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QUANTUM HERMITE-HADAMARD TYPE INEQUALITIES AND RELATED INEQUALITIES FOR SUBADDITIVE FUNCTIONS

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Abstract. In this work, Hermite-Hadamard type inequalities for subadditive functions via quantum integrals are established. Moreover, Hermite-Hadamard type inequalities for the product of two subadditive functions are also obtained. It is worth mentioning that some existing inequalities of Hermite-Hadamard type for subadditive functions are obtained by considering the limit of the real number $q \in (0, 1)$ as $q \rightarrow 1^-$ in the key results.

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1. INTRODUCTION

The main work on the general theory of subadditive functions is due to Hill and Phillips [9]. This link also contains part of Rosenbaum's work [14] on the subadditive function of several variables. Additivity, subadditivity, and superadditivity are important concepts in measure theory, as well as in various areas of mathematics and mathematical inequality. There are many examples of additive, subadditive, and superadditive functions, especially in various areas of mathematics such as norms, square roots, error functions, growth rates, differential equations, and integral mean. Inequalities, especially the theory of additive functions, is one of the most advanced areas of theoretical and applied mathematics, as well as physics and other applied sciences. Here we refer to the results of [5, 9, 11, 14] and the corresponding references cited therein.

Definition 1. A function $\mathcal{F}$ defined on a set H of real numbers and with the range contained in the set R^+ of all positive real numbers, is subadditive on H if, for all elements $\varkappa$ and γ of H such that $\varkappa + \gamma$ is an element of H

$$\mathcal{F}(\varkappa + \gamma) \leq \mathcal{F}(\varkappa) + \mathcal{F}(\gamma).$$

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If the equality holds, $\mathcal{F}$ is called additive; if the inequality is reversed, $\mathcal{F}$ is superadditive.

Remark 1. If $\mathcal{F}$ is convex and subadditive on H and if $\mathcal{F}(0) = 0$, then $\mathcal{F}$ is additive on H .

Definition 2. A function $\mathcal{F} : [0, \kappa_2] \rightarrow R$, $\kappa_2 > 0$ is said to be starshaped if for every $\varkappa \in [0, \kappa_2]$ and $\tau \in [0, 1]$ we have $\mathcal{F}(\tau\varkappa) \leq \tau\mathcal{F}(\varkappa)$.

According to the above definitions, if a subadditive function $\mathcal{F} : A \subset [0, \infty) \rightarrow R$ is also starshaped, then $\mathcal{F}$ is a convex function.

The following inequality is well known in the literature as the Hermite-Hadamard integral inequality (see, [7]):

$$\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2}\right) \leq \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa) d\varkappa \leq \frac{\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2)}{2}, \quad (1.1)$$

where $\mathcal{F} : I \subset \mathbb{R} \rightarrow \mathbb{R}$ is a convex function on the interval I of real numbers and $\kappa_1, \kappa_2 \in I$ with $\kappa_1 < \kappa_2$.

Inequality (1.1) has become an important pillar of mathematical analysis and optimization. Given the variety of settings, you can see that this inequality has many uses. Also, many inequalities are obtained in a special way for a given choice of the function $\mathcal{F}$. The Hermite-Hadamard inequality (1.1), for example, is important in its rich geometry, and many studies are being carried out to prove its proofs, improvements, extensions, and new generalizations. You can check [6, 8, 15] and the references included there.

2. PRELIMINARIES OF q -CALCULUS

In this section, we present some required definitions about q -calculus. Throughout the paper, we consider that $0 < q < 1$.

Definition 3 ([13]). Let $\mathcal{F} : [\kappa_1, \kappa_2] \rightarrow \mathbb{R}$ be a function. Then, the q_{κ_1} -definite integral on $[\kappa_1, \kappa_2]$ is defined as:

$$\begin{aligned} \int_{\kappa_1}^{\kappa_2} \mathcal{F}(\tau) {}_{\kappa_1}d_q\tau &= (1-q)(\kappa_2 - \kappa_1) \sum_{n=0}^{\infty} q^n \mathcal{F}(q^n\kappa_2 + (1-q^n)\kappa_1) \\ &= (\kappa_2 - \kappa_1) \int_0^1 \mathcal{F}((1-\tau)\kappa_1 + \tau\kappa_2) d_q\tau. \end{aligned} \quad (2.1)$$

Moreover, we will need to use the following lemma in our main results:

Lemma 1. For $a \in \mathbb{R} \setminus \{-1\}$ we have the equality

$$\int_{\kappa_1}^{\kappa_2} (\varkappa - \kappa_1)^a {}_{\kappa_1}d_q\varkappa = \frac{(\kappa_2 - \kappa_1)^{a+1}}{[a+1]_q}.$$

On the other side, Bermudo et al. gave the following new definition:

Definition 4 ([2]). Let $\mathcal{F} : [\kappa_1, \kappa_2] \rightarrow \mathbb{R}$ be a function. Then, the q^{κ_2} -definite integral on $[\kappa_1, \kappa_2]$ is defined as:

$$\begin{aligned} \int_{\kappa_1}^{\kappa_2} \mathcal{F}(\tau) {}_{\kappa_1} d_q \tau &= (1-q)(\kappa_2 - \kappa_1) \sum_{n=0}^{\infty} q^n \mathcal{F}(q^n \kappa_1 + (1-q^n) \kappa_2) \\ &= (\kappa_2 - \kappa_1) \int_0^1 \mathcal{F}(\tau \kappa_1 + (1-\tau) \kappa_2) d_q \tau. \end{aligned}$$

We have to give the following notations which will be used many times in the next sections (see, [10]):

$$[n]_q = \frac{q^n - 1}{q - 1}.$$

Recently, many studies have been conducted in quantum analysis. In 2018, one of these is the following q -Hermite-Hadamard inequality proved by Alp [1]:

Theorem 1 ([1]). Let $\mathcal{F} : [\kappa_1, \kappa_2] \rightarrow \mathbb{R}$ be a convex differentiable function on $[\kappa_1, \kappa_2]$ and $0 < q < 1$. Then we have

$$\mathcal{F}\left(\frac{q\kappa_1 + \kappa_2}{[2]_q}\right) \leq \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa) {}_{\kappa_1} d_q \varkappa \leq \frac{q\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2)}{[2]_q},$$

where $[2]_q = 1 + q$.

Theorem 2 ([2]). If $\mathcal{F} : [\kappa_1, \kappa_2] \rightarrow \mathbb{R}$ is a convex differentiable function on $[\kappa_1, \kappa_2]$. Then, we have the following q -Hermite-Hadamard inequalities

$$\mathcal{F}\left(\frac{\kappa_1 + q\kappa_2}{[2]_q}\right) \leq \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa) {}_{\kappa_2} d_q \varkappa \leq \frac{\mathcal{F}(\kappa_1) + q\mathcal{F}(\kappa_2)}{[2]_q}.$$

Recently, Alp et al. [1] obtained quantum estimations for q -midpoint and q -type inequalities and for more recent results one can read [3, 4, 17].

The main objective of this research is to offer q -Hermite-Hadamard type inequalities for subadditive functions by using the notions of q_{κ_1} and q^{κ_2} -integrals.

3. MAIN RESULTS

The q -Hermite-Hadamard type inequalities for subadditive functions are given in the following form:

Theorem 3. We assume that a continuous mapping $\mathcal{F} : I = [0, \infty) \rightarrow R$ is subadditive, $\kappa_1, \kappa_2 \in I^\circ$ with $\kappa_1 < \kappa_2$, then the following inequalities hold

$$\begin{aligned} \mathcal{F}(\kappa_1 + \kappa_2) &\leq \frac{1}{\kappa_2 - \kappa_1} \left[\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\mathcal{x}) \, {}^{\kappa_2}d_q \mathcal{x} + \int_{\kappa_1}^{\kappa_2} \mathcal{F}(\mathcal{x}) \, {}_{\kappa_1}d_q \mathcal{x} \right] \\ &\leq \frac{1}{\kappa_1} \left[\int_0^{\kappa_1} \mathcal{F}(\mathcal{x}) \, {}_0d_q \mathcal{x} + \int_0^{\kappa_1} \mathcal{F}(\mathcal{x}) \, {}^{\kappa_1}d_q \mathcal{x} \right] \\ &\quad + \frac{1}{\kappa_2} \left[\int_0^{\kappa_2} \mathcal{F}(\mathcal{x}) \, {}_0d_q \mathcal{x} + \int_0^{\kappa_2} \mathcal{F}(\mathcal{x}) \, {}^{\kappa_2}d_q \mathcal{x} \right], \end{aligned} \quad (3.1)$$

where $0 < q < 1$.

Proof. Since $\mathcal{F}$ is a subadditive function, we can write

$$\begin{aligned} \mathcal{F}(\kappa_1 + \kappa_2) &= \mathcal{F}(\tau\kappa_1 + (1 - \tau)\kappa_2 + \tau\kappa_2 + (1 - \tau)\kappa_1) \\ &\leq \mathcal{F}(\tau\kappa_1 + (1 - \tau)\kappa_2) + \mathcal{F}(\tau\kappa_2 + (1 - \tau)\kappa_1). \end{aligned} \quad (3.2)$$

q -integrating inequality (3.2) with respect to τ over $[0, 1]$, from Definitions 3 and 4, we have

$$\begin{aligned} \mathcal{F}(\kappa_1 + \kappa_2) &\leq \int_0^1 \mathcal{F}(\tau\kappa_1 + (1 - \tau)\kappa_2) \, d_q \tau + \int_0^1 \mathcal{F}(\tau\kappa_2 + (1 - \tau)\kappa_1) \, d_q \tau \\ &= \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \mathcal{F}(\mathcal{x}) \, {}^{\kappa_2}d_q \mathcal{x} + \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \mathcal{F}(\mathcal{x}) \, {}_{\kappa_1}d_q \mathcal{x} \end{aligned}$$

and the first inequality in (3.1) is proved. To prove the second inequality in (3.1), we use the subadditivity of $\mathcal{F}$, we have

$$\mathcal{F}(\tau\kappa_1 + (1 - \tau)\kappa_2) \leq \mathcal{F}(\tau\kappa_1) + \mathcal{F}((1 - \tau)\kappa_2) \quad (3.3)$$

and

$$\mathcal{F}(\tau\kappa_2 + (1 - \tau)\kappa_1) \leq \mathcal{F}(\tau\kappa_2) + \mathcal{F}((1 - \tau)\kappa_1). \quad (3.4)$$

Adding the inequalities (3.3) and (3.4), then q -integrating the resultant one with respect to τ over $[0, 1]$, we have

$$\begin{aligned} &\frac{1}{\kappa_2 - \kappa_1} \left[\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\mathcal{x}) \, {}^{\kappa_2}d_q \mathcal{x} + \int_{\kappa_1}^{\kappa_2} \mathcal{F}(\mathcal{x}) \, {}_{\kappa_1}d_q \mathcal{x} \right] \\ &\leq \frac{1}{\kappa_1} \left[\int_0^{\kappa_1} \mathcal{F}(\mathcal{x}) \, {}_0d_q \mathcal{x} + \int_0^{\kappa_1} \mathcal{F}(\mathcal{x}) \, {}^{\kappa_1}d_q \mathcal{x} \right] \\ &\quad + \frac{1}{\kappa_2} \left[\int_0^{\kappa_2} \mathcal{F}(\mathcal{x}) \, {}_0d_q \mathcal{x} + \int_0^{\kappa_2} \mathcal{F}(\mathcal{x}) \, {}^{\kappa_2}d_q \mathcal{x} \right], \end{aligned}$$

which completes the proof. $\square$

Remark 2. In Theorem 3, if we take the limit $q \rightarrow 1^-$, then we have the following inequality of Hermite-Hadamard type for subadditive functions (see, [16]):

$$\frac{1}{2} \mathcal{F}(\kappa_1 + \kappa_2) \leq \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa) d\varkappa \leq \frac{1}{\kappa_1} \int_0^{\kappa_1} \mathcal{F}(\varkappa) d\varkappa + \frac{1}{\kappa_2} \int_0^{\kappa_2} \mathcal{F}(\varkappa) d\varkappa.$$

Corollary 1. Under the conditions of Theorem 3, if we take $\mathcal{F}(\tau\varkappa) \leq \tau\mathcal{F}(\varkappa)$, then we get the following q -Hermite-Hadamard type inequality for convex functions (see, [2]):

$$\begin{aligned} \mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2}\right) &\leq \frac{1}{2} \mathcal{F}(\kappa_1 + \kappa_2) \\ &\leq \frac{1}{2(\kappa_2 - \kappa_1)} \left[\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa) {}^{\kappa_2}d_q\varkappa + \int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa) {}_{\kappa_1}d_q\varkappa \right] \\ &\leq \frac{\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2)}{2}. \end{aligned} \quad (3.5)$$

Remark 3. If we take the limit $q \rightarrow 1^-$ in Corollary 1, then we obtain the inequality (1.1) of Hermite-Hadamard for convex functions.

Theorem 4. We assume that the continuous mappings $\mathcal{F}, \mathcal{G} : I = [0, \infty) \rightarrow R$ are subadditive, $\kappa_1, \kappa_2 \in I^\circ$ with $\kappa_1 < \kappa_2$, then the following inequality holds:

$$\begin{aligned} &\mathcal{F}(\kappa_1 + \kappa_2) \mathcal{G}(\kappa_1 + \kappa_2) \\ &\leq \frac{1}{\kappa_2 - \kappa_1} \left[\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa) \mathcal{G}(\varkappa) {}^{\kappa_2}d_q\varkappa + \int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa) \mathcal{G}(\varkappa) {}_{\kappa_1}d_q\varkappa \right] \\ &\quad + \int_0^1 [\mathcal{F}(\tau\kappa_1) + \mathcal{F}((1-\tau)\kappa_2)] [\mathcal{G}(\tau\kappa_2) + \mathcal{G}((1-\tau)\kappa_1)] d_q\tau \\ &\quad + \int_0^1 [\mathcal{F}(\tau\kappa_2) + \mathcal{F}((1-\tau)\kappa_1)] [\mathcal{G}(\tau\kappa_1) + \mathcal{G}((1-\tau)\kappa_2)] d_q\tau \end{aligned} \quad (3.6)$$

where $0 < q < 1$.

Proof. By using the subadditivity of the functions $\mathcal{F}$ and $\mathcal{G}$, we can write

$$\begin{aligned} \mathcal{F}(\kappa_1 + \kappa_2) &= \mathcal{F}(\kappa_1\tau + (1-\tau)\kappa_2 + (1-\tau)\kappa_1 + \tau\kappa_2) \\ &\leq \mathcal{F}(\kappa_1\tau + (1-\tau)\kappa_2) + \mathcal{F}((1-\tau)\kappa_1 + \tau\kappa_2) \end{aligned}$$

and

$$\mathcal{G}(\kappa_1 + \kappa_2) \leq \mathcal{G}(\kappa_1\tau + (1-\tau)\kappa_2) + \mathcal{G}((1-\tau)\kappa_1 + \tau\kappa_2).$$

Multiplying the above inequalities, we have:

$$\begin{aligned} \mathcal{F}(\kappa_1 + \kappa_2) \mathcal{G}(\kappa_1 + \kappa_2) &\leq [\mathcal{F}(\kappa_1\tau + (1-\tau)\kappa_2) \mathcal{G}(\kappa_1\tau + (1-\tau)\kappa_2) \\ &\quad + \mathcal{F}((1-\tau)\kappa_1 + \tau\kappa_2) \mathcal{G}((1-\tau)\kappa_1 + \tau\kappa_2)] \\ &\quad + [\mathcal{F}(\kappa_1\tau + (1-\tau)\kappa_2) \mathcal{G}((1-\tau)\kappa_1 + \tau\kappa_2) \\ &\quad + \mathcal{F}((1-\tau)\kappa_1 + \tau\kappa_2) \mathcal{G}(\kappa_1\tau + (1-\tau)\kappa_2)] \end{aligned} \quad (3.7)$$

$$\begin{aligned}
&\leq [\mathcal{F}(\kappa_1\tau + (1-\tau)\kappa_2) \mathcal{G}(\kappa_1\tau + (1-\tau)\kappa_2) \\
&\quad + \mathcal{F}((1-\tau)\kappa_1 + \tau\kappa_2) \mathcal{G}((1-\tau)\kappa_1 + \tau\kappa_2)] \\
&\quad + [\mathcal{F}(\kappa_1\tau) + \mathcal{F}((1-\tau)\kappa_2)] [\mathcal{G}((1-\tau)\kappa_1) + \mathcal{G}(\tau\kappa_2)] \\
&\quad + [\mathcal{F}((1-\tau)\kappa_1) + \mathcal{F}(\tau\kappa_2)] [\mathcal{G}(\kappa_1\tau) + \mathcal{G}((1-\tau)\kappa_2)].
\end{aligned}$$

Taking the q -integral of both sides in (3.7) with respect to τ over $[0, 1]$ and by Definitions 3 and 4, we get

$$\begin{aligned}
&\mathcal{F}(\kappa_1 + \kappa_2) \mathcal{G}(\kappa_1 + \kappa_2) \\
&\leq \frac{1}{\kappa_2 - \kappa_1} \left[\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa) \mathcal{G}(\varkappa) {}^{\kappa_2}d_q\varkappa + \int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa) \mathcal{G}(\varkappa) {}_{\kappa_1}d_q\varkappa \right] \\
&\quad + \int_0^1 [\mathcal{F}(\tau\kappa_1) + \mathcal{F}((1-\tau)\kappa_2)] [\mathcal{G}(\tau\kappa_2) + \mathcal{G}((1-\tau)\kappa_1)] d_q\tau \\
&\quad + \int_0^1 [\mathcal{F}(\tau\kappa_2) + \mathcal{F}((1-\tau)\kappa_1)] [\mathcal{G}(\tau\kappa_1) + \mathcal{G}((1-\tau)\kappa_2)] d_q\tau,
\end{aligned}$$

which proves inequality (3.6). $\square$

Remark 4. In Theorem 4, if we set the limit $q \rightarrow 1^-$, then we have the following inequality of Hermite-Hadamard type for the product of two subadditive functions (see, [16]):

$$\begin{aligned}
&\frac{1}{2} \mathcal{F}(\kappa_1 + \kappa_2) \mathcal{G}(\kappa_1 + \kappa_2) \\
&\leq \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa) \mathcal{G}(\varkappa) d\varkappa \\
&\quad + \int_0^1 [\mathcal{F}(\kappa_1\tau) \mathcal{G}((1-\tau)\kappa_1) + \mathcal{F}((1-\tau)\kappa_2) \mathcal{G}(\tau\kappa_2)] d\tau \\
&\quad + \int_0^1 [\mathcal{F}(\kappa_1\tau) \mathcal{G}(\tau\kappa_2) d\tau + \mathcal{F}(\tau\kappa_2) \mathcal{G}(\tau\kappa_1)] d\tau.
\end{aligned}$$

Corollary 2. Under the conditions of Theorem 4, if we take $\mathcal{F}(\tau\kappa) \leq \tau\mathcal{F}(\kappa)$, then we get the following q -Hermite-Hadamard type inequality for the product of two convex functions:

$$\begin{aligned}
\mathcal{F}(\kappa_1 + \kappa_2) \mathcal{G}(\kappa_1 + \kappa_2) &\leq \frac{1}{\kappa_2 - \kappa_1} \left[\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa) \mathcal{G}(\varkappa) {}^{\kappa_2}d_q\varkappa + \int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa) \mathcal{G}(\varkappa) {}_{\kappa_1}d_q\varkappa \right] \\
&\quad + \frac{2q^2}{[2]_q [3]_q} M(\kappa_1, \kappa_2) + \left(\frac{1}{[3]_q} + \frac{q(1+q^2)}{[2]_q [3]_q} \right) N(\kappa_1, \kappa_2),
\end{aligned}$$

where

$$\begin{aligned}
M(\kappa_1, \kappa_2) &= \mathcal{F}(\kappa_1) \mathcal{G}(\kappa_1) + \mathcal{F}(\kappa_2) \mathcal{G}(\kappa_2) \\
N(\kappa_1, \kappa_2) &= \mathcal{F}(\kappa_1) \mathcal{G}(\kappa_2) + \mathcal{F}(\kappa_2) \mathcal{G}(\kappa_1).
\end{aligned} \tag{3.8}$$

Remark 5. In Corollary 2, if we take the limit $q \rightarrow 1^-$, then we obtain the inequality (2) of [12, Theorem 1].

Theorem 5. We assume that the continuous mappings $\mathcal{F}, \mathcal{G} : I = [0, \infty) \rightarrow R$ are subadditive, $\kappa_1, \kappa_2 \in I^\circ$ with $\kappa_1 < \kappa_2$, then the following inequality holds:

$$\begin{aligned} & \frac{1}{\kappa_2 - \kappa_1} \left[\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa) \mathcal{G}(\varkappa) {}^{\kappa_2} d_q \varkappa + \int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa) \mathcal{G}(\varkappa) {}_{\kappa_1} d_q \varkappa \right] \\ & \leq \frac{1}{\kappa_1} \left[\int_0^{\kappa_1} \mathcal{F}(\varkappa) \mathcal{G}(\varkappa) {}_0 d_q \varkappa + \int_0^{\kappa_1} \mathcal{F}(\varkappa) \mathcal{G}(\varkappa) {}^{\kappa_1} d_q \varkappa \right] \\ & \quad + \frac{1}{\kappa_2} \left[\int_0^{\kappa_2} \mathcal{F}(\varkappa) \mathcal{G}(\varkappa) {}^{\kappa_2} d_q \varkappa + \int_0^{\kappa_2} \mathcal{F}(\varkappa) \mathcal{G}(\varkappa) {}_0 d_q \varkappa \right] \\ & \quad + \int_0^1 [\mathcal{F}(\tau \kappa_1) \mathcal{G}((1-\tau) \kappa_2) + \mathcal{F}((1-\tau) \kappa_2) \mathcal{G}(\tau \kappa_1)] d_q \tau \\ & \quad + \int_0^1 [\mathcal{F}((1-\tau) \kappa_1) \mathcal{G}(\tau \kappa_2) + \mathcal{F}(\tau \kappa_2) \mathcal{G}((1-\tau) \kappa_1)] d_q \tau, \end{aligned} \quad (3.9)$$

where $0 < q < 1$.

Proof. Since $\mathcal{F}$ and $\mathcal{G}$ are subadditive functions, we can write

$$\begin{aligned} \mathcal{F}(\kappa_1 \tau + (1-\tau) \kappa_2) & \leq \mathcal{F}(\kappa_1 \tau) + \mathcal{F}((1-\tau) \kappa_2) \\ \mathcal{G}(\kappa_1 \tau + (1-\tau) \kappa_2) & \leq \mathcal{G}(\kappa_1 \tau) + \mathcal{G}((1-\tau) \kappa_2), \end{aligned}$$

and

$$\begin{aligned} \mathcal{F}((1-\tau) \kappa_1 + \tau \kappa_2) & \leq \mathcal{F}((1-\tau) \kappa_1) + \mathcal{F}(\tau \kappa_2) \\ \mathcal{G}((1-\tau) \kappa_1 + \tau \kappa_2) & \leq \mathcal{G}((1-\tau) \kappa_1) + \mathcal{G}(\tau \kappa_2). \end{aligned}$$

From the above inequalities, we get

$$\begin{aligned} & \mathcal{F}(\kappa_1 \tau + (1-\tau) \kappa_2) \mathcal{G}(\kappa_1 \tau + (1-\tau) \kappa_2) + \mathcal{F}((1-\tau) \kappa_1 + \tau \kappa_2) \mathcal{G}((1-\tau) \kappa_1 + \tau \kappa_2) \\ & \leq \mathcal{F}(\kappa_1 \tau) \mathcal{G}(\kappa_1 \tau) + \mathcal{F}(\kappa_1 \tau) \mathcal{G}((1-\tau) \kappa_2) \\ & \quad + \mathcal{F}((1-\tau) \kappa_2) \mathcal{G}(\kappa_1 \tau) + \mathcal{F}((1-\tau) \kappa_2) \mathcal{G}((1-\tau) \kappa_2) \\ & \quad + \mathcal{F}((1-\tau) \kappa_1) \mathcal{G}((1-\tau) \kappa_1) + \mathcal{F}((1-\tau) \kappa_1) \mathcal{G}(\tau \kappa_2) \\ & \quad + \mathcal{F}(\tau \kappa_2) \mathcal{G}((1-\tau) \kappa_1) + \mathcal{F}(\tau \kappa_2) \mathcal{G}(\tau \kappa_2) \end{aligned}$$

and taking the q -integral with respect to τ on $[0, 1]$, then using Definitions 3 and 4, we obtain inequality (3.9). $\square$

Remark 6. In Theorem 5, if we set the limit $q \rightarrow 1^-$, then we obtain the following inequality of Hermite-Hadamard type for the product of two subadditive functions (see, [16]):

$$\frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \mathcal{F}(\varkappa) \mathcal{G}(\varkappa) d\varkappa \leq \frac{1}{\kappa_1} \int_0^{\kappa_1} \mathcal{F}(\varkappa) \mathcal{G}(\varkappa) d\varkappa + \frac{1}{\kappa_2} \int_0^{\kappa_2} \mathcal{F}(\varkappa) \mathcal{G}(\varkappa) d\varkappa$$

$$+ \int_0^1 \mathcal{F}(\kappa_1 \tau) \mathcal{G}((1-\tau)\kappa_2) d\tau + \int_0^1 \mathcal{F}(\tau\kappa_2) \mathcal{G}((1-\tau)\kappa_1) d\tau.$$

Corollary 3. In Theorem 5, if we assume that $\mathcal{F}(\tau\kappa) \leq \tau\mathcal{F}(\kappa)$, then we have the following inequality of q -Hermite-Hadamard type for the product of two convex functions:

$$\begin{aligned} & \frac{1}{\kappa_2 - \kappa_1} \left[\int_{\kappa_1}^{\kappa_2} \mathcal{F}(\kappa) \mathcal{G}(\kappa) {}_{\kappa_2} d_q \kappa + \int_{\kappa_1}^{\kappa_2} \mathcal{F}(\kappa) \mathcal{G}(\kappa) {}_{\kappa_1} d_q \kappa \right] \\ & \leq \frac{2q^2}{[2]_q [3]_q} N(\kappa_1, \kappa_2) + \left(\frac{1}{[3]_q} + \frac{q(1+q^2)}{[2]_q [3]_q} \right) M(\kappa_1, \kappa_2), \end{aligned}$$

where $M(\kappa_1, \kappa_2)$ and $N(\kappa_1, \kappa_2)$ are defined by (3.8).

Remark 7. In Corollary 3, if we take the limit $q \rightarrow 1^-$, then we obtain the inequality (1) of [12, Theorem 1].

4. CONCLUSIONS

In this investigation, via the q_{κ_1} and q^{κ_2} -quantum integrals, we established q -Hermite-Hadamard type inequalities and some related inequalities for subadditive functions. It is also revealed that the findings shown here are a clear generalisation of those that have already been published. It is an important and new problem for the next researcher to be able to give similar inequalities in their future work for different integral operators.

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STURM-LIOUVILLE BOUNDARY VALUE PROBLEMS FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH p -LAPLACIAN OPERATOR VIA RIESZ-CAPUTO FRACTIONAL DERIVATIVES

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Abstract. In the current work, fractional differential equations (**FDEs**) with Sturm-Liouville boundary conditions (**BCs**) and p -Laplacian operator in terms of the Riesz-Caputo fractional derivatives are considered. With the aid of Schauder's and Banach's fixed point theorems, the existence and uniqueness results of the aforesaid problem are established. An explanatory example is set forth to make efficient the obtained results.

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1. INTRODUCTION

FDEs have acquired plentiful circulation and great significance thanks to their accurate description of some phenomena of real-world problems. For instance, we refer the reader to [21, 23, 30, 31, 33, 35] for detailed interpretations, and we refer to [1, 2, 6, 13–15, 28, 34] for the newest trends in the field of **FDEs**.

Boundary value problems (**BVPs**) for **FDEs** have become an interesting field of research. This is due to their strong presence in many significant problems in mathematical physics. In particular, Sturm-Liouville type **BVPs** have attracted the interest of many mathematicians and physicists due to their numerous applications in various scientific fields; see, for example, [4, 10, 11, 22, 37].

In [24], Leibenson introduced the p -Laplacian differential equation

$$(\varphi_p(u'(t)))' = f(t, u(t), u'(t)), \quad t \in (0, 1),$$

where $\varphi_p(r) = |r|^{p-2}r$, $p > 1$, for studying the turbulent flow problem in a porous medium. In the last few years, the topic was extended to the case of fractional differential equations using different types of fractional operators; see, for example, [18–20, 25, 29] and the references cited therein.

In 1892, Riesz [21] introduced a two-sided fractional operator including both left and right derivatives, which can reflect both the past and the future memory effects.

As a result of the two-sided feature of the Riesz fractional operator, it has arisen strongly in fractional modeling on finite domains and fractional variational problems; see [3, 5, 9, 17].

In despite of the significant contributions involving the numerical methods of the **FDEs** in terms of the Riesz-Caputo fractional derivatives [32, 36], the analytical results concerning the existence, uniqueness, and stability of the solutions are still scarce. For example, Chen et al. in [7] studied a class of **BVPs** for **FDEs** with the Riesz-Caputo derivative of the form

$$\begin{cases} {}^{RC}_0\mathfrak{D}_T^\gamma y(\tau) = g(\tau, y(\tau)), & t \in [0, T], \quad 0 < \gamma \leq 1, \\ y(0) = 0, \quad y(T) = y_T, \end{cases}$$

where ${}^{RC}_0\mathfrak{D}_T^\gamma$ is a Riesz-Caputo derivative and $g : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function. By means of new fractional Gronwall's inequalities and some fixed point theorems, they obtained the main results.

Further, Chen et al. [8] obtained some existence results of the anti-periodic **BVPs** for **FDEs** with Riesz-Caputo derivative

$$\begin{cases} {}^{RC}_0\mathfrak{D}_T^\gamma y(\tau) = g(\tau, y(\tau)), & t \in [0, T], \quad 1 < \gamma \leq 2, \\ y(0) + y(T) = 0, \quad y'(0) + y'(T) = 0, \end{cases}$$

where ${}^{RC}_0\mathfrak{D}_T^\gamma$ is a Riesz-Caputo derivative and $g : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function. Some existence results are established using Schaefer's and Schauder's fixed point theorems.

In [12], Gu et al. investigated the existence of positive solutions of the **FDEs** with Riesz-Caputo derivative

$$\begin{cases} {}^{RC}_0\mathfrak{D}_1^\alpha x(\tau) = h(\tau, x(\tau)), & t \in [0, 1], \quad 0 < \alpha \leq 1, \\ x(0) = x_0, \quad x(1) = x_1, \end{cases}$$

where ${}^{RC}_0\mathfrak{D}_T^\alpha$ is a Riesz-Caputo derivative and $h : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$ is a continuous function. Using Leray-Schauder and Krasnoselskii's fixed point theorems in a cone, they obtained trajectory of the solutions via the numerical discretization of the equivalent fractional integral equations. In this study, inspired by [7, 8], we investigate the existence and uniqueness of solutions of **FDEs** with Sturm-Liouville **BCs** and p -Laplacian operator of the form:

$$\begin{cases} {}^{RC}_0\mathfrak{D}_T^\alpha \left(\varphi_p \left({}^{RC}_0\mathfrak{D}_T^\beta v(t) \right) \right) = \mathcal{W}(t, v(t)), & t \in J := [0, T], \quad T < \infty, \\ a_1 \varphi_p \left({}^{RC}_0\mathfrak{D}_T^\beta v(t) \right) |_{t=0} - b_1 \left(\varphi_p \left({}^{RC}_0\mathfrak{D}_T^\beta v(t) \right) \right)' |_{t=0} = 0, \\ a_2 \varphi_p \left({}^{RC}_0\mathfrak{D}_T^\beta v(t) \right) |_{t=T} + b_2 \left(\varphi_p \left({}^{RC}_0\mathfrak{D}_T^\beta v(t) \right) \right)' |_{t=T} = 0, \\ c_1 v(0) - d_1 v'(0) = 0, \\ c_2 v(T) + d_2 v'(T) = 0, \end{cases} \quad (1.1)$$

where ${}^{RC}_0\mathfrak{D}_T^\alpha, {}^{RC}_0\mathfrak{D}_T^\beta$ denote the Riesz-Caputo fractional derivatives of order $\alpha, \beta \in (0, 1]$, $a_i, b_i, c_i, d_i \in \mathbb{R}, i = 1, 2$, $\mathcal{W} : J \times \mathbb{R} \rightarrow \mathbb{R}$ is a given continuous function, and $\Phi_p(r) = |r|^{p-2}r$, $p > 1$ is a p -Laplacian operator with $(\Phi_p)^{-1} = \Phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$.

To the best knowledge of the author, there are no publications have yet been treated with p -Laplacian **FDEs** with Sturm-Liouville **BCs** in terms of the Riesz-Caputo fractional derivative.

2. PRELIMINARIES

In this section, we recall some definitions, lemmas and properties of fractional calculus [21, 33] and Riesz-Caputo fractional derivative [3, 7].

Definition 1. (Riemann-Liouville fractional integrals)[33]

If $y \in C[0, T]$, then for $t \in [0, T]$, the left and the right Riemann-Liouville fractional integrals of order $\gamma > 0$ are defined by

$${}_0\mathfrak{J}_t^\gamma y(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} y(s) ds, \quad \gamma > 0, \quad (2.1)$$

$${}_t\mathfrak{J}_T^\gamma y(t) = \frac{1}{\Gamma(\gamma)} \int_t^T (s-t)^{\gamma-1} y(s) ds, \quad \gamma > 0, \quad (2.2)$$

respectively, where $\Gamma(\cdot)$ denotes the Gamma function.

Definition 2. (Riesz fractional integral)[21]

If $y \in C[0, T]$, then for $t \in [0, T]$, the Riesz fractional integral of order $\gamma > 0$ is defined by

$${}_0\mathfrak{J}_T^\gamma y(t) = \frac{1}{\Gamma(\gamma)} \int_0^t |t-s|^{\gamma-1} y(s) ds. \quad (2.3)$$

As a consequence of (2.1) and (2.2), it follows that

$${}_0\mathfrak{J}_T^\gamma y(t) = \frac{1}{2} ({}_0\mathfrak{J}_t^\gamma + {}_t\mathfrak{J}_T^\gamma) y(t). \quad (2.4)$$

Definition 3. (Caputo fractional derivatives)[33]

If $y \in C^n[0, T]$, then for $t \in [0, T]$, the left and the right Caputo fractional derivatives of order γ , $n-1 \leq \gamma < n$, are defined by

$${}_0\mathfrak{D}_t^\gamma y(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t (t-s)^{n-\gamma-1} \left(\frac{d}{ds} \right)^n y(s) ds, \quad (2.5)$$

$${}_t\mathfrak{D}_T^\gamma y(t) = \frac{1}{\Gamma(n-\gamma)} \int_t^T (s-t)^{n-\gamma-1} \left(-\frac{d}{ds} \right)^n y(s) ds, \quad (2.6)$$

respectively, where $n = [\gamma] + 1$, $n \in \mathbb{N}$.

Definition 4. (Riesz-Caputo fractional derivatives)[3]

If $y \in C^n[0, T]$, then for $t \in [0, T]$, the Riesz-Caputo fractional derivative of order γ , $n - 1 \leq \gamma < n$, is defined by

$${}^{RC}_0 \mathfrak{D}_T^\gamma y(t) = \frac{1}{\Gamma(n - \gamma)} \int_0^t |t - s|^{n - \gamma - 1} \left(\frac{d}{ds} \right)^n y(s) ds, \quad (2.7)$$

where $n = [\gamma] + 1$, $n \in \mathbb{N}$.

Using equations (2.5) and (2.6), it follows that

$${}^{RC}_0 \mathfrak{D}_T^\gamma y(t) = \frac{1}{2} ({}^C_0 \mathfrak{D}_t^\gamma + (-1)^n {}^C_t \mathfrak{D}_T^\gamma) y(t). \quad (2.8)$$

In particular, for $0 < \gamma < 1$ and $y(t) \in C(0, T)$, one has

$${}^{RC}_0 \mathfrak{D}_T^\gamma y(t) = \frac{1}{2} ({}^C_0 \mathfrak{D}_t^\gamma - {}^C_t \mathfrak{D}_T^\gamma) y(t). \quad (2.9)$$

Lemma 1. [21] If $y(t) \in C^n[0, T]$, then

$${}_0 \mathfrak{J}_t^\gamma {}^C_0 \mathfrak{D}_t^\gamma y(t) = y(t) - \sum_{m=0}^{n-1} \frac{y^{(m)}(0)}{m!} (t - 0)^m, \quad (2.10)$$

$${}_t \mathfrak{J}_T^\gamma {}^C_t \mathfrak{D}_T^\gamma y(t) = (-1)^n \left(y(t) - \sum_{m=0}^{n-1} \frac{(-1)^m y^{(m)}(T)}{m!} (T - t)^m \right). \quad (2.11)$$

In view of [7], equations (2.10) and (2.11) imply

$${}_0 \mathfrak{J}_T^\gamma {}^{RC}_0 \mathfrak{D}_T^\gamma y(t) = \frac{1}{2} ({}_0 \mathfrak{J}_t^\gamma {}^C_0 \mathfrak{D}_t^\gamma + (-1)^n {}_t \mathfrak{J}_T^\gamma {}^C_t \mathfrak{D}_T^\gamma) y(t). \quad (2.12)$$

In particular, for $0 < \gamma < 1$ and $y(t) \in C(0, T)$, one has

$${}_0 \mathfrak{J}_T^\gamma {}^{RC}_0 \mathfrak{D}_T^\gamma y(t) = y(t) - \frac{1}{2} (y(0) + y(T)) \quad (2.13)$$

To end this section, we recall some basic properties of the p -Laplacian operator [26, 27].

Definition 5. The p -Laplacian operator is defined as

$$\Phi_p(x) = |x|^{p-2} x, \quad p > 1,$$

at which $\Phi_p^{-1} = \Phi_q$ where $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 2. Let $\Phi_p(x)$, $p \geq 2$, be p -Laplacian operator and $|x|, |y| \leq M$, then

$$|\Phi_p(x) - \Phi_p(y)| \leq (p - 1) M^{p-2} |x - y|.$$

3. MAIN RESULTS

We consider the Banach space $C(J, \mathbb{R})$ of all real and continuous functions from J to $\mathbb{R}$ endowed with the supremum norm

$$\|\mathbf{v}\| = \sup_{t \in J} |\mathbf{v}(t)|.$$

Before investigating the main results, we consider the following auxiliary lemma.

Lemma 3. *Let $0 < \alpha, \beta \leq 1$ and $\mathbf{v} : J \rightarrow \mathbb{R}$ be a continuous function. Then the linear p -Laplacian BVP*

$${}^{RC}_0\mathfrak{D}_T^\alpha \left(\varphi_p \left({}^{RC}_0\mathfrak{D}_T^\beta \mathbf{v}(t) \right) \right) = \mathbf{v}(t), \quad t \in J \quad (3.1)$$

$$\begin{cases} a_1 \varphi_p \left({}^{RC}_0\mathfrak{D}_T^\beta \mathbf{v}(t) \right) |_{t=0} - b_1 \left(\varphi_p \left({}^{RC}_0\mathfrak{D}_T^\beta \mathbf{v}(t) \right) \right)' |_{t=0} = 0, \\ a_2 \varphi_p \left({}^{RC}_0\mathfrak{D}_T^\beta \mathbf{v}(t) \right) |_{t=T} + b_2 \left(\varphi_p \left({}^{RC}_0\mathfrak{D}_T^\beta \mathbf{v}(t) \right) \right)' |_{t=T} = 0, \end{cases} \quad (3.2)$$

$$\begin{cases} c_1 \mathbf{v}(0) - d_1 \mathbf{v}'(0) = 0, \\ c_2 \mathbf{v}(T) + d_2 \mathbf{v}'(T) = 0, \end{cases} \quad (3.3)$$

is equivalent to the integral equation

$$\begin{aligned} \mathbf{v}(t) = & \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \varphi_q(h_{\mathbf{v}}(s)) ds + \frac{1}{\Gamma(\beta)} \int_t^T (s-t)^{\beta-1} \varphi_q(h_{\mathbf{v}}(s)) ds \\ & - \frac{d_1}{2c_1\Gamma(\beta-1)} \int_0^T s^{\beta-2} \varphi_q(h_{\mathbf{v}}(s)) ds - \frac{1}{\Gamma(\beta)} \int_0^T s^{\beta-1} \varphi_q(h_{\mathbf{v}}(s)) ds \\ & - \frac{d_2}{2c_2\Gamma(\beta-1)} \int_0^T (T-s)^{\beta-2} \varphi_q(h_{\mathbf{v}}(s)) ds - \frac{1}{\Gamma(\beta)} \int_0^T (T-s)^{\beta-1} \varphi_q(h_{\mathbf{v}}(s)) ds, \end{aligned} \quad (3.4)$$

where

$$\begin{aligned} h_{\mathbf{v}}(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathbf{v}(s) ds + \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} \mathbf{v}(s) ds \\ & - \frac{b_1}{2a_1\Gamma(\alpha-1)} \int_0^T s^{\alpha-2} \mathbf{v}(s) ds - \frac{1}{\Gamma(\alpha)} \int_0^T s^{\alpha-1} \mathbf{v}(s) ds \\ & - \frac{b_2}{2a_2\Gamma(\alpha-1)} \int_0^T (T-s)^{\alpha-2} \mathbf{v}(s) ds - \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \mathbf{v}(s) ds. \end{aligned} \quad (3.5)$$

Proof. Let $\omega = \varphi_p \left({}^{RC}_0\mathfrak{D}_T^\beta \mathbf{v}(t) \right)$. Then the BVP (3.1)-(3.2) will be written as

$$\begin{cases} {}^{RC}_0\mathfrak{D}_T^\alpha \omega(t) & = \mathbf{v}(t), \quad t \in J, \\ a_1 \omega(0) - b_1 \omega'(0) & = 0, \\ a_2 \omega(T) + b_2 \omega'(T) & = 0, \end{cases} \quad (3.6)$$

From (2.3), (2.13) and the first equality of (3.6), one has

$$\begin{aligned}\omega(t) &= \frac{1}{2}(\omega(0) + \omega(T)) + \frac{1}{\Gamma(\alpha)} \int_0^T |t-s|^{\alpha-1} \mathbf{v}(s) ds \\ &= \frac{1}{2}(\omega(0) + \omega(T)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathbf{v}(s) ds + \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} \mathbf{v}(s) ds.\end{aligned}\quad (3.7)$$

Then

$$\omega'(t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \mathbf{v}(s) ds - \frac{1}{\Gamma(\alpha-1)} \int_t^T (s-t)^{\alpha-2} \mathbf{v}(s) ds.$$

Using the boundary conditions $a_1\omega(0) - b_1\omega'(0) = 0$ and $a_2\omega(T) + b_2\omega'(T) = 0$, we get

$$\begin{aligned}\omega(T) &= -\frac{b_1}{a_1\Gamma(\alpha-1)} \int_0^T s^{\alpha-2} \mathbf{v}(s) ds - \frac{2}{\Gamma(\alpha)} \int_0^T s^{\alpha-1} \mathbf{v}(s) ds, \\ \omega(0) &= -\frac{b_2}{a_2\Gamma(\alpha-1)} \int_0^T (T-s)^{\alpha-2} \mathbf{v}(s) ds - \frac{2}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \mathbf{v}(s) ds.\end{aligned}$$

Substituting the values of $\omega(0)$ and $\omega(T)$ into (3.7), we obtain that

$$\begin{aligned}\omega(t) &= \varphi_p \left({}^{RC}_0\mathfrak{D}_T^\beta \mathbf{v}(t) \right) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathbf{v}(s) ds + \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} \mathbf{v}(s) ds \\ &\quad - \frac{b_1}{2a_1\Gamma(\alpha-1)} \int_0^T s^{\alpha-2} \mathbf{v}(s) ds - \frac{1}{\Gamma(\alpha)} \int_0^T s^{\alpha-1} \mathbf{v}(s) ds \\ &\quad - \frac{b_2}{2a_2\Gamma(\alpha-1)} \int_0^T (T-s)^{\alpha-2} \mathbf{v}(s) ds \\ &\quad - \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \mathbf{v}(s) ds.\end{aligned}$$

Thus,

$${}^{RC}_0\mathfrak{D}_T^\beta \mathbf{v}(t) = \varphi_q(h_v(t)),$$

where the function h_v is defined in (3.5).

With a similar procedure, one can conclude that the **BVP**

$$\begin{cases} {}^{RC}_0\mathfrak{D}_T^\beta \mathbf{v}(t) &= \varphi_q(h_v(t)), \quad t \in J, \\ c_1 \mathbf{v}(0) - d_1 \mathbf{v}'(0) &= 0, \\ c_2 \mathbf{v}(T) + d_2 \mathbf{v}'(T) &= 0, \end{cases}\quad (3.8)$$

is equivalent to the integral equation

$$\begin{aligned}\mathbf{v}(t) &= \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \varphi_q(h_v(s)) ds + \frac{1}{\Gamma(\beta)} \int_t^T (s-t)^{\beta-1} \varphi_q(h_v(s)) ds \\ &\quad - \frac{d_1}{2c_1\Gamma(\beta-1)} \int_0^T s^{\beta-2} \varphi_q(h_v(s)) ds - \frac{1}{\Gamma(\beta)} \int_0^T s^{\beta-1} \varphi_q(h_v(s)) ds\end{aligned}$$

$$- \frac{d_2}{2c_2\Gamma(\beta-1)} \int_0^T (T-s)^{\beta-2} \Phi_q(h_v(s)) ds - \frac{1}{\Gamma(\beta)} \int_0^T (T-s)^{\beta-1} \Phi_q(h_v(s)) ds.$$

That is the desired integral equation (3.4). The proof is completed. $\square$

In view of Lemma 3, we deduce that the solution of the p -Laplacian BVP (1.1) is given by

$$\begin{aligned} v(t) = & \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \Phi_q(H_{\mathcal{W},v}(s)) ds + \frac{1}{\Gamma(\beta)} \int_t^T (s-t)^{\beta-1} \Phi_q(H_{\mathcal{W},v}(s)) ds \\ & - \frac{d_1}{2c_1\Gamma(\beta-1)} \int_0^T s^{\beta-2} \Phi_q(H_{\mathcal{W},v}(s)) ds - \frac{1}{\Gamma(\beta)} \int_0^T s^{\beta-1} \Phi_q(H_{\mathcal{W},v}(s)) ds \\ & - \frac{d_2}{2c_2\Gamma(\beta-1)} \int_0^T (T-s)^{\beta-2} \Phi_q(H_{\mathcal{W},v}(s)) ds \\ & - \frac{1}{\Gamma(\beta)} \int_0^T (T-s)^{\beta-1} \Phi_q(H_{\mathcal{W},v}(s)) ds, \end{aligned} \quad (3.9)$$

where

$$\begin{aligned} H_{\mathcal{W},v}(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathcal{W}(s, v(s)) ds + \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} \mathcal{W}(s, v(s)) ds \\ & - \frac{b_1}{2a_1\Gamma(\alpha-1)} \int_0^T s^{\alpha-2} \mathcal{W}(s, v(s)) ds - \frac{1}{\Gamma(\alpha)} \int_0^T s^{\alpha-1} \mathcal{W}(s, v(s)) ds \\ & - \frac{b_2}{2a_2\Gamma(\alpha-1)} \int_0^T (T-s)^{\alpha-2} \mathcal{W}(s, v(s)) ds \\ & - \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \mathcal{W}(s, v(s)) ds. \end{aligned} \quad (3.10)$$

For fulfillment the main results, the following assumptions will be imposed.

(A1): The function $\mathcal{W} : J \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous.

(A2): Two constants $L_{\mathcal{W}}, K_{\mathcal{W}} > 0$ exist such that

$$|\mathcal{W}(t, v_1) - \mathcal{W}(t, v_2)| \leq L_{\mathcal{W}} |v_1 - v_2|,$$

for each $t \in J$ and $v_1, v_2 \in \mathbb{R}$, along with

$$|\mathcal{W}(t, v)| \leq K_{\mathcal{W}}.$$

3.1. Existence result via Schauder's fixed point theorem

Lemma 4. (Schauder's fixed point theorem)[16] Let $\mathcal{U}$ be a Banach space with $\mathcal{B} \subset \mathcal{U}$ closed, bounded and convex, and $\mathcal{P} : \mathcal{B} \rightarrow \mathcal{B}$ is completely continuous. Then $\mathcal{P}$ has a fixed point in $\mathcal{U}$.

Theorem 1. Assume that (A1)–(A2) hold. Then the p -Laplacian BVP (1.1) possesses at least one solution on J .

Proof. The problem (1.1) will be transformed into the fixed point problem $\mathbf{v} = \mathcal{P}\mathbf{v}$, where the operator $\mathcal{P} : C(J, \mathbb{R}) \rightarrow C(J, \mathbb{R})$ is defined by

$$\begin{aligned} (\mathcal{P}\mathbf{v})(t) = & \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \Phi_q(H_{\mathcal{W},\mathbf{v}}(s)) ds + \frac{1}{\Gamma(\beta)} \int_t^T (s-t)^{\beta-1} \Phi_q(H_{\mathcal{W},\mathbf{v}}(s)) ds \\ & - \frac{d_1}{2c_1\Gamma(\beta-1)} \int_0^T s^{\beta-2} \Phi_q(H_{\mathcal{W},\mathbf{v}}(s)) ds - \frac{1}{\Gamma(\beta)} \int_0^T s^{\beta-1} \Phi_q(H_{\mathcal{W},\mathbf{v}}(s)) ds \\ & - \frac{d_2}{2c_2\Gamma(\beta-1)} \int_0^T (T-s)^{\beta-2} \Phi_q(H_{\mathcal{W},\mathbf{v}}(s)) ds \\ & - \frac{1}{\Gamma(\beta)} \int_0^T (T-s)^{\beta-1} \Phi_q(H_{\mathcal{W},\mathbf{v}}(s)) ds, \end{aligned} \quad (3.11)$$

where the function $H_{\mathcal{W},\mathbf{v}}$ is defined in (3.10).

Define $\mathcal{B}_k = \{\mathbf{v} \in C(J, \mathbb{R}) : \|\mathbf{v}\| \leq k\}$ with

$$k \geq \left[\frac{4T^\beta}{\Gamma(\beta+1)} + \frac{|d_1|T^{\beta-1}}{2|c_1|\Gamma(\beta)} + \frac{|d_2|T^{\beta-1}}{2|c_2|\Gamma(\beta)} \right] \Phi_q(M^*). \quad (3.12)$$

It is clear that the set $\mathcal{B}_k$ is a closed, bounded, convex subset of $C(J, \mathbb{R})$. Moreover, for $\mathbf{v}(t) \in \mathcal{B}_k$, the operator $\mathcal{P}$ is well defined.

We first show that $\mathcal{P}(\mathcal{B}_k) \subset \mathcal{B}_k$. For each $t \in J$ and $\mathbf{v} \in \mathcal{B}_k$, we get

$$\begin{aligned} |(\mathcal{P}\mathbf{v})(t)| \leq & \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} |\Phi_q(H_{\mathcal{W},\mathbf{v}}(s))| ds + \frac{1}{\Gamma(\beta)} \int_t^T (s-t)^{\beta-1} |\Phi_q(H_{\mathcal{W},\mathbf{v}}(s))| ds \\ & + \frac{|d_1|}{2|c_1|\Gamma(\beta-1)} \int_0^T s^{\beta-2} |\Phi_q(H_{\mathcal{W},\mathbf{v}}(s))| ds + \frac{1}{\Gamma(\beta)} \int_0^T s^{\beta-1} |\Phi_q(H_{\mathcal{W},\mathbf{v}}(s))| ds \\ & + \frac{|d_2|}{2|c_2|\Gamma(\beta-1)} \int_0^T (T-s)^{\beta-2} |\Phi_q(H_{\mathcal{W},\mathbf{v}}(s))| ds \\ & + \frac{1}{\Gamma(\beta)} \int_0^T (T-s)^{\beta-1} |\Phi_q(H_{\mathcal{W},\mathbf{v}}(s))| ds. \end{aligned} \quad (3.13)$$

From the assumption (A2), one has

$$\begin{aligned} |H_{\mathcal{W},\mathbf{v}}(t)| \leq & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |\mathcal{W}(s, \mathbf{v}(s))| ds + \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} |\mathcal{W}(s, \mathbf{v}(s))| ds \\ & + \frac{|b_1|}{2|a_1|\Gamma(\alpha-1)} \int_0^T s^{\alpha-2} |\mathcal{W}(s, \mathbf{v}(s))| ds + \frac{1}{\Gamma(\alpha)} \int_0^T s^{\alpha-1} |\mathcal{W}(s, \mathbf{v}(s))| ds \\ & + \frac{|b_2|}{2|a_2|\Gamma(\alpha-1)} \int_0^T (T-s)^{\alpha-2} |\mathcal{W}(s, \mathbf{v}(s))| ds \\ & + \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |\mathcal{W}(s, \mathbf{v}(s))| ds \end{aligned}$$

$$\begin{aligned}
&\leq \left[\frac{t^\alpha + (T-t)^\alpha + 2T^\alpha}{\Gamma(\alpha+1)} + \frac{|b_1|T^{\alpha-1}}{2|a_1|\Gamma(\alpha)} + \frac{|b_2|T^{\alpha-1}}{2|a_2|\Gamma(\alpha)} \right] K_{\mathcal{W}} \\
&\leq \left[\frac{4T^\alpha}{\Gamma(\alpha+1)} + \frac{|b_1|T^{\alpha-1}}{2|a_1|\Gamma(\alpha)} + \frac{|b_2|T^{\alpha-1}}{2|a_2|\Gamma(\alpha)} \right] K_{\mathcal{W}} := M^*.
\end{aligned} \tag{3.14}$$

Thus, by substitution the estimate M^* in (3.14) to (3.13), we get

$$\begin{aligned}
|(\mathcal{P}\mathbf{v})(t)| &\leq \left[\frac{t^\beta + (T-t)^\beta + 2T^\beta}{\Gamma(\beta+1)} + \frac{|d_1|T^{\beta-1}}{2|c_1|\Gamma(\beta)} + \frac{|d_2|T^{\beta-1}}{2|c_2|\Gamma(\beta)} \right] \Phi_q(M^*) \\
&\leq \left[\frac{4T^\beta}{\Gamma(\beta+1)} + \frac{|d_1|T^{\beta-1}}{2|c_1|\Gamma(\beta)} + \frac{|d_2|T^{\beta-1}}{2|c_2|\Gamma(\beta)} \right] \Phi_q(M^*) \\
&\leq k.
\end{aligned}$$

Thus, (3.12) implies that $\mathcal{P}(\mathcal{B}_k)$ maps into $\mathcal{B}_k$.

Now, we show that $\mathcal{P}$ is completely continuous on $\mathcal{B}_k$.

First, from $\mathcal{P}(\mathcal{B}_k) \subset \mathcal{B}_k$, it follows that $\|\mathcal{P}\mathbf{v}\| \leq k$ for any $\mathbf{v} \in \mathcal{B}_k$, and so $\{\mathbf{v} : \mathbf{v} \in \mathcal{P}\mathcal{B}_k\}$ is uniformly bounded.

Second, we show that $\mathcal{P}(\mathcal{B}_k)$ is equicontinuous. For each $t_1, t_2 \in J$, $t_1 < t_2$ and $\mathbf{v} \in \mathcal{B}_k$, using (3.14), we get

$$\begin{aligned}
&|(\mathcal{P}\mathbf{v})(t_2) - (\mathcal{P}\mathbf{v})(t_1)| \\
&\leq \left| \frac{1}{\Gamma(\beta)} \int_0^{t_2} (t_2-s)^{\beta-1} \Phi_q(H_{\mathcal{W},\mathbf{v}}(s)) ds \right. \\
&\quad \left. - \frac{1}{\Gamma(\beta)} \int_0^{t_1} (t_1-s)^{\beta-1} \Phi_q(H_{\mathcal{W},\mathbf{v}}(s)) ds \right| \\
&\quad + \left| \frac{1}{\Gamma(\beta)} \int_{t_2}^T (s-t_2)^{\beta-1} \Phi_q(H_{\mathcal{W},\mathbf{v}}(s)) ds \right. \\
&\quad \left. - \frac{1}{\Gamma(\beta)} \int_{t_1}^T (s-t_1)^{\beta-1} \Phi_q(H_{\mathcal{W},\mathbf{v}}(s)) ds \right| \\
&\leq \frac{1}{\Gamma(\beta)} \int_0^{t_2} \left| (t_2-s)^{\beta-1} - (t_1-s)^{\beta-1} \right| |\Phi_q(H_{\mathcal{W},\mathbf{v}}(s))| ds \\
&\quad + \frac{1}{\Gamma(\beta)} \int_{t_1}^{t_2} (t_1-s)^{\beta-1} |\Phi_q(H_{\mathcal{W},\mathbf{v}}(s))| ds \\
&\quad + \frac{1}{\Gamma(\beta)} \int_{t_2}^T \left| (s-t_2)^{\beta-1} - (s-t_1)^{\beta-1} \right| |\Phi_q(H_{\mathcal{W},\mathbf{v}}(s))| ds \\
&\quad + \frac{1}{\Gamma(\beta)} \int_{t_1}^{t_2} (s-t_1)^{\beta-1} |\Phi_q(H_{\mathcal{W},\mathbf{v}}(s))| ds \\
&\leq \frac{\Phi_q(M^*)}{\Gamma(\beta+1)} \left[(t_2^\beta - t_1^\beta) + ((T-t_2)^\beta - (T-t_1)^\beta) + 2(t_2-t_1)^\beta \right]
\end{aligned}$$

The right-hand side of the above inequality tends to zero as $t_2 \rightarrow t_1$. So we conclude that $\mathcal{P}(\mathcal{B}_k)$ is equicontinuous. As a consequence of the Arzelà-Ascoli Theorem, we infer that $\mathcal{P}(\mathcal{B}_k)$ is relatively compact in $C(J, \mathbb{R})$.

Finally, we show that $\mathcal{P}$ is continuous. Let $\{\mathfrak{v}_n\}$ be a sequence in $\mathcal{B}_k$ such that $\mathfrak{v}_n \rightarrow \mathfrak{v}$, as $n \rightarrow \infty$.

$$\begin{aligned}
& |(\mathcal{P}\mathfrak{v}_n)(t) - (\mathcal{P}\mathfrak{v})(t)| \\
& \leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} |\varphi_q(H_{\mathcal{W}, \mathfrak{v}_n}(s)) - \varphi_q(H_{\mathcal{W}, \mathfrak{v}}(s))| ds \\
& \quad + \frac{1}{\Gamma(\beta)} \int_t^T (s-t)^{\beta-1} |\varphi_q(H_{\mathcal{W}, \mathfrak{v}_n}(s)) - \varphi_q(H_{\mathcal{W}, \mathfrak{v}}(s))| ds \\
& \quad + \frac{|d_1|}{2|c_1|\Gamma(\beta-1)} \int_0^T s^{\beta-2} |\varphi_q(H_{\mathcal{W}, \mathfrak{v}_n}(s)) - \varphi_q(H_{\mathcal{W}, \mathfrak{v}}(s))| ds \\
& \quad + \frac{1}{\Gamma(\beta)} \int_0^T s^{\beta-1} |\varphi_q(H_{\mathcal{W}, \mathfrak{v}_n}(s)) - \varphi_q(H_{\mathcal{W}, \mathfrak{v}}(s))| ds \\
& \quad + \frac{|d_2|}{2|c_2|\Gamma(\beta-1)} \int_0^T (T-s)^{\beta-2} |\varphi_q(H_{\mathcal{W}, \mathfrak{v}_n}(s)) - \varphi_q(H_{\mathcal{W}, \mathfrak{v}}(s))| ds \\
& \quad + \frac{1}{\Gamma(\beta)} \int_0^T (T-s)^{\beta-1} |\varphi_q(H_{\mathcal{W}, \mathfrak{v}_n}(s)) - \varphi_q(H_{\mathcal{W}, \mathfrak{v}}(s))| ds. \quad (3.15)
\end{aligned}$$

Again, by virtue of the assumption **(A2)**, one has

$$\begin{aligned}
& |H_{\mathcal{W}, \mathfrak{v}_n}(t) - H_{\mathcal{W}, \mathfrak{v}}(t)| \\
& \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |\mathcal{W}(s, \mathfrak{v}_n(s)) - \mathcal{W}(s, \mathfrak{v}(s))| ds \\
& \quad + \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} |\mathcal{W}(s, \mathfrak{v}_n(s)) - \mathcal{W}(s, \mathfrak{v}(s))| ds \\
& \quad + \frac{|b_1|}{2|a_1|\Gamma(\alpha-1)} \int_0^T s^{\alpha-2} |\mathcal{W}(s, \mathfrak{v}_n(s)) - \mathcal{W}(s, \mathfrak{v}(s))| ds \\
& \quad + \frac{1}{\Gamma(\alpha)} \int_0^T s^{\alpha-1} |\mathcal{W}(s, \mathfrak{v}_n(s)) - \mathcal{W}(s, \mathfrak{v}(s))| ds \\
& \quad + \frac{|b_2|}{2|a_2|\Gamma(\alpha-1)} \int_0^T (T-s)^{\alpha-2} |\mathcal{W}(s, \mathfrak{v}_n(s)) - \mathcal{W}(s, \mathfrak{v}(s))| ds \\
& \quad + \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |\mathcal{W}(s, \mathfrak{v}_n(s)) - \mathcal{W}(s, \mathfrak{v}(s))| ds \\
& \leq \left[\frac{t^\alpha + (T-t)^\alpha + 2T^\alpha}{\Gamma(\alpha+1)} + \frac{|b_1|T^{\alpha-1}}{2|a_1|\Gamma(\alpha)} + \frac{|b_2|T^{\alpha-1}}{2|a_2|\Gamma(\alpha)} \right] \\
& \quad \times \|\mathcal{W}(\cdot, \mathfrak{v}_n(\cdot)) - \mathcal{W}(\cdot, \mathfrak{v}(\cdot))\|
\end{aligned}$$

$$\leq \left[\frac{4T^\alpha}{\Gamma(\alpha+1)} + \frac{|b_1|T^{\alpha-1}}{2|a_1|\Gamma(\alpha)} + \frac{|b_2|T^{\alpha-1}}{2|a_2|\Gamma(\alpha)} \right] \times \|\mathcal{W}(\cdot, \mathbf{v}_n(\cdot)) - \mathcal{W}(\cdot, \mathbf{v}(\cdot))\|. \quad (3.16)$$

Therefore, the continuity of the function $\mathcal{W}$ implies that the function $H_{\mathcal{W}, \mathbf{v}}$ is also continuous. Hence, by referring to (3.15), we infer the continuity of $\mathcal{P}$.

In the light of the above analysis, we realize that $\mathcal{P}$ is completely continuous. In consequence of Schauder's fixed point theorem (Lemma 4), $\mathcal{P}$ has a fixed point in $C(J, \mathbb{R})$, that is, it is a solution of the p -Laplacian **BVP** (1.1). The proof is finished. $\square$

3.2. Uniqueness result via Banach's fixed point theorem

Theorem 2. *Let $1 < p < 2$. Assume that the assumptions (A1) and (A2) are satisfied. Then the p -Laplacian **BVP** (1.1) possesses a unique solution on J as long as*

$$\Omega_{\alpha, \beta} := \left[\frac{4T^\beta}{\Gamma(\beta+1)} + \frac{|d_1|T^{\beta-1}}{2|c_1|\Gamma(\beta)} + \frac{|d_2|T^{\beta-1}}{2|c_2|\Gamma(\beta)} \right] \times \left[\frac{4T^\alpha}{\Gamma(\alpha+1)} + \frac{|b_1|T^{\alpha-1}}{2|a_1|\Gamma(\alpha)} + \frac{|b_2|T^{\alpha-1}}{2|a_2|\Gamma(\alpha)} \right] (q-1)M^{*q-2}L_{\mathcal{W}} < 1. \quad (3.17)$$

Proof. We consider the operator $\mathcal{P} : C(J, \mathbb{R}) \rightarrow C(J, \mathbb{R})$ defined by (3.11). We shall show that $\mathcal{P}$ is a contraction.

For each $t \in J$ and $\mathbf{v}_1, \mathbf{v}_2 \in C(J, \mathbb{R})$, using Lemma 2 and (3.14), we get

$$\begin{aligned} & |(\mathcal{P}\mathbf{v}_1)(t) - (\mathcal{P}\mathbf{v}_2)(t)| \\ & \leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} |\varphi_q(H_{\mathcal{W}, \mathbf{v}_1}(s)) - \varphi_q(H_{\mathcal{W}, \mathbf{v}_2}(s))| ds \\ & \quad + \frac{1}{\Gamma(\beta)} \int_t^T (s-t)^{\beta-1} |\varphi_q(H_{\mathcal{W}, \mathbf{v}_1}(s)) - \varphi_q(H_{\mathcal{W}, \mathbf{v}_2}(s))| ds \\ & \quad + \frac{|d_1|}{2|c_1|\Gamma(\beta-1)} \int_0^T s^{\beta-2} |\varphi_q(H_{\mathcal{W}, \mathbf{v}_1}(s)) - \varphi_q(H_{\mathcal{W}, \mathbf{v}_2}(s))| ds \\ & \quad + \frac{1}{\Gamma(\beta)} \int_0^T s^{\beta-1} |\varphi_q(H_{\mathcal{W}, \mathbf{v}_1}(s)) - \varphi_q(H_{\mathcal{W}, \mathbf{v}_2}(s))| ds \\ & \quad + \frac{|d_2|}{2|c_2|\Gamma(\beta-1)} \int_0^T (T-s)^{\beta-2} |\varphi_q(H_{\mathcal{W}, \mathbf{v}_1}(s)) - \varphi_q(H_{\mathcal{W}, \mathbf{v}_2}(s))| ds \\ & \quad + \frac{1}{\Gamma(\beta)} \int_0^T (T-s)^{\beta-1} |\varphi_q(H_{\mathcal{W}, \mathbf{v}_1}(s)) - \varphi_q(H_{\mathcal{W}, \mathbf{v}_2}(s))| ds \\ & \leq \frac{(q-1)M^{*q-2}}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} |H_{\mathcal{W}, \mathbf{v}_1}(s) - H_{\mathcal{W}, \mathbf{v}_2}(s)| ds \end{aligned}$$

$$\begin{aligned}
& + \frac{(q-1)M^{*q-2}}{\Gamma(\beta)} \int_t^T (s-t)^{\beta-1} |H_{\mathcal{W},v_1}(s) - H_{\mathcal{W},v_2}(s)| ds \\
& + \frac{|d_1|(q-1)M^{*q-2}}{2|c_1|\Gamma(\beta-1)} \int_0^T s^{\beta-2} |H_{\mathcal{W},v_1}(s) - H_{\mathcal{W},v_2}(s)| ds \\
& + \frac{(q-1)M^{*q-2}}{\Gamma(\beta)} \int_0^T s^{\beta-1} |H_{\mathcal{W},v_1}(s) - H_{\mathcal{W},v_2}(s)| ds \\
& + \frac{|d_2|(q-1)M^{*q-2}}{2|c_2|\Gamma(\beta-1)} \int_0^T (T-s)^{\beta-2} |H_{\mathcal{W},v_1}(s) - H_{\mathcal{W},v_2}(s)| ds \\
& + \frac{(q-1)M^{*q-2}}{\Gamma(\beta)} \int_0^T (T-s)^{\beta-1} |H_{\mathcal{W},v_1}(s) - H_{\mathcal{W},v_2}(s)| ds. \quad (3.18)
\end{aligned}$$

Using the assumption **(A2)**, one obtain

$$\begin{aligned}
& |H_{\mathcal{W},v_1}(t) - H_{\mathcal{W},v_2}(t)| \\
& \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |\mathcal{W}(s, v_1(s)) - \mathcal{W}(s, v_2(s))| ds \\
& \quad + \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} |\mathcal{W}(s, v_1(s)) - \mathcal{W}(s, v_2(s))| ds \\
& \quad + \frac{|b_1|}{2|a_1|\Gamma(\alpha-1)} \int_0^T s^{\alpha-2} |\mathcal{W}(s, v_1(s)) - \mathcal{W}(s, v_2(s))| ds \\
& \quad + \frac{1}{\Gamma(\alpha)} \int_0^T s^{\alpha-1} |\mathcal{W}(s, v_1(s)) - \mathcal{W}(s, v_2(s))| ds \\
& \quad + \frac{|b_2|}{2|a_2|\Gamma(\alpha-1)} \int_0^T (T-s)^{\alpha-2} |\mathcal{W}(s, v_1(s)) - \mathcal{W}(s, v_2(s))| ds \\
& \quad + \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |\mathcal{W}(s, v_1(s)) - \mathcal{W}(s, v_2(s))| ds \\
& \leq \left[\frac{4T^\alpha}{\Gamma(\alpha+1)} + \frac{|b_1|T^{\alpha-1}}{2|a_1|\Gamma(\alpha)} + \frac{|b_2|T^{\alpha-1}}{2|a_2|\Gamma(\alpha)} \right] L_{\mathcal{W}} \|v_1 - v_2\|. \quad (3.19)
\end{aligned}$$

By substitution from (3.19) into (3.18), we obtain that

$$\|\mathcal{P}v_1 - \mathcal{P}v_2\| \leq \Omega_{\alpha,\beta} \|v_1 - v_2\|.$$

By virtue of (3.17), we infer that $\mathcal{P}$ is a contraction. Hence, by the Banach's fixed point theorem, $\mathcal{P}$ has a unique fixed point which is the unique solution of the p -Laplacian **BVP** (1.1). The proof is completed. $\square$

4. EXPLANATORY EXAMPLE

In this section, we examine the applicability of the theoretical results.

Example 1. Consider the following **BVP**:

$$\begin{cases} {}^{RC}\mathfrak{D}_1^{\frac{1}{2}}\left(\varphi_{\frac{4}{3}}\left({}^{RC}\mathfrak{D}_1^{\frac{3}{4}}\mathfrak{v}(t)\right)\right) = \frac{1}{(1+5e^t)^2} \frac{|\mathfrak{v}(t)|}{1+|\mathfrak{v}(t)|}, & t \in [0, 1], \\ \varphi_{\frac{4}{3}}\left({}^{RC}\mathfrak{D}_1^{\frac{3}{4}}\mathfrak{v}(t)\right)|_{t=0} - \left(\varphi_{\frac{4}{3}}\left({}^{RC}\mathfrak{D}_1^{\frac{3}{4}}\mathfrak{v}(t)\right)\right)'|_{t=0} = 0, \\ \varphi_{\frac{4}{3}}\left({}^{RC}\mathfrak{D}_1^{\frac{3}{4}}\mathfrak{v}(t)\right)|_{t=1} + \left(\varphi_{\frac{4}{3}}\left({}^{RC}\mathfrak{D}_1^{\frac{3}{4}}\mathfrak{v}(t)\right)\right)'|_{t=1} = 0, \\ \mathfrak{v}(0) - \mathfrak{v}'(0) = 0, \\ \mathfrak{v}(1) + \mathfrak{v}'(1) = 0, \end{cases} \quad (4.1)$$

Here, $\alpha = \frac{1}{2}, \beta = \frac{3}{4}, T = 1, p = \frac{4}{3}, q = 4, a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = d_1 = d_2 = 1$.
Set

$$\mathcal{W}(t, \mathfrak{v}) = \frac{1}{(1+5e^t)^2} \frac{|\mathfrak{v}|}{1+|\mathfrak{v}|}.$$

For each $t \in [0, 1]$ and $\mathfrak{v} \in \mathbb{R}$, we get $|\mathcal{W}(t, \mathfrak{v})| \leq \frac{1}{36}$.

Additionally, for each $t \in [0, 1]$ and $\mathfrak{v}_1, \mathfrak{v}_2 \in \mathbb{R}$, one has

$$|\mathcal{W}(t, \mathfrak{v}_1) - \mathcal{W}(t, \mathfrak{v}_2)| \leq \frac{1}{36} |\mathfrak{v}_1 - \mathfrak{v}_2|.$$

Therefore, the assumptions **(A1)** and **(A2)** hold true with $L_{\mathcal{W}} = K_{\mathcal{W}} = \frac{1}{36}$.

Furthermore, one obtain that $M^* \approx 0.1410473959$ and $\Omega_{\alpha, \beta} \approx 0.0435 < 1$.

Hence, by Theorem 2, the p -Laplacian BVP (4.1) has a unique solution on $[0, 1]$.

Finally, by virtue of Definition 5, the inequality (3.12) proves that there exists a real number k satisfies

$$k \geq \left[\frac{4}{\Gamma(7/4)} + \frac{1}{\Gamma(3/4)} \right] \times (0.1410473959)^3,$$

which implies that $k \geq 0.01450$. Hence, by Theorem 1, the p -Laplacian BVP (4.1) has a solution on $[0, 1]$.

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NONLINEAR COUPLED LIOUVILLE–CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS WITH A NEW CLASS OF NONLOCAL BOUNDARY CONDITIONS

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Abstract. In this paper, we study a coupled system of nonlinear Liouville–Caputo fractional differential equations equipped with a new set of nonlocal boundary conditions involving an arbitrary strip together with two sets of nonlocal multi-points on either part of the strip on the given domain. We emphasize that the boundary conditions considered in this study are formulated with respect to the sum and difference of the unknown functions. We apply the well-known tools of the fixed point theory to derive the main results. Examples are presented for the illustration of the obtained results.

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Keywords: Coupled fractional differential equations, nonlocal multipoint boundary conditions, existence, fixed point

1. INTRODUCTION

Fractional calculus has evolved as an important area of research in view of its diverse applications in a variety of applied fields. Examples include immunology [8], chaotic systems [21], ecology [12], virology [7], economic model [20], neural networks [6], etc. A salient feature distinguishing fractional-order differential and integral operators from the classical ones is their nonlocal nature, which can trace the past history of the phenomena and processes under investigation. In the recent years, many researchers contributed to the development of fractional calculus, for example, see [3], [4], [5], [9], [11], [13], [22] and the references cited therein. One can also find a substantial material dealing with coupled systems of fractional differential equations in the recent literature, for instance see [1], [2], [10], [14] and the references cited therein.

In survey-cum-expository review articles [15–18], Srivastava described some recent developments on the subject of fractional calculus and its applications. In the survey [15], some recent developments involving different types of the Mittag-Leffler

type functions associated with generalized Riemann-Liouville and other related fractional derivative operators are reviewed. The article [16] contains a brief elementary and introductory overview of the theory and applications of the integral and derivative operators of fractional calculus. In [17], an overview of the fractional-calculus operators based on the general Fox-Wright function and its specialized forms is presented, while [18] deals with some variations of the operators of fractional calculus, related special functions and integral transformations.

In this paper, we consider a Liouville-Caputo type coupled system of nonlinear fractional differential equations supplemented with a new set of boundary conditions given by

$$\left\{ \begin{array}{l} {}^{LC}D^{\alpha}v(t) = \varphi(t, v(t), \omega(t)), \quad t \in [0, T], \\ {}^{LC}D^{\beta}\omega(t) = \psi(t, v(t), \omega(t)), \quad t \in [0, T], \\ (v + \omega)(0) = -(v + \omega)(T), \\ \int_{\eta}^{\zeta} (v - \omega)(s)ds - \sum_{i=1}^m a_i(v - \omega)(\sigma_i) - \sum_{j=1}^n b_j(v - \omega)(\delta_j) = A, \end{array} \right. \quad (1.1)$$

where ${}^{LC}D_{0+}^{\alpha}$ and ${}^{LC}D_{0+}^{\beta}$ denote the Liouville-Caputo fractional derivative operators of order $\alpha, \beta \in (0, 1]$, $0 < \sigma_i < \eta < \zeta < \delta_j < T$, $i = 1, \dots, m$, $j = 1, \dots, n$, and a_i, b_j, A are nonnegative constants, $\varphi, \psi : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuous functions. The first condition in (1.1) is anti-periodic one with respect to the sum of unknown functions v and ω , while the second condition describes that the contribution of the difference of the unknown functions v and ω on an arbitrary strip (η, ξ) within the given domain $[0, T]$ differs from the sum of such contributions due to arbitrary positions at $\sigma_i, i = 1, \dots, m$ and $\delta_j, j = 1, \dots, n$ (off the strip (η, ξ)) by a constant.

The present work is motivated by a recent article [2] in which the authors studied the system in (1.1) with boundary conditions of the form:

$$(v + \omega)(0) = -(v + \omega)(T), \quad \int_{\eta}^{\zeta} (v - \omega)(s)ds = A.$$

Here our objective is to enhance and generalize the study established in [2] by introducing more general nonlocal boundary conditions describing the changes within the given domain.

In the next section, we provide the related definitions of fractional integral and derivatives, and prove an auxiliary lemma. Our main results, presented in Section 3, rely on Schaefer like fixed point theorem and contraction mapping principle.

2. PRELIMINARIES

Let us commence this section with some definitions.

Definition 1. ([13]). For a function $v \in L_1[a, b]$, $-\infty < a < b < +\infty$, we define the Riemann-Liouville fractional integral $I_a^{\rho}v$ of order $\rho > 0$ existing almost everywhere

on $[a, b]$ as

$$I_a^\rho \mathfrak{v}(t) = \frac{1}{\Gamma(\rho)} \int_a^t (t-\tau)^{\rho-1} \mathfrak{v}(\tau) d\tau,$$

where Γ denotes the Euler gamma function.

Definition 2. [13]. The Riemann–Liouville fractional derivative $D_a^\rho \mathfrak{v}$ of order $\rho \in (p-1, p], p \in \mathbb{N}$ existing almost everywhere on $[a, b]$ is defined as

$$D_a^\rho \mathfrak{v}(t) = \frac{d^p}{dt^p} I_a^{p-\rho} \mathfrak{v}(t) = \frac{1}{\Gamma(p-\rho)} \frac{d^p}{dt^p} \int_a^t (t-\tau)^{p-1-\rho} \mathfrak{v}(\tau) d\tau,$$

where $\mathfrak{v}, \mathfrak{v}^{(p)} \in L_1[a, b]$.

The Liouville-Caputo fractional derivative ${}^{LC}D_a^\rho \mathfrak{v}$ of order $\rho \in (p-1, p], p \in \mathbb{N}$ is defined as

$${}^{LC}D_a^\rho \mathfrak{v}(t) = D_a^\rho \left[\mathfrak{v}(t) - \mathfrak{v}(a) - \mathfrak{v}'(a) \frac{(t-a)}{1!} - \dots - \mathfrak{v}^{(p-1)}(a) \frac{(t-a)^{p-1}}{(p-1)!} \right].$$

Remark 1. [13]. The Liouville-Caputo fractional derivative ${}^{LC}D_a^\rho \mathfrak{v}$ of order $\rho \in (p-1, p], p \in \mathbb{N}$ for a function $\mathfrak{v} \in AC^p[a, b]$, existing almost everywhere on $[a, b]$, is defined as

$${}^{LC}D_a^\rho \mathfrak{v}(t) = I_a^{p-\rho} \mathfrak{v}^{(p)}(t) = \frac{1}{\Gamma(p-\rho)} \int_a^t (t-\tau)^{p-1-\rho} \mathfrak{v}^{(p)}(\tau) d\tau.$$

In the follow lemma, we solve a linear variant of the problem (1.1).

Lemma 1. Let $F, G \in C[0, T]$. Then the solution of the following linear coupled boundary value problem:

$$\begin{cases} {}^{LC}D^\alpha \mathfrak{v}(t) = F(t), & 0 < \alpha < 1, \quad t \in [0, T], \\ {}^{LC}D^\beta \mathfrak{w}(t) = G(t), & 0 < \beta < 1, \quad t \in [0, T], \\ (\mathfrak{v} + \mathfrak{w})(0) = -(\mathfrak{v} + \mathfrak{w})(T), \\ \int_\eta^\zeta (\mathfrak{v} - \mathfrak{w})(s) ds - \sum_{i=1}^m a_i (\mathfrak{v} - \mathfrak{w})(\sigma_i) - \sum_{j=1}^n b_j (\mathfrak{v} - \mathfrak{w})(\delta_j) = A, \end{cases} \quad (2.1)$$

is given by

$$\begin{aligned} \mathfrak{v}(t) = & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F(s) ds \\ & + \frac{1}{2} \left\{ \frac{A}{\Lambda} - \frac{1}{2} \left(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} F(s) ds + \int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} G(s) ds \right) \right. \end{aligned} \quad (2.2)$$

$$\begin{aligned}
& -\frac{1}{\Lambda} \int_{\eta}^{\zeta} \left(\int_0^s \frac{(s-x)^{\alpha-1}}{\Gamma(\alpha)} F(x) dx - \int_0^s \frac{(s-x)^{\beta-1}}{\Gamma(\beta)} G(x) dx \right) ds \\
& + \frac{\sum_{i=1}^m a_i}{\Lambda} \left(\int_0^{\sigma_i} \frac{(\sigma_i-x)^{\alpha-1}}{\Gamma(\alpha)} F(x) dx - \int_0^{\sigma_i} \frac{(\sigma_i-x)^{\beta-1}}{\Gamma(\beta)} G(x) dx \right) \\
& + \frac{\sum_{j=1}^n b_j}{\Lambda} \left(\int_0^{\delta_j} \frac{(\delta_j-x)^{\alpha-1}}{\Gamma(\alpha)} F(x) dx - \int_0^{\delta_j} \frac{(\delta_j-x)^{\beta-1}}{\Gamma(\beta)} G(x) dx \right) \Bigg\}, \\
\omega(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} G(s) ds \\
& + \frac{1}{2} \left\{ \frac{-A}{\Lambda} - \frac{1}{2} \left(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} F(s) ds + \int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} G(s) ds \right) \right. \\
& + \frac{1}{\Lambda} \int_{\eta}^{\xi} \left(\int_0^s \frac{(s-x)^{\alpha-1}}{\Gamma(\alpha)} F(x) dx - \int_0^s \frac{(s-x)^{\beta-1}}{\Gamma(\beta)} G(x) dx \right) ds \\
& - \frac{\sum_{i=1}^m a_i}{\Lambda} \left(\int_0^{\sigma_i} \frac{(\sigma_i-x)^{\alpha-1}}{\Gamma(\alpha)} F(x) dx - \int_0^{\sigma_i} \frac{(\sigma_i-x)^{\beta-1}}{\Gamma(\beta)} G(x) dx \right) \\
& \left. - \frac{\sum_{j=1}^n b_j}{\Lambda} \left(\int_0^{\delta_j} \frac{(\delta_j-x)^{\alpha-1}}{\Gamma(\alpha)} F(x) dx - \int_0^{\delta_j} \frac{(\delta_j-x)^{\beta-1}}{\Gamma(\beta)} G(x) dx \right) \right\}, \tag{2.3}
\end{aligned}$$

where

$$\Lambda := \zeta - \eta - \sum_{i=1}^m a_i - \sum_{j=1}^n b_j \neq 0, \tag{2.4}$$

Proof. Applying the operators I^α and I^β on both sides of fractional differential equations in (2.1), respectively (for details, see [13]), we obtain

$$v(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h_1(s) ds + c_1, \tag{2.5}$$

$$\omega(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} h_2(s) ds + c_2, \tag{2.6}$$

where $c_1, c_2 \in \mathbb{R}$. Using (2.5) and (2.6) in the boundary conditions of the problem (2.1), we get

$$\begin{aligned}
c_1 + c_2 &= -\frac{1}{2} \left(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} F(s) ds + \int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} G(s) ds \right), \\
c_1 - c_2 &= \frac{1}{\Lambda} \left\{ A - \int_{\eta}^{\xi} \left(\int_0^s \frac{(s-x)^{\alpha-1}}{\Gamma(\alpha)} F(x) dx - \int_0^s \frac{(s-x)^{\beta-1}}{\Gamma(\beta)} G(x) dx \right) ds \right. \\
& \quad \left. + \sum_{i=1}^m a_i \left(\int_0^{\sigma_i} \frac{(\sigma_i-x)^{\alpha-1}}{\Gamma(\alpha)} F(x) dx - \int_0^{\sigma_i} \frac{(\sigma_i-x)^{\beta-1}}{\Gamma(\beta)} G(x) dx \right) \right\}
\end{aligned} \tag{2.7}$$

$$+ \sum_{j=1}^n b_j \left(\int_0^{\delta_j} \frac{(\delta_j - x)^{\alpha-1}}{\Gamma(\alpha)} F(x) dx - \int_0^{\delta_j} \frac{(\delta_j - x)^{\beta-1}}{\Gamma(\beta)} G(x) dx \right) \Bigg\}. \quad (2.8)$$

Solving (2.7) and (2.8) for c_1 and c_2 yields

$$\begin{aligned} c_1 = & \frac{1}{2} \left\{ \frac{A}{\Lambda} - \frac{1}{2} \left(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} F(s) ds + \int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} G(s) ds \right) \right. \\ & - \frac{1}{\Lambda} \int_{\eta}^{\xi} \left(\int_0^s \frac{(s-x)^{\alpha-1}}{\Gamma(\alpha)} F(x) dx - \int_0^s \frac{(s-x)^{\beta-1}}{\Gamma(\beta)} G(x) dx \right) ds \\ & + \frac{\sum_{i=1}^m a_i}{\Lambda} \left(\int_0^{\sigma_i} \frac{(\sigma_i - x)^{\alpha-1}}{\Gamma(\alpha)} F(x) dx - \int_0^{\sigma_i} \frac{(\sigma_i - x)^{\beta-1}}{\Gamma(\beta)} G(x) dx \right) \\ & \left. + \frac{\sum_{j=1}^n b_j}{\Lambda} \left(\int_0^{\delta_j} \frac{(\delta_j - x)^{\alpha-1}}{\Gamma(\alpha)} F(x) dx - \int_0^{\delta_j} \frac{(\delta_j - x)^{\beta-1}}{\Gamma(\beta)} G(x) dx \right) \right\}, \\ c_2 = & \frac{1}{2} \left\{ \frac{-A}{\Lambda} - \frac{1}{2} \left(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} F(s) ds + \int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} G(s) ds \right) \right. \\ & + \frac{1}{\Lambda} \int_{\eta}^{\xi} \left(\int_0^s \frac{(s-x)^{\alpha-1}}{\Gamma(\alpha)} F(x) dx - \int_0^s \frac{(s-x)^{\beta-1}}{\Gamma(\beta)} G(x) dx \right) ds \\ & - \frac{\sum_{i=1}^m a_i}{\Lambda} \left(\int_0^{\sigma_i} \frac{(\sigma_i - x)^{\alpha-1}}{\Gamma(\alpha)} F(x) dx - \int_0^{\sigma_i} \frac{(\sigma_i - x)^{\beta-1}}{\Gamma(\beta)} G(x) dx \right) \\ & \left. - \frac{\sum_{j=1}^n b_j}{\Lambda} \left(\int_0^{\delta_j} \frac{(\delta_j - x)^{\alpha-1}}{\Gamma(\alpha)} F(x) dx - \int_0^{\delta_j} \frac{(\delta_j - x)^{\beta-1}}{\Gamma(\beta)} G(x) dx \right) \right\}. \end{aligned}$$

Substituting the values of c_1 and c_2 in (2.5) and (2.6) yields the solution (2.2) and (2.3). By direct computation, one can prove the converse of this lemma. The proof is finished. $\square$

3. MAIN RESULTS

Let $\mathcal{X} = C([0, T], \mathbb{R}) \times C([0, T], \mathbb{R})$ be the Banach space equipped with the norm $\|(\mathbf{v}, \boldsymbol{\omega})\| = \|\mathbf{v}\| + \|\boldsymbol{\omega}\| = \sup_{t \in [0, T]} |\mathbf{v}(t)| + \sup_{t \in [0, T]} |\boldsymbol{\omega}(t)|$, for $(\mathbf{v}, \boldsymbol{\omega}) \in \mathcal{X}$. In order to transform the problem (1.1) into a fixed point problem, we define an operator $\Pi : \mathcal{X} \rightarrow \mathcal{X}$ via Lemma 1 as

$$\Pi(\mathbf{v}, \boldsymbol{\omega})(t) := (\Pi_1(\mathbf{v}, \boldsymbol{\omega})(t), \Pi_2(\mathbf{v}, \boldsymbol{\omega})(t)), \quad (3.1)$$

$$\begin{aligned} \Pi_1(\mathbf{v}, \boldsymbol{\omega})(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s, \mathbf{v}(s), \boldsymbol{\omega}(s)) ds \\ & + \frac{1}{2} \left\{ \frac{A}{\Lambda} - \frac{1}{2} \left(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi(s, \mathbf{v}(s), \boldsymbol{\omega}(s)) ds \right. \right. \end{aligned} \quad (3.2)$$

$$\begin{aligned}
& + \int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} \Psi(s, \mathbf{v}(s), \boldsymbol{\omega}(s)) ds \\
& - \frac{1}{\Lambda} \int_{\eta}^{\xi} \left(\int_0^s \frac{(s-x)^{\alpha-1}}{\Gamma(\alpha)} \Phi(x, \mathbf{v}(x), \boldsymbol{\omega}(x)) dx \right. \\
& - \int_0^s \frac{(s-x)^{\beta-1}}{\Gamma(\beta)} \Psi(x, \mathbf{v}(x), \boldsymbol{\omega}(x)) dx \Big) ds \\
& + \frac{\sum_{i=1}^m a_i}{\Lambda} \left(\int_0^{\sigma_i} \frac{(\sigma_i-x)^{\alpha-1}}{\Gamma(\alpha)} \Phi(x, \mathbf{v}(x), \boldsymbol{\omega}(x)) dx \right. \\
& - \int_0^{\sigma_i} \frac{(\sigma_i-x)^{\beta-1}}{\Gamma(\beta)} \Psi(x, \mathbf{v}(x), \boldsymbol{\omega}(x)) dx \Big) \\
& + \frac{\sum_{j=1}^n b_j}{\Lambda} \left(\int_0^{\delta_j} \frac{(\delta_j-x)^{\alpha-1}}{\Gamma(\alpha)} \Phi(x, \mathbf{v}(x), \boldsymbol{\omega}(x)) dx \right. \\
& - \int_0^{\delta_j} \frac{(\delta_j-x)^{\beta-1}}{\Gamma(\beta)} \Psi(x, \mathbf{v}(x), \boldsymbol{\omega}(x)) dx \Big) \Big\}
\end{aligned}$$

and

$$\begin{aligned}
\Pi_2(\mathbf{v}, \boldsymbol{\omega})(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \Psi(s, \mathbf{v}(s), \boldsymbol{\omega}(s)) ds \\
& + \frac{1}{2} \left\{ \frac{-A}{\Lambda} - \frac{1}{2} \left(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \Phi(s, \mathbf{v}(s), \boldsymbol{\omega}(s)) ds \right. \right. \\
& + \int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} \Psi(s, \mathbf{v}(s), \boldsymbol{\omega}(s)) ds \Big) \\
& + \frac{1}{\Lambda} \int_{\eta}^{\xi} \left(\int_0^s \frac{(s-x)^{\alpha-1}}{\Gamma(\alpha)} \Phi(x, \mathbf{v}(x), \boldsymbol{\omega}(x)) dx \right. \\
& - \int_0^s \frac{(s-x)^{\beta-1}}{\Gamma(\beta)} \Psi(x, \mathbf{v}(x), \boldsymbol{\omega}(x)) dx \Big) ds \\
& - \frac{\sum_{i=1}^m a_i}{\Lambda} \left(\int_0^{\sigma_i} \frac{(\sigma_i-x)^{\alpha-1}}{\Gamma(\alpha)} \Phi(x, \mathbf{v}(x), \boldsymbol{\omega}(x)) dx \right. \\
& - \int_0^{\sigma_i} \frac{(\sigma_i-x)^{\beta-1}}{\Gamma(\beta)} \Psi(x, \mathbf{v}(x), \boldsymbol{\omega}(x)) dx \Big) \\
& - \frac{\sum_{j=1}^n b_j}{\Lambda} \left(\int_0^{\delta_j} \frac{(\delta_j-x)^{\alpha-1}}{\Gamma(\alpha)} \Phi(x, \mathbf{v}(x), \boldsymbol{\omega}(x)) dx \right. \\
& - \int_0^{\delta_j} \frac{(\delta_j-x)^{\beta-1}}{\Gamma(\beta)} \Psi(x, \mathbf{v}(x), \boldsymbol{\omega}(x)) dx \Big) \Big\}.
\end{aligned} \tag{3.3}$$

In the sequel, we need the following assumptions.

(H₁) There exist continuous functions $\mu_i, \kappa_i \in C([0, T], \mathbb{R}^+), i = 1, 2, 3$, such that,
 $\forall(t, \mathbf{v}, \boldsymbol{\omega}) \in [0, T] \times \mathbb{R}^2$,

$$|\varphi(t, \mathbf{v}, \boldsymbol{\omega})| \leq \mu_1(t) + \mu_2(t)|\mathbf{v}| + \mu_3(t)|\boldsymbol{\omega}|;$$

$$|\psi(t, \mathbf{v}, \boldsymbol{\omega})| \leq \kappa_1(t) + \kappa_2(t)|\mathbf{v}| + \kappa_3(t)|\boldsymbol{\omega}|.$$

(H₂) There exist $C_i, K_i > 0, i = 1, 2$, such that, $\forall t \in [0, T], \mathbf{v}_i, \boldsymbol{\omega}_i \in \mathbb{R}, i = 1, 2$,

$$|\varphi(t, \mathbf{v}_1, \boldsymbol{\omega}_1) - \varphi(t, \mathbf{v}_2, \boldsymbol{\omega}_2)| \leq C_1 \|\mathbf{v}_1 - \mathbf{v}_2\| + C_2 \|\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2\|,$$

$$|\psi(t, \mathbf{v}_1, \boldsymbol{\omega}_1) - \psi(t, \mathbf{v}_2, \boldsymbol{\omega}_2)| \leq K_1 \|\mathbf{v}_1 - \mathbf{v}_2\| + K_2 \|\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2\|.$$

For the sake of computational convenience, we set the notation:

$$\begin{aligned} \Delta_1 = & \frac{T^\alpha}{4\Gamma(\alpha+1)} + \frac{1}{2|\Lambda|} \left[\frac{\zeta^{\alpha+1} - \eta^{\alpha+1}}{\Gamma(\alpha+2)} + \sum_{i=1}^m a_i \frac{\sigma_i^\alpha}{\Gamma(\alpha+1)} \right. \\ & \left. + \sum_{j=1}^n b_j \frac{\delta_j^\alpha}{\Gamma(\alpha+1)} \right], \end{aligned} \quad (3.4)$$

$$\begin{aligned} \Delta_2 = & \frac{T^\beta}{4\Gamma(\beta+1)} + \frac{1}{2|\Lambda|} \left[\frac{\zeta^{\beta+1} - \eta^{\beta+1}}{\Gamma(\beta+2)} + \sum_{i=1}^m a_i \frac{\sigma_i^\beta}{\Gamma(\beta+1)} \right. \\ & \left. + \sum_{j=1}^n b_j \frac{\delta_j^\beta}{\Gamma(\beta+1)} \right], \end{aligned} \quad (3.5)$$

and

$$\begin{aligned} M_0 = \min & \left\{ 1 - \left[\|\mu_2\| \left(2\Delta_1 + \frac{T^\alpha}{\Gamma(\alpha+1)} \right) + \|\kappa_2\| \left(2\Delta_2 + \frac{T^\beta}{\Gamma(\beta+1)} \right) \right], \right. \\ & \left. 1 - \left[\|\mu_3\| \left(2\Delta_1 + \frac{T^\alpha}{\Gamma(\alpha+1)} \right) + \|\kappa_3\| \left(2\Delta_2 + \frac{T^\beta}{\Gamma(\beta+1)} \right) \right] \right\}. \end{aligned}$$

Our first existence result for the problem (1.1) is based on the following fixed point theorem [19].

Lemma 2. Let $\mathcal{E}$ be a Banach space and $\mathcal{F} : \mathcal{E} \rightarrow \mathcal{E}$ be a completely continuous operator. If the set $S = \{x \in \mathcal{E} | x = \mu \mathcal{F} x, 0 < \mu < 1\}$ is bounded, then the operator $\mathcal{F}$ has a fixed point in $\mathcal{E}$.

Theorem 1. Suppose that the assumption (H₁) holds. Then there exists at least one solution for the problem (1.1) on $[0, T]$ if

$$\|\mu_2\| \left(2\Delta_1 + \frac{T^\alpha}{\Gamma(\alpha+1)} \right) + \|\kappa_2\| \left(2\Delta_2 + \frac{T^\beta}{\Gamma(\beta+1)} \right) < 1, \quad (3.6)$$

$$\|\mu_3\| \left(2\Delta_1 + \frac{T^\alpha}{\Gamma(\alpha+1)} \right) + \|\kappa_3\| \left(2\Delta_2 + \frac{T^\beta}{\Gamma(\beta+1)} \right) < 1,$$

where Δ_i ($i = 1, 2$) are given by (3.4)-(3.5).

Proof. Observe that continuity of functions ϕ and ψ implies that the operator $\Pi : X \rightarrow X$ is continuous. Let $\Omega_{\bar{r}} \subset X$ be bounded. Then we can find positive constants L_ϕ and L_ψ such that

$$|\phi(t, \mathbf{v}(t), \omega(t))| \leq L_\phi, \quad |\psi(t, \mathbf{v}(t), \omega(t))| \leq L_\psi, \quad \forall (\mathbf{v}, \omega) \in \Omega_{\bar{r}}.$$

So, for any $(\mathbf{v}, \omega) \in \Omega_{\bar{r}}$, $t \in [0, T]$, we get

$$\begin{aligned} |\Pi_1(\mathbf{v}, \omega)(t)| &\leq L_\phi \left(\frac{T^\alpha}{\Gamma(\alpha+1)} + \Delta_1 \right) + L_\psi \Delta_2 + \frac{A}{2|\Lambda|}, \\ |\Pi_2(\mathbf{v}, \omega)(t)| &\leq L_\phi \Delta_1 + L_\psi \left(\frac{T^\beta}{\Gamma(\beta+1)} + \Delta_2 \right) + \frac{A}{2|\Lambda|}. \end{aligned}$$

Thus

$$\begin{aligned} \|\Pi(\mathbf{v}, \omega)\| &= \|\Pi_1(\mathbf{v}, \omega)\| + \|\Pi_2(\mathbf{v}, \omega)\| \\ &\leq L_\phi \left(\frac{T^\alpha}{\Gamma(\alpha+1)} + 2\Delta_1 \right) + L_\psi \left(\frac{T^\beta}{\Gamma(\beta+1)} + 2\Delta_2 \right) + \frac{A}{|\Lambda|}, \end{aligned}$$

which implies that the operator Π is uniformly bounded.

Next it will be shown that Π maps bounded sets into equicontinuous sets of X . For $t_1, t_2 \in [0, T]$, $t_1 < t_2$, and $(\mathbf{v}, \omega) \in \Omega_{\bar{r}}$, we have

$$\begin{aligned} &|\Pi_1(\mathbf{v}, \omega)(t_2) - \Pi_1(\mathbf{v}, \omega)(t_1)| \\ &\leq \left| \frac{1}{\Gamma(\alpha)} \left(\int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] \phi(s, \mathbf{v}(s), \omega(s)) ds \right. \right. \\ &\quad \left. \left. + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \phi(s, \mathbf{v}(s), \omega(s)) ds \right) \right| \\ &\leq L_\phi \left(\frac{2(t_2 - t_1)^\alpha + t_2^\alpha - t_1^\alpha}{\Gamma(\alpha+1)} \right) \rightarrow 0 \text{ when } t_1 \rightarrow t_2, \end{aligned}$$

independently of $(\mathbf{v}, \omega) \in \Omega_{\bar{r}}$. Likewise

$$|\Pi_2(\mathbf{v}, \omega)(t_2) - \Pi_2(\mathbf{v}, \omega)(t_1)| \leq L_\psi \left(\frac{2(t_2 - t_1)^\beta + t_2^\beta - t_1^\beta}{\Gamma(\beta+1)} \right) \rightarrow 0$$

when $t_1 \rightarrow t_2$, independently of $(\mathbf{v}, \omega) \in \Omega_{\bar{r}}$. Thus it follows by the Arzelá-Ascoli theorem that the operator $\Pi : X \rightarrow X$ is completely continuous.

Next, we consider the set $\mathcal{U} = \{(v, \omega) \in X \mid u = \lambda \Pi(v, \omega), 0 < \lambda < 1\}$ and prove that it is bounded. Let $(v, \omega) \in \mathcal{U}$, then $(v, \omega) = \lambda \Pi(v, \omega)$, $0 < \lambda < 1$. For any $t \in [0, T]$, we have

$$v(t) = \lambda \Pi_1(v, \omega)(t), \quad \omega(t) = \lambda \Pi_2(v, \omega)(t).$$

As in the previous step, using Δ_i ($i = 1, 2$) given by (3.4)-(3.5), we find that

$$\begin{aligned} |v(t)| &= \lambda |\Pi_1(v, \omega)(t)| \leq \left(\|\mu_1\| + \|\mu_2\| \|v\| + \|\mu_3\| \|\omega\| \right) \left(\frac{T^\alpha}{\Gamma(\alpha+1)} + \Delta_1 \right) \\ &\quad + \left(\|\kappa_1\| + \|\kappa_2\| \|v\| + \|\kappa_3\| \|\omega\| \right) \Delta_2 + \frac{A}{2|\Lambda|}, \\ |\omega(t)| &= \lambda |\Pi_2(v, \omega)(t)| \leq \left(\|\mu_1\| + \|\mu_2\| \|v\| + \|\mu_3\| \|\omega\| \right) \Delta_1 \\ &\quad + \left(\|\kappa_1\| + \|\kappa_2\| \|v\| + \|\kappa_3\| \|\omega\| \right) \left(\frac{T^\beta}{\Gamma(\beta+1)} + \Delta_2 \right) + \frac{A}{2|\Lambda|}. \end{aligned}$$

In consequence, we get

$$\begin{aligned} \|v\| + \|\omega\| &\leq \|\mu_1\| \left(2\Delta_1 + \frac{T^\alpha}{\Gamma(\alpha+1)} \right) + \|\kappa_1\| \left(2\Delta_2 + \frac{T^\beta}{\Gamma(\beta+1)} \right) + \frac{A}{|\Lambda|} \\ &\quad + \left[\|\mu_2\| \left(2\Delta_1 + \frac{T^\alpha}{\Gamma(\alpha+1)} \right) + \|\kappa_2\| \left(2\Delta_2 + \frac{T^\beta}{\Gamma(\beta+1)} \right) \right] \|v\| \\ &\quad + \left[\|\mu_3\| \left(2\Delta_1 + \frac{T^\alpha}{\Gamma(\alpha+1)} \right) + \|\kappa_3\| \left(2\Delta_2 + \frac{T^\beta}{\Gamma(\beta+1)} \right) \right] \|\omega\|. \end{aligned}$$

By the condition (3.6), it follows that

$$\|(v, \omega)\| \leq \frac{\|\mu_1\| \left(2\Delta_1 + \frac{T^\alpha}{\Gamma(\alpha+1)} \right) + \|\kappa_1\| \left(2\Delta_2 + \frac{T^\beta}{\Gamma(\beta+1)} \right) + \frac{A}{|\Lambda|}}{M_0},$$

which implies that the set $\mathcal{U}$ is bounded. In consequence, we deduce that the operator Π has at least one fixed point by the conclusion of Lemma 2, which is a solution of the problem (1.1). $\square$

The statement of Theorem 1 reduces to the the following special form by fixing $\mu_2(t) = \mu_3(t) \equiv 0$ and $\kappa_2(t) = \kappa_3(t) \equiv 0$ in it.

Corollary 1. *Let $\varphi, \psi: J \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be continuous functions such that $|\varphi(t, v, \omega)| \leq \mu_1(t)$, $|\psi(t, v, \omega)| \leq \kappa_1(t)$, $\forall (t, v, \omega) \in [0, T] \times \mathbb{R}^2$, where $\mu_1, \kappa_1 \in C([0, T], \mathbb{R}^+)$. Then the problem (1.1) has at least one solution on $[0, T]$.*

Corollary 2. *If $\mu_i(t) = \lambda_i$, $\kappa_i(t) = \varepsilon_i$, $i = 1, 2, 3$, then the condition (H_1) becomes: (H'_1) there exist real constants $\lambda_i, \varepsilon_i > 0$, ($i = 1, 2$) such that*

$$\begin{aligned} |\varphi(t, v, \omega)| &\leq \lambda_1 + \lambda_2 |v| + \lambda_3 |\omega| \quad \forall (t, v, \omega) \in [0, T] \times \mathbb{R}^2; \\ |\psi(t, v, \omega)| &\leq \varepsilon_1 + \varepsilon_2 |v| + \varepsilon_3 |\omega| \quad \forall (t, v, \omega) \in [0, T] \times \mathbb{R}^2; \end{aligned}$$

and (3.6) takes the form:

$$\begin{aligned}\lambda_2 \left(2\Delta_1 + \frac{T^\alpha}{\Gamma(\alpha+1)} \right) + \varepsilon_2 \left(2\Delta_2 + \frac{T^\beta}{\Gamma(\beta+1)} \right) &< 1, \\ \lambda_3 \left(2\Delta_1 + \frac{T^\alpha}{\Gamma(\alpha+1)} \right) + \varepsilon_3 \left(2\Delta_2 + \frac{T^\beta}{\Gamma(\beta+1)} \right) &< 1.\end{aligned}$$

In the following result, we prove the uniqueness of solutions to the problem (1.1) via Banach contraction mapping principle.

Theorem 2. *If the assumption (H_2) holds, then the problem (1.1) has a unique solution on $[0, T]$ provided that*

$$C \left(\frac{T^\alpha}{\Gamma(\alpha+1)} + 2\Delta_1 \right) + \mathcal{K} \left(\frac{T^\beta}{\Gamma(\beta+1)} + 2\Delta_2 \right) < 1, \quad (3.7)$$

where $C = \max\{C_1, C_2\}$, $\mathcal{K} = \max\{K_1, K_2\}$ and Δ_i , $i = 1, 2$ are defined by (3.4)-(3.5).

Proof. Fix

$$r > \frac{M_1 \left(\frac{T^\alpha}{\Gamma(\alpha+1)} + 2\Delta_1 \right) + M_2 \left(\frac{T^\beta}{\Gamma(\beta+1)} + 2\Delta_2 \right) + \frac{A}{|\Lambda|}}{1 - \left(C \left(\frac{T^\alpha}{\Gamma(\alpha+1)} + 2\Delta_1 \right) + \mathcal{K} \left(\frac{T^\beta}{\Gamma(\beta+1)} + 2\Delta_2 \right) \right)},$$

where $M_1 = \sup_{t \in [0, T]} |\varphi(t, 0, 0)|$, and $M_2 = \sup_{t \in [0, T]} |\psi(t, 0, 0)|$. Then we show that $\Pi B_r \subset B_r$, where $\Pi : X \rightarrow X$ is defined by (3.1) and $B_r = \{(\mathbf{v}, \boldsymbol{\omega}) \in X : \|(\mathbf{v}, \boldsymbol{\omega})\| \leq r\}$. By the assumption (H_1) , for $(\mathbf{v}, \boldsymbol{\omega}) \in B_r$, $t \in [0, T]$, we have

$$|\varphi(t, \mathbf{v}(t), \boldsymbol{\omega}(t))| \leq |\varphi(t, \mathbf{v}(t), \boldsymbol{\omega}(t)) - \varphi(t, 0, 0)| + |\varphi(t, 0, 0)| \leq C(\|\mathbf{v}\| + \|\boldsymbol{\omega}\|) + M_1$$

and

$$|\psi(t, \mathbf{v}(t), \boldsymbol{\omega}(t))| \leq \mathcal{K}(\|\mathbf{v}\| + \|\boldsymbol{\omega}\|) + M_2.$$

In consequence, for $(\mathbf{v}, \boldsymbol{\omega}) \in B_r$, we obtain

$$\begin{aligned}|\Pi_1(\mathbf{v}, \boldsymbol{\omega})(t)| &\leq \frac{T^\alpha}{\Gamma(\alpha+1)} \left(C(\|\mathbf{v}\| + \|\boldsymbol{\omega}\|) + M_1 \right) \\ &\quad + \frac{1}{2} \left[\frac{A}{|\Lambda|} + \frac{1}{2} \left(\frac{T^\alpha}{\Gamma(\alpha+1)} \left(C(\|\mathbf{v}\| + \|\boldsymbol{\omega}\|) + M_1 \right) \right. \right. \\ &\quad \left. \left. + \frac{T^\beta}{\Gamma(\beta+1)} \left(\mathcal{K}(\|\mathbf{v}\| + \|\boldsymbol{\omega}\|) + M_2 \right) \right) \right] \\ &\quad + \frac{1}{|\Lambda|} \left(\frac{\zeta^{\alpha+1} - \eta^{\alpha+1}}{\Gamma(\alpha+2)} \left(C(\|\mathbf{v}\| + \|\boldsymbol{\omega}\|) + M_1 \right) \right. \\ &\quad \left. + \frac{\zeta^{\beta+1} - \eta^{\beta+1}}{\Gamma(\beta+2)} \left(\mathcal{K}(\|\mathbf{v}\| + \|\boldsymbol{\omega}\|) + M_2 \right) \right)\end{aligned}$$

$$\begin{aligned}
& + \frac{\sum_{i=1}^m a_i}{|\Lambda|} \left(\frac{\sigma_i^\alpha}{\Gamma(\alpha+1)} \left(C(\|\mathbf{v}\| + \|\boldsymbol{\omega}\|) + M_1 \right) \right. \\
& + \frac{\sigma_i^\beta}{\Gamma(\beta+1)} \left(\mathcal{K}(\|\mathbf{v}\| + \|\boldsymbol{\omega}\|) + M_2 \right) \Big) \\
& + \frac{\sum_{j=1}^n b_j}{|\Lambda|} \left(\frac{\delta_j^\alpha}{\Gamma(\alpha+1)} \left(C(\|\mathbf{v}\| + \|\boldsymbol{\omega}\|) + M_1 \right) \right. \\
& \left. \left. + \frac{\delta_j^\beta}{\Gamma(\beta+1)} \left(\mathcal{K}(\|\mathbf{v}\| + \|\boldsymbol{\omega}\|) + M_2 \right) \right) \right],
\end{aligned}$$

which, on taking the norm for $t \in [0, T]$, leads to

$$\begin{aligned}
\|\Pi_1(\mathbf{v}, \boldsymbol{\omega})\| & \leq \left(C \left(\frac{T^\alpha}{\Gamma(\alpha+1)} + \Delta_1 \right) + \mathcal{K}\Delta_2 \right) (\|\mathbf{v}\| + \|\boldsymbol{\omega}\|) \\
& + M_1 \left(\frac{T^\alpha}{\Gamma(\alpha+1)} + \Delta_1 \right) + M_2\Delta_2 + \frac{A}{2|\Lambda|}.
\end{aligned}$$

In the same way, for $(\mathbf{v}, \boldsymbol{\omega}) \in B_r$, one can obtain

$$\begin{aligned}
\|\Pi_2(\mathbf{v}, \boldsymbol{\omega})\| & \leq \left(C\Delta_1 + \mathcal{K} \left(\frac{T^\beta}{\Gamma(\beta+1)} + \Delta_2 \right) \right) (\|\mathbf{v}\| + \|\boldsymbol{\omega}\|) \\
& + M_1\Delta_1 + M_2 \left(\frac{T^\beta}{\Gamma(\beta+1)} + \Delta_2 \right) + \frac{A}{2|\Lambda|}.
\end{aligned}$$

Therefore, for any $(\mathbf{v}, \boldsymbol{\omega}) \in B_r$, we have

$$\begin{aligned}
\|\Pi(\mathbf{v}, \boldsymbol{\omega})\| & = \|\Pi_1(\mathbf{v}, \boldsymbol{\omega})\| + \|\Pi_2(\mathbf{v}, \boldsymbol{\omega})\| \\
& \leq \left(C \left(\frac{T^\alpha}{\Gamma(\alpha+1)} + 2\Delta_1 \right) + \mathcal{K} \left(\frac{T^\beta}{\Gamma(\beta+1)} + 2\Delta_2 \right) \right) (\|\mathbf{v}\| + \|\boldsymbol{\omega}\|) \\
& + M_1 \left(\frac{T^\alpha}{\Gamma(\alpha+1)} + 2\Delta_1 \right) + M_2 \left(\frac{T^\beta}{\Gamma(\beta+1)} + 2\Delta_2 \right) + \frac{A}{|\Lambda|} < r,
\end{aligned}$$

which shows that Π maps B_r into itself.

Now show that the operator Π is a contraction. For $(\mathbf{v}_1, \boldsymbol{\omega}_1), (\mathbf{v}_2, \boldsymbol{\omega}_2) \in X, t \in [0, T]$, we have

$$\begin{aligned}
& |\Pi_1(\mathbf{v}_1, \boldsymbol{\omega}_1)(t) - \Pi_1(\mathbf{v}_2, \boldsymbol{\omega}_2)(t)| \\
& \leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |\varphi(s, \mathbf{v}_1(s), \boldsymbol{\omega}_1(s)) - \varphi(s, \mathbf{v}_2(s), \boldsymbol{\omega}_2(s))| ds \\
& + \frac{1}{2} \left\{ \frac{1}{2} \left(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} |\varphi(s, \mathbf{v}_1(s), \boldsymbol{\omega}_1(s)) - \varphi(s, \mathbf{v}_2(s), \boldsymbol{\omega}_2(s))| ds \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + \int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} |\psi(s, \mathbf{v}_1(s), \boldsymbol{\omega}_1(s)) - \psi(s, \mathbf{v}_2(s), \boldsymbol{\omega}_2(s))| ds \\
& + \frac{1}{|\Lambda|} \int_{\eta}^{\xi} \left(\int_0^s \frac{(s-x)^{\alpha-1}}{\Gamma(\alpha)} |\varphi(x, \mathbf{v}_1(x), \boldsymbol{\omega}_1(s)) - \varphi(x, \mathbf{v}_2(x), \boldsymbol{\omega}_2(s))| dx \right. \\
& + \int_0^s \frac{(s-x)^{\beta-1}}{\Gamma(\beta)} |\psi(x, \mathbf{v}_1(x), \boldsymbol{\omega}_1(x)) - \psi(x, \mathbf{v}_2(x), \boldsymbol{\omega}_2(x))| dx \Big) ds \\
& + \frac{\sum_{i=1}^m a_i}{|\Lambda|} \left(\int_0^{\sigma_i} \frac{(\sigma_i-x)^{\alpha-1}}{\Gamma(\alpha)} |\varphi(x, \mathbf{v}_1(x), \boldsymbol{\omega}_1(s)) - \varphi(x, \mathbf{v}_2(x), \boldsymbol{\omega}_2(s))| dx \right. \\
& + \int_0^{\sigma_i} \frac{(\sigma_i-x)^{\beta-1}}{\Gamma(\beta)} |\psi(x, \mathbf{v}_1(x), \boldsymbol{\omega}_1(x)) - \psi(x, \mathbf{v}_2(x), \boldsymbol{\omega}_2(x))| dx \Big) \\
& + \frac{\sum_{j=1}^n b_j}{|\Lambda|} \left(\int_0^{\delta_j} \frac{(\delta_j-x)^{\alpha-1}}{\Gamma(\alpha)} |\varphi(x, \mathbf{v}_1(x), \boldsymbol{\omega}_1(s)) - \varphi(x, \mathbf{v}_2(x), \boldsymbol{\omega}_2(s))| dx \right. \\
& + \int_0^{\delta_j} \frac{(\delta_j-x)^{\beta-1}}{\Gamma(\beta)} |\psi(x, \mathbf{v}_1(x), \boldsymbol{\omega}_1(x)) - \psi(x, \mathbf{v}_2(x), \boldsymbol{\omega}_2(x))| dx \Big) \Big\} \\
& \leq \left\{ C \left(\frac{T^\alpha}{\Gamma(\alpha+1)} + \Delta_1 \right) + \mathcal{K} \Delta_2 \right\} (\|\mathbf{v}\| + \|\boldsymbol{\omega}\|),
\end{aligned}$$

and

$$\begin{aligned}
& |\Pi_2(\mathbf{v}_1, \boldsymbol{\omega}_1)(t) - \Pi_2(\mathbf{v}_2, \boldsymbol{\omega}_2)(t)| \\
& \leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |\psi(s, \mathbf{v}_1(s), \boldsymbol{\omega}_1(s)) - \psi(s, \mathbf{v}_2(s), \boldsymbol{\omega}_2(s))| ds \\
& + \frac{1}{2} \left\{ \frac{1}{2} \left(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} |\varphi(s, \mathbf{v}_1(s), \boldsymbol{\omega}_1(s)) - \varphi(s, \mathbf{v}_2(s), \boldsymbol{\omega}_2(s))| ds \right. \right. \\
& + \int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} |\psi(s, \mathbf{v}_1(s), \boldsymbol{\omega}_1(s)) - \psi(s, \mathbf{v}_2(s), \boldsymbol{\omega}_2(s))| ds \Big) \\
& + \frac{1}{|\Lambda|} \int_{\eta}^{\xi} \left(\int_0^s \frac{(s-x)^{\alpha-1}}{\Gamma(\alpha)} |\varphi(x, \mathbf{v}_1(x), \boldsymbol{\omega}_1(s)) - \varphi(x, \mathbf{v}_2(x), \boldsymbol{\omega}_2(s))| dx \right. \\
& + \int_0^s \frac{(s-x)^{\beta-1}}{\Gamma(\beta)} |\psi(x, \mathbf{v}_1(x), \boldsymbol{\omega}_1(x)) - \psi(x, \mathbf{v}_2(x), \boldsymbol{\omega}_2(x))| dx \Big) ds \\
& + \frac{\sum_{i=1}^m a_i}{|\Lambda|} \left(\int_0^{\sigma_i} \frac{(\sigma_i-x)^{\alpha-1}}{\Gamma(\alpha)} |\varphi(x, \mathbf{v}_1(x), \boldsymbol{\omega}_1(s)) - \varphi(x, \mathbf{v}_2(x), \boldsymbol{\omega}_2(s))| dx \right. \\
& + \int_0^{\sigma_i} \frac{(\sigma_i-x)^{\beta-1}}{\Gamma(\beta)} |\psi(x, \mathbf{v}_1(x), \boldsymbol{\omega}_1(x)) - \psi(x, \mathbf{v}_2(x), \boldsymbol{\omega}_2(x))| dx \Big)
\end{aligned}$$

$$\begin{aligned}
& + \frac{\sum_{j=1}^n b_j}{|\Lambda|} \left(\int_0^{\delta_j} \frac{(\delta_j - x)^{\alpha-1}}{\Gamma(\alpha)} |\varphi(x, v_1(x), \omega_1(x)) - \varphi(s, v_2(x), \omega_2(x))| dx \right. \\
& \left. + \int_0^{\delta_j} \frac{(\delta_j - x)^{\beta-1}}{\Gamma(\beta)} |\psi(x, v_1(x), \omega_1(x)) - \psi(s, v_2(x), \omega_2(x))| dx \right) \Bigg\} \\
& \leq \left\{ C\Delta_1 + \mathcal{K} \left(\frac{T^\beta}{\Gamma(\beta+1)} + \Delta_2 \right) \right\} (\|v\| + \|\omega\|).
\end{aligned}$$

In view of the foregoing inequalities, it follows that

$$\begin{aligned}
\|\Pi(v_1, \omega_1) - \Pi(v_2, \omega_2)\| &= \|\Pi_1(v_1, \omega_1) - \Pi_1(v_2, \omega_2)\| + \|\Pi_2(v_1, \omega_1) - \Pi_2(v_2, \omega_2)\| \\
&\leq \left\{ C \left(\frac{T^\alpha}{\Gamma(\alpha+1)} + 2\Delta_1 \right) + \mathcal{K} \left(\frac{T^\beta}{\Gamma(\beta+1)} + 2\Delta_2 \right) \right\} \|(v_1 - v_2, \omega_1 - \omega_2)\|,
\end{aligned}$$

which, by (3.7), implies that Π is a contraction mapping. Hence the conclusion of contraction mapping principle applies and Π has a unique fixed point, which is a unique solution of the problem (1.1). This finishes the proof. $\square$

Example 1. Consider the following problem

$$\begin{cases} {}^{LC}D^{1/2}v(t) = \varphi(t, v(t), \omega(t)), & t \in [0, 2], \\ {}^{LC}D^{2/3}\omega(t) = \psi(t, v(t), \omega(t)), & t \in [0, 2], \\ (v + \omega)(0) = -(v + \omega)(2), \\ \int_{3/4}^{3/2} (v - \omega)(s) ds - (v - \omega)(1/4) - 1/5(v - \omega)(1/3) - 4(v - \omega)(7/4) = 1, \end{cases} \quad (3.8)$$

where $\alpha = 1/2$, $\beta = 2/3$, $\eta = 3/4$, $\zeta = 3/2$, $a_1 = 1$, $a_2 = 1/5$, $b_1 = 4$, $\sigma_1 = 1/4$, $\sigma_2 = 1/3$, $\delta_1 = 7/4$, $A = 1$, $T = 2$, and $\varphi(t, v, \omega)$ and $\psi(t, v, \omega)$ will be fixed later.

Using the given values, it is found that

$$\Lambda = -4.45, \Delta_1 = 0.7477415389, \Delta_2 = 0.7168620998,$$

where Λ , Δ_1 and Δ_2 are respectively given by (2.4), (3.4) and (3.5). In order to illustrate Theorem 1, we consider

$$\varphi(t, v, \omega) = \frac{\cos t}{(2+t)^4} \left(\sin v + \frac{\omega}{2} + e^{-t} \right) \quad (3.9)$$

and

$$\psi(t, v, \omega) = \frac{e^{-t}}{2\sqrt{36+t}} \left(|v| \left(\frac{|v|}{2+|v|} \right) + t \sin \omega + 1 \right).$$

Clearly φ and ψ are continuous and satisfy the condition (H_1) with

$$\mu_1(t) = \frac{e^{-t} \cos t}{(2+t)^4}, \mu_2(t) = \frac{\cos t}{(2+t)^4}, \mu_3 = \frac{\cos t}{2(2+t)^4},$$

$$\kappa_1 = \kappa_2 = \frac{e^{-t}}{2\sqrt{36+t}}, \kappa_3 = \frac{te^{-t}}{2\sqrt{36+t}}.$$

Also

$$\|\mu_2\| \left(2\Delta_1 + \frac{T^\alpha}{\Gamma(\alpha+1)} \right) + \|\kappa_2\| \left(2\Delta_2 + \frac{T^\beta}{\Gamma(\beta+1)} \right) \approx 0.4592148800$$

and

$$\|\mu_3\| \left(2\Delta_1 + \frac{T^\alpha}{\Gamma(\alpha+1)} \right) + \|\kappa_3\| \left(2\Delta_2 + \frac{T^\beta}{\Gamma(\beta+1)} \right) \approx 0.3626132487 < 1.$$

Clearly the assumptions of Theorem 1 are satisfied. Therefore, the problem (3.8) with $\varphi(t, v, \omega)$ and $\psi(t, v, \omega)$ given by (3.9) has at least one solution on $[0, 2]$.

Now we explain Theorem 2 by taking

$$\varphi(t, v, \omega) = \frac{\tan^{-1} v + \omega}{\sqrt{900+t^2}} \text{ and } \psi(t, v, \omega) = \frac{1}{25(1+t^2)} \left(\frac{|v|}{2+|v|} + \sin \omega \right). \quad (3.10)$$

Observe that φ and ψ are continuous and satisfy the condition (H_2) with $C_1 = C_2 = 1/30 = C$ and $K_1 = 1/50, K_2 = 1/25$ and so, $\mathcal{K} = 1/25$. Also

$$C \left(\frac{T^\alpha}{\Gamma(\alpha+1)} + 2\Delta_1 \right) + \mathcal{K} \left(\frac{T^\beta}{\Gamma(\beta+1)} + 2\Delta_2 \right) \approx 0.2307273163 < 1.$$

Clearly the hypothesis of Theorem 2 is satisfied and hence its conclusion applies to the problem (3.8) with $\varphi(t, v, \omega)$ and $\psi(t, v, \omega)$ given by (3.10).

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VORONOVSKAJA-TYPE QUANTITATIVE DIFFERENCE ESTIMATES OF POSITIVE LINEAR OPERATORS WITH DIFFERENT BASIS FUNCTIONS

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Abstract. The present article is an extension of work [2] and provides Voronovskaja-type quantitative estimate for the difference of positive linear operators with different fundamental functions. We also present the applications of such results using Bernstein operators, Bernstein-Kantorovich operators, Bernstein-Durrmeyer operators, Generalized-Bernstein operators based on Pólya distribution and its Durrmeyer and Kantorovich-type modifications. We use Mathematica software to direct massive computations.

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1. INTRODUCTION

At the outset of the study of difference of positive linear operators, Lupaş [16] is the first to commence the study. Thenceforth, it became an important field of research in Approximation Theory. Aral et al. [6] generalized the work of A. Lupaş [16] on polynomial differences. Gupta [9] studied quantitative difference estimates of Lupaş operators with Lupaş-Szász operators and Lupaş-Kantorovich operators using modulus of continuity. Although researchers have addressed several convergence estimates in various forms, in this context, we highlight some of the seminal contributions of authors in this regard, including the linear preservance of general Srivastava-Gupta operators [13], the rate of convergence for the exponential operators [11], introduction and convergence estimation in weighted statistical convergence [7, 8], the construction of the α -Schurer-Kantorovich operators and the establishment of its approximation results [18]. Several studies were attracted to the operators due to Srivastava and collaborators [21, 22] in order to further extend the results on this operator, for instance [1, 4, 15, 19, 24].

The authors in [10] studied some generic estimates for the difference of operators associated with different basis functions. Very latterly, Acu et al. [2] established Voronovskaja-type quantitative results for the difference of positive linear operators

having the same fundamental functions. We consider here Voronovskaja-type quantitative difference estimates of positive linear operators having different fundamental functions.

We denote $J \subset \mathbb{R}$ as an interval and $\tilde{E}(J)$ as a space of continuous real valued functions determined on J containing the polynomials. Let us deal with two positive linear operators M_n and N_n having different basis functions as follows:

$$M_n(h, u) = \sum_{i \in K} F_i(h) p_{n,i}(u)$$

and

$$N_n(h, u) = \sum_{i \in K} G_i(h) q_{n,i}(u),$$

where $K = \mathbb{Z}^+ \cup \{0\}$, $u \in J$, F_i and G_i are positive linear functionals and $p_{n,i}$, $q_{n,i}$ denote the basis functions satisfying $\sum_{i \in K} p_{n,i} = e_0$ and $\sum_{i \in K} q_{n,i} = e_0$.

