1)**.** *Let $\mathbf{M}^{2n+1}$ be a paracontact metric manifold $R(X_1,Y_1)\zeta = -[\eta(Y_1)X_1 - \eta(X_1)Y_1]$, for all X_1, Y_1 on $\mathbf{M}^{2n+1}$. Then $\mathbf{M}^{2n+1}$ is para Sasakian if and only if ζ is a Killing vector field.*

Lemma 2. *Let $\mathbf{M}^{2n+1}$ be a $N(k)$-paracontact metric manifold and the Reeb vector field ζ is Killing. Then $\mathbf{M}^{2n+1}$ is a para-Sasakian manifold.*

Proof. In a $N(k)$-paracontact metric manifold,

$$R(X_1,Y_1)\zeta = k[\eta(Y_1)X_1 - \eta(X_1)Y_1]. \tag{2.5}$$

Contracting the above equation, we get

$$S(X_1,\zeta) = 2nk\eta(X_1),$$

which implies

$$Q\zeta = 2nk\zeta. \tag{2.6}$$

Now, if ζ is Killing in paracontact metric manifold, then it becomes a K-paracontact metric manifold. Hence $Q\zeta = -2n\zeta$. Therefore from (2.6), we get $k = -1$. Hence from (2.5), we obtain

$$R(X_1,Y_1)\zeta = -[\eta(Y_1)X_1 - \eta(X_1)Y_1].$$

Therefore, Lemma 1 implies that the manifold becomes a para-Sasakin manifold. This completes the proof. □

3. Conformal vector fields on $N(k)$-paracontact metric manifolds

Let us choose that the Reeb vector field ζ be a CVF on $\mathbf{M}^{2n+1}$. Then from (1.1), we get

$$(\pounds_\zeta g)(X_1,Y_1) = 2\rho g(X_1,Y_1), \tag{3.1}$$

which implies

$$g(\nabla_{X_1}\zeta,Y_1) + g(X_1,\nabla_{Y_1}\zeta) = 2\rho g(X_1,Y_1). \tag{3.2}$$

Contracting X_1 and Y_1 in (3.2) entails that

$$div\zeta = (2n+1)\rho. \tag{3.3}$$

Again, from (2.2), we infer

$$div\zeta = 0. \tag{3.4}$$

Equations (3.3) and (3.4) together imply

$$\rho = 0. \tag{3.5}$$

Using (3.5) in (3.1), we obtain

$$(\pounds_\zeta g)(X_1,Y_1) = 0,$$

which implies ζ is a Killing vector field. Hence from Lemma 2, we have:

Theorem 5. *If the Reeb vector field ζ of $\mathbf{M}^{2n+1}$ is a CVF, then $\mathbf{M}^{2n+1}$ becomes a para-Sasakian manifold.*

Suppose $\mathbf{V} = a\zeta$, where a is smooth function on $\mathbf{M}^{2n+1}$. Then from (2.1), we get

$$(\pounds_{a\zeta} g)(X_1,Y_1) = 2\rho g(X_1,Y_1), \tag{3.6}$$

which implies

$$g(\nabla_{X_1}a\zeta,Y_1) + g(X_1,\nabla_{Y_1}a\zeta) = 2\rho g(X_1,Y_1).$$

Using (2.2) in the above equation entails that

$$(X_1a)\eta(Y_1) + (Y_1a)\eta(X_1) - 2ag(h'X_1,2) = 2\rho g(X_1,Y_1). \tag{3.7}$$

Contracting (3.7), we provide

$$\zeta a = (2n+1)\rho. \tag{3.8}$$

Putting $Y_1 = \zeta$ in (3.7) and using (3.8) gives

$$X_1a = (1-2n)\rho\eta(X_1). \tag{3.9}$$

Above two equations together imply

$$\rho = 0.$$

Therefore from (3.9), we get $X_1a = 0$, which means that a is a constant. Again from (3.6), we get

$$(\pounds_{\mathbf{V}} g)(X_1,Y_1) = 0,$$

which implies $\mathbf{V}$ is Killing, that is, ζ is Killing. Hence we have:

Theorem 6. *If a CVF* $\mathbf{V}$ *in* $\mathbf{M}^{2n+1}$ *is pointwise collinear with the Reeb vector field* ζ*, then* $\mathbf{M}^{2n+1}$ *becomes a para-Sasakian manifold and* $\mathbf{V}$ *is a constant multiple of the Reeb vector field* ζ*.*

We assume that the vector field $\mathbf{V}$ in $\mathbf{M}^3$ is Killing. Then

$$(\pounds_{\mathbf{V}} g)(X_1, Y_1) = 0 \tag{3.10}$$

and

$$(\pounds_{\mathbf{V}} S)(X_1, Y_1) = 0. \tag{3.11}$$

Definition of Lie-derivative infers that

$$(\pounds_{\mathbf{V}} \eta)X_1 = \pounds_{\mathbf{V}} \eta(X_1) - \eta(\pounds_{\mathbf{V}} X_1).$$

Equation (3.10) implies

$$(\pounds_{\mathbf{V}} \eta)X_1 = g(\pounds_{\mathbf{V}} \zeta, X_1). \tag{3.12}$$

Also, we have

$$\eta(\pounds_{\mathbf{V}} \zeta) = 0 \qquad \text{and} \qquad (\pounds_{\mathbf{V}} \eta)\zeta = 0. \tag{3.13}$$

Now, we take Lie-derivative of the equation (2.4) along the Killing vector field $\mathbf{V}$ and get

$$\begin{aligned}(\pounds_{\mathbf{V}} S)(X_1, Y_1) =& \frac{Vr}{2}[g(X_1, Y_1) - \eta(X_1)\eta(Y_1)] \\ &+ (3k - \frac{r}{2})[(\pounds_{\mathbf{V}} \eta)X_1 \eta(Y_1) + (\pounds_{\mathbf{V}} \eta)Y_1 \eta(X_1)].\end{aligned} \tag{3.14}$$

Using (3.11) and (3.12) in (3.14), we provide

$$\begin{aligned}&\frac{Vr}{2}[g(X_1, Y_1) - \eta(X_1)\eta(Y_1)] \\ &+ (3k - \frac{r}{2})[g(X_1, \pounds_{\mathbf{V}} \zeta)\eta(Y_1) + g(\pounds_{\mathbf{V}} \zeta, Y_1)\eta(X_1)] = 0.\end{aligned} \tag{3.15}$$

Setting $Y_1 = \zeta$ in (3.15) and using (3.13) gives

$$(6k - r)g(\pounds_{\mathbf{V}} \zeta, X_1) = 0,$$

which implies either $r = 6k$ or, $g(\pounds_{\mathbf{V}} \zeta, X_1) = 0$.

Case I: If $r = 6k$, then (2.4) implies

$$S(X_1, Y_1) = 2kg(X_1, Y_1).$$

Hence, we get

$$R(X_1, Y_1)Z_1 = k[g(Y_1, Z_1)X_1 - g(X_1, Z_1)Y_1],$$

which means that it is a space of constant sectional curvature k.

Case II: If $g(\pounds_{\mathbf{V}} \zeta, X_1) = 0$, then (3.12) implies

$$(\pounds_{\mathbf{V}} \eta)X_1 = 0,$$

which means that the vector field $\mathbf{V}$ is an infinitesimal strict paracontact transformation. Hence we have:

Theorem 7. *If M^3 admits a Killing vector field* $\mathbf{V}$*, then it is either space of constant sectional curvature k or the vector field* $\mathbf{V}$ *is an infinitesimal strict paracontact transformation.*

4. PARA-KENMOTSU MANIFOLDS

An almost paracontact manifold with the structure (ϕ,ζ,η,g) is named an almost para-Kenmotsu manifold, if

$$d\eta = 0, \qquad d\Phi = 2\eta\wedge\Phi.$$

In a para-Kenmotsu manifold the following relations hold [6]:

$$\begin{aligned}
R(X_1,Y_1)\zeta &= \eta(X_1)Y_1 - \eta(Y_1)X_1,\\
R(X_1,\zeta)Y_1 &= g(X_1,Y_1)\zeta - \eta(Y_1)X_1,\\
R(\zeta,X_1)Y_1 &= \eta(Y_1)X_1 - g(X_1,Y_1)\zeta,\\
\eta(R(X_1,Y_1)Z_1) &= g(X_1,Z_1)\eta(Y_1) - g(Y_1,Z_1)\eta(X_1),\\
(\nabla_{X_1}\phi)Y_1 &= g(\phi X_1,Y_1)\zeta - \eta(Y_1)\phi X_1,\\
\nabla_{X_1}\zeta &= X_1 - \eta(X_1)\zeta,\\
S(X_1,\zeta) &= -2n\eta(X_1).
\end{aligned} \tag{4.1}$$

Also in a 3-dimensional para-Kenmotsu manifold, we have

$$QX_1 = (\frac{r}{2}+1)X_1 - (\frac{r}{2}+3)\eta(X_1)\zeta,$$

which implies

$$S(X_1,Y_1) = (\frac{r}{2}+1)g(X_1,Y_1) - (\frac{r}{2}+3)\eta(X_1)\eta(Y_1). \tag{4.2}$$

5. CONFORMAL VECTOR FIELDS ON PARA-KENMOTSU MANIFOLDS

We suppose that the Reeb vector field ζ is a CVF. Then from (1.1), we get

$$(\pounds_\zeta g)(X_1,Y_1) = 2\rho g(X_1,Y_1), \tag{5.1}$$

which implies

$$g(\nabla_{X_1}\zeta,Y_1) + g(X_1,\nabla_{Y_1}\zeta) = 2\rho g(X_1,Y_1). \tag{5.2}$$

Using (4.1) in (5.2), we provide

$$2[g(X_1,Y_1) - \eta(X_1)\eta(Y_1)] = 2\rho g(X_1,Y_1). \tag{5.3}$$

Setting $X_1 = Y_1 = \zeta$ in (5.3) entails that

$$\rho = 0. \tag{5.4}$$

Using (5.4) in (5.1), we obtain

$$(\pounds_\zeta g)(X_1,Y_1) = 0,$$

which implies ζ is a Killing vector field. However the Reeb vector field ζ is not a Killing vector field on para-Kenmotsu manifolds. Hence we have:

Theorem 8. *In a para-Kenmotsu manifold the Reeb vector field ζ can not be a CVF.*

We assume that the vector field $\mathbf{V}$ in $\mathbf{M}^3$ is Killing. Then

$$(\pounds_{\mathbf{V}} g)(X_1, Y_1) = 0. \tag{5.5}$$

From the definition of Lie-derivative, we infer

$$(\pounds_{\mathbf{V}} \eta)X_1 = \pounds_{\mathbf{V}} \eta(X_1) - \eta(\pounds_{\mathbf{V}} X_1).$$

Equation (5.5) implies

$$(\pounds_{\mathbf{V}} \eta)X_1 = g(\pounds_{\mathbf{V}} \zeta, X_1). \tag{5.6}$$

Also,

$$\eta(\pounds_{\mathbf{V}} \zeta) = 0 \qquad \text{and} \qquad (\pounds_{\mathbf{V}} \eta)\zeta = 0. \tag{5.7}$$

Taking the Lie-derivative of (4.2), we get

$$\begin{aligned}(\pounds_{\mathbf{V}} S)(X_1, Y_1) =& \frac{Vr}{2}[g(X_1, Y_1) - \eta(X_1)\eta(Y_1)] \\ & - (3 + \frac{r}{2})[g(X_1, \pounds_{\mathbf{V}} \zeta)\eta(Y_1) + g(Y_1, \pounds_{\mathbf{V}} \zeta)\eta(X_1)].\end{aligned}$$

Since, $\mathbf{V}$ is Killing vector field, then

$$(\pounds_{\mathbf{V}} S)(X_1, Y_1) = 0.$$

Above two equations together imply

$$\begin{aligned}&\frac{Vr}{2}[g(X_1, Y_1) - \eta(X_1)\eta(Y_1)] \\ &- (3 + \frac{r}{2})[g(X_1, \pounds_{\mathbf{V}} \zeta)\eta(Y_1) + g(Y_1, \pounds_{\mathbf{V}} \zeta)\eta(X_1)] = 0.\end{aligned} \tag{5.8}$$

Putting $X_1 = \zeta$ in (5.8) gives

$$(3 + \frac{r}{2})[\eta(\pounds_{\mathbf{V}} \zeta)\eta(Y_1) + g(\pounds_{\mathbf{V}} \zeta, Y_1)] = 0. \tag{5.9}$$

Using (5.7) in (5.9), we obtain

$$(r + 6)g(\pounds_{\mathbf{V}} \zeta, Y_1) = 0,$$

which implies either $r + 6 = 0$ or, $g(\pounds_{\mathbf{V}} \zeta, Y_1) = 0$.

Case I: If we take $r + 6 = 0$, then from (4.2) entails that

$$S(X_1, Y_1) = -2g(X_1, Y_1),$$

which means the manifold is Einstein. Hence we have

$$R(X_1, Y_1)Z_1 = -[g(Y_1, Z_1)X_1 - g(X_1, Z_1)Y_1],$$

which is a space of constant sectional curvature -1.

Case II: If $g(\pounds_{\mathbf{V}}\zeta, Y_1) = 0$, which implies $\pounds_{\mathbf{V}}\zeta = 0$. Hence from (5.6), we get

$$(\pounds_{\mathbf{V}}\eta)Y_1 = 0.$$

Hence from Definition 1 we have:

Theorem 9. *If M^3 admits a Killing vector field $\mathbf{V}$, then it is either locally isometric to $\mathbb{H}^3(1)$ or the vector field $\mathbf{V}$ is an infinitesimal strict paracontact transformation.*

Now, we like to state a lemma before proving the next theorem which is similar to the proof of the result of Sharma's paper [16].

Lemma 3. *If $\mathbf{V}$ is a CVF on a para-Kenmotsu manifold, then*

(i) $(\pounds_{\mathbf{V}}\eta)\zeta = \rho$ *and*
(ii) $\eta(\pounds_{\mathbf{V}}\zeta) = -\rho$.

Theorem 10. *Let $\mathbf{V}$ be a CVF on a para-Kenmotsu manifold $\mathbf{M}^{2n+1}$. If $\mathbf{V}$ is an infinitesimal paracontact transformation, then $\mathbf{V}$ is a homothetic vector field.*

Proof. Let $\mathbf{V}$ be a CVF, then by definition

$$(\pounds_{\mathbf{V}}g)(X_1, Y_1) = 2\rho g(X_1, Y_1),$$

which implies

$$\pounds_{\mathbf{V}}g(X_1, Y_1) - g(\pounds_{\mathbf{V}}X_1, Y_1) - g(X_1, \pounds_{\mathbf{V}}Y_1) = 2\rho g(X_1, Y_1). \tag{5.10}$$

Setting $Y = \zeta$ in (5.10), we provide

$$\pounds_{\mathbf{V}}\zeta = (\sigma - 2\rho)\zeta.$$

Using Lemma 3 in the foregoing equation gives

$$\sigma = \rho. \tag{5.11}$$

Equation (5.11) and Definition 1 together imply

$$\pounds_{\mathbf{V}}\eta = \rho\eta \qquad \text{and} \qquad \pounds_{\mathbf{V}}\zeta = -\rho\zeta.$$

The above equation implies

$$(\pounds_{\mathbf{V}}d\eta)(X_1, Y_1) = \frac{1}{2}[(X_1\rho)\eta(Y_1) - (Y_1\rho)\eta(X_1)] + \rho d\eta(X_1, Y_1). \tag{5.12}$$

Since in para-Kenmotsu manifold $d\eta = 0$, then (5.12) implies

$$(X_1\rho)\eta(Y_1) = (Y_1\rho)\eta(X_1),$$

which implies

$$X_1\rho = (\zeta\rho)\eta(X_1).$$

If we take $\zeta\rho = 0$, then the above equation implies $X_1\rho = 0$, which means ρ is a constant. Hence from (5.11), we get $\sigma = \rho =$ constant. This completes the proof. □

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CONVERGENCE, OPTIMAL POINTS AND APPLICATIONS

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Abstract. In this paper, we focus on the existence of the best proximity points in binormed linear spaces. As a consequence, we obtain some fixed point results. We also provide some illustrations to support our claims. As applications, we obtain the existence of a solution to split feasible and variational inequality problems.

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1. Introduction and preliminaries

In the fixed point approach, for two given non-empty subsets $\mathcal{E}_1$ and $\mathcal{E}_2$ of a metric space (χ, d), a non-self mapping $\Im : \mathcal{E}_1 \to \mathcal{E}_2$ the idea of a fixed point is not appropriate when the intersection of $\mathcal{E}_1$ and $\mathcal{E}_2$ is empty. If a mapping $\Im$ has a solution and the intersection of $\mathcal{E}_1$ and $\mathcal{E}_2$ is non-empty, then $\Im$ has a fixed point. Banach contraction theorem (BCT) plays an important role in nonlinear analysis. Due to its simplicity and applicability, it helps solve many kinds of nonlinear problems. This fact motivated researchers to try to extend and generalize BCT so that its area of applications should be as vast as possible. In 1968, Maia established a very interesting and beautiful generalization of BCT using assumptions on two comparable metrics defined on the set χ. Consider the case when the fixed point equation $\Im \grave{u} = \grave{u}$ has no solution in this case $d(\mathcal{E}_1, \mathcal{E}_2) > 0$. In this affair, it is interesting to find an approximate solution $\grave{u}$ such that the error $d(\grave{u}, \Im \grave{u})$ is minimum in some sense. For a nonself mapping $\Im : \mathcal{E}_1 \to \mathcal{E}_2$ a point $\grave{u}$, known as a best proximity point if satisfies the following condition

$$d(\grave{u}, \Im \grave{u}) = d(\mathcal{E}_1, \mathcal{E}_2) = \inf\{d(\grave{u}, \grave{v}) : \grave{u} \in \mathcal{E}_1, \grave{v} \in \mathcal{E}_2\}.$$

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In 1969, Fan [8] gave the classical best approximation theorem in the context of a Hausdorff locally convex topological vector space χ. After that, many authors studied the best proximity point problems in metric space or normed space (see [16–19] and references cited therein). In 2006, Eldred and Veeramani [7] proved the existence of a best proximity point for cyclic contraction mappings. After that, many authors extend the Eldred and Veeramani [7] result. In 2009, Suzuki et al. [20] proved the best proximity result using the property UC. In 2010, Kosuru et al. [12] investigated the best proximity pair result for cyclic maps using weak proximal normal structure. In 2013, Gabeleh and Abkar [2] proved the best proximity pair result for cyclic maps using proximal quasi normal structure. In 2019, Petruşel and Petruşel [15] obtained some coupled fixed point and best proximity point results satisfying orbital contraction condition. Recently, Hafshejani [9] study the existence and uniqueness of best proximity points for the generalized cyclic quasi-contraction mappings using a geometrical concept of ultrametric property.

In this paper, we investigate the existence of best proximity points in the context of binormed linear spaces. We also provide some illustrations to back up our work. As an application of our obtained results, we find the solution of split feasible and variational inequality problems.

To prove the main result of this paper, we need the following definition and lemmas in the sequel:

Definition 1. A normed vector space χ is said to be a uniformly convex Banach space [6], if for every $0 < \varepsilon \leq 2$ there is some $\delta > 0$ such that for any two vectors with $||\grave{u}|| = 1$ and $||\grave{v}|| = 1$, the condition

$$||\grave{u} - \grave{v}|| \geq \varepsilon,$$

implies

$$\left|\left|\frac{\grave{u} + \grave{v}}{2}\right|\right| \leq 1 - \delta.$$

Lemma 1 ([7])**.** *Let $\mathcal{E}_2$ be a nonempty closed subset and $\mathcal{E}_1$ be a nonempty convex and closed subset of a uniformly convex Banach space (UCBS) χ. Let $\{\grave{u}_n\}$ and $\{\grave{z}_n\}$ be sequences in $\mathcal{E}_1$ and $\{\grave{v}_n\}$ be a sequence in $\mathcal{E}_2$ satisfying*

(i) $||\grave{z}_n - \grave{v}_n|| \to d(\mathcal{E}_1, \mathcal{E}_2)$,

(ii) *for every $\varepsilon > 0$ there exists N_0 such that for all $m > n \geq N_0$,*

$$||\grave{u}_n - \grave{v}_n|| \leq d(\mathcal{E}_1, \mathcal{E}_2) + \varepsilon.$$

Then, for every $\varepsilon > 0$ there exists N_1 such that for all $m > n \geq N_1$, $||\grave{u}_n - \grave{z}_n|| \leq \varepsilon$.

Lemma 2 ([7])**.** *Let $\mathcal{E}_2$ be a nonempty closed subset and $\mathcal{E}_1$ be a nonempty convex and closed subset of a uniformly convex Banach space χ. Let $\{\grave{u}_n\}$ and $\{\grave{z}_n\}$ be sequences in $\mathcal{E}_1$ and $\{\grave{v}_n\}$ be a sequence in $\mathcal{E}_2$ satisfying*

(i) $||\grave{z}_n - \grave{v}_n|| \to d(\mathcal{E}_1, \mathcal{E}_2)$,

(ii) $||\grave{u}_n - \grave{v}_n|| \to d(\mathcal{E}_1, \mathcal{E}_2)$.

Then $||\grave{u}_n - \grave{z}_n||$ *converges to zero.*

2. Main results

First, we prove a very useful approximation result.

Proposition 1. *Let* $\mathcal{E}_2$ *and* $\mathcal{E}_1$ *be nonempty subsets of a metric space* (χ, d). *Suppose that* $\Im : \mathcal{E}_1 \cup \mathcal{E}_2 \to \mathcal{E}_1 \cup \mathcal{E}_2$ *is an operator fulfilling the following hypotheses:*

($\mathcal{T}_1$) $\Im(\mathcal{E}_1) \subseteq \mathcal{E}_2$ *and* $\Im(\mathcal{E}_2) \subseteq \mathcal{E}_1$,

($\mathcal{T}_2$) *there exist* $\alpha', \beta', \gamma', \delta'$ *are nonnegative real numbers with* $\alpha' + \beta' + \gamma' + 2\delta' < 1$ *such that*

$$\begin{aligned} d(\Im\grave{u}, \Im\grave{v}) \leq\ & \alpha' d(\grave{u}, \Im\grave{u}) + \beta' d(\grave{v}, \Im\grave{v}) + \gamma' d(\grave{u}, \grave{v}) + \delta'(d(\grave{u}, \Im\grave{v}) + d(\grave{v}, \Im\grave{u})) \\ & + (1 - \alpha' - \beta' - \gamma' - 2\delta')\, d(\mathcal{E}_1, \mathcal{E}_2), \end{aligned} \tag{2.1}$$

for all $\grave{u} \in \mathcal{E}_1$ *and* $\grave{v} \in \mathcal{E}_2$. *If* $\grave{u}_0 \in \mathcal{E}_1$ *and* $\grave{u}_{n+1} = \Im\grave{u}_n$ *where* $n \in \mathbb{N} \cup \{0\}$, *then* $d(\grave{u}_n, \Im\grave{u}_n) \to d(\mathcal{E}_1, \mathcal{E}_2)$.

Proof. Consider

$$\begin{aligned} d(\grave{u}_n, \grave{u}_{n+1}) =\ & d(\Im\grave{u}_{n-1}, \Im\grave{u}_n) \\ \leq\ & \alpha' d(\grave{u}_{n-1}, \Im\grave{u}_{n-1}) + \beta' d(\grave{u}_n, \Im\grave{u}_n) + \gamma' d(\grave{u}_{n-1}, \grave{u}_n) \\ & + \delta' d(\grave{u}_{n-1}, \Im\grave{u}_n) + \delta' d(\grave{u}_n, \Im\grave{u}_{n-1}) + (1 - \alpha' - \beta' - \gamma' - 2\delta')\, d(\mathcal{E}_1, \mathcal{E}_2) \\ =\ & \alpha' d(\grave{u}_{n-1}, \grave{u}_n) + \beta' d(\grave{u}_n, \grave{u}_{n+1}) + \gamma' d(\grave{u}_{n-1}, \grave{u}_n) + \delta' d(\grave{u}_{n-1}, \grave{u}_{n+1}) \\ & + \delta' d(\grave{u}_n, \grave{u}_n) + (1 - \alpha' - \beta' - \gamma' - 2\delta')\, d(\mathcal{E}_1, \mathcal{E}_2) \\ \leq\ & \alpha' d(\grave{u}_{n-1}, \grave{u}_n) + \beta' d(\grave{u}_n, \grave{u}_{n+1}) + \gamma' d(\grave{u}_{n-1}, \grave{u}_n) + \delta' d(\grave{u}_{n-1}, \grave{u}_n) \\ & + \delta' d(\grave{u}_n, \grave{u}_{n+1}) + (1 - \alpha' - \beta' - \gamma' - 2\delta')\, d(\mathcal{E}_1, \mathcal{E}_2). \end{aligned} \tag{2.2}$$

Rewriting equation (2.2), we have

$$\begin{aligned} d(\grave{u}_n, \grave{u}_{n+1}) \leq\ & \frac{\alpha' + \gamma' + \delta'}{(1 - \beta' - \delta')} d(\grave{u}_{n-1}, \grave{u}_n) + \frac{(1 - \alpha' - \beta' - \gamma' - 2\delta')}{(1 - \beta' - \delta')}\, d(\mathcal{E}_1, \mathcal{E}_2) \\ d(\grave{u}_n, \grave{u}_{n+1}) =\ & k d(\grave{u}_{n-1}, \grave{u}_n) + (1 - k)\, d(\mathcal{E}_1, \mathcal{E}_2)\ ;\ \ k = \frac{\alpha' + \gamma' + \delta'}{1 - \beta' - \delta'} < 1, \\ d(\grave{u}_n, \grave{u}_{n+1}) \leq\ & k d(\grave{u}_{n-1}, \grave{u}_n) + (1 - k) d(\grave{u}_{n-1}, \grave{u}_n) \\ d(\grave{u}_n, \grave{u}_{n+1}) \leq\ & d(\grave{u}_{n-1}, \grave{u}_n), \end{aligned} \tag{2.3}$$

for all $n \in \mathbb{N}$. Therefore, $\{d(\grave{u}_n, \grave{u}_{n+1})\}$ is a bounded below and decreasing sequence, so there exists $r \geq 0$ such that

$$r = \lim_{n \to \infty} d(\grave{u}_n, \grave{u}_{n+1}).$$

Again consider

$$\begin{aligned}
d(\grave{u}_n,\grave{u}_{n+1}) &= d(\Im\grave{u}_{n-1},\Im\grave{u}_n)\\
&\leq \alpha' d(\grave{u}_{n-1},\Im\grave{u}_{n-1})+\beta' d(\grave{u}_n,\Im\grave{u}_n)+\gamma'(\grave{u}_{n-1},\grave{u}_n)+\\
&\quad +\delta'(\grave{u}_{n-1},\Im\grave{u}_n)+\delta'(\grave{u}_n,\Im\grave{u}_{n-1})+(1-\alpha'-\beta'-\gamma'-2\delta')\,d(\mathcal{E}_1,\mathcal{E}_2)\\
&= \alpha' d(\grave{u}_{n-1},\grave{u}_n)+\beta' d(\grave{u}_n,\grave{u}_{n+1})+\gamma' d(\grave{u}_{n-1},\grave{u}_n)+\delta' d(\grave{u}_{n-1},\grave{u}_{n+1})\\
&\quad +\delta' d(\grave{u}_n,\grave{u}_n)+(1-\alpha'-\beta'-\gamma'-2\delta')\,d(\mathcal{E}_1,\mathcal{E}_2)\\
&\leq \alpha' d(\grave{u}_{n-1},\grave{u}_n)+\beta' d(\grave{u}_n,\grave{u}_{n+1})+\gamma' d(\grave{u}_{n-1},\grave{u}_n)+\delta' d(\grave{u}_{n-1},\grave{u}_n)\\
&\quad +\delta' d(\grave{u}_n,\grave{u}_{n+1})+(1-\alpha'-\beta'-\gamma'-2\delta')\,d(\mathcal{E}_1,\mathcal{E}_2)\\
&\leq \frac{\alpha'+\gamma'+\delta'}{(1-\beta'-\delta')}d(\grave{u}_{n-1},\grave{u}_n)+\frac{(1-\alpha'-\beta'-\gamma'-2\delta')}{(1-\beta'-\delta')}\,d(\mathcal{E}_1,\mathcal{E}_2)\\
&= kd(\grave{u}_{n-1},\grave{u}_n)+(1-k)\,d(\mathcal{E}_1,\mathcal{E}_2)\\
&= k^2 d(\grave{u}_{n-2},\grave{u}_{n-1})+(1-k^2)\,(\mathcal{E}_1,\mathcal{E}_2)\\
&\leq k^3 d(\grave{u}_{n-2},\grave{u}_{n-1})+(1-k^3)\,d(\mathcal{E}_1,\mathcal{E}_2)\\
&\ \ \vdots\\
&\leq k^n d(\grave{u}_0,\grave{u}_1)+(1-k^n)\,d(\mathcal{E}_1,\mathcal{E}_2). \qquad (2.4)
\end{aligned}$$

Since $k<1$, $k^n\to 0$ as $n\to+\infty$, we have

$$d(\grave{u}_n,\grave{u}_{n+1})\to d(\mathcal{E}_1,\mathcal{E}_2). \qquad (2.5)$$

□

Next, we prove an existence result for a best proximity point.

Theorem 1. *Let $\mathcal{E}_2$ and $\mathcal{E}_1$ be nonempty subsets of a metric space (χ,d). Suppose that $\Im:\mathcal{E}_1\cup\mathcal{E}_2\to\mathcal{E}_1\cup\mathcal{E}_2$ is an operator fulfilling the following hypotheses:*

($\mathcal{T}_1$) *$\Im(\mathcal{E}_1)\subseteq\mathcal{E}_2$ and $\Im(\mathcal{E}_2)\subseteq\mathcal{E}_1$,*

($\mathcal{T}_2$) *χ is complete,*

($\mathcal{T}_3$) *there exist $\alpha',\beta',\gamma',\delta'$ are nonnegative real numbers with $\alpha'+\beta'+\gamma'+2\delta'<1$ such that*

$$\begin{aligned}
d(\Im\grave{u},\Im\grave{v}) \leq\ & \alpha' d(\grave{u},\Im\grave{u})+\beta' d(\grave{v},\Im\grave{v})+\gamma' d(\grave{u},\grave{v})+\delta'(d(\grave{u},\Im\grave{v})+d(\grave{v},\Im\grave{u}))\\
&+(1-\alpha'-\beta'-\gamma'-2\delta')\,d(\mathcal{E}_1,\mathcal{E}_2), \qquad (2.6)
\end{aligned}$$

for all $\grave{u}\in\mathcal{E}_1$ and $\grave{v}\in\mathcal{E}_2$. Suppose $\grave{u}_0\in\mathcal{E}_1$ and $\grave{u}_{n+1}=\Im\grave{u}_n$ where $n\in\mathbb{N}\cup\{0\}$. If $\{u_{2n}\}$ has a convergent subsequence in $\mathcal{E}_1$ then $\Im$ has a best proximity point.

Proof. Let $\left\{\grave{u}_{2n(k')}\right\}$ be a subsequence of $\{u_{2n}\}$ which converges to a point $\grave{u}\in\mathcal{E}_1$. Now

$$d(\grave{u},\grave{u}_{2n(k')-1})\leq d(\grave{u},\grave{u}_{2n(k')})+d(\grave{u}_{2n(k')},\grave{u}_{2n(k')-1}). \qquad (2.7)$$

Taking $n \to \infty$ in (2.7), we get

$$d(\grave{u}, \grave{u}_{2n(k')-1}) \to d(\mathcal{E}_1, \mathcal{E}_2).$$

Since $d(\mathcal{E}_1, \mathcal{E}_2) \leq d(\grave{u}_{2n(k')}, \Im\grave{u}) \leq d(\grave{u}_{2n(k')-1}, u)$. Then $\Im$ has a best proximity point. □

Theorem 2. *Let $\mathcal{E}_2$ be a nonempty closed subset and $\mathcal{E}_1$ be a nonempty convex and closed subset of a uniformly convex binormed linear space $(\chi, ||.||_1, ||.||)$ with $||.||_1 \leq ||.||$. Suppose that $\Im : \mathcal{E}_1 \cup \mathcal{E}_2 \to \mathcal{E}_1 \cup \mathcal{E}_2$ be an operator fulfilling the following hypotheses:*

- $(\mathcal{T}_1)$ *$\Im(\mathcal{E}_1) \subseteq \mathcal{E}_2$ and $\Im(\mathcal{E}_2) \subseteq \mathcal{E}_1$,*
- $(\mathcal{T}_2)$ *χ is complete with respect to $||.||_1$,*
- $(\mathcal{T}_3)$ *there exist $\alpha', \beta', \gamma', \delta'$ are nonnegative real numbers with $\alpha' + \beta' + \gamma' + 2\delta' < 1$ such that*

$$\begin{aligned} ||\Im\grave{u} - \Im\grave{v}|| \leq\ & \alpha'||\grave{u} - \Im\grave{u}|| + \beta'||\grave{v} - \Im\grave{v}|| + \gamma'||\grave{u} - \grave{v}|| + \delta'(||\grave{u} - \Im\grave{v}|| + ||\grave{v} - \Im\grave{u}||) \\ & + (1 - \alpha' - \beta' - \gamma' - 2\delta')\, d(\mathcal{E}_1, \mathcal{E}_2), \end{aligned} \tag{2.8}$$

for all $\grave{u} \in \mathcal{E}_1$ and $\grave{v} \in \mathcal{E}_2$. If $\grave{u}_0 \in \mathcal{E}_1$ and $\grave{u}_{n+1} = \Im\grave{u}_n$ where $n \in \mathbb{N}$, then $\Im$ has the best proximity point in $\mathcal{E}_1$.

Proof. By Proposition 1, we have

$$||\grave{u}_{2n} - \grave{v}_{2n+1}|| \to d(\mathcal{E}_1, \mathcal{E}_2) \text{ and } ||\grave{u}_{2n+1} - \Im\grave{u}_{2n+1}|| \to d(\mathcal{E}_1, \mathcal{E}_2). \tag{2.9}$$

Since χ is a uniformly convex Banach space by Lemma (2), we get

$$||\grave{u}_{2n} - \grave{u}_{2(n+1)}|| \to 0 \text{ and } ||\Im\grave{u}_{2n+1} - \Im\grave{u}_{2n}|| \to 0. \tag{2.10}$$

We now show that for every $\varepsilon > 0$ there exists N_0 such that for all $m > n \geq N_0$, $||\grave{u}_{2m} - \Im\grave{u}_{2n}|| < d(\mathcal{E}_1, \mathcal{E}_2) + \varepsilon$. Suppose not, then there exists $\varepsilon > 0$ such that for all $k' \in N$, there exists $m(k') > n(k') \geq k'$, for $||\grave{u}_{2m(k')} - \Im\grave{u}_{2n(k')}|| \geq d(\mathcal{E}_1, \mathcal{E}_2) + \varepsilon$, this $m(k')$ can be chosen such that it is the least integer greater than $n(k')$ to satisfy the above inequality. Now

$$\begin{aligned} d(\mathcal{E}_1, \mathcal{E}_2) + \varepsilon &\leq ||\grave{u}_{2m(k')} - \Im\grave{u}_{2n(k')}|| \\ &\leq ||\grave{u}_{2m(k')} - \grave{u}_{2(m(k')-1)}|| + ||\grave{u}_{2(m(k')-1)} - \Im\grave{u}_{2n(k')}|| \\ &< ||\grave{u}_{2m(k')} - \grave{u}_{2(m(k')-1)}|| + d(\mathcal{E}_1, \mathcal{E}_2) + \varepsilon. \end{aligned}$$

Using equation (2.10) and taking $k' \to \infty$ in the above inequality we have

$$||\grave{u}_{2m(k')} - \Im\grave{u}_{2n(k')}|| = d(\mathcal{E}_1, \mathcal{E}_2) + \varepsilon.$$

Consider

$$\begin{aligned} ||\grave{u}_{2m(k')} - \Im\grave{u}_{2n(k')}|| &\leq ||\grave{u}_{2m(k')} - \grave{u}_{2(m(k')+1)}|| + ||\grave{u}_{2(m(k')+1)} - \Im\grave{u}_{2(n(k')+1)}|| \\ &\quad + ||\Im\grave{u}_{2(n(k')+1)} - \Im\grave{u}_{2n(k')}|| \\ &\leq ||\grave{u}_{2m(k')} - \grave{u}_{2(m(k')+1)}|| + ||\Im\grave{u}_{2(n(k')+1)} - \Im\grave{u}_{2n(k')}|| \end{aligned}$$

$$
\begin{aligned}
&+k^2||\grave{u}_{2m(k')}-\Im\grave{u}_{2n(k')}||+(1-k^2)d(\mathcal{E}_1,\mathcal{E}_2)\\
&\leq ||\grave{u}_{2m(k')}-\grave{u}_{2(m(k')+1)}||+||\Im\grave{u}_{2(n(k')+1)}-\Im\grave{u}_{2n(k')}||\\
&+k^2||\grave{u}_{2m(k')}-\Im\grave{u}_{2n(k')}||+(1-k^2)d(\mathcal{E}_1,\mathcal{E}_2). \qquad (2.11)
\end{aligned}
$$

Taking $k' \to \infty$ in equation (2.11) and using equation (2.10) we get

$$
\begin{aligned}
d(\mathcal{E}_1,\mathcal{E}_2)+\varepsilon &\leq k^2(d(\mathcal{E}_1,\mathcal{E}_2)+\varepsilon)+(1-k^2)d(\mathcal{E}_1,\mathcal{E}_2)\\
&= d(\mathcal{E}_1,\mathcal{E}_2)+k^2\varepsilon,
\end{aligned}
$$

which is a contradiction. Therefore $\{\grave{u}_{2n}\}$ is a Cauchy sequence in $\mathcal{E}$ with respect $||.||$. Because $||.||_1 \leq ||.||$, hence $\{u_{2n}\}$ is a Cauchy sequence in $\mathcal{E}_1$ with respect $||.||_1$. Since $\mathcal{E}_1$ is a closed subset of a complete metric space χ, then it is a complete subspace. By the completeness of $\mathcal{E}_1$, $\{\grave{u}_{2n}\}$ converges to a point $\grave{u}$ in $\mathcal{E}_1$, then by Theorem 1, we get $\Im$ has a best proximity point, $||\grave{u}-\Im\grave{u}||_1 = d_1(\mathcal{E}_1,\mathcal{E}_2)$ in $\mathcal{E}_1$. □

3. Consequences

Throughout in this section we use some notations which required in the sequel. Let $\mathcal{E}_2$ be a nonempty closed subset and $\mathcal{E}_1$ be a nonempty convex closed subset of a uniformly convex binormed linear space $(\chi, ||.||_1, ||.||)$ with $||.||_1 \leq ||.||$ and $\Im : \mathcal{E}_1 \cup \mathcal{E}_2 \to \mathcal{E}_1 \cup \mathcal{E}_2$ be an operator.

