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A CLASSIFICATION OF 2-DESIGNS WITH PRIME REPLICATION NUMBERS AND FLAG-TRANSITIVE AUTOMORPHISM GROUPS

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Abstract. In this article, we present a classification of 2-designs with prime replication numbers admitting flag-transitive automorphism groups. In conclusion, if G is a flag-transitive automorphism group of a 2-design $\mathcal{D}$ with prime replication number, then $G \leqslant \mathrm{A\Gamma L}_1(q)$, or $\mathcal{D}$ belongs to five infinite families of 2-designs or it is (isomorphic to) one of eight sporadic examples.

2010 *Mathematics Subject Classification:* 05B05; 05B25; 20B25

Keywords: automorphism group, flag-transitive, point-primitive, 2-design, affine group, almost simple group

1. Introduction

A 2-(v,k,λ) design $\mathcal{D}$ is a pair $(\mathcal{P},\mathcal{B})$ with a set $\mathcal{P}$ of v points and a set $\mathcal{B}$ of blocks such that each block is a k-subset of $\mathcal{P}$ and each two distinct points are contained in λ blocks. We say $\mathcal{D}$ is nontrivial if $2 < k < v-1$, and symmetric if $v = b$, where b is the number of blocks of $\mathcal{D}$. Each point of $\mathcal{D}$ is contained in exactly $r = bk/v$ blocks which is called the *replication number* of $\mathcal{D}$. An *automorphism* of $\mathcal{D}$ is a permutation of $\mathcal{P}$ which leaves $\mathcal{B}$ invariant. The full automorphism group $\mathrm{Aut}(\mathcal{D})$ of $\mathcal{D}$ is the group consisting of all automorphisms of $\mathcal{D}$. A *flag* of $\mathcal{D}$ is a point-block pair (α,B) such that $\alpha \in B$. For $G \leqslant \mathrm{Aut}(\mathcal{D})$, G is called *flag-transitive* if G acts transitively on the set of flags. The group G is said to be *point-primitive* if G acts primitively on $\mathcal{P}$. Further notation and definitions in both design theory and group theory are standard and can be found, for example, in [7, 9].

We know by [2, Proposition 2.1] that a flag-transitive automorphism group of a 2-design with prime replication number is a point-primitive group of almost simple or affine type (see also [23]). In [2], such 2-designs admitting a flag-transitive almost simple automorphism group have been studied (see also [1, 18, 19, 21, 22]). In this paper, we study the case where G is of affine type, and present a classification of 2-designs with prime replication number and flag-transitive automorphism groups:

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TABLE 1. Some nontrivial 2-design with prime replication number, almost simple type.

Line	v	b	r	k	λ	X	$G_\alpha \cap X$	G	**Design**	**References**
1	6	10	5	3	2	Alt_5	D_{10}	Alt_5	-	[6, 20]
2	7	7	3	3	1	$\mathrm{PSL}_2(7)$	Sym_4	$\mathrm{PSL}_2(7)$	$\mathrm{PG}(2,2)$	[3, 6, 17]
3	8	14	7	4	3	$\mathrm{PSL}_2(7)$	7:3	$\mathrm{PSL}_2(7)$	-	[6, 20]
4	11	11	5	5	2	$\mathrm{PSL}_2(11)$	Alt_5	$\mathrm{PSL}_2(11)$	Paley	[3, 6, 17]
5	12	22	11	6	5	M_{11}	$\mathrm{PSL}_2(11)$	M_{11}	-	[6, 19]
6	15	15	7	7	3	Alt_7	$\mathrm{PSL}_2(7)$	Alt_7	$\mathrm{PG}_2(3,2)$	[3, 6, 22]
7	15	35	7	3	1	Alt_7	$\mathrm{PSL}_2(7)$	Alt_7	$\mathrm{PG}(3,2)$	[6, 21]
8	15	35	7	3	1	Alt_8	$2^3{:}\mathrm{PSL}_3(2)$	Alt_8	$\mathrm{PG}(3,2)$	[6, 21]

Note: The last column addresses to references in which a design with the parameters in the line has been constructed.

Theorem 1. *Let $\mathcal{D}$ be a nontrivial 2-(v,k,λ) design with prime replication number r, and let α be a point of $\mathcal{D}$. If G is a flag-transitive automorphism group of $\mathcal{D}$, then one of the following holds:*

(a) *G is an almost simple group with socle X and one of the following holds:*
  (i) *$\lambda \in \{1,2,3,5\}$ and v, k, λ, X, $G_\alpha \cap X$ and G are as in one of the lines in* Table 1*;*
  (ii) *$\mathcal{D}$ is the Witt-Bose-Shrikhande space $\mathrm{W}(2^n)$ with parameters $v = 2^{n-1} \cdot (2^n-1)$, $b = 2^{2n}-1$, $r = 2^n+1$ Fermat prime, $k = 2^{n-1}$ and $\lambda = 1$, for $n = 2^{2^m} \geqslant 16$. Moreover, $G = X = \mathrm{PSL}_2(2^n)$ and $G_\alpha \cap X = \mathrm{D}_{2(2^n+1)}$;*
  (iii) *$X = \mathrm{PSL}_n(q)$, $G_\alpha \cap X = \hat{}[q^{n-1}]{:}\mathrm{SL}_{n-1}(q)\cdot(q-1)$, $v = (q^n-1)/(q-1)$ and r is a primitive divisor of $(q^{n-1}-1)/(q-1)$ with $n \geqslant 3$.*

(b) *G is an affine type group, $G_0 \leqslant \Gamma L_n(p^{\frac{d}{n}})$ and $v = p^d = q^n$ and one of the following holds:*
  (i) *$\mathcal{D}$ has $v = q = p^d$ points and $G \leqslant \mathrm{A}\Gamma\mathrm{L}_1(q)$;*
  (ii) *$\mathcal{D}$ is an affine space $\mathrm{AG}_n(q)$ $(n \geqslant 2)$ with parameters $v = q^n$, $b = \frac{q^{n-1}(q^n-1)}{q-1}$, $r = \frac{q^n-1}{q-1}$ prime, $k = q^d$ and $\lambda = 1$, and G is 2-transitive on points of $\mathcal{D}$ and the point-stabiliser G_0 contains $\mathrm{SL}_a(q^{\frac{n}{a}})$, $\mathrm{Sp}_a(q^{\frac{n}{a}})$ or $\mathrm{G}_2(q^{\frac{n}{6}})'$;*
  (iii) *$\mathcal{D}$ is a 2-design with parameters $v = p^d$, $b = \frac{p^d(p^d-1)}{p^u(p^{(u,d)}-1)}$, $r = \frac{p^d-1}{p^{(u,d)}-1}$ prime, $k = p^u$ and $\lambda = \frac{p^u-1}{p^{(u,d/n)}-1}$, where n is a divisor of d, $(u,h,d/n) = 1$, and either $(u,d) < u < d/n$, or $d - d/n \leqslant u < d$ and $(u,d) < u$;*
  (iv) *$\mathcal{D}$ is a 2-design with parameters $v = 2^d-1$, $b = 2(2^d-1)$, $r = 2^d-1$ Mersenne prime, $k = 2^{d-1}$, $\lambda = 2^{d-1}-1$ with $n = d \geqslant 3$ prime.*

A detailed information on the existence and constructions of the designs obtained in Theorem 1 can be found in [2, 4, 5, 16]. As mentioned above, in order to prove

TABLE 2. Some finite linear spaces admitting flag-transitive automorphism groups of affine type.

Line	v	b	r	k	λ	**Design**	**References**
1	81	9	10	9	1	Nearfield plane	[15, Example 1.2(ii)],[6, 11]
2	81	90	10	9	1	$AG_2(9)$	[15, Example 1.1(iii)],[6, 10]
3	81	1080	40	3	1	$AG(4,3)$	[15, Example 1.1(iv)],[6, 14]
4	121	132	12	11	1	$AG_2(11)$	[15, Examples 1.1(ii)-(iii)],[6, 10]
5	361	380	20	19	1	$AG_2(19)$	[15, Example 1.1(iii)],[6, 10]
6	529	552	24	23	1	$AG_2(23)$	[15, Example 1.1(ii)],[6, 10]
7	729	756	28	27	1	Hering's plane	[15, Example 1.2(iii)],[6, 12]
8	729	7371	91	9	1	Hering's design	[15, Example 1.3],[6, 13]
9	841	870	30	29	1	$AG_2(29)$	[15, Example 1.1(iii)],[6, 10]
10	3481	3540	60	59	1	$AG_2(59)$	[15, Example 1.1(iii)],[6, 10]

Note: The last column addresses to references in which a design with the parameters in the line has been constructed.

Theorem 1 in Section 2, by [2], we only need to treat the affine type case. At the time of preparing this paper, we have noticed that a classification of 2-designs with $(r,\lambda) = 1$ admitting flag-transitive affine type automorphism groups has been given in [16] (see also [4,5]). Our arguments were mostly the same as [16], and so in order to complete our project started a few years ago on the classification of 2-designs with replication numbers prime admitting flag-transitive automorphism groups (see [2]), we have decided to use the main results in [4, 5, 16] and to avoid repetition in the proof of Theorem 1.

2. PROOF OF THEOREM 1

Let $\mathcal{D}$ be a 2-design with prime replication number r admitting a flag-transitive automorphism group G. Since $r(k-1) = \lambda(v-1)$ and r is prime, $\gcd(r,\lambda) = 1$, and so by [8, 2.3.7(a)], we conclude that G is point-primitive. Moreover, [23, Theorem] implies that G is point-primitive of almost simple or affine type. If G is an almost simple group, then [2, Theorem 1.1] follows part (a). Therefore, we only need to consider the case where G is a primitive group of affine type. In this case, the points of $\mathcal{D}$ can be identified with the vectors in a vector space $V = V_d(p)$ of dimension d over prime field $\mathrm{GF}(p)$. If G_0 denotes the stabiliser of the zero vector $0 \in V$ in G, then G_0 is an irreducible subgroup of $\mathrm{GL}(V) \cong \mathrm{GL}_d(p)$. For each divisor n of d, the group $\Gamma L_n(p^{\frac{d}{n}})$ has a natural irreducible action on V. Since the group G_0 acts irreducibly on V, choose n to be minimal divisor of d such that $G_0 \leqslant \Gamma L_n(p^{\frac{d}{n}})$ in this action, and write $q = p^{\frac{d}{n}}$. Thus we have $G_0 \leqslant \Gamma L_n(p^{\frac{d}{n}})$ and $v = p^d = q^n$.

Suppose first that $\lambda = 1$. Then the possibilities for $(\mathcal{D}, G)$ can be read of from [15, Main Theorem] and therein references. In conclusion, $\mathcal{D}$ is a 2-design with parameters as in Table 2 or one of the following holds:

TABLE 3. Some 2-designs admitting affine type automorphism groups.

v	b	r	k	λ
9	12	8	6	5
25	30	24	20	19
25	50	24	12	11
25	75	24	8	7
25	100	24	6	5
49	56	8	7	1
49	56	48	42	41
49	98	48	24	23
49	196	48	12	11
49	294	48	8	7
49	392	48	6	5
64	192	21	7	2
81	120	40	27	13
81	162	80	40	39
81	540	80	12	11
81	810	80	7	1
81	810	80	8	7
81	1080	80	6	5
121	132	120	110	109
121	242	120	60	59
121	484	120	30	29
121	605	120	24	23
121	726	120	20	19
121	1210	120	12	11
121	1815	120	8	7
361	1083	360	120	119
361	5415	360	24	23
361	1805	360	72	71
361	7220	360	18	17
361	10830	360	12	11
361	16245	360	8	7
529	552	528	506	505
529	1058	528	264	263
529	1587	528	176	175
529	4232	528	66	65
529	5819	528	48	47
529	6348	528	44	43
529	11638	528	24	23
529	34914	528	8	7
529	46552	528	6	5
729	10206	728	52	51
729	20412	728	26	25
729	132678	728	4	3
841	4205	840	168	167
841	5046	840	140	139
841	8410	840	84	83
841	16820	840	42	41
841	29435	840	24	23
841	35322	840	20	19
841	58870	840	14	13
3481	34810	3480	348	347
3481	41772	3480	290	289
3481	69620	3480	174	173
3481	100949	3480	120	119
3481	504745	3480	24	23
3481	605694	3480	20	19
3481	1009490	3480	12	11
3481	1514235	3480	8	7

(1) $\mathcal{D}$ has $v = q = p^d$ points, and $G \leqslant \mathrm{A\Gamma L}_1(q)$;
(2) $\mathcal{D}$ is an affine space $\mathrm{AG}_n(q)$ and $G \leqslant \mathrm{A\Gamma L}_n(q)$ with $n \geqslant 2$, where the point-stabiliser G_0 contains $\mathrm{SL}_a(q^{\frac{n}{a}})$, $\mathrm{Sp}_a(q^{\frac{n}{a}})$ or $\mathrm{G}_2(q^{\frac{n}{6}})'$;
(3) $\mathcal{D}$ is the Lüneburg plane related to the Suzuki group $\mathrm{Sz}(q)$ with parameters $v = q^4$, $b = q^2(q^2+1)$, $r = q^2+1$ and $k = q^2$, for $q = 2^{2e+1} > 2$.

The possibilities recorded in Table 2 cannot occur as r must be prime. If $q = 2^{2e+1} > 2$, then $r = q^2+1$ is not prime, and hence case (3) does not hold. Note that in case (2), $r = (q^n-1)/(q-1)$ has to be prime. This follows part (b.i) or part (b.ii), as claimed.

Suppose now that $\lambda > 1$ and that G is not a subgroup of $\mathrm{A\Gamma L}_1(q)$. Then by [4], $\mathcal{D}$ is non-symmetric, and hence [5, Theorem 1] and [16, Theorem 1] imply that $\mathcal{D}$ is a 2-design with parameters as in Table 3 or one of the following holds:

(1) $\mathcal{D}$ is a 2-$\left(p^d, p^u, \dfrac{p^u-1}{p^{(u,d/n)}-1}\right)$ design, where n is a divisor of d and $u < d$ satisfying $(u,h,d/n) = 1$, and either $(u,d) < u < d/n$ or $d - d/n \leqslant u < d$ and $(u,d) < u$;
(2) $\mathcal{D}$ is a 2-$(p^d, p^u\omega, p^u\omega - 1)$ design, where $\omega = \dfrac{p^{(u,d/n)}-1}{\theta}$ and $(p,u,\theta) \neq (2,1,1)$ and $(\dfrac{p^d-1}{\omega}, p^u\omega - 1) = 1$ and either $0 \leqslant u < d/n$, or $d - d/n \leqslant u < d$.

Since r is prime, the possibilities in Table 3 can be ruled out. Moreover, in case (1), $r = \frac{p^d-1}{p^{(u,d)}-1}$ needs to be prime, and this follows part (b.iii). Now if $\mathcal{D}$ is a 2-design as in case (2), then $r = p^d - 1$ must be prime, which requires $p = 2$, and

hence $r = 2^d - 1$ is a Mersenne prime and d is prime. This together with the facts that $n \mid d$ and $n \neq 1$ implies that $n = d$. Then $\omega = (2^{(u,d/n)} - 1)/\theta = (2-1)/\theta$, and so $\omega = \theta = 1$. Note that either $0 \leqslant u < d/n$, or $d - d/n \leqslant u < d$. In the former case, we have that $u = 0$, and so $k = 2^u\omega = 1$, which is a contradiction. Therefore, $d - d/n \leqslant u < d$, and since $n = d$, it follows that $u = d-1$. Since $(p,u,\theta) \neq (2,1,1)$, we conclude that $u \geqslant 2$, or equivalently, $d \geqslant 3$. Therefore, $\mathcal{D}$ is a 2-design with parameters $(2^d - 1, 2^{d+1} - 2, 2^d - 1, 2^{d-1}, 2^{d-1} - 1)$ with $d \geqslant 3$ prime, this is part (b.iv). This completes the proof of Theorem 1.

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EXISTENCE AND ESTIMATION OF THE FIXED POINTS OF ENRICHED BERINDE NONEXPANSIVE MAPPINGS

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Abstract. In this article, inspired by Berinde (Approximating fixed points of enriched nonexpansive mappings by Krasnoselskii iteration in Hilbert spaces, Carpathian J. Math. 35(3), 2019, 293-304), we define and study a new enriched class of mappings which includes many other contractive type mappings. We prove an existence result for the fixed point of newly introduced mapping. We also estimate fixed points of the proposed mapping via newly modified Mann iteration. In the process, some convergence results are also obtained for the proposed class of mappings in Uniformly convex Banach space.

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1. Introduction

Fixed point theory is a fascinating subject, with an enormous number of applications in various fields of mathematics, especially in the existence and uniqueness theory. It includes classical results to form the existence and uniqueness theorems in ordinary differential equations, random differential equations, integral equations, functional equation, partial differential equations, variational inequalities, etc. In metric fixed point theory, Banach contraction [3] is one of the crucial results, which ensures that in a complete metric space every contraction mapping always have a unique fixed point. Banach theorem has been advanced and extended in many directions. This result is a very operative and common tool for guaranteeing the existence and uniqueness of the solution of certain nonlinear problems arising within and outside mathematics. Since the applications of Banach contraction principle are restricted to contraction mappings. Because of its limitation, we need other appropriate and nice mappings to study more nonlinear problems.

Throughout this paper, $\mathbb{R}$ denotes the set of all real numbers and $\mathbb{Z}_+$ denotes the set of all nonnegative integers. Let U be a nonempty subset of a Banach space $(X, ||\cdot||)$, $T : U \to U$ a mapping and $F(T) = \{t \in U : \ Tt = t\}$. A self map T on U is called

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nonexpansive if $\|Tp-Tu\| \leq \|p-u\|, \forall p,u \in U$. It is said to be a quasi nonexpansive if $F(T) \neq \varnothing$ and $\|Tp-t\| \leq \|p-t\|, \forall\ p \in U$ and $\forall\ \ t \in F(T)$.

In 2009, Chumpungam [7] studied Berinde's nonexpansive mappings and proved existence result for their fixed points. Furthermore, many authors approximated common fixed points and proved existence and convergence results for multivalued Berinde nonexpansive mappings (1.1) in linear spaces, e.g. [5,6].

Let X be a Banach space and U a nonempty closed convex subset of X. A self mapping T on U is said to be Berinde nonexpansive if there exists $\lambda \geq 0$ such that for all $p,u \in U$,

$$\|Tp-Tu\| \leq \|p-u\| + \lambda\|u-Tp\|. \tag{1.1}$$

Recently, Berinde [4] coined the notion of enriched nonexpansive mappings which is also a generalization of nonexpansive mapping and is defined as follows:

Definition 1. [4] Let X be a normed linear space. A mapping $T : X \to X$ is said to be an enriched nonexpansive mapping if there exists $\eta \in [0,\infty)$ such that

$$\|\eta(p-u)+Tp-Tu\| \leq (\eta+1)\|p-u\|,\ \forall\ p,u \in X. \tag{1.2}$$

It can easily be seen in [4] that every nonexpansive mapping is enriched nonexpansive, but the reverse is not true in general. Moreover, if T has at least one fixed point, then T need not be quasi nonexpansive mapping. The following interesting example is available in [4].

Example 1. Let $X = [\frac{1}{2},2]$ be a normed space endowed with usual norm and $T : X \to X$ a mapping determine as $T(p) = \frac{1}{p}$ for all $p \in X$. Then

(i) T is $\frac{3}{2}$-enriched nonexpansive mapping,
(ii) T is not nonexpansive,
(iii) $F(T) = \{1\}$ but T is not quasi nonexpansive.

Remark 1. It can be easily seen that any enriched nonexpansive mapping becomes nonexpansive when $\eta = 0$.

Inspired by the above, one can raise the following question:

Question. Does there exist a class of mappings which contains the mappings defined in (1.1) and (1.2)?

Here, we have a partial answer to this question, affirmatively. In fact, we propose a new class of mapping called enriched Berinde nonexpansive mapping. Further, we prove existence and convergence results for such mappings. We have also provided some numerical examples to substantiate the facts and results.

It is well known that by using Picard iteration process [15] we can approximate the fixed points of contraction mappings where the iteration $\{\xi_n\}$ is developed by an

arbitrary point $\xi_0 \in U$ as follows:

$$\xi_{n+1} = T\xi_n, \qquad n \in \mathbb{Z}_+. \tag{1.3}$$

But in case of nonexpansive mappings, if we take initial guess different from fixed points of mapping then Picard sequence may fail to converge to the fixed points of such mappings.

Therefore, in 1953, Mann [11] introduced an iteration process to approximate fixed point of nonexpansive mappings which is defined by an initial guess $\xi_0 \in U$ as follows:

$$\xi_{n+1} = (1-\theta_n)\xi_n + \theta_n T\xi_n, \quad n \in \mathbb{Z}_+, \tag{1.4}$$

where $\{\theta_n\}$ is a control sequence in $(0,1)$. In case of Pseudocontractive mappings Mann iteration need not converge to the fixed points of such mappings. Then, to approximate the fixed points of different non-linear mappings several authors introduced and studied remarkable iteration processes. However to approximate the fixed points of enriched class of mappings by Picard and Mann iterations or any other iteration present in the literature is a very difficult task.

2. Preliminaries

Definition 2. A Banach space X is said to have Opial's property [12] if for each weakly convergent sequence $\{\xi_n\}$ to $p \in X$, the following inequality

$$\lim_{n\to\infty} \inf \|\xi_n - p\| < \lim_{n\to\infty} \inf \|\xi_n - u\|$$

holds, for all $u \in X$ with $u \neq p$.

Lemma 1. *[16] Let X be a uniformly convex Banach space and $0 < a \leq \omega_n \leq b < 1$ for all $n \in \mathbb{N}$. Assume that $\{\xi_n\}$ and $\{\sigma_n\}$ are two sequences in X such that $\lim_{n\to\infty} \sup \|\xi_n\| \leq \omega$, $\lim_{n\to\infty} \sup \|\sigma_n\| \leq \omega$ and $\lim_{n\to\infty} \sup \|\omega_n\xi_n + (1-\omega_n)\sigma_n\| = \omega$ holds, for some $\omega \geq 0$. Then $\lim_{n\to\infty} \|\xi_n - \sigma_n\| = 0$.*

Definition 3. [10] (Condition$(*)$) A self map T on a nonempty subset U of a Banach space X is said to satisfy *Condition*$(*)$ if there exists a constant $\lambda \geq 0$ such that

$$\|Tp - Tu\| \leq \|p-u\| + \lambda\|p - Tp\| \qquad \forall\, p,u \in U. \tag{2.1}$$

Lemma 2. *[10] Assume that $T : U \to U$ is a Berinde nonexpansive mapping satisfying condition $(*)$, where U is a nonempty closed convex subset of Banach space X that satisfies the Opial's condition. Then, $I - T$ is demiclosed at zero.*

3. Enriched Berinde Non-Expansive Mapping and New Iterative Scheme

In this section, we define enriched Berinde nonexpansive mapping and modified Mann iteration, and also prove some basic results for such mapping.

Definition 4. Let U be a nonempty subset of Banach space X. A mapping $T : U \to U$ is said to be an enriched Berinde nonexpansive or η-enriched Berinde nonexpansive if there exist constants $\eta \in [0,\infty)$ and $\lambda \geq 0$ such that

$$\|\eta(p-u)+Tp-Tu\| \leq (\eta+1)\|p-u\|+\lambda\|(\eta+1)(u-p)+p-Tp\| \quad \forall\, p,u \in U. \tag{3.1}$$

Now we mention some facts and basic properties of enriched Berinde nonexpansive mapping.

Remark 2.

(1) If $\lambda = 0$ in inequality (3.1), then it reduces to enriched nonexpansive mapping (1.2).
(2) If $\lambda = 0$ and $\eta = 0$ in inequality (3.1), then it reduces to nonexpansive mapping.
(3) If $\eta = 0$ in inequality (3.1), then it reduces to Berinde nonexpansive mapping (1.1).

It can be easily seen that every nonexpansive mapping, Berinde nonexpansive mapping and enriched nonexpansive mapping are enriched Berinde nonexpansive mapping. The following examples show the converse is not true in general. An enriched Berinde nonexpansive mapping need not be quasi-nonexpansive.

Example 2. Define a self mapping T on $[0,3]$ by

$$T(p) = \begin{cases} 0, & \text{if } p \neq 3 \\ 1.9, & \text{if } p = 3. \end{cases}$$

Here T is enriched Berinde nonexpansive mapping for $\lambda > 0$ but not Berinde nonexpansive.

Verification.
Case I. If $p \in [0,1.1)$ and $u \in [0,3)$, then condition (4) reduces to

$$\begin{aligned} |\eta(p-u)+Tp-Tu| &= |\eta(p-u)| \\ &\leq (\eta+1)|p-u|+\lambda|(\eta+1)(u-p)+p-Tp|. \end{aligned}$$

Case II. If $p \in [1.1,\ 3]$, $u = 3$ and consider $p < u$, then condition (4) reduces to

$$\begin{aligned} |\eta(p-u)+Tp-Tu| &\leq \eta|p-u|+|Tp-Tu| = \eta|p-u|+1.9 \\ &\leq (\eta+1)|p-u|+\lambda|(\eta+1)(u-p)+p-Tp| \\ &\leq (\eta+1)|p-u|+\lambda\Big((\eta+1)|u-p|+|p|\Big) \\ &\leq (\eta+1)|p-u|+\lambda\Big((\eta+1)|p-u|+|p|\Big). \end{aligned}$$

Thus T is enriched Berinde nonexpansive mapping for $\lambda > 0$. Now take $p = 3, u = 1.9$, then condition (1.1) reduces to

$$\|Tp - Tu\| = 1.9 > 1.1 = \|p - u\|.$$

Thus T is not Berinde nonexpansive mapping, hence not nonexpansive.

Example 3. Let $U = [0,1] \subset \mathbb{R} = X$ and a map $T : U \to U$ defined by

$$T(p) = \begin{cases} p^2 & \text{if } p \in [0, \frac{1}{2}), \\ 1 & \text{if } p \in [\frac{1}{2}, 1]. \end{cases}$$

(a) T is enriched Berinde nonexpansive mapping (3.1) with $\lambda = 4$.
(b) T is not enriched nonexpansive mapping (1.2), hence not nonexpansive.
(c) T is not quasi nonexpansive.

Verification.

(a) To show that T is enriched Berinde nonexpansive mapping, we have the following cases.

Case I. If $p, u \in [0, \frac{1}{2})$, then we have,

$$|\eta(p-u) + (p^2 - u^2)| \leq |\eta(p-u)| + |(p^2 - u^2)| \leq (\eta + 1)|p - u|,$$

which is true for all $\eta \geq 0$ and $\lambda \geq 0$.

Case II. If $p \in [0, \frac{1}{2})$ and $u \in [\frac{1}{2}, 1]$, then we have,

$$\begin{aligned} |\eta(p-u) + p^2 - 1| &\leq \eta|p-u| + |p^2 - 1| \leq (\eta+1)|p-u| + 1 \\ &= (\eta+1)|p-u| + 4.\frac{1}{4} \\ &\leq (\eta+1)|p-u| + 4|(\eta+1)(u-p) + p - Tp|. \end{aligned}$$

Case III. If $p, u \in [\frac{1}{2}, 1]$, then we have,

$$|\eta(p-u) + 0| \leq (\eta+1)|p-u| + 4|(\eta+1)(u-p) + p - Tp|.$$

Case IV. If $p \in [\frac{1}{2}, 1]$ and $u \in [0, \frac{1}{2})$, then we have,

$$\begin{aligned} |\eta(p-u) + 1 - u^2| &\leq \eta|p-u| + |1 - u^2| \leq (\eta+1)|p-u| + 1 \\ &= (\eta+1)|p-u| + 4.\frac{1}{4} \leq (\eta+1)|p-u| + 4.\frac{1}{2} \\ &\leq (\eta+1)|p-u| + 4|(\eta+1)(u-p) + p - Tp|. \end{aligned}$$

Hence T is enriched Berinde nonexpansive mapping with $\lambda = 4$.

(b) Assume T is enriched nonexpansive mapping. Then

$$|\eta(p-u) + Tp - Tu| \leq (\eta+1)|p-u|, \qquad \forall\, p, u \in U.$$

For $p = 0.50$ and $u = 0.49$, leads to a contradiction

$$\frac{\eta}{100} + 0.7599 \leq \frac{\eta}{100} + 0.01.$$

Hence T is not enriched nonexpansive mapping.

(c) Assume that T is quasi nonexpansive mapping and $F(T) = \{0,1\}$. Then

$$|Tp - t| \leq |p - t|, \qquad \forall\, p \in U,\ t \in F(T).$$

Now take $p = \frac{1}{3}$ and $t = 1$, then

$$|Tp - t| = \left|\frac{1}{9} - 1\right| = \frac{8}{9} > \frac{2}{3} = \left|\frac{1}{3} - 1\right|.$$

Which is a contradiction. Hence T is not quasi nonexpansive mapping.

Definition 5. [2] Given a map $T : X \to X$, where X is a Banach space, the κ-Krasnoselskii map or averaged mapping is defined for $\kappa \in (0,1]$ as $T_\kappa : X \to X$,

$$T_\kappa(p) = (1-\kappa)p + \kappa Tp \qquad \forall\, p \in X.$$

Remark 3. For a self mapping T on a convex subset U of a Banach space X and for any $\kappa \in (0,1]$, we have

$$F(T_\kappa) = F(T).$$

Theorem 1. *Let X be a Banach space and $T : X \to X$ an enriched Berinde nonexpansive mapping. Then, κ-Krasnoselskii map $T_\kappa : X \to X$ is a Berinde nonexpansive mapping.*

Proof. Since T is an enriched Berinde nonexpansive mapping, so for all $p, u \in X$, we have

$$\|\eta(p-u) + Tp - Tu\| \leq (\eta+1)\|p-u\| + \lambda\|(\eta+1)(u-p) + p - Tp\|.$$

Using $\kappa = \frac{1}{\eta+1}$, we have

$$\|\left(\frac{1}{\kappa} - 1\right)(p-u) + Tp - Tu\| \leq \frac{1}{\kappa}\|p-u\| + \lambda\|\frac{1}{\kappa}(u-p) + p - Tp\|$$
$$\|(1-\kappa)(p-u) + \kappa Tp - \kappa Tu\| \leq \|p-u\| + \lambda\|u - T_\kappa p\|.$$

This gives

$$\|T_\kappa(p) - T_\kappa(u) \leq \|p-u\| + \lambda\|u - T_\kappa p\|.$$

Hence T_κ is a Berinde nonexpansive mapping with $\lambda \geq 0$. □

Theorem 2. *Let $T : X \to X$ be a Berinde nonexpansive mapping satisfying condition $(*)$, where X is a Banach space. Then T is quasi nonexpansive, i.e. for each $p \in X$ and $t \in F(T)$, we have*

$$\|Tp - t\| \leq \|p - t\|.$$

Proof. Since T is a Berinde nonexpansive map satisfying condition $(*)$, therefore there exists $\lambda \geq 0$ such that

$$\|Tp - Tu\| \leq \|p-u\| + \lambda\|p - Tp\|, \qquad \forall\, p, u \in X.$$

Now, for a fixed point t of T, we have

$$\|Tp-t\| = \|Tp-Tt\| \leq \|p-t\| + \lambda\|t-Tt\| \leq \|p-t\|.$$

□

Lemma 3. *Let $T : U \to U$ be an enriched Berinde nonexpansive map and $U \subset X$ be a nonempty closed and convex satisfying Opial's condition. Then, $I-T$ is demiclosed at zero.*

Proof. From Theorem 1, we know that T_κ is a Berinde nonexpansive map for $\kappa = \frac{1}{\eta+1}$. Now, let $\{\xi_n\}$ be a sequence that converges weakly to $p \in U$ and let $(I-T)\xi_n \to 0$ (strongly). This would mean $\lim_{n\to\infty} \|\xi_n - T(\xi_n)\| = 0$. However

$$\|\xi_n - T_\kappa(\xi_n)\| = \kappa\|\xi_n - T(\xi_n)\|,$$

so that

$$\lim_{n\to\infty} \|\xi_n - T_\kappa(\xi_n)\| = \kappa \lim_{n\to\infty} \|\xi_n - T(\xi_n)\| = 0.$$

By Lemma 2, $I - T_\kappa$ is demiclosed at zero. So we have

$$T_\kappa(p) = p.$$

Using the definition of T_κ and simplifying, we get

$$\kappa(p - T(p)) = 0.$$

However $\kappa \neq 0$, therefore $T(p) = p$. In other words, $I-T$ is demiclosed at zero. □

In some past years, many authors gave the generalizations and comparison of nonexpansive mappings such as Hardy and Rogers, Suzuki [18], Karapinar and Taş [9], Pant and Shukla [14], Pandey et al. [13], Ali et al. [1] etc. As Berinde [4] enriched the class of nonexpansive mappings and proved existence results on fixed points. But approximation of the fixed points of enriched class of nonexpansive mappings is very complicated by using the iteration schemes available in the literature, namely Picard, Mann, Ishikawa [8] and some others. Therefore, one can modify or define a new iteration to approximate the fixed points of such enriched mappings. To overcome this problem, we modify Mann iteration to approximate fixed points of enriched class of mappings. For any initial guess $\xi_0 \in U$ and $\eta \geq 0$ the modified Mann iteration can be determine as follows:

$$\xi_{n+1} = \frac{1}{\eta+1}[\eta\xi_n + T(\xi_n)], \qquad n \in \mathbb{Z}_+. \tag{3.2}$$

4. Existence Result

Theorem 3. *Let U be a bounded convex closed subset of a uniformly convex Banach space X and $T : U \to U$ be a η-enriched Berinde nonexpansive mapping. Then the set $F(T)$ is nonempty.*

Proof. Since T is a η-enriched Berinde nonexpansive mapping, by Definition 4, it follows that there exist constants $\eta \in [0,\infty)$ and $\lambda \geq 0$, such that

$$\|\eta(p-u)+Tp-Tu\| \leq (\eta+1)\|p-u\|+\lambda\|(\eta+1)(u-p)+p-Tp\| \qquad \forall\, p,u \in U.$$

By putting $\eta = \frac{1}{\kappa}-1$ for $\eta > 0$, it follows that $\kappa \in (0,1]$ and the previous inequality is equivalent to

$$\|(1-\kappa)(p-u)+\kappa Tp-\kappa Tu\| \leq \|p-u\|+\lambda\|(u-p)+\kappa p-\kappa Tp\| \qquad \forall\, p,u \in U. \tag{4.1}$$

Denote $T_\kappa(p) = (1-\kappa)p+\kappa Tp$. Then inequality (4.1) expresses the fact that

$$\|T_\kappa(p)-T_\kappa(u)\| \leq \|p-u\|+\lambda\|u-T_\kappa p\| \qquad \forall\, p,u \in U$$

i.e. T_κ is Berinde nonexpansive. So, by Chumpungam [7], it follows that $F(T_\kappa) \neq \varnothing$.

By Remark 3, we can see that $F(T)$=$F(T_\kappa)\neq\varnothing$. Hence T has at least one fixed point. □

In support of Theorem 3, one can see Example 3.

5. Convergence Results

Throughout this section, we presume that $T : U \to U$ is a η-enriched Berinde nonexpansive mapping, where U is a nonempty convex and closed subset of a uniformly convex Banach space X. Now, we prove the following useful lemma which will be used to prove the next results of this section.

Lemma 4. *Let $\{\xi_n\}$ be a sequence developed by the iteration process (3.2). Assume that $F(T) \neq \varnothing$. Then,*

(i) $\lim_{n\to\infty} \|\xi_n - t\|$ *exists for all* $t \in F(T)$.

(ii) $\lim_{n\to\infty} \|\xi_n - T(\xi_n)\| = 0$.

Proof. (i) Let $t \in F(T)$. Form Theorem 1, we know that for $\kappa = \frac{1}{\eta+1}$, T_κ is a Berinde nonexpansive map. Also from Theorem 2, we have $\|T_\kappa(p)-t\| \leq \|p-t\|$, for all $p \in U$. Therefore we have

$$\begin{aligned}\|\xi_{n+1}-t\| &= \|\frac{1}{\eta+1}[\eta\xi_n+T(\xi_n)]-t\| = \|\kappa[(\frac{1}{\kappa}-1)\xi_n+T(\xi_n)]-t\| \\ &= \|(1-\kappa)\xi_n+\kappa T(\xi_n)-t\| = \|T_\kappa(\xi_n)-t\| \leq \|\xi_n-t\|. \end{aligned} \tag{5.1}$$

Thus the sequence $\{\|\xi_n - t\|\}$ is bounded below and decreasing for all $t \in F(T)$. Hence $\lim_{n\to\infty} \|\xi_n - t\|$ exists.

(ii) Define $T_\kappa : U \to U$, for $\kappa = \frac{1}{\eta+1}$. Then from Theorem 1, we know that T_κ is Berinde nonexpansive. Also from Remark 3, we know that $F(T) = F(T_\kappa) \neq \varnothing$. Moreover, for the same initial guess $\xi_0 \in U$, the sequence generated by the modified Mann iteration process using T is the same as that generated by the Mann iteration

process using T_κ. Hence by Theorem 3.2 in [10], we have $\lim_{n\to\infty}\|\xi_n - T_\kappa(\xi_n)\| = 0$. By using the definition of T_κ, we get

$$\kappa\|\xi_n - T(\xi_n)\| = 0.$$

Since $\kappa \neq 0$, then $\lim_{n\to\infty}\|\xi_n - T(\xi_n)\| = 0$. □

Now, we prove that iteration process (3.2) converges weakly for the proposed mapping.

Theorem 4. *Let $\{\xi_n\}$ be defined by (3.2) and X satisfying Opial's property. Then, $\{\xi_n\}$ converges weakly to a fixed point of the mapping T.*

Proof. Let s and q be two weak sub-sequential limits of $\{\xi_{n_j}\}$ and $\{\xi_{n_k}\}$, respectively, where $\{\xi_{n_j}\}$ and $\{\xi_{n_k}\}$ are two subsequences of $\{\xi_n\}$. From Lemma 4(i), we get $\lim_{n\to\infty}\|\xi_n - t\|$ exists. Now our aim is to show that $s = q$ in $F(T)$. From Lemma 4(ii), $\lim_{n\to\infty}\|\xi_n - T(\xi_n)\| = 0$ and by Lemma 3, $I - T$ is demiclosed at zero. Thus $(I-T)s = 0$, that is, $s = Ts$. Similarly, $q = Tq$.

Now it is enough to show that $s = q$. If $s \neq q$, then applying Opial's property, we obtain

$$\begin{aligned}\lim_{n\to\infty}\|\xi_n - s\| &= \lim_{n_j\to\infty}\|\xi_{n_j} - s\| < \lim_{n_j\to\infty}\|\xi_{n_j} - q\| = \lim_{n\to\infty}\|\xi_n - q\| \\ &= \lim_{n_k\to\infty}\|\xi_{n_k} - q\| < \lim_{n_k\to\infty}\|\xi_{n_k} - s\| = \lim_{n\to\infty}\|\xi_n - s\|,\end{aligned}$$

which is a contradiction, Hence $s = q$. It concludes that $\{\xi_n\}$ converges weakly to a fixed point of T. □

Theorem 5. *The sequence $\{\xi_n\}$ developed by the iteration process (3.2) converges strongly to a fixed point of T if and only if $\lim_{n\to\infty}\inf d(\xi_n, F(T)) = 0$, where $d(\xi_n, F(T)) = \inf\{\|\xi_n - t\| : t \in F(T)\}$.*

Proof. First part is trivial. Now, we prove the converse part. Presume that $\lim_{n\to\infty}\inf d(\xi_n, F(T)) = 0$. From Lemma 4(i), $\lim_{n\to\infty}\|\xi_n - t\|$ exists, for all $t \in F(T)$ and by hypothesis $\lim_{n\to\infty} d(\xi_n, F(T)) = 0$. Now our assertion is that $\{\xi_n\}$ a Cauchy sequence in U. As $\lim_{n\to\infty} d(\xi_n, F(T)) = 0$, for a given $\rho > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$d(\xi_n, F(T)) < \frac{\rho}{2} \qquad \Longrightarrow \inf\{\|\xi_n - t\| : t \in F(T)\} < \frac{\rho}{2}.$$

Specifically, $\inf\{\|\xi_N - t\| : t \in F(T)\} < \frac{\rho}{2}$. So, there exists $t \in F(T)$ such that

$$\|\xi_N - t\| < \frac{\rho}{2}.$$

Now, for $m,n \geq N$,

$$\begin{aligned}\|\xi_{n+m}-\xi_n\| &\leq \|\xi_{n+m}-t\|+\|\xi_n-t\| \\ &\leq \|\xi_N-t\|+\|\xi_N-t\| = 2\|\xi_N-t\| < \rho.\end{aligned}$$

Thus, $\{\xi_n\}$ is a Cauchy sequence in U, so that there exists an element $\ell \in U$ such that $\lim_{n\to\infty} \xi_n = \ell$. Now, $\lim_{n\to\infty} d(\xi_n, F(T)) = 0$ implies $d(\ell, F(T)) = 0$, hence we get $\ell \in F(T)$. □

In 1974, Senter and Dotson [17] defined a condition on mappings called condition (I), using which we can get strong convergence.

Definition 6. [17] Let $\psi : [0,\infty) \to [0,\infty)$ be a nondecreasing function with $\psi(0) = 0$ and $\psi(z) > 0$, $\forall\ z > 0$. A self map T on U is said to satisfy condition (I), if $d(p,Tp) \geq \psi(d(p,F(T)))$, $\forall\ p \in U$.

Theorem 6. *Let condition (I) be satisfied by mapping T. Then strong convergence is achieved by iteration process (3.2) to a fixed point of T.*

Proof. We already proved in Lemma 4(ii) that

$$\lim_{n\to\infty} \|\xi_n - T(\xi_n)\| = 0. \tag{5.2}$$

By equation (5.2) and condition (I), we obtain

$$\begin{aligned}0 \leq \lim_{n\to\infty} \psi(d(\xi_n, F(T))) &\leq \lim_{n\to\infty} \|\xi_n - T(\xi_n)\| = 0 \\ \implies \lim_{n\to\infty} \psi(d(\xi_n, F(T))) &= 0. \\ \implies \lim_{n\to\infty} d(\xi_n, F(T)) &= 0.\end{aligned}$$

By Theorem 5, the sequence $\{\xi_n\}$ converges strongly to a fixed point of T. □

6. Conclusion

In this paper, we enriched the class of Berinde nonexpansive mapping and approximated the fixed points of proposed mapping by using modified Mann iteration process in uniformly convex Banach space. Our results are new and generalized several results in literature, in particular the results of Chumpungam [7].

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FRACTIONAL HERMITE-HADAMARD INEQUALITY AND ERROR ESTIMATES FOR SIMPSON'S FORMULA THROUGH CONVEXITY WITH RESPECT TO A PAIR OF FUNCTIONS

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Abstract. In this article, we establish two new and different versions of fractional Hermite-Hadamard type inequality for the convex functions with respect to a pair of functions. Moreover, we establish a new Simpson's type inequalities for differentiable convex functions with respect to a pair of functions. We also prove two more Simpson's type inequalities for differentiable convex functions with respect to a pair of functions using the power mean and Hölder's inequalities. It is also shown that the newly established inequalities are the extension of some existing results. Finally, we add some mathematical examples and their graphs to show the validity of newly established results.

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Keywords: Hermite-Hadamard inequality, Simpson's inequality, (g,h)-convex functions

1. Introduction

The interesting concept of inequalities has long been a source of debate among mathematicians. Only a few of the exciting applications include fractional calculus, quantum calculus, operator theory, numerical analysis, operator equations, network theory, and quantum information theory. Right now, this is a very dynamic research topic and the interaction between diverse fields have enriched it. Applied sciences need the use of numerical integration and definite integral estimate. Simpson's rules, which can be stated as follows, are fundamental among the numerical techniques:

(i) Simpson's $1/3$ rule

$$\int_{\omega_1}^{\omega_2} \mathfrak{F}(\varkappa)\,d\varkappa \approx \frac{\omega_2-\omega_1}{6}\left[\mathfrak{F}(\omega_1)+4\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+\mathfrak{F}(\omega_2)\right].$$

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Due to their importance in approximation theory, researchers have employed fractional calculus to establish many fractional integral inequalities. Examples of inequalities that can be used to discover the bounds of numerical integration formulas include Hermite-Hadamard, Simpson's, midpoint, Ostrowski's, and trapezoidal inequalities. The authors of [13] established the boundaries of the trapezoidal formula and inequality of the Hermite-Hadamard type using Riemann-Liouville fractional integrals. Set [15] derived fractional Ostrowski's type inequalities using differentiable convexity. Through Riemann-Liouville fractional integrals, İşcan and Wu [7] derived some bounds of numerical integration and inequality of Hermite-Hadamard type for reciprocal convex functions. Sarikaya and Yildrim established midpoint bounds and a new variant of fractional inequality of Hermite-Hadamard type in [14]. Sarikaya et al. [12] used general convexity and Riemann-Liouville fractional integral operators to find estimates for Simpson's $1/3$ formula. Using the Riemann-Liouville fractional integrals, the authors derived some new boundaries for Simpson's $1/3$ formula in [9]. An s-convexity was utilized by the authors of [3] to find various estimates for Simpson's $1/3$ formula. Sarikaya and Ertugral [11] introduced a new class of fractional integrals known as generalized fractional integrals in 2020, as well as Hermite-Hadamard type inequalities linked to the newly defined class of integrals. Zhao et al. used reciprocal convex functions and generalized fractional integral operators to establish various estimates for a trapezoidal formula in [18]. Budak et al. [2] used generalized fractional integrals to establish various estimates for Simpson's $1/3$ formula for differentiable convex functions.

Sitthiwirattham et al. [17] have discovered some estimates for Simpson's 3/8 formula using the Riemann-Liouville fractional integrals operators. For other inequalities involving fractional integral operators, see [1,6,16,19] and the references therein.

Motivated by the above-mentioned literature and applications of fractional inequalities, we use Riemann-Liouville fractional integrals to derive Hermite-Hadamard inequality for convex functions with respect to a pair of functions. In addition, we prove some important Simpson's type inequalities for differentiable convex functions with respect to a pair of functions. It is also shown that the results established in the paper are the generalization of the results proved in [4,10,11].

2. Fractional integrals and related inequalities

In this section, we review several fundamental fractional integral notations and concepts. We also address inequalities and various types of convexity. We use the following notations throughout the work for the sake of brevity:

- A convex subset of a real vector space X is denoted by C.
- I denotes an interval of $\mathbb{R}$.
- I° denotes the interior of I.

Definition 1. A mapping $\mathfrak{F}: C \to \mathbb{R}$ is called convex if

$$\mathfrak{F}(\tau\varkappa + (1-\tau)\gamma) \leq \tau\mathfrak{F}(\varkappa) + (1-\tau)\mathfrak{F}(\gamma),$$

where $\varkappa, \gamma \in C$ and $\tau \in [0,1]$.

The Hermite-Hadamard inequality (double inequality) is basic in the convex functions theory, providing a two-sided estimate of the (integral) the mean value of a convex function. In other words, if $\mathfrak{F}: I \to \mathbb{R}$ is convex with $\omega_1, \omega_2 \in I$ and $\omega_1 < \omega_2$, then

$$\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right) \leq \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2} \mathfrak{F}(\varkappa)\,d\varkappa \leq \frac{\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{2}. \tag{2.1}$$

B. Samet defined a new class of convex functions with respect to a pair of functions in [10], as follows:

Definition 2. Let $g, h: C \to \mathbb{R}$ be two mappings. A mapping $\mathfrak{F}: C \to \mathbb{R}$ is called (g,h)-convex, if the following inequality holds for $M(\varkappa,\gamma) = g(\varkappa)h(\gamma) + g(\gamma)h(\varkappa)$:

$$\mathfrak{F}(\tau\varkappa + (1-\tau)\gamma) \leq \tau^2\mathfrak{F}(\varkappa) + (1-\tau)^2\mathfrak{F}(\gamma) + \tau(1-\tau)M(\varkappa,\gamma) \tag{2.2}$$

for all $\tau \in [0,1]$ and $\varkappa, \gamma \in C$.

B. Samet also established the Hermite-Hadamard inequality (double inequality) for a new class of convex functions in terms of a pair of functions, which he expressed as:

Theorem 1 ([10])**.** *Let $\mathfrak{F}: I \subset \mathbb{R} \to \mathbb{R}$ be a (g,h)-convex function over $[\omega_1,\omega_2]$ with $g,h: [\omega_1,\omega_2] \to \mathbb{R}$. If $\mathfrak{F} \in L^1(\omega_1,\omega_2)$ and $g,h \in L^2(\omega_1,\omega_2)$, then we have the following fractional Hermite-Hadamard inequality:*

$$\begin{aligned}
&2\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right) - \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2} g(\omega_1+\omega_2-\varkappa)h(\varkappa)\,d\varkappa \\
&= 2\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right) - \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2} g(\varkappa)h(\omega_1+\omega_2-\varkappa)\,d\varkappa \\
&\leq \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2} \mathfrak{F}(\varkappa)\,d\varkappa \\
&\leq \frac{\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{3} + \frac{M(\omega_1,\omega_2)}{6}.
\end{aligned} \tag{2.3}$$

Definition 3 ([5,8])**.** Let $\mathfrak{F} \in L_1[\omega_1,\omega_2]$. The Riemann-Liouville fractional integrals (RLFIs) $J^{\alpha}_{\omega_1+}\mathfrak{F}$ and $J^{\alpha}_{\omega_2-}\mathfrak{F}$ of order $\alpha > 0$ with $\omega_1 \geq 0$ are defined as follows:

$$J^{\alpha}_{\omega_1+}\mathfrak{F}(\varkappa) = \frac{1}{\Gamma(\omega_1)}\int_{\omega_1}^{\varkappa} (\varkappa-\tau)^{\alpha-1}\mathfrak{F}(\tau)\,d\tau,\ \varkappa > \omega_1$$

and

$$J^{\alpha}_{\omega_2-}\mathfrak{F}(\varkappa) = \frac{1}{\Gamma(\alpha)}\int_{\varkappa}^{\omega_2} (\tau-\varkappa)^{\alpha-1}\mathfrak{F}(\tau)\,d\tau,\ \varkappa < \omega_2,$$

respectively, where Γ is the well-known Gamma function.

For the first time in 2013, Sarikaya et al. established the following fractional Hermite-Hadmard type inequality:

Theorem 2 ([13]). *For a positive convex function $\mathfrak{F}: I \subset \mathbb{R} \to \mathbb{R}$ with $\mathfrak{F} \in L_1[\omega_1, \omega_2]$ and $0 \leq \omega_1 < \omega_2$, the following inequality holds:*

$$\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(\omega_2-\omega_1)^\alpha}\left[J^\alpha_{\omega_1+}\mathfrak{F}(\omega_2)+J^\alpha_{\omega_2-}\mathfrak{F}(\omega_1)\right] \leq \frac{\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{2}.$$

The following modified version of the fractional Hermite-Hadamard inequality was then established by Sarikaya and Yildirim:

Theorem 3 ([14]). *For a positive convex function $\mathfrak{F}: I \subset \mathbb{R} \to \mathbb{R}$ with $\mathfrak{F} \in L_1[\omega_1, \omega_2]$, $0 \leq \omega_1 < \omega_2$ and $\omega_1, \omega_2 \in I$, the following inequality holds:*

$$\begin{aligned}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right) &\leq \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(\omega_2-\omega_1)^\alpha}\left[J^\alpha_{\left(\frac{\omega_1+\omega_2}{2}\right)+}\mathfrak{F}(\omega_2)+J^\alpha_{\left(\frac{\omega_1+\omega_2}{2}\right)-}\mathfrak{F}(\omega_1)\right]\\ &\leq \frac{\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{2}.\end{aligned}$$

Remark 1. If we set $\alpha = 1$ in inequalities (2.1) and (2.2), then we obtain the traditional Hermite-Hadamard inequality:

$$\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right) \leq \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(\varkappa)\,d\varkappa \leq \frac{\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{2}.$$

3. Hermite-Hadamard inequality

Using the Riemann-Liouville fractional integrals, we prove two new Hermite-Hadamard inequalities for (g,h)-convex functions in this section.

Theorem 4. *Let $\mathfrak{F}: I \subset \mathbb{R} \to \mathbb{R}$ be a (g,h)-convex function on $[\omega_1, \omega_2]$ with $g, h: [\omega_1, \omega_2] \to \mathbb{R}$. If $\mathfrak{F} \in L^1(\omega_1, \omega_2)$ and $g, h \in L^2(\omega_1, \omega_2)$, then we have the following fractional Hermite-Hadamard inequality:*

$$\begin{aligned}&2\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\omega_2-\omega_1)^\alpha}\left[J^\alpha_{\omega_1+}\Omega\left(\frac{\omega_1+\omega_2}{2}\right)+J^\alpha_{\omega_2-}\Omega\left(\frac{\omega_1+\omega_2}{2}\right)\right]\\ &\leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\omega_2-\omega_1)^\alpha}\left[J^\alpha_{\omega_1+}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+J^\alpha_{\omega_2-}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)\right]\\ &\leq \frac{[\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)](\alpha+1)}{2(\alpha+2)}+\frac{M(\omega_1,\omega_2)}{2(\alpha+2)},\end{aligned} \tag{3.1}$$

where $\Omega(\varkappa) = g(\varkappa)h(\omega_1+\omega_2-\varkappa)$.

Proof. Because (g,h)-convexity of $\mathfrak{F}$, we have

$$4\mathfrak{F}\left(\frac{\varkappa+\gamma}{2}\right) \leq \mathfrak{F}(\varkappa)+\mathfrak{F}(\gamma)+h(\varkappa)g(\gamma)+h(\gamma)g(\varkappa).$$

By letting $\varkappa = \frac{1-\tau}{2}\omega_1 + \frac{1+\tau}{2}\omega_2$ and $\gamma = \frac{1+\tau}{2}\omega_2 + \frac{1-\tau}{2}\omega_1$, we have

$$\begin{aligned} 4\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right) \leq{}& \mathfrak{F}\left(\frac{1-\tau}{2}\omega_1+\frac{1+\tau}{2}\omega_2\right)+\mathfrak{F}\left(\frac{1+\tau}{2}\omega_2+\frac{1-\tau}{2}\omega_1\right) \\ &+h\left(\frac{1-\tau}{2}\omega_1+\frac{1+\tau}{2}\omega_2\right)g\left(\frac{1+\tau}{2}\omega_2+\frac{1-\tau}{2}\omega_1\right) \\ &+h\left(\frac{1+\tau}{2}\omega_2+\frac{1-\tau}{2}\omega_1\right)g\left(\frac{1-\tau}{2}\omega_1+\frac{1+\tau}{2}\omega_2\right). \end{aligned} \tag{3.2}$$

Multiplying the last inequality by $\tau^{\alpha-1}$ and then integrating with respect to t over $[0,1]$, we have

$$\begin{aligned} \frac{4}{\alpha}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right) \leq{}& \frac{2^{\alpha}\Gamma(\alpha)}{(\omega_2-\omega_1)^{\alpha}}\left[J^{\alpha}_{\omega_1+}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+J^{\alpha}_{\omega_2-}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)\right] \\ &+\frac{2^{\alpha}\Gamma(\alpha)}{(\omega_2-\omega_1)^{\alpha}}\left[J^{\alpha}_{\omega_1+}\Omega\left(\frac{\omega_1+\omega_2}{2}\right)+J^{\alpha}_{\omega_2-}\Omega\left(\frac{\omega_1+\omega_2}{2}\right)\right], \end{aligned}$$

which implies that

$$\begin{aligned} &2\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\omega_2-\omega_1)^{\alpha}}\left[J^{\alpha}_{\omega_1+}\Omega\left(\frac{\omega_1+\omega_2}{2}\right)+J^{\alpha}_{\omega_2-}\Omega\left(\frac{\omega_1+\omega_2}{2}\right)\right] \\ &\leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\omega_2-\omega_1)^{\alpha}}\left[J^{\alpha}_{\omega_1+}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+J^{\alpha}_{\omega_2-}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)\right]. \end{aligned}$$

As a result, the first inequality in (3.1) is established. We again use the (g,h)-convexity to prove the second inequality as follows:

$$\mathfrak{F}\left(\frac{1-\tau}{2}\omega_1+\frac{1+\tau}{2}\omega_2\right) \leq \frac{(1-\tau)^2}{4}\mathfrak{F}(\omega_1)+\frac{(1+\tau)^2}{4}\mathfrak{F}(\omega_2)+\frac{1-\tau^2}{4}M(\omega_1,\omega_2) \tag{3.3}$$

and

$$\mathfrak{F}\left(\frac{1+\tau}{2}\omega_1+\frac{1-\tau}{2}\omega_2\right) \leq \frac{(1+\tau)^2}{4}\mathfrak{F}(\omega_1)+\frac{(1-\tau)^2}{4}\mathfrak{F}(\omega_2)+\frac{1-\tau^2}{4}M(\omega_1,\omega_2). \tag{3.4}$$

After multiplying the inequalities (3.3) and (3.4) with $\tau^{\alpha-1}$, we add them and integrate the resultant one with respect to t over $[0,1]$, we have

$$\begin{aligned} &\int_0^1 \tau^{\alpha-1}\mathfrak{F}\left(\frac{1-\tau}{2}\omega_1+\frac{1+\tau}{2}\omega_2\right)d\tau+\int_0^1 \tau^{\alpha-1}\mathfrak{F}\left(\frac{1+\tau}{2}\omega_1+\frac{1-\tau}{2}\omega_2\right)d\tau \\ &\leq \frac{\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{4}\int_0^1 \tau^{\alpha-1}\left((1+\tau)^2+(1-\tau)^2\right)d\tau \\ &\quad+\frac{M(\omega_1,\omega_2)}{2}\int_0^1 \tau^{\alpha-1}\left(1-\tau^2\right)d\tau. \end{aligned}$$

Thus, the required inequality is obtained with simple computations and changes in the integration variables. □

Remark 2. By setting $\alpha = 1$ in Theorem 4, we recapture the inequality presented in Theorem 1.

Remark 3. If we choose $h(\varkappa) = \mathfrak{F}(\varkappa)$ and $g(\varkappa) = 1$ or $h(\varkappa) = 1$ and $g(\varkappa) = \mathfrak{F}(\varkappa)$ in Theorem 4, then we obtain the Hermite-Hadamrd inequality given in Theorem 3.

Example 1. Let $[\omega_1, \omega_2] = [0,1]$ and define the functions $\mathfrak{F}, g, h : [0,1] \to \mathbb{R}$ by $\mathfrak{F}(\varkappa) = \varkappa^3$, $g(\varkappa) = \varkappa$ and $h(\varkappa) = \varkappa^2$ such that $\mathfrak{F}$ is (g,h)-convex on $[0,1]$. Under these assumptions, we have

$$\begin{aligned}
&2\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right) - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\omega_2-\omega_1)^{\alpha}}\left[J^{\alpha}_{\omega_1+}\Omega\left(\frac{\omega_1+\omega_2}{2}\right) + J^{\alpha}_{\omega_2-}\Omega\left(\frac{\omega_1+\omega_2}{2}\right)\right] \\
&= \frac{1}{4} - \frac{\alpha}{2^{1-\alpha}}\left[\int_0^{\frac{1}{2}}\left(\frac{1}{2}-\varkappa\right)^{\alpha-1}\varkappa(1-\varkappa)^2 d\varkappa + \int_{\frac{1}{2}}^{1}\left(\varkappa-\frac{1}{2}\right)^{\alpha-1}\varkappa(1-\varkappa)^2 d\varkappa\right] \\
&= \frac{1}{4} - \frac{\alpha}{2^{1-\alpha}}\left[\frac{2\alpha^2+6\alpha+3}{2^{\alpha+2}\alpha(\alpha+1)(\alpha+2)(\alpha+3)} + \frac{2\alpha+3}{2^{\alpha+2}\alpha(\alpha+1)(\alpha+2)(\alpha+3)}\right] \\
&= \frac{\alpha+1}{4(\alpha+2)},
\end{aligned}$$

$$\begin{aligned}
&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\omega_2-\omega_1)^{\alpha}}\left[J^{\alpha}_{\omega_1+}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right) + J^{\alpha}_{\omega_2-}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)\right] \\
&= \frac{\alpha}{2^{1-\alpha}}\left[\int_0^{\frac{1}{2}}\left(\frac{1}{2}-\varkappa\right)^{\alpha-1}\varkappa^3 d\varkappa + \int_{\frac{1}{2}}^{1}\left(\varkappa-\frac{1}{2}\right)^{\alpha-1}\varkappa^3 d\varkappa\right] \\
&= \frac{\alpha}{2^{1-\alpha}}\left[\frac{3}{2^{\alpha+2}\alpha(\alpha+1)(\alpha+2)(\alpha+3)} + \frac{(2\alpha+3)\left(2\alpha^2+6\alpha+3\right)}{2^{\alpha+2}\alpha(\alpha+1)(\alpha+2)(\alpha+3)}\right] \\
&= \frac{(2\alpha+3)\left(2\alpha^2+6\alpha+3\right)+3}{8(\alpha+1)(\alpha+2)(\alpha+3)},
\end{aligned}$$

and

$$\frac{[\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)](\alpha+1)}{2(\alpha+2)} + \frac{M(\omega_1,\omega_2)}{2(\alpha+2)} = \frac{\alpha+1}{2(\alpha+2)}.$$

We use MATLAB software to plot the function image of the above three functions, as shown in Figure 1. From the position relationship of the image, we can see that the inequalities relationship in Theorem 4 is tenable.

Theorem 5. *Let $\mathfrak{F} : I \subset \mathbb{R} \to \mathbb{R}$ be a (g,h)-convex function on $[\omega_1, \omega_2]$ with $g, h : [\omega_1, \omega_2] \to \mathbb{R}$. If $\mathfrak{F} \in L^1(\omega_1, \omega_2)$ and $g, h \in L^2(\omega_1, \omega_2)$, then we have the following*

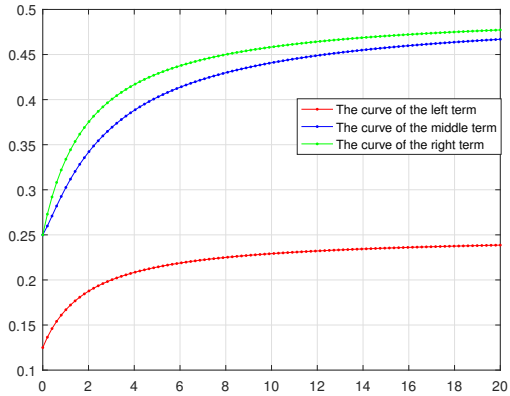

FIGURE 1. The image description for Theorem 4.

fractional Hermite-Hadamard inequality:

$$\begin{aligned}
&2\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\omega_2-\omega_1)^{\alpha}}\left[J^{\alpha}_{\frac{\omega_1+\omega_2}{2}+}\Omega(\omega_2)+J^{\alpha}_{\frac{\omega_1+\omega_2}{2}-}\Omega(\omega_1)\right]\\
&\leq\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\omega_2-\omega_1)^{\alpha}}\left[J^{\alpha}_{\frac{\omega_1+\omega_2}{2}+}\mathfrak{F}(\omega_2)+J^{\alpha}_{\frac{\omega_1+\omega_2}{2}-}\mathfrak{F}(\omega_1)\right]\\
&\leq\frac{[\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)]}{4}\left[2-\frac{\alpha(\alpha+3)}{(\alpha+1)(\alpha+2)}\right]+\frac{M(\omega_1,\omega_2)\alpha(\alpha+3)}{4(\alpha+1)(\alpha+2)},
\end{aligned}\tag{3.5}$$

where $\Omega(\varkappa)=g(\varkappa)h(\omega_1+\omega_2-\varkappa)$.

Proof. If we follow the steps used in the last theorem for $\varkappa=\frac{2-\tau}{2}\omega_1+\frac{\tau}{2}\omega_2$ and $\gamma=\frac{\tau}{2}\omega_1+\frac{2-\tau}{2}\omega_2$, then one can obtain the desired inequality. □

Remark 4. By setting $\alpha=1$ in Theorem 5, we recapture the inequality presented in Theorem 1.

Remark 5. If we choose $h(\varkappa)=\mathfrak{F}(\varkappa)$ and $g(\varkappa)=1$ or $h(\varkappa)=1$ and $g(\varkappa)=\mathfrak{F}(\varkappa)$ in Theorem 5, then we have the following Hermite-Hadamrd inequality proved by Dragomir et al. in [4]:

$$\begin{aligned}
\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)&\leq\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\omega_2-\omega_1)^{\alpha}}\left[J^{\alpha}_{\omega_1+}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+J^{\alpha}_{\omega_2-}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)\right]\\
&\leq\frac{\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)}{2}.
\end{aligned}$$

Example 2. Let $[\omega_1,\omega_2]=[0,1]$ and define the functions $\mathfrak{F},g,h:[0,1]\to\mathbb{R}$ by $\mathfrak{F}(\varkappa)=\varkappa^3$, $g(\varkappa)=\varkappa$ and $h(\varkappa)=\varkappa^2$ such that $\mathfrak{F}$ is (g,h)-convex on $[0,1]$. Under

these assumptions, we have

$$\begin{aligned}
&2\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\omega_2-\omega_1)^{\alpha}}\left[J^{\alpha}_{\frac{\omega_1+\omega_2}{2}+}\Omega(\omega_2)+J^{\alpha}_{\frac{\omega_1+\omega_2}{2}-}\Omega(\omega_1)\right]\\
&=\frac{1}{4}-\frac{\alpha}{2^{1-\alpha}}\left[\int_{\frac{1}{2}}^{1}(1-\varkappa)^{\alpha-1}\varkappa(1-\varkappa)^2d\varkappa+\int_0^{\frac{1}{2}}\varkappa^{\alpha-1}\varkappa(1-\varkappa)^2d\varkappa\right]\\
&=\frac{1}{4}-\frac{\alpha}{2^{1-\alpha}}\left[\frac{\alpha+4}{2^{\alpha+3}(\alpha+2)(\alpha+3)}+\frac{\alpha^2+7\alpha+14}{2^{\alpha+3}(\alpha+1)(\alpha+2)(\alpha+3)}\right]\\
&=\frac{1}{4}-\frac{\alpha}{16}\frac{2\alpha^2+12\alpha+18}{(\alpha+1)(\alpha+2)(\alpha+3)}\\
&=\frac{1}{4}-\frac{\alpha(\alpha+3)}{8(\alpha+1)(\alpha+2)},
\end{aligned}$$

$$\begin{aligned}
&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\omega_2-\omega_1)^{\alpha}}\left[J^{\alpha}_{\frac{\omega_1+\omega_2}{2}+}\mathfrak{F}(\omega_2)+J^{\alpha}_{\frac{\omega_1+\omega_2}{2}-}\mathfrak{F}(\omega_1)\right]\\
&=\frac{\alpha}{2^{1-\alpha}}\left[\int_{\frac{1}{2}}^{1}(1-\varkappa)^{\alpha-1}\varkappa^3d\varkappa+\int_0^{\frac{1}{2}}\varkappa^{\alpha-1}\varkappa^3d\varkappa\right]\\
&=\frac{\alpha}{2^{1-\alpha}}\left[\frac{(\alpha+4)\left(\alpha^2+5\alpha+12\right)}{2^{\alpha+3}\alpha(\alpha+1)(\alpha+2)(\alpha+3)}+\frac{1}{2^{\alpha+3}(\alpha+3)}\right]\\
&=\frac{\alpha^3+6\alpha^2+17\alpha+24}{8(\alpha+1)(\alpha+2)(\alpha+3)},
\end{aligned}$$

and

$$\begin{aligned}
&\frac{[\mathfrak{F}(\omega_1)+\mathfrak{F}(\omega_2)]}{4}\left[2-\frac{\alpha(\alpha+3)}{(\alpha+1)(\alpha+2)}\right]+\frac{M(\omega_1,\omega_2)\alpha(\alpha+3)}{4(\alpha+1)(\alpha+2)}\\
&=\frac{1}{4}\left[2-\frac{\alpha(\alpha+3)}{(\alpha+1)(\alpha+2)}\right].
\end{aligned}$$

We use MATLAB software to plot the function image of the above three functions, as shown in Figure 2. From the position relationship of the image, we can see that the inequalities relationship in Theorem 5 is tenable.

4. Simpson's type inequalities

We prove new Simpson's type inequalities for differentiable convex functions with respect to a pair of functions in this section. The following result and notations will be used to prove the inequalities in this section that are dependent on $\varpi=\left(\frac{2}{3}\right)^{\frac{1}{\alpha}}$:

$$A_1(\varpi,\alpha)=\int_0^1\left|\frac{\tau^{\alpha}}{2}-\frac{1}{3}\right|d\tau$$

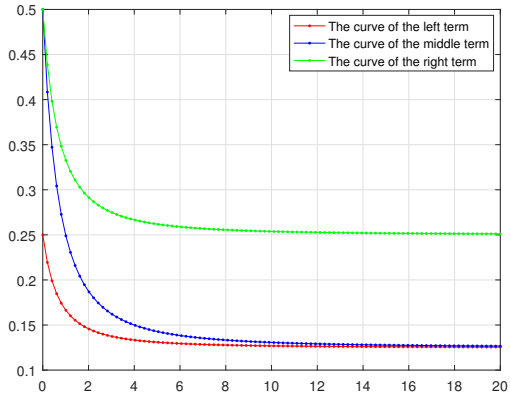

FIGURE 2. The image description for Theorem 5.

$$
\begin{aligned}
&= \int_0^{\varpi} \left(\frac{1}{3} - \frac{\tau^{\alpha}}{2}\right) d\tau + \int_{\varpi}^{1} \left(\frac{\tau^{\alpha}}{2} - \frac{1}{3}\right) d\tau \\
&= \frac{2}{3}\varpi - \frac{\varpi^{\alpha+1}}{\alpha+1} + \frac{1}{2(\alpha+1)} - \frac{1}{3},
\end{aligned}
$$

$$
\begin{aligned}
A_2(\varpi,\alpha) &= \int_0^1 \tau \left|\frac{\tau^{\alpha}}{2} - \frac{1}{3}\right| d\tau \\
&= \int_0^{\varpi} \tau \left(\frac{1}{3} - \frac{\tau^{\alpha}}{2}\right) d\tau + \int_{\varpi}^{1} \tau \left(\frac{\tau^{\alpha}}{2} - \frac{1}{3}\right) d\tau \\
&= \frac{\varpi^2}{3} - \frac{\varpi^{\alpha+2}}{\alpha+2} + \frac{1}{2(\alpha+2)} - \frac{1}{6},
\end{aligned}
$$

$$
\begin{aligned}
A_3(\varpi,\alpha) &= \int_0^1 \tau \left|\frac{\tau^{\alpha}}{2} - \frac{1}{3}\right| d\tau \\
&= \int_0^{\varpi} \tau^2 \left(\frac{1}{3} - \frac{\tau^{\alpha}}{2}\right) d\tau + \int_{\varpi}^{1} \tau^2 \left(\frac{\tau^{\alpha}}{2} - \frac{1}{3}\right) d\tau \\
&= \frac{2}{9}\varpi^3 - \frac{\varpi^{\alpha+3}}{\alpha+3} + \frac{1}{2(\alpha+3)} - \frac{1}{9}.
\end{aligned}
$$

Lemma 1 ([3]). *Let $\mathfrak{F} : [\omega_1, \omega_2] \to \mathbb{R}$ be a differentiable mapping over (ω_1, ω_2) and $\mathfrak{F}' \in L^1[\omega_1, \omega_2]$ with $\omega_1 < \omega_2$. Then, we have the following equality:*

$$
\frac{1}{6}\left[\mathfrak{F}(\omega_1) + 4\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right) + \mathfrak{F}(\omega_2)\right]
$$

$$-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\omega_2-\omega_1)^{\alpha}}\left[J^{\alpha}_{\omega_1+}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+J^{\alpha}_{\omega_2-}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)\right] \tag{4.1}$$
$$=\frac{(\omega_2-\omega_1)}{2}\left[\int_0^1\left(\frac{\tau^{\alpha}}{2}-\frac{1}{3}\right)\left[\begin{array}{c}\mathfrak{F}'\left(\frac{1+\tau}{2}\omega_2+\frac{1-\tau}{2}\omega_1\right)\\ -\mathfrak{F}'\left(\frac{1-\tau}{2}\omega_2+\frac{1+\tau}{2}\omega_1\right)\end{array}\right]d\tau\right].$$

Theorem 6. *In addition of Lemma 1, if $|\mathfrak{F}'|$ is (g,h)-mapping, then we have the following new fractional Simpson's inequality:*

$$\begin{aligned}&\left|\frac{1}{6}\left[\mathfrak{F}(\omega_1)+4\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+\mathfrak{F}(\omega_2)\right]\right.\\&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\omega_2-\omega_1)^{\alpha}}\left[J^{\alpha}_{\omega_1+}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+J^{\alpha}_{\omega_2-}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)\right]\right|\\&\leq\frac{\omega_2-\omega_1}{2}\left[\frac{|\mathfrak{F}'(\omega_1)|+|\mathfrak{F}'(\omega_2)|}{2}(A_1(\varpi,\alpha)+A_3(\varpi,\alpha))\right.\\&\left.+\frac{M(\omega_1,\omega_2)}{2}(A_1(\varpi,\alpha)-A_3(\varpi,\alpha))\right].\end{aligned} \tag{4.2}$$

Proof. From (4.1) and modulus properties, we have

$$\begin{aligned}&\left|\frac{1}{6}\left[\mathfrak{F}(\omega_1)+4\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+\mathfrak{F}(\omega_2)\right]\right.\\&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\omega_2-\omega_1)^{\alpha}}\left[J^{\alpha}_{\omega_1+}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+J^{\alpha}_{\omega_2-}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)\right]\right|\\&\leq\frac{(\omega_2-\omega_1)}{2}\left[\int_0^1\left|\frac{\tau^{\alpha}}{2}-\frac{1}{3}\right|\left[\begin{array}{c}\left|\mathfrak{F}'\left(\frac{1+\tau}{2}\omega_2+\frac{1-\tau}{2}\omega_1\right)\right|\\ +\left|\mathfrak{F}'\left(\frac{1-\tau}{2}\omega_2+\frac{1+\tau}{2}\omega_1\right)\right|\end{array}\right]d\tau\right].\end{aligned} \tag{4.3}$$

Using the (g,h)-convexity, we get

$$\begin{aligned}&\left|\frac{1}{6}\left[\mathfrak{F}(\omega_1)+4\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+\mathfrak{F}(\omega_2)\right]\right.\\&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\omega_2-\omega_1)^{\alpha}}\left[J^{\alpha}_{\omega_1+}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+J^{\alpha}_{\omega_2-}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)\right]\right|\\&\leq\frac{(\omega_2-\omega_1)}{2}\left[\int_0^1\left|\frac{\tau^{\alpha}}{2}-\frac{1}{3}\right|\left[\begin{array}{c}\frac{1+\tau^2+2\tau}{4}|\mathfrak{F}'(\omega_2)|\\ +\frac{1+\tau^2-2\tau}{4}|\mathfrak{F}'(\omega_1)|+\frac{1-\tau^2}{4}M(\omega_1,\omega_2)\end{array}\right]d\tau\right.\\&\left.+\int_0^1\left|\frac{\tau^{\alpha}}{2}-\frac{1}{3}\right|\left[\begin{array}{c}\frac{1+\tau^2+2\tau}{4}|\mathfrak{F}'(\omega_1)|\\ +\frac{1+\tau^2-2\tau}{4}|\mathfrak{F}'(\omega_2)|+\frac{1-\tau^2}{4}M(\omega_1,\omega_2)\end{array}\right]d\tau\right]\end{aligned}$$

$$= \frac{\omega_2-\omega_1}{2}\int_0^1 \left|\frac{\tau^\alpha}{2}-\frac{1}{3}\right| \left[\frac{|\mathfrak{F}'(\omega_1)|+|\mathfrak{F}'(\omega_2)|}{2}\left(1+\tau^2\right)+\frac{M(\omega_1,\omega_2)}{2}\left(1-\tau^2\right)\right] d\tau$$
$$= \frac{\omega_2-\omega_1}{2}\left[\frac{|\mathfrak{F}'(\omega_1)|+|\mathfrak{F}'(\omega_2)|}{2}\left(A_1(\varpi,\alpha)+A_3(\varpi,\alpha)\right)\right.$$
$$\left.+\frac{M(\omega_1,\omega_2)}{2}\left(A_1(\varpi,\alpha)-A_3(\varpi,\alpha)\right)\right].$$

Thus, the proof is finished. □

Remark 6. In Theorem 6, if we set $\alpha=1$, then we recapture the following Simpson's inequality:

$$\left|\frac{1}{6}\left[\mathfrak{F}(\omega_1)+4\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+\mathfrak{F}(\omega_2)\right]-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(\varkappa)\,d\varkappa\right|$$
$$\leq(\omega_2-\omega_1)\left[\frac{329}{7776}\left(\left|\mathfrak{F}'(\omega_1)\right|+\left|\mathfrak{F}'(\omega_2)\right|\right)+\frac{211}{7776}M(\omega_1,\omega_2)\right],$$

which is established by Samet in [10, Theorem 3.3].

Corollary 1. *If we choose $h(\varkappa)=|\mathfrak{F}'(\varkappa)|$ and $g(\varkappa)=1$ or $h(\varkappa)=1$ and $g(\varkappa)=|\mathfrak{F}'(\varkappa)|$ in Theorem 6, then we have the following inequality*

$$\left|\frac{1}{6}\left[\mathfrak{F}(\omega_1)+4\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+\mathfrak{F}(\omega_2)\right]\right.$$
$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\omega_2-\omega_1)^\alpha}\left[J^\alpha_{\omega_1+}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+J^\alpha_{\omega_2-}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)\right]\right|$$
$$\leq\frac{\omega_2-\omega_1}{2}A_1(\varpi,\alpha)\left[\left|\mathfrak{F}'(\omega_1)\right|+\left|\mathfrak{F}'(\omega_2)\right|\right].$$

Remark 7. If we take $\alpha=1$ in Corollary 1, then we have the following Simpson inequality

$$\left|\frac{1}{6}\left[\mathfrak{F}(\omega_1)+4\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+\mathfrak{F}(\omega_2)\right]-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\mathfrak{F}(\varkappa)\,d\varkappa\right|$$
$$\leq\frac{5(\omega_2-\omega_1)}{72}\left(\left|\mathfrak{F}'(\omega_1)\right|+\left|\mathfrak{F}'(\omega_2)\right|\right),$$

which is proved by Sarikaya et al. in [11].

Example 3. Let $[\omega_1,\omega_2]=[0,1]$ and define the functions $\mathfrak{F},g,h:[0,1]\to\mathbb{R}$ by $\mathfrak{F}(\varkappa)=\frac{\varkappa^4}{4}$, $g(\varkappa)=\varkappa$ and $h(\varkappa)=\varkappa^2$ such that $\mathfrak{F}'(\varkappa)=\varkappa^3$ and $|\mathfrak{F}'|$ is (g,h)-convex on $[0,1]$. Under these assumptions, we have

$$\left|\frac{1}{6}\left[\mathfrak{F}(\omega_1)+4\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+\mathfrak{F}(\omega_2)\right]\right.$$

$$
\begin{aligned}
&-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\omega_2-\omega_1)^{\alpha}}\left[J^{\alpha}_{\omega_1+}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+J^{\alpha}_{\omega_2-}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)\right]\Bigg| \\
&=\left|\frac{5}{96}-\frac{\alpha}{2^{3-\alpha}}\left[\int_0^{\frac{1}{2}}\left(\frac{1}{2}-\varkappa\right)^{\alpha-1}\varkappa^4 d\varkappa+\int_{\frac{1}{2}}^{1}\left(\varkappa-\frac{1}{2}\right)^{\alpha-1}\varkappa^4 d\varkappa\right]\right| \\
&=\left|\frac{5}{96}-\frac{\alpha}{2^{3-\alpha}}\left[\begin{array}{c}\frac{3}{2^{\alpha+1}(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)} \\ \\ +\frac{2\alpha(\alpha+4)(\alpha+2)^2+3}{2^{\alpha+1}\alpha(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}\end{array}\right]\right| \\
&=\left|\frac{5}{96}-\frac{\alpha(\alpha+4)(\alpha+2)^2+3}{8(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)}\right|
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\omega_2-\omega_1}{2}\left[\frac{|\mathfrak{F}'(\omega_1)|+|\mathfrak{F}'(\omega_2)|}{2}\left(A_1(\varpi,\alpha)+A_3(\varpi,\alpha)\right)\right. \\
&\quad\left.+\frac{M(\omega_1,\omega_2)}{2}\left(A_1(\varpi,\alpha)-A_3(\varpi,\alpha)\right)\right] \\
&=\frac{\left(A_1(\varpi,\alpha)+A_3(\varpi,\alpha)\right)}{4}.
\end{aligned}
$$

We use MATLAB software to plot the function image of the above three functions,

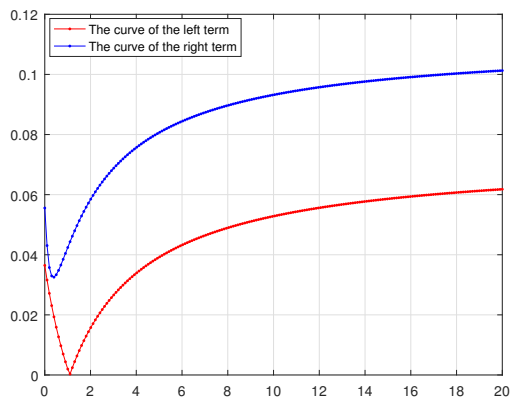

FIGURE 3. The image description for Theorem 6.

as shown in Figure 3. From the position relationship of the image, we can see that the inequalities relationship in Theorem 6 is tenable.

Theorem 7. *We assume that the conditions of Lemma 1. If $|\mathfrak{F}'|^q$, $q>1$ is (g,h)-mapping, then we have the following new fractional Simpson's inequality:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[\mathfrak{F}(\omega_1)+4\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+\mathfrak{F}(\omega_2)\right]\right.\\
&\quad\left.-\frac{\Gamma(\alpha+1)}{2(\omega_2-\omega_1)^{\alpha}}\left[J^{\alpha}_{\omega_1+}\mathfrak{F}(\omega_2)+J^{\alpha}_{\omega_2-}\mathfrak{F}(\omega_1)\right]\right|\\
&\leq\frac{(\omega_2-\omega_1)}{2}\Theta_1^{\frac{1}{p}}(\alpha,p)\times\left[\left(\frac{|\mathfrak{F}'(\omega_1)|^q+7|\mathfrak{F}'(\omega_2)|^q+2M(\omega_1,\omega_2)}{12}\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\frac{7|\mathfrak{F}'(\omega_1)|^q+|\mathfrak{F}'(\omega_2)|^q+2M(\omega_1,\omega_2)}{12}\right],
\end{aligned}$$

where

$$\Theta_1(\alpha,p)=\int_0^1\left|\frac{\tau^{\alpha}}{2}-\frac{1}{3}\right|^p d\tau$$

and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Taking modulus in (4.1) and using Hölder's inequality, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[\mathfrak{F}(\omega_1)+4\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+\mathfrak{F}(\omega_2)\right]\right.\\
&\quad\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\omega_2-\omega_1)^{\alpha}}\left[J^{\alpha}_{\omega_1+}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+J^{\alpha}_{\omega_2-}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)\right]\right|\\
&\leq\frac{(\omega_2-\omega_1)}{2}\left[\int_0^1\left|\frac{\tau^{\alpha}}{2}-\frac{1}{3}\right|\left[\begin{array}{c}\left|\mathfrak{F}'\left(\frac{1+\tau}{2}\omega_2+\frac{1-\tau}{2}\omega_1\right)\right|\\+\left|\mathfrak{F}'\left(\frac{1-\tau}{2}\omega_2+\frac{1+\tau}{2}\omega_1\right)\right|\end{array}\right]d\tau\right]\\
&\leq\frac{(\omega_2-\omega_1)}{2}\left(\int_0^1\left|\frac{\tau^{\alpha}}{2}-\frac{1}{3}\right|^p d\tau\right)^{\frac{1}{p}}\times\left[\begin{array}{c}\left(\int_0^1\left|\mathfrak{F}'\left(\frac{1+\tau}{2}\omega_2+\frac{1-\tau}{2}\omega_1\right)\right|^q\right)^{\frac{1}{q}}\\+\left(\int_0^1\left|\mathfrak{F}'\left(\frac{1-\tau}{2}\omega_2+\frac{1+\tau}{2}\omega_1\right)\right|^q\right)^{\frac{1}{q}}\end{array}\right].
\end{aligned}$$

Applying convexity of $|\mathfrak{F}'|^q$, we obtain

$$\begin{aligned}
&\left|\frac{1}{6}\left[\mathfrak{F}(\omega_1)+4\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+\mathfrak{F}(\omega_2)\right]\right.\\
&\quad\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\omega_2-\omega_1)^{\alpha}}\left[J^{\alpha}_{\omega_1+}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+J^{\alpha}_{\omega_2-}\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)\right]\right|\\
&\leq\frac{(\omega_2-\omega_1)}{2}\left[\int_0^1\left|\frac{\tau^{\alpha}}{2}-\frac{1}{3}\right|\left[\begin{array}{c}\left|\mathfrak{F}'\left(\frac{1+\tau}{2}\omega_2+\frac{1-\tau}{2}\omega_1\right)\right|\\+\left|\mathfrak{F}'\left(\frac{1-\tau}{2}\omega_2+\frac{1+\tau}{2}\omega_1\right)\right|\end{array}\right]d\tau\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{(\omega_2-\omega_1)}{2}\left(\int_0^1\left|\frac{\tau^\alpha}{2}-\frac{1}{3}\right|^p d\tau\right)^{\frac{1}{p}} \\
&\times\left[\left(\frac{(1+\tau)^2}{4}\left|\mathfrak{F}'(\omega_2)\right|^q+\frac{(1-\tau)^2}{4}\left|\mathfrak{F}'(\omega_1)\right|^q+\frac{1-\tau^2}{4}M(\omega_1,\omega_2)\right)\right. \\
&\left.\left(\frac{(1-\tau)^2}{4}\left|\mathfrak{F}'(\omega_2)\right|^q+\frac{(1+\tau)^2}{4}\left|\mathfrak{F}'(\omega_1)\right|^q+\frac{1-\tau^2}{4}M(\omega_1,\omega_2)\right)\right] \\
&\times\left[\begin{array}{c}\left(\int_0^1\left|\mathfrak{F}'\left(\frac{1+\tau}{2}\omega_2+\frac{1-\tau}{2}\omega_1\right)\right|^q\right)^{\frac{1}{q}} \\ \\ +\left(\int_0^1\left|\mathfrak{F}'\left(\frac{1-\tau}{2}\omega_2+\frac{1+\tau}{2}\omega_1\right)\right|^q\right)^{\frac{1}{q}}\end{array}\right].
\end{aligned}
$$

Thus, the proof is finished. □

Corollary 2. *In Theorem 7, if we set $\alpha=1$, then we obtain the following new Simpson's inequality:*

$$
\begin{aligned}
&\left|\frac{1}{6}\left[\mathfrak{F}(\omega_1)+4\mathfrak{F}\left(\frac{\omega_1+\omega_2}{2}\right)+\mathfrak{F}(\omega_2)\right]\right. \\
&\left.-\frac{\Gamma(\alpha+1)}{2(\omega_2-\omega_1)^\alpha}\left[J^\alpha_{\omega_1+}\mathfrak{F}(\omega_2)+J^\alpha_{\omega_2-}\mathfrak{F}(\omega_1)\right]\right| \\
&\leq \frac{\omega_2-\omega_1}{2^{1-\frac{1}{p}}}\left(\frac{1}{6^{p+1}(p+1)}+\frac{1}{3^{p+1}(p+1)}\right)^{\frac{1}{p}} \\
&\times\left[\left(\frac{|\mathfrak{F}'(\omega_2)|^q+7|\mathfrak{F}'(\omega_1)|^q+2M(\omega_1,\omega_2)}{12}\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{|\mathfrak{F}'(\omega_1)|^q+7|\mathfrak{F}'(\omega_2)|^q+2M(\omega_1,\omega_2)}{12}\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

5. Concluding Remarks

This paper introduces two fresh versions of Hermite-Hadamard type inequalities for convex functions in the context of fractional calculus, utilizing a pair of functions. Additionally, we have shown new Simpson's type inequalities for convex functions using fractional integrals. We have demonstrated through examples that these new inequalities are valid and a progression of existing ones. It is an intriguing challenge for future researchers to explore similar inequalities for other fractional integrals.

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ON ρ-STATISTICAL CONVERGENCE OF ORDER α OF SEQUENCES OF SETS

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Abstract. In this paper we introduce the concepts of Wijsman ρ-statistical convergence of order α and Wijsman strongly ρ-convergence of order α. In addition, some inclusion theorems are also presented.

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1. Introduction

The concept of statistical convergence was introduced by Steinhaus [30] and Fast [18]. Schoenberg [28] established some basic properties of statistical convergence and studied the concept as a summability method. Later on it was further investigated from the sequence space point of view and linked with summability theory by Altınok et al. [1], Aral et al. ([2,3]), Bhardwaj and Dhawan [5], Çakallı et al. ([6–8]), Caserta et al. [9], Çınar et al. [10], Connor [12], Çolak [11], Demirci et al. [13], Et et al. ([14–17]), Fridy [19], Gadjiev and Orhan [20], Işık and Akbaş [21], Salat [26], Savaş and Et [27], Şengül [29] and many others.

Let (X,σ) be a metric space. The distance $d(x,A)$ from a point x to a non-empty subset A of (X,σ) is defined to be

$$d(x,A) = \inf_{y\in A} \sigma(x,y).$$

If $\sup_k d(x,A_k) < \infty$ (for each $x \in X$), then we say that the sequence $\{A_k\}$ is bounded. The set of all bounded sequences of sets denoted $L_\infty(X)$.

The concepts of Wijsman statistical convergence for the sequence were given by Nuray and Rhoades [25] and the concept were generalized by Ulusu et al. ([31],[32]). In this paper we introduce the concepts of Wijsman ρ-statistical convergence of order α and Wijsman strongly ρ-convergence of order α.

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2. Results

Definition 1. Let (X,σ) be a metric space and $\alpha \in (0,1]$. For any non-empty closed subsets $A, A_k \subset X$, we say that the sequence $\{A_k\}$ is Wijsman ρ-statistical convergent to A of order α (or WS_ρ^α-convergent to A) if for each $\varepsilon > 0$ and $x \in X$,

$$\lim_{n\to\infty} \frac{1}{\rho_n^\alpha} \left|\{k \leqslant n : |d(x,A_k) - d(x,A)| \geq \varepsilon\}\right| = 0,$$

where $\rho = (\rho_n)$ is a non-decreasing sequence of positive real numbers tending to ∞ such that

$$\limsup_n \frac{\rho_n}{n} < \infty, \quad \Delta\rho_n = O(1) \quad \text{and} \quad \Delta\rho_n = \rho_{n+1} - \rho_n \tag{2.1}$$

for each positive integer n. In this case, we write $A_k \longrightarrow A(WS_\rho^\alpha)$. The set of all Wijsman ρ-statistical convergent sequences to A of order α will be denoted by WS_ρ^α.

If $\rho_n = n$, for all $n \in \mathbb{N}$, Wijsman ρ-statistical convergence of order α is coincided with Wijsman statistical convergence of order α denoted by $A_k \longrightarrow A\left((WS^\alpha)\right)$. If $\alpha = 1$, Wijsman ρ-statistical convergence of order α is coincided with Wijsman ρ-statistical convergence which were defined by Aral et al. [4]. In this case we write $A_k \to A(WS_\rho)$. If $\rho_n = n$ for all $n \in \mathbb{N}$ and $\alpha = 1$, Wijsman ρ-statistical convergence of order α is coincided with Wijsman statistical convergence which were defined by Nuray and Rhoades [25].

Definition 2. Let (X,σ) be a metric space and $\alpha \in (0,1]$. For any non-empty closed subsets $A, A_k \subset X$, we say that the sequence $\{A_k\}$ is said to be Wijsman strongly ρ-summable of order α to A, if for each $\varepsilon > 0$ and $x \in X$,

$$\lim_{n\to\infty} \frac{1}{\rho_n^\alpha} \sum_{k=1}^{n} |d(x,A_k) - d(x,A)| = 0.$$

The set of all Wijsman strongly ρ-summable sequences of order α to A will be denoted by W_ρ^α. In this case we write $A_k \longrightarrow A(W_\rho^\alpha)$. If $\rho_n = n$ for all $n \in \mathbb{N}$, Wijsman strong ρ-summability of order α is reduced to Wijsman strong summability of order α denoted by $A_k \longrightarrow A\left((W^\alpha)\right)$. If $\alpha = 1$, Wijsman strong ρ-summability of order α is reduced to Wijsman strong ρ-summability denoted by $A_k \longrightarrow A\left((W_\rho)\right)$ which were defined by Aral et al. [4]. If $\rho_n = n$ for all $n \in \mathbb{N}$ and $\alpha = 1$, Wijsman strong ρ-summability of order α is reduced to Wijsman strong summability which were defined by Nuray and Rhoades [25].

We give the following results without proof.

Theorem 1. *Let (X,σ) be a metric space and $\alpha \in (0,1]$. If $A_k \longrightarrow A(WS_\rho^\alpha)$, then A is unique.*

Theorem 2. *Let (X,σ) be a metric space, $\alpha \in (0,1]$ and A, A_k (for all $k \in \mathbb{N}$) be non-empty closed subsets of X, then*

(i) If $A_k \longrightarrow A(WS_\rho^\alpha)$ and $c \in \mathbb{C}$, then $cA_k \longrightarrow cA(WS_\rho^\alpha)$.
(ii) If $A_k \longrightarrow A(WS_\rho^\alpha)$ and $B_k \longrightarrow B(WS_\rho^\alpha)$, then $(A_k + B_k) \longrightarrow (A+B)(WS_\rho^\alpha)$.
(iii) If $A_k \longrightarrow A(W_\rho^\alpha)$ and $c \in \mathbb{C}$, then $cA_k \longrightarrow cA(W_\rho^\alpha)$.
(iv) If $A_k \longrightarrow A(W_\rho^\alpha)$ and $B_k \longrightarrow B(W_\rho^\alpha)$, then $(A_k + B_k) \longrightarrow (A+B)(W_\rho^\alpha)$.

Corollary 1. *In the case $\rho_n = n$ (for all $n \in \mathbb{N}$) (i)-(iv) hold in the above Theorem 2.*

Theorem 3. *Let (X,σ) be a metric space, $\alpha, \beta \in (0,1]$ such that $\alpha \leq \beta$ and A, A_k (for all $k \in \mathbb{N}$) be non-empty closed subsets of X, then $WS_\rho^\alpha \subset WS_\rho^\beta$ and the inclusion is strict.*

Proof. The inclusion part of the proof is easy. To prove the inclusion is strict define a sequence $\{A_k\}$ such that

$$A_k := \begin{cases} \{k,k\} & \text{if } k = n^2; \\ \{0,0\} & \text{otherwise.} \end{cases}$$

Then $\{A_k\} \in WS_\rho^\beta$ for $\frac{1}{2} < \beta \leqslant 1$ but $\{A_k\} \notin WS_\rho^\alpha$ for $0 < \alpha \leqslant \frac{1}{2}$. □

Corollary 2. *Let (X,σ) be a metric space, $\alpha \in (0,1]$ and A, A_k (for all $k \in \mathbb{N}$) be non-empty closed subsets of X. If $A_k \longrightarrow A(WS_\rho^\alpha)$, then $A_k \longrightarrow A(WS_\rho)$.*

Theorem 4. *Let (X,σ) be a metric space, $\alpha \in (0,1]$ and A, A_k (for all $k \in \mathbb{N}$) be non-empty closed subsets of X, then*

(i) $A_k \to A(W_\rho^\alpha) \Rightarrow A_k \to A(WS_\rho^\alpha)$ and W_ρ^α is a proper subset of WS_ρ^α.
(ii) $\{A_k\} \in L_\infty(X)$ and $A_k \to A(WS_\rho) \Rightarrow A_k \to A(W_\rho)$.
(iii) $WS_\rho \cap L_\infty(X) = W_\rho \cap L_\infty(X)$.

Proof.

(i) The inclusion part of the proof is easy. In order to show that the inclusion $W_\rho^\alpha \subseteq WS_\rho^\alpha$ is proper, we define a sequence $\{A_k\}$ as follows:

$$A_k = \begin{cases} \{\sqrt{k}\} & \text{if } k = n^2; \\ \{0\} & \text{otherwise.} \end{cases}$$

Let $X = \mathbb{R}$, $d(x,y) = |x-y|$ $(x,y \in X)$ and $\rho_n = n$ for all $n \in \mathbb{N}$. For $\varepsilon > 0$, $x > 0$ and $\frac{1}{2} < \alpha \leqslant 1$, we have

$$\frac{1}{\rho_n^\alpha} |\{k \leqslant n : |d(x,A_k) - d(x,\{0\})| \geq \varepsilon\}| \leq \frac{(\sqrt{n})}{n^\alpha} \to 0, \quad \text{as } n \to \infty$$

and so we get $A_k \to 0 \left(WS_\rho^\alpha\right)$.

On the other hand, for $0 < \alpha \leqslant 1$ and $x > 0$,

$$\frac{1}{\rho_n^\alpha} \sum_{k=1}^{n} |d(x,A_k) - d(x,\{0\})| = \frac{\sqrt{n}(\sqrt{n}+1)}{2n^\alpha} \to \infty$$

and for $\alpha = 1$

$$\frac{\sqrt{n}\left(\sqrt{n}+1\right)}{n^{\alpha}} \to 1.$$

So $A_k \nrightarrow 0\left(W_{\rho}^{\alpha}\right)$.

(ii) Omitted.

(iii) Omitted.

□

Theorem 5. *Let (X,σ) be a metric space, $\alpha \in (0,1]$ and A, A_k (for all $k \in \mathbb{N}$) be non-empty closed subsets of X. Then, $WS^{\alpha} \subset WS_{\rho}^{\alpha}$ if $\liminf_n \frac{\rho_n^{\alpha}}{n^{\alpha}} > 0$.*

Proof. Proof follows from the following inequality:

$$\frac{1}{n^{\alpha}}\left|\{k \leqslant n : |d(x,A_k) - d(x,A)| \geq \varepsilon\}\right| \geqslant \frac{{\rho_n}^{\alpha}}{n^{\alpha}}\frac{1}{\rho_n^{\alpha}}\left|\{k \leqslant n : |d(x,A_k) - d(x,A)| \geq \varepsilon\}\right|.$$

□

Theorem 6. *Let (X,σ) be a metric space, $\alpha \in (0,1]$ and A, A_k (for all $k \in \mathbb{N}$) be non-empty closed subsets of X. $WS \subset WS_{\rho}^{\alpha}$ if*

$$\liminf_{n\to\infty} \frac{\rho_n^{\alpha}}{n} > 0.$$

Proof. Omitted. □

Theorem 7. *Let (X,σ) be a metric space, $\rho = (\rho_n)$ and $\tau = (\tau_n)$ be two sequences satisfying (2.1) conditions such that $\rho_n \leqslant \tau_n$ for all $n \in \mathbb{N}$ and $0 < \alpha \leqslant \beta \leqslant 1$. If*

$$\lim_{n\to\infty} \frac{\rho_n^{\alpha}}{\tau_n^{\beta}} = a > 0, \tag{2.2}$$

then $WS_{\tau}^{\beta} \subseteq WS_{\rho}^{\alpha}$.

Proof. Omitted. □

Corollary 3. *Let (X,σ) be a metric space, $\rho = (\rho_n)$ and $\tau = (\tau_n)$ be two sequences satisfying (2.1) conditions such that $\rho_n \leqslant \tau_n$ for all $n \in \mathbb{N}$ and $0 < \alpha \leqslant 1$. If (2.2) holds, then*

(i) $WS_{\tau}^{\alpha} \subseteq WS_{\rho}^{\alpha}$.

(ii) $WS_{\tau} \subseteq WS_{\rho}^{\alpha}$.

Theorem 8. *Let (X,σ) be a metric space, $\rho = (\rho_n)$ and $\tau = (\tau_n)$ be two sequences satisfying (2.1) conditions such that $\rho_n \leqslant \tau_n$ for all $n \in \mathbb{N}$ and $0 < \alpha \leqslant \beta \leqslant 1$. If*

$$\limsup_{n\to\infty} \frac{\rho_n^{\alpha}}{\tau_n^{\beta}} < \infty, \tag{2.3}$$

then $WS_{\rho}^{\alpha} \subseteq WS_{\tau}^{\beta}$.

Proof. Let $A_k \to A(WS_\rho^\alpha)$ and for $\varepsilon > 0$. We have

$$\frac{1}{\tau_n^\beta}\left|\{k \leqslant n : |d(x,A_k) - d(x,A)| \geq \varepsilon\}\right| \leqslant \frac{\rho_n^\alpha}{\tau_n^\beta}\frac{1}{\rho_n^\alpha}\left|\{k \leqslant n : |d(x,A_k) - d(x,A)| \geq \varepsilon\}\right|.$$

Using the condition and $A_k \to A(WS_\rho^\alpha)$ we have $WS_\rho^\alpha \subseteq WS_\tau^\beta$.

□

Corollary 4. *Let (X,σ) be a metric space, $\rho = (\rho_n)$ and $\tau = (\tau_n)$ be two sequences satisfying (2.1) conditions such that $\rho_n \leqslant \tau_n$ for all $n \in \mathbb{N}$ and $0 < \alpha \leqslant 1$. If (2.3) holds, then*

(i) $WS_\rho^\alpha \subseteq WS_\tau^\alpha$.
(ii) $WS_\rho^\alpha \subseteq WS_\tau$.

3. Results related to Orlicz function

An Orlicz function is a function $M\colon [0,\infty) \to [0,\infty)$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

The study of Orlicz sequence spaces was initiated with a certain specific purpose in Banach space theory. Indeed, Lindberg [23] got interested in Orlicz sequence spaces in connection with finding Banach spaces with symmetric Schauder bases having complementary subspaces isomorphic to c_0 or ℓ_p $(0 \leq p < \infty)$. Subsequently, Lindenstrauss and Tzafriri [24] used the idea of Orlicz function to construct the sequence space

$$\ell_M = \left\{x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \text{ for some } \rho > 0\right\}.$$

The space ℓ_M with the norm

$$\|x\| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\}$$

becomes a Banach space, called an Orlicz sequence space. The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(x) = |x|^p$ for $1 \leq p < \infty$. Lindenstrauss and Tzafriri [24] proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to l_p $(1 \leq p < \infty)$. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [22].

It is well known that if M is a convex function and $M(0) = 0$, then $M(\lambda x) \leqslant \lambda M(x)$ for all λ with $0 < \lambda < 1$.

Definition 3. Let (X,σ) be a metric space. Let M be an Orlicz function, $p = (p_k)$ be a sequence of strictly positive real numbers, $\alpha \in (0,1]$, $\rho = (\rho_n)$ be a sequence

such as above and for $\lambda > 0$, now we define

$$(W_\rho^\alpha, [M,(p)]) = \left\{ \{A_k\} \in X : \frac{1}{\rho_n^\alpha} \sum_{k=1}^{n} \left[M\left(\frac{|d(x,A_k) - d(x,A)|}{\lambda} \right) \right]^{p_k} \to 0, \right.$$
$$\left. \text{for some } A \text{ and for } x \in X \right\}.$$

If $M(x) = x$ and $p_k = 1$ for all $k \in \mathbb{N}$ then we shall write $(W_\rho^\alpha, [M,(p)]) = W_\rho^\alpha$ and if $M(x) = x$, then we shall write $(W_\rho^\alpha, [M,(p)]) = (W_\rho^\alpha, (p))$. If $p_k = 1$ for all $k \in \mathbb{N}$ then we shall write $(W_\rho^\alpha, [M,(p)]) = \left(W_\rho^\alpha, [M]\right)$.

In the following theorems, assume that the sequence $p = (p_k)$ is bounded and $0 < h = \inf_k p_k \leqslant p_k \leqslant \sup_k p_k = H < \infty$.

Theorem 9. *Let $0 < \alpha \leqslant \beta \leqslant 1$, M be an Orlicz function and $\rho = (\rho_n)$ be a sequence such as above, then $(W_\rho^\alpha, [M,(p)]) \subset WS_\rho^\beta$.*

Proof. Let $\{A_k\} \in (W_\rho^\alpha, [M,(p)]$. Let $\varepsilon > 0$ be given. As $\rho_n^\alpha \leqslant \rho_n^\beta$ for each n we can write

$$\frac{1}{\rho_n^\alpha} \sum_{k=1}^{n} \left[M\left(\frac{|d(x,A_k) - d(x,A)|}{\lambda} \right) \right]^{p_k} =$$
$$\frac{1}{\rho_n^\alpha} \sum_{\substack{k=1 \\ |d(x,A_k)-d(x,A)| \geqslant \varepsilon}}^{n} \left[M\left(\frac{|d(x,A_k) - d(x,A)|}{\lambda} \right) \right]^{p_k}$$
$$+ \frac{1}{\rho_n^\alpha} \sum_{\substack{k=1 \\ |d(x,A_k)-d(x,A)| < \varepsilon}}^{n} \left[M\left(\frac{|d(x,A_k) - d(x,A)|}{\lambda} \right) \right]^{p_k}$$
$$\geqslant \frac{1}{\rho_n^\beta} \sum_{\substack{k=1 \\ |d(x,A_k)-d(x,A)| \geqslant \varepsilon}}^{n} \left[M\left(\frac{|d(x,A_k) - d(x,A)|}{\lambda} \right) \right]^{p_k}$$
$$+ \frac{1}{\rho_n^\alpha} \sum_{\substack{k=1 \\ |d(x,A_k)-d(x,A)| < \varepsilon}}^{n} \left[M\left(\frac{|d(x,A_k) - d(x,A)|}{\lambda} \right) \right]^{p_k}$$
$$\geqslant \frac{1}{\rho_n^\beta} \sum_{\substack{k=1 \\ |d(x,A_k)-d(x,A)| \geqslant \varepsilon}}^{n} \left[M\left(\frac{\varepsilon}{\lambda} \right) \right]^{p_k} \geqslant \frac{1}{\rho_n^\beta} \sum_{\substack{k=1 \\ |d(x,A_k)-d(x,A)| \geqslant \varepsilon}}^{n} \min\left([M(\varepsilon_1)]^h, [M(\varepsilon_2)]^H \right)$$
$$\geqslant \frac{1}{\rho_n^\beta} |\{k \leqslant n : |d(x,A_k) - d(x,A)| \geq \varepsilon\}| \min\left([M(\varepsilon_1)]^h, [M(\varepsilon_1)]^H \right).$$

where $\varepsilon_1 = \frac{\varepsilon}{\lambda}$. From the above inequality we have $\{A_k\} \in WS_\rho^\beta$.

□

Corollary 5.

(i) Let $0 < \alpha \leqslant 1$, *M be an Orlicz function and* $\rho = (\rho_n)$ *be a sequence as above, then* $(W_\rho^\alpha, [M,(p)]) \subset WS_\rho^\alpha$.

(ii) Let $\rho = (\rho_n)$ *be a sequence as above and* $0 < \alpha \leqslant \beta \leqslant 1$, *then* $(W_\rho^\alpha, [M]) \subseteq (WS_\rho^\beta)$.

Theorem 10. *Let M be an Orlicz function,*$\{A_k\}$ *be a sequence in* $L_\infty(X)$ *and* $\rho = (\rho_n)$ *be a sequence. If* $\lim_{n\to\infty} \frac{\rho_n}{\rho_n^\alpha} = 1$, *then* $WS_\rho^\alpha \subseteq (W_\rho^\alpha, [M,(p)])$.

Proof. Suppose that $\{A_k\}$ is in $L_\infty(X)$ and $A_k \longrightarrow A(WS_\rho^\alpha)$. As $\{A_k\} \in L_\infty(X)$ there exists $T > 0$ such that $|d(x,A_k) - d(x,A)| \leqslant T$ for all k. For given $\varepsilon > 0$ we have

$$\frac{1}{\rho_n^\alpha}\sum_{k=1}^{n}\left[M\left(\frac{|d(x,A_k)-d(x,A)|}{\lambda}\right)\right]^{p_k} =$$

$$\frac{1}{\rho_n^\alpha}\sum_{\substack{k=1\\ |d(x,A_k)-d(x,A)|\geqslant\varepsilon}}^{n}\left[M\left(\frac{|d(x,A_k)-d(x,A)|}{\lambda}\right)\right]^{p_k}$$

$$+\frac{1}{\rho_n^\alpha}\sum_{\substack{k=1\\ |d(x,A_k)-d(x,A)|<\varepsilon}}^{n}\left[M\left(\frac{|d(x,A_k)-d(x,A)|}{\lambda}\right)\right]^{p_k}$$

$$\leqslant \frac{1}{\rho_n^\alpha}\sum_{\substack{k=1\\ |d(x,A_k)-d(x,A)|\geqslant\varepsilon}}^{n}\max\left\{\left[M\left(\frac{T}{\lambda}\right)\right]^h, \left[M\left(\frac{T}{\lambda}\right)\right]^H\right\}$$

$$+\frac{1}{\rho_n^\alpha}\sum_{\substack{k=1\\ |d(x,A_k)-d(x,A)|<\varepsilon}}^{n}\left[M\left(\frac{\varepsilon}{\lambda}\right)\right]^{p_k}$$

$$\leqslant \max\left\{\left[M\left(\frac{T}{\lambda}\right)\right]^h, \left[M\left(\frac{T}{\lambda}\right)\right]^H\right\}\frac{1}{\rho_n^\alpha}\left|\{k\leqslant n : |d(x,A_k)-d(x,A)| \geq \varepsilon\}\right|$$

$$+\frac{\rho_n}{\rho_n^\alpha}\max\left\{\left[M\left(\frac{\varepsilon}{\lambda}\right)\right]^h, \left[M\left(\frac{\varepsilon}{\lambda}\right)\right]^H\right\}.$$

Therefore $\{A_k\} \in (W_\rho^\alpha, [M,(p)])$.

□

We give the following results without proof.

Theorem 11. *Let* $\rho = (\rho_n)$ *be a sequence as above and* $0 < \alpha \leqslant \beta \leqslant 1$, *then* $(W_\rho^\alpha, [M]) \subseteq (W_\rho^\beta, [M])$.

Theorem 12. *Let* $\rho = (\rho_n)$ *and* $\tau = (\tau_n)$ *be two sequences such that* $\rho_n \leqslant \tau_n$ *for all* $n \in \mathbb{N}$, $0 < \alpha \leqslant \beta \leqslant 1$. *If* $\lim_{n\to\infty} \frac{\rho_n^\alpha}{\tau_n^\beta} = a > 0$, *then* $(W_\tau^\beta, [M]) \subseteq (W_\rho^\alpha, [M])$.

Corollary 6. *Let* $\rho = (\rho_n)$ *and* $\tau = (\tau_n)$ *be two sequences such that* $\rho_n \leqslant \tau_n$ *for all* $n \in \mathbb{N}$. *If* $\lim_{n\to\infty} \frac{\rho_n^\alpha}{\tau_n^\beta} = a > 0$ *holds, then* $(W_\tau^\alpha, [M]) \subseteq (W_\rho^\alpha, [M])$ *for* $0 < \alpha \leqslant 1$.

Theorem 13. *Let* $\rho = (\rho_n)$ *and* $\tau = (\tau_n)$ *be two sequences such that* $\rho_n \leqslant \tau_n$ *for all* $n \in \mathbb{N}$, $0 < \alpha \leqslant \beta \leqslant 1$. *If* $\lim_{n\to\infty} \frac{\rho_n^\alpha}{\tau_n^\beta} = 1$, *then* $(W_\tau^\beta, [M]) \subseteq (WS_\rho^\alpha)$.

Theorem 14. *Let* M *be an Orlicz function and* $\inf_k p_k > 0$, *the limit of any sequence* $\{A_k\} \in (W_\rho^\alpha, [M, (p)])$ *is unique.*

Proof. Let $\lim_k p_k = s > 0$. Suppose that $A_k \to A_1((W_\rho^\alpha, [M, (p)]))$ and $A_k \to A(W_\rho^\alpha, [M, (p)])$. Then there exists $\lambda_1 > 0$ and $\lambda_2 > 0$ such that

$$\lim_{n\to\infty} \frac{1}{\rho_n^\alpha} \sum_{k=1}^{n} \left[M\left(\frac{|d(x, A_k) - d(x, A_1)|}{\lambda_1} \right) \right]^{p_k} = 0,$$

and

$$\lim_{n\to\infty} \frac{1}{\rho_n^\alpha} \sum_{k=1}^{n} \left[M\left(\frac{|d(x, A_k) - d(x, A_2)|}{\lambda_2} \right) \right]^{p_k} = 0.$$

Let $\lambda = \max\{2\lambda_1, 2\lambda_2\}$. As M is nondecreasing and convex, we have

$$\begin{aligned}
&\frac{1}{\rho_n^\alpha} \sum_{k=1}^{n} \left[M\left(\frac{|d(x, A_1) - d(x, A_2)|}{\lambda} \right) \right]^{p_k} \\
&\leqslant \frac{D}{\rho_n^\alpha} \sum_{k=1}^{n} \frac{1}{2^{p_k}} \left(\left[M\left(\frac{|d(x, A_k) - d(x, A_1)|}{\lambda_1} \right) \right]^{p_k} + \left[M\left(\frac{|d(x, A_k) - d(x, A_2)|}{\lambda_2} \right) \right]^{p_k} \right) \\
&\leqslant \frac{D}{\rho_n^\alpha} \sum_{k=1}^{n} \frac{1}{2^{p_k}} \left(\left[M\left(\frac{|d(x, A_k) - d(x, A_1)|}{\lambda_1} \right) \right]^{p_k} \right. \\
&\quad \left. + \frac{D}{\rho_n^\alpha} \sum_{k=1}^{n} \frac{1}{2^{p_k}} \left(\left[M\left(\frac{|d(x, A_k) - d(x, A_2)|}{\lambda_2} \right) \right]^{p_k} \right) \right) \to 0
\end{aligned}$$

as $n \to \infty$, where $\sup_k p_k = H$ and $D = \max(1, 2^{H-1})$. Therefore we get

$$\lim_{n\to\infty} \frac{1}{\rho_n^\alpha} \sum_{k=1}^{n} \left[M\left(\frac{|d(x, A_1) - d(x, A_2)|}{\lambda} \right) \right]^{p_k} = 0.$$

As $\lim_k p_k = s$, we have

$$\lim_{k\to\infty} \left[M\left(\frac{|d(x, A_1) - d(x, A_2)|}{\lambda} \right) \right]^{p_k} = \left[M\left(\frac{|d(x, A_1) - d(x, A_2)|}{\lambda} \right) \right]^{s}$$

and so $A_1 = A_2$. Thus, the limit is unique. □

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σ-CENTRALIZERS OF GENERALIZED MATRIX ALGEBRAS

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Abstract. In the present paper, we characterize Lie (Jordan) σ-centralizers of generalized matrix algebras. More precisely, we obtain some conditions under which every Lie σ-centralizer of a generalized matrix algebra can be expressed as the sum of a σ-centralizer and a center-valued mapping. Further, it is shown that under certain appropriate assumptions every Jordan σ-centralizer of a generalized matrix algebra is a σ-centralizer. Finally, the main results are applied to triangular algebras.

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1. Introduction

Let $\mathcal{A}$ be an algebra over a commutative unital ring $\mathcal{R}$ with center $\mathcal{Z}(\mathcal{A})$. For any $a,b \in \mathcal{A}$, $[a,b] = ab - ba$ (resp. $a \circ b = ab + ba$) denotes the Lie product (resp. Jordan product). Let σ be an automorphism of $\mathcal{A}$. An $\mathcal{R}$-linear mapping $L : \mathcal{A} \to \mathcal{A}$ is called a left σ-centralizer (resp. right σ-centralizer) if $L(ab) = L(a)\sigma(b)$ (resp. $L(ab) = \sigma(a)L(b)$) for all $a,b \in \mathcal{A}$. It is called a *σ-centralizer* if it is both a left σ-centralizer as well as a right σ-centralizer. An $\mathcal{R}$-linear mapping $L : \mathcal{A} \to \mathcal{A}$ is called a *Lie σ-centralizer* if $L([a,b]) = [L(a),\sigma(b)]$ (or $L([a,b]) = [\sigma(a),L(b)]$) for all $a,b \in \mathcal{A}$. An $\mathcal{R}$-linear mapping $L : \mathcal{A} \to \mathcal{A}$ is called a *Jordan σ-centralizer* if $L(a \circ b) = L(a) \circ \sigma(b)$ (or $L(a \circ b) = \sigma(a) \circ L(b)$) for all $a,b \in \mathcal{A}$. One can easily see that the conditions $L([a,b]) = [L(a),\sigma(b)]$ (resp. $L(a \circ b) = L(a) \circ \sigma(b)$) and $L([a,b]) = [\sigma(a),L(b)]$ (resp. $L(a \circ b) = \sigma(a) \circ L(b)$) are equivalent. Obviously, every σ-centralizer is a Lie σ-centralizer as well as a Jordan σ-centralizer but the converse statements are not true in general. If $d : \mathcal{A} \to \mathcal{A}$ is a σ-centralizer and $\ell : \mathcal{A} \to \mathcal{Z}(\mathcal{A})$ is a linear mapping, then $d + \ell$ is a Lie σ-centralizer on $\mathcal{A}$ if and only if $\ell([a,b]) = 0$ for all $a,b \in \mathcal{A}$. A Lie σ-centralizer is called *proper* if it can be written as the sum of a σ-centralizer and a center-valued mapping.

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Let $\mathcal{R}$ be a commutative ring with identity. A Morita context consists of two $\mathcal{R}$-algebras $\mathcal{A}$ and $\mathcal{B}$, two bimodules ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ and ${}_{\mathcal{B}}\mathcal{N}_{\mathcal{A}}$, and two bimodule homomorphisms called the pairings $\zeta_{\mathcal{M}\mathcal{N}} : \mathcal{M} \bigotimes_{\mathcal{B}} \mathcal{N} \to \mathcal{A}$ and $\psi_{\mathcal{N}\mathcal{M}} : \mathcal{N} \bigotimes_{\mathcal{A}} \mathcal{M} \to \mathcal{B}$ satisfying the following commutative diagrams:

$$\begin{tikzcd}
\mathcal{M} \underset{\mathcal{B}}{\otimes} \mathcal{N} \underset{\mathcal{A}}{\otimes} \mathcal{M} \arrow[r, "\zeta_{\mathcal{M}\mathcal{N}} \otimes I_{\mathcal{M}}"] \arrow[d, "I_{\mathcal{M}} \otimes \psi_{\mathcal{N}\mathcal{M}}"] & \mathcal{A} \underset{\mathcal{A}}{\otimes} \mathcal{M} \arrow[d, "\cong"] \\
\mathcal{M} \underset{\mathcal{B}}{\otimes} \mathcal{B} \arrow[r, "\cong"] & \mathcal{M}
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
\mathcal{N} \underset{\mathcal{A}}{\otimes} \mathcal{M} \underset{\mathcal{B}}{\otimes} \mathcal{N} \arrow[r, "\psi_{\mathcal{N}\mathcal{M}} \otimes I_{\mathcal{N}}"] \arrow[d, "I_{\mathcal{N}} \otimes \zeta_{\mathcal{M}\mathcal{N}}"] & \mathcal{B} \underset{\mathcal{B}}{\otimes} \mathcal{N} \arrow[d, "\cong"] \\
\mathcal{N} \underset{\mathcal{A}}{\otimes} \mathcal{A} \arrow[r, "\cong"] & \mathcal{N}.
\end{tikzcd}$$

We denote this Morita context by $(\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}, \zeta_{\mathcal{M}\mathcal{N}}, \psi_{\mathcal{N}\mathcal{M}})$. If $(\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}, \zeta_{\mathcal{M}\mathcal{N}}, \psi_{\mathcal{N}\mathcal{M}})$ is a Morita context, then the set

$$\begin{pmatrix} \mathcal{A} & \mathcal{M} \\ \mathcal{N} & \mathcal{B} \end{pmatrix} = \left\{ \begin{pmatrix} a & m \\ n & b \end{pmatrix} \middle| \, a \in \mathcal{A}, m \in \mathcal{M}, n \in \mathcal{N}, b \in \mathcal{B} \right\}$$

forms an $\mathcal{R}$-algebra under the usual matrix operations, where at least one of the two bimodules $\mathcal{M}$ and $\mathcal{N}$ is distinct from zero. Such an $\mathcal{R}$-algebra is called a *generalized matrix algebra* and it is denoted by $\mathcal{G} = G(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$.

Throughout the paper, we assume that $\mathcal{A}$ and $\mathcal{B}$ are unital algebras with identity elements $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$, respectively and $\mathcal{M}$ is a faithful $(\mathcal{A}, \mathcal{B})$-bimodule, but no any restrictions on $\mathcal{N}$. In view of [19, Lemma 3.1], the center of $\mathcal{G}$ is given by

$$\mathcal{Z}(\mathcal{G}) = \left\{ a \oplus b \;\middle|\; a \in \mathcal{A}, \;\; b \in \mathcal{B}, \;\; am = mb, \;\; na = bn \;\text{ for all }\; m \in \mathcal{M}, \;\; n \in \mathcal{N} \right\},$$

where $a \oplus b = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. Define two natural projections $\pi_{\mathcal{A}} : \mathcal{G} \to \mathcal{A}$ and $\pi_{\mathcal{B}} : \mathcal{G} \to \mathcal{B}$ by $\pi_{\mathcal{A}} \begin{pmatrix} a & m \\ n & b \end{pmatrix} = a$ and $\pi_{\mathcal{B}} \begin{pmatrix} a & m \\ n & b \end{pmatrix} = b$. By [19, Lemmas 3.1 and 3.2], it is easy to see that $\pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{G})) \subseteq \mathcal{Z}(\mathcal{A})$, $\pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{G})) \subseteq \mathcal{Z}(\mathcal{B})$ and there exists a unique algebra isomorphism $\xi : \pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{G})) \to \pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{G}))$ such that $am = m\xi(a)$ and $na = \xi(a)n$ for all $m \in \mathcal{M}$, $n \in \mathcal{N}$.

The aim of this paper is to characterize Lie (Jordan) σ-centralizers of generalized matrix algebras. Over the past few decades, a lot of work concerning characterizations of Lie (Jordan) mappings on various rings and algebras have been done (see [2, 3, 5, 9, 10, 15, 16, 21] and references therein). A related problem in this area is to characterize centralizers on various rings and algebras. Centralizers on different rings

and algebras have been broadly examined by many algebraists (see [6–8] and references therein). Zalar [20] introduced the notion of Jordan centralizers and proved that every Jordan centralizer on a 2-torsion free semiprime ring is a centralizer. Vukman and Kosi-Ulbl [17, 18] extensively studied centralizers mainly on semiprime rings. In the year 2019, Fošner and Jing [6] introduced the notion of Lie centralizers and investigated the additivity of Lie centralizers of triangular rings. Recently, in [14], Liu studied the structure of nonlinear Lie centralizers of generalized matrix algebras. Inspired by these results, in this paper we investigate Lie (Jordan) σ-centralizers of generalized matrix algebras. In fact, we prove that under certain restrictions every Lie σ-centralizer of a generalized matrix algebra is proper (Theorem 1), and every Jordan σ-centralizer of a generalized matrix algebra is a σ-centralizer (Theorem 2).

The study of group of automorphisms is an important key for understanding the underlying algebraic structure. Hence, the study of group of automorphisms of various kinds of algebraic structures have been extensively investigated in the literature (see [1, 4, 11–13] and references therein). However, it is not always possible to determine all the automorphisms of the given algebraic structure in the general case. For example, Boboc et al. in [4, Remark 4.8] explained that Morita context rings are the rings with non-trivial idempotents, and hence there is virtually no hope of finding all automorphisms of Morita context rings in the general case. Therefore, we shall use the following class of automorphisms of a generalized matrix algebra $\mathcal{G}$ which, under certain restrictions on $\mathcal{G}$, coincides with the group of all automorphisms of $\mathcal{G}$.

Lemma 1. *[4, Proposition 2.1] Let $(\gamma,\delta,u,v,m_0,n_0)$ be a 6-tuple such that $\gamma:\mathcal{A}\to\mathcal{A}$ and $\delta:\mathcal{B}\to\mathcal{B}$ are automorphisms, $u:\mathcal{M}\to\mathcal{M}$ is a (γ,δ)-bimodule isomorphism, $v:\mathcal{N}\to\mathcal{N}$ is a (δ,γ)-bimodule isomorphism, $m_0\in\mathcal{M}$ and $n_0\in\mathcal{N}$ are fixed elements, such that the following conditions are satisfied:*

(i) $m_0\mathcal{N}=0$ and $\mathcal{N}m_0=0$,
(ii) $\mathcal{M}n_0=0$ and $n_0\mathcal{M}=0$,
(iii) $u(m)v(n)=\gamma(mn)$ and $v(n)u(m)=\delta(nm)$ for all $m\in\mathcal{M}$ and $n\in\mathcal{N}$.

Then the mapping $\sigma:\mathcal{G}\to\mathcal{G}$ defined by

$$\sigma\begin{pmatrix} a & m \\ n & b \end{pmatrix}=\begin{pmatrix} \gamma(a) & \gamma(a)m_0-m_0\delta(b)+u(m) \\ n_0\gamma(a)-\delta(b)n_0+v(n) & \delta(b) \end{pmatrix}$$

is an automorphism.

Let us denote the set of all automorphisms of $\mathcal{G}$ defined in Lemma 1 by $Aut_0^0(\mathcal{G})$ and the group of all automorphisms of $\mathcal{G}$ by $Aut(\mathcal{G})$. It is easy to see that $Aut_0^0(\mathcal{G})$ is a subgroup $Aut(\mathcal{G})$ containing the identity automorphism. In [4, Theorem 4.7], Boboc et al. proved that if $\mathcal{G}=G(\mathcal{A},\mathcal{M},\mathcal{N},\mathcal{B})$ is a generalized matrix algebra such that $\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents and both the bilinear mappings $\zeta_{\mathcal{MN}},\psi_{\mathcal{NM}}$ are zero, then $Aut_0^0(\mathcal{G})\cup Aut_0^1(\mathcal{G})=Aut(\mathcal{G})$, where $Aut_0^1(\mathcal{G})$ is the set of all automorphisms of $\mathcal{G}$ as given in [4, Proposition 2.2]. Note that if $\mathcal{A}$ and $\mathcal{B}$ are not isomorphic,

then $Aut_0^1(\mathcal{G}) = \varnothing$. Therefore, if $\mathcal{G} = G(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$ is a generalized matrix algebra such that $\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents, $\mathcal{A}$ and $\mathcal{B}$ are not isomorphic and both the bilinear mappings $\zeta_{\mathcal{M}\mathcal{N}}, \psi_{\mathcal{N}\mathcal{M}}$ are zero, then $Aut_0^0(\mathcal{G})$ coincides with $Aut(\mathcal{G})$.

2. σ-CENTRALIZERS OF GENERALIZED MATRIX ALGEBRAS

In this section, we give the structure of a σ-centralizer $\Delta : \mathcal{G} \to \mathcal{G}$ with associated automorphism σ of $\mathcal{G}$ as given in Lemma 1.

Proposition 1. *Let $\Delta : \mathcal{G} \to \mathcal{G}$ be a σ-centralizer with associated automorphism σ of $\mathcal{G}$ as given in Lemma 1. Then Δ is of the form*

$$\Delta\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} A_{11}(a) & A_{11}(a)m_0 - m_0B_{22}(b) + C_{12}(m) \\ n_0A_{11}(a) - B_{22}(b)n_0 + D_{21}(n) & B_{22}(b) \end{pmatrix},$$

where $A_{11} : \mathcal{A} \to \mathcal{A}$, $C_{12} : \mathcal{M} \to \mathcal{M}$, $D_{21} : \mathcal{N} \to \mathcal{N}$ and $B_{22} : \mathcal{B} \to \mathcal{B}$ are $\mathcal{R}$-linear mappings satisfying the following conditions:

(*i*) *A_{11} is a γ-centralizer of $\mathcal{A}$, B_{22} is a δ-centralizer of $\mathcal{B}$;*
(*ii*) $A_{11}(mn) = C_{12}(m)v(n) = u(m)D_{21}(n)$, $B_{22}(nm) = D_{21}(n)u(m) = v(n)C_{12}(m)$;
(*iii*) $C_{12}(am) = A_{11}(a)u(m) = \gamma(a)C_{12}(m)$, $C_{12}(mb) = C_{12}(m)\delta(b) = u(m)B_{22}(b)$;
(*iv*) $D_{21}(na) = v(n)A_{11}(a) = D_{21}(n)\gamma(a)$, $D_{21}(bn) = \delta(b)D_{21}(n) = B_{22}(b)v(n)$.

Proof. Suppose that the σ-centralizer Δ is of the form

$$\Delta\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} A_{11}(a)+B_{11}(b)+C_{11}(m)+D_{11}(n) & A_{12}(a)+B_{12}(b)+C_{12}(m)+D_{12}(n) \\ A_{21}(a)+B_{21}(b)+C_{21}(m)+D_{21}(n) & A_{22}(a)+B_{22}(b)+C_{22}(m)+D_{22}(n) \end{pmatrix},$$

where $A_{11}, B_{11}, C_{11}, D_{11}$ are $\mathcal{R}$-linear mappings from $\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}$ to $\mathcal{A}$, respectively; $A_{12}, B_{12}, C_{12}, D_{12}$ are $\mathcal{R}$-linear mappings from $\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}$ to $\mathcal{M}$, respectively; A_{21}, B_{21},
C_{21}, D_{21} are $\mathcal{R}$-linear mappings from $\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}$ to $\mathcal{N}$, respectively; and A_{22}, B_{22}, C_{22}, D_{22} are $\mathcal{R}$-linear mappings from $\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}$ to $\mathcal{B}$, respectively.

Since Δ is a σ-centralizer, we have

$$\Delta(xy) = \Delta(x)\sigma(y) = \sigma(x)\Delta(y) \text{ for all } x, y \in \mathcal{G}. \tag{2.1}$$

Let us choose $x = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$ in (2.1). Then, we find that

$$\begin{aligned} \begin{pmatrix} C_{11}(am) & C_{12}(am) \\ C_{21}(am) & C_{22}(am) \end{pmatrix} &= \begin{pmatrix} 0 & A_{11}(a)u(m) \\ 0 & A_{21}(a)u(m) \end{pmatrix} \\ &= \begin{pmatrix} \gamma(a)C_{11}(m) & \gamma(a)C_{12}(m) + \gamma(a)m_0C_{22}(m) \\ n_0\gamma(a)C_{11}(m) & n_0\gamma(a)C_{12}(m) \end{pmatrix}. \end{aligned}$$

Thus,

$$\begin{aligned} &C_{11}(am) = 0 = \gamma(a)C_{11}(m), \\ &C_{12}(am) = A_{11}(a)u(m) = \gamma(a)C_{12}(m) + \gamma(a)m_0C_{22}(m), \\ &C_{21}(am) = 0 = n_0\gamma(a)C_{11}(m) \text{ and} \end{aligned}$$

$$C_{22}(am) = A_{21}(a)u(m) = n_0\gamma(a)C_{12}(m).$$

Putting $a = 1_{\mathcal{A}}$, we get

$$C_{11}(m) = 0, m_0C_{22}(m) = 0, C_{21}(m) = 0 \text{ and } C_{22}(m) = n_0C_{12}(m) = 0.$$

Hence, $C_{12}(am) = A_{11}(a)u(m) = \gamma(a)C_{12}(m)$. Similarly, choose $x = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$ in (2.1) to obtain $C_{12}(mb) = C_{12}(m)\delta(b) = u(m)B_{22}(b)$.

If we take $x = \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}$ and $y = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ in (2.1), then we have

$$\begin{aligned}\begin{pmatrix} D_{11}(na) & D_{12}(na) \\ D_{21}(na) & D_{22}(na) \end{pmatrix} &= \begin{pmatrix} D_{11}(n)\gamma(a) & D_{11}(n)\gamma(a)m_0 \\ D_{21}(n)\gamma(a) + D_{22}(n)n_0\gamma(a) & D_{21}(n)\gamma(a)m_0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ v(n)A_{11}(a) & v(n)A_{12}(a) \end{pmatrix}.\end{aligned}$$

This implies that $D_{11}(na) = D_{11}(n)\gamma(a) = 0$, $D_{12}(na) = D_{11}(n)\gamma(a)m_0 = 0$, $D_{21}(na) = D_{21}(n)\gamma(a) + D_{22}(n)n_0\gamma(a) = v(n)A_{11}(a)$ and $D_{22}(na) = D_{21}(n)\gamma(a)m_0 = v(n)A_{12}(a)$. Putting $a = 1_{\mathcal{A}}$, we get $D_{11}(n) = 0$, $D_{12}(n) = 0$, $D_{22}(n)n_0 = 0$ and $D_{22}(n) = D_{21}(n)m_0 = 0$. Hence, $D_{21}(na) = D_{21}(n)\gamma(a) = v(n)A_{11}(a)$. Similarly, by taking $x = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}$ in (2.1), we can obtain $D_{21}(bn) = B_{22}(b)v(n) = \delta(b)D_{21}(n)$.

Consider $x = \begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix}$, $y = \begin{pmatrix} a_2 & 0 \\ 0 & 0 \end{pmatrix}$ in (2.1). Then

$$\begin{aligned}\begin{pmatrix} A_{11}(a_1a_2) & A_{12}(a_1a_2) \\ A_{21}(a_1a_2) & A_{22}(a_1a_2) \end{pmatrix} &= \begin{pmatrix} A_{11}(a_1)\gamma(a_2) & A_{11}(a_1)\gamma(a_2)m_0 \\ A_{21}(a_1)\gamma(a_2) + A_{22}(a_1)n_0\gamma(a_2) & A_{21}(a_1)\gamma(a_2)m_0 \end{pmatrix} \\ &= \begin{pmatrix} \gamma(a_1)A_{11}(a_2) & \gamma(a_1)A_{12}(a_2) + \gamma(a_1)m_0A_{22}(a_2) \\ n_0\gamma(a_1)A_{11}(a_2) & n_0\gamma(a_1)A_{12}(a_2) \end{pmatrix}.\end{aligned}$$

From the above expression, we see that

$$\begin{aligned}&A_{11}(a_1a_2) = A_{11}(a_1)\gamma(a_2) = \gamma(a_1)A_{11}(a_2);\\ &A_{12}(a_1a_2) = A_{11}(a_1)\gamma(a_2)m_0 = \gamma(a_1)A_{12}(a_2) + \gamma(a_1)m_0A_{22}(a_2);\\ &A_{21}(a_1a_2) = A_{21}(a_1)\gamma(a_2) + A_{22}(a_1)n_0\gamma(a_2) = n_0\gamma(a_1)A_{11}(a_2);\\ &A_{22}(a_1a_2) = A_{21}(a_1)\gamma(a_2)m_0 = n_0\gamma(a_1)A_{12}(a_2).\end{aligned}$$

Putting $a_2 = 1_{\mathcal{A}}$, we get $A_{12}(a_1) = A_{11}(a_1)m_0$ and $A_{22}(a_1) = A_{21}(a_1)m_0 = 0$. Taking $a_1 = 1_{\mathcal{A}}$, we have $A_{21}(a_2) = n_0A_{11}(a_2)$, $A_{22}(a_2) = n_0A_{12}(a_2) = 0$. Since $a_1, a_2 \in \mathcal{A}$ are arbitrary, we conclude that A_{11} is a γ-centralizer of $\mathcal{A}$, $A_{12}(a) = A_{11}(a)m_0$, $A_{21}(a) = n_0A_{11}(a)$ and $A_{22}(a) = 0$ for all $a \in \mathcal{A}$. Repeating the same computational process

and choosing $x = \begin{pmatrix} 0 & 0 \\ 0 & b_1 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 \\ 0 & b_2 \end{pmatrix}$ in (2.1), we obtain $B_{22}(b_1b_2) = B_{22}(b_1)\delta(b_2) = \delta(b_1)B_{22}(b_2)$, for all $b_1, b_2 \in \mathcal{B}$, i.e., B_{22} is a δ-centralizer of $\mathcal{B}$ and $B_{11}(b) = 0$, $B_{12}(b) = -m_0B_{22}(b)$, $B_{21}(b) = -B_{22}(b)n_0$ for all $b \in \mathcal{B}$.

Putting $x = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}$ in (2.1), we arrive at

$$\begin{pmatrix} A_{11}(mn) & A_{12}(mn) \\ A_{21}(mn) & A_{22}(mn) \end{pmatrix} = \begin{pmatrix} C_{12}(m)v(n) & 0 \\ C_{22}(m)v(n) & 0 \end{pmatrix} = \begin{pmatrix} u(m)D_{21}(n) & u(m)D_{22}(n) \\ 0 & 0 \end{pmatrix}.$$

This yields $A_{11}(mn) = C_{12}(m)v(n) = u(m)D_{21}(n)$. Similarly, choosing $x = \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$ in (2.1), we get $B_{22}(nm) = D_{21}(n)u(m) = v(n)C_{12}(m)$. □

If $\mathcal{M}$ is a faithful $(\mathcal{A}, \mathcal{B})$-bimodule, then condition (i) in Proposition 1 become redundant and we obtain the following result:

Corollary 1. *Let $\mathcal{G} = G(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$ be a generalized matrix algebra with faithful $\mathcal{M}$ and $\Delta : \mathcal{G} \to \mathcal{G}$ be a σ-centralizer with associated automorphism σ of $\mathcal{G}$ as given in Lemma 1. Then Δ is of the form*

$$\Delta\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} A_{11}(a) & A_{11}(a)m_0 - m_0B_{22}(b) + C_{12}(m) \\ n_0A_{11}(a) - B_{22}(b)n_0 + D_{21}(n) & B_{22}(b) \end{pmatrix},$$

where $A_{11} : \mathcal{A} \to \mathcal{A}$, $C_{12} : \mathcal{M} \to \mathcal{M}$, $D_{21} : \mathcal{N} \to \mathcal{N}$ and $B_{22} : \mathcal{B} \to \mathcal{B}$ are $\mathcal{R}$-linear mappings satisfying the following conditions:

(i) $A_{11}(mn) = C_{12}(m)v(n) = u(m)D_{21}(n)$, $B_{22}(nm) = D_{21}(n)u(m) = v(n)C_{12}(m)$;
(ii) $C_{12}(am) = A_{11}(a)u(m) = \gamma(a)C_{12}(m)$, $C_{12}(mb) = C_{12}(m)\delta(b) = u(m)B_{22}(b)$;
(iii) $D_{21}(na) = v(n)A_{11}(a) = D_{21}(n)\gamma(a)$, $D_{21}(bn) = \delta(b)D_{21}(n) = B_{22}(b)v(n)$.

Proof. In view of Proposition 1, it is sufficient to show that if $\mathcal{M}$ is a faithful $(\mathcal{A}, \mathcal{B})$-bimodule, then A_{11} is a γ-centralizer of $\mathcal{A}$ and B_{22} is a δ-centralizer of $\mathcal{B}$. For all $a_1, a_2 \in \mathcal{A}$ and $m \in \mathcal{M}$, we see that

$$A_{11}(a_1a_2)u(m) = C_{12}(a_1a_2m) = A_{11}(a_1)u(a_2m) = A_{11}(a_1)\gamma(a_2)u(m).$$

Thus, $\{A_{11}(a_1a_2) - A_{11}(a_1)\gamma(a_2)\}\mathcal{M} = \{0\}$. Since $\mathcal{M}$ is a faithful left $\mathcal{A}$-module, we have $A_{11}(a_1a_2) = A_{11}(a_1)\gamma(a_2)$. Further,

$$A_{11}(a_1a_2)u(m) = C_{12}(a_1a_2m) = \gamma(a_1)C_{12}(a_2m) = \gamma(a_1)A_{11}(a_2)u(m)$$

for all $a_1, a_2 \in \mathcal{A}$, $m \in \mathcal{M}$. Hence $\{A_{11}(a_1a_2) - \gamma(a_1)A_{11}(a_2)\}\mathcal{M} = \{0\}$ which implies that $A_{11}(a_1a_2) = \gamma(a_1)A_{11}(a_2)$. Therefore, A_{11} is a γ-centralizer of $\mathcal{A}$. In a similar manner, one can prove that B_{22} is a δ-centralizer of $\mathcal{B}$. □

3. Lie σ-centralizers of generalized matrix algebras

In this section, we prove that under certain conditions every Lie σ-centralizer of a generalized matrix algebra is proper. To prove this, we first characterize a Lie σ-centralizer $\mathcal{L}:\mathcal{G}\to\mathcal{G}$ with associated automorphism σ as given in Lemma 1.

Proposition 2. *Let $\mathcal{L}:\mathcal{G}\to\mathcal{G}$ be a Lie σ-centralizer with associated automorphism σ of $\mathcal{G}$ as given in Lemma 1. Then $\mathcal{L}$ is of the form*

$$\mathcal{L}\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} R_{11}(a)+S_{11}(b) & (R_{11}(a)+S_{11}(b))m_0 - m_0(R_{22}(a)+S_{22}(b)) + T_{12}(m) \\ n_0(R_{11}(a)+S_{11}(b)) - (R_{22}(a)+S_{22}(b))n_0 + U_{21}(n) & R_{22}(a)+S_{22}(b) \end{pmatrix},$$

where $R_{11}:\mathcal{A}\to\mathcal{A}$, $S_{11}:\mathcal{B}\to\mathcal{Z}(\mathcal{A})$, $T_{12}:\mathcal{M}\to\mathcal{M}$, $U_{21}:\mathcal{N}\to\mathcal{N}$, $R_{22}:\mathcal{A}\to\mathcal{Z}(\mathcal{B})$ and $S_{22}:\mathcal{B}\to\mathcal{B}$ are $\mathcal{R}$-linear mappings satisfying the following conditions:

- (i) *R_{11} is a Lie γ-centralizer of $\mathcal{A}$, $R_{11}(mn)-S_{11}(nm)=T_{12}(m)v(n)=u(m)U_{21}(n)$;*
- (ii) *S_{22} is a Lie δ-centralizer of $\mathcal{B}$, $S_{22}(nm)-R_{22}(mn)=U_{21}(n)u(m)=v(n)T_{12}(m)$;*
- (iii) *$T_{12}(am)=R_{11}(a)u(m)-u(m)R_{22}(a)=\gamma(a)T_{12}(m)$, $T_{12}(mb)=T_{12}(m)\delta(b)=u(m)S_{22}(b)-S_{11}(b)u(m)$;*
- (iv) *$U_{21}(na)=v(n)R_{11}(a)-R_{22}(a)v(n)=U_{21}(n)\gamma(a)$, $U_{21}(bn)=\delta(b)U_{21}(n)=S_{22}(b)v(n)-v(n)S_{11}(b)$;*
- (v) *$R_{22}([a_1,a_2])=0$ and $S_{11}([b_1,b_2])=0$.*

Proof. Suppose that the Lie σ-centralizer $\mathcal{L}$ is of the form

$$\mathcal{L}\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} R_{11}(a)+S_{11}(b)+T_{11}(m)+U_{11}(n) & R_{12}(a)+S_{12}(b)+T_{12}(m)+U_{12}(n) \\ R_{21}(a)+S_{21}(b)+T_{21}(m)+U_{21}(n) & R_{22}(a)+S_{22}(b)+T_{22}(m)+U_{22}(n) \end{pmatrix},$$

where $R_{11},S_{11},T_{11},U_{11}$ are $\mathcal{R}$-linear mappings from $\mathcal{A},\mathcal{B},\mathcal{M},\mathcal{N}$ to $\mathcal{A}$, respectively; $R_{12},S_{12},T_{12},U_{12}$ are $\mathcal{R}$-linear mappings from $\mathcal{A},\mathcal{B},\mathcal{M},\mathcal{N}$ to $\mathcal{M}$, respectively; R_{21},S_{21}, T_{21},U_{21} are $\mathcal{R}$-linear mappings from $\mathcal{A},\mathcal{B},\mathcal{M},\mathcal{N}$ to $\mathcal{N}$, respectively; $R_{22},S_{22},T_{22},U_{22}$ are $\mathcal{R}$-linear mappings from $\mathcal{A},\mathcal{B},\mathcal{M},\mathcal{N}$ to $\mathcal{B}$, respectively.

Since $\mathcal{L}$ is a Lie σ-centralizer, we have

$$\mathcal{L}([x,y]) = [\mathcal{L}(x),\sigma(y)] = [\sigma(x),\mathcal{L}(y)] \text{ for all } x,y\in\mathcal{G}. \tag{3.1}$$

Choosing $x=\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $y=\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$ into (3.1), we get

$$\begin{pmatrix} T_{11}(am) & T_{12}(am) \\ T_{21}(am) & T_{22}(am) \end{pmatrix} = \begin{pmatrix} u(m)R_{21}(a) & R_{11}(a)u(m)-u(m)R_{22}(a) \\ 0 & R_{21}(a)u(m) \end{pmatrix}$$
$$= \begin{pmatrix} [\gamma(a),T_{11}(m)] & \gamma(a)T_{12}(m)+\gamma(a)m_0T_{22}(m)-T_{11}(m)\gamma(a)m_0 \\ n_0\gamma(a)T_{11}(m)-T_{21}(m)\gamma(a)-T_{22}(m)n_0\gamma(a) & n_0\gamma(a)T_{12}(m)-T_{21}(m)\gamma(a)m_0 \end{pmatrix}.$$

Thus, $T_{11}(am)=u(m)R_{21}(a)=[\gamma(a),T_{11}(m)]$, $T_{12}(am)=R_{11}(a)u(m)-u(m)R_{22}(a)=\gamma(a)T_{12}(m)+\gamma(a)m_0T_{22}(m)-T_{11}(m)\gamma(a)m_0$, $T_{21}(am)=0=n_0\gamma(a)T_{11}(m)-T_{21}(m)\gamma(a)$

$-T_{22}(m)n_0\gamma(a)$ and $T_{22}(am) = R_{21}(a)u(m) = n_0\gamma(a)T_{12}(m) - T_{21}(m)\gamma(a)m_0$. Putting $a = 1_{\mathcal{A}}$, we get $T_{11}(m) = 0$, $T_{21}(m) = 0$, and $T_{22}(m) = 0$. Thus, $T_{12}(am) = R_{11}(a)u(m) - u(m)R_{22}(a) = \gamma(a)T_{12}(m)$. Similarly, choose $x = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$ to obtain $T_{12}(mb) = u(m)S_{22}(b) - S_{11}(b)u(m) = T_{12}(m)\delta(b)$.

If we take $x = \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}$ and $y = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ in (3.1), then we have

$$\begin{aligned}
&\begin{pmatrix} U_{11}(na) & U_{12}(na) \\ U_{21}(na) & U_{22}(na) \end{pmatrix} \\
&= \begin{pmatrix} [U_{11}(n),\gamma(a)] & U_{11}(n)\gamma(a)m_0 - \gamma(a)U_{12}(n) - \gamma(a)m_0U_{22}(n) \\ U_{21}(n)\gamma(a) + U_{22}(n)n_0\gamma(a) - n_0\gamma(a)U_{11}(n) & U_{21}(n)\gamma(a)m_0 - n_0\gamma(a)U_{12}(n) \end{pmatrix} \\
&= \begin{pmatrix} -R_{12}(a)v(n) & 0 \\ v(n)R_{11}(a) - R_{22}(a)v(n) & v(n)R_{12}(a) \end{pmatrix}.
\end{aligned}$$

Thus,

$$\begin{aligned}
U_{11}(na) &= [U_{11}(n),\gamma(a)] = -R_{12}(a)v(n) \\
U_{12}(na) &= U_{11}(n)\gamma(a)m_0 - \gamma(a)U_{12}(n) - \gamma(a)m_0U_{22}(n) = 0 \\
U_{21}(na) &= U_{21}(n)\gamma(a) + U_{22}(n)n_0\gamma(a) - n_0\gamma(a)U_{11}(n) = v(n)R_{11}(a) - R_{22}(a)v(n)
\end{aligned}$$

and

$$\begin{aligned}
U_{22}(na) &= U_{21}(n)\gamma(a)m_0 - n_0\gamma(a)U_{12}(n) = v(n)R_{12}(a) \\
&= U_{21}(n)\gamma(a)m_0 - n_0\gamma(a)U_{12}(n).
\end{aligned}$$

Putting $a = 1_{\mathcal{A}}$, we get $U_{11}(n) = 0$, $U_{12}(n) = 0$ and $U_{22}(n) = 0$. Hence, $U_{21}(na) = U_{21}(n)\gamma(a) = v(n)R_{11}(a) - R_{22}(a)v(n)$. Similarly, take $x = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}$ to obtain $U_{21}(bn) = \delta(b)U_{21}(n) = S_{22}(b)v(n) - v(n)S_{11}(b)$.

Taking $x = \begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} a_2 & 0 \\ 0 & 0 \end{pmatrix}$ in (3.1), we get

$$\begin{aligned}
&\begin{pmatrix} R_{11}([a_1,a_2]) & R_{12}([a_1,a_2]) \\ R_{21}([a_1,a_2]) & R_{22}([a_1,a_2]) \end{pmatrix} \\
&= \begin{pmatrix} [R_{11}(a_1),\gamma(a_2)] & R_{11}(a_1)\gamma(a_2)m_0 - \gamma(a_2)R_{12}(a_1) - \gamma(a_2)m_0R_{22}(a_1) \\ R_{21}(a_1)\gamma(a_2) + R_{22}(a_1)n_0\gamma(a_2) - n_0\gamma(a_2)R_{11}(a_1) & 0 \end{pmatrix} \\
&= \begin{pmatrix} [\gamma(a_1),R_{11}(a_2)] & \gamma(a_1)R_{12}(a_2) + \gamma(a_1)m_0R_{22}(a_2) - R_{11}(a_2)\gamma(a_1)m_0 \\ n_0\gamma(a_1)R_{11}(a_2) - R_{21}(a_2)\gamma(a_1) - R_{22}(a_2)n_0\gamma(a_1) & 0 \end{pmatrix}.
\end{aligned}$$

Thus, $R_{11}([a_1,a_2]) = [R_{11}(a_1),\gamma(a_2)] = [\gamma(a_1),R_{11}(a_2)]$, i.e., R_{11} is a Lie γ-centralizer of $\mathcal{A}$, $R_{12}([a_1,a_2]) = R_{11}(a_1)\gamma(a_2)m_0 - \gamma(a_2)R_{12}(a_1) - \gamma(a_2)m_0R_{22}(a_1)$, $R_{21}([a_1,a_2]) =$

$R_{21}(a_1)\gamma(a_2)+R_{22}(a_1)n_0\gamma(a_2)-n_0\gamma(a_2)R_{11}(a_1)$, and $R_{22}([a_1,a_2])=0$. Taking $a_1=a$ and $a_2=1_{\mathcal{A}}$, we get $R_{12}(a)=R_{11}(a)m_0-m_0R_{22}(a)$, $R_{21}(a)=n_0R_{11}(a)-R_{22}(a)n_0$. Similarly, choosing $x=\begin{pmatrix} 0 & 0 \\ 0 & b_1 \end{pmatrix}$ and $y=\begin{pmatrix} 0 & 0 \\ 0 & b_2 \end{pmatrix}$ in (3.1), we get

$S_{11}([b_1,b_2])=0$, $S_{12}(b_1)=S_{11}(b_1)m_0-m_0S_{22}(b_1)$, $S_{21}(b_1)=n_0S_{11}(b_1)-S_{22}(b_1)n_0$ and $S_{22}([b_1,b_2])=[S_{22}(b_1),\delta(b_2)]=[\delta(b_1),S_{22}(b_2)]$, i.e., S_{22} is a Lie δ-centralizer of $\mathcal{B}$.

If we consider $x=\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $y=\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$ in (3.1), then we have

$$\begin{aligned}
&\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -R_{11}(a)m_0\delta(b)+R_{12}(a)\delta(b)+m_0\delta(b)R_{22}(b) \\ -R_{22}\delta(b)n_0+\delta(b)n_0R_{11}(a)-\delta(b)R_{21}(a) & [R_{22}(a),\delta(b)] \end{pmatrix} \\
&= \begin{pmatrix} [\gamma(a),S_{11}(b)] & \gamma(a)S_{12}(b)+\gamma(a)m_0S_{22}(b)-S_{11}(b)\gamma(a)m_0 \\ n_0\gamma(a)S_b-S_{21}(b)\gamma(a)-S_{22}(b)n_0\gamma(a) & 0 \end{pmatrix}.
\end{aligned}$$

This gives $[R_{22}(a),\delta(b)]=0$ and $[\gamma(a),S_{11}(b)]=0$ for all $a\in\mathcal{A}$, $b\in\mathcal{B}$. Since γ and δ are automorphisms of $\mathcal{A}$ and $\mathcal{B}$, respectively, we conclude $R_{22}(a)\in\mathcal{Z}(\mathcal{B})$ and $S_{11}(b)\in\mathcal{Z}(\mathcal{A})$.

Furthermore, if we take $x=\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$ and $y=\begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}$ in (3.1), then we have

$$\begin{aligned}
\begin{pmatrix} R_{11}(mn)-S_{11}(nm) & R_{12}(mn)-S_{12}(nm) \\ R_{21}(mn)-S_{21}(nm) & R_{22}(mn)-S_{22}(nm) \end{pmatrix} &= \begin{pmatrix} T_{12}(m)v(n) & 0 \\ 0 & -v(n)T_{12}(n) \end{pmatrix} \\
&= \begin{pmatrix} u(m)U_{21}(n) & 0 \\ 0 & -U_{21}(n)u(m) \end{pmatrix}.
\end{aligned}$$

This implies that $R_{11}(mn)-S_{11}(nm)=T_{12}(m)v(n)=u(m)U_{21}(n)$ and $R_{22}(mn)-S_{22}(nm)=-v(n)T_{12}(n)=-U_{21}(n)u(m)$. □

If $\mathcal{M}$ is a faithful $(\mathcal{A},\mathcal{B})$-bimodule, then condition (v) in the above theorem can be obtained using condition (iii). Thus, we have

Corollary 2. *Let $\mathcal{G}=G(\mathcal{A},\mathcal{M},\mathcal{N},\mathcal{B})$ be a generalized matrix algebra with faithful $\mathcal{M}$ and $\mathcal{L}:\mathcal{G}\to\mathcal{G}$ be a Lie σ-centralizer with associated automorphism σ of $\mathcal{G}$ as given in Lemma 1. Then $\mathcal{L}$ is of the form*

$$\mathcal{L}\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} R_{11}(a)+S_{11}(b) & n_0(R_{11}(a)+S_{11}(b))-(R_{22}(a)+S_{22}(b))n_0+U_{21}(n) \\ (R_{11}(a)+S_{11}(b))m_0-m_0(R_{22}(a)+S_{22}(b))+T_{12}(m) & R_{22}(a)+S_{22}(b) \end{pmatrix},$$

where $R_{11}:\mathcal{A}\to\mathcal{A}$, $S_{11}:\mathcal{B}\to\mathcal{Z}(\mathcal{A})$, $T_{12}:\mathcal{M}\to\mathcal{M}$, $U_{21}:\mathcal{N}\to\mathcal{N}$, $R_{22}:\mathcal{A}\to\mathcal{Z}(\mathcal{B})$ and $S_{22}:\mathcal{B}\to\mathcal{B}$ are $\mathcal{R}$-linear mappings satisfying the following conditions:

(i) R_{11} *is a Lie* γ*-centralizer of* $\mathcal{A}$, $R_{11}(mn) - S_{11}(nm) = T_{12}(m)v(n) = u(m)U_{21}(n)$;
(ii) S_{22} *is a Lie* δ*-centralizer of* $\mathcal{B}$, $S_{22}(nm) - R_{22}(mn) = U_{21}(n)u(m) = v(n)T_{12}(m)$;
(iii) $T_{12}(am) = R_{11}(a)u(m) - u(m)R_{22}(a) = \gamma(a)T_{12}(m)$, $T_{12}(mb) = T_{12}(m)\delta(b) = u(m)S_{22}(b) - S_{11}(b)u(m)$;
(iv) $U_{21}(na) = v(n)R_{11}(a) - R_{22}(a)v(n) = U_{21}(n)\gamma(a)$, $U_{21}(bn) = \delta(b)U_{21}(n) = S_{22}(b)v(n) - v(n)S_{11}(b)$.

The following proposition gives necessary and sufficient conditions for a Lie σ-centralizer with associated automorphism σ of $\mathcal{G}$ as given in Lemma 1 to be proper.

Proposition 3. *Let* $\mathcal{G} = G(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$ *be a generalized matrix algebra. A Lie* σ*-centralizer* $\mathcal{L} : \mathcal{G} \to \mathcal{G}$ *of the form presented in Proposition 2 is proper if and only if there exist linear mappings* $\ell_{\mathcal{A}} : \mathcal{A} \to \mathcal{Z}(\mathcal{A})$ *and* $\ell_{\mathcal{B}} : \mathcal{B} \to \mathcal{Z}(\mathcal{B})$ *satisfying the following conditions:*

(i) $R_{11} - \ell_{\mathcal{A}}$ *is a* γ*-centralizer of* $\mathcal{A}$ *and* $S_{22} - \ell_{\mathcal{B}}$ *is a* δ*-centralizer of* $\mathcal{B}$;
(ii) $\ell_{\mathcal{A}}(a) \oplus R_{22}(a) \in \mathcal{Z}(\mathcal{G})$ *and* $S_{11}(b) \oplus \ell_{\mathcal{B}}(b) \in \mathcal{Z}(\mathcal{G})$ *for all* $a \in \mathcal{A}$, $b \in \mathcal{B}$;
(iii) $\ell_{\mathcal{A}}(mn) = S_{11}(nm)$ *and* $R_{22}(mn) = \ell_{\mathcal{B}}(nm)$ *for all* $m \in \mathcal{M}$, $n \in \mathcal{N}$.

Proof. Assume that $\mathcal{L}$ is a Lie σ-centralizer of $\mathcal{G}$ of the form presented in Proposition 2 and there exist linear mappings $\ell_{\mathcal{A}} : \mathcal{A} \to \mathcal{Z}(\mathcal{A})$ and $\ell_{\mathcal{B}} : \mathcal{B} \to \mathcal{Z}(\mathcal{B})$ satisfying conditions $(i) - (iii)$. Define two mappings Δ and τ as follows:

$$\Delta\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} (R_{11} - \ell_{\mathcal{A}})(a) & (R_{11} - \ell_{\mathcal{A}})(a)m_0 - m_0(S_{22} - \ell_{\mathcal{B}})(b) + T_{12}(m) \\ n_0(R_{11} - \ell_{\mathcal{A}})(a) - (S_{22} - \ell_{\mathcal{B}})(b)n_0 + U_{21}(n) & (S_{22} - \ell_{\mathcal{B}})(b) \end{pmatrix}$$

and

$$\tau\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} \ell_{\mathcal{A}}(a) + S_{11}(b) & 0 \\ 0 & R_{22}(a) + \ell_{\mathcal{B}}(b) \end{pmatrix}.$$

It is easy to see that Δ and τ are $\mathcal{R}$-linear mappings and $\mathcal{L} = \Delta + \tau$. Moreover, it follows from Proposition 1 that Δ is a σ-centralizer of $\mathcal{G}$. It only remains to show that $\tau(\mathcal{G}) \subseteq \mathcal{Z}(\mathcal{G})$. Using assumption (ii), we have

$$(\ell_{\mathcal{A}}(a) + S_{11}(b))m = \ell_{\mathcal{A}}(a)m + S_{11}(b)m = mR_{22}(a) + m\ell_{\mathcal{B}}(b) = m(R_{22}(a) + \ell_{\mathcal{B}}(b))$$

for all $m \in \mathcal{M}$. Similarly, $n(\ell_{\mathcal{A}}(a) + S_{11}(b)) = (R_{22}(a) + \ell_{\mathcal{B}}(b))n$ for all $n \in \mathcal{N}$. Hence, it follows that $\tau(\mathcal{G}) \subseteq \mathcal{Z}(\mathcal{G})$.

Conversely, suppose that $\mathcal{L}$ is proper, that is, $\mathcal{L} = \Delta + \tau$, where Δ is a σ-centralizer and τ is a center-valued mapping. In view of the representations of $\mathcal{L}$ and Δ, the mapping $\tau = \mathcal{L} - \Delta$ has the following form:

$$\tau\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} (R_{11} - A_{11})(a) + S_{11}(b) & 0 \\ 0 & R_{22}(a) + (S_{22} - B_{22})(b) \end{pmatrix}.$$

Set $\ell_{\mathcal{A}} = R_{11} - A_{11}$ and $\ell_{\mathcal{B}} = S_{22} - B_{22}$. Then, it is straightforward to check that $\ell_{\mathcal{A}}$ and $\ell_{\mathcal{B}}$ are the desired mappings satisfying assumptions $(i) - (iii)$. $\square$

By Corollary 1, if $\mathcal{M}$ is a faithful $(\mathcal{A},\mathcal{B})$-bimodule, then condition (i) of the above proposition becomes superfluous. Thus, as a consequence of Proposition 3, we have the following corollary:

Corollary 3. *Let $\mathcal{G} = G(\mathcal{A},\mathcal{M},\mathcal{N},\mathcal{B})$ be a generalized matrix algebra and $\mathcal{L} : \mathcal{G} \to \mathcal{G}$ be a Lie σ-centralizer with associated automorphism σ of $\mathcal{G}$ as given in Lemma 1. If $\mathcal{L}$ is proper, then the following conditions hold:*

(i) $R_{22}(\mathcal{A}) \subseteq \pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{G}))$ *and* $S_{11}(\mathcal{B}) \subseteq \pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{G}))$;

(ii) $S_{11}(nm) \oplus R_{22}(mn) \in \mathcal{Z}(\mathcal{G})$ *for all* $m \in \mathcal{M}$, $n \in \mathcal{N}$.

The converse also holds provided $\mathcal{M}$ is faithful.

Proof. If $\mathcal{L}$ is a proper Lie σ-centralizer of $\mathcal{G}$, then the required conditions follow directly from Proposition 3. For the converse, suppose that $\mathcal{M}$ is a faithful $(\mathcal{A},\mathcal{B})$-bimodule and $\mathcal{L}$ is a Lie σ-centralizer of $\mathcal{G}$ of the form presented in Proposition 2 satisfying (i) and (ii). Since $\mathcal{M}$ is faithful, there exists a unique algebra isomorphism $\xi : \pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{G})) \to \pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{G}))$ such that $a \oplus \xi(a) \in \mathcal{Z}(\mathcal{G})$ for all $a \in \pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{G}))$. Define $\ell_{\mathcal{A}} : \mathcal{A} \to \mathcal{Z}(\mathcal{A})$ and $\ell_{\mathcal{B}} : \mathcal{B} \to \mathcal{Z}(\mathcal{B})$ by $\ell_{\mathcal{A}} = \xi^{-1} \circ R_{22}$ and $\ell_{\mathcal{B}} = \xi \circ S_{11}$, respectively. It is easy to verify that $\ell_{\mathcal{A}}$ and $\ell_{\mathcal{B}}$ are linear mappings satisfying the hypotheses of Proposition 3. Therefore, $\mathcal{L}$ is proper. □

Now we are in a position to give a sufficient condition for a Lie σ-centralizer with associated automorphism σ of $\mathcal{G}$ as given in Lemma 1 to be proper.

Corollary 4. *Let $\mathcal{G} = G(\mathcal{A},\mathcal{M},\mathcal{N},\mathcal{B})$ be a generalized matrix algebra with faithful $\mathcal{M}$. A Lie σ-centralizer with associated automorphism σ of $\mathcal{G}$ as given in Lemma 1 is proper if the following conditions hold:*

(i) $\pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{G})) = \mathcal{Z}(\mathcal{A})$, $\pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{G})) = \mathcal{Z}(\mathcal{B})$ *and*

(ii) *either $\mathcal{A}$ or $\mathcal{B}$ does not contain nonzero central ideals.*

Proof. Suppose that $\mathcal{L}$ is a Lie σ-centralizer of $\mathcal{G}$ of the form presented in Proposition 2. We shall use Corollary 3 to prove that $\mathcal{L}$ is proper. Obviously, the condition $R_{22}(\mathcal{A}) \subseteq \pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{G}))$ and $S_{11}(\mathcal{B}) \subseteq \pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{G}))$ of Corollary 3 follows from the assumption (i). We only need to show that $S_{11}(nm) \oplus R_{22}(mn) \in \mathcal{Z}(\mathcal{G})$ for all $m \in \mathcal{M}$, $n \in \mathcal{N}$. Without loss of generality, assume that $\mathcal{A}$ does not contain nonzero central ideals. Define $\ell_{\mathcal{A}}(a) = \xi^{-1}(R_{22}(a))$. Then $\ell_{\mathcal{A}}(a) \oplus R_{22}(a) \in \mathcal{Z}(\mathcal{G})$ for all $a \in \mathcal{A}$. Set $A_{11} = R_{11} - \ell_{\mathcal{A}}$ and $\nu(a,b) = \ell_{\mathcal{A}}(a) + S_{11}(b)$. Using Proposition 2(iii), one can easily show that A_{11} is a γ-centralizer. Again, using Proposition 2, we have

$$\begin{aligned}\nu(mn,-nm) = \ell_{\mathcal{A}}(mn) - S_{11}(nm) &= R_{11}(mn) - A_{11}(mn) - S_{11}(nm)\\ &= T_{12}(m)v(n) - A_{11}(mn)\end{aligned}$$

for all $m \in \mathcal{M}$, $n \in \mathcal{N}$. Thus, using the fact that A_{11} is a γ-centralizer, we have

$$\begin{aligned}\nu(amn,-nam) &= T_{12}(am)v(n) - A_{11}(amn) = \gamma(a)T_{12}(m)v(n) - \gamma(a)A_{11}(mn)\\ &= \gamma(a)(T_{12}(m)v(n) - A_{11}(mn)) = \gamma(a)\nu(mn,-nm)\end{aligned}$$

for all $a \in \mathcal{A}$. Since γ is an automorphism of $\mathcal{A}$, the set $\mathcal{A}\nu(mn,-nm)$ is a central ideal of $\mathcal{A}$ for each $m \in \mathcal{M}$, $n \in \mathcal{N}$. Hence, $\ell_{\mathcal{A}}(mn) - S_{11}(nm) = \nu(mn,-nm) = 0$. Therefore, $S_{11}(nm) \oplus R_{22}(mn) = \ell_{\mathcal{A}}(mn) \oplus R_{22}(mn) \in \mathcal{Z}(\mathcal{G})$. □

Recall that if $\mathcal{G} = G(\mathcal{A},\mathcal{M},\mathcal{N},\mathcal{B})$ is a generalized matrix algebra such that $\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents, $\mathcal{A}$ and $\mathcal{B}$ are not isomorphic and both the bilinear mappings $\zeta_{\mathcal{M}\mathcal{N}}, \psi_{\mathcal{N}\mathcal{M}}$ are zero, then the class of all automorphisms presented in Lemma 1 coincides with the group of all automorphisms of $\mathcal{G}$. Thus, in view of Corollary 4, we have the first main result of the paper which characterizes an arbitrary Lie σ-centralizer of a generalized matrix algebra.

Theorem 1. *Let $\mathcal{G} = G(\mathcal{A},\mathcal{M},\mathcal{N},\mathcal{B})$ be a generalized matrix algebra. Then every Lie σ-centralizer of $\mathcal{G}$ is proper if the following conditions hold:*

(i) $\pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{G})) = \mathcal{Z}(\mathcal{A})$ *and $\mathcal{M}$ is a faithful left $\mathcal{A}$-module;*
(ii) $\pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{G})) = \mathcal{Z}(\mathcal{B})$ *and $\mathcal{M}$ is a faithful right $\mathcal{B}$-module;*
(iii) *$\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents and $\mathcal{A}$ and $\mathcal{B}$ are not isomorphic;*
(iv) *either $\mathcal{A}$ or $\mathcal{B}$ does not contain nonzero central ideals;*
(v) *both the bilinear mappings $\zeta_{\mathcal{M}\mathcal{N}}, \psi_{\mathcal{N}\mathcal{M}}$ are zero.*

4. Jordan σ-Centralizer of generalized matrix algebras

In this section, we show that under certain restrictions every Jordan σ-centralizer of a generalized matrix algebra is a σ-centralizer. We begin this section with the following proposition which provides the structure of a Jordan σ-centralizer with associated automorphism σ as given in Lemma 1.

Proposition 4. *Let $\mathcal{G} = G(\mathcal{A},\mathcal{M},\mathcal{N},\mathcal{B})$ be a 2-torsion free generalized matrix algebra. Then a Jordan σ-centralizer $J : \mathcal{G} \to \mathcal{G}$ with associated automorphism σ of $\mathcal{G}$ as given in Lemma 1 is of the form*

$$J\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} R_{11}(a) & R_{11}(a)m_0 - m_0S_{22}(b) + T_{12}(m) \\ n_0R_{11}(a) - S_{22}(b)n_0 + U_{21}(n) & S_{22}(b) \end{pmatrix},$$

where $R_{11} : \mathcal{A} \to \mathcal{A}$, $S_{11} : \mathcal{B} \to \mathcal{Z}(\mathcal{A})$, $T_{12} : \mathcal{M} \to \mathcal{M}$, $U_{21} : \mathcal{N} \to \mathcal{N}$, $S_{11} : \mathcal{A} \to \mathcal{Z}(\mathcal{B})$ and $S_{22} : \mathcal{B} \to \mathcal{B}$ are $\mathcal{R}$-linear mappings satisfying the following conditions:

(i) *R_{11} is a Jordan γ-centralizer of $\mathcal{A}$, S_{22} is a Jordan δ-centralizer of $\mathcal{B}$;*
(ii) $R_{11}(mn) = T_{12}(m)v(n) = u(m)U_{21}(n)$, $S_{22}(nm) = U_{21}(n)u(m) = v(n)T_{12}(m)$;
(iii) $T_{12}(am) = R_{11}(a)u(m) = \gamma(a)T_{12}(m)$, $T_{12}(mb) = T_{12}(m)\delta(b) = u(m)S_{22}(b)$;
(iv) $U_{21}(na) = v(n)R_{11}(a) = U_{21}(n)\gamma(a)$, $U_{21}(bn) = \delta(b)U_{21}(n) = S_{22}(b)v(n)$.

Proof. Suppose that the Jordan σ-centralizer J is of the form

$$J\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} R_{11}(a)+S_{11}(b)+T_{11}(m)+U_{11}(n) & R_{12}(a)+S_{12}(b)+T_{12}(m)+U_{12}(n) \\ R_{21}(a)+S_{21}(b)+T_{21}(m)+U_{21}(n) & R_{22}(a)+S_{22}(b)+T_{22}(m)+U_{22}(n) \end{pmatrix},$$

where $R_{11}, S_{11}, T_{11}, U_{11}$ are $\mathcal{R}$-linear mappings from $\mathcal{A},\mathcal{B},\mathcal{M},\mathcal{N}$ to $\mathcal{A}$, respectively; $R_{12}, S_{12}, T_{12}, U_{12}$ are $\mathcal{R}$-linear mappings from $\mathcal{A},\mathcal{B},\mathcal{M},\mathcal{N}$ to $\mathcal{M}$, respectively; R_{21}, S_{21},

T_{21}, U_{21} are $\mathcal{R}$-linear mappings from $\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}$ to $\mathcal{N}$, respectively; $R_{22}, S_{22}, T_{22}, U_{22}$ are $\mathcal{R}$-linear mappings from $\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}$ to $\mathcal{B}$, respectively.

Since J is a Jordan σ-centralizer, we have

$$J(x \circ y) = J(x) \circ \sigma(y) = \sigma(x) \circ J(y) \text{ for all } x, y \in \mathcal{G}. \tag{4.1}$$

Let us choose $x = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$ in (4.1). Then

$$\begin{pmatrix} T_{11}(am) & T_{12}(am) \\ T_{21}(am) & T_{22}(am) \end{pmatrix} = \begin{pmatrix} u(m)R_{21}(a) & R_{11}(a)u(m) + u(m)R_{22}(a) \\ 0 & R_{21}(a)u(m) \end{pmatrix}$$
$$= \begin{pmatrix} \gamma(a) \circ T_{11}(m) & \gamma(a)T_{12}(m) + \gamma(a)m_0T_{22}(m) + T_{11}(m)\gamma(a)m_0 \\ n_0\gamma(a)T_{11}(m) + T_{21}(m)\gamma(a) + T_{22}(m)n_0\gamma(a) & n_0\gamma(a)T_{12}(m) + T_{21}(m)\gamma(a)m_0 \end{pmatrix}.$$

This yields $T_{11}(am) = u(m)R_{21}(a) = \gamma(a) \circ T_{11}(m)$, $T_{12}(am) = R_{11}(a)u(m) + u(m)R_{22}(a) = \gamma(a)T_{12}(m) + \gamma(a)m_0T_{22}(m) + T_{11}(m)\gamma(a)m_0$, $T_{21}(am) = 0 = n_0\gamma(a)T_{11}(m) + T_{21}(m)\gamma(a) + T_{22}(m)n_0\gamma(a)$ and $T_{22}(am) = R_{21}(a)u(m) = n_0\gamma(a)T_{12}(m) + T_{21}(m)\gamma(a)m_0$. Putting $a = 1_{\mathcal{A}}$, we get $T_{11}(m) = 0$, $T_{21}(m) = 0$, and $T_{22}(m) = 0$. Thus, $T_{12}(am) = R_{11}(a)u(m) + u(m)R_{22}(a) = \gamma(a)T_{12}(m)$. Similarly, choosing $x = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$ to obtain $T_{12}(mb) = u(m)S_{22}(b) + S_{11}(b)u(m) = T_{12}(m)\delta(b)$.

Taking $x = \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}$ and $y = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ in (4.1), we get

$$\begin{pmatrix} U_{11}(na) & U_{12}(na) \\ U_{21}(na) & U_{22}(na) \end{pmatrix}$$
$$= \begin{pmatrix} U_{11}(n) \circ \gamma(a) & U_{11}(n)\gamma(a)m_0 + \gamma(a)U_{12}(n) + \gamma(a)m_0U_{22}(n) \\ U_{21}(n)\gamma(a) + U_{22}(n)n_0\gamma(a) + n_0\gamma(a)U_{11}(n) & U_{21}(n)\gamma(a)m_0 + n_0\gamma(a)U_{12}(n) \end{pmatrix}$$
$$= \begin{pmatrix} R_{12}(a)v(n) & 0 \\ v(n)R_{11}(a) + R_{22}(a)v(n) & v(n)R_{12}(a) \end{pmatrix}.$$

Thus, $U_{11}(na) = U_{11}(n) \circ \gamma(a) = R_{12}(a)v(n)$, $U_{12}(na) = U_{11}(n)\gamma(a)m_0 + \gamma(a)U_{12}(n) + \gamma(a)m_0U_{22}(n) = 0$, $U_{21}(na) = U_{21}(n)\gamma(a) + U_{22}(n)n_0\gamma(a) + n_0\gamma(a)U_{11}(n) = v(n)R_{11}(a) + R_{22}(a)v(n)$ and $U_{22}(na) = U_{21}(n)\gamma(a)m_0 + n_0\gamma(a)U_{12}(n) = v(n)R_{12}(a) = U_{21}(n)\gamma(a)m_0 + n_0\gamma(a)U_{12}(n)$. Putting $a = 1_{\mathcal{A}}$, we get $U_{11}(n) = 0$, $U_{12}(n) = 0$ and $U_{22}(n) = 0$. Hence $U_{21}(na) = U_{21}(n)\gamma(a) = v(n)R_{11}(a) + R_{22}(a)v(n)$.

Similarly, by taking $x = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}$ in (4.1), one can obtain $U_{21}(bn) = \delta(b)U_{21}(n) = S_{22}(b)v(n) + v(n)S_{11}(b)$.

If we consider $x = \begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} a_2 & 0 \\ 0 & 0 \end{pmatrix}$ in (4.1). Then

$$\begin{pmatrix} R_{11}(a_1 \circ a_2) & R_{12}(a_1 \circ a_2) \\ R_{21}(a_1 \circ a_2) & R_{22}(a_1 \circ a_2) \end{pmatrix}$$

$$= \begin{pmatrix} R_{11}(a_1)\circ\gamma(a_2) & R_{11}(a_1)\gamma(a_2)m_0+\gamma(a_2)R_{12}(a_1)+\gamma(a_2)m_0R_{22}(a_1) \\ R_{21}(a_1)\gamma(a_2)+R_{22}(a_1)n_0\gamma(a_2)+n_0\gamma(a_2)R_{11}(a_1) & 0 \end{pmatrix}$$
$$= \begin{pmatrix} \gamma(a_1)\circ R_{11}(a_2) & \gamma(a_1)R_{12}(a_2)+\gamma(a_1)m_0R_{22}(a_2)+R_{11}(a_2)\gamma(a_1)m_0 \\ n_0\gamma(a_1)R_{11}(a_2)+R_{21}(a_2)\gamma(a_1)+R_{22}(a_2)n_0\gamma(a_1) & 0 \end{pmatrix}.$$

It follows from the above relation that $R_{11}(a_1\circ a_2) = R_{11}(a_1)\circ\gamma(a_2)+\gamma(a_1)\circ R_{11}(a_2)$, i.e., R_{11} is a Jordan γ-centralizer on $\mathcal{A}$, $R_{12}(a_1) = R_{11}(a_1)m_0+m_0R_{22}(a_1)$, $R_{21}(a_1) = n_0R_{11}(a_1)+R_{22}(a_1)n_0$, and $R_{22}(a_1\circ a_2) = 0$. Symmetrically, consider $x = \begin{pmatrix} 0 & 0 \\ 0 & b_1 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 \\ 0 & b_2 \end{pmatrix}$ in (4.1) to obtain $S_{11}(b_1\circ b_2) = 0$, $S_{12}(b_1) = S_{11}(b_1)m_0+m_0S_{22}(b_1)$, $S_{21}(b_1) = n_0S_{11}(b_1)+S_{22}(b_1)n_0$ and $S_{22}(b_1\circ b_2) = S_{22}(b_1)\circ\gamma(b_2)+\gamma(b_1)\circ S_{22}(b_2)$, i.e., S_{22} is a Jordan δ-centralizer on $\mathcal{B}$.

Considering $x = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$ in (4.1), we have

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -R_{11}(a)m_0\delta(b)+R_{12}(a)\delta(b)-m_0\delta(b)R_{22}(b) \\ -R_{22}\delta(b)n_0-\delta(b)n_0R_{11}(a)+\delta(b)R_{21}(a) & R_{22}(a)\circ\delta(b) \end{pmatrix}$$
$$= \begin{pmatrix} \gamma(a)\circ S_{11}(b) & \gamma(a)S_{12}(b)+\gamma(a)m_0S_{22}(b)+S_{11}(b)\gamma(a)m_0 \\ n_0\gamma(a)S_b+S_{21}(b)\gamma(a)+S_{22}(b)n_0\gamma(a) & 0 \end{pmatrix}.$$

It follows from the above equation that $R_{22}(a)\circ\delta(b) = 0$ and $\gamma(a)\circ S_{11}(b) = 0$ for all $a\in\mathcal{A}$, $b\in\mathcal{B}$. Putting $b = 1_{\mathcal{B}}$ and $a = 1_{\mathcal{A}}$ in the above equations, respectively, we get $2R_{22}(a) = 0$ and $2S_{11}(b) = 0$. Since $\mathcal{G}$ is 2-torsion free, we obtain $R_{22}(a) = 0$ and $S_{11}(b) = 0$.

Furthermore, if we choose $x = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}$ in (4.1), then we have

$$\begin{pmatrix} R_{11}(mn) & R_{12}(mn)+S_{12}(nm) \\ R_{21}(mn)+S_{21}(nm) & S_{22}(nm) \end{pmatrix} = \begin{pmatrix} T_{12}(m)v(n) & 0 \\ 0 & v(n)T_{12}(n) \end{pmatrix}$$
$$= \begin{pmatrix} u(m)U_{21}(n) & 0 \\ 0 & U_{21}(n)u(m) \end{pmatrix}.$$

Thus, $R_{11}(mn) = T_{12}(m)v(n) = u(m)U_{21}(n)$ and $S_{22}(nm) = v(n)T_{12}(n) = U_{21}(n)u(m)$. □

If $\mathcal{M}$ is a faithful $(\mathcal{A},\mathcal{B})$-bimodule, then condition (i) in Proposition 4 become superfluous.

Corollary 5. *Let $\mathcal{G} = G(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$ be a 2-torsion free generalized matrix algebra with faithful $\mathcal{M}$. Then a Jordan σ-centralizer $J : \mathcal{G} \to \mathcal{G}$ with associated automorphism σ of $\mathcal{G}$ as given in Lemma 1 is of the form*

$$J\begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} R_{11}(a) & R_{11}(a)m_0 - m_0S_{22}(b) + T_{12}(m) \\ n_0R_{11}(a) - S_{22}(b)n_0 + U_{21}(n) & S_{22}(b) \end{pmatrix},$$

where $R_{11} : \mathcal{A} \to \mathcal{A}$, $S_{11} : \mathcal{B} \to \mathcal{Z}(\mathcal{A})$, $T_{12} : \mathcal{M} \to \mathcal{M}$, $U_{21} : \mathcal{N} \to \mathcal{N}$, $S_{11} : \mathcal{A} \to \mathcal{Z}(\mathcal{B})$ and $S_{22} : \mathcal{B} \to \mathcal{B}$ are $\mathcal{R}$-linear mappings satisfying the following conditions:

(*i*) $R_{11}(mn) = T_{12}(m)v(n) = u(m)U_{21}(n)$, $S_{22}(nm) = U_{21}(n)u(m) = v(n)T_{12}(m)$;
(*ii*) $T_{12}(am) = R_{11}(a)u(m) = \gamma(a)T_{12}(m)$, $T_{12}(mb) = T_{12}(m)\delta(b) = u(m)S_{22}(b)$;
(*iii*) $U_{21}(na) = v(n)R_{11}(a) = U_{21}(n)\gamma(a)$, $U_{21}(bn) = \delta(b)U_{21}(n) = S_{22}(b)v(n)$.