Raşa [20] associated the positive linear integral operators to discrete operators and considered the link between the two. Since it is simple to work with the discrete operators associated with positive linear operator. Let D_M and D_N defines the discrete operators associated to positive linear operators M_n and N_n given by

$$D_M(h, u) = \sum_{i \in K} h(b^{F_i}) p_{n,i}(u)$$

and

$$D_N(h, u) = \sum_{i \in K} h(b^{G_i}) q_{n,i}(u),$$

where $b^F = F(e_1)$. To prove the main result of the paper, we need the following lemma.

2. AUXILIARY RESULT

Lemma 1 ([2]). *For arbitrary $u \in J$ and $h'' \in \tilde{E}(J)$ such that $\|h''\| := \sup_{u \in J} |h''(u)| < \infty$, we have*

$$\begin{aligned} & ((M - D_M)h)(u) - \sum_{j=1}^2 \frac{h^{(j)}(u)}{j!} ((M - D_M)(e_1 - ue_0)^j)(u) \\ &= \sum_{i \in K} \left\{ F_i(R_2(h, b^{F_i}, \cdot)) + \frac{1}{2} [h''(b^{F_i}) - h''(u)] \vartheta_2^{F_i} \right\} p_i(u), \end{aligned}$$

where $R_2(h, b^{F_i}, \cdot)$ is the remainder of Taylor's formula given by

$$\begin{aligned} R_2(h, b^F, u) &= h(u) - h(b^F) - h'(b^F)(u - b^F) - \frac{h''(b^F)}{2}(u - b^F)^2 \\ &= \frac{(u - b^F)^2}{2} (h''(\eta) - h''(b^F)); \eta \in (u, b^F), \end{aligned} \quad (2.1)$$

and $\vartheta_2^{F_i}$ is governed through $\vartheta_k^F = F(e_1 - b^F e_0)^k$, $k \in \mathbb{N}$.

Remark 1. For a positive linear functional F on $\tilde{E}(J)$ and $R_2(h, b^F, \cdot)$ as the remainder of Taylor's formula given by (2.1), we have

$$|R_2(h, b^F, u)| \leq \frac{(u - b^F)^2}{2} \omega(h'', |u - b^F|),$$

where $\omega(h, \cdot)$ denotes the first order modulus of continuity.

Furthermore, since $\vartheta_1^F := F(e_1 - b^F e_0) = 0$, we have

$$F(R_2(h, b^F, \cdot)) = F(h) - h(b^F) - \frac{h''(b^F)}{2} \vartheta_2^F.$$

Remark 2. Taking into account Remark 1 and the well known inequality for $\varepsilon > 0$ given by

$$\omega(f, |t - u|) \leq \left(1 + \frac{(t - u)^2}{\varepsilon^2}\right) \omega(f, \varepsilon),$$

we have

$$|R_2(h, b^F, u)| \leq \left((u - b^F)^2 + \frac{(u - b^F)^4}{\varepsilon^2}\right) \frac{\omega(h'', \varepsilon)}{2},$$

implying

$$|F(R_2(h, b^F, \cdot))| \leq \left(\vartheta_2^F + \frac{\vartheta_4^F}{\varepsilon^2}\right) \frac{\omega(h'', \varepsilon)}{2}.$$

Next, we shall study the quantitative Voronovskaja theorem for the difference $M_n - N_n$.

3. MAIN THEOREM

Theorem 1. For $u \in J$ and $h'' \in \tilde{E}(J)$ such that $\|h''\| := \sup_{u \in J} |h(u)| < \infty$, we have

$$\begin{aligned} & \left| ((M_n - N_n)h)(u) - \sum_{j=1}^2 \frac{h^{(j)}(u)}{j!} ((M_n - N_n)(e_1 - ue_0)^j)(u) \right| \\ & \leq \frac{[1 + \alpha(u)]}{2} \omega(h'', \sqrt{\beta(u)}) + \frac{[1 + ((M_n + N_n)(e_1 - ue_0)^2)(u)]}{2} \omega(h'', \sqrt{\gamma(u)}), \end{aligned}$$

where

$$\begin{aligned} \alpha(u) &= \sum_{i \in K} \left(\vartheta_2^{F_i} p_{n,i}(u) + \vartheta_2^{G_i} q_{n,i}(u) \right), \\ \beta(u) &= \sum_{i \in K} \left(\vartheta_4^{F_i} p_{n,i}(u) + \vartheta_4^{G_i} q_{n,i}(u) \right), \\ \gamma(u) &= \sum_{i \in K} \left((b^{F_i} - u)^2 \vartheta_2^{F_i}(u) p_{n,i}(u) + (b^{G_i} - u)^2 \vartheta_2^{G_i}(u) q_{n,i}(u) \right) \end{aligned}$$

and $\vartheta_k^F(u) = F(e_1 - ue_0)^k$, $k \in \mathbb{N}$.

Proof. Let $u \in J$ be arbitrary fixed point. Then, we have

$$\begin{aligned}
& ((M_n - N_n)h)(u) - \sum_{j=1}^2 \frac{h^{(j)}(u)}{j!} ((M_n - N_n)(e_1 - ue_0)^j)(u) \\
&= \sum_{i \in K} \left\{ \left(F_i(h) - \sum_{j=1}^2 \frac{h^{(j)}(u)}{j!} (F_i(e_1 - ue_0)^j) \right) p_{n,i}(u) \right. \\
&\quad \left. - \left(G_i(h) - \sum_{j=1}^2 \frac{h^{(j)}(u)}{j!} (G_i(e_1 - ue_0)^j) \right) q_{n,i}(u) \right\} \\
&= \sum_{i \in K} \left\{ \left(F_i(h) - h(b^{F_i}) - \sum_{j=1}^2 \frac{h^{(j)}(u)}{j!} (F_i(e_1 - ue_0)^j) \right) p_{n,i}(u) \right. \\
&\quad \left. - \left(G_i(h) - h(b^{G_i}) - \sum_{j=1}^2 \frac{h^{(j)}(u)}{j!} (G_i(e_1 - ue_0)^j) \right) q_{n,i}(u) \right\} \\
&\quad + \sum_{i \in K} \left(h(b^{F_i}) p_{n,i}(u) - h(b^{G_i}) q_{n,i}(u) \right).
\end{aligned}$$

Using the fact that $\sum_{i \in K} p_{n,i} = e_0$ and $\sum_{i \in K} q_{n,i} = e_0$, we get

$$\begin{aligned}
& ((M_n - N_n)h)(u) - \sum_{j=1}^2 \frac{h^{(j)}(u)}{j!} ((M_n - N_n)(e_1 - ue_0)^j)(u) \\
&= \sum_{i \in K} \left\{ [F_i(h) - h(b^{F_i})] p_{n,i}(u) - [G_i(h) - h(b^{G_i})] q_{n,i}(u) \right. \\
&\quad + \left[h(b^{F_i}) - h(u) - h'(u)F_i(e_1 - ue_0) - \frac{h''(u)}{2} F_i(e_1 - ue_0)^2 \right] p_{n,i}(u) \\
&\quad \left. - \left[h(b^{G_i}) - h(u) - h'(u)G_i(e_1 - ue_0) - \frac{h''(u)}{2} G_i(e_1 - ue_0)^2 \right] q_{n,i}(u) \right\} \\
&= \sum_{i \in K} \left\{ [F_i(h) - h(b^{F_i})] p_{n,i}(u) - [G_i(h) - h(b^{G_i})] q_{n,i}(u) \right. \\
&\quad + \left[h(b^{F_i}) - h(u) - h'(u)(b^{F_i} - u) - \frac{h''(u)}{2} \vartheta_2^{F_i}(u) \right] p_{n,i}(u) \\
&\quad \left. - \left[h(b^{G_i}) - h(u) - h'(u)(b^{G_i} - u) - \frac{h''(u)}{2} \vartheta_2^{G_i}(u) \right] q_{n,i}(u) \right\}. \tag{3.1}
\end{aligned}$$

Notice that, for a positive linear functional F , we have

$$\begin{aligned}
\vartheta_2^F(u) &= F(e_1 - ue_0)^2 = F(e_2) - 2ub^F + u^2 F(e_0) \\
&= F(e_2) - (b^F)^2 + (b^F)^2 - 2ub^F + u^2 = \vartheta_2^F + (b^F - u)^2. \tag{3.2}
\end{aligned}$$

Using (3.2) in (3.1) and taking into account (2.1), for fixed $u \in J$, we obtain

$$\begin{aligned}
& ((M_n - N_n)h)(u) - \sum_{j=1}^2 \frac{h^{(j)}(u)}{j!} ((M_n - N_n)(e_1 - ue_0)^j)(u) \\
&= \sum_{i \in K} \left\{ \left[F_i(h) - h(b^{F_i}) - \frac{h''(u)}{2} \mathfrak{D}_2^{F_i} \right] p_{n,i}(u) - \left[G_i(h) - h(b^{G_i}) - \frac{h''(u)}{2} \mathfrak{D}_2^{G_i} \right] q_{n,i}(u) \right. \\
&\quad \left. + R_2(h, u, b^{F_i}) p_{n,i}(u) - R_2(h, u, b^{G_i}) q_{n,i}(u) \right\} \\
&= \sum_{i \in K} \left\{ \left[F_i(h) - h(b^{F_i}) - \frac{h''(b^{F_i})}{2} \mathfrak{D}_2^{F_i} \right] p_{n,i}(u) - \left[G_i(h) - h(b^{G_i}) - \frac{h''(b^{G_i})}{2} \mathfrak{D}_2^{G_i} \right] q_{n,i}(u) \right. \\
&\quad + \left[\frac{h''(b^{F_i}) - h''(u)}{2} \right] \mathfrak{D}_2^{F_i} p_{n,i}(u) - \left[\frac{h''(b^{G_i}) - h''(u)}{2} \right] \mathfrak{D}_2^{G_i} q_{n,i}(u) \\
&\quad \left. + R_2(h, u, b^{F_i}) p_{n,i}(u) - R_2(h, u, b^{G_i}) q_{n,i}(u) \right\} \\
&= \sum_{i \in K} \left\{ F_i(R_2(h, b^{F_i}, \cdot)) p_{n,i}(u) - G_i(R_2(h, b^{G_i}, \cdot)) q_{n,i}(u) + \left[\frac{h''(b^{F_i}) - h''(u)}{2} \right] \mathfrak{D}_2^{F_i} p_{n,i}(u) \right. \\
&\quad \left. - \left[\frac{h''(b^{G_i}) - h''(u)}{2} \right] \mathfrak{D}_2^{G_i} q_{n,i}(u) + R_2(h, u, b^{F_i}) p_{n,i}(u) - R_2(h, u, b^{G_i}) q_{n,i}(u) \right\}.
\end{aligned}$$

Using Remark 2, we have

$$\begin{aligned}
& \left| ((M_n - N_n)h)(u) - \sum_{j=1}^2 \frac{h^{(j)}(u)}{j!} ((M_n - N_n)(e_1 - ue_0)^j)(u) \right| \\
&\leq \sum_{i \in K} \left\{ \left(\mathfrak{D}_2^{F_i} + \frac{\mathfrak{D}_4^{F_i}}{\beta(u)} \right) \frac{\omega(h'', \sqrt{\beta(u)})}{2} p_{n,i}(u) + \left(\mathfrak{D}_2^{G_i} + \frac{\mathfrak{D}_4^{G_i}}{\beta(u)} \right) \frac{\omega(h'', \sqrt{\beta(u)})}{2} q_{n,i}(u) \right. \\
&\quad + \frac{1}{2} \left[\omega(h'', |b^{F_i} - u|) \mathfrak{D}_2^{F_i} p_{n,i}(u) + \omega(h'', |b^{G_i} - u|) \mathfrak{D}_2^{G_i} q_{n,i}(u) \right] \\
&\quad \left. + (b^{F_i} - u)^2 \frac{\omega(h'', |b^{F_i} - u|)}{2} p_{n,i}(u) + (b^{G_i} - u)^2 \frac{\omega(h'', |b^{G_i} - u|)}{2} q_{n,i}(u) \right\} \\
&= \frac{1}{2} \sum_{i \in K} \left\{ \left[\left(\mathfrak{D}_2^{F_i} + \frac{\mathfrak{D}_4^{F_i}}{\beta(u)} \right) p_{n,i}(u) + \left(\mathfrak{D}_2^{G_i} + \frac{\mathfrak{D}_4^{G_i}}{\beta(u)} \right) q_{n,i}(u) \right] \omega(h'', \sqrt{\beta(u)}) \right. \\
&\quad \left. + \mathfrak{D}_2^{F_i}(u) \omega(h'', |b^{F_i} - u|) p_{n,i}(u) + \mathfrak{D}_2^{G_i}(u) \omega(h'', |b^{G_i} - u|) q_{n,i}(u) \right\} \\
&\leq \frac{\omega(h'', \sqrt{\beta(u)})}{2} \sum_{i \in K} \left\{ \mathfrak{D}_2^{F_i} p_{n,i}(u) + \mathfrak{D}_2^{G_i} q_{n,i}(u) + \frac{1}{\beta(u)} \left(\mathfrak{D}_4^{F_i} p_{n,i}(u) + \mathfrak{D}_4^{G_i} q_{n,i}(u) \right) \right\} \\
&\quad + \frac{\omega(h'', \sqrt{\gamma(u)})}{2} \sum_{i \in K} \left\{ \left(1 + \frac{(b^{F_i} - u)^2}{\gamma(u)} \right) \mathfrak{D}_2^{F_i}(u) p_{n,i}(u) + \left(1 + \frac{(b^{G_i} - u)^2}{\gamma(u)} \right) \mathfrak{D}_2^{G_i}(u) q_{n,i}(u) \right\}
\end{aligned}$$

$$= \frac{\omega(h'', \sqrt{\beta(u)})}{2} [1 + \alpha(u)] + \frac{\omega(h'', \sqrt{\gamma(u)})}{2} \left[1 + \sum_{i \in K} \left(\mathfrak{D}_2^{F_i}(u) p_{n,i}(u) + \mathfrak{D}_2^{G_i}(u) q_{n,i}(u) \right) \right].$$

Further, since we have

$$\begin{aligned} (M_n(e_1 - ue_0)^k)(u) &= \sum_{i \in K} \mathfrak{D}_k^{F_i}(u) p_{n,i}(u) \\ (N_n(e_1 - ue_0)^k)(u) &= \sum_{i \in K} \mathfrak{D}_k^{G_i}(u) q_{n,i}(u), \end{aligned}$$

the theorem is proved. $\square$

4. ILLUSTRATIONS

Now, we shall utilise Theorem 1 to estimate the Voronovskaja-type quantitative differences between some of the most often used positive linear operators in Approximation Theory, viz. Bernstein operators B_n , Bernstein Durrmeyer operators D_n , Bernstein Kantorovich operators K_n , Generalized-Bernstein operators based on Pólya distribution $P_n^{(1/n)}$, its Durrmeyer type integral modification $D_n^{(1/n)}$ and Kantorovich modification $K_n^{(1/n)}$.

Consider the well known Bernstein operators $B_n : C[0, 1] \rightarrow C[0, 1]$ given by

$$(B_n h)(u) = \sum_{i=0}^n p_{n,i}(u) h\left(\frac{i}{n}\right), \quad (4.1)$$

where $p_{n,i}(u) = \binom{n}{i} u^i (1-u)^{n-i}$, $u \in [0, 1]$ and $n \in \mathbb{N}$.

The Durrmeyer type integral modification of operators (4.1) for $h \in C[0, 1]$ introduced by [3] is given by

$$(D_n h)(u) = (n+1) \sum_{i=0}^n p_{n,i}(u) \int_0^1 p_{n,i}(\zeta) h(\zeta) d\zeta. \quad (4.2)$$

And the Bernstein-Kantorovich operators $K_n : L_1[0, 1] \rightarrow C[0, 1]$ proposed by [14] are defined as

$$(K_n h)(u) = (n+1) \sum_{i=0}^n p_{n,i}(u) \int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} h(\zeta) d\zeta. \quad (4.3)$$

As considered by [17], the Generalized-Bernstein operators $P_n^{(1/n)} : C[0, 1] \rightarrow C[0, 1]$ based on Pólya distributions are given by

$$(P_n^{(1/n)} h)(u) = \sum_{i=0}^n p_{n,i}^{(1/n)}(u) h\left(\frac{i}{n}\right), \quad (4.4)$$

where $p_{n,i}^{(1/n)}(u) = \frac{2(n)!}{(2n)!} \binom{n}{i} (nu)_i (n-nu)_{n-i}$ and $(n)_i$ is the rising factorial given by $(n)_i = n(n+1)\dots(n+i-1)$.

The Durrmeyer type integral modification of operators (4.4) for $h \in C[0, 1]$ introduced by [12] is given by

$$(D_n^{(1/n)}h)(u) = (n+1) \sum_{i=0}^n p_{n,i}^{(1/n)}(u) \int_0^1 p_{n,i}(\varsigma) h(\varsigma) d\varsigma. \quad (4.5)$$

Influenced by the work of [12], Agrawal et al. [5] proposed the Kantorovich modification of the operators (4.4) with $h \in C[0, 1]$ as

$$(K_n^{(1/n)}h)(u) = (n+1) \sum_{i=0}^n p_{n,i}^{(1/n)}(u) \int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} h(\varsigma) d\varsigma. \quad (4.6)$$

4.1. Difference estimates between operators B_n and $P_n^{(1/n)}$

Proposition 1. *Let $h'' \in C[0, 1]$. Then, for the Voronovskaja-type difference between B_n and $P_n^{(1/n)}$, we have*

$$\begin{aligned} & \left| ((B_n - P_n^{(1/n)})h)(u) - \sum_{j=1}^2 \frac{h^{(j)}(u)}{j!} ((B_n - P_n^{(1/n)})(e_1 - ue_0)^j)(u) \right| \\ & \leq \frac{1}{2} \left[1 + \frac{u(1-u)(3n+1)}{n(n+1)} \right] \omega(h'', \sqrt{\gamma(u)}), \end{aligned}$$

where

$$\begin{aligned} \gamma(u) &= \frac{u(1-u)}{n^3(n+1)(n+2)(n+3)} \\ & \cdot [3u(1-u)(5n^4 - 24n^3 - n^2 - 16n - 12) + (27n^3 + 4n^2 + 11n + 6)]. \end{aligned}$$

Proof. If we indicate $F_i(h) = h\left(\frac{i}{n}\right)$, then we can express the operators (4.1) and (4.4) as

$$(B_n h)(u) = \sum_{i=0}^n F_i(h) p_{n,i}(u)$$

and

$$(P_n^{(1/n)}h)(u) = \sum_{i=0}^n F_i(h) p_{n,i}^{(1/n)}(u).$$

Thus, we have

$$b^{F_i} = \frac{i}{n}, \quad \vartheta_2^{F_i} = 0 = \vartheta_4^{F_i}, \quad \vartheta_2^{F_i}(u) = \left(\frac{i}{n} - u \right)^2.$$

By simple computations, we get $\alpha(u) = 0$, $\beta(u) = 0$, and

$$\begin{aligned} \gamma(u) &= \frac{u(1-u)}{n^3(n+1)(n+2)(n+3)} \\ & \cdot [3u(1-u)(5n^4 - 24n^3 - n^2 - 16n - 12) + (27n^3 + 4n^2 + 11n + 6)]. \end{aligned}$$

Also, we have

$$\begin{aligned} [1 + ((B_n + P_n^{(1/n)})(e_1 - ue_0)^2)(u)] &= [1 + \mathfrak{D}_2^{F_i}(u)p_{n,i}(u) + \mathfrak{D}_2^{F_i}(u)p_{n,i}^{(1/n)}(u)] \\ &= 1 + \frac{u(1-u)(3n+1)}{n(n+1)}. \end{aligned}$$

The adoption of Theorem 1 leads us to the proof of the proposition. $\square$

4.2. Difference estimates between operators B_n and $D_n^{(1/n)}$

Proposition 2. *Let $h'' \in C[0, 1]$. Then, for the difference between B_n and $D_n^{(1/n)}$, we have*

$$\begin{aligned} &\left| ((B_n - D_n^{(1/n)})h)(u) - \sum_{j=1}^2 \frac{h^{(j)}(u)}{j!} ((B_n - D_n^{(1/n)})(e_1 - ue_0)^j)(u) \right| \\ &\leq \frac{1}{2} (1 + \alpha(u)) \omega(h'', \sqrt{\beta(u)}) \\ &\quad + \frac{1}{2} \left[\frac{u(1-u)(4n^3 + n^2 + 5n + 6) + n(n+1)(n^2 + 5n + 8)}{n(n+1)(n+2)(n+3)} \right] \omega(h'', \sqrt{\gamma(u)}), \end{aligned}$$

where

$$\begin{aligned} \alpha(u) &= \frac{n^2(n-1)u(1-u) + (n+1)^2}{(n+1)(n+2)^2(n+3)}, \\ \beta(u) &= \frac{3}{(n+1)(n+2)^5(n+3)^2(n+4)(n+5)} \\ &\quad \cdot [n^4u^3(u-2)(n-1)(n-2)(n-3)(n-4) + n^3u^2(n-1)(n^4 - 15n^3 \\ &\quad + 12n^2 - 60n - 4) + 2n^3u(n-1)(3n^3 + 7n^2 + 18n + 2) \\ &\quad + (n+1)^2(n+2)(n+3)(3n^2 + 5n + 4)], \\ \gamma(u) &= \frac{(u-1)}{n^3(n+1)^3(n+2)^2(n+3)^2} \\ &\quad \cdot \left[u^3 (17n^8 - 176n^7 - 334n^6 - 17n^5 - 141n^4 - 1011n^3 - 1446n^2 - 900n - 216) \right. \\ &\quad - u^2 (17n^8 - 203n^7 - 440n^6 - 76n^5 - 81n^4 - 975n^3 - 1446n^2 - 900n - 216) \\ &\quad - u (76n^7 + 189n^6 + 141n^5 + 120n^4 + 289n^3 + 325n^2 + 168n + 36) \\ &\quad \left. - 2n^3(n+1)(n+2)(n+3) \right]. \end{aligned}$$

Proof. If we denote $H_i(h) = (n+1) \int_0^1 p_{n,i}(\zeta)h(\zeta)d\zeta$, then by using the same functional notation as available in Proposition 1, we can express the operators (4.1) and

(4.5) as

$$(B_n h)(u) = \sum_{i=0}^n F_i(h) p_{n,i}(u)$$

and

$$(D_n^{(1/n)} h)(u) = \sum_{i=0}^n H_i(h) p_{n,i}^{(1/n)}(u).$$

Thus, we require the following extra parameters given below:

$$\begin{aligned} b^{H_i} &= \frac{i+1}{n+2}, \quad \mathfrak{d}_2^{H_i} = \frac{(i+1)(n-i+1)}{(n+2)^2(n+3)}, \\ \mathfrak{d}_4^{H_i} &= \frac{3(i+1)(n-i+1)(i(n-4)(n-i)+3n^2+5n+4)}{(n+2)^4(n+3)(n+4)(n+5)}, \\ \mathfrak{d}_2^{H_i}(u) &= u^2 - 2u \left(\frac{i+1}{n+2} \right) + \frac{(i+1)(i+2)}{(n+2)(n+3)}. \end{aligned}$$

Implication of Theorem 1 provides us the following:

$$\begin{aligned} \alpha(u) &= \frac{n^2(n-1)u(1-u) + (n+1)^2}{(n+1)(n+2)^2(n+3)}, \\ \beta(u) &= \frac{3}{(n+1)(n+2)^5(n+3)^2(n+4)(n+5)} \\ &\quad \cdot [n^4 u^3(u-2)(n-1)(n-2)(n-3)(n-4) + n^3 u^2(n-1)(n^4 - 15n^3 \\ &\quad + 12n^2 - 60n - 4) + 2n^3 u(n-1)(3n^3 + 7n^2 + 18n + 2) \\ &\quad + (n+1)^2(n+2)(n+3)(3n^2 + 5n + 4)], \\ \gamma(u) &= \frac{(u-1)}{n^3(n+1)^3(n+2)^2(n+3)^2} \\ &\quad \cdot \left[u^3 (17n^8 - 176n^7 - 334n^6 - 17n^5 - 141n^4 - 1011n^3 - 1446n^2 - 900n - 216) \right. \\ &\quad - u^2 (17n^8 - 203n^7 - 440n^6 - 76n^5 - 81n^4 - 975n^3 - 1446n^2 - 900n - 216) \\ &\quad - u (76n^7 + 189n^6 + 141n^5 + 120n^4 + 289n^3 + 325n^2 + 168n + 36) \\ &\quad \left. - 2n^3(n+1)(n+2)(n+3) \right]. \end{aligned}$$

Also, we have

$$\begin{aligned} [1 + ((B_n + D_n^{(1/n)})(e_1 - ue_0)^2)(u)] &= [1 + \mathfrak{d}_2^{F_i}(u) p_{n,i}(u) + \mathfrak{d}_2^{H_i}(u) p_{n,i}^{(1/n)}(u)] \\ &= \frac{u(1-u)(4n^3 + n^2 + 5n + 6) + n(n+1)(n^2 + 5n + 8)}{n(n+1)(n+2)(n+3)}. \end{aligned}$$

Thus, Proposition 2 is proved. $\square$

4.3. Difference estimates between operators B_n and $K_n^{(1/n)}$

Proposition 3. *Let $h'' \in C[0, 1]$. Then, for the difference between B_n and $K_n^{(1/n)}$, we have*

$$\begin{aligned} & \left| ((B_n - K_n^{(1/n)})h)(u) - \sum_{j=1}^2 \frac{h^{(j)}(u)}{j!} ((B_n - K_n^{(1/n)})(e_1 - ue_0)^j)(u) \right| \\ & \leq \frac{1}{2} \left(1 + \frac{1}{12(n+1)^2} \right) \omega \left(h'', \frac{1}{4\sqrt{5}(n+1)^2} \right) \\ & \quad + \frac{1}{2} \left[\frac{3u(1-u)(3n^3 + 2n^2 + 2n + 1) + n(n+1)(3n^2 + 6n + 4)}{3n(n+1)^3} \right] \omega \left(h'', \sqrt{\gamma(u)} \right), \end{aligned}$$

where

$$\begin{aligned} \gamma(u) = & \frac{1}{12n^3(n+1)^5(n+2)(n+3)} \\ & \cdot [12u^4(n-1)(4n^9 + 44n^8 + 223n^7 + 463n^6 + 783n^5 + 1085n^4 + 1041n^3 \\ & + 639n^2 + 228n + 36) - 24u^3(15n^8 - 120n^7 - 140n^6 - 42n^5 - 184n^4 - 426n^3 \\ & - 411n^2 - 192n - 36) - u^2(48n^10 + 480n^9 + 1788n^8 + 6266n^7 + 7911n^6 + 5310n^5 \\ & + 4891n^4 + 6738n^3 + 5964n^2 + 2724n + 504) + u(506n^7 + 711n^6 + 678n^5 + 1003n^4 \\ & + 1338n^3 + 1032n^2 + 420n + 72) + n^3(n+1)(n+2)(n+3)]. \end{aligned}$$

Proof. Denoting $G_i(h) = (n+1) \int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} h(\zeta) d\zeta$ and making use of the same functional notations wherever required, we can express the operators (4.1) and (4.6) as follows:

$$(B_n h)(u) = \sum_{i=0}^n F_i(h) p_{n,i}(u),$$

and

$$(K_n^{(1/n)} h)(u) = \sum_{i=0}^n G_i(h) p_{n,i}^{(1/n)}(u).$$

Taking into account Proposition 1, we need the following additional values:

$$\begin{aligned} b^{G_i} &= \frac{2i+1}{2(n+1)}, \quad \vartheta_2^{G_i} = \frac{1}{12(n+1)^2}, \\ \vartheta_4^{G_i} &= \frac{1}{80(n+1)^4}, \quad \vartheta_2^{G_i}(u) = \frac{1}{12(n+1)^2} + \left(\frac{2i+1}{2(n+1)} - u \right)^2. \end{aligned}$$

By simple calculations on Theorem 1, we obtain

$$\alpha(u) = \frac{1}{12(n+1)^2}, \quad \beta(u) = \frac{1}{80(n+1)^4},$$

and

$$\begin{aligned} \gamma(u) = & \frac{1}{12n^3(n+1)^5(n+2)(n+3)} \\ & \cdot [12u^4(n-1)(4n^9 + 44n^8 + 223n^7 + 463n^6 + 783n^5 + 1085n^4 + 1041n^3 \\ & + 639n^2 + 228n + 36) - 24u^3(15n^8 - 120n^7 - 140n^6 - 42n^5 - 184n^4 - 426n^3 \\ & - 411n^2 - 192n - 36) - u^2(48n^10 + 480n^9 + 1788n^8 + 6266n^7 + 7911n^6 + 5310n^5 \\ & + 4891n^4 + 6738n^3 + 5964n^2 + 2724n + 504) + u(506n^7 + 711n^6 + 678n^5 + 1003n^4 \\ & + 1338n^3 + 1032n^2 + 420n + 72) + n^3(n+1)(n+2)(n+3)]. \end{aligned}$$

Also, we have

$$\begin{aligned} & [1 + ((B_n + K_n^{(1/n)})(e_1 - ue_0)^2)(u)] \\ & = [1 + \mathfrak{D}_2^{F_i}(u)p_{n,i}(u) + \mathfrak{D}_2^{G_i}(u)p_{n,i}^{(1/n)}(u)] \\ & = \left[\frac{3u(1-u)(3n^3 + 2n^2 + 2n + 1) + n(n+1)(3n^2 + 6n + 4)}{3n(n+1)^3} \right]. \end{aligned}$$

This completes the proof of the proposition. $\square$

4.4. Difference between operators $P_n^{(1/n)}$ and K_n

Proposition 4. *Let $h'' \in C[0, 1]$. Then the difference between the positive linear operators $P_n^{(1/n)}$ and K_n is given by*

$$\begin{aligned} & \left| ((P_n^{(1/n)} - K_n)h)(u) - \sum_{j=1}^2 \frac{h^{(j)}(u)}{j!} ((P_n^{(1/n)} - K_n)(e_1 - ue_0)^j)(u) \right| \\ & \leq \frac{1}{2} \left(1 + \frac{1}{12(n+1)^2} \right) \omega \left(h'', \frac{1}{4\sqrt{5}(n+1)^2} \right) \\ & \quad + \frac{1}{2} \left[\frac{3u(1-u)(3n+1) + (3n^2 + 6n + 4)}{3(n+1)^2} \right] \omega \left(h'', \sqrt{\gamma(u)} \right), \end{aligned}$$

where

$$\begin{aligned} \gamma(u) = & \frac{1}{12n(n+1)^4(n+2)(n+3)} \\ & \cdot [12nu^3(u-2)(15n^4 - 53n^3 - 297n^2 - 355n - 78) + u^2(180n^5 - 1003n^4 \\ & - 4744n^3 - 5419n^2 - 1134n + 24) \\ & + u(367n^4 + 1180n^3 + 1159n^2 + 198n - 24) + n(n+2)(n+3)]. \end{aligned}$$

Proof. Using the same functional notations as available, we can express the operators (4.4) and (4.3) in the following manner:

$$(P_n^{(1/n)}h)(u) = \sum_{i=0}^n F_i(h)p_{n,i}^{(1/n)}(u),$$

and

$$(K_nh)(u) = \sum_{i=0}^n G_i(h)p_{n,i}(u).$$

Thus, we have $\alpha(u) = \frac{1}{12(n+1)^2}$, $\beta(u) = \frac{1}{80(n+1)^4}$, and

$$\begin{aligned} \gamma(u) = & \frac{1}{12n(n+1)^4(n+2)(n+3)} \\ & \cdot [12nu^3(u-2)(15n^4 - 53n^3 - 297n^2 - 355n - 78) + u^2(180n^5 - 1003n^4 \\ & - 4744n^3 - 5419n^2 - 1134n + 24) \\ & + u(367n^4 + 1180n^3 + 1159n^2 + 198n - 24) + n(n+2)(n+3)]. \end{aligned}$$

In addition, we have

$$\begin{aligned} [1 + ((P_n^{(1/n)} + K_n)(e_1 - ue_0)^2)(u)] &= [1 + \mathfrak{D}_2^{F_i}(u)p_{n,i}^{(1/n)}(u) + \mathfrak{D}_2^{G_i}(u)p_{n,i}(u)] \\ &= \left[\frac{3u(1-u)(3n+1) + (3n^2 + 6n + 4)}{3(n+1)^2} \right]. \end{aligned}$$

Hence the proof follows using Theorem 1. $\square$

4.5. Difference between operators $P_n^{(1/n)}$ and D_n

Proposition 5. Let $h'' \in C[0, 1]$. Then, for the difference between $P_n^{(1/n)}$ and D_n , we have

$$\begin{aligned} & \left| ((P_n^{(1/n)} - D_n)h)(u) - \sum_{j=1}^2 \frac{h^{(j)}(u)}{j!} ((P_n^{(1/n)} - D_n)(e_1 - ue_0)^j)(u) \right| \\ & \leq \frac{1}{2} (1 + \alpha(u)) \omega(h'', \sqrt{\beta(u)}) \\ & \quad + \frac{1}{2} \left[\frac{2u(1-u)(2n^2 + 3n + 3) + (n+1)(n^2 + 5n + 8)}{(n+1)(n+2)(n+3)} \right] \omega(h'', \sqrt{\gamma(u)}), \end{aligned}$$

where

$$\begin{aligned} \alpha(u) &= \frac{n(n-1)u(1-u) + (n+1)}{(n+2)^2(n+3)}, \\ \beta(u) &= \frac{3}{(n+2)^4(n+3)(n+4)(n+5)} \\ & \quad \cdot [nu^3(u-2)(n-1)(n-2)(n-3)(n-4) + nu^2(n-1)(n^3 - 14n^2 \end{aligned}$$

$$\begin{aligned}
& + 29n - 28) + nu(n-1)(5n^2 - 3n + 4) + (n+1)(3n^2 + 5n + 4)], \\
\gamma(u) &= \frac{2}{n(n+1)(n+2)^2(n+3)} \\
& \cdot [4nu^3(u-2)(2n^3 - 12n^2 - 41n - 39) + u^2(8n^4 - 69n^3 - 216n^2 \\
& - 197n + 4) + u(21n^3 + 52n^2 + 41n - 4) + n(n+1)].
\end{aligned}$$

Proof. If we use the above-mentioned functional notations to represent the operators (4.4) and (4.2), then we can express them as

$$(P_n^{(1/n)}h)(u) = \sum_{i=0}^n F_i(h)p_{n,i}^{(1/n)}(u),$$

and

$$(D_nh)(u) = \sum_{i=0}^n H_i(h)p_{n,i}(u).$$

By basic calculation and taking Proposition 1 and Proposition 2 into consideration, we have the following:

$$\begin{aligned}
\alpha(u) &= \frac{n(n-1)u(1-u) + (n+1)}{(n+2)^2(n+3)}, \\
\beta(u) &= \frac{3}{(n+2)^4(n+3)(n+4)(n+5)} \\
& \cdot [nu^3(u-2)(n-1)(n-2)(n-3)(n-4) + nu^2(n-1)(n^3 - 14n^2 \\
& + 29n - 28) + nu(n-1)(5n^2 - 3n + 4) + (n+1)(3n^2 + 5n + 4)], \\
\gamma(u) &= \frac{2}{n(n+1)(n+2)^2(n+3)} \\
& \cdot [4nu^3(u-2)(2n^3 - 12n^2 - 41n - 39) + u^2(8n^4 - 69n^3 - 216n^2 \\
& - 197n + 4) + u(21n^3 + 52n^2 + 41n - 4) + n(n+1)].
\end{aligned}$$

Also, we have

$$\begin{aligned}
[1 + ((P_n^{(1/n)} + D_n)(e_1 - ue_0)^2)(u)] &= [1 + \mathfrak{F}_2^{F_i}(u)p_{n,i}^{(1/n)}(u) + \mathfrak{F}_2^{H_i}(u)p_{n,i}(u)] \\
&= \frac{2u(1-u)(2n^2 + 3n + 3) + (n+1)(n^2 + 5n + 8)}{(n+1)(n+2)(n+3)}.
\end{aligned}$$

Hence the application of Theorem 1 leads to the proof of the proposition. $\square$

Remark 3. Here, we have studied the difference of Bernstein polynomials with generalized Bernstein based on Pólya, its Durrmeyer and Kantorovich variant. We also investigated the difference estimates between the operator having Pólya basis to Durrmeyer and Kantorovich operators having Bernstein basis. Similar efforts can be made to address the differences of a specific class of operators defined in [22, 23].

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ON EXISTENCE OF BOUNDED SOLUTIONS ON REAL AXIS $\mathbb{R}$ OF LINEAR SYSTEMS OF GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. Effective sufficient conditions are established for existence of bounded solutions satisfying the Nicoletti condition of the systems of linear generalized ordinary differential equations on the real axis. There are given the method of the construction of such solutions. The sufficient conditions of the existence of unique solution and of positiveness of that are established as well. As particular case, there are investigated the problem of existence of bounded solutions.,

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1. STATEMENT OF THE PROBLEM. BASIC NOTATION AND DEFINITIONS

For the linear system of the generalized differential equations

$$dx = dA(t) \cdot x + df(t) \quad \text{for } t \in \mathbb{R}, \quad (1.1)$$

consider the problem on the bounded on $\mathbb{R}$ solution

$$\sup\{\|x(t)\| : t \in \mathbb{R}\} < +\infty, \quad (1.2)$$

where $A = (a_{ik})_{i,k=1}^n \in \text{BV}_{loc}(\mathbb{R}, \mathbb{R}^{n \times n})$, and $f = (f_i)_{i=1}^n \in \text{BV}_{loc}(\mathbb{R}, \mathbb{R}^n)$.

The generalized ordinary differential equations was introduced by J. Kurzweil [11]. To a considerable extent, the interest to the theory has also been stimulated by the fact that this theory enabled one to investigate ordinary differential, impulsive differential and difference equations from a unified point of view (see [1–8], [12] and references therein). So, we can consider the ordinary differential, impulsive differential and difference equations as equations of the same type.

In this paper effective sufficient conditions are established for the existence of solutions of problem (1.1), (1.2). Analogous results are contained in [10], [9] (see also references therein) for the problem for systems of ordinary differential equations.

In the paper the use will be made of the following notation and definitions

$\mathbb{R} =]-\infty, +\infty[$, $\mathbb{R}_+ = [0, +\infty[$; $[a, b]$ and $]a, b[$ ($a, b \in \mathbb{R}$) are, respectively, closed and open intervals. I is an arbitrary finite or infinite interval from $\mathbb{R}$. $[t]$ is the integer part of $t \in \mathbb{R}$.

$\mathbb{R}^{n \times m}$ is the space of all real $n \times m$ matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm $\|X\| = \max_{j=1,\dots,m} \sum_{i=1}^n |x_{ij}|$. $\mathbb{R}^n = \mathbb{R}^{n \times 1}$ is the space of all real column n -vectors $x = (x_i)_{i=1}^n$.

If $X \in \mathbb{R}^{n \times n}$, then X^{-1} , $\det(X)$ and $r(X)$ are, respectively, the matrix inverse to X , the determinant of X and the spectral radius of X ; I_n is the identity $n \times n$ -matrix; δ_{ij} is the Kroneker symbol, i.e. $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$ ($i, j = 1, \dots$).

The inequalities between the matrices are understood componentwise.

$\overset{b}{\underset{a}{V}}(X)$ is the sum total variation of the components of the matrix-function $X : [a, b] \rightarrow \mathbb{R}^{n \times m}$. If $X = (x_{ij})_{i,j=1}^{n,m} : \mathbb{R} \rightarrow \mathbb{R}^{n \times m}$, then $V(X)(t) = \left(\overset{t}{\underset{0}{V}}(x_{ij}) \right)_{i,j=1}^{n,m}$.

$X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of X at the point t ($X(\alpha-) = X(\alpha)$ if $\alpha \in I$ and $X(\beta+) = X(\beta)$ if $\gamma \in I$; if α or β do not belong to I , then $X(t)$ is defined by continuity outside of I). $d_1X(t) = X(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$. $\|X\|_\infty = \sup \{\|X(t)\| : t \in I\}$.

$\text{BV}([a, b]; \mathbb{R}^{n \times m})$ is the set of all matrix-functions $X : [a, b] \rightarrow \mathbb{R}^{n \times m}$ such that $\overset{b}{\underset{a}{V}}(X) < \infty$. $\text{BV}_{loc}(\mathbb{R}; \mathbb{R}^{n \times m})$ is the set of all matrix-functions $X : \mathbb{R} \rightarrow \mathbb{R}^{n \times m}$ whose restrictions on every closed interval $[a, b]$ belong to $\text{BV}([a, b], \mathbb{R}^{n \times m})$.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its component is such.

s_1, s_2, s_c and $J : \text{BV}_{loc}(\mathbb{R}; \mathbb{R}) \rightarrow \text{BV}_{loc}(\mathbb{R}; \mathbb{R})$ are the operators defined by

$$\begin{aligned} s_1(x)(0) &= s_2(x)(0) = 0, \\ s_c(x)(0) &= x(0) \\ s_1(x)(t) &= s_1(x)(s) + \sum_{s < \tau \leq t} d_1x(\tau), \\ s_2(x)(t) &= s_2(x)(s) + \sum_{s \leq \tau < t} d_2x(\tau) \\ s_c(x)(t) &= s_c(x)(s) + x(t) - x(s) - \sum_{j=1}^2 (s_j(x)(t) - s_j(x)(s)) \quad \text{for } s < t; \\ J(x)(0) &= x(0), \\ J(x)(t) &= J(x)(s) + s_c(x)(t) - s_c(x)(s) \\ &\quad - \sum_{s < \tau \leq t} \ln|1 - d_1x(\tau)| + \sum_{s \leq \tau < t} \ln|1 + d_2x(\tau)| \quad \text{for } s < t. \end{aligned}$$

If $g \in \text{BV}([a, b]; \mathbb{R})$, $f : [a, b] \rightarrow \mathbb{R}$ and $a \leq s < t \leq b$, then we assume

$$\int_s^t x(\tau) dg(\tau) = (L - S) \int_{]s, t[} x(\tau) dg(\tau) + f(t) d_1 g(t) + f(s) d_2 g(s).$$

where $(L - S) \int_{]s, t[} f(\tau) dg(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s, t[$. It is known (see, [12]) that if the integral exists, then the right side of the integral equality equals to the Kurzweil–Stieltjes integral $(K - S) \int_s^t f(\tau) dg(\tau)$ and, therefore, $\int_s^t f(\tau) dg(\tau) = (K - S) \int_s^t f(\tau) dg(\tau)$. If $a = b$, then we assume $\int_a^b x(t) dg(t) = 0$.

$$\int_{-\infty}^a f(\tau) dg(\tau) = \lim_{t \rightarrow \infty} \int_t^a f(\tau) dg(\tau) \text{ and } \int_a^{+\infty} f(\tau) dg(\tau) = \lim_{t \rightarrow +\infty} \int_a^t f(\tau) dg(\tau)$$

if the last limits exist (finite or infinite).

If $G = (g_{ik})_{i,k=1}^n \in \text{BV}([a, b]; \mathbb{R}^{n \times n})$ and $x = (x_k)_k^n : [a, b] \rightarrow \mathbb{R}^n$, then

$$\int_a^b dG(\tau) \cdot x(\tau) = \left(\sum_{k=1}^n \int_a^b x_k(\tau) dg_{ik}(\tau) \right)_i^n.$$

We introduce the operator $\mathcal{A}(X, Y)$ in the following way:

if $X \in \text{BV}_{loc}(\mathbb{R}; \mathbb{R}^{n \times n})$, $\det(I_n + (-1)^j d_j X(t)) \neq 0$ for $t \in \mathbb{R}$ ($j = 1, 2$), and $Y \in \text{BV}_{loc}(\mathbb{R}; \mathbb{R}^{n \times m})$, then

$$\mathcal{A}(X, Y)(0) = O_{n \times m},$$

$$\begin{aligned} \mathcal{A}(X, Y)(t) &= \mathcal{A}(X, Y)(s) + Y(t) - Y(s) + \sum_{s < \tau \leq t} d_1 X(\tau) (I_n - d_1 X(\tau))^{-1} d_1 Y(\tau) \\ &\quad - \sum_{s \leq \tau < t} d_2 X(\tau) (I_n + d_2 X(\tau))^{-1} d_2 Y(\tau) \quad (s < t). \end{aligned}$$

Here the use will be made of the following formulas:

$$\begin{aligned} \int_a^b f(t) dg(t) &= \int_a^b f(t) dg(t+) + f(a) d_2 g(a) \\ &= \int_a^b f(t) dg(t-) + f(b) d_1 g(b), \\ \int_a^t x(\tau) dg(\tau) &= \int_a^{t-} x(\tau) dg(\tau) + x(t) d_1 g(t) \\ &= \int_a^{t+} x(\tau) dg(\tau) - x(t) d_2 g(t). \end{aligned}$$

$$\begin{aligned}
\int_a^b f(t) dg(t) + \int_a^b g(t) df(t) &= f(b)g(b) - f(a)g(a) + \sum_{a < t \leq b} d_1 f(t) \cdot d_1 g(t) \\
&\quad - \sum_{a \leq t < b} d_2 f(t) \cdot d_2 g(t) \\
&\quad \text{(integration-by-parts formula),} \\
\int_a^b h(t) d(f(t)g(t)) &= \int_a^b h(t)f(t) dg(t) + \int_a^b h(t)g(t) df(t) \\
&\quad - \sum_{a < t \leq b} h(t)d_1 f(t) \cdot d_1 g(t) \\
&\quad + \sum_{a \leq t < b} h(t)d_2 f(t) \cdot d_2 g(t) \\
&\quad \text{(general integration-by-parts formula),} \\
\int_a^b f(t) ds_1(g)(t) &= \sum_{a < t \leq b} f(t) d_1 g(t), \int_a^b f(t) ds_2(g)(t) \\
&= \sum_{a < t \leq b} f(t) d_2 g(t), \\
\int_a^b f(t) d\left(\int_a^s g(s) dh(s)\right) &= \int_a^b f(t)g(t) dh(t), \\
d_j\left(\int_a^t f(s) dg(s)\right) &= f(t) d_j g(t) \quad (j = 1, 2).
\end{aligned}$$

The proof of above formulas are given in [5, 6, 12] for example.

By a solution of system (1.1) we mean a vector-function $x \in \mathbf{BV}_{loc}(\mathbb{R}, \mathbb{R}^n)$ if

$$x(t) = x(s) + \int_s^t dA(\tau) \cdot x(\tau) + f(t) - f(s) \quad \text{for } s < t, s, t \in \mathbb{R}.$$

If $\alpha \in \mathbf{BV}_{loc}(\mathbb{R}, \mathbb{R})$ and $t_0 \in \mathbb{R}$ are such that $1 + (-1)^j d_j \alpha(t) \neq 0$ for $t \in \mathbb{R}, t \neq t_0$ ($j = 1, 2$). Then it is known the that (see [7, 8] the initial problem

$$d\xi = \xi d\alpha(t), \quad \xi(0) = 1$$

has the unique solution ξ_α and it is defined by

$$\xi_\alpha(t) = \begin{cases} \exp(s_c(\alpha)(t) - s_c(\alpha)(0)) \prod_{0 < \tau \leq t} (1 - d_1 \alpha(\tau))^{-1} \prod_{0 \leq \tau < t} (1 + d_2 \alpha(\tau)) & \text{for } t > 0, \\ \exp(s_c(\alpha)(t) - s_c(\alpha)(0)) \prod_{t < \tau \leq 0} (1 - d_1 \alpha(\tau)) \prod_{t \leq \tau < 0} (1 + d_2 \alpha(\tau))^{-1} & \text{for } t < 0. \end{cases}$$

Let $\gamma_\alpha(t, s) \equiv \xi_\alpha(t)\xi_\alpha^{-1}(s)$ be the Cauchy function of the problem. Then

$$\begin{aligned} \gamma_\alpha(t, s) &= \exp(J(\alpha)(t) - J(\alpha)(s)) \prod_{s < \tau \leq t} \operatorname{sgn}(1 - d_1\alpha(\tau)) \\ &\quad \times \prod_{s \leq \tau < t} \operatorname{sgn}(1 + d_2\alpha(\tau)) \text{ for } t > s, \\ \gamma_\alpha(t, s) &= \gamma_\alpha^{-1}(s, t) \text{ for } t < s. \end{aligned}$$

Note that the following equality holds (see, [5, 6])

$$d\xi_\alpha^{-1}(t, t_0) \equiv -\xi_\alpha^{-1}(t, t_0)d\mathcal{A}(\alpha, \alpha).(t) \quad (1.3)$$

Remark 1. Let $\alpha \in \operatorname{BV}([a, b], \mathbb{R})$ be such that $1 + (-1)^j d_j \alpha(t) > 0$ for $t \in [a, b]$ ($j = 1, 2$) and let one of the functions α , $J(\alpha)$ and $\mathcal{A}(\alpha, \alpha)$ be nondecreasing (nonincreasing). Then other two functions will be nondecreasing (nonincreasing), as well.

We introduce the operator

$$v(\zeta)(t) = \sup \left\{ \tau \geq t : \zeta(\tau) \leq \zeta(t+) + 1 \right\}$$

if $\xi : \mathbb{R} \rightarrow \mathbb{R}$ is a nondecreasing function, and

$$v(\zeta)(t) = \inf \left\{ \tau \leq t : \zeta(\tau) \leq \zeta(t-) + 1 \right\},$$

if $\zeta : \mathbb{R} \rightarrow \mathbb{R}$ is a nonincreasing function.

2. FORMULATION OF THE RESULTS

For every $t_i \in \mathbb{R} \cup \{-\infty, +\infty\}$ ($i = 1, \dots, n$) we put $\mathcal{N}_0(t_1, \dots, t_n) = \{i : t_i \in \mathbb{R}\}$. It is evident that $\mathcal{N}_0(t_1, \dots, t_n) = \{1, \dots, n\}$ if $t_i \in \mathbb{R}$ ($i = 1, \dots, n$), and $\mathcal{N}_0(t_1, \dots, t_n) = \emptyset$ if $t_i \in \{-\infty, +\infty\}$ ($i = 1, \dots, n$).

In the case, where $t_i = -\infty$ ($t_i = +\infty$), we assume $\operatorname{sgn}(t - t_i) = 1$ for $t \in \mathbb{R}$ ($\operatorname{sgn}(t - t_i) = -1$ for $t \in \mathbb{R}$).

Theorem 1. *Let*

$$1 + (-1)^j d_j a_{ii}(t) \neq 0 \text{ for } t \in \mathbb{R} \quad (j = 1, 2; i = 1, \dots, n) \quad (2.1)$$

and let there exist $t_i \in \mathbb{R} \cup \{-\infty, +\infty\}$ ($i = 1, \dots, n$) such that

$$\begin{aligned} s_{ik} &= \sup \left\{ \left| \int_{t_i}^t |\gamma_i(t, \tau)| dV(\mathcal{A}(a_{ii}, a_{ik}))(\tau) \right| : t \in \mathbb{R} \right\} \\ &< +\infty \quad (i \neq k; i, k = 1, \dots, n), \end{aligned} \quad (2.2)$$

$$\sup \left\{ \left| \int_{t_i}^t |\gamma_i(t, \tau)| dV(\mathcal{A}(a_{ii}, f_i))(\tau) \right| : t \in \mathbb{R} \right\} < +\infty \quad (i = 1, \dots, n) \quad (2.3)$$

and

$$\sup\{|\gamma_i(t, t_i)| : t \in \mathbb{R}\} < +\infty \text{ for } i \in \mathcal{N}_0(t_1, \dots, t_n), \quad (2.4)$$

where $\gamma_i(t, \tau) \equiv \gamma_{a_{ii}}(t, \tau)$ ($i = 1, \dots, n$). Let, moreover, the matrix $S = (s_{ik})_{i,k=1}^n$, where $s_{ii} = 0$ ($i = 1, \dots, n$), be such that

$$r(S) < 1. \quad (2.5)$$

Then for every $c_i \in \mathbb{R}$ ($i \in \mathcal{N}_0(t_1, \dots, t_n)$) system (1.1) has at last one a bounded on $\mathbb{R}$ solution satisfying the condition

$$x_i(t_i) = c_i \text{ for } i \in \mathcal{N}_0(t_1, \dots, t_n). \quad (2.6)$$

If the case, where $\mathcal{N}_0(t_1, \dots, t_n) = \emptyset$, conditions (2.4) and (2.6) be eliminated and the theorem has the following form.

Theorem 1'. Let conditions (2.1), (2.2) and (2.3) hold for some $t_i \in \{-\infty, +\infty\}$ ($i = 1, \dots, n$), where $\gamma_i(t, \tau) \equiv \gamma_{a_{ii}}(t, \tau)$ ($i = 1, \dots, n$), and the matrix $S = (s_{ik})_{i,k=1}^n$, where $s_{ii} = 0$ ($i = 1, \dots, n$), satisfy condition (2.5). Then system (1.1) has at last one solution bounded on $\mathbb{R}$.

Corollary 1. Let

$$1 + (-1)^j d_j a_{ii}(t) > 0 \text{ for } t \in \mathbb{R}, \quad (j = 1, 2; i = 1, \dots, n) \quad (2.7)$$

and let there exist $t_i \in \mathbb{R} \cup \{-\infty, +\infty\}$ ($i = 1, \dots, n$) such that conditions (2.2), (2.3), (2.4) and (2.5) hold, where $S = (s_{ik})_{i,k=1}^n$, $s_{ii} = 0$ ($i = 1, \dots, n$) and $\gamma_i(t, \tau) \equiv \gamma_{a_{ii}}(t, \tau)$ ($i = 1, \dots, n$). Let, moreover, the functions

$$\begin{aligned} \mathcal{A}(a_{ii}, a_{ik})(t) \operatorname{sgn}(t - t_i), \quad \mathcal{A}(a_{ii}, f_i)(t) \operatorname{sgn}(t - t_i) \\ (i \neq k; i, k = 1, \dots, n) \text{ are nondecreasing on } \mathbb{R}. \end{aligned} \quad (2.8)$$

Then for every $c_i \in \mathbb{R}_+$ ($i \in \mathcal{N}_0(t_1, \dots, t_n)$) system (1.1) has at last one nonnegative and bounded on $\mathbb{R}$ solution satisfying condition (2.6).

If $\mathcal{N}_0(t_1, \dots, t_n) = \emptyset$ then Corollary 1 has the following form.

Corollary 1'. Let conditions (2.7) and (2.8) hold and let there exist $t_i \in \{-\infty, +\infty\}$ ($i = 1, \dots, n$) such that conditions (2.2), (2.3) and (2.5) hold, where $S = (s_{ik})_{i,k=1}^n$, $s_{ii} = 0$ ($i = 1, \dots, n$) and $\gamma_i(t, \tau) \equiv \gamma_{a_{ii}}(t, \tau)$ ($i = 1, \dots, n$). Then system (1.1) has at last one a nonnegative and bounded on $\mathbb{R}$ solution.

Theorem 2. Let (2.1) hold and let there exist $t_i \in \mathbb{R} \cup \{-\infty, +\infty\}$ ($i = 1, \dots, n$) such that conditions (2.2), (2.3), (2.4) and (2.5) hold, where $S = (s_{ik})_{i,k=1}^n$, $s_{ii} = 0$ ($i = 1, \dots, n$) and $\gamma_i(t, \tau) \equiv \gamma_{a_{ii}}(t, \tau)$ ($i = 1, \dots, n$). Let, moreover,

$$\liminf_{t \rightarrow t_i} \gamma_i(0, t) = 0 \text{ for } i \in \{1, \dots, n\} \setminus \mathcal{N}_0(t_1, \dots, t_n). \quad (2.9)$$

Then for every $c_i \in \mathbb{R}$ ($i \in \mathcal{N}_0(t_1, \dots, t_n)$) system (1.1) has the unique and bounded on $\mathbb{R}$ solution $(x_i)_{i=1}^n$ satisfying condition (2.6) and

$$\sum_{i=1}^n |x_i(t) - x_{im}(t)| \leq \rho_0 \alpha^m \text{ for } t \in \mathbb{R} \text{ } (m = 1, 2, \dots), \quad (2.10)$$

where ρ_0 and α are the positive numbers independent of m , $(x_{im})_{i=1}^n$ ($m = 0, 1, \dots$) is the sequence of the vector-functions the components of which are defined by

$$x_{i0}(t) \equiv 0, \quad x_{im}(t) \equiv u_i(t) + \sum_{k=1, k \neq i}^n \int_{t_i}^t \gamma_i(t, \tau) d\mathcal{A}(a_{ii}, a_{ik}x_{km-1})(\tau) \quad (2.11)$$

($i = 1, \dots, n$; $m = 1, 2, \dots$), and the functions u_i ($i = 1, \dots, n$) are defined due to

$$u_i(t) \equiv c_i \gamma_i(t, t_i) + \int_{t_i}^t \gamma_i(t, \tau) d\mathcal{A}(a_{ii}, f_i)(\tau) \text{ for } i \in \mathcal{N}_0(t_1, \dots, t_n), \quad (2.12)$$

$$u_i(t) \equiv \int_{t_i}^t \gamma_i(t, \tau) d\mathcal{A}(a_{ii}, f_i)(\tau) \text{ for } i \in \{1, \dots, n\} \setminus \mathcal{N}_0(t_1, \dots, t_n). \quad (2.13)$$

Corollary 2. Let (2.7) hold and let there exist $t_i \in \mathbb{R} \cup \{-\infty, +\infty\}$ ($i = 1, \dots, n$) such that the functions $a_{ii}(t) \operatorname{sgn}(t - t_i)$ ($i = 1, \dots, n$) are non-increasing on $\mathbb{R}$,

$$\liminf_{t \rightarrow t_i} a_{ii}(t) = +\infty \text{ for } i \in \{1, \dots, n\} \setminus \mathcal{N}_0(t_1, \dots, t_n), \quad (2.14)$$

$$V(\mathcal{A}(a_{ii}, a_{ik}))(t) \leq -h_{ik} \operatorname{sgn}(t - t_i) \mathcal{A}(a_{ii}, a_{ii})(t) \quad (2.15)$$

for $t \in \mathbb{R}$ ($i \neq k$; $i, k = 1, \dots, n$) and

$$r(\mathcal{H}) < 1, \quad (2.16)$$

where h_{ik} ($i, k = 1, \dots, n$) are such that $\mathcal{H} = ((1 - \delta_{ik})h_{ik})_{i,k=1}^n$. Let, moreover,

$$\rho_i = \sup \left\{ \left| \bigvee_t^{\nu(\zeta_i)(t)} (\mathcal{A}(a_{ii}, f_i)) \right| : t \in \mathbb{R} \right\} < \infty \text{ } (i = 1, \dots, n), \quad (2.17)$$

where $\zeta_i(t) \equiv \xi_{a_{ii}} \operatorname{sgn}(t - t_i)$ ($i = 1, \dots, n$). Then conclusion of Theorem 2 is true.

Corollary 3. Let there exist the points $t_i \in \mathbb{R} \cup \{-\infty, +\infty\}$ ($i = 1, \dots, n$) and the functions $\alpha_i : \mathbb{R} \rightarrow \mathbb{R}$ ($i = 1, \dots, n$) such that $\alpha_i(t) \operatorname{sgn}(t - t_i)$ ($i = 1, \dots, n$) are non-decreasing on $\mathbb{R}$ and conditions

$$(s_c(a_{ii})(t) - s_c(a_{ii})(s)) \operatorname{sgn}(t - s) \leq \eta_{ii}(s_c(\alpha_i)(t) - s_c(\alpha_i)(s)) \quad (2.18)$$

for $(t - s)(s - t_i) > 0$ ($i = 1, \dots, n$),

$$d_1 a_{ii}(t) \leq \eta_{ii} d_1 \alpha_i(t) < 1, \quad -1 < d_2 a_{ii}(t) \leq \eta_{ii} d_2 \alpha_i(t) \text{ } (i = 1, \dots, n), \quad (2.19)$$

$$|s_c(a_{ik})(t) - s_c(a_{ik})(s)| \operatorname{sgn}(t - s) \leq \eta_{ik}(s_c(\alpha_i)(t) - s_c(\alpha_i)(s)) \quad (2.20)$$

for $(t-s)(s-t_i) > 0$ ($i \neq k; i, k = 1, \dots, n$) and

$$|d_j a_{ik}(t)| \leq \eta_{ik} |d_j \alpha_i(t)| \quad (j = 1, 2; i \neq k; i, k = 1, \dots, n) \quad (2.21)$$

hold on $\mathbb{R}$, where $\mathcal{H} = ((1 - \delta_{ik})\eta_{ik}|\eta_{ii}|^{-1})_{i,k=1}^n$. Let, moreover,

$$\rho_i = \sup \left\{ \left| \bigvee_t^{\vartheta_i(t)} (\mathcal{A}(\eta_i \alpha_i, f_i)) \right| : t \in \mathbb{R} \right\} < \infty \quad (i = 1, \dots, n), \quad (2.22)$$

where $\vartheta_i(t) \equiv \xi_{\eta_i a_i} \operatorname{sgn}(t - t_i)$ ($i = 1, \dots, n$). Then conclusion of Theorem 2 is true.

Theorem 2'. Let (2.1) hold and let there exist $t_i \in \{-\infty, +\infty\}$ ($i = 1, \dots, n$) such that conditions (2.2), (2.3) and (2.5) hold, where $S = (s_{ik})_{i,k=1}^n$, $s_{ii} = 0$ ($i = 1, \dots, n$) and $\gamma_i(t, \tau) \equiv \gamma_{a_{ii}}(t, \tau)$ ($i = 1, \dots, n$). Let, moreover,

$$\liminf_{t \rightarrow t_i} \gamma_i(0, t) = 0 \quad \text{for } i \in \{1, \dots, n\}.$$

Then system (1.1) has the unique and bounded on $\mathbb{R}$ solution $(x_i)_{i=1}^n$ and

$$\sum_{i=1}^n |x_i(t) - x_{im}(t)| \leq \rho_0 \alpha^m \quad \text{for } t \in \mathbb{R} \quad (m = 1, 2, \dots),$$

where ρ_0 and α are the positive numbers independent of m , $(x_{im})_{i=1}^n$ ($m = 0, 1, \dots$) is the sequence of the vector-functions the components of which are defined by

$$\begin{aligned} x_{i0}(t) &\equiv 0, \\ x_{im}(t) &\equiv \int_{t_i}^t \gamma_i(t, \tau) d\mathcal{A}(a_{ii}, f_i)(\tau) \\ &\quad + \sum_{k=1, k \neq i}^n \int_{t_i}^t \gamma_i(t, \tau) d\mathcal{A}(a_{ii}, a_{ik} x_{km-1})(\tau) \quad (i = 1, \dots, n; m = 1, 2, \dots). \end{aligned}$$

Corollary 2'. Let (2.7) hold and let there exist $t_i \in \{-\infty, +\infty\}$ ($i = 1, \dots, n$) such that the functions $a_{ii}(t) \operatorname{sgn}(t - t_i)$ ($i = 1, \dots, n$) are non-increasing on $\mathbb{R}$, conditions (2.15), (2.16), (2.17) and

$$\liminf_{t \rightarrow t_i} a_{ii}(t) = +\infty \quad \text{for } i \in \{1, \dots, n\} \quad (2.23)$$

hold, where $\zeta_i(t) \equiv \xi_{a_{ii}} \operatorname{sgn}(t - t_i)$ ($i = 1, \dots, n$), and the numbers h_{ik} ($i, k = 1, \dots, n$) are such that $\mathcal{H} = ((1 - \delta_{ik})h_{ik})_{i,k=1}^n$. Then conclusion of Theorem 2' is true.

Corollary 3'. Let there exist the points $t_i \in \{-\infty, +\infty\}$ ($i = 1, \dots, n$) and the functions $\alpha_i : \mathbb{R} \rightarrow \mathbb{R}$ ($i = 1, \dots, n$) such that $\alpha_i(t) \operatorname{sgn}(t - t_i)$ ($i = 1, \dots, n$) are nondecreasing on $\mathbb{R}$ and conditions (2.16), (2.18) – (2.22) hold on $\mathbb{R}$, where $\vartheta_i(t) \equiv \xi_{\eta_i a_i} \operatorname{sgn}(t - t_i)$ ($i = 1, \dots, n$), and the numbers $\eta_{ik}, \eta_{ii} < 0$ ($i, k = 1, \dots, n$) are such that $\mathcal{H} = ((1 - \delta_{ik})\eta_{ik}|\eta_{ii}|^{-1})_{i,k=1}^n$. Then conclusion of Theorem 2' is true.

Corollary 4. *Let the conditions of Theorem 2 or Corollary 2 or Corollary 3 are fulfilled. Let, in addition, condition (2.8) hold. Then for every $c_i \in \mathbb{R}_+$ ($i \in \mathcal{N}_0(t_1, \dots, t_n)$) system (1.1) has the unique and bounded on $\mathbb{R}$ solution satisfying condition (2.6) and it is nonnegative.*

Corollary 4'. *Let the conditions of Theorem 2' or Corollary 2' or Corollary 3' are fulfilled. Let, in addition, condition (2.8) hold. Then system (1.1) has the unique and bounded on $\mathbb{R}$ solution and it is nonnegative.*

3. PROOF OF THE RESULTS

Proof of Theorem 1. Let $c_i \in \mathbb{R}$ ($i \in \mathcal{N}_0(t_1, \dots, t_n)$) be an arbitrary fixed numbers. Consider the initial problems

$$du = u da_{ii}(t) + df_i(t) \text{ for } t \in \mathbb{R}, \quad u(t_i) = c_i$$

($i \in \mathcal{N}_0(t_1, \dots, t_n)$). By (2.1) the problem has the unique solution $u_i \in \text{BV}(\mathbb{R}; \mathbb{R})$ and according to modified variation of constant formulae (see [5]) it has form (2.12).

Consider the system of integral equations

$$x_i(t) = u_i(t) + \sum_{k=1, k \neq i}^n \int_{t_i}^t \gamma_i(t, \tau) d\mathcal{A}(a_{ii}, a_{ik}x_k)(\tau) \text{ for } t \in \mathbb{R} \quad (i = 1, \dots, n). \quad (3.1)$$

Due to modified variation of constant formulae (see [5]) we conclude that the vector-function $(x_i)_{i=1}^n$ is a solution of the system one. Moreover, it is evident the vector-function $(x_i)_{i=1}^n$ satisfy condition (2.6).

The solution of the last integral system we will find in the set of functions of bounded and local bounded variation on the set $\mathbb{R}$, i.e. in $\text{BV}(\mathbb{R}; \mathbb{R}^n)$.

Consider the sequences of the vector-functions $(x_{im})_{i=1}^n$ ($m = 0, 1, \dots$) defined by

$$x_{i0}(t) = 0, \quad x_{im}(t) = u_i(t) + \sum_{k=1, k \neq i}^n \int_{t_i}^t \gamma_i(t, \tau) d\mathcal{A}(a_{ii}, a_{ik}x_{k,m-1})(\tau) \text{ for } t \in \mathbb{R} \quad (i = 1, \dots, n).$$

In view of conditions (2.2) and (2.3), from (2.12) and (2.13) we get

$$(u_i)_{i=1}^n \in \text{BV}(\mathbb{R}; \mathbb{R}^n). \quad (3.2)$$

It is clear that $(x_i)_{i=1}^n \in \text{BV}(\mathbb{R}; \mathbb{R}^n)$. Now, if we assume that

$$(x_{i,m-1})_{i=1}^n \in \text{BV}(\mathbb{R}; \mathbb{R}^n) \quad (3.3)$$

for some m , then due (2.2) and (3.2), from (3.2) we get $(x_{im})_{i=1}^n \in \text{BV}_{loc}(\mathbb{R}; \mathbb{R}^n)$ and

$$\|x_{im}\|_\infty \leq \|u_i\|_\infty + \sum_{k=1, k \neq i}^n s_{ik} \|x_{k,m-1}\|_\infty < +\infty \quad (i = 1, \dots, n).$$

Therefore, condition (3.3) holds for every natural m .

Let us show that the sequence $(x_{im})_{i=1}^m$ ($m = 1, 2, \dots$) be uniformly stable on $\mathbb{R}$. For this sufficient to show that the functional series

$$\sum_{m=1}^{\infty} |x_{im}(t) - x_{im-1}(t)| \quad (i = 1, \dots, n) \quad (3.4)$$

are converged uniformly on $\mathbb{R}$.

According (2.2) and (3.3), from (2.11) it follows

$$(\|x_{im} - x_{im-1}\|_{\infty})_{i=1}^n \leq S(\|x_{im-1} - x_{im-2}\|_{\infty})_{i=1}^n \quad (m = 2, 3, \dots)$$

and, therefore,

$$(\|x_{im} - x_{im-1}\|_{\infty})_{i=1}^n \leq S^{m-1}(\|u_i\|_{\infty})_{i=1}^n \quad (m = 1, 2, \dots).$$

Due to (2.5) there exist numbers $\alpha \in]r(S), 1[$ and $\beta > 0$ such that

$$\|S^{m-1}\| \leq \beta \alpha^{m-1} \quad (m = 1, 2, \dots).$$

Therefore,

$$\|x_{im} - x_{im-1}\|_{\infty} \leq \beta_0 \alpha^m \quad (i = 1, \dots, n; m = 1, 2, \dots),$$

where $\beta_0 = \beta \alpha^{-1} \sum_{i=1}^n \|u_i\|_{\infty}$.

So that

$$\sum_{m=0}^{\infty} \beta_0 \alpha^m$$

is the convergence major numerical series for the functional series (3.4) on $\mathbb{R}$. From this, due to the Weierstrass theorem the sequence $(x_{im})_{i=1}^n$ ($m = 0, 1, \dots$) is uniformly convergence on $\mathbb{R}$.

Let

$$\lim_{m \rightarrow +\infty} x_{im}(t) = x_i(t) \quad \text{for } t \in \mathbb{R} \quad (i = 1, \dots, n). \quad (3.5)$$

Then due to Theorem I.4.17 ([12]) we conclude that $(x_i)_{i=1}^n$ will be a solution of system (3.1). Moreover, it is evident that $\|x_i\|_{\infty} < +\infty$ ($i = 1, \dots, n$) and by the equality (3.1) and estimates (2.2) – (2.4) we have $(x_i)_{i=1}^n \in \text{BV}(\mathbb{R}; \mathbb{R}^n)$. $\square$

Proof of Corollary 1. As we proof above, in conditions of Theorem 1, system (1.1) has the bounded solution on $\mathbb{R}$ satisfying condition (2.6) and it is obtained as the uniformly limits on $\mathbb{R}$ of the sequence of the vector-functions $(x_{im-1})_{i=1}^n \in \text{BV}(\mathbb{R}; \mathbb{R}^n)$ ($m = 0, 1, \dots$) the components of which are defined by (2.11), and u_i ($i = 1, \dots, n$) are defined by (2.12) and (2.13).

In view of (2.8), because $c_i \in \mathbb{R}_+$ ($i \in \mathcal{N}_0(t_1, \dots, t_n)$), it follows from (2.13) – (2.11) that $x_{im}(t) \geq 0$ and, that, $x_i(t) \geq 0$ for $t \in \mathbb{R}$ ($i = 1, \dots, n$). $\square$

Proof of Theorem 2. First we show that the every bounded on $\mathbb{R}$ solution $(x_i)_{i=1}^n$ of system (1.1), satisfying of condition (2.6) will be solution of the system of integral equations (3.1).

By (2.9), there exist a sequences t_{im} ($i = 1, \dots, n; m = 1, 2, \dots$) such that

$$\lim_{m \rightarrow +\infty} t_{im} = t_i, \quad \lim_{m \rightarrow +\infty} \gamma_i(0, t_{im}) = 0 \quad (i = 1, \dots, n). \quad (3.6)$$

We assume

$$t_{im} = t_i \quad \text{for } i \in \mathcal{N}_0(t_1, \dots, t_n) \quad (m = 1, 2, \dots). \quad (3.7)$$

By modified variation of constant formula and equalities (2.12), (3.7) we have

$$x_i(t) = u_{im}(t) + \sum_{k=1, k \neq i}^n \int_{t_{im}}^t \gamma_i(t, \tau) d\mathcal{A}(a_{ii}, a_{ik}x_k)(\tau) \quad (i = 1, \dots, n; m = 1, 2, \dots) \quad (3.8)$$

on $\mathbb{R}$, where

$$u_{im}(t) \equiv u_i(t) \quad \text{for } i \in \mathcal{N}_0(t_1, \dots, t_n) \quad (m = 1, 2, \dots), \quad (3.9)$$

$$u_{im}(t) \equiv x_{im}(t_{im})\gamma_i(t, t_{im}) + \int_{t_{im}}^t \gamma_i(t, \tau) d\mathcal{A}(a_{ii}, f_i)(\tau) \\ \text{for } i \in \{1, \dots, n\} \setminus \mathcal{N}_0(t_1, \dots, t_n) \quad (m = 1, 2, \dots). \quad (3.10)$$

Let $i \in \{1, \dots, n\} \setminus \mathcal{N}_0(t_1, \dots, t_n)$. Then because x_i is bounded, by conditions (2.3) and (3.6) from (2.13) and (3.10) we find

$$\lim_{m \rightarrow +\infty} u_{im}(t) = u_i(t) \quad \text{for } t \in \mathbb{R}.$$

On the hand, due (2.2) we get

$$\lim_{m \rightarrow +\infty} \int_{t_{im}}^t \gamma_i(t, \tau) d\mathcal{A}(a_{ii}, a_{ik}x_k)(\tau) = \int_{t_i}^t \gamma_i(t, \tau) d\mathcal{A}(a_{ii}, a_{ik}x_k)(\tau) \quad \text{for } t \in \mathbb{R}.$$

Therefore, from (3.8) we have

$$x_i(t) = u_i(t) + \sum_{k=1, k \neq i}^n \int_{t_i}^t \gamma_i(t, \tau) d\mathcal{A}(a_{ii}, a_{ik}x_k)(\tau) \quad \text{for } t \in \mathbb{R}.$$

Due to (3.7) – (3.9) the last equality is true for the case where $i \in \mathcal{N}_0(t_1, \dots, t_n)$, as well. So that it is proved that the vector-function $(x_i)_{i=1}^n$ is the solution of system (3.1).

In the proof of Theorem 1 we show that system (3.1) has solution $(x_i)_{i=1}^n$ and

$$\lim_{m \rightarrow +\infty} \|x_i - x_{im}\|_\infty = 0 \quad (i = 1, \dots, n).$$

In addition,

$$\|x_{im} - x_{im-1}\|_\infty \leq \beta_0 \alpha^m \quad (i = 1, \dots, n; m = 1, 2, \dots),$$

where β_0 and $\alpha \in]0, 1[$ are the numbers independent from m . From this, we get

$$\|x_{im+j} - x_{im}\|_\infty \leq \sum_{k=m+1}^{m+j} \|x_{ik} - x_{ik-1}\|_\infty \leq \beta_0 \sum_{k=m+1}^{m+j} \alpha^k < \beta_0 \frac{\alpha}{1-\alpha} \alpha^m$$

and

$$\|x_i - x_{im}\|_\infty \leq \beta_0 \frac{\alpha}{1-\alpha} \alpha^m \quad (j = 1, 2; i = 1, \dots, n; m = 1, 2, \dots).$$

So that estimate (2.10) holds for $\rho_0 = n\beta_0\alpha(1-\alpha)^{-1}$.

Finally, we show that system (3.1) has the unique solution $(x_i)_{i=1}^n$. Let $(\bar{x}_i)_{i=1}^n \in \text{BV}(\mathbb{R}; \mathbb{R}^n)$ and arbitrary solution of the system and let $y_i(t) \equiv \bar{x}_i(t) - x_i(t)$ ($i = 1, \dots, n$). Then

$$y_i(t) = \sum_{k=1, k \neq i}^n \int_{t_i}^t \gamma_i(t, \tau) d\mathcal{A}(a_{ii}, a_{ik}y_k)(\tau) \quad \text{for } t \in \mathbb{R}.$$

By this, in view of (2.2), $(\|y_i\|_\infty)_{i=1}^n \leq S(\|y_i\|_\infty)_{i=1}^n$, i.e., $(I_n - S)(\|y_i\|_\infty)_{i=1}^n \leq 0_n$. So that, because the matrix S is nonnegative, by condition (2.5) we have $(\|y_i\|_\infty)_{i=1}^n \leq 0_n$ and $\|y_i\|_\infty = 0$ ($i = 1, \dots, n$). Consequently, $\bar{x}_i(t) \equiv x_i(t)$ ($i = 1, \dots, n$). $\square$

Proof of Corollary 2. Let $\xi_i(t) \equiv \xi_{a_{ii}}(t)$, $\gamma_i(t, \tau) \equiv \xi_i(t)\xi_i^{-1}(\tau)$ and $v_i(t) \equiv v(\zeta_i)(t)$ ($i = 1, \dots, n$). Due condition (2.7) we have $\gamma_i(t, t_i) > 0$ ($i = 1, \dots, n$).

Let $i \in \{1, \dots, n\}$ be fixed. First, consider the case $t \geq t_i$. Then by (2.15) and equality (1.3) we find $d\xi_i^{-1}(t) \equiv -\xi_i^{-1}(t)d\mathcal{A}(a_{ii}, a_{ii})(t)$, and

$$\begin{aligned} \left| \int_{t_i}^t |\gamma_i(t, \tau)| dV(\mathcal{A}(a_{ii}, a_{ik}))(\tau) \right| &\leq -h_{ik}\xi_i(t) \int_{t_i}^t \xi_i^{-1}(\tau) d\mathcal{A}(a_{ii}, a_{ii})(\tau) \\ &= h_{ik}\xi_i(t) \int_{t_i}^t d\xi_i^{-1}(\tau) \leq h_{ik} \quad (i \neq k; i, k = 1, \dots, n). \end{aligned} \quad (3.11)$$

So that condition (2.2) hold. Beside, we have $s_{ik} \leq h_{ik}$ ($i, k = 1, \dots, n$) and, therefore, by (2.16) condition (2.5) hold.

If $i \in \mathcal{N}_0(t_1, \dots, t_n)$, then due Remark 1 the functions β_i ($i = 1, \dots, n$) are non-increasing. Hence the functions $\gamma_i(t, t_i) \equiv \gamma_{a_{ii}}(t, t_i)$ ($i = 1, \dots, n$) are non-increasing and so estimate (2.4) holds for $t \geq t_i$.

Let now $i \in \{1, \dots, n\} \setminus \mathcal{N}_0(t_1, \dots, t_n)$ be such that $t_i = -\infty$. Due to (2.7) we find $\gamma_i(0, t) = \exp(1 - \xi_i(t))$ for $t < 0$. Therefore, by (2.23) we conclude

$$\liminf_{t \rightarrow t_i} \gamma_i(0, t) = 0.$$

Consequently, condition (2.9) hold for the case.

Let us verify (2.3). Let $i \in \{1, \dots, n\}$, $t_i \in \mathbb{R} \cup \{-\infty, +\infty\}$ and $t > t_i$ be fixed. Then $\zeta_i(\tau) = -\xi_i(\tau)$ for $\tau > t_i$, $\zeta_i(t_i) = 0$. Then due to the conditions of theorem the function ζ_i is nondecreasing on the interval $[t_i, +\infty[$.

$$T_{ij} = \{\tau \in [t_i, t] : j \leq \zeta_i(\tau) < (j+1)\} \quad (j = 0, \dots, k_i(t) + 1),$$

where $k_i(t) \equiv [\zeta_i(t)]$ (the integer part) and let

$$\tau_{i0} = t_i, \quad \tau_{ij} = \begin{cases} \tau_{ij-1} & \text{if } T_{ij-1} = \emptyset, \\ \sup T_{ij-1} & \text{if } T_{ij-1} \neq \emptyset \quad (j = 1, \dots, k_i(t) + 1). \end{cases}$$

Let us show that

$$\tau_{ij+1} \leq v_i(\tau_{ij}) \quad (j = 0, \dots, k_i(t)). \quad (3.12)$$

Let $j \in \{0, \dots, k_i(t)\}$ be fixed. If $T_{ij} = \emptyset$, then (3.12) is evident. Let now $T_{ij} \neq \emptyset$. It suffices to show that

$$T_{ij} \subset Q_{ij}, \quad (3.13)$$

where $Q_{ij} = \{\tau \in [t_i, t] : \zeta_i(\tau) < \zeta_i(\tau_{ij-1}+) + 1\}$. It is easy to verify that

$$\zeta_i(\tau_{ij-1}+) \geq j. \quad (3.14)$$

Indeed, otherwise there exists $\delta > 0$ such that $\zeta_i(\tau_{ij-1} + s) < \zeta_i(\tau_{i0}) + j$ for $0 \leq s \leq \delta$. Next, by the definition of τ_{ij-1} , we have $\zeta_i(\tau_{i0}) + (j-1) \leq \zeta_i(\tau_{ij-1}-)$ and, therefore, $(j-1) \leq \zeta_i(\tau_{ij-1} + s) < j$ for $0 \leq s \leq \delta$. But this contradicts the definition of τ_{ij-1} . So that if $\tau \in T_{ij}$, then from (3.14) and the inequality $\zeta_i(\tau) < (j+1)$ we get $\zeta_i(\tau) < \zeta_i(\tau_{ij-1}+) + 1$ and hence $\tau \in Q_{ij}$. Therefore (3.12) is proved.

Due to (3.12) we find

$$v_i(t) \leq \int_{t_i}^t \exp(\xi_i(t) - \xi_i(\tau)) db_i(\tau) \leq \exp(\xi_i(t)) \sum_{j=1}^{k_i(t)+1} \int_{\tau_{ij-1}}^{\tau_{ij}} \exp(\zeta_i(\tau)) db_i(\tau), \quad (3.15)$$

where $v_i(t) \equiv \left| \int_{t_i}^t \gamma_i(t, \tau) db_i(\tau) \right|$ and $b_i(\tau) \equiv V(\mathcal{A}(a_{ii}, f_i))(\tau)$. On the other hand

$$\begin{aligned} \int_{\tau_{ij-1}}^{\tau_{ij}} \exp(\zeta_i(\tau)) db_i(\tau) &= \lim_{\varepsilon \rightarrow 0+} \int_{\tau_{ij-1}}^{\tau_{ij}-\varepsilon} \exp(\zeta_i(\tau)) db_i(\tau) + \exp(\zeta_i(\tau_{ij})) d_1 b_i(\tau_{ij}) \\ &\leq \exp(\zeta_i(\tau_{ij}-)) (b_i(\tau_{ij}-) - b_i(\tau_{ij-1})) + \exp(\zeta_i(\tau_{ij+1}-)) d_1 b_i(\tau_{ij}) \\ &\leq \exp(j) (b_i(\tau_{ij}-) - b_i(\tau_{ij-1})) + \exp(j+1) d_1 b_i(\tau_{ij}). \end{aligned}$$

Similarly we verify that

$$\begin{aligned} \int_{\tau_{ik_i(t)}}^t \exp(\zeta_i(\tau)) db_i(\tau) &\leq \exp(\zeta_i(t-)) (b_i(t-) - b_i(\tau_{ik_i(t)})) + \exp(\zeta_i(t)) d_1 b_i(t) \\ &\leq \exp(k_i(t) + 1) (b_i(t-) - b_i(\tau_{ik_i(t)})) + \exp(k_i(t) + 2) d_1 b_i(t). \end{aligned}$$

Taking account the last two estimates, by (3.15) we find

$$\begin{aligned} v_i(t) &\leq \exp(-k_i(t)) \sum_{j=1}^{k_i(t)+1} \left(\exp(j) (b_i(\tau_{ij}-) - b_i(\tau_{i,j-1}-)) + \exp(j+1) d_1 b_i(\tau_{ij}) \right) \\ &\leq \exp(-k_i(t)) \left(\exp(1) (b_i(\tau_{i1}-) - b_i(\tau_{i0})) \right. \\ &\quad \left. + \sum_{j=2}^{k_i(t)+1} (\exp(j) (b_i(\tau_{ij}-) - b_i(\tau_{i,j-1}-)) + \exp(k_i(t)+2) d_1 b_i(\tau_{i_{k_i(t)+2}})) \right). \end{aligned}$$

From this, due to (2.17) and (3.12) we have

$$\begin{aligned} v_i(t) &\leq 2\rho_i \exp(-k_i(t)) \sum_{j=1}^{k_i(t)+2} \exp(j) \\ &= 2\rho_i \exp(-k_i(t)) \exp(k_i(t)+2) \exp(1) (\exp(1) - 1)^{-1}. \end{aligned}$$

Consequently,

$$v_i(t) \leq \eta \rho \text{ for } t \geq t_i. \quad (3.16)$$

where $\eta = 2 \exp(3) (\exp(1) - 1)^{-1}$ and $\rho = \sum_{i=1}^n \rho_i$. So that, estimate (2.3) hold on the set $(-\infty, t]$. Similarly we show estimate (2.3) on the set $[t, +\infty)$. In the case we have $\zeta_i(\tau) = -\xi_i(\tau)$ for $\tau < t_i$, $\zeta_i(t_i) = 0$. and

$$\tilde{T}_{ij} = \{\tau \in [t, t_i] : j \leq \zeta_i(\tau) < (j+1)\} \quad (j = 0, \dots, k_i(t) + 1).$$

The function ζ_i is non-increasing on $(-\infty, t]$.

Similarly, as above, we conclude that the estimates

$$\tilde{\tau}_{i,j+1} \geq v_i(\tilde{\tau}_{ij}) \quad (j = 0, \dots, k_i(t)). \quad (3.17)$$

holds, where

$$\tilde{\tau}_{i0} = t_i, \quad \tau_{ij} = \begin{cases} \tilde{\tau}_{i,j-1} & \text{if } \tilde{T}_{i,j-1} = \emptyset, \\ \inf \tilde{T}_{i,j-1} & \text{if } \tilde{T}_{i,j-1} \neq \emptyset \quad (j = 1, \dots, k_i(t) + 1). \end{cases}$$

Similarly, we verify that (3.16) hold. So the corollary follows from Theorem 2. $\square$

Proof of Corollary 3. First, as in the proof of Corollary 2, due to condition (2.17) we show that estimates

$$\sup \left\{ \left| \int_{t_i}^t |\gamma_{\eta_i \alpha_{ii}}(t, \tau)| dV(\mathcal{A}(\eta_i \alpha_{ii}, f_i))(\tau) \right| : t \in \mathbb{R} \right\} < +\infty \quad (i = 1, \dots, n). \quad (3.18)$$

Let $\xi_i(t) \equiv \xi_{a_{ii}}(t)$ and $\gamma_i(t, \tau) \equiv \xi_i(t) \xi_i^{-1}(\tau)$ ($i = 1, \dots, n$). Due condition (2.19) we have $\gamma_i(t, t_i) > 0$ ($i = 1, \dots, n$).

In view of (2.18) and (2.19) it is not difficult to verify that

$$J(a_{ii})(t) - J(a_{ii})(s) \leq \text{sgn}(t-s) (J(\eta_{ii} \alpha_i)(t) - J(\eta_{ii} \alpha_i)(s)) \quad (3.19)$$

for $(t-s)(s-t_i) > 0$ ($i = 1, \dots, n$).

Let us show the estimates

$$V(\mathcal{A}(a_{ii}, a_{ik}))(t) \leq \frac{\eta_{ik}}{\eta_{ii}} \operatorname{sgn}(t-t_i) \mathcal{A}(\eta_{ii}\alpha_i, \eta_{ii}\alpha_i)(t) \quad (i \neq k; i, k = 1, \dots, n). \quad (3.20)$$

hold on $\mathbb{R}$. By definition of the operator $\mathcal{A}$ we have

$$\begin{aligned} \mathcal{A}(a_{ii}, a_{ik})(t) - \mathcal{A}(a_{ii}, a_{ik})(s) &= s_c(a_{ik})(t) - s_c(a_{ik})(s) \\ &+ \sum_{s < \tau \leq t} d_1 a_{ik}(\tau) + \sum_{s \leq \tau < t} d_2 a_{ik}(\tau) + \sum_{s < \tau \leq t} d_1 a_{ii}(\tau) (1 - d_1 a_{ii}(\tau))^{-1} d_1 a_{ik}(\tau) \\ &- \sum_{s \leq \tau < t} d_2 a_{ii}(\tau) (1 + d_2 a_{ii}(\tau))^{-1} d_2 a_{ik}(\tau) = s_c(a_{ik})(t) - s_c(a_{ik})(s) \\ &+ \sum_{s < \tau \leq t} (1 - d_1 a_{ii}(\tau))^{-1} d_1 a_{ik}(\tau) + \sum_{s \leq \tau < t} (1 + d_2 a_{ii}(\tau))^{-1} d_2 a_{ik}(\tau) \end{aligned}$$

for $s < t$ ($i \neq k; k = 1, \dots, n$). From this, thanks to (2.19), (2.20), (2.21), we find

$$\begin{aligned} |\mathcal{A}(a_{ii}, a_{ik})(t) - \mathcal{A}(a_{ii}, a_{ik})(s)| &\leq |s_c(a_{ik})(t) - s_c(a_{ik})(s)| \\ &+ \sum_{s < \tau \leq t} (1 - d_1 a_{ii}(\tau))^{-1} |d_1 a_{ik}(\tau)| + \sum_{s \leq \tau < t} (1 + d_2 a_{ii}(\tau))^{-1} |d_2 a_{ik}(\tau)| \\ &\leq \eta_{ik} (s_c(\alpha_i)(t) - s_c(\alpha_i)(s)) + \sum_{s < \tau \leq t} (1 - \eta_{ii} d_1 \alpha_i(\tau))^{-1} \eta_{ik} d_1 \alpha_i(\tau) \\ &+ \sum_{s \leq \tau < t} (1 + \eta_{ii} d_2 \alpha_i(\tau))^{-1} \eta_{ik} d_2 \alpha_i(\tau) \\ &= \frac{\eta_{ik}}{\eta_{ii}} (\mathcal{A}(\eta_{ii}\alpha_i, \eta_{ii}\alpha_i)(t) - \mathcal{A}(\eta_{ii}\alpha_i, \eta_{ii}\alpha_i)(s)) \end{aligned}$$

for $t_i \leq s < t$ ($i \neq k; k = 1, \dots, n$). So that estimate (3.20) hold for $t_i \leq t$. Similarly we show (3.20) for $t \leq t_i$ as well.

Moreover, using this way we conclude that for $t \in \mathbb{R}$ we have

$$V(\mathcal{A}(a_{ii}, f_i))(t) \leq \operatorname{sgn}(t-t_i) \mathcal{A}(\eta_{ii}\alpha_i, f_i)(t) \quad (i \neq k; i, k = 1, \dots, n). \quad (3.21)$$

Hence due to (3.18), (3.19) and (3.21) we conclude condition (2.3) and (2.4) hold.

On the other hand, by (1.3), (3.19) and (3.20) we get

$$d \exp(-J(\eta_{ii}\alpha_i)(t)) \equiv -\exp(-J(\eta_{ii}\alpha_i)(t)) d\mathcal{A}(\eta_{ii}\alpha_i, \eta_{ii}\alpha_i)(t)$$

and

$$\begin{aligned} &\left| \int_{t_i}^t |\gamma_i(t, \tau)| dV(\mathcal{A}(a_{ii}, a_{ik}))(\tau) \right| \\ &\leq \frac{\eta_{ik}}{\eta_{ii}} \exp(J(\eta_{ii}\alpha_i)(t)) \int_{t_i}^t \exp(-J(\eta_{ii}\alpha_i)(\tau)) d\mathcal{A}(\eta_{ii}\alpha_i, \eta_{ii}\alpha_i)(\tau) \end{aligned}$$

$$\begin{aligned}
&= \frac{\eta_{ik}}{|\eta_{ii}|} \exp(J(\eta_{ii}\alpha_i)(t)) \left(\exp(-J(\eta_{ii}\alpha_i)(t)) - \exp(-J(\eta_{ii}\alpha_i)(t_i)) \right) \\
&\leq \frac{\eta_{ik}}{|\eta_{ii}|} \text{ for } t \geq t_i \text{ (} i \neq k; k = 1, \dots, n \text{)}.
\end{aligned}$$

Similarly we show the estimate for $t \leq t_i$ as well. So that we have $s_{ik} \leq \eta_{ik}|\eta_{ii}|^{-1}$ ($i, k = 1, \dots, n$), where s_{ik} is the left hand of estimate (2.2).

Hence, inequality (2.5) follows from (2.16). By (2.21) condition (2.1) holds.

So conditions of Theorem 1 hold. The corollary follows from the theorem. $\square$

Theorem 2', Corollaries 2' and 3' are, respectively, the particular cases of Theorem 2, Corollaries 2 and 3 if we assume $\mathcal{N}_0(t_1, \dots, t_n) = \emptyset$ therein.

Corollaries 4 and 4' immediately follow from Theorems 2 and 2' and Corollaries 2, 3 and 2', 3'.

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HERMITE-HADAMARD TYPE INEQUALITIES FOR UNIFORMLY CONVEX FUNCTIONS WITH RESPECT TO GEODESIC IN HADAMARD SPACE

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Abstract. In this paper, we obtain some Hermite-Hadamard type inequalities for uniformly convex functions with respect to geodesic in Hadamard space. Also, we give some application of these functions in means.

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1. INTRODUCTION AND PRELIMINARIES

Convex analysis plays an important role in many areas of mathematics such inequality, optimization and monotone operator. The Hermite-Hadamard's inequalities give us an estimate of the (integral) mean value of a continuous convex function. Moreover, equality holds in either side only for affine functions (i.e., for functions of the form $ax + b$).

Recently, much attention have given to develop various inequalities for several classes of convex functions and their generalizations using novel ideas (see [1–3, 6, 8–10] and the references therein). Hermite-Hadamard's inequality is given as follows: Let $f: I \rightarrow \mathbb{R}$ be a convex function, and let $a, b \in I$ with $a < b$. Then the following inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2},$$

where I is an interval (finite or infinite) in $\mathbb{R}$. Throughout the paper we denote by I° the interior of I .

The Hermite-Hadamard inequality is expressed in terms of the concept of convexity. This concept can be generalized to a general metric space called Hadamard space. So, the inequalities related to the convex or uniformly convex functions are investigated in these spaces. In [5], the Hermite-Hadamard inequality is expressed on the

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Hadamard space for the convex functions. The purpose of this paper is to generalize the Hermite-Hadamard inequality for the uniformly convex functions on this space.

We consider the basic concepts and results, which are needed to obtain our main results. The following definitions can be found in [5, 7].

Definition 1. A geodesic is a rectifiable curve $\gamma: [0, 1] \rightarrow N$ such that the length of $\gamma|_{[t_1, t_2]}$ is $d(\gamma(t_1), \gamma(t_2))$ for all $0 \leq t_1 \leq t_2 \leq 1$.

Definition 2. A complete metric space $(N; d)$ is called a global NPC space (or Hadamard space) if for $x_1, x_2 \in N$ there exists a point $z \in N$ such that for each $x \in N$ we have

$$d(x, z)^2 \leq \frac{1}{2}d(x, x_1)^2 + \frac{1}{2}d(x, x_2)^2 - \frac{1}{4}d(x_1, x_2)^2.$$

Remark 1. The point z occurring in the preceding definition plays the role of a midpoint between x_1 and x_2 .

Remark 2. If $(N; d)$ is a global NPC and α, β are two geodesic arcs starting at $x \in X$, then the distance map $t \rightarrow d(\alpha(t), \beta(t))$ is a convex function.

Remark 3. The following classes of spaces are NPC: complete Riemannian manifolds with non-positive sectional curvature, Hilbert spaces, Bruhat Tits buildings, in particular metric trees.

Definition 3. A subset $C \subseteq B$ is called convex if for each geodesic $\gamma: [0, 1] \rightarrow B$ joining two arbitrary points in C holds that $\gamma([0, 1]) \subset C$.

2. MAIN RESULTS

In this section we give main results.

Let X be a vector space. For any two distinct a, b points in X we define the line segment connecting a to b by

$$\gamma(\lambda) := \{(1 - \lambda)a + \lambda b : \lambda \in [0, 1]\}.$$

If need to emphasise a, b we shall write $\gamma(\lambda) = \alpha_{a,b}(\lambda)$. We denote by $\alpha_{(a,b)}$ the set of all $(1 - \lambda)a + \lambda b$ such that $\lambda \in [0, 1]$.

Definition of uniformly convex functions found in [4, 5]. In the following definition we extended it in geodesic concept. Here, we refer to more restrictive versions of convexity introduced in reference [4].

Definition 4. Let $g: H \rightarrow (-\infty, +\infty]$ be proper and H be a Hilbert space. Then g is uniformly convex with modulus $\phi: [0, +\infty) \rightarrow [0, +\infty]$ if ϕ is increasing, ϕ vanishes only at zero and

$$g(tr + (1 - t)s) + t(1 - t)\phi(|r - s|) \leq tg(r) + (1 - t)g(s).$$

Using the above definition, we introduce the definition of uniformly convex with geodesic for a function which defined from a subset of a global NPC space to $\mathbb{R}$.

Definition 5. Suppose that (N, d) is a global NPC space, $C \subset N$ be a convex set. A function $f: C \subseteq N \rightarrow \mathbb{R}$ is called uniformly convex with modulus $\phi: [0, +\infty) \rightarrow [0, +\infty]$ with respect to γ (or simply γ -uniformly convex) if ϕ is increasing, ϕ vanishes only at 0, and also, the function $f \circ \gamma: [0, 1] \rightarrow \mathbb{R}$ is uniformly convex function (concept in 4), defined as

$$f(\gamma(tr + (1-t)s)) + t(1-t)\phi(|r-s|) \leq (1-t)f(\gamma(s)) + tf(\gamma(r)), \quad (2.1)$$

for each $t \in [0, 1]$. Where $\gamma: [0, 1] \rightarrow C$ is geodesic and $0 \leq r, s \leq 1$.

Theorem 1. Let (N, d) be a global NPC space, $C \subset N$ be a convex set and $f: C \rightarrow \mathbb{R}$ be a uniformly convex function. $\gamma: [0, 1] \rightarrow \alpha_{(a,b)}$ is geodesic with $\gamma(0) = a, \gamma(1) = b$, then

$$f\left(\gamma\left(\frac{1}{2}\right)\right) + \frac{1}{4} \int_0^1 \phi(t) dt \leq \int_0^1 f(\gamma(\lambda)) d\lambda \leq \alpha_{(f(a), f(b))}\left(\frac{1}{2}\right) - \frac{1}{6}\phi(1). \quad (2.2)$$

Proof. Since, $\gamma: [0, 1] \rightarrow \alpha_{(a,b)}$ is geodesic also, f is uniformly convex for $0 \leq r, s \leq 1$ we have

$$f(\gamma(tr + (1-t)s)) + t(1-t)\phi(|r-s|) \leq (1-t)f(\gamma(s)) + tf(\gamma(r))$$

Let $t = \frac{1}{2}$ in (2.1), we conclude that

$$f\left(\gamma\left(\frac{r+s}{2}\right)\right) + \frac{1}{4}\phi(|r-s|) \leq \frac{f(\gamma(r)) + f(\gamma(s))}{2}.$$

Also, if $s = 1 - r$ then

$$f\left(\gamma\left(\frac{1}{2}\right)\right) + \frac{1}{4}\phi(2r-1) \leq \frac{f(\gamma(r)) + f(\gamma(1-r))}{2}.$$

By integrating with respect to r from $r = 0$ to $r = 1$ we obtain

$$f\left(\gamma\left(\frac{1}{2}\right)\right) + \frac{1}{4} \int_0^1 \phi(u) du \leq \int_0^1 f(\gamma(r)) dr,$$

since

$$\begin{aligned} \int_0^1 f(\gamma(1-r)) dr &= \int_0^1 f(\gamma(r)) dr, \\ \int_0^1 \phi(|2r-1|) dr &= \int_0^1 \phi(u) du, \\ f\left(\gamma\left(\frac{1}{2}\right)\right) + \frac{1}{4} \int_0^1 \phi(u) du &\leq \int_0^1 f(\alpha_{(a,b)}(\lambda)) d\lambda. \end{aligned}$$

For second inequality from (2.1), if $r = 1, s = 0$ in (2.1) we have

$$f(\gamma(t)) + t(1-t)\phi(1) \leq tf(\gamma(1)) + (1-t)f(\gamma(0)).$$

Now, by integrating with respect t from $t = 0$ to $t = 1$ we have

$$\int_0^1 f(\gamma(t))dt + \frac{1}{6}\phi(1) \leq \frac{1}{2}f(\gamma(1)) + \frac{1}{2}f(\gamma(0)).$$

Therefore,

$$\int_0^1 f(\alpha_{(a,b)}(\lambda))d\lambda \leq \alpha_{(f(a),f(b))}\left(\frac{1}{2}\right) - \frac{1}{6}\phi(1).$$

□

Remark 4. Note that the first inequality, in (2.2), is stronger than the second, i.e.

$$\begin{aligned} 0 &\leq \int_0^1 f(\gamma(\lambda))d\lambda - f\left(\gamma\left(\frac{1}{2}\right)\right) - \frac{1}{4}\int_0^1 \phi(t)dt \\ &\leq \alpha_{(f(a),f(b))}\left(\frac{1}{2}\right) - \frac{1}{6}\phi(1) - \int_0^1 f(\gamma(\lambda))d\lambda. \end{aligned}$$

Proof. Let $m = m(a, b)$ be midpoint a and b then

$$\begin{aligned} 2 \int_0^1 f(\gamma_{a,b}(\lambda))d\lambda &= 2 \int_0^{\frac{1}{2}} f(\gamma_{a,b}(\lambda))d\lambda + 2 \int_{\frac{1}{2}}^1 f(\gamma_{a,b}(\lambda))d\lambda \\ &= \int_0^1 f(\gamma_{a,m}(\lambda))d\lambda + \int_0^1 f(\gamma_{m,b}(\lambda))d\lambda \\ &\leq \frac{f(\gamma_{a,m}(0)) + f(\gamma_{a,m}(1))}{2} - \frac{1}{6}\phi(1) \\ &\quad + \frac{f(\gamma_{m,b}(0)) + f(\gamma_{m,b}(1))}{2} - \frac{1}{6}\phi(1) \\ &= f\left(\gamma_{a,b}\left(\frac{1}{2}\right)\right) + \frac{f(\gamma_{a,b}(0)) + f(\gamma_{a,b}(1))}{2} - \frac{1}{3}\phi(1) \\ &\leq f\left(\gamma_{a,b}\left(\frac{1}{2}\right)\right) + \frac{f(\gamma_{a,b}(0)) + f(\gamma_{a,b}(1))}{2} \\ &\quad - \frac{1}{6}\phi(1) + \frac{1}{4}\int_0^1 \phi(t)dt, \end{aligned}$$

which completes the proof. □

Proposition 1. Let (N, d) be a global NPC space, $C \subset N$ be a convex set and $f: C \rightarrow \mathbb{R}$ a uniformly convex function, then

$$\begin{aligned} f\left(\gamma\left(\frac{1}{2}\right)\right) &\leq \frac{1}{2}\left(f\left(\gamma\left(\frac{1}{3}\right)\right) + f\left(\gamma\left(\frac{2}{3}\right)\right)\right) - \frac{1}{4}\phi\left(\frac{1}{3}\right) \\ &\leq \frac{1}{2}\left(f\left(\gamma\left(\frac{1}{4}\right)\right) + f\left(\gamma\left(\frac{3}{4}\right)\right)\right) - \frac{7}{18}\phi\left(\frac{1}{3}\right) \\ &\leq \int_0^1 f(\gamma(t))dt - \frac{1}{4}\int_0^1 \phi(t)dt - \frac{7}{18}\phi\left(\frac{1}{3}\right) \end{aligned}$$

$$\leq \Upsilon_{(f(\gamma(0)), f(\gamma(1)))} \left(\frac{1}{2} \right) - \frac{1}{8} \int_0^1 \phi(t) dt - \frac{43}{72} \phi \left(\frac{1}{2} \right).$$

Proof. In veiw of the following relation

$$f \left(\gamma \left(\frac{1}{2} \right) \right) + \frac{1}{4} \int_0^1 \phi(t) dt \leq \int_0^1 f(\gamma_{(a,b)}(\lambda)) d\lambda \leq \Upsilon_{(f(a), f(b))} \left(\frac{1}{2} \right) - \frac{1}{6} \phi(1)$$

we have

$$\begin{aligned} f \left(\gamma \left(\frac{1}{2} \right) \right) &= f \left(\gamma \left(\frac{1}{2} \left(\frac{1}{3} + \frac{2}{3} \right) \right) \right) \\ &\leq \frac{1}{2} \left(f \left(\gamma \left(\frac{1}{3} \right) \right) + f \left(\gamma \left(\frac{2}{3} \right) \right) \right) - \frac{1}{4} \phi \left(\frac{1}{3} \right) \\ &= \frac{1}{2} f \left(\gamma \left(\frac{5}{6} \cdot \frac{1}{4} + \frac{1}{6} \cdot \frac{3}{4} \right) \right) + \frac{1}{2} f \left(\gamma \left(\frac{1}{6} \cdot \frac{1}{4} + \frac{5}{6} \cdot \frac{3}{4} \right) \right) - \frac{1}{4} \phi \left(\frac{1}{3} \right) \\ &\leq \frac{5}{12} f \left(\gamma \left(\frac{1}{4} \right) \right) + \frac{1}{12} f \left(\gamma \left(\frac{3}{4} \right) \right) - \frac{5}{72} \phi \left(\frac{1}{2} \right) + \frac{1}{12} f \left(\gamma \left(\frac{1}{4} \right) \right) \\ &\quad + \frac{5}{12} f \left(\gamma \left(\frac{3}{4} \right) \right) - \frac{5}{72} \phi \left(\frac{1}{2} \right) - \frac{1}{4} \phi \left(\frac{1}{3} \right) \\ &\leq \frac{1}{2} f \left(\gamma \left(\frac{1}{4} \right) \right) + \frac{1}{2} f \left(\gamma \left(\frac{3}{4} \right) \right) - \frac{7}{18} \phi \left(\frac{1}{3} \right) \\ &\leq \int_0^1 f(\gamma(t)) dt - \frac{1}{4} \int_0^1 \phi(t) dt - \frac{7}{18} \phi \left(\frac{1}{3} \right) \\ &\leq \frac{1}{2} \left(f \left(\gamma \left(\frac{1}{2} \right) \right) + \frac{f(\gamma(0)) + f(\gamma(1))}{2} \right) - \frac{1}{6} \phi(1) + \frac{1}{4} \int_0^1 \phi(t) dt \\ &\quad - \frac{1}{4} \int_0^1 \phi(t) dt - \frac{7}{18} \phi \left(\frac{1}{3} \right) \\ &\leq \frac{1}{2} \left(\frac{f(\gamma(0)) + f(\gamma(1))}{2} - \frac{1}{4} \phi(1) \right) + \frac{f(\gamma(0)) + f(\gamma(1))}{4} - \frac{1}{12} \phi(1) \\ &\quad + \frac{1}{8} \int_0^1 \phi(t) dt - \frac{1}{4} \int_0^1 \phi(t) dt - \frac{7}{18} \phi \left(\frac{1}{2} \right) \\ &= \frac{1}{2} (f(\gamma(0)) + f(\gamma(1))) - \frac{5}{24} \phi(1) - \frac{1}{8} \int_0^1 \phi(t) dt - \frac{7}{18} \phi \left(\frac{1}{3} \right) \\ &\leq \Upsilon_{f(\gamma(0)), f(\gamma(1))} \left(\frac{1}{2} \right) - \frac{1}{8} \int_0^1 \phi(t) dt - \frac{43}{72} \phi \left(\frac{1}{3} \right). \end{aligned}$$

□

Lemma 1. *If $t_1, \dots, t_n \geq 0$, $\sum_{k=1}^n t_k = 1$, $\{x_i\}_{i=1}^n$ is an increasing sequence of $[0, 1]$, (N, d) be a global NPC space and $f: C \rightarrow \mathbb{R}$ be a uniformly convex function then*

$$f\left(\gamma\left(\sum_{k=1}^n t_k x_k\right)\right) + \sum_{i=1}^{n-1} t_i t_{i+1} \phi(x_{i+1} - x_i) \leq \sum_{k=1}^n t_k f(\gamma(x_k)).$$

Proof. The proof is by induction. By the use of Definition 5 first observe that the result clearly holds for $n = 2$. Assume it holds for n , $x_1 \leq x_2 \leq \dots \leq x_{n+1}$ and $\sum_{i=1}^{n+1} t_i = 1$, $t_i \geq 0$. So by the inductive hypothesis

$$\begin{aligned} f\left(\gamma\left(\sum_{i=1}^{n+1} t_i x_i\right)\right) &\leq \sum_{i=1}^{n-1} t_i f(\gamma(x_i)) + (t_n + t_{n+1}) f\left(\gamma\left(\frac{t_n x_n + t_{n+1} x_{n+1}}{t_n + t_{n+1}}\right)\right) \\ &\quad - t_1 t_2 \phi(x_2 - x_1) - \dots - t_{n-2} t_{n-1} \phi(x_{n-1} - x_{n-2}) \\ &\quad - t_{n-1} (t_n + t_{n+1}) \phi\left(\frac{t_n x_n + t_{n+1} x_{n+1}}{t_n + t_{n+1}} - x_{n-1}\right) \\ &\leq \sum_{i=1}^{n-1} t_i f(\gamma(x_i)) + (t_n + t_{n+1}) \left[\frac{t_n f(\gamma(x_n))}{t_n + t_{n+1}} \right. \\ &\quad \left. + \frac{t_{n+1} f(\gamma(x_{n+1}))}{t_n + t_{n+1}} - \frac{t_n t_{n+1}}{(t_n + t_{n+1})^2} \phi(x_{n+1} - x_n) \right] \\ &\quad - \sum_{i=1}^{n-2} t_i t_{i+1} \phi(x_{i+1} - x_i) - t_{n-1} t_n \phi(x_n - x_{n-1}) \\ &\leq \sum_{i=1}^{n+1} t_i f(\gamma(x_i)) - t_n t_{n+1} \phi(x_{n+1} - x_n) - t_n t_{n-1} \phi(x_n - x_{n-1}) \\ &\quad - \sum_{i=1}^{n-2} t_i t_{i+1} \phi(x_{i+1} - x_i) \\ &= \sum_{i=1}^{n+1} t_i f(\gamma(x_i)) - \sum_{i=1}^n t_i t_{i+1} \phi(x_{i+1} - x_i). \end{aligned}$$

Which completes the proof. $\square$

Theorem 2. *Let (N, d) be a global NPC space, $C \subseteq N$ be a convex set, $f: C \rightarrow \mathbb{R}$ be a uniformly convex function and k, p be positive integers, then*

$$\begin{aligned} &f\left(\gamma\left(\frac{1}{2}\right)\right) + \frac{1}{4} \int_0^1 \phi(t) dt + \frac{1}{k^p} \phi\left(\frac{1}{k^p}\right) \\ &\leq \frac{1}{k^p} \sum_{i=0}^{k^p-1} f\left(\gamma\left(\frac{2i+1}{2k^p}\right)\right) + \frac{1}{4} \int_0^1 \phi(t) dt \\ &\leq \int_0^1 f(\gamma(t)) dt \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2k^p} \sum_{i=0}^{k^p-1} \left[f\left(\gamma\left(\frac{i+1}{k^p}\right)\right) + f\left(\gamma\left(\frac{i}{k^p}\right)\right) \right] - \frac{1}{6}\phi(1) \\
&\leq \alpha_{(f(\gamma(0)), f(\gamma(1)))} \left(\frac{1}{2}\right) - \frac{1}{6}\phi(1).
\end{aligned}$$

Proof. Using Hermite-Hadamard inequality we have

$$\begin{aligned}
f\left(\frac{2i+1}{2k^p}\right) + \frac{1}{4} \int_0^1 \phi(t) dt &\leq \int_0^1 f(\gamma_{\frac{i}{k^p}, \frac{i+1}{k^p}}(t)) dt \\
&\leq \frac{f(\gamma(\frac{i}{k^p})) + f(\gamma(\frac{i+1}{k^p}))}{2} - \frac{1}{6}\phi(1),
\end{aligned}$$

if we add the above inequality from $i = 0$ to $i = k^p - 1$ we have

$$\begin{aligned}
\sum_{i=0}^{k^p-1} f\left(\gamma\left(\frac{2i+1}{2k^p}\right)\right) + \frac{k^p}{4} \int_0^1 \phi(t) dt &\leq \sum_{i=0}^{k^p-1} \int_0^1 f(\gamma_{\frac{i}{k^p}, \frac{i+1}{k^p}}(t)) dt \\
&\leq \sum_{i=0}^{k^p-1} \frac{f(\gamma(\frac{i}{k^p})) + f(\gamma(\frac{i+1}{k^p}))}{2} - \frac{k^p}{6}\phi(1).
\end{aligned}$$

Since,

$$\begin{aligned}
\sum_{i=0}^{k^p-1} \int_0^1 f(\gamma_{\frac{i}{k^p}, \frac{i+1}{k^p}}(t)) dt &= k^p \int_0^1 f(\gamma(t)) dt, \\
\frac{1}{k^p} \sum_{i=0}^{k^p-1} f\left(\gamma\left(\frac{2i+1}{2k^p}\right)\right) + \frac{1}{4} \int_0^1 \phi(t) dt &\leq \int_0^1 f(\gamma(t)) dt \\
&\leq \frac{1}{k^p} \sum_{i=0}^{k^p-1} \frac{f(\gamma(\frac{i}{k^p})) + f(\gamma(\frac{i+1}{k^p}))}{2} - \frac{1}{6}\phi(1).
\end{aligned}$$

Also, in view of Lemma 1 we have

$$\begin{aligned}
f\left(\gamma\left(\frac{1}{2}\right)\right) &= f\left(\gamma\left(\frac{1}{k^p} \sum_{i=0}^{k^p-1} \left(\frac{2i+1}{2k^p}\right)\right)\right) \\
&\leq \frac{1}{k^p} \sum_{i=0}^{k^p-1} f\left(\gamma\left(\frac{2i+1}{2k^p}\right)\right) - \frac{1}{k^p}\phi\left(\frac{1}{k^p}\right).
\end{aligned}$$

So,

$$\begin{aligned}
\frac{1}{k^p} \sum_{i=0}^{k^p-1} \frac{f(\gamma(\frac{i}{k^p})) + f(\gamma(\frac{i+1}{k^p}))}{2} &= \frac{1}{2k^p} \sum_{k=0}^{k^p-1} \left[f\left(\gamma\left(\frac{i}{k^p}\right)\right) + f\left(\gamma\left(\frac{i+1}{k^p}\right)\right) \right] \\
&= \frac{f(\gamma(0)) + f(\gamma(1))}{2}.
\end{aligned}$$

In view of the above relation the proof is completes. $\square$

Theorem 3. *Let (N, d) be a global NPC space, $C \subset N$ a convex set and $f: C \rightarrow \mathbb{R}$ be a uniformly convex function, then*

$$\begin{aligned} f\left(\gamma\left(\frac{1}{2}\right)\right) + \lambda(1-\lambda)\phi\left(\frac{1}{2}\right) &\leq \ell(\lambda) \leq \int_0^1 f(\gamma(t))dt - \frac{1}{4} \int_0^1 \phi(t)dt \\ &\leq L(\lambda) - \frac{1}{6}\phi(1) - \frac{1}{4} \int_0^1 \phi(t)dt \\ &\leq \alpha_{(f(\gamma(0)), f(\gamma(1)))}\left(\frac{1}{2}\right) - \frac{1}{6}\phi(1) - \frac{1}{4} \int_0^1 \phi(t)dt, \end{aligned}$$

where

$$\ell(\lambda) = \lambda f\left(\gamma\left(\frac{\lambda}{2}\right)\right) + (1-\lambda)f\left(\gamma\left(\frac{\lambda+1}{2}\right)\right)$$

and

$$L(\lambda) = \frac{1}{2}(f(\gamma(\lambda)) + \lambda f(\gamma(0)) + (1-\lambda)f(\gamma(1))).$$

Proof. In veiw of Theorem 1 we have

$$f\left(\gamma\left(\frac{1}{2}\right)\right) + \frac{1}{4} \int_0^1 \phi(t)dt \leq \int_0^1 f(\alpha_{(a,b)}(\lambda))d\lambda \leq \alpha_{(f(a), f(b))}\left(\frac{1}{2}\right) - \frac{1}{6}\phi(1).$$

Now, assume that f is a uniformly convex function on C . γ is a geodesic from $\gamma(0)$ to $\gamma(1)$ so

$$f\left(\gamma\left(\frac{\lambda}{2}\right)\right) + \frac{1}{4} \int_0^1 \phi(t)dt \leq \int_0^1 f(\gamma_{[0,\lambda]}(t))dt \leq \frac{f(\gamma(0)) + f(\gamma(1))}{2} - \frac{\phi(1)}{6}.$$

Also, if γ is a geodesic from $\gamma(\lambda)$ to $\gamma(1)$ we have

$$f\left(\gamma\left(\frac{1+\lambda}{2}\right)\right) + \frac{1}{4} \int_0^1 \phi(t)dt \leq \int_0^1 f(\gamma_{[\lambda,1]}(t))dt \leq \frac{f(\gamma(\lambda)) + f(\gamma(1))}{2} - \frac{\phi(1)}{6}.$$

Multiplying above inequality in λ , $1-\lambda$ repectivly and add them we have

$$\begin{aligned} \ell(\lambda) + \frac{1}{4} \int_0^1 \phi(t)dt &\leq \lambda \int_0^1 f(\gamma_{[0,\lambda]}(t))dt + (1-\lambda) \int_0^1 f(\gamma_{[\lambda,1]}(t))dt \\ &= \int_0^1 f(\gamma(t))dt \leq L(\lambda) - \frac{1}{6}\phi(1), \end{aligned}$$

and

$$\begin{aligned} f\left(\gamma\left(\frac{1}{2}\right)\right) &= f\left(\gamma\left(\lambda\frac{\lambda}{2} + (1-\lambda)\frac{1+\lambda}{2}\right)\right) \\ &\leq \lambda f\left(\gamma\left(\frac{\lambda}{2}\right)\right) + (1-\lambda)f\left(\gamma\left(\frac{1+\lambda}{2}\right)\right) - \lambda(1-\lambda)\phi\left(\frac{1}{2}\right) \\ &= \ell(\lambda) - \lambda(1-\lambda)\phi\left(\frac{1}{2}\right). \end{aligned}$$

Hence,

$$\begin{aligned} f\left(\gamma\left(\frac{1}{2}\right)\right) + \lambda(1-\lambda)\phi\left(\frac{1}{2}\right) &\leq \ell(\lambda) \leq \int_0^1 f(\gamma(t))dt - \frac{1}{4} \int_0^1 \phi(t)dt \\ &\leq L(\lambda) - \frac{1}{6}\phi(1) - \frac{1}{4} \int_0^1 \phi(t)dt \\ &\leq \alpha_{(f(\gamma(0)), f(\gamma(1)))}\left(\frac{1}{2}\right) - \frac{1}{6}\phi(1) - \frac{1}{4} \int_0^1 \phi(t)dt. \end{aligned}$$

□

Theorem 4. Let $p \in [2, +\infty)$ and γ be a geodesic. Then the following inequality holds.

$$\begin{aligned} &\left|\gamma\left(\frac{1}{2}\right)\right|^p + \frac{2^{-1-p}}{p+1} \min\{p2^{-\frac{p}{2}}, 1-2^{-\frac{p}{2}}\} \\ &\leq \left|\gamma\left(\frac{1}{2}\right)\right|^p + \frac{1}{4} \int_0^1 \phi(t)dt \\ &\leq \int_0^1 |\alpha_{a,b}(\lambda)|^p d\lambda \\ &\leq \frac{|\gamma(1)|^p + |\gamma(0)|^p}{2} - \frac{2^{-p}}{3} \min\{p2^{-\frac{p}{2}}, 1-2^{-\frac{p}{2}}\}. \end{aligned}$$

Proof. According to ([4], Proposition 10.13), since $|\cdot|^2$ is uniformly convex with modules of convexity $|\cdot|^2$. Hence for $p \in [2, +\infty)$ is uniformly convex with modules of convexity ϕ such that ϕ satisfying

$$\phi \geq 2^{1-p} \min\{p2^{-\frac{p}{2}}, 1-2^{-\frac{p}{2}}\} |\cdot|^p,$$

by integrating with respect t we have

$$\begin{aligned} \int_0^1 \phi(t)dt &\geq 2^{1-p} \min\{p2^{-\frac{p}{2}}, 1-2^{-\frac{p}{2}}\} \int_0^1 |t|^p dt \\ &= \frac{2^{1-p}}{p+1} \min\{p2^{-\frac{p}{2}}, 1-2^{-\frac{p}{2}}\}. \end{aligned}$$

Thus,

$$\begin{aligned} &\left|\gamma\left(\frac{1}{2}\right)\right|^p + \frac{2^{-1-p}}{p+1} \min\{p2^{-\frac{p}{2}}, 1-2^{-\frac{p}{2}}\} \\ &\leq \left|\gamma\left(\frac{1}{2}\right)\right|^p + \frac{1}{4} \int_0^1 \phi(t)dt \\ &\leq \int_0^1 |\alpha_{a,b}(\lambda)|^p d\lambda \\ &\leq \frac{|\gamma(1)|^p + |\gamma(0)|^p}{2} - \frac{1}{6}\phi(1) \end{aligned}$$

$$\leq \frac{|\gamma(1)|^p + |\gamma(0)|^p}{2} - \frac{2^{-p}}{3} \min\{p2^{-\frac{p}{2}}, 1 - 2^{-\frac{p}{2}}\}.$$

□

3. APPLICATIONS TO SPECIAL MEANS

Consider the following special means for two nonnegative real numbers $\alpha, \beta, \alpha \neq \beta$ as follows:

(1) The arithmetic mean:

$$A = A(\alpha, \beta) = \frac{\alpha + \beta}{2}, \quad \alpha, \beta \in \mathbb{R},$$

with $\alpha, \beta > 0$.

(1) The logarithmic mean:

$$\bar{L} = \bar{L}(\alpha, \beta) = \frac{\beta - \alpha}{\ln \beta - \ln \alpha}, \quad \alpha \neq \beta, \alpha, \beta \in \mathbb{R},$$

with $\alpha, \beta > 0$.

(2) The generalized logarithmic mean:

$$L_n(\alpha, \beta) = \left[\frac{\beta^{n+1} - \alpha^{n+1}}{(n+1)(\beta - \alpha)} \right]^{\frac{1}{n}}, \quad n \in \mathbb{Z} \setminus \{-1, 0\}, \alpha \neq \beta, \alpha, \beta \in \mathbb{R},$$

with $\alpha, \beta > 0$.

Proposition 2. Let $a, b \in \mathbb{R}$ with $a < b, a \neq 0$ and let p be even number. Then, the following inequality holds:

$$\begin{aligned} [A(a, b)]^p + \frac{2^{-1-p}}{p+1} \min\{p2^{-\frac{p}{2}}, 1 - 2^{-\frac{p}{2}}\} &\leq [L_p(a, b)]^p \\ &\leq A(a^p, b^p) - \frac{2^{-p}}{3} \min\{p2^{-\frac{p}{2}}, 1 - 2^{-\frac{p}{2}}\}. \end{aligned}$$

Proof. Put the geodesic $\alpha(\lambda) = \lambda a + (1 - \lambda)b$ in Theorem 4. □

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CANONICAL ALMOST GEODESIC MAPPINGS $\pi_2(e)$, $e = \pm 1$, OF SPACES WITH AFFINE CONNECTION ONTO m -SYMMETRIC SPACES

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Abstract. In the paper we consider canonical almost geodesic mappings $\pi_2(e)$, $e = \pm 1$, of spaces with affine connection onto 2-symmetric, 3-symmetric and m -symmetric spaces. The main equations for the mappings have been obtained as closed systems of PDEs of Cauchy type in covariant derivatives. We have found the maximum numbers of essential parameters which the general solutions of the systems depend on.

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1. INTRODUCTION

The paper is devoted to further study of the theory of almost geodesic mappings of affinely connected spaces. The theory goes back to the paper [14], by T. Levi-Civita, in which the problem on the search for Riemannian spaces with common geodesics was stated and solved in a special coordinate system. We note a remarkable fact that this problem is related to the study of equations of dynamics of mechanical systems.

The theory of geodesic mappings has been developed by T. Thomas, J. Thomas, H. Weyl, L. P. Eisenhart, P.A. Shirokov, A.S. Solodovnikov, N.S. Sinyukov, A.V. Aminova, J. Mikeš, and others, see [1, 16, 20].

Issues arisen by the exploration were studied by V.F. Kagan, D.V. Vedenyapin, G. Vrančeanu, Ya.L. Shapiro and others. The authors discover special classes of $(n - 2)$ -projective spaces.

In [17], A.Z. Petrov introduced the notion of quasi-geodesic mappings. In particular, holomorphically projective mappings of Kählerian spaces are special quasi-geodesic mappings; they were examined by T. Otsuki and Y. Tashiro, M. Prvanović, J. Mikeš, and others, see [16, 20].

A natural generalization of these classes of mappings is the class of almost geodesic mappings introduced by Sinyukov (see [19, 20]). He also specified three types of almost geodesic mappings π_1, π_2, π_3 .

The theory of almost geodesic mappings was developed by V.S. Sobchuk [22], N.Y. Yablonskaya [25], V.E. Berezovski, J. Mikeš [2–11, 15, 16], Lj.S. Velimirović, N. Vesić, M.S. Stanković, [24] et al.

In this paper we consider almost geodesic mappings of the second type $\pi_2(e)$, $e = \pm 1$, of spaces with affine connection onto 2-symmetric, 3-symmetric and m -symmetric spaces. The main equations for the mappings have been obtained as closed systems of PDEs of Cauchy type in covariant derivatives. Also we have found the maximum numbers of essential parameters which the solutions of the systems depend on.

2. BASIC DEFINITIONS OF ALMOST GEODESIC MAPPINGS OF SPACES WITH AFFINE CONNECTIONS.

Let us recall the basic definition, formulas and theorems of the theory presented in [4, 15, 16, 19, 21].

Consider a space A_n with an affine torsion-free connection $\Gamma_{ij}^h(x)$. The space is referred to a local coordinate system $x^1, x^2, \dots, x^n$.