If $\alpha' = \beta' = \gamma' = 0$ in Theorem 2, we get the following best proximity result.

Corollary 1. *Assume that $\Im$ is fulfilling the following hypotheses:*

($\mathcal{T}_1$) $\Im(\mathcal{E}_1) \subseteq \mathcal{E}_2$ *and* $\Im(\mathcal{E}_2) \subseteq \mathcal{E}_1$,

($\mathcal{T}_2$) χ *is complete with respect to* $||.||_1$,

($\mathcal{T}_3$) *there exist a nonnegative real number* δ' *with* $2\delta' < 1$ *such that*

$$||\Im\grave{u}-\Im\grave{v}|| \leq \delta'(||\grave{u}-\Im\grave{v}||+||\grave{v}-\Im\grave{u}||)+(1-2\delta')\, d(\mathcal{E}_1,\mathcal{E}_2),$$

for all $\grave{u} \in \mathcal{E}_1$ *and* $\grave{v} \in \mathcal{E}_2$. *If* $\grave{u}_0 \in \mathcal{E}_1$ *and* $\grave{u}_{n+1} = \Im\grave{u}_n$ *where* $n \in \mathbb{N}$, *then* $\Im$ *has the best proximity point in* $\mathcal{E}_1$.

If $\alpha' = \beta'$ and $\gamma' = \delta' = 0$ in Theorem 2, we get the following best proximity result.

Corollary 2. *Assume that $\Im$ is fulfilling the following hypotheses:*

($\mathcal{T}_1$) $\Im(\mathcal{E}_1) \subseteq \mathcal{E}_2$ *and* $\Im(\mathcal{E}_2) \subseteq \mathcal{E}_1$,

($\mathcal{T}_2$) χ *is complete with respect to* $||.||_1$,

($\mathcal{T}_3$) *there exist a nonnegative real number* α *with* $2\alpha' < 1$ *such that*

$$||\Im\grave{u}-\Im\grave{v}|| \leq \alpha'(||\grave{u}-\Im\grave{u}||+||\grave{v}-\Im\grave{v}||)+(1-2\alpha')\, d(\mathcal{E}_1,\mathcal{E}_2),$$

for all $\grave{u} \in \mathcal{E}_1$ *and* $\grave{v} \in \mathcal{E}_2$. *If* $\grave{u}_0 \in \mathcal{E}_1$ *and* $\grave{u}_{n+1} = \Im\grave{u}_n$ *where* $n \in \mathbb{N}$, *then* $\Im$ *has the best proximity point in* $\mathcal{E}_1$.

If $\alpha' = \beta' = \delta' = 0$ in Theorem 2, we get the following best proximity result.

Corollary 3. *Assume that $\Im$ is fulfilling the following hypotheses:*

($\mathcal{T}_1$) $\Im(\mathcal{E}_1) \subseteq \mathcal{E}_2$ *and* $\Im(\mathcal{E}_2) \subseteq \mathcal{E}_1$,
($\mathcal{T}_2$) χ *is complete with respect to* $||.||_1$,
($\mathcal{T}_3$) *there exist a nonnegative real number* γ' *with* $\gamma' < 1$ *such that*

$$||\Im\grave{u} - \Im\grave{v}|| \leq \gamma'||\grave{u} - \grave{v}|| + (1-\gamma')\, d(\mathcal{E}_1, \mathcal{E}_2),$$

for all $\grave{u} \in \mathcal{E}_1$ *and* $\grave{v} \in \mathcal{E}_2$. *If* $\grave{u}_0 \in \mathcal{E}_1$ *and* $\grave{u}_{n+1} = \Im\grave{u}_n$ *where* $n \in \mathbb{N}$, *then* $\Im$ *has the best proximity point in* $\mathcal{E}_1$.

If $\alpha' = \beta' = \gamma'$ and $\delta' = 0$ in Theorem 2, we get the following best proximity result.

Corollary 4. *Assume that* $\Im$ *is fulfilling the following hypotheses:*
($\mathcal{T}_1$) $\Im(\mathcal{E}_1) \subseteq \mathcal{E}_2$ *and* $\Im(\mathcal{E}_2) \subseteq \mathcal{E}_1$,
($\mathcal{T}_2$) χ *is complete with respect to* $||.||_1$,
($\mathcal{T}_3$) *there exist a nonnegative real number* α' *with* $3\alpha' < 1$ *such that*

$$||\Im\grave{u} - \Im\grave{v}|| \leq \alpha'(||\grave{u} - \Im\grave{u}|| + ||\grave{v} - \Im\grave{v}|| + ||\grave{u} - \grave{v}||) + (1-3\alpha')\, d(\mathcal{E}_1, \mathcal{E}_2),$$

for all $\grave{u} \in \mathcal{E}_1$ *and* $\grave{v} \in \mathcal{E}_2$. *If* $\grave{u}_0 \in \mathcal{E}_1$ *and* $\grave{u}_{n+1} = \Im\grave{u}_n$ *where* $n \in \mathbb{N}$, *then* $\Im$ *has the best proximity point in* $\mathcal{E}_1$.

Remark 1.
- By choosing different values of $\alpha', \beta', \gamma', \delta'$ in Theorem 2, we get many best proximity results.
- If we take $||.|| = ||.||_1$ in Corollaries 1, 2, 3 and 4, then we get the corresponding results of [7, 10, 14].

If $\mathcal{E}_1 = \mathcal{E}_2 = \chi$ in Theorem 2, we have the following fixed point result.

Corollary 5. *Assume that* $\Im$ *is fulfilling the following hypotheses:*
($\mathcal{T}_1$) χ *is complete with respect to* $||.||_1$,
($\mathcal{T}_2$) *there exist* $\alpha', \beta', \gamma', \delta'$ *are nonnegative real numbers with* $\alpha' + \beta' + \gamma' + 2\delta' < 1$ *such that*

$$||\Im\grave{u} - \Im\grave{v}|| \leq \alpha'||\grave{u} - \Im\grave{u}|| + \beta'||\grave{v} - \Im\grave{v}|| + \gamma'||\grave{u} - \grave{v}|| + \delta'(||\grave{u} - \Im\grave{v}|| + ||\grave{v} - \Im\grave{u}||), \tag{3.1}$$

for all $\grave{u}, \grave{v} \in \chi$. *If* $\grave{u}_0 \in \chi$ *and* $\grave{u}_{n+1} = \Im\grave{u}_n$ *where* $n \in \mathbb{N}$, *then* $\Im$ *has a fixed point.*

If we take $\mathcal{E}_1 = \mathcal{E}_2 = \chi$ in Corollaries 1, 2, 3, 4, we get the following fixed point results.

Corollary 6. *Assume that* $\Im$ *is fulfilling the following hypotheses:*
($\mathcal{T}_1$) χ *is complete with respect to* $||.||_1$,
($\mathcal{T}_2$) *there exist a nonnegative real number* δ' *with* $2\delta' < 1$ *such that*

$$||\Im\grave{u} - \Im\grave{v}|| \leq \delta'(||\grave{u} - \Im\grave{v}|| + ||\grave{v} - \Im\grave{u}||),$$

for all $\grave{u}, \grave{v} \in \chi$. *If* $\grave{u}_0 \in \chi$ *and* $\grave{u}_{n+1} = \Im\grave{u}_n$ *where* $n \in \mathbb{N}$, *then* $\Im$ *has a fixed point.*

Corollary 7. *Assume that $\Im$ is fulfilling the following hypotheses:*

($\mathcal{T}_1$) *χ is complete with respect to $||.||_1$,*
($\mathcal{T}_2$) *there exist a nonnegative real number α' with $2\alpha' < 1$ such that*

$$||\Im\grave{u} - \Im\grave{v}|| \leq \alpha'(||\grave{u} - \Im\grave{u}|| + ||\grave{v} - \Im\grave{v}||),$$

for all $\grave{u}, \grave{v} \in \chi$. If $\grave{u}_0 \in \chi$ and $\grave{u}_{n+1} = \Im\grave{u}_n$ where $n \in \mathbb{N}$, then $\Im$ has a fixed point.

Corollary 8 ([13])**.** *Assume that $\Im$ is fulfilling the following hypotheses:*

($\mathcal{T}_1$) *χ is complete with respect to $||.||_1$,*
($\mathcal{T}_2$) *there exist a nonnegative real number γ' with $\gamma' < 1$ such that*

$$||\Im\grave{u} - \Im\grave{v}|| \leq \gamma'||\grave{u} - \grave{v}||,$$

for all $\grave{u}, \grave{v} \in \chi$. If $\grave{u}_0 \in \chi$ and $\grave{u}_{n+1} = \Im\grave{u}_n$ where $n \in \mathbb{N}$, then $\Im$ has a fixed point.

Corollary 9. *Assume that $\Im$ is fulfilling the following hypotheses:*

($\mathcal{T}_1$) *χ is complete with respect to $||.||_1$,*
($\mathcal{T}_2$) *there exist a nonnegative real numbers α' with $3\alpha' < 1$ such that*

$$||\Im\grave{u} - \Im\grave{v}|| \leq \alpha'(||\grave{u} - \Im\grave{u}|| + ||\grave{v} - \Im\grave{v}|| + ||\grave{u} - \grave{v}||),$$

for all $\grave{u}, \grave{v} \in \chi$. If $\grave{u}_0 \in \chi$ and $\grave{u}_{n+1} = \Im\grave{u}_n$ where $n \in \mathbb{N}$, then $\Im$ has a fixed point.

Remark 2.

- By choosing different values of $\alpha', \beta', \gamma', \delta'$ in Corollary 5, we get many fixed point results.
- If we take $||.|| = ||.||_1$ in Corollaries 6, 7, 8 and 9, then we get the corresponding results of [3, 11].

Now, we provide several illustrations that support our findings in this section.

Example 1. Consider $\chi = \mathbb{R}$, define $||.||, ||.||_1 : \chi \to \mathbb{R}_+$ by

$$||\grave{u}|| = 2|\grave{u}| \text{ and } ||\grave{u}||_1 = |\grave{u}|$$

for all $\grave{u} \in \chi$. It is easy to see that $||\grave{u}||_1 < ||\grave{u}||$, for all $\grave{u} \in \chi$. Suppose $\mathcal{E}_1 = [\frac{1}{4}, \frac{1}{2}]$ and $\mathcal{E}_2 = [\frac{3}{4}, \frac{9}{8}]$ are two subsets of χ, then $d(\mathcal{E}_1, \mathcal{E}_2) = 0.5$ and $d_1(\mathcal{E}_1, \mathcal{E}_2) = 0.25$.

Define $\Im : \mathcal{E}_1 \cup \mathcal{E}_2 \to \mathcal{E}_1 \cup \mathcal{E}_2$ by $\Im(\grave{u}) = \begin{cases} \frac{\grave{u}}{32} + \frac{3}{4} \text{ if } \grave{u} \in [\frac{1}{4}, \frac{1}{2}) \\ \frac{3}{4} \text{ if } \grave{u} = \frac{1}{2} \\ \frac{1}{2} \text{ if } \grave{u} \in [\frac{3}{4}, \frac{9}{8}] \end{cases}$

for all $\grave{u} \in \mathcal{E}_1 \cup \mathcal{E}_2$. Next we prove that $\Im$ satisfies the following inequality,

$$\begin{aligned} ||\Im\grave{u} - \Im\grave{v}|| \leq\ & \alpha'||\grave{u} - \Im\grave{u}|| + \beta'||\grave{v} - \Im\grave{v}|| + \gamma'||\grave{u} - \grave{v}|| + \delta'(||\grave{u} - \Im\grave{v}|| + ||\grave{v} - \Im\grave{u}||) \\ & + (1 - \alpha' - \beta' - \gamma' - 2\delta')\, d(\mathcal{E}_1, \mathcal{E}_2), \end{aligned}$$

for all $\grave{u} \in \mathcal{E}_1$ and $\grave{v} \in \mathcal{E}_2$. Let $\alpha' = 0.488$, $\beta' = 0.01$, $\gamma' = \frac{1}{2}$, $\delta' = 0$ with $\alpha' + \beta' + \gamma' + 2\delta' < 1$.

$$||\Im\grave{u} - \Im\grave{v}|| = \left|\left|\frac{\grave{u}}{32} + \frac{3}{4} - \frac{1}{2}\right|\right| = \left|\left|\frac{\grave{u}}{32} + \frac{1}{4}\right|\right|.$$

If $\grave{u} \in \mathcal{E}_1$ and $\grave{v} \in \mathcal{E}_2$ then $\left|\left|\frac{\grave{u}}{32} + \frac{1}{4}\right|\right| \in [0.5156, 0.5312]$, $||\grave{u} - \grave{v}|| \in [0.5, 1.75]$, $||\grave{u} - \Im\grave{u}|| \in [0.75, 2]$ and $||\grave{v} - \Im\grave{v}|| \in [0.5, 1.25]$. This implies

$$\begin{aligned} ||\Im\grave{u} - \Im\grave{v}|| \leq\ & \alpha'||\grave{u} - \Im\grave{u}|| + \beta'||\grave{v} - \Im\grave{v}|| + \gamma'||\grave{u} - \grave{v}|| + \delta'(||\grave{u} - \Im\grave{v}|| + ||\grave{v} - \Im\grave{u}||) \\ & + (1 - \alpha' - \beta' - \gamma' - 2\delta')\, d(\mathcal{E}_1, \mathcal{E}_2), \end{aligned}$$

for all $\grave{u} \in \mathcal{E}_1$ and $\grave{v} \in \mathcal{E}_2$ and $\Im(\mathcal{E}_1) \subseteq \mathcal{E}_2$, $\Im(\mathcal{E}_2) \subseteq \mathcal{E}_1$. Since $||\grave{u}||_1 < ||\grave{u}||$, for all $\grave{u} \in \chi$, we have

$$\begin{aligned} ||\Im\grave{u} - \Im\grave{v}||_1 \leq\ & \alpha'||\grave{u} - \Im\grave{u}||_1 + \beta'||\grave{v} - \Im\grave{v}||_1 + \gamma'||\grave{u} - \grave{v}||_1 \\ & + \delta'(||\grave{u} - \Im\grave{v}||_1 + ||\grave{v} - \Im\grave{u}||_1) + (1 - \alpha' - \beta' - \gamma' - 2\delta')\, d_1(\mathcal{E}_1, \mathcal{E}_2). \end{aligned}$$

Starting with point $\grave{u}_0 = \frac{1}{4} \in \mathcal{E}_1$, we construct a sequence as

$\grave{u}_{n+1}$	$\grave{u}_0$	$\grave{u}_1$	$\grave{u}_2$	$\grave{u}_3$	$\grave{u}_4$	$\grave{u}_5$	$\grave{u}_6$	$\grave{u}_7$	$\cdots$
$\Im\grave{u}_n$	0.25	0.7578	0.50	0.75	0.50	0.75	0.50	0.75	$\cdots$

We found that $\{\grave{u}_{2n}\}$ has a subsequence (0.25, 0.5, 0.5, 0.5,...), which converges to $\frac{1}{2}$. All the conditions of Theorem 2 are satisfied, and $\Im$ has a best proximity point $\frac{1}{2}$.

Example 2. Consider $\chi = \mathbb{R}^2$ with usual metric defined as

$$||\grave{z}_1|| = ||\grave{z}_1||_1 = \sqrt{\grave{u}_1^2 + \grave{v}_1^2} \tag{3.2}$$

for all $\grave{z}_1 = (\grave{u}_1, \grave{v}_1) \in \mathbb{R}^2$. Suppose

$$\mathcal{E}_1 = \{(0, \grave{u}_1) : 0 \leq \grave{u}_1 \leq 3\} \text{ and } \mathcal{E}_2 = \{(0, \grave{u}_1) : 0 \leq \grave{u}_1 \leq 2\},$$

are two subsets of $\mathbb{R}^2$, then $d(\mathcal{E}_1, \mathcal{E}_2) = 0$. Define $\Im : \mathcal{E}_1 \to \mathcal{E}_2$ by

$$\Im(\grave{z}_1) = \frac{\grave{z}_1}{3} \tag{3.3}$$

for all $\grave{z}_1 \in \mathcal{E}_1$. Let $\grave{z}_1, \grave{z}_2 \in \mathcal{E}_1$. Next we prove that $\Im$ satisfies the following inequality,

$$\begin{aligned} ||\Im\grave{z}_1 - \Im\grave{z}_2|| \leq\ & \alpha'||\grave{z}_1 - \Im\grave{z}_1|| + \beta'||\grave{z}_2 - \Im\grave{z}_2|| + \gamma'||\grave{z}_1 - \grave{z}_2|| \\ & + \delta(||\grave{z}_1 - \Im\grave{z}_2|| + ||\grave{z}_2 - \Im\grave{z}_1||) + (1 - \alpha' - \beta' - \gamma' - 2\delta')\, d(\mathcal{E}_1, \mathcal{E}_2), \end{aligned}$$

for all $\grave{z}_1, \grave{z}_2 \in \mathcal{E}_1$. Let $\alpha' = \beta' = \delta' = 0$, $\gamma' = \frac{1}{2}$ with $\alpha' + \beta' + \gamma' + 2\delta' < 1$. Take $\grave{z}_1 = (0, \grave{u}_1), \grave{z}_2 = (0, \grave{u}_2)$ in $\mathcal{E}_1$ then

$$||\Im\grave{z}_1 - \Im\grave{z}_2|| = \left|\left|\frac{\grave{z}_1}{3} - \frac{\grave{z}_1}{3}\right|\right| = \left|\left|\frac{(0, \grave{u}_1)}{3} - \frac{(0, \grave{u}_2)}{3}\right|\right| = \sqrt{\left(\frac{\grave{u}_1}{3} - \frac{\grave{u}_2}{3}\right)^2}$$

$$= \sqrt{\frac{1}{3^2}(\grave{u}_1 - \grave{u}_2)^2} = \frac{1}{3}\sqrt{(\grave{u}_1 - \grave{u}_2)^2} = \frac{1}{3}||\grave{z}_1 - \grave{z}_2||,$$

$$||\Im\grave{z}_1 - \Im\grave{z}_2|| < \gamma||\grave{z}_1 - \grave{z}_2||.$$

This shows

$$||\Im\grave{z}_1 - \Im\grave{z}_2|| < \alpha'||\grave{z}_1 - \Im\grave{z}_1|| + \beta'||\grave{z}_2 - \Im\grave{z}_2|| + \gamma'||\grave{z}_1 - \grave{z}_2|| + \delta(||\grave{z}_1 - \Im\grave{z}_2|| + ||\grave{z}_2 - \Im\grave{z}_1||) + (1 - \alpha' - \beta' - \gamma' - 2\delta')\, d(\mathcal{E}_1, \mathcal{E}_2),$$

for all $\grave{z}_1, \grave{z}_2 \in \mathcal{E}_1$ and $\Im(\mathcal{E}_1) \subseteq \mathcal{E}_2$, $\Im(\mathcal{E}_2) \subseteq \mathcal{E}_1$. Starting with point $\grave{u}_0 = (0,1) \in \mathcal{E}_1$, we construct a sequence as

$\grave{u}_{n+1}$	$\grave{u}_0$	$\grave{u}_1$	$\grave{u}_2$	$\grave{u}_3$	$\grave{u}_4$	$\grave{u}_5$	$\grave{u}_6$	...
$\Im\grave{x}_n$	$(0,1)$	$(0,0.3333)$	$(0,0.1111)$	$(0,0.0366)$	$(0,0.0122)$	$(0,0.0040)$	$(0,0.0013)$	...

We found that $\{\grave{u}_{2n}\}$ has a subsequence ((0,1) (0,0.1111), (0,0.0122), (0,0.0013), (0,0.0001),(0,0.0000)...), which converges to $(0,0)$. All the conditions of Theorem 2 are satisfied, and $\Im$ has a best proximity point $(0,0)$.

4. Applications

4.1. *Solving split feasibility problems*

The split feasibility problem (SFP), which is mathematically formulated as:

$$\text{find a point } \grave{u} \in \mathcal{C} \text{ such that } \mathcal{E}\grave{u} \in \mathcal{D}, \tag{4.1}$$

where $\mathcal{C}$ and $\mathcal{D}$ are non-empty convex and closed subsets of the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and $\mathcal{E} : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator. This problem was first proposed by Censor and Elfving [5] in Euclidean spaces. Assume that the SFP (4.1) has at least one solution and $\mathcal{S}$ its solution set, then (see [1]) $\grave{u} \in \mathcal{C}$ is a solution of (4.1) if and only if it is a solution of the fixed-point problem (FPP),

$$\grave{u} = \mathcal{P}_{\mathcal{C}}(I - \omega\mathcal{E}^*(I - \mathcal{P}_{\mathcal{D}})\mathcal{E})\grave{u},$$

where $\mathcal{P}_{\mathcal{C}}$ and $\mathcal{P}_{\mathcal{D}}$ are the near by point projections onto $\mathcal{C}$ and $\mathcal{D}$, respectively, $\omega > 0$, and $\mathcal{E}^*$ is the adjoint operator of $\mathcal{E}$. It has been shown in [4] if κ is the spectral radius of $\mathcal{E}^*\mathcal{E}$ and $\omega \in (0, \frac{2}{\kappa})$, then the operator

$$\Im = \mathcal{P}_{\mathcal{C}}(I - \omega\mathcal{E}^*(I - \mathcal{P}_{\mathcal{D}})\mathcal{E})$$

is nonexpansive and averaged and the so-called $\mathcal{CQ}$ algorithm,

$$\grave{u}_{n+1} = \mathcal{P}_{\mathcal{C}}(I - \omega\mathcal{E}^*(I - \mathcal{P}_{\mathcal{D}})\mathcal{E})\grave{u}_n,\ n \geq 0,$$

converges weakly to a solution of the SFP. Now we generlaize this using assumptions on two norm $||.||$ and $||.||_1$, defined on the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ such that $||.||_1 \leq ||.||$ and proved the convergence of the iterative algorithm to the solution of a SFP.

Theorem 3. *Assume that $\mathcal{C}$ is convex and closed subset of a $\mathcal{H}_1$ endowed with two norm $||.||$ and $||.||_1$ such that $||.||_1 \leq ||.||$ and $\mathcal{P}_{\mathcal{C}}(I-\omega\mathcal{E}^*(I-\mathcal{P}_{\mathcal{D}})\mathcal{E})$ satisfying (3.1), $\omega \in (0, \frac{2}{\kappa})$. If SFP (4.1) has at least a solution and $\mathcal{C}$ is complete with respect $||.||_1$, then*

$$\grave{u}_{n+1} = \mathcal{P}_{\mathcal{C}}(I-\omega\mathcal{E}^*(I-\mathcal{P}_{\mathcal{D}})\mathcal{E})(\grave{u}_n),$$

converges to the unique solution $\grave{u}$ of the SFP.

Proof. Given $\mathcal{C}$ is convex and closed, we can take $\mathcal{C} = \chi$ and

$$\Im = \mathcal{P}_{\mathcal{C}}(I-\omega\mathcal{E}^*(I-\mathcal{P}_{\mathcal{D}})\mathcal{E})$$

and using Corollary 5, we get the result. □

4.2. *Solving variational inequality problems*

Assume that $\mathcal{H}$ is a real Hilbert space and $\mathcal{C} \subset \mathcal{H}$ be a convex and closed. A function $\mathcal{S}'$ defined on Hilbert space $\mathcal{H}$ is said to be monotone if

$$\left\langle \mathcal{S}'\grave{u} - \mathcal{S}'\grave{v}, \grave{u} - \grave{v} \right\rangle \geq 0, \text{ for all } \grave{u}, \grave{v} \in \mathcal{H}.$$

The variational inequality problem (VIP) with respect to $\mathcal{S}'$ and $\mathcal{C}$, symbolized by $VIP(\mathcal{S}', \mathcal{C})$, is to discover $\grave{u}^* \in \mathcal{C}$ such that

$$\left\langle \mathcal{S}'\grave{u}^*, \grave{u} - \grave{u}^* \right\rangle \geq 0, \text{ for all } \grave{u} \in \mathcal{H}.$$

It is popular if $\omega > 0$, then $\grave{u}^* \in \mathcal{C}$ is a solution of $VIP(\mathcal{S}', \mathcal{C})$ if and only if $\grave{u}^*$ is a solution of the FPP

$$\grave{u} = \mathcal{P}_{\mathcal{C}}(I-\omega\mathcal{S}')\grave{u},$$

where $\mathcal{P}_{\mathcal{C}}$ is the near by point projection onto $\mathcal{C}$. It was confirmed by others results (see [4]), if $\mathcal{P}_{\mathcal{C}}(I-\omega\mathcal{S}')$ and $(I-\omega\mathcal{S}')$ are averaged nonexpansive mappings, then, under few more hypotheses, the iterative algorithm defined by,

$$\grave{u}_{n+1} = \mathcal{P}_{\mathcal{C}}(I-\omega\mathcal{S}')\grave{u}_n, n \in \mathbb{N},$$

converges weakly to a solution of $VIP(\mathcal{S}', \mathcal{C})$, if such solutions exist. Now we generalize this using assumptions on two norms $||.||$ and $||.||_1$, defined on the Hilbert space $\mathcal{H}$ such that $||.||_1 \leq ||.||$ and proved the convergence of iterative algorithm to the solution of a VIP.

Theorem 4. *Assume that $\mathcal{C}$ is convex and closed subset of a $\mathcal{H}$ endowed with two norms $||.||$ and $||.||_1$ such that $||.||_1 \leq ||.||$ and $\mathcal{P}_{\mathcal{C}}(I-\omega\mathcal{S}')$ satisfying (3.1) on $\mathcal{C}$, $\omega > 0$. If $\mathcal{C}$ is complete with respect $||.||_1$, then*

$$\grave{u}_{n+1} = \mathcal{P}_{\mathcal{C}}(I-\omega\mathcal{S}')\grave{u}_n,$$

converges to the unique solution $\grave{u}$ of the $VIP(S', C)$.

Proof. Given $\mathcal{C}$ is convex and closed, we can take $\mathcal{C} = \chi$ and $\Im = \mathcal{P}_{\mathcal{C}}(I-\omega\mathcal{S}')$ and using Corollary 5, we get result. □

5. Conclusion

We find some novel best proximity point results in binormed linear spaces using Hardy-Roger type contraction mappings. One can obtain interesting results by choosing different values of $\alpha', \beta', \gamma', \delta'$. Many known results in the literature are generalized and extended by our findings.

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MITTAG-LEFFLER FUNCTION FOR REAL INDEX AND ITS APPLICATION IN SOLVING DIFFERENCE AND DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we derive a Mittag-Leffler function for real index and establish solutions of special type of fractional order differential equations (FDEs). The same concept is extended to discrete case by replacing polynomials into factorial polynomials and differentiation into ℓ-difference operator. Moreover, numerical examples of our results are stated to validate our findings. The acquired results here have the ability to generate a wide range of formulas in relation to newer results.

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Keywords: fractional difference equations, Mittag-Leffler function, inverse difference operator, Mittag-Leffler factorial function, binomial expansion

1. Introduction

Fractional calculus, like standard integer calculus, is one of the oldest branches of mathematics. It is nowadays utilized in the vast areas of science, modelling, bio-engineering, computational optimization, control analysis of systems. We have different structures of fractional differential equations (FDEs) containing various special functions in their kernels which have main importance in analyzing mathematical models. For instances, the application of these kernels can be seen in some works including [1–3, 5, 10, 18, 21, 22]. In addition to these papers, the Mittag-Leffler function is used as the kernel of some new operators [6, 8, 11]. In fact, in last twenty years, the use of Mittag-Leffler function has gained its momentum in the investigation and solving different boundary problems and dynamical systems in life sciences [13]. The Mittag-Leffler function with two parameters is defined by [24] as

$$E_{\beta,\gamma}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\gamma+\beta j)}, \quad R(\beta), R(\gamma) > 0, \beta, \gamma, z \in \mathbb{C}. \tag{1.1}$$

This function is applicable to solve the fractional Kinetic equations [13]. Regarding Mittag-Leffler function and its properties, we can refer to Hilfer [14] and Saxena

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[23]. As exponential function naturally is appeared from solving ordinary differential equations, the Mittag-Leffler function plays a similar role in solving (FDEs) fractional differential equations.

On the other hand, Diaz *et al.* [9] introduced the notion of the fractional difference by the rather usual method which permits the index of difference, in the standard representation of the n-th difference, to be each arbitrary real/complex number. After that, by using Taylor's series, Hirota [15] introduced the difference operator ∇^q in the fractional settings in which $q \in \mathbb{R}$. Nagai [20] indicated another kind of this definition for the mentioned difference operator in view of the modification of the Hirota's definition. Accordingly, Deekshitulu *et al.* [7] changed the definition presented by Nagai in [20] for $q \in (0,1)$ so that the expression for ∇^q does not possess any difference operator. Recently, in the field of discrete fractional calculus, delta (Δ^q) and nabla (∇^q) operators ($q \in \mathbb{R}$) play a key role in the modelings [12, 19].

In this paper, we develop ν^{th} real order Mittag-Leffler function by replacing z by $(\lambda\varkappa)^\nu$, $\gamma = 1, \beta = \nu$ and $\varkappa$ by r in (1.1). The motive of initiating this new type fractional or real order Mittag-Leffler function is for solving the fractional or real order differential and difference equations. The solution of fractional order differential and difference equations helps us to obtain applications in the context of fractional calculus. The applications related to this function clearly yield connection between Binomial expansion and Mittag-Leffler Factorial Function.

2. Mittag-Leffler function in the fractional settings

For this section, we go through some important definitions, concepts of factorial polynomials, Extorial function, Mittag-Leffler function and also necessary theorems can be exploit in the later sections. Here we use the domain set $J_\ell \subset \mathbb{R}$ for which $t \in J_\ell$ gives $t \pm \ell \in J_\ell$ and $\mathbb{N}(a) = \{a, a+1, a+2, \cdots\}$.

Definition 1. [4, 17] The ℓ-Delta operator and the inverse for a function $u : J_\ell \to \ell$ are respectively defined as

$$\Delta_\ell u(\varkappa) = u(\varkappa + \ell) - u(\varkappa),\ \varkappa \in [0,\infty),\ \ell \in (0,\infty), \tag{2.1}$$

$$\text{if } \Delta_\ell v(\varkappa) = u(\varkappa),\ \text{then } v(\varkappa) \ = \ \Delta_\ell^{-1} u(\varkappa) + c, \tag{2.2}$$

where the constant c is obtained by substituting suitable value for $\varkappa$.

Definition 2. [17] Let $\ell > 0$, $\nu \in (-\infty,\infty)$ and $\Gamma(\frac{\varkappa}{\ell}+1)$ be the Gamma function. Then, the ℓ - factorial polynomial in $\varkappa$ for real index ν is defined by

$$\varkappa_\ell^{(\nu)} = \ell^\nu \frac{\Gamma(\varkappa/\ell+1)}{\Gamma(\varkappa/\ell+1-\nu)},\ \varkappa/\ell+1, (\varkappa/\ell+1-\nu) \notin -N(0) = \{0,-1,\cdots\}, \tag{2.3}$$

where the Gamma function is defined by

$$\Gamma(z) = \int_0^\infty s^{z-1} e^{-s} \mathrm{d}s,\ \ Re(z) > 0.$$

Special Cases:

(i) when $\ell = -1$, $\varkappa_{-1}^{(n)} = \varkappa(\varkappa+1)(\varkappa+2)\cdots(\varkappa+n-1) = \prod_{r=1}^{n}(\varkappa+r-1), n \in \mathbb{Z}^+$.

(ii) when $\ell = 1$, $\varkappa_{1}^{(n)} = \varkappa(\varkappa-1)(\varkappa-2)\cdots(\varkappa-n+1) = \prod_{r=1}^{n}(\varkappa-r+1) = \varkappa^{(n)}, n \in \mathbb{Z}^+$.

Definition 3. The nth order Mittag-Leffler function $e_n(\lambda,\varkappa)$, with $n \in \mathbb{Z}^+$, $\varkappa \in (-\infty,\infty)$ is defined by

$$e_n(\lambda,\varkappa) = 1 + \frac{\lambda^n \varkappa^n}{(n)!} + \frac{\lambda^{2n}\varkappa^{2n}}{(2n)!} + \frac{\lambda^{3n}\varkappa^{3n}}{(3n)!} + \cdots + \infty = \sum_{r=0}^{\infty} \frac{\lambda^{rn}\varkappa^{rn}}{(rn)!}. \tag{2.4}$$

Note that $e_1(\lambda,\varkappa) = e^{\lambda\varkappa}$. Since the function defined in (2.4) is a sub series of $e^{\lambda\varkappa}$ for fixed λ, which is finite.

Definition 4. The νth order Mittag-Leffler Function, for non integer real number ν, with the condition $r\nu + 1 \notin -\mathbb{N}(0)$ define by,

$$e_\nu(\lambda,\varkappa) = \sum_{r=0}^{\infty} \lambda^{r\nu} \frac{\varkappa^{r\nu}}{\Gamma(r\nu+1)}. \tag{2.5}$$

Lemma 1. *[25] We have the identities for any $r \in N(1)$ and $\ell > 0$:*

(i) $\Delta_\ell \varkappa_\ell^{(r)} = r\ell\varkappa_\ell^{(r-1)}$,

(ii) $\Delta_\ell \varkappa_{-\ell}^{(r)} = r\ell(\varkappa+\ell)_{-\ell}^{(r-1)}$,

(iii) $\Delta_\ell \varkappa_\ell^{(-r)} = -r\ell(\varkappa+\ell)_\ell^{-(r+1)}$,

and

(iv) $\Delta_\ell \varkappa_{-\ell}^{(-r)} = -r\ell(\varkappa+\ell)_{-\ell}^{(-(r+1))}$.

We use our n^{th} order Mittag-Leffler Function to solve certain type of linear differential equation.

Theorem 1. *For $\lambda \neq 0$, the function defined by $e_n(\lambda,\varkappa)$ given in (2.4) is considered as a solution of the $(n-1)^{th}$-linear non homogeneous DE*

$$\frac{1}{\lambda^{(n-1)}}\frac{d^{n-1}}{d\varkappa^{n-1}}u(\varkappa) + \frac{1}{\lambda^{(n-2)}}\frac{d^{n-2}}{d\varkappa^{n-2}}u(\varkappa) + \cdots + \frac{1}{\lambda}\frac{d}{d\varkappa}u(\varkappa) + u(\varkappa) = e^{\lambda\varkappa}.$$

Proof. From the Definition 4, we have

$$e_n(\lambda,\varkappa) = 1 + \frac{\lambda^n\varkappa^n}{(n)!} + \frac{\lambda^{2n}\varkappa^{2n}}{(2n)!} + \frac{\lambda^{3n}\varkappa^{3n}}{(3n)!} + \cdots + \infty.$$

Differentiating successively w.r.t $\varkappa$, we obtain

$$\frac{1}{\lambda}\frac{d}{d\varkappa}e_n(\lambda,\varkappa) = \frac{\lambda^{n-1}\varkappa^{n-1}}{(n-1)!} + \frac{\lambda^{2n-1}\varkappa^{2n-1}}{(2n-1)!} + \frac{\lambda^{3n-1}\varkappa^{3n-1}}{(3n-1)!} + \cdots + \infty,$$

$$\frac{1}{\lambda^2}\frac{d^2}{d\varkappa^2}e_n(\lambda,\varkappa)=\frac{\lambda^{n-2}\varkappa^{n-2}}{(n-2)!}+\frac{\lambda^{2n-2}\varkappa^{2n-2}}{(2n-2)!}+\frac{\lambda^{3n-2}\varkappa^{3n-2}}{(3n-2)!}\cdots+\infty.$$

In general,

$$\frac{1}{\lambda^r}\frac{d^r}{d\varkappa^r}e_n(\lambda,\varkappa)=\frac{\lambda^{n-r}\varkappa^{n-r}}{(n-r)!}+\frac{\lambda^{2n-r}\varkappa^{2n-r}}{(2n-r)!}+\frac{\lambda^{3n-r}\varkappa^{3n-r}}{(3n-r)!}\cdots+\infty. \tag{2.6}$$

Adding (2.6) for $r=0,1,2,3,\cdots,n-1, d^0=1$, we get

$$\sum_{r=0}^{n-1}\frac{1}{\lambda^r}\frac{d^r}{d\varkappa^r}e_n(\lambda,\varkappa)=e_1(\lambda,\varkappa),$$

and hence we get the proof by taking $u(\varkappa)=e_n(\lambda,\varkappa)$. □

For the succeeding theorem, we use the notation $\lim_{t\to\infty}\Delta_\ell^{-1}u(t)=\Delta_\ell^{-1}u(-\infty)$.