Proof. In view of Proposition 4, it suffices to show that if $\mathcal{M}$ is a faithful $(\mathcal{A}, \mathcal{B})$-bimodule, then R_{11} is a Jordan γ-centralizer of $\mathcal{A}$ and S_{22} is a Jordan δ-centralizer of $\mathcal{B}$. For any $a_1, a_2 \in \mathcal{A}$ and $m \in \mathcal{M}$, we have

$$\begin{aligned} R_{11}(a_1 \circ a_2)m &= T_{12}((a_1 \circ a_2)m) = T_{12}(a_1a_2m + a_2a_1m) \\ &= R_{11}(a_1)u(a_2m) + \gamma(a_2)T_{12}(a_1m) \\ &= R_{11}(a_1)\gamma(a_2)u(m) + \gamma(a_2)R_{11}(a_1)u(m) = (R_{11}(a_1) \circ \gamma(a_2))u(m). \end{aligned}$$

This implies that $\{R_{11}(a_1 \circ a_2) - R_{11}(a_1) \circ \gamma(a_2)\}\mathcal{M} = \{0\}$. Since $\mathcal{M}$ is faithful as a left $\mathcal{A}$-module, we conclude $R_{11}(a_1 \circ a_2) = R_{11}(a_1) \circ \gamma(a_2)$. Thus, R_{11} is a Jordan γ-centralizer of $\mathcal{A}$. Similarly, we can show that S_{22} is a Jordan δ-centralizer of $\mathcal{B}$. □

Combining Corollaries 1 and 5, we get the following proposition:

Proposition 5. *Let $\mathcal{G} = G(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$ be a 2-torsion free generalized matrix algebra with faithful $\mathcal{M}$. Then a Jordan σ-centralizer of $\mathcal{G}$ with associated automorphism σ of $\mathcal{G}$ as given in Lemma 1 is a σ-centralizer.*

In view of Proposition 5, we now obtain the second main result of the paper which characterizes an arbitrary Jordan σ-centralizer of a generalized matrix algebra.

Theorem 2. *Let $\mathcal{G} = G(\mathcal{A}, \mathcal{M}, \mathcal{N}, \mathcal{B})$ be a 2-torsion free generalized matrix algebra with faithful $\mathcal{M}$. Then every Jordan σ-centralizer of $\mathcal{G}$ is a σ-centralizer if the following conditions hold:*

(*i*) *$\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents;*
(*ii*) *$\mathcal{A}$ and $\mathcal{B}$ are not isomorphic;*
(*iii*) *both the bilinear mappings $\zeta_{\mathcal{MN}}, \psi_{\mathcal{NM}}$ are zero.*

5. Applications

In this section, we apply Theorems 1 and 2 to triangular algebras. Recall that a triangular algebra is an algebra

$$\mathfrak{A} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B}) = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \middle| \, a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B} \right\}$$

with the usual matrix operations consisting of two unital algebras $\mathcal{A}$, $\mathcal{B}$ and an $(\mathcal{A},\mathcal{B})$-bimodule $\mathcal{M}$. Triangular algebras are classical example of generalized matrix algebras. Indeed, if we take $\mathcal{N}=\{0\}$ in the definition of generalized matrix algebra $\mathcal{G}=G(\mathcal{A},\mathcal{M},\mathcal{N},\mathcal{B})$, then $\mathcal{G}$ exactly degenerates to a triangular algebra $\mathfrak{A}=\mathrm{Tri}(\mathcal{A},\mathcal{M},\mathcal{B})$. It is easy to see that the conditions "either $\mathcal{A}$ or $\mathcal{B}$ does not contain nonzero central ideals" and "both the bilinear mappings $\zeta_{\mathcal{M}\mathcal{N}},\psi_{\mathcal{N}\mathcal{M}}$ are zero" are not required in case of triangular algebras. Further, in the case where $\mathfrak{A}=\mathrm{Tri}(\mathcal{A},\mathcal{M},\mathcal{B})$ is a triangular algebras such that $\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents, we have $Aut_0^0(\mathfrak{A})=Aut(\mathcal{G})$. Hence the condition "$\mathcal{A}$ and $\mathcal{B}$ are not isomorphic" in Theorems 1 and 2 also become redundant. Therefore, as an application of Theorems 1 and 2, we have the following results:

Corollary 6. *Let $\mathfrak{A}=\mathrm{Tri}(\mathcal{A},\mathcal{M},\mathcal{B})$ be a triangular algebra. Suppose that $\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents and $\mathcal{M}$ is a faithful $(\mathcal{A},\mathcal{B})$-bimodule. Then every Lie σ-centralizer of $\mathfrak{A}$ is proper if $\pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{G}))=\mathcal{Z}(\mathcal{A})$ and $\pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{G}))=\mathcal{Z}(\mathcal{B})$.*

Corollary 7. *Let $\mathfrak{A}=\mathrm{Tri}(\mathcal{A},\mathcal{M},\mathcal{B})$ be a 2-torsion free triangular algebra. Suppose that $\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents and $\mathcal{M}$ is a faithful $(\mathcal{A},\mathcal{B})$-bimodule. Then every Jordan σ-centralizer of $\mathfrak{A}$ is a σ-centralizer.*

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A NEW GENERALIZED LAPLACE TRANSFORM AND ITS APPLICATIONS TO FRACTIONAL BAGLEY-TORVIK AND FRACTIONAL HARMONIC VIBRATION PROBLEMS

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Abstract. In this paper, a new generalized Laplace transform is defined and its certain properties are given. By using the new transform, the solutions of fractional Bagley-Torvik and fractional harmonic vibration problems are obtained, as application. Also transformations of some elementary functions and the relationships between the new transform with other generalized Laplace transforms are given in separate tables.

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Keywords: generalized Laplace transform, generalized inverse Laplace transform, Caputo fractional derivative, fractional order differential equations

1. Introduction

Integral transformations are very powerful tools used to solve differential and integral equations that arise in various fields of science. These transformations usually consist of integrating the equation with a weight function of two variables, which will result a simplification of the considered particular problem. Many problems in the field of oscillation theory, thermal conductivity, neutron diffusion, hydrodynamics, theory of elasticity and physical kinetics can be solved with the help of these transformations.

The one of the most popular integral transformation is the Laplace transform [4] which defined as

$$\mathfrak{L}\{\kappa(\xi)\} = \int_0^\infty \exp(-\xi\eta)\kappa(\xi)d\xi,$$

where $\Re(\eta) > 0$, $\xi \geq 0$ and $\kappa(\xi)$ is a piecewise continuous and α-exponential function. There are many generalizations of Laplace transform in the literature such as, Aboodh [1], *HY* [2], Novel [3], Elzaki [5], Gupta [7], Jafari [8], Kamal [9], Kashuri [10], *N* [12], *G* [13], Mahgoub [15], Shedu [16], α-Laplace [17], Mohand [18], Sawi [19], ARA [21], Sadik [22], $\mathbb{M}$ [23], Upadhyaya [24], Sumudu [25] and ZZ [26] transforms (see Table 1 for details).

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In this paper, we defined a new generalization of Laplace transform, which has a more general form than the transforms mentioned above. We determined some of its properties such as, linearity, convolution and derivative formula. We also calculate generalized Laplace transforms of some elementary functions and Caputo fractional derivative. Finally, we used the generalized Laplace transform for obtaining the solutions of fractional Bagley-Torvik and fractional harmonic vibration problems.

2. Generalized Laplace transform and its properties

In this section, we give the definitions of generalized Laplace ($\mathfrak{L}_n$) and inverse Laplace ($\mathfrak{L}_n^{-1}$) transforms and some of their properties.

Definition 2.1. Let $n \in \mathbb{R} - \{0\}$, $\Re(s^n) > 0$, $t \geq 0$ and $f(t)$ is a piecewise continuous and α-exponential function. Then, the generalized Laplace and the inverse Laplace transforms are defined respectively

$$\mathfrak{L}_n\{f(t);s\} := \hat{f}_n(s) = s^{n-1}\int_0^\infty \exp(-s^n t) f(t)dt,$$

and

$$\mathfrak{L}_n^{-1}\left\{\hat{f}_n(s);t\right\} := f(t).$$

Remark 2.1. For the special value $n = 1$, the generalized Laplace transform converts to the classical Laplace transform. Also, the special cases of $\mathfrak{L}_n$ transform are listed in Table 2.

Theorem 2.1. *If a function $f(t)$ is continuous or piecewise continuous and α-exponential in every finite interval $(0,T)$, then the function $f(t)$ has a generalized Laplace transform for all s^n with $\Re(s^n) > \alpha$.*

Proof. Let rewrite the generalized Laplace transform as follows:

$$\begin{aligned}\mathfrak{L}_n\{f(t);s\} &= \left(s^{n-1}\int_0^T \exp(-s^n t) f(t)dt\right) + \left(s^{n-1}\int_T^\infty \exp(-s^n t) f(t)dt\right)\\ &= I_1 + I_2.\end{aligned}$$

The I_1 integral is convergent, since it can be written as the sum of integrals over intervals where $\exp(-s^n t) f(t)$ is continuous. Considering $|f(t)| \leq M\exp(\alpha t)$ for the positive constant M and the entire $t > T$ in the range $0 \leq t < \infty$, we have

$$\begin{aligned}\left|I_2\right| &\leq \left|s^{n-1}\right| \int_T^\infty \exp(-s^n t)\,|f(t)|\,dt\\ &\leq \left|s^{n-1}\right| M \int_0^\infty \exp\left(-t(s^n - \alpha)\right)dt\\ &= \frac{M\left|s^{n-1}\right|}{s^n - \alpha}, \quad \left(\Re(s^n) > \alpha\right).\end{aligned}$$

□

Throughout the paper, unless otherwise stated, we assumed that the functions $f(t)$ and $g(t)$ have $\mathfrak{L}_n$ transforms with $n \in \mathbb{R} - \{0\}$, $\Re(s^n) > 0$, $t \geq 0$.

Theorem 2.2. *Let $f(t)$ and $g(t)$ be functions whose $\mathfrak{L}_n$ transforms exist, and let k_1 and k_2 be constants. Then*

$$\mathfrak{L}_n\{k_1 f(t) + k_2 g(t); s\} = k_1 \mathfrak{L}_n\{f(t); s\} + k_2 \mathfrak{L}_n\{g(t); s\}.$$

Proof. Applying the $\mathfrak{L}_n$ transform, we have

$$\mathfrak{L}_n\{f(t); s\} = s^{n-1} \int_0^\infty \exp(-s^n t) f(t) dt, \quad \big(\Re(s^n) > \alpha_1\big),$$
$$\mathfrak{L}_n\{g(t); s\} = s^{n-1} \int_0^\infty \exp(-s^n t) g(t) dt, \quad \big(\Re(s^n) > \alpha_2\big).$$

With $\Re(s^n) > \max\{\alpha_1, \alpha_2\}$, we have

$$\begin{aligned}
\mathfrak{L}_n\{k_1 f(t) + k_2 g(t); s\} &= s^{n-1} \int_0^\infty \exp(-s^n t)\big(k_1 f(t) + k_2 g(t)\big) dt \\
&= k_1 s^{n-1} \int_0^\infty \exp(-s^n t) f(t) dt + k_2 s^{n-1} \int_0^\infty \exp(-s^n t) g(t) dt \\
&= k_1 \mathfrak{L}_n\{f(t); s\} + k_2 \mathfrak{L}_n\{g(t); s\}.
\end{aligned}$$

□

Example 2.1. Consider the function $f(t) = \exp(at)$ for $t > 0$. If we apply the $\mathfrak{L}_n$ transform, we get

$$\begin{aligned}
\mathfrak{L}_n\{\exp(at); s\} &= s^{n-1} \int_0^\infty \exp(-s^n t) \exp(at) dt \\
&= s^{n-1} \left(\lim_{A \to \infty} \int_0^A \exp\big(-t(s^n - a)\big) dt \right) \\
&= \frac{s^{n-1}}{s^n - a}, \quad \text{for all } \Re(s^n) > a.
\end{aligned}$$

Example 2.2. If we apply $\mathfrak{L}_n$ transform to the function $f(t) = t^m$, where m is a positive integer, then we get

$$\mathfrak{L}_n\{t^m; s\} = \frac{m!}{s^{mn+1}}.$$

To see this, we use induction. For $m = 1$, we have

$$\begin{aligned}
\mathfrak{L}_n\{t; s\} &= s^{n-1} \int_0^\infty \exp(-s^n t) t dt \\
&= s^{n-1} \lim_{A \to \infty} \left(\frac{t \exp(-s^n t)}{-s^n} \bigg|_0^A + \frac{1}{s^n} \int_0^A \exp(-s^n t) dt \right) \\
&= \frac{1}{s^{n+1}}.
\end{aligned}$$

For $m = k$, let the following equation be true:

$$\mathfrak{L}_n\left\{t^k;s\right\} = \frac{k!}{s^{kn+1}}. \tag{2.1}$$

For $m = k+1$, considering equation (2.1), we have

$$\begin{aligned}
\mathfrak{L}_n\left\{t^{k+1};s\right\} &= s^{n-1}\int_0^\infty \exp(-s^n t)t^{k+1}dt \\
&= s^{n-1}\lim_{A\to\infty}\left(\frac{t^{k+1}\exp(-s^n t)}{-s^n}\bigg|_0^A + \frac{(k+1)}{s^n}\int_0^A \exp(-s^n t)t^k dt\right) \\
&= \frac{(k+1)}{s^n}\frac{k!}{s^{nk+1}} \\
&= \frac{(k+1)!}{s^{n(k+1)+1}}.
\end{aligned}$$

Definition 2.2. Let $f(t)$ and $g(t)$ be two functions that are piecewise continuous on every finite closed interval $0 \le t \le b$ and of exponential order. The function denoted by $f * g$ which defined by

$$f(t) * g(t) = s^{n-1}\int_0^t f(t-\tau)g(\tau)d\tau$$

is called the convolution of the functions $f(t)$ and $g(t)$.

Theorem 2.3 (Convolution Theorem)**.** *Denoting $\mathfrak{L}_n$ transforms of $f(t)$ by $\hat{f}_n(s)$ and $g(t)$ by $\hat{g}_n(s)$, we have*

$$\mathfrak{L}_n\left\{f(t) * g(t);s\right\} = \hat{f}_n(s)\hat{g}_n(s).$$

Proof. By using the definitions of convolution and the $\mathfrak{L}_n$ transform, we have

$$\begin{aligned}
\mathfrak{L}_n\left\{f(t) * g(t);s\right\} &= s^{n-1}\int_0^\infty \exp(-s^n t)s^{n-1}\int_0^t f(t-\tau)g(\tau)d\tau dt \\
&= \left(s^{n-1}\int_0^\infty \exp(-s^n\tau)g(\tau)d\tau\right)\left(s^{n-1}\int_0^\infty \exp(-s^n x)f(x)dx\right) \\
&= \hat{g}_n(s)\hat{f}_n(s).
\end{aligned}$$

□

Theorem 2.4 (Translation Property)**.** *Suppose $f(t)$ is a function such that $\mathfrak{L}_n\left\{f(t)\right\}$ exist for $\Re(s^n) > \alpha$. For any constant a,*

$$\mathfrak{L}_n\left\{\exp(at)f(t);s\right\} = \frac{s^{n-1}}{(s^n-a)^{\frac{n-1}{n}}}\mathfrak{L}_n\left\{f(t);(s^n-a)^{\frac{1}{n}}\right\}$$

for $\Re\left(s^n\right) > \alpha + a$.

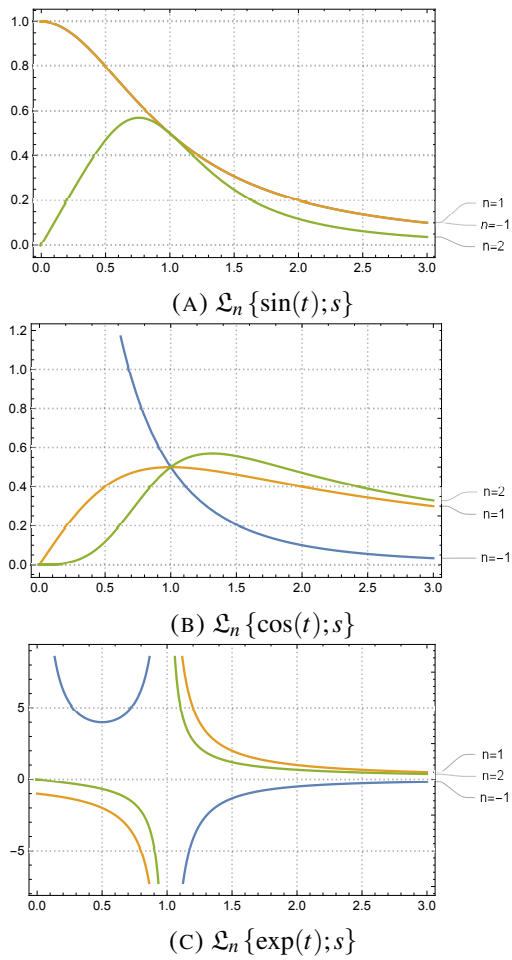
Proof. Using the $\mathfrak{L}_n$ transform, we have

$$\mathfrak{L}_n\left\{\exp(at)f(t);s\right\} = s^{n-1}\int_0^\infty \exp(-s^n t)\exp(at)f(t)dt$$

$$= \frac{s^{n-1}}{(s^n-a)^{\frac{n-1}{n}}} (s^n-a)^{\frac{n-1}{n}} \int_0^\infty \exp\left(-\left((s^n-a)^{\frac{1}{n}}\right)^n t\right) f(t)dt$$

$$= \frac{s^{n-1}}{(s^n-a)^{\frac{n-1}{n}}} \mathfrak{L}_n\left\{f(t);(s^n-a)^{\frac{1}{n}}\right\}. \qquad \square$$

The new generalized Laplace transform of some functions, which can be obtained by direct calculations, are listed in Table 3. Also the graphics of the generalized Laplace transform of the functions $\sin(t)$, $\cos(t)$ and $\exp(t)$ can be found in Figure 1 for the values of $n=-1, n=1$ and $n=2$.



(A) $\mathfrak{L}_n\{\sin(t);s\}$

(B) $\mathfrak{L}_n\{\cos(t);s\}$

(C) $\mathfrak{L}_n\{\exp(t);s\}$

FIGURE 1. The graphics of the $\mathfrak{L}_n$ transforms of the functions $\sin(t), \cos(t)$ and $\exp(t)$ for $n=-1, n=1$ and $n=2$.

Theorem 2.5 ($\mathfrak{L}_n$ Transforms of Derivatives)**.** *The $\mathfrak{L}_n$ transform of the function $f^{(k)}(t)$ is obtained as:*

$$\mathfrak{L}_n\left\{f^{(k)}(t);s\right\} = s^{kn}\hat{f}_n(s) - s^{n-1}\sum_{m=0}^{k-1}(s^n)^{k-1-m}f^{(m)}(0), \tag{2.2}$$

where $f^{(i)}(t)\exp(-s^n t) \to 0$, $(i = 0,1,\ldots,r-1)$ as $t \to \infty$.

Proof. We again use induction. For $k = 1$, we have

$$\begin{aligned}\mathfrak{L}_n\left\{f'(t);s\right\} &= s^{n-1}\int_0^\infty \exp(-s^n t)f'(t)dt \\ &= s^{n-1}\lim_{A\to\infty}\left(\exp(-s^n t)f(t)\Big|_0^A + s^n\int_0^A \exp(-s^n t)f(t)dt\right) \\ &= s^n\hat{f}_n(s) - s^{n-1}f(0),\end{aligned}$$

in which we assumed $f(t)\exp(-s^n t) \to 0$ as $t \to \infty$.
For $k = r$, let the following equation be true

$$\mathfrak{L}_n\left\{f^{(r)}(t);s\right\} = s^{rn}\hat{f}_n(s) - s^{n-1}\sum_{m=0}^{r-1}(s^n)^{r-1-m}f^{(m)}(0). \tag{2.3}$$

For $k = r+1$, considering equation (2.3), we have

$$\begin{aligned}\mathfrak{L}_n\left\{f^{(r+1)}(t);s\right\} &= s^{n-1}\int_0^\infty \exp(-s^n t)f^{(r+1)}(t)dt \\ &= s^{n-1}\left(-f^{(r)}(0) + s^n\int_0^\infty \exp(-s^n t)f^{(r)}(t)dt\right) \\ &= s^{(r+1)n}\hat{f}_n(s) - s^{n-1}\sum_{m=0}^{r}(s^n)^{r-m}f^{(m)}(0),\end{aligned}$$

in which we assumed $f^{(i)}(t)\exp(-s^n t) \to 0$, $(i = 0,1,\ldots,r)$ as $t \to \infty$. □

Theorem 2.6. *The $\mathfrak{L}_n$ transform of the Caputo fractional derivative of order α is obtained as:*

$$\mathfrak{L}_n\left\{{}_0^cD_t^\alpha f(t);s\right\} = s^{\alpha n}\hat{f}_n(s) - \sum_{k=0}^{m-1}s^{\alpha n-nk-1}f^{(k)}(0). \tag{2.4}$$

Proof. The Caputo fractional derivative [20] for $\Re(\alpha) > 0$, $m-1 < \Re(\alpha) < m$, $m \in \mathbb{N}$ is given by

$${}_0^cD_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}f^{(m)}(\tau)d\tau.$$

For $g(\tau) = f^{(m)}(\tau)$, we have

$$ {}_0^cD_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} g(\tau) d\tau. $$

Multiplied each sides by $\frac{s^{n-1}}{s^{n-1}}$, we have

$$ {}_0^cD_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)s^{n-1}} \left(s^{n-1} \int_0^t (t-\tau)^{m-\alpha-1} g(\tau) d\tau \right). $$

Considering the Convolution Theorem and applying the $\mathfrak{L}_n$ transform to both sides of the equation, we have

$$ \mathfrak{L}_n \left\{ {}_0^cD_t^\alpha f(t); s \right\} = s^{-(m-\alpha)n} \hat{g}_n(s). $$

Using equation (2.2), we have

$$ \mathfrak{L}_n \left\{ {}_0^cD_t^\alpha f(t); s \right\} = s^{\alpha n} \hat{f}_n(s) - \sum_{k=0}^{m-1} s^{\alpha n - nk - 1} f^{(k)}(0). \qquad \square $$

3. Applications of $\mathfrak{L}_n$ transform to differential problems

Problem 3.1. *Let the fractional Bagley-Torvik differential equation be given:*

$$ y''(x) + {}_0^cD_x^{\frac{3}{2}} y(x) + y(x) = x + 1, \quad y(0) = y'(0) = 1. $$

Considering equation (2.4) for $m = 2$ and applying the $\mathfrak{L}_n$ transform to the Bagley-Torvik differential equation, we have

$$ \mathfrak{L}_n \left\{ y''(x) \right\} + \mathfrak{L}_n \left\{ {}_0^cD_x^{\frac{3}{2}} y(x) \right\} + \mathfrak{L}_n \left\{ y(x) \right\} = \mathfrak{L}_n \left\{ x \right\} + \mathfrak{L}_n \left\{ 1 \right\} $$

$$ s^{2n} \hat{y}_n(s) - s^{2n-1} - s^{n-1} + s^{\frac{3n}{2}} \hat{y}_n(s) - s^{\frac{3n}{2}-1} - s^{\frac{3n}{2}-n-1} + \hat{y}_n(s) = \frac{1}{s^{n+1}} + \frac{1}{s} $$

$$ \hat{y}_n(s) \left(s^{2n} + s^{\frac{3n}{2}} + 1 \right) - s^{2n-1} - s^{n-1} - s^{\frac{3n}{2}-1} - s^{\frac{3n}{2}-n-1} = \frac{1}{s^{n+1}} + \frac{1}{s}. $$

Then

$$ \hat{y}_n(s) = \frac{s^{-n-1} + s^{-1} + s^{2n-1} + s^{n-1} + s^{\frac{3n}{2}-1} + s^{\frac{3n}{2}-n-1}}{s^{2n} + s^{\frac{3n}{2}} + 1} = \frac{1}{s^{n+1}} + \frac{1}{s}. $$

By applying the $\mathfrak{L}_n^{-1}$ transform, we have

$$ \mathfrak{L}_n^{-1} \left\{ \hat{y}_n(s) \right\} = \mathfrak{L}_n^{-1} \left\{ \frac{1}{s^{n+1}} \right\} + \mathfrak{L}_n^{-1} \left\{ \frac{1}{s} \right\} $$

$$ y(x) = x + 1. $$

Problem 3.2. *Let $1 < \Re(\alpha) < 2$ and the fractional harmonic vibration differential equation be given:*

$$ {}_0^cD_x^\alpha y(x) + w^2 y(x) = 0, \quad y(0) = c_0, \; y'(0) = c_1. $$

Considering equation (2.4) for $m=2$ and applying the $\mathfrak{L}_n$ transform to the harmonic vibration differential equation, we have

$$\begin{aligned}\mathfrak{L}_n\left\{{}_0^cD_x^\alpha y(x)\right\}+w^2\mathfrak{L}_n\left\{y(x)\right\}&=0\\ s^{\alpha n}\hat{y}_n(s)-s^{\alpha n-1}y(0)-s^{\alpha n-n-1}y'(0)+w^2\hat{y}_n(s)&=0\\ s^{\alpha n}\hat{y}_n(s)-c_0s^{\alpha n-1}-c_1s^{\alpha n-n-1}+w^2\hat{y}_n(s)&=0.\end{aligned}$$

Then

$$\begin{aligned}\hat{y}_n(s)&=\frac{c_0s^{\alpha n-1}}{s^{\alpha n}+w^2}+\frac{c_1s^{\alpha n-n-1}}{s^{\alpha n}+w^2}\\ &=\frac{c_0s^{-1}}{1+w^2s^{-\alpha n}}+\frac{c_1s^{-n-1}}{1+w^2s^{-\alpha n}}\\ &=c_0\sum_{k=0}^{\infty}(-1)^kw^{2k}s^{-\alpha kn-1}+c_1\sum_{k=0}^{\infty}(-1)^kw^{2k}s^{-\alpha kn-n-1}.\end{aligned}$$

By applying the $\mathfrak{L}_n^{-1}$ transform, we obtain

$$\begin{aligned}\mathfrak{L}_n^{-1}\left\{\hat{y}_n(s)\right\}&=c_0\sum_{k=0}^{\infty}(-1)^kw^{2k}\mathfrak{L}_n^{-1}\left\{s^{-\alpha kn-1}\right\}+c_1\sum_{k=0}^{\infty}(-1)^kw^{2k}\mathfrak{L}_n^{-1}\left\{s^{-\alpha kn-n-1}\right\}\\ y(x)&=c_0\sum_{k=0}^{\infty}\frac{\left(-w^2x^\alpha\right)^k}{\Gamma(\alpha k+1)}+c_1x\sum_{k=0}^{\infty}\frac{\left(-w^2x^\alpha\right)^k}{\Gamma(\alpha k+2)}\\ &=c_0E_{\alpha,1}\left(-w^2x^\alpha\right)+c_1xE_{\alpha,2}\left(-w^2x^\alpha\right),\end{aligned}$$

where $E_{\alpha,\beta}(x)$ is the Mittag-Leffler function with two parameters [6], which defined as

$$E_{\alpha,\beta}(x)=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+\beta)}.$$

Remark 3.1. For the graphics of the approximate solution of Problem 3.2, we define the function $y_p(x)$ with finite sums as

$$y_p(x)=c_0\sum_{k=0}^{p}\frac{\left(-w^2x^\alpha\right)^k}{\Gamma(\alpha k+1)}+c_1x\sum_{k=0}^{p}\frac{\left(-w^2x^\alpha\right)^k}{\Gamma(\alpha k+2)}.$$

The graphics of the approximate solution $y_2(x)$ of the fractional harmonic vibration problem for the special values $c_0=c_1=w=1$ with $\alpha=\frac{6}{5},\frac{7}{5},\frac{8}{5},\frac{9}{5}$ can be found in Figure 2.

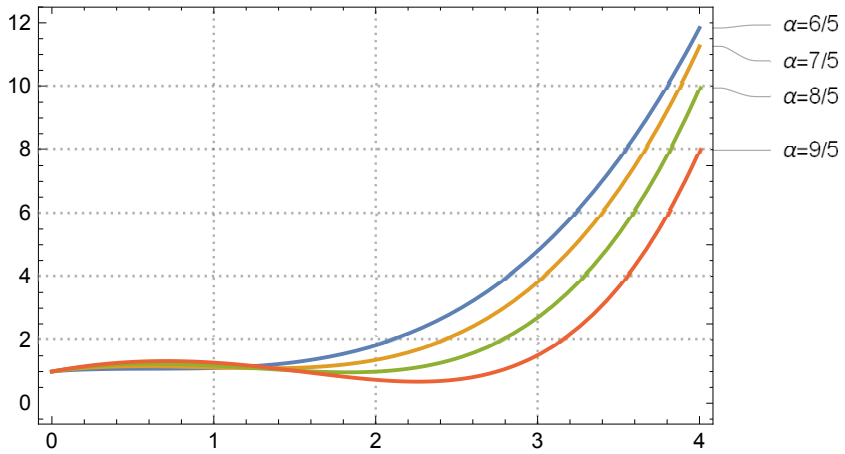


FIGURE 2. The behavior of approximate solution $y_2(x)$ of Problem 3.2 for different values of α, where $1 < \Re(\alpha) < 2$.

4. CONCLUSIONS

In this article, we introduced the generalized Laplace $\mathfrak{L}_n$ and inverse Laplace $\mathfrak{L}_n^{-1}$ transforms and examine their certain properties. Our motivation in doing this was to define an integral transform that has a more general form than many Laplace-like integral transforms found in the literature. We also presented a transformation table for $\mathfrak{L}_n$ transform (Table 1), and another table containing the relationships with other transforms (Table 2). Then, we use generalized Laplace and inverse Laplace transforms to arrive the analytical solutions of fractional Bagley-Torvik and harmonic vibration problems. Finally, we examined the behavior for different alpha values of the approximate solutions of the fractional harmonic vibration problem on a graph (Figure 2).

We should say that the solutions of the application problems we obtained with the generalized Laplace transform are in full agreement with the results obtained in the literature (see for example [11, 14]). As a result of these applications, we saw that the newly defined generalized Laplace transform is much more general and quite compatible with fractional problems. We conclude this research by stating that the generalized Laplace transform plays a very important role in finding analytical solutions of fractional problems, and therefore the results presented in this article are very important for application.

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TABLE 1. Generalized Laplace transforms

Aboodh Transform	[1]: $\mathbf{A}\{\kappa(\eta)\} = \frac{1}{\xi}\int_0^\infty \exp(-\xi\eta)\kappa(\eta)d\eta$
HY Transform	[2]: $HY\{\kappa(\eta)\} = \xi\int_0^\infty \exp\left(-\xi^2\eta\right)\kappa(\eta)d\eta$
Novel Transform	[3]: $M_n\{\kappa(\eta)\} = \int_0^\infty \exp(-\xi\eta)\eta^n\kappa(\eta)d\eta$
Laplace Transform	[4]: $\mathfrak{L}\{\kappa(\eta)\} = \int_0^\infty \exp(-\xi\eta)\kappa(\eta)d\eta$
Elzaki Transform	[5]: $E\{\kappa(\eta)\} = \xi\int_0^\infty \exp\left(-\frac{\eta}{\xi}\right)\kappa(\eta)d\eta$
Gupta Transform	[7]: $\dot{R}\{\kappa(\eta)\} = \frac{1}{\xi^3}\int_0^\infty \exp(-\xi\eta)\kappa(\eta)d\eta$
Kamal Transform	[9]: $K\{\kappa(\eta)\} = \int_0^\infty \exp\left(-\frac{\eta}{\xi}\right)\kappa(\eta)d\eta$
Kashuri Transform	[10]: $\mathcal{K}\{\kappa(\eta)\} = \frac{1}{\xi}\int_0^\infty \exp\left(-\frac{\eta}{\xi^2}\right)\kappa(\eta)d\eta$
N-Transform	[12]: $N\{\kappa(\eta)\} = \int_0^\infty \exp(-\xi\eta)\kappa(u\eta)d\eta$
G-Transform	[13]: $G\{\kappa(\eta)\} = \xi^\alpha\int_0^\infty \exp\left(-\frac{\eta}{\xi}\right)\kappa(\eta)d\eta$
Mahgoub Transform	[15]: $M\{\kappa(\eta)\} = \xi\int_0^\infty \exp(-\xi\eta)\kappa(\eta)d\eta$
Shedu Transform	[16]: $\mathbb{S}\{\kappa(\eta)\} = \int_0^\infty \exp\left(-\frac{\xi\eta}{u}\right)\kappa(\eta)d\eta$
α-Laplace Transform	[17]: $\mathfrak{L}_\alpha\{\kappa(\eta)\} = \int_0^\infty \exp\left(-\xi^{\frac{1}{\alpha}}\eta\right)\kappa(\eta)d\eta$
Mohand Transform	[18]: $\mathbf{M}\{\kappa(\eta)\} = \xi^2\int_0^\infty \exp(-\xi\eta)\kappa(\eta)d\eta$
Sawi Transform	[19]: $\mathbf{S}\{\kappa(\eta)\} = \frac{1}{\xi^2}\int_0^\infty \exp\left(-\frac{\eta}{\xi}\right)\kappa(\eta)d\eta$
ARA Transform	[21]: $G_n\{\kappa(\eta)\} = \xi\int_0^\infty \exp(-\xi\eta)\eta^{n-1}\kappa(\eta)d\eta$
Sadik Transform	[22]: $S\{\kappa(\eta)\} = \frac{1}{\xi^\beta}\int_0^\infty \exp(-\xi^\alpha\eta)\kappa(\eta)d\eta$
$\mathbb{M}$-Transform	[23]: $\mathbb{M}_{\rho,m}\{\kappa(\eta)\}(u,v) = \int_0^\infty \frac{\exp(-u\eta)\kappa(v\eta)}{(\eta^m+v^m)^\rho}d\eta$
Upadhyaya Transform	[24]: $U\{\kappa(\eta)\} = \lambda_1\int_0^\infty \exp(-\lambda_2\eta)\kappa(\lambda_3\eta)d\eta$
Sumudu Transform	[25]: $\mathscr{S}\{\kappa(\eta)\} = \frac{1}{\xi}\int_0^\infty \exp\left(-\frac{\eta}{\xi}\right)\kappa(\eta)d\eta$
ZZ Transform	[26]: $H\{\kappa(\eta)\} = \xi\int_0^\infty \exp(-\xi\eta)\kappa(u\eta)d\eta$

TABLE 2. Relationships of $\mathfrak{L}_n$ transform with other transforms

Laplace Transform	$\mathfrak{L}_1\{\kappa(\eta)\} = \mathfrak{L}\{\kappa(\eta)\}$
Sumudu Transform	$\xi\mathfrak{L}_{-1}\{\kappa(\eta)\} = \mathscr{S}\{\kappa(\eta)\}$
N-Transform	$\mathfrak{L}_1\{\kappa(u\eta)\} = N\{\kappa(\eta)\}$
Elzaki Transform	$\xi^3\mathfrak{L}_{-1}\{\kappa(\eta)\} = E\{\kappa(\eta)\}$
Aboodh Transform	$\frac{1}{\xi}\mathfrak{L}_1\{\kappa(\eta)\} = \mathbf{A}\{\kappa(\eta)\}$
Kashuri Transform	$\xi^2\mathfrak{L}_{-2}\{\kappa(\eta)\} = \mathcal{K}\{\kappa(\eta)\}$
Novel Transform	$\mathfrak{L}_1\{\eta^n\kappa(\eta)\} = M_n\{\kappa(\eta)\}$
$\mathbb{M}$-Transform	$\mathfrak{L}_1\left\{\frac{\kappa(\nu\eta)}{(\eta^m+\nu^m)^\rho}\right\} = \mathbb{M}_{\rho,m}\{\kappa(\eta)\}$
α-Laplace Transform	$\xi^{\frac{\alpha-1}{\alpha}}\mathfrak{L}_{\frac{1}{\alpha}}\{\kappa(\eta)\} = \mathfrak{L}_\alpha\{\kappa(\eta)\}$
Kamal Transform	$\xi^2\mathfrak{L}_{-1}\{\kappa(\eta)\} = K\{\kappa(\eta)\}$
ZZ Transform	$\xi\mathfrak{L}_1\{\kappa(u\eta)\} = H\{\kappa(\eta)\}$
Mahgoub Transform	$\xi\mathfrak{L}_1\{\kappa(\eta)\} = M\{\kappa(\eta)\}$
G-Transform	$\xi^{\alpha+2}\mathfrak{L}_{-1}\{\kappa(\eta)\} = G\{\kappa(\eta)\}$
Mohand Transform	$\xi^2\mathfrak{L}_1\{\kappa(\eta)\} = \mathbf{M}\{\kappa(\eta)\}$
Sadik Transform	$\xi^{1-\alpha-\beta}\mathfrak{L}_\alpha\{\kappa(\eta)\} = S\{\kappa(\eta)\}$
HY Transform	$\mathfrak{L}_2\{\kappa(\eta)\} = HY\{\kappa(\eta)\}$
Sawi Transform	$\mathfrak{L}_{-1}\{\kappa(\eta)\} = \mathbf{S}\{\kappa(\eta)\}$
Shedu Transform	$\mathfrak{L}_1\left\{\kappa(\eta);\frac{\xi}{u}\right\} = \mathbb{S}\{\kappa(\eta)\}$
Upadhyaya Transform	$\lambda_1\mathfrak{L}_1\{\kappa(\lambda_3\eta);\lambda_2\} = U\{\kappa(\eta)\}$
ARA Transform	$\xi\mathfrak{L}_1\{\eta^{n-1}\kappa(\eta)\} = G_n\{\kappa(\eta)\}$
Gupta Transform	$\frac{1}{\xi^3}\mathfrak{L}_1\{\kappa(\eta)\} = \dot{R}\{\kappa(\eta)\}$

TABLE 3. $\mathfrak{L}_n$ transform table

$f(t)$	$\mathfrak{L}_n\{f(t);s\} = s^{n-1}\int_0^\infty \exp(-s^n t)f(t)dt$
1	$\frac{1}{s}$
$\exp(at)$	$\frac{s^{n-1}}{s^n-a}$
$\sin(at)$	$\frac{as^{n-1}}{s^{2n}+a^2}$
$\cos(at)$	$\frac{s^{2n-1}}{s^{2n}+a^2}$
$\sinh(at)$	$\frac{as^{n-1}}{s^{2n}-a^2}$
$\cosh(at)$	$\frac{s^{2n-1}}{s^{2n}-a^2}$
t	$\frac{1}{s^{n+1}}$
t^2	$\frac{2}{s^{2n+1}}$
t^m	$\frac{m!}{s^{mn+1}}$
$\exp(at)t^m$	$\frac{m!s^{n-1}}{(s^n-a)^{m+1}}$
$\exp(at)\sin(bt)$	$\frac{bs^{n-1}}{(s^n-a^2)+b^2}$
$\exp(at)\cos(bt)$	$\frac{s^{n-1}(s^n-a)}{(s^n-a)^2+b^2}$
$f'(t)$	$s^n\hat{f}_n(s) - s^{n-1}f(0)$
$f''(t)$	$s^{2n}\hat{f}_n(s) - s^{2n-1}f(0) - s^{n-1}f'(0)$
$f'''(t)$	$s^{3n}\hat{f}_n(s) - s^{3n-1}f(0) - s^{2n-1}f'(0) - s^{n-1}f''(0)$
$f^{(k)}(t)$	$s^{kn}\hat{f}_n(s) - s^{n-1}\sum_{m=0}^{k-1}(s^n)^{k-1-m}f^{(m)}(0)$
${}_0^cD_t^\alpha f(t)$	$s^{\alpha n}\hat{f}_n(s) - \sum_{m=0}^{k-1}s^{\alpha n-nm-1}f^{(m)}(0)$

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CONSTRUCTION OF VECTOR LYAPUNOV FUNCTION FOR NONLINEAR LARGE-SCALE SYSTEM WITH PERIODIC SUBSYSTEMS

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Abstract. A new approach for constructing vector Lyapunov function for nonlinear nonautonomous large-scale systems is proposed. It is assumed that independent subsystems are linear periodic systems. The components of the vector Lyapunov function are chosen as a quadratic form with a variable matrix. This matrix is an approximate solution of the Lyapunov matrix differential equation. This solution is constructed using the discretization method and the representation of the evolution operator proposed by Magnus. Sufficient conditions for the asymptotic stability of a trivial solution of a nonlinear large-scale system are established. The effectiveness of obtained results are illustrated by the example of stability investigation for coupled systems.

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Keywords: vector Lyapunov function, asymptotic stability, discretization method, comparison method, nonlinear nonautonomous large-scale systems

1. Introduction

Vector Lyapunov functions (VLF) method is widely used to study the stability of complex nonlinear large-scale systems. The main reason for the emergence of this method is the complexity of constructing the Lyapunov function for systems of large dimension. In this situation, the VLF is a more convenient tool for studying the stability of systems than the scalar Lyapunov function, and has been first presented in [4, 12]. Method of VLF assumes a preliminary decomposition of a large-scale system into subsystems of smaller dimensions. Further, independent subsystems are singled out, for which the problem of constructing the Lyapunov function is much easier to solve than for the whole system. The constructed (scalar) Lyapunov functions for independent subsystems are combined into VLF. It should be noted that the application of VLF method is possible only if the independent subsystems are asymptotically stable.

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There are two approaches to study the stability of large-scale systems based on VLF. The first [12] approach involves obtaining a system of differential inequalities for VLF and applying the comparison principle. The second [13] one involves constructing a scalar function by convolution of the constructed VLF with a positive vector and then applying the Lyapunov theorem for scalar function.

The methods of constructing VLF for nonlinear systems with asymptotically stable linear subsystems are presented in [3]. Advances in the development of computational mathematics and computer calculations methods stimulated the development of discretization method for constructing Lyapunov functions. In [1], the discretization method is used to study the robust stability of a switched linear differential system. In [18], a continuous switched system with all unstable subsystems is considered. To study asymptotic stability, the direct Lyapunov method is used, based on the idea of time discretization of solutions of the Lyapunov matrix differential equation.

Vector Lyapunov functions method is widely used in various branches of control theory and systems theory. In [15], the concept of decentralized control is proposed for various classes of autonomous control systems and, based on the method of vector functions, algorithms for stabilization of nonlinear large-scale systems are proposed. The efficiency of these algorithms is primarily due to the fact that nonlinear systems with linear autonomous subsystems are considered, for which there is an explicit procedure for constructing the Lyapunov function.

In [14], a generalized vector Lyapunov function is developed to analyze the stability of nonlinear dynamical systems using the generalized comparison principle. In particular, a cascade of nonlinear dynamical system with continuous mapping is considered. A generalization of the inverse Lyapunov theorem, which establishes the existence of a Lyapunov vector function for the asymptotic stability of equilibrium of a nonlinear dynamical system, is presented. This result is used to establish an equivalence between asymptotic stabilizability and the existence of a vector control function. The problem of feedback control is also considered and the concept of a vector control function is introduced as a generalization of the Lyapunov control function.

In [2, 16], using the Matrosov comparison method, in combination with methods of convex geometry, the approaches of constructing VLF for studying the stability of fixed points of dynamical systems in the space of convex compact sets are proposed. As a result, for a dynamical system in the space of convex compact sets, a comparison system in a finite-dimensional space has been obtained and investigated. The cases of nonlinear systems with nilpotent and idempotent operators are also considered, and the case on the plane is studied in detail. A construction of an infinite-dimensional VLF for cascades in the space of convex compact sets is proposed in [2].

Vector Lyapunov function is widely used in the theory of nonlinear control systems [9], the main result of which shows that the existence of vector Lyapunov control

function is a necessary and sufficient condition for the existence of a smooth globally stabilizing feedback.

Note also that VLF is applicable to the study of impulsive nonlinear systems [7, 8]. In [6], this method is used to study the critical equilibrium states of nonlinear impulsive large-scale systems in the case when continuous and discrete dynamics are both unstable.

In the theory of stability of autonomous nonlinear systems, the concept of input-to-state stability (ISS) and the based on it stability theorems for coupled nonlinear systems (small-gain theorems) (see [5]) is a reliable alternative to the method of vector Lyapunov functions. This approach as well as VLF method assumes the existence of the property of asymptotic stability of independent subsystems.

The main contribution of our article is to develop a new method for constructing the VLF for studying the stability of a nonlinear nonautonomous coupled system of differential equations with linear periodic subsystems. To study the stability of the zero solution, the method of VLF and the comparison principle are used.

For independent subsystems, a new method for constructing scalar Lyapunov functions, which are used as components of the vector Lyapunov function is proposed. The methods of commutator calculus [10, 11] in combination with the discretization method are used. Our approach develops the ideas from [17], which is proposed to use the identities of the commutator calculus to study the asymptotic stability of linear periodic systems.

A linear impulsive comparison system is obtained and investigated. The asymptotic stability of this comparison systems implies the stability of the equilibrium of the considered nonlinear system. An illustrative example of the study of asymptotic stability of nonlinear system consisting of two coupled subsystems is given. For each linear independent subsystem, the Lyapunov function is constructed as a quadratic form with a variable matrix. In this example, there is no Lyapunov function of a quadratic form with a constant matrix.

The paper consists of 8 Sections and is organized as follows. Section 2 provides basic notation and information from linear algebra. In Section 3, the problem statement of stability of the equilibrium of a nonlinear large-scale system, which consists of coupled linear periodic systems, is presented. In Section 4 a new method for constructing the vector Lyapunov function is proposed and an appropriate comparison system is constructed. The main result of the paper is formulated and proved in Section 5, and in Section 6, the problem of stability of linear impulsive comparison systems is discussed. Section 7 gives an example of constructing a vector Lyapunov function. Section 8 is devoted to a discussion of the results and their possible development.

2. Preliminaries

Let $\mathbb{Z}$ be a set of integers, $\mathbb{Z}_+$ — set of non-negative integers, $\mathbb{R}$ — set of real numbers, $\mathbb{R}_+$ — set of real non-negative numbers.

Let $\mathbb{R}^n$ denote the n -dimensional Euclidean space with standard scalar product, $\mathbb{R}^{n\times m}$ is a linear space of $n\times m$ matrices, $\mathbb{R}^{n\times n}$ is the Banach algebra with a norm $\|A\| = \lambda_{\max}^{1/2}(A^{\mathrm{T}}A)$. Let $\mathbb{S}^n \subset \mathbb{R}^{n\times n}$ be the subspace of symmetric matrices, for $P \in \mathbb{S}^n$ the inequality $P \succ 0$ means that the matrix P is positive definite, let $\sigma(A)$ and $r_\sigma(A)$ denote the spectrum and spectral radius for the matrix $A \in \mathbb{R}^{n\times n}$ respectively. If for A the spectrum $\sigma(A) \subset \mathbb{R}$, then $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ are the smallest and largest eigenvalues, respectively, $I_n \in \mathbb{R}^{n\times n}$ is an identity matrix.

$$\mathbb{C}^- = \{z \in \mathbb{C}, \quad \operatorname{Re} z < 0\}, \quad \mathbb{C}^+ = \{z \in \mathbb{C}, \quad \operatorname{Re} z > 0\},$$

$\mathbb{C}$ is a set of complex numbers. The Kronecker symbol δ_{ij} is defined as $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$.

Next we present some information from the commutator calculus that is important for further research [11].

The commutator of two matrices $A, B \in \mathbb{R}^{n\times n}$ which is defined as

$$[A,B] = AB - BA$$

introduces the structure of a Lie algebra in $\mathbb{R}^{n\times n}$. The commutation operator ad_A, $A \in \mathbb{R}^{n\times n}$ is defined as a linear mapping

$$\mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}, \quad Y \mapsto \mathrm{ad}_A(Y) = [A,Y], \quad Y \in \mathbb{R}^{n\times n}.$$

Let X, Y and Z be the independent matrix variables, $F(X,Y)$ is a formal series of variables X and Y, $\lambda \in \mathbb{R}$, then the polarization identity

$$F(X+\lambda Z, Y) = F(X,Y) + \lambda F_1(X,Y,Z) + \lambda^2 F_2(X,Y,Z) + \ldots$$

defines the Hausdorff derivative $\left(Z\frac{\partial}{\partial X}\right)F(X,Y) \stackrel{df}{=} F_1(X,Y,Z)$.

We define recursively the following Lie polynomials of matrix variables X and Y (definition and more details about Lie elements see in [10])

$$\{Y, X^0\} = Y, \quad \{Y, X^{l+1}\} = [\{Y, X^l\}, X], \quad l \in \mathbb{Z}_+.$$

It is easy to see that

$$\mathrm{ad}_X^l(Y) = (-1)^l \{Y, X^l\}.$$

For the further discussion we need the following F. Hausdorff's identities

$$\begin{aligned} e^{-X}\left(\left(Y\frac{\partial}{\partial X}\right)e^X\right) &= Y + \sum_{k=1}^{\infty} \frac{1}{(k+1)!}\{Y, X^k\}, \\ \left(\left(Y\frac{\partial}{\partial X}\right)e^X\right)e^{-X} &= Y + \sum_{k=1}^{\infty} \frac{(-1)^k}{(k+1)!}\{Y, X^k\}. \end{aligned} \tag{2.1}$$

We also need a chain rule for differentiating the composition of mappings. Let $P(X)$ be a series of matrix variable X, $A\colon (a,b) \to \mathbb{R}^{n\times n}$ is a mapping differentiable at the point $t_0 \in (a,b)$ and $B(t) := P(A(t))$ exists in a neighborhood of the point t_0, then we obtain the chain rule

$$\frac{dB}{dt}\Big|_{t=t_0} = \Big(Y\frac{\partial}{\partial X}\Big)P(X), \quad Y = \frac{dA}{dt}\Big|_{t=t_0}, \quad X = A(t_0). \tag{2.2}$$

3. Problem statement

Consider a coupled nonautonomous system of differential equations consisting of r subsystems

$$\dot{\mathbf{x}}_i(t) = A_i(t)\mathbf{x}_i(t) + \mathbf{f}_i(t,\mathbf{x}_1(t),\dots,\mathbf{x}_r(t)), \tag{3.1}$$

where $\mathbf{x}_i \in \mathbb{R}^{n_i}$, $i = 1,\dots,r$, $A_i\colon \mathbb{R} \to \mathbb{R}^{n_i\times n_i}$ is a piecewise continuous mapping. Suppose $A_i(t)$ are θ -periodic functions, $A_i(t+\theta) = A_i(t)$ for all $t \in \mathbb{R}$, $\mathbf{f}_i : \mathbb{R}\times\Omega \to \mathbb{R}^{n_i}$, $\Omega \subset \mathbb{R}^n$ is an open connected set containing $\mathbf{x} = 0$. If

$$\mathbf{x} = (\mathbf{x}_1^{\mathrm{T}},\dots,\mathbf{x}_r^{\mathrm{T}})^{\mathrm{T}}, \quad \mathbf{f} = (\mathbf{f}_1^{\mathrm{T}},\dots,\mathbf{f}_r^{\mathrm{T}})^{\mathrm{T}},$$

then $\mathbf{f} : \mathbb{R}\times\Omega \to \mathbb{R}^n$, $n = \sum_{k=1}^r n_k$. Suppose that the mapping $\mathbf{f}$ satisfies the conditions guaranteeing the existence and uniqueness of solutions of the Cauchy problem for the system (3.1) with the initial conditions $\mathbf{x}_i(t_0) = \mathbf{x}_{i0}$, $i = 1,\dots,r$ and $\mathbf{f}(t,0) \equiv 0$. This condition guarantees that $\mathbf{x} = 0$ is a solution of the system (3.1).

Let N be a fixed positive integer, $h = \frac{\theta}{N}$. Let us define the positive constants a_{mi}, b_{mi}, c_{mi}, where $m = 0,\dots,N-1$, $i = 1,\dots,r$, so that the following inequalities are satisfied

$$\|A_i(t)\| \le a_{mi}, \quad \Big\|\int_{mh}^{t} A_i(s)\,ds\Big\| \le c_{mi}(t-mh), \tag{3.2}$$

$$\Big\|\Big[A_i(t), \int_{mh}^{t} A_i(\tau)\,d\tau\Big]\Big\| \le b_{mi}(t-mh) \tag{3.3}$$

for all $t \in (mh,(m+1)h]$.

Regarding the function $\mathbf{f}$ let us make the following assumptions:

(i) there exist piecewise continuous functions $f_{ij} : \mathbb{R} \to \mathbb{R}_+$, $i,j = 1,\dots,r$ such that for all $\mathbf{x} \in \Omega \subset \mathbb{R}^n$ the following inequalities hold

$$\|\mathbf{f}_i(t,\mathbf{x})\| \le \sum_{j=1}^{r} f_{ij}(t)\|\mathbf{x}_j\|; \tag{3.4}$$

(ii) there exist finite piecewise continuous functions $\widehat{f}_{ij} : (0,\theta] \to \mathbb{R}_+$ such that

$$\sup\{f_{ij}(t+k\theta) \,:\, k \in \mathbb{Z}_+\} \le \widehat{f}_{ij}(t). \tag{3.5}$$

Remark 1. The θ - periodic continuation of the functions $\widehat{f}_{ij}$ to $\mathbb{R}$ will also be denoted by $\widehat{f}_{ij}$.

Let $\mathbf{x}(t,t_0,\mathbf{x}_0)$ be a solution of the Cauchy problem for the system (3.1) with the initial condition $\mathbf{x}(t_0,t_0,\mathbf{x}_0)=\mathbf{x}_0$.

Definition 1. The equilibrium state $\mathbf{x}=0$ of the system of differential equations (3.1) is called

1) Lyapunov stable if for any $\varepsilon>0$, $t_0\in\mathbb{R}$ there exists $\delta=\delta(\varepsilon,t_0)>0$ such that inequality $\|\mathbf{x}_0\|<\delta$ implies $\|\mathbf{x}(t,t_0,\mathbf{x}_0)\|<\varepsilon$ for all $t\geq t_0$;
2) asymptotically stable in the sense of Lyapunov if it is stable and for any $t_0\in\mathbb{R}$ there exists a number $\rho=\rho(t_0)>0$ such that from condition $\|\mathbf{x}_0\|<\rho$ it follows that $\lim\limits_{t\to+\infty}\|\mathbf{x}(t,t_0,\mathbf{x}_0)\|=0$;
3) uniformly asymptotically stable, if $\delta=\delta(\varepsilon,t_0)$ and $\rho(t_0)$ can be chosen independent of t_0.

Remark 2. Using the θ - periodicity of the $A_i(t)$ and inequalities (3.4)–(3.5), one can show that the asymptotic stability of the equilibrium state $\mathbf{x}=0$ is equivalent to the uniform asymptotic stability of $\mathbf{x}=0$.

The aim of the paper is to prove sufficient conditions for the stability of the equilibrium state $\mathbf{x}=0$ of the large-scale system (3.1).

4. Construction of the vector Lyapunov function

Without loss of generality, it is assumed that $t_0=0$. To study the stability, we will use the VLF method. Let $v:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^r_+$ be a vector Lyapunov function. We choose the components of this function as quadratic forms $v_i(t,\mathbf{x}_i)=\mathbf{x}_i^{\mathrm{T}}P_i(t)\mathbf{x}_i$, where $P_i:\mathbb{R}\to\mathbb{S}^{n_i}$ is left continuous and θ -periodic mapping.

Obviously, it is sufficient to define the functions $P_i(t)$, $i=1,\dots,r$ on the period $(0,\theta]$. If N is certain positive integer, $h=\frac{\theta}{N}$, than for $t\in(mh,(m+1)h]$ we can define the matrices

$$\widehat{A}_{mi}(t)=\int\limits_{mh}^{t}A_i(s)\,ds,\quad \widetilde{A}_{mi}=\frac{1}{h}\widehat{A}_{mi}((m+1)h)=\frac{1}{h}\int\limits_{mh}^{(m+1)h}A_i(s)\,ds,$$

$$\Phi_i=e^{h\widetilde{A}_{N-1,i}}\dots e^{h\widetilde{A}_{0,i}}.$$

Suppose that the number N can be chosen so that $r_\sigma(\Phi_i)<1$. If this condition is satisfied, then we choose a symmetric positive definite matrix P_{0i} that satisfies the linear matrix inequality

$$\Phi_i^{\mathrm{T}}P_{0i}\Phi_i-P_{0i}\prec 0.$$

Next, we define the matrices

$$P_{m+1,i}=e^{-h\widetilde{A}_{mi}^{\mathrm{T}}}P_{mi}e^{-h\widetilde{A}_{mi}},\quad m=0,\dots,N-1$$

and, finally, the function $P_i(t)$ is defined as

$$P_i(t)=e^{-\widehat{A}_{mi}^{\mathrm{T}}(t)}P_{mi}e^{-\widehat{A}_{mi}(t)},\quad t\in(mh,(m+1)h].$$

Using the inequalities (3.2)– (3.5) we can obtain the estimate of the total derivative of the constructed vector function $v(t,\mathbf{x})$ along the trajectories of the large-scale system (3.1).

Consider the total derivative of the function $v_i(t,\mathbf{x}_i)$ along the trajectory of the large-scale system (3.1)

$$\dot{v}_i(t,\mathbf{x}_i(t)) = \mathbf{x}_i^{\mathrm{T}}(\dot{P}_i(t)+A_i^{\mathrm{T}}(t)P_i(t)+P_i(t)A_i(t))\mathbf{x}_i + 2\mathbf{x}_i^{\mathrm{T}}P_i(t)\mathbf{f}_i(t,\mathbf{x}).$$

For $t \in (mh,(m+1)h]$, we need to compute the value of the expressions $\dot{P}_i(t)+A_i^{\mathrm{T}}(t)P_i(t)+P_i(t)A_i(t)$, $i=1,\dots,r$. Let us first consider the expression for $\frac{d}{dt}e^{-\widehat{A}_{mi}(t)}$. Applying the chain rule of differentiation (2.2), we obtain

$$\begin{aligned}\frac{d}{dt}e^{-\widehat{A}_{mi}(t)} &= -\Big(\frac{d\widehat{A}_{mi}(t)}{dt}\frac{\partial}{\partial X}\Big)e^{X}\Big|_{X=-\widehat{A}_{mi}(t)}\\ &= -\Big(A(t)\frac{\partial}{\partial X}\Big)e^{X}\Big|_{X=-\widehat{A}_{mi}(t)}.\end{aligned}$$

From the Hausdorff identity (2.1), it follows

$$\frac{d}{dt}e^{-\widehat{A}_{mi}(t)} = -e^{-\widehat{A}_{mi}(t)}\Big(A_i(t)+\sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)!}\{A_i(t),\widehat{A}_{mi}^k(t)\}\Big).$$

Consequently, we obtain

$$\frac{d}{dt}e^{-\widehat{A}_{mi}^{\mathrm{T}}(t)} = -\Big(A_i^{\mathrm{T}}(t)+\sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)!}\{A_i(t),\widehat{A}_{mi}^k(t)\}^{\mathrm{T}}\Big)e^{-\widehat{A}_{mi}^{\mathrm{T}}(t)}$$

and therefore

$$\begin{aligned}\dot{P}_i(t)+A_i^{\mathrm{T}}(t)P_i(t)+P_i(t)A_i(t) = &-\Big(\sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)!}\{A_i(t),\widehat{A}_{mi}^k(t)\}\Big)^{\mathrm{T}}P_i(t)\\ &-P_i(t)\Big(\sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)!}\{A_i(t),\widehat{A}_{mi}^k(t)\}\Big).\end{aligned}$$

From (3.2)– (3.3) it follows the estimates for $t \in (mh,(m+1)h]$

$$\|\{A_i(t),\widehat{A}_{mi}^k(t)\}\| \le b_{mi}(2c_{mi})^{k-1}(t-mh)^k,$$

$$\Big\|\sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)!}\{A_i(t),\widehat{A}_{mi}^k(t)\}\Big\| \le b_{mi}\sum_{k=1}^{\infty}\frac{(2c_{mi})^{k-1}(t-mh)^k}{(k+1)!} \le \eta_{mi}(t-mh),$$

where $\eta_{mi} = b_{mi}\sum_{k=1}^{\infty}\frac{(2c_{mi}h)^{k-1}}{(k+1)!}$, so we obtain

$$\mathbf{x}_i^{\mathrm{T}}(\dot{P}_i(t)+A_i^{\mathrm{T}}(t)P(t)+P(t)A_i(t))\mathbf{x}_i \le 2\|P_i^{-1/2}(t)\|\|P_i^{1/2}(t)\|(t-mh)\eta_{mi}\mathbf{x}_i^{\mathrm{T}}P_i(t)\mathbf{x}_i(t).$$

Using the following estimates

$$\|P_i^{1/2}(t)\| = \lambda_{\max}^{1/2}(P_i(t)) = \|P_i(t)\|^{1/2} = \|e^{-\widehat{A}_{mi}^{\mathrm{T}}(t)}P_{mi}e^{-\widehat{A}_{mi}(t)}\|^{1/2}$$

$$\leq e^{c_{mi}(t-mh)}\|P_{mi}\|^{1/2},$$

$$\lambda_{\min}(P_i(t)) = \min_{\|\mathbf{x}_i\|=1} \mathbf{x}_i^{\mathrm{T}} P_i(t)\mathbf{x}_i$$

$$= \min_{\|\mathbf{x}_i\|=1} (e^{-\widehat{A}_{mi}(t)}\mathbf{x}_i)^{\mathrm{T}} P_{mi} e^{-\widehat{A}_{mi}(t)}\mathbf{x}_i \geq \lambda_{\min}(P_{mi}) \min_{\|\mathbf{x}_i\|=1} \|e^{-\widehat{A}_{mi}(t)}\mathbf{x}_i\|^2$$

$$\geq \min_{\|\mathbf{x}_i\|=1} \lambda_{\min}(P_{mi}) e^{-2c_{mi}(t-mh)}\|\mathbf{x}_i\|^2 = \lambda_{\min}(P_{mi}) e^{-2c_{mi}(t-mh)}$$

and

$$\|P_i^{-1/2}(t)\| = \lambda_{\max}^{1/2}(P_i^{-1}(t)) = \lambda_{\min}^{-1/2}(P_i(t)) \leq \lambda_{\min}^{-1/2}(P_{mi}) e^{c_{mi}(t-mh)},$$

which are valid for $t \in (mh,(m+1)h]$, we obtain

$$\mathbf{x}_i^{\mathrm{T}}(\dot{P}_i(t) + A_i^{\mathrm{T}}(t)P_i(t) + P_i(t)A_i(t))\mathbf{x}_i \leq 2e^{2c_{mi}(t-mh)}\sqrt{\frac{\lambda_{\max}(P_{mi})}{\lambda_{\min}(P_{mi})}}(t-mh)\eta_{mi}v_i(t,\mathbf{x}_i)$$

for all $t \in (mh,(m+1)h)$.

Let us now consider the problem of estimation the expression $\mathbf{x}_i^{\mathrm{T}} P_i(t)\mathbf{f}_i(t,\mathbf{x})$. Applying the Cauchy-Bunyakovsky inequality and the assumptions (3.4)– (3.5) about the function $\mathbf{f}$, for $t \in (mh,(m+1)h]$, we obtain

$$\mathbf{x}_i^{\mathrm{T}} P_i(t)\mathbf{f}_i(t,\mathbf{x}) = (P_i^{1/2}(t)\mathbf{x}_i)^{\mathrm{T}} P_i^{1/2}(t)\mathbf{f}_i(t,\mathbf{x}) \leq \sqrt{v_i(t,\mathbf{x}_i)}\|P_i^{1/2}(t)\| \sum_{j=1}^{r} f_{ij}(t)\|\mathbf{x}_j\|$$

$$\leq \sqrt{v_i(t,\mathbf{x}_i)}\|P_i^{1/2}(t)\| \sum_{j=1}^{r} \widehat{f}_{ij}(t)\|P_j^{-1/2}(t)\|\sqrt{v_j(t,\mathbf{x}_j)}$$

$$\leq \sqrt{v_i(t,\mathbf{x}_i)} \sum_{j=1}^{r} \widehat{f}_{ij}(t) e^{(c_{mi}+c_{mj})(t-mh)} \sqrt{\frac{\lambda_{\max}(P_{mi})}{\lambda_{\min}(P_{mj})}}\sqrt{v_j(t,\mathbf{x}_j)}.$$

Let us define the matrix $\Gamma(t) = (\gamma_{ij}(t))_{i,j=1}^{r}$, where

$$\gamma_{ij}(t) = e^{2c_{mi}(t-mh)}\sqrt{\frac{\lambda_{\max}(P_{mi})}{\lambda_{\min}(P_{mi})}}(t-mh)\eta_{mi}\delta_{ij} + \widehat{f}_{ij}(t)e^{(c_{mi}+c_{mj})(t-mh)}\sqrt{\frac{\lambda_{\max}(P_{mi})}{\lambda_{\min}(P_{mj})}}.$$

Then we get the estimate

$$\dot{v}_i(t,\mathbf{x}_i(t)) \leq 2\gamma_{ii}(t)v_i(t,\mathbf{x}_i) + 2\sqrt{v_i(t,\mathbf{x}_i)} \sum_{j=1, j\neq i}^{r} \gamma_{ij}(t)\sqrt{v_j(t,\mathbf{x}_i)}$$

for all $t \in (0,\theta]$. If $t = \theta$, then based on the periodicity of the functions $P_i(t)$, we obtain

$$v_i(\theta+0,\mathbf{x}_i(\theta+0)) = \mathbf{x}_i^{\mathrm{T}}(\theta+0)P_i(\theta+0)\mathbf{x}_i(\theta+0) = \mathbf{x}_i^{\mathrm{T}}(\theta)P_i(0+0)\mathbf{x}_i(\theta)$$

$$= \mathbf{x}_i^{\mathrm{T}}(\theta)P_{0i}\mathbf{x}_i(\theta) = (P_{Ni}^{1/2}\mathbf{x}_i(\theta))^{\mathrm{T}} P_{Ni}^{-1/2}P_{0i}P_{Ni}^{-1/2}P_{Ni}^{1/2}\mathbf{x}_i(\theta)$$

$$\leq \lambda_{\max}(P_{0i}P_{Ni}^{-1})\mathbf{x}_i^{\mathrm{T}}(\theta)P_{Ni}\mathbf{x}_i(\theta) = \lambda_{\max}(P_{0i}P_{Ni}^{-1})\mathbf{x}_i^{\mathrm{T}}(\theta)P_i(\theta)\mathbf{x}_i(\theta)$$

$$= \lambda_{\max}(P_{0i}P_{Ni}^{-1})v_i(\theta,\mathbf{x}_i(\theta)) := \delta_i v_i(\theta,\mathbf{x}_i(\theta)).$$

Extending the function $\Gamma(t)$ to $\mathbb{R}_+$ in a periodic manner, we conclude that the components of the vector Lyapunov function satisfy the inequalities

$$\begin{gathered}\dot{v}_i(t,\mathbf{x}_i(t)) \le 2\gamma_{ii}(t)v_i(t,\mathbf{x}_i) + 2\sqrt{v_i(t,\mathbf{x}_i)}\sum_{j=1,j\neq i}^{r}\gamma_{ij}(t)\sqrt{v_j(t,\mathbf{x}_i)}, \quad t\neq k\theta,\\ v_i(t+0,\mathbf{x}_i(t+0)) \le \delta_i v_i(t,\mathbf{x}_i(t)), \quad t = k\theta.\end{gathered} \tag{4.1}$$

Applying comparison theorems for impulsive differential inequalities, we get the estimates

$$v_i(t,\mathbf{x}_i(t)) \le u_i^+(t), \quad t\ge 0, \quad i=1,\dots,r,$$

where $u^+(t) = (u_1(t),\dots,u_r(t))^{\mathrm{T}}$ is the maximum solution of the Cauchy problem ($u_0 = (v_1(0+0,\mathbf{x}_{10}),\dots,v_r(0+0,\mathbf{x}_{r0}))^{\mathrm{T}}$) for a system of impulsive differential equations

$$\begin{gathered}\dot{u}_i(t) = 2\gamma_{ii}(t)u_i(t) + 2\sqrt{u_i(t)}\sum_{j=1,j\neq i}^{r}\gamma_{ij}(t)\sqrt{u_j(t)}, \quad t\neq k\theta,\\ u_i(t+0) = \delta_i u_i(t), \quad t = k\theta, \quad u(0+0) = u_0.\end{gathered} \tag{4.2}$$

It is easy to see that for comparison system (4.2) if $u_0 > 0$, $u_0 \neq 0$, then $u(t) > 0$ for $t \ge 0$.

In this case, by changing variables $\zeta_i = u_i^{1/2}$, $i = 1,\dots,r$ the system of comparison (4.2) is reduced to a linear system of impulsive differential equations

$$\begin{gathered}\dot{\zeta}(t) = \Gamma(t)\zeta(t), \quad t\neq k\theta,\\ \zeta(t+0) = \Delta^{1/2}\zeta(t), \quad t = k\theta,\end{gathered} \tag{4.3}$$

where $\zeta = (\zeta_1,\dots,\zeta_r)^{\mathrm{T}}$, $\Delta = \operatorname{diag}\{\delta_1,\dots,\delta_r\}$.

5. Stability Theorem

The VLF constructed in the previous section and the obtained estimates of its derivative allow us to reduce the study of the stability of the equilibrium state of a nonlinear nonautonomous system to the study of the stability of a linear impulsive system (4.3) (linear comparison system).

Theorem 1. *Assume the linear impulsive system* (4.3) *is asymptotically stable in the sense of Lyapunov. Then the equilibrium state* $\mathbf{x} = 0$ *of the system* (3.1) *is asymptotically stable in the sense of Lyapunov.*

Proof. Let $\mathbf{x}_0 = (\mathbf{x}_{10}^{\mathrm{T}},\dots,\mathbf{x}_{r0}^{\mathrm{T}})^{\mathrm{T}}$ be an initial conditions for the solution $\mathbf{x}(t)$ of the system of differential equations (3.1). Denote by $\eta > 0$ an arbitrary number. Consider the solution of the Cauchy problem for the differential comparison equation (4.2) with

the initial conditions $u_{i0} = v_i(0+0, \mathbf{x}_{i0}) + \eta$. Then, by the comparison principle, the following inequalities hold

$$v_i(t, \mathbf{x}_i(t)) \le u_i(t).$$

Since, $u_0 > 0$, then $u_i(t) = \zeta_i^2(t, \zeta_0)$, where $\zeta(t) = (\zeta_1(t), \dots, \zeta_r(t))^{\mathrm{T}}$ is the solution of the Cauchy problem for (4.3) with initial condition $\zeta(0+0) = \zeta_0$, $\zeta_0 = (u_{10}^{1/2}, \dots, u_{r0}^{1/2})^{\mathrm{T}}$ and

$$v_i(t, \mathbf{x}_i(t)) \le \zeta_i^2(t).$$

Passing to the limit $\eta \to 0+$ and taking into account that the solutions of the linear impulsive differential equation continuously depend on the initial conditions, we obtain the estimate

$$v_i(t, \mathbf{x}_i(t)) \le \hat{\zeta}_i^2(t), \quad \text{for all} \quad t \ge 0, \quad i = 1, \dots, r,$$

where $\hat{\zeta}_i^2(t)$ is the solution of the Cauchy problem for (4.2) with initial condition $\hat{\zeta}(0) = (\hat{\zeta}_{10}, \dots, \hat{\zeta}_{r0})^{\mathrm{T}}$, $\hat{\zeta}_{i0} = (\mathbf{x}_{i0}^{\mathrm{T}} P_{0i} \mathbf{x}_{i0})^{1/2}$.

In particular, for $t = k\theta + 0$ we obtain

$$\lambda_{\min}(P_{0i}) \|\mathbf{x}_i(k\theta)\|^2 \le v_i(k\theta+0, \mathbf{x}_i(k\theta+0)) \le \hat{\zeta}_i^2(k\theta+0)$$
$$\text{for all} \quad k \in \mathbb{Z}_+, \quad i = 1, \dots, r.$$

From asymptotic stability of the comparison system (4.3) it follows that for a given $\varepsilon_1 > 0$ there exists $\delta_1(\varepsilon_1) > 0$ such that inequalities $\hat{\zeta}_{i0} < \delta_1(\varepsilon_1)$ imply $\hat{\zeta}_i(t) < \varepsilon_1$ for all $t \ge 0$. Let ε be an arbitrary positive number,

$$\varepsilon_1 = \frac{1}{\sqrt{r}} \varepsilon \min_{i=1,\dots,r} \{\sqrt{\lambda_{\min}(P_{0i})}\}, \quad \delta(\varepsilon) = \min_{i=1,\dots,r} \Big\{ \frac{\delta_1(\varepsilon_1)}{\sqrt{\lambda_{\max}(P_{0i})}} \Big\}.$$

If $\|\mathbf{x}_{i0}\| < \delta(\varepsilon)$, then $\sqrt{\mathbf{x}_{i0}^{\mathrm{T}} P_{0i} \mathbf{x}_{i0}} < \delta_1(\varepsilon_1)$ and

$$\lambda_{\min}(P_{0i}) \|\mathbf{x}_i(k\theta)\|^2 \le v_i(k\theta+0, \mathbf{x}_i(k\theta+0)) \le \hat{\zeta}_i^2(k\theta+0) \le \frac{\varepsilon^2}{r} \lambda_{\min}(P_{0i})$$
$$\text{for all} \quad k \in \mathbb{Z}_+, \quad i = 1, \dots, r.$$

Hence, we have the estimate $\|\mathbf{x}_i(k\theta)\| < \frac{\varepsilon}{\sqrt{r}}$ for all $k \in \mathbb{Z}_+$ and $\|\mathbf{x}(k\theta)\| < \varepsilon$.

Using the Cauchy-Bunyakovsky inequality, we obtain

$$\begin{aligned} \frac{d}{dt} \|\mathbf{x}_i(t)\|^2 &= 2\mathbf{x}_i^{\mathrm{T}}(t) A_i(t) \mathbf{x}_i(t) + 2\mathbf{x}_i^{\mathrm{T}}(t) f_i(t, \mathbf{x}_1(t), \dots, \mathbf{x}_r(t)) \\ &\le 2\|A_i(t)\| \|\mathbf{x}_i(t)\|^2 + 2\|\mathbf{x}_i(t)\| \sum_{j=1}^{r} \widehat{f}_{ij}(t) \|\mathbf{x}_j(t)\|. \end{aligned}$$

Denote

$$a = \max_{i=1,\dots,r} \sup_{t \in [0,\theta]} \|A_i(t)\|, \quad f^* = \max_{i=1,\dots,r} \sup_{t \in [0,\theta]} \sqrt{\sum_{k=1}^{r} \widehat{f}_{ij}^2},$$

then, again applying the Cauchy-Bunyakovsky inequality, we get

$$\frac{d}{dt}\|\mathbf{x}_i(t)\|^2 \le 2a\|\mathbf{x}_i(t)\|^2 + 2f^*\|\mathbf{x}_i(t)\|\|\mathbf{x}(t)\| \le (2a+f^*)\|\mathbf{x}_i(t)\|^2 + f^*\|\mathbf{x}(t)\|^2.$$

Hence, it follows that

$$\frac{d}{dt}\|\mathbf{x}(t)\|^2 \le (2a+f^*)\|\mathbf{x}(t)\|^2 + rf^*\|\mathbf{x}(t)\|^2 = (2a+(r+1)f^*)\|\mathbf{x}(t)\|^2 := L\|\mathbf{x}(t)\|^2.$$

And therefore, we have the estimate

$$\|\mathbf{x}(t)\| \le \|\mathbf{x}(k\theta)\|e^{\frac{L}{2}(t-k\theta)} \le e^{\frac{L}{2}\theta}\|\mathbf{x}(k\theta)\|, \quad \text{for all} \quad t \in [k\theta,(k+1)\theta].$$

As proved above, for $\varepsilon e^{-L\theta}$ there exists $\delta = \delta(\varepsilon) > 0$ such that from inequality $\|\mathbf{x}_0\| < \delta$ it follows $\sup\limits_{k\in\mathbb{Z}_+}\|\mathbf{x}(k\theta)\| < \varepsilon e^{-\frac{L}{2}\theta}$. So, we obtain the following estimate

$$\sup_{t\ge 0}\|\mathbf{x}(t)\| \le e^{\frac{L}{2}\theta}\sup_{k\in\mathbb{Z}_+}\|\mathbf{x}(k\theta)\| < \varepsilon,$$

which proves the stability of the equilibrium state $\mathbf{x} = 0$. To prove the attraction property of the equilibrium state $\mathbf{x} = 0$, note that from inequality

$$\lambda_{\min}(P_{0i})\|\mathbf{x}_i(k\theta)\|^2 \le \hat{\zeta}_i^2(k\theta+0) \quad \text{for all} \quad k \in \mathbb{Z}_+, \quad i = 1,\dots,r$$

it follows that $\lim_{k\to\infty}\|\mathbf{x}(k\theta)\| = 0$, and therefore

$$\lim_{t\to\infty}\|\mathbf{x}(t)\| \le e^{\frac{L}{2}\theta}\lim_{k\to\infty}\|\mathbf{x}(k\theta)\| = 0.$$

□

6. Stability of comparison system

Theorem 1 reduces the problem of stability of the equilibrium state $\mathbf{x} = 0$ of the original nonautonomous system (3.1) to the study of the linear comparison system (4.3). The study of the comparison system is easier, since it, as a rule, has a lower dimension and has an important property of positivity with respect to the cone $\mathcal{K} = \mathbb{R}_+^r$. The last property makes it possible to significantly simplify the study of this system and to obtain sufficient conditions for the stability of the linear comparison system (4.3).

Let us define the matrices $\Gamma_m = (\gamma_{ij}^{(m)})_{i,j=1}^r$, $m = 0,\dots,N-1$, where

$$\gamma_{ij}^{(m)} = e^{2c_{mi}h}\sqrt{\frac{\lambda_{\max}(P_{mi})}{\lambda_{\min}(P_{mi})}}h\eta_{mi}\delta_{ij} + \sup_{t\in(mh,(m+1)h]}\widehat{f}_{ij}(t)e^{(c_{mi}+c_{mj})h}\sqrt{\frac{\lambda_{\max}(P_{mi})}{\lambda_{\min}(P_{mj})}}.$$

It is obvious that $\Gamma(t) \le \Gamma_m$ for all $t \in (mh,(m+1)h]$. So, we get

$$\zeta(t) \le \widetilde{\zeta}(t), \tag{6.1}$$

where $\zeta(t)$ is a solution of the Cauchy problem for the linear comparison system (4.3) with the initial condition $\zeta(0) = \zeta_0 \geq 0$, $\widetilde{\zeta}(t)$ is a solution of Cauchy problem for system

$$\begin{aligned} \frac{d\widetilde{\zeta}}{dt} &= \widetilde{\Gamma}(t)\widetilde{\zeta}, \quad t \neq k\theta, \quad \widetilde{\zeta}(0) = \zeta_0, \\ \widetilde{\zeta}(t+0) &= \Delta^{1/2}\widetilde{\zeta}(t), \quad t = k\theta, \end{aligned} \tag{6.2}$$

where $\widetilde{\Gamma}(t) = \Gamma_m$ for $t \in (mh, (m+1)h]$. From the inequality (6.1) it follows that the asymptotic stability of the linear comparison system (6.2) implies the asymptotic stability of the linear comparison system (4.3) and the equilibrium state $\mathbf{x} = 0$ of the nonlinear nonautonomous system (3.1). Thus, we obtain the following sufficient conditions for the asymptotic stability of the equilibrium state $\mathbf{x} = 0$ of the nonlinear nonautonomous system (3.1).

Proposition 1. *Assume that the positive integer N is chosen large enough so that $r_\sigma(\Phi_i) < 1$ and the following inequality hold*

$$r_\sigma\Big(\prod_{k=0}^{N-1} e^{\Gamma_k h}\Delta^{1/2}\Big) < 1.$$

Then the equilibrium state $\mathbf{x} = 0$ of the nonlinear nonautonomous system (3.1) is asymptotically stable.

For the *proof* it is necessary to use the main result of the Floquet-Lyapunov theory.

7. Numerical example

Consider a nonlinear coupled system

$$\begin{aligned} \dot{x}_1(t) &= (-0.8 + \cos(20\pi t))x_1(t) - \sin(20\pi t)x_2 + f_1(t, \mathbf{x}_2(t)), \\ \dot{x}_2(t) &= -\sin(20\pi t)x_1 + (-0.8 - \cos(20\pi t))x_2(t) + f_2(t, \mathbf{x}_2(t)), \\ \dot{x}_3(t) &= (-0.8 + \cos(20\pi t))x_3(t) - \sin(20\pi t)x_4 + f_3(t, \mathbf{x}_1(t)), \\ \dot{x}_4(t) &= -\sin(20\pi t)x_3 + (-0.8 - \cos(20\pi t))x_4(t) + f_4(t, \mathbf{x}_1(t)), \end{aligned} \tag{7.1}$$

where $x_i \in \mathbb{R}$, $\mathbf{x} = (x_1, x_2, x_3, x_4)^{\mathrm{T}}$, $\mathbf{x}_1 = (x_1, x_2)^{\mathrm{T}}$, $\mathbf{x}_2 = (x_3, x_4)^{\mathrm{T}}$ and the functions $\mathbf{f}_1 = (f_1, f_2)^{\mathrm{T}}$, $\mathbf{f}_2 = (f_3, f_4)^{\mathrm{T}}$ satisfy the estimates

$$\|\mathbf{f}_1(t, \mathbf{x}_2)\| \leq \widehat{f}_{12}\|\mathbf{x}_2\|, \quad \|\mathbf{f}_2(t, \mathbf{x}_1)\| \leq \widehat{f}_{21}\|\mathbf{x}_1\|.$$

If $N = 1$, $P_{0i} = I$, $i = 1, 2$, then the matrices Γ_0 and Δ have the form

$$\Gamma = \begin{pmatrix} 0.6432 & 1.5834\widehat{f}_{12} \\ 1.5834\widehat{f}_{21} & 0.64312 \end{pmatrix},$$

$$\Delta = 0.9231I.$$

The sufficient conditions for the asymptotic stability of the equilibrium state $\mathbf{x} = 0$ of the nonlinear system (7.1) are of the form $\widehat{f}_{12}\widehat{f}_{21} < 0.00981$.

Let $\widehat{f}_{12} = \widehat{f}_{21} = 0.6$, $N = 10$, $P_{10} = P_{20} = I_2$, $a_{mi} = 2.2978$, $b_{mi} = 6.9282$, $\eta_{mi} = 3.5178$, $\delta_i = 0.8578$, $m = 0,\dots,9$, $i = 1,2$, $r_\sigma(\prod_{k=0}^{9} e^{\Gamma_k h}\Delta^{1/2}) = 0.99276 < 1$. Therefore, the equilibrium state $\mathbf{x} = 0$ of the nonlinear system (7.1) is asymptotically stable.

It should be noted that matrices

$$A_{11}(t) = A_{22}(t) = \begin{pmatrix} -0.8+\cos(20\pi t) & -\sin(20\pi t) \\ -\sin(20\pi t) & -0.8-\cos(20\pi t) \end{pmatrix}$$

have a spectrum $\sigma(A_{ii}(t)) = \{-1.8, 0.2\}$. Therefore, for each independent subsystem there is no Lyapunov function of quadratic form with a constant (independent of t) matrix.

8. Conclusion

The example given in Section 7 shows that the choice of the Lyapunov function for independent subsystems of a large-scale system is far from obvious. However, the proposed algorithm allows one to construct Lyapunov functions for independent subsystems and obtain sufficient stability conditions for large-scale system. These conditions are analogous to the well-known small-gain conditions in the theory of stability of coupled systems. Algorithm for constructing the Lyapunov function can be easily implemented using modern computing tools such as Maple or Matlab. It is important to note that the implementation of this algorithm requires calculating of a finite number of matrices and checking a finite number of inequalities. For high frequency systems, it requires little computation. In what follows, it is of interest to generalize the results obtained for some classes of infinite-dimensional systems and to study the possibility of abandoning the requirement for asymptotic stability of independent subsystems.

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AN ENUMERATION APPROACH TO NETWORK EVOLUTION

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Abstract. A simple theoretical model of network evolution is discussed here. In each step, we add a new vertex to the graph and it is allowed to connect it to maximum degree vertices (hubs) only. Given a constant p, the probability of such a connection is p for any hub. The initial (non-random) graph G_1 is arbitrary but here we investigate mostly the case when G_1 has one vertex.

We solve here some particular cases of the problem, using enumeration methods. We obtain not limit theorems but exact results for the parameters discussed.

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1. Introduction

Network evolution is the subject of very intensive research since the starting of World Wide Web [4]. The literature of random graphs is extremely diverse [5]. In [3] the authors examine the size of large subgraphs of the binomial random graphs. From [15] we get a general insight how random graphs model large networks. In [16] a sequential metric dimension was examined in random graphs. Móri and Backhausz [1] study the degree distribution in the lower levels of the uniform recursive tree. Among the numerous further models and applications we mention here the solution of problems in physics [7].

In [8] the random graph dynamics appears, and [12] examines the adversarial deletion in a scale-free random graph process. In [17], the random graphs and the complex networks are examined. [13] actually reports the same results as Erdős and Rényi [9]. In Grenander's book [14], the descriptions of [10] is generalized. [11] details the probability theory information used in the article.

A simple theoretical model of such processes is investigated here. As we know, for numerous processes in large networks, Web, pandemic, and so on, the new vertex will be adjacent mostly to vertices of large degrees. According to that, we define a sequence of random graphs such that the rules of developing yields a model for

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the properties above. A very frequent problem is to determine the distribution of degrees. We have partial results, concerning that. The model is far from the Erdős-Rényi one ([9], [10]), even the simplest questions are difficult here, thus often it is worth assuming that we start with the one-vertex graph and in every step, a new vertex appears.

Paper of Bollobás ([6]) is also significant on the subject.

Note that even now, for many networks, different from Web, the 'old' model can be applied successfully. In [2], for example, we solve domination problems, using Erdős-Rényi model. Here we would have tried it in vain.

2. Definitions and notation

Here random graphs will be denoted by boldface characters. Let the random graph $\mathbf{G}_N$ be the output of the following process, called Majority Process. Given a constant non-random graph G_1, the vertex set of the t'th graph $\mathbf{G}_t$ is $V(G_1) \cup \{x_2, \ldots, x_t\}$. The maximum possible value of t is denoted by N. $\mathbf{G}_N$, the final graph, represents the entire network. We call $V(\mathbf{G_N})$ by V for brevity. A vertex of a (random or deterministic) graph is called *hub* if it has maximum degree in the underlying graph. In general, G_1 is an arbitrary fixed graph, but we begin the work by the special case $G_1 \cong K_1$, where K_n is the clique on n vertices.

Supposed that the t'th graph is $\mathbf{G}_t$, we define $\mathbf{G}_{t+1}$ in the following way:

For any hub x, v_{t+1} is adjacent to x with probability p. These decisions are completely independent. For any other vertex x, xv_{t+1} is defined to be a non-edge.

Remark 1. At this step, we can see already that the model differs from the Erdős-Rényi one.

Remark 2. N and p are the only non-random values here.

The maximum vertex degree of a graph G is denoted by $\Delta(G)$. We denote the set of hubs in G by $B(G)$, $|B(G)|$ by $b(G)$. A vertex of a graph is *universal*, if it is adjacent to all the other vertices in the graph. For a non-empty graph G, omitting the isolated points, we obtain graph G'. A vertex is *quasi-universal* in G if it is universal in G'. A graph is *primitive* if it consists of a clique and isolated points. A graph is (γ,ε)-*primitive* if the clique has γ vertices and the number of isolated points is ε. Empty graphs and cliques are primitive, by definition. For a graph G_t in the process, the probability of being (γ,ε)-primitive, will be denoted $\pi(t,\gamma,\varepsilon)$.

A cycle (path) on n vertices is denoted by C_n (P_n). We need a basic definition concerning the general model.

Definition 1. A graph is G_1-*relevant* if it appears as $\mathbf{G}_t$ somewhere during the Majority Process, beginning with G_1. G_1 is G_1-relevant, by definition. If $G_1 \cong K_1$, we often write *relevant* for short.

Example 1. C_4 and P_4 are not [K_1-]relevant.

3. Results and proofs for the general case

As we shall see, in the special case the number of hubs and the maximum degree are in thorough connection. (See Corollary 1, for example.) The following statement shows that for the general case, on the contrary, the pair (b, Δ) can be almost arbitrary.

Proposition 1. *Let $b_0 \geq 1$ and $\Delta_0 \geq 0$ be arbitrary integers, except the case "b_0 odd, $\Delta_0 = 1$". Then there exists some graph G_1 with $\Delta(G_1) = \Delta_0$ and $b(G_1) = b_0$.*

Proof. We begin by the case $\Delta_0 \geq 2$. Let H be any graph on $b_0 \geq 3$ vertices with all degrees at most Δ_0. (Such a graph exists, say, the cycle on b_0 vertices.) For a vertex x in H, we join $\Delta_0 - deg_H(x)$ pendant edges to x. For $b_0 = 2$, a P_4 fits.

For $\Delta_0 = 1$, the construction for an even b_0 is a set of independent edges and isolates, which always yields an even number of hubs. The case $\Delta_0 = 0$ is trivial. □

The following statement is valid for every graph G_1.

Proposition 2. *Suppose for a G_1-relevant graph G_j that $|B(G_j)| = \Delta(G_j) + 1$, in words, the number of hubs is exactly one greater than the maximum degree. Then, starting from this graph, there exists even an infinite sequence of increasing b and Δ, keeping their distance* 1.

Proof. We may connect the new vertex with every hub, in every step. □

Remark 3. Clearly, the probability of this phenomenon tends to zero.

We will use the statements below also in the next section.

Proposition 3. *Suppose v_j is universal in $\mathbf{G}_j$. Then $\mathbf{G}_{j-1}$ is a regular graph.*

Proof. In this case, by definition, each vertex in $\mathbf{G}_{j-1}$ is a hub. Thus, it is regular. □

We present here a surprising fact.

Proposition 4. *If a G_1-relevant graph is regular, then it is a clique or an empty graph.*

Proof. Let G be a graph in the statement of the Lemma and let G be r-regular. Let us omit the last vertex y of G. Suppose, by way of contradiction, that y does have both neighbors and non-neighbors. (This is equivalent for the whole graph to be neither a clique nor an empty graph.) Take a neighbor z of y. Clearly, $r \geq 1$ and the degree of z is $r-1$. The non-neighbors of y have degree r in $G-y$, thus z is not a hub in $G-y$. We have got a contradiction since a non-hub must be nonadjacent to y, by definition. □

4. Results and proofs for the special case

Let us give now some further definitions and notation. Let x be an arbitrary element of V and let $\Pi(k)$ be the probability that x has degree k in $\mathbf{G}_N$.

Remark 4. This probability depends on N, p and G_1 but they may be considered as constants.

Theorem 1. *For the probability of being universal in $\mathbf{G}_N$, the following result is valid.*

$$\Pi(N-1) = (p^{N-1}/N)\sum_{j=1}^{N}(p^{\lambda_j} + (1-p)^{\lambda_j})$$

where

$$\lambda_j = (j-1)(j-2)/2.$$

First we prepare the proof of Theorem 1 by six statements.

For a given element u of V, for any j between 1 and N, $\mathbb{P}(u = v_j) = 1/N$. Let us introduce now the following auxiliary variables:

$$a_j := \mathbb{P}(\mathbf{G}_{j-1} \text{ is a clique})$$
$$b_j := \mathbb{P}(v_j \text{ is universal in } \mathbf{G_j})$$
$$c_j := \mathbb{P}(\text{For every } l > j,\ v_l \text{ and } v_j \text{ are adjacent})$$
$$d_j := \mathbb{P}(\mathbf{G}_{j-1} \text{ is an empty graph})$$

The two variables below are of more importance:

$$q_j := \mathbb{P}(\mathbf{G}_{j-1} \text{ is a clique and } v_j \text{ is universal in } \mathbf{G_N})$$
$$r_j := \mathbb{P}(\mathbf{G}_{j-1} \text{ is an empty graph and } v_j \text{ is universal in } \mathbf{G_N})$$

Proposition 5.

$$a_j = p^{(j-1)(j-2)/2}$$

Proof.

$$\mathbb{P}(\mathbf{G}_{t+1} \text{ is a clique} \mid \mathbf{G}_t \text{ is a clique}) = p^t$$

for every $t < j$ since each vertex of $\mathbf{G}_t$ is a hub. Moreover, $\sum_{i=1}^{j-2} i = (j-1)(j-2)/2$. □

□

Proposition 6.

$$d_j = (1-p)^{(j-1)(j-2)/2}$$

Proof.

$$\mathbb{P}(\mathbf{G}_{t+1} \text{ is an empty graph} \mid \mathbf{G}_t \text{ is an empty graph}) = (1-p)^t$$

for every $t \leq j-2$ since each vertex of $\mathbf{G}_t$ is a hub. We continue similarly as in the proof of Proposition 5. □

Proposition 7.

$$b_j = p^{j-1}$$

Proof. Each vertex of $\mathbf{G}_{j-1}$ is a hub. □

□

Proposition 8.

$$c_j = p^{N-j}$$

Proof. For each $l > j$, v_j is a hub in $\mathbf{G}_{l-1}$, thus $\mathbb{P}(v_l$ and v_j are adjacent$)= p$. □

□

Proposition 9.

$$q_j = p^{(j-1)(j-2)/2+N-1}$$

Proof. Clearly, $q_j = a_j b_j c_j$. From Proposition 5 and Proposition 8, we obtain the result. □

□

Proposition 10.

$$r_j = (1-p)^{(j-1)(j-2)/2} p^{N-1}$$

Proof. Clearly, $r_j = d_j b_j c_j$. From *Proposition* 7 and *Proposition* 6, the statement follows. □

Proof of Theorem 1. Let us apply the statements above, to prove Theorem 1. We denote $\mathbb{P}(v_j$ is universal in $\mathbf{G}_N)$ by s_j. Clearly,

$$s_j = q_j + r_j$$

Finally, it can be easily seen that $\Pi(N-1) = 1/N \sum_{j=2}^{N} s_j$ The proof of Theorem 1 has been established. □

Theorem 2. *Suppose $G_1 \cong K_1$. Then in any non-empty relevant graph, the hubs are quasi-universal.*

Proof. We use induction on $|V(G)|$. The first step is obvious. By the induction hypothesis, the hubs are universal in G'_{j-1}. Let v_j be a hub in G_j. If v_j is isolated in G_j then we are done. Otherwise, taking a neighbor s of v_j, it is a hub of G_{j-1}, by definition. That is, s is universal in G'_{j-1}. The degree of v_j in G_j is at least the degree of s. Consequently, it is quasi-universal in G_j. □

Corollary 1. *For the special case, the number of hubs is at most one more than the maximum degree.*

(This is an example for the difference between special and general case.)
Before stating the theorem below, we give a definition.

Definition 2. If a graph consists of a clique and isolated points, we call it *primitive*.

Theorem 3. *Suppose $G_1 \cong K_1$. Then, in the Majority Process, two phases can be distinguished.*

Phase 1 Every graph in this first sequence is primitive. In some steps the clique is growing, in other steps new isolates appear.

Phase 2 In this phase, we find two types of steps.

Step type a) The set B of hubs (which is always a clique, by Theorem 2) is replaced by one of its proper subsets (but it remains non-empty of course).

Step type b) The set B does not change.

Proof. Suppose G_{j-1} is not primitive. We state that in this case, the step from $j-1$ to j is in *Phase 2* and from this subscript on, we stay in this phase. From the supposition, there exist vertices which are neither isolates nor quasi-universal. As we know from Theorem 2, even they are not hubs. Consequently, v_j is not adjacent to them and thus it is not universal.

If v_j is adjacent to each point of B_{j-1} then this step is of type b). Otherwise, it is of type a). We stay in this second phase since the new graph G_j is not primitive. □

Theorems 2 and 3 show that for the special case the structure of the graphs occurring in the process is too poor. However, the calculation of $\Pi(0)$ is even more difficult than that of $\Pi(N-1)$. Let

$$S_j := \mathbb{P}(v_j \text{ is isolated in } G_N)$$

Obviously

$$\Pi(0) = 1/N \sum_{j=1}^{N} S_j$$

Our aim is to determine S_j for all j's.

We begin by two cases, $j=1$ and $j=N$. The general solution will be a mixture of them.

Proposition 11.

$$S_1 = (1-p)^{\binom{N}{2}} + \sum_{\iota=2}^{N-1} (1-p)^{\binom{\iota}{2}+1} - \sum_{\iota=2}^{N-1} (1-p)^{\binom{\iota+1}{2}}$$

Proof. Let A_ι be the event that G_ι is an empty graph and $G_{\iota+1}$ is not. (Clearly, in this case all the graphs G_ν with $\nu \le \iota$ are empty graphs as well.)

Let F_ι be the product of the events A_ι and "v_1 is isolated in G_N". Clearly

Claim 1.

$$S_1 = (1-p)^{\binom{N}{2}} + \sum_{\iota=2}^{N-1} \mathbb{P}(F_\iota)$$

Proof. The events F_ι and the event "G_N is an empty graph" are pairwise exclusive. □

Take G_ι and the neighborhood of $v_{\iota+1}$ as random variables. By the definition of the process, they are independent. The neighborhood does not contain v_1 but it contains at least one vertex v_μ with $2 \le \mu \le \iota$. This implies

$$\mathbb{P}(F_\iota) = (1-p)^{\binom{\iota}{2}}((1-p)-(1-p)^\iota)$$

and, using $\binom{\iota}{2}+\iota = \binom{\iota+1}{2}$,

$$S_1 = (1-p)^{\binom{N}{2}} + \sum_{\iota=2}^{N-1}(1-p)^{\binom{\iota}{2}+1} - \sum_{\iota=2}^{N-1}(1-p)^{\binom{\iota+1}{2}}$$

The calculation of S_1 has been achieved. □

We need

$$B_i := \{x \in V(G_i) \mid x \text{ hub in } G_i\},$$

to define a quantity helping in the calculation of the S_j's.

$$\kappa(i,h) := \mathbb{P}(|B_i| = h)$$

Remark 5. By Theorem 2, instead of 'hub', we could write 'quasi-universal vertex'.

First, we can easily show the fact

Proposition 12.

$$S_N = \sum_{z=1}^{N-1} \kappa(N-1,z)(1-p)^z$$

Proof.

$$S_N = \sum_{z=1}^{N-1} \mathbb{P}(v_N \text{ is isolated in } G_N \text{ and } |B_{N-1}| = z)$$

□

We need further notation.

$$\theta(\iota, j) := \mathbb{P}(v_j \text{ is isolated in } G_N \text{ and } A_\iota \text{ occurs})$$

Now we are in the position to state the theorem determining S_j for arbitrary j. (S_0 and S_N are determined already.)

Theorem 4. *For* $3 \le j \le N-1$, $S_j = T_j + U_j + (1-p)^{\binom{N}{2}}$ *where*

$$T_j = \sum_{z=1}^{j-1} \kappa(j-1,z)(1-p)^z$$

and

$$U_j = \sum_{\iota=j}^{N-1} \theta(\iota, j)$$

Moreover, for $j = 2$, $T_2 = 0$ but the equaity for U_2 remains valid.

Proof. For the subscript of the effectively occurring event A_ι, there are three cases. $\iota \le j-1$, $j \le \iota \le N-1$ or $\iota = N$. According to that, S_j is the sum of three probabilities. □

Claim 2. For $\iota \le j-2$, if v_j is isolated in G_j then it is isolated in G_N as well.

Proof. G_j is not an empty graph, thus, having degree zero, v_j is not a hub in any further graph and consequently it will not get any new neighbor. □

Proposition 13. *Let $j \le \iota \le N-1$. Then*

$$\theta(\iota, j) = (1-p)^{\binom{\iota}{2}}((1-p) - (1-p)^\iota))$$

Proof. The new vertex is not adjacent to v_j but it has to be adjacent to some other vertex. □

We will present a recursion formula for κ.

Let us add one vertex to a graph in the process. The number of hubs may increase, namely, when the original graph is a specific primitive one. This is the reason that the recursion formula for κ is in connection also with primitive graphs.

Proposition 14.

$$\kappa(i,h) = \mathbb{P}(G_{i-1} \textit{ is } (h-1, t-h)\textit{-primitive})p^{h-1} + \sum_{z=h}^{N} \kappa(i-1, z)\binom{z}{h} p^h (1-p)^{z-h}$$

To obtain the probability of being primitive, a recursion formula is needed. This can be obtained by finding a recursion for the probability of being (γ, ε)- primitive. Let the underlying graph have t vertices. Thus $\gamma + \varepsilon = t$.

For $\gamma = 0$ we have an empty graph and for $\varepsilon = 0$ we have a clique. The calculation of the probabilities in these cases is straightforward. We consider them as starting values. In most of the cases, the following recursion can be used.

Proposition 15. *For $\gamma \ge 3$ and $\varepsilon \ge 1$*

$$\begin{aligned}\mathbb{P}(G_t \textit{ is } (\gamma,\varepsilon)\textit{-primitive}) = {} & \mathbb{P}(G_{t-1} \textit{ is } (\gamma-1,\varepsilon)\textit{-primitive})p^{\gamma-1} \\ & + \mathbb{P}(G_{t-1} \textit{ is } (\gamma,\varepsilon-1)\textit{-primitive})(1-p)^\gamma\end{aligned}$$

Proof. We can obtain the large graph in two ways: by increasing the clique or the empty part. The two addends here represent exclusive events. □

Instead of $(1,\varepsilon)$-primitiveness, $(0,\varepsilon+1)$-primitiveness will be used. But this means that the recursion formula for $\gamma = 2$ must have an exceptional form:

Proposition 16.

$$\mathbb{P}(G_t \textit{ is } (2,\varepsilon)\textit{-primitive}) = (t-1)\,\mathbb{P}(G_{t-1} \textit{ is } (0,t-1)\textit{-primitive})p(1-p)^{t-2} + \mathbb{P}(G_{t-1} \textit{ is } (2,t-3)\textit{-primitive})(1-p)^2$$

Proof. In the first graph at the right hand side, we have to choose exactly one neighbor for v_t. □

Summarising, we expressed the S_j's by two quantities, θ and κ. An explicit formula for θ has been found, while for κ a recursion only, moreover, we needed another auxiliary parameter, not explicitly known but using a recursion again.

5. Concluding Remarks

In the subject, the most frequent question is the degree distribution of the network. For general degrees, to obtain exact formulae like here for $\Pi(0)$ and $\Pi(N-1)$ is not hopeful. Inspite of that, we ask:

Open problem How does the degree distribution depend on the initial graph G_1?

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QUALITATIVE STUDY FOR IMPULSIVE PANTOGRAPH FRACTIONAL INTEGRO-DIFFERENTIAL EQUATION VIA ψ-HILFER DERIVATIVE

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Abstract. In this paper, we study the existence and stability of solutions for impulsive pantograph fractional integro-differential equation via ψ-Hilfer fractional derivative in a appropriate Banach space. Our approach is based on fixed point theorems of Darbo's and Mönch via Kuratowski measure of non-compactness. An example is given to illustrate our approach.

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Keywords: measure of non-compactness, fixed point theorem, ψ-Hilfer fractional derivative

1. Introduction

Fractional differential equations are one of the most successful and interesting branches of mathematics. Its results can be used to prove some important properties in many fields of science and engineering [18, 20, 22, 24].

Many effective theoretical studies published by several researchers which reside on the result of existence, uniqueness and the stability for differential equations involving a fractional derivative with various conditions, see [10, 11]. The class of impulsive fractional differential equations is distinguished from others by the modeling of phenomena which undergo distortions, in particular in the field of medicine and physics, see for example [6].

In the book [13], Fečkan et al. found a revised formula for the solutions of an impulsive differential equation involving the Caputo derivative. In the references [4, 5, 8, 12], the authors are interested in the study of impulsive differential equations involving the derivative of Riemann or that of Hilfer. One of the properties of solutions is the stability in the sense of Hyers which is introduced in [16] by the study of a question posed by Ulam.

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Recently, the stability in the sense of Ulam-Hyers and Ulam-Hyers-Rassias have been studied by many researchers for certain differential problems considered, see [1,2,23].

In view of the above considerations, we consider the following impulsive pantograph fractional integro-differential equation

$$(\mathfrak{P}) \quad \begin{cases} {}^{H}\mathcal{D}^{\alpha,\beta,\psi}_{t_k^+} y(t) = f(t, y(t), \mathfrak{I}^{\delta,\psi}_{t_k^+} y(\sigma_k(t))), \quad t \in (t_k, t_{k+1}],\ k = 0, \dots, m, \\ \Delta_{\gamma,\psi} y|_{t_k} = J_k(y(t_k^-)),\ k = 0, \dots, m, \\ \mathfrak{I}^{1-\gamma,\psi}_{c^+} y(c^+) = M\Gamma(\gamma), \end{cases}$$

where ${}^{H}\mathcal{D}^{\alpha,\beta,\psi}_{t_k^+}$ denote the left-sided $\psi-$Hilfer fractional derivative of order $0 < \alpha < 1$ and of type β, $0 \le \beta \le 1$, $\gamma = \alpha + \beta(1-\alpha)$, $\delta > 1-\gamma$. The operator $\mathfrak{I}^{1-\gamma,\psi}_{t_k^+}$ denotes the left-sided $\psi-$Riemann-Liouville fractional integral of order $\varsigma \in \{\delta, 1-\gamma\}$, $f : (c, L] \times E^2 \to E$ a function satisfying some specified conditions, t_k, $k = 0, \dots, m$ are pre-fixed points satisfying $t_0 = c < t_1 \le \cdots \le t_m < t_{m+1} = L$ and, $J_k : E \to E$, $\Delta_{\gamma,\psi} y|_{t_k} = \frac{\mathfrak{I}^{1-\gamma,\psi} y(t_k^+)}{\Gamma(\gamma)} - y(t_k^-)$, where $y(t_k^-) = \lim_{t \longrightarrow t_k^-} y(t)$, $k = 1, \dots, m$, $M \in E$, $\sigma_k(t) = t_k + \sigma(t - t_k), k = 0, \dots, m$ with $0 < \sigma \le 1$ and $\psi \in \mathcal{C}^1([c, L], \mathbb{R}^+)$ satisfies $\psi'(t) > 0$, for all $t \in [c, L]$. The $\psi-$Hilfer fractional derivative is in a general form and in particular cases, it covers special cases. Pantograph equations arise in electrodynamics [21]. Our conditions are also related of other systems mentioned above, and in particular to the following recent papers. Initial value problems for two different classes of implicit ϕ-Hilfer fractional pantograph differential equations are considered in [3]. ϕ-Caputo differential inclusion boundary value problems are studied in [7] supplemented with mixed integro-derivative conditions in the frame of the ϕ-Riemann-Liouville operators. Nonlinear impulsive pantograph fractional BVPs under Caputo proportional fractional derivative are investigated in [17]. The significance of our impulsive and initial conditions in $(\mathfrak{P})$ compare with the above results relies on the fact that they are nonlocal.

This paper is organized in the following way. In Section 2 we give some general results and preliminaries, in Section 3, we show the existence results for the problem $(\mathfrak{P})$ based on fixed point theorems of Darbo's and Mönch and in Section 4, we present a result about the stability in the sense of Ulam-Hyers-Rassias of Problem $(\mathfrak{P})$. Finally an illustrative example will be presented in Section 5.

2. Preliminary results

In this section, we introduce some notation and technical results which are used throughout this paper. Let $I = [a, b]$, $b > a$ and $(E, \|\cdot\|)$ be a Banach space. $\mathcal{C}(I, E)$ be the space of continuous functions on I with the norm

$$\|u\|_\infty = \sup\{\|u(t)\|,\ t \in I\}.$$

$L^1(I,E)$ is the space of E-valued Bochner integrable functions on I with the norm

$$\|h\|_{L^1} = \int_a^b \|h(t)\|dt.$$

For all $\eta > -1$ and $s,r \in [0,L]$ with $r > s$, we pose $\mathrm{N}_{\eta,\psi}(r,s) = (\psi(r)-\psi(s))^{\eta}$. We consider the Banach spaces of functions

$$\mathcal{C}_{1-\gamma,\psi}([a,b]) = \{y \in \mathcal{C}((a,b],E) : \lim_{t\to a^+} \mathrm{N}_{1-\gamma,\psi}(t,a)y(t)) \text{ exists and finite}\}.$$

A norm in this space is given by

$$\|y\|_{\gamma,\psi} = \sup_{t\in[a,b]} \mathrm{N}_{1-\gamma,\psi}(t,a)\|y(t)\|,$$

and

$$\begin{aligned}\mathbb{PC}_{1-\gamma,\psi}([c,L]) \quad &= \Big\{y : (c,L] \to E : \ y_k \in \mathcal{C}_{1-\gamma,\psi}([t_k,t_{k+1}],E) \\ &\quad \text{with } y(t_k) = y(t_k^-), \text{ for all } k = 0,\dots,m, \Big\},\end{aligned}$$

with the norm

$$\|y\|_{\mathbb{PC}_{\gamma,\psi}} = \max_{k=0,\dots,m.} \|y\|_{\gamma,\psi},$$

where y_k is the restriction of y to $(t_k,t_{k+1}]$, $k = 0,\dots,m$. Let us now give the definition of the measure of non-compactness in the sense of Kuratowski and its properties. For all $G \subseteq E$, we denote by $S_b(G)$ the set of all bounded subsets of G.

Definition 1 ([9,15])**.** Let $D \in S_b(E)$. The Kuratowski measure of non-compactness ϑ of the subset D is defined as follows:

$$\vartheta(D) = \inf\{e > 0 : \Omega \text{ admits a finite cover by sets of diameter} \leq e\}.$$

Lemma 1 ([9,15])**.** *Let $A,B \in S_b(E)$. The following properties hold:*

(i_1) $\vartheta(A) = 0$ *if and only if A is relatively compact,*
(i_2) $\vartheta(A) = \vartheta(\overline{A})$, *where $\overline{A}$ denotes the closure of A,*
(i_3) $\vartheta(A+B) \leq \vartheta(A) + \vartheta(B)$,
(i_4) $A \subset B$ *implies* $\vartheta(A) \leq \vartheta(B)$,
(i_5) $\vartheta(a.A) = |a|.\vartheta(A)$ *for all* $a \in \mathbb{R}$,
(i_6) $\vartheta(\{a\} \cup A) = \vartheta(A)$ *for all* $a \in E$,
(i_7) $\vartheta(A) = \vartheta(Conv(A))$, *where $Conv(A)$ is the smallest convex that contains A.*

Lemma 2 ([15])**.** *If D is a equicontinuous and bounded subset of $\mathcal{C}([a,b],E)$, then* $\vartheta(D(.)) \in \mathcal{C}([a,b],\mathbb{R}^+)$

$$\vartheta_{\mathcal{C}}(D) = \max_{t\in[a,b]} \vartheta(D(t)), \ \vartheta\left(\left\{\int_a^b w(t)dt : w \in D\right\}\right) \leq \int_a^b \vartheta(D(t))dt,$$

where $D(t) = \{w(t) : w \in D\}$ and $\vartheta_{\mathcal{C}}$ is the non-compactness measure on the space $\mathcal{C}([a,b],E)$.

We denote by $\vartheta^k_{\gamma,\psi}$ and $\vartheta_{\gamma,\psi}$ the Kuratowski measures of non-compactness defined respectively on $\mathcal{C}_{1-\gamma,\psi}([t_k,t_{k+1}])$, $k=0,\dots,m$ and $\mathbb{PC}_{1-\gamma,\psi}([c,L])$.

Lemma 3 ([15]). *For all bounded subset D of $\mathbb{PC}_{1-\gamma,\psi}([c,L])$, we have*

$$\vartheta_{\gamma,\psi}(D)=\max_{k=0,\dots,m}\vartheta^k_{\gamma,\psi}(D_k),$$

where D_k is the restriction of D on $(t_k,t_{k+1}]$.

Theorem 1 ([15]). *Let ρ the Kuratowski measure of non-compactness on Banach space E and G a closed, bounded and convex subset of E which contains the 0. Let Δ be an operator from G to G, assume that Δ is continuous and satisfied, for every subset V of G, we have the following implication:*

$$V=\Delta(V)\cup\{0\}\ \text{ or }\ V=\overline{conv}\,\Delta(V)\ \implies\ \rho(V)=0.$$

Then the set $\{w\in G:\Delta(w)=w\}$ is nonempty.

Theorem 2 ([14]). *Let ρ the Kuratowski measure of non-compactness on Banach space E, G a nonempty, closed, bounded and convex subset of E and Δ be an continuous operator from G to G such that, for all nonempty subset V of G:*

$$\rho(\Delta(V))\le\varsigma\rho(V),$$

where $0\le\varsigma<1$. Then Δ has a fixed point in G.

We begin with some definitions from the theory of fractional calculus.

Definition 2 ([18,25]). Let ℓ be an integrable function defined on $(a,b]$,

(i) the ψ-Riemann- Liouville fractional integral of order $\alpha>0$ of the function ℓ is defined by

$$\mathfrak{I}^{\alpha,\psi}_{a^+}\ell(t)=\frac{1}{\Gamma(\alpha)}\int_a^t\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t,s)\ell(s)ds,$$

(ii) the ψ-Riemann- Liouville fractional derivative of order $\alpha>0$ of the function ℓ is defined by

$${}^{RL}\mathcal{D}^{\alpha,\psi}_{a^+}h(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n\left(\int_a^t\psi'(s)\mathrm{N}_{n-\alpha-1,\psi}(t,s)\ell(s)ds\right),$$

where Γ is the gamma function and $n=[\alpha]+1$ ($[\alpha]$ represents the integer part of the real number α).

Definition 3 ([18,25]). Let $\psi\in\mathcal{C}^1([a,b],E)$ a functions such that $\psi'(t)>0$, for all $t\in[a,b]$. The ψ-Hilfer fractional derivative of a function ℓ of order $0<\alpha<1$ and type $0\le\beta\le1$ is given by

$${}^{H}\mathcal{D}^{\alpha,\beta,\psi}_{a^+}\ell(t)=\mathfrak{I}^{\beta(1-\alpha),\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)\mathfrak{I}^{(1-\beta)(1-\alpha),\psi}\ell(t)=\mathfrak{I}^{1-\gamma,\psi}{}^{RL}D^{\gamma,\psi}\ell(t),$$

where $\gamma=\alpha+\beta(1-\alpha)$.

Lemma 4 ([18]). *Let $\alpha,\rho \in \mathbb{R}_+^*$ and $t > a$. We have then*

(i_1) $\mathfrak{I}_{a^+}^{\alpha,\psi}\mathrm{N}_{\rho-1,\psi}(t,a) = \frac{\Gamma(\rho)}{\Gamma(\alpha+\rho)}\mathrm{N}_{\alpha+\rho-1,\psi}(t,a)$.

(i_2) ${}^{H}\mathcal{D}_{a^+}^{\alpha,\rho,\psi}\mathrm{N}_{\rho-1,\psi}(t,a) = \frac{\Gamma(\rho)}{\Gamma(\rho-\alpha)}\mathrm{N}_{\rho-\alpha-1,\psi}(t,a), 0<\alpha<1,\ \rho>1$.

We consider the spaces

$$\mathcal{C}_{1-\gamma,\psi}^{\gamma}([a,b]) = \Big\{u \in \mathcal{C}_{1-\gamma,\psi}([a,b]),\ {}^{RL}\mathcal{D}_{a^+}^{\gamma}u \in \mathcal{C}_{1-\gamma,\psi}([a,b])\Big\},$$

$$\mathbb{PC}_{1-\gamma,\psi}^{\gamma}([c,L]) = \Big\{u \in \mathbb{PC}_{1-\gamma,\psi}([c,L]) :\ {}^{RL}\mathcal{D}_{t_k^+}^{\gamma,\psi}u_k \in \mathcal{C}_{1-\gamma}([t_k,t_{k+1}]),\ k=0,\dots,m\Big\},$$

and

$$\mathbb{PC}_{1-\gamma,\psi}^{\alpha,\beta}([c,L]) = \Big\{u \in \mathbb{PC}_{1-\gamma,\psi}([c,L]) :\ {}^{H}\mathcal{D}_{t_k^+}^{\alpha,\beta,\psi}u_k \in \mathcal{C}_{1-\gamma}([t_k,t_{k+1}]),\ k=0,\dots,m\Big\}.$$

Lemma 5 ([19]). *Let $0<\alpha<1,\ 0\le\beta\le 1$ and $\gamma=\alpha+\beta-\alpha\beta$. If $f \in \mathcal{C}_{1-\gamma}^{\gamma}([a,b])$, then*

$$\mathfrak{I}_{a^+}^{\gamma,\psi}\mathcal{D}_{a^+}^{\gamma,\psi}f = \mathfrak{I}_{a^+}^{\alpha,\psi}\mathcal{D}_{a^+}^{\alpha,\beta,\psi}f \text{ and } \mathcal{D}_{a^+}^{\gamma,\psi}\mathfrak{I}_{a^+}^{\alpha,\psi}f = \mathcal{D}_{a^+}^{\beta(1-\alpha)}f.$$

Lemma 6 ([19]). *Let $\omega : (a,b] \to E$ be a function such that $\omega(.) \in \mathcal{C}_{1-\gamma,\psi}([a,b])$. Then, a function $y \in \mathcal{C}_{1-\gamma,\psi}^{\gamma}([a,b])$ is a solution of linear fractional differential problem:*

$$\begin{cases} {}^{H}\mathcal{D}_{a^+}^{\alpha,\beta,\psi}y(t) = \omega(t), & 0<\alpha<1,\ 0\le\beta\le 1; \\ \mathfrak{I}_{a^+}^{1-\gamma,\psi}y(a^+) = \omega_0, & \gamma=\alpha+\beta-\alpha\beta. \end{cases}$$

if and only if y satisfies the following integral equation:

$$y(t) = \frac{\omega_0\mathrm{N}_{\gamma-1,\psi}(t,a)}{\Gamma(\gamma)} + \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)\mathrm{N}_{\alpha-1,\psi}(t,s)\omega(s)ds.$$

For any $k \in \{1,\dots,m\}$, we define the constants $\Xi_{j=1}^{k-i}N_{\gamma-1,\psi}(t_{k-j+1},t_{k-j})$, $i=1,\dots,k$ by

$$\Xi_{j=1}^{k-i}N_{\gamma-1,\psi}(t_{k-j+1},t_{k-j}) = \begin{cases} 1, & i=k; \\ \prod_{j=1}^{k-i}\mathrm{N}_{\gamma-1,\psi}(t_{k-j+1},t_{k-j}), & i=1,\dots,k-1, \end{cases}$$

we also put

$$T^* = \frac{\Gamma(\gamma)}{\Gamma(\delta+\gamma)}\max\Big\{\sup_{t\in(t_k,t_{k+1}]}\Big(\frac{\mathrm{N}_{\gamma-1,\psi}(t,t_k)}{\mathrm{N}_{\gamma-1,\psi}(\sigma_k(t),t_k)}\Big), k=0,\dots,m\Big\} \text{ and}$$

$$T_\eta = \max\Big\{1, \mathrm{N}_{\eta,\psi}(t_{k+1},t_k), k=k=0,\dots,m\Big\},\ \eta>-1.$$

Lemma 7. *Let $f : (c,L]\times E^2 \to E$ be a function such that $f(.,y(.),\mathfrak{I}_{t_k}^{\delta,\psi}y(\sigma_k(.))) \in \mathcal{C}_{1-\gamma,\psi}([t_k,t_{k+1}])$ $k=0,\dots,m$, for all $y \in \mathbb{PC}_{1-\gamma,\psi}([c,L])$. If $y \in \mathbb{PC}_{1-\gamma,\psi}^{\gamma}([c,L])$. Then,*

y is a solution of Problem $(\mathfrak{P})$ if and only if y satisfies the following integral equation:

$$y(t)=\begin{cases} M\mathrm{N}_{\gamma-1,\psi}(t,c)+\frac{1}{\Gamma(\alpha)}\int_c^t \psi'(s)\mathrm{N}_{\gamma-1,\psi}(t,s)f\big(s,y(s),\mathfrak{I}_{t_k}^{\delta,\psi}y(\sigma_k(s))\big)ds, & \text{if } t\in I_0,\\ \mathrm{N}_{\gamma-1,\psi}(t,t_k)\Big[M\prod_{i=1}^k \mathrm{N}_{\gamma-1,\psi}(t_i,t_{i-1}) \\ +\sum_{i=1}^k \Xi_{j=1}^{k-i}N_{\gamma-1,\psi}(t_{k-j+1},t_{k-j})J_i(y(t_i^-)) \\ +\sum_{i=1}^k \Xi_{j=1}^{k-i}N_{\gamma-1,\psi}(t_{k-j+1},t_{k-j})\mathfrak{I}_{t_{i-1}^+}^{\alpha,\psi}f\big(t_i,y(t_i),\mathfrak{I}_{t_{i-1}}^{\delta,\psi}y(\sigma_{t_{i-1}}(t_i))\big)\Big] \\ +\frac{1}{\Gamma(\alpha)}\int_{t_k}^t \psi'(s)\mathrm{N}_{\alpha-1,\psi}(t,s)f\big(s,y(s),\mathfrak{I}_{t_k}^{\delta,\psi}y(\sigma_k(s))\big)ds, & \text{if } t\in I_k,\end{cases} \tag{2.1}$$

where $I_k=(t_k,t_{k+1}], k=1,\dots,m$.

Proof. First, we prove the necessity. Let $y\in\mathbb{PC}^{\gamma}_{1-\gamma,\psi}([c,L])$ be a solution of $(\mathfrak{P})$. If $t\in(a,t_1]$, we have ${}^H\mathcal{D}_{t_0}^{\alpha,\beta,\psi}y(t)=f\big(t,y(t),\mathfrak{I}_{t_k}^{\delta,\psi}y(\sigma_k(t))\big)$, from Lemma 6, we get

$$y(t)=M\mathrm{N}_{\gamma-1,\psi}(t,c)+\frac{1}{\Gamma(\alpha)}\int_c^t \psi'(s)\mathrm{N}_{\alpha-1,\psi}(t,s)f\big(s,y(s),\mathfrak{I}_{t_k}^{\delta,\psi}y(\sigma_k(s))\big)ds.$$

If $t\in(t_1,t_2]$, then, from Lemma 6, we get

$$y(t)=\frac{\mathfrak{I}_{t_1^+}^{1-\gamma,\psi}y(t_1^+)}{\Gamma(\gamma)}\mathrm{N}_{\gamma-1,\psi}(t,t_1)+\frac{1}{\Gamma(\alpha)}\int_{t_1}^t \psi'(s)\mathrm{N}_{\alpha-1,\psi}(t,s)f\big(s,y(s),\mathfrak{I}_{t_1}^{\delta,\psi}y(\sigma_1(s))\big)ds.$$

By using the condition $\frac{\mathfrak{I}_{t_1^+}^{1-\gamma,\psi}y(t_1^+)}{\Gamma(\gamma)}=y(t_1^-)+J_1(y(t_1^-))$, we obtain

$$\begin{aligned} y(t)&=\big(y(t_1)+J_1(y(t_1^-))\big)\mathrm{N}_{\gamma-1,\psi}(t,t_1)\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^t \psi'(s)\mathrm{N}_{\alpha-1,\psi}(t,s)f(s,y(s),\mathfrak{I}_{t_1}^{\delta,\psi}y(\sigma_1(s)))ds\\ &=\Big[J_1(y(t_1^-))+M\mathrm{N}_{\gamma-1,\psi}(t_1,a)+\mathfrak{I}_{c^+}^{\alpha,\psi}f\big(t_1^-,y(t_1^-),\mathfrak{I}_{t_0}^{\delta,\psi}y(\sigma_1(t_1))\big)\Big]\mathrm{N}_{\gamma-1,\psi}(t,t_1)\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^t \psi'(s)\mathrm{N}_{\alpha-1,\psi}(t,s)f\big(s,y(s),\mathfrak{I}_{t_1}^{\delta,\psi}y(\sigma_1(s))\big)ds. \end{aligned}$$

If $t\in(t_2,t_3]$, by utilizing Lemma 6 and the condition $\frac{\mathfrak{I}_{t_2^+}^{1-\gamma,\psi}y(t_2^+)}{\Gamma(\gamma)}=y(t_2^-)+J_2(y(t_2^-))$, we arrive

$$\begin{aligned} y(t)&=\frac{\mathfrak{I}_{t_2^+}^{1-\gamma,\psi}y(t_2^+)}{\Gamma(\gamma)}\mathrm{N}_{\gamma-1,\psi}(t,t_2)\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_{t_2}^t \psi'(s)\mathrm{N}_{\alpha-1,\psi}(t,s)f\big(s,y(s),\mathfrak{I}_{t_2}^{\delta,\psi}y(\sigma_2(s))\big)ds\\ &=\big(J_2(y(t_2^-))+y(t_2^-)\big)\mathrm{N}_{\gamma-1,\psi}(t,t_2) \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\int_{t_2}^{t}\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t,s)f\big(s,y(s),\mathfrak{I}_{t_2}^{\delta,\psi}y(\sigma_2(s))\big)ds\\
&=\mathrm{N}_{\gamma-1,\psi}(t,t_2)\Big[M\mathrm{N}_{\gamma-1,\psi}(t_1,t_0)\mathrm{N}_{\gamma-1,\psi}(t_2,t_1)+\mathrm{N}_{\gamma-1,\psi}(t_2,t_1)J_1(y(t_1^-))+J_2(y(t_2^-))\\
&\quad+\mathrm{N}_{\gamma-1,\psi}(t_2,t_1)\mathfrak{I}_{c^+}^{\alpha}f\big(t_1^-,y(t_1^-),\mathfrak{I}_{t_0}^{\delta,\psi}y(\sigma_0(t_1))\big)+\mathfrak{I}_{t_1}^{\alpha,\psi}f\big(t_2^-,y(t_2^-),\mathfrak{I}_{t_1}^{\delta,\psi}y(\sigma_1(t_2))\big)\Big]\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_2}^{t}\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t,s)f\big(s,y(s),\mathfrak{I}_{t_2}^{\delta,\psi}y(\sigma_2(s))\big)ds.
\end{aligned}
$$

If $t\in(t_k,t_{k+1}]$, we continue the procedure and again using Lemmas 6 and the condition $\frac{\mathfrak{I}_{t_k^+}^{1-\gamma,\psi}y(t_k^+)}{\Gamma(\gamma)}=y(t_k^-)+J_k(y(t_k^-))$, we find

$$
\begin{aligned}
y(t)=\mathrm{N}_{\gamma-1,\psi}(t,t_k)&\Bigg[M\prod_{i=1}^{k}\mathrm{N}_{\gamma-1,\psi}(t_i,t_{i-1})+\sum_{i=1}^{k}\Xi_{j=1}^{k-i}N_{\gamma-1,\psi}(t_{k-j+1},t_{k-j})J_i(y(t_i^-))\\
&+\sum_{i=1}^{k}\Xi_{j=1}^{k-i}N_{\gamma-1,\psi}(t_{k-j+1},t_{k-j})\mathfrak{I}_{t_{i-1}^+}^{\alpha,\psi}f\big(t_i,y(t_i),\mathfrak{I}_{t_{i-1}}^{\delta,\psi}y(\sigma_{t_{i-1}}(t_i))\big)\Bigg]\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t,s)f\big(s,y(s),\mathfrak{I}_{t_k}^{\delta,\psi}y(\sigma_k(s))\big)ds.
\end{aligned}
$$

Conversely, assume that y satisfies the impulsive equation (2.1). If $t\in(t_0,t_1]$, by using 6, we get

$$
\mathfrak{I}_{c^+}^{1-\gamma,\psi}y(t)=M \text{ and } {}^{H}\mathcal{D}_{t_0}^{\alpha,\beta,\psi}y(t)=f\big(t,y(t),\mathfrak{I}_{t_0}^{\delta,\psi}y(\sigma_{t_0}(t))\big),\quad \text{for each } t\in(t_0,t_1].
$$

By recurrence, if $t\in(t_k,t_{k+1}]$, $k=1,\dots,m$ and according to Lemma 6, we get

$$
{}^{H}\mathcal{D}_{t_k}^{\alpha,\beta,\psi}y(t)=f\big(t,y(t),\mathfrak{I}_{t_k}^{\delta,\psi}y(\sigma_{t_k}(t))\big),\quad \text{for each } t\in(t_k,t_{k+1}].
$$

And, we can easily show that

$$
\Delta_{\alpha,\psi}y|_{t=t_k}=J_k(y(t_k^+)).
$$

□

3. Existence of the solution

Suppose that the function $f:(c,L]\times E^2\to E$ verifies

$$
f(.,u(.),v(.))\in\mathbb{PC}_{1-\gamma,\psi}^{\gamma}([c,L]),
$$

for all $u(.),v(.)\in\mathbb{PC}_{1-\gamma}([c,L])$, $f(.,0,0)\in\mathcal{C}([c,L],E)$ and there exists $A,B\in\mathbb{R}^+$ and $\lambda\geq 1-\alpha$ such that

($\mathbf{H_1}$) For any $u,v,\overline{u},\overline{v}\in E$ and for all $t\in I_k$, $k=1,\dots,m$:

$$
\|f(t,u,v)-f(t,\overline{u},\overline{v})\|\leq A\mathrm{N}_{\lambda,\psi}(t,t_k)\|u-\overline{u}\|+B\|v-\overline{v}\|.
$$

($\mathbf{H_2}$) For each nonempty, bounded set $\Omega \subset \mathbb{PC}_{1-\gamma,\psi}([c,L])$, for all $t \in I_k, k = 0,\dots,m$, we have

$$\vartheta(f(t,\Omega(t),\mathfrak{I}_{t_k}^{\delta,\psi}\Omega(\sigma_k(t))) \leq A\mathrm{N}_{\lambda,\psi}(t,t_k)\vartheta(\Omega(t))) + B\vartheta(\mathfrak{I}_{t_k}^{\delta,\psi}\Omega(\sigma_k(t))),$$

where $\mathfrak{I}_{t_k}^{\delta,\psi}\Omega(\sigma_k(t)) = \{\mathfrak{I}_{t_k}^{\delta,\psi}y(\sigma_k(t)), y \in \mathbb{PC}_{1-\gamma,\psi}([c,L])\},\ k = 0,\dots,m.$

Suppose that the functions $J_k : E \to E,\ k = 1,\dots,m$, are continuous and there exists $C \in \mathbb{R}^+$ such that

($\mathbf{H_3}$) For any $u \in E$:

$$\|J_k(u)\| \leq C\|u\|,\ k = 1,\dots,m.$$

($\mathbf{H_4}$) For each nonempty, bounded set $\Omega \subset \mathbb{PC}_{1-\gamma,\psi}([c,L])$, we have

$$\vartheta(J_k(\Omega(t))) \leq C\vartheta(\Omega(t)),\ k = 0,\dots,m.$$

($\mathbf{H_5}$) $\overline{T}_{\gamma-1}^{m} T_\alpha \Big(mC\Gamma(\alpha+1) + (m+1)(AT_\lambda + BT^* T_\delta) \Big) < \Gamma(\alpha+1)$,
where $\overline{T}_{\gamma-1} = \max\left\{T_{1-\gamma}, T_{\gamma-1}\right\}$.

Our first result concerning the existence of solutions of the problem ($\mathfrak{P}$) for which we have used the fixed point theorem of Mönch's is as follows:

Theorem 3. *We assume that the hypotheses from* ($\mathbf{H_1}$) *to* ($\mathbf{H_5}$) *are satisfied, then problem* ($\mathfrak{P}$) *has at least one solution in* $\mathbb{PC}_{1-\gamma,\psi}^{\gamma}([c,L])$.

Proof. Consider the operator $\Lambda : \mathbb{PC}_{1-\gamma,\psi}([c,L]) \to \mathbb{PC}_{1-\gamma,\psi}([c,L])$ defined by

$$\begin{aligned}\Lambda y(t) = \mathrm{N}_{\gamma-1,\psi}(t,t_k) \Bigg[& M\prod_{i=1}^{k}\mathrm{N}_{\gamma-1,\psi}(t_i,t_{i-1}) + \sum_{i=1}^{k}\Xi_{j=1}^{k-i}N_{\gamma-1,\psi}(t_{k-j+1},t_{k-j})J_i(y(t_i^-)) \\ & + \sum_{i=1}^{k}\Xi_{j=1}^{k-i}N_{\gamma-1,\psi}(t_{k-j+1},t_{k-j})\mathfrak{I}_{t_{i-1}^+}^{\alpha,\psi} f\big(t_i, y(t_i), \mathfrak{I}_{t_{i-1}}^{\delta,\psi}y(\sigma_{t_{i-1}}(t_i))\big)\Bigg] \\ & + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t,s) f\big(s, y(s), \mathfrak{I}_{t_k}^{\delta,\psi}y(\sigma_k(s))\big)ds,\end{aligned}$$

for any $t \in I_k, k = 1,\dots,m$. From the definition of the operator Λ and Lemma 7, we see that the fixed points of Λ are solutions of problem ($\mathfrak{P}$). For this reason, it suffices to verify the axioms of Theorem 1, it is done in four steps.

First step. We start to prove that Λ is continuous. Let $\varepsilon > 0$ and $\{y_n\}_{n\in\mathbb{N}} \to y$ in $\mathbb{PC}_{1-\gamma,\psi}([c,L])$. The hypothesis ($\mathbf{H_1}$) and ($\mathbf{H_3}$) confirm the existence of an integer $n_1 \in \mathbb{N}$ such that, for all $n \geq n_1$ and $t \in I_k$, $k = 0,\dots,m$, we have

$$\|f\big(t, y_n(t), \mathfrak{I}_{t_k}^{\delta,\psi}y_n(\sigma_k(t))\big) - f(t, y(t), \mathfrak{I}_{t_k}^{\delta,\psi}y(\sigma_k(t)))\big)\| < \frac{\Gamma(\alpha+1)\varepsilon}{(2m+1)(AT_\lambda + BT^*T_\delta)T_\alpha\overline{T}_{\gamma-1}^{m}} \tag{3.1}$$

and

$$\|J_k(y_n(t_k^-)) - J_k(y(t_k^-))\| < \frac{\varepsilon}{2m\overline{T}^m_{\gamma-1}}. \tag{3.2}$$

Thus, for all $t \in I_k,\ k = 0,\dots,m$, we have

$$\begin{aligned}
&\mathrm{N}_{\gamma-1,\psi}(t,t_k)\|\Lambda y_n(t) - \Lambda y(t)\| \le \sum_{i=1}^{k} T^m_{\gamma-1}\|J_i(y_n(t_i^-)) - J_i(y(t_i^-))\| \\
&+ \sum_{i=1}^{k} T^m_{\gamma-1}\mathfrak{I}^{\gamma,\psi}_{t_i}\|f\big(t_{i+1}, y_n(t_{i+1}), \mathfrak{I}^{\delta,\psi}_{t_i} y_n(\sigma_i(t_{i+1}))\big) - f\big(t_{i+1}, y(t_{i+1}), \mathfrak{I}^{\delta,\psi}_{t_i} y(\sigma_i(t_{i+1}))\big)\| \\
&+ \frac{T_{1-\gamma}}{\Gamma(\alpha)}\int_{t_k}^{t} \psi'(s)\mathrm{N}_{1-\alpha}(t,s)\|f\big(s, y_n(s), \mathfrak{I}^{\delta,\psi}_{t_k} y_n(\sigma_k(s))\big) - f\big(s, y(s), \mathfrak{I}^{\delta,\psi}_{t_k} y(\sigma_k(s))\big)\|ds.
\end{aligned}$$

By equations (3.1) and (3.2), we get

$$\|\Lambda y_n - \Lambda y\|_{\mathbb{PC}_{1-\gamma,\psi}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, Λ is continuous on $\mathbb{PC}_{1-\gamma,\psi}([c,L])$.

Second step. Now we will prove that Λ is bounded. Let $y \in \mathbb{PC}_{1-\gamma,\psi}([c,L])$, from $(\mathbf{H_1})$ it is easy to deduce that $Ny \in \mathbb{PC}_{1-\gamma,\psi}([c,L])$. Using $(\mathbf{H_1})$ and $(\mathbf{H_3})$, for all $y \in D_\kappa = \{y \in \mathbb{PC}_{1-\gamma,\psi}([c,L]) : \|y\|_{\mathbb{PC}_{\gamma,\psi}} < \kappa\}$ and $t \in I_k$, we get

$$\begin{aligned}
\|\mathrm{N}_{1-\gamma,\psi}(t,t_k)\Lambda y(t)\| &\le \|M\|T^m_{\gamma-1} + T^m_{\gamma-1}\sum_{i=1}^{k}\|\mathrm{N}_{1-\gamma,\psi}(t_i,t_{i-1})J_i(y(t_i^-))\| \\
&\quad + T^m_{\gamma-1}\sum_{i=1}^{k}\mathrm{N}_{1-\gamma,\psi}(t_i,t_{i-1})\mathfrak{I}^{\alpha,\psi}_{t_{i-1}^+}\|f\big(t_i, y(t_i), \mathfrak{I}^{\delta,\psi}_{t_{i-1}} y(\sigma_{t_{i-1}}(t_i))\big)\| \\
&\quad + \frac{\mathrm{N}_{1-\gamma,\psi}(t,t_k)}{\Gamma(\alpha)}\int_{t_k}^{t}\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t,s)\|f\big(s, y(s), \mathfrak{I}^{\delta,\psi}_{t_k} y(\sigma_{t_k}(s))\big)\|ds \\
&\le T^m_{\gamma-1}\big(\|M\| + Cm\kappa\big) + \frac{(m+1)T_\alpha\overline{T}^m_{\gamma-1}}{\Gamma(\alpha+1)}\Big(f^* + \kappa\big(AT_\lambda + BT^*T_\delta\big)\Big),
\end{aligned}$$

where $f^* = \sup_{t\in[c,L]}\big(\|f(t,0,0)\|\big)$.

Third step. We prove that $(\Lambda D)_k$ is equicontinuous for all bounded subset D of $\mathbb{PC}_{1-\gamma,\psi}([c,L])$, $k = 1,\dots,m$, where $(\Lambda D)_k$ the restriction of ΛD on the interval I_k, let D_κ be the subset which was previously defined. It suffices to prove that $(\Lambda D_\kappa)_k$ is equicontinuous in $\mathcal{C}_{1-\gamma,\psi}([t_k,t_{k+1}])$. Let $y \in (D_\kappa)_k$ and $t_1,t_2 \in I_k$ with $t_1 < t_2$, from $(\mathbf{H_1})$, we have

$$\begin{aligned}
&\|\mathrm{N}_{1-\gamma,\psi}(t_2,t_k)\Lambda y(t_2) - \mathrm{N}_{1-\gamma,\psi}(t_1,t_k)\Lambda y(t_1)\| \\
&\le \frac{\mathrm{N}_{1-\gamma,\psi}(t_1,t_k)}{\Gamma(\alpha)}\int_{t_k}^{t_1}\psi'(s)[\mathrm{N}_{\alpha-1,\psi}(t_1,s) - \mathrm{N}_{\alpha-1,\psi}(t_2,s)]\|f\big(s, y(s), \mathfrak{I}^{\delta,\psi}_{t_k} y(\sigma_k(s))\big)\|ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{[\mathrm{N}_{1-\gamma,\psi}(t_2,t_k)-\mathrm{N}_{1-\gamma,\psi}(t_1,t_k)]}{\Gamma(\alpha)}\int_{t_k}^{t_1}\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t_2,s)\|f(s,y(s,\mathfrak{I}_{t_k}^{\delta,\psi}y(\sigma_k(s)))\|ds\\
&+\frac{\mathrm{N}_{1-\gamma,\psi}(t_2,t_k)}{\Gamma(\alpha)}\int_{t_1}^{t_2}\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t_2,s)\|f(s,y(s),\mathfrak{I}_{t_k}^{\delta,\psi}y(\sigma_k(s)))\|ds\\
\leq&\frac{f^*+A\kappa T_\lambda+B\kappa T^*T_\delta}{\Gamma(\alpha)}\int_{t_k}^{t_1}\psi'(s)[\mathrm{N}_{\alpha-1,\psi}(t_1,s)-\mathrm{N}_{\alpha-1,\psi}(t_2,s)]ds\\
&+\frac{f^*[\mathrm{N}_{1-\gamma,\psi}(t_2,t_k)-\mathrm{N}_{1-\gamma,\psi}(t_1,t_k)]}{\Gamma(\alpha)}\int_{t_k}^{t_1}\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t_2,s)ds\\
&+\frac{A\kappa[\mathrm{N}_{1-\gamma,\psi}(t_2,t_k)-\mathrm{N}_{1-\gamma,\psi}(t_1,t_k)]}{\Gamma(\alpha)}\int_{t_k}^{t_1}\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t_2,s)\mathrm{N}_{\lambda+\alpha-1,\psi}(s,t_k)ds\\
&+\frac{B\kappa T^*[\mathrm{N}_{1-\gamma,\psi}(t_2,t_k)-\mathrm{N}_{1-\gamma,\psi}(t_1,t_k)]}{\Gamma(\alpha)}\int_{t_k}^{t_1}\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t_2,s)\mathrm{N}_{\delta+\gamma-1,\psi}(s,t_k)ds\\
&+\frac{f^*+A\kappa T_\lambda+B\kappa T^*T_\delta}{\Gamma(\alpha)}\int_{t_1}^{t_2}\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t_2,s)ds\\
\leq&\frac{f^*+A\kappa\Psi_\lambda^*}{\Gamma(\alpha+1)}[2\mathrm{N}_{\alpha,\psi}(t_2,t_1)+\mathrm{N}_{\alpha,\psi}(t_1,t_k)-\mathrm{N}_{\alpha,\psi}(t_2,t_k)]\\
&+\frac{f^*T_\alpha+A\kappa T_{2\alpha+\lambda-1}+B\kappa T^*T_{\alpha+\delta+\lambda-1}}{\Gamma(\alpha+1)}[\Psi_{1-\alpha}(t_2,t_k)-\Psi_{1-\alpha}(t_1,t_k)],
\end{aligned}
$$

Taking t_2 tends towards t_1, we get that, the last inequality tends to zero. Then $(\Lambda D_\kappa)_k$ is equicontinuous in $\mathcal{C}_{1-\gamma,\psi}([t_k,t_{k+1}])$, $k=0,\dots,m$.

Final step. We verify that Λ satisfies the assumptions of theorem 1. We pose

$$D=\{y\in\mathbb{PC}_{1-\gamma,\psi}([c,L]):\|y\|_{\mathbb{PC}_{\gamma,\psi}}\leq R\},$$

where R is a real number verifies the following equality

$$R>\frac{\overline{T}_{\gamma-1}^m\Big(\Gamma(\alpha+1)\|M\|+(m+1)T_\alpha f^*\Big)}{\Gamma(\alpha+1)-\overline{T}_{\gamma-1}^mT_\alpha\Big(mC\Gamma(\alpha+1)+(m+1)(AT_\lambda+BT^*T_\delta)\Big)}. \tag{3.3}$$

First, we now show that Λ is defined from D to D, Indeed, for any $y\in D$, by above conditions $(\mathbf{H_2}),(\mathbf{H_5})$ and by according to a little calculation, for all $t\in I_k$, we have

$$
\begin{aligned}
\|\mathrm{N}_{1-\gamma,\psi}\Lambda y(t)\leq&\frac{\overline{T}_{\gamma-1}^m}{\Gamma(\alpha+1)}\Big(\Gamma(\alpha+1)\|M\|+(m+1)T_\alpha f^*\Big)\\
&+\frac{\overline{T}_{\gamma-1}^mT_\alpha}{\Gamma(\alpha+1)}\Big(mC\Gamma(\alpha+1)+(m+1)(AT_\lambda+BT^*T_\delta)\Big)R.
\end{aligned}
$$

From inequality (3.3), we obtain

$$\forall y\in D:\|\Lambda y\|_{\mathbb{PC}_{\gamma,\psi}}<R.$$

Then Λ remains defined from D to D. Note that D is bounded, convex and closed subset of $\mathbb{PC}_{\gamma,\psi}([a,L])$ and Λ is continuous on D, we can easily show the following equality

$$\vartheta^k_{\gamma,\psi}((NV)_k) = \sup\left\{\vartheta\left(\mathrm{N}_{1-\gamma,\psi}(t,t_k)\Lambda V(t)\right),\ t\in I_k\right\},$$

for all $V\subset D,\ k=0,\dots,m$. Next, we need to prove the following implication

$$V\subset \overline{conv}\{\Lambda(V)\cup\{0\}\} \Longrightarrow \vartheta_{\gamma,\psi}(V)=0,\ \text{for any } V\subset D.$$

Let $V\subset D$ such that $V\subset \overline{conv}\{\Lambda(V)\cup\{0\}\}$. From $(\mathbf{H_2})$, $(\mathbf{H_4})$, Lemmas 1-2 and the previous steps, for all $t\in I_k$, we have

$$\begin{aligned}\vartheta\left(\mathrm{N}_{1-\gamma,\psi}(\Lambda V)(t)\right) &\le \frac{AT_\lambda T^m_{\gamma-1}}{\Gamma(\alpha)}\sum_{i=0}^{k}\int_{t_i}^{t_{i+1}}\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t_{i+1},s)\vartheta^i_{\gamma,\psi}((\Lambda V)_i)ds\\ &+\frac{BT^*T_\delta T^m_{\gamma-1}}{\Gamma(\alpha)}\sum_{i=0}^{k}\int_{t_i}^{t_{i+1}}\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t_{i+1},s)\vartheta^i_{\gamma,\psi}((\Lambda V)_i)ds+CT^m_{\gamma-1}\sum_{i=0}^{k-1}\vartheta^i_{\alpha,\psi}((\Lambda V)_i)\\ &+\frac{AT_\lambda+BT^*T_\delta}{\Gamma(\alpha)}\int_{t_k}^{t}\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t,s)\vartheta^k_{\gamma,\psi}((\Lambda V)_k)ds.\end{aligned}$$

Thus,

$$\vartheta_{\gamma,\psi}(\Lambda V)\le \frac{T^m_{\gamma-1}\left[(m+1)AT_{\lambda+\alpha}+(m+1)BT^*T_{\delta+\alpha}+mC\Gamma(\alpha+1)\right]}{\Gamma(\alpha+1)}\vartheta_{\gamma,\psi}(\Lambda V).$$

By condition $(\mathbf{H_5})$, we get $\vartheta_{\gamma,\psi}(\Lambda V)=0$, that is $\vartheta_{\gamma,\psi}(V)=0$. From Theorem 1, Λ has a fixed point $\bar{y}\in D$ which is a solution of Problem $(\mathfrak{P})$. Let us now show that the fixed point of Λ is included in $\mathbb{PC}^{\gamma}_{1-\gamma,\psi}([c,L])$, Let $t\in I_k,\ k=0,\dots,m$. So, we have

$$\begin{aligned}\bar{y}(t) = \mathrm{N}_{\gamma-1,\psi}(t,t_k)\Bigg[&M\prod_{i=1}^{k}\mathrm{N}_{\gamma-1,\psi}(t_i,t_{i-1})+\sum_{i=1}^{k}\Xi^{k-i}_{j=1}N_{\gamma-1,\psi}(t_{k-j+1},t_{k-j})J_i(\bar{y}(t_i^-))\\ &+\sum_{i=1}^{k}\Xi^{k-i}_{j=1}N_{\gamma-1,\psi}(t_{k-j+1},t_{k-j})\mathfrak{I}^{\alpha,\psi}_{t^+_{i-1}}f\left(t_i,\bar{y}(t_i),\mathfrak{I}^{\delta,\psi}_{t_{i-1}}\bar{y}(\sigma_{t_{i-1}}(t_i))\right)\Bigg]\\ &+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t,s)f\left(s,\bar{y}(s),\mathfrak{I}^{\delta,\psi}_{t_k}\bar{y}(\sigma_k(s))\right)ds.\end{aligned}$$

By entering ${}^{RL}\mathcal{D}^{\gamma}_{t_k^+}$ on both sides and utilizing Lemma 6, we find

$$\begin{aligned}{}^{RL}\mathcal{D}^{\gamma}_{t_k^+}\bar{y}(t) &={}^{RL}\mathcal{D}^{\gamma}_{t_k^+}\mathfrak{I}^{\alpha,\psi}_{t_k^+}f\left(t,\bar{y}(t),\mathfrak{I}^{\delta,\psi}_{t_k}\bar{y}(\sigma_{t_k}(t))\right)\\ &={}^{RL}\mathcal{D}^{\beta(1-\alpha)}_{t_k^+}f\left(t,\bar{y}(t),\mathfrak{I}^{\delta,\psi}_{t_k}\bar{y}(\sigma_{t_k}(t))\right).\end{aligned}$$

Thus, according to the hypotheses on f, we deduce that ${}^{RL}\mathcal{D}^{\gamma}_{t_k^+}\bar{y}(t) \in \mathcal{C}^{\gamma}_{1-\gamma}(I_k)$, $k = 0,\dots,m$, from the definition of $\mathbb{PC}^{\gamma}_{1-\gamma,\psi}([c,L])$, we conclude that the fixed point $\bar{y}$ of Λ is an element of such space. □

Our present result is based on the Darbo's fixed point theorem.

Theorem 4. *Suppose that the conditions* $(\mathbf{H_1})-(\mathbf{H_5})$ *are valid. Then, the problem* $(\mathfrak{P})$ *has at least one solution. Moreover its solutions belong to* $\mathbb{PC}^{\gamma}_{1-\gamma,\psi}([c,L]) \subset \mathbb{PC}^{\alpha,\beta}_{1-\gamma,\psi}([c,L])$.

Proof. By Lemma 7, the solutions of Problem $(\mathfrak{P})$ and fixed points of operator Λ are coincident. We will prove that Λ satisfies the conditions of Darbo's fixed point Theorem 2. According to what precedes the operator Λ is defined from D to D, continuous, bounded and that ΛD is equicontinuous, it suffices to prove that there exists a real $0<\xi<1$ such that

$$\vartheta_{\gamma,\psi}(\Lambda V) \le \xi \vartheta_{\gamma,\psi}(V), \text{ for all } V \subset D.$$

Let $V \subset D$ and $t \in I_k$, $k = 0,\dots,m$. From $(\mathbf{H_2})$, $(\mathbf{H_4})$ and by using Lemmas 1-3, we have

$$\begin{aligned}
\vartheta\left(\mathrm{N}_{1-\gamma,\psi}(t,t_k)(\Lambda V)(t)\right) &= \vartheta\left(\left\{\mathrm{N}_{1-\gamma,\psi}(t,t_k)\Lambda u(t),\ u \in V\right\}\right)\\
&\le \frac{AT_\lambda T^m_{\gamma-1}}{\Gamma(\alpha)}\sum_{i=0}^{k}\int_{t_i}^{t_{i+1}} \psi'(s)\mathrm{N}_{\alpha-1,\psi}(t_{i+1},s)\vartheta^i_{\gamma,\psi}((\Lambda V)_i)ds\\
&\quad + \frac{BT^*T_\delta T^m_{\gamma-1}}{\Gamma(\alpha)}\sum_{i=0}^{k}\int_{t_i}^{t_{i+1}} \psi'(s)\mathrm{N}_{\alpha-1,\psi}(t_{i+1},s)\vartheta^i_{\gamma,\psi}((\Lambda V)_i)ds\\
&\quad + CT^m_{\gamma-1}\sum_{i=0}^{k-1}\vartheta^i_{\alpha,\psi}((\Lambda V)_i)\\
&\quad + \frac{AT_\lambda + BT^*T_\delta}{\Gamma(\alpha)}\int_{t_k}^{t}\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t,s)\vartheta^k_{\gamma,\psi}((\Lambda V)_k)ds.
\end{aligned}$$

So, from $(\mathbf{H_5})$ there exists a real $0<\xi<1$ such that

$$\vartheta_{\gamma,\psi}(\Lambda V) \le \xi \vartheta_{\gamma,\psi}(V), \text{ for all } V \subset D,$$

where $\xi = \frac{T^m_{\gamma-1}}{\Gamma(\alpha+1)}\left[(m+1)AT_{\lambda+\alpha} + (m+1)BT^*T_{\delta+\alpha} + mC\Gamma(\alpha+1)\right]$. So, Theorem 2 assures us that the operator has at least one fixed point $\bar{y}$. Method similar to that of the last step of our first result, we find that its fixed points belong to $\mathbb{PC}^{\gamma}_{1-\gamma,\psi}([c,L]) \subset \mathbb{PC}^{\alpha,\beta}_{1-\gamma,\psi}([c,L])$. □

4. Stability

For any $\varepsilon > 0$, $\zeta > 0$ and $\theta : (c,L] \to \mathbb{R}^+$ be a continuous function, we consider the following system of inequalities

$$(\mathfrak{S}) \quad \begin{cases} \|{}^H\mathcal{D}^{\alpha,\beta,\psi}_{t_k^+} y(t) - f(t, y(t), \mathfrak{I}^{\delta,\psi}_{t_k^+} y(\sigma_k(t)))\| \leq \varepsilon\theta(t), \quad t \in I_k,\ k = 0,\dots,m, \\ \|\Delta_{\gamma,\psi} y|_{t_k} - J_k(y(t_k^-))\| \leq \varepsilon\zeta, k = 1,\dots,m. \end{cases}$$

Definition 4. Problem $(\mathfrak{P})$ is said to be stable in the sense of Ulam-Hyers-Rassias according to (θ,ζ) if there is a real number $\chi_{(\theta,\zeta)}$, for all solution $\varpi \in \mathbb{PC}_{1-\gamma,\psi}([c,L])$ of problem $(\mathfrak{S})$ there exists a solution $y \in \mathbb{PC}_{1-\gamma,\psi}([c,L])$ of problem $(\mathfrak{P})$ such that

$$\|y(t) - \varpi(t)\| \leq \varepsilon\chi_{(\theta,\zeta)}\big(\theta(t) + \zeta\big), \text{ for all } t \in (c,L].$$

Remark 1. A function ϖ of $\mathbb{PC}_{1-\gamma,\psi}([c,L])$ is called solution of problem $(\mathfrak{S})$ if there exists a function $\varphi \in \mathcal{C}([c,L))$ and constants $\rho_k \in E,\ k = 0,\dots,m$ satisfies $\|\varphi(t)\| \leq \varepsilon\theta(t)$ and $\|\rho_k\| \leq \varepsilon\zeta,\ t \in I_k,\ k = 0,\dots,m$, such that ϖ is a solution of the following problem

$$\begin{cases} {}^H\mathcal{D}^{\alpha,\beta,\psi}_{t_k^+} y(t) = f(t, y(t), \mathfrak{I}^{\delta,\psi}_{t_k^+} y(\sigma_k(t))) + \varphi(t), \quad t \in I_k,\ k = 0,\dots,m, \\ \Delta_{\gamma,\psi} y|_{t_k} = J_k(y(t_k^-)) + \rho_k,\ k = 1,\dots,m. \end{cases}$$

In the following we give a result about the stability in the sense of Ulam-Hyers-Rassias of Problem $(\mathfrak{P})$. We are interested in studying the case where $\theta : (c,L] \to \mathbb{R}^+$ is a constant function.

Theorem 5. *Suppose that the conditions* $(\mathbf{H_1}) - (\mathbf{H_5})$ *are valid. Then, the problem* $(\mathfrak{P})$ *is stable in the sense of Ulam-Hyers-Rassias according to* (θ,ζ).

Proof. Let $\varepsilon > 0$, $\varpi \in \mathbb{PC}_{1-\gamma,\psi}([c,L])$ be any solution of Problem $(\mathfrak{S})$ and y be the solution of the following problem

$$(\mathfrak{P}) \quad \begin{cases} {}^H\mathcal{D}^{\alpha,\beta,\psi}_{t_k^+} y(t) = f(t, y(t), \mathfrak{I}^{\delta,\psi}_{t_k^+} y(\sigma_k(t))), \quad t \in I_k,\ k = 0,\dots,m, \\ \Delta_{\gamma,\psi} y|_{t_k} = J_k(y(t_k^-)), k = 1,\dots,m, \\ \mathfrak{I}^{1-\gamma,\psi}_{t_k^+} y(t_k^+) = \mathfrak{I}^{1-\gamma,\psi}_{t_k^+} \varpi(t_k^+), k = 0,\dots,m. \end{cases}$$

From Lemma 6 the solution y of the previous problem is written in the following form

$$y(t) = \frac{\mathfrak{I}^{1-\gamma,\psi}_{t_k^+} y(t_k^+)}{\Gamma(\gamma)} \mathrm{N}_{\gamma-1,\psi}(t,t_k) + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \psi'(s) \mathrm{N}_{\alpha-1,\psi}(t,s) f\big(s, y(s), \mathfrak{I}^{\delta,\psi}_{t_k} y(\sigma_k(s))\big) ds.$$

Since ϖ is a solution of Problem $(\mathfrak{S})$ and by utilizing the remark 1, we have

$$\begin{cases} {}^H\mathcal{D}^{\alpha,\beta,\psi}_{t_k^+} \varpi(t) = f(t, \varpi(t), \mathfrak{I}^{\delta,\psi}_{t_k^+} \varpi(\sigma_k(t))) + \varphi(t), \quad t \in I_k = (t_k, t_{k+1}],\ k = 0,\dots,m, \\ \Delta_{\gamma,\psi} \varpi|_{t_k} = J_k(\varpi(t_k^-)) + \rho_k,\ k = 0,\dots,m. \end{cases}$$

Thus, for all $t \in I_k,\ k = 0,\dots,m$, ϖ is given by

$$\varpi(t) = \frac{I_{t_k^+}^{1-\gamma,\psi}\varpi(t_k^+)}{\Gamma(\gamma)}\mathrm{N}_{\gamma-1,\psi}(t,t_k) + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\psi'(s)\mathrm{N}_{\alpha-1,\psi}(t,s)\Big(\varphi(t) + f\big(s,\varpi(s),\mathfrak{I}_{t_k}^{\delta,\psi}\varpi(\sigma_k(s))\big)\Big)ds.$$

Since $\mathfrak{I}_{t_k^+}^{1-\gamma,\psi}y(t_k^+) = \mathfrak{I}_{t_k^+}^{1-\gamma,\psi}\varpi(t_k^+)$. So, for all $t \in (t_k,t_{k+1}],\ k = 0,\dots,m$, we have

$$\|\varpi(t) - y(t)\| \le \mathfrak{I}_{t_k^+}^{\alpha,\psi}\|f\big(t,\varpi(t),\mathfrak{I}_{t_k}^{\delta,\psi}\varpi(\sigma_k(t))\big) - f\big(t,y(t),\mathfrak{I}_{t_k}^{\delta,\psi}y(\sigma_k(t))\big)\| + \mathfrak{I}_{t_k^+}^{\alpha,\psi}\|\varphi(t)\|.$$

From $(\mathbf{H_2})$, we get

$$\left(1 - \frac{T_\alpha(AT_\lambda + BT^*T_\delta)}{\Gamma(\alpha+1)}\right)\|\varpi - y\|_{\mathbb{PC}_{1-\gamma,\psi}} \le \frac{\varepsilon T_\alpha}{\Gamma(\alpha+1)}\theta.$$

Thus, for all $t \in (c,L]$, we obtain

$$\|\varpi(t) - y(t)\| \le \frac{T_{\gamma-1}T_\alpha}{\Gamma(\alpha+1) - T_\alpha(AT_\lambda + BT^*T_\delta)}(\theta+\zeta)\varepsilon.$$

Thus, the proof is completed. □

5. Example

We pose $\psi(t) = t$, $m = 1$, $t_0 = 0$, $t_1 = 0.5$, $t_2 = 1, \sigma = 1$, $\alpha = \beta = \lambda = \delta = 0.5$ and

$$E = \{(y_1,y_2,\dots,y_n,\dots) : \sup_n |y_n| < \infty\},\ \text{with } \|y\| = \sup_n |y_n|.$$

We take the following problem

$$^H\mathcal{D}_{t_k}^{\alpha,\beta,\psi}y(t) = \Big(f_n(t,y(t),\mathfrak{I}_{t_k}^{\delta,\psi}y(\sigma_k(t)))\Big)_{n=1}^{\infty},\ t \in (t_k,t_{k+1}] \subset (0,1],\ k = 0,1 \quad (5.1)$$

$$\mathfrak{I}_{0^+}^{1-\gamma,\psi}y(0^+) = (1,0,\dots,0,\dots). \quad (5.2)$$

$$\Delta_{\gamma,\psi}y|_{t=\frac{1}{2}} = J_1(y(\frac{1}{2}^-)), \quad (5.3)$$

with

$$f_n(t,y(t),\mathfrak{I}_{t_k}^{\delta,\psi}y(\sigma_k(t))) = \frac{\mathfrak{I}_{t_k}^{\delta,\psi}y_n(\sigma_k(t))}{10+nt^2} + \frac{\sqrt{t-t_i}}{10+t+t^2}y_n(t), k = 0,1, n \in \mathbb{N}^* \text{ and}$$

$$J_1(u) = \frac{1}{10}u,\ \text{for all } u \in E.$$

We can easily see that $f : (t_k,t_{k+1}] \times E \to E,\ k = 0,1$ and $J_1 : E \to E$ are continuous and there exists $A = B = C = \frac{1}{10}$ such that

$$\|f(t,u,v) - f(t,\overline{u},\overline{v})\| \le A\sqrt{t-t_k}\|u-\overline{u}\| + B\|v-\overline{v}\|,\ \text{for all } t \in I_k \text{ and } u,v,\overline{u},\overline{v} \in E \text{ and}$$

$$\|J_1(u)\| = C\|u\|,\ \text{for all } u \in E.$$

So, $(\mathbf{H_2})$ and $(\mathbf{H_4})$ are valid. Next, let Ω be a bounded subset of $\mathbb{PC}_{1\gamma,\psi}([0,1])$, we have

$$\vartheta\Big(f(t,\Omega(t),\mathfrak{I}_{t_k}^{\delta,\psi}\Omega(\sigma_k(t)))\Big) \leq \frac{1}{10}\Big(\sqrt{t-t_k}\vartheta\Big(\Omega(t)\Big)+\vartheta\Big(\mathfrak{I}_{t_k}^{\delta,\psi}\Omega(\sigma_k(t))\Big),\ t\in I_k$$

and

$$\vartheta\Big(J_1(\Omega(t)\Big) \leq \frac{1}{10}\Big(\Omega(t)\Big).$$

Thus, $(\mathbf{H_1})$ and $(\mathbf{H_3})$ are satisfied. A quick calculation gives us

$$\overline{T}_{\gamma-1}^{m}T_{\alpha}\Big(mC\Gamma(\alpha+1)+(m+1)(AT_{\lambda}+BT^{*}T_{\delta})\Big) < \Gamma(\alpha+1).$$

So, $(\mathbf{H_5})$ holds. By virtue of Theorem 3 or 4 the problem (5.1)-(5.3) has at least one solution. Moreover, from Theorem 5, we have for any constant function $\theta:(0,1]\to[0,\infty)$ and $\zeta>0$, the problem (5.1)-(5.3) is stable in the sense of Ulam-Hyers-Rassias according to (θ,ζ).

6. Conclusion

In this paper, we study the existence of a solution and its Ulam-Hyears-Rassias stability for certain pantograph fractional integro-differential equations with impulsive conditions. The significance of our work is that these conditions are nonlocal. The future consideration will be to consider non-instantaneous impulsive conditions.

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NUMERICAL RADIUS INEQUALITIES OF OPERATOR MATRICES FROM A NEW NORM ON $\mathcal{B}(\mathcal{H})$

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Abstract. This paper is a continuation of a recent work on a new norm, christened the (α,β)-norm, on the space of bounded linear operators on a Hilbert space. We obtain some upper bounds for the said norm of $n\times n$ operator matrices. As an application of the present study, we estimate bounds for the numerical radius and the usual operator norm of $n\times n$ operator matrices, which generalize the existing ones.

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1. Introduction

The purpose of the present article is to study the bounds for the newly introduced [15] (α,β)-norm of $n\times n$ operator matrices, from which we obtain bounds for the numerical radius of $n\times n$ operator matrices. Let us first introduce the following notations and terminologies to be used throughout the article.

Let $\mathcal{H}_i,\mathcal{H}_j$ be two complex Hilbert spaces with usual inner product $\langle .,.\rangle$ and let $\mathcal{B}(\mathcal{H}_i,\mathcal{H}_j)$ denote the space of all bounded linear operators from $\mathcal{H}_i$ to $\mathcal{H}_j$. If $\mathcal{H}_i=\mathcal{H}_j=\mathcal{H}$ then we write $\mathcal{B}(\mathcal{H},\mathcal{H})=\mathcal{B}(\mathcal{H})$. For $T\in\mathcal{B}(\mathcal{H})$, we write $Re(T)$ and $Im(T)$ for the real part of T and the imaginary part of T, respectively, i.e., $Re(T)=\frac{T+T^*}{2}$ and $Im(T)=\frac{T-T^*}{2i}$. Let T^* denote the adjoint of T and let $|T|$ be the positive operator $(T^*T)^{\frac{1}{2}}$. Let $\sigma(T)$ denote the spectrum of T. The spectral radius of T, denoted by $r(T)$, is defined by $r(T)=\sup\{|\lambda|:\lambda\in\sigma(T)\}$. The numerical range of T, denoted by $W(T)$, is defined as $W(T)=\{\langle Tx,x\rangle : x\in\mathcal{H},\|x\|=1\}$. The usual operator norm and the numerical radius of T, denoted by $\|T\|$ and $w(T)$, respectively, are defined as $\|T\|=\sup\{\|Tx\|:x\in\mathcal{H},\|x\|=1\}$ and $w(T)=\sup\{|c|:c\in W(T)\}$. Let M_T denote the usual operator norm attainment set of T, i.e., $M_T=\{x\in\mathcal{H}:\|Tx\|=\|T\|,\|x\|=1\}$.

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It is well-known that the numerical radius defines a norm on $\mathcal{B}(\mathcal{H})$ and is equivalent to the usual operator norm, satisfying that for $T \in \mathcal{B}(\mathcal{H})$,

$$\frac{1}{2}\|T\| \leq w(T) \leq \|T\|.$$

The study of the numerical range of an operator and the associated numerical radius inequalities are an important area of research in operator theory and it has attracted many mathematicians [1, 3, 5–7, 12] over the years. Recently, some generalizations for the concept of numerical radius have been introduced in [2, 4, 14, 16, 18]. One of these generalizations is the A-numerical radius of an operator $T \in \mathcal{B}(\mathcal{H})$ defined by $w_A(T) = \sup\{|\langle ATx, x\rangle| : x \in \mathcal{H}, \langle Ax, x\rangle = 1\}$, see, e.g., [8, 10, 17]. Here, A is a positive bounded linear operator on $\mathcal{H}$. With an aim to develop better upper and lower bounds for the numerical radius, a new norm named as the (α, β)-norm, was introduced on $\mathcal{B}(\mathcal{H})$ in [15]. For $T \in \mathcal{B}(\mathcal{H})$, the (α, β)-norm of T, denoted by $\|T\|_{\alpha,\beta}$, is defined as:

$$\|T\|_{\alpha,\beta} = \sup\left\{\sqrt{\alpha|\langle Tx, x\rangle|^2 + \beta\|Tx\|^2} : x \in \mathcal{H}, \|x\| = 1\right\},$$

where α, β are real positive constants with $(\alpha, \beta) \neq (0, 0)$. We note that if $\alpha = 1, \beta = 0$ then $\|T\|_{\alpha,\beta} = w(T)$, and if $\alpha = 0, \beta = 1$ then $\|T\|_{\alpha,\beta} = \|T\|$. Also, if we consider $\alpha = \beta = 1$, then we have the modified Davis-Wielandt radius of T, that is, $\|T\|_{\alpha,\beta} = dw^*(T)$, (see [9]). In this article, we consider $\alpha + \beta = 1$, i.e., $\beta = 1 - \alpha$ and explore the α-norm of $n \times n$ operator matrices, where the α-norm of T is defined as:

$$\|T\|_{\alpha} = \sup\left\{\sqrt{\alpha|\langle Tx, x\rangle|^2 + (1-\alpha)\|Tx\|^2} : x \in \mathcal{H}, \|x\| = 1\right\}.$$

We compute the exact value of the α-norm of 2×2 operator matrices in $\mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ of the form $\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$, where $X \in \mathcal{B}(\mathcal{H})$. We obtain some upper bounds for the α-norm of $n \times n$ operator matrices, which generalize the existing numerical radius inequalities and the usual operator norm inequalities of $n \times n$ operator matrices. As an application of our results, we estimate new upper bounds for the numerical radius and the usual operator norm of $n \times n$ operator matrices.

2. Main results

We begin this section with the following proposition, the proof of which follows from the weakly unitarily invariant property of the α-norm, i.e., for $T \in \mathcal{B}(\mathcal{H})$, $\|U^*TU\|_{\alpha} = \|T\|_{\alpha}$ for every unitary operator $U \in \mathcal{B}(\mathcal{H})$ (see [15, Prop. 2.6]).

Proposition 1. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then the following results hold:*

(a) $\left\|\begin{pmatrix} 0 & A \\ e^{i\theta}B & 0 \end{pmatrix}\right\|_{\alpha} = \left\|\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\right\|_{\alpha}$ *for every* $\theta \in \mathbb{R}$.

(b) $\left\|\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\right\|_\alpha = \left\|\begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix}\right\|_\alpha.$

(c) $\left\|\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right\|_\alpha = \left\|\begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix}\right\|_\alpha.$

(d) $\left\|\begin{pmatrix} A & B \\ B & A \end{pmatrix}\right\|_\alpha = \left\|\begin{pmatrix} A-B & 0 \\ 0 & A+B \end{pmatrix}\right\|_\alpha.$

Next, we estimate upper and lower bounds for the α-norm of 2×2 operator matrices in $\mathcal{B}(\mathcal{H}\oplus\mathcal{H})$ of the form $\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$, where $X,Y \in \mathcal{B}(\mathcal{H})$. Let us first note the following inequality for $X \in \mathcal{B}(\mathcal{H})$,

$$\alpha|\langle Xx,x\rangle|^2 + (1-\alpha)\|Xx\|^2 \le \|X\|_\alpha^2\|x\|^2 \text{ for all } x \in \mathcal{H} \text{ with } \|x\| \le 1.$$

Theorem 1. *Let $X,Y \in \mathcal{B}(\mathcal{H})$. Then the following inequalities hold:*

$$\begin{aligned}
(i)\quad \max\{\|X\|_\alpha, \|Y\|_\alpha\} &\le \left\|\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}\right\|_\alpha \\
&\le \max\left\{\sqrt{\|X\|_\alpha^2 + \alpha w^2(X)}, \sqrt{\|Y\|_\alpha^2 + \alpha w^2(Y)}\right\} \\
&\le \sqrt{2}\max\{\|X\|_\alpha, \|Y\|_\alpha\}.
\end{aligned}$$

$$(ii)\quad \left\|\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}\right\|_\alpha \le \sqrt{\max\{\|X\|_\alpha^2, \|Y\|_\alpha^2\} + \alpha w(X)w(Y)}.$$

$$(iii)\quad \left\|\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}\right\|_\alpha \le \|X\|_\alpha + \|Y\|_\alpha.$$

Proof. (i) Let $T = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$. Let $x \in \mathcal{H}$ with $\|x\| = 1$ and let $\tilde{x} = \begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathcal{H}\oplus\mathcal{H}$. Clearly, $\|\tilde{x}\| = 1$. Therefore, we have,

$$\sqrt{\alpha|\langle Xx,x\rangle|^2 + (1-\alpha)\|Xx\|^2} = \sqrt{\alpha|\langle T\tilde{x},\tilde{x}\rangle|^2 + (1-\alpha)\|T\tilde{x}\|^2} \le \|T\|_\alpha.$$

Taking supremum over all unit vector in $\mathcal{H}$, we get,

$$\|X\|_\alpha \le \|T\|_\alpha.$$

Similarly, it can be proved that

$$\|Y\|_\alpha \le \|T\|_\alpha.$$

Combining the above two inequalities, we get the first inequality in (i). Let us now prove the second inequality in (i). Let $z = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{H}\oplus\mathcal{H}$ with $\|z\| = 1$, i.e., $\|x\|^2 + \|y\|^2 = 1$. Then we have,

$$\alpha|\langle Tz,z\rangle|^2 + (1-\alpha)\|Tz\|^2$$

$$
\begin{aligned}
&= \alpha\,|\langle Xx,x\rangle + \langle Yy,y\rangle|^2 + (1-\alpha)(\|Xx\|^2 + \|Yy\|^2)\\
&\leq \alpha\,(|\langle Xx,x\rangle| + |\langle Yy,y\rangle|)^2 + (1-\alpha)\left(\|Xx\|^2 + \|Yy\|^2\right)\\
&\leq \alpha|\langle Xx,x\rangle|^2 + (1-\alpha)\|Xx\|^2 + \alpha|\langle Yy,y\rangle|^2 + (1-\alpha)\|Yy\|^2\\
&\qquad + \alpha\left(|\langle Xx,x\rangle|^2 + |\langle Yy,y\rangle|^2\right)\\
&\leq \|X\|_\alpha^2\|x\|^2 + \|Y\|_\alpha^2\|y\|^2\\
&\qquad + \alpha\left(w^2(X)\|x\|^2 + w^2(Y)\|y\|^2\right) \quad (\text{since } \|x\| \leq 1, \|y\| \leq 1)\\
&= \left(\|X\|_\alpha^2 + \alpha w^2(X)\right)\|x\|^2 + \left(\|Y\|_\alpha^2 + \alpha w^2(Y)\right)\|y\|^2\\
&\leq \max\left\{\|X\|_\alpha^2 + \alpha w^2(X), \|Y\|_\alpha^2 + \alpha w^2(Y)\right\}.
\end{aligned}
$$

Therefore, taking supremum over all unit vectors in $\mathcal{H}\oplus\mathcal{H}$, we get the second inequality in (i). The remaining inequality in (i) follows from the inequalities $\alpha w^2(X) \leq \|X\|_\alpha^2$ and $\alpha w^2(Y) \leq \|Y\|_\alpha^2$. This completes the proof of (i).

(ii) From

$$
\alpha|\langle Tz,z\rangle|^2 + (1-\alpha)\|Tz\|^2 \leq \alpha\,(|\langle Xx,x\rangle| + |\langle Yy,y\rangle|)^2 + (1-\alpha)\left(\|Xx\|^2 + \|Yy\|^2\right),
$$

we get

$$
\begin{aligned}
&\alpha|\langle Tz,z\rangle|^2 + (1-\alpha)\|Tz\|^2\\
&\qquad \leq \alpha|\langle Xx,x\rangle|^2 + (1-\alpha)\|Xx\|^2 + \alpha|\langle Yy,y\rangle|^2 + (1-\alpha)\|Yy\|^2\\
&\qquad\qquad + 2\alpha|\langle Xx,x\rangle|\ \ |\langle Yy,y\rangle|\\
&\qquad \leq \alpha|\langle Xx,x\rangle|^2 + (1-\alpha)\|Xx\|^2 + \alpha|\langle Yy,y\rangle|^2 + (1-\alpha)\|Yy\|^2\\
&\qquad\qquad + 2\alpha w(X)w(Y)\|x\|^2\|y\|^2\\
&\qquad \leq \|X\|_\alpha^2\|x\|^2 + \|Y\|_\alpha^2\|y\|^2\\
&\qquad\qquad + 2\alpha w(X)w(Y)\|x\|\|y\| \quad (\text{since } \|x\| \leq 1, \|y\| \leq 1)\\
&\qquad \leq \max\left\{\|X\|_\alpha^2, \|Y\|_\alpha^2\right\} + \alpha w(X)w(Y).
\end{aligned}
$$

Taking supremum over all unit vectors in $\mathcal{H}\oplus\mathcal{H}$, we get the inequality in (ii).

(iii) The inequality in (iii) follows from the triangle inequality of the α-norm, and by using the inequality in (ii). □

In the following theorem, we obtain the exact value of the α-norm of 2×2 operator matrices in $\mathcal{B}(\mathcal{H}\oplus\mathcal{H})$ of the form $\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$, where $X \in \mathcal{B}(\mathcal{H})$.

Theorem 2. *Let $X \in \mathcal{B}(\mathcal{H})$. Then*

$$
\left\|\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}\right\|_\alpha = \begin{cases} \frac{1}{2\sqrt{\alpha}}\,\|X\| & \text{if } \alpha > \frac{1}{2} \\ \sqrt{1-\alpha}\,\|X\| & \text{if } \alpha \leq \frac{1}{2}. \end{cases}
$$

Proof. Let $T = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$. Let $z = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{H} \oplus \mathcal{H}$ with $\|z\| = 1$, i.e., $\|x\|^2 + \|y\|^2 = 1$. Then $\langle Tz, z\rangle = \langle Xy, x\rangle$ and $\|Tz\| = \|Xy\|$. Now we have,

$$\begin{aligned}
\|T\|_\alpha^2 &= \sup_{\|z\|=1} (\alpha|\langle Tz, z\rangle|^2 + (1-\alpha)\|Tz\|^2) \\
&= \sup_{\|x\|^2+\|y\|^2=1} (\alpha|\langle Xy, x\rangle|^2 + (1-\alpha)\|Xy\|^2) \\
&\leq \sup_{\|x\|^2+\|y\|^2=1} (\alpha\|X\|^2\|y\|^2\|x\|^2 + (1-\alpha)\|X\|^2\|y\|^2) \\
&= \sup_{\theta\in[0,\frac{\pi}{2}]} \|X\|^2 \sin^2\theta(\alpha\cos^2\theta + (1-\alpha)).
\end{aligned}$$

First we consider the case $\alpha > \frac{1}{2}$. Then

$$\sup_{\theta\in[0,\frac{\pi}{2}]} \|X\|^2 \sin^2\theta(\alpha\cos^2\theta + (1-\alpha)) = \frac{1}{4\alpha}\|X\|^2.$$

Therefore, $\|T\|_\alpha^2 \leq \frac{1}{4\alpha}\|X\|^2$. We claim that there exists a sequence $\{z_n\}$ in $\mathcal{H} \oplus \mathcal{H}$ with $\|z_n\| = 1$ such that

$$\lim_{n\to\infty} (\alpha|\langle Tz_n, z_n\rangle|^2 + (1-\alpha)\|Tz_n\|^2) = \frac{1}{4\alpha}\|X\|^2.$$

Clearly, there exists a sequence $\{y_n\}$ in $\mathcal{H}$ with $\|y_n\| = 1$ such that $\lim_{n\to\infty}\|Xy_n\| = \|X\|$. Let $z_n = \frac{1}{\sqrt{\|Xy_n\|^2+k^2}} \begin{pmatrix} Xy_n \\ ky_n \end{pmatrix}$, where $k = \sqrt{\frac{1}{2\alpha-1}}\|X\|$. Then

$$\lim_{n\to\infty} \alpha|\langle Tz_n, z_n\rangle|^2 + (1-\alpha)\|Tz_n\|^2 = \frac{1}{4\alpha}\|X\|^2.$$

Therefore, $\|T\|_\alpha = \frac{1}{2\sqrt{\alpha}}\|X\|$ if $\alpha > \frac{1}{2}$.

Next we consider the case $\alpha \leq \frac{1}{2}$. Then

$$\sup_{\theta\in[0,\frac{\pi}{2}]} \|X\|^2 \sin^2\theta(\alpha\cos^2\theta + (1-\alpha)) = (1-\alpha)\|X\|^2$$

Therefore, $\|T\|_\alpha^2 \leq (1-\alpha)\|X\|^2$. Proceeding as before, we can show that there exists a sequence $\{z_n\}, \|z_n\| = 1$ such that $\lim_{n\to\infty}(\alpha|\langle Tz_n, z_n\rangle|^2 + (1-\alpha)\|Tz_n\|^2) = (1-\alpha)\|X\|^2$. Therefore, $\|T\|_\alpha = \sqrt{(1-\alpha)}\|X\|$ if $\alpha \leq \frac{1}{2}$. □

Remark 1. It follows from Proposition 1 (b) that $\left\|\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}\right\|_\alpha = \left\|\begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix}\right\|_\alpha$. Also, it follows from Theorem 1 that $\left\|\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}\right\|_\alpha = \left\|\begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix}\right\|_\alpha = \|X\|_\alpha$.

Our next goal is to obtain upper bounds for the α-norm of $n \times n$ operator matrices in $\mathcal{B}(\oplus_{i=1}^n \mathcal{H}_i)$. We require the following lemmas for our purpose.

Lemma 1. *([11, p. 44]) Let $T = (t_{ij}) \in M_n(\mathbb{C})$ with $t_{ij} \geq 0$ for all i, j. Then*

$$w(T) = r(Re(T)) = \|Re(T)\|.$$

Lemma 2. *([13]) Let $T \in \mathcal{B}(\mathcal{H})$ be self-adjoint and let $x \in \mathcal{H}$. Then*

$$|\langle Tx, x\rangle| \leq \langle |T|x, x\rangle.$$

Lemma 3. *([13]) Let $T \in \mathcal{B}(\mathcal{H})$ with $T \geq 0$ and let $x \in \mathcal{H}$ with $\|x\| = 1$. Then*

$$\langle Tx, x\rangle^p \leq \langle T^p x, x\rangle \text{ for all } p \geq 1.$$

Lemma 4. *([15, Th. 2.1]) Let $T \in \mathcal{B}(\mathcal{H})$. Then the following inequalities hold:*

$$w(T) \leq \|T\|_\alpha \leq \sqrt{4 - 3\alpha}\, w(T),$$

$$\max\left\{\frac{1}{2}, \sqrt{(1-\alpha)}\right\} \|T\| \leq \|T\|_\alpha \leq \|T\|.$$

Now we are in a position to prove the following inequality.

Theorem 3. *Let $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n$ be Hilbert spaces. Let $T = (T_{ij})$ be an $n \times n$ operator matrix, where $T_{ij} \in \mathcal{B}(\mathcal{H}_j, \mathcal{H}_i)$. Then*

$$\|T\|_\alpha \leq \sqrt{\|\alpha |R|^2 + (1-\alpha)|S|^2\|},$$

where $R = (r_{ij})_{n\times n}$, $r_{ij} = \begin{cases} w(T_{ij}) & \text{if } i = j \\ \frac{1}{2}(\|T_{ij}\| + \|T_{ji}\|) & \text{if } i \neq j \end{cases}$
and $S = (s_{ij})_{n\times n}$, $s_{ij} = \|T_{ij}\|$.