A curve $l: x^h = x^h(t)$ in the space A_n is a *geodesic* if its tangent vector $\lambda^h(t) = dx^h(t)/dt$ satisfies the equations

$$\lambda_1^h = \rho(t) \cdot \lambda^h,$$

where

$$\lambda_1^h \equiv \lambda_{1,\alpha}^h \lambda^\alpha = d\lambda^h(t)/dt + \Gamma_{\alpha\beta}^h(x(t)) \lambda^\alpha(t) \lambda^\beta(t),$$

and $\rho(t)$ is a function of t . We denote by comma “,” the covariant derivative with respect to the connection of the space A_n .

A curve $l: x^h = x^h(t)$ in the space A_n ($n > 2$) is an *almost geodesic* if its tangent vector $\lambda^h(t)$ satisfies the equations

$$\lambda_2^h = a(t) \cdot \lambda^h + b(t) \cdot \lambda_1^h,$$

where $\lambda_2^h \equiv \lambda_{1,\alpha}^h \lambda^\alpha$, $a(t)$ and $b(t)$ are functions of t .

We say that a mapping $f: A_n \rightarrow \bar{A}_n$ is an *almost geodesic mapping* if any geodesic curve of A_n is mapped under f onto an almost geodesic curve in $\bar{A}_n$.

Suppose, that a space A_n with affine connection $\Gamma_{ij}^h(x)$ admits a mapping f onto a space $\bar{A}_n$ with affine connection $\bar{\Gamma}_{ij}^h(x)$, and the spaces are referred to a common coordinate system $x^1, x^2, \dots, x^n$.

The tensor

$$P_{ij}^h(x) = \bar{\Gamma}_{ij}^h(x) - \Gamma_{ij}^h(x) \quad (2.1)$$

is called a *deformation tensor* of the connections $\Gamma_{ij}^h(x)$ and $\bar{\Gamma}_{ij}^h(x)$ with respect to the mapping f . The symbols $\Gamma_{ij}^h(x)$ and $\bar{\Gamma}_{ij}^h(x)$ are components of affine connections of the spaces A_n and $\bar{A}_n$ respectively. The components are expressed in the common local coordinate system.

It is known [16, 20, 21] that a necessary and sufficient condition for the mapping of a space A_n onto a space $\bar{A}_n$ to be almost geodesic is that the deformation tensor $P_{ij}^h(x)$ of the mapping f in the common coordinate system $x^1, x^2, \dots, x^n$ has to satisfy the condition

$$A_{\alpha\beta\gamma}^h \lambda^\alpha \lambda^\beta \lambda^\gamma = a \cdot P_{\alpha\beta}^h \lambda^\alpha \lambda^\beta + b \cdot \lambda^h,$$

where λ^h is an arbitrary vector, a and b are certain functions of variables $x^1, x^2, \dots, x^n$ and $\lambda^1, \lambda^2, \dots, \lambda^n$. The tensor A_{ijk}^h is defined as

$$A_{ijk}^h \stackrel{\text{def}}{=} P_{ij,k}^h + P_{ij}^\alpha P_{\alpha k}^h.$$

According to the character of a and b , i. e. depending on how the functions involve the coordinates $\lambda^1, \lambda^2, \dots, \lambda^n$ of the vector λ , N.S. Sinyukov [16, 20] distinguished three kinds of almost geodesic mappings, namely π_1 , π_2 and π_3 .

A mapping $f: A_n \rightarrow \bar{A}_n$ is called *almost geodesic of type π_1* , if the conditions

$$A_{(ijk)}^h = \delta_{(i}^h a_{jk)} + b_{(i} P_{jk)}^h$$

are satisfied, where a_{ij} is a symmetric tensor, b_i is a covariant vector, and δ_i^h is the Kronecker delta. We denote by the round parentheses an operation called *symmetrization* without division with respect to the indices i, j and k .

A mapping $f: A_n \rightarrow \bar{A}_n$ is called *almost geodesic of type π_2* , if the conditions

$$P_{ij}^h = \delta_{(i}^h \psi_{j)} + F_{(i}^h \phi_{j)}, \quad (2.2)$$

$$F_{(i,j)}^h + F_{\alpha}^h F_{(i}^\alpha \phi_{j)} = \delta_{(i}^h \mu_{j)} + F_{(i}^h \rho_{j)} \quad (2.3)$$

holds. Here $\psi_i, \phi_i, \mu_i, \rho_i$ are some covectors, F_i^h is a tensor of type $(1, 1)$.

We consider mappings $\pi_2: A_n \rightarrow \bar{A}_n$ characterized locally in a common coordinate system, by the equations (2.2) and (2.3) as corresponding to $F_i^h(x)$.

A mapping π_2 satisfies the *mutuality condition* if the inverse mapping is also an almost geodesic of type π_2 and corresponding to the same affinor $F_i^h(x)$.

The mappings π_2 satisfying the mutuality condition will be denoted as $\pi_2(e)$, where $e = -1, 0, 1$.

As it was proved in [23], in the case when $e = \pm 1$ the basic equations of the mappings $\pi_2(e)$ can be written as (2.2), the differential equations

$$F_{i,j}^h = F_{ij}^h, \quad F_{ij,k}^h = \overset{6}{\Theta}_{ijk}^h, \quad \mu_{i,j} = \mu_{ij}, \quad \mu_{ij,k} = \overset{7}{\Theta}_{ijk}, \quad (2.4)$$

and algebraic equations

$$F_{(ij)}^h = F_{(i}^h \mu_{j)} - \delta_{(i}^h F_{j)}^\alpha \mu_\alpha, \quad F_\alpha^h F_i^\alpha = e \delta_i^h, \quad \mu_{(ij)} = \overset{5}{\Theta}_{ij}, \quad (2.5)$$

where

$$\begin{aligned}
\Theta_{ijk}^1 &\equiv \Theta_{ijk}^2 + \Theta_{kji}^2 - \Theta_{jki}^2 + 2F_\alpha^h R^\alpha \alpha_{kji} - F_i^\alpha R_{\alpha jk}^h + F_j^\alpha R_{\alpha ik}^h + F_k^\alpha R_{\alpha ij}^h, \\
\Theta_{ijk}^2 &\equiv \mu_{(i} F_{j)k}^h - \delta_{(i}^h F_{j)k}^\alpha \mu_\alpha, \\
\Theta_{ijk}^3 &\equiv \Theta_{ijk}^2 - \Theta_{kji}^2 + F_j^\alpha R_{\alpha ik}^h - F_\alpha^h R_{jik}^\alpha, \\
\Theta_{jk}^4 &\equiv F_\beta^\alpha \Theta_{\alpha jk}^\beta + 2F_{\beta j}^\alpha F_{\alpha k}^\beta, \\
\Theta_{jk}^5 &\equiv \frac{1}{(n-1-F_\alpha^\alpha)^2-1} \left((n-1-F_\alpha^\alpha) \Theta_{ij}^4 + \Theta_{\alpha\beta}^4 F_i^\alpha F_j^\beta \right), \\
\Theta_{ijk}^6 &\equiv \frac{1}{2} (F_i^h \mu_{(jk)} + F_j^h \mu_{[ik]} + F_k^h \mu_{[ij]} - \delta_i^h m_{(jk)} - \delta_j^h m_{[ik]} - \delta_k^h m_{[ij]} + \Theta_{ikj}^1), \\
\Theta_{ijk}^7 &\equiv \mu_\alpha R_{kji}^\alpha + \frac{1}{2} (\Theta_{ij,k}^5 + \Theta_{ik,j}^5 - \Theta_{jk,i}^5), \quad m_{ij} \equiv F_i^\alpha \mu_{\alpha j},
\end{aligned}$$

$F_i^h(x)$, $F_{ij}^h(x)$, $\mu_i(x)$, $\mu_{ij}(x)$ are unknown functions, $R_{ijk}^h(x)$ is the Riemann tensor of the space A_n . We denote by the brackets $[ik]$ an operation called *antisymmetrization* (or, *alternation*) without division with respect to the indices i and k .

Obviously, right hand sides of the equations (2.4) depend on unknown functions $F_i^h(x)$, $F_{ij}^h(x)$, $\mu_i(x)$, $\mu_{ij}(x)$, and on the components $\Gamma_{ij}^h(x)$ of the space A_n .

The equations (2.4) and (2.5) form a closed mixed system of differential equations of Cauchy type in covariant derivatives with respect to the functions $F_i^h(x)$, $F_{ij}^h(x)$, $\mu_i(x)$, $\mu_{ij}(x)$. Also the mapping $\pi_2(e)$ depends on unknown functions $\psi_i(x)$, $\varphi_j(x)$ (see the equations (2.2)).

An almost geodesic mapping π_2 for which $\psi_i \equiv 0$ is called *canonical*. It is known [21] that any almost geodesic mapping π_2 can be written as the composition of a canonical almost geodesic mapping of type π_2 and a geodesic mapping. The latter may be referred to as a trivial almost geodesic mapping.

Hence canonical almost geodesic mappings $\pi_2(e)$, $e = \pm 1$, are determined by the equations

$$P_{(ij)}^h = F_{(i}^h \Phi_{j)}, \quad (2.6)$$

and also by the equations (2.4) and (2.5).

A mapping $f : A_n \rightarrow \bar{A}_n$ is *almost geodesic of type π_3* , if the conditions

$$P_{ij}^h = \delta_{(i}^h \psi_{j)} + \theta^h a_{ij}, \quad \theta_{,i}^h = \rho \cdot \delta_i^h + \theta^h a_i$$

holds. Here θ^h is a certain vector, ψ_i , a_i are certain covectors, a_{ij} is a certain symmetric tensor and ρ is a certain function.

The types of almost geodesic mappings π_1 , π_2 , π_3 can intersect. The problem of completeness of classification had long remained unresolved. V. Berezovsky and

J. Mikeš [7] proved that for $n > 5$ other types of almost geodesic mappings except π_1 , π_2 , and π_3 do not exist.

3. CANONICAL ALMOST GEODESIC MAPPINGS $\pi_2(e)$, $e = \pm 1$, OF SPACES A_n WITH AFFINE CONNECTIONS ONTO 2-SYMMETRIC SPACES

A space $\bar{A}_n$ with an affine connection is called *2-symmetric* if its Riemann tensor $\bar{R}_{ijk}^h$ satisfies the condition

$$\bar{R}_{ijk|mp}^h = 0. \quad (3.1)$$

By the symbol “|” we denote covariant derivative respect to the connection of the space $\bar{A}_n$.

We recall that *symmetric* spaces $\bar{A}_n$ are characterized by $\bar{R}_{ijk|m}^h = 0$. Symmetric spaces were introduced by P.A. Shirokov [18] and É. Cartan [12], see also S. Helgason [13].

Let us consider the canonical almost geodesic mappings of type $\pi_2(e)$, $e = \pm 1$, of spaces A_n with affine connection onto 2-symmetric spaces $\bar{A}_n$, which are determined by the equations (2.4), (2.6) and (2.5). Suppose, that the spaces A_n and $\bar{A}_n$ are referred to a common coordinate system $x^1, x^2, \dots, x^n$.

Since

$$\bar{R}_{ijk|m}^h = \frac{\partial \bar{R}_{ijk}^h}{\partial x^m} + \bar{\Gamma}_{m\alpha}^h \bar{R}_{ijk}^\alpha - \bar{\Gamma}_{mi}^\alpha \bar{R}_{\alpha jk}^h - \bar{\Gamma}_{mj}^\alpha \bar{R}_{i\alpha k}^h - \bar{\Gamma}_{mk}^\alpha \bar{R}_{ij\alpha}^h,$$

then taking account of (2.1) we can obtain

$$\bar{R}_{ijk|m}^h = \bar{R}_{ijk,m}^h + P_{m\alpha}^h \bar{R}_{ijk}^\alpha - P_{mi}^\alpha \bar{R}_{\alpha jk}^h - P_{mj}^\alpha \bar{R}_{i\alpha k}^h - P_{mk}^\alpha \bar{R}_{ij\alpha}^h. \quad (3.2)$$

Since according to the definition of covariant derivative

$$(\bar{R}_{ijk|m}^h)_{,p} = \frac{\partial \bar{R}_{ijk|m}^h}{\partial x^p} + \Gamma_{\alpha p}^h \bar{R}_{ijk|m}^\alpha - \Gamma_{ip}^\alpha \bar{R}_{\alpha jk|m}^h - \Gamma_{jp}^\alpha \bar{R}_{i\alpha k|m}^h - \Gamma_{kp}^\alpha \bar{R}_{ij\alpha|m}^h - \Gamma_{mp}^\alpha \bar{R}_{ijk|\alpha}^h,$$

then taking account of (2.1), we have

$$\begin{aligned} (\bar{R}_{ijk|m}^h)_{,p} &= \bar{R}_{ijk|mp}^h - P_{\alpha p}^h \bar{R}_{ijk|m}^\alpha + P_{ip}^\alpha \bar{R}_{\alpha jk|m}^h + P_{jp}^\alpha \bar{R}_{i\alpha k|m}^h \\ &\quad + P_{kp}^\alpha \bar{R}_{ij\alpha|m}^h + P_{mp}^\alpha \bar{R}_{ijk|\alpha}^h. \end{aligned} \quad (3.3)$$

Differentiating (3.2) with respect to x^p in the space A_n , we get

$$\begin{aligned} (\bar{R}_{ijk|m}^h)_{,p} &= \bar{R}_{ijk,mp}^h + P_{m\alpha,p}^h \bar{R}_{ijk}^\alpha + P_{m\alpha}^h \bar{R}_{ijk,\rho}^\alpha - P_{mi,p}^\alpha \bar{R}_{\alpha jk}^h - P_{mi}^\alpha \bar{R}_{\alpha jk,p}^h \\ &\quad - P_{mj,p}^\alpha \bar{R}_{i\alpha k}^h - P_{mj}^\alpha \bar{R}_{i\alpha k,p}^h - P_{mk,p}^\alpha \bar{R}_{ij\alpha}^h - P_{mk}^\alpha \bar{R}_{ij\alpha,p}^h. \end{aligned} \quad (3.4)$$

Substituting in (3.3) from (3.4), we have

$$\begin{aligned} \bar{R}_{ijk,mp}^h &= \bar{R}_{ijk|mp}^h - P_{\alpha p}^h \bar{R}_{ijk|m}^\alpha + P_{ip}^\alpha \bar{R}_{\alpha jk|m}^h + P_{jp}^\alpha \bar{R}_{i\alpha k|m}^h + P_{kp}^\alpha \bar{R}_{ij\alpha|m}^h \\ &\quad + P_{mp}^\alpha \bar{R}_{ijk|\alpha}^h - P_{m\alpha,p}^h \bar{R}_{ijk}^\alpha - P_{m\alpha}^h \bar{R}_{ijk,\rho}^\alpha + P_{mi,p}^\alpha \bar{R}_{\alpha jk}^h + P_{mi}^\alpha \bar{R}_{\alpha jk,p}^h \\ &\quad + P_{mj,p}^\alpha \bar{R}_{i\alpha k}^h + P_{mj}^\alpha \bar{R}_{i\alpha k,p}^h - P_{mk,p}^\alpha \bar{R}_{ij\alpha}^h - P_{mk}^\alpha \bar{R}_{ij\alpha,p}^h. \end{aligned}$$

$$+ P_{mj,\rho}^\alpha \bar{R}_{i\alpha k}^h + P_{mj}^\alpha \bar{R}_{i\alpha k,\rho}^h + P_{mk,\rho}^\alpha \bar{R}_{ij\alpha}^h + P_{mk}^\alpha \bar{R}_{ij\alpha,\rho}^h \quad (3.5)$$

Suppose that the space $\bar{A}_n$ is a 2-symmetric space. Then the identity (3.1) holds. Hence from (3.5) we obtain

$$\begin{aligned} \bar{R}_{ijk,m\rho}^h = & -P_{\alpha\rho}^h \bar{R}_{ijk|m}^\alpha + P_{i\rho}^\alpha \bar{R}_{\alpha jk|m}^h + P_{j\rho}^\alpha \bar{R}_{i\alpha k|m}^h + P_{k\rho}^\alpha \bar{R}_{ij\alpha|m}^h \\ & + P_{m\rho}^\alpha \bar{R}_{ijk|\alpha}^h - P_{m\alpha,\rho}^h \bar{R}_{ijk}^\alpha - P_{m\alpha}^h \bar{R}_{ijk,\rho}^\alpha + P_{mi,\rho}^\alpha \bar{R}_{\alpha jk}^h + P_{mi}^\alpha \bar{R}_{\alpha jk,\rho}^h \\ & + P_{mj,\rho}^\alpha \bar{R}_{i\alpha k}^h + P_{mj}^\alpha \bar{R}_{i\alpha k,\rho}^h + P_{mk,\rho}^\alpha \bar{R}_{ij\alpha}^h + P_{mk}^\alpha \bar{R}_{ij\alpha,\rho}^h. \end{aligned} \quad (3.6)$$

Let us introduce the tensor $\bar{R}_{ijk m}^h$ defined by

$$\bar{R}_{ijk,m}^h = \bar{R}_{ijk m}^h. \quad (3.7)$$

It is known [16, 20] that the Riemann tensors of the spaces A_n and $\bar{A}_n$ are related to each other by the equations

$$\bar{R}_{ijk}^h = R_{ijk}^h + P_{ik,j}^h - P_{ij,k}^h + P_{ik}^\alpha P_{\alpha j}^h - P_{ij}^\alpha P_{\alpha k}^h. \quad (3.8)$$

Since the deformation tensor of the mapping P_{ij}^h is represented by the equations (2.6), it follows from (3.8) that

$$\varphi_{i,j} F_k^h + \varphi_{k,j} F_i^h - \varphi_{i,k} F_j^h - \varphi_{j,k} F_i^h = C_{ijk}^h, \quad (3.9)$$

where

$$\begin{aligned} C_{ijk}^h = & \bar{R}_{ijk}^h - R_{ijk}^h - \varphi_i (F_{kj}^h + \varphi_\alpha F_k^\alpha F_j^h + e \delta_k^h \varphi_j - F_{jk}^h - \varphi_\alpha F_j^\alpha F_k^h - e \delta_j^h \varphi_k) \\ & + \varphi_k (F_{ij}^h + \varphi_\alpha F_i^\alpha F_j^h) - \varphi_j (F_{ik}^h + \varphi_\alpha F_i^\alpha F_k^h). \end{aligned} \quad (3.10)$$

Let us multiply (3.9) by F_ρ^m and contract for ρ and h . Hence we have

$$\delta_k^m \varphi_{i,j} + \delta_i^m \varphi_{k,j} - \delta_j^m \varphi_{i,k} - \delta_i^m \varphi_{j,k} = e C_{ijk}^\alpha F_\alpha^m. \quad (3.11)$$

Contracting the equations (3.11) for m and i we get

$$\varphi_{k,j} - \varphi_{j,k} = \frac{e}{n+1} C_{\beta jk}^\alpha F_\alpha^\beta. \quad (3.12)$$

Again, contracting the equations (3.11) for m and k we get

$$n \varphi_{i,j} - \varphi_{j,i} = e C_{i j \beta}^\alpha F_\alpha^\beta. \quad (3.13)$$

Taking account of (3.12) the equations (3.13) can be written as

$$\varphi_{i,j} = \frac{e}{n-1} (C_{ij\beta}^\alpha - \frac{1}{n+1} C_{\beta ji}^\alpha) F_\alpha^\beta. \quad (3.14)$$

And finally, taking account of (2.4), (2.6) and (3.7) the equations (3.6) can be written as

$$\bar{R}_{ijk m,\rho}^h = \Theta_{ijk m \rho}^h, \quad (3.15)$$

where

$$\begin{aligned}\Theta_{ijkmp}^h = & -F_{(\alpha}^h \Phi_{\rho)} \bar{R}_{ijk|m}^\alpha + F_{(i}^\alpha \Phi_{\rho)} \bar{R}_{\alpha jk|m}^h + F_{(j}^\alpha \Phi_{\rho)} \bar{R}_{i\alpha k|m}^h + F_{(k}^\alpha \Phi_{\rho)} \bar{R}_{ij\alpha|m}^h \\ & + F_{(m}^\alpha \Phi_{\rho)} \bar{R}_{ijk|\alpha}^h - (F_{(m|\rho|}^h \Phi_\alpha) + F_{(m}^h \Phi_{\alpha),\rho}) \bar{R}_{ijk}^\alpha - F_{(m}^h \Phi_\alpha) \bar{R}_{ijk\rho}^\alpha + (F_{(m|\rho|}^\alpha \Phi_i) \\ & + F_{(m}^\alpha \Phi_{i),\rho}) \bar{R}_{\alpha jk}^h + F_{(m}^\alpha \Phi_i) \bar{R}_{\alpha jk\rho}^h + (F_{(m|\rho|}^\alpha \Phi_j) + F_{(m}^\alpha \Phi_{j),\rho}) \bar{R}_{i\alpha k}^h \\ & + F_{(m}^\alpha \Phi_j) \bar{R}_{i\alpha k\rho}^h + (F_{(m|\rho|}^\alpha \Phi_k) + F_{(m}^\alpha \Phi_{k),\rho}) \bar{R}_{ij\alpha}^h + F_{(m}^\alpha \Phi_k) \bar{R}_{ij\alpha\rho}^h.\end{aligned}$$

Suppose, that in the above formula the tensors $\bar{R}_{ijk|m}^h$ and $\Phi_{i,j}$ are expressed according to (3.2) and (3.14). Also we suppose that $\bar{R}_{ijk,m}^h = \bar{R}_{ijkm}^h$. Obviously, in the space A_n the equations (2.4), (3.7), (3.14) and (3.15) form a system of PDE's of Cauchy type in covariant derivatives with respect to functions $F_i^h(x)$, $F_{ij}^h(x)$, $\mu_i(x)$, $\mu_{ij}(x)$, $\bar{R}_{ijk}^h(x)$, $\bar{R}_{ijkm}^h(x)$, $\Phi_i(x)$, and the functions $F_i^h(x)$, $F_{ij}^h(x)$, $\mu_i(x)$, $\mu_{ij}(x)$ must satisfy the algebraic conditions (2.5). The algebraic conditions for the functions $\bar{R}_{ijk}^h(x)$ are

$$\bar{R}_{i(jk)}^h = 0, \quad \bar{R}_{(ijk)}^h = 0. \quad (3.16)$$

Since $\bar{R}_{i(jk|m)}^h = 0$ and taking account of (2.6) and (3.2), we obtain another algebraic condition

$$\begin{aligned}\bar{R}_{i(jkm)}^h = & -F_{(m}^h \Phi_\alpha) \bar{R}_{ijk}^\alpha - F_{(k}^h \Phi_\alpha) \bar{R}_{imj}^\alpha - F_{(j}^h \Phi_\alpha) \bar{R}_{ikm}^\alpha \\ & + F_{(m}^\alpha \Phi_i) \bar{R}_{\alpha jk}^h + F_{(k}^\alpha \Phi_i) \bar{R}_{\alpha mj}^h + F_{(j}^\alpha \Phi_i) \bar{R}_{\alpha km}^h.\end{aligned} \quad (3.17)$$

Hence we have the following theorem.

Theorem 1. *In order that a space A_n with affine connection admit almost geodesic mapping of type $\pi_2(e)$, $e = \pm 1$, onto a 2-symmetric space $\bar{A}_n$, it is necessary and sufficient that the closed mixed system of differential equations of Cauchy type in covariant derivatives (2.4), (3.7), (3.14), (3.15), (2.5), (3.16), (3.17) have a solution with respect to functions $F_i^h(x)$, $F_{ij}^h(x)$, $\mu_i(x)$, $\mu_{ij}(x)$, $\bar{R}_{ijk}^h(x)$, $\bar{R}_{ijkm}^h(x)$, $\Phi_i(x)$.*

Consequence 1. *The general solution of the closed mixed system of Cauchy type (2.4), (3.7), (3.14), (3.15), (2.5), (3.16) and (3.17) depends on no more than*

$$n(n^2 + 2n + 2) + \frac{1}{3}n^2(n+1)(n^2 - 1)$$

essential parameters.

4. CANONICAL ALMOST GEODESIC MAPPINGS $\pi_2(e)$, $e = \pm 1$, OF SPACES WITH AFFINE CONNECTIONS ONTO 3-SYMMETRIC SPACES

A space $\bar{A}_n$ with an affine connection is called 3-symmetric if its Riemann tensor $\bar{R}_{ijk}^h$ satisfies the condition

$$\bar{R}_{ijk|mpl}^h = 0. \quad (4.1)$$

We differentiate the equations (3.5) covariantly with respect to x^l and the connection of A_n . Then in the left-hand side we express the covariant derivative with respect to the connection of A_n in terms of the covariant derivative with respect to the connection of $\bar{A}_n$, using the formula

$$\begin{aligned} (\bar{R}_{ijk|mp}^h)_{,l} &= \bar{R}_{ijk|mpl}^h - P_{\alpha l}^h \bar{R}_{ijk|mp}^\alpha + P_{il}^\alpha \bar{R}_{\alpha jk|mp}^h + P_{jl}^\alpha \bar{R}_{i\alpha k|mp}^h \\ &\quad + P_{kl}^\alpha \bar{R}_{ij\alpha|mp}^h + P_{ml}^\alpha \bar{R}_{ijk|\alpha p}^h + P_{\rho l}^\alpha \bar{R}_{ijk|m\alpha}^h. \end{aligned}$$

Let us introduce the tensor $\bar{R}_{ijkmp}^h$ defined by

$$\bar{R}_{ijkm,\rho}^h = \bar{R}_{ijkmp}^h. \quad (4.2)$$

Suppose that the space $\bar{A}_n$ is a 3-symmetric space. Then from the obtained equation, if we take account of the equations (4.1) and (4.2), we have

$$\bar{R}_{ijkmp,l}^h = \Theta_{ijkmp,l}^h, \quad (4.3)$$

where

$$\begin{aligned} \Theta_{ijkmp,l}^h &= -P_{\alpha l}^h \bar{R}_{ijk|mp}^\alpha + P_{il}^\alpha \bar{R}_{\alpha jk|mp}^h + P_{jl}^\alpha \bar{R}_{i\alpha k|mp}^h + P_{kl}^\alpha \bar{R}_{ij\alpha|mp}^h + P_{ml}^\alpha \bar{R}_{ijk|\alpha p}^h \\ &\quad + P_{\rho l}^\alpha \bar{R}_{ijk|m\alpha}^h + (-P_{\alpha p}^h \bar{R}_{ijk|m}^\alpha + P_{ip}^\alpha \bar{R}_{\alpha jk|m}^h + P_{jp}^\alpha \bar{R}_{i\alpha k|m}^h \\ &\quad + P_{kp}^\alpha \bar{R}_{ij\alpha|m}^h + P_{mp}^\alpha \bar{R}_{ijk|\alpha}^h - P_{m\alpha,p}^h \bar{R}_{ijk}^\alpha - P_{m\alpha}^h \bar{R}_{ijk,p}^\alpha + P_{mi,p}^\alpha \bar{R}_{\alpha jk}^h + P_{mi}^\alpha \bar{R}_{\alpha jk,p}^h \\ &\quad + P_{mj,p}^\alpha \bar{R}_{i\alpha k}^h + P_{mj}^\alpha \bar{R}_{i\alpha k,p}^h + P_{mk,p}^\alpha \bar{R}_{ij\alpha}^h + P_{mk}^\alpha \bar{R}_{ij\alpha,p}^h)_{,l}. \end{aligned}$$

We have assumed that in the last formula the covariant derivatives of the tensors are expressed according to the formulas (3.5), (3.4), (3.3), (3.2), (2.6) and (3.14).

Obviously, in the space A_n the equations (2.4), (3.7), (3.14), (4.2) and (4.3) form a system of PDEs of Cauchy type with respect to the functions $F_i^h(x)$, $F_{ij}^h(x)$, $\mu_i(x)$, $\mu_{ij}(x)$, $\bar{R}_{ijk}^h(x)$, $\bar{R}_{ijkm}^h(x)$, $\bar{R}_{ijkmp}^h(x)$, $\varphi_i(x)$. The functions $F_i^h(x)$, $F_{ij}^h(x)$, $\mu_i(x)$, $\mu_{ij}(x)$ must satisfy the algebraic conditions (2.5). In turn the functions $\bar{R}_{ijk}^h(x)$ and $\bar{R}_{ijkm}^h(x)$ must satisfy the algebraic conditions (3.16) and (3.17).

Hence we proved the theorem:

Theorem 2. *In order that a space A_n with affine connection admit almost geodesic mapping of type $\pi_2(e)$, $e = \pm 1$, onto a 3-symmetric space $\bar{A}_n$, it is necessary and sufficient that the closed mixed system of differential equations of Cauchy type in covariant derivatives (2.4), (3.7), (3.14), (4.2), (4.3), (2.5), (3.16), (3.17) have a solution with respect to functions $F_i^h(x)$, $F_{ij}^h(x)$, $\mu_i(x)$, $\mu_{ij}(x)$, $\bar{R}_{ijk}^h(x)$, $\bar{R}_{ijkm}^h(x)$, $\bar{R}_{ijkmp}^h(x)$, $\varphi_i(x)$.*

Consequence 2. *The general solution of the closed mixed system of Cauchy type (2.4), (3.7), (3.14), (4.2), (4.3), (2.5), (3.16), (3.17) depends on no more than*

$$n(n^2 + 2n + 2) + \frac{1}{3}n^2(n + 1)(n^3 - 1)$$

essential parameters.

5. CANONICAL ALMOST GEODESIC MAPPINGS $\pi_2(e)$, $e = \pm 1$, OF SPACES WITH AFFINE CONNECTIONS ONTO m -SYMMETRIC SPACES

A space $\bar{A}_n$ with an affine connection is called *m-symmetric* if its Riemann tensor $\bar{R}_{ijk}^h$ satisfies the condition

$$\bar{R}_{ijk|\rho_1\rho_2\dots\rho_m}^h = 0. \quad (5.1)$$

Of course 2-symmetric spaces and 3-symmetric spaces are special cases of *m*-symmetric spaces.

Let us introduce the tensors $\bar{R}_{ijk\rho_1\rho_2\rho_3}^h, \dots, \bar{R}_{ijk\rho_1\rho_2\rho_3\dots\rho_{m-2}\rho_{m-1}}^h$ defined by

$$\begin{aligned} \bar{R}_{ijk\rho_1\rho_2\rho_3}^h &= \bar{R}_{ijk\rho_1\rho_2\rho_3}^h, \\ &\dots \\ \bar{R}_{ijk\rho_1\rho_2\rho_3\dots\rho_{m-2}\rho_{m-1}}^h &= \bar{R}_{ijk\rho_1\rho_2\rho_3\dots\rho_{m-2}\rho_{m-1}}^h. \end{aligned} \quad (5.2)$$

We differentiate (4.3) covariantly $(m - 2)$ times with respect to the connection of the space A_n and in the left-hand side express the covariant derivatives with respect to the connection of A_n in terms of the covariant derivatives with respect to the connection of $\bar{A}_n$, using the formula

$$\begin{aligned} (\bar{R}_{ijk|\rho_1\dots\rho_{\tau-2}\rho_{\tau-1}}^h)_{,\rho_\tau} &= \bar{R}_{ijk|\rho_1\dots\rho_{\tau-2}\rho_{\tau-1}\rho_\tau}^h - P_{\alpha\rho_\tau}^h \bar{R}_{ijk|\rho_1\dots\rho_{\tau-2}\rho_{\tau-1}}^\alpha + P_{i\rho_\tau}^\alpha \bar{R}_{\alpha jk|\rho_1\dots\rho_{\tau-2}\rho_{\tau-1}}^h \\ &\quad + P_{j\rho_\tau}^\alpha \bar{R}_{i\alpha k|\rho_1\dots\rho_{\tau-2}\rho_{\tau-1}}^h + P_{k\rho_\tau}^\alpha \bar{R}_{ij\alpha|\rho_1\dots\rho_{\tau-2}\rho_{\tau-1}}^h \\ &\quad + P_{\rho_1\rho_\tau}^\alpha \bar{R}_{ijk|\alpha\dots\rho_{\tau-2}\rho_{\tau-1}}^h + \dots + P_{\rho_{\tau-1}\rho_\tau}^\alpha \bar{R}_{ijk|\rho_1\dots\rho_{\tau-2}\alpha}^h. \end{aligned}$$

This equation was obtained making use of (2.1).

Let us assume that the space $\bar{A}_n$ is *m*-symmetric ($m > 2$). Hence, from the obtained equation because of (5.3), using substitutions and transformations, taking account of (5.1), we get

$$\bar{R}_{ijk\rho_1\dots\rho_{m-2}\rho_{m-1}\rho_m}^h = \Theta_{ijk\rho_1\dots\rho_{m-1}\rho_m}^h, \quad (5.3)$$

where $\Theta_{ijk\rho_1\dots\rho_{m-1}\rho_m}^h$ is a tensor which involves unknown tensors $F_i^h, F_{ij}^h, \mu_i, \mu_{ij}, \varphi_i, \bar{R}_{ijk}, \bar{R}_{ijk\rho_1}^h, \dots, \bar{R}_{ijk\rho_1\dots\rho_{m-1}}^h$. The tensor $\Theta_{ijk\rho_1\dots\rho_{m-1}\rho_m}^h$ also involves some given tensors.

Obviously, in the space A_n the equations (2.4), (3.7), (3.14), (4.2), (5.2), (5.3) form a closed system of PDEs of Cauchy type with respect to the functions $F_i^h(x), F_{ij}^h(x)$,

$\mu_i(x)$, $\mu_{ij}(x)$, $\varphi_i(x)$, $\bar{R}_{ijk}^h(x)$, $\bar{R}_{ijkp_1}^h(x)$, $\dots$, $\bar{R}_{ijkp_1\dots p_{m-1}}^h(x)$. The functions must satisfy the algebraic conditions (2.5), (3.16) and (3.17).

Finally, we obtain the following theorem.

Theorem 3. *In order that a space A_n with an affine connection admit almost geodesic mapping of type $\pi_2(e)$, $e = \pm 1$, onto an m -symmetric space $\bar{A}_n$, it is necessary and sufficient that the closed mixed system of differential equations of Cauchy type in covariant derivatives (2.4), (3.7), (3.14), (4.2), (5.2), (5.3), (2.5), (3.16) and (3.17) have a solution with respect to functions $F_i^h(x)$, $F_{ij}^h(x)$, $\mu_i(x)$, $\mu_{ij}(x)$, $\varphi_i(x)$, $\bar{R}_{ijk}^h(x)$, $\bar{R}_{ijkp_1}^h(x)$, $\dots$, $\bar{R}_{ijkp_1\dots p_{m-1}}^h(x)$.*

Consequence 3. *The general solution of the closed mixed system of Cauchy type (2.4), (3.7), (3.14), (4.2), (5.2), (5.3), (2.5), (3.16) and (3.17) depends on no more than*

$$n(n^2 + 2n + 2) + \frac{1}{3}n^2(n + 1)(n^m - 1)$$

essential parameters.

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ON A REDUCTION OF A NONLINEAR AUTONOMOUS BOUNDARY-VALUE PROBLEM TO A CRITICAL CASE OF THE FIRST ORDER

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Abstract. We found constructive conditions of solvability and the scheme for constructing solutions of the nonlinear autonomous boundary-value problem in the case of multiple solutions of the equation for generating constants. The convergent iterative scheme was constructed for finding approximations to solutions of the nonlinear autonomous boundary-value problem for the system of ordinary differential equations in the case of multiple solutions of the equation for generating constants. We found the approximations to solutions of the periodic boundary-value problem for autonomous equation of Duffing type as an example of applying the constructed iterative scheme. We applied residuals in the original equation for controlling the accuracy of the found approximations to solutions of the periodic boundary-value problem for the autonomous equation of Duffing type.

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1. INTRODUCTION

Traditionally, the study of periodic and Noether boundary-value problems in critical cases was associated with the assumption that the differential equation, as well as the boundary condition, are known and fixed [1, 3, 17]. As a rule, the study of periodic problems in the case of parametric resonance was limited to the study of stability issues [18, 20]. Studies of linear autonomous boundary-value problems lead to the study of boundary-value problems in the case of parametric resonance, since the replacement of the independent variable [1, 3, 17] used in the critical case defines a non-autonomous boundary-value problem with an additional unknown. The aims of this article is to use the results, obtained in the study of nonlinear boundary-value problems in the case of parametric resonance, to solve linear autonomous boundary-value problems. We investigate the problem of finding the solutions [1, 3, 17]

$$z(t, \varepsilon) : z(\cdot, \varepsilon) \in \mathbb{C}^1[a, b(\varepsilon)], \quad z(t, \cdot) \in \mathbb{C}[0, \varepsilon_0], \quad b(\varepsilon) \in \mathbb{C}[0, \varepsilon_0]$$

of autonomous system of ordinary differential equations

$$dz(t, \varepsilon)/dt = Az(t, \varepsilon) + f + \varepsilon Z(z(t, \varepsilon), \varepsilon), \quad (1.1)$$

satisfying the boundary condition

$$\ell z(\cdot, \varepsilon) = \alpha + \varepsilon J(z(\cdot, \varepsilon), \varepsilon), \quad \alpha \in \mathbb{R}^m. \quad (1.2)$$

We will seek the solutions of boundary-value problem (1.1), (1.2) in a small neighborhood of a solution

$$z_0(t) \in \mathbb{C}^1[a, b_0], \quad b_0 = b(0)$$

of the generating Noether ($m \neq n$) boundary-value problem

$$dz_0/dt = Az_0 + f, \quad f \in \mathbb{R}^n, \quad \ell z_0(\cdot) = \alpha. \quad (1.3)$$

Here, A is an $(n \times n)$ constant matrix, $Z(z, \varepsilon)$ is a nonlinear function which is twice continuously differentiable with respect to the unknown $z(t, \varepsilon)$ and continuously differentiable with respect to a small parameter ε in a small neighborhood of a solution of the generating problem and on the segment $[0, \varepsilon_0]$; $\ell z(\cdot, \varepsilon)$ and $J(z(\cdot, \varepsilon), \varepsilon)$ are, respectively, a linear and nonlinear vector functionals $\ell z(\cdot, \varepsilon), J(z(\cdot, \varepsilon), \varepsilon) : \mathbb{C}[a, b(\varepsilon)] \rightarrow \mathbb{R}^m$. Moreover, the second functional is twice continuously differentiable with respect to the unknown $z(t, \varepsilon)$ and continuously differentiable with respect to the small parameter ε in a small neighborhood of a solution of the generating problem and on the segment $[0, \varepsilon_0]$. The problem posed continues the study of a nonlinear autonomous boundary-value problem for a system of ordinary differential equations, including in the case of parametric resonance [1, 3, 6, 17, 18, 20]. In the critical case, $(P_{Q^*} \neq 0)$ provided

$$P_{Q_d^*} \{\alpha - \ell K[f](\cdot)\} = 0 \quad (1.4)$$

the generating problem (1.3) possesses a family of solutions [3]

$$z_0(t, c_r) = X_r(t)c_r + G[f; \alpha](t), \quad X_r(t) = X(t)P_{Q_r}, \quad c_r \in \mathbb{R}^r.$$

Here, $Q := \ell X(\cdot)$ is an $(m \times n)$ matrix, $\text{rank } Q := n_1$, $n - n_1 = r$, $P_{Q^*} - (m \times m)$ is a matrix-orthoprojector

$$P_{Q^*} : \mathbb{R}^m \rightarrow \mathbb{N}(Q^*),$$

$X(t)$ is the normal ($X(a) = I_n$) fundamental matrix of the homogeneous part of differential system (1.3); P_{Q_r} is the $(n \times r)$ matrix formed by r linearly independent columns of the $(n \times n)$ matrix-orthoprojector $P_Q : \mathbb{R}^n \rightarrow \mathbb{N}(Q)$;

$$G[f; \alpha](t) = X(t)Q^+ \{\alpha - \ell K[f](\cdot)\} + K[f](t)$$

is the generalized Green operator of the boundary-value problem (1.3), Q^+ is the Moore–Penrose pseudoinverse matrix [3],

$$K[f](t) = X(t) \int_a^t X^{-1}(s)f ds$$

is the Green operator of the Cauchy problem for differential system (1.3), I_n is the identity $(n \times n)$ matrix; $P_{Q_d^*}$ is the matrix formed by d linearly independent rows of the matrix-orthoprojector P_{Q^*} .

Suppose that the equation for generating constants for a nonlinear autonomous boundary-value problem for a system of ordinary differential equations (1.1), (1.2) has multiple roots [1, 3, 17]. In this case, the study for an autonomous boundary-value problem for the system of ordinary differential equations (1.1), (1.2) with respect to scheme [1, 3] is not possible. Along with the autonomous boundary-value problem for the system of ordinary differential equations (1.1), (1.2) we consider the problem of finding solutions [1, 3, 17]

$$z(t, \varepsilon) : z(\cdot, \varepsilon) \in \mathbb{C}^1[a, b(\varepsilon)], \quad z(t, \cdot) \in \mathbb{C}[0, \varepsilon_0], \quad b(\varepsilon), h(\varepsilon) \in \mathbb{C}[0, \varepsilon_0]$$

of an autonomous system of ordinary differential equations in the case of parametric resonance [6, 18]

$$dz(t, \varepsilon)/dt = Az(t, \varepsilon) + f + \varepsilon Z(z(t, \varepsilon), h(\varepsilon), \varepsilon), \quad (1.5)$$

satisfying the boundary condition

$$\ell z(\cdot, \varepsilon) = \alpha + \varepsilon J(z(\cdot, \varepsilon), h(\varepsilon), \varepsilon), \quad \alpha \in \mathbb{R}^m. \quad (1.6)$$

We will seek the solutions of boundary-value problem (1.5), (1.6) in a small neighborhood of a solution

$$z_0(t) \in \mathbb{C}^1[a, b_0], \quad b_0 = b(0), \quad h_0 = h(0) \in \mathbb{R}^q$$

of the generating Noether $(m \neq n)$ boundary-value problem

$$dz_0(t)/dt = Az_0(t) + f, \quad f \in \mathbb{R}^n, \quad \ell z_0(\cdot) = \alpha. \quad (1.7)$$

Here, $B(z(t, \varepsilon))$ is a nonlinear $(n \times q)$ matrix which is twice continuously differentiable with respect to the unknown $z(t, \varepsilon)$ in a small neighborhood of the generating solution

$$Z(z(t, \varepsilon), h(\varepsilon), \varepsilon) := Z(z(t, \varepsilon), \varepsilon) + B(z(t, \varepsilon))h(\varepsilon),$$

is a nonlinear function which is twice continuously differentiable with respect to the unknowns $z(t, \varepsilon)$ and $h(\varepsilon)$ in a small neighborhood of the generating solution and continuously differentiable with respect to the small parameter ε on the segment $[0, \varepsilon_0]$; $\ell z(\cdot, \varepsilon)$ and $J(z(\cdot, \varepsilon), h(\varepsilon), \varepsilon)$ are, respectively, a linear and nonlinear vector functionals $\ell z(\cdot, \varepsilon), J(z(\cdot, \varepsilon), h(\varepsilon), \varepsilon) : \mathbb{C}[a, b(\varepsilon)] \rightarrow \mathbb{R}^m$. Moreover, the second functional is twice continuously differentiable with respect to the unknown $z(t, \varepsilon)$ in a small neighborhood of a solution of the generating problem (1.7) and continuously differentiable with respect to the small parameter ε in a small neighborhood of a solution of the generating problem and on the segment $[0, \varepsilon_0]$. Note that the solutions of the generating problem (1.7) and problem (1.3), as well as the problems themselves, coincide. In the critical case, the problem (1.5), (1.6) differs significantly from similar non-autonomous boundary-value problems; unlike the latter, the right end $b(\varepsilon)$

in the interval $[a, b(\varepsilon)]$, on which we are looking for a solution to the problem (1.5), (1.6), unknown and to be determined in the process of constructing a solution.

The change of independent variables [1]

$t = a + (\tau - a)(1 + \varepsilon\beta(\varepsilon))$, $b(\varepsilon) = b^* + \varepsilon(b^* - a)\beta(\varepsilon)$, $\beta(\varepsilon) \in \mathbb{C}[0, \varepsilon_0]$, $\beta(0) = \beta^*$, in problem (1.5), (1.6) we obtain the problem of finding a solution

$$z(\cdot, \varepsilon) \in \mathbb{C}^1[a, b_0], \quad z(\tau, \cdot) \in \mathbb{C}[0, \varepsilon_0], \quad \beta(\varepsilon), \quad h(\varepsilon) \in \mathbb{C}[0, \varepsilon_0]$$

to a system of differential equations

$$\begin{aligned} dz(\tau, \varepsilon)/d\tau &= Az(\tau, \varepsilon) + f + \varepsilon Z(z(\tau, \varepsilon), h(\varepsilon), \varepsilon) \\ &\quad + \varepsilon\beta(\varepsilon)[Az(\tau, \varepsilon) + \varepsilon Z(z(\tau, \varepsilon), h(\varepsilon), \varepsilon)], \end{aligned}$$

that satisfy the boundary condition

$$\ell z(\cdot, \varepsilon) = \alpha + \varepsilon J(z(\cdot, \varepsilon), h(\varepsilon), \varepsilon) + \varepsilon\beta(\varepsilon)[\alpha + \varepsilon J(z(\cdot, \varepsilon), h(\varepsilon), \varepsilon)].$$

Next, we change the variable

$$z(\tau, \varepsilon) = z_0(\tau, c_r) + x(\tau, \varepsilon), \quad h(\varepsilon) = h_0 + \mu(\varepsilon), \quad \beta(\varepsilon) = \beta_0 + \eta(\varepsilon),$$

and arrive at the problem of finding a solution

$$x(\cdot, \varepsilon) \in \mathbb{C}^1[a, b_0], \quad x(\tau, \cdot) \in \mathbb{C}[0, \varepsilon_0], \quad \mu(\varepsilon), \quad \eta(\varepsilon) \in \mathbb{C}[0, \varepsilon_0]$$

to a system of differential equations

$$dx(\tau, \varepsilon)/d\tau = Ax(\tau, \varepsilon) + \varepsilon Y(z_0(\tau, c_r) + x(\tau, \varepsilon), h_0 + \mu(\varepsilon), \beta_0 + \eta(\varepsilon), \varepsilon), \quad (1.8)$$

that satisfy the boundary condition

$$\ell x(\cdot, \varepsilon) = \varepsilon H(z_0(\cdot, c_r) + x(\cdot, \varepsilon), h_0 + \mu(\varepsilon), \beta_0 + \eta(\varepsilon), \varepsilon). \quad (1.9)$$

Here,

$$\begin{aligned} &Y(z_0(\tau, c_r) + x(\tau, \varepsilon), h_0 + \mu(\varepsilon), \beta_0 + \eta(\varepsilon), \varepsilon) \\ &:= (1 + \varepsilon\beta(\varepsilon)) \times Z(z_0(\tau, c_r) + x(\tau, \varepsilon), h_0 + \mu(\varepsilon), \varepsilon) + \beta(\varepsilon)A(z_0(\tau, c_r) + x(\tau, \varepsilon)), \\ &H(z_0(\cdot, c_r) + x(\cdot, \varepsilon), h_0 + \mu(\varepsilon), \beta_0 + \eta(\varepsilon), \varepsilon) \\ &:= (1 + \varepsilon\beta(\varepsilon)) \times J(z_0(\cdot, c_r) + x(\cdot, \varepsilon), h_0 + \mu(\varepsilon), \varepsilon) + \alpha\beta(\varepsilon) : \mathbb{C}[a, b_0] \rightarrow \mathbb{R}^m. \end{aligned}$$

In the critical case, ($P_{Q^*} \neq 0$) provided that (1.4) the boundary-value problem (1.8), (1.9) solvable under the condition

$$\begin{aligned} &P_{Q_d^*}\{H(z_0(\cdot, c_r) + x(\cdot, \varepsilon), h_0 + \mu(\varepsilon), \eta(\varepsilon), \varepsilon) \\ &\quad - \ell K[Y(z_0(\tau, c_r) + x(\tau, \varepsilon), h_0 + \mu(\varepsilon), \beta_0 + \eta(\varepsilon), \varepsilon)](\cdot)\} = 0. \end{aligned} \quad (1.10)$$

We denote vectors

$$\check{c}_0 := \begin{bmatrix} c_r \\ \beta_0 \\ h_0 \end{bmatrix} \in \mathbb{R}^{r+q+1}, \quad \check{c}_0^* := \begin{bmatrix} c_r^* \\ \beta_0^* \\ h_0^* \end{bmatrix} \in \mathbb{R}^{r+q+1}, \quad \check{c}(\varepsilon) := \begin{bmatrix} c_r(\varepsilon) \\ \mu(\varepsilon) \\ \eta(\varepsilon) \end{bmatrix} \in \mathbb{R}^{r+q+1}.$$

Since the nonlinear function $Y(z, \beta(\epsilon), h(\epsilon), \tau, \epsilon)$ and the nonlinear vector functional $H(z(\cdot, \epsilon), h(\epsilon), \beta(\epsilon), \epsilon)$ are continuous with respect to z , β and h in a small neighborhood of the generating problem (1.7), in a small neighborhood of the initial value h_0 of the function $h(\epsilon)$ and the initial value β_0 of the function $\beta(\epsilon)$, we arrive at the following equation

$$\mathcal{F}(\check{c}_0) := P_{Q_d^*} \left\{ \alpha \beta_0 + J(z_0(\cdot, c_r), h_0, 0) - \ell K [\beta_0 A z_0(s, c_r) + Z(z_0(s, c_r), h_0, s, 0)](\cdot) \right\} = 0.$$

Thus, the necessary conditions of the existence of a solution to the nonlinear autonomous boundary-value problem (1.5), (1.6) in the case of parametric resonance is determined by the following lemma. This lemma is a generalization of the corresponding statements from [1, 3] on the case of parametric resonance and from [6] on the case of an autonomous boundary-value problem (1.5), (1.6).

Lemma 1. *Suppose that the boundary-value problem (1.5), (1.6) corresponds to the critical case ($P_{Q^*} \neq 0$) and the solvability condition of generating Noether ($m \neq n$) problem (1.7) is satisfied condition (1.4). Also, suppose that in a small neighborhood of the generating solution*

$$z_0(t, c_r^*) \in \mathbb{C}^1[a, b_0^*], \quad b_0^* = b(0), \quad h_0^* = h(0) \in \mathbb{R}^q$$

the weakly nonlinear boundary-value problem (1.5), (1.6) has a solution

$$z(t, \epsilon) : z(\cdot, \epsilon) \in \mathbb{C}^1[a, b(\epsilon)], \quad z(t, \cdot) \in \mathbb{C}[0, \epsilon_0], \quad b(\epsilon), h(\epsilon) \in \mathbb{C}[0, \epsilon_0].$$

Moreover, in a sufficiently small neighborhood of the vector h_0^ there exists an eigenfunction $h(\epsilon) \in \mathbb{C}[0, \epsilon_0]$. Then the equality*

$$\mathcal{F}(\check{c}_0^*) = 0. \tag{1.11}$$

is satisfied.

By analogy with weakly nonlinear boundary-value problems in the critical case [1, 3] and periodic boundary-value problems [20], we say that equation (1.11) is an equation for generating constants of the boundary-value problem (1.5), (1.6) in the case of parametric resonance. Assume that equation (1.11) has real root. We fix one of the solutions $\check{c}_0 \in \mathbb{R}^{r+q+1}$ of equation (1.11) and arrive at the problem of finding the solutions

$$x(\cdot, \epsilon) \in \mathbb{C}^1[a, b_0], \quad x(\tau, \cdot) \in \mathbb{C}[0, \epsilon_0], \quad \mu(\epsilon), \eta(\epsilon) \in \mathbb{C}[0, \epsilon_0]$$

of the problem (1.8), (1.9) in the neighborhood of the generating solution

$$z_0(t, c_r^*) = X_r(t) c_r^* + G[f; \alpha](t), \quad c_r^* \in \mathbb{R}^r,$$

and finding functions

$$h(\epsilon) := h_0^* + \mu(\epsilon), \quad \beta(\epsilon) = \beta_0^* + \eta(\epsilon), \quad \mu(\epsilon), \eta(\epsilon) \in C[0, \epsilon_0]$$

in the neighborhood of points h_0^* and β_0^* .

2. ON A REDUCTION OF AN AUTONOMOUS NOETHER BOUNDARY-VALUE PROBLEM TO A CRITICAL CASE OF THE FIRST ORDER

In the critical case ($P_{Q^*} \neq 0$) under condition (1.4) to construct an operator system that is used to solve the boundary-value problem (1.8), (1.9), the linearization of the solvability condition was previously applied (1.10). Wherein, in the case of simplicity of the roots [3, 12] by the equations of the generating constants (1.11)

$$P_{C_0^*(\check{c}_0^*)} P_{Q_d^*} = 0, \quad C_0 \left(\check{c}_0^* \right) = \mathcal{F}'_{\check{c}(\epsilon)} \left(\check{c}(\epsilon) \right) \Big|_{\check{c}(\epsilon) = \check{c}_0^*}$$

solution were constructed using the method of simple iterations [3, 6]. Note that the solutions of the boundary-value problem (1.5), (1.6) in the case of parametric resonance and the problem (1.1), (1.2), in the case of their existence, as well as the problems themselves, coincide for $h(\epsilon) = 0$. To find a solution

$$\begin{aligned} x(\tau, \epsilon) &= X_r(\tau) c_r(\epsilon) + x^{(1)}(\tau, \epsilon), \quad h(\epsilon) = h_0^* + \mu(\epsilon), \\ x^{(1)}(\tau, \epsilon) &= \epsilon \cdot G \left[Y(z_0(s, c_r^*) + x(s, \epsilon), h_0^* + \mu(\epsilon), \beta_0^* + \eta(\epsilon), \epsilon); \right. \\ &\quad \left. H(z_0(\cdot, c_r^*) + x(\cdot, \epsilon), h_0^* + \mu(\epsilon), \beta_0^* + \eta(\epsilon), \epsilon) \right] (\tau) \end{aligned}$$

of the boundary value-problem (1.8), (1.9) the Newton–Kantorovich method can be used [8, 13].

Lemma 2. *Suppose that the following conditions are fulfilled for the equation*

$$\mathcal{F}(\check{c}(\epsilon)) = 0. \quad (2.1)$$

1. *The nonlinear vector function $\mathcal{F}(\check{c}(\epsilon)) : \mathbb{R}^{r+q+1} \rightarrow \mathbb{R}^d$ is twice continuously differentiable in a neighborhood of zero and has the root $\check{c}(\epsilon)$.*
2. *In a neighborhood of zero, the following inequalities hold:*

$$\left\| J_j^+ \right\| \leq \sigma_1(j), \quad \left\| d^2 \mathcal{F}(\xi_j; \check{c}(\epsilon) - \check{c}_j(\epsilon)) \right\| \leq \sigma_2(j) \cdot \|\check{c}(\epsilon) - \check{c}_j(\epsilon)\|.$$

3. *There is a constant*

$$\theta := \sup_{j \in N} \left\{ \frac{\sigma_1(j) \sigma_2(j)}{2} \right\}.$$

Then, under the conditions

$$P_{J_j^*} = 0, \quad J_j := \mathcal{F}'(\check{c}_j(\epsilon)) \in \mathbb{R}^{d \times (r+q+1)}, \quad \theta \cdot |\check{c}(\epsilon)| < 1 \quad (2.2)$$

or finding a solution $\check{c}(\epsilon)$ of the equation (2.1) an iterative scheme can be used

$$\check{c}_{j+1}(\epsilon) = \check{c}_j(\epsilon) - J_j^+ \mathcal{F}(\check{c}_j(\epsilon)), \quad j = 0, 1, 2, \dots \quad (2.3)$$

In this case, the rate of convergence of the sequence $\{\check{c}_j(\epsilon)\}$ to the solution $\check{c}(\epsilon)$ of the equation (2.1) will be quadratic. Here, $P_{J_j^*}: \mathbb{R}^d \rightarrow \mathbb{N}(J_j^*)$ is a matrix-orthoprojector J_k^* .

Note that the condition (2.2) is equivalent to the requirement of completeness of the rank of the matrix J_k and is possible only in the case $d \leq p + q + 1$.

Theorem 1. Suppose that for the generating problem (1.7) corresponds to the critical case $P_{Q^*} \neq 0$ and the condition of its solvability is satisfied (1.4). We also assume that the conditions of Lemma 2 are satisfied for the equation (2.1). Then, for each root $c_r^* \in \mathbb{R}^r$, $h_0^* \in \mathbb{R}^q$ of the equation for the generating constants (1.11) in the neighborhood generating solution $z_0(\epsilon, c_r^*)$, and in the neighborhood of the points h_0^* and β_0^* to find at least one solution

$$z(\cdot, \epsilon) \in \mathbb{C}^1[a, b(\epsilon)], \quad z(\tau, \cdot) \in \mathbb{C}[0, \epsilon_0], \quad h(\epsilon), \beta(\epsilon) \in \mathbb{C}[0, \epsilon_0]$$

of the problem (1.5), (1.6) in the case of parametric resonance, an iterative scheme can be used

$$\begin{aligned} z_{k+1}(\tau, \epsilon) &= z_0(\tau, c_r^*) + x_{k+1}(\tau, \epsilon), \quad \beta_{k+1}(\epsilon) = \beta_0^* + \eta_{k+1}(\epsilon), \quad h_{k+1}(\epsilon) = h_0^* + \mu_{k+1}(\epsilon), \\ x_{k+1}(\tau, \epsilon) &= X_r(\tau) c_{r_{k+1}}(\epsilon) + x_{k+1}^{(1)}(\tau, \epsilon), \quad \mathcal{F}(\check{c}_{k+1}(\epsilon)) = 0, \quad k = 0, 1, 2, \dots, \\ x_{k+1}^{(1)}(\tau, \epsilon) &= \epsilon \cdot G \left[Y(z_0(s, c_r^*) + x_k(s, \epsilon), h_0^* + \mu_k(\epsilon), \beta_0^* + \eta_k(\epsilon), \epsilon); \right. \\ &\quad \left. H(z_0(\cdot, c_r^*) + x_k(\cdot, \epsilon), h_0^* + \mu_k(\epsilon), \beta_0^* + \eta_k(\epsilon), \epsilon) \right](\tau). \end{aligned} \quad (2.4)$$

Returning to the nonlinear autonomous boundary-value problem for the system of ordinary differential equations (1.1), (1.2), suppose that the equation for generating constants for an autonomous Noether boundary-value problem for a system of ordinary differential equations (1.1), (1.2) has multiples roots [1, 3, 17], while equations for generating constants (1.11) for the boundary-value problem (1.5), (1.6) in the case of parametric resonance has simple roots:

$$P_{C_0^*(c_0^*)} P_{Q_d^*} = 0,$$

among which $\check{c}_0^* \in \mathbb{R}^{r+q+1}$, $h_0^* = 0$. Suppose also that at each step of the iterative scheme (2.4) among the roots of the equation $\mathcal{F}(\check{c}_k(\epsilon)) = 0$ is a root for which $\mu_k(\epsilon) = 0$. In this case, the iteration scheme (2.4) can be used to find solutions an autonomous boundary-value problem (1.1), (1.2). Therefore, solutions to the boundary-value problem (1.5), (1.6) in the case of parametric resonance and the problem (1.1), (1.2), if they exist, coincide for $h(\epsilon) = 0$.

Corollary 1. Assume that, to find at least one solution

$$z(\cdot, \epsilon) \in \mathbb{C}^1[a, b(\epsilon)], \quad z(\tau, \cdot) \in \mathbb{C}[0, \epsilon_0], \quad h(\epsilon), \beta(\epsilon) \in \mathbb{C}[0, \epsilon_0]$$

of the problem (1.5), (1.6) in the case of parametric resonance, an iterative scheme can be used (2.4). Assume also that, among the roots of the equation for generating constants (1.11) for the boundary-value problem (1.5), (1.6) in the case of parametric resonance is the simple root

$$\check{c}_0^* \in \mathbb{R}^{r+q+1}, \quad h_0^* = 0.$$

Suppose also that, at each step of the iterative scheme (2.4) among the roots of the equation

$$\mathcal{F}(\check{c}_k(\epsilon)) = 0, \quad k = 0, 1, 2, \dots$$

there is a root for which $\mu_k(\epsilon) = 0$. The iterative scheme (2.4) can be used to know the solutions autonomous boundary-value problem (1.1), (1.2) in this case.

If the equation for generating constants for an autonomous boundary-value problem for a system of ordinary differential equations (1.1), (1.2) has multiple roots [1, 3, 17], for this equation for generating constants (1.11) for the boundary-value problem (1.5), (1.6) in the case of parametric resonance has simple roots, among which $\check{c}_0^* \in \mathbb{R}^{r+q+1}$, $h_0^* = 0$. The conditions of the proved corollary are also satisfied, similarly [1, 3, 9, 15], then we will say that an autonomous Noether boundary-value problem (1.1), (1.2) is reduced to a critical case of the first order.

3. AUTONOMOUS PERIODIC PROBLEM FOR THE DUFFING EQUATION

The conditions of the proven corollary are satisfied in the case of an autonomous periodic problem for the Duffing equation [17]

$$y'' + y = \epsilon y^3. \quad (3.1)$$

The generating periodic problem for the Duffing equation (3.1) is solvable and, with an appropriate fixation of the origin of the independent variable, has a general solution

$$z_0(t, c_r) = X_r(t) c_r, \quad X_r(t) = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}, \quad c_r \in \mathbb{R}^1.$$

Equation for generating amplitudes in the case of an autonomous periodic problem for the Duffing equation (3.1)

$$\frac{\pi c_r}{4} \begin{pmatrix} 3c_r^2 - 8\beta_0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

obviously has only multiple roots:

$$B_0 = \frac{\pi c_r^*}{2} \begin{pmatrix} 3c_r^* & -4 \\ 0 & 0 \end{pmatrix}, \quad \beta^* = \frac{3c_r^*}{8}.$$

Along with the autonomous periodic problem for the equation (3.1) consider the problem of finding periodic solutions to the equation Duffing-type with parametric perturbation

$$y'' + y = \epsilon y^3 + \epsilon h(\epsilon) (y' + y'^3). \quad (3.2)$$

Note that, in contrast to the article [6] the Duffing-type equation (3.2) with parametric perturbation is autonomous. The equation (3.2) is reduced to the form (1.5) for

$$z(t, \varepsilon) = \begin{bmatrix} z^{(a)}(t, \varepsilon) \\ z^{(b)}(t, \varepsilon) \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$Z(z(t, \varepsilon), h(\varepsilon), t, \varepsilon) = \begin{bmatrix} 0 \\ h(\varepsilon)z^{(b)}(t, \varepsilon) + h(\varepsilon) \left(z^{(b)}(t, \varepsilon) \right)^3 + \left(z^{(a)}(t, \varepsilon) \right)^3 \end{bmatrix}.$$

The generating periodic problem for a Duffing-type equation (3.2) with parametric perturbation is also solvable with a corresponding fixation of the origin of the independent variable and has a general solution [17]

$$z_0(t, c_r) = X_r(t) c_r, \quad X_r(t) = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}, \quad c_r \in \mathbb{R}^1.$$

The equation for generating constants (1.11), in the case of a periodic problem for an equation of Duffing-type (3.2) with parametric perturbation

$$\pi c_r (3 c_r^2 - 8 \beta_0) = 0, \quad \pi c_r h_0 (3 c_r^2 + 4) = 0,$$

has a simple root [1, 3, 6, 12]

$$c_r^* = \frac{1}{10}, \quad \beta_0^* = \frac{3}{800}, \quad h_0^* = 0, \quad C_0(\check{c}_0^*) = \frac{\pi}{4000} \begin{bmatrix} 0 & -800 & 0 \\ 403 & 0 & 403 \end{bmatrix}.$$

At the first step of the iterative scheme (2.4) due to the completeness of the rank of the matrix J_0 the condition (2.2) is satisfied. Assuming that $\varepsilon := 0, 1$. The first approximation to the solution of the periodic problem for the equation (3.2)

$$y_1(\tau, \varepsilon) = c_{r_1}(\varepsilon) \cos \tau + y_1^{(1)}(\tau, \varepsilon), \quad h_1(\varepsilon) = h_0^* + \mu_1(\varepsilon)$$

define functions

$$y_1^{(1)}(\tau, \varepsilon) = \frac{\varepsilon}{32\,000} (\cos \tau - \cos 3\tau),$$

as well as

$$c_{r_1}(\varepsilon) \approx -\frac{8413}{25\,079\,771\,422}, \quad \beta_1(\varepsilon) \approx \frac{7\,140\,587}{1\,903\,038\,646}, \quad h_1(\varepsilon) = \mu_1(\varepsilon) = 0.$$

At the second step of the iterative scheme (2.4) due to the completeness of the rank of the matrix J_1 the condition (2.2) is also satisfied, wherein

$$y_2(\tau, \varepsilon) = c_{r_2}(\varepsilon) \cos \tau + y_2^{(1)}(\tau, \varepsilon), \quad h_2(\varepsilon) = h_0^* + \mu_2(\varepsilon);$$

here

$$y_2^{(1)}(\tau, \varepsilon) \approx \frac{155\,839 \cos \tau}{49\,833\,136\,807} - \frac{1\,12138 \cos 3\tau}{35\,857\,607\,281} + \frac{67 \cos 5\tau}{685\,548\,603\,382} - \frac{\cos 7\tau}{654\,850\,200\,560\,485} - \frac{\cos 9\tau}{104\,778\,954\,869\,870\,365\,088},$$

as well as

$$c_{r_2}(\varepsilon) \approx -\frac{34\,312}{102\,284\,783\,173}, \quad \beta_2(\varepsilon) \approx \frac{31\,802\,653}{8\,475\,728\,456}, \quad h_2(\varepsilon) = \mu_2(\varepsilon) = 0.$$

At the third step of the iterative scheme (2.4) due to the completeness of the rank of the matrix J_2 the condition (2.2) is also satisfied, wherein

$$y_3(\tau, \varepsilon) = c_{r_3}(\varepsilon) \cos \tau + y_3^{(1)}(\tau, \varepsilon), \quad h_3(\varepsilon) = h_0^* + \mu_3(\varepsilon);$$

here

$$y_3^{(1)}(\tau, \varepsilon) \approx \frac{149\,943 \cos \tau}{47\,947\,755\,026} - \frac{106\,169 \cos 3\tau}{33\,948\,940\,663} + \frac{447 \cos 5\tau}{4\,570\,493\,090\,387} \\ + \frac{\cos 7\tau}{327\,064\,274\,993\,900} + \frac{\cos 9\tau}{12\,308\,773\,007\,388\,528\,938},$$

as well as

$$c_{r_3}(\varepsilon) \approx -\frac{39\,176}{116\,784\,468\,121}, \quad \beta_3(\varepsilon) \approx \frac{1\,318\,533}{351\,402\,371}, \quad h_3(\varepsilon) = \mu_3(\varepsilon) = 0.$$

The found zero-order and first three approximations to the periodic solution of the Duffing-type equation (3.2) with parametric perturbation and the function $h(\varepsilon)$ are characterized by the discrepancies

$$\Delta_k(\varepsilon) = \left\| y_k''(t, \varepsilon) + y_k(t, \varepsilon) - \varepsilon y_k^3(t, \varepsilon) - \varepsilon h_k(\varepsilon) y_k'(t, \varepsilon) - \varepsilon h_k(\varepsilon) y_k^3(t, \varepsilon) \right\|_{\mathbb{C}[0; 2\pi(1+\varepsilon\beta_k(\varepsilon))]}, \quad k = 0, 1, 2, 3.$$

In particular, for $\varepsilon = 0, 1$ we get

$$\Delta_0(0, 1) \approx 0,0000\,25, \quad \Delta_1(0, 1) \approx 1,99\,328 \times 10^{-8}, \\ \Delta_2(0, 1) \approx 1,72\,064 \times 10^{-12}, \quad \Delta_3(0, 1) \approx 1,03\,067 \times 10^{-16}.$$

The found zero-order and first three approximations to the periodic solution of the Duffing equation (3.1) characterize the discrepancies

$$\delta_k(\varepsilon) = \left\| y_k''(t, \varepsilon) + y_k(t, \varepsilon) - \varepsilon y_k^3(t, \varepsilon) \right\|_{\mathbb{C}[0; 2\pi(1+\varepsilon\beta_k(\varepsilon))]}, \quad k = 0, 1, 2, 3.$$

In particular, for $\varepsilon = 0, 1$ we get

$$\delta_0(0, 1) \approx 0,000\,025\,061, \quad \delta_1(0, 1) \approx 1,99\,328 \times 10^{-8}, \\ \delta_2(0, 1) \approx 1,72\,065 \times 10^{-12}, \quad \delta_3(0, 1) \approx 1,249 \times 10^{-16}.$$

We now compare the obtained zero-order and first three approximations to the periodic solution of the Duffing-type equation (3.1) with the corresponding zero-order

$$y_{0p}(\tau, c_r^*) = \frac{\cos \tau}{10}, \quad \beta_0^* = \frac{3}{800}, \quad h_0^* = 0,$$

and first three approximations obtained by the Poincare method used to find the periodic solution of the same Duffing-type equation (3.1)

$$\begin{aligned} y_{1p}(\tau, \varepsilon) &= y_0(\tau, c_r^*) + \varepsilon w_1(\tau), \quad \beta_{1p}(\varepsilon) = \frac{3}{800} + \frac{57\varepsilon}{2\,560\,000}, \\ y_{2p}(\tau, \varepsilon) &= y_0(\tau, c_r^*) + \varepsilon w_1(\tau) + \varepsilon^2 w_2(\tau), \quad \beta_{2p}(\varepsilon) = \frac{3}{800} + \frac{57\varepsilon}{2\,560\,000} + \frac{63\varepsilon^2}{409\,600\,000}, \\ y_{3p}(\tau, \varepsilon) &= y_0(\tau, c_r^*) + \varepsilon w_1(\tau) + \varepsilon^2 w_2(\tau) + \varepsilon^3 w_3(\tau), \\ \beta_{3p}(\varepsilon) &= \frac{3}{800} + \frac{57\varepsilon}{2\,560\,000} + \frac{63\varepsilon^2}{409\,600\,000} + \frac{6\,069\varepsilon^3}{5\,242\,880\,000\,000}; \end{aligned}$$

here

$$\begin{aligned} w_1(\tau) &= \frac{1}{32\,000} (\cos \tau - \cos 3\tau), \\ w_2(\tau) &= \frac{1}{102\,400\,000} (23 \cos \tau - 24 \cos 3\tau + \cos 5\tau), \\ w_3(\tau) &= \frac{1}{327\,680\,000\,000} (547 \cos \tau - 594 \cos 3\tau + 48 \cos 5\tau - \cos 7\tau). \end{aligned}$$

Note that the zero-order and first three approximations obtained by the Poincare method for the periodic solution of the Duffing-type equation (3.1) are characterized by the discrepancies

$$\delta_{kp}(\varepsilon) = \left\| y_{kp}''(t, \varepsilon) + y_{kp}(t, \varepsilon) - \varepsilon y_{kp}^3(t, \varepsilon) \right\|_{C[0; 2\pi(1+\varepsilon\beta_{kp}(\varepsilon))]}, \quad k = 0, 1, 2, 3.$$

In particular, for $\varepsilon = 0, 1$ we get

$$\begin{aligned} \delta_{0p}(0, 1) &\approx 0,000\,025\,061, & \delta_{1p}(0, 1) &\approx 2,01\,363 \times 10^{-8}, \\ \delta_{2p}(0, 1) &\approx 1,63\,890 \times 10^{-11}, & \delta_{3p}(0, 1) &\approx 1,45\,439 \times 10^{-14}. \end{aligned}$$

Thus, the zero-order and first three approximations obtained according to the iterative scheme (2.4) aimed at finding the periodic solution of the Duffing-type equation (3.1) are much more exact than the first three approximations given by the Poincare method for the periodic solution of the same Duffing-type equation.

By analogy [2, 5, 7], the proposed scheme of investigation of the nonlinear autonomous boundary-value problems in the case of parametric resonance can be generalized to the matrix boundary-value problems.

The proposed scheme for studying nonlinear autonomous boundary-value problems in the case of parametric resonance can be used in electronics [18], geodesy [16], plasma theory [19], nonlinear optics, mechanics [4] and machine-tool industry [14], along with finding solutions, it is necessary to calculate the eigenfunction of the corresponding differential equation, and similarly [10, 11] can be transferred to nonlinear Noether boundary-value problems unsolved with respect to the derivative.

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BEST PROXIMITY POINTS OF MULTIVALUED GERAGHTY CONTRACTIONS

PRADIP DEBNATH

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Abstract. There are situations when we have to resort to the approximate optimal solution of equations of the type $g(t) = t$ when g is not a self-map, because exact solution of that equation does not exist. The existence of such optimal solutions are ensured by best proximity point theorems. In this paper, we define multivalued Geraghty contraction (MVG) in a complete metric space and establish the corresponding best proximity point (BPP) result. Our result extends the famous result due to Geraghty on fixed points.

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1. PRELIMINARIES

In 1969, Nadler [13] presented some very interesting fixed point results for multivalued maps by considering the distance between two arbitrary sets.

Let $(\mathcal{W}, \eta)$ be a complete metric space (MS) and let $\Lambda(\mathcal{W})$ be the collection of all nonempty closed and bounded subsets of $\mathcal{W}$. Then for $\mathcal{X}, \mathcal{Y} \in \Lambda(\mathcal{W})$, define the map $\mathcal{H} : \Lambda(\mathcal{W}) \times \Lambda(\mathcal{W}) \rightarrow [0, \infty)$ by

$$\mathcal{H}(\mathcal{X}, \mathcal{Y}) = \max \left\{ \sup_{\xi \in \mathcal{Y}} \Delta(\xi, \mathcal{X}), \sup_{\delta \in \mathcal{X}} \Delta(\delta, \mathcal{Y}) \right\},$$

where $\Delta(\delta, \mathcal{Y}) = \inf_{\xi \in \mathcal{Y}} \eta(\delta, \xi)$. Consequently, $(\Lambda(\mathcal{W}), \mathcal{H})$ is an MS induced by η .

Let $\mathcal{X}, \mathcal{Y}$ be two non-empty subsets of the MS $(\mathcal{W}, \eta)$. The following notations will be used throughout:

$$\mathcal{X}_{\mathcal{Y}} = \{\theta \in \mathcal{X} : \eta(\theta, \xi) = \eta(\mathcal{X}, \mathcal{Y}) \text{ for some } \xi \in \mathcal{Y}\},$$

$$\mathcal{Y}_{\mathcal{X}} = \{\xi \in \mathcal{Y} : \eta(\theta, \xi) = \eta(\mathcal{X}, \mathcal{Y}) \text{ for some } \theta \in \mathcal{X}\},$$

where $\eta(\mathcal{X}, \mathcal{Y}) = \inf\{\eta(\theta, \xi) : \theta \in \mathcal{X}, \xi \in \mathcal{Y}\}$.

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For $X, \mathcal{Y} \in \Lambda(\mathcal{W})$, we have

$$\eta(X, \mathcal{Y}) \leq H(X, \mathcal{Y}).$$

$\theta \in \mathcal{W}$ is called a BPP of the multivalued map $\Phi : \mathcal{W} \rightarrow \Lambda(\mathcal{W})$ if $\Delta(\theta, \Phi\theta) = \eta(X, \mathcal{Y})$. $v \in \mathcal{W}$ is said to be a fixed point of Φ if $v \in \Phi v$.

- Remark 1.* (1) In the MS $(\Lambda(\mathcal{W}), \mathcal{H})$, $v \in \mathcal{W}$ is a fixed point of Φ if and only if $\Delta(v, \Phi v) = 0$.
 (2) If $X, \mathcal{Y}$ are two closed sets with $X \cap \mathcal{Y} \neq \emptyset$, we obtain $\eta(X, \mathcal{Y}) = 0$. Then a fixed point and a BPP coincide.
 (3) The function Δ is continuous, for if $\theta_n \rightarrow \theta$ as $n \rightarrow \infty$, then $\Delta(\theta_n, X) \rightarrow \Delta(\theta, X)$ as $n \rightarrow \infty$ for any $X \subseteq \mathcal{W}$.

The next two lemmas are important for the sequel.

Lemma 1 ([3, 5]). *Let $(\mathcal{W}, \eta)$ be a MS and $X, \mathcal{Y}, Z \in \Lambda(\mathcal{W})$. Then*

- (1) $\Delta(\theta, \mathcal{Y}) \leq \eta(\theta, \xi)$ for any $\xi \in \mathcal{Y}$ and $\theta \in \mathcal{W}$;
- (2) $\Delta(\theta, \mathcal{Y}) \leq \mathcal{H}(X, \mathcal{Y})$ for any $\theta \in X$.

Lemma 2 ([13]). *Let $X, \mathcal{Y} \in \Lambda(\mathcal{W})$ and let $\theta \in X$, then for any $s > 0$, there exists $\xi \in \mathcal{Y}$ satisfying*

$$\eta(\theta, \xi) \leq \mathcal{H}(X, \mathcal{Y}) + s.$$

A point $\xi \in \mathcal{Y}$ may not exist satisfying

$$\eta(\theta, \xi) \leq \mathcal{H}(X, \mathcal{Y}).$$

If $\mathcal{Y}$ is compact, then we obtain a point ξ satisfying $\eta(\theta, \xi) \leq \mathcal{H}(X, \mathcal{Y})$.

The concept of $\mathcal{H}$ -continuity plays a significant role as discussed next.

Definition 1 ([7]). Let $(\mathcal{W}, \eta)$ be a MS. A multivalued map $\Phi : \mathcal{W} \rightarrow \Lambda(\mathcal{W})$ is said to be $\mathcal{H}$ -continuous at a point θ_0 , if for each sequence $\{\theta_n\} \subset \mathcal{W}$, such that $\lim_{n \rightarrow \infty} \eta(\theta_n, \theta_0) = 0$, we have $\lim_{n \rightarrow \infty} \mathcal{H}(\Phi\theta_n, \Phi\theta_0) = 0$ (i.e., if $\theta_n \rightarrow \theta_0$, then $\Phi\theta_n \rightarrow \Phi\theta_0$ as $n \rightarrow \infty$).

Definition 2 ([13]). Let $\Phi : \mathcal{W} \rightarrow \Lambda(\mathcal{W})$ be a multivalued map. Φ is said to be a multivalued contraction if $\mathcal{H}(\Phi\theta, \Phi\xi) \leq \kappa\eta(\theta, \xi)$ for all $\theta, \xi \in \mathcal{W}$, where $\kappa \in [0, 1)$.

- Remark 2.* (1) When Φ is continuous on every point of X , it is said to be $\mathcal{H}$ -continuous on X .
 (2) If Φ is a multivalued contraction, then it is $\mathcal{H}$ -continuous.

Sankar Raj [14] put forward the notion of P -property. The idea of weak P property was introduced by Zhang et al. [19] which enhanced the results of Caballero et al. [4] on Geraghty-contractions.

Definition 3 ([14]). Let $(\mathcal{W}, \eta)$ be a MS and $\mathcal{X}, \mathcal{Y}$ be two non-empty subsets of $\mathcal{W}$ such that $\mathcal{X}_{\mathcal{Y}} \neq \emptyset$. The pair $(\mathcal{X}, \mathcal{Y})$ satisfies the P -property if and only if $\eta(\theta_1, \xi_1) = \eta(\mathcal{X}, \mathcal{Y}) = \eta(\theta_2, \xi_2)$ implies $\eta(\theta_1, \theta_2) = \eta(\xi_1, \xi_2)$, where $\theta_1, \theta_2 \in \mathcal{X}_{\mathcal{Y}}$ and $\xi_1, \xi_2 \in \mathcal{Y}_{\mathcal{X}}$.

Definition 4 ([19]). Let $(\mathcal{W}, \eta)$ be a MS and $\mathcal{X}, \mathcal{Y}$ be two non-empty subsets of $\mathcal{W}$ such that $\mathcal{X}_{\mathcal{Y}} \neq \emptyset$. The pair $(\mathcal{A}, \mathcal{Y})$ satisfies the weak P -property if and only if $\eta(\theta_1, \theta_1) = \eta(\mathcal{X}, \mathcal{Y}) = \eta(\theta_2, \xi_2)$ implies $\eta(\theta_1, \theta_2) \leq \eta(\xi_1, \xi_2)$, where $\theta_1, \theta_2 \in \mathcal{X}_{\mathcal{Y}}$ and $\xi_1, \xi_2 \in \mathcal{Y}_{\mathcal{A}}$.

Srivastava et al. [15, 16] presented Krasnosel'skii type hybrid fixed point theorems which significantly improved the study of fractional integral equations. Recently, Debnath and Srivastava [9] studied common BPPs for multivalued contractive pairs of mappings. Debnath and Srivastava [10] also proved new extensions of Kannan's and Reich's theorems. Another Kannan-type contraction for multivalued asymptotic regular maps were presented by Debnath et al. [8]. Further, a very significant application of fixed points of $F(\psi, \phi)$ -contractions to fractional differential equations was recently provided by Srivastava et al. [18]. Srivastava et al. [17] studied implicit functional differential inclusions of arbitrary fractional order.

Geraghty [11] used a particular family of functions that generalized the Banach's fixed point theorem. Let $\mathcal{G}$ be the collection of mappings $g : [0, \infty) \rightarrow [0, 1)$ satisfying the condition: $g(t_n) \rightarrow 1$ implies $t_n \rightarrow 0$. An example of such a map is $g(t) = (1+t)^{-1}$ for all $t > 0$ and $g(0) \in [0, 1)$.

Many authors have studied different generalized Geraghty-type contractions [1, 2, 12], but the exact multivalued analogue of Geraghty contraction [11] has not been established yet. Recently, Debnath [6] presented a new technique of studying BPPs for multivalued F -contractions, which generalized and extended several existing results in literature. In the current paper, we observe that this new technique is also helpful for establishing the multivalued analogue of the Geraghty contraction.

2. BEST PROXIMITY POINT OF MVGC

In this section, we extend the famous result on fixed points by Geraghty to its multivalued analogue in terms of BPPs. However, additional assumptions such as weak P -property and compactness of the images of the multivalued map under consideration have been made.

First, we define a MVGC.

Definition 5. Let $(\mathcal{W}, \eta)$ be a MS and $\mathcal{X}, \mathcal{Y}$ be two non-empty subsets of $\mathcal{W}$. The mapping $\Phi : \mathcal{X} \rightarrow \Lambda(\mathcal{Y})$ is said to be a multivalued Geraghty contraction (MVGC) if there exists $g \in \mathcal{G}$ such that

$$\mathcal{H}(\Phi\theta, \Phi\xi) \leq g(\eta(\theta, \xi))\eta(\theta, \xi)$$

for all $\theta, \xi \in \mathcal{X}$.

Remark 3. Since $g : [0, \infty) \rightarrow [0, 1)$, it is easy to see that $\mathcal{H}(\Phi\theta, \Phi\xi) < \eta(\theta, \xi)$ for all $\theta, \xi \in \mathcal{X}$ with $\theta \neq \xi$. Therefore, every MVGC is a multivalued contractive mapping and hence $\mathcal{H}$ -continuous.

Theorem 1. Let $(\mathcal{W}, \eta)$ be a complete MS and $\mathcal{X}, \mathcal{Y}$ be two non-empty closed subsets of $\mathcal{W}$ such that $\mathcal{X}_{\mathcal{Y}} \neq \emptyset$ and that the pair $(\mathcal{X}, \mathcal{Y})$ satisfies the weak P -property. Let $\Phi : \mathcal{X} \rightarrow \Lambda(\mathcal{Y})$ be a MVGC such that $\Phi\theta$ is compact for each $\theta \in \mathcal{X}$ and $\Phi\theta \subseteq \mathcal{Y}_{\mathcal{X}}$ for all $\theta \in \mathcal{X}_{\mathcal{Y}}$. Then Φ has a BPP.

Proof. Fix $\theta_0 \in \mathcal{X}_{\mathcal{Y}}$ and choose $\xi_0 \in \Phi\theta_0 \subseteq \mathcal{Y}_{\mathcal{X}}$. By the definition of $\mathcal{Y}_{\mathcal{X}}$, we can select $\theta_1 \in \mathcal{X}_{\mathcal{Y}}$ such that

$$\eta(\theta_1, \xi_0) = \eta(\mathcal{X}, \mathcal{Y}). \quad (2.1)$$

If $\xi_0 \in \Phi\theta_1$, then

$$\eta(\mathcal{X}, \mathcal{Y}) \leq \Delta(\theta_1, \Phi\theta_1) \leq \eta(\theta_1, \xi_0) = \eta(\mathcal{X}, \mathcal{Y}).$$

Thus $\eta(\mathcal{X}, \mathcal{Y}) = \Delta(\theta_1, \Phi\theta_1)$, i.e., θ_1 is a BPP of Φ . Therefore, assume that $\xi_0 \notin \Phi\theta_1$.

Since $\Phi\theta_1$ is compact, by Lemma 2, there exists $\xi_1 \in \Phi\theta_1$ such that

$$0 < \eta(\xi_0, \xi_1) \leq \mathcal{H}(\Phi\theta_0, \Phi\theta_1) \leq g(\eta(\theta_0, \theta_1))\eta(\theta_0, \theta_1). \quad (2.2)$$

Since $\xi_1 \in \Phi\theta_1 \subseteq \mathcal{Y}_{\mathcal{X}}$, there exists $\theta_2 \in \mathcal{X}_{\mathcal{Y}}$ such that

$$\eta(\theta_2, \xi_1) = \eta(\mathcal{X}, \mathcal{Y}). \quad (2.3)$$

Using (2.1), (2.3) and the weak P -property, we have that

$$\eta(\theta_1, \theta_2) \leq \eta(\xi_0, \xi_1). \quad (2.4)$$

(2.2) and (2.4) imply that

$$\eta(\theta_1, \theta_2) \leq \eta(\xi_0, \xi_1) \leq \mathcal{H}(\Phi\theta_0, \Phi\theta_1) \leq g(\eta(\theta_0, \theta_1))\eta(\theta_0, \theta_1). \quad (2.5)$$

If $\xi_1 \in \Phi\theta_2$, then

$$\eta(\mathcal{A}, \mathcal{B}) \leq \Delta(\theta_2, \Phi\theta_2) \leq \eta(\theta_2, \xi_1) = \eta(\mathcal{A}, \mathcal{B}),$$

i.e., $\eta(\mathcal{A}, \mathcal{B}) = \Delta(\theta_2, \Phi\theta_2)$, and hence, θ_2 is a BPP of Φ . So, assume that $\xi_1 \notin \Phi\theta_2$.

Since $\Phi\theta_2$ is compact, by Lemma 2, there exists $\xi_2 \in \Phi\theta_2$ such that

$$0 < \eta(\xi_1, \xi_2) \leq \mathcal{H}(\Phi\theta_1, \Phi\theta_2) \leq g(\eta(\theta_1, \theta_2))\eta(\theta_1, \theta_2). \quad (2.6)$$

Since $\xi_2 \in \Phi\theta_2 \subseteq \mathcal{Y}_{\mathcal{X}}$, there exists $\theta_3 \in \mathcal{X}_{\mathcal{Y}}$ such that

$$\eta(\theta_3, \xi_2) = \eta(\mathcal{X}, \mathcal{Y}). \quad (2.7)$$

From (2.3) and (2.7) and using weak P -property, we have that

$$\eta(\theta_2, \theta_3) \leq \eta(\xi_1, \xi_2). \quad (2.8)$$

From (2.6) and (2.8), we have

$$\eta(\theta_2, \theta_3) \leq \eta(\xi_1, \xi_2) \leq \mathcal{H}(\Phi\theta_1, \Phi\theta_2) \leq g(\eta(\theta_1, \theta_2))\eta(\theta_1, \theta_2). \quad (2.9)$$

Continuing in this manner, we obtain two sequences $\{\theta_n\}$ and $\{\xi_n\}$ in $\mathcal{X}_{\mathcal{Y}}$ and $\mathcal{Y}_{\mathcal{X}}$ respectively, satisfying

- (A) $\xi_n \in \Phi\theta_n \subseteq \mathcal{Y}_{\mathcal{X}}$,
- (B) $\eta(\theta_{n+1}, \xi_n) = \eta(\mathcal{X}, \mathcal{Y})$,
- (C) $\eta(\theta_n, \theta_{n+1}) \leq \eta(\xi_{n-1}, \xi_n) \leq g(\eta(\theta_{n-1}, \theta_n))\eta(\theta_{n-1}, \theta_n)$.

for each $n = 0, 1, 2, \dots$

From (C) we have that

$$\eta(\theta_n, \theta_{n+1}) < \eta(\theta_{n-1}, \theta_n) \text{ for all } n \in \mathbb{N}. \quad (2.10)$$

If $\eta(\theta_{n_0}, \theta_{n_0+1}) = 0$ for some $n_0 \in \mathbb{N}$, then $\eta(\theta_{n_0+1}, \theta_{n_0+2}) < \eta(\theta_{n_0}, \theta_{n_0+1}) = 0$, which is a contradiction. Thus, $\eta(\theta_n, \theta_{n+1}) > 0$ for all $n \in \mathbb{N}$.

Hence, $\{\eta(\theta_n, \theta_{n+1})\}$ is a decreasing sequence of positive real numbers, and therefore, there exists $r \geq 0$ such that $\lim_{n \rightarrow \infty} \eta(\theta_n, \theta_{n+1}) = 0$.

We claim that $r = 0$.

If $r > 0$, then from (C) we have that

$$0 < \frac{\eta(\theta_n, \theta_{n+1})}{\eta(\theta_{n-1}, \theta_n)} \leq g(\eta(\theta_{n-1}, \theta_n)) < 1 \text{ for all } n \in \mathbb{N}. \quad (2.11)$$

Taking limit in (2.11) as $n \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} \frac{\eta(\theta_n, \theta_{n+1})}{\eta(\theta_{n-1}, \theta_n)} = \frac{r}{r} = 1 \leq \lim_{n \rightarrow \infty} g(\eta(\theta_{n-1}, \theta_n)) \leq 1. \quad (2.12)$$

Therefore, $\lim_{n \rightarrow \infty} g(\eta(\theta_{n-1}, \theta_n)) = 1$.

Since $g \in$, we have $\lim_{n \rightarrow \infty} g(\eta(\theta_{n-1}, \theta_n)) = \lim_{n \rightarrow \infty} g(\eta(\theta_n, \theta_{n+1})) = r = 0$.

Next, we prove that $\{\theta_n\}$ is a Cauchy sequence in $\mathcal{X}_{\mathcal{Y}}$.

From (C), for fixed $p, q \in \mathbb{N}$, we have that

$$\eta(\theta_{p+1}, \xi_p) = \eta(\mathcal{X}, \mathcal{Y})$$

and

$$\eta(\theta_{q+1}, \xi_q) = \eta(\mathcal{X}, \mathcal{Y}).$$

By the weak P -property of $(\mathcal{X}, \mathcal{Y})$, we have

$$\eta(\theta_{p+1}, \theta_{q+1}) \leq \eta(\xi_p, \xi_q).$$

Let $\{\theta_n\}$ be not Cauchy and that $\limsup_{m, n \rightarrow \infty} \eta(\theta_n, \theta_m) > 0$. Then we have

$$\begin{aligned} \eta(\theta_n, \theta_m) &\leq \eta(\theta_n, \theta_{n+1}) + \eta(\theta_{n+1}, \theta_{m+1}) + \eta(\theta_{m+1}, \theta_m) \\ &\leq \eta(\theta_n, \theta_{n+1}) + \eta(\xi_n, \xi_m) + \eta(\theta_{m+1}, \theta_m) \\ &\leq \eta(\theta_n, \theta_{n+1}) + \mathcal{H}(\Phi\xi_n, \Phi\xi_m) + \eta(\theta_{m+1}, \theta_m), \\ &\quad (\text{since } \xi_n \in \Phi\theta_n, \xi_m \in \Phi\theta_m) \\ &\leq \eta(\theta_n, \theta_{n+1}) + g(\eta(\theta_n, \theta_m))\eta(\theta_n, \theta_m) + \eta(\theta_{m+1}, \theta_m) \end{aligned}$$

$$\implies \eta(\theta_n, \theta_m) \leq \frac{\eta(\theta_n, \theta_{n+1}) + \eta(\theta_m, \theta_{m+1})}{1 - g(\eta(\theta_n, \theta_m))}.$$

Using the fact that $\limsup_{m,n \rightarrow \infty} g(\eta(\theta_n, \theta_m)) > 0$ and $r = 0$, from the last inequality, we have that

$$\limsup_{m,n \rightarrow \infty} [1 - g(\eta(\theta_n, \theta_m))] = 0.$$

Hence, $\limsup_{m,n \rightarrow \infty} g(\eta(\theta_n, \theta_m)) = 1$.

Since $g \in \mathcal{G}$, we have that $\limsup_{m,n \rightarrow \infty} \eta(\theta_n, \theta_m) = 0$, which is a contradiction, and therefore, $\{\theta_n\}$ is Cauchy in $\mathcal{X}_{\mathcal{Y}} \subseteq \mathcal{X}$.

Since $(\mathcal{W}, \eta)$ is complete and $\mathcal{X}$ is closed, we have $\lim_{n \rightarrow \infty} \theta_n = \theta$ for some $\theta \in \mathcal{X}$.

Since Φ is $\mathcal{H}$ -continuous (for it is an MVGC), we have

$$\lim_{n \rightarrow \infty} \mathcal{H}(\Phi\theta_n, \Phi\theta) = 0. \quad (2.13)$$

Exactly in the similar manner as above, using (C), we can prove that $\{\xi_n\}$ is Cauchy in $\mathcal{Y}$ and since $\mathcal{Y}$ is closed, there exists $\xi \in \mathcal{Y}$ such that $\lim_{n \rightarrow \infty} \xi_n = \xi$.

Since $\eta(\theta_{n+1}, \xi_n) = \eta(\mathcal{X}, \mathcal{Y})$ for all $n \in \mathbb{N}$, we have $\lim_{n \rightarrow \infty} \eta(\theta_{n+1}, \xi_n) = \eta(\theta, \xi) = \eta(\mathcal{X}, \mathcal{Y})$.

We claim that $\xi \in \Phi\theta$. Indeed, since $\xi_n \in \Phi\theta_n$ for all $n \in \mathbb{N}$, we have

$$\lim_{n \rightarrow \infty} \Delta(\xi_n, \Phi\theta) \leq \lim_{n \rightarrow \infty} \mathcal{H}(\Phi\theta_n, \Phi\theta) = 0.$$

Therefore, $\Delta(\xi, \Phi\theta) = 0$. Since $\Phi\theta$ is closed, we have $\xi \in \Phi\theta$.

Now,

$$\eta(\mathcal{X}, \mathcal{Y}) \leq \Delta(\theta, \Phi\theta) \leq \eta(\theta, \xi) = \eta(\mathcal{X}, \mathcal{Y}).$$

Hence $\Delta(\theta, \Phi\theta) = \eta(\mathcal{X}, \mathcal{Y})$, i.e., θ is a BPP of Φ . $\square$

Example 1. Let $\mathcal{W} = \mathbb{R}$ and $\eta(\theta, \xi) = |\theta - \xi|$ for all $\theta, \xi \in \mathbb{R}$. Then $(\mathcal{W}, \eta)$ is a complete MS. Let $\mathcal{X} = [7, 8]$ and $\mathcal{Y} = [-8, -7]$. Then $\mathcal{X}_{\mathcal{Y}} = \{7\}$ and $\mathcal{Y}_{\mathcal{X}} = \{-7\}$.

Define $\Phi : \mathcal{X} \rightarrow \Lambda(\mathcal{Y})$ by

$$\Phi\theta = \left[\frac{-\theta - 7}{2}, -7 \right] \text{ for all } \theta \in \mathcal{X}.$$

Also, consider the function $g : [0, \infty) \rightarrow [0, 1)$ by $g(t) = \frac{1}{1+t}$ for all $t > 0$ and $g(0) = 0$.

Now, $\Phi(7) = \{-7\}$ (i.e., $\Phi\theta \subseteq \mathcal{Y}_{\mathcal{X}}$ for all $\theta \in \mathcal{X}_{\mathcal{Y}}$).

If $\theta \neq \xi$, then

$$\begin{aligned} \mathcal{H}(\Phi\theta, \Phi\xi) &= \mathcal{H}\left(\left[\frac{-\theta - 7}{2}, -7\right], \left[\frac{-\xi - 7}{2}, -7\right]\right) \\ &= \left| \left(\frac{-\theta - 7}{2}\right) - \left(\frac{-\xi - 7}{2}\right) \right| \\ &= \frac{|\xi - \theta|}{2} \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{1 + \eta(\theta, \xi)} \cdot \eta(\theta, \xi), (\text{ since } \sup\{\eta(\theta, \xi) : \theta, \xi \in [7, 8]\} = 1) \\ &= g(\eta(\theta, \xi))\eta(\theta, \xi). \end{aligned}$$

Thus, Φ is an MVGC and all conditions of Theorem 1 are satisfied. We see that $\Delta(7, \Phi(7)) = \eta(\mathcal{X}, \mathcal{Y}) = 14$. Hence 7 is a BPP of Φ .

Corollary 1. *Let $(\mathcal{W}, \eta)$ be a complete MS and $\mathcal{X}$ be a non-empty closed subset of $\mathcal{W}$. Let $\Phi : \mathcal{X} \rightarrow \Lambda(\mathcal{X})$ be a MVGC such that $\Phi\theta$ is compact for each $\theta \in \mathcal{X}$. Then Φ has a fixed point.*

Proof. The proof follows if we assume $\mathcal{X} = \mathcal{Y}$ in Theorem 1. $\square$

Remark 4. In Corollary 1, if we take $\mathcal{X} = \mathcal{W}$, we obtain the multivalued analogue of Geraghty's result [11].

Next, we present an example for Corollary 1.

Example 2. Let $\mathcal{W} = [0, 1]$ and $\eta(\theta, \xi) = |\theta - \xi|$ for all $\theta, \xi \in \mathbb{R}$. Then $(\mathcal{W}, \eta)$ is a complete MS.

Define $\Phi : \mathcal{W} \rightarrow \Lambda(\mathcal{W})$ by $\Phi\theta = [0, \frac{\theta}{11}]$ for all $\theta \in [0, 1]$. Figure 1 describes the plot of the multivalued mapping $\Phi(\theta)$.

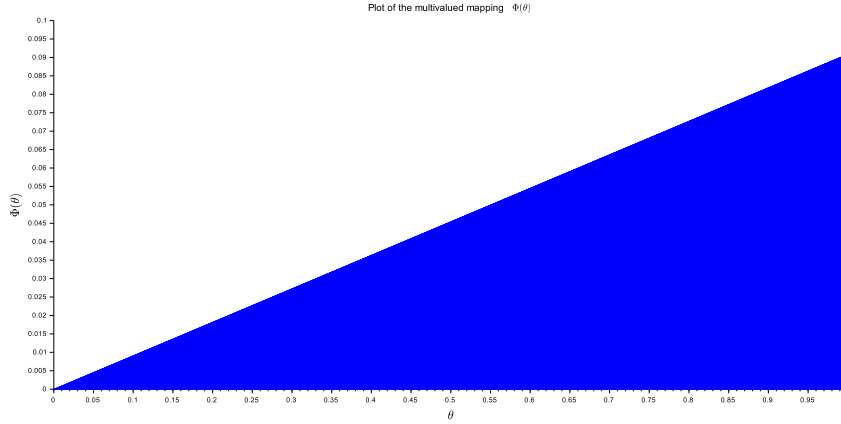


FIGURE 1. Plot of the multivalued mapping Φ .

If $\theta \neq \xi$, we have that

$$\begin{aligned} \mathcal{H}(\Phi\theta, \Phi\xi) &= \left| \frac{\theta}{11} - \frac{\xi}{11} \right| \\ &= \frac{|\xi - \theta|}{11} = \frac{1}{11}\eta(\theta, \xi) \end{aligned}$$

$$\leq \frac{e^{-\eta(\theta, \xi)}}{2} \cdot \eta(\theta, \xi) = g(\eta(\theta, \xi)) \cdot \eta(\theta, \xi),$$

where $g(t) = \frac{e^{-t}}{2}$ for $t > 0$ and $g(0) = 0$ and hence $g \in \mathcal{G}$.

We observe that $0 \in \Phi(0)$, i.e., 0 is a fixed point of Φ .

Conclusion. We have proved the main result with a strong condition that images of the MVGC are compact sets. Relaxation of this compactness criterion is a suggested future work. Adopting a new technique, we extended the famous fixed point result due to Geraghty to its exact multivalued counterpart in terms of BPP. The corresponding fixed point theorem followed as a consequence.

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HADAMARD HOMOMORPHISMS AND HADAMARD DERIVATIONS ON BANACH ALGEBRAS

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Abstract. In this paper, we prove the stability of Hadamard homomorphisms and Hadamard derivations in Banach algebras. This is applied to investigate Hadamard isomorphisms between Banach algebras.

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Keywords: 2-additive functional equation, Hadamard homomorphism, Hadamard derivation, 2-linear mapping

1. INTRODUCTION

In 1941, D. H. Hyers [3] gave the first partial solution to Ulam's question for the case of approximate additive mappings under the assumption that G and G' are Banach spaces. In 1978, Th. M. Rassias [14] generalized the theorem of Hyers [3] by considering the stability problem with unbounded Cauchy differences. In 1991, Z. Gajda [1], following the same approach as that by Th. M. Rassias [14] gave an affirmative solution to this question for $p > 1$. It was shown by Z. Gajda [1] as well as by Th. M. Rassias and P. Šemrl [15], that one cannot prove a Rassias-type theorem when $p = 1$. P. Găvruta [2] obtained the generalized result of the Rassias theorem which allows the Cauchy difference to be controlled by a general unbounded function.

The stability problems for several functional equations or inequalities have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [4, 5, 7]). The method provided by D. H. Hyers [3] which produces the additive function will be called a direct method. This method is the most important and useful tool to study the stability of different functional equations.

During the last two decades, a number of articles and research monographs have been published on various generalizations and applications of the Hyers-Ulam stability to a number of functional equations and mappings, for example, Cauchy-Jensen mappings, k -additive mappings, invariant means, multiplicative mappings, bounded

n th differences, convex functions, generalized orthogonality mappings, Euler-Lagrange functional equations, differential equations, and Navier-Stokes equations (see [9, 16–20]).

The theory of stability is a significant branch of the qualitative theory of differential equations and dynamical systems. During the recent decades many interesting results have been studied on some types differential equations (see [10, 11, 21]).

The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results, containing ternary homomorphisms and ternary derivations, concerning this problem (see [8, 13]).

Let $(A, \|\cdot\|)$ be a normed linear space over real or complex number field. For $X = (x_1, x_2) \in A^2$, we define

$$\|(x_1, x_2)\|_\infty := \max\{\|x_1\|, \|x_2\|\}.$$

Let A and B be complex vector spaces. A mapping $f : A \times A \longrightarrow B \times B$ is 2-additive if

$$f(x_1 + x_2, y_1 + y_2) = \sum_{i,j=1}^2 f(x_i, y_j)$$

for all $x_1, y_1, x_2, y_2 \in A$, and a mapping $f : A \times A \rightarrow B \times B$ is called a $\mathbb{C}$ -2-linear mapping if f is $\mathbb{C}$ -linear for each variable.

Lemma 1. [12] *Let A and B be real or complex normed linear spaces. Then the mapping $f : A \times A \longrightarrow B \times B$ is 2-additive if and only if*

$$f(x_1 + x_2, y_1 - y_2) + f(x_1 - x_2, y_1 + y_2) = 2f(x_1, y_1) - 2f(x_2, y_2) \quad (1.1)$$

for all $x_1, x_2, y_1, y_2 \in A$.

2. STABILITY OF HADAMARD HOMOMORPHISMS AND HADAMARD ISOMORPHISMS IN COMPLEX BANACH ALGEBRAS

Let A and B be complex Banach algebras. Throughout this section, $A \times A$ (resp. $B \times B$) will be complex Banach algebra with norm $\|\cdot\|_\infty$ and unital (e, e) (resp. (e', e')).

Definition 1. Let A be a complex Banach algebra. For $X = (x_1, y_1), Y = (x_2, y_2) \in A^2$, the *inner Hadamard product* (entry-wise product) of X and Y is defined by

$$X \cdot_H Y = (x_1, y_1) \cdot_H (x_2, y_2) := (x_1 x_2, y_1 y_2).$$

Definition 2. Let A and B be two complex Banach algebras. A $\mathbb{C}$ -2-linear mapping $\varphi : A \times A \rightarrow B \times B$ is called a *Hadamard homomorphism* if it satisfies

$$\varphi(X \cdot_H Y) = \varphi(X) \cdot_H \varphi(Y)$$

for all $X = (x_1, y_1), Y = (x_2, y_2) \in A^2$.

Example 1. Let $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $\phi(x, y) = (xy, xy)$. Then ϕ is a Hadamard homomorphism.

Example 2. Let M_{mn} be the set of all $m \times n$ real matrices, and $\phi : M_{mn} \rightarrow M_{mn}$ be defined by $\phi(X, Y) = (X \cdot_H Y, 0)$ for $X = [x_{ij}], Y = [y_{ij}] \in M_{mn}$, where the Hadamard product (entry-wise product) of X and Y is defined by

$$X \cdot_H Y = Z = [z_{ij}], \quad \text{where } z_{ij} = x_{ij}y_{ij} \quad \text{for all } 1 \leq i \leq m, 1 \leq j \leq n.$$

Then ϕ is a Hadamard homomorphism.

Lemma 2. [12] Let A and B be complex Banach algebras and $f : A \times A \rightarrow B \times B$ be a 2-additive mapping such that $f(\mu x, \nu y) = \mu \nu f(x, y)$ for all $\mu, \nu \in \mathbb{T}^1 := \{\xi \in \mathbb{C} : |\xi| = 1\}$ and all $x, y \in A$. Then f is 2-linear over $\mathbb{C}$.

Proof. Let $\lambda, \beta \in \mathbb{C}$ be nonzero numbers and M, N be two integers such that $M > 4|\lambda|$ and $N > 4|\beta|$. Then $|\frac{\lambda}{M}| < \frac{1}{4} < 1 - \frac{2}{3}$ and $|\frac{\beta}{N}| < \frac{1}{4} < 1 - \frac{2}{3}$. By [6] Theorem 1, there exist $\mu_1, \mu_2, \mu_3, \nu_1, \nu_2, \nu_3 \in \mathbb{T}^1$ such that $3\frac{\lambda}{M} = \mu_1 + \mu_2 + \mu_3$ and $3\frac{\beta}{N} = \nu_1 + \nu_2 + \nu_3$. Since f is 2-additive, $f(x, y) = f(3 \cdot \frac{1}{3}x, 3 \cdot \frac{1}{3}y) = 9f(\frac{1}{3}x, \frac{1}{3}y)$ for all $x, y \in A$. Thus $f(\frac{1}{3}x, \frac{1}{3}y) = \frac{1}{9}f(x, y)$. So

$$\begin{aligned} f(\lambda x, \beta y) &= f\left(\frac{M}{3} \cdot 3\frac{\lambda}{M}x, \frac{N}{3} \cdot 3\frac{\beta}{N}y\right) = MNf\left(\frac{1}{3} \cdot 3\frac{\lambda}{M}x, \frac{1}{3} \cdot 3\frac{\beta}{N}y\right) \\ &= \frac{MN}{9}f\left(3\frac{\lambda}{M}x, 3\frac{\beta}{N}y\right) = \frac{MN}{9}f(\mu_1x + \mu_2x + \mu_3x, \nu_1y + \nu_2y + \nu_3y) \\ &= \frac{MN}{9}(\mu_1 + \mu_2 + \mu_3)(\nu_1 + \nu_2 + \nu_3)f(x, y) = \frac{MN}{9} \cdot 3\frac{\lambda}{M} \cdot 3\frac{\beta}{N}f(x, y) \\ &= \lambda\beta f(x, y) \end{aligned}$$

for all $x, y \in A$. Therefore the mapping $f : A \times A \rightarrow B \times B$ is 2-linear over $\mathbb{C}$. $\square$

Lemma 3. Let A and B be complex Banach algebras and $f : A \times A \rightarrow B \times B$ be a mapping such that

$$f(\mu(x_1 + x_2), \nu(y_1 - y_2)) + f(\mu(x_1 - x_2), \nu(y_1 + y_2)) = 2\mu\nu f(x_1, y_1) - 2\mu\nu f(x_2, y_2)$$

for all $\mu, \nu \in \mathbb{T}^1$ and all $x, y \in A$, then f is 2-linear over $\mathbb{C}$.

Proof. By the use of Lemma 1 with $\mu = \nu = 1$ and Lemma 2 with $x_2 = y_2 = 0$, the mapping f is 2-linear over $\mathbb{C}$. $\square$

Theorem 1. Let A and B be complex Banach algebras. Let $p + q < 2, r + s < 2$ and η be positive real numbers. Suppose that $f : A \times A \rightarrow B \times B$ is a mapping satisfying $f(0, 0) = 0$ and the inequality

$$\begin{aligned} &||f(\lambda(x_1 + x_2), \mu(y_1 - y_2)) + f(\lambda(x_1 - x_2), \mu(y_1 + y_2)) - 2\lambda\mu f(x_1, y_1) \\ &\quad + 2\lambda\mu f(x_2, y_2)||_\infty \leq \eta\{||x_1||^p ||x_2||^q + ||y_1||^r ||y_2||^s\} \end{aligned} \quad (2.1)$$

for all $\lambda, \mu \in \mathbb{T}^1$ and $x_1, y_1, x_2, y_2 \in A$. If the mapping $f : A \times A \rightarrow B \times B$ satisfies

$$\|f(X \cdot_H Y) - f(X) \cdot_H f(Y)\|_\infty \leq \eta \|x_1\|^p \|x_2\|^q \|y_1\|^r \|y_2\|^s \quad (2.2)$$

for all $X = (x_1, y_1), Y = (x_2, y_2) \in A^2$, then there exists a unique Hadamard homomorphism $\varphi : A \times A \rightarrow B \times B$ such that

$$\|\varphi(x, y) - f(x, y)\|_\infty \leq 3\eta \left(\frac{\|x\|^{p+q}}{4 - 2^{p+q}} + \frac{\|y\|^{r+s}}{4 - 2^{r+s}} \right) \quad (2.3)$$

for all $x, y \in A$.

Proof. Letting $\lambda = \mu = 1$, $x_1 = x_2$ and $y_2 = -y_1$ in (2.1), we have

$$\|f(2x_1, 2y_1) - 2f(x_1, y_1) + 2f(x_1, -y_1)\|_\infty \leq \eta (\|x_1\|^{p+q} + \|y_1\|^{r+s}). \quad (2.4)$$

for all $x_1, y_1 \in A$. Setting $\lambda = \mu = 1$, $x_1 = y_1 = 0$ in (2.1), we get

$$f(x_2, -y_2) + f(-x_2, y_2) + 2f(x_2, y_2) = 0$$

for all $x_2, y_2 \in A$. Replacing x_2 by x_1 and y_2 by y_1 in the above inequality, we obtain

$$f(x_1, -y_1) + f(-x_1, y_1) + 2f(x_1, y_1) = 0 \quad (2.5)$$

for all $x_1, y_1 \in A$. Putting $\lambda = \mu = 1$, $x_2 = -x_1$ and $y_1 = y_2$ in (2.1), we obtain

$$\|f(2x_1, 2y_1) - 2f(x_1, y_1) + 2f(-x_1, y_1)\|_\infty \leq \eta (\|x_1\|^{p+q} + \|y_1\|^{r+s}) \quad (2.6)$$

for all $x_1, y_1 \in A$. By (2.4) and (2.5), we get

$$\|f(2x_1, 2y_1) - 4f(x_1, y_1) + f(x_1, -y_1) - f(-x_1, y_1)\|_\infty \leq 2\eta (\|x_1\|^{p+q} + \|y_1\|^{r+s}),$$

for all $x_1, y_1 \in A$. By (2.4) and (2.6), we get

$$\|f(x_1, -y_1) - f(-x_1, y_1)\|_\infty \leq \eta (\|x_1\|^{p+q} + \|y_1\|^{r+s})$$

for all $x_1, y_1 \in A$. By the above two inequalities, we have

$$\|f(2x_1, 2y_1) - 4f(x_1, y_1)\|_\infty \leq 3\eta (\|x_1\|^{p+q} + \|y_1\|^{r+s}) \quad (2.7)$$

for all $x_1, y_1 \in A$. Replacing x_1 by $2^n x_1$ and y_1 by $2^n y_1$ and dividing 4^{n+1} in (2.7), we obtain that

$$\begin{aligned} & \left\| \frac{1}{4^{n+1}} f(2^{n+1} x_1, 2^{n+1} y_1) - \frac{1}{4^n} f(2^n x_1, 2^n y_1) \right\|_\infty \\ & \leq \frac{3\eta}{4^{n+1}} \left(2^{n(p+q)} \|x_1\|^{p+q} + 2^{n(r+s)} \|y_1\|^{r+s} \right) \end{aligned}$$

for all $x_1, y_1 \in A$ and all $n = 0, 1, 2, \dots$. For given integers k, m ($0 \leq k \leq m$), we get

$$\begin{aligned} & \left\| \frac{1}{4^m} f(2^m x_1, 2^m y_1) - \frac{1}{4^k} f(2^k x_1, 2^k y_1) \right\|_\infty \\ & \leq \sum_{n=k}^{m-1} \frac{3\eta}{4^{n+1}} \left(2^{n(p+q)} \|x_1\|^{p+q} + 2^{n(r+s)} \|y_1\|^{r+s} \right) \end{aligned} \quad (2.8)$$

for all $x_1, y_1 \in A$. By the use of (2.8), the sequence $\{\frac{1}{4^n}f(2^n x_1, 2^n y_1)\}$ is a Cauchy sequence for all $x_1, y_1 \in A$. Since $B \times B$ is complete, the sequence $\{\frac{1}{4^n}f(2^n x_1, 2^n y_1)\}$ converges for all $x_1, y_1 \in A$. Define $\varphi : A \times A \longrightarrow B \times B$ by

$$\varphi(x_1, y_1) := \lim_{n \rightarrow \infty} \frac{1}{4^n} f(2^n x_1, 2^n y_1)$$

for all $x_1, y_1 \in A$, we have

$$\begin{aligned} & \|\varphi(x_1 + x_2, y_1 - y_2) + \varphi(x_1 - x_2, y_1 + y_2) - 2\varphi(x_1, y_1) + 2\varphi(x_2, y_2)\|_\infty \\ &= \lim_{n \rightarrow \infty} \left\| \frac{1}{4^n} f(2^n(x_1 + x_2), 2^n(y_1 + y_2)) + \frac{1}{4^n} f(2^n(x_1 - x_2), 2^n(y_1 + y_2)) \right. \\ & \quad \left. - \frac{2}{4^n} f(2^n x_1, 2^n y_1) + \frac{2}{4^n} f(2^n x_2, 2^n y_2) \right\|_\infty \\ &\leq \lim_{n \rightarrow \infty} \frac{\eta}{4^n} [2^{n(p+q)} \|x_1\|^p \|x_2\|^q + 2^{n(r+s)} \|y_1\|^r \|y_2\|^s] \\ &\leq \lim_{n \rightarrow \infty} \frac{2^{\zeta n}}{4^n} \eta [\|x_1\|^p \|x_2\|^q + \|y_1\|^r \|y_2\|^s] \end{aligned}$$

where $\zeta = \max\{p+q, r+s\}$, $x_1, x_2, y_1, y_2 \in A$ and $n = 0, 1, 2, \dots$. As $n \rightarrow \infty$, by Lemma 3, the mapping $\varphi : A \times A \rightarrow B \times B$ is 2-linear. Setting $k = 0$ and taking $m \rightarrow \infty$ in (2.8), one can obtain the inequality (2.3). Let $X = (x_1, y_1), Y = (x_2, y_2)$. From (2.2) we have

$$\begin{aligned} & \|\varphi(X \cdot_H Y) - \varphi(X) \cdot_H \varphi(Y)\|_\infty \\ &= \lim_{n \rightarrow \infty} \frac{1}{16^n} \|f(2^n x_1, 2^n x_2, 2^n y_1, 2^n y_2) - f(2^n x_1, 2^n y_1) \cdot_H f(2^n x_2, 2^n y_2)\| \\ &\leq \lim_{n \rightarrow \infty} \frac{2^{n(p+q+r+s)}}{16^n} \|x_1\|^p \|x_2\|^q \|y_1\|^r \|y_2\|^s = 0, \end{aligned}$$

for all $x_1, y_1, x_2, y_2 \in A$, since $p+q+r+s < 4$. Thus

$$\varphi(X \cdot_H Y) = \varphi(X) \cdot_H \varphi(Y).$$

Now, let $\psi : A \times A \rightarrow B \times B$ be another 2-additive mapping satisfying (2.3). Then we have

$$\begin{aligned} \|\varphi(x, y) - \psi(x, y)\|_\infty &= \frac{1}{4^n} \|\varphi(2^n x, 2^n y) - \psi(2^n x, 2^n y)\|_\infty \\ &\leq \frac{1}{4^n} (\|\varphi(2^n x, 2^n y) - f(2^n x, 2^n y)\|_\infty \\ & \quad + \|f(2^n x, 2^n y) - \psi(2^n x, 2^n y)\|_\infty) \\ &\leq \frac{3\eta}{4^n} \left(\frac{2^{n(p+q)} \|x\|^{p+q}}{4 - 2^{p+q}} + \frac{2^{n(r+s)} \|y\|^{r+s}}{4 - 2^{r+s}} \right) \end{aligned}$$

$$\leq \frac{3\eta \times 2^{n\zeta}}{4^n(4-2^\zeta)}(\|x\|^{p+q} + \|y\|^{r+s}),$$

where $\zeta = \max\{p+q, r+s\}$. This tends to zero as $n \rightarrow \infty$ for all $x, y \in A$. So we can conclude that $\varphi(x, y) = \psi(x, y)$ for all $x, y \in A$. This proves the uniqueness of φ . $\square$

Theorem 2. *Let A and B be complex Banach algebras. Let $p+q < 2, r+s < 2$ and η be positive real numbers. Let $f : A \times A \rightarrow B \times B$ be a bijective mapping satisfying (2.1) such that*

$$f(X \cdot_H Y) = f(X) \cdot_H f(Y) \quad (2.9)$$

for all $X, Y \in A^2$. If $f(\alpha x, \beta y)$ is continuous in $\alpha, \beta \in \mathbb{C}$ for each fixed $X = (x, y) \in A^2$ and $\frac{1}{4^n}f(2^n e, 2^n e) = (e', e')$, then the mapping $f : A \times A \rightarrow B \times B$ is a Hadamard isomorphism.

Proof. Since $f(X \cdot_H Y) = f(X) \cdot_H f(Y)$ for all $X, Y \in A^2$, the mapping $f : A \times A \rightarrow B \times B$ satisfies (2.1). By Theorem 1, there exists a Hadamard homomorphism $\varphi : A^2 \rightarrow B^2$ satisfying (2.3). The mapping $\varphi : A^2 \rightarrow B^2$ is defined by

$$\varphi(x, y) = \lim_{n \rightarrow \infty} \frac{1}{4^n} f(2^n x, 2^n y)$$

for all $x, y \in A$.

It follows from (2.9) that

$$\begin{aligned} \varphi(x, y) &= \varphi((e, e) \cdot_H (x, y)) = \lim_{n \rightarrow \infty} \frac{1}{4^n} f(2^n ex, 2^n ey) \\ &= \lim_{n \rightarrow \infty} \frac{1}{4^n} f((2^n e, 2^n e) \cdot_H (x, y)) = \lim_{n \rightarrow \infty} \frac{1}{4^n} (f(2^n e, 2^n e) \cdot_H f(x, y)) \\ &= \left(\lim_{n \rightarrow \infty} \frac{1}{4^n} f(2^n e, 2^n e) \right) \cdot_H f(x, y) = (e', e') \cdot_H f(x, y) = f(x, y) \end{aligned}$$

for all $x, y \in A$. So the bijective mapping $f : A \times A \rightarrow B \times B$ is a Hadamard isomorphism. $\square$

3. STABILITY OF HADAMARD DERIVATIONS ON COMPLEX BANACH ALGEBRAS

In this section, we prove the stability of Hadamard derivations on complex Banach algebras associated with the 2-additive functional equation.

Definition 3. Let A be a complex Banach algebra. A $\mathbb{C}$ -2-linear mapping $D : A \times A \rightarrow A \times A$ is a *Hadamard derivation* if D satisfies

$$D(X \cdot_H Y) = D(X) \cdot_H Y^2 + X^2 \cdot_H D(Y)$$

for all $X, Y \in A^2$.

Theorem 3. Let A be a complex Banach algebra. Let $p + q < 2, r + s < 2$ and η be positive real numbers. Let $f : A \times A \rightarrow A \times A$ be a mapping satisfying (2.1) such that

$$\|f(X, {}_H Y) - f(X), {}_H Y^2 - X^2 \cdot {}_H f(Y)\|_\infty \leq \eta \|x_1\|^p \|x_2\|^q \|y_1\|^r \|y_2\|^s \quad (3.1)$$

for all $x_1, y_1, x_2, y_2 \in A$. Then there exists a unique Hadamard derivation $\varphi : A \times A \rightarrow A \times A$ such that

$$\|D(x, y) - f(x, y)\| \leq 3\eta \left(\frac{\|x\|^{p+q}}{4 - 2^{p+q}} + \frac{\|y\|^{r+s}}{4 - 2^{r+s}} \right). \quad (3.2)$$

Proof. By (2.8), the sequence $\{\frac{1}{4^n} f(2^n x_1, 2^n y_1)\}$ is a Cauchy sequence for all $x_1, y_1 \in A$. Since A is complete, the sequence $\{\frac{1}{4^n} f(2^n x_1, 2^n y_1)\}$ converges. So one can define the mapping $D : A \times A \rightarrow A \times A$ by

$$D(x, y) = \lim_{n \rightarrow \infty} \frac{1}{4^n} f(2^n x_1, 2^n y_1)$$

for all $x_1, y_1 \in A$. By the same reasoning as in the proof of Theorem 1, we get the fact that D is 2-bilinear.

It follows from (3.1) that

$$\begin{aligned} & \|D(X, {}_H Y) - D(X), {}_H Y^2 - X^2 \cdot {}_H D(Y)\|_\infty \\ &= \lim_{n \rightarrow \infty} \left\| \frac{1}{4^n} f(2^n x_1 x_2, 2^n y_1 y_2) - \frac{1}{4^n} f(2^n x_1, 2^n y_1) \cdot {}_H (x_2, y_2)^2 \right. \\ & \quad \left. - \frac{1}{4^n} (x_1, y_1)^2 \cdot {}_H f(2^n x_2, 2^n y_2) \right\|_\infty \\ &= \lim_{n \rightarrow \infty} \frac{1}{2^{4n}} \left\| f(2^n x_1 \cdot 2^n x_2, 2^n y_1 \cdot 2^n y_2) - f(2^n x_1, 2^n y_1) \cdot {}_H (2^n x_2, 2^n y_2)^2 \right. \\ & \quad \left. - (2^n x_1, 2^n y_1)^2 \cdot {}_H f(2^n x_2, 2^n y_2) \right\|_\infty \\ &\leq \lim_{n \rightarrow \infty} \frac{2^{n(p+q+r+s)}}{2^{4n}} \|x_1\|^p \|x_2\|^q \|y_1\|^r \|y_2\|^s = 0 \end{aligned}$$

for all $x_1, y_1, x_2, y_2 \in A$. So

$$D(X \cdot {}_H Y) = D(X), {}_H Y^2 + X^2 \cdot {}_H D(Y)$$

for all $x_1, y_1, x_2, y_2 \in A$. Now, let Ψ be another mapping satisfying (3.2). Then we have

$$\begin{aligned} \|D(x, y) - \Psi(x, y)\|_\infty &= \lim_{n \rightarrow \infty} \frac{1}{4^n} \|D(2^n x, 2^n y) - \Psi(2^n x, 2^n y)\|_\infty \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{4^n} (\|D(2^n x, 2^n y) - f(2^n x, 2^n y)\|_\infty + \|f(2^n x, 2^n y) - \Psi(2^n x, 2^n y)\|_\infty) \\ &\leq \lim_{n \rightarrow \infty} \frac{3\eta}{4^n} \left(\frac{2^{n(p+q)} \|x\|^{p+q}}{4 - 2^{p+q}} + \frac{2^{n(r+s)} \|y\|^{r+s}}{4 - 2^{r+s}} \right) \end{aligned}$$

$$\leq \lim_{n \rightarrow \infty} \frac{3\eta \times 2^{n\zeta}}{4^n(4 - 2^\zeta)} (||x||^{p+q} + ||y||^{r+s}).$$

By the same argument as in the proof of Theorem 1, the mapping $D : A \times A \longrightarrow A \times A$ is a unique Hadamard derivation satisfying (3.2). $\square$

4. CONCLUSION

In this work, we have proved the Hyers-Ulam stability of Hadamard homomorphisms and Hadamard derivations in Banach algebras associated to the bi-additive functional equation (1.1). This has been applied to investigate Hadamard isomorphisms between Banach algebras.

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ANSWERS TO GEORGE-RADENović-RESHMA-SHUKLA QUESTIONS IN RECTANGULAR b -METRIC SPACES

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Abstract. In this paper, we prove two new general fixed point theorems in rectangular b -metric spaces. As applications, we answer two open questions in rectangular b -metric spaces posed in [12].

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1. INTRODUCTION AND PRELIMINARIES

There have been many generalizations of a metric. One of the most interesting generalizations is a quasi-distance D on a set X , where the triangle inequality is replaced by

$$D(x, z) \leq \kappa[D(x, y) + D(y, z)]$$

for all $x, y, z \in X$ and some constant $\kappa \geq 1$, see for example [21, page 257]. After that Czerwik used the name b -metric space for a set with a quasi-distance [6, 7]. In 2010 Khamsi and Hussain [17] reintroduced the notion of a b -metric under the name *metric-type*. Khamsi [16] also introduced another definition of a metric-type, that was called an s -relaxed _{p} metric in [11, Definition 4.2]. The first fixed point theorems in b -metric spaces were proved by Bakhtin [1], and Czerwik [6]. Kirk and Shahzad essentially used Czerwik's technique to prove a general fixed point theorem [18, Theorem 12.2]. Recently, Kajántó and Lukács [15] pointed out and corrected an inaccuracy in the proof of [6, Theorem 1]. Kirk-Shahzad theorem was also used to answer the early stated question on transforming fixed point theorems in metric spaces to fixed point theorems in b -metric spaces [9, Theorem 2.1]. Bessenyei and Páles [2] introduced the notion of a triangle function and extended the Banach contraction principle in this spirit for such complete semi-metric spaces that fulfil an extra regularity property. Kirk and Shahzad [19] introduced a strengthening of a b -metric space, called a *strong b -metric space*, and examined instances in which this notion plays a critical role. Miculescu and Mihail also proved a fixed point theorem for ϕ -contraction but their main result requires the continuity of the given

map [22, Theorem 3.1]. In fact, a very general result was proved by Bessenyei and Páles [2] in regular semi-metric spaces.