Theorem 2. *Let $\ell,\nu>0$ and $\varkappa\in\mathbb{R}$ and $u:\mathbb{R}\to\mathbb{R}$. If $\sum_{r=0}^{\infty}(r+\nu-1)$ is convergent, then we have*

$$\Delta_\ell^{-\nu}u(\varkappa)=\sum_{r=0}^{\infty}\frac{(r+\nu-1)_1^{(\nu-1)}}{\Gamma(\nu+1)}u(\varkappa-(r+\nu-1)\ell). \tag{2.7}$$

Proof. Since, $\Delta_\ell^{-1}u(\varkappa)=v(\varkappa)$, we have $u(\varkappa)=v(\varkappa+\ell)-v(\varkappa)$.
Replacing $\varkappa$ by $\varkappa-\ell$, it becomes

$$v(\varkappa)=u(\varkappa-\ell)+v(\varkappa-\ell). \tag{2.8}$$

Again replacing $\varkappa$ by $\varkappa-\ell,\varkappa-2\ell$, $\varkappa-3\ell$, $\cdots$, $\varkappa-(n-1)\ell$ in (2.8), we arrive

$$v(\varkappa)=u(\varkappa-\ell)+u(\varkappa-2\ell)+u(\varkappa-3\ell)+\cdots+u(\varkappa-n\ell)+v(\varkappa-n\ell).$$

Taking $\lim_{n\to\infty}$ in the above equation along with $v(-\infty)=0, u(-\infty)=0$, we obtain

$$\Delta_\ell^{-1}u(\varkappa)=\sum_{r=0}^{\infty}u(\varkappa-(r+1)\ell). \tag{2.9}$$

Taking Δ_ℓ^{-1} on both sides of (2.9), we obtain

$$\Delta_\ell^{-2}u(\varkappa)=\Delta_\ell^{-1}u(\varkappa-\ell)+\Delta_\ell^{-1}u(\varkappa-2\ell)+\Delta_\ell^{-1}u(\varkappa-3\ell)+\Delta_\ell^{-1}u(\varkappa-4\ell)+\cdots.$$

Expanding all the inverse function in above expression by the use of equation (2.9) and then arranging the terms, we obtain

$$\Delta_\ell^{-2}u(\varkappa)=u(\varkappa-2\ell)+2u(\varkappa-3\ell)+3u(\varkappa-4\ell)+4u(\varkappa-5\ell)+5u(\varkappa-6\ell)+\cdots,$$

which is similar to

$$\Delta_\ell^{-2}u(\varkappa)=\sum_{r=0}^{\infty}\frac{(r+1)^{(1)}}{1!}u(\varkappa-(r+2)\ell). \tag{2.10}$$

Proceeding like this, we get

$$\Delta_{\ell}^{-m}u(\varkappa)=\sum_{r=0}^{\infty}\frac{(r+m-1)_1^{(\nu-1)}}{(m-1)!}u(\varkappa-(r+m-1)\ell),m\in\mathbb{N},\ell\in\mathbb{R}. \tag{2.11}$$

For any $\nu>0$, we obtain the result by replacing m by ν. □

Theorem 3. *If* $\Delta_{\ell}^{-1}u(-\infty)=0$ *and* $\sum_{r=0}^{\infty}u(t+r\ell)$ *is convergent, then*

$$\Delta_{-\ell}^{-1}u(t)\Big|_t^{\infty}=\sum_{r=1}^{\infty}u(t+r\ell). \tag{2.12}$$

Proof. Replacing $\varkappa$ by t, ℓ by $-\ell$ in (2.9), we get the proof. □

Corollary 1. *Keeping* $\varkappa$ *as a constant, t as a variable, taking* Δ_{-1}^{-1} *with respect to t, then*

$$\Delta_{-1}^{-1}\frac{(\lambda\varkappa)^{mt}}{T_{-1}^{(mt)}}\Big|_{t=0}=\frac{(\lambda\varkappa)^{0}}{T_{-1}^{(0)}}+\frac{(\lambda\varkappa)^{m.1}}{T_{-1}^{(m.1)}}+\frac{(\lambda\varkappa)^{m.2}}{T_{-1}^{(m.2)}}+\cdots,\textit{for } T_{-1}^{(mt)}\neq 0. \tag{2.13}$$

Proof. Taking $u(t)=\dfrac{(\lambda\varkappa)^{mt}}{(T)_{-1}^{(mt)}}$ gives the proof, where

$$T_{-1}^{(m.t)}=T(T+1)\cdots(T+(mt-1))=\sum_{r=0}^{mt-1}(T+r),$$

in (2.12). □

3. Application of Fractional Order Mittag-Leffler Function in Numerical Methods

This section describes about the relation among Fractional order Mittag-Leffler Function, inverse of Δ_{-1} and raising factorial polynomials and some identities in numerical methods.

Theorem 4. *For* $mt\in\mathbb{N}$, *by denoting* $(s+1)_{-1}^{(mt)}=(s+1)(s+2)\cdots(s+mt)$, *we have the identity*

$$e_1(\lambda,\varkappa)-e_m(\lambda,\varkappa)=\sum_{s=1}^{m-1}\frac{(\lambda\varkappa)^s}{s!}\Delta_{-1}^{-1}\frac{(\lambda\varkappa)^{mt}}{(s+1)_{-1}^{(mt)}}\Big|_{t=0}. \tag{3.1}$$

Proof. From the Definition 3 and taking $n=1$, we have

$$e_1(\lambda,\varkappa)=1+\frac{\lambda\varkappa}{1!}+\frac{\lambda^2\varkappa^2}{2!}+\frac{\lambda^3\varkappa^3}{3!}+\frac{\lambda^4\varkappa^4}{4!}+\cdots+\infty.$$

Arranging the terms into m disjoint groups, since the series is absolutely convergent, we obtain

$$\begin{aligned} e_1(\lambda,\varkappa) = 1 &+ \frac{\lambda^m \varkappa^m}{m!} + \frac{\lambda^{2m}\varkappa^{2m}}{(2m)!} + \frac{\lambda^{3m}\varkappa^{3m}}{(3m)!} + \cdots + \infty \\ &+ \frac{\lambda\varkappa}{1!} + \frac{\lambda^{m+1}\varkappa^{m+1}}{(m+1)!} + \frac{\lambda^{2m+1}\varkappa^{2m+1}}{(2m+1)!} + \cdots + \infty \\ &+ \frac{\lambda^2\varkappa^2}{2!} + \frac{\lambda^{m+2}\varkappa^{m+2}}{(m+2)!} + \frac{\lambda^{2m+2}\varkappa^{2m+2}}{(2m+2)!} + \cdots + \infty \\ &+ \cdots \quad \cdots \quad \cdots \\ &+ \frac{\lambda^{m-1}\varkappa^{m-1}}{(m-1)!} + \frac{\lambda^{2m-1}\varkappa^{2m-1}}{(2m-1)!} + \frac{\lambda^{3m-1}\varkappa^{3m-1}}{(3m-1)!} + \cdots + \infty, \end{aligned}$$

which is the same as, by taking $n = m$ in (2.4),

$$\begin{aligned} e_1(\lambda,\varkappa) = e_m(\lambda,\varkappa) &+ \frac{\lambda\varkappa}{1!}\left[1 + \frac{(\lambda\varkappa)^{m.1}}{2_{-1}^{(m)}} + \frac{(\lambda\varkappa)^{m.2}}{2_{-1}^{(m.2)}} + \cdots\right] \\ &+ \frac{(\lambda\varkappa)^2}{2!}\left[1 + \frac{(\lambda\varkappa)^{m.1}}{3_{-1}^{(m)}} + \frac{(\lambda\varkappa)^{m.2}}{3_{-1}^{(m.2)}} + \cdots\right] \\ &+ \frac{(\lambda\varkappa)^3}{3!}\left[1 + \frac{(\lambda\varkappa)^{m.1}}{4_{-1}^{(m)}} + \frac{(\lambda\varkappa)^{m.2}}{4_{-1}^{(m.2)}} + \cdots\right]. \end{aligned} \tag{3.2}$$

Hence to obtain (3.1), apply (2.13) in (3.2), and this ends the proof. □

Theorem 5. *Suppose that* $\nu m = 1$, *where* $m \in N$. *Then*

$$e_\nu(\lambda,\varkappa) - e_1(\lambda,\varkappa) = \sum_{s=1}^{m-1} \frac{(\lambda\varkappa)^{\nu s}}{\Gamma(1+\nu s)} \Delta_{-1}^{-1} \frac{(\lambda\varkappa)^{\nu t}}{(1+\nu s)_{-1}^{(\nu t)}}\Big| t = 0. \tag{3.3}$$

Proof. By rearranging into four disjoint groups in Theorem 4, we get (3.3). □

The following example is for the demonstration of Theorem 5.

Example 1. From (2.4), by taking $\nu = 0.25, m = 4$ we have,

$$e_{0.25}(\lambda,\varkappa) = 1 + \frac{\lambda^{0.25}\varkappa^{0.25}}{\Gamma(1.25)} + \frac{\lambda^{0.5}\varkappa^{0.5}}{\Gamma(1.5)} + \frac{\lambda^{0.75}\varkappa^{0.75}}{\Gamma(1.75)} + \frac{\lambda^{1}\varkappa^{1}}{\Gamma(2)} + \cdots + \infty.$$

$$e_{0.25}(\lambda,\varkappa) = \frac{\lambda^{0.25}\varkappa^{0.25}}{\Gamma 1.25} + \frac{\lambda^{1.25}\varkappa^{1.25}}{\Gamma 2.25} + \frac{\lambda^{2.25}\varkappa^{2.25}}{\Gamma 3.25} + \cdots + \infty$$

$$+\frac{\lambda^{0.5}\varkappa^{0.5}}{\Gamma 1.5}+\frac{\lambda^{1.5}\varkappa^{1.5}}{\Gamma 2.5}+\frac{\lambda^{3.5}\varkappa^{3.5}}{\Gamma 3.5}+\cdots+\infty$$
$$+\frac{\lambda^{0.75}\varkappa^{0.75}}{\Gamma 1.75}+\frac{\lambda^{1.75}\varkappa^{1.75}}{\Gamma 2.75}+\frac{\lambda^{2.75}\varkappa^{2.75}}{\Gamma 3.75}+\cdots+\infty$$
$$+\frac{\lambda^{0}\varkappa^{0}}{\Gamma 1}+\frac{\lambda^{1}\varkappa^{1}}{\Gamma 2}+\frac{\lambda^{2}\varkappa^{2}}{\Gamma 3}+\cdots+\infty$$

which is the same as

$$\begin{aligned}e_{0.25}(\lambda,\varkappa)=&\frac{\lambda^{0.25}\varkappa^{0.25}}{\Gamma 1.25}\left[1+\frac{\lambda\varkappa}{(1.25)}+\frac{\lambda^2\varkappa^2}{(1.25)(2.25)}+\cdots\right]\\&+\frac{\lambda^{0.5}\varkappa^{0.5}}{\Gamma 1.5}\left[1+\frac{\lambda\varkappa}{(1.5)}+\frac{\lambda^2\varkappa^2}{(1.5)(2.5)}+\cdots\right]\\&+\frac{\lambda^{0.75}\varkappa^{0.75}}{\Gamma 1.75}\left[1+\frac{\lambda\varkappa}{(1.75)}+\frac{\lambda^2\varkappa^2}{(1.75)(2.75)}+\cdots\right]\\&+\frac{\lambda^{0}\varkappa^{0}}{\Gamma 1}\left[1+\frac{\lambda\varkappa}{(1)}+\frac{\lambda^2\varkappa^2}{(1)(2)}+\frac{\lambda^3\varkappa^3}{(1)(2)(3)}+\cdots\right]\end{aligned}$$

From the special cases of Definition 2 and taking Δ_{-1}^{-1} with respect to t, we get

$$\begin{aligned}e_{0.25}(\lambda,\varkappa)=&\frac{\lambda^{0.25}\varkappa^{0.25}}{\Gamma 1.25}\Delta_{-1}^{-1}\frac{(\lambda\varkappa)^t}{(1.25)_{-1}^{(t)}}\,|_{t=0}+\frac{\lambda^{0.5}\varkappa^{0.5}}{\Gamma 1.5}\Delta_{-1}^{-1}\frac{(\lambda\varkappa)^t}{(1.5)_{-1}^{(t)}}\,|_{t=0}\\&+\frac{\lambda^{0.75}\varkappa^{0.75}}{\Gamma 1.75}\Delta_{-1}^{-1}\frac{(\lambda\varkappa)^t}{(1.75)_{-1}^{(t)}}\,|_{t=0}+\frac{\lambda^{0}\varkappa^{0}}{\Gamma 1}\Delta_{-1}^{-1}\frac{(\lambda\varkappa)^t}{(1)_{-1}^{(t)}}\,|_{t=0}\end{aligned}$$

which can be expressed as

$$e_{0.25}(\lambda,\varkappa)-e_1(\lambda,\varkappa)=\sum_{s=1}^{3}+\frac{(\lambda\varkappa)^{(0.25)s}}{\Gamma 1+s(0.25)}\Delta_{-1}^{-1}\frac{(\lambda\varkappa)^t}{(1+0.25s)_{-1}^{(t)}}\,|_{t=0}\,.$$

Theorem 6. *By defining* $s_1(\lambda,\varkappa)=\frac{\lambda\varkappa}{1!}+\frac{(\lambda\varkappa)^3}{3!}+\frac{(\lambda\varkappa)^5}{5!}+\cdots$ *and* $c_1(\lambda,\varkappa)=1+\frac{(\lambda\varkappa)^2}{2!}+\frac{(\lambda\varkappa)^4}{4!}+\cdots$, *we get the following identities*

(i) $e_1(\lambda,\varkappa)=c_1(\lambda,\varkappa)+s_1(\lambda,\varkappa)$.

(ii) $\frac{d}{d\varkappa}e_2(\lambda,\varkappa)=\lambda s_1(\lambda,\varkappa)$.

(iii) $\frac{d^2}{d\varkappa^2}e_2(\lambda,\varkappa)=\lambda^2 c_1(\lambda,\varkappa)$.

Proof. Keeping λ as a constant and taking differentiation with respect to $'\varkappa'$ termwise,

$$\frac{d}{d\varkappa}e_2(\lambda,\varkappa)=\lambda\left[\frac{\lambda\varkappa}{1!}+\frac{(\lambda\varkappa)^3}{3!}+\frac{(\lambda\varkappa)^5}{5!}+\cdots\right]=\lambda s_1(\lambda,\varkappa).$$

Again taking derivative with respect to $\varkappa$, we get,

$$\frac{d^2}{d\varkappa^2}e_2(\lambda,\varkappa)=\lambda\Big[\lambda+\frac{\lambda^3\varkappa^2}{2!}+\frac{\lambda^5\varkappa^4}{4!}+\cdots\Big]=\lambda^2c_1(\lambda,\varkappa).$$

□

4. Fractional order Mittag-Leffler factorial function

Here, we extend the Fractional order Mittag-Leffler Function into Fractional order Mittag-Leffler Factorial Function by replacing the polynomials into factorial polynomials in the Fractional order Mittag-Leffler Function.

Definition 5. [16] For $|\lambda|,|\ell|<1$ and $c\in[0,1]$, if $\varkappa-(a+\ell)+cj\ell\nu$ is defined for $j\nu+1\notin N(0)$, $\varkappa\in\mathbb{R}$, then the extended Mittag-Leffler factorial (EMLF) function is defined as

$$e_\nu(\lambda,\varkappa_\ell,c)=\sum_{j=0}^{\infty}\frac{\lambda^j}{\Gamma(j\nu+1)}\Big(\varkappa-(a+\ell)+cj\ell\nu\Big)_\ell^{(j\nu)}. \tag{4.1}$$

$e_\nu(\lambda,\varkappa_{-\ell},1)$ is defined for $|\lambda|\geq 1$, $\ell>0$ whenever $\varkappa-(a+\ell)+cj\ell\nu$ is positive and a multiple of ℓ.

Also, we have two special cases in the following form:

(i) $e_1(1,\varkappa_0,c)=e^{\varkappa}$ if $\ell=0$, $\lambda=\nu=1$, $a=-1$.

(i') $e_\nu(1,\varkappa_\ell,0)=1+\dfrac{\varkappa_\ell^{(1\nu)}}{\Gamma(1\nu+1)}+\dfrac{\varkappa_\ell^{(2\nu)}}{\Gamma(2\nu+1)}+\dfrac{\varkappa_\ell^{(3\nu)}}{\Gamma(3\nu+1)}+\cdots+\dfrac{\varkappa_\ell^{(n\nu)}}{\Gamma(n\nu+1)}+\cdots$.
(It is the extorial function).

(i") $e_\nu(\lambda,\varkappa_0,1)$ is the same Mittag-Leffler factorial function.

Theorem 7. *The extorial function illustrated by $e_\nu(1,\varkappa_\ell,0)$ (special case of Mittag-Leffler Factorial Function) is convergent if $|\ell|<1$.*

Proof. In view of the definition of extorial function, we have

$$e(1,\varkappa_\ell,0)=1+\frac{\varkappa_\ell^{(1)}}{1!}+\frac{\varkappa_\ell^{(2)}}{2!}+\frac{\varkappa_\ell^{(3)}}{3!}+\cdots+\frac{\varkappa_\ell^{(n)}}{n!}+\cdots.$$

Consider the term $a_n=\dfrac{\varkappa_\ell^{(n)}}{n!}=\dfrac{\varkappa(\varkappa-\ell)\cdots(\varkappa-(n-1)\ell)}{n!}$. Then

$$|a_n|\leq\frac{|\varkappa|(|\varkappa|+|\ell|)(|\varkappa|+2|\ell|)\cdots(|\varkappa|+(n-1)|\ell|)}{n!}.$$

Since $|\ell|<1$, choose N such that $|\varkappa|(|\varkappa|+|\ell|)\cdots(|\varkappa|+(n-1)|\ell|)<N!$ and hence

$$|a_N|\leq\frac{|\varkappa|(|\varkappa|+|\ell|)\cdots(|\varkappa|+(n-1)|\ell|)}{N!}<\rho_N<1,$$

which is possible, since in between two real numbers, there exists another real number.

Since $\rho_N > \rho_{N+1} > \cdots, \lim_{n\to\infty} \sqrt[n]{|a_n|} < \sqrt[N]{\rho_N} < 1$, thus, by root test, $e(\varkappa_\ell)$ is convergent. □

Theorem 8. *For rational $\varkappa > 0$ and $|\lambda\ell| < 1$, we have $(1+\lambda\ell)^\varkappa = e_1(\lambda, (\ell\varkappa)_\ell, 0)$, where $e_1(\lambda, (\ell\varkappa)_\ell, 0)$ is the Mittag-Leffler Factorial Function (MLFF).*

Proof. From the binomial expansion for rational index,

$$(1+\lambda\ell)^\varkappa = 1 + \frac{(\lambda\ell)\varkappa}{1!} + \frac{(\lambda\ell)^2\varkappa(\varkappa-1)}{2!} + \frac{(\lambda\ell)^3\varkappa(\varkappa-1)(\varkappa-2)}{3!} + \cdots,$$

which can be expressed as

$$(1+\lambda\ell)^\varkappa = 1 + \lambda\frac{(\ell\varkappa)}{1!} + \lambda^2\frac{(\ell\varkappa)(\ell\varkappa-\ell)}{2!} + \lambda^3\frac{(\ell\varkappa)(\ell\varkappa-\ell)(\ell\varkappa-2\ell)}{3!} + \cdots,$$

and we get the identity from the definition of $e_1(\lambda, (\ell\varkappa)_\ell, 0)$. □

Theorem 9. *For positive integer m and for $\varkappa > 0$, we have the identity*

$$\Delta_\ell^{-m} e_1(\lambda, (\ell\varkappa)_\ell, 0) = (\lambda\ell)^m e_1(\lambda, (\ell\varkappa)_\ell, 0), \text{ if } (\lambda\ell)^{-1} = 0. \tag{4.2}$$

Proof. By Theorem 8, $e_1(\lambda, (\ell\varkappa)_\ell, 0) = (1+\lambda\ell)^\varkappa$. By taking Δ_ℓ on both sides, we find

$$\Delta_\ell e_1(\lambda, (\ell\varkappa)_\ell, 0) = (1+\lambda\ell)^{\varkappa+1} - (1+\lambda\ell)^\varkappa = (1+\lambda\ell)^\varkappa(1+\lambda\ell-1),$$

which yields

$$\Delta_\ell^{-1}(1+\ell)^\varkappa = \frac{e_1(\lambda, (\ell\varkappa)_\ell, 0)}{\lambda\ell}.$$

By Definition 1 of inverse difference operator, we have

$$\Delta_\ell^{-1}(e_1(\lambda, (\ell\varkappa)_\ell, 0)) = (\lambda\ell)^{-1} e_1(\lambda, (\ell\varkappa)_\ell, 0),$$
$$\Delta_\ell^{-2}(e_1(\lambda, (\ell\varkappa)_\ell, 0)) = (\lambda\ell)^{-2} e_1(\lambda, (\ell\varkappa)_\ell, 0).$$

In general, by induction on m, $\Delta_\ell^{-m} e_1(\lambda, (\ell\varkappa)_\ell, 0) = (\lambda\ell)^{-m} e_1(\lambda, (\ell\varkappa)_\ell, 0)$.
Now (4.2) is followed by replacing m by $-\nu$ and the proof is ended. □

Corollary 2. *For any $\nu > 0$ and $|\ell| < 1$, if we take $\varkappa = \frac{m}{\ell}$ in (4.2), we have*

$$\Delta_\ell^{-\nu} e_1(\lambda, (\varkappa)_\ell, 0) = (\lambda\ell)^{-\nu} e_1(\lambda, (\varkappa)_\ell, 0). \tag{4.3}$$

Proof. Replacing $\varkappa$ by $\frac{m}{\ell}$ in (4.2), then relabeling m as $\varkappa$, we get (4.3). □

To illustrate Corollary 2, we give the following example.

Example 2. Consider

$$\Delta_\ell^{-\nu} e_1(\lambda, (\ell\varkappa)_\ell, 0) = (\lambda\ell)^{-\nu} e_1(\lambda, (\ell\varkappa)_\ell, 0). \tag{4.4}$$

Taking $\ell = 0.2, \nu = 0.5, a = -\ell, c = 0, \varkappa = 20, \lambda = 2, \nu = 1$ and $u(\varkappa) = e(\lambda, \varkappa_\ell, 0)$ in (4.4) and by using Theorem 2 for $\nu = 1$, we get

$$\sum_{r=0}^{\infty} e_1(\lambda, (\ell(\varkappa - r\ell))_\ell, 0) = (\lambda\ell)^{-1} e_1(\lambda, (\ell\varkappa)_\ell, 0),$$

where the generalized Mittag-Leffler factorial function $e_\nu(\lambda, \varkappa_\ell, 0)$ is given in (4.1) and so

$$\begin{aligned}
\sum_{r=0}^{\infty} e_1(\lambda, (\ell(\varkappa - r\ell))_\ell, 0) &= \sum_{j=0}^{\infty} \frac{\lambda^j (\ell\varkappa - \ell)_{0.2}^{(j)}}{j!} + \sum_{j=0}^{\infty} \frac{\lambda^j (\ell\varkappa - 2\ell)_{0.2}^{(j)}}{j!} \\
&\quad + \sum_{j=0}^{\infty} \frac{\lambda^j (\ell\varkappa - 3\ell)_{0.2}^{(j)}}{j!} + \sum_{j=0}^{\infty} \frac{\lambda^j (\ell\varkappa - 4\ell)_{0.2}^{(j)}}{j!} + \sum_{j=0}^{\infty} \frac{\lambda^j (\ell\varkappa - 5\ell)_{0.2}^{(j)}}{j!} \\
&\quad + \sum_{j=0}^{\infty} \frac{\lambda^j (\ell\varkappa - 6\ell)_{0.2}^{(j)}}{j!} + \cdots \\
&= 597.61 + 426.82 + 304.79 + 217.75 + 155.55 + 111.10 \\
&\quad + 79.33 + 56.62 + 40.48 + 28.93 + 20.06 + 14.73 + 10.55 + 7.52 \\
&\quad + 5.38 + 3.85 + 2.74 + 1.96 + 1.4 + 1 + 1 + \cdots = 2089.16,
\end{aligned}$$

and

$$\begin{aligned}
(\lambda\ell)^{-1} e_1(\lambda, (\ell\varkappa)_\ell, 0) &= \frac{1}{0.4}\Big[\sum_{j=0}^{\infty} \frac{(\ell\varkappa)_\ell^{(j)}}{j!}\Big] = 1 + 8 + 30.40 + 72.96 \\
&\quad + 124.03 + 158.76 + 158.76 + 127.01 + 82.56 + 44.03 \\
&\quad + 19.37 + 7.04 + 2.11 + 0.52 + \cdots = \frac{1}{0.4}(836.55) = 2091.38.
\end{aligned}$$

Corollary 3. *If $|\ell| < 1$, then $\Delta_\ell^{-\nu}(1+\lambda\ell)^{\frac{\varkappa}{\ell}} = (\lambda\ell)^{-\nu}(1+\lambda\ell)^{\frac{\varkappa}{\ell}} = \Delta_\ell^{-\nu} e_1(\lambda, \varkappa_\ell, 0)$.*

Proof. We know that

$$e_1(\lambda, \varkappa_\ell, 0) = 1 + \frac{\lambda\varkappa}{1!} + \frac{\lambda^2\varkappa(\varkappa - \ell)}{2!} + \frac{\lambda^3\varkappa(\varkappa - \ell)(\varkappa - 2\ell)}{3!} + \cdots.$$

Replacing k by $m\ell$, we obtain

$$\begin{aligned}
e_1(\lambda, m\ell_\ell, 0) &= 1 + \frac{\lambda m\ell}{1!} + \frac{\lambda^2 m\ell(m\ell - \ell)}{2!} + \frac{\lambda^3 m\ell(m\ell - \ell)(m\ell - 2\ell)}{3!} + \cdots \\
&= 1 + \frac{\lambda m\ell}{1!} + \frac{\lambda^2 m(m-1)}{2!}\ell^2 + \frac{\lambda^3 m(m-1)(m-2)}{3!}\ell^3 + \cdots \\
&= (1+\lambda\ell)^m = (1+\lambda\ell)^{\frac{\varkappa}{\ell}}.
\end{aligned}$$

The proof is concluded by taking $\Delta_\ell^{-\nu}$ on both sides and applying (4.3). □

5. Application of the Mittag-Leffler factorial function

This section, revolves around the solution of a specific kind of higher–order linear non-homogeneous difference equation.

Theorem 10. *For a positive integer n, the Mittag-Leffler Factorial Function $e_n(\lambda, \varkappa_\ell, 0)$ satisfies the $(n-1)^{th}$-linear non homogeneous difference equation*

$$\Delta_\ell^{n-1}\frac{u(\varkappa)}{\ell^{n-1}}+\Delta_\ell^{n-2}\frac{u(\varkappa)}{\ell^{n-2}}+\cdots+\Delta_\ell\frac{u(\varkappa)}{\ell}+u(\varkappa)=e_1(\varkappa_\ell)=(1+\lambda\ell)^{\frac{\varkappa}{\ell}}. \tag{5.1}$$

Proof. From the special case of Definition 5,

$$e_n(\lambda,\varkappa_\ell,0)=1+\frac{\lambda^n\varkappa_\ell^{(n)}}{n!}+\frac{\lambda^{2n}\varkappa_\ell^{(2n)}}{2n!}+\frac{\lambda^{3n}\varkappa_\ell^{(3n)}}{3n!}+\cdots+\infty.$$

Since $\Delta_\ell\varkappa_\ell^{(r)}=r\ell\varkappa_\ell^{r-1}$, we have

$$\frac{1}{\lambda\ell}\Delta_\ell^1 e_n(\lambda,\varkappa_\ell,0)=1+\frac{\lambda^{n-1}\ell\varkappa_\ell^{(n-1)}}{(n-1)!}+\frac{\lambda^{2n-1}\ell\varkappa_\ell^{(2n-1)}}{(2n-1)!}+\frac{\lambda^{3n-1}\ell\varkappa_\ell^{(3n-1)}}{(3n-1)!}+\cdots+\infty,$$

$$\frac{1}{(\lambda\ell)^2}\Delta_\ell^2 e_n(\lambda,\varkappa_\ell,0)=1+\frac{\lambda^{n-2}\ell^2\varkappa_\ell^{(n-2)}}{(n-2)!}+\frac{\lambda^{2n-2}\ell^2\varkappa_\ell^{(2n-2)}}{(2n-2)!}+\frac{\lambda^{3n-2}\ell^2\varkappa_\ell^{(3n-2)}}{(3n-2)!}+\cdots+\infty,$$

and in general, we find

$$\frac{1}{(\lambda\ell)^r}\Delta_\ell^r e_n(\lambda,\varkappa_\ell,0)=1+\frac{\lambda^{n-r}\ell^r\varkappa_\ell^{(n-r)}}{(n-r)!}+\frac{\lambda^{2n-r}\ell^r\varkappa_\ell^{(2n-r)}}{(2n-r)!}+\frac{\lambda^{3n-r}\ell^r\varkappa_\ell^{(3n-r)}}{(3n-r)!}+\cdots+\infty. \tag{5.2}$$

Adding (5.2) for $r=0,1,2,3,\cdots,n-1,\Delta^0=1$, we get

$$\sum_{r=0}^{n-1}\frac{1}{(\lambda\ell)^r}\Delta_\ell^r e_n(\lambda,\varkappa_\ell,0)=e_1(\lambda,\varkappa_\ell,0),$$

and the proof follows $u(\varkappa)=e_n(\lambda,\varkappa_\ell,0)$. □

Example 3. For $n=2$, the equation (5.1) becomes the extorial function $e_2(\lambda,\varkappa_\ell,0)$ which satisfies the difference equation $\frac{1}{\lambda\ell}\Delta_\ell u(\varkappa)+u(\varkappa)=(1+\lambda\ell)^{\frac{\varkappa}{\ell}}, |\ell\lambda|<1$.

Proof. By taking Δ_ℓ on $e_2(\lambda,\varkappa_\ell,0)=1+\frac{\lambda^2\varkappa_\ell^{(2)}}{2!}+\frac{\lambda^4\varkappa_\ell^{(4)}}{4!}+\frac{\lambda^6\varkappa_\ell^{(6)}}{6!}+\cdots$ and applying

$$\Delta_\ell\varkappa_\ell^{(m)}=n\ell\varkappa_\ell^{(n-1)},$$

we get

$$\Delta_\ell e_2(\lambda,\varkappa_\ell,0)=\frac{\lambda^2 2\ell\varkappa_\ell^{(1)}}{1!}+\frac{\lambda^4 4\ell\varkappa_\ell^{(3)}}{4!}+\cdots,$$

which is the same as

$$\frac{1}{\lambda\ell}\Delta_\ell e_2(\lambda,\varkappa_\ell,0)=\frac{\lambda\varkappa_\ell^{(1)}}{1!}+\frac{\lambda^3\varkappa_\ell^{(3)}}{3!}+\cdots,$$

and yields that $e_2(\lambda,\varkappa_\ell,0)+\frac{1}{\lambda\ell}\Delta_\ell e_2(\lambda,\varkappa_\ell,0)=e_1(\lambda,\varkappa_\ell,0)=(1+\lambda\ell)^{\frac{\varkappa}{\ell}}$. □

6. Conclusion

Through this research, we derived a special kind of Mittag-Leffler function and Mittag-Leffler Factorial Functions. These functions are applied to find solution of higher order linear differential and difference equations. These solutions will generate a large number of relations in the vast area of fractional calculus.

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A NEW q-HERMITE-HADAMARD'S INEQUALITY AND ESTIMATES FOR MIDPOINT TYPE INEQUALITIES FOR CONVEX FUNCTIONS

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Abstract. This paper proves a new q-Hermite-Hadamard inequality for convex functions using quantum integrals. We also prove some new midpoint-type inequalities for q-differentiable convex functions. Moreover, we present some examples to illustrate our established results, supplemented with graphs.

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1. Introduction

A function $\varphi : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$ is called convex if it satisfies the inequality

$$\varphi(tx+(1-t)y) \leq t\varphi(x)+(1-t)\varphi(y)$$

where $x, y \in I$ and $t \in [0,1]$.

It is also well known that φ is convex if and only if it satisfies the Hermite-Hadamard inequality, stated below (see, [6]):

$$\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right) \leq \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\varphi(x)dx \leq \frac{\varphi(\alpha_1)+\varphi(\alpha_2)}{2}, \tag{1.1}$$

where $\varphi : I \to \mathbb{R}$ is a convex function and $\alpha_1, \alpha_2 \in I$ with $\alpha_1 < \alpha_2$. In [7, 11], authors proved some bounds for the right and left sides of the inequality (1.1) which are called trapezoid and midpoint inequalities, respectively.

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On the other hand, in [2], Alp et al. proved the following version of quantum Hermite-Hadamard type for convex functions using the left quantum integrals:

$$\varphi\left(\frac{q\alpha_1+\alpha_2}{[2]_q}\right) \leq \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\varphi(x)\ {}_{\alpha_1}d_qx \leq \frac{q\varphi(\alpha_1)+\varphi(\alpha_2)}{[2]_q}. \tag{1.2}$$

Recently, Bermudo et al. [3] used the right quantum integrals and proved the following variant of Hermite-Hadamard type inequality for convex functions:

$$\varphi\left(\frac{\alpha_1+q\alpha_2}{[2]_q}\right) \leq \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\varphi(x)\ {}^{\alpha_2}d_qx \leq \frac{\varphi(\alpha_1)+q\varphi(\alpha_2)}{[2]_q}. \tag{1.3}$$

Several papers are devoted to finding bounds for the left and right sides of inequalities (1.2) and (1.3). In [12], Noor et al. proved several bounds for the right-hand side of the inequality of (1.2). Alp et al., by using convex functions, established some new results for the left side of the inequality (1.2) in [2]. In [1, 4], quantum Simpson's and Newton's type inequalities for convex and co-ordinated convex functions were proved. For more recent results, one can consult [5, 10].

Inspired by the ongoing studies, we prove a new version of quantum Hermite–Hadamard inequalities for convex functions. We also prove some new corresponding quantum midpoint inequalities for q-differentiable convex functions. We present many examples to illustrate our results, supplemented with graphs.

2. Basics of q-calculus

In this section, we recall some basics of quantum calculus, and throughout this paper, let $0<q<1$ be a constant.

The q-number or q-analogue of $n\in\mathbb{N}$ is given by (see [9])

$$[n]_q = \frac{1-q^n}{1-q} = 1+q+q^2+\cdots+q^{n-1}. \tag{2.1}$$

The q-Jackson integral for the function φ over $[0,\alpha_1]$ is defined as (see, [8]):

$$\int_0^{\alpha_2}\varphi(x)\,d_qx = (1-q)\,\alpha_2\sum_{n=0}^{\infty}q^n\varphi(\alpha_2q^n) \tag{2.2}$$

and q-Jackson integral for a function φ over $[\alpha_1,\alpha_2]$ is as follows (see, [8]):

$$\int_{\alpha_1}^{\alpha_2}\varphi(x)\,d_qx = \int_0^{\alpha_2}\varphi(x)\,d_qx - \int_0^{\alpha_1}\varphi(x)\,d_qx. \tag{2.3}$$

Definition 1 ([15])**.** Let $\varphi:[\alpha_1,\alpha_2]\to\mathbb{R}$ be a continuous function. Then the left q-derivative of function φ at $x\in[\alpha_1,\alpha_2]$ is defined by

$${}_{\alpha_1}D_q\varphi(x) = \begin{cases} \dfrac{\varphi(x)-\varphi(qx+(1-q)\alpha_1)}{(1-q)(x-\alpha_1)}, & \text{if } x\neq\alpha_1; \\ \lim\limits_{x\to\alpha_1}{}_{\alpha_1}D_q\varphi(x), & \text{if } x=\alpha_1. \end{cases} \tag{2.4}$$

The function φ is said to be q-differentiable function on $[\alpha_1,\alpha_2]$ if ${}_{\alpha_1}D_q\varphi(x)$ exists for all $x\in[\alpha_1,\alpha_2]$.

Definition 2 ([15])**.** Let $\varphi:[\alpha_1,\alpha_2]\to\mathbb{R}$ be a continuous function. Then the left q-integral of function φ at $z\in[\alpha_1,\alpha_2]$ is defined by

$$\int_{\alpha_1}^{z}\varphi(x)_{\alpha_1}d_qx=(1-q)(z-\alpha_1)\sum_{n=0}^{\infty}q^n\varphi\left(q^nz+(1-q^n)\alpha_1\right). \tag{2.5}$$

The function φ is said to be q-integrable function on $[\alpha_1,\alpha_2]$ if $\int_{\alpha_1}^{z}\varphi(x)_{\alpha_1}d_qx$ exists for all $z\in[\alpha_1,\alpha_2]$.

On the other hand, Bermudo et al. defined new quantum derivatives and quantum integral which are called right q-derivative and right q-integral:

Definition 3 ([3])**.** The right q-derivative of mapping $\varphi:[\alpha_1,\alpha_2]\to\mathbb{R}$ is defined as:

$${}^{\alpha_2}D_q\varphi(x)=\frac{\varphi(qx+(1-q)\alpha_2)-\varphi(x)}{(1-q)(\alpha_2-x)},\quad x\neq\alpha_2.$$

If $x=\alpha_2$, we define ${}^{\alpha_2}D_q\varphi(\alpha_2)=\lim_{x\to\alpha_2}{}^{\alpha_2}D_q\varphi(x)$ if it exists and it is finite.

Definition 4 ([3])**.** The right q-definite integral of mapping $\varphi:[\alpha_1,\alpha_2]\to\mathbb{R}$ at $z\in[\alpha_1,\alpha_2]$ is defined as:

$$\int_{z}^{\alpha_2}\varphi(x)^{\alpha_2}d_qx=(1-q)(\alpha_2-z)\sum_{k=0}^{\infty}q^k\varphi\left(q^kz+\left(1-q^k\right)\alpha_2\right).$$

Lemma 1 ([13])**.** *For continuous functions $\varphi,h:[\alpha_1,\alpha_2]\to\mathbb{R}$, the following equality is true:*

$$\begin{aligned}&\int_0^c h(t)\ {}^{\alpha_2}D_q\varphi(t\alpha_1+(1-t)\alpha_2)\,d_qt\\&=\frac{1}{\alpha_2-\alpha_1}\int_0^c D_qh(t)\varphi(qt\alpha_1+(1-qt)\alpha_2)\,d_qt-\left.\frac{h(t)\varphi(t\alpha_1+(1-t)\alpha_2)}{\alpha_2-\alpha_1}\right|_0^c.\end{aligned}$$

Lemma 2 ([14])**.** *For continuous functions $\varphi,h:[\alpha_1,\alpha_2]\to\mathbb{R}$, the following equality is true:*

$$\begin{aligned}&\int_0^c h(t)\,{}_{\alpha_1}D_q\varphi(t\alpha_2+(1-t)\alpha_1)\,d_qt\\&=\left.\frac{h(t)\varphi(t\alpha_2+(1-t)\alpha_1)}{\alpha_2-\alpha_1}\right|_0^c-\frac{1}{\alpha_2-\alpha_1}\int_0^c D_qh(t)\varphi(qt\alpha_2+(1-qt)\alpha_1)\,d_qt.\end{aligned}$$

3. q-HERMITE-HADAMARD INEQUALITY

In this section, we prove the new quantum Hermite-Hadamard inequality and give an example to illustrate the obtained inequality.

Theorem 1. *Let $\varphi:[\alpha_1,\alpha_2]\to\mathbb{R}$ be a convex mapping, then we have the following inequality:*

$$\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)\leq\frac{1}{\alpha_2-\alpha_1}\left[\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}\varphi(x)\ {}_{\frac{\alpha_1+\alpha_2}{2}}d_qx+\int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}\varphi(x)\ {}^{\frac{\alpha_1+\alpha_2}{2}}d_qx\right]\leq\frac{\varphi(\alpha_1)+\varphi(\alpha_2)}{2}. \tag{3.1}$$

Proof. Since φ is a convex function, we have

$$\begin{aligned}\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)&=\varphi\left(\frac{1}{2}\left(\frac{1-t}{2}\alpha_1+\frac{1+t}{2}\alpha_2\right)+\frac{1}{2}\left(\frac{1+t}{2}\alpha_1+\frac{1-t}{2}\alpha_2\right)\right)\\&\leq\frac{1}{2}\left[\varphi\left(\frac{1-t}{2}\alpha_1+\frac{1+t}{2}\alpha_2\right)+\varphi\left(\frac{1+t}{2}\alpha_1+\frac{1-t}{2}\alpha_2\right)\right]\\&\leq\frac{\varphi(\alpha_1)+\varphi(\alpha_2)}{2}.\end{aligned} \tag{3.2}$$

q-integrating the inequality (3.2) over $[0,1]$, we have

$$\begin{aligned}\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)&\leq\frac{1}{2}\int_0^1\varphi\left(\frac{\alpha_1+\alpha_2}{2}+t\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)\right)d_qt\\&\quad+\frac{1}{2}\int_0^1\varphi\left(\frac{\alpha_1+\alpha_2}{2}+t\left(\alpha_1-\frac{\alpha_1+\alpha_2}{2}\right)\right)d_qt\\&=\frac{1}{\alpha_2-\alpha_1}\left[\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}\varphi(x)\ {}_{\frac{\alpha_1+\alpha_2}{2}}d_qx+\int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}\varphi(x)\ {}^{\frac{\alpha_1+\alpha_2}{2}}d_qx\right].\end{aligned}$$

It is obvious from Definition 2 and Definition 4 that

$$\int_0^1\varphi\left(\frac{\alpha_1+\alpha_2}{2}+t\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)\right)d_qt=\frac{2}{\alpha_2-\alpha_1}\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}\varphi(x)\ {}_{\frac{\alpha_1+\alpha_2}{2}}d_qx$$

and

$$\int_0^1\varphi\left(\frac{\alpha_1+\alpha_2}{2}+t\left(\alpha_1-\frac{\alpha_1+\alpha_2}{2}\right)\right)d_qt=\frac{2}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}\varphi(x)\ {}^{\frac{\alpha_1+\alpha_2}{2}}d_qx.$$

This completes the proof. □

Remark 1. In Theorem 1, if we set the limit as $q\to1^-$, then we obtain the classical Hermite-Hadamard inequality (1.1).