Proof. Let $x = (x_1, x_2, \ldots, x_n) \in \oplus_{i=1}^n \mathcal{H}_i$ with $\|x\| = 1$ and let $\tilde{x} = (\|x_1\|, \|x_2\|, \ldots, \|x_n\|)$. Clearly, $\tilde{x}$ is a unit vector in $\mathbb{C}^n$. Now,

$$\begin{aligned}
|\langle Tx, x\rangle| &= \left|\sum_{i,j=1}^n \langle T_{ij}x_j, x_i\rangle\right| \leq \sum_{i,j=1}^n |\langle T_{ij}x_j, x_i\rangle| \\
&\leq \sum_{i=1}^n |\langle T_{ii}x_i, x_i\rangle| + \sum_{i,j=1; i\neq j}^n |\langle T_{ij}x_j, x_i\rangle| \\
&\leq \sum_{i=1}^n w(T_{ii})\|x_i\|^2 + \sum_{i,j=1; i\neq j}^n \|T_{ij}\|\|x_j\|\|x_i\| \\
&= \sum_{i,j=1}^n \tilde{t_{ij}}\|x_j\|\|x_i\| = \langle \tilde{T}\tilde{x}, \tilde{x}\rangle = \langle Re(\tilde{T})\tilde{x}, \tilde{x}\rangle + i\langle Im(\tilde{T})\tilde{x}, \tilde{x}\rangle,
\end{aligned}$$

where $\tilde{T} = (\tilde{t_{ij}})$, $\tilde{t_{ij}} = \begin{cases} w(T_{ij}) & \text{if } i = j \\ \|T_{ij}\| & \text{if } i \neq j. \end{cases}$

Clearly, $\langle Im(\tilde{T})\tilde{x}, \tilde{x}\rangle = 0$. So by using Lemma 2 and Lemma 3, we get

$$|\langle Tx, x\rangle| \leq \langle Re(\tilde{T})\tilde{x}, \tilde{x}\rangle \leq \langle |Re(\tilde{T})|\tilde{x}, \tilde{x}\rangle$$

$$\Rightarrow |\langle Tx,x\rangle|^2 \leq \langle |Re(\tilde{T})|\tilde{x},\tilde{x}\rangle^2 \leq \langle |Re(\tilde{T})|^2\tilde{x},\tilde{x}\rangle = \langle |R|^2\tilde{x},\tilde{x}\rangle.$$

Also,

$$\begin{aligned}\|Tx\|^2 = |\langle Tx,Tx\rangle| &= \left|\sum_{i,j,k=1}^{n} \langle T_{kj}x_j, T_{ki}x_i\rangle\right| \\ &\leq \sum_{i,j,k=1}^{n} |\langle T_{kj}x_j, T_{ki}x_i\rangle| \leq \sum_{i,j,k=1}^{n} |\langle T_{ki}^*T_{kj}x_j, x_i\rangle| \\ &\leq \sum_{i,j,k=1}^{n} \|T_{ki}\|\|T_{kj}\|\|x_j\|\|x_i\| = \langle |S|^2\tilde{x},\tilde{x}\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}\alpha|\langle Tx,x\rangle|^2 + (1-\alpha)\|Tx\|^2 &\leq \alpha\langle |R|^2\tilde{x},\tilde{x}\rangle + (1-\alpha)\langle |S|^2\tilde{x},\tilde{x}\rangle \\ &= \langle \left(\alpha|R|^2 + (1-\alpha)|S|^2\right)\tilde{x},\tilde{x}\rangle \leq \left\|\alpha|R|^2 + (1-\alpha)|S|^2\right\|.\end{aligned}$$

Taking supremum over all unit vectors in $\oplus_{i=1}^{n}\mathcal{H}_i$, we get the desired inequality. □

As a consequence of Theorem 3, the following numerical radius inequality and the usual operator norm inequality can be proved quite easily.

Corollary 1. *Let $\mathcal{H}_1,\mathcal{H}_2,\ldots,\mathcal{H}_n$ be Hilbert spaces. Let $T=(T_{ij})$ be an $n\times n$ operator matrix, where $T_{ij}\in\mathcal{B}(\mathcal{H}_j,\mathcal{H}_i)$. Then*

$$(i)\ \ w(T) \leq \min_{0\leq\alpha\leq 1}\sqrt{\|\alpha|R|^2+(1-\alpha)|S|^2\|} \leq w(\tilde{T}),$$

$$(ii)\ \ \|T\| \leq \min_{0\leq\alpha\leq 1}\frac{1}{\max\left\{\frac{1}{2},\sqrt{1-\alpha}\right\}}\sqrt{\|\alpha|R|^2+(1-\alpha)|S|^2\|} \leq \|S\|,$$

where $\tilde{T}=(\tilde{t_{ij}})_{n\times n}$, $\tilde{t_{ij}} = \begin{cases} w(T_{ij}) & \text{if } i=j \\ \|T_{ij}\| & \text{if } i\neq j\end{cases}$ and R,S are same as described in Theorem 3.

We would like to note that the inequalities in [1, Th. 1] and [12, Th. 1.1] follow from (i) and (ii) of Corollary 1, respectively.

In our next result, we obtain an upper bound for the α-norm of $n\times n$ operator matrices in terms of non-negative continuous functions on $[0,\infty)$. First we need the following lemma.

Lemma 5. *([13, Th. 5]) Let $T\in\mathcal{B}(\mathcal{H})$ and let f and g be two non-negative continuous functions on $[0,\infty)$ such that $f(t)g(t)=t,\ \forall\ t\in[0,\infty)$. Then*

$$|\langle Tx,y\rangle| \leq \|f(|T|)x\|\|g(|T^*|)y\|,\ \ \forall\ x,y\in\mathcal{H}.$$

Theorem 4. *Let $T=(T_{ij})$ be an $n\times n$ operator matrix, where $T_{ij}\in\mathcal{B}(\mathcal{H})$. Let f and g be two non-negative continuous functions on $[0,\infty)$ such that $f(t)g(t)=t$, $\forall\, t\geq 0$. Then*

$$\|T\|_\alpha \leq \sqrt{\|\alpha|R|^2+(1-\alpha)|S|^2\|},$$

where $R=(r_{ij})_{n\times n}$, $r_{ij}=\frac{1}{2}\left(\|f^2(|T_{ij}|)\|^{\frac{1}{2}}\|g^2(|T_{ij}^*|)\|^{\frac{1}{2}}+\|f^2(|T_{ji}|)\|^{\frac{1}{2}}\|g^2(|T_{ji}^*|)\|^{\frac{1}{2}}\right)$ and $S=(s_{ij})_{n\times n}$, $s_{ij}=\|T_{ij}\|$.

Proof. Let $x=(x_1,x_2,\ldots,x_n)\in\oplus_{i=1}^n\mathcal{H}$ with $\|x\|=1$ and let $\tilde{x}=(\|x_1\|,\|x_2\|,\ldots,\|x_n\|)$. Clearly, $\tilde{x}$ is a unit vector in $\mathbb{C}^n$. Using Lemma 5, we get that

$$\begin{aligned}
|\langle Tx,x\rangle| &= \left|\sum_{i,j=1}^n\langle T_{ij}x_j,x_i\rangle\right| \leq \sum_{i,j=1}^n|\langle T_{ij}x_j,x_i\rangle| \\
&\leq \sum_{i,j=1}^n\|f(|T_{ij}|)x_j\|\|g(|T_{ij}^*|)x_i\| = \sum_{i,j=1}^n\langle f^2(|T_{ij}|)x_j,x_j\rangle^{\frac{1}{2}}\langle g^2(|T_{ij}^*|)x_i,x_i\rangle^{\frac{1}{2}} \\
&\leq \sum_{i,j=1}^n\|f^2(|T_{ij}|)\|^{\frac{1}{2}}\|g^2(|T_{ij}^*|)\|^{\frac{1}{2}}\|x_i\|\|x_j\| = \sum_{i,j=1}^n\tilde{t_{ij}}\|x_j\|\|x_i\| \\
&= \langle\tilde{T}\tilde{x},\tilde{x}\rangle = \langle Re(\tilde{T})\tilde{x},\tilde{x}\rangle + i\langle Im(\tilde{T})\tilde{x},\tilde{x}\rangle,
\end{aligned}$$

where $\tilde{T}=(\tilde{t_{ij}})$, $\tilde{t_{ij}}=\|f^2(|T_{ij}|)\|^{\frac{1}{2}}\|g^2(|T_{ij}^*|)\|^{\frac{1}{2}}$.

Proceeding similarly as in the proof of Theorem 3, we get

$$|\langle Tx,x\rangle|^2\leq\langle|R|^2\tilde{x},\tilde{x}\rangle \quad\text{and}\quad \|Tx\|^2\leq\langle|S|^2\tilde{x},\tilde{x}\rangle.$$

Therefore,

$$\alpha|\langle Tx,x\rangle|^2+(1-\alpha)\|Tx\|^2\leq\left\|\alpha|R|^2+(1-\alpha)|S|^2\right\|.$$

Taking supremum over all unit vectors in $\oplus_{i=1}^n\mathcal{H}$, we get the desired inequality. □

The following numerical radius inequality is an easy consequence of Theorem 4 .

Corollary 2. *Let $T=(T_{ij})$ be an $n\times n$ operator matrix, where $T_{ij}\in\mathcal{B}(\mathcal{H})$. Let f and g be non-negative continuous functions on $[0,\infty)$ such that $f(t)g(t)=t$, $\forall\, t\geq 0$. Then*

$$w(T)\leq\min_{0\leq\alpha\leq 1}\sqrt{\|\alpha|R|^2+(1-\alpha)|S|^2\|}\leq w(\tilde{T}),$$

where $\tilde{T}=(\tilde{t_{ij}})_{n\times n}$, $\tilde{t_{ij}}=\|f^2(|T_{ij}|)\|^{\frac{1}{2}}\|g^2(|T_{ij}^*|)\|^{\frac{1}{2}}$ and R,S are same as described in Theorem 4.

We would like to note that the inequality in [7, Th. 3.1] follows from Corollary 2.

In our next theorem, we obtain an upper bound for the α-norm of $n\times n$ operator matrices.

Theorem 5. *Let $T=(T_{ij})$ be an $n\times n$ operator matrix, where $T_{ij}\in\mathcal{B}(\mathcal{H})$. Let f and g be two non-negative continuous functions on $[0,\infty)$ such that $f(t)g(t)=t$, $\forall\, t\geq 0$. If $p\geq 1$, then*

$$\|T\|_{\alpha}^{p}\leq\sqrt{\|\alpha|R|^{2p}+(1-\alpha)|S|^{2p}\|},$$

where $R=(r_{ij})_{n\times n}$,

$$r_{ij}=\begin{cases}\frac{1}{2}\left\|f^2(|T_{ii}|)+g^2(|T_{ii}^*|)\right\| & \text{if } i=j\\ \frac{1}{2}\left(\|f^2(|T_{ij}|)\|^{\frac{1}{2}}\|g^2(|T_{ij}^*|)\|^{\frac{1}{2}}+\|f^2(|T_{ji}|)\|^{\frac{1}{2}}\|g^2(|T_{ji}^*|)\|^{\frac{1}{2}}\right) & \text{if } i\neq j\end{cases}$$

and $S=(s_{ij})_{n\times n}$, $s_{ij}=\|T_{ij}\|$.

Proof. Let $x=(x_1,x_2,\ldots,x_n)\in\oplus_{i=1}^{n}\mathcal{H}$ with $\|x\|=1$ and let $\tilde{x}=(\|x_1\|,\|x_2\|,\ldots,\|x_n\|)$. Clearly, $\tilde{x}$ is a unit vector in $\mathbb{C}^n$. Using Lemma 5, we get that

$$\begin{aligned}
|\langle Tx,x\rangle| &= \left|\sum_{i,j=1}^{n}\langle T_{ij}x_j,x_i\rangle\right|\leq\sum_{i,j=1}^{n}|\langle T_{ij}x_j,x_i\rangle|\\
&\leq\sum_{i,j=1}^{n}\|f(|T_{ij}|)x_j\|\|g(|T_{ij}^*|)x_i\|=\sum_{i,j=1}^{n}\langle f^2(|T_{ij}|)x_j,x_j\rangle^{\frac{1}{2}}\langle g^2(|T_{ij}^*|)x_i,x_i\rangle^{\frac{1}{2}}\\
&\leq\sum_{i=1}^{n}\frac{1}{2}\left(\langle f^2(|T_{ii}|)x_i,x_i\rangle+\langle g^2(|T_{ii}^*|)x_i,x_i\rangle\right)\\
&\quad+\sum_{i,j=1,i\neq j}^{n}\langle f^2(|T_{ij}|)x_j,x_j\rangle^{\frac{1}{2}}\langle g^2(|T_{ij}^*|)x_i,x_i\rangle^{\frac{1}{2}}\\
&=\sum_{i=1}^{n}\frac{1}{2}\langle\left(f^2(|T_{ii}|)+g^2(|T_{ii}^*|)\right)x_i,x_i\rangle\\
&\quad+\sum_{i,j=1,i\neq j}^{n}\langle f^2(|T_{ij}|)x_j,x_j\rangle^{\frac{1}{2}}\langle g^2(|T_{ij}^*|)x_i,x_i\rangle^{\frac{1}{2}}\\
&\leq\sum_{i=1}^{n}\frac{1}{2}\left\|f^2(|T_{ii}|)+g^2(|T_{ii}^*|)\right\|\|x_i\|^2\\
&\quad+\sum_{i,j=1,i\neq j}^{n}\|f^2(|T_{ij}|)\|^{\frac{1}{2}}\|g^2(|T_{ij}^*|)\|^{\frac{1}{2}}\|x_i\|\|x_j\|\\
&=\sum_{i,j=1}^{n}\tilde{t_{ij}}\|x_j\|\|x_i\|=\langle\tilde{T}\tilde{x},\tilde{x}\rangle=\langle Re(\tilde{T})\tilde{x},\tilde{x}\rangle+i\langle Im(\tilde{T})\tilde{x},\tilde{x}\rangle,
\end{aligned}$$

where $\tilde{T}=(\tilde{t_{ij}})_{n\times n}$,

$$\tilde{t_{ij}}=\begin{cases}\frac{1}{2}\left\|f^2(|T_{ii}|)+g^2(|T_{ii}^*|)\right\| & \text{if } i=j\\ \|f^2(|T_{ij}|)\|^{\frac{1}{2}}\|g^2(|T_{ij}^*|)\|^{\frac{1}{2}} & \text{if } i\neq j.\end{cases}$$

Clearly, $\langle Im(\tilde{T})\tilde{x},\tilde{x}\rangle = 0$, and so using Lemma 2 and Lemma 3, we get that

$$\begin{aligned}|\langle Tx,x\rangle| \leq \langle Re(\tilde{T})\tilde{x},\tilde{x}\rangle &\Rightarrow |\langle Tx,x\rangle| \leq \langle |Re(\tilde{T})|\tilde{x},\tilde{x}\rangle \\ &\Rightarrow |\langle Tx,x\rangle|^{2p} \leq \langle |Re(\tilde{T})|\tilde{x},\tilde{x}\rangle^{2p} \Rightarrow |\langle Tx,x\rangle|^{2p} \leq \langle |Re(\tilde{T})|^{2p}\tilde{x},\tilde{x}\rangle \\ &\Rightarrow |\langle Tx,x\rangle|^{2p} \leq \langle |R|^{2p}\tilde{x},\tilde{x}\rangle.\end{aligned}$$

Now proceeding similarly as in the proof of Theorem 3 and using Lemma 3, we obtain

$$\|Tx\|^{2p} \leq \langle |S|^2\tilde{x},\tilde{x}\rangle^p \leq \langle |S|^{2p}\tilde{x},\tilde{x}\rangle.$$

By convexity of $t^p, p \geq 1$, it follows that

$$\begin{aligned}\left(\alpha|\langle Tx,x\rangle|^2 + (1-\alpha)\|Tx\|^2\right)^p &\leq \left(\alpha|\langle Tx,x\rangle|^{2p} + (1-\alpha)\|Tx\|^{2p}\right) \\ &\leq \left(\alpha\langle |R|^{2p}\tilde{x},\tilde{x}\rangle + (1-\alpha)\langle |S|^{2p}\tilde{x},\tilde{x}\rangle\right) \\ &= \langle \left(\alpha|R|^{2p} + (1-\alpha)|S|^{2p}\right)\tilde{x},\tilde{x}\rangle \\ &\leq \left\|\alpha|R|^{2p} + (1-\alpha)|S|^{2p}\right\|.\end{aligned}$$

Therefore, taking supremum over all unit vectors in $\oplus_{i=1}^{n}\mathcal{H}$, we get the desired inequality. □

We simply state the following result and omit its proof, as it can be completed using similar arguments as given in the proof of Theorem 5.

Theorem 6. *Let $T = (T_{ij})$ be an $n \times n$ operator matrix, where $T_{ij} \in \mathcal{B}(\mathcal{H})$. Let f and g be two non-negative continuous functions on $[0,\infty)$ such that $f(t)g(t) = t$, $\forall\, t \geq 0$. Then*

$$\|T\|_\alpha \leq \sqrt{\|\alpha|R|^2 + (1-\alpha)|S|^2\|},$$

where $R = (r_{ij})_{n\times n}$,

$$r_{ij} = \begin{cases} \frac{1}{2}\left\|f^2(|T_{ii}|) + g^2(|T_{ii}^*|)\right\| & \text{if } i = j \\ \frac{1}{2}\left(\|f^2(|T_{ij}|)\|^{\frac{1}{2}}\|g^2(|T_{ij}^*|)\|^{\frac{1}{2}} + \|f^2(|T_{ji}|)\|^{\frac{1}{2}}\|g^2(|T_{ji}^*|)\|^{\frac{1}{2}}\right) & \text{if } i \neq j \end{cases}$$

and $S = (s_{ij})_{n\times n}$, $s_{ij} = \|T_{ij}\|$.

The following numerical radius inequality follows easily from Theorem 6 by using Lemma 4.

Corollary 3. *Let $T = (T_{ij})$ be an $n \times n$ operator matrix, where $T_{ij} \in \mathcal{B}(\mathcal{H})$. Let f and g be two non-negative continuous functions on $[0,\infty)$ such that $f(t)g(t) = t$, $\forall\, t \geq 0$. Then*

$$w(T) \leq \min_{0\leq\alpha\leq 1} \sqrt{\|\alpha|R|^2 + (1-\alpha)|S|^2\|},$$

where R, S are same as described in Theorem 6.

Remark 2. In particular, if we consider $\alpha = 1$ in Corollary 3 then using Lemma 1, we get

$$w(T) \leq \min_{0\leq\alpha\leq 1} \sqrt{\|\alpha|R|^2 + (1-\alpha)|S|^2\|} \leq w(\tilde{T}),$$

where $\tilde{T} = (\tilde{t_{ij}})_{n\times n}$, $\tilde{t_{ij}} = \begin{cases} \frac{1}{2}\left\|f^2(|T_{ii}|) + g^2(|T_{ii}^*|)\right\| & \text{if } i = j \\ \|f^2(|T_{ij}|)\|^{\frac{1}{2}}\|g^2(|T_{ij}^*|)\|^{\frac{1}{2}} & \text{if } i \neq j. \end{cases}$ Note that the existing inequality in [7, Th. 3.3] follows from Corollary 3.

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RAD$-\oplus-$SUPPLEMENTED LATTICES

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Abstract. In this work, we define Rad$-\oplus-$supplemented and strongly Rad$-\oplus-$supplemented lattices and give some properties of these lattices. We generalize some properties of Rad$-\oplus-$supplemented modules to lattices. Let L be a lattice and $1 = a_1 \oplus a_2 \oplus \ldots \oplus a_n$ with $a_1, a_2, \ldots, a_n \in L$. If $a_i/0$ is Rad$-\oplus-$ supplemented for every $i = 1, 2, \ldots, n$, then L is also Rad$-\oplus-$ supplemented. Let L be a distributive Rad$-\oplus-$supplemented lattice. Then $1/u$ is Rad$-\oplus-$supplemented for every $u \in L$. We also define completely Rad$-\oplus-$supplemented lattices and prove that every Rad$-\oplus-$supplemented lattice with SSP property is completely Rad$-\oplus-$ supplemented.

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Keywords: lattices, radical, supplemented Lattices, generalized (radical) supplemented lattices

1. Introduction

In this paper, every lattice is complete modular lattice with the smallest element 0 and the greatest element 1. Let L be a lattice, $x, y \in L$ and $x \leq y$. A sublattice $\{a \in L | x \leq a \leq y\}$ is called a *quotient sublattice* and denoted by y/x. An element y of a lattice L is called a *complement* of x in L if $x \wedge y = 0$ and $x \vee y = 1$, this case we denote $1 = x \oplus y$ (in this case we call x and y are *direct summands* of L). L is said to be *complemented* if each element has at least one complement in L. An element x of L is said to be *small* or *superfluous* and denoted by $x \ll L$ if $y = 1$ for every $y \in L$ such that $x \vee y = 1$. The meet of all the maximal $(\neq 1)$ elements of a lattice L is called the *radical* of L and denoted by $r(L)$. An element a of L is called a *supplement* of b in L if it is minimal for $a \vee b = 1$. a is a supplement of b in a lattice L if and only if $a \vee b = 1$ and $a \wedge b \ll a/0$. A lattice L is called a *supplemented lattice* if every element of L has a supplement in L. If every element of L has a supplement that is a direct summand in L, then L is called a $\oplus-$*supplemented lattice*. We say that an element y of L *lies above* an element x of L if $x \leq y$ and $y \ll 1/x$. L is said to be *hollow* if every element distinct from 1 is superfluous in L, and L is said to be *local* if L has the greatest element $(\neq 1)$. An element $x \in L$ has *ample supplements* in L if for every $y \in L$ with $x \vee y = 1$, x has a supplement z in L with $z \leq y$. L is said to be *amply supplemented*, if every element of L has ample supplements in L. It is clear that every amply supplemented lattice is supplemented. A lattice L is said to

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be *distributive* if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for every $x, y, z \in L$. An element y of a lattice L is called a *generalized* (*radical*) *supplement* (or briefly, *Rad-supplement*) of x in L if $1 = x \vee y$ and $x \wedge y \leq r(y/0)$. A lattice L is said to be *generalized* (*radical*) *supplemented* (or briefly, *Rad-supplemented*) if every element of L has a generalized (radical) supplement in L.

Let L be a lattice. Consider the following conditions.

(D1) For every element x of L, there exist $x_1, x_2 \in L$ such that $1 = x_1 \oplus x_2$, $x_1 \leq x$ and $x_2 \wedge x \ll x_2/0$.

(D3) If x_1 and x_2 are direct summands of L and $1 = x_1 \vee x_2$, then $x_1 \wedge x_2$ is also a direct summand of L.

More informations about (amply) supplemented lattices are in [1,2,8]. The definition of $\oplus-$supplemented lattices and some informations about these lattices are in [5]. More results about (amply) supplemented modules are in [11]. The definition of generalized supplemented lattices and some important properties of them are in [6]. Some important properties of Rad$-\oplus-$supplemented modules are in [4,7,10]. The definition of β_* relation on lattices and some properties of this relation are in [9]. The definition of β^* relation on modules and some properties of this relation are in [3].

Lemma 1. *Let L be a lattice, $a, b \in L$ and a be a Rad-supplement of b in L. Then $r(a/0) = a \wedge r(L)$.*

Proof. See [6, Lemma 2(b)]. □

Lemma 2. *Let L be a lattice and y be a Rad-supplement of x in L. Then for $a \leq x$, $a \vee y$ is a Rad-supplement of x in $1/a$.*

Proof. See [6, Lemma 5]. □

Lemma 3. *Let L be a lattice and $a, b \in L$. If x is a Rad-supplement of $a \vee b$ in L and y is a Rad-supplement of $a \wedge (b \vee x)$ in $a/0$, then $x \vee y$ is a Rad-supplement of b in L.*

Proof. See [6], the proof of Lemma 7. □

2. Rad$-\oplus-$Supplemented Lattices

Definition 1. Let L be a lattice. If every element of L has a Rad-supplement that is a direct summand in L, then L is called a Rad$-\oplus-$ supplemented (or generalized $\oplus-$ supplemented) lattice.

It is clear that every Rad$-\oplus-$supplemented lattice is Rad-supplemented, but the converse is not true in general (see Example 1 and Example 2). It is also clear that every $\oplus-$supplemented lattice is Rad$-\oplus-$ supplemented, but the converse is not true in general (See Example 3). Hence Rad$-\oplus-$supplemented lattices are more general than $\oplus-$ supplemented lattices. Hollow and local lattices are Rad$-\oplus-$supplemented.

Lemma 4. *Let L be a lattice, $a_1, a_2 \in L$ and $1 = a_1 \oplus a_2$. If $a_1/0$ and $a_2/0$ are Rad$-\oplus-$supplemented, then L is also Rad$-\oplus-$supplemented.*

Proof. Let x be any element of L. Then 0 is a Rad-supplement of $a_1 \vee a_2 \vee x$ in L. Since $a_1/0$ is Rad$-\oplus-$supplemented, $a_1 \wedge (a_2 \vee x)$ has a Rad-supplement y that is a direct summand in $a_1/0$. Then by Lemma 3, $y = y \vee 0$ is a Rad-supplement of $a_2 \vee x$ in L. Since $a_2/0$ is Rad$-\oplus-$ supplemented, $a_2 \wedge (x \vee y)$ has a Rad-supplement z that is a direct summand in $a_2/0$. Then by Lemma 3, $y \vee z$ is a Rad-supplement of x in L. Since y is a direct summand of $a_1/0$ and z is a direct summand of $a_2/0$ and $1 = a_1 \oplus a_2$, $y \vee z = y \oplus z$ is a direct summand of L. Hence L is Rad$-\oplus-$supplemented. □

Corollary 1. *Let $a_1, a_2, \ldots, a_n \in L$ and $1 = a_1 \oplus a_2 \oplus \ldots \oplus a_n$. If $a_i/0$ is Rad$-\oplus-$supplemented for every $i = 1, 2, \ldots, n$, then L is also Rad$-\oplus-$supplemented.*

Proof. Clear form Lemma 4. □

Lemma 5. *Let L be a Rad$-\oplus-$supplemented lattice, $u \in L$ and $u = (u \wedge a) \vee (u \wedge b)$ for every $a, b \in L$ with $1 = a \oplus b$. Then;*

(i) *$1/u$ is Rad$-\oplus-$supplemented.*
(ii) *If u is a direct summand of L, then $u/0$ is also Rad $-\oplus-$supplemented.*

Proof.

(i) Let $x \in 1/u$. Since L is Rad$-\oplus-$supplemented, x has a Rad-supplement y that is a direct summand in L. By Lemma 2, $y \vee u$ is a Rad-supplement of x in $1/u$. Since y is a direct summand of L, there exists $z \in L$ such that $1 = y \oplus z$. Here $1 = (y \vee u) \vee (z \vee u)$. Since $u = (u \wedge y) \vee (u \wedge z)$, $(y \vee u) \wedge (z \vee u) = (y \vee (u \wedge y) \vee (u \wedge z)) \wedge (z \vee (u \wedge y) \vee (u \wedge z)) = (y \vee (u \wedge z)) \wedge (z \vee (u \wedge y)) = (y \wedge (z \vee (u \wedge y))) \vee (u \wedge z) = (y \wedge z) \vee (u \wedge y) \vee (u \wedge z) = 0 \vee (u \wedge y) \vee (u \wedge z) = (u \wedge y) \vee (u \wedge z) = u$. Hence $1/u$ is Rad $-\oplus-$supplemented.
(ii) Let u be a direct summand of L and $x \in u/0$. Since L is Rad$-\oplus-$supplemented, there exist $y, z \in L$ such that $1 = x \vee y$, $x \wedge y \leq r(y/0) \leq r(L)$ and $1 = y \oplus z$. By hypothesis $u = (u \wedge y) \oplus (u \wedge z)$. Since u is a direct summand of L, $u \wedge y$ is also a direct summand of L. Since $1 = x \vee y$ and $x \leq u$, by modularity, $u = x \vee (u \wedge y)$. Since $u \wedge y$ is a direct summand of L, by Lemma 1, $r((u \wedge y)/0) = u \wedge y \wedge r(L)$. Since $x \wedge u \wedge y = x \wedge y \leq r(L)$ and $x \wedge u \wedge y \leq u \wedge y$, $x \wedge u \wedge y \leq u \wedge y \wedge r(L) = r((u \wedge y)/0)$. Hence $u/0$ is Rad$-\oplus-$supplemented.

□

Corollary 2. *Let L be a distributive and Rad$-\oplus-$supplemented lattice. Then $1/u$ Rad$-\oplus-$supplemented for every $u \in L$.*

Proof. Clear from Lemma 5. □

Lemma 6. *Let L be a Rad$-\oplus-$supplemented lattice with $(D3)$ property. Then for every direct summand u of L, $u/0$ is Rad$-\oplus-$supplemented.*

Proof. Let u be a direct summand of L and $x \in u/0$. Since L is Rad$-\oplus-$ supplemented, there exists a direct summand y of L such that $1 = x \vee y$ and $x \wedge y \leq r(y/0)$. Since $u \vee y = 1$ and L has $(D3)$ property, $u \wedge y$ is a direct summand of L. Hence $u \wedge y$ is a direct summand of $u/0$. By modularity, $u = x \vee (u \wedge y)$. Since $x \wedge u \wedge y = x \wedge y \leq r(y/0) \leq r(L)$ and $x \wedge u \wedge y \leq u \wedge y$, $x \wedge u \wedge y \leq u \wedge y \wedge r(L)$. Since $u \wedge y$ is a direct summand of L, by Lemma 1, $u \wedge y \wedge r(L) = r((u \wedge y)/0)$. Therefore, $u/0$ is Rad$-\oplus-$supplemented. □

Proposition 1. *Let $1 = a \oplus b$ with $a, b \in L$. Then $b/0$ is Rad$-\oplus-$supplemented if and only if for every $x \in 1/a$, there exists a direct summand y of L such that $y \in b/0$, $1 = x \vee y$ and $x \wedge y \leq r(L)$.*

Proof. $(\Longrightarrow)$ Let $x \in 1/a$. Then $x \wedge b \in b/0$ and since $b/0$ is Rad$-\oplus-$ supplemented, $x \wedge b$ has a Rad-supplement y that is a direct summand in $b/0$. Here $b = (x \wedge b) \vee y$ and $x \wedge y = x \wedge b \wedge y \leq r(y/0) \leq r(L)$. Since y is a direct summand of $b/0$, there exists $z \in b/0$ such that $b = y \oplus z$. Then $1 = a \oplus b = a \oplus y \oplus z$ and y is a direct summand of L. Since $a \leq x$ and $b = (x \wedge b) \vee y$, $1 = a \vee b = x \vee b = x \vee (x \wedge b) \vee y = x \vee y$.

$(\Longleftarrow)$ Let $x \in b/0$. Then $a \vee x \in 1/a$ and by hypothesis, there exists a direct summand y of L such that $y \in b/0$, $1 = a \vee x \vee y$ and $(a \vee x) \wedge y \leq r(L)$. Then we have $b = b \wedge 1 = b \wedge (a \vee x \vee y) = (a \wedge b) \vee x \vee y = x \vee y$ and $x \wedge y \leq (a \vee x) \wedge y \leq r(L)$. Since y is a direct summand of L, there exists $z \in L$ with $1 = y \oplus z$. Here $b = b \wedge 1 = b \wedge (y \oplus z) = y \oplus (b \wedge z)$ and y is a direct summand of $b/0$. By Lemma 1, $r(y/0) = y \wedge r(L)$. Since $x \wedge y \leq r(L)$ and $x \wedge y \leq y$, $x \wedge y \leq y \wedge r(L) = r(y/0)$. Hence y is a Rad-supplement of x in $b/0$ and $b/0$ is Rad$-\oplus-$ supplemented. □

Proposition 2. *Let L be a Rad$-\oplus-$supplemented lattice, a be a direct summand of L and for every direct summand t of L with $1 = t \vee a$, $t \wedge a$ be a direct summand of $a/0$. Then $a/0$ is Rad$-\oplus-$supplemented.*

Proof. Since a is a direct summand of L, there exists $b \in L$ with $1 = a \oplus b$. Let $x \in a/0$. Since L is Rad$-\oplus-$supplemented, there exist $y, z \in L$ such that $1 = x \vee y$, $x \wedge y \leq r(y/0)$ and $1 = y \oplus z$. By $x \leq a$, $1 = x \vee y = a \vee y$. By hypothesis, $a \wedge y$ is a direct summand of $a/0$ and since a is a direct summand of L, $a \wedge y$ is a direct summand of L. By Lemma 1, $r((a \wedge y)/0) = a \wedge y \wedge r(L)$. Here $x \wedge y \leq y \wedge r(L)$ and $x \wedge a \wedge y \leq a \wedge y \wedge r(L) = r((a \wedge y)/0)$. Since $1 = x \vee y$ and $x \leq a$, $a = a \wedge 1 = a \wedge (x \vee y) = x \vee (a \wedge y)$. Hence $a/0$ is Rad$-\oplus-$supplemented. □

Let $x, y \in L$. It is defined a relation β_* on the elements of L by $x\beta_* y$ if and only if for every $t \in L$ with $x \vee t = 1$ then $y \vee t = 1$ and for every $k \in L$ with $y \vee k = 1$ then $x \vee k = 1$. (See [9, Definition 1])

Lemma 7. *Let L be a Rad-supplemented lattice. If every Rad-supplement element in L is β_* equivalent to a direct summand of L, then L is Rad$-\oplus-$supplemented.*

Proof. Let x be any element of L and y be a Rad-supplement of x in L. By hypothesis, there exists a direct summand a of L such that $y\beta_* a$. Since $x \vee y = 1$, $x \vee a = 1$.

Assume $x\wedge a\nleq r(L)$. Then there exists a maximal $(\neq 1)$ element t of L with $x\wedge a\nleq t$. Here $(x\wedge a)\vee t=1$. By [9, Lemma 2], $a\vee(x\wedge t)=1$ and since $y\beta_* a$, $y\vee(x\wedge t)=1$. Since $x\vee t=1$, by [9, Lemma 2], $(x\wedge y)\vee t=1$. Since $x\wedge y\leq r(y/0)\leq r(L)\leq t$, $t=(x\wedge y)\vee t=1$. This contradicts with $t\neq 1$. Hence $x\wedge a\leq r(L)$. Since a is a direct summand of L, by Lemma 1, $x\wedge a\leq a\wedge r(L)=r(a/0)$. Hence a is a Rad-supplement of x in L and L is Rad $-\oplus-$ supplemented. □

Corollary 3. *Let L be a Rad-supplemented lattice. If every Rad-supplement element in L lies above a direct summand of L, then L is Rad$-\oplus-$supplemented.*

Proof. Clear from Lemma 7. □

Definition 2. Let L be a lattice. If $a/0$ is Rad$-\oplus-$supplemented for every direct summand a of L, then L is called a completely Rad$-\oplus-$ supplemented lattice.

Clearly we can see that every completely Rad$-\oplus-$supplemented lattice is Rad$-\oplus-$supplemented.

Proposition 3. *Let L be a Rad$-\oplus-$supplemented lattice with $(D3)$ property. Then L is completely Rad$-\oplus-$supplemented.*

Proof. Clear from Lemma 6. □

Definition 3. Let L be a lattice. L is said to have SSP property if $a\vee b$ is a direct summand for every direct summands a and b of L.

Proposition 4. *Let L be a Rad$-\oplus-$supplemented lattice with SSP property. Then L is completely Rad$-\oplus-$supplemented.*

Proof. Let a be a direct summand of L. Then there exists $b\in L$ such that $1=a\oplus b$. let $x\in 1/b$. Since L is Rad$-\oplus-$supplemented, there exists a direct summand y of L such that $x\vee y=1$ and $x\wedge y\leq r(y/0)$. Here $b\vee y$ is a Rad-supplement of x in $1/b$, by Lemma 2. Since b and y are direct summands of L and L has SSP property, $b\vee y$ is a direct summand of L and there exists $z\in L$ such that $1=(b\vee y)\oplus z$. Here $1=(b\vee y)\vee(b\vee z)$ and $(b\vee y)\wedge(b\vee z)=b\vee((b\vee y)\wedge z)=b\vee 0=b$ and $b\vee y$ is a direct summand of $1/b$. Hence $1/b$ is Rad$-\oplus-$ supplemented and since $\frac{a}{0}=\frac{a}{a\wedge b}\cong\frac{a\vee b}{b}=\frac{1}{b}$, $a/0$ also Rad$-\oplus-$supplemented. □

Definition 4. Let L be a Rad-supplemented lattice. If every Rad-supplement element in L is a direct summand of L, then L is called a strongly Rad$-\oplus-$ supplemented lattice.

It is clear that every strongly $\oplus-$supplemented lattice is Rad$-\oplus-$ supplemented. Since every lattice with $(D1)$ property is strongly $\oplus-$supplemented, these lattices are Rad $-\oplus-$supplemented too. Every strongly Rad$-\oplus-$supplemented lattice is Rad$-\oplus-$supplemented, but the converse is not true in general (See Example 3).

Lemma 8. *Let $1=a\oplus b$ in L and $x,y\in b/0$. Then y is a Rad-supplement of x in $b/0$ if and only if y is a Rad-supplement of $a\vee x$ in L.*

Proof. ($\Longrightarrow$) Since y is a Rad-supplement of x in $b/0$, $b = x \vee y$ and $x \wedge y \leq r(y/0)$. Then $1 = a \oplus b = a \vee x \vee y$ and $(a \vee x) \wedge y = (a \vee x) \wedge b \wedge y = ((a \wedge b) \vee x) \wedge y = x \wedge y \leq r(y/0)$. Hence y is a Rad-supplement of $a \vee x$ in L.

($\Longleftarrow$) Since y is a Rad-supplement of $a \vee x$ in L, $1 = a \vee x \vee y$ and $(a \vee x) \wedge y \leq r(y/0)$. Then $b = 1 \wedge b = (a \vee x \vee y) \wedge b = (a \wedge b) \vee x \vee y = x \vee y$ and $x \wedge y \leq (a \vee x) \wedge y \leq r(y/0)$. Hence y is a Rad-supplement of x in $b/0$. □

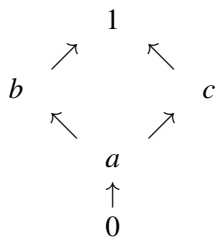
Lemma 9. *Let L be a strongly Rad$-\oplus-$supplemented lattice. Then $a/0$ is strongly Rad$-\oplus-$supplemented for every direct summand a of L.*

Proof. Let a be a direct summand of L and $1 = a \oplus b$ with $b \in L$. Let y be a Rad-supplement of x in $a/0$. By Lemma 8, y is a Rad-supplement of $b \vee x$ in L. Since L is strongly Rad$-\oplus-$ supplemented, y is a direct summand of L. By this, there exists $z \in L$ with $1 = y \oplus z$. By modularity, $a = a \wedge 1 = a \wedge (y \oplus z) = y \oplus (a \wedge z)$. Hence y is a direct summand of $a/0$ and $a/0$ is strongly Rad$-\oplus-$supplemented. □

Corollary 4. *Every strongly Rad$-\oplus-$supplemented lattice is completely Rad$-\oplus-$supplemented.*

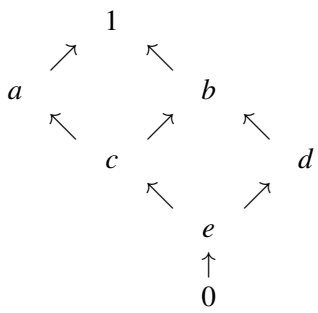
Proof. Clear from Lemma 9. □

Example 1. Consider the lattice $L = \{1, a, b, c, 0\}$ given by the following diagram;



Then L is Rad-supplemented but not Rad$-\oplus-$supplemented.

Example 2. Consider the lattice $L = \{1, a, b, c, d, e, 0\}$ given by the following diagram;



Then L is Rad-supplemented but not Rad$-\oplus-$supplemented.

Example 3. Consider the interval $[0,1]$ with natural topology. Let P be the set of all closed subsets of $[0,1]$. P is complete modular lattice by the inclusion (See[1, Example 2.10]). Here $\underset{i\in I}{\wedge} C_i = \underset{i\in I}{\cap} C_i$ and $\underset{i\in I}{\vee} C_i = \overline{\underset{i\in I}{\cup} C_i}$ for every $C_i \in P\,(i \in I)$ $\left(\overline{\underset{i\in I}{\cup} C_i}\right.$ is the closure of $\left.\underset{i\in I}{\cup} C_i\right)$. By [5, Example 3], P is amply supplemented but not $\oplus$ -supplemented. Since P is amply supplemented, then it is Rad-supplemented too. Since P is not $\oplus-$supplemented, it is not strongly Rad$-\oplus-$supplemented too. It is clear that $r(P) = [0,1]$ and hence P is Rad$-\oplus-$ supplemented.

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SOLUTIONS OF CERTAIN CLASS OF NON-LINEAR TIME-FRACTIONAL DIFFUSION EQUATIONS VIA THE FRACTIONAL DIFFERENTIAL TRANSFORM METHOD

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Abstract. In this paper, the generalized differential transform method (GDTM) is used to solve certain class of nonlinear time-fractional diffusion equation. An efficient recurrence relation is obtained to solve this problem. Some numerical examples are given for different class of derivative orders which are analyzed numerically for the specified examples.

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Keywords: numerical methods, analytical solution, time-fractional diffusion equation, generalized differential transform method, Caputo fractional derivative, differential equations.

1. Introduction

Fractional Differential Equations have been the subject of intensive research of recent years because of their applications in physics and engineering especially in electromagnetic, electrochemistry, material science, bioengineering, quantum mechanics, finance applied problems including diffusion equations (see [22], [9], [5], [10], [13], [14], [15], [25], [2], [11], [23] and [26]).

Several numerical methods have been generalized to solve this kind of applied problems such as Variational Iteration Method [8], Adomian Decomposition Method [17], [27], [20],[16], Fractional Differential Transform Method [3], [21] etc. The basic idea of differential transform method was first proposed by Zhou [28]. This method is a numerical method based on the Taylor series expansion which constructs an analytical solution in the form of a polynomial. It's different from the traditional higher order Taylor series method because it requires symbolic computation. In 2007, Arikoglu and Ozkol [4] introduced Fractional Differential Transform Method (FDTM) as a new analytical technique for solving fractional type differential equations. In [19], the authors developed a new generalization of the two dimensional differential transform method which is called the Generalized Differential Transform Method (GDTM) (also see [7], [24], [18] and [8]).

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It is pointed out by Bervillier [6] that the limited improvements of the DTM over the well-known Taylor-series solution of ODE's is because of their great similarity. Beside this, in problems involving fractional derivatives major contribution of the DTM can be found.

Now, let us give the following necessary definitions and properties which is needed to summarize the FDTM.

Let $\Omega = [a,b]$ be a finite interval on the real axis $\mathbb{R}$ and the Riemann-Liouville fractional integral [12] of order α is defined by

$$(I_{a^+}^{\alpha} f)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)dt}{(x-t)^{1-\alpha}}, \ (x > a; \ \mathrm{Re}(\alpha) > 0), \tag{1.1}$$

where $\Gamma(\alpha)$ is the gamma function defined by

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt. \tag{1.2}$$

The Riemann-Liouville fractional derivative [12] of order α is defined by

$$(D_{a^+}^{\alpha} f)(x) := \left(\frac{d}{dx}\right)^n \left(I_{a^+}^{n-\alpha} f\right)(x), \ n = [\mathrm{Re}(\alpha)] + 1; \ x > a. \tag{1.3}$$

The following relation holds true for the power function (see p. 418 of [12])

$$\left(D_{a^+}^{\alpha} (t-a)^{\beta}\right)(x) := \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} (x-a)^{\beta-\alpha}, \ x > a, \mathrm{Re}(\alpha) > 0, \ \mathrm{Re}(\beta) > -1. \tag{1.4}$$

The Caputo fractional derivative [12] is defined by

$$(C_{a^+}^{\alpha} f)(x) = (I_{a^+}^{n-\alpha} f^{(n)})(x), \ x > a, \ n-1 < \alpha \leq n, \ n \in \mathbb{N}. \tag{1.5}$$

The sequential fractional derivative [1] for a sufficiently smooth function $f(t)$ due to Miller-Ross [15] is defined by

$$D^{\delta} f(t) = D^{\delta_1} D^{\delta_2} ... D^{\delta_k} f(t), \tag{1.6}$$

where $\delta = (\delta_1 ... \delta_k)$ is a multi-index.

In general, the operator D^{δ} in (1.6) can either be Riemann-Liouville or Caputo or any other kind of integro-differential operator.

In [24] Rida, et al. considered the reaction diffusion equation

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = K \frac{\partial^2 u}{\partial x^2} + f(u), 0 < \alpha \leq 1, t > 0, x \in \mathbb{R} \tag{1.7}$$

where K is the diffusion coefficient and they provide the extension of the GDTM in order to give its numerical solution. It was noteworthy that for $f(u) = 6u(1-u)$, Eq. (1.7) reduces to the time-fractional Fisher equation.

Subsequently, in the year 2013, Cetinkaya and Kiymaz [7], considered the following time-fractional diffusion equation

$$D_t^{\beta} u = \lambda \frac{\partial^2 u}{\partial x^2} - \frac{\partial}{\partial x}\left(F(x)u(x,t)\right), 0 < \beta \leq 1, x, t > 0 \tag{1.8}$$

with initial condition $u(x,0) = f(x)$. Here λ is a positive constant, $F(x)$ is the external force, $u(x,t)$ represents the probability density function of finding a particle at the point x in the time t and $D_t^{\beta} u(x,t) = I_0^{1-\beta}\left[\frac{\partial}{\partial t} u(x,t)\right]$. A general recurrence relation for (1.8) was obtained with the GDTM.

Our main concern in this paper is the following nonlinear time-fractional diffusion equation. Through the above definitions

$$D_t^{\beta} u(x,t) = \lambda C_x^{(2\alpha} u(x,t) - C_x^{\alpha}\left[F(x)u(x,t)\left(1-\mu u(x,t)\right)\right], 0 < \alpha, \beta \leq 1, x, t > 0 \tag{1.9}$$

with initial condition $u(x,0) = f(x)$. Here $C_x^{(2\alpha}$ represent the Caputo sequential fractional derivatives of order 2α. Also λ and μ is a positive constant, $F(x)$ is the external force, $u(x,t)$ represents the probability density function of finding a particle at the point x in the time t and $D_t^{\beta} u(x,t) = \frac{\partial}{\partial t}\left[I_0^{1-\beta} u(x,t)\right]$.

Remark 1. If we set $\alpha = 1$ and $\mu = 0$ in (1.9), we have the time-fractional diffusion equation which is given by (1.8).

The paper is organized as follows: In Section 2, the GDTM is summarized. In Section 3, the solution of the general problem is derived. In Section 4, the method is implemented to a couple of examples and some special cases of the examples are obtained numerically and graphically. Conclusion is given in the last section.

2. Generalized Differential Transform

In this section we shall summarize the GDTM. Consider a function of two variables $u(x,t)$ and suppose that it can be represented as a product of two single-variable function, i.e.,

$$u(x,t) = f(x)g(t). \tag{2.1}$$

Based on the properties of generalized two-dimensional differential transform, the function $u(x,t)$ can be represented as

$$\begin{aligned} u(x,t) &= \sum_{i=0}^{\infty} F_{\alpha}(i)(x-x_0)^{i\alpha} \sum_{j=0}^{\infty} G_{\beta}(j)(t-t_0)^{j\beta} \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} U_{\alpha,\beta}(i,j)(x-x_0)^{i\alpha}(t-t_0)^{j\beta} \end{aligned} \tag{2.2}$$

where $0 < \alpha, \beta \leq 1$ and $U_{\alpha,\beta}(i,j) = F_{\alpha}(i)G_{\beta}(j)$ is the spectrum of $u(x,t)$.

If the function $u(x,t)$ is analytic and differentiable continuously with respect to time t, then the GDT of the function $u(x,t)$ is defined as follows:

$$U_{\alpha,\beta}(i,j)=\frac{1}{\Gamma(i\alpha+1)\Gamma(j\beta+1)}\left[\left(C_{x_0}^{(i\alpha)}\right)\left(C_{t_0}^{(j\beta)}\right)u(x,t)\right]_{(x_0,t_0)} \tag{2.3}$$

where $\left(C_{x_0}^{(i\alpha)}\right)=C_{x_0}^{\alpha}C_{x_0}^{\alpha}...C_{x_0}^{\alpha}$, $i-times$. Besides, equation (2.2) is called the generalized inverse differential transform of $U_{\alpha,\beta}(i,j)$. In the case of $\alpha=\beta=1$, the generalized two-dimensional differential transform (2.3) reduces to the classical two-dimensional transform.

Theorem 1. *[19] (see also [7]) Suppose that $U_{\alpha,\beta}(i,j), V_{\alpha,\beta}(i,j)$ and $W_{\alpha,\beta}(i,j)$ are the differential transformations of the function $u(x,t)$, $v(x,t)$ and $w(x,t)$, respectively. Then the following statements hold true:*

(1) *If $u(x,t)=v(x,t)\pm w(x,t)$ then $U_{\alpha,\beta}(i,j)=V_{\alpha,\beta}(i,j)\pm W_{\alpha,\beta}(i,j)$.*
(2) *If $u(x,t)=\lambda v(x,t), \lambda\in\mathbb{R}$ then $U_{\alpha,\beta}(i,j)=\lambda V_{\alpha,\beta}(i,j)$.*
(3) *If $u(x,t)=v(x,t)w(x,t)$ then*

$$U_{\alpha,\beta}(i,j)=\sum_{r=0}^{i}\sum_{s=0}^{j}V_{\alpha,\beta}(r,j-s)W_{\alpha,\beta}(i-r,s).$$

(4) *If $u(x,t)=(x-x_0)^{n\alpha}(t-t_0)^{m\beta}$ then $U_{\alpha,\beta}(i,j)=\delta(i-n)\delta(j-m)$.*
(5) *If $u(x,t)=f(x)g(t)$ and the function $f(x)=x^{\lambda}h(x)$ where $\lambda>-1$, $h(x)$ has the generalized Taylor series expansion*

$$h(x)=\sum_{m=0}^{\infty}a_m(x-x_0)^{m\alpha}$$

and

(a): *$\beta<\lambda+1$ and α arbitrary or*
(b): *$\beta\geq\lambda+1$, α arbitrary and $a_m=0$ for $m=0,1,\ldots,n-1$ where $(n-1)<\beta<n$, then the generalized differential transform (2.3) becomes*

$$U_{\alpha,\beta}(i,j)=\frac{1}{\Gamma(i\alpha+1)\Gamma(j\beta+1)}\left[C_{x_0}^{(i\alpha}\left(C_{t_0}^{(j\beta}\right)u(x,t)\right]_{(x_0,t_0)}.$$

(6) *If $u(x,t)=C_{x_0}^{\gamma}v(x,t), m-1<\gamma\leq m$ and $v(x,t)=f(x)g(t)$ then*

$$U_{\alpha,\beta}(i,j)=\frac{\Gamma(i\alpha+\gamma+1)}{\Gamma(i\alpha+1)}V_{\alpha,\beta}(i+\frac{\gamma}{\alpha},j).$$

(7) *If $u(x,t)=C_{t_0}^{\gamma}v(x,t), m-1<\gamma\leq m$ and $v(x,t)=f(x)g(t)$ then*

$$U_{\alpha,\beta}(i,j)=\frac{\Gamma(j\beta+\gamma+1)}{\Gamma(j\beta+1)}V_{\alpha,\beta}(i,j+\frac{\gamma}{\beta}).$$

(8) *If* $u(x,t) = C_{x_0}^{\gamma} C_{t_0}^{\mu} v(x,t), m-1 < \gamma \leq m, n-1 < \mu \leq n$ *and* $v(x,t) = f(x)g(t)$ *where the function* $f(x)$ *and* $g(t)$ *satisfy the condition given in* 5 *above then*

$$U_{\alpha,\beta}(i,j) = \frac{\Gamma(i\alpha+\gamma+1)}{\Gamma(i\alpha+1)} \frac{\Gamma(j\beta+\gamma+1)}{\Gamma(j\beta+1)} V_{\alpha,\beta}(i+\frac{\gamma}{\alpha}, j+\frac{\mu}{\beta}).$$

Proof. The proof is straighthforward using (2.2) and (2.3) and it was given in detail in [19]. □

3. The Solution of the Main Problem

In this section, we obtain a recurrence relation for the coefficients of the solution $u(x,t)$ of the Eq. (1.9).

Theorem 2. *If the function* $F(x)$ *has the series expansion* $F(x) = \sum_{n=0}^{\infty} a_n (x - x_o)^{n\alpha}$ *with a radius of convergence* $R > 0$ *where* $a_n = \frac{1}{\Gamma(n\alpha+1)} \left[C_{x_0}^{(n\alpha} F(x_0) \right]$ *for* $n = 0,1,2,\ldots,$ *then the GDT of the solution of Eq.* (1.9) *satisfies the following recurrence relation*

$$\begin{aligned} U_{\alpha,\beta}(i,j+1) = \frac{\Gamma(j\beta+1)}{\Gamma((j+1)\beta+1)} \Bigg[& \lambda \frac{\Gamma((i+2)\alpha+1)}{\Gamma(i\alpha+1)} U_{\alpha,\beta}(i+2,j) \\ & - \frac{\Gamma((i+1)\alpha+1)}{\Gamma(i\alpha+1)} \sum_{m=0}^{i+1} a_{i+1-m} U_{\alpha,\beta}(m,j) \\ & + \mu \frac{\Gamma((i+1)\alpha+1)}{\Gamma(i\alpha+1)} \sum_{m=0}^{i+1} \sum_{r=0}^{m} \sum_{s=0}^{j} a_{i+1-m} U_{\alpha,\beta}(r,j-s) U_{\alpha,\beta}(m-r,s) \Bigg]. \end{aligned} \tag{3.1}$$

Proof. Suppose that the solution $u(x,t)$ can be represented as a product of two single-variable functions. Applying the generalized two-dimensional differential transform to both sides of Eq. (1.9) and using theorem (1), Eq. (1.9) transforms to

$$\begin{aligned} \frac{\Gamma((j+1)\beta+1)}{\Gamma(j\beta+1)} & U_{\alpha,\beta}(i,j+1) \\ & = \lambda \frac{\Gamma((i+2)\alpha+1)}{\Gamma(i\alpha+1)} U_{\alpha,\beta}(i+2,j) \\ & \quad - \frac{\Gamma((i+1)\alpha+1)}{\Gamma(i\alpha+1)} \sum_{m=0}^{i+1} a_{i+1-m} U_{\alpha,\beta}(m,j) \\ & \quad + \mu \frac{\Gamma((i+1)\alpha+1)}{\Gamma(i\alpha+1)} \sum_{m=0}^{i+1} \sum_{r=0}^{m} \sum_{s=0}^{j} a_{i+1-m} U_{\alpha,\beta}(r,j-s) U_{\alpha,\beta}(m-r,s) \end{aligned}$$

which can be written as

$$U_{\alpha,\beta}(i,j+1) = \frac{\Gamma(j\beta+1)}{\Gamma((j+1)\beta+1)} \Bigg[\lambda \frac{\Gamma((i+2)\alpha+1)}{\Gamma(i\alpha+1)} U_{\alpha,\beta}(i+2,j)$$

$$-\frac{\Gamma((i+1)\alpha+1)}{\Gamma(i\alpha+1)}\sum_{m=0}^{i+1}a_{i+1-m}U_{\alpha,\beta}(m,j)$$
$$+\mu\frac{\Gamma((i+1)\alpha+1)}{\Gamma(i\alpha+1)}\sum_{m=0}^{i+1}\sum_{r=0}^{m}\sum_{s=0}^{j}a_{i+1-m}U_{\alpha,\beta}(r,j-s)U_{\alpha,\beta}(m-r,s)\Bigg].$$

□

We should note that the generalized two-dimensional transform of the initial condition $u(x,0)$ is given by

$$U_{\alpha,\beta}(i,0)=\frac{1}{\Gamma(i\alpha+1)}\left[\left(C_{x_0}^{(i\alpha}\right)u(x,0)\right],i=0,1,2,\ldots. \tag{3.2}$$

4. Numerical Examples

In this section, we have selected two examples which show the simplicity and effectiveness of the proposed general recurrence relation (3.1) in order to give the solution of the main problem. We also present the numerical results for each example.

Example 1. Consider the following time-fractional diffusion equation

$$D_t^{\beta}u(x,t)=\lambda C_x^{(2\alpha}u(x,t)-C_x^{\alpha}\left[-(x^{\alpha})u(x,t)\left(1-\mu u(x,t)\right)\right],x,t>0 \tag{4.1}$$

where $0<\alpha,\beta\leq 1$, $\lambda=1$, $\mu=1$ and subject to the initial condition

$$u(x,0)=x^{\alpha}. \tag{4.2}$$

Since $F(x)=-(x)^{\alpha}$, we have $a_1=-1$ and $a_n=0$ for $n\neq 1$. In addition to this, substituting $f(x)=x^{\alpha}$ in (3.2), we have

$$U_{\alpha,\beta}(1,0)=1,U_{\alpha,\beta}(i,0)=0 \text{ for } i\neq 1. \tag{4.3}$$

Applying the recurrence relation (3.1) and using the transformed initial condition (4.3), the first few components of $U_{\alpha,\beta}(i,j)$ can be calculated as follows:

$$U_{\alpha,\beta}(1,1)=\frac{\Gamma(2\alpha+1)}{\Gamma(\beta+1)\Gamma(\alpha+1)},\ U_{\alpha,\beta}(i,1)=0 \text{ for } i\neq 1, \tag{4.4}$$
$$U_{\alpha,\beta}(1,2)=\frac{(\Gamma(2\alpha+1))^2}{\Gamma(2\beta+1)\left(\Gamma(\alpha+1)\right)^2},\ U_{\alpha,\beta}(i,2)=0 \text{ for } i\neq 1,$$
$$U_{\alpha,\beta}(1,3)=\frac{(\Gamma(2\alpha+1))^3}{\Gamma(3\beta+1)\left(\Gamma(\alpha+1)\right)^3},\ U_{\alpha,\beta}(i,3)=0 \text{ for } i\neq 1,$$
$$U_{\alpha,\beta}(1,4)=\frac{(\Gamma(2\alpha+1))^4}{\Gamma(4\beta+1)\left(\Gamma(\alpha+1)\right)^4},\ U_{\alpha,\beta}(i,4)=0 \text{ for } i\neq 1,$$
$$\vdots$$

$$U_{\alpha,\beta}(1,j) = \frac{(\Gamma(2\alpha+1))^j}{\Gamma(j\beta+1)\,(\Gamma(\alpha+1))^j},\ U_{\alpha,\beta}(i,j) = 0 \text{ for } i \neq 1.$$

So the solution $u(x,t)$ of Eq. (4.1) is obtained by

$$\begin{aligned} u(x,t) &= x^\alpha \left[1 + \frac{\Gamma(2\alpha+1)}{\Gamma(\beta+1)\Gamma(\alpha+1)} t^\beta + \frac{(\Gamma(2\alpha+1))^2}{\Gamma(2\beta+1)\,(\Gamma(\alpha+1))^2} t^{2\beta} \right. \\ &\left. + \frac{(\Gamma(2\alpha+1))^3}{\Gamma(3\beta+1)\,(\Gamma(\alpha+1))^3} t^{3\beta} + \frac{(\Gamma(2\alpha+1))^4}{\Gamma(4\beta+1)\,(\Gamma(\alpha+1))^4} t^{4\beta} + \dots \right] \\ &= x^\alpha \sum_{k=0}^{\infty} \frac{(\Gamma(2\alpha+1))^k}{\Gamma(k\beta+1)\,(\Gamma(\alpha+1))^k} t^{k\beta} \\ &= x^\alpha E_\beta \left(\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} t^\beta \right), \end{aligned}$$

		$u(x,t)$			
t	x	$\alpha = 0.85$, $\beta = 0.9$	$\alpha = 0.9$, $\beta = 0.95$	$\alpha = 0.95$ $\beta = 0.99$	$\alpha = 1$, $\beta = 1$
0.00	0.25	0.3078	0.2872	0.2679	0.2500
	0.50	0.5548	0.5359	0.5176	0.5000
	0.75	0.7831	0.7719	0.7609	0.7500
	1.00	1.0000	1.0000	1.000	1.0000
0.25	0.25	0.5072	0.4651	0.4312	0.4122
	0.50	0.9142	0.8678	0.8330	0.8244
	0.75	1.2904	1.2500	1.2244	1.2365
	1.00	1.6479	1.6194	1.6092	1.6487
0.50	0.25	0.7953	0.7352	0.6906	0.6796
	0.50	1.4336	1.3719	1.3341	1.3591
	0.75	2.0235	1.9760	1.9610	2.0387
	1.00	2.5841	2.5600	2.5773	2.7183
0.75	0.25	1.2347	1.1563	1.1049	1.1204
	0.50	2.2255	2.1577	2.1345	2.2408
	0.75	3.1413	3.1079	3.1374	3.3613
	1.00	4.0115	4.0264	4.1235	4.4817
1.00	0.25	1.9088	1.8148	1.7670	1.8473
	0.50	3.4406	3.3866	3.4136	3.6945
	0.75	4.8564	4.8780	5.0176	5.5418
	1.00	6.2018	6.3196	6.5946	7.3891

TABLE 1. Approximate solution for u(x,t).

where

$$E_{\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\beta k + 1)}$$

is the usual Mittag-Leffler function.

It is seen from Figure 1 that $u(x,t)$ increase as t increases for all α and β. However for fixed t, $u(x,t)$ is found to decrease with the increase in α and β.

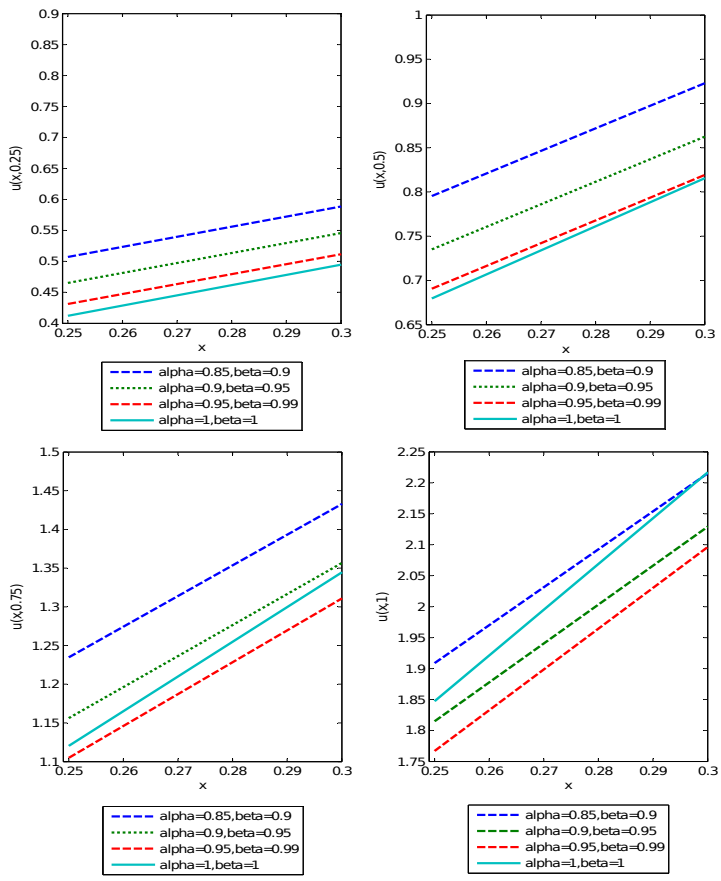

FIGURE 1. The solution u(x,t) vs. point x.

It is also seen from the $3-D$ figures which are described through Figure 2 that the variations of $u(x,t)$ are linear with x. However it becomes exponential with t for different values of α and β as stated in the caption of the figures.

Remark 2. Setting $\mu = 0$ and $\alpha = 1$ in (4.1) and (4.2) we get the same results given in [7].

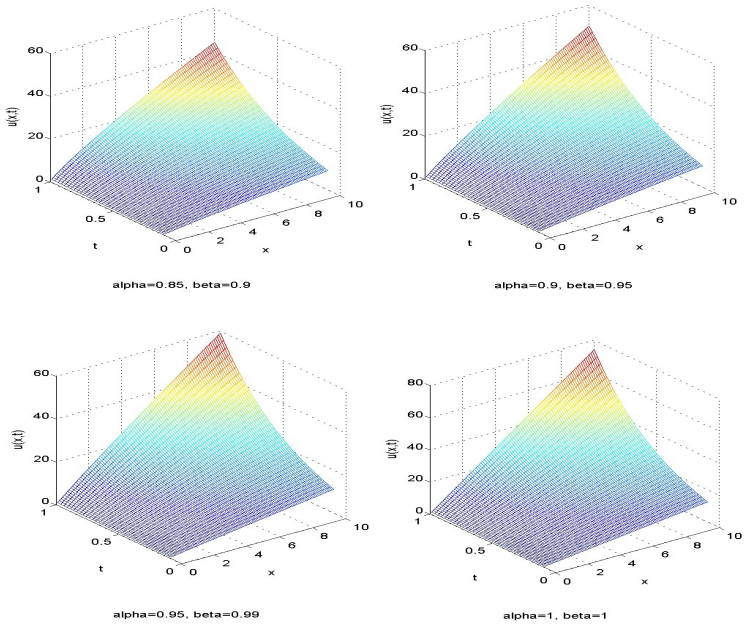

FIGURE 2. The solution u(x,t) with respect to x and time t.

Example 2. Consider the following time-fractional reaction diffusion equation

$$D_t^{\beta} u(x,t) = \lambda C_x^{(2\alpha} u(x,t) - C_x^{\alpha}\left[E_{\alpha}(-x^{\alpha})u(x,t)\left(1-\mu u(x,t)\right)\right], x,t>0 \tag{4.5}$$

where $0<\alpha,\beta\leq 1$, $\lambda=2$, $\mu=-1$ subject to the initial condition

$$u(x,0) = 1+\sum_{k=1}^{\infty}\frac{x^{k\alpha}}{\Gamma(k\alpha+1)} = E_{\alpha}(x^{\alpha}). \tag{4.6}$$

Since $F(x)=\sum_{k=0}^{\infty}\frac{(-1)^k(x)^{k\alpha}}{\Gamma(k\alpha+1)}$, then $a_0=1,\ a_1=\frac{-1}{\Gamma(\alpha+1)},\ a_2=\frac{1}{\Gamma(2\alpha+1)},\ldots$ In addition to this, substituting $f(x)=\sum_{k=0}^{\infty}\frac{(x)^{k\alpha}}{\Gamma(k\alpha+1)}$ in (3.2), we have

$$U_{\alpha,\beta}(i,0) = \frac{1}{\Gamma(i\alpha+1)},\ i=0,1,2,\ldots. \tag{4.7}$$

Applying the recurrence relation (3.1) and using the transformed initial condition (4.7), the first few components of $U_{\alpha,\beta}(i,j)$ can be calculated as follows:

$$U_{\alpha,\beta}(0,0)=1,\ U_{\alpha,\beta}(1,0)=\frac{1}{\Gamma(\alpha+1)},\ U_{\alpha,\beta}(2,0)=\frac{1}{\Gamma(2\alpha+1)},\ldots$$

$$U_{\alpha,\beta}(0,1)=\frac{1}{\Gamma(\beta+1)},U_{\alpha,\beta}(0,2)=\frac{1}{\Gamma(2\beta+1)},\ldots$$

$$\vdots$$

$$U_{\alpha,\beta}(i,j)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{\Gamma(i\alpha+1)\Gamma(j\beta+1)},\ i,j=0,1,2,\ldots$$

So the solution $u(x,t)$ of Eq. (4.5) is obtained by

$$u(x,t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{\Gamma(i\alpha+1)\Gamma(j\beta+1)}x^{i\alpha}t^{j\beta}=E_{\alpha}(x^{\alpha})E_{\beta}(t^{\beta}).$$

Table 2 shows the approximate solution for Eq. (4.5) obtained for different values of α and β.

		$u(x,t)$			
t	x	$\alpha=0.85$, $\beta=0.9$	$\alpha=0.9$, $\beta=0.95$	$\alpha=0.95$, $\beta=0.99$	$\alpha=1$, $\beta=1$
0.00	0.25	1.3961	1.3540	1.3169	1.2840
	0.50	1.8456	1.7725	1.7073	1.6487
	0.75	2.4092	2.3010	2.2042	2.1170
	1.00	3.1255	2.9749	2.8400	2.7183
0.25	0.25	1.8903	1.7831	1.6992	1.6487
	0.50	2.4990	2.3342	2.2029	2.1170
	0.75	3.2620	3.0302	2.8441	2.7183
	1.00	4.2319	3.9177	3.6644	3.4903
0.50	0.25	2.4746	2.3116	2.1860	2.1170
	0.50	3.2714	3.0261	2.8339	2.7183
	0.75	4.2703	3.9284	3.6588	3.4903
	1.00	5.5399	5.0790	4.7142	4.4817
0.75	0.25	3.2124	2.9845	2.8099	2.7183
	0.50	4.2468	3.9070	3.6428	3.4903
	0.75	5.5436	5.0719	4.7032	4.4817
	1.00	7.1919	6.5574	6.0598	5.7546
1.00	0.25	4.1533	3.8454	3.6105	3.4903
	0.50	5.4906	5.0339	4.6807	4.4817
	0.75	7.1671	6.5349	6.0432	5.7546
	1.00	9.2982	8.4488	7.7863	7.3891

TABLE 2. Approximate solution for u(x,t).

It is seen from Figure 3 that $u(x,t)$ increase with the increase in t for all α and β. However it is found to decrease with the increase in α and β.

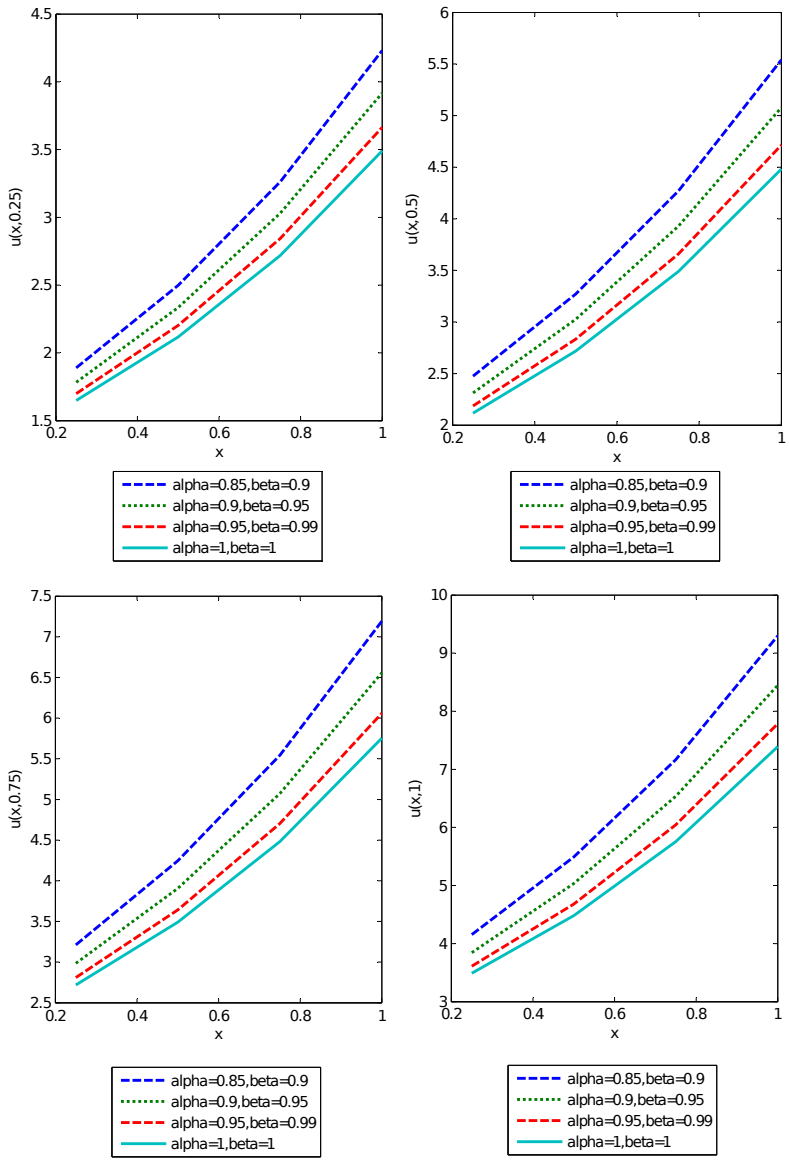

FIGURE 3. The solution u(x,t) vs. point x.

It is also seen from the $3-D$ figures which are described through Figure 2 that the variations of $u(x,t)$ becomes exponential with x and t for different values of α and β as stated in the caption of the figures.

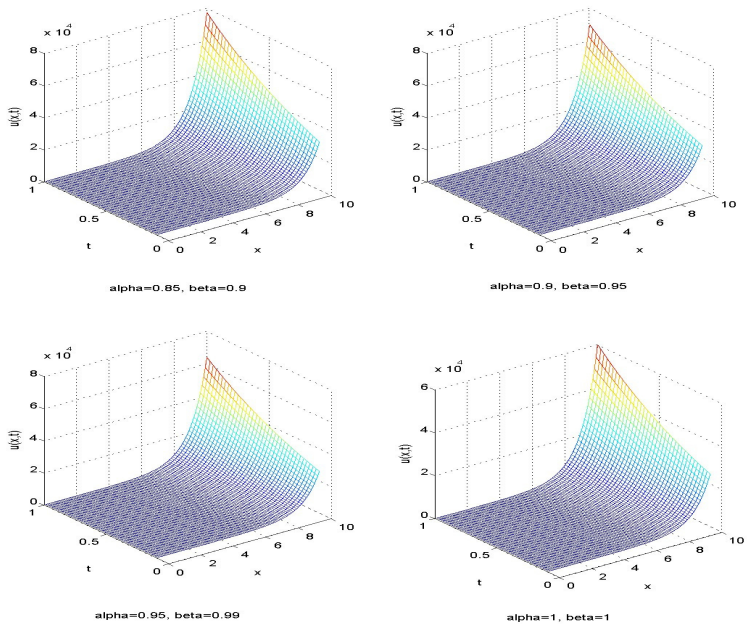

FIGURE 4. The solution u(x,t) with respect to x and time t.

Remark 3. Setting $\mu = 0$, $\alpha = 1$ and $\lambda = 1$ in (4.5) and (4.6) we get the same results given in [7].

5. CONCLUSION

In this paper, we consider a non-linear time-fractional diffusion equation given in (1.9) and we applied the GDTM for its solution. The approximate solutions resulting from the method are shown graphically. Results also show that the numerical scheme is very effective and convenient for solving non-linear time-fractional diffusion equation of fractional order. Numerical computations associated with the two examples discussed above were performed by using Matlab. Another important contribution of this paper is the observation that, the presence of external force increases the rate of diffusion in the nonlinear system but the rate becomes slower in the presence of nonlinear term.

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REPDIGITS AS SUMS OF THREE HALF-COMPANION PELL NUMBERS

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Abstract. In this paper, we find all repdigits which can be expressed as the sum of three Half-companion Pell numbers. To prove our main result, we use the combined approach of lower bounds for linear forms in logarithms of algebraic numbers and a version of the Baker Davenport reduction method.

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1. Introduction

Diophantine equations involving recurrence sequences have been studied for a long time. One of the most interesting of these equations is the equations involving repdigits.

A repdigit, short for "repeated digit", T is a natural number composed of repeated instances of the same digit in its decimal expansion. That is, T is of the form

$$x \cdot \left(\frac{10^t - 1}{9} \right)$$

for some positive integers x, t with $t \geq 1$ and $1 \leq x \leq 9$.

Some of the most recent papers related to the Diophantine equations and repdigits with well known recurrence sequences are [2–5, 7, 8, 10, 11]. In this note, we use Half-companion Pell sequence in our main result.

The companion Pell numbers or Pell–Lucas numbers are defined by the recurrence relation

$$Q_n = \begin{cases} 2 & \text{if } n = 0, \\ 2 & \text{if } n = 1, \\ 2Q_{n-1} + Q_{n-2} & \text{otherwise.} \end{cases}$$

The first few terms of the sequence are (sequence A002203 in the OEIS): $2, 2, 6, 14, 34, 82, 198, 478, \ldots$.

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Binet's formula for Companion Pell numbers numbers is

$$Q_n = \varphi^n + \psi^n$$

where $\varphi = \left(1+\sqrt{2}\right)$ (the silver ratio) and $\psi = \left(1-\sqrt{2}\right)$ are roots of the characteristic equation $x^2 - 2x - 1 = 0$. From this formula, one can easiliy prove that

$$\varphi^{n-2} \leq Q_n \leq 2\varphi^{n-1}.$$

by the induction on n.

Half-companion Pell numbers H_n that play an important role in obtaning Lucas-Balancing numbers C_n as $C_n = H_{2n}$ are defined by $H_n = Q_n/2$. So, it is obvious that

$$H_n = \frac{\varphi^n + \psi^n}{2}.$$

Further, for every positive integer $n \geq 2$,

$$\varphi^{n-1} \leq 2H_n \leq \varphi^{n+1}. \tag{1.1}$$

In this study, our main result is the following:

Theorem 1.1. *All nonnegative integer solutions (n,m,k,x,t) with $1 \leq x \leq 9$ satisfying the Diophantine equation*

$$N = H_n + H_m + H_k = x \cdot \left(\frac{10^t - 1}{9}\right) \tag{1.2}$$

as follows:

$$\begin{aligned}(n,m,k,x,t) \in \{&(0,1,1,3,1),(0,1,2,5,1),(0,1,3,9,1),(0,2,1,5,1),(0,2,2,7,1),\\ &(0,2,3,1,2),\ (0,3,1,9,1),(0,3,2,1,2),(1,1,1,3,1),(1,1,2,5,1),\\ &(1,1,3,9,1),(1,2,1,5,1),\ (1,2,2,7,1),(1,2,3,1,2),(1,3,1,9,1),\\ &(1,3,2,1,2),(2,1,1,5,1),(2,1,2,7,1),\ (2,1,3,1,2),(2,2,1,7,1),\\ &(2,2,2,9,1),(2,3,1,1,2),(3,1,1,9,1),(3,1,2,1,2),\ (3,2,1,1,2),\\ &(3,3,5,5,2),(3,5,3,5,2),(4,5,5,9,2),(5,3,3,5,2),(5,4,5,9,2),\\ &(5,5,4,9,2)\}\end{aligned}$$

2. Preliminaries

Before proceeding with the proof of our main result, let us give some necessary information for proof. We give the definition of the logarithmic height of an algebraic number and its some properties.

Definition 2.1. Let z be an algebraic number of degree $d(z)$ with minimal polynomial

$$f(x) = a_0 x^{d(z)} + a_1 x^{d(z)-1} + \cdots + a_{d(z)} = a_0 \cdot \prod_{i=1}^{d(z)} (x - z_i)$$

where a_i's are relatively prime integers with $a_0 > 0$ and z_i's are conjugates of z. Then

$$h(z) = \frac{1}{d(z)} \left(\log a_0 + \sum_{i=1}^{d(z)} \log(\max\{|z_i|, 1\}) \right)$$

is called *the logarithmic height of* z. The following proposition gives some properties of logarithmic height that can be found in [12].

Proposition 2.1. *Let $z, z_1, z_2, \ldots, z_t$ be elements of an algebraic closure of $\mathbb{Q}$ and $m \in \mathbb{Z}$. Then*

(1) $h(z_1 \cdots z_t) \leq \sum_{i=1}^{t} h(z_i)$
(2) $h(z_1 + \cdots + z_t) \leq \log t + \sum_{i=1}^{t} h(z_i)$
(3) $h(z^m) = |m| h(z)$.

We will use the following theorem (see [9] or Theorem 9.4 in [1]) for proving our results.

Theorem 2.1. *Let $z_1, z_2, \ldots, z_s$ be nonzero elements of algebraic number field $\mathbb{F}$ of degree D over $\mathbb{Q}$ and let $b_1, b_2, \ldots, b_s \in \mathbb{Z}$. Set*

$$B := \max\{|b_1|, \ldots, |b_s|\}$$

and

$$\Lambda := z_1^{b_1} \ldots z_s^{b_s} - 1.$$

If Λ is nonzero, then

$$\log|\Lambda| > -3 \cdot 30^{s+4} \cdot (s+1)^{5.5} \cdot D^2 \cdot (1 + \log D) \cdot (1 + \log(sB)) \cdot A_1 \cdots A_s$$

where

$$A_i \geq \max\{D \cdot h(z_i), |\log z_i|, 0.16\}$$

for all $1 \leq i \leq s$. If $\mathbb{F} = \mathbb{R}$, then

$$\log|\Lambda| > -1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot D^2 \cdot (1 + \log D) \cdot (1 + \log B) \cdot A_1 \cdots A_s.$$

Another main tool for our proof is a variant of Baker and Davenport reduction method due to [6].

Let $\psi_1, \psi_2, \varepsilon \in \mathbb{R}$ be given and let $x_1, x_2 \in \mathbb{Z}$ be unknowns. Let

$$\Gamma = \varepsilon + x_1 \psi_1 + x_2 \psi_2. \tag{2.1}$$

Let c, δ be positive constants. Set $X = \max\{|x_1|, |x_2|\}$. Let X_0, Y be positive. Assume that

$$|\Gamma| < c \cdot \exp(-\delta \cdot Y), \tag{2.2}$$

$$Y \leq X \leq X_0. \tag{2.3}$$

Set $\psi = -\psi_1/\psi_2$. Let the continued fraction expansion of ψ be given by $[a_0, a_1, a_2, \ldots]$ and let the kth convergent of ψ be p_k/q_k for $k = 0, 1, 2, \ldots$. Without loss of the generality, we may assume that $|\psi_1| < |\psi_2|$ and that $x_1 > 0$. We have the following two result from [6].

Lemma 2.1. *Let*

$$A = \max_{0 \leq k \leq Y_0} a_{k+1}.$$

If (2.2) *and* (2.3) *hold for* x_1, x_2 *and* $\varepsilon = 0$ *in* (2.1)*, then*

$$Y < \frac{1}{\delta} \log \left(\frac{c\,(A+2)\,X_0}{|\psi_2|} \right).$$

If $\varepsilon \neq 0$*, then setting* $\phi = \varepsilon/\psi_2$ *we have that*

$$\frac{\Gamma}{\psi_2} = \phi - x_1 \psi + x_2.$$

Lemma 2.2. *Let* p/q *be a convergent of* ψ *with* $q > X_0$*. Suppose that*

$$\|q\phi\| > \frac{2X_0}{q}$$

where $\|\cdot\|$ *denotes the distance from the nearest integer. Then, the solutions of* (2.2) *and* (2.3) *satisfy*

$$Y < \frac{1}{\delta} \log \left(\frac{q^2 c}{|\psi_2| X_0} \right).$$

3. The Proof of Theorem 1.1

Let us assume that $n \geq m \geq k$, $t > 0$ and $x \in \{1, 2, \ldots, 9\}$. A quick search in Mathematica reveals that the solutions of equation (1.2) for $0 \leq n \leq 500$ are as stated in Theorem 1.1. Exactly, the solutions of equation (1.2) are

$$N \in \{3, 5, 7, 9, 11, 55, 99\}.$$

We will assume $n > 500$ for the remainder of the work. Further, since Half-companion Pell numbers are all odd, (1.2) has no solution for even x. Thus, with $A = \{1, 3, 5, 7, 9\}$, $x \in A$.

The equations (1.1) and (1.2) imply that

$$H_{501} \leq H_n \leq H_n + H_m + H_k = x \cdot \left(\frac{10^t - 1}{9} \right) \leq 10^t - 1.$$

It follows that

$$191 \leq \frac{\log(1 + H_{501})}{\log(10)} \leq t. \tag{3.1}$$

Now,

$$10^{t-1} \leq x \cdot \left(\frac{10^t - 1}{9}\right) = H_n + H_m + H_k \leq 3H_n$$
$$\leq \frac{3}{2}\left(\varphi^n + |\psi|^n\right) < \frac{3}{2}\left(\varphi^n + 1\right) < \frac{3}{2}\varphi^n < \varphi^{n+0.47}$$

which means that

$$2.61\,(t-1) < (t-1)\,\frac{\log 10}{\log \varphi} < n + 0.47.$$

It can be easily seen from the last inequality that

$$t < 2.61t - 3.08 < n$$

and from (3.1) that

$$191 < t < n.$$

Let us rewrite the equation (1.2) as

$$\frac{\varphi^n + \psi^n}{2} + \frac{\varphi^m + \psi^m}{2} + \frac{\varphi^k + \psi^k}{2} = x \cdot \left(\frac{10^t - 1}{9}\right). \tag{3.2}$$

The equality (3.2) can be simply converted to

$$\frac{\varphi^n + \psi^n}{2} + \frac{\varphi^m + \psi^m}{2} + \frac{\varphi^k + \psi^k}{2} - \frac{x \times 10^t}{9} = -\frac{x}{9}. \tag{3.3}$$

In order to obtain a bound on n, we examine equation (3.3) in three different cases.

Case 1. *We have that*

$$\frac{\varphi^n}{2}\left(1 + \varphi^{m-n} + \varphi^{k-n}\right) - \frac{x \times 10^t}{9} = -\frac{x}{9} - \frac{\psi^n}{2} - \frac{\psi^m}{2} - \frac{\psi^k}{2}.$$

This yields

$$\left|\frac{\varphi^n}{2}\left(1 + \varphi^{m-n} + \varphi^{k-n}\right) - \frac{x \times 10^t}{9}\right| \leq \frac{x}{9} + \frac{|\psi|^n}{2} + \frac{|\psi|^m}{2} + \frac{|\psi|^k}{2} < \frac{1}{2}\left(2 + 3\right).$$

From this inequality, we get

$$\left|\frac{\varphi^n}{2}\left(1 + \varphi^{m-n} + \varphi^{k-n}\right) - \frac{x \times 10^t}{9}\right| < \frac{\varphi^2}{2}. \tag{3.4}$$

By multiplying both sides of the inequality (3.4) *by*

$$\frac{2\varphi^{-n}}{1 + \varphi^{m-n} + \varphi^{k-n}}$$

we obtain

$$\left|1 - \varphi^{-k} 10^t \left(\frac{2x}{9\left(\varphi^{n-k} + \varphi^{m-k} + 1\right)}\right)\right| < \frac{\varphi^{2-n}}{1 + \varphi^{m-n} + \varphi^{k-n}} < \varphi^{2-n}. \tag{3.5}$$

Now let us check whether the obtained inequality (3.5) conforms to the hypothesis in Theorem 2.1. Set

$$\Delta_1 := 1 - \varphi^{-k} 10^t \left(\frac{2x}{9\left(\varphi^{n-k} + \varphi^{m-k} + 1\right)} \right).$$

Suppose that $\Delta_1 = 0$. Then, we have

$$\varphi^n + \varphi^m + \varphi^k = \frac{10^t \times 2x}{9}$$

and hence, $\varphi^n + \varphi^m + \varphi^k \in \mathbb{Q}$, which is not possible for any $n, m, k > 0$. Therefore $\Delta_1 \neq 0$. Thus, we can apply Theorem 2.1 to the inequality (3.5). Take,

$$z_1 = \varphi, \quad z_2 = 10, \quad z_3 = \frac{2x}{9\left(\varphi^{n-k} + \varphi^{m-k} + 1\right)}, \quad b_1 = -k, \quad b_2 = t, \quad b_3 = 1$$

where $z_1, z_2, z_3 \in \mathbb{Q}\sqrt{2}$ and $b_1, b_2, b_3 \in \mathbb{Z}$. Since $z_i \in \mathbb{Q}\sqrt{2}$ for $i \in \{1,2,3\}$ $D = 2$. So, we can take

$$\begin{aligned} 0.9 = A_1 &\geq 2 \cdot h(\varphi) = 2 \cdot \frac{1}{2} \log(\varphi) = \log(\varphi) \sim 0.88 \\ 4.7 = A_2 &\geq 2 \cdot h(10) < 2 \cdot \log(10) \sim 4.6. \end{aligned}$$

Let us compute A_3. We have

$$z_3 = \frac{2x}{9\left(\varphi^{n-k} + \varphi^{m-k} + 1\right)} < 2$$

and

$$z_3^{-1} = \frac{9\left(\varphi^{n-k} + \varphi^{m-k} + 1\right)}{2x} < \frac{27}{2} \varphi^{n-k}.$$

So, $|\log(z_3)| < 3 + (n-k)\log\varphi$. Also,

$$\begin{aligned} h(z_3) &\leq h(2x) + h(9) + h\left(\varphi^{n-k} + \varphi^{m-k} + 1\right) \\ &\leq h(18) + h(9) + h\left(\varphi^{n-k} + \varphi^{m-k}\right) + \log 2 \\ &\leq h(18) + h(9) + \log 2 + h\left(\varphi^{m-k}\left(\varphi^{n-m} + 1\right)\right) \\ &\leq \log 18 + \log 9 + 2\log 2 + h\left(\varphi^{m-k}\right) + h\left(\varphi^{n-m}\right) \\ &\leq \log 18 + \log 9 + 2\log 2 + (m-k)\, h(\varphi) + (n-m)\, h(\varphi) \\ &= \log 18 + \log 9 + 2\log 2 + \frac{1}{2}(n-k)\log\varphi. \end{aligned}$$

Therefore, $2h(z_3) \leq 13 + (n-k)\log\varphi$. This yields

$$13 + (n-k)\log\varphi = A_3 \geq \max\left\{2h(z_3), |\log(z_3)|, 0.16\right\}.$$

Finally, $B = \max\{k,t,1\} < n$. *Set* $K_1 = 1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 \cdot (1+\log 2) \cdot A_1 \cdot A_2$. *By applying Theorem 2.1 to* Δ_1, *we get*

$$\log|\Delta_1| > -K_1 \cdot (1+\log(n)) \cdot (13+(n-k)\log\varphi)$$

and by using the inequality (3.5), we obtain

$$n\log\varphi < 2\log\varphi + K_1 \cdot (1+\log(n)) \cdot (13+(n-k)\log\varphi). \tag{3.6}$$

Case 2. *Let us rewrite the equation (3.3) as*

$$\frac{\varphi^n}{2}\left(1+\varphi^{m-n}\right) - \frac{x\times 10^t}{9} = -\frac{x}{9} - \frac{\psi^n}{2} - \frac{\psi^m}{2} - \frac{\varphi^k}{2} - \frac{\psi^k}{2}. \tag{3.7}$$

This equation yields

$$\begin{aligned}\left|\frac{\varphi^n}{2}\left(1+\varphi^{m-n}\right) - \frac{x\times 10^t}{9}\right| &\le \frac{x}{9} + \frac{|\psi|^n}{2} + \frac{|\psi|^m}{2} + \frac{\varphi^k}{2} + \frac{|\psi|^k}{2} \\ &\le 1 + \frac{1}{2} + \frac{1}{2} + \frac{\varphi^k}{2} + \frac{1}{2} \\ &\le \frac{1}{2}(2+4)\varphi^k\end{aligned}$$

and from the last inequality we have that

$$\left|\frac{\varphi^n}{2}\left(1+\varphi^{m-n}\right) - \frac{x\times 10^t}{9}\right| < \frac{\varphi^{k+3}}{2}. \tag{3.8}$$

By multiplying both sides of the inequality (3.8) by

$$\frac{2\varphi^{-n}}{1+\varphi^{m-n}}$$

we get

$$\left|1-\varphi^{-m}10^t\left(\frac{2x}{9\left(1+\varphi^{n-m}\right)}\right)\right| < \frac{\varphi^{k-n+3}}{1+\varphi^{m-n}}.$$

Finally, we can write

$$\left|1-\varphi^{-m}10^t\left(\frac{2x}{9\left(1+\varphi^{n-m}\right)}\right)\right| < \varphi^{k-n+3}. \tag{3.9}$$

Set

$$\Delta_2 := 1-\varphi^{-m}10^t\left(\frac{2x}{9\left(1+\varphi^{n-m}\right)}\right).$$

Similar to the proof given in the previous case, it can be seen easily that $\Delta_2 \neq 0$. *Using the notations in Theorem 2.1, we deduce that*

$$z_1 = \varphi, \qquad z_2 = 10, \qquad z_3 = \frac{2x}{9\left(1+\varphi^{n-m}\right)}, \qquad b_1 = -m, \qquad b_2 = t, \qquad b_3 = 1$$

where $z_1, z_2, z_3 \in \mathbb{Q}\sqrt{2}$ *and* $b_1, b_2, b_3 \in \mathbb{Z}$. *Since* $z_i \in \mathbb{Q}\sqrt{2}$ *for* $i \in \{1,2,3\}$ $D = 2$ *and* $B = \max\{m,t,1\} < n$. *So, we can take* $A_1 = 0.9$ *and* $A_2 = 4.7$ *as in the previous case. Let us compute* A_3. *We have*

$$z_3 = \frac{2x}{9(1+\varphi^{n-m})} < 2$$

and

$$z_3^{-1} = \frac{9(1+\varphi^{n-m})}{2x} < \frac{18}{2}\varphi^{n-m},$$

so $|\log(z_3)| < 3 + (n-m)\log\varphi$. *Also,*

$$\begin{aligned} h(z_3) &\le h(2x) + h(9) + h\left(1+\varphi^{n-m}\right) \\ &\le h(18) + h(9) + \log 2 + (n-m)h(\varphi) \\ &= h(18) + h(9) + \log 2 + \frac{1}{2}(n-m)\log\varphi. \end{aligned}$$

Therefore, $2h(z_3) \le 12 + (n-m)\log\varphi$, *and so*

$$12 + (n-m)\log\varphi = A_3 \ge \max\{2h(z_3), |\log(z_3)|, 0.16\}.$$

Applying Theorem 2.1 to Δ_2 *gives us*

$$\log|\Delta_2| > -K_1 \cdot (1+\log(n)) \cdot (12 + (n-m)\log\varphi)$$

with K_1 *as given in the previous case. By using the inequality (3.9), we get*

$$(n-k)\log\varphi < 3\log\varphi + K_1 \cdot (1+\log(n)) \cdot (12 + (n-m)\log\varphi). \tag{3.10}$$

Case 3. *Let us consider equation (3.3) for the last time as follows:*

$$\frac{\varphi^n}{2} - \frac{x \times 10^t}{9} = -\frac{x}{9} - \frac{\psi^n}{2} - \frac{\varphi^m}{2} - \frac{\psi^m}{2} - \frac{\varphi^k}{2} - \frac{\psi^k}{2}. \tag{3.11}$$

This yields

$$\begin{aligned} \left|\frac{\varphi^n}{2} - \frac{x \times 10^t}{9}\right| &\le \frac{x}{9} + \frac{|\psi|^n}{2} + \frac{\varphi^m}{2} + \frac{|\psi|^m}{2} + \frac{\varphi^k}{2} + \frac{|\psi|^k}{2} \\ &\le 1 + \frac{1}{2} + \frac{\varphi^m}{2} + \frac{1}{2} + \frac{\varphi^k}{2} + \frac{1}{2} \le \frac{1}{2}(2+5)\varphi^m \end{aligned}$$

and so

$$\left|\frac{\varphi^n}{2} - \frac{x \times 10^t}{9}\right| < \frac{\varphi^{m+3}}{2}. \tag{3.12}$$

Multiplying through (3.12) by $2\varphi^{-n}$, *we obtain*

$$\left|1 - \varphi^{-n}10^t\left(\frac{2x}{9}\right)\right| < \varphi^{m-n+3}. \tag{3.13}$$

Set

$$\Delta_3 := 1 - \varphi^{-n}10^t\left(\frac{2x}{9}\right).$$

Again it is clear that $\Delta_3 \neq 0$. *We apply Theorem 2.1 with the data*

$$z_1 = \varphi, \qquad z_2 = 10, \qquad z_3 = \frac{2x}{9}, \qquad b_1 = -n, \qquad b_2 = t, \qquad b_3 = 1$$

where $z_1, z_2, z_3 \in \mathbb{Q}\sqrt{2}$ *and* $b_1, b_2, b_3 \in \mathbb{Z}$. *Since* $z_i \in \mathbb{Q}\sqrt{2}$ *for* $i \in \{1,2,3\}$ $D = 2$. *We have the same* A_1 *and* A_2 *as in the previous cases. Let us compute* A_3.

$$z_3 = \frac{2x}{9} \leq 2$$

and

$$z_3^{-1} = \frac{9}{2x} \leq \frac{9}{2},$$

so $|\log z_3| < 1.51$. *Also,*

$$h(z_3) \leq h(2x) + h(9) \leq h(18) + h(9)$$

which implies that

$$2h(z_3) < 10.2.$$

Therefore, we obtain

$$10.2 = A_3 \geq \max\{2h(z_3), |\log z_3|, 0.16\}.$$

By applying Theorem 2.1 to (3.13), we get

$$(n-m)\log\varphi < 3\log\varphi + 10.2 \cdot K_1 \cdot (1 + \log n) < 10.4 \cdot K_1 \cdot (1 + \log n).$$

Combining this result with the inequality (3.10), we get

$$\begin{aligned}(n-k)\log\varphi &< 3\log\varphi + K_1 \cdot (1 + \log(n)) \cdot (12 + 10.4 \cdot K_1 \cdot (1 + \log(n))) \\ &< 10.6 \cdot K_1^2 \cdot (1 + \log(n))^2.\end{aligned}$$

The last result and the inequality (3.6) show that

$$\begin{aligned}n\log\varphi &< 2\log\varphi + K_1 \cdot (1 + \log(n)) \cdot \left(13 + 10.6 \cdot K_1^2 \cdot (1 + \log(n))^2\right) \\ &< 10.8 \cdot K_1^3 \cdot (1 + \log(n))^3.\end{aligned}$$

Simplifying the last inequality we obtain

$$n \leq 9.68 \times 10^{44}.$$

Now let us try to lower this bound. Set

$$\Gamma_1 = -n\log\varphi + t\log 10 + \log\left(\frac{2x}{9}\right).$$

If we insert the Γ_1 into the equation (3.11), we get

$$\frac{\varphi^n}{2}\left(1 - e^{\Gamma_1}\right) = -\frac{x}{9} - \frac{\psi^n}{2} - H_m - H_k < 0$$

and so $\Gamma_1 > 0$. Thus, we obtain

$$0 < \Gamma_1 < e^{\Gamma_1} - 1 < \varphi^{m-n+3}$$

from the inequality (3.13). It follows that

$$|\Gamma_1| < \varphi^{3.1}\exp\left(-0.88\,(n-m)\right)$$

with $n-m \leq n \leq 9.68\times 10^{44}$. If we divide Γ_1 by $\log 10$, we get

$$\frac{\Gamma_1}{\log 10} = \frac{\log\left(2x/9\right)}{\log 10} - n\frac{\log\varphi}{\log 10} + t.$$

Hence, we can take

$$c=\varphi^{3.1}, \qquad \delta = 0.88, \qquad X_0 = 9.68\times 10^{44}, \qquad \phi = \frac{\log\left(2x/9\right)}{\log 10}$$

$$\psi = \frac{\log\varphi}{\log 10}, \qquad \psi_1 = -\log\varphi, \qquad \psi_2 = \log 10, \qquad \varepsilon = \log\left(2x/9\right).$$

For all values of $x\in\{1,2,\ldots,9\}$, $q=q_{99}$ satisfies the conditions $q>X_0$ and $\|q\phi\| > \frac{2X_0}{q}$ given in the Lemma 2.2. So, applying Lemma 2.2, we obtain

$$n-m<128.$$

Set

$$\Gamma_2 = -m\log\varphi + t\log 10 + \log\left(\frac{2x}{9\left(\varphi^{n-m}+1\right)}\right).$$

From (3.7), we have that

$$\frac{\varphi^n}{2}\left(1+\varphi^{m-n}\right)\left(1-e^{\Gamma_2}\right) = -\frac{x}{9} - \frac{\psi^n}{2} - \frac{\psi^m}{2} - H_k < 0$$

which implies $\Gamma_2>0$. From the inequality (3.9), we obtain

$$0<\Gamma_2<e^{\Gamma_2}-1<\varphi^{k-n+3}.$$

Hence,

$$|\Gamma_2| < \varphi^{3.1}\exp\left(-0.88\,(n-k)\right)$$

where $n-k\leq n\leq 9.68\times 10^{44}$. $q=q_{103}$ satisfies the hypothesis of Lemma 2.2 and applying the Lemma 2.2, we obtain $n-k\leq 141$. Finally, set

$$\Gamma_3 = -k\log\varphi + t\log 10 + \log\left(\frac{2x}{9\left(\varphi^{n-k}+\varphi^{m-k}+1\right)}\right).$$

It is easy to see from previous trials that $\Gamma_3>0$. So, the inequality (3.5) implies that

$$0<\Gamma_3<e^{\Gamma_3}-1<\varphi^{2-n}<\varphi^{2.1}\cdot\exp\left(-0.88\cdot n\right)$$

which yields

$$|\Gamma_3| < \varphi^{2.1}\cdot\exp\left(-0.88\cdot n\right).$$

We see that hypothesis of Lemma 2.2 holds for $q=q_{109}$. Thus, applying Lemma 2.2 we obtain $n<155$ which contradicts our assumption that $n>500$.

This completes our proof.

4. Conclusion

In this paper we have given all solutions of equation (1.2). A similar work can be done for the quadruple sum.

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Conflict of Interest

The author declare no conflicts of interest.

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TAUBERIAN CONDITIONS UNDER WHICH STATISTICAL CONVERGENCE FOLLOWS FROM STATISTICAL SUMMABILITY BY WEIGHTED MEAN METHOD IN TWO-NORMED SPACES

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Abstract. In this paper, we introduce the statistical summability by the weighted mean method in a two-normed space and establish necessary and sufficient Tauberian conditions under which statistical convergence of a sequence (x_n) follows from its statistical summability by weighted mean method in a two-normed space. In particular, our Tauberian conditions are satisfied if (x_n) is statistically slowly oscillating in a two-normed space. The main theorem of this paper is the statistical extension of a Tauberian theorem in Çanak et al. [Acta Comment. Univ. Tartu. Math. 24, 49-57 (2020)].

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1. Introduction

The concept of statistical convergence as an extension to the concept of ordinary convergence was independently introduced by Fast [4] and Schoenberg [19].

A sequence (x_n) of real or complex numbers is said to be statistically convergent to some number l if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n+1} |\{k \leq n : |x_k - l| \geq \varepsilon\}| = 0,$$

where we denote by $|\mathcal{S}|$ the number of the elements in the set $\mathcal{S}$ of positive integers. In symbol, we write

$$st - \lim_{n\to\infty} x_n = l. \tag{1.1}$$

The concept of two-normed spaces was first defined by S. Gähler [8] and has been studied extensively by Cho, Diminnie, Freese, Gähler, Kim, White and many others ([3, 5, 6, 12, 13]).

Let X be a real vector space with $\dim X \geq 2$. A two-norm on X is a function $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ which satisfies

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(i) $\|x,y\| = 0$ if and only if x and y are linearly dependent;
(ii) $\|x,y\| = \|y,x\|$ for each $x,y \in X$;
(iii) $\|\alpha x,y\| = |\alpha|\|x,y\|$ for each $x,y \in X$ and $\alpha \in \mathbb{R}$;
(iv) $\|x,y+z\| \leq \|x,y\| + \|x,z\|$ for each $x,y,z \in X$.

The pair $(X,\|\cdot,\cdot\|)$ is then called a two-normed space ([7]).

A standard example of a two-normed space is $X = \mathbb{R}^2$ being equipped with the two-norm $\|x,y\| :=$ area of the parallelogram spanned by the vectors x and y, which may be given explicitly by

$$\|x,y\| = |x_1y_2 - x_2y_1|, \text{ where } x = (x_1,x_2) \text{ and } y = (y_1,y_2).$$

Let (p_n) be a sequence of nonnegative numbers with $p_0 > 0$ such that

$$P_n = \sum_{k=0}^{n} p_k \to \infty, \quad n \to \infty. \tag{1.2}$$

The weighted means of a sequence (x_n) are defined by

$$\sigma_n = \frac{1}{P_n}\sum_{k=0}^{n} p_k x_k \quad (n = 0,1,2,\ldots).$$

A sequence (x_n) is said to be summable to $l \in X$ by the weighted mean method determined by the sequence $p = (p_n)$ (or briefly summable $(\overline{N},p)$ to $l \in X$) if for every $y \in X$,

$$\lim_{n\to\infty} \|\sigma_n - l, y\| = 0. \tag{1.3}$$

We note that if $p_n = 1$ for all $n = 0,1,2,\ldots$, $(\overline{N},p)$ summability method reduces to the Cesàro summability method.

Savaş and Sezer [18] introduced the concept of Cesàro summability method and obtained necessary and sufficient Tauberian conditions for Cesàro summability method in two-normed spaces. Following Savaş and Sezer [18], Çanak et al. [2] introduced the concept of weighted mean method of summability and established necessary and sufficient Tauberian conditions for the weighted mean summability of sequences in two-normed spaces.

Gürdal and Pehlivan [9] defined the concept of statistical convergence and obtained some properties of statistical convergence in two-normed spaces. A sequence (x_n) in two-normed space $(X,\|\cdot,\cdot\|)$ is said to be statistically convergent to some number l if for every $\varepsilon > 0$ and for every $z \in X$,

$$\lim_{n\to\infty} \frac{1}{n+1} |\{k \leq n : \|x_k - l, z\| \geq \varepsilon\}| = 0.$$

In symbol, we write

$$st - \lim_{n\to\infty} \|x_n, z\| = \|l, z\|. \tag{1.4}$$

A sequence (x_n) is said to be statistically summable to l in two-normed space $(X, \|\cdot,\cdot\|)$ by the weighted mean method determined by the sequence $p = (p_n)$ (or briefly statistically summable $(\overline{N}, p)$ to l in $(X, \|\cdot,\cdot\|)$ if for every $z \in X$,

$$st - \lim_{n\to\infty} \|\sigma_n, z\| = \|l, z\|. \tag{1.5}$$

Notice that statistical summability $(\overline{N}, p)$ reduces to the statistical summability $(C,1)$ if $p_n = 1$ for all $n = 0,1,2,...$; to the statistical logarithmic summability $(\ell,1)$ if $p_n = 1/(n+1)$ for all $n = 0,1,2,....$

Kolk [14] obtained necessary and sufficient Tauberian conditions under which statistical convergence of a bounded sequence implies its statistical summability by weighted means. But the converse implication is not true in general. Móricz and Orhan [16] obtained Tauberian conditions under which the converse implication is satisfied for sequences for real and complex numbers.

In 2020, Jena et al. [11] proved some Tauberian theorems for Cesàro summability of double sequences of fuzzy numbers. Subsequently, Parida et al. [17] established some new Tauberian theorems based on post-quantum calculus by means of statistical Cesàro summability mean of real-valued continuous functions of one variable under some appropriate conditions.

On the other hand, Srivastava et al. [21] obtained a new Korovkin-type approximation theorem on a Banach space and verified that their theorem both extends and improves most of the previous results. Later on, Srivastava et al. [25] introduced the notions of statistical convergence and statistical summability for martingale sequences of random variables by means of deferred Cesàro mean and proved new Korovkin-type approximation theorems with algebraic test functions for a martingale sequence over a Banach space via these notions.

For recent developments in Korovkin and Voronovskaya type approximation theorems given in different contents, we refer to [1, 20, 22–24, 26].

In Çanak et al. [2], Tauberian theorems for summability by weighted mean method were obtained in two-normed spaces. Recently, Loku et al. [15] established necessary and sufficient Tauberian conditions under which ordinary convergence of a sequence follows from its summability by Nörlund mean method in two-normed spaces. Although a special case of summability by Nörlund mean method is summability by weighted mean method, this present paper is roughly a statistical extension of the results in Çanak et al. [2]. Therefore, the results obtained in Loku et al. [15] and this present paper are completely independent of each other. But one can easily obtain a statistical extension of the results in Loku et al. [15] by using the techniques in this present paper.

In this paper, we investigate the problem under what conditions (1.4) follows from (1.5) in two-normed spaces.

2. Main Result

Theorem 1. *Let (p_n) be a sequence of nonnegative numbers such that $p_0 > 0$ and*

$$st - \liminf_{n\to\infty} \frac{P_{\lambda_n}}{P_n} > 1 \quad \text{for every} \quad \lambda > 1, \tag{2.1}$$

where $\lambda_n := [\lambda n]$ denotes the integer part of the product λn, and let (x_n) be a sequence in $(X, ||\cdot,\cdot||)$ which is statistically summable $(\overline{N}, p)$ to $l \in X$. Then (x_n) is statistically convergent to l if and only if for every $z \in X$, one of the following two conditions is satisfied:

$$\inf_{\lambda>1} \limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \le N : \left\| \frac{1}{P_{\lambda_n} - P_n} \sum_{k=n+1}^{\lambda_n} p_k(x_k - x_n), z \right\| \ge \varepsilon \right\} \right| = 0, \tag{2.2}$$

$$\inf_{0<\lambda<1} \limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \le N : \left\| \frac{1}{P_n - P_{\lambda_n}} \sum_{k=\lambda_n+1}^{n} p_k(x_n - x_k), z \right\| \ge \varepsilon \right\} \right| = 0. \tag{2.3}$$

Following Hardy [10], a sequence (x_n) is said to be statistically slowly oscillating in two-norm if for every $\varepsilon > 0$ and for every $z \in X$,

$$\inf_{\lambda>1} \limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \le N : \max_{n<k\le\lambda_n} \|x_k - x_n, z\| \ge \varepsilon \right\} \right| = 0 \tag{2.4}$$

or equivalently,

$$\inf_{0<\lambda<1} \limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \le N : \max_{\lambda_n<k\le n} \|x_n - x_k, z\| \ge \varepsilon \right\} \right| = 0. \tag{2.5}$$

These conditions imply conditions (2.2) and (2.3), respectively.

Thus, the following corollary of Theorem 1 is obvious.

Corollary 1. *Let (p_n) be a sequence of nonnegative numbers such that $p_0 > 0$ and condition (2.1) is satisfied. If (1.5) and (2.4) hold, then (1.4) also holds.*

It is clear that (2.4) is satisfied if there exists a positive constant H such that for every $z \in X$, $n\|x_n - x_{n-1}, z\| \le H$ holds for every n large enough, say $n > n_1$.

For each $1 < n_1 \le n < k \le \lambda_n$ and each $z \in X$, we have

$$\|x_k - x_n, z\| \le \sum_{j=n+1}^{k} \|x_j - x_{j-1}, z\| \le H \sum_{j=n+1}^{k} \frac{1}{j} \le H\left(\frac{k-n}{n}\right) \le H(\lambda - 1). \tag{2.6}$$

It follows from (2.6) that

$$\max_{n<k\le\lambda_n} \|x_k - x_n, z\| \le \varepsilon \tag{2.7}$$

for each $\varepsilon > 0$ and $1 < \lambda \le 1 + \frac{\varepsilon}{H}$. For $n > n_1$ and for $z \in X$, the set

$$\left\{ n_1 < n \le N : \max_{n<k\le\lambda_n} \|x_k - x_n, z\| \ge \varepsilon \right\}$$

is empty. This shows that (2.4) is satisfied.

3. Auxiliary Results

Lemma 1 ([16]). *If (P_n) is a nondecreasing sequence of positive numbers, then conditions (2.1) and*

$$st-\liminf_{n\to\infty}\frac{P_n}{P_{\lambda_n}}>1 \quad \textit{for every} \quad 0<\lambda<1 \tag{3.1}$$

are equivalent.

Lemma 2. *Let (p_n) be a sequence of nonnegative numbers such that $p_0>0$ and condition (2.1) is satisfied, and let (x_n) be a sequence in X which is statistically summable $(\overline{N},p)$ to l. Then for every $\lambda>0$ and for every $z\in X$,*

$$st-\lim_{n\to\infty}\|\sigma_{\lambda_n},z\|=\|l,z\|. \tag{3.2}$$

Lemma 2 is given for sequences of complex numbers in [16]. Since the proof of Lemma 2 is similar to that of proof of Lemma 1 in [16], we omit the proof of it.

Lemma 3 ([9]). *Let (x_n) and (y_n) be sequences in 2-normed space $(X,\|\cdot,\cdot\|)$ and $L,L'\in X$ and $a\in\mathbb{R}$. If st-$\lim_{n\to\infty}\|x_n,z\|=\|L,z\|$ and st-$\lim_{n\to\infty}\|y_n,z\|=\|L',z\|$, for every nonzero $z\in X$, then*

(i) *st-$\lim_{n\to\infty}\|x_n+y_n,z\|=\|L+L',z\|$, for each nonzero $z\in X$ and*
(ii) *st-$\lim_{n\to\infty}\|ax_n,z\|=\|aL,z\|$, for each nonzero $z\in X$.*

Lemma 4. *Let (p_n) be a sequence of nonnegative numbers such that $p_0>0$ and condition (2.1) is satisfied, and let (x_n) be a sequence in X which is statistically summable $(\overline{N},p)$ to l. Then,*

$$st-\lim_{n\to\infty}\left\|\frac{1}{P_{\lambda_n}-P_n}\sum_{k=n+1}^{\lambda_n}p_kx_k,z\right\|=\|l,z\| \tag{3.3}$$

for every $\lambda>1$ and for every $z\in X$,

$$st-\lim_{n\to\infty}\left\|\frac{1}{P_n-P_{\lambda_n}}\sum_{k=\lambda_n+1}^{n}p_kx_k,z\right\|=\|l,z\| \tag{3.4}$$

for every $0<\lambda<1$ and for every $z\in X$.

Proof. Case $\lambda>1$. By definition, we have

$$\left\|\frac{1}{P_{\lambda_n}-P_n}\sum_{k=n+1}^{\lambda_n}p_kx_k-l,z\right\|=\left\|\frac{1}{P_{\lambda_n}-P_n}\sum_{k=n+1}^{\lambda_n}p_kx_k+\sigma_{\lambda_n}-\sigma_{\lambda_n}-l,z\right\|$$

$$=\left\|\frac{1}{P_{\lambda_n}-P_n}\sum_{k=0}^{\lambda_n}p_kx_k-\frac{1}{P_{\lambda_n}-P_n}\sum_{k=0}^{n}p_kx_k+\frac{1}{P_{\lambda_n}}\sum_{k=0}^{\lambda_n}p_kx_k-\frac{1}{P_{\lambda_n}}\sum_{k=0}^{\lambda_n}p_kx_k-l,z\right\|$$

$$=\left\|\frac{P_n}{P_{\lambda_n}-P_n}\frac{1}{P_{\lambda_n}}\sum_{k=0}^{\lambda_n}p_kx_k-\frac{P_n}{P_{\lambda_n}-P_n}\frac{1}{P_n}\sum_{k=0}^{n}p_kx_k+\frac{1}{P_{\lambda_n}}\sum_{k=0}^{\lambda_n}p_kx_k-l,z\right\|$$

$$\leq \frac{P_n}{P_{\lambda_n}-P_n}\left\|\sigma_{\lambda_n}-\sigma_n,z\right\|+\left\|\sigma_{\lambda_n}-l,z\right\|.$$

Thus we find

$$\left\|\frac{1}{P_{\lambda_n}-P_n}\sum_{k=n+1}^{\lambda_n}p_kx_k-l,z\right\|\leq \frac{P_n}{P_{\lambda_n}-P_n}\left\|\sigma_{\lambda_n}-\sigma_n,z\right\|+\left\|\sigma_{\lambda_n}-l,z\right\|. \tag{3.5}$$

By (2.1), we have

$$st-\limsup_{n\to\infty}\frac{P_n}{P_{\lambda_n}-P_n}=\left(st-\liminf_{n\to\infty}\frac{P_{\lambda_n}}{P_n}-1\right)^{-1}<\infty. \tag{3.6}$$

Now, (3.3) follows from (1.5), (3.2) and (3.5).

Case $0<\lambda<1$. By definition, we have

$$\begin{aligned}
&\left\|\frac{1}{P_n-P_{\lambda_n}}\sum_{k=\lambda_n+1}^{n}p_kx_k-l,z\right\|=\left\|\frac{1}{P_n-P_{\lambda_n}}\sum_{k=\lambda_n+1}^{n}p_kx_k+\sigma_n-\sigma_n-l,z\right\|\\
&=\left\|\frac{1}{P_n-P_{\lambda_n}}\sum_{k=0}^{n}p_kx_k-\frac{1}{P_n-P_{\lambda_n}}\sum_{k=0}^{\lambda_n}p_kx_k+\frac{1}{P_n}\sum_{k=0}^{n}p_kx_k-\frac{1}{P_n}\sum_{k=0}^{n}p_kx_k-l,z\right\|\\
&=\left\|\frac{P_{\lambda_n}}{P_n-P_{\lambda_n}}\frac{1}{P_n}\sum_{k=0}^{n}p_kx_k-\frac{P_{\lambda_n}}{P_n-P_{\lambda_n}}\frac{1}{P_{\lambda_n}}\sum_{k=0}^{\lambda_n}p_kx_k+\frac{1}{P_n}\sum_{k=0}^{n}p_kx_k-l,z\right\|\\
&\leq\frac{P_{\lambda_n}}{P_n-P_{\lambda_n}}\left\|\sigma_n-\sigma_{\lambda_n},z\right\|+\left\|\sigma_n-l,z\right\|.
\end{aligned}$$

Thus we find

$$\left\|\frac{1}{P_n-P_{\lambda_n}}\sum_{k=\lambda_n+1}^{n}p_kx_k-l,z\right\|\leq\frac{P_{\lambda_n}}{P_n-P_{\lambda_n}}\left\|\sigma_n-\sigma_{\lambda_n},z\right\|+\left\|\sigma_n-l,z\right\|. \tag{3.7}$$

By (3.1), we have

$$st-\limsup_{n\to\infty}\frac{P_{\lambda_n}}{P_n-P_{\lambda_n}}=\left(st-\liminf_{n\to\infty}\frac{P_n}{P_{\lambda_n}}-1\right)^{-1}<\infty. \tag{3.8}$$

Now, (3.4) follows from (1.5), (3.2) and (3.7). □

4. Proof of the Main Result

Proof of Theorem 1. Necessity. Assume that (1.4) and (1.5) are satisfied. Let $\lambda>1$. By Lemmas 3 and 4, we have

$$st-\lim_{n\to\infty}\left\|\frac{1}{P_{\lambda_n}-P_n}\sum_{k=n+1}^{\lambda_n}p_k(x_k-x_n),z\right\|$$

$$\leq st-\lim_{n\to\infty}\left\|\frac{1}{P_{\lambda_n}-P_n}\sum_{k=n+1}^{\lambda_n}p_kx_k-l,z\right\|+st-\lim_{n\to\infty}\|x_n-l,z\|=0$$

for every $z\in X$. This proves (2.2) even in a stronger form.

Let $0<\lambda<1$. We obtain in an analogous way that for every $z\in X$,

$$st-\lim_{n\to\infty}\left\|\frac{1}{P_n-P_{\lambda_n}}\sum_{k=\lambda_n+1}^{n}p_k(x_n-x_k),z\right\|=0,$$

which is stronger than (2.3).

Sufficiency. Assume that (1.5), (2.1) and one of conditions (2.2) and (2.3) are satisfied. To this end, let $\varepsilon>0$ be given. In case $\lambda>1$, we rewrite the difference x_n-l in the following form:

$$\begin{aligned}x_n-l&=\left[\frac{1}{P_{\lambda_n}-P_n}\sum_{k=n+1}^{\lambda_n}p_kx_k-l\right]-\left[\frac{1}{P_{\lambda_n}-P_n}\sum_{k=n+1}^{\lambda_n}p_k(x_k-x_n)\right]\\&=A+B,\ \text{say.}\end{aligned}$$

Then, we have

$$\{n\leq N:\|x_n-l,z\|\geq\varepsilon\}\subseteq\left\{n\leq N:\|A,z\|\geq\frac{\varepsilon}{2}\right\}\cup\left\{n\leq N:\|B,z\|\geq\frac{\varepsilon}{2}\right\}$$

for every $z\in X$. Hence,

$$|\{n\leq N:\|x_n-l,z\|\geq\varepsilon\}|\leq\left|\left\{n\leq N:\|A,z\|\geq\frac{\varepsilon}{2}\right\}\right|+\left|\left\{n\leq N:\|B,z\|\geq\frac{\varepsilon}{2}\right\}\right| \tag{4.1}$$

for every $z\in X$.

In case $0<\lambda<1$, we rewrite the difference x_n-l in the following form:

$$\begin{aligned}x_n-l&=\left[\frac{1}{P_n-P_{\lambda_n}}\sum_{k=\lambda_n+1}^{n}p_kx_k-l\right]+\left[\frac{1}{P_n-P_{\lambda_n}}\sum_{k=\lambda_n+1}^{n}p_k(x_n-x_k)\right]\\&=A'+B',\ \text{say.}\end{aligned}$$

Then, we have

$$\{n\leq N:\|x_n-l,z\|\geq\varepsilon\}\subseteq\left\{n\leq N:\|A',z\|\geq\frac{\varepsilon}{2}\right\}\cup\left\{n\leq N:\|B',z\|\geq\frac{\varepsilon}{2}\right\}$$

for every $z\in X$. Hence,

$$|\{n\leq N:\|x_n-l,z\|\geq\varepsilon\}|\leq\left|\left\{n\leq N:\|A',z\|\geq\frac{\varepsilon}{2}\right\}\right|+\left|\left\{n\leq N:\|B',z\|\geq\frac{\varepsilon}{2}\right\}\right| \tag{4.2}$$

for every $z \in X$. By (2.2), for every $\eta > 0$ there exists some $\lambda > 1$ such that

$$\limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \le N : \left\| \frac{1}{P_{\lambda_n} - P_n} \sum_{k=n+1}^{\lambda_n} p_k(x_k - x_n), z \right\| \ge \frac{\varepsilon}{2} \right\} \right| \le \eta, \tag{4.3}$$

for every $z \in X$ or by (2.3), for every $\eta > 0$ there exists some $0 < \lambda < 1$ such that

$$\limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \le N : \left\| \frac{1}{P_n - P_{\lambda_n}} \sum_{k=\lambda_n+1}^{n} p_k(x_n - x_k), z \right\| \ge \frac{\varepsilon}{2} \right\} \right| \le \eta \tag{4.4}$$

for every $z \in X$. Combining (4.1) and (4.4), in both cases we have by Lemma 4

$$\lim_{N\to\infty} \frac{1}{N+1} \left| \{ n \le N : \|x_n - l, z\| \ge \varepsilon \} \right| \le \eta$$

for every $z \in X$. Since $\eta > 0$ is arbitrary, we necessarily have for every $\varepsilon > 0$,

$$\lim_{N\to\infty} \frac{1}{N+1} \left| \{ n \le N : \|x_n - l, z\| \ge \varepsilon \} \right| = 0$$

for every $z \in X$. This completes the proof of the Theorem.

5. Concluding Remark

Theorem 1 is the statistical extension of the Tauberian theorem given in [2]. Condition (2.2) is equivalent to the following: For given $\varepsilon > 0$ and $\eta > 0$, there exists some $\lambda > 1$ such that

$$\limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \le N : \left\| \frac{1}{P_{\lambda_n} - P_n} \sum_{k=n+1}^{\lambda_n} p_k(x_k - x_n), z \right\| \ge \varepsilon \right\} \right| \le \eta.$$

The equivalent form of condition (2.3) can be similarly given.

If conditions (1.4), (1.5) and (2.1) are satisfied, then we necessarily have

$$st - \lim_{n\to\infty} \left\| \frac{1}{P_{\lambda_n} - P_n} \sum_{k=n+1}^{\lambda_n} p_k(x_k - x_n), z \right\| = 0 \tag{5.1}$$

for every $\lambda > 1$ and for every $z \in X$, and

$$st - \lim_{n\to\infty} \left\| \frac{1}{P_n - P_{\lambda_n}} \sum_{k=\lambda_n+1}^{n} p_k(x_n - x_k), z \right\| = 0 \tag{5.2}$$

for every $0 < \lambda < 1$ and for every $z \in X$.

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CONVERGENCE THEOREM AND CONVERGENCE RATE OF A NEW FASTER ITERATION METHOD FOR CONTINUOUS FUNCTIONS ON AN ARBITRARY INTERVAL

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Abstract. The aim of this paper is to propose a new faster iterative method, called the MN-iteration process, for approximating a fixed point of continuous functions on an arbitrary interval. Then, a necessary and sufficient condition for the convergence of the MN-iteration of continuous functions on an arbitrary interval is established. We also compare the rate of convergence between the proposed iteration and some other iteration processes in the literature. Specifically, our main result shows that MN-iteration converges faster than NSP-iteration to the fixed point. We finally give numerical examples to compare the result with Mann, Ishikawa, Noor, SP and NSP iterations. Our findings improve corresponding results in the contemporary literature.

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1. Introduction

Let C be a closed interval on the real line and let $f\colon C \to C$ be a continuous function. A point $x \in C$ is called a fixed point of f if $f(x) = x$.

Iteration procedures are used in nearly every branch of applied mathematics. There are many iterative methods for finding a fixed point of f. In computational mathematics, it is important to compare the iterative schemes with regard to their rate of convergence.

The classical iteration process was introduced by Mann [7] which is formulated as follows:

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n f(u_n) \tag{1.1}$$

for all $n \geq 1$, where $\alpha_n \in [0,1]$. Such an iteration process is known as *Mann iteration.* In 1991, Borwein and Borwein [3] proved the convergence theorem for a continuous function on the closed and bounded interval in the real line by using iteration (1.1).

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The Ishikawa iterative scheme, usually called the two-step iteration method, due to Ishikawa [6] is given by $s_1 \in C$ and

$$\begin{aligned} t_n &= (1-\beta_n)s_n + \beta_n f(s_n), \\ s_{n+1} &= (1-\alpha_n)s_n + \alpha_n f(t_n) \end{aligned} \tag{1.2}$$

for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1]. Such iterative method is called *Ishikawa iteration.* In 2006, Qing and Qihou [11] proved the convergence theorem of the sequence generated by iteration (1.2) for a continuous function on the closed interval in the real line (see also [15]).

In 2000, Noor [8] defined the following iterative scheme by $l_1 \in C$ and

$$\begin{aligned} m_n &= (1-\mu_n)l_n + \mu_n f(l_n), \\ v_n &= (1-\beta_n)l_n + \beta_n f(m_n), \\ l_{n+1} &= (1-\alpha_n)l_n + \alpha_n f(v_n) \end{aligned} \tag{1.3}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\mu_n\}$ are sequences in [0,1], which is called *Noor iteration* for continuous functions on an arbitrary interval in the real line.

Clearly, the Mann and Ishikawa iteration processes are special cases of the Noor iteration. However, there are only a few articles concerning comparison of those iterative methods in order to establish which one converges faster. For more details, we orient the reader to [1, 2, 10, 12, 14] and references therein.

Rhoades [13] introduced the concept to compare iterative methods which one converges faster as follows.

Definition 1 ([9, Definition 3.1])**.** Let C be a closed interval on the real line and let $f\colon C \to C$ be a continuous mapping. Suppose that $\{x_n\}$ and $\{w_n\}$ are two iterations which converge to the fixed point p of f. Then $\{x_n\}$ is said to converge faster than $\{w_n\}$ if

$$|x_n - p| \leq |w_n - p|$$

for all $n \geq 1$.

In 2011, Phuengrattana and Suantai [9] introduced and studied the SP-iteration as follows: $h_1 \in C$ and

$$\begin{aligned} e_n &= (1-\mu_n)h_n + \mu_n f(h_n), \\ d_n &= (1-\beta_n)e_n + \beta_n f(e_n), \\ h_{n+1} &= (1-\alpha_n)d_n + \alpha_n f(d_n) \end{aligned} \tag{1.4}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\mu_n\}$ are sequences in [0,1]. They showed that (1.4) converges to a fixed point of f. Moreover, the rate of convergence is better than those of Mann (1.1), Ishikawa (1.2) and Noor (1.3) in the sense of Rhoades [13].

Clearly Mann iteration is special cases of SP-iteration. Some interesting results concerning fixed point theory of continuous functions can be found in [5].

Recently, by combining the SP-iteration and Noor iteration, Cholamjiak and Pholasa [4] proposed the NSP-iteration as follows: $w_1 \in C$ and

$$\begin{aligned} r_n &= (1-\mu_n)w_n + \mu_n f(w_n), \\ q_n &= (1-\tau_n-\beta_n)w_n + \tau_n r_n + \beta_n f(r_n), \\ w_{n+1} &= (1-\gamma_n-\alpha_n)w_n + \gamma_n q_n + \alpha_n f(q_n) \end{aligned} \tag{1.5}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\mu_n\}$, $\{\tau_n\}$ and $\{\gamma_n\}$ are sequences in [0,1]. They proved some convergence theorems of such iterations for continuous functions on an arbitrary interval. Also, they compared the rate of convergence of Mann, Ishikawa, Noor and NSP iterations by numerical examples and concluded that NSP-iteration converges faster than all of them.

Inspired and motivated by these facts, we introduce and study a new modified Noor-iteration process for solving a fixed point problem for continuous function on an arbitrary interval in the real line. The scheme is defined as follows.

Let C be a closed interval on the real line and $f\colon C \to C$ given mapping. Then for an arbitrary $x_1 \in C$, the following iteration scheme is studied:

$$\begin{aligned} z_n &= (1-\mu_n)x_n + \mu_n f(x_n), \\ y_n &= (1-\tau_n-\beta_n)x_n + \tau_n f(x_n) + \beta_n f(z_n), \\ x_{n+1} &= (1-\gamma_n-\alpha_n)x_n + \gamma_n f(z_n) + \alpha_n f(y_n),\ n \geq 1, \end{aligned} \tag{1.6}$$

where, $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ are appropriate real sequences in $[0,1]$. The iterative scheme (1.6) is called the modified Noor iteration for continuous functions (abbreviate MN-iteration).

The first purpose of this article is to give a necessary and sufficient condition for the strong convergence of the MN-iteration of continuous functions on an arbitrary interval. The second purpose is to improve the rate of convergence compared to previous work. Specifically, our main result shows that MN-iteration converges faster than NSP-iteration to the fixed point. Numerical examples are also presented to compare the result with Mann, Ishikawa, Noor, SP and NSP iterations.

Consequently, we have that MN-iteration converges faster than the other schemes in the same category.

2. Convergence theorem

In this section, we provide the convergence theorem of MN-iteration (1.6) for continuous functions on an arbitrary closed interval. Now, we will give some crucial lemmas for proofs of our main results.

Lemma 1. *Let C be a closed interval on the real line (can be unbounded) and let $f\colon C \to C$ be a continuous function. Let $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0,1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \beta_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$*

and $\sum_{n=1}^{\infty} \tau_n < \infty$. *From an arbitrary initial guess* $x_1 \in C$, *define the sequence* $\{x_n\}$ *using (1.6). If* $x_n \to a$, *then* a *is a fixed point of* f.

Proof. Let $x_n \to a$, and suppose $a \neq f(a)$. Then $\{x_n\}$ is bounded. So, $\{f(x_n)\}$ is bounded by the continuity of f. So are $\{y_n\}, \{z_n\}, \{f(y_n)\}$ and $\{f(z_n)\}$. Moreover, $z_n \to a$ since $x_n \to a$ and $\mu_n \to 0$. We also have $y_n \to a$ since $x_n \to a$, $\beta_n \to 0$ and $\tau_n \to 0$. From (1.6), we get

$$\begin{aligned} x_{n+1} &= (1-\gamma_n-\alpha_n)x_n + \gamma_n f(z_n) + \alpha_n f(y_n) \\ &= x_n + \gamma_n(f(z_n)-x_n) + \alpha_n(f(y_n)-x_n). \end{aligned} \tag{2.1}$$

Let $p_k = f(z_k) - x_k, q_k = f(y_k) - x_k$. Then, we have

$$\begin{aligned} \lim_{k\to\infty} p_k &= \lim_{k\to\infty}(f(z_k)-x_k) = f(a)-a \neq 0, \\ \lim_{k\to\infty} q_k &= \lim_{k\to\infty}(f(y_k)-x_k) = f(a)-a \neq 0. \end{aligned}$$

From (2.1) we get

$$\begin{aligned} x_n &= x_1 + \sum_{k=1}^{n} \gamma_k(f(z_k)-x_k) + \sum_{k=1}^{n} \alpha_k(f(y_k)-x_k) \\ &= x_1 + \sum_{k=1}^{n} \gamma_k p_k + \sum_{k=1}^{n} \alpha_k q_k. \end{aligned}$$

It is worth noting here that $\sum_{k=1}^{\infty} \gamma_k p_k < \infty$ since $\lim_{k\to\infty} p_k \neq 0$ and $\sum_{k=1}^{\infty} \gamma_k < \infty$. This shows that $\{x_n\}$ is a divergent sequence since $\lim_{k\to\infty} q_k \neq 0$ and $\sum_{k=1}^{\infty} \alpha_k = \infty$. This contradicts to the convergence of $\{x_n\}$. Hence $f(a) = a$ and a is fixed point of f. □

Lemma 2. *Let* C *be a closed interval on the real line (can be unbounded) and let* $f\colon C \to C$ *be a continuous function. Let* $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ *and* $\{\tau_n\}$ *be sequences in* $[0,1]$ *such that* $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \beta_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ *and* $\sum_{n=1}^{\infty} \tau_n < \infty$. *From an arbitrary initial guess* $x_1 \in C$, *define the sequence* $\{x_n\}$ *using (1.6). If* $\{x_n\}$ *is bounded, then* $\{x_n\}$ *is convergent.*

Proof. Suppose $\{x_n\}$ is not convergent. Let $a = \liminf_n x_n$ and $b = \limsup_n x_n$. Then $a < b$. We first show that if $a < m < b$, then $f(m) = m$. Suppose $f(m) \neq m$. Without loss of generality, we suppose $f(m) - m > 0$. Since f is continuous, there exists δ with $0 < \delta < b - a$ such that for $|x - m| \leq \delta$,

$$f(x) - x > 0.$$

By continuity of f and $\{x_n\}$ is bounded we have that $\{f(x_n)\}$ is bounded, so $\{z_n\}$, $\{y_n\}$, $\{f(z_n)\}$ and $\{f(y_n)\}$ are bounded sequences. Using

$$\begin{aligned} x_{n+1}-x_n &= \gamma_n(f(z_n)-x_n)+\alpha_n(f(y_n)-x_n),\\ y_n-x_n &= \tau_n(f(x_n)-x_n)+\beta_n(f(z_n)-x_n),\\ z_n-x_n &= \mu_n(f(x_n)-x_n), \end{aligned}$$

we can easily show that $|z_n-x_n|\to 0$, $|y_n-x_n|\to 0$ and $|x_{n+1}-x_n|\to 0$. Thus, there exists a positive integer N such that

$$|x_{n+1}-x_n|<\frac{\delta}{2},\ |y_n-x_n|<\frac{\delta}{2},\ |z_n-x_n|<\frac{\delta}{2}, \forall n>N. \tag{2.2}$$

Since $b=\limsup_n x_n>m$, there exists $k_1>N$ such that $x_{n_{k_1}}>m$. Let $n_{k_1}=k$, then $x_k>m$. For x_k, there exist two cases as follows:

(i) $x_k>m+\frac{\delta}{2}$, then $x_{k+1}>x_k-\frac{\delta}{2}\geq m$ using (2.2). So, we have $x_{k+1}>m$.

(ii) $m<x_k<m+\frac{\delta}{2}$, then $m-\frac{\delta}{2}<y_k<m+\delta$ and $m-\frac{\delta}{2}<z_k<m+\delta$ by (2.2). So, we obtain $|x_k-m|<\frac{\delta}{2}<\delta, |y_k-m|<\delta, |z_k-m|<\delta$. Hence

$$f(x_k)-x_k>0,\ f(y_k)-y_k>0,\ f(z_k)-z_k>0. \tag{2.3}$$

From (2.1) and (2.3), we have

$$\begin{aligned} x_{k+1} &= x_k+\gamma_k(f(z_k)-x_k)+\alpha_k(f(y_k)-x_k)\\ &= x_k+\gamma_k(f(z_k)-z_k)+\gamma_k(z_k-x_k)+\alpha_k(f(y_k)-y_k)+\alpha_k(y_k-x_k)\\ &= x_k+\gamma_k(f(z_k)-z_k)+\gamma_k\mu_k(f(x_k)-x_k)+\alpha_k(f(y_k)-y_k)\\ &\quad+\alpha_k\tau_k(f(x_k)-x_k)+\alpha_k\beta_k(f(z_k)-x_k)\\ &= x_k+\gamma_k(f(z_k)-z_k)+\gamma_k\mu_k(f(x_k)-x_k)+\alpha_k(f(y_k)-y_k)\\ &\quad+\alpha_k\tau_k(f(x_k)-x_k)+\alpha_k\beta_k(f(z_k)-z_k)+\alpha_k\beta_k(z_k-x_k)\\ &= x_k+\gamma_k(f(z_k)-z_k)+\gamma_k\mu_k(f(x_k)-x_k)+\alpha_k(f(y_k)-y_k)\\ &\quad+\alpha_k\tau_k(f(x_k)-x_k)+\alpha_k\beta_k(f(z_k)-z_k)+\alpha_k\beta_k\mu_k(f(x_k)-x_k)\\ &> x_k. \end{aligned}$$

Thus $x_{k+1}>x_k>m$. This together with (i) and (ii), imply $x_{k+1}>m$. Similarly, we get that $x_{k+2}>m$, $x_{k+3}>m,\ldots$. Thus we have $x_n>m$ for all $n>k=n_{k_1}$. So $a=\lim_{k\to\infty}x_{n_k}\geq m$, which is a contradiction with $a<m$. Thus $f(m)=m$.

We next consider the following two cases.

(i) There exists x_M such that $a<x_M<b$. Then $f(x_M)=x_M$. It follows that

$$z_M=(1-\mu_M)x_M+\mu_M f(x_M)=x_M$$

and

$$\begin{aligned} y_M &= (1-\tau_M-\beta_M)z_M+\tau_M f(x_M)+\beta_M f(z_M) \\ &= (1-\tau_M-\beta_M)x_M+\tau_M f(x_M)+\beta_M f(x_M) \\ &= x_M. \end{aligned}$$

It follows that

$$\begin{aligned} x_{M+1} &= (1-\gamma_M-\alpha_M)y_M+\gamma_M f(z_M)+\alpha_M f(y_M) \\ &= (1-\tau_M-\gamma_M)x_M+\gamma_M f(x_M)+\alpha_M f(x_M) \\ &= x_M. \end{aligned}$$

Similarly, we obtain $x_M = x_{M+1} = x_{M+2} = \dots$. It clear that $x_n \to x_M$. Since there exists $x_{n_k} \to a, x_M = a$. This shows that $x_n \to a$, which is a contradiction.

(ii) For all $n, x_n \le a$ or $x_n \ge b$. Since $b-a>0$ and $\lim_{n\to\infty}|x_{n+1}-x_n| = 0$, there exists $\tilde{N}$ such that $|x_{n+1}-x_n| < \frac{(b-a)}{2}$ for $n > \tilde{N}$. So, it is seen that $x_n \le a$ for $n > \tilde{N}$, or it is always that $x_n \ge b$ for $n > \tilde{N}$. If $x_n \le a$ for $n > \tilde{N}$, then $b = \lim_{j\to\infty} x_{n_j} \le a$, which is a contradiction with $a < b$. If $x_n \ge b$ for $n > \tilde{N}$, then $a = \lim_{k\to\infty} x_{n_k} \ge b$, which is a contradiction with $a < b$. Thus we conclude that $x_n \to a$. The proof is completed.

□

We are now ready to prove the main theorem.

Theorem 1. *Let C be a closed interval on the real line (can be unbounded) and let $f\colon C \to C$ be a continuous function. Let $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0,1]$ such that $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\beta_n < \infty$, $\sum_{n=1}^{\infty}\mu_n < \infty$, $\sum_{n=1}^{\infty}\gamma_n < \infty$ and $\sum_{n=1}^{\infty}\tau_n < \infty$. From an arbitrary initial guess $x_1 \in C$, define the sequence $\{x_n\}$ using (1.6). Then $\{x_n\}$ is bounded if and only if it converges to a fixed point of f.*

Proof. Sufficiency is obvious. It suffices to show that if $\{x_n\}$ is bounded, then $\{x_n\}$ converges to a fixed point. Let $\{x_n\}$ be a bounded sequence. Using Lemma 2, we have $\{x_n\}$ is a convergent sequence. Hence, by Lemma 1, it converges to a fixed point of f. □

When $C = [a,b]$ in Theorem 1, we obtain the following result.

Corollary 1. *Let $f\colon [a,b] \to [a,b]$ be a continuous function. Let $\{\alpha_n\},\{\beta_n\}$, $\{\mu_n\},\{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0,1]$. Let $\{x_n\}$ be a sequence generated iteratively by $x_1 \in [a,b]$ and*

$$\begin{aligned} z_n &= (1-\mu_n)x_n + \mu_n f(x_n), \\ y_n &= (1-\tau_n-\beta_n)x_n + \tau_n f(x_n) + \beta_n f(z_n), \\ x_{n+1} &= (1-\gamma_n-\alpha_n)x_n + \gamma_n f(z_n) + \alpha_n f(y_n),\ n \geq 1, \end{aligned}$$

where $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \beta_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \tau_n < \infty$. Then $\{x_n\}$ converges to a fixed point of f.

3. Rate of convergence

In this section, we provide a theoretical estimation proof of the rate of convergence of the sequence $\{x_n\}$ defined by (1.6). We compare the convergence rate of (1.6) with the NSP-iteration proposed in [4]. We show that the MN-iteration (1.6) converges faster than the NSP-iteration (1.5) for the class of continuous nondecreasing functions on an arbitrary interval in the sense of Rhoades [13].

We next prove some crucial lemmas which will be used in the sequel.

Lemma 3. *Let C be a closed interval on the real line and let $f\colon C \to C$ be a continuous and nondecreasing function. Let $\{\alpha_n\},\{\beta_n\},\{\mu_n\},\{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0,1)$. Let $\{w_n\}$ and $\{x_n\}$ be sequences defined by (1.5) and (1.6), respectively. Then the following hold:*

(i) If $f(w_1) < w_1$, then $f(w_n) < w_n$ for all $n \geq 1$ and $\{w_n\}$ is nonincreasing.
(ii) If $f(w_1) > w_1$, then $f(w_n) > w_n$ for all $n \geq 1$ and $\{w_n\}$ is nondecreasing.
(iii) If $f(x_1) < x_1$, then $f(x_n) < x_n$ for all $n \geq 1$ and $\{x_n\}$ is nonincreasing.
(iv) If $f(x_1) > x_1$, then $f(x_n) > x_n$ for all $n \geq 1$ and $\{x_n\}$ is nondecreasing.

Proof.

(i) Let $f(w_1) < w_1$. Then $f(w_1) < r_1 \leq w_1$. Since f is nondecreasing, we have $f(r_1) \leq f(w_1) < r_1 \leq w_1$. This implies $f(r_1) < q_1 \leq w_1$. Thus $f(q_1) \leq f(w_1) < r_1 \leq w_1$. For q_1, we consider the following two cases.

Case 1: $f(r_1) < q_1 \leq r_1$. Then $f(q_1) \leq f(r_1) < q_1 \leq r_1 \leq w_1$. This implies $f(q_1) < w_2 \leq w_1$. Thus $f(w_2) \leq f(w_1) < r_1 \leq w_1$. It follows that if $f(q_1) < w_2 \leq q_1$, then $f(w_2) \leq f(q_1) < w_2$, if $q_1 < w_2 \leq r_1$, then $f(w_2) \leq f(r_1) < q_1 < w_2$ and if $r_1 < w_2 \leq w_1$, then $f(w_2) \leq f(w_1) < r_1 < w_2$. Thus we have $f(w_2) < w_2$.

Case 2: $r_1 < q_1 \leq w_1$. Then $f(q_1) \leq f(w_1) < r_1 \leq w_1$. This implies $f(q_1) < w_2 \leq w_1$. Thus $f(w_2) \leq f(w_1) < r_1 < q_1 \leq w_1$. It follows that if $f(q_1) <$

$w_2 \leq q_1$, then $f(w_2) \leq f(q_1) < w_2$ and if $q_1 < w_2 \leq w_1$, then $f(w_2) \leq f(w_1) < q_1 < w_2$. Hence, we have $f(w_2) < w_2$.

In conclusion by Case 1 and Case 2, we have $f(w_2) < w_2$. By continuing in this way, we can show that $f(w_n) < w_n$ for all $n \geq 1$. This implies $r_n \leq w_n$ for all $n \geq 1$. Since f is nondecreasing, we have $f(r_n) \leq f(w_n) < w_n$ for all $n \geq 1$. Thus $q_n \leq w_n$ for all $n \geq 1$, then $f(q_n) \leq f(w_n) < w_n$ for all $n \geq 1$. Hence, we have $w_{n+1} \leq w_n$ for all $n \geq 1$, that is $\{w_n\}$ is nonincreasing.

(ii) By using the same argument as in (i), we obtain the desired result.

(iii) Let $f(x_1) < x_1$. Then $f(x_1) < z_1 \leq x_1$. Since f is nondecreasing, we have $f(z_1) \leq f(x_1) < z_1 \leq x_1$. This implies $f(z_1) < y_1 \leq x_1$. Thus $f(y_1) \leq f(x_1) < z_1 \leq x_1$. For y_1, we consider the following two cases.

Case 1: $f(z_1) < y_1 \leq z_1$. Then $f(y_1) \leq f(z_1) < z_1 < x_1$. It follows that if $f(y_1) < x_2 \leq y_1$, then $f(x_2) \leq f(y_1) < x_2$, if $y_1 < x_2 \leq z_1$, then $f(x_2) \leq f(z_1) < y_1 < x_2$ and if $z_1 < x_2 \leq x_1$, then $f(x_2) \leq f(x_1) < z_1 < x_2$. Thus we have $f(x_2) < x_2$.

Case 2: $z_1 < y_1 \leq x_1$. Then $f(y_1) \leq f(x_1) < z_1 \leq x_1$. This implies $f(y_1) < x_2 \leq x_1$. Thus $f(x_2) \leq f(x_1) < z_1 < y_1 \leq x_1$. It follows that if $f(y_1) < x_2 \leq y_1$, then $f(x_2) \leq f(y_1) < x_2$ and if $y_1 < x_2 \leq x_1$, then $f(x_2) \leq f(x_1) < y_1 < x_2$. Hence, we have $f(x_2) < x_2$.

In conclusion by Case 1 and Case 2, we have $f(x_2) < x_2$. By continuing in this way, we can show that $f(x_n) < x_n$ for all $n \geq 1$. This implies $z_n \leq x_n$ for all $n \geq 1$. Since f is nondecreasing, we have $f(z_n) \leq f(x_n) < x_n$ for all $n \geq 1$. Thus $y_n \leq x_n$ for all $n \geq 1$, then $f(y_n) \leq f(x_n) < x_n$ for all $n \geq 1$. Hence, we have $x_{n+1} \leq x_n$ for all $n \geq 1$, that is $\{x_n\}$ is nonincreasing.

(iv) Following the proof line as in (iii), we obtain the desired result.

□

Lemma 4. *Let C be a closed interval on the real line and let $f\colon C \to C$ be a continuous and nondecreasing function. Let $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0,1)$. For $w_1 = x_1 \in C$, let $\{w_n\}$ and $\{x_n\}$ be sequences defined by the NSP-iteration (1.5) and MN-iteration (1.6), respectively. Then the following are satisfied:*

(i) If $f(w_1) < w_1$, then $x_n \leq w_n$ for all $n \geq 1$.
(ii) If $f(w_1) > w_1$, then $x_n \geq w_n$ for all $n \geq 1$.

Proof.

(i) Let $f(w_1) < w_1$. Then $f(x_1) < x_1$ since $w_1 = x_1$. From (1.6), we get $f(x_1) < z_1 \leq x_1$. Since f is nondecreasing, we obtain $f(z_1) \leq f(x_1) < z_1 \leq x_1$. Hence $f(z_1) < y_1 \leq z_1$. Using the NSP-iteration (1.5) and MN-iteration (1.6), we obtain the following estimation: $z_1 - r_1 = (1-\mu_1)(x_1 - w_1) + \mu_1(f(x_1) - f(w_1)) = 0$. So, $z_1 = r_1$, and so $y_1 - q_1 = (1-\tau_1-\beta_1)(x_1 - w_1) + \tau_1(f(x_1) - r_1) + \beta_1(f(z_1) - f(r_1)) \leq 0$. Hence, we have $y_1 \leq q_1$. Since

f is nondecreasing, we have $f(y_1) \leq f(q_1)$. We next obtain $x_2 - w_2 = (1 - \gamma_1 - \alpha_1)(x_1 - w_1) + \gamma_1(f(z_1) - q_1) + \alpha_1(f(y_1) - f(q_1)) \leq 0$, so, $x_2 \leq w_2$. Assume that $x_k \leq w_k$. Thus $f(x_k) \leq f(w_k)$. From Lemma 3 (i) and Lemma 3 (iii), we get $f(w_k) < w_k$ and $f(x_k) < x_k$. It follows that $f(x_k) < z_k \leq x_k$ and $f(z_k) \leq f(x_k) < z_k$. Thus $z_k - r_k = (1-\mu_k)(x_k - w_k) + \mu_k(f(x_k) - f(w_k)) \leq 0$. So, $z_k \leq r_k$. Since $f(z_k) \leq f(r_k)$, we have $y_k - q_k = (1 - \tau_k - \beta_k)(x_k - w_k) + \tau_k(f(x_k) - r_k) + \beta_k(f(z_k) - f(r_k)) \leq 0$, so, $y_k \leq q_k$, which yields $f(y_k) \leq f(q_k)$. In addition, $f(z_k) \leq f(x_k) < z_k \leq x_k$, using (1.6), we have

$f(z_k) - y_k = (1 - \tau_k - \beta_k)(f(z_k) - x_k) + \tau_k(f(z_k) - f(x_k)) + \beta_k(f(z_k) - f(z_k)) \leq 0$. So, $f(z_k) - q_k = (f(z_k) - y_k) + (y_k - q_k) \leq 0$. This shows that $x_{k+1} - w_{k+1} = (1 - \gamma_k - \alpha_k)(x_k - w_k) + \gamma_k(f(z_k) - q_k) + \alpha_k(f(y_k) - f(q_k)) \leq 0$, which gives, $x_{k+1} \leq w_{k+1}$. By induction, we conclude that $x_n \leq w_n$ for all $n \geq 1$.

(ii) From Lemma 3 (ii), Lemma 3 (iv) and the same argument as in (i), we can show that $x_n \geq w_n$ for all $n \geq 1$.

□

For convenience, we write algorithm (1.6) by $MN(x_1, \alpha_n, \beta_n, \mu_n, \gamma_n, \tau_n, f)$.

Proposition 1. *Let C be a closed interval on the real line and let $f\colon C \to C$ be a continuous and nondecreasing function such that $F(f)$ is nonempty and bounded with $x_1 > \sup\{p \in C\colon p = f(p)\}$. Let $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0,1)$. If $f(x_1) > x_1$, then $\{x_n\}$ defined by $NSP(x_1, \alpha_n, \beta_n, \mu_n, \gamma_n, \tau_n, f)$ and $MN(x_1, \alpha_n, \beta_n, \mu_n, \gamma_n, \tau_n, f)$ do not converge to a fixed point of f.*

Proof. From Lemma 3 $((ii), (iv))$, we know that $\{x_n\}$ is nondecreasing. Since the initial point $x_1 > \sup\{p \in C\colon p = f(p)\}$, it follows that $\{x_n\}$ does not converge to a fixed point of f. □

Proposition 2. *Let C be a closed interval on the real line and let $f\colon C \to C$ be a continuous and nondecreasing function such that $F(f)$ is nonempty and bounded with $x_1 < \inf\{p \in C\colon p = f(p)\}$. Let $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0,1)$. If $f(x_1) < x_1$, then $\{x_n\}$ defined by $NSP(x_1, \alpha_n, \beta_n, \mu_n, \gamma_n, \tau_n, f)$ and $MN(x_1, \alpha_n, \beta_n, \mu_n, \gamma_n, \tau_n, f)$ do not converge to a fixed point of f.*

Proof. From Lemma 3 $((i), (iii))$, we know that $\{x_n\}$ is nonincreasing. Since the initial point $x_1 < \inf\{p \in C\colon p = f(p)\}$, it follows that $\{x_n\}$ does not converge to a fixed point of f. □

Next, we compare the rate of convergence of MN-iteration with NSP-iteration.

Theorem 2. *Let C be a closed interval on the real line and let $f\colon C \to C$ be a continuous and nondecreasing function such that $F(f)$ is nonempty and bounded. Let $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0,1)$. For $w_1 = x_1 \in C$, let $\{w_n\}$ and $\{x_n\}$ be sequences defined by the NSP-iteration (1.5) and the MN-iteration (1.6),*

respectively. If the NSP-iteration $\{w_n\}$ converges to $p \in F(f)$, then the MN-iteration $\{x_n\}$ converges to p. Moreover, the MN-iteration (1.6) converges faster than the NSP-iteration (1.5).

Proof. Assume that the NSP-iteration $\{w_n\}$ converges to $p \in F(f)$. Put $L = \inf\{p \in C\colon p = f(p)\}$ and $U = \sup\{p \in C\colon p = f(p)\}$. For $w_1 = x_1$, we divide our proof into the following three cases:

Case 1: $w_1 = x_1 > U$, Case 2: $w_1 = x_1 < L$, Case 3: $L \leq w_1 = x_1 \leq U$.

Case 1: $w_1 = x_1 > U$. By Proposition 1, we get $f(w_1) < w_1$ and $f(x_1) < x_1$. So, by Lemma 4 (i), we have $x_n \leq w_n$ for all $n \geq 1$. By induction, we can show that $U \leq x_n$ for all $n \geq 1$. Then, we have $0 \leq x_n - p \leq w_n - p$, which yields $|x_n - p| \leq |w_n - p|$ for all $n \geq 1$. This shows that $x_n \to p$. By Definition 1, we conclude that the MN-iteration $\{x_n\}$ converges faster than the NSP-iteration $\{w_n\}$.

Case 2: $w_1 = x_1 < L$. By Proposition 2, we get $f(w_1) > w_1$ and $f(x_1) > x_1$. This implies, by Lemma 4 (ii), that $x_n \geq w_n$ for all $n \geq 1$. So, by induction, we can show that $x_n \leq L$ for all $n \geq 1$. Then, we have $|x_n - p| \leq |w_n - p|$ for all $n \geq 1$. It follows that $x_n \to p$ and the MN-iteration $\{x_n\}$ converges faster than the NSP-iteration $\{w_n\}$.

Case 3: $L \leq w_1 = x_1 \leq U$. Suppose that $f(w_1) \neq w_1$. If $f(w_1) < w_1$, we have, by Lemma 3 (i), that $\{w_n\}$ is nonincreasing with limit p. Lemma 4 (i) gives $p \leq x_n \leq w_n$ for all $n \geq 1$. It follows that $|x_n - p| \leq |w_n - p|$ for all $n \geq 1$. Therefore $x_n \to p$ and the result follows. If $f(w_1) > w_1$, by Lemma 3 (ii) and Lemma 4 (ii), then we can also show that the result holds.

□

4. Numerical examples

In this section, some numerical examples are given to demonstrate the convergence of the algorithm defined in this paper. For convenience, we call the iteration (1.6) the MN-iteration.

Example 1. $f\colon [-1,4] \to [-1,4]$ defined by $f(x) = \frac{x^3+x-3}{19}$. The fixed point of the function is $p = -0.166925$. Initial point is $x_1 = 4$ and control conditions are $\alpha_n = \frac{1}{(n+1)^{0.5}}$, $\beta_n = \frac{1}{(n+1)^{1.7}}$, $\mu_n = \frac{1}{(n+1)^{2.3}}$, $\gamma_n = \frac{1}{(n+1)^{1.5}}$ and $\tau_n = \frac{1}{(n+1)^{1.1}}$. The stopping criteria is $|x_n - p| < 10^{-8}$.

Example 2. $f\colon [1,\infty] \to [1,\infty]$ defined by $f(x) = x^{0.3} - (\sqrt{log(x+9)-1})^3$. The fixed point of the function is $p = 1$. Initial point is $x_1 = 9$ and control conditions are $\alpha_n = \frac{1}{(n+1)^{0.5}}$, $\beta_n = \frac{1}{(n+1)^{2.0}}$, $\mu_n = \frac{1}{(n+1)^{3.6}}$, $\gamma_n = \frac{1}{(n+1)^{2.5}}$ and $\tau_n = \frac{1}{(n+1)^{1.1}}$. The stopping criteria is $|x_n - p| < 10^{-6}$.

n	Mann	Ishikawa	Noor	NSP	SP	MN-iteration	
	u_n	s_n	l_n	w_n	h_n	x_n	$\lvert x_n - p\rvert$
1	4	4	4	4	4	4	4.166925
5	1.393239	0.753340	0.628636	0.451696	0.293228	0.016801	0.183726
10	0.046198	-0.049731	-0.066200	-0.090005	-0.119193	-0.150974	0.015950
15	-0.120743	-0.141524	-0.145086	-0.150271	-0.157307	-0.163938	0.015950
20	-0.153846	-0.159732	-0.160740	-0.162211	-0.164309	-0.166147	0.000777
25	-0.162557	-0.164523	-0.164860	-0.165351	-0.166074	-0.166679	0.000245
30	-0.165290	-0.166026	-0.166152	-0.166336	-0.166612	-0.166836	0.000088
35	-0.166258	-0.166558	-0.166610	-0.166685	-0.166799	-0.166890	0.000034
No. of iterations	133	126	124	121	113	99	

TABLE 1. Mann, Ishikawa, Noor, NSP, SP and MN iterations for $f(x) = \frac{x^3+x-3}{19}$.

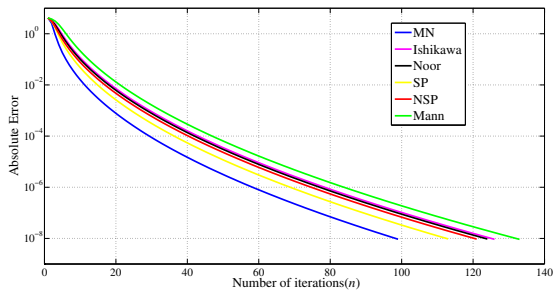

FIGURE 1. Error values obtained from MN, Ishikawa, Noor, SP, NSP and Mann iterations for $f(x) = \frac{x^3+x-3}{19}$.

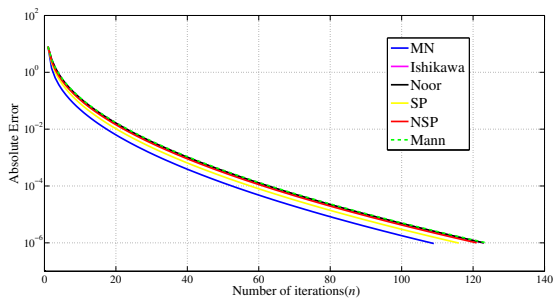

FIGURE 2. Mann, Ishikawa, Noor, SP, NSP and MN iterations for given $x_1 = 9$ of $f(x) = x^{0.3} - (\sqrt{log(x+9)} - 1)^3$.

n	Mann	Ishikawa	Noor	NSP	SP	MN-iteration	
	u_n	s_n	l_n	w_n	h_n	x_n	$\lvert x_n - p\rvert$
1	9	9	9	9	9	9	8
5	1.590135	1.567476	1.567411	1.493424	1.382242	1.235851	0.235851
10	1.127788	1.122291	1.122277	1.107154	1.079598	1.048813	0.048813
15	1.042340	1.040461	1.040456	1.035520	1.025965	1.015841	0.015841
20	1.017031	1.016264	1.016262	1.014287	1.010351	1.006290	0.006290
25	1.007706	1.007356	1.007355	1.006463	1.004656	1.002821	0.002821
30	1.003782	1.003609	1.003609	1.003172	1.002276	1.001375	0.001375
35	1.001972	1.001882	1.001882	1.001654	1.001183	1.000714	0.000714
40	1.001078	1.001029	1.001029	1.000904	1.000645	1.000389	0.000389
45	1.000613	1.000584	1.000584	1.000514	1.000366	1.000220	0.000220
No. of iterations	124	123	123	121	116	109	

TABLE 2. Mann, Ishikawa, Noor, NSP, SP and MN iterations for $x_1 = 9$ and $f(x) = x^{0.3} - (\sqrt{log(x+9)-1})^3$.

Table 1, Table 2, Figure 1 and Figure 2 show the behavior of six comparative methods consisting of Mann iteration, Ishikawa iteration, Noor iteration, NSP-iteration, SP-iteration and MN-iteration in converging to the fixed point of the numerical experiments. The results of the both examples indicates that the MN-iteration converges faster than the other methods.

Next, we will consider on the rate of convergence between the MN-iteration and the algorithm defined in this paper. The Definition 1 will be used to indicate the rate of convergence in the numerical aspects and results are scoped only on the Example 1 and Example 2.

	Mann	Ishikawa	Noor	NSP	SP	MN
n	$\lvert u_n - p\rvert$	$\lvert s_n - p\rvert$	$\lvert l_n - p\rvert$	$\lvert w_n - p\rvert$	$\lvert h_n - p\rvert$	$\lvert x_n - p\rvert$
1	4.166925	4.166925	4.166925	4.166925	4.166925	4.166925
...	...	...	...	...	...	...
22	8.29793E-03	4.56306E-03	3.92351E-03	2.99014E-03	1.63974E-03	4.81195E-04
23	6.66600E-03	3.66550E-03	3.15176E-03	2.40187E-03	1.31039E-03	3.82269E-04
24	5.38267E-03	2.95970E-03	2.54489E-03	1.93931E-03	1.05299E-03	3.05476E-04
...	...	...	...	...	...	...
58	2.15202E-05	1.18252E-05	1.01680E-05	7.74561E-06	3.91320E-06	1.03354E-06
59	1.88782E-05	1.03734E-05	8.91967E-06	6.79466E-06	3.42936E-06	9.04423E-07
60	1.65801E-05	9.11051E-06	7.83376E-06	5.96744E-06	3.00896E-06	7.92411E-07

TABLE 3. The rate of convergence of Mann, Ishikawa, Noor, NSP, SP and MN iterations for $f(x) = \frac{x^3+x-3}{19}$ given in Example 1.

We also give a graphic to compare the rates of convergence of the iterations mentioned in Example 1 visually.

Rate of convergence between two sequences

n	$\frac{\lvert x_n-p\rvert}{\lvert u_n-p\rvert}$	$\frac{\lvert x_n-p\rvert}{\lvert s_n-p\rvert}$	$\frac{\lvert x_n-p\rvert}{\lvert l_n-p\rvert}$	$\frac{\lvert x_n-p\rvert}{\lvert w_n-p\rvert}$	$\frac{\lvert x_n-p\rvert}{\lvert h_n-p\rvert}$
1	1.0000	1.0000	1.0000	1.0000	1.0000
5	0.1178	0.1996	0.2309	0.2970	0.3993
10	0.0748	0.1361	0.1584	0.2074	0.3342
20	0.0595	0.1081	0.1257	0.1650	0.2973
30	0.0540	0.0981	0.1141	0.1498	0.2823
40	0.0510	0.0928	0.1079	0.1417	0.2735
50	0.0491	0.0894	0.1040	0.1365	0.2676
60	0.0478	0.0870	0.1012	0.1328	0.2634

TABLE 4. Convergence comparison of sequences generated by Mann iteration, Ishikawa iteration, Noor iteration, NSP-iteration and SP-iteration with MN-iteration (see in Table 3) for numerical experiment of Example 1.

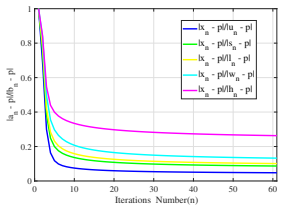

FIGURE 3. Convergence comparison of sequence generated by Mann iteration (u_n), Ishikawa iteration (s_n), Noor iteration (l_n), NSP-iteration (w_n) and SP-iteration (h_n) with MN-iteration (x_n) for Example 1.

n	Mann $\lvert u_n-p\rvert$	Ishikawa $\lvert s_n-p\rvert$	Noor $\lvert l_n-p\rvert$	NSP $\lvert w_n-p\rvert$	SP $\lvert h_n-p\rvert$	MN $\lvert x_n-p\rvert$
1	8	8	8	8	8	8
...	...	...	...	...	...	...
35	1.97255E-03	1.88222E-03	1.88202E-03	1.65419E-03	1.18342E-03	7.14044E-04
36	1.74222E-03	1.66239E-03	1.66221E-03	1.46101E-03	1.04470E-03	6.30133E-04
37	1.54156E-03	1.47089E-03	1.47074E-03	1.29272E-03	9.23939E-04	5.57109E-04
...	...	...	...	...	...	...
86	1.44878E-05	1.38175E-05	1.38160E-05	1.21448E-05	8.59461E-06	5.13795E-06
87	1.34005E-05	1.27804E-05	1.27791E-05	1.12333E-05	7.94884E-06	4.75143E-06
88	1.24005E-05	1.18267E-05	1.18254E-05	1.03950E-05	7.35501E-06	4.39603E-06

TABLE 5. The rate of convergence of Mann, Ishikawa, Noor, NSP, SP and MN iterations for $f(x) = x^{0.3} - (\sqrt{log(x+9)-1})^3$ given in Example 2.

Rate of convergence between two sequences

n	$\frac{\lvert x_n-p\rvert}{\lvert u_n-p\rvert}$	$\frac{\lvert x_n-p\rvert}{\lvert s_n-p\rvert}$	$\frac{\lvert x_n-p\rvert}{\lvert l_n-p\rvert}$	$\frac{\lvert x_n-p\rvert}{\lvert w_n-p\rvert}$	$\frac{\lvert x_n-p\rvert}{\lvert h_n-p\rvert}$
1	1.0000	1.0000	1.0000	1.0000	1.0000
5	0.3997	0.4156	0.4157	0.4780	0.6170
10	0.3820	0.3992	0.3992	0.4555	0.6132
20	0.3694	0.3868	0.3868	0.4403	0.6077
40	0.3606	0.3779	0.3780	0.4300	0.6024
60	0.3570	0.3743	0.3744	0.4259	0.5998
80	0.3551	0.3551	0.3723	0.4236	0.5982
100	0.3538	0.3710	0.3710	0.4221	0.5971

TABLE 6. Convergence comparison of sequences generated by Mann iteration, Ishikawa iteration, Noor iteration, NSP-iteration and SP-iteration with MN-iteration (see in Table 5) for numerical experiment of Example 2.

We also give a graphic to compare the rates of convergence of the iterations mentioned in Example 2 visually.

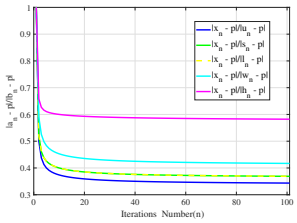

FIGURE 4. Convergence comparison of sequence generated by Mann iteration (u_n), Ishikawa iteration (s_n), Noor iteration (l_n), NSP-iteration (w_n) and SP-iteration (h_n) with MN-iteration (x_n) for Example 2.

Table 3 and 5 show the absolute errors of Mann, Ishikawa, Noor, NSP, SP and MN iterations of the Example 1 and Example 2, respectively. Table 4 and Table 6 show ratios between the absolute error of MN-iteration and those of other methods and graphs of Table 4 and Table 6 are represented on Figure 3 and Figure 4. Clearly, the graphs on both figures converge to constants less than 1. It indicates that the sequences of absolute error of MN-iteration are less than those sequences of other methods. By Definition 1, we can conclude that MN-iteration converges to the fixed point faster than other method. These results verify the proof on the section 3 which show that MN-iteration converge faster than Mann, Ishikawa, Noor, NSP, and SP iterations.

ACKNOWLEDGEMENT

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TOLERANCES ON POSETS

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Abstract. The concept of a tolerance relation, shortly called tolerance, was studied on various algebras since the seventies of the twentieth century by B. Zelinka and the first author (see e.g. [6] and the monograph [1] and the references therein). Since tolerances need not be transitive, their blocks may overlap and hence in general the set of all blocks of a tolerance cannot be converted into a quotient algebra in the same way as in the case of congruences. However, G. Czédli ([7]) showed that lattices can be factorized by means of tolerances in a natural way, and J. Grygiel and S. Radeleczki ([8]) proved some variant of an Isomorphism Theorem for tolerances on lattices. The aim of the present paper is to extend the concept of a tolerance on a lattice to posets in such a way that results similar to those obtained for tolerances on lattices can be derived.

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Keywords: poset, tolerance relation, congruence on a poset, block, directed, convex, relatively complemented poset, quotient poset by a tolerance, Isomorphism Theorem

Tolerance relations play an important role both in algebra and in applications. For corresponding results, examples and references see the monograph [1]. Tolerances were studied by several authors within the last decades. Up to now, tolerances were treated on various algebras, in particular on lattices, but not on posets. The aim of this paper is to extend this investigation also to posets.

Let $\mathbf{L} = (L, \vee, \wedge)$ be a lattice. A *tolerance* on $\mathbf{L}$ is a reflexive and symmetric binary relation T on L satisfying the following condition:

- If $(x,y),(z,u) \in T$ then $(x \vee z, y \vee u), (x \wedge z, y \wedge u) \in T$.

The congruences on $\mathbf{L}$ are exactly the transitive tolerances on $\mathbf{L}$.

G. Czédli ([7]) showed that for every lattice $\mathbf{L} = (L, \vee, \wedge)$ and each tolerance T on $\mathbf{L}$, the set L/T of all blocks of T forms a lattice again, the so-called quotient lattice of $\mathbf{L}$ by T. His famous result in this paper is that every lattice can be embedded into the quotient lattice of a distributive lattice by a suitable tolerance.

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Unfortunately, quotients by tolerances cannot be introduced in a similar way even for semilattices since the join of two blocks of a tolerance on a semilattice need not exist, see e.g. [2].

Now there arises the question if results similar to those by G. Czédli can be obtained also for posets. In other words, we ask if tolerances on posets can be defined in such a way that

(i) in case of lattices this concept coincides with that for lattices as introduced above,
(ii) the set of all blocks of a tolerance forms a poset again.

In accordance with [6], every block of a non-trivial tolerance should be convex and directed.

It was shown in [5] that every tolerance on a relatively complemented lattice is a congruence. The corresponding result for posets should hold for our new concept.

Let us note that results on tolerances on algebras can be found in the monograph [1], see also [2, 3] and [8]. It is worth noticing that a similar attempt for defining congruences on posets was introduced and treated in [4].

Now let $\mathbf{P} = (P, \leq)$ be a poset. A *tolerance* on $\mathbf{P}$ is a reflexive and symmetric binary relation T on P satisfying the following conditions:

(1) If $(x,y),(z,u) \in T$ and $x \vee z$ and $y \vee u$ exist then $(x \vee z, y \vee u) \in T$,
(2) if $(x,y),(z,u) \in T$ and $x \wedge z$ and $y \wedge u$ exist then $(x \wedge z, y \wedge u) \in T$,
(3) if $x,y,z \in P$ and $(x,y),(y,z) \in T \neq P^2$ then there exist $u,v \in P$ with $u \leq x,y,z \leq v$ and $(u,y),(y,v) \in T$,
(4) if $(x,y) \in T \neq P^2$ then there exists some $(z,u) \in T$ with both $z \leq x,y \leq u$ and $(v,z),(v,u) \in T$ for all $v \in P$ with $(v,x),(v,y) \in T$.

Conditions (3) and (4) are quite natural since they are satisfied by every tolerance on a lattice. In condition (3) one can take $u := x \wedge y \wedge z$ and $v := x \vee y \vee z$, and in condition (4) one can take $z := x \wedge y$ and $u := x \vee y$.

Let $\operatorname{Tol}\mathbf{P}$ denote the set of all tolerances on $\mathbf{P}$. Obviously, $\bigcup_{x \in P} \{x\}^2$ is the smallest tolerance on $\mathbf{P}$ and P^2 the greatest one. These *tolerances* are called the *trivial* ones. A *block* of a tolerance T on $\mathbf{P}$ is a maximal subset B of P satisfying $B^2 \subseteq T$. Let P/T denote the set of all blocks of T. Clearly, $T = \bigcup_{B \in P/T} B^2$. A *congruence* on $\mathbf{P}$ is a transitive tolerance on $\mathbf{P}$. Let $\operatorname{Con}\mathbf{P}$ denote the set of all congruences on $\mathbf{P}$.

Let $A \subseteq P$. Then A is called *directed* if for every $x,y \in A$ there exist $z,u \in A$ with $z \leq x,y \leq u$. Further, A is called *convex* if for all $x,y \in A$ with $x \leq y$ we have $[x,y] \subseteq A$. If $\mathbf{P}$ has a bottom element a and a top element b then $\mathbf{P}$ is called *complemented* if for every $x \in P$ there exists some $y \in P$ satisfying $x \vee y = b$ and $x \wedge y = a$. Every such element y is called a *complement* of x. The poset $\mathbf{P}$ is called *relatively complemented* if for all $x,y \in P$ with $x \leq y$ the poset $([x,y], \leq)$ is complemented.

At first, we investigate intervals in blocks of tolerances.

Proposition 1. *Let* $\mathbf{P} = (P, \leq)$ *be a poset,* $T \in \operatorname{Tol}\mathbf{P}$ *and* $a, b \in P$ *with* $a \leq b$*. Then the following assertions hold:*

(i) *If* $(a,b) \in T$ *then* $[a,b]^2 \subseteq T$*,*
(ii) *if* $B \in P/T$ *has bottom element* a *and top element* b *then* $B = [a,b]$*.*

Proof. The case $T = P^2$ is trivial. Hence assume $T \neq P^2$.

(i) Assume $(a,b) \in T$ and let $c, d \in [a,b]$. Now

$$(a,b),(d,d) \in T \text{ implies } (a,d) = (a \wedge d, b \wedge d) \in T,$$
$$(b,a),(c,c) \in T \text{ implies } (c,a) = (b \wedge c, a \wedge c) \in T.$$

Hence $(a,d),(c,a) \in T$ which implies $(c,d) = (a \vee c, d \vee a) \in T$ showing $[a,b]^2 \subseteq T$.

(ii) Assume $B \in P/T$ to have bottom element a and top element b. Then $B \subseteq [a,b]$. If $B \neq [a,b]$ then B would not be a maximal subset C of P satisfying $C^2 \subseteq T$. Therefore $B = [a,b]$ and hence B is convex. □

Using the previous result, we can show that blocks of non-trivial tolerances on posets share the essential properties of blocks of tolerances on lattices.

Theorem 1. *Every block of a non-trivial tolerance on a poset is directed and convex.*

Proof. Let $\mathbf{P} = (P, \leq)$ be a poset, T a non-trivial tolerance on $\mathbf{P}$ and $B \in P/T$. We first prove that B is directed. Let $a, b \in B$. Then $(a,b) \in T$. According to (4) there exists some $(c,d) \in T$ with both $c \leq a, b \leq d$ and $(x,c),(x,d) \in T$ for all $x \in P$ with $(x,a),(x,b) \in T$. Now let $e \in B$. Since $(e,a),(e,b) \in T$ we have $(e,c),(e,d) \in T$. This shows $(B \cup \{c\})^2 \cup (B \cup \{d\})^2 \subseteq T$. If $c \notin B$ then B would not be a maximal subset C of P satisfying $C^2 \subseteq T$. Hence $c \in B$. Analogously, we obtain $d \in B$. This shows that B is directed. Now let $f, g \in B$ with $f \leq g$ and $h \in [f,g]$. Further let $i \in B$. Since B is directed there exist $j, k \in B$ with $j \leq f, i$ and $g, i \leq k$. Now we have $j \leq f \leq h \leq g \leq k$, $j \leq i \leq k$ and $(j,k) \in T$. According to Proposition 1 (i), $(h,i) \in [j,k]^2 \subseteq T$. This means that $(h,x) \in T$ for all $x \in B$ and hence $(B \cup \{h\})^2 \subseteq T$. If $h \notin B$ then B would not be a maximal subset C of P satisfying $C^2 \subseteq T$. Hence $h \in B$ showing the convexity of B. □

A poset $\mathbf{P} = (P, \leq)$ is said to satisfy the *Ascending Chain Condition* (ACC) if there do not exist infinite ascending chains in $\mathbf{P}$, and it is said to satisfy the *Descending Chain Condition* (DCC) if there do not exist infinite descending chains in $\mathbf{P}$. In particular, every finite poset satisfies both conditions ACC and DCC, respectively.

Corollary 1. *Every block of a non-trivial tolerance on a poset satisfying the* ACC *and the* DCC*, respectively, especially every block of a non-trivial tolerance on a finite poset is an interval of the form* $[a,b]$ *with* $a \leq b$*.*

The following two examples show that the intersection of two tolerances need not be a tolerance and that the posets $(\operatorname{Tol}\mathbf{P},\subseteq)$ and $(\operatorname{Con}\mathbf{P},\subseteq)$ need not be lattices.

Example 1. Consider the poset $\mathbf{P}$ depicted in Figure 1:

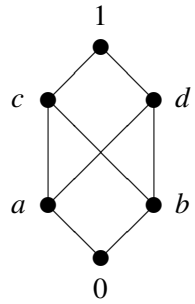

FIGURE 1.

If

$$T_1 := \{0,a,b,c\}^2 \cup \{b,c,d,1\}^2,$$
$$T_2 := \{0,a,b,d\}^2 \cup \{a,c,d,1\}^2$$

then $T_1, T_2 \in \operatorname{Tol}\mathbf{P}$ are not congruences on $\mathbf{P}$ and

$$T_1 \cap T_2 = \{0,a,b\}^2 \cup \{a,c\}^2 \cup \{b,d\}^2 \cup \{c,d,1\}^2,$$

but $T_1 \cap T_2 \notin \operatorname{Tol}\mathbf{P}$ since $\{0,a,b\}$ and $\{c,d,1\}$ are not directed.

Example 2. Consider the poset $\mathbf{P}$ visualized in Figure 2:

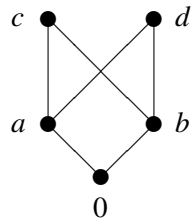

FIGURE 2.

If

$$T_1 := \{0,a\}^2 \cup \{b\}^2 \cup \{c\}^2 \cup \{d\}^2,$$
$$T_2 := \{0,b\}^2 \cup \{a\}^2 \cup \{c\}^2 \cup \{d\}^2,$$
$$T_3 := \{0,a,b,c\}^2 \cup \{d\}^2,$$
$$T_4 := \{0,a,b,d\}^2 \cup \{c\}^2$$

then $T_1, T_2, T_3, T_4 \in \operatorname{Tol}\mathbf{P}$ and T_3 and T_4 are minimal upper bounds of $\{T_1, T_2\}$ in $(\operatorname{Tol}\mathbf{P},\subseteq)$ and hence $T_1 \vee T_2$ does not exist, i.e. $(\operatorname{Tol}\mathbf{P},\subseteq)$ is not a lattice. Observe that $T_1,\dots,T_4 \in \operatorname{Con}\mathbf{P}$. Hence, also $(\operatorname{Con}\mathbf{P},\subseteq)$ need not be a lattice.

As shown by Theorem 1, non-trivial blocks of tolerances on posets have similar properties as those on lattices. Hence our next task is to show when the set P/T of all blocks of a tolerance T on a poset $\mathbf{P}$ is again a poset. In other words, we ask if there can be introduced a partial order relation on the set P/T which in the case that $\mathbf{P}$ is a lattice coincides with the partial order relation induced by the lattice $(P/T,\vee,\wedge)$ introduced by G. Czédli ([7]). For this, we need the following.

Let $\mathbf{P}=(P,\leq)$ be a poset, $T\in\operatorname{Tol}\mathbf{P}$ and $B_1,B_2\in P/T$. We define

$$B_1\sqsubseteq B_2 \text{ if for every } b_1\in B_1 \text{ there exists some } b_2'\in B_2 \text{ with } b_1\leq b_2' \text{ and}$$
$$\text{for every } b_2\in B_2 \text{ there exists some } b_1'\in B_1 \text{ with } b_1'\leq b_2.$$

In the proof of Theorem 3 we will use the following famous theorem by G. Czédli showing that for any lattice $\mathbf{L}$ and any $T\in\operatorname{Tol}\mathbf{L}$ the set L/T forms again a lattice in some natural way.

Theorem 2 ([7])**.** *Let $\mathbf{L}=(L,\vee,\wedge)$ be a lattice, $T\in\operatorname{Tol}\mathbf{L}$ and $B_1,B_2\in L/T$. Then there exist unique $B_3,B_4\in L/T$ such that $b_1\vee b_2\in B_3$ and $b_1\wedge b_2\in B_4$ for all $b_1\in B_1$ and all $b_2\in B_2$. Put $B_1\vee B_2:=B_3$ and $B_1\wedge B_2:=B_4$. Then $(L/T,\vee,\wedge)$ is again a lattice.*

Now we show that the ordering of blocks introduced above for arbitrary posets extends the lattice ordering mentioned in Theorem 2 from lattices to posets. In the proofs of the following two theorems we frequently use Theorem 1.

Theorem 3. *Let $\mathbf{L}=(L,\vee,\wedge)$ be a lattice and $T\in\operatorname{Tol}\mathbf{L}$. Then the relation $\sqsubseteq$ defined above is the partial order relation induced by the lattice $(L/T,\vee,\wedge)$.*

Proof. Let $B_1,B_2\in L/T$ and let $\leq$ denote the partial order relation induced by the lattice $(L/T,\vee,\wedge)$. Then the following are equivalent:

(i) $B_1\leq B_2$,
(ii) $B_1\vee B_2=B_2$ and $B_1\wedge B_2=B_1$,
(iii) $b_1\vee b_2\in B_2$ and $b_1\wedge b_2\in B_1$ for all $b_1\in B_1$ and all $b_2\in B_2$,
(iv) for every $b_1\in B_1$ there exists some $b_2'\in B_2$ with $b_1\leq b_2'$, and for every $b_2\in B_2$ there exists some $b_1'\in B_1$ with $b_1'\leq b_2$,
(v) $B_1\sqsubseteq B_2$.

(i) $\Leftrightarrow$ (ii):
This is clear.
(ii) $\Leftrightarrow$ (iii):
This follows from Theorem 2.
(iii) $\Rightarrow$ (iv):
If $b_1\in B_1$ and $b_2\in B_2$ then $b_1\leq b_1\vee b_2\in B_2$ and $b_2\geq b_1\wedge b_2\in B_1$.
(iv) $\Rightarrow$ (iii):
Let $b_1\in B_1$ and $b_2\in B_2$. Then there exists some $b_2'\in B_2$ with $b_1\leq b_2'$ and there exists some $b_1'\in B_!$ with $b_1'\leq b_2$. Since B_2 is directed there exists some $b_2''\in B_2$

with $b_2, b_2' \leq b_2''$. Now $b_2 \leq b_1 \vee b_2 \leq b_2''$. Since B_2 is convex we obtain $b_1 \vee b_2 \in B_2$. Analogously, since B_1 is directed there exists some $b_1'' \in B_1$ with $b_1'' \leq b_1, b_1'$. Now $b_1'' \leq b_1 \wedge b_2 \leq b_1$. Since B_1 is convex we obtain $b_1 \wedge b_2 \in B_1$.
(iv) $\Leftrightarrow$ (v): This follows from the definition of $\sqsubseteq$. $\square$

Now we can state and prove our result revealing the structure of P/T for $T \in \operatorname{Tol}\mathbf{P}$.

Theorem 4. *Let* $\mathbf{P} = (P, \leq)$ *be a poset and* $T \in \operatorname{Tol}\mathbf{P}$. *Then* $(P/T, \sqsubseteq)$ *is again a poset, called the* quotient poset $\mathbf{P}/T$ of $\mathbf{P}$ with respect to T.