In 2000, Branciari [3] introduced the notion of a v -generalized metric space. A 2-generalized metric space was also called a *generalized metric space*, or for short, *g.m.s* [3, Definition 1.1], or *rectangular metric space* [10, Definition 1]. v -generalized metric spaces were investigated and fixed point theorems in such spaces were stated, see [13, 14] and references therein. For v -generalized metrics that being not metrics, see [3, 3. An example], [8, Examples 1 & 2], [13, Examples 2.1 & 4.1].

Motivated by b -metric spaces and rectangular metric spaces, George *et al.* [12] introduced the notion of a rectangular b -metric space. This notion was also introduced independently by Roshan *et al.* in [25]. The convergence, Cauchy sequence and completeness in rectangular b -metric spaces were defined similarly to that in metric spaces.

Recently some results in b -metric spaces and in rectangular b -metric spaces were stated, see [5, 24] and the references therein. However there are open questions relating to such spaces, see Question 1 and Question 2 below. Note that Question 1 was answered positively recently in [23, Theorem 2.1] by direct proof. Also, a partial answer to Question 2 was presented on [20, Theorem 3.2] that an analogue of Reich contraction in rectangular b -metric spaces was proved. In the proof on [20, page 85], the author claimed $\lim_{n \rightarrow \infty} d(x_{n+1}, Tx^*) = d(x^*, Tx^*)$ provided that $\lim_{n \rightarrow \infty} x_n = x^*$. Unfortunately, this claim does not hold since the rectangular b -metric is not continuous. A similar flaw also appeared in the proof of [20, Theorem 3.1] for the case b -metric. Indeed, the conclusion in [20, Theorem 3.1] does not hold which was proved in [9, Remark 2.7].

In this paper, we are interested in studying fixed point theorems in rectangular b -metric spaces. We prove two fixed point theorems in rectangular b -metric spaces. Using these theorems, we give answers to Question 1 and Question 2.

Now we recall notions and properties which are useful in the latter.

Definition 1 ([7], page 263). Let X be a nonempty set, $\kappa \geq 1$ and $D: X \times X \rightarrow [0, \infty)$ be a function such that for all $x, y, z \in X$,

- (1) $D(x, y) = 0$ if and only if $x = y$.
- (2) $D(x, y) = D(y, x)$.
- (3) $D(x, z) \leq \kappa[D(x, y) + D(y, z)]$.

Then D is called a b -metric on X and (X, D, κ) is called a b -metric space.

Definition 2 ([16], Definition 2.7). Let X be a nonempty set, $\kappa \geq 1$ and $D: X \times X \rightarrow [0, \infty)$ be a function such that for all $n \in \mathbb{N}$ and all $x, y_1, \dots, y_n, z \in X$,

- (1) $D(x, y) = 0$ if and only if $x = y$.
- (2) $D(x, y) = D(y, x)$.
- (3) $D(x, z) \leq \kappa[D(x, y_1) + \dots + D(y_n, z)]$.

Then D is called a $metric$ -type on X and (X, D, κ) is called a $metric$ -type space.

Definition 3 ([12], Definition 1.3). Let X be a nonempty set, $\kappa \geq 1$ and $D: X \times X \rightarrow [0, \infty)$ be a function such that for all $x, y \in X$, all distinct points $u, v \notin \{x, y\}$,

- (1) $D(x, y) = 0$ if and only if $x = y$.
- (2) $D(x, y) = D(y, x)$.
- (3) $D(x, y) \leq \kappa[D(x, u) + D(u, v) + D(v, y)]$.

Then D is called a *rectangular b -metric* on X and (X, D, κ) is called a *rectangular b -metric space*.

Note that κ in Definitions 1-3 is always assumed to be the smallest possible value, and it is also called the *coefficient* of the corresponding distance function.

Definition 4 ([12], Definition 1.6). Let (X, D, κ) be a rectangular b -metric space.

- (1) A sequence $\{x_n\}$ is called *convergent* to x , written as $\lim_{n \rightarrow \infty} x_n = x$, if $\lim_{n \rightarrow \infty} D(x_n, x) = 0$.
- (2) A sequence $\{x_n\}$ is called *Cauchy* if $\lim_{n, m \rightarrow \infty} D(x_n, x_m) = 0$.
- (3) (X, D, κ) is called *complete* if each Cauchy sequence is a convergent sequence.

Theorem 1 ([12], Theorem 2.1). Let (X, D, κ) be a complete rectangular b -metric space and $f: X \rightarrow X$ be a map such that for all $x, y \in X$ and for some $\lambda \in [0, \frac{1}{\kappa})$,

$$D(f(x), f(y)) \leq \lambda D(x, y)$$

Then f has a unique fixed point $x^* \in X$.

Question 1 ([12], Open problem (1) on page 1012). In Theorem 1, can we extend the range of λ to the case $\frac{1}{\kappa} \leq \lambda < 1$?

Question 2 ([12], Open problem (2) on page 1012). Prove the analogue of Chatterjea contraction, Reich contraction, Ćirić contraction and Hardy-Rogers contraction in rectangular b -metric spaces.

Definition 5 ([2], page 516). Let (X, d) be a semi-metric space. A function $\Phi: [0, \infty] \times [0, \infty] \rightarrow [0, \infty]$ is called a *triangle function* for d if Φ is increasing in each of its variables, $\Phi(0, 0) = 0$ and for all $x, y, z \in X$,

$$d(x, y) \leq \Phi(d(x, z), d(z, y)).$$

Lemma 1 ([2], page 516). Let (X, d) be a semi-metric space and for all $u, v \in [0, \infty]$,

$$\Phi_d(u, v) = \sup\{d(x, y) : \exists p \in X, d(p, x) \leq u, d(p, y) \leq v\}.$$

Then Φ_d is a triangle function for d . Moreover, if Φ is a triangle function for d then $\Phi_d \leq \Phi$.

Definition 6 ([2], page 516). Let (X, d) be a semi-metric space. Then the triangle function Φ_d defined as in Lemma 1 is called the *basic triangle function* and (X, d) is called *regular* if Φ_d is continuous at $(0, 0)$.

Remark 1 ([2], page 516).

- (1) Every metric space is a semi-metric space with the triangle function $\Phi(u, v) = u + v$.
- (2) Every ultrametric space is a semi-metric space with the triangle function $\Phi(u, v) = \max\{u, v\}$.
- (3) Every b -metric space is a semi-metric space with the triangle function $\Phi(u, v) = \kappa(u + v)$.

Theorem 2 ([2], Theorem 1). Let (X, D) be a complete regular semi-metric space and $f: X \rightarrow X$ be a map such that for all $x, y \in X$,

$$D(f(x), f(y)) \leq \varphi(D(x, y))$$

where $\varphi: [0, \infty) \rightarrow [0, \infty)$ is an increasing function and for each $t \in [0, \infty)$

$$\lim_{n \rightarrow \infty} \varphi^n(t) = 0.$$

Then f has a unique fixed point $x^* \in X$ and $\lim_{n \rightarrow \infty} f^n(x) = x^*$ for each $x \in X$.

Replacing regular semi-metric spaces in Theorem 2 by b -metric spaces we get the following result.

Theorem 3 ([18], Theorem 12.2). Let (X, D, κ) be a complete b -metric space and $f: X \rightarrow X$ be a map such that for all $x, y \in X$,

$$D(f(x), f(y)) \leq \varphi(D(x, y))$$

where $\varphi: [0, \infty) \rightarrow [0, \infty)$ is an increasing function and for each $t \in [0, \infty)$

$$\lim_{n \rightarrow \infty} \varphi^n(t) = 0.$$

Then f has a unique fixed point $x^* \in X$ and $\lim_{n \rightarrow \infty} f^n(x) = x^*$ for each $x \in X$.

2. MAIN RESULTS

We prove an analogue of Theorem 3 in rectangular b -metric spaces. Note that in the spirit of Remark 1, every rectangular b -metric space may not be a semi-metric space since the right side of Definition 3.(3) contains three terms while Φ is a two-variable function. So the following result may not be deduced directly from Theorem 2.

Theorem 4. Let (X, D, κ) be a complete rectangular b -metric space and $f: X \rightarrow X$ be a map such that for all $x, y \in X$,

$$D(f(x), f(y)) \leq \varphi(D(x, y)) \tag{2.1}$$

where $\varphi: [0, \infty) \rightarrow [0, \infty)$ is an increasing function and for each $t \in [0, \infty)$

$$\lim_{n \rightarrow \infty} \varphi^n(t) = 0. \quad (2.2)$$

Then f has a unique fixed point $x^* \in X$ and $\lim_{n \rightarrow \infty} f^n(x) = x^*$ for each $x \in X$.

Proof. First we prove that $\varphi(t) < t$ for all $t > 0$. Indeed, if there exists $t_0 > 0$ such that $t_0 \leq \varphi(t_0)$, then from the increasing property of φ we have for all n ,

$$0 < t_0 \leq \varphi(t_0) \leq \varphi^2(t_0) \leq \dots \leq \varphi^n(t_0).$$

It follows from $\lim_{n \rightarrow \infty} \varphi^n(t_0) = 0$ that $t_0 = 0$. It is a contradiction to $t_0 > 0$. So $\varphi(t) < t$ for all $t > 0$.

Now we prove that

$$\lim_{t \rightarrow 0^+} \varphi(t) = 0. \quad (2.3)$$

Indeed, since φ is increasing, there exists $\lim_{t \rightarrow 0^+} \varphi(t) = l \geq 0$. If $\lim_{t \rightarrow 0^+} \varphi(t) = l > 0$ then $\varphi(t) \geq l$ for all $t > 0$. In particular, $\varphi(l) \geq l$, a contradiction. So (2.3) holds. Then there exists n_0 such that

$$\varphi^{n_0}(1) < \frac{1}{3\kappa}. \quad (2.4)$$

Let $x \in X$. Put $g = f^{n_0}$ and put $x_m = g^m(x)$ for all $m \in \mathbb{N}$. By (2.1) we deduce that

$$D(x_{m+1}, x_m) = D(g^m(g(x)), g^m(x)) \leq \dots \leq \varphi^m(D(g(x), x)) \quad (2.5)$$

and

$$D(x_{m+2}, x_m) = D(g^m(g^2(x)), g^m(x)) \leq \dots \leq \varphi^m(D(g^2(x), x)). \quad (2.6)$$

Letting $m \rightarrow \infty$ in (2.5) and (2.6) we get

$$\lim_{m \rightarrow \infty} D(x_{m+1}, x_m) = \lim_{m \rightarrow \infty} D(x_{m+2}, x_m) = 0.$$

So there exists m_0 such that for all $m \geq m_0$,

$$D(x_{m+1}, x_m) < \frac{1}{3\kappa} \text{ and } D(x_{m+2}, x_m) < \frac{1}{3\kappa}. \quad (2.7)$$

Now for each $u \in B[x_{m_0}, 1]$ and by (2.4) we have

$$D(g(u), g(x_{m_0})) = D(f^{n_0}(u), f^{n_0}(x_{m_0})) \leq \varphi^{n_0}(D(u, x_{m_0})) \leq \varphi^{n_0}(1) < \frac{1}{3\kappa}. \quad (2.8)$$

We first show that there exists k such that g^k has a fixed point x^* . On the contrary, we have $g(x_{m_0}) \neq g(g(x_{m_0})) \neq x_{m_0}$. Let $u \in B[x_{m_0}, 1]$. If $g(x_{m_0}) = g(u)$ or $g(g(x_{m_0})) = g(u)$ then by (2.7) we get $D(g(u), x_{m_0}) < \frac{1}{3\kappa} < 1$. So $g(u) \in B[x_{m_0}, 1]$.

So we may assume that $g(x_{m_0}) \neq g(g(x_{m_0})) \notin \{x_{m_0}, g(u)\}$. In this case, from (2.7) and (2.8) we find that

$$D(g(u), x_{m_0}) \leq \kappa[D(g(u), g(x_{m_0})) + D(g(x_{m_0}), g(g(x_{m_0}))) + D(g(g(x_{m_0})), x_{m_0})]$$

$$\leq \kappa \left[\frac{1}{3\kappa} + \frac{1}{3\kappa} + \frac{1}{3\kappa} \right] = 1.$$

So $g(u) \in B[x_{m_0}, 1]$.

Then we conclude that $g: B[x_{m_0}, 1] \rightarrow B[x_{m_0}, 1]$. For all $n, m \geq m_0$ and from the contrary assumption we get $x_{n+1} \neq x_{m_0} \notin \{x_n, x_m\}$. By (2.7) we have

$$\begin{aligned} D(x_n, x_m) &\leq \kappa[D(x_n, x_{n+1}) + D(x_{n+1}, x_{m_0}) + D(x_{m_0}, x_m)] \\ &\leq \kappa \left[\frac{1}{3\kappa} + 1 + 1 \right] = \frac{1+6\kappa}{3}. \end{aligned}$$

By (2.1) we find that for $m \geq n \geq m_0$,

$$\begin{aligned} D(x_n, x_m) &= D(g^n(x), g^m(x)) \\ &= D(g^{n-m_0}g^{m_0}(x), g^{m-m_0}g^{m_0}(x)) \\ &= D(g^{n-m_0}(x_{m_0}), g^{m-m_0}(x_{m_0})) \\ &= D(f^{(n-m_0)n_0}(x_{m_0}), f^{(m-m_0)n_0}(x_{m_0})) \\ &\leq \varphi(D(f^{(n-m_0)n_0-1}(x_{m_0}), f^{(m-m_0)n_0-1}(x_{m_0}))) \\ &\leq \varphi^{(n-m_0)n_0}(D(x_{m_0}, f^{(m-m_0)n_0-(n-m_0)n_0}(x_{m_0}))) \\ &= \varphi^{(n-m_0)n_0}(D(x_{m_0}, f^{(m-n)n_0}(x_{m_0}))) \\ &= \varphi^{(n-m_0)n_0}(D(x_{m_0}, x_{m-n+m_0})) \\ &\leq \varphi^{(n-m_0)n_0} \left(\frac{1+6\kappa}{3} \right). \end{aligned} \tag{2.9}$$

Letting $n, m \rightarrow \infty$ in (2.9) and using (2.2) we find that $\lim_{n, m \rightarrow \infty} D(x_n, x_m) = 0$. So $\{x_m\}$ is a Cauchy sequence in (X, D) . Since (X, D) is complete, there exists $\lim_{m \rightarrow \infty} x_m = x^*$.

It follows from (2.1) and (2.3) that f is continuous in the sense it preserves the limit of sequences. Therefore g is continuous. Then

$$x^* = \lim_{m \rightarrow \infty} x_m = \lim_{m \rightarrow \infty} x_{m+1} = \lim_{m \rightarrow \infty} g(x_m) = g(x^*).$$

So g has a fixed point. It is a contradiction to the contrary assumption.

Therefore, there exists k such that g^k has a fixed point x^* . From (2.1) we get

$$\begin{aligned} D(x^*, g^{km}(f(x))) &= D(g^{km}(x^*), g^{km}(f(x))) \\ &= D(f^{n_0 km}(x^*), f^{n_0 km}(f(x))) \\ &\leq \varphi^{n_0 km}(D(x^*, f(x))) \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} D(x^*, g^{km}(x)) &= D(g^{km}(x^*), g^{km}(x)) \\ &= D(f^{n_0 km}(x^*), f^{n_0 km}(x)) \\ &\leq \varphi^{n_0 km}(D(x^*, x)). \end{aligned} \quad (2.11)$$

Letting $m \rightarrow \infty$ in (2.10) and (2.11) we get

$$\lim_{m \rightarrow \infty} D(x^*, g^{km}(f(x))) = \lim_{m \rightarrow \infty} D(x^*, g^{km}(x)) = 0.$$

Then $\lim_{m \rightarrow \infty} g^{km}(f(x)) = \lim_{m \rightarrow \infty} g^{km}(x) = x^*$ in (X, D) . By the continuity of f we have

$$f(x^*) = \lim_{m \rightarrow \infty} f(g^{km}(x)) = \lim_{m \rightarrow \infty} f(f^{n_0 km}(x)) = \lim_{m \rightarrow \infty} g^{km}(f(x)) = x^*.$$

This proves that x^* is a fixed point of f .

We next prove the uniqueness of fixed points of f . On the contrary, let x^* and y^* be two distinct fixed points of f . Then $D(x^*, y^*) > 0$. Therefore

$$D(x^*, y^*) = D(f(x^*), f(y^*)) \leq \varphi(D(x^*, y^*)) < D(x^*, y^*).$$

It is a contradiction.

Finally, we show that $\lim_{n \rightarrow \infty} f^n(x) = x^*$. Note that $\lim_{m \rightarrow \infty} g^{km}(y) = x^*$ for all $y \in X$. For each $n \in \mathbb{N}$, there exists l_n such that $n = l_n k n_0 + r_n$ with $0 \leq r_n \leq k n_0 - 1$. So

$$f^n(x) = f^{l_n k n_0 + r_n}(x) = g^{l_n k}(f^{r_n}(x)).$$

Fix $r_n = r \in [0, k n_0 - 1]$. Then

$$\lim_{l_n \rightarrow \infty} f^{l_n k n_0 + r}(x) = \lim_{l_n \rightarrow \infty} g^{l_n k}(f^r(x)) = x^*.$$

It implies that $\lim_{n \rightarrow \infty} f^n(x) = x^*$. □

Now by using Theorem 4 with $\varphi(t) = \lambda t$ with $t \geq 0$ we get a positive answer to Question 1. Note that this question was answered recently in [23, Theorem 2.1] but by a different proof.

Corollary 1. *Let (X, D, κ) be a complete rectangular b -metric space and $f: X \rightarrow X$ be a map such that for all $x, y \in X$ and for some $\lambda \in [0, 1)$,*

$$D(f(x), f(y)) \leq \lambda D(x, y)$$

Then f has a unique fixed point $x^ \in X$ and $\lim_{n \rightarrow \infty} f^n(x) = x^*$ for each $x \in X$.*

In 1974, Ćirić proved a very general fixed point theorem in metric spaces, see [4, Theorem 1]. Next, we prove Ćirić type fixed point theorem in rectangular b -metric spaces. The proof in rectangular b -metric spaces is more complicated than that in metric spaces since the inequality is only used for distinct points.

Theorem 5 (Ćirić type fixed point theorem in rectangular b -metric spaces). *Let (X, D, κ) be a complete rectangular b -metric space and $f: X \rightarrow X$ be a map such that for some $\lambda \in [0, \frac{1}{\kappa})$ and all $x, y \in X$,*

$$\begin{aligned} D(f(x), f(y)) \\ \leq \lambda \max \{D(x, y), D(x, f(x)), D(y, f(y)), D(x, f(y)), D(y, f(x))\}. \end{aligned} \quad (2.12)$$

Then f has a unique fixed point x^ and $\lim_{n \rightarrow \infty} f^n(x) = x^*$ for all $x \in X$.*

Proof. For each $x \in X$ and $m \leq n$ put

$$\begin{aligned} f(m, n)(x) &= \{f^i(x) : m \leq i \leq n\} \\ f(m, \infty)(x) &= \{f^i(x) : m \leq i\} \\ D(f(m, n)(x)) &= \sup\{D(u, v) : u, v \in f(m, n)(x)\} \\ D(f(m, \infty)(x)) &= \sup\{D(u, v) : u, v \in f(m, \infty)(x)\} \end{aligned}$$

where f^0 is the identity map on X . For $m \leq i \leq n-1$ and $m \leq j \leq n$, from (2.12) we find that

$$\begin{aligned} D(f^i(x), f^j(x)) &= D(ff^{i-1}(x), ff^{j-1}(x)) \\ &\leq \lambda \max \{D(f^{i-1}(x), f^{j-1}(x)), D(f^{i-1}(x), ff^{j-1}(x)), \\ &\quad D(f^{j-1}(x), ff^{j-1}(x)), D(f^{i-1}(x), ff^{j-1}(x)), \\ &\quad D(f^{j-1}(x), ff^{i-1}(x))\} \\ &= \lambda \max \{D(f^{i-1}(x), f^{j-1}(x)), D(f^{i-1}(x), f^i(x)), D(f^{j-1}(x), f^j(x)), \\ &\quad D(f^{i-1}(x), f^j(x)), D(f^{j-1}(x), f^i(x))\}. \end{aligned} \quad (2.13)$$

From (2.13), we get

$$D(f(m, n)(x)) \leq \lambda D(f(m-1, n)(x)). \quad (2.14)$$

Since $0 \leq \lambda < 1$, we see that

$$D(f(0, n)(x)) = \max\{D(x, f^i(x)) : 1 \leq i \leq n\}. \quad (2.15)$$

We now consider the following two cases.

Case 1: There exists $m < n$ such that $f^m(x) = f^n(x)$. Note that $f^m(x) = f^n(x)$, so we have

$$\begin{aligned} D(f(m+1, n)(x)) &= \sup\{D(f^i(x), f^j(x)) : m+1 \leq i, j \leq n\} \\ &= \sup\{D(f^i(x), f^j(x)) : m \leq i, j \leq n-1\} \\ &= D(f(m, n-1)(x)). \end{aligned} \quad (2.16)$$

Similarly, we have

$$\begin{aligned} D(f(m, n)(x)) &= \sup\{D(f^i(x), f^j(x)) : m \leq i, j \leq n\} \\ &= \sup\{D(f^i(x), f^j(x)) : m \leq i, j \leq n-1\} \\ &= D(f(m, n-1)(x)). \end{aligned} \quad (2.17)$$

It follows from (2.14), (2.16) and (2.17) that

$$\begin{aligned} D(f(m, n-1)(x)) &= D(f(m+1, n)(x)) \\ &\leq \lambda D(f(m, n)(x)) \\ &= \lambda D(f(m, n-1)(x)). \end{aligned}$$

Since $0 \leq \lambda < 1$, we get $D(f(m, n-1)(x)) = 0$ for all $n > m$. For $n = m+2$ we deduce that $D(f(m, m+1)(x)) = 0$. Then $x^* = f^m(x)$ is a fixed point of f . Moreover $D(f(m, \infty)(x)) = 0$. So $\lim_{n \rightarrow \infty} f^n(x) = x^*$.

Case 2: $f^m(x)$'s are all distinct. For each $n > 2$, by (2.15) there exists $1 \leq k_n(x) \leq n$ such that $D(x, f^{k_n(x)}(x)) = D(f(0, n)(x))$. If $k_n(x) \geq 3$ then by (2.14) we get

$$\begin{aligned} D(f(0, n)(x)) &= D(x, f^{k_n(x)}(x)) \\ &\leq \kappa[D(x, f(x)) + D(f(x), f^2(x)) + D(f^2(x), f^{k_n(x)}(x))] \\ &\leq 2\kappa D(f(0, 2)(x)) + \kappa\lambda D(f(1, k_n(x))(x)) \\ &\leq 2\kappa D(f(0, 2)(x)) + \kappa\lambda^2 D(f(0, k_n(x))(x)) \\ &\leq 2\kappa D(f(0, 2)(x)) + \kappa\lambda^2 D(f(0, n)(x)). \end{aligned}$$

Then

$$D(f(0, n)(x)) \leq \frac{2\kappa}{1 - \kappa\lambda^2} D(f(0, 2)(x)). \quad (2.18)$$

If $k_n(x) \leq 2$ then (2.18) obviously holds. Therefore $\{D(f(0, n)(x))\}$ is bounded. So $D(f(0, \infty)(x)) < \infty$. By (2.14) we have

$$D(f(m, \infty)(x)) \leq \lambda D(f(m-1, \infty)(x)) \leq \dots \leq \lambda^m D(f(0, \infty)(x)).$$

Then $\lim_{m \rightarrow \infty} D(f(m, \infty)(x)) = 0$. Therefore the sequence $\{f^n(x)\}$ is a Cauchy sequence. Since X is complete, there exists $x^* \in X$ such that

$$\lim_{n \rightarrow \infty} f^n(x) = x^*. \quad (2.19)$$

By (2.12) we get

$$\begin{aligned} D(f^{n+1}(x), f(x^*)) &= D(f f^n(x), f(x^*)) \\ &\leq \lambda \max \{D(f^n(x), x^*), D(f^n(x), f^{n+1}(x)), D(x^*, f(x^*)), \\ &\quad D(f^n(x), f(x^*)), D(x^*, f^{n+1}(x))\}. \end{aligned}$$

Using (2.19) and $\{x_n\}$ being a Cauchy sequence, we obtain

$$\begin{aligned} & \liminf_{n \rightarrow \infty} D(f^{n+1}(x), f(x^*)) \\ & \leq \lambda \max \{0, 0, D(x^*, f(x^*)), \liminf_{n \rightarrow \infty} D(f^n(x), f(x^*)), 0\} \\ & = \lambda \max \{D(x^*, f(x^*)), \liminf_{n \rightarrow \infty} D(f^n(x), f(x^*))\}. \end{aligned} \quad (2.20)$$

From (2.20), we consider two following subcases.

Subcase 2.1: $\liminf_{n \rightarrow \infty} D(f^{n+1}(x), f(x^*)) \leq \lambda \liminf_{n \rightarrow \infty} D(f^n(x), f(x^*))$.

From $\liminf_{n \rightarrow \infty} D(f^n(x), f(x^*)) = \liminf_{n \rightarrow \infty} D(f^{n+1}(x), f(x^*))$ and $0 \leq \lambda < \frac{1}{\kappa}$ we have $\liminf_{n \rightarrow \infty} D(f^n(x), f(x^*)) = 0$. So there exists a subsetsequence $\{f^{k_n}(x)\}$ of $\{f^n(x)\}$ such that

$$\lim_{n \rightarrow \infty} f^{k_n}(x) = f(x^*). \quad (2.21)$$

Note that all $f^n(x)$'s are distinct. So for n large enough we have

$$\begin{aligned} & D(x^*, f(x^*)) \\ & \leq \kappa [D(x^*, f^n(x)) + D(f^n(x), f^{k_n}(x)) + D(f^{k_n}(x), f(x^*))]. \end{aligned} \quad (2.22)$$

Letting $n \rightarrow \infty$ in (2.22) and using (2.21), (2.19) we obtain $D(x^*, f(x^*)) = 0$. Then $x^* = f(x^*)$.

Subcase 2.2: $\liminf_{n \rightarrow \infty} D(f^{n+1}(x), f(x^*)) \leq \lambda D(x^*, f(x^*))$.

For n large enough we have $f^n(x)$'s are distinct and different from $f(x^*)$ and x^* . So we find that

$$\begin{aligned} & D(x^*, f(x^*)) \\ & \leq \kappa [D(x^*, f^n(x)) + D(f^n(x), f^{n+1}(x)) + D(f^{n+1}(x), f(x^*))]. \end{aligned} \quad (2.23)$$

From (2.19) and (2.23) we deduce that

$$\liminf_{n \rightarrow \infty} D(f^{n+1}(x), f(x^*)) \geq \frac{1}{\kappa} D(x^*, f(x^*)). \quad (2.24)$$

On the contrary, suppose that $x^* \neq f(x^*)$. Note that $0 \leq \lambda < \frac{1}{\kappa}$. Then

$$\liminf_{n \rightarrow \infty} D(f^{n+1}(x), f(x^*)) \leq \lambda D(x^*, f(x^*)) < \frac{1}{\kappa} D(x^*, f(x^*)).$$

This is a contradiction with (2.24). Therefore $x^* = f(x^*)$.

By above Subcase 2.1 and Subcase 2.2, f has a fixed point x^* and by (2.19), $\lim_{n \rightarrow \infty} f^n(x) = x^*$.

By Case 1 and Case 2, f has a fixed point x^* and $\lim_{n \rightarrow \infty} f^n(x) = x^*$. We next show that the fixed point of f is unique. Indeed, let x^*, y^* be two fixed points of f . From (2.12)

we have

$$\begin{aligned} D(x^*, y^*) &= D(f(x^*), f(y^*)) \\ &\leq \lambda \max \{D(x^*, y^*), D(x^*, f(x^*)), D(y^*, f(y^*)), D(x^*, f(y^*)), D(y^*, f(x^*))\} \\ &= \lambda D(x^*, y^*). \end{aligned}$$

Since $\lambda \in [0, \frac{1}{\kappa})$, we obtain $D(x^*, y^*) = 0$, that is, $x^* = y^*$. Then the fixed point of f is unique. $\square$

From Theorem 5 we get the following corollaries since contraction conditions (2.25), (2.26), (2.27), (2.28) are particular cases of the contraction condition (2.12). Moreover, Theorem 5, Corollary 2, Corollary 3 and Corollary 4 are analogues of Ćirić contraction, Hardy-Rogers contraction, Reich contraction and Chatterjea contraction in rectangular b -metric spaces respectively, that are answers to Question 2.

Corollary 2 (Hardy-Rogers type fixed point theorem in rectangular b -metric spaces). *Let (X, D, κ) be a complete rectangular b -metric space and $f: X \rightarrow X$ be a map such that there exist $a_i \geq 0$, $i = 1, \dots, 5$, $\sum_{i=1}^5 a_i < \frac{1}{\kappa}$ and for all $x, y \in X$,*

$$\begin{aligned} D(f(x), f(y)) &\leq a_1 D(x, y) + a_2 D(x, f(x)) + a_3 D(y, f(y)) + a_4 D(x, f(y)) + a_5 D(y, f(x)). \end{aligned} \quad (2.25)$$

Then f has a unique fixed point x^ and $\lim_{n \rightarrow \infty} f^n(x) = x^*$ for all $x \in X$.*

Corollary 3 (Reich type fixed point theorem in rectangular b -metric spaces). *Let (X, D, κ) be a complete rectangular b -metric space and $f: X \rightarrow X$ be a map such that there exist $a, b, c \geq 0$, $a + b + c < \frac{1}{\kappa}$ and for all $x, y \in X$,*

$$D(f(x), f(y)) \leq aD(x, y) + bD(x, f(x)) + cD(y, f(y)). \quad (2.26)$$

Then f has a unique fixed point x^ and $\lim_{n \rightarrow \infty} f^n(x) = x^*$ for all $x \in X$.*

Corollary 4 (Chatterjea type fixed point theorem in rectangular b -metric spaces). *Let (X, D, κ) be a complete rectangular b -metric space and $f: X \rightarrow X$ be a map such that for some $a \in [0, \frac{1}{2\kappa})$ and for all $x, y \in X$,*

$$D(f(x), f(y)) \leq a[D(x, f(y)) + D(y, f(x))]. \quad (2.27)$$

Then f has a unique fixed point x^ and $\lim_{n \rightarrow \infty} f^n(x) = x^*$ for all $x \in X$.*

Corollary 5 (Kannan type fixed point theorem in rectangular b -metric spaces). *Let (X, D, κ) be a complete rectangular b -metric space and $f: X \rightarrow X$ be a map such that for some $a \in [0, \frac{1}{2\kappa})$ and for all $x, y \in X$,*

$$D(f(x), f(y)) \leq a[D(x, f(x)) + D(y, f(y))]. \quad (2.28)$$

Then f has a unique fixed point x^ and $\lim_{n \rightarrow \infty} f^n(x) = x^*$ for all $x \in X$.*

Finally, the following example shows that the domain of contraction constant $[0, \frac{1}{\kappa})$ in Corollary 3 may not be relaxed to $[0, 1)$. Then so may not the domains in Theorem 5.

Example 1. Let $X = \{0, 1, \frac{1}{2}, \dots, \frac{1}{n}, \dots\}$, and

$$D(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y \in \{0, 1\}, \\ |x - y| & \text{if } x \neq y \in \{0\} \cup \{\frac{1}{2n} : n = 1, 2, \dots\}, \\ \frac{1}{4} & \text{otherwise,} \end{cases}$$

and let $f: X \rightarrow X$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{1}{10n} & \text{if } x = \frac{1}{n}, n = 1, 2, \dots \end{cases}$$

Then

- (1) (X, D, κ) is a complete rectangular b -metric space with the coefficient $\kappa = 4$.
- (2) There exist $a, b, c \geq 0$, $\frac{1}{\kappa} \leq a + b + c < 1$ such that the contraction condition (2.26) holds for all $x, y \in X$.
- (3) f is fixed point free.

Proof. By [9, Example 2.6], (X, D, κ) is a complete metric-type space with the coefficient $\kappa = 4$. Then (X, D, κ) is also a complete rectangular b -metric on X with the coefficient $\kappa = 4$. The remaining conclusions were proved in [9, Example 2.6]. $\square$

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S-COPURE SUBMODULES OF A MODULE

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Abstract. Let R be a commutative ring with identity, S be a multiplicatively closed subset of R , and M be an R -module. The aim of this paper is to introduce the notion of S -copure submodules and investigate some properties of this class of modules. We say that a submodule N of M is S -copure if there exists an $s \in S$ such that $s(N :_M I) \subseteq N + (0 :_M I)$ for every ideal I of R .

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1. Introduction

Throughout this paper R will denote a commutative ring with identity and $\mathbb{Z}$ will denote the ring of integers.

It is well known that the notions of purity and copurity with their different generalizations play a fundamental role in theory of module categories. In [2], Anderson and Fuller defined a submodule N of an R -module M a *pure submodule* if $IN = N \cap IM$ for every ideal I of R . In [4], H. Ansari-Toroghy and F. Farshadifar introduced the dual notion of pure submodules (that is copure submodules). A submodule N of an R -module M is said to be *copure* if $(N :_M I) = N + (0 :_M I)$ for every ideal I of R [4], where $(N :_M I) = \{x \in M : Ix \subseteq N\}$. An R -module M is said to be *fully pure* (resp. *fully copure*) if every submodule of M is pure (resp. copure) [6].

Let S be a multiplicatively closed subset of R . In [7], F. Farshadifar introduced and investigated the concept of S -pure submodules of modules as a generalization of pure submodules. A submodule N of an R -module M is said to be *S -pure* if there exists an $s \in S$ such that $s(N \cap IM) \subseteq IN$ for every ideal I of R [7]. Also, an R -module M is said to be *fully S -pure* if every submodule of M is S -pure [7].

Let S be a multiplicatively closed subset of R and M be an R -module. In this paper, we introduce the notion of S -copure submodules of M as a generalisation of copure submodules. Also, this notion can be regarded as a dual notion of S -pure submodules. We provide some useful information concerning the is new class of modules.

2. Main results

Throughout this section, S is a multiplicatively closed subset of R , that is, $1 \in S$ and $s_1 s_2 \in S$ for any $s_1 \in S$ and any $s_2 \in S$.

Definition 1. We say that a submodule N of an R -module M is S -copure if there exists an $s \in S$ such that $s(N :_M I) \subseteq N + (0 :_M I)$ for every ideal I of R .

Definition 2. We say that an R -module M is *fully S -copure* if every submodule of M is S -copure.

Example 1. Let M be an R -module with $\text{Ann}_R(M) \cap S \neq \emptyset$. Then clearly, M is a fully S -copure R -module.

Proposition 1. Every fully copure R -module is a fully S -copure R -module. The converse is true if $S \subseteq U(R)$, where $U(R)$ is the set of units in R .

Proof. This is clear. □

The following example shows that the converse of Proposition 1 is not true in general.

Example 2. Clearly, for a prime number p , the submodule $p\mathbb{Z}$ of the $\mathbb{Z}$ -module $\mathbb{Z}$ is not copure. Take the multiplicatively closed subset $S = \{p^n : n \in \mathbb{N} \cup \{0\}\}$ of $\mathbb{Z}$. Then for each $k \in \mathbb{N}$, $p(p\mathbb{Z} :_{\mathbb{Z}} k\mathbb{Z}) \subseteq p\mathbb{Z} + (0 :_{\mathbb{Z}} k\mathbb{Z})$ implies that $p\mathbb{Z}$ is an S -copure submodule of $\mathbb{Z}$.

Theorem 1. Let M be an R -module, and let N and K be submodules of M such that $N \subseteq K \subseteq M$. Then we have the following.

- (a) If K is an S -copure submodule of M and N is a S -copure submodule of K , then N is an S -copure submodule of M .
- (b) If N is an S -copure submodule of M , then N is an S -copure submodule of K .
- (c) If K is an S -copure submodule of M , then K/N is an S -copure submodule of M/N .
- (d) If N is an S -copure submodule of M and K/N is an S -copure submodule of M/N , then K is an S -copure submodule of M .
- (e) If N is an S -copure submodule of M , then there is a bijection between the S -copure submodules of M containing N and the S -copure submodules of M/N .

Proof. (a) Let I be an ideal of R . Then since K is an S -copure submodule of M , there exists an $s \in S$ such that

$$\begin{aligned} s(N :_M I) &= s(N \cap K :_M I) = s((N :_M I) \cap (K :_M I)) \\ &\subseteq (N :_M I) \cap (K + (0 :_M I)) = (N :_K I) + (0 :_M I). \end{aligned}$$

Now since N is an S -copure submodule of K , there exists an $t \in S$ such that

$$st(N :_M I) \subseteq t(N :_K I) + t(0 :_M I) \subseteq N + (0 :_K I) + (0 :_M I) = N + (0 :_M I).$$

(b) Let I be an ideal of R . Then as N is an S -copure submodule of M , there exists an $s \in S$ such that

$$\begin{aligned} s(N :_K I) &= s(K \cap (N :_M I)) \subseteq K \cap (N + (0 :_M I)) \\ &= K \cap N + K \cap (0 :_M I) \subseteq N + (0 :_M I). \end{aligned}$$

(c) Let I be an ideal of R . Then there exists an $s \in S$ such that

$$s(K :_K I) \subseteq K + (0 :_M I).$$

Thus

$$\begin{aligned} s(K/N :_{M/N} I) &= s((K :_M I)/N) = s((K :_M I) + K \cap (N :_M I))/N \\ &\subseteq (K + (0 :_M I) + K \cap (N :_M I))/N \\ &= K/N + ((N :_M I) \cap (K + (0 :_M I)))/N \\ &\subseteq K/N + ((N :_M I) \cap (K :_M I))/N \\ &= K/N + (N :_M I)/N = K/N + (0 :_{M/N} I). \end{aligned}$$

(d) Let I be an ideal of R . Since N is an S -copure submodule of M , there exists an $s \in S$ such that $s(N :_M I) \subseteq N + (0 :_M I)$. We have

$$s(0 :_{M/N} I) = s(N :_M I)/N \subseteq ((0 :_M I) + N)/N.$$

Now since K/N is an S -copure submodule of M/N , there exists an $t \in S$ such that

$$t(K/N :_{M/N} I) \subseteq K/N + (0 :_{M/N} I).$$

Therefore,

$$\begin{aligned} st(K :_M I)/N &= st(K/N :_{M/N} I) \subseteq sK/N + s(0 :_{M/N} I) \\ &\subseteq K/N + ((0 :_M I) + N)/N = (K + N + (0 :_M I))/N \\ &= (K + (0 :_M I))/N. \end{aligned}$$

Thus $st(K :_M I) \subseteq K + (0 :_M I)$, as desired.

(e) This follows from parts (c) and (d). $\square$

Recall that the saturation S^* of S is defined as $S^* = \{x \in R : x/1 \text{ is a unit of } S^{-1}R\}$. It is obvious that S^* is a multiplicatively closed subset of R containing S [9].

A multiplicatively closed subset S of R is said to satisfy the *maximal multiple condition* if there exists an $s \in S$ such that $t \mid s$ for each $t \in S$.

Proposition 2. *Let M be an R -module. Then we have the following.*

- (a) *If $S_1 \subseteq S_2$ are multiplicatively closed subsets of R and M is a fully S_1 -copure R -module, then M is a fully S_2 -copure R -module.*
- (b) *M is a fully S -copure R -module if and only if M is a fully S^* -copure R -module.*
- (c) *If N is an S -copure submodule of M , then sN is an S -copure submodule of M for each $s \in S$.*

- (d) If $f : M \rightarrow M$ is an endomorphism and there exists an $s \in S$ such that $sf(x) = f^2(x)$ for each $x \in M$, then $\text{Ker}(f)$ is an S -copure submodule of M .
- (e) If N and K are submodules of M such that $N \cap K$ and $N + K$ are S -copure submodules of M . Then N is an S -copure submodule of M .
- (f) If S is satisfying the maximal multiple condition (e.g., S is finite or $S \subseteq U(R)$) and $\{M_\lambda\}_\Lambda$ is a family of submodules of M with S -copure submodules $N_\lambda \subseteq M_\lambda$, then $\bigoplus_{\lambda \in \Lambda} N_\lambda$ is an S -copure submodule of $\bigoplus_{\lambda \in \Lambda} M_\lambda$.

Proof. (a) This is clear.

(b) Let M be a fully S -copure R -module. Since $S \subseteq S^*$, by part (a), M is a fully S^* -copure R -module. For the converse, assume that M is a fully S^* -copure module, N is a submodule of M , and I is an ideal of R . Then there exists an $x \in S^*$ such that $x(N :_M I) \subseteq N + (0 :_M I)$. As $x \in S^*$, $x/1$ is a unit of $S^{-1}R$ and so $(x/1)(a/s) = 1$ for some $a \in R$ and $s \in S$. This yields that $us = uxa$ for some $u \in S$. Thus we have

$$us(N :_M I) = uxa(N :_M I) \subseteq x(N :_M I) \subseteq N + (0 :_M I).$$

Therefore, M is a fully S -copure R -module.

(c) Let $s \in S$. As N is S -copure, there is an $t \in S$ such that $t(N :_M I) \subseteq N + (0 :_M I)$ for each ideal I of R . Therefore,

$$ts(sN :_M I) \subseteq ts(N :_M I) \subseteq sN + s(0 :_M I) \subseteq sN + (0 :_M I).$$

(d) Let I be an ideal of R and $x \in (\text{Ker}(f) :_M I)$. Then $xI \subseteq \text{Ker}(f)$. It follows that $f(x) \in (0 :_M I)$. As $sf = f^2$, we have $sx - f(x) \in \text{Ker}(f)$. Therefore, $sx = sx - f(x) + f(x) \in \text{Ker}(f) + (0 :_M I)$. This implies that

$$s(\text{Ker}(f) :_M I) \subseteq \text{Ker}(f) + (0 :_M I).$$

(e) Let I be an ideal of R and let $m \in (N :_M I)$. Since $N + K$ is an S -copure submodule of M , there exists an $s \in S$ such that $s(N + K :_M I) \subseteq N + K + (0 :_M I)$. Then $Im \subseteq N + K$ implies that $sm = x + y + t$ for some $x \in N$, $y \in K$ and $t \in (0 :_M I)$. Thus $msI = xI + yI$. This implies that $yI \subseteq N \cap K$. Since $N \cap K$ is an S -copure submodule of M , there exists an $h \in S$ such that $h(N \cap K :_M I) \subseteq N \cap K + (0 :_M I)$. Thus $hy = \acute{x} + \acute{t}$ for some $\acute{x} \in N \cap K$ and $\acute{t} \in (0 :_M I)$. It follows that $shm \in N + (0 :_M I)$. Therefore, $sh(N :_M I) \subseteq N + (0 :_M I)$, as desired.

(f) Let I be an ideal of R . Then there exists an $s \in S$ such that $s(N_\lambda :_{M_\lambda} I) \subseteq N_\lambda + (0 :_{M_\lambda} I)$ for each $\lambda \in \Lambda$. Now one can see that $s(\bigoplus_{\lambda \in \Lambda} N_\lambda :_{\bigoplus_{\lambda \in \Lambda} M_\lambda} I) \subseteq \bigoplus_{\lambda \in \Lambda} N_\lambda + (0 :_{\bigoplus_{\lambda \in \Lambda} M_\lambda} I)$. $\square$

Definition 3. We say that a submodule N of an R -module M is an S -direct summand of M if there exist a submodule K of M and $s \in S$ such that $sM = N + K$ (d.s.).

Definition 4. We say that an R -module M is an S -semisimple module if every submodule of M is an S -direct summand of M .

Proposition 3. Let M be an S -semisimple R -module. Then M is a fully S -copure R -module.

Proof. Let N be a submodule of M . Then there exist a submodule K of M and $s \in S$ such that $sM = N + K$ (d.s.). Now for each ideal I of R , we have

$$s(N :_M I) = (N :_{sM} I) = (N :_K I) + (N :_N I) \subseteq (0 :_K I) + N \subseteq (0 :_M I) + N.$$

□

Proposition 4. *Let R be a principal ideal domain and M be an R -module. Then every submodule of M is an S -pure submodule if and only if it is an S -copure submodule.*

Proof. First suppose that N is an S -pure submodule of M and $r \in R$. Then there exists an $s \in S$ such that $s(N \cap rM) \subseteq rN$. Now let $rm \in N$. Then $srM = srN$, for some $n \in N$. Thus, $sm = s(m - n) + sn \in (0 :_M r) + N$. So, $s(N :_M r) \subseteq N + (0 :_M r)$ and N is S -copure. Now suppose that N is an S -copure submodule of M and $r \in R$.

Then there exists an $s \in S$ such that $s(N :_M r) \subseteq N + (0 :_M r)$. Suppose that $rm \in N$. Then $sm = n_1 + m_1$, where $n_1 \in N$ and $rm_1 = 0$. Thus $srM = rn_1 \in rN$. This shows that N is an S -pure submodule of M . □

Theorem 2. *Let M be a distributive R -module. Then the following hold.*

- (a) *A submodule N of M is S -copure if and only if there exists an $s \in S$ such that for each $a \in R$ we have*

$$s(N :_M a) \subseteq N + (0 :_M a).$$

- (b) *A submodule N of M is S -pure if and only if there exists an $s \in S$ such that for each $a \in R$ we have*

$$s(N \cap aM) \subseteq aN.$$

- (c) *A submodule N of M is an S -pure submodule if and only if it is an S -copure submodule.*

Proof. (a) First assume that there exists an $s \in S$ such that for each $a \in R$ we have $s(N :_M a) \subseteq N + (0 :_M a)$. Suppose that I is an ideal of R . Then we have

$$s(N :_M I) = s(N :_M \sum_{a \in I} Ra) = s \bigcap_{a \in I} (N :_M a) \subseteq \bigcap_{a \in I} (N + (0 :_M a)).$$

Now as M is distributive, we have

$$\bigcap_{a \in I} (N + (0 :_M a)) = N + \bigcap_{a \in I} (0 :_M a) = N + (0 :_M I).$$

Therefore, N is an S -copure submodule of M . The converse is clear.

(b) First suppose that exists an $s \in S$ such that for each $a \in R$ we have $s(N \cap aM) \subseteq aN$. Let I be an ideal of R . Then as M is a distributive R -module, we have

$$\begin{aligned} IN &= (\sum_{a \in I} Ra)N \supseteq \sum_{a \in I} s(RaM \cap N) \supseteq s \sum_{a \in I} (RaM \cap N) \\ &= s((\sum_{a \in I} Ra)M \cap N) = s(IM \cap N). \end{aligned}$$

Hence, N is an S -pure submodule of M . The converse is clear.

(c) This follows from parts (a), (b), and Proposition 4. $\square$

Proposition 5. *Let R be a Noetherian ring and let M be an R -module. Then the following hold.*

- (a) *If N is an S -copure submodule of M , then for each prime ideal $\mathfrak{p}$ of R , $N_{\mathfrak{p}}$ is an $S_{\mathfrak{p}}$ -copure submodule of $M_{\mathfrak{p}}$ as an $R_{\mathfrak{p}}$ -module.*
- (b) *If $N_{\mathfrak{m}}$ is an $S_{\mathfrak{m}}$ -copure submodule of an $R_{\mathfrak{m}}$ -module $M_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m}$ of R , then N is an S -copure submodule of M .*

Proof. (a) This follows from the fact that by [10, 9.13], if I is a finitely generated ideal of R , then $(N :_M I)_{\mathfrak{p}} = (N_{\mathfrak{p}} :_{M_{\mathfrak{p}}} I_{\mathfrak{p}})$.

(b) Suppose that I is an ideal of R . As R is a Noetherian ring, I is finitely generated ideal of R . Hence by [10, 9.13], for any maximal ideal $\mathfrak{m}$ of R , $(N :_M I)_{\mathfrak{m}} = (N_{\mathfrak{m}} :_{M_{\mathfrak{m}}} I_{\mathfrak{m}})$. Thus by assumption, for any maximal ideal $\mathfrak{m}$ of R , there is an $h/a \in S_{\mathfrak{m}}$ such that

$$(h(N :_M I))_{\mathfrak{m}} = h/a(N_{\mathfrak{m}} :_{M_{\mathfrak{m}}} I_{\mathfrak{m}}) \subseteq N_{\mathfrak{m}} + (0 :_{M_{\mathfrak{m}}} I_{\mathfrak{m}}) = (N + (0 :_M I))_{\mathfrak{m}}.$$

It follows that

$$h(N :_M I) \subseteq N + (0 :_M I).$$

$\square$

Let M be an R -module. M is said to be a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$ [3]. M satisfies the *double annihilator conditions* (DAC for short) if for each ideal I of R , we have $I = \text{Ann}_R((0 :_M I))$. M is said to be a *strong comultiplication module* if M is a comultiplication R -module which satisfies the double annihilator condition [4]. M is said to be an *S -comultiplication module* if for each submodule N of M , there exist $s \in S$ and an ideal I of R such that $s(0 :_M I) \subseteq N \subseteq (0 :_M I)$ [1].

Definition 5. We say that an R -module M satisfies the *S -double annihilator condition* (S -DAC for short) if for each ideal I of R there exists an $s \in S$ such that $s\text{Ann}_R((0 :_M I)) \subseteq I$.

Definition 6. We say that an R -module M is an *S -strong comultiplication module* if M is an S -comultiplication R -module which satisfies the S -double annihilator condition.

Lemma 1. *Let M be an S -strong comultiplication R -module. Then we have the following.*

- (a) *If I and J are ideals of R with $(0 :_M I) \subseteq (0 :_M J)$, then there exists an $s \in S$ such that $sJ \subseteq I$.*
- (b) *For ideals I and J of R there exists an $t \in S$ such that*

$$t(0 :_M I \cap J) \subseteq (0 :_M I) + (0 :_M J).$$

Proof. (a) Let I and J be ideals of R with $(0 :_M I) \subseteq (0 :_M J)$. Then $\text{Ann}_R((0 :_M J)) \subseteq \text{Ann}_R((0 :_M I))$. As M satisfies the S -double annihilator conditions, there exists an $s \in S$ such that $s\text{Ann}_R((0 :_M I)) \subseteq I$. Thus $sJ \subseteq s\text{Ann}_R((0 :_M J)) \subseteq I$.

(b) As M satisfies the S -double annihilator condition, there exist $s, t \in S$ such that $s\text{Ann}_R((0 :_M I)) \subseteq I$ and $t\text{Ann}_R((0 :_M J)) \subseteq J$. Thus $s\text{Ann}_R((0 :_M I)) \cap t\text{Ann}_R((0 :_M J)) \subseteq I \cap J$. It follows that

$$(0 :_M I \cap J) \subseteq (0 :_M st(\text{Ann}_R((0 :_M I)) \cap \text{Ann}_R((0 :_M J)))).$$

Since M is an S -comultiplication module, there exists an $h \in S$ such that

$$h(0 :_M \text{Ann}_R((0 :_M I) + (0 :_M J))) \subseteq (0 :_M I) + (0 :_M J).$$

Therefore, we have

$$hst(0 :_M I \cap J) \subseteq h(0 :_M \text{Ann}_R((0 :_M I) + (0 :_M J))) \subseteq (0 :_M I) + (0 :_M J).$$

□

Theorem 3. *Let M be an S -strong comultiplication R -module. Then we have the following.*

- (a) *N is an S -copure submodule of M if and only if $\text{Ann}_R(N)$ is an S -pure ideal of R .*
- (b) *An ideal I of R is S -pure if and only if $(0 :_M I)$ is an S -copure submodule of M .*

Proof. (a) Let N be an S -copure submodule of M and let I be an ideal of R . As M is an S -comultiplication R -module, there exists an $t \in S$ such that $t(0 :_M \text{Ann}_R(N)) \subseteq N$ and so $(0 :_M \text{Ann}_R(N)) \subseteq (N :_M t)$. It follows that

$$(0 :_M \text{Ann}_R(N)I) = ((0 :_M \text{Ann}_R(N)) :_M I) \subseteq (N :_M tI).$$

Since N is an S -copure submodule of M , there exists an $s \in S$ such that

$$s(N :_M tI) \subseteq N + (0 :_M tI) \subseteq (0 :_M \text{Ann}_R(N)) + (0 :_M tI) \subseteq (0 :_M \text{Ann}_R(N) \cap tI).$$

Therefore, $(0 :_M \text{Ann}_R(N)I) \subseteq (0 :_M (s(\text{Ann}_R(N) \cap tI)))$. This in turn implies that $hst(\text{Ann}_R(N) \cap I) \subseteq \text{Ann}_R(N)I$ for some $h \in S$ by using Lemma 1 (a). Conversely, assume that N is a submodule of M such that $\text{Ann}_R(N)$ is an S -pure ideal of R and I be an ideal of R . Then there exists an $s \in S$ such that $s(\text{Ann}_R(N) \cap I) \subseteq \text{Ann}_R(N)I$. Now we have

$$(N :_M I) \subseteq (0 :_M \text{Ann}_R(N)I) \subseteq (0 :_M s(\text{Ann}_R(N) \cap I)) = ((0 :_M \text{Ann}_R(N) \cap I) :_M s).$$

This implies that $s(N :_M I) \subseteq (0 :_M \text{Ann}_R(N) \cap I)$. By Lemma 1 (b), there exists an $t \in S$ such that $t(0 :_M \text{Ann}_R(N) \cap I) \subseteq (0 :_M \text{Ann}_R(N)) + (0 :_M I)$. As M is an S -comultiplication R -module, there exists an $h \in S$ such that $h(0 :_M \text{Ann}_R(N)) \subseteq N$. Therefore, we have

$$tsh(N :_M I) \subseteq th(0 :_M \text{Ann}_R(N) \cap I) \subseteq h(0 :_M \text{Ann}_R(N)) + h(0 :_M I) \subseteq N + (0 :_M I),$$

as desired.

(b) Let I be an S -pure ideal of R . Then there is an $s \in S$ such that $s(I \cap J) \subseteq IJ$ for each ideal J of R . Now we have

$$((0 :_M I) :_M J) = (0 :_M IJ) \subseteq (0 :_M s(I \cap J)).$$

It follows that $s((0 :_M I) :_M J) \subseteq (0 :_M I \cap J)$. By Lemma 1 (b), there exists an $t \in S$ such that $t(0 :_M I \cap J) \subseteq (0 :_M I) + (0 :_M J)$. Therefore,

$$ts((0 :_M I) :_M J) \subseteq (0 :_M I) + (0 :_M J).$$

Conversely, assume that $(0 :_M I)$ is an S -copure submodule of M and J is an ideal of R . Then by part (a), $\text{Ann}_R((0 :_M I))$ is an S -pure ideal of R . Thus there exists an $s \in S$ such that $s(\text{Ann}_R((0 :_M I)) \cap I) \subseteq \text{Ann}_R((0 :_M I))I$ for each ideal I of R . Hence we have

$$s(I \cap J) = s(I \cap \text{Ann}_R((0 :_M I)) \cap J) \subseteq \text{Ann}_R((0 :_M I))(I \cap J)$$

As M satisfies the S -double annihilator condition, there exists an $t \in S$ such that $t\text{Ann}_R((0 :_M I)) \subseteq I$. Hence, $st(I \cap J) \subseteq I(I \cap J) \subseteq I^2 \cap IJ \subseteq IJ$, as needed. $\square$

A proper submodule N of an R -module M is said to be *completely irreducible* if $N = \bigcap_{i \in I} N_i$, where $\{N_i\}_{i \in I}$ is a family of submodules of M , implies that $N = N_i$ for some $i \in I$. It is easy to see that every submodule of M is an intersection of completely irreducible submodules of M [8].

Remark 1. Let N and K be two submodules of an R -module M . To prove $N \subseteq K$, it is enough to show that if L is a completely irreducible submodule of M such that $K \subseteq L$, then $N \subseteq L$ [5].

A family $\{N_i\}_{i \in I}$ of submodules of an R -module M is said to be an *inverse family of submodules of M* if the intersection of two of its submodules again contains a module in $\{N_i\}_{i \in I}$. Also, M satisfies the property $AB5^*$ if for every submodule K of M and every inverse family $\{N_i\}_{i \in I}$ of submodules of M , $K + \bigcap_{i \in I} N_i = \bigcap_{i \in I} (K + N_i)$ [11].

Theorem 4. Let S be a multiplicatively closed subset of R which satisfies the maximal multiple condition (e.g., S is finite or $S \subseteq U(R)$) and M be an R -module which satisfies the property $AB5^*$. Then we have the following.

- (a) If $\{N_\lambda\}_{\lambda \in \Lambda}$ is a chain of S -copure submodules of M , then $\bigcap_{\lambda \in \Lambda} N_\lambda$ is S -copure.
- (b) If N is a submodule of M , then there is a submodule K of M minimal with respect to $N \subseteq K$ and K is an S -copure submodule of M .

Proof. (a) Let I be an ideal of R . Let L be a completely irreducible submodule of M such that $\bigcap_{\lambda \in \Lambda} N_\lambda + (0 :_M I) \subseteq L$. Then $\bigcap_{\lambda \in \Lambda} N_\lambda + (0 :_M I) + L = L$. Since M satisfies the property $AB5^*$, we have

$$\bigcap_{\lambda \in \Lambda} (N_\lambda + (0 :_M I) + L) = L.$$

Now as L is a completely irreducible submodule of M , there exists an $\alpha \in \Lambda$ such that $N_\alpha + (0 :_M I) + L = L$. Now as S satisfies the maximal multiple condition, there exists

an $s \in S$ such that $s(N_\alpha :_M I) + L \subseteq L$ since N_α is an S -copure submodule of M . Thus $s(N_\alpha :_M I) \subseteq L$. Hence, $s(\cap_{\lambda \in \Lambda} N_\lambda :_M I) \subseteq L$. This implies that

$$s(\cap_{\lambda \in \Lambda} N_\lambda :_M I) \subseteq \cap_{\lambda \in \Lambda} N_\lambda + (0 :_M I),$$

by Remark 1.

(b) Let

$$\Sigma = \{N \leq H \mid H \text{ is a } S\text{-copure submodule of } M\}.$$

Then $M \in \Sigma \neq \emptyset$. Let $\{N_\lambda\}_{\lambda \in \Lambda}$ be a totally ordered subset of Σ . Then $N \leq \cap_{\lambda \in \Lambda} N_\lambda$ and by part (a), $\cap_{\lambda \in \Lambda} N_\lambda$ is an S -copure submodule of M . Thus by using Zorn's Lemma, one can see that Σ has a minimal element, K say as needed. $\square$

Let R_i be a commutative ring with identity and M_i be an R_i -module, for $i = 1, 2$. Let $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is an R -module. Clearly, every submodule of M is in the form of $N = N_1 \times N_2$ for some submodules N_1 of M_1 and N_2 of M_2 . Also, if S_i is a multiplicatively closed subset of R_i for each $i = 1, 2$, then $S = S_1 \times S_2$ is a multiplicatively closed subset of R .

Theorem 5. *Let M_i be an R_i -module and $S_i \subseteq R_i$ be a multiplicatively closed subset for $i = 1, 2$. Assume that $M = M_1 \times M_2$, $R = R_1 \times R_2$, and $S = S_1 \times S_2$. Then M is a fully S -copure module if and only if M_i is a fully S_i -copure module for $i = 1, 2$.*

Proof. For only if part, without loss of generality we will show M_1 is a fully S_1 -copure R_1 -module. Take a submodule N_1 of M_1 and ideal I_1 of R_1 . Then $N_1 \times \{0\}$ is a submodule of M and $I_1 \times \{0\}$ is an ideal of R . Since M is a fully S -copure R -module, there exists $s = (s_1, s_2) \in S_1 \times S_2$ such that

$$(s_1, s_2)(N_1 \times \{0\} :_M I_1 \times \{0\}) \subseteq N_1 \times \{0\} + (0 :_M I_1 \times \{0\}).$$

By focusing on first coordinate, we have $s_1(N_1 :_{M_1} I_1) \subseteq N_1 + (0 :_{M_1} I_1)$. So M_1 is a fully S_1 -copure R_1 -module. Now assume that M_1 is a fully S_1 -copure module and M_2 is a fully S_2 -copure module. Take a submodule N of M and ideal I of R . Then N must be in the form of $N_1 \times N_2$ and $I = I_1 \times I_2$, where $N_1 \subseteq M_1, N_2 \subseteq M_2$ and $I_1 \subseteq R_1, I_2 \subseteq R_2$. Since M_1 is a fully S_1 -copure R_1 -module, there exists an $s_1 \in S_1$ such that $s_1(N_1 :_{M_1} I_1) \subseteq N_1 + (0 :_{M_1} I_1)$. Similarly, there exists an element $s_2 \in S_2$ such that $s_2(N_2 :_{M_2} I_2) \subseteq N_2 + (0 :_{M_2} I_2)$. Now, put $s = (s_1, s_2) \in S$. Then we get

$$\begin{aligned} (s_1, s_2)(N :_M I) &= (s_1, s_2)(N_1 \times N_2 :_{M_1 \times M_2} I_1 \times I_2) \\ &= s_1(N_1 :_{M_1} I_1) \times s_2(N_2 :_{M_2} I_2) \\ &\subseteq (N_1 + (0 :_{M_1} I_1)) \times (N_2 + (0 :_{M_2} I_2)) \\ &= N_1 \times N_2 + (0 :_{M_1 \times M_2} I_1 \times I_2) = N + (0 :_M I). \end{aligned}$$

Hence, M is a fully S -copure R -module. $\square$

In the following theorem, we characterize the fully copure R -modules.

Theorem 6. *Let M be an R -module. Then the following statements are equivalent:*

- (a) M is a fully copure R -module;
- (b) M is a fully $(R - \mathfrak{p})$ -copure R -module for each prime ideal $\mathfrak{p}$ of R ;
- (c) M is a fully $(R - \mathfrak{m})$ -copure R -module for each maximal ideal $\mathfrak{m}$ of R ;
- (d) M is a fully $(R - \mathfrak{m})$ -copure R -module for each maximal ideal $\mathfrak{m}$ of R with $M_{\mathfrak{m}} \neq 0_{\mathfrak{m}}$.

Proof. (a) $\Rightarrow$ (b) Let M be a fully copure R -module and $\mathfrak{p}$ be a prime ideal of R . Then $R - \mathfrak{p}$ is multiplicatively closed set of R and so M is a fully $(R - \mathfrak{p})$ -copure R -module by Proposition 1.

(b) $\Rightarrow$ (c) Since every maximal ideal is a prime ideal, the result follows from the part (b).

(c) $\Rightarrow$ (d) This is clear.

(d) $\Rightarrow$ (a) Let N be a submodule of M and I be an ideal of R . Take a maximal ideal $\mathfrak{m}$ of R with $M_{\mathfrak{m}} \neq 0_{\mathfrak{m}}$. As M is a fully $(R - \mathfrak{m})$ -copure module, there exists an $s \notin \mathfrak{m}$ such that $s(N :_M I) \subseteq N + (0 :_M I)$. This implies that

$$(N :_M I)_{\mathfrak{m}} = (s(N :_M I))_{\mathfrak{m}} \subseteq N_{\mathfrak{m}} + (0 :_M I)_{\mathfrak{m}}.$$

Now we have $(N :_M I)_{\mathfrak{m}} \subseteq N_{\mathfrak{m}} + (0 :_M I)_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m}$ of R . It follows that $(N :_M I) \subseteq N + (0 :_M I)$, as needed. $\square$

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AVERAGE CONSENSUS STABILITY ANALYSIS FOR STOCHASTIC MULTI-AGENT SYSTEMS WITH COMPOUND NOISES UNDER MARKOVIAN SWITCHING TOPOLOGIES

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Abstract. In this paper, a new class of multi-agent systems with additive and multiplicative measurement noises under Markovian switching topologies is introduced. Some sufficient conditions of average consensus to the systems are also established under consideration. First, based on the continuous-time Markov chain with finite modes, the time-varying topologies of the considering system is figured. Then, the mean square average consensus for the multi-agent systems with a time-varying gain under the time-varying topologies and the Markov chain is studied. Finally, the example and simulation results are given to illustrate the effectiveness of the obtained theoretical results.

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1. INTRODUCTION

The mathematical models subjected to the phenomenon of uncertain environment for collective behavior have been widely introduced and analyzed (see e.g., [9, 16, 17], and its special cases, flocking [1, 2] and consensus [5–7, 10, 11, 15–18]). Noises are ubiquitous stochastic phenomena in the real world which are usually modeled as additive noise [5, 7, 16] or multiplicative noises [1, 6, 11]. Recently, Li et al. considered the additive noise [7] and multiplicative noises [6] for continuous-time models under fixed topologies, respectively. Besides, random link failures often happen in the various systems which motives to study the dynamics with stochastic switching. The average consensus for a linear system with Markov switching topologies has been developed by Matei et al. [10]. After that it was extended to a more general linear system in [15]. Under the assumption that the systems have additive noises, the sufficient conditions for average consensus were proposed by Zhang et al. [16]. Then,

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Li et al. proposed the weaker necessary and sufficient conditions to obtain weaker consensus results. Since the measurements of multiple sensors are often influenced by compound noises[3], Wang et al.[13, 14] gave measurement models with compound noises and constant control gains, and Zong et al.[17] studied a first-order consensus problem with additive and multiplicative noises to seek consensus. If the topology is fixed, the necessary and sufficient conditions for mean square weak and strong consensus were developed with a time-varying gain by Zong et al.[17]. Furthermore, when time-varying topologies were considered in the systems, they also obtained sufficient conditions for weak consensus and average consensus. However, the time-varying topologies were not figured explicitly and no numerical simulations were presented. This is the main motivation of the paper.

To fill this gap, the goal of this paper is to establish the asymptotic consensus for a class of first-order continuous-time multi-agent systems with additive and multiplicative noises under Markovian random graphs. Towards the asymptotic consensus, we transform the dynamics of agents into a stochastic error systems. In order to decrease the errors, we combine with stochastic analysis and the tools of symmetrized graph in the theory of stochastic Lyapunov to systemically form a good expression for the ease of analysis. To ensure the connectivity of the network to some extent, we assume that the graph resulting from the union of graphs which are switched by ergodic Markov chain is contained into a spanning tree. Under the assumption, we can show that the error systems in the mean square sense tend to zero as time goes on making sure that all agents may asymptotically agree on their states. Moreover, we prove that the state of each agent converges to the average of initial states under some convergence and robust conditions.

The paper is organized as follows. In Section 2, we recall some related concepts and basic results on graphs and matrices, and present the setup and formulation of the continuous-time systems under consideration. Section 3 is devoted to state our main convergence theorem for average consensus. In Section 4, the example and simulation results are given to illustrate the effectiveness of the obtained theoretical results. Section 5 gives a brief summary of the paper.

2. PRELIMINARY AND PROBLEM FORMULATION

In this section, we will give some related preliminaries and formulate the considered problem in this work.

Notation 1. In this paper, unless otherwise specified, the following symbols will be used in next sections. For any positive integers N , I_N and $e_{N,i}$ represents N -by- N identity matrix and its i th column, $\mathbb{R}^N$ denotes the N dimensional Euclidean space, $\mathbb{R}^{m \times n}$ denotes the set of all $m \times n$ real matrices and 1_N denotes the N dimensional column vector with all ones. For any given vector x or matrix X , $\|x\|$ denotes the Euclid norm and $\|X\|_F$ denotes the Frobenius norm, $\mathbb{E}x$ denotes the mathematical expectation of x and $\text{Var}x$ denotes the variance of x . For any given matrices $A \in \mathbb{R}^{n \times n}$

and $B \in \mathbb{R}^{m \times m}$, A^T denotes the transpose of A , $A \otimes B$ denotes the Kronecker product. Set $A \oplus B = (I_m \otimes A) + (B \otimes I_n) \in \mathbb{R}^{mn \times mn}$. The Kronecker product has the properties: $(A \otimes B)^T = A^T \otimes B^T$ and $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.

2.1. Preliminary

The generator matrix of a continuous-time Markov chain $\{\theta(t)\}_{t \geq 0}$ is denoted by $\Gamma = (\gamma_{ij})_{s \times s}$, so that for a sufficiently small $h > 0$, the probability of mode k jumps to mode l is determined by the generator matrix

$$\mathbb{P}\{\theta(t+h) = l | \theta(t) = k\} = \begin{cases} \gamma_{kl}h + o(h), & k \neq l \\ 1 + \gamma_{kk}h + o(h), & k = l \end{cases},$$

where $o(h)$ is a high-order infinitesimal with respect to infinitesimal h , and $\gamma_{lk} \geq 0$ is the transition rate from state l to state k if $k \neq l$. Let $\lambda_{kk} = -\sum_{k \neq l} \lambda_{kl}$. From [9], for almost every sample path of $\theta(t)$ in any finite interval of $\mathbb{R}_+ := [0, +\infty)$, we know there are finite simple jumps.

Throughout this paper, we associate to a dynamic system with a communication graph within N agents. Let the set of nodes $\mathcal{V} = \{1, 2, \dots, N\}$ be the set of agents with node i representing the i th agent. The interactive communication topology among the agents is modeled by a time-dependent and weighted digraph $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t), \mathcal{A}(t))$ with no self-loops and $\mathcal{E}(t) \subseteq \mathcal{V} \times \mathcal{V}$. An ordered pair (j, i) belongs to the edge set $\mathcal{E}(t)$, which means that the i th agent can receive information from the j th agent directly at t . If $(k_1, k_2), (k_2, k_3), \dots, (k_{m-1}, k_m) \in \mathcal{E}(t)$, then there is a directed path from node k_1 to k_m at t . The weighted adjacency matrix is defined by $\mathcal{A}(t) = [a_{ij}(t)] \in \mathbb{R}^{N \times N}$. For any $i, j \in \mathcal{V}$, $a_{ij}(t) \geq 0$, and $a_{ij}(t) > 0$, i.e., $(j, i) \in \mathcal{E}$, the graph corresponding to a given adjacency matrix $\mathcal{A}(t)$ is denoted by $\mathcal{G}_{\mathcal{A}(t)}$. The Laplacian matrix is given by $\mathcal{L}(t) = \mathcal{D}(t) - \mathcal{A}(t)$, where $\mathcal{D}(t) = \text{diag}(\sum_{j=1}^N a_{ij}(t), i = 1, \dots, N)$. A graph $\mathcal{G}$ is called to be balanced digraph, if $\sum_{j=1}^N a_{ji}(t) = \sum_{j=1}^N a_{ij}(t)$, $i = 1, 2, \dots, N$. It is clear that an undirected graph is a balanced digraph. If there exists a directed path from some agent of the systems to the rest of agents, then we say that the digraph $\mathcal{G}$ contains a spanning tree.

Definition 1. Let $\mathbb{S} = \{1, 2, \dots, s\}$ be the index set of modes corresponding to all possible network topologies, and let $\{A(i) \in \mathbb{R}^{N \times N}, i \in \mathbb{S}\}$ be a set of matrices with respect to a set of graphs $\{\mathcal{G}_{A(i)}, i = 1, \dots, s\}$. We say that the graph $\mathcal{G}_{\mathcal{A}}$ corresponds to the set $\mathcal{A}$ if it is given by the union of graphs, i.e.,

$$\mathcal{G}_{\mathcal{A}} \triangleq \bigcup_{i \in \mathbb{S}} \mathcal{G}_{A(i)} = \left(\mathcal{V}, \bigcup_{i \in \mathbb{S}} \mathcal{E}_i, \mathcal{A} \right),$$

where $\mathcal{A} = \sum_{i=1}^s A(i)$.

Lemma 1. [7] *If the fixed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ is undirected, then the corresponding Laplacian matrix $\mathcal{L}$ is symmetric with N real eigenvalues, and the eigenvalues are in an ascending order:*

$$0 = v_1 \leq v_2 \leq \dots \leq v_N,$$

and

$$\min_{x \neq 0, \mathbb{1}^T x = 0} \frac{x^T \mathcal{L} x}{\|x\|^2} = v_2.$$

We call that $v_2(\mathcal{L})$ is the algebraic connectivity of $\mathcal{G}$. Moreover, if $\mathcal{G}$ is connected, then $v_2 > 0$.

2.2. Problem formulation

We consider the consensus problem for the multi-agent systems with the following distributed control coordination:

$$\dot{x}_i(t) = c(t) \sum_{j=1}^N a_{ij}(t)(y_{ji}(t) - x_i(t)), \quad t \geq 0, \quad i = 1, 2, \dots, N, \quad (2.1)$$

where $x_i(t) \in \mathbb{R}$, and $c(t) : [0, +\infty) \rightarrow [0, +\infty)$ is a time-varying gain. The communication information from j th agent to i th agent is measured by additive and multiplicative noises as the form:

$$y_{ji}(t) = x_j(t) + \alpha_{ji} \mathfrak{W}_{1ji}(t) + f_{ji}(x_j(t) - x_i(t)) \mathfrak{W}_{2ji}(t), \quad (2.2)$$

where $\alpha_{ji} \geq 0$ in the additive noisy intensity, $f_{ji}(\cdot) : \mathbb{R} \rightarrow \mathbb{R}_+$ is the multiplicative noisy intensity function, $\{\mathfrak{W}_{nji}(t), i, j = 1, 2, \dots, N\}$, $n = 1, 2$ are the noise processes. In this work, $\alpha_{ji} \mathfrak{W}_{1ji}(t) + f_{ji}(x_j(t) - x_i(t)) \mathfrak{W}_{2ji}(t)$ is the noise term including two case:

- $\alpha_{ji} \mathfrak{W}_{1ji}(t)$ are additive and independent of the agents' states;
- $f_{ji}(x_j(t) - x_i(t)) \mathfrak{W}_{2ji}(t)$ are multiplicative depending on the relative states.

Remark 1. The measurements and information transmission are often disturbed by additive and multiplicative noises [12]. Generally speaking, $x_j(t) - x_i(t)$ is an ideal measurement and cannot be obtained accurately, because noises appear everywhere. Li et al. considered the additive noise [7] and multiplicative noises [6] for continuous-time models under fixed topologies, respectively. Zong et al. [17] studied the consensus problem for multi-agent systems with additive noise and fixed topology under weaker sufficient conditions obtained by [7]. Li et al. [5] generalized the additive results to a Markovian switching topologies case. Since the measurements to multiple sensors are often influenced by compound noises [3], Wang et al. [13, 14] proposed measurement models with compound noises and constant control gains, and Zong et al. [17] also considered a first-order consensus problem with additive and multiplicative noises and designed a time-varying control gain to seek consensus.

Assumption 1. The noise processes $\{\mathfrak{w}_{nji}(t), i, j = 1, 2, \dots, N\}$, $n = 1, 2$ satisfying $\int_0^t \mathfrak{w}_{nji}(s)ds = w_{nji}(t)$, $t \geq 0$ and $\{w_{nji}(t), i, j = 1, 2, \dots, N\}$ are scalar independent Brownian motions.

From Assumption 1, (2.1) and (2.2), the dynamic behavior of the i th agent is given by

$$\begin{aligned} dx_i(t) = & c(t) \sum_{j=1}^N a_{ij}(t) (x_j(t) - x_i(t)) dt + c(t) \sum_{j=1}^N a_{ij}(t) \alpha_{ji} dw_{1ji}(t) \\ & + c(t) \sum_{j=1}^N a_{ij}(t) f_{ji}(x_j(t) - x_i(t)) dw_{2ji}(t), \end{aligned} \quad (2.3)$$

where $f_{ji}(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is the noise intensity function.

Let $\eta_i(t)^T$ be the first row of $\mathcal{A}(t)$, $\Sigma_i(t) = \eta_i^T(t) \text{diag}(\alpha_{1i}, \alpha_{2i}, \dots, \alpha_{Ni})$, $\alpha^{(t)} = \text{diag}(\Sigma_1(t), \dots, \Sigma_N(t))_{N \times N^2}$ and $dW_1 = (dw_{111} \dots dw_{1N1} \dots dw_{11N} \dots dw_{1NN})^T$. Then, (2.3) can be rewritten as

$$\begin{aligned} dx(t) = & -c(t) \mathcal{L}(t)x(t)dt + c(t) \alpha^{(t)} dW_1(t) \\ & + c(t) \sum_{i=1}^N \sum_{j=1}^N a_{ij}(t) e_{N,i} f_{ji}(x_j(t) - x_i(t)) dw_{2ji}(t), \end{aligned} \quad (2.4)$$

where $x(t) = (x_1(t), \dots, x_N(t))^T$.

Next, we give the definition of consensus as follows.

Definition 2. The agents reach asymptotically unbiased mean square average consensus (AUMSAC), if there is a random vector $x^* \in \mathbb{R}$ such that $\mathbb{E}[x^*] = \frac{1}{N} \sum_{j=1}^N x_j(0)$, $\mathbb{E}[x^*]^2 < \infty$, and $\lim_{t \rightarrow \infty} \mathbb{E} \|x(t) - x^* 1_N\|^2 = 0$.

Remark 2. Definition 2 follows from [18], and we use the mean square consensus to figure such asymptotical behavior.

3. CONSENSUS ANALYSIS

In the section, we are going to give several sufficient conditions for asymptotically unbiased mean square average-consensus to the multi-agent systems. To this end, we make the following assumptions.

Assumption 2. All digraphs $\mathcal{G}_{\mathcal{A}(k)}$, $k \in \mathbb{S}$ are assumed to be balanced and the union graph of all available digraphs contains a spanning tree.

Assumption 3. The homogeneous, finite-state Markov chain $\{\theta(t)\}_{t \geq 0}$ is right continuous-time, and ergodic with generator matrix $\Gamma = (\gamma_{ij})_{s \times s}$ satisfies $\sum_{j \neq i}^s \gamma_{ij} = \sum_{j \neq i}^s \gamma_{ji}$ and initial probability distribution has $\{q_i, i \in \mathbb{S}\}$, $\mathbb{S} = \{1, 2, \dots, s\}$.

Assumption 4. $c(t) \geq 0$, $\int_0^\infty c(t) = \infty$, $\int_0^\infty c^2(t) < \infty$.

Assumption 5. The initial data $x(0)$ satisfies $\mathbb{E}\|x(0)\|^2 < \infty$. Besides, $x(0)$, $\{\theta(t)\}_{t \geq 0}$, $\{\varpi_{nji}(t), i, j = 1, 2, \dots, N\}$ are independent.

Remark 3. The Assumptions above are sufficient for seeking consensus of the considered systems in this work.

- (1) Assumption 2 is often used to analysis the dynamics for systems with switching topologies (see [5]).
- (2) Assumption 3 makes the generator matrix Γ to be doubly stochastic and to correspond to a balanced digraph[10]. Besides $Q := \frac{\Gamma^T + \Gamma}{2}$ is irreducible and negative semidefinite.
- (3) The two integrals in Assumption 4 are convergence condition and robust condition, respectively (see [7]). And it is easy to find that $c(t) \rightarrow 0$, $t \rightarrow \infty$.

Theorem 1. Suppose that Assumptions 1, 2, 3, 4 hold. If $f_{ji}(x) = \beta_{ji}x$, $0 \leq \beta_{ji} \leq \kappa_\beta$, $i, j = 1, 2, \dots, N$ and $a_{ij}(k) = 1$ or 0 , $i, j = 1, \dots, N$, $k \in \mathbb{S}$, then the multi-agent systems achieve unbiased mean square average consensus under the distributed control coordination (2.1) with measurements (2.2), that is,

$$\lim_{t \rightarrow \infty} \|x(t) - x^* 1_N\|^2 = 0,$$

where

$$x^* = \frac{1}{N} 1_N^T x(0) + \frac{1}{N} \int_0^\infty c(t) 1_N^T \alpha^{(\theta(t))} dW_1(t) + \frac{1}{N} \sum_{i,j=1}^N \int_0^\infty c(t) \beta_{ji} 1_N^T S_{ij}^{(\theta(t))} x(t) dw_{2ji}(t).$$

Proof. Since $f_{ji}(x) = \beta_{ji}x$, $i, j = 1, 2, \dots, N$, let $S_{ij}^{(t)} = (s_{ij}(t))_{N \times N}$ be an N -th-order square matrix with $s_{ij}(t) = a_{ij}(t)$, $s_{ii}(t) = -a_{ij}(t)$ and the rest elements being zero, $i, j = 1, 2, \dots, N$. Then, (2.4) can be rewritten as

$$dx(t) = -c(t) \mathcal{L}(t) x(t) dt + c(t) \alpha^{(t)} dW_1(t) + c(t) \sum_{i=1}^N \sum_{j=1}^N \beta_{ji} S_{ij}^{(t)} x(t) dw_{2ji}(t). \quad (3.1)$$

Let $\delta(t) = (I_N - J_N) x(t)$ and $J_N = (1/N) 1_N 1_N^T$. Note that $(I_N - J_N) \mathcal{L}(t) = \mathcal{L}(t) = \mathcal{L}(t) (I_N - J_N)$,

$$\begin{aligned} d\delta(t) &= -c(t) \mathcal{L}(t) \delta(t) dt + c(t) (I_N - J_N) \alpha^{(t)} dW_1(t) \\ &\quad + c(t) \sum_{i=1}^N \sum_{j=1}^N \beta_{ji} (I_N - J_N) S_{ij}^{(t)} \delta(t) dw_{2ji}(t), \end{aligned}$$

we integrate both sides to the equality $d[\delta(t) \delta(t)^T \chi_{\{\theta(t)=k\}}] = \chi_{\{\theta(t)=k\}} d\delta(t) \delta(t)^T + \delta(t) \delta(t)^T d\chi_{\{\theta(t)=k\}}$, to yield

$$\begin{aligned} \delta(t) \delta(t)^T \chi_{\{\theta(t)=k\}} &= \delta^T(0) \delta(0) \chi_{\{\theta(0)=k\}} + \int_0^t \chi_{\{\theta(s)=k\}} d\delta(s) \delta^T(s) \\ &\quad + \int_0^t \delta(s) \delta^T(s) d\chi_{\{\theta(s)=k\}}. \end{aligned}$$

Hence, we obtain

$$\begin{aligned}\mathbb{E}[\delta^T \delta \chi_{\{\theta(t)=k\}}] &= \mathbb{E}[\delta^T(0) \delta(0) \chi_{\{\theta(0)=k\}}] + \int_0^t \mathbb{E} \chi_{\{\theta(s)=k\}} d\delta(s) \delta^T(s) \\ &\quad + \int_0^t \mathbb{E} \delta(s) \delta^T(s) d\chi_{\{\theta(s)\}}.\end{aligned}$$

Denote $V_k(t) = \mathbb{E}[\delta(t) \delta^T(t) \chi_{\{\theta(t)=k\}}]$, we use Ito's formula and Lemma 4.2 in [4] to obtain

$$\begin{aligned}\frac{dV_k(t)}{dt} &= \frac{d\mathbb{E}[\delta(t) \delta^T(t) \chi_{\{\theta(t)=k\}}]}{dt} \\ &= -c(t) (V_k(t) \mathcal{L}(k)^T + \mathcal{L}(k) V_k(t)) + c^2(t) (I_N - J_N) \alpha^{(k)} \alpha^{(k)T} (I_N - J_N) p_k(t) \\ &\quad + c^2(t) \sum_{i,j=1} \beta_{ji}^2 (I_N - J_N) S_{ij}^{(k)} V_k(t) S_{ij}^{(k)T} (I_N - J_N)^T + \sum_{l=1} \gamma_{lk} V_l(t).\end{aligned}\quad (3.2)$$

But, $V(t) = \sum_{k=1}^s V_k(t)$ deduces $V(t) = \mathbb{E}(\delta(t) \delta(t)^T)$. Let $\hat{V}(t) = [V_1(t) \dots V_s(t)] \in \mathbb{R}^{N \times Ns}$, and $\psi(V)$ ($V \in \mathbb{R}^{N \times N}$) be an N^2 dimensional vector obtained by “stacking” V 's columns, that is,

$$\psi(V) = \begin{pmatrix} V(:,1) \\ \vdots \\ V(:,N) \end{pmatrix}, \quad \text{and then, } \psi(\hat{V}(t)) = \begin{pmatrix} \psi(V_1(t)) \\ \vdots \\ \psi(V_s(t)) \end{pmatrix}.$$

Hence,

$$\begin{aligned}\frac{d\psi(V_k(t))}{dt} &= -c(t) \psi(V_k(t) \mathcal{L}(k)^T + \mathcal{L}(k) V_k(t)) + \sum_{l=1} \gamma_{lk} \psi(V_l(t)) \\ &\quad + c^2(t) \psi \left((I_N - J_N) \alpha^{(k)} \alpha^{(k)T} (I_N - J_N) p_k(t) \right) \\ &\quad + c^2(t) \sum_{i,j=1} \beta_{ji}^2 \psi \left((I_N - J_N) S_{ij}^{(k)} V_k(t) S_{ij}^{(k)T} (I_N - J_N)^T \right) \\ &= -c(t) (\mathcal{L}(k) \oplus \mathcal{L}(k)) \psi(V_k(t)) + (\Gamma_k^T \otimes I_{N^2}) \psi(\hat{V}(t)) + c^2(t) \psi(Z_k(t)) \\ &\quad + c^2(t) \sum_{i,j=1} \beta_{ji}^2 \left((I_N - J_N) S_{ij}^{(k)} \otimes (I_N - J_N) S_{ij}^{(k)} \right) \psi(V_k(t)),\end{aligned}\quad (3.3)$$

where $Z_k(t) = (I_N - J_N) \alpha^{(k)} \alpha^{(k)T} (I_N - J_N) p_k(t)$. Let $Z(t) = [Z_1(t) \dots Z_s(t)]$, $\tilde{\mathcal{L}} = \text{diag}(\mathcal{L}(k) \oplus \mathcal{L}(k), k = 1, \dots, s)$, $H = \text{diag}(\sum_{i,j=1} \beta_{ji}^2 ((I_N - J_N) S_{ij}^{(k)} \otimes (I_N - J_N) S_{ij}^{(k)}), k = 1, \dots, s)$, and $\psi(\hat{V}(0)) = (q_1 \psi(V(0))^T, \dots, q_s \psi(V(0))^T)^T$. Using the above notation, we have

$$\frac{d\psi(\hat{V}(t))^T \psi(\hat{V}(t))}{dt} = 2\psi(\hat{V}(t))^T (-c(t) \tilde{\mathcal{L}} + \Gamma^T \otimes I_{N^2}) \psi(\hat{V}(t)) \quad (3.4)$$

$$+ 2c^2(t)\Psi(\hat{V}(t))^T \Psi(Z(t)) + 2c^2(t)\Psi(\hat{V}(t))^T \Psi(\hat{V}(t)).$$

Let $L := \text{diag}(L(k) \oplus L(k), k = 1, 2, \dots, s)$. Applying Assumption 2, we know that $L(k) = \frac{\mathcal{L}^T(k) + \mathcal{L}(k)}{2}$ is symmetric. From Assumption 3, we can see that $Q = \frac{\Gamma^T + \Gamma}{2}$ is irreducible, since $\{\theta(t)\}_{t \geq 0}$ is ergodic and Γ is double stochastic. Therefore, (3.4) could be rewritten as

$$\begin{aligned} \frac{d\Psi(\hat{V}(t))^T \Psi(\hat{V}(t))}{dt} &= 2\Psi(\hat{V}(t))^T (-c(t)L + Q \otimes I_{N^2}) \Psi(\hat{V}(t)) \\ &\quad + 2c^2(t)\Psi(\hat{V}(t))^T \Psi(Z(t)) + 2c^2(t)\Psi(\hat{V}(t))^T \Psi(\hat{V}(t)). \end{aligned} \quad (3.5)$$

From Assumptions 2, 3 and Lemma 3.13 in [10], we know that matrix $(-c(t)L + Q \otimes I_{N^2})$ has an eigenvalue zero with algebraic multiplicity one, and the matrix can be viewed as the Laplacian matrix corresponding to a strongly undirect graph with the second smallest eigenvalue $v_2 > 0$, due to Lemma 1. From Assumption 4, there exists t_0 such that $c(t) \leq \min\{1, \frac{v_2}{1+\kappa_H}\}, t \geq t_0$, where $\kappa_H = \max\{0, \lambda_{\max}(H)\}$ and $\lambda_{\max}(H)$ is the maximum eigenvalue of H . The latter combined with Assumption 3 finds that for $t \geq t_0$ it holds

$$\begin{aligned} \frac{d\Psi(\hat{V}(t))^T \Psi(\hat{V}(t))}{dt} &\leq 2\Psi(\hat{V}(t))^T (-c(t)L + Q \otimes I_{N^2}) \Psi(\hat{V}(t)) + 2c^2(t)\lambda_{\max}(H)\|\Psi(\hat{V}(t))\|^2 \\ &\quad + c^2(t)(\|\Psi(\hat{V}(t))\|^2 + \|\Psi(Z(t))\|^2) \\ &\leq (-2v_2c(t) + c^2(t) + 2\lambda_{\max}(H)c^2(t))\|\Psi(\hat{V}(t))\|^2 + c^2(t)\|\Psi(Z(t))\|^2 \\ &\leq -v_2c(t)\|\Psi(\hat{V}(t))\|^2 + c^2(t)\|\Psi(Z(t))\|^2. \end{aligned} \quad (3.6)$$

In (3.6), we use the comparison principle of differential equations, and the becomes

$$\|\Psi(\hat{V}(t))\|^2 \leq \|\Psi(\hat{V}(t_0))\|^2 I_1(t) + I_2(t),$$

where

$$I_1(t) = e^{-v_2 \int_{t_0}^t c(s) ds}, \quad I_2(t) = \int_{t_0}^t e^{-v_2 \int_s^t c(u) du} c^2(s) \|\Psi(Z(s))\|^2 ds.$$

Applying Assumptions 4, 5 and $v_2 > 0$ implies

$$\lim_{t \rightarrow \infty} \|\Psi(\hat{V}(t_0))\|^2 I_1(t) = \lim_{t \rightarrow \infty} \|\Psi(\hat{V}(t_0))\|^2 e^{-v_2 \int_{t_0}^t c(s) ds} = 0.$$

From Assumption 3, we know that the Markov chain $\{\theta(t)\}_{t \geq 0}$ is ergodic. So, there exists a stationary distribution $\{\pi_k > 0, k = 1, \dots, s\}$. Let $\bar{Z} = [Z_1 \dots Z_s]$ with $Z_k = (I_N - J_N)\alpha^{(k)}\alpha^{(k)T}(I_N - J_N)\pi_k, k \in \mathbb{S}$. Then, for any given $\varepsilon_1 > 0$, there is $t_1 > 0$ such that

$$|\|\Psi(Z(s))\|^2 - \|\Psi(\bar{Z}(s))\|^2| < \varepsilon_1, \quad t \geq t_1. \quad (3.7)$$

From Assumption 4, for any given $\varepsilon_2 > 0$, there exists $t_2 > 0$ such that $\int_{t_2}^{\infty} c^2(t) dt < \varepsilon_2$, $t \geq t_2$. Set $t_3 = \max\{t_1, t_2\}$. We estimate I_2 to get

$$\begin{aligned}
I_2(t) &= \int_{t_0}^{t_3} e^{-\nu_2 \int_s^t c(u) du} c^2(s) \|\Psi(Z(s))\|^2 ds + \int_{t_3}^t e^{-\nu_2 \int_s^t c(u) du} c^2(s) \|\Psi(Z(s))\|^2 ds \\
&\leq e^{-\nu_2 \int_{t_3}^t c(u) du} \int_{t_0}^{t_3} c^2(s) \|\Psi(Z(s))\|^2 ds + \int_{t_3}^t c^2(s) \|\Psi(Z(s))\|^2 ds \\
&\leq e^{-\nu_2 \int_{t_3}^t c(u) du} \int_{t_0}^{t_3} c^2(s) \|\Psi(Z(s))\|^2 ds + (\|\Psi(\bar{Z})\|^2 + \varepsilon_1) \int_{t_3}^t c^2(s) ds \\
&\leq e^{-\nu_2 \int_{t_3}^t c(u) du} \int_{t_0}^{t_3} c^2(s) \|\Psi(Z(s))\|^2 ds + (\|\Psi(\bar{Z})\|^2 + \varepsilon_1) \varepsilon_2 \\
&\rightarrow (\|\Psi(\bar{Z})\|^2 + \varepsilon_1) \varepsilon_2, \quad t \rightarrow \infty.
\end{aligned} \tag{3.8}$$

Since ε_2 is arbitrary, we have $I_2(t) \rightarrow 0$, $t \rightarrow \infty$. Therefore, we conclude that $\lim_{t \rightarrow \infty} \|\Psi(\hat{V}(t))\| = 0$. This leads to $\lim_{t \rightarrow \infty} \|V(t)\|_F = 0$. Because of $\mathbb{E}\|\delta(t)\|^2 = \text{tr}(V(t))$, we get $\lim_{t \rightarrow \infty} \mathbb{E}\|\delta(t)\|^2 = 0$. Hence, there exists $\kappa_1 > 0$ such that $\mathbb{E}\|\delta(t)\|^2 \leq \kappa_1$, $\forall t \geq 0$.

We use (3.1) and Assumption 2 again to obtain

$$d(J_N x(t) - J_N x(0)) = c(t) J_N \alpha^{(\theta(t))} dW_1(t) + c(t) \sum_{i=1}^N \sum_{j=1}^N \beta_{ji} J_N S_{ij}^{(\theta(t))} x(t) dw_{2ji}(t).$$

Integrating the equality above, we have

$$J_N x(t) = J_N x(0) + \int_0^t c(s) J_N \alpha^{(\theta(s))} dW_1(s) + \int_0^t c(s) \sum_{i=1}^N \sum_{j=1}^N \beta_{ji} J_N S_{ij}^{(\theta(s))} \delta(s) dw_{2ji}(s).$$

Let

$$x^* = \frac{1}{N} 1_N^T x(0) + \frac{1}{N} \int_0^\infty c(s) 1_N^T \alpha^{(s)} dW_1(s) + \frac{1}{N} \int_0^\infty c(s) \sum_{i=1}^N \sum_{j=1}^N \beta_{ji} 1_N^T S_{ij}^{(\theta(s))} \delta(s) dw_{2ji}(s).$$

From Lemma 5.4 in [8], we obtain

$$\begin{aligned}
&\mathbb{E}\|J_N x(t) - x^* 1_N\|^2 \\
&= \mathbb{E} \left\| \int_t^\infty c(s) J_N \alpha^{(\theta(s))} dW_1(s) + \int_t^\infty c(s) \sum_{i=1}^N \sum_{j=1}^N \beta_{ji} J_N S_{ij}^{(\theta(s))} \delta(s) dw_{2ji}(s) \right\|^2 \\
&= \mathbb{E} \left\| \int_t^\infty c(s) J_N \alpha^{(\theta(s))} dW_1(s) \right\|^2 + \mathbb{E} \left\| \int_t^\infty c(s) \sum_{i=1}^N \sum_{j=1}^N \beta_{ji} J_N S_{ij}^{(\theta(s))} \delta(s) dw_{2ji}(s) \right\|^2 \\
&= \mathbb{E} \int_t^\infty \|J_N \alpha^{(\theta(s))}\|_F^2 c^2(s) ds + \sum_{i,j=1}^N \beta_{ji}^2 \mathbb{E} \int_t^\infty \|J_N S_{ij}^{(\theta(s))} \delta(s)\|^2 c^2(s) ds \\
&:= I_3(t) + I_4(t).
\end{aligned}$$

We apply Assumptions 2, 4 and $I_3(t)$ to find

$$I_3(t) \leq \max_{1 \leq k \leq s} \|J_N \alpha^{(k)}\|_F^2 \int_t^\infty c^2(s) ds = o(1), \quad t \rightarrow \infty, \quad (3.9)$$

and

$$\begin{aligned} I_4(t) &= \sum_{i,j=1}^N \beta_{ji}^2 \mathbb{E} \int_t^\infty \|J_N S_{ij}^{(\theta(s))} \delta(s)\|^2 c^2(s) ds \\ &= \sum_{i,j=1}^N \beta_{ji}^2 \mathbb{E} \int_t^\infty \|a_{ij}(\theta(s))(\delta_j(s) - \delta_i(s))1_N\|^2 c^2(s) ds \\ &= \sum_{i,j=1}^N \beta_{ji}^2 \mathbb{E} \int_t^\infty a_{ij}^2(\theta(s))(\delta_j(s) - \delta_i(s))^2 N c^2(s) ds \\ &\leq N \kappa_\beta^2 \int_t^\infty \mathbb{E} \sum_{i,j=1}^N (\delta_j(s) - \delta_i(s))^2 c^2(s) ds \\ &= N \kappa_\beta^2 \int_t^\infty N \mathbb{E} \|\delta(s)\|^2 c^2(s) ds \\ &\leq N^2 \kappa_1 \kappa_\beta^2 \int_t^\infty c^2(s) ds = o(1), \quad t \rightarrow \infty. \end{aligned} \quad (3.10)$$

Taking account of (3.9) and (3.10), we have

$$\lim_{t \rightarrow \infty} \mathbb{E} \|J_N x(t) - x^* 1_N\|^2 = 0.$$

Hence,

$$\mathbb{E}[x^*] = \frac{1}{N} 1_N^T x(0), \quad \text{Var}[x^*] < \infty.$$

Consequently, by using $\lim_{t \rightarrow \infty} \mathbb{E} \|\delta(t)\|^2 = 0$, we complete the proof of the theorem. $\square$

Remark 4. If the considered system in this work is noiseless and $c(t) \equiv 1$, it generates to a linear system with Markov switching topologies whose average consensus has been developed by Matei et al. [10], and the consensus results of [10] were extended to a more general linear systems in [15]. Under assumptions that the systems has only additive noises working in a stochastic switching topologies, Zhang et al. [16] obtained the sufficient conditions for average consensus. Unlike the consensus results in [16], Li et al. ([7]) proposed weaker necessary and sufficient conditions to obtain weaker consensus results under a fixed topology. If the topology is fixed, then the systems (2.1) with information measurement (2.2) degenerate to a general stochastic differential equations with compound noises, the necessary conditions and sufficient conditions for mean square weak and strong consensus were developed with a weaker time-varying gain comparing to Assumption 4 by Zong et al. [17]. Furthermore,

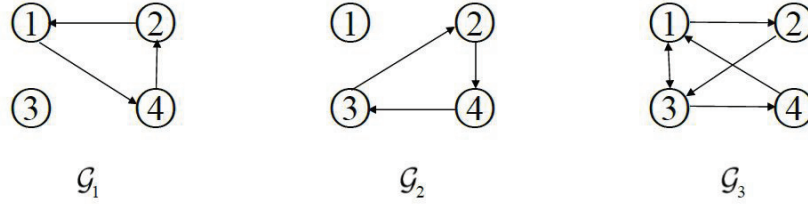


FIGURE 1. All the possible topologies

when a regular time-varying topologies were considered in the systems, they proposed sufficient conditions for weak consensus and average consensus.

4. SIMULATIONS

In this section, in order to analyze the influence of noises and switching topologies to the dynamics of the coupled system (2.1) with information measurement (2.2), we now consider a dynamic of (2.1) with four agents where three different directed graphs are as follows: Obviously, the corresponding topology graphs are balanced and the union graph of the three topology graphs has a spanning tree. If the switching topologies are compelled by a Markov chain in which the generator is chosen as

$$\Gamma = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix},$$

then we know that it will jump between the three topologies and reach every mode in the given time. Given $x(0) = (8, 1, -6, 3)$. Let $\alpha_{11} = \alpha_{12} = \alpha_{13} = \alpha_{14} = \alpha_{21} = \alpha_{41} = 0.1$, $\beta_{12} = \beta_{13} = \beta_{31} = 1$, and $c(t) = \frac{1}{1+t}$. The dynamic curve of the four agents (see Figure 3) shows a sample path of the consensus seeking process corresponding to a ergodic Markov chain in Figure 2. This also illustrates the average consensus which is achieved and consistent with Theorem 1

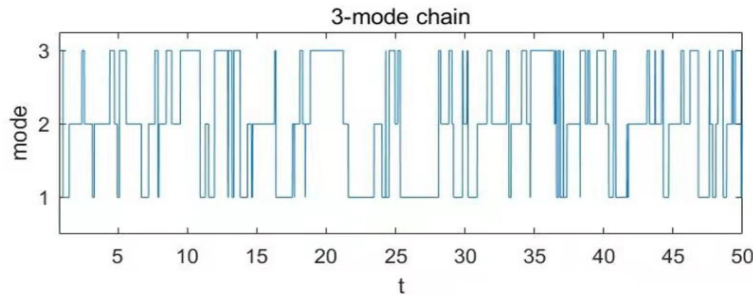


FIGURE 2. Ergodic Markov chain

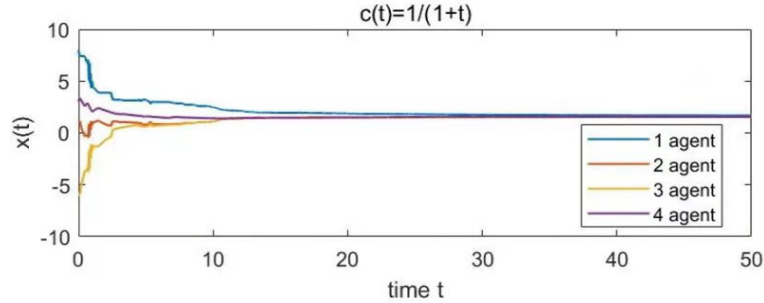


FIGURE 3. dynamic curve of all agents

5. CONCLUSION

We have investigated considered stochastic consensus of the first-order continuous-time multi-agent systems with additive and multiplicative noises under time-varying switching topologies. To analysis the time-varying topologies, the time-varying topologies is figured by using a continuous-time Markov chain with finite modes which corresponds to different directed graphs in the topological structure of the systems. Besides, for each fixed mode, the noise effecting on an agent is measured under the compound noises in the fixed topology corresponding the mode with fixed noise intensities. By investigating the connectivity of the directed and balanced graphs, conditions on the time-vary control gain and the irreducible Markov chain, we obtain a time-asymptotic mean square average consensus for the considered systems. A simulation example is operated to demonstrate our theoretical results.

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A STUDY OF q -ANALOGUE OF THE ANALYTIC CHARACTERIZATION OF LIMAÇON FUNCTIONS

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Abstract. The investigation of Ma and Minda classes of functions associated with q -calculus has been on increase in recent times. Newly, limaçon functions, the classes of starlike and convex limaçon functions were initiated and investigated in the literature. As a result, the present article is aimed to present the q -analogue of the limaçon functions and utilize it to establish the classes of starlike and convex limaçon functions that are correlated with quantum calculus. To this end, the geometric characterization of these functions is examined. Moreover, radius, subordination and some other related results with these novel classes are verified. Overall, some consequences of our investigations are also illustrated.

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1. INTRODUCTION AND PRELIMINARIES

The study of univalent functions is a significant aspect of Complex analysis. Due to the complexity and difficulty involved in analyzing the geometric properties of these functions as a whole, numerous subclasses of them have been emerging in the literature. To unify and classify this concept of the subfamily of univalent functions, Ma and Minda [7] gave an encyclopedic classification of those univalent functions that map the open unit disc U onto convex and starlike domains. As a result, these functions are scattered in the literature in different directions and perspectives. For more details on Ma and Minda functions, see [5, 8, 9, 12–14, 16, 19, 20] and the references therein.

In recent years, the novel associated area of Complex analysis showed up and created several interesting outcomes with potential utilities. The most fascinating aspect of it is the introduction of q -calculus (also known as quantum calculus). Jackson was the first mathematician who demonstrated the concept of q -integration and differentiation in a systematic way, see [2, 3]. The development of this concept in Complex analysis is traced to the work of Ismail et al. [1]. As such, many subclasses of Ma and Minda classes correlated with the idea of quantum calculus have been illustrated in various distinct directions (see [6, 10, 15, 21, 22]). For example, Srivastava et

al. [23] introduce certain higher-order q -derivatives, which were used to study new subclasses of multivalent q -starlike functions that are associated with the Janowski functions and involving certain conic domains. Also in [6], Bilal et al., utilized the idea of q -calculus to study the class of starlike functions which mapped U onto a conic domains of Janowski type. They obtained some useful results, such as, the sufficient conditions, closure results, the Fekete-Szegő type inequalities and distortion theorems. q -Srivastava-Attiya operator was implemented by Srivastava et al. [24] to define and investigate classes of analytic-bi univalent function, which are subordinate to Horadam polynomials. In this direction, the second and third coefficients of functions associated with these classes were derived.

Srivastava [6] uncovered that the purported (p, q) -calculus extension of the classical Quantum calculus is irrelevant, trivial, unimportant and inconsequential since the p -parameter remained insignificant, redundant and contributed nothing to the so-called generalization.

Recently, Kanas and Masih [8] initiated analytic characterization of limaçon domains. The geometric properties of this function were studied and used to present convex and starlike limaçon classes denoted by $CV_{\mathcal{L}}(s)$ and $ST_{\mathcal{L}}(s)$, respectively. Furthermore, Afis et al. [16] continued with the investigation of these classes and proved many interesting results associated with them.

Motivated with these new work, our interest in this paper is to present and study q -analogue of the analytic classification of the limaçon functions. Besides, the classes q -starlike limaçon (denoted by $ST_{\mathcal{L}_q}(s)$) and q -convex limaçon (depicted by $CV_{\mathcal{L}_q}(s)$) are introduced. Moreover, radius, subordination and some other related results with these novel classes are verified.

To put our findings in a clear perspectives, the following preliminaries and definitions are presented.

Let $\mathcal{A}$ denotes the class of normalized analytic functions $f(\zeta)$ of the form

$$f(\zeta) = \zeta + \sum_{n=2}^{\infty} a_n \zeta^n, \quad \zeta \in U. \quad (1.1)$$

The subclass of $\mathcal{A}$ which are univalent in U is depicted by S . Symbolized by $\mathcal{W}$ is the class of analytic functions

$$w(\zeta) = \sum_{n=1}^{\infty} w_n \zeta^n, \quad \zeta \in U \quad (1.2)$$

such that $w(0) = 0$ and $|w(\zeta)| < 1$. These functions are known as Schwarz functions. If $f(\zeta)$ and $g(\zeta)$ are analytic functions in U , then $f(\zeta)$ is subordinate to $g(\zeta)$ (written as $f(\zeta) \prec g(\zeta)$) if there exists $w(\zeta) \in \mathcal{W}$ such that $f(\zeta) = g(w(\zeta))$, $\zeta \in U$.

Janowski [4] introduced the class $P(A, B)$, $-1 \leq B < A \leq 1$ of functions $p(\zeta)$ satisfying the subordination condition

$$p(\zeta) \prec \frac{1+A\zeta}{1+B\zeta},$$

or equivalently, satisfying the inequality

$$\left| p(\zeta) - \frac{1-ABr^2}{1-B^2r^2} \right| \leq \frac{(A-B)r}{1-B^2r^2}, \quad 0 < r < 1. \quad (1.3)$$

Also, $p \in P(A, B)$ if and only if

$$\left| \frac{p(\zeta) - 1}{A - Bp(\zeta)} \right| < 1, \quad \zeta \in U. \quad (1.4)$$

As a special cases, $P(1, -1) \equiv P$ and $P(1 - 2\beta, -1) \equiv P(\beta)$ ($0 \leq \beta < 1$) are the classes of functions of positive real part and those functions whose real parts are greater than β , respectively.

Definition 1. Let $q \in (0, 1)$. Then the q -number $[n]_q$ is given as

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & n \in \mathbb{C}, \\ \sum_{l=0}^{n-1} q^l = 1 + q + q^2 + \dots + q^{n-1}, & n \in \mathbb{N}, \\ n, & \text{as } q \rightarrow 1^-. \end{cases} \quad (1.5)$$

and the q -derivative of a complex valued function $f(\zeta)$ in U is given by

$$D_q f(\zeta) = \begin{cases} \frac{f(q\zeta) - f(\zeta)}{(q-1)\zeta}, & \zeta \neq 0 \\ f'(0), & \zeta = 0, \\ f'(\zeta), & \text{as } q \rightarrow 1^-. \end{cases} \quad (1.6)$$

From the above explanation, it is easy to see that for $f(\zeta)$ given by (1.1),

$$D_q f(\zeta) = 1 + \sum_{n=2}^{\infty} [n]_q a_n \zeta^n. \quad (1.7)$$

Let $f, g \in \mathcal{A}$, we have the following rules for q -difference operator D_q .

- (i) $D_q(f(\zeta)g(\zeta)) = f(q\zeta)D_q g(\zeta) + g(\zeta)D_q f(\zeta)$;
- (ii) $D_q(\sigma f(\zeta) \pm \delta g(\zeta)) = \sigma D_q f(\zeta) \pm \delta D_q g(\zeta)$, for $\sigma, \delta \in \mathbb{C} \setminus \{0\}$;
- (iii)

$$D_q \left(\frac{f(\zeta)}{g(\zeta)} \right) = \frac{g(\zeta)D_q f(\zeta) - f(\zeta)D_q g(\zeta)}{f(\zeta)g(q\zeta)}, \quad f(\zeta) \neq 0, g(q\zeta) \neq 0;$$

$$(iv) \quad D_q(\log f(\zeta)) = \log q^{\frac{1}{q-1}} \frac{D_q f(\zeta)}{f(\zeta)}.$$

As a right inverse, Jackson [2] presented the q -integral of the analytic function $f(\zeta)$ as

$$\int f(\zeta) d_q \zeta = (1-q)\zeta \sum_{j=0}^{\infty} q^j f(q^j \zeta).$$

For example, $f(\zeta) = \zeta^n$ has q -anti derivative as

$$\int \zeta^n d_q \zeta = \frac{\zeta^{n+1}}{[n+1]_q}, n \neq -1.$$

Using the concept of q -calculus, the q -Caratheodory class of functions $P(q)$ was formulated [1, 10] as follow: Let $p(\zeta)$ be analytic in U with $p(0) = 1$. Then $p \in P(q)$ if and only if

$$p(\zeta) \prec \frac{1+\zeta}{1-q\zeta}, \quad \zeta \in U, q \in (0, 1).$$

or

$$\left| p(\zeta) - \frac{1+qr^2}{1-q^2r^2} \right| \leq \frac{(1+q)r}{1-q^2r^2}. \quad (1.8)$$

. If $f \in \mathcal{A}$ and $p(\zeta) = \frac{\zeta D_q f(\zeta)}{f(\zeta)}$ in (1.8), then $f \in ST_q$. Also, with $p(\zeta) = \frac{D_q(\zeta D_q f(\zeta))}{D_q f(\zeta)}$, then $f \in CV_q$. These are the classes of q -starlike and convex functions respectively (see [3, 10]). We say a function $f \in \mathcal{A}$ is said to be starlike of order β ($0 \leq \beta < 1$) if and only if

$$\operatorname{Re} \frac{\zeta f'(\zeta)}{f(\zeta)} > \beta, \quad \zeta \in U.$$

The class of this function is denoted by $S^*(\beta)$ and was introduced by Robertson [11]. For $\zeta f'(\zeta) \in S^*(\beta)$, $f(\zeta)$ is called a convex function of order β and the class containing such functions is symbolized with $C(\beta)$. These concept may be generalized by replacing the ordinary derivative with q -derivative, and the classes are tagged as $S_q^*(\beta)$ and $C_q(\beta)$, respectively.

Definition 2. [8] Let $p(\zeta) = 1 + \sum_{n=1}^{\infty} c_n \zeta^n$. Then $p \in P(\mathbb{L}_s)$ if and only if

$$p(\zeta) \prec (1+s\zeta)^2, \quad 0 < s \leq \frac{1}{\sqrt{2}}, \quad \zeta \in U,$$

or equivalently, if $p(\zeta)$ satisfies the inequality

$$|p(\zeta) - 1| < 1 - (1-s)^2.$$

Presented in [8], was the inclusion relation

$$\{w \in \mathbb{C} : |w-1| < 1 - (1-s)^2\} \subset \mathbb{L}_s(U) \subset \{w \in \mathbb{C} : |w-1| < (1+s)^2 - 1\}. \quad (1.9)$$

Definition 3. [8] Let $f \in \mathcal{A}$. Then $f \in ST_{\mathcal{L}}(s)$ if and only if

$$\frac{\zeta f'(\zeta)}{f(\zeta)} \in P(\mathbb{L}_s), \quad 0 < s \leq \frac{1}{\sqrt{2}}.$$

Also, $f \in CV_{\mathcal{L}}(s)$ if and only if

$$\zeta f'(\zeta) \in ST_{\mathcal{L}}(s), \quad 0 < s \leq \frac{1}{\sqrt{2}}.$$

Inspired with these definitions and the notion of q -calculus, we introduce the following novel classes of functions.

Definition 4. Let $p(\zeta) = 1 + \sum_{n=1}^{\infty} c_n \zeta^n$. Then $p \in P(\mathbb{L}_s(q))$ if and only if

$$p(\zeta) \prec \left(\frac{2(1+s\zeta)}{2+s(1-q)\zeta} \right)^2 := \mathbb{L}_{q,s}(\zeta), \quad 0 < q < 1, 0 < s \leq \frac{1}{\sqrt{2}}, \zeta \in U.$$

Definition 5. Let $f \in \mathcal{A}$. Then $f \in ST_{\mathcal{L}_q}(s)$ if and only if

$$\frac{\xi D_q f(\zeta)}{f(\zeta)} \in P(\mathbb{L}_s(q)), \quad 0 < q < 1, 0 < s \leq \frac{1}{\sqrt{2}}.$$

Also, $f \in CV_{\mathcal{L}_q}(s)$ if and only if

$$\xi D_q f(\zeta) \in ST_{\mathcal{L}_q}(s), \quad 0 < q < 1, 0 < s \leq \frac{1}{\sqrt{2}}.$$

In particular, as $q \rightarrow 1^-$, we are back to Definitions 2 and 3.

2. A PRELIMINARY LEMMA

Lemma 1. [25] Let $w \in \mathcal{W}$. If $|w(\zeta)|$ attains its maximum value on the circle $|\zeta| = r$ at a point $\zeta_0 \in U$, then we have $\zeta_0 D_q w(\zeta_0) = kw(\zeta_0)$, for some $k \geq 1$.

The main results of this manuscript are presented in subsequent sections with the condition that analytic functions $p(\zeta)$ in U satisfies $p(0) = 1$.

3. PROPERTIES ASSOCIATED WITH THE CLASS $P(\mathbb{L}_s(q))$

Theorem 1. Let $p \in P(\mathbb{L}_s(q))$. Then

$$4 \left(\frac{1-sr}{2-s(1-q)r} \right)^2 \leq |p(\zeta)| \leq 4 \left(\frac{1+sr}{2+s(1-q)r} \right)^2, \quad z = re^{i\theta}, 0 < r < 1.$$

Proof. Assume $A = s$ and $B = \frac{s(1-q)}{2}$. Since $p \in P(\mathbb{L}_s(q))$, then

$$p(\zeta) \prec \left(\frac{1+A\zeta}{1+B\zeta} \right)^2.$$

Taking the principal values of the square root and applying (1.3), we obtain the result. $\square$

Remark 1. As $q \rightarrow 1^-$, in the above theorem, we obtain the bounds for $p(\zeta) \in P(\mathbb{L}_s)$ in [8]. Also, we can infer from the theorem that the smallest disc with center $(0, 1)$ that contains $\mathbb{L}_{q,s}(U)$ and the largest disc with center $(0, 1)$ that contained in $\mathbb{L}_{q,s}(U)$ are related with the following relation

$$\left\{ w \in \mathbb{C} : |w(\zeta) - 1| < 1 - 4 \left(\frac{1-s}{2-s(1-q)} \right)^2 \right\} \subset \mathbb{L}_{q,s}(U) \quad (3.1)$$

and

$$\mathbb{L}_{q,s}(U) \subset \left\{ w \in \mathbb{C} : |w(\zeta) - 1| < 4 \left(\frac{1+s}{2+s(1-q)} \right)^2 - 1 \right\}. \quad (3.2)$$

Lastly, $f \in \mathcal{A}$ belongs to $ST_{\mathcal{L}_q}(s)$ if and only

$$\left| \frac{\zeta D_q f(\zeta)}{f(\zeta)} \right| < 1 - 4 \left(\frac{1-s}{2-s(1-q)} \right).$$

Theorem 2. If $p(\zeta)$ is analytic in U and satisfies the condition

$$\left| \frac{\zeta D_q p(\zeta)}{p(\zeta)} \right| < \frac{2(1+q)s}{(1+s)(2+s(1-q))}, \quad (3.3)$$

then for $0 < s \leq \frac{1}{\sqrt{2}}$, $p \in P(\mathbb{L}_s(q))$.

Proof. Consider the function

$$p(\zeta) = \left(\frac{2(1+sw(\zeta))}{2+s(1-q)w(\zeta)} \right)^2,$$

where $w(\zeta)$ is analytic in U with $w(0) = 0$. To determine our result, it is enough to show that $|w(\zeta)| < 1$ in U . By q -logarithmic differentiation, it is easy to see that

$$\frac{\zeta D_q p(\zeta)}{p(\zeta)} = \frac{2(1+q)s \zeta D_q w(\zeta)}{(1+sw(\zeta))(2+s(1-q)w(\zeta))}.$$

If there exists $\zeta_0 \in U$ such that $\max_{|\xi| \leq |\zeta_0|} |w(\xi)| = 1$, then by Lemma 1,

$\zeta_0 D_q w(\zeta_0) = kw(\zeta_0)$ for some $k \geq 1$. Setting $w(\zeta_0) = e^{i\theta}$ ($-\pi \leq \theta \leq \pi$), we obtain

$$\begin{aligned} \left| \frac{\zeta_0 D_q p(\zeta_0)}{p(\zeta_0)} \right| &= 2s(1+q) \left| \frac{\zeta_0 D_q w(\zeta_0)}{(1+sw(\zeta_0))(2+s(1-q)w(\zeta_0))} \right| \geq \frac{2ks(1+q)}{(1+s)(2+s(1-q))} \\ &\geq \frac{2s(1+q)}{(1+s)(2+s(1-q))}, \end{aligned}$$

which contradicts (3.3) and so, there is no ζ_0 in U such that $|w(\zeta_0)| = 1$. Thus, $|w(\zeta)| < 1$ for all $\zeta \in U$. Hence, if (3.3) is satisfied, $p \in P(\mathbb{L}_s(q))$. $\square$

Let $f(\zeta) \in \mathcal{A}$ and setting $p(\zeta) = \frac{\zeta D_q f(\zeta)}{f(\zeta)}$ and $p(\zeta) = \frac{D_q(\zeta D_q f(\zeta))}{D_q f(\zeta)}$ in Theorem 2, respectively, we are left with the following corollary.

Corollary 1. *If $f \in \mathcal{A}$, then*

$$\left| 1 + \frac{q\zeta D_q^2 f(\zeta)}{D_q f(\zeta)} - \frac{\zeta D_q f(\zeta)}{f(\zeta)} \right| < \frac{2(1+q)s}{(1+s)(2+s(1-q))} \implies f \in ST_{\mathcal{L}_q}(s)$$

and similarly

$$\left| 1 + \frac{q^2 \zeta^2 D_q^3 f(\zeta) + \zeta D_q^2 f(\zeta) - D_q f(\zeta)}{q\zeta D_q^2 f(\zeta) + D_q f(\zeta)} - \frac{\zeta D_q^2 f(\zeta)}{D_q f(\zeta)} \right| < \frac{2(1+q)s}{(1+s)(2+s(1-q))}$$

implies $f \in CV_{\mathcal{L}_q}(s)$.

Theorem 2 reduces to the following result as $q \rightarrow 1^-$.

Corollary 2. *If $p(\zeta)$ is analytic in U and satisfies the condition*

$$\left| \frac{\xi p'(\zeta)}{p(\zeta)} \right| < \frac{2s}{(1+s)}, \quad (3.4)$$

then for $0 < s \leq \frac{1}{\sqrt{2}}$, $p \in P(\mathbb{L}_s)$.

Theorem 3. *Let $0 < s \leq \frac{1}{\sqrt{2}}$. Then*

$$\begin{aligned} \max_{|\xi|=1} \operatorname{Re}(\mathbb{L}_{q,s}(\zeta)) &= 4 \left(\frac{1+s}{2+s(1-q)} \right), \\ \min_{|\xi|=1} \operatorname{Re}(\mathbb{L}_{q,s}(\zeta)) &:= \mu_q(s) = \begin{cases} 4 \left(\frac{1-s}{2-s(1-q)} \right), & 0 < s \leq s_0, \\ \frac{2[s^4(1-q)^2 - 2s^2(3+q) + 4]}{(4-s^2(1-q)^2)^2}, & s_0 \leq s \leq \frac{1}{\sqrt{2}}, \end{cases} \end{aligned} \quad (3.5)$$

where

$$s_0 = \frac{\sqrt{3+q^2} - (1+q)}{1-q}, \quad q \in (0, 1). \quad (3.6)$$

Proof. We can write $\mathbb{L}_{q,s}(\zeta) = \left(\frac{1+A\xi}{1+B\xi} \right)^2$, where $A = s, B = \frac{s(1-q)}{2}$. Then

$$\operatorname{Re}(\mathbb{L}_{q,s}(\zeta)) = \frac{(1+AB)^2 + 2AB + 2(1+AB)(A+B)\cos\theta + (A^2+B^2)\cos 2\theta}{(1+2B\cos\theta+B^2)^2},$$

with $-\pi \leq \theta \leq \pi$. Let $\phi(\theta) = \operatorname{Re}(\mathbb{L}_{q,s}(\zeta))$. Then

$$\phi'(\theta) = \frac{2(A-B) \left[(2B^3 - 2A)\cos\theta + AB^3 - 3AB - 1 + 3B^2 \right] \sin\theta}{(1+2B\cos\theta+B^2)^3}.$$

Because $\phi(\theta)$ is an even function, we consider $\theta \in [0, \pi]$. Therefore, the maximum of $\phi(\theta)$ is attained at the stationary point, that is

$$2(A - B) \left[(2B^3 - 2A) \cos \theta + AB^3 - 3AB - 1 + 3B^2 \right] \sin \theta = 0,$$

which are $\theta = 0, \pi$ and the solution of the equation

$$\cos \theta = \frac{3AB + 1 - AB^3 - 3B^2}{2B^3 - 2A}.$$

Similarly,

$$\phi''(\theta) = \frac{\gamma_1 \cos^3 \theta + \gamma_2 \cos^2 \theta + \gamma_3 \cos \theta + C}{(1 + 2B \cos \theta + B^2)^4},$$

where

$$\begin{aligned} \gamma_1 &= 2(A - B) (4AB - 4B^4), \\ \gamma_2 &= 2(A - B) (4B^5 - 4AB^4 - 8B^3 + 8AB^2 + 4B - 4A) \\ \gamma_3 &= 2(A - B) (AB^5 + 11B^4 - 2AB^3 + 2B^2 - 11AB - 1) \\ C &= 2(A - B) (-2B^5 + 6AB^4 + 16B^3 - 16AB^2 - 6B + 2A). \end{aligned}$$

At $\theta = 0$,

$$\phi''(0) = -\frac{8s(1+q) [s^2(q-1) + 2s(1+q) + 2]}{(2 + s(1-q))^4} < 0.$$

This implies that $\phi(\theta)$ attain maximum at $\theta = 0$. Similarly for $\theta = \pi$,

$$\phi''(\pi) = \frac{8s(1+q) [s^2(q-1) - 2s(1+q) + 2]}{(2 - s(1-q))^4}.$$

Therefore, $\phi''(\pi) \leq 0$ for $s_0 \leq s \leq \frac{1}{\sqrt{2}}$ and $\phi''(\pi) \geq 0$ for $0 < s \leq s_0$, where s_0 is given by (3.6). These imply that $\phi(\theta)$ attains its maximum when $s_0 \leq s \leq \frac{1}{\sqrt{2}}$ and minimum when $0 < s \leq s_0$. At

$$\begin{aligned} \cos \theta &= \frac{3AB + 1 - AB^3 - 3B^2}{2B^3 - 2A}, \\ \phi''(\theta) &= -\frac{4[s^2(q-1) + 2s(1+q) + 2][s^2(q-1) - 2s(1+q) + 2][s^2(q-1)^3 + 8]^2}{(2 + s(1-q))^4 (2 - s(1-q))^4 (1+q)}, \end{aligned}$$

which means that $\phi(\theta)$ attain maximum when $0 < s \leq s_0$ and minimum when $s_0 \leq s \leq \frac{1}{\sqrt{2}}$.

Overall, if $s_0 \leq s \leq \frac{1}{\sqrt{2}}$, $\phi(\theta)$ attains maximum at $\theta = 0, \pi$ and minimum when at

$$\cos \theta = \frac{3AB + 1 - AB^3 - 3B^2}{2B^3 - 2A}$$

Also, for $0 < s \leq s_0$, $\phi(\theta)$ attains maximum at $\theta = 0$ and minimum at $\theta = \pi$. Hence, for $s_0 \leq s \leq \frac{1}{\sqrt{2}}$,

$$\max_{|\xi|=1} \operatorname{Re}(\mathbb{L}_{q,s}(\zeta)) = 4 \left(\frac{1+s}{2+s(1-q)} \right),$$

and

$$\min_{|\xi|=1} \operatorname{Re}(\mathbb{L}_{q,s}(\zeta)) = \frac{2[s^4(1-q)^2 - 2s^2(3+q) + 4]}{(4-s^2(1-q)^2)^2}.$$

For $0 < s \leq s_0$,

$$\min_{|\xi|=1} \operatorname{Re}(\mathbb{L}_{q,s}(\zeta)) = 4 \left(\frac{1-s}{2-s(1-q)} \right).$$

□

Remark 2. As $q \rightarrow 1^-$ in Theorem 3, we are led to the result of Kanas and Masih in [8, Theorem 1].