Example 1. The function $\varphi(x) = e^x$ is convex on $\mathbb{R}$ so that it also convex on $[0,1]$. For this convex function, we have

$$\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2} \varphi(x) \ {}_{\frac{\alpha_1+\alpha_2}{2}} d_q x = \int_{\frac{1}{2}}^{1} e^x \ {}_{\frac{1}{2}} d_q x = \frac{(1-q)\, e^{1/2}}{2} \sum_{n=0}^{\infty} q^n e^{\frac{q^n}{2}}$$

and

$$\int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}} \varphi(x) \ {}_{\frac{\alpha_1+\alpha_2}{2}} d_q x = \int_{0}^{\frac{1}{2}} e^x \ {}_{\frac{1}{2}} d_q x = \frac{(1-q)\, e^{1/2}}{2} \sum_{n=0}^{\infty} q^n e^{-\frac{q^n}{2}}.$$

Hence, we have

$$\frac{1}{\alpha_2-\alpha_1}\left[\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2} \varphi(x) \ {}_{\frac{\alpha_1+\alpha_2}{2}} d_q x + \int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}} \varphi(x) \ {}_{\frac{\alpha_1+\alpha_2}{2}} d_q x\right]$$
$$= \frac{(1-q)\, e^{1/2}}{2}\left[\sum_{n=0}^{\infty} q^n e^{\frac{q^n}{2}} + \sum_{n=0}^{\infty} q^n e^{-\frac{q^n}{2}}\right] := \Upsilon_{mid}.$$

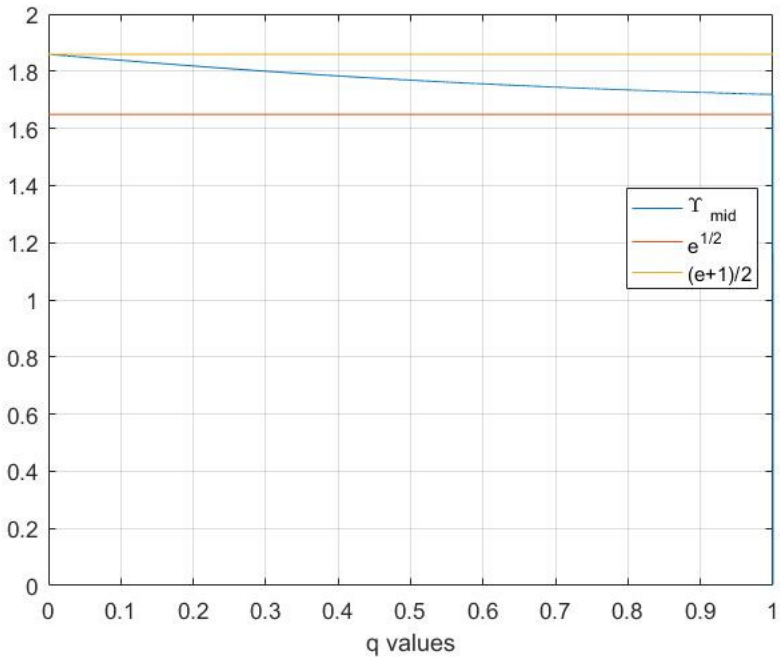

FIGURE 1. Example 1

We also have

$$\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right) = \varphi\left(\frac{1}{2}\right) = e^{1/2}$$

and

$$\frac{\varphi(\alpha_1)+\varphi(\alpha_2)}{2}=\frac{\varphi(0)+\varphi(1)}{2}=\frac{e+1}{2}.$$

It is obvious from Figure 1 that we have the inequality

$$e^{1/2}\leq\Upsilon_{mid}\leq\frac{e+1}{2}$$

for $q\in(0,1)$. On the other hand Figure 1 shows that $\Upsilon_{mid}\to\int_0^1 e^x dx=e-1$ for $q\to 1$.

4. Midpoint inequalities

In this section, we prove some left-estimates of the newly proved Hermite-Hadamard inequality (3.1) using the q-differentiability of the function.

Let us start with the following lemma.

Lemma 3. *Let $\varphi:[\alpha_1,\alpha_2]\subset\mathbb{R}\to\mathbb{R}$ be a function. If the functions ${}_{\alpha_1}D_q\varphi$ and ${}^{\alpha_2}D_q\varphi$ are continuous and integrable over $[\alpha_1,\alpha_2]$, then we have the following new equality:*

$$\begin{aligned}&\frac{1}{\alpha_2-\alpha_1}\left[\begin{array}{c}\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}\varphi(x)\ {}_{\frac{\alpha_1+\alpha_2}{2}}d_qx\\ +\int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}\varphi(x)\ {}^{\frac{\alpha_1+\alpha_2}{2}}d_qx\end{array}\right]-\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)\\&=\frac{(\alpha_2-\alpha_1)}{4}\left[\int_0^1(1-qt)\ {}_{\alpha_1}D_q\varphi\left(\frac{\alpha_1+\alpha_2}{2}+t\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)\right)d_qt\right.\\&\left.-\int_0^1(1-qt)\ {}^{\alpha_2}D_q\varphi\left(\frac{\alpha_1+\alpha_2}{2}+t\left(\alpha_1-\frac{\alpha_1+\alpha_2}{2}\right)\right)d_qt\right].\end{aligned}\tag{4.1}$$

Proof. Let

$$\begin{aligned}&\frac{(\alpha_2-\alpha_1)}{4}\left[\int_0^1(1-qt)\ {}_{\alpha_1}D_q\varphi\left(t\alpha_2+(1-t)\frac{\alpha_1+\alpha_2}{2}\right)d_qt\right.\\&\left.-\int_0^1(1-qt)\ {}^{\alpha_2}D_q\varphi\left(t\alpha_1+(1-t)\frac{\alpha_1+\alpha_2}{2}\right)d_qt\right]\\&=\frac{(\alpha_2-\alpha_1)}{4}[I_1-I_2].\end{aligned}\tag{4.2}$$

By applying Lemma 1 to calculate the integral I_1, we have

$$\begin{aligned}I_1&=\int_0^1(1-qt)_{\alpha_1}D_q\,\varphi\left(\frac{\alpha_1+\alpha_2}{2}+t\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)\right)d_qt\\&=\int_0^1{}_{\alpha_1}D_q\,\varphi\left(\frac{\alpha_1+\alpha_2}{2}+t\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)\right)d_qt\end{aligned}\tag{4.3}$$

$$
\begin{aligned}
&-\int_0^1 qt_{\alpha_1}D_q\,\varphi\left(\frac{\alpha_1+\alpha_2}{2}+t\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)\right)d_qt\\
&=\left.\frac{\varphi\left(\frac{\alpha_1+\alpha_2}{2}+t\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)\right)}{\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)}\right|_0^1-\left.\frac{qt\,\varphi\left(\frac{\alpha_1+\alpha_2}{2}+t\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)\right)}{\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)}\right|_0^1\\
&+\int_0^1\frac{q\,\varphi\left(\frac{\alpha_1+\alpha_2}{2}+tq\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)\right)}{\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)}d_qt\\
&=\frac{\varphi\left(\frac{\alpha_1+\alpha_2}{2}+\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)}{\left(\frac{\alpha_2-\alpha_1}{2}\right)}-\frac{\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)}{\left(\frac{\alpha_2-\alpha_1}{2}\right)}-\frac{q\,\varphi\left(\frac{\alpha_1+\alpha_2}{2}+\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)}{\left(\frac{\alpha_2-\alpha_1}{2}\right)}\\
&+\frac{2q}{b-a}(1-q)(1-0)\sum_{n=0}^{\infty}q^n\varphi\left(\frac{\alpha_1+\alpha_2}{2}+q^{n+1}\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)\right)\\
&=\frac{2\varphi(\alpha_2)}{(\alpha_2-\alpha_1)}-\frac{2\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)}{(\alpha_2-\alpha_1)}-\frac{2q\,\varphi(\alpha_2)}{(\alpha_2-\alpha_1)}\\
&+\frac{2q}{b-a}(1-q)\sum_{n=0}^{\infty}q^n\varphi\left(\frac{\alpha_1+\alpha_2}{2}+q^{n+1}\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)\right)\\
&=\frac{2(1-q)\varphi(\alpha_2)}{(\alpha_2-\alpha_1)}-\frac{2\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)}{(\alpha_2-\alpha_1)}\\
&+\frac{2^2}{(\alpha_2-\alpha_1)^2}(1-q)\left(\frac{\alpha_2-\alpha_1}{2}\right)\sum_{n=0}^{\infty}q^{n+1}\varphi\left(\frac{\alpha_1+\alpha_2}{2}+q^{n+1}\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)\right)\\
&=\frac{2(1-q)\varphi(\alpha_2)}{(\alpha_2-\alpha_1)}-\frac{2\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)}{(\alpha_2-\alpha_1)}\\
&+\frac{4}{(\alpha_2-\alpha_1)^2}(1-q)\left(\frac{\alpha_2-\alpha_1}{2}\right)\sum_{n=1}^{\infty}q^{n}\varphi\left(\frac{\alpha_1+\alpha_2}{2}+q^{n}\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)\right)\\
&=\frac{2(1-q)\varphi(\alpha_2)}{(\alpha_2-\alpha_1)}-\frac{2\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)}{(\alpha_2-\alpha_1)}\\
&+\frac{4}{(\alpha_2-\alpha_1)^2}(1-q)\left(\frac{\alpha_2-\alpha_1}{2}\right)\sum_{n=0}^{\infty}q^{n}\varphi\left(\frac{\alpha_1+\alpha_2}{2}+q^{n}\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)\right)\\
&+\frac{4}{(\alpha_2-\alpha_1)^2}(1-q)\left(\frac{\alpha_2-\alpha_1}{2}\right)\varphi(\alpha_2)\\
&=-\frac{2}{(\alpha_2-\alpha_1)}\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)+\frac{4}{(\alpha_2-\alpha_1)^2}\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}\varphi(x)\;{}_{\frac{\alpha_1+\alpha_2}{2}}d_qx.
\end{aligned}
$$

Similarly, by using Lemma 2 to calculate the integral I_2, we obtain

$$I_2 = \int_0^1 (1-qt)\ {}^{\alpha_2}D_q\varphi\left(\frac{\alpha_1+\alpha_2}{2}+t\left(\alpha_1-\frac{\alpha_1+\alpha_2}{2}\right)\right)d_qt \tag{4.4}$$
$$= \frac{2}{(\alpha_2-\alpha_1)}\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)-\frac{4}{(\alpha_2-\alpha_1)^2}\int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}\varphi(x)\ {}^{\frac{\alpha_1+\alpha_2}{2}}d_qx.$$

Thus, we establish the desired equality by putting (4.3) and (4.4) in (4.2). The proof is completed. □

Theorem 2. *We assume that the conditions Lemma 3 hold. If the functions* $\left|{}_{\alpha_1}D_q\varphi\right|$ *and* $\left|{}^{\alpha_2}D_q\varphi\right|$ *are convex, then the following inequality holds:*

$$\left|\frac{1}{\alpha_2-\alpha_1}\left[\begin{array}{c}\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}\varphi(x)\ {}^{\frac{\alpha_1+\alpha_2}{2}}d_qx\\ +\int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}\varphi(x)\ {}^{\frac{\alpha_1+\alpha_2}{2}}d_qx\end{array}\right]-\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)\right| \tag{4.5}$$
$$\leq \frac{(\alpha_2-\alpha_1)}{8[2]_q[3]_q}\left[q[2]_q\left|{}_{\alpha_1}D_q\varphi(\alpha_1)\right|+\left(1+[3]_q\right)\left|{}_{\alpha_1}D_q\varphi(\alpha_2)\right|\right.$$
$$\left.+\left(1+[3]_q\right)\left|{}^{\alpha_2}D_q\varphi(\alpha_1)\right|+q[2]_q\left|{}^{\alpha_2}D_q\varphi(\alpha_2)\right|\right].$$

Proof. By taking modulus in (4.1), we have

$$\left|\frac{1}{\alpha_2-\alpha_1}\left[\begin{array}{c}\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}\varphi(x)\ {}^{\frac{\alpha_1+\alpha_2}{2}}d_qx\\ +\int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}\varphi(x)\ {}^{\frac{\alpha_1+\alpha_2}{2}}d_qx\end{array}\right]-\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)\right| \tag{4.6}$$
$$\leq \frac{(\alpha_2-\alpha_1)}{4}\left[\int_0^1(1-qt)\left|{}_{\alpha_1}D_q\varphi\left(\frac{\alpha_1+\alpha_2}{2}+t\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)\right)\right|d_qt\right.$$
$$\left.+\int_0^1(1-qt)\left|{}^{\alpha_2}D_q\varphi\left(\frac{\alpha_1+\alpha_2}{2}+t\left(\alpha_1-\frac{\alpha_1+\alpha_2}{2}\right)\right)\right|d_qt\right].$$

Since the functions $\left|{}_{\alpha_1}D_q\varphi\right|$ and $\left|{}^{\alpha_2}D_q\varphi\right|$ are convex, we have

$$\left|\frac{1}{\alpha_2-\alpha_1}\left[\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}\varphi(x)\ {}^{\frac{\alpha_1+\alpha_2}{2}}d_qx+\int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}\varphi(x)\ {}^{\frac{\alpha_1+\alpha_2}{2}}d_qx\right]-\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|$$
$$\leq \frac{(\alpha_2-\alpha_1)}{4}\left[\int_0^1(1-qt)\left(\left(\frac{1-t}{2}\right)\left|{}_{\alpha_1}D_q\varphi(\alpha_1)\right|+\left(\frac{1+t}{2}\right)\left|{}_{\alpha_1}D_q\varphi(\alpha_2)\right|\right)d_qt\right.$$
$$\left.+\int_0^1(1-qt)\left(\left(\frac{1+t}{2}\right)\left|{}^{\alpha_2}D_q\varphi(\alpha_1)\right|+\left(\frac{1-t}{2}\right)\left|{}^{\alpha_2}D_q\varphi(\alpha_2)\right|\right)d_qt\right]$$

$$= \frac{(\alpha_2-\alpha_1)}{4}\left[\frac{q}{2[3]_q}\left|{}_{\alpha_1}D_q\varphi(\alpha_1)\right| + \frac{1+[3]_q}{2[2]_q[3]_q}\left|{}_{\alpha_1}D_q\varphi(\alpha_2)\right| + \frac{1+[3]_q}{2[2]_q[3]_q}\left|{}^{\alpha_2}D_q\varphi(\alpha_1)\right| + \frac{q}{2[3]_q}\left|{}^{\alpha_2}D_q\varphi(\alpha_2)\right|\right].$$

Thus, the proof is completed. □

Remark 2. In Theorem 2, if we set the limit as $q \to 1^-$, then we obtain [11, Theorem 2.2].

Theorem 3. *We assume that the conditions Lemma 3 hold. If the functions* $\left|{}_{\alpha_1}D_q\varphi\right|^s$ *and* $\left|{}^{\alpha_2}D_q\varphi\right|^s$, $s \geq 1$ *are convex, then the following inequality holds:*

$$\left|\frac{1}{\alpha_2-\alpha_1}\left[\begin{array}{c}\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}\varphi(x)\ {}_{\frac{\alpha_1+\alpha_2}{2}}d_qx \\ +\int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}\varphi(x)\ {}^{\frac{\alpha_1+\alpha_2}{2}}d_qx\end{array}\right] - \varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)\right| \tag{4.7}$$

$$\leq \frac{(\alpha_2-\alpha_1)}{4}\left(\frac{1}{[2]_q}\right)^{1-\frac{1}{s}}\left[\left(\frac{q[2]_q\left|{}_{\alpha_1}D_q\varphi(\alpha_1)\right|^s + \left(1+[3]_q\right)\left|{}_{\alpha_1}D_q\varphi(\alpha_2)\right|^s}{2[2]_q[3]_q}\right)^{\frac{1}{s}} + \left(\frac{\left(1+[3]_q\right)\left|{}^{\alpha_2}D_q\varphi(\alpha_1)\right|^s + q[2]_q\left|{}^{\alpha_2}D_q\varphi(\alpha_2)\right|^s}{2[2]_q[3]_q}\right)^{\frac{1}{s}}\right].$$

Proof. By using the power mean inequality in (4.6), we have

$$\left|\frac{1}{\alpha_2-\alpha_1}\left[\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}\varphi(x)\ {}_{\frac{\alpha_1+\alpha_2}{2}}d_qx + \int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}\varphi(x)\ {}^{\frac{\alpha_1+\alpha_2}{2}}d_qx\right] - \varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|$$
$$\leq \frac{(\alpha_2-\alpha_1)}{4}\left(\int_0^1(1-qt)\,d_qt\right)^{1-\frac{1}{s}}$$
$$\times\left[\left(\int_0^1(1-qt)\left|{}_{\alpha_1}D_q\varphi\left(\frac{\alpha_1+\alpha_2}{2}+t\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)\right)\right|^s d_qt\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\int_0^1(1-qt)\left|{}^{\alpha_2}D_q\varphi\left(\frac{\alpha_1+\alpha_2}{2}+t\left(\alpha_1-\frac{\alpha_1+\alpha_2}{2}\right)\right)\right|^s d_qt\right)^{\frac{1}{s}}\right].$$

By using the convexity of the functions $\left|{}_{\alpha_1}D_q\varphi\right|^s$ and $\left|{}^{\alpha_2}D_q\varphi\right|^s$, we have

$$\left|\frac{1}{\alpha_2-\alpha_1}\left[\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}\varphi(x)\ {}_{\frac{\alpha_1+\alpha_2}{2}}d_qx + \int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}\varphi(x)\ {}^{\frac{\alpha_1+\alpha_2}{2}}d_qx\right] - \varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|$$

$$
\begin{aligned}
&\leq \frac{(\alpha_2-\alpha_1)}{4}\left(\int_0^1 (1-qt)\,d_qt\right)^{1-\frac{1}{s}} \\
&\quad\times\left[\left(\int_0^1 (1-qt)\left(\frac{1-t}{2}\left|{}_{\alpha_1}D_q\varphi(\alpha_1)\right|^s+\frac{1+t}{2}\left|{}_{\alpha_1}D_q\varphi(\alpha_2)\right|^s\right)d_qt\right)^{\frac{1}{s}}\right. \\
&\quad\left.+\left(\int_0^1 (1-qt)\left(\frac{1+t}{2}\left|{}^{\alpha_2}D_q\varphi(\alpha_1)\right|^s+\frac{1-t}{2}\left|{}^{\alpha_2}D_q\varphi(\alpha_2)\right|^s\right)d_qt\right)^{\frac{1}{s}}\right] \\
&=\frac{(\alpha_2-\alpha_1)}{4}\left(\frac{1}{[2]_q}\right)^{1-\frac{1}{s}}\left[\left(\frac{q}{2[3]_q}\left|{}_{\alpha_1}D_q\varphi(\alpha_1)\right|^s+\frac{1+[3]_q}{2[2]_q[3]_q}\left|{}_{\alpha_1}D_q\varphi(\alpha_2)\right|^s\right)^{\frac{1}{s}}\right. \\
&\quad\left.+\left(\frac{1+[3]_q}{2[2]_q[3]_q}\left|{}^{\alpha_2}D_q\varphi(\alpha_1)\right|^s+\frac{q}{2[3]_q}\left|{}^{\alpha_2}D_q\varphi(\alpha_2)\right|^s\right)^{\frac{1}{s}}\right].
\end{aligned}
$$

Thus, the proof is completed. □

Theorem 4. *We assume that the conditions Lemma 3 hold. If the functions* $\left|{}_{\alpha_1}D_q\varphi\right|^s$ *and* $\left|{}^{\alpha_2}D_q\varphi\right|^s$, $s>1$ *are convex, then the following inequality holds:*

$$
\begin{aligned}
&\left|\frac{1}{\alpha_2-\alpha_1}\left[\begin{array}{c}\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}\varphi(x)\ {}_{\frac{\alpha_1+\alpha_2}{2}}d_qx \\ +\int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}\varphi(x)\ {}^{\frac{\alpha_1+\alpha_2}{2}}d_qx\end{array}\right]-\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)\right| \qquad (4.8)\\
&\leq\frac{(\alpha_2-\alpha_1)}{4}\left(\frac{1-(1-q)^{r+1}}{q[r+1]_q}\right)^{\frac{1}{r}}\left[\left(\frac{q\left|{}_{\alpha_1}D_q\varphi(\alpha_1)\right|^s+\left([2]_q+1\right)\left|{}_{\alpha_1}D_q\varphi(\alpha_2)\right|^s}{2[2]_q}\right)^{\frac{1}{s}}\right. \\
&\quad\left.+\left(\frac{\left([2]_q+1\right)\left|{}^{\alpha_2}D_q\varphi(\alpha_1)\right|^s+q\left|{}^{\alpha_2}D_q\varphi(\alpha_2)\right|^s}{2[2]_q}\right)^{\frac{1}{s}}\right],
\end{aligned}
$$

where $s^{-1}+r^{-1}=1$.

Proof. By applying Hölder's inequality in (4.6), we have

$$
\begin{aligned}
&\left|\frac{1}{\alpha_2-\alpha_1}\left[\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}\varphi(x)\ {}_{\frac{\alpha_1+\alpha_2}{2}}d_qx+\int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}\varphi(x)\ {}^{\frac{\alpha_1+\alpha_2}{2}}d_qx\right]-\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)\right| \\
&\leq\frac{(\alpha_2-\alpha_1)}{4}\left(\int_0^1(1-qt)^r\,d_qt\right)^{\frac{1}{r}}
\end{aligned}
$$

$$\times\left[\left(\int_0^1\left|{}_{\alpha_1}D_q\varphi\left(\frac{\alpha_1+\alpha_2}{2}+t\left(\alpha_2-\frac{\alpha_1+\alpha_2}{2}\right)\right)\right|^s d_qt\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\int_0^1\left|{}^{\alpha_2}D_q\varphi\left(\frac{\alpha_1+\alpha_2}{2}+t\left(\alpha_1-\frac{\alpha_1+\alpha_2}{2}\right)\right)\right|^s d_qt\right)^{\frac{1}{s}}\right].$$

Since the functions $\left|{}_{\alpha_1}D_q\varphi\right|^s$ and $\left|{}^{\alpha_2}D_q\varphi\right|^s$ are convex, we have

$$\left|\frac{1}{\alpha_2-\alpha_1}\left[\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}\varphi(x)\ {}_{\frac{\alpha_1+\alpha_2}{2}}d_qx+\int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}\varphi(x)\ {}^{\frac{\alpha_1+\alpha_2}{2}}d_qx\right]-\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|$$
$$\leq\frac{(\alpha_2-\alpha_1)}{4}\left(\int_0^1(1-qt)^r d_qt\right)^{\frac{1}{r}}$$
$$\times\left[\left(\int_0^1\left(\frac{1-t}{2}\left|{}_{\alpha_1}D_q\varphi(\alpha_1)\right|^s+\frac{1+t}{2}\left|{}_{\alpha_1}D_q\varphi(\alpha_2)\right|^s\right)d_qt\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\int_0^1\left(\frac{1+t}{2}\left|{}^{\alpha_2}D_q\varphi(\alpha_1)\right|^s+\frac{1-t}{2}\left|{}^{\alpha_2}D_q\varphi(\alpha_2)\right|^s\right)d_qt\right)^{\frac{1}{s}}\right]$$
$$=\frac{(\alpha_2-\alpha_1)}{4}\left(\frac{1-(1-q)^{r+1}}{q[r+1]_q}\right)^{\frac{1}{r}}\left[\left(\frac{q}{2[2]_q}\left|{}_{\alpha_1}D_q\varphi(\alpha_1)\right|^s+\frac{[2]_q+1}{2[2]_q}\left|{}_{\alpha_1}D_q\varphi(\alpha_2)\right|^s\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\frac{[2]_q+1}{2[2]_q}\left|{}^{\alpha_2}D_q\varphi(\alpha_1)\right|^s+\frac{q}{2[2]_q}\left|{}^{\alpha_2}D_q\varphi(\alpha_2)\right|^s\right)^{\frac{1}{s}}\right].$$

Hence, the proof is completed. □

5. Example

In this section, we give an example to support one of the newly established inequalities.

Example 2. For a convex functions $\varphi:[1,3]\to\mathbb{R}$ is defined by $\varphi(x)=x+2$. From Theorem 2 with $q=\frac{1}{4}$, the left hand side of inequality (4.5) becomes

$$\left|\frac{1}{\alpha_2-\alpha_1}\left[\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}\varphi(x)\ {}_{\frac{\alpha_1+\alpha_2}{2}}d_qx+\int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}\varphi(x)\ {}^{\frac{\alpha_1+\alpha_2}{2}}d_qx\right]-\varphi\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|$$
$$=\left|\frac{1}{2}\left[\int_2^3(x+2)\ {}_2d_{\frac{1}{4}}x+\int_1^2(x+2)\ {}^2d_{\frac{1}{4}}x\right]-4\right|=\left|\frac{1}{2}(4+4)-4\right|=0$$

and the right side becomes

$$\begin{aligned}
&\frac{(\alpha_2-\alpha_1)}{8[2]_q[3]_q}\Big[q[2]_q\left|{}_{\alpha_1}D_q\varphi(\alpha_1)\right|+\left(1+[3]_q\right)\left|{}_{\alpha_1}D_q\varphi(\alpha_2)\right| \\
&\qquad\qquad +\left(1+[3]_q\right)\left|{}^{\alpha_2}D_q\varphi(\alpha_1)\right|+q[2]_q\left|{}^{\alpha_2}D_q\varphi(\alpha_2)\right|\Big] \\
&=\frac{2}{8(1+q)(q^2+q+1)}\cdot\left[\left(2+q^2+q\right)+q(q+1)+q(q+1)+\left(2+q^2+q\right)\right] \\
&=\frac{1}{4(1+q)(q^2+q+1)}\cdot 4\left(1+q^2+q\right)=\frac{1}{q+1}=\frac{4}{5}=0.8.
\end{aligned}$$

It is clear that

$$0<0.8$$

which demonstrates inequality (4.5).

6. Concluding remarks

In this paper, we have established some new Hermite-Hadamard inequality for convex functions in the context of quantum calculus. Moreover, we have proved some quantum estimates for midpoint-type inequalities for q -differentiable convex functions. It is also has proven by some mathematical examples that the newly established inequalities are valid for any convex function. It is an interesting and new problem that the upcoming researchers can obtain similar inequalities for coordinated convex functions.

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NJ-SEMICOMMUTATIVE RINGS

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Abstract. We call a ring R NJ-semicommutative if $wh \in N(R)$ implies $wRh \subseteq J(R)$ for any $w, h \in R$. The class of NJ-semicommutative rings is large enough that it contains semicommutative rings, left (right) quasi-duo rings, J-clean rings, and J-quasipolar rings. We provide some conditions for NJ-semicommutative rings to be reduced. We also observe that if $R/J(R)$ is reduced, then R is NJ-semicommutative, and therefore we provide some conditions for NJ-semicommutative ring R for which $R/J(R)$ is reduced. We also study some extensions of NJ-semicommutative rings wherein, among other results, we prove that the polynomial ring over an NJ-semicommutative ring need not be NJ-semicommutative.

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1. Introduction

All rings considered in this paper are associative with identity unless otherwise mentioned. R represents a ring, and all modules are unital. The symbols $Z(R)$, $E(R)$, $J(R)$, $N(R)$, $U(R)$, $T_n(R)$, $M_n(R)$, $N^*(R)$, and $N_*(R)$ respectively denote the set of all central elements of R, the set of all idempotent elements of R, the Jacobson radical of R, the set of all nilpotent elements of R, the set of all units of R, the ring of upper triangular matrices of order $n \times n$ over R, the ring of all $n \times n$ matrices over R, the upper nil radical of R, and the lower nil radical of R. For any $a \in R$, the notation $l(a)$ $(r(a))$ stands for the left (right) annihilator of a.

Recall that R is said to be:

(1) *reduced* if $N(R) = 0$.
(2) *semicommutative* ([8]) if $wh = 0$ implies $wRh = 0$ for any $w,\ h \in R$.
(3) *abelian* if $E(R) \subseteq Z(R)$.
(4) *directly finite* if $xy = 1$ implies $yx = 1$, where $x,\ y \in R$.
(5) *left (right) quasi-duo* ([7]) if every maximal left (right) ideal of R is an ideal of R.
(6) *2-primal* if $N(R) = N_*(R)$.

Let $ME_l(R) = \{e \in E(R) \mid Re \text{ is a minimal left ideal of } R\}$. An element $e \in E(R)$ is said to be *left (right) semicentral* if $re = ere$ $(er = ere)$ for any $r \in R$. Following

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[13], R is called *left min-abel* if every element of $ME_l(R)$ is left semicentral in R. R is called left $MC2$ ([13]) if $aRe = 0$ implies $eRa = 0$ for any $a \in R,\ e \in ME_l(R)$.

2. Results

Definition 1. We call R an $NJ-semicommutative$ if $a, b \in R$ and $ab \in N(R)$, then $aRb \subseteq J(R)$.

Evidently, semicommutative rings are NJ-semicommutative. However, the converse is not true (see the following example).

Example 1. In view of Corollary 10, $T_2(\mathbb{Z})$ is NJ-semicommutative, where $\mathbb{Z}$ is the ring of integers. Note that $e_{11}e_{22} = 0$, $e_{11}e_{12}e_{22} = e_{12}$, where e_{ij} denote the matrix units in $T_2(\mathbb{Z})$ whose $(i, j)^{th}$ entry is 1 and zero elsewhere. So, $T_2(\mathbb{Z})$ is not semicommutative.

Theorem 1. *Left (right) quasi-duo rings are NJ-semicommutative.*

Proof. Let R be a left quasi-duo ring, and M be a maximal left ideal of R. Let $w, h \in R$ be such that $wh \in N(R)$. If $w \notin M$, then $M + Rw = R$. This implies that $m + rw = 1$ for some $m \in M, r \in R$. So $mh + rwh = h$. By [14, Lemma 2.3], $wh \in J(R) \subseteq M$. Since M is an ideal, $h \in M$. Therefore, either $w \in M$ or $h \in M$. This yields that $wRh \subseteq M$, and hence $wRh \subseteq J(R)$, that is, R is NJ-semicommutative. Similarly, it can be shown that R is an NJ-semicommutative ring whenever R is a right quasi-duo ring. □

However, the converse is not true (see the following example).

Example 2. By [5, Example 2(ii)], $\mathbb{H}[x]$ is not right quasi-duo, where $\mathbb{H}$ is the Hamilton quaternion over the field of real numbers. Since $\mathbb{H}[x]$ is reduced, it is NJ-semicommutative.

R is said to be *J-clean* if for each $w \in R$, $w = e + j$ for some $e \in E(R)$ and $j \in J(R)$.

Theorem 2. *J-clean rings are NJ-semicommutative.*

Proof. Let R be a J-clean ring, and $w,\ h \in R$ be such that $wh \in N(R)$. By hypothesis, for any $r \in R$, there exists $e \in E(R)$ such that $wrh - e \in J(R)$. We prove that $e = 0$. Observe that

$$(wrh - e)^2 = wrhwrh - wrhe - e(wrh - e). \tag{2.1}$$

Since $wh \in N(R)$, $1 - hw \in U(R)$. As R is J-clean, $1 - hw - e_1 \in J(R)$ for some $e_1 \in E(R)$. Therefore, $1 - (1 - hw)^{-1}e_1 \in J(R)$. This yields that $(1 - hw)^{-1}e_1 \in U(R)$ and hence $e_1 \in U(R)$, that is, $e_1 = 1$. This implies that $hw \in J(R)$. So, from Equation (2.1), $wrhe \in J(R)$. Note that $wrh - e = j$ for some $j \in J(R)$. Hence $e = wrhe - je \in J(R)$, and so $e = 0$. Thus, $wrh \in J(R)$. □

However, the converse of Theorem 2 is not true; for example, any commutative ring which is not J-clean (for example any field with more than two elements).

Following [6], for any $w \in R$, the *commutant* of w is defined by $comm(w) = \{y \in R \mid yw = wy\}$ and $comm^2(w) = \{x \in R \mid yx = xy \text{ for all } y \in comm(w)\}$ is called the *double commutant* of w. According to [4], R is called $J-quasipolar$, if for any $w \in R$, $w + f \in J(R)$ for some $f^2 = f \in comm^2(w)$.

The proof of the following proposition is similar to that of Theorem 2.

Proposition 1. *J-quasipolar rings are NJ-semicommutative.*

Theorem 3. *If $R/J(R)$ is NJ-semicommutative, then R is NJ-semicommutative.*

Proof. Let $a,\ b \in R$ and $ab \in N(R)$. Clearly, $\bar{a}\bar{b} \in N(R/J(R))$. Since $R/J(R)$ is NJ-semicommutative , $\bar{a}\bar{r}\bar{b} \in J(R/J(R))$. As $R/J(R)$ is semiprimitive, $arb \in J(R)$. □

The converse of Theorem 3 is not true (see the following example), and hence a homomorphic image of an NJ-semicommutative ring need not be NJ-semicommutative.

Example 3. Let $R = \mathbb{Z}_{(3\mathbb{Z})}$ be the localization of $\mathbb{Z}$ at $3\mathbb{Z}$ and S the set of quaternions over the ring R. Observe that S is a noncommutative domain. So, S is an NJ-semicommutative ring. Observe that $J(S) = 3S$ and $S/3S \cong M_2(\mathbb{Z}_3)$ via the isomorphism Ψ defined by $\Psi((x_0/y_0) + (x_1/y_1)i + (x_2/y_2)j + (x_3/y_3)k + 3S) =$

$$\begin{pmatrix} \overline{x_0y_0^{-1} + x_1y_1^{-1} - x_2y_2^{-1}} & \overline{x_1y_1^{-1} + x_2y_2^{-1} - x_3y_3^{-1}} \\ \overline{x_1y_1^{-1} + x_2y_2^{-1} + x_3y_3^{-1}} & \overline{x_0y_0^{-1} - x_1y_1^{-1} + x_2y_2^{-1}} \end{pmatrix}$$

for any $(x_0/y_0) + (x_1/y_1)i + (x_2/y_2)j + (x_3/y_3)k) + 3S \in S/3S$. Take $A = \begin{pmatrix} \bar{0} & \bar{1} \\ \bar{0} & \bar{0} \end{pmatrix}$ and $B = \begin{pmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{1} \end{pmatrix}$. Observe that $A^2 \in N(M_2(\mathbb{Z}_3))$ and $ABA \notin J(M_2(\mathbb{Z}_3))$ as $J(M_2(\mathbb{Z}_3)) = 0$. So, $S/J(S)$ is not NJ-semicommutative.

Proposition 2. *Let R be an NJ-semicommutative ring. Then*

(1) *R is left-min abel.*
(2) *If $ab \in N(R)$ then, either $a \in M$ or $b \in M$ for any maximal left ideal M of R.*
(3) *R is directly finite.*

Proof.

(1) Let $e \in ME_l(R)$ and $a \in R$. Take $w = ae - eae$. Then $ew = 0$, $we = w$ and $w^2 = 0$. Since R is NJ-semicommutative, $wRw \subseteq J(R)$. As $J(R)$ is semiprime, $w \in J(R)$. If $w = 0$, then we are done. Assume, if possible, that $w \neq 0$. Since

Re is minimal left ideal of R, $Re = Rw$. As $w \in J(R)$, $Re = Rw \subseteq J(R)$, a contradiction. Thus, R is left-min abel.

(2) Let M be a maximal left ideal of R, and $ab \in N(R)$. Suppose $a \notin M$. Then, $Ra + M = R$. So, $ra + m = 1$ for some $r \in R$ and $m \in M$. Thus, $bra + bm = b$. Since R is NJ-semicommutative, $bra \in J(R)$. So, $b \in M$.

(3) Let $wh = 1$. Observe that $(1 - hw)w1 \in N(R)$. Since R is NJ-semicommutative, $(1 - hw)wh1 \in J(R)$, that is, $1 - hw \in J(R) \cap E(R)$. Hence $hw = 1$.

□

A left R-module M is called *Wnil-injective* ([12]) if for each $a\ (\neq 0) \in N(R)$, there exists a positive integer n such that $a^n \neq 0$ and each left R-homomorphism from Ra^n to M can be extended to one from R to M.

It is evident that reduced rings are NJ-semicommutative. By Example 1, $T_2(\mathbb{Z})$ is NJ-semicommutative, but it is not reduced. In this context, we have the following result.

Proposition 3. *If R is an NJ-semicommutative ring, then R is reduced in each of the following cases:*

(1) *R is semiprimitive.*

(2) *R is a left MC2 NJ-semicommutative ring and each simple singular left R module is Wnil-injective.*

Proof. (1) Suppose $h^2 = 0$. Since R is NJ-semicommutative, $hRh \subseteq J(R)$. As $J(R)$ is semiprime, $h \in J(R) = 0$.

(2) Suppose $h^2 = 0$ for some $h\ (\neq 0) \in R$. Then, $l(h) \subseteq M$ for some maximal left ideal M of R. Assume, if possible, that M is not an essential left ideal of R, then $M = l(e)$ for some $e \in ME_l(R)$. By Proposition 2 (1), R is left min-abel, and since R is left $MC2$, by [13, Theorem 1.8], $e \in Z(R)$. So, $eh = 0$. Thus, $e \in l(h) \subseteq M = l(e)$, a contradiction. Therefore, M is an essential left ideal of R and R/M is simple singular left R module. By hypothesis, R/M is Wnil-injective. Define a left R-homomorphism $\Psi : Rh \to R/M$ via $\Psi(rh) = r + M$. As R/M is Wnil-injective, $1 - ht \in M$ for some $t \in R$. Since R is NJ-semicommutative, and $h^2 = 0$, $hRh \subseteq J(R)$. Therefore, $h \in J(R)$. So $1 - ht \in U(R)$, a contradiction. Hence $h = 0$.