Proof. If $T = \bigcup_{x \in P} \{x\}^2$ and $a, b \in P$ then $\{a\} \sqsubseteq \{b\}$ if and only if $a \leq b$. Hence $(P/T, \sqsubseteq)$ is a poset in this case. If $T = P^2$ then $(P/T, \sqsubseteq)$ is the one-element poset. Hence assume T to be non-trivial. We are going to show that $\sqsubseteq$ is a partial order relation on P/T. For this purpose let $B_1, B_2, B_3 \in P/T$.
Reflexivity of $\sqsubseteq$ is trivial.
Now assume $B_1 \sqsubseteq B_2 \sqsubseteq B_1$. Let $b_1 \in B_1$ and $b_2 \in B_2$. Then there exist $b_1', b_1'' \in B_1$ and $b_2', b_2'' \in B_2$ with $b_1 \leq b_2'$, $b_1' \leq b_2$, $b_2 \leq b_1''$ and $b_2'' \leq b_1$. Together we obtain $b_2'' \leq b_1 \leq b_2'$ and $b_1' \leq b_2 \leq b_1''$ which because of the convexity of B_2 and B_1 yields $b_1 \in B_2$ and $b_2 \in B_1$. This shows $B_1 \subseteq B_2$ and $B_2 \subseteq B_1$, i.e. $B_1 = B_2$, and $\sqsubseteq$ is antisymmetric.
Finally, assume $B_1 \sqsubseteq B_2 \sqsubseteq B_3$. Let $b_1 \in B_1$ and $b_3 \in B_3$. Since $B_1 \sqsubseteq B_2$ there exists some $b_2 \in B_2$ with $b_1 \leq b_2$. Since $B_2 \sqsubseteq B_3$ there exists some $b_3' \in B_3$ with $b_2 \leq b_3'$. Together we obtain $b_3' \in B_3$ and $b_1 \leq b_2 \leq b_3'$. Since $b_3 \in B_3$ and $B_2 \sqsubseteq B_3$ there exists some $b_2' \in B_2$ with $b_2' \leq b_3$. Since $B_1 \sqsubseteq B_2$ there exists some $b_1' \in B_1$ with $b_1' \leq b_2'$. Together we obtain $b_1' \in B_1$ and $b_1' \leq b_2' \leq b_3$, i.e. $B_1 \sqsubseteq B_3$, and $\sqsubseteq$ is transitive. $\square$

Example 3. Consider the poset $\mathbf{P}$ depicted in Figure 1 and the following tolerances on $\mathbf{P}$:

$$T_1 = \{0, a, b, c\}^2 \cup \{b, c, d, 1\}^2 = B_1^2 \cup B_2^2,$$
$$T_2 = \{0, a\}^2 \cup \{b, c\}^2 \cup \{d, 1\}^2 = C_1^2 \cup C_2^2 \cup C_3^2.$$

Then the quotient posets $\mathbf{P}/T_i$ $(i = 1, 2)$ are visualized in Figure 3.

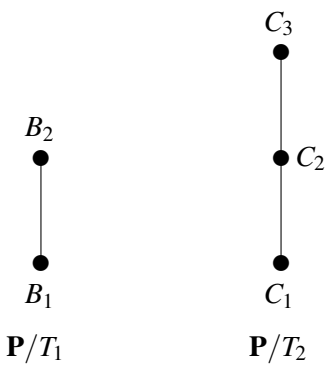

FIGURE 3.

Remark 1. If B_1 and B_2 are intervals of the form $[a,b]$ and $[c,d]$, respectively, then $[a,b] \sqsubseteq [c,d]$ if and only if $a \leq c$ and $b \leq d$.

It was shown in [5] that every tolerance on a relatively complemented lattice $\mathbf{L}$ is a congruence. We are going to prove an analogous result also for posets.

Theorem 5. *Let* $\mathbf{P} = (P, \leq)$ *be a relatively complemented poset. Then* $\operatorname{Tol}\mathbf{P} = \operatorname{Con}\mathbf{P}$.

Proof. Let $a,b,c \in P$ and $T \in \operatorname{Tol}\mathbf{P}$ and assume $(a,b),(b,c) \in T$. If $T = P^2$ then $T \in \operatorname{Con}\mathbf{P}$. Hence assume $T \neq P^2$. According to (3) there exist $d,e \in P$ with $d \leq a,b,c \leq e$ and $(d,b),(b,e) \in T$. Since $\mathbf{P}$ is relatively complemented, there exists some complement f of b in $[d,e]$. Now $d \vee f$, $b \vee f$, $f \wedge b$ and $e \wedge e$ exist, thus

$$(d,b),(f,f) \in T \text{ implies } (f,e) = (d \vee f, b \vee f) \in T,$$
$$(f,e),(b,e) \in T \text{ implies } (d,e) = (f \wedge b, e \wedge e) \in T.$$

Since $(d,e) \in T$, from Proposition 1 (i) we obtain $(a,c) \in [d,e]^2 \subseteq T$, i.e. T is transitive and hence $T \in \operatorname{Con}\mathbf{P}$. □

Example 4. The poset depicted in Figure 4 is relatively complemented, but not a lattice:

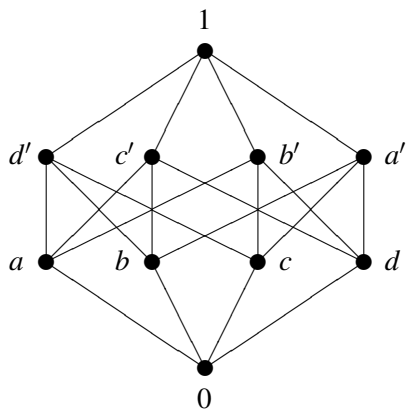

FIGURE 4.

Example 5. Another example of a relatively complemented poset $\mathbf{P}$ which is not directed is visualized in Figure 5:

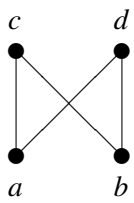

FIGURE 5.

The list of congruences of $\mathbf{P}$ is as follows:

$$\begin{aligned}
C_1 &= \{a\}^2 \cup \{b\}^2 \cup \{c\}^2 \cup \{d\}^2,\\
C_2 &= \{a\}^2 \cup \{c\}^2 \cup \{b,d\}^2,\\
C_3 &= \{a\}^2 \cup \{d\}^2 \cup \{b,c\}^2,\\
C_4 &= \{b\}^2 \cup \{c\}^2 \cup \{a,d\}^2,\\
C_5 &= \{b\}^2 \cup \{d\}^2 \cup \{a,c\}^2,\\
C_6 &= \{a,c\}^2 \cup \{b,d\}^2,\\
C_7 &= \{a,d\}^2 \cup \{b,c\}^2,\\
C_8 &= \{a,b,c,d\}^2.
\end{aligned}$$

The poset $(\operatorname{Con}\mathbf{P}, \subseteq)$ is depicted in Figure 6:

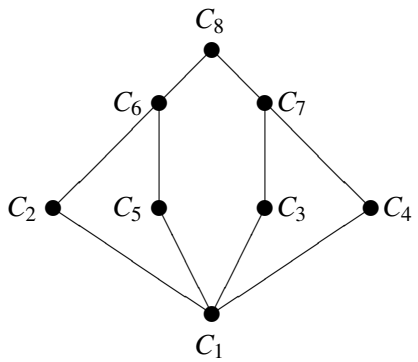

FIGURE 6.

It is easy to see that $\operatorname{Con}\mathbf{P} = \operatorname{Tol}\mathbf{P}$.

J. Grygiel and S. Radeleczki ([8]) showed that a partial order relation $\leq$ on the set $\operatorname{Tol}\mathbf{L}$ of tolerances on a lattice $\mathbf{L}$ can be introduced in such a way that for $S, T \in \operatorname{Tol}\mathbf{L}$ with $S \leq T$ a tolerance T/S on the quotient lattice $\mathbf{L}/S$ can be defined such that the *Isomorphism Theorem for tolerances*

$$(\mathbf{L}/S)/(T/S) \cong \mathbf{L}/T$$

holds. For posets, we proceed as follows.

Let $\mathbf{P} = (P, \leq)$ be a poset and $S, T \in \operatorname{Tol}\mathbf{P}$. We say that $S \leq T$ if the following conditions hold:

- If $B_1 \in P/S$ then there exists exactly one $B_2 \in P/T$ with $B_1 \subseteq B_2$,
- every block of T is a union of blocks of S.

Note that the first condition implies $S \subseteq T$. It is easy to see that $\leq$ is reflexive and antisymmetric.

In case $S \leq T$ for $S,T \in \operatorname{Tol}\mathbf{P}$ define a binary relation T/S on P/S as follows: For $B_1,B_2 \in P/S$ we have $(B_1,B_2) \in T/S$ if there exists some $B_3 \in P/T$ with $B_1,B_2 \subseteq B_3$.

The following lemma is obvious.

Lemma 1. *Let* $\mathbf{P} = (P,\leq)$ *be a poset and* $S,T \in \operatorname{Tol}\mathbf{P}$ *and assume* $S \leq T$. *Then* T/S *is reflexive and symmetric.*

Proof. If $B_1 \in P/S$ then there exists some $B_2 \in P/T$ with $B_1 \subseteq B_2$ which shows $(B_1,B_1) \in T/S$. Symmetry of T/S is clear. □

Example 7 shows that the Isomorphism Theorem as it was mentioned above in the case of lattices does not hold for tolerances on posets. However, for tolerances S and T on a poset $\mathbf{P} = (P,\leq)$ such that $S \leq T$ and $T/S \in \operatorname{Tol}(\mathbf{P}/S)$ we can construct an injective mapping from P/T to $(P/S)/(T/S)$, see the following theorem.

Theorem 6. *Let* $\mathbf{P} = (P,\leq)$ *be a poset and* $S,T \in \operatorname{Tol}\mathbf{P}$ *and assume* $S \leq T$. *Further assume that* T/S *satisfies* (1) – (4). *Then*

(i) $T/S \in \operatorname{Tol}(\mathbf{P}/S)$,
(ii) $|(P/S)/(T/S)| \geq |P/T|$.

Proof. (i) follows from Lemma 1.

(ii) Define $f(B_1) := \{B_2 \in P/S \mid B_2 \subseteq B_1\}$ for all $B_1 \in P/T$. Observe that because of $S \leq T$, $f(B_1) \neq \varnothing$ for all $B_1 \in P/T$. We show that f is an injective mapping from P/T to $(P/S)/(T/S)$. Let $B_1 \in P/T$. Then $(B_2,B_3) \in T/S$ for all $B_2,B_3 \in f(B_1)$. Now let $B_4 \in P/S$ and assume $(B_4,B_2) \in T/S$ for all $B_2 \in f(B_1)$. Let $B_5 \in f(B_1)$. Then there exists some $B_6 \in P/T$ with $B_4,B_5 \subseteq B_6$. Now $B_5 \in P/S$, $B_1,B_6 \in P/T$ and $B_5 \subseteq B_1,B_6$. Since $S \leq T$ we conclude $B_1 = B_6$. This shows $B_4 \subseteq B_6 = B_1$, i.e. $B_4 \in f(B_1)$. Therefore $f(B_1) \in (P/S)/(T/S)$. Injectivity of f follows from the fact that because of $S \leq T$ we have $B_1 = \bigcup_{B_2 \in f(B_1)} B_2$ for all $B_1 \in P/T$.

□

The following example demonstrates that in some cases not only $|(P/S)/(T/S)| \geq |P/T|$, but $|(P/S)/(T/S)| = |P/T|$.

Example 6. Let $\mathbf{P}$ be the poset visualized in Figure 2 and put

$$S := \{0,a\}^2 \cup \{b,c\}^2 \cup \{d\}^2 = B_1^2 \cup B_2^2 \cup B_3^2,$$
$$T := \{0,a,b,c\}^2 \cup \{d\}^2 = C_1^2 \cup C_2^2.$$

Then $S,T \in \operatorname{Tol}\mathbf{P}$, $S \leq T$ and $\mathbf{P}/S$ and $\mathbf{P}/T$ look as follows:

FIGURE 7.

Further,

$$T/S = \{B_1, B_2\}^2 \cup \{B_3\}^2 = D_1^2 \cup D_2^2 \in \mathrm{Tol}(\mathbf{P}/S),$$

and $(\mathbf{P}/S)/(T/S)$ looks as follows:

$(\mathbf{P}/S)/(T/S)$

FIGURE 8.

This shows $\mathbf{P}/T \cong (\mathbf{P}/S)/(T/S)$.

If in addition to the assumptions of Theorem 6 we assume $S, T \in \mathrm{Con}\,\mathbf{P}$ then we can show that there exists even a bijection from $(P/S)/(T/S)$ to P/T which is, moreover, order-preserving. Unfortunately, in general this bijection is not an order-isomorphism since its inverse need not be order-preserving, see Example 7.

Theorem 7. *Let* $\mathbf{P} = (P, \leq)$ *be a poset and* $S, T \in \mathrm{Con}\,\mathbf{P}$ *and assume* $S \leq T$ *and* $T/S \in \mathrm{Tol}(\mathbf{P}/S)$. *Then there exists a bijective order-preserving mapping from* $(\mathbf{P}/S)/(T/S)$ *onto* $\mathbf{P}/T$.

Proof. Since $S, T \in \mathrm{Con}\,\mathbf{P}$, the blocks of S as well as the blocks of T form a partition of P and every block of T is a disjoint union of blocks of S. Hence two blocks of S are in relation T/S if and only if they are contained in the same block of T. It is clear that the blocks of T/S are exactly the sets of the form $\{C \in P/S \mid C \subseteq B\}$ with $B \in P/T$. Define a mapping f from $(P/S)/(T/S)$ to P/T by $f(B) := \bigcup_{C \in B} C$ for all $B \in (P/S)/(T/S)$. Since every block of T/S is the set of all blocks of S that are contained in a fixed block of T, f is a bijection. We show that it is order-preserving. Let $B_1, B_2 \in (P/S)/(T/S)$ and assume $B_1 \sqsubseteq B_2$. Let $a \in f(B_1)$. Then there exists some $B_3 \in B_1$ with $a \in B_3$. Since $B_1 \sqsubseteq B_2$ there exists some $B_4 \in B_2$ with $B_3 \sqsubseteq B_4$. Hence there exists some $b \in B_4$ with $a \leq b$. Therefore $b \in f(B_2)$ and $a \leq b$. Conversely, let $b \in f(B_2)$. Then there exists some $B_4 \in B_2$ with $b \in B_4$. Since $B_1 \sqsubseteq B_2$ there exists some $B_3 \in B_1$ with $B_3 \sqsubseteq B_4$. Hence there exists some $a \in B_3$ with $a \leq b$. Therefore $a \in f(B_1)$ and $a \leq b$. Together, we have proved that $f(B_1) \sqsubseteq f(B_2)$, i.e. f is order-preserving. □

If we apply the proof of Theorem 7 to Example 6 we obtain $f(D_i) = C_i$ for $i = 1, 2$.

That the inverse of the bijective order-preserving mapping mentioned in Theorem 7 need not be order-preserving is demonstrated by the following example.

Example 7. Let $\mathbf{P}$ be the poset depicted in Figure 1 and put

$$S := \{0, a\}^2 \cup \{b\}^2 \cup \{c\}^2 \cup \{d, 1\}^2 = B_1^2 \cup B_2^2 \cup B_3^2 \cup B_4^2,$$
$$T := \{0, a\}^2 \cup \{b, c\}^2 \cup \{d, 1\}^2 = C_1^2 \cup C_2^2 \cup C_3^2.$$

Then $S, T \in \mathrm{Tol}\,\mathbf{P}$, $S \leq T$ and $\mathbf{P}/S$ and $\mathbf{P}/T$ look as follows:

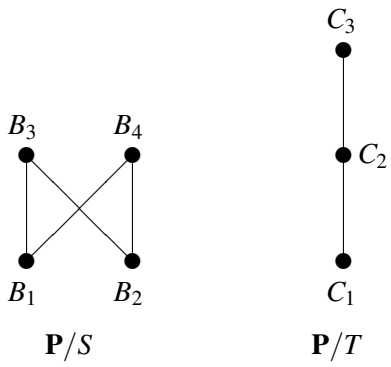

FIGURE 9.

Further,

$$T/S = \{B_1\}^2 \cup \{B_2, B_3\}^2 \cup \{B_4\}^2 = D_1^2 \cup D_2^2 \cup D_3^2 \in \mathrm{Tol}(\mathbf{P}/S),$$

and $(\mathbf{P}/S)/(T/S)$ looks as follows:

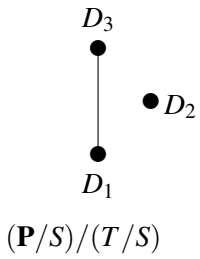

FIGURE 10.

The mapping f from the proof of Theorem 7 maps D_i onto C_i for $i = 1,2,3$. Since $C_1 \sqsubseteq C_2$, but $f^{-1}(C_1) = D_1 \not\sqsubseteq D_2 = f^{-1}(C_2)$, the mapping f^{-1} is not order-preserving. Even more, there does not exist a bijective order-preserving mapping from $\mathbf{P}/T$ to $(\mathbf{P}/S)/(T/S)$. Hence, the Isomorphism Theorem for tolerances on posets does not hold in general even in the case when the tolerances in question are congruences.

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EXISTENCE OF SOLUTIONS TO MULTIVALUED PROBLEMS IN INCOMPLETE METRIC SPACES WITH APPLICATIONS

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Abstract. In this paper, we obtain some existence results for multivalued contraction mappings in the context of an O-complete orthogonal metric space (not necessarily complete metric space). Also, we provide a partial solution to Reich's problem for multivalued orthogonal contraction mappings and extend Mizoguchi-Takahashi's point theorem. In addition, we give an example to demonstrate the applicability of our established results. We study the solution of a differential equation and its Ulam's stability as an application of the obtained results.

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Keywords: O-complete, fixed point, orthogonal multivalued contraction, orthogonal metric space

1. Introduction and preliminaries

Gordji et al. [12] established the notion of orthogonal sets and subsequently extended the Banach fixed point theorem in the so-called orthogonal-complete metric spaces to substitute the metric space completeness in fixed point results. Several authors have investigated this concept as not necessarily complete metric spaces (see [6, 7, 10, 23]).

With thanks to Nadler [18], Gordji et al. [12], the goal of this study is to prove some results on Mizoguchi-Takahashi type contraction mappings in the framework of an orthogonal metric space (which is not necessarily complete) and to adequate criteria for the existence of fixed points for such class of mappings.

There are many results on the existence of solutions for multivalued problems (see [1, 10, 15, 22, 23]). In 1974, Reich (see [19, 20]) questioned whether we might use a nonempty closed and bounded set instead of a nonempty compact set. Despite the fact that many fixed point theorists have researched this topic, it has not been entirely solved. There are several partial affirmative answers to this problem; for example, recently, Azé and Corvellec [5] give a partial positive answer to the conjecture using a

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simple variational approach and Mizoguchi et al. [17] generalize Nadler's conclusion in the establishment of metric space frameworks.

Daffer et al. [11] proved that $\alpha(t) = 1 - at^{b-1}$, where $a > 0$, for some $b \in (1,2)$ on some interval $[0,s]$, $0 < s < a^{-\frac{1}{b-1}}$, is a class of functions that satisfy all of Reich's conjecture's conditions and used this class to get their results. Suzuki [24] remarked that the corresponding result of [17] is a real generalization of the corresponding result of Nadler [18].

Recently, Gordji et al. [12] established the concept of an orthogonal set (also known as an O-set) and proved several fixed point theorems in the context of orthogonal metric spaces.

Definition 1. Let $\perp \subset X \times X$ be a binary relation, where $X \neq \varnothing$. If $\perp$ satisfies the following condition:
There exists $x_0 \in X$ such that (for all $y \in X$, $y \perp x_0$) or (for all $y \in X$, $x_0 \perp y$), then it is called an orthogonal set (briefly O-set). We denote this O-set by $(X, \perp)$.

If d is a metric on O-set $(X, \perp)$, then $(X, \perp, d)$ is an orthogonal metric space (OMS).

Example 1 ([12]). Let $X = \mathbb{Z}$. Define $m \perp n$ if there exists $k \in \mathbb{Z}$ such that $m = k\,n$. It is easy to see that $0 \perp n$ for all $n \in \mathbb{Z}$. Hence $(X, \perp)$ is an O-set.

Example 2. Let (X, d) be a metric space and $T : X \to X$ be a Picard operator, that is, T has a unique fixed point $x^* \in X$ and $\lim_{n \to +\infty} T_n(y) = x^*$ for all $y \in X$. We define the binary relation $\perp$ on X by $x \perp y$ if

$$\lim_{n \to \infty} d(x,\ T_n(y)) = 0.$$

Then, $(X, \perp)$ is an O-set (see [12]).

Example 3. Define the binary relation $\perp$ on an inner product space $(X, \langle .,. \rangle)$ by $x \perp y$ if $\langle x, y \rangle = 0$. Here, $0 \perp x$ for all $x \in X$. Hence, $(X, \perp)$ is an O-set (see [12]).

For other definitions viz. O-sequence, Cauchy O-sequence, O-complete, $\perp$-continuous, and $\perp$-preserving we refer to [12].

It is easy to see that every complete metric space is O-complete but the converse is not true (see [12]).

Let H be a Hausdorff-Pompeiu metric induced by metric d on a set X. Denote $\mathcal{CB}(X)$ the family of all nonempty, closed and bounded subsets of X.

$H : \mathcal{CB}(X) \times \mathcal{CB}(X) \to \mathbb{R}$ defined by, for every $A, B \in \mathcal{CB}(X)$,

$$H(A,B) = \max \left\{ \sup_{x \in A} d(x,B), \sup_{y \in B} d(y,A) \right\},$$

where $d(x,A) = \inf\{d(x,y) : y \in A\}$.

Now, we have the following open question:

Let (X,d) be a O-complete OMS (not necessarily complete metric space) and $T: X \to \mathcal{CB}(X)$ satisfies the following

$$H(Tx, Ty) \leq \alpha(d(x,y))\, d(x,y), \tag{1.1}$$

for all $x,y \in X$ where α is a function from $[0,\infty)$ into $[0,1)$ satisfying $\limsup_{s\to t^+} \alpha(s) < 1$ for all $t \in [0,\infty)$. Does T has a fixed point?

In the context of OMS, we propose a partial solution to Reich's original problem using weaker multivalued orthogonal contraction mappings, that is (1.1).

2. Main results

We get certain findings for multivalued orthogonal Mizoguchi-Takahashi type contractions in the context of O-complete OMS (not necessarily complete metric space) and then construct fixed points of mappings fulfilling these contractions in this section.

We denote $\mathcal{P}$ be the family of all the functions $\alpha : [0,+\infty) \to [0,1)$ such that $\limsup_{r\to t^+} \alpha(r) < 1$, for all $r \in [0,+\infty)$.

Here, we define the orthogonal relation between two nonempty subsets of an orthogonal set.

Definition 2. Let $(X,\perp)$ an orthogonal set. Suppose that A and B are two nonempty subsets of X. The set A is orthogonal to set B given by symbol $\perp_1$ and defined as follows: $A \perp_1 B$, if for every $a \in A$ and $b \in B$, $a \perp b$.

We are now ready to provide our first outcome.

Theorem 1. *Let $T: X \to \mathcal{CB}(X)$ be a multivalued mapping on an O-complete OMS $(X,\perp,d)$. Assume that the following conditions hold:*

(i) *for all $x,y \in X$, $x \perp y$ implies $Tx \perp_1 Ty$,*
(ii) *there exists $x_0 \in X$ such that $\{x_0\} \perp_1 Tx_0$ or $Tx_0 \perp_1 \{x_0\}$,*
(iii) *if $\{x_n\}$ is O-sequence in X such that $x_n \to x^* \in X$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ with $x_{n(k)} \perp x^*$ or $x^* \perp x_{n(k)}$ for all $n \in \mathbb{N}$,*
(iv) *there exists $\alpha \in \mathcal{P}$ such that*

$$H(Tx,Ty) \leq \alpha(d(x,y))\, d(x,y),$$

for all $x,y \in X$ with $x \perp y$.

Then there exists $x \in X$ such that $x \in Tx$, that is, a fixed point of T.

Proof. By assumption (i), there exists $x_1 \in Tx_0$ such that $x_0 \perp x_1$ or $x_1 \perp x_0$. By assumption (ii), we get $Tx_0 \perp_1 Tx_1$, that is there exists $x_2 \in Tx_1$ such that $x_1 \perp x_2$ or $x_2 \perp x_1$.

Define a function $\beta \in \mathcal{P}$ such that $\beta(t) = \frac{\alpha(t)+1}{2}$. Then $\beta(t) < \alpha(t)$ and for all $t \in [0,\infty)$ $\limsup_{r\to t+0}\beta(r) < 1$. Therefore, using (iv), we get

$$d(x_1,x_2) \leq H(Tx_0, Tx_1) \leq \beta(d(x_0,x_1))\, d(x_0,x_1). \tag{2.1}$$

Continuing this process inductively, we can construct an orthogonal sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$, for all $n \in \mathbb{N}\cup\{0\}$. Thus we have $x_n \perp x_{n+1}$ or $x_{n+1} \perp x_n$ for all $n \in \mathbb{N}\cup\{0\}$.

$$d(x_{n+1},x_{n+2}) \leq \beta(d(x_n,x_{n+1}))\, d(x_n,x_{n+1}), \tag{2.2}$$

since $\beta(t) < 1$, for all $t \in [0,\infty)$ and $\{d(x_{n+1},x_n)\}$ is strictly non-increasing sequence. Suppose that $t_n = d(x_{n+1},x_n) \to t$, for some $t \geq 0$.
Since $\limsup_{s\to t+0}\beta(s) < 1$ and $\beta(t) < 1$, there exist $r \in [0,1)$ and $\varepsilon > 0$ such that $\beta(s) \leq r$ for all $s \in [t,t+\varepsilon]$. Now, we can choose some $m \in \mathbb{N}\cup\{0\}$ such that $t \leq t_n \leq t+\varepsilon$ for all $n \in \mathbb{N}\cup\{0\}$ with $n \geq m$. Here it is to note that

$$t_{n+1} \leq \beta(t_n)\, t_n \leq r\, t_n,$$

and thus by ratio test, we get

$$\sum_{n=1}^{\infty} t_n < \infty.$$

This shows that $\{x_n\}$ is a Cauchy orthogonal sequence. Since X is an O-complete, there exists $x^* \in X$ such that $\lim_{n\to+\infty} x_n = x^*$.

Now, we claim that $x^* \in Tx^*$. Assume to the contrary that $x^* \notin Tx^*$. Hence there exists $n_1 \in \mathbb{N}$ such that $x^* \notin \{x_n\}_{n\geq n_1}$ and $d(x_{n(k)}, Tx^*) > 0$. By using our assumption (iii), we have $x_{n(k)} \perp x^*$ or $x^* \perp x_{n(k)}$. Using (iv), we get

$$\begin{aligned} d(x_{n(k)+1}, Tx^*) &\leq H(Tx_{n(k)}, Tx^*) \\ &\leq \beta(d(x_{n(k)},x^*))\, d(x_{n(k)},x^*). \end{aligned}$$

Taking $k \to +\infty$, we get $x^* \in \overline{Tx^*} = Tx^*$. Hence we get the result. □

Remark 1. In 2012, Karapinar and Samet [16] gave some results using α-admissible mappings. Later various generalizations have done using this (see [15,22]). It is interesting to see the main result without using orthogonality on α-admissible mappings for multivalued operators in the setting of orthogonal metric space.

3. Consequences

3.1. *Single valued result*

As a consequence of Theorem 1, we have the following result for single valued mappings by replacing condition (iii) with T is $\perp$-continuous.

Theorem 2. *Let $T : X \to X$ be a self mapping on an O-complete OMS $(X,\perp,d)$. Assume that the following conditions hold:*

(i) *for all* $x, y \in X$, $x \perp y$ *implies* $Tx \perp Ty$,
(ii) *there exists* $x_0 \in X$ *such that* $x_0 \perp Tx_0$ *or* $Tx_0 \perp x_0$,
(iii) T *is* $\perp$*-continuous,*
(iv) *there exists* $\alpha \in \mathcal{P}$ *such that*

$$d(Tx, Ty) \leq \alpha(d(x,y))\, d(x,y),$$

for all $x, y \in X$ *with* $x \perp y$.

Then T *has a fixed point.*

Proof. In this case, we may choose T as a multivalued mapping by assuming that Tx is a singleton set for every $x \in X$. Also, $\{x_n\}$ is a Cauchy orthogonal sequence and $\lim_{n\to\infty} x_n = x^*$, by arguing on the same lines of Theorem 1. As T is $\perp$-continuous, we have

$$d(x^*, Tx^*) = \lim_{n\to+\infty} d(Tx_n, Tx^*) = 0,$$

that is, x^* is a fixed point of T. □

3.2. *Coupled fixed point*

Now we'll explain how our findings enable us to construct coupled fixed point theorems in O-complete OMS. We begin by recalling the following definition.

Let $G : X \times X \to X$ be a given mapping. We say that $(x,y) \in X \times X$ is a coupled fixed point of G if $G(x,y) = x$ and $G(y,x) = y$.

Our result is based on the following simple lemma which tells us when a coupled fixed point is a fixed point (see Samet [21]).

Lemma 1. *Let* $G : X \times X \to X$ *be a given mapping, where* $X \neq \varnothing$. *Define the mapping* $T : Y = X \times X \to Y = X \times X$ *by* $T(x,y) = (G(x,y), G(y,x))$, *for all* $(x,y) \in X \times X$. *Then,* (x,y) *is a fixed point of* T *if and only if* (x,y) *is a coupled fixed point of* G.

Theorem 3. *Let* $(X, \perp, d)$ *be an O-complete OMS and* $G : X \times X \to X$ *be a self mapping on* X. *Assume that the following conditions hold:*

(i) *there exists* $\alpha \in \mathcal{P}$ *such that for all* $x, y, u, v \in X$ *with* $x \perp y, u \perp v$,

$$d(G(x,y), G(u,v)) \leq \alpha(d((x,y),(u,v)))\, d((x,y),(u,v)), \tag{3.1}$$

(ii) G *is* $\perp$*-preserving,*
(iii) *there exists* $(x_0, y_0) \in X$ *such that* $x_0 \perp G(x_0, y_0)$ *or* $G(x_0, y_0) \perp x_0$ *and* $y_0 \perp G(y_0, x_0)$ *or* $G(y_0, x_0) \perp y_0$,
(iv) G *is* $\perp$*-continuous;*

Then G *has a coupled fixed point.*

Proof. Here take $(Y = X \times X, d)$ is O-complete orthogonal metric space. Define the mapping $T : Y \to Y$ by $T(x,y) = (G(x,y), G(y,x))$, for all $(x,y) \in X \times X$. From (3.1), we have

$$d(T(\xi,\eta)) \leq \alpha(d(\xi,\eta))\, d(\xi,\eta),$$

for all $\xi = (\xi_1,\xi_2), \eta = (\eta_1,\eta_2) \in Y$. So using Theorem 2, we get the result. □

Remark 2. On the same lines of Theorem 3, we can prove other coupled fixed point results.

3.3. *Illustration*

In this section, we provide an example to show the usability of our obtained results.

Example 4. Let

$$X = \{-\frac{1}{2},\ldots,-\frac{1}{2^n},\ldots\} \cup \{0\} \cup \{\frac{1}{2},\ldots,\frac{1}{2^n},\ldots\}$$

and $d : X \times X \to [0,\infty)$ be a mapping defined by $d(x,y) = |x-y|$ for all $x,y \in X$. Define a relation $\perp$ on X by $x \perp y$ if and only if $xy \in \{x,y\} \subseteq X$.

Thus $(X,\perp,d)$ is an O-complete orthogonal metric space. Now, we define a mapping $T : X \to \mathcal{CB}(X)$ by

$$Tx = \begin{cases} \{0\}, & x = 0 \\ \{\frac{1}{2^{2n}}, \frac{1}{2^{2n+1}}\}, & x = -\frac{1}{2^n}, \frac{1}{2^n}, n \geq 1. \end{cases}$$

Here T satisfies all the hypothesis of Theorem 1 for

$$\alpha(t) = \begin{cases} \frac{1}{2}, t > 0, \\ 0, t = 0 \end{cases}.$$

Hence T has a fixed point.

4. Applications

Many researchers worked on different problems and obtained the solution using fixed point approach, see [2, 3, 23]. Ulam [14, 25] stability has attracted attention of several authors in fixed point theory, see [4, 8, 9, 13, 23]. In this section, we investigate the existence of a solution to a differential equation and its Ulam stability as an application of the results presented in previous sections.

4.1. *Existence of solution for differential equations*

The purpose of this section is to find the existence of solution for a first order boundary value problem using the results proved in the paper.

In this section, we examine the following boundary value problem:

$$\begin{cases} u'(t) = \mathscr{T}(t,u(t)), t \in I = [0,T], T > 0 \\ u(0) = u(T), \end{cases} \tag{4.1}$$

where $u \in C(I,\mathbb{R})$ and $N : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Let $X := C([0,1],\mathbb{R})$ be the set of continuous real valued functions defined on I, endowed with the metric $d : X \times X \to [0,\infty)$ defined as $d(u,v) = \sup_{t\in[0,1]} |u(t) - v(t)|$, for all $u,v \in X$. Define an orthogonal relation $u \perp v$ if and only if $uv \geq 0$, for all $u,v \in X$. Then $(X,\perp,d)$ is OMS.

Clearly, (4.1) is equivalent to the following linear first order equation

$$u'(t) + \mu u(t) = L(t,u(t)), t \in I, u(0) = u(T), \tag{4.2}$$

where $L(t,u(t)) = \mathscr{T}(t,u(t)) + \mu u(t)$ and $\mu > 0$. Also, a solution of the equation (4.2) is a fixed point of an integral equation

$$\begin{aligned} \mathscr{T}u(t) = &\frac{1}{e^{\mu T} - 1} \int_0^t e^{\mu(T+s-t)} L(s,u(s))\, ds \\ &+ \frac{1}{e^{\mu T} - 1} \int_t^T e^{\mu(s-t)} L(s,u(s))\, ds, \quad \text{for some } \mu > 0. \end{aligned} \tag{4.3}$$

Also, $u \mapsto L(t,u(t))$ is $\perp$-continuous.

Theorem 4. *Assume that there is $\mu > 0$ such that for all $u,v \in C(I,\mathbb{R})$ with $u \perp v$,*

$$|L(s,u(s)) - L(s,v(s))| \leq e^{-\mu}\, \mu\, |u(s) - v(s)|,$$

for each $s \in [0,T]$. Then equation (4.1) with given boundary conditions has a solution in $C(I,\mathbb{R})$.

Proof. Define $\mathscr{T} : X \to X$ as in (4.3). So $\mathscr{T}$ is $\perp$-continuous. First, we show that $\mathscr{T}$ is $\perp$-preserving, let $u(t) \perp v(t)$ for all $t \in [0,1]$. Now, from (4.3) we have

$$\begin{aligned} \mathscr{T}u(t) = &\frac{1}{e^{\mu T} - 1} \int_0^t e^{\mu(T+s-t)} L(s,u(s))\, ds \\ &+ \frac{1}{e^{\mu T} - 1} \int_t^T e^{\mu(s-t)} L(s,u(s))\, ds > 0, \end{aligned}$$

for some $\mu > 0$. It implies that $\mathscr{T}u \perp \mathscr{T}v$.

Now, we have to show that $\mathscr{T}$ satisfies (iv) of Theorem 2 for $\alpha(r) = e^{-r}, r > 0$. For all $t \in [0,1]$, $u(t) \perp v(t)$, we have

$$\begin{aligned} |\mathscr{T}u(t) - \mathscr{T}v(t)| = &|\frac{1}{e^{\mu T} - 1} \int_0^t e^{\mu(T+s-t)} L(s,u(s))\, ds \\ &+ \frac{1}{e^{\mu T} - 1} \int_t^T e^{\mu(s-t)} L(s,u(s))\, ds \end{aligned}$$

$$
\begin{aligned}
&- \Big(\frac{1}{e^{\mu T}-1}\int_0^t e^{\mu(T+s-t)}\, L(s,v(s))\, ds \\
&+ \frac{1}{e^{\mu T}-1}\int_t^T e^{\mu(s-t)}\, L(s,v(s))\, ds\Big)| \\
\leq\ & \frac{1}{e^{\mu T}-1}\int_0^t e^{\mu(T+s-t)}|\, L(s,u(s)) - L(s,v(s))|\, ds \\
&+ \frac{1}{e^{\mu T}-1}\int_t^T e^{\mu(s-t)}|\, L(s,u(s)) - L(s,v(s))|\, ds \\
\leq\ & \frac{1}{e^{\mu T}-1}\int_0^t e^{\mu(T+s-t)} e^{-\mu}\, \mu \sup_{s\in[0,1]} |u(s)-v(s)|]\, ds \\
&+ \frac{1}{e^{\mu T}-1}\int_t^T e^{\mu(s-t)} e^{-\mu}\, \mu \sup_{s\in[0,1]} |u(s)-v(s)|]\, ds \\
\leq\ & [e^{-\mu}\, \mu \sup_{s\in[0,1]} |u(s)-v(s)|] \times \sup_{t\in[0,1]} \Big\{\frac{1}{e^{\mu T}-1}\int_0^t e^{\mu(T+s-t)}\, ds \\
&+ \int_t^T e^{\mu(s-t)}\, ds\Big\} \\
\leq\ & e^{-\mu} \sup_{s\in[0,1]} |u(s)-v(s)|] \\
=\ & e^{-\mu} d(u,v),
\end{aligned}
$$

for all $u, v \in X$. Therefore, the condition (iv) of Theorem 2 holds. Accordingly all axioms of Theorem 2 are verified and $\mathscr{T}$ has a unique fixed point. It yields that the differential equation (4.1) possesses a unique solution. □

Remark 3. Theorem 4 will used to analyse some of the following real life applications:

1. To study the flow of current in the electric circuit in the engineering problems, for example: $LI'(t) + RI(t) = E$, where $I(t)$ is the current in the circuit at time t, L is inductance, R is resistance, E is electromotive force.

2. To study the velocity of the sky driver at any time. Once the sky diver jumps from an airplane, there are two forces that determine his motion : The pull of the earth's gravity acting down ward and the opposing force of air resistance acting upward. At high speeds, the strength of the air resistance force (the drag force) is proportional to the square of velocity, so the upward resistance force due to the air resistance can be expressed as $v'(t) = g - \frac{k}{m}v^2$, where $v(t)$ is velocity of sky-driver at any time t, g is acceleration due to gravity, the air resistance force has strength kv.

4.2. *Ulam stability of differential equation*

In this section, we discuss the Ulam stability of the differential equation (4.1).

Equation (4.1) is called Ulam stable if it satisfies the following condition:

(AI) there is a constant $\delta > 0$, for each $\varepsilon > 0$ and for every solution $u \in X$ satisfying

$$|u'(t) - L(t,u(t))| \leq \varepsilon, \tag{4.4}$$

that is,

$$|\mathscr{T}u(t) - \frac{1}{e^{\mu T}-1}\int_0^t e^{\mu(T+s-t)}\, L(s,u(s))\, ds - \frac{1}{e^{\mu T}-1}\int_t^T e^{\mu(s-t)}\, L(s,u(s))\, ds| \leq \varepsilon,$$

there exists some $v \in X$ satisfying $v \perp x$ and

$$\mathscr{T}v(t) = \frac{1}{e^{\mu T}-1}\int_0^t e^{\mu(T+s-t)}\, L(s,v(s))\, ds + \frac{1}{e^{\mu T}-1}\int_t^T e^{\mu(s-t)}\, L(s,v(s))\, ds, \tag{4.5}$$

such that

$$|v - x| \leq \delta\varepsilon. \tag{4.6}$$

Theorem 5. *Under the hypothesis of Theorem 4, the differential equation (4.1), is Ulam stable.*

Proof. On the account of Theorem 4, we guarantee a unique $v^* \in X$ such that $\mathscr{T}v^*(t) = \frac{1}{e^{\mu T}-1}\int_0^t e^{\mu(T+s-t)}\, L(s,v^*(s))\, ds + \frac{1}{e^{\mu T}-1}\int_t^T e^{\mu(s-t)}\, L(s,v^*(s))\, ds$, that is, $v^* \in X$ forms a solution of differential equation (4.1). Let $\varepsilon > 0$ and $u^* \in X$ be an ε-solution, that is,

$$\begin{aligned} |u^* - \mathscr{T}u^*| &= |u^*(t) - \frac{1}{e^{\mu T}-1}\int_0^t e^{\mu(T+s-t)}\, L(s,u^*(s))\, ds \\ &\quad - \frac{1}{e^{\mu T}-1}\int_t^T e^{\mu(s-t)}\, L(s,u^*(s))\, ds| \\ &\leq \varepsilon. \end{aligned}$$

Using Theorem 4, for $v^* \perp u^*$ we have

$$\begin{aligned} |v^* - u^*| &= |\mathscr{T}v^* - u^*| \\ &\leq |\mathscr{T}v^* - \mathscr{T}u^*| + |\mathscr{T}u^* - u^*| \\ &\leq e^{-\mu}|v^* - u^*| + \varepsilon. \end{aligned}$$

Hence, $|v^* - u^*| \leq \frac{1}{1-e^{-\mu}}\varepsilon = \delta\varepsilon$, where $\delta = \frac{1}{1-e^{-\mu}} > 0$. Therefore, equation (4.1) is Ulam stable.

□

4.3. *Ulam stability of fixed point problem*

On OMS $(X, \perp, d)$, $T : X \to X$, we investigate the fixed point equation

$$Tv = v \tag{4.7}$$

and the inequality (for $\varepsilon > 0$)

$$d(Tx, x) \leq \varepsilon. \tag{4.8}$$

Equation (4.7) is called Ulam stable if it satisfies the following condition:
(A) there is a constant $\delta > 0$, for each $\varepsilon > 0$ and for every solution x^* of the inequality (4.8), there is a solution $v^* \in X$ of the equation (4.7) with $v^* \perp x^*$ such that

$$d(v^*, x^*) \leq \delta\varepsilon. \tag{4.9}$$

Theorem 6. *Under the hypothesis of Theorem 2, the fixed point equation (4.7) is Ulam stable.*

Proof. On the account of Theorem 2, we guarantee a unique $v^* \in X$ such that $v^* = Tv^*$, that is, $v^* \in X$ forms a solution of (4.7). Let $\varepsilon > 0$ and $x^* \in X$ be an ε-solution, that is,

$$d(Tx^*, x^*) \leq \varepsilon.$$

Using Theorem 2, for $v^* \perp x^*$ we have

$$\begin{aligned} d(v^*, x^*) = d(Tv^*, x^*) &\leq d(Tv^*, Tx^*) + d(Tx^*, x^*) \\ &\leq \alpha(d(v^*, x^*))\, d(v^*, x^*) + \varepsilon. \end{aligned}$$

Hence, $d(v^*, x^*) \leq \frac{1}{1-\alpha(d(v^*,x^*))}\varepsilon = k\varepsilon$, where $k = \frac{1}{1-\alpha(d(v^*,x^*))} > 0$. Therefore, equation (4.7) is Ulam stable. □

Conclusions. In this manuscript, we provide a partial answer to Reich's problem in the context of incomplete metric spaces.

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BLOW-UP OF WEAK SOLUTIONS FOR A HIGHER-ORDER HEAT EQUATION WITH LOGARITHMIC NONLINEARITY

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Abstract. This paper deal with the initial boundary value problem for a higher-order heat equation with logarithmic source term

$$u_t + (-\Delta)^m u - \Delta u_t = u^{k-2} u \ln |u| .$$

We obtain blow-up of weak solutions in the finite time, by employing potential well technique and concave technique. In addition, the upper bound of blow-up time is considered. This improves and extends some previous studies.

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1. Introduction

In this article, we consider the blow up of solutions for the higher-order heat equation with logarithmic nonlinearity

$$\begin{cases} u_t + Au - \Delta u_t = |u|^{k-2} u \ln |u| , & x \in \Omega, \quad t > 0; \\ D^\gamma u(x,t) = 0, \quad |\gamma| \leq m-1, & x \in \partial\Omega, \quad t > 0; \\ u(x,0) = u_0(x), & x \in \Omega; \end{cases} \tag{1.1}$$

where $A = (-\Delta)^m$, $m \geq 1$ a positive integer, Ω is a bound domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ is multi-index, γ_i $(i = 1, 2, \ldots, n)$ are nonnegative integers, $|\gamma| = \gamma_1 + \gamma_2 + \cdots + \gamma_n$, $D^\gamma = \frac{\partial^{|\gamma|}}{\partial x_1^{\gamma_1} \partial x_2^{\gamma_2} \ldots \partial x_n^{\gamma_n}}$ are multi-index derivative operator, $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator. The parameter k satisfies

$$\begin{cases} 2 \leq k \leq +\infty, & n \leq 2m, \\ 2 \leq k \leq \frac{2n}{n-2m}, & n > 2m. \end{cases}$$

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Peng and Zhou [11] studied the following parabolic equation with logarithmic source term

$$u_t - \Delta u = u^{k-2} u \ln |u| .$$

They studied by employing energy technique and potential well technique, the global existence of solutions and blow-up in finite time. In addition, the upper bound of blow-up time is considered under appropriate conditions.

When $m = 1$, in the equation (1.1), becomes a heat equation as follows

$$u_t - \Delta u - \Delta u_t = u^{k-2} u \ln |u| ,$$

where $2 \leq k$, was considered by many authors [2–4, 7, 16]. In the case of $k = 2$, Chen and Tian [3] studied by employing the logarithmic Sobolev inequality and potantial well method, the global existence of solutions and blow-up of solutions at $+\infty$. In the case of $2 < k$, Ding and Zhou [4] studied the blow-up of solutions in finite time, by using eigenfunction method. Also, the upper bound of the blow-up time is studied under appropriate conditions.

Recently many other authors investigated higher-order hyperbolic and parabolic type equation [5, 6, 8, 10, 13–15, 17, 18]. Ishige et al. [8] studied the Cauchy problem for nonlinear higher-order heat equation as follows

$$u_t + (-\Delta)^m u = |u|^k .$$

They obtained existence of solutions of the Cauchy problem by introducing a new majorizing kernel. In addition, they studied the local existence of solutions under the different conditions. Xiao and Li [15] considered the following initial boundary value problem for nonlinear higher-order heat equations

$$u_t + (-\Delta)^m u_t + (-\Delta)^m u = f(u).$$

They established the existence of weak solution to the static problem, via the potential well technique.

Motivated by the above studies, in this work, we investigate the finite time blow-up of weak solutions for the Eq. (1.1).

The remainder of our work is organized as follows. In Section2, some important Lemmas are given. In Section 3, the main result is proved.

2. Preliminaries

We material needed for proving the main result is introduced. Let $\|u\|_{H^m(\Omega)} = \left(\sum_{|\gamma| \leq m} \|D^\gamma u\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}$ denote $H^m(\Omega)$ norm, let $H^m_0(\Omega)$ denote the closure in $H^m(\Omega)$ of $C^\infty_0(\Omega)$. Let $\|.\|_r$ and $\|.\|$ denote the usual $L^r(\Omega)$ norm and $L^2(\Omega)$ norm (see [1, 12], for details).

For $u \in H_0^m(\Omega)\backslash\{0\}$, we define the energy functional

$$J(u) = \frac{1}{2}\left\|A^{\frac{1}{2}}u\right\|^2 - \frac{1}{k}\int_\Omega |u|^k \ln|u|\,dx + \frac{1}{k^2}\|u\|_k^k, \tag{2.1}$$

and Nehari functional

$$I(u) = \left\|A^{\frac{1}{2}}u\right\|^2 - \int_\Omega |u|^k \ln|u|\,dx. \tag{2.2}$$

Combining (2.1) and (2.2), we obtain

$$J(u) = \frac{1}{k}I(u) + \left(\frac{1}{2} - \frac{1}{k}\right)\left\|A^{\frac{1}{2}}u\right\|^2 + \frac{1}{k^2}\|u\|_k^k. \tag{2.3}$$

Let

$$\mathcal{N} = \{u \in H_0^m(\Omega)\backslash\{0\} : I(u) = 0\},$$

be the Nehari manifold. Also, we may define

$$d = \inf_{u\in\mathcal{N}} J(u), \tag{2.4}$$

and

$$V = \{u \in H_0^m(\Omega) \mid J(u) < d,\ I(u) < 0\}.$$

We refer to V as the potential well and d as the depth of the well.

Lemma 1 (Theorem 4.31 in [1]). *Let q be a number with $2 \le q < +\infty$, $n \le 2m$ and $2 \le q \le \frac{2n}{n-2m}$, $n > 2m$. Then there is a constant C depending*

$$\|u\|_q \le C\left\|A^{\frac{1}{2}}u\right\|, \quad \forall u \in H_0^m(\Omega).$$

Lemma 2. *$J(t)$ is a nonincreasing function for $t \ge 0$ and*

$$J'(u) = -\int_\Omega \left(u_t^2 + \nabla u_t^2\right)dx \le 0.$$

Proof. Multiplying the equation (1.1) by u_t and integrating on Ω, we get

$$\int_\Omega u_t^2 dx + \int_\Omega Auu_t dx + \int_\Omega \nabla u_t^2 dx = \int_\Omega u^{k-1}u_t \ln|u|\,dx.$$

By straightforward calculation, we obtain

$$\int_\Omega u_t^2 dx + \frac{1}{2}\frac{d}{dt}\left\|A^{\frac{1}{2}}u\right\|^2 + \int_\Omega \nabla u_t^2 dx = \frac{1}{k}\frac{d}{dt}\int_\Omega |u|^k \ln|u|\,dx - \frac{1}{k^2}\frac{d}{dt}\|u\|_k^k,$$

which yields that

$$\frac{1}{2}\frac{d}{dt}\left\|A^{\frac{1}{2}}u\right\|^2 - \frac{1}{k}\frac{d}{dt}\int_\Omega |u|^k \ln|u|\,dx + \frac{1}{k^2}\frac{d}{dt}\|u\|_k^k = -\int_\Omega u_t^2 dx - \int_\Omega \nabla u_t^2 dx.$$

Thus we get

$$\frac{d}{dt}\left(\frac{1}{2}\left\|A^{\frac{1}{2}}u\right\|^2-\frac{1}{k}\int_\Omega |u|^k\ln|u|\,dx+\frac{1}{k^2}\|u\|_k^k\right)=-\int_\Omega\left(u_t^2+\nabla u_t^2\right)dx. \tag{2.5}$$

By (2.1) and (2.5), we obtain

$$\frac{d}{dt}J(u)=-\int_\Omega\left(u_t^2+\nabla u_t^2\right)dx. \tag{2.6}$$

Moreover, Integrating (2.6) with respect to t on $[0,t]$, we obtain

$$\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)}\,ds+J(u(t))=J(u_0).$$

□

Lemma 3. *Let $u\in H_0^m(\Omega)\backslash\{0\}$. We contract the function $h\colon \beta\to J(\beta u)$ for $\beta>0$. Then we get*

(i) $\lim_{\beta\to 0^+}h(\beta)=0$ *and* $\lim_{\beta\to+\infty}h(\beta)=-\infty$;

(ii) *there is a unique $\overline{\beta}_1>0$ such that $h'(\overline{\beta}_1)=0$;*

(iii) *$h(\beta)$ is increasing on $(0,\overline{\beta}_1)$, decreasing on $(\overline{\beta}_1,+\infty)$ and taking the maximum at $\overline{\beta}_1$; $I(\beta u)=\beta h'(\beta)$ and*

$$I(\beta u)\begin{cases}>0, & 0<\beta<\overline{\beta}_1,\\ =0, & \beta=\overline{\beta}_1,\\ <0, & \overline{\beta}_1<\beta<+\infty.\end{cases}$$

Proof. By the definition of h, for $u\in H_0^m(\Omega)\backslash\{0\}$, we have

$$\begin{aligned}h(\beta)&=\frac{1}{2}\left\|A^{\frac{1}{2}}(\beta u)\right\|^2-\frac{1}{k}\int_\Omega|\beta u|^k\ln|\beta u|\,dx+\frac{1}{k^2}\|\beta u\|_k^k\\&=\frac{\beta^2}{2}\left\|A^{\frac{1}{2}}u\right\|^2-\frac{\beta^k}{k}\int_\Omega|u|^k\ln|u|\,dx-\frac{\beta^k}{k}\ln\beta\,\|u\|_k^k+\frac{\beta^k}{k^2}\|u\|_k^k.\end{aligned}$$

We see that *(i)* holds due to $\|u\|_k^k\neq 0$. We obtain

$$\begin{aligned}\frac{d}{d\beta}h(\beta)&=\beta\left\|A^{\frac{1}{2}}u\right\|^2-\beta^{k-1}\int_\Omega|u|^k\ln|u|\,dx\\&\quad-\beta^{k-1}\ln\beta\,\|u\|_k^k-\frac{\beta^{k-1}}{k}\|u\|_k^k+\frac{\beta^{k-1}}{k}\|u\|_k^k\\&=\beta\left(\left\|A^{\frac{1}{2}}u\right\|^2-\beta^{k-2}\int_\Omega|u|^k\ln|u|\,dx-\beta^{k-2}\ln\beta\,\|u\|_k^k\right).\end{aligned}$$

Let $\zeta(\beta) = \beta^{-1}h'(\beta)$, thus we get

$$\begin{aligned}\zeta(\beta) &= \beta^{-1}h'(\beta)\\ &= \beta^{-1}\beta\left(\left\|A^{\frac{1}{2}}u\right\|^2 - \beta^{k-2}\int_\Omega |u|^k \ln|u|\,dx - \beta^{k-2}\ln\beta\,\|u\|_k^k\right)\\ &= \left\|A^{\frac{1}{2}}u\right\|^2 - \beta^{k-2}\int_\Omega |u|^k \ln|u|\,dx - \beta^{k-2}\ln\beta\,\|u\|_k^k.\end{aligned}$$

Then

$$\zeta'(\beta) = -(k-2)\beta^{k-3}\int_\Omega |u|^k \ln|u|\,dx - (k-2)\beta^{k-3}\ln\beta\,\|u\|_k^k - \beta^{k-3}\,\|u\|_k^k,$$

which yields that there exists a $\overline{\beta}_1 > 0$ such that $\zeta'(\beta) > 0$ on $(0,\overline{\beta}_1)$, $\zeta'(\beta) < 0$ on $(\overline{\beta}_1,+\infty)$ and $\zeta'(\beta) = 0$. So $\zeta(\beta)$ is increasing on $(0,\overline{\beta}_1)$, decreasing on $(\overline{\beta}_1,+\infty)$. Since $\lim_{\beta\to 0^+}\zeta(\beta) = \left\|A^{\frac{1}{2}}u\right\|^2 > 0$, $\lim_{\beta\to+\infty}\zeta(\beta) = -\infty$, there exists a unique $\overline{\beta}_1 > 0$ such that $\zeta(\overline{\beta}_1) = 0$, i.e. $h'(\overline{\beta}_1) = 0$. So *(ii)* holds. Then, $h'(\beta) = \beta\zeta(\beta)$ is positive on $(0,\overline{\beta}_1)$, negative on $(\overline{\beta}_1,+\infty)$. Thus $h(\beta)$ is increasing on $(0,\overline{\beta}_1)$, decreasing on $(\overline{\beta}_1,+\infty)$ and taking the maximum at $\overline{\beta}_1$. From (2.2), we get

$$\begin{aligned}I(\beta u) &= \left\|A^{\frac{1}{2}}(\beta u)\right\|^2 - \int_\Omega |\beta u|^k \ln|\beta u|\,dx\\ &= \beta^2\left\|A^{\frac{1}{2}}u\right\|^2 - \beta^k\int_\Omega |u|^k\ln|u|\,dx - \beta^k\ln\beta\,\|u\|_k^k\\ &= \beta\left(\beta\left\|A^{\frac{1}{2}}u\right\|^2 - \beta^{k-1}\int_\Omega |u|^k\ln|u|\,dx - \beta^{k-1}\ln\beta\,\|u\|_k^k\right)\\ &= \beta h'(\beta).\end{aligned}$$

Thus $I(\beta u) > 0$ for $0 < \beta < \overline{\beta}_1$, $I(\beta u) < 0$ for $\overline{\beta}_1 < \beta < +\infty$ and $I(\overline{\beta}_1 u) = 0$. So *(iii)* holds. For this reason, the proof is completed. □

Lemma 4. *d defined by (2.4) is positive and there exists a positive function $u \in \mathcal{N}$ such that $J(u) = d$.*

Proof. By (2.4), we suppose $\{u_r\}_r^\infty \subset N$ is a minimizing sequence of J. Since $\{u_r\}_r^\infty \subset N$ is also a minimizing sequence of J, we consider the case where $u_r > 0$ a.e. in Ω, $r \in N$ without loss of generality. Thus

$$\lim_{r\to\infty} J(u_r) = d, \tag{2.7}$$

which implies that $\{J(u_r)\}_r^\infty$ is bounded, i.e. there exists a constant $C_1 > 0$ such that $|J(u_r)| \le C_1$. Using (2.3), $I(u_r) = d$ and $|J(u_r)| \le C_1$, we obtain

$$\left(\frac{1}{2} - \frac{1}{k}\right)\left\|A^{\frac{1}{2}}u_r\right\|^2 + \frac{1}{k^2}\,\|u_r\|_k^k \le C_1. \tag{2.8}$$

From (2.8), we get

$$\left\|A^{\frac{1}{2}}u_r\right\|^2 \leq \left(\frac{1}{2}-\frac{1}{k}\right)^{-1} C_1. \tag{2.9}$$

By (2.9) and Lemma 1, we obtain

$$\|u_r\|^2 \leq C\left\|A^{\frac{1}{2}}u_r\right\|^2 \leq \left(\frac{1}{2}-\frac{1}{k}\right)^{-1} C_1.$$

Moreover, we have already observed that J is coercive on $\mathcal{N}$ which satisfies that $\{u_r\}_r^\infty$ is bounded in $H_0^m(\Omega)$. Let $\mu > 0$ be small enough such that $k+\mu < \frac{2n}{n-2m}$. Since $H_0^m(\Omega) \hookrightarrow L^{k+\mu}(\Omega)$ is compact, so there exists a function u and a subsequence of $\{u_r\}_r^\infty$, still denote by $\{u_r\}_r^\infty$, such that

$$u_r \to u \text{ weakly in } H_0^m(\Omega),$$

$$u_r \to u \text{ strongly in } L^{k+\mu}(\Omega),$$

$$u_r(x) \to u(x) \text{ a.e. in } \Omega.$$

Also, $u \geq 0$ a.e. in Ω. First, we prove $u \neq 0$. From the dominated convergence theorem, we have

$$\int_\Omega |u|^k \ln|u|\,dx = \lim_{r\to\infty}\int_\Omega |u_r|^k \ln|u_r|\,dx \tag{2.10}$$

and

$$\int_\Omega |u|^k\,dx = \lim_{r\to\infty}\int_\Omega |u_r|^k\,dx. \tag{2.11}$$

From the weak lower semicontinuity of $H_0^m(\Omega)$, we get

$$\left\|A^{\frac{1}{2}}u\right\|^2 \leq \liminf_{r\to\infty}\left\|A^{\frac{1}{2}}u_r\right\|^2. \tag{2.12}$$

Then it follows from (2.1), (2.7), (2.10), (2.11) and (2.12) that

$$\begin{aligned} J(u) &= \frac{1}{2}\left\|A^{\frac{1}{2}}u\right\|^2 - \frac{1}{k}\int_\Omega |u|^k \ln|u|\,dx + \frac{1}{k^2}\|u\|_k^k \\ &\leq \liminf_{r\to\infty}\frac{1}{2}\left\|A^{\frac{1}{2}}u_r\right\|^2 - \lim_{r\to\infty}\frac{1}{k}\int_\Omega |u_r|^k \ln|u_r|\,dx + \lim_{r\to\infty}\frac{1}{k^2}\|u_r\|_k^k \\ &= \liminf_{r\to\infty}\left(\frac{1}{2}\left\|A^{\frac{1}{2}}u_r\right\|^2 - \frac{1}{k}\int_\Omega |u_r|^k \ln|u_r|\,dx + \frac{1}{k^2}\|u_r\|_k^k\right) \\ &= \liminf_{r\to\infty} J(u_r) = d. \end{aligned} \tag{2.13}$$

Using (2.2), (2.10) and (2.12), we have

$$\begin{aligned} I(u) &= \left\|A^{\frac{1}{2}}u\right\|^2 - \int_\Omega |u|^k \ln|u|\,dx \\ &\le \liminf_{r\to\infty}\left\|A^{\frac{1}{2}}u_r\right\|^2 - \lim_{r\to\infty}\int_\Omega |u_r|^k \ln|u_r|\,dx \\ &= \liminf_{r\to\infty}\left(\left\|A^{\frac{1}{2}}u_r\right\|^2 - \int_\Omega |u_r|^k \ln|u_r|\,dx\right) \\ &= \liminf_{r\to\infty} I(u_r) = 0. \end{aligned} \tag{2.14}$$

Since $u_r \in \mathcal{N}$, we have $I(u_r) = 0$. So by using Lemma 1 and the fact $x^{-\mu}\ln x \le (e\mu)^{-1}$ for $x \ge 1$, we get

$$\begin{aligned} \left\|A^{\frac{1}{2}}u_r\right\|^2 &= \int_\Omega |u_r|^k \ln|u_r|\,dx \\ &\le (e\mu)^{-1}\int_\Omega |u_r|^{k+\mu}\,dx \\ &= (e\mu)^{-1}\|u_r\|_{k+\mu}^{k+\mu} \\ &\le C\left\|A^{\frac{1}{2}}u_r\right\|_2^{k+\mu}, \end{aligned}$$

where C is Sobolev embedding constant. This satisfies that

$$\int_\Omega |u_r|^k \ln|u_r|\,dx = \left\|A^{\frac{1}{2}}u_r\right\|^2 \ge C. \tag{2.15}$$

By (2.10) and (2.15), we have

$$\int_\Omega |u|^k \ln|u|\,dx \ge C.$$

Thus we have $u \in H_0^m(\Omega)\backslash\{0\}$.

If $I(u_r) < 0$, from Lemma 3, there exists a $\overline{\beta}_1$ such that $I(\overline{\beta}_1 u) = 0$ and $0 < \overline{\beta}_1 < 1$. Thus $\overline{\beta}_1 u \in \mathcal{N}$. It follows from (2.3), (2.4), (2.11) and (2.12) that

$$\begin{aligned} d &\le J(\overline{\beta}_1 u) \\ &= \frac{1}{k}I(\overline{\beta}_1 u) + \left(\frac{1}{2}-\frac{1}{k}\right)\left\|A^{\frac{1}{2}}\left(\overline{\beta}_1 u\right)\right\|^2 + \frac{1}{k^2}\left\|\overline{\beta}_1 u\right\|_k^k \\ &= \left(\frac{1}{2}-\frac{1}{k}\right)\left\|A^{\frac{1}{2}}\left(\overline{\beta}_1 u\right)\right\|^2 + \frac{1}{k^2}\left\|\overline{\beta}_1 u\right\|_k^k \\ &= (\overline{\beta}_1)^2\left(\frac{1}{2}-\frac{1}{k}\right)\left\|A^{\frac{1}{2}}u\right\|^2 + (\overline{\beta}_1)^k\left(\frac{1}{k^2}\right)\|u\|_k^k \\ &\le (\overline{\beta}_1)^2\left[\left(\frac{1}{2}-\frac{1}{k}\right)\left\|A^{\frac{1}{2}}u\right\|^2 + \frac{1}{k^2}\|u\|_k^k\right] \end{aligned}$$

$$
\begin{aligned}
&\leq (\overline{\beta}_1)^2 \liminf_{r\to\infty} \left[\left(\frac{1}{2} - \frac{1}{k} \right) \left\| A^{\frac{1}{2}} u_r \right\|^2 + \frac{1}{k^2} \|u_r\|_k^k \right] \\
&= (\overline{\beta}_1)^2 \liminf_{r\to\infty} J(u_r) \\
&= (\overline{\beta}_1)^2 d,
\end{aligned}
$$

which indicates $\overline{\beta}_1 \geq 1$ by $d > 0$. It contradicts $0 < \overline{\beta}_1 < 1$. By (2.14), we have $I(u) = 0$. For this reason, $u \in \mathcal{N}$. From (2.7), we have $J(u) \geq d$. From (2.13), we have $J(u) \leq d$. So $J(u) = d$. □

Lemma 5 (Theorem 2 in [9]). *Let $\phi(t)$ be a nonnegative function C^2, which satisfies, for $t > 0$, inequality*

$$\phi(t)\phi''(t) - (1+\gamma)\left[\phi'(t)\right]^2 \geq 0,$$

with some $\gamma > 0$. If $\phi(0) > 0$ and $\phi'(0) > 0$, then there exist a time

$$T \leq \frac{\phi(0)}{\beta\phi'(0)},$$

such that

$$\lim_{t\to T^-} \phi(t) = \infty.$$

3. Main results

Definition 1. (Weak Solution). We say that function $u(t)$ is weak solution of the problem (1.1) on $\Omega \times [0,T]$, if $u \in L^\infty(0,T;H_0^m(\Omega))$ with $u_t \in L^2(0,T;H_0^m(\Omega))$ and implies the initial condition $u(0) = u_0 \in H_0^m(\Omega)\backslash\{0\}$, and the follow equality

$$(u_t, w) + \left(A^{\frac{1}{2}}u, A^{\frac{1}{2}}w\right) + (\nabla u_t, \nabla w) = \left(|u|^{r-2} u \ln|u|, w\right),$$

for all $w \in H_0^m(\Omega)$ holds for a.e. $t \in [0,T]$, and $(.,.)$ means the inner product $(.,.)_{L^2(\Omega)}$, that is

$$(\eta, \xi) = \int_\Omega \eta(x)\xi(x)dx.$$

Theorem 1 (Blow up). *Suppose that $u_0 \in V$. Then $u(t)$ blows up at finite time in the sence of $T_* > 0$ and*

$$\lim_{t\to T_*} \|u(t)\|_{H_0^1(\Omega)}^2 = \infty.$$

Furthermore, the upper bound for blow up time T_* is given by

$$T_* \leq \frac{4\|u_0\|_{H_0^1(\Omega)}^2}{(k-2)^2 (d - J(u_0))}.$$

Proof. Let $u(t) \in V$ for $t \in [0, T_{\max}]$. We prove that $u(t)$ blows up in the finite time. By employing contradiction, we assume that $u(t)$ is global. We consider a function $P\colon [0,T) \to \mathbb{R}^+$, and

$$P(t) = \int_0^t \|u(s)\|^2_{H_0^1(\Omega)}\, ds + (T_* - t)\, \|u_0\|^2_{H_0^1(\Omega)} + \phi\,(t+\psi)^2, \quad t \in [0,T), \tag{3.1}$$

where ϕ, ψ are two positive fixed which will be specified later.

Then, for any $t \in [0,T)$, a straightforward calculation gives

$$\begin{aligned} P'(t) &= \|u(t)\|^2_{H_0^1(\Omega)} - \|u_0\|^2_{H_0^1(\Omega)} + 2\phi\,(t+\psi) \\ &= 2\int_0^t \int_\Omega (u_s u + \nabla u_s \nabla u)\, dx ds + 2\phi\,(t+\psi), \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} P''(t) &= 2\int_\Omega (u_s u + \nabla u_s \nabla u)\, dx + 2\phi \\ &= 2\int_\Omega u\,(u_s - \Delta u_s)\, dx + 2\phi \\ &= 2\int_\Omega |u|^k \ln|u| - 2\left\|A^{\frac{1}{2}} u\right\|^2 + 2\phi \\ &= -2I(u) + 2\phi. \end{aligned} \tag{3.3}$$

By (3.3) and $I(u) < 0$, we obtain $P''(t) > 0$. From (2.3) and (3.3) that it follows

$$\begin{aligned} P''(t) &= -2I(u) + 2\phi \\ &\geq -2kJ(u) + (k-2)\left\|A^{\frac{1}{2}} u\right\|^2 + \frac{2}{k}\|u\|_k^k \\ &\geq -2kJ(u_0) + 2k\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)}\, ds + (k-2)\left\|A^{\frac{1}{2}} u\right\|^2 + \frac{2}{k}\|u\|_k^k. \end{aligned} \tag{3.4}$$

Since $u(t) \in V$, $t \in [0,T]$, so $I(u) < 0$. By Lemma 3, there exists a $\overline{\beta}_1 \in (0,1)$ such that $I(\overline{\beta}_1 u(t)) = 0$. By (2.3) and the definition of d, we obtain

$$\begin{aligned} d &= \inf_{u \in \mathcal{N}} J(u) \leq J(\overline{\beta}_1 u(t)) \\ &= \frac{1}{k} I(\overline{\beta}_1 u) + \left(\frac{1}{2} - \frac{1}{k}\right)\left\|A^{\frac{1}{2}}\left(\overline{\beta}_1 u\right)\right\|^2 + \frac{1}{k^2}\left\|\overline{\beta}_1 u\right\|_k^k \\ &= \left(\frac{1}{2} - \frac{1}{k}\right)\left\|A^{\frac{1}{2}}\left(\overline{\beta}_1 u\right)\right\|^2 + \frac{1}{k^2}\left\|\overline{\beta}_1 u\right\|_k^k \\ &= (\overline{\beta}_1)^2\left(\frac{1}{2} - \frac{1}{k}\right)\left\|A^{\frac{1}{2}} u\right\|^2 + (\overline{\beta}_1)^k\left(\frac{1}{k^2}\right)\|u\|_k^k \\ &\leq \left(\frac{1}{2} - \frac{1}{k}\right)\left\|A^{\frac{1}{2}} u\right\|^2 + \frac{1}{k^2}\|u\|_k^k. \end{aligned} \tag{3.5}$$

From (3.4) and (3.5), we obtain

$$\begin{aligned} P''(t) &\geq -2kJ(u_0) + 2k\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)}\,ds + (k-2)\left\|A^{\frac{1}{2}}u\right\|^2 + \frac{2}{k}\|u\|_k^k \\ &= -2kJ(u_0) + 2k\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)}\,ds + 2k\left[\frac{k-2}{2k}\left\|A^{\frac{1}{2}}u\right\|^2 + \frac{1}{k^2}\|u\|_k^k\right] \qquad (3.6) \\ &\geq 2k\left(d - J(u_0)\right) + 2k\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)}\,ds. \end{aligned}$$

Thus we have

$$P(t) \geq P(0) > 0, \quad t \in [0, T_*].$$

Let

$$\rho(t) := \left(\int_0^t \|u(s)\|^2_{H_0^1(\Omega)}\,ds\right)^{\frac{1}{2}}, \quad \sigma(t) := \left(\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)}\,ds\right)^{\frac{1}{2}}.$$

By employing Hölder's inequality, we get

$$\begin{aligned} &\left[\int_0^t \|u(s)\|^2_{H_0^1(\Omega)}\,ds + \phi\,(t+\psi)^2\right]\left[\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)}\,ds + \phi\right] \\ &- \left[\frac{1}{2}\left(\|u\|^2_{H_0^1(\Omega)} - \|u_0\|^2_{H_0^1(\Omega)}\right) + \phi\,(t+\psi)\right]^2 \\ &= \left[\rho^2(t) + \phi\,(t+\psi)^2\right]\left[\sigma^2(t) + \phi\right] - \left[\frac{1}{2}\int_0^t \frac{d}{ds}\|u\|^2_{H_0^1(\Omega)}\,ds + \phi\,(t+\psi)\right]^2 \\ &\geq \left[\rho^2(t) + \phi\,(t+\psi)^2\right]\left[\sigma^2(t) + \phi\right] - \left[\int_0^t \|u\|_{H_0^1(\Omega)}\|u_s\|_{H_0^1(\Omega)}\,ds + \phi\,(t+\psi)\right]^2 \\ &= \left[\sqrt{\phi}\rho(t)\right]^2 - 2\phi\,(t+\psi)\,\rho(t)\sigma(t) + \left[\sqrt{\phi}\,(t+\psi)\,\sigma(t)\right]^2 \\ &= \left[\sqrt{\phi}\rho(t) - \sqrt{\phi}\,(t+\psi)\,\sigma(t)\right]^2 \geq 0. \end{aligned}$$

Then, by (3.2), we get

$$\begin{aligned} \frac{1}{4}\left(P'(t)\right)^2 &= \left[\frac{1}{2}\left(\|u(t)\|^2_{H_0^1(\Omega)} - \|u_0\|^2_{H_0^1(\Omega)}\right) + \phi\,(t+\psi)\right]^2 \\ &= \left[\frac{1}{2}\left(\|u(t)\|^2_{H_0^1(\Omega)} - \|u_0\|^2_{H_0^1(\Omega)}\right) + \phi\,(t+\psi)\right]^2 \\ &\quad + \left(\int_0^t \|u(\tau)\|^2_{H_0^1(\Omega)}\,d\tau + \phi\,(t+\psi)^2\right)\left(\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)}\,ds + \phi\right) \qquad (3.7) \\ &\quad - \left[P(t) - (T_* - t)\,\|u_0\|^2_{H_0^1(\Omega)}\right]\left(\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)}\,ds + \phi\right) \\ &\leq P(t)\left(\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)}\,ds + \phi\right). \end{aligned}$$

So it follows from (3.6) and (3.7) that

$$\begin{aligned} P(t)P''(t)-\frac{k}{2}\left(P'(t)\right)^2 &\geq P(t)\left[P''(t)-2k\left(\int_0^t \|u_s(s)\|^2_{H_0^1(\Omega)}\,ds+\phi\right)\right] \\ &\geq P(t)\left[2k\left(d-J(u_0)\right)-2k\phi\right]. \end{aligned}$$

We choose ϕ sufficiently small, such that

$$\phi \in \left(0,\frac{\mu}{2k}\right], \tag{3.8}$$

where

$$\mu := 2k\left(d-J(u_0)\right) > 0,$$

then it follows that

$$P(t)P''(t)-\frac{k}{2}\left(P'(t)\right)^2 \geq 0.$$

Let $\omega(t)=P(t)^{-\frac{k-2}{2}}$ for $t\in[t_0,T]$, then by $P(t)>0$, $P'(t)>0$, $k>2$ and the definition of $\omega(t)$, we have

$$\omega'(t) = -\frac{k-2}{2}P(t)^{-\frac{k}{2}}P'(t). \tag{3.9}$$

By (3.9), we obtain

$$\begin{aligned} \omega''(t) &= \frac{k(k-2)}{4}P(t)^{-\frac{k+2}{2}}P'(t)^2-\frac{k-2}{2}P(t)^{-\frac{k}{2}}P''(t) \\ &= \frac{k-2}{2}P(t)^{-\frac{k+2}{2}}\left[\frac{k}{2}P'(t)^2-P(t)P''(t)\right] < 0 \quad \text{for all } t\in[t_0,T]. \end{aligned} \tag{3.10}$$

We see that, for any large enough $T>t_0$, $\omega(t)$ is a concave function in $[t_0,T]$. Since $\omega(t_0)>0$ and $\omega''(t_0)<0$, there exists a finite time T_* such that

$$\lim_{t\to T_*^-}\omega(t)=0,$$

which yields

$$\lim_{t\to T_*^-}P(t)=\infty.$$

Moreover, we obtain

$$\lim_{t\to T_*^-}\|u(s)\|^2_{H_0^1(\Omega)}=\infty.$$

This is a contradiction to our assumption. Thus $u(t)$ blows up at finite time.

Now, we give an upper bound estimate of T_*. By (3.10) and $\omega''(t)\leq 0$, we obtain

$$\omega(T)-\omega(0)=T\int_0^1\omega'(\theta T)d\theta \leq \omega'(0)T. \tag{3.11}$$

From (3.1) and the definition of $\omega(t)$, we obtain

$$\omega(0)=P(0)^{-\frac{k-2}{2}}>0,$$

$$\omega(T)=P(T)^{-\frac{k-2}{2}}>0,$$

$$\omega'(0) = -\frac{k-2}{2}P(0)^{-\frac{k}{2}}P'(0) < 0.$$

It follows from (3.11) that

$$T \leq \frac{\omega(T)-\omega(0)}{\omega'(0)} < -\frac{\omega(0)}{\omega'(0)} = \frac{2P(0)}{(k-2)\,P'(0)}. \tag{3.12}$$

By (3.1) and (3.2), we obtain

$$P(0) = T\,\|u_0\|^2_{H_0^1(\Omega)} + \phi\psi^2$$

and

$$P'(0) = 2\phi\psi.$$

By Lemma 5 and (3.12), we get

$$T_* \leq \frac{T\,\|u_0\|^2_{H_0^1(\Omega)}}{(k-2)\,\phi\psi} + \frac{\psi}{(k-2)} \qquad \text{for all } T \in [0,T_*). \tag{3.13}$$

Moreover, letting $T \to T_*$, we obtain

$$T_* \leq \frac{\|u_0\|^2_{H_0^1(\Omega)}}{(k-2)\,\phi\psi}T_* + \frac{\psi}{(k-2)}.$$

Let ψ be sufficiently large such that

$$\psi \in \left(\frac{\|u_0\|^2_{H_0^1(\Omega)}}{(k-2)\,\phi}, +\infty\right). \tag{3.14}$$

From (3.13), we obtain

$$T_* \leq \frac{\phi\psi^2}{(k-2)\,\phi\psi - \|u_0\|^2_{H_0^1(\Omega)}}. \tag{3.15}$$

With respect to (3.8) and (3.14), we define

$$\begin{aligned}\varphi &= \left\{(\phi,\psi) : \phi \in \left(0, \frac{\mu}{2k}\right], \psi \in \left(\frac{\|u_0\|^2_{H_0^1(\Omega)}}{(k-2)\,\phi}, +\infty\right)\right\} \\ &= \left\{(\psi,\phi) : \psi \in \left(\frac{2k\,\|u_0\|^2_{H_0^1(\Omega)}}{(k-2)\,\mu}, +\infty\right), \phi \in \left(\frac{\|u_0\|^2_{H_0^1(\Omega)}}{(k-2)\,\psi}, \frac{\mu}{2k}\right]\right\}\end{aligned}$$

and then

$$T_* \leq \inf \frac{\phi\psi^2}{(k-2)\,\phi\psi - \|u_0\|^2_{H_0^1(\Omega)}}.$$

Let $\eta = \phi\psi$ (see [11, Theorem 2.8]) and

$$f(\psi,\eta) := \frac{\eta\psi}{(k-2)\,\eta - \|u_0\|^2_{H_0^1(\Omega)}}.$$

We see that $f(\psi,\eta)$ is decreasing with η and we get

$$\begin{aligned} T_* &\leq \inf_{\psi\in\left(\frac{2k\|u_0\|^2_{H_0^1(\Omega)}}{(k-2)\mu},+\infty\right)} f\left(\psi,\frac{\mu\psi}{2k}\right) \\ &= \inf_{\psi\in\left(\frac{2k\|u_0\|^2_{H_0^1(\Omega)}}{(k-2)\mu},+\infty\right)} \frac{\mu\psi^2}{(k-2)\mu\psi-2k\|u_0\|^2_{H_0^1(\Omega)}} \\ &= \frac{\mu\psi^2}{(k-2)\mu\psi-2k\|u_0\|^2_{H_0^1(\Omega)}}\Big|_{\psi=\frac{4k\|u_0\|^2_{H_0^1(\Omega)}}{(k-2)\mu}} \\ &= \frac{8k\|u_0\|^2_{H_0^1(\Omega)}}{(k-2)^2\mu}. \end{aligned} \tag{3.16}$$

Moreover, by (3.16) and the definition of μ, we obtain

$$T_* \leq \frac{4\|u_0\|^2_{H_0^1(\Omega)}}{(k-2)^2(d-J(u_0))}.$$

□

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RICH DYNAMICS OF A DISCRETE-TIME PREY-PREDATOR MODEL

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Abstract. A newly-disclosed non-standard finite difference method has been used to discretize a prey-predator model to investigate the critical normal form coefficients of bifurcations for both one-parameter and two-parameter bifurcations. The discrete-time prey-predator model exhibits variety of local bifurcations such as period-doubling, Neimark-Sacker, and strong resonances. Critical normal form coefficients are determined to reveal dynamical scenario corresponding to each bifurcation point bifurcation. We also investigates the complex dynamics of the model numerically using by MATLAB package MATCONTM based on numerical continuation technique.

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Keywords: non-standard finite difference method, prey-predator model, normal form coefficient, bifurcation, chaos

1. INTRODUCTION

Let us consider the following continues prey-predator model

$$\begin{cases} \frac{dx_{pp}}{d\tau} = ax_{pp} - bx_{pp}y_{pp}, \\ \frac{dy_{pp}}{d\tau} = y_{pp}\left(c - d\frac{y_{pp}}{x_{pp}}\right), \end{cases} \tag{1.1}$$

where prey (predator) population at time τ is denoted by x_{pp} (y_{pp}). Furthermore, a represents the intrinsic growth rate of prey, b represents the functional response, c represents the growth rate of predator and d represents the number of prey required to support one predator at equilibrium, see [25].

In ecology, population dynamics are generally determined by both discrete-time and continuous-time dynamics. The study of discrete-time biology systems has received a great deal of attention in recent years, see [1, 3–8, 12, 13, 18]. Discrete models provide more realistic representations when the generation takes place in nonoverlapping spaces, and these models also provide more efficient computational models for numerical simulations when compared to continuous-time models, see [2,9,11,16,19–22,24]. In the present paper, we are interested in examining the behavior of discrete-time version obtained from system (1.1). There is an expectation that the discrete-time model will be dynamically compatible with the continuous-time

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model. Zhang et [23] also point out that if there are no overlapping generations in a biological population, then one must obtain discrete-time systems from a dynamical model of continuous population dynamics. As a result, using a nonstandard scheme, the first one achieves the discrete form of (1.1). Follow these steps to discretize model (1.1). At first the model (1.1) is considered on a finite interval $[0,\Theta]$ and discretize this interval by a uniform mesh as follows

$$0=\tau_0<\tau_1<\cdots<\tau_{N-1}<\tau_N=\Theta,$$

and let $h=\tau_{j+1}-\tau_j$ for $j=1,2,\cdots,N$. Our non-standard finite difference (NSFD) schemes for model (1.1) are based on Mickens' theory as follows:

$$\begin{cases} \frac{x_{pp}^{n+1}-x_{pp}^{n}}{\varphi(h)}=ax_{pp}-bx_{pp}y_{pp}, \\ \frac{y_{pp}^{n+1}-y_{pp}^{n}}{\varphi(h)}=y_{pp}\left(c-d\frac{y_{pp}}{x_{pp}}\right), \end{cases} \tag{1.2}$$

where $\varphi(h)=1-e^{-h}$. The model

$$\begin{cases} x_{pp}\mapsto -b\left(1-\mathrm{e}^{-h}\right)x_{pp}y_{pp}+a\left(1-\mathrm{e}^{-h}\right)x_{pp}+x_{pp}, \\ y_{pp}\mapsto -\frac{x_{pp}y_{pp}}{c\left(1-\mathrm{e}^{-h}\right)x_{pp}-d\left(1-\mathrm{e}^{-h}\right)y_{pp}-x_{pp}}, \end{cases} \tag{1.3}$$

is the discrete model obtained from (1.2).

2. Existence and feasibility of fixed points

The system (1.2) can be considered as the following map

$$\begin{pmatrix} x_{pp} \\ y_{pp} \end{pmatrix}\mapsto \mathcal{M}^{pp}(\Psi,\Omega)=\begin{pmatrix} -b\left(1-\mathrm{e}^{-h}\right)x_{pp}y_{pp}+a\left(1-\mathrm{e}^{-h}\right)x_{pp}+x_{pp} \\ -\frac{x_{pp}y_{pp}}{c\left(1-\mathrm{e}^{-h}\right)x_{pp}-d\left(1-\mathrm{e}^{-h}\right)y_{pp}-x_{pp}} \end{pmatrix}, \tag{2.1}$$

where $\Psi=(x_{pp},y_{pp})^T$, $\Omega=(a,b,c,d,h)^T$.

In order to find the fixed points of the map (2.1), we solve the following system

$$\begin{cases} -b\left(1-\mathrm{e}^{-h}\right)x_{pp}y_{pp}+a\left(1-\mathrm{e}^{-h}\right)x_{pp}+x_{pp} & =x_{pp}, \\ -\frac{x_{pp}y_{pp}}{c\left(1-\mathrm{e}^{-h}\right)x_{pp}-d\left(1-\mathrm{e}^{-h}\right)y_{pp}-x_{pp}} & =y_{pp}. \end{cases}$$

The unique positive fixed point

$$\Psi_*^{pp}=\left(\frac{da}{cb},\frac{a}{b}\right),$$

are clearly visible in the map (2.1).

Let us consider

$$\mathcal{M}^{pp}(\Psi,\Omega)=\mathcal{J}_1(\Psi,\Omega)\Psi+\frac{1}{2!}\mathcal{J}_2(\Psi,\Psi)+\frac{1}{3!}\mathcal{J}_3(\Psi,\Psi,\Psi)+O(\|\Psi\|^4),$$

in which

$$\mathcal{J}_1(\Psi,\Omega)=\mathcal{M}_{\Psi}^{pp}(\Psi,\Omega)$$

$$= \begin{pmatrix} by_{pp}\mathrm{e}^{-h} - a\mathrm{e}^{-h} - by_{pp} + a + 1 & b\left(-1+\mathrm{e}^{-h}\right)x_{pp} \\ -\frac{{y_{pp}}^2 d\left(-1+\mathrm{e}^{-h}\right)}{\left(cx_{pp}\mathrm{e}^{-h} - dy_{pp}\mathrm{e}^{-h} - cx_{pp} + dy_{pp} + x_{pp}\right)^2} & \frac{{x_{pp}}^2\left(c\mathrm{e}^{-h} - c + 1\right)}{\left(cx_{pp}\mathrm{e}^{-h} - dy_{pp}\mathrm{e}^{-h} - cx_{pp} + dy_{pp} + x_{pp}\right)^2} \end{pmatrix},$$

$$\mathcal{J}_2(\Psi,\Psi) = \begin{pmatrix} \mathcal{J}_{21}(\Psi,\Psi) \\ \mathcal{J}_{22}(\Psi,\Psi) \end{pmatrix}, \quad \mathcal{J}_3(\Psi,\Psi,\Psi) = \begin{pmatrix} \mathcal{J}_{31}(\Psi,\Psi,\Psi) \\ \mathcal{J}_{32}(\Psi,\Psi,\Psi) \end{pmatrix},$$

and

$$\mathcal{J}_{2i}(\Gamma,\Sigma) = \sum_{j,k=1}^{2} \frac{\partial^2 \mathcal{M}_i^{pp}(\Psi,\Omega)}{\partial\Psi_j \partial\Psi_k}\gamma_j\sigma_k,$$

$$\mathcal{J}_{3i}(\Gamma,\Sigma,\Upsilon) = \sum_{j,k,l=1}^{2} \frac{\partial^3 \mathcal{M}_i^{pp}(\Psi,\Omega)}{\partial\Psi_j \partial\Psi_k \partial\Psi_l}\gamma_j\sigma_k\upsilon_l,$$

where

$$\Gamma = (\gamma_1,\gamma_2)^T,\ \Sigma = (\sigma_1,\sigma_2)^T,\ \Upsilon = (\upsilon_1,\upsilon_2)^T.$$

3. Bifurcations of positive fixed point Ψ_*^{pp}

3.1. *One parameter bifurcations*

The parameter h is considered in this part to be a bifurcation parameter.

Theorem 1. *The critical value*

$$a = a_{PD,*} = -2\,\frac{c\mathrm{e}^{-h} - c + 2}{c\left(\left(\mathrm{e}^{-h}\right)^2 - 2\,\mathrm{e}^{-h} + 1\right)},$$

causes a period-doubling bifurcation of Ψ_*^{pp}.

Proof. For $a = a_{PD,*}$ the matrix

$$\mathcal{J}_1(\Psi_{PD,*}^{pp},\Omega_{PD,*}) = \begin{pmatrix} 1 & -2\,\frac{\left(c\mathrm{e}^{-h}-c+2\right)d}{\left(-1+\mathrm{e}^{-h}\right)c^2} \\ -\frac{\left(-1+\mathrm{e}^{-h}\right)c^2}{d} & c\mathrm{e}^{-h} - c + 1 \end{pmatrix},$$

$$\Omega_{PD,*} = (a_{PD,*},b,c,d,h)^T, \quad \Psi_{PD,*}^{pp} = \begin{pmatrix} -2\,\frac{\left(c\mathrm{e}^{-h}-c+2\right)d}{c^2\left(\mathrm{e}^{-2h}-2\,\mathrm{e}^{-h}+1\right)b} \\ -2\,\frac{c\mathrm{e}^{-h}-c+2}{c\left(\mathrm{e}^{-2h}-2\,\mathrm{e}^{-h}+1\right)b} \end{pmatrix},$$

has the multipliers

$$\lambda_{PD,*}^1 = -1, \qquad \lambda_{PD,*}^2 = c\mathrm{e}^{-h} - c + 3.$$

In the case where $\lambda_{PD,*}^2 \neq \pm 1$, a period-doubling bifurcation may occur on the curve

$$\mathcal{T}_{PD,*}^{pp} = \left\{(x_{pp},y_{pp},r,h,b,c,d);\, a = a_{PD,*}\right\}.$$

It is possible to consider map $\mathcal{M}^{pp}(\Psi^{pp}_{PD,*},\Omega_{PD,*})$ as follows:

$$\eta_{PD,*} \mapsto -\eta_{PD,*} + \frac{1}{6}\widehat{\beta^{pp}_{PD,*}}\eta^3_{PD,*} + O(\eta^4_{PD,*}).$$

We can derive the normal form coefficient $\widehat{\beta^{pp}_{PD,*}}$ as follows

$$\begin{aligned}\widehat{\beta^{pp}_{PD,*}} = \frac{1}{6}\,\Big\langle & w_{PD,*}, \mathcal{J}_3(v_{PD,*},v_{PD,*},v_{PD,*}) \\ & +3\mathcal{J}_2\Big(v_{PD,*},(I_2-\mathcal{J}_1(\Psi_{PD,*},\Omega_{PD,*}))^{-1}\,\mathcal{J}_2(v_{PD,*},v_{PD,*})\Big)\Big\rangle,\end{aligned}$$

where

$$\begin{aligned}\mathcal{J}_1(\Psi_{PD,*},\Omega_{PD,*})v_{PD,*} &= v_{PD,*},\\ \mathcal{J}_1^T(\Psi_{PD,*},\Omega_{PD,*})w_{PD,*} &= w_{PD,*}, \quad \langle w_{PD,*},v_{PD,*}\rangle = 1.\end{aligned}$$

As a result, we have

$$v_{PD,*} = \begin{pmatrix} \frac{\left(c\mathrm{e}^{-h}-c+2\right)d}{\left(-1+\mathrm{e}^{-h}\right)c^2} \\ 1 \end{pmatrix}, \quad w_{PD,*} = \begin{pmatrix} \frac{\left(-1+\mathrm{e}^{-h}\right)c^2}{\left(c\mathrm{e}^{-h}-c+4\right)d} \\ 2\left(c\mathrm{e}^{-h}-c+4\right)^{-1} \end{pmatrix},$$

and

$$\widehat{\beta^{pp}_{PD,*}} = \frac{b^2\left(\mathrm{e}^{-3h}c-3\,\mathrm{e}^{-2h}c+\mathrm{e}^{-2h}+3\,c\mathrm{e}^{-h}-2\,\mathrm{e}^{-h}-c+1\right)}{c\mathrm{e}^{-h}-c+4}.$$

If $\widehat{\beta^{pp}_{PD,*}} \neq 0$ there is a generic period doubling bifurcation on the curve $\mathcal{T}^{pp}_{PD,*}$. When $\widehat{\beta^{pp}_{PD,*}} > 0$ ($\widehat{\beta^{pp}_{PD,*}} < 0$) the bifurcated period-2 cycles is stable (unstable) and the period doubling bifurcation is supercritical (subcritical). □

Theorem 2. *The critical value $a = a_{NS,*} = -\left(-1+\mathrm{e}^{-h}\right)^{-1}$ causes a Neimark-Sacker bifurcation of Ψ^{pp}_*.*

Proof. For $a = a_{NS,*}$ the matrix

$$\mathcal{J}_1(\Psi^{pp}_{NS,*},\Omega_{NS,*}) = \begin{pmatrix} 1 & -\frac{d}{c} \\ -\frac{c^2\left(-1+\mathrm{e}^{-h}\right)}{d} & c\,\mathrm{e}^{-h}-c+1 \end{pmatrix},$$

$$\Omega_{NS,*} = (a_{NS,*},b,c,d,h)^T, \qquad \Psi^{pp}_{NS,*} = \begin{pmatrix} -\frac{d}{c\left(-1+\mathrm{e}^{-h}\right)b} \\ -\frac{1}{\left(-1+\mathrm{e}^{-h}\right)b} \end{pmatrix},$$

has two complex multipliers

$$\lambda^{1,2}_{NS,*} = e^{\pm i\theta_0} = 1+1/2\,c\,\mathrm{e}^{-h}-c/2 \pm i/2\sqrt{-\left(\mathrm{e}^{-h}\right)^2c^2+2\,\mathrm{e}^{-h}c^2-4\,c\,\mathrm{e}^{-h}-c^2+4\,c}.$$

On the curve,

$$\mathcal{T}^{pp}_{NS,*} = \{(x_{pp}, y_{pp}, a, b, c, d, h)\,; a = a_{NS,*}\}\,.$$

the Neimark-Sacker bifurcation will occur.

It is possible to write $\mathcal{M}^{pp}(\Psi_{NS,*}, \Omega_{NS,*})$ as follows:

$$\eta_{NS,*} \mapsto e^{i\theta_0}\eta_{NS,*} + \widehat{\delta^{pp}_{NS,*}}\eta^2_{NS,*}\overline{\eta_{NS,*}} + O(|\eta_{NS,*}|^4),$$

The normal form coefficient $\widehat{\delta^{pp}_{NS,*}}$ can be derived as

$$\begin{aligned}\widehat{\delta^{pp}_{NS,*}} = \frac{1}{2}\langle & w_{NS,*}, \mathcal{J}_3\left(v_{NS,*}, v_{NS,*}, \overline{v_{NS,*}}\right) \\ & + 2\mathcal{J}_2\left(v_{NS,*}, (I_2 - \mathcal{J}_1(\Psi_{NS,*}, \Omega_{NS,*}))^{-1}\,\mathcal{J}_2(v_{NS,*}, \overline{v_{NS,*}})\right) \\ & + \mathcal{J}_2\left(\overline{v_{NS,*}}, \left(e^{2i\theta_0}I_2 - \mathcal{J}_1(\Psi_{NS,*}, \Omega_{NS,*})\right)^{-1}\mathcal{J}_2(v_{NS,*}, v_{NS,*})\right)\rangle,\end{aligned}$$

where

$$\begin{aligned} \mathcal{J}_1(\Psi_{NS,*}, \Omega_{NS})v_{NS,*} &= e^{i\theta_0}v_{NS,*}, & \mathcal{J}_1(\Psi_{NS,*}, \Omega_{NS,*})\overline{v_{NS,*}} &= e^{-i\theta_0}\overline{v_{NS,*}}, \\ \mathcal{J}_1^T(\Psi_{NS,*}, \Omega_{NS,*})w_{NS,*} &= e^{-i\theta_0}w_{NS,*}, & \mathcal{J}_1^T(\Psi_{NS,*}, \Omega_{NS,*})\overline{w_{NS,*}} &= e^{i\theta_0}\overline{w_{NS,*}}, \\ \langle w_{NS,*}, v_{NS,*}\rangle &= 1, \end{aligned}$$

and

$$v_{NS,*} = \begin{pmatrix} -2\,\frac{d}{c\left(c\,\mathrm{e}^{-h} + i\sqrt{c\left(2c\,\mathrm{e}^{-h} - \mathrm{e}^{-2h}c - 4\,\mathrm{e}^{-h} - c + 4\right)} - c\right)} \\ 1 \end{pmatrix},$$

$$w_{NS,*} = \frac{1}{\overline{wv}}\begin{pmatrix} 2\,\frac{c^2\left(-1 + \mathrm{e}^{-h}\right)}{d\left(-c\,\mathrm{e}^{-h} + i\sqrt{c\left(2c\,\mathrm{e}^{-h} - \mathrm{e}^{-2h}c - 4\,\mathrm{e}^{-h} - c + 4\right)} + c\right)} \\ 1 \end{pmatrix},$$

where $wv =$

$$= 2\,\frac{c\left(\left(i\mathrm{e}^{-h} - i\right)\sqrt{-\left((-2c+4)\,\mathrm{e}^{-h} + \mathrm{e}^{-2h}c + c - 4\right)c} + (-2c+4)\,\mathrm{e}^{-h} + \mathrm{e}^{-2h}c + c - 4\right)}{\left(i\sqrt{-\left((-2c+4)\,\mathrm{e}^{-h} + \mathrm{e}^{-2h}c + c - 4\right)c} + c\left(-1 + \mathrm{e}^{-h}\right)\right)^2}.$$

If $\widehat{\delta^{pp}_{NS,*}} \neq 0$ there is a generic Neimark-Sacker on the curve $\mathcal{T}^{pp}_{NS,*}$. When

$$\widehat{\sigma^{pp}_{NS,*}} = \Re\left(e^{-i\theta_0}\widehat{\delta^{pp}_{NS,*}}\right) \neq 0,$$

a unique closed invariant curve for $\mathcal{M}^{pp}(\Psi_{NS,*}, \Omega_{NS,*})$ appears around the $\Psi^{pp}_{NS,*}$, when a crosses $a_{NS,*}$. The sign of $\widehat{\sigma^{pp}_{NS,*}}$ indicates the stability of the closed invariant curve. □

3.2. *Two parameters bifurcations*

Each bifurcation on the bifurcation curves mentioned in Section 3.1 is examined within a two-parameter space (a,h).

Theorem 3. *The critical values $a = a_{R_2,*} = c/4$ and $h = h_{R_2,*} = -\ln\left(\frac{-4+c}{c}\right)$ causes a strong resonance 1:2 bifurcation Ψ_*^{pp}*

Proof. For $r = r_{R_2,*}$ and $h = h_{R_2,*}$ the matrix

$$\mathcal{J}_1(\Psi^{pp}_{R_2,*},\Omega_{R_2,*}) = \begin{pmatrix} 1 & -\frac{d}{c} \\ 4\,\frac{c}{d} & -3 \end{pmatrix},$$

$$\Psi^{pp}_{R_2,*} = \begin{pmatrix} 1/4\,\frac{d}{b} \\ 1/4\,\frac{c}{b} \end{pmatrix}, \quad \Omega_{R_2,*} = (a_{R_2,*}, b, c, d, h_{R_2,*}),$$

has two repeated multipliers $\lambda^{1,2}_{R_2,*} = -1$. On the curve

$$\mathcal{T}^{pp}_{R_2,*} = \{(x_{pp}, y_{pp}, a, b, c, d, h), a = a_{R_2,*},\, h = h_{R_2,*}\}.$$

the resonance 1:2 bifurcation will occur.

The map $\mathcal{M}^{pp}(\Psi_{R_2,*},\Omega_{R_2,*})$ can be written as

$$\begin{pmatrix} \eta_{R_2,*} \\ \zeta_{R_2,*} \end{pmatrix} \mapsto \begin{pmatrix} -\eta_{R_2,*} + \zeta_{R_2,*} \\ \zeta_{R_2,*} + \widehat{\sigma^{pp}_{R_2,*}}\eta^3_{R_2,*} + \widehat{\delta^{pp}_{R_2,*}}\eta^2_{R_2,*}\zeta_{R_2,*} \end{pmatrix}.$$

The normal form coefficients $\widehat{\sigma^{pp}_{R_2,*}}$ and $\widehat{\delta^{pp}_{R_2,*}}$ can be derived as

$$\begin{aligned}
\widehat{\sigma^{pp}_{R_2,*}} &= \frac{1}{6}\langle w^0_{R_2}, \mathcal{J}_3(v^0_{R_2,*}, v^0_{R_2,*}, v^0_{R_2,*}) \\
&\quad + 3\mathcal{J}_2\left(v^0_{R_2,*}, \left(I_2 - \mathcal{J}_1(\Psi^{pp}_{R_2,*},\Omega_{R_2,*})\right)^{-1}\right)\mathcal{J}_2(v^0_{R_2,*}, v^0_{R_2,*})\rangle, \\
\widehat{\delta^{pp}_{R_2,*}} &= \frac{1}{2}\langle w^0_{R_2,*}, \mathcal{J}_3(v^0_{R_2,*}, v^0_{R_2,*}, v^1_{R_2,*}) + 2\mathcal{J}_2(v^0_{R_2,*}, h^{11}_{R_2,*}) + \mathcal{J}_2(v^1_{R_2,*}, h^{20}_{R_2,*})\rangle \\
&\quad + \frac{1}{2}\langle w^1_{R_2,*}, \mathcal{J}_3(v^0_{R_2,*}, v^0_{R_2,*}, v^0_{R_2,*}) + 2\mathcal{J}_2(v^0_{R_2,*}, h^{20}_{R_2,*})\rangle, \\
h^{20}_{R_2,*} &= \left(I_2 - \mathcal{J}_1(\Psi^{pp}_{R_2,*},\Omega_{R_2,*})\right)^{-1}\left(\mathcal{J}_2(v^0_{R_2,*}, v^0_{R_2,*})\right), \\
h^{11}_{R_2,*} &= \left(I_2 - \mathcal{J}_1(\Psi^{pp}_{R_2,*},\Omega_{R_2,*})\right)^{-1}\left(\mathcal{J}_2(v^0_{R_2,*}, v^1_{R_2}) + h^{20}_{R_2,*}\right).
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{J}_1(\Psi^{pp}_{R_2,*},\Omega_{R_2,*})v^0_{R_2,*} &= -v^0_{R_2,*}, & \mathcal{J}_1^T(\Psi^{pp}_{R_2,*},\Omega_{R_2,*})w^1_{R_2,*} &= -w^1_{R_2,*} + w^0_{R_2,*}, \\
\mathcal{J}_1^T(\Psi^{pp}_{R_2,*},\Omega_{R_2,*})w^0_{R_2,*} &= -w^0_{R_2,*}, & \left\langle w^0_{R_2,*}, v^1_{R_2,*}\right\rangle &= \left\langle w^1_{R_2,*}, v^0_{R_2,*}\right\rangle = 1, \\
\mathcal{J}_1(\Psi^{pp}_{R_2,*},\Omega_{R_2,*})v^1_{R_2,*} &= -v^1_{R_2,*} + v^0_{R_2,*}, & \left\langle w^0_{R_2,*}, v^0_{R_2,*}\right\rangle &= \left\langle w^1_{R_2,*}, v^1_{R_2,*}\right\rangle = 0.
\end{aligned}$$

As a result, we have

$$v^0_{R_2,*} = \begin{pmatrix} 1/2\,\frac{d}{c} \\ 1 \end{pmatrix}, \qquad v^1_{R_2,*} = \begin{pmatrix} 1/4\,\frac{d^3}{c\,(9c^2+d^2)} \\ 3/4\,\frac{d^2}{9c^2+d^2} \end{pmatrix},$$

$$w^0_{R_2,*} = \begin{pmatrix} -8\,\frac{c\left(9c^2+d^2\right)}{d^3} \\ 4\,\frac{9c^2+d^2}{d^2} \end{pmatrix}, \qquad w^1_{R_2,*} = \begin{pmatrix} 6\,\frac{c}{d} \\ -2 \end{pmatrix}.$$

Consequently, we get

$$\widehat{\sigma^{pp}_{R_2,*}} = -96\,\frac{\left(9\,c^2+d^2\right)b^2}{d^2c^2}, \quad \widehat{\delta^{pp}_{R_2,*}} = -16\,\frac{b^2\left(360\,c^2+29\,d^2\right)}{d^2c^2}.$$

If $\widehat{\sigma^{pp}_{R_2,*}} \neq 0$ and $-2\widehat{\delta^{pp}_{R_2,*}} \neq -6\,\widehat{\sigma^{pp}_{R_2,*}}$ there is a generic resonance 1:2 bifurcation on the curve $\mathcal{T}^{pp}_{R_2,*}$. □

Theorem 4. *The critical values $a = a_{R_3,*} = c/3$ and $h = h_{R_3,*} = -\ln\left(\frac{-3+c}{c}\right)$ causes a strong resonance 1:3 bifurcation Ψ^{pp}_*.*

Proof. For $r = r_{R_3,*}$ and $h = h_{R_3,*}$ the matrix

$$\mathcal{J}_1(\Psi^{pp}_{R_3,*}, \Omega_{R_3,*}) = \begin{pmatrix} 1 & -\frac{d}{c} \\ 3\,\frac{c}{d} & -2 \end{pmatrix},$$

$$\Psi^{pp}_{R_3,*} = \begin{pmatrix} 1/3\,\frac{d}{b} \\ 1/3\,\frac{c}{b} \end{pmatrix}, \quad \Omega_{R_3,*} = (a_{R_3,*}, b, c, d, h_{R_3,*})^T,$$

has multipliers $\lambda^{1,2}_{R_3,*} = \cos(\frac{2\pi}{3}) \pm i\,\sin(\frac{2\pi}{3})$. The resonance 1:3 bifurcation will occur on the curve

$$\mathcal{T}^{pp}_{R_3,*} = \{(x_{pp}, y_{pp}, a, b, c, d, h), a = a_{R_3,*},\, h = h_{R_3,*}\}.$$

The map $\mathcal{M}^{pp}(\Psi^{pp}_{R_3,*}, \Omega_{R_3,*})$ can be written as

$$\eta_{R_3,*} \mapsto \left(\cos(\frac{2\pi}{3}) \pm i\,\sin(\frac{2\pi}{3})\right)\eta_{R_3,*} + \widehat{\beta^{pp}_{R_3,*}}\eta^2_{R_3,*}\overline{\eta_{R_3,*}} + \widehat{\sigma^{pp}_{R_3,*}}\,\overline{\eta_{R_3,*}}^3 + O(|\eta_{R_3,*}|^4).$$

The normal form coefficients $\widehat{\beta^{pp}_{R_3,*}}$ and $\widehat{\sigma^{pp}_{R_3,*}}$ can be derived as

$$\widehat{\beta^{pp}_{R_3,*}} = \frac{1}{2}\langle w_{R_3,*}, \mathcal{J}_2(\overline{v_{R_3,*}}, \overline{v_{R_3,*}})\rangle,$$

$$\widehat{\sigma^{pp}_{R_3,*}} = \frac{1}{2}\langle w_{R_3,*}, \mathcal{J}_3(v_{R_3,*}, v_{R_3,*}, \overline{v_{R_3,*}}) + 2\mathcal{J}_2(v_{R_3,*}, h^{11}_{R_3,*}) - \mathcal{J}_2\left(\overline{v_{R_3,*}}, h^{20}_{R_3,*}\right)\rangle,$$

$$h^{11}_{R3,*} = \left(I_2 - \mathcal{J}_1(\Psi^{pp}_{R_3,*}, \Omega_{R_3,*})\right)^{-1}\mathcal{J}_2(v_{R_3,*}, \overline{v_{R_3,*}}),$$

$$h^{20}_{R_3,*} = \left(e^{\frac{4\pi}{3}i}I_2 - \mathcal{J}_1(\Psi^{pp}_{R_3,*}, \Omega_{R_3,*})\right)^{-INV}\left(\overline{\widehat{\beta^{pp}_{R_3,*}}}\,\overline{v_{R_3,*}} - \mathcal{J}_2(v_{R_3,*}, v_{R_3,*})\right),$$

where

$$\mathcal{J}_1(\Psi^{pp}_{R_{3,*}},\Omega_{R_{3,*}})v_{R_{3,*}} = e^{\frac{2\pi}{3}i}v_{R_{3,*}}, \qquad \mathcal{J}_1(\Psi^{pp}_{R_{3,*}},\Omega_{R_{3,*}})\overline{v_{R_{3,*}}} = e^{\frac{-2\pi}{3}i}\overline{v_{R_{3,*}}},$$
$$\mathcal{J}_1^T(\Psi^{pp}_{R_{3,*}},\Omega_{R_{3,*}})w_{R_{3,*}} = e^{\frac{-2\pi}{3}i}w_{R_{3,*}}, \qquad \mathcal{J}_1^T(\Psi^{pp}_{R_{3,*}},\Omega_{R_{3,*}})\overline{w_{R_{3,*}}} = e^{\frac{2\pi}{3}i}\overline{w_{R_{3,*}}},$$
$$\langle w_{R_{3,*}}, v_{R_{3,*}}\rangle = 1.$$

As a result, we have

$$v_{R_{3,*}} = \begin{pmatrix} -2\,\frac{d}{c\left(i\sqrt{3}-3\right)} \\ 1 \end{pmatrix}, \quad w_{R_{3,*}} = \begin{pmatrix} -12\,\frac{c}{\left(i\sqrt{3}+1\right)d\left(i\sqrt{3}+3\right)} \\ 2\left(i\sqrt{3}+1\right)^{-1} \end{pmatrix}.$$

Consequently, we get

$$\widehat{\beta^{pp}_{R_{3,*}}} = 3/2\,\frac{b\left(3i\sqrt{3}-1\right)}{c}, \quad \widehat{\sigma^{pp}_{R_{3,*}}} = -21\,\frac{b^2\left(2i\sqrt{3}+3\right)}{c^2}.$$

There is a generic resonance 1:3 bifurcation on the curve $\mathcal{T}^{pp}_{R_{3,*}}$, provided that $\widehat{\beta^{pp}_{R_{3,*}}} \neq 0$ and $\widehat{\sigma^{pp}_{R_{3,*}}} \neq 0$. □

Theorem 5. *The critical values $a = a_{R_{4,*}} = c/2$ and $h = h_{R_{4,*}} = -\ln\left(\frac{-2+c}{c}\right)$ causes a strong resonance 1:4 bifurcation Ψ^{pp}_**

Proof. For $a = a_{R_{4,*}}$ and $h = h_{r_{4,*}}$ the matrix

$$\mathcal{J}_1(\Psi^{pp}_{R_{4,*}},\Omega_{R_{4,*}}) = \begin{pmatrix} 1 & -\frac{d}{c} \\ 2\,\frac{c}{d} & -1 \end{pmatrix},$$
$$\Psi^{pp}_{R_{4,*}} = \begin{pmatrix} 1/2\,\frac{d}{b} \\ 1/2\,\frac{c}{b} \end{pmatrix}, \quad \Omega_{R_{4,*}} = (a_{R_{4,*}}, b.c.d, h_{R_{4,*}})^T,$$

has multipliers $\lambda^{1,2}_{R_{4,*}} = \cos(\frac{\pi}{2}) \pm i\sin(\frac{\pi}{2})$. The resonance 1:4 bifurcation will occur on the curve

$$\mathcal{T}^{pp}_{R_{4,*}} = \{(x_{pp}, y_{pp}, a, b, c, d, h),\, a = a_{R_{4,*}},\, h = h_{R_{4,*}}\}.$$

The map $\mathcal{M}^{pp}(\Psi^{pp}_{R_{4,*}},\Omega_{R_{4,*}})$ can be written as

$$\eta_{R_{4,*}} \mapsto \left(\cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2})\right)\eta_{R_{4,*}} + \widehat{\sigma^{pp}_{R_{4,*}}}\eta^2_{R_{4,*}}\overline{\eta_{R_{4,*}}} + \widehat{\delta^{pp}_{R_{4,*}}}\overline{\eta_{R_{4,*}}}^3 + O(|\eta_{R_{4,*}}|^4).$$

The normal form coefficients $\widehat{\sigma^{pp}_{R_{4,*}}}$ and $\widehat{\delta^{pp}_{R_{4,*}}}$ can be derived as

$$\widehat{\sigma^{pp}_{R_{4,*}}} = \frac{1}{2}\langle w_{R_{4,*}}, \mathcal{J}_3(v_{R_{4,*}}, v_{R_{4,*}}, \overline{v_{R_{4,*}}}) + 2\mathcal{J}_2\left(v_{R_{4,*}}, h^{11}_{R_{4,*}}\right) - \mathcal{J}_2\left(\overline{v_{R_{4,*}}}, h^{20}_{R_{4,*}}\right)\rangle,$$
$$\widehat{\sigma^{pp}_{R_{4,*}}} = \frac{1}{6}\langle w_{R_{4,*}}, \mathcal{J}_3(\overline{v_{R_{4,*}}}, \overline{v_{R_{4,*}}}, \overline{v_{R_{4,*}}}) - 3\mathcal{J}_2\left(\overline{v_{R_{4,*}}}, h^{02}_{R_{4,*}}\right)\rangle,$$
$$h^{11}_{R_{4,*}} = \left(I_2 - \mathcal{J}_1(\Psi^{pp}_{R_{4,*}},\Omega_{R_{4,*}})\right)^{-1}\mathcal{J}_2\left(v_{R_{4,*}}, \overline{v_{R_{4,*}}}\right),$$

$$h^{20}_{R_4,*} = \left(I_2 + \mathcal{J}_1(\Psi^{pp}_{R_4,*}, \Omega_{R_4,*})\right)^{-1} \mathcal{J}_2\left(v_{R_4,*}, v_{R_4,*}\right),$$
$$h^{02}_{R_4,*} = \left(I_2 + \mathcal{J}_1(\Psi^{pp}_{R_4,*}, \Omega_{R_4,*})\right)^{-1} \mathcal{J}_2(\overline{v_{R_4,*}}, \overline{v_{R_4,*}}).$$

where

$$\mathcal{J}_1(\Psi^{pp}_{R_4,*}, \Omega_{R_4,*})v_{R_4,*} = e^{\frac{\pi}{2}i}v_{R_4,*}, \qquad \mathcal{J}_1(\Psi^{pp}_{R_4,*}, \Omega_{R_4,*})\overline{v_{R_4,*}} = e^{\frac{-\pi}{2}i}\overline{v_{R_4,*}},$$
$$\mathcal{J}_1^T(\Psi^{pp}_{R_4,*}, \Omega_{R_4,*})w_{R_4,*} = e^{\frac{-\pi}{2}i}w_{R_4,*}, \qquad \mathcal{J}_1^T(\Psi^{pp}_{R_4,*}, \Omega_{R_4,*})\overline{w_{R_4,*}} = e^{\frac{\pi}{2}i}\overline{w_{R_4,*}},$$
$$\langle w_{R_4,*}, v_{R_4,*}\rangle = 1.$$

As a result, we have

$$v_{R_4,*} = \begin{pmatrix} \frac{(1/2+i/2)d}{c} \\ 1 \end{pmatrix}, \quad w_{R_4,*} = \begin{pmatrix} \frac{ic}{d} \\ 1/2 - i/2 \end{pmatrix}.$$

Consequently, we get

$$\widehat{\sigma^{pp}_{R_4,*}} = \frac{(-24-14\,i)\,b^2}{c^2}, \quad \widehat{\delta^{pp}_{R_4,*}} = \frac{(6-8\,i)\,b^2}{c^2}.$$

There is a generic resonance 1:4 bifurcation on the curve $\mathcal{T}^{pp}_{R_4,*}$ provided that $\widehat{\sigma^{pp}_{R_4,*}} \neq 0$ and $\widehat{\delta^{pp}_{R_4,*}} \neq 0$.

The bifurcation scenario in the neighbourhood of the curve $\mathcal{T}^{pp}_{R_4,*}$ is determined by the coefficient $\widehat{A^{pp}_{R_4,*}} = -\frac{i\,\widehat{\sigma^{pp}_{R_4,*}}}{|\widehat{\delta^{pp}_{R_4,*}}|}$, provided that $\widehat{\delta^{pp}_{R_4,*}} \neq 0$. If $|\widehat{A^{pp}_{R_4,*}}| > 1$, we conclude there are two fold curves of cycles with four times the original period, see [15, 17]. □

4. Numerical bifurcation analysis

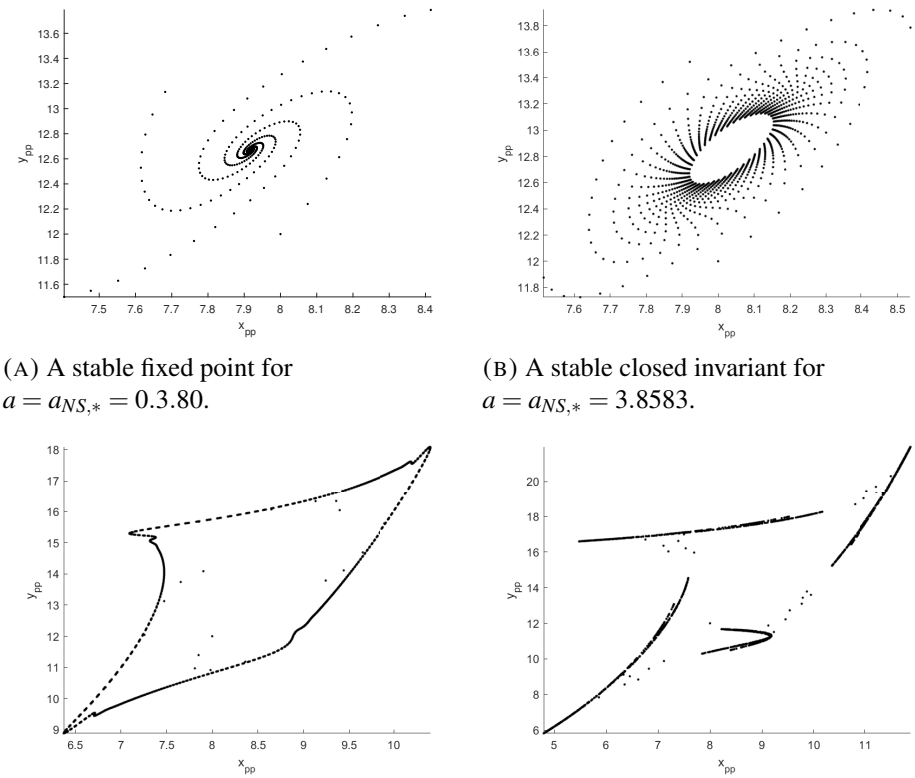
To confirm the obtained results in Section 3 and investigate further complex behaviour of $\mathcal{M}^{pp}(\Psi,\Omega)$, we use MatContM which is a MATLAB interactive toolbox for the numerical study of iterated discrete dynamical systems, [10, 14, 17].

In this case we consider fixed parameters $b = 0.3$, $c = 8$, $d = 5$ and $h = 0.3$ as a free parameter. By varying the free parameter a, the continuation method produces the following one parameter bifurcations:

i) A Neimark-Sacker bifurcation (NS) occurs at the point $\Psi^{pp}_{NS,*} = (8.038116, 12.860986)$ for $a = a_{NS,*} = 3.858296$ with $\widehat{\sigma^{pp}_{NS,*}} = -2.225899 \times 10^{-2}$.

ii) A period-doubling bifurcation (PD) occurs at the point $\Psi^{pp}_{PD,*} = (0.569517, 0.911227)$ for $a = a_{PD,*} = 0.273368$ with $\widehat{\beta^{pp}_{PD,*}} = -3.366997 \times 10^{-3}$.

Since $\widehat{\sigma^{pp}_{NS,*}} < 0$ the bifurcation is supercritical and the bifurcated closed invariant is stable, see Fig. 6. The numerical simulation shown in these graphs indicates the dynamical behavior of the map $\mathcal{M}^{pp}(\Psi,\Omega)$ near $\Psi^{pp}_{NS,*}$.



(A) A stable fixed point for $a = a_{NS,*} = 0.3.80$.

(B) A stable closed invariant for $a = a_{NS,*} = 3.8583$.

(C) The broken invariant closed curve after $\Psi^{pp}_{NS,*}$ for $a = a_{NS,*} = 4.2$.

(D) The chaotic attractor for $a = a_{NS,*} = 4.5$.

FIGURE 1. Behavior of the map $\mathcal{M}^{pp}(\Psi,\Omega)$ near the NS point.

Remark 1. Fig. 2 illustrates the maximum Lyapunov exponent for $a \succeq 4.45$, indicating that chaos exists. Positive Lyapunov exponents are generally considered to be a sign of chaos.

Since $\widehat{\beta^{pp}_{PD,*}} < 0$ the bifurcation is sub-critical and the bifurcated period-2 cycles is unstable.

In Fig. 3, we can see the stability region for the positive fixed point Ψ^{pp}_{*}.

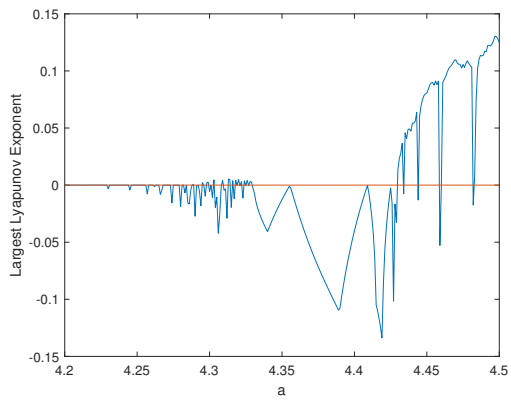


FIGURE 2. The maximum Lyapunov exponent corresponding to Fig. 6.

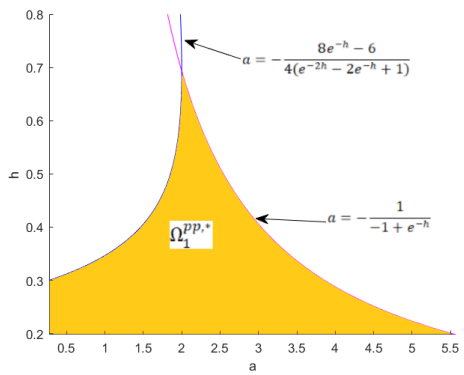


FIGURE 3. The stability region for the positive fixed point Ψ_*^{pp} in space (a,h).

The following two-parameter bifurcations can be obtained with the selected $\Psi_{NS,*}^{pp}$ and continuation with two free parameters, r, and h:

i) A resonance 1:3 bifurcation (R3) occurs at the point $\Psi_{R_3,*}^{pp} = (5.555624, 8.888998)$ for $a = a_{R_3,*} = 2.666699$ and $h = h_{R_3,*} = 0.0.470006$ with $\Re\left(\frac{1}{3}\left(e^{\frac{4\pi}{3}i}\widehat{\sigma_{R_3,*}^{pp}}/|\widehat{\beta_{R_3,*}^{pp}}|^2 - 1\right)\right) = -4.999943 \times 10^{-1}$.

ii) A resonance 1:4 bifurcation (R4) occurs at the point $\Psi_{R_4,*}^{pp} = (8.333333, 13.333333)$ for $a = a_{R_4,*} = 4.000000$ and $h = h_{R_4,*} = 0.287682$ with $\widehat{A_{R_4,*}^{pp}} = -1.600000 + 6.000000 \times 10^{-1}\, i$.

iii) A resonance 1:2 bifurcation (R2) occurs at the point $\Psi^{pp}_{R_2,*} = (4.166667, 6.666667)$ for $a = a_{R_2,*} = 2.000000$ and $h = h_{R_2,*} = 0.0.693147$ with $\widehat{\sigma^{pp}_{R_2,*}} = 2.459786 \times 10^1$ and $\widehat{\delta^{pp}_{R_2,*}} = -1.639858$.

The following two-parameter bifurcation can be obtained with the selected PD point and continuation with two free parameters, r and h:

i) A resonance 1:2 bifurcation (R2) occurs at the point $\Psi^{pp}_{R_2,*} = (4.166667, 6.666667)$ for $a = a_{R_2,*} = 2.000000$ and $h = h_{R_2,*} = 0.0.693147$ with $\widehat{\sigma^{pp}_{R_2,*}} = 2.459786 \times 10^1$ and $\widehat{\delta^{pp}_{R_2,*}} = -1.639858$.

Fig. 4 illustrates these results.

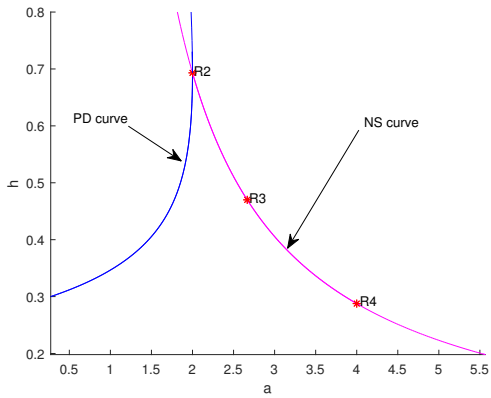


FIGURE 4. The Neimark-Sacker and period doubling bifurcations curves of $\mathcal{M}^{pp}(\Psi, \Omega)$ in the (r, h) space.

According to Theorem 5, since $|\widehat{A^{pp}_{R_4,*}}| > 1$, there are two fold curves of cycles with period 4, see Fig. 5.

For $a = 5.51841$ and $h = 0.240474$, $C^{pp}_4 = \{C^{pp}_{1,4}, C^{pp}_{2,4}, C^{pp}_{3,4}, C^{pp}_{4,4}\}$ gives a stable four-cycle where $C^{pp}_{1,4} = (8.846315, 22.650752)$, see Fig. 6a. In Fig. 6b we can see the stability region for C^{pp}_4.

5. DISCUSSION AND ECOLOGICAL IMPLICATIONS

A discrete-time prey-predator model using non-standard finite difference discretization method is examined in terms of its complex dynamics. We demonstrate that model (1.3) has unique interior fixed point (positive) Ψ^{pp}_*. Ψ^{pp}_* may bifurcate in many ways, as shown in Section (3). In Section (4), the curves of fixed points and one-parameter bifurcations of cycles up to the fourth order are computed. The analytical predictions and numerical observations obtained in Sections (3) and (4) are

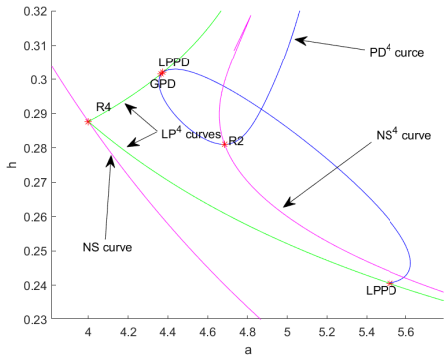
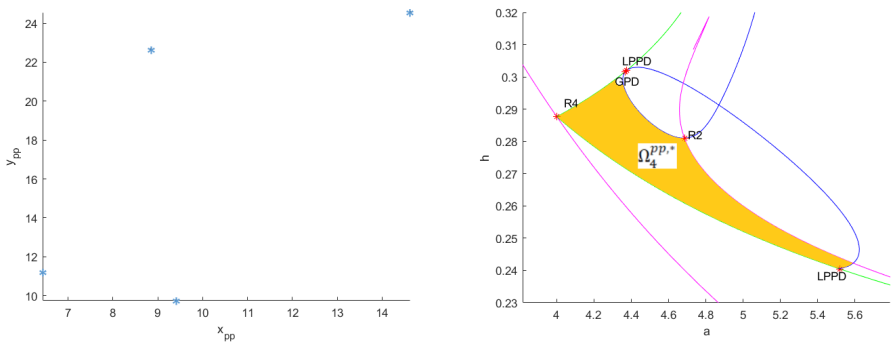


FIGURE 5. Two fold (LP) curves of the fourth iterate emanate from $\Psi^{pp}_{R_4,*}$ of $\mathcal{M}^{pp}(\Psi,\Omega)$ and the Period doubling and Neimark-Sacker bifurcations curves of the fourth iterate of $\mathcal{M}^{pp}(\Psi,\Omega)$ in the (r,h) space.



(A) Stable four-cycle C_4^{pp}. (B) Stability region for C_4^{pp}.

FIGURE 6. The four-cycle with its stability region.

in excellent agreement. The complicated dynamics of model (1.3) are depicted via MatcontM and continuation technique. Period-doubling, Neimark-Sacker, and strong resonance bifurcation of Ψ^{pp}_* have been demonstrated.

A Neimark-Sacker bifurcation, implies that both prey and predator populations can oscillate around some mean values of intrinsic growth rate a and that these oscillations will continue as long as $\widehat{\sigma^{pp}_{NS,*}} < 0$ is constant. In ecology, an invariant closed curve is bifurcated, which means both predator and prey can live together and produce their own densities. A periodic or quasi-periodic dynamics may be present

on the invariant curve. A period-doubling bifurcation is shown in the model, indicating that the prey and predator populations change over time. In this model, there are strong resonances bifurcating around some mean values of intrinsic growth rate and step size h when predator and prey coexist. If certain conditions are met, this coexistence is definitely possible.

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CONNECTION OF BALANCING NUMBERS WITH SOLUTION OF A SYSTEM OF TWO HIGHER-ORDER DIFFERENCE EQUATIONS

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Abstract. We provide some theoretical justifications pertaining to the representation for the solution of the system of the higher-order rational difference equations

$$x_{n+1} = \frac{1}{6-y_{n-k}}, \quad y_{n+1} = \frac{1}{6-x_{n-k}}, \qquad n,k \in \mathbb{N}_0.$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and the initial conditions x_{-k}, $x_{-k+1}, \ldots$, x_0, y_{-k}, $y_{-k+1}, \ldots$, y_0 are non zero real numbers such that their solution is related to Balancing numbers. We also study the stability character and asymptotic behavior of this system.

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Keywords: general solution, Balancing numbers, stability, system of difference equations

1. Introduction

The notion of Balancing numbers was first suggested by Behera and Panda [4] in relation to a Diophantine equation in 1999. A positive integer n is known as a Balancing number if

$$1+2+\ldots+(n-1) = (n+1)+(n+2)+\ldots+(n+r)$$

for some $r \in \mathbb{N}$. Here r is known as the balancer corresponding to the Balancing number n. Behera and Panda [4] demonstrated in a joint study that the Balancing numbers satisfy the following recurrence relation

$$B_{n+2} = 6B_{n+1} - B_n, \quad n \in \mathbb{N},$$

where $B_1 = 1$ and $B_2 = 6$. The Binet's formula for Balancing numbers is given by

$$B_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \in \mathbb{N}_0,$$

where

$$\alpha = 3+2\sqrt{2}, \quad \beta = 3-2\sqrt{2}.$$

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We always have the quotient of two consecutive terms of Balancing numbers tends to α, that is

$$\lim_{n\to\infty}\frac{B_{n+1}}{B_n}=\alpha.$$

Many papers have been published previously focusing on the forme of the solution of system of difference equations, see for example [1,2,5–8,10,12,15,19,20]. Curiously, some of the solution forms of these systems can even be expressed in terms of well-known sequences such as Fibonacci numbers, Horadam numbers, Padovan numbers, Lucas numbers and Pell numbers, see for example [3, 9, 11, 13, 14, 16–18] .

In this paper we provide some theoretical justifications pertaining to the representation for the solution of the system of the higher-order rational difference equations

$$x_{n+1}=\frac{1}{6-y_{n-k}},\quad y_{n+1}=\frac{1}{6-x_{n-k}},\qquad n,k\in\mathbb{N}_0. \tag{1.1}$$

where $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, and the initial conditions x_{-k}, $x_{-k+1},\ldots,$ x_0, y_{-k}, $y_{-k+1},\ldots,$ y_0 are non zero real numbers such that their solution is related to Balancing numbers.

2. Auxiliary Results

To prove our main results in Sections 3 and 4, we will need the following two lemmas.

Lemma 1. *Consider the two homogenous second order linear autonomous difference equations*

$$y_{n+2}-6y_{n+1}+y_n=0,\quad n\in\mathbb{N}_0, \tag{2.1}$$

$$S_{n+2}+6S_{n+1}+S_n=0,\quad n\in\mathbb{N}_0. \tag{2.2}$$

Then, we have for all $n\in\mathbb{N}_0$

$$y_n=-y_0B_{n-1}+y_1B_n,\qquad S_n=(-1)^n(S_0B_{n-1}+S_1B_n).$$

Proof. As is well-known, the equation

$$y_{n+2}-6y_{n+1}+y_n=0,\quad n\in\mathbb{N}_0,$$

where y_0 and $y_1\in\mathbb{R}$, is usually solved by using the characteristic roots α and β of the characteristic polynomial $\varphi^2-6\varphi+1$, so

$$\alpha=3+2\sqrt{2},\qquad \beta=3-2\sqrt{2},$$

and the formulas of general solution is

$$y_n=c_1\alpha^n+c_2\beta^n.$$

Using the initial conditions y_0 and y_1 with some calculations we get

$$c_1=\frac{y_0\beta-y_1}{\beta-\alpha},\qquad c_2=\frac{y_1-y_0\alpha}{\beta-\alpha}.$$

So

$$y_n = y_0 \frac{\alpha^{n-1} - \beta^{n-1}}{-(\alpha - \beta)} + y_1 \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

Hence

$$y_n = -y_0 B_{n-1} + y_1 B_n.$$

By the same argument, we get

$$S_n = (-1)^n (S_0 B_{n-1} + S_1 B_n).$$

□

Lemma 2. *Consider the linear system of second order linear autonomous difference equations*

$$t_{n+2} = 6p_{n+1} - t_n, \quad p_{n+2} = 6t_{n+1} - p_n, \quad n \in \mathbb{N}_0. \tag{2.3}$$

Then, we have for all $n \in \mathbb{N}_0$

$$\begin{cases} t_{2n} = -p_0 B_{2n-1} + t_1 B_{2n}, & t_{2n+1} = -t_0 B_{2n} + p_1 B_{2n+1}, \\ p_{2n} = -t_0 B_{2n-1} + p_1 B_{2n}, & p_{2n+1} = -p_0 B_{2n} + t_1 B_{2n+1}. \end{cases}$$

Proof. From (2.3) we get the following system

$$\begin{cases} t_{n+2} + p_{n+2} = 6p_{n+1} - t_n + 6t_{n+1} - p_n, \\ t_{n+2} - p_{n+2} = 6p_{n+1} - t_n - 6t_{n+1} + p_n. \end{cases}$$

So

$$\begin{cases} t_{n+2} + p_{n+2} = 6(p_{n+1} + t_{n+1}) - (t_n + p_n), \\ t_{n+2} - p_{n+2} = -6(t_{n+1} - p_{n+1}) - (t_{n+} - p_n). \end{cases} \tag{2.4}$$

By posing the following changes of the variables

$$R_n = t_n + p_n, \quad S_n = t_n - p_n. \tag{2.5}$$

System (2.4) becomes

$$R_{n+2} = 6R_{n+1} - R_n, \qquad S_{n+2} = -6S_{n+1} - S_n.$$

which are in the form of equations (2.1) and (2.2). Then it follows from Lemma 1 that

$$\begin{cases} R_{2n} = -R_0 B_{n-1} + R_1 B_n, & R_{2n+1} = -R_0 B_{n-1} + R_1 B_n, \\ S_{2n} = S_0 B_{n-1} + S_1 B_n, & S_{2n+1} = -S_0 B_{n-1} - S_1 B_n. \end{cases}$$

From (2.5), we have

$$t_n = \frac{1}{2}(R_n - S_n), \quad p_n = \frac{1}{2}(R_n + S_n).$$

So

$$t_{2n} = \frac{1}{2}(R_{2n} - S_{2n}), \quad t_{2n+1} = \frac{1}{2}(R_{2n+1} - S_{2n+1}),$$

$$p_{2n} = \frac{1}{2}(R_{2n} + S_{2n}), \quad p_{2n+1} = \frac{1}{2}(R_{2n+1} + S_{2n+1}).$$

Hence

$$\begin{aligned} t_{2n} &= \frac{1}{2}\left(B_{2n-1}(-R_0 - S_0) + B_{2n}(R_1 - S_1)\right), \\ t_{2n+1} &= \frac{1}{2}\left(B_{2n}(-R_0 + S_0) + B_{2n+1}(R_1 + S_1)\right), \\ p_{2n} &= \frac{1}{2}\left(B_{2n-1}(-R_0 + S_0) + B_{2n}(R_1 + S_1)\right), \\ p_{2n+1} &= \frac{1}{2}\left(B_{2n}(-R_0 - S_0) + B_{2n+1}(R_1 - S_1)\right). \end{aligned}$$

So

$$\begin{cases} t_{2n} = -p_0 B_{2n-1} + t_1 B_{2n}, \\ t_{2n+1} = -t_0 B_{2n} + p_1 B_{2n+1}, \\ p_{2n} = -t_0 B_{2n-1} + p_1 B_{2n}, \\ p_{2n+1} = -p_0 B_{2n} + t_1 B_{2n+1}. \end{cases}$$

□

3. Main result

In this section we give the explicit formula of solution to system (1.1) in terms of Balancing numbers.

From (1.1), we can write

$$\begin{cases} x_{(k+1)(n+1)-j} = & \dfrac{1}{6 - y_{(k+1)n-j}}, \\ y_{(k+1)(n+1)-j} = & \dfrac{1}{6 - x_{(k+1)n-j}}. \end{cases}$$

Let

$$x_n^{(j)} = x_{(k+1)n-j}, \quad y_n^{(j)} = y_{(k+1)n-j}, \quad j = 0, 1, \ldots, k-1.$$

So, the system (1.1) becomes

$$x_{n+1}^{(j)} = \frac{1}{6 - y_n^{(j)}}, \quad y_{n+1}^{(j)} = \frac{1}{6 - x_n^{(j)}}, \quad n \in \mathbb{N}_0. \tag{3.1}$$

To find the form of the solution of the system (3.1) we consider the following changes of variables

$$x_n^{(j)} = \frac{W_n}{U_{n+1}}, \quad y_n^{(j)} = \frac{U_n}{W_{n+1}}.$$

So

$$x_{n+1}^{(j)} = \frac{W_{n+1}}{U_{n+2}}, \quad y_{n+1}^{(j)} = \frac{U_{n+1}}{W_{n+2}}.$$

Hence

$$\frac{W_{n+1}}{U_{n+2}} = \frac{1}{6-\dfrac{U_n}{W_{n+1}}}, \quad \frac{W_{n+1}}{U_{n+2}} = \frac{W_{n+1}}{6W_{n+1}-U_n},$$
$$\frac{U_{n+1}}{W_{n+2}} = \frac{1}{6-\dfrac{W_n}{U_{n+1}}}, \quad \frac{U_{n+1}}{W_{n+2}} = \frac{U_{n+1}}{6U_{n+1}-W_n}.$$

So the system (3.1) becomes

$$U_{n+2} = 6W_{n+1} - U_n, \quad W_{n+2} = 6U_{n+1} - W_n.$$

Then it follows from Lemma 2 that

$$\begin{cases} U_{2n} = -W_0B_{2n-1} + U_1B_{2n}, \\ U_{2n+1} = -U_0B_{2n} + W_1B_{2n+1}, \\ W_{2n} = -U_0B_{2n-1} + W_1B_{2n}, \\ W_{2n+1} = -W_0B_{2n} + U_1B_{2n+1}. \end{cases}$$

So,

$$x_{2n+1}^{(j)} = \frac{W_{2n+1}}{U_{2n+2}} = \frac{-W_0B_{2n} + U_1B_{2n+1}}{-W_0B_{2n+1} + U_1B_{2n+2}} = \frac{\dfrac{-W_0B_{2n} + U_1B_{2n+1}}{U_1}}{\dfrac{-W_0B_{2n+1} + U_1B_{2n+2}}{U_1}}.$$

Hence

$$x_{2n+1}^{(j)} = \frac{-x_0^{(j)}B_{2n} + B_{2n+1}}{-x_0^{(j)}B_{2n+1} + B_{2n+2}},$$

and

$$x_{2n}^{(j)} = \frac{W_{2n}}{U_{2n+1}} = \frac{-U_0B_{2n-1} + W_1B_{2n}}{-U_0B_{2n} + W_1B_{2n+1}} = \frac{\dfrac{-U_0B_{2n-1} + W_1B_{2n}}{W_1}}{\dfrac{-U_0B_{2n} + W_1B_{2n+1}}{W_1}}.$$

So

$$x_{2n}^{(j)} = \frac{-y_0^{(j)}B_{2n-1} + B_{2n}}{-y_0^{(j)}B_{2n} + B_{2n+1}}.$$

By a similar calculation, we find $y_n^{(j)}$. We have

$$y_{2n+1}^{(j)} = \frac{U_{2n+1}}{W_{2n+2}} = \frac{-U_0B_{2n} + W_1B_{2n+1}}{-U_0B_{2n+1} + W_1B_{2n+2}} = \frac{\dfrac{-U_0B_{2n} + W_1B_{2n+1}}{W_1}}{\dfrac{-U_0B_{2n+1} + W_1B_{2n+2}}{W_1}}.$$

So

$$y_{2n+1}^{(j)} = \frac{-y_0^{(j)} B_{2n} + B_{2n+1}}{-y_0^{(j)} B_{2n+1} + B_{2n+2}},$$

and

$$y_{2n}^{(j)} = \frac{U_{2n}}{W_{2n+1}} = \frac{-W_0 B_{2n-1} + U_1 B_{2n}}{-W_0 B_{2n} + U_1 B_{2n+1}} = \frac{\dfrac{-W_0 B_{2n-1} + U_1 B_{2n}}{U_1}}{\dfrac{-W_0 B_{2n} + U_1 B_{2n+1}}{U_1}}.$$

Hence

$$y_{2n}^{(j)} = \frac{-x_0^{(j)} B_{2n-1} + B_{2n}}{-x_0^{(j)} B_{2n} + B_{2n+1}}.$$

According to all the above, we have the following theorem.

Theorem 1. *Let* $\left\{x_n^{(j)}, y_n^{(j)}\right\}_{n\geq 0}$ *be the solution to system* (3.1). *Then for* $n \in \mathbb{N}$ *and* $j = 0, 1 \ldots, k$

$$\begin{cases} x_{2n+1}^{(j)} = \dfrac{-x_0^{(j)} B_{2n} + B_{2n+1}}{-x_0^{(j)} B_{2n+1} + B_{2n+2}}, & x_{2n}^{(j)} = \dfrac{-y_0^{(j)} B_{2n-1} + B_{2n}}{-y_0^{(j)} B_{2n} + B_{2n+1}}, \\ y_{2n+1}^{(j)} = \dfrac{-y_0^{(j)} B_{2n} + B_{2n+1}}{-y_0^{(j)} B_{2n+1} + B_{2n+2}}, & y_{2n}^{(j)} = \dfrac{-x_0^{(j)} B_{2n-1} + B_{2n}}{-x_0^{(j)} B_{2n} + B_{2n+1}}. \end{cases}$$

The following corollary is our main result which give the explicit formula of solution to system (1.1).

Corollary 1. *Let* $\{x_n, y_n\}_{n\geq 0}$ *be the solution to system* (1.1). *Then for* $n \in \mathbb{N}$ *and* $j = 0, 1 \ldots, k$

$$\begin{cases} x_{(k+1)(2n+1)-j} = \dfrac{-x_{-j} B_{2n} + B_{2n+1}}{-x_{-j} B_{2n+1} + B_{2n+2}}, \\ x_{(k+1)(2n)-j} = \dfrac{-y_{-j} B_{2n-1} + B_{2n}}{-y_{-j} B_{2n} + B_{2n+1}}, \\ y_{(k+1)(2n+1)-j} = \dfrac{-y_{-j} B_{2n} + B_{2n+1}}{-y_{-j} B_{2n+1} + B_{2n+2}}, \\ y_{(k+1)(2n)-j} = \dfrac{-x_{-j} B_{2n-1} + B_{2n}}{-x_{-j} B_{2n} + B_{2n+1}}. \end{cases}$$

Proof. We have

$$x_n^{(j)} = x_{(k+1)n-j}, \quad j \in \{0, 1, 2 \ldots, k\}.$$

So

$$x_{2n+1}^{(j)} = x_{(k+1)(2n+1)-j},$$

and

$$x_0^{(j)} = x_{-j}.$$

Then

$$x_{2n+1}^{(j)} = x_{(k+1)(2n+1)-j} = \frac{-x_{-j}B_{2n} + B_{2n+1}}{-x_{-j}B_{2n+1} + B_{2n+2}},$$

and

$$x_{2n}^{(j)} = x_{(k+1)(2n)-j} = \frac{-y_{-j}B_{2n-1} + B_{2n}}{-y_{-j}B_{2n} + B_{2n+1}}.$$

We have

$$y_n^{(j)} = y_{(k+1)n-j}.$$

So

$$y_{2n+1}^{(j)} = y_{(k+1)(2n+1)-j},$$

and

$$y_0^{(j)} = y_{-j}.$$

Hence

$$y_{2n+1}^{(j)} = y_{(k+1)(2n+1)-j} = \frac{-y_{-j}B_{2n} + B_{2n+1}}{-y_{-j}B_{2n+1} + B_{2n+2}},$$

and

$$y_{2n}^{(j)} = y_{(k+1)(2n)-j} = \frac{-x_{-j}B_{2n-1} + B_{2n}}{-x_{-j}B_{2n} + B_{2n+1}}.$$

□

4. Global stability of the solutions to system (1.1)

In this section we study the global stability character of the solutions of system (1.1). It is easy to show that (1.1) has two real equilibrium points given by

$$\overline{M} = (\overline{x}, \overline{y}) = \left(3 + 2\sqrt{2}, 3 + 2\sqrt{2}\right), \quad \overline{M'} = (\overline{x'}, \overline{y'}) = \left(3 - 2\sqrt{2}, 3 - 2\sqrt{2}\right).$$

Theorem 2. *The equilibrium point* $\overline{M} = \left(3 + 2\sqrt{2}, 3 + 2\sqrt{2}\right)$ *is locally asymptotically stable.*

Proof. The linearized system about the equilibrium point $\overline{M} = \left(3 + 2\sqrt{2}, 3 + 2\sqrt{2}\right)$ is given by

$$X_{n+1} = JX_n,$$

where

$$X_n = (x_n, x_{n-1}, \ldots, x_{n-k}, y_n, y_{n-1}, \ldots, y_{n-k})^t$$

and

$$J=\begin{pmatrix} 0 & 0 & \dots & 0 & 17-12\sqrt{2} \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & \dots & 17-12\sqrt{2} & \dots & 0\dots \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}.$$

The characteristic polynomial of J is

$$P(\lambda)=(-\lambda)^{2k+2}-\left(17-12\sqrt{2}\right)^2.$$

Now, consider the two functions defined by

$$\varphi(\lambda)=(-\lambda)^{2k+2}, \quad \phi(\lambda)=\left(17-12\sqrt{2}\right)^2.$$

We have

$$|\phi(\lambda)|<|\varphi(\lambda)|, \quad \forall\lambda\in C:|\lambda|=1$$

So, according to Rouche's Theorem φ and $P=\varphi+\phi$ have the same number of zeros in the unit disc $|\lambda|<1$, and since φ admits as root $\lambda=0$ of multiplicity $2(k+1)$, then all the roots of P are in the disc $|\lambda|<1$. Thus, the equilibrium point is locally asymptotically stable. □

Corollary 2. *The equilibrium point $\overline{M}$ is globally asymptotically stable.*

Proof. Theorem 2 states that, $\overline{M}$ is locally asymptotically stable therefore, it must be demonstrated that this point is globally attractive. To do this we use the Corollary 1.

We have

$$\lim_{n\to\infty} x_{(k+1)(2n)-j}=\lim_{n\to\infty}\frac{-y_{-j}B_{2n-1}+B_{2n}}{-y_{-j}B_{2n}+B_{2n+1}}=\lim_{n\to\infty}\frac{-y_{-j}\dfrac{B_{2n-1}}{B_{2n}}+1}{-y_{-j}+\dfrac{B_{2n+1}}{B_{2n}}}.$$

Using the following two limits

$$\lim_{n\to\infty}\left(\frac{B^{2n+1}}{B_{2n}}\right)=\alpha, \quad \lim_{n\to\infty}\left(\frac{B^{2n-1}}{B_{2n}}\right)=\frac{1}{\alpha}=\beta,$$

we get

$$\begin{aligned}\lim_{n\to\infty} x_{(k+1)(2n)-j}&=\frac{-y_{-j}\beta+1}{-y_{-j}+\alpha}=\frac{-y_{-j}(3-2\sqrt{2})+1}{-y_{-j}+3+2\sqrt{2}}\\ &=\frac{-3y_{-j}+2\sqrt{2}y_{-j}+1}{-y_{-j}+3+2\sqrt{2}}\end{aligned}$$

$$
\begin{aligned}
&= \frac{\left(-3y_{-j}+2\sqrt{2}y_{-j}+1\right)\left(-y_{-j}+3-2\sqrt{2}\right)}{(y_{-j})^2-6y_{-j}+1} \\
&= \frac{3\left((y_{-j})^2-6y_{-j}+1\right)-2\sqrt{2}\left((y_{-j})^2-6y_{-j}+1\right)}{(y_{-j})^2-6y_{-j}+1} \\
&= 3-2\sqrt{2}=\bar{x}.
\end{aligned}
$$

However, we have

$$
\lim_{n\to\infty} x_{(k+1)(2n+1)-j} = \lim_{n\to\infty} \frac{-x_{-j}B_{2n}+B_{2n+1}}{-x_{-j}B_{2n+1}+B_{2n+2}} = \lim_{n\to\infty} \frac{-x_{-j}\dfrac{B_{2n}}{B_{2n+1}}+1}{-x_{-j}+\dfrac{B_{2n+2}}{B_{2n+1}}} = \frac{-x_{-j}\beta+1}{-x_{-j}+\alpha},
$$

Using the following two limits

$$
\lim_{n\to\infty}\left(\frac{B^{2n+1}}{B_{2n}}\right)=\alpha, \quad \lim_{n\to\infty}\left(\frac{B^{2n-1}}{B_{2n}}\right)=\frac{1}{\alpha}=\beta,
$$

we get

$$
\lim_{n\to\infty} x_{(k+1)(2n+1)-j} = 3-2\sqrt{2}=\bar{x}.
$$

So $\lim\limits_{n\to\infty} x_{(k+1)n-j}=\bar{x}$. By a similar argument, it can be shown that $\lim\limits_{n\to\infty} y_{(k+1)n-j}=\bar{y}$. Hence

$$
\lim_{n\to\infty}\left(x_{(k+1)n-j}, y_{(k+1)n-j}\right)=(\bar{x},\bar{y}).
$$

□

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A NEW APPROACH TO STATISTICAL RIEMANN-STIELTJES INTEGRALS

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Abstract. This paper aims to introduce various concepts of statistical Riemann-Stieltjes sum via the deferred Nörlund summability mean for the sequence of functions as well as the sequence of distribution functions. We first establish some fundamental limit theorems for statistical Riemann-Stieltjes sum in the sequence space. Then over the probability sequence space, we establish some more advanced results. Finally, over both the spaces we establish some inclusion theorems via our proposed mean in association with statistical Riemann-Stieltjes integral.

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1. Introduction and Motivation

Let $[a,b] \subset \mathbb{R}$, and for all $k \in \mathbb{N}$ there is a function $g_k : [a,b] \to \mathbb{R}$ and it is called a sequence (g_k) of functions on $[a,b]$.

We now define the Riemann-Stieltjes sum of a sequence (g_k) of functions along with a tagged partition $\dot{\mathcal{P}}$ which is given by

$$\delta(g_k;\dot{\mathcal{P}}) := \sum_{i=1}^{k} g_i(\gamma_i)[\alpha(r_i) - \alpha(r_{i-1})],$$

where α is a monotonic increasing function.

Let $\mathcal{F}$ be a real-valued monotone transformation on $I \subset \mathbb{R}$,

$$\mathcal{F} : I \to \mathcal{F}(I).$$

We next assume that (h_k) is a real-valued sequence of step functions on the interval I,

$$h_k(t) = \sum_{i=1}^{k} c_i 1(t \in I_i),$$

where c_i's are constants and I_i is a partition with $I = \cup_{i=1}^{n} I_i$.

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We now recall the Riemann-Stieltjes integral of $(h_k(u))$ sequence of functions as

$$\int h_k(u)d\mathcal{F}(u) = \sum_{i=1}^{k} c_i|\mathcal{F}(I_i)|.$$

Note 1. Let $|\mathcal{F}(I)| = \mathcal{F}(\max(I_i)) - \mathcal{F}(\min(I_i))$. If $I_i = (a_i, b_i)$, then

$$|\ \mathcal{F}(I_i)\ | = \mathcal{F}(b_i) - \mathcal{F}(a_i),$$

and the integral value of $(h_k(u))$ is

$$\int_I h_k(u)d\mathcal{F}(u) = \sum_{i=1}^{k} c_i(\mathcal{F}(b_i) - \mathcal{F}(a_i)).$$

Accordingly, we make the following definition.

Definition 1. Let $\mathcal{F}$ be an increasing function and (g_k) be a sequence of functions defined over the interval I. The given sequence (g_k) of functions is Riemann-Stieltjes integrable with respect to $\mathcal{F}$, if for every $\varepsilon > 0$, there exists a sequence of step functions h'_k and h''_k such that

$$\int_I h''_k(u)d\mathcal{F}(u) - \int_I h'_k(u)d\mathcal{F}(u) < \varepsilon \quad (h'_k < g_k < h''_k)$$

and

$$\int_I g_k(u)d\mathcal{F}(u) = \sup \int_I h_k(u)d\mathcal{F}(u),$$

where $h_k < g_k$ and h_k is a sequence of step functions.

Note 2. The Riemann integral is a special case of Riemann-Stieltjes integral, when

$$\mathcal{F}(x) = x$$

is the identity transformation. Thus,

$$\int g_k(t)dF(t) = \int g_k(t)dt \qquad (\text{because}\,\mathcal{F}(t) = t).$$

The study of *convergence* on sequence space is one of the most important and fascinating aspects in the domain of real and functional analysis. The gradual improvement in this study leads to the development of *statistical convergence* which is more general than the usual convergence. The credit for independently defining this beautiful concept goes to both Fast [3] and Steinhaus [20]. Nowadays, this potential notion of statistical convergence has been a field of interest of many researchers and becoming an active research area in various fields of pure and applied Mathematics. In particular, it is very much useful in the study of Machine Learning, Soft Computing, Number Theory, Measure theory, Probability theory, and so on. For some recent research works in this direction, see [1], [4], [5], [6], [7], [8], [9], [12], [14], [19] and [21].

Suppose $\mathcal{Y} \subseteq \mathbb{N}$, and let $\mathcal{Y}_k = \{\zeta : \zeta \leq k \quad \text{and} \quad \zeta \in \mathcal{Y}\}$. Then the natural density $d(\mathcal{Y})$ of $\mathcal{Y}$ is defined by

$$d(\mathcal{Y}) = \lim_{k\to\infty} \frac{|\mathcal{Y}_k|}{k} = \rho,$$

where the number ρ is real and finite, and $|\mathcal{Y}_k|$ is the cardinality of $\mathcal{Y}_k$.

A given sequence (y_k) is statistically convergent to a if, for each $\varepsilon > 0$,

$$\mathcal{Y}_\varepsilon = \{\zeta : \zeta \in \mathbb{N} \quad \text{and} \quad |y_\zeta - a| \geq \varepsilon\}$$

has zero natural density (see [3] and [20]). Thus, for each $\varepsilon > 0$,

$$d(\mathcal{Y}_\varepsilon) = \lim_{k\to\infty} \frac{|\mathcal{Y}_\varepsilon|}{k} = 0.$$

We write

$$\text{stat} \lim_{k\to\infty} y_k = a.$$

In the year 2002, Móricz [11] studied and introduced the notion of statistical Cesàro summability. Subsequently, Karakaya and Chishti [10] introduced the idea of weighted statistical convergence. Recently, Srivastava *et al.* [16] introduced and studied the notion of statistical probability convergence via the deferred Nörlund mean and its applications to approximation theorems, and in the same year Srivastava *et al.* [14] also studied a result on deferred weighted statistical probability convergence together with its applications to approximation theorems. Furthermore, Jena and Paikray [5] also proved a result on the product of deferred Cesàro and deferred weighted statistical probability convergence and its applications to Korovkin-type theorems. Subsequently, Srivastava *et al.* [18] established another result on statistical product convergence of martingale sequences and its applications to Korovkin-type approximation theorems. Recently, Jena and Paikray [6] demonstrated statistical convergence of martingale difference sequence via deferred weighted mean and proved a Korovkin-type theorem. Moreover, such concepts have been generalized in many aspects. In this direction, the interested readers may refer, [17], [15] and [13].

We now make the following definition of statistical Riemann-Stieltjes integrablility for sequence of functions.

Definition 2. Let $\mathcal{F}$ be an increasing function, and let (g_k) be a sequence of functions defined over the interval I. The given sequence (g_k) of functions is statistical Riemann-Stieltjes integrable to g with respect to $\mathcal{F}$, if for every $\varepsilon > 0$, there exists a sequence of step functions h'_k and h''_k such that

$$\int_I h''_k(u)d\mathcal{F}(u) - \int_I h'_k(u)d\mathcal{F}(u) < \varepsilon \quad (h'_k < g_k < h''_k)$$

and the set

$$\mathcal{Y}_\varepsilon = \left\{\zeta : \zeta \in \mathbb{N} \quad \text{and} \quad \left|\int_I g_k(u)d\mathcal{F}(u) - g\right| \geq \varepsilon\right\}$$

has zero natural density (see [3] and [20]). Thus, for each $\varepsilon > 0$,

$$d(\mathcal{Y}_\varepsilon) = \lim_{k\to\infty} \frac{|\mathcal{Y}_\varepsilon|}{k} = 0.$$

We write

$$\text{stat}_{\text{RS}} \lim_{k\to\infty} \int_I g_k(u) d\mathcal{F}(u) = g.$$

By making use of Definitions 1 and 2, we setup an example (below), every Riemann-Stieltjes integrability implies statistical Riemann-Stieltjes integrability. However, the converse is not true.

Example 1. Let $g_k : [0,1] \to \mathbb{R}$ be a sequence of functions defined by

$$g_k(x) = \begin{cases} \frac{1}{2} & (x \in \mathbb{Q} \cap [0,1];\ k = j^2,\ j \in \mathbb{N}) \\ \\ \frac{n}{n+1} & (\text{otherwise}). \end{cases} \tag{1.1}$$

It is easy to see that the sequence (g_k) of functions is statistically Riemann-Stieltjes integrable to 1 over $[0,1]$, but not Riemann-Stieltjes integrable (in the ordinary sense) over $[0,1]$.

Next, in view of Definition 2, we establish the following theorem.

Theorem 1. *Let $\mathcal{F}$ be a bounded and increasing function on I, and let the sequence of functions (g_k) be bounded and uniformly convergent on I. Then (g_k) is statistically Riemann-Stieltjes integrable on I.*

Proof. Given $\mathcal{F}$ be a bounded and increasing function on $I = [a,b]$, that is,

$$-\infty < \ell = \mathcal{F}(a) = \inf_I \mathcal{F} \le \sup_I \mathcal{F} = \mathcal{F}(b) = L < +\infty.$$

Suppose that I is finite. Since (g_k) is uniformly convergent to a function g and also bounded on I, so for each $\varepsilon > 0$ there exists a natural number $N(\varepsilon)$ with $k \ge N(\varepsilon)$ such that

$$\|g_k(t) - g(t)\| < \varepsilon \qquad (\forall\, t \in I).$$

Next, let I_i be a partition with $I = \cup_{i=1}^n I_i$ and $|\mathcal{F}(I_i)| \le \delta$ $(\delta > 0)$, and let

$$m_i = \inf_{t\in I_i} g_k(t) \quad \text{and} \quad M_i = \sup_{t\in I_i} g_k(t),$$

where $M_i - m_i < \varepsilon$.

Further, the sequence of step functions are

$$h_k'(t) = \sum_{i=1}^k m_i 1\{t \in I_i\} \quad \text{and} \quad h_k''(t) = \sum_{i=1}^k M_i 1\{t \in I_i\}$$

with $h_k' < g_k < h_k''$.

Thus,

$$\int_I h'_k(u)d\mathcal{F}(u) = \sum_{i=1}^{n} m_i|\mathcal{F}(I_i)| \leq \sum_{i=1}^{n} M_i|\mathcal{F}(I_i)| = \int_I h''_k(u)d\mathcal{F}(u),$$

which implies that

$$\begin{aligned}\int_I h''_k(u)d\mathcal{F}(u) - \int_I h'_k(u)d\mathcal{F}(u) &\leq \sum_{i=1}^{n}(M_i - m_i)|\mathcal{F}(I_i)| \\ &\leq \varepsilon \sum_{i=1}^{n} |\mathcal{F}(I_i)| = \varepsilon|F(I)|,\end{aligned}$$

where $\mathcal{F}(I_i)$'s are disjoint.

Thus, for each $\varepsilon > 0$, (h'_k) and (h''_k) are sequences of step functions, and we get (g_k) is statistically Riemann-Stieltjes integrable on I.

Next, for I is infinite, and $\mathcal{F}$ being increasing and continuous (piecewise), for every $\varepsilon > 0$ there exists a finite interval $\tilde{I}$ with $\tilde{I} \subseteq I$ such that

$$\max(\sup_I \mathcal{F} - \sup_{\tilde{I}} \mathcal{F}; \inf_{\tilde{I}} \mathcal{F} - \inf_I \mathcal{F}) < \varepsilon.$$

Also, since (g_k) is bounded

$$\sup_{I\setminus\tilde{I}} |g_k| \leq \mathcal{G}.$$

This implies

$$\int_{I\setminus\tilde{I}} \mathcal{G}d\mathcal{F}(u) - \int_{I\setminus\tilde{I}} -\mathcal{G}d\mathcal{F}(u) \leq 2\mathcal{G}\varepsilon,$$

assuming finite number of points of I and $\tilde{I}$. Successively, we choose the sequences of step functions $\tilde{h}'_k = (-\mathcal{G}, h'_k, -\mathcal{G})$ and $\tilde{h}''_k = (-\mathcal{G}, h'_k, -\mathcal{G})$ such that $\tilde{h}'_k < (g_k) < \tilde{h}''_k$, that is, (g_k) is bounded over I, and as such

$$\begin{aligned}\int_I \tilde{h}''_k(u)d\mathcal{F}(u) - \int_I \tilde{h}'_k(u)d\mathcal{F}(u) &= \int_{\tilde{I}} h''_k(u)d\mathcal{F}(u) - \int_{\tilde{I}} h'_k(u)d\mathcal{F}(u) \\ &\quad + \int_{I\setminus\tilde{I}} \mathcal{G}d\mathcal{F}(u) - \int_{I\setminus\tilde{I}} -\mathcal{G}d\mathcal{F}(u) \\ &\leq \varepsilon|\mathcal{F}(\tilde{I})| + 2\mathcal{G}\varepsilon.\end{aligned}$$

Hence, (g_k) is statistically Riemann-Stieltjes integrable over I. □

We next discuss the following two special cases of statistically Riemann-Stieltjes integrals.

Corollary 1. *Let $\mathcal{F}$ be an increasing step function on I such that*

$$\mathcal{F}(t) = \sum_{i=1}^{k} c_i 1\{t \leq t_i\},$$

with $\min(I) = t_0 < t_1 < \cdots < t_k = \max(I)$, *and* $c_i > 0$. *Then* (g_k) *is uniformly convergent, and*

$$\text{stat}_{\text{RS}} \lim_{k\to\infty} \int_I g_k(t)d\mathcal{F}(t) = \sum_{i=1}^{k} g_k(\lambda_i)c_i.$$

Proof. In view of statistical Riemann-Stieltjes sums

$$\text{stat}_{\text{RS}} \sum_{i=1}^{k} g_k(\gamma_i)[\mathcal{F}(t_i) - \mathcal{F}(t_{i-1})]$$

with

$$\mathcal{F}(t_i) - \mathcal{F}(t_{i-1}) = \begin{cases} c_k & (t_{i-1} < \lambda_k < t_i; t_i - t_{i-1} < \varepsilon) \\ 0 & (\lambda_k \notin (t_i - t_{i-1}) \forall\, k), \end{cases} \tag{1.2}$$

we have,

$$\text{stat}_{\text{RS}} \sum_{i=1}^{k} g_k(\gamma_i)[\mathcal{F}(t_i) - \mathcal{F}(t_{i-1})] = \sum_{i=1}^{k} g_k(\gamma_i)c_i.$$

Next, (g_k) is uniformly convergent with $\gamma_i \to \lambda_i$, that is $g_k(\gamma_i) \to g_k(\lambda_i)$. Thus,

$$\begin{aligned} \lim_{n\to\infty} \sum_{i=1}^{k} g_k(\gamma_i)[\mathcal{F}(t_i) - \mathcal{F}(t_{i-1})] &= \lim_{n\to\infty} \sum_{i=1}^{k} g_k(\gamma_i)c_i \\ &= \lim_{n\to\infty} \sum_{i=1}^{k} g_k(t_i)c_i. \end{aligned}$$

□

Corollary 2. *Let* $\mathcal{F}$ *be differentiable with* $\mathcal{F} = f$ *(continuous). Then* (g_k) *is integrable, and*

$$\text{stat}_{\text{RS}} \int_I g_k(t)d\mathcal{F}(t) = \text{stat}_{\text{RS}} \int_I g_k(t)f(t)dt.$$

Proof. In view of the statistical Riemann-Stieltjes sums, we have

$$\text{stat}_{\text{RS}} \sum_{i=1}^{k} g_k(\gamma_i)[\mathcal{F}(t_i) - \mathcal{F}(t_{i-1})]. \tag{1.3}$$

Following mean value theorem,

$$\mathcal{F}(t_i) - \mathcal{F}(t_{i-1}) = f(\mu_i)(t_i - t_{i-1})$$

for some $\mu_i \in (t_{i-1} - t_i)$. Consequently from (1.3), we obtain

$$\text{stat}_{\text{RS}} \sum_{i=1}^{k} g_k(\gamma_i)[\mathcal{F}(t_i) - \mathcal{F}(t_{i-1})] = \text{stat}_{\text{RS}} \sum_{i=1}^{k} g_k(\gamma_i)f(\mu_i)(t_i - t_{i-1}),$$

which ends the proof of the Corollary 2. □

Motivated chiefly essentially by the above-mentioned investigations and developments, we introduce various concepts of statistical Riemann-Stieltjes sum via the deferred Nörlund summability mean for the sequence of functions as well as the sequence of distribution functions. We first establish some fundamental limit theorems for statistical Riemann-Stieltjes sum in the sequence space. Then over the probability sequence space, we establish some more advance results. Finally, over both the spaces we establish some inclusion theorems via our proposed mean in association with statistical Riemann-Stieltjes integral.

2. Riemann-Stieltjes Integrability Over a Probability Space

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space, and let $(X_n)_{n\in\mathbb{N}}$ be the sequence of random variables defined on Ω, such that

$$\{\omega \in \Omega : X_n(\omega) \leq x\} \in \mathbb{F}$$

for each $x \in \mathbb{R}$. Also, it is known as *measurability*.

Accordingly, we can define the sequence of distribution functions $(F_n(x))$ of (X_n), that is,

$$F_n(x) = \mathbb{P}\{\omega \in \Omega : X_n(\omega) \leq x\}.$$

Definition 3. Let $(X_n)_{n\in\mathbb{N}}$ be the sequence of random variables with sequence of distribution functions $(F_n(x))$. Then the *expectation* $E(X_n)$ of (X_n) is

$$E(X_n) = \int x dF_n(x).$$

We next present the statistical version of Definition 3.

Definition 4. Let $E(X_n)$ be the expectation of (X_n). Then, for each $\varepsilon > 0$, we define the statistical expectation $E(X_n)$, that is,

$$\mathcal{Y}_\varepsilon = \{\zeta : \zeta \in \mathbb{N} \quad \text{and} \quad |E(X_n) - h| \geq \varepsilon\}$$

has zero natural density (see [3] and [20]). Thus,

$$d(\mathcal{Y}_\varepsilon) = \lim_{k\to\infty} \frac{|\mathcal{Y}_\varepsilon|}{k} = 0.$$

We write

$$\text{stat}_{\text{SE}} \lim_{k\to\infty} E(X_n) = h.$$

We interpret the expectation $E(X_n)$ in the form of Riemann-Stieltjes integral, that is,

$$E(X_n;\xi) = \sum_{i=1}^{n} \xi_i [F_n(x_i) - F_n(x_{i-1})] \tag{2.1}$$

where $\xi \in (x_{i-1}, x_i]$.

We next present below the statistical version of (2.1).

Definition 5. Let $F_n(x)$ be a sequence of distribution function, and let $E(X_n)$ be the expectation of (X_n). Then, for each $\varepsilon > 0$, we define the statistical Riemann-Stieltjes integral of $E(X_n)$, that is,

$$\mathcal{Y}_\varepsilon = \{\zeta : \zeta \in \mathbb{N} \quad \text{and} \quad |E(X_n;\xi) - h| \geq \varepsilon\}$$

has zero natural density (see [3] and [20]). Thus,

$$d(\mathcal{Y}_\varepsilon) = \lim_{k\to\infty} \frac{|\mathcal{Y}_\varepsilon|}{k} = 0.$$

We write

$$\text{stat}_{\text{ERS}} \lim_{k\to\infty} E(X_n;\xi) = h.$$

We now easily capable to adopt the following propositions from the earlier established Corollaries 1 and 2, and Definition 5.

Proposition 1. *If $(\mathcal{F}_i)$ is a sequence of step functions with jump at (x_i), then*

$$\text{stat}_{\text{ERS}} E(X_n;\xi) = \text{stat}_{\text{ERS}} \sum_{i=1}^{n} x_i[\mathcal{F}_n(x_i) - \mathcal{F}_n(x_{i-1})].$$

Proposition 2. *If $(\mathcal{F}_i)$ is differentiable with $\mathcal{F}_i^{'} = f_i$, then*

$$\text{stat}_{\text{ERS}} E(X_n) = \text{stat}_{\text{ERS}} \int x f_n(x) dx.$$

Let f be a function on $\mathbb{R}$ with

$$\int f(x) dF_n(x) < \infty$$

such that

$$\{\omega : f(X_n(\omega)) \leq u\} \in \mathcal{F}, \quad u \in \mathbb{R}^2.$$

We can define the distribution function F of Y,

$$F_n(y) = \mathbb{P}\{\omega : Y_n(\omega) \leq y\}.$$

Then the expectation of (Y_n) is

$$E(Y_n) = \int y dF_n(y)$$

(exists and finite).

We next define below the statistical version of $E(Y_n)$.

Definition 6. Let $E(Y_n)$ be the expectation of (Y_n). Then, for each $\varepsilon > 0$, we define the statistical expectation of $E(Y_n)$, that is,

$$\mathcal{Y}_\varepsilon = \{\zeta : \zeta \in \mathbb{N} \quad \text{and} \quad |E(Y_n) - h| \geq \varepsilon\}$$

has zero natural density (see [3] and [20]). Thus,

$$d(\mathcal{Y}_\varepsilon) = \lim_{k\to\infty} \frac{|\mathcal{Y}_\varepsilon|}{k} = 0.$$

We write

$$\text{stat}_{\text{SE}'} \lim_{k\to\infty} E(Y_n) = h.$$

We now reform the expectation $E(Y_n)$ in the form of Riemann-Stieltjes integral,

$$E(Y_n;\xi) = \sum_{i=1}^{n} \xi_i [F_n(y_i) - F_n(y_{i-1})], \tag{2.2}$$

where $\xi \in (y_{i-1}, y_i]$.

We next present the statistical version of (2.2).

Definition 7. Let $(F_n(y))$ be a sequence of distribution function, and let $E(Y_n)$ be the expectation of (Y_n). Then, for each $\varepsilon > 0$, we define the statistical Riemann-Stieltjes integral of $E(Y_n)$, that is,

$$\mathcal{Y}_\varepsilon = \{\zeta : \zeta \in \mathbb{N} \quad \text{and} \quad |E(Y_n;\xi) - h| \geq \varepsilon\}$$

has zero natural density (see [3] and [20]). Thus,

$$d(\mathcal{Y}_\varepsilon) = \lim_{k\to\infty} \frac{|\mathcal{Y}_\varepsilon|}{k} = 0.$$

We write

$$\text{stat}_{\text{E}'\text{RS}} \lim_{k\to\infty} E(Y_n;\xi) = h.$$

In view of Definition 7, we establish the following theorem.

Theorem 2. *Let X_n is a sequence of random variables, and let f be a function on $\mathbb{R}$. Then the sequence of random variables $(Y_n) = f(X_n)$ has the statistical expectation*

$$\text{stat}_{\text{SE}'} E(Y_n) = \text{stat}_{\text{SE}'} \int f(x) dF_n(x).$$

Proof. Following the Riemann-Stieltjes integral,

$$\begin{aligned}
\text{stat}_{\text{E}'\text{RS}} \sum_{i=1}^{n} \xi_i [F_n(y_i) - F_n(y_{i-1}, y_i)] &= \text{stat}_{\text{E}'\text{RS}} \sum_{i=1}^{n} \xi_i \mathbb{P}(Y_n \in (y_{i-1}, y_i]) \\
&= \text{stat}_{\text{E}'\text{RS}} \sum_{i=1}^{n} \xi_i \mathbb{P}(f(X_n) \in (y_{i-1}, y_i]) \\
&= \text{stat}_{\text{E}'\text{RS}} \sum_{i=1}^{n} \xi_i \mathbb{P}(X_n \in f^{-1}(y_{i-1}, y_i]),
\end{aligned}$$

where $\xi_i \in (y_{i-1}, y_i)]$. Also, recall that

$$\begin{aligned}
\xi_i \in (y_{i-1}, y_i)] &\Longleftrightarrow \eta_i = f^{-1}(\xi_i) \in f^{-1}\{(y_{i-1}, y_i)]\} \\
&\Longleftrightarrow f(\eta_i) \in (y_{i-1}, y_i)].
\end{aligned}$$

Consequently, we have

$$\text{stat}_{\text{SE}'} \sum_{i=1}^{n} f(\eta_i)\mathbb{P}(X_n \in f^{-1}(y_{i-1}, y_i]) \tag{2.3}$$

with $\eta_i \in f^{-1}\{(y_{i-1}, y_i)\}$.

Furthermore, if the intervals $(y_{i-1}, y_i]$ form a partition, then the intervals $(x_{i-1}, x_i] = f^{-1}\{(y_{i-1}, y_i]\}$ also form a partition.

Thus, (2.3) can be written as

$$\text{stat}_{\text{SE}'} \sum_{i=1}^{n} f(\eta_i)\mathbb{P}(X_n \in f(x_{i-1}, x_i]), \tag{2.4}$$

where $\eta_i \in (x_{i-1}, x_i)$, which is the Riemann-Stieltjes sum. □

3. Statistical Riemann-Stieltjes Integrability via Deferred Nörlund Mean

Let (ϕ_k) and $(\varphi_k) \in \mathbb{Z}^{0+}$ with $\phi_k < \varphi_k$ and $\lim_{k\to\infty} \varphi_k = +\infty$, and let (p_i) be a sequence of non-negative real numbers with

$$P_k = \sum_{i=\phi_k+1}^{\varphi_k} p_{\varphi_k - i}.$$

Accordingly, we define the deferred Nörlund summability mean for the Riemann-Stieltjes sum $\delta(g_k; \dot{\mathcal{P}})$ of a sequence of functions with tagged partition $\dot{\mathcal{P}}$ of the form

$$z_k = \frac{1}{P_k} \sum_{\rho=\phi_k+1}^{\varphi_k} p_{\varphi_k - \rho}\, \delta(g_\rho; \dot{\mathcal{P}}). \tag{3.1}$$

We now present the notions of statistical Riemann-Stieltjes integrability and statistical Riemann-Stieltjes summability of a sequence of functions via the deferred Nörlund mean.

Definition 8. Let (ϕ_k) and $(\varphi_k) \in \mathbb{Z}^{0+}$, and let (p_k) be a sequence of non-negative real numbers. Also let $\mathcal{F}$ be an increasing function, and let (g_k) be a sequence of functions defined over the interval I. The given sequence (g_k) of functions is deferred Nörlund statistically Riemann-Stieltjes integrable to g with respect to $\mathcal{F}$, if for every $\varepsilon > 0$, there exists a sequence of step functions h'_k and h''_k such that

$$\int_I h''_k(u)d\mathcal{F}(u) - \int_I h'_k(u)d\mathcal{F}(u) < \varepsilon \quad (h'_k < g_k < h''_k)$$

and the set

$$\{\zeta : \zeta \le P_k \quad \text{and} \quad p_{\varphi_k - \zeta}|\delta(g_\zeta; \dot{\mathcal{P}}) - g| \ge \varepsilon\}$$

has zero natural density. This implies that,

$$\lim_{k\to\infty} \frac{|\{\zeta : \zeta \le P_k \quad \text{and} \quad p_{\varphi_k - \zeta}|\delta(g_\zeta; \dot{\mathcal{P}}) - g| \ge \varepsilon\}|}{P_k} = 0.4$$

We write

$$\mathrm{DNRS}_{\mathrm{stat}} \lim_{k\to\infty} \delta(g_k;\dot{\mathcal{P}}) = g.$$

Definition 9. Let (ϕ_k) and $(\varphi_k) \in \mathbb{Z}^{0+}$, and let (p_k) be a sequence of non-negative real numbers. Also let $\mathcal{F}$ be an increasing function, and let (g_k) be a sequence of functions defined over the interval I. The given sequence (g_k) of functions is statistically deferred Nörlund Riemann-Stieltjes summable to g with respect to $\mathcal{F}$, if for every $\varepsilon > 0$, there exists a sequence of step functions h'_k and h''_k such that

$$\int_I h''_k(u)d\mathcal{F}(u) - \int_I h'_k(u)d\mathcal{F}(u) < \varepsilon \quad (h'_k < g_k < h''_k),$$

and the set

$$\{\zeta : \zeta \leq \mathbb{N} \quad \text{and} \quad |z_\zeta - g| \geq \varepsilon\}$$

has zero natural density. This implies that,

$$\lim_{k\to\infty} \frac{|\{\zeta : \zeta \leq P_k \quad \text{and} \quad |z_\zeta - g| \geq \varepsilon\}|}{P_k} = 0.$$

We write

$$\mathrm{stat}_{\mathrm{DNRS}} \lim_{k\to\infty} z_k = g.$$

We now establish an inclusion theorem between these two new potentially useful notions.

Theorem 3. *Let (ϕ_k) and (φ_k) be sequences of non-negative integers and let (p_k) be a sequence of non-negative real numbers. If a sequence $(g_k)_{k\in\mathbb{N}}$ of functions is deferred Nörlund statistically Riemann-Stieltjes integrable to a function g on $[a,b]$, then it is statistically deferred Nörlund Riemann-Stieltjes summable to the same function g on $[a,b]$, but not conversely.*

Proof. Suppose $(g_k)_{k\in\mathbb{N}}$ is deferred Nörlund statistically Riemann-Stieltjes integrable to a function g on $[a,b]$, then by Definition 8, we have

$$\lim_{k\to\infty} \frac{|\{\zeta : \zeta \leq P_k \quad \text{and} \quad p_{\varphi_k-\zeta}|\delta(g_\zeta;\dot{\mathcal{P}}) - g| \geq \varepsilon\}|}{P_k} = 0.$$

Now assuming two sets as follows:

$$\mathcal{J}_\varepsilon = \{\zeta : \zeta \leq P_k \quad \text{and} \quad p_{\varphi_k-\zeta}|\delta(g_\zeta;\dot{\mathcal{P}}) - g| \geq \varepsilon\}$$

and

$$\mathcal{J}^c_\varepsilon = \{\zeta : \zeta \leq P_k \quad \text{and} \quad p_{\varphi_k-\zeta}|\delta(g_\zeta;\dot{\mathcal{P}}) - g| \geq \varepsilon\},$$

we have

$$|z_k - g| = \left| \frac{1}{P_k} \sum_{\rho=\phi_k+1}^{\varphi_k} p_{\varphi_k-\rho}\delta(g_\rho;\dot{\mathcal{P}}) - g \right|$$

$$
\begin{aligned}
&\leq \left| \frac{1}{P_k} \sum_{\rho=\phi_k+1}^{\varphi_k} p_{\varphi_k-\rho} \left[\delta(g_\rho; \dot{\mathcal{P}}) - g \right] \right| + \left| \frac{1}{P_k} \sum_{\rho=\phi_k+1}^{\varphi_k} p_{\varphi_k-\rho} g - g \right| \\
&\leq \frac{1}{P_k} \sum_{\substack{\rho=\phi_k+1 \\ (\zeta\in\mathcal{J}_\varepsilon)}}^{\varphi_k} p_{\varphi_k-\rho} \left| \delta(g_\rho; \dot{\mathcal{P}}) - g \right| + \frac{1}{P_k} \sum_{\substack{\rho=\phi_k+1 \\ (\zeta\in\mathcal{J}_\varepsilon^c)}}^{\varphi_k} p_{\varphi_k-\rho} \left| \delta(g_\rho; \dot{\mathcal{P}}) - g \right| \\
&\quad + |g| \left| \frac{1}{P_k} \sum_{\rho=\phi_k+1}^{\varphi_k} p_{\varphi_k-\rho} - 1 \right| \\
&\leq \frac{1}{P_k} |\mathcal{J}_\varepsilon| + \frac{1}{P_k} |\mathcal{J}_\varepsilon^c| = 0.
\end{aligned}
$$

This implies that

$$|z_k - g| < \varepsilon.$$

Thus, the sequence of functions (g_k) is statistically deferred Nörlund Riemann-Stieltjes summable to the function g on $[a,b]$.

Next, in view of the non-validity of the converse statement, we present the following example. □

Example 2. Let $\phi_k = 2k$, $\varphi_k = 4k$ and $p_{\varphi_k-\zeta} = 1$ and let $g_k : [0,1] \to \mathbb{R}$ be a sequence of functions of the form

$$
g_k(x) = \begin{cases} 0 & (x \in \mathbb{Q} \cap [0,1];\ k = 2m : m \in \mathbb{N}) \\ \\ 1 & (x \in \mathbb{R} - \mathbb{Q} \cap [0,1];\ k = 2m+1 : m \in \mathbb{N}). \end{cases} \tag{3.2}
$$

The given sequence (g_k) of functions trivially indicates that, it is neither Riemann-Stieltjes integrable nor deferred Nörlund statistically Riemann-Stieltjes integrable. However, as per our proposed mean (3.1), it is easy to see that, the sequence (g_k) of functions has deferred Nörlund Riemann-Stieltjes sum $\frac{1}{2}$ under the tagged partition $\dot{\mathcal{P}}$. Therefore, the sequence (g_k) of functions is statistically deferred Nörlund Riemann-Stieltjes summable to $\frac{1}{2}$ over $[0,1]$ but it is not deferred Nörlund statistically Riemann-Stieltjes integrable.

We now present the notions of statistical Riemann-Stieltjes integrability and statistical Riemann-Stieltjes summability of a sequence of random variables via the deferred Nörlund mean.