Theorem 4. Let $0 < s \leq \frac{1}{\sqrt{2}}$. Then

$$\frac{\mathbb{L}_{q,s}(\zeta) - 1}{s(1+q)} \in S_q^*(\beta_1),$$

where

$$\beta_1 = \frac{(q-1)s^2 - 2(1+q)s + 3 + q^2}{4 - s(3-q)}.$$

Proof. Let

$$g(\zeta) = \frac{\mathbb{L}_{q,s}(\zeta) - 1}{s(1+q)}.$$

Then by q -logarithmic differentiation, we have

$$\begin{aligned} \operatorname{Re} \frac{\zeta D_q g(\zeta)}{g(\zeta)} &= \operatorname{Re} \frac{\zeta D_q (\mathbb{L}_{q,s}(\zeta))}{\mathbb{L}_{q,s}(\zeta) - 1} \\ &= \operatorname{Re} \frac{1}{(1+Bq\zeta)^2} + \operatorname{Re} \frac{q[(A+B)\zeta + 2AB\zeta^2]}{(2+(A+B)\zeta)(1+Bq\zeta)^2}, \quad A=s, B=s(1-q)/2 \\ &= (1+q) \operatorname{Re} \frac{1}{(1+Bq\zeta)^2} + 2q \operatorname{Re} \frac{1-AB\zeta^2}{(2+(A+B)\zeta)(1+Bq\zeta)^2} \\ &\geq \frac{1+q}{(1+Bq)^2} - 2q \left| \frac{1-AB\zeta^2}{(2+(A+B)\zeta)(1+Bq\zeta)^2} \right| \\ &\geq \frac{1+q}{(1+Bq)^2} - \frac{2q(1+AB)}{(2-(A+B))(1-Bq)^2} \\ &= \frac{(q-1)s^2 - 2(1+q)s + 3 + q^2}{4 - s(3-q)}. \end{aligned}$$

□

Corollary 3. [8] Let $0 < s \leq \sqrt{2}$. Then as $q \rightarrow 1^-$,

$$\frac{\mathbb{L}_{q,s}(\zeta) - 1}{2s} \in S^* \left(\frac{2-2s}{2-s} \right). \quad (3.7)$$

Theorem 5. Let $p \in P(q)$. Then $p \in P(\mathbb{L}_s(q))$ for z in the disc

$$|\xi| < \frac{(1+q)(4-(3-q)s)s}{(1+2q)(2-s(1-q))^2 - 4q(1-s)^2}. \quad (3.8)$$

Proof. Since $p \in P(q)$, then

$$\left| p(\zeta) - \left(\frac{1+qr^2}{1-q^2r^2} \right) \right| < \frac{(1+q)r}{1-q^2r^2}.$$

Now,

$$\begin{aligned} |p(\zeta) - 1| &\leq \left| p(\zeta) - \left(\frac{1+qr^2}{1-q^2r^2} \right) \right| + \frac{q(1+q)r^2}{1-q^2r^2} \\ &< \frac{(1+q)r}{1-q^2r^2} + \frac{q(1+q)r^2}{1-q^2r^2} = \frac{(1+q)r}{1-qr}. \end{aligned}$$

Therefore, $p \in \mathbb{L}_s(q)$ if

$$\frac{(1+q)r}{1-qr} < 1 - \left(\frac{2(1-s)}{2+s(1-q)} \right)^2$$

and from this, we obtain the required result. $\square$

Corollary 4. Let $0 < s \leq \frac{1}{\sqrt{2}}$. Then

$$C_q \subset CV_{\mathbb{L}_q}(s) \quad \text{and} \quad S_q^* \subset ST_{\mathbb{L}_q}(s),$$

for all z in the disc given by (3.8).

4. SUBORDINATION RESULTS

In this section, we assume the parameter $\alpha \neq 0$.

Theorem 6. Let $p(\zeta)$ be analytic in U and Suppose it satisfies the differential subordination

$$1 + \alpha \frac{\zeta D_q p(\zeta)}{p(\zeta)} \prec \mathbb{L}_{q,s}(\zeta).$$

If

$$|\alpha| \geq \frac{(1+s)(4-s(3-q))}{2(2-s(1-q))}, \quad (4.1)$$

then $p(\zeta) \prec \mathbb{L}_{q,s}(\zeta)$.

Proof. Suppose

$$p(\zeta) = \left(\frac{2(1+sw(\zeta))}{2+s(1-q)w(\zeta)} \right)^2, \quad (4.2)$$

where $w(\zeta)$ is analytic in U with $w(0) = 0$. Let

$$H_1(\zeta) = 1 + \alpha \frac{\zeta D_q p(\zeta)}{p(\zeta)}.$$

Then

$$|H_1(\zeta) - 1| = |\alpha| \left| \frac{\zeta D_q p(\zeta)}{p(\zeta)} \right| < 1 - \left(\frac{2(1+s)}{2+s(1-q)} \right)^2. \quad (4.3)$$

By q -logarithmic differential equation of (4.2), we have

$$\frac{\zeta D_q p(\zeta)}{p(\zeta)} = \frac{2(1+q)s \zeta D_q w(\zeta)}{(1+sw(\zeta))(2+s(1-q)w(\zeta))}.$$

Therefore,

$$|H_1(\zeta) - 1| = \frac{2|\alpha|(1+q)s|\zeta D_q w(\zeta)|}{|1+sw(\zeta)||2+s(1-q)w(\zeta)|}.$$

To show that $p(\zeta) \prec \mathbb{L}_{q,s}(\zeta)$, it is enough to prove that $|w(\zeta)| < 1$ for all $\zeta \in U$. Assume there exists $\zeta_0 \in U$ such that $\max_{|\xi| \leq |\zeta_0|} |w(\xi)| = |w(\zeta_0)| = 1$, then by Lemma 1,

$\zeta_0 D_q w(\zeta_0) = kw(\zeta_0)$, for some $k \geq 1$. Thus, for $w(\zeta_0) = e^{i\theta}$, $-\pi \leq \theta \leq \pi$,

$$\begin{aligned} |H_1(\zeta_0) - 1| &= \frac{2|\alpha|(1+q)s|\zeta_0 D_q w(\zeta_0)|}{|1+sw(\zeta_0)||2+s(1-q)w(\zeta_0)|} \\ &= \frac{2|\alpha|(1+q)sk}{|1+sw(\zeta_0)||2+s(1-q)w(\zeta_0)|} \geq \frac{2|\alpha|(1+q)sk}{|(1+s)(2+s(1-q))|}. \end{aligned}$$

In view of (4.1), we have a contradiction. Hence, there exists no $\zeta_0 \in U$ such that $|w(\zeta_0)| = 1$. Thus, $|w(\zeta)| < 1$ for all $\zeta \in U$. This completes the proof. $\square$

Let $f(\zeta) \in \mathcal{A}$ and setting $p(\zeta) = \frac{\zeta D_q f(\zeta)}{f(\zeta)}$ and $p(\zeta) = \frac{D_q(\zeta D_q f(\zeta))}{D_q f(\zeta)}$ in Theorem 6, respectively, we are led to the following result.

Corollary 5. Let $f \in \mathcal{A}$ and suppose

$$|\alpha| \geq \frac{(1+s)(4-s(3-q))}{2(2-s(1-q))},$$

then

$$1 + \alpha \left(\frac{q \zeta D_q^2 f(\zeta)}{D_q f(\zeta)} + 1 - \frac{\zeta D_q f(\zeta)}{f(\zeta)} \right) \prec \mathbb{L}_{q,s}(\zeta) \implies f \in ST_{L_q}(s),$$

and similarly

$$1 + \alpha \left(\frac{q^2 \zeta^2 D_q^3 f(\zeta) + \zeta D_q^2 f(\zeta) - D_q f(\zeta)}{q \zeta D_q^2 f(\zeta) + D_q f(\zeta)} + 1 - \frac{\zeta D_q^2 f(\zeta)}{D_q f(\zeta)} \right) \prec \mathbb{L}_{q,s}(\zeta) \implies f \in CV_{L_q}(s).$$

Theorem 7. Assume that

$$|\alpha| \geq \frac{2(4+s(q-3))(1+s)^3}{(2+s(1-q))(2-s(1-q))^2}, \quad (4.4)$$

and $p(\zeta)$ is an analytic function in U obeying the subordination condition

$$1 + \alpha \frac{\zeta D_q p(\zeta)}{p^2(\zeta)} \prec \mathbb{L}_{q,s}(\zeta).$$

Then $p(\zeta) \prec \mathbb{L}_{q,s}(\zeta)$.

Proof. Consider

$$p(\zeta) = \left(\frac{2(1+sw(\zeta))}{(2+s(1-q)w(\zeta))} \right)^2, \quad (4.5)$$

where $w(\zeta)$ is analytic in U with $w(0) = 0$. Then

$$H_2(\zeta) := 1 + \alpha \frac{\zeta D_q p(\zeta)}{p^2(\zeta)} = 1 + \frac{\alpha s(1+q)(2+s(1-q)w(\zeta))\zeta D_q w(\zeta)}{2(1+sw(\zeta))^3}.$$

To show that $p(\zeta) \prec \mathbb{L}_{q,s}(\zeta)$, it suffices to establish that $|w(\zeta)| < 1$ for all $\zeta \in U$. If otherwise, there is a $\zeta_0 \in U$ such that $\max_{|\xi| \leq |\zeta_0|} |w(\xi)| = |w(\zeta_0)| = 1$. By Lemma 1, there is a

$k \geq 1$ such that $\zeta_0 D_q w(\zeta_0) = kw(\zeta_0)$. Let $w(\zeta_0) = e^{i\theta}$, $-\pi \leq \theta \leq \pi$, then

$$\begin{aligned} |H_2(\zeta_0) - 1| &= \frac{|\alpha|(1+q)s|2+s(1-q)e^{i\theta}|k}{2|1+se^{i\theta}|^3} = \frac{s|\alpha|(1+q)}{2} \left[\frac{4+(4s\cos\theta+s^2)(1-q)}{(1+2s\cos\theta+s^2)^3} \right]^{\frac{1}{2}} \\ &= \frac{s|\alpha|(1+q)}{2} \varphi(\theta), \end{aligned}$$

where

$$\varphi(\theta) = \sqrt{\frac{4+(4s\cos\theta+s^2)(1-q)}{(1+2s\cos\theta+s^2)^3}}.$$

Since $\varphi(\theta)$ is an even function, it is enough to analyze $\varphi(\theta)$ for $\theta \in [0, \pi]$. Therefore,

$$\varphi'(\theta) = \frac{[(3q^2-4q+1)s^2+8(1-q)s+2(q+s)]s\sin\theta}{(2s\cos\theta+1+s^3)^3\sqrt{4+4(1-q)s\cos\theta+(1-q)^2s^2}}.$$

It is simple to see that $\varphi'(\theta) = 0$ when $\theta = 0, \pi$. Similarly, we find that $\varphi''(0) = g(s) > 0$ and $\varphi''(\pi) = g(-s) < 0$, where

$$g(s) = \frac{s[10+2q+8(1-q)s+(1-q)(1-3q)s^2]}{(2-s(1-q))(1+s)^5}.$$

Hence, $\varphi(\theta) > \varphi(0)$. Therefore,

$$|H_2(\zeta_0) - 1| \geq \frac{|\alpha|(1+q)}{2} \varphi(0)$$

$$= \frac{|\alpha|(1+q)(2+s(1-q))}{2(1+s)^3}.$$

Utilizing (4.4), we arrive at a contradiction. Thus, there exists no $\zeta_0 \in U$ such that $|w(\zeta_0)| = 1$. Consequently, $|w(\zeta)| < 1$ for all $\zeta \in U$. Hence, $p(\zeta) \prec \mathbb{L}_{q,s}(\zeta)$.

Let $f(\zeta) \in \mathcal{A}$ and putting $p(\zeta) = \frac{\zeta D_q f(\zeta)}{f(\zeta)}$ and $p(\zeta) = \frac{D_q(\zeta D_q f(\zeta))}{D_q f(\zeta)}$ in Theorem 7, respectively, we arrive at the following corollary. $\square$

Corollary 6. *Let $f \in \mathcal{A}$ and suppose*

$$|\alpha| \geq \frac{2(4+s(q-3))(1+s)^3}{(2+s(1-q))(2-s(1-q))^2},$$

then

$$1 + \alpha \frac{f(\zeta)}{\zeta D_q f(\zeta)} \left(\frac{q \zeta D_q^2 f(\zeta)}{D_q f(\zeta)} + 1 - \frac{\zeta D_q f(\zeta)}{f(\zeta)} \right) \prec \mathbb{L}_{q,s}(\zeta) \implies f \in ST_{\mathcal{L}_q}(s)$$

and similarly

$$1 + \alpha \frac{D_q f(\zeta)}{D_q(\zeta D_q f(\zeta))} \left(\frac{q^2 \zeta^2 D_q^3 f(\zeta) + \zeta D_q^2 f(\zeta) - D_q f(\zeta)}{q \zeta D_q^2 f(\zeta) + D_q f(\zeta)} + 1 - \frac{\zeta D_q^2 f(\zeta)}{D_q f(\zeta)} \right) \prec \mathbb{L}_{q,s}(\zeta)$$

implies $f \in CV_{\mathcal{L}_q}(s)$.

Theorem 8. *If*

$$|\alpha| \geq \frac{(A-B)(1+s)(2+s(1-q))}{2s(1+q)(1-|B|)}, \quad -1 < B < A \leq 1, \quad (4.6)$$

and $p(\zeta)$ is analytic in U satisfying

$$1 + \alpha \frac{\zeta D_q p(\zeta)}{p(\zeta)} \prec \frac{1+A\zeta}{1+B\zeta}$$

then $p(\zeta) \prec \mathbb{L}_{q,s}(\zeta)$.

Proof. Following the proof of Theorem 6, we have

$$H_3(\zeta) := 1 + \alpha \frac{\zeta D_q p(\zeta)}{p(\zeta)} = 1 + \alpha \frac{2(1+q)s\zeta D_q w(\zeta)}{(1+sw(\zeta))(2+s(1-q)w(\zeta))}.$$

Hence,

$$\left| \frac{H_3(\zeta) - 1}{A - BH_3(\zeta)} \right| < 1.$$

But

$$\left| \frac{H_3(\zeta) - 1}{A - BH_3(\zeta)} \right| = \left| \frac{2\alpha(1+q)s\zeta D_q w(\zeta)}{(A-B)(1+sw(\zeta))(2+s(1-q)w(\zeta)) - 2B\alpha(1+q)s\zeta D_q w(\zeta)} \right|.$$

To establish our result, we need to show that $|w(\zeta)| < 1$ in U . Suppose there is $\zeta_0 \in U$ such that $\max_{|\xi| \leq |\zeta_0|} |w(\zeta)| = |w_{\zeta_0}| = 1$. Using Lemma 1, $\zeta_0 D_q w(\zeta_0) = k w(\zeta_0)$, we have for $w(\zeta_0) = e^{i\theta}$, $\theta \in [-\pi, \pi]$,

$$\begin{aligned} \left| \frac{H_3(\zeta_0) - 1}{A - BH_3(\zeta_0)} \right| &= \left| \frac{2\alpha(1+q)ske^{i\theta}}{(A-B)(1+se^{i\theta})(2+s(1-q)e^{i\theta}) - 2B\alpha(1+q)ske^{i\theta}} \right| \\ &\geq \frac{2\alpha(1+q)sk}{(A-B)(1+s)(2+s(1-q)) + 2|B||\alpha|(1+q)sk} \\ &:= \varphi(k), \end{aligned}$$

where $\varphi(k)$ is an increase function of k on $[1, \infty)$. Thus

$$\left| \frac{H_3(\zeta_0) - 1}{A - BH_3(\zeta_0)} \right| \geq \varphi(1) = \frac{2|\alpha|(1+q)s}{(A-B)(1+s)(2+s(1-q)) + 2|\alpha||B|(1+q)s}.$$

Applying the condition (4.6) leads to a contradiction. Hence, we have the desire result. $\square$

Let $f(\zeta) \in \mathcal{A}$ and setting $p(\zeta) = \frac{\zeta D_q f(\zeta)}{f(\zeta)}$ and $p(\zeta) = \frac{D_q(\zeta D_q f(\zeta))}{D_q f(\zeta)}$ in Theorem 8, respectively, we have the following result.

Corollary 7. *Let $f \in \mathcal{A}$ and suppose*

$$|\alpha| \geq \frac{(A-B)(1+s)(2-s(1-q))}{2s(1+q)(1-|B|)}, \quad -1 < B < A \leq 1.$$

then

$$1 + \alpha \left(\frac{q\zeta D_q^2 f(\zeta)}{D_q f(\zeta)} + 1 - \frac{\zeta D_q f(\zeta)}{f(\zeta)} \right) \prec \mathbb{L}_{q,s}(\zeta) \implies f \in ST_{L_q}(s)$$

and similarly

$$1 + \alpha \left(\frac{q^2 \zeta^2 D_q^3 f(\zeta) + \zeta D_q^2 f(\zeta) - D_q f(\zeta)}{q\zeta D_q^2 f(\zeta) + D_q f(\zeta)} + 1 - \frac{\zeta D_q^2 f(\zeta)}{D_q f(\zeta)} \right) \prec \mathbb{L}_{q,s}(\zeta) \implies f \in CV_{L_q}(s).$$

Theorem 9. *If*

$$|\alpha| > \frac{2(A-B)(1+s)^3}{(1+q)s(2-s(1-q)(1-|B|))} \quad (4.7)$$

and

$$1 + \alpha \frac{\zeta D_q p(\zeta)}{p^2(\zeta)} \prec \frac{1+A\zeta}{1+B\zeta}, \quad -B < A \leq 1,$$

then $p(\zeta) \prec \mathbb{L}_{q,s}(\zeta)$.

Proof. The proof follows the same techniques of Theorem 7. $\square$

5. CONCLUSION

In this work, we had successfully studied subfamilies of Ma and Mind classes of functions that were endowed with q -calculus. At first, we introduced a q -Limaçon function and investigated some of its geometric properties. Furthermore, the classes of q -Limaçon starlike and convex functions were initiated. The radius of inclusion relationship between these classes and that of q starlike and convex functions were established. Few differential subordination results associated with these new families were proved. Also, we presented a few consequences of our findings.

Moreover, to have more new theorems under present examinations, new generalization and applications can be explored with some positive and novel outcomes in various fields of science, especially, in applied mathematics. These new surveys will be presented in future research work being processed by the authors of the present paper.

However, the purported trivial (p, q) -calculus extension was clearly demonstrated to be relatively insignificant and inconsequential variation of the classical q -calculus, the extra parameter p being redundant or superfluous (see, for details, [17, p. 340] and [18, pp. 1511-1512]). This observation by Srivastava (see [17] and [18]) will indeed apply also to any future attempt to produce the rather straightforward (p, q) -variants of the results which we have presented in this paper.

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ON THE VON KARMAN'S EQUATION IN THE NONLINEAR THEORY OF GAS DYNAMICS

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Abstract. In this paper, by using invariants of Riemann and general solutions of Euler–Poisson–Darboux–Riemann equations, a new class of exact solutions of von Karman's equation in the nonlinear theory of gas dynamics is given.

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In the plane of independent variables x and y , consider quasilinear von Karman's equation arising in a variety of physical problems such as nonlinear vibrations and irrotational transonic flows of barotropic gas (see [1–5, 7, 13])

$$(u_x)^\alpha u_{xx} - u_{yy} = 0. \quad (1)$$

Equation (1) is considered in the class of hyperbolic solutions, which in this case is determined by the condition

$$u_x > 0. \quad (2)$$

Let

$$m := \frac{\alpha}{2(\alpha+2)}, \quad -2 \neq \alpha \in \mathbb{R} := (-\infty, +\infty). \quad (3)$$

Theorem 1. *If the condition $m \in \mathbb{N} := \{1, 2, 3, \dots\}$ is fulfilled, then the general classical solution $u \in C^2$ of equation (1) is given by the formulas*

$$\begin{cases} x = (X - Y)^{2m+1} \frac{\partial^{2m}}{\partial X^m \partial Y^m} \frac{F(X) - G(Y)}{X - Y}, \\ y = m[2(1 - 2m)]^{2m} \frac{\partial^{2m-2}}{\partial X^{m-1} \partial Y^{m-1}} \frac{F'(X) - G'(Y)}{X - Y}, \\ u = m[2(1 - 2m)]^{2m} \left[\left(\frac{m-1}{2m-1} X + \frac{m}{2m-1} Y \right) \frac{\partial^{2m-2}}{\partial X^{m-1} \partial Y^{m-1}} \frac{F'(X) - G'(Y)}{X - Y} \right. \\ \left. - \frac{m-1}{2m-1} \frac{\partial^{2m-3}}{\partial X^{m-2} \partial Y^{m-1}} \frac{F'(X) - G'(Y)}{X - Y} \right] \text{ for } m = 2, 3, \dots \end{cases} \quad (4)$$

and

$$\begin{cases} x = -2[F(X) - G(Y)] + [F'(X) + G'(Y)](X - Y), \\ y = \frac{4[F'(X) - G'(Y)]}{X - Y}, \\ u = \frac{4[YF'(X) - XG'(Y)]}{X - Y} \text{ for } m = 1. \end{cases} \quad (5)$$

Here $F, G \in C^{m+1}$ are arbitrary functions with respect to the variables X and Y , respectively.

Proof. Let us find the Riemann invariants, that is, what is conserved along the corresponding characteristic curves of equation (1) (see [8, 9]). For any $u \in C^2$, classical solution of equation (1), introduce the designation of Monge-Ampere:

$$p := u_x, \quad q := u_y, \quad r := u_{xx}, \quad s := u_{xy}, \quad t := u_{yy}.$$

In this notation equation (1) has the form

$$t = p^\alpha r. \quad (6)$$

Due to (2), equation (1) along each regular solution is hyperbolic and therefore two families of characteristic curves pass through any fixed point of the plane of independent variables x and y . Along one of these curves, for example, the one whose equation has the form

$$dx = -p^{\frac{\alpha}{2}} dy,$$

by virtue of equality (6), we have

$$dp = rdx + sdy = (-p^{\frac{\alpha}{2}} r + s)dy$$

and

$$dq = sdx + tdy = (-p^{\frac{\alpha}{2}} s + p^\alpha r)dy.$$

Whence it follows that

$$dq + p^{\frac{\alpha}{2}} dp = 0$$

and hence

$$q + \frac{2}{\alpha + 2} p^{\frac{\alpha+2}{2}} = C_1.$$

Similarly, along the family of other characteristic curves, whose equation has the form

$$dx = p^{\frac{\alpha}{2}} dy,$$

we obtain

$$q - \frac{2}{\alpha + 2} p^{\frac{\alpha+2}{2}} = C_2.$$

Here $C_i, i = 1, 2$, are arbitrary constants.

Let us introduce the Riemann invariants of equation (1) as independent variables

$$\begin{cases} X = q + \frac{2}{\alpha+2} p^{\frac{\alpha+2}{2}}, \\ Y = q - \frac{2}{\alpha+2} p^{\frac{\alpha+2}{2}}, \end{cases} \quad (7)$$

in terms of which equation (1) can be rewritten in the form of a system of equations of the first order [5, 8, 9]

$$\begin{cases} X_y + X_x \frac{dx}{dy} = X_y - p^{\frac{\alpha}{2}} X_x = 0, \\ Y_y + Y_x \frac{dx}{dy} = Y_y + p^{\frac{\alpha}{2}} Y_x = 0. \end{cases} \quad (8)$$

In system (8), we choose X and Y as independent variables, while $x(X, Y)$ and $y(X, Y)$ as unknown functions. Applying the formulas of differentiation of implicit functions of two variables

$$x_X = DY_y, \quad x_Y = -DX_y, \quad y_X = -DY_x, \quad y_Y = DX_x,$$

where $D := \frac{D(x,y)}{D(X,Y)}$ is the Jacobian of transformation, from system (8) we obtain

$$\begin{cases} x_Y + p^{\frac{\alpha}{2}} y_Y = 0, \\ x_X - p^{\frac{\alpha}{2}} y_X = 0. \end{cases} \quad (9)$$

Here

$$p^{\frac{\alpha}{2}} = \left[\frac{X - Y}{2(1 - 2m)} \right]^{2m}, \quad (10)$$

due to (2), (3) and (7).

Eliminating the function $y(X, Y)$ from system (9), we receive that the function $x(X, Y)$ satisfies the Euler–Poisson–Darboux–Riemann equation (see [4, 11])

$$x_{XY} + \frac{m}{X - Y} x_X - \frac{m}{X - Y} x_Y = 0. \quad (11)$$

By a similar way, for the function $y(X, Y)$ we get

$$y_{XY} - \frac{m}{X - Y} y_X + \frac{m}{X - Y} y_Y = 0. \quad (12)$$

First, let us get the general solution of equation

$$w_{XY} + \frac{n}{X - Y} w_X - \frac{m}{X - Y} w_Y = 0 \quad (13)$$

for all $n, m \in \mathbb{N}$.

Denote by $z(\alpha, \beta)$ a solution of equation

$$z_{XY} - \frac{\beta}{X - Y} z_X + \frac{\alpha}{X - Y} z_Y = 0,$$

then we have

$$z(\alpha, \beta) = (X - Y)^{1-\alpha-\beta} z(1-\beta, 1-\alpha). \quad (14)$$

(see, for example, [10, 11]).

Putting $\alpha = \alpha' + m$, $\beta = \beta' + n$ in (14) and taking into account the equality

$$z(\alpha' + m, \beta' + n) = \frac{\partial^{m+n} z(\alpha', \beta')}{\partial X^m \partial Y^n},$$

we get

$$(X - Y)^{1-m-n-\alpha'-\beta'} z(1 - \beta' - n, 1 - \alpha' - m) = \frac{\partial^{m+n} z(\alpha', \beta')}{\partial X^m \partial Y^n}.$$

Let us use again formula (14), then we have

$$(X - Y)^{1-m-n-\alpha'-\beta'} z(1 - \beta' - n, 1 - \alpha' - m) = \frac{\partial^{m+n}}{\partial X^m \partial Y^n} \left[\frac{z(1 - \beta', 1 - \alpha')}{(X - Y)^{\alpha'+\beta'-1}} \right].$$

Replacing α' , β' , m , n , respectively, by $1 - \beta'$, $1 - \alpha'$, n , m , we obtain

$$z(\alpha' - m, \beta' - n) = (X - Y)^{m+n+1-\alpha'-\beta'} \frac{\partial^{m+n}}{\partial X^m \partial Y^n} \left[\frac{z(\alpha', \beta')}{(X - Y)^{1-\alpha'-\beta'}} \right].$$

Putting $\alpha' = \beta' = 0$, we get the solution of equation (13) by the formula

$$w(X, Y) = z(-m, -n) = (X - Y)^{m+n+1} \frac{\partial^{m+n}}{\partial X^m \partial Y^n} \left[\frac{F_1(X) - G_1(Y)}{X - Y} \right], \quad (15)$$

where $F_1(X)$ and $G_1(Y)$ are the arbitrary functions of its arguments.

Let us now get a general solution of equation

$$w_{XY} - \frac{n}{X - Y} w_X + \frac{m}{X - Y} w_Y = 0. \quad (16)$$

First, we obtain a general solution to the equation

$$(X - Y)v_{XY} - v_X + v_Y = 0. \quad (17)$$

Proposition 1. *The general solution of equation (17) can be represented by the formula*

$$v(X, Y) = \frac{F_2(X) - G_2(Y)}{X - Y}, \quad (18)$$

where $F_2(X)$ and $G_2(Y)$ are the arbitrary functions of its arguments.

Proof. Indeed, let us introduce the function $\chi := (X - Y)v$. Then by (17), the function χ satisfies the equation $\chi_{XY} = 0$, from which (18) immediately follows. $\square$

Proposition 2. *The general solution of equation (16) has the form*

$$w(X, Y) = \frac{\partial^{m+n-2}}{\partial X^{m-1} \partial Y^{n-1}} \frac{F_2(X) - G_2(Y)}{X - Y}. \quad (19)$$

Proof. It is easy to show that the following equalities are valid:

$$\frac{\partial^m}{\partial X^m} [(X - Y)v] = m \frac{\partial^{m-1} v}{\partial X^{m-1}} + (X - Y) \frac{\partial^m v}{\partial X^m} \quad (20)$$

and

$$\frac{\partial^n}{\partial Y^n}[(X - Y)v] = -n \frac{\partial^{n-1}v}{\partial Y^{n-1}} + (X - Y) \frac{\partial^n v}{\partial Y^n}. \quad (21)$$

Indeed, let us prove equality (20) by induction. For $m = 1$, we have that

$$\frac{\partial}{\partial X}[(X - Y)v] = v + (X - Y) \frac{\partial v}{\partial X}$$

is true.

Suppose that (20) is true for m , and show for $m + 1$. We have

$$\begin{aligned} \frac{\partial^{m+1}}{\partial X^{m+1}}[(X - Y)v] &= \frac{\partial}{\partial X} \left[m \frac{\partial^{m-1}v}{\partial X^{m-1}} + (X - Y) \frac{\partial^m v}{\partial X^m} \right] \\ &= m \frac{\partial^m v}{\partial X^m} + \frac{\partial^m v}{\partial X^m} + (X - Y) \frac{\partial^{m+1}v}{\partial X^{m+1}} \\ &= (m + 1) \frac{\partial^m v}{\partial X^m} + (X - Y) \frac{\partial^{m+1}v}{\partial X^{m+1}} \end{aligned}$$

and therefore (20) is true. The validity of (21) can be shown analogously.

Differentiating equation (17) $(m - 1)$ times with respect to the variable x and $(n - 1)$ times with respect to the variable y , respectively, by virtue of (20) and (21), we have

$$\begin{aligned} 0 &= \frac{\partial^{m+n-2}}{\partial X^{m-1} \partial Y^{n-1}} [(X - Y)v_{XY} - v_X + v_Y] \\ &= \frac{\partial^{n-1}}{\partial Y^{n-1}} \left[(m - 1) \frac{\partial^{m-2}v_{XY}}{\partial X^{m-2}} + (X - Y) \frac{\partial^{m-1}v_{XY}}{\partial X^{m-1}} \right] - \frac{\partial^{m+n-2}v_X}{\partial X^{m-1} \partial Y^{n-1}} + \frac{\partial^{m+n-2}v_Y}{\partial X^{m-1} \partial Y^{n-1}} \\ &= (m - 1) \frac{\partial^{m+n-3}v_{XY}}{\partial X^{m-2} \partial Y^{n-1}} - (n - 1) \frac{\partial^{m+n-3}v_{XY}}{\partial X^{m-1} \partial Y^{n-2}} + (X - Y) \frac{\partial^{m+n-2}v_{XY}}{\partial X^{m-1} \partial Y^{n-1}} \\ &\quad - \frac{\partial^{m+n-2}v_X}{\partial X^{m-1} \partial Y^{n-1}} + \frac{\partial^{m+n-2}v_Y}{\partial X^{m-1} \partial Y^{n-1}} \\ &= m \frac{\partial^{m+n-2}v_Y}{\partial X^{m-1} \partial Y^{n-1}} - n \frac{\partial^{m+n-2}v_X}{\partial X^{m-1} \partial Y^{n-1}} + (X - Y) \frac{\partial^{m+n-2}v_{XY}}{\partial X^{m-1} \partial Y^{n-1}}. \end{aligned}$$

Introduce the notation

$$w := \frac{\partial^{m+n-2}v}{\partial X^{m-1} \partial Y^{n-1}}. \quad (22)$$

Due the last equality, it is easy to see that the function $w(X, Y)$, taking into account (18), can be represented by formula (19) and satisfies equation (16).

Finally, considering (15), (18) and (22) for $n = m$, we get that the general solutions of equations (11) and (12) under the conditions of Theorem 1 have the following form (see [10, 12])

$$x = (X - Y)^{2m+1} \frac{\partial^{2m}}{\partial X^m \partial Y^m} \frac{F_1(X) - G_1(Y)}{X - Y}, \quad (23)$$

$$y = \frac{\partial^{2m-2}}{\partial X^{m-1} \partial Y^{m-1}} \frac{F_2(X) - G_2(Y)}{X - Y}, \quad (24)$$

respectively. Here $F_1, G_1 \in C^{m+2}$ and $F_2, G_2 \in C^{m+1}$ are arbitrary functions. $\square$

The following statement is true.

Proposition 3. *For equalities (9) to hold, it is necessary and sufficient that the equalities*

$$F_2(X) = m[2(1-2m)]^{2m} F_1'(X), \quad G_2(Y) = m[2(1-2m)]^{2m} G_1'(Y) \quad (25)$$

were satisfied.

Proof. Necessity. Obviously, the equalities

$$\frac{\partial^{2m}}{\partial X^m \partial Y^m} \frac{F_1(X) - G_1(Y)}{X - Y} = \frac{\partial^{2m}}{\partial X^m \partial Y^m} \frac{F_1(X)}{X - Y} + \frac{\partial^{2m}}{\partial Y^m \partial X^m} \frac{G_1(Y)}{Y - X} \quad (26)$$

and

$$\frac{\partial^j}{\partial X^j} \frac{1}{(X - Y)^k} = \frac{(-1)^j k(k+1) \cdots (k+j-1)}{(X - Y)^{k+j}} \quad \forall j, k \in \mathbb{N} \quad (27)$$

are valid.

In particular, from (27) it follows that for $j = m, k = 1$,

$$\frac{\partial^m}{\partial X^m} \frac{1}{X - Y} = \frac{(-1)^m m!}{(X - Y)^{m+1}} \quad (28)$$

and for $j = m - i, k = m + 1$,

$$\frac{\partial^{m-i}}{\partial X^{m-i}} \frac{1}{(X - Y)^{m+1}} = \frac{(-1)^{m-i} (m+1)(m+2) \cdots (2m-i)}{(X - Y)^{2m-i+1}}. \quad (29)$$

Further, due to (28) and (29), we obtain

$$\begin{aligned} \frac{\partial^{2m}}{\partial X^m \partial Y^m} \frac{F_1(X)}{X - Y} &= - \frac{\partial^m}{\partial X^m} \left[F_1(X) \frac{\partial^m}{\partial Y^m} \frac{1}{Y - X} \right] = m! \frac{\partial^m}{\partial X^m} \left[F_1(X) \frac{1}{(X - Y)^{m+1}} \right] \\ &= m! \sum_{i=0}^m C_m^i F_1^{(i)}(X) \frac{\partial^{m-i}}{\partial X^{m-i}} \frac{1}{(X - Y)^{m+1}} \\ &= \sum_{i=0}^m C_m^i (-1)^{m-i} (2m-i)! \frac{F_1^{(i)}(X)}{(X - Y)^{2m-i+1}}. \end{aligned} \quad (30)$$

Analogously,

$$\frac{\partial^{2m}}{\partial Y^m \partial X^m} \frac{G_1(Y)}{Y - X} = (-1)^{m+1} \sum_{i=0}^m C_m^i (2m-i)! \frac{G_1^{(i)}(Y)}{(X - Y)^{2m-i+1}}. \quad (31)$$

Therefore, due to (23), (24), (26), (30) and (31), we have

$$\begin{aligned} x(X, Y) = & \sum_{i=0}^m C_m^i (-1)^{m-i} (2m-i)! F_1^{(i)}(X) (X-Y)^i \\ & + (-1)^{m+1} \sum_{i=0}^m C_m^i (2m-i)! G_1^{(i)}(Y) (X-Y)^i \end{aligned} \quad (32)$$

and

$$\begin{aligned} y(X, Y) = & \sum_{i=0}^{m-1} C_{m-1}^i (-1)^{m-i-1} (2m-i-2)! \frac{F_2^{(i)}(X)}{(X-Y)^{2m-i-1}} \\ & + (-1)^m \sum_{i=0}^{m-1} C_{m-1}^i (2m-i-2)! \frac{G_2^{(i)}(Y)}{(X-Y)^{2m-i-1}}. \end{aligned} \quad (33)$$

Differentiating equalities (32) and (33) with respect to the variable X , we get

$$\begin{aligned} x_X = & \sum_{i=0}^m C_m^i (-1)^{m-i} (2m-i)! F_1^{(i+1)}(X) (X-Y)^i \\ & + \sum_{i=0}^{m-1} C_m^{i+1} (-1)^{m-i-1} (i+1) (2m-i-1)! F_1^{(i+1)}(X) (X-Y)^i \\ & + (-1)^{m+1} \sum_{i=0}^{m-1} C_m^{i+1} (i+1) (2m-i-1)! G_1^{(i+1)}(Y) (X-Y)^i \end{aligned} \quad (34)$$

and

$$\begin{aligned} y_X = & \sum_{i=0}^{m-1} C_{m-1}^i (-1)^{m-i-1} (2m-i-2)! \frac{F_2^{(i+1)}(X)}{(X-Y)^{2m-i-1}} \\ & + \sum_{i=0}^{m-1} C_{m-1}^i (-1)^{m-i} (2m-i-1)! \frac{F_2^{(i)}(X)}{(X-Y)^{2m-i}} \\ & + (-1)^{m+1} \sum_{i=0}^{m-1} C_{m-1}^i (2m-i-1)! \frac{G_2^{(i)}(Y)}{(X-Y)^{2m-i}}, \end{aligned} \quad (35)$$

respectively.

Substituting (34) and (35) in the second equality of (9), we obtain

$$\begin{aligned}
& \sum_{i=0}^m C_m^i (-1)^{m-i} (2m-i)! F_1^{(i+1)}(X) (X-Y)^i \\
& + \sum_{i=0}^{m-1} C_m^{i+1} (-1)^{m-i-1} (i+1) (2m-i-1)! F_1^{(i+1)}(X) (X-Y)^i \\
& + (-1)^{m+1} \sum_{i=0}^{m-1} C_m^{i+1} (i+1) (2m-i-1)! G_1^{(i+1)}(Y) (X-Y)^i \\
& = [2(1-2m)]^{-2m} \left[\sum_{i=0}^{m-1} C_{m-1}^i (-1)^{m-i-1} (2m-i-2)! F_2^{(i+1)}(X) (X-Y)^{i+1} \right. \\
& + \sum_{i=0}^{m-1} C_{m-1}^i (-1)^{m-i} (2m-i-1)! F_2^{(i)}(X) (X-Y)^i \\
& \left. + (-1)^{m+1} \sum_{i=0}^{m-1} C_{m-1}^i (2m-i-1)! G_2^{(i)}(Y) (X-Y)^i \right]. \tag{36}
\end{aligned}$$

First, equating in (36) the expressions before the functions F_1 and F_2 and their derivatives, we obtain

$$\begin{aligned}
& \sum_{i=0}^m C_m^i (-1)^{m-i} (2m-i)! F_1^{(i+1)}(X) (X-Y)^i \\
& + \sum_{i=0}^{m-1} C_m^{i+1} (-1)^{m-i-1} (i+1) (2m-i-1)! F_1^{(i+1)}(X) (X-Y)^i \\
& = [2(1-2m)]^{-2m} \left[\sum_{i=1}^m C_{m-1}^{i-1} (-1)^{m-i} (2m-i-1)! F_2^{(i)}(X) (X-Y)^i \right. \\
& \left. + \sum_{i=0}^{m-1} C_{m-1}^i (-1)^{m-i} (2m-i-1)! F_2^{(i)}(X) (X-Y)^i \right].
\end{aligned}$$

From here

$$\begin{aligned}
& C_m^i (-1)^{m-i} (2m-i)! F_1^{(i+1)}(X) + C_m^{i+1} (-1)^{m-i-1} (i+1) (2m-i-1)! F_1^{(i+1)}(X) \\
& = [2(1-2m)]^{-2m} \left[C_{m-1}^{i-1} (-1)^{m-i} (2m-i-1)! F_2^{(i)}(X) \right. \\
& \left. + C_{m-1}^i (-1)^{m-i} (2m-i-1)! F_2^{(i)}(X) \right],
\end{aligned}$$

for $i = 1, 2, \dots, m-1$.

Consequently,

$$F_2^{(i)}(X) = m[2(1-2m)]^{2m} F_1^{(i+1)}(X), \quad i = 1, 2, \dots, m-1. \tag{37}$$

Let us now equate together the present members for $i = 0$ and $i = m$, we get

$$\begin{aligned} & C_m^0(-1)^m(2m)!F_1'(X) + C_m^m m!F_1^{(m+1)}(X)(X-Y)^m + C_m^1(-1)^{m-1}(2m-1)!F_1'(X) \\ &= [2(1-2m)]^{-2m} \left[C_{m-1}^{m-1}(m-1)!F_2^{(m)}(X)(X-Y)^m + C_{m-1}^0(-1)^m(2m-1)!F_2(X) \right]. \end{aligned}$$

Hence

$$F_2(X) = m[2(1-2m)]^{2m}F_1'(X) \quad \text{and} \quad F_2^{(m)}(X) = m[2(1-2m)]^{2m}F_1^{(m+1)}(X).$$

Let us equate the expressions before the functions G_1 and G_2 and their derivatives, we obtain

$$\begin{aligned} & \sum_{i=0}^{m-1} C_m^{i+1}(i+1)(2m-i-1)!G_1^{(i+1)}(Y)(X-Y)^i \\ &= [2(1-2m)]^{-2m} \sum_{i=0}^{m-1} C_{m-1}^i(2m-i-1)!G_2^{(i)}(Y)(X-Y)^i, \end{aligned}$$

this immediately implies that

$$G_2^{(i)}(Y) = m[2(1-2m)]^{2m}G_1^{(i+1)}(Y), \quad i = 0, 1, \dots, m-1.$$

Consequently, taking into account (37) and the last equality, (25) is shown.

Remark 1. We obtain the same equality if we differentiate equalities (32) and (33) with respect to the variable Y and substitute them in the first equality of (9), and equate the expressions before the functions F_1, F_2 and G_1, G_2 and their derivatives, respectively.

Sufficiency. Let conditions (25) be fulfilled and show that equalities (9) hold. Indeed, by virtue of the obvious equality

$$\frac{F_1'(X) - G_1'(Y)}{X-Y} = \frac{\partial}{\partial X} \frac{F_1(X) - G_1(Y)}{X-Y} + \frac{\partial}{\partial Y} \frac{F_1(X) - G_1(Y)}{X-Y},$$

(10) and conditions (23)–(25), we have

$$\begin{aligned} x_X - p^{\frac{\alpha}{2}} y_X &= (X-Y)^{2m} \left[(2m+1) \frac{\partial^{2m}}{\partial X^m \partial Y^m} \frac{F_1(X) - G_1(Y)}{X-Y} \right. \\ &\quad \left. + (X-Y) \frac{\partial^{2m+1}}{\partial X^{m+1} \partial Y^m} \frac{F_1(X) - G_1(Y)}{X-Y} - m \frac{\partial^{2m-1}}{\partial X^m \partial Y^{m-1}} \frac{F_1'(X) - G_1'(Y)}{X-Y} \right] \end{aligned}$$

$$\begin{aligned}
&= (X-Y)^{2m} \left[(m+1) \frac{\partial^{2m}}{\partial X^m \partial Y^m} \frac{F_1(X) - G_1(Y)}{X-Y} \right. \\
&\quad \left. - m \frac{\partial^{2m}}{\partial X^{m+1} \partial Y^{m-1}} \frac{F_1(X) - G_1(Y)}{X-Y} + (X-Y) \frac{\partial^{2m+1}}{\partial X^{m+1} \partial Y^m} \frac{F_1(X) - G_1(Y)}{X-Y} \right] \\
&= (X-Y)^{2m} \frac{\partial}{\partial X} \left[(X-Y) \frac{\partial^2}{\partial X \partial Y} \frac{\partial^{2m-2}}{\partial X^{m-1} \partial Y^{m-1}} \frac{F_1(X) - G_1(Y)}{X-Y} \right. \\
&\quad \left. - m \frac{\partial}{\partial X} \frac{\partial^{2m-2}}{\partial X^{m-1} \partial Y^{m-1}} \frac{F_1(X) - G_1(Y)}{X-Y} \right. \\
&\quad \left. + m \frac{\partial}{\partial Y} \frac{\partial^{2m-2}}{\partial X^{m-1} \partial Y^{m-1}} \frac{F_1(X) - G_1(Y)}{X-Y} \right] \\
&= 0,
\end{aligned}$$

by virtue of equality (24), if we replace there formally the functions F_2 and G_2 with the functions F_1 and G_1 , respectively.

The first equality in (9) can be proved in a similar way. Thus Proposition 3 is proved. $\square$

Further, to obtain the final form of the function u , due (3), (7), (9) and (10), we have

$$\begin{aligned}
du &= pdx + qdy = (px_X + qy_X)dX + (px_Y + qy_Y)dY \\
&= \left(q + p^{\frac{\alpha+2}{2}} \right) y_X dX + \left(q - p^{\frac{\alpha+2}{2}} \right) y_Y dY \\
&= \left(\frac{m-1}{2m-1} X + \frac{m}{2m-1} Y \right) y_X dX + \left(\frac{m}{2m-1} X + \frac{m-1}{2m-1} Y \right) y_Y dY,
\end{aligned}$$

whence

$$U_X = \left(\frac{m-1}{2m-1} X + \frac{m}{2m-1} Y \right) y_X, \quad U_Y = \left(\frac{m}{2m-1} X + \frac{m-1}{2m-1} Y \right) y_Y. \quad (38)$$

By virtue of the first equality of (38), we obtain

$$\begin{aligned}
U(X, Y) &= \frac{m-1}{2m-1} \int X y_X dX + \frac{m}{2m-1} Y y + \varphi(Y) \\
&= \frac{m-1}{2m-1} \left(X y - \int y dX \right) + \frac{m}{2m-1} Y y + \varphi(Y), \quad (39)
\end{aligned}$$

where φ is an arbitrary function.

According to the second equality from (38), for definition of the function φ , we get

$$\frac{m-1}{2m-1} \left(X y_Y - \int y_Y dX \right) + \frac{m}{2m-1} (y + Y y_Y) + \varphi'(Y) = \left(\frac{m}{2m-1} X + \frac{m-1}{2m-1} Y \right) y_Y. \quad (40)$$

By virtue of (12), we receive

$$m \int y_Y dX = \int \left[(Y - X)y_{XY} + my_X \right] dX = (Y - X)y_Y + \int y_Y dX + my.$$

Thus

$$\int y_Y dX = \frac{Y - X}{m - 1} y_Y + \frac{m}{m - 1} y, \quad \text{for } m = 2, 3, \dots$$

Taking into account the latter equality, from (40) we obtain

$$\varphi'(Y) \equiv 0, \Rightarrow \varphi = \text{const} \quad \text{for } m = 2, 3, \dots \quad (41)$$

Analogously, from (39) for $m = 1$, we get

$$U(X, Y) = Yy + \varphi(Y). \quad (42)$$

According to the second equality from (38), in this case, for definition of the function φ , we receive

$$\varphi'(Y) = (X - Y)y_Y - y. \quad (43)$$

Since

$$\left\{ (X - Y)y_Y - y \right\}_X = y_Y + (X - Y)y_{XY} - y_X = 0,$$

by virtue of equation (12) for $m = 1$, from (43) implies that

$$\varphi'(Y) = -G'_2(Y),$$

therefore

$$\varphi(Y) = -G_2(Y). \quad (44)$$

Now, introducing the notation $F := F_1$, $G := G_1$ and taking into account (23)–(25), (41), (42), (44), we obtain (4) and (5), respectively. $\square$

Remark 2. In the case $m = 1$, i.e., for $\alpha = -4$, solution (5) of equation (1) has been obtained in [6] by the method of Lee's group.

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DYNAMICAL COMPLEXITIES IN A DISCRETE-TIME PREDATOR-PREY SYSTEM AS CONSEQUENCES OF THE WEAK ALLEE EFFECT ON PREY

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Abstract. In this paper, a two dimensional discrete-time predator-prey system with weak Allee effect, affecting the prey population, is considered. The existence of the positive fixed points of the system and topological classification of coexistence positive fixed point are examined. By using the bifurcation theory, it is shown that the discrete-time predator-prey system with Allee effect undergoes flip and Neimark-Sacker bifurcations depending on the parameter a . The parametric conditions for existence and direction of bifurcations are investigated. Numerical simulations including bifurcation diagrams, phase portraits and maximum Lyapunov exponents of the system are performed to validate analytical results. The computation of the maximum Lyapunov exponents confirm the presence of chaotic behaviour in the considered system. Finally, the OGY feedback control method is implemented to stabilize chaos existing in the system.

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Keywords: predator-prey system, fixed point, stability, flip bifurcation, Neimark-Sacker bifurcation, chaotic behavior, OGY feedback control method

1. INTRODUCTION

The predator-prey systems which show interactions between two species are very dominant phenomenon in bio-mathematical literature. The first and the simplest of predator prey system is formulated by Lotka-Volterra [2, 22]. Since this system has neglected many real situations, significant changes in system by adding ecological factors such as functional responses, emigration, immigration, time delays, diffusion, Allee effect, etc. have been made by many researchers. In recent times, there have been increasing interests on study of the complex dynamical behaviors of predator prey systems [1, 10–19, 24].

The Allee effect can appear thanks to widespread range of biological phenomena, such as reduced anti-predator vigilance, genetic trends, mating difficulty and feeding deficiency at low population densities. The Allee effect introduced by Warder Clyde Allee can be classified into two types as a weak Allee effect and strong Allee effect to the basis of per capita growth rate at low density. The weak Allee effect represents

that the unit growth rate is smaller when the population is small; but it will not be negative [23]. The strong Allee effect is called that the sign of the per capita growth rate in the limit of low density is negative. The strong Allee effect which is known as the Allee effect have been investigated intensively, but the researches on the weak Allee effect are relatively rare. Nowadays, it is well known that the Allee effect plays an important and fundamental role to understand the biological systems. So, many scholars have payed attention on Allee effect since it exhibits a rich dynamics [5, 6, 8, 9, 12, 20, 21, 23].

In [3], the author has investigated the following discrete-time predator-prey model

$$X_{t+1} = aX_t(1 - X_t) - X_tY_t \quad Y_{t+1} = \frac{1}{\beta}X_tY_t \quad (1.1)$$

where X_t and Y_t represent prey and predator population respectively with t th generation. The parameters a and β are positive real numbers. The authors have analyzed stability and Neimark-Sacker bifurcation of the discrete-time predator-prey system. In this paper, our aim is to discuss the dynamics of the following modified discrete-time predator-prey system (1.1) with adding weak Allee effect on prey population:

$$X_{t+1} = aX_t(1 - X_t)(X_t - \theta) - X_tY_t \quad Y_{t+1} = \frac{1}{\beta}X_tY_t \quad (1.2)$$

In the system (1.2), the term $aX_t(1 - X_t)(X_t - \theta)$ represents the growth rate of the prey population with $-1 < \theta < 1$. In the above the system (1.2) if $0 < \theta < 1$ or $-1 < \theta < 0$, the Allee effect is considered to be strong or weak, respectively. In this study, we have investigated only type of weak Allee of discrete-time predator prey system on prey population [8, 12, 20]. We studied type of strong Allee effect of the presented model on prey population in our another paper.

The aim of the present paper is to compare dynamical behaviors of the discrete-time predator- prey system with and without Allee effect and discuss richer and complex behaviors of the system (1.2) with weak Allee effect on prey population. Thereby we will investigate stability, flip and Neimark-Sacker bifurcations, and chaos control analyses of the system (1.2) at the coexistence fixed point. In Section 2, the conditions of the existence and stability of the fixed points are discussed. In Section 3, flip and Neimark-Sacker bifurcation analysis are studied by choosing a parameter as a bifurcation parameter. Furthermore, directions of both flip bifurcation and Neimark-Sacker bifurcation are obtained by using normal form theory [7]. In Section 4, OGY feedback control method is implemented for chaos control due to emergence of Neimark-Sacker bifurcation. Finally, some numerical simulations are carried out both to illustrate the analytic finding and to display new complex dynamical behaviors in Section 5.

2. THE FIXED POINTS, THEIR EXISTENCE AND STABILITY ANALYSIS

In this section, we present the existence and stability conditions of the fixed points of the system (1.2). So, we can obtain the following Lemma 1 about the existence of the fixed points of the considered system:

Lemma 1. *For the system (1.2), the following statements hold true:*

- (1) *The trivial fixed point $E_0 = (0, 0)$ is always feasible.*
- (2) *The axial fixed points $E_1 = \left(\frac{a(\theta+1) \pm \sqrt{a^2(\theta-1)^2 - 4a}}{2a}, 0 \right)$ are feasible if $-\frac{1}{a} < \theta < 0$.*
- (3) *The unique positive fixed point $E_2 = (\beta, a(1-\beta)(\beta-\theta)-1)$ is feasible if $a > \frac{1}{(1-\beta)(\beta-\theta)}$ and $0 < \beta < 1$.*

Proof. To find the fixed points of the system (1.2), we can solve Equation (2.1)

$$X^* = aX^*(1-X^*)(X^*-\theta) - X^*Y^*, \quad Y^* = \frac{1}{\beta}X^*Y^*. \quad (2.1)$$

We first observe that for $X^* = 0$, we have the extinction fixed point $(0, 0)$ for any values of parameters. For $X^* \neq 0$ and $Y^* = 0$, we have the solution

$$E_1 = \left(\frac{a(\theta+1) \pm \sqrt{a^2(\theta-1)^2 - 4a}}{2a}, 0 \right).$$

The fixed point E_1 has only biological meaning if $-\frac{1}{a} < \theta < 0$. For $X^* \neq 0$ and $Y^* \neq 0$, from the second equation of the system (1.2), we obtain $X^* = \beta$. Substituting $X^* = \beta$ into the function (2.1); it is obtained $Y^* = a(1-\beta)(\beta-\theta)-1 > 0$ if $a > \frac{1}{(1-\beta)(\beta-\theta)}$ and $0 < \beta < 1$. □

Now, we analyze the stability of the coexistence positive fixed point E_2 of the system (1.2) only. The Jacobian matrix of the system (1.2) evaluated at the unique positive fixed point E_2 is

$$J(E_2) = \begin{pmatrix} a\beta - 2a\beta^2 + a\beta\theta + 1 & -\beta \\ \frac{a\beta - a\beta^2 - a\theta + a\beta\theta - 1}{\beta} & 1 \end{pmatrix}.$$

Then characteristic polynomial of $J(E_2)$ is

$$F(\lambda) = \lambda^2 + (-2 - a\beta + 2a\beta^2 - a\beta\theta)\lambda + 2a\beta - 3a\beta^2 + 2a\beta\theta - a\theta. \quad (2.2)$$

In order to investigate the dynamics of a unique positive fixed point E_2 of the system (1.2), we give the following Definition 1 and Lemma 2 [3, 10, 24]:

Definition 1. Assume that λ_1 and λ_2 be roots of the characteristic polynomial at the positive fixed point (x, y) . An fixed point (x, y) is called

- (1) sink if $|\lambda_1| < 1$ and $|\lambda_2| < 1$, and it is locally asymptotically stable,

- (2) source if $|\lambda_1| > 1$ and $|\lambda_2| > 1$, and it is locally unstable,
- (3) saddle if $|\lambda_1| > 1$ and $|\lambda_2| < 1$ or $|\lambda_1| < 1$ and $|\lambda_2| > 1$,
- (4) non-hyperbolic if either $|\lambda_1| = 1$ or $|\lambda_2| = 1$.

Lemma 2. Let $F(\lambda) = \lambda^2 + B\lambda + C$, where B and C are two real constants and let $F(1) > 0$. Suppose λ_1 and λ_2 are two roots of $F(\lambda) = 0$. Then the following statements hold true:

- (1) $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $F(-1) > 0$ and $C < 1$,
- (2) $|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $F(-1) > 0$ and $C > 1$,
- (3) $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$) if and only if $F(-1) < 0$,
- (4) λ_1 and λ_2 are a pair of conjugate complex roots and $|\lambda_1| = |\lambda_2| = 1$ if and only if $B^2 - 4C < 0$ and $C = 1$,
- (5) $\lambda_1 = -1$ and $|\lambda_2| \neq 1$ if and only if $F(-1) = 0$ and $C \neq \pm 1$.

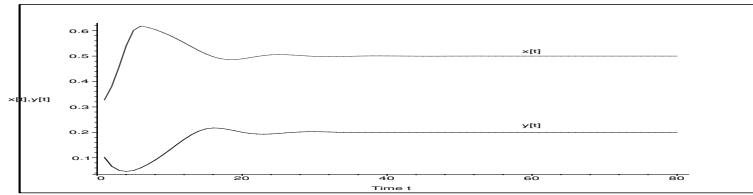
Now we will discuss the topological classification of the unique positive fixed point E_2 of the system (1.2) and we will apply Lemma 2 to prove the following Lemma 3. From Lemma 2, we have $F(1) = a\beta - a\theta - a\beta^2 + a\beta\theta - 1 > 0$. Since $a > \frac{1}{(1-\beta)(\beta-\theta)}$ and $0 < \beta < 1$, $F(1) > 0$. $F(-1) = Ka + 3$ and $F(0) = Sa$ where $K = 3\beta - 5\beta^2 + 3\beta\theta - \theta$, $S = -3\beta^2 + 2\beta\theta - \theta + 2\beta$.

Lemma 3. Assume that $a > a_3$ and $0 < \beta < 1$ then for unique positive fixed point E_2 of the system (1.2) the following holds true.

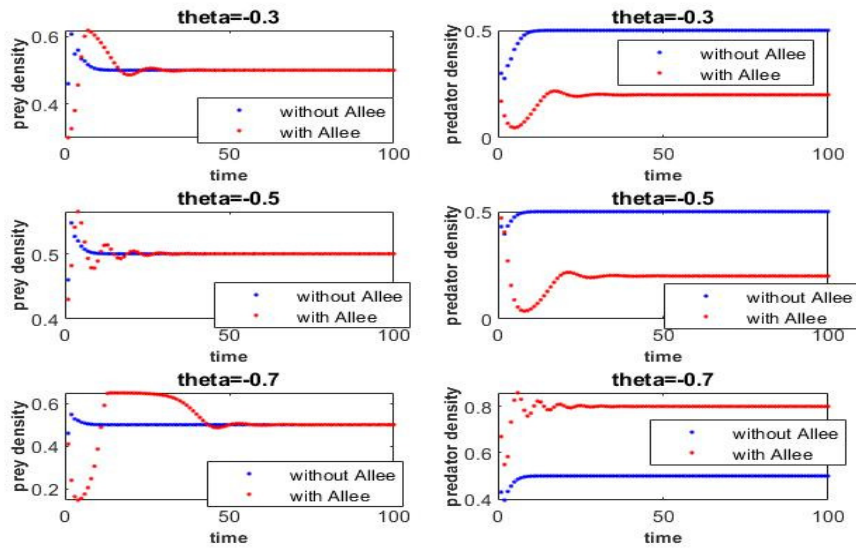
- (1) E_2 is a sink fixed point if the following conditions hold:
 - (i) $K > 0$, $S > 0$ and $\max\{a_1, a_3\} < a < a_2$,
 - (ii) $K < 0$, $S < 0$ and $\max\{a_2, a_3\} < a < a_1$,
 - (iii) $K < 0$, $S > 0$ and $a_3 < a < \min\{a_1, a_2\}$,
- (2) E_2 is a source fixed point if the following conditions hold:
 - (i) $K > 0$, $S > 0$ and $a > \max\{a_1, a_2, a_3\}$,
 - (ii) $K < 0$, $S < 0$ and $a_3 < a < \min\{a_1, a_2\}$,
 - (iii) $K < 0$, $S > 0$ and $\max\{a_2, a_3\} < a < a_1$,
- (3) E_2 is a saddle fixed point if the following conditions hold:
 - (i) $K > 0$, $S > 0$ and $a_3 < a < a_1$,
 - (ii) $K < 0$, $S < 0$ and $\min\{a_1, a_3\} < a$,
 - (iii) $K < 0$, $S > 0$ and $a > \max\{a_1, a_3\}$,
- (4) The roots of Equation (2.2) are complex with modules one if and only if $a = a_2$, $a > a_3$, $-1 < \theta < 0$ and $0 < \beta < 1$, $K > 0$, $S > 0$,
- (5) E_2 non-hyperbolic fixed point if the following conditions hold:
 - $a = a_1$ and $a \neq \pm \frac{1}{S}$ where $a_1 = \frac{-3}{K}$, $a_2 = \frac{1}{S}$, $a_3 = \frac{1}{(1-\beta)(\beta-\theta)}$, $K = 3\beta - 5\beta^2 + 3\beta\theta - \theta$, $S = -3\beta^2 + 2\beta\theta - \theta + 2\beta$.

Example 1. For the parameter values $a = 3$, $\beta = 0.5$, $\theta = -0.3$ and initial condition $(X_0, Y_0) = (0.3, 0.17)$, the positive fixed point of the system (1.2) is obtained as $(X^*, Y^*) = (0.5, 0.2)$. Figure 1a is shown that the fixed point (X^*, Y^*) of the system

(1.2) is local asymptotically stable where $x(t)$ prey and $y(t)$ predator population represent, respectively. For showing the impact of the Allee effect on the stability of the model, Figure 1b is plotted for different values of θ . From Figure 1b, it is clearly that if $-1 < \theta < -0.5$, then the predator density of the model subject to Allee effect increases, otherwise ($-0.5 \leq \theta < 0$) it decreases.



(A) A stable fixed point for the system (1.2).



(B) Time-population density graphs of systems with and without Allee effect for different values of θ .

FIGURE 1

3. BIFURCATION ANALYSIS

In this section, by choosing the parameter a as a bifurcation parameter, we analyze the existence conditions and directions of both Neimark-Sacker bifurcation and flip bifurcation at unique positive fixed point E_2 of the system (1.2).

3.1. Neimark-Sacker Bifurcation at the Fixed Point E_2

From Lemma 3 (4), we can write NSB_{E_2} as follow:

$$NSB_{E_2} = \{(a, \beta, \theta) : a > a_3, a_{NS} = a_2, -1 < \theta < 0 \text{ and } 0 < \beta < 1, K > 0, S > 0\}.$$

The two eigenvalues of the Jacobian matrix $J(E_2)$ are complex conjugate pairs with modulus one, if the parameters change in small neighborhood of NSB_{E_2} . This means that Neimark-Sacker bifurcation will occur at the fixed point E_2 . The eigenvalues of the system (1.2) under these conditions are given by

$$\lambda, \bar{\lambda} = \frac{M \pm i\sqrt{Q}}{-2S},$$

where

$$M = 8\beta^2 - 5(1 + \beta\theta) + 2\theta,$$

$$Q = -28\beta^4 + 32(1 + \theta)\beta^3 + (-9 - 26\theta - 9\theta^2)\beta^2 + 4\theta(1 + \theta)\beta.$$

One gets that $|\lambda| = |\bar{\lambda}| = 1$. The transversality condition imply that

$$\left. \frac{d|\lambda_i(a)|}{da} \right|_{a=a_{NS}} = -3\beta^2 + 2\beta\theta - \theta + 2\beta \neq 0 \quad i = 1, 2.$$

The nonresonance condition $trJ_{E_2}(a_{NS}) \neq 0, -1$ namely

$$a_2 \neq 2R, 3R \quad (3.1)$$

where $R = \frac{1}{\beta(2\beta - \theta - 1)}$. Then, we have $\lambda^k(a_{NS}) \neq 1$ for $k = 1, 2, 3, 4$.

Assume that $q, p \in \mathbb{C}^2$ are two eigenvectors of $J(NSB_{E_2})$ and transposed matrix $J^T(NSB_{E_2})$ corresponding to λ and $\bar{\lambda}$, respectively. We have

$$q \sim \left(1, \frac{F - 2G}{2\beta} - i\frac{\sqrt{Q}}{2\beta}\right)^T \text{ and } p \sim \left(-\frac{F - 2G}{2\beta} + i\frac{\sqrt{Q}}{2\beta}, 1\right)^T.$$

where $F = a\beta(\theta + 1)$, $G = a\beta^2$, $Q = 4(F - G + 1 + a\theta) - (F - 2G)^2$. To achieve the normalization $\langle p, q \rangle = 1$ where $\langle, \rangle$ means the standard scalar product in $\mathbb{C}^2$, we can take the normalized eigenvectors as

$$q = \left(1, \frac{F - 2G}{2\beta} - i\frac{\sqrt{Q}}{2\beta}\right)^T, \quad p = \left(\frac{1}{2} + \frac{i(F - 2G)}{2\sqrt{Q}}, \frac{-i\beta}{\sqrt{Q}}\right)^T.$$

Let $x_t = X_t - x^*$, $y_t = Y_t - y^*$ and $J(E_*) = J(x^*, y^*)$. We transform the fixed point E_2 of the system (1.2) into the origin $(0, 0)$. From Taylor expansion, the system (1.2) convert to

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} \rightarrow J(E_*) \begin{pmatrix} x_t \\ y_t \end{pmatrix} + \begin{pmatrix} F_1(x_t, y_t) \\ F_2(x_t, y_t) \end{pmatrix}, \quad (3.2)$$

where $F_1(x_t, y_t) = -ax_t^3 + (a\theta - 3a\beta + a)x_t^2 - y_t x_t + O(X_t^4)$, $F_2(x_t, y_t) = \frac{x_t y_t}{\beta} + O(X_t^4)$, $X_t = (x_t, y_t)^T$.

The system (3.2) can be expressed as

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = J(E_*) \begin{pmatrix} x_t \\ y_t \end{pmatrix} + \frac{1}{2}B(x_t, x_t) + \frac{1}{6}C(x_t, x_t, x_t) + O(X_t^4),$$

where $B(x, y) = \begin{pmatrix} B_1(x, y) \\ B_2(x, y) \end{pmatrix}$ and $C(x, y, u) = \begin{pmatrix} C_1(x, y, u) \\ C_2(x, y, u) \end{pmatrix}$ are symmetric multi-linear vector functions of $x, y, u \in \mathbb{R}^2$. These functions are defined by as follows:

$$\begin{aligned} B_1(x, y) &= \sum_{j,k=1}^2 \frac{\partial^2 F_1}{\partial \xi_j \partial \xi_k} \Big|_{\xi=0} x_j y_k = (2a\theta - 6a\beta + 2a)x_1 y_1 - x_1 y_2 - x_2 y_1, \\ B_2(x, y) &= \sum_{j,k=1}^2 \frac{\partial^2 F_2}{\partial \xi_j \partial \xi_k} \Big|_{\xi=0} x_j y_k = \frac{x_1 y_2 + x_2 y_1}{\beta}, \\ C_1(x, y, u) &= \sum_{j,k,l=1}^2 \frac{\partial^3 F_1}{\partial \xi_j \partial \xi_k \partial \xi_l} \Big|_{\xi=0} x_j y_k u_l = 6ax_1 y_1 u_1, \\ C_2(x, y, u) &= \sum_{j,k,l=1}^2 \frac{\partial^3 F_2}{\partial \xi_j \partial \xi_k \partial \xi_l} \Big|_{\xi=0} x_j y_k u_l = 0. \end{aligned}$$

Now, we decompose vector $X \in \mathbb{R}^2$ as $X = zq + \bar{z}\bar{q}$, for r near to a_{NS} and $z \in \mathbb{C}$. The explicit formula of z is determined as $z = \langle p, X \rangle$. The system (3.2) can be transformed for all sufficiently small $|a|$ into the form $z \rightarrow \lambda(a)z + g(z, \bar{z}, a)$ where $\lambda(a) = (1 + \varphi(a))e^{i\arctan(a)}$ with $\varphi(a_{NS}) = 0$ and $g(z, \bar{z}, a)$ is smooth complex-valued function. After Taylor expression of g with respect to $(z, \bar{z})$, we obtain

$$g(z, \bar{z}, a) = \sum_{k+l \geq 2} \frac{1}{k!l!} g_{kl}(a) z^k \bar{z}^l, \text{ with } g_{kl} \in \mathbb{C}, l = 0, 1, \dots$$

By symmetric multi-linear vector functions, the Taylor coefficients g_{kl} can be expressed by the formulas

$$\begin{aligned} g_{20}(a_{NS}) &= \langle p, B(q, q) \rangle, \\ g_{11}(a_{NS}) &= \langle p, B(q, \bar{q}) \rangle, \\ g_{02}(a_{NS}) &= \langle p, B(\bar{q}, \bar{q}) \rangle, \\ g_{21}(a_{NS}) &= \langle p, C(q, q, \bar{q}) \rangle. \end{aligned}$$

The coefficient $\beta_2(a_{NS})$, which determines the direction of the appearance of the invariant curve in a generic system exhibiting the Neimark-Sacker bifurcation, can

be calculated via

$$\beta_2(a_{NS}) = \operatorname{Re} \left(\frac{e^{-i\theta(a_{NS})} g_{21}}{2} \right) - \operatorname{Re} \left(\frac{(1 - 2e^{i\theta(a_{NS})}) e^{-2i\theta(a_{NS})}}{2(1 - e^{i\theta(a_{NS})})} g_{20} g_{11} \right) - \frac{1}{2} |g_{11}|^2 - \frac{1}{4} |g_{02}|^2$$

where $e^{i\theta(a_{NS})} = \lambda(a_{NS})$.

One gets that the following result which gives parametric conditions for existence and direction of the Neimark-Sacker bifurcation for the positive fixed point E_2 of system (1.2):

Theorem 1. *Suppose that E_2 is a positive unique fixed point of the system (1.2). If (3.1) holds, $\beta_2(a_{NS}) \neq 0$ and the parameter a changes its value in small vicinity of NSB_{E_2} then, the system (1.2) passes through a Neimark-Sacker bifurcation at only fixed point E_2 . Moreover, if $\beta_2(a_{NS}) < 0$ (respectively $\beta_2(a_{NS}) > 0$), then the Neimark-Sacker bifurcation of the system (1.2) at $a = a_{NS}$ is supercritical (respectively subcritical) and there exists a unique closed invariant curve bifurcates from E_2 for $a = a_{NS}$, which is attracting (respectively repelling).*

3.2. Flip Bifurcation at the Fixed Point E_2

Let us consider the term of FB_{E_2} as follows:

$$FB_{E_2} = \{(a, \beta, \theta) : a > a_3, a_F = a_1, -1 < \theta < 0 \text{ and } 0 < \beta < 1, K < 0, S < 0\}.$$

We consider the system (1.2) at fixed point E_2 with parameters lie in FB_{E_2} . The eigenvalues of the system (1.2) under these conditions are obtained as $\lambda_1(a_F) = -1$, $\lambda_2(a_F) = \frac{3S}{K}$.

The condition $|\lambda_2(a_F)| \neq 1$ leads to

$$K \neq \pm 3R, \quad (3.3)$$

where $R = 3\beta^2 + \theta - 2\beta\theta - 2\beta$.

Let $x_t = X_t - x^*$, $y_t = Y_t - y^*$ and $J(E_*) = J(x^*, y^*)$. We transform the fixed point E_2 of the system (1.2) into the origin $(0, 0)$. By first-order Taylor expansion, the system (1.2) can be written

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} \rightarrow J(E_*) \begin{pmatrix} x_t \\ y_t \end{pmatrix} + \begin{pmatrix} F_1(x_t, y_t, a) \\ F_2(x_t, y_t, a) \end{pmatrix}, \quad (3.4)$$

where $F_1(x_t, y_t) = -a_1 x_t^3 + (a_1 - 3a_1\beta + a_1\theta)x_t^2 - y_t x_t + O(X_t^4)$, $F_2(x_t, y_t) = \frac{x_t y_t}{\beta} + O(X_t^4)$.

The system (3.4) can be expressed as

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = J(E_*) \begin{pmatrix} x_t \\ y_t \end{pmatrix} + \frac{1}{2} B(x_t, x_t) + \frac{1}{6} C(x_t, x_t, x_t) + O(x_t^4),$$

where $B(x, y) = \begin{pmatrix} B_1(x, y) \\ B_2(x, y) \end{pmatrix}$ and $C(x, y, u) = \begin{pmatrix} C_1(x, y, u) \\ C_2(x, y, u) \end{pmatrix}$ are symmetric multilinear vector functions of $x, y, u \in R^2$. These functions are defined by as follows:

$$\begin{aligned} B_1(x, y) &= \sum_{j,k=1}^2 \frac{\partial^2 F_1}{\partial \xi_j \partial \xi_k} \Big|_{\xi=0} x_j y_k = (2a_1 - 6a_1\beta + 2a_1\theta)x_1 y_1 - x_1 y_2 - x_2 y_1, \\ B_2(x, y) &= \sum_{j,k=1}^2 \frac{\partial^2 F_2}{\partial \xi_j \partial \xi_k} \Big|_{\xi=0} x_j y_k = \frac{x_1 y_2 + x_2 y_1}{\beta}, \\ C_1(x, y, u) &= \sum_{j,k,l=1}^2 \frac{\partial^3 F_1}{\partial \xi_j \partial \xi_k \partial \xi_l} \Big|_{\xi=0} x_j y_k u_l = -6a_1 x_1 y_1 u_1, \\ C_2(x, y, u) &= \sum_{j,k,l=1}^2 \frac{\partial^3 F_2}{\partial \xi_j \partial \xi_k \partial \xi_l} \Big|_{\xi=0} x_j y_k u_l = 0. \end{aligned}$$

Now, we assume that $q, p \in R^2$ are two eigenvectors of $J(FB_{E_2})$ for eigenvalue $\lambda_1(a_{FB}) = -1$ such that $J(FB_{E_2})q = -q$ and $J^T(FB_{E_2})p = -p$. then by direct calculation we get

$$q \sim \left(\frac{(-3\beta + 5 - 3\beta\theta + \theta)\beta}{-3\beta + 2\theta + 4\beta^2 - 3\beta\theta}, 1 \right)^T \text{ and } p \sim \left(1, \frac{\beta}{2} \right)^T.$$

To achieve the normalization $\langle p, q \rangle = 1$ where $\langle, \rangle$ means the standard scalar product in R^2 , we can take the normalized vectors as

$$\begin{aligned} q &= \left(\frac{(-3\beta + 5 - 3\beta\theta + \theta)\beta}{-3\beta + 2\theta + 4\beta^2 - 3\beta\theta}, 1 \right)^T, \\ p &= \left(\frac{2(-3\beta + 2\theta + 4\beta^2 - 3\beta\theta)}{\beta(-9\beta + 14\beta^2 - 9\beta\theta + 4\theta)}, \frac{-3\beta + 2\theta + 4\beta^2 - 3\beta\theta}{-9\beta + 14\beta^2 - 9\beta\theta + 4\theta} \right)^T. \end{aligned}$$

We see that $\langle p, q \rangle = 1$. The direction of the flip bifurcation is determined by sign $c(a_F)$ and is computed by

$$c(a_F) = \frac{1}{6} \langle p, C(q, q, q) \rangle - \frac{1}{2} \langle p, B(q, (A - I)^{-1} B(q, q)) \rangle. \quad (3.5)$$

From the above obtained results, one gets the following theorem for direction and existence of flip bifurcation for the positive fixed point E_2 of the system (1.2):

Theorem 2. Suppose that E_2 is a positive unique fixed point of the system (1.2). If (3.3) holds, $c(a_F) \neq 0$ and the parameter a varies its value in a small vicinity of FB_{E_2} , the system (1.2) passes through a flip bifurcation at positive fixed point E_2 . Moreover, if $c(a_F) < 0$ (respectively $c(a_F) > 0$), then there exists unstable (respectively stable) period-2 orbits bifurcate from E_2 .

4. CHAOS CONTROL

The theory of bifurcation and chaos control have significant characteristics in biological species. So, in this section, we display how chaos can be ruled out from the irregular complex dynamics of the system (1.2) such as bifurcations and chaotic attractors. There is chaotic behavior in many areas like physics, biochemistry, econometrics, cardiology, communications, biology, and engineering. As a matter of fact, it is wished that the system be optimized with respect to some performance criterion and chaos be avoided in dynamical systems. The aim of chaos control is to make chaotic behavior more predible and stable. Chaos control is a technique of stabilization via the help of small perturbations. There are many techniques for chaos control in literature [10, 11]. The chaotic motion of the considered system (1.2) is controlled on the stable orbit through OGY control strategy, taking a as a control parameter. In order to apply OGY method introduced by Ott et al [4], we rewrite system (1.2) in the following form:

$$\begin{aligned} X_{t+1} &= aX_t(1 - X_t)(X_t - \theta) - X_tY_t = f(X_t, Y_t, a), \\ Y_{t+1} &= \frac{1}{\beta}X_tY_t = g(X_t, Y_t, a), \end{aligned} \quad (4.1)$$

where a denotes parameter for chaos control. Furthermore, a is restricted to lie in some small interval $|a - a_0| < \mu$ with $\mu > 0$ and a_0 represents the nominal value belong to chaotic region. We apply the stabilizing feedback control strategy in order to move the trajectory towards the desired orbit. Suppose that (X^*, Y^*) be unstable fixed point of the system (1.2) in chaotic region produced by the emergence of Neimark-Sacker bifurcation, then the system (4.1) can be approximated in the neighbourhood of the unstable fixed point (X^*, Y^*) by the following linear map:

$$\begin{bmatrix} X_{t+1} - X^* \\ Y_{t+1} - Y^* \end{bmatrix} \approx A \begin{bmatrix} X_t - X^* \\ Y_t - Y^* \end{bmatrix} + B[a - a_0], \quad (4.2)$$

where

$$A = \begin{bmatrix} \frac{\partial f(X^*, Y^*, a_0)}{\partial X} & \frac{\partial f(X^*, Y^*, a_0)}{\partial Y} \\ \frac{\partial g(X^*, Y^*, a_0)}{\partial X} & \frac{\partial g(X^*, Y^*, a_0)}{\partial Y} \end{bmatrix} = \begin{bmatrix} a_0\beta - 2a_0\beta^2 + a_0\beta\theta + 1 & -\beta \\ -\frac{a_0\beta + a_0\theta + a_0\beta^2 - a_0\beta\theta + 1}{\beta} & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} \frac{\partial f(X^*, Y^*, a_0)}{\partial a} \\ \frac{\partial g(X^*, Y^*, a_0)}{\partial a} \end{bmatrix} = \begin{bmatrix} \beta^2 - \beta\theta - \beta^3 + \beta^2\theta \\ 0 \end{bmatrix}.$$

In order to check that the system (4.1) is controllable, the following matrix is computed:

$$C = [B : AB] = \begin{bmatrix} \beta^2 - \beta\theta - \beta^3 + \beta^2\theta & (a_0\beta - 2a_0\beta^2 + a_0\beta\theta + 1)(\beta^2 - \beta\theta - \beta^3 + \beta^2\theta) \\ 0 & -\frac{(a_0\beta + a_0\theta + a_0\beta^2 - a_0\beta\theta + 1)(\beta^2 - \beta\theta - \beta^3 + \beta^2\theta)}{\beta} \end{bmatrix}.$$

C is of rank 2 at positive fixed point. We suppose that $[a - a_0] = -K \begin{bmatrix} X_t - X^* \\ Y_t - Y^* \end{bmatrix}$, where $K = [\rho_1 \ \rho_2]$, then the system (4.2) can be written as follows:

$$\begin{bmatrix} X_{t+1} - X^* \\ Y_{t+1} - Y^* \end{bmatrix} \approx [A - BK] \begin{bmatrix} X_t - X^* \\ Y_t - Y^* \end{bmatrix}.$$

The corresponding controlled system of (1.2) can be written as:

$$\begin{aligned} X_{t+1} &= (a_0 - \rho_1(X_t - X^*) - \rho_2(Y_t - Y^*))X_t(1 - X_t)(X_t - \theta) - X_tY_t, \\ Y_{t+1} &= \frac{1}{\beta}X_tY_t. \end{aligned} \quad (4.3)$$

In addition, if the modulus the matrix $A - BK$'s eigenvalues is less than, the fixed point (X^*, Y^*) of the system (4.3) is locally asymptotically stable. The Jacobian matrix $A - BK$ of the controlled system (4.3) can be written as follows:

$$[A - BK] = \begin{bmatrix} a & -\beta - (\beta^2 - \beta\theta - \beta^3 + \beta^2\theta)\rho_2 \\ -\frac{a_0\beta + a_0\theta + a_0\beta^2 - a_0\beta\theta + 1}{\beta} & 1 \end{bmatrix}$$

where

$$a = a_0\beta - 2a_0\beta^2 + a_0\beta\theta + 1 - (\beta^2 - \beta\theta - \beta^3 + \beta^2\theta)\rho_1.$$

The characteristic equation of the Jacobian matrix $A - BK$ is given by

$$P(\lambda) = \lambda^2 - (tr(A - BK))\lambda + \det(A - BK) = 0. \quad (4.4)$$

Let λ_1 and λ_2 are roots of characteristic equation (4.4), then

$$\begin{aligned} \lambda_1 + \lambda_2 &= (-\beta^2 + \beta\theta + \beta^3 - \beta^2\theta)\rho_1 + 2 + a_0\beta - 2a_0\beta^2 + a_0\beta\theta, \\ \lambda_1\lambda_2 &= ((\beta - \beta^2)\theta - \beta^2 + \beta^3)\rho_1 + ((a_0\beta^2 + a_0 - 2a_0\beta)\theta^2 \\ &\quad + (1 - 2a_0\beta^3 + 4a_0\beta^2 + (-1 - 2a_0)\beta)\theta + \beta^4a_0 - 2a_0\beta^3 \\ &\quad + (a_0 + 1)\beta^2 - \beta)\rho_2 + (2a_0\beta - a_0)\theta + 2a_0\beta - 3a_0\beta^2 \end{aligned} \quad (4.5)$$

are valid. In order to obtain the lines of marginal stability we must solve equations $\lambda_1 = \pm 1$ and $\lambda_1\lambda_2 = 1$. These restrictions make sure that λ_1 and λ_2 have absolute less than 1. Assume that $\lambda_1\lambda_2 = 1$, then second part of Equation (4.5) implies that:

$$\begin{aligned} L_1 &:= (\beta^3 + (-1 - \theta)\beta^2 + \beta\theta)\rho_1 + (\beta^4a_0 + (2a_0(-1 - \theta)\beta^3 \\ &\quad + (a_0 + 4a_0\theta + \theta^2a_0 + 1)\beta^2 + (-1 - 2\theta^2a_0 + (-1 - 2a_0)\theta)\beta + \theta + \theta^2a_0)\rho_2 \\ &\quad - 3a_0\beta^2 + (2a_0(1 + \theta))\beta - a_0 - 1 = 0 \end{aligned}$$

Moreover, we suppose that $\lambda_1 = 1$, then using the equations (4.5) yield that:

$$\begin{aligned} L_2 := & (\beta^4 a_0 + (-2a_0 - 2a_0\theta)\beta^3 + (a_0 + 4a_0\theta + \theta^2 a_0 + 1)\beta^2 \\ & + (-1 - 2\theta^2 a_0 + (-1 - 2a_0)\theta)\beta + \theta + \theta^2 a_0)\rho_2 + \theta + \theta^2 a_0)\rho_2\beta^2 \\ & + a_0(1 + \theta)\beta - a_0\theta - 1 = 0 \end{aligned}$$

Finally, taking $\lambda_1 = -1$ and using equations (4.5) we get

$$\begin{aligned} L_3 := & ((-2\beta^2 + 2\beta)\theta - 2\beta^2 + 2\beta^3)\rho_1 + (a_0\beta^2 + a_0 - 2a_0\beta)\theta^2 \\ & + (1 - 2a_0\beta^3 + 4a_0\beta^2 + (-1 - 2a_0)\beta)\theta + \beta^4 a_0\beta^3 - 2a_0\beta^3\beta^3 \\ & + (a + 1)\beta^2 - \beta)\rho_2 + (-a_0 + 3a_0\beta)\theta + 3 - 5a_0\beta^2 + 3a_0\beta = 0 \end{aligned}$$

Then, stable eigenvalues lie within the triangular region in $\rho_1\rho_2$ plane bounded by the straight lines L_1, L_2, L_3 for particular parametric values.

5. NUMERICAL ANALYSIS

In this section, numerical simulations including bifurcation diagrams, phase portraits, and maximum Lyapunov exponents are presented both to illustrate the results of theoretical analyses and to exhibit complex and new dynamical behaviours.

Example 2. By taking the parameters values $(\beta, \theta) = (0.5, -0.1)$, the critical value of Neimark-Sacker bifurcation point is $a_{NS} = 4$. Also, the positive fixed point of the system (1.2) is evaluated as $E_2 = (0.5, 0.2)$. Using these parameter values, we get the Jacobian matrix as $J_{E_2}(a_{NS}) = \begin{bmatrix} 0.8 & -0.5 \\ 0.4 & 1 \end{bmatrix}$. Also, we have

$$\begin{aligned} \lambda_{1,2} &= 0.9 \pm 0.4358898944i, \\ g_{20}(a_{NS}) &= 0.276346573 + 0.795433216i, \\ g_{11}(a_{NS}) &= 2.774882688 + 4.072599295i, \\ g_{02}(a_{NS}) &= 7.749765376 + 4.635245515i, \\ g_{21}(a_{NS}) &= -15.000000 - 23.93149824i, \\ \beta_2(a_{NS}) &= -41.52149634 < 0. \end{aligned}$$

Therefore, Neimark-Sacker bifurcation is supercritical and it shows the correctness of Theorem 1. The bifurcation diagram, maximum Lyapunov exponents and the phase portraits of the system (1.2) are shown in Figure 2 and Figure 3, respectively.

The bifurcation diagrams shown in Figure 2a and Figure 2b show that the stability of E_2 happens for $a < 4$ and loses its stability at $a = 4$ and an attracting invariant curve appears if $r > 4$. We compute the maximum Lyapunov exponents for detecting the presence of chaos in the model. The existence of chaotic regions in the parameter space is clearly visible in Figure 2c.

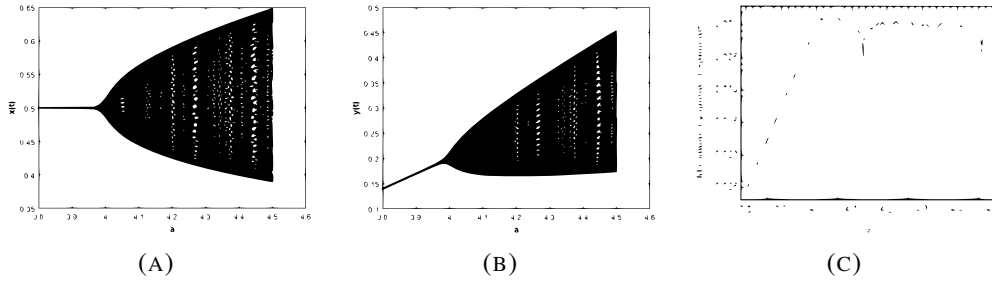


FIGURE 2. Bifurcation diagrams and MLE for the system (1.2) with values of $\beta = 0.5, \theta = -0.1$ and $a \in [3.8, 4.6]$ and initial value $(x_0, y_0) = (0.4, 0.1)$.

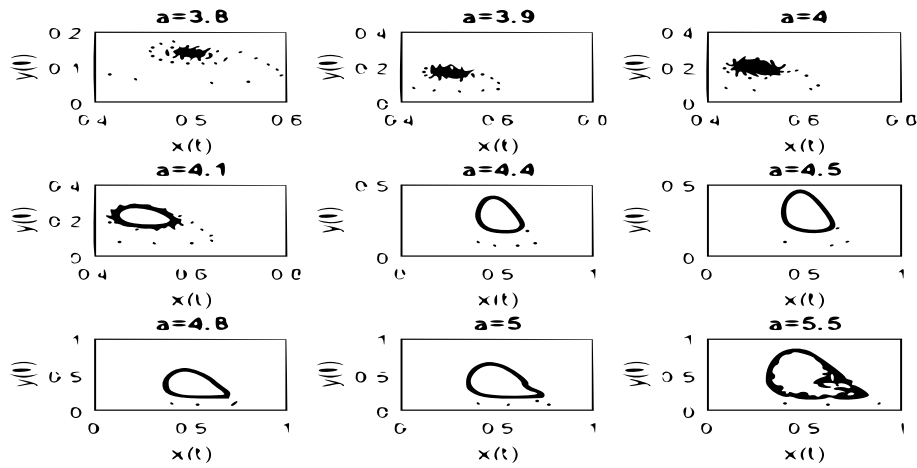


FIGURE 3. Phase portraits of the system (1.2) for different values of a .

The phase portraits for different values of a are presented in Figure 3, which clearly depicts the process of how a smooth invariant circle bifurcates from the stable fixed point $E_2 = (0.5, 0.2)$. When a exceeds 4 there appears a circular curve enclosing the fixed point E_2 , and its radius becomes larger with respect to the growth of a .

Example 3. For the parameters values $\theta = -0.8, \beta = 0.7$, the critical value of flip bifurcation point is obtained as $a_F = 2.439024390$ and the positive fixed point of the system (1.2) is evaluated as $E_2 = (0.6999999998, 0.09756097611)$. The Jacobian matrix of the system (1.2) is $J_{E_2}(a_F) = \begin{bmatrix} -1.048780486 & -0.6999999998 \\ 0.1393728231 & 1.0000000000 \end{bmatrix}$.

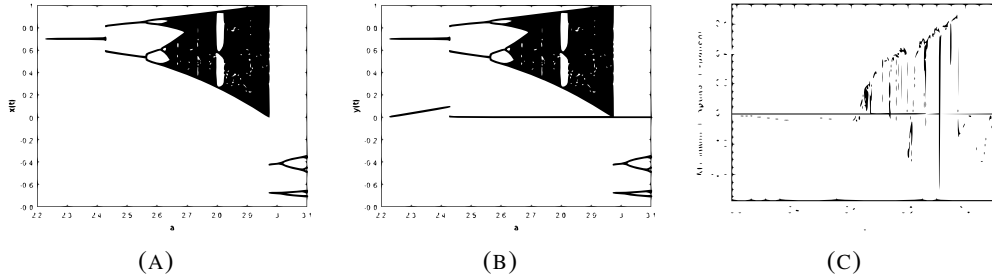


FIGURE 4. Bifurcation diagrams and MLE for the system (1.2) with values of $\theta = -0.8, \beta = 0.7, a_F = 2.439024390$ and initial value $(x_0, y_0) = (0.67, 0.08)$.

The eigenvalues are evaluated as $\lambda_1 = -1, \lambda_2 = 0.9512195120$. Direct computation shows

$$q \sim (-0.9975807099, 0.06951781980)^T,$$

$$p \sim (-0.9438583564, -0.3303504248)^T.$$

To obtain the normalization $\langle p, q \rangle = 1$, we can take normalized vectors as

$$q = (-0.9975807099, 0.06951781980)^T$$

$$p = (-1.027485787, -0.3596200254)^T.$$

Then, symmetric multi-linear vector functions are obtained as follows

$$B_1(x, y) = -9.268292682 x_1 y_1 - x_1 y_2 - x_2 y_1,$$

$$B_2(x, y) = 1.428571429 x_1 y_2 + 1.428571429 x_2 y_1,$$

$$C_1(x, y, u) = -14.63414634 x_1 y_1 u_1,$$

$$C_2(x, y, u) = 0.$$

From (3.5), the critical part is obtained as $c(a_F) = -1.091637569 < 0$. Therefore, the flip bifurcation is unstable period-2 orbits bifurcate from E_2 and it shows the correctness of Theorem 2. The bifurcations diagram, maximum Lyapunov exponents and the phase portraits of the system (1.2) are shown in Figure 4.

From Figures 4a and 4b, we see that for $a < 2.439024390$ the fixed point E_2 is stable, and loses its stability at the flip bifurcation parameter value $a = 2.439024390$. We also observe that if let $a > 2.439024390$, the system (1.2) with Allee effect displays chaotic dynamics through flip bifurcation in Figure 4a and 4b. After the stable fixed point, periodic oscillations are observed with periods 2, 4, 8 and eventually leading to chaos with gradually increasing the parameter a . Moreover, the maximum Lyapunov exponents are computed and the existence of chaotic regions in the parameter space is clearly visible in Figure 4c.

Example 4. In order to discuss the OGY feedback control method for system (1.2), we take $a_0 = 4.2$ and $(\beta, \theta) = (0.5, -0.1)$. In this case system (1.2) has unique positive fixed point $(X^*, Y^*) = (0.5, 0.260)$ which is unstable. Then corresponding controlled system is given by:

$$\begin{aligned} X_{t+1} &= (4.2 - \rho_1(X_t - 0.5) - \rho_2(Y_t - 0.260))X_t(1 - X_t)(X_t + 0.1) - X_t Y_t \\ Y_{t+1} &= \frac{1}{\beta} X_t Y_t \end{aligned} \quad (5.1)$$

where $K = [\rho_1 \ \rho_2]$ be gain matrix and $(X^*, Y^*) = (0.5, 0.26)$ is unstable fixed point of the system (1.2). We have

$$A = \begin{bmatrix} 0.790 & -0.5 \\ 0.520 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.150 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0.150 & 0.11850 \\ 0 & 0.07800 \end{bmatrix}.$$

Then it is easy to check that rank of C matrix is 2. Therefore, the system (5.1) is controllable. Then, Jacobian matrix $A - BK$ of the controlled system (5.1) is given by

$$[A - BK] = \begin{bmatrix} 0.790 - 0.150\rho_1 & -0.5 - 0.150\rho_2 \\ 0.520000 & 1 \end{bmatrix}$$

Moreover, the lines L_1 , L_2 and L_3 of marginal stability are given by:

$$\begin{aligned} L_1 &= 0.05 - 0.15\rho_1 + 0.0780\rho_2 = 0, \\ L_2 &= 0.26 + 0.0780\rho_2 = 0, \\ L_3 &= 3.84 - 0.3\rho_1 + 0.078\rho_2 = 0. \end{aligned}$$

Then, the stable triangular region bounded by marginal lines L_1 , L_2 and L_3 for the controlled system (5.1) is shown in Figure 5.

6. DISCUSSIONS

In this paper, we extend the system (1.1) by introducing weak Allee effect and investigate the dynamical behaviors of the modified the system (1.2) around coexistence fixed point. The system (1.2) obtained by modified from of the system (1.1) has different dynamics properties compared to the system (1.1). While the system (1.1) has two fixed points, model (1.2) has three fixed points. One of them is the same as in the system (1.1) without Allee effect, E_0 and the others have a new expressions E_1 and E_2 . Because of its biological significance, we focused on the coexistence fixed point E_2 and analyze the topological classifications of this fixed point of the discrete-time predator-prey system with weak Allee effect. In Figure 1b, we have compared the local stability analysis of the coexistence fixed point of the predator-prey model with and without Allee effect. One can see that the system with Allee effect reaches the fixed solution more slowly than in the system without the Allee effect. Moreover, if $-1 < \theta < -0.5$, then the predator density of the model subject to Allee effect increases, otherwise ($-0.5 \leq \theta < 0$) it decreases.

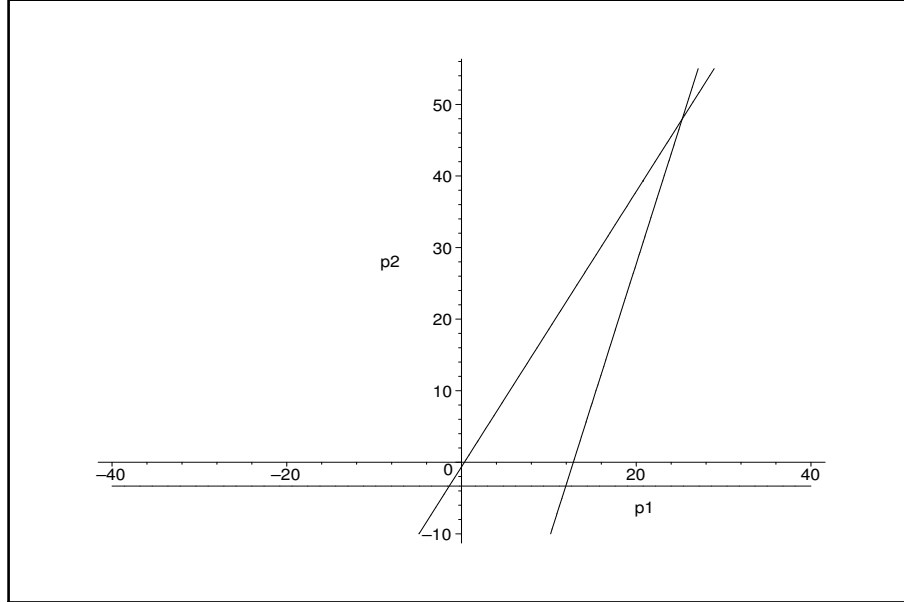


FIGURE 5. Triangular stability region bounded by L_1, L_2 and L_3 for the controlled system (5.1).

We display that flip and Neimark-Sacker bifurcations happen at certain bifurcation parameter a and some conditions on parameters θ and β . The directions of both flip bifurcation and Neimark-Sacker bifurcation are given. In order to support to obtained theoretical results and show the complex dynamical behaviors of the system (1.2), we present bifurcation diagrams, phase portraits and maximum Lyapunov exponents. Maximum Lyapunov exponents exhibit the existences of periodic orbits and chaos as parameter a increases.

In study [3], while the system (1.1) undergoes Neimark-Sacker bifurcation around the coexistence fixed point, the system (1.2) undergoes both flip and Neimark-Sacker bifurcations. Under the influence of Neimark-Sacker bifurcation unstable invariant closed curves are produced. Moreover, when the system (1.1) undergoes flip bifurcation, we can observe the existence many attractors such as steady state, period-2 orbit, period-4 orbit, period-8 orbit, chaos etc as parameter a varies. These results show that the system (1.2) has far richer dynamics compared to the system (1.1).

On the other hand, Neimark-Sacker bifurcation is successfully controlled with OGY control strategy. From our numerical investigation, it is clear that OGY method based on feedback control strategy can restore the stability. This controlling method is effective in order to advance or completely vanish the chaos due to emergence of Neimark-Sacker bifurcation.

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HERSTEIN'S THEOREM FOR PAIR OF GENERALIZED DERIVATIONS ON PRIME RINGS WITH INVOLUTION

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Abstract. The aim of this paper is to study the $*$ -identities with a pair of generalized derivations on $*$ -ideals of prime rings with involution. In particular, we prove that if a noncommutative prime $*$ -ring admit two generalized derivations $\mathcal{F}$ and G such that $[\mathcal{F}(x), G(x^*)] = 0$ for all $x \in I$, where I is a nonzero $*$ -ideal of $\mathcal{R}$, then there exists $\lambda \in C$ such that $\mathcal{F} = \lambda G$. Finally, we provide an example which shows that the primeness of $\mathcal{R}$ is crucial in our results.

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Keywords: prime ring, $*$ -ideal, involution, generalized derivation

1. NOTATIONS AND INTRODUCTION

This research has been motivated by the work of Ali et al. [2]. In all that follows, unless specially stated, $\mathcal{R}$ always denotes an associative ring with center $Z(\mathcal{R})$. As usual the symbols $s \circ t$ and $[s, t]$ will denote the anti-commutator $st + ts$ and commutator $st - ts$, respectively. Given an integer $n \geq 2$, a ring $\mathcal{R}$ is said to be n -torsion free if $nx = 0$ (where $x \in \mathcal{R}$) implies that $x = 0$. A ring $\mathcal{R}$ is called prime if $a\mathcal{R}b = (0)$ (where $a, b \in \mathcal{R}$) implies $a = 0$ or $b = 0$. We denote by Q_r, Q_l, Q_s and C right, left, symmetric Martindale ring of quotients and extended centroid of a semiprime ring $\mathcal{R}$, respectively (see [5, Chapter 2]). An additive map $x \mapsto x^*$ of $\mathcal{R}$ into itself is called an involution if (i) $(xy)^* = y^*x^*$ and (ii) $(x^*)^* = x$ hold for all $x, y \in \mathcal{R}$. A ring equipped with an involution is called ring with involution or $*$ -ring. An ideal I of $\mathcal{R}$ is called $*$ -ideal if $I^* = I$. An element x in a ring with involution is said to be hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. The sets of all hermitian and skew-hermitian elements of $\mathcal{R}$ will be denoted by $\mathcal{H}(\mathcal{R})$ and $\mathcal{S}(\mathcal{R})$, respectively. The involution is called the first kind if $Z(\mathcal{R}) \subseteq \mathcal{H}(\mathcal{R})$, otherwise it is said to be of the second kind. In the later case $\mathcal{S}(\mathcal{R}) \cap Z(\mathcal{R}) \neq (0)$.

An additive mapping $d : \mathcal{R} \rightarrow \mathcal{R}$ is said to be a derivation of $\mathcal{R}$ if $d(st) = d(s)t + sd(t)$ for all $s, t \in \mathcal{R}$. An additive mapping $\mathcal{F} : \mathcal{R} \rightarrow \mathcal{R}$ is called a generalized derivation of $\mathcal{R}$ if there exists a derivation d of $\mathcal{R}$ such that $\mathcal{F}(st) = \mathcal{F}(s)t + sd(t)$ for all $s, t \in \mathcal{R}$. Obviously, any derivation is a generalized derivation, but the converse is not

true in general. A significative example is a map of the form $\mathcal{F}(s) = as + sb$ for some $a, b \in \mathcal{R}$; such generalized derivations are called inner. Over the last few decades, several authors have studied on rings with involution involving additive mappings like derivations, cenralizers, generalized derivations etc. (viz.; [1–4] and references therein).

In [8], Herstein proved the following result: If $\mathcal{R}$ is a prime ring of characteristic not two admitting a nonzero derivation d such that $[d(x), d(y)] = 0$ for all $x, y \in \mathcal{R}$, then $\mathcal{R}$ is commutative. In [10], Lanski prove that if L is a noncommutative Lie ideal of a 2-torsion free prime ring $\mathcal{R}$ and d, h are nonzero derivations of $\mathcal{R}$ such that $[d(x), h(x)] \in C$ for all $x \in L$, then $h = \lambda d$, where $\lambda \in C$. Very recently, Ali et al. [2] prove the following result: Let $\mathcal{R}$ be a noncommutative prime ring with involution of the second kind such that $\text{char}(\mathcal{R}) \neq 2$. If $\mathcal{R}$ admits a nonzero generalized derivation $\mathcal{F}$ such that $[\mathcal{F}(x), \mathcal{F}(x^*)] = 0$ for all $x \in \mathcal{R}$, then $\mathcal{R}$ is an order in a central simple algebra of dimension at most 4 over its center and $\mathcal{F}(x) = ax + xb$ for all $x \in \mathcal{R}$ and fixed $a, b \in Q$ such that $a - b \in C$. In the same paper, they also proved that: Let $\mathcal{R}$ be a prime ring with involution of the second kind such that $\text{char}(\mathcal{R}) \neq 2$. If $\mathcal{R}$ admits a generalized derivation $\mathcal{F}$ such that $\mathcal{F}(x) \circ \mathcal{F}(x^*) = 0$ for all $x \in \mathcal{R}$, then $\mathcal{F} = 0$.

The main objective of this paper is to study the above mentioned results for a pair of generalized derivations in prime rings with involution. Throughout the paper we assume that $\mathcal{S}(\mathcal{R}) \cap \mathcal{Z}(\mathcal{R}) \neq (0)$.

2. THE RESULTS

We shall do a great deal of calculations with commutators and anti-commutators, routinely using the following basic identities. For all $x, y, z \in \mathcal{R}$;

$$\begin{aligned} [xy, z] &= x[y, z] + [x, z]y, \\ [x, yz] &= [x, y]z + y[x, z], \\ x \circ (yz) &= (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z, \\ (xy) \circ z &= x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]. \end{aligned}$$

We start our investigation with some well known facts in rings which will be used frequently throughout the text.

Fact 1. *If $\mathcal{R}$ is a prime ring and $0 \neq b \in \mathcal{Z}(\mathcal{R})$ and $ab \in \mathcal{Z}(\mathcal{R})$, then $a \in \mathcal{Z}(\mathcal{R})$.*

Fact 2. *Let $\mathcal{R}$ be a prime ring with involution $'^*$ of second kind such that $\text{char}(\mathcal{R}) \neq 2$. Let d be a nonzero derivation of $\mathcal{R}$ such that $d(h) = 0$ for all $h \in \mathcal{H}(\mathcal{R}) \cap \mathcal{Z}(\mathcal{R})$. Then $d(x) = 0$ for all $x \in \mathcal{Z}$.*

Theorem 1. *Let $\mathcal{R}$ be a 2-torsion free noncommutative prime $*$ -ring and I be a nonzero $*$ -ideal of $\mathcal{R}$. If $\mathcal{R}$ admit nonzero generalized derivations $\mathcal{F}$ and $\mathcal{G}$ associated with derivations d and g respectively such that $[\mathcal{F}(x), \mathcal{G}(x^*)] = 0$ for all $x \in I$, then $\mathcal{F} = \lambda \mathcal{G}$ for some $\lambda \in C$.*

Proof. By the assumption, we have

$$[\mathcal{F}(x), \mathcal{G}(x^*)] = 0 \text{ for all } x \in I.$$

Taking $x = x + y$, we get

$$[\mathcal{F}(x), \mathcal{G}(y^*)] + [\mathcal{F}(y), \mathcal{G}(x^*)] = 0 \text{ for all } x, y \in I. \quad (2.1)$$

Replacing y by yh (where $h \in \mathcal{Z}(\mathcal{R}) \cap \mathcal{H}(\mathcal{R})$) in (2.1), we obtain

$$[\mathcal{F}(x), y^*]g(h) + [y, \mathcal{G}(x^*)]d(h) = 0 \text{ for all } x, y \in I \text{ and } h \in \mathcal{Z}(\mathcal{R}) \cap \mathcal{H}(\mathcal{R}). \quad (2.2)$$

Replacing y by yk (where $k \in \mathcal{Z}(\mathcal{R}) \cap \mathcal{S}(\mathcal{R})$) in (2.2), we get

$$(-[\mathcal{F}(x), y^*]g(h) + [y, \mathcal{G}(x^*)]d(h))k = 0 \text{ for all } x, y \in I.$$

Since $\mathcal{Z}(\mathcal{R}) \cap \mathcal{S}(\mathcal{R}) \neq \{0\}$, so application of Fact 1 yields that

$$-[\mathcal{F}(x), y^*]g(h) + [y, \mathcal{G}(x^*)]d(h) = 0 \text{ for all } x, y \in I \text{ and } h \in \mathcal{Z}(\mathcal{R}) \cap \mathcal{H}(\mathcal{R}). \quad (2.3)$$

Combining (2.2) and (2.3), we obtain that $2[\mathcal{F}(x), y^*]g(h) = 0$. Since $\mathcal{R}$ is a 2-torsion free prime ring, the last relation gives that either $[\mathcal{F}(x), y^*] = 0$ or $g(h) = 0$.

Consider the first case $[\mathcal{F}(x), y^*] = 0$ for all $x, y \in I$. This implies that $[\mathcal{F}(x), t] = 0$ for all $x, t \in I$. Since I and $\mathcal{R}$ satisfy the same differential identities (see [11, Theorem 2]), hence $[\mathcal{F}(x), t] = 0$ for all $x, t \in \mathcal{R}$. Therefore, in view of Lemma 3 in [9], $\mathcal{F} = 0$. This leads a contradiction.

Now, consider the second case $g(h) = 0$, then (2.2) gives that $[y, \mathcal{G}(x^*)]d(h) = 0$. Using the same arguments as we have used above, we get $d(h) = 0$. Thus, the conclusion of both cases are $d(h) = g(h) = 0$ for all $h \in \mathcal{Z}(\mathcal{R}) \cap \mathcal{H}(\mathcal{R})$. By Fact 2, $d(z) = g(z) = 0$ for all $z \in \mathcal{Z}(\mathcal{R})$. Substitute $y = yk$ in (2.1), we get

$$(-[\mathcal{F}(x), \mathcal{G}(y^*)] + [\mathcal{F}(y), \mathcal{G}(x^*)])k = 0 \text{ for all } x, y \in I.$$

This gives

$$-[\mathcal{F}(x), \mathcal{G}(y^*)] + [\mathcal{F}(y), \mathcal{G}(x^*)] = 0 \text{ for all } x, y \in I. \quad (2.4)$$

Combining (2.1) and (2.4), we obtain $2[\mathcal{F}(y), \mathcal{G}(x^*)] = 0$ for all $x, y \in I$. Taking $x^* = t$ and using the fact that $\mathcal{R}$ is 2-torsion free, we conclude that $[\mathcal{F}(y), \mathcal{G}(t)] = 0$ for all $y, t \in I$. Since I and $\mathcal{R}$ satisfy the same differential identities (see [11, Theorem 2]), so we have $[\mathcal{F}(y), \mathcal{G}(t)] = 0$ for all $y, t \in \mathcal{R}$. By [9, Theorem 2] there exists $\lambda \in C$ such that $\mathcal{F} = \lambda\mathcal{G}$. This completes the proof of the theorem. $\square$

The immediate consequences of the above theorem are the following results:

Corollary 1. *Let $\mathcal{R}$ be a 2-torsion free noncommutative prime $\ast$ -ring. If $\mathcal{R}$ admit nonzero generalized derivations $\mathcal{F}$ and $\mathcal{G}$ such that $[\mathcal{F}(x), \mathcal{G}(x^*)] = 0$ for all $x \in \mathcal{R}$, then $\mathcal{F} = \lambda\mathcal{G}$ for some $\lambda \in C$.*

Corollary 2. *Let $\mathcal{R}$ be a 2-torsion free noncommutative prime $\ast$ -ring. If $\mathcal{R}$ admit nonzero generalized derivations $\mathcal{F}$ and $\mathcal{G}$ such that $[\mathcal{F}(x), \mathcal{G}(y^*)] = 0$ for all $x, y \in \mathcal{R}$, then $\mathcal{F} = \lambda\mathcal{G}$ for some $\lambda \in C$.*

Corollary 3. [3, Main Theorem] *Let $\mathcal{R}$ be a 2-torsion free noncommutative prime $*$ -ring. If $\mathcal{R}$ admit nonzero derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ such that $[\mathcal{D}_1(x), \mathcal{D}_2(x^*)] = 0$ for all $x \in \mathcal{R}$, then $\mathcal{D}_1 = \lambda \mathcal{D}_2$ for some $\lambda \in C$.*

Theorem 2. *Let $\mathcal{R}$ be a 2-torsion free noncommutative prime $*$ -ring and I be a nonzero $*$ -ideal of $\mathcal{R}$. If $\mathcal{R}$ admit nonzero generalized derivations $\mathcal{F}$ and $\mathcal{G}$ associated with derivations d and g respectively such that $[\mathcal{F}(x), \mathcal{G}(x^*)] = \mathcal{G}([x, x^*])$ for all $x \in I$, then there exists $\lambda \in C$ such that $\mathcal{F} = \lambda \mathcal{G}$.*

Proof. Linearize the given condition, we have

$$[\mathcal{F}(x), \mathcal{G}(y^*)] + [\mathcal{F}(y), \mathcal{G}(x^*)] = \mathcal{G}([x, y^*]) + \mathcal{G}([y, x^*]) \text{ for all } x, y \in I. \quad (2.5)$$

Substitute yh in place of y , we reach at

$$[\mathcal{F}(x), y^*]g(h) + [y, \mathcal{G}(x^*)]d(h) = 0 \text{ for all } x, y \in I.$$

This is same as (2.2) and thus following the same technique, we get $d(z) = g(z) = 0$ for all $z \in \mathcal{Z}(\mathcal{R})$. Now replacing y by yk in (2.5), we get

$$(-[\mathcal{F}(x), \mathcal{G}(y^*)] + [\mathcal{F}(y), \mathcal{G}(x^*)])k = (-\mathcal{G}([x, y^*]) + \mathcal{G}([y, x^*]))k \text{ for all } x, y \in I.$$

Since $\mathcal{Z}(\mathcal{R}) \cap \mathcal{S}(\mathcal{R}) \neq \{0\}$ and $\mathcal{R}$ is prime, we have

$$-[\mathcal{F}(x), \mathcal{G}(y^*)] + [\mathcal{F}(y), \mathcal{G}(x^*)] = -\mathcal{G}([x, y^*]) + \mathcal{G}([y, x^*]) \text{ for all } x, y \in I. \quad (2.6)$$

Combining (2.5) and (2.6), we obtain that $[\mathcal{F}(x), \mathcal{G}(y^*)] = \mathcal{G}([x, y^*])$ for all $x, y \in I$. In particular, for $y^* = x$, we get $[\mathcal{F}(x), \mathcal{G}(x)] = 0$ for all $x, y \in I$. Thus, in view of [9, Theorem 2] there exists $\lambda \in C$ such that $\mathcal{F} = \lambda \mathcal{G}$. This completes the proof of the theorem. $\square$

Corollary 4. *Let $\mathcal{R}$ be a 2-torsion free noncommutative prime $*$ -ring. If $\mathcal{R}$ admit nonzero generalized derivations $\mathcal{F}$ and $\mathcal{G}$ such that $[\mathcal{F}(x), \mathcal{G}(x^*)] = \mathcal{G}([x, x^*])$ for all $x \in \mathcal{R}$, then there exists $\lambda \in C$ such that $\mathcal{F} = \lambda \mathcal{G}$.*

Theorem 3. *Let $\mathcal{R}$ be a 2-torsion free prime $*$ -ring and I be a nonzero $*$ -ideal of $\mathcal{R}$. If $\mathcal{R}$ admit generalized derivations $\mathcal{F}$ and $\mathcal{G}$ such that $\mathcal{F}(x) \circ \mathcal{G}(x^*) = 0$ or $\mathcal{F}(x^*) \circ \mathcal{G}(x) = 0$ for all $x \in I$, then in both cases either $\mathcal{F} = 0$ or $\mathcal{G} = 0$.*

Proof. Suppose d and g are associated derivations of $\mathcal{F}$ and $\mathcal{G}$ respectively. Assume the relation

$$\mathcal{F}(x) \circ \mathcal{G}(x^*) = 0 \text{ for all } x \in I.$$

Linearize the above equation, we get

$$\mathcal{F}(x) \circ \mathcal{G}(y^*) + \mathcal{F}(y) \circ \mathcal{G}(x^*) = 0 \text{ for all } x, y \in I. \quad (2.7)$$

Putting yh (where $h \in \mathcal{Z}(\mathcal{R}) \cap \mathcal{H}(\mathcal{R})$) in place of y and we obtain

$$(\mathcal{F}(x) \circ y^*)g(h) + (y \circ \mathcal{G}(x^*))d(h) = 0 \text{ for all } x, y \in I. \quad (2.8)$$

Taking $y = yk$, we get

$$(-(\mathcal{F}(x) \circ y^*)g(h) + (y \circ \mathcal{G}(x^*))d(h))k = 0 \text{ for all } x, y \in I.$$

This implies that

$$-(\mathcal{F}(x) \circ y^*)g(h) + (y \circ \mathcal{G}(x^*))d(h) = 0 \text{ for all } x, y \in I. \quad (2.9)$$

Combining (2.8) and (2.9), we find that $(\mathcal{F}(x) \circ y^*)g(h) = 0$ for all $x, y \in I$. By the primeness of $\mathcal{R}$, we have $\mathcal{F}(x) \circ y^* = 0$ or $g(h) = 0$. Firstly, if $\mathcal{F}(x) \circ y^* = 0$, then $\mathcal{F}(x) \circ x = 0$ for all $x \in I$. Therefore, in view of [6, Theorem 1], $\mathcal{F}(x) = 0$ for all $x \in I$. For any $r \in \mathcal{R}$, $0 = \mathcal{F}(xr) = \mathcal{F}(x)r + xd(r) = xd(r)$. This implies $d = 0$ on $\mathcal{R}$. In the other way, for any $r \in \mathcal{R}$, $0 = \mathcal{F}(rx) = \mathcal{F}(r)x + rd(x) = \mathcal{F}(r)x$ and hence $\mathcal{F} = 0$ on $\mathcal{R}$. Now, consider the second case $g(h) = 0$. From (2.8), we have $(y \circ \mathcal{G}(x^*))d(h) = 0$ for all $x, y \in I$. Primeness of $\mathcal{R}$ yields that $y \circ \mathcal{G}(x^*) = 0$ or $d(h) = 0$. If $y \circ \mathcal{G}(x^*) = 0$, then $\mathcal{G} = 0$ on $\mathcal{R}$. If $d(h) = 0$, then by Fact 2, we have $d(z) = g(z) = 0$ for all $z \in Z(\mathcal{R})$. Substituting $y = yk$ in (2.7), we obtain

$$-\mathcal{F}(x) \circ \mathcal{G}(y^*) + \mathcal{F}(y) \circ \mathcal{G}(x^*) = 0 \text{ for all } x, y \in I. \quad (2.10)$$

Subtracting (2.10) from (2.7), we get

$$2(\mathcal{F}(x) \circ \mathcal{G}(y^*)) = 0 \text{ for all } x, y \in I.$$

Since $\mathcal{R}$ is 2-torsion free and taking $y = y^*$, we obtain that

$$\mathcal{F}(x) \circ \mathcal{G}(y) = 0 \text{ for all } x, y \in I.$$

Since I and $\mathcal{R}$ satisfy the same differential identities (see [11, Theorem 2]), so $\mathcal{F}(x) \circ \mathcal{G}(y) = 0$ for all $x, y \in \mathcal{R}$. Thus in view of [7, Theorem 2.7], we conclude that either $\mathcal{F} = 0$ or $\mathcal{G} = 0$ on $\mathcal{R}$. The proof in case we have the relation $\mathcal{F}(x^*) \circ \mathcal{G}(x) = 0$ for all $x \in I$ goes through similarly. $\square$

Following corollaries are the immediate consequences of the above theorem.

Corollary 5. *Let $\mathcal{R}$ be a 2-torsion free prime *-ring. If $\mathcal{R}$ admit generalized derivations $\mathcal{F}$ and $\mathcal{G}$ such that $\mathcal{F}(x) \circ \mathcal{G}(x^*) = 0$ or $\mathcal{F}(x^*) \circ \mathcal{G}(x) = 0$ for all $x \in \mathcal{R}$, then in both cases either $\mathcal{F} = 0$ or $\mathcal{G} = 0$.*

Corollary 6. *Let $\mathcal{R}$ be a 2-torsion free prime *-ring and I be a nonzero *-ideal of $\mathcal{R}$. If $\mathcal{R}$ admit a derivation $\mathcal{D}$ and a generalized derivation $\mathcal{F}$ such that $\mathcal{F}(x) \circ \mathcal{D}(x^*) = 0$ or $\mathcal{F}(x^*) \circ \mathcal{D}(x) = 0$ for all $x \in I$, then in both cases either $\mathcal{F} = 0$ or $\mathcal{D} = 0$.*

Corollary 7. *Let $\mathcal{R}$ be a 2-torsion free prime *-ring. If $\mathcal{R}$ admit derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ such that $\mathcal{D}_1(x) \circ \mathcal{D}_2(x^*) = 0$ or $\mathcal{D}_1(x^*) \circ \mathcal{D}_2(x) = 0$ for all $x \in \mathcal{R}$, then in both cases either $\mathcal{D}_1 = 0$ or $\mathcal{D}_2 = 0$.*

Corollary 8. *Let $\mathcal{R}$ be a 2-torsion free prime *-ring and I be a nonzero *-ideal of $\mathcal{R}$. If $\mathcal{R}$ admits generalized derivation $\mathcal{F}$ such that $\mathcal{F}(x) \circ \mathcal{F}(x^*) = 0$ for all $x \in I$, then $\mathcal{F} = 0$.*

Corollary 9. [2, Theorem 1.2.] *Let $\mathcal{R}$ be a 2-torsion free prime *-ring. If $\mathcal{R}$ admits generalized derivation $\mathcal{F}$ such that $\mathcal{F}(x) \circ \mathcal{F}(x^*) = 0$ for all $x \in \mathcal{R}$, then $\mathcal{F} = 0$.*

Theorem 4. *Let $\mathcal{R}$ be a 2-torsion free prime $*$ -ring and I be a nonzero $*$ -ideal of $\mathcal{R}$. If $\mathcal{R}$ admit nonzero generalized derivations $\mathcal{F}$ and $\mathcal{G}$ associated with derivations d and g respectively such that $\mathcal{F}(x)\mathcal{G}(x^*) = 0$ or $\mathcal{F}(x^*)\mathcal{G}(x) = 0$ for all $x \in I$, then in both cases $\mathcal{F}$ and $\mathcal{G}$ are of the form $\mathcal{F}(x) = xp$ and $\mathcal{G}(x) = qx$ for all $x \in \mathcal{R}$, where $p \in Q_l$, $q \in Q_r$ are fixed elements such that $pq = 0$.*

Proof. Firstly, we suppose that

$$\mathcal{F}(x)\mathcal{G}(x^*) = 0 \text{ for all } x \in I.$$

Linearization of the above condition gives that

$$\mathcal{F}(x)\mathcal{G}(y^*) + \mathcal{F}(y)\mathcal{G}(x^*) = 0 \text{ for all } x, y \in I. \quad (2.11)$$

Replacing y by yh , we obtain

$$\mathcal{F}(x)y^*g(h) + y\mathcal{G}(x^*)d(h) = 0 \text{ for all } x, y \in I. \quad (2.12)$$

Write yk in place of y , we get

$$(-\mathcal{F}(x)y^*g(h) + y\mathcal{G}(x^*)d(h))k = 0 \text{ for all } x, y \in I.$$

Primeness of $\mathcal{R}$ yields that

$$-\mathcal{F}(x)y^*g(h) + y\mathcal{G}(x^*)d(h) = 0 \text{ for all } x, y \in I.$$

Combining the last relation with (2.12), we can find that $\mathcal{F}(x)y^*g(h) = 0$ and hence primeness of $\mathcal{R}$ yields that $\mathcal{F}(x)y^* = 0$ or $g(h) = 0$. Firstly, consider the case $\mathcal{F}(x)y^* = 0$ for all $x, y \in I$. This implies that $\mathcal{F}(x) = 0$ for all $x \in I$, which contradicts the fact that $\mathcal{F} \neq 0$. Now, consider the second case $g(h) = 0$ for all $h \in \mathcal{Z}(\mathcal{R}) \cap \mathcal{H}(\mathcal{R})$. From (2.12), we have $y\mathcal{G}(x^*)d(h) = 0$. This further implies that $y\mathcal{G}(x^*) = 0$ or $d(h) = 0$. If $y\mathcal{G}(x^*) = 0$ for all $x, y \in I$, then again we get a contradiction. Thus, there is the only possible case $d(h) = 0$ and $g(h) = 0$ for all $h \in \mathcal{Z}(\mathcal{R}) \cap \mathcal{H}(\mathcal{R})$ and hence $d(k) = g(k) = 0$ for all $k \in \mathcal{Z}(\mathcal{R}) \cap \mathcal{S}(\mathcal{R})$. Now, replacing y by yk in (2.11), we arrive at

$$(-\mathcal{F}(x)\mathcal{G}(y^*) + \mathcal{F}(y)\mathcal{G}(x^*))k = 0 \text{ for all } x, y \in I.$$

This implies that

$$-\mathcal{F}(x)\mathcal{G}(y^*) + \mathcal{F}(y)\mathcal{G}(x^*) = 0 \text{ for all } x, y \in I.$$

Subtracting the last relation from (2.11) and using the fact that $\mathcal{R}$ is 2-torsion free, we get that $\mathcal{F}(x)\mathcal{G}(y^*) = 0$ for all $x, y \in I$. In particular, for $y^* = x$ this gives that $\mathcal{F}(x)\mathcal{G}(x) = 0$ for all $x \in I$. Since I and $\mathcal{R}$ satisfy the same differential identities (see [11, Theorem 2]), so $\mathcal{F}(x)\mathcal{G}(x) = 0$ for all $x \in \mathcal{R}$. Therefore, by the Theorem 2.2 in [7], we get the required result. Similarly, we can prove the result for the case $\mathcal{F}(x^*)\mathcal{G}(x) = 0$ for all $x \in I$. Therefore the proof of the theorem is completed. $\square$

Corollary 10. *Let $\mathcal{R}$ be a 2-torsion free prime $*$ -ring. If $\mathcal{R}$ admit a derivation $\mathcal{D}$ and a generalized derivation $\mathcal{F}$ such that $\mathcal{D}(x)\mathcal{F}(x^*) = 0$ or $\mathcal{F}(x^*)\mathcal{D}(x) = 0$ for all $x \in \mathcal{R}$, then $\mathcal{F} = 0$ or $\mathcal{D} = 0$.*

Proof. For the first case $\mathcal{D}(x)\mathcal{F}(x^*) = 0$ for all $x \in \mathcal{R}$. On the contrary, suppose that $\mathcal{D} \neq 0$ and $\mathcal{F} \neq 0$. Then by the Theorem 4, there exists $p \in Q_l$ such that $\mathcal{D}(x) = xp$ for all $x \in \mathcal{R}$. Therefore $\mathcal{D}(xy) = xyp = \mathcal{D}(x)y + x\mathcal{D}(y)$. This gives that $\mathcal{D}(x)y = 0$ for all $x, y \in \mathcal{R}$. This implies that $\mathcal{D} = 0$, which leads a contradiction to our supposition. Similar proof for the case $\mathcal{F}(x^*)\mathcal{D}(x) = 0$ for all $x \in \mathcal{R}$. $\square$

Corollary 11. *Let $\mathcal{R}$ be a 2-torsion free prime *-ring. If $\mathcal{R}$ admit derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ such that $\mathcal{D}_1(x)\mathcal{D}_2(x^*) = 0$ or $\mathcal{D}_1(x^*)\mathcal{D}_2(x) = 0$ for all $x \in \mathcal{R}$, then $\mathcal{D}_1 = 0$ or $\mathcal{D}_2 = 0$.*

The following example justifies the fact that Theorems 1, 3 & 4 are not true for semiprime rings.

Example 1. Let $\mathcal{R} = \left\{ \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \mid c_1, c_2, c_3, c_4 \in \mathbb{C} \right\}$, where $\mathbb{C}$ is a ring of complex numbers. Of course, $\mathcal{R}$ with matrix addition and matrix multiplication is a non-commutative prime ring. Define mappings $*_1, d_1, \mathcal{F}_1 : \mathcal{R} \rightarrow \mathcal{R}$ such that

$$\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}^{*_1} = \begin{pmatrix} \bar{c}_4 & \bar{c}_2 \\ \bar{c}_3 & \bar{c}_1 \end{pmatrix}, \quad d_1 \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = \begin{pmatrix} 0 & -c_2 \\ c_3 & 0 \end{pmatrix}$$

$$\text{and } \mathcal{F}_1 \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = \begin{pmatrix} c_1 & 2c_2 \\ -c_3 & 0 \end{pmatrix}.$$

It can be easily checked that $*_1$ is an involution of the second kind and $\mathcal{F}_1$ is a generalized derivation of $\mathcal{R}$ associated with the derivation d_1 . Let $\mathbb{H}$ be a ring of real quaternions. Clearly, a mapping $*_2$ such that $q^{*2} = \alpha - i\beta - j\gamma - k\delta$ is an involution on $\mathbb{H}$. Next, define a mapping $\mathcal{G}_1$ on $\mathbb{H}$ such that $\mathcal{G}_1(q) = 2iq - qi$. Then $\mathcal{G}_1$ is a generalized derivation of $\mathbb{H}$ associated with the derivation $g_1 = g_i$ (where g_i is an inner derivation on $\mathbb{H}$ determined by $i \in \mathbb{H}$).