□

R is said to be:

(1) *semiperiodic* ([2]) if for each $w \in R \setminus (J(R) \cup Z(R))$, $w^q - w^p \in N(R)$ for some integers p and q of opposite parity.

(2) *left* (*right*) *SF* ([10]) if all simple left (right) R-modules are flat.

Theorem 4. *If $R/J(R)$ is reduced, then R is NJ-semicommutative. The converse holds if R is:*

(1) *semiperiodic.*
(2) *left SF.*

Proof. Suppose $R/J(R)$ is reduced. Let $ab \in N(R)$. Clearly, $\bar{b}\bar{a} \in N(R/J(R))$. Since $R/J(R)$ is reduced, $ba \in J(R)$. For any $r \in R$, $(arb)^2 = arbarb \in J(R)$, that is, $\bar{a}\bar{r}\bar{b} \in N(R/J(R)) = 0$ and so, $arb \in J(R)$.
Conversely;

(1) Suppose R is an NJ-semicommutative semiperiodic ring. Write $\bar{R} = R/J(R)$ and let $\bar{w} \in \bar{R}$ with $\bar{w}^2 = 0$. Then by [2, Lemma 2.6], $w^2 \in J(R) \subseteq N(R) \cup Z(R)$. If $w^2 \in N(R)$, then $wRw \subseteq J(R)$ (since R is NJ-semicommutative). As $J(R)$ is semiprime, $\bar{w} = 0$. Suppose $w^2 \notin N(R)$, then $w^2 \in Z(R)$. If $w \in Z(R)$, then $\bar{w}\bar{R}\bar{w} = 0$. As $\bar{R}$ is semiprime, $\bar{w} = 0$. Assume, if possible, that $\bar{w} \notin Z(\bar{R})$ then $w \notin J(R) \cup Z(R)$. By [2, Lemma 2.3(iii)], there exist $e \in E(R)$ and a positive integer p satisfying $w^p = w^p e$ and $e = wy$ for some $y \in R$. Hence $e = ewy = ew(1-e)y + ewey = ew(1-e)y + ew^2y^2$. As R is NJ-semicommutative, $eR(1-e) \subseteq J(R)$. Hence $e \in J(R)$, that is, $e = 0$. This yields that $w^p = 0$ and so $w \in N(R)$, a contradiction to $w^2 \notin N(R)$. Therefore $\bar{w} \in Z(\bar{R})$ and so $\bar{w} = 0$. Thus, $\bar{R}$ is reduced.
(2) Suppose R is an NJ-semicommutative left SF ring. By [11, Proposition 3.2], $R/J(R)$ is left SF. Let $w^2 \in J(R)$ such that $w \notin J(R)$. Assume, if possible, $Rr(w) + J(R) = R$, then $1 = x + \sum^{finite} r_i s_i$, $x \in J(R)$, $r_i \in R$, $s_i \in r(w)$. Then $w = xw + \sum^{finite} r_i s_i w$. Observe that $s_i w \in N(R)$. As R is NJ-semicommutative, $s_i R w \in J(R)$. This implies that $w \in J(R)$, a contradiction. Hence $Rr(w) + J(R) \neq R$. There exist some maximal left ideal H satisfying $Rr(w) + J(R) \subseteq H$. Note that $w^2 \in H$. By [11, Lemma 3.14], $w^2 = w^2 x$ for some $x \in H$, that is, $w - wx \in r(w) \subseteq H$. So, $w \in H$. Hence there exists $y \in H$ satisfying $w = wy$, that is, $1 - y \in r(w) \subseteq H$. This implies that $1 \in H$, a contradiction. Therefore, $R/J(R)$ is reduced.

□

Corollary 1. *If R is an NJ-semicommutative semiperiodic ring, then $R/J(R)$ is commutative.*

Proof. Since $R/J(R)$ is semiperiodic, by Theorem 4 (1) and [2, Theorem 4.4], $R/J(R)$ is commutative. □

Corollary 2. *If R is an NJ-semicommutative left SF, then R is strongly regular.*

Proof. By Theorem 4, $R/J(R)$ is reduced. Hence $R/J(R)$ is strongly regular by [11, Remark 3.13]. This implies that R is left quasi-duo, and hence by [11, Theorem 4.10], R is strongly regular. □

As an immediate consequence of Corollary 1 and Corollary 2, the following corollary is obtained.

Corollary 3. *If R is an NJ-semicommutative, semiperiodic, left SF ring, then R is commutative regular ring.*

The proof of the following proposition is trivial.

Proposition 4. *Suppose $\{R_\delta\}_{\delta\in\Delta}$ is a family of rings, and Δ represents an index set. Then $\Pi_{\delta\in\Delta}R_\delta$ is NJ-semicommutative if and only if R_δ is NJ-semicommutative for each $\delta\in\Delta$.*

Corollary 4. *eR and $(1-e)R$ are NJ-semicommutative for some central idempotent $e\in R$ if and only if R is NJ-semicommutative.*

Proposition 5. *R is NJ-semicommutative if and only if eRe is NJ-semicommutative for all $e\in E(R)$.*

Proof. Suppose R is NJ-semicommutative. Let $eae,\ ebe\in eRe$ with $(eae)(ebe)\in N(eRe)$. Since R is NJ-semicommutative, $(eae)(ere)(ebe)\in J(R)$ for all $r\in R$. Since $eJ(R)e=J(eRe)$, $(eae)(ere)(ebe)\in J(eRe)$. Hence, eRe is NJ-semicommutative. Whereas the converse is trivial. □

Proposition 6. *Let I be an ideal of an NJ-semicommutative ring W and R a subring of W with $I\subseteq R$. If R/I is NJ-semicommutative, then so is R.*

Proof. Let $x,\ y\in R$ and $xy\in N(R)$. Since W is NJ-semicommutative, $xr_0y\in J(W)$ for any $r_0\in R$. Therefore, for any $r\in R$, $1-xr_0yr\in U(W)$. There exists $w\in W$ such that $w(1-xr_0yr)=1=(1-xr_0yr)w$. Note that $\bar{x}\bar{y}\in N(R/I)$. Since R/I is NJ-semicommutative, $\bar{x}\bar{r_0}\bar{y}\in J(R/I)$. This implies that $\bar{1}-\bar{x}\bar{r_0}\bar{y}\bar{r}\in U(R/I)$. So there exists $\bar{t}\in R/I$ such that $\bar{t}(\bar{1}-\bar{x}\bar{r_0}\bar{y}\bar{r})=\bar{1}$. This implies that $1-t(1-xr_0yr)\in I$. Hence, $w-t(1-xr_0yr)w\in R$, that is, $w\in R$. Hence $xr_0y\in J(R)$. □

Corollary 5. *Let I be an ideal of an NJ-semicommutative ring W and R an NJ-semicommutative subring of W. Then, I+R is NJ-semicommutative.*

Proof. Follows directly from Proposition 6. □

Corollary 6. *Every finite subdirect product of NJ-semicommutative rings is NJ-semicommutative.*

Proof. Let R/K and R/L be NJ-semicommutative rings for some ideals K and L of R with $K\cap L=0$. Define $\Psi:R\to R/K\bigoplus R/L$ via $\Psi(x)=(x+K,x+L)$. So $R\cong Im(\Psi)$. By hypothesis, $Im(\Psi)/\Psi(K)\cong R/K$ is NJ-semicommutative. Observe that $\Psi(K)\subseteq Im(\Psi)\subseteq R/K\bigoplus R/L$. By Proposition 6, R is NJ-semicommutative. □

Corollary 7. *Let K and L be ideals of R such that R/K and R/L are NJ-semicommutative. Then, $R/(K\cap L)$ is NJ-semicommutative.*

Proof. Define $\Psi:R/(K\cap L)\to R/K$ and $\Phi:R/(K\cap L)\to R/L$ via $\Psi(r+K\cap L)=r+K$ and $\Phi(r+K\cap L)=r+L$, respectively. Clearly, Ψ and Φ are epimorphism with $ker(\Psi)\cap ker(\Phi)=0$. So, $R/(K\cap L)$ is the subdirect product of R/K and R/L. By Corollary 6, $R/(K\cap L)$ is NJ-semicommutative. □

Lemma 1. *Let I be a nil ideal of R such that R/I is an NJ-semicommutative ring. Then, R is NJ-semicommutative.*

Proof. Let $w,\ h \in R$ and $wh \in N(R)$. Clearly, $\bar{w}\bar{h} \in N(R/I)$. Since R/I is NJ-semicommutative, $\bar{w}\bar{r}_0\bar{h} \in J(R/I)$ for any $r_0 \in R$. So $\bar{1} - \bar{w}\bar{r}_0\bar{h}\bar{r} \in U(R/I)$ for all $r \in R$. So $(\bar{1} - \bar{w}\bar{r}_0\bar{h}\bar{r})\bar{s} = \bar{1} = \bar{s}(\bar{1} - \bar{w}\bar{r}_0\bar{h}\bar{r})$ for some $s \in R$. This implies that $1 - (1 - wr_0hr)s \in I$. Since I is nil, $(1 - wr_0hr)s \in U(R)$ and hence $1 - wr_0hr \in U(R)$. Therefore $wr_0h \in J(R)$, that is, R is NJ-semicommutative. □

Proposition 7. *If K and L are ideals of R such that R/K and R/L are NJ-semicommutative, then R/KL is NJ-semicommutative.*

Proof. Observe that $KL \subseteq K \cap L$ and $R/(K \cap L) \cong (R/KL)/((K \cap L)/KL)$. Clearly, $((K \cap L)/KL)^2 = 0$, and by Corollary 7, $R/(K \cap L)$ is NJ-semicommutative. By Lemma 1, R/KL is NJ-semicommutative. □

The following result is an immediate consequence of Proposition 7.

Corollary 8. *The following are equivalent for an ideal I of R.*

(1) *R/I is NJ-semicommutative.*
(2) *R/I^n is NJ-semicommutative for all positive integer n.*

A *Morita context* ([9]) is a 4-tuple $\begin{pmatrix} R_1 & M \\ P & R_2 \end{pmatrix}$, where R_1, R_2 are rings, M is (R_1, R_2)-bimodule and P is (R_2, R_1)-bimodule, and there exists a context product $M \times P \to R_1$ and $P \times M \to R_2$ written multiplicatively as $(m, p) \mapsto mp$ and $(p, m) \mapsto pm$. Clearly, $\begin{pmatrix} R_1 & M \\ P & R_2 \end{pmatrix}$ is an associative ring with the usual matrix operations.

A Morita context $\begin{pmatrix} R_1 & M \\ P & R_2 \end{pmatrix}$ is said to be trivial if the context products are trivial, that is, $MP = 0$ and $PM = 0$.

Proposition 8. *Suppose $R = \begin{pmatrix} R_1 & M \\ P & R_2 \end{pmatrix}$ is a trivial Morita context. Then R is NJ-semicommutative if and only if R_1 and R_2 are NJ-semicommutative.*

Proof. Suppose R is NJ-semicommutative. By Proposition 5, eRe is NJ-semicommutative. So R_1 and R_2 are NJ-semicommutative. Conversely, assume that R_1 and R_2 are NJ-semicommutative and $\alpha = \begin{pmatrix} a_1 & m_1 \\ p_1 & b_1 \end{pmatrix}, \beta = \begin{pmatrix} a_0 & m_0 \\ p_0 & b_0 \end{pmatrix} \in R$ be such that $\alpha\beta \in N(R)$. Then $a_1a_0 \in N(R_1)$ and $b_1b_0 \in N(R_2)$. Let $\gamma = \begin{pmatrix} a & m \\ p & b \end{pmatrix}$ be any element of R. Since R_1 and R_2 are NJ-semicommutative rings, $a_1aa_0 \in J(R_1)$ and $b_1bb_0 \in J(R_1)$. Therefore $\alpha\beta\gamma \in J(R)$. Hence R is NJ-semicommutative. □

Let R_1 and R_2 be any rings, M a (R_1,R_2)-bimodule and $R=\begin{pmatrix} R_1 & M \\ 0 & R_2 \end{pmatrix}$, the formal triangular matrix ring. It is well known that $J(R)=\begin{pmatrix} J(R_1) & M \\ 0 & J(R_2) \end{pmatrix}$.

Corollary 9. *Let R_1 and R_2 be any rings and M a (R_1,R_2)- bimodule. Then $\begin{pmatrix} R_1 & M \\ 0 & R_2 \end{pmatrix}$ is NJ-semicommutative if and only if R_1 and R_2 are NJ-semicommutative.*

Corollary 10. *R is NJ-semicommutative if and only if $T_n(R)$ is NJ-semicommutative.*

Proposition 9. *The following are equivalent.*

(1) *R is NJ-semicommutative.*

(2) $R_n=\left\{\begin{pmatrix} a & a_{12} & \dots & a_{1(n-1)} & a_{1n} \\ 0 & a & \dots & a_{2(n-1)} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & a & a_{(n-1)n} \\ 0 & 0 & \dots & 0 & a \end{pmatrix} : a, a_{ij} \in R,\ i<j\right\}$ *is NJ-semicommutative.*

Proof. (1) $\implies$ (2) Let $I=\left\{\begin{pmatrix} 0 & a_{12} & \dots & a_{1n} \\ 0 & 0 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} : a_{ij} \in R\right\} \subseteq R_n$. Note that I is an ideal of R_n and $I^n=0$. Also $R_n/I \cong R$. By Lemma 1, R_n is NJ-semicommutative.

(2) $\implies$ (1) It follows from Proposition 5. □

Corollary 11. *The following are equivalent.*

(1) *R is NJ-semicommutative.*

(2) *$R[x]/<x^n>$ is NJ-semicommutative for any positive integer n, where $<x^n>$ is the ideal generated by x^n in $R[x]$.*

Proof. Observe that

$R[x]/<x^n> \cong \left\{\begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_{n-1} & a_n \\ 0 & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \dots & a_1 & a_2 \\ 0 & 0 & 0 & \dots & 0 & a_1 \end{pmatrix} : a_i \in R\right\}$. So, the proof follows from the proof of Proposition 9. □

Let A be a (R,R)-bimodule which is a general ring (not necessarily with unity) in which $(aw)r=a(wr)$, $(ar)w=a(rw)$ and $(ra)w=r(aw)$ hold for all $a,w \in A$ and $r \in R$. Then *ideal-extension* (also called *Dorroh extension*) $I(R;A)$ of R by A is defined

to be the additive abelian group $I(R;A) = R \oplus A$ with multiplication $(r,a)(s,w) = (rs, rw + as + aw)$.

Proposition 10. *Let A be an (R,R)-bimodule which is a general ring (not necessarily with unity) in which $(aw)r = a(wr)$, $(ar)w = a(rw)$ and $(ra)w = r(aw)$ hold for all $a,w \in A$ and $r \in R$. Suppose that for any $a \in A$ there exists $w \in A$ such that $a + w + aw = 0$. Then the following are equivalent.*

(1) *R is NJ-semicommutative.*
(2) *Dorroh extension $S = I(R;A)$ is NJ-semicommutative.*

Proof. (1) $\Longrightarrow$ (2) Suppose R is NJ-semicommutative and $\alpha = (r,v)$, $\beta = (p,w) \in S$ be such that $\alpha\beta \in N(S)$. Let $\gamma = (s,u)$ be any element of S. Since R is NJ-semicommutative $rsp \in J(R)$. We claim that $\alpha\gamma\beta \in J(S)$. Now, let $(0,a) \in (0,A)$. For any $(r_1,a_1) \in S$, we have, $(1,0) - (0,a)(r_1,a_1) = (1, -ar_1 - aa_1)$. By hypothesis, there exists $a_2 \in A$ such that $(1, -ar_1 - aa_1)(1,a_2) = (1,0)$. Therefore $(0,A) \subseteq J(S)$. Note that $\alpha\gamma\beta = (rsp, w)$ for some $w \in A$. So if we show $(rsp,0) \in J(S)$ then we are done. Let (r_1,v_1) be any element of S. Then $(1,0) - (r_1,v_1)(rsp,0) = (1 - r_1rsp, -v_1rsp) \in U(S)$, as $(1 - r_1rsp, -v_1rsp) = (1 - r_1rsp, 0)(1, (1 - r_1rsp)^{-1}(-v_1rsp))$ and $(1, (1 - r_1rsp)^{-1}(-v_1rsp)) = (1,0) + (0, (1 - r_1rsp)^{-1}(-v_1rsp)) \in U(S)$. Thus $(rsp,0) \in J(S)$ and hence $\alpha\gamma\beta \in J(S)$. Therefore S is NJ-semicommutative.

(2) $\Longrightarrow$ (1) Let $a,\ b \in R$ and $ab \in N(R)$. Clearly, $(a,0)(b,0) \in N(S)$. Since S is NJ-semicommutative ring, $(a,0)(r,0)(b,0) \in J(S)$ for all $r \in R$. Hence $arb \in J(R)$, that is, R is an NJ-semicommutative ring. □

Let $\Psi : R \to R$ be a ring homomorphism , $R[[x,\Psi]]$ represents the ring of skew formal power series over R, that is, all formal power series in x with coefficients from R and multiplication is defined with respect to the rule $xr = \Psi(r)x$ for all $r \in R$. It is well known that $J(R[[x,\Psi]]) = J(R) + < x >$, $< x >$ is the ideal of $R[[x,\Psi]]$ generated by x. Since $R[[x,\Psi]] \cong I(R; < x >)$, the following result is an immediate consequence of Proposition 10.

Corollary 12. *Let $\Psi : R \to R$ be a ring homomorphism. Then the following are equivalent.*

(1) *R is NJ-semicommutative.*
(2) *$R[[x,\Psi]]$ is NJ-semicommutative.*

Corollary 13. *Then the following are equivalent.*

(1) *R is NJ-semicommutative.*
(2) *$R[[x]]$ is NJ-semicommutative.*

It is a natural question to ask whether the polynomial ring over an NJ-semicommutative ring is NJ-semicommutative. However, the following example gives the answer in negative.

Example 4. For any countable field K, there exists a nonzero nil algebra S over K such that $N^*(S[x]) = 0$ (see the proof of Lemma 3.7 in [3]). Let $R = K + S$. Observe that R is a local ring with $J(R) = S$. Hence, R is an NJ-semicommutative ring and $N^*(R[x]) = N^*(S[x])$. If $R[x]$ is not NJ-semicommutative, then we are done. If $R[x]$ is NJ-semicommutative, then we show that $(R[x])[y]$ is not NJ-semicommutative. Assume, if possible, that $(R[x])[y]$ is NJ-semicommutative. By [1, Theorem 1], $J((R[x])[y]) = I[y]$ for some nil ideal I of $R[x]$ which is $N^*(R[x]) = N^*(S[x]) = 0$. Therefore, $J((R[x])[y]) = 0$. So, $(R[x])[y]$ is semicommutative, which further implies that $R[x]$ is a semicommutative ring. Hence $R[x]$ is 2-primal, and so, $N(R[x]) = N_*(R[x])$. But, this is a contradiction to the fact that $0 \neq N(R) = S \subseteq N(R[x])$ and $N_*(R[x]) \subseteq N^*(R[x]) = N^*(S[x]) = 0$.

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VISCOSITY S-ITERATION ALGORITHM FOR FINDING COMMON FIXED POINT OF NONEXPANSIVE MAPPINGS AND APPLICATION TO NONEXPANSIVE SEMIGROUPS

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Abstract. We study the problem of finding the common element of the set of fixed points of two nonexpansive mappings in Hilbert spaces. Some previous attempts in this direction make some restrictive assumptions which may be difficult to check in practice on the generated sequence. In this paper, we introduce a viscosity S-iteration scheme for finding the common fixed point of two nonexpansive mappings. Under some very mild conditions, we obtain a strong convergence theorem for the sequence generated by our algorithm. We apply our main result to approximating common fixed points of nonexpansive semigroups. We also provide numerical examples to support our main results and illustrate the efficiency and effectiveness of our algorithm by comparing with some existing algorithms in literature. This work generalizes and improves some existing works in the literature in this direction.

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1. Introduction

Let H be a real Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$ and C a nonempty closed and convex subset of H. A mapping $T: C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for every $x, y \in C$. A mapping $f: C \to C$ is called a contraction if there exists $\theta \in [0,1)$ such that $\|f(x) - f(y)\| \leq \theta \|x - y\|$ for every $x, y \in C$. A point $x \in C$ is called the fixed point of T if $Tx = x$. We shall denote

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the fixed point set of T by $\mathrm{Fix}(T)$. The study of fixed point theory for nonexpansive mappings has flourished in recent years due to its vast applications in fields like compressed sensing, economics and other applied sciences (see for instance, [4] and some of the references therein). In particular, some problems such as convex feasibility problems, convex optimization problems, monotone inclusion problems and image restoration problems can be seen as finding the fixed points of nonexpansive mapping (see [**?**, 6, 8, 20]). In the past and recent years, researchers have put considerable efforts in the study and in the formulation of algorithms to approximate the fixed points of nonexpansive mappings and related optimization problems, see [**?**, **?**, 3, 15, 16, 19, 24]. We mention few of these algorithms:

The Mann iterative scheme is defined as follows:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n \geq 1, \end{cases} \tag{1.1}$$

where $\{\alpha_n\} \subset [0,1]$. It is known that the Mann iterative scheme (1.1) converges weakly to a fixed point of T provided that $\{\alpha_n\} \subset [0,1]$ satisfies

$$\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = +\infty.$$

Halpern [13] proposed the following recursive formula:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \; n \geq 0, \tag{1.2}$$

where $\{\alpha_n\} \subset [0,1]$ and $u \in C$. Halpern shown that under the following control conditions:

(H1) $\lim_{n\to\infty} \alpha_n = 0$;
(H2) $\sum_{n=1}^{\infty} \alpha_n = +\infty$;

the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Oftentimes, in many real world problems arising in infinite dimensional spaces, strong convergence is much more desirable than weak convergence (see [7] and the references therein). Both Mann's and Halpern's algorithms have received considerable research efforts recently. Lions [17] proved that the Halpern iterative scheme (1.2) converges strongly to a fixed point of T provided $\{\alpha_n\}$ satisfies the following control conditions:

(A1) $\lim_{n\to\infty} \alpha_n = 0$;
(A2) $\sum_{n=1}^{\infty} \alpha_n = +\infty$;
(A3) $\lim_{n\to\infty} \frac{|\alpha_{n+1}-\alpha_n|}{\alpha_{n+1}^2} = 0$.

One can see that in [17], the sequence $\{\alpha_n\}$ excluded canonical choice like $\alpha_n = \frac{1}{n+1}$. Wittmann [26] considered the iterative scheme (1.2) where $\{\alpha_n\}$ satisfies the following control conditions:

(B1) $\lim_{n\to\infty} \alpha_n = 0$;

(B2) $\sum_{n=1}^{\infty} \alpha_n = +\infty$;
(B3) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < +\infty$,

and proved that $\{x_n\}$ converges strongly to $x^* \in F(T)$ such that $x^* = P_{F(T)}u$.

Moudafi [19] proposed the viscosity approximation method for finding the fixed points of nonexpansive mapping T: Let $x_1 \in C$ be arbitrary and define

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \quad n \geq 1, \tag{1.3}$$

where $f\colon C \to C$ is a contraction and $\{\alpha_n\} \subset [0,1]$ satisfying the following control conditions:

(M1) $\lim_{n\to\infty} \alpha_n = 0$;
(M2) $\sum_{n=1}^{\infty} \alpha_n = +\infty$;
(M3) $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$.

It is obvious that (1.3) extends (1.2). Moudafi [19] proved the following result in Hilbert spaces:

Theorem 1 ([19, Theorem 2.1])**.** *If $\{\alpha_n\}$ satisfies the conditions (M1) - (M3) as above, then the sequence $\{x_n\}$ generated by (1.3) converges strongly to a fixed point x^* of T, which also solves the variational inequality problem:*

$$\textit{find } x^* \in F(T) \textit{ such that } \langle f(x^*) - x^*, x - x^* \rangle \leq 0, \quad x \in F(T).$$

For other recent works on viscosity method for approximating fixed point of nonexpansive mappings, please see [16].

Agarwal et al. [1] proposes the following iterative scheme called the S-iteration process:

Algorithm 1. *Let C be a convex subset of a linear space X and T a mapping of C onto itself. Let $x_1 \in C$ and generate the sequence $\{x_n\}$ by*

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n T y_n, \quad n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$ satisfying the condition:

$$\sum_{n=1}^{\infty} \alpha_n \beta_n (1-\beta_n) = \infty.$$

It is known that Algorithm 1 does not reduce to the Mann iterative scheme (1.1).

Let $T, U\colon C \to C$ be two nonexpansive mappings, our aim in this paper is to find the common fixed points of T and U. Problems of this kind have been studied by authors very recently (see [9, 23] and some of the references therein).

Suparatulatorn et al. [23] introduced a modified S-iteration process defined as follows:

$$\begin{cases} x_0 \in C, \\ y_n = (1-\beta_n)x_n + \beta_n S_1 x_n, \\ x_{n+1} = (1-\alpha_n)S_1 x_n + \alpha_n S_2 y_n, \quad n \geq 0, \end{cases}$$

where C is a nonempty subset of a real Banach space, the two sequences $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ and $S_1, S_2 \colon C \to C$ are G-nonexpansive mappings and under some conditions proved weak and strong convergence theorems for finding common fixed point of the two G-nonexpansive mappings in a uniformly convex Banach space.

Also very recently, Ahmad et al. [2] studied the problem of finding the common fixed point of two nonexpansive mappings $T, U \colon C \to C$ in Hilbert spaces and proved the following theorem:

Theorem 2 ([2, Theorem 2.1])**.** *Let $T, U \colon C \to C$ be two nonexpansive mappings with $\Gamma := \mathrm{Fix}(T) \cap \mathrm{Fix}(U) \neq \varnothing$. Also let $f \colon C \to C$ be a contraction with coefficient $\theta \in [0,1)$. Assume that the sequence $\{x_n\}$ in C generated by* (1.4)

$$x_{n+1} = \alpha_n f(x_n) + \beta_n T(x_n) + \gamma_n U(x_n), \quad n \geq 1, \tag{1.4}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ satisfying

(1) $\alpha_n + \beta_n + \gamma_n = 1$,
(2) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} = \infty$,
(3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *and* $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,
(4) $\lim_{n\to\infty} \|U(x_n) - T(x_n)\| = 0$.

Then $\{x_n\}$ converges strongly to $x^* \in \Gamma$, which satisfies the variational inequality

$$\langle (1-f)x^*, x - x^* \rangle \geq 0 \qquad \forall\, x \in \Gamma.$$

Looking at iterative scheme (1.4), one will see that the condition (4) is restrictive. This is because one will have to first check that $\lim_{n\to\infty} \|U(x_n) - T(x_n)\| = 0$ before the implementation. This then brought about the following question:

Question 1. Is it possible to modify (1.4) so that the conditions (1) - (4) are relaxed and also obtain strong convergence?

We answer this question in the affirmative.

In this paper, motivated by the works of Ahmad et al. [2], the above works and the ongoing research interest in this direction, we propose a viscosity S-iteration for finding the common fixed points of two nonexpansive mappings in Hilbert space. Under some mild conditions, we prove a strong convergence theorem and give some consequence of our main result. We apply our main results to nonexpansive semigroups.

This paper is organised as follows. In Section 2, we give some useful definitions, notations and lemmas which are needed for our algorithm's analysis. In Section 3, the algorithm and its strong convergence theorem is presented. In Section 4, we apply our main result to nonexpansive semigroups. In Section 5, we give numerical

example to illustrate our algorithms and compare it with some existing algorithms in literature. We conclude in Section 6.

2. Preliminaries

Throughout this article, we let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\|\cdot\|$. C is a nonempty closed convex subset of H and $I\colon H \to H$ is the identity mapping on H. We denote by '$x_n \rightharpoonup x$' and '$x_n \to x$', the weak and the strong convergence of $\{x_n\}$ to a point x respectively.

We recall the following definitions:

Definition 1 ([8, Definition 4.1, Definition 22.1])**.** An operator $T\colon C \to C$ is said to be:

(i) Lipschitzian if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\| \qquad \forall\, x, y \in C;$$

(ii) firmly nonexpansive if

$$\|Tx - Ty\|^2 + \|(I - T)x - (I - T)y\|^2 \leq \|x - y\|^2 \qquad \forall\, x, y \in C,$$

or equivalently,

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle \qquad \forall\, x, y \in C;$$

(iii) β-strongly monotone if there exists $\beta > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \beta\|x - y\|^2 \qquad \forall\, x, y \in C.$$

One can verify that if the mapping $h\colon C \to C$ is a contraction with contractive constant $\tau \in [0, 1)$, then $I - h$ is $2(1 + \tau^2)$-Lipschitzian and $(1 - \tau)$-strongly monotone. It can be deduced from the definition above that every firmly nonexpansive mapping is nonexpansive. It is known that the set of fixed points of nonexpansive mapping in Hilbert space is closed and convex. The metric projection of H onto C (see [**?**]), denotes as P_C, is the mapping that assigns every point $x \in H$ to its unique nearest point in C i.e.,

$$\|x - P_C x\| \leq \|x - y\| \qquad \forall\, y \in C.$$

The metric projection is characterized by $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x \rangle \leq 0 \qquad \forall y \in C.$$

Moreover, P_C is nonexpansive and $\mathrm{Fix}(P_C) = C$.

Let $h\colon C \to C$ be a nonlinear operator. The Variational Inequality Problem (VIP) is to

$$\text{find } x^* \in C \text{ such that } \langle h(x^*), x - x^* \rangle \geq 0 \qquad \forall\, x \in C.$$

Lemma 1 ([28, Proposition 2.7])**.** *Let H be a real Hilbert space. Suppose that $h\colon H \to H$ is κ- Lipschitzian and β- strongly monotone over a closed convex subset $C \subset H$. Then, the following VIP*

$$\langle h(u^*), v - u^* \rangle \geq 0 \qquad \forall\, v \in C$$

has its unique solution $u^* \in C$.

Lemma 2. *[12] Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \to C$ be a nonexpansive mapping such that $\mathrm{Fix}(T) \neq \varnothing$. If a sequence $\{x_n\}$ in C is such that $x_n \rightharpoonup x^*$ and $\|x_n - Tx_n\| \to 0$, then $x^* = Tx^*$.*

Lemma 3 ([11, Page 1])**.** *Let H be a real Hilbert space, then the following assertions hold:*

(i) $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y\rangle + \|y\|^2$ *for all $x, y \in H$;*
(ii) *for all $x, y, z \in H$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$, we have*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2.$$

Lemma 4 ([27, Lemma 2.1])**.** *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$s_{n+1} \leq (1 - t_n)s_n + t_n\rho_n, \quad n \geq n_0,$$

where $\{t_n\} \subset (0,1)$ and $\{\rho_n\} \subset \mathbb{R}$ satisfying the following conditions: $\lim_{n\to\infty} t_n = 0$, $\sum_{n=1}^{\infty} t_n = \infty$, and $\limsup_{n\to\infty} \rho_n \leq 0$. Then $s_n \to 0$ as $n \to \infty$.

Lemma 5 ([18, Lemma 3.1])**.** *Let $\{a_n\}$ be sequence of real numbers such that there exists a subsequence $\{a_{n_i}\}_{i\geq 0}$ of $\{a_n\}$ with $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then, there exists an increasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \quad \text{and} \quad a_k \leq a_{m_k+1}.$$

In fact, m_k is the largest number n in the set $\{1, 2, \cdots k\}$ such that the condition $a_n \leq a_{n+1}$ holds.

3. Main results

In this section, we present our algorithm and study the convergence analysis.

Algorithm 2. *Let $T, U : C \to C$ be two nonexpansive mappings with $\Omega := \mathrm{Fix}(T) \cap \mathrm{Fix}(U) \neq \varnothing$. Also let $f : C \to C$ be a contraction with coefficient $\nu \in [0,1)$. Choose $x_0 \in C$ and let $\{x_n\}$ be the sequence generated by (3.1)*

$$\begin{cases} u_n = (1 - \alpha_n)x_n + \alpha_n Tx_n, \\ x_{n+1} = \beta_n f(x_n) + \gamma_n Tx_n + \xi_n Uu_n, \quad n \geq 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\xi_n\}$are sequences in $(0,1)$ satisfying

(i) $\beta_n + \gamma_n + \xi_n = 1$;

(ii) $\liminf_{n\to\infty}\gamma_n\xi_n > 0$, $\liminf_{n\to\infty}\xi_n\alpha_n(1-\alpha_n) > 0$;
(iii) $\lim_{n\to\infty}\beta_n = 0$, $\sum_{n=1}^{\infty}\beta_n = \infty$.

Lemma 6. *The sequence $\{x_n\}$ generated by Algorithm 2 is bounded.*

Proof. Let $p \in \Omega$. Then

$$\begin{aligned}\|u_n - p\| &= \|(1-\alpha_n)x_n + \alpha_n T x_n - p\| \\ &\le (1-\alpha_n)\|x_n - p\| + \alpha_n\|Tx_n - p\| \\ &\le (1-\alpha_n)\|x_n - p\| + \alpha_n\|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{3.2}$$

Also,

$$\begin{aligned}\|x_{n+1} - p\| &= \|\beta_n[f(x_n) - p] + \gamma_n[Tx_n - p] + \xi_n[Uu_n - p]\| \\ &\le \beta_n\|f(x_n) - p\| + \gamma_n\|x_n - p\| + \xi_n\|u_n - p\| \\ &\le \beta_n\nu\|x_n - p\| + \beta_n\|f(p) - p\| + \gamma_n\|x_n - p\| + \xi_n\|u_n - p\|. \end{aligned} \tag{3.3}$$

Substituting (3.2) into (3.3) gives

$$\begin{aligned}\|x_{n+1} - p\| &\le (1 - \beta_n + \beta_n\nu)\|x_n - p\| + \beta_n\|f(p) - p\| \\ &= [1 - \beta_n(1-\nu)]\|x_n - p\| + \frac{\beta_n(1-\nu)\|f(p) - p\|}{1-\nu} \\ &\le \max\left\{\|x_n - p\|, \frac{\|f(p) - p\|}{1-\nu}\right\} \\ &\vdots \\ &\le \max\left\{\|x_0 - p\|, \frac{\|f(p) - p\|}{1-\nu}\right\}. \end{aligned}$$

Showing that $\{\|x_{n+1} - p\|\}$ is bounded. Hence $\{Tx_n\}$, $\{Uu_n\}$ and $\{x_n\}$ are bounded. □

Theorem 3. *The sequence $\{x_n\}$ generated by Algorithm 2 converges strongly to $p \in \Omega$, where p is the unique solution of the following VIP (3.4): Find $p \in \Omega$ such that*

$$\langle (I - f)p, x - p\rangle \ge 0 \qquad \forall\, x \in \Omega. \tag{3.4}$$

Proof. Since f is a ν-contraction mapping, it follows that $I - f$ is $2(1+\nu^2)$-Lipschitzian and $(1-\nu)$-strongly monotone ([**?**]). Therefore by Lemma 1, it follows that the VIP (3.4) has a unique solution $p \in \Omega$.