Definition 10. Let (ϕ_k) and $(\varphi_k) \in \mathbb{Z}^{0+}$, and let (p_k) be a sequence of non-negative real numbers. Let $F_n(x)$ be a sequence of distribution function, and let $E(X_n)$ be the expectation of (X_n). Then, for each $\varepsilon > 0$, $E(X_n)$ is deferred Nörlund statistically Riemann-Stieltjes integrable to the function g, if

$$\{\zeta : \zeta \leq P_k \quad \text{and} \quad p_{\varphi_k-\zeta} |\delta(E(X_\zeta; \zeta); \dot{\mathcal{P}}) - g| \geq \varepsilon\}$$

has zero natural density. This implies that,

$$\lim_{k\to\infty} \frac{|\{\zeta : \zeta \leq P_k \quad \text{and} \quad p_{\varphi_k-\zeta}|\delta(E(X_\zeta;\zeta);\dot{\mathcal{P}}) - g| \geq \varepsilon\}|}{P_k} = 0.$$

We write

$$\text{DNERS}_{\text{stat}} \lim_{k\to\infty} \delta(E(X_k;k);\dot{\mathcal{P}}) = g.$$

Definition 11. Let (ϕ_k) and $(\varphi_k) \in \mathbb{Z}^{0+}$, and let (p_k) be a sequence of non-negative real numbers. Let $F_n(x)$ be a sequence of distribution function, and let $E(X_n)$ be the expectation of (X_n). Then, for each $\varepsilon > 0$, $E(X_n)$ is statistically deferred Nörlund Riemann-Stieltjes summable to g, if

$$\{\zeta : \zeta \leq \mathbb{N} \quad \text{and} \quad |z_n(E(X_\zeta;\zeta) - g| \geq \varepsilon\}$$

has zero natural density. This implies that,

$$\lim_{k\to\infty} \frac{|\{\zeta : \zeta \leq \mathbb{N} \quad \text{and} \quad |z_\zeta(E(X_\zeta;\zeta) - g| \geq \varepsilon\}|}{P_k} = 0.$$

We write

$$\text{stat}_{\text{DNERS}} \lim_{k\to\infty} z_k(E(X_k;k) = g.$$

We now establish an inclusion theorem between these two new potentially useful notions.

Theorem 4. *Let (ϕ_k) and (φ_k) be sequences of non-negative integers and let (p_k) be a sequence of non-negative real numbers. If $(F_n(x))$ be the sequence of distribution functions, and let $E(X_n)$ be the expectation of (X_n), then $E(X_n)$ is deferred Nörlund statistically Riemann-Stieltjes integrable to a function g on $[a,b]$ implies, it is statistically deferred Nörlund Riemann-Stieltjes summable to the same function g on $[a,b]$, but not conversely.*

Proof. The proof of Theorem 4 follows in the similar lines from the proof of the Theorem 3. Thus, we choose skip the details. □

4. Conclusion

In this study, we have introduced the notion of statistical Riemann-Stieltjes sum on the sequence space via the deferred Nörlund summability mean and established some fundamental limit theorems. Next, considering the probability space, we also established some basic new results based on the Riemann-Stieltjes integral for the sequence of distribution functions. Finally, over both the spaces we established some inclusion theorems via our proposed deferred Nörlund summability means associated with statistical Riemann-Stieltjes integral for the sequence of functions as well as the sequence of distribution functions.

Many researchers have considered different summability means on the sequence spaces to prove several approximation results. A list of some articles has been mentioned in the references. Further combining the existing ideas and direction of the sequence spaces associated with our proposed mean, many new Korvokin-type approximation theorems can be proved under different settings of algebraic and trigonometric functions.

Influenced by a recently-published article by Srivastava [14], we draw the awareness of the interested reader's toward the prospect of establishing some Korvokin-type approximation theorems over the Banach space as well as the probability space. Furthermore, in view of a recent result of Das *et al.* [2], the attention of the curious readers is also drawn for further researches towards fuzzy approximation theorems.

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BOUNDARY VALUE PROBLEMS OF HIGHER ORDER FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS INVOLVING GRONWALL'S INEQUALITY IN BANACH SPACES

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Abstract. Using the Caputo derivative of order $q \in (\alpha-1,\alpha)$, we examine boundary value problems for fractional integro-differential equations in Banach spaces. Using an a priori estimate technique, the Holder's inequality, a suitable singular Gronwall's inequality, and the fixed point theorem are utilised to prove the existence and uniqueness of solutions.

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Keywords: boundary value problems, existence and uniqueness, fractional order integro-differential equations, fixed point theorem, singular Gronwall inequality.

1. Introduction

When representing natural events, differential and integro-differential equations and inclusions are more realistic, and they can be found in a variety of applications (refer[10]-[9], and the cited references). ODEs and integration to arbitrary non integer order are generalised to fractional differential equations(FDEs).

Differential equations with fractional order derivatives have recently proven to be effective tools for modelling a wide range of physical phenomena as well as in a variety of scientific and technical domains. In ordinary and partial FDEs with fractional order, there has been a lot of advancement in recent years; look into the monographs of Abbas *et al.*[2], Baleanu *et al.* [8], the papers by Abbas *et al.* (refer [1]-[3]). Many applications exist in control, electrochemistry, porous media, electromagnetism, viscoelastic, and other fields (see [15]-[21]).

Miller and Ross have a comprehensive bibliography on this subject. [18]. As a result, the theory of FDEs has been extensively developed. The monographs of Kilbas et al. [16], Lakshmikantham et al. [17]. Particularly, Agarwal et al. [4] provide necessary criteria for the existence and uniqueness of solutions for many classes of starting and boundary value problems utilising the Caputo fractional derivative for FDEs and inclusions in $\mathbb{R}$. Some FDEs and optimum controllers in abstract Banach spaces have recently been investigated by Balachandran et al. [6, 7], Dong et al.[12],

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El-Borai [13], Henderson and Ouahab [14], Hernández [15], Wang et al. ([20]-[21]), and Zhou et al. ([23]-[24]). Chalishajar and Karthikeyan [11] have recently proved the existence of impulsive fractional order integro-differential equations in Banach spaces using a combination of generalised Grownwall's inequality, Caputo derivative, and the fixed point approach. The writers have looked at an abstract boundary condition in this paper. In this paper, we use the generalised boundary condition to generalise the conclusion found in [11].

The remainder of this work is laid out as follows: The problem is explicitly defined with motivations in Section 2. We make some notes and recall certain concepts and findings from the preparation. Theorems 2 and 3 are presented in Section 3, with the first based on the Banach contraction principle and the second based on Schaefer's fixed point theorem.

2. Preliminaries

The goal of this paper is to build on previous research [5,22] on FBVPs, for FDEs in $\mathbb{R}$ to U is an abstract Banach space of type

$$\begin{cases} {}^cD^q z(v) &= \phi(v, z(v), (Az)(v)),\ v \in I = [0,T],\ q \in (\alpha-1, \alpha), \\ z(0) &= z_0,\ z'(0) = z_0^1,\ z''(0) = z_0^2, \cdots,\ z^{(\alpha-1)}(0) = z_0^{\alpha-2}, \\ z^{(\alpha-1)}(T) &= z_T \end{cases} \tag{2.1}$$

where ${}^cD^q$ is the Caputo fractional derivative of order q, $\phi : I \times U \times U \to U$ is a given function satisfying F(z) a few assumptions to be detailed later, and $z_0,\ z_0^j$ ($j = 1, 2, \cdots, \alpha-2, \alpha \geq 4, \alpha$ is a integer), z_T are some elements of U and S is a nonlinear integral operator given by

$$(Az)(v) = \int_0^v \psi(v,s) z(s) ds,$$

with $\varphi_0 = \max\{\int_0^v \psi(v,s)ds : (v,s) \in I \times I\}$ where $\psi \in C(I \times I, \mathbb{R}^+)$.

Some existence and uniqueness results for the fractional BVP (2.1) are shown using Holder's inequality, a suitable singular Gronwall's inequality, and the fixed point method. There are at least three discrepancies between the preceding results obtained in [5]: To establish the priori bounds, (i) the work space is not $\mathbb{R}$ but the abstract Banach space X; (ii) f is not necessarily jointly continuous and fulfils some weaker constraints; (iii) another singular Gronwall's inequality is presented.

The study of such problems has gotten a lot of attention in recent years, both theoretically and practically. The following recent works on this topic will be mentioned in [7, 14], without making an attempt to be exhaustive The following conditions are frequently used by authors: for the existence of solutions, the nonlinear term ϕ must satisfy F(z) that there exist functions.$u_1, u_2 \in C([0,1],[0,\infty))$ as a result $1 \geq v \geq 0$ and each $x \in \mathbb{R}$,

$$|\phi(v,x)| \leq u_1(v)|x| + u_2(v),$$

In terms of uniqueness, they believe that the nonlinear term ϕ satisfies the requirement that there exist functions. $u_1, u_2 \in C([0,1],[0,\infty))$ as if for each $1 \geq v \geq 0$ and any $x, z \in \mathbb{R}$,

$$|\phi(v,x) - \phi(v,y)| \leq u_1(v)|x-y|.$$

Now we'll go over the notation, terminology, and preliminary findings that will be used throughout the work. We denote $C(I,U)$ all continuous functions from the Banach space$I$ into U with the norm $\|z\|_\infty := \sup\{\|z(v)\| : v \in I\}$. For measurable functions $\mu : I \to \mathbb{R}$, define the norm $\|\mu\|_{L^p(I,\mathbb{R})} = \left(\int_I |\mu(v)|^p dt\right)^{\frac{1}{p}}$, $1 \leq p < \infty$. We denote $L^p(I,\mathbb{R})$ the Banach space of all Lebesgue measurable functions μ with $\|\mu\|_{L^p(I,\,\mathbb{R})} < \infty$.

To follow the contents of this paper, we'll require the following essential definitions and properties of fractional calculus theory. For more information, see, for instance, [16].

Definition 1. ([19]) The Caputo fractional order derivative of order q of h for a suitable function h given on the interval $[c,d]$ is defined by

$$({}^cD^q_{a+}h)(v) = \frac{1}{\Gamma(\alpha-q)}\int_a^v (v-s)^{\alpha-q-1}h^{(\alpha)}(s)ds,$$

Lemma 1. *Let $q > 0$, then the differential equation ${}^cD^q h(v) = 0$ has the following general solution*

$$h(v) = d_0 + d_1 v + d_2 v^2 + \cdots + d_{\alpha-1}v^{\alpha-1},$$

where $d_j \in \mathbb{R}$, $j = 0,1,2,\cdots,\alpha-1,\alpha$, with $\alpha = -[-q]$.

Lemma 2. *Let $q > 0$, then $I^q({}^cD^q h)(v) = h(v) + d_0 + d_1 v + d_2 v^2 + \cdots + d_{\alpha-1}v^{\alpha-1}$, for some $d_j \in \mathbb{R}$, $j = 0,1,2,\cdots,\alpha-1$, $\alpha = -[-q]$.*

We introduce the concept of a solution of the fractional BVP (2.1), which is similar to Definition 3.7 in [4].

Definition 2. A function $z \in C(I,U)$ with its q-derivative existing on I is said to be a solution of the fractional BVP (2.1) if y satisfies the equation ${}^cD^q z(v) = \phi(v, z(v), (Az)(v))$ a.e. on I, and the conditions $z(0) = z_0$, $z'(0) = z_0^1$, $z''(0) = z_0^2$, $\cdots$, $z^{(\alpha-1)}(0) = z_0^{\alpha-2}$, $z^{(\alpha-1)}(T) = z_T$.

The following auxiliary lemma is required for the existence of solutions for the fractional BVP (2.1).

Lemma 3. *Let $\bar{\phi} : I \to U$ be continuous. A function $z \in C(I,U)$ is a solution of the fractional integral equation*

$$z(v) = \frac{1}{\Gamma(q)}\int_0^v (v-s)^{q-1}\bar{\phi}(s)ds$$

$$-\frac{v^{\alpha-1}}{(\alpha-1)!\Gamma(q-\alpha+1)}\int_0^T (T-s)^{q-\alpha}\bar{\phi}(s)ds$$
$$+z_0+z_0^1 v+\frac{z_0^2}{2!}v^2+\cdots+\frac{z_0^{\alpha-2}}{(\alpha-2)!}v^{\alpha-2}+\frac{z_T}{(\alpha-1)!}v^{\alpha-1},$$

if and only if z is a solution of the following fractional BVP

$$\begin{cases} {}^cD^q z(v) &= \bar{\phi}(v),\ v\in I=[0,T],\ q\in(\alpha-1,\alpha), \\ z(0) &= z_0,\ z'(0)=z_0^1,\ z''(0)=z_0^2,\cdots,\ z^{(\alpha-1)}(0)=z_0^{\alpha-2}, \\ z^{(\alpha-1)}(T) &= z_T. \end{cases} \tag{2.2}$$

Proof. The proof can be completed quickly using Lemma 3.8 in [4] and mathematical induction. □

We get the following result as a result of Lemma 3, which is relevant in the next section.

Lemma 4. *Let $\phi : I\times U\times U\to U$ be continuous function $z\in C(I,U)$ is a solution of the fractional integral equation*

$$z(v)=\frac{1}{\Gamma(q)}\int_0^v (v-s)^{q-1}\phi(s,z(s),(Sz)(s))ds$$
$$-\frac{v^{\alpha-1}}{(\alpha-1)!\Gamma(q-\alpha+1)}\int_0^T (T-s)^{q-\alpha}\phi(s,z(s),(Sz)(s))ds$$
$$+z_0+z_0^1 v+\frac{z_0^2}{2!}v^2+\cdots+\frac{z_0^{\alpha-2}}{(\alpha-2)!}v^{\alpha-2}+\frac{z_T}{(\alpha-1)!}v^{\alpha-1},$$

if and only if z is a solution of the fractional BVP (2.1)

Lemma 5. (Bochner theorem) *A measurable function ϕ: $I\to U$ is Bochner integrable if $\|\phi\|$ is Lebesgue integrable.*

Lemma 6. (Mazur lemma) *If $\mathcal{U}$ is a compact subset of U, then its convex closure $\overline{conv}\ \mathcal{U}$ is compact.*

Lemma 7. (Ascoli-Arzela theorem) *Let $\mathcal{S}=\{s(v)\}$ is a function family of continuous mappings $s:[c,d]\to X$. If $\mathcal{S}$ is uniformly bounded and equicontinuous, and for any $v^*\in[c,d]$, the set $\{s(v^*)\}$ is relatively compact, then there exists a uniformly convergent function sequence $\{s_\alpha(v)\}$*
$(\alpha=1,2,\cdots,v\in[c,d])$ in $\mathcal{S}$.

Theorem 1. (Schaefer's fixed point theorem) *Let $F:U\to U$ completely continuous operator. If the set $E(F)=\{z\in U : z=\rho^* F(z)\ for\ some\ \rho^*\in[0,1]\}$ is bounded, then F has fixed points.*

3. MAIN RESULTS

We introduce the following hypotheses, before stating and proving the main results. Our first result is based on the principle of Banach contraction. The following assumptions are made:

(H1) The function $\phi : I \times U \times U \to U$ is measurable with respect to v on I.

(H2) There exists a constant $q_1 \in (0, q-\alpha+1)$ and real-valued functions $m_1(v)$, $m_2(v) \in L^{\frac{1}{q_1}}(I,U)$ such that

$$\|\phi(v,z(v),(Sz)(v)) - \phi(v,z(v),(Az)(v))\| \leq m_1(v)\|y-z\| + m_2(v)\|Sy-Sz\|,$$

for each $v \in I$, and all $y,z \in U$.

(H3) There exists a constant $q_2 \in (0, q-\alpha+1)$ and real-valued function $h(v) \in L^{\frac{1}{q_2}}(I,U)$ such that $\|\phi(v,z,Sz)\| \leq h(v)$, for each $v \in I$, and all $z \in U$.

For brevity, let $M = \|m_1 + \gamma_0 m_2\|_{L^{\frac{1}{q_1}}(I,U)}$ and $H = \|h\|_{L^{\frac{1}{q_2}}(I,U)}$.

Our second finding is based on the well known Schaefer's fixed point theorem. The following assumptions are made:

(H4) There exist constants $\rho \in [0, 1-\frac{1}{p})$ for some $1 < p < \frac{1}{\alpha-q}$ and $N > 0$ such that

$$\|\phi(v,x,Sx)\| \leq N(1+\varphi_0\|x\|^{\rho}) \ for\ each\ v \in I\ and\ all\ x \in U.$$

(H5) For every $v \in I$, the set

$$K_1 = \left\{(v-s)^{q-1}\phi(s,z(s),(Sz)(s)) : z \in C(I,U),\ s \in [0,v]\right\}$$

and

$$K_2 = \left\{(v-s)^{q-\alpha}\phi(s,z(s),(Sz)(s)) : z \in C(I,X),\ s \in [0,v]\right\}$$

are relatively compact.

Theorem 2. *Assume that* (H1)-(H3) *hold. If*

$$\Phi_{q,T,\alpha} = \frac{M}{\Gamma(q)}\frac{T^{q-q_1}}{(\frac{q-q_1}{1-q_1})^{1-q_1}} + \frac{M}{(\alpha-1)!\Gamma(q-\alpha+1)}\frac{T^{q-q_1}}{(\frac{q-q_1-\alpha+1}{1-q_1})^{1-q_1}} < 1, \tag{3.1}$$

then the fractional BVP (2.1) *has a unique solution on I.*

Proof. For each $v \in I$, we have

$$\begin{aligned}\int_0^v \left\|(v-s)^{q-1}\phi(s,z(s),(Sz)(s))\right\| ds &\leq \left(\int_0^v (v-s)^{\frac{q-1}{1-q_2}} ds\right)^{1-q_2}\left(\int_0^v (h(s))^{\frac{1}{q_2}} ds\right)^{q_2} \\ &\leq \left(\int_0^v (v-s)^{\frac{q-1}{1-q_2}} ds\right)^{1-q_2}\left(\int_0^T (h(s))^{\frac{1}{q_2}} ds\right)^{q_2} \\ &\leq \frac{T^{q-q_2}H}{\left(\frac{q-q_2}{1-q_2}\right)^{1-q_2}}.\end{aligned}$$

Thus $\|(v-s)^{q-1}\phi(s,z(s),(Sz)(s))\|$ is Lebesgue integrable with respect to $s\in[0,v]$ for all $v\in I$ and $z\in C(I,U)$. Then $(v-s)^{q-1}\phi(s,z(s),(Sz)(s))$ is Bochner integrable with respect to $s\in[0,v]$ for all $v\in I$ due to Lemma 5,

$$\int_0^T \left\|(v-s)^{q-1}\phi(s,z(s),(Sz)(s))\right\|ds \le \left(\int_0^T (T-s)^{\frac{q-\alpha}{1-q_2}}ds\right)^{1-q_2}\left(\int_0^T (h(s))^{\frac{1}{q_2}}ds\right)^{q_2}$$
$$\le \frac{T^{q-q_2-\alpha+1}H}{(\frac{q-q_2-\alpha+1}{1-q_2})^{1-q_2}}.$$

Thus, $\|(T-s)^{q-\alpha}\phi(s,z(s),(Sz)(s))\|$ is Lebesgue integrable with respect to $s\in[0,T]$ for all $v\in I$ and $z\in C(I,X)$. Then $(T-s)^{q-\alpha}\phi(s,z(s),(Sz)(s))$ is Bochner integrable with respect to $s\in[0,T]$ for all $v\in I$ due to Lemma 5.

Hence, the FBVP (1) is equivalent to the following fractional integral equation

$$\begin{aligned} z(v) &= \frac{1}{\Gamma(q)}\int_0^v (v-s)^{q-1}\phi(s,z(s),(Sz)(s))ds \\ &\quad - \frac{v^{\alpha-1}}{(\alpha-1)!\Gamma(q-\alpha+1)}\int_0^T (T-s)^{q-\alpha}\phi(s,z(s),(Sz)(s))ds \\ &\quad + z_0 + y_0^1 v + \frac{z_0^2}{2!}v^2 + \cdots + \frac{z_0^{\alpha-2}}{(\alpha-2)!}v^{\alpha-2} + \frac{z_T}{(\alpha-1)!}v^{\alpha-1}, \quad v\in I \end{aligned}$$

Let

$$\begin{aligned} r &\ge \frac{HT^{q-q_2}}{\Gamma(q)(\frac{q-q_2}{1-q_2})^{1-q_2}} + \frac{HT^{q-q_1}}{(\alpha-1)!\Gamma(q-\alpha+1)}(\frac{q-q_2-\alpha+1}{1-q_2})^{1-q_2} \\ &\quad + \|z_0\| + \|z_0^1\|T + \frac{\|z_0^2\|}{2!}T^2 + \cdots + \frac{\|z_0^{\alpha-2}\|}{(\alpha-2)!}T^{\alpha-2} + \frac{\|y_T\|}{(\alpha-1)!}T^{\alpha-1}. \end{aligned}$$

Now we define the operator $\aleph$ on $B_r := \{z\in C(I,U) : \|z\|\le r\}$ as follows

$$\begin{aligned} (\aleph(z))(v) &= \frac{1}{\Gamma(q)}\int_0^v (v-s)^{q-1}\phi(s,z(s),(Sz)(s))ds \\ &\quad - \frac{v^{\alpha-1}}{(\alpha-1)!\Gamma(q-\alpha+1)}\int_0^T (T-s)^{q-\alpha}\phi(s,z(s),(Sz)(s))ds \\ &\quad + z_0 + z_0^1 v + \frac{v_0^2}{2!}t^2 + \cdots + \frac{z_0^{\alpha-2}}{(\alpha-2)!}v^{\alpha-2} + \frac{z_T}{(\alpha-1)!}v^{\alpha-1}, \quad v\in I. \end{aligned} \tag{3.2}$$

As a result, the existence of a fractional BVP (2.1) solution equates to the existence of a fixed point for the operator $\aleph$ on Br. The Banach contraction principle will be used to show that $\aleph$ has a fixed point. There are two parts to the proof.

Part 1. $\aleph(z)\in B_r$ for every $y\in B_r$.

For every $z\in B_r$ and any $\theta>0$, by (H3) and Holder's inequality, we get

$$\|(\aleph(z))(v+\theta)-(\aleph(z))(v)\|$$

$$
\begin{aligned}
&\leq \Big\| \frac{1}{\Gamma(q)} \int_0^{v+\theta} (v+\theta-s)^{q-1} \phi(s,z(s),(Sz)(s))ds \\
&\quad - \frac{1}{\Gamma(q)} \int_0^{v} (v-s)^{q-1} \phi(s,z(s),(Sz)(s))ds \Big\| \\
&\quad + \Big\| \frac{(v+\theta)^{\alpha-1}}{(\alpha-1)!\Gamma(q-\alpha+1)} \int_0^{T} (T-s)^{q-\alpha} \phi(s,z(s),(Sz)(s))ds \\
&\quad + \frac{(v+\theta)^{\alpha-1}}{(\alpha-1)!\Gamma(q-\alpha+1)} \int_0^{T} (T-s)^{q-\alpha} \phi(s,z(s),(Sz)(s))ds \Big\| \\
&\quad + \Big\| z_0^1(v+\theta-t) + \frac{y_0^2}{2!}[(t+\theta)^2 - v^2] + \dots \\
&\quad + \frac{z_0^{\alpha-2}}{(\alpha-2)!}[(v+\theta)^{\alpha-2} - v^{\alpha-2}] \\
&\quad + \frac{z_T}{(\alpha-1)!}[(v+\theta)^{\alpha-1} - v^{\alpha-1}] \Big\| \\
&\leq \frac{H}{\Gamma(q)} \left(\frac{(v+\theta)^{\frac{q-q_2}{1-q_2}}}{\frac{q-q_2}{1-q_2}} - \frac{\theta^{\frac{q-q_2}{1-q_2}}}{\frac{q-q_2}{1-q_2}} - \frac{v^{\frac{q-q_2}{1-q_2}}}{\frac{q-q_2}{1-q_2}} \right)^{1-q_2} \\
&\quad + \frac{H}{\Gamma(q)} \left(\frac{\theta^{\frac{q-q_2}{1-q_2}}}{\frac{q-q_2}{1-q_2}} \right)^{1-q_2} + \frac{[(v+\theta)^{\alpha-1} - v^{\alpha-1}]}{(\alpha-1)!\Gamma(q-\alpha+1)} \frac{T^{q-q_2-\alpha+1}H}{(\frac{q-q_2-\alpha+1}{1-q_2})^{1-q_2}} \\
&\quad + \|z_0^1\|(v+\theta-v) + \frac{\|z_0^2\|}{2!}[(v+\theta)^2 - v^2] + \dots \\
&\quad + \frac{\|z_0^{\alpha-2}\|}{(\alpha-2)!}[(v+\theta)^{\alpha-2} - v^{\alpha-2}] \\
&\quad + \frac{zT\|}{(\alpha-1)!}[(v+\theta)^{\alpha-1} - v^{\alpha-1}].
\end{aligned}
$$

It is self-evident that the right-hand side of the inequality above tends to zero.$\theta \to 0$. Therefore, $\aleph$ is continuous on I, i.e., $\aleph(z) \in C(I,X)$. Moreover, for $z \in B_r$ and all $t \in I$, we get

$$
\begin{aligned}
\|(\aleph(z))(v)\| &\leq \frac{1}{\Gamma(q)} \int_0^{v} (v-s)^{q-1} \|\phi(s,z(s),(Sz)(s))\| ds \\
&\quad + \frac{v^{\alpha-1}}{(\alpha-1)!\Gamma(q-\alpha+1)} \int_0^{T} (T-s)^{q-\alpha} \|\phi(s,z(s),(Sz)(s))\| ds \\
&\quad + \|z_0\| + \|z_0^1\|T + \frac{\|z_0^2\|}{2!}T^2 + \dots + \frac{\|z_0^{\alpha-2}\|}{(\alpha-2)!}T^{\alpha-2} + \frac{\|z_T\|}{(\alpha-1)!}T^{\alpha-1} \\
&\leq r,
\end{aligned}
$$

which implies that $\|\aleph(z)\|_\infty \le r$, Thus, we can conclude that for all $z \in B_r$, $\aleph(z) \in B_r$, i.e., $\aleph : B_r \to B_r$.

Step 2. F is a contraction mapping on B_r.

For $y, z \in B_r$ and any $v \in I$, using (H2) and Holder's inequality, we get

$$\begin{aligned}
&\|(\aleph(z))(v) - (\aleph(z))(v)\| \\
&\quad \le \frac{1}{\Gamma(q)} \int_0^v (v-s)^{q-1} \|\phi(s, x(s), (Sx)(s)) - \phi(s, z(s), (Sz)(s))\| ds \\
&\qquad + \frac{T^{\alpha-1}}{(\alpha-1)!\Gamma(q-\alpha+1)} \int_0^T (T-s)^{q-\alpha} \|\phi(s, y(s), (Sy)(s)) \\
&\qquad\quad - \phi(s, z(s), (Sz)(s))\| ds \\
&\quad \le \frac{\|y-z\|_\infty}{\Gamma(q)} \left(\int_0^v (v-s)^{\frac{q-1}{1-q_1}} ds \right)^{1-q_1} \left(\int_0^v (m_1(s) + \gamma_0 m_2(s))^{\frac{1}{q_1}} ds \right)^{q_1} \\
&\qquad + \frac{[(v+\theta)^{\alpha-1} - v^{\alpha-1}]}{(\alpha-1)!\Gamma(q-\alpha+1)} \left(\int_0^T (T-s)^{\frac{q-\alpha}{1-q_1}} ds \right)^{1-q_1} \\
&\qquad\quad \times \left(\int_0^T (m_1(s) + \gamma_0 m_2(s))^{\frac{1}{q_1}} ds \right)^{q_1}
\end{aligned}$$

So we obtain

$$\|\aleph(y) - \aleph(z)\|_\infty \le \Phi_{q,T,\alpha} \|y-z\|_\infty.$$

Thus, $\aleph$ is contraction due to the condition (3.1). By Banach contraction principle, we can deduce that $\aleph$ has an unique fixed point which is the unique solution of the fractional BVP (2.1). □

Our second results is based on the well known Schaefer's fixed point theorem.

We adopt following assumptions:

Theorem 3. *Let* (H1), (H4), *and* (H5) *assumptions are satisfied. Then the fractional* BVP (1) *has at least one solution on* I.

Proof. Let us transform the FBVP (2.1) into a fixed point problem. Consider the operator $\aleph : C(I,U) \to C(I,U)$ defined as (.). Because of (3.2), Holder's inequality, and Lemma 5, it is clear that F is clearly defined.

We've divided the proof into multiple steps for ease of understanding.

Step 1. $\aleph$ is a continuous operator.

Let $\{z_\alpha\}$ be a sequence such that $z_\alpha \to z$ in $C(I,U)$. Then for each $v \in I$, using the continuity of f, we have

$$\begin{aligned}
&\|\aleph(z)_\alpha - \aleph(z)\|_\infty \\
&\quad \le \left(\frac{T^q}{\Gamma(q+1)} + \frac{T^q}{(\alpha-1)!\Gamma(q-\alpha+2)} \right) \|\phi(\cdot, z_\alpha(\cdot), (Sz_\alpha)(\cdot)) \\
&\qquad - \phi(\cdot, z(\cdot), (Sz)(\cdot))\|_\infty \to 0
\end{aligned}$$

as $\alpha \to \infty$.

Step 2. $\aleph$ maps bounded sets into bounded sets in $C(I,U)$.

Indeed, it is enough to show that for any $\xi^* > 0$, there exists a $l > 0$ such that for each $z \in B_{\xi^*} = \{z \in C(I,U) : \|z\|_\infty \le \xi^*\}$, we have $\|z\|_\infty \le l$.
For each $v \in I$, by (H4), we get

$$\begin{aligned}\|(\aleph(z))(v)\| \le{}& \left(\frac{1}{\Gamma(q+1)} + \frac{1}{(\alpha-1)!\Gamma(q-\alpha+2)}\right) T^q \gamma_0 N(1+(\xi^*)^\rho) \\ &+ \|z_0\| + \|z_0^1\| T + \frac{\|z_0^2\|}{2!} T^2 + \cdots + \frac{\|z_0^{\alpha-2}\|}{(\alpha-2)!} T^{\alpha-2} + \frac{\|z_T\|}{(\alpha-1)!} T^{\alpha-1} := l\end{aligned}$$

which implies that $\|\aleph(z)\|_\infty \le l$.

Step 3. $\aleph$ maps bounded sets into equicontinuous sets of $C(I,U)$.

Let $0 \le v_1 < v_2 \le T$, $z \in B_{\xi^*}$. Using (H4) again, we have

$$\begin{aligned}&\|(\aleph(z))(v_2) - (\aleph(z))(v_1)\| \\ &\quad\le \frac{\gamma_0 N(1+(\xi^*)^\rho)}{\Gamma(q+1)}(v_2^q - v_1^q) + \frac{T^{q-\alpha+1}\gamma_0 N(1+(\xi^*)^\rho)}{(\alpha-1)!\Gamma(q-\alpha+2)}(v_2^2 - v_1^2) \\ &\quad+ \|z_0^1\|(v_2 - v_1) + \frac{\|z_0^2\|}{2!}(v_2^2 - v_1^2) + \cdots + \frac{\|z_0^{\alpha-2}\|}{(\alpha-2)!}(v_2^{\alpha-2} - v_1^{\alpha-2}) \\ &\quad+ \frac{\|z_T\|}{(\alpha-1)!}(v_2^{\alpha-1} - v_1^{\alpha-1}).\end{aligned}$$

As $v_2 \to v_1$, the right-hand side of the above inequality tends to zero, therefore $\aleph$ is equicontinuous.

Now, let $\{z_\alpha\}$, $\alpha = 1, 2, \cdots$ be a sequence on B_{ξ^*}, and

$$(\aleph(z)_\alpha)(v) = (\aleph_1 z_\alpha)(v) + (\aleph_2 z_\alpha)(v) + (\aleph_3 z)(v), \quad v \in I,$$

where

$$(\aleph_1 z_\alpha)(v) = \frac{1}{\Gamma(q)} \int_0^v (v-s)^{q-1} \phi(s, z_\alpha(s), (Sz_\alpha)(s)) ds, \quad v \in I,$$

$$(\aleph_2 z_\alpha)(v) = -\frac{v^{\alpha-1}}{(\alpha-1)!\Gamma(q-\alpha+1)} \int_0^T (T-s)^{q-\alpha} f(s, z_\alpha(s), (Sz_\alpha)(s)) ds, \quad v \in I,$$

$$(\aleph_3 z)(v) = z_0 + z_0^1 v + \frac{z_0^2}{2!} v^2 + \cdots + \frac{z_0^{\alpha-2}}{(\alpha-2)!} v^{\alpha-2} + \frac{z_T}{(\alpha-1)!} v^{\alpha-1}, \quad v \in I.$$

In view of the condition (H5) and Lemma 6, we known that $\overline{conv}\, K_1$ is compact. For any $v^* \in I$,

$$(\aleph_1 z_\alpha)(v^*) = \frac{1}{\Gamma(q)} \lim_{k\to\infty} \sum_{j=1}^{k} \frac{v^*}{k} \left(v^* - \frac{it^*}{k}\right)^{q-1} \phi\left(\frac{it^*}{k}, z_\alpha\left(\frac{it^*}{k}\right), (Sz_\alpha)\left(\frac{it^*}{k}\right)\right)$$

$$= \frac{z^*}{\Gamma(q)}\omega_{\alpha 1},$$

where $$\omega_{\alpha 1} = \lim_{k\to\infty}\sum_{j=1}^{k}\frac{1}{k}\left(v^* - \frac{it^*}{k}\right)^{q-1} f\left(\frac{it^*}{k}, z_\alpha\left(\frac{it^*}{k}\right), (Sz_\alpha)\left(\frac{it^*}{k}\right)\right).$$

Since $\overline{conv}\, K_1$ is convex and compact, we known that $\omega_{\alpha 1} \in \overline{conv}\, K_1$. Hence, for any $v^* \in I$, the set $\{(\aleph_1 z_\alpha)(v^*)\}$ is relatively compact. From Lemma 7, every $\{(\aleph_1 z_\alpha)(v)\}$ contains a uniformly convergent subsequence $\{(\aleph_1 z_{\alpha_k})(v)\},\ k = 1,2,\cdots$ on I. Thus, the set $\{\aleph_1 z : z \in B_{\xi^*}\}$ is relatively compact.

Set

$$(\overline{\aleph}_2 z_\alpha)(v) = -\frac{v^{\alpha-1}}{(\alpha-1)!\Gamma(q-\alpha+1)}\int_0^v (v-s)^{q-\alpha}\phi(s, z_\alpha(s), (Sz_\alpha)(s))ds, \quad v \in I.$$

For any $v^* \in I$,

$$\begin{aligned}(\overline{\aleph}_2 z_\alpha)(v^*) &= -\frac{(v^*)^{\alpha-1}}{(\alpha-1)!\Gamma(q-\alpha+1)} \\ &\quad \times \lim_{k\to\infty}\sum_{j=1}^{k}\frac{v^*}{k}\left(v^* - \frac{it^*}{k}\right)^{q-\alpha}\phi\left(\frac{it^*}{k}, y_\alpha\left(\frac{it^*}{k}\right), (Sy_\alpha)\left(\frac{it^*}{k}\right)\right) \\ &= -\frac{(v^*)^{\alpha}}{(\alpha-1)!\Gamma(q-\alpha+1)}\omega_{\alpha 2},\end{aligned}$$

where

$$\omega_{\alpha 2} = \lim_{k\to\infty}\sum_{j=1}^{k}\frac{1}{k}\left(v^* - \frac{it^*}{k}\right)^{q-\alpha}\phi\left(\frac{it^*}{k}, z_\alpha\left(\frac{it^*}{k}\right), (Sz_\alpha)\left(\frac{it^*}{k}\right)\right).$$

Since $\overline{conv}\, K_2$ is convex and compact, we known that $\omega_{\alpha 2} \in \overline{conv}\, K_2$. Hence, for any $v^* \in I$, the set $\{(\overline{\aleph}_2 z_\alpha)(v^*)\}$ is relatively compact. From Lemma 7, every $\{(\overline{\aleph}_2 z_\alpha)(v)\}$ contains a uniformly convergent subsequence $\{(\overline{\aleph}_2 z_{\alpha_k})(v)\},\ k = 1,2,\cdots$ on I. Particularly, $\{(\aleph_2 z_\alpha)(v)\}$ contains a uniformly convergent subsequence $\{(\aleph_2 y_{\alpha_k})(v)\},\ k = 1,2,\cdots$ on I. Thus, the set $\{\aleph_2 z : z \in B_{\xi^*}\}$ is relatively compact.

Obviously, the set $\{\aleph_3 z : z \in B_{\xi^*}\}$ is relatively compact. As a result, the set $\{\aleph(z) : z \in B_{\xi^*}\}$ is relatively compact.

As a consequence of Step 1-3, we conclude that F is continuous and hence completely continuous.

Step 4. A priori bounds.

Now it remains to show that the set

$$E(\aleph) = \{z \in C(I,U) : z = \rho^*\aleph(z), \text{ for some } \rho^* \in [0,1]\}$$

is bounded.

Let $z \in E(\aleph)$, then $z = \rho^* \aleph(z)$ for some $\rho^* \in [0,1]$. Thus, for each $v \in I$, we have

$$\begin{aligned} z(v) = \rho^* \Bigg(& \frac{1}{\Gamma(q)} \int_0^v (v-s)^{q-1} \phi(s, z_\alpha(s), (Sz_\alpha)(s)) ds \\ & - \frac{v^{\alpha-1}}{(\alpha-1)!\Gamma(q-\alpha+1)} \int_0^T (T-s)^{q-\alpha} \phi(s, z_\alpha(s), (Sz_\alpha)(s)) ds \\ & + z_0 + z_0^1 v + \frac{z_0^2}{2!} v^2 + \cdots + \frac{z_0^{\alpha-2}}{(\alpha-2)!} v^{\alpha-2} + \frac{z_T}{(\alpha-1)!} v^{\alpha-1} \Bigg). \end{aligned}$$

For each $v \in I$, we have

$$\begin{aligned} \|z(v)\| &\leq \|(\aleph(z))(v)\| \\ &\leq \frac{\gamma_0 N T^q}{\Gamma(q+1)} + \frac{\gamma_0 N T^q}{(\alpha-1)!\Gamma(q-\alpha+1)} + \|z_0\| + \|z_0^1\| T \\ &\quad + \frac{\|z_0^2\|}{2!} T^2 + \cdots + \frac{\|z_0^{\alpha-2}\|}{(\alpha-2)!} T^{\alpha-2} + \frac{\|z_T\|}{(\alpha-1)!} T^{\alpha-1} \\ &\quad + \frac{\gamma_0 N}{\Gamma(q)} \int_0^v (v-s)^{q-1} \|z(s)\|^\rho ds \\ &\quad + \frac{\gamma_0 N T^{\alpha-1}}{(\alpha-1)!\Gamma(q-\alpha+1)} \int_0^T (T-s)^{q-\alpha} \|z(s)\|^\rho ds. \end{aligned}$$

By Lemma 2.9 in [9], there exists a $M^* > 0$ such that $\|z(v)\| \leq M^*, \ \ v \in I$.

Thus for every $v \in I$, we have $\|z\|_\infty \leq M^*$.

This establishes the boundedness of the set $E(\aleph)$. We derive that $\aleph$ has a fixed point that is a solution of the fractional BVP as a result of Schaefer's fixed point theorem. □

4. Examples

We provide two examples in this part to demonstrate the utility of our main results.

Example 1. Let us consider the following FBVPs,

$$\begin{cases} {}^cD^q z(v) = \frac{e^{-pv}|z(v)|}{(1+ke^v)(1+|z(v)|)} + \int_0^v \frac{e^{-pv}}{16} s \frac{|z(s)|}{1+|z(s)|} ds, \\ \qquad\qquad y \in [0,1], \ v \in I = [0,T], \ q \in (3,4), \ k > 0, \\ z(0,y) = 0, \ z'(0,y) = 0, \ z''(0,y) = 0 \ z'''(T,y) = 0, \ y \in [0,1], \\ z(v,0) = z(v,1) = 0 \quad v > 0. \end{cases} \tag{4.1}$$

where $z > 0$ is a constant. Set

$$\phi(v,y) = \frac{e^{-pv} y}{(1+e^v)(1+y)}, \ (v,y) \in I_1 \times [0,\infty), \ k(v,s) = \frac{e^{-pv}}{16} s.$$

Let $z_1, z_2 \in [0,\infty)$ and $v \in I_1$. Then we have

$$|\phi(v,z_1,Sz_1) - \phi(v,z_2,Sz_2)| \leq \frac{9e^{-pv}}{16}|z_1 - z_2|.$$

Obviously, for all $z \in [0,\infty)$ and each $v \in I_1$,

$$|f(v,z,Sz)| \leq \frac{9e^{-pt}}{16}.$$

For $v \in I_1$, $\zeta \in (0,q-3)$, let $m_1(v) = m_2(v) = h(v) = \frac{e^{-pv-s}}{32} \in L^{\frac{1}{\zeta}}(I_1,\mathbb{R})$, $M = \left\| \frac{9e^{-pv}}{16} \right\|_{L^{\frac{1}{\zeta}}(I_1,\mathbb{R})}$. Choosing some $z > 0$ large enough and suitable $\zeta \in (0,q-3)$, one can arrive at the following inequality

$$\Omega_{q,T} = \frac{M}{\Gamma(q)} \frac{T^{q-\zeta}}{(\frac{q-\zeta}{1-\zeta})^{1-\zeta}} + \frac{M}{2!\Gamma(q-2)} \frac{T^{q-\zeta}}{(\frac{q-\zeta-2}{1-\zeta})^{1-\zeta}} < 1,$$

Clearly, all of the assumptions in Theorem 2 are met. Our findings can be used to solve the issue (4.1).

Example 2. Let us consider the another FBVPs,

$$\begin{cases} {}^cD^q z(v) &= \frac{v^{p+3}|z(v)|^\rho}{(1+e^v)(1+z(v)|)} + \int_0^v \frac{v^{p+3}}{16} s \frac{|z(s)|^\rho}{1+|z(s)|} ds, \\ & \qquad y \in [0,1],\ q \in I = [0,T],\ q \in (3,4), \\ z &> -q,\ \ \rho \in [0,-\frac{1}{p}),\ \ 1 < p < \frac{1}{4-q}, \\ z(0,y) &= 0,\ \ z'(0,y) = 0,\ \ z''(0,y) = 0\ \ z'''(T,y) = 0,\ \ y \in [0,1], \\ z(v,0) &= z(v,1) = 0 \quad t > 0. \end{cases} \tag{4.2}$$

Set

$$\phi(v,u) = \frac{v^{p+3\,\rho}}{u}\ (1+e^v)(1+z),\ \ v \in I \times [0,\infty),\ \ k(v,s) = \frac{v^{p+3}}{16}s.$$

Obviously, for all $z \in U$ and each $v \in I$.

$$|\phi(v,z)| \leq \frac{T^{p+3}}{16}|z|^\rho.$$

Since $z > -q$ and $p+3 > -(q-3)$. As a result of satisfying all of Theorem 3 assumptions, we may apply our findings to the Problem (4.2)

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AN ENGEL CONDITION WITH b-GENERALIZED DERIVATIONS IN PRIME RINGS

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Abstract. Let $\mathcal{R}$ be a prime ring, I be a nonzero ideal of $\mathcal{R}$, Q be its maximal right ring of quotients and C be its extended centroid. The aim of this paper is to show that if $\mathcal{R}$ admits a nonzero b-generalized derivation $\mathcal{F}$ such that $[\mathcal{F}(x^m)x^n + x^n\mathcal{F}(x^m), x^r]_k = 0$ for all $x \in I$, where m,n,r,k are fixed positive integers, then there exists $\lambda \in C$ such that $\mathcal{F}(x) = \lambda x$ unless $\mathcal{R} \cong \mathrm{M}_2(\mathrm{GF}(2))$, the 2×2 matrix ring over the Galois field $\mathrm{GF}(2)$ of two elements.

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1. Introduction

In all that follows, unless specially stated, $\mathcal{R}$ always denotes an associative ring with center $\mathrm{Z}(\mathcal{R})$. A ring $\mathcal{R}$ is called prime if $a\mathcal{R}b = (0)$ (where $a,b \in \mathcal{R}$) implies $a = 0$ or $b = 0$. We denote by Q maximal right ring of quotients of $\mathcal{R}$ and C is the center of Q which is called the extended centroid of $\mathcal{R}$ see [5, Chapter 2] for more details. As usual the symbol $[x,y]$ will denote the commutator $xy - yx$. Given $x,y \in \mathcal{R}$ set $[x,y]_1 = xy - yx$ and inductively $[x,y]_k = [[x,y]_{k-1}, y]$ for $k > 1$. Note that Engel condition is a polynomial $[x,y]_k = \sum_{i=0}^{k} (-1)^i \binom{k}{i} y^i x y^{k-i}$ in noncommutative indeterminates x,y and $[x+y,z]_k = [x,z]_k + [y,z]_k$.

An additive mapping $d\colon \mathcal{R} \to \mathcal{R}$ is said to be a derivation of $\mathcal{R}$ if $d(xy) = d(x)y + xd(y)$ for all $x,y \in \mathcal{R}$. An additive mapping $G\colon \mathcal{R} \to \mathcal{R}$ is called a generalized derivation of $\mathcal{R}$ if there exists a derivation d of $\mathcal{R}$ such that $G(xy) = G(x)y + xd(y)$ for all $x,y \in \mathcal{R}$. Obviously, any derivation is a generalized derivation, but the converse is not true in general. A significative example is a map of the form $G(x) = ax + xb$ for some $a,b \in \mathcal{R}$; such generalized derivations are called inner. Over the last few decades, several authors have studied on rings with generalized derivations (viz.; [1, 2, 6–8, 10, 15] and references therein).

In a recent paper [12], Koşan and Lee proposed the following new definition. Let $d\colon \mathcal{R} \to Q$ be an additive mapping and $b \in Q$. An additive map $\mathcal{F}\colon \mathcal{R} \to Q$ is called a

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left b-generalized derivation, with associated mapping d, if $\mathcal{F}(xy) = \mathcal{F}(x)y + bxd(y)$, for all $x, y \in \mathcal{R}$. In the same paper, it is proved that, if $\mathcal{R}$ is a prime ring, then d is a derivation of $\mathcal{R}$. For simplicity of notation, this mapping $\mathcal{F}$ will be called a b-generalized derivation with associated pair (b, d). Clearly, any generalized derivation with associated derivation d is a b-generalized derivation with associated pair $(1, d)$. Similarly, the mapping $x \mapsto ax + b[x, c]$, for $a, b, c \in Q$, is a b-generalized derivation with associated pair $(b, ad(c))$, where $ad(c)(x) = [x, c]$ denotes the inner derivation of $\mathcal{R}$ induced by the element c. More generally, the mapping $x \mapsto ax + qxc$, for $a, q, c \in Q$, is a b-generalized derivation with associated pair $(q, ad(c))$. This mapping is called inner b-generalized derivation.

Recently, Alahmadi et al. [1] proved the following result:

Theorem 1 ([1, Theorem 1.1])**.** *Let $\mathcal{R}$ be a noncommutative prime ring with extended centroid C and k, m, n, r be fixed positive integers. If there exists a generalized derivation G of $\mathcal{R}$ such that $[G(x^m)x^n + x^n G(x^m), x^r]_k = 0$ for all $x \in \mathcal{R}$, then there exists $\lambda \in C$ such that $G(x) = \lambda x$ for all $x \in \mathcal{R}$.*

In this paper, we investigate the above result for b-generalized derivation.

Theorem 2. *Let $\mathcal{R}$ be a noncommutative prime ring and I be a nonzero ideal of $\mathcal{R}$. If $\mathcal{R}$ admits a nonzero b-generalized derivation $\mathcal{F}$ associated with the map d such that $[\mathcal{F}(x^m)x^n + x^n \mathcal{F}(x^m), x^r]_k = 0$ for all $x \in I$, where m, n, k, r are fixed positive integers, then there exists $\lambda \in C$ such that $\mathcal{F}(x) = \lambda x$ for all $x \in \mathcal{R}$ unless $\mathcal{R} \cong \mathrm{M}_2(\mathrm{GF}(2))$, the 2×2 matrix ring over the Galois field $\mathrm{GF}(2)$ of two elements.*

The following example shows that any b-generalized derivation $\mathcal{F}$ may satisfy all conditions of Theorem 2 on the ring $\mathcal{R}$ which is isomorphic to the 2×2 matrix ring over the Galois field $\mathrm{GF}(2)$ of two elements, but $\mathcal{F}$ may not be of the form in Theorem 2.

Example 1. *Let $\mathcal{R} = \mathrm{M}_2(\mathrm{GF}(2))$ be the 2×2 matrix ring over the Galois field $\mathrm{GF}(2)$ of two elements. The set of matrix units in $\mathcal{R}$ will be denoted by $\{e_{ij} \mid 1 \leq i, j \leq 2\}$. Define a mapping $\mathcal{F} : \mathcal{R} \to \mathcal{R}$ such that $\mathcal{F}(X) = aX + bXc$ for all $X \in \mathcal{R}$, where $a = e_{11} + e_{21}, b = e_{21}$ and $c = e_{11} + e_{12}$. Clearly, $\mathcal{F}$ is a b-generalized derivation of $\mathcal{R}$. A simple calculation gives that $\mathcal{F}(X)X + X\mathcal{F}(X) = 0$ for $X \in \{0, e_{11}, e_{21}, e_{22}, e_{11} + e_{22}, e_{11} + e_{21}, e_{11} + e_{12}, e_{12} + e_{22}, e_{21} + e_{22}\}$ and $X^6 \in \mathrm{Z}(\mathcal{R})$ for the remaining elements of $\mathcal{R}$. Thus, for any positive integers r and k, it can be easily verified that $[\mathcal{F}(X)X + X\mathcal{F}(X), X^{6r}]_k = 0$ for all $X \in \mathcal{R}$. However, $\mathcal{F}$ is not of the form described in Theorem 2.*

Let ρ be an automorphism of $\mathcal{R}$. It is well known that any automorphism ρ of $\mathcal{R}$ can be uniquely extended to an automorphism of Q. The automorphism ρ of $\mathcal{R}$ is said to be X-inner if there exists a unit $a \in Q$ such that $\rho(x) = axa^{-1}$ for all $x \in \mathcal{R}$. An additive mapping $d : \mathcal{R} \to \mathcal{R}$ is called a ρ-derivation of $\mathcal{R}$ if $d(xy) = d(x)y + \rho(x)d(y)$ for all $x, y \in \mathcal{R}$. An additive mapping $\mathcal{F} : \mathcal{R} \to \mathcal{R}$ is said to be a

generalized ρ-derivation of $\mathcal{R}$ if there exists a ρ-derivation d of $\mathcal{R}$ such that $\mathcal{F}(xy) = \mathcal{F}(x)y + \rho(x)d(y)$ for all $x, y \in \mathcal{R}$. In particular, for X-inner automorphism ρ induced by $a \in Q$, any generalized ρ-derivation $\mathcal{F}$ of $\mathcal{R}$ becomes an a-generalized derivation of $\mathcal{R}$ with associated map $a^{-1}d$. Because of the above observations, we have the following result, which is an application of Theorem 2.

Corollary 1. *Let $\mathcal{R}$ be a noncommutative prime ring and I be a nonzero ideal of $\mathcal{R}$. If $\mathcal{R}$ admits a nonzero ρ-generalized derivation $\mathcal{F}$ associated with an X-inner automorphism ρ of $\mathcal{R}$ such that $[\mathcal{F}(x^m)x^n + x^n\mathcal{F}(x^m), x^r]_k = 0$ for all $x \in I$, where m, n, k, r are fixed positive integers, then there exists $\lambda \in C$ such that $\mathcal{F}(x) = \lambda x$ for all $x \in \mathcal{R}$ unless $\mathcal{R} \cong \mathrm{M}_2(\mathrm{GF}(2))$, the 2×2 matrix ring over the Galois field $\mathrm{GF}(2)$ of two elements.*

2. The lemmas

To prove Theorem 2, we prove the following sequence of lemmas

Lemma 1. *Let V be an infinite dimensional vector space over a field F and let $\mathcal{R}$ be a dense subring of $End({}_FV)$. Suppose that $a, b, c \in End({}_FV)$ such that $[(ax^m + bx^mc)x^n + x^n(ax^m + bx^mc), x^r]_k = 0$ for all $x \in \mathcal{R}$, where m, n, r, k are fixed positive integers. If $b \neq 0$, then $c \in F \cdot I_V$ and $a + bc \in F \cdot I_V$, where I_V denotes the identity transformation of V.*

Proof. We have

$$[(ax^m + bx^mc)x^n + x^n(ax^m + bx^mc), x^r]_k = 0 \text{ for all } x \in \mathcal{R}. \tag{2.1}$$

Claim 1: $c \in F \cdot I_V$. Assume on the contrary that $c \notin F \cdot I_V$. By [4, Lemma 7.1], there is $v \in V$ such that v and cv are linearly independent over F. We divide the proof into two cases.

Case 1: $bV \neq F \cdot v$. Since $b \neq 0$ and $\dim_F V = \infty$, there exists $u \in V$ such that u, v, cv are linearly independent over F and bu, v are linearly independent over F. Write $bu = \alpha cv + \beta v + \gamma u + \delta w$, where $\alpha, \beta, \gamma, \delta \in F$ and $w \notin F \cdot cv + F \cdot v + F \cdot u$. Clearly, α, γ, δ are not all zero. Choose $v_{-1}, v_0, v_1, \ldots, v_{m+n+rk+1} \in V$ such that $v_{-1}, v_0, v_1, \ldots, v_{m+n+rk+1}$ are linearly independent over F and $v_{-1} = v, v_0 = cv, v_m = u, v_{m+1} = w$. Then $v = v_{-1}, cv_{-1} = v_0$ and $bv_m = \alpha v_0 + \beta v_{-1} + \gamma v_m + \delta v_{m+1}$. By the density of $\mathcal{R}$, there exist $x, r \in \mathcal{R}$ such that $xv_{-1} = 0$ and $xv_i = v_{i+1}$ for all $i = 0, 1, 2, \ldots, m + n + rk$. From (2.1), we have

$$\begin{aligned}
0 &= [(ax^m + bx^mc)x^n + x^n(ax^m + bx^mc), x^r]_k v \\
&= \sum_{i=0}^{k} {}^kC_i(-1)^i x^{ri}\{(ax^m + bx^mc)x^n + x^n(ax^m + bx^mc)\}x^{rk-ri}v_{-1} \\
&= (-1)^k x^{kr+n}bx^mcv_{-1} = (-1)^k x^{kr+n}bx^mcv
\end{aligned}$$

$$= (-1)^k x^{kr+n} b x^m v_0 = (-1)^k x^{kr+n} b v_m$$
$$= (-1)^k x^{kr+n} (\alpha v_0 + \beta v_{-1} + \gamma v_m + \delta v_{m+1})$$
$$= (-1)^k (\alpha v_{kr+n} + \gamma v_{m+n+kr} + \delta v_{m+n+kr+1}),$$

which gives a contradiction. Thus, $c \in F \cdot I_V$.

Case 2: $bV = F \cdot v$. Choose $w \in V$ such that $w \notin F \cdot v + F \cdot cv$. Then w, v, cv are linearly independent over F. Suppose first that w and cw are linearly independent over F. Clearly, $bV = F \cdot v \neq F \cdot w$. By the same proof of Case 1, there exists $x \in \mathcal{R}$ such that $[(ax^m + bx^m c)x^n + x^n(ax^m + bx^m c), x^r]_k w \neq 0$, a contradiction. Suppose next that w and cw are linearly dependent over F. Write $cw = \alpha w$, for some $\alpha \in F$. Then $c(v + w) = cv + cw = cv + \alpha w$. Hence, $c(v + w)$ and $v + w$ are linearly independent over F as w, v, cv are linearly independent over F. Clearly, $bV = F \cdot v \neq F \cdot (w + v)$. By the same proof of Case 1, there exists $x \in \mathcal{R}$ such that $[(ax^m + bx^m c)x^n + x^n(ax^m + bx^m c), x^r]_k (w + v) \neq 0$, a contradiction. This proves Claim 1.

Claim 2: $a + bc \in F \cdot I_V$. By Claim 1, (2.1) reduces to

$$[(a+bc)x^{m+n} + x^n(a+bc)x^m, x^r]_k = 0 \text{ for all } x \in \mathcal{R}. \tag{2.2}$$

Suppose on contrary that $a + bc \notin F \cdot I_V$. Then by [4, Lemma 7.1], $v, (a+bc)v$ are linearly independent over F for some $v \in V$. Choose $v_1, \ldots, v_{m+2} \in V$ such that $v_1, \ldots, v_{m+2}$ are linearly independent over F and $v_{m+1} = v$, $v_{m+2} = (a + bc)v$. By the density of $\mathcal{R}$ there exists $x \in \mathcal{R}$ such that $xv_i = v_{i+1}$ for all $i = 1, 2, \ldots, m$, $xv_{m+1} = 0$, $xv_{m+2} = v_{m+2}$. Therefore from (2.2), we have

$$0 = [(a+bc)x^{m+n} + x^n(a+bc)x^m, x^r]_k v_1$$
$$= \sum_{i=0}^{k} {}^kC_i (-1)^i x^{ri} \left\{(a+bc)x^{m+n} + x^n(a+bc)x^m\right\} x^{r(k-i)} v_1$$
$$= (-1)^k x^{rk+n}(a+bc)x^m v_1 = (-1)^k x^{rk+n}(a+bc)v_{m+1}$$
$$= (-1)^k x^{rk+n}(a+bc)v = (-1)^k x^{rk+n} v_{m+2} = (-1)^k v_{m+2}$$
$$= (-1)^k (a+bc)v,$$

which gives a contradiction, and hence $a + bc \in F \cdot I_V$.

□

Lemma 2. *Let $\mathcal{R} = \mathrm{M}_s(F)$ be the $s \times s$ matrix ring over a field F, where $s \geq 3$ is an integer. Suppose $a, b, c \in \mathcal{R}$ such that $[(ax^m + bx^m c)x^n + x^n(ax^m + bx^m c), x^r]_k = 0$ for all $x \in \mathcal{R}$, where m, n, r, k are fixed positive integers. If $b \neq 0$, then $c \in F \cdot I_s$ and $a + bc \in F \cdot I_s$, where I_s denotes the identity matrix of $\mathcal{R}$.*

Proof. By the assumption we have

$$[(ax^m + bx^m c)x^n + x^n(ax^m + bx^m c), x^r]_k = 0 \text{ for all } x \in \mathcal{R}. \tag{2.3}$$

Let ψ be an F-linear automorphism of $\mathcal{R}$. Then

$$[(\psi(a)y^m+\psi(b)y^m\psi(c))y^n+y^n(\psi(a)y^m+\psi(b)y^m\psi(c)),y^r]_k=0 \text{ for all } y\in\mathcal{R}. \tag{2.4}$$

Claim 1: $c\in F\cdot I_s$. Write $c=\Sigma_{i,j=1}^{s}c_{ij}e_{ij}$ and $b=\Sigma_{i,j=1}^{s}b_{ij}e_{ij}$, where $b_{ij},c_{ij}\in F$. It can be easily conclude that $[(ax^m+bx^mc)x^n+x^n(ax^m+bx^mc),x^r]_{2k+1}=0$ for all $x\in\mathcal{R}$. So we may assume that k is an odd integer. Also, for any idempotent e of $\mathcal{R}$, $[x,e]_3=[x,e]$ for all $x\in\mathcal{R}$ Therefore $[x,e]_{2k+1}=[x,e]$ for all $x\in\mathcal{R}$. Putting $x=e_{ii}$, where i is an integer with $1\le i\le s$ in (2.3) and using the fact that $[x,e]_{2k+1}=[x,e]$, we get

$$\begin{aligned}0&=[(ae_{ii}^m+be_{ii}^mc)e_{ii}^n+e_{ii}^n(ae_{ii}^m+be_{ii}^mc),e_{ii}^r]_k\\&=[(ae_{ii}+be_{ii}c)e_{ii}+e_{ii}(ae_{ii}+be_{ii}c),e_{ii}]_k\\&=[ae_{ii}+be_{ii}ce_{ii}+e_{ii}ae_{ii}+e_{ii}be_{ii}c,e_{ii}]\\&=ae_{ii}+be_{ii}ce_{ii}-e_{ii}ae_{ii}-e_{ii}be_{ii}c.\end{aligned}$$

Now, multiplying by e_{jj} from right, we get

$$e_{ii}be_{ii}ce_{jj}=0. \tag{2.5}$$

This implies that

$$b_{ii}c_{ij}=0 \text{ for all } j\neq i \text{ and } 1\le i,j\le s. \tag{2.6}$$

Again, putting $x=e_{ii}+e_{ji}$, where i,j are distinct integers with $1\le i,j\le s$ in (2.3), we obtain

$$\begin{aligned}0&=[(a(e_{ii}+e_{ji})^m+b(e_{ii}+e_{ji})^mc)(e_{ii}+e_{ji})^n\\&\quad+(e_{ii}+e_{ji})^n(a(e_{ii}+e_{ji})^m+b(e_{ii}+e_{ji})^mc),(e_{ii}+e_{ji})^r]_k\\&=[(a(e_{ii}+e_{ji})+b(e_{ii}+e_{ji})c)(e_{ii}+e_{ji})\\&\quad+(e_{ii}+e_{ji})(a(e_{ii}+e_{ji})+b(e_{ii}+e_{ji})c),(e_{ii}+e_{ji})]\\&=a(e_{ii}+e_{ji})+b(e_{ii}+e_{ji})c(e_{ii}+e_{ji})\\&\quad-(e_{ii}+e_{ji})a(e_{ii}+e_{ji})-(e_{ii}+e_{ji})b(e_{ii}+e_{ji})c\\&=a(e_{ii}+e_{ji})+b(e_{ii}+e_{ji})c(e_{ii}+e_{ji})\\&\quad-(e_{ii}+e_{ji})a(e_{ii}+e_{ji})-(e_{ii}+e_{ji})b(e_{ii}+e_{ji})c.\end{aligned}$$

Multiplying by e_{ll} from right, we get

$$(e_{ii}+e_{ji})b(e_{ii}+e_{ji})ce_{ll}=0.$$

Again multiply by e_{ii} from left and using (2.5), we conclude that

$$0=e_{ii}be_{ji}ce_{ll}=b_{ij}c_{il}e_{jl} \text{ for all } j\neq i.$$

This implies that

$$b_{ij}c_{il}=0 \text{ for all } j\neq i. \tag{2.7}$$

Thus, from (2.6) and (2.7), we conclude that

$$\text{if } c_{il} \neq 0 \text{ for some } i \neq l, \text{ then } b_{ij} = 0 \text{ for all } j = 1, 2, \ldots s. \tag{2.8}$$

First we need to show that c is a diagonal matrix. Suppose c is not a diagonal matrix and assume that $c_{21} \neq 0$. Then by [16, Lemma 2.1], there exists an inner automorphism ψ of $\mathcal{R}$ induced by q such that $\psi(c) = qcq^{-1} = \Sigma_{i,j=1}^{s} c_{ij}^{\psi} e_{ij}$, where $c_{ij}^{\psi} \in F$ and

$$c_{21}^{\psi} \neq 0,\ c_{31}^{\psi} \neq 0, \ldots, c_{s1}^{\psi} \neq 0,\ c_{1s}^{\psi} \neq 0. \tag{2.9}$$

Write $\psi(b) = \Sigma_{i,j=1}^{s} b_{ij}^{\psi} e_{ij}$, where $b_{ij}^{\psi} \in F$. Combining (2.4), (2.8) and (2.9), we find that $b_{ij}^{\psi} = 0$ for all i, j with $1 \leq i, j \leq s$. Therefore $\psi(b) = 0$ and hence $b = 0$, which is a contradiction. Thus c is a diagonal matrix that is, $c = \Sigma_{i=1}^{s} c_{ii} e_{ii}$. Let j be an integer with $2 \leq j \leq s$ and let ϕ be an F-linear automorphism of $\mathcal{R}$ defined by $\phi(x) = (I_s + e_{1j})x(I_s - e_{1j})$ for all $x \in \mathcal{R}$. Then $\phi(c) = (c_{jj} - c_{11})e_{1j} + \Sigma_{i=1}^{s} c_{ii} e_{ii}$. Since $b \neq 0$, so $\phi(b) \neq 0$. In view of (2.4) and using the same arguments as we have used above, we find that $\phi(c)$ is a diagonal matrix. Thus $c_{jj} - c_{11} = 0$ for all $2 \leq j \leq s$. This implies that $c \in F \cdot I_s$.

Claim 2: $a + bc \in F \cdot I_s$. From (2.3) and Claim 1, we have $[(a+bc)x^{m+n} + x^n(a+bc)x^m, x^r]_k = 0$ for all $x \in \mathcal{R}$. This implies that

$$\begin{aligned} 0 &= [(x^{m+n})^t(a+bc)^t + (x^m)^t(a+bc)^t(x^n)^t, (x^r)^t]_k \\ &= [(x^t)^{m+n}(a+bc)^t + (x^t)^m(a+bc)^t(x^t)^n, (x^t)^r]_k, \end{aligned}$$

where x^t denotes the usual matrix transpose of x in $\mathcal{R}$. Substituting x^t for x and using the fact that $(x^t)^t = x$, we get $[b'x^m c' x^n + x^n b' x^m c', x^r]_k = 0$ for all $x \in \mathcal{R}$, where $c' = (a+bc)^t$ and $b' = I_s$, the identity matrix of $\mathcal{R}$. Again from (2.3) and by the same arguments as above we have used, we get $(a+bc)^t \in F \cdot I_s$. This implies that $a + bc \in F \cdot I_s$.

□

Lemma 3. *Let $\mathcal{R} = \mathrm{M}_2(F)$ be the 2×2 matrix ring over a field F and $a, b, c \in \mathcal{R}$. Suppose that $b \neq 0$ and $[(ax^m + bx^m c)x^n + x^n(ax^m + bx^m c), x^r]_k = 0$ for all $x \in \mathcal{R}$, where m, n, r, k are fixed positive integers. If c is not a diagonal matrix, then b is not an invertible matrix and $F = \{0, 1\}$.*

Proof. We have

$$[(ax^m + bx^m c)x^n + x^n(ax^m + bx^m c), x^r]_k = 0 \tag{2.10}$$

for all $x \in \mathcal{R}$. Let ϕ be a F-linear automorphism of $\mathcal{R}$. Then from (2.10), we have

$$[(\psi(a)y^m + \psi(b)y^m\psi(c))y^n + y^n(\psi(a)y^m + \psi(b)y^m\psi(c)), y^r]_k = 0$$

for all $y \in \mathcal{R}$. It follows from (2.6) that

$$\text{if } c_{il} \neq 0 \text{ for some } i \neq l, \text{ then } b_{ij} = 0 \text{ for all } i, j \in \{1,2\}. \tag{2.11}$$

Let $c, b \in \mathrm{M}_2(F)$. Then $c = \Sigma_{i,j=1}^{s} c_{ij}e_{ij}$ and $b = \Sigma_{i,j=1}^{s} b_{ij}e_{ij}$, where $b_{ij}, c_{ij} \in F$ and $i, j \in \{1,2\}$. By hypothesis, c is not a diagonal matrix, we may assume $c_{12} \neq 0$. Thus, from (2.11), we have $b_{11} = b_{12} = 0$. If $c_{21} \neq 0$, then $b_{21} = b_{22} = 0$. This implies $b = 0$, a contradiction. Hence, we have $c_{21} = 0$. So

$$c_{12} \neq 0,\ c_{21} = 0,\ b_{11} = b_{12} = 0. \tag{2.12}$$

It is clear from above that b is not an invertible matrix. Let us define, for $\alpha \in F$, ϕ_α be an F-linear automorphism of $\mathcal{R}$ such that $\phi_\alpha(x) = (I_2 + \alpha e_{21})x(I_2 - \alpha e_{21})$ for all $x \in \mathcal{R}$. If $\phi_\alpha(b) = \Sigma_{i,j=1}^{s} b_{ij}^{\alpha} e_{ij}$ and $\phi_\alpha(c) = \Sigma_{i,j=1}^{s} c_{ij}^{\alpha} e_{ij}$, where $b_{ij}^{\alpha}, c_{ij}^{\alpha} \in F$, then it follows from above that $b_{21}^{\alpha} = b_{21} - \alpha b_{22}$ and $c_{21}^{\alpha} = \alpha(c_{11} - c_{22}) - \alpha^2 c_{12}$. If $c_{21}^{\alpha} \neq 0$, then we see from (2.11) that $b_{21}^{\alpha} = b_{21} - \alpha b_{22} = 0$ and $b_{22}^{\alpha} = b_{22} = 0$. Thus, $b_{21} = b_{22} = 0$. Now, from (2.12), we have $b = 0$. This leads to a contradiction. Therefore,

$$c_{21}^{\alpha} = \alpha(c_{11} - c_{22}) - \alpha^2 c_{12} = 0 \tag{2.13}$$

for all $\alpha \in F$. Suppose, if F has more than two elements then from (2.13), we can conclude that $c_{12} = 0$, which gives a contradiction $b = 0$. Therefore, F can not have more than two elements i.e., $F = \{0,1\}$. This completes the proof of Lemma. □

Lemma 4. *Let $\mathcal{R} = \mathrm{M}_2(F)$ be the 2×2 matrix ring over a field F and $a, b, c \in \mathcal{R}$ such that $b \neq 0$ and $[(ax^m + bx^m c)x^n + x^n(ax^m + bx^m c), x^r]_k = 0$ for all $x \in \mathcal{R}$, where m, n, r, k are fixed positive integers. If $b \neq 0$, then $c \in F.I_2$ and $a + cb \in F.I_2$, unless $F \cong \mathrm{GF}(2)$, the Galois field of two elements.*

Proof. Assume that $F \ncong \mathrm{GF}(2)$. In view of Lemma 3, we have c is a diagonal matrix, and hence $c = \Sigma_{i=1}^{2} c_{ii}e_{ii}$, where $c_{ii} \in F$. Let ϕ be a F-linear automorphism of $\mathcal{R}$, defined by $\phi(x) = (I_2 + e_{12})x(I_2 - e_{12})$ for all $x \in \mathcal{R}$. Hence, $\phi(c) = \Sigma_{i=1}^{2} c_{ii}e_{ii} + (c_{22} - c_{11})e_{12}$. Obviously, $\phi(b) \neq 0$ and $[(\phi(a)x^m + \phi(b)x^m\phi(c))x^n + x^n(\phi(a)x^m + \phi(b)x^m\phi(c)), x^r]_k = 0$ for all $x \in \mathcal{R}$. It follows from Lemma 3 that $\phi(c)$ is a diagonal matrix. This gives $c_{22} - c_{11} = 0$, and hence $c = c_{11}I_2 \in F.I_2$. Now, by the assumption, we have $[(a + bc)x^{m+n} + x^n(a + bc)x^m, x^r]_k = 0$ for all $x \in \mathcal{R}$. This implies that

$$\begin{aligned} 0 &= [(x^{m+n})^t(a + bc)^t + (x^m)^t(a + bc)^t(x^n)^t, (x^r)^t]_k \\ &= [(x^t)^{m+n}(a + bc)^t + (x^t)^m(a + bc)^t(x^t)^n, (x^t)^r]_k, \end{aligned}$$

where x^t denotes the usual matrix transpose of x in $\mathcal{R}$. Substituting x^t for x and using the fact that $(x^t)^t = x$, we get $[b'x^m c'x^n + x^n b'x^m c', x^r]_k = 0$ for all $x \in \mathcal{R}$, where $c' = (a + bc)^t$ and $b' = I_2$, the identity matrix of $\mathcal{R}$. Again by using the same arguments as we have used in the above, we get $c' = (a + bc)^t \in F \cdot I_2$. This implies that $a + bc \in F \cdot I_2$. □

3. Proof of Theorem 2

Now, we are in position to prove our theorem.
Suppose first that $b=0$. Then $\mathcal{F}(xy)=\mathcal{F}(x)y$ for all $x,y\in\mathcal{R}$. In view of [3, Lemma 2.3], there is $a\in Q$ such that $\mathcal{F}(x)=ax$ for all $x\in\mathcal{R}$. In this case, by the hypothesis, we have

$$[ax^{m+n}+x^n ax^m,x^r]_k=0$$

for all $x\in I$ and hence for all $x\in\mathcal{R}$. In view of [1, Corollary 1.7], we get $a\in C$, which gives the required result.
Now, we assume that $b\neq 0$. By [12, Theorem 2.3], $d\colon\mathcal{R}\to Q$ is a derivation and there exists $a'\in Q$ such that $\mathcal{F}(x)=a'x+bd(x)$ for all $x\in\mathcal{R}$. It is known that d can be uniquely extended to a derivation of Q [14, Lemma 2]. By the assumption we have

$$[\mathcal{F}(x^m)x^n+x^n\mathcal{F}(x^m),x^r]_k=0 \text{ for all } x\in I. \tag{3.1}$$

We divide the proof into two cases.

Case 1: d is Q-inner. That is, there exists $c'\in Q$ such that $d(x)=[c',x]$ for all $x\in\mathcal{R}$. So $\mathcal{F}(x)=a'x+bd(x)=a'x+b[c',x]=ax+bxc$ for all $x\in\mathcal{R}$, where $a=a'+bc'$ and $c=-c'$. By (3.1), we have

$$[(ax^m+bx^mc)x^n+x^n(ax^m+bx^mc),x^r]_k=0 \text{ for all } x\in I. \tag{3.2}$$

Since I, $\mathcal{R}$ and Q satisfies the same polynomial identities by [5, Theorem 6.4.4]. Therefore, $[(ax^m+bx^mc)x^n+x^n(ax^m+bx^mc),x^r]_k=0$ for all $x\in Q$. If $c\in C$ and $a+cb\in C$, then $g(x)=\lambda x$ for all $x\in\mathcal{R}$, where $\lambda=a+cb$ and $d=0$ as $c=-c'\in C$, proving the theorem. Now we assume that $c\notin C$ or $a+cb\notin C$. Let $\varphi(x)=[(ax^m+bx^mc)x^n+x^n(ax^m+bx^mc),x^r]_k$ Clearly, if $c\in C$ and $a+cb\notin C$, then $\varphi(x)$ is a nonzero GPI of Q by (3.2). Suppose $c\notin C$. Then by (3.2) $\varphi(x)$ is a nonzero GPI of Q as $b\neq 0$. So we conclude that $\varphi(x)$ is a nonzero GPI of Q. Let F be the algebraic closure of C if C is infinite and set $F=C$ for C finite. Clearly, the map $r\in Q\mapsto r\otimes 1\in Q\otimes_C F$ gives a ring embedding. So we may assume Q is a subring of $Q\otimes_C F$. By [13, Proposition], $\varphi(x)$ is also a nonzero GPI of $Q\otimes_C F$. Moreover, in view of [9, Theorem 3.5], $Q\otimes_C F$ is a prime ring with F as its extended centroid. Thus, $\overline{Q}=Q\otimes_C F$ is a prime ring satisfies a nonzero GPI $\varphi(x)$ and its extended centroid F is either an algebraically closed field or a finite field. By Martindale's Theorem [5, Theorem 6.1.6], $\overline{Q}$ is a primitive ring having nonzero socle with F as its associated division ring. Moreover, $\overline{Q}$ is a dense subring of $\mathrm{End}({}_FV)$, where V is a vector space over F. If $\dim_F V=1$, then $\overline{Q}$ is commutative and hence $\mathcal{R}$ is commutative, a contradiction. So $\dim_F V\geq 2$. Firstly, we suppose $\dim_F V=2$. Then $\overline{Q}=\mathrm{End}({}_FV)\cong \mathrm{M}_2(F)$. Application of Lemma 4 yields that $c\in F$ and $a+bc\in F$ or $F\cong \mathrm{GF}(2)$. In first case, we have $\mathcal{F}(x)=\lambda x$, where $\lambda=a+bc\in C$ in the other case, we have $C=F\cong\mathrm{GF}(2)$ and hence $\mathcal{R}=Q\cong\mathrm{M}_2(\mathrm{GF}(2))$. This proves the theorem. Next, if $\dim_F V\geq 3$, then by

Lemma 2 and Lemma 1, we have $c \in F$ and $a+bc \in F$. Therefore, $-c' \in C$ and $a+bc \in C$ and hence $\mathcal{F}(x) = \lambda x$ for all $x \in \mathcal{R}$, where $\lambda = a+bc \in C$, which is the required form of $\mathcal{F}$.

Case 2: d is not Q-inner. From (3.1), we have

$$[a'x^{m+n} + bd(x^m)x^n + x^n a'x^m + x^n bd(x^m), x^r]_k = 0$$

for all $x, y \in I$ and thus for all $x, y \in \mathcal{R}$ [14, Theorem 2]. Now, we set $h(Y,X) = \sum_{i=0}^{m-1} X^i Y X^{m-1-i}$, a polynomial in two non commuting variables X and Y. Note that $d(x^m) = h(d(x),x)$. Then the above expression becomes as $[a'x^{m+n} + bh(d(x),x)x^n + x^n a'x^m + x^n bh(d(x),x), x^r]_k = 0$ for all $x \in \mathcal{R}$.

Applying Kharchenko's theorem [11], we obtain

$$[a'x^{m+n} + bh(y,x)x^n + x^n a'x^m + x^n bh(y,x), x^r]_k = 0$$

for all $x, y \in \mathcal{R}$ and hence for all $x, y \in Q$ by [5, Theorem 6.4.4]. In particular choose $u \notin C$ and take $y = [u,x]$ and using the fact that $h([u,x],x) = [u,x^m]$ in the above expression we get In particular for $y = 0$, we have

$$[a'x^{m+n} + b[u,x^m]x^n + x^n a'x^m + x^n b[u,x^m], x^r]_k = 0 \text{ for all } x \in Q.$$

This can be rewritten as

$$[(ax^m + bx^m c)x^n + x^n(ax^m + bx^m c), x^r]_k = 0 \text{ for all } x \in \mathcal{R}, \tag{3.3}$$

where $a = a' + bu$ and $c = -u$. Since $c = -u \notin C$, so in view of (3.2) and (3.3), we obtain the required result from Case 1. Thereby the proof is completed. □

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EXISTENCE OF TWO WEAK SOLUTIONS FOR SOME ELLIPTIC PROBLEMS INVOLVING $p(x)$-BIHARMONIC OPERATOR

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Abstract. In this paper, we establish the existence of at least two distinct weak solutions for fourth-order PDEs with variable exponents, subject to Navier boundary conditions in a smooth bounded domain in $\mathbb{R}^N$, under a suitable subcritical growth condition with the classical Ambrosetti-Rabinowitz condition. The approach is based on variational methods and critical point theory.

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1. Introduction

Differential equations and variational problems with variable exponents growth conditions have been studied more in the last few years. These problems are connected to modeling of nonlinear electrorheological fluids and elastic mechanics. In addition, the study of these problems has become an important subject by progress in physics and other topics. In this sense, we refer the reader to [1, 4, 7, 12, 16, 17, 20]. Fourth-order differential equations become visible in many applications such as, micro-electro-mechanical systems, surface diffusion on solids, flow in Hele-Shaw cells [11]. The existence of solutions of $p(x)$-biharmonic problems has been studied by several authors (see [2, 3, 8, 13, 14]).

For instance, El Amrouss *et al.* [8] studied a class of $p(x)$-biharmonic of the form

$$\begin{cases} \Delta^2_{p(x)} u = \lambda |u|^{p(x)-2} u + f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geq 1$, $\lambda \leq 0$, $\Delta^2_{p(x)} u := \Delta(|\Delta u|^{p(x)-2}\Delta u)$ is the $p(x)$-biharmonic operator, p is a continuous function on $\overline{\Omega}$ with $\inf_{x\in\overline{\Omega}} p(x) > 1$ and $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Using

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the Mountain Pass Theorem, they obtained the existence of at least one solution and the existence of infinitely many solutions of this problem.

Recently, motivated by this interest, in [2], the authors established the existence and multiplicity of solutions to the following problem

$$\begin{cases} \Delta^2_{p(x)}u + |u|^{p(x)-2}u = \lambda|u|^{q(x)-2}u + \mu|u|^{\gamma(x)-2}u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geq 1$, p, q and γ are continuous functions on $\overline{\Omega}$ with $\inf_{x\in\overline{\Omega}} p(x) > 1$, $\inf_{x\in\overline{\Omega}} q(x) > 1$, $\inf_{x\in\overline{\Omega}} \gamma(x) > 1$ and λ and μ are parameters such that $\lambda^2 + \mu^2 \neq 0$.

In this paper, we want to consider the following fourth-order elliptic equation with Navier boundary conditions

$$\begin{cases} \Delta^2_{p(x)}u + |u|^{p(x)-2}u = \lambda f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geq 1$, $p(\cdot) \in C(\overline{\Omega})$ such that $1 < p^- := \min_{x\in\overline{\Omega}} p(x) \leq p(x) \leq p^+ := \max_{x\in\overline{\Omega}} p(x) < +\infty$, $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying

(f_1) $|f(x,t)| \leq a_1 + a_2|t|^{q(x)-1}, \quad \forall (x,t) \in \Omega \times \mathbb{R}$,

for some non-negative constants a_1, a_2, and $q(x)$ is a continuous function on $\overline{\Omega}$ with $1 < q(x) < p_2^*(x)$ for each $x \in \overline{\Omega}$, where

$$p_2^*(x) := \begin{cases} \frac{Np(x)}{N-2p(x)}, & 2p(x) < N, \\ +\infty, & 2p(x) \geq N. \end{cases}$$

In this work, our goal is to obtain the existence of two distinct weak solutions for problem (1.1).

Recall that a function $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is said to be a Carathéodory function, if

(C_1) the function $x \to f(x,t)$ is measurable for every $t \in \mathbb{R}$;
(C_2) the function $t \to f(x,t)$ is continuous for almost every $x \in \Omega$.

2. PRELIMINARIES AND BASIC DEFINITIONS

To study problem (1.1), we need some some theories on spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$. Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Set

$$C_+(\overline{\Omega}) := \left\{h : h \in C(\overline{\Omega}), h(x) > 1, \ \ \forall x \in \overline{\Omega}\right\}.$$

For any $h \in C_+(\overline{\Omega})$, we define

$$h^+ := \max\left\{h(x) : x \in \overline{\Omega}\right\}, \qquad h^- := \min\left\{h(x) : x \in \overline{\Omega}\right\}.$$

For any $p \in C_+(\overline{\Omega})$, we define the *variable exponent Lebesgue space*

$$L^{p(x)}(\Omega) := \left\{ u\colon \Omega \to \mathbb{R} : u \text{ is measurable and} \int_\Omega |u(x)|^{p(x)} dx \ < +\infty \right\}.$$

We can introduce the so-called *Luxemburg norm* on $L^{p(x)}(\Omega)$ by

$$\|u\|_{L^{p(x)}(\Omega)} := \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\},$$

and $\left(L^{p(x)}(\Omega), \|\cdot\|_{L^{p(x)}(\Omega)} \right)$ becomes a Banach space.

Proposition 1 (Theorems 1.6 and 1.10 of [10])**.** *The space* $\left(L^{p(x)}(\Omega), \|\cdot\|_{L^{p(x)}(\Omega)} \right)$ *is separable, uniformly convex, reflexive Banach space and its conjugate space is* $L^{q(x)}(\Omega)$*, where*

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1 \quad \forall x \in \Omega.$$

For $u \in L^{p(x)}(\Omega)$ *and* $v \in L^{q(x)}(\Omega)$*, we have*

$$\left| \int_\Omega uv \, dx \right| \le \left(\frac{1}{p^-} + \frac{1}{q^-} \right) \|u\|_{L^{p(x)}(\Omega)} \|v\|_{L^{q(x)}(\Omega)} \le 2\|u\|_{L^{p(x)}(\Omega)} \|v\|_{L^{q(x)}(\Omega)}.$$

The Sobolev space with variable exponent $W^{k,p(x)}(\Omega)$ is defined as

$$W^{k,p(x)}(\Omega) := \left\{ u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \le k \right\},$$

where

$$D^\alpha u := \frac{\partial^{|\alpha|}}{\partial {x_1}^{\alpha_1} \partial {x_2}^{\alpha_2} \cdots \partial {x_N}^{\alpha_N}} u,$$

with $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a multi-index and $|\alpha| := \sum\limits_{i=1}^{N} \alpha_i$.

The space $W^{k,p(x)}(\Omega)$ endowed with the norm

$$\|u\|_{k,p(x)} = \sum_{|\alpha| \le k} \|D^\alpha u\|_{L^{p(x)}(\Omega)}$$

also becomes a separable and reflexive Banach space (Theorem 2.1 of [10]). For more details, we refer the reader to [9, 10, 15, 22].

Denote

$$p_k^*(x) := \begin{cases} \frac{Np(x)}{N - kp(x)}, & kp(x) < N, \\ +\infty, & kp(x) \ge N \end{cases}$$

for any $x \in \overline{\Omega}$, $k \ge 1$.

Proposition 2 (Theorem 2.3 of [10]). *For $p, r \in C_+(\overline{\Omega})$ such that $r(x) \leq p_k^*(x)$ for all $x \in \overline{\Omega}$, there is a continuous embedding*

$$W^{k,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega).$$

If we replace $\leq$ with $<$, the embedding is compact.

By $W_0^{k,p(x)}(\Omega)$, we denote the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$. Further, denote by X the space $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ endowed with the norm

$$\|u\| := \inf\left\{\mu > 0 : \int_\Omega \left(\left|\frac{\Delta u(x)}{\mu}\right|^{p(x)} + \left|\frac{u(x)}{\mu}\right|^{p(x)}\right) dx \leq 1\right\}. \tag{2.1}$$

Remark 1.

(1) According to [23], the norm $\|.\|_{2,p(x)}$ is equivalent to the norm $\|\Delta\cdot\|_{L^{p(x)}(\Omega)}$ in the space X. Consequently, the norms $\|\cdot\|_{2,p(x)}$, $\|\cdot\|$ and $\|\Delta\cdot\|_{L^{p(x)}(\Omega)}$ are equivalent.

(2) By the above remark and Proposition 2, there is a continuous and compact embedding of X into $L^{q(x)}(\Omega)$, where $q \in C(\overline{\Omega})$ and $1 \leq q(x) < p_2^*(x)$ for all $x \in \overline{\Omega}$.

In the sequel, we will denote by c_q the best constant for which one has

$$\|u\|_{L^{q(x)}(\Omega)} \leq c_q \|u\| \tag{2.2}$$

for all $u \in X$.

Proposition 3 (Proposition 3.2 of [8]). *If we denote*

$$\rho(u) := \int_\Omega \left(|\Delta u(x)|^{p(x)} + |u(x)|^{p(x)}\right) dx,$$

then, for $u, u_n \in X$, we have

(1) $\|u\| < 1$ *(respectively $= 1$; > 1)* $\Leftrightarrow \rho(u) < 1$ *(respectively $= 1$; > 1);*
(2) $\|u\| \leq 1 \Rightarrow \|u\|^{p^+} \leq \rho(u) \leq \|u\|^{p^-}$;
(3) $\|u\| \geq 1 \Rightarrow \|u\|^{p^-} \leq \rho(u) \leq \|u\|^{p^+}$;
(4) $\|u_n\| \to 0$ *(respectively $\to +\infty$)* $\Leftrightarrow \rho(u_n) \to 0$ *(respectively $\to +\infty$).*

Let us define $F(x,\xi) := \int_0^\xi f(x,t)dt$ for every (x,ξ) in $\Omega \times \mathbb{R}$. Moreover, we introduce the functional $I_\lambda : X \to \mathbb{R}$ associated with (1.1),

$$I_\lambda(u) := \Phi(u) - \lambda\Psi(u),$$

for every $u \in X$, where

$$\Phi(u) := \int_\Omega \frac{1}{p(x)} \left(|\Delta u(x)|^{p(x)} + |u(x)|^{p(x)}\right) dx, \qquad \Psi(u) := \int_\Omega F(x,u(x))dx. \tag{2.3}$$

Fixing the real parameter λ, a function $u\colon \Omega \to \mathbb{R}$ is said to be a *weak solution* of (1.1) if $u \in X$ and

$$\int_\Omega |\Delta u(x)|^{p(x)-2}\Delta u(x)\Delta v(x)dx + \int_\Omega |u(x)|^{p(x)-2}u(x)v(x)dx \\ -\lambda \int_\Omega f(x,u(x))v(x)dx = 0,$$

for every $v \in X$. Hence, the critical points of I_λ are exactly the weak solutions of (1.1).

Definition 1. Let Φ and Ψ be two continuously Gâteaux differentiable functionals defined on a real Banach space X and fix $r \in \mathbb{R}$. The functional $I := \Phi - \Psi$ is said to verify the Palise-Smale condition (in short (PS)-condition) if any sequence $\{u_n\}$ in X such that

(a) $\{I(u_n)\}$ is bounded,
(b) $\lim_{n\to\infty} \|I'(u_n)\|_{X^*} = 0$,

has a convergent subsequence.

Our main tool is the following critical points theorem.

Theorem 1 (Theorem 3.2 of [5])**.** *Let X be a real Banach space and let $\Phi, \Psi\colon X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$. Fix $r > 0$ such that $\sup_{\{\Phi(u)<r\}} \Psi(u) < +\infty$ and assume that, for each $\lambda \in \left]0, \frac{r}{\sup_{\{\Phi(u)<r\}}\Psi(u)}\right[$, the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies* (PS)*-condition and it is unbounded from below. Then, for each $\lambda \in \left]0, \frac{r}{\sup_{\{\Phi(u)<r\}}\Psi(u)}\right[$, the functional I_λ admits two distinct critical points.*

3. Main results

In this section we establish the main abstract result of this paper. We recall that c_q is the constant of the embedding $X \hookrightarrow L^{q(x)}(\Omega)$ for each $q \in C(\overline{\Omega})$ and $1 \le q(x) < p_2^*(x)$ for all $x \in \overline{\Omega}$, and c_1 stands for c_q with $q = 1$ (see (2.2)).

Before introducing our result, we observe that putting

$$[\alpha]^h := \max\left\{\alpha^{h^-}, \alpha^{h^+}\right\}, \qquad [\alpha]_h := \min\left\{\alpha^{h^-}, \alpha^{h^+}\right\},$$

It is easy to verify that

$$[\alpha]^{\frac{1}{h}} := \max\left\{\alpha^{\frac{1}{h^-}}, \alpha^{\frac{1}{h^+}}\right\}, \qquad [\alpha]_{\frac{1}{h}} := \min\left\{\alpha^{\frac{1}{h^-}}, \alpha^{\frac{1}{h^+}}\right\}.$$

Theorem 2. *Let $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that condition* (f_1) *holds. Moreover, assume that*

(f_2) *there exist $\theta > p^+$ and $M > 0$ such that*

$$0 < \theta F(x,t) \le t f(x,t)$$

for each $x \in \Omega$ and $|t| \geq M$. Then, for each $\lambda \in]0, \lambda^*[$, problem (1.1) admits at least two distinct weak solutions, where

$$\lambda^* := \frac{q^-}{q^- a_1 c_1 (p^+)^{\frac{1}{p^-}} + a_2 [c_q]^q (p^+)^{\frac{q^+}{p^-}}}.$$

Proof. Our aim is to apply Theorem 1 to problem (1.1) in the case $r = 1$ to the space X with the norm $\|\cdot\|$ defined in (2.1) and to the functionals $\Phi, \Psi \colon X \to \mathbb{R}$ defined in (2.3) for all $u \in X$. Clearly, $\Phi(0) = \Psi(0) = 0$. The functional Φ is in $C^1(X, \mathbb{R})$ and $\Phi' \colon X \to X^*$ is a homeomorphism (see Theorem 3.4 of [8]). Moreover, thanks to condition (f_1) and to the compact embedding $X \hookrightarrow L^{q(x)}(\Omega)$, Ψ is in $C^1(X, \mathbb{R})$ and has compact derivative and

$$\Psi'(u)(v) = \int_\Omega f(x, u(x)) v(x) dx,$$

for every $v \in X$. Now we prove that $I_\lambda = \Phi - \lambda \Psi$ satisfies (PS)-condition for every $\lambda > 0$. Namely, we will prove that any sequence $\{u_n\} \subset X$ satisfying

$$d := \sup_n I_\lambda(u_n) < +\infty, \qquad \left\| I'_\lambda(u_n) \right\|_{X^*} \to 0, \tag{3.1}$$

contains a convergent subsequence. Thus it is sufficient to verify that $\{\|u_n\|\}$ is bounded. Assume $\|u_n\| > 1$ for convenience. For n large enough, we have by (3.1)

$$d \geq I_\lambda(u_n) = \int_\Omega \frac{1}{p(x)} \left(|\Delta u_n(x)|^{p(x)} + |u_n(x)|^{p(x)} \right) dx - \lambda \int_\Omega F(x, u_n(x)) dx,$$

then, by (f_2) and Proposition 3,

$$\begin{aligned}
I_\lambda(u_n) &\geq \frac{1}{p^+} \int_\Omega \left(|\Delta u_n(x)|^{p(x)} + |u_n(x)|^{p(x)} \right) dx - \frac{\lambda}{\theta} \int_\Omega f(x, u_n(x)) u_n(x) dx \\
&= \left(\frac{1}{p^+} - \frac{1}{\theta} \right) \int_\Omega \left(|\Delta u_n(x)|^{p(x)} + |u_n(x)|^{p(x)} \right) dx \\
&\quad + \frac{1}{\theta} \left[\int_\Omega \left(|\Delta u_n(x)|^{p(x)} + |u_n(x)|^{p(x)} \right) dx - \lambda \int_\Omega f(x, u_n(x)) u_n(x) dx \right] \\
&\geq \left(\frac{1}{p^+} - \frac{1}{\theta} \right) \|u_n\|^{p^-} + \frac{1}{\theta} \langle I'_\lambda(u_n), u_n \rangle.
\end{aligned}$$

Due to (3.1), we can actually assume that $\left| \frac{1}{\theta} \langle I'_\lambda(u_n), u_n \rangle \right| \leq \|u_n\|$. Thus

$$d + \|u_n\| \geq I_\lambda(u_n) - \frac{1}{\theta} \langle I'_\lambda(u_n), u_n \rangle \geq \left(\frac{1}{p^+} - \frac{1}{\theta} \right) \|u_n\|^{p^-}.$$

Since $\theta > p^+$ and $p^- > 1$, it follows from this quadratic inequality that $\{\|u_n\|\}$ is bounded. By the Eberlian-Smulyan theorem, passing to a subsequence if necessary,

we can assume that $u_n \rightharpoonup u$. Then $\Psi'(u_n) \to \Psi'(u)$ because of compactness. Since $I'_\lambda(u_n) = \Phi'(u_n) - \lambda\Psi'(u_n) \to 0$, then we gain the following convergence

$$\Phi'(u_n) \to \lambda\Psi'(u).$$

Since Φ' is a homeomorphism, then $u_n \to u$ and so I_λ satisfies (PS)-condition.

At this step we prove that there is a positive constant C such that

$$F(x,t) \geq C|t|^\theta \tag{3.2}$$

for all $x \in \Omega$ and $|t| > M$. For this, setting $a(x) := \min_{|\xi|=M} F(x,\xi)$ and

$$\varphi_t(s) := F(x,st) \quad \forall s > 0, \tag{3.3}$$

by (f_2), for every $x \in \Omega$ and $|t| > M$ one has

$$0 < \theta\varphi_t(s) = \theta F(x,st) \leq st \cdot f(x,st) = s\varphi'_t(s) \quad \forall s > \frac{M}{|t|}.$$

Therefore,

$$\int_{M/|t|}^1 \frac{\varphi'_t(s)}{\varphi_t(s)} ds \geq \int_{M/|t|}^1 \frac{\theta}{s} ds.$$

Then

$$\varphi_t(1) \geq \varphi_t\left(\frac{M}{|t|}\right) \frac{|t|^\theta}{M^\theta}.$$

Taking into account of (3.3), we obtain

$$F(x,t) \geq F\left(x, \frac{M}{|t|}t\right) \frac{|t|^\theta}{M^\theta} \geq a(x)\frac{|t|^\theta}{M^\theta} \geq C|t|^\theta,$$

where $C > 0$ is a constant. Thus, (3.2) is proved.

Fixed $u_0 \in X\backslash\{0\}$ for each $t > 1$ one has

$$I_\lambda(tu_0) \leq \frac{1}{p^-} t^{p^+} \|u_0\|^p - \lambda C t^\theta \int_\Omega |u_0(x)|^\theta dx.$$

Since $\theta > p^+$, this condition guarantees that I_λ is unbounded from below. Fixed $\lambda \in]0,\lambda^*[$ for each $u \in X$ such that $u \in \Phi^{-1}(]-\infty,1[)$, thanks to Proposition 3, one has

$$\|u\| \leq \left[p^+\Phi(u)\right]^{\frac{1}{p}} < \left[p^+\right]^{\frac{1}{p}} = \left(p^+\right)^{\frac{1}{p^-}}. \tag{3.4}$$

By Theorem 1.3 of [10] and from the compact embedding $X \hookrightarrow L^{q(x)}(\Omega)$, we have

$$\int_\Omega |u(x)|^{q(x)} dx \leq \left[\|u\|_{L^{q(x)}(\Omega)}\right]^q \leq [c_q \|u\|]^q, \tag{3.5}$$

for each $u \in X$. Moreover, the compact embedding $X \hookrightarrow L^1(\Omega)$, (f_1), (3.4) and (3.5) imply that for each $u \in \Phi^{-1}(]-\infty,1[)$, we have

$$\Psi(u) \leq a_1 \int_\Omega |u(x)| dx + \frac{a_2}{q^-} \int_\Omega |u(x)|^{q(x)} \leq a_1 c_1 \|u\| + \frac{a_2}{q^-} [c_q \|u\|]^q$$

$$\leq a_1 c_1 \left(p^+\right)^{\frac{1}{p^-}} + \frac{a_2}{q^-}[c_q]^q \left(p^+\right)^{\frac{q^+}{p^-}},$$

and so

$$\sup_{\Phi(u)<1} \Psi(u) \leq a_1 c_1 \left(p^+\right)^{\frac{1}{p^-}} + \frac{a_2}{q^-}[c_q]^q \left(p^+\right)^{\frac{q^+}{p^-}} = \frac{1}{\lambda^*} < \frac{1}{\lambda}. \tag{3.6}$$

From (3.6) one has

$$\lambda \in]0, \lambda^*[\subseteq \left]0, \frac{r}{\sup_{\{\Phi(u)<r\}} \Psi(u)}\right[.$$

So, all hypotheses of Theorem 1 are verified. Therefore for each $\lambda \in]0, \lambda^*[$ the functional I_λ admits two distinct critical points that are weak solutions of problem (1.1). □

Remark 2. We observe that, if f is non-negative and $f(x,0) \neq 0$ in Ω, then Theorem 2 ensures the existence of two positive weak solutions for problem (1.1) (see, e.g., Theorem 11.1 of [18]).

A special case of Theorem 2 reads as follows.

Theorem 3. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function with $f(0) \neq 0$, satisfying for some $q \in (p, p_2^*)$,*

$$\lim_{|t|\to\infty} \frac{f(t)}{|t|^{q-1}} = 0,$$

where $p > 1$ and

$$p_2^* := \begin{cases} \frac{Np}{N-2p}, & 2p < N, \\ +\infty, & 2p \geq N. \end{cases}$$

Then, there exists $\lambda^* > 0$, such that, for any $\lambda \in]0, \lambda^*[$ the following problem

$$\begin{cases} \Delta_p^2 u + |u|^{p-2} u = \lambda f(u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

admits two positive weak solutions.

Remark 3. Thanks to Talenti's inequality, it is possible to obtain an estimate of the embedding's constants c_1, c_q. By the Sobolev embedding theorem, there exists a positive constant c such that (see Proposition B.7 of [19])

$$\|u\|_{L^{p^{-*}}(\Omega)} \leq c\,\|u\| \quad (\forall u \in X). \tag{3.7}$$

The best constant that appears in (3.7) is (see [21])

$$c := \frac{1}{N\sqrt{\pi}} \left(\frac{N!\Gamma\left(\frac{N}{2}\right)}{2\Gamma\left(\frac{N}{p^-}\right)\Gamma\left(N+1-\frac{N}{p^-}\right)} \right)^{\frac{1}{N}} \eta^{1-\frac{1}{p^-}}, \tag{3.8}$$

where

$$\eta := \frac{N(p^- - 1)}{N - p^-}.$$

Due to (3.8), as a simple consequence of Hölder's inequality, it follows that

$$c_q \leq \frac{\text{meas}(\Omega)^{\frac{p^{-*}-q^+}{p^{-*}q^+}}}{N\sqrt{\pi}} \left(\frac{N!\Gamma\left(\frac{N}{2}\right)}{2\Gamma\left(\frac{N}{p^-}\right)\Gamma\left(N+1-\frac{N}{p^-}\right)} \right)^{\frac{1}{N}} \eta^{1-\frac{1}{p^-}},$$

where "meas(Ω)" denotes the Lebesgue measure of the set Ω.

In conclusion, we present a concrete example of application of Theorem 2 whose construction is motivated by Example 4.1 of [6].

Example 1. We consider the function f defined by

$$f(x,t) := \begin{cases} c + dqt^{q(x)-1} & \text{if } x \in \Omega,\ t \geq 0, \\ c - dq(-t)^{q(x)-1} & \text{if } x \in \Omega,\ t < 0. \end{cases}$$

for each $(x,t) \in \Omega \times \mathbb{R}$, where $p, q \in C(\overline{\Omega})$ verify the condition $1 < p^+ < q^- \leq q(x) < p^*(x)$ for each $x \in \Omega$ and c, d are two positive constants. Fixed $p^+ < \theta < q^-$ and

$$r > \max\left\{ \left[\frac{(\theta-1)c}{d(q^- - \theta)}\right]^h, \left[\frac{c}{d}\right]^h \right\},$$

with $h(\cdot) = \frac{1}{q(\cdot)-1}$. We prove that f verifies the assumptions requested in Theorem 2. condition (f_1) of Theorem 2 is easily verified. We observe that

$$F(x,t) = ct + d|t|^{q(x)},$$

for each $(x,t) \in \Omega \times \mathbb{R}$. Taking into account that, condition (f_2) is verified (see Example 4.1 of [6]) and clearly $f(x,0) \neq 0$ in Ω, problem (1.1) has at least two nontrivial weak solutions for every $\lambda \in]0, \lambda^*[$, where λ^* is the constant introduced in the statement of Theorem 2.

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A NEW OSCILLATION RESULT FOR NONLINEAR DIFFERENTIAL EQUATIONS WITH NONMONOTONE DELAY

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Abstract. In this article, our objective is to study the oscillation of first order nonlinear delay differential equation

$$x'(t)+p(t)f(x(\tau(t)))=0,\ t\geq t_0,$$

where the functions $p(t)$ and $\tau(t)$ are functions of nonnegative real numbers, $\tau(t)$ is not necessarily monotone such that $\tau(t)\leq t$ for $t\geq t_0$, $\lim_{t\to\infty}\tau(t)=\infty$, $f\in C(\mathbb{R},\mathbb{R})$ and $xf(x)>0$ for $x\neq 0$. Also, we establish a new oscillation condition involving both limsup and liminf. Finally, we present two examples to demonstrate the main result.

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1. Introduction

The paper deals with the first order nonlinear delay differential equation

$$x'(t)+p(t)f(x(\tau(t)))=0,\ t\geq t_0, \tag{1.1}$$

where the functions $p(t)$ and $\tau(t)$ are functions of nonnegative real numbers, $\tau(t)$ is not necessarily monotone such that

$$\tau(t)\leq t \ \text{ for } t\geq t_0,\ \lim_{t\to\infty}\tau(t)=\infty, \tag{1.2}$$

$$f\in C(\mathbb{R},\mathbb{R}) \ \text{ and } \ xf(x)>0 \ \text{ for } \ x\neq 0 \tag{1.3}$$

and

$$0<\widetilde{N}:=\limsup_{x\to 0}\frac{x}{f(x)}<\infty. \tag{1.4}$$

By a solution of (1.1), we mean a continuously differentiable function defined on $[\tau(T_0),\infty)$ for some $T_0\geq t_0$ such that (1.1) holds for $t\geq T_0$. A solution of (1.1) is called *oscillatory* if it has arbitrarily large zeros. Otherwise, it is called *nonoscillatory*.

The question of obtaining new sufficient conditions for the oscillatory solutions of these equations has attracted many researchers (see the references section). Furthermore, delay differential equations have numerous applications in the field of applied sciences and engineering. See the studies in [1,2,10,22,26,27] for more details. The

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reader is referred to monograph [11] for the general information about oscillation theory.

When $f(x) = x$, we have the linear form of (1.1)

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0. \tag{1.5}$$

The first study about the oscillation of all solutions of (1.5) was examined by Myshkis in 1950. Later, Ladas et al. [19], Koplatadze and Chanturija [16], Ladas and Stavroulakis [20], Fukagai and Kusano [9], Ladde et al. [21], Győri and Ladas [11] and Erbe et al. [7] studied (1.5) and obtained some well-known oscillation criteria.

Now, let α and β be defined by

$$\alpha := \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds \tag{1.6}$$

and

$$\beta := \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds. \tag{1.7}$$

In 1972, Ladas et al. [19] proved that if $\tau(t)$ is nondecreasing and

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > 1, \tag{1.8}$$

then all solutions of (1.5) are oscillatory.

Also, in 1982, Koplatadze and Chanturija [16] established the following result. If $\tau(t)$ is not necessarily monotone and

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > \frac{1}{e}, \tag{1.9}$$

then all solutions of (1.5) are oscillatory.

When the researchers encountered with the equations which do not satisfy the well-known oscillation criteria (1.8) and (1.9), they tried to obtain new conditions including both $\liminf$ and $\limsup$ conditions for the oscillatory solutions of these equations. The first successful attempt was carried out by Erbe and Zhang [8] in 1988. They established the following condition by using the upper bound of the ratio $\frac{x(\tau(t))}{x(t)}$ for the nonoscillatory solutions $x(t)$ of (1.5).

If $0 < \alpha \leq \frac{1}{e}$ and $\tau(t)$ is nondecreasing,

$$\beta > 1 - \left(\frac{\alpha}{2}\right)^2, \tag{1.10}$$

then all solutions of (1.5) are oscillatory.

Since then, many authors have tried to obtain better results by improving the upper bound for $\frac{x(\tau(t))}{x(t)}$. Also, in 1991, Chao [4] obtained the following condition for (1.5) with nondecreasing argument.

$$\beta > 1 - \frac{\alpha^2}{2(1-\alpha)}. \tag{1.11}$$

In 1992 Yu and Wang [30] and Yu et al. [31] found out the following result. If $0 < \alpha \leq \frac{1}{e}$ and $\tau(t)$ is nondecreasing,

$$\beta > 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, \tag{1.12}$$

then all solutions of (1.5) are oscillatory.

When $0 < \alpha \leq \frac{1}{e}$ and $\tau(t)$ is nondecreasing, in 1990, Elbert and Stavroulakis [6] and in 1991, Kwong [18] established the following criteria for the oscillatory solutions of (1.5) by using different techniques, respectively.

$$\beta > 1 - \left(1 - \frac{1}{\sqrt{\lambda_1}}\right)^2 \tag{1.13}$$

and

$$\beta > \frac{\ln\lambda_1 + 1}{\lambda_1}, \tag{1.14}$$

where λ_1 is the smaller root of equation $\lambda = e^{\alpha\lambda}$.

In 1994, Koplatadze and Kvinikadze [17] improved (1.12). Furthermore, in 1998 Philos and Sficas [25], in 1999 Jaros and Stavroulakis [12], in 2000 Kon et al. [15] and in 2003 Sficas and Stavroulakis [28] established the following conditions for oscillatory solutions of (1.5) when $0 < \alpha \leq \frac{1}{e}$ and $\tau(t)$ is nondecreasing.

$$\beta > 1 - \frac{\alpha^2}{2(1-\alpha)} - \frac{\alpha^2}{2}\lambda_1, \tag{1.15}$$

$$\beta > \frac{\ln\lambda_1 + 1}{\lambda_1} - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, \tag{1.16}$$

$$\beta > 2\alpha + \frac{2}{\lambda_1} - 1 \tag{1.17}$$

and

$$\beta > \frac{\ln\lambda_1 - 1 + \sqrt{5 - 2\lambda_1 + 2\alpha\lambda_1}}{\lambda_1}, \tag{1.18}$$

where λ_1 is the smaller root of equation $\lambda = e^{\alpha\lambda}$.

Now, we define the function

$$h(t) = \sup_{s \leq t}\{\tau(s)\},\ t \geq 0. \tag{1.19}$$

Clearly, $h(t)$ is nondecreasing and $\tau(t) \leq h(t)$ for all $t \geq 0$.

In 2011, Braverman and Karpuz [3] established the following oscillation condition for (1.5). If $\tau(t)$ is not necessarily monotone and

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s)ds > 1, \tag{1.20}$$

where $h(t)$ is defined by (1.19), then all solutions of (1.5) are oscillatory.

Moreover, in 2014, Stavroulakis [29] improved the condition (1.20) to the following condition for the oscillatory solutions of (1.5). If $0 < \alpha \leq \frac{1}{e}$, $\tau(t)$ is not necessarily monotone and

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s)ds > 1 - \left(\frac{\alpha}{2}\right)^2, \tag{1.21}$$

where $h(t)$ is defined by (1.19), then all solutions of (1.5) are oscillatory. It can be seen that the right side of (1.21) is smaller than (1.20). Hence, (1.21) improves (1.20).

When the delay argument $\tau(t)$ is not necessarily monotone, the results which were presented by Chatzarakis and Pėics [5] and Kılıç [13] include (1.16) and (1.17), respectively.

On the other hand, in 2017 and 2020, Öcalan et al. [23, 24] obtained the following criteria for the oscillatory solution of (1.1). Assume that (1.2) and (1.3) hold. If $\tau(t)$ is not necessarily monotone and

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s)ds > \widetilde{N},\ 0 < \widetilde{N} < \infty, \tag{1.22}$$

or

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > \frac{\widetilde{N}}{e},\ 0 \leq \widetilde{N} < \infty, \tag{1.23}$$

where $\widetilde{N}$ and $h(t)$ are defined by (1.4) and (1.19), respectively, then all solutions of (1.1) are oscillatory.

The conditions (1.22) and (1.23) which are established for nonlinear delay differential equations can be considered as equivalent conditions to (1.8) and (1.9) which are obtained for linear delay differential equations.

As seen above, there are few papers dealing with oscillation of (1.1). In 1984, Fukagai and Kusano [9] studied (1.1) with nondecreasing delay. They obtained that if (1.23) holds, then all solutions of (1.1) are oscillatory. Also, see the results in Ladde et al. [21] for some oscillation criteria for the solutions of (1.1).

In view of this, an important question that arises in the case $\tau(t)$ is not necessarily monotone and (1.22) and (1.23) are not satisfied, is whether we can obtain new oscillation condition for (1.1). In the present paper, we will give the positive answer to

this question. Also, the main purpose of this paper is to modify the condition (1.21) for the nonlinear delay differential equations. Especially, the present article will be the first study involving both $\liminf$ and $\limsup$ integral conditions in the literature for the nonlinear differential equations with nonmonotone delay by using the ratio $\frac{x(h(t))}{x(t)}$.

2. Main Results

In this section, we present a new sufficient condition for the oscillation of all solutions of (1.1), under the assumption that the delay argument $\tau(t)$ is not necessarily monotone. The following lemmas will be useful to establish our result.

Lemma 1 ([7], Lemma 2.1.1)**.** *Assume that* (1.19) *holds and*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > 0.$$

Then

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds = \liminf_{t\to\infty} \int_{h(t)}^{t} p(s)ds. \tag{2.1}$$

Lemma 2 ([14], Lemma 2.2)**.** *Assume that $x(t)$ is an eventually positive solution of* (1.1)*. If*

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s)ds > 0, \tag{2.2}$$

then $\lim_{t\to\infty} x(t) = 0$, where $h(t)$ is defined by (1.19)*.*

Also, assume that $x(t)$ is an eventually negative solution of (1.1)*. If* (2.2) *holds, then $\lim_{t\to\infty} x(t) = 0$.*

Lemma 3. *Assume that x(t) is an eventually positive solution of* (1.1) *and*

$$\alpha := \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > 0. \tag{2.3}$$

Then, we have

$$\limsup_{t\to\infty} \frac{x(h(t))}{x(t)} \le \left(\frac{2N}{\alpha}\right)^2, \tag{2.4}$$

where $h(t)$ is defined by (1.19) *and N is a constant with $\widetilde{N} < N$.*

Also, assume that x(t) is an eventually negative solution of (1.1)*. If* (2.3) *holds, then we get* (2.4)*.*

Proof. Let $x(t)$ be an eventually positive solution of (1.1). Then, there exists $t_1 > t_0$ such that $x(t),\ x(\tau(t)),\ x(h(t)) > 0$ for all $t \geq t_1$. Thus, from (1.1) we have

$$x'(t) = -p(t)f(x(\tau(t))) \leq 0$$

for all $t \geq t_1$, which means that $x(t)$ is an eventually nonincreasing function. Also, with the help of Lemma 1, (2.3) implies (2.2), then from Lemma 2, we know that $\lim_{t\to\infty} x(t) = 0$. Then from (1.4), we can choose $t_2 > t_1$ and there exists N with $\tilde{N} < N$ such that

$$f(x(\tau(t))) > \frac{1}{N}x(\tau(t)) \text{ for } t \geq t_2. \tag{2.5}$$

Using the fact that $x(t)$ is nonincreasing, $\tau(t) \leq h(t)$ and the inequality (2.5), from (1.1) we obtain

$$x'(t) + \frac{1}{N}p(t)x(h(t)) < 0. \tag{2.6}$$

Moreover, from (2.3) and Lemma 1, we have

$$\int_{h(t)}^{t} p(s)ds \geq \alpha - \varepsilon,\ \varepsilon \in (0,\alpha), \tag{2.7}$$

therefore there exists a $t^* > t$ such that

$$\int_{h(t^*)}^{t} p(s)ds \geq \frac{\alpha-\varepsilon}{2} \text{ and } \int_{t}^{t^*} p(s)ds \geq \frac{\alpha-\varepsilon}{2}. \tag{2.8}$$

Then, integrating (2.6) from $h(t^*)$ to t and using the fact that $x(t)$ is nonincreasing, $h(t)$ is nondecreasing and (2.8), we obtain

$$x(t) - x(h(t^*)) + \frac{1}{N}\int_{h(t^*)}^{t} p(s)x(h(s))\,ds < 0$$

or

$$x(t) - x(h(t^*)) + \frac{1}{N}x(h(t))\int_{h(t^*)}^{t} p(s)ds < 0$$

and

$$x(h(t^*)) > \frac{1}{N}x(h(t))\frac{\alpha-\varepsilon}{2}. \tag{2.9}$$

By using the same facts as above, integrating (2.6) from t to t^*, we have

$$x(t^*) - x(t) + \frac{1}{N}\int_{t}^{t^*} p(s)x(h(s))\,ds < 0$$

or

$$x(t^*) - x(t) + \frac{1}{N} x(h(t^*)) \int_t^{t^*} p(s)ds < 0$$

and

$$x(t) > \frac{1}{N} x(h(t^*)) \frac{\alpha - \varepsilon}{2}. \tag{2.10}$$

Finally, combining (2.9) and (2.10), we get

$$x(t) > \frac{\alpha - \varepsilon}{2N} x(h(t^*)) > \frac{(\alpha - \varepsilon)}{2N} \frac{(\alpha - \varepsilon)}{2N} x(h(t))$$

or

$$\frac{x(h(t))}{x(t)} < \left(\frac{2N}{(\alpha - \varepsilon)} \right)^2.$$

Hence

$$\limsup_{t \to \infty} \frac{x(h(t))}{x(t)} \leq \left(\frac{2N}{(\alpha - \varepsilon)} \right)^2,$$

because of ε is arbitrary, by letting $\varepsilon \to 0$, we obtain (2.4) and this completes the proof. □

Theorem 1. *Assume that* (1.2) *and* (1.3) *hold. If* $\tau(t)$ *is not necessarily monotone,* $0 < \alpha \leq \frac{\widetilde{N}}{e}$ *and*

$$\beta := \limsup_{t \to \infty} \int_{h(t)}^{t} p(s)ds > N \left[1 - \left(\frac{\alpha}{2N} \right)^2 \right], \tag{2.11}$$

then all solutions of (1.1) *are oscillatory, where* $h(t)$ *is defined by* (1.19) *and* N *is a constant with* $\widetilde{N} < N$.

Proof. Assume, for the sake of contradiction, that there exists an eventually positive solution $x(t)$ of (1.1). If $x(t)$ is an eventually negative solution of (1.1), the proof of the theorem can be done similarly. Then, there exists $t_1 > t_0$ such that $x(t)$, $x(\tau(t))$, $x(h(t)) > 0$ for all $t \geq t_1$. Thus, from (1.1) we have

$$x'(t) = -p(t) f(x(\tau(t))) \leq 0$$

for all $t \geq t_1$, which means, that $x(t)$ is an eventually nonincreasing function. Lemma 2 and condition (2.11) imply that $\lim_{t \to \infty} x(t) = 0$. Then from (1.4), we can choose $t_2 > t_1$ and there exists N with $\widetilde{N} < N$ such that

$$f(x(\tau(t))) > \frac{1}{N} x(\tau(t)) \text{ for } t \geq t_2. \tag{2.12}$$

Using inequality (2.12), the fact that $x(t)$ is nonincreasing and $\tau(t) \leq h(t)$, from (1.1) we obtain

$$x'(t) + \frac{1}{N} p(t) x(h(t)) < 0. \tag{2.13}$$

Integrating (2.13) from $h(t)$ to t, we have

$$x(t) - x(h(t)) + \frac{1}{N} \int_{h(t)}^{t} p(s) x(h(s)) ds < 0,$$

$$x(t) - x(h(t)) + \frac{1}{N} x(h(t)) \int_{h(t)}^{t} p(s) ds < 0$$

and

$$x(t) + x(h(t)) \left[\frac{1}{N} \int_{h(t)}^{t} p(s) ds - 1 \right] < 0$$

or

$$\frac{1}{N} \int_{h(t)}^{t} p(s) ds < 1 - \frac{x(t)}{x(h(t))}$$

and by Lemma 3,

$$\limsup_{t \to \infty} \int_{h(t)}^{t} p(s) ds \leq N \left[1 - \left(\frac{\alpha}{2N} \right)^2 \right], \tag{2.14}$$

which contradicts to (2.11), so this completes the proof. □

Example 1. We consider the nonlinear delay differential equation

$$x'(t) + \frac{1}{t} x\left(\frac{t}{2.7}\right) \ln\left(\left| x\left(\frac{t}{2.7}\right) \right| + e \right) = 0. \tag{2.15}$$

Then,

$$\widetilde{N} = \limsup_{x \to 0} \frac{x}{f(x)} = \limsup_{x \to 0} \frac{x}{x \ln(|x| + e)} = 1 < N = 1.01$$

and using this, we have

$$\beta = \limsup_{t \to \infty} \int_{\tau(t)}^{t} p(s) ds = \limsup_{t \to \infty} \int_{\frac{t}{2.7}}^{t} \frac{1}{s} ds = \ln(2.7) \approx 0.99325 < 1 = \widetilde{N},$$

so condition (1.22) doesn't hold. But,

$$\beta \approx 0.99325 > N \left[1 - \left(\frac{\alpha}{2N} \right)^2 \right] \approx 0.76580.$$

We see that condition (2.11) is satisfied and therefore, all solutions of (2.15) are oscillatory.

Example 2. We consider the nonlinear delay differential equation

$$x'(t)+(0.09)x(\tau(t))\ln(|x(\tau(t))|+19.02)=0, \tag{2.16}$$

where

$$\tau(t)=\begin{cases} -t+12k-2, & t\in[6k,6k+1], \\ 4t-18k-7, & t\in[6k+1,6k+2], \\ 0.5t+3k, & t\in[6k+2,6k+4], \\ -6t+42k+26, & t\in[6k+4,6k+5], \\ 8t-42k-44, & t\in[6k+5,6k+6], \end{cases} \quad k\in\mathbb{N}_0,$$

and with the help of (1.19), we obtain

$$h(t):=\sup_{s\le t}\{\tau(s)\}=\begin{cases} 6k-2, & t\in[6k,6k+1.25], \\ 4t-18k-7, & t\in[6k+1.25,6k+2], \\ 0.5t+3k, & t\in[6k+2,6k+4], \\ 6k+2, & t\in[6k+4,6k+5.75], \\ 8t-42k-44, & t\in[6k+5.75,6k+6], \end{cases} \quad k\in\mathbb{N}_0.$$

Then, from (1.4), we have

$$\tilde{N}=\limsup_{x\to 0}\frac{x}{f(x)}=\limsup_{x\to 0}\frac{x}{x\ln(|x|+19.02)}\approx 0.3395<N=0.3396.$$

Also,

$$\alpha=\liminf_{t\to\infty}\int_{\tau(t)}^{t}p(s)ds=\liminf_{t\to\infty}\int_{6k+1}^{6k+2}0.09ds=0.09<\frac{\tilde{N}}{e}\approx 0.12489$$

and

$$\beta=\limsup_{t\to\infty}\int_{h(t)}^{t}p(s)ds=\limsup_{t\to\infty}\int_{6k+2}^{6k+5.75}0.09ds=0.3375<\tilde{N}\approx 0.3395,$$

so, the conditions (1.22) and (1.23) don't hold.

However,

$$\beta=0.3375>N\left[1-\left(\frac{\alpha}{2N}\right)^2\right]\approx 0.33363,$$

which means, the conditions of Theorem 1 hold. So, all solutions of (2.16) are oscillatory.

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AMPLY ESSENTIAL G-SUPPLEMENTED MODULES

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Abstract. Let M be an $R-$module. If every essential submodule of M has a g-supplement in M, then M is called an essential g-supplemented (or briefly, eg-supplemented) module. If every essential submodule has ample g-supplements in M, then M is called an amply essential g-supplemented (or briefly, amply eg-supplemented) module. In this work, some properties of these modules are investigated. It is proved that every factor module and every homomorphic image of an amply eg-supplemented module are amply eg-supplemented. Let M be a projective and eg-supplemented module. Then every finitely $M-$ generated $R-$module is amply eg-supplemented.

2010 *Mathematics Subject Classification:* 16D10; 16D70

Keywords: g-small submodules, generalized radical, essential submodules, g-supplemented modules

1. Introduction

Throughout this paper all rings are associative with identity and all modules are unital left modules.

Let R be a ring and M be an $R-$module. We denote a submodule N of M by $N \leq M$. Let M be an $R-$module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a *small* (or *superfluous*)submodule of M and denoted by $N \ll M$. A submodule N of an R -module M is called an *essential* submodule, denoted by $N \trianglelefteq M$, in case $K \cap N \neq 0$ for every submodule $K \neq 0$, or equivalently, $N \cap L = 0$ for $L \leq M$ implies that $L = 0$. Let M be an $R-$module and K be a submodule of M. K is called a *generalized small* (briefly, *g-small*) submodule of M if for every essential submodule T of M with the property $M = K + T$ implies that $T = M$, we denote this by $K \ll_g M$. It is clear that every small submodule is generalized small but the converse is not true generally. Let M be an $R-$module and $U, V \leq M$. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M. M is said to be *supplemented* if every submodule of M has a supplement in M. M is said to be *essential supplemented* (briefly, *e-supplemented*) if every essential submodule of M has a supplement in M.

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Let M be an $R-$module and $U \leq M$. If for every $V \leq M$ such that $M = U + V$, U has a supplement $V^{'}$ with $V^{'} \leq V$, we say U has *ample supplements* in M. M is said to be *amply supplemented* if every submodule of M has ample supplements in M. M is said to be *amply essential supplemented* (briefly, *amply e-supplemented*) if every essential submodule of M has ample supplements in M. Let M be an $R-$module and $U, V \leq M$. If $M = U + V$ and $M = U + T$ with $T \trianglelefteq V$ implies that $T = V$, or equivalently, $M = U + V$ and $U \cap V \ll_g V$, then V is called a *g-supplement* of U in M. M is said to be *g-supplemented* if every submodule of M has a g-supplement in M. Let M be an $R-$module and $U \leq M$. If for every $V \leq M$ such that $M = U + V$, U has a g-supplement $V^{'}$ with $V^{'} \leq V$, we say U has *ample g-supplements* in M. M is said to be *amply g-supplemented* if every submodule of M has ample g-supplements in M. The intersection of maximal submodules of an R-module M is called the *radical* of M and denoted by $RadM$. If M have no maximal submodules, then we denote $RadM = M$. The intersection of essential maximal submodules of an R-module M is called the *generalized radical* (briefly, *g-radical*) of M and denoted by Rad_gM. If M have no essential maximal submodules, then we denote $Rad_gM = M$. An $R-$module M is said to be *g-semilocal* if M/Rad_gM is semisimple (see [13]). The sum of all simple submodules of an $R-$module M is called the *socle* of M and denoted by $SocM$. Let M be an $R-$module and $K \leq V \leq M$. We say V *lies above* K in M if $V/K \ll M/K$.

More details about (amply) supplemented modules are in [2, 9, 15]. More details about (amply) essential supplemented modules are in [8, 12]. More informations about g-small submodules and g-supplemented modules are in [3, 4, 10].

Lemma 1. *Let M be an R -module and $K \leq N \leq M$. If K is a generalized small submodule of N, then K is a generalized small submodule in every submodule of M which contains N.*

Proof. See [3, Lemma 1 (2)]. □

Lemma 2. *Let M be an R -module. Then $Rad_gM = \sum_{L<<_gM} L$.*

Proof. See [3, Lemma 5 and Corollary 5]. □

Lemma 3. *The following assertions are hold.*

(1) *If M is an $R-$module, then $Rm \ll_g M$ for every $m \in Rad_gM$.*
(2) *If $N \leq M$, then $Rad_gN \leq Rad_gM$.*

Proof. See [4, Lemma 3]. □

2. Essential g-Supplemented Modules

Definition 1. Let M be an $R-$module. If every essential submodule of M has a g-supplement in M, then M is called an essential g-supplemented (or briefly, eg-supplemented) module. (See also [6, 7, 11]).

Definition 2 ([11]). Let M be an $R-$module and $X \leq M$. If X is a g-supplement of an essential submodule of M, then X is called an eg-supplement submodule in M.

Proposition 1. *Let M be an eg-supplemented module. Then M is g-semilocal.*

Proof. Since M is eg-supplemented, by [11, Proposition 2.5], M/Rad_gM have no proper essential submodules. Then by [15, 21.1], $Soc\,(M/Rad_gM) = M/Rad_gM$. Hence M/Rad_gM is semisimple and M is g-semilocal. □

Lemma 4. *Let M be an $R-$module, V be an eg-supplement in M and $K \ll_g M$. Then $K \cap V \ll_g V$.*

Proof. Since V is an eg-supplement in M, there exists $U \trianglelefteq M$ such that V is a g-supplement of U in M. Let $K \cap V + T = V$ with $T \trianglelefteq V$. Then $M = U + V = U + T + K \cap V$, and since $K \cap V \leq K \ll_g M$ and $(U + T) \trianglelefteq M$, $U + T = M$. By V being a g-supplement of U in M and $T \trianglelefteq V$, $T = V$. Hence $K \cap V \ll_g V$. □

Corollary 1 ([10, Lemma 2.3]). *Let M be an $R-$module, V be an eg-supplement in M and $K \leq V$. Then $K \ll_g M$ if and only if $K \ll_g V$.*

Proof. Clear from Lemma 1 and Lemma 4. □

Corollary 2 ([10, Theorem 2.4]). *Let M be an $R-$module and V be an eg-supplement in M. Then $Rad_gV = V \cap Rad_gM$.*

Proof. By Lemma 3, $Rad_gV \leq V \cap Rad_gM$. Let $y \in V \cap Rad_gM$. Then $y \in V$ and $y \in Rad_gM$. Since $y \in Rad_gM$, by Lemma 3, $Ry \ll_g M$ and by Lemma 4, $Ry \ll_g V$. By Lemma 2, $Ry \leq Rad_gV$ and $y \in Rad_gV$. Hence $V \cap Rad_gM \leq Rad_gV$. Therefore, $Rad_gV = V \cap Rad_gM$. □

We can also prove this corollary as follows:

Proof. Since V is an eg-supplement in M, there exists $U \trianglelefteq M$ such that V is a g-supplement of U in M. Here $M = U + V$ and $U \cap V \ll_g V$. Let K be a generalized maximal submodule of V. Since $U \cap V \ll_g V$, by Lemma 2, $U \cap V \leq Rad_gV \leq K$. By $\frac{M}{U+K} = \frac{U+K+V}{U+K} \cong \frac{V}{V\cap(U+K)} = \frac{V}{U\cap V+K} = \frac{V}{K}$ and $U + K \trianglelefteq M$, $U + K$ is a generalized maximal submodule of M and $Rad_gM \leq U + K$. This case $V \cap Rad_gM \leq V \cap (U + K) = U \cap V + K = K$ and $V \cap Rad_gM \leq Rad_gV$. By Lemma 3, $Rad_gV \leq V \cap Rad_gM$. Hence $Rad_gV = V \cap Rad_gM$. □

Lemma 5 ([10, Lemma 2.5]). *Let M be an $R-$module, V be a g-supplement of U in M and $K \trianglelefteq V$. Then for $T \leq V$, T is a g-supplement of K in V if and only if T is a g-supplement of $U + K$ in M.*

Lemma 6 ([11, Lemma 2.11]). *Let M be an eg-supplemented module. Then every finitely $M-$generated $R-$module is eg-supplemented.*

3. Amply Essential g-Supplemented Modules

Definition 3. Let M be an $R-$module. If every essential submodule of M has ample g-supplements in M, then M is called an amply essential g-supplemented (or briefly, amply eg-supplemented) module. (See also [5]).

Clearly, every amply essential g-supplemented module is essential g-supplemented.

Lemma 7. *Let M be an amply eg-supplemented module. Then M/Rad_gM have no proper essential submodules.*

Proof. Since M is amply eg-supplemented, then M is eg-supplemented. Then by [11, Proposition 2.5], M/Rad_gM have no proper essential submodules. □

Corollary 3. *If M is amply eg-supplemented, then M is g-semilocal.*

Proof. By Lemma 7, M/Rad_gM have no proper essential submodules. Then by [15, 21.1], $Soc\,(M/Rad_gM) = M/Rad_gM$. Hence M/Rad_gM is semisimple and M is g-semilocal. □

Proposition 2. *Let M be an amply eg-supplemented module. Then every eg-supplement submodule in M is amply eg-supplemented.*

Proof. Let V be an eg supplement submodule in M. Then there exists $U \trianglelefteq M$ such that V is a g-supplement of U in M. Let $V = K + X$ with $K \trianglelefteq V$ and $X \leq V$. Here $M = U + K + X$. Since M is amply eg-supplemented and $U + K \trianglelefteq M$, $U + K$ has a g-supplement T in M such that $T \leq X$. By Lemma 5, T is a g-supplement of K in V. Moreover, $T \leq X$. Hence V is amply eg-supplemented. □

Lemma 8. *Let M be an amply eg-supplemented module. Then every factor module of M is amply eg-supplemented.*

Proof. Let $\frac{M}{K}$ be any factor module of M. Let $\frac{U}{K} \trianglelefteq \frac{M}{K}$ and $\frac{M}{K} = \frac{U}{K} + \frac{V}{K}$. Then $U \trianglelefteq M$ and $M = U + V$. Since M is amply eg-supplemented, U has a g-supplement X in M with $X \leq V$. Since $K \leq U$, by [3, Lemma 4], $\frac{X+K}{K}$ is a g-supplement of $\frac{U}{K}$ in $\frac{M}{K}$. Moreover, $\frac{X+K}{K} \leq \frac{V}{K}$. Hence $\frac{M}{K}$ is amply eg-supplemented. □

Corollary 4. *Every homomorphic image of an amply eg-supplemented module is amply eg-supplemented.*

Proof. Clear from Lemma 8. □

Lemma 9. *If M is a π-projective and eg-supplemented module, then M is an amply eg-supplemented module.*

Proof. Let $U \trianglelefteq M$, $M = U + V$ and X be a g-supplement of U in M. Here $M = U + X$ and $U \cap X \ll_g X$. Since M is π -projective and $M = U + V$, there exists an R -module homomorphism $f : M \to M$ such that $Imf \subset V$ and $Im\,(1 - f) \subset U$. So, we have $M = f\,(M) + (1 - f)\,(M) = f\,(U) + f\,(X) + U = U + f\,(X)$. Suppose

that $a \in U \cap f(X)$. Since $a \in f(X)$, then there exists $x \in X$ such that $a = f(x)$. Since $a = f(x) = f(x) - x + x = x - (1-f)(x)$ and $(1-f)(x) \in U$, we have $x = a + (1-f)(x) \in U$. Thus $x \in U \cap X$ and so, $a = f(x) \in f(U \cap X)$. Therefore, we have $U \cap f(X) \leq f(U \cap X)$. Since $U \cap X \ll_g X$, $f(U \cap X) \ll_g f(X)$. Hence $U \cap f(X) \ll_g f(X)$ and since $M = U + f(X)$, $f(X)$ is a g-supplement of U in M. Moreover, $f(X) \subset V$. Therefore, M is amply eg-supplemented. □

Corollary 5. *If M is a projective and eg-supplemented module, then M is amply eg-supplemented.*

Proof. Clear from Lemma 9. □

Lemma 10. *Let M be a π-projective $R-$module. If every essential submodule of M is β_g^* equivalent to an eg-supplement submodule in M, then M is amply eg-supplemented. (The definition of β_g^* relation and some properties of this relation are in [14].)*

Proof. By [11, Lemma 2.13], M is eg-supplemented. Then by Lemma 9, M is amply eg-supplemented. □

Corollary 6. *Let M be a projective $R-$module. If every essential submodule of M is β_g^* equivalent to an eg-supplement submodule in M, then M is amply eg-supplemented.*

Proof. Clear from Lemma 10. □

Corollary 7. *Let M be a π-projective $R-$module. If every essential submodule of M is β^* equivalent to an eg-supplement submodule in M, then M is amply eg-supplemented. (The definition of β^* relation and some properties of this relation are in [1].)*

Proof. Clear from Lemma 10. □

Corollary 8. *Let M be a π-projective $R-$module. If every essential submodule of M lies above an eg-supplement submodule in M, then M is amply eg-supplemented.*

Proof. Clear from Corollary 7. □

Corollary 9. *Let M be a projective $R-$module. If every essential submodule of M is β^* equivalent to an eg-supplement submodule in M, then M is amply eg-supplemented.*

Proof. Clear from Corollary 7. □

Corollary 10. *Let M be a projective $R-$module. If every essential submodule of M lies above an eg-supplement submodule in M, then M is amply eg-supplemented.*

Proof. Clear from Corollary 9. □

Lemma 11. *Let Λ be a finite index set and $\{M_\lambda\}_\Lambda$ be a family of projective $R-$modules. If M_λ is eg-supplemented for every $\lambda \in \Lambda$, then $\underset{\lambda\in\Lambda}{\oplus} M_\lambda$ is amply eg-supplemented.*

Proof. Since M_λ is eg-supplemented for every $\lambda \in \Lambda$, by [11, Corollary 2.8], $\underset{\lambda\in\Lambda}{\oplus} M_\lambda$ is eg-supplemented. Since M_λ is projective for every $\lambda \in \Lambda$, by [15, 18.1], $\underset{\lambda\in\Lambda}{\oplus} M_\lambda$ is projective. Since $\underset{\lambda\in\Lambda}{\oplus} M_\lambda$ is projective and eg-supplemented, by Corollary 5, $\underset{\lambda\in\Lambda}{\oplus} M_\lambda$ is amply eg-supplemented. □

Corollary 11. *Let M be a projective $R-$module. If M is eg-supplemented, then $M^{(\Lambda)}$ is amply eg-supplemented for every finite index set Λ.*

Proof. Clear from Lemma 11. □

Corollary 12. *Let M be a projective $R-$module. If M is eg-supplemented, then every finitely $M-$generated $R-$module is amply eg-supplemented.*

Proof. Let N be a finitely $M-$generated $R-$module. Then there exist a finite index set Λ and an $R-$module epimorphism $f : M^{(\Lambda)} \longrightarrow N$. Since M is projective and eg-supplemented, by Corollary 11, $M^{(\Lambda)}$ is amply eg-supplemented. Then by Corollary 4, N is amply eg-supplemented. □

Lemma 12. *Let M be an $R-$module. If every submodule of M is eg-supplemented, then M is amply eg-supplemented.*

Proof. Let $U \trianglelefteq M$ and $M = U + V$ with $V \leq M$. Since $U \trianglelefteq M$, $U \cap V \trianglelefteq V$. By hypothesis, V is eg-supplemented. Then $U \cap V$ has a g-supplement X in V. By this, $V = U \cap V + X$ and $U \cap X = U \cap V \cap X \ll_g X$. Then $M = U + V = U + U \cap V + X = U + X$ and $U \cap X \ll_g X$. Moreover, $X \leq V$. Hence M is amply eg-supplemented. □

Proposition 3. *Let R be any ring. Then every $R-$module is eg-supplemented if and only if every $R-$module is amply eg-supplemented.*

Proof. $(\Longrightarrow)$ Let M be any $R-$module. Since every $R-$module is eg-supplemented, every submodule of M is eg-supplemented. Then by Lemma 12, M is amply eg-supplemented.
$(\Longleftarrow)$ Clear. □

Proposition 4. *Let R be a ring. The following assertions are equivalent.*

(*i*) *${}_RR$ is eg-supplemented.*
(*ii*) *${}_RR$ is amply eg-supplemented.*
(*iii*) *Every finitely generated $R-$module is eg-supplemented.*
(*iv*) *Every finitely generated $R-$module is amply eg-supplemented.*

Proof. $(i) \Longleftrightarrow (ii)$ Clear from Corollary 5, since ${}_RR$ is projective.

$(i) \Longrightarrow (iii)$ Clear from Lemma 6.

$(iii) \Longrightarrow (iv)$ Let M be a finitely generated $R-$module. Since every finitely generated $R-$module is eg-supplemented, ${}_RR$ is eg-supplemented. Then by Corollary 12, M is amply eg-supplemented.

$(iv) \Longrightarrow (i)$ Clear. □

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BIVARIATE k-MITTAG-LEFFLER FUNCTIONS WITH 2D-k-LAGUERRE-KONHAUSER POLYNOMIALS AND CORRESPONDING k-FRACTIONAL OPERATORS

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Abstract. In this paper, we first introduce new class of 2D-k-Laguerre-Konhauser polynomials, ${}_{\delta}L_{k,n}^{(\alpha,\beta)}(x,y)$, which generalizes the 2D-Laguerre-Konhauser polynomials (see [18]). Then, we define a new family of bivariate k-Mittag-Leffler functions $E_{k,\alpha,\beta,\delta}^{(\gamma)}(x,y)$ and establish the k-Riemann-Liouville double fractional integral and derivative of the functions $E_{k,\alpha,\beta,\delta}^{(\gamma)}(x,y)$. Moreover, we introduce an integral operator ${}_{k}\varepsilon_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}^{(\gamma)}$ which contains the bivariate k-Mittag-Leffler functions $E_{k,\alpha,\beta,\delta}^{(\gamma)}(x,y)$ in the kernel and investigate the semigroup property of this operator. Finally, the left inverse operator of the integral operator ${}_{k}\varepsilon_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}^{(\gamma)}$ is constructed.

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Keywords: Laguerre and Konhauser polynomials, k-Gamma function, k-Mittag-Leffer function, k-fractional integral, k-fractional derivative, double Laplace transform, convolution integral equation

1. Introduction

In recent years, fractional calculus and its wide application have been paid to an ever increasing extent attentions. In mathematical analysis, the fractional calculus is a very important device to achive differentiation and integration with the real number or complex number of the differential or intergral operator (see [2, 9, 10]). Miller and Rose [13] and Kiryakova [12] defined a well known number of different fractional calculus operators along with their properties and applications.

In recent decades, the fractional integral and derivative operators in terms of special functions have started the use of important applications in different areas of mathematical analysis. The Mittag-Leffler function which includes the Gamma function is one of the most famous special funtion which is used in the solution of fractional order integral equations or fractional order differential equations.

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Apart from fractional calculus, the Mittag-Leffler function also plays a significant role in various braches of applied mathematics and engineering sciences, such as chemistry, biology, statistics, thermodynamics, mechanics, quantum physics, informatics, signal processing and others.

In this paper, we define and study a new class of 2D-k-Laguerre Konhauser polynomials ${}_{\delta}L_{k,n}^{(\alpha,\beta)}(x,y)$ which generalizes the 2D-Laguerre-Konhauser polynomials (see [3]). Also, we introduce a new family of k-Mittag-Leffler functions, $E_{k,\alpha,\beta,\delta}^{(\gamma)}(x,y)$. We calculate k-Riemann-Liouville double fractional integral and derivative of the functions $E_{k,\alpha,\beta,\delta}^{(\gamma)}(x,y)$. Moreover, we introduce an integral operator ${}_{k}\varepsilon_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}^{(\gamma)}$ with respect to two variable k-Mittag-Leffler function, $E_{k,\alpha,\beta,\delta}^{(\gamma)}(x,y)$ in the kernel. We are also able to define fractional derivative operators, making use of the semigroup property.

In 2006, Diaz and Pariguan [6, 17] defined the Pochhammer k-symbol and k-gamma function as

$$(\gamma)_{n,k} := \begin{cases} \frac{\Gamma_k(\gamma+nk)}{\Gamma_k(\gamma)} & \text{if } k \in \mathbb{R};\ \gamma \in \mathbb{C}\setminus\{0\}; \\ \gamma(\gamma+k)\dots(\gamma+(n-1)k) & \text{if } n \in \mathbb{N};\ \gamma \in \mathbb{C}; \end{cases} \tag{1.1}$$

where k-Gamma function Γ_k has a relation with the classical Euler's gamma function as:

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right) \qquad (\mathrm{Re}(x) > 0,\ k \in \mathbb{R}^+). \tag{1.2}$$

Knowing that

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \qquad (\mathrm{Re}(x) > 0).$$

When $k = 1$, (1.2) reduces to the classical Pochhammer symbol and Euler's gamma function, respectively. Integral representation of k-Gamma function [6, 17] is given by

$$\Gamma_k(x) = \int_0^\infty e^{-\frac{t^k}{k}}t^{x-1}dt \qquad (\mathrm{Re}(x) > 0,\ k \in \mathbb{R}^+).$$

From (1.1) and (1.2), we have

$$(x)_{rq,k} = k^{rq}\left(\frac{x}{k}\right)_{rq} \qquad (k \in \mathbb{R}^+,\ x,r,q \in \mathbb{C}).$$

As particular case

$$(x)_{r,k} = k^r\left(\frac{x}{k}\right)_r.$$

The k-Beta function B_k [6] is defined by

$$B_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}dt \qquad (k \in \mathbb{R}^+,\ \mathrm{Re}(x) > 0,\ \mathrm{Re}(y) > 0)$$

and we have

$$B_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x,y)} = \frac{1}{k}B\left(\frac{m}{k},\frac{n}{k}\right).$$

The generalized k-Mittag-Leffler function is introduced by Chand et al. [5] as

$$E_{k,\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k}}{\Gamma_k(\alpha n+\beta)} \frac{z^n}{n!} \tag{1.3}$$
$$(k \in \mathbb{R},\ \alpha,\beta,\gamma \in \mathbb{C},\ \mathrm{Re}(\alpha) > 0,\ \mathrm{Re}(\beta) > 0,\ \mathrm{Re}(\gamma) > 0,\ q \in \mathbb{R}^+).$$

Special Cases of $E_{k,\alpha,\beta}^{\gamma,q}(z)$:

(1) For $q = 1$, (1.3) reduces to k-Mittag-Leffler function (see [8]) such that

$$E_{k,\alpha,\beta}^{\gamma,1}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n+\beta)} \frac{z^n}{n!} = E_{k,\alpha,\beta}^{\gamma}(z).$$

(2) For $k = 1$, (1.3) yields Mittag-Leffler function (see [4]) defined as

$$E_{1,\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq}}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!} = E_{\alpha,\beta}^{\gamma,q}(z).$$

(3) For $q = 1$ and $k = 1$, (1.3) gives Mittag-Leffler function presented by Prabhakar [19]

$$E_{1,\alpha,\beta}^{\gamma,1}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n}}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!} = E_{\alpha,\beta}^{\gamma}(z).$$

(4) For $q = 1$, $k = 1$ and $\gamma = 1$, (1.3) reduces the following Mittag-Leffler function (see [21])

$$E_{1,\alpha,\beta}^{1,1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)} = E_{\alpha,\beta}(z).$$

(5) For $q = 1$, $k = 1$, $\gamma = 1$ and $\beta = 1$, (1.3) gives Mittag-Leffler function introduced by Gösta Mittag-Leffler (see [14, 15])

$$E_{1,\alpha,1}^{1,1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)} = E_{\alpha}(z).$$

As an initial motivation, we consider the following class of generalized Laguerre-Konhauser polynomials were introduced by Bin-Saad [3]

$${}_{\delta}L_n^{(\alpha,\beta)}(x,y) = n!\sum_{s=0}^{n}\sum_{r=0}^{n-s} \frac{(-1)^{s+r}x^{r+\alpha}y^{\delta s+\beta}}{s!r!(n-s-r)!\Gamma(\alpha+r+1)\Gamma(\delta s+\beta+1)} \tag{1.4}$$
$$(\alpha,\beta \in \mathbb{R},\ \delta = 1,2,\cdots).$$

Recently, Özarslan and Kürt (see [18]) have considered Laguerre-Konhauser polynomials ${}_{\delta}L_n^{(\alpha,\beta)}(x,y)$ and defined a bivariate Mittag-Leffler functions $E_{\alpha,\beta,\delta}^{(\gamma)}(x,y)$ by

$$E_{\alpha,\beta,\delta}^{(\gamma)}(x,y)=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{r+s}}{\Gamma(\alpha+r)\Gamma(\beta+\delta s)}\frac{x^r}{r!}\frac{y^{\delta s}}{s!} \tag{1.5}$$
$$(\alpha,\beta,\gamma\in\mathbb{C},\ \mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\delta)>0).$$

Remark 1. According to the convergence conditions investigated by Srivastava and Daoust ([20], pp. 155) for the generalized Lauricella series in two variables, the series in (1.5) converges absolutely for $\mathrm{Re}(\delta)>0$.

From (1.4) and (1.5), we see

$${}_{\delta}L_n^{(\alpha,\beta)}(x,y)=x^{\alpha}y^{\beta}E_{\alpha+1,\beta+1,\delta}^{(-n)}(x,y).$$

In a view of the above definitions and motivations, in this paper, we introduce a new class of 2D-k-Laguerre-Konhauser polynomials by

$$\begin{aligned}{}_{\delta}L_{k,n}^{(\alpha,\beta)}(x,y)&=\sum_{s=0}^{n}\sum_{r=0}^{n-s}\frac{(-nk)_{k,r+s}x^{r+\alpha}y^{\delta s+\beta}}{\Gamma_k(\alpha+r+1)\Gamma_k(\delta s+\beta+1)}\frac{x^{r+\alpha}}{r!}\frac{y^{\delta s+\beta}}{s!}\\&=n!\sum_{s=0}^{n}\sum_{r=0}^{n-s}\frac{(-1)^{r+s}k^{r+s}}{(n-s-r)!\Gamma_k(\alpha+r+1)\Gamma_k(\delta s+\beta+1)}\frac{x^{r+\alpha}}{r!}\frac{y^{\delta s+\beta}}{s!}\end{aligned} \tag{1.6}$$

In the case $k=1$, relation (1.6) coincides with the 2D-Laguerre-Konhauser polynomials (1.4).

According to the inverse operator $\hat{D}{}_x^{-\alpha}$ i.e.

$$\hat{D}{}_x^{-\alpha}f(x):=\frac{1}{\Gamma(\alpha)}\int_0^x f(t)dt,\qquad \mathrm{Re}(\alpha)>0;$$

we can rewrite ${}_{\delta}L_{k,n}^{(\alpha,\beta)}(x,y)$ in the operational representation as

$${}_{\delta}L_{k,n}^{(\alpha,\beta)}(x,y)=\sum_{s=0}^{n}\sum_{r=0}^{n-s}(-nk)_{k,r+s}\,{}_k\hat{D}_x^{-r}\,{}_k\hat{D}_y^{-\delta s}\left\{\frac{x^{\frac{\alpha}{k}}y^{\frac{\beta}{k}}}{\Gamma_k(\alpha+1)\Gamma_k(\beta+1)}\right\}$$

which further yields the Rodrigues-type relation

$$E_{\alpha,\beta,\delta}^{(\gamma)}(x,y)=\left(1-\hat{D}{}_x^{-1}\hat{D}{}_y^{-\delta}\right)^{-\gamma}\left\{\frac{x^{\alpha-1}y^{\beta-1}}{\Gamma_k(\alpha)\Gamma_k(\beta)}\right\}.$$

We also motivate by the above results and define the following extended type k-Mittag-Leffler function of two variables:

$$E_{k,\alpha,\beta,\delta}^{(\gamma)}(x,y)=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{r+s,k}}{\Gamma_k(\alpha+r)\Gamma_k(\beta+\delta s)}\frac{x^r}{r!}\frac{y^{\delta s}}{s!} \tag{1.7}$$
$$\left(k\in\mathbb{R}^+,\ \alpha,\beta,\gamma\in\mathbb{C},\ \mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\delta)>0\right).$$

Remark 2. According to the convergence conditions investigated by Srivastava and Daoust ([20], pp. 155) for the generalized Lauricella series in two variables, the series in (1.5) converges absolutely for $\mathrm{Re}(\delta) > 0$.

Special Cases of $E^{(\gamma)}_{k,\alpha,\beta,\delta}(x,y)$:

(1) For $k = 1$, (1.7) is a particular case of (1.5).

(2) When $x \to 0$ and $r = 0$, we get a k-Mittag-Leffler function of one variable

$$E^{(\gamma)}_{k,\alpha,\beta,\delta}(0,y) = \frac{1}{\Gamma_k(\alpha)} E^{\gamma}_{k,\delta,\beta}(y^{\delta}).$$

Comparing (1.6) and (1.7), we see that

$$_{\delta}L^{(\alpha,\beta)}_{k,n}(x,y) = x^{\alpha}y^{\beta}E^{(-nk)}_{k,\alpha+1,\beta+1,\delta}(x,y). \tag{1.8}$$

This paper is organised as follows: In Section 2, we investigate the Riemann-Liouville double k-fractional integral and derivative of $E^{(\gamma)}_{k,\alpha,\beta,\delta}(x,y)$ and $_{\delta}L^{(\alpha,\beta)}_{k,n}(x,y)$. In Section 3, we obtain the double Laplace transform of $E^{(\gamma)}_{k,\alpha,\beta,\delta}(x,y)$ and $_{\delta}L^{(\alpha,\beta)}_{k,n}(x,y)$. Also, we find the linear generating function for the polynomials $_{\delta}L^{(\alpha,\beta)}_{k,n}(x,y)$ by means of $E^{(\gamma)}_{k,\alpha,\beta,\delta}(x,y)$. In Section 4, we introduce an integral operator and establish its fundamental properties.

2. Fractional calculus approach

In this section, we study the Riemann-Liouville double k-fractional integral and derivative properties of $E^{(\gamma)}_{k,\alpha,\beta,\delta}(x,y)$ and $_{\delta}L^{(\alpha,\beta)}_{k,n}(x,y)$.

Definition 1 (see [16])**.** Let $\alpha \in \mathbb{R}^+$ and $n \in \mathbb{N}$ such that $n-1 < \alpha < n$, $f \in L_1([0,\infty)$. Then the Riemann-Liouville integral of f is

$$I^{\alpha}_{k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_0^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt.$$

Clearly, when $k = 1$ then $I^{\alpha}_{k} f(x)$ reduces to the Riemann-Liouville fractional integration formula (see [11]).

Similarly, k-Riemann-Liouville double fractional integral of $f(x,y)$ is defined as follows:

$$_{k,y}I^{\lambda}_{b^+}\ {}_{k,x}I^{\mu}_{a^+} f(x,y) = \frac{1}{k^2\Gamma_k(\mu)\Gamma_k(\lambda)} \int_b^y \int_a^x (y-\tau)^{\frac{\lambda}{k}-1}(x-t)^{\frac{\mu}{k}-1} f(t,\tau)dtd\tau$$
$$(x > a,\ y > b,\ \mathrm{Re}\,(\lambda) > 0,\ \mathrm{Re}\,(\mu) > 0).$$

Theorem 1. *For* $\alpha,\beta,\gamma,\delta \in \mathbb{C}$, $\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\delta),\mathrm{Re}(\gamma),\mathrm{Re}(\lambda),\mathrm{Re}(\mu) > 0$, *we have*

$$_{k,x}I^{\lambda}_{a^+}\ {}_{k,y}I^{\mu}_{b^+}\left[(x-a)^{\frac{\alpha}{k}-1}(y-b)^{\frac{\beta}{k}-1}E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1(x-a)^{\frac{1}{k}},\omega_2(y-b)^{\frac{1}{k}})\right]$$

$$= \frac{(x-a)^{\frac{\alpha+\lambda}{k}-1}(y-b)^{\frac{\beta+\mu}{k}-1}}{k^2} E^{(\gamma)}_{k,\alpha+\lambda,\beta+\mu,\delta}(\omega_1(x-a)^{\frac{1}{k}}, \omega_2(y-b)^{\frac{1}{k}}).$$

Proof. Interchanging the order of series and fractional integral operators yields

$$\begin{aligned}
&{}_{k,x}I^{\lambda}_{a^+}\, {}_{k,y}I^{\mu}_{b^+}\left[(x-a)^{\frac{\alpha}{k}-1}(y-b)^{\frac{\beta}{k}-1}E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1(x-a)^{\frac{1}{k}}, \omega_2(y-b)^{\frac{1}{k}})\right] \\
&= \int_a^x \int_b^y (x-t)^{\frac{\lambda}{k}-1}(t-a)^{\frac{\alpha}{k}-1}(y-\tau)^{\frac{\mu}{k}-1}(\tau-b)^{\frac{\beta}{k}-1}E^{(\gamma)}_{k,\alpha,\beta,\delta} \\
&\quad (\omega_1(t-a)^{\frac{1}{k}}, \omega_2(\tau-b)^{\frac{1}{k}})d\tau dt \\
&= \frac{1}{k\Gamma_k(\lambda)}\frac{1}{k\Gamma_k(\mu)}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{r+s}}{\Gamma_k(\alpha+r)\Gamma_k(\beta+\delta s)}\frac{\omega_1^r}{r!}\frac{\omega_2^{\delta s}}{s!}\times \\
&\quad \int_a^x (x-t)^{\frac{\lambda}{k}-1}(t-a)^{\frac{\alpha}{k}+\frac{r}{k}-1}dt \int_b^y (y-\tau)^{\frac{\mu}{k}-1}(\tau-b)^{\frac{\beta}{k}+\frac{\delta s}{k}-1}d\tau \\
&= \frac{(x-a)^{\frac{\alpha+\lambda}{k}-1}}{k}\frac{(y-b)^{\frac{\beta+\mu}{k}-1}}{k}\times \\
&\quad \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{k,r+s}}{\Gamma_k(\alpha+\lambda+r)\Gamma_k(\beta+\mu+\delta s)}\frac{\left[\omega_1(x-a)^{\frac{1}{k}}\right]}{r!}\frac{\left[\omega_2(y-b)^{\frac{1}{k}}\right]^{\delta s}}{s!} \\
&= \frac{(x-a)^{\frac{\alpha+\lambda}{k}-1}}{k}\frac{(y-b)^{\frac{\beta+\mu}{k}-1}}{k}E^{(\gamma)}_{k,\alpha+\lambda,\beta+\mu,\delta}(\omega_1(x-a)^{\frac{1}{k}}, \omega_2(y-b)^{\frac{1}{k}}),
\end{aligned}$$

which completes the desired proof. □

Corollary 1. *As a consequence of* (1.8) *and Theorem 1, we have*

$${}_{k,x}I^{\lambda}_{a^+}\, {}_{k,y}I^{\mu}_{b^+}\left[{}_{\delta}L^{(\alpha,\beta)}_{k,n}(\omega_1(x-a), \omega_2(y-b))\right] = {}_{\delta}L^{(\alpha+\lambda,\beta+\mu)}_{k,n}(\omega_1(x-a), \omega_2(y-b)),$$

where $\alpha,\beta,\gamma,\delta \in \mathbb{C}$, $\mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(\delta), \mathrm{Re}(\gamma), \mathrm{Re}(\lambda), \mathrm{Re}(\mu) > 0$.

Definition 2 (see [7])**.** Let $k, \alpha \in \mathbb{R}^+$ and $n \in \mathbb{N}$ such that $n-1 < \alpha < n$, $f \in L_1([0,\infty)$. Then the Riemann-Liouville derivative of f is

$$D^{\alpha}_k f(x) = \left(\frac{d}{dx}\right)^n k^n\, {}_{k,x}I^{nk-\alpha}_{a^+} f(x).$$

Obviously, if $k = 1$ then $D^{\alpha}_k f(x)$ coincides with the Riemann-Liouville fractional derivative (see [11]).

Similarly, k-Riemann-Liouville double fractional derivative of $f(x,y)$ is defined as follows:

$${}_{k,y}D^{\lambda}_{b^+}\, {}_{k,x}D^{\mu}_{a^+} f(x,y) = \left(\frac{\partial}{\partial x}\right)^n \left(\frac{\partial}{\partial y}\right)^m k^n k^m\, {}_{k,x}I^{nk-\alpha}_{a^+}\, {}_{k,y}I^{mk-\mu}_{b^+} f(x,y)$$

$$(x > a, qy > b,\ \mathrm{Re}\,(\lambda) > 0,\ \mathrm{Re}\,(\mu) > 0).$$

Theorem 2. *For* $\alpha,\beta,\gamma,\delta \in \mathbb{C},\ \mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\delta),\mathrm{Re}(\gamma),\mathrm{Re}(\lambda),\mathrm{Re}(\mu) > 0$*, we have*

$$\begin{aligned}
&{}_{k,x}D^{\lambda}_{a^+}\ {}_{k,y}D^{\mu}_{b^+}\left[(x-a)^{\frac{\alpha}{k}-1}(y-b)^{\frac{\beta}{k}-1}E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1(x-a)^{\frac{1}{k}},\omega_2(y-b)^{\frac{1}{k}})\right]\\
&= (x-a)^{\frac{\alpha-\lambda}{k}-1}(y-b)^{\frac{\beta-\mu}{k}-1}E^{(\gamma)}_{k,\alpha-\lambda,\beta-\mu,\delta}(\omega_1(x-a)^{\frac{1}{k}},\omega_2(y-b)^{\frac{1}{k}}).
\end{aligned}$$

Proof. Interchanging the order of series and fractional integral operators yields

$$\begin{aligned}
&{}_{k,x}D^{\lambda}_{a^+}\ {}_{k,y}D^{\mu}_{b^+}\left[(x-a)^{\frac{\alpha}{k}-1}(y-b)^{\frac{\beta}{k}-1}E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1(x-a)^{\frac{1}{k}},\omega_2(y-b)^{\frac{1}{k}})\right]\\
&= \left(\frac{\partial}{\partial x}\right)^n\left(\frac{\partial}{\partial y}\right)^m k^n k^m\\
&=_{k,n} I^{nk-\lambda}_{a^+}\ {}_{k,m}I^{mk-\mu}_{b^+}\left[(x-a)^{\frac{\alpha}{k}-1}(y-b)^{\frac{\beta}{k}-1}E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1(x-a)^{\frac{1}{k}},\omega_2(y-b)^{\frac{1}{k}})\right]\\
&= \left(\frac{\partial}{\partial x}\right)^n\left(\frac{\partial}{\partial y}\right)^m k^n k^m \frac{1}{k\Gamma_k(nk-\lambda)}\frac{1}{k\Gamma_k(mk-\mu)}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{k,r+s}\omega_1^r\omega_2^{\delta s}}{r!s!\Gamma_k(\alpha+r)\Gamma_k(\beta+\delta s)}\\
&\quad\times\int_a^x (x-t)^{\frac{nk-\lambda}{k}-1}(t-a)^{\frac{\alpha+r}{k}-1}dt\int_b^x (y-\tau)^{\frac{mk-\mu}{k}-1}(\tau-b)^{\frac{\beta+\delta s}{k}-1}d\tau\\
&= \left(\frac{\partial}{\partial x}\right)^n\left(\frac{\partial}{\partial y}\right)^m k^n k^m \frac{1}{k\Gamma_k(nk-\lambda)}\frac{1}{k\Gamma_k(mk-\mu)}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{k,r+s}\omega_1^r\omega_2^{\delta s}}{r!s!\Gamma_k(\alpha+r)\Gamma_k(\beta+\delta s)}\\
&\quad\times\frac{\Gamma_h(nk-\lambda)\Gamma_k(\alpha+r)}{\Gamma_k(nk-\lambda+\alpha+r)}\frac{\Gamma_k(mk-\mu)\Gamma_k(\beta+\delta s)}{\Gamma_k(mk-\mu+\beta+\delta s)}\\
&\quad\times(x-a)^{\frac{nk-\lambda+\alpha}{k}+\frac{r}{k}-1}(y-b)^{\frac{mk-\mu+\beta}{k}+\frac{\delta s}{k}-1}
\end{aligned}$$

Since $\Gamma_k(nk-\lambda+\alpha+r) = k^{\frac{nk-\lambda+\alpha+r}{k}-1}\Gamma(\frac{nk-\lambda+\alpha+r}{k})$ and $\Gamma_k(mk-\mu+\beta+\delta s) = k^{\frac{mk-\mu+\beta+\delta s}{k}-1}\Gamma(\frac{mk-\mu+\beta+\delta s}{k})$, we have

$$\begin{aligned}
&{}_{k,x}D^{\lambda}_{a^+}\ {}_{k,y}D^{\mu}_{b^+}\left[(x-a)^{\frac{\alpha}{k}-1}(y-b)^{\frac{\beta}{k}-1}E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1(x-a)^{\frac{1}{k}},\omega_2(y-b)^{\frac{1}{k}})\right]\\
&= (x-a)^{\frac{\alpha-\lambda}{k}-1}(y-b)^{\frac{\beta-\mu}{k}-1}\\
&\quad\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{k,r+s}\omega_1^r\omega_2^{\delta s}}{\Gamma_k(\alpha-\lambda+r)\Gamma_k(\beta-\mu+\delta s)}\frac{(x-a)^{\frac{r}{k}-1}}{r!}\frac{(y-b)^{\frac{\delta s}{k}-1}}{s!}\\
&= (x-a)^{\frac{\alpha-\lambda}{k}-1}(y-b)^{\frac{\beta-\mu}{k}-1}E^{(\gamma)}_{k,\alpha-\lambda,\beta-\mu,\delta}(\omega_1(x-a)^{\frac{1}{k}},\omega_2(y-b)^{\frac{1}{k}}).
\end{aligned}$$

Hence, the result is achieved. □

Corollary 2. *As a consequence of* (1.8) *and Theorem 2, we have*

$$\begin{aligned} &{}_{k,x}D^{\lambda}_{a^+}\,{}_{k,y}D^{\mu}_{b^+}\left[{}_{\delta}L^{(\alpha,\beta)}_{k,n}(\omega_1(x-a)^{\frac{1}{k}},\omega_2(y-b)^{\frac{1}{k}})\right] \\ &= {}_{\delta}L^{(\alpha-\lambda,\beta-\mu)}_{k,n}(\omega_1(x-a)^{\frac{1}{k}},\omega_2(y-b)^{\frac{1}{k}}), \end{aligned}$$

where $\alpha,\beta,\gamma,\delta\in\mathbb{C},\ \mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\delta),\mathrm{Re}(\gamma),\mathrm{Re}(\lambda),\mathrm{Re}(\mu)>0.$

3. Laplace transform and generating function

In this section, we obtain the double Laplace transform of $E^{(\gamma)}_{k,\alpha,\beta,\delta}(x,y)$ and ${}_{\delta}L^{(\alpha,\beta)}_{k,n}(x,y)$. Moreover, we derive the linear generating function for the polynomials ${}_{\delta}L^{(\alpha,\beta)}_{k,n}(x,y)$ in terms of $E^{(\gamma)}_{k,\alpha,\beta,\delta}(x,y)$. The double Laplace transform (see [1]) of $f(x,y)$ is defined as

$$\mathbb{L}_2[f(x,y)](p,q)=\int_0^\infty\int_0^\infty e^{-(px+qy)}f(x,y)dxdy \qquad (\mathrm{Re}(p)>0,\ \mathrm{Re}(q)>0).$$

Theorem 3. *Let* $\alpha,\beta,\gamma,\delta\in\mathbb{C},\ \mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\delta),\mathrm{Re}(\gamma),\mathrm{Re}(p),\mathrm{Re}(q)>0,$ $\left|\frac{\omega_2^{\delta}}{q^{\delta}}\right|<1$ *and* $\left|\frac{\omega_1 q^{\delta}}{p(q^{\delta}-\omega_2^{\delta})}\right|<1$. *Then we have*

$$\begin{aligned} &\mathbb{L}_2\left(x^{\frac{\alpha}{k}-1}y^{\frac{\beta}{k}-1}E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1x^{\frac{1}{k}},\omega_2y^{\frac{1}{k}})\right)(p,q) \\ &= p^{-\frac{\alpha}{k}}q^{-\frac{\beta}{k}}k^2k^{-\frac{\alpha}{k}}k^{-\frac{\beta}{k}}\left(1-\frac{k\omega_2^{\delta}(pk)^{\frac{1}{k}}+\omega_1(qk)^{\frac{\delta}{k}}}{(pk)^{\frac{1}{k}}(qk)^{\frac{\delta}{k}}}\right)^{-\frac{\gamma}{k}}. \end{aligned}$$

Proof. Interchanging the order of series and fractional integral operators yields

$$\begin{aligned} &\mathbb{L}_2\left(x^{\frac{\alpha}{k}-1}y^{\frac{\beta}{k}-1}E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1x^{\frac{1}{k}},\omega_2y^{\frac{1}{k}})\right)(p,q) \\ &=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{k,r+s}\omega_1^r\omega_2^{\delta s}}{\Gamma_k(\alpha+r)\Gamma_k(\beta+\delta s)r!s!}\int_0^\infty e^{-px}x^{\frac{\alpha}{k}+\frac{r}{k}-1}dx\int_0^\infty e^{-qy}y^{\frac{\beta}{h}+\frac{\delta s}{k}-1}dy \\ &=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{k,r+s}}{\Gamma_k(\alpha+r)\Gamma_k(\beta+\delta s)r!s!}\frac{\omega_1^r}{p^{\frac{\alpha+r}{k}}}\frac{\omega_2^{\delta s}}{q^{\frac{\beta+\delta s}{k}}}\int_0^\infty e^{-u}u^{\frac{\alpha+r}{k}-1}du\int_0^\infty e^{-v}v^{\frac{\beta+\delta s}{k}-1}dv \\ &=\frac{1}{p^{\frac{\alpha}{k}}}\frac{1}{q^{\frac{\beta}{h}}}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{k,r+s}}{\Gamma_h(\alpha+r)\Gamma_h(\beta+\delta s)r!s!}\Gamma\left(\frac{\alpha+r}{h}\right)\Gamma\left(\frac{\beta+\delta s}{k}\right)\left(\frac{\omega_1}{p}\right)^{\frac{r}{k}}\left(\frac{\omega_2}{q}\right)^{\frac{\delta s}{k}} \end{aligned}$$

Since $\frac{\Gamma\left(\frac{\alpha+r}{k}\right)}{\Gamma_k(\alpha+r)}=\frac{1}{k^{\frac{\alpha+r}{k}-1}}$ and $\frac{\Gamma\left(\frac{\beta+\delta s}{k}\right)}{\Gamma_k(\beta+\delta s)}=\frac{1}{k^{\frac{\beta+\delta s}{k}-1}}$, it follows

$$\mathbb{L}_2\left(x^{\frac{\alpha}{k}-1}y^{\frac{\beta}{k}-1}E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1x^{\frac{1}{k}},\omega_2y^{\frac{1}{k}})\right)(p,q)$$

$$
\begin{aligned}
&= \frac{1}{p^{\frac{\alpha}{h}}}\frac{1}{q^{\frac{\beta}{k}}}k^2\frac{1}{k^{\frac{\alpha}{k}}}\frac{1}{k^{\frac{\beta}{k}}}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\gamma)_{k,r+s}\omega_1^r\omega_2^{\delta s}}{r!s!\left(k^{\frac{1}{k}}\right)^r\left(k^{\frac{\delta}{k}}\right)^s}\left(\frac{1}{p^{\frac{1}{k}}}\right)^r\left(\frac{1}{q^{\frac{\delta}{k}}}\right)^s\\
&= \frac{1}{p^{\frac{\alpha}{k}}}\frac{1}{q^{\frac{\beta}{k}}}k^2\frac{1}{k^{\frac{\alpha}{k}}}\frac{1}{k^{\frac{\beta}{k}}}\sum_{r=0}^{\infty}\frac{(\gamma)_{k,r}\omega_1^r}{r!\left(k^{\frac{1}{k}}\right)^r}\left(\frac{1}{p^{\frac{1}{h}}}\right)^r\sum_{s=0}^{\infty}\frac{(\gamma+kr)_{h,s}\omega_2^{\delta s}}{s!\left(k^{\frac{\delta}{k}}\right)^s}\left(\frac{1}{q^{\frac{\delta}{k}}}\right)^s\\
&= \frac{1}{p^{\frac{\alpha}{k}}}\frac{1}{q^{\frac{\beta}{k}}}k^2\frac{1}{k^{\frac{\alpha}{k}}}\frac{1}{k^{\frac{\beta}{k}}}\sum_{r=0}^{\infty}\frac{(\gamma)_{k,r}\omega_1^r}{r!\left(k^{\frac{1}{k}}\right)^r}\left(\frac{1}{p^{\frac{1}{k}}}\right)^r\left(1-\frac{k\omega_2^{\delta}}{(qk)^{\frac{\delta}{k}}}\right)^{-\left(\frac{\gamma+kr}{k}\right)}\\
&= \frac{1}{p^{\frac{\alpha}{k}}}\frac{1}{q^{\frac{\beta}{k}}}k^2\frac{1}{k^{\frac{\alpha}{k}}}\frac{1}{k^{\frac{\beta}{k}}}\left(1-\frac{k\omega_2^{\delta}}{(qk)^{\frac{\delta}{k}}}\right)^{-\frac{\gamma}{k}}\sum_{r=0}^{\infty}\frac{(\gamma)_{k,r}\omega_1^r}{r!\left(k^{\frac{1}{k}}\right)^r}\left(1-\frac{k\omega_2^{\delta}}{(qk)^{\frac{\delta}{k}}}\right)^{-r}\\
&= \frac{1}{p^{\frac{\alpha}{k}}}\frac{1}{q^{\frac{\beta}{k}}}k^2\frac{1}{k^{\frac{\alpha}{k}}}\frac{1}{k^{\frac{\beta}{k}}}\left(1-\frac{k\omega_2^{\delta}}{(qk)^{\frac{\delta}{k}}}\right)^{-\frac{\gamma}{k}}\left(1-\frac{\omega_1(qk)^{\frac{\delta}{k}}}{(pk)^{\frac{1}{k}}\left((qk)^{\frac{\delta}{k}}-k\omega_2^{\delta}\right)}\right)^{-\frac{\gamma}{k}}\\
&= p^{-\frac{\alpha}{k}}q^{-\frac{\beta}{k}}k^2k^{-\frac{\alpha}{k}}k^{-\frac{\beta}{k}}\left(1-\frac{k\omega_2^{\delta}(pk)^{\frac{1}{k}}+\omega_1(qk)^{\frac{\delta}{k}}}{(pk)^{\frac{1}{k}}(qk)^{\frac{\delta}{k}}}\right)^{-\frac{\gamma}{k}}.
\end{aligned}
$$

Whence the result. □

Corollary 3. *For the polynomials* ${}_{\delta}L_{k,n}^{(\alpha,\beta)}(x,y)$*, we have*

$$
\mathbb{L}_2\left({}_{\delta}L_{k,n}^{(\alpha,\beta)}(\omega_1x,\omega_2y)\right)(p,q)=\frac{\omega_1^{\alpha}}{p^{\alpha+1}}\frac{\omega_2^{\beta}}{q^{\beta+1}}\left(\frac{pq^{\delta}-(\omega_2^{\delta}p+\omega_1q^{\delta})}{pq)^{\delta}}\right)^n.
$$

In the next theorem, we give the linear generating function for the polynomials ${}_{\delta}L_{k,n}^{(\alpha,\beta)}(x,y)$ in terms of the bivariate k-Mittag-Leffler function $E_{k,\alpha,\beta,\delta}^{(\gamma)}(x,y)$.

Theorem 4. *For* $\alpha,\beta,\sigma\in\mathbb{C}$ *and* $|kt|<1$*, we have*

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(\sigma)_{k,n}}{n!}\,{}_{\delta}L_{k,n}^{(\alpha,\beta)}(x,y)t^n\\
&=x^{\alpha}y^{\beta}(1-kt)^{-\frac{\sigma}{k}}E_{k,\alpha+1,\beta+1,\delta}^{(\sigma)}\left(\frac{xt}{kt-1},y\left(\frac{t}{kt-1}\right)^{\frac{1}{\delta}}\right).
\end{aligned}
\tag{3.1}
$$

When $k=1$, (3.1) *reduces to Theorem 22 in [18].*

Proof. Using the Cauchy product of the series, we get

$$
\sum_{n=0}^{\infty}\frac{(\sigma)_{k,n}}{n!}\,{}_{\delta}L_{k,n}^{(\alpha,\beta)}(x,y)t^n
$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty}\sum_{s=0}^{n}\sum_{r=0}^{n-s} \frac{(\sigma)_{k,n}(-1)^{s+r}k^{s+r}x^{\alpha+r}y^{\beta+\delta s}}{s!r!(n-s-r)!\Gamma_k(\alpha+r+1)\Gamma_k(\beta+\delta s+1)}t^n \\
&= \sum_{n=0}^{\infty}\sum_{s=0}^{\infty}\sum_{r=0}^{n} \frac{(\sigma)_{k,n+s}(-1)^{s+r}k^{s+r}x^{\alpha+r}y^{\beta+\delta s}}{s!r!(n-r)!\Gamma_k(\alpha+r+1)\Gamma_k(\beta+\delta s+1)}t^{n+s} \\
&= \sum_{n=0}^{\infty}\sum_{s=0}^{\infty}\sum_{r=0}^{\infty} \frac{(\sigma)_{k,n+r+s}(-1)^{s+r}k^{s+r}x^{\alpha+r}y^{\beta+\delta s}}{s!r!\Gamma_k(\alpha+r+1)\Gamma_k(\beta+\delta s+1)}\frac{t^{n+r+s}}{n!}.
\end{aligned}
$$

From $(\sigma)_{k,n+r+s} = (\sigma)_{k,r+s}(\sigma+rk+k)_{k,n}$, we get

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(\sigma)_n}{n!}\ {}_{\delta}L_{k,n}^{(\alpha,\beta)}(x,y)t^n \\
&= \sum_{s=0}^{\infty}\sum_{r=0}^{\infty} \frac{(-1)^{s+r}k^{s+r}(\sigma)_{delta,r+s}x^{\alpha+r}y^{\beta+\delta s}t^{s+r}}{s!r!\Gamma_k(\alpha+r+1)\Gamma_k(\beta+\delta s+1)}\sum_{n=0}^{\infty}(\sigma+r+s)_{k,n}\frac{t^n}{n!} \\
&= x^{\alpha}y^{\beta}(1-kt)^{-\frac{\sigma}{k}} \\
&\quad\times \sum_{s=0}^{\infty}\sum_{r=0}^{\infty} \frac{(-1)^{s+r}k^{s+r}(\sigma)_{k,r+s}}{s!r!\Gamma_k(\alpha+r+1)\Gamma_k(\beta+\delta s+1)}\left(\left(\frac{xt}{kt-1}\right)^r\left(\frac{y^{\delta}t}{kt-1}\right)^s\right) \\
&= x^{\alpha}y^{\beta}(1-kt)^{-\frac{\sigma}{k}}E_{k,\alpha+1,\beta+1,\delta}^{(\sigma)}\left(\frac{xt}{kt-1}, y\left(\frac{t}{kt-1}\right)^{\frac{1}{\delta}}\right),
\end{aligned}
$$

where we have interchanged the order of summations which is guaranteed because of the uniform convergence of the series under the condition $|kt| < 1$. □

4. k-FRACTIONAL CALCULUS OPERATORS

In this section, we consider the following double (fractional) k-integral operator

$$
\begin{aligned}
&\left({}_k\varepsilon_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}^{(\gamma)}\varphi\right)(x,y) \\
&= \int_c^y\int_a^x \frac{(x-t)^{\frac{\alpha}{k}-1}}{k}\frac{(y-\tau)^{\frac{\beta}{k}-1}}{k}E_{k,\alpha,\beta,\delta}^{(\gamma)}[\omega_1(x-t),\omega_2(y-\tau)]\times\varphi(t,\tau)dtd\tau \\
&(x>a,\ y>c).
\end{aligned}
$$

In the case $\gamma = 0$, the integral operator ${}_k\varepsilon_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}^{(\gamma)}$ reduces to the left-sided k-Riemann-Liouville double fractional integral operator, i.e.

$$
\left({}_h\varepsilon_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}^{(0)}\varphi\right)(x,y) = \left({}_{h,y}I_{c^+}^{\beta}\ {}_{h,x}I_{a^+}^{\alpha}\varphi\right)(x,y).
$$

In the following theorem, we show the integral operator ${}_k\varepsilon^{(\gamma)}_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}$ is bounded in the space $L((a,b)\times(c,d))$ of Lebesgue measurable functions where

$$L((a,b)\times(c,d)) = \left\{ \|f\|_1 := \int_a^b \int_c^d |f(x,y)|\,dydx < \infty \right\}$$

Theorem 5. *In the space $L((a,b)\times(c,d))$ of Lebesgue measurable functions, we show the integral operator ${}_k\varepsilon^{(\gamma)}_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}$ is bounded, i.e.*

$$\left\| {}_k\varepsilon^{(\gamma)}_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}\varphi \right\|_1 \leq A\,\|\varphi\|_1$$

where the constant A $(0<A<\infty)$ is given by

$$\begin{aligned} A = {} & \frac{(b-a)^{\mathrm{Re}(\frac{\alpha}{k})}(d-c)^{\mathrm{Re}(\frac{\beta}{k})}}{k^2} \\ & \times \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{|(\gamma)_{k,r+s}|}{\left\{\mathrm{Re}(\frac{\alpha}{k})+r\right\}|\Gamma_k(\alpha+r)|\left\{\mathrm{Re}(\frac{\beta}{k})+\delta s\right\}|\Gamma_k(\beta+\delta s)|} \\ & \times \frac{|\omega_1(b-a)|^r}{r!}\frac{|\omega_2(d-c)|^{\delta s}}{s!} < \infty. \end{aligned}$$

Proof. By using the Fubini's Theorem, we get

$$\begin{aligned} & \left\| {}_k\varepsilon^{(\gamma)}_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}\varphi \right\|_1 \leq \\ & \int_a^b\int_c^d |\varphi(t,\tau)| \times \left(\int_t^b\int_\tau^d \frac{(x-t)^{\mathrm{Re}(\frac{\alpha}{k})-1}}{k}\frac{(y-\tau)^{\mathrm{Re}(\frac{\beta}{k})-1}}{k} \right. \\ & \left. \left| E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1(x-t),\omega_2(y-\tau)) \right| dydx \right) d\tau dt \\ = {} & \int_a^b\int_c^d |\varphi(t,\tau)| \left(\int_0^{b-t}\int_0^{d-\tau} \frac{u^{\mathrm{Re}(\frac{\alpha}{k})-1}}{k}\frac{v^{\mathrm{Re}(\frac{\beta}{k})-1}}{k} \left| E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1 u,\omega_2 v) \right| dudv \right) d\tau dt \\ \leq {} & \int_a^b\int_c^d |\varphi(t,\tau)| \left(\int_0^{b-a}\int_0^{d-c} \frac{u^{\mathrm{Re}(\frac{\alpha}{k})-1}}{k}\frac{v^{\mathrm{Re}(\frac{\beta}{k})-1}}{k} \left| E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1 u,\omega_2 v) \right| dudv \right) d\tau dt \\ \leq {} & \frac{1}{k^2}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{|(\gamma)_{k,r+s}|}{|\Gamma_k(\alpha+r)|\,|\Gamma_k(\beta+\delta s)|}\frac{|\omega_1|^r}{r!}\frac{|\omega_2|^{\delta s}}{s!} \\ & \int_0^{b-a} u^{\mathrm{Re}(\frac{\alpha}{h})+r-1}du \int_0^{d-c} v^{\mathrm{Re}(\frac{\beta}{k})+\delta s-1}dv\,\|\varphi\|_1 = A\,\|\varphi\|_1 . \end{aligned}$$

Whence the result. □

In the next theorem, we calculate double integrals which consist the product of bivariate k-Mittag-Leffler functions $E^{(\gamma)}_{k,\alpha,\beta,\delta}(x,y)$ in the integrand.