Let $\mathcal{L} = \mathcal{R} \times \mathbb{H}$. Then $\mathcal{L}$ is a 2-torsion free noncommutative semiprime ring. Now define an involution $*$ on $\mathcal{L}$, as $(x, y)^* = (x^{*_1}, y^{*_2})$. Clearly, $*$ is an involution of the second kind. Further, we define the mappings $\mathcal{F}$ and $\mathcal{G}$ from $\mathcal{L}$ to $\mathcal{L}$ such that $\mathcal{F}(x, y) = (\mathcal{F}_1(x), 0)$ and $\mathcal{G}(x, y) = (0, \mathcal{G}_2(x))$ for all $(x, y) \in \mathcal{L}$. It can be easily checked that $\mathcal{F}$ and $\mathcal{G}$ are nonzero generalized derivations on $\mathcal{L}$ and satisfying $[\mathcal{F}(X), \mathcal{G}(X^*)] = 0$, $\mathcal{F}(X) \circ \mathcal{G}(X^*) = 0$ and $\mathcal{F}(X)\mathcal{G}(X^*) = 0$ for all $X \in \mathcal{L}$, but the conclusions of Theorems 1, 3 & 4 are not held. Hence, in these results the hypothesis of primeness is essential.

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FIXED POINT THEOREMS FOR UPPER SEMICONTINUOUS MULTIVALUED OPERATORS IN LOCALLY CONVEX SPACES

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Abstract. In the paper, we will discuss some τ -topological properties of the set $\mathcal{F}(S_0, T, H) := \{x \in X : S_0x \in Tx + Hx\}$, where T is multivalued operator, S_0 and H are two single valued operators acting on a Hausdorff locally convex space X and τ is a weaker Hausdorff locally convex topology on X . Moreover, when X is a τ -angelic space with the so-called τ -Krein-Šmulian property, we will provide some new variants of fixed point theorems for multivalued operators. Our results are formulated in terms of τ -upper semicontinuity, τ - S_0 -demicompactness and families of axiomatic measures of noncompactness.

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1. INTRODUCTION

In 1966, W. V. Petryshyn [16] introduced the concept of demicompact nonlinear operators and gave an iterative methods for the construction of fixed points on Hilbert space. In [11, 17] the authors used this notion to give new fixed point theorems for demicompact- k -set contractions defined on Banach space X . In other direction the demicompactness concept was used to provide several results on Fredholm and spectral theories (see [2, 6, 18]). In 2014 B. Krichen [13] gave a generalization of this concept by introducing the class of relative demicompact linear operator with respect to a given linear operator. This definition asserts that if X is a Banach space $T: \mathcal{D}(T) \subset X \rightarrow X$, and $S_0: \mathcal{D}(S_0) \subset X \rightarrow X$ are two linear operator with $\mathcal{D}(T) \subset \mathcal{D}(S_0)$. Then T is said to be S_0 -demicompact (or relative demicompact with respect to S_0) if every bounded $(x_n)_n$ in $\mathcal{D}(T)$ such that $(S_0x_n - Tx_n)$ converges in X , have a convergent subsequence. In [14] B. Krichen and D. O'Regan studied some topological properties of the set $\mathcal{F}(S_0, T, z) := \{x \in X, S_0x \in Tx + z\}$, where T is a nonlinear multivalued mapping and S_0 is single valued mapping on Banach space X . Their results are formulated in term of the concept of weakly relative demicompactness.

In this paper we generalize the results of [14], to discuss some topological properties of the set

$$\mathcal{F}(S_0, T, H) := \{x \in X, S_0x \in Tx + Hx\},$$

where T is a multivalued operator, S_0 and H are two single valued operators acting on a Hausdorff locally convex space X such that $D(T) \subset D(S_0) \cap D(H)$. Then, we present some new fixed point theorems for τ -upper semicontinuous multivalued operators on X , where τ is a weaker Hausdorff locally convex topology on X .

Firstly, let us recall some notations and basic concepts. Let $(X, \{|\cdot|_p\}_{p \in \Lambda})$ be a Hausdorff locally convex vector space endowed with a family of seminorms $\{|\cdot|_p\}_{p \in \Lambda}$ generating its topology and let τ be a weaker Hausdorff locally convex topology on X . We denote by $\xrightarrow{\tau}$ the convergence in (X, τ) and by $\rightarrow$ the convergence in $(X, \{|\cdot|_p\}_{p \in \Lambda})$. We mean by τ -compact, (resp. τ -closed) sets, compact, (resp. closed) sets with respect to the topology τ . We also denote by $\mathcal{B}(X)$ the family of all non-empty bounded subsets of X (with respect to the topology generated by $\{|\cdot|_p\}_{p \in \Lambda}$).

Now, let us consider the following axiomatic definition of a family of measures of noncompactness in a Hausdorff locally convex vector space.

Definition 1 (Definition 2.1 in [4]). A family of functions $\phi_{p_\tau}: \mathcal{B}(X) \rightarrow \mathbb{R}^+$, ($p \in \Lambda$) is said to be a Φ_Λ^τ -measures of noncompactness in X (Φ_Λ^τ -MNC, in short) if for each $p \in \Lambda$, it satisfies the following conditions:

- (i) $\phi_{p_\tau}(\overline{\text{conv}}^\tau(M)) \leq \phi_{p_\tau}(M)$ for each $M \in \mathcal{B}(X)$, where $\overline{\text{conv}}^\tau(M)$ is the closure of the convex hull of M in (X, τ) ;
- (ii) $M_1 \subseteq M_2 \Rightarrow \phi_{p_\tau}(M_1) \leq \phi_{p_\tau}(M_2)$, where $M_1, M_2 \in \mathcal{B}(X)$;
- (iii) $\phi_{p_\tau}(\{x\} \cup M) = \phi_{p_\tau}(M)$ for any $x \in X$ and $M \in \mathcal{B}(X)$;
- (iv) $\phi_{p_\tau}(M) = 0$ implies M is relatively τ -compact in X ;
- (v) if $(M_n)_n$ is a sequence of τ -closed sets of $\mathcal{B}(X)$ such that $M_{n+1} \subset M_n$, $n = 1, 2, \dots$ and $\lim_{n \rightarrow \infty} \phi_{p_\tau}(M_n) = 0$ for each $p \in \Lambda$, then $M_\infty = \bigcap_{n=1}^\infty M_n$ is nonempty relatively τ -compact subset of X .

The family Φ_Λ^τ -MNC is called:

- Positively homogeneous, if for each $p \in \Lambda$, $\phi_{p_\tau}(\lambda M) = \lambda \phi_{p_\tau}(M)$, $\lambda > 0$, where $M \in \mathcal{B}(X)$.
- Subadditive, if for each $p \in \Lambda$, $\phi_{p_\tau}(M_1 + M_2) \leq \phi_{p_\tau}(M_1) + \phi_{p_\tau}(M_2)$, where $M_1, M_2 \in \mathcal{B}(X)$.

Example 1 ([7]). The family of measures of weak noncompactness in a locally convex space X , which is defined by:

$$\omega_p(M) = \inf\{r > 0 : \exists W \in \mathcal{W}(X) \text{ such that } M \subseteq W + B_p(0, r)\}, \quad p \in \Lambda$$

is positively homogeneous and subadditive $\Phi_\Lambda^{\sigma(X, X^*)}$ -MNC. Here, $B_p(0, r)$ is the closed ball centered at 0 with radius r , $\sigma(X, X^*)$ is the weak topology of X and $\mathcal{W}(X)$ is the set of all nonempty relatively weakly compact subsets of X . This formula is

based on the notion of single measure of weak noncompactness introduced by De Blasi [7].

Definition 2. Let M be a nonempty subset of X and let $\Phi_\Lambda^\tau := \{\phi_{p_\tau}, p \in \Lambda\}$ be a family of Φ_Λ^τ -MNC in X . An operator $A: M \rightarrow X$ is said Φ_Λ^τ -contraction if for any bounded subset S of M , $A(S) \in \mathcal{B}(X)$, and for each $p \in \Lambda$, there exists a constant $\beta_p \in [0, 1)$ such that $\phi_{p_\tau}(A(S)) \leq \beta_p \phi_{p_\tau}(S)$. The operator A is called Φ_Λ^τ -condensing if for any bounded subset S of M , $A(S) \in \mathcal{B}(X)$, and for each $p \in \Lambda$ such that $\phi_{p_\tau}(S) > 0$, $\phi_{p_\tau}(A(S)) < \phi_{p_\tau}(S)$.

Definition 3 (Definition p. 30 in [8]). A topological (Hausdorff) space X is called angelic (or has countably determined compactness) if for every relatively countably compact subset M of X , the following holds:

- (i) M is relatively compact.
- (ii) For each $x \in \overline{M}$, there is a sequence in M which converges to x .

Remark 1.

- (i) Note that all metrizable locally convex spaces endowed with their weak topology are angelic. (See Eberlein-Šmulian theorem [15].)
- (ii) In angelic spaces the classes of compact, countably compact, and sequentially compact sets coincide (see [8, p. 31]).

Definition 4 (Definition 2.4 in [4]). Let M be a nonempty subset of X . An operator $A: M \rightarrow X$ is said to be τ -closed on M if for each sequence $(x_n)_n \in M$ such that $x_n \xrightarrow{\tau} x$ and $Ax_n \xrightarrow{\tau} y$, then $x \in M$ and $y = Ax$.

Definition 5 (Definition 2.5 in [5]). Let M be a nonempty subset of X , $A: M \rightarrow X$ be an operator. We say that A is τ -sequentially continuous on M if for each sequence $(x_n)_n \subset M$ with $x_n \xrightarrow{\tau} x$ and $x \in M$, we have that $Ax_n \xrightarrow{\tau} Ax$.

Remark 2.

- (i) Clearly, every τ -sequentially continuous operator is τ -closed, but the converse is not true.
- (ii) If X is angelic, then any τ -sequentially continuous operator on a τ -compact set is τ -continuous.

Definition 6 (Definition 2.6 in [4]). We say that X has the τ -Krein-Šmulian property (τ -KS, in short) if the closed convex hull of a τ -compact set is τ -compact.

Remark 3.

- (i) Each Fréchet space X has the τ -KS property, here τ is the weak topology $\sigma(X, X^*)$ defined on X , particularly for each Banach space.
- (ii) Let X be a Hausdorff locally convex space with the τ -KS property, and let M be a nonempty relatively τ -compact subset of X . Using property (ii) of Definition 1, we obtain that, $\phi_{p_\tau}(M) \leq \phi_{p_\tau}(\overline{\text{conv}}^\tau(M))$, hence $\phi_{p_\tau}(M) = 0$. In this case, we say that Φ_Λ^τ -MNC is regular.

Let X be a Hausdorff locally convex space, we define

$$\mathcal{P}(X) = \{M \subseteq X : M \text{ is nonempty}\}.$$

$$\mathcal{P}_{cl,cv}(X) = \{M \subseteq X : M \text{ is nonempty closed convex}\}.$$

$$\mathcal{P}_{\tau-cl,cv}(X) = \{M \subseteq X : M \text{ is nonempty } \tau\text{-closed convex}\}.$$

Let M be a nonempty subset of X and $A: M \rightarrow \mathcal{P}(X)$ be a multivalued mapping. We denote by $\mathcal{R}(A) = \bigcup_{y \in M} A(y)$ and $Gr(A) = \{(x, y) \in M \times X, x \in A(y)\}$ the range and the graph of A respectively.

Definition 7. Let $A: M \rightarrow \mathcal{P}(X)$ be a multivalued operator. We say that

- (i) A is τ -compact if the set $\mathcal{R}(A)$ is relatively τ -compact in X .
- (ii) A is τ -closed if its graph $Gr(T)$ is τ -closed in $X \times X$.
- (iii) A has a τ -sequentially closed graph if for every sequence $(x_n)_n \subset M$ with $x_n \xrightarrow{\tau} x$ in M and for every sequence $y_n \in A(x_n)$, $y_n \xrightarrow{\tau} y$ in X implies $y \in A(x)$.

Definition 8. Let $A: M \rightarrow \mathcal{P}(X)$ be a multivalued operator.

- (i) A is called τ -upper semicontinuous if for each τ -closed set $C \subset M$, the set $A^{-1}(C)$ is τ -closed in X , such that

$$A^{-1}(C) = \{x \in X, A(x) \cap C \neq \emptyset\}.$$

- (ii) We say that A is sequentially τ -upper semicontinuous if $A^{-1}(C)$ is sequentially τ -closed.

Remark 4. Clearly, every τ -upper semicontinuous multivalued operator is sequentially τ -upper semicontinuous, but the converse is not true.

The following lemma provides a sequential characterization of an upper semicontinuous multi-valued mapping.

Lemma 1. A multi-valued map T is upper semi-continuous at a point $x \in X$ if, and only if, for every sequence $\{x_n\}_{n=0}^{\infty}$ in X which converges to x , and for any open set $V \subset X$ such that $T(x) \subset V$, there exists $n_0 \in \mathbb{N}$ with $T(x_n) \subset V$, for all $n \geq n_0$.

The following lemma [10] will be used later.

Lemma 2. Assume that $T: X \rightarrow \mathcal{P}(X)$ is an upper semi-continuous multi-valued operator. Then the graph $Gr(T)$ is a closed subset of $X \times X$.

2. RELATIVE τ -DEMICOMPACT NONLINEAR OPERATORS

In this section X is a Hausdorff locally convex topological vector space, τ is a weaker Hausdorff locally convex vector topology on X and S_0 is a single-valued operator from $\mathcal{D}(S_0) \subset X$ into X . Now, we introduce a new concept of τ - S_0 -demicompact.

Definition 9. Let X be a topological vector space. Let $T: \mathcal{D}(T) \subset X \longrightarrow X$ be a single-valued operator with $\mathcal{D}(T) \subset \mathcal{D}(S_0)$. Now T is called τ - S_0 -demicompact if whenever $S_0x_n - Tx_n$ is τ -convergent and $(x_n)_n$ is contained in a bounded set of X , then the sequence $(x_n)_n$ has a τ -convergent subsequence in $\mathcal{D}(T)$. If $S_0 = I$, T is simply said to be τ -demicompact.

Definition 10. A single-valued operator $T: \mathcal{D}(T) \subset X \longrightarrow X$ with $\mathcal{D}(T) \subset \mathcal{D}(S_0)$ is said to be τ - S_0 -semiclosed if for any τ -closed subset $V \subset$ of X , the set $(S_0 - T)V$ is τ -closed.

Remark 5. We note that there is no relationship between the concepts of τ - S_0 -semiclosedness and τ - S_0 -demicompactness. Consider the map

$$\begin{cases} T: \mathbb{R} & \longrightarrow \mathbb{R} \\ x & \longrightarrow \frac{x|x|}{1+|x|} \end{cases}$$

Obviously, $(Id_{\mathbb{R}} - T)(\mathbb{R}) =]-1, 1[$, so it follows that T is not $Id_{\mathbb{R}}$ -semiclosed. However, we know that in finite dimensional spaces, every bounded sequence has a weakly convergent subsequence. Therefore, the operator T is weakly $Id_{\mathbb{R}}$ -demicompact.

For the following, let T be multivalued operator, S_0 and H are two single-valued operators from $\mathcal{D}(S_0), \mathcal{D}(H)$ into X respectively, with $\mathcal{D}(S_0) \subset \mathcal{D}(H)$ and $\mathcal{D}(T) \subset \mathcal{D}(S_0) \cap \mathcal{D}(H)$. We denote by $\mathcal{F}(S_0, T, H)$ the set of solutions of

$$S_0x \in Tx + Hx, \quad (2.1)$$

where, $Tx + Hx = \{y + Hx : y \in Tx\}$, and x is a solution if the relation (2.1) holds. Note we do not assume existence or uniqueness so the set $\mathcal{F}(S_0, T, H)$ might be empty or contain many elements.

Remark 6. Let X be a Hausdorff locally convex topological vector space. Assume that S_0 and H are continuous single-valued operators. Then, the subset $\mathcal{F}(S_0, T, H)$ is closed.

If X is endowed with its τ -topology, then we have the following.

Theorem 1. Let X be a Hausdorff topological vector space (locally convex topological vector space). Assume that T is a multi-valued operator with τ -sequentially closed graph and S_0 and H are τ -sequentially continuous operators. Then, the subset $\mathcal{F}(S_0, T, H)$ is τ -sequentially closed.

Proof. Let $(x_n)_n$ be a sequence of $\mathcal{F}(S_0, T, H)$ such that $x_n \xrightarrow{\tau} x$. Then, there exists a sequence $(y_n)_n$ of $(Tx_n)_n$ such that $S_0x_n = y_n + Hx_n$ for every $n \in \mathbb{N}$. From the τ -sequential continuity of S_0 and H it follows that $S_0x_n \xrightarrow{\tau} S_0x$ and $Hx_n \xrightarrow{\tau} Hx$ and then $(x_n, y_n) \xrightarrow{\tau} (x, S_0x - Hx)$. Since T has a τ -sequentially closed graph, it follows that $x \in \mathcal{F}(S_0, T, H)$ and so, $\mathcal{F}(S_0, T, H)$ is τ -sequentially closed. $\square$

Notice that if S_0 is only τ -closed, then we have the following:

Theorem 2. *Let X be a Hausdorff topological vector space (locally convex topological vector space). Assume that compact sets in (X, τ) are angelic. If T is a τ -compact multi-valued operator and $S_0 - H$ is a τ -closed operator, then the subset $\mathcal{F}(S_0, T, H)$ is τ -sequentially closed.*

Proof. Let $(x_n)_n \subset \mathcal{D}(T)$ be a sequence of $\mathcal{F}(S_0, T, H)$ such that $x_n \xrightarrow{\tau} x$. Then, there exists a sequence $(y_n)_n$ of $(Tx_n)_n$ such that $S_0x_n = y_n + Hx_n$ for every $n \in \mathbb{N}$. Since $\{y_n : n \in \mathbb{N}\} \subset \mathcal{R}(T)$, it follows that $\phi_{p_\tau}(\{y_n : n \in \mathbb{N}\}) = 0$. Since compact sets in (X, τ) are angelic, there exists a subsequence $(y_{\phi(n)})_n$ such that $(S_0 - H)x_{\phi(n)} \xrightarrow{\tau} y$. The τ -closedness of $S_0 - H$ shows that $x \in \mathcal{D}(T)$ and $S_0x \in Tx + Hx$. Then, $x \in \mathcal{F}(S_0, T, H)$ and consequently, $\mathcal{F}(S_0, T, H)$ is τ -sequentially closed. $\square$

Theorem 3. *Let X be a Hausdorff locally convex topological vector space such that compact sets in (X, τ) are angelic and let T be a τ -compact operator with τ -sequentially closed graph. Assume that $S_0 - H$ is τ -closed and H is a S_0 - τ -demicompact mapping. Then the set $\mathcal{F}(S_0, T, H)$ is relatively τ -compact.*

Proof. Let $(x_n)_n \subset \mathcal{D}(T)$ be a sequence of $\mathcal{F}(S_0, T, H)$. Then, $S_0x_n - Hx_n \in Tx_n \subset \mathcal{R}(T)$. Since $\mathcal{R}(T)$ is τ -compact, there exists a subsequence $x_{\phi(n)} \subset \mathcal{D}(T)$ such that $S_0x_n - Hx_n \xrightarrow{\tau} y$, $y \in X$. Using the S_0 - τ -demicompactness of H , we deduce the existence of a subsequence $(x_{\phi \circ \psi(n)})_n$ of $(x_{\phi(n)})_n$ such that $x_{\phi \circ \psi(n)}$ τ -converges to some $x \in X$. From the τ -closedness of $S_0 - H$, we obtain $x \in \mathcal{D}(T)$ and $y = S_0x - Hx$. Now, since $Gr(T)$ is τ -sequentially closed, we deduce that $x \in \mathcal{F}(S_0, T, H)$ and so the result follows from the angelicity of compact sets in (X, τ) . $\square$

Now, we give a sufficient condition to an nonlinear operator to be τ -relative demicompact with respect to a given nonlinear operator.

Theorem 4. *Let $T : \mathcal{D}(T) \subset X \longrightarrow X$, $S_0 : \mathcal{D}(S_0) \subset X \longrightarrow X$ be two single-valued operators with $\mathcal{D}(T) \subset \mathcal{D}(S_0)$ such that $S_0 - T$ have a τ -sequentially closed graph. Assume that for every τ -closed convex, bounded set D , the multi-valued map*

$$F_D : X \longrightarrow \mathcal{P}(X)$$

$$y \longmapsto D \cap (S_0 - T)^{-1}y,$$

is compact-valued and τ -upper-semicontinuous. Then, the operator T is τ - S_0 -demicompact.

Proof. Let $y_n := (S_0 - T)x_n$ be a τ -converging sequence and assume that $(x_n)_n$ is included in the τ -closed, and bounded set D . The τ -upper-semicontinuity of $y \longmapsto D \cap (S_0 - T)^{-1}y$ implies that $(x_n)_n$ τ -converges to the set $D \cap (S_0 - T)^{-1}y_0$ where $y_0 = \lim y_n$, with respect to the topology τ . The τ -compactness of $D \cap (S_0 - T)^{-1}y_0$ implies that there exists a subsequence of $(x_n)_n$ that converges τ -to an element $x \in D \cap (S_0 - T)^{-1}y_0$. Therefore, T is τ - S_0 -demicompact. $\square$

Remark 7. The above results extended the result in [14] in Hausdorff locally convex space endowed with topology τ .

3. FIXED POINT THEOREMS

Let us give the following two important lemmas needed in our considerations.

Lemma 3. *Let (X, τ) be an angelic space, and let M be a τ -compact subset of X . Then, any sequentially τ -upper semicontinuous multivalued mapping $A : M \rightarrow \mathcal{P}(X)$ is τ -upper semicontinuous.*

Proof. Let C be a τ -closed subset of X . We have that A is sequentially τ -upper semicontinuous, then $A^{-1}(C)$ is τ -sequentially closed in X . Since C is τ -compact and $A^{-1}(C)^\tau \subset C$, so $A^{-1}(C)^\tau$ is τ -compact.

Now, we show that $A^{-1}(C)^\tau$ is τ -closed. Let $x \in \overline{A^{-1}(C)^\tau}$, by the angelicity of X , there exists a sequence $(x_n)_n \in A^{-1}(C)$ such that $x_n \xrightarrow{\tau} x$. Also, since $A^{-1}(C)$ is τ -sequentially closed, we have $x \in A^{-1}(C)$. Hence, $\overline{A^{-1}(C)^\tau} = A^{-1}(C)$. Then, $A^{-1}(C)$ is τ -closed and the multivalued operator A is τ -upper semicontinuous. $\square$

Lemma 4. *Assume that (X, τ) is angelic space. Let K and M be two τ -compact subsets of X . Then, every τ -sequentially closed multivalued mapping $A : K \rightarrow \mathcal{P}(M)$ is τ -upper semicontinuous.*

Proof. In view of Lemma 3 it is sufficient to prove that $A : K \rightarrow \mathcal{P}(M)$ is sequentially τ -upper semicontinuous. We have that $G_r(A) \subset K \times M$. Thus, $\overline{G_r(A)}^\tau \subset K \times M$. By hypothesis, we obtain that $G_r(A)$ is τ -compact. Using the angelicity of the space X , we get $G_r(A)$ is sequentially τ -compact. Consequentially, $G_r(A)$ is sequentially τ -closed.

Let $C \subset K$ be a τ -closed set. Now, we claim that $A^{-1}(C)$ is sequentially τ -closed. To prove that, let $(x_n)_n \in A^{-1}(C)$ with $x_n \xrightarrow{\tau} x$. Since $A(x_n) \cap C \neq \emptyset$, there exists $(y_n)_n \in A(x_n) \cap C$ and $y_n \in A(x_n) \subset M$. Since M is τ -compact, we may assume without loss of generality that $y_n \xrightarrow{\tau} y$, $y \in C$. Then, for every n , we have $(x_n, y_n)_n \in G_r(A)$ and

$$(x_n, y_n) \xrightarrow{\tau} (x, y),$$

since $G_r(A)$ is sequentially τ -closed subset of $X \times X$, so $(x, y) \in G_r(A)$. Thus, $y \in A(x) \cap C$. $x \in A^{-1}(C)$ and so $y \in A^{-1}(C)$. This prove that $A^{-1}(C)$ is sequentially τ -closed. $\square$

Now, we are ready to stat our first result.

Theorem 5. *Let X be a Hausdorff locally convex space. Assume that (X, τ) is angelic space. Let M be a τ -compact convex subset of X . Suppose that the multivalued operator $A : M \rightarrow \mathcal{P}_{cv, \tau-cl}(M)$ is sequentially τ -upper semicontinuous. Then, A has a fixed point in M .*

Proof. From Lemma 3 it follows that A is τ -upper semicontinuous and Fan-Glicksberg fixed point theorem [9] guarantees that A has a fixed point in M . $\square$

Remark 8. Note that Theorem 5 generalize Arino-Gautier-Penot fixed point theorem [3, see Remark 1] and Theorem 2.1 [1].

The following result is a version of Himmelberg fixed point theorem [12] for sequentially τ -upper semicontinuous multivalued operator in Hausdorff locally convex space.

Theorem 6. *Let X be a Hausdorff locally convex space. Assume that (X, τ) is angelic space with τ -KS property. Let M be a nonempty τ -closed convex subset of X . Assume that $A : M \rightarrow \mathcal{P}_{cv, \tau-cl}(M)$ is sequentially τ -upper semicontinuous multivalued operator. If $A(M)$ is relatively τ -compact, then A has a fixed point in M .*

Proof. Since $A(M)$ is relatively τ -compact and by the τ -KS property of X , we infer that the subset $K := \overline{\text{conv}^\tau(A(M))}$ is also τ -compact and $A(K) \subset K$. Using Theorem 5, we deduce that A has a fixed point in $K \subset M$. $\square$

Corollary 1. *Let X be a Hausdorff locally convex space. Assume that (X, τ) is angelic space with τ -KS property. Let M be a τ -closed, convex subset of X . Let $A : M \rightarrow \mathcal{P}_{cv, \tau-cl}(M)$ be a multivalued operator with a τ -sequentially closed graph and $A(M)$ is relatively τ -compact. Then, A has a fixed point in M .*

Proof. Let $K := \overline{\text{conv}^\tau(A(M))}$ and by hypothesis is a τ -compact subset of M and $A(K) \subset K$. Then, by Lemma 4 A is sequentially τ -upper semicontinuous. By Theorem 5 the multivalued operator A has a fixed point in $K \subset M$. $\square$

Corollary 2. *(X, τ) is angelic with τ -KS property. Let M be a τ -closed, bounded and convex subset of X . Assume that $A : M \rightarrow \mathcal{P}_{cv, \tau-cl}(M)$ has a τ -sequentially closed graph and Φ_λ^τ -contraction (or Φ_λ^τ -condensing). Then, A has a fixed point.*

Now, we will study the existence of solution of Eq. (2.1) under some conditions for the operators T, S_0 , and H .

Theorem 7. *Let X be a Hausdorff locally convex space. Assume that (X, τ) is angelic and M is a τ -compact convex subset of X . Assume that $T : M \rightarrow \mathcal{P}_{cv}(X)$ is a multivalued operator, and S_0, H are two operators on X , such that $M \subset D(S_0) \cap D(H)$. Suppose that:*

- (i) T have a τ -sequentially closed graph,
- (ii) S_0 is invertible,
- (iii) S_0 and H are τ -sequentially continuous, and
- (iv) $(T + H)(M) \subset S_0(M)$.

Then, $\mathcal{F}(S_0, T, H)$ is nonempty τ -sequentially closed subset of X .

Proof. In view of assumptions (ii) and (iv) the multivalued operator $\psi : M \rightarrow \mathcal{P}_{cv}(M)$ defined by $\psi(x) = S_0^{-1}(T + H)(x)$, $x \in M$ is well defined. Now, we show that

ψ has a τ -sequentially closed graph. Let $(x_n)_n$ be a sequence in M with $x_n \xrightarrow{\tau} x$, and let $y_n \in \psi(x_n)$ with $y_n \xrightarrow{\tau} y$. Since M is τ -compact then $x \in M$, and $y_n = S_0^{-1}(T + H)x_n$ which imply that $S_0 y_n = T x_n + H x_n$, and by assumption (iii) we get

$$T x_n \xrightarrow{\tau} S_0 y - H x.$$

Since T is τ -sequentially closed graph, we get $S_0 y - H x \in T x$ and so $y \in \psi(x)$. This prove that ψ has τ -sequentially closed graph. Hence, by Lemma 4 and Theorem 5 the multivalued operator $\psi : M \rightarrow \mathcal{P}_{cv, \tau-cl}(M)$ has a fixed point in M i.e, there exists $x \in M$ such that $S_0 x \in T x + H x$. Hence, $\mathcal{F}(S_0, T, H)$ is nonempty and by Theorem 1 it is τ -sequentially closed. This achieve the proof. $\square$

Theorem 8. *Let X be a Hausdorff locally convex space. Assume that (X, τ) is angelic space with τ -KS property. Let M is τ -closed convex subset of X . Assume that $T : M \rightarrow \mathcal{P}_{cv}(M)$ is a multivalued operator on X and S_0, H are two singled valued operators on X , such that $M \subset \mathcal{D}(S_0) \cap \mathcal{D}(H)$. Suppose that the following assumptions verified*

- (i) T have a τ -sequentially closed graph and τ -compact,
- (ii) S_0 is invertible,
- (iii) S_0 and H are τ -sequentially continuous, and
- (iv) $(T + H)(C) \subset S_0(C)$ for each τ -compact subset C of M .

Then, $\mathcal{F}(S_0, T, H)$ is nonempty sequentially τ -closed subset of X .

Proof. Let $K := \overline{\text{conv}}^\tau(T(M))$. The τ -KS property guarantees that K is τ -compact. Using assumptions (ii) and (iv), we show that the multivalued operator $\psi : K \rightarrow \mathcal{P}_{cv}(K)$ by $\psi(x) = S_0^{-1}(T + H)(x)$, $x \in K$ is well defined, and as we shown in the proof of Theorem 7, ψ has a τ -sequentially closed graph. By Corollary 1, the operator ψ has a fixed point in K , so there exists $x \in K$ such that $x = S_0^{-1}(T + H)x$. Then, $\mathcal{F}(S_0, T, H)$ is nonempty and by Theorem 2 it is τ -sequentially closed subset of X . $\square$

Theorem 9. *Let X be a Hausdorff locally convex space. Assume that (X, τ) is angelic with τ -KS property. Consider $\Phi_\Lambda^\tau = \{\phi_{p_\tau}, p \in \Lambda\}$ is a regular and subadditive Φ_Λ^τ -MNC. Let M be a nonempty, τ -closed, bounded and convex subset of X , and let T and H be two single valued operators such that $M \subset D(T) \subset D(H)$. If T is Φ_Λ^τ -condensing and H is τ -compact, then the following assertions hold*

- (i) For every bounded subset D of X , $D \cap \mathcal{F}(I, T, H)$ is relatively τ -compact.
- (ii) If the operators T and H are τ -closed, then $\mathcal{F}(I, T, H)$ is nonempty subset of X .

Proof.

- (i) Let D be a bounded subset of X . Since $x \in \mathcal{F}(I, T, H)$ means that $x = T x + H x$, it follows that

$$\mathcal{F}(I, T, H) \cap D \subset T(\mathcal{F}(I, T, H) \cap D) + H(\mathcal{F}(I, T, H) \cap D).$$

From the τ -compactness of H and properties of Φ_Λ^τ -MNC, we deduce that

$$\phi_{p_\tau}(\mathcal{F}(I, T, H) \cap D) \leq \phi_{p_\tau}(T(\mathcal{F}(I, T, H) \cap D)).$$

If $\phi_{p_\tau}(\mathcal{F}(I, T, H) \cap D) > 0$, then since T is Φ_Λ^τ -condensing, we have

$$\phi_{p_\tau}(\mathcal{F}(I, T, H) \cap D) \leq \phi_{p_\tau}(T(\mathcal{F}(I, T, H) \cap D)) < \phi_{p_\tau}(\mathcal{F}(I, T, H) \cap D),$$

which is a contradiction. Thus $\phi_{p_\tau}(\mathcal{F}(I, T, H) \cap D) = 0$ so $\mathcal{F}(I, T, H) \cap D$ is relatively τ -compact.

- (ii) Clearly, the operator $T + H$ has a τ -sequentially closed graph. Let B be a bounded subset of M . Then, we have

$$\phi_{p_\tau}(T + H)(B) \leq \phi_{p_\tau}(T(B)) + \phi_{p_\tau}(H(B)),$$

Since H is τ -compact, we get

$$\phi_{p_\tau}(T + H)(B) < \phi_{p_\tau}(H(B)),$$

Hence, if $\phi_{p_\tau}(B) > 0$, we obtain

$$\phi_{p_\tau}(T + H)(B) < \phi_{p_\tau}(B),$$

this show that $T + H$ is Φ_Λ^τ -condensing. Using Corollary 1, there exists $x \in M$ such that $x = Tx + Hx$. So, $\mathcal{F}(I, T, H)$ is nonempty.

□

Theorem 10. Assume that (X, τ) is angelic with τ -KS property. Let M be a nonempty τ -closed, convex of X , and let $T : \mathcal{D}(T) \subset X \longrightarrow X$, $S_0 : \mathcal{D}(S_0) \subset X \longrightarrow X$ be two single-valued τ -closed operators with $M \subset \mathcal{D}(T) \subset \mathcal{D}(S_0)$ such that $S_0 - T$ is τ -closed. If T is τ - S_0 -demicompact, then for every τ -closed, convex bounded set D of M , the multi-valued map

$$\begin{aligned} F_D : M &\longrightarrow \mathcal{P}(M), \\ y &\longrightarrow D \cap (S_0 - T)^{-1}y, \end{aligned}$$

has a fixed point on M .

Proof. Suppose T is τ - S_0 -demicompact. First we show F_D is τ -upper-semicontinuous. Let $(y_n)_n$ be a sequence with $y_n \xrightarrow{\tau} y$ and Q a τ -open set such that $F_D(y) \subset Q$. From Lemma 1 it suffices to show the existence of $n_0 \in \mathbb{N}$ such that $F_D(y_n) \subset Q$, $\forall n \geq n_0$. If not, then there exists a subsequence $(x_{\varphi(n)})_n$ of $(x_n)_n$ (here $x_n \in D \cap (S_0 - T)^{-1}y_n$) such that $x_{\varphi(n)} \in D \cap (S_0 - T)^{-1}y_n$ and $x_{\varphi(n)} \notin Q$. Note $S_0 x_{\varphi(n)} - T x_{\varphi(n)}$ τ -converges to y . Since T is τ - S_0 -demicompact we infer that there exists a subsequence $(x_{\varphi \circ \psi(n)})_n$ of $(x_{\varphi(n)})_n$ such that $(x_{\varphi \circ \psi(n)})_n \xrightarrow{\tau} x$, $x \in X$. Since D is a τ -closed and convex subset of X we deduce that $x \in D$. Moreover, taking into account that $(S_0 - T)x_{\varphi \circ \psi(n)} \xrightarrow{\tau} y$ and $S_0 - T$ is a τ -closed mappings, we deduce that $y = (S_0 - T)x$. Consequently, $x \in D \cap (S_0 - T)^{-1}y = F_D(y)$. Therefore, $x_{\varphi \circ \psi(n)} \in Q$

for n large enough, a contradiction to the construction of $(x_{\varphi(n)})_n$. Thus F_D is τ -upper-semicontinuous. Next fix $y \in X$. Since T is τ - S_0 -demicompact then if $(x_n)_n$ is a sequence in $D \cap (S_0 - T)^{-1}y$, then it has a τ -converging subsequence, and so the τ -closedness of $S_0 - T$ implies that the limit is also in $D \cap (S_0 - T)^{-1}y$. Thus, $D \cap (S_0 - T)^{-1}y$ is τ -compact. By Himmelberg fixed point theorem [12], there exist $x \in M$ such that $x \in F_D(x)$. Then $\mathcal{F}(S_0, T, I)$ is nonempty subset. $\square$

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WEIGHTED HERMITE-HADAMARD TYPE INEQUALITIES FOR DIFFERENTIABLE PREINVEX AND PREQUASIINVEX MAPPINGS

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Abstract. In this paper, new weighted Hermite-Hadamard type inequalities for differentiable preinvex and prequasiinvex functions are proved. These results generalize many results proved in earlier works for these classes of functions.

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1. INTRODUCTION

The theory of convex functions depends on the definition of convex functions as stated below:

A function $\varphi : \emptyset \neq I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be convex on a convex set I if inequality

$$\varphi(\eta s + (1 - \eta)y) \leq \eta\varphi(s) + (1 - \eta)\varphi(y)$$

holds for all $s, y \in I$ and $\eta \in [0, 1]$.

Now, we evoke that the concept of quasi-convex functions generalizes the concept of convex functions. More accurately, a function $\varphi : \emptyset \neq I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said quasi-convex on I if

$$\varphi(\eta s + (1 - \eta)y) \leq \max \{ \varphi(s), \varphi(y) \}$$

for all $s, y \in I$ and $\eta \in [0, 1]$. Evidently, any convex function is a quasi-convex function.

Many researchers have developed a substantial material by using the convex and quasi-convex functions in the past three decades but the most profoundly studied inequalities for convex functions is the Hermite-Hadamard inequality.

The Hermite-Hadamard states that a function $\varphi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a convex on $[z, u]$, where $z, u \in I$ with $z < u$ if and only if the inequalities (see [6])

$$\varphi\left(\frac{z+u}{2}\right) \leq \frac{1}{u-z} \int_z^u \varphi(s) ds \leq \frac{\varphi(z) + \varphi(u)}{2} \quad (1.1)$$

hold. The inequalities in (1.1) are reversed if ϕ is a concave function.

The subject of Hermite-Hadamard type inequalities has been in the focus of young and experienced scholars world-wide working in this domain. A gargantuan study has also been accomplished by a number of mathematicians all over the world.

The notions convex sets and convex functions have been continuously generalized and extended in diverse directions. As a result of unceasing advances in convex sets and convex functions, many new proofs, stimulating extensions, noticeable generalizations, refinements, new Hermite-Hadamard-type inequalities and plentiful applications of the inequalities (1.1) have appeared in the literature of mathematical inequalities and in different branches of mathematics, see for instance [2–5, 7–13, 15, 16, 16–18] and the references quoted in these papers.

Let us now recall the important generalizations of the convex sets, convex functions and quasi-convex functions which are acknowledged as invex sets, preinvex functions and prequasiinvex functions respectively.

Definition 1.1. [19] Let K be a non-empty subset in $\mathbb{R}^n$ and $\beta : K \times K \rightarrow \mathbb{R}^n$. Let $s \in K$, then the set K is said to be invex at s with respect to $\beta(\cdot, \cdot)$, if

$$s + \eta\beta(y, s) \in K, \forall s, y \in K, \eta \in [0, 1].$$

The set K is said to be an invex set with respect to β if K is invex at each $s \in K$. The invex set K is also called an β -connected set.

Definition 1.2. [19] A function $\phi : K \rightarrow \mathbb{R}$ on an invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect to β , if

$$\phi(s + \eta\beta(y, s)) \leq (1 - \eta)\phi(s) + \eta\phi(y), \forall s, y \in K, \eta \in [0, 1].$$

The function ϕ is said to be preincave if and only if $-\phi$ is preinvex.

Every convex function is preinvex with respect to the map $\beta(s, y) = s - y$ but the converse is not true see for instance [1].

Definition 1.3. [20] A function $\phi : K \rightarrow \mathbb{R}$ on an invex set $K \subseteq \mathbb{R}^n$ is said to be prequasiinvex with respect to β , if

$$\phi(s + \eta\beta(y, s)) \leq \max\{\phi(s), \phi(y)\}, \forall s, y \in K, \eta \in [0, 1].$$

Quasi-convex functions are prequasiinvex with respect to the map $\beta(y, s) = y - s$ but the converse does not hold, see for example [20].

Hermite-Hadamard type inequalities for the preinvex functions were initially proved by Noor [15]. Later on, many results on inequalities for preinvex and prequasiinvex functions which are connected to Hermite-Hadamard type inequalities proved in [15] are established, see for instance the articles [2, 9–13, 17, 18] and the references mentioned in them.

In section 2, we prove some integral identities involving a differentiable mapping and a symmetric function with respect to $z + \frac{1}{2}\beta(u, z)$ defined over an interval

$[z, z + \beta(u, z)]$. In section 2, an important inequality for positive linear functional on $C([z, z + \beta(u, z)])$ and a preinvex function are proved in order to obtain some very fascinating results of this manuscript. Section 3 contains some new weighted Hermite-Hadamard type integral inequalities related with the left and right parts of Hermite-Hadamard type inequalities for the preinvex given in [15]. The results of section 3 provide weighted generalization of the results proved so far in the field of mathematical inequalities for differentiable preinvex and prequasiinvex functions.

2. SOME AUXILIARY RESULTS

Let $w(s) : [z, z + \beta(u, z)] \rightarrow [0, \infty)$ be a continuous function such that

$$\int_z^{z+\beta(u,z)} w(s) ds = 1.$$

Let us denote the integral $\int_z^{z+\beta(u,z)} sw(s) ds$ by z_1 , i.e.

$$z_1 = \int_z^{z+\beta(u,z)} sw(s) ds.$$

Note that $w(s) : [z, z + \beta(u, z)] \rightarrow [0, \infty)$ is symmetric with respect to $z + \frac{1}{2}\beta(u, z)$ if the equality holds

$$w(2z + \beta(u, z) - s) = w(s).$$

Now we present a result in which the function $w(s)$ is symmetric with respect to $z + \frac{1}{2}\beta(u, z)$.

Lemma 2.1. *If $w(s) : [z, z + \beta(u, z)] \rightarrow [0, \infty)$ is symmetric with respect to $z + \frac{1}{2}\beta(u, z)$. Then*

$$z_1 = z + \frac{1}{2}\beta(u, z).$$

Proof. Since w is symmetric with respect to $z + \frac{1}{2}\beta(u, z)$, we have

$$\begin{aligned} \int_z^{z+\beta(u,z)} sw(s) ds &= \int_{z+\beta(u,z)}^z (2z + \beta(u, z) - s) w(2z + \beta(u, z) - s) (-ds) \\ &= \int_z^{z+\beta(u,z)} (2z + \beta(u, z) - s) w(s) ds. \end{aligned}$$

That is

$$2 \int_z^{z+\beta(u,z)} sw(s) ds = (2z + \beta(u, z)) \int_z^{z+\beta(u,z)} w(s) ds$$

which gives the desired result since

$$\int_z^{z+\beta(u,z)} w(s) ds = 1.$$

□

Lemma 2.2. Let $\varphi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° and $\varphi' \in L([z, z + \beta(u, z)])$, where $[z, z + \beta(u, z)] \subseteq I^\circ$ with $\beta(u, z) > 0$. Let $w : [z, z + \beta(u, z)] \rightarrow [0, \infty)$ be a continuous mapping. Then

$$\begin{aligned} & \frac{1}{\beta(u, z)} \left(\varphi(z) \int_z^{z+\beta(u, z)} (z-s+\beta(u, z)) w(s) ds \right) \\ & \quad + \varphi(z+\beta(u, z)) \int_z^{z+\beta(u, z)} (s-z) w(s) ds - \int_z^{z+\beta(u, z)} \varphi(s) w(s) ds \\ & = \frac{\beta(z_1, z)}{\beta(u, z)} \int_0^1 H(w, z, z_1; y) \varphi'(z+y\beta(z_1, z)) dy \\ & \quad + \frac{\beta(u, z_1)}{\beta(u, z)} \int_0^1 H(w, z_1, u; y) \varphi'(z_1+y\beta(u, z_1)) dy, \end{aligned} \quad (2.1)$$

where

$$G(w, \alpha, \beta; y) = \int_z^{\alpha+y\beta(\beta, \alpha)} (s-z) w(s) ds - \int_{\alpha+y\beta(\beta, \alpha)}^{z+y\beta(u, z)} (z+\beta(u, z)-s) w(s) ds$$

and $\alpha, \beta \in [z, z + \beta(u, z)]$.

Proof. The following identities hold

$$\varphi(s) - \varphi(z) = \int_z^{z+\beta(u, z)} \sigma(s-\eta) \varphi'(\eta) d\eta, \quad (2.2)$$

$$\varphi(s) - \varphi(z+\beta(u, z)) = - \int_z^{z+\beta(u, z)} \sigma(\eta-s) \varphi'(\eta) d\eta, \quad (2.3)$$

where $\sigma(\cdot)$ is the Heavyside function defined by

$$\sigma(s) = \begin{cases} 0, & s < 0 \\ 1 & s > 0. \end{cases}$$

From (2.2), we have

$$\begin{aligned} & \int_z^{z+\beta(u, z)} (z-s+\beta(u, z)) w(s) \varphi(s) ds - \varphi(z) \int_z^{z+\beta(u, z)} (z-s+\beta(u, z)) w(s) ds \\ & = \int_z^{z+\beta(u, z)} \left(\int_\eta^{z+\beta(u, z)} (z-s+\beta(u, z)) w(s) ds \right) \varphi'(\eta) d\eta. \end{aligned} \quad (2.4)$$

Similarly, we also have

$$\begin{aligned} & \int_z^{z+\beta(u, z)} (s-z) w(s) \varphi(s) ds - \varphi(u) \int_z^{z+\beta(u, z)} (s-z) w(s) ds \\ & = - \int_z^{z+\beta(u, z)} \left(\int_z^\eta (s-z) w(s) ds \right) \varphi'(\eta) d\eta. \end{aligned} \quad (2.5)$$

From (2.4) and (2.5), we get

$$\begin{aligned}
& \frac{1}{\beta(u, z)} \left(\varphi(z) \int_z^{z+\beta(u, z)} (z-s+\beta(u, z)) w(s) ds \right. \\
& \quad \left. + \varphi(z+\beta(u, z)) \int_z^{z+\beta(u, z)} (s-z) w(s) ds \right) - \int_z^{z+\beta(u, z)} \varphi(s) w(s) ds \\
&= \frac{1}{\beta(u, z)} \int_z^{z+\beta(u, z)} \left[\int_z^\eta (s-z) w(s) ds - \int_\eta^{z+\beta(u, z)} (z-s+\beta(u, z)) w(s) ds \right] \\
& \quad \times \varphi'(\eta) d\eta \\
&= \frac{1}{\beta(u, z)} \int_z^{z_1} \left[\int_z^\eta (s-z) w(s) ds - \int_\eta^{z+\beta(u, z)} (z-s+\beta(u, z)) w(s) ds \right] \\
& \quad \times \varphi'(\eta) d\eta + \frac{1}{\beta(u, z)} \int_{z_1}^{z+\beta(u, z)} \left[\int_z^\eta (s-z) w(s) ds \right. \\
& \quad \left. - \int_\eta^{z+\beta(u, z)} (z-s+\beta(u, z)) w(s) ds \right] \varphi'(\eta) d\eta. \tag{2.6}
\end{aligned}$$

In the last identity, we set $\eta = z + y\beta(z_1, z)$ for the first integral and $\eta = z_1 + y\beta(u, z_1)$ by for the second integral and we obtain (2.1). $\square$

Remark 2.3. If we take $w(s) = \frac{1}{\beta(u, z)}$, for all $s \in [z, z + \beta(u, z)]$, then (2.1) reduces to the identity

$$\begin{aligned}
& \frac{\varphi(z) + \varphi(z + \beta(u, z))}{2} - \frac{1}{\beta(u, z)} \int_z^{z+\beta(u, z)} \varphi(s) ds = \frac{\beta(u, z)}{4} \\
& \quad \times \left[\int_0^1 y \varphi' \left(z + \frac{1+y}{2} \beta(u, z) \right) dy - \int_0^1 y \varphi' \left(z + \frac{1-y}{2} \beta(u, z) \right) dy \right]. \tag{2.7}
\end{aligned}$$

Corollary 2.4. If the function $w(s)$ is symmetric with respect to $z + \frac{1}{2}\beta(u, z)$ on $[z, z + \beta(u, z)]$, then

$$\begin{aligned}
& \frac{\varphi(z) + \varphi(z + \beta(u, z))}{2} - \int_z^{z+\beta(u, z)} \varphi(s) w(s) ds \\
&= \int_0^1 G(w, z, u; y) \varphi'(z + (1-y)\beta(u, z)) dy, \tag{2.8}
\end{aligned}$$

where

$$G(w, z, u; y) = \int_z^{z + \frac{1+y}{2}\beta(u, z)} (s-z) w(s) ds - \int_{z + \frac{1+y}{2}\beta(u, z)}^{z+\beta(u, z)} (z-s+\beta(u, z)) w(s) ds$$

for all $y \in [0, 1]$.

Proof. Since the function $w(s)$ is symmetric with respect to $z + \frac{1}{2}\beta(u, z)$ on $[z, z + \beta(u, z)]$, we have

$$\int_z^{z+\beta(u, z)} w(s) ds = 1 \text{ and } \int_z^{z+\beta(u, z)} sw(s) ds = z + \frac{1}{2}\beta(u, z).$$

Moreover

$$\begin{aligned} G(w, z, z_1; y) &= G(w, z_1, u; y) = G(w, z, u; y) = \int_z^{z+\frac{1+y}{2}\beta(u, z)} (s-z)w(s) ds \\ &\quad - \int_{z+\frac{1+y}{2}\beta(u, z)}^{z+\beta(u, z)} (z-s+\beta(u, z))w(s) ds. \end{aligned}$$

Hence from (2.1), we get the required identity (2.8). $\square$

In the next result we need the following fact for the function β proved in [14].

Condition C. Let $\beta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$. The function β is said to satisfy Condition C if for any $s, y \in \mathbb{R}^n$ and $\eta \in [0, 1]$, the following equalities hold

$$\beta(s, s + \eta\beta(y, s)) = -\eta\beta(y, s)$$

and

$$\beta(y, s + \eta\beta(y, s)) = (1 - \eta)\beta(y, s).$$

Lemma 2.5. Let $A : C([z, z + \beta(u, z)]) \rightarrow \mathbb{R}$ be a positive linear functional on $C([z, z + \beta(u, z)])$ and let e_i be monomials $e_i(s) = s^i$, $s \in [z, z + \beta(u, z)]$, $i \in \mathbb{N}$. Let g be a preinvex function on $[z, z + \beta(u, z)]$, then the following inequality holds

$$A(g(e_1)) \leq \left(\frac{z + \beta(u, z) - e_1}{\beta(u, z)} \right) g(z) + \left(\frac{e_1 - z}{\beta(u, z)} \right) g(z + \beta(u, z)). \quad (2.9)$$

Proof. By using the preinvexity of g on $[z, z + \beta(u, z)]$ and Condition C, we can write

$$e_1 = z + \left(\frac{e_1 - z}{\beta(u, z)} \right) \beta(z + \beta(u, z), z).$$

Thus

$$\begin{aligned} g(e_1) &= g\left(z + \left(\frac{e_1 - z}{\beta(u, z)} \right) \beta(z + \beta(u, z), z)\right) \\ &\leq \left(\frac{z + \beta(u, z) - e_1}{\beta(u, z)} \right) g(z) + \left(\frac{e_1 - z}{\beta(u, z)} \right) g(z + \beta(u, z)). \end{aligned} \quad (2.10)$$

Since A is a positive linear functional, we get the inequality (2.9) by applying A on both sides of (2.10). $\square$

3. MAIN RESULTS

The following theorem generalizes the result given by

Theorem 3.1. *Let $\varphi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° and $\varphi' \in L([z, z + \beta(u, z)])$, where $[z, z + \beta(u, z)] \subseteq I^\circ$ with $\beta(u, z) > 0$. If $w : [z, z + \beta(u, z)] \rightarrow [0, \infty)$ is a continuous mapping and $|\varphi'|$ is preinvex on $[z, z + \beta(u, z)]$, then the following inequality holds*

$$\left| \int_z^{z+\beta(u,z)} \varphi(s) w(s) ds - \varphi(z_1) \right| \leq \frac{\mathcal{A}_\beta(z, z_1, u) |\varphi'(z)| + \mathcal{B}_\beta(z, z_1, u) |\varphi'(z + \beta(u, z))|}{\beta(u, z)}, \quad (3.1)$$

where

$$\begin{aligned} \mathcal{A}_\beta(z, z_1, u) = & \frac{1}{2} \int_z^{z_1} (z_1 - s)^2 w(s) ds - \frac{1}{2} \int_{z_1}^{z+\beta(u,z)} (z_1 - s)^2 w(s) ds \\ & + (z + \beta(u, z) - z_1) \left[\int_z^{z_1} (z_1 - s) w(s) ds - \int_{z_1}^{z+\beta(u,z)} (z_1 - s) w(s) ds \right] \end{aligned}$$

and

$$\begin{aligned} \mathcal{B}_\beta(z, z_1, u) = & \frac{1}{2} \int_{z_1}^{z+\beta(u,z)} (s - z_1)^2 w(s) ds - \frac{1}{2} \int_z^{z_1} (s - z_1)^2 w(s) ds \\ & + (z_1 - z) \left[\int_{z_1}^{z+\beta(u,z)} (s - z_1) w(s) ds - \int_z^{z_1} (s - z_1) w(s) ds \right]. \end{aligned}$$

Proof. We can write

$$\varphi(s) - \varphi(z_1) = \int_z^{z+\beta(u,z)} [\sigma(s - \eta) - \sigma(z_1 - \eta)] \varphi'(\eta) d\eta \quad (3.2)$$

From (3.2), we obtain

$$\begin{aligned} \int_z^{z+\beta(u,z)} \varphi(s) w(s) ds - \varphi(z_1) &= \int_z^{z+\beta(u,z)} \left(\int_\eta^{z+\beta(u,z)} w(s) ds - \sigma(z_1 - \eta) \right) \varphi'(\eta) d\eta \quad (3.3) \end{aligned}$$

Taking absolute value on both sides of (3.3) and applying Lemma 2.5, we have

$$\begin{aligned} \left| \int_z^{z+\beta(u,z)} \varphi(s) w(s) ds - \varphi(z_1) \right| &\leq \int_z^{z+\beta(u,z)} \left| \int_\eta^{z+\beta(u,z)} w(s) ds - \sigma(z_1 - \eta) \right| |\varphi'(\eta)| d\eta \end{aligned}$$

$$\begin{aligned}
&\leq \frac{|\varphi'(z)| \int_z^{z+\beta(u,z)} \left| \int_{\eta}^{z+\beta(u,z)} w(s) ds - \sigma(z_1 - \eta) \right| (z + \beta(u, z) - \eta) d\eta}{\beta(u, z)} \\
&+ \frac{|\varphi'(z + \beta(u, z))| \int_z^{z+\beta(u,z)} \left| \int_{\eta}^{z+\beta(u,z)} w(s) ds - \sigma(z_1 - \eta) \right| (\eta - z) d\eta}{\beta(u, z)}. \quad (3.4)
\end{aligned}$$

We notice that

$$\begin{aligned}
&\int_z^{z+\beta(u,z)} \left| \int_{\eta}^{z+\beta(u,z)} w(s) ds - \sigma(z_1 - \eta) \right| (z + \beta(u, z) - \eta) d\eta \\
&= \int_z^{z_1} \left| \int_{\eta}^{z+\beta(u,z)} w(s) ds - \int_z^{z+\beta(u,z)} w(s) ds \right| (z + \beta(u, z) - \eta) d\eta \\
&\quad + \int_{z_1}^{z+\beta(u,z)} \left(\int_{\eta}^{z+\beta(u,z)} w(s) ds \right) (z + \beta(u, z) - \eta) d\eta \\
&= \int_z^{z_1} \left(\int_z^{\eta} w(s) ds \right) (z + \beta(u, z) - \eta) d\eta \\
&\quad + \int_{z_1}^{z+\beta(u,z)} \left(\int_{\eta}^{z+\beta(u,z)} w(s) ds \right) (z + \beta(u, z) - \eta) d\eta. \quad (3.5)
\end{aligned}$$

Using integration by parts

$$\begin{aligned}
&\int_z^{z_1} \left(\int_z^{\eta} w(s) ds \right) (z + \beta(u, z) - \eta) d\eta \\
&= -\frac{(z + \beta(u, z) - z_1)^2}{2} \int_z^{z_1} w(s) ds + \frac{1}{2} \int_z^{z_1} (z + \beta(u, z) - s)^2 w(s) ds. \quad (3.6)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{z_1}^{z+\beta(u,z)} \left(\int_{\eta}^{z+\beta(u,z)} w(s) ds \right) (z + \beta(u, z) - \eta) d\eta \quad (3.7) \\
&= \frac{(z + \beta(u, z) - z_1)^2}{2} \int_{z_1}^{z+\beta(u,z)} w(s) ds \\
&\quad - \frac{1}{2} \int_{z_1}^{z+\beta(u,z)} (z + \beta(u, z) - s)^2 w(s) ds.
\end{aligned}$$

A combination of (3.5), (3.6) and (3.7), we obtain

$$\begin{aligned}
\mathcal{A}_{\beta}(z, z_1, u) &= \int_z^{z+\beta(u,z)} \left| \int_{\eta}^{z+\beta(u,z)} w(s) ds - \sigma(z_1 - \eta) \right| (z + \beta(u, z) - s) d\eta \quad (3.8) \\
&= \frac{(z + \beta(u, z) - z_1)^2}{2} \left[\int_{z_1}^{z+\beta(u,z)} w(s) ds - \int_z^{z_1} w(s) ds \right] \\
&\quad + \frac{1}{2} \int_z^{z_1} (z + \beta(u, z) - s)^2 w(s) ds
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2} \int_{z_1}^{z+\beta(u,z)} (z+\beta(u,z)-s)^2 w(s) ds \\
& = \frac{(z+\beta(u,z)-z_1)^2}{2} \left[\int_{z_1}^{z+\beta(u,z)} w(s) ds - \int_z^{z_1} w(s) ds \right] \\
& \quad + \frac{1}{2} \int_z^{z_1} (z+\beta(u,z)-z_1+z_1-s)^2 w(s) ds \\
& \quad - \frac{1}{2} \int_{z_1}^{z+\beta(u,z)} (z+\beta(u,z)-z_1+z_1-s)^2 w(s) ds \\
& = \frac{1}{2} \int_z^{z_1} (z_1-s)^2 w(s) ds - \frac{1}{2} \int_{z_1}^{z+\beta(u,z)} (z_1-s)^2 w(s) ds \\
& \quad + (z+\beta(u,z)-z_1) \left[\int_z^{z_1} (z_1-s) w(s) ds \right. \\
& \quad \left. - \int_{z_1}^{z+\beta(u,z)} (z_1-s) w(s) ds \right].
\end{aligned}$$

In a similar way

$$\begin{aligned}
\mathcal{B}_\beta(z, z_1, u) & = \int_z^{z+\beta(u,z)} \left| \int_\eta^{z+\beta(u,z)} w(s) ds - \sigma(z_1 - \eta) \right| (\eta - z) d\eta \\
& = \frac{1}{2} \int_{z_1}^{z+\beta(u,z)} (s - z_1)^2 w(s) ds - \frac{1}{2} \int_z^{z_1} (s - z_1)^2 w(s) ds \\
& \quad + (z_1 - z) \left[\int_{z_1}^{z+\beta(u,z)} (s - z_1) w(s) ds - \int_z^{z_1} (s - z_1) w(s) ds \right].
\end{aligned}$$

This concludes the proof of the theorem. $\square$

Corollary 3.2. Suppose that the assumptions of Theorem 3.1 are satisfied and that $w(s)$ is symmetric with respect to $z + \frac{1}{2}\beta(u, z)$ on $[z, z + \beta(u, z)]$. Then the following inequality holds:

$$\begin{aligned}
& \left| \int_z^{z+\beta(u,z)} \varphi(s) w(s) ds - \varphi\left(z + \frac{1}{2}\beta(u, z)\right) \right| \\
& \leq [|\varphi'(z)| + |\varphi'(z + \beta(u, z))|] \int_z^{z+\frac{1}{2}\beta(u,z)} \left(z + \frac{1}{2}\beta(u, z) - s\right) w(s) ds. \quad (3.9)
\end{aligned}$$

Proof. Since the function $w(s)$ is symmetric with respect to $z + \frac{1}{2}\beta(u, z)$ on $[z, z + \beta(u, z)]$ so $z_1 = z + \frac{1}{2}\beta(u, z)$ and the function $(s - z_1)^2 w(s)$ is symmetric with respect to $z + \frac{1}{2}\beta(u, z)$ on $[z, z + \beta(u, z)]$. This fact gives

$$\int_z^{z_1} (s - z_1)^2 w(s) ds = \int_{z_1}^{z+\beta(u,z)} (s - z_1)^2 w(s) ds = \frac{1}{2} \int_z^{z+\beta(u,z)} (s - z_1)^2 w(s) ds.$$

Thus

$$\mathcal{A}_{\beta}(z, z_1, u) = \mathcal{B}_{\beta}(z, z_1, u) = \beta(u, z) \int_z^{z+\frac{1}{2}\beta(u, z)} \left(z + \frac{1}{2}\beta(u, z) - s\right) w(s) ds.$$

□

Corollary 3.3. *If we take $w(s) = \frac{g(s)}{\int_z^{z+\beta(u, z)} g(s) ds}$ in (3.1) and $g(s)$ is symmetric with respect to $z + \frac{1}{2}\beta(u, z)$. Then the following inequality holds*

$$\begin{aligned} & \left| \int_z^{z+\beta(u, z)} \varphi(s) g(s) ds - \varphi\left(z + \frac{1}{2}\beta(u, z)\right) \int_z^{z+\beta(u, z)} g(s) ds \right| \\ & \leq \beta(u, z) \left[\frac{|\varphi'(z)| + |\varphi'(z + \beta(u, z))|}{2} \right] \int_0^1 \left(\int_z^{z+\frac{1-\eta}{2}\beta(u, z)} g(s) ds \right) d\eta. \quad (3.10) \end{aligned}$$

Theorem 3.4. *Let $\varphi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° and $\varphi' \in L([z, z + \beta(u, z)])$, where $[z, z + \beta(u, z)] \subseteq I^\circ$ with $\beta(u, z) > 0$. If $w : [z, z + \beta(u, z)] \rightarrow [0, \infty)$ is a continuous mapping and $|\varphi'|^q$ is preinvex on $[z, z + \beta(u, z)]$ for $q \geq 1$, then the following inequality holds*

$$\begin{aligned} & \left| \int_z^{z+\beta(u, z)} \varphi(s) w(s) ds - \varphi(z_1) \right| \leq \left(2 \int_{z_1}^{z+\beta(u, z)} w(s) ds \right)^{1-\frac{1}{q}} \\ & \quad \times \left(\frac{\mathcal{A}_{\beta}(z, z_1, u) |\varphi'(z)|^q + \mathcal{B}_{\beta}(z, z_1, u) |\varphi'(\beta(u, z))|^q}{\beta(u, z)} \right)^{\frac{1}{q}}, \quad (3.11) \end{aligned}$$

where $\mathcal{A}_{\beta}(z, z_1, u)$ and $\mathcal{B}_{\beta}(z, z_1, u)$ are as defined in Theorem 3.1.

Proof. Application of Hölder inequality in (3.3) yields that

$$\begin{aligned} & \left| \int_z^{z+\beta(u, z)} \varphi(s) w(s) ds - \varphi(z_1) \right| \\ & \leq \int_z^{z+\beta(u, z)} \left| \int_{\eta}^{z+\beta(u, z)} w(s) ds - \sigma(z_1 - \eta) \right| |\varphi'(\eta)|^q d\eta \\ & \leq \left(\int_z^{z+\beta(u, z)} \left| \int_{\eta}^{z+\beta(u, z)} w(s) ds - \sigma(z_1 - \eta) \right| d\eta \right)^{1-\frac{1}{q}} \\ & \quad \times \left(\int_z^{z+\beta(u, z)} \left| \int_{\eta}^{z+\beta(u, z)} w(s) ds - \sigma(z_1 - \eta) \right| |\varphi'(\eta)|^q d\eta \right)^{\frac{1}{q}}. \quad (3.12) \end{aligned}$$

Applying Lemma 2.5, we have

$$\int_z^{z+\beta(u, z)} \left| \int_{\eta}^{z+\beta(u, z)} w(s) ds - \sigma(z_1 - \eta) \right| |\varphi'(\eta)|^q d\eta \quad (3.13)$$

$$\leq \frac{\mathcal{A}_{\beta}(z, z_1, u) |\varphi'(z)|^q + \mathcal{B}_{\beta}(z, z_1, u) |\varphi'(z + \beta(u, z))|^q}{\beta(u, z)}.$$

On the other hand, we have

$$\begin{aligned} & \int_z^{z+\beta(u, z)} \left| \int_{\eta}^{z+\beta(u, z)} w(s) ds - \sigma(z_1 - \eta) \right| d\eta \\ &= \int_z^{z_1} \left(\int_z^{\eta} w(s) ds \right) d\eta + \int_{z_1}^{z+\beta(u, z)} \left(\int_{\eta}^{z+\beta(u, z)} w(s) ds \right) d\eta \\ &= 2 \int_{z_1}^{z+\beta(u, z)} w(s) ds. \end{aligned} \quad (3.14)$$

A combination of (3.12)-(3.14) gives (3.11). This completes the proof of the theorem. $\square$

Corollary 3.5. *If $w(s)$ is symmetric with respect to $z + \frac{1}{2}\beta(u, z)$ on $[z, z + \beta(u, z)]$, then from (3.11), we obtain the following inequality*

$$\begin{aligned} & \left| \int_z^{z+\beta(u, z)} \varphi(s) w(s) ds - \varphi\left(z + \frac{1}{2}\beta(u, z)\right) \right| \\ & \leq 2 \left(\frac{|\varphi'(z)|^q + |\varphi'(z + \beta(u, z))|^q}{2} \right)^{\frac{1}{q}} \\ & \quad \times \int_z^{z+\frac{1}{2}\beta(u, z)} \left(z + \frac{1}{2}\beta(u, z) - s \right) w(s) ds. \end{aligned} \quad (3.15)$$

Corollary 3.6. *If $w(s) = \frac{g(s)}{\int_z^{z+\beta(u, z)} \frac{g(s)}{s} ds}$ and $g(s)$ is symmetric with respect to $z + \frac{1}{2}\beta(u, z)$ on $[z, z + \beta(u, z)]$, then the following inequality holds:*

$$\begin{aligned} & \left| \int_z^{z+\beta(u, z)} \varphi(s) g(s) ds - \varphi\left(z + \frac{1}{2}\beta(u, z)\right) \int_z^{z+\beta(u, z)} g(s) ds \right| \\ & \leq 2 \left(\frac{|\varphi'(z)|^q + |\varphi'(z + \beta(u, z))|^q}{\beta(u, z)} \right)^{\frac{1}{q}} \\ & \quad \times \int_z^{z+\frac{1}{2}\beta(u, z)} \left(z + \frac{1}{2}\beta(u, z) - s \right) g(s) ds. \end{aligned} \quad (3.16)$$

For our next results, we use the following notations.

$$\begin{aligned} \varphi(w, \varphi) &:= \frac{1}{\beta(u, z)} \left(\varphi(z) \int_z^{z+\beta(u, z)} (z + \beta(u, z) - s) ds \right. \\ & \quad \left. + \varphi(z + \beta(u, z)) \int_z^{z+\beta(u, z)} (s - z) w(s) ds \right) \end{aligned}$$

$$- \int_z^{z+\beta(u,z)} \varphi(s) w(s) ds. \quad (3.17)$$

It is clear from (3.17) that

$$\varphi\left(\frac{1}{\beta(u,z)}, \varphi\right) := \frac{\varphi(z) + \varphi(z + \beta(u,z))}{2} - \frac{1}{\beta(u,z)} \int_z^{z+\beta(u,z)} \varphi(s) ds. \quad (3.18)$$

The next result gives upper bound of $|\varphi(w, \varphi)|$ when the function $\varphi(s)$ is prequasiinvex.

Theorem 3.7. *Let $\varphi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° and $\varphi' \in L([z, z + \beta(u, z)])$, where $[z, z + \beta(u, z)] \subseteq I^\circ$ with $\beta(u, z) > 0$. If $w : [z, z + \beta(u, z)] \rightarrow [0, \infty)$ be a continuous mapping and $|\varphi'|$ is prequasiinvex on $[z, z + \beta(u, z)]$, then the following inequality holds*

$$\begin{aligned} |\varphi(w, \varphi)| &\leq \frac{\beta(z_1, z)}{\beta(u, z)} (\sup \{|\varphi'(z)|, |\varphi'(z + \beta(z_1, z))|\}) \int_0^1 |G(w, z, z_1; y)| dy \\ &\quad + \frac{\beta(u, z_1)}{\beta(u, z)} (\sup \{|\varphi'(z_1)|, |\varphi'(z_1 + \beta(u, z_1))|\}) \int_0^1 |G(w, z_1, u; y)| dy, \end{aligned} \quad (3.19)$$

where $G(w, \alpha, \beta; y)$ is defined as in Lemma 2.2.

Proof. Since $|\varphi'|$ is prequasiinvex on $[z, z + \beta(u, z)]$, we have

$$|\varphi'(z + y\beta(z_1, z))| \leq \sup \{|\varphi'(z)|, |\varphi'(z + \beta(z_1, z))|\}$$

and

$$|\varphi'(z_1 + y\beta(u, z_1))| \leq \sup \{|\varphi'(z_1)|, |\varphi'(z_1 + \beta(u, z_1))|\}.$$

for all $y \in [0, 1]$. Hence the inequality (3.19) follows from (2.1). $\square$

Theorem 3.8. *Let $\varphi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° and $\varphi' \in L([z, z + \beta(u, z)])$, where $[z, z + \beta(u, z)] \subseteq I^\circ$ with $\beta(u, z) > 0$, then the following inequality holds*

$$\begin{aligned} |\varphi(w, \varphi)| &\leq \left(\int_z^{z+\frac{1}{2}\beta(u,z)} (\eta - z) w(\eta) d\eta \right) \left[\sup \left\{ |\varphi'(z)|, \left| \varphi' \left(z + \frac{1}{2}\beta(u, z) \right) \right| \right\} \right. \\ &\quad \left. + \sup \left\{ \left| \varphi' \left(z + \frac{1}{2}\beta(u, z) \right) \right|, |\varphi'(z + \beta(u, z))| \right\} \right] \end{aligned} \quad (3.20)$$

Proof. The symmetry of $w(s)$ with respect to $z + \frac{1}{2}\beta(u, z)$ on $[z, z + \beta(u, z)]$ gives

$$\varphi(w, \varphi) = \frac{\varphi(z) + \varphi(z + \beta(u, z))}{2} - \int_z^{z+\beta(u,z)} \varphi(s) w(s) ds.$$

We also observe that

$$\frac{\beta(z_1, z)}{\beta(u, z)} \int_0^1 |G(w, z, z_1; y)| dy = \frac{1}{\beta(u, z)} \int_z^{z_1} \left| \int_z^\eta (s - z) w(s) ds - \int_\eta^{z+\beta(u, z)} (z + \beta(u, z) - s) w(s) ds \right| d\eta$$

and

$$\frac{\beta(u, z_1)}{\beta(u, z)} \int_0^1 |G(w, z_1, u; y)| dy = \frac{1}{\beta(u, z)} \int_{z_1}^{z+\beta(u, z)} \left| \int_z^\eta (s - z) w(s) ds - \int_\eta^{z+\beta(u, z)} (z + \beta(u, z) - s) w(s) ds \right| d\eta$$

Consider the function $p : [z, z + \beta(u, z)] \rightarrow \mathbb{R}$ defined by

$$p(\eta) = \int_z^\eta (s - z) w(s) ds - \int_\eta^{z+\beta(u, z)} (z + \beta(u, z) - s) w(s) ds.$$

Then

$$p'(\eta) = \frac{\beta(u, z) w(\eta)}{\eta} > 0, \eta \in [z, z + \beta(u, z)]$$

This shows that $p(\eta)$ is an increasing function on $[z, z + \beta(u, z)]$ and

$$p\left(z + \frac{1}{2}\beta(u, z)\right) = 0.$$

Now it is easy to see that

$$\begin{aligned} \frac{\beta(z_1, z)}{\beta(u, z)} \int_0^1 |G(w, z, z_1; y)| dy &= \frac{1}{\beta(u, z)} \int_z^{z_1} \left(\int_z^\eta (s - z) w(s) ds - \int_\eta^{z+\beta(u, z)} (z + \beta(u, z) - s) w(s) ds \right) d\eta \\ &= \int_z^{z_1} (\eta - z) w(\eta) d\eta = \int_z^{z+\frac{1}{2}\beta(u, z)} (\eta - z) w(\eta) d\eta \end{aligned}$$

and

$$\begin{aligned} \frac{\beta(u, z_1)}{\beta(u, z)} \int_0^1 |G(w, z_1, u; y)| dy &= \frac{1}{\beta(u, z)} \int_{z_1}^{z+\beta(u, z)} \left(\int_z^\eta (s - z) w(s) ds - \int_\eta^{z+\beta(u, z)} (z + \beta(u, z) - s) w(s) ds \right) d\eta \\ &= \int_z^{z_1} (\eta - z) w(\eta) d\eta = \int_z^{z+\frac{1}{2}\beta(u, z)} (\eta - z) w(\eta) d\eta. \end{aligned}$$

Hence the inequality (3.20) follows from the inequality (3.19), the above equalities and using the Property C. Since

$$\beta\left(z + \frac{1}{2}\beta(u, z), z\right) = \frac{1}{2}\beta(u, z)$$

and

$$z + \frac{1}{2}\beta(u, z) + \beta\left(u, z + \frac{1}{2}\beta(u, z)\right) = z + \beta(u, z).$$

□

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NONLINEAR INVERSE PROBLEM FOR IDENTIFYING A COEFFICIENT OF THE LOWEST TERM IN HYPERBOLIC EQUATION WITH NONLOCAL CONDITIONS

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Abstract. In this paper, a nonlinear inverse boundary value problem for the second-order hyperbolic equation with nonlocal conditions is studied. To investigate the solvability of the original problem, we first consider an auxiliary inverse boundary value problem and prove its equivalence (in a certain sense) to the original problem. Then using the Fourier method and contraction mappings principle, the existence and uniqueness theorem for auxiliary problem is proved. Further, on the basis of the equivalency of these problems the existence and uniqueness theorem for the classical solution of the considered inverse coefficient problem is proved for the smaller value of time.

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1. INTRODUCTION AND PROBLEM STATEMENT

Let $0 < T < +\infty$ be some fixed number and D_T be a rectangular region in the xt -plane defined by $0 \leq x \leq 1$, $0 \leq t \leq T$. Consider the problem of determining the unknown functions $u = u(x, t)$ and $a = a(t)$ such that the pair $\{u, a\}$ satisfies the following hyperbolic equation of second order

$$u_{tt}(x, t) - u_{xx}(x, t) = a(t)u(x, t) + f(x, t) \quad (x, t) \in D_T, \quad (1.1)$$

with the nonlocal initial conditions

$$u(x, 0) + \delta_1 u(x, T) = \varphi(x), \quad u_t(x, 0) + \delta_2 u_t(x, T) = \psi(x), \quad 0 \leq x \leq 1, \quad (1.2)$$

the boundary conditions

$$u(0, t) = \beta u(1, t), \quad 0 \leq t \leq T, \quad (1.3)$$

$$\int_0^1 u(x, t) dx = 0, \quad 0 \leq t \leq T, \quad (1.4)$$

and overdetermination condition

$$u\left(\frac{1}{2}, t\right) = h(t), \quad 0 \leq t \leq T, \quad (1.5)$$

in which $\delta_1, \delta_2 \geq 0$ and $\beta \neq \pm 1$ are given numbers, $f(x, t)$, $\varphi(x)$, and $\psi(x)$ are known functions of $x \in [0, 1]$ and $t \in [0, T]$.

In the present work we investigate a nonlinear inverse problem for identifying a coefficient of the lowest term in hyperbolic equation from the overdetermination data. Such problems are called inverse problems in mathematical physics. The applied importance of inverse problems is so great (seismology, mineral exploration, biology, medicine, desalination of seawater, movement of liquid in a porous medium, acoustics, electromagnetics, fluid dynamics, calculating the density of the Earth from measurements of its gravity field, for example) which puts them a series of the most actual problems of modern mathematics. Strictly speaking, the inverse problems for hyperbolic/wave equations are of prime interest in seismology. Besides, vibrations of structures (as buildings and beams) are modeled by hyperbolic differential equations. However, it should be noted that an inverse problem is called linear if the recovery function enters the given equation linearly, and nonlinear otherwise.

Nowadays, inverse problems for hyperbolic equations have been well studied by many authors using different methods (see, e.g., [2, 5–7, 9, 10, 19–22], and the references given therein). Moreover, in [15–18] the authors present a regularity result for solutions of partial differential equations in the framework of mixed Morrey spaces.

A distinctive feature of presented article is the investigation of an inverse hyperbolic problem with both spatial and time nonlocal conditions.

This article is based on ideas close to those used in [1, 3, 8, 13].

Definition 1. A pair of functions $\{u(x, t), a(t)\}$ is said to be a classical solution of problem (1.1)–(1.5) if all three of the following conditions are satisfied:

- The function $u(x, t)$ with the derivatives $u_{xx}(x, t)$ and $u_{tt}(x, t)$ are continuous in the domain D_T .
- The function $a(t)$ is continuous on the interval $[0, T]$.
- The Eq. (1.1) and conditions (1.2)–(1.5) are satisfied in the classical (usual) sense.

Now, to study problem (1.1)–(1.5), we consider the following auxiliary inverse boundary value problem: it is required to find a pair of functions $u(x, t) \in C^2(D_T)$, $a(t) \in C[0, T]$ from (1.1)–(1.3) and

$$u_x(0, t) = u_x(1, t), \quad 0 \leq t \leq T, \quad (1.6)$$

$$h''(t) - u_{xx}\left(\frac{1}{2}, t\right) = a(t)h(t) + f\left(\frac{1}{2}, t\right), \quad 0 \leq t \leq T. \quad (1.7)$$

Similarly (see [12], Theorem 2.2, p.4) it can be proved that

Theorem 1. Suppose that $\varphi(x), \psi(x) \in C[0, 1]$, $h(t) \in C^2[0, T]$, $h(t) \neq 0$, $f(x, t) \in C(D_T)$, $\int_0^1 f(x, t) dx = 0$, $0 \leq t \leq T$ and the compatibility conditions

$$\int_0^1 \varphi(x) dx = 0, \quad \int_0^1 \psi(x) dx = 0, \\ h(0) + \delta_1 h(T) = \varphi\left(\frac{1}{2}\right), \quad h'(0) + \delta_2 h'(T) = \psi\left(\frac{1}{2}\right),$$

hold. Then the following statements are true:

- (i) Each classical solution $\{u(x, t), a(t)\}$ of problem (1.1)–(1.5) is the solution of problem (1.1)–(1.3), (1.6), (1.7), as well
- (ii) each solution $\{u(x, t), a(t)\}$ of problem (1.1)–(1.3), (1.6), (1.7) is a classical solution to the problem (1.1)–(1.5), if

$$\frac{(1 + 2\delta_1 + 3\delta_2 + \delta_1\delta_2)T^2}{2(1 + \delta_1)(1 + \delta_2)} \|a(t)\|_{C[0, T]} < 1.$$

2. AUXILIARY FACTS AND DENOTATIONS

It is known that sequences of functions [14]

$$X_0(x) = px + q, \quad X_{2k-1}(x) = (px + q) \cos \lambda_k x, \quad X_{2k}(x) = \sin \lambda_k x, \quad k = 1, 2, \dots, \quad (2.1)$$

$$Y_0(x) = 2, \quad Y_{2k-1}(x) = 4 \sin \lambda_k x, \quad Y_{2k}(x) = q(1 - q - px) \cos \lambda_k x, \quad k = 1, 2, \dots, \quad (2.2)$$

form a biorthogonal system and system (2.1) forms a Riesz basis in $L_2(0, 1)$ for $\lambda_k = 2k\pi$ ($k = 1, 2, \dots$). Here p and q denotes, in turn, the numbers

$$p = \frac{1 - \beta}{1 + \beta} \neq 0, \quad q = \frac{\beta}{1 + \beta}.$$

We state the following lemmas without proof.

Lemma 1. (see [4, 11]) For any function $v(x)$ with the properties:

$$v(x) \in C^{2i-1}[0, 1], \quad v^{(2i)}(x) \in L_2(0, 1), \\ v^{(2s)}(0) = \beta v^{(2s)}(1), \quad v^{(2s+1)}(0) = v^{(2s+1)}(1) \quad (i \geq 1; \quad s = \overline{0, i}),$$

the estimates are valid

$$\left(\sum_{k=0}^{\infty} (\lambda_k^{2i} v_{2k-1})^2 \right)^{\frac{1}{2}} \leq 2\sqrt{2} \|v^{(2i)}(x)\|_{L_2(0, 1)},$$

$$\left(\sum_{k=1}^{\infty} (\lambda_k^{2i} v_{2k})^2 \right)^{\frac{1}{2}} \leq 2\sqrt{2} \left\| v^{(2i)}(x)(1-q-px) - 2ipv^{(2i-1)}(x) \right\|_{L_2(0,1)}, \quad (2.3)$$

where

$$v_k = \int_0^1 v(x) Y_k(x) dx, \quad k = 0, 1, \dots$$

Lemma 2. (see [4, 11]) Under the assumptions:

$$\begin{aligned} v(x) &\in C^{2i}[0, 1], \quad v^{(2i+1)}(x) \in L_2(0, 1), \\ v^{(2s)}(0) &= \beta v^{(2s)}(1), \quad v^{(2s+1)}(0) = v^{(2s+1)}(1) \quad (i \geq 1; \quad s = \overline{0, i}), \end{aligned}$$

we establish the validity of the estimates

$$\begin{aligned} \left(\sum_{k=1}^{\infty} (\lambda_k^{2i+1} v_{2k-1})^2 \right)^{\frac{1}{2}} &\leq 2\sqrt{2} \left\| v^{(2i+1)}(x) \right\|_{L_2(0,1)}, \\ \left(\sum_{k=1}^{\infty} (\lambda_k^{2i+1} v_{2k-1})^2 \right)^{\frac{1}{2}} &\leq 2\sqrt{2} \left\| v^{(2i+1)}(x)(1-q-px) - (2i+1)pv^{(2i)}(x) \right\|_{L_2(0,1)}. \end{aligned} \quad (2.4)$$

We now look at the following functional spaces:

$B_{2,T}^3$ [11] denotes a set of all functions of the form

$$u(x, t) = \sum_{k=1}^{\infty} u_k(t) X_k(x),$$

considered in D_T , where $u_k(t) \in C[0, T]$ and

$$\begin{aligned} J(u) &\equiv \|u_0(t)\|_{C[0,T]} + \left\{ \sum_{k=1}^{\infty} (\lambda_k^3 \|u_{2k-1}(t)\|_{C[0,T]})^2 \right\}^{\frac{1}{2}} \\ &+ \left\{ \sum_{k=1}^{\infty} (\lambda_k^3 \|u_{2k}(t)\|_{C[0,T]})^2 \right\}^{\frac{1}{2}} < +\infty. \end{aligned}$$

Actually, the functions $X_k(x)$ ($k = 0, 1, \dots$) defined by the relation (2.1). The norm on the set $J(u)$ is established as follows

$$\|u(x, t)\|_{B_{2,T}^3} = J(u).$$

Let E_T^3 denote the space consisting of the topological product $B_{2,T}^3 \times C[0, T]$, which is the norm of the element $z = \{u, a\}$ defined by the formula

$$\|z(x, t)\|_{E_T^3} = \|u(x, t)\|_{B_{2,T}^3} + \|a(t)\|_{C[0,T]}.$$

3. CLASSICAL SOLVABILITY OF INVERSE BOUNDARY VALUE PROBLEM

Suppose that the data of problem (1.1)–(1.3), (1.6), (1.7) satisfies the following conditions:

- $C_1.$ $\delta_1, \delta_2 \geq 0, 1 + \delta_1 \delta_2 \geq \delta_1 + \delta_2$;
- $C_2.$ $\varphi(x) \in C^2[0, 1], \varphi'''(x) \in L_2(0, 1), \varphi(0) = \beta\varphi(1),$
 $\varphi'(0) = \varphi'(1), \varphi''(0) = \beta\varphi''(1)$;
- $C_3.$ $\psi(x) \in C^1[0, 1], \psi''(x) \in L_2(0, 1), \psi(0) = \beta\psi(1), \psi'(0) = \psi'(1)$;
- $C_4.$ $f(x, t), f_x(x, t) \in C(D_T), f_{xx}(x, t) \in L_2(D_T), f(0, t) = \beta f(1, t),$
 $f_x(0, t) = f_x(1, t), 0 \leq t \leq T$;
- $C_5.$ $h(t) \in C^2[0, T], h(t) \neq 0, 0 \leq t \leq T$.

Since the system (2.1) forms a Riesz basis in $L_2(0, 1)$. Then each solution to problem (1.1)–(1.3), (1.6), (1.7) can be sought in the form:

$$u(x, t) = \sum_{k=0}^{\infty} u_k(t) X_k(x), \quad (3.1)$$

where

$$u_k(t) = \int_0^1 u(x, t) Y_k(x) dx, \quad k = 0, 1, \dots$$

Moreover, $X_k(x)$ ($k = 0, 1, \dots$) and $Y_k(x)$ ($k = 0, 1, \dots$) are defined by relations (2.1) and (2.2), respectively.

To determine of the desired functions $u_k(t)$ ($k = 0, 1, 2, \dots$), using separation of variables and by (1.1) and (1.2), we get

$$u_0''(t) = F_0(t; u, a), \quad 0 \leq t \leq T, \quad (3.2)$$

$$u_{2k-1}''(t) + \lambda_k^2 u_{2k-1}(t) = F_{2k-1}(t; u, a), \quad k = 1, 2, \dots; \quad 0 \leq t \leq T, \quad (3.3)$$

$$u_{2k}''(t) + \lambda_k^2 u_{2k}(t) = F_{2k}(t; u, a) - 2p\lambda_k u_{2k-1}(t), \quad k = 1, 2, \dots; \quad 0 \leq t \leq T, \quad (3.4)$$

$$u_k(0) + \delta_1 u_k(T) = \varphi_k, \quad u_k'(0) + \delta_2 u_k'(T) = \psi_k, \quad k = 0, 1, 2, \dots, \quad (3.5)$$

where

$$F_k(t; u, a) = a(t)u_k(t) + f_k(t), \quad f_k(t) = \int_0^1 f(x, t) Y_k(x) dx, \quad k = 0, 1, \dots$$

$$\varphi_k = \int_0^1 \varphi(x) Y_k(x) dx, \quad \psi_k = \int_0^1 \psi(x) Y_k(x) dx, \quad k = 0, 1, 2, \dots$$

Solving problem (3.2)–(3.5), we find

$$u_0(t) = \frac{\varphi_0}{1 + \delta_1} + \frac{t - \delta_1(T - t)}{(1 + \delta_1)(1 + \delta_2)} \psi_0 + \int_0^T G_0(t, \tau) F_0(\tau; u, a) d\tau, \quad (3.6)$$

$$u_{2k-1}(t) = \frac{1}{\rho_k(T)} [\varphi_{2k-1}(\cos \lambda_k t + \delta_2 \cos \lambda_k(T-t)) + \frac{\Psi_{2k-1}}{\lambda_k}(\sin \lambda_k t - \delta_1 \sin \lambda_k(T-t))] + \int_0^T G_k(t, \tau) F_{2k-1}(\tau; u, a) d\tau, \quad (3.7)$$

$$\begin{aligned} u_{2k}(t) = & \frac{1}{\rho_k(T)} \left[\varphi_{2k}(\cos \lambda_k t + \delta_2 \cos \lambda_k(T-t)) + \frac{\Psi_{2k}}{\lambda_k}(\sin \lambda_k t - \delta_1 \sin \lambda_k(T-t)) \right] \\ & + \int_0^T G_k(t, \tau) (F_{2k}(\tau; u, a)) d\tau \\ & - \varphi_{2k-1} \left\{ -\frac{1}{\rho_k^2(T)} \frac{1}{\lambda_k} \left[\delta_1 \cos \lambda_k t \left(\frac{T}{2} \sin \lambda_k T + \delta_2 \frac{1}{4} (1 - \cos 2\lambda_k T) \right) \right. \right. \\ & + \delta_2 \sin \lambda_k t \left(\frac{1}{2\lambda_k} \sin \lambda_k T + \frac{T}{2} \cos \lambda_k T + \delta_2 \left(\frac{T}{2} + \frac{1}{4\lambda_k} \sin 2\lambda_k T \right) \right) \\ & + \delta_1 \delta_2 \left(\frac{1}{4\lambda_k} (\cos \lambda_k(2T-t) - \cos \lambda_k t) + \frac{T}{2} \sin \lambda_k t \right. \\ & + \delta_2 \left(-\frac{T}{2} \sin \lambda_k(T-t) + \frac{1}{4\lambda_k} (\cos \lambda_k(T-t) - \cos \lambda_k(T+t)) \right) \left. \left. \right) \right] \\ & + \frac{t}{2} \sin \lambda_k t + \delta_2 \left(-\frac{t}{2} \sin \lambda_k(T-t) \right. \\ & + \left. \left. \frac{1}{4\lambda_k} (\cos \lambda_k(T-t) - \cos \lambda_k(T+t)) \right) \right\} \\ & + \frac{\Psi_{2k-1}}{\lambda_k} \left\{ -\frac{1}{\rho_k^2(T)} \cdot \frac{1}{\lambda_k} \left[\delta_1 \cos \lambda_k t \left(\frac{1}{2\lambda_k} \sin \lambda_k T - \frac{T}{2} \cos \lambda_k T \right) \right. \right. \\ & - \delta_1 \left(\frac{T}{2} - \frac{1}{4\lambda_k} \sin 2\lambda_k T \right) + \delta_2 \sin \lambda_k t \left(\frac{T}{2} \sin \lambda_k T - \frac{\delta_1}{4\lambda_k} (1 - \cos 2\lambda_k T) \right) \\ & + \delta_1 \delta_2 \left(\frac{1}{4\lambda_k} (\sin \lambda_k(2T-t) + \sin \lambda_k t) - \frac{T}{2} \cos \lambda_k t \right. \\ & - \delta_1 \left(\frac{T}{2} \cos \lambda_k(T-t) - \frac{1}{4\lambda_k} (\sin \lambda_k(T-t) + \sin \lambda_k(T+t)) \right) \left. \left. \right) \right] \\ & + \left(\frac{1}{2\lambda_k} \sin \lambda_k t - \frac{t}{2} \cos \lambda_k t - \delta_1 \left(\frac{t}{2} \cos \lambda_k(T-t) + \frac{1}{4\lambda_k} (\sin \lambda_k(T-t) \right. \right. \\ & \left. \left. - \sin \lambda_k(T+t)) \right) \right) \left. \right\} + \int_0^T G_k(t, \tau) \left(\int_0^T G_k(\tau, \xi) F_{2k-1}(\xi; u, a) d\xi \right) d\tau, \quad (3.8) \end{aligned}$$

where

$$G_0(t, \tau) = \begin{cases} -\frac{\delta_2 t + \delta_1(T-\tau) + \delta_1 \delta_2(t-\tau)}{(1+\delta_1)(1+\delta_2)}, & t \in [0, \tau], \\ -\frac{\delta_2 t + \delta_1(T-\tau) - (1+\delta_1+\delta_2)(t-\tau)}{(1+\delta_1)(1+\delta_2)}, & t \in [\tau, T], \end{cases}$$

$$G_k(t, \tau) = \begin{cases} -\frac{1}{\rho_k(T)} \cdot \frac{1}{\lambda_k} [\delta_1 \sin \lambda_k(T - \tau) \cos \lambda_k t + \delta_2 \cos \lambda_k(T - \tau) \sin \lambda_k t \\ \quad + \delta_1 \delta_2 \sin \lambda_k(t - \tau)], \quad t \in [0, \tau], \\ -\frac{1}{\rho_k(T)} \cdot \frac{1}{\lambda_k} [\delta_1 \sin \lambda_k(T - \tau) \cos \lambda_k t + \delta_2 \cos \lambda_k(T - \tau) \sin \lambda_k t \\ \quad + \delta_1 \delta_2 \sin \lambda_k(t - \tau)] + \frac{1}{\lambda_k} \sin \lambda_k(t - \tau), \quad t \in [\tau, T], \end{cases}$$

and

$$\rho_k(T) = 1 + (\delta_1 + \delta_2) \cos \lambda_k T + \delta_1 \delta_2.$$

Substituting the expressions of (3.6), (3.7), and (3.8) into (3.1), we find the component $u(x, t)$ of the classical solution to problem (1.1)–(1.3), (1.6), (1.7) to be

$$\begin{aligned} u(x, t) = & \left\{ \frac{\varphi_0}{1 + \delta_1} + \frac{t - \delta_1(T - t)}{(1 + \delta_1)(1 + \delta_2)} \psi_0 + \int_0^T G_0(t, \tau) F_0(\tau; u, a) d\tau \right\} X_0(x) \\ & + \sum_{k=1}^{\infty} \left\{ \frac{1}{\rho_k(T)} [\varphi_{2k-1}(\cos \lambda_k t + \delta_2 \cos \lambda_k(T - t)) \right. \\ & \left. + \frac{\Psi_{2k-1}}{\lambda_k} (\sin \lambda_k t - \delta_1 \sin \lambda_k(T - t))] + \int_0^T G_k(t, \tau) F_{2k-1}(\tau; u, a) d\tau \right\} X_{2k-1}(x) \\ & + \sum_{k=1}^{\infty} \left\{ \frac{1}{\rho_k(T)} [\varphi_{2k}(\cos \lambda_k t + \delta_2 \cos \lambda_k(T - t)) + \right. \\ & \left. + \frac{\Psi_{2k}}{\lambda_k} (\sin \lambda_k t - \delta_1 \sin \lambda_k(T - t))] + \int_0^T G_k(t, \tau) F_{2k}(\tau; u, a) d\tau \right. \\ & - \varphi_{2k-1} \left\{ -\frac{1}{\rho_k^2(T)} \cdot \frac{1}{\lambda_k} \left[\delta_1 \cos \lambda_k t \left(\frac{T}{2} \sin \lambda_k T + \delta_2 \frac{1}{4} (1 - \cos 2\lambda_k T) \right) \right. \right. \\ & \left. \left. + \delta_2 \sin \lambda_k t \left(\frac{1}{2\lambda_k} \sin \lambda_k T + \frac{T}{2} \cos \lambda_k T + \delta_2 \left(\frac{T}{2} + \frac{1}{4\lambda_k} \sin 2\lambda_k T \right) \right) \right. \right. \\ & \left. \left. + \delta_1 \delta_2 \left(\frac{1}{4\lambda_k} (\cos \lambda_k(2T - t) - \cos \lambda_k t) + \frac{T}{2} \sin \lambda_k t \right. \right. \right. \\ & \left. \left. \left. + \delta_2 \left(-\frac{T}{2} \sin \lambda_k(T - t) + \frac{1}{4\lambda_k} (\cos \lambda_k(T - t) - \cos \lambda_k(T + t)) \right) \right) \right] \right\} \\ & \left. + \frac{t}{2} \sin \lambda_k t + \delta_2 \left(-\frac{t}{2} \sin \lambda_k(T - t) + \frac{1}{4\lambda_k} (\cos \lambda_k(T - t) - \cos \lambda_k(T + t)) \right) \right\} \\ & + \frac{\Psi_{2k-1}}{\lambda_k} \left\{ -\frac{1}{\rho_k^2(T)} \cdot \frac{1}{\lambda_k} \left[\delta_1 \cos \lambda_k t \left(\frac{1}{2\lambda_k} \sin \lambda_k T - \frac{T}{2} \cos \lambda_k T \right) \right. \right. \\ & \left. \left. - \delta_1 \left(\frac{T}{2} - \frac{1}{4\lambda_k} \sin 2\lambda_k T \right) + \delta_2 \sin \lambda_k t \left(\frac{T}{2} \sin \lambda_k T - \frac{\delta_1}{4\lambda_k} (1 - \cos 2\lambda_k T) \right) \right. \right. \\ & \left. \left. + \delta_1 \delta_2 \left(\frac{1}{4\lambda_k} (\sin \lambda_k(2T - t) + \sin \lambda_k t) - \frac{T}{2} \cos \lambda_k t \right) \right] \right\} \end{aligned}$$

$$\begin{aligned}
& -\delta_1 \left(\frac{T}{2} \cos \lambda_k(T-t) - \frac{1}{4\lambda_k} (\sin \lambda_k(T-t) + \sin \lambda_k(T+t)) \right) \Bigg] \\
& + \left(\frac{1}{2\lambda_k} \sin \lambda_k t - \frac{t}{2} \cos \lambda_k t - \delta_1 \left(\frac{t}{2} \cos \lambda_k(T-t) + \frac{1}{4\lambda_k} (\sin \lambda_k(T-t) \right. \right. \\
& \left. \left. - \sin \lambda_k(T+t)) \right) \right) + \int_0^T G_k(t, \tau) \left(\int_0^T G_k(\tau, \xi) F_{2k-1}(\xi; u, a) d\xi \right) d\tau \Bigg\} X_{2k}(x).
\end{aligned} \tag{3.9}$$

Now, using (3.1) from (1.7) we have

$$a(t) = [h(t)]^{-1} \left\{ h''(t) - f\left(\frac{1}{2}, t\right) - \frac{1}{2} \sum_{k=1}^{\infty} (-1)^k \lambda_k^2 u_{2k-1}(t) \right\}. \tag{3.10}$$

Substituting expressions $u_{2k-1}(t)$ ($k = 1, 2, \dots$) from (3.8) into (3.10), immediately yields:

$$\begin{aligned}
a(t) &= [h(t)]^{-1} \\
&\times \left\{ h''(t) - f\left(\frac{1}{2}, t\right) - \frac{1}{2} \sum_{k=1}^{\infty} (-1)^k \lambda_k^2 \left\{ \frac{1}{\rho_k(T)} [\varphi_{2k-1}(\cos \lambda_k t + \delta_2 \cos \lambda_k(T-t)) \right. \right. \\
&\left. \left. + \frac{\Psi_{2k-1}}{\lambda_k} (\sin \lambda_k t - \delta_1 \sin \lambda_k(T-t)) \right] + \int_0^T G_k(t, \tau) F_{2k-1}(\tau; u, a) d\tau \right\} \Bigg\}.
\end{aligned} \tag{3.11}$$

Thus, the solution of problem (1.1)–(1.3), (1.6), (1.7) is reduced to the solution of system (3.9), (3.11) with respect to unknown functions $u(x, t)$ and $a(t)$.

Using the same discussion in [11], one can prove

Lemma 3. *If $\{u(x, t), a(t)\}$ is any solution to problem (1.1)–(1.3), (1.6), (1.7), then the functions*

$$u_k(t) = \int_0^1 u(x, t) Y_k(x) dx, \quad k = 0, 1, \dots,$$

satisfy the countable system (3.6), (3.7) and (3.8) on an interval $[0, T]$.

Obviously, if $u_k(t) = \int_0^1 u(x, t) Y_k(x) dx$, $k = 0, 1, \dots$ is a solution to system (3.6),

(3.7) and (3.8), then the pair $\{u(x, t), a(t)\}$ of functions $u(x, t) = \sum_{k=0}^{\infty} u_k(t) X_k(x)$ and $a(t)$ is also a solution to system (3.9), (3.11).

It follows from the Lemma 3 that

Corollary 1. *Let system (3.9), (3.11) have a unique solution. Then problem (1.1)–(1.3), (1.6), (1.7) cannot have more than one solution, i.e. if the problem (1.1)–(1.3), (1.6), (1.7) has a solution, then it is unique.*

Now consider the operator

$$\Phi(u, a) = \{\Phi_1(u, a), \Phi_2(u, a)\},$$

in the space E_T^3 , where

$$\Phi_1(u, a) = \tilde{u}(x, t) \equiv \tilde{u}_0(t)X_0(x) + \sum_{k=0}^{\infty} \tilde{u}_{2k-1}(t)X_{2k-1}(x) + \sum_{k=1}^{\infty} \tilde{u}_{2k}(t)X_{2k}(x),$$

$$\Phi_2(u, a) = \tilde{a}(t),$$

and the functions $\tilde{u}_0(t), \tilde{u}_{2k-1}(t), \tilde{u}_{2k}(t)$ ($k = 1, 2, \dots$), and $\tilde{a}(t)$ are equal correspondingly to the right sides of (3.6), (3.7), (3.8), and (3.11).

It is clear that under conditions $\delta_1, \delta_2 \geq 0$, $1 + \delta_1\delta_2 > \delta_1 + \delta_2$ and using the inequality

$$\rho_k(T) \geq 1 - (\delta_1 + \delta_2) + \delta_1\delta_2$$

we can write

$$\frac{1}{\rho_k(T)} \leq \frac{1}{1 - (\delta_1 + \delta_2) + \delta_1\delta_2} \equiv \rho > 0.$$

Using this relation, from (3.6), (3.7), (3.8), and (3.11) we obtain

$$\begin{aligned} \|\tilde{u}_0(t)\|_{C[0,T]} &\leq \frac{1}{1 + \delta_1} |\varphi_0| + \frac{T}{1 + \delta_2} \psi_0 + \frac{1 + 3\delta_1 + 3\delta_2}{(1 + \delta_1)(1 + \delta_2)} T \\ &\quad \times \left(\sqrt{T} \left(\int_0^T |f_0(\tau)|^2 d\tau \right)^{\frac{1}{2}} + T \|a(t)\|_{C[0,T]} \|u_0(t)\|_{C[0,T]} \right), \end{aligned} \quad (3.12)$$

$$\begin{aligned} \left\{ \sum_{k=1}^{\infty} (\lambda_k^3 \|\tilde{u}_{2k-1}(t)\|_{C[0,T]})^2 \right\}^{\frac{1}{2}} &\leq 2\rho(1 + \delta_2) \left(\sum_{k=1}^{\infty} (\lambda_k^3 |\varphi_{2k-1}|)^2 \right)^{\frac{1}{2}} \\ &\quad + 2\rho(1 + \delta_1) \left(\sum_{k=1}^{\infty} (\lambda_k^2 |\psi_{2k-1}|)^2 \right)^{\frac{1}{2}} \\ &\quad + 2(1 + 2\rho(\delta_1 + \delta_2 + \delta_1\delta_2)) \sqrt{T} \left(\int_0^T \sum_{k=1}^{\infty} (\lambda_k^2 |f_{2k-1}(\tau)|)^2 d\tau \right)^{\frac{1}{2}} \\ &\quad + 2(1 + 2\rho(\delta_1 + \delta_2 + \delta_1\delta_2)) T \|a(t)\|_{C[0,T]} \left(\sum_{k=1}^{\infty} (\lambda_k^3 \|u_{2k-1}(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}}, \end{aligned} \quad (3.13)$$

$$\begin{aligned} \left\{ \sum_{k=1}^{\infty} (\lambda_k^3 \|\tilde{u}_{2k}(t)\|_{C[0,T]})^2 \right\}^{\frac{1}{2}} &\leq 2\sqrt{2}\rho(1 + \delta_2) \left(\sum_{k=1}^{\infty} (\lambda_k^3 |\varphi_{2k}|)^2 \right)^{\frac{1}{2}} \\ &\quad + 2\sqrt{2}\rho(1 + \delta_1) \left(\sum_{k=1}^{\infty} (\lambda_k^2 |\psi_{2k}|)^2 \right)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
& + 2\sqrt{2}(1 + 2\rho(\delta_1 + \delta_2 + \delta_1\delta_2))\sqrt{T} \left(\int_0^T \sum_{k=1}^{\infty} (\lambda_k^2 |f_{2k}(\tau)|)^2 d\tau \right)^{\frac{1}{2}} \\
& + 2\sqrt{2}(1 + 2\rho(\delta_1 + \delta_2 + \delta_1\delta_2))T \|a(t)\|_{C[0,T]} \left(\sum_{k=1}^{\infty} (\lambda_k^3 \|u_{2k}(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \\
& + 2\sqrt{2}\rho_1 \left(\sum_{k=1}^{\infty} (\lambda_k^3 |\varphi_{2k-1}|)^2 \right)^{\frac{1}{2}} + 2\sqrt{2}\rho_2 \left(\sum_{k=1}^{\infty} (\lambda_k^2 |\psi_{2k-1}|)^2 \right)^{\frac{1}{2}} \\
& + 2\sqrt{2}(1 + 2\rho(\delta_1 + \delta_2 + \delta_1\delta_2))^2 T \left[\sqrt{T} \left(\int_0^T \sum_{k=1}^{\infty} (\lambda_k^2 |f_{2k-1}(\tau)|)^2 d\tau \right)^{\frac{1}{2}} \right. \\
& \left. + T \|a(t)\|_{C[0,T]} \left(\sum_{k=1}^{\infty} (\lambda_k^3 \|u_{2k-1}(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \right], \tag{3.14}
\end{aligned}$$

$$\begin{aligned}
\|\tilde{a}(t)\|_{C[0,T]} & \leq \| [h(t)]^{-1} \|_{C[0,T]} \left\{ \left\| h''(t) - f\left(\frac{1}{2}, t\right) \right\|_{C[0,T]} + \frac{1}{2} \left(\sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \right. \\
& \times \left[\rho(1 + \delta_2) \left(\sum_{k=1}^{\infty} (\lambda_k^3 |\varphi_{2k-1}|)^2 \right)^{\frac{1}{2}} + \rho(1 + \delta_1) \left(\sum_{k=1}^{\infty} (\lambda_k^2 |\psi_{2k-1}|)^2 \right)^{\frac{1}{2}} \right. \\
& + (1 + 2\rho(\delta_1 + \delta_2 + \delta_1\delta_2))\sqrt{T} \left(\int_0^T \sum_{k=1}^{\infty} (\lambda_k^2 |f_{2k-1}(\tau)|)^2 d\tau \right)^{\frac{1}{2}} \\
& \left. \left. + (1 + 2\rho(\delta_1 + \delta_2 + \delta_1\delta_2))T \|a(t)\|_{C[0,T]} \left(\sum_{k=1}^{\infty} (\lambda_k^3 \|u_{2k-1}(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \right] \right\}, \tag{3.15}
\end{aligned}$$

where

$$\begin{aligned}
\rho_1 & = \rho^2 \left[\delta_1 \left(\frac{\delta_2}{2} + \frac{T}{2} \right) + \delta_2 \left(\frac{1}{2} + \frac{T}{2} + \delta_2 \left(\frac{1}{4} + \frac{T}{2} \right) \right) \right. \\
& \quad \left. + \delta_1\delta_2 \left(\frac{1}{2} + \frac{T}{2} + \delta_2 \left(\frac{1}{2} + \frac{T}{2} \right) \right) \right] + \frac{T}{2} + \delta_2 \left(\frac{1}{2} + \frac{T}{2} \right), \\
\rho_2 & = \rho^2 \left[\delta_1 \left(\frac{1}{2} + \frac{T}{2} \right) + \delta_1 \left(\frac{1}{4} + \frac{T}{2} \right) + \delta_2 \left(\frac{\delta_1}{2} + \frac{T}{2} \right) \right. \\
& \quad \left. + \delta_1\delta_2 \left(\frac{1}{2} + \frac{T}{2} + \delta_1 \left(\frac{1}{2} + \frac{T}{2} \right) \right) \right] + \frac{1}{2} + \frac{T}{2} + \delta_1 \left(\frac{1}{2} + \frac{T}{2} \right).
\end{aligned}$$

Then from (3.12), (3.13), (3.14), and (3.15), taking into account (2.3), (2.4), respectively, we find:

$$\begin{aligned}
 \|\tilde{u}_0(t)\|_{C[0,T]} &\leq \frac{2}{1+\delta_1} \|\varphi(x)\|_{L_2(0,1)} + \frac{2T}{1+\delta_2} \|\psi(x)\|_{L_2(0,1)} + \frac{1+3\delta_1+3\delta_2}{(1+\delta_1)(1+\delta_2)} T \\
 &\quad \times \left(2\sqrt{T} \|f(x,t)\|_{L_2(D_T)} + T \|a(t)\|_{C[0,T]} \|u(x,t)\|_{B_{2,T}^3} \right), \\
 \left\{ \sum_{k=1}^{\infty} (\lambda_k^3 \|\tilde{u}_{2k-1}(t)\|_{C[0,T]})^2 \right\}^{\frac{1}{2}} &\leq 4\sqrt{2}\rho(1+\delta_2) \|\varphi'''(x)\|_{L_2(0,1)} \\
 &\quad + 4\sqrt{2}\rho(1+\delta_1) \|\psi''(x)\|_{L_2(0,1)} \\
 &\quad + 4(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))\sqrt{2T} \|f_{xx}(x,t)\|_{L_2(D_T)} \\
 &\quad + 2(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))T \|a(t)\|_{C[0,T]} \|u(x,t)\|_{B_{2,T}^3}, \\
 \left\{ \sum_{k=1}^{\infty} (\lambda_k^3 \|\tilde{u}_{2k}(t)\|_{C[0,T]})^2 \right\}^{\frac{1}{2}} &\leq 8\rho(1+\delta_2) \|\varphi'''(x)(1-q-px) - 3p\varphi''(x)\|_{L_2(0,1)} \\
 &\quad + 8\rho(1+\delta_1) \|\psi''(x)(1-q-px) - 2p\psi'(x)\|_{L_2(0,1)} \\
 &\quad + 8(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))\sqrt{T} \|f_{xx}(x,t)(1-q-px) - 2pf_x(x,t)\|_{L_2(D_T)} \\
 &\quad + 2\sqrt{2}(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))T \|a(t)\|_{C[0,T]} \|u(x,t)\|_{B_{2,T}^3} \\
 &\quad + 8\rho_1 \|\varphi'''(x)\|_{L_2(0,1)} + 8\rho_2 \|\psi''(x)\|_{L_2(0,1)} \\
 &\quad + 2\sqrt{2}(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))^2 T [2\sqrt{2T} \|f_{xx}(x,t)\|_{L_2(D_T)} \\
 &\quad + T \|a(t)\|_{C[0,T]} \|u(x,t)\|_{B_{2,T}^3}], \\
 \|\tilde{a}(t)\|_{C[0,T]} &\leq \| [h(t)]^{-1} \|_{C[0,T]} \\
 &\quad \times \left\{ \left\| h''(t) - f\left(\frac{1}{2}, t\right) \right\|_{C[0,T]} + \frac{1}{2} \left(\sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} [2\sqrt{2}\rho(1+\delta_2) \|\varphi'''(x)\|_{L_2(0,1)} \right. \right. \\
 &\quad + 2\sqrt{2}\rho(1+\delta_1) \|\psi''(x)\|_{L_2(0,1)} \\
 &\quad + (1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))2\sqrt{2T} \|f_{xx}(x,t)\|_{L_2(D_T)} \\
 &\quad \left. \left. + (1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))T \|a(t)\|_{C[0,T]} \|u(x,t)\|_{B_{2,T}^3} \right] \right\},
 \end{aligned}$$

or

$$\|\tilde{u}_0(t)\|_{C[0,T]} \leq A_1(T) + B_1(T) \|a(t)\|_{C[0,T]} \|u(x,t)\|_{B_{2,T}^3}, \quad (3.16)$$

$$\left\{ \sum_{k=1}^{\infty} (\lambda_k^3 \|\tilde{u}_{2k-1}(t)\|_{C[0,T]})^2 \right\}^{\frac{1}{2}} \leq A_2(T) + B_2(T) \|a(t)\|_{C[0,T]} \|u(x,t)\|_{B_{2,T}^3}, \quad (3.17)$$

$$\left\{ \sum_{k=1}^{\infty} (\lambda_k^3 \|\tilde{u}_{2k}(t)\|_{C[0,T]})^2 \right\}^{\frac{1}{2}} \leq A_3(T) + B_3(T) \|a(t)\|_{C[0,T]} \|u(x,t)\|_{B_{2,T}^3}, \quad (3.18)$$

$$\|\tilde{a}(t)\|_{C[0,T]} \leq A_4(T) + B_4(T) \|a(t)\|_{C[0,T]} \|u(x,t)\|_{B_{2,T}^3}, \quad (3.19)$$

where

$$\begin{aligned} A_1(T) &= \frac{2}{1+\delta_1} \|\varphi(x)\|_{L_2(0,1)} \\ &\quad + \frac{2T}{1+\delta_2} \|\psi(x)\|_{L_2(0,1)} + \frac{2(1+3\delta_1+3\delta_2)}{(1+\delta_1)(1+\delta_2)} T\sqrt{T} \|f(x,t)\|_{L_2(D_T)}, \\ B_1(T) &= \frac{1+3\delta_1+3\delta_2}{(1+\delta_1)(1+\delta_2)} T^2, \\ A_2(T) &= 4\sqrt{2}\rho(1+\delta_2) \|\varphi'''(x)\|_{L_2(0,1)} + 4\sqrt{2}\rho(1+\delta_1) \|\psi''(x)\|_{L_2(0,1)} \\ &\quad + 4(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))\sqrt{2T} \|f_{xx}(x,t)\|_{L_2(D_T)}, \\ B_2(T) &= 2(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))T, \\ A_3(T) &= 8\rho(1+\delta_2) \|\varphi'''(x)(1-q-px) - 3p\varphi''(x)\|_{L_2(0,1)} \\ &\quad + 8\rho(1+\delta_1) \|\psi''(x)(1-q-px) - 2p\psi'(x)\|_{L_2(0,1)} \\ &\quad + 8(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))\sqrt{T} \|f_{xx}(x,t)(1-q-px) - 2pf_x(x,t)\|_{L_2(D_T)} \\ &\quad + 8\rho_1 \|\varphi'''(x)\|_{L_2(0,1)} + 8\rho_2 \|\psi''(x)\|_{L_2(0,1)} \\ &\quad + 8(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))^2 T\sqrt{T} \|f_{xx}(x,t)\|_{L_2(D_T)}, \\ B_3(T) &= 2\sqrt{2}(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))T + 2\sqrt{2}(1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))^2 T^2, \\ A_4(T) &= \|[h(t)]^{-1}\|_{C[0,T]} \\ &\quad \times \left\{ \left\| h''(t) - f\left(\frac{1}{2}, t\right) \right\|_{C[0,T]} + \frac{1}{2} \left(\sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} [2\sqrt{2}\rho(1+\delta_2) \|\varphi'''(x)\|_{L_2(0,1)} \right. \\ &\quad + 2\sqrt{2}\rho(1+\delta_1) \|\psi''(x)\|_{L_2(0,1)} \\ &\quad \left. + (1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))2\sqrt{2T} \|f_{xx}(x,t)\|_{L_2(D_T)}] \right\}, \\ B_4(T) &= \frac{1}{2} \|[h(t)]^{-1}\|_{C[0,T]} \left(\sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} (1+2\rho(\delta_1+\delta_2+\delta_1\delta_2))T. \end{aligned}$$

Now, from (3.16)–(3.18) we obtain:

$$\|\tilde{u}(x,t)\|_{B_{2,T}^3} \leq A_5(T) + B_5(T) \|a(t)\|_{C[0,T]} \|u(x,t)\|_{B_{2,T}^3}, \quad (3.20)$$

where

$$A_5(T) = A_1(T) + A_2(T) + A_3(T), \quad B_5(T) = B_1(T) + B_2(T) + B_3(T).$$

Finally, from (3.19) and (3.20) we conclude:

$$\|\tilde{u}(x, t)\|_{B_{2,T}^3} + \|\tilde{a}(t)\|_{C[0,T]} \leq A(T) + B(T) \|a(t)\|_{C[0,T]} \|u(x, t)\|_{B_{2,T}^3}, \quad (3.21)$$

where

$$A(T) = A_4(T) + A_5(T), \quad B(T) = B_4(T) + B_5(T).$$

So, let us prove the following theorem.

Theorem 2. *Let conditions $C_1 - C_5$ be satisfied and the inequality*

$$B(T)(A(T) + 2)^2 < 1 \quad (3.22)$$

holds. Then problem (1.1)–(1.3), (1.6), (1.7) has a unique solution in the ball $K = K_R(\|z\|_{E_T^3} \leq R \leq A(T) + 2)$ of the space E_T^3 .

Remark 1. Inequality (3.22) is satisfied for sufficiently small values of T .

Proof. Let us denote $z = [u(x, t), a(t)]^T$ and rewrite the system of equations (3.9) and (3.11) in the following operator equation

$$z = \Phi z, \quad (3.23)$$

where $\Phi = [\varphi_1, \varphi_2]^T$ and $\varphi_1(z), \varphi_2(z)$ defined by the right sides of (3.9) and (3.11), respectively.

Consider the operator $\Phi(u, a)$ in the ball $K = K_R$ of the space E_T^3 . Similar to (3.21) we obtain that for any $z_1, z_2, z \in K_R$ the following estimates hold:

$$\|\Phi z\|_{E_T^3} \leq A(T) + B(T) \|a(t)\|_{C[0,T]} \|u(x, t)\|_{B_{2,T}^3} \leq A(T) + B(T)(A(T) + 2)^2, \quad (3.24)$$

$$\|\Phi z_1 - \Phi z_2\|_{E_T^3} \leq B(T)R(\|a_1(t) - a_2(t)\|_{C[0,T]} + \|u_1(x, t) - u_2(x, t)\|_{B_{2,T}^3}). \quad (3.25)$$

Then taking into account (3.22) in (3.24) and (3.25), it follows that the operator Φ acts in the ball $K = K_R$ and is contractive. Therefore, in the ball $K = K_R$ the operator Φ has a unique fixed point $\{z\} = \{u, a\}$ that is a unique solution of (3.23) in the ball $K = K_R$; i.e. it is a unique solution of system (3.9), (3.11) in the ball $K = K_R$. $\square$

Thus, we obtain that the function $u(x, t)$ as an element of the space $B_{2,T}^3$ is continuous and has continuous derivatives $u_x(x, t)$ and $u_{xx}(x, t)$ in D_T .

Analogous to [14] one can show that the function $u_{tt}(x, t)$ is continuous in the region D_T .

It is easy to verify that Eq.(1.1) and conditions (1.2), (1.3), (1.6), and (1.7) are satisfied in the ordinary sense. Consequently, $\{u(x, t), a(t)\}$ is a solution of problem (1.1)–(1.3), (1.6), (1.7) and by Remark 3.1 this solution is unique in the ball $K = K_R$. The theorem is proved.

From Theorem 1 and Theorem 2 immediately imply that the original problem (1.1)–(1.5) has a unique classical solution.

Theorem 3. *Suppose that all the conditions of Theorem 2 are satisfied and*

$$\int_0^1 f(x, t) dx = 0, \quad 0 \leq t \leq T, \quad \int_0^1 \varphi(x) dx = 0, \quad \int_0^1 \psi(x) dx = 0,$$

$$h(0) + \delta_1 h(T) = \varphi\left(\frac{1}{2}\right), \quad h'(0) + \delta_2 h'(T) = \psi\left(\frac{1}{2}\right),$$

$$\frac{(1 + 2\delta_1 + 3\delta_2 + \delta_1\delta_2)T^2(A(T) + 2)}{2(1 + \delta_1)(1 + \delta_2)} < 1.$$

Then problem (1.1)–(1.5) has a unique classical solution in the ball $K = K_R(\|z\|_{E_T^3} \leq R \leq A(T) + 2)$ of the space E_T^3 .

4. CONCLUSION

In the work, the classical solvability of a nonlinear inverse boundary value problem for a second-order hyperbolic equation with nonlocal conditions was studied. The considered problem was reduced to an auxiliary inverse boundary value problem in a certain sense and its equivalence to the original problem is shown. Then using the Fourier method and contraction mappings principle, the existence and uniqueness theorem for auxiliary problem is proved. Further, on the basis of the equivalency of these problems, the existence and uniqueness theorem for the classical solution of the original inverse coefficient problem is established for the smaller value of time.

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ON FEJÉR-HERMITE-HADAMARD INEQUALITIES FOR FUNCTIONS INVOLVING FRACTIONAL INTEGRALS

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Abstract. In this paper, some new estimations of Hermite-Hadamard-Fejér type inequalities for higher order differentiable preinvex functions are established via fractional integral operators. The main motivation point of the study is that our findings, which include generalized preinvex functions for Hermite-Hadamard-Fejér type inequalities by means of fractional integral operators, are the rare results in the literature. The coincidence of the special cases of the our main theorems with the earlier works in the literature is also demonstrated as a verification of our results.

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1. INTRODUCTION

Many researchers have concentrated on the concept of convexity, which has an important place in mathematics as well as in many other disciplines with its applications and features. We will start by recalling two basic inequalities built on the concept of convexity, which is the task of establishing the strong link between inequality theory and convex analysis. The Hermite-Hadamard inequality, which produces bounds for the Cauchy mean value of a convex function, and the Hermite-Hadamard-Fejér inequality, which is a weighted version of this inequality, have very useful and effective applications in mathematical analysis, engineering, statistics, physics and numerical integration.

Assume that g is a convex mapping such that $g : V \subseteq \mathbb{R} \rightarrow \mathbb{R}$ and $\ell_1, \ell_2 \in V$ with $\ell_1 < \ell_2$, then the inequality

$$g\left(\frac{\ell_1 + \ell_2}{2}\right) \leq \frac{1}{\ell_2 - \ell_1} \int_{\ell_1}^{\ell_2} g(x) dx \leq \frac{g(\ell_1) + g(\ell_2)}{2}. \quad (1.1)$$

is known as the celebrated **Hermite-Hadamard inequality** in the literature.

In [12], a motivated generalization of inequality (1.1) is established by Fejér as follow:

$$g\left(\frac{\ell_1 + \ell_2}{2}\right) \int_{\ell_1}^{\ell_2} f(x) dx \leq \int_{\ell_1}^{\ell_2} f(x) g(x) dx \leq \frac{g(\ell_1) + g(\ell_2)}{2} \int_{\ell_1}^{\ell_2} f(x) dx, \quad (1.2)$$

holds, where $f: [\ell_1; \ell_2] \rightarrow \mathbb{R}$ is a nonnegative, integrable function and it is symmetric about $x = \frac{\ell_1 + \ell_2}{2}$.

Although fractional analysis is historically as old as classical analysis, it has not only rapidly increased its popularity in recent years, but also the properties and applications of new concepts that have been gained to the literature by fractional analysis have made a breakthrough especially in applied sciences. Fractional analysis makes it attractive for many researchers, with its advantages and perfect compatibility with classical analysis for some concepts whose deficiencies are discussed according to classical analysis. Defining fractional derivatives and integral operators, revealing their properties, using them in modeling and solving equations dealing with various natural phenomena, memory effect caused by locality and singularity of kernel structures, and effective results obtained in applications are the factors that carry the fractional analysis forward. To provide further on fractional analysis, see the papers [1–4, 6–11, 13, 14, 18, 21, 23–25, 27]. Now let's get to know one of the cornerstones of fractional analysis, the Riemann-Liouville fractional integral integrals, and the fractional version of the Hadamard inequality that includes these famous operators.

Theorem 1. (See [22]) Let $g: [\ell_1, \ell_2] \subseteq \mathbb{R} \rightarrow \mathbb{R}$ with $0 \leq \ell_1 < \ell_2$ and $g \in L[\ell_1, \ell_2]$. If g is positive and convex function on $[\ell_1, \ell_2]$, then the following inequalities hold:

$$g\left(\frac{\ell_1 + \ell_2}{2}\right) \leq \frac{\Gamma(\alpha + 1)}{2(\ell_2 - \ell_1)^\alpha} [J_{\ell_2}^\alpha g(\ell_1) + J_{\ell_1}^\alpha g(\ell_2)] \leq \frac{g(\ell_1) + g(\ell_2)}{2}.$$

with $\beta > 0$.

Here, the left and right-sided Riemann-Liouville fractional integrals of the order $\beta \in \mathbb{R}^+$ denoted by the symbols $J_{\ell_1}^\beta$ and $J_{\ell_2}^\beta$ that are defined in [15]

$$J_{\ell_1}^\beta g(x) = \frac{1}{\Gamma(\beta)} \int_{\ell_1}^x (x-t)^{\beta-1} g(t) dt, \quad 0 \leq \ell_1 < x \leq \ell_2$$

and

$$J_{\ell_2}^\beta g(x) = \frac{1}{\Gamma(\beta)} \int_x^{\ell_2} (t-x)^{\beta-1} g(t) dt, \quad 0 \leq \ell_1 \leq x < \ell_2.$$

If we choose $\beta = 1$, the fractional integral becomes the classical integral.

Researchers who have worked on the concept of convexity have introduced many new types of convex functions, made definitions of convexity in multidimensional spaces, and contributed to the spread of the concept to a wide spectrum from operator

convexity to convexity defined in n -dimensional space. As a result of these efforts, many new classes of functions, their properties and countless inequalities built on them have been revealed. In this context, we will continue with the concept of preinvexity, which is a generalization of convexity. But first, let's remind the definition of invex sets.

Definition 1. (See [5]) A set $V \subseteq \mathbb{R}$ is invex with respect to the map $\mathfrak{S} : V \times V \rightarrow \mathbb{R}$ if for every $\ell_1, \ell_2 \in V$ and $t \in [0, 1]$, $\ell_2 + t\mathfrak{S}(\ell_1, \ell_2) \in V$. The invex set V is also called an $\mathfrak{S}$ -connected set. Every convex set is an invex set but its converse is not true.

Definition 2. (See [29]) Let $V \subseteq \mathbb{R}$ be an invex set and a function $g : V \rightarrow \mathbb{R}$ is said to be preinvex w.r.t. $\mathfrak{S}$ if

$$g(\ell_2 + t\mathfrak{S}(\ell_1, \ell_2)) \leq tg(\ell_1) + (1-t)g(\ell_2).$$

$\forall \ell_1, \ell_2 \in V$ and $t \in [0, 1]$.

If $\mathfrak{S}(\ell_1, \ell_2) = \ell_1 - \ell_2$, then in classical sense, the preinvex functions coincide convex functions. Recently, the following identity has been proved by Sikander et. al in [20] for $\check{n}$ -times differentiable preinvex functions.

Lemma 1. Let $V \subseteq [0, \infty)$ be an open invex subset with respect to $\mathfrak{S} : V \times V \rightarrow \mathbb{R}$. Suppose $g : V \rightarrow \mathbb{R}$ is a function such that $g^{(\check{n})}$ exists on V and $g^{(\check{n})}$ is integrable on $[\ell_1, \ell_1 + \mathfrak{S}(\ell_2, \ell_1)]$ for $\check{n} \in \mathbb{N}, \check{n} \geq 1$, then for every $\ell_1, \ell_2 \in V$ with $\mathfrak{S}(\ell_2, \ell_1) > 0$, the following equality holds:

$$\begin{aligned} & \frac{g(\ell_1) + g(\ell_1 + \mathfrak{S}(\ell_2, \ell_1))}{2} \\ & - \frac{\Gamma(\beta + 1)}{2(\mathfrak{S}(\ell_2, \ell_1))^\beta} \left[J_{\ell_1^+}^\beta g(\ell_1 + \mathfrak{S}(\ell_2, \ell_1)) + J_{(\ell_1 + \mathfrak{S}(\ell_2, \ell_1))^-}^\beta g(\ell_1) \right] \\ & = \sum_{\kappa=1}^{\check{n}-1} \frac{\Gamma(\beta + 1)(\mathfrak{S}(\ell_2, \ell_1))^\kappa}{2\Gamma(\beta + \kappa + 1)} \left[(-1)^{\kappa-1} g^{(\kappa)}(\ell_1 + \mathfrak{S}(\ell_2, \ell_1)) - g^{(\kappa)}(\ell_1) \right] \\ & - \frac{(\mathfrak{S}(\ell_2, \ell_1))^{\check{n}} \Gamma(\beta + 1)}{2\Gamma(\beta + \check{n})} \\ & \times \int_0^1 \left[(1-t)^{\beta+\check{n}-1} + (-1)^{\check{n}} t^{\beta+\check{n}-1} \right] g^{(\check{n})}(\ell_1 + t\mathfrak{S}(\ell_2, \ell_1)) dt, \end{aligned}$$

for $\beta > 0$ and $\check{n} \geq 1$

Similar results can be found in [16, 17, 26, 28, 30].