Set $K(x_n, u_n) = \theta_n T x_n + (1-\theta_n)Uu_n$, where $\theta_n = \frac{\gamma_n}{1-\beta_n}$, $1 - \theta_n = \frac{\xi_n}{1-\beta_n}$. Note that by Lemma 3 (ii)

$$\|K(x_n, u_n) - p\|^2 = \|\theta_n T x_n + (1-\theta_n)Uu_n - p\|^2$$

$$\begin{aligned}
&\leq \theta_n\|Tx_n - p\|^2 + (1-\theta_n)\|Uu_n - p\|^2 \\
&\leq \theta_n\|x_n - p\|^2 + (1-\theta)\|u_n - p\|^2 \\
&\leq \|x_n - p\|^2. \qquad (3.5)
\end{aligned}$$

Also

$$\begin{aligned}
\langle f(x_n) - p, K(x_n,u_n) - p\rangle &= \langle f(x_n) - f(p), K(x_n,u_n) - p\rangle \\
&\quad + \langle f(p) - p, K(x_n,u_n) - p\rangle \\
&\leq \|f(x_n) - f(p)\|\|K(x_n,u_n) - p\| \\
&\quad + \langle f(p) - p, K(x_n,u_n) - p\rangle \\
&\leq \frac{1}{2}[\|f(x_n) - f(p)\|^2 + \|K(x_n,u_n) - p\|^2] \\
&\quad + \langle f(p) - p, K(x_n,u_n) - p\rangle \\
&\leq \frac{1}{2}[\nu^2\|x_n - p\|^2 + \|x_n - p\|^2] \\
&\quad + \langle f(p) - p, K(x_n,u_n) - p\rangle. \qquad (3.6)
\end{aligned}$$

Then we have by (3.5) and (3.6) that

$$\begin{aligned}
\|x_{n+1} - p\|^2 =& \|\beta_n f(x_n) + (1-\beta_n)K(x_n,u_n) - p\|^2 \\
=& \beta_n^2\|f(x_n) - p\|^2 + 2\beta_n(1-\beta_n)\langle f(x_n) - p, K(x_n,u_n) - p\rangle \\
&+ (1-\beta_n)^2\|K(x_n,u_n) - p\|^2 \\
\leq& \beta_n^2\|f(x_n) - p\|^2 + \beta_n(1-\beta_n)[(\nu^2+1)\|x_n - p\|^2] \\
&+ (1-\beta_n)^2\|x_n - p\|^2 \\
&+ 2\beta_n(1-\beta_n)\langle f(p) - p, K(x_n,u_n) - p\rangle \\
=& (1-\beta_n(1-(1-\beta_n)\nu^2))\|x_n - p\|^2 \\
&+ \beta_n[\beta_n\|f(x_n) - p\|^2 + 2(1-\beta_n)\langle f(p) - p, K(x_n,u_n) - p\rangle] \\
=& (1-\lambda_n)\Gamma_n(p) + \lambda_n\delta_n, \qquad (3.7)
\end{aligned}$$

where $\lambda_n = \beta_n(1-(1-\beta_n)\nu^2)$, $\Gamma_n(p) = \|x_n - p\|^2$ and $\delta_n = \frac{[\beta_n\|f(x_n)-p\|^2 + 2(1-\beta_n)\langle f(p)-p, K(x_n,u_n)-p\rangle]}{1-(1-\beta_n)\nu^2}$. We now consider the following cases:

Case 1: Suppose $\{\Gamma_n(p)\}$ is monotonically non-increasing for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. Since $\{\Gamma_n(p)\}$ is bounded and assumed to be monotonically non-increasing, it then implies that it converges. So $\Gamma_n(p) - \Gamma_{n+1}(p) \to 0$ as $n \to \infty$. From (3.1) we derive that

$$\begin{aligned}
\|x_{n+1} - p\|^2 =& \|\beta_n f(x_n) + \gamma_n Tx_n + \xi_n Uu_n - p\|^2 \\
\leq& \beta_n\|f(x_n) - p\|^2 + \gamma_n\|Tx_n - p\|^2
\end{aligned}$$

$$
\begin{aligned}
&+\xi_n\|Uu_n-p\|^2-\gamma_n\xi_n\|Tx_n-Uu_n\|^2\\
\leq&\beta_n\|f(x_n)-p\|^2+\gamma_n\|x_n-p\|^2+\xi_n\|x_n-p\|^2\\
&-\gamma_n\xi_n\|Tx_n-Uu_n\|^2-\xi_n\alpha_n(1-\alpha_n)\|Tx_n-x_n\|^2\\
=&\beta_n\|f(x_n)-p\|^2+(1-\beta_n)\|x_n-p\|^2\\
&-\gamma_n\xi_n\|Tx_n-Uu_n\|^2-\xi_n\alpha_n(1-\alpha_n)\|Tx_n-x_n\|^2.
\end{aligned}
$$

Then, we get that

$$
\gamma_n\xi_n\|Tx_n-Uu_n\|^2+\xi_n\alpha_n(1-\alpha_n)\|Tx_n-x_n\|^2\leq \beta_n\|f(x_n)-p\|^2+(1-\beta_n)\Gamma_n(p)-\Gamma_{n+1}(p). \tag{3.8}
$$

Taking the limit in (3.8) as $n\to\infty$ and using the control conditions, we get

$$
\|Tx_n-Uu_n\|\to 0 \tag{3.9}
$$

and

$$
\|Tx_n-x_n\|\to 0. \tag{3.10}
$$

Therefore from (3.1) and (3.10), we obtain that

$$
\begin{aligned}
\|Ux_n-Uu_n\|&=\|Ux_n-U((1-\alpha_n)x_n+\alpha_nTx_n)\|\\
&\leq\|x_n-[(1-\alpha_n)x_n+\alpha_nTx_n]\|\\
&=\alpha_n\|x_n-Tx_n\|\to 0 \text{ as } n\to\infty.
\end{aligned} \tag{3.11}
$$

Thus from (3.9) and (3.11), we obtain that

$$
\|Tx_n-Ux_n\|\leq\|Tx_n-Uu_n\|+\|Uu_n-Ux_n\|\to 0 \text{ as } n\to\infty. \tag{3.12}
$$

And from (3.10) and (3.12), we get

$$
\|Ux_n-x_n\|\leq\|Ux_n-Tx_n\|+\|Tx_n-x_n\|\to 0 \text{ as } n\to\infty. \tag{3.13}
$$

We next prove that $\|u_n-x_n\|\to 0$ and $\|x_{n+1}-x_n\|\to 0$ as $n\to\infty$. Indeed, from (3.1) and (3.10) we see that

$$
\|u_n-x_n\|=\alpha_n\|x_n-Tx_n\|\to 0 \text{ as } n\to\infty.
$$

Also from (3.10) and (3.13), we obtain that

$$
\begin{aligned}
&\|x_{n+1}-x_n\|=\|\beta_nf(x_n)+\gamma_nTx_n+\xi_nUu_n-x_n\|\\
&\leq\beta_n\|f(x_n)-x_n\|+\gamma_n\|Tx_n-x_n\|+\xi_n\|Uu_n-x_n\|\to 0 \text{ as } n\to\infty.
\end{aligned}
$$

By the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup x^*$ as $k\to\infty$. By the demiclosedness of T and U and (3.10) (see Lemma 2) and (3.13), it then implies that $x^*\in\mathrm{Fix}(T)\cap\mathrm{Fix}(U)$. We will next show that $\limsup_{n\to\infty}\delta_n\leq 0$ and establish the strong convergence of the sequence $\{x_n\}$. First observe that from (3.10), (3.11) and (3.12), we have

$$
K(x_{n_k},u_{n_k})=\theta_{n_k}Tx_{n_k}+(1-\theta_{n_k})Uu_{n_k}
$$

$$\begin{aligned}&\to \theta_{n_k} x_{n_k} + (1-\theta_{n_k}) x_{n_k}\\ &\rightharpoonup x^* \quad \text{as} \quad k \to \infty.\end{aligned}$$

Therefore,

$$\begin{aligned}&\limsup_{n\to\infty} \langle f(p)-p, K(x_n,u_n)-p\rangle\\ &\le \limsup_{k\to\infty} \langle f(p)-p, K(x_{n_k},u_{n_k})-p\rangle\\ &\le \langle f(p)-p, x^*-p\rangle \le 0, \end{aligned} \tag{3.14}$$

where (3.14) follows from (3.4). Furthermore, it is easy to see that $\lim_{n\to\infty}\lambda_n = 0$ and $\sum_{n=1}^{\infty}\lambda_n = \infty$. Hence, we conclude by Lemma 4 that $\lim_{n\to\infty}\Gamma_n(p) = 0$. Therefore $x_n \to p$ as $n \to \infty$.

Case 2: Suppose $\{\Gamma_n(p)\}$ is not monotonically decreasing. Then there exists a subsequence $\{n_r\}$ of $\{n\}$ such that $\Gamma_{n_r}(p) < \Gamma_{n_r+1}(p)$ for all $r \in \mathbb{N}$. By Lemma 5, there exists an increasing sequence $\{m_r\} \subset \mathbb{N}$ such that $m_r \to \infty$ and

$$0 \le \Gamma_{m_r}(p) \le \Gamma_{m_r+1}(p) \text{ for all } r \in \mathbb{N}. \tag{3.15}$$

Then using similar argument as in Case 1, we see that as $m_r \to \infty$,

$$\|x_{m_r} - Tx_{m_r}\| \to 0\ , \|Ux_{m_r} - x_{m_r}\| \to 0$$

and

$$\limsup_{n\to\infty} \langle f(p)-p, K(x_{m_r},u_{m_r})-p\rangle \le 0.$$

It then follows from (3.7) and (3.15) that

$$\begin{aligned}0 &\le (1-\lambda_{m_r})\Gamma_{m_r}(p) - \Gamma_{m_r+1}(p) + \lambda_{m_r}\delta_{m_r}\\ &\le (1-\lambda_{m_r})\Gamma_{m_r+1}(p) - \Gamma_{m_r}(p) + \lambda_{m_r}\delta_{m_r}\\ &= -\lambda_{m_r}\Gamma_{m_r+1}(p) + \lambda_{m_r}\delta_{m_r}.\end{aligned}$$

Therefore,

$$\Gamma_{m_r+1}(p) \le \delta_{m_r}. \tag{3.16}$$

Note that from the definition of δ_{m_r} we have $\limsup_{r\to\infty}\delta_{m_r} \le 0$. Hence taking the limit of (3.16) as $r \to \infty$, we get $\Gamma_{m_r+1}(p) \to 0$. Consequently, $\Gamma_{m_r}(p) \to 0$. By applying Lemma 5, we then get

$$0 \le \Gamma_r(p) \le \max\{\Gamma_r(p), \Gamma_{m_r}(p)\} \le \Gamma_{m_r+1}(p).$$

Hence $\lim_{r\to\infty}\Gamma_r(p) = 0$. Therefore, $\{x_n\}$ converges strongly to $p \in \Omega$.

So in both cases, we obtain that $x_n \to p \in \Omega$.

□

Taking $f(x) = u$ for all $x \in C$, we obtain the following Halpern S-iteration as a consequence of Theorem 3.

Corollary 1. *Let $T,U : C \to C$ be two nonexpansive mappings with $\Omega := \mathrm{Fix}(T) \cap \mathrm{Fix}(U) \neq \varnothing$ and $u \in C$ be arbitrary. Choose $x_0 \in C$ and let $\{x_n\}$ be the sequence generated by* (3.17)

$$\begin{cases} u_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ x_{n+1} = \beta_n u + \gamma_n T x_n + \xi_n U u_n, \quad n \geq 1, \end{cases} \tag{3.17}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\xi_n\}$are sequences in $(0,1)$ satisfying

(i) $\beta_n + \gamma_n + \xi_n = 1$;
(ii) $\liminf_{n\to\infty} \gamma_n \xi_n > 0$, $\liminf_{n\to\infty} \xi_n \alpha_n (1-\alpha_n) > 0$;
(iii) $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$.

Then, the sequence $\{x_n\}$ generated by Algorithm (3.17) *converges strongly to $p \in \Omega$, where $p = P_\Omega u$.*

4. Application to common fixed point of nonexpansive semigroups

Let H be a Hilbert space and C a nonempty closed convex subset of H. The one parameter family $\mathcal{T} := \{T(t) : 0 \leq t < \infty\}$ of mappings from C to C is said to be nonexpansive semigroup, if the following conditions are satisfied:

(i) $T(0)x = x$ for all $x \in C$;
(ii) $T(s+t) = T(s)T(t)$ for all $s,t \geq 0$;
(iii) for each $x \in C$, the mapping $t \rightarrowtail T(t)x$ is continuous;
(iv) $\|T(t)x - T(t)y\| \leq \|x-y\|$.

If $F(\mathcal{T}) \neq \varnothing$, it is known that $F(\mathcal{T})$ is closed and convex. An example of a one-parameter nonexpansive semigroup is given below.

Example 1 ([10]). Let $H = \mathbb{R}$ and $\mathcal{T} := \{T(t) : 0 \leq t < \infty\}$, where $T(t)x = (\frac{1}{10^t})x$ for all $x \in H$. Then $\mathcal{T}$ is a one-parameter nonexpansive semigroup.

The following lemma was proved in Shimizu and Takahashi [22]; see also [5,21].

Lemma 7. *Let C be a nonempty bounded closed convex subset of a real Hilbert space H and let $\mathcal{T} := \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C. Then, for any $h \geq 0$,*

$$\lim_{t\to\infty} \sup_{x\in C} \left\| \frac{1}{t}\int_0^t T(s)x ds - T(h)\left(\frac{1}{t}\int_0^t T(s)x ds\right) \right\| = 0.$$

We next apply our main result to find the common fixed point of two nonexpansive semigroups.

Theorem 4. *Let $\mathcal{T} := \{T(s) : 0 \leq s < \infty\}$ and $\mathcal{U} := \{U(s) : 0 \leq s < \infty\}$ be two families of nonexpansive semigroups on C. Also let $f : C \to C$ be a contraction with*

coefficient $\nu \in [0,1)$. *Assume* $\Omega := \mathrm{Fix}(\mathcal{T}) \cap \mathrm{Fix}(\mathcal{U}) \neq \varnothing$. *Choose* $x_0 \in C$ *and let* $\{x_n\}$ *be the sequence generated by the algorithm*

$$\begin{cases} u_n = (1-\alpha_n)x_n + \alpha_n \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \\ x_{n+1} = \beta_n f(x_n) + \gamma_n \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds + \xi_n \frac{1}{t_n}\int_0^{t_n} U(s)u_n ds, \quad n \geq 1, \end{cases} \tag{4.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\xi_n\} \subset (0,1)$ *are sequences such that*

(i) $\beta_n + \gamma_n + \xi_n = 1$;
(ii) $\liminf_{n\to\infty} \gamma_n \xi_n > 0$, $\liminf_{n\to\infty} \xi_n \alpha_n (1-\alpha_n) > 0$;
(iii) $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$;
(iv) $\{t_n\}$ *is a sequence of positive numbers such that* $t_n \to \infty$ *as* $n \to \infty$.

Then $x_n \to p \in \Omega$, *where* p *is the unique solution of the following VIP* (4.2)*: Find* $p \in \Omega$ *such that*

$$\langle (I-f)p, x-p \rangle \geq 0 \qquad \forall\, x \in \Omega. \tag{4.2}$$

Proof. Since f is ν-contraction mapping, it follows that $I-f$ is $2(1+\nu^2)$-Lipschitzian and $(1-\nu)$-strongly monotone. Therefore by Lemma 1, it follows that the VIP (4.2) has a unique solution $p \in \Omega$.

We divide the remaining part of the proof into two steps.

Step 1: The sequence $\{x_n\}$ is bounded.

To see this, let $p \in \Omega$. Then from (4.1)

$$\begin{aligned} \|u_n - p\| &= \left\| (1-\alpha_n)x_n + \alpha_n \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - p \right\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n \left\| \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - p \right\| \\ &= (1-\alpha_n)\|x_n - p\| + \alpha_n \left\| \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - \frac{1}{t_n}\int_0^{t_n} T(s)p ds \right\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n \frac{1}{t_n}\int_0^{t_n} \|T(s)x_n - T(s)p\| ds \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n \|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{4.3}$$

Thus from (4.1) and (4.3) we get

$$\begin{aligned} \|x_{n+1} - p\| &= \left\| \beta_n f(x_n) + \gamma_n \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds + \xi_n \frac{1}{t_n}\int_0^{t_n} U(s)u_n ds - p \right\| \\ &\leq \beta_n \|f(x_n) - p\| + \gamma_n \left\| \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - p \right\| \\ &\quad + \xi_n \left\| \frac{1}{t_n}\int_0^{t_n} U(s)u_n ds - p \right\| \\ &\leq \beta_n \nu \|x_n - p\| + \beta_n \|f(p) - p\| + \gamma_n \|x_n - p\| + \xi_n \|u_n - p\| \end{aligned}$$

$$
\begin{aligned}
&\leq \beta_n \nu \|x_n - p\| + \beta_n \|f(p) - p\| + \gamma_n \|x_n - p\| + \xi_n \|x_n - p\| \\
&\quad = [1 - \beta_n(1-\nu)]\|x_n - p\| + \beta_n \|f(p) - p\| \\
&\leq \max\left\{ \|x_n - p\|, \frac{\|f(p) - p\|}{1-\nu} \right\} \\
&\quad \vdots \\
&\leq \max\left\{ \|x_0 - p\|, \frac{\|f(p) - p\|}{1-\nu} \right\}.
\end{aligned}
$$

Which shows that $\{\|x_n - p\|\}$ is bounded and hence $\{x_n\}$ is bounded.

Step 2: We claim that $x_n \to p \in \Omega$. Indeed, following similar procedure as in the derivation of (3.7), we have that

$$\Gamma_{n+1}(p) \leq (1-\lambda_n)\Gamma_n(p) + \lambda_n \eta_n,$$

where $\lambda_n = \beta_n(1-(1-\beta_n)\nu^2)$, $\Gamma_n(p) = \|x_n - p\|$,

$$\eta_n = \frac{[\beta_n \|f(x_n) - p\|^2 + 2(1-\beta_n)\langle f(p) - p, M(x_n, u_n) - p\rangle]}{1-(1-\beta_n)\nu^2}$$

and

$$M(x_n, u_n) = \frac{\gamma_n}{(1-\beta_n)t_n} \int_0^{t_n} T(s)x_n ds + \frac{\xi_n}{(1-\beta_n)t_n} \int_0^{t_n} U(s)u_n ds.$$

Similar to the proof of Theorem 3, we consider two cases.

Case 1: Suppose the sequence $\{\Gamma_n(p)\}$ is monotonically non-increasing for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. From (4.1), (4.3) and by applying Lemma 3 (ii), we get that

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &= \left\| \beta_n f(x_n) + \gamma_n \frac{1}{t_n} \int_0^{t_n} T(s)x_n ds + \xi_n \frac{1}{t_n} \int_0^{t_n} U(s)u_n ds - p \right\|^2 \\
&\leq \beta_n \|f(x_n) - p\|^2 + \gamma_n \left\| \frac{1}{t_n} \int_0^{t_n} T(s)x_n ds - p \right\|^2 \\
&\quad + \xi_n \left\| \frac{1}{t_n} \int_0^{t_n} U(s)u_n ds - p \right\|^2 \\
&\quad - \gamma_n \xi_n \left\| \frac{1}{t_n} \int_0^{t_n} (T(s)x_n - U(s)u_n) ds \right\|^2 \\
&\leq \beta_n \|f(x_n) - p\|^2 + \gamma_n \|x_n - p\|^2 + \xi_n \|u_n - p\|^2 \\
&\quad - \gamma_n \xi_n \left\| \frac{1}{t_n} \int_0^{t_n} (T(s)x_n - U(s)u_n) ds \right\|^2 \\
&\quad - \xi_n \alpha_n (1-\alpha_n) \left\| \frac{1}{t_n} \int_0^{t_n} T(s)x_n ds - x_n \right\|^2
\end{aligned}
$$

$$\begin{aligned}
&\leq \beta_n \|f(x_n)-p\|^2 + (1-\beta_n)\|x_n-p\|^2 \\
&\quad - \gamma_n \xi_n \left\| \frac{1}{t_n}\int_0^{t_n} (T(s)x_n - U(s)u_n)ds \right\|^2 \\
&\quad - \xi_n \alpha_n (1-\alpha_n) \left\| \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - x_n \right\|^2. \qquad (4.4)
\end{aligned}$$

From (4.4), we then have that

$$\begin{aligned}
&\xi_n \alpha_n (1-\alpha_n) \left\| \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - x_n \right\|^2 + \gamma_n \xi_n \left\| \frac{1}{t_n}\int_0^{t_n} (T(s)x_n - U(s)u_n)ds \right\|^2 \\
&\quad \leq \beta_n \|f(x_n)-p\|^2 + (1-\beta_n)\|x_n-p\|^2 - \|x_{n+1}-p\|^2. \qquad (4.5)
\end{aligned}$$

Taking the limit as $n \to \infty$ in (4.5) gives

$$\left\| \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - x_n \right\| \to 0 \qquad (4.6)$$

and

$$\left\| \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - \frac{1}{t_n}\int_0^{t_n} U(s)u_n ds \right\| \to 0 \qquad (4.7)$$

as $n \to \infty$. Therefore from (4.1) and (4.6) we get

$$\|u_n - x_n\| = \alpha_n \left\| \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - x_n \right\| \to 0,$$

from (4.6) and (4.7), we get

$$\begin{aligned}
\left\| \frac{1}{t_n}\int_0^{t_n} U(s)u_n ds - x_n \right\| &\leq \left\| \frac{1}{t_n}\int_0^{t_n} U(s)u_n ds - \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds \right\| \\
&\quad + \left\| \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - x_n \right\| \to 0, \qquad (4.8)
\end{aligned}$$

and from (4.1) and (4.8), we get

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq \beta_n \|f(x_n) - x_n\| + \gamma_n \left\| \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - x_n \right\| \\
&\quad + \xi_n \left\| \frac{1}{t_n}\int_0^{t_n} U(s)u_n ds - x_n \right\| \to 0
\end{aligned}$$

as $n \to \infty$. Moreover, for all $h \in [0,\infty)$, we obtain that

$$\begin{aligned}
\|T(h)x_n - x_n\| &\leq \left\| T(h)x_n - T(h)(\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds) \right\| \\
&\quad + \left\| T(h)(\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds) - \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds \right\|
\end{aligned}$$

$$
\begin{aligned}
&+\left\|\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - x_n\right\| \\
\leq &2\left\|\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - x_n\right\| \\
&+\left\|T(h)(\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds) - \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds\right\|. \qquad (4.9)
\end{aligned}
$$

By applying Lemma 7, (4.6) and taking the limit as $n \to \infty$ of (4.9) we derive that

$$\|T(h)x_n - x_n\| \to 0 \text{ as } n \to \infty.$$

Furthermore, using Lemma 7, (4.6) and (4.7), we get

$$
\begin{aligned}
\|U(h)x_n - x_n\| \leq &\left\|U(h)x_n - U(h)(\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds)\right\| \\
&+\left\|U(h)(\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds) - U(h)(\frac{1}{t_n}\int_0^{t_n} U(s)u_n ds)\right\| \\
&+\left\|U(h)(\frac{1}{t_n}\int_0^{t_n} U(s)u_n ds) - \frac{1}{t_n}\int_0^{t_n} U(s)u_n ds\right\| \\
&+\left\|\frac{1}{t_n}\int_0^{t_n} U(s)u_n ds - \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds\right\| + \left\|\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - x_n\right\| \\
\leq &2\left\|\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - x_n\right\| \\
&+2\left\|\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - \frac{1}{t_n}\int_0^{t_n} U(s)u_n ds\right\| \\
&+\left\|U(h)(\frac{1}{t_n}\int_0^{t_n} U(s)u_n ds) - \frac{1}{t_n}\int_0^{t_n} U(s)u_n ds\right\| \to 0 \text{ as } n \to \infty.
\end{aligned}
$$

Since $\{x_n\}$ is bounded, there exists subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^* \in C$. Therefore by the demiclosedness of $T(h)$ and $U(h)$ (see Lemma 2), we have that $x^* \in \mathrm{Fix}(\mathcal{T}) \cap \mathrm{Fix}(\mathcal{U})$. Note also that from (4.6) and (4.8),

$$
\begin{aligned}
M(x_{n_k}, u_{n_k}) = &\frac{\gamma_{n_k}}{(1-\beta_{n_k})t_{n_k}}\int_0^{t_{n_k}} T(s)x_{n_k} ds + \frac{\xi_{n_k}}{(1-\beta_{n_k})t_{n_k}}\int_0^{t_{n_k}} U(s)u_{n_k} ds \\
&\to \frac{\gamma_{n_k}}{(1-\beta_{n_k})}x_{n_k} + \frac{\xi_{n_k}}{(1-\beta_{n_k})}x_{n_k} \\
&\rightharpoonup x^* \text{ as } k \to \infty. \qquad (4.10)
\end{aligned}
$$

We will next show that $\limsup_{n\to\infty} \eta_n \leq 0$ and establish the strong convergence of the sequence $\{x_n\}$. Indeed, using (4.10)

$$\limsup_{n\to\infty} \eta_n \leq \limsup_{k\to\infty} \langle f(p) - p, M(x_{n_k}, u_{n_k}) - p\rangle$$

$$\begin{aligned} &\leq \langle f(p)-p, x^*-p\rangle \\ &\leq 0, \end{aligned} \tag{4.11}$$

where (4.11) follows from (4.2). Furthermore, it is easy to see that $\lim_{n\to\infty}\lambda_n = 0$ and $\sum_{n=1}^{\infty}\lambda_n = \infty$. Hence, we conclude by Lemma 4 that $\lim_{n\to\infty}\Gamma_n(p) = 0$. Therefore $x_n \to p$ as $n \to \infty$.

Case 2: Suppose $\{\Gamma_n(p)\}$ is not monotonically decreasing. It is easy to conclude from the proof of Case 2 of Theorem 3 and Case 1 of Theorem 4 that $x_n \to p$ as $n \to \infty$.

So in both cases, we obtain that $x_n \to p \in \Omega$.

□

Taking $f(x) = u$ for all $x \in C$, we obtain the following Halpern S-iteration as a consequence of Theorem 4.

Corollary 2. *Let $\mathcal{T} := \{T(s) : 0 \leq s < \infty\}$ and $\mathcal{U} := \{U(s) : 0 \leq s < \infty\}$ be two families of nonexpansive semigroups on C and $u \in C$ be arbitrary. Assume $\Omega := \mathrm{Fix}(\mathcal{T}) \cap \mathrm{Fix}(\mathcal{U}) \neq \varnothing$. Let $\{x_n\}$ be the sequence generated by the algorithm*

$$\begin{cases} u_n = (1-\alpha_n)x_n + \alpha_n \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \\ x_{n+1} = \beta_n u + \gamma_n \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds + \xi_n \frac{1}{t_n}\int_0^{t_n} U(s)u_n ds, \quad n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\xi_n\} \subset (0,1)$ are sequences such that

(i) $\beta_n + \gamma_n + \xi_n = 1$;
(ii) $\liminf_{n\to\infty}\gamma_n\xi_n > 0$, $\liminf_{n\to\infty}\xi_n\alpha_n(1-\alpha_n) > 0$;
(iii) $\lim_{n\to\infty}\beta_n = 0$, $\sum_{n=1}^{\infty}\beta_n = \infty$;
(iv) *$\{t_n\}$ is a sequence of positive numbers such that $t_n \to \infty$ as $n \to \infty$.*

Then $x_n \to p \in \Omega$, where $p = P_\Omega u$.

5. Numerical Examples

Example 2. Let $H = \ell_2(\mathbb{R})$, where $\ell_2(\mathbb{R}) := \{\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots), \sigma_i \in \mathbb{R} : \sum_{i=1}^{\infty}|\sigma_i|^2 < \infty\}$, $\|\sigma\| = (\sum_{i=1}^{\infty}|\sigma_i|^2)^{\frac{1}{2}}$, $\forall\, \sigma \in H$. Let $C := \{\sigma \in H : \|\sigma\| \leq 3\}$. We define $T, U : C \to C$ by

$$Tx = \frac{x}{2} + (1,0,0,\cdots) \qquad \forall\, x \in C$$

and

$$Ux = \frac{x}{4} + (\frac{3}{2},0,0,\cdots) \qquad \forall\, x \in C,$$

respectively. It is easy to see that $T(2,0,0,\cdots) = U(2,0,0,\cdots) = (2,0,0,\cdots)$. Therefore $\mathrm{Fix}(T) \cap \mathrm{Fix}(U) \neq \varnothing$. We choose $f(x) = \frac{x}{1.02}$ for every $x \in C$, $\beta_n = \frac{1}{n+1}$,

$\gamma_n = \frac{2n}{3(n+1)}$, $\xi_n = \frac{n}{3(n+1)}$ and $\alpha_n = \frac{n+1}{2n-1}$. In this case, (3.1) and (1.4) gives

$$\begin{cases} u_n = \frac{n-2}{2n-1}x_n + \frac{n+1}{2n-1}Tx_n, \\ x_{n+1} = \frac{x_n}{1.02(n+1)} + \frac{2n}{3(n+1)}Tx_n + \frac{n}{3(n+1)}Uu_n, \quad n \geq 1, \end{cases}$$

and

$$x_{n+1} = \frac{x_n}{1.02(n+1)} + \frac{2n}{3(n+1)}Tx_n + \frac{n}{3(n+1)}Ux_n, n \geq 1,$$

respectively.

We choose different initial values as follows:

Case Ia: $x_1 = (0.3, 0.8, -0.4, -0.2, 0, 0, 0, \dots)$;
Case Ib: $x_1 = (-2, 1, 1, -0.2, 0, 0, 0, \dots)$;
Case Ic: $x_1 = (1.3, 0.4, 0, -0.2, 0, 0, 0, \dots)$;
Case Id: $x_1 = (0.4, 0, 1, -0.8, 0, 0, 0, \dots)$.

Using MATLAB 2017(b), we compare the performance of Algorithm 2 with Algorithm (1.4) of Ahmad et al. [2]. The stopping criterion used for our computation is $\frac{\|x_{n+1}-x_n\|^2}{\|x_2-x_1\|^2} < 10^{-7}$. We plot the graphs of errors against the number of iterations in each case.

TABLE 1. Numerical results for Example 2.

		Alg. (1.4)	Alg. 3.1
Case Ia	CPU time (sec)	0.0014	9.5022e-4
	No of Iter.	43	37
Case Ib	CPU time (sec)	0.0016	9.3929e-4
	No. of Iter.	45	39
Case Ic	CPU time (sec)	0.0014	9.3664e-4
	No of Iter.	41	35
Case Id	CPU time (sec)	0.0013	9.5949-4
	No of Iter.	43	37

The next example is inspired by Vong and Liu [25] and He et al. [14].

Example 3. Let $\mathbb{R}^2$ be the two dimensional Euclidean space with the usual inner product $\langle x, y\rangle = x_1y_1 + x_2y_2$ for all $x = (x_1, x_2)^\intercal, y = (y_1, y_2)^\intercal \in \mathbb{R}^2$ and the norm $\|x\| = \sqrt{x_1^2 + x_2^2}$. Let $C := [-1, 1] \times [-1, 1]$ and define $T : C \to C$ by

$$T : x = (x_1, x_2)^\intercal \mapsto \left(\frac{1}{2}\sin(\frac{x_1+x_2}{\sqrt{2}}), \frac{1}{2}(\cos(\frac{x_1+x_2}{\sqrt{2}}) - 1)\right).$$

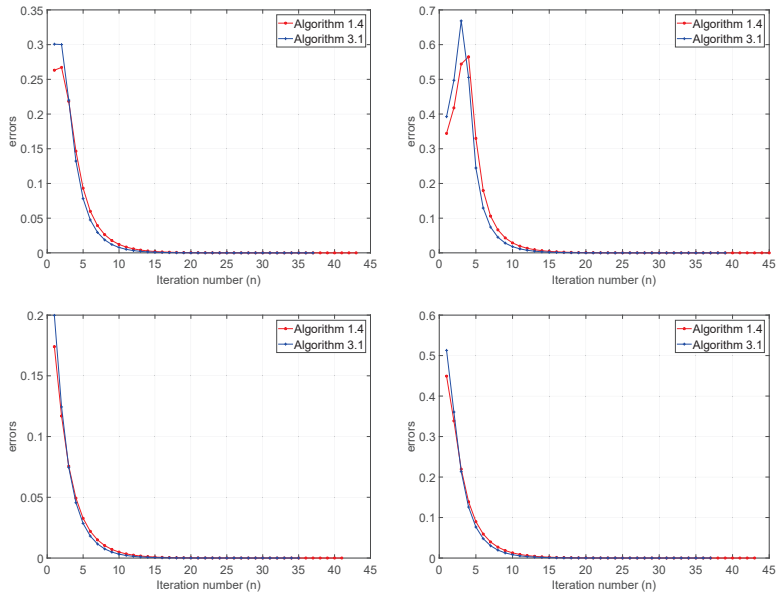

FIGURE 1. Example 2: Top left: Case Ia; Top right: Case Ib; Bottom left: Case Ic; Bottom right: Case Id.

For each $x = (x_1, x_2) \in \mathbb{R}^2$, T is a Fréchet differentiable and

$$T'(x) = \frac{1}{2\sqrt{2}} \begin{pmatrix} \cos(\frac{x_1+x_2}{\sqrt{2}}) & \cos(\frac{x_1+x_2}{\sqrt{2}}) \\ -\sin(\frac{x_1+x_2}{\sqrt{2}}) & -\sin(\frac{x_1+x_2}{\sqrt{2}}) \end{pmatrix}. \tag{5.1}$$

It is known that $T'(x)$ is a bounded linear operator from $\mathbb{R}^2$ to itself. The norm of $T'(x)$ can be derived by the formula:

$$\|T'(x)\| = \sqrt{\lambda} \qquad \forall\, x \in \mathbb{R}^2,$$

where λ is the maximal eigenvalue of the matrix $(T'(x))^\intercal T'(x)$. From (5.1), it is easy to see that

$$(T'(x))^\intercal = \frac{1}{2\sqrt{2}} \begin{pmatrix} \cos(\frac{x_1+x_2}{\sqrt{2}}) & -\sin(\frac{x_1+x_2}{\sqrt{2}}) \\ \cos(\frac{x_1+x_2}{\sqrt{2}}) & -\sin(\frac{x_1+x_2}{\sqrt{2}}) \end{pmatrix} \tag{5.2}$$

and therefore from (5.1) and (5.2)

$$(T'(x))^\intercal T'(x) = \frac{1}{8} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{5.3}$$

It therefore follows from (5.3) that $\|T'(x)\| = \frac{1}{2}$, $\forall\, x \in \mathbb{R}^2$. Then, using Mean Value Theorem, for any $x, y \in \mathbb{R}^2$, there is a constant $\tau \in (0,1)$ such that

$$\|Tx - Ty\| = \|T'(\tau x + (1-\tau)y)(x-y)\|$$

$$\leq \|T'(\tau x + (1-\tau)y)\|\|(x-y)\|$$
$$\leq \|x-y\|.$$

Which shows that T is a nonexpansive mapping. We define $U : C \to C$ by $Ux = \frac{x}{2}$ for all $x \in C$. Then it is easy to see that $(0,0) \in \mathrm{Fix}(T) \cap \mathrm{Fix}(U)$. We choose $f(x) = \frac{x}{1.05}$ for every $x \in C$, $\beta_n = \frac{1}{n+1}$, $\gamma_n = \frac{2n}{3(n+1)}$, $\xi_n = \frac{n}{3(n+1)}$ and $\alpha_n = \frac{n+1}{2n-1}$. In this case, (3.1) and (1.4) gives

$$\begin{cases} u_n = \frac{n-2}{2n-1}x_n + \frac{n+1}{2n-1}Tx_n, \\ x_{n+1} = \frac{x_n}{1.05(n+1)} + \frac{2n}{3(n+1)}Tx_n + \frac{n}{3(n+1)}Uu_n, \quad n \geq 1, \end{cases}$$

and

$$x_{n+1} = \frac{x_n}{1.05(n+1)} + \frac{2n}{3(n+1)}Tx_n + \frac{n}{3(n+1)}Ux_n, \quad n \geq 1,$$

respectively.

We choose different initial values as follows:

Case IIa: $x_1 = (0.05, -0.4)$;
Case IIb: $x_1 = (-0.9, 0)$;
Case IIc: $x_1 = (0.9, 0.4)$;
Case IId: $x_1 = (-0.5, -0.8)$.

Using MATLAB 2017(b), we compare the performance of Algorithm 2 with Algorithm (1.4) of Ahmad et al. [2]. The stopping criterion used for our computation is $\|x_{n+1} - x_n\|^2 < 10^{-7}$. We plot the graphs of $||x_{n+1} - x_n||^2$ against the number of iterations in each case.

TABLE 2. Numerical results for Example 2.

		Alg. (1.4)	Alg. 3.1
Case IIa	CPU time (sec)	0.0013	8.6347e-4
	No of Iter.	26	17
Case IIb	CPU time (sec)	0.0014	8.7407e-4
	No. of Iter.	27	18
Case IIc	CPU time (sec)	0.0027	7.9229e-4
	No of Iter.	28	18
Case IId	CPU time (sec)	0.0011	9.3366e-4
	No of Iter.	27	18

Remark 1. From the computational results, it can be inferred that our algorithm performs better in both number of iterations and computation time taken. In addition, the choices of different initial values do not have significant effects on the output of the Algorithm (3.1) in terms of the number of iterations and computation time taken.

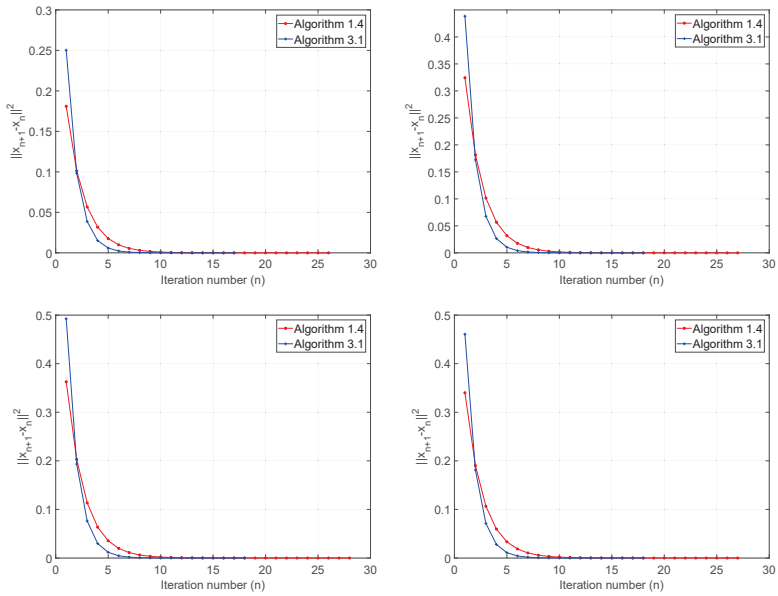

FIGURE 2. Example 3: Top left: Case IIa; Top right: Case IIb; Bottom left: Case IIc; Bottom right: Case IId.

6. CONCLUSION

In this paper, we present a viscosity-S iteration for finding the common fixed points of two nonexpansive mappings in real Hilbert spaces. We apply our results to nonexpansive semigroup of operators. We also illustrate the efficiency and effectiveness of our algorithm using examples from a finite dimensional Hilbert space and an infinite dimensional Hilbert space and by comparing with a similar existing algorithm in literature.

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Declaration

The authors declare that they have no competing interests.

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APPLICATIONS OF HORADAM POLYNOMIALS ON A NEW FAMILY OF BI-PRESTARLIKE FUNCTIONS

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Abstract. In this article, we introduce and investigate a new family of analytic and bi-prestarlike functions by using the Horadam polynomials defined in the open unit disk U. We determine upper bounds for the first two coefficients $|a_2|$ and $|a_3|$ and solve Fekete-Szegő problem of functions that belong to this family. Also, we point out several certain special cases for our results.

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1. Introduction and preliminaries

Indicate by $\mathcal{A}$ the collection of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ that have the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Further, let S stand for the subclass of $\mathcal{A}$ containing of functions in U satisfying (1.1) which are univalent in U.

A function $f \in \mathcal{A}$ is called starlike of order θ $(0 \leq \theta < 1)$, if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \theta, \quad (z \in U).$$

For $f \in \mathcal{A}$ given by (1.1) and $g \in \mathcal{A}$ defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

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the Hadamard product of f and g is defined (as usual) by

$$(f*g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad (z \in U).$$

Ruscheweyh [14] defined and considered the family of prestarlike functions of order θ, which are the functions f such that $f * I_\theta$ is a starlike function of order θ, where

$$I_\theta(z) = \frac{z}{(1-z)^{2(1-\theta)}}, \quad (0 \le \theta < 1, z \in U).$$

The function I_θ can be written in the form:

$$I_\theta(z) = z + \sum_{n=2}^{\infty} \varphi_n(\theta) z^n,$$

where

$$\varphi_n(\theta) = \frac{\prod_{i=2}^{n}(i-2\theta)}{(n-1)!}, \quad n \ge 2.$$

We note that $\varphi_n(\theta)$ is a decreasing function in θ and satisfies

$$\lim_{n\to\infty} \varphi_n(\theta) = \begin{cases} \infty, & if\, \theta < \frac{1}{2} \\ 1, & if\, \theta = \frac{1}{2} \\ 0, & if\, \theta > \frac{1}{2} \end{cases}.$$

According to the Koebe one-quarter theorem (see [6]) every function $f \in S$ has an inverse f^{-1} which satisfies $f^{-1}(f(z)) = z$, $(z \in U)$ and $f(f^{-1}(w)) = w$, $(|w| < r_0(f), r_0(f) \ge \frac{1}{4})$, where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots. \quad (1.2)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. Let Σ stands for the class of bi-univalent functions in U given by (1.1). Srivastava et al. [19] revived the study of analytic and bi-univalent functions in recent years, was followed by such works as those by Bulut [4], Adegani and et al. [1], Caglar et al. [5] and others (see, for example [15, 17, 18, 20]). We notice that the class Σ is not empty. For example, the functions z, $\frac{z}{1-z}$, $-\log(1-z)$ and $\frac{1}{2}\log\frac{1+z}{1-z}$ are members of Σ. However, the Koebe function is not a member of Σ. Until now, the coefficient estimate problem for each of the following Taylor-Maclaurin coefficients $|a_n|$, $(n = 3, 4, \cdots)$ for functions $f \in \Sigma$ is still an open problem.