Theorem 6. *Let* $\alpha,\beta,\delta,\zeta,\sigma,\gamma,\eta\in\mathbb{C}, \mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\delta),\mathrm{Re}(\gamma),\mathrm{Re}(\sigma)>0$. *Then*

$$\int_0^x\int_0^y (x-t)^{\frac{\alpha}{k}-1}(y-\tau)^{\frac{\beta}{k}-1}E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1(x-t)^{\frac{1}{k}},\omega_2(y-\tau)^{\frac{1}{k}})t^{\frac{\zeta}{k}-1}\tau^{\frac{\sigma}{k}-1}$$
$$\times E^{(\eta)}_{k,\zeta,\sigma,\delta}(\omega_1 t^{\frac{1}{k}},\omega_2\tau^{\frac{1}{k}})d\tau dt=\frac{1}{k^2}x^{\frac{\alpha+\zeta}{k}-1}y^{\frac{\beta+\sigma}{k}-1}E^{(\gamma+\eta)}_{k,\alpha+\zeta,\beta+\sigma,\delta}(\omega_1 x^{\frac{1}{k}},\omega_2 y^{\frac{1}{k}}).$$

Proof. Using the double convolution theorem for the double Laplace transform, we have,

$$\begin{aligned}
&\mathbb{L}_2\left\{\int_0^x\int_0^y (x-t)^{\frac{\alpha}{k}-1}(y-\tau)^{\frac{\beta}{k}-1}E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1(x-t)^{\frac{1}{k}},\omega_2(y-\tau)^{\frac{1}{k}})\times\right.\\
&\qquad\left. t^{\frac{\zeta}{k}-1}\tau^{\frac{\sigma}{k}-1}E^{(\eta)}_{h,\zeta,\sigma,\delta}(\omega_1 t^{\frac{1}{k}},\omega_2\tau^{\frac{1}{k}})d\tau dt\right\}(p,q)\\
&=\mathbb{L}_2\left\{x^{\frac{\alpha}{k}-1}y^{\frac{\beta}{k}-1}E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1 x^{\frac{1}{k}},\omega_2 y^{\frac{1}{k}})\right\}(p,q)\\
&\quad\times\mathbb{L}_2\left\{x^{\frac{\zeta}{k}-1}y^{\frac{\sigma}{k}-1}E^{(\eta)}_{k,\zeta,\sigma,\delta}(\omega_1 x^{\frac{1}{k}},\omega_2 y^{\frac{1}{k}})\right\}(p,q).
\end{aligned}$$

From Theorem 3, we have for $\mathrm{Re}(p),\mathrm{Re}(q)>0$, $\left|\frac{k\omega_2^\delta}{(qk)^{\frac{\delta}{k}}}\right|<1$, $\left|\frac{\omega_1(qk)^{\frac{\delta}{k}}}{(pk)^{\frac{1}{k}}\left(qk)^{\frac{\delta}{k}}-k\omega_2^\delta\right)}\right|<1$,

$$\begin{aligned}
&\mathbb{L}_2\left\{\int_0^x\int_0^y (x-t)^{\frac{\alpha}{k}-1}(y-\tau)^{\frac{\beta}{k}-1}E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1(x-t)^{\frac{1}{k}},\omega_2(y-\tau)^{\frac{1}{k}})\times\right.\\
&\qquad\left. t^{\frac{\zeta}{k}-1}\tau^{\frac{\sigma}{k}-1}E^{(\eta)}_{k,\zeta,\sigma,\delta}(\omega_1 t^{\frac{1}{k}},\omega_2\tau^{\frac{1}{k}})d\tau dt\right\}(p,q)\\
&=\mathbb{L}_2\left\{x^{\frac{\alpha}{k}-1}y^{\frac{\beta}{k}-1}E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1 x^{\frac{1}{k}},\omega_2 y^{\frac{1}{k}})\right\}(p,q)\\
&\quad\times\mathbb{L}_2\left\{x^{\frac{\zeta}{k}-1}y^{\frac{\sigma}{k}-1}E^{(\eta)}_{k,\zeta,\sigma,\delta}(\omega_1 x^{\frac{1}{k}},\omega_2 y^{\frac{1}{k}})\right\}(p,q).\\
&=\frac{1}{p^{\frac{\alpha}{k}}}\frac{1}{q^{\frac{\beta}{k}}}k^2\frac{1}{k^{\frac{\alpha}{k}}}\frac{1}{k^{\frac{\beta}{k}}}\left(1-\frac{k\omega_2^\delta(pk)^{\frac{1}{k}}+\omega_1(qk)^{\frac{\delta}{k}}}{(pk)^{\frac{1}{k}}(qk)^{\frac{\delta}{k}}}\right)^{-\frac{\gamma}{k}}\\
&\quad\times\frac{1}{p^{\frac{\zeta}{k}}}\frac{1}{q^{\frac{\sigma}{k}}}k^2\frac{1}{k^{\frac{\zeta}{k}}}\frac{1}{k^{\frac{\sigma}{k}}}\left(1-\frac{k\omega_2^\delta(pk)^{\frac{1}{k}}+\omega_1(qk)^{\frac{\delta}{k}}}{(pk)^{\frac{1}{k}}(qk)^{\frac{\delta}{k}}}\right)^{-\frac{\eta}{k}}
\end{aligned}\tag{4.1}$$

$$= \frac{1}{p^{\frac{\alpha+\zeta}{k}}}\frac{1}{q^{\frac{\beta+\sigma}{k}}}k^4\frac{1}{k^{\frac{\alpha+\zeta}{k}}}\frac{1}{k^{\frac{\beta+\sigma}{k}}}\left(1-\frac{k\omega_2^{\delta}(pk)^{\frac{1}{k}}+\omega_1(qk)^{\frac{\delta}{k}}}{(pk)^{\frac{1}{k}}(qk)^{\frac{\delta}{k}}}\right)^{-(\frac{\gamma+\eta}{k})}$$
$$= \mathbb{L}_2\left\{x^{\frac{\alpha+\zeta}{k}-1}y^{\frac{\beta+\sigma}{k}-1}E^{(\gamma+\eta)}_{k,\alpha+\zeta,\beta+\sigma,\delta}(\omega_1 x^{\frac{1}{k}},\omega_2 y^{\frac{1}{k}})\right\}(p,q).$$

Taking inverse Laplace on both sides of (4.1), we get

$$\int_0^x\int_0^y (x-t)^{\frac{\alpha}{k}-1}(y-\tau)^{\frac{\beta}{k}-1}E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1(x-t)^{\frac{1}{k}},\omega_2(y-\tau)^{\frac{1}{k}})\times$$
$$t^{\frac{\zeta}{k}-1}\tau^{\frac{\sigma}{k}-1}E^{(\eta)}_{h,\zeta,\sigma,\delta}(\omega_1 t^{\frac{1}{k}},\omega_2\tau^{\frac{1}{k}})d\tau dt$$
$$= \frac{1}{k^2}x^{\frac{\alpha+\zeta}{k}-1}y^{\frac{\beta+\sigma}{k}-1}E^{(\gamma+\eta)}_{k,\alpha+\zeta,\beta+\sigma,\delta}(\omega_1 x^{\frac{1}{k}},\omega_2 y^{\frac{1}{k}}).$$

□

In the following theorem, we state the semigroup property of the operator ${}_k\varepsilon^{(\gamma)}_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}$.

Theorem 7. *Let* $\alpha,\beta,\delta,\gamma,\zeta,\eta,\sigma,\omega_1,\omega_2\in\mathbb{C}$, $\mathrm{Re}(\gamma),\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\sigma),\mathrm{Re}(\zeta)$, $\mathrm{Re}(\eta)$, $\mathrm{Re}(\delta)>0$, *then the relation*

$$\left({}_k\varepsilon^{(\gamma)}_{\alpha,\beta;\omega_1,\omega_2;a^+,c^+}\ {}_k\varepsilon^{(\sigma)}_{\zeta,\eta;\omega_1,\omega_2;a^+,c^+}\varphi\right)(x,y)=\left({}_k\varepsilon^{(\gamma+\sigma)}_{\alpha+\zeta,\beta+\eta;\omega_1,\omega_2;a^+,c^+}\varphi\right)(x,y)$$

is valid for any summable function $\varphi\in L((a,b)\times(c,d))$. *In particular*

$$\left({}_k\varepsilon^{(\gamma)}_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}\ {}_k\varepsilon^{(-\gamma)}_{\zeta,\eta,\delta;\omega_1,\omega_2;a^+,c^+}\varphi\right)(x,y)=\left({}_{k,y}I^{\beta+\eta}_{c^+}\ {}_{k,x}I^{\alpha+\zeta}_{a^+}\varphi\right)(x,y).$$

Proof. Direct calculations yield

$$\left({}_k\varepsilon^{(\gamma)}_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}\ {}_k\varepsilon^{(\sigma)}_{\zeta,\eta,\delta;\omega_1,\omega_2;a^+,c^+}\varphi\right)(x,y)$$
$$= \frac{1}{k^2}\int_c^y\int_a^x (x-t)^{\frac{\alpha}{k}-1}(y-\tau)^{\frac{\beta}{k}-1}E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1(x-t),\omega_2(y-\tau))$$
$$\times{}_k\varepsilon^{(\sigma)}_{(\zeta,\eta,\delta;\omega_1,\omega_2;a^+,c^+)}\varphi(t,\tau)dtd\tau$$
$$= \frac{1}{k^4}\int_c^y\int_a^x\int_c^\tau\int_a^t (x-t)^{\frac{\alpha}{k}-1}(y-\tau)^{\frac{\beta}{k}-1}E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1(x-t),\omega_2(y-\tau))$$
$$\times(t-u)^{\frac{\zeta}{k}-1}(\tau-v)^{\frac{\eta}{k}-1}\times E^{(\sigma)}_{k,\zeta,\eta,\delta}(\omega_1(t-u),\omega_2(\tau-v))\varphi(u,v)du\,dv\,dt\,d\tau$$
$$= \frac{1}{k^4}\int_c^y\int_a^x\int_v^y\int_u^x (x-t)^{\frac{\alpha}{k}-1}(y-\tau)^{\frac{\beta}{k}-1}E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1(x-t),\omega_2(y-\tau))$$

$$\times (t-u)^{\frac{\zeta}{k}-1}(\tau-v)^{\frac{\eta}{k}-1}E^{(\sigma)}_{k,\zeta,\eta,\delta}(\omega_1(t-u),\omega_2(\tau-v))\varphi(u,v)dt\,d\tau\,du\,dv$$

$$= \int_c^y\int_a^x\int_0^{y-v}\int_0^{x-u}(x-m-u)^{\frac{\alpha}{k}-1}(y-n-v)^{\frac{\beta}{k}-1}$$

$$\times E^{(\gamma)}_{k,\alpha,\beta,\delta}(\omega_1(x-m-u),\omega_2(y-n-v))k^{\frac{\zeta}{k}-1}l^{\frac{\eta}{k}-1}$$

$$\times E^{(\sigma)}_{k,\zeta,\eta,\delta}(\omega_1 m,\omega_2 n)\varphi(u,v)dm\,dn\,du\,dv.$$

Now, using Theorem 6, we obtain

$$\left({}_k\varepsilon^{(\gamma)}_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}\,{}_k\varepsilon^{(\sigma)}_{\zeta,\eta,\delta;\omega_1,\omega_2;a^+,c^+}\varphi\right)(x,y)$$

$$= \frac{1}{k^2}\int_c^y\int_a^x(x-u)^{\frac{\alpha+\zeta}{k}-1}(y-v)^{\frac{\beta+\eta}{k}-1}E^{(\gamma+\sigma)}_{k,\alpha+\zeta,\beta+\eta,\delta}(\omega_1(x-u)^{\frac{1}{k}},\omega_2(y-v)^{\frac{1}{k}})\varphi(u,v)dudv$$

$$= \left({}_k\varepsilon^{(\gamma+\sigma)}_{\alpha+\zeta,\beta+\eta,\delta;\omega_1,\omega_2;a^+,c^+}\varphi\right)(x,y),$$

which is the desired result. When $\sigma=-\gamma$, the above theorem yields h-Riemann-Liouville double fractional integral of $\varphi(x,y)$. □

By using the above theorem, we can enable to define an inverse to the fractional integral operator and hence we give the following theorem to find the corresponding fractional derivative operator.

Theorem 8. *Let* $\alpha,\beta,\gamma,\delta,\omega_1,\omega_2\in\mathbb{C}$ *and let* $\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\delta)>0$. *For any* $\zeta,\eta\in\mathbb{C}$ *with* $\mathrm{Re}(\zeta),\mathrm{Re}(\eta)>0$, *then the following left inverse operator of* ${}_k\varepsilon^{(\gamma)}_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}$ *is defined by*

$${}_k\mathcal{D}^{(\gamma)}_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}\,\varphi(x,y) = {}_{k,y}D^{\beta+\eta}_{c^+}\,{}_{k,x}D^{\alpha+\zeta}_{a^+}\,{}_k\varepsilon^{(-\gamma)}_{\zeta,\eta;\omega_1,\omega_2;a^+,c^+}\varphi(x,y).$$

Proof. Consider $f\in L^1((a,b)\times(c,d))$. Then

$$\varphi(x,y) := \left({}_k\varepsilon^{(\gamma)}_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}f\right)(x,y). \tag{4.2}$$

By Theorem 5, the operator ${}_k\varepsilon^{(\gamma)}_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}$ is bounded on the space L^1space. It follows that $\varphi\in L^1$. Now, we contruct the operator ${}_k\varepsilon^{(-\gamma)}_{\zeta,\eta,\delta;\omega_1,\omega_2;a^+,c^+}$ and we apply it to (4.2) and we get

$$\begin{aligned}{}_k\varepsilon^{(-\gamma)}_{\zeta,\eta,\delta;\omega_1,\omega_2;a^+,c^+}\varphi(x,y) &= \left({}_k\varepsilon^{(-\gamma)}_{\zeta,\eta,\delta;\omega_1,\omega_2;a^+,c^+}\,{}_k\varepsilon^{(\gamma)}_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}f\right)(x,y) \\ &= \left({}_{k,y}I^{\beta+\eta}_{c^+}\,{}_{k,x}I^{\alpha+\zeta}_{a^+}f\right)(x,y).\end{aligned} \tag{4.3}$$

Since $\mathrm{Re}(\alpha+\zeta)>0$ and $\mathrm{Re}(\beta+\eta)>0$, by the Theorem 5, ${}_k\varepsilon^{(-\gamma)}_{\zeta,\eta,\delta;\omega_1,\omega_2;a^+,c^+}\in L^1$. Applying the double fractional $k-$derivative on both sides of (4.3), we get

$$\begin{aligned}\left({}_{k,y}D^{\beta+\eta}_{c^+}\,{}_{k,x}D^{\alpha+\zeta}_{a^+}\,{}_k\varepsilon^{(-\gamma)}_{\zeta,\eta,\delta;\omega_1,\omega_2;a^+,c^+}\varphi\right)(x,y) &= \left({}_{k,y}D^{\beta+\eta}_{c^+}\,{}_{k,x}D^{\alpha+\zeta}_{a^+}\,{}_{k,y}I^{\beta+\eta}_{c^+}\,{}_{k,x}I^{\alpha+\zeta}_{a^+}f\right)(x,y)\\ &= f(x,y).\end{aligned}$$

In other words,

$${}_k\mathcal{D}^{(\gamma)}_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}\,\varphi(x,y)=f(x,y).$$

So, we have shown that, ${}_k\mathcal{D}^{(\gamma)}_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}$ is the left-inverse of ${}_k\varepsilon^{(\gamma)}_{\alpha,\beta,\delta;\omega_1,\omega_2;a^+,c^+}$. □

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OPTIMAL FEEDBACK CONTROL OF HILFER FRACTIONAL EVOLUTION INCLUSIONS INVOLVING HISTORY-DEPENDENT OPERATORS

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Abstract. The main purpose of this paper is to study the feedback control systems governed by Hilfer fractional evolution inclusions involving history-dependent operators. We first show a priori estimates of the solutions to the fractional feedback control system. Then, by using the well-known Bohnenblust-Karlin fixed point theorem, we prove an existence theorem for the fractional feedback control system. Finally, we consider an optimal control problem driven by the fractional feedback control system, and establish its solvability.

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1. Introduction

In recent years, with the development of computer technology and scientific computing methods, the theory and application of fractional differential evolution equations have developed rapidly. The fractional differential evolution equations in infinite dimensional spaces have been applied in many fields such as economy, mechanics, physics and so on. The fractional differential evolution equations with Caputo type and Riemann-Liouville type have also been widely studied by a large number of scholars(see [9, 11, 17, 19]). Hilfer type fractional derivatives include both Caputo type and Riemann-Liouville type fractional derivatives, which have been widely studied by a large number of scholars (cf. [2]). For example, Gu-Trujillo [2] studied the existence of mild solutions of Hilfer fractional differential equations and the references therein.

Feedback control is a very important concept in control theory. Feedback control refers to the process of sending the output information of the system back to the input end, comparing with the input information, and using the deviation of the

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two to control. Its characteristic is that the control function affects the state function, and the state function will affect the control function in turn, so as to achieve the effect of feedback. In recent years, feedback control systems have been widely used in many fields, such as spacecraft, robot operation and greenhouse regulation (cf. [5, 10, 20–22]. In [20], Wang et al. considered optimal feedback control problems of fractional evolution equations with Caputo fractional derivatives. In [21, 22], the authors studied the feedback control problems of impulsive fractional evolution equations with Riemann-Liouville type. However, there are few studies on optimal feedback control problems of fractional evolution equations with Hilfer type fractional derivatives. This is one of the motivations of this paper. Based on the above considerations, we first study the existence of feasible pairs of Hilfer fractional evolution inclusion with feedback control of the following form:

$$\begin{cases} D_t^{\nu,\mu}x(t) \in Ax(t) + (\mathcal{R}x)(t) + F(t,x(t)) + B(t,x(t))u(t), & t \in (0,b], \\ u(t) \in U(t,t^{1-\alpha}x(t)), \qquad \text{a.e. } t \in (0,b), \\ I_{0^+}^{(1-\nu)(1-\mu)}x(t)\mid_{t=0} = x_0, \end{cases} \tag{1.1}$$

where $D_t^{\nu,\mu}$ denotes the Hilfer fractional derivative, $\nu \in [0,1]$, $\mu \in (0,1)$.

Let $\alpha = \nu + \mu - \nu\mu$, then $1-\alpha = (1-\nu)(1-\mu) \geq 0$. A: $D(A) \subseteq X \to X$ is the infinitesimal generator of a uniformly bounded C_0-semigroup $\{T(t)\}_{t\geq 0}$ on a reflexive Banach space X. $\mathcal{R}$ is a history-dependent operator, F: $[0,b] \times X \to \mathcal{P}(X)$, B: $[0,b] \times X \to \mathcal{L}(V,X)$, where $\mathcal{L}(V,X)$ represents the space of all bounded linear operators from V to X with the standard norm $\|\cdot\|_{\mathcal{L}(V,X)}$, $U : [0,b] \times X \rightrightarrows V$ is a feedback multifunction.

An outline of this paper is organized as follows. In Section 2, we will present some basic definitions and preliminary facts which will be used throughout the following sections. In Section 3, we present a priori estimates of the solutions to the fractional feedback control system and by using the well-known Bohnenblust-Karlin fixed point theorem, we prove an existence result of feasible pairs of the system (1.1). In Section 4, we will study an optimal control problem driven by the feedback control system and establish its solvability.

2. Preliminaries

In order to study the feedback control systems of Hilfer fractional evolution inclusions involving history-dependent operators, we introduce the following basic definitions and preparatory knowledge. The norm of a Banach space X will be denoted by $\|\cdot\|_X$. In the sequel, we assume that V is a separable reflexive Banach space. For a uniformly bounded C_0-semigroup $T(t)(t \geq 0)$, we set $M := \sup_{t\in[0,\infty)} \|T(t)\|_{\mathcal{L}(X,X)}$. $C_{1-\alpha}([0,b],X) = \{x\colon\ y(t) = t^{1-\alpha}x(t),\ y \in C([0,b],X)\}$ with the norm $\|x\|_{C_{1-\alpha}} = \sup\{t^{1-\alpha}\|x(t)\|_X\colon t \in [0,b]\}$, where $0 < \alpha \leq 1$.

In the sequel, we denote by $\mathcal{P}(Y)$ [$\mathcal{P}_f(Y)$, $\mathcal{P}_{fc}(Y)$, $\mathcal{P}_{fbc}(Y)$,$\mathcal{P}_{(w)cp}(Y)$] the collections of all nonempty [respectively, nonempty closed, nonempty closed and convex,

nonempty closed,bounded and convex, nonempty (weakly) compact] subsets of a Banach space Y.

Now, let us recall the following basic definitions and properties related to fractional calculus that will be used in the sequel.

Definition 1 ([6, 19])**.** For a function $x(t)$ given in the interval $[0,\infty)$, the integral

$$I_t^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)ds, \qquad \alpha > 0,$$

is called the Riemann-Liouville fractional integral of order α, where Γ is the Gamma function.

Definition 2 ([6, 19])**.** For a function $x(t)$ given in the interval $[0,\infty)$, the integral

$${}^{L}D_t^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-\alpha-1}x(s)ds, \qquad \alpha > 0,$$

where $n = [\alpha]+1$, $[\alpha]$ denotes the integer part of number α, is called the Riemann-Liouville fractional derivative of order α.

Definition 3 ([6, 19])**.** The Caputo fractional derivative of order $\alpha > 0$ is defined as

$${}^{C}D_t^{\alpha}x(t) = {}^{L}D_t^{\alpha}[x(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}x^{(k)}(0)], \qquad n-1 < \alpha < n.$$

If $x \in C^n[0,+\infty)$, then

$${}^{C}D_t^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}x^{(n)}(s)ds, \quad n-1 < \alpha < n.$$

Definition 4 ([17])**.** The Hilfer fractional derivative $D_t^{\nu,\mu}x(t)$ of order $\nu \in [0,1]$ and $\mu \in (0,1)$ is defined as

$$D_t^{\nu,\mu}x(t) = I_t^{\nu(1-\mu)}\frac{d}{dt}I_t^{(1-\nu)(1-\mu)}x(t),$$

provided the right side is point-wise defined on $[0,\infty)$.

Remark 1 ([17]). The properties of Hilfer fractional derivatives are as follows.

(i) For $\nu = 0$, $\mu \in (0,1)$, we have

$$D_t^{0,\mu}x(t) = \frac{d}{dt}I_t^{1-\mu}x(t),$$

which means the Hilfer fractional derivative $D_t^{0,\mu}x(t)$ is the Riemann-Liouville fractional derivative.

(ii) For $\nu = 1$, $\mu \in (0,1)$, we have

$$D_t^{1,\mu}x(t) = I_t^{1-\mu}\frac{d}{dt}x(t) = {}^{C}D_t^{\mu}x(t),$$

which means the Hilfer fractional derivative $D_t^{1,\mu}x(t)$ is the Caputo fractional derivative.

Lemma 1 ([2]). *Let $\nu \in [0,1]$ and $\mu \in (0,1)$, $\alpha = \nu + \mu - \nu\mu$, $h \in L^p(J,X)(p > \frac{1}{\mu})$. If $x \in C_{1-\alpha}(J,X)$ and x is a solution of the following problem*

$$\begin{cases} D_t^{\nu,\mu}x(t) = Ax(t) + h(t), & t \in (0,b], \\ I_{0^+}^{(1-\nu)(1-\mu)}x(t)\mid_{t=0} = x_0 \in X, \end{cases} \tag{2.1}$$

then, x satisfies the following equation:

$$x(t) = S_{\nu,\mu}(t)x_0 + \int_0^t (t-s)^{\mu-1}P_\mu(t-s)h(s)ds, \ \ t \in (0,b],$$

$$S_{\nu,\mu}(t) = I_t^{\nu(1-\mu)}T_\mu(t), \ T_\mu(t) = t^{\mu-1}P_\mu(t), \ P_\mu(t) = \mu\int_0^\infty \theta M_\mu(\theta)T(t^\mu\theta)d\theta,$$

where $M_\mu(\theta) = \sum\limits_{n=1}^{\infty} \frac{(-\theta)^{n-1}}{(n-1)!\Gamma(1-n\mu)}$ $(0 < \mu < 1)$ is the Wright function and satisfies

$$\int_0^\infty \theta^\delta M_\mu(\theta)d\theta = \frac{\Gamma(1+\delta)}{\Gamma(1+\mu\delta)}, \ \delta \geq 0.$$

According to Lemma 1, we give the following definition.

Definition 5. A function $x \in C_{1-\alpha}([0,b],X)$ is called a mild solution of system (1.1), if there exists $u(t) \in U(t,t^{1-\alpha}x(t))$ and $\xi(t) \in F(t,x(t))$ a.e. $t \in [0,b]$ such that x satisfies the following fractional integral equation

$$x(t) = S_{\nu,\mu}(t)x_0 + \int_0^t (t-s)^{\mu-1}P_\mu(t-s)\big[(\mathcal{R}x)(s) + \xi(s) + B(s,x(s))u(s)\,ds, \ t \in [0,b]. \tag{2.2}$$

Obviously, from [2, 18], we have

Lemma 2. *Assume that $T(t)$ is strongly continuous and there exists $M > 1$ such that $\sup_{t\in[0,\infty)} \|T(t)\| \leq M$, we have the following properties.*

(i) *$P_\mu(t)$, $T_\mu(t)$ and $S_{\nu,\mu}(t)$ are linear and bounded operators, i.e. for all $t > 0$, $x \in X$ we can obtain:*

$$\|P_\mu(t)x\| \ \leq \ \frac{M\|x\|}{\Gamma(\mu)}, \qquad \|T_\mu(t)x\| \ \leq \frac{Mt^{\mu-1}\|x\|}{\Gamma(\mu)},$$

and

$$\|S_{\nu,\mu}(t)x\| \leq \frac{Mt^{\alpha-1}\|x\|}{\Gamma(\alpha)}, \quad \alpha = \nu + \mu - \nu\mu.$$

(ii) *Operators $P_\mu(t)$, $T_\mu(t)$ and $S_{\nu,\mu}(t)$ are strongly continuous.*

(iii) *For each $t > 0, S_{\nu,\mu}(t)$ and $P_\mu(t)$ are compact operators if $T(t)$ is compact.*

At the end of this section, we state the following Bohnenblust-Karlin fixed point theorem, which will play an important role to obtain the existence of solutions of feedback control system (1.1).

Theorem 1. *Let $\mathfrak{D}$ be a nonempty subset of Banach space X, which is bounded, closed and convex. Suppose $G:\mathfrak{D}\to\mathcal{P}(X)$ is u.s.c. with closed, convex values, and such that $G(\mathfrak{D})\subseteq\mathfrak{D}$ and $\overline{G(\mathfrak{D})}$ is compact (i.e., $G(\mathfrak{D})$ is relatively compact). Then G has a fixed point.*

3. The existence of feasible pairs

In this section, we first study the existence of feasible pairs for feedback control system (1.1). At first, we need the following assumptions on the data of our problems.

$H(T)$: $T(t)$ is compact for every $t>0$.

$H(R)$ $\mathcal{R}:L^p([0,b],X)\to L^p([0,b],X)$ is a history-dependent operator, i.e., there exists a constant $c_{\mathcal{R}}>0$ such that

$$\|(\mathcal{R}x_1)(t)-(\mathcal{R}x_2)(t)\|_X\le c_{\mathcal{R}}\int_0^t\|x_1(s)-x_2(s)\|_X ds,$$
$$\text{for a.e. } t\in[0,b],\ \text{all } x_1,x_2\in L^p([0,b],X).$$

Remark 2. Obviously, if we denote $(\mathcal{R}0)(t)$ by $\varphi(t)$, then, one has from $H(R)$

$$\varphi(t)\in L^p([0,b]) \text{ and } \|(\mathcal{R}x)(t)\|_X\le\varphi(t)+c_{\mathcal{R}}\int_0^t\|x(s)\|_X ds, \tag{3.1}$$
$$\text{for a.e. } t\in[0,b],\ \text{all } x\in L^p([0,b],X).$$

$H(F)$: $[0,b]\times X\to\mathcal{P}_{fc}(X)$ is such that

(i) $t\mapsto F(t,x)$ is measurable on $[0,b]$ for all $x\in X$;

(ii) $F(t,\cdot)$ admits a strongly-weakly closed graph for a.e. $t\in[0,b]$;

(iii) there are a function $\varphi_F\in L^p([0,b])$ for some $p>\dfrac{1}{\mu}$ and a constant $c_F>0$ such that

$$\|F(t,x)\|_X=\sup_{z\in F(t,x)}\|z\|_X\le\varphi_F(t)+c_F t^{1-\alpha}\|x\|_X\quad\text{for all } x\in X \text{ and a.e. } t\in[0,b].$$

Remark 3. If we assume that $J:[0,b]\times X\to\mathbb{R}$ is such that

(i) $t\mapsto J(t,x)$ is measurable on $[0,b]$ for all $x\in X$;

(ii) $x\mapsto J(t,x)$ is locally Lipschitz on X for a.e. $t\in[0,b]$;

(iii) there are a function $\varphi_J\in L^p([0,b])$ for some $p>\dfrac{1}{\mu}$ and a constant $c_J>0$ such that

$$\|\partial J(t,x)\|_{X^*}\le\varphi_J(t)+c_J t^{1-\alpha}\|x\|_X\quad\text{for all } x\in X \text{ and a.e. } t\in[0,b],$$

where $\partial J(t,\cdot)$ stands for the generalized Clarke subdifferential of the local Lipschitz functional $J(t,\cdot)$ on X (cf.[1,15]). Then, ∂J satisfies all the conditions of $H(F)$. Thus, if we replace F by the generalized Clarke subdifferential ∂J, some recent problems involving hemivariational inequalities are special cases of our problems studied here [7, 12–14, 16, 23–25].

$H(B)$: $B\colon [0,b]\times X \to \mathcal{L}(V,X)$ is such that

(i) $t \mapsto B(t,x)u$ is measurable on $[0,b]$ for any $(x,u)\in X\times V$;
(ii) $x \mapsto B^*(t,x)y$ is continuous for all $y\in X^*$ and a.e. $t\in[0,b]$;
(iii) there exists a constant $c_B>0$ such that for any $x\in X$ and a.e. $t\in[0,b]$

$$\|B(t,x)\|_{\mathcal{L}(U,X)} \le c_B.$$

$H(U)$: the feedback multifunction $U\colon [0,b]\times X \to \mathcal{P}_{fc}(V)$ is such that

(i) the map $t\to U(t,x)$ is measurable for all $x\in X$;
(ii) there exist a function $\varphi_U(\cdot)\in L^p([0,b],\mathbb{R}^+)$ $(p>\frac{1}{\mu})$ and a constant $c_U>0$ such that

$$\|U(t,x)\| = \sup_{z\in U(t,x)} \|z\|_V \le \varphi_U(t) + c_U t^{1-\alpha}\|x\|_X, \quad \text{for all } (t,x)\in[0,b]\times X.$$

(iii) for a.e. $t\in[0,b]$, the function $x\to U(t,x)$ is upper semicontinuous.

Using the same ideas of the proof of Lemma 3.2 in [8, p.104], one easily obtain

Lemma 3. *If $H(T)$ holds, then the operator π: $L^p([0,b],X)\to C([0,b],X)$ for some $p>\frac{1}{\mu}$, given by*

$$(\pi h)(\cdot) = \int_0^{\cdot} (\cdot - s)^{\mu-1} P_\mu(\cdot - s) h(s)\,ds, \quad \forall h\in L^p(J,X)$$

is compact.

Definition 6. A pair $(x,u)\in C_{1-\alpha}([0,b],X)\times L^p([0,b],V)$ is said to be feasible if (x,u) satisfies (1.1).

For the sake of conveniences, we denote by $\mathcal{S}$ the collection of all the feasible pairs of (1.1). Now we will start to prove an existence result of mild solutions for feedback control system (1.1).

Proposition 1. *If $H(T)$, $H(R)$, $H(F)$, $H(B)$ and $H(U)$ hold and $(x,u)\in\mathcal{S}$, then there exist two constants $R_1,R_2>0$ such that the following inequalities hold*

$$\|x\|_{C_{1-\alpha}([0,b],X)} \le R_1, \quad \|u\|_{L^p([0,b],V)} \le R_2. \tag{3.2}$$

Proof. For the sake of convenience, in the sequel, we introduce an equivalent weighted norm on the Banach space $C_{1-\alpha}([0,b],X)$ as follows

$$\|x\|_r = \sup_{t\in[0,b]} t^{1-\mu}\|x(t)\|_X e^{-rt}, \tag{3.3}$$

where r will be specified later. Let $(x,u) \in \mathcal{S}$. Then there exists $f(s) \in F(t,x(t))$ such that

$$\begin{cases} x(t) = S_{\nu,\mu}(t)x_0 + \int_0^t (t-s)^{\mu-1} P_\mu(t-s)\big[(\mathcal{R}x)(s) + f(s) + B(s,x(s))u(s))\big]\,ds, \\ u(t) \in U(t, t^{1-\alpha}x(t)), \quad \text{a.e. } t \in [0,b]. \end{cases} \tag{3.4}$$

Notice that

$$\int_0^t (t-s)^{\alpha-1} e^{rs} ds = r^{-\alpha} e^{rt} \int_0^{rt} z^{\alpha-1} e^{-z} dz \le r^{-\alpha} e^{rt} \Gamma(\alpha), \quad \forall r > 0, 0 < \alpha \le 1. \tag{3.5}$$

From the assumption $H(R)$, $H(F)$, $H(B)$, the formula (3.4) and the Hölder inequality, we have

$$\begin{aligned} t^{1-\alpha}\|x(t)\|_X &\le t^{1-\alpha}\|S_{\nu,\mu}(t)x_0\|_X \\ &\quad + t^{1-\alpha}\Big\|\int_0^t (t-s)^{\mu-1} P_\mu(t-s)\big[(\mathcal{R}x)(s) + f(s) + B(s,x(s))u(s))\big]\,ds\Big\|_X \\ &\le \frac{M}{\Gamma(\alpha)}\|x_0\|_X + \frac{Mb^{1-\alpha}}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1}\big[\varphi(s) + \varphi_F(s) + c_B\varphi_U(s)\big]\,ds \\ &\quad + \frac{Mt^{1-\alpha}[c_F + c_B c_U]}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1} s^{1-\alpha} e^{rs} e^{-rs}\|x(s)\|_X ds \\ &\quad + \frac{Mt^{1-\alpha}c_{\mathcal{R}}}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1}\int_0^s \tau^{1-\alpha}\tau^{\alpha-1} e^{r\tau} e^{-r\tau}\|x(\tau)\|_X\,d\tau\big]\,ds \\ &\le \frac{M}{\Gamma(\alpha)}\|x_0\|_X \\ &\quad + \frac{Mb^{1-\alpha+\mu-\frac{1}{p}}}{\Gamma(\mu)}\Big(\frac{p-1}{\mu p-1}\Big)^{\frac{p-1}{p}}[\|\varphi\|_{L^p} + \|\varphi_F\|\|_{L^p} + c_B\|\varphi_U\|\|_{L^p}] \\ &\quad + \big[Mt^{1-\alpha}(c_F + c_B c_U) + \frac{Mtc_R}{\alpha}\big] r^{-\mu} e^{rt}\|x\|_r \\ &\le w + \big[Mb^{1-\alpha}(c_F + c_B c_U) + \frac{Mbc_R}{\alpha}\big] r^{-\mu} e^{rt}\|x\|_r, \end{aligned} \tag{3.6}$$

where

$$w = \frac{M}{\Gamma(\alpha)}\|x_0\|_X + \frac{Mb^{1-\alpha+\mu-\frac{1}{p}}}{\Gamma(\mu)}\Big(\frac{p-1}{\mu p-1}\Big)^{\frac{p-1}{p}}[\|\varphi\|_{L^p} + \|\varphi_F\|\|_{L^p} + c_B\|\varphi_U\|\|_{L^p}].$$

Let us choose

$$r \ge \big[2(Mb^{1-\alpha}(c_F + c_B c_U) + \frac{Mbc_R}{\alpha})\big]^{\frac{1}{\mu}}. \tag{3.7}$$

Then from the above inequality, one has

$$\|x\|_r = \sup_{t\in[0,b]} t^{1-\alpha}\|x(t)\|_X e^{-rt} \le R_1 (:= 2w).$$

By $H(U)$, there exists a constant $R_2 > 0$ such that

$$\|u\|_{L^p([0,b],V)} \leq R_2 (:= \|\varphi_U\|_{L^p([0,b],\mathbb{R}^+)} + c_U b R_1),$$

which completes the proof. □

From Proposition 1, if $(x,u) \in \mathcal{S}$, then we have from $H(B)$, $H(F)$ and $H(U)$

$$\|F(t,x(t))\|_X + \|B(t,x(t))u(t)\|_X \leq \varphi_F(t) + c_B\varphi_U(t) + (c_F + c_B c_U)t^{1-\alpha}\|x(t)\|_X,$$

a.e. $t \in [0,b]$, and set

$$\mathcal{G} := \{g \in L^p([0,b],X) | \|f\|_X \leq \varphi_F(t) + c_B\varphi_U(t) + (c_F + c_B c_U)R_1, \text{ a.e. } t \in [0,b]\}. \tag{3.8}$$

We now consider the following Cauchy problem:

$$\begin{cases} D_t^{\nu,\mu}x(t) \in Ax(t) + (\mathcal{R}x)(t) + g(t), & t \in (0,b], \\ I_{0^+}^{(1-\nu)(1-\mu)}x(t)\,|_{t=0} = x_0. \end{cases} \tag{3.9}$$

Next we begin to prove the existence and uniqueness of mild solutions for (3.9).

Theorem 2. *If $H(T)$ and $H(R)$ hold, then for any $g \in \mathcal{G}$, the system (3.9) has a unique mild solution on $C_{1-\alpha}([0,b],X)$. Furthermore, the map $S\colon \mathcal{G} \to C_{1-\alpha}([0,b],X)$ defined by*

$$S(g) = x(g) \quad \forall g \in K,$$

where $x(g)$ is the unique solution of problem (3.9) corresponding to $g \in \mathcal{G}$, is continuous from $w - L^p([0,b],X)$ into $C_{1-\alpha}([0,b],X)$.

Proof. For any $g \in \mathcal{G}$, define the operator $G\colon C_{1-\alpha}(J,X) \to C_{1-\alpha}([0,b],X)$ by

$$(Gx)(t) = S_{\nu,\mu}(t)x_0 + \int_0^t (t-s)^{\mu-1}P_\mu(t-s)[(\mathcal{R}x)(s) + g(s)]ds, \quad t \in [0,b].$$

Here we also use the same equivalent weighted norm

$$\|x\|_r = \sup_{t\in(0,T]} t^{1-\alpha}\|x(t)\|_X e^{-rt}$$

defined by (3.3) in the Banach space $C_{1-\alpha}([0,b],X)$ and r also satisfies the inequality (3.7).

We show that the operator G is a contraction operator on $C_{1-\alpha}([0,b],X)$.

For any $x,y \in C_{1-\alpha}(J,X)$, if $t \in [0,b]$, we have

$$(Gx)(t) = S_{\nu,\mu}(t)x_0 + \int_0^t (t-s)^{\mu-1}P_\mu(t-s)[(\mathcal{R}x)(s) + g(s)]ds,$$

$$(Gy)(t) = S_{\nu,\mu}(t)x_0 + \int_0^t (t-s)^{\mu-1}P_\mu(t-s)[(\mathcal{R}y)(s) + g(s)]ds.$$

So, we obtain from $H(R)$

$$t^{1-\alpha}\|(Gx)(t) - (Gy)(t)\|_X \leq \frac{Mc_{\mathcal{R}}b}{\alpha}e^{rt}r^{-\mu}\|x-y\|_r,$$

which implies that from (3.7)

$$\|(Gx)-(Gy)\|_r = \sup_{t\in[0,t_1]} t^{1-\alpha}\|(Gx)(t)-(Gy)(t)\|_X e^{-rt} \le \frac{Mc_{\mathcal{R}}b}{\alpha r^{\mu}}\|x-y\|_r \le \frac{1}{2}\|x-y\|_r.$$

Therefore, G is a contraction operator on $C_{1-\alpha}([0,b],X)$. According to the Banach's fixed theorem, G has a unique fixed point on $C_{1-\alpha}([0,b],X)$ and this fixed point is the mild solution of system (3.9).

In the sequel, we show that the map $S\colon \mathcal{G} \to C_{1-\alpha}([0,b],X)$ is continuous from $w-L^p([0,b],X)$ into $C_{1-\alpha}([0,b],X)$.

Let $\{g_n\} \subset \mathcal{G}$ and $\{x_n\} \subset C_{1-\alpha}([0,b],X)$ be such that $x_n = S(g_n)$ and $g_n \rightharpoonup g$ in $L^p([0,b],X)$.

For each $n \in \mathbb{N}$, one has

$$x_n(t) = S_{\nu,\mu}(t)x_0 + \int_0^t (t-s)^{\mu-1}P_\mu(t-s)[(\mathcal{R}x_n)(s)+g_n(s)]ds, \quad t\in[0,b], \tag{3.10}$$

$$x(t) = S_{\nu,\mu}(t)x_0 + \int_0^t (t-s)^{\mu-1}P_\mu(t-s)[(\mathcal{R}x)(s)+g(s)]ds, \quad t\in[0,b], \tag{3.11}$$

for any $n \in \mathbb{N}$, which implies that

$$t^{1-\alpha}\|x_n(t)-x(t)\|_X \le \frac{Mc_{\mathcal{R}}b}{\alpha}e^{rt}r^{-\mu}\|x_n-x\|_r + \int_0^t (t-s)^{\mu-1}P_\mu(t-s)[g_n(s)-g(s)]\,ds,$$

$$\|x_n-x\|_r \le 2\sup_{t\in[0,b]}\int_0^t (t-s)^{\mu-1}P_\mu(t-s)[g_n(s)-g(s)]\,ds. \tag{3.12}$$

Notice that by Lemma 3, the following holds

$$\int_0^{\cdot} (\cdot-s)^{\mu-1}P_\mu(\cdot-s)[g_n(s)-g(s)]\,ds \to 0 \quad \text{in } C([0,b],X) \quad \text{as } n\to\infty. \tag{3.13}$$

Thus, we obtain

$$x_n \to x \text{ in } C_{1-\alpha}([0,b],X) \quad \text{as } n\to\infty.$$

The proof is complete. □

Theorem 3. *If $H(T)$, $H(R)$, $H(F)$, $H(B)$ and $H(U)$ hold, then the solution set $\mathcal{S}$ of feedback control system 1.1 is nonempty.*

Proof. We first consider the multifunction $S_U : C_{1-\alpha}([0,b],X) \to \mathcal{P}(L^p([0,b],V))$ defined by

$$S_U(x) := \{u \in L^p([0,b],V) \mid u(t) \in U(t,x(t)) \quad \text{for a.e. } t\in[0,b]\}.$$

From the Yankov-von Neumann-Aumann selection theorem (see Hu-Papageorgiou [3, p.158]) and hypotheses $H(U)$, we easily see that $S_U(x) \in \mathcal{P}_{fc}(L^p([0,b],V))$ for any $x \in C_{1-\alpha}([0,b],X)$.

We show that S_U is strongly-weakly u.s.c.,i.e., S_U is u.s.c. from $C_{1-\alpha}([0,b],X)$ to $w-L^p([0,b],V)$. To this end, we need to prove that for any weakly closed set $\mathcal{C}$ in $L^p([0,b],V)$ the set $S_U^-(\mathcal{C})$ is closed in $C_{1-\alpha}([0,b],X)$. Let $\{x_n\} \subset S_U^-(\mathcal{C})$ be such that

$$x_n \to x \text{ in } C_{1-\alpha}([0,b],X) \text{ for some } x \in C_{1-\alpha}([0,b],X). \tag{3.14}$$

So, there exists a sequence $\{u_n\} \subset L^p([0,b],V)$ with $u_n \in S_U(x_n) \cap \mathcal{C}$ for each $n \in \mathbb{N}$ such that

$$u_n(t) \in U(t,x_n(t)) \text{ for a.e. } t \in [0,b].$$

Condition $H(U)$(ii) implies that the sequence $\{u_n\}$ is bounded in $L^p([0,b],V)$. Passing to a subsequence if necessary, we may assume that

$$u_n \to u \text{ weakly in } L^p([0,b],V) \quad \text{for some } u \in L^p([0,b],V). \tag{3.15}$$

Applying Mazur Theorem, see e.g. [8, Chapter 2, Corollary 2.8], we have that there is a sequence $a_{il} \geq 0$ with $\sum_{i\geq 1} a_{il} = 1$ such that

$$\overline{u}_l(\cdot) := \sum_{i\geq 1} a_{il} u_{i+l}(\cdot) \to u \text{ strongly in } L^p([0,b],V).$$

Hence, we may assume that

$$\overline{u}_l(t) \to u(t) \text{ in } V \text{ for a.e. } t \in [0,b]. \tag{3.16}$$

Notice that $x_n \to x$ in $C_{1-\alpha}([0,b],X)$, we have

$$t^{1-\alpha} x_n(t) \to t^{1-\alpha} x(t) \text{ in } X \text{ for all } t \in [0,b]. \tag{3.17}$$

Since $x \mapsto U(t,x)$ is u.s.c., then for any $\varepsilon > 0$, there exists $k > 0$ large enough such that

$$u_k(t) \in U(t,x_k(t)) \subset U(t,x(t)) + B_\varepsilon \text{ for a.e. } t \in [0,b],$$

where B_ε is an open ball with radius $\varepsilon > 0$ centered at 0_V. So, if l is large enough, one also has

$$\overline{u}_l(t) \in U(t,x(t)) + B_\varepsilon \text{ for a.e. } t \in [0,b],$$

due to the convexity of $U(t,x(t)) + B_\varepsilon$. This combined with the convergence (3.16) deduces

$$u(t) \in \overline{U(t,x(t)) + B_\varepsilon} \text{ for a.e. } t \in [0,b].$$

Letting $\varepsilon \to 0$, we have

$$u(t) \in \overline{U(t,x(t))} \text{ for a.e. } t \in [0,b].$$

Since U has closed values, so, we have $u(t) \in \overline{U(t,x(t))} = U(t,x(t))$ for a.e. $t \in [0,b]$. This means that $u \in S_U(x)$. But, the weak closedness of $\mathcal{C}$ infers that $u \in \mathcal{C}$, thus is, $x \in S_U^-(\mathcal{C})$. So, S_U is u.s.c. from $C_{1-\alpha}([0,b],X)$ to $w-L^p([0,b],V)$.

Let us consider the multifunction $\Lambda\colon C_{1-\alpha}([0,b],X) \to 2^{C_{1-\alpha}([0,b],X)}$ defined by

$$\Lambda(x) = S(\mathcal{F}_B(x)) \text{ for all } x \in C_{1-\alpha}([0,b],X), \tag{3.18}$$

where S is the solution map defined in Theorem 2 and

$$\mathcal{F}_B\colon C_{1-\alpha}([0,b],X) \to 2^{L^p([0,b],X)}$$

is given by

$$\mathcal{F}_B(x) := \{F(\cdot,x(\cdot)) + B(\cdot,x(\cdot))u(\cdot) \mid u \in S_U(x)\} \text{ for all } x \in C_{1-\alpha}([0,b],X). \quad (3.19)$$

It is easy to see that for each $g \in \mathcal{F}_B(x)$, there exist $f(t) \in F(t,x(t)$ and $u \in S_U(x)$ such that

$$g(t) = f(t) + B(t,x(t))u(t) \text{ for a.e. } t \in [0,b]$$

and

$$\begin{aligned}\|g(t)\|_X &= \|f(t) + B(t,x(t))u(t)\|_X \le \|f(t)\|_X + c_B\|u(t)\|_V \\ &\le \varphi_F(t) + c_F t^{1-\alpha}\|x(t)\|_X + c_B(a_U(t) + c_U t^{1-\alpha}\|x(t)\|_X) \text{ for a.e. } t \in [0,b].\end{aligned}$$

Obviously, for each $x \in C_{1-\alpha}([0,b],X)$, $\mathcal{F}_B(x)$ is a bounded, closed and convex subset of $L^p([0,b],X)$, due to the closedness and convexity of $F(\cdot,x(\cdot))$ and $S_U(x)$ in $L^p([0,b],X)$ and $L^p([0,b],V)$, respectively. Besides, we also say that $\mathcal{F}_B$ is u.s.c. from $C_{1-\alpha}([0,b],X)$ to $w-L^p([0,b],X)$. For any weakly closed set $\mathcal{D}$ in $L^p([0,b],X)$, let $\{x_n\} \subset \mathcal{F}_B^-(\mathcal{D})$ be such that

$$x_n \to x \text{ in } C_{1-\alpha}([0,b],X) \text{ for some } x \in C_{1-\alpha}([0,b],X).$$

So, for each $n \in \mathbb{N}$, there exist $f_n(\cdot) \in F(\cdot,x_n(\cdot))$ and $u_n \in S_U(x_n)$ such that

$$f_n(\cdot) + B(\cdot,x_n(\cdot))u_n(\cdot) \in \mathcal{D} \cap \mathcal{F}_B(x_n).$$

Conditions $H(F)$(iii) and $H(U)$(ii) point out that $\{F(\cdot,x_n(\cdot))\}$ and $\{u_n(\cdot)\}$ are bounded in $L^p([0,b],X)$ and $L^p([0,b],V)$, respectively. Thus we may assume that $f_n \rightharpoonup f$ and $u_n \rightharpoonup u$ in $L^p([0,b],X)$ and $L^p([0,b],V)$,respectively. By $H(F)$(ii), one has $f(\cdot) \in F(\cdot,x(\cdot))$. Using the same arguments as the proof of the upper semicontinuity of S_U, we obtain $u \in S_U(x)$. For any $y \in [L^p([0,b],X)]^*(= L^{p'}([0,b],X^*))$, it has

$$\int_0^b \langle B(t,x_n(t))u_n(t),y(t)\rangle\,dt = \int_0^b \langle u_n(t),B^*(t,x_n(t))y(t)\rangle\,dt.$$

The continuity of $x \mapsto B^*(t,x)$ and Lebesgue dominated convergence theorem entail

$$\begin{aligned}\lim_{n\to\infty}\int_0^b \langle B(t,x_n(t))u_n(t),y(t)\rangle\,dt &= \lim_{n\to\infty}\int_0^b \langle u_n(t),[B^*(t,x_n(t)) - B^*(t,x(t))]y(t)\rangle\,dt \\ &\quad + \lim_{n\to\infty}\int_0^b \langle u_n(t),B^*(t,x(t))y(t)\rangle\,dt \\ &= \int_0^b \langle u(t),B^*(t,x(t))y(t)\rangle\,dt \\ &= \int_0^b \langle B(t,x(t))u(t),y(t)\rangle\,dt,\end{aligned}$$

so, $B(\cdot,x_n(\cdot))u_n(\cdot) \to B(\cdot,x(\cdot))u(\cdot)$ weakly in $L^p([0,b],X)$. The fact $u \in S_U(x)$ turns out $f(\cdot)+B(\cdot,x(\cdot))u(\cdot) \in \mathcal{F}_B(x)$. This combined with the weak closedness of $\mathcal{D}$ implies

$$f(\cdot)+B(\cdot,x(\cdot))u(\cdot) \in \mathcal{F}_B(x) \cap \mathcal{D},$$

that is, $x \in \mathcal{F}_B^-(\mathcal{D})$. So, $\mathcal{F}_B$ is strongly-weakly u.s.c.

The continuity of S and $\mathcal{F}_B$ concludes that the multifunction $\Lambda\colon C_{1-\alpha}([0,b],X) \to 2^{C_{1-\alpha}([0,b],X)}$ is u.s.c. as well (see e.g. [4, Theorem 1.2.8]). Besides, it is easy to prove that Λ has closed and convex values. Set

$$\mathcal{B}(R_1) := \{x \in C_{1-\alpha}([0,b],X) \mid \|x(t)\|_{C_{1-\alpha}([0,b],X)} \le R_1 \text{ for all } t \in [0,b]\}. \tag{3.20}$$

$$\mathcal{G} := \{g \in L^p([0,b],X) \mid \|g(t)\|_X \le \varphi_F(t)+c_B a_U(t)+(c_F+c_B c_U)R_1 \quad \text{for a.e. } t \in [0,b]\} \tag{3.21}$$

It is easy to see that for each $g \in \mathcal{F}_B(x)$, there exist $f(t) \in F(t,x(t)$ and $u \in S_U(x)$ such that

$$g(t) = f(t)+B(t,x(t))u(t) \text{ for a.e. } t \in [0,b].$$

Now, we prove that Λ maps $\mathcal{B}(R_1)$ into itself. For any $x \in \mathcal{B}(R_1)$ and $y \in \Lambda(x)$, there exist $f(\cdot) \in F(\cdot,x(\cdot))$ and $u(\cdot) \in S_U(x)$ such that $y = S(f(\cdot)+B(\cdot,x(\cdot))u(\cdot))$, i.e.,

$$y(t) = S_{\nu,\mu}(t)x_0 + \int_0^t (t-s)^{\mu-1}P_\mu(t-s)[(\mathcal{R}y)(s)+f(s)+B(s,x(s)u(s)]\,ds.$$

Using the same arguments in the proof of Proposition 1, we can also show that $\|y(t)\|_{C_{1-\alpha}([0,b],X)} \le R_1$. So, Λ maps $\mathcal{B}(R_1)$ into itself. Moreover, condition $H(U)$(ii) can imply the inclusion $F(\cdot,\mathcal{B}(R_1))+B(\cdot,\mathcal{B}(R_1))S_U(\mathcal{B}(R_1)) \subset \mathcal{G}$. But, Theorem 2 turns out that the set $\Lambda(\mathcal{B}(R_1))$ is relatively compact in $C_{1-\alpha}([0,b],X)$.

Since all the conditions of Theorem 1 are satisfied, we apply this theorem to conclude that Λ has a fixed point, i.e., there exists $x \in \mathcal{B}(R_1)$ such that $x \in \Lambda(x)$. Then, we can find $f(\cdot) \in F(\cdot,x(\cdot))$,$u \in S_U(x)$ such that $x = \Gamma(f(\cdot)+B(\cdot,x(\cdot))u(\cdot))$, namely,

$$\begin{cases} D_t^{\nu,\mu}x(t) \in Ax(t)+(\mathcal{R}x)(t)+F(t,x(t))+B(t,x(t))u(t), & t \in (0,b], \\ u(t) \in U(t,t^{1-\alpha}x(t)), & \text{a.e. } t \in (0,b), \\ I_{0^+}^{(1-\nu)(1-\mu)}x(t)\,|_{t=0} = x_0. \end{cases} \tag{3.22}$$

Consequently, $(x,u) \in C_{1-\alpha}([0,b],X) \times L^p([0,b],V)$ is a solution of feedback control system 1.1. □

4. Existence of optimal feedback control pairs

In this section, we study the existence of solutions for the following optimal feedback control problem: find $(x^*,u^*) \in \mathcal{S}$ such that

$$I(x^*,u^*) = \inf_{(x,u)\in\mathcal{S}} I(x,u), \tag{4.1}$$

where $\mathcal{S}$ is the collection of all the feasible pairs of feedback control system (1.1) and I is defined by

$$I(x,u) = \int_0^b h(t,x(t),u(t))\,dt \text{ for all } (x,u) \in C_{1-\alpha}(J,X) \times L^p(J,V).$$

In the sequel, we need the following hypotheses $H(h)$: $h\colon [0,b] \times X \times V \to \mathbb{R}$ is such that

(i) for all $(x,u) \in X \times V$, the map $t \mapsto h(t,x,u)$ is measurable;
(ii) there exist $k_1, k_2 \in L^1([0,b],\mathbb{R})$ and $c_h \geq 0$ such that

$$|h(t,x,u)| \leq k_1(t) + k_2(t)t^{1-\alpha}\|x\|_X + c_h\|u\|_V \quad \text{for all } (x,u) \in X \times V \text{ and a.e. } t \in [0,b];$$

(iii) for each bounded set $\mathcal{D} \subset L^p([0,b],V)$, there exists a function $a_D \in L^p([0,b],\mathbb{R})$ such that

$$|h(t,x,u(t)) - h(t,y,u(t))| \leq a_D(t)t^{1-\alpha}\|x-y\|_X \quad \text{for all } x,y \in X,\ u \in \mathcal{D} \text{ and a.e. } t \in [0,b];$$

(iv) for a.e. $t \in [0,b]$ and $x \in X$, the function $u \mapsto h(t,x,u)$ is lower semicontinuous and convex.

Theorem 4. *If $H(T)$, $H(R)$, $H(F)$, $H(B)$, $H(U)$ and $H(h)$ hold, then the optimal feedback control problem (4.1) has at least one solution.*

Proof. For any $(x,u) \in C_{1-\alpha}([0,b],X) \times L^p([0,b],V)$, using $H(h)$(ii) and Hölder inequality, one has

$$\begin{aligned} I(x,u) &= \int_0^b h(t,x(t),u(t))\,dt \\ &\geq -\int_0^b \left(k_1(t) + k_2(t)t^{1-\alpha}\|x(t)\|_X + c_h\|u(t)\|_V\right)dt \\ &\geq -(\|k_1\|_{L^1} + \|x\|_{C_{1-\alpha}([0,b],X)}\|k_2\|_{L^1} + c_h b^{1-\frac{1}{p}}\|u\|_{L^p([0,b],V)}), \end{aligned}$$

which implies that the functional I is bounded from below on $\mathcal{S}$, due to the boundedness of $\mathcal{S}$ in $C_{1-\alpha}([0,b],X) \times L^p([0,b],V)$ (recall Proposition 1).

Let $\{(x_n,u_n)\} \subset \mathcal{S}$ be a minimizing sequence of optimal feedback control problem 4.1, i.e.,

$$\inf = \lim_{n\to\infty} I(x_n,u_n). \tag{4.2}$$

For each $n \in \mathbb{N}$, there exist $f_n(\cdot) \in F(\cdot,x_n(\cdot))$, $u_n(\cdot) \in S_U(x_n)$ such that $\forall t \in [0,b]$

$$x_n(t) = S_{\nu,\mu}(t)x_0 + \int_0^t (t-s)^{\mu-1}P_\mu(t-s)[(\mathcal{R}x_n)(s) + f_n(s) + B(s,x_n(s))u_n(s)]ds. \tag{4.3}$$

The boundedness of $\{u_n\}$, $H(F)$(iii) and $H(B)$(iii) allow us to assume that

$$u_n \to u \text{ weakly in } L^p([0,b],V) \text{ for some } u \in L^p([0,b],V) \tag{4.4}$$

$$f_n \to f \text{ weakly in } L^p([0,b],X) \text{ for some } f \in L^p([0,b],X) \tag{4.5}$$

$$B(\cdot,x_n(\cdot))u_n(\cdot) \to z \text{ weakly in } L^p([0,b],X) \text{ for some } z \in L^p([0,b],X). \tag{4.6}$$

However, the continuity of S (see Theorem 2) implies

$$x_n = S(f_n(\cdot) + B(\cdot,x_n(\cdot))u_n) \to S(f+z) := x \text{ in } C_{1-\alpha}([0,b],X).$$

Obviously, by $H(F)$(ii), we have

$$f(t) \in F(t,x(t)) \quad \text{a.e. } t \in [0,b]. \tag{4.7}$$

We shall show that $z = B(\cdot,x(\cdot))u(\cdot)$. To this end, for any $y \in \big(L^p([0,b],X)\big)^*$, we have from $H(B)$(ii)

$$\begin{aligned}\lim_{n\to\infty}\int_0^b \langle B(t,x_n(t))u_n(t),y(t)\rangle_X\,dt &= \int_0^b \langle u(t),B^*(t,S(f+z)(t))y(t)\rangle\,dt\\ &= \int_0^b \langle B(t,S(f+z)(t))u(t),y(t)\rangle\,dt.\end{aligned}$$

Hence, $B(\cdot,x_n(\cdot))u_n(\cdot) \to z = B(t,S(f+z)(\cdot))u(\cdot) = B(t,x(\cdot))u(\cdot)$ weakly in $L^p([0,b],X)$. From the definition of S and $x = S(f+z)$, we can see that $(x,u) \in C_{1-\alpha}([0,b],X) \times L^p([0,b],V)$ is a solution of feedback control system (1.1), namely, $(x,u) \in \mathcal{S}$.

In what follows, we prove that $(x,u) \in \mathcal{S}$ is also a solution of optimal feedback control problem 4.1. Now, we show that $u \mapsto I(x,u)$ is weakly lower semicontinuous.

Firstly, we claim that for any $x \in C_{1-\alpha}([0,b],X)$, $u \mapsto I(x,u)$ is convex. In fact, for any $u,v \in L^p([0,b],V)$ and $s \in [0,1]$, we have

$$\begin{aligned}I(x,su+(1-s)v) &= \int_0^b h(t,x(t),su(t)+(1-s)v(t))\,dt\\ &\le s\int_0^b h(t,x(t),u(t))\,dt + (1-s)\int_T h(t,x(t),v(t))\,dt\\ &= sI(x,u)+(1-s)I(x,v),\end{aligned}$$

i.e., $u \mapsto I(x,u)$ is convex.

Let $\{u_n\} \to u$ in $L^p([0,b],V)$ for some $u \in L^p([0,b],V)$. Without any loss of generality, we may assume that $u_n(t) \to u(t)$ in V for a.e. $t \in [0,b]$. It follows from $H(h)$(iv) and Fatou lemma that

$$\begin{aligned}\liminf_{n\to\infty} I(x,u_n) &= \liminf_{n\to\infty}\int_0^b h(t,x(t),u_n(t))\,dt\\ &\ge \int_0^b \liminf_{n\to\infty} h(t,x(t),u_n(t))\,dt \ge \int_0^b h(t,x(t),u(t))\,dt.\end{aligned}$$

So, $u \mapsto I(x,u)$ is l.s.c for any $x \in C_{1-\alpha}([0,b],X)$. Moreover, it is weakly lower semicontinuous by the convexity of $u \mapsto I(x,u)$.

Let $\{(x_n,u_n)\} \subset C_{1-\alpha}([0,b],X) \times L^p([0,b],V)$ be such that $x_n \to x$ in $C_{1-\alpha}([0,b],X)$ for some $x \in C_{1-\alpha}([0,b],X)$ and $u_n \rightharpoonup u$ in $L^p([0,b],V)$ for some $u \in L^p([0,b],V)$. It follows from $H(h)$(ii), (iii), (iv) and Fatou lemma that

$$\begin{aligned}\liminf_{n\to\infty} I(x_n,u_n) &\geq \liminf_{n\to\infty}[I(x_n,u_n) - I(x,u_n)] + \liminf_{n\to\infty} I(x,u_n)\\ &\geq -\limsup_{n\to\infty}\int_0^b a_D(t)t^{1-\alpha}\|x_n(t)-x(t)\|_X\,dt + \liminf_{n\to\infty} I(x,u_n) \geq I(x,u),\end{aligned}$$

where the weak lower semicontinuity of $u \mapsto I(x,u)$. The above estimates and (4.2) conclude that $\inf \leq I(x,u) \leq \liminf_{n\to\infty} I(x_n,u_n) = \inf$, due to $(x,u) \in \mathcal{S}$. So, $(x,u) \in \mathcal{S}$ is also a solution of optimal feedback control problem 4.1. □

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MONOTONICITY PROPERTIES AND FUNCTIONAL INEQUALITIES FOR THE BARNES MITTAG-LEFFLER FUNCTION

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Abstract. In this paper, our main focus is to establish complete monotonicity properties, convexity properties and some interesting inequalities for the Barnes Mittag-Leffler function. Furthermore, monotonicity properties of ratios of Barnes Mittag-Leffler functions are derived. Moreover, Turán type inequalities and several new inequalities are obtained for this function as application. Results obtained in this work are new and their importance is illustrated by several attractive consequences and corollaries.

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1. Introduction

Mittag-Leffler function is one of the important special functions, which plays vital role in mathematical physics, fractional calculus, approximation theory and various fields of science and engineering. It appears in the differential equations of fractional order and study of complex system. Mittag-Leffler function was introduced by Gosta Mittag-Leffler in 1903 [15], defined as

$$E_\kappa(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\kappa n+1)}, \quad z \in \mathbb{C},\ \kappa \geq 0, \tag{1.1}$$

where Γ is the well-known Euler's gamma function. From (1.1), it can be observed that $E_\kappa(z)$ interpolates between $\exp(z)$ and $1/(1-z)$ for $0<\kappa<1$. In 1905, Wiman [24] studied a generalization of $E_\kappa(z)$, defined as

$$E_{\kappa,\nu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\kappa n+\nu)}, \quad \kappa,\nu \in \mathbb{C},\ \Re(\kappa)>0, \Re(\nu)>0. \tag{1.2}$$

The generalized Mittag-Leffler function $E_{\kappa,\nu}(z)$ is also known as Wiman's function. There are several generalizations of Mittag-Leffler function available in the literature. One of the most important generalizations of Mittag-Leffler function is the Barnes

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Mittag-Leffler function $E^{(a)}_{\kappa,\nu}(s;z)$, which is defined [6] as

$$E^{(a)}_{\kappa,\nu}(s;z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\kappa n+\nu)(n+a)^s}, \quad a,\nu \in \mathbb{C}\setminus\mathbb{Z}_0^-, \Re(\kappa)>0, s,z\in\mathbb{C}. \tag{1.3}$$

It can be noted that $E^{(a)}_{\kappa,\nu}(0;z) = E_{\kappa,\nu}(z)$, $E_{\kappa,1}(z) = E_\kappa(z)$ and $E_1(z) = \exp(z)$. For further information on Mittag-Leffler function, we refer to [7] and references cited therein.

In terms of the extended Hurwitz-Lerch zeta function $\Phi^{\rho,\sigma,\kappa}_{\lambda,\mu,\nu}(z,s,a)$, defined by [19, p. 503, Eq. (6.2)]

$$\begin{aligned}\Phi^{(\lambda_j,\rho_j;p)}_{(\mu_j,\sigma_j;q)}(z,s,a) &= \Phi^{(\rho_1,\ldots,\rho_p;\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}(z,s,a)\\ &= \frac{\prod_{j=1}^q \Gamma(\mu_j)}{\prod_{j=1}^p \Gamma(\lambda_j)} \sum_{k=0}^{\infty} \frac{\prod_{j=1}^p \Gamma(\lambda_j+k\rho_j)}{\prod_{j=1}^q \Gamma(\mu_j+k\sigma_j)} \frac{z^k}{k!(k+a)^s},\end{aligned} \tag{1.4}$$

we see that

$$E^{(a)}_{\kappa,\nu}(s;z) = \frac{1}{\Gamma(\nu)} \Phi^{(1,1;1)}_{(\nu,\kappa;1)}(z,s,a).$$

From [19, Theorem 8], we deduce that the Barnes Mittag-Leffler function $E^{(a)}_{\kappa,\nu}(s;z)$ possesses the following integral representation

$$E^{(a)}_{\kappa,\nu}(s;z) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-at} E_{\kappa,\nu}(ze^{-t})dt, \ \ (\min(\Re(a),\Re(s),\kappa)>0). \tag{1.5}$$

Monotonicity properties and functional inequalities play vital role in the theory of various inequalities and equilibrium problems [2, 3, 5, 23, 25]. Nowadays most of the researchers use Turán-type inequalities to find monotonicity properties of special functions. Turán's inequality was first introduced by the Hungarian mathematician Paul Turán [21] in 1950 as follows:

$$\frac{P_n(x)P_{n+2}(x)}{P^2_{n+1}(x)} \le 1, \quad x\in[-1,1], \quad n\in\mathbb{N}\cup\{0\}, \tag{1.6}$$

where $P_n(x)$ are Legendre polynomials. Several Turán-type inequalities for various special functions can be found in [3, 4, 11, 12] and references cited therein. It will be interesting to see from (1.6) that stronger results on monotonicity of ratios of functions of the form (1.6) with upper or lower constants as unity can be obtained using these types of inequalities. These inequalities have applications in the theory of transmutation operators for estimating transmulation kernels and norms [18] and also in the problems of function expansions by system of integer shifts of Gaussians [10].

This paper is organized as follows. In the next Section, we present some useful lemmas which are required to complete the proofs of the main results. Using the

integral representation (1.5) of the Barnes Mittag-Leffler function, complete monotonicity property and several interesting inequalities and their consequences are established in Section 3. Turán type inequalities for $E_{\kappa,\nu}^{(a)}(s;z)$ are obtained in Section 4. Moreover, monotonicity, convexity and log-convexity properties for the ratios of Barnes Mittag-Leffler function are also studied in this section. Section 5 is devoted to derive further interesting inequalities for $E_{\kappa,\nu}^{(a)}(s;z)$.

2. Useful lemmas

Before proving the main results in this paper, we need the following preliminary lemmas.

Lemma 1 ([12])**.** *Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be two sequences of real numbers. If $b_n > 0$ for $n \geq 0$ and if the sequence (a_n/b_n) is increasing (decreasing), then $\left(\frac{a_1+\cdots+a_n}{b_1+\cdots+b_n}\right)_{n\geq 0}$ is increasing (decreasing).*

Lemma 2 ([17])**.** *Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be two sequences of real numbers and let the power series $f(x) = \sum_{n\geq 0} a_n x^n$ and $g(x) = \sum_{n\geq 0} b_n x^n$ be convergent for all $|x| < r$. If $b_n > 0$ for $n \geq 0$ and if the sequence (a_n/b_n) is increasing (decreasing), then the function $x \mapsto f(x)/g(x)$ is increasing (decreasing) on $(0,r)$.*

Lemma 3 ([1])**.** *Suppose that $f,g : [a,b] \to \mathbb{R}$ are continuous functions, which are differentiable on (a,b). Moreover, assume that $g'(x) \neq 0$ on (a,b). If f'/g' is increasing (or decreasing) on (a,b), then the following ratios*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \quad \text{and} \quad \frac{f(x)-f(b)}{g(x)-g(b)}$$

are also increasing (or decreasing) on (a,b).

3. Complete monotonicity property and related inequalities

In the next theorem, we will study complete monotonicity property of the function $E_{\kappa,\nu}^{(a)}(s;-z)$. A C^∞ function $f(x)$ is said to be completely monotonic on an interval I if

$$(-1)^{n-1} f^{n-1}(x) \geq 0,$$

for any $n \in \mathbb{N}$ and $x \in I$.

Theorem 1. *Assume that* $\min(\Re(a), \Re(s)) > 0, \kappa > 0$. *Under the following conditions*

$$(\kappa,\nu \in (0,1) \ \ \nu \geq \kappa) \ \text{ or } \ (\kappa \in (0,1] \text{ and } \nu \geq \kappa),$$

the function $z \mapsto E_{\kappa,\nu}^{(a)}(s;-z)$ is completely monotonic on $(0,\infty)$. Moreover, the following inequalities

$$E_{\kappa,\nu}^{(a)}(s;-z_1)E_{\kappa,\nu}^{(a)}(s;-z_2) \leq E_{\kappa,\nu}^{(a)}(s;-z_1-z_2), \ (\min(z_1,z_2) > 0), \tag{3.1}$$

$$E^{(a)}_{\kappa,\nu}(s;-(z_1+z_2)/2) \le \sqrt{E^{(a)}_{\kappa,\nu}(s;-z_1)E^{(a)}_{\kappa,\nu}(s;-z_2)},\ (\min(z_1,z_2)>0) \tag{3.2}$$

$$\frac{1}{a^s\Gamma(\nu)}\exp\left(-\frac{a^s\Gamma\nu)}{(a+1)^s\Gamma(\nu+\kappa)}z\right) \le E^{(a)}_{\kappa,\nu}(s;-z),\ z>0, \tag{3.3}$$

hold true under the given hypothesis.

Proof. By using (1.5) with [20, Corollary 3] and [8, Theorem 2.1], we establish that $E^{(a)}_{\kappa,\nu}(s;-z)$ is completely monotonic on $(0,\infty)$, under the given hypothesis. However, we see that the function $z \mapsto a^s\Gamma(\nu)E^{(a)}_{\kappa,\nu}(s;-z)$ is completely monotonic on $(0,\infty)$ and maps $(0,\infty)$ to $(0,1)$. According to [9, Theorem 3], we get the result (3.1). Let us now focus on the inequality (3.2). By using the fact that every completely monotonic function is log-convex [22, p. 167], we deduce that the function $E^{(a)}_{\kappa,\nu}(s;-z)$ is log-convex and consequently the inequality (3.2) holds true. It remains to prove the inequality (3.3). Let

$$F(z) = \log\left(a^s\Gamma(\nu)E^{(a)}_{\kappa,\nu}(s;-z)\right) =: \log\tilde{E}^{(a)}_{\kappa,\nu}(s;z) \text{ and } G(z)=z.$$

Observe that the function $z \mapsto (\tilde{E}^{(a)}_{\kappa,\nu}(s;z))'/\tilde{E}^{(a)}_{\kappa,\nu}(s;z)$ is increasing on $(0,\infty)$ and consequently the function

$$z \mapsto \frac{F(z)-F(0)}{G(z)-G(0)} =: \Delta(z)$$

is also increasing on $(0,\infty)$ by Lemma 3. Furthermore,

$$\lim_{z\to 0}\Delta(z) = \lim_{z\to 0}\frac{(\tilde{E}^{(a)}_{\kappa,\nu}(s;z))'}{\tilde{E}^{(a)}_{\kappa,\nu}(s;z)} = -\frac{a^s\Gamma(\nu)}{(a+1)^s\Gamma(\nu+\kappa)},$$

which completes the proof of Theorem 1. □

Theorem 2. *Under the assumptions of Theorem 1, the following inequality holds true:*

$$E_{\kappa,\nu}(e^{-\frac{s}{a}}z) \le E^{(a)}_{\kappa,\nu}(s;z) \le \frac{(a+1)^s-\Gamma(s)}{(a+1)^s\Gamma(\kappa+\nu)} + \frac{\Gamma(s)}{(a+1)^s}E_{\kappa,\nu}(z). \tag{3.4}$$

Proof. We recall the Jensen's integral inequality [14, Chap. I, Eq. (7.15)],

$$\varphi\left(\int_a^b f(s)d\sigma(s)\Big/\int_a^b d\sigma(s)\right) \le \int_a^b \varphi(f(s))d\sigma(s)\Big/\int_a^b d\sigma(s), \tag{3.5}$$

where, φ is convex and f is integrable with respect to a probability measure σ. Let

$$\varphi_z(t) = E_{\kappa,\nu}(ze^{-t}),\ f(t)=t \text{ and } d\sigma(t) = \frac{1}{\Gamma(s)}t^{s-1}e^{-at}dt.$$

Using an elementary property of Gamma-function:

$$\frac{\Gamma(s)}{a^s} = \int_0^\infty t^{s-1}e^{-at}dt,\ (\Re(a)>0,\ \Re(a)>0), \tag{3.6}$$

and the representation (1.5), we obtain

$$\int_a^b d\sigma(t) = \frac{1}{a^s}, \quad \int_a^b f(t)d\sigma(t) = \frac{s}{a^{s+1}} \text{ and } \int_a^b \varphi(f(t))d\sigma(t) = E_{\kappa,\nu}^{(a)}(s;z).$$

Again, from (3.5), we have

$$E_{\kappa,\nu}(e^{-\frac{s}{a}}z) \leq E_{\kappa,\nu}^{(a)}(s;z),$$

which proves the lower bound of (3.4). In order to demonstrate the upper bound, we will apply the converse Jensen inequality, due to Lah and Ribarić, which reads as follows. Set

$$A(f) = \int_m^M f(s)d\sigma(s) \Big/ \int_m^M d\sigma(s),$$

where σ is a non-negative measure and f is a continuous function. If $-\infty < m < M < \infty$ and φ is convex on $[m,M]$, then according to [16, Theorem 3.37], we have

$$(M-m)A(\varphi(f)) \leq (M-A(f))\varphi(m) + (A(f)-m)\varphi(M). \tag{3.7}$$

Rewriting the representation (1.5) as follows:

$$E_{\kappa,\nu}^{(a)}(s;z) = \frac{1}{\Gamma(s)} \int_0^1 \log^{s-1}(1/u)u^{a-1}E_{\kappa,\nu}(zu)du, \tag{3.8}$$

and taking

$$[m,M] = [0,1], \varphi(u) = E_{\kappa,\nu}(zu), f(u) = u \text{ and } d\sigma(u) = \frac{\log^{s-1}(1/u)u^{a-1}du}{\Gamma(s)},$$

in (3.7), in view of (3.6) and (3.8), we obtain

$$\frac{E_{\kappa,\nu}^{(a)}(s;z)}{a^s} \leq \frac{(a(a+1))^s - \Gamma(s)}{(a(a+1))^s\Gamma(\kappa+\nu)} + \frac{\Gamma(s)}{(a(a+1))^s}E_{\kappa,\nu}(z).$$

□

Theorem 3. *The following inequalities hold true:*

(a) *Let a,ν,s,z be positive real numbers such that $s \leq 1$ and $a \geq \nu$. Then the following inequality*

$$\frac{E_{1,\nu+1}(z)}{(a-\nu+1)^s} \leq E_{1,\nu}^{(a)}(s,z) \tag{3.9}$$

holds true. Furthermore, the above inequality is reversed if $s \geq 1$ and $\nu - 1 < a \leq \nu$.

(b) *Let $a,\nu,s,z > 0$ be such that $a > \nu$ and $0 < s < 1$. Then the inequality*

$$E_{1,\nu}^{(a)}(s,z)E_{1,\nu}^{(a)}(s+2,z) \leq \frac{2^{2s}s\Gamma(s+1/2)}{\sqrt{\pi}\Gamma(s+2)} E_{1,\nu+1}(z)E_{1,\nu}^{(2a-\nu)}(2s+1,z) \tag{3.10}$$

is valid.

Proof. (**a**) First of all, we recall the Chebyshev integral inequality [13, p. 40]. If $f,g:[a,b]\longrightarrow\mathbb{R}$ are integrable functions, both increasing or both decreasing, and $p:[a,b]\longrightarrow\mathbb{R}$ is a positive integrable function, then

$$\int_a^b p(t)f(t)dt\int_a^b p(t)g(t)dt\leq\int_a^b p(t)dt\int_a^b p(t)f(t)g(t)dt. \tag{3.11}$$

It can be noted that if one of the functions f or g is decreasing and the other is increasing, then (3.11) is reversed. We will use (3.8) and (3.11) to prove (3.9). We set

$$p(t)=1,\ f(t)=\frac{t^{a-\nu}}{\Gamma(s)}\log^{s-1}(1/t)\ \text{ and }\ g(t)=t^{\nu-1}E_{1,\nu}(zt).$$

From the differentiation formula [7, Eq. (4.3.1), p. 51]

$$\left(\frac{d}{dz}\right)^m\left[z^{\nu-1}E_{\kappa,\nu}(z^\kappa)\right]=z^{\nu-m-1}E_{\kappa,\nu-m}(z^\kappa),\ \ (m\geq 1),$$

we see that the function g is increasing on $(0,1]$. Moreover, the function f is decreasing (increasing) if $s\geq 1$ and $\nu-1<a\geq\nu$ $(0<s\leq 1$ and $a\geq\nu)$ on $(0,1]$. Furthermore, we have

$$\begin{aligned}\int_0^1 p(t)f(t)dt&=\frac{t^{a-\nu}}{\Gamma(s)}\int_0^1\log^{s-1}(1/t)dt\\&=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-(a-\nu+1)t}dt=\frac{1}{(a-\nu+1)^s},\ \ (\text{by }(3.6)).\end{aligned}$$

Taking into account [7, Eq. (4.4.4), p. 61]

$$\int_0^z t^{\nu-1}E_{\kappa,\nu}(\lambda t^\kappa)dt=z^\kappa E_{\kappa,\nu+1}(\lambda z^\kappa)\ \ (\nu>0),$$

we obtain

$$\int_0^1 p(t)g(t)dt=E_{1,\nu+1}(z).$$

By applying (3.11), we obtain the desired bound (3.9).

(**b**) We set

$$p(t)=t^{\nu-1}E_{1,\nu}(zt),\ \ f(t)=\frac{t^{a-\nu}}{\Gamma(s)}\log^{s-1}(1/t)\ \text{ and }g(t)=\frac{t^{a-\nu}}{\Gamma(s+2)}\log^{s+1}(1/t).$$

Moreover, by (3.8) we obtain

$$\int_0^1 p(t)f(t)dt=E_{1,\nu}^{(a)}(s;z),\quad\int_0^1 p(t)g(t)dt=E_{1,\nu}^{(a)}(s+2;z),$$

and

$$\int_0^1 p(t)g(t)f(t)dt=\frac{\Gamma(2s+1)}{\Gamma(s)\Gamma(s+2)}E_{1,\nu}^{(2a-\nu)}(s+2;z).$$

Observe that the function f and g are decreasing if $0<s<1$ and $\nu-1<a<\nu$. Then, using (3.11) and the Legendre duplication formula, we obtain (3.10). □

On setting $\nu = 1$ in the above theorem, in view of the fact that

$$E_{1,2}(z) = \frac{e^z - 1}{z},$$

we obtain the following result.

Corollary 1.

(a) *Let a, s, z be positive real numbers such that $s \leq 1$ and $a \geq 1$. Then*

$$\frac{e^z - 1}{za^s} \leq E_{1,1}^{(a)}(z). \tag{3.12}$$

Moreover, the inequality (3.12) is reversed if $a \leq 1$ and $s \geq 1$.

(b) *Let $a, s, z > 0$ be such that $a > 1$ and $0 < s < 1$. Then the following inequality holds:*

$$E_{1,1}^{(a)}(s,z)E_{1,1}^{(a)}(s+2,z) \leq \frac{2^{2s}s\Gamma(s+1/2)(e^z-1)}{\sqrt{\pi}\Gamma(s+2)z}E_{1,1}^{(2a-1)}(2s+1,z). \tag{3.13}$$

Theorem 4. *Suppose that p, q, r, s, κ and ν are positive real numbers such that $\frac{1}{r} + \frac{1}{s} = 1$ with $r > 1$. Then the following inequality holds true:*

$$E_{\kappa,\nu}^{(a)}\left(\frac{p}{r} + \frac{q}{s} + 1; z\right) \leq \frac{\Gamma^{1/r}(p+1)\Gamma^{1/s}(q+1)}{\Gamma\left(\frac{p}{r} + \frac{q}{s} + 1\right)}\left[E_{\kappa,\nu}^{(a)}(p+1;z)\right]^{1/r}\left[E_{\kappa,\nu}^{(a)}(q+1;z)\right]^{1/s}. \tag{3.14}$$

Proof. Using Hölder's inequality and (1.5), we have

$$\begin{aligned}
E_{\kappa,\nu}^{(a)}\left(\frac{p}{r} + \frac{q}{s} + 1; z\right) &= \frac{1}{\Gamma\left(\frac{p}{r} + \frac{q}{s} + 1\right)}\int_0^\infty t^{\left(\frac{p}{r}+\frac{q}{s}\right)}e^{-at}E_{\kappa,\nu}(ze^{-t})dt \\
&= \frac{1}{\Gamma\left(\frac{p}{r} + \frac{q}{s} + 1\right)}\int_0^\infty \{t^p e^{-at}E_{\kappa,\nu}(ze^{-t})\}^{1/r}\{t^q e^{-at}E_{\kappa,\nu}(ze^{-t})\}^{1/s}dt \\
&\leq \frac{1}{\Gamma\left(\frac{p}{r} + \frac{q}{s} + 1\right)}\left(\int_0^\infty t^p e^{-at}E_{\kappa,\nu}(ze^{-t})dt\right)^{1/r}\left(\int_0^\infty t^q e^{-at}E_{\kappa,\nu}(ze^{-t})dt\right)^{1/s} \\
&= \frac{\Gamma^{1/r}(p+1)\Gamma^{1/s}(q+1)}{\Gamma\left(\frac{p}{r} + \frac{q}{s} + 1\right)}\left[E_{\kappa,\nu}^{(a)}(p+1;z)\right]^{1/r}\left[E_{\kappa,\nu}^{(a)}(q+1;z)\right]^{1/s},
\end{aligned}$$

which completes the proof of the theorem. □

Remark 1. Setting $r = s = 2$ in (3.14) of Theorem 4, we obtain the following inequality:

$$\left[\Gamma\left(\frac{p+q+2}{2}\right)E_{\kappa,\nu}^{(a)}\left(\frac{p+q+2}{2};z\right)\right]^2 \leq \Gamma(p+1)\Gamma(q+1)E_{\kappa,\nu}^{(a)}(p+1;z)E_{\kappa,\nu}^{(a)}(q+1;z).$$

Theorem 5. *Let a,b,p,q,κ and ν be positive real numbers satisfying the conditions $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$. Then the following inequality holds true:*

$$E_{\kappa,\nu}^{\left(\frac{a}{p}+\frac{b}{q}\right)}(s;z) \leq \left[E_{\kappa,\nu}^{(a)}(s;z)\right]^{1/p}\left[E_{\kappa,\nu}^{(b)}(s;z)\right]^{1/q}. \tag{3.15}$$

Proof. Using Hölder's inequality and (1.5), obtain

$$\begin{aligned}
E_{\kappa,\nu}^{\left(\frac{a}{p}+\frac{b}{q}\right)}(s;z) &= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-\left(\frac{a}{p}+\frac{b}{q}\right)t}E_{\kappa,\nu}(ze^{-t})dt \\
&= \int_0^\infty \left(\frac{t^{(s-1)}e^{-at}E_{\kappa,\nu}(ze^{-t})}{\Gamma(s)}\right)^{1/p}\left(\frac{t^{(s-1)}e^{-bt}E_{\kappa,\nu}(ze^{-t})}{\Gamma(s)}\right)^{1/q}dt \\
&\leq \left(\frac{1}{\Gamma(s)}\int_0^\infty t^{(s-1)}e^{-at}E_{\kappa,\nu}(ze^{-t})dt\right)^{1/p}\left(\frac{1}{\Gamma(s)}\int_0^\infty t^{(s-1)}e^{-bt}E_{\kappa,\nu}(ze^{-t})dt\right)^{1/q} \\
&= \left[E_{\kappa,\nu}^{(a)}(s;z)\right]^{1/p}\left[E_{\kappa,\nu}^{(b)}(s;z)\right]^{1/q},
\end{aligned}$$

which proves the theorem. □

Remark 2. Setting $p=q=2$ in (3.15) of Theorem 5, we obtain the following inequality:

$$\left[E_{\kappa,\nu}^{\left(\frac{a+b}{2}\right)}(s;z)\right]^2 \leq E_{\kappa,\nu}^{(a)}(s;z)E_{\kappa,\nu}^{(b)}(s;z).$$

4. Turán type inequalities for Barnes Mittag-Leffler function

In this section, we discuss the monotonicity property of the following Turán ratio, which will be useful to establish the sharp Turán type inequality for $E_{\kappa,\nu}^{(a)}(s;z)$:

$$\mathcal{E}_{\kappa,\nu,\mu}^{(a)}(s;z) = \frac{E_{\kappa,\nu}^{(a)}(s;z)E_{\kappa,\mu}^{(a)}(s;z)}{\left[E_{\kappa,(\nu+\mu)/2}^{(a)}(s;z)\right]^2}. \tag{4.1}$$

Theorem 6. *Let ν,μ,κ be three distinct positive real numbers and $s\geq 0$. If $\max(\kappa,\nu)<\mu$, then the function $z\mapsto \mathcal{E}_{\kappa,\nu,\mu}^{(a)}(s;z)$ defined by (4.1) is increasing on $(0,\infty)$. Furthermore, the following sharp Turán type inequality*

$$\left[E_{\kappa,\nu+1}^{(a)}(s;z)\right]^2 - E_{\kappa,\nu}^{(a)}(s;z)E_{\kappa,\nu+2}^{(a)}(s;z) \leq \left(\frac{1}{\nu+1}\right)\left[E_{\kappa,\nu+1}^{(a)}(s;z)\right]^2, \tag{4.2}$$

holds for all $\nu>0$.

Proof. By means of the Cauchy product, we have

$$E_{\kappa,\nu}^{(a)}(s;z)E_{\kappa,\mu}^{(a)}(s;z) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}A_{j,k}z^k,$$

$$[E^{(a)}_{\kappa,(\nu+\mu)/2}(s;z)]^2 = \sum_{k=0}^{\infty}\sum_{j=0}^{k} B_{j,k} z^k,$$

where

$$A_{j,k} = \frac{1}{\Gamma(\kappa j+\nu)\Gamma(\kappa(k-j)+\mu)(j+a)^s(k-j+a)^s},$$
$$B_{j,k} = \frac{1}{\Gamma(\kappa j+(\nu+\mu)/2)\Gamma(\kappa(k-j)+(\nu+\mu)/2)(j+a)^s(k-j+a)^s}.$$

Now, we define the sequence $(C_{j,k})$ by $C_{j,k} = A_{j,k}/B_{j,k}$. Therefore,

$$\frac{C_{j+1,k}}{C_{j,k}} = \left[\frac{\Gamma(\kappa j+(\nu+\mu)/2+\kappa)\Gamma(\kappa j+\nu)}{\Gamma(\kappa j+\nu+\kappa)\Gamma(\kappa j+(\nu+\mu)/2)}\right]\left[\frac{\Gamma(\kappa(k-j)+\mu)\Gamma(\kappa(k-j)+(\nu+\mu)/2-\kappa)}{\Gamma(\kappa(k-j)+\mu-\kappa)\Gamma(\kappa(k-j)+(\nu+\mu)/2)}\right]. \tag{4.3}$$

In virtue of the following inequality for the gamma function

$$\Gamma(x+a)\Gamma(x+b) \le \Gamma(x)\Gamma(x+a+b),\ a>0, b>0, x>0, \tag{4.4}$$

and choosing $x = \kappa(k-j)+(\nu+\mu)/2-\kappa, a = (\mu-\nu)/2$ and $b = \kappa$, we obtain

$$\frac{\Gamma(\kappa(k-j)+\mu)\Gamma(\kappa(k-j)+(\nu+\mu)/2-\kappa)}{\Gamma(\kappa(k-j)+\mu-\kappa)\Gamma(\kappa(k-j)+(\nu+\mu)/2)} \ge 1. \tag{4.5}$$

On the other hand, uppon setting $x = \kappa j+\nu, a = (\mu-\nu)/2$ and $b = \kappa$ in (4.4), we get

$$\frac{\Gamma(\kappa j+(\nu+\mu)/2+\kappa)\Gamma(\kappa j+\nu)}{\Gamma(\kappa j+\nu+\kappa)\Gamma(\kappa j+(\nu+\mu)/2)} \ge 1. \tag{4.6}$$

Combining (4.3), (4.5) and (4.6), we deduce that the sequence $(C_{j,k})_{j\ge 0}$ is increasing and consequently the sequence $(S_k)_{k\ge 0}$ defined by $S_k = \left(\sum_{j=0}^{k} A_{j,k}/\sum_{j=0}^{k} B_{j,k}\right)$ is increasing too, by means of Lemma 1. Further, using Lemma 2, we deduce that the function $z \mapsto \mathcal{E}^{(a)}_{\kappa,\nu,\mu}(s;z)$ is increasing on $(0,\infty)$. Moreover,

$$\mathcal{E}^{(a)}_{\kappa,\nu,\mu}(s;z) \ge \mathcal{E}^{(a)}_{\kappa,\nu,\mu}(s;0) = \frac{\Gamma^2((\nu+\mu)/2)}{\Gamma(\nu)\Gamma(\mu)}.$$

Finally, putting $\mu = \nu+2$ in the above inequality, we get (4.2) for all $\nu > 0$, which completes the proof of the theorem. □

Corollary 2. *The following assertions are true.*

(i) *Let* $\kappa > 0$, $s \ge 0$, $a > 0$ *and* $z > 0$, *then the function* $\nu \mapsto \frac{E^{(a)}_{\kappa,\nu}(s;z)}{E^{(a)}_{\kappa,\nu+1}(s;z)}$, *is convex on* $(0,\infty)$.

(ii) *Under the hypothesis of Theorem 6, the function* $\nu \mapsto \frac{E^{(a)}_{\kappa,\nu}(s;z_1)}{E^{(a)}_{\kappa,\nu}(s;z_2)}$ *is log-convex on* $(0,\infty)$, *for all* $0 < z_2 < z_1$.

Proof. (i) From Theorem 6, we deduce that the function $z \mapsto \log \mathcal{E}^{(a)}_{\kappa,\nu,\mu}(s;z)$ is increasing on $(0,\infty)$. We have,

$$\frac{\partial}{\partial z}\log \mathcal{E}^{(a)}_{\kappa,\nu,\mu}(s;z) = \frac{\frac{\partial}{\partial z}E^{(a)}_{\kappa,\nu}(s;z)}{E^{(a)}_{\kappa,\nu}(s;z)} + \frac{\frac{\partial}{\partial z}E^{(a)}_{\kappa,\mu}(s;z)}{E^{(a)}_{\kappa,\mu}(s;z)} - 2\frac{\frac{\partial}{\partial z}E^{(a)}_{\kappa,(\nu+\mu)/2}(s;z)}{E^{(a)}_{\kappa,(\nu+\mu)/2}(s;z)} \geq 0,$$

and consequently, the function $\nu \mapsto \frac{\frac{\partial}{\partial z}E^{(a)}_{\kappa,\nu}(s;z)}{E^{(a)}_{\kappa,\nu}(s;z)}$ is convex on $(0,\infty)$. In view of (4.9), we get

$$\frac{\frac{\partial}{\partial z}E^{(a)}_{\kappa,\nu+1}(s;z)}{E^{(a)}_{\kappa,\nu+1}(s;z)} = \frac{E^{(a)}_{\kappa,\nu}(s;z)}{\kappa z E^{(a)}_{\kappa,\nu+1}(s;z)} - \frac{\nu}{\kappa z},$$

which implies that the function $\nu \mapsto \frac{E^{(a)}_{\kappa,\nu}(s;z)}{E^{(a)}_{\kappa,\nu+1}(s;z)}$ is convex on $(0,\infty)$.

(ii) Again, by means of Theorem 6, we have $\mathcal{E}^{(a)}_{\kappa,\nu,\mu}(s;z_1) > \mathcal{E}^{(a)}_{\kappa,\nu,\mu}(s;z_2)$ for $0 < z_2 < z_1$, which yields

$$\frac{E^{(a)}_{\kappa,\nu}(s;z_1)}{E^{(a)}_{\kappa,\nu}(s;z_2)}\frac{E^{(a)}_{\kappa,\mu}(s;z_1)}{E^{(a)}_{\kappa,\mu}(s;z_2)} \geq \frac{[E^{(a)}_{\kappa,(\nu+\mu)/2}(s;z_1)]^2}{[E^{(a)}_{\kappa,(\nu+\mu)/2}(s;z_2)]^2},$$

which implies that the function $\nu \mapsto \frac{E^{(a)}_{\kappa,\nu}(s;z_1)}{E^{(a)}_{\kappa,\nu}(s;z_2)}$ is log-convex on $(0,\infty)$. □

Theorem 7. *Suppose that* μ, ν_1, ν_2 *and* a *are positive real numbers and* $s \geq 0$. *If* $\nu_1 < \nu_2$, *then the function* $z \mapsto \frac{E^{(a)}_{\kappa,\nu_2}(s;z)}{E^{(a)}_{\kappa,\nu_1}(s;z)}$ *is decreasing on* $(0,\infty)$. *Moreover, the following Turán type inequality*

$$[E^{(a)}_{\kappa,\nu+1}(s;z)]^2 - E^{(a)}_{\kappa,\nu}(s;z)E^{(a)}_{\kappa,\nu+2}(s;z) \leq (\kappa+1)[E^{(a)}_{\kappa,\nu+1}(s;z)]^2, \tag{4.7}$$

holds true for all $z > 0$.

Proof. From (1.3), we have

$$\frac{E^{(a)}_{\kappa,\nu_1}(s;z)}{E^{(a)}_{\kappa,\nu_2}(s;z)} = \frac{\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\kappa n+\nu_1)(n+a)^s}}{\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\kappa n+\nu_2)(n+a)^s}} = \frac{\sum_{n=0}^{\infty}a_n z^n}{\sum_{n=0}^{\infty}b_n z^n}, \tag{4.8}$$

where

$$a_n = \frac{1}{\Gamma(\kappa n+\nu_1)(n+a)^s}, \quad b_n = \frac{1}{\Gamma(\kappa n+\nu_2)(n+a)^s}.$$

Now, we define the sequence $(c_n)_{n\geq 0}$ by $c_n = a_n/b_n$. Therefore,

$$c_{n+1} - c_n = \frac{\Gamma(\kappa n+\nu_2+\kappa)\Gamma(\kappa n+\nu_1) - \Gamma(\kappa n+\nu_1+\kappa)\Gamma(\kappa n+\nu_2)}{\Gamma(\kappa n+\nu_1+\kappa)\Gamma(\kappa n+\nu_1)}.$$

Setting $z = \kappa n+\nu_1, a = \kappa$ and $b = \nu_2-\nu_1$ in (4.4), we get

$$\Gamma(\kappa n+\nu_2+\kappa)\Gamma(\kappa n+\nu_1) \geq \Gamma(\kappa n+\nu_1+\kappa)\Gamma(\kappa n+\nu_2).$$

This yields that the sequence $(c_n)_{n\geq 0}$ is increasing, and consequently the function $z \mapsto \frac{E^{(a)}_{\kappa,\nu_2}(s;z)}{E^{(a)}_{\kappa,\nu_1}(s;z)}$ is decreasing on $(0,\infty)$ if $\nu_2 > \nu_1$. Now, setting $\nu_2 = \nu+2$ and $\nu_1 = \nu+1$, we deduce that the function $z \mapsto \frac{zE^{(a)}_{\kappa,\nu+1}(s;z)}{E^{(a)}_{\kappa,\nu+2}(s;z)}$ is increasing on $(0,\infty)$. This implies that

$$\frac{\partial}{\partial z}\frac{zE^{(a)}_{\kappa,\nu+1}(s;z)}{E^{(a)}_{\kappa,\nu+2}(s;z)} \geq 0.$$

Using the recurrence formula

$$\frac{\partial}{\partial z}E^{(a)}_{\kappa,\nu+1}(s;z) = \frac{E^{(a)}_{\kappa,\nu}(s;z) - \nu E^{(a)}_{\kappa,\nu+1}(s;z)}{\kappa z}, \tag{4.9}$$

we obtain

$$\begin{aligned}[E^{(a)}_{\kappa,\nu+2}(s;z)]^2\frac{\partial}{\partial z}\frac{zE^{(a)}_{\kappa,\nu+1}(s;z)}{E^{(a)}_{\kappa,\nu+2}(s;z)} &= \left(1+\frac{1}{\kappa}\right)E^{(a)}_{\kappa,\nu+1}(s;z)E^{(a)}_{\kappa,\nu+2}(s;z)\\ &\quad + \frac{E^{(a)}_{\kappa,\nu}(s;z)E^{(a)}_{\kappa,\nu+2}(s;z) - [E^{(a)}_{\kappa,\nu+1}(s;z)]^2}{\kappa} \geq 0,\end{aligned} \tag{4.10}$$

which yields

$$[E^{(a)}_{\kappa,\nu+1}(s;z)]^2 - E^{(a)}_{\kappa,\nu}(s;z)E^{(a)}_{\kappa,\nu+2}(s;z) \leq (\kappa+1)E^{(a)}_{\kappa,\nu+1}(s;z)E^{(a)}_{\kappa,\nu+2}(s;z). \tag{4.11}$$

Moreover, we observe that

$$E^{(a)}_{\kappa,\nu+2}(s;z) \leq E^{(a)}_{\kappa,\nu+1}(s;z) \text{ for all } z > 0.$$

Combining the above inequality with (4.11), we obtain (4.7). □

5. Further inequalities for Barnes Mittag-Leffler function

This section is devoted to study further new inequalities for Barnes Mittag-Leffler function.

Theorem 8. *Let $\nu > 0$, $\mu > 0$ and $s \geq 0$. If $\mu < \nu$, then the following inequality*

$$\left[E^{(a)}_{\kappa,\nu+1}(s;z)\right]^{\frac{1}{\nu}} \leq \frac{a^{\frac{s(\nu-\mu)}{\nu\mu}}[\Gamma(\mu+1)]^{\frac{1}{\mu}}}{[\Gamma(\nu+1)]^{\frac{1}{\nu}}}\left[E^{(a)}_{\kappa,\mu+1}(s;z)\right]^{\frac{1}{\mu}}, \tag{5.1}$$

holds true for all $z > 0$.

Proof. Suppose that $0 < \mu < \nu$ and $s \geq 0$ and the function $F^{(a)}_{\kappa,\nu,\mu} : [0,\infty) \to \mathbb{R}$ is defined by

$$F^{(a)}_{\kappa,\nu,\mu}(s;z) = \frac{\mu}{\nu}\log \tilde{E}^{(a)}_{\kappa,\nu+1}(s;z) - \log \tilde{E}^{(a)}_{\kappa,\mu+1}(s;z),$$

where

$$\tilde{E}^{(a)}_{\kappa,\nu}(s;z) = a^s\Gamma(\nu)E^{(a)}_{\kappa,\nu}(s;z).$$

In view of the recurrence relation (4.9), we have

$$\begin{aligned}\frac{\partial}{\partial z}F^{(a)}_{\kappa,\nu,\mu}(s;z) &= \frac{1}{\kappa z}\left[\frac{\mu}{\nu}\frac{E^{(a)}_{\kappa,\nu}(s;z)}{E^{(a)}_{\kappa,\nu+1}(s;z)} - \frac{E^{(a)}_{\kappa,\mu}(s;z)}{E^{(a)}_{\kappa,\mu+1}(s;z)}\right] \\ &= \frac{\mu}{\kappa z}\left[\frac{\tilde{E}^{(a)}_{\kappa,\nu}(s;z)}{\tilde{E}^{(a)}_{\kappa,\nu+1}(s;z)} - \frac{\tilde{E}^{(a)}_{\kappa,\mu}(s;z)}{\tilde{E}^{(a)}_{\kappa,\mu+1}(s;z)}\right].\end{aligned} \tag{5.2}$$

Moreover, the function $\nu \mapsto \tilde{E}^{(a)}_{\kappa,\nu}(s;z)$ is log-convex on $(0,\infty)$. Indeed, for convenience, let us write

$$\tilde{E}^{(a)}_{\kappa,\nu}(s;z) = \sum_{n=0}^{\infty} a_n(\nu)z^n, \text{ where } a_n(\nu) = \frac{a^s\Gamma(\nu)}{\Gamma(\kappa n+\nu)(n+a)^s}, \ n \geq 0.$$

Using the fact that the sum of log-convex functions is log-convex too, we just need to prove the log-convexity of $a_n(\nu)$. Hence, for all $n \geq 0$, we have

$$\frac{\partial^2}{\partial \nu^2}\log(a_n(\nu)) = \psi'(\nu) - \psi'(\kappa n+\nu),$$

where $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ is the digamma function. Since ψ is concave, we deduce that $\frac{\partial^2}{\partial \nu^2}\log(a_n(\nu)) \geq 0$, and consequently $\nu \to a_n(\nu)$ is log-convex on $(0,\infty)$, which implies that the function $\nu \mapsto \tilde{E}^{(a)}_{\kappa,\nu}(s;z)$ is log-convex on $(0,\infty)$. Therefore, the function $\nu \mapsto \log \tilde{E}^{(a)}_{\kappa,\nu+1}(s;z) - \log \tilde{E}^{(a)}_{\kappa,\nu}(s;z)$ is increasing on $(0,\infty)$ and consequently the function $\nu \mapsto \frac{\tilde{E}^{(a)}_{\kappa,\nu+1}(s;z)}{\tilde{E}^{(a)}_{\kappa,\nu}(s;z)}$ is increasing on $(0,\infty)$. Combining this fact with (5.2), we obtain $\frac{\partial}{\partial z}F^{(a)}_{\kappa,\nu,\mu}(s;z) \leq 0$. Therefore, $F^{(a)}_{\kappa,\nu,\mu}(s;z) \leq F^{(a)}_{\kappa,\nu,\mu}(s;0) = 0$, which proves the desired result. □

Corollary 3. *Under the assumption of Theorem 8, the following inequality*

$$\frac{a^{\frac{s(\nu-\mu)}{\mu}}[\Gamma(\mu+1)]^{\frac{\nu}{\mu}}}{[\Gamma(\nu+1)]}\left[E^{(a)}_{\kappa,\mu+1}(s;z)\right]^{\frac{\nu-\mu}{\mu}}+\frac{E^{(a)}_{\kappa,\mu+1}(s;z)}{E^{(a)}_{\kappa,\nu+1}(s;z)}\geq 2, \tag{5.3}$$

holds true for all $z>0$.

Proof. Using (5.1), we obtain

$$\frac{a^{\frac{s(\nu-\mu)}{\mu}}[\Gamma(\mu+1)]^{\frac{\nu}{\mu}}}{[\Gamma(\nu+1)]}\left[E^{(a)}_{\kappa,\mu+1}(s;z)\right]^{\frac{\nu}{\mu}}\left[E^{(a)}_{\kappa,\nu+1}(s;z)\right]^{-1}\geq 1.$$

With the help of the above inequality and the arithmetic-geometric mean inequality, we get

$$\frac{1}{2}\left(\frac{a^{\frac{s(\nu-\mu)}{\mu}}[\Gamma(\mu+1)]^{\frac{\nu}{\mu}}}{\Gamma(\nu+1)}\left[E^{(a)}_{\kappa,\mu+1}(s;z)\right]^{\frac{\nu-\mu}{\mu}}+\frac{E^{(a)}_{\kappa,\mu+1}(s;z)}{E^{(a)}_{\kappa,\nu+1}(s;z)}\right)$$
$$\geq\sqrt{\frac{a^{\frac{s(\nu-\mu)}{\mu}}[\Gamma(\mu+1)]^{\frac{\nu}{\mu}}}{\Gamma(\nu+1)}\frac{\left[E^{(a)}_{\kappa,\mu+1}(s;z)\right]^{\frac{\nu}{\mu}}}{E^{(a)}_{\kappa,\nu+1}(s;z)}}\geq 1,$$

which proves (5.3). □

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ON THE COMPARISON OF THE MARCINKIEWICZ AND LUXEMBURG NORMS OF THE EXPONENTIAL SPACE

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Abstract. The exponential space, which is a Banach function space, can be defined with two very differently looking, but equivalent norms. In this paper, we give estimates for the best constants of the ratio of these two norms. Our result answers a question of C. Bennett, and R. Sharpley.

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1. Introduction

1.1. *The $L_{\exp}$ space*

In 1928, A. Zygmund and E. C. Titchmarsh independently introduced the $L\log L$ *logarithm space*, and the $L_{\exp}$ *exponential space* during their studies on Fourier analysis [5–7] (see also the historical remarks on p. 288 of [1]). Later, these spaces turned out to be important in the theory of interpolation of operators, too [1]. The exponential space can be defined by a Marcinkiewicz space norm, and via an Orlicz space norm. These norms, although are guaranteed to be equivalent, look very differently. The authors of [1] wrote that "the exact relationship" between the two norms "is not clear" (p. 271). Our aim in this paper is to clarify this relation by giving approximations of the best constants for the ratio of the two norms. The logarithm space was treated in our previous paper [3].

The $L_{\exp} = L_{\exp}[0,1]$ space is the complete vector space of measurable functions on $[0,1]$ with finite norm, where the norm is given by the expression

$$\|f\|_{L_{\exp}} = \sup_{0<t<1} \frac{f^{**}(t)}{1+\log\left(\frac{1}{t}\right)}. \tag{1.1}$$

Here

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\mathrm{d}s$$

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is the *maximal function* of f, and $f^*(s)$ is the *decreasing rearrangement* of f. The reader can find more on these notions in the classical text [2] or in the newer [1, 4]. Norms of the type (1.1) are called Marcinkiewicz-type norms [4, Chapter 11].

The exponential space can be viewed as an Orlicz space, too. In this case $f \in L_{\exp}$ is equipped with the following norm, called *Luxemburg norm* [4, Notes on p. 234]:

$$\|f\|_\Phi = \inf\left\{\lambda > 0 : \int_0^1 \Phi\left(\frac{|f(t)|}{\lambda}\right) \mathrm{d}t \le 1\right\}, \tag{1.2}$$

where

$$\Phi(t) = \begin{cases} t, & 0 \le t \le 1; \\ e^{t-1}, & t > 1. \end{cases} \tag{1.3}$$

The fact that (1.1) and (1.2) are equivalent can be verified by showing that the two corresponding spaces contain the same set of functions (see [1, p. 271] for more details, and detailed explanation). It therefore follows that there are two positive, finite constants c_1 and c_2 such that $c_1 \le \frac{\|f\|_{L_{\exp}}}{\|f\|_\Phi} \le c_2$. In the following theorem we give estimates for these constants.

Theorem 1. *There exist constants c_1 and c_2, such that for all $f \in L_{\exp}$*

$$c_1 \le \frac{\|f\|_{L_{\exp}}}{\|f\|_\Phi} \le c_2 \tag{1.4}$$

with

$$0 < c_1 \le 1, \quad \textit{and} \quad 1 + \frac{1}{e} \le c_2 \le 2(1 + \log 2).$$

Before proving our theorem, a remark and a question are in order.

Question. The constants in our theorem unfortunately do not seem to be optimal, so we put the following question: what are the optimal constants in (1.4)?

2. The proof of Theorem 1

2.1. *The estimate of c_1*

We will consider step functions[1] f which are decreasing and right-continuous (so that, conveniently, $f^* = f$):

$$f(t) = \sum_{k=0}^{n} a_k \chi_{]i_k, i_{k+1}]}(t) \tag{2.1}$$

[1]Note that $L_{\exp}$ is a non-separable space. The theory of Marcinkiewicz spaces says that if the fundamental function ϕ of the space is so that $\phi(0+) = 0$, and $\phi'(0+) = \infty$, then the given Marcinkiewicz space is non-separable [4, p. 164]. The fundamental function of the exponential space is

$$\phi(t) = \frac{1}{1 + \log\left(\frac{1}{t}\right)},$$

which behaves at zero exactly as we said above. Therefore, the step functions do not form a dense set in $L_{\exp}$.

with $0 = i_0 < i_1 < \cdots i_{n+1} = 1$ and $a_0 > a_1 > \cdots > a_n \geq 0$. (Here χ_I is the characteristic function of the interval I.)

It turns out, that on one-step functions the norms (1.1) and (1.2) agree. To see this, let f be a one-step function. Then

$$f^{**}(t) = \frac{1}{t}\int_0^{i_1} a_0 \mathrm{d}s = \begin{cases} a_0, & \text{if } 0 < t \leq i_1; \\ \frac{a_0 i_1}{t}, & \text{if } t > i_1, \end{cases}$$

whence

$$\|f\|_{L_{\exp}} = \max\left\{ \sup_{0<t\leq i_1} \frac{a_0}{1+\log\left(\frac{1}{t}\right)}, \sup_{i_1<t<1} \frac{a_0 i_1/t}{1+\log\left(\frac{1}{t}\right)} \right\} = \frac{a_0}{1+\log\left(\frac{1}{i_1}\right)}. \tag{2.2}$$

On the other hand,

$$\|f\|_{\Psi} = \inf\left\{ \lambda > 0 \,\middle|\, \int_0^{i_1} \Phi\left(\frac{a_0}{\lambda}\right) \mathrm{d}t \leq 1 \right\} = \inf\left\{ \lambda > 0 \,\middle|\, i_1 \Phi\left(\frac{a_0}{\lambda}\right) \leq 1 \right\}.$$

By (1.3),

$$\Phi\left(\frac{a_0}{\lambda}\right) = \begin{cases} \frac{a_0}{\lambda}, & \lambda \geq a_0; \\ \exp(a_0/\lambda - 1), & 0 < \lambda < a_0. \end{cases}$$

It is seen, that $\Phi\left(\frac{a_0}{\lambda}\right) = 1$ has a unique solution:

$$\lambda^* = \frac{a_0}{1+\log\left(\frac{1}{i_1}\right)}. \tag{2.3}$$

And thus we have that

$$\|f\|_{\Phi} = \frac{a_0}{1+\log\left(\frac{1}{i_1}\right)}. \tag{2.4}$$

This indeed agrees with (2.2). It therefore follows that

$$c_1 = \inf_{f\in L_{\exp}} \frac{\|f\|_{L_{\exp}}}{\|f\|_{\Phi}} \leq \inf_{f\in S_1} \frac{\|f\|_{L_{\exp}}}{\|f\|_{\Phi}} = 1.$$

(That c_1 is positive is a trivial consequence of the equality of the norms.) Above S_1 is the set of one-step functions. The first part of our theorem is proved.

2.2. *The estimate of c_2*

To find lower estimate for

$$c_2 = \sup_{f\in L_{\exp}} \frac{\|f\|_{L_{\exp}}}{\|f\|_{\Phi}},$$

we consider step functions of special form. Let S be a class of step functions, for which the norm (1.2) is taken already on the first step, i.e., $\|f\|_{\Phi} \geq a_1$. In view of

(2.3), a step function f can belong to S only if

$$a_1 < \frac{a_0}{1+\log\left(\frac{1}{i_1}\right)} \tag{2.5}$$

(otherwise we would have that $\lambda < a_1$, and a_1 would therefore be needed to calculate $\|f\|_\Phi$).

Seeking for the supremum of $\frac{\|f\|_{L_{\exp}}}{\|f\|_\Phi}$, we can raise the steps of f and thus making the nominator bigger, while the denominator remains unchanged. We do this so that our function stays in S. Clearly, in limit, any rear steps in f become as high as a_1, and a_1 itself will be equal to $\frac{a_0}{1+\log\left(\frac{1}{i_1}\right)}$ (see (2.5)). For simplicity, let us denote i_1 by α. Thus, in the set S, the extremal ratio of the norms is attained for

$$f(t) = a_0\chi_{]0,\alpha}(t) + a_1\chi_{]\alpha,1]}(t) \tag{2.6}$$

with some $0<\alpha<1$, and $a_1 = \frac{a_0}{1+\log\left(\frac{1}{\alpha}\right)}$. For this f, (2.4) is still valid (with $i_1=\alpha$), which gives us the Luxemburg norm of f. We still need to determine the Marcinkiewicz norm of f. This can be calculated as follows:

$$\begin{aligned}\|f\|_{L_{\exp}} &= \max\left\{\sup_{0<t<\alpha}\frac{a_0}{1+\log\left(\frac{1}{t}\right)}, \sup_{\alpha\le t<1}\frac{a_0\alpha+a_1(t-\alpha)}{t\left(1+\log\left(\frac{1}{t}\right)\right)}\right\}\\ &= \max\left\{\frac{a_0}{1+\log\left(\frac{1}{\alpha}\right)}, \sup_{\alpha\le t<1}\frac{(a_0-a_1)\alpha+a_1t}{t\left(1+\log\left(\frac{1}{t}\right)\right)}\right\}.\end{aligned}$$

If the maximum was the first value, we would be in the case of one step functions. Otherwise, we have that

$$\frac{\|f\|_{L_{\exp}}}{\|f\|_\Phi} = \frac{\sup_{\alpha\le t<1}\frac{(a_0-a_1)\alpha+a_1t}{t\left(1+\log\left(\frac{1}{t}\right)\right)}}{a_1} = \sup_{\alpha\le t<1}\frac{\left(\frac{a_0}{a_1}-1\right)\frac{\alpha}{t}+1}{1+\log\left(\frac{1}{t}\right)}.$$

Since $a_0/a_1 = 1+\log\left(\frac{1}{\alpha}\right)$, it comes that

$$\frac{\|f\|_{L_{\exp}}}{\|f\|_\Phi} = \sup_{\alpha\le t<1}\frac{\log\left(\frac{1}{\alpha}\right)\frac{\alpha}{t}+1}{1+\log\left(\frac{1}{t}\right)} \tag{2.7}$$

for our particular f in (2.6). For any $\alpha\in]0,1[$, the function on the right-hand-side inside the supremum decreases, and then increases in t, so its extremal values in $t\in[\alpha,1]$ are taken in the points $t=\alpha$ and $t=1$. In $t=\alpha$ its value is one, and at $t=1$ it is $\log\left(\frac{1}{\alpha}\right)\alpha+1$. We choose α such that this expression is maximal. The maximum is reached when $\alpha=\frac{1}{e}$.

From these considerations and from (2.7), we altogether have that

$$\frac{\|f\|_{L_{\exp}}}{\|f\|_\Phi} = \frac{1}{e}+1.$$

Since the set S is obviously smaller than the whole set $L_{\exp}$, we expect the supremum of the ratio on $L_{\exp}$ larger than it is on S. That is,

$$c_2 \geq \frac{1}{e} + 1.$$

The only statement that still needs to be proven is that $c_2 \leq 2(1+\log 2)$. To prove this inequality, we recall that the finiteness of

$$K := \int_0^1 \exp(\lambda f^*(t))\mathrm{d}t \tag{2.8}$$

implies that

$$f^*(t) \leq c\left(1+\log\left(\frac{1}{t}\right)\right) \quad (0<t<1). \tag{2.9}$$

Here $c = \frac{1}{\lambda}\max\{\log K, 1\}$, and λ is positive. See the argument on p. 244-245 of [1].

At this point we suppose that f belongs to the unit sphere of $(L_{\exp}, \|\cdot\|_\Phi)$ in the expression $\frac{\|f\|_{L_{\exp}}}{\|f\|_\Phi}$. This causes no lost of generality, because of the homogeneity of the norms. We therefore have that

$$\|f\|_\Phi = \int_A |f(t)|\mathrm{d}t + \int_B \exp(|f(t)|-1)\mathrm{d}t = 1. \tag{2.10}$$

Here

$$A = \{t \in [0,1] : |f(t)| \leq 1\}, \quad B = [0,1]\setminus A.$$

Next, we estimate the constant K in (2.8) (with $\lambda = 1$):

$$\int_0^1 \exp(f^*(t))\mathrm{d}t = \int_B \exp(|f(t)|)\mathrm{d}t + \int_A \exp(|f(t)|)\mathrm{d}t.$$

From (2.10) it comes that $\int_B \exp(|f(t)|)\mathrm{d}t \leq e$. Moreover, by the definition of the set A,

$$\int_A \exp(|f(t)|)\mathrm{d}t \leq \int_A \exp(1)\mathrm{d}t \leq e.$$

Hence

$$K = \int_0^1 \exp(f^*(t))\mathrm{d}t \leq 2e,$$

and this yields that in (2.9) the constant c is

$$c = \frac{1}{1}\max\{\log(2e), 1\} = 1+\log 2.$$

To estimate $\|f\|_{L_{\exp}}$, however, we need an estimate for $f^{**}(t)$. But this is easy after having (2.9) at hand:

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\mathrm{d}s \leq \frac{1+\log 2}{t}\int_0^t \left(1+\log\frac{1}{s}\right)\mathrm{d}s \leq 2(1+\log 2)\left(1+\log\frac{1}{t}\right).$$

Now, (1.1) immediately yields that, if f is on the unit sphere of $(L_{\exp}, \|\cdot\|_\Phi)$, then

$$\|f\|_{L_{\exp}} \leq 2(1+\log 2).$$

Our proof is done.

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AN OPTIMIZED MODEL FOR NEUTROSOPHIC MULTI-CHOICE GOAL PROGRAMMING

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Abstract. Goal Programming (GP), as a generalization of linear programming, is a proper technique for handling Multi-Objective Decision-Making (MODM) problems. Using this technique, we can determine multiple aspiration levels for each objective. The Neutrosophic Set (NS) theory is a powerful tool to deal with indeterminacy and inconsistent data. This paper provides a new idea to formulate Neutrosophic Multi-Choice Goal Programming (NMCGP) problems to simulate real-life situations more realistically. Also, to show the low computational complexity of the proposed model, the Space and Time complexity criteria are used in the Python language. Eventually, a practical mathematical example is solved to present the effectiveness of the proposed model.

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Keywords: neutrosophic, goal programming, multi-objective decision making, aspiration level

1. Introduction

Goal Programming (GP) is a helpful technique to obtain the optimal solution to Multi-Objective Decision-Making (MODM) problems with an aspiration level for each target. GP could also be considered an extension of Linear Programming (LP) to process multiple, usually conflicting, objective problems. It enables us to set our aspiration levels for each goal.

The GP problems can be incorporated into two different groups: (1) GP models for crisp Decision-Making (DM) problems (classical goal programming) and (2) GP models for decision-making problems under uncertainty. Most of the existing problems in GP are proposed based on classical goal programming. These kinds of problems were firstly offered by Charnce and Cooper [5] and then developed by Lee [11], Ignazio [7], Tamiz et al.[29], Romero [25], and others [3, 12, 20, 22–24]. The main purpose of GP is to minimize the deviations between the achievement of goals and their aspiration levels. It can be represented as follows:

$$\begin{array}{ll} Min & \sum_{j=1}^{n} |f_i(X) - g_i|, \\ s.t. & AX = b, \ X \geq 0, \end{array} \tag{1.1}$$

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where g_i is the aspiration level of the ith goal. Due to imprecision in determining aspiration levels of real-world problems, decision-makers are unable to consider a crisp amount of aspiration level for goal programming problems. In this situation, we require the approaches that belong to the second group of GP problems. This group includes the developed approach for goal programming problems under uncertainty. To overcome the current uncertainty, several methods based on fuzzy sets and their extensions have been proposed. Bankian et al. [1], Hocine et al. [6], Kouaissah and Hocine [10], Kamal et al. [8], Maity et al. [13], Keshteli and Nasseri [9], and many others proposed different techniques to deal with fuzzy goal programming problems. However, the fuzzy sets are still unable to handle indeterminate and inconsistent information that commonly exists in natural systems. From this point of view, Smarandache in [28] suggested a concept of Neutrosophic Set (NS) theory. As another deployment of FS theory, NS theory is a powerful tool to handle uneven, indistinctive, and defective data. The degree of indeterminacy in neutrosophic theory is considered as an independent factor between zero and one, which has a significant contribution in decision making. A NS N can be characterized by three membership functions: truth, indeterminacy, and falsity membership functions that are entirely independent of each other. Wang et al. [30] proposed the Single-Valued Neutrosophic Sets (SVNSs) as an instance of the NSs to handle practical problems from a scientific and engineering perspective. Recently, SVNS has become an important research topic and attracted much attention in decision-making problems. By applying a bidirectional measure on decision-making problems, Ye [32] proposed a new approach to handle MAGDM problems under a neutrosophic environment. Zhang et al. [33] presented two aggregation operators based on neutrosophic values, such as neutrosophic weighted arithmetic operator and neutrosophic weighted geometric operator, and applied them to multi-criteria decision-making problems. Moreover, Ye in [31] proposed a correlation coefficient of SVNSs and the cross-entropy measure of SVNSs and used them for decision-making under the Neutrosophic environment. By introducing the cosine measures for evaluating linguistic neutrosophic numbers, Shi and Ye [27] proposed a novel strategy to solve neutrosophic DM problems. Ranking of the NS is an essential part of the MAGDM process. By defining a score function for ranking the SVNNs, Sahin [26] proposed a strategy for Multi-Attribute Group Decision-Making (MAGDM) under the neutrosophic environment. By highlighting the shortcomings of the existing score functions, Nancy and Garg [19] introduced a modified score function for ranking the SVNNs and then applied it to neutrosophic decision-making problems. Using a bidirectional projection measure, Pramanik et al. [21] proposed a novel neutrosophic approach to handle teacher selection problems. Nafei et al. [17] proposed a novel approach for decision-making with group recommendations based on NSs. Meanwhile, Nafei et al. [14], by developing a new Hamming distance between SVNNs, proposed an extension of the TOPSIS method for MAGDM based on SVNSs where the information about attributes are expressed

by decision-makers based on neutrosophic numbers. This research presents a new method for solving Neutrosophic Multi-Choice Goal Programming (NMCGP) problems. The advantage of using NMCGP is its ability to simulate all aspects of real-life situations in our models. The remainder of this research is marshaled as follows. Section 2 presents some practical definitions of Neutrosophic Sets and other essential concepts. In Section 3, the Neutrosophic Multi-Choice Goal Programming formulation is described. In Section 4, we first explain our method using a numerical example and then present the calculation processing of the model. The conclusions are discussed in Section 5.

2. Preliminaries

This section briefly reviews some necessary backgrounds and preliminaries of neutrosophic sets, single-valued neutrosophic sets, and other essential details.

Definition 1 ([18])**.** A neutrosophic set U in domain X $(x \in X)$ is characterized by truth $T_U(x) : X \to]0^-, 1^+[$, indeterminacy $I_U(x) : X \to]0^-, 1^+[$ and falsity $F_U(x) : X \to]0^-, 1^+[$ membership functions. There is no restriction on the sum of $T_U(x), I_U(x)$, and $F_U(x)$. Therefore, $\overline{0} \leq T_N(x) + I_N(x) + F_N(x) \leq 3^+$.

Definition 2 ([2, 15])**.** Assume that X is a domain. A Single-Valued Neutrosophic Set (SVNS) U through X taking the form $U = \{x, T_U(x), I_U(x), F_U(x); x \in X\}$, where $T_U(x) : X \to [0,1], I_U(x) : X \to [0,1]$ and $F_U(x) : X \to [0,1]$ with $0 \leq T_U(x) + I_U(x) + F_U(x) \leq 3$ for all $x \in X$. Also, $T_U(x), I_U(x)$ and $F_U(x)$ represent the truth membership degree, the indeterminacy membership degree and the falsity membership degree, respectively, of x to U.

Definition 3 ([16])**.** Assume that U and V are two NSs. For all $x \in X$, U is included in V, if and only if:

$$\begin{aligned} &Inf\ T_U(x) \leq Inf\ T_V(x), \\ &Sup\ T_U(x) \leq Sup\ T_V(x), \\ &Inf\ I_U(x) \geq Inf\ I_V(x), \\ &Sup\ I_U(x) \geq Sup\ I_V(x), \\ &Inf\ F_U(x) \geq Inf\ F_V(x), \\ &Sup\ F_U(x) \geq Sup\ F_V(x). \end{aligned} \tag{2.1}$$

Definition 4 ([15])**.** Let U be a neutrosophic set. The complement of U is interpreted by U^c and it could be defined as

$$T_U^c(x) = \{1^+\} - T_U(x), I_U^c(x) = \{1^+\} - I_U(x), F_U^c(x) = \{1^+\} - F_U(x),\ \forall x \in X.$$

Therefore, $U^c = \{x, F_U(x), 1 - I_U(x), T_U(x); x \in X\}$.

3. NEUTROSOPHIC MULTI-CHOICE GOAL PROGRAMMING

Decision-Making is a substantial and essential part of daily life that can be applied in various areas, such as society, economics, management, military, and engineering technology. In most cases, it is intricate for decision-makers to accurately reveal a preference when solving Multi Attribute Decision Making (MADM) problems with imprecise, vague, and incomplete information.

Multi-Objective Decision-Making as a subset of decision science is an essential part of daily life that can be applied to various problems. The Goal Programming approach is a powerful technique to address these kinds of problems. Sometimes, decision-makers prefer to have more than one aspiration level for each goal because of some specific situations. To handle this problem, Chang in [4] proposed a Multi-Choice Goal Programming model that is characterized based on crisp decision-making problems. In some conditions, the objectives can have an ambiguous level. In this case, we present the Neutrosophic Multi-Choice Goal Grogramming (NMCGP) model approach that is a powerful tool to solve this kind of unclear difficulties. A NMCGP problem could be formulated as follows:

$$\begin{aligned} &Min \ \ \textstyle\sum_{i=1}^{n} \left| f_i(X) - g_{i1} \text{ or } g_{i2} \ \text{ or } \ldots g_{im} \right|, \\ &s.t. \quad AX = b, \ X \geq 0, \end{aligned} \tag{3.1}$$

where $f_i(X)$ is a linear function of x_i $(1 \leq i \leq n)$ for the ith goal and g_{ij} $(1 \leq i \leq n$ and $1 \leq j \leq m)$ is the jth aspiration level of the ith goal.

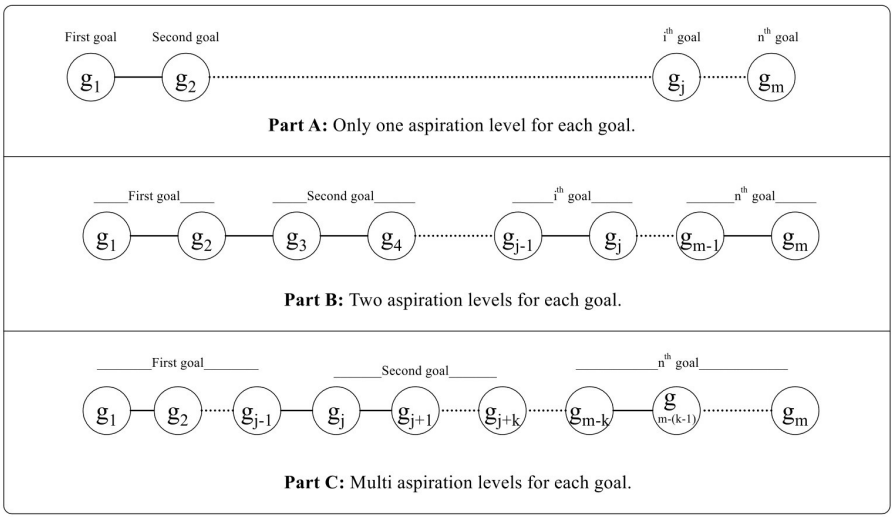

FIGURE 1. Neutrosophic aspirational levels for Multi-Objective Goal Programming problems

Three different classes for NMCGP problems can be considered as shown in (Fig. 1):

(I) Problems with one neutrosophic aspiration level for each goal (Part A),
(II) Problems with two neutrosophic aspiration levels for each goal (Part B),
(III) Problems with more than two (multiple) neutrosophic aspiration levels for each goal (Part C).

In order to model these problems, we present the truth, the indeterminacy, and the falsity membership functions for the ith neutrosophic goal as follows (Fig. 2):

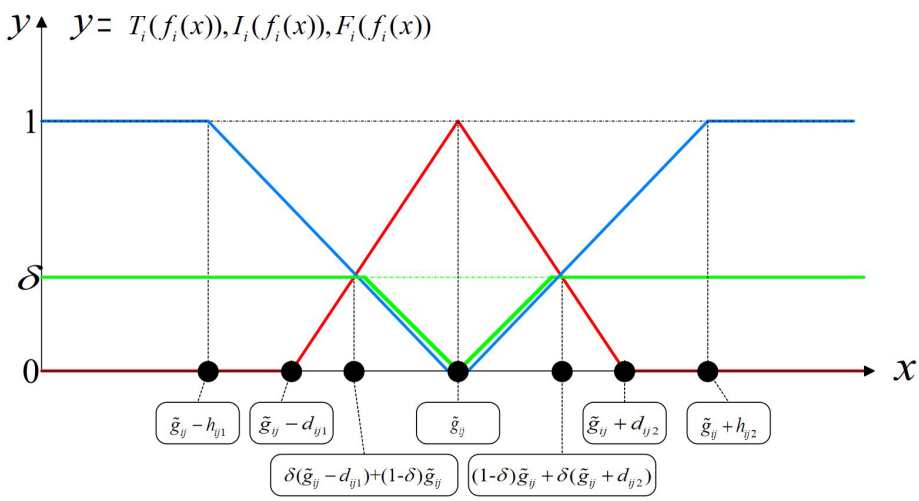

FIGURE 2. Truth, indeterminacy, and falsity membership functions for NMCGP

$$T_i(f_i(x)) = \begin{cases} \sum_{j=1}^{m} \frac{f_i(x)-\tilde{g}_{ij}}{d_{ij1}} S_{ij}(B), & \tilde{g}_{ij} - d_{ij1} \le f_i(x) \le \tilde{g}_{ij}, \\ 1, & f_i(x) = \tilde{g}_{ij}, \\ \sum_{j=1}^{m} \frac{\tilde{g}_{ij}-f_i(x)}{d_{ij2}} S_{ij}(B), & \tilde{g}_{ij} \le f_i(x) \le \tilde{g}_{ij} + d_{ij2}, \\ 0, & O.W. \end{cases} \tag{3.2}$$

$$I_i(f_i(x)) = \begin{cases} \sum_{j=1}^{m} \frac{\tilde{g}_{ij}-f_i(x)}{d_{ij2}} S_{ij}(B), & \delta(\tilde{g}_{ij} - d_{ij1}) + (1-\delta)\tilde{g}_{ij} \le f_i(x) \le \tilde{g}_{ij}, \\ \delta, & f_i(x) = \tilde{g}_{ij}, \\ \sum_{j=1}^{m} \frac{f_i(x)-\tilde{g}_{ij}}{d_{ij2}} S_{ij}(B), & \tilde{g}_{ij} \le f_i(x) \le (1-\delta)\tilde{g}_{ij} + \delta(\tilde{g}_{ij} + d_{ij2}), \\ \delta, & O.W. \end{cases} \tag{3.3}$$

$$F_i(f_i(x)) = \begin{cases} \sum_{j=1}^{m} \frac{\tilde{g}_{ij} - f_i(x)}{h_{ij1}} S_{ij}(B), & \tilde{g}_{ij} - h_{ij1} \leq f_i(x) \leq \tilde{g}_{ij}, \\ 0, & f_i(x) = \tilde{g}_{ij}, \\ \sum_{j=1}^{m} \frac{f_i(x) - \tilde{g}_{ij}}{h_{ij2}} S_{ij}(B) & \tilde{g}_{ij} \leq f_i(x) \leq \tilde{g}_{ij} + h_{ij2}, \\ 0, & O.W. \end{cases} \tag{3.4}$$

where d_{ij1} and d_{ij2} are the negative and positive tolerances for the truth membership function in the jth aspiration level of the ith goal, respectively. While h_{ij1} and h_{ij2} are the negative and positive tolerances for the falsity membership function in the jth aspiration level of the ith goal, respectively. $S_{ij}(B)$ is a function of binary serial numbers (i.e., a concept of height-balanced binary tree) attached to different aspiration levels. As shown in Fig. 3, a binary tree with n binary variables can represent 2^n aspiration levels.

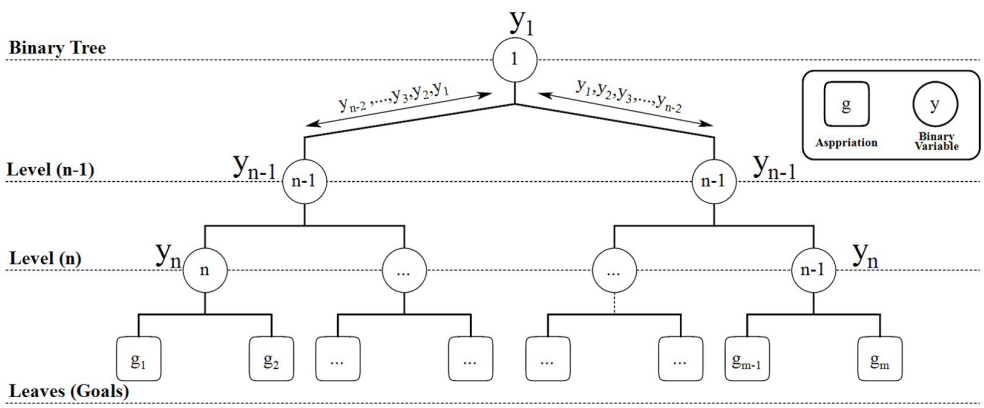

FIGURE 3. The concept of height-balanced binary tree

Nevertheless, the Neutrosophic Multi-Choice Poal Programming problem can be modeled as follows:

$$\begin{aligned} & Max\, T_i(f_i(x)),\ Min\, I_i(f_i(x)),\ Min\, F_i(f_i(x)), \\ & s.t. \\ & \qquad T_i(f_i(x)) \geq F_i(f_i(x)), \\ & \qquad T_i(f_i(x)) \geq I_i(f_i(x)), \\ & \qquad 0 \leq T_i(f_i(x)) + F_i(f_i(x)) + I_i(f_i(x)) \leq 3, \\ & \qquad T_i(f_i(x)), F_i(f_i(x)), I_i(f_i(x)) \geq 0, \\ & \qquad AX \geq b, \\ & \qquad X \geq 0, x \in X, X \in F,\ \ i = 1, \ldots, n. \end{aligned}$$

The previous NMCGP model is reformulated as follows:

$$
\begin{aligned}
&Max\, T_i(f_i(x)),\ Min\, I_i(f_i(x)),\ Min\, F_i(f_i(x)),\\
&s.t.\\
&\qquad T_i(f_i(x)) \geq F_i(f_i(x)),\\
&\qquad T_i(f_i(x)) \geq I_i(f_i(x)),\\
&\qquad 0 \leq T_i(f_i(x)) + F_i(f_i(x)) + I_i(f_i(x)) \leq 3,\\
&\qquad T_i(f_i(x)), F_i(f_i(x)), I_i(f_i(x)) \geq 0,\\
&\qquad AX \geq b,\\
&\qquad X \geq 0, x \in X, X \in F,\ \ i = 1, \ldots, n.
\end{aligned}
$$

Because $Max\ \alpha_i$ is equivalent to $Min\ (1-\alpha_i)$ where $0 \leq \alpha_i \leq 1$, the above model can be described as below:

$$
\begin{aligned}
&Max\, T_i(f_i(x)),\ Min\, I_i(f_i(x)),\ Min\, F_i(f_i(x)),\\
&s.t.\\
&\qquad T_i(f_i(x)) \geq F_i(f_i(x)),\\
&\qquad T_i(f_i(x)) \geq I_i(f_i(x)),\\
&\qquad 0 \leq T_i(f_i(x)) + F_i(f_i(x)) + I_i(f_i(x)) \leq 3,\\
&\qquad T_i(f_i(x)), F_i(f_i(x)), I_i(f_i(x)) \geq 0,\\
&\qquad AX \geq b,\\
&\qquad X \geq 0, x \in X, X \in F,\ \ i = 1, \ldots, n,
\end{aligned}
$$

where $\theta_i = 1 - \alpha_i$. The summary of the proposed model is shown in Fig. 4. To present a clear description of the proposed approach for solving the NMCGP problems, assume that we have three different goals with.

1. Only one neutrosophic aspiration level for each goal. This case can be modeled as:

$$
\begin{aligned}
&Min\ Z = \sum_{i=1}^{3} (\theta_i + \beta_i + \gamma_i),\\
&s.t.\\
&\qquad \theta_i \geq 1 - \frac{f_1(x) - \tilde{g}_1}{d_1^-},\ \theta_i \geq 1 - \frac{\tilde{g}_1 - f_1(x)}{d_1^+},\ \beta_1 \geq \frac{\tilde{g}_1 - f_1(x)}{d_{\bar{1}}},\\
&\qquad \beta_1 \geq \frac{f_1(x) - \tilde{g}_1}{d_1^+},\ \gamma_1 \geq \frac{\tilde{g}_1 - f_1(x)}{h_1^-},\ \gamma_1 \geq \frac{f_1(x) - \tilde{g}_1}{h_1^+},\\
&\qquad \alpha_2 \leq \frac{f_2(x) - \tilde{g}_2}{d_2^-},\ \alpha_2 \leq \frac{\tilde{g}_2 - f_2(x)}{d_2^+},\ \beta_2 \geq \frac{\tilde{g}_2 - f_2(x)}{d_{\bar{2}}},
\end{aligned}
$$

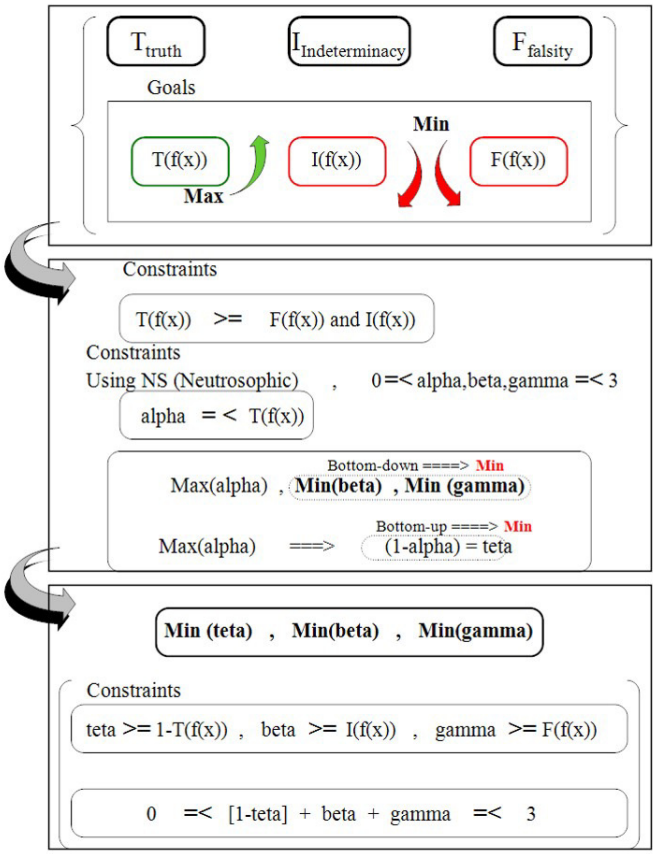

FIGURE 4. The framework of the proposed model

$$\beta_2 \geq \frac{f_2(x)-\tilde{g}_2}{d_2^+},\ \gamma_2 \geq \frac{\tilde{g}_2-f_2(x)}{h_2^-},\ \gamma_2 \geq \frac{f_2(x)-\tilde{g}_2}{h_2^+},$$
$$\alpha_3 \leq \frac{f_3(x)-\tilde{g}_3}{d_3^-},\ \alpha_3 \leq \frac{\tilde{g}_3-f_3(x)}{d_3^+},\ \beta_3 \geq \frac{\tilde{g}_3-f_3(x)}{d_3^-},$$
$$\beta_3 \geq \frac{f_3(x)-\tilde{g}_3}{d_3^+},\ \gamma_3 \geq \frac{\tilde{g}_3-f_3(x)}{h_3^-},\ \gamma_3 \geq \frac{f_3(x)-\tilde{g}_3}{h_3^+},$$
$$AX \geq b,\ 0 \leq (1-\theta_1)+\beta_1+\gamma_1 \leq 3,\ 0 \leq (1-\theta_2)+\beta_2+\gamma_2 \leq 3,$$
$$0 \leq (1-\theta_3)+\beta_3+\gamma_3 \leq 3,\ 1-\beta_1 \geq \theta_1,\ 1-\gamma_1 \geq \theta_1,$$
$$1-\beta_2 \geq \theta_2,\ 1-\gamma_2 \geq \theta_2,\ 1-\beta_3 \geq \theta_3,$$
$$1-\gamma_3 \geq \theta_3,\ 0 \leq \theta_1,\beta_1,\gamma_1 \leq 1,\ 0 \leq \theta_2,\beta_2,\gamma_2 \leq 1,\ 0 \leq \theta_3,\beta_3,\gamma_3 \leq 1,$$
$$X \geq 0,\ x \in X,\ X \in F\ (F \text{ is a feasible set}).$$

2. Two neutrosophic aspiration levels for each goal. It belongs to a case of neutrosophic goal programming problems with an either-or selection. Assume that the target in the first goal is to select an appropriate Neutrosophic aspiration level from either $\tilde{g}_1$ or $\tilde{g}_5$. Also, the target in the second goal is to select an appropriate neutrosophic aspiration level from either $\tilde{g}_2$ or $\tilde{g}_4$. Finally, the third goal's target is to select an appropriate neutrosophic aspiration level from either $\tilde{g}_3$ or $\tilde{g}_6$. This case can be modeled as follows:

$$
\begin{aligned}
&Min\, Z = (\theta_1+\beta_1+\gamma_1)+(\theta_2+\beta_2+\gamma_2)+(\theta_3+\beta_3+\gamma_3),\\
&s.t.\\
&\qquad \theta_1 \geq 1-\left(\frac{f_1(x)-\tilde{g}_1}{d_{11}^-}y_1+\frac{f_1(x)-\tilde{g}_5}{d_{15}^-}(1-y_1)\right),\\
&\qquad \theta_1 \geq 1-\left(\frac{\tilde{g}_1-f_1(x)}{d_{11}^+}y_1+\frac{\tilde{g}_5-f_1(x)}{d_{15}^+}(1-y_1)\right),\\
&\qquad \beta_1 \geq \frac{\tilde{g}_1-f_1(x)}{d_{11}^-}y_1+\frac{\tilde{g}_5-f_1(x)}{d_{15}^-}(1-y_1),\\
&\qquad \beta_1 \geq \frac{f_1(x)-\tilde{g}_1}{d_{11}^+}y_1+\frac{f_1(x)-\tilde{g}_5}{d_{15}^+}(1-y_1),\\
&\qquad \gamma_1 \geq \frac{\tilde{g}_1-f_1(x)}{h_{11}^-}y_1+\frac{\tilde{g}_5-f_1(x)}{h_{15}^-}(1-y_1),\\
&\qquad \gamma_1 \geq \frac{f_1(x)-\tilde{g}_1}{h_{11}^+}y_1+\frac{f_1(x)-\tilde{g}_5}{h_{15}^+}(1-y_1),\\
&\qquad \theta_2 \geq 1-\left(\frac{f_2(x)-\tilde{g}_2}{d_{22}^-}y_2+\frac{f_2(x)-\tilde{g}_4}{d_{24}^-}(1-y_2)\right),\\
&\qquad \theta_2 \geq 1-\left(\frac{\tilde{g}_2-f_2(x)}{d_{22}^+}y_2+\frac{\tilde{g}_4-f_2(x)}{d_{24}^+}(1-y_2)\right),\\
&\qquad \beta_2 \geq \frac{\tilde{g}_2-f_2(x)}{d_{22}^-}y_2+\frac{\tilde{g}_4-f_2(x)}{d_{24}^-}(1-y_2),\\
&\qquad \beta_2 \geq \frac{f_2(x)-\tilde{g}_2}{d_{22}^+}y_2+\frac{f_2(x)-\tilde{g}_4}{d_{24}^+}(1-y_2),\\
&\qquad \gamma_2 \geq \frac{\tilde{g}_2-f_2(x)}{h_{22}^-}y_2+\frac{\tilde{g}_4-f_2(x)}{h_{24}^-}(1-y_2),\\
&\qquad \gamma_2 \geq \frac{f_2(x)-\tilde{g}_2}{h_{22}^+}y_2+\frac{f_2(x)-\tilde{g}_4}{h_{24}^+}(1-y_2),\\
&\qquad \theta_3 \geq 1-\left(\frac{f_3(x)-\tilde{g}_3}{d_{33}^-}y_3+\frac{f_3(x)-\tilde{g}_6}{d_{36}^-}(1-y_3)\right),
\end{aligned}
$$

$$
\begin{aligned}
&\theta_3 \geq 1-\left(\frac{\tilde{g}_3-f_3(x)}{d_{33}^{+}}y_3+\frac{\tilde{g}_6-f_3(x)}{d_{36}^{+}}(1-y_3)\right),\\
&\beta_3 \geq \frac{\tilde{g}_3-f_3(x)}{d_{33}^{-}}y_3+\frac{\tilde{g}_6-f_3(x)}{d_{36}^{-}}(1-y_3),\\
&\beta_3 \geq \frac{f_3(x)-\tilde{g}_3}{d_{33}^{+}}y_3+\frac{f_3(x)-\tilde{g}_6}{d_{36}^{+}}(1-y_3),\\
&\gamma_3 \geq \frac{\tilde{g}_3-f_3(x)}{h_{33}^{-}}y_3+\frac{\tilde{g}_6-f_3(x)}{h_{36}^{-}}(1-y_3),\\
&\gamma_3 \geq \frac{f_3(x)-\tilde{g}_3}{h_{33}^{+}}y_3+\frac{f_3(x)-\tilde{g}_6}{h_{36}^{+}}(1-y_3),\\
&AX \geq b,\\
&0 \leq (1-\theta_1)+\beta_1+\gamma_1 \leq 3,\\
&0 \leq (1-\theta_2)+\beta_2+\gamma_2 \leq 3,\\
&0 \leq (1-\theta_3)+\beta_3+\gamma_3 \leq 3,\\
&1-\beta_1 \geq \theta_1,\ 1-\gamma_1 \geq \theta_1,\ 1-\beta_2 \geq \theta_2,\\
&1-\gamma_2 \geq \theta_2,\ 1-\beta_3 \geq \theta_3,\ 1-\gamma_3 \geq \theta_3,\\
&0 \leq \theta_1,\beta_1,\gamma_1 \leq 1,\ 0 \leq \theta_2,\beta_2,\gamma_2 \leq 1,\\
&0 \leq \theta_3,\beta_3,\gamma_3 \leq 1, X \geq 0, x \in X,\\
&X \in F\ (F \text{ is a feasible set}),
\end{aligned}
$$

where y_1, y_2 and y_3 represent the binary variables.

3. Multi neutrosophic aspiration levels for each goal. It belongs to the case of neutrosophic goal programming problems with multi-choice selection. Here we arbitrarily considered three neutrosophic aspiration levels for each goal. Assume that the target in the first goal is to select an appropriate neutrosophic aspiration level between $\tilde{g}_1$, $\tilde{g}_5$ and $\tilde{g}_8$. Also, the target in the second goal is to select an appropriate neutrosophic aspiration level between $\tilde{g}_2$, $\tilde{g}_4$ and $\tilde{g}_7$. Finally, the target in the third goal is to select an appropriate neutrosophic aspiration level between $\tilde{g}_3$, $\tilde{g}_6$ and $\tilde{g}_9$. This case can be modeled as follows:

$$
\begin{aligned}
&Min\ Z=(\theta_1+\beta_1+\gamma_1)+(\theta_2+\beta_2+\gamma_2)+(\theta_3+\beta_3+\gamma_3),\\
&s.t.\\
&\qquad \theta_1 \geq 1-\left(\frac{f_1(x)-\tilde{g}_1}{d_{11}^{-}}y_1y_2+\frac{f_1(x)-\tilde{g}_5}{d_{15}^{-}}y_1(1-y_2)+\frac{f_1(x)-\tilde{g}_8}{d_{18}^{-}}y_2(1-y_1)\right),\\
&\qquad \theta_1 \geq 1-\left(\frac{\tilde{g}_1-f_1(x)}{d_{11}^{+}}y_1y_2+\frac{\tilde{g}_5-f_1(x)}{d_{15}^{+}}y_1(1-y_2)+\frac{\tilde{g}_8-f_1(x)}{d_{18}^{+}}y_2(1-y_1)\right),
\end{aligned}
$$

$$\beta_1 \geq \frac{\tilde{g}_1 - f_1(x)}{d_{11}^-} y_1 y_2 + \frac{\tilde{g}_5 - f_1(x)}{d_{15}^-} y_1(1-y_2) + \frac{\tilde{g}_8 - f_1(x)}{d_{18}^-} y_2(1-y_1),$$

$$\beta_1 \geq \frac{f_1(x) - \tilde{g}_1}{d_{11}^+} y_1 y_2 + \frac{f_1(x) - \tilde{g}_5}{d_{15}^+} y_1(1-y_2) + \frac{f_1(x) - \tilde{g}_8}{d_{18}^+} y_2(1-y_1),$$

$$\gamma_1 \geq \frac{\tilde{g}_1 - f_1(x)}{h_{11}^-} y_1 y_2 + \frac{\tilde{g}_5 - f_1(x)}{h_{15}^-} y_1(1-y_2) + \frac{\tilde{g}_8 - f_1(x)}{h_{18}^-} y_2(1-y_1),$$

$$\gamma_1 \geq \frac{f_1(x) - \tilde{g}_1}{h_{11}^+} y_1 y_2 + \frac{f_1(x) - \tilde{g}_5}{h_{15}^+} y_1(1-y_2) + \frac{f_1(x) - \tilde{g}_8}{h_{18}^+} y_2(1-y_1),$$

$$\theta_2 \geq 1 - \left(\frac{f_2(x) - \tilde{g}_2}{d_{22}^-} y_3 y_4 + \frac{f_2(x) - \tilde{g}_4}{d_{24}^-} y_3(1-y_4) + \frac{f_2(x) - \tilde{g}_7}{d_{27}^-} y_4(1-y_3) \right),$$

$$\theta_2 \geq 1 - \left(\frac{\tilde{g}_2 - f_2(x)}{d_{22}^+} y_3 y_4 + \frac{\tilde{g}_4 - f_2(x)}{d_{24}^+} y_3(1-y_4) + \frac{\tilde{g}_7 - f_2(x)}{d_{27}^+} y_4(1-y_3) \right),$$

$$\beta_2 \geq \frac{\tilde{g}_2 - f_2(x)}{d_{22}^-} y_3 y_4 + \frac{\tilde{g}_4 - f_2(x)}{d_{24}^-} y_3(1-y_4) + \frac{\tilde{g}_7 - f_2(x)}{d_{27}^-} y_4(1-y_3),$$

$$\beta_2 \geq \frac{f_2(x) - \tilde{g}_2}{d_{22}^+} y_3 y_4 + \frac{f_2(x) - \tilde{g}_4}{d_{24}^+} y_3(1-y_4) + \frac{f_2(x) - \tilde{g}_7}{d_{27}^+} y_4(1-y_3),$$

$$\gamma_2 \geq \frac{\tilde{g}_2 - f_2(x)}{h_{22}^-} y_3 y_4 + \frac{\tilde{g}_4 - f_2(x)}{h_{24}^-} y_3(1-y_4) + \frac{\tilde{g}_7 - f_2(x)}{h_{27}^-} y_4(1-y_3),$$

$$\gamma_2 \geq \frac{f_2(x) - \tilde{g}_2}{h_{22}^+} y_3 y_4 + \frac{f_2(x) - \tilde{g}_4}{h_{24}^+} y_3(1-y_4) + \frac{f_2(x) - \tilde{g}_7}{h_{27}^+} y_4(1-y_3),$$

$$\theta_3 \geq 1 - \left(\frac{f_3(x) - \tilde{g}_3}{d_{33}^-} y_5 y_6 + \frac{f_3(x) - \tilde{g}_6}{d_{36}^-} y_5(1-y_6) + \frac{f_3(x) - \tilde{g}_9}{d_{39}^-} y_6(1-y_5) \right),$$

$$\theta_3 \geq 1 - \left(\frac{\tilde{g}_3 - f_3(x)}{d_{33}^+} y_5 y_6 + \frac{\tilde{g}_6 - f_3(x)}{d_{36}^+} y_5(1-y_6) + \frac{\tilde{g}_9 - f_3(x)}{d_{39}^+} y_6(1-y_5) \right),$$

$$\beta_3 \geq \frac{\tilde{g}_3 - f_3(x)}{d_{33}^-} y_5 y_6 + \frac{\tilde{g}_6 - f_3(x)}{d_{36}^-} y_5(1-y_6) + \frac{\tilde{g}_9 - f_3(x)}{d_{39}^-} y_6(1-y_5),$$

$$\beta_3 \geq \frac{f_3(x) - \tilde{g}_3}{d_{33}^+} y_5 y_6 + \frac{f_3(x) - \tilde{g}_6}{d_{36}^+} y_5(1-y_6) + \frac{f_3(x) - \tilde{g}_9}{d_{39}^+} y_6(1-y_5),$$

$$\gamma_3 \geq \frac{\tilde{g}_3 - f_3(x)}{h_{33}^-} y_5 y_6 + \frac{\tilde{g}_6 - f_3(x)}{h_{36}^-} y_5(1-y_6) + \frac{\tilde{g}_9 - f_3(x)}{h_{39}^-} y_6(1-y_5),$$

$$\gamma_3 \geq \frac{f_3(x) - \tilde{g}_3}{h_{33}^+} y_5 y_6 + \frac{f_3(x) - \tilde{g}_6}{h_{36}^+} y_5(1-y_6) + \frac{f_3(x) - \tilde{g}_9}{h_{39}^+} y_6(1-y_5),$$

$$AX \geq b,$$

$$y_1 + y_2 \geq 1,\ y_3 + y_4 \geq 1,\ y_5 + y_6 \geq 1,$$

$$0 \leq (1-\theta_1)+\beta_1+\gamma_1 \leq 3,\ 0 \leq (1-\theta_2)+\beta_2+\gamma_2 \leq 3,$$
$$0 \leq (1-\theta_3)+\beta_3+\gamma_3 \leq 3,$$
$$1-\beta_1 \geq \theta_1, 1-\gamma_1 \geq \theta_1,\ 1-\beta_2 \geq \theta_2,$$
$$1-\gamma_2 \geq \theta_2,\ 1-\beta_3 \geq \theta_3,\ 1-\gamma_3 \geq \theta_3,$$
$$0 \leq \theta_1, \beta_1, \gamma_1 \leq 1,\ 0 \leq \theta_2, \beta_2, \gamma_2 \leq 1,\ 0 \leq \theta_3, \beta_3, \gamma_3 \leq 1,$$
$$X \geq 0,\ x \in X,\ X \in F\ (F \text{ is a feasible set}).$$

4. Numerical illustration

A factory produces three different products such as P_1, P_2 and P_3. To produce each product we need three resources as Z_1, Z_2, and Z_3. The value of these resources is shown in Table 1. Assume that there exist six different markets willing to buy some products. The information about products and demand for them is presented in Table 2. We expect that the profit earned by selling the products should be at least 2000 dollars. The factory's policy is to select only one applicant to sell each product.

Remark 1. In this example, the negative and positive tolerances for acceptance and rejection of each goal are considered equal values.

Table 1. The values of required resources to produce a product

Products	**Markets**	**Demands**	***d***	***h***	**Profit**
P1	Market 1	80	7	5	20
	Market 2	90	5	2	
	Market 3	50	4	1	
P2	Market 4	28	4	9	70
	Market 5	26	6	7	
P3	Market 6	10	5	4	85

Table 2. Information about products

Resources	**Products**		
	P1	**P2**	**P3**
Z1	4	5	9
Z2	7	6	2
Z3	3	8	7
Capacity	500	600	400

This NMCGP problem can be formulated as follows:

$$Min\ Z = (\theta_1+\beta_1+\gamma_1)+(\theta_2+\beta_2+\gamma_2)+(\theta_3+\beta_3+\gamma_3)$$

$s.t.$

$$\theta_1 \geq 1-\left(\frac{P_1-80}{7}y_1y_2+\frac{P_1-90}{5}y_1(1-y_2)+\frac{P_1-50}{4}y_2(1-y_1)\right),$$
$$\theta_1 \geq 1-\left(\frac{80-P_1}{7}y_1y_2+\frac{90-P_1}{5}y_1(1-y_2)+\frac{50-P_1}{4}y_2(1-y_1)\right),$$
$$\beta_1 \geq \frac{80-P_1}{7}y_1y_2+\frac{90-P_1}{5}y_1(1-y_2)+\frac{50-P_1}{4}y_2(1-y_1),$$
$$\beta_1 \geq \frac{P_1-80}{7}y_1y_2+\frac{P_1-90}{5}y_1(1-y_2)+\frac{P_1-50}{4}y_2(1-y_1),$$
$$\gamma_1 \geq \frac{80-P_1}{5}y_1y_2+\frac{90-P_1}{2}y_1(1-y_2)+\frac{50-P_1}{1}y_2(1-y_1),$$
$$\gamma_1 \geq \frac{P_1-80}{5}y_1y_2+\frac{P_1-90}{2}y_1(1-y_2)+\frac{P_1-50}{1}y_2(1-y_1),$$
$$\theta_2 \geq 1-\left(\frac{P_2-28}{4}y_3+\frac{P_2-26}{6}(1-y_3)\right),$$
$$\theta_2 \geq 1-\left(\frac{28-P_2}{4}y_3+\frac{26-P_2}{6}(1-y_3)\right),$$
$$\beta_2 \geq \frac{28-P_2}{4}y_3+\frac{26-P_2}{6}(1-y_3),$$
$$\beta_2 \geq \frac{P_2-28}{4}y_3+\frac{P_2-26}{6}(1-y_3),$$
$$\gamma_2 \geq \frac{28-P_2}{9}y_3+\frac{26-P_2}{7}(1-y_3),$$
$$\gamma_2 \geq \frac{P_2-28}{9}y_3+\frac{P_2-26}{7}(1-y_3),$$
$$\theta_3 \geq 1-\left(\frac{P_3-10}{5}\right),\ \theta_3 \geq 1-\left(\frac{10-P_3}{5}\right),\ \beta_3 \geq \frac{10-P_3}{5},$$
$$\beta_3 \geq \frac{P_3-10}{5},\ \gamma_3 \geq \frac{10-P_3}{4},\ \gamma_3 \geq \frac{P_3-10}{4},$$
$$20P_1+70P_2+85P_3 \geq 2000,$$
$$x_{11}+x_{12}+x_{13} \leq 500,\ x_{21}+x_{22}+x_{23} \leq 600,\ x_{31}+x_{32}+x_{33} \leq 400,$$
$$P_1 \leq \frac{x_{11}}{4}, P_1 \leq \frac{x_{12}}{7}, P_1 \leq \frac{x_{13}}{3},\ P_2 \leq \frac{x_{21}}{5}, P_2 \leq \frac{x_{22}}{6}, P_2 \leq \frac{x_{23}}{8},$$
$$P_3 \leq \frac{x_{31}}{9}, P_3 \leq \frac{x_{32}}{2}, P_3 \leq \frac{x_{33}}{7},$$
$$y_1+y_2 \geq 1,\ 0 \leq (1-\theta_1)+\beta_1+\gamma_1 \leq 3,$$
$$0 \leq (1-\theta_2)+\beta_2+\gamma_2 \leq 3,\ 0 \leq (1-\theta_3)+\beta_3+\gamma_3 \leq 3,$$
$$1-\beta_1 \geq \theta_1,\ 1-\gamma_1 \geq \theta_1,\ 1-\beta_2 \geq \theta_2,$$

$$1-\gamma_2 \geq \theta_2,\ 1-\beta_3 \geq \theta_3,\ 1-\gamma_3 \geq \theta_3,$$
$$0 \leq \theta_1, \beta_1, \gamma_1 \leq 1,\ 0 \leq \theta_2, \beta_2, \gamma_2 \leq 1,\ 0 \leq \theta_3, \beta_3, \gamma_3 \leq 1,$$
$$X \geq 0,\ x \in X.$$

The optimal solutions of the given problem are $P_1 = 0.20, P_2 = 5.13, P_3 = 22.22$, $y_1 = 2.87, y_2 = 1.23$, and $y_3 = 0.36$. In this regard, the obtained degrees for truth, indeterminacy, and falsity of first, second, and third neutrosophic goals are respectively obtained as $(\theta_1, \beta_1, \gamma_1) = (0.7, 0.4, 0.5), (\theta_2, \beta_2, \gamma_2) = (0.3, 0.5, 0.5)$, $(\theta_3, \beta_3, \gamma_3) = (0.1, 0.8, 0.8)$. Subsequently, $Z = 4.6$.

4.1. *Calculation Complexity Analysis*

The Space Complexity of our proposed model indicates how much memory space is required. This is intended to solve computational problems by considering input functions and typescripts and the amount of space that needs to be run is in Gigabytes. It is usually used for complex problems with many goals.

The proposed method is implemented on MathPy 2.7.3 on Python 3.03, and Ubuntu 20.0 OS on a virtual machine in VMbox (VM). The simulation machine has 16 G RAM, Intel i4 processor running at 3.2 GHz. All tests are based on three blocks with a block size of the 128 MB distributed file system. To investigate the Space Complexity (SC) of the proposed method, we have used Calculation Processing (CP) in each part of the method. To run the simulation. In this case, the evaluation based on SC to organize different products has been provided. We have considered different cases for P_1, P_2,P_3 and their combinations to evaluate the SC based on memory usage. The meaning of simulation steps is finishing each part of CP. As shown in Part A, Fig. 5, the details of memory usage are illustrated. The SCP (Space of CP) for various P_i $(1 \leq i \leq 3)$ is 3 GHz memory and computational execution time is 12 minutes. In this case, we considered 10 steps of calculations for simulation. With increasing the steps of simulation from 10 to 20, we can observe the increases in SCP and time complexity (Part B, Fig. 5). Subsequently, by increasing the steps of simulation from 20 to 25steps, we can observe the increases in SC to 10 G and increases in TC to 32 min (Part C, Fig. 5). In summary, this computational process shows that the proposed method can be used on ordinary machines by finding many targets and indicates a low computational Memory and CPU space.

5. Conclusion

This paper proposed a new model and a new method for solving Neutrosophic multi-choice goal programming problems to increase accuracy and reduce doubts and mistakes in the decision-making process. NMCGP, as a generalization of classical MCGP and Fuzzy MCGP, is a helpful technique to mark indeterminacy as an independent component in the Decision-Making process. Using NMCGP, we can consider all aspects of real-world problems in determining our goals' aspiration levels.

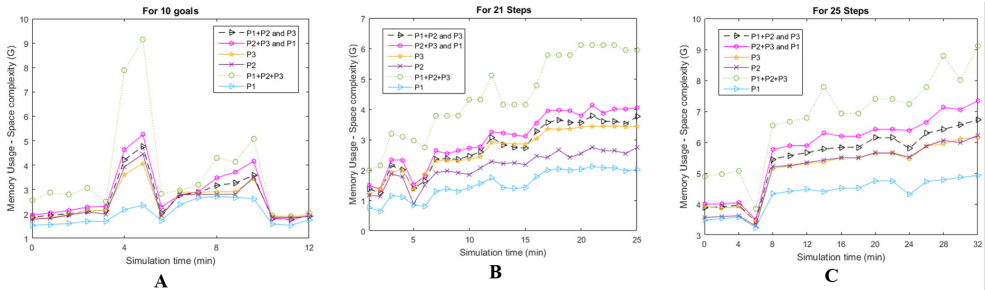

FIGURE 5. Different steps of calculations for simulation

The proposed approach can help us solve problems in all fields, including transportation, management, production, marketing, etc. In this respect, a practical example is presented to illustrate the applicability of the proposed method. The computational process shows that the proposed model can be used on ordinary machines by finding many targets and indicates a low computational memory and CPU space.

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ON CO-ANNIHILATORS IN EQUALITY ALGEBRAS

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Abstract. In this paper, we introduce the notion of co-annihilator on a lattice equality algebra $\mathbb{E}$ and investigate some basic properties of them. Specially, by using the generated filters in $\mathbb{E}$, we prove that the set of all filters of $\mathbb{E}$ forms a bounded distributive pseudo-complemented lattice, which pseudo-complemented of any filter is co-annihilator of it. Finally, we construct a Boolean algebra by the set of all co-annihilators of $\mathbb{E}$.

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1. Introduction

A more general algebraic structure in logic without contractions is residuated lattices [14]. Among logical algebras, residuated lattices have received the most attention due to their interesting properties and including two important sub-classes: BL-algebras and MV-algebras. Fuzzy type theory was developed as a higher order fuzzy logic. Novák and De Beats generalized residuated lattices and proposed EQ-algebra [10]. Because by replacing the product operation with a lesser or equal operation, we get an EQ-algebra again, Jenei [7] introduced a new algebra, called equality algebra. Since equality algebra can be a good alternative to possible algebraic semantics for fuzzy type theory, the study of equality algebra is very valuable.

In [3], it was proved that equality algebras and BCK-meet semilattices (under distributivity condition) correspond to each other. Because different filters have natural expressions as diverse sets of provable formulas, filter theory has a significant impact on the study of logical algebras. For this, in [1], Borzooei et al. introduced some types of filters in equality algebras. For more recent studies about equality algebras, you can see [5, 6, 11, 12].

Many works have been done with respect to the co-annihilators and annihilators. Davey studied the relationship between minimal prime ideals conditions and annihilators conditions on distributive lattices [4]. Turunen defined co-annihilator of a non-empty subset of a BL-algebra and proved some of its properties [13]. Leustean

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introduced the notion of co-annihilator relative to a filter F on pseudo BL-algebras. Then Meng introduced generalized co-annihilators in BL-algebras and gave characterizations of prime and minimal prime filters [9]. Also, Zou et al introduced the notion of annihilators in BL-algebras and investigated some related properties of them in [16].

In this paper, we introduce the notion of co-annihilator in equality algebras. We study basic properties of co-annihilators and investigate relationship between them and filters. Also, we get that the lattice of all filters of an equality algebra forms a pseudo-complemented lattice. Moreover, the collection of co-annihilators of an equality algebra forms a Boolean algebra. The paper is organized as follows: In Section 2, we gather the basic notions and results on equality algebras. In Section 3, we study generated filters in equality algebras and get some interesting results about them. In Section 4, we introduce the co-annihilators on a lattice equality algebra and investigate some related properties of them.

2. Preliminaries

In this section, we gather some basic notions relevant to equality algebra which will be needed in the next sections.

Definition 1 ([7, Definition 1.1])**.** An algebraic structure $\mathbb{E} = (\mathbb{E}; \wedge, \sim, 1)$ of type $(2,2,0)$ is called an *equality algebra*, if, for all $\alpha, \gamma, \eta \in \mathbb{E}$, it satisfies the following conditions:

$(E1)$ $(\mathbb{E}, \wedge, 1)$ is a commutative idempotent integral monoid,
$(E2)$ $\alpha \sim \gamma = \gamma \sim \alpha$,
$(E3)$ $\alpha \sim \alpha = 1$,
$(E4)$ $\alpha \sim 1 = \alpha$,
$(E5)$ $\alpha \leq \gamma \leq \eta$ implies $\alpha \sim \eta \leq \gamma \sim \eta$ and $\alpha \sim \eta \leq \alpha \sim \gamma$,
$(E6)$ $\alpha \sim \gamma \leq (\alpha \wedge \eta) \sim (\gamma \wedge \eta)$,
$(E7)$ $\alpha \sim \gamma \leq (\alpha \sim \eta) \sim (\gamma \sim \eta)$.

The operation $\wedge$ is called *meet* and $\sim$ is an *equality* operation.

Notation 1. From now on, we let $(\mathbb{E}, \sim, \wedge, 0, 1)$ or $\mathbb{E}$ be an equality algebra, unless otherwise state.

On $\mathcal{E}$ we write $\alpha \leq \gamma$ if and only if $\alpha \wedge \gamma = \alpha$. Clearly, $(\mathbb{E}, \leq)$ is a poset. Also, other two derived operations are defined, as the following, and we call them *implication* and *equivalence*, respectively:

$$\alpha \twoheadrightarrow \gamma = \alpha \sim (\alpha \wedge \gamma) \quad \text{and} \quad \alpha \leftrightarrow \gamma = (\alpha \twoheadrightarrow \gamma) \wedge (\gamma \twoheadrightarrow \alpha).$$

Proposition 1 ([7, Proposition 2], [15, Proposition 3.1])**.** *The following statements hold on $\mathbb{E}$, for all $\alpha, \gamma, \eta \in \mathbb{E}$:*

(i) $\alpha \twoheadrightarrow \gamma = 1$ *if and only if* $\alpha \leq \gamma$,

(ii) $1 \twoheadrightarrow \alpha = \alpha$, $\alpha \twoheadrightarrow 1 = 1$, *and* $\alpha \twoheadrightarrow \alpha = 1$,
(iii) $\alpha \leq \gamma \to \alpha$,
(iv) $\alpha \leq (\alpha \twoheadrightarrow \gamma) \twoheadrightarrow \gamma$,
(v) $\alpha \twoheadrightarrow (\gamma \twoheadrightarrow \eta) = \gamma \twoheadrightarrow (\alpha \twoheadrightarrow \eta)$,
(vi) $\alpha \leq \gamma$ *implies* $\gamma \twoheadrightarrow \eta \leq \alpha \twoheadrightarrow \eta$ *and* $\eta \to \alpha \leq \eta \to \gamma$,
(vii) $\alpha \twoheadrightarrow \gamma = \alpha \twoheadrightarrow (\alpha \wedge \gamma)$,
(viii) $\alpha \to \gamma \leq (\eta \twoheadrightarrow \alpha) \twoheadrightarrow (\eta \twoheadrightarrow \gamma)$.

A *bounded* equality algebra $\mathbb{E}$ is an equality algebra with the least element $0 \in \mathbb{E}$ such that $0 \leq \alpha$, for all $\alpha \in \mathbb{E}$. If $\mathbb{E}$ is bounded, then we define the operation negation "$^-$ " on $\mathbb{E}$ by, $\alpha^- = \alpha \twoheadrightarrow 0 = \alpha \sim 0$, for all $\alpha \in \mathbb{E}$. If $\alpha^{--} = \alpha$, for all $\alpha \in \mathbb{E}$, then the bounded equality algebra $\mathbb{E}$ is called *involutive*. Also, $\mathbb{E}$ is called prelinear, if 1 is the unique upper bound of the set $\{\alpha \twoheadrightarrow \gamma,\ \gamma \twoheadrightarrow \alpha\}$ for all $\alpha, \gamma \in \mathbb{E}$. A *lattice equality algebra* is an equality algebra which is a lattice.

Proposition 2 ([15, Proposition 3.7])**.** *If* $\mathbb{E}$ *is a lattice, then for all* $\alpha, \gamma, \eta \in \mathbb{E}$*, the following statements hold:*

(i) $(\alpha \vee \gamma) \twoheadrightarrow \eta = (\alpha \twoheadrightarrow \eta) \wedge (\gamma \twoheadrightarrow \eta)$,
(ii) $\alpha \twoheadrightarrow \gamma = (\alpha \vee \gamma) \twoheadrightarrow \gamma$.

Definition 2 ([8, Definition 3])**.** Let $\varnothing \neq \mathbb{F} \subseteq \mathbb{E}$. Then $\mathbb{F}$ is called a *deductive system* or *filter* of $\mathbb{E}$, if for all $\alpha, \gamma \in \mathbb{E}$, we have

(i) $\alpha \in \mathbb{F}$ and $\alpha \leq \gamma$ imply $\gamma \in \mathbb{F}$;
(ii) $\alpha \in \mathbb{F}$ and $\alpha \sim \gamma \in \mathbb{F}$ imply $\gamma \in \mathbb{F}$.

Proposition 3 ([8, Proposition 3])**.** *Let* $\mathbb{F} \subseteq \mathbb{E}$*. Then* $\mathbb{F} \in \mathcal{F}(\mathbb{E})$ *if and only if, for all* $\alpha, \gamma \in \mathbb{E}$

(i) $1 \in \mathbb{F}$,
(ii) *if* $\alpha \in \mathbb{F}$ *and* $\alpha \twoheadrightarrow \gamma \in \mathbb{F}$*, then* $\gamma \in \mathbb{F}$.

The set of all filters of $\mathbb{E}$ is denoted by $\mathcal{F}(\mathbb{E})$. Clearly, $1 \in \mathbb{F}$, for all $\mathbb{F} \in \mathcal{F}(\mathbb{E})$. A filter $\mathbb{F}$ of $\mathbb{E}$ is called *proper* if $\mathbb{F} \neq \mathbb{E}$. Obviously, if $\mathbb{E}$ is bounded, then a filter is proper if and only if $0 \notin \mathbb{F}$.

Definition 3 ([1, Definition 3.1])**.** Let $\mathbb{F} \subseteq \mathbb{E}$ such that $1 \in \mathbb{F}$. Then $\mathbb{F}$ is called a *positive implicative filter* of $\mathbb{E}$ if $\alpha \twoheadrightarrow (\gamma \twoheadrightarrow \eta) \in \mathbb{F}$ and $\alpha \twoheadrightarrow \gamma \in \mathbb{F}$ imply $\alpha \twoheadrightarrow \eta \in \mathbb{F}$, for all $\alpha, \gamma, \eta \in \mathbb{E}$.

Definition 4 ([11, Definition 4.1])**.** Let $(\mathbb{E}; \wedge_{\mathbb{E}}, \sim_{\mathbb{E}}, 1_e)$ and $(\mathbb{Q}; \wedge_{\mathbb{Q}}, \sim_{\mathbb{Q}}, 1_{\mathbb{Q}})$ be two equality algebras. Then a map $f: \mathbb{E} \twoheadrightarrow \mathbb{Q}$ is called an *equality homomorphism*, if for all $\alpha, \gamma \in \mathbb{E}$, the following conditions hold:

$$f(\alpha \wedge_{\mathbb{E}} \gamma) = f(\alpha) \wedge_{\mathbb{Q}} f(\gamma) \qquad \text{and} \qquad f(\alpha \sim_{\mathbb{E}} \gamma) = f(\alpha) \sim_{\mathbb{Q}} f(\gamma).$$

Moreover, if $\mathbb{E}$ and $\mathbb{Q}$ are two bounded, then an equality homomorphism f is called bounded, if $f(0_{\mathbb{E}}) = 0_{\mathbb{Q}}$. An equality homomorphism f is called an equality *(epimorphism) monomorphism*, if f is an (onto)one-to-one mapping and an equality

homomorphism f is called an *equality isomorphism*, if f is a one-to-one and onto mapping.

Definition 5 ([2, Definition 5.1])**.** Consider X as a non-empty set. A mapping $\phi : \mathcal{P}(X) \to \mathcal{P}(X)$ is called a *closure map* on X, if for all $Y, Z \in \mathcal{P}(X)$, the following statements hold:

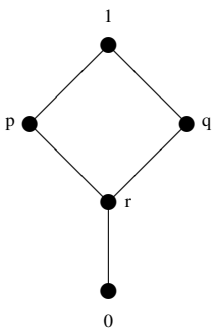
$(C1)$ $Y \subseteq \phi(Y)$,
$(C2)$ $\phi^2(Y) = \phi(Y)$,
$(C3)$ $Y \subseteq Z$ implies $\phi(Y) \subseteq \phi(Z)$.

3. Some results on filters of equality algebras

In this section, we investigate some properties of generated filter on equality algebras, that these results will be used in the next section to prove some theorems related to co-annihilators.

Definition 6. Let $\mathcal{X} \subseteq \mathbb{E}$. The smallest filter of $\mathbb{E}$ containing $\mathcal{X}$ is called *the generated filter by $\mathcal{X}$ in* $\mathbb{E}$ which is denoted by $\langle \mathcal{X} \rangle$. Indeed, $\langle \mathcal{X} \rangle = \bigcap_{\mathcal{X} \subseteq \mathbb{F} \in \mathcal{F}(\mathbb{E})} \mathbb{F}$.

Example 1. Let $\mathbb{E} = \{0, p, q, r, 1\}$ be a poset with the following Hasse diagram. Define operation "$\sim$" on $\mathbb{E}$ as follows:



$\sim$	0	r	p	q	1
0	1	0	0	0	1
r	0	1	q	p	r
p	0	q	1	r	p
q	0	p	r	1	q
1	0	r	p	q	1

$\twoheadrightarrow$	0	r	p	q	1
0	1	1	1	1	1
r	0	1	1	1	1
p	0	q	1	q	1
q	0	p	p	1	1
1	0	r	p	q	1

Then $(\mathbb{E}, \sim, \wedge, 1)$ is an equality algebra. If $\mathcal{X} = \{p, q\}$, then $\langle \mathcal{X} \rangle = \{p, q, r, 1\}$.

Proposition 4. *Let $\varnothing \neq \mathcal{X} \subseteq \mathbb{E}$. Then*

$$\langle \mathcal{X} \rangle = \{\alpha \in \mathbb{E} \mid \textit{for some } n \in \mathbb{N} \textit{ and } p_1, \ldots, p_n \in \mathcal{X}, \\ p_1 \to (p_2 \to (\cdots \to (p_n \to \alpha) \ldots)) = 1\}.$$

In particular, for any element $p \in \mathbb{E}$, we have

$$\langle p \rangle = \{\alpha \in \mathbb{E} \mid \textit{for some } n \in \mathbb{N},\ p \to^n \alpha = 1\},$$

where $\alpha \twoheadrightarrow^0 \gamma = \gamma$ and $\alpha \twoheadrightarrow^n \gamma = \alpha \twoheadrightarrow (\alpha \twoheadrightarrow^{n-1} \gamma)$.

Proof. Suppose

$$B = \{\alpha \in \mathbb{E} \mid \text{ for some } n \in \mathbb{N} \text{ and } p_1, \dots, p_n \in \mathcal{X}, \\ p_1 \to (p_2 \to (\cdots \to (p_n \to \alpha) \dots)) = 1\}.$$

Since $p \twoheadrightarrow 1 = 1$, we have $1 \in B$, and so $B \neq \varnothing$. In addition, since $p \twoheadrightarrow p = 1$, for all $p \in \mathcal{X}$, we get $p \in B$, thus $\mathcal{X} \subseteq B$. Now, we prove $B \in \mathcal{F}(\mathbb{E})$. Let $\alpha, \alpha \twoheadrightarrow \gamma \in B$. Then there exist $m, n \in \mathbb{N}$, $p_1, \dots, p_n \in \mathcal{X}$ and $q_1, \dots, q_m \in \mathcal{X}$ such that

$$p_1 \twoheadrightarrow (p_2 \twoheadrightarrow (\dots \twoheadrightarrow (p_n \to \alpha) \dots)) = 1.$$

$$q_1 \twoheadrightarrow (q_2 \twoheadrightarrow (\dots \twoheadrightarrow (q_m \twoheadrightarrow (\alpha \twoheadrightarrow \gamma)) \dots)) = 1.$$

Thus by Proposition 1(viii), we get

$$\begin{aligned}
\alpha \twoheadrightarrow \gamma &\leq (p_n \twoheadrightarrow \alpha) \twoheadrightarrow (p_n \twoheadrightarrow \gamma) \\
&\leq (p_{n-1} \twoheadrightarrow (p_n \twoheadrightarrow \alpha)) \twoheadrightarrow (p_{n-1} \twoheadrightarrow (p_n \twoheadrightarrow \gamma)) \\
&\;\;\vdots \\
&\leq \underbrace{[p_1 \twoheadrightarrow (p_2 \twoheadrightarrow (\dots (p_n \twoheadrightarrow \alpha) \dots))]}_{1} \twoheadrightarrow \underbrace{[p_1 \twoheadrightarrow (p_2 \twoheadrightarrow (\dots (p_n \twoheadrightarrow \gamma) \dots))]}_{Y} \\
&= 1 \twoheadrightarrow Y \\
&= Y.
\end{aligned}$$

Since $\alpha \twoheadrightarrow \gamma \leq Y$, by Proposition 1(vi), we get $q_m \twoheadrightarrow (\alpha \twoheadrightarrow \gamma) \leq q_m \twoheadrightarrow Y$. By repeating this method, we conclude that

$$q_1 \twoheadrightarrow (q_2 \twoheadrightarrow (\dots \twoheadrightarrow (q_m \twoheadrightarrow (\alpha \twoheadrightarrow \gamma)) \dots)) \leq q_1 \twoheadrightarrow (q_2 \twoheadrightarrow (\dots \twoheadrightarrow (q_m \twoheadrightarrow Y)) \dots).$$

Since $q_1 \twoheadrightarrow (q_2 \twoheadrightarrow (\dots \twoheadrightarrow (q_m \twoheadrightarrow (\alpha \twoheadrightarrow \gamma)) \dots)) = 1$, we have $q_1 \twoheadrightarrow (q_2 \twoheadrightarrow (\dots \twoheadrightarrow (q_m \twoheadrightarrow Y) \dots)) = 1$ and so

$$q_1 \twoheadrightarrow (q_2 \twoheadrightarrow (\dots \twoheadrightarrow (q_m \twoheadrightarrow (p_1 \twoheadrightarrow (\dots \twoheadrightarrow (p_n \twoheadrightarrow \gamma)))) \dots)) = 1.$$

Hence $\gamma \in B$. Therefore, $B \in \mathcal{F}(\mathbb{E})$ containing $\mathcal{X}$. It is enough to prove that B is the smallest filter of $\mathbb{E}$ containing $\mathcal{X}$. Suppose $\mathbb{F} \in \mathcal{F}(\mathbb{E})$ containing $\mathcal{X}$ and $\alpha \in B$. Then there exist $p_1, \dots, p_n \in \mathcal{X}$ such that $p_1 \twoheadrightarrow (p_2 \twoheadrightarrow (\dots \twoheadrightarrow (p_n \to \alpha) \dots)) = 1$. Since $\mathbb{F} \in \mathcal{F}(\mathbb{E})$, $1 \in \mathbb{F}$ and $p_i \in \mathcal{X} \subseteq \mathbb{F}$, we get $\alpha \in \mathbb{F}$. Hence $B \subseteq \mathbb{F}$, and so B is the smallest filter of $\mathbb{E}$ containing $\mathcal{X}$. Therefore, $\langle \mathcal{X} \rangle = B$. □

Proposition 5. *If $\mathbb{F} \in \mathcal{F}(\mathbb{E})$ and $p \in \mathbb{E} \setminus \mathbb{F}$, then*

$$\langle \mathbb{F} \cup \{p\} \rangle = \{\alpha \in \mathbb{E} \mid \textit{for some } n \in \mathbb{N},\ p \to^n \alpha \in \mathbb{F}\}.$$

Proof. Let $C = \{\alpha \in \mathbb{E} \mid \text{ for some } n \in \mathbb{N},\ p \to^n \alpha \in \mathbb{F}\}$. Clearly, $1 \in C$. By Propositions 1(iii), for all $f \in \mathbb{F}$, we have $f \leq p \twoheadrightarrow f$ then $p \twoheadrightarrow f \in \mathbb{F}$ i.e. $f \in C$ and so $\mathbb{F} \subseteq C$. Also, since $p \twoheadrightarrow p = 1 \in \mathbb{F}$, we get $p \in C$. Hence $\mathbb{F} \cup \{p\} \subseteq C$.