The main purpose of this study is to create integral identities that include Riemann-Liouville integral operators that have the potential to produce Hermite-Hadamard-Fejér type inequalities for $\check{n}$ -times differentiable preinvex functions and to obtain integral inequalities that present new approaches based on these identities.

2. MAIN RESULTS

In the sequel of the paper, we will denote

$$\|f\|_{\infty} = \sup_{x \in [\ell_1, \ell_1 + \mathfrak{I}(\ell_2, \ell_1)]} |f(x)|,$$

where $f : [\ell_1; \ell_1 + \mathfrak{I}(\ell_2, \ell_1)] \rightarrow \mathbb{R}$ is a continuous function, $g^{(\check{n})}$ is the $\check{n}$ -th derivative of g w.r.t. variable t and $\mathbb{L}[\ell_1, \ell_2]$ is the classes of all real-valued Riemann integrable mappings defined on $[\ell_1, \ell_2]$.

Lemma 2. Suppose that $V \subseteq \mathbb{R}$ is an open invex set and $\mathfrak{I} : V \times V \rightarrow \mathbb{R}$ is a mapping. Assume that $g : V \rightarrow \mathbb{R}$ is a differentiable mapping such that $g^{(\check{n})} \in L[\ell_1, \ell_1 + \mathfrak{I}(\ell_2, \ell_1)]$ where $\mathfrak{I}(\ell_2, \ell_1) > 0$. If $w : [\ell_1, \ell_1 + \mathfrak{I}(\ell_2, \ell_1)] \rightarrow [0, \infty)$ is an integrable mapping, then we have the following equality:

$$\begin{aligned} & \sum_{\kappa=0}^{\check{n}-1} \frac{g^{(\kappa)}(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1))}{2^{\kappa}\mathfrak{I}(\ell_2, \ell_1)^{\beta+\check{n}-\kappa}} \\ & \quad \times \left[(-1)^{\check{n}-\kappa-1} J_{(\ell_1+\frac{1}{2}\mathfrak{I}(\ell_2, \ell_1))^-}^{\beta} f(\ell_1) + (-1)^{\check{n}+1} J_{(\ell_1+\frac{1}{2}\mathfrak{I}(\ell_2, \ell_1))^+}^{\beta} f(\ell_1 + \mathfrak{I}(\ell_2, \ell_1)) \right] \\ & \quad + (-1)^{\check{n}} \frac{1}{(\mathfrak{I}(\ell_2, \ell_1))^{\beta+\check{n}}} \\ & \quad \times \left[J_{(\ell_1+\frac{1}{2}\mathfrak{I}(\ell_2, \ell_1))^-}^{\beta} (gf)(\ell_1) + J_{(\ell_1+\frac{1}{2}\mathfrak{I}(\ell_2, \ell_1))^+}^{\beta} (gf)(\ell_1 + \mathfrak{I}(\ell_2, \ell_1)) \right] \\ & = \frac{1}{\Gamma(\beta)} \int_0^1 w(t) g^{(\check{n})}(\ell_1 + t\mathfrak{I}(\ell_2, \ell_1)) dt, \end{aligned} \quad (2.1)$$

for $\forall \ell_1, \ell_2 \in V$, where

$$w(t) = \begin{cases} \underbrace{\int_0^t \int_0^t \dots \int_0^t}_{\check{n}-\text{integrals}} u^{\beta-1} f(\ell_1 + u\mathfrak{I}(\ell_2, \ell_1)) (du)^{\check{n}}, & t \in [0, \frac{1}{2}). \\ \underbrace{\int_1^t \int_1^t \dots \int_1^t}_{\check{n}-\text{integrals}} (1-u)^{\beta-1} f(\ell_1 + u\mathfrak{I}(\ell_2, \ell_1)) (du)^{\check{n}}, & t \in [\frac{1}{2}, 1]. \end{cases}$$

Proof. Consider

$$\begin{aligned} & \int_0^1 w(t) g^{(\check{n})}(\ell_1 + t\mathfrak{I}(\ell_2, \ell_1)) dt \\ & = \int_0^{\frac{1}{2}} \left(\underbrace{\int_0^t \int_0^t \dots \int_0^t}_{\check{n}-\text{integrals}} u^{\beta-1} f(\ell_1 + u\mathfrak{I}(\ell_2, \ell_1)) (du)^{\check{n}} \right) \\ & \quad \times g^{(\check{n})}(\ell_1 + t\mathfrak{I}(\ell_2, \ell_1)) dt \end{aligned}$$

$$\begin{aligned}
& + \int_{\frac{1}{2}}^1 \left(\underbrace{\int_1^t \int_1^t \dots \int_1^t}_{\check{n}-\text{integrals}} (1-u)^{\mathfrak{B}-1} f(\ell_1 + u\mathfrak{I}(\ell_2, \ell_1)) (du)^{\check{n}} \right) \\
& \times g^{(\check{n})}(\ell_1 + t\mathfrak{I}(\ell_2, \ell_1)) dt = I_1 + I_2
\end{aligned}$$

From the first integral, we have

$$\begin{aligned}
I_1 &= \int_0^{\frac{1}{2}} \left(\underbrace{\int_0^t \int_0^t \dots \int_0^t}_{\check{n}-\text{integrals}} u^{\mathfrak{B}-1} f(\ell_1 + u\mathfrak{I}(\ell_2, \ell_1)) (du)^{\check{n}} \right) \\
& \times g^{(\check{n})}(\ell_1 + t\mathfrak{I}(\ell_2, \ell_1)) dt \\
&= \frac{1}{\mathfrak{I}(\ell_2, \ell_1)} \left(\underbrace{\int_0^t \int_0^t \dots \int_0^t}_{\check{n}-1 \text{ integrals}} u^{\mathfrak{B}-1} f(\ell_1 + u\mathfrak{I}(\ell_2, \ell_1)) (du)^{\check{n}-1} \right) \\
& \times g^{(\check{n}-1)}(\ell_1 + t\mathfrak{I}(\ell_2, \ell_1)) \Big|_0^{\frac{1}{2}} \\
& - \frac{1}{\mathfrak{I}(\ell_2, \ell_1)} \int_0^{\frac{1}{2}} \left(\underbrace{\int_0^t \int_0^t \dots \int_0^t}_{\check{n}-1 \text{ integrals}} u^{\mathfrak{B}-1} f(\ell_1 + u\mathfrak{I}(\ell_2, \ell_1)) (du)^{\check{n}} \right) \\
& \times g^{(\check{n}-1)}(\ell_1 + t\mathfrak{I}(\ell_2, \ell_1)) dt \\
&= \frac{g^{(\check{n}-1)}(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1))}{\mathfrak{I}(\ell_2, \ell_1)} \\
& \times \left(\underbrace{\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \dots \int_0^{\frac{1}{2}}}_{\check{n} \text{ integrals}} t^{\mathfrak{B}-1} f(\ell_1 + t\mathfrak{I}(\ell_2, \ell_1)) (dt)^{\check{n}} \right) \\
& - \frac{1}{\mathfrak{I}(\ell_2, \ell_1)} \int_0^{\frac{1}{2}} \left(\underbrace{\int_0^t \int_0^t \dots \int_0^t}_{\check{n}-1 \text{ integrals}} u^{\mathfrak{B}-1} f(\ell_1 + u\mathfrak{I}(\ell_2, \ell_1)) (du)^{\check{n}-1} \right) \\
& \times g^{(\check{n}-1)}(\ell_1 + t\mathfrak{I}(\ell_2, \ell_1)) dt.
\end{aligned}$$

On generalizing the result, we have

$$\begin{aligned}
I_1 = & \frac{g^{(\check{n}-1)}(\ell_1 + \frac{1}{2}\mathfrak{S}(\ell_2, \ell_1))}{\mathfrak{S}(\ell_2, \ell_1)} \left(\underbrace{\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \dots \int_0^{\frac{1}{2}}}_{\check{n}-\text{integrals}} t^{\mathfrak{B}-1} f(\ell_1 + t\mathfrak{S}(\ell_2, \ell_1)) (dt)^{\check{n}} \right) \\
& - \frac{g^{(\check{n}-2)}(\ell_1 + \frac{1}{2}\mathfrak{S}(\ell_2, \ell_1))}{(\mathfrak{S}(\ell_2, \ell_1))^2} \left(\underbrace{\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \dots \int_0^{\frac{1}{2}}}_{\check{n}-1 \text{ integrals}} t^{\mathfrak{B}-1} f(\ell_1 + t\mathfrak{S}(\ell_2, \ell_1)) (dt)^{\check{n}-1} \right) \\
& \vdots \\
& + (-1)^{\check{n}-2} \frac{g'(\ell_1 + \frac{1}{2}\mathfrak{S}(\ell_2, \ell_1))}{(\mathfrak{S}(\ell_2, \ell_1))^{\check{n}-1}} \left(\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} t^{\mathfrak{B}-1} f(\ell_1 + t\mathfrak{S}(\ell_2, \ell_1)) (dt)^2 \right) \\
& + (-1)^{\check{n}-1} \frac{g(\ell_1 + \frac{1}{2}\mathfrak{S}(\ell_2, \ell_1))}{(\mathfrak{S}(\ell_2, \ell_1))^{\check{n}}} \left(\int_0^{\frac{1}{2}} t^{\mathfrak{B}-1} f(\ell_1 + t\mathfrak{S}(\ell_2, \ell_1)) dt \right) \\
& + (-1)^{\check{n}} \frac{1}{(\mathfrak{S}(\ell_2, \ell_1))^{\check{n}}} \left(\int_0^{\frac{1}{2}} t^{\mathfrak{B}-1} g(\ell_1 + t\mathfrak{S}(\ell_2, \ell_1)) f(\ell_1 + t\mathfrak{S}(\ell_2, \ell_1)) dt \right).
\end{aligned}$$

After simplification

$$\begin{aligned}
I_1 = & \frac{g^{(\check{n}-1)}(\ell_1 + \frac{1}{2}\mathfrak{S}(\ell_2, \ell_1))}{2^{\check{n}-1} \mathfrak{S}(\ell_2, \ell_1)} \left(\int_0^{\frac{1}{2}} t^{\mathfrak{B}-1} f(\ell_1 + t\mathfrak{S}(\ell_2, \ell_1)) dt \right) \\
& - \frac{g^{(\check{n}-2)}(\ell_1 + \frac{1}{2}\mathfrak{S}(\ell_2, \ell_1))}{2^{\check{n}-2} (\mathfrak{S}(\ell_2, \ell_1))^2} \left(\int_0^{\frac{1}{2}} t^{\mathfrak{B}-1} f(\ell_1 + t\mathfrak{S}(\ell_2, \ell_1)) dt \right) \\
& \vdots \\
& + (-1)^{\check{n}-2} \frac{g'(\ell_1 + \frac{1}{2}\mathfrak{S}(\ell_2, \ell_1))}{2 (\mathfrak{S}(\ell_2, \ell_1))^{\check{n}-1}} \left(\int_0^{\frac{1}{2}} t^{\mathfrak{B}-1} f(\ell_1 + t\mathfrak{S}(\ell_2, \ell_1)) dt \right) \\
& + (-1)^{\check{n}-1} \frac{g(\ell_1 + \frac{1}{2}\mathfrak{S}(\ell_2, \ell_1))}{(\mathfrak{S}(\ell_2, \ell_1))^{\check{n}}} \left(\int_0^{\frac{1}{2}} t^{\mathfrak{B}-1} f(\ell_1 + t\mathfrak{S}(\ell_2, \ell_1)) dt \right) \\
& + (-1)^{\check{n}} \frac{1}{(\mathfrak{S}(\ell_2, \ell_1))^{\check{n}}} \left(\int_0^{\frac{1}{2}} t^{\mathfrak{B}-1} g(\ell_1 + t\mathfrak{S}(\ell_2, \ell_1)) f(\ell_1 + t\mathfrak{S}(\ell_2, \ell_1)) dt \right).
\end{aligned}$$

On substituting $x = \ell_1 + t\mathfrak{I}(\ell_2, \ell_1)$, we get

$$\begin{aligned}
 I_1 &= \frac{g^{(\check{n}-1)}(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1))}{2^{\check{n}-1}\mathfrak{I}(\ell_2, \ell_1)^{\mathfrak{B}+1}} \int_{\ell_1}^{\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1)} (x - \ell_1)^{\mathfrak{B}-1} f(x) dx \\
 &\quad - \frac{g^{(\check{n}-2)}(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1))}{2^{\check{n}-2}(\mathfrak{I}(\ell_2, \ell_1))^{\mathfrak{B}+2}} \int_{\ell_1}^{\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1)} (x - \ell_1)^{\mathfrak{B}-1} f(x) dx \\
 &\quad \vdots \\
 &\quad + (-1)^{\check{n}-2} \frac{g'(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1))}{2(\mathfrak{I}(\ell_2, \ell_1))^{\mathfrak{B}+\check{n}-1}} \int_{\ell_1}^{\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1)} (x - \ell_1)^{\mathfrak{B}-1} f(x) dx \\
 &\quad + (-1)^{\check{n}-1} \frac{g(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1))}{(\mathfrak{I}(\ell_2, \ell_1))^{\mathfrak{B}+\check{n}}} \int_{\ell_1}^{\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1)} (x - \ell_1)^{\mathfrak{B}-1} f(x) dx \\
 &\quad + (-1)^{\check{n}} \frac{1}{(\mathfrak{I}(\ell_2, \ell_1))^{\mathfrak{B}+\check{n}}} \int_{\ell_1}^{\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1)} (x - \ell_1)^{\mathfrak{B}-1} g(x) f(x) dx.
 \end{aligned}$$

Using the definition of Riemann-Liouville fractional integrals, we have

$$\begin{aligned}
 I_1 &= \frac{g^{(\check{n}-1)}(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1))\Gamma(\mathfrak{B})}{2^{\check{n}-1}\mathfrak{I}(\ell_2, \ell_1)^{\mathfrak{B}+1}} J_{(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1))^-}^{\mathfrak{B}} f(\ell_1) \\
 &\quad - \frac{g^{(\check{n}-2)}(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1))\Gamma(\mathfrak{B})}{2^{\check{n}-2}(\mathfrak{I}(\ell_2, \ell_1))^{\mathfrak{B}+2}} J_{(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1))^-}^{\mathfrak{B}} f(\ell_1) \\
 &\quad \vdots \\
 &\quad + (-1)^{\check{n}-2} \frac{g'(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1))\Gamma(\mathfrak{B})}{2(\mathfrak{I}(\ell_2, \ell_1))^{\mathfrak{B}+\check{n}-1}} J_{(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1))^-}^{\mathfrak{B}} f(\ell_1) \\
 &\quad + (-1)^{\check{n}-1} \frac{g(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1))\Gamma(\mathfrak{B})}{(\mathfrak{I}(\ell_2, \ell_1))^{\mathfrak{B}+\check{n}}} J_{(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1))^-}^{\mathfrak{B}} f(\ell_1) \\
 &\quad + (-1)^{\check{n}} \frac{\Gamma(\mathfrak{B})}{(\mathfrak{I}(\ell_2, \ell_1))^{\mathfrak{B}+\check{n}}} J_{(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1))^-}^{\mathfrak{B}} (gf)(\ell_1).
 \end{aligned}$$

After summing the above expressions, we get

$$\begin{aligned}
 I_1 &= \sum_{\kappa=0}^{\check{n}-1} (-1)^{\check{n}-\kappa-1} \frac{g^{(\kappa)}(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1))}{2^{\kappa}\mathfrak{I}(\ell_2, \ell_1)^{\mathfrak{B}+\check{n}-\kappa}} \Gamma(\mathfrak{B}) J_{(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1))^-}^{\mathfrak{B}} f(\ell_1) \\
 &\quad + (-1)^{\check{n}} \frac{1}{(\mathfrak{I}(\ell_2, \ell_1))^{\mathfrak{B}+\check{n}}} \Gamma(\mathfrak{B}) J_{(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1))^-}^{\mathfrak{B}} (gf)(\ell_1). \quad (2.2)
 \end{aligned}$$

By a similar argument for the second integral, we have

$$\begin{aligned} I_2 = & (-1)^{\check{n}+1} \sum_{\kappa=0}^{\check{n}-1} \frac{g^{(\kappa)}(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1))}{2^\kappa \mathfrak{I}(\ell_2, \ell_1)^{\beta+\check{n}-\kappa}} \Gamma(\beta) J_{(\ell_1+\frac{1}{2}\mathfrak{I}(\ell_2, \ell_1))+}^\beta f(\ell_1 + \mathfrak{I}(\ell_2, \ell_1)) \\ & + (-1)^{\check{n}} \frac{1}{(\mathfrak{I}(\ell_2, \ell_1))^{\beta+\check{n}}} \Gamma(\beta) J_{(\ell_1+\frac{1}{2}\mathfrak{I}(\ell_2, \ell_1))+}^\beta (gf)(\ell_1 + \mathfrak{I}(\ell_2, \ell_1)). \end{aligned} \quad (2.3)$$

Upon adding (2.1) and (2.2), we get the required result. Thus, the lemma is proved. $\square$

Remark 1. If we set $\check{n} = 1$, we get Lemma 1 of [19].

$$\begin{aligned} & \frac{g(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1))}{\mathfrak{I}(\ell_2, \ell_1)^{\beta+1}} \\ & \times \left[J_{(\ell_1+\frac{1}{2}\mathfrak{I}(\ell_2, \ell_1))-}^\beta f(\ell_1) + J_{(\ell_1+\frac{1}{2}\mathfrak{I}(\ell_2, \ell_1))+}^\beta f(\ell_1 + \mathfrak{I}(\ell_2, \ell_1)) \right] \\ & - \frac{1}{(\mathfrak{I}(\ell_2, \ell_1))^{\beta+1}} \\ & \times \left[J_{(\ell_1+\frac{1}{2}\mathfrak{I}(\ell_2, \ell_1))-}^\beta (gf)(\ell_1) + J_{(\ell_1+\frac{1}{2}\mathfrak{I}(\ell_2, \ell_1))+}^\beta (gf)(\ell_1 + \mathfrak{I}(\ell_2, \ell_1)) \right] \\ & = \frac{1}{\Gamma(\beta)} \int_0^1 w(t) g'(\ell_1 + t\mathfrak{I}(\ell_2, \ell_1)) dt, \end{aligned}$$

where

$$w(t) = \begin{cases} \int_0^t u^{\beta-1} f(\ell_1 + u\mathfrak{I}(\ell_2, \ell_1)) du, & t \in [0, \frac{1}{2}) \\ \int_1^t (1-u)^{\beta-1} f(\ell_1 + u\mathfrak{I}(\ell_2, \ell_1)) du, & t \in [\frac{1}{2}, 1] \end{cases}.$$

Lemma 3. If $g : [\ell_1, \ell_1 + \mathfrak{I}(\ell_2, \ell_1)] \rightarrow \mathbb{R}$ is an integrable function which is also symmetric about $\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1)$ with $\ell_1 < \ell_1 + \mathfrak{I}(\ell_2, \ell_1)$, then the following identity holds

$$\begin{aligned} J_{\ell_1+g}^\beta g(\ell_1 + \mathfrak{I}(\ell_2, \ell_1)) &= J_{(\ell_1+\mathfrak{I}(\ell_2, \ell_1))-}^\beta g(\ell_1) \\ &= \frac{1}{2} \left[J_{\ell_1+g}^\beta g(\ell_1 + \mathfrak{I}(\ell_2, \ell_1)) + J_{(\ell_1+\mathfrak{I}(\ell_2, \ell_1))-}^\beta g(\ell_1) \right], \end{aligned} \quad (2.4)$$

where $\beta > 0$

Proof. Since g is symmetric about $\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1)$, we have $g(2\ell_1 + \mathfrak{I}(\ell_2, \ell_1) - x) = g(x)$, for all $x \in [\ell_1, \ell_1 + \mathfrak{I}(\ell_2, \ell_1)]$. Taking into account $2\ell_1 + \mathfrak{I}(\ell_2, \ell_1) - t = x$, we can write

$$\begin{aligned} J_{\ell_1+g}^\beta g(\ell_1 + \mathfrak{I}(\ell_2, \ell_1)) &= \frac{1}{\Gamma(\beta)} \int_{\ell_1}^{\ell_1+\mathfrak{I}(\ell_2, \ell_1)} [(\ell_1 + \mathfrak{I}(\ell_2, \ell_1) - t)^{\beta-1} g(t) dt \\ &= \frac{1}{\Gamma(\beta)} \int_{\ell_1}^{\ell_1+\mathfrak{I}(\ell_2, \ell_1)} (x - \ell_1)^{\beta-1} g(2\ell_1 + \mathfrak{I}(\ell_2, \ell_1) - x) dx \end{aligned}$$

$$= \frac{1}{\Gamma(\beta)} \int_{\ell_1}^{\ell_1 + \mathfrak{I}(\ell_2, \ell_1)} (x - \ell_1)^{\beta-1} g(x) dx = J_{(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))^-}^{\beta} g(\ell_1).$$

The proof is completed. $\square$

Lemma 4. Let $V \subseteq \mathbb{R}$ be an open invex set and $\mathfrak{I} : V \times V \rightarrow \mathbb{R}$ be a mapping. Suppose that $g : V \rightarrow \mathbb{R}$ is a differentiable mapping such that $g^{(\check{n})} \in L[\ell_1, \ell_1 + \mathfrak{I}(\ell_2, \ell_1)]$ where $\mathfrak{I}(\ell_2, \ell_1) > 0$. If $w : [\ell_1, \ell_1 + \mathfrak{I}(\ell_2, \ell_1)] \rightarrow [0, \infty)$ is an integrable mapping, then we have the following equality:

$$\begin{aligned} & \sum_{\kappa=0}^{\check{n}-1} \left[\frac{(-1)^{\check{n}+1} g^{(\kappa)}(\ell_1) + (-1)^{\check{n}-\kappa-1} g^{(\kappa)}(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))}{2(\mathfrak{I}(\ell_2, \ell_1))^{\beta+\check{n}-\kappa}} \right] \\ & \quad \times \left[J_{(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))^-}^{\beta} f(\ell_1) + J_{\ell_1+}^{\beta} f(\ell_1 + \mathfrak{I}(\ell_2, \ell_1)) \right] \\ & \quad + (-1)^{\check{n}} \left[\frac{J_{(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))^-}^{\beta} (gf)(\ell_1) + J_{\ell_1+}^{\beta} (gf)(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))}{(\mathfrak{I}(\ell_2, \ell_1))^{\beta+\check{n}}} \right] \\ & = \frac{1}{\Gamma(\beta)} \int_0^1 w(t) g^{(\check{n})}(\ell_1 + t\mathfrak{I}(\ell_2, \ell_1)) dt, \end{aligned} \quad (2.5)$$

for $\forall \ell_1, \ell_2 \in V$, where

$$\begin{aligned} w(t) &= \underbrace{\int_0^t \int_0^t \dots \int_0^t}_{\check{n} \text{ integrals}} u^{\beta-1} f(\ell_1 + u\mathfrak{I}(\ell_2, \ell_1)) (du)^{\check{n}} \\ & \quad + \underbrace{\int_1^t \int_1^t \dots \int_1^t}_{\check{n} \text{ integrals}} (1-u)^{\beta-1} f(\ell_1 + u\mathfrak{I}(\ell_2, \ell_1)) (du)^{\check{n}}, \quad t \in [0, 1]. \end{aligned}$$

Proof. We will start with

$$\begin{aligned} & \int_0^1 w(t) g^{(\check{n})}(\ell_1 + t\mathfrak{I}(\ell_2, \ell_1)) dt \\ & = \int_0^1 \left(\underbrace{\int_0^t \int_0^t \dots \int_0^t}_{\check{n} \text{ integrals}} u^{\beta-1} f(\ell_1 + u\mathfrak{I}(\ell_2, \ell_1)) (du)^{\check{n}} \right) \\ & \quad \times g^{(\check{n})}(\ell_1 + t\mathfrak{I}(\ell_2, \ell_1)) dt \\ & \quad + \int_0^1 \left(\underbrace{\int_1^t \int_1^t \dots \int_1^t}_{\check{n} \text{ integrals}} (1-u)^{\beta-1} f(\ell_1 + u\mathfrak{I}(\ell_2, \ell_1)) (du)^{\check{n}} \right) \end{aligned}$$

$$\times g^{(\check{n})}(\ell_1 + t\mathfrak{S}(\ell_2, \ell_1))dt = I_1 + I_2.$$

By evaluating the first integral, we have

$$\begin{aligned} I_1 &= \int_0^1 \left(\underbrace{\int_0^t \int_0^t \dots \int_0^t}_{\check{n} \text{ integrals}} u^{\mathfrak{B}-1} f(\ell_1 + u\mathfrak{S}(\ell_2, \ell_1)) (du)^{\check{n}} \right) \\ &\quad \times g^{(\check{n})}(\ell_1 + t\mathfrak{S}(\ell_2, \ell_1))dt \\ &= \frac{1}{\mathfrak{S}(\ell_2, \ell_1)} \left(\underbrace{\int_0^t \int_0^t \dots \int_0^t}_{\check{n}-1 \text{ integrals}} u^{\mathfrak{B}-1} f(\ell_1 + u\mathfrak{S}(\ell_2, \ell_1)) (du)^{\check{n}} \right) \\ &\quad \times g^{(\check{n}-1)}(\ell_1 + t\mathfrak{S}(\ell_2, \ell_1)) \Big|_0^1 \\ &\quad - \frac{1}{\mathfrak{S}(\ell_2, \ell_1)} \int_0^1 \left(\underbrace{\int_0^t \int_0^t \dots \int_0^t}_{\check{n}-1 \text{ integrals}} u^{\mathfrak{B}-1} f(\ell_1 + u\mathfrak{S}(\ell_2, \ell_1)) (du)^{\check{n}} \right) \\ &\quad \times g^{(\check{n}-1)}(\ell_1 + t\mathfrak{S}(\ell_2, \ell_1))dt. \end{aligned}$$

Namely,

$$\begin{aligned} I_1 &= \frac{g^{(\check{n}-1)}(\ell_1 + \mathfrak{S}(\ell_2, \ell_1))}{\mathfrak{S}(\ell_2, \ell_1)} \left(\underbrace{\int_0^1 \int_0^1 \dots \int_0^1}_{\check{n} \text{ integrals}} t^{\mathfrak{B}-1} f(\ell_1 + t\mathfrak{S}(\ell_2, \ell_1)) (dt)^{\check{n}} \right) \\ &\quad - \frac{1}{\mathfrak{S}(\ell_2, \ell_1)} \int_0^1 \left(\underbrace{\int_0^t \int_0^t \dots \int_0^t}_{\check{n}-1 \text{ integrals}} u^{\mathfrak{B}-1} f(\ell_1 + u\mathfrak{S}(\ell_2, \ell_1)) (du)^{\check{n}-1} \right) \\ &\quad \times g^{(\check{n}-1)}(\ell_1 + t\mathfrak{S}(\ell_2, \ell_1))dt. \end{aligned}$$

On generalizing the result, we have

$$I_1 = \frac{g^{(\check{n}-1)}(\ell_1 + (\ell_2, \ell_1))}{\mathfrak{S}(\ell_2, \ell_1)} \left(\underbrace{\int_0^1 \int_0^1 \dots \int_0^1}_{\check{n} \text{ integrals}} t^{\mathfrak{B}-1} f(\ell_1 + t\mathfrak{S}(\ell_2, \ell_1)) (dt)^{\check{n}} \right)$$

$$\begin{aligned}
& - \frac{g^{(\check{n}-2)}(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))}{(\mathfrak{I}(\ell_2, \ell_1))^2} \left(\underbrace{\int_0^1 \int_0^1 \dots \int_0^1}_{\check{n}-1 \text{ integrals}} t^{\mathfrak{B}-1} f(\ell_1 + t\mathfrak{I}(\ell_2, \ell_1)) (dt)^{\check{n}-1} \right) \\
& + \frac{g^{(\check{n}-3)}(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))}{(\mathfrak{I}(\ell_2, \ell_1))^3} \left(\underbrace{\int_0^1 \int_0^1 \dots \int_0^1}_{\check{n}-2 \text{ integrals}} t^{\mathfrak{B}-1} f(\ell_1 + t\mathfrak{I}(\ell_2, \ell_1)) (dt)^{\check{n}-2} \right) \\
& \vdots \\
& + (-1)^{\check{n}-2} \frac{g'(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))}{(\mathfrak{I}(\ell_2, \ell_1))^{\check{n}-1}} \left(\int_0^1 \int_0^1 t^{\mathfrak{B}-1} f(\ell_1 + t\mathfrak{I}(\ell_2, \ell_1)) (dt)^2 \right) \\
& + (-1)^{\check{n}-1} \frac{g(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))}{(\mathfrak{I}(\ell_2, \ell_1))^{\check{n}}} \left(\int_0^1 t^{\mathfrak{B}-1} f(\ell_1 + t\mathfrak{I}(\ell_2, \ell_1)) dt \right) \\
& + (-1)^{\check{n}} \frac{1}{(\mathfrak{I}(\ell_2, \ell_1))^{\check{n}}} \left(\int_0^1 t^{\mathfrak{B}-1} g(\ell_1 + t\mathfrak{I}(\ell_2, \ell_1)) f(\ell_1 + t\mathfrak{I}(\ell_2, \ell_1)) dt \right).
\end{aligned}$$

After simplifications, we obtain

$$\begin{aligned}
I_1 &= \frac{g^{(\check{n}-1)}(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))}{\mathfrak{I}(\ell_2, \ell_1)} \left(\int_0^1 t^{\mathfrak{B}-1} f(\ell_1 + t\mathfrak{I}(\ell_2, \ell_1)) dt \right) \\
& - \frac{g^{(\check{n}-2)}(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))}{(\mathfrak{I}(\ell_2, \ell_1))^2} \left(\int_0^1 t^{\mathfrak{B}-1} f(\ell_1 + t\mathfrak{I}(\ell_2, \ell_1)) dt \right) \\
& + \frac{g^{(\check{n}-3)}(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))}{(\mathfrak{I}(\ell_2, \ell_1))^3} \left(\int_0^1 t^{\mathfrak{B}-1} f(\ell_1 + t\mathfrak{I}(\ell_2, \ell_1)) dt \right) \\
& \vdots \\
& + (-1)^{\check{n}-2} \frac{g'(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))}{(\mathfrak{I}(\ell_2, \ell_1))^{\check{n}-1}} \left(\int_0^1 t^{\mathfrak{B}-1} f(\ell_1 + t\mathfrak{I}(\ell_2, \ell_1)) dt \right) \\
& + (-1)^{\check{n}-1} \frac{g(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))}{(\mathfrak{I}(\ell_2, \ell_1))^{\check{n}}} \left(\int_0^1 t^{\mathfrak{B}-1} f(\ell_1 + t\mathfrak{I}(\ell_2, \ell_1)) dt \right) \\
& + (-1)^{\check{n}} \frac{1}{(\mathfrak{I}(\ell_2, \ell_1))^{\check{n}}} \left(\int_0^1 t^{\mathfrak{B}-1} g(\ell_1 + t\mathfrak{I}(\ell_2, \ell_1)) f(\ell_1 + t\mathfrak{I}(\ell_2, \ell_1)) dt \right).
\end{aligned}$$

After substituting $x = \ell_1 + t\mathfrak{I}(\ell_2, \ell_1)$, we have

$$I_1 = \frac{g^{(\check{n}-1)}(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))}{\mathfrak{I}(\ell_2, \ell_1)^{\mathfrak{B}+1}} \int_{\ell_1}^{\ell_1 + \mathfrak{I}(\ell_2, \ell_1)} (x - \ell_1)^{\mathfrak{B}-1} f(x) dx$$

$$\begin{aligned}
& - \frac{g^{(\check{n}-2)}(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))}{(\mathfrak{I}(\ell_2, \ell_1))^{\mathfrak{B}+2}} \int_{\ell_1}^{\ell_1 + \mathfrak{I}(\ell_2, \ell_1)} (x - \ell_1)^{\mathfrak{B}-1} f(x) dx \\
& + \frac{g^{(\check{n}-3)}(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))}{(\mathfrak{I}(\ell_2, \ell_1))^{\mathfrak{B}+3}} \int_{\ell_1}^{\ell_1 + \mathfrak{I}(\ell_2, \ell_1)} (x - \ell_1)^{\mathfrak{B}-1} f(x) dx \\
& \vdots \\
& + (-1)^{\check{n}-2} \frac{g'(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))}{(\mathfrak{I}(\ell_2, \ell_1))^{\mathfrak{B}+\check{n}-1}} \int_{\ell_1}^{\ell_1 + \mathfrak{I}(\ell_2, \ell_1)} (x - \ell_1)^{\mathfrak{B}-1} f(x) dx \\
& + (-1)^{\check{n}-1} \frac{g(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))}{(\mathfrak{I}(\ell_2, \ell_1))^{\mathfrak{B}+\check{n}}} \int_{\ell_1}^{\ell_1 + \mathfrak{I}(\ell_2, \ell_1)} (x - \ell_1)^{\mathfrak{B}-1} f(x) dx \\
& + (-1)^{\check{n}} \frac{1}{(\mathfrak{I}(\ell_2, \ell_1))^{\mathfrak{B}+\check{n}}} \int_{\ell_1}^{\ell_1 + \mathfrak{I}(\ell_2, \ell_1)} (x - \ell_1)^{\mathfrak{B}-1} g(x) f(x) dx.
\end{aligned}$$

Using the definition of Riemann-Liouville fractional integrals, we can easily see that

$$\begin{aligned}
I_1 &= \frac{g^{(\check{n}-1)}(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))\Gamma(\mathfrak{B})}{\mathfrak{I}(\ell_2, \ell_1)^{\mathfrak{B}+1}} J_{(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))}^{\mathfrak{B}} f(\ell_1) \\
& - \frac{g^{(\check{n}-2)}(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))\Gamma(\mathfrak{B})}{(\mathfrak{I}(\ell_2, \ell_1))^{\mathfrak{B}+2}} J_{(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))}^{\mathfrak{B}} f(\ell_1) \\
& + \frac{g^{(\check{n}-3)}(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))\Gamma(\mathfrak{B})}{(\mathfrak{I}(\ell_2, \ell_1))^{\mathfrak{B}+3}} J_{(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))}^{\mathfrak{B}} f(\ell_1) \\
& \vdots \\
& + (-1)^{\check{n}-2} \frac{g'(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))\Gamma(\mathfrak{B})}{(\mathfrak{I}(\ell_2, \ell_1))^{\mathfrak{B}+\check{n}-1}} J_{(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))}^{\mathfrak{B}} f(\ell_1) \\
& + (-1)^{\check{n}-1} \frac{g(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))\Gamma(\mathfrak{B})}{(\mathfrak{I}(\ell_2, \ell_1))^{\mathfrak{B}+\check{n}}} J_{(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))}^{\mathfrak{B}} f(\ell_1) \\
& + (-1)^{\check{n}} \frac{\Gamma(\mathfrak{B})}{(\mathfrak{I}(\ell_2, \ell_1))^{\mathfrak{B}+\check{n}}} J_{(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))}^{\mathfrak{B}} (gf)(\ell_1).
\end{aligned}$$

After summing the above expressions, we get

$$\begin{aligned}
I_1 &= \sum_{\kappa=0}^{\check{n}-1} (-1)^{\check{n}-\kappa-1} \frac{g^{(\kappa)}(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))}{\mathfrak{I}(\ell_2, \ell_1)^{\mathfrak{B}+\check{n}-\kappa}} \Gamma(\mathfrak{B}) J_{(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))}^{\mathfrak{B}} f(\ell_1) \\
& + (-1)^{\check{n}} \frac{1}{(\mathfrak{I}(\ell_2, \ell_1))^{\mathfrak{B}+\check{n}}} \Gamma(\mathfrak{B}) J_{(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))}^{\mathfrak{B}} (gf)(\ell_1). \tag{2.6}
\end{aligned}$$

By proceeding a similar method for the second integral, we deduce

$$I_2 = (-1)^{\check{n}+1} \sum_{\kappa=0}^{\check{n}-1} \frac{g^{(\kappa)}(\ell_1)}{\mathfrak{S}(\ell_2, \ell_1)^{\beta+\check{n}-\kappa}} \Gamma(\beta) J_{\ell_1+}^{\beta} f(\ell_1 + \mathfrak{S}(\ell_2, \ell_1)) \\ + (-1)^{\check{n}} \frac{1}{(\mathfrak{S}(\ell_2, \ell_1))^{\beta+\check{n}}} \Gamma(\beta) J_{\ell_1+}^{\beta} (gf)(\ell_1 + \mathfrak{S}(\ell_2, \ell_1)). \quad (2.7)$$

Upon adding (2.6) and (2.7) and using lemma 3, we get the required result. The lemma is proved. $\square$

Remark 2. If we set $\check{n} = 1$, the identity reduces to the Lemma 3 of [19].

$$\left[\frac{g(\ell_1) + g(\ell_1 + \mathfrak{S}(\ell_2, \ell_1))}{2(\mathfrak{S}(\ell_2, \ell_1))^{\beta+1}} \right] \\ \times \left[J_{(\ell_1+\mathfrak{S}(\ell_2, \ell_1))^-}^{\beta} f(\ell_1) + J_{\ell_1+}^{\beta} f(\ell_1 + \mathfrak{S}(\ell_2, \ell_1)) \right] \\ - \left[\frac{J_{(\ell_1+\mathfrak{S}(\ell_2, \ell_1))^-}^{\beta} (gf)(\ell_1) + J_{\ell_1+}^{\beta} (gf)(\ell_1 + \mathfrak{S}(\ell_2, \ell_1))}{(\mathfrak{S}(\ell_2, \ell_1))^{\beta+1}} \right] \\ = \frac{1}{\Gamma(\beta)} \int_0^1 w(t) g'(\ell_1 + t\mathfrak{S}(\ell_2, \ell_1)) dt,$$

where

$$w(t) = \int_0^t u^{\beta-1} f(\ell_1 + u\mathfrak{S}(\ell_2, \ell_1)) du \\ + \int_1^t (1-u)^{\beta-1} f(\ell_1 + u\mathfrak{S}(\ell_2, \ell_1)) du, \quad t \in [0, 1].$$

Theorem 2. Let $V \subseteq \mathbb{R}$ be an open invex set and $\mathfrak{S} : V \times V \rightarrow \mathbb{R}^{\beta}$ be a mapping. Suppose that $g : V \rightarrow \mathbb{R}$ is a differentiable mapping such that $g^{(\check{n})} \in L[\ell_1, \ell_1 + \mathfrak{S}(\ell_2, \ell_1)]$ where $\mathfrak{S}(\ell_2, \ell_1) > 0$. If $f : [\ell_1, \ell_1 + \mathfrak{S}(\ell_2, \ell_1)] \rightarrow [0, \infty)$ is an integrable mapping and it is also symmetric w.r.t. $\ell_1 + \frac{1}{2}\mathfrak{S}(\ell_2, \ell_1)$ and $|g^{(\check{n})}|$ is preinvex function on V , then the following inequality holds:

$$\left| \sum_{\kappa=0}^{\check{n}-1} \frac{g^{(\kappa)}(\ell_1 + \frac{1}{2}\mathfrak{S}(\ell_2, \ell_1))}{2^{\kappa}\mathfrak{S}(\ell_2, \ell_1)^{\beta+\check{n}-\kappa}} \right. \\ \times \left[(-1)^{\check{n}-\kappa-1} J_{(\ell_1+\frac{1}{2}\mathfrak{S}(\ell_2, \ell_1))^-}^{\beta} f(\ell_1) + (-1)^{\check{n}+1} J_{(\ell_1+\frac{1}{2}\mathfrak{S}(\ell_2, \ell_1))+}^{\beta} f(\ell_1 + \mathfrak{S}(\ell_2, \ell_1)) \right] \\ + (-1)^{\check{n}} \frac{1}{(\mathfrak{S}(\ell_2, \ell_1))^{\beta+\check{n}}} \\ \times \left[J_{(\ell_1+\frac{1}{2}\mathfrak{S}(\ell_2, \ell_1))^-}^{\beta} (gf)(\ell_1) + J_{(\ell_1+\frac{1}{2}\mathfrak{S}(\ell_2, \ell_1))+}^{\beta} (gf)(\ell_1 + \mathfrak{S}(\ell_2, \ell_1)) \right] \Big|$$

$$\begin{aligned}
&\leq \frac{\|f\|_\infty}{\Gamma(\beta + \check{n} + 1)} \frac{1}{2^{\beta + \check{n}}} \left[\left| g^{(\check{n})}(\ell_1) \right| + \left| g^{(\check{n})}(\ell_2) \right| \right], \quad \text{when } \check{n} \text{ is odd integer} \\
&\leq \frac{\|f\|_\infty}{\Gamma(\beta + \check{n} + 2)} \frac{1}{2^{\beta + \check{n}}} \left[\left| g^{(\check{n})}(\ell_1) \right| - \left| g^{(\check{n})}(\ell_2) \right| \right], \quad \text{when } \check{n} \text{ is even integer}
\end{aligned}$$

for $\forall \ell_1, \ell_2 \in V$.

Proof. Applying modulus on both sides of (2.1), we can write

$$\begin{aligned}
&\left| \sum_{\kappa=0}^{\check{n}-1} \frac{g^{(\kappa)}(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1))}{2^\kappa \mathfrak{I}(\ell_2, \ell_1)^{\beta + \check{n} - \kappa}} \right. \\
&\quad \times \left[(-1)^{\check{n} - \kappa - 1} J_{(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1)) -}^\beta f(\ell_1) + (-1)^{\check{n} + 1} J_{(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1)) +}^\beta f(\ell_1 + \mathfrak{I}(\ell_2, \ell_1)) \right] \\
&\quad + (-1)^{\check{n}} \frac{1}{(\mathfrak{I}(\ell_2, \ell_1))^{\beta + \check{n}}} \left[J_{(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1)) -}^\beta (gf)(\ell_1) \right. \\
&\quad \left. \left. + J_{(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1)) +}^\beta (gf)(\ell_1 + \mathfrak{I}(\ell_2, \ell_1)) \right] \right| \\
&= \frac{1}{\Gamma(\beta)} \left| \int_0^{1/2} w(t) g^{(\check{n})}(\ell_1 + t\mathfrak{I}(\ell_2, \ell_1)) dt + \frac{1}{\Gamma(\beta)} \int_{1/2}^1 w(t) g^{(\check{n})}(\ell_1 + t\mathfrak{I}(\ell_2, \ell_1)) dt \right|.
\end{aligned}$$

From preinvexity of $|g^{(\check{n})}|$ on V and using the fact $\|f\|_\infty = \sup_{x \in [\ell_1, \ell_2]} |f(x)|$, we have

$$\begin{aligned}
&\left| \sum_{\kappa=0}^{\check{n}-1} \frac{g^{(\kappa)}(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1))}{2^\kappa \mathfrak{I}(\ell_2, \ell_1)^{\beta + \check{n} - \kappa}} \right. \\
&\quad \times \left[(-1)^{\check{n} - \kappa - 1} J_{(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1)) -}^\beta f(\ell_1) + (-1)^{\check{n} + 1} J_{(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1)) +}^\beta f(\ell_1 + \mathfrak{I}(\ell_2, \ell_1)) \right] \\
&\quad + (-1)^{\check{n}} \frac{1}{(\mathfrak{I}(\ell_2, \ell_1))^{\beta + \check{n}}} \left[J_{(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1)) -}^\beta (gf)(\ell_1) \right. \\
&\quad \left. \left. + J_{(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1)) +}^\beta (gf)(\ell_1 + \mathfrak{I}(\ell_2, \ell_1)) \right] \right| \\
&\leq \frac{\|f\|_\infty}{\Gamma(\beta)} \int_0^{1/2} \left(\underbrace{\int_0^t \int_0^t \dots \int_0^t}_{\check{n} \text{ integrals}} u^{\beta-1} (du)^{\check{n}} \right) \left[(1-t) \left| g^{(\check{n})}(\ell_1) \right| + t \left| g^{(\check{n})}(\ell_2) \right| \right] dt \\
&\quad + \frac{\|f\|_\infty}{\Gamma(\beta)} \int_{1/2}^1 \left(\underbrace{\int_1^t \int_1^t \dots \int_1^t}_{\check{n} \text{ integrals}} (1-u)^{\beta-1} (du)^{\check{n}} \right) \left[(1-t) \left| g^{(\check{n})}(\ell_1) \right| + t \left| g^{(\check{n})}(\ell_2) \right| \right] dt \\
&= I_1 + I_2 \tag{2.8}
\end{aligned}$$

From the first term of (2.8), we get

$$\begin{aligned}
 I_1 &= \frac{\|f\|_\infty}{\Gamma(\mathfrak{B})} \int_0^{1/2} \left(\underbrace{\int_0^t \int_0^t \dots \int_0^t}_{\check{n}\text{-integrals}} u^{\mathfrak{B}-1} (du)^{\check{n}} \right) \left[(1-t) \left| g^{(\check{n})}(\ell_1) \right| + t \left| g^{(\check{n})}(\ell_2) \right| \right] dt \\
 &= \frac{\|f\|_\infty}{\Gamma(\mathfrak{B} + \check{n} - 1)} \int_0^{1/2} u^{\mathfrak{B} + \check{n} - 2} \int_u^{1/2} \left[(1-t) \left| g^{(\check{n})}(\ell_1) \right| + t \left| g^{(\check{n})}(\ell_2) \right| \right] dt du \\
 &= \frac{\|f\|_\infty}{\Gamma(\mathfrak{B} + \check{n} - 1)} \int_0^{1/2} u^{\mathfrak{B} + \check{n} - 2} \left[\left| g^{(\check{n})}(\ell_1) \right| \left(\frac{(1-u)^2}{2} - \frac{1}{8} \right) + \left| g^{(\check{n})}(\ell_2) \right| \left(\frac{1}{8} - \frac{u^2}{2} \right) \right] du.
 \end{aligned}$$

By making use the change of variable $x = \ell_1 + u\mathfrak{I}(\ell_2, \ell_1)$ for $u \in [0, 1]$, it is obvious to see that

$$\begin{aligned}
 I_1 &= \frac{\|f\|_\infty}{\Gamma(\mathfrak{B} + \check{n} - 1)} \frac{\left| g^{(\check{n})}(\ell_1) \right|}{\mathfrak{I}(\ell_2, \ell_1)} \\
 &\quad \times \int_{\ell_1}^{\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1)} \left(\frac{1}{2} \left(1 - \frac{x - \ell_1}{\mathfrak{I}(\ell_2, \ell_1)} \right)^2 - \frac{1}{8} \right) \left(\frac{x - \ell_1}{\mathfrak{I}(\ell_2, \ell_1)} \right)^{\mathfrak{B} + \check{n} - 2} dx \\
 &\quad + \frac{\|f\|_\infty}{\Gamma(\mathfrak{B} + \check{n} - 1)} \frac{\left| g^{(\check{n})}(\ell_2) \right|}{\mathfrak{I}(\ell_2, \ell_1)} \\
 &\quad \times \int_{\ell_1}^{\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1)} \left(\frac{1}{8} - \frac{1}{2} \left(\frac{x - \ell_1}{\mathfrak{I}(\ell_2, \ell_1)} \right)^2 \right) \left(\frac{x - \ell_1}{\mathfrak{I}(\ell_2, \ell_1)} \right)^{\mathfrak{B} + \check{n} - 2} dx. \quad (2.9)
 \end{aligned}$$

From the second term of (2.8), we have

$$\begin{aligned}
 I_2 &= \frac{\|f\|_\infty}{\Gamma(\mathfrak{B})} \int_{1/2}^1 \left(\underbrace{\int_1^t \int_1^t \dots \int_1^t}_{\check{n}\text{-integrals}} (1-u)^{\mathfrak{B}-1} (du)^{\check{n}} \right) \left[(1-t) \left| g^{(\check{n})}(\ell_1) \right| + t \left| g^{(\check{n})}(\ell_2) \right| \right] dt \\
 &= \frac{(-1)^{\check{n}-1} \|f\|_\infty}{\Gamma(\mathfrak{B} + \check{n} - 1)} \int_{1/2}^1 \left(\int_1^t (1-u)^{\mathfrak{B} + \check{n} - 2} du \right) \left[(1-t) \left| g^{(\check{n})}(\ell_1) \right| + t \left| g^{(\check{n})}(\ell_2) \right| \right] dt \\
 &= \frac{(-1)^{\check{n}-1} \|f\|_\infty}{\Gamma(\mathfrak{B} + \check{n} - 1)} \|f\|_\infty \int_{1/2}^1 (1-u)^{\mathfrak{B} + \check{n} - 2} \left(\int_{1/2}^u \left[(1-t) \left| g^{(\check{n})}(\ell_1) \right| + t \left| g^{(\check{n})}(\ell_2) \right| \right] dt \right) du \\
 &= \frac{(-1)^{\check{n}-1} \|f\|_\infty}{\Gamma(\mathfrak{B} + \check{n} - 1)} \|f\|_\infty \int_{1/2}^1 (1-u)^{\mathfrak{B} + \check{n} - 2} \left[\left| g^{(\check{n})}(\ell_1) \right| \left(\frac{1}{8} - \frac{(1-u)^2}{2} \right) \right. \\
 &\quad \left. + \left| g^{(\check{n})}(\ell_2) \right| \left(\frac{u^2}{2} - \frac{1}{8} \right) \right] du.
 \end{aligned}$$

By the change of variable $x = \ell_1 + (1-u)\mathfrak{I}(\ell_2, \ell_1)$, we get

$$\begin{aligned}
 I_2 &= \frac{(-1)^{\check{n}-1} \|f\|_\infty}{\Gamma(\mathfrak{B} + \check{n} - 1)} \frac{|g^{(\check{n})}(\ell_1)|}{\mathfrak{I}(\ell_2, \ell_1)} \\
 &\quad \times \int_{\ell_1}^{\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1)} \left(\frac{1}{8} - \frac{1}{2} \left(\frac{x - \ell_1}{\mathfrak{I}(\ell_2, \ell_1)} \right)^2 \right) \left(\frac{x - \ell_1}{\mathfrak{I}(\ell_2, \ell_1)} \right)^{\mathfrak{B} + \check{n} - 2} dx \\
 &\quad + \frac{(-1)^{\check{n}-1} \|f\|_\infty}{\Gamma(\mathfrak{B} + \check{n} - 1)} \|f\|_\infty \frac{|g^{(\check{n})}(\ell_2)|}{\mathfrak{I}(\ell_2, \ell_1)} \\
 &\quad \times \int_{\ell_1}^{\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1)} \left(\frac{1}{2} \left(1 - \frac{x - \ell_1}{\mathfrak{I}(\ell_2, \ell_1)} \right)^2 - \frac{1}{8} \right) \left(\frac{x - \ell_1}{\mathfrak{I}(\ell_2, \ell_1)} \right)^{\mathfrak{B} + \check{n} - 2} dx. \quad (2.10)
 \end{aligned}$$

Case (i) When n is odd integer.

Adding (2.9) and (2.10) based on (2.8), we have

$$\begin{aligned}
 &\left| \sum_{\kappa=0}^{\check{n}-1} \frac{g^{(\kappa)}(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1))}{2^\kappa \mathfrak{I}(\ell_2, \ell_1)^{\mathfrak{B} + \check{n} - \kappa}} \right. \\
 &\quad \times \left[(-1)^{\check{n}-\kappa-1} J_{(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1)) -}^{\mathfrak{B}} f(\ell_1) + (-1)^{\check{n}+1} J_{(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1)) +}^{\mathfrak{B}} f(\ell_1 + \mathfrak{I}(\ell_2, \ell_1)) \right] \\
 &\quad + (-1)^{\check{n}} \frac{1}{(\mathfrak{I}(\ell_2, \ell_1))^{\mathfrak{B} + \check{n}}} \left[J_{(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1)) -}^{\mathfrak{B}} (gf)(\ell_1) \right. \\
 &\quad \left. \left. + J_{(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1)) +}^{\mathfrak{B}} (gf)(\ell_1 + \mathfrak{I}(\ell_2, \ell_1)) \right] \right| \\
 &\leq \frac{\|f\|_\infty}{\Gamma(\mathfrak{B} + \check{n} + 1)} \frac{1}{2^{\mathfrak{B} + \check{n}}} \left[|g^{(\check{n})}(\ell_1)| + |g^{(\check{n})}(\ell_2)| \right].
 \end{aligned}$$

Case (ii) When n is even integer.

Adding (2.9) and (2.10) based on (2.8), we have

$$\begin{aligned}
 &\left| \sum_{\kappa=0}^{\check{n}-1} \frac{g^{(\kappa)}(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1))}{2^\kappa \mathfrak{I}(\ell_2, \ell_1)^{\mathfrak{B} + \check{n} - \kappa}} \right. \\
 &\quad \times \left[(-1)^{\check{n}-\kappa-1} J_{(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1)) -}^{\mathfrak{B}} f(\ell_1) + (-1)^{\check{n}+1} J_{(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1)) +}^{\mathfrak{B}} f(\ell_1 + \mathfrak{I}(\ell_2, \ell_1)) \right] \\
 &\quad + (-1)^{\check{n}} \frac{1}{(\mathfrak{I}(\ell_2, \ell_1))^{\mathfrak{B} + \check{n}}} \left[J_{(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1)) -}^{\mathfrak{B}} (gf)(\ell_1) \right. \\
 &\quad \left. \left. + J_{(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1)) +}^{\mathfrak{B}} (gf)(\ell_1 + \mathfrak{I}(\ell_2, \ell_1)) \right] \right| \\
 &\leq \frac{\|f\|_\infty}{\Gamma(\mathfrak{B} + \check{n} + 2)} \frac{1}{2^{\mathfrak{B} + \check{n}}} \left[|g^{(\check{n})}(\ell_1)| - |g^{(\check{n})}(\ell_2)| \right].
 \end{aligned}$$

The proof is completed. $\square$

Remark 3. If we choose $\check{n} = 1$, the result coincides with the Theorem 3 of [19].

$$\begin{aligned} & \left| \frac{g(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1))}{\mathfrak{I}(\ell_2, \ell_1)^{\beta+1}} \right. \\ & \quad \times \left[J_{(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1))^-}^{\beta} f(\ell_1) + J_{(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1))^+}^{\beta} f(\ell_1 + \mathfrak{I}(\ell_2, \ell_1)) \right] \\ & \quad \left. + \frac{1}{(\mathfrak{I}(\ell_2, \ell_1))^{\beta+2}} \left[J_{(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1))^-}^{\beta} (gf)(\ell_1) + J_{(\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1))^+}^{\beta} (gf)(\ell_1 + \mathfrak{I}(\ell_2, \ell_1)) \right] \right| \\ & \leq \frac{\|f\|_{\infty}}{\Gamma(\beta+2)} \frac{1}{2^{\beta+1}} [|g'(\ell_1)| + |g'(\ell_2)|]. \end{aligned}$$

Theorem 3. Let $V \subseteq \mathbb{R}$ be an open invex set and $\mathfrak{I} : V \times V \rightarrow \mathbb{R}^{\beta}$ be a mapping. Suppose that $g : V \rightarrow \mathbb{R}$ is a differentiable mapping such that $g^{(\check{n})} \in L[\ell_1, \ell_1 + \mathfrak{I}(\ell_2, \ell_1)]$ where $\mathfrak{I}(\ell_2, \ell_1) > 0$. If $f : [\ell_1, \ell_1 + \mathfrak{I}(\ell_2, \ell_1)] \rightarrow [0, \infty)$ is an integrable mapping and it is also symmetric w.r.t. $\ell_1 + \frac{1}{2}\mathfrak{I}(\ell_2, \ell_1)$ and $|g^{(\check{n})}|$ is preinvex function on V , then the following inequality holds:

$$\begin{aligned} |J| &= \left| \sum_{\kappa=0}^{\check{n}-1} \left[\frac{(-1)^{\check{n}+1} g^{(\kappa)}(\ell_1) + (-1)^{\check{n}-\kappa-1} g^{(\kappa)}(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))}{2(\mathfrak{I}(\ell_2, \ell_1))^{\beta+\check{n}-\kappa}} \right] \right. \\ & \quad \times \left[J_{(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))^-}^{\beta} f(\ell_1) + J_{\ell_1+}^{\beta} f(\ell_1 + \mathfrak{I}(\ell_2, \ell_1)) \right] \\ & \quad \left. + (-1)^{\check{n}} \left[\frac{J_{(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))^-}^{\beta} (gf)(\ell_1) + J_{\ell_1+}^{\beta} (gf)(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))}{(\mathfrak{I}(\ell_2, \ell_1))^{\beta+\check{n}}} \right] \right| \\ & \leq \frac{\|f\|_{\infty}}{(\beta + \check{n})\Gamma(\beta + \check{n} - 1)} [|g^{(\check{n})}(\ell_1)| + |g^{(\check{n})}(\ell_2)|] \quad ; \text{ when } \check{n} \text{ is odd integer} \\ & \leq \frac{\|f\|_{\infty}}{(\beta + \check{n})(\beta + \check{n} + 1)\Gamma(\beta + \check{n} - 1)} [|g^{(\check{n})}(\ell_1)| - |g^{(\check{n})}(\ell_2)|]; \\ & \hspace{15em} \text{when } \check{n} \text{ is even integer} \end{aligned}$$

for $\forall \ell_1, \ell_2 \in V$.

Proof. Applying modulus on both sides of (2.5), one can write

$$\begin{aligned} & \left| \sum_{\kappa=0}^{\check{n}-1} \left[\frac{(-1)^{\check{n}+1} g^{(\kappa)}(\ell_1) + (-1)^{\check{n}-\kappa-1} g^{(\kappa)}(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))}{2(\mathfrak{I}(\ell_2, \ell_1))^{\beta+\check{n}-\kappa}} \right] \right. \\ & \quad \times \left[J_{(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))^-}^{\beta} f(\ell_1) + J_{\ell_1+}^{\beta} f(\ell_1 + \mathfrak{I}(\ell_2, \ell_1)) \right] \\ & \quad \left. + (-1)^{\check{n}} \left[\frac{J_{(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))^-}^{\beta} (gf)(\ell_1) + J_{\ell_1+}^{\beta} (gf)(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))}{(\mathfrak{I}(\ell_2, \ell_1))^{\beta+\check{n}}} \right] \right| \end{aligned}$$

$$= \left| \frac{1}{\Gamma(\mathfrak{B})} \int_0^1 \left(\underbrace{\int_0^t \int_0^t \dots \int_0^t}_{\check{n} \text{ integrals}} u^{\mathfrak{B}-1} f(\ell_1 + u\mathfrak{I}(\ell_2, \ell_1)) (du)^{\check{n}} \right. \right. \\ \left. \left. + \underbrace{\int_1^t \int_1^t \dots \int_1^t}_{\check{n} \text{ integrals}} (1-u)^{\mathfrak{B}-1} f(\ell_1 + u\mathfrak{I}(\ell_2, \ell_1)) (du)^{\check{n}} \right) g^{(\check{n})}(\ell_1 + t\mathfrak{I}(\ell_2, \ell_1)) dt \right|.$$

From preinvexity of $|g^{(\check{n})}|$ on V , we have

$$|J| = \left| \sum_{\kappa=0}^{\check{n}-1} \left[\frac{(-1)^{\check{n}+1} g^{(\kappa)}(\ell_1) + (-1)^{\check{n}-\kappa-1} g^{(\kappa)}(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))}{2(\mathfrak{I}(\ell_2, \ell_1))^{\mathfrak{B}+\check{n}-\kappa}} \right] \right. \\ \times \left[J_{(\ell_1+\mathfrak{I}(\ell_2, \ell_1))}^{\mathfrak{B}} f(\ell_1) + J_{\ell_1+}^{\mathfrak{B}} f(\ell_1 + \mathfrak{I}(\ell_2, \ell_1)) \right] \\ \left. + (-1)^{\check{n}} \left[\frac{J_{(\ell_1+\mathfrak{I}(\ell_2, \ell_1))}^{\mathfrak{B}} (gf)(\ell_1) + J_{\ell_1+}^{\mathfrak{B}} (gf)(\ell_1 + \mathfrak{I}(\ell_2, \ell_1))}{(\mathfrak{I}(\ell_2, \ell_1))^{\mathfrak{B}+\check{n}}} \right] \right| \\ = \frac{1}{\Gamma(\mathfrak{B})} \int_0^1 \left(\underbrace{\int_0^t \int_0^t \dots \int_0^t}_{\check{n} \text{ integrals}} u^{\mathfrak{B}-1} |f(\ell_1 + u\mathfrak{I}(\ell_2, \ell_1))| (du)^{\check{n}} \right. \\ \left. + \underbrace{\int_1^t \int_1^t \dots \int_1^t}_{\check{n} \text{ integrals}} (1-u)^{\mathfrak{B}-1} |f(\ell_1 + u\mathfrak{I}(\ell_2, \ell_1))| (du)^{\check{n}} \right) \\ \left((1-t) |g^{(\check{n})}(\ell_1)| + t |g^{(\check{n})}(\ell_2)| \right) dt$$

After simplification and letting $\|f\|_{\infty} = \sup_{x \in [\ell_1, \ell_2]} |f(x)|$, we have

$$|J| \leq \frac{\|f\|_{\infty}}{\Gamma(\mathfrak{B}+\check{n}-1)} \int_0^1 u^{\mathfrak{B}+\check{n}-2} \left(\int_0^u \left((1-t) |g^{(\check{n})}(\ell_1)| + t |g^{(\check{n})}(\ell_2)| \right) dt \right) du \\ + \frac{(-1)^{\check{n}-1} \|f\|_{\infty}}{\Gamma(\mathfrak{B}+\check{n}-1)} \int_0^1 (1-u)^{\mathfrak{B}+\check{n}-2} \left(\int_u^1 \left((1-t) |g^{(\check{n})}(\ell_1)| + t |g^{(\check{n})}(\ell_2)| \right) dt \right) du$$

Case (i) n is odd integer.

Adding (2.9) and (2.10) based on (2.8), we have

$$|J| \leq \frac{\|f\|_{\infty}}{(\mathfrak{B}+\check{n})\Gamma(\mathfrak{B}+\check{n}-1)} \left[|g^{(\check{n})}(\ell_1)| + |g^{(\check{n})}(\ell_2)| \right].$$

Case (ii) n is even integer.

Adding (2.9) and (2.10) based on (2.8), we have

$$|J| \leq \frac{\|f\|_{\infty}}{(\beta + \check{n})(\beta + \check{n} + 1)\Gamma(\beta + \check{n} - 1)} \left[|g^{(\check{n})}(\ell_1)| - |g^{(\check{n})}(\ell_2)| \right].$$

The proof is completed. $\square$

Corollary 1. *If we set $\check{n} = 1$, we have the following result*

$$\begin{aligned} & \left| \left[\frac{g(\ell_1) + g(\ell_1 + \Im(\ell_2, \ell_1))}{2(\Im(\ell_2, \ell_1))^{\beta+1}} \right] \left[J_{(\ell_1 + \Im(\ell_2, \ell_1))^-}^{\beta} f(\ell_1) + J_{\ell_1+}^{\beta} f(\ell_1 + \Im(\ell_2, \ell_1)) \right] \right. \\ & \quad \left. - \left[\frac{J_{(\ell_1 + \Im(\ell_2, \ell_1))^-}^{\beta} (gf)(\ell_1) + J_{\ell_1+}^{\beta} (gf)(\ell_1 + \Im(\ell_2, \ell_1))}{(\Im(\ell_2, \ell_1))^{\beta+1}} \right] \right| \\ & \leq \frac{\|f\|_{\infty}}{(\beta + 1)\Gamma(\beta)} [|g'(\ell_1)| + |g'(\ell_2)|]. \end{aligned}$$

3. CONCLUSION

We have obtained new lower and upper bounds of Hermite-Hadamard-Fejér type inequalities for $\check{n}$ -times differentiable preinvex functions. We have also elaborated the results with special cases. Authors expect that the results of our paper will be inspiring for interested readers to prove new integral inequalities via some certain fractional integral operators.

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SOME NORM INEQUALITIES FOR SOME POSITIVE BLOCK MATRICES

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Abstract. We review Lin's inequality of numerical range widths and prove that if $M = [X_{i,j}]_{i,j=1}^n$ is positive semi-definite then $\bigoplus_{i=1}^n (X_{i,i} - \sum_{j \neq i} X_{j,j}) \leq M^{\tau} \leq I_n \otimes \sum_{i=1}^n X_{i,i}$, where M^{τ} is the partial transpose of M . Some classical results are also discussed in terms of permutations.

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1. INTRODUCTION AND PRELIMINARIES

Let $\mathbb{M}_n(\mathbb{M}_m)$ denote the space of $n \times n$ complex matrices with entries in $\mathbb{M}_m(\mathbb{C})$. We write $\mathbb{M}_n^H(\mathbb{M}_m)$ and $\mathbb{M}_n^+(\mathbb{M}_m)$ for the set of Hermitian and positive semi-definite matrices in $\mathbb{M}_n(\mathbb{M}_m)$, respectively. Let $Im(X) := \frac{X - X^*}{2i}$ and $Re(X) := \frac{X + X^*}{2}$ be the imaginary part and the real part of a matrix X , respectively. If $W(X)$ denotes the field of values of X then $W(Re(X)) = \Re(W(X))$ and $W(Im(X)) = \Im(W(X))$ see [14]. A positive semi-definite matrix A is denoted by $A \geq 0$ and a norm $\|\cdot\|$ over the space of matrices is a symmetric norm if $\|UAV\| = \|A\|$ for all A and all unitaries U and V .

For a matrix $M = [X_{i,j}] \in \mathbb{M}_n(\mathbb{M}_m)$ we denote M^{τ} the matrix $[X_{i,j}^*]$ where each block is replaced by its adjoint. We define $\text{Tr}_1(M) := \sum_{i=1}^n X_{i,i}$, $\text{Tr}_2(M) := [\text{Tr}(X_{i,j})]$ for $\text{Tr}_1(M)$ (resp. $\text{Tr}_2(M)$) is an $m \times m$ matrix (resp. an $n \times n$ complex matrix). The matrix M is positive partial transpose (P.P.T.) if $M \geq 0$ and $M^{\tau} \geq 0$. The matrix I_m (or I) is the identity matrix of dimension m and a 0 block-entry is an all zero submatrix. We finally denote $\lambda_i(M)$ and $\sigma_i(M)$, $i = 1, \dots, n$ the eigenvalues and the singular values of M , respectively, arranged in decreasing order.

We need the following notations used latter in the article:

For $X \in \mathbb{M}_n$, the vertical width $\omega_v(X)$ and the width $\omega(X)$ of X are the smallest possible real numbers v and w such that $W(X)$ is contained in a vertical strip of width v and in a strip (rectangle) of width w , respectively. In particular, $\omega(X) \leq \omega_v(X)$ and

we may write

$$\min_{(a,b) \in \mathbb{R}^2 \mid W(X) \subset \{x \in \mathbb{C}; a \leq \Re(x) \leq b\}} |b - a| = \omega_v(X).$$

Notice that if the width of $W(A)$ is ω , then one can find a $\theta \in [0, \pi]$ such that

$$rI_m \leq \operatorname{Re}(e^{i\theta}A) \leq (r + \omega)I_m$$

for some $r \in \mathbb{R}$ and $\omega = \omega_v(e^{i\theta}A)$.

If $S_i = \{M_k, 1 \leq k \leq i, M_k \in \mathbb{M}_m\}$ of cardinality i , we let $\{S_i\}$ denote the set of matrices $\{M_1 \pm \dots \pm M_i\}$ and an element in $\{S_i\}$ is denoted by $S_i^{[\sigma]}$. We denote by $\omega_v(\{S_i\})$ the 2^{i-1} -tuple of $\omega_v(S_i^{[\sigma]})$ for each $S_i^{[\sigma]}$.

Finally if $M := [X_{i,j}]_{i,j=1}^n \in \mathbb{M}_n(\mathbb{M}_m)$ with blocks $X_{i,j}$ we write $M = [{}_M[x_{k,l}^{i,j}]]$, where ${}_M[x_{k,l}^{i,j}]$ is the complex number belonging to block $X_{i,j}$ on its line k and column l . For $k, l: 1 \dots m$ and $i, j: 1 \dots n$ one has

$${}_M[x_{k,l}^{i,j}] = {}_M[x_{l,k}^{j,i}] \quad ; \quad {}_M[x_{k,l}^{i,j}] = \overline{{}_M[x_{l,k}^{i,j}]}.$$

The matrix $\tilde{M}$ defined in $\mathbb{M}_m(\mathbb{M}_n)$ by ${}_{\tilde{M}}[x_{i,j}^{k,l}] = {}_M[x_{k,l}^{i,j}]$ is known to be congruent to M , that is, there exist a permutation $\mathcal{P}_M$ such that $\mathcal{P}_M^T M \mathcal{P}_M = \tilde{M}$, see [3, Theorem 7] and [9, Theorem 2.4] with $(\tilde{M})^\tau = (M^\tau)^T$.

We sketch $\mathcal{P}_M$ as follows: An index in $[1, mn]$ is denoted by $\mathfrak{t}$, upon completing M by $m \times m$ zero blocks or extending each block by 0's we consider the completed matrix $M_0 \in \mathbb{M}_r(\mathbb{M}_r)$ with $\max(m, n) = r$. Take the transposition T permuting for every s , $0 \leq s \leq r-2$ and $sr+2+s \leq \mathfrak{t} \leq sr+r$, $\mathfrak{t} \leftrightarrow (\mathfrak{t} - sr - 1)r + 1 + s$, $TM_0T = \tilde{M}_0$. Applying block swapping (Proposition 1.1) one can show that there is some permutation P such that

$$P^T \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} P = \begin{pmatrix} \tilde{M} & 0 \\ 0 & 0 \end{pmatrix}.$$

Assume that M is invertible then so is $\tilde{M}$, writing P by blocks $P = \begin{pmatrix} L & \star \\ \star & \star \end{pmatrix}$ we see that $L^T M L = \tilde{M}$ which implies that L is invertible and thus a permutation matrix. By a continuity argument we can take $\mathcal{P}_M = L$, see [5] and the references therein for further reading.

Proposition 1.1. *Let $A = [A_{i,j}] \in \mathbb{M}_s(\mathbb{M}_p)$ and let $B = [B_{i,j}] \in \mathbb{M}_s(\mathbb{M}_q)$ two complex matrices written by blocks. For $C = [C_{i,j}] \in \mathbb{M}_s(\mathbb{M}_{p+q})$, where $C_{i,j} = \begin{pmatrix} A_{i,j} & 0 \\ 0 & B_{i,j} \end{pmatrix}$ there is a permutation P such that $P^T C P = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$.*

Proof. A (u, v, r) block swap B_{uvr} is a permutation matrix (here of dimension $s(p+q)$) of the form $\begin{pmatrix} I_u & 0 & 0 & 0 \\ 0 & 0 & I_v & 0 \\ 0 & I_r & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$. Say $s \geq 2$ for $i = 1 \dots s-1$, take $P = \prod_{i=1}^{s-1} B_{u_i v_i r_i}$, where $u_i = ip, v_i = iq$ and $r_i = p$. $\square$

2. MAIN RESULTS

If $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \in \mathbb{M}_2^H(\mathbb{M}_m)$ (not necessarily ≥ 0) with positive semi-definite diagonal blocks then it is easy to see that $|\lambda_n(M)| \leq \lambda_1(M)$ (to see this consider the eigenvalues of $\begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}$). In particular if $M \leq I \otimes S$ then $\|M\|_s \leq \|S\|_s$, where $\|\cdot\|_s$ is the spectral norm. This is not true for $n > 2$ with $n = 4$ taking for example $\begin{pmatrix} 0 & 3 & -1 & 0 \\ 3 & 0 & 2 & 0 \\ -1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, where $\lambda_4 \approx -4.1$ and $\lambda_1 \approx 3.22$. In fact if M is an entry wise non-negative matrix we know by Perron-Frobenius theorem that $|\lambda_n(M)| \leq \lambda_1(M)$.

2.1. Lin's Inequality

In [1] the following norm inequality is proved:

Theorem 2.1. [1] Let $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \in \mathbb{M}_2^+(\mathbb{M}_m)$. Then for all symmetric norms

$$\|M\| \leq \|A + B + \omega I_m\|,$$

where ω is the width of a strip containing the numerical range of X .

The previous theorem has been generalized in [12] for 2×2 block matrices. In [11] a generalization to any number of blocks was presented, here we refine Lin's idea mainly from Theorem's 2.1 proof (the base case):

As M is positive, we may write $M = \begin{pmatrix} X_1^* \\ X_2^* \\ \vdots \\ X_n^* \end{pmatrix} (X_1 \ X_2 \ \dots \ X_n)$ for some $X_i \in \mathbb{M}_{mn \times m}$ so that $M_{i,i} = X_i^* X_i$ and $\|M\|_k = \left\| \sum_{i=1}^n X_i X_i^* \right\|_k$ ($\|\cdot\|_k$ are Ky-Fan k -norms). Identifying the n -tuple X to the set $\{X_i, 1 \leq i \leq n\}$ we have (by a direct induction) $\sum_{i=1}^n X_i X_i^* = \frac{1}{2^{n-1}} (\sum_{\sigma} (X^{[\sigma]})(X^{[\sigma]})^*)$, applying Ky-Fan k -norms triangular inequality we get

$$\|M\|_k = \frac{1}{2^{n-1}} \left\| \sum_{\sigma} (X^{[\sigma]})(X^{[\sigma]})^* \right\|_k \leq \frac{1}{2^{n-1}} \sum_{\sigma} \left\| (X^{[\sigma]})^* (X^{[\sigma]}) \right\|_k. \quad (2.1)$$

Theorem 2.2. [11] Let $M = [X_{i,j}] \in \mathbb{M}_n^+(\mathbb{M}_m)$ and let $S_i = \{X_{i,k}, i+1 \leq k \leq n\}$ for $i = 1, \dots, n-1$. Then for all symmetric norms

$$\|M\| \leq \left\| \sum_{i=1}^n X_{i,i} + \sum_{i=1}^{n-1} \mathfrak{v}_i I_m \right\|,$$

where $\mathfrak{v}_i$ is the average of $\omega_v(\{S_i\})$.

Proof. The proof follows from [11] or one can use (2.1), each term in the σ sum is a positive semi-definite matrix that can be written as

$$\sum_{i=1}^n X_i^* X_i + \sum_{i=1}^{n-1} \pm \operatorname{Re}(X_i^* (\pm X_{i+1} \cdots \pm X_n)),$$

we can bound each term of the second sum by scalar identities and splitting the quantities we get the result by a counting argument for Ky-Fan k -norms. Ky-Fan dominance theorem completes the proof ([14] -Sec 10.7-). The same last step can be seen in [1, Theorem 2.1's proof]. $\square$

Corollary 2.3. Let $M = [X_{i,j}] \in \mathbb{M}_n^+(\mathbb{M}_m)$. Then for all symmetric norms

$$\|M\| \leq \left\| \sum_{i=1}^n X_{i,i} + \sum_{i=2}^n \mathfrak{v}_i I_m \right\|,$$

where $\mathfrak{v}_i$ ($i = 2, \dots, n$) is the average of $\omega_v(\{S_i\})$ and $S_i = \{X_{k,i}, 1 \leq k \leq i-1\}$.

Proof. Consider the matrix $\mathcal{L}M\mathcal{L}$ in Theorem 2.2 with $\mathcal{L}$ the anti-diagonal permutation matrix for blocks $\left(\mathcal{L} = \begin{pmatrix} 0 & \cdots & I_m \\ \vdots & \ddots & \vdots \\ I_m & \cdots & 0 \end{pmatrix} \right)$. $\square$

Corollary 2.4. Let $M = [X_{i,j}] \in \mathbb{M}_n^+(\mathbb{M}_m)$ such that for $i < j$, $r_{i,j}I_m \leq \operatorname{Re}(X_{i,j}) \leq r_{i,j}I_m + \delta_{i,j}I_m$. Then $\|M\| \leq \left\| \sum_{i=1}^n X_{i,i} + (\sum_{i < j} \delta_{i,j})I_m \right\|$ for all symmetric norms.

Proof. The proof follows from Theorem 2.2 by writing each $X_{i,j} = \operatorname{Re}(X_{i,j}) + i\operatorname{Im}(X_{i,j})$ for $i < j$ in the average expression and bounding the real parts we get the stated inequality. $\square$

The case $\operatorname{Re}(X_{i,j}) = 0 \cdot I_m$ (skew-Hermitian blocks) is the main result of [13]. We also point out that in Theorem 2.1 of [11] the author replaced (by a miss-justification when $n > 2$ as per our private communication) vertical widths by smallest widths ω and proposed the same replacement in Corollary 2.4 as Question 2.5.

2.2. Around Hiroshima majorization

In this section we prove some classical results concerning matrices in $\mathbb{M}_n^H(\mathbb{M}_m)$, related to some permutations and P.P.T. matrices. The approach is close (dual) to that in [4].

Corollary 2.5. [4] Let $M \in \mathbb{M}_n^H(\mathbb{M}_m)$ such that $M^\tau \leq \text{Tr}_2(M^\tau) \otimes I_m$. Then there exist a permutation $\mathcal{P}_M$ with:

$$N = \mathcal{P}_M^T M^\tau \mathcal{P}_M \leq \mathcal{P}_M^T (\text{Tr}_2(M^\tau) \otimes I_m) \mathcal{P}_M = I_m \otimes \text{Tr}_1(N) \quad (2.2)$$

for $N \in \mathbb{M}_m^H(\mathbb{M}_n)$, $\text{Tr}_1(N) = \text{Tr}_2(M^\tau)$ is an $n \times n$ matrix. Furthermore if $M \geq 0$ then $N^\tau \geq 0$.

Remark 2.6. Set $M \in \mathbb{M}_n(\mathbb{M}_m)$, U unitary and $U^*MU \in \mathbb{M}_n(\mathbb{M}_m)$. Define the block $m \times m$ permutation matrix $P_M \in \mathbb{M}_n(\mathbb{M}_m)$ having zero or identity $m \times m$ blocks; if $U = P_M$ is a transposition block matrix then it is easy to see that $(P_M M P_M)^\tau = P_M M^\tau P_M$ and thus upon composition for any block $m \times m$ permutation matrix P_M , $P_M (P_M^T M P_M)^\tau P_M^T = M^\tau$.

Let $M \in \mathbb{M}_n^H(\mathbb{M}_m)$ such that $M^\tau \geq 0$ and U unitary, for $U^*MU \in \mathbb{M}_n^H(\mathbb{M}_m)$ the matrix $(U^*MU)^\tau$ may not be positive semi-definite in general; consider $M = U N U^* \in$

$\mathbb{M}_3^+(\mathbb{M}_2)$ in Example 2.12 with $U = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$, however this is true if $n = m$

and $U = \mathcal{P}_M$, this is also true for $M \in \mathbb{M}_2^H(\mathbb{M}_m)$ and $U^* = \begin{pmatrix} e^{i\zeta} \cos(\theta) I & \sin(\theta) I \\ -e^{i\zeta} \sin(\theta) I & \cos(\theta) I \end{pmatrix}$ ([12]).

From Quantum Physics theory we know that if M is P.P.T. then $M \leq I_n \otimes \text{Tr}_1(M)$ and $M \leq \text{Tr}_2(M) \otimes I_m$ see for example [7, 8] and more recently [2, 4]. Here we present a different proof as follows:

Lemma 2.7. If $M = [X_{i,j}] \in \mathbb{M}_n^+(\mathbb{M}_m)$ then

$$\bigoplus_{i=1}^n (X_{i,i} - \sum_{j \neq i} X_{j,j}) \leq M^\tau \leq I_n \otimes \sum_{i=1}^n X_{i,i}.$$

Proof. We prove the right side inequality as the lower bound has a similar proof by noticing that $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \geq 0 \iff \begin{pmatrix} A & -X \\ -X^* & B \end{pmatrix} \geq 0$. We write $E = I_n \otimes \sum_{i=1}^n X_{i,i} - M^\tau$

as the sum of positive semi-definite matrices of the form $E_{k,j} = \begin{pmatrix} X_{j,j(k)} & -X_{k,j}^* \\ -X_{k,j} & X_{k,k(j)} \end{pmatrix}$; here the matrix $E_{k,j}$ with $k < j$ refers to the $mn \times mn$ block matrix having all blocks zero except $-X_{k,j}$ and $-X_{k,j}^*$ taken on the same position as $X_{k,j}$ and $X_{k,j}^*$ in E , respectively, $X_{j,j(k)}$ and $X_{k,k(j)}$, where $X_{j,j(k)}$ is the block $X_{j,j}$ on the k^{th} diagonal entry of E (as of $E_{k,j}$). Since $M \geq 0$, up to the congruence $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ each $E_{k,j}$ is a principal block submatrix of M and $E = \sum_{k < j} E_{k,j} \geq 0$. □

Remark 2.8. Let $M = [X_{i,j}] \in \mathbb{M}_n^+(\mathbb{M}_m)$ one can check following the previous proof that $M \leq \bigoplus_{i=1}^n n X_{i,i}$.

The last lemma and corollary imply the following:

Corollary 2.9. [4] If $M = [X_{i,j}] \in \mathbb{M}_n^+(\mathbb{M}_m)$ then $M^{\mathfrak{r}} \leq \text{Tr}_2(M^{\mathfrak{r}}) \otimes I_m$.

In [6]-Theorem 1- the following (Hiroshima majorization) is proved:

Theorem 2.10. [6] Let $M = [X_{i,j}] \in \mathbb{M}_n^+(\mathbb{M}_m)$ such that $M \leq \text{Tr}_2(M) \otimes I_m$. Then for all $k \in [1, mn]$:

$$\sum_{i=1}^k \lambda_i(M) \leq \sum_{i=1}^k \lambda_i(\text{Tr}_2(M)).$$

Corollary 2.11. Let $M = [X_{i,j}] \in \mathbb{M}_n^H(\mathbb{M}_m)$ such that

$$0 \leq M \leq I_n \otimes \sum_{i=1}^n X_{i,i}. \quad (2.3)$$

Then for all $k \in [1, m]$:

$$\sum_{i=1}^k \sigma_i(M) \leq \sum_{i=1}^k \sigma_i \left(\sum_{i=1}^n X_{i,i} \right)$$

equivalently $\|M\| \leq \left\| \sum_{i=1}^n X_{i,i} \right\|$ for all symmetric norms.

Upon completing $\sum_{i=1}^n X_{i,i}$ by zeros we can take $k; 1 \leq k \leq mn$ to get the last inequality from Ky-Fan dominance theorem. In [10] another direct proof of Corollary 2.11 is given with the fact that Corollary 2.11 and Theorem 2.10 are equivalent (see Corollary 2.5).

Example 2.12. It is well known that a matrix in $\mathbb{M}_n^+(\mathbb{M}_m)$ ($n > 2$) satisfying (2.3) need not be P.P.T. as for example $N \in \mathbb{M}_3^+(\mathbb{M}_2) = \begin{pmatrix} 4 & 0 & 0 & 2 & 0 & 0 \\ 0 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

The next corollary can be proved from the double inequality of Lemma 2.7.

Corollary 2.13. If $M \in \mathbb{M}_n^+(\mathbb{M}_m)$ then $\|M^{\mathfrak{r}}\|_s \leq \|\text{Tr}_1(M)\|_s$.

The positive condition in Corollary 2.11 seems necessary as this example shows:

Example 2.14. We have $M = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 4.5 & 0 & 0 \\ 0 & 0 & 0.9 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix} \in \mathbb{M}_2(\mathbb{M}_2)$ and $\sigma_1(M) + \sigma_2(M) = 4.5 + 3 > 7.4$

For a matrix $D = [D_{i,j}] \in \mathbb{M}_n^+(\mathbb{M}_m)$ with all blocks diagonal, it is easily seen that D is P.P.T. as $D^\tau = \bar{D}$ ($\bar{D}$ is D with conjugate entries). Upon considering the rearrangement of D taking $\mathcal{P}_D^T D \mathcal{P}_D$, we see that each diagonal entry of $\text{Tr}_1(D)$ is the sum of n eigenvalues of D which implies: $\|D\| \leq \|\text{Tr}_1(D)\|$ for all symmetric norms.

Corollary 2.15. [15] Let $M = [X_{i,j}] \in \mathbb{M}_n^+(\mathbb{M}_m)$. Then $M \leq m\text{Tr}_2(M) \otimes I_m$.

Proof. Take $M_{\mathcal{P}_M} = \mathcal{P}_M^T M \mathcal{P}_M = [X'_{i,j}] \in \mathbb{M}_m^+(\mathbb{M}_n)$, applying Remark 2.8 $M_{\mathcal{P}_M} \leq \bigoplus_{i=1}^m mX'_{i,i}$, reversing the rearrangement $M \leq mD$ where $D = [D_{i,j}] \in \mathbb{M}_n^+(\mathbb{M}_m)$ and $D_{i,j}$ is the (main) diagonal of $X_{i,j}$. Since D is P.P.T. we get the result from Corollary 2.9. $\square$

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ASYMPTOTIC EXPANSION OF THE SOLUTION FOR SINGULARLY PERTURBED BOUNDARY VALUE PROBLEM WITH BOUNDARY JUMPS

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Abstract. We consider the two-point integral boundary value problem with boundary jumps for third order linear integro-differential equation with the small parameter at two highest derivatives. An asymptotic expansion of the solution of the integral boundary value problem with any degree of accuracy with respect to a small parameter was constructed. A justification of the asymptotic is provided.

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1. INTRODUCTION

Singularly perturbed equations act as mathematical models in many applied problems related to diffusion, heat and mass transfer, chemical kinetics and combustion, heat propagation in thin bodies, semiconductor theory, gyroscope motion, quantum mechanics, biology and biophysics, and many other branches of science and technology. There are a number of introductory as well as advanced books on asymptotic methods that discuss some aspects of different perturbation techniques and their applications. Among them, let us mention, e.g., the monographs and textbooks by Thikhonov [22], Chang and Howes [3], Kevorkian and Cole [14], Murdock [15], Hinch [9], Wasow [27], Vishik and Lyusternik [26], Bogoliubov and Mitropolskii [2], O'Malley [17], Van Dyke [23], Nayfeh [16], Smith [21], Eckhaus [8], Hoppensteadt [10], Sanders [20], Vasil'eva and Bytsov [24, 25] and others.

We should mention that the boundary function, although widely used, is not universal. It is applied most successfully to problems whose solutions exhibit boundary

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layer types of behavior. There exist problems where this method does not work. However, for those problems to which this method can be applied, it works in a most effective and easy way, which makes it advantageous as compared with the other methods. The boundary function method makes it possible to construct the asymptotic solution in a form that provides a uniform asymptotic approximation for the solution of the original problem over the domain of interest and allows one to estimate the remainder term, which justifies the algorithm. Conditions that allow one to verify for each particular problem, whether the boundary function method can or cannot be used to obtain the asymptotic solution are discussed in the book [24, 25] for quite general classes of ordinary and partial differential equations. Initial and boundary value problems with initial jumps for singularly perturbed ordinary and integro-differential equations were considered [4, 5, 11–13]. Singularly perturbed differential equations with piece-wise constant argument of generalized type were considered [1]. Boundary-value problems by using parametrization were investigated in [18, 19].

In the articles [6, 7] local and integral boundary value problems for singularly perturbed integro-differential equations were considered, showing phenomena of initial jumps at both boundaries of a given interval. For example, the solutions of the boundary value problems have at both points $t = 0$ and $t = 1$ phenomena of initial jumps respectively of the first and zero orders. Such boundary value problems are called boundary value problems with boundary jumps. In these papers asymptotic estimates of solutions were obtained and in the case of a local boundary value problem the passage to the limit from the solution of assumed singularly perturbed problem to the solution of the corresponding degenerate problem was described. An a similar way the passage to the limits can be shown with some changes in the case of integral boundary value problems. However, we note that these passages to the limit are non-uniform and have $O(\epsilon)$ degree of accuracy with respect to the small parameter.

Therefore, the aim of this work is to construct a uniform asymptotic expansion of the solutions for the integral boundary value problem with any degree of accuracy with respect to the small parameter. The scientific novelty lies in the fact that a modification of the method of boundary functions to boundary value problems with boundary jumps is proposed. Here the orders of the initial jumps at the ends of the considered segment are taken into account.

2. MAIN RESULTS

Consider the singularly perturbed integro-differential equation

$$L_\epsilon y \equiv \epsilon^2 y''' + \epsilon A_0(t) y'' + A_1(t) y' + A_2(t) y = F(t) + \sum_{i=0}^1 \int_0^1 H_i(t, x) y^{(i)}(x, \epsilon) dx, \quad (2.1)$$

with integral boundary conditions

$$y(0, \varepsilon) = \alpha, \quad y'(0, \varepsilon) = \beta, \quad y(1, \varepsilon) = \gamma + \sum_{i=0}^1 \int_0^1 a_i(x) y^{(i)}(x, \varepsilon) dx, \quad (2.2)$$

where $\varepsilon > 0$ is a small parameter, α, β, γ are known constants independent of ε .

We will need the following assumptions:

- C1) $A_i(t), F(t), a_j(x) \in C^{N+3}[0, 1]$, $i = \overline{0, 2}$, $j = 0, 1$, $H_0(t, x), H_1(t, x)$ are sufficiently smooth functions defined in the domain $D = \{0 \leq t \leq 1, 0 \leq x \leq 1\}$ and also $H_1(t, 1) \neq 0$.
- C2) The roots $\mu_i(t)$, $i = 1, 2$ of "the additional characteristic equation" $\mu^2 + A_0(t)\mu + A_1(t) = 0$ satisfy the inequalities $\mu_1(t) < -\gamma_1 < 0$, $\mu_2(t) > \gamma_2 > 0$.
- C3) $a_1(1) \neq 1$.

In the work [19] a theorem was proved about asymptotic estimates of the solution for the problem (2.1), (2.2).

In this paper, we will construct uniformly asymptotic expansion of the boundary value problem (2.1), (2.2) on the interval $0 \leq t \leq 1$. We will look for the asymptotic expansion of the solution of the problem (2.1), (2.2):

$$y(t, \varepsilon) = y_\varepsilon(t) + \varepsilon w_\varepsilon(\tau_1) + v_\varepsilon(\tau_2), \quad \tau_1 = \frac{t}{\varepsilon}, \quad \tau_2 = \frac{t-1}{\varepsilon}, \quad (2.3)$$

where

$$y_\varepsilon(t) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \dots, \quad (2.4)$$

$$w_\varepsilon(\tau_1) = w_0(\tau_1) + \varepsilon w_1(\tau_1) + \varepsilon^2 w_2(\tau_1) + \dots, \quad (2.5)$$

$$v_\varepsilon(\tau_2) = v_0(\tau_2) + \varepsilon v_1(\tau_2) + \varepsilon^2 v_2(\tau_2) + \dots, \quad (2.6)$$

(2.4) is called the regular part of the asymptotic and (2.5), (2.6) are called the boundary layer parts of the asymptotic.

Substituting (2.3) into the equation (2.1), we obtain the following equality

$$\begin{aligned} & \varepsilon^2 \left(y_\varepsilon'''(t) + \frac{1}{\varepsilon^2} \ddot{w}_\varepsilon(\tau_1) + \frac{1}{\varepsilon^3} \ddot{v}_\varepsilon(\tau_2) \right) + \varepsilon A_0(t) \left(y_\varepsilon''(t) + \frac{1}{\varepsilon} \ddot{w}_\varepsilon(\tau_1) + \frac{1}{\varepsilon^2} \ddot{v}_\varepsilon(\tau_2) \right) \\ & + A_1(t) \left(y_\varepsilon'(t) + \dot{w}_\varepsilon(\tau_1) + \frac{1}{\varepsilon} \dot{v}_\varepsilon(\tau_2) \right) + A_2(t) (y_\varepsilon(t) + \varepsilon w_\varepsilon(\tau_1) + v_\varepsilon(\tau_2)) \\ & = F(t) + \int_0^1 \sum_{i=0}^1 H_i(t, x) \left(y_\varepsilon^{(i)}(x) + \varepsilon^{1-i} \dot{w}_\varepsilon^{(i)}\left(\frac{x}{\varepsilon}\right) + \frac{1}{\varepsilon^i} v_\varepsilon^{(i)}\left(\frac{x-1}{\varepsilon}\right) \right) dx. \end{aligned} \quad (2.7)$$

We make the substitution $\frac{x}{\varepsilon} = s_1$, $\frac{x-1}{\varepsilon} = s_2$. Then the expression in integral of the right-hand side of (2.7) and the integral expressions have the form

$$\begin{aligned} J_1(t, \varepsilon) &= \int_0^{\frac{1}{\varepsilon}} \sum_{i=0}^1 \varepsilon^{2-i} H_i(t, \varepsilon s_1) \overset{(i)}{w}_\varepsilon(s_1) ds_1 \\ &= \int_0^\infty \sum_{i=0}^1 \varepsilon^{2-i} H_i(t, \varepsilon s_1) \overset{(i)}{w}_\varepsilon(s_1) ds_1 - \int_{\frac{1}{\varepsilon}}^\infty \sum_{i=0}^1 \varepsilon^{2-i} H_i(t, \varepsilon s_1) \overset{(i)}{w}_\varepsilon(s_1) ds_1, \end{aligned} \quad (2.8)$$

$$\begin{aligned} J_2(t, \varepsilon) &= \int_{-\frac{1}{\varepsilon}}^0 \sum_{i=0}^1 \varepsilon^{1-i} H_i(t, \varepsilon s_2 + 1) \overset{(i)}{v}_\varepsilon(s_2) ds_2 \\ &= \int_{-\infty}^0 \sum_{i=0}^1 \varepsilon^{1-i} H_i(t, \varepsilon s_2 + 1) \overset{(i)}{v}_\varepsilon(s_2) ds_2 - \int_{-\infty}^{-\frac{1}{\varepsilon}} \sum_{i=0}^1 \varepsilon^{1-i} H_i(t, \varepsilon s_2 + 1) \overset{(i)}{v}_\varepsilon(s_2) ds_2. \end{aligned} \quad (2.9)$$

The improper integrals in (2.8), (2.9) converge and the second term in (2.8), (2.9) vanishes, because $O\left(e^{(-\gamma_1 \frac{t}{\varepsilon})}\right)$, $O\left(e^{(-\gamma_2 \frac{1-t}{\varepsilon})}\right)$ are less than any power of ε , as $\varepsilon \rightarrow 0$. Writing separately for coefficients depending on t and on τ_1 , τ_2 , we get the equations for functions $y_\varepsilon(t)$, $w_\varepsilon(\tau_1)$, $v_\varepsilon(\tau_2)$:

$$\begin{aligned} &\varepsilon^2 y_\varepsilon'''(t) + \varepsilon A_0(t) y_\varepsilon''(t) + A_1(t) y_\varepsilon'(t) + A_2(t) y_\varepsilon(t) \\ &= F(t) + \int_0^1 \sum_{i=0}^1 H_i(t, x) y_\varepsilon^{(i)}(x) dx + \int_0^\infty \sum_{i=0}^1 \varepsilon^{2-i} H_i(t, \varepsilon s_1) \overset{(i)}{w}_\varepsilon(s_1) ds_1 \\ &\quad + \int_{-\infty}^0 \sum_{i=0}^1 \varepsilon^{1-i} H_i(t, \varepsilon s_2 + 1) \overset{(i)}{v}_\varepsilon(s_2) ds_2, \end{aligned} \quad (2.10)$$

$$\ddot{w}_\varepsilon(\tau_1) + A_0(\varepsilon \tau_1) \ddot{w}_\varepsilon(\tau_1) + A_1(\varepsilon \tau_1) \dot{w}_\varepsilon(\tau_1) + \varepsilon A_2(\varepsilon \tau_1) w_\varepsilon(\tau_1) = 0, \quad (2.11)$$

$$\ddot{v}_\varepsilon(\tau_2) + A_0(\varepsilon \tau_2 + 1) \ddot{v}_\varepsilon(\tau_2) + A_1(\varepsilon \tau_2 + 1) \dot{v}_\varepsilon(\tau_2) + \varepsilon A_2(\varepsilon \tau_2 + 1) v_\varepsilon(\tau_2) = 0. \quad (2.12)$$

Let us expand the functions $A_i(\varepsilon \tau_1)$, $i = 0, 1, 2$ and $H_j(t, \varepsilon s_1)$, $j = 0, 1$ into Taylor power series in ε in a neighborhood of the point 0:

$$\begin{aligned} A_i(\varepsilon \tau_1) &= A_i(0) + \varepsilon \tau_1 A_i'(0) + \frac{(\varepsilon \tau_1)^2}{2!} A_i''(0) + \dots + \frac{(\varepsilon \tau_1)^k}{k!} A_i^{(k)}(0) \dots, \quad i = 0, 1, 2, \\ H_j(t, \varepsilon s_1) &= H_j(t, 0) + \varepsilon s_1 H_j'(t, 0) + \dots + \frac{(\varepsilon s_1)^k}{k!} H_j^{(k)}(t, 0) \dots, \quad j = 0, 1. \end{aligned} \quad (2.13)$$

Expand the functions $A_i(\varepsilon\tau_2 + 1)$, $i = 0, 1, 2$ and $H_j(t, \varepsilon s_2 + 1)$, $j = 0, 1$ into Taylor power series in ε in a neighborhood of the point 1:

$$A_i(\varepsilon\tau_2 + 1) = A_i(1) + \varepsilon\tau_2 A_i'(1) + \frac{(\varepsilon\tau_2)^2}{2!} A_i''(1) + \dots + \frac{(\varepsilon\tau_2)^k}{k!} A_i^{(k)}(1) \dots, \quad i = 0, 1, 2,$$

$$H_j(t, \varepsilon s_2 + 1) = H_j(t, 1) + \varepsilon s_2 H_j'(t, 1) + \dots + \frac{(\varepsilon s_2)^k}{k!} H_j^{(k)}(t, 1) \dots, \quad j = 0, 1. \quad (2.14)$$

Substituting the formulas (2.4)-(2.6), (2.13), (2.14) into (2.10)-(2.12) and equating the coefficients at the corresponding powers of ε on both sides of (2.10)-(2.12), we obtain the sequence of equations for the functions $y_k(t)$, $w_k(\tau_1)$, $v_k(\tau_2)$.

For $y_k(t)$, $k = 0, 1, \dots$ we have the linear integro-differential equations

$$A_1(t)y_k'(t) + A_2(t)y_k(t) = F_k(t) + \int_0^1 \sum_{i=0}^1 H_i(t, x) y_k^{(i)}(x) dx + \Delta_k(t), \quad (2.15)$$

where $F_0(t) = F(t)$,

$$F_k(t) = \int_0^\infty \left(\sum_{i=0}^{k-2} \frac{s_1^i}{i!} H_0^{(i)}(t, 0) w_{k-2-i}(s_1) + \sum_{i=0}^{k-1} \frac{s_1^i}{i!} H_1^{(i)}(t, 0) \dot{w}_{k-1-i}(s_1) \right) ds_1$$

$$+ \int_{-\infty}^0 \left(\sum_{i=0}^{k-1} \frac{s_2^i}{i!} H_0^{(i)}(t, 1) v_{k-1-i}(s_2) + \sum_{i=0}^{k-1} \frac{s_2^{i+1}}{(i+1)!} H_1^{(i+1)}(t, 1) \dot{v}_{k-1-i}(s_2) \right) ds_2$$

$$- y_{k-2}'''(t) - A_0(t)y_{k-1}''(t), \quad k = 1, 2, \dots$$

$$\Delta_k(t) = \int_{-\infty}^0 H_1(t, 1) \dot{v}_k(s_2) ds_2, \quad k = 0, 1, 2, \dots \quad (2.16)$$

The values $\Delta_k(t)$, $k = 0, 1, 2, \dots$ are called *the initial jumps of the integral term*.

For $w_k(\tau_1)$, $k = 0, 1, 2, \dots$ we have the linear differential equations with constant coefficients

$$\ddot{w}_k(\tau_1) + A_0(0)\ddot{w}_k(\tau_1) + A_1(0)\dot{w}_k(\tau_1) = \Phi_k(\tau_1), \quad (2.17)$$

where $\Phi_0(\tau_1) = 0$,

$$\Phi_k(\tau_1) = - \sum_{i=1}^k \frac{\tau_1^{i-1}}{(i-1)!} A_2^{(i-1)}(0) w_{k-i}(\tau_1) - \sum_{i=1}^k \frac{\tau_1^i}{i!} A_1^{(i)}(0) \dot{w}_{k-i}(\tau_1)$$

$$- \sum_{i=1}^k \frac{\tau_1^i}{i!} A_0^{(i)}(0) \ddot{w}_{k-i}(\tau_1), \quad k = 1, 2, \dots \quad (2.18)$$

For $v_k(\tau_2)$, $k = 0, 1, 2, \dots$ we have the linear differential equations with constant coefficients

$$\ddot{v}_k(\tau_2) + A_0(1)\ddot{v}_k(\tau_2) + A_1(1)\dot{v}_k(\tau_2) = P_k(\tau_2), \quad (2.19)$$

where $P_0(\tau_2) = 0$,

$$\begin{aligned} P_k(\tau_2) = & - \sum_{i=1}^k \frac{\tau_2^{i-1}}{(i-1)!} A_2^{(i-1)}(1) v_{k-i}(\tau_2) - \sum_{i=1}^k \frac{\tau_2^i}{i!} A_1^{(i)}(1) \dot{v}_{k-i}(\tau_2) \\ & - \sum_{i=1}^k \frac{\tau_2^i}{i!} A_0^{(i)}(1) \ddot{v}_{k-i}(\tau_2), \quad k = 1, 2, \dots \end{aligned} \quad (2.20)$$

Substituting the boundary conditions (2.2) into the asymptotic expansion (2.3), we get

$$\sum_{i=0}^{\infty} \varepsilon^i y_i(0) + \varepsilon \sum_{i=0}^{\infty} \varepsilon^i w_i(0) + \sum_{i=0}^{\infty} \varepsilon^i v_i \left(-\frac{1}{\varepsilon} \right) = \alpha, \quad (2.21)$$

$$\sum_{i=0}^{\infty} \varepsilon^i y_i'(0) + \sum_{i=0}^{\infty} \varepsilon^i \dot{w}_i(0) + \frac{1}{\varepsilon} \sum_{i=0}^{\infty} \varepsilon^i \dot{v}_i \left(-\frac{1}{\varepsilon} \right) = \beta, \quad (2.22)$$

$$\begin{aligned} \sum_{i=0}^{\infty} \varepsilon^i y_i(1) + \varepsilon \sum_{i=0}^{\infty} \varepsilon^i w_i \left(\frac{1}{\varepsilon} \right) + \sum_{i=0}^{\infty} \varepsilon^i v_i(0) = & \gamma + \int_0^1 \sum_{i=0}^1 a_i(x) \left(\sum_{j=0}^{\infty} \varepsilon^j y_j^{(i)}(x) \right) dx \\ & + \int_0^1 \sum_{i=0}^1 a_i(x) \left(\varepsilon^{1-i} \sum_{j=0}^{\infty} \varepsilon^j w_j^{(i)} \left(\frac{x}{\varepsilon} \right) \right) dx + \int_0^1 \sum_{i=0}^1 a_i(x) \left(\frac{1}{\varepsilon^i} \sum_{j=0}^{\infty} \varepsilon^j v_j^{(i)} \left(\frac{x-1}{\varepsilon} \right) \right) dx. \end{aligned} \quad (2.23)$$

We make the substitution $\frac{x}{\varepsilon} = s_1$, $\frac{x-1}{\varepsilon} = s_2$. Then the expression in integral of the right-hand side of (2.23) and the integral expressions have the form

$$\begin{aligned} I_1(\varepsilon) = & \int_0^{\frac{1}{\varepsilon}} \sum_{i=0}^1 \varepsilon^{2-i} a_i(\varepsilon s_1) w_{\varepsilon}^{(i)}(s_1) ds_1 \\ = & \int_0^{\infty} \sum_{i=0}^1 \varepsilon^{2-i} a_i(\varepsilon s_1) w_{\varepsilon}^{(i)}(s_1) ds_1 - \int_{\frac{1}{\varepsilon}}^{\infty} \sum_{i=0}^1 \varepsilon^{2-i} a_i(\varepsilon s_1) w_{\varepsilon}^{(i)}(s_1) ds_1, \end{aligned} \quad (2.24)$$

$$\begin{aligned} I_2(\varepsilon) = & \int_{-\frac{1}{\varepsilon}}^0 \sum_{i=0}^1 \varepsilon^{1-i} a_i(\varepsilon s_2 + 1) v_{\varepsilon}^{(i)}(s_2) ds_2 \\ = & \int_{-\infty}^0 \sum_{i=0}^1 \varepsilon^{1-i} a_i(\varepsilon s_2 + 1) v_{\varepsilon}^{(i)}(s_2) ds_2 - \int_{-\infty}^{-\frac{1}{\varepsilon}} \sum_{i=0}^1 \varepsilon^{1-i} a_i(\varepsilon s_2 + 1) v_{\varepsilon}^{(i)}(s_2) ds_2. \end{aligned} \quad (2.25)$$

The improper integrals in (2.24), (2.25) converges and the second term in (2.24), (2.25) vanishes, because $O\left(e^{(-\gamma_1 \frac{t}{\varepsilon})}\right)$, $O\left(e^{(-\gamma_2 \frac{1-t}{\varepsilon})}\right)$ are less than any power of ε , as

$\varepsilon \rightarrow 0$. Since $w_k^{(i)}(\frac{1}{\varepsilon}), v_k^{(i)}(-\frac{1}{\varepsilon}), i = 0, 1$ are boundary layer functions, hence $w_k^{(i)}(\frac{1}{\varepsilon}) \rightarrow 0, v_k^{(i)}(-\frac{1}{\varepsilon}) \rightarrow 0, i = 0, 1, k = 0, 1, 2, \dots$ as $\varepsilon \rightarrow 0$.

Expand the functions $a_i(\varepsilon s_1), i = 0, 1$ into Taylor power series in ε in a neighborhood of the point 0:

$$a_i(\varepsilon s_1) = a_i(0) + \varepsilon s_1 a_i'(0) + \frac{(\varepsilon s_1)^2}{2!} a_i''(0) + \dots + \frac{(\varepsilon s_1)^k}{k!} a_i^{(k)}(0) \dots, \quad i = 0, 1. \quad (2.26)$$

Expand the functions $a_i(\varepsilon s_2 + 1), i = 0, 1$ into Taylor power series in ε in a neighborhood of the point 1:

$$a_i(\varepsilon s_2 + 1) = a_i(1) + \varepsilon s_2 a_i'(1) + \frac{(\varepsilon s_2)^2}{2!} a_i''(1) + \dots + \frac{(\varepsilon s_2)^k}{k!} a_i^{(k)}(1) \dots, \quad i = 0, 1. \quad (2.27)$$

In view of (2.24)-(2.27), equating the coefficients at the corresponding powers of ε on both sides of (2.21)-(2.23), we obtain the following conditions

$$y_0(0) = \alpha, \quad y_k(0) + w_{k-1}(0) = 0, \quad k = 1, 2, \dots \quad (2.28)$$

$$y_0'(0) + \dot{w}_0(0) = \beta, \quad y_k'(0) + \dot{w}_k(0) = 0, \quad k = 1, 2, \dots \quad (2.29)$$

$$y_0(1) = \gamma + (a_1(1) - 1)v_0(0) + \int_0^1 \sum_{i=0}^1 a_i(x) y_0^{(i)}(x) dx; \quad (2.30)$$

$$y_k(1) = \gamma_k + (a_1(1) - 1)v_k(0) + \int_0^1 \sum_{i=0}^1 a_i(x) y_k^{(i)}(x) dx; \quad k = 1, 2, \dots, \quad (2.31)$$

where

$$\begin{aligned} \gamma_k = & \int_0^\infty \left(\sum_{i=0}^{k-2} \frac{s_1^i}{i!} a_0^{(i)}(0) w_{k-2-i}(s_1) + \sum_{i=0}^{k-1} \frac{s_1^i}{i!} a_1^{(i)}(0) \dot{w}_{k-1-i}(s_1) \right) ds_1 \\ & + \int_{-\infty}^0 \left(\sum_{i=0}^{k-1} \frac{s_2^i}{i!} a_0^{(i)}(1) v_{k-1-i}(s_2) + \sum_{i=0}^{k-1} \frac{s_2^{i+1}}{(i+1)!} a_1^{(i+1)}(1) \dot{v}_{k-1-i}(s_2) \right) ds_2, \quad k = 1, 2, \dots \end{aligned} \quad (2.32)$$

Calculating the improper integrals (2.16) and using the condition $v_k(-\infty) = 0, k = 0, 1, 2, \dots$, we get

$$\Delta_k(t) = \int_{-\infty}^0 H_1(t, 1) \dot{v}_k(s_2) ds_2 = H_1(t, 1) (v_k(0) - v_k(-\infty)) = H_1(t, 1) v_k(0).$$

The function $y_0(t)$ is a solution of the following problem

$$A_1(t) y_0'(t) + A_2(t) y_0(t) = F(t) + \int_0^1 \sum_{i=0}^1 H_i(t, x) y_0^{(i)}(x) dx + H_1(t, 1) v_0(0),$$

$$y_0(0) = \alpha, y_0(1) = \gamma + (a_1(1) - 1)v_0(0) + \int_0^1 \sum_{i=0}^1 a_i(x)y_0^{(i)}(x)dx. \quad (2.33)$$

We can find $y_0(t)$ and $v_0(0)$ from problem (2.33).

Thus, for determining the boundary functions $w_0(\tau_1)$, $v_0(\tau_1)$, we consider the case $k = 0$. The differential equation (2.19) is of the third order; this equation requires three conditions. We construct the characteristic equation of

$$\mu^3 + A_0(1)\mu^2 + A_1(1)\mu = 0. \quad (2.34)$$

The numbers $\mu_1(1)$, $\mu_2(1)$ are the roots of $\mu^2 + A_0(1)\mu + A_1(1) = 0$ and satisfy $\mu_1(1) < 0$, $\mu_2(1) > 0$, $\mu_3 = 0$. Then the general solution of (2.19) is

$$v_0(\tau_2) = C_1 e^{\mu_1(1)\tau_2} + C_2 e^{\mu_2(1)\tau_2} + C_3. \quad (2.35)$$

Using the initial condition (2.30) in (2.35) and the equality $v_0(-\infty) = 0$, we get

$$C_1 = C_3 = 0, \quad C_2 = \frac{1}{1 - a_1(1)} \left(\gamma - y_0(1) + \int_0^1 \sum_{i=0}^1 a_i(x)y_0^{(i)}(x)dx \right).$$

As a result,

$$v_0(\tau_2) = \frac{1}{1 - a_1(1)} \left(\gamma - y_0(1) + \int_0^1 \sum_{i=0}^1 a_i(x)y_0^{(i)}(x)dx \right) e^{\mu_2(1)\tau_2}. \quad (2.36)$$

Considering the derivative of the solution (2.36) taken twice with respect to τ_2 and estimating the value at the point $\tau_2 = 0$, we obtain

$$\begin{aligned} \dot{v}_0(0) &= \frac{\mu_2(1)}{1 - a_1(1)} \left(\gamma - y_0(1) + \int_0^1 \sum_{i=0}^1 a_i(x)y_0^{(i)}(x)dx \right), \\ \ddot{v}_0(0) &= \frac{\mu_2^2(1)}{1 - a_1(1)} \left(\gamma - y_0(1) + \int_0^1 \sum_{i=0}^1 a_i(x)y_0^{(i)}(x)dx \right). \end{aligned} \quad (2.37)$$

The boundary layer function $v_0(\tau_2)$ is a solution of the problem

$$\begin{aligned} \ddot{v}_0(\tau_2) + A_0(1)\dot{v}_0(\tau_2) + A_1(1)v_0(\tau_2) &= 0, \\ v_0(0) &= \frac{1}{1 - a_1(1)} \left(\gamma - y_0(1) + \int_0^1 \sum_{i=0}^1 a_i(x)y_0^{(i)}(x)dx \right), \\ \dot{v}_0(0) &= \frac{\mu_2(1)}{1 - a_1(1)} \left(\gamma - y_0(1) + \int_0^1 \sum_{i=0}^1 a_i(x)y_0^{(i)}(x)dx \right), \end{aligned}$$

$$\ddot{v}_0(0) = \frac{\mu_2^2(1)}{1 - a_1(1)} \left(\gamma - y_0(1) + \int_0^1 \sum_{i=0}^1 a_i(x) y_0^{(i)}(x) dx \right).$$

By this means, we construct the characteristic equation of (2.17)

$$\mu^3 + A_0(0)\mu^2 + A_1(0)\mu = 0. \quad (2.38)$$

The numbers $\mu_1(0)$, $\mu_2(0)$ are the roots of $\mu^2 + A_0(0)\mu + A_1(0) = 0$ and satisfy $\mu_1(0) < 0$, $\mu_2(0) > 0$, $\mu_3 = 0$. Then the general solution of (2.17) is

$$w_0(\tau_1) = C_1 e^{\mu_1(0)\tau_1} + C_2 e^{\mu_2(0)\tau_1} + C_3. \quad (2.39)$$

Using the initial condition (2.29) in (2.39) and the equality $w_0(\infty) = 0$, we get

$$C_1 = \frac{\beta - y_0'(0)}{\mu_1(0)} \quad C_2 = C_3 = 0,$$

and as a result,

$$w_0(\tau_1) = \frac{\beta - y_0'(0)}{\mu_1(0)} e^{\mu_1(0)\tau_1}. \quad (2.40)$$

Considering the derivative of the solution (2.40) taken twice with respect to τ_1 and estimating the value at the point $\tau_1 = 0$, we obtain

$$w_0(0) = \frac{\beta - y_0'(0)}{\mu_1(0)}, \quad \ddot{w}_0(0) = \mu_1(0)(\beta - y_0'(0)). \quad (2.41)$$

The boundary layer function $w_0(\tau_1)$ is a solution of the problem

$$\ddot{w}_0(\tau_1) + A_0(0)\dot{w}_0(\tau_1) + A_1(0)w_0(\tau_1) = 0,$$

$$w_0(0) = \frac{\beta - y_0'(0)}{\mu_1(0)}, \quad \dot{w}_0(0) = \beta - y_0'(0), \quad \ddot{w}_0(0) = \mu_1(0)(\beta - y_0'(0)).$$

Thus, the zeroth approximation of the asymptotic expansion is completely constructed.

The function $y_k(t)$ is a solution of the problem

$$A_1(t)y_k'(t) + A_2(t)y_k(t) = F_k(t) + \int_0^1 \sum_{i=0}^1 H_i(t, x) y_k^{(i)}(x) dx + H_1(t, 1)v_k(0),$$

$$y_k(0) = -w_{k-1}(0), \quad y_k(1) = \gamma_k + (a_1(1) - 1)v_k(0) + \int_0^1 \sum_{i=0}^1 a_i(x) y_k^{(i)}(x) dx, \quad (2.42)$$

where the functions $F_k(t)$ and constants γ_k are defined by the formulas (2.16) and (2.32). We can find $y_k(t)$ and $v_k(0)$ from the problem (2.42).

Let us consider $k = 1, 2, \dots$. The general solution of (2.19) has the form

$$v_k(\tau_2) = C_1 e^{\mu_1(1)\tau_2} + C_2 e^{\mu_2(1)\tau_2} + C_3 + \int_{-\infty}^{\tau_2} K_3(\tau_2, s) P_k(s) ds, \quad (2.43)$$

where $K_3(\tau_2, s)$ is the Cauchy function, which has the form

$$K_3(\tau_2, s) = \frac{1}{A_1(1)} - \frac{1}{\mu_1(1)(\mu_2(1) - \mu_1(1))} e^{\mu_1(1)(\tau_2 - s)} + \frac{1}{\mu_2(1)(\mu_2(1) - \mu_1(1))} e^{\mu_2(1)(\tau_2 - s)}$$

and $P_k(s)$ is given by the formula (2.20). Using the initial condition (2.32) in (2.43) and the equality $v_k(-\infty) = 0$, we get $C_1 = C_3 = 0$,

$$C_2 = \frac{1}{1 - a_1(1)} \left(\gamma_k - y_k(1) + \int_0^1 \sum_{i=0}^1 a_i(x) y_k^{(i)}(x) dx \right) - \int_{-\infty}^0 K_3(0, s) P_k(s) ds.$$

As a result,

$$v_k(\tau_2) = \left(\frac{1}{1 - a_1(1)} \left(\gamma_k - y_k(1) + \int_0^1 \sum_{i=0}^1 a_i(x) y_k^{(i)}(x) dx \right) - \int_{-\infty}^0 K_3(0, s) P_k(s) ds \right) e^{\mu_2(1)\tau_2} + \int_{-\infty}^{\tau_2} K_3(\tau_2, s) P_k(s) ds. \quad (2.44)$$

Considering the derivative of the solution (2.44) taken twice with respect to τ_2 and estimating the value at the point $\tau_2 = 0$, we obtain

$$\begin{aligned} \dot{v}_k(0) &= \left(\frac{1}{1 - a_1(1)} \left(\gamma_k - y_k(1) + \int_0^1 \sum_{i=0}^1 a_i(x) y_k^{(i)}(x) dx \right) - \int_{-\infty}^0 K_3(0, s) P_k(s) ds \right) \mu_2(1) \\ &\quad + \int_{-\infty}^0 K_3'(0, s) P_k(s) ds, \end{aligned} \quad (2.45)$$

$$\begin{aligned} \ddot{v}_k(0) &= \left(\frac{1}{1 - a_1(1)} \left(\gamma_k - y_k(1) + \int_0^1 \sum_{i=0}^1 a_i(x) y_k^{(i)}(x) dx \right) - \int_{-\infty}^0 K_3(0, s) P_k(s) ds \right) \mu_2^2(1) \\ &\quad + \int_{-\infty}^0 K_3''(0, s) P_k(s) ds. \end{aligned} \quad (2.46)$$

From the initial value problem (2.19), (2.32), (2.45), (2.46), we can define the boundary layer functions $v_k(\tau_2)$, $k = 1, 2, \dots$

In this way, the general solution of (2.17) has the form

$$w_k(\tau_1) = C_1 e^{\mu_1(0)\tau_1} + C_2 e^{\mu_2(0)\tau_1} + C_3 + \int_{\tau_1}^{\infty} K_3(\tau_1, s) \Phi_k(s) ds, \quad (2.47)$$

where $K_3(\tau_1, s)$ is the Cauchy function of the form

$$K_3(\tau_1, s) = \frac{1}{A_1(0)} - \frac{1}{\mu_1(0)(\mu_2(0) - \mu_1(0))} e^{\mu_1(0)(\tau_1 - s)} + \frac{1}{\mu_2(0)(\mu_2(0) - \mu_1(0))} e^{\mu_2(0)(\tau_1 - s)}$$

and $\Phi_k(s)$ is given by the formula (2.18). Using the initial condition (2.29) in (2.47) and the equality $w_k(\infty) = 0$, we get $C_2 = C_3 = 0$,

$$C_1 = -\frac{1}{\mu_1(0)} \left(y'_k(0) - \int_0^{\infty} K'_3(0, s) \Phi_k(s) ds \right).$$

As a result,

$$w_k(\tau_1) = -\frac{1}{\mu_1(0)} \left(y'_k(0) - \int_0^{\infty} K'_3(0, s) \Phi_k(s) ds \right) e^{\mu_1(0)\tau_1} - \int_{\tau_1}^{\infty} K_3(\tau_1, s) \Phi_k(s) ds. \quad (2.48)$$

Considering the derivative of the solution (2.48) taken twice with respect to τ_1 and estimating the value at the point $\tau_1 = 0$, we obtain

$$w_k(0) = -\frac{1}{\mu_1(0)} \left(y'_k(0) - \int_0^{\infty} K'_3(0, s) \Phi_k(s) ds \right) - \int_0^{\infty} K_3(0, s) \Phi_k(s) ds, \quad (2.49)$$

$$\ddot{w}_k(0) = -\mu_1(0) \left(y'_k(0) - \int_0^{\infty} K'_3(0, s) \Phi_k(s) ds \right) - \int_0^{\infty} K''_3(0, s) \Phi_k(s) ds. \quad (2.50)$$

From the initial value problem (2.17), (2.31), (2.49), (2.50), we can define the boundary layer functions $w_k(\tau_1)$, $k = 1, 2, \dots$

Thus, the k th approximation of the asymptotic expansion is completely constructed. The following theorem is true.

Theorem 1. *Let assumptions (C1)-(C3) hold. Then for sufficiently small ε the boundary value problem (2.1), (2.2) has a unique solution on $[0, 1]$ which can be expressed by the formula*

$$y(t, \varepsilon) = y_N(t, \varepsilon) + R_N(t, \varepsilon),$$

where $y_N(t, \varepsilon)$ is defined by the relation

$$y_N(t, \varepsilon) = \sum_{i=0}^N \varepsilon^i y_i(t) + \varepsilon \sum_{i=0}^{N+1} \varepsilon^i w_i(\tau_1) + \sum_{i=0}^{N+2} \varepsilon^i v_i(\tau_2), \quad (2.51)$$

and the following estimates for the remainder term $R_N(t, \varepsilon)$ are true

$$\left| R_N^{(i)}(t, \varepsilon) \right| \leq C \varepsilon^{N+1}, \quad i = 0, 1, 2, \quad 0 \leq t \leq 1,$$

where $C > 0$ is a constant independent of ε and $N \geq 0$ is an integer number.

Proof. Let us construct the N th partial sum (2.51) of the expansion (2.3), here $N \geq 0$ is an integer number, the functions $y_i(t)$, $w_i(\tau_1)$, $v_i(\tau_2)$ are regular and boundary layer parts of the asymptotic.

Substituting (2.51) into equation (2.1), we obtain

$$\varepsilon^2 y_N''' + \varepsilon A_0(t) y_N'' + A_1(t) y_N' + A_2(t) y_N = F(t) + O(\varepsilon^{N+1}) + \int_0^1 \sum_{i=0}^1 H_i(t, x) y_N^{(i)}(x, \varepsilon) dx. \quad (2.52)$$

Hence, the function $y_N(t, \varepsilon)$ satisfies equation (2.1) with the accuracy $O(\varepsilon^{N+1})$. Using (2.51) in (2.2), we obtain the conditions

$$y_N(0, \varepsilon) = \sum_{i=0}^N \varepsilon^i y_i(0) + \varepsilon \sum_{i=0}^{N+1} \varepsilon^i w_i(0) + \sum_{i=0}^{N+2} \varepsilon^i v_i \left(-\frac{1}{\varepsilon} \right) = \alpha + O(\varepsilon^{N+1}), \quad (2.53)$$

$$y_N'(0, \varepsilon) = \sum_{i=0}^N \varepsilon^i y_i'(0) + \sum_{i=0}^{N+1} \varepsilon^i \dot{w}_i(0) + \frac{1}{\varepsilon} \sum_{i=0}^{N+2} \varepsilon^i \dot{v}_i \left(-\frac{1}{\varepsilon} \right) = \beta + O(\varepsilon^{N+1}), \quad (2.54)$$

$$\begin{aligned} y_N(1, \varepsilon) - \int_0^1 \sum_{i=0}^1 a_i(x) y_N^{(i)}(x) dx &= \sum_{i=0}^N \varepsilon^i y_i(1) + \varepsilon \sum_{i=0}^{N+1} \varepsilon^i w_i \left(\frac{1}{\varepsilon} \right) + \sum_{i=0}^{N+2} \varepsilon^i v_i(0) \\ &\quad - \int_0^1 \left[a_0(x) \left(\sum_{i=0}^N \varepsilon^i y_i(x) + \varepsilon \sum_{i=0}^{N+1} \varepsilon^i w_i \left(\frac{x}{\varepsilon} \right) + \sum_{i=0}^{N+2} \varepsilon^i v_i \left(\frac{x-1}{\varepsilon} \right) \right) \right. \\ &\quad \left. + a_1(x) \left(\sum_{i=0}^N \varepsilon^i y_i'(x) + \sum_{i=0}^{N+1} \varepsilon^i \dot{w}_i \left(\frac{x}{\varepsilon} \right) + \frac{1}{\varepsilon} \sum_{i=0}^{N+2} \varepsilon^i \dot{v}_i \left(\frac{x-1}{\varepsilon} \right) \right) \right] dx = \gamma + O(\varepsilon^{N+1}), \end{aligned} \quad (2.55)$$

Since $w_k^{(i)} \left(\frac{1}{\varepsilon} \right)$, $v_k^{(i)} \left(-\frac{1}{\varepsilon} \right)$, $i = 0, 1$ are boundary layer functions in (2.53)-(2.55), it follows that $w_k^{(i)} \left(\frac{1}{\varepsilon} \right) \rightarrow 0$, $v_k^{(i)} \left(-\frac{1}{\varepsilon} \right) \rightarrow 0$, $i = 0, 1$, $k = 0, 1, 2, \dots$ as $\varepsilon \rightarrow 0$.

Let us set

$$R_N(t, \varepsilon) = y(t, \varepsilon) - y_N(t, \varepsilon) \implies y(t, \varepsilon) = R_N(t, \varepsilon) + y_N(t, \varepsilon), \quad (2.56)$$

here $R_N(t, \varepsilon)$ is called the remainder term of the asymptotic.