Let the functions f and g be analytic in U. We say that the function f is said to be subordinate to g, if there exists a Schwarz function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f(z) = g(w(z))$. This subordination is denoted by $f \prec g$ or $f(z) \prec g(z)$ $(z \in U)$. It is well known (see [13]) that if the function g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

The Horadam polynomials $h_n(r)$ are defined by the following repetition relation (see [8]):

$$h_n(r) = prh_{n-1}(r) + qh_{n-2}(r) \quad (r \in \mathbb{R}, n \in \mathbb{N} = \{1,2,3,\cdots\}), \tag{1.3}$$

with $h_1(r) = a \quad and \quad h_2(r) = br$, for some real constant a, b, p and q. The characteristic equation of repetition relation (1.3) is $t^2 - prt - q = 0$. This equation has two real roots $x = \frac{pr+\sqrt{p^2r^2+4q}}{2}$ and $y = \frac{pr-\sqrt{p^2r^2+4q}}{2}$.

Remark 1. For particular values of a, b, p and q, the Horadam polynomial $h_n(r)$ reduces to several known polynomials listed below:

(1) the Fibonacci polynomials $F_n(r)$ $(a = b = p = q = 1)$.
(2) the Lucas polynomials $L_n(r)$ $(a = 2$ and $b = p = q = 1)$.
(3) the Pell polynomials $P_n(r)$ $(a = q = 1$ and $b = p = 2)$.
(4) the Pell-Lucas polynomials $Q_n(r)$ $(a = b = p = 2$ and $q = 1)$.
(5) the Chebyshev polynomials $T_n(r)$ of the first kind $(a = b = 1,\ p = 2$ and $q = -1)$.
(6) the Chebyshev polynomials $U_n(r)$ of the second kind $(a = 1,\ b = p = 2$ and $q = -1)$.

These polynomials, the families of orthogonal polynomials and other special polynomials as well as their generalizations are potentially important in a variety of disciplines in many of sciences, specially in mathematics, statistics and physics. For more information associated with these polynomials see [7, 8, 10, 11, 21].

The generating function of the Horadam polynomials $h_n(r)$ (see [9]) is given by

$$\Pi(r,z) = \sum_{n=1}^{\infty} h_n(r) z^{n-1} = \frac{a + (b - ap)rz}{1 - prz - qz^2}. \tag{1.4}$$

2. Main results

We begin this section by defining the subclass $\mathcal{N}_{\Sigma}(\delta,\lambda,\theta,r)$ as follows:

Definition 1. For $\delta \geq 0$, $0 \leq \lambda \leq 1$, $0 \leq \theta < 1$ and $r \in \mathbb{R}$, a function $f \in \Sigma$ is said to be in the class $\mathcal{N}_{\Sigma}(\delta,\lambda,\theta,r)$ if it satisfies the subordinations

$$(1-\delta)\left[(1-\lambda)\frac{z(f*I_\theta)'(z)}{(f*I_\theta)(z)} + \lambda\left(1 + \frac{z(f*I_\theta)''(z)}{(f*I_\theta)'(z)}\right)\right] + \delta\frac{\lambda z^2(f*I_\theta)''(z) + z(f*I_\theta)'(z)}{\lambda z(f*I_\theta)'(z) + (1-\lambda)(f*I_\theta)(z)} \prec \Pi(r,z) + 1 - a$$

and

$$(1-\delta)\left[(1-\lambda)\frac{w(g*I_\theta)'(w)}{(g*I_\theta)(w)} + \lambda\left(1 + \frac{w(g*I_\theta)''(w)}{(g*I_\theta)'(w)}\right)\right]$$

$$+\delta\frac{\lambda w^2\left(g*I_\theta\right)''(w)+w\left(g*I_\theta\right)'(w)}{\lambda w\left(g*I_\theta\right)'(w)+(1-\lambda)\left(g*I_\theta\right)(w)}\prec\Pi(r,w)+1-a,$$

where a is a real constant and the function $g=f^{-1}$ is given by (1.2).

Theorem 1. *For $\delta\geq 0$, $0\leq\lambda\leq 1$, $0\leq\theta<1$ and $r\in\mathbb{R}$, let $f\in\mathcal{A}$ be in the class $\mathcal{N}_\Sigma(\delta,\lambda,\theta,r)$. Then*

$$|a_2|\leq\frac{|br|\sqrt{|br|}}{\sqrt{2\left|\left[(1-\theta)\left[2\lambda\delta(1-\theta)(1-\lambda)+2\theta\lambda+1\right]b-2p\left(1-\theta\right)^2\left(\lambda+1\right)^2\right]br^2-2qa\left(1-\theta\right)^2\left(\lambda+1\right)^2\right|}}$$

and

$$|a_3|\leq\frac{|br|}{2(1-\theta)(3-2\theta)(2\lambda+1)}+\frac{b^2r^2}{4\left(1-\theta\right)^2\left(\lambda+1\right)^2}.$$

Proof. Let $f\in\mathcal{N}_\Sigma(\delta,\lambda,\theta,r)$. Then there are two analytic functions $u,v:U\longrightarrow U$ given by

$$u(z)=u_1z+u_2z^2+u_3z^3+\cdots\quad(z\in U)\tag{2.1}$$

and

$$v(w)=v_1w+v_2w^2+v_3w^3+\cdots\quad(w\in U),\tag{2.2}$$

with $u(0)=v(0)=0$, $|u(z)|<1$, $|v(w)|<1$, $z,w\in U$ such that

$$(1-\delta)\left[(1-\lambda)\frac{z\left(f*I_\theta\right)'(z)}{\left(f*I_\theta\right)(z)}+\lambda\left(1+\frac{z\left(f*I_\theta\right)''(z)}{\left(f*I_\theta\right)'(z)}\right)\right]$$
$$+\delta\frac{\lambda z^2\left(f*I_\theta\right)''(z)+z\left(f*I_\theta\right)'(z)}{\lambda z\left(f*I_\theta\right)'(z)+(1-\lambda)\left(f*I_\theta\right)(z)}=\Pi(r,u(z))+1-a$$

and

$$(1-\delta)\left[(1-\lambda)\frac{w\left(g*I_\theta\right)'(w)}{\left(g*I_\theta\right)(w)}+\lambda\left(1+\frac{w\left(g*I_\theta\right)''(w)}{\left(g*I_\theta\right)'(w)}\right)\right]$$
$$+\delta\frac{\lambda w^2\left(g*I_\theta\right)''(w)+w\left(g*I_\theta\right)'(w)}{\lambda w\left(g*I_\theta\right)'(w)+(1-\lambda)\left(g*I_\theta\right)(w)}=\Pi(r,v(w))+1-a.$$

Or, equivalently

$$(1-\delta)\left[(1-\lambda)\frac{z\left(f*I_\theta\right)'(z)}{\left(f*I_\theta\right)(z)}+\lambda\left(1+\frac{z\left(f*I_\theta\right)''(z)}{\left(f*I_\theta\right)'(z)}\right)\right]$$
$$+\delta\frac{\lambda z^2\left(f*I_\theta\right)''(z)+z\left(f*I_\theta\right)'(z)}{\lambda z\left(f*I_\theta\right)'(z)+(1-\lambda)\left(f*I_\theta\right)(z)}=1+h_1(r)+h_2(r)u(z)+h_3(r)u^2(z)+\cdots\tag{2.3}$$

and

$$(1-\delta)\left[(1-\lambda)\frac{w\left(g*I_\theta\right)'(w)}{\left(g*I_\theta\right)(w)}+\lambda\left(1+\frac{w\left(g*I_\theta\right)''(w)}{\left(g*I_\theta\right)'(w)}\right)\right]$$

$$+\delta\frac{\lambda w^2\left(g*I_\theta\right)''(w)+w\left(g*I_\theta\right)'(w)}{\lambda w\left(g*I_\theta\right)'(w)+(1-\lambda)\left(g*I_\theta\right)(w)}=1+h_1(r)+h_2(r)v(w)+h_3(r)v^2(w)+\cdots. \tag{2.4}$$

Combining (2.1), (2.2), (2.3) and (2.4) yields

$$(1-\delta)\left[(1-\lambda)\frac{z\left(f*I_\theta\right)'(z)}{\left(f*I_\theta\right)(z)}+\lambda\left(1+\frac{z\left(f*I_\theta\right)''(z)}{\left(f*I_\theta\right)'(z)}\right)\right]$$
$$+\delta\frac{\lambda z^2\left(f*I_\theta\right)''(z)+z\left(f*I_\theta\right)'(z)}{\lambda z\left(f*I_\theta\right)'(z)+(1-\lambda)\left(f*I_\theta\right)(z)}=1+h_2(r)u_1z+\left[h_2(r)u_2+h_3(r)u_1^2\right]z^2+\cdots \tag{2.5}$$

and

$$(1-\delta)\left[(1-\lambda)\frac{w\left(g*I_\theta\right)'(w)}{\left(g*I_\theta\right)(w)}+\lambda\left(1+\frac{w\left(g*I_\theta\right)''(w)}{\left(g*I_\theta\right)'(w)}\right)\right]$$
$$+\delta\frac{\lambda w^2\left(g*I_\theta\right)''(w)+w\left(g*I_\theta\right)'(w)}{\lambda w\left(g*I_\theta\right)'(w)+(1-\lambda)\left(g*I_\theta\right)(w)}=1+h_2(r)v_1w+\left[h_2(r)v_2+h_3(r)v_1^2\right]w^2+\cdots. \tag{2.6}$$

It is quite well-known that if $|u(z)|<1$ and $|v(w)|<1$, $z,w\in U$, then

$$|u_i|\leq 1 \quad and \quad |v_i|\leq 1\ forall\ i\in\mathbb{N}. \tag{2.7}$$

Comparing the corresponding coefficients in (2.5) and (2.6), after simplifying, we have

$$2(1-\theta)(\lambda+1)a_2=h_2(r)u_1, \tag{2.8}$$

$$2(1-\theta)(3-2\theta)(2\lambda+1)a_3-4\left(1-\theta\right)^2\left(\lambda\delta(\lambda-1)+3\lambda+1\right)a_2^2$$
$$=h_2(r)u_2+h_3(r)u_1^2, \tag{2.9}$$

$$-2(1-\theta)(\lambda+1)a_2=h_2(r)v_1 \tag{2.10}$$

and

$$2(1-\theta)(3-2\theta)(2\lambda+1)(2a_2^2-a_3)-4\left(1-\theta\right)^2\left(\lambda\delta(\lambda-1)+3\lambda+1\right)a_2^2$$
$$=h_2(r)v_2+h_3(r)v_1^2. \tag{2.11}$$

It follows from (2.8) and (2.10) that

$$u_1=-v_1 \tag{2.12}$$

and

$$8\left(1-\theta\right)^2\left(\lambda+1\right)^2a_2^2=h_2^2(r)(u_1^2+v_1^2). \tag{2.13}$$

If we add (2.9) to (2.11), we find that

$$4(1-\theta)\left[2\lambda\delta(1-\theta)(1-\lambda)+2\theta\lambda+1\right]a_2^2=h_2(r)(u_2+v_2)+h_3(r)(u_1^2+v_1^2). \tag{2.14}$$

Substituting the value of $u_1^2+v_1^2$ from (2.13) in the right hand side of (2.14), we deduce that

$$a_2^2=\frac{h_2^3(r)(u_2+v_2)}{4\left[h_2^2(r)(1-\theta)\left[2\lambda\delta(1-\theta)(1-\lambda)+2\theta\lambda+1\right]-2h_3(r)\left(1-\theta\right)^2\left(\lambda+1\right)^2\right]}. \tag{2.15}$$

If we make further computations using (1.3), (2.7) and (2.15), we obtain

$$|a_2|\leq\frac{|br|\sqrt{|br|}}{\sqrt{2\left|\left[(1-\theta)\left[2\lambda\delta(1-\theta)(1-\lambda)+2\theta\lambda+1\right]b-2p\left(1-\theta\right)^2\left(\lambda+1\right)^2\right]br^2-2qa\left(1-\theta\right)^2\left(\lambda+1\right)^2\right|}}.$$

Next, if we subtract (2.11) from (2.9), we can easily see that

$$4(1-\theta)(3-2\theta)(2\lambda+1)(a_3-a_2^2)=h_2(r)(u_2-v_2)+h_3(r)(u_1^2-v_1^2). \tag{2.16}$$

In view of (2.12) and (2.13), we get from (2.16)

$$a_3=\frac{h_2(r)(u_2-v_2)}{4(1-\theta)(3-2\theta)(2\lambda+1)}+\frac{h_2^2(r)(u_1^2+v_1^2)}{8\left(1-\theta\right)^2\left(\lambda+1\right)^2}.$$

Thus applying (1.3), we obtain

$$|a_3|\leq\frac{|br|}{2(1-\theta)(3-2\theta)(2\lambda+1)}+\frac{b^2r^2}{4\left(1-\theta\right)^2\left(\lambda+1\right)^2}.$$

This completes the proof of Theorem 1. □

In the next theorem, we discuss the Fekete-Szegő problem for the subclass $\mathcal{N}_\Sigma(\delta,\lambda,\theta,r)$.

Theorem 2. *For $\delta\geq 0$, $0\leq\lambda\leq 1$, $0\leq\theta<1$ and $r,\mu\in\mathbb{R}$, let $f\in\mathcal{A}$ be in the class $\mathcal{N}_\Sigma(\delta,\lambda,\theta,r)$. Then $\left|a_3-\mu a_2^2\right|\leq$*

$$\leq\begin{cases}\dfrac{|br|}{2(1-\theta)(3-2\theta)(2\lambda+1)};\\[2ex] \text{for } |\mu-1|\leq\dfrac{\left|\left[(1-\theta)\left[2\lambda\delta(1-\theta)(1-\lambda)+2\theta\lambda+1\right]b-2p\left(1-\theta\right)^2\left(\lambda+1\right)^2\right]br^2-2qa\left(1-\theta\right)^2\left(\lambda+1\right)^2\right|}{b^2r^2(1-\theta)(3-2\theta)(2\lambda+1)},\\[2ex] \dfrac{|br|^3\,|\mu-1|}{\left|\left[(1-\theta)\left[2\lambda\delta(1-\theta)(1-\lambda)+2\theta\lambda+1\right]b-2p\left(1-\theta\right)^2\left(\lambda+1\right)^2\right]br^2-2qa\left(1-\theta\right)^2\left(\lambda+1\right)^2\right|};\\[2ex] \text{for } |\mu-1|\geq\dfrac{\left|\left[(1-\theta)\left[2\lambda\delta(1-\theta)(1-\lambda)+2\theta\lambda+1\right]b-2p\left(1-\theta\right)^2\left(\lambda+1\right)^2\right]br^2-2qa\left(1-\theta\right)^2\left(\lambda+1\right)^2\right|}{b^2r^2(1-\theta)(3-2\theta)(2\lambda+1)}.\end{cases}$$

Proof. It follows from (2.15) and (2.16) that

$$\begin{aligned}a_3-\mu a_2^2&=\frac{h_2(r)(u_2-v_2)}{4(1-\theta)(3-2\theta)(2\lambda+1)}+(1-\mu)\,a_2^2\\&=\frac{h_2(r)(u_2-v_2)}{4(1-\theta)(3-2\theta)(2\lambda+1)}\end{aligned}$$

$$+\frac{h_2^3(r)(u_2+v_2)(1-\mu)}{4\left[h_2^2(r)(1-\theta)[2\lambda\delta(1-\theta)(1-\lambda)+2\theta\lambda+1]-2h_3(r)(1-\theta)^2(\lambda+1)^2\right]}$$
$$=\frac{h_2(r)}{4}\left[\left(\psi(\mu,r)+\frac{1}{(1-\theta)(3-2\theta)(2\lambda+1)}\right)u_2\right.$$
$$\left.+\left(\psi(\mu,r)-\frac{1}{(1-\theta)(3-2\theta)(2\lambda+1)}\right)v_2\right],$$

where

$$\psi(\mu,r)=\frac{h_2^2(r)(1-\mu)}{h_2^2(r)(1-\theta)[2\lambda\delta(1-\theta)(1-\lambda)+2\theta\lambda+1]-2h_3(r)(1-\theta)^2(\lambda+1)^2}.$$

According to (1.3), we find that

$$\left|a_3-\mu a_2^2\right|\leq\begin{cases}\dfrac{|br|}{2(1-\theta)(3-2\theta)(2\lambda+1)}, & 0\leq|\psi(\mu,r)|\leq\dfrac{1}{(1-\theta)(3-2\theta)(2\lambda+1)},\\[2ex] \dfrac{1}{2}|br||\psi(\mu,r)|, & |\psi(\mu,r)|\geq\dfrac{1}{(1-\theta)(3-2\theta)(2\lambda+1)}.\end{cases}$$

After some computations, we obtain $\left|a_3-\mu a_2^2\right|\leq$

$$\leq\begin{cases}\dfrac{|br|}{2(1-\theta)(3-2\theta)(2\lambda+1)};\ for\ |\mu-1|\leq\\[2ex] \leq\dfrac{\left|\left[(1-\theta)[2\lambda\delta(1-\theta)(1-\lambda)+2\theta\lambda+1]b-2p(1-\theta)^2(\lambda+1)^2\right]br^2-2qa(1-\theta)^2(\lambda+1)^2\right|}{b^2r^2(1-\theta)(3-2\theta)(2\lambda+1)},\\[2ex] \dfrac{|br|^3|\mu-1|}{\left|\left[(1-\theta)[2\lambda\delta(1-\theta)(1-\lambda)+2\theta\lambda+1]b-2p(1-\theta)^2(\lambda+1)^2\right]br^2-2qa(1-\theta)^2(\lambda+1)^2\right|};\\[2ex] for\ |\mu-1|\geq\\[2ex] \geq\dfrac{\left|\left[(1-\theta)[2\lambda\delta(1-\theta)(1-\lambda)+2\theta\lambda+1]b-2p(1-\theta)^2(\lambda+1)^2\right]br^2-2qa(1-\theta)^2(\lambda+1)^2\right|}{b^2r^2(1-\theta)(3-2\theta)(2\lambda+1)}.\end{cases}$$

□

Putting $\mu=1$ in Theorem 2, we obtain the following result:

Corollary 1. *For $\delta\geq0$, $0\leq\lambda\leq1$, $0\leq\theta<1$ and $r\in\mathbb{R}$, let $f\in\mathcal{A}$ be in the class $\mathcal{N}_\Sigma(\delta,\lambda,\theta,r)$. Then*

$$\left|a_3-a_2^2\right|\leq\frac{|br|}{2(1-\theta)(3-2\theta)(2\lambda+1)}.$$

Remark 2. Special cases are shown below:

(1) If we put $\lambda = 0$ and $\theta = \frac{1}{2}$ in our Theorems, we have the results for well-known class $S^*_\Sigma(r)$ of bi-starlike functions which was studied recently by Srivastava et al. [16].
(2) If we put $\lambda = 1$ and $\theta = \frac{1}{2}$ in our Theorems, we have the results for the class $\mathcal{K}_\Sigma(r)$ which was considered recently by Magesh et al. [12].
(3) If we put $\delta = 0$ and $\theta = \frac{1}{2}$ in our Theorems, we have the results for the class $M_\Sigma(\lambda, r)$ which was investigated recently by Magesh et al. [12].
(4) If we put $\lambda = 0$, $\theta = \frac{1}{2}$, $a = 1$, $b = p = 2$, $q = -1$ and $r \longrightarrow t$ in our Theorems, we obtain the results for the class $S^*_\Sigma(t)$ of bi-starlike functions which was introduced recently by Altınkaya and Yalçin [3].
(5) If we put $\delta = 0$, $\theta = \frac{1}{2}$, $a = 1$, $b = p = 2$, $q = -1$ and $r \longrightarrow t$ in our Theorems, we have the results for the class $\mathcal{M}^*_\Sigma(\alpha, t)$ which was considered recently by Altınkaya and Yalçin [2].

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CURVATURE PROPERTIES OF PROJECTIVE SEMI-SYMMETRIC LINEAR CONNECTIONS

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Abstract. We study a projective semi-symmetric linear connection on a differentiable manifold $\mathcal{M}$ endowed with a Riemannian metric g. We start with linearly independent curvature tensors $\underset{\theta}{R}$, $\theta = 0,1,\dots,5$ and derive the tensors $\underset{\theta}{W}$ for $\theta = 0,1,\dots,5$ that, as we show, coincide with the Weyl tensor of projective curvature W^g. This confirms the well-known fact that there does not exist a generalization of the Weyl projective curvature tensor W^g.

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1. Introduction

Let $\mathcal{M}$ be a differentiable manifold of dimension $n > 2$ endowed with a Riemannian metric g. Let us consider a linear connection $\overset{1}{\nabla}$ whose torsion tensor is given by

$$\overset{1}{T}(X,Y) = \overset{1}{\nabla}_X Y - \overset{1}{\nabla}_Y X - [X,Y].$$

For a linear connection $\overset{1}{\nabla}$ on a differentiable manifold $\mathcal{M}$ there exists a unique *dual (conjugate) linear connection* $\overset{2}{\nabla}$ given by

$$\overset{2}{\nabla}_X Y = \overset{1}{\nabla}_Y X + [X,Y], \tag{1.1}$$

where X and Y are smooth vector fields on $\mathcal{M}$. The last equation is equivalent with the equation

$$\overset{2}{\nabla}_X Y = \overset{1}{\nabla}_X Y - \overset{1}{T}(X,Y).$$

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By virtue of the connection $\overset{1}{\nabla}$ and its dual $\overset{2}{\nabla}$, one can define a torsion-free (symmetric) linear connection (see for instance [16])

$$\overset{0}{\nabla}_X Y = \frac{1}{2}(\overset{1}{\nabla}_X Y + \overset{2}{\nabla}_X Y).$$

Consequently, the torsion-free linear connection $\overset{0}{\nabla}$ can be expressed via linear connections $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$, respectively, and the torsion tensor $\overset{1}{T}$ as follows:

$$\overset{0}{\nabla}_X Y = \overset{1}{\nabla}_X Y - \frac{1}{2}\overset{1}{T}(X,Y), \qquad \overset{0}{\nabla}_X Y = \overset{2}{\nabla}_X Y + \frac{1}{2}\overset{1}{T}(X,Y).$$

Let ∇^g stands for the Levi-Civita connection of metric g and R^g denotes corresponding Riemannian curvature tensor given by

$$R^g(X,Y)Z = \nabla^g_X \nabla^g_Y Z - \nabla^g_Y \nabla^g_X Z - \nabla^g_{[X,Y]} Z.$$

Let us note that in the available literature the linear connections $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$ are sometimes denoted by $\nabla^{\pm}$. Since these linear connections have non-vanishing torsion tensors the following two curvature tensors can defined

$$\underset{1}{R}(X,Y)Z = \overset{1}{\nabla}_X \overset{1}{\nabla}_Y Z - \overset{1}{\nabla}_Y \overset{1}{\nabla}_X Z - \overset{1}{\nabla}_{[X,Y]} Z, \tag{1.2}$$

$$\underset{2}{R}(X,Y)Z = \overset{2}{\nabla}_X \overset{2}{\nabla}_Y Z - \overset{2}{\nabla}_Y \overset{2}{\nabla}_X Z - \overset{2}{\nabla}_{[X,Y]} Z. \tag{1.3}$$

U. P. Singh in [26] introduced the third curvature tensor given by

$$\underset{3}{R}(X,Y)Z = \overset{2}{\nabla}_X \overset{1}{\nabla}_Y Z - \overset{1}{\nabla}_Y \overset{2}{\nabla}_X Z + \overset{2}{\nabla}_{\overset{1}{\nabla}_Y X} Z - \overset{1}{\nabla}_{\overset{2}{\nabla}_X Y} Z. \tag{1.4}$$

M. Prvanović in [24] proved the existence of the fourth curvature tensor

$$\underset{4}{R}(X,Y)Z = \overset{2}{\nabla}_X \overset{1}{\nabla}_Y Z - \overset{1}{\nabla}_Y \overset{2}{\nabla}_X Z + \overset{2}{\nabla}_{\overset{2}{\nabla}_Y X} Z - \overset{1}{\nabla}_{\overset{1}{\nabla}_X Y} Z. \tag{1.5}$$

As shown in [15] there exist six linearly independent curvature tensors, while the rest can be expressed as linear combinations of these six curvature tensors. We will consider the following six linearly independent curvature tensors $\underset{1}{R}, \underset{2}{R}, \underset{3}{R}, \underset{4}{R}$ given by (1.2 - 1.5) and $\underset{0}{R}$ and $\underset{5}{R}$ that are respectively given by

$$\underset{0}{R}(X,Y)Z = \overset{0}{\nabla}_X \overset{0}{\nabla}_Y Z - \overset{0}{\nabla}_Y \overset{0}{\nabla}_X Z - \overset{0}{\nabla}_{[X,Y]} Z \tag{1.6}$$

and

$$\underset{5}{R}(X,Y)Z = \frac{1}{2}\big(\overset{1}{\nabla}_X \overset{1}{\nabla}_Y Z - \overset{2}{\nabla}_Y \overset{1}{\nabla}_X Z + \overset{2}{\nabla}_X \overset{2}{\nabla}_Y Z - \overset{1}{\nabla}_Y \overset{2}{\nabla}_X Z - \overset{1}{\nabla}_{[X,Y]} Z - \overset{2}{\nabla}_{[X,Y]} Z\big). \tag{1.7}$$

The curvature tensor $\underset{1}{R}$ determined only by the linear connection $\overset{1}{\nabla}$ with torsion $\overset{1}{T}$ has been widely studied in many papers devoted to investigation of linear connections with torsion. Linear connections with torsion are particularly important in study of Kähler, para-Kähler, Hermitian and para-Hermitian geometries and because of their applications in string theory (see for instance S. Ivanov [9–11]).

Let us consider a local coordinate frame $\{\frac{\partial}{\partial x_s}, s = 1,\dots,dim(\mathcal{M})\}$ and set

$$X = \frac{\partial}{\partial x_i}, \quad Y = \frac{\partial}{\partial x_j}, \quad Z = \frac{\partial}{\partial x_k}.$$

In the following sections, we will express the curvature tensors $\underset{\theta}{R}$, $\theta = 0,1,\dots,5$, in terms of the Riemannian curvature tensor R^g and the projective semi-symmetric connection.

As is well-known the curvature tensor $\underset{0}{R}$ is skew-symmetric with respect to the first two arguments, and satisfies the first Bianchi identity:

$$\underset{0}{R}(X,Y)Z = -\underset{0}{R}(Y,X)Z, \qquad \sigma_{XYZ}\underset{0}{R}(X,Y)Z = 0.$$

From the definitions of curvature tensors (1.2 - 1.7) one can conclude that [16]

$$\underset{\theta}{R}(X,Y)Z = -\underset{\theta}{R}(Y,X)Z, \quad \theta = 1,2, \quad \text{and} \quad \sigma_{XYZ}\underset{\theta}{R}(X,Y)Z = 0, \quad \theta = 4,5.$$

The (0,4) curvature tensors corresponding to $\underset{\theta}{R}(X,Y)Z$, $\theta = 0,1,\dots,5$, and $R^g(X,Y)Z$ are respectively given by

$$\underset{\theta}{R}(X,Y,Z,W) := g(\underset{\theta}{R}(X,Y)Z,W), \qquad \theta = 0,1,\dots,5$$

and

$$R^g(X,Y,Z,W) := g(R^g(X,Y)Z,W).$$

The corresponding Ricci tensors are given by

$$\underset{\theta}{Ric}(Y,Z) := \text{trace}[X \to \underset{\theta}{R}(X,Y)Z] \qquad \theta = 0,1,\dots,5$$

and

$$Ric^g(Y,Z) := \text{trace}[X \to R^g(X,Y)Z].$$

The scalar curvatures are defined as the trace of $\underset{\theta}{Ric}(Y,Z)$, $\theta = 0,1,\dots,5$, and $Ric^g(Y,Z)$, and will be denoted by $\underset{\theta}{s}$, $\theta = 0,1,\dots,5$, and s^g, respectively.

The Weyl projective curvature tensor W^g of a Riemannian space $(\mathcal{M},g)$ given by

$$W^g(X,Y)Z = R^g(X,Y)Z + \frac{1}{n-1}\big(Ric^g(X,Z)Y - Ric^g(Y,Z)X\big), \tag{1.8}$$

is invariant with respect to a geodesic mapping between Riemannian spaces (see for instance [13, 14]). If we consider a geodesic mapping between some special manifolds with non-symmetric linear connection one can prove that there exist analogous expressions of the Weyl projective curvature tensor with respect to the curvature tensors $\underset{\theta}{R}(X,Y)Z$, $\theta = 0,1,\ldots,5$, but some assumptions are necessary [21]. Therefore, there does not exist any generalization of the Weyl projective curvature tensor, but the tensors that have a similar role might be useful as replacements in some applications. The aim of the present paper is to derive such tensors with respect to the metric semi-symmetric linear connection. Some authors thoroughly studied semi-symmetric linear connections, see for instance [3–5, 8, 19, 20, 23, 25, 32, 33, 35]. Recently, some applications of semi-symmetric metric connection in the theories of gravity and electromagnetism has been shown in [6]. As mentioned in [7], Y. X. Liang [12] proved that the projective curvature tensor of semi-symmetric metric connection coincides with the Weyl projective curvature tensor of Levi-Civita connection if and only if the characteristic vector is proportional to the Riemannian metric g. P. Zhao and H. Song in [36] introduced the projective semi-symmetric connection and obtained the invariant under transformation of the projective semi-symmetric connection. In [1] S. K. Chaubey et. al. studied projective semi-symmetric connection with a parallel unit vector field and proved that the projective curvature tensor of this connection coincides with the Weyl projective curvature tensor of Levi-Civita connection. In [7] Y. Han et. al. considered the projective transformation of the quarter-symmetric metric connection and found some invariants under that transformation.

2. Projective Semi-Symmetric Linear Connections

Let $(\mathcal{M},g)$ be a Riemannian space of dimension $n>2$ and $\overset{1}{\nabla}$ be a *projective semi-symmetric linear connection* on $\mathcal{M}$ given by [34, 36]

$$\overset{1}{\nabla}_X Y = \nabla^g_X Y + \psi(Y)X + \psi(X)Y + \sigma(Y)X - \sigma(X)Y, \tag{2.1}$$

where ∇^g is the Levi-Civita connection of metric g, whereas ψ and σ are 1-forms given by

$$\psi(X) = \frac{n-1}{2(n+1)}\pi(X), \qquad \sigma(X) = \frac{1}{2}\pi(X).$$

From (2.1) we have that the (symmetric) linear connection $\overset{0}{\nabla}$ is related with the Levi-Civita connection ∇^g by [14]

$$\overset{0}{\nabla}_X Y = \nabla^g_X Y + \psi(Y)X + \psi(X)Y.$$

As is well-known a linear connection $\overset{2}{\nabla}$ which is dual to $\overset{1}{\nabla}$ is given by (1.1) and via linear connections $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$ one can determine the curvature tensors $\underset{\theta}{R}(X,Y)Z$,

$\theta = 0,1,\ldots,5$ that are related with the Riemannian curvature tensor $R^g(X,Y)Z$ as in Proposition 1.

Proposition 1. *Let $(\mathcal{M},g)$ be a Riemannian space of dimension $n>2$ and $\overset{1}{\nabla}$ be a projective semi-symmetric linear connection given by (2.1). The curvature tensors $\underset{\theta}{R}(X,Y)Z$, $\theta = 0,1,2,\ldots,5$ are related with the curvature tensor $R^g(X,Y)Z$ by:*

$$\begin{aligned}\underset{0}{R}(X,Y)Z &= R^g(X,Y)Z + \frac{n-1}{2(n+1)}\big((\nabla^g_X\pi)(Y) - (\nabla^g_Y\pi)(X)\big)Z \\ &\quad + \frac{n-1}{2(n+1)}\big((\nabla^g_X\pi)(Z)Y - (\nabla^g_Y\pi)(Z)X\big) \\ &\quad + \frac{(n-1)^2}{4(n+1)^2}\pi(Z)\big(\pi(Y)X - \pi(X)Y\big),\end{aligned} \tag{2.2}$$

$$\begin{aligned}\underset{1}{R}(X,Y)Z &= R^g(X,Y)Z - \frac{1}{n+1}\big((\nabla^g_X\pi)(Y) - (\nabla^g_Y\pi)(X)\big)Z \\ &\quad + \frac{n}{n+1}\big((\nabla^g_X\pi)(Z)Y - (\nabla^g_Y\pi)(Z)X\big) \\ &\quad + \frac{n^2}{(n+1)^2}\pi(Z)\big(\pi(Y)X - \pi(X)Y\big),\end{aligned} \tag{2.3}$$

$$\begin{aligned}\underset{2}{R}(X,Y)Z &= R^g(X,Y)Z + \frac{n}{n+1}\big((\nabla^g_X\pi)(Y) - (\nabla^g_Y\pi)(X)\big)Z \\ &\quad - \frac{1}{n+1}\big((\nabla^g_X\pi)(Z)Y - (\nabla^g_Y\pi)(Z)X\big) \\ &\quad + \frac{1}{(n+1)^2}\pi(Z)\big(\pi(Y)X - \pi(X)Y\big),\end{aligned} \tag{2.4}$$

$$\begin{aligned}\underset{3}{R}(X,Y)Z &= R^g(X,Y)Z + \frac{n}{n+1}\big((\nabla^g_X\pi)(Z)Y - (\nabla^g_Y\pi)(X)Z\big) \\ &\quad + \frac{1}{n+1}\big((\nabla^g_Y\pi)(Z)X - (\nabla^g_X\pi)(Y)Z\big) - \frac{n^2}{(n+1)^2}\pi(X)\pi(Z)Y \\ &\quad + \frac{1}{(n+1)^2}\pi(Y)\pi(Z)X + \frac{n-1}{n+1}\pi(Y)\pi(X)Z,\end{aligned} \tag{2.5}$$

$$\begin{aligned}\underset{4}{R}(X,Y)Z &= R^g(X,Y)Z + \frac{n}{n+1}\big((\nabla^g_X\pi)(Z)Y - (\nabla^g_Y\pi)(X)Z\big) \\ &\quad + \frac{1}{n+1}\big((\nabla^g_Y\pi)(Z)X - (\nabla^g_X\pi)(Y)Z\big) + \frac{2n+1}{(n+1)^2}\pi(X)\pi(Z)Y \\ &\quad - \frac{n^2+2n}{(n+1)^2}\pi(Y)\pi(Z)X + \frac{n-1}{n+1}\pi(Y)\pi(X)Z,\end{aligned} \tag{2.6}$$

$$\underset{5}{R}(X,Y)Z = R^g(X,Y)Z + \frac{n-1}{2(n+1)}\big((\nabla^g_X\pi)(Y) - (\nabla^g_Y\pi)(X)\big)Z$$

$$
\begin{aligned}
&+\frac{n-1}{2(n+1)}\big((\nabla^g_X\pi)(Z)Y-(\nabla^g_Y\pi)(Z)X\big) \qquad (2.7)\\
&+\frac{(n-1)^2}{4(n+1)^2}\pi(Z)\big(\pi(Y)X-\pi(X)Y\big)\\
&-\frac{1}{4}\pi(Z)\big(\pi(Y)X+\pi(X)Y\big)+\frac{1}{2}\pi(X)\pi(Y)Z.
\end{aligned}
$$

By taking the trace with respect to the vector field X in the equations (2.2 - 2.7) from Proposition 1. we obtain the corresponding relations between the Ricci tensors $\underset{\theta}{Ric}$, $\theta=0,1,\dots,5$ and Ric^g.

Proposition 2. *Let $(\mathcal{M},g)$ be a Riemannian space of dimension $n>2$ and $\overset{1}{\nabla}$ be a projective semi-symmetric linear connection given by (2.1). The Ricci tensors $\underset{\theta}{Ric}(Y,Z)$, $\theta=0,1,2,\dots,5$ are related with the Ricci tensor $Ric^g(Y,Z)$ by:*

$$
\begin{aligned}
\underset{0}{Ric}(Y,Z)&=Ric^g(Y,Z)-\frac{n(n-1)}{2(n+1)}(\nabla^g_Y\pi)(Z)+\frac{n-1}{2(n+1)}(\nabla^g_Z\pi)(Y)\\
&\quad+\frac{(n-1)^3}{4(n+1)^2}\pi(Y)\pi(Z),\\
\underset{1}{Ric}(Y,Z)&=Ric^g(Y,Z)-\frac{n^2-n-1}{n+1}(\nabla^g_Y\pi)(Z)-\frac{1}{n+1}(\nabla^g_Z\pi)(Y)\\
&\quad+\frac{n^2(n-1)}{(n+1)^2}\pi(Y)\pi(Z),\\
\underset{2}{Ric}(Y,Z)&=Ric^g(Y,Z)-\frac{1}{n+1}(\nabla^g_Y\pi)(Z)+\frac{n}{n+1}(\nabla^g_Z\pi)(Y)\\
&\quad+\frac{n-1}{(n+1)^2}\pi(Y)\pi(Z),\\
\underset{3}{Ric}(Y,Z)&=Ric^g(Y,Z)+\frac{n}{n+1}(\nabla^g_Y\pi)(Z)-\frac{1}{n+1}(\nabla^g_Z\pi)(Y)\\
&\quad+\frac{n-1}{(n+1)^2}\pi(Y)\pi(Z),\\
\underset{4}{Ric}(Y,Z)&=Ric^g(Y,Z)+\frac{n}{n+1}(\nabla^g_Y\pi)(Z)-\frac{1}{n+1}(\nabla^g_Z\pi)(Y)\\
&\quad-\frac{n(n^2+n-2)}{(n+1)^2}\pi(Y)\pi(Z),\\
\underset{5}{Ric}(Y,Z)&=Ric^g(Y,Z)-\frac{n(n-1)}{2(n+1)}(\nabla^g_Y\pi)(Z)+\frac{n-1}{2(n+1)}(\nabla^g_Z\pi)(Y)\\
&\quad+\frac{n(n-1)}{(n+1)^2}\pi(Y)\pi(Z).
\end{aligned}
$$

Symmetry properties of Ricci tensors $\underset{\theta}{Ric}$, $\theta = 0,1,2,\dots,5$ are related with closedness of the 1-form π as given in Proposition 3.

Proposition 3 (see [22, Theorem 3.6]). *Let $(\mathcal{M},g)$ be a Riemannian space of dimension $n > 2$ and $\overset{1}{\nabla}$ be a projective semi-symmetric linear connection given by (2.1) then for arbitrary $\theta \in \{0,1,2,\dots,5\}$ the Ricci tensor $\underset{\theta}{Ric}$ is symmetric if and only if the 1-form π is closed.*

Starting with a projective semi-symmetric linear connection $\overset{1}{\nabla}$ given by (2.1) we will apply the method for deriving the Weyl projective curvature tensors on the linearly independent curvature tensor $\underset{\theta}{R}(X,Y)Z$ for $\theta = 0,1,\dots,5$ to derive the tensors $\underset{\theta}{W}(X,Y)Z$, $\theta = 0,1,\dots,5$ corresponding to the projective semi-symmetric linear connection $\overset{1}{\nabla}$ and its dual connection $\overset{2}{\nabla}$.