Now, let $\alpha, \alpha \twoheadrightarrow \gamma \in C$. Then there exist $m, n \in \mathbb{N}$ such that $p \twoheadrightarrow^m \alpha = f \in \mathbb{F}$ and $p \twoheadrightarrow^n (\alpha \twoheadrightarrow \gamma) \in \mathbb{F}$. By m-times using of Proposition 1(viii), we have

$$\begin{aligned}
\alpha \twoheadrightarrow \gamma &\leq (p \twoheadrightarrow \alpha) \twoheadrightarrow (p \twoheadrightarrow \gamma) \\
&\vdots \\
&\leq \underbrace{[f \twoheadrightarrow (p \twoheadrightarrow^m \alpha)]}_{1} \twoheadrightarrow [f \twoheadrightarrow (p \twoheadrightarrow^m \gamma)] \\
&= 1 \twoheadrightarrow (f \twoheadrightarrow (p \twoheadrightarrow^m \gamma)) \quad \text{by Proposition 1(ii)}, \\
&= f \twoheadrightarrow (p \twoheadrightarrow^m \gamma).
\end{aligned}$$

Then $\alpha \twoheadrightarrow \gamma \leq f \twoheadrightarrow (p \twoheadrightarrow^m \gamma)$. Also, by Propositions 1(vi) and (v), we get

$$p \twoheadrightarrow (\alpha \twoheadrightarrow \gamma) \leq p \twoheadrightarrow (f \twoheadrightarrow (p \twoheadrightarrow^m \gamma)) = f \twoheadrightarrow (p \twoheadrightarrow^{m+1} \gamma).$$

By the similar way and n-times using the above argument, we conclude that

$$p \twoheadrightarrow^n (\alpha \twoheadrightarrow \gamma) \leq f \twoheadrightarrow (p \twoheadrightarrow^{m+n} \gamma).$$

Since $p \twoheadrightarrow^n (\alpha \twoheadrightarrow \gamma) \in \mathbb{F}$ and $\mathbb{F} \in \mathcal{F}(\mathbb{E})$, then $f \twoheadrightarrow (p \twoheadrightarrow^{m+n} \gamma) \in \mathbb{F}$. Again, since $f \in \mathbb{F}$, we obtain $p \twoheadrightarrow^{m+n} \gamma \in \mathbb{F}$ and so $\gamma \in C$. Hence, $C \in \mathcal{F}(\mathbb{E})$ containing $\mathbb{F} \cup \{p\}$. Let $\mathbb{G}$ be an arbitrary filter of $\mathbb{E}$ such that containing $\mathbb{F} \cup \{p\}$. If $\alpha \in C$, then there exists $n \in \mathbb{N}$ such that $p \twoheadrightarrow^n \alpha \in \mathbb{F}$. Since $\mathbb{G} \in \mathcal{F}(\mathbb{E})$ and $\mathbb{F} \cup \{p\} \subseteq \mathbb{G}$, we have $\alpha \in \mathbb{G}$. Hence $C \subseteq \mathbb{G}$ and so C is the smallest filter of $\mathbb{E}$ containing $\mathbb{F} \cup \{p\}$. Therefore, $\langle \mathbb{F} \cup \{p\} \rangle = C$. □

Corollary 1. *If $\mathbb{F}$ and $\mathbb{G}$ are two filters of equality algebra $\mathbb{E}$, then*

$$\begin{aligned}
\langle \mathbb{F} \cup \mathbb{G} \rangle &= \{\alpha \in \mathbb{E} \mid \textit{for some } g \in \mathbb{G},\ g \twoheadrightarrow \alpha \in \mathbb{F}\} \\
&= \{\alpha \in \mathbb{E} \mid \textit{for some } f \in \mathbb{F},\ f \twoheadrightarrow \alpha \in \mathbb{G}\}.
\end{aligned}$$

Proof. Let $D = \{\alpha \in \mathbb{E} \mid \text{ for some } g \in \mathbb{G},\ g \twoheadrightarrow \alpha \in \mathbb{F}\}$. If $\alpha \in D$, then there exists $g \in \mathbb{G}$ such that $g \twoheadrightarrow \alpha \in \mathbb{F}$. Thus by Proposition 5, we get $\alpha \in \langle \mathbb{F} \cup \{g\} \rangle \subseteq \langle \mathbb{F} \cup \mathbb{G} \rangle$. Thus $D \subseteq \langle \mathbb{F} \cup \mathbb{G} \rangle$. It is enough to show that $\langle \mathbb{F} \cup \mathbb{G} \rangle \subseteq D$. Let $\alpha \in \langle \mathbb{F} \cup \mathbb{G} \rangle$. Then by Proposition 4, there exist $f_1, \ldots, f_m \in \mathbb{F}$ and $g_1, \ldots, g_n \in \mathbb{G}$ such that $f_1 \twoheadrightarrow (\ldots (f_m \twoheadrightarrow (g_1 \twoheadrightarrow (\ldots (g_n \twoheadrightarrow \alpha))))) = 1$. Since $1, f_i \in \mathbb{F}$ and $\mathbb{F} \in \mathcal{F}(\mathbb{E})$, then $g_1 \twoheadrightarrow (\ldots (g_n \twoheadrightarrow \alpha) \ldots) \in \mathbb{F}$. Hence there exists $f \in \mathbb{F}$ such that $g_1 \twoheadrightarrow (\ldots (g_n \twoheadrightarrow \alpha)) = f$ and so by Proposition 1(v),

$$1 = f \twoheadrightarrow f = f \twoheadrightarrow (g_1 \twoheadrightarrow (\ldots (g_n \twoheadrightarrow \alpha)) \ldots) = g_1 \twoheadrightarrow (\ldots (g_n \twoheadrightarrow (f \twoheadrightarrow \alpha))).$$

Since $1, g_j \in \mathbb{G}$ and $\mathbb{G} \in \mathcal{F}(\mathbb{E})$, we have $f \twoheadrightarrow \alpha \in \mathbb{G}$. Thus there exists $g \in \mathbb{G}$ such that $g = f \twoheadrightarrow \alpha$ and so by Proposition 1(ii) and (v),

$$1 = g \twoheadrightarrow g = g \twoheadrightarrow (f \twoheadrightarrow \alpha) = f \twoheadrightarrow (g \twoheadrightarrow \alpha).$$

In addition, from $1, f \in \mathbb{F}$ and $\mathbb{F} \in \mathcal{F}(\mathbb{E})$, we get $g \twoheadrightarrow \alpha \in \mathbb{F}$ i.e. $\alpha \in D$. Therefore, $\langle \mathbb{F} \cup \mathbb{G} \rangle = D$. The proof of other case is similar. □

Proposition 6. *Let $\mathbb{F}$ and $\mathbb{G}$ be two proper filters of $\mathbb{E}$. Then for all $\mathcal{X},\mathcal{Y}\subseteq\mathbb{E}$ and $\alpha,p,q\in\mathbb{E}$, the following statements hold:*

(i) if $\mathcal{X}\subseteq\mathcal{Y}$, then $\langle\mathcal{X}\rangle\subseteq\langle\mathcal{Y}\rangle$,
(ii) if $\mathbb{F}\subseteq\mathbb{G}$, then $\langle\mathbb{F}\cup\{\alpha\}\rangle\subseteq\langle\mathbb{G}\cup\{\alpha\}\rangle$,
(iii) if $p\leq q$, then $\langle q\rangle\subseteq\langle p\rangle$,
(iv) if $\mathbb{F}$ is a positive implicative filter, then $\langle\mathbb{F}\cup\{p\}\rangle=\{\gamma\in\mathbb{E}\mid\ p\twoheadrightarrow\gamma\in\mathbb{F}\}$,
(v) $\langle p\rangle\cup\langle q\rangle\subseteq\langle p\wedge q\rangle$.

Proof.

(i) Let $\alpha\in\langle\mathcal{X}\rangle$. Then there exist $p_1,\dots,p_n\in\mathcal{X}$ such that $p_1\twoheadrightarrow(\dots(p_n\twoheadrightarrow\alpha)\dots)=1$. Since $\mathcal{X}\subseteq\mathcal{Y}$, we get $p_i\in\mathcal{Y}$ for all $1\leq i\leq n$ and so $\alpha\in\langle\mathcal{Y}\rangle$. Hence $\langle\mathcal{X}\rangle\subseteq\langle\mathcal{Y}\rangle$.

(ii) By (i) the proof is clear.

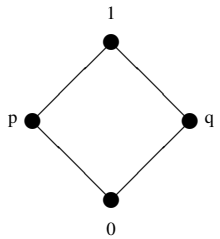
(iii) If $p\leq q$, then by Proposition 1(ii), $p\twoheadrightarrow q=1$ and so $q\in\langle p\rangle$. Let $\alpha\in\langle q\rangle$. Then $q\twoheadrightarrow^n\alpha=1$, for some $n\in\mathbb{N}$ i.e. $q\twoheadrightarrow(q\twoheadrightarrow\cdots(q\twoheadrightarrow\alpha))=1\in\langle p\rangle$. Since $q,1\in\langle p\rangle$ and $\langle p\rangle\in\mathcal{F}(\mathbb{E})$, we have $\alpha\in\langle p\rangle$. Hence $\langle q\rangle\subseteq\langle p\rangle$.

(iv) Let $\mathbb{F}$ be a positive implicative filter of $\mathbb{E}$ and $\gamma\in\langle\mathbb{F}\cup\{p\}\rangle$. Then by Proposition 5, $p\twoheadrightarrow^n\gamma\in\mathbb{F}$ for some $n\in\mathbb{N}$ i. e. $p\twoheadrightarrow(p\twoheadrightarrow(\cdots(p\twoheadrightarrow\gamma)))\in\mathbb{F}$. Since $p\twoheadrightarrow p=1\in\mathbb{F}$ and $\mathbb{F}$ is a positive implicative filter of $\mathbb{E}$, we get $p\twoheadrightarrow^{n-1}\gamma\in\mathbb{F}$. If we use this argument $(n-1)-$times, then we conclude that $p\twoheadrightarrow\gamma\in\mathbb{F}$. Conversely, if $p\twoheadrightarrow\gamma\in\mathbb{F}$, then by Proposition 5, $\gamma\in\langle\mathbb{F}\cup\{p\}\rangle$.

(v) Since $p\wedge q\leq p,q$, by (iii) we get $\langle p\rangle\cup\langle q\rangle\subseteq\langle p\wedge q\rangle$.

□

Example 2. Let $E=\{0,p,q,1\}$ be a poset with the following Hasse diagram. Define the operation $\sim$ on $\mathbb{E}$ as follows:



$\sim$	0	p	q	1
0	1	q	p	0
p	q	1	0	p
q	p	0	1	q
1	0	p	q	1

$\rightarrow$	0	p	q	1
0	1	1	1	1
p	q	1	q	1
q	p	p	1	1
1	0	p	q	1

Then $(\mathbb{E},\wedge,\rightarrow,1)$ is an equality algebra. Since

$$\langle p\wedge q\rangle=\langle 0\rangle=\mathbb{E}\nsubseteq\langle p\rangle\cup\langle q\rangle=\{p,1\}\cup\{q,1\}=\{p,q,1\},$$

so the converse of Proposition 6(v), is not true in general.

Theorem 1. *Let $\mathbb{E}$ be a lattice and $p,q,\alpha\in\mathbb{E}$. If $p\twoheadrightarrow^n\alpha=1=q\twoheadrightarrow^n\alpha$ for some $n\in\mathbb{N}$, then there exists $k\in\mathbb{N}$ such that $(p\vee q)\twoheadrightarrow^k\alpha=1$.*

Proof. By using induction on n, we have, if $n=1$, then $p\twoheadrightarrow\alpha=1=q\twoheadrightarrow\alpha$. Thus by Proposition 2(i), $(p\vee q)\twoheadrightarrow\alpha=(p\twoheadrightarrow\alpha)\wedge(q\twoheadrightarrow\alpha)=1\wedge 1=1$.

Suppose the statement hold for n. Let $p \twoheadrightarrow^{n+1} \alpha = 1 = q \twoheadrightarrow^{n+1} \alpha$. Then by Proposition 1(ii), $q \twoheadrightarrow^{n} (p \twoheadrightarrow^{n+1} \alpha) = 1 = p \twoheadrightarrow^{n} (q \twoheadrightarrow^{n+1} \alpha)$. Thus by Proposition 1(v), we have

$$p \twoheadrightarrow (q \twoheadrightarrow^{n} (p \twoheadrightarrow^{n} \alpha)) = q \twoheadrightarrow^{n} (p \twoheadrightarrow (p \twoheadrightarrow^{n} \alpha)) = q \twoheadrightarrow^{n} (p \twoheadrightarrow^{n+1} \alpha) = 1, \tag{3.1}$$

$$q \twoheadrightarrow (q \twoheadrightarrow^{n} (p \twoheadrightarrow^{n} \alpha)) = q \twoheadrightarrow^{n+1} (p \twoheadrightarrow^{n} \alpha) = p \twoheadrightarrow^{n} (q \twoheadrightarrow^{n+1} \alpha) = 1. \tag{3.2}$$

By Proposition 2(i), (3.1) and (3.2) we get

$$\begin{aligned}(p \vee q) \twoheadrightarrow [q \twoheadrightarrow^{n} (p \twoheadrightarrow^{n} \alpha)] &= [p \twoheadrightarrow (q \twoheadrightarrow^{n} (p \twoheadrightarrow^{n} \alpha))] \wedge [q \twoheadrightarrow (q \twoheadrightarrow^{n} (p \twoheadrightarrow^{n} \alpha))] \\ &= 1 \wedge 1 = 1. \end{aligned} \tag{3.3}$$

By (3.3) and Proposition 1(v), we have

$$q \twoheadrightarrow [(p \vee q) \twoheadrightarrow (q \twoheadrightarrow^{n-1} (p \twoheadrightarrow^{n} \alpha))] = 1. \tag{3.4}$$

Also, by Propositions 2(i), 1(v) and (ii), we have

$$\begin{aligned}
&p \twoheadrightarrow [(p \vee q) \twoheadrightarrow (q \twoheadrightarrow^{n-1} (p \twoheadrightarrow^{n} \alpha))] \\
&= p \twoheadrightarrow [\, [p \twoheadrightarrow (q \twoheadrightarrow^{n-1} (p \twoheadrightarrow^{n} \alpha))] \\
&\quad \wedge [q \twoheadrightarrow (q \twoheadrightarrow^{n-1} (p \twoheadrightarrow^{n} \alpha))] \,] \\
&= p \twoheadrightarrow [\, [q \twoheadrightarrow^{n-1} \underbrace{(p \twoheadrightarrow^{n+1} \alpha)}_{1}] \wedge [q \twoheadrightarrow^{n} (p \twoheadrightarrow^{n} \alpha)] \,] \\
&= p \twoheadrightarrow [\, \underbrace{(q \twoheadrightarrow^{n-1} 1)}_{1} \wedge (q \twoheadrightarrow^{n} (p \twoheadrightarrow^{n} \alpha)) \,] \\
&= p \twoheadrightarrow [q \twoheadrightarrow^{n} (p \twoheadrightarrow^{n} \alpha)] \\
&= q \twoheadrightarrow^{n} \underbrace{(p \twoheadrightarrow^{n+1} \alpha)}_{1} \\
&= 1.
\end{aligned}$$

So we get

$$p \twoheadrightarrow [(p \vee q) \twoheadrightarrow (q \twoheadrightarrow^{n-1} (p \twoheadrightarrow^{n} \alpha))] = 1. \tag{3.5}$$

Now, by Proposition 2(i), (3.4) and (3.5), we get

$$(p \vee q) \twoheadrightarrow^{2} (q \twoheadrightarrow^{n-1} (p \twoheadrightarrow^{n} \alpha)) = 1. \tag{3.6}$$

So, if we repeat this argument (from (3.1) to (3.6)) $(n+1)-$times, then we conclude

$$(p \vee q) \twoheadrightarrow^{n+1} (p \twoheadrightarrow^{n} \alpha) = 1, \tag{3.7}$$

thus by Proposition 1(v), and (3.7)

$$p \twoheadrightarrow^{n} \underbrace{((p \vee q) \twoheadrightarrow^{n+1} \alpha)}_{\eta} = 1. \tag{3.8}$$

By the similar way,

$$(p \vee q) \twoheadrightarrow^{n+1} (q \twoheadrightarrow^{n} \alpha) = 1. \tag{3.9}$$

By Proposition 1(v), and (3.9), we have

$$q \twoheadrightarrow^{n} \underbrace{((p \vee q) \twoheadrightarrow^{n+1} \alpha)}_{\eta} = 1. \tag{3.10}$$

Hence, by induction hypothesis for (3.8) and (3.10), there exists $t \in \mathbb{N}$ such that $(p \vee q) \twoheadrightarrow^{t} \eta = 1$. Thus

$$1 = (p \vee q) \twoheadrightarrow^{t} \eta = (p \vee q) \twoheadrightarrow^{t} ((p \vee q) \twoheadrightarrow^{n+1} \alpha) = (p \vee q) \twoheadrightarrow^{t+n+1} \alpha.$$

□

Theorem 2. *Let $\mathbb{F} \in \mathcal{F}(\mathbb{E})$. If $p \vee q \in \mathbb{F}$, then*

$$\mathbb{F} = \langle \mathbb{F} \cup \{p\} \rangle \cap \langle \mathbb{F} \cup \{q\} \rangle,$$

for all $p, q \in \mathbb{E}$.

Proof. Let $p \vee q \in \mathbb{F}$. Obviously,$\mathbb{F} \subseteq \langle \mathbb{F} \cup \{p\} \rangle \cap \langle \mathbb{F} \cup \{q\} \rangle$. Conversely, if $\alpha \in \langle \mathbb{F} \cup \{p\} \rangle \cap \langle \mathbb{F} \cup \{q\} \rangle$, then by Proposition 5, there exist $m, n \in \mathbb{N}$ such that $p \twoheadrightarrow^{m} \alpha$, $q \twoheadrightarrow^{n} \alpha \in \mathbb{F}$. Let $l = \max\{m, n\}$. Then by Proposition 1(vi), $p \twoheadrightarrow^{m} \alpha \leq p \twoheadrightarrow^{l} \alpha$ and $q \twoheadrightarrow^{n} \alpha \leq q \twoheadrightarrow^{l} \alpha$. Since $\mathbb{F} \in \mathcal{F}(\mathbb{E})$, we get $p \twoheadrightarrow^{l} \alpha, q \twoheadrightarrow^{l} \alpha \in \mathbb{F}$. Hence there exist $f_1, f_2 \in \mathbb{F}$ such that $f_1 \twoheadrightarrow (p \twoheadrightarrow^{l} \alpha) = 1 = f_2 \twoheadrightarrow (q \twoheadrightarrow^{l} \alpha)$ and so by Proposition 1(viii),

$$p \twoheadrightarrow^{l} (f_1 \twoheadrightarrow \alpha) = 1 \quad \text{and} \quad q \twoheadrightarrow^{l} (f_2 \twoheadrightarrow \alpha) = 1.$$

Also, by Proposition 1(iii),

$$f_2 \twoheadrightarrow (p \twoheadrightarrow^{l} (f_1 \twoheadrightarrow \alpha)) = 1, \quad \text{and} \quad f_1 \twoheadrightarrow (q \twoheadrightarrow^{l} (f_2 \twoheadrightarrow \alpha)) = 1. \tag{3.11}$$

Again, by Proposition 1(viii), and (3.11), we get

$$p \twoheadrightarrow^{l} \underbrace{[f_2 \twoheadrightarrow (f_1 \twoheadrightarrow \alpha)]}_{\eta} = 1 \quad \text{and} \quad q \twoheadrightarrow^{l} \underbrace{[f_2 \twoheadrightarrow (f_1 \twoheadrightarrow \alpha)]}_{\eta} = 1. \tag{3.12}$$

By Theorem 1, and (3.12) there exists $k \in \mathbb{N}$ such that $(p \vee q) \twoheadrightarrow^{k} [f_2 \twoheadrightarrow (f_1 \twoheadrightarrow \alpha)] = 1$. Since $p \vee q, f_1, f_2, 1 \in \mathbb{F}$ and $\mathbb{F} \in \mathcal{F}(\mathbb{E})$, we get $\alpha \in \mathbb{F}$. Hence $\langle \mathbb{F} \cup \{p\} \rangle \cap \langle \mathbb{F} \cup \{q\} \rangle \subseteq F$. Therefore, $\mathbb{F} = \langle \mathbb{F} \cup \{p\} \rangle \cap \langle \mathbb{F} \cup \{q\} \rangle$, for all $p, q \in \mathbb{E}$. □

Corollary 2. *If $\mathbb{E}$ is a lattice, then $\langle p \vee q \rangle = \langle p \rangle \cap \langle q \rangle$.*

Proof. Since $p, q \leq p \vee q$, by Proposition 6(iii), $\langle p \vee q \rangle \subseteq \langle p \rangle \cap \langle q \rangle$. Conversely, from $p \in \langle \langle p \vee q \rangle \cup \{p\} \rangle$ we have $\langle p \rangle \subseteq \langle \langle p \vee q \rangle \cup \{p\} \rangle$ and similarly $\langle q \rangle \subseteq \langle \langle p \vee q \rangle \cup \{q\} \rangle$. In Theorem 2, take $\mathbb{F} := \langle p \vee q \rangle$, then

$$\langle p \rangle \cap \langle q \rangle \subseteq \langle \langle p \vee q \rangle \cup \{p\} \rangle \cap \langle \langle p \vee q \rangle \cup \{q\} \rangle = \langle p \vee q \rangle.$$

Therefore, $\langle p \vee q \rangle = \langle p \rangle \cap \langle q \rangle$. □

Definition 7. Let $\mathbb{E}$ be a lattice and $\mathbb{F}$ be a proper filter of $\mathbb{E}$. Then $\mathbb{F}$ is called a $\vee-$irreducible filter of $\mathbb{E}$, if $\alpha\vee\gamma\in\mathbb{F}$ implies $\alpha\in\mathbb{F}$ or $\gamma\in\mathbb{F}$, for all $\alpha,\gamma\in\mathbb{E}$.

Example 3. Let $(\mathbb{E},\wedge,\sim,1)$ be the equality algebra as in Example 2, $\mathbb{F}=\{p,1\}$ and $\mathbb{G}=\{1\}$. Then $\mathbb{F}$ is a $\vee-$irreducible filter. Since $p\vee q=1\in\mathbb{G}$ but $p,q\notin\mathbb{G}$, we get $\mathbb{G}$ is not a $\vee-$irreducible filter of $\mathbb{E}$.

Theorem 3. *Let $\mathbb{F}\in\mathcal{F}(\mathbb{E})$. Then for each $p\notin\mathbb{F}$, there exists an $\vee$-irreducible filter P containing $\mathbb{F}$ such that $p\notin P$.*

Proof. Let $\Sigma=\{\mathbb{G}\in\mathcal{F}(\mathbb{E})|\ \mathbb{F}\subseteq\mathbb{G},\ p\notin\mathbb{G}\}$. Since $\mathbb{F}\in\Sigma$, we have $\Sigma\neq\varnothing$. Since any chain of elements in Σ has an upper bound in Σ, by Zorn' Lemma on $(\Sigma,\subseteq)$, there exists a maximal element $\mathbb{P}\in\Sigma$ such that $\mathbb{F}\subseteq\mathbb{P}$ and $p\notin\mathbb{P}$. Now, we prove that $\mathbb{P}$ is $\vee$-irreducible. Let $\alpha\vee\gamma\in\mathbb{P}$ and $\alpha,\gamma\notin\mathbb{P}$. Since $\mathbb{P}$ is maximal, we get $\langle\mathbb{P}\cup\{\alpha\}\rangle,\langle\mathbb{P}\cup\{\gamma\}\rangle\notin\Sigma$ and so $p\in\langle\mathbb{P}\cup\{\alpha\}\rangle\cap\langle\mathbb{P}\cup\{\gamma\}\rangle$. Moreover, by Theorem 2, we have $p\notin\mathbb{P}=\langle\mathbb{P}\cup\{\alpha\}\rangle\cap\langle\mathbb{P}\cup\{\gamma\}\rangle$ which is a contradiction. Therefore, $\mathbb{P}$ is a $\vee-$irreducible filter of $\mathbb{E}$. □

4. Co-annihilators in equality algebras

In this section, we introduce the notion of co-annihilators on a lattice equality algebra $\mathbb{E}$ and investigate some related properties of them. Specially, by using the results of last section, we prove that the set of all filters of E is a bounded distributive pseudo-complemented lattice, which pseudo-complemented of any filter is co-annihilator of it. Finally, we will construct a Boolean algebra by the set of all co-annihilators of $\mathbb{E}$.

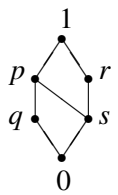
Notation 2. From now on, we let $\mathbb{E}=(\mathbb{E},\sim,\wedge,1)$ or $\mathbb{E}$ be a lattice equality algebra (there exists the operation $\vee$ on $\mathbb{E}$), unless otherwise state.

Definition 8. Let $\varnothing\neq\mathcal{X}\subseteq\mathbb{E}$. Then the set

$$\mathcal{X}^{\top}=\{\alpha\in\mathbb{E}\mid p\vee\alpha=1,\ \text{for all } p\in\mathcal{X}\},$$

is called a *co-annihilator* of $\mathcal{X}$. In particular, if $\mathcal{X}=\{\alpha\}$, then $\mathcal{X}^{\top}$ is denoted by $\alpha^{\top}$, for short.

Example 4. Let $(\mathbb{E}=\{0,p,q,r,s,1\},\leq)$ be a lattice with the following Hasse diagram. Define the operation "$\sim$" on $\mathbb{E}$ as follows:



$\sim$	0	p	q	r	s	1
0	1	s	r	q	p	0
p	s	1	p	s	r	p
q	r	p	1	0	s	q
r	q	s	0	1	p	r
s	p	r	s	p	1	s
1	0	p	q	r	s	1

$\twoheadrightarrow$	0	p	q	r	s	1
0	1	1	1	1	1	1
p	s	1	p	r	r	1
q	r	1	1	r	r	1
r	q	p	q	1	p	1
s	p	1	p	1	1	1
1	0	p	q	r	s	1

Then $(\mathbb{E}, \sim, \wedge, 1)$ is a lattice equality algebra. Let $\mathcal{X} = \{q, s\}$. Clearly, $\mathcal{X}^\top = \{1\}$ and $q^\top = \{r, 1\}$.

Proposition 7. *If $\varnothing \neq \mathcal{X} \subseteq \mathbb{E}$, then $\mathcal{X}^\top \in \mathcal{F}(\mathbb{E})$.*

Proof. Consider $\varnothing \neq \mathcal{X} \subseteq \mathbb{E}$. Since for any $p \in \mathcal{X}$, $p \vee 1 = 1$, we have $1 \in \mathcal{X}^\top$. Suppose $\alpha, \alpha \twoheadrightarrow \gamma \in \mathcal{X}^\top$. Then for all $p \in \mathcal{X}$, we have $\alpha \vee p = 1$ and $(\alpha \twoheadrightarrow \gamma) \vee p = 1$. If $\gamma \vee p := t$, then $p \leq t$ and so $1 = \alpha \vee p \leq t \vee \alpha$. Thus $\alpha \vee t = 1$ and by Proposition 2(ii), we get

$$\alpha \twoheadrightarrow t = (\alpha \vee t) \twoheadrightarrow t = 1 \twoheadrightarrow t = t.$$

Moreover, since $\gamma \leq t$, by Proposition 1(vi), we have $\alpha \twoheadrightarrow \gamma \leq \alpha \twoheadrightarrow t = t$. Thus $1 = (\alpha \twoheadrightarrow \gamma) \vee p \leq t \vee p$ and so $t \vee p = 1$. On the other hand, $\gamma \vee p = (\gamma \vee p) \vee p = t \vee p = 1$. Hence, $\gamma \in \mathcal{X}^\top$. Therefore, $\mathcal{X}^\top \in \mathcal{F}(\mathbb{E})$. □

Proposition 8. *Suppose $\mathcal{X} \subseteq \mathbb{E}$ and $p, q \in \mathbb{E}$. Then the following statements hold:*

(i) $\mathbb{E}^\top = \{1\}$ *and* $1^\top = \mathbb{E}$. *If $\mathbb{E}$ is bounded, then* $0^\top = \{1\}$,
(ii) if $p \leq q$, then $p^\top \subseteq q^\top$,
(iii) $(p \wedge q)^\top \subseteq p^\top \cap q^\top$,
(iv) $p^\top \cup q^\top \subseteq (p \vee q)^\top$,
(v) if $\alpha \in p^\top$ and $\gamma \in q^\top$, then $\alpha \vee \gamma \in (p \vee q)^\top$ and $\alpha \vee \gamma \in (p^\top \cap q^\top)$,
(vi) If $f : \mathbb{E}_1 \to \mathbb{E}_2$ is a lattice-equality isomorphism, then $f(p^\top) = \{f(p)\}^\top$, for any $p \in \mathbb{E}_1$.

Proof.

(i) By definition of co-annihilator, we have

$$\mathbb{E}^\top = \{\alpha \in \mathbb{E} \mid \alpha \vee p = 1, \text{ for any } p \in \mathbb{E}\} = \{\alpha \in \mathbb{E} \mid \alpha \vee \alpha = 1\} = \{1\}.$$

Also,

$$1^\top = \{\alpha \in \mathbb{E} \mid \alpha \vee 1 = 1\} = \mathbb{E}.$$

In addition, if $\mathbb{E}$ is bounded, then

$$0^\top = \{\alpha \in \mathbb{E} \mid \alpha \vee 0 = 1\} = \{\alpha \in \mathbb{E} \mid \alpha = 1\} = \{1\}.$$

(ii) Let $p \leq q$ and $\alpha \in p^\top$. Then $1 = \alpha \vee p \leq \alpha \vee q$. Thus $\alpha \vee q = 1$ and so $\alpha \in q^\top$. Hence $p^\top \subseteq q^\top$.

(iii) Since $p \wedge q \leq p, q$, by (ii) we get $(p \wedge q)^\top \subseteq p^\top, q^\top$. Hence $(p \wedge q)^\top \subseteq p^\top \cap q^\top$.

(iv) From $p, q \leq p \vee q$, by (ii) we have, $p^\top, q^\top \subseteq (p \vee q)^\top$. Hence $p^\top \cup q^\top \subseteq (p \vee q)^\top$.

(v) Let $\alpha \in p^\top$ and $\gamma \in q^\top$. Then

$$(\alpha \vee \gamma) \vee (p \vee q) = (\alpha \vee p) \vee (\gamma \vee q) = 1 \vee 1 = 1.$$

Thus $(\alpha \vee \gamma) \in (p \vee q)^\top$. Also, since $\alpha, \gamma \leq \alpha \vee \gamma$ and $p^\top, q^\top$ are filters of $\mathbb{E}$, we have $\alpha \vee \gamma \in (p^\top \cap q^\top)$.

(vi) Let $\gamma \in f(p^\top)$. Then there exists $\alpha \in p^\top$ such that $\gamma = f(\alpha)$. Since $\alpha \vee p = 1$, we have

$$\gamma \vee f(p) = f(\alpha) \vee f(p) = f(\alpha \vee p) = f(1) = 1.$$

Thus $\gamma \in (f(p))^\top$ and so $f(p^\top) \subseteq (f(p))^\top$. Conversely, let $\gamma \in f(p)^\top$. Then $\gamma \vee f(p) = 1$. Since f is onto, there exists $\alpha \in \mathbb{E}_1$ such that $f(\alpha) = \gamma$ and so

$$f(1) = 1 = \gamma \vee f(p) = f(\alpha) \vee f(p) = f(\alpha \vee p).$$

By f is one-one, we get $\alpha \vee p = 1$. Thus $\alpha \in p^\top$ and so $\gamma = f(\alpha) \in f(p^\top)$.

□

Proposition 9. *Let $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{E}$. Then the following statements hold:*

(i) $\mathcal{X}^\top = \bigcap_{p \in \mathcal{X}} p^\top$,
(ii) $\mathcal{X} \cap \mathcal{X}^\top = \{1\}$,
(iii) $\mathcal{X}^\top \cap \mathcal{X}^{\top\top} = \{1\}$,
(iv) *if* $\mathcal{X} \subseteq \mathcal{Y}$, *then* $\mathcal{Y}^\top \subseteq \mathcal{X}^\top$,
(v) $\mathcal{X}^\top \cap \mathcal{Y}^\top = (\mathcal{X} \cup \mathcal{Y})^\top$,
(vi) $\mathcal{X}^\top \cup \mathcal{Y}^\top \subseteq (\mathcal{X} \cap \mathcal{Y})^\top$,
(vii) $\mathcal{X} \subseteq \mathcal{X}^{\top\top}$,
(viii) $\mathcal{X}^\top = \mathcal{X}^{\top\top\top}$.

Proof.

(i) By definition, the proof is clear.
(ii) Let $\alpha \in \mathcal{X} \cap \mathcal{X}^\top$. Then $\alpha \vee p = 1$, for all $p \in \mathcal{X}$. Since $\alpha \in \mathcal{X}$, we get $\alpha \vee \alpha = 1$ and so $\alpha = 1$.
(iii) By (ii) the proof is straightforward.
(iv) Let $\mathcal{X} \subseteq \mathcal{Y}$ and $\alpha \in \mathcal{Y}^\top$. Then for any $p \in \mathcal{Y}$, $\alpha \vee p = 1$. Since $\mathcal{X} \subseteq \mathcal{Y}$, we get for any $p \in \mathcal{X}$, $\alpha \vee p = 1$ and so $\alpha \in \mathcal{X}^\top$.
(v) Since $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{X} \cup \mathcal{Y}$, by (iv) we get $(\mathcal{X} \cup \mathcal{Y})^\top \subseteq \mathcal{X}^\top, \mathcal{Y}^\top$ and so $(\mathcal{X} \cup \mathcal{Y})^\top \subseteq \mathcal{X}^\top \cap \mathcal{Y}^\top$. Conversely, let $\alpha \in \mathcal{X}^\top \cap \mathcal{Y}^\top$. Then for all $p \in \mathcal{X} \cup \mathcal{Y}$, we have $p \in \mathcal{X}$ or $p \in \mathcal{Y}$ and so $\alpha \vee p = 1$. Hence $\alpha \in (\mathcal{X} \cup \mathcal{Y})^\top$.
(vi) Since $\mathcal{X} \cap \mathcal{Y} \subseteq \mathcal{X}, \mathcal{Y}$, by (iv) we get $\mathcal{X}^\top, \mathcal{Y}^\top \subseteq (\mathcal{X} \cap \mathcal{Y})^\top$ and so $\mathcal{X}^\top \cup \mathcal{Y}^\top \subseteq (\mathcal{X} \cap \mathcal{Y})^\top$.
(vii) By definition of co-annihilator for $\mathcal{X}^\top$,

$$\mathcal{X}^{\top\top} = \{\alpha \in \mathbb{E} \mid \gamma \vee \alpha = 1, \text{ for all } \gamma \in \mathcal{X}^\top\}.$$

If $p \in \mathcal{X}$, then $p \vee \gamma = 1$, for all $\gamma \in \mathcal{X}^\top$. Thus $p \in \mathcal{X}^{\top\top}$ and so $\mathcal{X} \subseteq \mathcal{X}^{\top\top}$.
(viii) By (vii) $\mathcal{X} \subseteq \mathcal{X}^{\top\top}$ and so by (iv) we get $\mathcal{X}^{\top\top\top} \subseteq \mathcal{X}^\top$. Conversely, let $\mathcal{X}^\top = \mathcal{Y}$. Then by (vii), $\mathcal{Y} \subseteq \mathcal{Y}^{\top\top}$ and so $\mathcal{X}^\top \subseteq \mathcal{X}^{\top\top\top}$. Hence, $\mathcal{X}^\top = \mathcal{X}^{\top\top\top}$.

□

In the following example we show that the converse relation of (vi) and (vii) of Proposition 9, may not be true, in general.

Example 5. Let $\mathbb{E}$ be lattice equality algebra as in Example 1. If $\mathcal{X} = \{p,q\}$ and $\mathcal{Y} = \{q,r\}$, then $\mathcal{X}^\top = \{1\} = \mathcal{Y}^\top$, but

$$(\mathcal{X} \cap \mathcal{Y})^\top = \{q\}^\top = \{p,1\} \not\subseteq \{1\} = \mathcal{X}^\top \cup \mathcal{Y}^\top.$$

Also, since $\mathcal{X}^{\top\top} = \{1\}^\top = \mathbb{E}$, we get $\mathcal{X}^{\top\top} \not\subseteq \mathcal{X}$.

Proposition 10. *If $\varnothing \neq \mathcal{X} \subseteq \mathbb{E}$, then*

$$\mathcal{X}^\top \subseteq \{\alpha \in \mathbb{E} \mid \alpha \to p = p, \textit{ for all } p \in \mathcal{X}\} \cap \{\alpha \in \mathbb{E} \mid \ p \to \alpha = \alpha, \textit{ for all } p \in \mathcal{X}\}.$$

Proof. Let $\mathcal{Y} := \{\alpha \in \mathbb{E} \mid \alpha \to p = p, \text{ for all } p \in \mathcal{X}\}$ and $\mathcal{Z} := \{\alpha \in \mathbb{E} \mid \ p \to \alpha = \alpha, \text{ for all } p \in \mathcal{X}\}$. If $\alpha \in \mathcal{X}^\top$, then $\alpha \vee p = 1$, for all $p \in \mathcal{X}$. Thus by Propositions 2(ii) and 1(iii), we have

$$\alpha \twoheadrightarrow p = (\alpha \vee p) \twoheadrightarrow p = 1 \twoheadrightarrow p = p.$$

Hence $\alpha \twoheadrightarrow p = p$ and so $\alpha \in \mathcal{Y}$. Moreover, by Propositions 1(iii) and 2(i), we get,

$$\alpha = 1 \twoheadrightarrow \alpha = (\alpha \vee p) \twoheadrightarrow \alpha = (\alpha \twoheadrightarrow \alpha) \wedge (p \twoheadrightarrow \alpha) = 1 \wedge (p \twoheadrightarrow \alpha) = p \twoheadrightarrow \alpha.$$

Thus $p \twoheadrightarrow \alpha = \alpha$ and so $\alpha \in \mathcal{Z}$. Therefore, $\mathcal{X}^\top \subseteq \mathcal{Y} \cap \mathcal{Z}$. □

In the following example we show that the converse of Proposition 10 does not hold, in general.

Example 6. Let $\mathbb{E}$ be equality algebra as in Example 4, and $\mathcal{X} = \{r\}$. Then $\mathcal{X}^\top = \{p,1\}$ and

$$\begin{aligned}\{\alpha \in \mathbb{E} \mid \alpha \to p = p, \text{ for all } p \in \mathcal{X}\} \cap \{\alpha \in \mathbb{E} \mid \ p \to \alpha = \alpha, \text{ for all } p \in \mathcal{X}\} \\ = \{p,q,1\} \cap \{p,q,1\}.\end{aligned}$$

So $\{p,q,1\} \not\subseteq \{p,1\} = \mathcal{X}^\top$.

Proposition 11. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two non-empty subsets of $\mathbb{E}$ and $\mathbb{F}, \mathbb{G} \in \mathcal{F}(\mathbb{E})$. Then*

(i) $\langle \mathcal{X} \rangle \cap \mathcal{X}^\top = \{1\}$,
(ii) $\mathbb{F} \cap \mathbb{G} = \{1\}$ *if and only if* $\mathbb{F} \subseteq \mathbb{G}^\top$ *and* $\mathbb{G} \subseteq \mathbb{F}^\top$,
(iii) $\mathcal{X}^\top \cap \mathcal{Y}^\top = \{1\}$ *if and only if* $\mathcal{X}^\top \subseteq \mathcal{Y}^{\top\top}$ *and* $\mathcal{Y}^\top \subseteq \mathcal{X}^{\top\top}$.

Proof.

(i) Let $\alpha \in \langle \mathcal{X} \rangle \cap \mathcal{X}^\top$. Since $\alpha \in \langle \mathcal{X} \rangle$, there exist $p_1, \dots, p_n \in \mathcal{X}$ such that $p_1 \twoheadrightarrow (p_2 \twoheadrightarrow (\dots (p_n \twoheadrightarrow \alpha) \dots)) = 1$. Also, since $\alpha \in \mathcal{X}^\top$, by Proposition 10, for any $1 \leq i \leq n$, $p_i \twoheadrightarrow \alpha = \alpha$. So

$$\begin{aligned}1 = p_1 \twoheadrightarrow (p_2 \twoheadrightarrow (\dots \twoheadrightarrow \underbrace{(p_n \twoheadrightarrow \alpha)}_{\alpha} \dots)) &= p_1 \twoheadrightarrow (p_2 \twoheadrightarrow (\dots \underbrace{(p_{n-1} \twoheadrightarrow \alpha)}_{\alpha} \dots)) \\ &= \quad \dots \quad = p_1 \twoheadrightarrow \alpha = \alpha.\end{aligned}$$

Thus $\alpha = 1$ and so $\langle \mathcal{X} \rangle \cap \mathcal{X}^\top = \{1\}$.

(ii) Let $\mathbb{F}\cap\mathbb{G}=\{1\}$. Then for any $f\in\mathbb{F}$ and $g\in\mathbb{G}$, $f,g\leq f\vee g$. Since $\mathbb{F},\mathbb{G}$ are filters of $\mathbb{E}$, we have $f\vee g\in\mathbb{F}\cap\mathbb{G}=\{1\}$. Thus $f\in\mathbb{G}^\top$ and $g\in\mathbb{F}^\top$. Hence $\mathbb{F}\subseteq\mathbb{G}^\top$ and $\mathbb{G}\subseteq\mathbb{F}^\top$. Conversely, if $\mathbb{F}\subseteq\mathbb{G}^\top$, then $\mathbb{F}\cap\mathbb{G}\subseteq\mathbb{G}^\top\cap\mathbb{G}=\{1\}$.

(iii) By Proposition 7, $\mathcal{X}^\top,\mathcal{Y}^\top\in\mathcal{F}(\mathbb{E})$ then by (ii), the proof is clear.

□

Proposition 12. *Let $\varnothing\neq\mathbb{F}\subseteq\mathbb{E}$. Then there exists $\varnothing\neq\mathcal{X}\subseteq\mathbb{E}$ such that $\mathbb{F}=\mathcal{X}^\top$ if and only if $\mathbb{F}$ is upset, $\mathbb{F}^{\top\top}=\mathbb{F}$, $\mathbb{F}\cap\mathcal{X}^{\top\top}=\{1\}$ and $\mathcal{X}^\top\cap\mathbb{F}^\top=\{1\}$.*

Proof. If there exists $\varnothing\neq\mathcal{X}\subseteq\mathbb{E}$ such that $\mathbb{F}=\mathcal{X}^\top$, then by Proposition 7, $\mathbb{F}=\mathcal{X}^\top\in\mathcal{F}(\mathbb{E})$, then $\mathbb{F}$ is upset. Also, by Proposition 9(viii), $\mathbb{F}^{\top\top}=\mathcal{X}^{\top\top\top}=\mathcal{X}^\top=\mathbb{F}$. Moreover, by Proposition 9(iii) and (vi), we have $\mathbb{F}\cap\mathcal{X}^{\top\top}=\mathcal{X}^\top\cap\mathcal{X}^{\top\top}=\{1\}$ and finally, $\mathcal{X}^\top\cap\mathbb{F}^\top=\mathcal{X}^\top\cap\mathcal{X}^{\top\top}=\{1\}$.

Conversely, if $\alpha\in\mathbb{F}$, then for all $p\in\mathcal{X}$, $\alpha,p\leq\alpha\vee p$. Since $\mathbb{F}$ and $\mathcal{X}^{\top\top}$ are upset, we have $p\vee\alpha\in\mathbb{F}\cap\mathcal{X}^{\top\top}=\{1\}$. Thus $p\vee\alpha=1$ and so $\alpha\in\mathcal{X}^\top$. Hence $\mathbb{F}\subseteq\mathcal{X}^\top$. On the other hand, since $\mathbb{F}^\top\cap\mathcal{X}^\top=\{1\}$, by Proposition 11(ii), we have $\mathcal{X}^\top\subseteq\mathbb{F}^{\top\top}=\mathbb{F}$. Therefore, $\mathcal{X}^\top=\mathbb{F}$. □

Theorem 4. *Let $\mathcal{X}\subseteq\mathbb{E}$. Then $\langle\mathcal{X}\rangle^\top=\mathcal{X}^\top$.*

Proof. Since $\mathcal{X}\subseteq\langle\mathcal{X}\rangle$, by Proposition 9(iv), we have $\langle\mathcal{X}\rangle^\top\subseteq\mathcal{X}^\top$. Conversely, we show that $\mathcal{X}^\top\subseteq\langle\mathcal{X}\rangle^\top$: Let $\alpha\in\mathcal{X}^\top$ and $\gamma\in\langle\mathcal{X}\rangle$. Then $\alpha\vee p=1$, for all $p\in\mathcal{X}$ and there exist $p_1,\dots,p_n\in\mathcal{X}$ such that $p_1\twoheadrightarrow(p_2\twoheadrightarrow(\dots(p_n\twoheadrightarrow\gamma)\dots))=1$. If $\alpha\vee\gamma\neq1$, then since $\mathbb{F}=\{1\}$ and by Theorem 3, we get there exists an $\vee$-irreducible filter of $\mathbb{E}$ as $\mathbb{P}\in\mathcal{F}(\mathbb{E})$ such that $\gamma\vee\alpha\notin\mathbb{P}$. Moreover, since $\alpha\vee p_i=1\in\mathbb{P}$, for all $1\leq i\leq n$, and definition of $\vee$-irreducible filter, we get $\alpha\in\mathbb{P}$ or $p_i\in\mathbb{P}$. If $\alpha\in\mathbb{P}$, then from $\alpha\leq\alpha\vee\gamma$ and $\mathbb{P}\in\mathcal{F}(\mathbb{E})$, we get $\alpha\vee\gamma\in\mathbb{P}$, which is a contradiction. Thus $p_i\in\mathbb{P}$ for all $1\leq i\leq n$ and so $p_1\twoheadrightarrow(p_2\twoheadrightarrow(\dots(p_n\twoheadrightarrow\gamma)\dots))=1\in\mathbb{P}$ implies $\gamma\in\mathbb{P}$, which is a contradiction. Therefore, $\alpha\vee\gamma=1$ and $\alpha\in\langle\mathcal{X}\rangle^\top$. □

Definition 9 ([16, Proposition3.17])**.** In a lattice L with bottom element 0, an element $\alpha\in L$ is said to have a pseudo-complement element if there exists the greatest element $\alpha^*\in L$, disjoint from α, with the property that $\alpha\wedge\alpha^*=0$. More formally, $\alpha^*=\max\{\gamma\in L\mid\alpha\wedge\gamma=0\}$. The lattice L itself is called a *pseudo-complemented lattice* if every element of L has a pseudo-complement element.

Theorem 5. *The algebraic structure $(\mathcal{F}(\mathbb{E}),\subseteq,\wedge,\vee,\{1\},\mathbb{E})$ is a bounded distributive pseudo complemented lattice, where for any $\mathbb{F},\mathbb{G}\in\mathcal{F}(\mathbb{E})$, $\mathbb{F}\wedge\mathbb{G}=\mathbb{F}\cap\mathbb{G},\mathbb{F}\vee\mathbb{G}=\langle\mathbb{F}\cup\mathbb{G}\rangle$, and $\mathbb{F}^\top$ is pseudo-complement of $\mathbb{F}$.*

Proof. Clearly, $\mathcal{F}(\mathbb{E})$ is closed under $\wedge$ and $\vee$. So, $(\mathcal{F}(\mathbb{E}),\subseteq,\wedge,\vee,\{1\},\mathbb{E})$ is a bounded lattice. Now, we prove distribution i.e., for any $\mathbb{F},\mathbb{G},\mathbb{H}\in\mathcal{F}(\mathbb{E})$,

$$\mathbb{F}\wedge(\mathbb{G}\vee\mathbb{H})=(\mathbb{F}\wedge\mathbb{G})\vee(\mathbb{F}\wedge\mathbb{H}).$$

Since $\mathbb{G},\mathbb{H}\subseteq\mathbb{G}\vee\mathbb{H}$, we have $\mathbb{F}\cap\mathbb{G},\mathbb{F}\cap\mathbb{H}\subseteq\mathbb{F}\cap(\mathbb{G}\vee\mathbb{H})$ and so

$$(\mathbb{F}\cap\mathbb{G})\vee(\mathbb{F}\cap\mathbb{H})=(\mathbb{F}\wedge\mathbb{G})\vee(\mathbb{F}\wedge\mathbb{H})\subseteq\mathbb{F}\cap(\mathbb{G}\vee\mathbb{H})=\mathbb{F}\wedge(\mathbb{G}\vee\mathbb{H}). \tag{4.1}$$

Conversely, if $\alpha\in\mathbb{F}\wedge(\mathbb{G}\vee\mathbb{H})$, then $\alpha\in\mathbb{F}$ and $\alpha\in\mathbb{G}\vee\mathbb{H}=\langle\mathbb{G}\cup\mathbb{H}\rangle$. By Corollary 1, there exists $g\in\mathbb{G}$ such that $g\twoheadrightarrow\alpha\in\mathbb{H}$. Thus there exists $h\in\mathbb{H}$ such that $g\twoheadrightarrow\alpha=h$. Since $\alpha\leq g\twoheadrightarrow\alpha$ and $\mathbb{F}$ is upset, we get $h=g\twoheadrightarrow\alpha\in\mathbb{F}\cap\mathbb{H}$. Moreover, by Proposition 1(iv), $g\leq(g\twoheadrightarrow\alpha)\twoheadrightarrow\alpha=h\twoheadrightarrow\alpha$ and from $\mathbb{G}$ is upset, we have $h\twoheadrightarrow\alpha\in\mathbb{G}$. Also, by Proposition 1(iii), $\alpha\leq h\twoheadrightarrow\alpha$, and $\mathbb{F}$ is upset, we get $h\twoheadrightarrow\alpha\in\mathbb{F}\cap\mathbb{G}$. Hence $\alpha\in\langle(\mathbb{F}\cap\mathbb{H})\cup(\mathbb{F}\cap\mathbb{G})\rangle$. Thus

$$\mathbb{F}\wedge(\mathbb{G}\vee\mathbb{H})\subseteq\langle(\mathbb{F}\cap\mathbb{H})\cup(\mathbb{F}\cap\mathbb{G})\rangle=(\mathbb{F}\wedge\mathbb{H})\vee(\mathbb{F}\wedge\mathbb{G}). \tag{4.2}$$

By (4.1) and (4.2), we get

$$\mathbb{F}\wedge(\mathbb{G}\vee\mathbb{H})=(\mathbb{F}\wedge\mathbb{H})\vee(\mathbb{F}\wedge\mathbb{G}).$$

Hence $(\mathcal{F}(\mathbb{E}),\subseteq,\wedge,\vee,\{1\},\mathbb{E})$ is a bounded distributive lattice.

Now, by Propositions 9(ii), $\mathbb{F}\cap\mathbb{F}^{\top}=\{1\}$, for any $\mathbb{F}\in\mathcal{F}(\mathcal{X})$. If $\mathbb{G}\in\mathcal{F}(\mathbb{E})$ such that $\mathbb{G}\cap\mathbb{F}=\{1\}$, then by Proposition 11(ii), $\mathbb{G}\subseteq\mathbb{F}^{\top}$. Hence $\mathbb{F}^{\top}$ is the largest filter of $\mathbb{E}$ such that $\mathbb{G}\cap\mathbb{F}=\{1\}$. Then $\mathbb{F}^{\top}$ is a pseudo-complement of $\mathbb{F}$. Therefore, $(\mathcal{F}(\mathbb{E}),\subseteq,\wedge,\vee,\{1\},\mathbb{E})$ is a bounded distributive pseudo-complemented lattice. □

Notation 3. Let $Co_{ann}(\mathbb{E})=\{\mathcal{X}^{\top}\mid\varnothing\neq\mathcal{X}\subseteq\mathbb{E}\}$ be the set of all co-annihilators of $\mathbb{E}$. Since $\mathcal{X}^{\top}=\langle\mathcal{X}\rangle^{\top}$, by Proposition 4, then $Co_{ann}(\mathbb{E})=\{\mathbb{F}^{\top}\mid\mathbb{F}\in\mathcal{F}(\mathbb{E})\}$. Thus $Co_{ann}(\mathbb{E})$ is the set of all pseudo-complements of the pseudo-complement lattice $\mathcal{F}(\mathbb{E})$.

Proposition 13. *For any $\varnothing\neq\mathcal{X}\subseteq\mathbb{E}$ and $\mathbb{F},\mathbb{G}\in\mathcal{F}(\mathbb{E})$, the following statements hold:*

(i) $\{1\},\mathbb{E}\in Co_{ann}(\mathbb{E})$,
(ii) $\mathbb{F}\in Co_{ann}(\mathbb{E})$ *if and only if* $\mathbb{F}^{\top\top}=\mathbb{F}$,
(iii) a map $\top\top:\mathcal{F}(\mathbb{E})\twoheadrightarrow\mathcal{F}(\mathbb{E})$ *defined by* $\mathbb{F}\to\mathbb{F}^{\top\top}$ *is a closure map,*
(iv) $(\mathbb{F}\cap\mathbb{G})^{\top\top}=\mathbb{F}^{\top\top}\cap\mathbb{G}^{\top\top}$,
(v) $(\mathbb{F}\vee\mathbb{G})^{\top}=\mathbb{F}^{\top}\cap\mathbb{G}^{\top}$,
(vi) if $\mathbb{F},\mathbb{G}\in Co_{ann}(\mathbb{E})$, *then* $\mathbb{F}\bigvee_{Co_{ann}(\mathbb{E})}\mathbb{G}=(\mathbb{F}^{\top}\cap\mathbb{G}^{\top})^{\top}$.

Proof.

(i) By Proposition 8(i), $\{1\}^{\top}=\mathbb{E}$ and $\mathbb{E}^{\top}=\{1\}$. Thus $\{1\},\mathbb{E}\in Co_{ann}(\mathbb{E})$.

(ii) Let $\mathbb{F}\in Co_{ann}(\mathbb{E})$. Then there exists $\varnothing\neq\mathcal{X}\subseteq\mathbb{E}$ such that $\mathbb{F}=\mathcal{X}^{\top}$. Thus, by Proposition 9(viii),

$$\mathbb{F}^{\top\top}=\mathcal{X}^{\top\top\top}=\mathcal{X}^{\top}=\mathbb{F}$$

Conversely, if $\mathbb{F}^{\top\top}=\mathbb{F}$, then $\mathbb{F}=(\mathbb{F}^{\top})^{\top}\in Co_{ann}(\mathbb{E})$.

(iii) By applying Definition 5, Proposition 9(vii), (viii) and (iv) the proof is clear.

(iv) Since $\mathbb{F}\cap\mathbb{G}\subseteq\mathbb{F},\mathbb{G}$, by two times using of Proposition 9(iv), $(\mathbb{F}\cap\mathbb{G})^{\top\top}\subseteq\mathbb{F}^{\top\top}\cap\mathbb{G}^{\top\top}$. On the other hand, by Proposition 9(ii), $(\mathbb{F}\cap\mathbb{G})\cap(\mathbb{F}\cap\mathbb{G})^{\top}=\{1\}$. Then by Propositions 11(ii) and 9(viii), we get

$$\mathbb{F}\cap(\mathbb{F}\cap\mathbb{G})^{\top}\subseteq\mathbb{G}^{\top}=\mathbb{G}^{\top\top\top}$$

So $\mathbb{F}\cap\mathbb{G}^{\top\top}\cap(\mathbb{F}\cap\mathbb{G})^{\top}=\{1\}$. By the similar way, we can see that $\mathbb{G}^{\top\top}\cap(\mathbb{F}\cap\mathbb{G})^{\top}\subseteq\mathbb{F}^{\top}=\mathbb{F}^{\top\top\top}$, and so $\mathbb{F}^{\top\top}\cap\mathbb{G}^{\top\top}\cap(\mathbb{F}\cap\mathbb{G})^{\top}=\{1\}$. Thus, $\mathbb{F}^{\top\top}\cap\mathbb{G}^{\top\top}\subseteq(\mathbb{F}\cap\mathbb{G})^{\top\top}$. Therefore,

$$(\mathbb{F}\cap\mathbb{G})^{\top\top}=\mathbb{F}^{\top\top}\cap\mathbb{G}^{\top\top}.$$

(v) Since $\mathbb{F},\mathbb{G}\subseteq\mathbb{F}\vee\mathbb{G}$, we ahve

$$\begin{aligned}(\mathbb{F}\vee\mathbb{G})^{\top}&\subseteq\mathbb{F}^{\top}\cap\mathbb{G}^{\top}\quad\text{by Proposition 9(iv)},\\&=\mathbb{F}^{\top\top\top}\cap\mathbb{G}^{\top\top\top}\quad\text{by Proposition 9(viii)},\\&=(\mathbb{F}^{\top}\cap\mathbb{G}^{\top})^{\top\top}\quad\text{by (iv)}.\end{aligned}$$

So $(\mathbb{F}\vee\mathbb{G})^{\top}\subseteq(\mathbb{F}^{\top}\cap\mathbb{G}^{\top})^{\top\top}$. Moreover, $\mathbb{F}^{\top}\cap\mathbb{G}^{\top}\subseteq\mathbb{F}^{\top}$ and by Proposition 9(vii), (iv) $\mathbb{F}\subseteq\mathbb{F}^{\top\top}\subseteq(\mathbb{F}^{\top}\cap\mathbb{G}^{\top})^{\top}$. Similarly, $\mathbb{G}\subseteq(\mathbb{F}^{\top}\cap\mathbb{G}^{\top})^{\top}$, and by definition of "$\vee$", we get $\mathbb{F}\vee\mathbb{G}\subseteq(\mathbb{F}^{\top}\cap\mathbb{G}^{\top})^{\top}$. Thus

$$(\mathbb{F}^{\top}\cap\mathbb{G}^{\top})^{\top\top}\subseteq(\mathbb{F}\vee\mathbb{G})^{\top}\subseteq(\mathbb{F}^{\top}\cap\mathbb{G}^{\top})^{\top\top},$$

and so, $(\mathbb{F}\vee\mathbb{G})^{\top}=(\mathbb{F}^{\top}\cap\mathbb{G}^{\top})^{\top\top}$. Hence, by (iv)

$$(\mathbb{F}\vee\mathbb{G})^{\top}=(\mathbb{F}^{\top}\cap\mathbb{G}^{\top})^{\top\top}=\mathbb{F}^{\top\top\top}\cap\mathbb{G}^{\top\top\top}=\mathbb{F}^{\top}\cap\mathbb{G}^{\top}.$$

(vi) By (ii) and the proof of (v), $\mathbb{F}\bigvee_{Co_{ann}(\mathbb{E})}\mathbb{G}=(\mathbb{F}\vee\mathbb{G})^{\top\top}=(\mathbb{F}^{\top}\cap\mathbb{G}^{\top})^{\top}$.

□

Theorem 6. *The structure* $(Co_{ann}(\mathbb{E}),\wedge,\vee,\top,\{1\},\mathbb{E})$ *is a Boolean algebra.*

Proof. By Proposition 13(i), $Co_{ann}(\mathbb{E})\neq\varnothing$ and it is bounded. By Proposition 13(iv), (v) and (vi), for any $\mathbb{F},\mathbb{G}\in Co_{ann}(\mathbb{E})$,

$$\mathbb{F}\wedge_{Co_{ann}(\mathbb{E})}\mathbb{G}=\mathbb{F}\cap\mathbb{G},\quad\mathbb{F}\vee_{Co_{ann}(\mathbb{E})}\mathbb{G}=(\mathbb{F}^{\top}\cap\mathbb{G}^{\top})^{\top}.$$

Thus, $(Co_{ann}(\mathbb{E}),\wedge,\vee,\{1\},\mathbb{E})$ is a bounded lattice. For distribution, let $\mathbb{F},\mathbb{G},\mathbb{H}\in Co_{ann}(\mathbb{E})$. Then by Proposition 13(ii), we have

$$\mathbb{F}\wedge\mathbb{G}\subseteq(\mathbb{F}\wedge\mathbb{G})\vee(\mathbb{F}\wedge\mathbb{H})=[\underbrace{(\mathbb{F}\cap\mathbb{G})\vee(\mathbb{F}\cap\mathbb{H})}_{\mathbb{K}}]^{\top\top}=\mathbb{K}^{\top\top}.\tag{4.3}$$

By (4.3) and Proposition 11(ii), we get $(\mathbb{F}\cap\mathbb{G})\cap\mathbb{K}^{\top}=\{1\}$. By using one more time of Proposition 11(ii), $\mathbb{F}\cap\mathbb{K}^{\top}\subseteq\mathbb{G}^{\top}$. Similarly, we get $\mathbb{F}\cap\mathbb{K}^{\top}\subseteq\mathbb{H}^{\top}$ and so by Proposition 13(ii),

$$\mathbb{F}\cap\mathbb{K}^{\top}\subseteq\mathbb{G}^{\top}\cap\mathbb{H}^{\top}=(\mathbb{G}^{\top}\cap\mathbb{H}^{\top})^{\top\top}.\tag{4.4}$$

Then by (4.4) and Proposition 11(ii), we get $(\mathbb{F}\cap\mathbb{K}^{\top})\cap(\mathbb{G}^{\top}\cap\mathbb{H}^{\top})=\{1\}$ which implies that

$$\mathbb{F}\cap(\mathbb{G}^{\top}\cap\mathbb{H}^{\top})^{\top}\subseteq\mathbb{K}^{\top\top}=\mathbb{K}. \tag{4.5}$$

Hence, by (4.5) and Proposition 13(vi)

$$\mathbb{F}\wedge(\mathbb{G}\vee\mathbb{H})=\mathbb{F}\cap(\mathbb{G}^{\top}\cap\mathbb{H}^{\top})^{\top}\subseteq\mathbb{K}=(\mathbb{F}\wedge\mathbb{G})\vee(\mathbb{F}\wedge\mathbb{H}). \tag{4.6}$$

On the other hand, since $\mathbb{F}\cap\mathbb{G}\subseteq\mathbb{G}$ and $\mathbb{F}\cap\mathbb{H}\subseteq\mathbb{H}$, by Proposition 9(iv), we have, $\mathbb{G}^{\top}\subseteq(\mathbb{F}\cap\mathbb{G})^{\top}$ and $\mathbb{H}^{\top}\subseteq(\mathbb{F}\cap\mathbb{H})^{\top}$. Thus

$$\mathbb{G}^{\top}\cap\mathbb{H}^{\top}\subseteq(\mathbb{F}\cap\mathbb{G})^{\top}\cap(\mathbb{F}\cap\mathbb{H})^{\top}$$

and so by Proposition 9(iv)

$$[(\mathbb{F}\cap\mathbb{G})^{\top}\cap(\mathbb{F}\cap\mathbb{H})^{\top}]^{\top}\subseteq(\mathbb{G}^{\top}\cap\mathbb{H}^{\top})^{\top}. \tag{4.7}$$

Also, since $\mathbb{F}\cap\mathbb{G},\mathbb{F}\cap\mathbb{H}\subseteq\mathbb{F}$, by two times using of Propositions 9(iv) and 13(ii) we get

$$[(\mathbb{F}\cap\mathbb{G})^{\top}\cap(\mathbb{F}\cap\mathbb{H})^{\top}]^{\top}\subseteq\mathbb{F}^{\top\top}=\mathbb{F}. \tag{4.8}$$

So by (4.7) and (4.8),

$$[(\mathbb{F}\cap\mathbb{G})^{\top}\cap(\mathbb{F}\cap\mathbb{H})^{\top}]^{\top}\subseteq\mathbb{F}\cap(\mathbb{G}^{\top}\cap\mathbb{H}^{\top})^{\top}.$$

Hence, by Proposition 13(iv),

$$(\mathbb{F}\wedge\mathbb{G})\vee(\mathbb{F}\wedge\mathbb{H})=[(\mathbb{F}\cap\mathbb{G})^{\top}\cap(\mathbb{F}\cap\mathbb{H})^{\top}]^{\top}\subseteq\mathbb{F}\cap(\mathbb{G}^{\top}\cap\mathbb{H}^{\top})^{\top}=\mathbb{F}\wedge(\mathbb{G}\vee\mathbb{H}). \tag{4.9}$$

Now, by (4.6) and (4.9), we get

$$\mathbb{F}\wedge(\mathbb{G}\vee\mathbb{H})=(\mathbb{F}\wedge\mathbb{G})\vee(\mathbb{F}\wedge\mathbb{H}).$$

Thus $Co_{ann}(\mathbb{E})$ is a distributive lattice. Since for all $\mathbb{F}\in Co_{ann}(\mathbb{E})$,

$$\mathbb{F}\cap\mathbb{F}^{\top}=\{1\}\ \text{ and }\ \mathbb{F}\vee\mathbb{F}^{\top}=(\mathbb{F}^{\top}\cap\mathbb{F}^{\top\top})^{\top}=(\mathbb{F}^{\top}\cap\mathbb{F})^{\top}=\{1\}^{\top}=\mathbb{E},$$

we get $\mathbb{F}^{\top}$ is the complement of $\mathbb{F}$. Therefore, $(Co_{ann}(\mathbb{E}),\wedge,\vee,\top,\{1\},\mathbb{E})$ is a Boolean algebra. □

5. Conclusions and future works

In this paper we introduced the notion of co-annihilator of a subset $\mathcal{X}$ in an equality algebra $\mathbb{E}$. We investigated many important properties of the co-annihilators. Finally, we define the $Co_{ann}(\mathbb{E})$ to be the set of all co-annihilators of $\mathbb{E}$, then we show that it can be made as a Boolean algebra.

In our future work, we will continue our study of algebraic properties of this special sets and define the notion of minimal prime filter and ideals on equality algebras, and we investigate the relation between these new notions with co-annihilators in equality algebras.

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LYAPUNOV AND HARTMAN-TYPE INEQUALITIES FOR HIGHER-ORDER DISCRETE FRACTIONAL BOUNDARY VALUE PROBLEMS

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Abstract. By employing Green's function, we obtain new Lyapunov and Hartman-type inequalities for higher-order discrete fractional boundary value problems. Reported results essentially generalize some theorems existing in the literature. As an application, we discuss the corresponding eigenvalue problems.

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1. Introduction

Lyapunov inequality was first introduced by the Russian Mathematician Liapounoff [13] who has demonstrated significant effect in the qualitative theory of differential equations such as oscillation and Sturmian theory, asymptotic theory, disconjugacy, boundary value problems, eigenvalue problems and numerous other applications. Due to its importance, Lyapunov inequality has been improved and generalized in many forms, see for instance [2–4, 8, 9, 16–20]. For the sake of convenience, we recall the following some versions of Lyapunov inequality.

Consider the Hill equation

$$z''(\rho)+\omega(\rho)z(\rho)=0, \qquad \rho_1<\rho<\rho_2, \tag{1.1}$$

where $\omega(\rho)\in L^1[\rho_1,\rho_2]$ is a real-valued function. Let z be a non-trivial solution of (1.1) satisfying the boundary conditions:

$$z(\rho_1)=z(\rho_2)=0,$$

where $\rho_1,\rho_2\in\mathbb{R}$ $(\rho_1<\rho_2)$. If ρ_1 and ρ_2 are consecutive zeros of z, then the inequality

$$\int_{\rho_1}^{\rho_2}|\omega(\rho)|d\rho>\frac{4}{\rho_2-\rho_1} \tag{1.2}$$

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holds. In 1951, Wintner [21] (and then many authors) showed that the function "$|\omega(\rho)|$" in (1.2) can be replaced by the function "$\omega^+(\rho)$" so that (1.2) turns out to be the inequality

$$\int_{\rho_1}^{\rho_2} \omega^+(\rho)d\rho > \frac{4}{\rho_2-\rho_1}, \tag{1.3}$$

where $\omega^+(\rho) = \max\{\omega(\rho), 0\}$ and the constant "4" in the right-hand-side of inequalities (1.2) and (1.3) is the best possible largest number (see [10]). Since then, inequality (1.3) has been known as *Lyapunov inequality* in the literature.

In 1964, Hartman [10] obtained the inequality

$$\int_{\rho_1}^{\rho_2} (\rho_2-\rho)(\rho-\rho_1)\omega^+(\rho)d\rho > \rho_2-\rho_1$$

which is known as *Hartman inequality* in the literature and it is the best Lyapunov-type inequality for being stronger than both inequalities (1.2) and (1.3).

Although Lyapunov-type inequalities have been obtained by some authors [12, 14, 15] for continuous fractional boundary value problems, there are few relevant results, see for example the paper by Ferreira [5] for the right-focal discrete boundary value problem of the form

$$\begin{cases} \Delta^v z(\rho) + q(\rho+v-1)z(\rho+v-1) = 0, & \rho \in \mathbb{N}_0^{b+1}, \\ z(v-2) = 0 = \Delta z(v+b), \end{cases} \tag{1.4}$$

and the inequality

$$\sum_{s=0}^{b+1} |q(s+v-1)| > \frac{1}{\Gamma(v-1)(b+2)}$$

is obtained. However the estimation for the corresponding Green's function of Prb. (1.4) obtained in [5] is not true. The corrected result is published as an addendum version by the same author [6].

For the sake of improving and generalizing existing results, we derive new Lyapunov and Hartman-type inequalities for higher-order discrete fractional BVP of the form

$$\begin{cases} \Delta^v_{v-n_0} z(\rho) + q(\rho)z(\rho+v-1) = 0, & \rho \in \mathbb{N}_0^{b+m_0}, \\ \Delta^i z(v-n_0) = 0, & i \in \mathbb{N}_0^{n_0-2}, \\ \Delta^\mu_{v-n_0} z(b+m_0+v-\mu) = 0, \end{cases} \tag{1.5}$$

where $q(\rho)$ is real-valued function defined on $\mathbb{N}_0^{b+m_0}$, $b, n_0, m_0 \in \mathbb{N}$, $n_0-1 < v \le n_0$, $m_0-1 < \mu \le m_0$ and

- $1 \le \mu < v$ when $v \ge 2$;
- $\mu = 1$ when $1 < v < 2$.

2. Preliminaries

In this section we assemble some basic definitions on delta fractional calculus. The terms and statements of this section are adopted for the remarkable monograph [7].

The functions that we consider in this paper is defined on a set

$$\mathbb{N}_c = \{c, c+1, c+2, \ldots\}, \qquad c \in \mathbb{R}$$

or a set

$$\mathbb{N}_c^d = \{c, c+1, c+2, \ldots, d\}, \qquad c, d \in \mathbb{R}.$$

It is well known that the forward difference operator Δ is defined by

$$\Delta w(\rho) = w(\rho+1) - w(\rho).$$

Further, the operator Δ^n is recursively defined by

$$\Delta^n w(\rho) = \Delta(\Delta^{n-1} w(\rho)), \qquad n = 1, 2, \ldots$$

and Δ^0 denotes the identity operator, that is, $\Delta^0 w(\rho) = w(\rho)$.

Definition 1. The (generalized) falling function is defined by

$$\rho^{\underline{r}} := \frac{\Gamma(\rho+1)}{\Gamma(\rho-r+1)},$$

where $\Gamma(*)$ is the *Gamma function*. Note that the definition is valid if and only if the right-hand side of this equation is well-defined.

Definition 1 can be extended by prevalent convention that $\rho^{\underline{r}} = 0$ when $\rho - r + 1$ and $\rho + 1$ are not non-positive integers.

Definition 2. The v-th fractional Taylor monomial based at s is defined by

$$h_v(\rho, s) := \frac{(\rho - s)^{\underline{v}}}{\Gamma(v+1)} \tag{2.1}$$

whenever the right-hand side is well-defined.

Now, we can define v-th fractional sum.

Definition 3. Assume $w : \mathbb{N}_c \to \mathbb{R}$ and $v > 0$. Then the v-th fractional sum of w (based at c) is defined by

$$\Delta_c^{-v} w(\rho) := \sum_{\tau=c}^{\rho-v} h_{v-1}(\rho, \sigma(\tau)) w(\tau), \qquad \rho \in \mathbb{N}_{c+v},$$

where the function h_* is defined in (2.1) and σ is forward jump operator defined by $\sigma(\rho) = \rho + 1$.

Next we define the fractional difference in terms of the fractional sum.

Definition 4. Assume $w : \mathbb{N}_c \to \mathbb{R}$ and $v > 0$. Take a positive integer n_0 such that $n_0 - 1 < v \leq n_0$. Then as known, v-th fractional difference is defined by

$$\Delta_c^v w(\rho) := \Delta^{n_0} \Delta_c^{-(n_0-v)} w(\rho), \qquad \rho \in \mathbb{N}_{c+n_0-v}.$$

Note that

$$\Delta_c^v w(\rho) := \Delta^{n_0} \Delta_c^{-(n_0-v)} w(\rho) = \Delta^{n_0} \Delta_c^0 w(\rho) = \Delta^{n_0} w(\rho) \qquad (\rho \in \mathbb{N}_c)$$

for any $v = n_0 \in \mathbb{N}_0$.

Lemma 1. *Assume $w : \mathbb{N}_c \to \mathbb{R}$, $k \in \mathbb{N}_0$ and $v, \mu > 0$ and $n_0 - 1 < \mu \leq n_0$. Then*

$$\Delta_c^{-v} \Delta^k w(\rho) = \Delta_c^{k-v} w(\rho) - \sum_{j=0}^{k-1} h_{v-k+j}(\rho, c) \Delta^j w(c)$$

for $\rho \in \mathbb{N}_{c+v}$, and

$$\Delta_{c+n_0-\mu}^{-v} \Delta_c^{\mu} w(\rho) = \Delta_c^{\mu-v} w(\rho) - \sum_{j=0}^{n_0-1} h_{v-n_0+j}(\rho - n_0 + \mu, c) \Delta_c^{j-(n_0-\mu)} w(c + n_0 - \mu)$$

for $\rho \in \mathbb{N}_{c+n_0-\mu+v}$, where the function h_* is defined in (2.1).

3. Main Results

Before proceeding, we present some lemmas that will be used to prove our main results. Next, we obtain Lyapunov and Hartman-type inequalities for Prb. (1.5). Finally, an example is given.

To prove the main results, we will employ the following lemmas.

Lemma 2. *If g is defined on $\mathbb{N}_0^{b+m_0}$, then the solution of the BVP*

$$\begin{cases} -\Delta_{v-n_0}^v y(\rho) = g(\rho), & \rho \in \mathbb{N}_0^{b+m_0}, \\ \Delta^i y(v - n_0) = 0, & i \in \mathbb{N}_0^{n_0-2}, \\ \Delta_{v-n_0}^{\mu} y(b + m_0 + v - \mu) = 0 \end{cases} \tag{3.1}$$

can be represented by

$$y(\rho) = \sum_{s=0}^{b+m_0} \psi(\rho, s) g(s), \qquad \rho \in \mathbb{N}_{v-n_0}^{b+m_0+v}, \tag{3.2}$$

where

$$\psi(\rho, s) := \frac{1}{\Gamma(v)} \begin{cases} u_1(\rho, s), & 0 \leq \rho - v + 1 \leq s \leq b + m_0, \\ u_2(\rho, s), & 0 \leq s \leq \rho - v \leq b + m_0, \\ 0, & v - n_0 \leq \rho \leq v - 2, \, 0 \leq s \leq b + m_0 \end{cases} \tag{3.3}$$

is the Green function for Prb. (3.1) with

$$u_1(\rho, s) = \frac{[b + m_0 + v - \mu - \sigma(s)]^{\underline{v-\mu-1}}}{(b + m_0 + v - \mu)^{\underline{v-\mu-1}}} \rho^{\underline{v-1}}$$

and

$$u_2(\rho,s)=\frac{[b+m_0+v-\mu-\sigma(s)]^{\underline{v-\mu-1}}}{(b+m_0+v-\mu)^{\underline{v-\mu-1}}}\rho^{\underline{v-1}}-[\rho-\sigma(s)]^{\underline{v-1}}.$$

The proof of Lemma 2 can be found in [11]. It is also shown in [11] that

$$\min_{\rho\in\mathbb{N}_{v-1}^{b+m_0+v}}\psi(\rho,s)>0$$

and

$$\max_{\rho\in\mathbb{N}_{v-n_0}^{b+m_0+v}}\psi(\rho,s)=\psi(b+m_0+v,s) \tag{3.4}$$

for $s\in\mathbb{N}_0^{b+m_0}$.

Lemma 3. *The inequality*

$$\psi(\rho,s)\le\frac{1}{\Gamma(v)}f_1(s)f_2(s);\qquad (\rho,s)\in\mathbb{N}_{v-n_0}^{b+m_0+v}\times\mathbb{N}_0^{b+m_0} \tag{3.5}$$

holds, where $\psi(\rho,s)$ is the Green function of Prb. (3.1), and the functions $f_1(s)$ and $f_2(s)$ are defined by

$$f_1(s):=[b+m_0+v-\mu-\sigma(s)]^{\underline{v-\mu-1}} \tag{3.6}$$

and

$$f_2(s):=(b+m_0+v)^{\underline{\mu}}-[b+m_0+v-\sigma(s)]^{\underline{\mu}} \tag{3.7}$$

for $s\in\mathbb{N}_0^{b+m_0}$, respectively.

Proof. Let $\psi(\rho,s)$ be the Green's function of Prb. (3.1). Then equations (3.3) and (3.4) yield that

$$\begin{aligned}\psi(\rho,s)&\le\psi(b+m_0+v,s)\\&=\frac{1}{\Gamma(v)}\left\{\frac{[b+m_0+v-\mu-\sigma(s)]^{\underline{v-\mu-1}}}{(b+m_0+v-\mu)^{\underline{v-\mu-1}}}\times(b+m_0+v)^{\underline{v-1}}\right.\\&\qquad\left.-[b+m_0+v-\mu-\sigma(s)]^{\underline{v-1}}\right\}\end{aligned} \tag{3.8}$$

for $(\rho,s)\in\mathbb{N}_{v-1}^{b+m_0+v}\times\mathbb{N}_0^{b+m_0}$.

If we multiply both sides of (3.8) by $\Gamma(v)$, then it turns out to be

$$\begin{aligned}\Gamma(v)\psi(\rho,s)\le{}&\frac{[b+m_0+v-\mu-\sigma(s)]^{\underline{v-\mu-1}}}{(b+m_0+v-\mu)^{\underline{v-\mu-1}}}\\&\times(b+m_0+v)^{\underline{v-1}}-[b+m_0+v-\sigma(s)]^{\underline{v-1}}\\={}&[b+m_0+v-\mu-\sigma(s)]^{\underline{v-\mu-1}}\times\frac{\Gamma(b+m_0+v+1)}{\Gamma(b+m_0+v-\mu+1)}\\&-[b+m_0+v-\sigma(s)]^{\underline{v-1}}\end{aligned}$$

$$
\begin{aligned}
&= [b+m_0+v-\mu-\sigma(s)]^{\underline{v-\mu-1}} \times (b+m_0+v)^{\underline{\mu}} - [b+m_0+v-\sigma(s)]^{\underline{v-1}} \\
&= \frac{\Gamma(b+m_0+v-\mu-s)}{\Gamma(b+m_0-s+1)} \times (b+m_0+v)^{\underline{\mu}} - \frac{\Gamma(b+m_0+v-s)}{\Gamma(b+m_0-s+1)} \\
&= \frac{\Gamma(b+m_0+v-\mu-s)}{\Gamma(b+m_0-s+1)} \times \left[(b+m_0+v)^{\underline{\mu}} - \frac{\Gamma(b+m_0+v-s)}{\Gamma(b+m_0+v-\mu-s)}\right] \\
&= [b+m_0+v-\mu-\sigma(s)]^{\underline{v-\mu-1}} \times \left\{(b+m_0+v)^{\underline{\mu}} - [b+m_0+v-\sigma(s)]^{\underline{\mu}}\right\} \\
&= f_1(s)f_2(s)
\end{aligned}
$$

which implies the desired inequality (3.5). □

The following is our first main result.

Theorem 1 (Hartman-type inequality)**.** *Let z be a non-trivial solution of Prb. (1.5). If $z(\rho+v-1) \neq 0$ for $\rho \in \mathbb{N}_0^{b+m_0}$, then the inequality*

$$
\sum_{s=0}^{b+m_0} f_1(s)f_2(s)|q(s)| \geq \Gamma(v) \tag{3.9}
$$

holds, where the functions $f_1(s)$ and $f_2(s)$ are defined in (3.6) and (3.7), respectively.

Proof. Let z be a non-trivial solution of Prb. (1.5) such that $z(\rho+v-1) \neq 0$ for $\rho \in \mathbb{N}_0^{b+m_0}$. Then the solution $z(\rho)$ can be explained as

$$
z(\rho) = \sum_{s=0}^{b+m_0} \psi(\rho,s)q(s)z(s+v-1) \tag{3.10}
$$

by (3.2) in Lemma 2, where $\psi(\rho,s)$ is the Green function of Prb. (3.1).

Let $|z(c)| = \max_{\rho\in\mathbb{N}_{v-n_0}^{b+m_0+v}} |z(\rho)|$. Then, taking into account Eq. (3.10) together with inequality (3.5), it turns out the inequality

$$
\begin{aligned}
|z(c)| &= \left|\sum_{s=0}^{b+m_0} \psi(c,s)q(s)z(s+v-1)\right| \\
&\leq \sum_{s=0}^{b+m_0} |\psi(c,s)||q(s)||z(s+v-1)| \\
&\leq |z(c)| \sum_{s=0}^{b+m_0} |\psi(c,s)||q(s)| \\
&\leq \frac{|z(c)|}{\Gamma(v)} \sum_{s=0}^{b+m_0} f_1(s)f_2(s)|q(s)|
\end{aligned} \tag{3.11}
$$

for $c \in \mathbb{N}_{v-n_0}^{b+m_0+v}$, where the functions $f_1(s)$ and $f_2(s)$ are defined in (3.6) and (3.7), respectively. If we multiply the both sides of inequality (3.11) by $\Gamma(v)/|z(c)| > 0$, then we obtain the desired inequality (3.9). This completes the proof. □

Corollary 1 (Lyapunov-type inequality)**.** *Let z be a non-trivial solution of Prb. (1.5). If $z(\rho+v-1)\neq 0$ for $\rho\in\mathbb{N}_0^{b+m_0}$, then the inequality*

$$\sum_{s=0}^{b+m_0}|q(s)|\geq\frac{1}{M_0}\Gamma(v-\mu)\left[\frac{(b+m_0+v)^{\underline{b+m_0+1}}}{(b+m_0+v-\mu)^{\underline{b+m_0+1}}}-1\right]^{-1} \tag{3.12}$$

holds, where

$$M_0=\max\left\{\Gamma(v-\mu),\frac{\Gamma(b+m_0+v-\mu)}{\Gamma(b+m_0+1)}\right\}. \tag{3.13}$$

Proof. Let z be a non-trivial solution of Prb. (1.5) such that $z(\rho+v-1)\neq 0$ for $\rho\in\mathbb{N}_0^{b+m_0}$. Then by Theorem 2, we have the Hartman-type inequality (3.9) for Prb. (1.5).

Now, we will find the maximum value of the functions $f_1(s)$ and $f_2(s)$ on $[0,b+m_0]$, where the functions $f_1(s)$ and $f_2(s)$ are defined in (3.6) and (3.7), respectively. For that purpose, applying the difference operator to them, we find

$$\begin{aligned}\Delta f_1(s)&=-(v-\mu-1)\,[b+m_0+v-\mu-1-\sigma(s)]^{\underline{v-\mu-2}}\\&=-(v-\mu-1)\times\frac{\Gamma(b+m_0+v-\mu-s-1)}{\Gamma(b+m_0-s+1)}\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}\Delta f_2(s)&=\mu\times[b+m_0+v-1-\sigma(s)]^{\underline{\mu-1}}\\&=\mu\times\frac{\Gamma(b+m_0+v-s-1)}{\Gamma(b+m_0+v-\mu-s)}.\end{aligned} \tag{3.15}$$

Thus, considering Eq. (3.14), we find that if $v-\mu<1$, then the function $f_1(s)$ is increasing and if $v-\mu>1$, then it is decreasing on $[0,b+m_0]$, that is, if $v-\mu<1$, then

$$f_1(s)\leq f(b+m_0)=(v-\mu-1)^{\underline{v-\mu-1}}=\Gamma(v-\mu), \tag{3.16}$$

and that if $v-\mu>1$, then

$$f_1(s)\leq f(0)=(b+m_0+v-\mu-1)^{\underline{v-\mu-1}}=\frac{\Gamma(b+m_0+v-\mu)}{\Gamma(b+m_0+1)}. \tag{3.17}$$

on $[0,b+m_0]$. From inequalities (3.16) and (3.17), we deduce that

$$f_1(s)\leq\max\left\{\Gamma(v-\mu),\frac{\Gamma(b+m_0+v-\mu)}{\Gamma(b+m_0+1)}\right\}=M_0. \tag{3.18}$$

Similarly, noticing Eq. (3.15), we see that the function $f_2(s)$ is increasing on $[0,b+m_0]$ and hence we have

$$\begin{aligned}f_2(s)&\leq g(b+m_0)\\&=(b+m_0+v)^{\underline{\mu}}-(v-1)^{\underline{\mu}}\end{aligned}$$

$$
\begin{aligned}
&= \frac{\Gamma(b+m_0+v+1)}{\Gamma(b+m_0+v-\mu+1)} - \frac{\Gamma(v)}{\Gamma(v-\mu)} \\
&= \frac{(b+m_0+v)\times\cdots\times v\times\Gamma(v)}{(b+m_0+v-\mu)\times\cdots\times(v-\mu)\times\Gamma(v-\mu)} - \frac{\Gamma(v)}{\Gamma(v-\mu)} \\
&= \frac{\Gamma(v)}{\Gamma(v-\mu)}\left[\frac{(b+m_0+v)^{\underline{b+m_0+1}}}{(b+m_0+v-\mu)^{\underline{b+m_0+1}}} - 1\right]. \qquad (3.19)
\end{aligned}
$$

Now, using inequalities (3.18) and (3.19), inequality (3.9) turns out to be

$$
\Gamma(v) \le \frac{M_0\Gamma(v)}{\Gamma(v-\mu)} \times \left[\frac{(b+m_0+v)^{\underline{b+m_0+1}}}{(b+m_0+v-\mu)^{\underline{b+m_0+1}}} - 1\right] \times \sum_{s=0}^{b+m_0} |q(s)|.
$$

Dividing both sides of the inequality above by $\Gamma(v) > 0$, we get the desired Lyapunov-type inequality (3.12) which completes the proof of Corollary 1. □

Corollary 2. *Let $v = n_0 \in \mathbb{N}_2$ and $\mu = 1$. If z is a non-trivial solution of Prb.* (1.5) *and $z(\rho+n_0-1) \neq 0$ for $\rho \in \mathbb{N}_0^{b+1}$, then the inequality*

$$
\begin{aligned}
\sum_{s=0}^{b+1} |q(s)| \ge \frac{(n_0-1)!}{(b+2)(b+n_0-1)^{\underline{n_0-2}}} &= \frac{\Gamma(b+2)}{b+2} \times \frac{(n_0-1)!}{\Gamma(b+n_0)} \\
&= \frac{(n_0-1)!}{b+2} \times \prod_{p=2}^{n_0-1} (p+b)^{-1}
\end{aligned}
$$

holds.

We remark that a lower bound for the eigenvalues of an eigenvalue problem can be obtained via Lyapunov-type inequality for the same problem. In this sense, we have the following example as an application of the Lyapunov-type inequality.

Example 1. Consider the eigenvalue problem

$$
\begin{cases}
\Delta^{v}_{v-n_0} z(\rho) + \lambda z(\rho+v-1) = 0, & \rho \in \{0,\dots,b+m_0\}, \\
\Delta^{i} z(v-n_0) = 0, & i \in \{0,\dots,n_0-2\}, \\
\Delta^{\mu}_{v-n_0} z(b+m_0+v-\mu) = 0
\end{cases} \qquad (3.20)
$$

and let z be a non-trivial solution of Prb. (3.20). If $z(\rho+v-1) \neq 0$ for $\rho \in \{0,\dots,b+m_0\}$, then the eigenvalues λ of Prb. (3.20) satisfy the inequality

$$
\begin{aligned}
|\lambda| &\ge \frac{\Gamma(v-\mu)}{M_0(b+m_0)} \times \left[\frac{(b+m_0+v)^{\underline{b+m_0+1}}}{(b+m_0+v-\mu)^{\underline{b+m_0+1}}} - 1\right]^{-1} \\
&= \frac{\Gamma(v-\mu)}{M_0(b+m_0)} \times \left[\frac{\Gamma(b+m_0+v+1)\times\Gamma(v-\mu)}{\Gamma(b+m_0+v-\mu+1)\times\Gamma(v)} - 1\right]^{-1},
\end{aligned}
$$

where the constant M_0 is defined in (3.13).

3.1. *Concluding Remarks*

We conclude this paper by clarifying some highlights. Lyapunov-type inequalities have so many applications as we explained in the beginning of the paper. For example, with the help of these inequalities, an idea can be obtained about the existence or non-existence of the real zeros of Mittag-Leffler functions well-known in the study of these inequalities. A lower bound can be found for eigenvalues of eigenvalue problems generated by the corresponding boundary value problems. Furthermore, by using them, stability criteria for some certain systems and equations can be acquired. We refer the reader to the monograph by Agarwal et al. [1] which includes much more information on these issues.

In the light of these information, the inequalities obtained in this paper are valuable since they can be applied on many fields. These inequalities presents a lower bound for eigenvalues of eigenvalue problem which is obtained from the higher-order discrete fractional boundary value problem as seen in the Example 1. As an application of our study, bounds for eigenvalues of special cases of our problem can be considered. Also, so-called inequalities can be used to obtain an interval in which the Mittag-Leffler function in question has no real zeros. Moreover, the inequalities can help to establish conditions that guarantee the stability of some related systems and equations.

Such applications and more are left to the researchers. It will be of interest to study on these application. On the other hand, considering that the idea of studying the Hartman-type inequalities for higher-order discrete boundary value problems is new, we think that these inequalities are promising for applications and we hope that the area of discrete fractional equations will be developed with their use.

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EXISTENCE OF WEAK SOLUTIONS FOR A DOUBLE PHASE VARIABLE EXPONENT PROBLEM WITH A GRADIENT DEPENDENT REACTION TERM

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Abstract. In the present paper, we study the existence of at least one weak solution for a class of double phase variable exponent problem with a reaction term depending on the gradient and on two real parameters. By using the topological degree theory for a class of demicontinuous operators of generalized (S_+) and the theory of the variable exponent Sobolev spaces, we obtain the existence of at least one weak solution of this problem.

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1. Introduction and motivation

The study of differential equations with with nonstandard $p(x)$-Laplacian operator $\big((p(x),q(x))$-Laplacian operator$\big)$ is an attractive topic and has been the object of considerable attention in recent years. Perhaps the impulse for this comes from the new search field that reflects a new type of physical phenomenon is a class of nonlinear problems with variable exponents. In the subject of fluid mechanics, for example, Rajagopal and M. Ruzicka recently developed a very interesting model for these fluids in [21] (see also [22]). Other applications relate to image processing [1], elasticity problems [18–20, 24, 26], the flow in porous media [3], and problems in the calculus of variations involving variational integrals with nonstandard growth [2].

Here and in the sequel, we will assume that Ω is a bounded domain in $\mathbb{R}^n (n > 1)$, with a Lipschitz boundary denoted by $\partial\Omega$, and let μ and λ be two real parameters.

In this paper, we consider the following double phase variable exponent problem:

$$\begin{cases} -\Delta_{p(x),q(x)}(u) = \mu g(x,u) + \lambda f(x,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where

$$\Delta_{p(x),q(x)}(u) := \operatorname{div}\Big(|\nabla u|^{p(x)-2}\nabla u + a(x)|\nabla u|^{q(x)-2}\nabla u\Big).$$

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In this problem, the coefficient $a\colon \overline{\Omega} \to \mathbb{R}^+$ is Lipschitz continuous function, $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ and $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are Carathéodory functions that satisfy the assumption of growth, and the variables exponents $p(\cdot), q(\cdot) \in C_+(\overline{\Omega})$ are assumed to satisfy the following assumption:

$$1 < q^- \leq q \leq q^+ < p^- \leq p \leq p^+ < +\infty.$$

The double phase operator has been used in the modelling of strongly anisotropic materials [26] and in Lavrentiev's phenomenon [27]. In the one hand, we have the physical motivation; since the double phase operator has been used to model the steady-state solutions of reaction-diffusion problems, that arise in biophysic, plasma-physic and in the study of chemical reactions. In the other hand, these operators provide a useful paradigm for describing the behaviour of strongly anisotropic materials, whose hardening properties are linked to the exponent governing the growth of the gradient change radically with the point, where the coefficient $a(\cdot)$ determines the geometry of a composite made of two different materials (see [4, 28] and the references given there).

Let us recall some known results on problem (1.1). For example, Fan and Zhang [12], based on the theory of the spaces $L^{p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$, present several sufficient conditions for the existence of solutions for the problem (1.1) with $\mu = 1$ and $a = \lambda = 0$.

R. Alsaedi [9] establishes sufficient conditions for the existence of nontrivial weak solutions for the problem (1.1) when $\mu = 1$, $a = 0$, $g(x,u) = |u|^{q(x)-2}u$, $\lambda > 0$ and $f(x,u) = |u|^{p(x)-2}u$.

Problems related to (1.1) in case $p(x) \equiv p$ and $q(x) \equiv q$ have been studied by many scholars, for example, Liu et al. [17] study the problem (1.1) when $\lambda = 0$ and $\mu = 1$, and Wang et al. [23] showed, by using the topological degree theory for a class of demicontinuous operators, the existence of at least one weak solution of (1.1) with $\mu = 0$ and $\lambda = 1$.

We would like to draw attention to the fact that the $p(x)$-laplacian operator has more complicated nonlinearity than the p-laplacian operator. For example, they are non-homogeneous, which prove that our problem is more difficult than the operators p-Laplacian type.

Motivated by the aforementioned works, in the present paper, we will generalize these works. By using a topological degree for a class of demicontinuous operators of generalized (S_+) type of [5] and the theory of the generalized Sobolev spaces, we establish the existence of weak solutions u in $W_0^{1,p(x)}(\Omega)$ for the problem (1.1).

The paper is designed as follows: in Section 2, we review some fundamental preliminaries about the functional framework where we will treat our problems. In Section 3, we introduce some classes of operators, as well as the topological degree theory for a class of demicontinuous operators of generalized (S_+). Finally, Section 4 is devoted to discussing the existence of weak solutions to (1.1).

2. Preliminaries

In this section, we recall the most important and relevant properties and notations about generalized Sobolev spaces $W^{1,p(x)}(\Omega)$, that we will need in our analysis of the problem (1.1), by that, referring to [6–8, 10, 11, 13, 16] for more details.

Let Ω be a smooth bounded domain in $\mathbb{R}^n (n > 1)$, with a Lipschitz boundary denoted by $\partial\Omega$. Set

$$C_+(\overline{\Omega}) = \Big\{ p : p \in C(\overline{\Omega}) \text{ such that } p(x) > 1 \text{ for any } x \in \overline{\Omega} \Big\}.$$

For each $p \in C_+(\overline{\Omega})$, we define

$$p^+ := \max\Big\{ p(x),\ x \in \overline{\Omega} \Big\} \ \text{ and } \ p^- := \min\Big\{ p(x),\ x \in \overline{\Omega} \Big\}.$$

For every $p \in C_+(\overline{\Omega})$, we define

$$L^{p(x)}(\Omega) = \Big\{ u\colon \Omega \to \mathbb{R} \ \text{ is measurable such that } \int_\Omega |u(x)|^{p(x)} dx < +\infty \Big\},$$

equipped with the Luxemburg norm

$$|u|_{p(x)} = \inf\Big\{ \lambda > 0 : \rho_{p(x)}\Big(\frac{u}{\lambda}\Big) \le 1 \Big\},$$

where

$$\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)} dx,\ \forall\, u \in L^{p(x)}(\Omega).$$

Proposition 1 ([13, Theorem 1.3 and Theorem 1.4])**.** *Let (u_n) and $u \in L^{p(\cdot)}(\Omega)$, then*

$$|u|_{p(x)} < 1\big(resp. = 1; > 1\big) \Leftrightarrow \rho_{p(x)}(u) < 1\big(resp. = 1; > 1\big)$$

$$|u|_{p(x)} > 1 \ \Rightarrow\ |u|_{p(x)}^{p^-} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^+}, \tag{2.1}$$

$$|u|_{p(x)} < 1 \ \Rightarrow\ |u|_{p(x)}^{p^+} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^-}, \tag{2.2}$$

$$\lim_{n\to\infty} |u_n - u|_{p(x)} = 0 \ \Leftrightarrow\ \lim_{n\to\infty} \rho_{p(x)}\big(u_n - u\big) = 0. \tag{2.3}$$

Remark 1. According to (2.1) and (2.2), we have

$$|u|_{p(x)} \le \rho_{p(x)}(u) + 1, \tag{2.4}$$

$$\rho_{p(x)}(u) \le |u|_{p(x)}^{p^-} + |u|_{p(x)}^{p^+}. \tag{2.5}$$

Proposition 2 ([16, Theorem 2.5 and Corollary 2.7])**.** *The spaces $L^{p(x)}(\Omega)$ is a separable and reflexive Banach spaces.*

Proposition 3 ([16, Theorem 2.1])**.** *The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$ where $\frac{1}{p(x)}+\frac{1}{p'(x)}=1$ for all $x\in\Omega$. For any $u\in L^{p(x)}(\Omega)$ and $v\in L^{p'(x)}(\Omega)$, we have the following Hölder-type inequality*

$$\Big|\int_\Omega uv\,dx\Big|\le\left(\frac{1}{p^-}+\frac{1}{p'^-}\right)|u|_{p(x)}|v|_{p'(x)}\le 2|u|_{p(x)}|v|_{p'(x)}. \tag{2.6}$$

Remark 2 ([13, Theorem 1.11]). If $p_1,\,p_2\in C_+(\overline{\Omega})$ with $p_1(x)\le p_2(x)$ for any $x\in\overline{\Omega}$, then there exists the continuous embedding $L^{p_2(x)}(\Omega)\hookrightarrow L^{p_1(x)}(\Omega)$.

Now, let $p\in C_+(\overline{\Omega})$ and we define $W^{1,p(x)}(\Omega)$ as

$$W^{1,p(x)}(\Omega)=\Big\{u\in L^{p(x)}(\Omega)\text{ such that }|\nabla u|\in L^{p(x)}(\Omega)\Big\},$$

equipped with the norm

$$||u||=|u|_{p(x)}+|\nabla u|_{p(x)}.$$

We also define $W_0^{1,p(\cdot)}(\Omega)$ as the subspace of $W^{1,p(\cdot)}(\Omega)$, which is the closure of $C_0^\infty(\Omega)$ with respect to the norm $||\cdot||$.

Proposition 4 ([14, Theorem 4.3])**.** *If the exponent $p(\cdot)$ satisfies the log-Hölder continuity condition, i.e. there is a constant $b>0$ such that for every $x,\,y\in\Omega$, $x\neq y$ with $|x-y|\le\dfrac{1}{2}$ one has*

$$|p(x)-p(y)|\le\frac{b}{-\log|x-y|}, \tag{2.7}$$

then we have the Poincaré inequality, i.e. the exists a constant $C>0$ depending only on Ω and the function p such that

$$|u|_{p(x)}\le C|\nabla u|_{p(x)},\ \forall\,u\in W_0^{1,p(\cdot)}(\Omega). \tag{2.8}$$

In this paper we will use the following equivalent norm on $W_0^{1,p(\cdot)}(\Omega)$

$$|u|_{1,p(x)}=|\nabla u|_{p(x)},$$

which is equivalent to $||\cdot||$.

Furthermore, we have the compact embedding $W_0^{1,p(\cdot)}(\Omega)\hookrightarrow\hookrightarrow L^{p(\cdot)}(\Omega)$(see [16]).

Proposition 5 ([16, Theorem 3.1])**.** *The spaces $\Big(W^{1,p(x)}(\Omega),||\cdot||\Big)$ and $\Big(W_0^{1,p(x)}(\Omega),|\cdot|_{1,p(x)}\Big)$ are separable and reflexive Banach spaces.*

Remark 3. The dual space of $W_0^{1,p(x)}(\Omega)$ denoted $W^{-1,p'(x)}(\Omega)$, is equipped with the norm

$$|u|_{-1,p'(x)}=\inf\Big\{|u_0|_{p'(x)}+\sum_{i=1}^N|u_i|_{p'(x)}\Big\},$$

where the infinimum is taken on all possible decompositions $u = u_0 - \mathrm{div}F$ with $u_0 \in L^{p'(x)}(\Omega)$ and $F = (u_1, \dots, u_N) \in (L^{p'(x)}(\Omega))^N$.

3. A REVIEW ON SOME CLASSES OF MAPPINGS AND TOPOLOGICAL DEGREE THEORY

We start by defining some classes of mappings. In what follows, let X be a real separable reflexive Banach space and X^* be its dual space with dual pairing $\langle \cdot, \cdot \rangle$ and given a nonempty subset Ω of X. Strong (weak) convergence is represented by the symbol $\to$ ($\rightharpoonup$).

Definition 1. Let Y be another real Banach space. A operator $F\colon \Omega \subset X \to Y$ is said to be :

(1) bounded, if it takes any bounded set into a bounded set.
(2) demicontinuous, if for any sequence $(u_n) \subset \Omega$, $u_n \to u$ implies that $F(u_n) \rightharpoonup F(u)$.
(3) compact, if it is continuous and the image of any bounded set is relatively compact.

Definition 2. A mapping $F\colon \Omega \subset X \to X^*$ is said to be :

(1) of class (S_+), if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$ and $\limsup\limits_{n\to\infty} \langle Fu_n, u_n - u\rangle \le 0$, we have $u_n \to u$.
(2) quasimonotone, if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, we have $\limsup\limits_{n\to\infty} \langle Fu_n,\ u_n - u\rangle \ge 0$.

Definition 3. Let $T\colon \Omega_1 \subset X \to X^*$ be a bounded operator such that $\Omega \subset \Omega_1$. For any operator $F\colon \Omega \subset X \to X$, we say that

(1) F is of class $(S_+)_T$, if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$ and $\limsup\limits_{n\to\infty} \langle Fu_n,\ y_n - y\rangle \le 0$, we have $u_n \to u$.
(2) F has the property $(QM)_T$, if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u,\ y_n := Tu_n \rightharpoonup y$, we have $\limsup\limits_{n\to\infty} \langle Fu_n, y - y_n\rangle \ge 0$.

In the sequel, we consider the following classes of operators:

$$\mathcal{F}_1(\Omega) := \Big\{F\colon \Omega \to X^* : F \text{ is bounded, demicontinuous and of class } (S_+)\Big\},$$

$$\mathcal{F}_T(\Omega) := \Big\{F\colon \Omega \to X : F \text{ is demicontinuous and of class } (S_+)_T\Big\},$$

$$\mathcal{F}_{T,B}(\Omega) := \Big\{F \in \mathcal{F}_T(\Omega) : F \text{ is bounded }\Big\},$$

for any $\Omega \subset D(F)$, where $D(F)$ denotes the domain of F, and any $T \in \mathcal{F}_1(\Omega)$.

Now, let O be the collection of all bounded open sets in X and we define

$$\mathcal{F}(X) := \Big\{F \in \mathcal{F}_T(\overline{E}) : E \in O,\ \mathrm{T} \in \mathcal{F}_1(\overline{\mathrm{E}})\Big\},$$

where, $\mathrm{T} \in \mathcal{F}_1(\overline{\mathrm{E}})$ is called an essential inner map to F.

Lemma 1 ([15, Lemma 2.3])**.** *Let $T \in \mathcal{F}_1(\overline{E})$ be continuous and $S\colon D(S) \subset X^* \to X$ be demicontinuous such that $T(\overline{E}) \subset D(S)$, where E is a bounded open set in a real reflexive Banach space X. Then the following statements are true :*

(1) *If S is quasimonotone, then $I + S \circ T \in \mathcal{F}_T(\overline{E})$, where I denotes the identity operator.*
(2) *If S is of class (S_+), then $S \circ T \in \mathcal{F}_T(\overline{E})$.*

Definition 4. Suppose that E is bounded open subset of a real reflexive Banach space X, $T \in \mathcal{F}_1(\overline{E})$ is continuous and $F, S \in \mathcal{F}_T(\overline{E})$. Then the affine homotopy $\mathcal{H}\colon [0,1] \times \overline{E} \to X$ defined by

$$\mathcal{H}(t,u) := (1-t)Fu + tSu, \quad \text{for} \quad (t,u) \in [0,1] \times \overline{E}$$

is called an admissible affine homotopy with the common continuous essential inner map T.

Remark 4 ([15, Lemma 2.5]). *The above affine homotopy is of class $(S_+)_T$.*

As in [15] we give the topological degree for the class $\mathcal{F}(X)$.

Theorem 1. *Let*

$$M = \Big\{(F,E,h) : E \in O,\ T \in \mathcal{F}_1(\overline{E}),\ F \in \mathcal{F}_{T,B}(\overline{E}),\ h \notin F(\partial E)\Big\}.$$

Then, there exists a unique degree function $d\colon M \to \mathbb{Z}$ that satisfies the following properties:

(1) (Normalization.) *For any $h \in F(E)$, we have*

$$d(I,E,h) = 1.$$

(2) (Additivity.) *Let $F \in \mathcal{F}_{T,B}(\overline{E})$. If E_1 and E_2 are two disjoint open subsets of E such that $h \notin F(\overline{E} \backslash (E_1 \cup E_2))$, then we have*

$$d(F,E,h) = d(F,E_1,h) + d(F,E_2,h).$$

(3) (Homotopy invariance). *If $\mathcal{H}\colon [0,1] \times \overline{E} \to X$ is a bounded admissible affine homotopy with a common continuous essential inner map and $h\colon [0,1] \to X$ is a continuous path in X such that $h(t) \notin \mathcal{H}(t, \partial E)$ for all $t \in [0,1]$, then*

$$d(\mathcal{H}(t,\cdot),E,h(t)) = \text{const} \textit{ for all } t \in [0,1].$$

(4) (Existence.) *If $d(F,E,h) \neq 0$, then the equation $Fu = h$ has a solution in E.*

Definition 5 ([15, Definition 3.3])**.** The above degree is defined as follows:

$$d(F,E,h) := d_B(F|_{\overline{E}_0}, E_0, h),$$

where d_B is the Berkovits degree [5] *and E_0 is any open subset of E with $F^{-1}(h) \subset E_0$ and F is bounded on $\overline{E}_0$.*

4. Existence of weak solutions

In this section, we will discuss the existence of weak solutions of (1.1). For this, we list our assumptions associated with our problem to show the existence result.

From new on, we always assume that $\Omega \subset \mathbb{R}^n (n > 1)$ is a bounded domain with a Lipschitz boundary $\partial\Omega$, $p \in C_+(\overline{\Omega})$ satisfy the log-Hölder continuity condition (2.7), $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ and $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are functions such that :

(A_1)**:** f is a Carathéodory function.

(A_2)**:** There exists $\rho > 0$ and $\gamma \in L^{p'(x)}(\Omega)$ such that

$$|f(x,\zeta,\xi)| \leq \rho(\gamma(x) + |\zeta|^{q(x)-1} + |\xi|^{q(x)-1}).$$

(A_3)**:** g is a Carathéodory function.

(A_4)**:** There are $\sigma > 0$ and $\nu \in L^{p'(x)}(\Omega)$ such that

$$|g(x,\zeta)| \leq \sigma(\nu(x) + |\zeta|^{s(x)-1}),$$

for a.e. $x \in \Omega$ and all $(\zeta,\xi) \in \mathbb{R} \times \mathbb{R}^n$, where $k, s \in C_+(\overline{\Omega})$ with $2 \leq s^- \leq s(x) \leq s^+ < p^-$.

Remark 5.

- Note that for all $\varphi \in W_0^{1,p(x)}(\Omega)$

$$\int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u \nabla\varphi + a(x)|\nabla u|^{q(x)-2}\nabla u \nabla\varphi\Big)dx$$

is well defined (see [12, 17]).

- $\mu g(x,u)$ and $\lambda f(x,u,\nabla u)$ are belongs to $L^{p'(x)}(\Omega)$ under $u \in W_0^{1,p(x)}(\Omega)$, the assumptions (A_2) and (A_4) and the given hypotheses about the exponents p, α, q and s because: $r(x) = (q(x)-1)p'(x) \in C_+(\overline{\Omega})$ with $r(x) < p(x)$, and $\kappa(x) = (s(x)-1)p'(x) \in C_+(\overline{\Omega})$ with $\kappa(x) < p(x)$.

  Then, by Remark 2 we can conclude that

$$L^{p(x)} \hookrightarrow L^{r(x)} \text{ and } L^{p(x)} \hookrightarrow L^{\kappa(x)}.$$

  Hence, since $\varphi \in L^{p(x)}(\Omega)$, we have

$$\Big(\mu g(x,u) + \lambda f(x,u,\nabla u)\Big)\varphi \in L^1(\Omega).$$

  This implies that, the integral

$$\int_\Omega \Big(\mu g(x,u) + \lambda f(x,u,\nabla u)\Big)\varphi dx$$

  exist.

Then, we shall use the definition of weak solution for (1.1) in the following sense:

Definition 6. We say that a function $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of (1.1), if for any $\varphi \in W_0^{1,p(x)}(\Omega)$, it satisfies the following:

$$\int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u \nabla \varphi + a(x)|\nabla u|^{q(x)-2}\nabla u \nabla \varphi\Big)dx = \int_\Omega \Big(\mu g(x,u) + \lambda f(x,u,\nabla u)\Big)\varphi dx.$$

Before giving the existence result for the problem (1.1), we first give two lemmas that will be used in the proof of this result.

Let us consider the following functional:

$$\mathcal{J}(u) := \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_\Omega \frac{a(x)}{q(x)}|\nabla u|^{q(x)}dx.$$

From [12,17], it is obvious that the derivative operator of the functional $\mathcal{J}$ in the weak sense at the point $u \in W_0^{1,p(x)}(\Omega)$ is the functional $\mathcal{T}(u) := \mathcal{J}'(u) \in W^{-1,p'(x)}(\Omega)$, given by

$$\langle \mathcal{T}u, \varphi\rangle = \int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u \nabla \varphi + a(x)|\nabla u|^{q(x)-2}\nabla u \nabla \varphi\Big)dx,$$

for all $u, \varphi \in W_0^{1,p(x)}(\Omega)$ where $\langle \cdot, \cdot \rangle$ means the duality pairing between $W^{-1,p'(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$. Furthermore, the properties of the operator $\mathcal{T}$ are summarized in the following lemma (see [12, Theorem 3.1] and [17, Proposition 3.1]).

Lemma 2. *The operator $\mathcal{T}\colon W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ defined by*

$$\langle \mathcal{T}u, \varphi\rangle = \int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u \nabla \varphi + a(x)|\nabla u|^{q(x)-2}\nabla u \nabla \varphi\Big)dx, \tag{4.1}$$

is a continuous, bounded, strictly monotone operator, and is of class (S_+).

Lemma 3. *If $(A_1)-(A_4)$ hold, then the operator $\mathcal{S}\colon W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ defined by*

$$\langle \mathcal{S}u, \varphi\rangle = -\int_\Omega \Big(\mu g(x,u) + \lambda f(x,u,\nabla u)\Big)\varphi dx, \tag{4.2}$$

for all $u, \varphi \in W_0^{1,p(x)}(\Omega)$, is compact.

Proof. In order to prove this lemma, we proceed in three steps.

Step 1 : Let $\Psi_1\colon W^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ be an operator defined by

$$\Psi_1 u(x) := -\mu g(x,u).$$

In this step, we prove that the operator Ψ_1 is bounded and continuous.

First, let $u \in W^{1,p(x)}(\Omega)$, bearing (A_4) in mind and using (2.4) and (2.5), we infer

$$\begin{aligned}
|\Psi_1 u|_{p'(x)} &\leq \rho_{p'(x)}(\Psi_1 u) + 1 \\
&= \int_\Omega |\mu g(x,u(x))|^{p'(x)} dx + 1 \\
&= \int_\Omega |\mu|^{p'(x)} |g(x,u(x)|^{p'(x)} dx + 1 \\
&\leq \left(|\mu|^{p'^-} + |\mu|^{p'^+}\right) \int_\Omega |\sigma\left(\nu(x) + |u|^{s(x)-1}\right)|^{p'(x)} dx + 1 \\
&\leq \text{const}\left(|\mu|^{p'^-} + |\mu|^{p'^+}\right) \int_\Omega \left(|\nu(x)|^{p'(x)} + |u|^{\kappa(x)}\right) dx + 1 \\
&\leq \text{const}\left(|\mu|^{p'^-} + |\mu|^{p'^+}\right) \left(\rho_{p'(x)}(\nu) + \rho_{\kappa(x)}(u)\right) + 1 \\
&\leq \text{const}\left(|\nu|_{p(x)}^{p'^+} + |u|_{\kappa(x)}^{\kappa^+} + |u|_{\kappa(x)}^{\kappa^-}\right) + 1.
\end{aligned}$$

Then, we deduce from (2.8) and $L^{p(x)} \hookrightarrow L^{\kappa(x)}$, that

$$|\Psi_1 u|_{p'(x)} \leq \text{const}\left(|\nu|_{p(x)}^{p'^+} + |u|_{1,p(x)}^{\kappa^+} + |u|_{1,p(x)}^{\kappa^-}\right) + 1,$$

that means Ψ_1 is bounded on $W^{1,p(x)}(\Omega)$.

Second, we show that the operator Ψ_1 is continuous. To this purpose let $u_n \to u$ in $W^{1,p(x)}(\Omega)$. We need to show that $\Psi_1 u_n \to \Psi_1 u$ in $L^{p'(x)}(\Omega)$. We will apply the Lebesgue's theorem. Note that if $u_n \to u$ in $W^{1,p(x)}(\Omega)$, then $u_n \to u$ in $L^{p(x)}(\Omega)$. Hence there exist a subsequence (u_k) of (u_n) and ϕ in $L^{p(x)}(\Omega)$ such that

$$u_k(x) \to u(x) \ \text{ and } \ |u_k(x)| \leq \phi(x), \tag{4.3}$$

for a.e. $x \in \Omega$ and all $k \in \mathbb{N}$.

Hence, from (A_2) and (4.3), we have

$$|g(x,u_k(x))| \leq \sigma(\nu(x) + |\phi(x)|^{s(x)-1}),$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$.

On the other hand, thanks to (A_3) and (4.3), we get, as $k \longrightarrow \infty$

$$g(x,u_k(x)) \to g(x,u(x)) \ \text{ a.e. } \ x \in \Omega.$$

Seeing that $\nu + |\phi|^{s(x)-1} \in L^{p'(x)}(\Omega)$ and

$$\rho_{p'(x)}(\Psi_1 u_k - \Psi_1 u) = \int_\Omega |g(x,u_k(x)) - g(x,u(x))|^{p'(x)} dx,$$

then, from the Lebesgue's theorem and the equivalence (2.3), we have

$$\Psi_1 u_k \to \Psi_1 u \ \text{ in } \ L^{p'(x)}(\Omega),$$

and consequently

$$\Psi_1 u_n \to \Psi_1 u \ \text{ in } \ L^{p'(x)}(\Omega),$$

that is, Ψ_1 is continuous.

Step 2 :: Let us define the operator $\Psi_2 \colon W^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ by

$$\Psi_2 u(x) := -\lambda f(x, u(x), \nabla u(x)).$$

We will show that Ψ_2 is bounded and continuous. Let $u \in W^{1,p(x)}(\Omega)$. According to (A_2) and the inequalities (2.4) and (2.5), we obtain

$$\begin{aligned}
|\Psi_2 u|_{p'(x)} &\leq \rho_{p'(x)}(\Psi_2 u) + 1 \\
&= \int_\Omega |\lambda f(x, u(x), \nabla u(x))|^{p'(x)} dx + 1 \\
&= \int_\Omega |\lambda|^{p'(x)} |f(x, u(x), \nabla u(x))|^{p'(x)} dx + 1 \\
&\leq \Big(|\lambda|^{p'^-} + |\lambda|^{p'^+} \Big) \int_\Omega |\rho \Big(\gamma(x) + |u|^{q(x)-1} + |\nabla u|^{q(x)-1} \Big)|^{p'(x)} dx + 1 \\
&\leq \text{const} \Big(|\lambda|^{p'^-} + |\lambda|^{p'^+} \Big) \int_\Omega \Big(|\gamma(x)|^{p'(x)} + |u|^{r(x)} + |\nabla u|^{r(x)} \Big) dx + 1 \\
&\leq \text{const} \Big(|\lambda|^{p'^-} + |\lambda|^{p'^+} \Big) \Big(\rho_{p'(x)}(\gamma) + \rho_{r(x)}(u) + \rho_{r(x)}(\nabla u) \Big) + 1 \\
&\leq \text{const} \Big(|\gamma|_{p(x)}^{p'^+} + |u|_{r(x)}^{r^+} + |u|_{r(x)}^{r^-} + |\nabla u|_{r(x)}^{r^+} + |\nabla u|_{r(x)}^{r^-} \Big) + 1.
\end{aligned}$$

Taking into account that $L^{p(x)} \hookrightarrow L^{r(x)}$ and (2.8), we have then

$$|\Psi_2 u|_{p'(x)} \leq \text{const} \Big(|\gamma|_{p(x)}^{p'^+} + |u|_{1,p(x)}^{r^+} + |u|_{1,p(x)}^{r^-} \Big) + 1,$$

and consequently Ψ_2 is bounded on $W^{1,p(x)}(\Omega)$.

It remains to show that Ψ_2 is continuous. Let $u_n \to u$ in $W^{1,p(x)}(\Omega)$, we need to show that $\Psi_2 u_n \to \Psi_2 u$ in $L^{p'(x)}(\Omega)$. We will apply the Lebesgue's theorem.

Note that if $u_n \to u$ in $W^{1,p(x)}(\Omega)$, then $u_n \to u$ in $L^{p(x)}(\Omega)$ and $\nabla u_n \to \nabla u$ in $(L^{p(x)}(\Omega))^N$. Hence, there exist a subsequence (u_k) and ϕ in $L^{p(x)}(\Omega)$ and ψ in $(L^{p(x)}(\Omega))^N$ such that

$$u_k(x) \to u(x) \ \text{ and } \ \nabla u_k(x) \to \nabla u(x), \tag{4.4}$$

$$|u_k(x)| \leq \phi(x) \ \text{ and } \ |\nabla u_k(x)| \leq |\psi(x)|, \tag{4.5}$$

for a.e. $x \in \Omega$ and all $k \in \mathbb{N}$.

Hence, thanks to (A_1) and (4.4), we get, as $k \longrightarrow \infty$

$$f(x, u_k(x), \nabla u_k(x)) \to f(x, u(x), \nabla u(x)) \ \text{ a.e. } \ x \in \Omega.$$

On the other hand, from (A_2) and (4.5), we can deduce the estimate

$$|f(x, u_k(x), \nabla u_k(x))| \leq \rho(\gamma(x) + |\phi(x)|^{q(x)-1} + |\psi(x)|^{q(x)-1}),$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$.

Seeing that

$$\gamma + |\phi|^{q(x)-1} + |\psi(x)|^{q(x)-1} \in L^{p'(x)}(\Omega),$$

and taking into account the equality

$$\rho_{p'(x)}(\Psi_2 u_k - \Psi_2 u) = \int_\Omega |f(x,u_k(x),\nabla u_k(x)) - f(x,u(x),\nabla u(x))|^{p'(x)} dx,$$

then, we conclude from the Lebesgue's theorem and (2.3) that

$$\Psi_2 u_k \to \Psi_2 u \ \text{ in } \ L^{p'(x)}(\Omega),$$

and consequently

$$\Psi_2 u_n \to \Psi_2 u \ \text{ in } \ L^{p'(x)}(\Omega),$$

and then Ψ_2 is continuous.

Step 3: Let $I^* \colon L^{p'(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ be the adjoint operator of the operator $I \colon W^{1,p(x)}(\Omega) \to L^{p(x)}(\Omega)$.

We then define

$$I^* \circ \Psi_1 \colon W^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega),$$

and

$$I^* \circ \Psi_2 \colon W^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega).$$

On another side, taking into account that I is compact, then I^* is compact. Thus, the compositions $I^* \circ \Psi_1$ and $I^* \circ \Psi_2$ are compact, that means $\mathcal{S} = I^* \circ \Psi_1 + I^* \circ \Psi_2$ is compact. With this last step the proof of Lemma 3 is completed.

□

We are now in the position to get the existence result of weak solution for (1.1).

Theorem 2. *Assume that the assumptions* $(A_1) - (A_4)$ *hold, then the problem* (1.1) *possesses at least one weak solution* u *in* $W_0^{1,p(x)}(\Omega)$.

Proof. The basic idea of our proof is to reduce the problem (1.1) to a new one governed by a Hammerstein equation, and apply the theory of topological degree introduced in Section 3 to show the existence of weak solutions to the state problem.

First, for all $u, \varphi \in W_0^{1,p(x)}(\Omega)$, we define the operators $\mathcal{T}$ and $\mathcal{S}$, as defined in (4.1) and (4.2) respectively. Consequently, the problem (1.1) is equivalent to the equation

$$\mathcal{T}u + \mathcal{S}u = 0, \quad u \in W_0^{1,p(x)}(\Omega). \tag{4.6}$$

Taking into account that, by Lemma 2, the operator $\mathcal{T}$ is a continuous, bounded, strictly monotone and of class (S_+), then, by [25, Theorem 26 A], the inverse operator

$$\mathcal{L} := \mathcal{T}^{-1} \colon W^{-1,p'(x)}(\Omega) \to W_0^{1,p(x)}(\Omega),$$

is also bounded, continuous, strictly monotone and of class (S_+).

On another side, according to Lemma 3, we have that the operator $\mathcal{S}$ is bounded, continuous and quasimonotone. Consequently, following Zeidler's terminology [25], the equation (4.6) is equivalent to the following abstract Hammerstein equation

$$u = \mathcal{L}\varphi \ \text{ and } \ \varphi + \mathcal{S}\circ\mathcal{L}\varphi = 0, \ \ u \in W_0^{1,p(x)}(\Omega) \text{ and } \varphi \in W^{-1,p'(x)}(\Omega). \tag{4.7}$$

Seeing that (4.6) is equivalent to (4.7), then to solve (4.6) it is thus enough to solve (4.7). In order to solve (4.7), we will apply the Berkovits topological degree introducing in Section 3. First, let us set

$$\mathcal{B} := \Big\{\varphi \in W^{-1,p'(x)}(\Omega) : \exists\, t \in [0,1] \ \text{ such that } \ \varphi + t\mathcal{S}\circ\mathcal{L}\varphi = 0\Big\}.$$

Next, we show that $\mathcal{B}$ is bounded in $\in W^{-1,p'(x)}(\Omega)$. Let us put $u := \mathcal{L}\varphi$ for all $\varphi \in \mathcal{B}$. Taking into account that $|\mathcal{L}\varphi|_{1,p(x)} = |\nabla u|_{p(x)}$, then we have the following two cases:

First case: If $|\nabla u|_{p(x)} \le 1$, then $|\mathcal{L}\varphi|_{1,p(x)} \le 1$, that means $\Big\{\mathcal{L}\varphi : \varphi \in \mathcal{B}\Big\}$ is bounded.

Second case: If $|\nabla u|_{p(x)} > 1$, then, we deduce from (2.1), (A_2) and (A_4), the inequalities (2.6) and (2.5) and the Young's inequality that

$$\begin{aligned}
|\mathcal{L}\varphi|_{1,p(x)}^{p^-} &= |\nabla u|_{p(x)}^{p^-} \\
&\le \rho_{p(x)}(\nabla u) \\
&\le \langle \mathcal{T}u,\ u\rangle \\
&\le \langle \varphi,\ \mathcal{L}\varphi\rangle \\
&= -t\langle \mathcal{S}\circ\mathcal{L}\varphi,\ \mathcal{L}\varphi\rangle \\
&= t\int_\Omega \Big(\mu g(x,u) + \lambda f(x,u,\nabla u)\Big) u dx \\
&\le \text{const}\Big(\int_\Omega |\nu(x)u(x)|dx + \int_\Omega |u(x)|^{s(x)}dx + \int_\Omega |\gamma(x)u(x)|dx \\
&\qquad\qquad + \int_\Omega |u(x)|^{q(x)}dx + \int_\Omega |\nabla u|^{q(x)-1}|u|dx\Big) \\
&\le \text{const}\Big(\int_\Omega |\nu(x)u(x)|dx + \int_\Omega |\gamma(x)u(x)|dx + \rho_{s(x)}(u) + \rho_{q(x)}(u) \\
&\qquad\qquad + \int_\Omega |\nabla u|^{q(x)-1}|u|dx\Big) \\
&\le \text{const}\Big(|\nu|_{p'(x)}|u|_{p(x)} + |\gamma|_{p'(x)}|u|_{p(x)} + |u|_{s(x)}^{s^+} + |u|_{s(x)}^{s^-} + |u|_{q(x)}^{q^+} + |u|_{q(x)}^{q^-} \\
&\qquad\qquad + \frac{1}{q'^-}\rho_{q(x)}(\nabla u) + \frac{1}{q^-}\rho_{q(x)}(u)\Big) \\
&\le \text{const}\Big(|u|_{p(x)} + |u|_{s(x)}^{s^+} + |u|_{s(x)}^{s^-} + |u|_{q(x)}^{q^+} + |u|_{q(x)}^{q^-} + |\nabla u|_{q(x)}^{q^+}\Big),
\end{aligned}$$

then, according to $L^{p(x)} \hookrightarrow L^{s(x)}$ and $L^{p(x)} \hookrightarrow L^{q(x)}$, we get

$$|\mathcal{L}\varphi|_{1,p(x)}^{p^-} \leq \text{const}\Big(|\mathcal{L}\varphi|_{1,p(x)} + |\mathcal{L}\varphi|_{1,p(x)}^{s^+} + |\mathcal{L}\varphi|_{1,p(x)}^{q^+}\Big),$$

what implies that $\Big\{\mathcal{L}\varphi : \varphi \in \mathcal{B}\Big\}$ is bounded.

On the other hand, we have that the operator is $\mathcal{S}$ is bounded, then $\mathcal{S}o\mathcal{L}\varphi$ is bounded. Thus, thanks to (4.7), we have that $\mathcal{B}$ is bounded in $W^{-1,p'(x)}(\Omega)$. However, $\exists\, R > 0$ such that

$$|\varphi|_{-1,p'(x)} < R \ \text{ for all } \ \varphi \in \mathcal{B},$$

which leads to

$$\varphi + t\mathcal{S} \circ \mathcal{L}\varphi \neq 0, \ \ \varphi \in \partial\mathcal{B}_R(0) \ \text{ and } \ t \in [0,1],$$

where $\mathcal{B}_R(0)$ is the ball of center 0 and radius R in $W^{-1,p'(x)}(\Omega)$. Moreover, by Lemma 1, we conclude that

$$I + \mathcal{S} \circ \mathcal{L} \in \mathcal{F}_{\mathcal{L}}(\overline{\mathcal{B}_R(0)}) \ \text{ and } \ I = \mathcal{T} \circ \mathcal{L} \in \mathcal{F}_{\mathcal{L}}(\overline{\mathcal{B}_R(0)}).$$

On another side, taking into account that I, $\mathcal{S}$ and $\mathcal{L}$ are bounded, then $I + \mathcal{S}o\mathcal{L}$ is bounded. Hence, we infer that

$$I + \mathcal{S} \circ \mathcal{L} \in \mathcal{F}_{\mathcal{L},B}(\overline{\mathcal{B}_R(0)}) \ \text{ and } \ I = \mathcal{T} \circ \mathcal{L} \in \mathcal{F}_{\mathcal{L},B}(\overline{\mathcal{B}_R(0)}).$$

Next, we define the homotopy

$$\mathcal{H}\colon [0,1] \times \overline{\mathcal{B}_R(0)} \to W^{-1,p'(x)}(\Omega) \tag{4.8}$$

$$(t,\varphi) \mapsto \mathcal{H}(t,\varphi) := \varphi + t\mathcal{S} \circ \mathcal{L}\varphi. \tag{4.9}$$

Applying the homotopy invariance and normalization property of the degree d seen in Theorem 1, we obtain

$$d(I + \mathcal{S} \circ \mathcal{L}, \mathcal{B}_R(0), 0) = d(I, \mathcal{B}_R(0),\ 0) = 1 \neq 0.$$

Since $d(I + \mathcal{S} \circ \mathcal{L}, \mathcal{B}_R(0), 0) \neq 0$, then by the existence property of the degree d stated in Theorem 1, we conclude that there exists $\varphi \in \mathcal{B}_R(0)$ which verifies

$$\big(I + \mathcal{S} \circ \mathcal{L}\big)(\varphi) = 0 \Leftrightarrow \varphi + \mathcal{S} \circ \mathcal{L}\varphi = 0 \Leftrightarrow \mathcal{T} \circ \mathcal{L}\varphi + \mathcal{S} \circ \mathcal{L}\varphi = 0.$$

Finally, we infer that $u = \mathcal{L}\varphi$ is a weak solution of (1.1). The proof is completed.

□

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STRONGLY $\oplus - g-$RADICAL SUPPLEMENTED MODULES

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Abstract. In this work, strongly $\oplus - g-$radical supplemented modules are defined and some properties of these modules are investigated. Every ring has unity and every module is unital left module in this work. It is proved that every direct summand of a strongly $\oplus - g-$radical supplemented module is strongly $\oplus - g-$radical supplemented. Let $f : M \longrightarrow N$ be an $R-$module epimorphism and $Ker(f)$ be a direct summand of M. If M is strongly $\oplus - g-$radical supplemented, then N is also strongly $\oplus - g-$radical supplemented. Let M be a strongly $\oplus - g-$ radical supplemented $R-$module and $K \leq M$. If V is a g-radical supplement submodule in M for every g-radical supplement submodule V/K in M/K, then M/K is strongly $\oplus - g-$radical supplemented.

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1. Introduction

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let R be a ring and M be an $R-$module. We will denote a submodule N of M by $N \leq M$. Let M be an $R-$module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a *small submodule* of M and denoted by $N \ll M$. Let M be an $R-$module and $N \leq M$. If there exists a submodule K of M such that $M = N + K$ and $N \cap K = 0$, then N is called a *direct summand* of M and it is denoted by $M = N \oplus K$. For any $R-$module M, we have $M = M \oplus 0$. The intersection of all maximal submodules of M is called the *radical* of M and denoted by $RadM$. If M have no maximal submodules, then it is defined $RadM = M$. M is said to be *semilocal* if $M/RadM$ is semisimple. A submodule N of an $R-$module M is called an *essential submodule* of M and denoted by $N \trianglelefteq M$ in case $K \cap N \neq 0$ for every submodule $K \neq 0$, or equivalently, $K = 0$ for every $K \leq M$ with $N \cap K = 0$. Let M be an $R-$module and K be a submodule of M. K is called a *generalized small* (or briefly, $g-$*small*) *submodule* of M if for every essential submodule T of M with the property $M = K + T$ implies that $T = M$, then we write $K \ll_g M$ (in [15], it is called

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an *e-small submodule* of M and denoted by $K \ll_e M$). It is clear that every small submodule is a generalized small submodule but the converse is not true in general. Let M be an R -module. M is called a *hollow module* if every proper submodule of M is small in M. M is called a *local module* if M has the largest submodule, i.e. a proper submodule which contains all other proper submodules. Let U and V be submodules of M. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M. M is said to be *supplemented* if every submodule of M has a supplement in M. If every submodule of M has a supplement that is a direct summand in M, then M is called a $\oplus-$*supplemented module*. Let M be a supplemented $R-$module. If every supplement submodule of M is a direct summand of M, then M is called a *strongly* $\oplus-$*supplemented module*. Let M be an R $-$module and $U, V \leq M$. If $M = U + V$ and $M = U + T$ with $T \trianglelefteq V$ implies that $T = V$, or equivalently, $M = U + V$ and $U \cap V \ll_g V$, then V is called a *g-supplement* of U in M. M is said to be *g-supplemented* if every submodule of M has a g-supplement in M. M is said to be $\oplus - g-$*supplemented* if every submodule of M has a g-supplement that is a direct summand in M (see [8]). Let M be an R-module and $U, V \leq M$. If $M = U + V$ and $U \cap V \leq RadV$, then V is called a *generalized* (*radical*) *supplement* (briefly, *Rad-supplement*) of U in M. M is said to be *generalized* (*radical*) *supplemented* (briefly, *Rad-supplemented*) if every submodule of M has a Rad-supplement in M. M is said to be *generalized* (*radical*) $\oplus-$*supplemented* (briefly, *Rad*$- \oplus -$*supplemented*) if every submodule of M has a Rad-supplement that is a direct summand in M. M said to be *strongly Rad*$-\oplus$*-supplemented* if M is Rad-supplemented and every Rad-supplement submodule in M is a direct summand of M (see [12]). The intersection of all essential maximal submodules of an $R-$module M is called the *generalized radical* of M and denoted by Rad_gM (in [15], it is denoted by Rad_eM). If M have no essential maximal submodules, then we denote $Rad_gM = M$. An $R-$module M is said to be *g-semilocal* if M/Rad_gM is semisimple (see [9]). Let M be an $R-$module and $U, V \leq M$. If $M = U + V$ and $U \cap V \leq Rad_gV$, then V is called a *generalized radical supplement* (or briefly, *g-radical supplement*) of U in M. M is said to be *generalized radical supplemented* (briefly, *g-radical supplemented*) if every submodule of M has a g-radical supplement in M. M is said to be $\oplus - g - Rad-$*supplemented* if every submodule of M has a g-radical supplement that is a direct summand in M (see [10]).

More informations about supplemented modules are in [2, 14]. More results about $\oplus-$supplemented modules are in [4]. More details about strongly $\oplus-$supplemented modules are in [7]. More details about generalized (radical) supplemented modules are in [13]. More details about generalized (radical) $\oplus-$supplemented modules are in [1, 3]. More informations about g-supplemented modules are in [5]. More informations about g-radical supplemented modules are in [6]. Now we will give some important properties of the generalized radical of any module.

Lemma 1. *Let M be an $R-$module. The following conditions hold.*

(1) $Rad_gM = \sum_{L \ll_g M} L$.
(2) $Rm \ll_g M$ *for every* $m \in Rad_gM$.
(3) *If* $N \leq M$, *then* $Rad_gN \leq Rad_gM$.
(4) *If* $K, L \leq M$, *then* $Rad_gK + Rad_gL \leq Rad_g(K+L)$.
(5) *Let* N *be an* $R-$*module and* $f : M \longrightarrow N$ *be an* $R-$*module homomorphism. Then* $f(Rad_gM) \leq Rad_gN$.
(6) *If* $K, L \leq M$, *then* $\frac{Rad_gK+L}{L} \leq Rad_g\frac{K+L}{L}$.
(7) *If* $M = \bigoplus_{i\in I} M_i$, *then* $Rad_gM = \bigoplus_{i\in I} Rad_gM_i$.

Proof. See [6, Lemma 2, Lemma 3 and Lemma 4]. □

2. Strongly $\oplus-g-$Radical Supplemented Modules

Definition 1. Let M be a g-supplemented $R-$module. If every g-supplement submodule is a direct summand in M, then M is called a strongly $\oplus-g-$ supplemented module.

Definition 2. Let M be a g-radical supplemented $R-$module. If every g-radical supplement submodule is a direct summand in M, then M is called a strongly $\oplus-$ generalized radical supplemented (briefly, strongly $\oplus-g-$radical supplemented) module. (See also [11])

Clearly we can see that every strongly $\oplus-g-$radical supplemented module is $\oplus-g-$radical supplemented. But the converse of this statement is not true in general (see Example 1, Example 2 and Example 3). Let M be a strongly $\oplus-g-$radical supplemented $R-$module. Then M is g-radical supplemented and by [6, Theorem 1], M is g-semilocal.

Proposition 1. *Let* M *be a strongly* $\oplus-g-$*radical supplemented* $R-$*module. If* $Rad_gM \ll_g M$, *then* M *is strongly* $\oplus-g-$*supplemented.*

Proof. Let $U \leq M$ and V be any g-radical supplement of U in M. Here $M = U+V$ and $U \cap V \leq Rad_gV$. Since M is strongly $\oplus-g-$radical supplemented, V is a direct summand of M. By Lemma 1, $Rad_gV \leq Rad_gM \ll_g M$. Since V is a direct summand of M, we can see that $Rad_gV \ll_g V$. Hence $U \cap V \ll_g V$ and V is a g-supplement of U in M. Therefore, M is stronlgy $\oplus-g-$ supplemented. □

Proposition 2. *Let* M *be a strongly* $\oplus-g-$*radical supplemented* $R-$*module. If* $Rad_gM \ll M$, *then* M *is strongly* $\oplus-$*supplemented.*

Proof. Let $U \leq M$ and V be any g-radical supplement of U in M. Here $M = U+V$ and $U \cap V \leq Rad_gV$. Since M is strongly $\oplus-g-$radical supplemented, V is a direct summand of M. By Lemma 1, $Rad_gV \leq Rad_gM \ll M$. Since V is a direct summand of M, we can see that $Rad_gV \ll V$. Hence $U \cap V \ll V$ and V is a supplement of U in M. Therefore, M is stronlgy $\oplus-$supplemented. □

Lemma 2. *Let $M = M_1 \oplus M_2$ and $X, Y \leq M_2$. Then Y is a g-radical supplement of X in M_2 if and only if Y is a g-radical supplement of $M_1 + X$ in M.*

Proof. ($\Longrightarrow$)Since Y is a g-radical supplement of X in M_2, $M_2 = X + Y$ and $X \cap Y \leq Rad_g Y$. Then $M = M_1 + M_2 = M_1 + X + Y$ and by Modular Law, $(M_1 + X) \cap Y = (M_1 + X) \cap M_2 \cap Y = (M_1 \cap M_2 + X) \cap Y = (0 + X) \cap Y = X \cap Y \leq Rad_g Y$. Hence Y is a g-radical supplement of $M_1 + X$ in M.

($\Longleftarrow$) Since Y is a g-radical supplement of $M_1 + X$ in M, $M = M_1 + X + Y$ and $(M_1 + X) \cap Y \leq Rad_g Y$. Then by Modular Law, $M_2 = M_2 \cap M = M_2 \cap (M_1 + X + Y) = M_1 \cap M_2 + X + Y = 0 + X + Y = X + Y$ and $X \cap Y \leq (M_1 + X) \cap Y \leq Rad_g Y$. Hence Y is a g-radical supplement of X in M_2. □

Lemma 3. *Let M be a strongly $\oplus - g-$radical supplemented module. Then every direct summand of M is strongly $\oplus - g-$radical supplemented.*

Proof. Let N be a direct summand of M and $M = N \oplus K$ with $K \leq M$. Since M is g-radical supplemented, by [6, Lemma 9], M/K is g-radical supplemented. By $\frac{M}{K} = \frac{N \oplus K}{K} \cong \frac{N}{N \cap K} = \frac{N}{0} \cong N$, N is also g-radical supplemented. Let $X \leq N$ and Y be any g-radical supplement of X in N. Since $M = N \oplus K$, by Lemma 2, Y is a g-radical supplement of $K + X$ in M. Since M is strongly $\oplus - g-$radical supplemented, every g-radical supplement submodule in M is a direct summand of M and hence Y is a direct summand of M. By $Y \leq N$, Y is also a direct summand of N. Hence N is strongly $\oplus - g-$radical supplemented. □

Corollary 1. *Let M be a strongly $\oplus - g-$radical supplemented module. Then M/K is strongly $\oplus - g-$radical supplemented for every direct summand K of M.*

Proof. Let K be any direct summand of M and $M = K \oplus X$ with $X \leq M$. By Lemma 3, X is strongly $\oplus - g-$radical supplemented. Then by $M/K = (K + X)/K \cong X/(K \cap X) = X/0 \cong X$, M/K is strongly $\oplus - g-$radical supplemented. □

Corollary 2. *Let $f : M \longrightarrow N$ be an $R-$module epimorphism and $Ker(f)$ be a direct summand of M. If M is strongly $\oplus - g-$radical supplemented, then N is also strongly $\oplus - g-$radical supplemented.*

Proof. By Corollary 1, $M/Ker(f)$ is strongly $\oplus - g-$radical supplemented. Since $M/Ker(f) \cong f(M) = N$, N is also strongly $\oplus - g-$radical supplemented. □

Proposition 3. *Let M be a strongly $\oplus - g-$radical supplemented $R-$module and $K \leq M$. If V is a g-radical supplement submodule in M for every g-radical supplement submodule V/K in M/K, then M/K is strongly $\oplus - g-$ radical supplemented.*

Proof. By [6, Lemma 9], M/K is g-radical supplemented. Let V/K be any g-radical supplement submodule in M/K. By hypothesis, V is a g-radical supplement submodule in M. Since M is strongly $\oplus - g-$radical supplemented, V is a direct summand of M. Then there exists $X \leq M$ with $M = V \oplus X$. Since $M = V \oplus X$,

$M/K = V/K + (X+K)/K$ and $V/K \cap (X+K)/K = (X \cap V + K)/K = (0+K)/K = 0$. Hence $M/K = V/K \oplus (X+K)/K$. Thus M/K is strongly $\oplus-g-$radical supplemented. □

Lemma 4. *Let M be a strongly $\oplus-g-$radical supplemented $R-$module, $K \leq M$ and $K = (K \cap M_1) \oplus (K \cap M_2)$ for every $M_1, M_2 \leq M$ with $M = M_1 \oplus M_2$. Then M/K is $\oplus-g-$radical supplemented.*

Proof. Let $U/K \leq M/K$. Since M is strongly $\oplus-g-$radical supplemented, U has a g-radical supplement V that is a direct summand in M. Here there exists $X \leq M$ such that $M = V \oplus X$. By hypothesis, $K = (K \cap V) \oplus (K \cap X)$. Since V is a g-radical supplement of U in M and $K \leq U$, by [6, Lemma 8], $\frac{V+K}{K}$ is a g-radical supplement of U/K in M/K. Since $M = V \oplus X$, $\frac{M}{K} = \frac{V+K}{K} + \frac{X+K}{K}$. Here $\frac{V+K}{K} \cap \frac{X+K}{K} = \frac{(V+K)\cap(X+K)}{K} = \frac{(V+K)\cap X+K}{K} = \frac{(V+K\cap V+K\cap X)\cap X+K}{K} = \frac{(V+K\cap X)\cap X+K}{K} = \frac{V\cap X+K\cap X+K}{K} = \frac{0+K}{K} = \frac{K}{K} = 0$. Hence $\frac{M}{K} = \frac{V+K}{K} \oplus \frac{X+K}{K}$. Thus M/K is $\oplus-g-$radical supplemented. □

Corollary 3. *Let M be a strongly $\oplus-g-$radical supplemented $R-$module, $f : M \longrightarrow N$ be an $R-$module epimorphism with N be an $R-$module and $Ker(f) = (Ker(f) \cap M_1) \oplus (Ker(f) \cap M_2)$ for every $M_1, M_2 \leq M$ with $M = M_1 \oplus M_2$. Then N is $\oplus-g-$radical supplemented.*

Proof. Clear from Lemma 4, since $M/Ker(f) \cong Im(f) = N$. □

Proposition 4. *Let M be a strongly $\oplus-g-$radical supplemented $R-$module and $K \leq M$. If $\frac{X+K}{K}$ is a direct summand of $\frac{M}{K}$ for every direct summand X of M, then $\frac{M}{K}$ is $\oplus-g-$radical supplemented.*

Proof. Let U/K be any submodule of M/K. Since M is strongly $\oplus-g-$ radical supplemented, U has a g-radical supplement X in M and X is a direct summand of M. Since X is a g-radical supplement of U in M and $K \leq U$, by [6, Lemma 8], $\frac{X+K}{K}$ is a g-radical supplement of U/K in M/K. Since X is a direct summand of M, by hypothesis, $\frac{X+K}{K}$ is a direct summand of M/K. Hence M/K is $\oplus-g-$radical supplemented. □

Remark 1. Let M be a Rad-supplemented module. If M is strongly $\oplus-g-$radical supplemented, then M is strongly $Rad-\oplus-$supplemented. But the converse of this statement is not true in general (see Example 2 and Example 3).

Remark 2. Let M be a supplemented module. If M is strongly $\oplus-g-$radical supplemented, then M is strongly $\oplus-$supplemented. But the converse of this statement is not true in general (see Example 1, Example 2 and Example 3).

Example 1. Let M be a local module with $Rad_g(Rad_g M) = Rad_g M = Rad M \neq 0$. Then M is supplemented and since the supplement submodules in M are only 0 and M, M is strongly $\oplus-$ supplemented. Also M is $\oplus-g-$radical supplemented. Here $Rad_g M$ is the largest submodule of M. Since $Rad_g M$ is a g-radical supplement of

M in M and Rad_gM is not a direct summand of M, M is not strongly $\oplus-g-$radical supplemented.

Example 2. Consider the $\mathbb{Z}-$module $\mathbb{Z}_4$. Here $\mathbb{Z}_4$ is local and $Rad_g\left(Rad_g\mathbb{Z}_4\right) = Rad_g\mathbb{Z}_4 = Rad\mathbb{Z}_4 \neq 0$. By Example 1, $\mathbb{Z}_4$ is strongly $\oplus-$supplemented and $\oplus-g-$radical supplemented. But $\mathbb{Z}_4$ is not strongly $\oplus-g-$radical supplemented. Since the Rad-supplement submodules in $\mathbb{Z}_4$ are only 0 and $\mathbb{Z}_4$, $\mathbb{Z}_4$ is strongly $Rad-\oplus-$supplemented.

Example 3. Consider the $\mathbb{Z}-$module $\mathbb{Z}_8$. Here $\mathbb{Z}_8$ is local, $Rad_g\mathbb{Z}_8 = Rad\mathbb{Z}_8 = \left\{\overline{0},\overline{2},\overline{4},\overline{6}\right\}$ and $Rad_g\left(Rad_g\mathbb{Z}_8\right) = \left\{\overline{0},\overline{4}\right\}$. Since the supplement submodules in $\mathbb{Z}_8$ are only 0 and $\mathbb{Z}_8$, $\mathbb{Z}_8$ is strongly $\oplus-$supplemented. Also $\mathbb{Z}_8$ is $\oplus-g-$radical supplemented. Since $\left\{\overline{0},\overline{4}\right\}$ is a g-radical supplement of $\mathbb{Z}_8$ in $\mathbb{Z}_8$ and $\left\{\overline{0},\overline{4}\right\}$ is not a direct summand of $\mathbb{Z}_8$, $\mathbb{Z}_8$ is not strongly $\oplus-g-$radical supplemented. Since the Rad-supplement submodules in $\mathbb{Z}_8$ are only 0 and $\mathbb{Z}_8$, $\mathbb{Z}_8$ is strongly $Rad-\oplus-$supplemented.

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ENTROPY SOLUTIONS FOR SOME ELLIPTIC PROBLEMS INVOLVING THE GENERALIZED $p(u)$-LAPLACIAN OPERATOR

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Abstract. In this paper, we study the existence of entropy solutions for some generalized elliptic $p(u)$-Laplacian problem when $p(u)$ is a local quantity. We get the results by assuming the right-hand side function f to be an integrable function, and by using the regularization approach combined with the theory of Sobolev spaces with variable exponents.

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1. Introduction

In this paper, we study the existence of entropy solutions for some generalized elliptic equations with exponents p which may depends on the unknown solution u. We consider the case where the dependency of p on u is a local quantity. Namely, we study the following problem

$$\begin{cases} -\operatorname{div}(|\nabla u-\Theta(u)|^{p(u)-2}(\nabla u-\Theta(u))) = f & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N, N\geq 2$, f is a given data, $p\colon \mathbb{R}\to[p_-,p_+]$ is a real continuous function, $1<p_-\leq p_+<+\infty$ and $p'(z)=\frac{p(z)}{p(z)-1}$ is the conjugate exponent of $p(z)$, with

$$p_- := \operatorname*{ess\,inf}_{z\in\mathbb{R}} p(z), \quad p_+ := \operatorname*{ess\,sup}_{z\in\mathbb{R}} p(z).$$

The problem (1.1) models several natural phenomena which appear in area of oceanography, turbulent fluid flows, induction heating and electrochemical problems. We cite for example the following parabolic model:

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- Fluid flow through porous media. This model is governed by the following equation

$$\frac{\partial \theta}{\partial t} - \operatorname{div}\left(|\nabla\varphi(\theta) - K(\theta)e|^{p-2}(\nabla\varphi(\theta) - K(\theta)e)\right) = 0,$$

  where θ is the volumetric content of moisture, $K(\theta)$ the hydraulic conductivity, $\varphi(\theta)$ the hydrostatic potential and e is the unit vector in the vertical direction.

In the classical cases, when $p(u) = p(x)$ or $p(u) = p$ many authors have studied the problem (1.1) by proving the existence and the uniqueness of several types of solutions, and by different approaches ([1, 7, 8, 15]).

The novelty of this work is to study some problems involving the generalized p-Laplacian operator in the case when the variable exponents p depend on the unknown solution u. The motivation to study these kind of problems relies in the fact that, in reality the measurements of some physical quantities are not made pointwise but through some local averages. The situation where the variable exponents p depend on the unknown solution u is non-standard as in the classical case (see [2–4, 7, 8, 10, 14, 15, 19]). This kind of problems appear in the applications of some numerical techniques for the total variation image restoration method that have been used in some restoration problems of mathematical image processing and computer vision [11, 12, 18]. Türola, J. in [18] have presented several numerical examples suggesting that the consideration of exponents $p = p(u)$ preserves the edges and reduces the noise of the restored images u. A numerical example suggesting a reduction of noise in the restored images u when the exponent of the regularization term is $p = p(|\nabla u|)$ is presented in [11]. Many authors have considered the problem (1.1) in the case where $\Theta = 0$ and especially the study of questions of existence and uniqueness of weak or entropy solutions to the problem (1.1). In this case, M. Chipot and H. B. de Oliveira in [13] have proved the existence of weak solutions for some $p(u)$-Laplacian problems, the existence proofs of [13] are based on the Schauder fixed-point theorem. C. Allalou, K. Hilal and S. A. Temghart in [5], extended the results established in [13] by proving some existence results for some local and nonlocal problems. Andreianov et al. [6], have studied the following prototype problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(u)-2}\nabla u) + u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

C. Zhang and X. Zhang in [20] have proved the existence of entropy solutions to problem (1.1) in the case where $\Theta = 0$ and they have provided some positive answers for the two questions proposed by Chipot and de Oliveira in [13]. S. Ouaro and N. Sawadogo in [16] and [17] considered the following nonlinear Fourier boundary

value problem

$$\begin{cases} b(u) - \operatorname{div} a(x,u,\nabla u) = f & \text{in } \Omega \\ a(x,u,\nabla u)\cdot\eta + \lambda u = g & \text{on } \partial\Omega. \end{cases}$$

The existence and uniqueness results of entropy and weak solutions are established by an approximation method and convergent sequences in terms of Young measure.

In this paper, we first show that the approximated problems admits a sequence of weak solutions by applying the variational method combined with a special type of operators. In the second step, we will prove that the sequence of weak solutions converges to some function u and by using some a priori estimates, we will show that this function u is an entropy solution of elliptic problem (1.1).

This paper is organized as follow. In Section 2 we introduce the basic assumptions and we recall some definitions, basic properties of generalised Sobolev spaces that we will be used later. The Section 3 is devoted to show the existence of entropy solutions to the local problem (1.1).

2. Preliminaries

The exponent function p depends on the solution u and therefore it depends on the space variable x. This allows us to look for the entropy solutions to the problem (1.1) in a Sobolev space with variable exponents.

Let Ω be a bounded domain of $\mathbb{R}^N$, $N \geq 2$, we say that a real-valued continuous function $h(.)$ is log-Hölder continuous in Ω if

$$\exists C > 0 : \ |h(x) - h(y)| \leq \frac{C}{\ln\left(\frac{1}{|x-y|}\right)} \qquad \forall x, y \in \Omega, \ \ |x-y| < \frac{1}{2}. \tag{2.1}$$

For any Lebesgue-measurable function $h\colon \Omega \to [1, \infty)$, we define

$$h_- := \operatorname*{ess\,inf}_{x\in\Omega} h(x) \ , \quad h_+ := \operatorname*{ess\,sup}_{x\in\Omega} h(x),$$

and we introduce the variable exponent Lebesgue space by:

$$L^{h(\cdot)}(\Omega) = \{\, u\colon \Omega \to \mathbb{R} \ / \ \rho_{h(\cdot)}(u) \ := \int_\Omega |u(x)|^{h(x)} dx < \infty \}.$$

Equipped with the Luxembourg norm

$$\|u\|_{h(\cdot)} \ := \inf\left\{\lambda > 0 \ : \ \rho_{h(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1\right\}, \tag{2.2}$$

$L^{h(\cdot)}(\Omega)$ becomes a Banach space. If

$$1 < h_- \leq h_+ < \infty, \tag{2.3}$$

$L^{h(\cdot)}(\Omega)$ is separable and reflexive. The dual space of $L^{h(\cdot)}(\Omega)$ is $L^{h'(\cdot)}(\Omega)$, where $h'(x)$ is the generalised Hölder conjugate of $h(x)$,

$$\frac{1}{h(x)}+\frac{1}{h'(x)}=1.$$

From the definitions of the modular $\rho_{h(\cdot)}(u)$ and the norm (2.2), it can be proved that if (2.3) holds then

$$\min\left\{\|u\|_{h(\cdot)}^{h_-},\ \|u\|_{h(\cdot)}^{h_+}\right\}\leq \rho_{h(\cdot)}(u)\leq \max\left\{\|u\|_{h(\cdot)}^{h_-},\ \|u\|_{h(\cdot)}^{h_+}\right\}. \tag{2.4}$$

A very useful consequence very useful of (2.4) is the following:

$$\|u\|_{h(\cdot)}^{h_-}-1\leq \rho_{h(\cdot)}(u)\leq \|u\|_{h(\cdot)}^{h_+}+1.$$

For any functions $u\in L^{h(\cdot)}(\Omega)$ and $v\in L^{h'(\cdot)}(\Omega)$, the generalized Hölder inequality hold:.

$$\int_\Omega uvdx\left(\frac{1}{h_-}+\frac{1}{h'_-}\right)\|u\|_{h(\cdot)}\|v\|_{h'(\cdot)}\leq 2\|u\|_{h(\cdot)}\|v\|_{h'(\cdot)}.$$

We define also the generalized Sobolev space by

$$W^{1,h(\cdot)}(\Omega):=\{u\in L^{h(\cdot)}(\Omega):\nabla u\in L^{h(\cdot)}(\Omega)\},$$

which is a Banach space with the norm

$$\|u\|_{1,h(\cdot)}:=\|u\|_{h(\cdot)}+\|\nabla u\|_{h(\cdot)}.$$

The space $W^{1,h(\cdot)}(\Omega)$ is separable and is reflexive when (2.3) is satisfied. We also have

$$W^{1,h(\cdot)}(\Omega)\hookrightarrow W^{1,r(\cdot)}(\Omega)\text{ whenever } h(x)\geq r(x)\quad \text{for a.e. } x\in\Omega.$$

Now, we introduce the following function space

$$W_0^{1,h(\cdot)}(\Omega):=\{u\in \mathrm{W}_0^{1,1}(\Omega):\nabla u\in L^{h(\cdot)}(\Omega)\},$$

endowed with the following norm

$$\|u\|_{W_0^{1,h(\cdot)}(\Omega)}:=\|u\|_1+\|\nabla u\|_{h(\cdot)}.$$

If $h\in C(\overline{\Omega})$, then the norm in $W_0^{1,h(\cdot)}(\Omega)$ is equivalent to $\|\nabla u\|_{h(\cdot)}$. $C_0^\infty(\Omega)$ is dense in $W_0^{1,h(.)}(\Omega)$, when h is log-Hölder continuous. If h is a measurable function in Ω satisfying $1\leq h_-\leq h_+<d$ and the Log-Hölder continuity property (2.1), then

$$\|u\|_{h^*(\cdot)}\leq C\|\nabla u\|_{h(\cdot)}\quad \forall u\in W_0^{1,h(\cdot)}(\Omega),$$

for some positive constant C, where

$$h^*(x):=\begin{cases}\frac{Nh(x)}{N-h(x)} & \text{if } h(x)<N,\\ \infty & \text{if } h(x)\geq N.\end{cases}$$

On the other hand, if h satisfies (2.1) and $h_- > N$, then

$$\|u\|_\infty \leq C\|\nabla u\|_{h(\cdot)} \quad \forall u \in W_0^{1,h(\cdot)}(\Omega)\ ,$$

where C is another positive constant.

Let T_k denote the truncation function at height $k \geqslant 0$:

$$T_k(r) = \min\{k, \max\{r, -k\}\} = \begin{cases} k & \text{if } r \geqslant k \\ r & \text{if } |r| < k, \\ -k & \text{if } r \leqslant -k. \end{cases}$$

Next, we define the very weak gradient of a measurable function u with $T_k(u) \in W_0^{1,p(u)}(\Omega)$. The proof follows from [9, Lemma 2.1] due to the fact that $W_0^{1,p(\cdot)}(\Omega) \subset W_0^{1,p_-}(\Omega)$ (for more details we recommend readers see [6, Proposition 3.5]).

Proposition 1. *For every measurable function u with $T_k(u) \in W_0^{1,p(u)}(\Omega)$, there exists a unique measurable function $v\colon \Omega \to \mathbb{R}^N$, which we call the very weak gradient of u and denote $v = \nabla u$, such that*

$$\nabla T_k(u) = v\chi_{\{|u|<k\}} \quad \textit{for a.e. } x \in \Omega \textit{ and for every } k > 0,$$

where χ_E denotes the characteristic function of a measurable set E.

Moreover, if u belongs to $W_0^{1,1}(\Omega)$, then v coincides with the weak gradient of u.

Lemma 1 ([7, Lemma 2.1])**.** *For $\xi, \eta \in \mathbb{R}^N$ and $1 < p < \infty$, we have*

$$\frac{1}{p}|\xi|^p - \frac{1}{p}|\eta|^p \leq |\xi|^{p-2}\xi(\xi - \eta).$$

Lemma 2. *For $a \geq 0, b \geq 0$ and $1 \leq p < \infty$, we have*

$$(a+b)^p \leq 2^{p-1}\left(a^p + b^p\right).$$

3. Main results

In this section, we prove the existence of entropy solutions of problem (1.1). Firstly, we state the following assumptions:

(H_1) $f \in L^1(\Omega)$.

(H_2) $\Theta\colon \mathbb{R} \to \mathbb{R}^N$ is a continuous function such that $\Theta(0) = 0$ and $|\Theta(x) - \Theta(y)| \leq \lambda|x-y|$ for all $x, y \in \mathbb{R}$, where λ is a positive constant such that $\lambda < \dfrac{1}{2C_0}$, and C_0 is the constant given by the Poincaré inequality.

We define the set where we are going to look for the solutions to problem (1.1) as

$$W_0^{1,p(u)}(\Omega) := \left\{ u \in W_0^{1,1}(\Omega) : \int_\Omega |\nabla u|^{p(u)} \mathrm{d}x < \infty \right\}.$$

Now, we give a definition of entropy solutions for the elliptic problem (1.1).

Definition 1. A measurable function u with $T_k(u) \in W_0^{1,p(u)}(\Omega)$ is said to be an entropy solution for the problem (1.1), if

$$\int_\Omega \Phi\left(\nabla u - \Theta(u)\right) \nabla T_k(u-\varphi) dx \leqslant \int_\Omega f T_k(u-\varphi) \mathrm{d}x,$$

for all $\varphi \in C_0^1(\Omega)$ and for every $k > 0$ with

$$\Phi(\xi) = |\xi|^{p(u)-2}\xi \quad \forall \xi \in \mathbb{R}^N.$$

Theorem 1. *Let (H_1) and (H_2) be satisfied. Then, there exists at least one entropy solution of the problem* (1.1).

Proof. The proof of Theorem 1 is divided into several steps.

Part 1: The approximate problem.

Let f_n be a sequence of $C_0^\infty(\Omega)$ functions strongly converging to f in L^1 such that $\|f_n\|_{L^1} \leq \|f\|_{L^1}$.

We consider the following problem

$$(\mathcal{P}_n) \begin{cases} -\mathrm{div}(\Phi\left(\nabla u_n - \Theta(u_n)\right)) = f_n & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$\Phi(\xi) = |\xi|^{p(u_n)-2}\xi \quad \forall \xi \in \mathbb{R}^N.$$

We define the operator A by

$$\langle Au, v\rangle = \int_\Omega \Phi(\nabla u - \Theta(u)) \nabla v dx \quad \text{with} \quad u, v \in W_0^{1,p(\cdot)}(\Omega).$$

We will prove that A is coercive. From Lemma 1, we obtain

$$\begin{aligned} \langle Au, u\rangle &= \int_\Omega \Phi(\nabla u - \Theta(u)) \nabla u dx \\ &= \int_\Omega |\nabla u - \Theta(u)|^{p(u)-2} (\nabla u - \Theta(u)) \nabla u dx \\ &\geq \int_\Omega^1 \frac{1}{p(u)} |\nabla u - \Theta(u)|^{p(u)} dx - \int_\Omega \frac{1}{p(u)} |\Theta(u)|^{p(u)} dx. \end{aligned}$$

Since

$$(a+b)^p \leq 2^{p-1}\left(|a|^p + |b|^p\right),$$

we have

$$\frac{1}{2^{p_+-1}} |\nabla u|^{p(u)} = \frac{1}{2^{p_+-1}} |\nabla u - \Theta(u) + \Theta(u)|^{p(u)} \leq |\nabla u - \Theta(u)|^{p(u)} + |\Theta(u)|^{p(u)},$$

then

$$\frac{1}{2^{p_+-1}} |\nabla u|^{p(u)} - |\Theta(u)|^{p(u)} \leq |\nabla u - \Theta(u)|^{p(u)}.$$

Therefore, from Poincaré inequality we get

$$\begin{aligned}\langle Au,u\rangle &\geq \int_\Omega \frac{1}{p(u)}\left[\frac{1}{2^{p_+-1}}|\nabla u|^{p(u)}-|\Theta(u)|^{p(u)}\right]dx-\int_\Omega \frac{1}{p(u)}|\Theta(u)|^{p(u)}dx\\ &\geq \int_\Omega \frac{1}{p(u)}\frac{1}{2^{p_+-1}}|\nabla u|^{p(u)}dx-\int_\Omega \frac{2}{p(u)}|\Theta(u)|^{p(u)}dx\\ &\geq \int_\Omega \frac{1}{p(u)}\frac{1}{2^{p_+-1}}|\nabla u|^{p(u)}dx-\int_\Omega \frac{2}{p(u)}\lambda^{p(u)}|u|^{p(u)}dx\\ &\geq \int_\Omega \frac{1}{p_+}\frac{1}{2^{p_+-1}}|\nabla u|^{p(u)}dx-\int_\Omega \frac{2\lambda^{p(u)}}{p_-}C_0^{p(u)}|\nabla u|^{p(u)}dx.\end{aligned}$$

So the choice of the constant λ in (H_2) implies that

$$\langle Au,u\rangle \geq \left(\frac{1}{p_+}\frac{1}{2^{p_+-1}}-\frac{1}{p_-}\frac{1}{2^{p_--1}}\right)\int_\Omega |\nabla u|^{p(u)}dx.$$

Consequently

$$\frac{\langle Au,u\rangle}{\|u\|_{W_0^{1,p(\cdot)}(\Omega)}}\longrightarrow\infty \quad \text{as} \quad \|u\|_{W_0^{1,p(\cdot)}(\Omega)}\to\infty.$$

We deduce that the operator A is coercive. Besides, that the operator A is bounded and hemi-continuous. Then, the problem $(\mathcal{P}_n)$ admits at least one weak solution $u_n\in W_0^{1,p(\cdot)}(\Omega)$ in the following sense

$$\int_\Omega |\nabla u_n-\Theta(u_n)|^{p(u_n)-2}\left(\nabla u_n-\Theta(u_n)\right))\cdot\nabla\varphi dx=\int_\Omega f_n\varphi dx \tag{3.1}$$

for all $\varphi\in W_0^{1,p(\cdot)}(\Omega)$.

Our aim is to prove that a subsequence of these approximate solutions $\{u_n\}$ converges to a measurable function u, which is an entropy solution to (1.1).

Part 1: A priori estimate.

Lemma 3. $(\nabla T_k(u_n))_{n\in\mathbb{N}}$ *is bounded in* $L^{p_-}(\Omega)$.

Proof. We take $\varphi=T_k(u_n)$ as a test function in (3.1), we obtain

$$\int_\Omega \Phi\left(\nabla u_n-\Theta(u_n)\right)\nabla T_k(u_n)dx\leqslant k\|f\|_{L^1(\Omega)}.$$

By the same way as in the proof of the coerciveness, we get

$$\rho_{1,p(u_n)}(T_k(u_n))\leqslant Ck.$$

Therefore

$$\|T_k(u_n)\|_{W_0^{1,p(u_n)}}\leqslant 1+(Ck)^{\frac{1}{p_-}},$$

we deduce that for any $k>0$, the sequence $(T_k(u_n))_{n\in\mathbb{N}}$ is uniformly bounded in $W_0^{1,p(u_n(\cdot))}(\Omega)$ and also in $W_0^{1,p_-}(\Omega)$. Then, up to a subsequence still denoted $T_k(u_n)$,

we can assume that for any $k>0$, $T_k(u_n)$ weakly converges to ν_k in $W_0^{1,p_-}(\Omega)$ and also $T_k(u_n)$ strongly converges to ν_k in $L^{p_-}(\Omega)$. □

Lemma 4. *$(u_n)_{n\in\mathbb{N}}$ converges in measure to some measurable function u.*

Proof. Firstly we prove that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in measure. For every fixed $\delta>0$ and every positive integer $k>0$, we know that

$$\begin{aligned}\operatorname{meas}\{|u_n-u_m|>\delta\} \leq \operatorname{meas}\{|u_n|>k\}+\operatorname{meas}\{|u_m|>k\}\\ +\operatorname{meas}\{|T_k(u_n)-T_k(u_m)|>\delta\}.\end{aligned}$$

Choosing $T_k(u_n)$ as a test function in (3.1), we get

$$\rho_{1,p(u_n)}(T_k(u_n)) \leqslant k\|f_n\|_{L^1(\Omega)} \leqslant k\|f\|_{L^1(\Omega)}.$$

It follows that

$$\int_{\{|u_n|>k\}} k^{p(u_n)}dx \leqslant k\|f\|_{L^1(\Omega)}.$$

Therefore

$$\operatorname{meas}\{|u_n|>k\} \leqslant k^{1-p_-}\|f\|_{L^1(\Omega)}.$$

Let $\varepsilon>0$, we choose $k=k(\varepsilon)$ such that

$$\operatorname{meas}\{|u_n|>k\} \leqslant \frac{\varepsilon}{3} \quad \text{and} \quad \operatorname{meas}\{|u_m|>k\} \leqslant \frac{\varepsilon}{3}.$$

Since $\{T_k(u_n)\}$ converges strongly in $L^{p_-}(\Omega)$, then it is a Cauchy sequence. Thus

$$\operatorname{meas}\{|T_k(u_n)-T_k(u_m)|>\delta\} \leqslant \frac{\varepsilon}{3},$$

for all $n,m \geqslant n_0(\delta,\varepsilon)$.
Finally, we obtain

$$\operatorname{meas}\{|u_n-u_m|>\delta\} \leqslant \varepsilon,$$

for all $n,m \geqslant n_0(\delta,\varepsilon)$. Hence

$$\limsup_{n,m\to\infty} \operatorname{meas}\{|u_n-u_m|>\delta\}=0,$$

which proves that the sequence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in measure and then converges almost everywhere to some measurable function u.

$$u_n \to u \quad \text{a.e in} \quad \Omega.$$

Therefore

$$T_k(u_n) \rightharpoonup T_k(u) \text{ in } W_0^{1,p_-}(\Omega)$$
$$T_k(u_n) \longrightarrow T_k(u) \text{ in } L^{p_-}(\Omega) \text{ and a.e. in } \Omega.$$

□

Lemma 5. *$(\nabla u_n)_{n\in\mathbb{N}}$ converges almost everywhere in Ω to ∇u.*

Proof. We first prove that $\{\nabla u_n\}$ is a Cauchy sequence in measure. Let δ, h, ε are positive real numbers, obviously we have

$$\{x \in \Omega : |\nabla u_n - \nabla u_m| > \delta\} \subset \{x \in \Omega : |\nabla u_n| > h\} \cup \{x \in \Omega : |\nabla u_m| > h\} \\ \cup \{x \in \Omega : |u_n - u_m| > 1\} \cup E,$$

where

$$E := \{x \in \Omega : |\nabla u_n| \leqslant h, |\nabla u_m| \leqslant h, |u_n - u_m| \leqslant 1, |\nabla u_n - \nabla u_m| > \delta\}.$$

For $k > 0$, we can write

$$\{x \in \Omega : |\nabla u_n| \geqslant h\} \subset \{x \in \Omega : |u_n| \geqslant k\} \cup \{x \in \Omega : |\nabla T_k(u_n)| \geqslant h\},$$

then by using the same method as in Lemma 4, we obtain for k sufficiently large

$$\operatorname{meas}\{\{x \in \Omega : |\nabla u_n| > h\} \cup \{x \in \Omega : |\nabla u_m| > h\} \cup \{x \in \Omega : |u_n - u_m| > 1\}\} \leqslant \frac{\varepsilon}{2}.$$

Notice that the application

$$\mathcal{G} : (s, t, \xi_1, \xi_2) \mapsto (\Phi(\xi_1 - \Theta(s)) - \Phi(\xi_2 - \Theta(t)))(\xi_1 - \xi_2)$$

is continuous and the set

$$\mathcal{H} := \left\{(s, t, \xi_1, \xi_2) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N, |s| \leq h, |t| \leq h, |\xi_1| \leq h, |\xi_2| \leq h, |\xi_1 - \xi_2| > \delta\right\}$$

is compact and

$$(\Phi(\xi_1 - \Theta(s)) - \Phi(\xi_2 - \Theta(t)))(\xi_1 - \xi_2) > 0 \quad \forall \xi_1 \neq \xi_2.$$

Then the application $\mathcal{G}$ has its minimum on $\mathcal{H}$. Therefore, there exists a real valued function $\beta(h, \delta) > 0$ such that

$$\begin{aligned}
\beta(h,\delta)\operatorname{meas}(E) \leq \int_E \Big[& |\nabla u_n - \Theta(u_n)|^{p(u_n)-2}(\nabla u_n - \Theta(u_n)) \\
& - |\nabla u_m - \Theta(u_m)|^{p(u_n)-2}(\nabla u_m - \Theta(u_m))\Big][\nabla u_n - \nabla u_m]\,dx, \\
= \int_E \Big[& |\nabla u_m - \Theta(u_m)|^{p(u_m)-2}(\nabla u_m - \Theta(u_m)) \\
& - |\nabla u_m - \Theta(u_m)|^{p(u_n)-2}(\nabla u_m - \Theta(u_m))\Big][\nabla u_n - \nabla u_m]\,dx \\
& + \int_E \Big[|\nabla u_n - \Theta(u_n)|^{p(u_n)-2}(\nabla u_n - \Theta(u_n)) \\
& - |\nabla u_m - \Theta(u_m)|^{p(u_m)-2}(\nabla u_m - \Theta(u_m))\Big][\nabla u_n - \nabla u_m]\,dx.
\end{aligned}$$

We take $T_1(u_n - u_m)$ as a test function in (3.1) to get

$$\begin{aligned}
\beta(h,\delta)\operatorname{meas}(E) \leqslant \int_E \Big[& |\nabla u_m - \Theta(u_m)|^{p(u_m)-2}(\nabla u_m - \Theta(u_m)) \\
& - |\nabla u_m - \Theta(u_m)|^{p(u_n)-2}(\nabla u_m - \Theta(u_m))\Big][\nabla u_n - \nabla u_m]\,dx
\end{aligned}$$

$$+\int_E [f_n - f_m]\, T_1(u_n - u_m)\, dx.$$

By using the mean value theorem, there exists η taking values between $p(u_n)$ and $p(u_m)$ such that

$$\beta(h,\delta)\,\mathrm{meas}(E) \leqslant \int_E |\nabla u_m - \Theta(u_m)|^{\eta-1} \cdot \left|\log|\nabla u_m - \Theta(u_m)|\right| \cdot |\nabla u_n - \nabla u_m| \cdot |p(u_m) - p(u_n)|\, dx + \|f_n - f_m\|_{L^1(\Omega)}.$$

By using Lemma 2, (H_2), the facts that $h \gg 1$ and the definition of E, we get

$$\beta(h,\delta)\,\mathrm{meas}(E) \leqslant 2^{p^+} h^{p^+} \left(1+\lambda^{\eta-1}\right) \log((1+\lambda)h) \cdot \int_\Omega |p(u_m) - p(u_n)|\, dx + \|f_n - f_m\|_{L^1(\Omega)} := \alpha_{n,m}.$$

From Lebesgue dominated convergence theorem, we obtain

$$\mathrm{meas}(E) \leqslant \frac{\alpha_{n,m}}{\beta(h,\delta)} \leqslant \frac{\varepsilon}{2},$$

for all $n,m \geqslant N_2(\varepsilon,\delta)$. Consequently combining the previous results we get

$$\mathrm{meas}\{x \in \Omega : |\nabla u_n - \nabla u_m| > \delta\} \leqslant \varepsilon \quad \text{for all } n,m \geqslant \max\{N_1, N_2\}.$$

Hence $\{\nabla u_n\}$ is a Cauchy sequence in measure. Then we can choose a subsequence (denoted it by the original sequence) such that

$$\nabla u_n \to v \quad \text{a.e. in } \Omega.$$

Thus, using Proposition 1 and the fact that $\nabla T_k(u_n) \to \nabla T_k(u)$ in $(L^{p^-}(\Omega))^N$, we deduce that v coincides with the very weak gradient of u almost everywhere. Therefore, we have

$$\nabla u_n \to \nabla u \quad \text{a.e. in } \Omega.$$

□

Part 3: Passing to the limit.

We choose $T_k(u_n - \phi)$ as a test function in (3.1) for $\phi \in C_0^1(\Omega)$. Then

$$\int_\Omega |\nabla u_n - \Theta(u_n)|^{p(u_n)-2} (\nabla u_n - \Theta(u_n)) \cdot \nabla T_k(u_n - \phi)\, dx = \int_\Omega f_n T_k(u_n - \phi)\, dx. \quad (3.2)$$

We now focus our attention on the left-hand side of (3.2).
We note that, if $L = k + \|\phi\|_{L^\infty(\Omega)}$

$$\int_\Omega |\nabla u_n - \Theta(u_n)|^{p(u_n)-2} (\nabla u_n - \Theta(u_n)) \cdot \nabla T_k(u_n - \phi)\, dx = \int_\Omega |\nabla T_L(u_n) - \Theta(T_L(u_n))|^{p(u_n)-2}$$

$$
\begin{aligned}
&(\nabla T_L(u_n) - \Theta(T_L(u_n))) \cdot \nabla T_k(T_L(u_n) - \phi)\, dx \\
= &\int_\Omega |\nabla T_L(u_n) - \Theta(T_L(u_n))|^{p(u_n)-2} \\
&(\nabla T_L(u_n) - \Theta(T_L(u_n))) \cdot \nabla T_L(u_n) \chi_{\{|T_L(u_n)-\phi|\leqslant k\}} dx \\
&- \int_\Omega |\nabla T_L(u_n) - \Theta(T_L(u_n))|^{p(u_n)-2} \\
&(\nabla T_L(u_n) - \Theta(T_L(u_n))) \cdot \nabla \varphi \chi_{\{|T_L(u_n)-\phi|\leqslant k\}} dx.
\end{aligned}
$$

From (3.2), we have

$$
\begin{aligned}
&\int_\Omega \Big[|\nabla T_L(u_n) - \Theta(T_L(u_n))|^{p(u_n)-2} (\nabla T_L(u_n) - \Theta(T_L(u_n))) \cdot \nabla T_L(u_n) \\
&\qquad + \frac{1}{p_-} |\Theta(T_L(u_n)|^\gamma \Big] \cdot \chi_{\{|T_L(u_n)-\phi|\leqslant k\}} dx \\
&- \int_\Omega |\nabla T_L(u_n) - \Theta(T_L(u_n))|^{p(u_n)-2} (\nabla T_L(u_n) - \Theta(T_L(u_n))) \nabla \varphi \chi_{\{|T_L(u_n)-\phi|\leqslant k\}} dx \\
&= \int_\Omega f_n T_k(u_n - \phi)\, dx + \int_\Omega \frac{1}{p_-} |\Theta(T_L(u_n)|^\gamma \chi_{\{|T_L(u_n)-\phi|\leqslant k\}} dx,
\end{aligned}
\tag{3.3}
$$

where

$$
\gamma = \begin{cases} p_+ & \text{if } |\Theta(T_L(u_n)| \leq 1, \\ p_- & \text{if } |\Theta(T_L(u_n)| > 1. \end{cases}
$$

Since $\{\nabla T_L(u_n)\}$ is bounded in $\left(L^{p'(u_n)}(\Omega)\right)^N \subset \left(L^{p'_+}(\Omega)\right)^N$, then from the hypothesis (H_3), the sequence $\{\Theta(T_L(u_n)\}$ is also bounded in $\left(L^{p(u_n)}(\Omega)\right)^N \subset (L^{p_-}(\Omega))^N$, which implies that $\left\{|\nabla T_L(u_n) - \Theta(T_L(u_n))|^{p(u_n)-2} (\nabla T_L(u_n) - \Theta(T_L(u_n)))\right\}$ is bounded in $\left(L^{p'(u_n)}(\Omega)\right)^N \subset \left(L^{p'_+}(\Omega)\right)^N$.

As $u_n \to u$ a.e. in Ω and $\nabla u_n \to \nabla u$ a.e. in Ω, it follows that

$$
\Theta(T_L(u_n)) \longrightarrow \Theta(T_L(u)) \text{ a.e. in } \Omega \tag{3.4}
$$

and

$$
\nabla T_L(u_n) \longrightarrow \nabla T_L(u) \text{ a.e. in } \Omega.
$$

Hence,

$$
\begin{aligned}
&|\nabla T_L(u_n) - \Theta(T_L(u_n))|^{p(u_n)-2} (\nabla T_L(u_n) - \Theta(T_L(u_n))) \\
&\rightharpoonup |\nabla T_L(u) - \Theta(T_L(u))|^{p(u)-2} (\nabla T_L(u) - \Theta(T_L(u))) \quad \text{in } \left(L^{p'_+}(\Omega)\right)^N.
\end{aligned}
$$

As $\phi \in C_0^1(\Omega)$, we get that

$$
\int_\Omega |\nabla T_L(u_n) - \Theta(T_L(u_n))|^{p(u_n)-2} (\nabla T_L(u_n) - \Theta(T_L(u_n))) \cdot \nabla \varphi \chi_{\{|T_L(u_n)-\phi|\leqslant k\}} dx
$$

$$\underset{n\to+\infty}{\longrightarrow}\int_\Omega |\nabla T_L(u)-\Theta(T_L(u))|^{p(u)-2}(\nabla T_L(u)-\Theta(T_L(u)))\cdot\nabla\varphi\chi_{\{|T_L(u)-\phi|\leqslant k\}}dx.$$

From (3.4) and the Lebesgue dominated convergence theorem, we obtain

$$\int_\Omega \frac{1}{p_-}|\Theta(T_L(u_n)|^\gamma\chi_{\{|T_L(u_n)-\phi|\leqslant k\}}dx \to \int_\Omega \frac{1}{p_-}|\Theta(T_L(u)|^\gamma\chi_{\{|T_L(u)-\phi|\leqslant k\}}dx.$$

On the other hand, by using Lemma 1, we have

$$\left[|\nabla T_L(u_n)-\Theta(T_L(u_n))|^{p(u_n)-2}(\nabla T_L(u_n)-\Theta(T_L(u_n)))\cdot\nabla T_L(u_n)\right.$$
$$\left.+\frac{1}{p_-}|\Theta(T_L(u_n)|^\gamma\right]\cdot\chi_{\{|T_L(u_n)-\phi|\leqslant k\}}\geqslant 0 \quad \text{a.e. in } \Omega.$$

By using Fatou's Lemma, we get

$$\int_\Omega\left[|\nabla T_L(u)-\Theta(T_L(u))|^{p(u)-2}(\nabla T_L(u)-\Theta(T_L(u)))\cdot\nabla T_L(u)\right.$$
$$\left.+\frac{1}{p_-}|\Theta(T_L(u)|^\gamma\right]\chi_{\{|T_L(u)-\phi|\leqslant k\}}dx$$
$$\leq \liminf_{n\to\infty}\int_\Omega\left[|\nabla T_L(u_n)-\Theta(T_L(u_n))|^{p(u_n)-2}(\nabla T_L(u_n)-\Theta(T_L(u_n)))\cdot\nabla T_L(u_n)\right.$$
$$\left.+\frac{1}{p_-}|\Theta(T_L(u_n)|^\gamma\right]\cdot\chi_{\{|T_L(u_n)-\phi|\leqslant k\}}dx.$$

Now, we consider the first term in the right hand side of (3.3). Since $f_n\to f$ in $L^1(\Omega)$, then

$$\lim_{n\to\infty}\int_\Omega f_nT_k(u_n-\phi)\,dx=\int_\Omega fT_k(u-\phi)\,dx.$$

Finally, by using the above results we can pass to the limit as $n\to\infty$ in the equality (3.3) to concludes that u is an entropy solution to the problem (1.1). □

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ON SHARP SOLUTIONS TO MAJORIZATION AND FEKETE-SZEGŐ PROBLEMS FOR STARLIKE FUNCTIONS

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Abstract. This paper is concerned with sharp solutions to majorization and Fekete-Szegő problems for several relatively new normalized subfamilies of starlike functions. Regarding starlike functions which are subordinated to Sine and Cosine functions, a number of new results are established. The results provide improvements to some recent results. As applications, some interesting deductions are also listed. The findings of this study are thoroughly supported by examples wherever necessary.

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Keywords: majorization problem, Fekete-Szegő problem, starlike function, sine and cosine function, subordination

1. Introduction

Analytic function theory of one complex variable is a very rich area of study and an elegant subject of classical mathematics. Univalent and multivalent functions appear as fascinating aspect of this theory with the interplay of analytic structures and geometric behaviors of analytic functions. Some early results can be considered as the corner stone of this theory that emerged in the beginning of the 20th century. For example, from Koebe's investigation in 1907, the Gronwall's proof of the area theorem in 1914, and the famous Bieberbach's estimate of the second coefficient in 1916. Applications and extensions of univalent and multivalent function theory have been used in fields such as ordinary and partial differential equations, fractional calculus, operator theory, mathematical physics, and differential subordinations (for details, see [7]).

After the settlement of Bieberbach conjecture by de-Branges [11] concerning modulus of nth coefficients of normalized univalent functions, (see e.g. equation (1.1)), this field has observed exponential growth. A number of basic subfamilies of univalent functions such as starlike, convex, close-to-convex and spiral-like functions were thoroughly investigated. Results such as coefficient bounds, growth, distortion

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and covering theorems, majorization and radius problems were reported for many of these and other closely related subfamilies [1,3–6,8,9,12–14,17,25]. For some other related results on these subfamilies, the reader can refer to [15,16].

Generally speaking, analytic (geometric) function theory observes close connection with geometry and analysis, and therefore has attracted the attention of many function theorists since the beginning of 20th century to recent times. Most of the results of analytic function theory are available over the open unit disk of the complex plane $\mathbb{C}$ that are characterized by the fact that such a function provides one-to-one mapping onto its image domain.

Let us take $H(\mathcal{E})$ as the family of all analytic functions defined on the open unit disk $\mathcal{E} = \{z \in \mathbb{C} : |z| < 1\}$. Take $\mathcal{A} \subset H(\mathcal{E})$ as the family of all normalized analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathcal{E}) \tag{1.1}$$

that satisfy the conditions

$$f(0) = f'(0) - 1 = 0.$$

Also, we let $\mathcal{U}$ denote the subfamily of all functions $f \in \mathcal{A}$ that are univalent in $\mathcal{E}$. As usual, let $\mathcal{ST}$ denote the familiar subfamily of all starlike functions (see [13]), defined as

$$\mathcal{ST} = \left\{ f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in \mathcal{E}) \right\}.$$

Furthermore, by $\mathcal{ST}(\alpha)$, we mean the subfamily of starlike functions of order $\alpha \in [0,1)$, which is given as follows:

$$\mathcal{ST}(\alpha) = \left\{ f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathcal{E}) \right\}.$$

It is interesting to point out that by using the Alexander relationship (see [13]), we can define the subfamilies $\mathcal{CV}$ and $\mathcal{CV}(\alpha)$, which are, respectively, the families of all convex and convex functions of order $\alpha \in [0,1)$. More precisely, let $F(z) = zf'(z)$, where $f'(z) \neq 0$ for $z \in \mathcal{E}$, then

$$f(z) \in \mathcal{CV} \Longleftrightarrow F(z) \in \mathcal{ST},$$

and for $\alpha \in [0,1)$,

$$f(z) \in \mathcal{CV}(\alpha) \Longleftrightarrow F(z) \in \mathcal{ST}(\alpha).$$

For relevant advances in this direction, one can consult references [2,9,15,16].

In what follows, we recall several definitions involving subordination and majorization.

Definition 1 ([13])**.** If $f, g \in \mathcal{A}$, then f is subordinated to g, written as $f \prec g$, if for some analytic function $w(z)$, the relationship

$$f(z) = g(w(z)) \quad (z \in \mathcal{E})$$

holds, where $w(z)$ is the Schwarz function satisfies $w(0)=0$ and $|w(z)|\leq|z|<1$ for each $z\in\mathcal{E}$.

In addition, if $g(z)$ is univalent in $\mathcal{E}$, that is $g\in\mathcal{U}$, then $f\prec g$ can be written equivalently as (see [13])

$$f(0)=g(0) \quad \text{and} \quad f(\mathcal{E})\subset g(\mathcal{E}) \ \ (z\in\mathcal{E}).$$

Definition 2 ([19])**.** If $f,g\in\mathcal{A}$, we say that f is majorized by g, written as $f\ll g$, if for some analytic function $\phi(z)$, the relationship

$$f(z)=\phi(z)g(z) \quad (z\in\mathcal{E})$$

holds, where $\phi(z)$ satisfies the condition $|\phi(z)|\leq 1$ for each $z\in\mathcal{E}$.

It should be mentioned that majorization theory was initiated by MacGregor [19], and subsequent results for starlike and convex functions were obtained by the authors [4,5].

In 1992, by using the subordination technique, Ma and Minda [18] presented a unified treatment for the subfamilies of starlike and convex functions. In particular, they took a convex analytic function $\psi(z)$ in $\mathcal{E}$ such that $\psi(0)=1$, $\psi'(0)>0$ and $\Re\big(\psi(z)\big)>0$. In other words, the image domain of $\psi(\mathcal{E})$ is a star-shaped domain with respect to 1, and symmetric with real axis for each $z\in\mathcal{E}$. The following analytic description for the subfamilies of all starlike and convex functions were given by Ma and Minda [18]:

$$\mathcal{ST}(\psi)=\left\{f\in\mathcal{A}:\frac{zf'(z)}{f(z)}\prec\psi(z)\right\},\ \ \mathcal{CV}(\psi)=\left\{f\in\mathcal{A}:\frac{(zf'(z))'}{f'(z)}\prec\psi(z)\right\}.$$

For recent applications of Ma-Minda function classes in complex analysis, we refer the reader to [26,27].

As a special case, if we let

$$\psi(z):=\psi_\alpha(z)=\frac{1+(1-2\alpha)z}{1-z} \quad (0\leq\alpha<1),$$

then the above mentioned classes reduce to usual subclasses of starlike and convex functions of order α, respectively, see e.g. [2]. Similarly, the choice $\psi_0(z)=(1+z)/(1-z)$ (see Fig. 1 (top left)) gives the usual classes of starlike and convex functions, respectively.

In an analogous manner, by taking $\psi(z)=1+\sin z$ and $\psi(z)=1+\cos z$, the following subfamilies of starlike functions were introduced by Tang et al. [25]:

$$\mathcal{S}_s^*=\left\{f\in\mathcal{A}:\frac{zf'(z)}{f(z)}\prec 1+\sin z\right\},\ \ \mathcal{S}_c^*=\left\{f\in\mathcal{A}:\frac{zf'(z)}{f(z)}\prec 1+\cos z\right\}.$$

The same authors in [25] have investigated majorization properties for functions in these families. However, the subfamily S_s^* was earlier introduced and investigated by Cho et al. [10]. Geometrically, by the above function families, we mean that the

image domain of $|z| < 1$ under $zf'(z)/f(z)$ in each class represent an eight shaped and circular region as shown in Fig. 1 (top right, bottom).

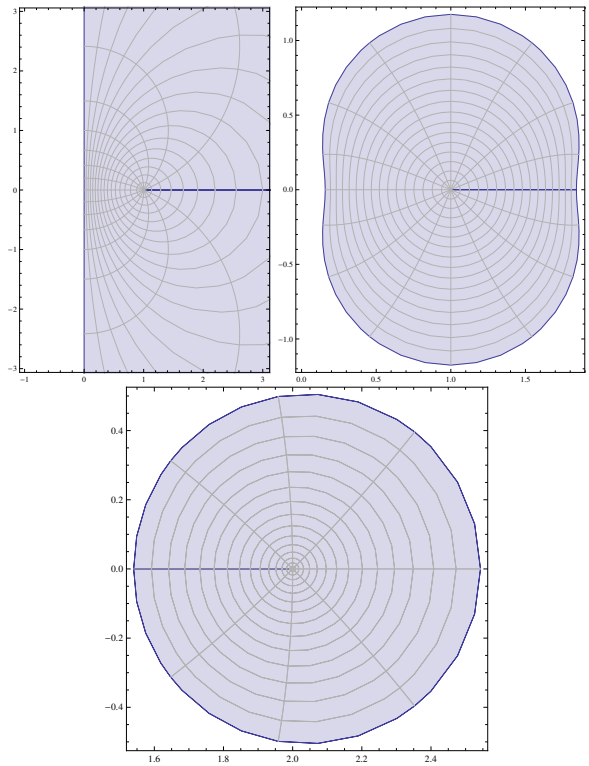

FIGURE 1. Graphs of analytic functions $\psi_0(z) = (1+z)/(1-z)$ (top left), $\psi(z) = 1+\sin z$ (top right), $\psi(z) = 1+\cos z$ (bottom).

Motivated essentially by these facts, and from the reported works in [24, 25], we introduce the subfamilies $\mathcal{ST}_s(\alpha)$ and $\mathcal{ST}_c(\alpha)$ of starlike functions of order $\alpha \in [0,1)$ by subordinating each function in the member class to

$$\psi_s(z) = 1+\sin z \ \text{ and } \ \psi_c(z) = 1+z\cos z,$$

respectively. From onward now, unless otherwise stated, we take $\mathcal{S}_c^*$ as the subfamily subordinated to $\psi_c(z)$. We now give the following definition.

Definition 3. Let $f \in \mathcal{A}$. We say that $f \in \mathcal{ST}_s(\alpha)$, if it satisfies the following condition:

$$\frac{1}{1-\alpha}\left(\frac{zf'(z)}{f(z)} - \alpha\right) \prec \psi_s(z).$$

In an analogous manner, we say that $f \in \mathcal{ST}_c(\alpha)$, if the condition

$$\frac{1}{1-\alpha}\left(\frac{zf'(z)}{f(z)}-\alpha\right) \prec \psi_c(z)$$

is satisfied.

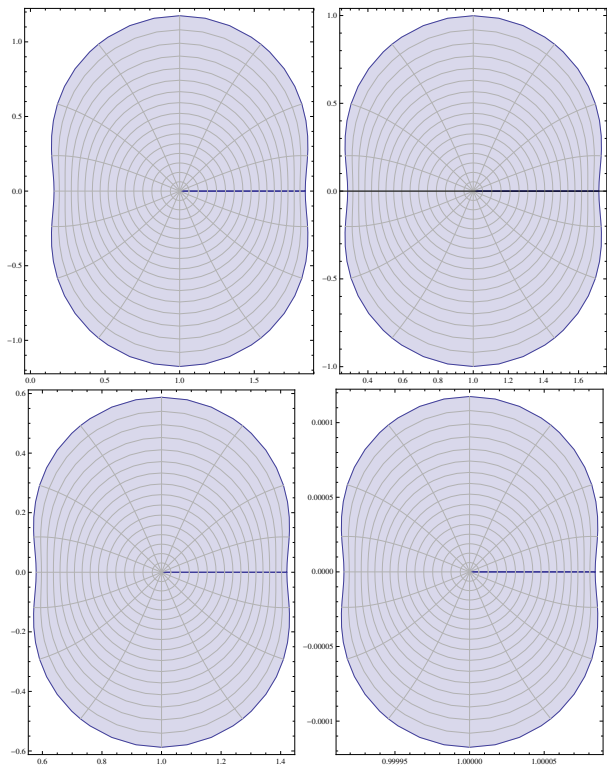

FIGURE 2. Graphs of analytic functions $\psi_s^{\alpha}(z) = 1+(1-\alpha)\sin z$ for $\alpha = 0.0$ (top left), $\alpha = 0.15$ (top right), $\alpha = 0.5$ (bottom left), and $\alpha \to 1$ (bottom right).

It is now obvious that the function $\psi_c(z) = 1+z\cos z$ satisfies the requirements as provided by Ma and Minda [18]. In particular, $\psi_c(0) = 1,\ \psi_c'(0) > 0$, and the image domain is star-shaped region with respect to 1.

Alternatively, if one consider the functions $\psi_s^{\alpha}(z) = \alpha+(1-\alpha)\psi_s(z)$ and $\psi_c^{\alpha}(z) = \alpha+(1-\alpha)\psi_c(z)$, that is

$$\psi_s^{\alpha}(z) = 1+(1-\alpha)\sin z \ \text{ and } \ \psi_c^{\alpha}(z) = 1+(1-\alpha)z\cos z \ (0 \le \alpha < 1),$$

then the subfamilies $\mathcal{ST}_s(\alpha)$ and $\mathcal{ST}_c(\alpha)$ can be defined equivalently as:

$$\mathcal{ST}_s(\alpha)=\left\{f\in\mathcal{A}:\frac{zf'(z)}{f(z)}\prec\psi_s^{\alpha}(z)\right\},\ \ \mathcal{ST}_c(\alpha)=\left\{f\in\mathcal{A}:\frac{zf'(z)}{f(z)}\prec\psi_c^{\alpha}(z)\right\}.$$

As an immediate deduction, one can see that

$$\mathcal{ST}_s(0)\equiv\mathcal{S}_s^*\ \text{ and }\ \mathcal{ST}_c(0)\equiv\mathcal{S}_c^*.$$

Remark 1. It can be seen from Figs. 2 and 3 that the image domains of $\psi_s^{\alpha}(z)$ and $\psi_c^{\alpha}(z)$ become smaller and smaller as $\alpha\to 1$. In addition, these image domains are star-shaped with respect to 1, and lie in the right half plane implying that $\Re\left(\psi_s^{\alpha}(z)\right)>0$ and $\Re\left(\psi_c^{\alpha}(z)\right)>0$ (cf. Fig 1 (top left)). Moreover, it can be seen that the following inclusion relationships

$$\psi_s^{\alpha}(\mathcal{E})\subset\psi_s(\mathcal{E})\subset\psi(\mathcal{E})\ \text{ and }\ \psi_c^{\alpha}(\mathcal{E})\subset\psi_c(\mathcal{E})\subset\psi(\mathcal{E})\ (0\le\alpha<1)$$

hold, where

$$\psi(z):=\psi_0(z)=\frac{1+z}{1-z}$$

maps the open unit disk to right-half plane for each $z\in\mathcal{E}$, see Fig. 1 (top left). This in turn implies that

$$\mathcal{ST}_s(\alpha)\subset\mathcal{S}_s^*\subset\mathcal{ST}\ \text{ and }\ \mathcal{ST}_c(\alpha)\subset\mathcal{S}_c^*\subset\mathcal{ST}\ \ (0\le\alpha<1).$$

Remark 2. We would like to point out that, it is an open question to establish inclusion relationships of the classes $\mathcal{ST}_s(\alpha)$, $\mathcal{ST}_c(\alpha)$ and $\mathcal{ST}(\alpha)$. More precisely, we pose the following problems:

$$\mathcal{ST}_s(\alpha)\subset\mathcal{ST}_c(\alpha)\subset\mathcal{ST}(\alpha)\ \text{ or }\ \mathcal{ST}_c(\alpha)\subset\mathcal{ST}_s(\alpha)\subset\mathcal{ST}(\alpha)\ \ (0\le\alpha<1).$$

The main objective of this study is to solve majorization and Fekete-Szegő problems for functions in the subfamilies $\mathcal{ST}_s(\alpha)$ and $\mathcal{ST}_c(\alpha)$. This is due to the recent studies conducted so far in this direction, see e.g. [3,6,9,14–17,20–23]. The obtained results thus comprehend and improve the already existed results in a unified manner, especially those in references [24,25], and will also continue to hold for various subfamilies of analytic functions. We point out that these results are obtained without involving any linear/nonlinear differential or integral operators, as compared to those in [14–17] and the references cited therein.

The rest of discussions are arranged as follows. In Section 2, we recall some useful results as lemmas. Section 3 is devoted to the main results and their demonstrations. Section 4 describes several applications of the main results. In Section 5, we end the discussion with a concise conclusion.

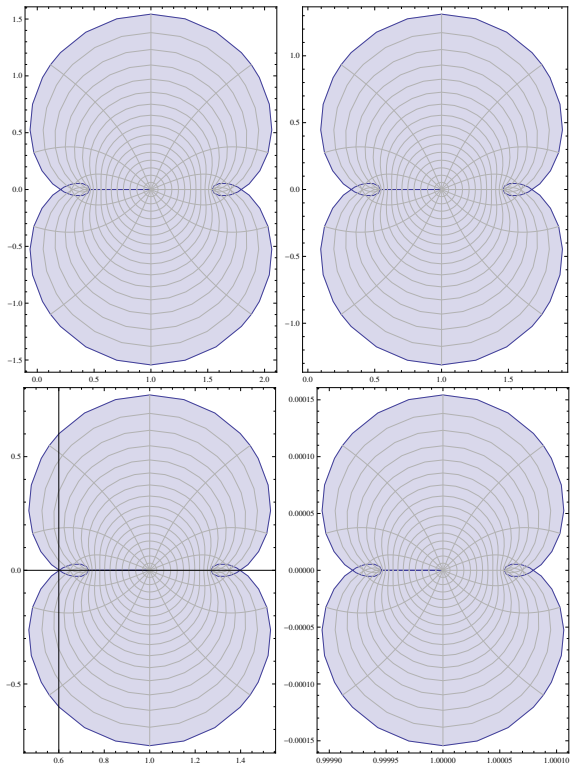

FIGURE 3. Graphs of analytic functions $\psi_c^{\alpha}(z) = 1 + (1-\alpha)z\cos z$ for $\alpha = 0.0$ (top left), $\alpha = 0.15$ (top right), $\alpha = 0.5$ (bottom left), and $\alpha \to 1$ (bottom right)

2. PRELIMINARY RESULTS

In this section, we present some helpful results that are essential in the proof of main results.

Lemma 1 ([10]). *The analytic function $\phi(z)$ for each $|z| = r < 1$ satisfies the following inequality*

$$|\phi'(z)| \leq \frac{1-|\phi(z)|^2}{1-|z|^2} = \frac{1-|\phi(z)|^2}{1-r^2} \quad (z \in \mathcal{E}).$$

For some recent applications of this lemma, see [24, 25].

Lemma 2 ([3]). *Let*

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots \quad (z \in \mathcal{E})$$

be a Schwarz function. Then for any real number β,

$$\left|w_2-\beta w_1^2\right| \leq \begin{cases} -\beta & (\beta<-1), \\ 1 & (-1 \leq \beta \leq 1), \\ \beta & (\beta>1). \end{cases}$$

These estimates are sharp and attain for $\beta>1$ *or* $\beta<-1$, *iff* $w(z)=z$ *or one of its rotations. If* $-1<\beta<1$, *then equality occurs iff* $w(z)=z^2$ *or one of its rotations. Equality also occurs for* $\beta=-1$, *iff*

$$w(z)=\frac{z(z+\lambda)}{1+\lambda z} \quad (0 \leq \lambda \leq 1)$$

or one of its rotations, while for $\beta=1$, *iff*

$$w(z)=-\frac{z(z+\lambda)}{1+\lambda z} \quad (0 \leq \lambda \leq 1)$$

or one of its rotations.

Lemma 3 ([3])**.** *Let*

$$w(z)=w_1 z+w_2 z^2+w_3 z^3+\cdots \quad (z \in \mathcal{E})$$

be a Schwarz function. Then for any complex number β,

$$\left|w_2-\beta w_1^2\right| \leq \max\{1,|\beta|\}.$$

This estimate is sharp and attains equality by $w(z)=z$ *or* $w(z)=z^2$.

3. Main results

In this section, we investigate the majorization problems for functions in the classes $\mathcal{ST}_s(\alpha)$ and $\mathcal{ST}_c(\alpha)$. The Fekete-Szegő type problems are also solved for each function in $\mathcal{ST}_s(\alpha)$ and $\mathcal{ST}_c(\alpha)$. Throughout the discussion, it is assumed that $0 \leq \alpha<1$.

3.1. *Majorization problems*

We begin by stating the following result.

Theorem 1. *Let* $f \in \mathcal{A}$ *and* $g \in \mathcal{ST}_s(\alpha)$. *If* f *is majorized by* g *for each* $z \in \mathcal{E}$, *then for* $|z|=r \leq r_s$,

$$|f'(z)| \leq |g'(z)|,$$

where r_s *is the least positive root of the following equation*

$$[1-(1-\alpha)\sin r](1-r^2)-2r=0. \tag{3.1}$$

Proof. For $g \in \mathcal{ST}_s(\alpha)$, by using the subordination principle, we can write

$$\frac{zg'(z)}{g(z)}=1+(1-\alpha)\sin(w(z)) \quad (z \in \mathcal{E}), \tag{3.2}$$

where $w(z)$ is a Schwarz function. Now, by setting $w(z) = \rho\exp(it)$ such that $\rho \leq |z| = r < 1$ and $t \in [-\pi, \pi]$. Then for $\alpha < 1$, it can be seen that

$$\begin{aligned}|\sin(w(z))|^2 &= |\sin(\rho\exp(it))|^2 \\ &= \cos^2(\rho\cos t)\sinh^2(\rho\sin t) + \sin^2(\rho\cos t)\cosh^2(\rho\sin t).\end{aligned}$$

Thus, for the function

$$\vartheta(t) = \cos^2(\rho\cos t)\sinh^2(\rho\sin t) + \sin^2(\rho\cos t)\cosh^2(\rho\sin t),$$

one can readily verify that

$$\vartheta'(t) = \sinh(2\rho\sin t) - \sin(2\rho\cos t) = 0$$

admits five real roots within the interval $[-\pi, \pi]$, namely, $0, \pm\frac{\pi}{2}, \pm\pi$. In addition, we see that $\vartheta(-t) = \vartheta(t)$, so we consider $t \in [0, \pi]$ to simplify the analysis. A simple exercise reveals that

$$\max\left\{\vartheta(0), \vartheta\left(\frac{\pi}{2}\right), \vartheta(\pi)\right\} = \vartheta\left(\frac{\pi}{2}\right) = \sinh^2(\rho).$$

Now, it follows that

$$|\sin(w(z))| = |\sin(\rho\exp(it))| \leq \sinh\rho \leq \sinh r. \tag{3.3}$$

Moreover, we note that

$$\sin r \leq \sin\rho \leq |\sin(w(z))|. \tag{3.4}$$

Combining (3.3) and (3.4), then (3.2) gives

$$\left|\frac{g(z)}{g'(z)}\right| = \frac{|z|}{|1 + (1-\alpha)\sin(w(z))|} \leq \frac{|z|}{1 - (1-\alpha)|\sin(w(z))|} \leq \frac{r}{1 - (1-\alpha)\sin r}. \tag{3.5}$$

From the assumption, on the other hand, we have $f(z) \ll g(z)$ for $z \in \mathcal{E}$, which is equivalent to the statement

$$f(z) = \phi(z)g(z) \quad (z \in \mathcal{E}).$$

Then by differentiation, we obtain

$$f'(z) = \phi'(z)g(z) + \phi(z)g'(z) = \left(\phi(z) + \phi'(z)\frac{g(z)}{g'(z)}\right)g'(z),$$

which implies that

$$|f'(z)| \leq \left(|\phi(z)| + |\phi'(z)|\left|\frac{g(z)}{g'(z)}\right|\right)|g'(z)|. \tag{3.6}$$

By setting $|\phi(z)| = \tau$ ($\tau \in [0,1]$), and using Lemma 2 together with (3.5), inequality (3.6) yields

$$|f'(z)| \leq \left(\tau + \frac{r(1-\tau^2)}{[1-(1-\alpha)\sin r](1-r^2)}\right)|g'(z)|.$$

In what follows, we define the following function

$$\Lambda_\alpha(r,\tau) := \tau + \frac{r(1-\tau^2)}{[1-(1-\alpha)\sin r](1-r^2)}.$$

Then we will reach to the required result, if

$$\Lambda_\alpha(r,\tau) \le 1 \quad (\forall \tau \in [0,1]; \alpha \in [0,1)),$$

i.e.,

$$r_s = \max\{r, \alpha \in [0,1) : \Lambda_\alpha(r,\tau) \le 1; \forall \tau \in [0,1]\}.$$

This is equivalent to the following statement

$$r_s = \max\{r, \alpha \in [0,1) : \Pi_\alpha(r,\tau) := 1 - \Lambda_\alpha(r,\tau) \ge 0 \quad (\forall \tau \in [0,1])\},$$

where

$$\Pi_\alpha(r,\tau) = [1-(1-\alpha)\sin r](1-r^2) - (1+\tau)r.$$

Now, following the proof of Tang et al. [25], we can verify that the minima of $\Pi_\alpha(r,\tau)$ occurs at $\tau = 1$, that is

$$\min\{\Pi_\alpha(r,\tau),\ \forall \tau \in [0,1]\} = \Pi_\alpha(r,1) =: \Upsilon_\alpha(r),$$

where

$$\Upsilon_\alpha(r) = [1-(1-\alpha)\sin r](1-r^2) - 2r.$$

By using the fact that

$$\Upsilon_\alpha(0) = 1 > 0 \ \text{ and } \ \Upsilon_\alpha(1) = -2 < 0,$$

we arrive at the conclusion that there exists r_s for which

$$\Upsilon_\alpha(r) \ge 0 \quad (\forall r \in [0, r_s]),$$

where r_s is the least positive root of equation (3.1). □

Remark 3. A search for the least positive root immediately yields $r_s = 0.372342$ at $\alpha = 0.627658$ whenever $0 < \alpha < 1$.

Remark 4. We would like to mention that the obtained results are sharper than those earlier reported ones in [25]. Indeed, the estimate in Theorem 1 is sharper than [25, Theorem 2.1]. This is due to the fact that

$$\sin r \le \sinh r \quad (r \in [0,1]).$$

Theorem 2. *Let $f \in \mathcal{A}$ and $g \in \mathcal{ST}_c(\alpha)$. If f is majorized by g for each $z \in \mathcal{E}$, then for $|z| = r \le r_c$,*

$$|f'(z)| \le |g'(z)|,$$

where r_c is the least positive root of the following equation

$$[1-(1-\alpha)r\cos r](1-r^2) - 2r = 0. \tag{3.7}$$

Proof. For $g \in \mathcal{ST}_c(\alpha)$, by using the subordination principle, we can write

$$\frac{zg'(z)}{g(z)} = 1 + (1-\alpha)z\cos(w(z)) \quad (z \in \mathcal{E}), \tag{3.8}$$

where $w(z)$ is a Schwarz function. By setting $w(z) = \rho\exp(it)$ such that $\rho \leq |z| = r < 1$ and $t \in [0,\pi]$, we get the following estimates:

$$\cos r \leq \cos\rho \leq |\cos(w(z))| \leq |\cos(\rho\exp(it))| \leq \cosh\rho \leq \cosh r.$$

Now, following the proof of Theorem 1 and (3.8), for $0 \leq \alpha < 1$, we obtain

$$\left|\frac{g(z)}{g'(z)}\right| \leq \frac{r}{1-(1-\alpha)r\cos r}. \tag{3.9}$$

Also, for $f(z) \ll g(z)$, we can write

$$f(z) = \phi(z)g(z) \quad (z \in \mathcal{E}),$$

which implies that

$$|f'(z)| \leq \left(|\phi(z)| + |\phi'(z)|\left|\frac{g(z)}{g'(z)}\right|\right)|g'(z)|. \tag{3.10}$$

By setting $|\phi(z)| = \tau$ ($\tau \in [0,1]$), and using Lemma 2 together with (3.9), inequality (3.10) becomes

$$|f'(z)| \leq \left(\tau + \frac{r(1-\tau^2)}{[1-(1-\alpha)r\cos r](1-r^2)}\right)|g'(z)|.$$

For the function

$$\Xi_\alpha(r,\tau) = \tau + \frac{r(1-\tau^2)}{[1-(1-\alpha)r\cos r](1-r^2)},$$

we reach to the required result if $\Xi_\alpha(r,\tau) \leq 1$ for $\forall\tau \in [0,1]$, i.e.,

$$r_s = \max\{r, \alpha \in [0,1) : \Xi_\alpha(r,\tau) \leq 1,\ \forall\tau \in [0,1]\}.$$

Alternatively, we need to show that

$$r_s = \max\{r, \alpha \in [0,1) : \Omega_\alpha(r,\tau) \geq 0,\ \forall\tau \in [0,1]\},$$

where

$$\Omega_\alpha(r,\tau) = (1-r^2)[1-(1-\alpha)r\cos r] - (1+\tau)r.$$

As in the proof of Theorem 1, it is easy to verify that $\Omega_\alpha(r,\tau)$ takes minima at $\tau = 1$, namely,

$$\min\{\Omega_\alpha(r,\tau),\ \forall\tau \in [0,1]\} = \Omega_\alpha(r,1) := \Theta_\alpha(r),$$

where

$$\Theta_\alpha(r) = (1-r^2)[1-(1-\alpha)r\cos r] - 2r.$$

Also, we can see that

$$\Theta_\alpha(0) = 1 > 0 \ \text{ and } \ \Theta_\alpha(1) = -2 < 0,$$

so there exists r_c for which

$$\Theta_\alpha(r) \geq 0 \quad (\forall r \in [0, r_c]),$$

where r_c is the least positive root of equation (3.7). □

Remark 5. As illustrated earlier, a search for the least positive root yields $r_c = 0.374041$ at $\alpha = 0.625959$ whenever $0 < \alpha < 1$.

3.2. *Fekete-Szegő type problems for the classes $\mathcal{ST}_s(\alpha)$ and $\mathcal{ST}_c(\alpha)$*

In this subsection, the Fekete-Szegő type problems are solved for functions in the classes $\mathcal{ST}_s(\alpha)$ and $\mathcal{ST}_c(\alpha)$.

Theorem 3. *Let $f \in \mathcal{ST}_s(\alpha)$ and $0 \leq \alpha < 1$. Then for any real number β,*

$$\left|a_3 - \beta a_2^2\right| \leq \begin{cases} \left(\frac{\alpha-1}{2}\right)\sigma & (\beta < \sigma_1), \\ \frac{1-\alpha}{2} & (\sigma_1 \leq \beta \leq \sigma_2), \\ \left(\frac{1-\alpha}{2}\right)\sigma & (\beta > \sigma_2), \end{cases} \tag{3.11}$$

where

$$\sigma = (2\beta - 1)(1-\alpha), \quad \sigma_1 = \frac{\alpha(\alpha-1)}{2}, \quad \sigma_2 = \frac{(1-\alpha)(2-\alpha)}{2}. \tag{3.12}$$

Proof. By substituting

$$w(z) = 1 + w_1 z + w_2 z^2 + \cdots$$

and

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$$

into (3.2), we see that

$$a_2 z + (2a_3 - a_2^2)z^2 + \cdots = (1-\alpha)w_1 z + (1-\alpha)w_2 z^2 + \cdots. \tag{3.13}$$

Equating coefficients of the same powers of z in (3.13), we find that

$$a_2 = (1-\alpha)w_1, \quad a_3 = \frac{1}{2}\left[(1-\alpha)w_2 + (1-\alpha)^2 w_1^2\right].$$

Now, it is a simple exercise to get

$$\left|a_3 - \beta a_2^2\right| \leq \left(\frac{1-\alpha}{2}\right)\left|w_2 - \sigma w_1^2\right|, \tag{3.14}$$

where σ is given by (3.12).

Hence, from Lemma 2, the first inequality in (3.11) is established, whenever

$$(2\beta - 1)(1-\alpha) < -1 \Longleftrightarrow \beta < \sigma_1 = \frac{\alpha(\alpha-1)}{2}.$$

Similarly, an application of Lemma 2 gives the third inequality in (3.11), whenever

$$(2\beta - 1)(1-\alpha) > 1 \Longleftrightarrow \beta > \sigma_2 = \frac{(1-\alpha)(2-\alpha)}{2}.$$

Finally, the second inequality in (3.11) follows immediately by Lemma 2 for

$$\sigma_1 \leq \beta \leq \sigma_2.$$

□

Theorem 4. *Let $f \in \mathcal{ST}_s(\alpha)$ and $0 \leq \alpha < 1$. Then for any complex number β,*

$$\left|a_3 - \beta a_2^2\right| \leq \frac{1-\alpha}{2} \max\{1, |\sigma|\}, \tag{3.15}$$

where σ is given by (3.12).

Proof. By applying Lemma 3 to inequality (3.14) for the complex number β, the desired inequality (3.15) follows. □

Theorem 5. *Let $f \in \mathcal{ST}_c(\alpha)$ and $0 \leq \alpha < 1$. Then for any real number β,*

$$\left|a_3 - \beta a_2^2\right| \leq \begin{cases} \frac{\alpha-1}{2}\sigma & (\beta < \sigma_1), \\ \frac{1-\alpha}{2} & (\sigma_1 \leq \beta \leq \sigma_2), \\ \frac{1-\alpha}{2}\sigma & (\beta > \sigma_2), \end{cases}$$

where

$$\sigma = (2\beta - 1)(1-\alpha), \quad \sigma_1 = \frac{\alpha(\alpha-1)}{2}, \quad \sigma_2 = \frac{(1-\alpha)(2-\alpha)}{2}.$$

Proof. The proof is similar to that of Theorem 3. □

Theorem 6. *Let $f \in \mathcal{ST}_c(\alpha)$ and $0 \leq \alpha < 1$. Then for any complex number β,*

$$\left|a_3 - \beta a_2^2\right| \leq \frac{1-\alpha}{2} \max\{1, |\sigma|\},$$

where σ is given by (3.12).

Proof. The proof is similar to that of Theorem 4. □

Remark 6. The results provided in Theorems 3, 4, 5 and 6 are sharp. Indeed, in Theorems 3 and 4, equalities occur for the function $f_s(z)$ given by

$$f_s(z) = z\exp\left(\int_0^z \frac{\psi_s^\alpha(t) - 1}{t}\right) \mathrm{d}t = z + (1-\alpha)z^2 + \frac{(1-\alpha)^2}{2}z^3 + \cdots.$$

Also, in Theorems 5 and 6, equalities occur for the function $f_c(z)$ given by

$$f_c(z) = z\exp\left(\int_0^z \frac{\psi_c^\alpha(t) - 1}{t}\right) \mathrm{d}t = z + (1-\alpha)z^2 + \frac{(1-\alpha)^2}{2}z^3 + \cdots.$$

4. Applications

In this section, our aim is to highlight some interesting applications of the main results (Theorems 1, 2, 3, 4, 5, 6) in the form of the following corollaries and remarks.

Corollary 1. *Let $f \in \mathcal{A}$ and $g \in \mathcal{ST}_s(0) \equiv S_s^*$. If f is majorized by g for each $z \in \mathcal{E}$, then for $|z| = r \leq r_s$,*

$$|f'(z)| \leq |g'(z)|$$

where r_s is the least positive root of the following equation

$$(1-r^2)(1-\sin r) - 2r = 0.$$

Remark 7. Corollary 1 is sharper than the one obtained in [25, Theorem 2.1].

Corollary 2. *Let $f \in \mathcal{A}$ and $g \in \mathcal{ST}_c(0) \equiv S_c^*$. If f is majorized by g for each $z \in \mathcal{E}$, then for $|z| = r \leq r_c$,*

$$|f'(z)| \leq |g'(z)|$$

where r_c is the least positive root of the following equation

$$\left(1-r^2\right)(1-r\cos r) - 2r = 0.$$

Remark 8. In view of Theorems 3, 4, 5, 6, one can make the following deductions easily.

Corollary 3. *Let $f \in \mathcal{ST}_s(0) \equiv S_s^*$ or $f \in \mathcal{ST}_c(0) \equiv S_c^*$. Then for any real number β,*

$$\left|a_3 - \beta a_2^2\right| \leq \begin{cases} -\frac{2\beta-1}{2} & (\beta < 0), \\ \frac{1}{2} & (0 \leq \beta \leq 1), \\ \frac{2\beta-1}{2} & (\beta > 1). \end{cases}$$

Corollary 4. *Let $f \in \mathcal{ST}_s(0) \equiv S_s^*$ or $f \in \mathcal{ST}_c(0 \equiv S_c^*$. Then for any complex number β,*

$$\left|a_3 - \beta a_2^2\right| \leq \max\left\{\frac{1}{2}, \frac{|2\beta-1|}{2}\right\}.$$

Remark 9. A direct application of Lemma 2 also shows that (see (3.13))

$$|a_2| \leq 1-\alpha, \quad |a_3| \leq \frac{(1-\alpha)^2}{2}, \tag{4.1}$$

and

$$|2a_3 - a_2^2| \leq 1-\alpha \Longrightarrow \left|a_3 - \frac{a_2^2}{2}\right| \leq \frac{1-\alpha}{2}, \tag{4.2}$$

which holds wherever $f \in \mathcal{ST}_s(\alpha)$ or $f \in \mathcal{ST}_c(\alpha)$. The last inequality is the same as, if one take $\beta = 1/2$ in Theorems 3 and 5 (cf. inequality (3.11)). We note that the coefficient bounds for a_2 and a_3 are sharp, and equality occurs for the functions defined in Remark 6.

5. Conclusion

In this paper, we introduce some new subfamilies of starlike functions of order $\alpha \in [0,1)$, by subordinated to sine and cosine functions. For functions in these subfamilies, sharp solutions of majorization and Fekete-Szegő problems have been derived. Applications of the obtained results for some new and known families of analytic functions have also been demonstrated for further illustration. The obtained results have unified and improved some recent results in the literature, especially those in [24, 25]. Based on Remark 2, it will be an interesting research problem to find conditions on α $(0 \leq \alpha < 1)$ or to find radius of a certain disk within $|z| < 1$ so that the inclusion results hold.

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CONVERGENCE RELATION BETWEEN SOLUTIONS OF A P(X)-LANDAU-LIFSCHITZ TYPE AND P(X)-HARMONIC MAPS

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Abstract. The author studies the asymptotic behavior of solutions u_ε of a $p(x)$-Landau-Lifschitz equation as ε tends to zero. Several kinds of convergence to the $p(x)$-harmonic map are presented in different senses.

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1. Introduction

Let $G \subset \mathbb{R}^2$ be a bounded and simply connected domain with smooth boundary ∂G, and $B_1 = \{x \in \mathbb{R}^2; x_1^2 + x_2^2 < 1\}$. Denote $\mathbb{S}^1 = \{x \in \mathbb{R}^3; x_1^2 + x_2^2 = 1, x_3 = 0\}$ and $\mathbb{S}^2 = \{x \in \mathbb{R}^3; x_1^2 + x_2^2 + x_3^2 = 1\}$. The vector value function can be denoted as $u = (u_1, u_2, u_3) = (u', u_3)$. Let $g = (g', 0)$ be a smooth map from ∂G into $\mathbb{S}^1$. The p-Landau-Lifschitz-type energy functional is

$$E^{LL}(u) = \frac{1}{p}\int_G |\nabla u|^p dx + \frac{1}{2\varepsilon^p}\int_G u_3^2 dx$$

with a small parameter $\varepsilon > 0$. When $p = 2$, it was introduced in the study of some simplified model of high-energy physics, which controls the statics of planner ferromagnets and antiferromagnets (see [11] and [17]). The asymptotic behavior of minimizers of $E^{LL}(u)$ had been studied by Hang and Lin in [9]. When $p > 1$, the corresponding asymptotic properties were studied in [12] and [21]. These works show that the minimizers of $E_\varepsilon(u)$ converge to the p-harmonic maps with $\mathbb{S}^1$-value.

When $p = 2$, if the term $\frac{u_3^2}{2\varepsilon^2}$ replaced by $\frac{(1-|u|^2)^2}{4\varepsilon^2}$ and $\mathbb{S}^2$ replaced by $\mathbb{R}^2$, the problem becomes the simplified model of the Ginzburg-Landau theory for superconductors and was well studied in [3] and [4]. The energy functional is

$$E^{GL}(u) = \frac{1}{2}\int_G |\nabla u|^2 dx + \frac{1}{4\varepsilon^2}\int_G (1-|u|^2)^2 dx.$$

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It is also showed that the properties of the harmonic maps can be studied via researching the minimizers of the functional with Ginzburg-Landau-type penalization term. Indeed, Chen and Struwe used the penalty method to establish the global existence of partial regular weak solutions of the harmonic map flow (see [5] and [7]). They also generalized the results to the case of $p > 1$ (cf. [6]). Afterwards, many papers proved that the limit of minimizers of p-Ginzburg-Landau functional is p-harmonic maps (cf. [1, 2, 13, 14, 18, 19, 22] and references therein). Misawa studied the p-harmonic maps by using the same idea of the penalty method in [16]. In 2009, Lei generalized the results to the case that p is a bounded function (cf. [13]). Now, the functional with penalization term is

$$E^{GL}(u,G) = \int_G \left[\frac{1}{p(x)} |\nabla u|^{p(x)} dx + \frac{1}{4\varepsilon^{p(x)}} (1-|u|^2)^2 \right] dx.$$

The main result is the convergence relation between the minimizers and $p(x)$-harmonic maps. Other results of $p(x)$-harmonic maps can be found in [8, 15] and [20].

In this paper, we are concerned with the $p(x)$-Landau-Lifschitz functional

$$E_\varepsilon(u,G) = \int_G \left[\frac{1}{p(x)} |\nabla u|^{p(x)} dx + \frac{1}{2\varepsilon^{p(x)}} u_3^2 \right] dx.$$

Here

$$2 < p_* = \min_{\overline{G}} p(x) \le \max_{\overline{G}} p(x) = p^* < \infty.$$

From the direct method in the calculus of variations, it is easy to see that the functional achieves its minimum in the function class

$$W_g^{1,p(x)}(G,\mathbb{S}^2) := \{u \in W^{1,p(x)}(G,\mathbb{S}^2); u-g \in W_0^{1,p(x)}(G,\mathbb{R}^3)\}.$$

Without loss of generality, we assume $u_3 \ge 0$, otherwise we may consider $|u_3|$ in view of the expression of the functional. We call u_ε a minimizer of $E_\varepsilon(u,G)$ in $W_g^{1,p(x)}(G,\mathbb{S}^2)$, if

$$E_\varepsilon(u_\varepsilon,G) = \min\{E_\varepsilon(u,G); u \in W_g^{1,p(x)}(G,\mathbb{S}^2)\}.$$

We will research the asymptotic properties of minimizers in $W_g^{1,p(x)}(G,\mathbb{S}^2)$ when $\varepsilon \to 0$, and shall prove the limit is the $p(x)$-harmonic map.

Theorem 1. *Let u_ε be a minimizer of $E_\varepsilon(u,G)$ in $W_g^{1,p(x)}(G,\mathbb{S}^2)$. Assume*

$$\deg(g',\partial G) = 0.$$

Then

$$\lim_{\varepsilon\to 0} u_\varepsilon = (u_p, 0), \quad \text{in} \quad W^{1,p(x)}(G,\mathbb{S}^2),$$

where u_p is the minimizer of $\int_G |\nabla u|^{p(x)} dx$ in $W_g^{1,p(x)}(G,\partial B_1)$.

Comparing with the assumption of Theorem 1, we will consider the problem under some weaker conditions.

Theorem 2. *Assume u_ε is a critical point of $E_\varepsilon(u,G)$ in $W_g^{1,p(x)}(G,\mathbb{S}^2)$. If*

$$E_\varepsilon(u_\varepsilon,K)\le C \tag{1.1}$$

for some subdomain $K\subseteq G$. Then there exists a subsequence u_{ε_k} of u_ε such that as $k\to\infty$,

$$u_{\varepsilon_k}\to(u_p,0),\quad \textit{weakly}\quad \textit{in}\quad W^{1,p(x)}(K,\mathbb{R}^3), \tag{1.2}$$

where u_p is a critical point of $\int_K|\nabla u|^{p(x)}dx$ in $W^{1,p(x)}(K,\partial B_1)$ (which is named $p(x)$-harmonic map on K). Moreover, for any $\zeta\in C_0^\infty(K)$, when $\varepsilon\to 0$,

$$\int_K|\nabla u_{\varepsilon_k}|^{p(x)}\zeta dx\to\int_K|\nabla u_p|^{p(x)}\zeta dx, \tag{1.3}$$

$$\int_K\frac{1}{\varepsilon_k^{p(x)}}u_{\varepsilon_k 3}\zeta dx\to 0. \tag{1.4}$$

2. Proof of Theorem 1

A vector-valued function $u\in W_g^{1,p(x)}(G,\partial B_1)$ is named $p(x)$-harmonic map, if it is the critical point of $\int_G|\nabla u|^{p(x)}dx$. Namely, it is the weak solution of

$$-div(|\nabla u|^{p(x)-2}\nabla u)=u|\nabla u|^{p(x)} \tag{2.1}$$

on G, or for any $\phi\in C_0^\infty(G,\mathbb{R}^2)$, it satisfies

$$\int_G|\nabla u|^{p(x)-2}\nabla u\nabla\phi dx=\int_G u|\nabla u|^{p(x)}\phi dx. \tag{2.2}$$

By the argument of the weak low semi-continuity of the functional, we can deduce the strong convergence in $W^{1,p(x)}$ sense for some subsequence of the minimizer u_ε. To improve the conclusion of the convergence for all u_ε, we need the uniqueness of $p(x)$-harmonic maps. Therefore, we always assume $deg(g',\partial G)=0$ in this section.

From $deg(g',\partial G)=0$ and the smoothness of ∂G and g, we see that there is a smooth function $\phi_0:\partial G\to\mathbb{R}$ such that

$$g=(\cos\phi_0,\sin\phi_0),\ \text{ on }\ \partial G. \tag{2.3}$$

Consider the Dirichlet problem

$$-div(|\nabla\Phi|^{p(x)-2}\nabla\Phi)=0,\ \text{ in }\ G, \tag{2.4}$$

$$\Phi|_{\partial G}=\phi_0. \tag{2.5}$$

According to Proposition 2.4 in [13], there exists the unique weak solution Φ of (2.4) and (2.5) in $W^{1,p(x)}(G,\mathbb{R})$. Set

$$u_p=(\cos\Phi,\sin\Phi),\ \text{ on }\ \overline{G}. \tag{2.6}$$

Clearly, u_p is a $p(x)$-harmonic map on G.

Since $W_g^{1,p(x)}(G,\partial B_1)\neq\varnothing$ when $\deg(g',\partial G)=0$, we may consider the minimization problem

$$Min\left\{\int_G|\nabla u|^{p(x)}dx;u\in W_g^{1,p(x)}(G,\partial B_1)\right\}. \tag{2.7}$$

The solution is called the $p(x)$-energy minimizer. By the direct method, the solution of (2.7) exists. Obviously, the $p(x)$-energy minimizer is a $p(x)$-harmonic map. According to Proposition 2.5 in [13], the $p(x)$-harmonic map is unique in $W_g^{1,p(x)}(G,\partial B_1)$. So the $p(x)$-energy minimizer is also unique in $W_g^{1,p(x)}(G,\partial B_1)$.

In general, u_p is the unique $p(x)$-harmonic map as well as the unique $p(x)$-energy minimizer.

Proof of Theorem 1. Noticing that u_ε is the minimizer, we have

$$E_\varepsilon(u_\varepsilon,G)\leq E_\varepsilon((u_p,0),G)\leq C \tag{2.8}$$

with $C>0$ independent of ε. This means

$$\int_G|\nabla u_\varepsilon|^{p(x)}dx\leq C, \tag{2.9}$$

$$\int_G u_{\varepsilon 3}^2dx\leq C\varepsilon^{p_*}. \tag{2.10}$$

Using (2.9), $|u_\varepsilon|=1$ and the embedding theorem, we see that there exists a subsequence u_{ε_k} of u_ε and $u_*\in W^{1,p(x)}(G,\mathbb{R}^3)$, such that as $\varepsilon_k\to 0$,

$$u_{\varepsilon_k}\to u_*,\ \text{ weakly }\ \text{ in }\ W^{1,p(x)}(G,\mathbb{S}^2), \tag{2.11}$$

$$u_{\varepsilon_k}\to u_*,\ \text{ in }\ C^\alpha(\overline{G},\mathbb{S}^2),\quad \alpha\in(0,1-2/p_*). \tag{2.12}$$

Obviously, (2.10) and (2.12) lead to $u_*\in W_g^{1,p(x)}(G,\mathbb{S}^1)$.

Applying (2.11) and the weak low semi-continuity of $\int_G|\nabla u|^{p(x)}dx$, we have

$$\int_G|\nabla u_*|^{p(x)}dx\leq\underline{\lim}_{\varepsilon_k\to 0}\int_G|\nabla u_{\varepsilon_k}|^{p(x)}dx.$$

On the other hand, (2.8) implies

$$\int_G|\nabla u_{\varepsilon_k}|^{p(x)}dx\leq\int_G|\nabla(u_p,0)|^{p(x)}dx.$$

Thus,

$$\int_G|\nabla u_*'|^{p(x)}dx\leq\int_G|\nabla u_p|^{p(x)}dx.$$

This means that u_*' is also a $p(x)$-energy minimizer. Noting the uniqueness, we see $u_*=u_p$. Thus

$$\int_G|\nabla u_p|^{p(x)}dx\leq\underline{\lim}_{\varepsilon_k\to 0}\int_G|\nabla u_{\varepsilon_k}|^{p(x)}dx\leq\overline{\lim}_{\varepsilon_k\to 0}\int_G|\nabla u_{\varepsilon_k}|^{p(x)}dx\leq\int_G|\nabla u_p|^{p(x)}dx.$$

When $\varepsilon_k\to 0$,

$$\int_G|\nabla u_{\varepsilon_k}|^{p(x)}\to\int_G|\nabla u_p|^{p(x)}.$$

Combining this with (2.11) yields

$$\lim_{k\to\infty} \nabla u_{\varepsilon_k} = \nabla(u_p, 0), \ \text{ in } \ L^{p(x)}(G, \mathbb{S}^2).$$

In addition, (2.12) implies that as $\varepsilon \to 0$,

$$u_{\varepsilon_k} \to (u_p, 0), \ \text{ in } \ L^{p(x)}(G, \mathbb{S}^2).$$

Then

$$\lim_{k\to\infty} u_{\varepsilon_k} = (u_p, 0), \ \text{ in } \ W^{1,p(x)}(G, \mathbb{S}^2).$$

Noticing the uniqueness of $(u_p, 0)$, we see the convergence above also holds for all u_ε.

3. Proof of Theorem 2

In this section, we always assume that u_ε is the critical point of the functional, and $E_\varepsilon(u_\varepsilon, K) \le C$ for some subdomain $K \subseteq G$, where C is independent of ε. The assumption is weaker than that of Theorem 1. So, all the results in this section will be derived in the weak sense.

The method in the calculus of variations shows that the minimizer u_ε of $E_\varepsilon(u, G)$ in $W_g^{1,p(x)}(G, S^2)$ is a weak solution of

$$-div(|\nabla u|^{p(x)-2}\nabla u) = u|\nabla u|^{p(x)} + \frac{1}{\varepsilon^{p(x)}}(uu_3^2 - u_3 e_3), \ \text{ on } \ G, \tag{3.1}$$

where $e_3 = (0,0,1)$. Namely, for any $\psi \in W_0^{1,p(x)}(G, \mathbb{R}^3)$, u_ε satisfies

$$\int_G |\nabla u|^{p(x)-2}\nabla u \nabla \psi dx = \int_G u\psi|\nabla u|^{p(x)} dx + \frac{1}{\varepsilon^{p(x)}}\int_G \psi(uu_3^2 - u_3 e_3) dx. \tag{3.2}$$

Proof of (1.2). $E_\varepsilon(u_\varepsilon, K) \le C$ means

$$\int_K |\nabla u_\varepsilon|^{p(x)} dx \le C, \tag{3.3}$$

$$\int_K u_{\varepsilon 3}^2 dx \le C\varepsilon^{p_*}, \tag{3.4}$$

where C is independent of ε. Combining the fact $|u_\varepsilon| = 1$ a.e. on $\overline{G}$ with (3.3) we know that there exist $u_p \in W^{1,p(x)}(K, \partial B_1)$ and a subsequence u_{ε_k} of u_ε, such that as $\varepsilon_k \to 0$,

$$u_{\varepsilon_k} \to (u_p, 0), \ \text{ weakly in } \ W^{1,p(x)}(K), \tag{3.5}$$

$$u_{\varepsilon_k} \to (u_p, 0), \ \text{ in } \ C^\alpha(\overline{K}), \tag{3.6}$$

for some $\alpha \in (0, 1 - \frac{2}{p_*})$. In the following we will prove that u_p is a weak solution of (2.1).

Let $B = B(x,3R) \subset K$. $\phi \in C_0^\infty(B(x,3R);[0,1])$, $\phi = 1$ on $B(x,R)$, $\phi = 0$ on $B \setminus B(x,2R)$ and $|\nabla\phi| \le C$, where C is independent of ε. Denote $u = u_{\varepsilon_k}$ in (3.2) and take $\psi = (0,0,\phi)$. Thus

$$\int_B |\nabla u|^{p(x)-2}\nabla u_3 \nabla\phi dx + \int_B \frac{1}{\varepsilon_k^{p(x)}}|u'|^2\phi u_3 dx = \int_B u_3\phi|\nabla u|^{p(x)}dx.$$

Applying (3.3) we can derive that

$$\int_B \frac{1}{\varepsilon_k^{p(x)}}|u'|^2\phi|u_3|dx \le \int_B |\nabla u|^{p(x)}\phi dx + \int_B |\nabla u|^{p(x)-1}|\nabla\phi|dx \le C. \tag{3.7}$$

From (3.6) it follows $|u'| \ge 1/2$ when ε_k is sufficiently small. Noting $\phi = 1$ on $B(x,R)$, we have

$$\int_{B(x,R)} \frac{1}{\varepsilon_k^{p(x)}}|u_3|dx \le C. \tag{3.8}$$

Taking $\frac{1}{k} = \varepsilon_k$, $F_k = \frac{1}{\varepsilon_k^{p(x)}}(u_{\varepsilon_k}u_{\varepsilon_k 3}^2 - u_{\varepsilon_k 3}e_3)$ in Lemma 3.11 of [10] (the proof is similar to Theorem 2.1 in [6]), noting $|F_k| = \frac{1}{\varepsilon_k^{p(x)}}|u_3|^2|u'|$ and applying (3.5) and (3.8), we obtain that

$$\lim_{\varepsilon_k\to 0}\nabla u_{\varepsilon_k} = \nabla u_p, \text{ in } L^q(B(x,R)),\ \forall q \in (1,p(x)).$$

Since $B(x,R)$ is an arbitrary disc in K, we can see that, for any $\xi \in C_0^\infty(B,\mathbb{R}^3)$ there holds

$$\lim_{\varepsilon_k\to 0}\int_B |\nabla u_{\varepsilon_k}|^{p(x)-2}\nabla u_{\varepsilon_k}\nabla\xi dx = \int_B |\nabla u_p|^{p(x)-2}\nabla u_p\nabla\xi dx. \tag{3.9}$$

Now, denote $u' = u'_{\varepsilon_k} = (u_1,u_2)$. Taking $\psi = (u_2,0,0)\zeta$ and $\psi = (0,u_1,0)\zeta$ in (3.2) respectively, where $\zeta \in C_0^\infty(B,\mathbb{R})$, we have

$$\begin{aligned}\int_B \frac{1}{\varepsilon_k^{p(x)}}u_3^2u_1u_2\zeta dx + \int_B u_1u_2\zeta|\nabla u|^{p(x)}dx &= \int_B |\nabla u|^{p(x)-2}\nabla u_1\nabla u_2\zeta dx \\ &\quad + \int_B u_2|\nabla u|^{p(x)-2}\nabla u_1\nabla\zeta dx.\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}\int_B \frac{1}{\varepsilon_k^{p(x)}}u_3^2u_2u_1\zeta dx + \int_B u_2u_1\zeta|\nabla u|^{p(x)}dx &= \int_B |\nabla u|^{p(x)-2}\nabla u_2\nabla u_1\zeta dx \\ &\quad + \int_B u_1|\nabla u|^{p(x)-2}\nabla u_2\nabla\zeta dx.\end{aligned} \tag{3.11}$$

Equation (3.10) subtracts (3.11), then

$$0 = \int_B |\nabla u|^{p(x)-2}(u\wedge\nabla u)\nabla\zeta dx, \tag{3.12}$$

where $u \wedge \nabla u = u_1 \nabla u_2 - u_2 \nabla u_1$. On the other hand, since

$$\int_B u_2 |\nabla u|^{p(x)-2} \nabla u_1 \nabla \zeta dx - \int_B u_{p2} |\nabla u_p|^{p(x)-2} \nabla u_{p1} \nabla \zeta dx$$
$$= \int_B (|\nabla u|^{p(x)-2} \nabla u_1 - |\nabla u_p|^{p(x)-2} \nabla u_{p1}) u_{p2} \nabla \zeta dx + \int_B |\nabla u|^{p(x)-2} \nabla u_1 \nabla \zeta (u_2 - u_{p2}) dx,$$

using (3.3), (3.6) and (3.9), we obtain that

$$\lim_{\varepsilon_k \to 0} \int_B u_2 |\nabla u|^{p(x)-2} \nabla u_1 \nabla \zeta dx \to \int_B u_{p2} |\nabla u_p|^{p(x)-2} \nabla u_{p1} \nabla \zeta dx. \tag{3.13}$$

Similarly, we may also get that

$$\lim_{\varepsilon_k \to 0} \int_B u_1 |\nabla u|^{p(x)-2} \nabla u_2 \nabla \zeta dx = \int_B u_{p1} |\nabla u_p|^{p(x)-2} \nabla u_{p2} \nabla \zeta dx. \tag{3.14}$$

Clearly, (3.14) subtracting (3.13) yields

$$\lim_{\varepsilon_k \to 0} \int_B |\nabla u|^{p(x)-2} (u \wedge \nabla u) \nabla \zeta dx = \int_B |\nabla u_p|^{p(x)-2} (u_p \wedge \nabla u_p) \nabla \zeta dx.$$

Combining this with (3.12), we have

$$\int_B |\nabla u_p|^{p(x)-2} (u_p \wedge \nabla u_p) \nabla \zeta dx = 0. \tag{3.15}$$

Let $u_* = u_{p1} + i u_{p2} : B \to \mathbb{C}$, where $\mathbb{C}$ is the complex plane. Thus

$$|\nabla u_*|^2 = |\nabla u_p|^2. \tag{3.16}$$

It is easy to see that

$$\overline{u_*} \nabla u_* = \nabla (|u_*|^2) + (u_* \wedge \nabla u_*) i = 0 + (u_* \wedge \nabla u_*) i$$

since $|u_*|^2 = |u_{p1}|^2 + |u_{p2}|^2 = 1$. Substituting this into (3.15) yields

$$-i \int_B |\nabla u_*|^{p(x)-2} \overline{u_*} \nabla u_* \nabla \zeta dx = 0$$

for any $\zeta \in C_0^\infty(B, \mathbb{R})$. Taking $\zeta = Re(u_* \phi_j)$ and $\zeta = Im(u_* \phi_j)$ $(j = 1, 2)$ respectively, where $\phi = (\phi_1, \phi_2) \in C_0^\infty(B, \mathbb{R}^2)$, we can see that

$$\int_B |\nabla u_*|^{p(x)-2} \overline{u_*} \nabla u_* \nabla Re(u_* \phi) dx + i \int_B |\nabla u_*|^{p(x)-2} \overline{u_*} \nabla u_* \nabla Im(u_* \phi) dx = 0.$$

Namely

$$0 = \int_G |\nabla u_*|^{p(x)-2} \overline{u_*} \nabla u_* \nabla (u_* \phi) dx.$$

Noting $\overline{u_*} \nabla u_* = -u_* \nabla \overline{u_*}$, we obtain

$$0 = \int_B |\nabla u_*|^{p(x)-2} \nabla u_* \nabla \phi dx - \int_B |\nabla u_*|^{p(x)-2} u_* \nabla \overline{u_*} \nabla u_* \phi dx$$
$$= \int_B |\nabla u_*|^{p(x)-2} \nabla u_* \nabla \phi dx - \int_B |\nabla u_*|^p (x) u_* \phi dx := J.$$

By using (3.16) and $Re(J)=0$, $Im(J)=0$, we have

$$\int_B |\nabla u_p|^{p(x)-2}\nabla u_{p1}\nabla\phi dx = \int_B |\nabla u_p|^{p(x)} u_{p1}\phi dx \tag{3.17}$$

and

$$\int_B |\nabla u_p|^{p(x)-2}\nabla u_{p2}\nabla\phi dx = \int_B |\nabla u_p|^{p(x)} u_{p2}\phi dx.$$

Combining this with (3.17) yields that for any $\phi \in C_0^\infty(B,\mathbb{R}^3)$,

$$\int_B |\nabla u_p|^{p(x)-2}\nabla u_p\nabla\phi dx = \int_B |\nabla u_p|^{p(x)} u_p\phi dx.$$

This shows that u_p is a weak solution of (2.5). From (3.5), we know that (1.2) is proved.

Proof of (1.3). For simplification, we drop ε and ε_k from u_ε and u_{ε_k}. From (3.3) and (3.6) it is deduced that as $\varepsilon \to 0$,

$$\left|\int_K u_3^2\zeta|\nabla u|^{p(x)}dx\right| \le \sup_K(1-|u'|^2)\cdot\int_K |\nabla u|^{p(x)}dx \to 0, \tag{3.18}$$

$$\begin{aligned}\left|\int_K u'u_p\zeta|\nabla u|^{p(x)}dx - \int_K \zeta|\nabla u|^{p(x)}dx\right| &= \left|\int_K (u'u_p - u_pu_p)\zeta|\nabla u|^{p(x)}dx\right| \\ &\le \sup_K|u'-u_p|\cdot\left|\int_K u_p|\nabla u|^{p(x)}dx\right| \to 0,\end{aligned} \tag{3.19}$$

and

$$\left|\int_K (u-(u_p,0))\zeta|\nabla u|^{p(x)}dx\right| \le \sup_K|u-(u_p,0)|\cdot\left|\int_K u_p|\nabla u|^{p(x)}dx\right| \to 0. \tag{3.20}$$

Similarly, (3.4) and (3.6) imply that as $\varepsilon \to 0$,

$$\left|\int_K \frac{1}{\varepsilon^{p(x)}}u_3^2\zeta dx - \int_K \frac{1}{\varepsilon^{p(x)}}u_3^2\zeta(1-u_3^2)dx\right| \le \sup_K|1-|u'|^2\cdot\left|\int_K \frac{1}{\varepsilon^{p(x)}}u_3^2dx\right| \to 0, \tag{3.21}$$

and

$$\left|\int_K \frac{1}{\varepsilon^{p(x)}}u_p\zeta u'u_3^2dx - \int_K \frac{1}{\varepsilon^{p(x)}}\zeta u_3^2dx\right| \le \sup_K|u'-u_p|\cdot\left|\int_K \frac{1}{\varepsilon^{p(x)}}u_pu_3^2dx\right| \to 0. \tag{3.22}$$

Letting $\varepsilon \to 0$ in (3.2) we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_K \left[u\psi|\nabla u|^{p(x)} + \frac{1}{\varepsilon^{p(x)}}\psi(uu_3^2 - u_3e_3)\right]dx &= \int_K |\nabla u_p|^{p(x)-2}\nabla(u_p,0)\nabla\psi dx \\ &= \int_G (u_p,0)\psi|\nabla u_p|^{p(x)}dx.\end{aligned} \tag{3.23}$$

Take $\psi = (0,0,u_3\zeta)$ where $\zeta \in C_0^\infty(K)$ we have

$$\lim_{\varepsilon\to 0}\int_K \left[u_3^2\zeta|\nabla u|^{p(x)} + \frac{1}{\varepsilon^{p(x)}}u_3^2\zeta(u_3^2-1)\right]dx = 0.$$

Combining this with (3.18) we derive

$$\lim_{\varepsilon\to 0}\int_K \frac{1}{\varepsilon^{p(x)}}u_3^2\zeta(u_3^2-1)dx=0.$$

Substituting this into (3.21) yields

$$\lim_{\varepsilon\to 0}\int_K \frac{1}{\varepsilon^{p(x)}}u_3^2\zeta dx=0. \tag{3.24}$$

Hence, as $\varepsilon\to 0$,

$$\left|\int_K \frac{1}{\varepsilon^{p(x)}}uu_3^2\zeta dx\right|\le \int_K \frac{1}{\varepsilon^{p(x)}}u_3^2\zeta dx\to 0.$$

Thus, for any $\psi\in W_0^{1,p}(K,\mathbb{R}^3)$, there holds

$$\lim_{\varepsilon\to 0}\int_K \frac{1}{\varepsilon^{p(x)}}uu_3^2\psi dx=0. \tag{3.25}$$

In addition, substituting (3.24) into (3.22) leads to

$$\lim_{\varepsilon\to 0}\int_K \frac{1}{\varepsilon^{p(x)}}u_p\zeta u'u_3^2 dx=0. \tag{3.26}$$

Take $\psi=(u_p\zeta,0)$ in (3.23) we have

$$\lim_{\varepsilon\to 0}\int_K\left[u'u_p\zeta|\nabla u|^{p(x)}dx+\frac{1}{\varepsilon^{p(x)}}u_p\zeta u'u_3^2\right]dx=\int_K|\nabla u_p|^{p(x)}\zeta dx,$$

which, together with (3.26), implies

$$\lim_{\varepsilon\to 0}\int_K u'u_p\zeta|\nabla u|^{p(x)}dx=\int_K|\nabla u_p|^{p(x)}\zeta dx.$$

Combining this with (3.19) we can see (1.3) at last.
Proof of (1.4). Obviously, (3.20) and (1.3) show that as $\varepsilon\to 0$,

$$\begin{aligned}&\left|\int_K u|\nabla u|^{p(x)}\psi dx-\int_K(u_p,0)|\nabla u_p|^{p(x)}\psi dx\right|\\ &\le\left|\int_K(u-(u_p,0))|\nabla u|^{p(x)}\psi dx\right|+\left|\int_K(u_p,0)(|\nabla u|^{p(x)}-|\nabla u_p|^{p(x)})\psi dx\right|\to 0.\end{aligned}$$

Substituting this and (3.25) into (3.23), we see that the left hand side of (3.23) becomes

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_K\left[u\psi|\nabla u|^{p(x)}+\frac{1}{\varepsilon^{p(x)}}\psi(uu_3^2-u_3e_3)\right]dx\\ &=\int_K(u_p,0)|\nabla u_p|^{p(x)}\psi dx-\lim_{\varepsilon\to 0}\int_K\frac{1}{\varepsilon^{p(x)}}\psi u_3e_3dx.\end{aligned}$$

Comparing this with the right hand side of (3.23), we have

$$\lim_{\varepsilon\to 0}\int_K\frac{1}{\varepsilon^{p(x)}}\psi u_3e_3dx=0.$$

This is (1.4). Theorem 2 is proved.

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ALMOST QUASI-YAMABE SOLITONS ON CONTACT METRIC MANIFOLDS

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Abstract. In this article, we study contact metric manifolds admitting almost quasi-Yamabe solitons (g,V,m,λ). First we prove that there does not exist a nontrivial almost quasi-Yamabe soliton whose potential vector field V is pointwise collinear with the Reeb vector field ξ on a contact metric manifold. For V being orthogonal to ξ, we consider the three dimensional cases. Next we consider a non-Sasakian contact metric (κ,μ)-manifold admitting a nontrivial closed almost quasi-Yamabe soliton and give a classification. Finally, for a closed almost quasi-Yamabe soliton on K-contact manifolds, we prove that either the soliton is trivial or $r-\lambda=m$ if $r-\lambda$ is nonnegative and attains a maximum on M, where r is the scalar curvature.

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1. Introduction

Yamabe soliton, introduced by R. Hamilton, is a Riemannian metric g of a complete Riemannian manifold (M,g) satisfying

$$\frac{1}{2}\mathcal{L}_V g=(r-\lambda)g \tag{1.1}$$

for $\lambda\in\mathbb{R}$ and a smooth vector field V, where $\mathcal{L}_V$ is the Lie derivative along V and r is the scalar curvature of M. For $\lambda=0$ the Yamabe soliton is steady, for $\lambda<0$ is expanding, and for $\lambda>0$ is shrinking. In particular, if the potential vector field V is a gradient field, the Yamabe soliton is said to be a Yamabe gradient soliton. Yamabe solitons have been studied under some conditions (cf.[4,8,11,16,18]). In the Yamabe soliton equation (1.1), if λ is a smooth function, (g,V,λ) is called an almost Yamabe soliton, introduced by E. Barbosa and E. Ribeiro in [1], and T. Seko and S. Maeta in [23] completely classified almost Yamabe solitons on hypersurfaces in Euclidean spaces.

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Later many researchers generalized the notion of Yamabe soliton. For instance, Huang and Li [17] proposed the concept of quasi-Yamabe gradient soliton, namely the Riemannian metric g satisfies the equation

$$\nabla^2 f - \frac{1}{m} df \otimes df = (r-\lambda)g \tag{1.2}$$

for some $f \in C^\infty(M)$, $\lambda \in \mathbb{R}$ and a constant $m > 0$. Such a soliton is also considered by Neto [20] and Wang [27]. V. Pirhadi and A. Razavi in [22] modified λ to be a smooth function and obtained some formulas and a necessary and sufficient condition under which an arbitrary compact almost Yamabe soliton is necessarily gradient. Furthermore, Neto-Oliveira [21] defined the generalized quasi Yamabe gradient soliton by replacing $\frac{1}{m}$ by a smooth function in equation (1.2). Recently, Blaga [2] and Chen-Deshmukh [9] studied more generalized quasi Yamabe solitons. In this article, we consider almost quasi-Yamabe soliton, which is defined as follows:

Definition 1. A Riemannian metric is said to be *almost quasi-Yamabe soliton* if there exist a constant $m > 0$, a smooth vector field V and a C^∞ function λ such that

$$\frac{1}{2}\mathcal{L}_V g - \frac{1}{m} V^\flat \otimes V^\flat = (r-\lambda)g \tag{1.3}$$

holds, where $V^\flat$ is the 1-form associated to V and r stands for the scalar curvature. Denote the almost quasi Yamabe soliton by (g, V, m, λ).

If the 1-form $V^\flat$ is closed, the almost quasi-Yamabe soliton (g, V, m, λ) is said to be *closed*. Using the terminology of Yamabe solitons, we call an almost quasi-Yamabe soliton *shrinking, steady or expanding*, respectively, if $\lambda < 0, \lambda = 0$, or $\lambda > 0$. When $V \equiv 0$, an almost quasi-Yamabe soliton is said to be *trivial*. Otherwise, it will be called *nontrivial*. It is mentioned that an almost quasi-Yamabe soliton (g, V, m, λ) is reduced to an almost Yamabe soliton when $m = \infty$. If $V = Df$ is a gradient vector field, it is called an *almost quasi-Yamabe gradient soliton*, denoted by (g, f, m, λ). Notice that equation (1.2) recovers the Yamabe gradient soliton when $m = \infty$.

For the odd-dimensional manifold, we notice that Sharma [24] proved that a 3-dimensional Sasakian manifold with a Yamabe soliton has constant scalar curvature, and V is Killing. Venkatesha-Naik [26] further generalized Sharma's results to a 3-dimensional contact metric manifold with commuting Ricci operator. For other results the reader can see [12, 13, 25, 28].

In the present paper, we consider almost quasi-Yamabe solitons on contact metric manifolds and it is organized as follows: In Section 2, we recall some definitions and related conclusions on contact metric manifolds. In Section 3, we first prove an nonexistence for a general contact metric manifold with a nontrivial almost quasi-Yamabe soliton whose potential vector field is pointwise collinear with the Reeb vector field. For V being orthogonal to the Reeb vector field, we also obtain two

results. In the following Section 4 and Section 5, we study respectively contact metric (κ,μ)-manifolds and K-contact manifolds admitting closed almost quasi-Yamabe solitons.

2. Preliminaries

Let M^{2n+1} be a $(2n+1)$-dimensional smooth manifold. If there exists a global 1-form η (called contact form) on M such that $\eta\wedge(d\eta)^n\neq 0$ everywhere, M^{2n+1} is said to be a *contact manifold*. The contact form induces a unique vector field ξ, called *Reeb vector field*, satisfying $\eta(\xi)=1$ and $d\eta(\xi,\cdot)=0$. Every contact manifold has an almost contact structure (ϕ,ξ,η), where ϕ is a $(1,1)$-tensor field such that $\phi^2=-I+\eta\otimes\xi$, $\eta\circ\phi=0$, $\phi\circ\xi=0$.

A Riemannian metric g on M can be defined by

$$d\eta(X,Y)=g(\phi X,Y),\quad g(X,\xi)=\eta(X)$$

for any $X,Y\in\mathfrak{X}(M)$. We note that the Riemannian metric g, ϕ and contact form η can be related each other by

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y).$$

We refer to (ϕ,ξ,η,g) as a contact metric structure and to the manifold M^{2n+1} carrying such a structure as a *contact metric manifold*.

We define the tensor $h=\frac{1}{2}\mathcal{L}_\xi\phi$, where $\mathcal{L}$ denotes the Lie derivative and satisfies

$$\operatorname{trace}(h)=0,\quad h\xi=0,\quad \phi h=-h\phi,\quad g(hX,Y)=g(X,hY), \tag{2.1}$$

$$\operatorname{trace}(\phi h)=0. \tag{2.2}$$

Furthermore, we also have

$$\nabla_X\xi=-\phi X-\phi hX \tag{2.3}$$

and $\nabla_\xi\phi=0$. A contact metric manifold $(M^{2n+1},\phi,\xi,\eta,g)$ for which Reeb vector field ξ is Killing, i.e. $\mathcal{L}_\xi g=0$, is called a *K-contact manifold*. If $h=0$ then we have $\mathcal{L}_\xi g=0$, that means that M^{2n+1} is a K-contact manifold. For a K-contact manifold $(M^{2n+1},\phi,\xi,\eta,g)$ the following equations were proved in [3]:

$$Q\xi=2n\xi, \tag{2.4}$$

$$R(X,\xi)\xi=-\phi^2X \tag{2.5}$$

for any vector field X on M. An almost contact structure (ϕ,ξ,η) is said to be *normal* if the corresponding complex structure J on $M\times\mathbb{R}$ is integrable. A normal contact metric manifold is said to be a *Sasakian manifold*. A contact metric manifold is Sasakian if and only if $R(X,Y)\xi=\eta(Y)X-\eta(X)Y$ for all vector fields X,Y on the manifold.

In addition, Blair-Koufogiorgos-Papantoniou [6] defined the notion of *contact metric* (κ,μ)*-manifold*, i.e. the curvature tensor of a contact metric manifold satisfies

$$R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) \tag{2.6}$$

for any vector fields X,Y, where κ,μ are constants.

The following two lemmas will be used in the sequel proofs.

Lemma 1. *For an almost quasi-Yamabe gradient soliton* (M,g,f,m,λ)*, the curvature tensor* R *can be expressed as*

$$R(X,Y)Df = -\frac{r-\lambda}{m}\{X(f)Y - Y(f)X\} + X(r-\lambda)Y - Y(r-\lambda)X \tag{2.7}$$

for any vector fields X,Y *on* M*.*

Proof. Since equation (1.2) may be exhibited as

$$\nabla_Y Df = \frac{1}{m}Y(f)Df + (r-\lambda)Y, \tag{2.8}$$

we get

$$\nabla_X\nabla_Y Df = \frac{1}{m}\{X(Y(f))Df + Y(f)\nabla_X Df)\} + X(r-\lambda)Y + (r-\lambda)\nabla_X Y.$$

Using the previous two equations, a direct calculation gives

$$\begin{aligned} R(X,Y)Df &= \nabla_X\nabla_Y Df - \nabla_Y\nabla_X Df - \nabla_{[X,Y]}Df \\ &= \frac{r-\lambda}{m}\{Y(f)X - X(f)Y\} + X(r-\lambda)Y - Y(r-\lambda)X. \end{aligned}$$

□

Lemma 2 ([10])**.** *For an almost quasi-Yamabe gradient soliton* (M^{2n+1},g,f,m,λ)*, the following equation holds:*

$$\Delta(r-\lambda) = \frac{2}{m}g(Df,D(r-\lambda)) + \frac{2n+1}{m}(r-\lambda)^2 - \frac{1}{4n}g(Df,Dr) - \frac{1}{2n}(r-\lambda)r. \tag{2.9}$$

3. Contact metric manifolds with almost quasi-Yamabe solitons

Theorem 1. *There does not exist a nontrivial almost quasi-Yamabe soliton* (g,V,m,λ) *with* $V = \eta(V)\xi$ *on a contact metric manifold.*

Proof. We set $V = F\xi$ for a non-zero function F. By (2.3), we have

$$\nabla_X V = X(F)\xi - F(\phi X + \phi hX). \tag{3.1}$$

Using (3.1), formula (1.3) becomes

$$(r-\lambda)g(X,Y) - \frac{1}{2}(X(F)\eta(Y) + Y(F)\eta(X)) + Fg(\phi hX,Y) + \frac{F^2}{m}\eta(X)\eta(Y) = 0. \tag{3.2}$$

Now replacing X and Y by ϕX and ϕY, respectively, implies

$$(r-\lambda)\phi X - FhX = 0.$$

Taking the inner product of the above relation with ϕX and contracting over X, we get $r-\lambda = 0$ by (2.2), which further implies $h = 0$ by the previous relation.

Now letting $Y = \xi$ in (3.2) gives

$$\Big(-\xi(F) + \frac{2F^2}{m}\Big)\eta(X) = X(F).$$

Further putting $X = \xi$ implies $\xi(F) = \frac{F^2}{m}$. Thus the above relation yields $DF = \frac{F^2}{m}\xi$. For any vector fields X, Y, it follows from (2.3) that

$$g(\nabla_X DF, Y) = g(2F\frac{X(F)}{m}\xi - \frac{F^2}{m}\phi X, Y).$$

Since $g(\nabla_X DF, Y) = g(\nabla_Y DF, X)$, we have

$$2F\frac{X(F)}{m}\eta(Y) - 2F\frac{Y(F)}{m}\eta(X) = \frac{2F^2}{m}g(\phi X, Y).$$

Replacing X and Y by ϕX and ϕY, respectively, we deduce $F = 0$, which is a contradiction. We thus complete the proof. □

For V being orthogonal to the Reeb vector field ξ, we intend to consider a three dimensional non-Sasakian contact metric manifold (i.e. $h \neq 0$). It is well-known that there exits a local orthonormal frame field $\mathcal{E} = \{e, \phi e, \xi\}$ such that $he = \nu e$ and $h\phi e = -\nu\phi e$, where ν is a positive non-vanishing smooth function of M.

First of all, we have the following lemma:

Lemma 3 ([15]). *In the open subset U, the Levi-Civita connection ∇ is given by*

$$\begin{aligned}
&\nabla_\xi e = a\phi e, && \nabla_\xi \phi e = -ae, && \nabla_\xi \xi = 0,\\
&\nabla_e \xi = -(1+\nu)\phi e, && \nabla_e e = b\phi e, && \nabla_e \phi e = -be + (1+\nu)\xi,\\
&\nabla_{\phi e}\xi = (1-\nu)e, && \nabla_{\phi e}\phi e = ce, && \nabla_{\phi e} e = -c\phi e + (\nu-1)\xi,
\end{aligned}$$

where a is a smooth function,

$$b = \frac{1}{2\nu}[\phi e(\nu) + A] \quad \textit{with} \quad A = Ric(e,\xi), \tag{3.3}$$

$$c = \frac{1}{2\nu}[e(\nu) + B] \quad \textit{with} \quad B = Ric(\phi e,\xi). \tag{3.4}$$

The components of Ricci operator Q are given by

$$\begin{cases}
Qe = \Big(\frac{1}{2}r - 1 + \nu^2 - 2a\nu\Big)e + \xi(\nu)\phi e + A\xi,\\
Q\phi e = \xi(\nu)e + \Big(\frac{1}{2}r - 1 + \nu^2 + 2a\nu\Big)\phi e + B\xi,\\
Q\xi = Ae + B\phi e + 2(1-\nu^2)\xi.
\end{cases} \tag{3.5}$$

The scalar curvature

$$r = \mathrm{trace}(Q) = 2(1-\nu^2-b^2-c^2+2a+e(c)+\phi e(b)). \tag{3.6}$$

Moreover, it follows from Lemma 3 that

$$\begin{cases} [e,\phi e] = \nabla_e \phi e - \nabla_{\phi e} e = -be + c\phi e + 2\xi, \\ [e,\xi] = \nabla_e \xi - \nabla_\xi e = -(a+\nu+1)\phi e, \\ [\phi e,\xi] = \nabla_{\phi e}\xi - \nabla_\xi \phi e = (a-\nu+1)e. \end{cases} \tag{3.7}$$

Theorem 2. *If a non-Sasakian contact metric manifold admits a non-trivial almost quasi-Yamabe gradient soliton (g,f,m,λ) whose potential vector field is orthogonal to the Reeb vector field, then (g,f,m,λ) is a steady quasi-Yamabe gradient soliton and M is locally isometric to $E(2)$.*

Proof. Since the potential vector field Df is orthogonal to ξ, we may write $Df = f_1 e + f_2 \phi e$, where f_1, f_2 are two smooth functions on M. For any vector field X, equation (1.2) may be expressed as

$$\nabla_X Df - \frac{1}{m} X(f) Df = (r-\lambda)X. \tag{3.8}$$

Choosing $X = \xi$ in (3.8) and using Lemma 3, we have

$$(\xi(f_1) - f_2 a)e + (\xi(f_2) + f_1 a)\phi e = (r-\lambda)\xi.$$

This shows

$$r = \lambda, \quad \xi(f_1) - f_2 a = 0, \quad \xi(f_2) + f_1 a = 0. \tag{3.9}$$

Similarly, putting $X = e$ in (3.8) and using Lemma 3, we obtain

$$e(f_1) - b f_2 - \frac{1}{m} f_1^2 = 0, \tag{3.10}$$

$$e(f_2) + b f_1 - \frac{1}{m} f_1 f_2 = 0, \tag{3.11}$$

$$(1+\nu) f_2 = 0. \tag{3.12}$$

Putting $X = \phi e$ in (3.8) and using Lemma 3, we obtain

$$\phi e(f_1) + c f_2 - \frac{1}{m} f_1 f_2 = 0, \tag{3.13}$$

$$\phi e(f_2) - c f_1 - \frac{1}{m} f_2^2 = 0, \tag{3.14}$$

$$0 = (\nu - 1) f_1. \tag{3.15}$$

Since Df is nonzero and $\nu > 0$, we know $f_2 = 0$ and $\nu = 1$ from (3.12) and (3.15). Moreover, we deduce from (3.10), (3.14) and the third term of (3.9) that $a = b = c = 0$. Because $\nu = 1$, it follows from (3.3) and (3.4) that $A = B = 0$. This implies $Q\xi = 0$. Making use of (3.6) we obtain $r = 0$. Moreover, (3.20) becomes

$$[e,\phi e] = 2\xi, \quad [\phi e,\xi] = 0, \quad [\xi, e] = 2\phi e.$$

We complete the proof by Milnor's classification theorem ([19]). □

Theorem 3. *Let M^3 be non-Sasakian contact metric manifold with $Q\phi = \phi Q$. If (g,V,m,λ) is a nontrivial almost quasi-Yamabe soliton whose potential vector field V is orthogonal to ξ, then (g,V,m,λ) is a steady quasi-Yamabe soliton and M is flat.*

Proof. We write $V = f_1 e + f_2 \phi e$, where f_1, f_2 are two smooth functions on M. By the assumpitons, we have $A = B = 0$ and $\xi(\nu) = a = 0$ (see [14, Proposition 2.5]). Therefore (3.5) becomes

$$\begin{cases} Qe = \left(\frac{1}{2}r - 1 + \nu^2\right)e, \\ Q\phi e = \left(\frac{1}{2}r - 1 + \nu^2\right)\phi e, \\ Q\xi = 2(1-\nu^2)\xi. \end{cases}$$

Next making use of the above formulas and Lemma 3 we compute

$$\begin{aligned} (\nabla_\xi Q)\xi &= -2\xi(\nu^2)\xi = 0, \\ (\nabla_e Q)e &= \nabla_e(Qe) - Q\nabla_e e = e\left(\frac{1}{2}r - 1 + \nu^2\right)e, \\ (\nabla_{\phi e} Q)\phi e &= \nabla_{\phi e}(Q\phi e) - Q\nabla_{\phi e}\phi e = \phi e\left(\frac{1}{2}r - 1 + \nu^2\right)\phi e. \end{aligned}$$

Since $\frac{1}{2}Dr = \mathrm{div}Q$, we obtain

$$e(\nu) = 0, \quad \phi e(\nu) = 0.$$

That shows $b = c = 0$ from (3.3) and (3.4).

For any vector fields X, Y, equation (1.3) may be expressed as

$$g(\nabla_X V, Y) + g(\nabla_Y V, X) - \frac{2}{m}V(X)V(Y) = 2(r-\lambda)g(X,Y). \tag{3.16}$$

Letting $X = Y = \xi$ in (3.16) gives $r = \lambda$. Putting $X = e$ and $Y = \xi$ in (3.16) and using Lemma 3, we obtain

$$\xi(f_1) + f_2(1+\nu) = 0. \tag{3.17}$$

Putting $X = \phi e$ and $Y = \xi$ and using Lemma 3, we obtain

$$\xi(f_2) + f_1(\nu - 1) = 0. \tag{3.18}$$

Choosing $X = e$ and $Y = \phi e$ we get

$$\phi e(f_1) + e(f_2) - \frac{2}{m}f_1 f_2 = 0. \tag{3.19}$$

On the other hand, since $a = b = c = 0$, the Lie bracket (3.7) may be expressed as

$$[e,\xi] = -(1+\nu)\phi e, \quad [\phi e, \xi] = (1-\nu)e. \tag{3.20}$$

Applying the first term of (3.20) on f_2 and using (3.18), we obtain

$$\xi(e(f_2)) - (1+\nu)\phi e(f_2) = -e(f_1)(\nu - 1).$$

Applying the second term of (3.20) on f_1 and using (3.17), we obtain

$$\xi(\phi e(f_1)) + (1-\nu)e(f_1) = \phi e(f_2)(-1-\nu).$$

Therefore, the previous two equations together with (3.19) give

$$\xi(f_1 f_2) = 0.$$

Using (3.17) and (3.18), we thus derive

$$f_2^2(\nu+1) + f_1^2(\nu-1) = 0.$$

Differentiating this along ξ and using (3.17) and (3.18) again, we have

$$\nu^2 - 1 = 0.$$

This shows that $\nu = 1$ and (3.6) yields $\lambda = r = 0$. Moreover, it is clear that $Q\xi = 0$. We complete the proof by [5, Remark 3.1]. □

4. Contact metric (κ,μ)-manifolds with closed almost quasi-Yamabe solitons

In this section we suppose that $(M^{2n+1}, \phi, \xi, \eta, g)$ is a contact metric (κ,μ)-manifold, namely the curvature tensor satisfies (2.6). Furthermore, the following relations are provided (see [6]) :

$$QX = (2(n-1) - n\mu)X + (2(n-1)+\mu)hX + (n(2\kappa+\mu) - 2(n-1))\eta(X)\xi, \quad (4.1)$$

$$h^2 = (\kappa-1)\phi^2. \quad (4.2)$$

Using (2.1), it follows from (4.1) that the scalar curvature $r = 2n(2(n-1)+\kappa-n\mu)$ and $Q\xi = 2n\kappa\xi$. By (4.2), we find easily that $\kappa \leq 1$ and $\kappa = 1$ if and only if M is a Sasakian manifold. In particular, for $\kappa = \mu = 0$, Blair proved the following result.

Theorem 4 ([3, Theorem 7.5]). *A contact metric manifold M^{2n+1} satisfying $R(X,Y)\xi = 0$ is locally isometric to $E^{n+1} \times S^n(4)$ for $n > 1$ and flat for $n = 1$.*

For a non-Sasakian (κ,μ)-manifold M, Boeckx [7] introduced an invariant

$$I_M = \frac{1-\frac{\mu}{2}}{\sqrt{1-\kappa}}$$

and proved the following conclusion:

Theorem 5 ([7, Corollary 5]). *Let M be a non-Sasakian (κ,μ)-manifold. Then it is locally isometric, up to a D-homothetic transformation, to the unit tangent sphere bundle of some space of constant curvature (different from 1) if and only if $I_M > -1$.*

Making use of the above theorems we obtain

Theorem 6. *Let M^{2n+1} be a non-Sasakian (κ,μ)-manifold. If M admits a non-trivial closed almost quasi-Yamabe soliton (g,V,m,λ), then M is flat for $n=1$ and for $n>1$, M is either locally isometric, up to a D-homothetic transformation, to the unit tangent sphere bundle of some space of constant curvature, or locally isometric to $E^{n+1}\times S^n(4)$.*

Proof. In view of equation (1.3), we obtain

$$\nabla_Y V=(r-\lambda)Y+\frac{1}{m}g(V,Y)V \tag{4.3}$$

for any vector Y. Since the scalar curvature $r=2n(2(n-1)+\kappa-n\mu)$ is constant, using (4.3) we compute

$$\begin{aligned}R(X,Y)V&=\nabla_X\nabla_Y V-\nabla_Y\nabla_X V-\nabla_{[X,Y]}V\\&=-X(\lambda)Y+\frac{1}{m}g(\nabla_X V,Y)V+\frac{1}{m}g(V,Y)\nabla_X V\\&\quad+Y(\lambda)X-\frac{1}{m}g(\nabla_Y V,X)V-\frac{1}{m}g(V,X)\nabla_Y V\\&=Y(\lambda)X-X(\lambda)Y+\frac{r-\lambda}{m}g(V,Y)X-\frac{r-\lambda}{m}g(V,X)Y.\end{aligned} \tag{4.4}$$

Taking an inn product of the above formula with ξ and using (2.6), we have

$$\begin{aligned}&-\kappa(\eta(Y)g(X,V)-\eta(X)g(Y,V))-\mu(\eta(Y)g(hX,V)-\eta(X)g(hY,V))\\&=\frac{r-\lambda}{m}[g(V,Y)\eta(X)-g(V,X)\eta(Y)]+Y(\lambda)\eta(X)-X(\lambda)\eta(Y).\end{aligned}$$

Now replacing X and Y by ϕX and ξ, respectively, yields

$$\kappa g(\phi X,V)+\mu g(h\phi X,V)=\phi X(\lambda)+\frac{r-\lambda}{m}g(\phi X,V)$$

for any vector field X. This is equivalent to

$$\Big(\kappa-\frac{r-\lambda}{m}\Big)\phi V+\mu\phi hV=\phi D\lambda. \tag{4.5}$$

On the other hand, contracting (4.4) over Y and using (4.1) we obtain

$$\begin{aligned}&(2(n-1)-n\mu)V+(2(n-1)+\mu)hV+(n(2\kappa+\mu)-2(n-1))\eta(V)\xi\\&=2n\Big(D\lambda+\frac{r-\lambda}{m}V\Big).\end{aligned} \tag{4.6}$$

Now applying ϕ in this formula implies

$$(2(n-1)-n\mu)\phi V+(2(n-1)+\mu)\phi hV=2n\Big(\phi D\lambda+\frac{r-\lambda}{m}\phi V\Big), \tag{4.7}$$

which, combining with (4.5), gives

$$\Big\{(2(n-1)-n\mu)\mu-(2(n-1)+\mu)\Big(\kappa-\frac{r-\lambda}{m}\Big)-2n\mu\frac{r-\lambda}{m}\Big\}\phi V$$

$$= \Big(2n\mu - 2(n-1) - \mu\Big)\phi D\lambda, \tag{4.8}$$

implying

$$\Big\{(2(n-1)-n\mu)\mu - (2(n-1)+\mu)\Big(\kappa - \frac{r-\lambda}{m}\Big) - 2n\mu\frac{r-\lambda}{m}\Big\}V$$
$$-\Big(2n\mu - 2(n-1) - \mu\Big)D\lambda \in \mathbb{R}\xi.$$

Case I. If $d := 2n\mu - 2(n-1) - \mu = 0$, i.e. $\mu = \frac{2(n-1)}{2n-1}$, then from (4.8) we know

$$(2(n-1)-n\mu)\mu - (2(n-1)+\mu)\Big(\kappa - \frac{r-\lambda}{m}\Big) - 2n\mu\frac{r-\lambda}{m} = 0.$$

That is, $\kappa = \frac{(n-1)^2}{n(2n-1)}$ for $n > 1$. Clearly, in this case we have

$$I_M = \frac{1-\frac{\mu}{2}}{\sqrt{1-\kappa}} = \frac{1-\frac{2(n-1)}{2n-1}}{\sqrt{1-\kappa}} = \frac{1}{(2n-1)\sqrt{1-\kappa}} > -1.$$

For $n = 1$, we have $\mu = 0$. Equation (4.6) is simplified as

$$\kappa\eta(V)\xi = D\lambda + \frac{r-\lambda}{m}V,$$

which, combining with (4.5), yields $\kappa = 0$. Here we have used the conclusion that $\phi V \neq 0$, which is obtained by Theorem 1.

Case II. If $d = 2n\mu - 2(n-1) - \mu \neq 0$ then we can write

$$D\lambda = cV + s\xi, \tag{4.9}$$

where

$$c = \frac{1}{d}\Big\{(2(n-1)-n\mu)\mu - (2(n-1)+\mu)\Big(\kappa - \frac{r-\lambda}{m}\Big) - 2n\mu\frac{r-\lambda}{m}\Big\}$$

and s is a smooth function. By (4.6), we have

$$(2(n-1)+\mu)hV + (n(2\kappa+\mu) - 2(n-1))\eta(V)\xi$$
$$= \Big\{2n\Big(c + \frac{r-\lambda}{m}\Big) - (2(n-1)-n\mu)\Big\}V + 2ns\xi. \tag{4.10}$$

Now applying h in this formula and recalling (4.2) imply

$$(2(n-1)+\mu)(\kappa-1)\phi^2 V = \Big\{2n\Big(c + \frac{r-\lambda}{m}\Big) - (2(n-1)-n\mu)\Big\}hV. \tag{4.11}$$

Combining (4.10) with (4.11) we get

$$\Big[\Big\{2n\Big(c + \frac{r-\lambda}{m}\Big) - (2(n-1)-n\mu)\Big\}\Big(n(2\kappa+\mu) - 2(n-1)\Big)$$
$$+ (2(n-1)+\mu)^2(\kappa-1)\Big]\eta(V)\xi - 2ns\Big\{2n\Big(c + \frac{r-\lambda}{m}\Big) - (2(n-1)-n\mu)\Big\}\xi$$

$$= \Big[\Big\{2n\Big(-c+\frac{r-\lambda}{m}\Big)-(2(n-1)-n\mu)\Big\}^2+(2(n-1)+\mu)^2(\kappa-1)\Big]V. \tag{4.12}$$

Since $\phi V \neq 0$, this implies that

$$\Big\{2n\Big(c+\frac{r-\lambda}{m}\Big)-(2(n-1)-n\mu)\Big\}^2+(2(n-1)+\mu)^2(\kappa-1)=0, \tag{4.13}$$

then λ is constant. Hence equation (4.5) becomes

$$\Big(\kappa-\frac{r-\lambda}{m}\Big)\phi V+\mu\phi hV=0. \tag{4.14}$$

Furthermore, from (4.9) we find $s=c=0$ since $V\notin\mathbb{R}\xi$. Thus it follows from (4.12) that

$$\Big\{2n\Big(\frac{r-\lambda}{m}\Big)-(2(n-1)-n\mu)\Big\}\Big(\kappa-\frac{r-\lambda}{m}\Big)\eta(V)=0. \tag{4.15}$$

II(a). When $2(n-1)+\mu\neq 0$, it follows from (4.13) and (4.15) that

$$\Big(\kappa-\frac{r-\lambda}{m}\Big)\eta(V)=0. \tag{4.16}$$

Consequently, either $\kappa-\frac{r-\lambda}{m}=0$ or $\eta(V)=0$. If $\kappa-\frac{r-\lambda}{m}=0$ then (4.14) yields $\mu=0$. This implies $n>1$ and equation (4.13) becomes

$$n^2\kappa^2-(n^2-1)\kappa=0,$$

i.e. $\kappa=0$ or $\kappa=\frac{n^2-1}{n^2}$. Since $\kappa<1$, relation $\kappa=\frac{n^2-1}{n^2}$ does not hold.

If $\eta(V)=0$, we differentiate this along ξ and obtain $r=\lambda$ by (4.3). Hence equations (4.14) and (4.7) respectively become

$$\kappa\phi V+\mu\phi hV=0 \quad \text{and} \quad (2(n-1)-n\mu)\phi V+(2(n-1)+\mu)\phi hV=0.$$

Using ϕ to act on the above relations yields

$$\begin{aligned}&\kappa V+\mu hV=0,\\&(2(n-1)-n\mu)V+(2(n-1)+\mu)hV=0.\end{aligned}$$

Thus

$$\begin{cases}\kappa+\mu\sqrt{1-\kappa}=0,\\(2(n-1)-n\mu)+(2(n-1)+\mu)\sqrt{1-\kappa}=0.\end{cases}$$

or

$$\begin{cases}\kappa-\mu\sqrt{1-\kappa}=0,\\(2(n-1)-n\mu)-(2(n-1)+\mu)\sqrt{1-\kappa}=0.\end{cases}$$

Here $\sqrt{1-\kappa}$ is an eigenvalue of h. The above two cases shall lead to

$$\frac{2(n-1)(1+\sqrt{1-\kappa})}{\sqrt{1-\kappa}-n}=\frac{\kappa}{\sqrt{1-\kappa}},$$

i.e.

$$x^3+(n-2)x^2+(2n-3)x+n=0,$$

where $x=\sqrt{1-\kappa}$. Clearly, the above relation does not hold for $n>1$ since $x>0$.

II(b). When $2(n-1)+\mu=0$ then equation (4.13) yields

$$n^2-1=n\frac{r-\lambda}{m}. \tag{4.17}$$

Substituting this into (4.14) yields

$$-(1-\kappa)-\frac{n^2-n-1}{n}=2(n-1)\sqrt{1-\kappa}.$$

If $n=1$ then $\kappa=\mu=0$. For $n>1$, the above relation is impossible since $\kappa<1$. In this case, M is flat.

Summing up the above discussion, we proved that $\kappa=\mu=0$ for $n=1$ and for $n>1$, either $\kappa=\mu=0$ or $I_M>-1$. Therefore we complete of the proof by Theorem 4 and Theorem 5. □

When $V=Df$, it is clear that $V^\flat$ is closed, thus we have

Corollary 1. *A non-Sasakian contact metric (κ,μ)-manifold M^{2n+1}, admitting a nontrivial quasi-Yamabe gradient soliton (g,f,m,λ), is locally isomorphic to $E^{n+1}\times S^n(4)$ for $n>1$ and flat for $n=1$.*

Proof. Since $r=2n(2(n-1)+\kappa-n\mu)$ and λ are constants, by (2.9) we have

$$0=\frac{2n+1}{m}(r-\lambda)^2-\frac{1}{2n}(r-\lambda)r.$$

If $\lambda\neq r$ then

$$0=2n(2n+1)(r-\lambda)-mr.$$

Equation (4.5) becomes

$$\Big(\kappa-\frac{r}{2n(2n+1)}\Big)\phi V+\mu\phi hV=0. \tag{4.18}$$

For Case I in the proof of Theorem 6, $\kappa=\frac{(n-1)^2}{n(2n-1)}$ and $\mu=\frac{2(n-1)}{2n-1}$. A direct computation yields $\kappa-\frac{r}{2n(2n+1)}=0$. This implies from (4.18) that $\kappa=\mu=0$ and $n=1$. For Case II(a), since $r\neq\lambda$ we see $\eta(V)\neq 0$. Thus in this case we also have $\kappa=\mu=0$.

When $r=\lambda$, for Case I in the proof of Theorem 6, equation (4.7) is simplified as

$$(n-1)V+2nhV=0.$$

Using h to act on this and recalling (4.2), we obtain

$$(n-1)hV+2n\Big(1-\frac{(n-1)^2}{n(2n-1)}\Big)V=0.$$

Thus the previous two formulas yields

$$(n-1)^2-4n^2\Big(1-\frac{(n-1)^2}{n(2n-1)}\Big)=0,\ \text{i.e. } 2n^3+9n^2-8n+1=0.$$

Obviously, it is impossible.

For Case II, it follows from (4.5) that if $\kappa = 0$ then $\mu = 0$. If $\kappa \neq 0$ then $\eta(V) = 0$ by (4.16). Thus from Case II(a) in the proof of Theorem 6, we know that is impossible.

Summarizing the above discussion, we know $\kappa = \mu = 0$. Thus the desired conclusion is proved by Theorem 4. □

5. K-CONTACT MANIFOLDS WITH CLOSED ALMOST QUASI-YAMABE SOLITONS

In this section we study a K-contact manifold admitting a closed almost quasi-Yamabe soliton.

Theorem 7. *Let (g,V,m,λ) be a closed almost quasi-Yamabe soliton on a K-contact manifold M^{2n+1}. If $r-\lambda$ is nonnegative and attains a maximum in M then either the soliton is trivial or $r-\lambda = m$. Moreover, if M is compact, then (g,V,m,λ) is trivial under the condition that $r-\lambda$ is nonnegative.*

Proof. As $V^\flat$ is closed, equation (1.3) is equivalent to

$$\nabla_Y V = (r-\lambda)Y + \frac{1}{m}g(V,Y)V. \tag{5.1}$$

Via this formula one derives easily

$$\begin{aligned} R(X,Y)V &= \nabla_X\nabla_Y V - \nabla_Y\nabla_X V - \nabla_{[X,Y]}V \\ &= X(r-\lambda)Y + \frac{1}{m}g(V,Y)(r-\lambda)X - Y(r-\lambda)X - \frac{1}{m}g(V,X)(r-\lambda)Y. \end{aligned} \tag{5.2}$$

By (2.5), taking an inner product of (5.2) with ξ gives

$$X(r-\lambda) + \frac{r-\lambda}{m}\eta(V)\eta(X) - \xi(r-\lambda)\eta(X) - \frac{r-\lambda}{m}g(V,X) = g(\phi^2 X, V).$$

Now replacing X by ϕX yields

$$\phi D(r-\lambda) - \frac{r-\lambda-m}{m}\phi V = 0, \quad \text{i.e.} \quad D(r-\lambda) - \frac{r-\lambda-m}{m}V \in \mathbb{R}\xi.$$

We write $D(r-\lambda) - \frac{r-\lambda-m}{m}V = c\xi$ for some function c on M.

On the other hand, contracting (5.2) over X yields

$$2n\Big(-D(r-\lambda) + \frac{r-\lambda}{m}V\Big) = QV.$$

Since $Q\xi = 2n\xi$, the previous two formulas imply $c = g(D(r-\lambda) - \frac{r-\lambda-m}{m}V, \xi) = 0$, i.e.

$$D(r-\lambda) = \frac{r-\lambda-m}{m}V. \tag{5.3}$$

Differentiating (5.3) along X yields

$$\frac{X(r-\lambda)}{m}V + \frac{r-\lambda-m}{m}\nabla_X V = \nabla_X D(r-\lambda).$$

Further, contracting this over X gives

$$\frac{V(r-\lambda)}{m}+\frac{r-\lambda-m}{m}\mathrm{div}V=\Delta(r-\lambda). \tag{5.4}$$

Since $\Delta(r-\lambda-m)^2=2|D(r-\lambda-m)|^2+2(r-\lambda-m)\Delta(r-\lambda-m)$ and

$$\mathrm{div}V=(2n+1)(r-\lambda)+\frac{1}{m}|V|^2 \tag{5.5}$$

obtained from (5.1), it follows from (5.3) and (5.4) that

$$\Delta(r-\lambda-m)^2=6|D(r-\lambda-m)|^2+2(r-\lambda-m)^2\frac{(2n+1)(r-\lambda)}{m}.$$

If $r-\lambda$ is nonnegative and attains a maximum in M then $r-\lambda=m$ or $r-\lambda=0$. For $r-\lambda=0$, equation (5.3) yields $V=0$, i.e. the soliton is trivial.

If M is compact, it is easy to get from (5.5) that $V=0$ under the assumption that $r-\lambda$ is nonnegative. □

If $V=Df$, equation (5.3) becomes

$$\frac{r-\lambda-m}{m}Df=D(r-\lambda). \tag{5.6}$$

In view of (2.8), we know

$$\Delta f=\frac{1}{m}|Df|^2+(2n+1)(r-\lambda).$$

Inserting this into equation (5.4), we get

$$\frac{1}{m}g(Df,D(r-\lambda))+\frac{r-\lambda-m}{m}\left[(2n+1)(r-\lambda)+\frac{1}{m}|Df|^2\right]=\Delta(r-\lambda). \tag{5.7}$$

Making use of (2.9) and (5.6), it follows from (5.7) that

$$(4n(2n+1)-2r)(r-\lambda)=g(Df,Dr). \tag{5.8}$$

Thus the following conclusion is clear from (5.8) and (5.6).

Corollary 2. *Let M^{2n+1} be a K-contact manifold with an almost quasi-Yamabe gradient soliton (g,f,m,λ). If the scalar curvature r is constant then either (g,f,m,λ) is trivial, or $r=2n(2n+1)$.*

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SOME ALGEBRAIC PROPERTIES OF GENERALIZED $q-$ CESÀRO MATRIX $C_g(q)$

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Abstract. In this paper, we define the generalized q-Cesàro matrix by using generalized Cesàro matrix and q-Cesàro matrix. We investigate normality, self-adjointedness and Hilbert-Schmidt properties of generalized q-Cesàro matrices.

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1. Introduction

In [1], Bustoz and Gordillo defined the q-Cesàro matrix. For $0<q<1$, q-Cesàro matrix $C_1(q)$ is defined by

$$c_{nk}=\begin{cases} \frac{q^{n-k}}{1+q+\cdots+q^n} & , \quad 0<n\leq k \\ 0 & , \quad n<k \end{cases}.$$

Since $c_{nk}(q)\to\dfrac{1}{n+1}$, for $q\to 1$, the q-analog of $C_1=(c_{nk})$ is matrix $C_1(q)$ where the entries of Cesàro matrix are

$$c_{nk}=\begin{cases} \frac{1}{n+1} & , \quad 0<k\leq n \\ 0 & , \quad n<k \end{cases}$$

and $q-$analog of the integer n is

$$[n]_q=\frac{1-q^n}{1-q}, \quad (q\neq 1).$$

In [3] Durna and Türkay obtained the spectrum of q-Cesàro matrix $C_1(q)$ on the space of convergent sequences c. Also, they calculated spectral decomposition in the sense of Goldberg. In [6] El Shabrawy investigated boundedness, compactness and spectra of q-Cesàro matrix $C_1(q)$.

Let us denote the space $H(\mathbb{D})=\{f:\mathbb{D}\to\mathbb{C}\text{: } f \text{ is an analytic function}\}$, on $\mathbb{D}$ the standard Hardy space H^p $(1\leq p<\infty)$ where $\mathbb{D}$ is a unit disk, and p-summable

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complex-valued sequences on the set of non-negative integers ℓ^p. You can see [7] for details.

Previously, some authors have generalized various matrix transformations. For example, Young generalized the Cesàro matrix in his PhD thesis ([10]). For all $f \in H^2$ the generalized Cesàro operator with the symbol g is defined as

$$C_g(f)(z) = \frac{1}{z}\int_0^z f(t)\,g(t)\,dt.$$

Here, g is analytic in $\mathbb{D}$ and has Taylor series representation $g(z) = \sum_{j=0}^{\infty} a_j z^j$, where $\left\{z^{n-1}\right\}_{n=1}^{\infty}$ is the standard basis of H^2. The matrix form of C_g in the standard basis is

$$C_g = \begin{pmatrix} a_0 & 0 & 0 & 0 & \cdots \\ \frac{a_1}{2} & \frac{a_0}{2} & 0 & 0 & \cdots \\ \frac{a_2}{3} & \frac{a_1}{3} & \frac{a_0}{3} & 0 & \cdots \\ \frac{a_3}{4} & \frac{a_2}{4} & \frac{a_1}{4} & \frac{a_0}{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

And so, one can express entries of matrix C_g as follows:

$$(c_g)_{nj} = \begin{cases} \frac{a_{n-j}}{n} & , \quad n \geq j \\ 0 & , \quad n < j \end{cases}.$$

For all $0 < q < 1$ we can define the generalized $q-$Cesàro matrix $C_g(q)$ by

$$C_g(q) = \begin{pmatrix} a_0 & 0 & 0 & 0 & \cdots \\ \frac{q}{1+q}a_1 & \frac{1}{1+q}a_0 & 0 & 0 & \cdots \\ \frac{q^2}{1+q+q^2}a_2 & \frac{q}{1+q+q^2}a_1 & \frac{1}{1+q+q^2}a_0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and we have

$$(c_q(q))_{nk} = \begin{cases} \frac{q^{n-k}}{1+q+\cdots+q^{n-1}}a_{n-k} & , \quad 0 \leq k \leq n \\ 0 & , \quad n < k \end{cases}. \tag{1.1}$$

Since $\lim_{q\to 1^-}(c_g(q))_{nk} = \frac{a_{n-k}}{n}$, matrix $C_g(q)$ is a $q-$analog of generalized Cesàro matrix C_g. If we take $g(z) = \frac{1}{1-z}$ and limit for $q \to 1$, then we obtain $C_g(q) = C(q)$. Thus generalized q-Cesàro matrix contains both of generalized Cesàro matrix and ordinary Cesàro matrix.

The conjugate transpose of the matrix $C_g(q)$ is

$$C_g^*(q) = \begin{pmatrix} \overline{a_0} & \frac{q}{1+q}\overline{a_1} & \frac{q^2}{1+q+q^2}\overline{a_2} & \cdots \\ 0 & \frac{1}{1+q}\overline{a_0} & \frac{q}{1+q+q^2}\overline{a_1} & \cdots \\ 0 & 0 & \frac{1}{1+q+q^2}\overline{a_0} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and so,

$$\left(c_g^*(q)\right)_{nk} = \begin{cases} \frac{q^{n-k}}{1+q+\cdots+q^{n-1}}\overline{a_{n-k}} & , \quad 0 \le n \le k \\ 0 & , \quad k < n \end{cases} . \tag{1.2}$$

Inspired by the relationship between Rhaly and Cesàro matrices, Durna and Yıldırım in [4] defined the generalized Rhaly matrix using the generalized Cesàro matrix. They also investigated the topological properties of the generalized Rhaly matrix. In [5], Durna and Yıldırım show that if an infinite matrix B is commutative with every generalized terraced matrix, then $B = kI$, where I is a unit matrix and $k \in \mathbb{C}$. Also, they investigated normality and self-adjointedness of generalized terraced matrix. Now, let us examine normality and self-adjointedness of generalized q-Cesàro matrix $C_g(q)$.

2. Results

Theorem 1. *$C_g(q)$ is a normal matrix iff for $c \in \mathbb{C}$, $g(z) = c$.*

Proof. We consider $(1,1)$ - entries of $C_g(q)C_g^*(q)$ and $C_g^*(q)C_g(q)$:

$$\begin{aligned} \left[C_g(q)C_g^*(q)\right]_{11} &= \sum_{k=1}^{\infty} (c_g(q))_{1k} \left(c_g^*(q)\right)_{k1} \\ &= (c_g(q))_{11} \left(c_g^*(q)\right)_{11} \\ &= a_0\overline{a_0} = |a_0|^2 \end{aligned}$$

and

$$\begin{aligned} \left[C_g^*(q)C_g(q)\right]_{11} &= \sum_{k=1}^{\infty} \left(c_g^*(q)\right)_{k1} (c_g(q))_{1k} \\ &= a_0\overline{a_0} + \sum_{k=2}^{\infty} \frac{q^{k-1}}{1+q+\cdots+q^{k-1}}\overline{a_{k-1}} \frac{q^{k-1}}{1+q+\cdots+q^{k-1}} a_{k-1} \\ &= |a_0|^2 + \sum_{k=2}^{\infty} \left(\frac{q^{k-1}}{1+q+\cdots+q^{k-1}}\right)^2 |a_{k-1}|^2 . \end{aligned}$$

Since $C_g(q)$ is a normal matrix, we get $\sum_{k=2}^{\infty} \left(\frac{q^{k-1}}{1+q+\cdots q^{k-1}}\right)^2 |a_{k-1}|^2 = 0$. From here, we obtain that for all $i > 1$, $a_i = 0$. And so, $g(z) = a_0$, $a_0 \in \mathbb{C}$.

Conversely, if $g(z) = a_0$, $a_0 \in \mathbb{C}$ then we have

$$C_g(q) = \begin{pmatrix} a_0 & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{1+q}a_0 & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{1+q+q^2}a_0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

hence $C_g(q)$ is normal. □

Corollary 1. *C_1 is not normal.*

Corollary 2. *$C_1(q)$ is not normal.*

Theorem 2. *$C_g(q)$ is a self-adjoint matrix iff for $c \in \mathbb{R}$, $g(z) = c$.*

Proof. From (1.1) and (1.2), $a_0 = \overline{a_0}$ and

$$\frac{q}{1+q}a_1 = \frac{q^2}{1+q+q^2}a_2 = \frac{q^3}{1+q+q^2+q^3}a_3 \cdots = 0.$$

Since, for all $n \in \mathbb{N}$, $\frac{q^n}{1+q+\cdots+q^n} \neq 0$, then

$$a_0 = \overline{a_0}\,,\ a_1 = a_2 = a_3 = \cdots = 0.$$

Hence $a_0 \in \mathbb{R}$ and $g(z) = a_0 \in \mathbb{R}$.

The other direction is obvious. □

Corollary 3. *C_1 is not self-adjoint.*

Corollary 4. *$C_1(q)$ is not self-adjoint.*

Note that if $n \geq 0$ and if $T_{ij} = 0$ for $j < i+n$, we say that T is an n-triangular operator matrix ([9]).

Lemma 1 ([9, Theorem 2])**.** *Let $T = (T_{ij}) \in \mathcal{L}\left(\bigoplus_{n=0}^{\infty} \mathcal{H}\right)$ be a $1-$triangular operator matrix. Suppose that $T_{i,i+1}$ has dense range for all i. Then T is cyclic.*

Lemma 2 ([2, Proposition 3.6])**.** *Let T be a triangular operator whose diagonal entries with respect to some orthonormal basis for $\mathcal{H}$ are distinct. Then T is cyclic.*

Theorem 3. *$C_g^*(q)$ is cyclic for all $g \in \mathcal{B}$, where*

$$\mathcal{B} = \left\{ g : \mathbb{D} \to \mathbb{C} : \int_0^z g(t)\,dt \ \text{ is a bounded mean oscillation function} \right\}.$$

Proof. Suppose that $g(0) = 0$. From Lemma 1, we obtain the state of the theorem.

Let $g(0) \neq 0$. Therefore the diagonal entries of matrix $C_g^*(q)$ are distinct. And so, from Lemma 2, $C_g^*(q)$ is cyclic. □

Theorem 4. *If $g_\beta(z) = g(\beta z)$ with $|\beta| = 1$, then $C_{g_\beta}(q)$ is unitarily equivalent to $C_g(q)$. We denote this by $C_{g_\beta}(q) \approxeq C_g(q)$.*

Proof. Define the map $U_\beta : H^2 \to H^2$ by $U_\beta(f)(z) = f(\beta z)$. It is easy to see that U_β is unitary with $U_\beta^* = U_{\overline{\beta}}$.

Now, to show the unitary equivalence, we must prove that $U_\beta^* C_{g_\beta}(q) U_\beta = C_g(q)$.

The matrix representation of U_β in the basis $\left\{z^{n-1}\right\}_{n=1}^{\infty}$ is the diagonal matrix $diag\{\beta^n\}$. Moreover, we know that $\left(U_\beta\right)^* = U_{\overline{\beta}} = \left(U_\beta\right)^{-1}$. Using these matrix representations we have $U_\beta^* C_{g_\beta}(q) U_\beta = C_g(q)$ and consequently $C_{g_\beta}(q) \approxeq C_g(q)$ is obtained. □

Corollary 5. *If $\beta \in \partial(\mathbb{D})$, then the spectrum of $C_{\frac{1}{1-\beta z}}(q)$ is*

$$\sigma\left(C_{\frac{1}{1-\beta z}}(q)\right) = \sigma(C(q)) = \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{1+q}\right| \leq \frac{q}{1+q}\right\}.$$

Proof. This is immediate from the unitary equivalence and [6], Theorem 2.2. □

In [8] Mursalenn et al. studied the spectrum and Hilbert Schmidt properties of generalized Rhaly matrices. Now, let us show that $C_g(q)$ is a Hilbert-Schmidt operator. For this, we shall give the following lemma.

Lemma 3 ([8])**.** *If $\alpha \in \mathbb{Z}$, $B_{j-1} = 0$, $B_k = \sum_{k=j}^{n} \beta^k$ for $k = j, \ldots, n$ and (a_k) is a positive decreasing sequence, then*

$$\left|\sum_{k=j}^{n} \frac{\beta^k}{(a_k)^\alpha}\right| \leq \frac{2}{|1-\beta|}\left(\frac{1}{a_n^\alpha} + \left|\frac{1}{a_j^\alpha} - \frac{1}{a_n^\alpha}\right|\right).$$

The set of all Hilbert Schmidt operators on H is denoted by $B_2(H)$.

Theorem 5. *If $\beta_1 \neq \beta_2$, then $C_\beta(q) C(q)$ is a Hilbert-Schmidt operator.*

Proof. Notice that

$$C_{\beta_1}(q) C_{\beta_2}(q) \in B_2\left(H^2\right) \Leftrightarrow U_{\overline{\beta_2}} C_{\beta_1}(q) U_{\beta_2} U_{\overline{\beta_2}} C_{\beta_2}(q) U_{\beta_2} \in B_2\left(H^2\right).$$

Since $U_{\overline{\beta_2}} C_{\beta_2}(q) U_{\beta_2} = C(q)$, then we can assume $\beta_2 = 1$. Rewrite β_1 as β. Since $C_\beta(q) = C_{\frac{1}{1-\beta z}}(q)$, we have

$$\left[C_\beta(q)\right]_{nj} = \begin{cases} \frac{(\beta q)^{n-j}}{1+q+\cdots+q^{n-1}}, & n \geq j \\ 0, & n < j \end{cases}, \; n, j = 1, 2, \ldots$$

Since $[C(q)]_{nk} = \begin{cases} \frac{q^{n-k}}{1+q+\cdots+q^{n-1}}, & n \geq k \\ 0, & n < k \end{cases}$, $n, k = 1, 2, \ldots$, we get that

$$\left[C_\beta(q) C(q)\right]_{nj} = \begin{cases} \frac{1}{1+q+\cdots+q^{n-1}} \sum_{k=j}^{n} (\beta q)^{n-k} \frac{q^{k-j}}{1+\cdots+q^{k-1}}, & n \geq j \\ 0, & n < j \end{cases}, \; n, j = 1, 2, \ldots$$

$$= \begin{cases} \frac{\beta^n q^n q^{-j}}{1+q+\cdots+q^{n-1}} \sum_{k=j}^{n} \frac{\overline{\beta}^k}{1+\cdots+q^{k-1}}, & n \geq j \\ 0, & n < j \end{cases}, \; n, j = 1, 2, \ldots$$

From here,

$$\begin{aligned} \left\| C_\beta(q) C(q) \right\|_{H.S}^2 &= \sum_{n,j} \left| \left[C_\beta(q) C(q) \right]_{nj} \right|^2 \\ &= \sum_{n=1}^{\infty} \sum_{j=1}^{n} \left| \left[C_\beta(q) C(q) \right]_{nj} \right|^2 \\ &= \sum_{n=1}^{\infty} \frac{\beta^{2n} q^{2n}}{(1+q+\cdots+q^{n-1})^2} \sum_{j=1}^{n} q^{-2j} \left| \frac{\overline{\beta}^k}{1+q+\cdots+q^{k-1}} \right|^2. \end{aligned}$$

From Lemma 3,

$$\begin{aligned} &\left| \sum_{k=j}^{n} \frac{\overline{\beta}^k}{1+\cdots+q^{k-1}} \right| \\ &\leq \frac{2}{\left|1-\overline{\beta}\right|} \left(\frac{1}{1+q+\cdots+q^{n-1}} + \left| \frac{1}{1+\cdots+q^{j-1}} - \frac{1}{1+q+\cdots+q^{n-1}} \right| \right) \\ &\leq \frac{2}{\left|1-\overline{\beta}\right| (1+q+\cdots+q^{j-1})}. \end{aligned}$$

We use this in the last equation,

$$\begin{aligned} \left\| C_\beta(q) C(q) \right\|_{H.S}^2 &= \sum_{n=1}^{\infty} \frac{\beta^{2n} q^{2n}}{(1+q+\cdots+q^{n-1})^2} \sum_{j=1}^{n} q^{-2j} \left| \frac{\overline{\beta}^k}{1+q+\cdots+q^{k-1}} \right|^2 \\ &\leq \frac{4}{\left|1-\overline{\beta}\right|^2} \sum_{n=1}^{\infty} \frac{\beta^{2n} q^{2n}}{(1+q+\cdots+q^{n-1})^2} \sum_{j=1}^{n} \frac{q^{-2j}}{|1+q+\cdots+q^{k-1}|^2} \\ &\leq \frac{4}{\left|1-\overline{\beta}\right|^2} \sum_{n=1}^{\infty} \frac{\beta^{2n} q^{2n}}{(1+q+\cdots+q^{n-1})^2} < \infty. \end{aligned}$$

Thus $C_\beta(q) C(q)$ is a Hilbert-Schmidt operator. □

Theorem 6. *If $\beta_1 \neq \beta_2$, then $\left(C_{\beta_1}(q)\right)^* C_{\beta_2}(q)$ and $C_{\beta_1}(q) \left(C_{\beta_2}(q)\right)^*$ are Hilbert-Schmidt operators.*

Proof. Without loss of generality, let us take $\beta_1 = \beta$. Firstly, we will show that $\left(C_{\beta_1}(q)\right)^* C_{\beta_2}(q) \in B_2\left(H^2\right)$.

$$\left[\left(C_\beta(q)\right)^*\right]_{nj} = \begin{cases} 0, & n > j \\ \frac{\overline{\beta}^{n-j} q^{n-j}}{1+q+\cdots+q^{n-1}}, & n \leq j \end{cases}, \; n, j = 1, 2, \ldots$$

Therefore,

$$\left[\left(C_\beta(q)\right)^* C(q)\right]_{nj} = \frac{\overline{\beta}^n q^n q^{-j}}{1+q+\cdots+q^{n-1}} \sum_{k=\max\{n,j\}}^{\infty} \frac{\beta^k q^k}{1+q+\cdots+q^{k-1}}, \; n, j = 1, 2, \ldots$$

Since

$$\begin{aligned} &\sum_{k=n}^{m} \frac{\beta^k q^k}{1+q+\cdots+q^{k-1}} \\ &\leq \frac{2}{|1-q\beta|}\left(\frac{q^{n-1}}{1+q+\cdots+q^{n-1}} + \left|\frac{q^{j-1}}{1+q+\cdots+q^{j-1}} - \frac{q^{n-1}}{1+q+\cdots+q^{n-1}}\right|\right) \\ &\leq \frac{2q^{j-1}}{|1-q\beta|\left(1+q+\cdots+q^{j-1}\right)}, \end{aligned}$$

we have

$$\begin{aligned} \left[\left(C_\beta(q)\right)^* C(q)\right]_{nj} &= \sum_{n,j=1}^{\infty} \left|\sum_{k=j}^{\infty} \frac{\beta^k q^k}{1+q+\cdots+q^{k-1}}\right|^2 \\ &\leq \sum_{n,j=1}^{\infty} \frac{4}{|1-q\beta|^2} \frac{q^{2j-2}}{\left(1+q+\cdots+q^{k-1}\right)^2} \\ &= \frac{4}{|1-q\beta|^2} \sum_{n,j=1}^{\infty} \frac{q^{2j-2}}{\left(1+q+\cdots+q^{k-1}\right)^2} < \infty. \end{aligned}$$

Thus $\left(C_{\beta_1}(q)\right)^* C_{\beta_2}(q)$ is a Hilbert-Shcmidt operator.

Now, let us show that $C_\beta(q)\left(C(q)\right)^* \in B_2\left(H^2\right)$.

$$\begin{aligned} \left[C_\beta(q)\left(C(q)\right)^*\right]_{nj} &= \sum_{k=1}^{\min\{n,j\}} \frac{\beta^{n-k} q^{n-k}}{1+q+\cdots+q^{n-1}} \frac{q^{k-j}}{1+q+\cdots+q^{k-1}} \\ &= \frac{\beta^n q^{n-j}}{1+q+\cdots+q^{n-1}} \sum_{k=1}^{\min\{n,j\}} \frac{\overline{\beta}^k}{1+q+\cdots+q^{k-1}} \end{aligned}$$

and so,

$$\left|\frac{\beta^n q^{n-j}}{1+q+\cdots+q^{n-1}} \sum_{k=1}^{\min\{n,j\}} \frac{\overline{\beta}^k}{1+q+\cdots+q^{k-1}}\right| \leq \frac{q^{n-j}}{1+q+\cdots+q^{n-1}} \sum_{k=1}^{\min\{n,j\}} \left|\overline{\beta}^k\right|$$

$$= \frac{q^{n-j}}{1+q+\cdots+q^{n-1}} \left|\overline{\beta}\right| \frac{\left|1-\overline{\beta}^{\min\{n,j\}}\right|}{\left|1-\overline{\beta}\right|} \leq \frac{2q^{n-j}}{\left(1+q+\cdots+q^{n-1}\right)\left|1-\overline{\beta}\right|}.$$

Therefore we obtain that

$$\begin{aligned}\left\|C_{\beta}(q)\left(C(q)\right)^{*}\right\|_{H.S} &= \sum_{n,j=1}^{\infty}\left|\frac{\beta^{n}q^{n-j}}{1+\ldots+q^{n-1}}\sum_{k=1}^{\min\{n,j\}}\frac{\overline{\beta}^{k}}{1+\ldots+q^{k-1}}\right|^{2} \\ &\leq \frac{4}{\left|1-\overline{\beta}\right|^{2}}\sum_{n,j=1}^{\infty}\frac{q^{2n}q^{-2j}}{\left(1+\ldots+q^{n-1}\right)^{2}} < \infty.\end{aligned}$$

Thus $C_{\beta}(q)\left(C(q)\right)^{*} \in B_{2}\left(H^{2}\right)$ is a Hilbert-Shcmidt operator. □

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EXISTENCE AND EXPONENTIAL DECAY OF A LOGARITHMIC WAVE EQUATION WITH DISTRIBUTED DELAY

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Abstract. In this article, we deal with a logarithmic wave equation with distributed delay. Firstly, we establish the well-posedness by utilizing the semigroup theory. Later, we obtain the global existence of solutions by using the well-depth method. Moreover, under appropriate assumptions on the weight of the distributed delay and that of strong damping, we get the exponential decay results.

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1. Introduction

In this paper, we deal with the following logarithmic wave equation with distributed delay:

$$\begin{cases} u_{tt} - \Delta u - \mu_1 \Delta u_t(x,t) - \int_{\tau_1}^{\tau_2} \mu_2(s) \Delta u_t(x,t-s)\, ds & \\ = u|u|^{p-2} \ln |u|^k, & \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,\infty), \\ u_t(x,-t) = f_0(x,-t), & 0 < t \le \tau_2, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded and regular domain of R^n. k, μ_1 are positive constants, the integral term denotes the distributed delay for $\tau_1 < \tau_2$ and $\mu_2 : [\tau_1, \tau_2] \to R$ is a bounded function and u_0, u_1, f_0 are the functions of the initial data to be specified later.

The study of the asymptotic behavior of wave equation is an old and large areas because it is important in the applications, for this reasons wave equation has been taking different forms and names according to the phenomena it describes and also the material using in the experiment. In this paper we will study the nonlinear wave equation (1.1) with time delay. Also, in the same equation, we add the logarithmic source term which seems in inflation cosmology, nuclear physics, geophysics and optics, for instance (see [3–5,7,12]). To give a problem describe a phenomena enough perfect, we must take a consideration the delay which appears in practical phenomena like physical, biological, economic and some of themes. Furthermore, delay term

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influenced on the stability and instability of the studied system and many papers has been discussed the both situations, for example [8–10, 16].

During the past decades, many authors considered extensively on existence, nonexistence, stability and blow-up of solutions for the strongly damped wave equations with source term as follows:

$$u_{tt} - \Delta u - \omega \Delta u_t + \mu u_t = f(u). \tag{1.2}$$

Firstly, Sattinger [28] studied the existence of local as well as global solutions for the equation (1.2) with $\omega = \mu = 0$ by introducing the concepts of stable and unstable sets. In [13], Ikehata looked into the decay and blow-up of solutions for the equation (1.2) with linear damping that is $\omega = 0$ and $\mu > 0$. In [11], Gazzola and Squassina obtained the global existence and blow-up results for the equation (1.2) with both weak and strong damping ($\omega > 0$). In the case of logarithmic source term $f(u) = u \ln |u|^k$, Ma and Fang [20] obtained the existence and blow-up results for the equation (1.2) with $\omega = 1$, $\mu = 0$ and $k = 2$. In [18], Lian and Xu established the global existence, energy decay and blow-up results in the case $\omega \geq 0$ and $\mu > -\omega\lambda_1$, here λ_1 is the first eigenvalue of the operator $-\Delta$ under homogeneous Dirichlet boundary conditions.

It is well known that, in the n-dimensional case, the equation:

$$u_{tt} - \Delta u + a_0 u_t(x,t) + a u_t(x, t-\tau) = 0, \tag{1.3}$$

is exponentially stable without delay ($a = 0$, $a_0 > 0$), (see [19, 35]). With delay term ($a > 0$), Nicaise and Pignotti [23] considered the equation (1.3) and established, under the condition that the weight of the feedback is larger than the weight of the delay ($a < a_0$), that the energy of (1.3) is exponentially stable. If the delay term in the equation (1.3) is replaced by the distributed delay:

$$\int_{\tau_1}^{\tau_2} a(s) u_t(x, t-s)\, ds,$$

exponential stability of solutions have been proved in Ref. [22] under the assumption:

$$\int_{\tau_1}^{\tau_2} a(s)\, ds < a_0.$$

Without logarithmic nonlinearity ($u|u|^{p-2} \ln |u|^k$), the equation (1.1) takes the following form:

$$u_{tt} - \Delta u - \mu_1 \Delta u_t(x,t) - \int_{\tau_1}^{\tau_2} \mu_2(s) \Delta u_t(x, t-s)\, ds = 0, \tag{1.4}$$

Messaoudi et al. [21], established the well-posedness and obtained an exponential decay results under a suitable assumption on the weight of the damping and the weight of the delay of the equation (1.4). Furthermore, in [21], they extended the results obtained to the case of strong delay ($\mu_2 \Delta u_t(x, t-\tau)$) in same work. Some other researchers considered delayed hyperbolic-type equations (see [2,14,15,25,27,29–34]).

Motivated by previous works and in the presence of strong damping $(-\mu_1 \Delta u_t(x,t))$, distributed delay $(\int_{\tau_1}^{\tau_2} \mu_2(s) \Delta u_t(x,t-s)\, ds)$ and logarithmic $(u|u|^{p-2}\ln|u|^k)$ nonlinearity, we consider the well-posedness, global existence and exponential decay for the logarithmic wave equation (1.1) with strong damping and distributed delay.

The contents of this work is organized as follows: In Section 2, firstly, we obtain the well-posedness by utilizing the semigroup theory. Then, in Section 3, we establish the global existence results by the well-depth method. Finally, in Section 4, we get the exponential decay of solutions.

2. Well-posedness

As usual, the notation $\|.\|_p$ denotes L^p norm and $(.,.)$ is the L^2 inner product. In particular, we write $\|.\|$ instead of $\|.\|_2$ (see [1, 26], for details). Similar to the [23], we introduce the new variable

$$z(x,\rho,s,t) = u_t(x,t-\rho s), \text{ in } \Omega \times (0,1) \times (\tau_1,\tau_2) \times (0,\infty).$$

Hence, we get

$$s z_t(x,\rho,s,t) + z_\rho(x,\rho,s,t) = 0 \text{ in } \Omega \times (0,1) \times (\tau_1,\tau_2) \times (0,\infty).$$

Therefore, the problem (1.1) becomes the form:

$$\begin{cases} u_{tt} - \Delta u - \mu_1 \Delta u_t(x,t) - \int_{\tau_1}^{\tau_2} \mu_2(s) \Delta z(x,1,s,t)\, ds & \\ \qquad = u|u|^{p-2}\ln|u|^k, & \text{in } \Omega \times (0,+\infty) \\ s z_t(x,\rho,s,t) + z_\rho(x,\rho,s,t) = 0, & \text{in } \Omega \times (0,1) \times (\tau_1,\tau_2) \times (0,+\infty) \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & \text{in } \Omega \\ z(x,\rho,s,0) = f_0(x,-\rho s), & \text{in } \Omega \times (0,1) \times (\tau_1,\tau_2). \end{cases} \tag{2.1}$$

We establish the local existence utilizing the semigroup theory [17, 24]. Let $\upsilon = u_t$ and denote by

$$\Phi = (u,\upsilon,z)^T,\ \Phi(0) = \Phi_0 = (u_0,u_1,f_0(\cdot,\rho s))^T \text{ and } J(\Phi) = \left(0, u|u|^{p-2}\ln|u|^k, 0\right)^T.$$

Hence, (2.1) can be written as an initial-value problem:

$$\begin{cases} \partial_t \Phi + \mathcal{A}\Phi &= J(\Phi), \\ \Phi(0) &= \Phi_0, \end{cases} \tag{2.2}$$

where the linear operator $\mathcal{A} : D(\mathcal{A}) \longrightarrow \mathcal{H}$ is defined by

$$\mathcal{A}\Phi = \begin{pmatrix} -\upsilon \\ -\Delta u - \mu_1 \Delta \upsilon - \int_{\tau_1}^{\tau_2} \mu_2(s) \Delta z(x,1,s,t)\, ds \\ \frac{1}{s} z_\rho \end{pmatrix},$$

where $D(\mathcal{A})$ and $\mathcal{H}$ are introduced below.

The setting space of Φ is the Hilbert space

$$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega) \times L^2\left((0,1) \times (\tau_1,\tau_2); H_0^1(\Omega)\right),$$

equipped with the inner product

$$\left\langle \Phi, \widetilde{\Phi} \right\rangle_{\mathcal{H}} = \int_{\Omega} (\nabla u \nabla \widetilde{u} + \upsilon \widetilde{\upsilon})\, dx + \int_{\Omega} \int_{\tau_1}^{\tau_2} s\, |\mu_2(s)| \int_0^1 \nabla z \nabla \widetilde{z}\, d\rho\, ds\, dx,$$

for all $\Phi = (u, \upsilon, z)^T$ and $\widetilde{\Phi} = (\widetilde{u}, \widetilde{\upsilon}, \widetilde{z})^T$ in $\mathcal{H}$. The domain of $\mathcal{A}$ is

$$D(\mathcal{A}) = \left\{ \begin{array}{c} \Phi \in \mathcal{H} : \left(u + \mu_1 \upsilon + \left(\int_{\tau_1}^{\tau_2} \mu_2(s) z\right)\right) \in H^2(\Omega), \\ z, z_\rho \in L^2\left((0,1) \times (\tau_1, \tau_2); H_0^1(\Omega)\right), z(\cdot, 0, \cdot) = \upsilon \end{array} \right\}.$$

Now, we obtain the local existence and uniqueness of solutions as follows:

Theorem 1. *Suppose that* $\mu_1 > \int_{\tau_1}^{\tau_2} \mu_2(s)\, ds$ *and*

$$\left\{ \begin{array}{ll} p > 2 & \text{if } n = 1, 2, \\ 2 < p < \frac{2(n-1)}{n-2} & \text{if } n \geq 3. \end{array} \right. \tag{2.3}$$

Then, for any $\Phi_0 \in \mathcal{H}$*, problem (2.2) has a unique weak solution*

$$\Phi \in C([0,T); \mathcal{H}).$$

Furthermore, if $\Phi_0 \in D(\mathcal{A})$*, the solution of (2.2) satisfies*

$$\Phi \in C^1([0,T), \mathcal{H}) \cap C([0,T), D(\mathcal{A})).$$

Proof. We will utilize the Hille-Yoside theorem [6, 17]. For this aim, we indicate that $\mathcal{A}$ is monotone. Thus, for all $\Phi \in D(\mathcal{A})$, we obtain

$$\begin{aligned} \langle \mathcal{A}\Phi, \Phi \rangle_{\mathcal{H}} = & -\int_{\Omega} \nabla \upsilon \nabla u\, dx \\ & - \int_{\Omega} \upsilon \left[\Delta u + \mu_1 \Delta \upsilon + \int_{\tau_1}^{\tau_2} \mu_2(s) \Delta z(x, 1, s, t)\, ds \right] dx \\ & + \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_2(s)| \int_0^1 \nabla z \nabla z_\rho\, d\rho\, ds\, dx \\ = & \mu_1 \int_{\Omega} |\nabla \upsilon|^2\, dx + \int_{\Omega} \nabla \upsilon \left(\int_{\tau_1}^{\tau_2} \mu_2(s) \nabla z(x, 1, s, t)\, ds \right) dx \\ & + \frac{1}{2} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_2(s)| \, |\nabla z(x, 1, s, t)|^2\, ds\, dx \\ & - \frac{1}{2} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\mu_2(s)| \, |\nabla \upsilon|^2\, ds\, dx. \end{aligned} \tag{2.4}$$

By using the Young's inequality, the estimate (2.4) becomes the form:

$$\begin{aligned} \langle \mathcal{A}\Phi, \Phi \rangle_{\mathcal{H}} \geq & \mu_1 \int_{\Omega} |\nabla \upsilon|^2\, dx - \frac{|\mu_2|}{2} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\nabla \upsilon|^2\, ds\, dx - \frac{|\mu_2|}{2} \int_{\Omega} \int_{\tau_1}^{\tau_2} |\nabla \upsilon|^2\, ds\, dx \\ = & \left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2|(s)\, ds \right) \int_{\Omega} |\nabla \upsilon|^2\, dx \geq 0. \end{aligned}$$

Therefore, $\mathcal{A}$ is a monotone operator.

To indicate that $\mathcal{A}$ is maximal, we establish that for each:

$$F=(f,g,h)^T\in\mathcal{H},$$

there exists $V=(u,\upsilon,z)^T\in D(\mathcal{A})$ such that $(I+\mathcal{A})V=F$. That is,

$$\begin{cases} u-\upsilon &= f \\ \upsilon-\Delta u-\mu_1\Delta\upsilon-\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,\Delta z(x,1,s,t)\,ds &= g, \\ sz+z_\rho &= sh. \end{cases} \tag{2.5}$$

As $z(.,0,.)=u-f=\upsilon$, by the third equation of (2.5), we infer that

$$z(.,\rho,.)=(u-f)\,e^{-\rho s}+se^{-\rho s}\int_0^\rho h(.,\gamma,s)\,e^{\gamma s}d\gamma. \tag{2.6}$$

By substituting (2.6) in the second equation of (2.5), we have

$$u-k\Delta u=G,$$

where

$$k=1+\mu_1+\int_{\tau_1}^{\tau_2}e^{-s}\,|\mu_2(s)|\,ds>0,$$

$$G=g+f-\left(\mu_1+\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,e^{-s}ds\right)\Delta f \tag{2.7}$$

$$+s\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,e^{-s}\int_0^1\Delta h(\gamma,x)\,e^{\gamma s}d\gamma ds\in H^{-1}(\Omega). \tag{2.8}$$

We define, over $H_0^1(\Omega)$, the bilinear and linear forms:

$$B(u,w)=\int_\Omega uwdx+k\int_\Omega\nabla u\nabla wdx,\ L(w)=\langle G,w\rangle_{H^{-1}\times H_0^1}.$$

We see that B is coercive and continuous, and L is continuous on $H_0^1(\Omega)$.

Then, Lax-Milgram theorem specifies that the equation

$$B(u,w)=L(w),\ \forall w\in H_0^1(\Omega), \tag{2.9}$$

has a unique solution $u\in H_0^1(\Omega)$. Thus, $\upsilon=u-f\in H_0^1(\Omega)$.

As a result, by (2.6), we obtain

$$z,z_\rho\in L^2(\Omega\times(0,1)\times(\tau_1,\tau_2))\ \text{and}\ z(.,0,.)=\upsilon.$$

Replacing Δf by $\Delta(u-\upsilon)$ and $se^{-s}\int_0^1\Delta h(\gamma,x)\,e^{\gamma s}d\gamma$ by $\Delta z(x,1,s)-e^{-s}\Delta\upsilon$ in the right-hand side of (2.9) and utilizing the Green's formula, we get

$$\int_\Omega u\upsilon+\int_\Omega\nabla\left(u+\mu_1\upsilon+\int_{\tau_1}^{\tau_2}\mu_2(s)\,e^{-s}z(x,1,s)\,ds\right)\nabla w=\int_\Omega(f+g)\,w,\ \forall w\in H_0^1(\Omega).$$

From the standard elliptic regularity theory [6], we have

$$u+\mu_1\upsilon+\int_{\tau_1}^{\tau_2}\mu_2(s)\,e^{-s}z(.,1,s)\,ds\in H^2(\Omega).$$

Hence,

$$\Delta\left(u+\mu_1\upsilon+\int_{\tau_1}^{\tau_2}\mu_2(s)e^{-s}z(.,1,s)\,ds\right)=g+f-u\in L^2(\Omega).$$

Thus,

$$V=(u,\upsilon,z)^T\in D(\mathcal{A}).$$

As a result, $I+\mathcal{A}$ is surjective and then $\mathcal{A}$ is maximal.

Finally, we denote that $J:\mathcal{H}\longrightarrow\mathcal{H}$ is locally Lipschitz. Hence, if we set:

$$f(s)=|s|^{p-2}s\ln|s|^k,\ |s|\neq 0 \text{ and } f(s)=0,\ |s|=0,$$

then

$$f'(s)=k\left[1+(p-1)\ln|s|\right]|s|^{p-2},\ |s|\neq 0 \text{ and } f'(s)=0,\ |s|=0.$$

Thus,

$$\begin{aligned}\left\|J(\Phi)-J\left(\widetilde{\Phi}\right)\right\|_{\mathcal{H}}^2&=\left\|\left(0,|u|^{p-2}u\ln|u|^k-|\widetilde{u}|^{p-2}\widetilde{u}\ln|\widetilde{u}|^k,0\right)\right\|_{\mathcal{H}}^2\\&=\left\||u|^{p-2}u\ln|u|^k-|\widetilde{u}|^{p-2}\widetilde{u}\ln|\widetilde{u}|^k\right\|_{L^2}^2\\&=\|f(u)-f(\widetilde{u})\|_{L^2}^2.\end{aligned}$$

By the mean value theorem, we obtain, for some $\theta\in[0,1]$,

$$\begin{aligned}|f(u)\ -f(\widetilde{u})|&=\left|f'(\theta u+(1-\theta)\widetilde{u})(u-\widetilde{u})\right|\\&=k\left|\left[1+(p-1)\ln|\theta u+(1-\theta)\widetilde{u}|\right]|\theta u+(1-\theta)\widetilde{u}|^{p-2}(u-\widetilde{u})\right|\\&\leq k\left[1+(p-1)\left|\ln|\theta u+(1-\theta)\widetilde{u}|\right|\right]|\theta u+(1-\theta)\widetilde{u}|^{p-2}|u-\widetilde{u}|\\&=k(p-1)\left|\ln|\theta u+(1-\theta)\widetilde{u}|\right||\theta u+(1-\theta)\widetilde{u}|^{p-2}|u-\widetilde{u}|\\&\quad+k|\theta u+(1-\theta)\widetilde{u}|^{p-2}|u-\widetilde{u}|.\end{aligned}$$

To control the logarithmic term

$$\ln|\theta u+(1-\theta)\widetilde{u}|\,|\theta u+(1-\theta)\widetilde{u}|^{p-2},$$

seems in the last inequality we remind that, for any $\varepsilon>0$,

$$\lim_{|s|\to+\infty}\frac{\ln|s|}{|s|^{\varepsilon}}=0.$$

Then, there exists $B>0$ such that

$$\frac{\ln|s|}{|s|^{\varepsilon}}<1,\ \forall|s|>B.$$

Hence, whenever $|\theta u+(1-\theta)\widetilde{u}|>B$, we get

$$\ln|\theta u+(1-\theta)\widetilde{u}|\leq|\theta u+(1-\theta)\widetilde{u}|^{\varepsilon},$$

and

$$\ln|\theta u+(1-\theta)\widetilde{u}|\,|\theta u+(1-\theta)\widetilde{u}|^{p-2}\leq|\theta u+(1-\theta)\widetilde{u}|^{p-2+\varepsilon}.$$

Since $p>2$, then for some $A>0$ and $|\theta u+(1-\theta)\widetilde{u}|\leq B$, we obtain

$$\ln|\theta u+(1-\theta)\widetilde{u}|\,|\theta u+(1-\theta)\widetilde{u}|^{p-2}\leq A.$$

Thus, we get

$$\ln|\theta u+(1-\theta)\widetilde{u}|\,|\theta u+(1-\theta)\widetilde{u}|^{p-2}\leq A+|\theta u+(1-\theta)\widetilde{u}|^{p-2+\varepsilon}.$$

Then, we obtain the following estimation for

$$\begin{aligned}|f(u)-f(\widetilde{u})|&\leq k(p-1)|\theta u+(1-\theta)\widetilde{u}|^{p-2+\varepsilon}|u-\widetilde{u}|\\&\quad+k|\theta u+(1-\theta)\widetilde{u}|^{p-2}|u-\widetilde{u}|+kA(p-1)|u-\widetilde{u}|\\&\leq k(p-1)(|u|+|\widetilde{u}|)^{p-2+\varepsilon}|u-\widetilde{u}|\\&\quad+k(|u|+|\widetilde{u}|)^{p-2}|u-\widetilde{u}|+kA(p-1)|u-\widetilde{u}|.\end{aligned}\tag{2.10}$$

As $u,\widetilde{u}\in H_0^1(\Omega)$, by utilizing Hölder's inequality, (2.3) and the Sobolev embedding

$$H_0^1(\Omega)\hookrightarrow L^r(\Omega),\ 1\leq r\leq\frac{2n}{n-2},$$

to have

$$\begin{aligned}\int_\Omega\left[(|u|+|\widetilde{u}|)^{p-2}|u-\widetilde{u}|\right]^2dx&=\int_\Omega(|u|+|\widetilde{u}|)^{2(p-2)}|u-\widetilde{u}|^2dx\\&\leq C\left(\int_\Omega(|u|+|\widetilde{u}|)^{2(p-1)}dx\right)^{\frac{p-2}{p-1}}\left(\int_\Omega|u-\widetilde{u}|^{2(p-1)}dx\right)^{\frac{1}{p-1}}\\&\leq C\left[\|u\|^{2(p-1)}_{L^{2(p-1)}(\Omega)}+\|\widetilde{u}\|^{2(p-1)}_{L^{2(p-1)}(\Omega)}\right]^{\frac{p-2}{p-1}}\|u-\widetilde{u}\|^2_{L^{2(p-1)}(\Omega)}\\&\leq C\left[\|u\|^{2(p-1)}_{H_0^1(\Omega)}+\|\widetilde{u}\|^{2(p-1)}_{H_0^1(\Omega)}\right]^{\frac{p-2}{p-1}}\|u-\widetilde{u}\|^2_{H_0^1(\Omega)}.\end{aligned}$$

Similarly,

$$\begin{aligned}\int_\Omega\left[(|u|+|\widetilde{u}|)^{p-2+\varepsilon}|u-\widetilde{u}|\right]^2dx&=\int_\Omega(|u|+|\widetilde{u}|)^{2(p-2+\varepsilon)}|u-\widetilde{u}|^2dx\\&\leq\left(\int_\Omega(|u|+|\widetilde{u}|)^{\frac{2(p-2+\varepsilon)(p-1)}{p-2}}dx\right)^{\frac{p-2}{p-1}}\left(\int_\Omega|u-\widetilde{u}|^{2(p-1)}dx\right)^{\frac{1}{p-1}}\\&\leq\left(\int_\Omega(|u|+|\widetilde{u}|)^{2(p-1)+\frac{2\varepsilon(p-1)}{p-2}}dx\right)^{\frac{p-2}{p-1}}\|u-\widetilde{u}\|^2_{L^{2(p-1)}(\Omega)}.\end{aligned}$$

Since $p<\frac{2(n-1)}{n-2}$, by choosing $\varepsilon>0$ small enough, such that

$$p^*=2(p-1)+\frac{2\varepsilon(p-1)}{p-2}\leq\frac{2n}{n-2}.$$

Thus, we get

$$\int_{\Omega}(|u|+|\widetilde{u}|)^{2(p-2+\varepsilon)}|u-\widetilde{u}|^2\,dx \leq C\left[\|u\|_{L^{p^*}(\Omega)}^{p^*}+\|\widetilde{u}\|_{L^{p^*}(\Omega)}^{p^*}\right]^{\frac{p-2}{p-1}}\|u-\widetilde{u}\|_{L^{2(p-1)}(\Omega)}^2$$
$$\leq C\left[\|u\|_{H_0^1(\Omega)}^{p^*}+\|\widetilde{u}\|_{H_0^1(\Omega)}^{p^*}\right]^{\frac{p-2}{p-1}}\|u-\widetilde{u}\|_{H_0^1(\Omega)}^2.$$

Thus, by combining the last three estimations, we get

$$\left\|J(\Phi)-J\left(\widetilde{\Phi}\right)\right\|_{\mathcal{H}}^2 \leq \left[k^2(p-1)^2A^2\right]\|u-\widetilde{u}\|_{H_0^1(\Omega)}^2$$
$$+C\left[\left(\|u\|_{H_0^1(\Omega)}^{2(p-1)}+\|\widetilde{u}\|_{H_0^1(\Omega)}^{2(p-1)}\right)^{\frac{p-2}{p-1}}\right.$$
$$\left.+\left(\|u\|_{H_0^1(\Omega)}^{p^*}+\|\widetilde{u}\|_{H_0^1(\Omega)}^{p^*}\right)^{\frac{p-2}{p-1}}\right]\|u-\widetilde{u}\|_{H_0^1(\Omega)}^2$$
$$\leq C\left(\|u\|_{H_0^1(\Omega)},\|\widetilde{u}\|_{H_0^1(\Omega)}\right)\|u-\widetilde{u}\|_{H_0^1(\Omega)}^2.$$

Hence, J is locally Lipschitz. Then, as a consequence of the Theorem 1.2 page 184, Pazy [24] (see also a remark in the beginning of page 118, Komornik [17]), we complete the proof. □

3. Global existence

In this section, we establish that the solution of (2.1) is uniformly bounded and global in time. The energy functional for the problem (2.1) is,

$$E(t)=\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\nabla u\|^2+\frac{k}{p^2}\|u\|_p^p-\frac{1}{p}\int_{\Omega}|u|^p\ln|u|^k\,dx$$
$$+\frac{1}{2}\int_{\Omega}\int_0^1\int_{\tau_1}^{\tau_2}s\,|\mu_2(s)|\,|\nabla z(x,\rho,s,t)|^2\,dsd\rho dx. \tag{3.1}$$

The following lemma indicates that the energy functional is nonincreasing:

Lemma 1. *The energy $E(t)$ satisfies, along the solution (u,z) of (2.1), the estimate*

$$E'(t)\leq-\left[\mu_1-\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)\right]\int_{\Omega}|\nabla u_t|^2\,dx<0. \tag{3.2}$$

Proof. Multiply the first equation in (2.1) by u_t and integrate over Ω and the second equation in (2.1) by $-|\mu_2|\Delta z$ and integrate over $\Omega\times(0,1)\times(\tau_1,\tau_2)$, sum up, we obtain

$$\frac{d}{dt}\left[\begin{array}{c}\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\nabla u\|^2+\frac{k}{p^2}\|u\|_p^p-\frac{1}{p}\int_{\Omega}|u|^p\ln|u|^k\,dx\\ +\frac{1}{2}\int_{\Omega}\int_0^1\int_{\tau_1}^{\tau_2}s\,|\mu_2(s)|\,|\nabla z(x,\rho,s,t)|^2\,dsd\rho dx\end{array}\right]$$
$$=-\mu_1\int_{\Omega}|\nabla u_t|^2\,dx-\int_{\Omega}\nabla u_t\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,\nabla z(x,1,s,t)\,dsdx$$

$$-\left|\mu_2(s)\right|\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\nabla z\nabla z_\rho(x,\rho,s,t)\,dsd\rho dx. \tag{3.3}$$

Now, we handle the last two terms of the right-hand side for (3.3) as:

$$\begin{aligned}
&-\left|\mu_2(s)\right|\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\nabla z\nabla z_\rho(x,\rho,s,t)\,dsd\rho dx\\
&\quad=-\frac{\left|\mu_2(s)\right|}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\frac{\partial}{\partial\rho}\left[\left|\nabla z(x,\rho,s,t)\right|^2\right]dsd\rho dx\\
&\quad=\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right|ds\right)\int_\Omega\left|\nabla u_t\right|^2dx\\
&\qquad-\frac{\left|\mu_2(s)\right|}{2}\int_\Omega\int_{\tau_1}^{\tau_2}\left|\nabla z(x,1,s,t)\right|^2dsdx
\end{aligned}$$

and

$$\begin{aligned}
&-\int_\Omega\nabla u_t\int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right|\nabla z(x,1,s,t)\,dsdx\\
&\quad\le\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right|ds\int_\Omega\left|\nabla u_t\right|^2dx+\left|\mu_2(s)\right|\int_\Omega\int_{\tau_1}^{\tau_2}\left|\nabla z(x,1,s,t)\right|^2dsdx\right).
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\frac{dE(t)}{dt}\le&-\mu_1\int_\Omega\left|\nabla u_t\right|^2dx+\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right|ds\right)\int_\Omega\left|\nabla u_t\right|^2dx\\
&-\frac{\left|\mu_2(s)\right|}{2}\int_\Omega\int_{\tau_1}^{\tau_2}\left|\nabla z(x,1,s,t)\right|^2dsdx\\
&+\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right|ds\int_\Omega\left|\nabla u_t\right|^2dx+\left|\mu_2(s)\right|\int_\Omega\int_{\tau_1}^{\tau_2}\left|\nabla z(x,1,s,t)\right|^2dsdx\right).
\end{aligned}$$

Therefore, we get

$$E'(t)\le-\left[\mu_1-\left(\int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right|ds\right)\right]\int_\Omega\left|\nabla u_t\right|^2dx\le 0.$$

□

Firstly, we set

$$\begin{aligned}
I(t)&=\|\nabla u\|^2-\int_\Omega|u|^p\ln|u|^k\,dx,\\
J(t)&=\frac{1}{2}\int_\Omega|\nabla u|^2dx+\frac{k}{p^2}\|u\|_p^p-\frac{1}{p}\int_\Omega|u|^p\ln|u|^k\,dx\\
&\quad+\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}s\left|\mu_2(s)\right|\left|\nabla z(x,\rho,s,t)\right|^2dsd\rho dx.
\end{aligned}\tag{3.4}$$

Thus, we obtain

$$E(t) = J(t) + \frac{1}{2}\|u_t\|^2.$$

Lemma 2. *Assume that the initial data u_0, $u_1 \in H_0^1(\Omega) \times L^2(\Omega)$ satisfying*

$$I(0) > 0 \text{ and } \beta = kC_{p+l}\left(\frac{2pE(0)}{p-2}\right)^{\frac{p-2+l}{2}} < 1. \tag{3.5}$$

Then, $I(t) > 0$, for any $t \in [0,T]$.

Proof. Since $I(0) > 0$, we infer from continuity that there exists $T^* \leq T$ such that $I(t) \geq 0$ for all $t \in [0,T^*]$. This implies that, for all $t \in [0,T^*]$,

$$\begin{aligned} J(t) = {} & \frac{p-2}{2p}\|\nabla u\|^2 + \frac{k}{p^2}\|u\|_p^p + \frac{1}{p}I(t) \\ & + \frac{1}{2}\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\,|\nabla z(x,\rho,s,t)|^2\,ds\,d\rho\,dx. \end{aligned}$$

Thus, we get

$$J(t) \geq \frac{p-2}{2p}\|\nabla u\|^2.$$

Therefore,

$$\|\nabla u\|^2 \leq \frac{2p}{p-2}J(t) \leq \frac{2p}{p-2}E(t) \leq \frac{2p}{p-2}E(0). \tag{3.6}$$

On the other hand, by using the fact that $\ln|u| < |u|^l$, we obtain

$$\int_\Omega |u|^p \ln|u|\,dx \leq \int_\Omega |u|^{p+l}\,dx, \tag{3.7}$$

where l is chosen to be $0 < l < \frac{2}{n-2}$, such that

$$p + l < \frac{2n-2}{n-2} + l < \frac{2n}{n-2}.$$

Hence, the embedding $H_0^1(\Omega) \hookrightarrow L^{p+l}(\Omega)$, satisfies

$$\begin{aligned} \int_\Omega |u|^p \ln|u|\,dx & \leq C_{p+l}\|\nabla u\|^{p+l} \\ & = C_{p+l}\|\nabla u\|^2\|\nabla u\|^{p-2+l} = C_{p+l}\|\nabla u\|^2\left(\|\nabla u\|^2\right)^{\frac{p-2+l}{2}} \\ & \leq C_{p+l}\left(\frac{2pE(0)}{p-2}\right)^{\frac{p-2+l}{2}}\|\nabla u\|^2, \end{aligned} \tag{3.8}$$

where C_{p+l} is the embedding constant.

As a result, from (3.4) and (3.5), we have

$$I(t) > \|\nabla u\|^2 - \beta\|\nabla u\|^2 > 0, \forall t \in [0,T^*]. \tag{3.9}$$

Therefore, T^* can be extended to T. □

Theorem 2. *Assume that the initial data u_0, u_1 satisfy the conditions of Lemma 2, then the solution of* (2.1) *is uniformly bounded and global in time.*

Proof. It suffices to indicate that $\|\nabla u\|^2+\|u_t\|^2$ is bounded independently of t. We have,

$$\begin{aligned} E(0)\geq E(t)&=\frac{1}{2}\|u_t\|^2+J(t)\geq\frac{1}{2}\|u_t\|^2+\frac{k}{p^2}\|u\|_p^p+\frac{1}{p}I(t)\\ &\quad+\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}s|\mu_2(s)||\nabla z(x,\rho,s,t)|^2dsd\rho dx\\ &\geq\frac{1}{2}\|u_t\|^2+\frac{1}{p}(1-\beta)\|\nabla u\|^2. \end{aligned}$$

Therefore,

$$\|\nabla u\|^2+\|u_t\|^2\leq CE(0),$$

where C is a positive constant depending only on k, p and C_{p+1}. □

4. Exponential decay

In this section, we establish the exponential decay results. Firstly, we give the lemmas as follows:

Lemma 3. *[21] The functional*

$$F_1(t)=\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}se^{-s\rho}|\mu_2(s)||\nabla z(x,\rho,s,t)|^2dsd\rho dx,$$

satisfies, along the solution of (2.1),

$$\begin{aligned} F_1'(t)\leq&\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds\right)\int_\Omega|\nabla u_t|^2dx-\int_\Omega\int_{\tau_1}^{\tau_2}e^{-s}|\mu_2(s)||\nabla z(x,1,s,t)|^2dsdx\\ &-\int_\Omega\int_{\tau_1}^{\tau_2}s|\mu_2(s)|\int_0^1e^{-s\rho}|\nabla z(x,\rho,s,t)|^2d\rho dsdx. \end{aligned}\tag{4.1}$$

Lemma 4. *The functional*

$$F_2(t)=NE(t)+\varepsilon\int_\Omega uu_tdx+\frac{\varepsilon\mu_1}{2}\int_\Omega|\nabla u|^2dx,$$

satisfies, along the solution of (2.1),

$$\begin{aligned} F_2'(t)\leq&-\left\{N\left[\mu_1-\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds\right)\right]-\varepsilon\right\}\int_\Omega|\nabla u_t|^2dx\\ &-\varepsilon(1-\beta-\delta)\|\nabla u\|^2+\varepsilon\frac{c_*}{4\delta}\int_\Omega\int_{\tau_1}^{\tau_2}|\nabla z(x,1,s,t)|^2dsdx, \end{aligned}\tag{4.2}$$

where N, α and ε are positive constants.

Proof. A direct differentiation, from the equations in (2.1), satisfies

$$\begin{aligned} F_2'(t) \leq & -N\left[\mu_1 - \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\right)\right] \int_\Omega |\nabla u_t|^2 dx \\ & + \varepsilon \left(\int_\Omega |u_t|^2 dx - \int_\Omega |\nabla u|^2 dx + \int_\Omega |u|^p \ln |u|^k dx\right) \\ & - \varepsilon \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| \nabla u \nabla z(x,1,s,t)\, ds dx. \end{aligned} \tag{4.3}$$

From the Young's inequality and the boundness property of $\mu_2(s)$, we get, for any $\delta > 0$ and some $c_* > 0$,

$$\begin{aligned} & -\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| \nabla u \nabla z(x,1,s,t)\, ds dx \\ & \qquad \leq \delta \|\nabla u\|^2 + \frac{c_*}{4\delta} \int_\Omega \int_{\tau_1}^{\tau_2} |\nabla z(x,1,s,t)|^2 ds dx. \end{aligned} \tag{4.4}$$

Combining (3.7), (4.3) and (4.4), the result follows: □

Theorem 3. *Assume that* (3.5) *holds. Then, there exist two positive constants* c_3 *and* c_4 *such that*

$$E(t) \leq c_3 e^{-c_4 t}.$$

Proof. By setting

$$F_3(t) = F_1(t) + F_2(t).$$

It is easy to see, for ε small enough, that

$$F_3(t) \sim E(t). \tag{4.5}$$

From the (4.1) and (4.2), we get

$$\begin{aligned} F_3'(t) \leq & -\left\{N\left[\mu_1 - \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\right)\right] - \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\right) - \varepsilon\right\} \|\nabla u_t\|^2 \\ & - \varepsilon(1-\beta-\delta)\|\nabla u\|^2 - \left(e^{-s}|\mu_2(s)| - \varepsilon \frac{c_*}{4\delta}\right) \int_\Omega \int_{\tau_1}^{\tau_2} |\nabla z(x,1,s,t)|^2 ds dx \\ & - \int_\Omega \int_{\tau_1}^{\tau_2} s|\mu_2(s)| \int_0^1 e^{-s\rho} |\nabla z(x,\rho,s,t)|^2 d\rho ds dx. \end{aligned} \tag{4.6}$$

Since $\beta < 1$, by choosing δ small enough, such that $\alpha = 1 - \beta - \delta > 0$.

For some $\omega > 0$, the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ satisfies

$$\begin{aligned} \|u\|_p^p \leq & C\|\nabla u\|_2^p \leq C\left(\|\nabla u\|^2\right)^{\frac{p-2}{2}} \|\nabla u\|^2 \leq C(E(0))^{\frac{p-2}{2}} \|\nabla u\|^2 \\ & \leq \omega \|\nabla u\|^2, \end{aligned}$$

or

$$-\frac{\varepsilon\alpha\omega^{-1}}{2}\|u\|_p^p \geq -\frac{\varepsilon\alpha}{2}\|\nabla u\|_2^2.$$

Therefore, (4.6) becomes the form

$$\begin{aligned} F_3'(t) \leq & -\left\{N\left[\mu_1 - \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\right)\right] - \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\right) - \varepsilon\right\} \|\nabla u_t\|^2 \\ & - \frac{\varepsilon\alpha}{2}\|\nabla u\|^2 - \frac{\varepsilon\alpha\omega^{-1}}{2}\|u\|_p^p - \left(e^{-s}|\mu_2(s)| - \varepsilon\frac{c_*}{4\delta}\right)\int_\Omega\int_{\tau_1}^{\tau_2} |\nabla z(x,1,s,t)|^2\, dsdx \\ & - \int_\Omega\int_{\tau_1}^{\tau_2} s|\mu_2(s)|\int_0^1 e^{-s\rho}|\nabla z(x,\rho,s,t)|^2\, d\rho dsdx. \end{aligned} \tag{4.7}$$

Whence δ is fixed, by choosing N to be large enough, such that

$$N\left[\mu_1 - \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\right)\right] - \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds\right) - \varepsilon > 0 \text{ and } e^{-s}|\mu_2(s)| - \varepsilon\frac{c_*}{4\delta} > 0.$$

Hence, (4.7) takes the form, for some $C > 0$,

$$\begin{aligned} F_3'(t) \leq & -C\left[\|\nabla u_t\|^2 + \|\nabla u\|^2 + \|u\|_p^p \right. \\ & \left. + \int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|\,|\nabla z(x,\rho,s,t)|^2\, dsd\rho dx\right] \\ \leq & -CE(t). \end{aligned}$$

From the equivalence relation (4.5) and taking a simple integration over $(0,t)$, the result is established. □

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n-FINE RINGS

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Abstract. A ring R is said to be n-fine if every nonzero element in R can be written as a sum of a nilpotent and n units in R. The class of these rings contains fine rings and n-good rings in which each element is a sum of n units. Fundamental properties of such rings are obtained. One of the main results of this paper is that the $m \times m$ matrix ring $M_m(R)$ over any arbitrary ring R is 2-fine. Furthermore, the $m \times m$ matrix ring $M_m(R)$ over a n-fine ring R is n-fine.

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1. Introduction

All rings in this paper are assumed to be associative with identity. For a ring R, our terminology and notations are mainly in agreement with [4]. For instance, $U(R)$ is the multiplicative group of units of R, $Nil(R)$ is the set of all nilpotent elements of R and $Id(R)$ is the set of all idempotents of R. If R is commutative, then $Nil(R) = N(R)$ is the nil-radical of R. We denote by $M_m(R)$ the ring of $m \times m$ matrices over R with the identity I_m.

In the last four decades, an additive theory has emerged in the study of these three interesting sets. A ring R is called n-good if every element of R is a sum of n units. Many mathematicians for instance Vámos and Ashrafi studied 2-good rings extensively (see [1, 6, 7]). In 1977, Nicholson defined a ring element $a \in R$ to be clean if it can be written in the form $e + u$ where $e \in Id(R)$ and $u \in U(R)$ [5]. If every $a \in R$ is clean, R is said to be a clean ring. The interest in the clean property of rings stems from its close connection to exchange rings, since clean is a concise property that implies exchange. Prompted by this, Xiao and Tong [9] called a ring R n-clean if every element of R is the sum of an idempotent and n units (see also [8]). The class of these rings contains clean rings and n-good rings. Recently, G. Călugăreanu and T. Y. Lam defined a property in [2] related to 2-good in the following way: a nonzero element $a \in R$ is fine if $a = t + u$ where $t \in Nil(R)$ and $u \in U(R)$. The ring R is called fine if every nonzero element of R is fine. It is shown that fine rings form a proper class of simple ring [2, Theorem 2.3].

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Guided by these definitions, we introduce in this work the following definition. Given a positive integer n, we call a ring R n-fine if every nonzero element of R can be written as the sum of a nilpotent and n units in R. It is clear that fine rings are 1-fine. In section 2, some fundamental properties of n-fine rings are studied. We shall prove every n-fine ring is $(n+1)$-good. Furthermore, we prove that the class of n-fine rings is closed under factor rings and direct products. The main result of this section states that a ring R is n-fine if and only if every factor ring of R is n-fine if and only if every indecomposable factor ring of R is n-fine. Then in Section 3 we will look at matrix rings and, more generally, endomorphism rings of free modules of infinite rank. In fact, over any ring R, we give an explicit 2-fine decomposition for generic matrices of orders 2 and 3. In the main theorem, we prove that over any ring R, the matrix ring $M_m(R)$ is 2-fine for each $m \geq 2$. As a consequence, we will revisit Henriksen's result that a proper matrix ring over any ring has unit sum number at most 3. An example shows that there exists a 2-fine ring that is not fine. This shows that n-fine rings are a proper generalization of fine rings. We also show that if R is n-fine, then so is the matrix ring $M_m(R)$ for any integer $m \geq 1$. Moreover, we prove that for any ring R, the endomorphism ring of a free R-module of rank at least 2 is 2-fine.

2. Basic properties of n-fine rings

Definition 1. Let n be a positive integer. A nonzero element x of R is called n-fine if $x = t + u_1 + \ldots + u_n$ where t is a nilpotent element of R and $u_1, \ldots, u_n$ are units in R. A ring R is called n-fine if every nonzero element of R is n-fine.

Proposition 1. *Let R be a ring. Then the following statements hold:*

(1) *If R is n-fine, then it is also m-fine for all $m \geq n$.*
(2) *If $Nil(R)$ is additively closed (in particular, if R is a commutative ring), then the sum of n-fine and m-fine elements of R is $(n+m)$-fine.*
(3) *Every n-good ring is n-fine; if R is n-fine, then R is $(n+1)$-good.*

Proof.

(1) Let r be a nonzero element of R and let $m > n$. Then, we can write $r = (r-(m-n).1) + (m-n).1$ and expressing $(r-(m-n).1)$ as a sum of a nilpotent element and n units of R gives a representation of r as a sum of a nilpotent and m units.
(2) It suffices to notice that if $Nil(R)$ is additively closed, then the sum of two nilpotent elements is a nilpotent element.
(3) It is clear that every n-good ring is n-fine. For the second statement, let r be a nonzero element of R. By hypothesis, $r - 1 = t + u_1 + \ldots + u_n$ where $t \in Nil(R)$ and $u_1, \ldots, u_n \in U(R)$. Hence, $r = (1+t) + u_1 + \ldots + u_n$. Since, $(1+t) \in U(R)$. Then, r is $(n+1)$-good.

□

In the next two lemmas we consider the effect of some ring operations on our invariants

Lemma 1. *Let R be a ring, I an ideal of R and let $J(R)$ denote the Jacobson radical of R. If $a \in R$ is n-fine, then so is $\overline{a} \in R/I$. The converse also holds if $I \subseteq J(R)$ and $J(R)$ is nil.*

Proof. The first part of the statement is clear since the image of a unit (resp., a nilpotent) is again a unit (resp., a nilpotent).
Assume now that $I \subseteq J(R)$ and $J(R)$ is nil. Let $a \in R$ be such that $\overline{a}$ is n-fine in R/I. Then there are unit elements $\overline{u_i} \in R/I$, $1 \leq i \leq n$ and a nilpotent element $\overline{t} \in R/I$ such that $\overline{a} = \overline{t} + \overline{u_1} + \ldots + \overline{u_n}$. Then $u_i \in U(R)$ for all $1 \leq i \leq n$, and $h = a - (t + u_1 + \ldots + u_n) \in J(R)$. So, $u_1 + h \in U(R)$. Also, $t^m \in I$ for some integer $m \geq 2$. Hence, there is an integer $m' \geq 2$ such that $t^{mm'} = 0$ since $I \subseteq J(R)$ is nil; that is t is a nilpotent element of R. It follows that $a = t + (u_1 + h) + u_2 + \ldots + u_n$. This shows that a is n-fine. □

Lemma 2. *Let n be a positive integer. The following hold:*

(1) *A homomorphic image of a n-fine ring is n-fine.*
(2) *A direct product $\prod R_\alpha$ of rings $\{R_\alpha\}$ is n-fine if and only if so is each $\{R_\alpha\}$.*

Proof.

(1) The proof of (1) is clear.
(2) Suppose that each $\{R_\alpha\}$ is n-fine. Let $x = (x_\alpha) \in \prod R_\alpha$. For each α, write $x_\alpha = t_\alpha + u_\alpha^1 + \ldots + u_\alpha^n$, where $u_\alpha^i \in U(R_\alpha)$ for all $1 \leq i \leq n$ and $t_\alpha \in Nil(R_\alpha)$. Then, $x = t + u^1 + \ldots + u^n$, where $u_i = (u_\alpha^i) \in U(\prod R_\alpha)$ for all $1 \leq i \leq n$ and $t = (t_\alpha) \in Nil(\prod R_\alpha)$. Hence, $\prod R_\alpha$ is n-fine.
The converse immediately follows from Lemma 1.

□

We next determine when a polynomial ring or power series ring is a n-fine ring.

Proposition 2. *Let R be a nonzero commutative ring and n be a positive integer.*

(1) *$R[X]$ is never a n-fine ring.*
(2) *$R[[x]]$ is n-fine if and only if so is R.*

Proof.

(1) Note that $Nil(R[X]) = \{r_0 + r_1X + \ldots + r_nX^n |\ r_i \in \sqrt{0}\ (i = 0, \ldots, n)\}$ and $U(R[X]) = \{r_0 + r_1X + \ldots + r_nX^n |\ r_0 \in U(R), r_i \in \sqrt{0}\ (i = 1, \ldots, n)\}$. If X is n-fine, we may let

$$X = t + (u_1 + r_1X + \ldots) + (u_2 + r_2X + \ldots) + \ldots + (u_n + r_nX + \ldots),$$

where $t \in Nil(R)$, $u_1, \ldots, u_n \in U(R)$ and $r_1, \ldots, r_n \in \sqrt{0} \subseteq J(R)$ Jacobson radical of R, for each $1 \leq i \leq n$. Then $\sum_{i=1}^n r_i = 1 \in J(R)$, which is a contradiction. Thus $R[X]$ is not a n-fine ring.

(2) By Lemma 2(1), $R[[x]]$ is a n-fine ring, and this gives that $R = R[[x]]/(x)$ is a n-fine ring. Conversely, suppose that R is n-fine. Let $f = \sum_{i=0}^{\infty} r_i x^i \in R[[x]]$. Write $r_0 = t + u_1 + \ldots + u_n$, where $t \in Nil(R)$ and $u_1, \ldots, u_n$ are units in R. Then, $f = t + (u_1 + r_1 x + r_2 x^2 + \ldots) + u_2 + \ldots + u_n$, where $t \in Nil(R) \subseteq Nil(R[[x]])$ and $(u_1 + r_1 x + r_2 x^2 + \ldots) \in U(R[[x]])$, $u_i \in U(R) \subseteq U(R[[x]])$ for all $2 \leq i \leq n$. Thus, $R[[x]]$ is n-fine.

□

The next result shows that the n-fine property needs to be checked only for indecomposable rings.

Theorem 1. *Let R be a nonzero ring. Then the following are equivalent:*

(1) *R is n-fine;*
(2) *Every factor ring of R is n-fine;*
(3) *Every indecomposable factor ring of R is n-fine.*

Proof.

(1) ⇒ (2) Follows from Lemma 1.

(2) ⇒ (3) It is clear.

(3) ⇒ (1) Suppose that (3) holds and R is not n-fine. Then there is an element $a \in R$ which is not n-fine. Now let S be the set of all proper ideals I of R such $\overline{a}$ is not n-fine in R/I. Clearly $0 \in S$, and the set S is not empty. If $\{I_\alpha, \alpha \in A\}$ is a chain in S, let $I = \bigcup_{\alpha \in A} I_\alpha$. We prove that $\overline{a}$ is not n-fine in R/I. Suppose that $\overline{a}$ is n-fine in R/I. Then there are $\overline{u_1}, \ldots, \overline{u_n} \in U(R/I)$ (with inverses $\overline{v_1}, \ldots, \overline{v_n}$ respectively) and $\overline{t} \in Nil(R/I)$ such that $\overline{a} = \overline{t} + \overline{u_1} + \ldots + \overline{u_n}$. Note that $t^m \in \bigcup_{\alpha \in A} I_\alpha$ for some positive integer $m \geq 2$ and $u_i v_i - 1$, $v_i u_i - 1 \in \bigcup_{\alpha \in A} I_\alpha$, hence $t^m \in I_{\alpha_0}$, $u_i v_i \in I_{\alpha_i}$ and $v_i u_i \in I_{\alpha_i'}$ for α_0, α_i and $\alpha_i' \in A$. Now, we can use Zorn's lemma to pick an ideal I_0 of R maximal with respect to the property that a is not n-fine in R/I. Then R/I is decomposable as a ring by (3): $R/I_0 = R/I_1 \bigoplus R/I_2$, where both the ideals I_1, I_2 strictly contain I_0 and so by the choice of I_0, $\overline{a}$ is n-fine in R/I_1 and R/I_2. But then $\overline{a}$ is n-fine in R/I_0 by Lemma 2(2), a contradiction.

□

3. n-FINENESS OF MATRIX RINGS

The purpose of this section is to investigate n-fine property of matrices over any arbitrary ring and of the endomorphism ring of a free R-module of rank at least 2. First we give the following interesting decomposition.

Theorem 2. *Over any ring, the 2×2 and 3×3 matrices are 2-fine.*

Proof. Let R be a ring and let $M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2(R) \setminus \{0\}$. Put $N = \begin{pmatrix} a_{11}-1 & 1-a_{11} \\ a_{11}-1 & 1-a_{11} \end{pmatrix}$. It is checked easily that then $N^2 = 0$. Thus we have

$$M - N = \begin{pmatrix} 1 & a_{12}+a_{11}-1 \\ a_{21}-a_{11}+1 & a_{22}-a_{11}-1 \end{pmatrix}.$$

Now there exist invertible matrices P and Q such that

$$P(M-N)Q = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -1 & c \end{pmatrix}$$

for an appropriate c and thus is a sum of two units. Hence M is 2-fine. Now, let

$$M = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

be 3×3 matrix over $R \setminus \{0\}$. We first construct an nilpotent in order to show 2-fineness of M. Set

$$N = \begin{pmatrix} b_{11}-1 & b_{22}-1 & 2-b_{11}-b_{22} \\ b_{11}-1 & b_{22}-1 & 2-b_{11}-b_{22} \\ b_{11}-1 & b_{22}-1 & 2-b_{11}-b_{22} \end{pmatrix}.$$

It may be directly verified that $N^2 = 0$. Thus

$$M - N = \begin{pmatrix} 1 & b_{12}-b_{22}+1 & b_{13}+b_{11}+b_{22}-2 \\ b_{21}-b_{11}+1 & 1 & b_{23}+b_{11}+b_{22}-2 \\ b_{31}-b_{11}+1 & b_{32}-b_{22}+1 & b_{33}+b_{11}+b_{22}-2 \end{pmatrix}.$$

We only need to show that $M-N$ is 2-good. Now there exist invertible matrices P and Q such that

$$P(M-N)Q = \begin{pmatrix} c_1 & 0 & c_2 \\ c_3 & 1 & 0 \\ 0 & c_4 & c_5 \end{pmatrix} = \begin{pmatrix} 0 & 1 & c_2 \\ 0 & 0 & 1 \\ 1 & c_4 & c_5 \end{pmatrix} + \begin{pmatrix} c_1 & -1 & 0 \\ c_3 & 1 & -1 \\ -1 & 0 & 0 \end{pmatrix}$$

for an appropriate c_i where $i \in \{1, \ldots, 5\}$ and thus is a sum of two units. Hence M is 2-fine. This completes the proof. □

Remark 1.

(1) For any ring R, R can be embedded in the 2×2 matrix ring $M_2(R)$. That is, all rings can be embedded in a 2-fine ring by Theorem 2.

(2) It is known that fine rings are 2-fine rings. However, the converse is not true. For example, taking $R = M_2(\mathbb{Z})$, then R is a 2-fine ring by Theorem 2. Let $M = \begin{pmatrix} 3 & 5 \\ 0 & 0 \end{pmatrix}$, then M is not fine in R by [2, Corollary 5.4], that is, R is not a fine ring.

To facilitate the proof of Theorem 3, we isolate the following lemma, which is of some independent interest.

Lemma 3. *Let R be a ring, $n,m \geq 1$ and $k \geq 2$. If the matrix rings $M_n(R)$ and $M_m(R)$ are both k-fine, then so is the matrix ring $M_{n+m}(R)$.*

Proof. Let $M \in M_{n+m}(R)$ which we will write in the block decomposition form

$$M = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where $A_{11} \in M_n(R)$, $A_{22} \in M_m(R)$ and A_{12}, A_{21} are appropriately sized rectangular matrices. By hypothesis, there exist invertible $n \times n$, $m \times m$ matrices $U_1, U_2, \ldots, U_k$ and $V_1, V_2, \ldots, V_k$, and nilpotent matrices N_1, N_2 such that $A_{11} = N_1 + U_1 + U_2 + \ldots + U_k$ and $A_{22} = N_2 + V_1 + V_2 + \ldots + V_k$. Thus the decomposition

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} U_1 & A_{12} \\ 0 & V_1 \end{pmatrix} + \begin{pmatrix} U_2 & 0 \\ A_{21} & V_2 \end{pmatrix} + \ldots + \begin{pmatrix} U_k & 0 \\ 0 & V_k \end{pmatrix} + \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix}$$

shows that M is k-fine. □

We now present the titular result.

Theorem 3. *Over any ring R, the matrix ring $M_m(R)$ is 2-fine for any positive integer $m \geq 2$.*

Proof. The result follows directly by combining Theorem 2 and Lemma 3. □

Remark 2. Commenting on Theorem 3, [2, Proposition 2.12] ensures that over any ring S, the ring $R = M_n(S)$ where $n \geq 2$ is 3-fine. Indeed, if M is a matrix of R, then $M - I_n = T_1 + T_2$ where T_1 and T_2 are two fine matrices. Then, there are $N_1, N_2 \in Nil(R)$ and $U_1, U_2 \in U(R)$ such that $M - I_n = (N_1 + U_1) + (N_2 + U_2)$. Hence, $M = N_1 + (N_2 + I_n) + U_1 + U_2$ where $N_2 + I_n \in U(R)$, this implies that M is 3-fine.

Again, combining Proposition 1 and Theorem 3 we recover a result due to Henriksen [3, Theorem 3].

Corollary 1. *If R is any nonzero ring with identity, and $m \geq 2$, then every matrix in $M_m(R)$ is the sum of three invertible matrices in $M_m(R)$.*

Theorem 4. *Let $n \geq 1$. If R is a n-fine ring, then so is the matrix ring $M_m(R)$ for any positive integer m.*

Proof. For $n = 1$, the result follows from [2, Theorem 3.1]. For $n \geq 2$, it is clear by induction and by Lemma 3. □

We will conclude this section by considering the 2-fineness of the endomorphism ring of a free R-module of rank at least 2.

Proposition 3. *Let R be a ring and let the free R-module F be (isomorphic to) the direct sum of $\alpha \geq 2$ copies of R where α is a cardinal number. Then the ring of endomorphisms E of F is 2-fine.*

Proof. Assume first that $\alpha \geq 2$ is finite, so $E \cong M_\alpha(R)$. Then E is 2-fine for $\alpha = 2,3$ by Theorem 2 and the values $\alpha < \omega$ for which E is 2-fine are closed under addition by Theorem 3. So E is 2-fine for all finite α.
Assume now that α is infinite. Then from $F \cong F \oplus F$, $E \cong M_2(E)$, and so E is 2-fine by Theorem 2. □

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