2.1. *Curvature tensor of the zero kind*

The relation (2.2) of Proposition 1 can be written as

$$\underset{0}{R}(X,Y)Z = R^g(X,Y)Z + \underset{0}{\alpha}(X,Z)Y - \underset{0}{\alpha}(Y,Z)X + \underset{0}{\beta}(X,Y)Z, \tag{2.8}$$

where

$$\underset{0}{\alpha}(X,Y) = \frac{n-1}{2(n+1)}(\nabla^g_X\pi)(Y) - \frac{(n-1)^2}{4(n+1)^2}\pi(X)\pi(Y)$$

and

$$\underset{0}{\beta}(X,Y) = \underset{0}{\alpha}(X,Y) - \underset{0}{\alpha}(Y,X).$$

By taking the trace with respect to the vector field X in (2.8) we obtain

$$\underset{0}{Ric}(Y,Z) = Ric^g(Y,Z) - (n-1)\underset{0}{\alpha}(Y,Z) + \underset{0}{\beta}(Z,Y). \tag{2.9}$$

After skew-symmetrization in the last relation with respect to the vector fields Y and Z we obtain

$$\underset{0}{Ric}(Y,Z) - \underset{0}{Ric}(Z,Y) = -(n-1)\underset{0}{\beta}(Y,Z) + 2\underset{0}{\beta}(Z,Y),$$

i.e.

$$\underset{0}{\beta}(Z,Y) = \frac{1}{n+1}(\underset{0}{Ric}(Y,Z) - \underset{0}{Ric}(Z,Y)), \tag{2.10}$$

which after plugging into (2.9) yields

$$\underset{0}{\alpha}(Y,Z) = -\frac{n}{n^2-1}\underset{0}{Ric}(Y,Z) - \frac{1}{n^2-1}\underset{0}{Ric}(Z,Y) + \frac{1}{n-1}Ric^g(Y,Z). \tag{2.11}$$

Now, by using (2.10) and (2.11) into (2.8) we obtain

$$\underset{0}{W}(X,Y)Z = W^g(X,Y)Z,$$

where

$$\begin{aligned}\underset{0}{W}(X,Y)Z =& \underset{0}{R}(X,Y)Z + \frac{n}{n^2-1}\big(\underset{0}{Ric}(X,Z)Y - \underset{0}{Ric}(Y,Z)X\big)\\ &+ \frac{1}{n^2-1}\big(\underset{0}{Ric}(Z,X)Y - \underset{0}{Ric}(Z,Y)X\big)\\ &+ \frac{1}{n+1}\big(\underset{0}{Ric}(X,Y) - \underset{0}{Ric}(Y,X)\big)Z\end{aligned} \tag{2.12}$$

and $W^g(X,Y)Z$ is the Weyl projective curvature tensor given by (1.8). The tensor $\underset{0}{W}(X,Y)Z$ given by (2.12) is independent of a 1-form π and it is known as the Weyl projective curvature tensor of symmetric linear connection (for example see [14]).

Proposition 4. *Let $(\mathcal{M},g)$ be a Riemannian space of dimension $n>2$, ∇^g be the Levi-Civita connection of g and $\overset{1}{\nabla}$ be a projective semi-symmetric linear connection given by (2.1), $\underset{0}{W}(X,Y)Z$ be the Weyl projective curvature tensor given by (2.12) and $W^g(X,Y)Z$ be the Weyl projective curvature tensor given by (1.8) then:*

(*i*) $\underset{0}{W}(X,Y)Z = W^g(X,Y)Z$,

(*ii*) *If the Ricci tensor $\underset{0}{Ric}(X,Y)$ vanishes then $\underset{0}{W}(X,Y)Z = \underset{0}{R}(X,Y)Z$,*

(*iii*) *If the Ricci tensor $\underset{0}{Ric}(X,Y)$ is symmetric then the tensor $\underset{0}{W}(X,Y)Z$ given by (2.12) takes form*

$$\underset{0}{W}(X,Y)Z = \underset{0}{R}(X,Y)Z + \frac{1}{n-1}\big(\underset{0}{Ric}(X,Z)Y - \underset{0}{Ric}(Y,Z)X\big).$$

2.2. *Curvature tensor of the first kind*

The relation (2.3) of Proposition 1 can be written as

$$\underset{1}{R}(X,Y)Z = R^g(X,Y)Z + \underset{1}{\alpha}(X,Z)Y - \underset{1}{\alpha}(Y,Z)X + \underset{1}{\beta}(X,Y)Z \tag{2.13}$$

where $\underset{1}{\alpha}$ and $\underset{1}{\beta}$ are bilinear forms given by

$$\underset{1}{\alpha}(X,Y) = \frac{n}{n+1}(\nabla^g_X\pi)(Y) - \frac{n^2}{(n+1)^2}\pi(X)\pi(Y)$$

and

$$\underset{1}{\beta}(X,Y) = \frac{1}{n+1}\big((\nabla^g_Y\pi)(X) - (\nabla^g_X\pi)(Y)\big) = \frac{1}{n}\big(\underset{1}{\alpha}(Y,X) - \underset{1}{\alpha}(X,Y)\big). \tag{2.14}$$

Obviously, $\underset{1}{\beta}(X,Y)$ is a skew-symmetric bilinear form, i.e. $\underset{1}{\beta}(X,Y) = -\underset{1}{\beta}(Y,X)$ which will be used in what follows.

By taking the trace with respect to the vector field X in the equation (2.13) we get

$$\underset{1}{Ric}(Y,Z) = Ric^g(Y,Z) - (n-1)\underset{1}{\alpha}(Y,Z) + \underset{1}{\beta}(Z,Y), \tag{2.15}$$

which after skew-symmetrization with respect to the vector fields Y and Z reads

$$\underset{1}{Ric}(Y,Z) - \underset{1}{Ric}(Z,Y) = n\underset{1}{\beta}(Z,Y) - n^2\underset{1}{\beta}(Z,Y) + 2\underset{1}{\beta}(Z,Y), \tag{2.16}$$

where we used that the Ricci tensor $Ric^g(Y,Z)$ is symmetric and $\underset{1}{\beta}$ is a skew-symmetric bilinear form given by (2.14).

Further, the equation (2.16) can be rewritten as

$$\underset{1}{Ric}(Y,Z) - \underset{1}{Ric}(Z,Y) = (n^2-n-2)\underset{1}{\beta}(Y,Z), \tag{2.17}$$

which after plugging into (2.15) yields

$$\begin{aligned}(n-1)\underset{1}{\alpha}(Y,Z) =& Ric^g(Y,Z) - \underset{1}{Ric}(Y,Z)\\ &- \frac{1}{(n^2-n-2)}\big(\underset{1}{Ric}(Y,Z) - \underset{1}{Ric}(Z,Y)\big).\end{aligned} \tag{2.18}$$

By replacing (2.17) and (2.18) into (2.13) we get

$$\begin{aligned}\underset{1}{R}(X,Y)Z =& R^g(X,Y)Z + \frac{1}{n-1}\big(Ric^g(X,Z) - \underset{1}{Ric}(X,Z)\big)Y\\ &- \frac{1}{n-1}\big(Ric^g(Y,Z) - \underset{1}{Ric}(Y,Z)\big)X\\ &- \frac{1}{(n-1)(n^2-n-2)}\big(\underset{1}{Ric}(X,Z) - \underset{1}{Ric}(Z,X)\big)Y\\ &+ \frac{1}{(n-1)(n^2-n-2)}\big(\underset{1}{Ric}(Y,Z) - \underset{1}{Ric}(Z,Y)\big)X\\ &+ \frac{1}{n^2-n-2}\big(\underset{1}{Ric}(X,Y) - \underset{1}{Ric}(Y,X)\big)Z\end{aligned}$$

or in a simplified way

$$\underset{1}{W}(X,Y)Z = W^g(X,Y)Z,$$

where

$$\begin{aligned}\underset{1}{W}(X,Y)Z =& \underset{1}{R}(X,Y)Z + \frac{n^2-n-1}{(n-1)(n^2-n-2)}\big(\underset{1}{Ric}(X,Z)Y - \underset{1}{Ric}(Y,Z)X\big)\\ &- \frac{1}{(n-1)(n^2-n-2)}\big(\underset{1}{Ric}(Z,X)Y - \underset{1}{Ric}(Z,Y)X\big)\\ &- \frac{1}{n^2-n-2}\big(\underset{1}{Ric}(X,Y) - \underset{1}{Ric}(Y,X)\big)Z\end{aligned} \tag{2.19}$$

and $W^g(X,Y)Z$ is given by (1.8). We note that the tensor $\underset{1}{W}(X,Y)Z$ given by (2.19) is independent of a 1-form π and it is obtained by P. Zhao and H. Song in [36].

Proposition 5. *Let $(\mathcal{M},g)$ be a Riemannian space of dimension $n>2$, ∇^g be the Levi-Civita connection of g and $\overset{1}{\nabla}$ be a projective semi-symmetric linear connection given by (2.1), $\underset{1}{W}(X,Y)Z$ be the tensor given by (2.19) and $W^g(X,Y)Z$ be the Weyl projective curvature tensor given by (1.8) then:*

(i) $\underset{1}{W}(X,Y)Z = W^g(X,Y)Z$,

(ii) *If the Ricci tensor $\underset{1}{Ric}(X,Y)$ vanishes then $\underset{1}{W}(X,Y)Z = \underset{1}{R}(X,Y)Z$,*

(iii) *If the Ricci tensor $\underset{1}{Ric}(X,Y)$ is symmetric, then the tensor $\underset{1}{W}(X,Y)Z$ given by (2.19) takes form*

$$\underset{1}{W}(X,Y)Z = \underset{1}{R}(X,Y)Z + \frac{1}{n-1}\left(\underset{1}{Ric}(X,Z)Y - \underset{1}{Ric}(Y,Z)X\right).$$

2.3. *Curvature tensor of the second kind*

The relation (2.4) of Proposition 1 can be written as

$$\underset{2}{R}(X,Y)Z = R^g(X,Y)Z - \gamma(X,Z)Y + \gamma(Y,Z)X + n(\gamma(X,Y) - \gamma(Y,X))Z, \tag{2.20}$$

where γ is a bilinear form given by

$$\gamma(X,Y) = \frac{1}{n+1}(\nabla^g_X\pi)(Y) + \frac{1}{(n+1)^2}\pi(X)\pi(Y).$$

Contracting of the vector field X in the equation (2.20) yields the following relation between the corresponding Ricci tensors

$$\underset{2}{Ric}(Y,Z) = Ric^g(Y,Z) - \gamma(Y,Z) + n\gamma(Z,Y). \tag{2.21}$$

On the other side, contracting the vector field Z in the equation (2.20), we have

$$\frac{1}{n^2-1}{}'\underset{2}{R}(X,Y) = \gamma(X,Y) - \gamma(Y,X), \tag{2.22}$$

where we denoted ${}'\underset{2}{R}(X,Y) = \mathrm{trace}[Z \to \underset{2}{R}(X,Y)Z]$ and used $\mathrm{trace}[Z \to R^g(X,Y)Z] = 0$.

If we put $Y = X$ and $Z = Y$ in (2.21) we get

$$\underset{2}{Ric}(X,Y) = Ric^g(X,Y) - \gamma(X,Y) + n\gamma(Y,X). \tag{2.23}$$

Adding (2.22) to (2.23) we obtain

$$\gamma(Y,X) = \frac{1}{n-1}\left(\underset{2}{Ric}(X,Y) - Ric^g(X,Y) + \frac{1}{n^2-1}{}'\underset{2}{R}(X,Y)\right). \tag{2.24}$$

Replacing (2.22) and (2.24) into (2.20) we obtain

$$\underset{2}{W}(X,Y)Z = W^g(X,Y)Z,$$

where

$$\begin{aligned}\underset{2}{W}(X,Y)Z = \underset{2}{R}(X,Y)Z &+ \frac{1}{(n-1)(n^2-1)}\big((n^2-1)\underset{2}{Ric}(Z,X) + {}'\underset{2}{R}(Z,X)\big)Y \\ &- \frac{1}{(n-1)(n^2-1)}\big((n^2-1)\underset{2}{Ric}(Z,Y) + {}'\underset{2}{R}(Z,Y)\big)X \\ &- \frac{n}{n^2-1}{}'\underset{2}{R}(X,Y)Z\end{aligned} \tag{2.25}$$

and $W^g(X,Y)Z$ is given by (1.8). The tensor $\underset{2}{W}(X,Y)Z$ given by (2.25) is independent of a 1-form π.

By skew-symmetrization in the equation (2.21) with respect to the vector fields Y and Z we obtain

$$\underset{2}{Ric}(Y,Z) - \underset{2}{Ric}(Z,Y) = -(n+1)(\gamma(Y,Z) - \gamma(Z,Y))$$

and after comparing it with equation (2.22), we get

$$\underset{2}{Ric}(Y,Z) - \underset{2}{Ric}(Z,Y) = -\frac{1}{n-1}{}'\underset{2}{R}(Y,Z),$$

from which we see that the tensor ${}'\underset{2}{R}(Y,Z)$ is skew-symmetric. Finally, using the above, we can prove the following theorem.

Theorem 1. *Let $(\mathcal{M},g)$ be a Riemannian space of dimension $n>2$, ∇^g be the Levi-Civita connection of g and $\overset{1}{\nabla}$ be a projective semi-symmetric linear connection given by (2.1), $\underset{2}{W}(X,Y)Z$ be the tensor given by (2.25) and $W^g(X,Y)Z$ be the Weyl projective curvature tensor given by (1.8) then:*

(i) $\underset{2}{W}(X,Y)Z = W^g(X,Y)Z$,

(ii) *If the Ricci tensor $\underset{2}{Ric}(X,Y)$ vanishes then $\underset{2}{W}(X,Y)Z = \underset{2}{R}(X,Y)Z$,*

(iii) *If the Ricci tensor $\underset{2}{Ric}(X,Y)$ is symmetric then ${}'\underset{2}{R}(X,Y) = 0$ and the tensor $\underset{2}{W}(X,Y)Z$ given by (2.25) takes form*

$$\underset{2}{W}(X,Y)Z = \underset{2}{R}(X,Y)Z + \frac{1}{n-1}\big(\underset{2}{Ric}(X,Z)Y - \underset{2}{Ric}(Y,Z)X\big).$$

2.4. *Curvature tensor of the third kind*

We start with the equation (2.5) of Proposition 1 which reads

$$\underset{3}{R}(X,Y)Z = R^g(X,Y)Z + \frac{n}{n+1}\big((\nabla^g_X\pi)(Z)Y - (\nabla^g_Y\pi)(X)Z\big)$$

$$+\frac{1}{n+1}\big((\nabla^g_Y\pi)(Z)X-(\nabla^g_X\pi)(Y)Z\big)-\frac{n^2}{(n+1)^2}\pi(X)\pi(Z)Y \tag{2.26}$$
$$+\frac{1}{(n+1)^2}\pi(Y)\pi(Z)X+\frac{n-1}{n+1}\pi(Y)\pi(X)Z.$$

Contracting of the vector field X in the equation (2.26) yields a relation between the corresponding Ricci tensors which is

$$\begin{aligned}\underset{3}{Ric}(Y,Z)=&Ric^g(Y,Z)+\frac{n}{n+1}(\nabla^g_Y\pi)(Z)-\frac{1}{n+1}(\nabla^g_Z\pi)(Y)\\&+\frac{n-1}{(n+1)^2}\pi(Y)\pi(Z).\end{aligned} \tag{2.27}$$

On the other side, contracting of the vector field Z in the equation (2.26), we have

$$\begin{aligned}{}'\underset{3}{R}(X,Y):=&\operatorname{trace}[Z\to\underset{3}{R}(X,Y)Z]\\=&(n-1)\Big(\frac{n-1}{n+1}\pi(X)\pi(Y)-(\nabla^g_Y\pi)(X)\Big).\end{aligned} \tag{2.28}$$

Combining (2.27) and (2.28), we have

$$\begin{aligned}\frac{1}{n+1}(\nabla^g_Y\pi)(Z)=&\frac{1}{n}\Big(\underset{3}{Ric}(Y,Z)-Ric^g(Y,Z)-\frac{1}{n^2-1}{}'\underset{3}{R}(Y,Z)\Big),\\\frac{1}{n+1}\pi(Y)\pi(Z)=&\frac{n+1}{n(n-1)}\underset{3}{Ric}(Y,Z)-\frac{n+1}{n(n-1)}Ric^g(Y,Z)-\frac{1}{n(n-1)^2}{}'\underset{3}{R}(Y,Z)\\&+\frac{1}{(n-1)^2}{}'\underset{3}{R}(Z,Y).\end{aligned}$$

Substituting the last two equalities in (2.26), we obtain

$$\underset{3}{W}(X,Y)Z=W^g(X,Y)Z,$$

where

$$\begin{aligned}\underset{3}{W}(X,Y)Z=&\underset{3}{R}(X,Y)Z+\frac{1}{n-1}\big(\underset{3}{Ric}(X,Z)Y-\underset{3}{Ric}(Y,Z)X\big)\\&-\big(\underset{3}{Ric}(X,Y)-\underset{3}{Ric}(Y,X)\big)Z\\&-\frac{1}{(n^2-1)(n-1)}\big({}'\underset{3}{R}(X,Z)-n^2{}'\underset{3}{R}(Z,X)\big)Y\\&+\frac{1}{(n^2-1)(n-1)}\big({}'\underset{3}{R}(Y,Z)-{}'\underset{3}{R}(Z,Y)\big)X\\&+\frac{1}{n^2-1}\big({}'\underset{3}{R}(X,Y)-(n+2){}'\underset{3}{R}(Y,X)\big)Z\end{aligned} \tag{2.29}$$

and $W^g(X,Y)Z$ is given by (1.8). The tensor $\underset{3}{W}(X,Y)Z$ given by (2.29) is independent of a 1-form π.

By comparing the results obtained by skew-symmetrization in the equation (2.27) and by skew-symmetrization in the equation (2.28), we get

$$\underset{3}{Ric}(Y,Z)-\underset{3}{Ric}(Z,Y)=\frac{1}{n-1}({}'\underset{3}{R}(Y,Z)-{}'\underset{3}{R}(Z,Y)).$$

These conclusions are stated in Theorem 2.

Theorem 2. *Let $(\mathcal{M},g)$ be a Riemannian space of dimension $n>2$, ∇^g be the Levi-Civita connection of g and $\overset{1}{\nabla}$ be a projective semi-symmetric linear connection given by (2.1), $\underset{3}{W}(X,Y)Z$ be the tensor given by (2.29) and $W^g(X,Y)Z$ be the Weyl projective curvature tensor given by (1.8) then:*

(i) $\underset{3}{W}(X,Y)Z=W^g(X,Y)Z$,

(ii) *If the tensors $\underset{3}{Ric}(X,Y)$ and ${}'\underset{3}{Ric}(X,Y)$ vanish then $\underset{3}{W}(X,Y)Z=\underset{3}{R}(X,Y)Z$,*

(iii) *If the Ricci tensor $\underset{3}{Ric}(X,Y)$ is symmetric then the tensor ${}'\underset{3}{R}(X,Y)$ is symmetric and the tensor $\underset{3}{W}(X,Y)Z$ given by (2.29) takes form*

$$\begin{aligned}\underset{3}{W}(X,Y)Z=&\underset{3}{R}(X,Y)Z+\frac{1}{n-1}\big(\underset{3}{Ric}(X,Z)Y-\underset{3}{Ric}(Y,Z)X\big)\\&+\frac{1}{n-1}\big({}'\underset{3}{R}(X,Z)Y-{}'\underset{3}{R}(X,Y)Z\big).\end{aligned}$$

2.5. *Curvature tensor of the fourth kind*

The relation (2.6) of Proposition 1 reads

$$\begin{aligned}\underset{4}{R}(X,Y)Z=&R^g(X,Y)Z+\frac{n}{n+1}\big((\nabla^g_X\pi)(Z)Y-(\nabla^g_Y\pi)(X)Z\big)\\&+\frac{1}{n+1}\big((\nabla^g_Y\pi)(Z)X-(\nabla^g_X\pi)(Y)Z\big)+\frac{2n+1}{(n+1)^2}\pi(X)\pi(Z)Y\\&-\frac{n^2+2n}{(n+1)^2}\pi(Y)\pi(Z)X+\frac{n-1}{n+1}\pi(Y)\pi(X)Z.\end{aligned}\tag{2.30}$$

Contracting the vector field X in the equation (2.30) we obtain the relation

$$\begin{aligned}\underset{4}{Ric}(Y,Z)=&Ric^g(Y,Z)+\frac{n}{n+1}(\nabla^g_Y\pi)(Z)-\frac{1}{n+1}(\nabla^g_Z\pi)(Y)\\&-\frac{n^3+n^2-2n}{(n+1)^2}\pi(Y)\pi(Z).\end{aligned}\tag{2.31}$$

Contracting of the vector field Z in the equation (2.30) we obtain

$${}'\underset{4}{R}(X,Y)=-(n-1)(\nabla^g_Y\pi)(X)+\frac{(n-1)^2}{n+1}\pi(X)\pi(Y),\tag{2.32}$$

where

$${}'\underset{4}{R}(X,Y)=\mathrm{trace}[Z\to\underset{4}{R}(X,Y)Z].$$

From equation (2.32) we have

$$(\nabla^g_Y\pi)(X) = -\frac{1}{n-1}{}'\underset{4}{R}(X,Y) + \frac{n-1}{n+1}\pi(X)\pi(Y) \tag{2.33}$$

and by substituting equation (2.33) into (2.31) we get

$$\begin{aligned}\pi(Y)\pi(Z) = \frac{(n+1)^2}{n^3-1}\Bigg(&Ric^g(Y,Z) - \underset{4}{Ric}(Y,Z)\\ &+\frac{1}{n^2-1}{}'\underset{4}{R}(Y,Z) - \frac{n}{n^2-1}{}'\underset{4}{R}(Z,Y)\Bigg).\end{aligned} \tag{2.34}$$

Combining equations (2.33) and (2.34), we obtain

$$\begin{aligned}(\nabla^g_Y\pi)(Z) =& \frac{n+1}{n^2+n+1}Ric^g(Y,Z) - \frac{n+1}{n^2+n+1}\underset{4}{Ric}(Y,Z) + \frac{1}{n^3-1}{}'\underset{4}{R}(Y,Z)\\ &-\frac{(n+1)^2}{n^3-1}{}'\underset{4}{R}(Z,Y).\end{aligned}$$

Substituting the last two equalities in (2.30) we get

$$\underset{4}{W}(X,Y)Z = W^g(X,Y)Z,$$

where

$$\begin{aligned}\underset{4}{W}(X,Y)Z =& \underset{4}{R}(X,Y)Z + \frac{1}{n-1}\big(\underset{4}{Ric}(X,Z)Y - \underset{4}{Ric}(Y,Z)X\big)\\ &+\frac{n}{n^2+n+1}\big(\underset{4}{Ric}(X,Y) - \underset{4}{Ric}(Y,X)\big)Z\\ &-\frac{1}{(n^2-1)(n-1)}\big({}'\underset{4}{R}(X,Z) - n^2{}'\underset{4}{R}(Z,X)\big)Y\\ &+\frac{1}{(n^2-1)(n-1)}\big({}'\underset{4}{R}(Y,Z) - {}'\underset{4}{R}(Z,Y)\big)X\\ &-\frac{n(n^2+2n+2)}{(n^3-1)(n+1)}{}'\underset{4}{R}(X,Y)Z\\ &-\frac{1}{(n^3-1)(n+1)}{}'\underset{4}{R}(Y,X)Z\end{aligned} \tag{2.35}$$

and $W^g(X,Y)Z$ is given by (1.8). The tensor $\underset{4}{W}(X,Y)Z$ given by (2.35) is independent of a 1-form π.

By skew-symmetrization in the equation (2.34) with respect to the vector fields Y and Z we obtain

$$\underset{4}{Ric}(Y,Z) - \underset{4}{Ric}(Z,Y) = \frac{1}{n-1}\big({}'\underset{4}{R}(Y,Z) - {}'\underset{4}{R}(Z,Y)\big)$$

and if we replace the last equation in (2.35), then the tensor $\underset{4}{W}(X,Y)Z$ can be rewritten as

$$\begin{aligned}\underset{4}{W}(X,Y)Z =& \underset{4}{R}(X,Y)Z + \frac{1}{n-1}\big(\underset{4}{Ric}(X,Z)Y - \underset{4}{Ric}(Y,Z)X\big)\\ &- \frac{1}{n^2-1}\big(n\,{}'\underset{4}{R}(X,Y) + {}'\underset{4}{R}(Y,X)\big)Z\\ &- \frac{1}{(n^2-1)(n-1)}\big({}'\underset{4}{R}(X,Z) - n^2\,{}'\underset{4}{R}(Z,X)\big)Y\\ &+ \frac{1}{(n^2-1)(n-1)}\big({}'\underset{4}{R}(Y,Z) - {}'\underset{4}{R}(Z,Y)\big)X.\end{aligned} \tag{2.36}$$

These conclusions are stated in Theorem 3.

Theorem 3. *Let $(\mathcal{M},g)$ be a Riemannian space of dimension $n>2$, ∇^g be the Levi-Civita connection of g and $\overset{1}{\nabla}$ be a projective semi-symmetric linear connection given by (2.1), $\underset{4}{W}(X,Y)Z$ be the tensor given by (2.36) and $W^g(X,Y)Z$ be the Weyl projective curvature tensor given by (1.8) then:*

- (i) $\underset{4}{W}(X,Y)Z = W^g(X,Y)Z$,
- (ii) *If the tensors $\underset{4}{Ric}(X,Y)$ and ${}'\underset{4}{R}(X,Y)$ vanish, then $\underset{4}{W}(X,Y)Z = \underset{4}{R}(X,Y)Z$,*
- (iii) *If the Ricci tensor $\underset{4}{Ric}(X,Y)$ is symmetric then ${}'\underset{4}{R}(X,Y)$ is symmetric tensor and the tensor $\underset{4}{W}(X,Y)Z$ given by (2.36) takes form*

$$\begin{aligned}\underset{4}{W}(X,Y)Z =& \underset{4}{R}(X,Y)Z + \frac{1}{n-1}\big(\underset{4}{Ric}(X,Z)Y - \underset{4}{Ric}(Y,Z)X\big)\\ &+ \frac{1}{n-1}\big({}'\underset{4}{R}(X,Z)Y - {}'\underset{4}{R}(X,Y)Z\big).\end{aligned}$$

2.6. *Curvature tensor of the fifth kind*

We start with the relation (2.7) of Proposition 1:

$$\begin{aligned}\underset{5}{R}(X,Y)Z =& R^g(X,Y)Z + \frac{n-1}{2(n+1)}\big((\nabla^g_X\pi)(Y) - (\nabla^g_Y\pi)(X)\big)Z\\ &+ \frac{n-1}{2(n+1)}\big((\nabla^g_X\pi)(Z)Y - (\nabla^g_Y\pi)(Z)X\big)\\ &+ \frac{(n-1)^2}{4(n+1)^2}\pi(Z)\big(\pi(Y)X - \pi(X)Y\big)\\ &- \frac{1}{4}\pi(Z)\big(\pi(Y)X + \pi(X)Y\big) + \frac{1}{2}\pi(X)\pi(Y)Z.\end{aligned} \tag{2.37}$$

If we contract the last relation with respect to X, Y and Z, respectively, we obtain

$$\begin{aligned}\underset{5}{Ric}(Y,Z) =& Ric^g(Y,Z) + \frac{n-1}{2(n+1)}(\nabla^g_Z\pi)(Y) - \frac{n(n-1)}{2(n+1)}(\nabla^g_Y\pi)(Z)\\ &- \frac{n(n-1)}{(n+1)^2}\pi(Y)\pi(Z),\end{aligned} \tag{2.38}$$

$$\begin{aligned}{}''\underset{5}{R}(X,Z) =& -Ric^g(X,Z) + \frac{n(n-1)}{2(n+1)}(\nabla^g_X\pi)(Z) - \frac{n-1}{2(n+1)}(\nabla^g_Z\pi)(X)\\ &- \frac{(n-1)(n^2+1)}{2(n+1)^2}\pi(X)\pi(Z)\end{aligned} \tag{2.39}$$

and

$${}'\underset{5}{R}(X,Y) = \frac{n-1}{2}(\nabla^g_X\pi)(Y) - \frac{n-1}{2}(\nabla^g_Y\pi)(X) + \frac{n-1}{2}\pi(X)\pi(Y), \tag{2.40}$$

where ${}''\underset{5}{R}(X,Z) = \mathrm{trace}[Y \to \underset{5}{R}(X,Y)Z]$ and ${}'\underset{5}{R}(X,Y) = \mathrm{trace}[Z \to \underset{5}{R}(X,Y)Z]$.

If we choose $Y = X$ and $Z = Y$ into (2.38) we get

$$\begin{aligned}\underset{5}{Ric}(X,Y) =& Ric^g(X,Y) + \frac{n-1}{2(n+1)}(\nabla^g_Y\pi)(X) - \frac{n(n-1)}{2(n+1)}(\nabla^g_X\pi)(Y)\\ &- \frac{n(n-1)}{(n+1)^2}\pi(X)\pi(Y).\end{aligned} \tag{2.41}$$

On the other hand, by putting $Z = Y$ into (2.39) we obtain

$$\begin{aligned}{}''\underset{5}{R}(X,Y) =& -Ric^g(X,Y) + \frac{n(n-1)}{2(n+1)}(\nabla^g_X\pi)(Y) - \frac{n-1}{2(n+1)}(\nabla^g_Y\pi)(X)\\ &- \frac{(n-1)(n^2+1)}{2(n+1)^2}\pi(X)\pi(Y).\end{aligned} \tag{2.42}$$

Adding the last two relations we obtain

$$\frac{n-1}{2}\pi(X)\pi(Y) = -\underset{5}{Ric}(X,Y) - {}''\underset{5}{R}(X,Y). \tag{2.43}$$

By substituting equation (2.43) into (2.41) and (2.40) we have the next relations

$$\begin{aligned}\frac{n^2+1}{(n+1)^2}\underset{5}{Ric}(X,Y) - \frac{2n}{(n+1)^2}{}''\underset{5}{R}(X,Y) =& Ric^g(X,Y) - \frac{n(n-1)}{2(n+1)}(\nabla^g_X\pi)(Y)\\ &+ \frac{n-1}{2(n+1)}(\nabla^g_Y\pi)(X)\end{aligned} \tag{2.44}$$

and

$$\underset{5}{Ric}(X,Y) + {}''\underset{5}{R}(X,Y) + {}'\underset{5}{R}(X,Y) = \frac{n-1}{2}(\nabla^g_X\pi)(Y) - \frac{n-1}{2}(\nabla^g_Y\pi)(X). \tag{2.45}$$

Combining equations (2.44) and (2.45) we get

$$\begin{aligned}\frac{n-1}{2(n+1)}(\nabla^g_X\pi)(Y) =& \frac{1}{n-1}Ric^g(X,Y) - \frac{n^2+n+2}{(n^2-1)(n+1)}\underset{5}{Ric}(X,Y)\\ &- \frac{1}{n^2-1}{}'\underset{5}{R}(X,Y) + \frac{1}{(n+1)^2}{}''\underset{5}{R}(X,Y).\end{aligned} \tag{2.46}$$

Finally, putting (2.43) and (2.46) into (2.37) we obtain

$$\underset{5}{W}(X,Y)Z = W^g(X,Y)Z,$$

where

$$\begin{aligned}\underset{5}{W}(X,Y)Z =& \underset{5}{R}(X,Y)Z + \frac{1}{n^2-1}\big(\underset{5}{Ric}(X,Z)Y - (n+2)\underset{5}{Ric}(Y,Z)X\big)\\ &+ \frac{1}{(n+1)^2}\big(\underset{5}{Ric}(Y,X) - {}''\underset{5}{R}(X,Y)\big)Z\\ &+ \frac{1}{n^2-1}\big({}'\underset{5}{R}(X,Y) - {}'\underset{5}{R}(Y,X)\big)Z\\ &+ \frac{1}{n^2-1}\big({}'\underset{5}{R}(X,Z)Y - {}'\underset{5}{R}(Y,Z)X\big)\\ &+ \frac{n^2+n+2}{(n^2-1)(n+1)}\underset{5}{Ric}(X,Y)Z\\ &+ \frac{n^2+3n}{(n^2-1)(n+1)}{}''\underset{5}{R}(Y,X)Z\\ &- \frac{1}{n^2-1}\big(n{}''\underset{5}{R}(X,Z)Y + {}''\underset{5}{R}(Y,Z)X\big)\end{aligned} \tag{2.47}$$

and $W^g(X,Y)Z$ is the Weyl projective curvature tensor given by (1.8). The tensor $\underset{5}{W}(X,Y)Z$ given by (2.47) is independent of a 1-form π.

By using skew-symmetrization of equations (2.40), (2.41) and (2.42) we obtain

$$\underset{5}{Ric}(X,Y) - \underset{5}{Ric}(Y,X) = -\frac{1}{2}\big({}'\underset{5}{R}(X,Y) - {}'\underset{5}{R}(Y,X)\big) = -{}''\underset{5}{R}(X,Y) + {}''\underset{5}{R}(Y,X),$$

from which

$$\underset{5}{Ric}(X,Y) - {}''\underset{5}{R}(Y,X) = \underset{5}{Ric}(Y,X) - {}''\underset{5}{R}(X,Y).$$

If we use the last equation then the tensor $\underset{5}{W}(X,Y)Z$ given by (2.47) can be written in the form

$$\begin{aligned}\underset{5}{W}(X,Y)Z =& \underset{5}{R}(X,Y)Z + \frac{1}{n^2-1}\big(\underset{5}{Ric}(X,Z)Y - (n+2)\underset{5}{Ric}(Y,Z)X\big)\\ &+ \frac{1}{n-1}\big(\underset{5}{Ric}(X,Y) + {}''\underset{5}{R}(Y,X)\big)Z\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{n^2-1}\big({}'\underset{5}{R}(X,Y)-{}'\underset{5}{R}(Y,X)\big)Z \\
&+\frac{1}{n^2-1}\big({}'\underset{5}{R}(X,Z)Y-{}'\underset{5}{R}(Y,Z)X\big) \\
&-\frac{1}{n^2-1}\big(n{}''\underset{5}{R}(X,Z)Y+{}''\underset{5}{R}(Y,Z)X\big).
\end{aligned}
\tag{2.48}
$$

The obtained conclusions will be formulated in the following theorem.

Theorem 4. *Let $(\mathcal{M},g)$ be a Riemannian space of dimension $n>2$, ∇^g be the Levi-Civita connection of g and $\overset{1}{\nabla}$ be a projective semi-symmetric linear connection given by (2.1), $\underset{5}{W}(X,Y)Z$ be the tensor given by (2.48) and $W^g(X,Y)Z$ be the Weyl projective curvature tensor given by (1.8) then:*

(i) $\underset{5}{W}(X,Y)Z=W^g(X,Y)Z$,

(ii) *If the tensors $\underset{5}{Ric}(X,Y)$, ${}'\underset{5}{R}(X,Y)$ and ${}''\underset{5}{R}(X,Y)$ vanish then $\underset{5}{W}(X,Y)Z=\underset{5}{R}(X,Y)Z$,*

(iii) *If the Ricci tensor $\underset{5}{Ric}(X,Y)$ is symmetric then the tensors ${}'\underset{5}{R}(X,Y)$, ${}''\underset{5}{R}(X,Y)$ are symmetric and the tensor $\underset{5}{W}(X,Y)Z$ given by (2.48) takes form*

$$
\begin{aligned}
\underset{5}{W}(X,Y)Z=&\underset{5}{R}(X,Y)Z+\frac{1}{n^2-1}\big(\underset{5}{Ric}(X,Z)Y-(n+2)\underset{5}{Ric}(Y,Z)X\big) \\
&+\frac{1}{n-1}\big(\underset{5}{Ric}(X,Y)+{}''\underset{5}{R}(Y,X)\big)Z+\frac{1}{n^2-1}\big({}'\underset{5}{R}(X,Z)Y-{}'\underset{5}{R}(Y,Z)X\big) \\
&-\frac{1}{n^2-1}\big(n{}''\underset{5}{R}(X,Z)Y+{}''\underset{5}{R}(Y,Z)X\big).
\end{aligned}
$$

3. Conclusion

The method for obtaining the Weyl projective curvature tensors did not allow us to make any additional assumptions in order to derive the tensors $\underset{\theta}{W}(X,Y)Z$, $\theta=0,1,\dots,5$. By comparing our results with the results of related papers that deal with projective semi-symmetric linear connections we concluded that our approach which is based on the curvature tensors $\underset{\theta}{R}(X,Y)Z$, for $\theta=3,4,5$ complements the existing results in the current literature. It would be interesting to see some applications of tensors $\underset{\theta}{W}(X,Y)Z$, $\theta=0,1,\dots,5$ in the theories of gravity that includes the torsion tensor. Also, we can conclude that the Riemannian metric g is projectively flat if and only if for arbitrary $\theta\in\{0,1,\dots,5\}$ the tensor $\underset{\theta}{W}(X,Y)Z$ identically vanishes.

Remark 1. In this paper, using the projective semi-symmetric linear connection, we obtained that the corresponding tensors $\underset{\theta}{W},\theta=0,1,\dots,5$, are coincident with the

Weyl projective tensor W^g of the Levi-Civita connection. Also, papers [22, 23, 36] determined tensors coincident with the Weyl projective tensor W^g in a similar way. Papers [1, 2, 12] dealt with similar problems, as they also determined tensors coincident with Weyl projective tensor W^g. If we consider a geodesic mapping between some special manifolds with non-symmetric linear connection one can prove that there exist analogous expressions of the Weyl projective curvature tensor with respect to the curvature tensors $\underset{\theta}{R}(X,Y)Z$, $\theta = 0,1,\ldots,5$, but some assumptions are necessary [21]. In papers [17, 18, 27–31, 37], the authors obtained some geometric objects that are different than the Weyl tensor of projective curvature and invariant with respect to the so-called equitorsion geodesic mapping. These invariant geometric objects were called ET-projective parameters. In case when some of them were tensors, then they were called ET-projective curvature tensors. The relations between the ET-projective parameters and the Weyl projective tensor with respect to the symmetric connection are presented in the paper [28] (in the present paper, the tensor $\underset{0}{W}$ given by (2.12) is the Weyl projective tensor of symmetric connection). Using the idea from that paper, the relations between the tensors $\underset{\theta}{W}$, $\theta = 0,1,\ldots,5$ and the corresponding ET-projective parameters can be presented analogously.

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