Substituting (2.56) into (2.1), considering that the function $y_N(t, \varepsilon)$ satisfies the equation (2.52) and the boundary conditions (2.53)-(2.55), we get the problem for the remainder term $R_N(t, \varepsilon)$:

$$\varepsilon^2 R_N''' + \varepsilon A_0(t) R_N'' + A_1(t) R_N' + A_2(t) R_N = O(\varepsilon^{N+1}) + \int_0^1 \sum_{i=0}^1 H_i(t, x) R_N^{(i)}(x, \varepsilon) dx, \quad (2.57)$$

$$R_N(0, \varepsilon) = O(\varepsilon^{N+1}), \quad ; R_N'(0, \varepsilon) = O(\varepsilon^{N+1}),$$

$$R_N(1, \varepsilon) = \int_0^1 \sum_{i=0}^1 a_i(x) R_N^{(i)}(x, \varepsilon) dx + O(\varepsilon^{N+1}).$$

The problem (2.57) is of the same type as the problem (2.1), (2.2). By applying asymptotic estimates of the solution [19] for the solution of the problem (2.57), we obtain

$$\begin{aligned} |R_N(t, \varepsilon)| &\leq C\varepsilon^{N+1} + C\varepsilon^{N+2} e^{-\gamma_1 \frac{t}{\varepsilon}} + C\varepsilon^{N+1} e^{\gamma_2 \frac{t-1}{\varepsilon}} \leq C\varepsilon^{N+1}, \\ |R_N'(t, \varepsilon)| &\leq C\varepsilon^{N+1} + C\varepsilon^{N+1} e^{-\gamma_1 \frac{t}{\varepsilon}} + C\varepsilon^N e^{\gamma_2 \frac{t-1}{\varepsilon}}, \\ |R_N''(t, \varepsilon)| &\leq C\varepsilon^{N+1} + C\varepsilon^N e^{-\gamma_1 \frac{t}{\varepsilon}} + C\varepsilon^{N-1} e^{\gamma_2 \frac{t-1}{\varepsilon}}. \end{aligned} \quad (2.58)$$

The estimate of the remainder term $R_{N+2}(t, \varepsilon)$ in the interval $[0, 1]$ is similar to (2.58):

$$\begin{aligned} |R_{N+2}(t, \varepsilon)| &\leq C\varepsilon^{N+3} + C\varepsilon^{N+4} e^{-\gamma_1 \frac{t}{\varepsilon}} + C\varepsilon^{N+3} e^{\gamma_2 \frac{t-1}{\varepsilon}} \leq C\varepsilon^{N+3}, \\ |R_{N+2}'(t, \varepsilon)| &\leq C\varepsilon^{N+3} + C\varepsilon^{N+3} e^{-\gamma_1 \frac{t}{\varepsilon}} + C\varepsilon^{N+2} e^{\gamma_2 \frac{t-1}{\varepsilon}}, \\ |R_{N+2}''(t, \varepsilon)| &\leq C\varepsilon^{N+3} + C\varepsilon^{N+2} e^{-\gamma_1 \frac{t}{\varepsilon}} + C\varepsilon^{N+1} e^{\gamma_2 \frac{t-1}{\varepsilon}}. \end{aligned} \quad (2.59)$$

So, the equality

$$y(t, \varepsilon) = y_N(t, \varepsilon) + R_N(t, \varepsilon) = y_{N+2}(t, \varepsilon) + R_{N+2}(t, \varepsilon) \quad (2.60)$$

holds.

From (2.60) it follows that

$$R_N^{(i)}(t, \varepsilon) = y_{N+2}^{(i)}(t, \varepsilon) - y_N^{(i)}(t, \varepsilon) + R_{N+2}^{(i)}(t, \varepsilon), \quad i = 0, 1, 2. \quad (2.61)$$

Using the estimates (2.58), (2.59) for the remainder term (2.61), we obtain

$$\begin{aligned} R_N(t, \varepsilon) &= \varepsilon^{N+1} y_{N+1}(t) + \varepsilon^{N+2} y_{N+2}(t) + \varepsilon^{N+3} w_{N+2}(\tau_1) + \varepsilon^{N+4} w_{N+3}(\tau_1) \\ &\quad + \varepsilon^{N+3} v_{N+3}(\tau_2) + \varepsilon^{N+4} v_{N+4}(\tau_2) + R_{N+2}(t, \varepsilon) \\ &\Rightarrow |R_N(t, \varepsilon)| \leq C\varepsilon^{N+1}, \end{aligned} \quad (2.62)$$

$$\begin{aligned} R_N'(t, \varepsilon) &= \varepsilon^{N+1} y_{N+1}'(t) + \varepsilon^{N+2} y_{N+2}'(t) + \varepsilon^{N+2} \dot{w}_{N+2}(\tau_1) + \varepsilon^{N+3} \dot{w}_{N+3}(\tau_1) \\ &\quad + \varepsilon^{N+2} \dot{v}_{N+3}(\tau_2) + \varepsilon^{N+3} \dot{v}_{N+4}(\tau_2) + R_{N+2}'(t, \varepsilon) \end{aligned}$$

$$\Rightarrow |R'_N(t, \varepsilon)| \leq C\varepsilon^{N+1}, \quad (2.63)$$

$$\begin{aligned} R''_N(t, \varepsilon) &= \varepsilon^{N+1} y''_{N+1}(t) + \varepsilon^{N+2} y''_{N+2}(t) + \varepsilon^{N+1} \ddot{w}_{N+2}(\tau_1) + \varepsilon^{N+2} \ddot{w}_{N+3}(\tau_1) \\ &\quad + \varepsilon^{N+1} \ddot{v}_{N+3}(\tau_2) + \varepsilon^{N+2} \ddot{v}_{N+4}(\tau_2) + R''_{N+2}(t, \varepsilon) \\ &\Rightarrow |R''_N(t, \varepsilon)| \leq C\varepsilon^{N+1}. \end{aligned} \quad (2.64)$$

The theorem is proved. $\square$

3. CONCLUSION

The boundary value problem with boundary jumps for singularly perturbed integro-differential equation was considered. To determine the zeroth approximation of the regular part $y_0(t)$, we have problem (2.33). Problem (2.33) is different from the usual degenerate problem obtained from (1) as $\varepsilon = 0$. Here by virtue of the integral term, an additional term occurs, $\Delta(t) = H_1(t, 1)v_0(0)$, called the initial jump of the integral term. The integral boundary condition is also changed: $\gamma + (a_1(1) - 1)v_0(0)$ is taken instead of γ , where $(a_1(1) - 1)v_0(0)$ is called the initial jump of the solution. A similar situation is observed for the k th approximation of the regular part $y_k(t)$ (see (2.42)). An algorithm of asymptotic expansion of the solution with any degree of accuracy with respect to a small parameter was constructed. The problems for determining the regular and boundary layer parts were completely obtained. A theorem about uniformly asymptotic expansion is proved.

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ON GOLDIE*-SUPPLEMENTED MODULES

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Abstract. Let M be a left R -module. X, Y of M are β^* equivalent, $X\beta^*Y$, if and only if $\frac{X+Y}{X}$ is small in $\frac{M}{X}$ and $\frac{X+Y}{Y}$ is small in $\frac{M}{Y}$. A module M is called G^* -supplemented if for every submodule X of M there is a supplement submodule S of M such that $X\beta^*S$. In this work some new properties of β^* are given and G^* -supplemented modules are studied. Also completely G^* -supplemented modules and G^* -radical supplemented modules are defined.

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1. INTRODUCTION

Throughout this paper R denotes an associative ring with unity and all R -modules are unital left R -modules. Let M be an R -module. A submodule S is called a *small* submodule of M if for every proper submodule A of M , $M \neq A + S$. We will use the notation $S \ll M$ to indicate that a submodule S is small in M .

Let M be an R -module. Let N be a submodule of M . A *supplement* of N in M is a submodule K of M minimal with respect to the property $M = N + K$, equivalently, $M = N + K$ and $N \cap K \ll K$. An R -module M is called a *supplemented* module if every submodule of M has a supplement in M . A submodule N of M has *ample supplements* in M if every submodule L such that $M = N + L$ contains a supplement of N in M . The module M is called *amply supplemented* if every submodule of M has ample supplements in M . More generally, a submodule N of M has a *weak supplement* L in M if $M = N + L$ and $N \cap L \ll M$ and M is called *weakly supplemented* if every submodule of M has a weak supplement in M . The R -module M is called $\oplus$ -*supplemented* if every submodule of M has a supplement that is a direct summand of M . M is called *completely $\oplus$ -supplemented* if every direct summand of M is $\oplus$ -supplemented.

Let M be an R -module and $K \leq U \leq M$. If $\frac{U}{K} \ll \frac{M}{K}$ then we say U lies above K . It is well known that, U lies above a submodule K of M if and only if $K \leq U$ and for every submodule T of M with $U + T = M$, then $K + T = M$. Let M be an R -module. M satisfies (D1) if for every submodule N of M there exist submodules K_1 and K_2 of

M such that $M = K_1 \oplus K_2$, $K_1 \leq N$ and $N \cap K_2 \ll K_2$. Furthermore M satisfies (D1) iff every submodule of M lies above a direct summand of M .

Other terminologies and notations can be found in [2, 4, 8].

2. THE β^* RELATION

Let M be an R -module. The relation " β^* " on the set of submodules of M is defined by $X\beta^*Y$ if and only if $\frac{X+Y}{X}$ is small in $\frac{M}{X}$ and $\frac{X+Y}{Y}$ is small in $\frac{M}{Y}$. Moreover, β^* is an equivalence relation [1].

Lemma 1. *Let M be an R -module and $X, Y \leq M$. The following are equivalent:*

- (i) $X\beta^*Y$.
- (ii) *For each $A \leq M$ such that $X + Y + A = M$ then $X + A = M$ and $Y + A = M$.*
- (iii) *If $K \leq M$ with $X + K = M$ then $Y + K = M$ and if $H \leq M$ with $Y + H = M$ then $X + H = M$.*

Proof. See [1, Theorem 2.3]. □

Lemma 2. *Let V be a supplement of U in M . If $X, Y \leq V$ such that $X\beta^*Y$ on the set of submodules of M , then $X\beta^*Y$ on the set of submodules of V .*

Proof. Let $X + K = V$, for some $K \leq V$. Then $X + K + U = M$ and since $X\beta^*Y$ on the set of submodules of M , $Y + K + U = M$. Then by V being a supplement of U in M , $Y + K = V$. Similarly, $Y + H = V$ with $H \leq V$ then $X + H = V$. Thus $X\beta^*Y$ on the set of submodules of V . □

Corollary 1. *Let $M = A \oplus B$ be an R -module. If $X, Y \leq A$ such that $X\beta^*Y$ on the set of submodules of M , then $X\beta^*Y$ on the set of submodules of A .*

Proof. Clear from Lemma 2. □

Lemma 3. *Let M be an R -module, $X, Y \leq M$ and $\text{Rad}(M/X) = 0$. If $X\beta^*Y$, then $Y \leq X$.*

Proof. Since $X\beta^*Y$, $\frac{X+Y}{X} \ll M/X$. Then $\frac{X+Y}{X} \leq \text{Rad}(M/X) = 0$. Hence $X + Y = X$ and $Y \leq X$. □

Corollary 2. *Let M be an R -module, $X, Y \leq M$, $\text{Rad}(M/X) = 0$ and $\text{Rad}(M/Y) = 0$. Then $X\beta^*Y$ if and only if $X = Y$.*

Proof. Clear from Lemma 3. □

Theorem 1. *Let M be a semisimple R -module and $X, Y \leq M$. Then $X\beta^*Y$ if and only if $X = Y$.*

Proof. Since M is semisimple, M/X and M/Y are semisimple. Then $\text{Rad}(M/X) = 0$ and $\text{Rad}(M/Y) = 0$. The rest is obvious by Corollary 2. □

Theorem 2. *Let M be an R -module. M is hollow if and only if all proper submodules of M are equivalent to each other with β^* .*

Proof. Clear. $\square$

Let M be an R -module. M is called *distributive* if for arbitrary submodules K, L, N of M , $N + (K \cap L) = (N + K) \cap (N + L)$ this equivalent to $N \cap (K + L) = (N \cap K) + (N \cap L)$.

Theorem 3. *Let M be a distributive module and $X \leq M$. If $M = M_1 \oplus M_2$ and $M_1 \beta^* X$ then $M_1 \leq X$ and $M_2 \cap X \ll M$.*

Proof. Since $M = M_1 \oplus M_2$ and $M_1 \beta^* X$, $M = X + M_2$. Thus $M_1 = M_1 \cap M = M_1 \cap (X + M_2) = M_1 \cap X + M_1 \cap M_2 = M_1 \cap X$ and $M_1 \leq X$. Since $M_1 \leq X$ and $M_1 \beta^* X$, X lies above M_1 and $M_2 \cap X \ll M$. $\square$

Corollary 3. *Let M be a distributive module. Assume that, for a submodule X of M , there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq X$ and $M_2 \cap X \ll M$. If $X \beta^* Y$, then the decomposition $M = M_1 \oplus M_2$ exists for Y such that $M_1 \leq Y$ and $M_2 \cap Y \ll M$.*

Proof. By hypothesis, $M_1 \beta^* X$. Since $X \beta^* Y$, $M_1 \beta^* Y$. From Theorem 3, $M_1 \leq Y$ and $M_2 \cap Y \ll M$. $\square$

Theorem 4. *Let M be an R -module. M is weakly supplemented if and only if for each $X \leq M$, there exists a weak supplement W in M such that $X \beta^* W$.*

Proof.

- ($\Rightarrow$) Assume that, M is weakly supplemented. Then every submodule of M is a weak supplement. Since $X \beta^* X$ for each $X \leq M$, every submodule of M is β^* equivalent to a weak supplement.
- ($\Leftarrow$) Let $X \leq M$. By hypothesis there exists a weak supplement W in M such that $X \beta^* W$. Since W is a weak supplement in M , there exists $A \leq M$ such that $W + A = M$ and $W \cap A \ll M$. Hence, A is a weak supplement of X by [1, Theorem 2.6]. So M is weakly supplemented. $\square$

Corollary 4. *Let M be an R -module. M is weakly supplemented if and only if for each $X \leq M$ there exists a weak supplement W and a small submodule H of M such that $X + H = W + H = X + W$.*

Proof.

- ($\Rightarrow$) Let $X \leq M$. Since M is weakly supplemented, by Theorem 4, there exists a weak supplement W in M such that $X \beta^* W$. Hence there exists $A \leq M$ such that $W + A = M$ and $W \cap A \ll M$. From [1, Proposition 2.11], $X \beta^* (X + W)$ and $W \beta^* (X + W)$. By [1, Theorem 2.6], A is a weak supplement of X and

$X + W$. By the modular law, $X + H = W + H = X + W$, where $H = (X + W) \cap A \ll M$.

($\Leftarrow$) It can be seen easily. □

3. G^* -SUPPLEMENTED MODULES

Let M be an R -module. M is called G^* -supplemented (G^* -lifting = H -supplemented) if for every submodule X of M , there is a supplement submodule S (direct summand D) of M such that $X \beta^* S$ ($X \beta^* D$). [1]

Theorem 5. *Let M be a G^* -supplemented module and $X \leq M$. If for every supplement submodule S of M , $\frac{(X+S)}{X}$ is a supplement submodule of $\frac{M}{X}$, then $\frac{M}{X}$ is G^* -supplemented.*

Proof. Let $\frac{N}{X} \leq \frac{M}{X}$. Since M is G^* -supplemented, there exists a supplement submodule S of M such that $N \beta^* S$. Then by [1, Proposition 2.9(i)], $\frac{N}{X} \beta^* \frac{S+X}{X}$. Since S is a supplement submodule of M , then by hypothesis, $\frac{S+X}{X}$ is a supplement submodule of $\frac{M}{X}$. Hence $\frac{M}{X}$ is G^* -supplemented. □

Corollary 5. *Let M be a G^* -supplemented module. If M is a distributive module, then $\frac{M}{X}$ is G^* -supplemented for every submodule X of M .*

Proof. Let S be a supplement submodule of M . There exists a submodule S' of M such that $M = S + S'$ and $S \cap S' \ll S$. Then $\frac{(S+X)}{X} + \frac{(S'+X)}{X} = \frac{M}{X}$. Let $\left[\frac{(S+X)}{X} \cap \frac{(S'+X)}{X} \right] + \frac{K}{X} = \frac{(S+X)}{X}$ for some $\frac{K}{X} \leq \frac{(S+X)}{X}$. Then $[X + (S \cap S')] + K = S + X$. Since $S \cap S' \ll S$, $K = S + X$. So $\frac{(S+X)}{X}$ is a supplement submodule of $\frac{M}{X}$. Thus $\frac{M}{X}$ is G^* -supplemented by Theorem 5. □

Definition 1. Let M be an R -module and $K \leq M$. We say that a submodule $\bar{T}$ of $\frac{M}{K}$ lifts to a submodule T of M , if under the natural morphism $\pi : M \rightarrow \frac{M}{K}$, $\pi(T) = \bar{T}$.

Theorem 6. *Let M be an R -module. If $\text{Rad}(M) \ll M$, then M is G^* -supplemented if and only if $\bar{M} = \frac{M}{\text{Rad}(M)}$ is semisimple and each submodule of $\bar{M}$ lifts to a supplement submodule of M .*

Proof.

($\Rightarrow$) Suppose that M is G^* -supplemented and $\bar{A} \leq \bar{M}$. Then M is a supplemented module by [1, Theorem 3.6]. So the full inverse image A of $\bar{A}$ has a supplement B in M . Then $A \cap B$ is small in B hence in M . Therefore $A \cap B \leq \text{Rad}(M)$ and consequently $\bar{A} \oplus \bar{B} = \bar{M}$. We conclude that $\bar{M}$ is semisimple.

Returning to $\bar{A} \leq \bar{M}$, we have a supplement submodule $S \leq M$ such that $A \beta^* S$. Then $\bar{A} \beta^* \frac{(S+\text{Rad}(M))}{\text{Rad}(M)}$. Since $\bar{M}$ is semisimple, $\bar{A} = \frac{(S+\text{Rad}(M))}{\text{Rad}(M)}$ by Theorem 1. Consequently $\bar{A}$ lifts to the supplement submodule S .

($\Leftarrow$) If $N \leq M$ is given, there exists a supplement submodule S of M such that $\bar{S} = \bar{N}$. Since $\text{Rad}(M) \ll M$, $N\beta^*S$. Thus M is G^* -supplemented. $\square$

Theorem 7. *Let R be a complete discrete valuation ring. Then every R -module is G^* -supplemented if and only if every R -module is amply supplemented.*

Proof.

($\Rightarrow$) Let M be an R -module. Then M is G^* -supplemented. By [1, Theorem 3.6], M is supplemented and hence by [9, Theorem 2.2(c)], M is amply supplemented.

($\Leftarrow$) Let M be an R -module. Then M is amply supplemented. From [1, Proposition 3.11], M is G^* -supplemented. $\square$

Theorem 8. *Let R be any ring. Then R is left perfect if and only if every projective R -module is G^* -supplemented.*

Proof.

($\Rightarrow$) Let R be a left perfect ring and M be a projective R -module. By [4, Theorem 4.41], M is supplemented. Since M is projective, M is G^* -supplemented by [1, Proposition 3.12].

($\Leftarrow$) Let M be a left R -module and $f : P \rightarrow M$ be an epimorphism with a projective R -module P . By assumption, P is G^* -supplemented. Then P is supplemented by [1, Theorem 3.6]. Since P is projective, P is π -projective. By [8, Theorem 41.16], every supplement submodule of P is a direct summand of P . Since P is G^* -supplemented, there exists a supplement submodule S in P such that $\text{Ker}(f)\beta^*S$ and also S is a direct summand. Let $P = S \oplus S'$ for some submodules S' of P . Then $P = \text{Ker}(f) + S'$ and $\text{Ker}(f) \cap S' \ll S'$. Let $g = f|_{S'}$. Then $g : S' \rightarrow M$ is a projective cover of M . Hence R is left perfect. $\square$

It is unknown whether every direct summand of a G^* -supplemented (G^* -lifting) module is G^* -supplemented (G^* -lifting).

Definition 2. Let M be an R -module. M is called completely G^* -supplemented (G^* -lifting) if every direct summand of M is G^* -supplemented (G^* -lifting).

It is clear that lifting and completely G^* -lifting modules are completely G^* -supplemented.

Theorem 9. *Suppose that M is G^* -supplemented and distributive R -module. If the intersection of any two supplement submodules of M is again a supplement submodule in M , then M is completely G^* -supplemented.*

Proof. Let D be a direct summand of M . There exists a direct summand D' of M such that $M = D \oplus D'$. Let A be a submodule of D . Since M is G^* -supplemented and $A \leq M$, there exists a supplement submodule S of M such that $A \beta^* S$. Since M is distributive, $A \beta^* (D \cap S)$. Also $D \cap S$ is a supplement submodule in D . Consequently D is G^* -supplemented and hence M is completely G^* -supplemented. $\square$

The SSP Property. A module M is said to have the *summand sum property*, if the sum of any two direct summands of M is again a direct summand of M .

Theorem 10. *Let M be a projective module with SSP. The followings are equivalent.*

- (1) M is supplemented.
- (2) M is quasi-discrete.
- (3) M is discrete.
- (4) M is lifting.
- (5) M is G^* -lifting (= H -supplemented).
- (6) M is completely G^* -lifting module.
- (7) M is amply supplemented.
- (8) M is $\oplus$ -supplemented.
- (9) M is completely $\oplus$ -supplemented.
- (10) M is G^* -supplemented.
- (11) M is completely G^* -supplemented.
- (12) M is semiperfect.

Proof. Result of [3, Theorem 2.11] and [1, Proposition 3.12]. $\square$

4. G^* -RADICAL SUPPLEMENTED MODULES

For submodules U and V of a module M , the submodule V is said to be a *radical supplement* (or briefly *Rad-supplement*) of U in M or U is said to have a *radical supplement* V in M if $U + V = M$ and $U \cap V \leq \text{Rad}(V)$. A module M is called a *radical supplemented* (or briefly *Rad-supplemented*) module if every submodule of M has a *Rad-supplement* in M (according to [5], a *generalized supplemented module*) and it is called *amply Rad-supplemented* in case $M = A + B$ implies that A has a *Rad-supplement* $B' \leq B$.

Lemma 4. *Let M be an R -module and $U, V \leq M$. V is a radical supplement of U in M if and only if $U + V = M$ and for every $m \in U \cap V$, $Rm \ll V$.*

Proof. See [6, Proposition 4]. $\square$

Lemma 5. *Let M be an R -module. If T is a *Rad-supplement* in M and $a \in T$ then $Ra \ll T$ iff $Ra \ll M$.*

Proof.

($\Rightarrow$) Clear.

($\Leftarrow$) Assume $Ra \ll M$. Let $X \leq T$ such that $Ra + X = T$. Since T is a *Rad*-supplement in M , there exists a submodule A of M such that $A + T = M$ and $A \cap T \leq \text{Rad}(T)$. Hence $Ra + X + A = M$. Since $Ra \ll M$, $X + A = M$. By the modular law, $X + (T \cap A) = T$. Since $a \in T$, there exist $a_1 \in X$ and $a_2 \in T \cap A$ such that $a = a_1 + a_2$. Thus $Ra \leq Ra_1 + Ra_2$. Since $Ra + X = T$ and $a_1 \in X$, $T = Ra_2 + X$. By Lemma 4, $Ra_2 \ll T$ and so $X = T$. Finally $Ra \ll T$. □

Theorem 11. *Let M be an R -module and $S, T \leq M$. If S is a radical supplement of T in M and T is a radical supplement in M , then T is a radical supplement of S in M .*

Proof. By hypothesis, there exist $A \leq M$ such that $A + T = M$ and $A \cap T \leq \text{Rad}(T)$ and also $S + T = M$ and $S \cap T \leq \text{Rad}(S)$. Let $X \leq T$ and a is an arbitrary element of $S \cap T$ such that $Ra + X = T$. Then $Ra + X + A = M$. Since $a \in S \cap T \leq \text{Rad}(S)$, $Ra \ll S$. Thus $Ra \ll M$. Since $a \in T$ and $Ra \ll M$, $Ra \ll T$ by Lemma 5. Hence T is a *Rad*-supplement of S in M by Lemma 4. □

Theorem 12. *Let $X, Y \leq M$ such that $X \beta^* Y$. Then, for $U \leq M$, U is a *Rad*-supplement of X in M if and only if U is a *Rad*-supplement of Y in M .*

Proof. Assume U is a *Rad*-supplement of X in M . Then $X + U = M$ and $X \cap U \leq \text{Rad}(U)$. Since $X \beta^* Y$, $Y + U = M$ by Lemma 1. Let $a \in U \cap Y$ and $T \leq U$ such that $Ra + T = U$. Then $Y + Ra + T = M$ and by $Ra \leq Y$, $Y + T = M$. Since $X \beta^* Y$, $X + T = M$ by Lemma 1. Hence $U = T + (U \cap X)$. So we can write $a = a_1 + a_2$ such that $a_1 \in T$ and $a_2 \in U \cap X$. Since $Ra \leq Ra_1 + Ra_2 \leq U$, $Ra_1 + Ra_2 + T = U$ and since $a_2 \in U \cap X$, $Ra_2 \ll U$ by Lemma 4. Thus $Ra_1 + T = U$ and since $Ra_1 \leq T$, $U = T$. So $Ra \ll U$. By Lemma 4, U is a *Rad*-supplement of Y . □

Theorem 13. *Let M be an R -module. If M is amply *Rad*-supplemented then its submodules which have the same *Rad*-supplements are equivalent with β^* .*

Proof. Let U and V be submodules of M . Assume that they have the same *Rad*-supplements and $U + T = M$ with $T \leq M$. Since M is amply *Rad*-supplemented, there exists a submodule $T' \leq T$ such that T' is a *Rad*-supplement of U in M . Since U and V have the same *Rad*-supplements, T' is a *Rad*-supplement of V too. Thus $V + T' = M$ and since $T' \leq T$, $V + T = M$. Similarly, if $V + K = M$ with $K \leq M$, then $U + K = M$. Therefore, from Lemma 1, $U \beta^* V$. □

Corollary 6. *Let M be an R -module. If M is amply supplemented then its submodules which have the same *Rad*-supplements are equivalent with β^* .*

Proof. Since M is amply supplemented, M is amply *Rad*-supplemented. So the proof is clear by Theorem 13. □

Definition 3. Let M be an R -module. M is called G^* -Radical supplemented (or briefly $G^* - \text{Rad}$ -supplemented) if for every submodule X of M , there is a Rad -supplement submodule U in M such that $X \beta^* U$.

Theorem 14. Let M be an R -module. Consider the following conditions:

- (1) M is G^* -supplemented.
- (2) M is $G^* - \text{Rad}$ -supplemented.
- (3) M is Rad -supplemented.

Then $(1) \Rightarrow (2) \Rightarrow (3)$.

Proof.

- (1) $\Rightarrow$ (2) Let M be G^* -supplemented. Since every supplement submodule is a Rad -supplement submodule, M is $G^* - \text{Rad}$ -supplemented.
- (2) $\Rightarrow$ (3) Let M be $G^* - \text{Rad}$ -supplemented. For every submodule X of M , there exists $V \leq M$ such that V is a Rad -supplement submodule in M and $X \beta^* V$. Hence, there exists $T \leq M$ such that $V + T = M$ and $V \cap T \leq \text{Rad}(V)$. Since M is $G^* - \text{Rad}$ -supplemented, there exists a submodule K of M such that K is a Rad -supplement submodule in M and $K \beta^* T$. Hence, V is Rad -supplement of K in M by Theorem 12. Since K is Rad -supplement submodule in M , K is a Rad -supplement of V in M by Theorem 11. Since $X \beta^* V$, K is Rad -supplement of X in M by Theorem 12. So M is Rad -supplemented.

□

Example 1. Let K be the quotient field of a Dedekind domain R which is not local. Let $M = K \oplus K$. Since $\text{Rad}(M) = M$, M is Rad -supplemented but not G^* -supplemented. To see this, assume that M is G^* -supplemented. Then M is supplemented by [1, Theorem 3.6]. Hence K is supplemented. Since K is supplemented, R is local by [10, Remark 2 in the proof of Theorem 3.1], a contradiction.

Theorem 15. Let M be an R -module. Consider the following conditions:

- (1) M is lifting.
- (2) M is G^* -lifting(= H -supplemented).
- (3) M is G^* -supplemented.
- (4) M is $G^* - \text{Rad}$ -supplemented.
- (5) M is Rad -supplemented.

Then $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$ hold. If every Rad -supplement submodule of M is projective then $(5) \Rightarrow (1)$ also holds.

Proof. $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$ is clear by [1, Theorem 3.6] and Theorem 14.

- (5) $\Rightarrow$ (1) Let M be Rad -supplemented and every Rad -supplement submodule of M be projective. Let $U < M$. Since M is Rad -supplemented, there is a Rad -supplement V of U in M . So $M = U + V$ and $U \cap V \leq \text{Rad}(V)$ and also $V \neq 0$. By hypothesis and $V \neq 0$, there is a maximal submodule T of V . Since $M =$

$U + V$ and $U \cap V \leq \text{Rad}(V) \leq T$, $\frac{M}{(U+T)} = \frac{(U+V)}{(U+T)} \cong \frac{V}{[V \cap (U+T)]} = \frac{V}{U \cap V + T} = \frac{V}{T}$. So $U + T$ is a maximal submodule of M and every proper submodule of M is contained in a maximal submodule in M . Hence $\text{Rad}(M) \ll M$. Let $X \leq M$. Since M is Rad -supplemented, there is a Rad -supplement Y of X in M . Then $M = X + Y$ and $X \cap Y \leq \text{Rad}(Y)$. So Y is a direct summand in M by [5, Lemma II.1]. Finally Y is a supplement of X in M and so M is supplemented. Since M is supplemented and projective, M is also lifting. $\square$

Let R be a ring. R is called a Bass ring, if every R -module has a maximal submodule. Also, a ring R is a Bass ring if and only if for every R -module M , $\text{Rad}(M) \ll M$ [2].

Theorem 16. *Let R be a Bass ring. Then an R -module M is G^* – Rad -supplemented if and only if every R -module M is G^* -supplemented.*

Proof. Since R is a Bass ring, every Rad -supplement submodule is a supplement submodule. Therefore, proof is clear. $\square$

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SOME PROPERTIES OF $\oplus - g$ -SUPPLEMENTED MODULES

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Abstract. In this work $\oplus - g$ -supplemented modules are defined and some properties of these modules are investigated. It is proved that the finite direct sum of $\oplus - g$ -supplemented modules is also $\oplus - g$ -supplemented. Let M be a distributive and $\oplus - g$ -supplemented R -module. Then every factor module and homomorphic image of M are $\oplus - g$ -supplemented. Let M be a $\oplus - g$ -supplemented R -module with *SSP* property. Then every direct summand of M is $\oplus - g$ -supplemented.

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1. INTRODUCTION

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let R be a ring and M be an R -module. We will denote a submodule N of M by $N \leq M$. Let M be an R -module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a *small submodule* of M and denoted by $N \ll M$. Let M be an R -module and $N \leq M$. If there exists a submodule K of M such that $M = N + K$ and $N \cap K = 0$, then N is called a *direct summand* of M and it is denoted by $M = N \oplus K$. For any R -module M , we have $M = M \oplus 0$. The intersection of all maximal submodules of M is called the *radical* of M and denoted by $RadM$. A submodule N of an R -module M is called an *essential submodule* and denoted by $N \leq_e M$ in case $K \cap N \neq 0$ for every submodule $K \neq 0$. Let M be an R -module and K be a submodule of M . K is called a *generalized small* (or briefly, *g -small*) *submodule* of M if for every essential submodule T of M with the property $M = K + T$ implies that $T = M$, then we write $K \ll_g M$ (in [10], it is called an *e -small submodule* of M and denoted by $K \ll_e M$). It is clear that every small submodule is a generalized small submodule but the converse is not true in general. Let M be an R -module. M is called a *hollow module* if every proper submodule of M is small in M . M is called a *generalized hollow* (or briefly, *g -hollow*) *module* if every proper submodule of M is g -small in M . Here it is clear that every hollow module is generalized hollow.

The converse of this statement is not always true. M is called a *local module* if M has the largest submodule, i.e. a proper submodule which contains all other proper submodules. Let U and V be submodules of M . If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M . M is called a *supplemented module* if every submodule of M has a supplement in M . If every submodule of M has a supplement that is a direct summand in M , then M is called a $\oplus$ -*supplemented module*. Let M be an R -module and $U, V \leq M$. If $M = U + V$ and $M = U + T$ with $T \trianglelefteq V$ implies that $T = V$, or equivalently, $M = U + V$ and $U \cap V \ll_g V$, then V is called a *g-supplement* of U in M . If every submodule of M has a g-supplement in M , then M is called a *g-supplemented module*. A module M is said to have the *Summand Sum Property* (SSP) if the sum of two direct summands of M is again a direct summand of M (see [1]). We say that a module M has (D3) property if $M_1 \cap M_2$ is a direct summand of M for every direct summands M_1 and M_2 of M with $M = M_1 + M_2$ (see [7]). The intersection of all essential maximal submodules of an R -module M is called the *generalized radical* of M and denoted by $\text{Rad}_g M$ (in [10], it is denoted by $\text{Rad}_e M$). If M have no essential maximal submodules, then we denote $\text{Rad}_g M = M$.

More information about supplemented modules are in [2, 9]. More results about $\oplus$ -supplemented modules are in [3, 4, 7]. More information about g-supplemented modules are in [5].

Now we will give some important properties of generalized small submodules.

Lemma 1. *Let M be an R -module and $K, N \leq M$. Consider the following conditions.*

- (1) *If $K \leq N$ and N is a generalized small submodule of M , then K is a generalized small submodule of M .*
- (2) *If K is contained in N and a generalized small submodule of N , then K is a generalized small submodule in submodules of M which contains N .*
- (3) *Let T be an R -module and $f : M \rightarrow T$ be an R -module homomorphism. If $K \ll_g M$, then $f(K) \ll_g T$.*
- (4) *If $K \ll_g L$ and $N \ll_g T$ with $L, T \leq M$, then $K + N \ll_g L + T$.*

Proof. See [5, Lemma 1]. □

Corollary 1. *Let M be an R -module and $K \leq N \leq M$. If $N \ll_g M$, then $N/K \ll_g M/K$. [5]*

Corollary 2. *Let M be an R -module, $K \ll_g M$ and $L \leq M$. Then $(K + L)/L \ll_g M/L$. [5]*

2. $\oplus$ - g -SUPPLEMENTED MODULES

Definition 1. Let M be an R -module. If every submodule of M has a g -supplement that is a direct summand of M , then M is called a $\oplus$ - g -supplemented module. (See also [8])

Clearly we can see that every $\oplus$ - g -supplemented module is g -supplemented. We also clearly can see that every $\oplus$ -supplemented and every generalized hollow modules are $\oplus$ - g -supplemented.

Lemma 2. Let M be an R -module, V be a supplement of U in M and $X, Y \leq V$. Then X is a g -supplement of Y in V if and only if X is a g -supplement of $U + Y$ in M .

Proof. ($\implies$) Let $M = U + Y + T$ with $T \leq X$. Since V is a supplement of U in M and $Y + T \leq V$, $V = Y + T$ and since X is a g -supplement of Y in V , then $T = X$. Hence X is a g -supplement of $U + Y$ in M .

($\impliedby$) Let $V = Y + T$ with $T \leq X$. Since V is a supplement of U in M , $M = U + V = U + Y + T$ and since X is a g -supplement of $U + Y$ in M , then $T = X$. Hence X is a g -supplement of Y in V . $\square$

Corollary 3. Let $M = M_1 \oplus M_2$ and $X, Y \leq M_2$. Then X is a g -supplement of Y in M_2 if and only if X is a g -supplement of $M_1 + Y$ in M .

Proof. Clear from Lemma 2. $\square$

Lemma 3. Let M be an R -module and $M = M_1 \oplus M_2$. If M_1 and M_2 are $\oplus$ - g -supplemented, then M is also $\oplus$ - g -supplemented.

Proof. Let U be any submodule of M . Since M_2 is $\oplus$ - g -supplemented, $(M_1 + U) \cap M_2$ has a g -supplement X that is a direct summand of M_2 . Since X is a g -supplement of $(M_1 + U) \cap M_2$ in M_2 , $M_2 = (M_1 + U) \cap M_2 + X$ and $(M_1 + U) \cap X = (M_1 + U) \cap M_2 \cap X \ll_g X$. By $M_2 = (M_1 + U) \cap M_2 + X$, $M = M_1 \oplus M_2 = M_1 + (M_1 + U) \cap M_2 + X = M_1 + U + X$. Since M_1 is $\oplus$ - g -supplemented, $(U + X) \cap M_1$ has a g -supplement Y that is a direct summand of M_1 . Since Y is a g -supplement of $(U + X) \cap M_1$ in M_1 , $M_1 = (U + X) \cap M_1 + Y$ and $(U + X) \cap Y = (U + X) \cap M_1 \cap Y \ll_g Y$. By $M_1 = (U + X) \cap M_1 + Y$, $M = M_1 + U + X = (U + X) \cap M_1 + Y + U + X = U + X + Y$. Since $(M_1 + U) \cap X \ll_g X$ and $(U + X) \cap Y \ll_g Y$, by Lemma 1, $U \cap (X + Y) \leq (U + Y) \cap X + (U + X) \cap Y \leq (M_1 + U) \cap X + (U + X) \cap Y \ll_g X + Y$. Hence $X + Y$ is a g -supplement of U in M . Since X is a direct summand of M_2 and Y is a direct summand of M_1 , $X + Y$ is a direct summand of $M = M_1 \oplus M_2$. Hence M is $\oplus$ - g -supplemented. $\square$

Corollary 4. Let M be an R -module and $M = M_1 \oplus M_2 \oplus \dots \oplus M_n$. If M_i is $\oplus$ - g -supplemented for every $i = 1, 2, \dots, n$, then M is also $\oplus$ - g -supplemented.

Proof. Clear from Lemma 3. $\square$

Proposition 1. *Let M be a $\oplus - g$ -supplemented module. If every g -supplement submodule in M is a direct summand of M , then every direct summand of M is $\oplus - g$ -supplemented.*

Proof. Let N be a direct summand of M and $M = N \oplus K$ with $K \leq M$. Since M is g -supplemented, by [5, Theorem 2], M/K is g -supplemented. By $\frac{M}{K} = \frac{N \oplus K}{K} \cong \frac{N}{N \cap K} = \frac{N}{0} \cong N$, N is also g -supplemented. Let $X \leq N$ and Y be a g -supplement of X in N . Since $M = N \oplus K$, by Corollary 3, Y is a g -supplement of $K + X$ in M . Since every g -supplement submodule in M is a direct summand of M , Y is a direct summand of M . By $Y \leq N$, Y is also a direct summand of N . Hence N is $\oplus - g$ -supplemented. $\square$

Lemma 4. *Let M be a $\oplus - g$ -supplemented R -module and $K \leq M$. If $\frac{X+K}{K}$ is a direct summand of $\frac{M}{K}$ for every direct summand X of M , then $\frac{M}{K}$ is $\oplus - g$ -supplemented.*

Proof. Let U/K be any submodule of M/K . Since M is $\oplus - g$ -supplemented, U has a g -supplement X in M that is a direct summand in M . Since X is a g -supplement of U in M and $K \leq U$, by [5, Lemma 4], $\frac{X+K}{K}$ is a g -supplement of U/K in M/K . Since X is a direct summand of M , by hypothesis, $\frac{X+K}{K}$ is a direct summand of M/K . Hence M/K is $\oplus - g$ -supplemented. $\square$

Lemma 5. *Let M be a distributive and $\oplus - g$ -supplemented R -module. Then every factor module of M is $\oplus - g$ -supplemented.*

Proof. Let $K \leq M$ and X be a direct summand of M . Since X is a direct summand of M , there exists $Y \leq M$ such that $M = X \oplus Y$. Since $M = X \oplus Y$, $\frac{M}{K} = \frac{X+K}{K} \oplus \frac{Y+K}{K}$. Since M is distributive, $(X+K) \cap (Y+K) = K$ and $\frac{X+K}{K} \cap \frac{Y+K}{K} = \frac{(X+K) \cap (Y+K)}{K} = \frac{K}{K} = 0$. Hence $\frac{M}{K} = \frac{X+K}{K} \oplus \frac{Y+K}{K}$ and by Lemma 4, M/K is $\oplus - g$ -supplemented. $\square$

Corollary 5. *Let M be a distributive and $\oplus - g$ -supplemented R -module. Then every homomorphic image of M is $\oplus - g$ -supplemented.*

Proof. Clear from Lemma 5. $\square$

Lemma 6. *Let M be a $\oplus - g$ -supplemented R -module with $(D3)$ property. Then every direct summand of M is $\oplus - g$ -supplemented.*

Proof. Let K be any direct summand of M . Then there exists $T \leq M$ such that $M = K \oplus T$. Let $U \leq K$. Since M is $\oplus - g$ -supplemented, U has a g -supplement X that is a direct summand in M . Here $M = U + X$ and $U \cap X \ll_g X$. Since $U \leq K$, $M = U + X = K + X$ and since M has $(D3)$ property, $K \cap X$ is a direct summand of M . Then there exists $Y \leq M$ such that $M = (K \cap X) \oplus Y$. Here $K = (K \cap X) \oplus (K \cap Y)$. Since $M = U + X$ and $U \leq K$, by Modular Law, $K = U + (K \cap X)$. Let $\pi: M \rightarrow K \cap X$ be a canonical projection. Since $U \cap X \ll_g X \leq M$, by Lemma 1, $U \cap K \cap X = U \cap X = \pi(U \cap X) \ll_g K \cap X$. Hence K is $\oplus - g$ -supplemented. $\square$

Corollary 6. *Let M be a $\oplus - g$ -supplemented R -module with $(D3)$ property. Then M/X is $\oplus - g$ -supplemented for every direct summand X of M .*

Proof. Let X be any direct summand of M . Then there exists $Y \leq M$ such that $M = X \oplus Y$. By Lemma 6, Y is $\oplus - g$ -supplemented. Then by $\frac{M}{X} = \frac{X+Y}{X} \cong \frac{Y}{X \cap Y} = \frac{Y}{0} \cong Y$, M/X is also $\oplus - g$ -supplemented. $\square$

Corollary 7. *Let M be a $\oplus - g$ -supplemented R -module with (D3) property and $f : M \rightarrow N$ be an R -module epimorphism with N is an R -module and $\text{Ker}(f)$ is a direct summand of M . Then N is $\oplus - g$ -supplemented.*

Proof. Clear from Corollary 6, since $M/\text{Ker}(f) \cong \text{Im}(f) = N$. $\square$

Lemma 7. *Let M be a $\oplus - g$ -supplemented R -module, $K \leq M$ and $K = (K \cap M_1) \oplus (K \cap M_2)$ for every $M_1, M_2 \leq M$ with $M = M_1 \oplus M_2$. Then M/K is $\oplus - g$ -supplemented.*

Proof. Let $U/K \leq M/K$. Since M is $\oplus - g$ -supplemented, U has a g -supplement V that is a direct summand in M . Here there exists $X \leq M$ such that $M = V \oplus X$. By hypothesis, $K = (K \cap V) \oplus (K \cap X)$. Since V is a g -supplement of U in M and $K \leq U$, by [5, Lemma 4], $\frac{V+K}{K}$ is a g -supplement of U/K in M/K . Since $M = V \oplus X$, $\frac{M}{K} = \frac{V+K}{K} + \frac{X+K}{K}$. Here $\frac{V+K}{K} \cap \frac{X+K}{K} = \frac{(V+K) \cap (X+K)}{K} = \frac{(V+K) \cap X+K}{K} = \frac{(V+K \cap V+K \cap X) \cap X+K}{K} = \frac{(V+K \cap X) \cap X+K}{K} = \frac{V \cap X+K \cap X+K}{K} = \frac{0+K}{K} = \frac{K}{K} = 0$. Hence $\frac{M}{K} = \frac{V+K}{K} \oplus \frac{X+K}{K}$. Thus M/K is $\oplus - g$ -supplemented. $\square$

Corollary 8. *Let M be a $\oplus - g$ -supplemented R -module, $f : M \rightarrow N$ be and R -module epimorphism with N be an R -module and $\text{Ker}(f) = (\text{Ker}(f) \cap M_1) \oplus (\text{Ker}(f) \cap M_2)$ for every $M_1, M_2 \leq M$ with $M = M_1 \oplus M_2$. Then N is $\oplus - g$ -supplemented.*

Proof. Clear from Lemma 7, since $M/\text{Ker}(f) \cong \text{Im}(f) = N$. $\square$

Proposition 2. *Let M be a $\oplus - g$ -supplemented R -module. Then there exist $M_1, M_2 \leq M$ such that $M = M_1 \oplus M_2$, $\text{Rad}_g M_1 \ll_g M_1$ and $\text{Rad}_g M_2 = M_2$.*

Proof. Since M is $\oplus - g$ -supplemented, $\text{Rad}_g M$ has a g -supplement M_1 in M such that M_1 is a direct summand of M . Since M_1 is a direct summand of M , there exists $M_2 \leq M$ such that $M = M_1 \oplus M_2$. Since M_1 is a g -supplement of $\text{Rad}_g M$ in M , $M = \text{Rad}_g M + M_1$ and $M_1 \cap \text{Rad}_g M \ll_g M_1$. Since $M = M_1 \oplus M_2$, by [6, Lemma 4], $\text{Rad}_g M = \text{Rad}_g M_1 \oplus \text{Rad}_g M_2$. Hence $\text{Rad}_g M_1 = M_1 \cap \text{Rad}_g M \ll_g M_1$. Since $\text{Rad}_g M = \text{Rad}_g M_1 \oplus \text{Rad}_g M_2$, $M = \text{Rad}_g M + M_1 = \text{Rad}_g M_1 + \text{Rad}_g M_2 + M_1 = M_1 \oplus \text{Rad}_g M_2$. Hence $M_2 = M_2 \cap M = M_2 \cap (M_1 \oplus \text{Rad}_g M_2) = (M_2 \cap M_1) \oplus \text{Rad}_g M_2 = 0 \oplus \text{Rad}_g M_2 = \text{Rad}_g M_2$. $\square$

Proposition 3. *Let M be a $\oplus - g$ -supplemented R -module. Then there exist $M_1, M_2 \leq M$ such that $M = M_1 \oplus M_2$, $\text{Rad} M_1 \ll_g M_1$ and $\text{Rad} M_2 = M_2$.*

Proof. We can also prove this similar to proof of the previous Proposition. But we prove by different way. Since M is $\oplus - g$ -supplemented, $\text{Rad} M$ has a g -supplement

M_1 in M such that M_1 is a direct summand of M . Since M_1 is a direct summand of M , there exists $M_2 \leq M$ such that $M = M_1 \oplus M_2$. Since M_1 is a g -supplement of $\text{Rad}M$ in M , $M = \text{Rad}M + M_1$ and $\text{Rad}M_1 = M_1 \cap \text{Rad}M \ll_g M_1$. Assume that X be a maximal submodule of M_2 . Since $\frac{M}{M_1+X} = \frac{M_1+M_2}{M_1+X} \cong \frac{M_2}{M_2 \cap (M_1+X)} = \frac{M_2}{M_2 \cap M_1+X} = \frac{M_2}{X}$, $M_1 + X$ is a maximal submodule of M . Then $M = \text{Rad}M + M_1 \leq M_1 + X$. This is a contradiction. Hence $\text{Rad}M_2 = M_2$. $\square$

Lemma 8. *Let M be a $\oplus - g$ -supplemented R -module with SSP property. Then M/K is $\oplus - g$ -supplemented for every direct summand K of M .*

Proof. Let K be any direct summand of M and $U/K \leq M/K$. Since M is $\oplus - g$ -supplemented, U has a g -supplement V in M such that V is a direct summand of M . By [5, Lemma 4], $\frac{V+K}{K}$ is a g -supplement of U/K in M/K . Since K and V are direct summands of M and M has SSP property, $K + V$ is also a direct summand of M . Hence there exists $T \leq M$ such that $M = (K + V) \oplus T$. Since $M = (K + V) \oplus T$, $\frac{M}{K} = \frac{K+V+T}{K} = \frac{V+K}{K} + \frac{T+K}{K}$. Since $(V + K) \cap T = 0$, $\frac{V+K}{K} \cap \frac{T+K}{K} = \frac{(V+K) \cap (T+K)}{K} = \frac{(V+K) \cap T+K}{K} = \frac{0+K}{K} = 0$. Hence $\frac{M}{K} = \frac{V+K}{K} \oplus \frac{T+K}{K}$ and M/K is $\oplus - g$ -supplemented. $\square$

Corollary 9. *Let M be a $\oplus - g$ -supplemented R -module with SSP property. Then every direct summand of M is $\oplus - g$ -supplemented.*

Proof. Let T be any direct summand of M . Then there exists a submodule K of M such that $M = T \oplus K$. By Lemma 8, M/K is $\oplus - g$ -supplemented. Since $\frac{M}{K} = \frac{T+K}{K} \cong \frac{T}{T \cap K} = \frac{T}{0} \cong T$, T is also $\oplus - g$ -supplemented. $\square$

Remark 1. Let M be an R -module which has only four proper submodules 0 , A , B , C with $C \leq A$, $C \leq B$, $A \not\leq B$ and $B \not\leq A$. Then M is g -supplemented but not $\oplus - g$ -supplemented.

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OSTROWSKI TYPE INEQUALITIES FOR FUNCTIONS OF TWO VARIABLES VIA GENERALIZED FRACTIONAL INTEGRALS

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Abstract. In this paper, we establish some Ostrowski type integral inequalities for functions of two variables involving generalized fractional integrals. The results presented here provide extensions of those given in earlier works.

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1. INTRODUCTION

The study of various types of integral inequalities has been the focus of great attention for well over a century by a number of mathematicians, interested both in pure and applied mathematics. One of the many fundamental mathematical discoveries of A. M. Ostrowski [20] is the following classical integral inequality associated with the differentiable mappings:

Theorem 1. *Let $f: [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) whose derivative $f': (a, b) \rightarrow \mathbb{R}$ is bounded on (a, b) , i.e. $\|f'\|_\infty = \sup_{t \in (a, b)} |f'(t)| < +\infty$. Then, we have the following inequality:*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty,$$

for all $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible.

The Ostrowski inequality has applications in quadrature, probability and optimization theory, stochastic, statistics, information and integral operator theory. Until now, a large number of research papers and books have been written on Ostrowski inequalities and their numerous applications, see [1–3, 5–15, 17–19, 21–23].

Definition 1. Let $f \in L_1[a, b]$. The Riemann–Liouville fractional integrals $J_{a+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha > 0$ are defined by

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b,$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

Hadamard fractional integrals given by as follows:

Definition 2 ([16, Page 110]). Let $f \in L_1([a, b])$. The Hadamard fractional integrals $\mathbf{H}_{a+}^\alpha f$ and $\mathbf{H}_{b-}^\alpha f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$\mathbf{H}_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} \frac{f(t)}{t} dt, \quad x > a$$

and

$$\mathbf{H}_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{t}{x}\right)^{\alpha-1} \frac{f(t)}{t} dt, \quad x < b,$$

respectively.

Definition 3 ([16, Page 99-100]). Let $g: [a, b] \rightarrow \mathbb{R}$ be a positive increasing function on $(a, b]$, having a continuous derivative $g'(x)$ on (a, b) . The left-side ($I_{a+;g}^\alpha f(x)$) and right-side ($I_{b-;g}^\alpha f(x)$) fractional integral of f with respect to the function g on $[a, b]$ of order $\alpha > 0$ are defined by

$$I_{a+;g}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t)f(t)}{[g(x)-g(t)]^{1-\alpha}} dt, \quad x > a$$

and

$$I_{b-;g}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{g'(t)f(t)}{[g(t)-g(x)]^{1-\alpha}} dt, \quad x < b,$$

respectively.

Hadamard fractional integrals of a function with two variables can be given as follows:

Definition 4. Let $\Delta = [a, b] \times [c, d]$ and $f \in L_1(\Delta)$. The Hadamard fractional integrals $\mathbf{J}_{a+,c+}^{\alpha,\beta} f$, $\mathbf{J}_{a+,d-}^{\alpha,\beta} f$, $\mathbf{J}_{b-,c+}^{\alpha,\beta} f$ and $\mathbf{J}_{b-,d-}^{\alpha,\beta} f$ of order $\alpha, \beta > 0$ with $a, c \geq 0$ are defined by

$$\begin{aligned}\mathbf{J}_{a+,c+}^{\alpha,\beta} f(x,y) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_c^y \left(\ln \frac{x}{t}\right)^{\alpha-1} \left(\ln \frac{y}{s}\right)^{\beta-1} \frac{f(t,s)}{ts} ds dt, \quad x > a, \quad y > c, \\ \mathbf{J}_{a+,d-}^{\alpha,\beta} f(x,y) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_y^d \left(\ln \frac{x}{t}\right)^{\alpha-1} \left(\ln \frac{s}{y}\right)^{\beta-1} \frac{f(t,s)}{ts} ds dt, \quad x > a, \quad y < d, \\ \mathbf{J}_{b-,c+}^{\alpha,\beta} f(x,y) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_c^y \left(\ln \frac{t}{x}\right)^{\alpha-1} \left(\ln \frac{y}{s}\right)^{\beta-1} \frac{f(t,s)}{ts} ds dt, \quad x < b, \quad y > c\end{aligned}$$

and

$$\mathbf{J}_{b-,d-}^{\alpha,\beta} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_y^d \left(\ln \frac{t}{x}\right)^{\alpha-1} \left(\ln \frac{s}{y}\right)^{\beta-1} \frac{f(t,s)}{ts} ds dt, \quad x < b, \quad y < d,$$

respectively.

Now, we give the following generalized fractional integral operators:

Definition 5 ([4, Definition 6]). Let $g: [a, b] \rightarrow \mathbb{R}$ be a positive increasing function on $(a, b]$, having a continuous derivative $g'(x)$ on (a, b) and let $w: [c, d] \rightarrow \mathbb{R}$ be a positive increasing function on $(c, d]$, having a continuous derivative $w'(y)$ on (c, d) . If $f \in L_1(\Delta)$, then for $\alpha, \beta > 0$, the generalized fractional integral operators for functions of two variables are defined by

$$\begin{aligned}\mathcal{J}_{a+,c+;g,w}^{\alpha,\beta} f(x,y) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_c^y \frac{g'(t)}{[g(x) - g(t)]^{1-\alpha}} \frac{w'(s)}{[w(y) - w(s)]^{1-\beta}} f(t,s) ds dt, \\ &\quad x > a, \quad y > c; \\ \mathcal{J}_{a+,d-;g,w}^{\alpha,\beta} f(x,y) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_y^d \frac{g'(t)}{[g(x) - g(t)]^{1-\alpha}} \frac{w'(s)}{[w(s) - w(y)]^{1-\beta}} f(t,s) ds dt, \\ &\quad x > a, \quad y < d; \\ \mathcal{J}_{b-,c+;g,w}^{\alpha,\beta} f(x,y) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_c^y \frac{g'(t)}{[g(t) - g(x)]^{1-\alpha}} \frac{w'(s)}{[w(y) - w(s)]^{1-\beta}} f(t,s) ds dt, \\ &\quad x < b, \quad y > c\end{aligned}$$

and

$$J_{b-,d-;g,w}^{\alpha,\beta} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_y^d \frac{g'(t)}{[g(t)-g(x)]^{1-\alpha}} \frac{w'(s)}{[w(s)-w(y)]^{1-\beta}} f(t,s) ds dt,$$

$$x < b, \quad y < d$$

respectively.

The aim of this study is to establish Ostrowski type integral inequalities for functions of two variables involving generalized fractional integrals. The results presented in this paper provide extensions of those given in earlier works.

2. SOME OSTROWSKI TYPE INEQUALITIES FOR GENERALIZED FRACTIONAL INTEGRALS

Firstly, we define the following functions which will be used frequently:

$$M_g^\alpha(a,b;x) = \frac{[g(x)-g(a)]^\alpha + [g(b)-g(x)]^\alpha}{\Gamma(\alpha+1)}$$

and

$$N_w^\beta(c,d;y) = \frac{[w(y)-w(c)]^\beta + [w(d)-w(y)]^\beta}{\Gamma(\beta+1)},$$

for $(x,y) \in \Delta$. Introduce $M_g^\alpha(a,b)$ and $N_w^\beta(c,d)$ by acting

$$M_g^\alpha(a,b) = M_g^\alpha(a,b;a) = M_g^\alpha(a,b;b) = \frac{[g(b)-g(a)]^\alpha}{\Gamma(\alpha+1)}$$

and

$$N_w^\beta(c,d) = N_w^\beta(c,d;c) = N_w^\beta(c,d;d) = \frac{[w(d)-w(c)]^\beta}{\Gamma(\beta+1)}.$$

By choosing $g(t) = \ln t$, $t \in [a,b]$ and $w(s) = \ln s$, $s \in [c,d]$, we get representations

$$M_{\ln}^\alpha(a,b;x) = \frac{[\ln \frac{x}{a}]^\alpha + [\ln \frac{b}{x}]^\alpha}{\Gamma(\alpha+1)} \quad \text{and} \quad N_{\ln}^\beta(c,d;y) = \frac{[\ln \frac{y}{c}]^\beta + [\ln \frac{d}{y}]^\beta}{\Gamma(\beta+1)}$$

and

$$M_{\ln}^\alpha(a,b) = \frac{[\ln \frac{b}{a}]^\alpha}{\Gamma(\alpha+1)} \quad \text{and} \quad N_{\ln}^\beta(c,d) = \frac{[\ln \frac{d}{c}]^\beta}{\Gamma(\beta+1)}.$$

Throughout this paper, we denote the second partial derivative $\frac{\partial^2 f}{\partial t \partial s}$ by f_{ts} . We assume also that $g: [a,b] \rightarrow \mathbb{R}$ is a positive increasing function on $(a,b]$, having a continuous derivative $g'(x)$ on (a,b) and $w: [c,d] \rightarrow \mathbb{R}$ is a positive increasing function on $(c,d]$, having a continuous derivative $w'(y)$ on (c,d) .

Now, we are in position to prove the following identity:

Lemma 1. Let $f: \Delta \rightarrow \mathbb{R}$ be a twice partial differentiable mapping on Δ° with $a < b, c < d$. If $\frac{\partial^2 f}{\partial t \partial s} \in L(\Delta)$, then for $\alpha, \beta > 0$ we have the following equality:

$$\begin{aligned} & f(x, y) - \frac{1}{M_g^\alpha(a, b; x)} \left[J_{x^-; g}^\alpha f(a, y) + J_{x^+; g}^\alpha f(b, y) \right] \\ & - \frac{1}{N_w^\beta(c, d; y)} \left[J_{y^-; w}^\beta f(x, c) + J_{y^+; w}^\beta f(x, d) \right] + \frac{1}{M_g^\alpha(a, b; x) N_w^\beta(c, d; y)} \\ & \times \left[J_{x^- y^-; g, w}^{\alpha, \beta} f(a, c) + J_{x^- y^+; g, w}^{\alpha, \beta} f(a, d) + J_{x^+ y^-; g, w}^{\alpha, \beta} f(b, c) + J_{x^+ y^+; g, w}^{\alpha, \beta} f(b, d) \right] \\ & = \frac{1}{\Gamma(\alpha + 1) \Gamma(\beta + 1) M_g^\alpha(a, b; x) N_w^\beta(c, d; y)} [I_1 - I_2 - I_3 + I_4], \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_a^x \int_c^y (g(t) - g(a))^\alpha (w(s) - w(c))^\beta f_{ts}(t, s) ds dt, \\ I_2 &= \int_a^x \int_y^d (g(t) - g(a))^\alpha (w(d) - w(s))^\beta f_{ts}(t, s) ds dt, \\ I_3 &= \int_x^b \int_c^y (g(b) - g(t))^\alpha (w(s) - w(c))^\beta f_{ts}(t, s) ds dt \end{aligned}$$

and

$$I_4 = \int_x^b \int_y^d (g(b) - g(t))^\alpha (w(d) - w(s))^\beta f_{ts}(t, s) ds dt.$$

Proof. By integration by parts, we have

$$\begin{aligned} I_1 &= \int_a^x \int_c^y (g(t) - g(a))^\alpha (w(s) - w(c))^\beta f_{ts}(t, s) ds dt \\ &= \int_a^x (g(t) - g(a))^\alpha \left[(w(s) - w(c))^\beta f_t(t, s) \Big|_c^y \right. \\ &\quad \left. - \beta \int_c^y (w(s) - w(c))^{\beta-1} w'(s) f_t(t, s) ds \right] dt \\ &= \int_a^x (g(t) - g(a))^\alpha (w(y) - w(c))^\beta f_t(t, y) dt \end{aligned} \tag{2.1}$$

$$\begin{aligned}
& -\beta \int_a^x \int_c^y (g(t) - g(a))^\alpha (w(s) - w(c))^{\beta-1} w'(s) f_t(t, s) ds dt \\
& = (w(y) - w(c))^\beta \\
& \quad \times \left[(g(t) - g(a))^\alpha f(t, y) - \alpha \int_a^x (g(t) - g(a))^{\alpha-1} g'(t) f(t, y) dt \right] \\
& \quad - \beta \int_c^y (w(s) - w(c))^{\beta-1} w'(s) \\
& \quad \times \left[(g(t) - g(a))^\alpha f(t, s) \Big|_a^x - \alpha \int_a^x (g(t) - g(a))^{\alpha-1} g'(t) f(t, s) dt \right] ds \\
& = (w(y) - w(c))^\beta (g(x) - g(a))^\alpha f(x, y) - \Gamma(\alpha + 1) (w(y) - w(c))^\beta J_{x^-; g}^\alpha f(a, y) \\
& \quad - \Gamma(\beta + 1) (g(x) - g(a))^\alpha J_{y^-; w}^\beta f(x, c) + \Gamma(\alpha + 1) \Gamma(\beta + 1) J_{x^- y^-; g, w}^{\alpha, \beta} f(a, c).
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
I_2 = & -(g(x) - g(a))^\alpha (w(d) - w(y))^\beta f(x, y) \\
& + \Gamma(\alpha + 1) (w(d) - w(y))^\beta J_{x^-; g}^\alpha f(a, y) \\
& + \Gamma(\beta + 1) (g(x) - g(a))^\alpha J_{y^+; w}^\beta f(x, d) \\
& - \Gamma(\alpha + 1) \Gamma(\beta + 1) J_{x^- y^+; g, w}^{\alpha, \beta} f(a, d),
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
I_3 = & -(g(b) - g(x))^\alpha (w(y) - w(c))^\beta f(x, y) \\
& + \Gamma(\alpha + 1) (w(y) - w(c))^\beta J_{x^+; g}^\alpha f(b, y) \\
& + \Gamma(\beta + 1) (g(b) - g(x))^\alpha J_{y^-; w}^\beta f(x, c) \\
& - \Gamma(\alpha + 1) \Gamma(\beta + 1) J_{x^+ y^-; g, w}^{\alpha, \beta} f(b, c)
\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}
I_4 = & (g(b) - g(x))^\alpha (w(d) - w(y))^\beta f(x, y) \\
& - \Gamma(\alpha + 1) (w(d) - w(y))^\beta J_{x^+; g}^\alpha f(b, y) \\
& - \Gamma(\beta + 1) (g(b) - g(x))^\alpha J_{y^+; w}^\beta f(x, d) \\
& + \Gamma(\alpha + 1) \Gamma(\beta + 1) J_{x^+ y^+; g, w}^{\alpha, \beta} f(b, d).
\end{aligned} \tag{2.4}$$

By equalities (2.1)–(2.4), we establish

$$\begin{aligned} I_1 - I_2 - I_3 + I_4 &= \Gamma(\alpha + 1)\Gamma(\beta + 1)M_g^\alpha(a, b; x)N_w^\beta(c, d; y)f(x, y) \\ &\quad - \Gamma(\alpha + 1)\Gamma(\beta + 1)N_w^\beta(c, d; y)\left[J_{x^-;g}^\alpha f(a, y) + J_{x^+;g}^\alpha f(b, y)\right] \\ &\quad - \Gamma(\alpha + 1)\Gamma(\beta + 1)M_g^\alpha(a, b; x)\left[J_{y^-;w}^\beta f(x, c) + J_{y^+;w}^\beta f(x, d)\right] \\ &\quad + \Gamma(\alpha + 1)\Gamma(\beta + 1) \\ &\quad \times \left[J_{x^-y^-;g,w}^{\alpha,\beta} f(a, c) + J_{x^-y^+;g,w}^{\alpha,\beta} f(a, d) + J_{x^+y^-;g,w}^{\alpha,\beta} f(b, c) + J_{x^+y^+;g,w}^{\alpha,\beta} f(b, d)\right], \end{aligned}$$

which gives the desired result. $\square$

Theorem 2. Under assumptions of Lemma 1, if the function $\frac{f_{ts}}{g'w'} \in L_\infty(\Delta)$, then we have the following Ostrowski type inequality for the generalized fractional integrals:

$$\begin{aligned} &\left| f(x, y) - \frac{1}{M_g^\alpha(a, b; x)} \left[J_{x^-;g}^\alpha f(a, y) + J_{x^+;g}^\alpha f(b, y) \right] \right. \\ &\quad - \frac{1}{N_w^\beta(c, d; y)} \left[J_{y^-;w}^\beta f(x, c) + J_{y^+;w}^\beta f(x, d) \right] + \frac{1}{M_g^\alpha(a, b; x)N_w^\beta(c, d; y)} \\ &\quad \times \left[J_{x^-y^-;g,w}^{\alpha,\beta} f(a, c) + J_{x^-y^+;g,w}^{\alpha,\beta} f(a, d) + J_{x^+y^-;g,w}^{\alpha,\beta} f(b, c) + J_{x^+y^+;g,w}^{\alpha,\beta} f(b, d) \right] \Big| \\ &\leq \frac{M_g^{\alpha+1}(a, b; x)N_w^{\beta+1}(c, d; y)}{M_g^\alpha(a, b; x)N_w^\beta(c, d; y)} \left\| \frac{f_{ts}}{g'w'} \right\|_{\Delta, \infty} \end{aligned}$$

where

$$\left\| \frac{f_{ts}}{g'w'} \right\|_{\Delta, \infty} = \sup_{(t,s) \in \Delta} \left| \frac{f_{ts}(t, s)}{g'(t)w'(s)} \right|.$$

Proof. Using Lemma 1, we get

$$\begin{aligned} &\left| f(x, y) - \frac{1}{M_g^\alpha(a, b; x)} \left[J_{x^-;g}^\alpha f(a, y) + J_{x^+;g}^\alpha f(b, y) \right] \right. \\ &\quad - \frac{1}{N_w^\beta(c, d; y)} \left[J_{y^-;w}^\beta f(x, c) + J_{y^+;w}^\beta f(x, d) \right] + \frac{1}{M_g^\alpha(a, b; x)N_w^\beta(c, d; y)} \\ &\quad \times \left[J_{x^-y^-;g,w}^{\alpha,\beta} f(a, c) + J_{x^-y^+;g,w}^{\alpha,\beta} f(a, d) + J_{x^+y^-;g,w}^{\alpha,\beta} f(b, c) + J_{x^+y^+;g,w}^{\alpha,\beta} f(b, d) \right] \Big| \\ &\leq \frac{1}{\Gamma(\alpha + 1)\Gamma(\beta + 1)M_g^\alpha(a, b; x)N_w^\beta(c, d; y)} [|I_1| + |I_2| + |I_3| + |I_4|]. \end{aligned} \tag{2.5}$$

By the assumptions of Theorem 2, we have

$$\begin{aligned}
 |I_1| &= \left| \int_a^x \int_c^y (g(t) - g(a))^\alpha (w(s) - w(c))^\beta f_{ts}(t, s) ds dt \right| \\
 &\leq \int_a^x \int_c^y (g(t) - g(a))^\alpha (w(s) - w(c))^\beta |f_{ts}(t, s)| ds dt \\
 &= \int_a^x \int_c^y (g(t) - g(a))^\alpha (w(s) - w(c))^\beta \left| \frac{f_{ts}(t, s)}{g'(t)w'(s)} \right| g'(t)w'(s) ds dt \\
 &\leq \left\| \frac{f_{ts}}{g'w'} \right\|_{[a,x] \times [c,y], \infty} \int_a^x \int_c^y (g(t) - g(a))^\alpha (w(s) - w(c))^\beta g'(t)w'(s) ds dt \\
 &= \frac{(g(x) - g(a))^{\alpha+1} (w(y) - w(c))^{\beta+1}}{(\alpha+1)(\beta+1)} \left\| \frac{f_{ts}}{g'w'} \right\|_{[a,x] \times [c,y], \infty}
 \end{aligned} \tag{2.6}$$

and

$$\begin{aligned}
 |I_2| &= \left| \int_a^x \int_y^d (g(t) - g(a))^\alpha (w(d) - w(s))^\beta f_{ts}(t, s) ds dt \right| \\
 &\leq \int_a^x \int_y^d (g(t) - g(a))^\alpha (w(d) - w(s))^\beta |f_{ts}(t, s)| ds dt \\
 &= \int_a^x \int_y^d (g(t) - g(a))^\alpha (w(d) - w(s))^\beta |f_{ts}(t, s)| ds dt \\
 &= \int_a^x \int_y^d (g(t) - g(a))^\alpha (w(d) - w(s))^\beta \left| \frac{f_{ts}(t, s)}{g'(t)w'(s)} \right| g'(t)w'(s) ds dt \\
 &\leq \left\| \frac{f_{ts}}{g'w'} \right\|_{[a,x] \times [y,d], \infty} \int_a^x \int_y^d (g(t) - g(a))^\alpha (w(d) - w(s))^\beta g'(t)w'(s) ds dt \\
 &= \frac{(g(x) - g(a))^{\alpha+1} (w(d) - w(y))^{\beta+1}}{(\alpha+1)(\beta+1)} \left\| \frac{f_{ts}}{g'w'} \right\|_{[a,x] \times [y,d], \infty}.
 \end{aligned}$$

In a similar way, we obtain

$$|I_3| = \left| \int_x^b \int_c^y (g(b) - g(t))^\alpha (w(s) - w(c))^\beta f_{t,s}(t, s) ds dt \right| \quad (2.7)$$

$$\leq \frac{(g(b) - g(x))^{\alpha+1} (w(y) - w(c))^{\beta+1}}{(\alpha+1)(\beta+1)} \left\| \frac{f_{ts}}{g'w'} \right\|_{[x,b] \times [c,y], \infty}$$

and

$$|I_4| = \left| \int_x^b \int_y^d (g(b) - g(t))^\alpha (w(d) - w(s))^\beta f_{t,s}(t, s) ds dt \right| \quad (2.8)$$

$$\leq \frac{(g(b) - g(x))^{\alpha+1} (w(d) - w(y))^{\beta+1}}{(\alpha+1)(\beta+1)} \left\| \frac{f_{ts}}{g'w'} \right\|_{[x,b] \times [y,d], \infty}.$$

By using the inequalities (2.6)–(2.8) in (2.5), we get

$$\begin{aligned} & \left| f(x, y) - \frac{1}{M_g^\alpha(a, b; x)} \left[J_{x^-; g}^\alpha f(a, y) + J_{x^+; g}^\alpha f(b, y) \right] \right. \\ & - \frac{1}{N_w^\beta(c, d; y)} \left[J_{y^-; w}^\beta f(x, c) + J_{y^+; w}^\beta f(x, d) \right] + \frac{1}{M_g^\alpha(a, b; x) N_w^\beta(c, d; y)} \\ & \times \left[J_{x^- y^-; g, w}^{\alpha, \beta} f(a, c) + J_{x^- y^+; g, w}^{\alpha, \beta} f(a, d) + J_{x^+ y^-; g, w}^{\alpha, \beta} f(b, c) + J_{x^+ y^+; g, w}^{\alpha, \beta} f(b, d) \right] \Big| \\ & \leq \frac{1}{\Gamma(\alpha+1) \Gamma(\beta+1) M_g^\alpha(a, b; x) N_w^\beta(c, d; y)} \\ & \times \left[\frac{(g(x) - g(a))^{\alpha+1} (w(y) - w(c))^{\beta+1}}{(\alpha+1)(\beta+1)} \left\| \frac{f_{ts}}{g'w'} \right\|_{[a,x] \times [c,y], \infty} \right. \\ & + \frac{(g(x) - g(a))^{\alpha+1} (w(d) - w(y))^{\beta+1}}{(\alpha+1)(\beta+1)} \left\| \frac{f_{ts}}{g'w'} \right\|_{[a,x] \times [y,d], \infty} \\ & + \frac{(g(b) - g(x))^{\alpha+1} (w(y) - w(c))^{\beta+1}}{(\alpha+1)(\beta+1)} \left\| \frac{f_{ts}}{g'w'} \right\|_{[x,b] \times [c,y], \infty} \\ & \left. + \frac{(g(b) - g(x))^{\alpha+1} (w(d) - w(y))^{\beta+1}}{(\alpha+1)(\beta+1)} \left\| \frac{f_{ts}}{g'w'} \right\|_{[x,b] \times [y,d], \infty} \right] \\ & \leq \frac{1}{\Gamma(\alpha+2) \Gamma(\beta+2) M_g^\alpha(a, b; x) N_w^\beta(c, d; y)} \left((g(x) - g(a))^{\alpha+1} + (g(b) - g(x))^{\alpha+1} \right) \\ & \times \left((w(y) - w(c))^{\beta+1} + (w(d) - w(y))^{\beta+1} \right) \left\| \frac{f_{ts}}{g'w'} \right\|_{[a,b] \times [b,d], \infty}, \end{aligned}$$

which completes the proof. $\square$

By choosing $g(t) = t$, $t \in [a, b]$ and $w(s) = s$, $s \in [c, d]$ in Theorem 2, we get the following Ostrowski type inequality for Riemann–Liouville fractional integrals.

Corollary 1. *The following inequality holds:*

$$\begin{aligned} & \left| f(x, y) - \frac{\Gamma(\alpha + 1)}{(x - a)^\alpha + (b - x)^\alpha} [J_{x^-}^\alpha f(a, y) + J_{x^+}^\alpha f(b, y)] \right. \\ & - \frac{\Gamma(\beta + 1)}{(y - c)^\beta + (d - y)^\beta} [J_{y^-}^\beta f(x, c) + J_{y^+}^\beta f(x, d)] \\ & + \frac{\Gamma(\alpha + 1) \Gamma(\beta + 1)}{[(x - a)^\alpha + (b - x)^\alpha] [(y - c)^\beta + (d - y)^\beta]} \\ & \times \left[J_{x^-y^-}^{\alpha, \beta} f(a, c) + J_{x^-y^+}^{\alpha, \beta} f(a, d) + J_{x^+y^-}^{\alpha, \beta} f(b, c) + J_{x^+y^+}^{\alpha, \beta} f(b, d) \right] \Big| \\ & \leq \frac{[(x - a)^{\alpha+1} + (b - x)^{\alpha+1}] [(y - c)^{\beta+1} + (d - y)^{\beta+1}]}{(\alpha + 1)(\beta + 1) [(x - a)^\alpha + (b - x)^\alpha] [(y - c)^\beta + (d - y)^\beta]} \|f_{ts}\|_{\Delta, \infty}. \end{aligned} \quad (2.9)$$

Remark 1. If we take $\alpha = \beta = 1$ in Corollary 1, then inequality (2.9) reduces to the inequality established in [3, Theorem 2.1].

By choosing $g(t) = \ln t$, $t \in [a, b]$ and $w(s) = \ln s$, $s \in [c, d]$ in Theorem 2, we get the following Ostrowski type inequalities for Hadamard fractional integrals.

Corollary 2. *The following inequality holds:*

$$\begin{aligned} & \left| f(x, y) - \frac{1}{M_{\ln}^\alpha(a, b; x)} [J_{x^-; g}^\alpha f(a, y) + J_{x^+; g}^\alpha f(b, y)] \right. \\ & - \frac{1}{N_{\ln}^\beta(c, d; y)} [J_{y^-; w}^\beta f(x, c) + J_{y^+; w}^\beta f(x, d)] + \frac{1}{M_{\ln}^\alpha(a, b; x) N_{\ln}^\beta(c, d; y)} \\ & \times \left[J_{x^-y^-}^{\alpha, \beta} f(a, c) + J_{x^-y^+; g, w}^{\alpha, \beta} f(a, d) + J_{x^+y^-; g, w}^{\alpha, \beta} f(b, c) + J_{x^+y^+; g, w}^{\alpha, \beta} f(b, d) \right] \Big| \\ & \leq \frac{M_{\ln}^{\alpha+1}(a, b; x) N_{\ln}^{\beta+1}(c, d; y)}{M_{\ln}^\alpha(a, b; x) N_{\ln}^\beta(c, d; y)} \left\| \frac{f_{ts}}{g'w'} \right\|_{\Delta, \infty}. \end{aligned}$$

Theorem 3. *Under assumptions of Lemma 1, if the function $\frac{f_{ts}}{g'w'} \in L_p(\Delta)$, then we have the following Ostrowski type inequality for generalized fractional integrals:*

$$\begin{aligned} & \left| f(x, y) - \frac{1}{M_g^\alpha(a, b; x)} [J_{x^-; g}^\alpha f(a, y) + J_{x^+; g}^\alpha f(b, y)] \right. \\ & - \frac{1}{N_w^\beta(c, d; y)} [J_{y^-; w}^\beta f(x, c) + J_{y^+; w}^\beta f(x, d)] + \frac{1}{M_g^\alpha(a, b; x) N_w^\beta(c, d; y)} \end{aligned}$$

$$\begin{aligned}
& \times \left[J_{x^-y^-;g,w}^{\alpha,\beta} f(a,c) + J_{x^-y^+;g,w}^{\alpha,\beta} f(a,d) + J_{x^+y^-;g,w}^{\alpha,\beta} f(b,c) + J_{x^+y^+;g,w}^{\alpha,\beta} f(b,d) \right] \Bigg| \\
& \leq \frac{\left[(g(x) - g(a))^{\alpha+\frac{1}{q}} + (g(b) - g(x))^{\alpha+\frac{1}{q}} \right]}{(\alpha q + 1)^{\frac{1}{q}} \Gamma(\alpha + 1) M_g^\alpha(a, b; x)} \\
& \quad \times \frac{\left[(w(y) - w(c))^{\beta+\frac{1}{q}} + (w(d) - w(y))^{\beta+\frac{1}{q}} \right]}{(\beta q + 1)^{\frac{1}{q}} \Gamma(\beta + 1) N_w^\beta(c, d; y)} \left\| \frac{f_{ts}}{(g'w')^{\frac{1}{q}}} \right\|_{[a,b] \times [c,d], p},
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and

$$\|F\|_{\Delta, p} = \left(\int_a^b \int_c^d |F(t, s)|^p ds dt \right)^{\frac{1}{p}} < +\infty.$$

Proof. By Hölder inequality and properties of the modulus, we have

$$\begin{aligned}
|I_1| & \leq \int_a^x \int_c^y (g(t) - g(a))^\alpha (w(s) - w(c))^\beta |f_{ts}(t, s)| ds dt \tag{2.10} \\
& = \int_a^x \int_c^y (g(t) - g(a))^\alpha (w(s) - w(c))^\beta \left| \frac{f_{ts}(t, s)}{g'(t)w'(s)} \right| (g'(t))^{\frac{1}{p} + \frac{1}{q}} (w'(s))^{\frac{1}{p} + \frac{1}{q}} ds dt \\
& \leq \left(\int_a^x \int_c^y \left| \frac{f_{ts}(t, s)}{g'(t)w'(s)} \right|^p g'(t)w'(s) ds dt \right)^{\frac{1}{p}} \\
& \quad \times \left(\int_a^x \int_c^y (g(t) - g(a))^{\alpha q} (w(s) - w(c))^{\beta q} g'(t)w'(s) ds dt \right)^{\frac{1}{q}} \\
& \leq \left\| \frac{f_{ts}}{(g'w')^{\frac{1}{q}}} \right\|_{[a,x] \times [c,y], p} \left(\int_a^x (g(t) - g(a))^{\alpha q} g'(t) dt \right)^{\frac{1}{q}} \\
& \quad \times \left(\int_c^y (w(s) - w(c))^{\beta q} w'(s) ds \right)^{\frac{1}{q}} \\
& = \left(\frac{(g(x) - g(a))^{\alpha q + 1}}{\alpha q + 1} \right)^{\frac{1}{q}} \left(\frac{(w(y) - w(c))^{\beta q + 1}}{\beta q + 1} \right)^{\frac{1}{q}} \left\| \frac{f_{ts}}{(g'w')^{\frac{1}{q}}} \right\|_{[a,x] \times [c,y], p}
\end{aligned}$$

$$= \frac{(g(x) - g(a))^{\alpha + \frac{1}{q}} (w(y) - w(c))^{\beta + \frac{1}{q}}}{(\alpha q + 1)^{\frac{1}{q}} (\beta q + 1)^{\frac{1}{q}}} \left\| \frac{f_{ts}}{(g'w')^{\frac{1}{q}}} \right\|_{[a,x] \times [c,y],p}$$

Similarly, we get

$$|I_2| \leq \frac{(g(x) - g(a))^{\alpha + \frac{1}{q}} (w(d) - w(y))^{\beta + \frac{1}{q}}}{(\alpha q + 1)^{\frac{1}{q}} (\beta q + 1)^{\frac{1}{q}}} \left\| \frac{f_{ts}}{(g'w')^{\frac{1}{q}}} \right\|_{[a,x] \times [y,d],p}, \quad (2.11)$$

$$|I_3| \leq \frac{(g(b) - g(x))^{\alpha + \frac{1}{q}} (w(y) - w(c))^{\beta + \frac{1}{q}}}{(\alpha q + 1)^{\frac{1}{q}} (\beta q + 1)^{\frac{1}{q}}} \left\| \frac{f_{ts}}{(g'w')^{\frac{1}{q}}} \right\|_{[x,b] \times [c,y],p} \quad (2.12)$$

and

$$|I_4| \leq \frac{(g(b) - g(x))^{\alpha + \frac{1}{q}} (w(d) - w(y))^{\beta + \frac{1}{q}}}{(\alpha q + 1)^{\frac{1}{q}} (\beta q + 1)^{\frac{1}{q}}} \left\| \frac{f_{ts}}{(g'w')^{\frac{1}{q}}} \right\|_{[x,b] \times [y,d],p}. \quad (2.13)$$

By substituting the inequalities (2.10)–(2.13) in (2.5), we establish

$$\begin{aligned} & \left| f(x, y) - \frac{1}{M_g^\alpha(a, b; x)} \left[J_{x^-; g}^\alpha f(a, y) + J_{x^+; g}^\alpha f(b, y) \right] \right. \\ & - \frac{1}{N_w^\beta(c, d; y)} \left[J_{y^-; w}^\beta f(x, c) + J_{y^+; w}^\beta f(b, y) \right] + \frac{1}{M_g^\alpha(a, b; x) N_w^\beta(c, d; y)} \\ & \times \left[J_{x^- y^-; g, w}^{\alpha, \beta} f(a, c) + J_{x^- y^+; g, w}^{\alpha, \beta} f(a, d) + J_{x^+ y^-; g, w}^{\alpha, \beta} f(b, c) + J_{x^+ y^+; g, w}^{\alpha, \beta} f(b, d) \right] \Big| \\ & \leq \frac{1}{\Gamma(\alpha + 1) \Gamma(\beta + 1) M_g^\alpha(a, b; x) N_w^\beta(c, d; y)} \\ & \times \left[\frac{(g(x) - g(a))^{\alpha + \frac{1}{q}} (w(y) - w(c))^{\beta + \frac{1}{q}}}{(\alpha q + 1)^{\frac{1}{q}} (\beta q + 1)^{\frac{1}{q}}} \left\| \frac{f_{ts}}{(g'w')^{\frac{1}{q}}} \right\|_{[a,x] \times [c,y],p} \right. \\ & + \frac{(g(x) - g(a))^{\alpha + \frac{1}{q}} (w(d) - w(y))^{\beta + \frac{1}{q}}}{(\alpha q + 1)^{\frac{1}{q}} (\beta q + 1)^{\frac{1}{q}}} \left\| \frac{f_{ts}}{(g'w')^{\frac{1}{q}}} \right\|_{[a,x] \times [y,d],p} \\ & + \frac{(g(b) - g(x))^{\alpha + \frac{1}{q}} (w(y) - w(c))^{\beta + \frac{1}{q}}}{(\alpha q + 1)^{\frac{1}{q}} (\beta q + 1)^{\frac{1}{q}}} \left\| \frac{f_{ts}}{(g'w')^{\frac{1}{q}}} \right\|_{[x,b] \times [c,y],p} \\ & \left. + \frac{(g(b) - g(x))^{\alpha + \frac{1}{q}} (w(d) - w(y))^{\beta + \frac{1}{q}}}{(\alpha q + 1)^{\frac{1}{q}} (\beta q + 1)^{\frac{1}{q}}} \left\| \frac{f_{ts}}{(g'w')^{\frac{1}{q}}} \right\|_{[x,b] \times [y,d],p} \right] \\ & \leq \frac{1}{(\alpha q + 1)^{\frac{1}{q}} (\beta q + 1)^{\frac{1}{q}} \Gamma(\alpha + 1) \Gamma(\beta + 1) M_g^\alpha(a, b; x) N_w^\beta(c, d; y)} \left\| \frac{f_{ts}}{(g'w')^{\frac{1}{q}}} \right\|_{[a,b] \times [c,d],p} \end{aligned}$$

$$\begin{aligned} & \times \left[(g(x) - g(a))^{\alpha + \frac{1}{q}} + (g(b) - g(x))^{\alpha + \frac{1}{q}} \right] \\ & \times \left[(w(y) - w(c))^{\beta + \frac{1}{q}} + (w(d) - w(y))^{\beta + \frac{1}{q}} \right]. \end{aligned}$$

This completes the proof. $\square$

By choosing $g(t) = t$, $t \in [a, b]$ and $w(s) = s$, $s \in [c, d]$ in Theorem 3, we get the following Ostrowski type inequality for Riemann–Liouville fractional integrals.

Corollary 3. *The following inequality holds:*

$$\begin{aligned} & \left| f(x, y) - \frac{\Gamma(\alpha + 1)}{(x - a)^\alpha + (b - x)^\alpha} [J_{x^-}^\alpha f(a, y) + J_{x^+}^\alpha f(b, y)] \right. \\ & - \frac{\Gamma(\beta + 1)}{(y - c)^\beta + (d - y)^\beta} [J_{y^-}^\beta f(x, c) + J_{y^+}^\beta f(x, d)] \\ & + \frac{\Gamma(\alpha + 1)\Gamma(\beta + 1)}{[(x - a)^\alpha + (b - x)^\alpha][(y - c)^\beta + (d - y)^\beta]} \\ & \times [J_{x^-y^-}^{\alpha, \beta} f(a, c) + J_{x^-y^+}^{\alpha, \beta} f(a, d) + J_{x^+y^-}^{\alpha, \beta} f(b, c) + J_{x^+y^+}^{\alpha, \beta} f(b, d)] \Big| \\ & \leq \frac{[(x - a)^{\alpha + \frac{1}{q}} + (b - x)^{\alpha + \frac{1}{q}}][(y - c)^{\beta + \frac{1}{q}} + (d - y)^{\beta + \frac{1}{q}}]}{(\alpha q + 1)^{\frac{1}{q}}(\beta q + 1)^{\frac{1}{q}}[(x - a)^\alpha + (b - x)^\alpha][(y - c)^\beta + (d - y)^\beta]} \|f_{ts}\|_{\Delta, p}. \end{aligned} \quad (2.14)$$

Remark 2. If we take $\alpha = \beta = 1$ in Corollary 3, then we have

$$\begin{aligned} & \left| f(x, y) - \frac{1}{b - a} \int_a^b f(t, y) dt - \frac{1}{d - c} \int_c^d f(x, s) ds \right. \\ & \left. + \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(t, s) ds dt \right| \\ & \leq \frac{[(x - a)^{1 + \frac{1}{q}} + (b - x)^{1 + \frac{1}{q}}][(y - c)^{1 + \frac{1}{q}} + (d - y)^{1 + \frac{1}{q}}]}{(b - a)(d - c)(q + 1)^{\frac{2}{q}}} \|f_{ts}\|_{\Delta, p}. \end{aligned}$$

By choosing $g(t) = \ln t$, $t \in [a, b]$ and $w(s) = \ln s$, $s \in [c, d]$ in Theorem 3, we the following Ostrowski type inequality for Hadamard fractional integrals.

Corollary 4. *The following inequality holds:*

$$\begin{aligned} & \left| f(x, y) - \frac{1}{M_{\ln}^\alpha(a, b; x)} [J_{x^-; g}^\alpha f(a, y) + J_{x^+; g}^\alpha f(b, y)] \right. \\ & - \frac{1}{N_{\ln}^\beta(c, d; y)} [J_{y^-; w}^\beta f(x, c) + J_{y^+; w}^\beta f(x, d)] + \frac{1}{M_{\ln}^\alpha(a, b; x) N_{\ln}^\beta(c, d; y)} \end{aligned}$$

$$\begin{aligned} & \times \left[\mathbf{J}_{x^-y^-;g,w}^{\alpha,\beta} f(a,c) + \mathbf{J}_{x^-y^+;g,w}^{\alpha,\beta} f(a,d) + \mathbf{J}_{x^+y^-;g,w}^{\alpha,\beta} f(b,c) + \mathbf{J}_{x^+y^+;g,w}^{\alpha,\beta} f(b,d) \right] \Bigg| \\ & \leq \frac{\left[\left(\ln \frac{x}{a} \right)^{\alpha+\frac{1}{q}} + \left(\ln \frac{b}{x} \right)^{\alpha+\frac{1}{q}} \right] \left[\left(\ln \frac{y}{c} \right)^{\beta+\frac{1}{q}} + \left(\ln \frac{d}{y} \right)^{\beta+\frac{1}{q}} \right]}{(\alpha q + 1)^{\frac{1}{q}} (\beta q + 1)^{\frac{1}{q}} \Gamma(\alpha + 1) \Gamma(\beta + 1) M_{\ln}^{\alpha}(a, b; x) N_{\ln}^{\beta}(c, d; y)} \left\| \frac{f_{ts}}{(h)^{\frac{1}{q}}} \right\|_{[a,b] \times [c,d], p}, \end{aligned}$$

where

$$h(t, s) = \frac{1}{st}.$$

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FUZZY APPROXIMATION OF AN ADDITIVE (ρ_1, ρ_2) -RANDOM OPERATOR INEQUALITY

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Abstract. We study and solve an additive (ρ_1, ρ_2) -random operator inequality in which $\rho_1, \rho_2 \in \mathbb{C}$ are fixed and $\max\{\sqrt{2}|\rho_1|, |\rho_2|\} < 1$.

Finally, we get a fuzzy approximation of the mentioned additive (ρ_1, ρ_2) -random operator inequality.

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1. INTRODUCTION

Let $(\Omega, \mathfrak{U}, \mu)$ be a probability measure space. Assume that $(U, \mathfrak{B}_U)$ and $(V, \mathfrak{B}_V)$ are Borel measurable spaces, in which U and V are complete *fuzzy normed spaces* (in short, *FN-spaces*) and $T: \Omega \times U \rightarrow V$ is a random operator. In FN-spaces, first we solve the (ρ_1, ρ_2) -random operator inequality

$$\begin{aligned} & \eta(T(\omega, u+v) - T(\omega, u) - T(\omega, v), t) \\ & \geq \kappa_M \left(\eta(\rho_1(T(\omega, u+v) + T(\omega, u-v) - 2T(\omega, u)), t), \right. \\ & \quad \left. \eta\left(\rho_2\left(2T\left(\omega, \frac{u+v}{2}\right) - T(\omega, u) - T(\omega, v)\right), t\right) \right), \end{aligned} \quad (1.1)$$

in which $\rho_1, \rho_2 \in \mathbb{C}$ are fixed and $\max\{\sqrt{2}|\rho_1|, |\rho_2|\} < 1$.

Next, we get a fuzzy approximation of the (ρ_1, ρ_2) -random operator inequality (1.1). Fuzzy approximation of additive random operator inequality has some applications for generating secret keys in client–server environment [11].

2. PRELIMINARIES

In this paper, we let $I = [0, 1]$ and $J = (0, 1]$.

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Definition 1 ([7, Page 15] and [17, Definition 5.5.1]). A *continuous triangular norm* (shortly, a *ct-norm*) is a continuous mapping κ from I^2 to I such that

- (a) $\kappa(\tau, \upsilon) = \kappa(\upsilon, \tau)$ and $\kappa(\tau, \kappa(\upsilon, \vartheta)) = \kappa(\kappa(\tau, \upsilon), \vartheta)$ for all $\tau, \upsilon, \vartheta \in I$;
- (b) $\kappa(\tau, 1) = \tau$ for all $\tau \in I$;
- (c) $\kappa(\tau, \upsilon) \leq \kappa(\vartheta, \mathfrak{t})$ whenever $\tau \leq \vartheta$ and $\upsilon \leq \mathfrak{t}$ for all $\tau, \upsilon, \vartheta, \mathfrak{t} \in I$.

Some examples of the t -norms are:

- (1) $\kappa_P(\tau, \upsilon) = \tau\upsilon$;
- (2) $\kappa_M(\tau, \upsilon) = \min\{\tau, \upsilon\}$;
- (3) $\kappa_L(\tau, \upsilon) = \max\{\tau + \upsilon - 1, 0\}$ (the Lukasiewicz t -norm).

Definition 2 ([16, Definition 2] and [2, Definition 2.4]). Suppose that κ is a ct -norm, V is a linear space and η is a fuzzy set from V to J . In this case, the ordered tuple (V, η, κ) is called a FN-space if the following conditions are satisfied:

- (FN1) $\eta(v, t) = 1$ for all $t > 0$ if and only if $v = 0$;
- (FN2) $\eta(\alpha v, t) = \eta\left(v, \frac{t}{|\alpha|}\right)$ for all $v \in V$ and $\alpha \in \mathbb{C}$ with $\alpha \neq 0$;
- (FN3) $\eta(u + v, t + s) \geq \kappa(\eta(u, t), \eta(v, s))$ for all $u, v \in V$ and $t, s \geq 0$.
- (FN4) $\eta(u, \cdot) : (0, \infty) \rightarrow J$ is continuous.

Let $(V, \|\cdot\|)$ be a linear normed space. Then

$$\eta(v, s) = \exp\left(-\frac{\|v\|}{s}\right)$$

for all $s > 0$ defines a fuzzy norm and the ordered tuple (V, η, κ_M) is a FN-space.

Let $(\Omega, \mathfrak{U}, \mu)$ be a probability measure space. Assume that $(U, \mathfrak{B}_U)$ and $(V, \mathfrak{B}_V)$ are Borel measurable spaces, in which U and V are complete FN-spaces. A mapping $T : \Omega \times U \rightarrow V$ is said to be a random operator if $\{\omega : T(\omega, u) \in B\} \in \mathfrak{U}$ for all u in U and $B \in \mathfrak{B}_V$. Also, T is random operator, if $T(\omega, u) = v(\omega)$ be a V -valued random variable for every u in U . A random operator $T : \Omega \times U \rightarrow V$ is called *linear* if $T(\omega, \alpha u_1 + \beta u_2) = \alpha T(\omega, u_1) + \beta T(\omega, u_2)$ almost everywhere for each u_1, u_2 in U and scalars $\alpha, \beta \in \mathbb{C}$, and *bounded* if there exists a nonnegative real-valued random variable $M(\omega)$ such that

$$\eta(T(\omega, u_1) - T(\omega, u_2), M(\omega)t) \geq \eta(u_1 - u_2, t),$$

almost everywhere for each u_1, u_2 in U and $t > 0$.

Theorem 1 ([3, Proposition 1.3] and [5, Page 306]). Consider a complete generalized metric space (Γ, Δ) and a strictly contractive function $L : \Gamma \rightarrow \Gamma$ with Lipschitz constant $\beta < 1$. Then, for every given element $\gamma \in \Gamma$, either

$$\Delta(L^m \gamma, L^{m+1} \gamma) = \infty$$

for each $m \in \mathbb{N}$ or there is $m_0 \in \mathbb{N}$ such that

$$(1) \Delta(L^m \gamma, L^{m+1} \gamma) < \infty, \forall m \geq m_0;$$

- (2) the fixed point ϖ^* of L is the limit of sequence $\{L^m \gamma\}$;
 (3) in the set $\Upsilon = \{\varpi \in \Gamma \mid \Delta(L^{m_0} \gamma, \varpi) < \infty\}$, ϖ^* is the unique fixed point of L ;
 (4) $(1 - \beta)\Delta(\varpi, \varpi^*) \leq \Delta(\varpi, L\varpi)$ for every $\varpi \in \Upsilon$.

Recently, some authors have published several papers on approximation of functional equations in different spaces by the direct technique and the fixed point technique, for example, fuzzy Menger normed algebras [10], fuzzy metric spaces [15], fuzzy normed spaces [13], non-Archimedean random Lie C^* -algebras [8], non-Archimedean random normed spaces [18], random multi-normed space [1], see also [4], [6] and [14].

3. APPROXIMATION OF ADDITIVE (ρ_1, ρ_2) -RANDOM OPERATOR INEQUALITY

Now, we are ready to get a fuzzy approximation of the (ρ_1, ρ_2) -random operator inequality (1.1) as a generalization of Park's results [12].

Lemma 1. Assume that (V, η, κ_M) is a FN-space. Let $T: \Omega \times U \rightarrow V$ be a random operator, where $T(\omega, 0) = 0$ and satisfies (1.1), then T is additive.

Proof. Replacing v by u in (1.1) and using (FN2), we get

$$\eta(T(\omega, 2u) - 2T(\omega, u), t) \geq \eta\left(T(\omega, 2u) - 2T(\omega, u), \frac{t}{|\rho_1|}\right).$$

Since $|\rho_1| < 1$, $T(\omega, 2u) = 2T(\omega, u)$ for each $u \in U$ and so

$$T\left(\omega, \frac{u}{2}\right) = \frac{1}{2}T(\omega, u) \quad (3.1)$$

almost everywhere for each $u \in U$ and $\omega \in \Omega$.

Using (1.1) and (3.1) imply that

$$\begin{aligned} & \eta(T(\omega, u+v) - T(\omega, u) - T(\omega, v), t) \\ & \geq \kappa_M \left(\eta\left(T(\omega, u+v) + T(\omega, u-v) - 2T(\omega, u), \frac{t}{|\rho_1|}\right), \right. \\ & \quad \left. \eta\left(T(\omega, u+v) - T(\omega, u) - T(\omega, v), \frac{t}{|\rho_2|}\right) \right), \end{aligned}$$

and so

$$\begin{aligned} & \eta(T(\omega, u+v) - T(\omega, u) - T(\omega, v), t) \\ & \geq \eta\left(T(\omega, u+v) + T(\omega, u-v) - 2T(\omega, u), \frac{t}{|\rho_1|}\right) \end{aligned} \quad (3.2)$$

almost everywhere for each $u, v \in U$, $\omega \in \Omega$ and $t > 0$.

Putting $z = u + v$ and $w = u - v$ in (3.2), we get

$$\begin{aligned} & \eta(T(\omega, z + w) + T(\omega, z - w) - 2T(\omega, z), t) \\ & \geq \eta\left(T(\omega, z + w) - T(\omega, z) - T(\omega, w), \frac{t}{2|\rho_1|}\right) \end{aligned} \quad (3.3)$$

almost everywhere for every $z, w \in U$ and $\omega \in \Omega$.

Now, (3.2), (3.3) and (FN2) imply that

$$\begin{aligned} & \eta(T(\omega, u + v) - T(\omega, u) - T(\omega, v), t) \\ & \geq \eta\left(T(\omega, u + v) - T(\omega, u) - T(\omega, v), \frac{t}{2|\rho_1|^2}\right) \end{aligned}$$

almost everywhere for all $u, v \in U$, $\omega \in \Omega$ and $t > 0$. Since $|\rho_1| < \frac{\sqrt{2}}{2}$, $T(\omega, u + v) = T(\omega, u) + T(\omega, v)$ almost everywhere for all $u, v \in U$ and $\omega \in \Omega$, which implies that T is additive. $\square$

We use fixed point technique to get an approximation of the additive (ρ_1, ρ_2) -random operator inequality (1.1) in FN-space.

Theorem 2. Consider the complete FN-space (V, η, κ_M) . Assume that $\psi: U^2 \times (0, \infty) \rightarrow J$ is a fuzzy set such that there exists an $\beta < 1$ with

$$\psi\left(\frac{u}{2}, \frac{v}{2}, \frac{\beta t}{2}\right) \geq \psi(u, v, t) \quad \text{and} \quad \lim_{p \rightarrow \infty} \psi\left(\frac{u}{2^p}, \frac{v}{2^p}, \frac{t}{2^p}\right) = 1$$

for each $u, v \in U$ and $t > 0$. Suppose that $T: \Omega \times U \rightarrow V$ is a random operator, where $T(\omega, 0) = 0$ and

$$\begin{aligned} & \eta(T(\omega, u + v) - T(\omega, u) - T(\omega, v), t) \\ & \geq \kappa_M\left(\eta(\rho_1(T(\omega, u + v) + T(\omega, u - v) - 2T(\omega, u)), t), \right. \\ & \quad \left. \eta\left(\rho_2\left(2T\left(\omega, \frac{u + v}{2}\right) - T(\omega, u) - T(\omega, v)\right), t\right), \psi(u, v, t)\right), \end{aligned} \quad (3.4)$$

in which $\rho_1, \rho_2 \in \mathbb{C}$ are fixed and $\max\{\sqrt{2}|\rho_1|, |\rho_2|\} < 1$. Then, there is a unique additive random operator $S: \Omega \times U \rightarrow V$ such that

$$\eta(T(\omega, u) - S(\omega, u), t) \geq \psi\left(u, u, \frac{2(1 - \beta)}{\beta}t\right)$$

almost everywhere for each $u \in U$, $\omega \in \Omega$ and $t > 0$.

Proof. Putting $u = v$ in (3.4), implies that

$$\eta\left(2T\left(\omega, \frac{u}{2}\right) - T(\omega, u), t\right) \geq \psi\left(\frac{u}{2}, \frac{u}{2}, t\right) \quad (3.5)$$

almost everywhere for each $u \in U$, $\omega \in \Omega$ and $t > 0$.

On

$$\Gamma := \{H: \Omega \times U \rightarrow V, H(\omega, 0) = 0\}$$

we define the following generalized metric:

$$\Delta(G, H) = \inf\{\alpha \in \mathbb{R}_+ : \eta(G(\omega, u) - H(\omega, u), \alpha t) \geq \psi(u, u, t), \forall u \in U, t > 0\}.$$

In [9], Mihet and Radu proved that (Γ, Δ) is a complete generalized metric.

We define the linear function $L: \Gamma \rightarrow \Gamma$ as

$$LG(\omega, u) := 2G\left(\omega, \frac{u}{2}\right)$$

almost everywhere for each $u \in U$ and $\omega \in \Omega$. Consider $G, H \in \Gamma$ such that $\Delta(G, H) = \varepsilon$. Then,

$$\eta(G(\omega, u) - H(\omega, u), \varepsilon t) \geq \psi(u, u, t)$$

almost everywhere for each $u \in U$, $\omega \in \Omega$ and $t > 0$. Also,

$$\begin{aligned} \eta(LG(\omega, u) - LH(\omega, u), \beta \varepsilon t) &= \eta\left(G\left(\omega, \frac{u}{2}\right) - H\left(\omega, \frac{u}{2}\right), \frac{\beta \varepsilon t}{2}\right) \\ &\geq \psi\left(\frac{u}{2}, \frac{u}{2}, \frac{\beta t}{2}\right) \geq \psi(u, u, t) \end{aligned}$$

almost everywhere for each $u \in U$, $\omega \in \Omega$ and $t > 0$. Then, from $\Delta(G, H) = \varepsilon$ we conclude that $\Delta(LG, LH) \leq \beta \varepsilon$ and so

$$\Delta(LG, LH) \leq \beta \Delta(G, H)$$

for each $G, H \in \Gamma$.

By (3.5) we have that

$$\eta\left(2T\left(\omega, \frac{u}{2}\right) - T(\omega, u), \frac{\beta t}{2}\right) \geq \psi(u, u, t)$$

almost everywhere for each $u \in U$ and $t > 0$, which implies that $\Delta(T, LT) \leq \frac{\beta}{2}$.

Theorem 1 implies that, there exists a random operator $S: \Omega \times U \rightarrow V$ such that:

(1) A fixed point for function L , is S ,

$$S(\omega, u) = 2S\left(\omega, \frac{u}{2}\right)$$

almost everywhere for each $u \in U$ and $\omega \in \Omega$, which is unique in the set

$$\Upsilon = \{G \in \Gamma : \Delta(G, H) < \infty\};$$

(2) $\Delta(L^p T, S) \rightarrow 0$ as $p \rightarrow \infty$, which implies that

$$\lim_{p \rightarrow \infty} 2^p T\left(\omega, \frac{u}{2^p}\right) = S(\omega, u) \quad (3.6)$$

almost everywhere for each $u \in U$ and $\omega \in \Omega$;

(3) $\Delta(T, S) \leq \frac{1}{1-\beta} \Delta(T, LT)$, which implies

$$\eta(T(\omega, u) - S(\omega, u), t) \geq \Psi\left(u, u, \frac{2(1-\beta)}{\beta}t\right)$$

almost everywhere for each $u \in U$, $\omega \in \Omega$ and $t > 0$.

Using (3.4) and (3.6) imply that

$$\begin{aligned} & \eta(S(\omega, u+v) - S(\omega, u) - S(\omega, v), t) \\ &= \lim_{p \rightarrow \infty} \eta\left(T\left(\omega, \frac{u+v}{2^p}\right) - T\left(\omega, \frac{u}{2^p}\right) - T\left(\omega, \frac{v}{2^p}\right), \frac{t}{2^p}\right) \\ &\geq \lim_{p \rightarrow \infty} \kappa_M\left(\eta\left(\rho_1\left(T\left(\omega, \frac{u+v}{2^p}\right) + T\left(\omega, \frac{u-v}{2^p}\right) - 2T\left(\omega, \frac{u}{2^p}\right)\right), \frac{t}{2^p}\right), \right. \\ &\quad \left. \eta\left(\rho_2\left(2T\left(\omega, \frac{u+v}{2^{p+1}}\right) - T\left(\omega, \frac{u}{2^p}\right) - T\left(\omega, \frac{v}{2^p}\right)\right), \frac{t}{2^p}\right), \right. \\ &\quad \left. \Psi\left(\frac{u}{2^p}, \frac{v}{2^p}, \frac{t}{2^p}\right)\right) \\ &= \kappa_M\left(\eta(\rho_1(S(\omega, u+v) + S(\omega, u-v) - 2S(\omega, u)), t), \right. \\ &\quad \left. \eta\left(\rho_2\left(2S\left(\omega, \frac{u+v}{2}\right) - S(\omega, u) - S(\omega, v)\right), t\right)\right) 2 \end{aligned}$$

almost everywhere for each $u, v \in U$, $\omega \in \Omega$ and $t > 0$. Then

$$\begin{aligned} & \eta(S(\omega, u+v) - S(\omega, u) - S(\omega, v), t) \\ &\geq \kappa_M\left(\eta(\rho_1(S(\omega, u+v) + S(\omega, u-v) - 2S(\omega, u)), t), \right. \\ &\quad \left. \eta\left(\rho_2\left(2S\left(\omega, \frac{u+v}{2}\right) - S(\omega, u) - S(\omega, v)\right), t\right)\right) \end{aligned}$$

almost everywhere for each $u, v \in U$, $\omega \in \Omega$ and $t > 0$. Now, Lemma 1, implies that S is an additive random operator. $\square$

Corollary 1. Let (V, η, κ_M) be a FN-space, $\rho > 1$ and $\tau > 0$. Suppose that $T: \Omega \times U \rightarrow V$ is a random operator, where $T(\omega, 0) = 0$ and

$$\begin{aligned} & \eta(T(\omega, u+v) - T(\omega, u) - T(\omega, v), t) \\ &\geq \kappa_M\left(\eta(\rho_1(T(\omega, u+v) + T(\omega, u-v) - 2T(\omega, u)), t), \right. \\ &\quad \left. \eta\left(\rho_2\left(2T\left(\omega, \frac{u+v}{2}\right) - T(\omega, u) - T(\omega, v)\right), t\right), \right. \\ &\quad \left. \frac{t}{t + \tau(\|u\|^\rho + \|v\|^\rho)}\right), \end{aligned} \tag{3.7}$$

in which $\rho_1, \rho_2 \in \mathbb{C}$ are fixed and $\max\{\sqrt{2}|\rho_1|, |\rho_2|\} < 1$. Then, there is a unique additive random operator $S: \Omega \times U \rightarrow V$ such that

$$\eta(T(\omega, u) - S(\omega, u), t) \geq \frac{(2^{\rho-1} - 1)t}{(2^{\rho-1} - 1)t + \tau\|u\|^\rho}$$

for each $u \in U$, $\omega \in \Omega$ and $t > 0$.

Proof. In Theorem 2, we put $\psi(u, v, t) = \frac{t}{t + \tau(\|u\|^\rho + \|v\|^\rho)}$ for each $u, v \in U$, $t > 0$ and $\beta = 2^{1-\rho}$. Now, we show that the fuzzy control function ψ satisfies in the conditions of Theorem 2.

Let $u, v \in U$, $t > 0$ and $\beta = 2^{1-\rho}$. Then we have

$$\begin{aligned} \psi\left(\frac{u}{2}, \frac{v}{2}, \frac{\beta t}{2}\right) &= \frac{2^{-\rho}t}{2^{-\rho}t + \tau(\|\frac{u}{2}\|^\rho + \|\frac{v}{2}\|^\rho)} = \frac{2^{-\rho}t}{2^{-\rho}t + \frac{1}{2^\rho}\tau(\|u\|^\rho + \|v\|^\rho)} \\ &= \frac{t}{t + \tau(\|u\|^\rho + \|v\|^\rho)} = \psi(u, v, t), \end{aligned}$$

also

$$\begin{aligned} \lim_{n \rightarrow \infty} \psi\left(\frac{u}{2^n}, \frac{v}{2^n}, \frac{t}{2^n}\right) &= \frac{2^{1-\rho} \frac{t}{2^n}}{2^{1-\rho} \frac{t}{2^n} + \tau(\|\frac{u}{2^n}\|^\rho + \|\frac{v}{2^n}\|^\rho)} \\ &= \lim_{n \rightarrow \infty} \frac{2^{1-\rho} 2^{n(\rho-1)} t}{2^{1-\rho} 2^{n(\rho-1)} t + \tau(\|u\|^\rho + \|v\|^\rho)} \\ &= \lim_{n \rightarrow \infty} \frac{2^{1-\rho} 2^{n(\rho-1)} t}{2^{1-\rho} 2^{n(\rho-1)} t} = 1. \end{aligned}$$

□

Theorem 3. Consider the complete FN-space (V, η, κ_M) . Suppose that $\psi: U^2 \times (0, \infty) \rightarrow J$ is a fuzzy set such that there exists an $\beta < 1$ with

$$\psi(2u, 2v, 2\beta t) \geq \psi(u, v, t) \quad \text{and} \quad \lim_{p \rightarrow \infty} \psi(2^p u, 2^p v, 2^p t) = 1$$

for all $u, v \in U$ and $t > 0$. Suppose that $T: \Omega \times U \rightarrow V$ be a random operator in which $T(\omega, 0) = 0$ and satisfies in (3.4). So, there is a unique additive random operator $S: \Omega \times U \rightarrow V$ such that

$$\eta(T(\omega, u) - S(\omega, u), t) \geq \psi(u, u, 2(1 - \beta)t)$$

almost everywhere for each $u \in U$ and $t > 0$.

Proof. Let (Γ, Δ) be same as proof of Theorem 2. We define the linear function $L: \Gamma \rightarrow \Gamma$ as

$$LG(\omega, u) := \frac{1}{2}G(\omega, 2u)$$

almost everywhere for each $u \in U$. Using (3.5) implies that

$$\eta\left(\frac{T(\omega, 2u)}{2} - T(\omega, u), t\right) \geq \psi\left(\frac{u}{2}, \frac{u}{2}, \frac{t}{\beta}\right)$$

almost everywhere for each $u \in U$ and $t > 0$.

By similar method of proof last theorem, the proof will be completed. $\square$

Corollary 2. Let (V, η, κ_M) be a FN-space, $\rho < 1$ and $\tau > 0$. Suppose that $T: \Omega \times U \rightarrow V$ is a random operator in which $T(\omega, 0) = 0$ and satisfies (3.7). Then there exists a unique additive random operator $S: \Omega \times U \rightarrow V$ such that

$$\eta(T(\omega, u) - S(\omega, u), t) \geq \frac{(2 - 2^\rho)t}{(2 - 2^\rho)t + 2\tau\|u\|^\rho}$$

for each $u \in U$ and $t > 0$.

Proof. In Theorem 3, we put $\psi(u, v, t) = \frac{t}{t + \tau(\|u\|^\rho + \|v\|^\rho)}$ for each $u, v \in U, t > 0$ and $\beta = 2^{\rho-1}$. Now, we show that the fuzzy control function ψ satisfies in the conditions of Theorem 3.

Let $u, v \in U, t > 0$ and $\beta = 2^{\rho-1}$. Then we have

$$\begin{aligned} \psi(2u, 2v, 2\beta t) &= \frac{2^{\rho-1}2t}{2^{\rho-1}2t + \tau(\|2u\|^\rho + \|2v\|^\rho)} = \frac{2^\rho t}{2^\rho t + 2^\rho \tau(\|u\|^\rho + \|v\|^\rho)} \\ &= \frac{t}{t + \tau(\|u\|^\rho + \|v\|^\rho)} = \psi(u, v, t), \end{aligned}$$

also

$$\begin{aligned} \lim_{n \rightarrow \infty} \psi(2^n u, 2^n v, 2^n t) &= \frac{2^{\rho-1}2^n t}{2^{\rho-1}2^n t + \tau(\|2^n u\|^\rho + \|2^n v\|^\rho)} \\ &= \lim_{n \rightarrow \infty} \frac{2^n 2^{\rho-1} t}{2^n 2^{\rho-1} t + 2^{n\rho} \tau(\|u\|^\rho + \|v\|^\rho)} \\ &= \lim_{n \rightarrow \infty} \frac{2^{n(1-\rho)} 2^{\rho-1} t}{2^{n(1-\rho)} 2^{\rho-1} t + \tau(\|u\|^\rho + \|v\|^\rho)} \\ &= \lim_{n \rightarrow \infty} \frac{2^{n(1-\rho)} 2^{\rho-1} t}{2^{n(\rho-1)} 2^{\rho-1} t} = 1. \end{aligned}$$

$\square$

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APPROXIMATION BY APOSTOL-GENOCCHI SUMMATION-INTEGRAL TYPE OPERATORS

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Abstract. In the present paper, we proposed a new sequence of summation-integral type operators involving Apostol-Genocchi polynomials. We study some approximation results of the proposed operators using first and second-order modulus of continuity, the global rate of convergence using Voronovskaja-type asymptotic theorem, Lipschitz-type space, and Ditzian-Totik modulus of smoothness. Lastly, we study the weighted approximation.

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Keywords: Apostol-Genocchi polynomials, rate of convergence, Ditzian-Totik modulus of smoothness, Voronovskaja-type asymptotic formula, Lipschitz-type space, weighted approximation

1. INTRODUCTION

The Appell polynomial $A_n(x)$ is defined by

$$e^{xt}f(t) = \sum_{n=0}^{\infty} A_n(x) \frac{t^n}{n!},$$

where $f(t)$ be a formal power series in t . This polynomial found stunning applications in several field of mathematics, see [7, 21, 22]. One special case known as Genocchi polynomials $G_n(x)$ are defined by

$$\frac{2te^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}, \quad |t| < \pi.$$

In this case the Genocchi numbers G_n have many applications in number theory, special functions, combinatorics and numerical analysis, where

$$\frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, \quad |t| < \pi.$$

The Apostol-Genocchi polynomials $G_k^\alpha(x; \beta)$, $\beta \in \mathbb{C}$, of order α (non negative integers) are defined and found some results in [7].

Recently, Prakash et al. [15] proposed a sequence of operators involving Apostol-Genocchi polynomials given in (1.1). They studied some approximation results using

second-order modulus of continuity, Voronovskaja-type theorem and weighted approximation theorem for these operators. They also proposed the Kantorovich form of (1.1), and discussed some approximation results. Many researchers have worked on new modifications of various operators and deliberate the approximation properties ([1, 3, 9, 10, 16, 17, 19, 22–24]). The operators discussed in [15] are given as follows. For $f \in C[0, \infty)$,

$$\mathcal{M}_n^{\alpha, \beta}(f; x) = e^{-nx} \left(\frac{1 + e\beta}{2} \right)^\alpha \sum_{k=0}^{\infty} \frac{G_k^\alpha(nx; \beta)}{k!} f\left(\frac{k}{n}\right), \quad (1.1)$$

where $G_k^\alpha(x; \beta)$ are the generalized Apostol-Genocchi polynomials, with the generating function of the form

$$\left(\frac{2t}{1 + \beta e^t} \right)^\alpha e^{xt} = \sum_{k=0}^{\infty} G_k^\alpha(x; \beta) \frac{t^k}{k!}, \quad (\beta \in \mathbb{C}, \alpha \in \mathbb{N} \cup 0, |t| < \pi).$$

For more information about the Apostol-Genocchi polynomials and their properties, readers should refer articles [6, 8, 11, 12, 14, 18, 20].

In 2007, Srivastava et al. proposed a family of summation-integral type operators and estimated rate of convergence and function having derivative of bounded variation [20]. Here we refer to some more articles related to summation-integral type operators for the readers ([4, 5, 13]). Motivated by the above work, for non negative integer α and $f \in L_1[0, \infty)$ we propose the integral-type generalization of the operators (1.1) as follows:

$$T_{n, \rho}^{\alpha, \beta}(f, x) = e^{-nx} \left(\frac{1 + e\beta}{2} \right)^\alpha \left(G_0^\alpha(nx; \beta) f(0) + \sum_{k=1}^{\infty} \frac{G_k^\alpha(nx; \beta)}{k!} \int_0^\infty \Theta_{n, k}^\rho(t, c) f(t) dt \right), \quad (1.2)$$

where $\rho > 0$, and

$$\Theta_{n, k}^\rho(t, c) = \begin{cases} \frac{n\rho}{\Gamma(k\rho)} e^{-n\rho t} (n\rho t)^{k\rho-1}, & c = 0, \\ \frac{\Gamma(\frac{n\rho}{c} + k\rho)}{\Gamma(k\rho)\Gamma(\frac{n\rho}{c})} \frac{c^{k\rho} t^{k\rho-1}}{(1 + ct)^{\frac{n\rho}{c} + k\rho}}, & c = 1, 2, 3, \dots \end{cases}$$

It can be easily observed by simple computation that

$$\int_0^\infty \Theta_{n, k}^\rho(t, c) t^r dt = \begin{cases} \frac{\Gamma(k\rho + r)}{\Gamma(k\rho)} \frac{1}{\prod_{i=1}^r (n\rho - ic)}, & r \neq 0 \\ 1, & r = 0. \end{cases} \quad (1.3)$$

2. LEMMAS

In this section we deal with some useful lemmas and results.

Lemma 1 ([15]). For $\mathcal{M}_n^{\alpha, \beta}(t^i; x)$, $i = \overline{0, 4}$, we have $\mathcal{M}_n^{\alpha, \beta}(1; x) = 1$;

$$\mathcal{M}_n^{\alpha, \beta}(t; x) = x + \frac{\alpha}{n(1 + e\beta)};$$

$$\begin{aligned}
\mathcal{M}_n^{\alpha,\beta}(t^2;x) &= x^2 + \frac{(1+2\alpha+e\beta)}{n(1+e\beta)}x + \frac{\alpha^2-2\alpha e\beta-\alpha e^2\beta^2}{n^2(1+e\beta)^2}; \\
\mathcal{M}_n^{\alpha,\beta}(t^3;x) &= x^3 + \frac{(3+3\alpha+3e\beta)}{n(1+e\beta)}x^2 + \frac{(3\alpha^2+3\alpha+e^2\beta^2-3\alpha e^2\beta^2-3\alpha e\beta+2e\beta+1)}{n^2(1+e\beta)^2}x \\
&\quad + \frac{(\alpha^3-6\alpha^2e\beta-3\alpha^2e^2\beta^2-5\alpha e\beta-4\alpha e^2\beta^2-\alpha e^3\beta^3)}{n^3(1+e\beta)^3}; \\
\mathcal{M}_n^{\alpha,\beta}(t^4;x) &= x^4 + \frac{(3+2\alpha+3e\beta)}{n(1+e\beta)}x^3 + \frac{(6\alpha^2+25e^2\beta^2-50e\beta-6\alpha e^2\beta^2+12\alpha+25)}{n^2(1+e\beta)^2}x^2 \\
&\quad + \frac{x}{n^3(1+e\beta)^3}(4\alpha^3+6\alpha^2+42e\beta+48e\alpha\beta-18e\alpha^2\beta+42e^2\beta^2+6e^2\alpha\beta^2 \\
&\quad -12e^2\alpha^2\beta^2+14e^3\beta^3-10e^3\alpha\beta^2+40\alpha+14) \\
&\quad + \frac{1}{n^4(1+e\beta)^4}(16\alpha^4-1056e\alpha\beta+256e\alpha^2\beta-192e\alpha^3\beta-1888e^2\alpha\beta^2 \\
&\quad +224e^2\alpha^2\beta^2-96e^2\alpha^3\beta^2-1312e^3\alpha\beta^3+128e^3\alpha^2\beta^3-304e^4\alpha\beta^4 \\
&\quad +48e^4\alpha^2\beta^4+288\alpha^2-80\alpha).
\end{aligned}$$

Lemma 2. *The moments of the proposed operator $T_{n,\rho}^{\alpha,\beta}(t^i, x)$, $i = \overline{0, 3}$, we have*

$$\begin{aligned}
T_{n,\rho}^{\alpha,\beta}(1, x) &= 1; \\
T_{n,\rho}^{\alpha,\beta}(t, x) &= \frac{n\rho}{(n\rho-c)}x + \frac{\alpha\rho}{(n\rho-c)(1+e\beta)}; \\
T_{n,\rho}^{\alpha,\beta}(t^2, x) &= \frac{n^2\rho^2}{(n\rho-c)(n\rho-2c)}x^2 + \frac{n\rho^2+2n\alpha\rho^2+n\rho^2e\beta}{(n\rho-c)(n\rho-2c)(1+e\beta)}x \\
&\quad + \frac{\alpha^2\rho^2-2\alpha\rho^2e\beta-\alpha\rho^2e^2\beta^2}{(n\rho-c)(n\rho-2c)(1+e\beta)^2}; \\
T_{n,\rho}^{\alpha,\beta}(t^3, x) &= \frac{n^3\rho^3}{(n\rho-c)(n\rho-2c)(n\rho-3c)}x^3 \\
&\quad + \frac{3n^2\rho^2+3n^2\rho^3+3\alpha n^2\rho^3+3n^2\rho^2e\beta+3n^2\rho^3e\beta}{(n\rho-c)(n\rho-2c)(n\rho-3c)(1+e\beta)}x^2 \\
&\quad + \frac{x}{(n\rho-c)(n\rho-2c)(n\rho-3c)(1+e\beta)^2} [n\alpha^2\rho^3+9n\alpha\rho^2 \\
&\quad +n\rho^2e^2\beta^2(3-2n\rho)+3n\alpha\rho^2e\beta+n\rho(1+e\beta)^2+8n\rho^2e\beta+3n\rho^2+n\rho] \\
&\quad + \frac{1}{(n\rho-c)(n\rho-2c)(n\rho-3c)(1+e\beta)^3} [\alpha^3\rho^3+3\alpha^2\rho^2e\beta(1-2\rho-\rho e\beta) \\
&\quad +3\alpha^2\rho^2-5\alpha\rho^3e\beta-\alpha\rho^2e^2\beta^2(4\rho+\rho e\beta+9e\beta+3)-6\alpha\rho^2e\beta+2\alpha\rho \\
&\quad +2\alpha\rho e^2\beta^2+4\alpha\rho e\beta].
\end{aligned}$$

The proof the above lemma follows from (1.2), (1.3) and Lemma 1.

Remark 1. From Lemma 2 and simple estimation, we have

$$\begin{aligned} T_{n,\rho}^{\alpha,\beta}(t-x, x) &= \frac{cx}{(n\rho - c)} + \frac{\alpha\rho}{(n\rho - c)(1 + e\beta)}; \\ T_{n,\rho}^{\alpha,\beta}((t-x)^2, x) &= \frac{n\rho c + 2c^2}{(n\rho - c)(n\rho - 2c)}x^2 + \frac{n\rho^2 + n\rho^2 e\beta + 4c\alpha\rho}{(n\rho - c)(n\rho - 2c)(1 + e\beta)}x \\ &\quad + \frac{\alpha^2\rho^2 - 2\alpha\rho^2 e\beta - \alpha\rho^2 e^2\beta^2}{(n\rho - c)(n\rho - 2c)(1 + e\beta)^2}. \end{aligned}$$

Remark 2. For the central moments $T_{n,\rho}^{\alpha,\beta}((t-x)^m; x)$ for $m = 1, 2$, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} nT_{n,\rho}^{\alpha,\beta}((t-x); x) &= \frac{xc}{\rho} + \frac{\alpha}{(1 + e\beta)}; \\ \lim_{n \rightarrow \infty} nT_{n,\rho}^{\alpha,\beta}((t-x)^2; x) &= \frac{x(cx + \rho)}{\rho}. \end{aligned}$$

For $n \in \mathbb{N}$, $T_{n,\rho}^{\alpha,\beta}((t-x)^2; x) \leq \frac{2\phi^2(x)}{n\rho}$, where $\phi(x) = \sqrt{x(cx + \rho)}$.

Let $C_B[0, \infty)$ be the set of all continuous and bounded functions on $[0, \infty)$ endowed with the norm $\|f\| = \sup\{|f(x)| : x \in [0, \infty)\}$.

Lemma 3. Let $f \in C_B[0, \infty)$, $0 \leq x < \infty$ and $n \in \mathbb{N}$, then

$$|T_{n,\rho}^{\alpha,\beta}(f; x)| \leq \|f\|,$$

where $\|\cdot\|$ is the uniform norm on $[0, \infty)$.

Proof. We have $T_{n,\rho}^{\alpha,\beta}(e_0; x) = 1$, so $|T_{n,\rho}^{\alpha,\beta}(f; x)| \leq T_{n,\rho}^{\alpha,\beta}(e_0; x)\|f\| = \|f\|$. $\square$

3. DIRECT RESULTS

In this section we establish the uniform convergence of the operators (1.2) using the Bohman-Korovkin theorem, the rate of convergence with the aid of different kind of modulus of smoothness and for functions in Lipschitz-type spaces, and a Voronovskaja-type asymptotic theorem.

Theorem 1. Let $f \in C[0, \infty)$ and an adequately large n , then the sequence $\{T_{n,\rho}^{\alpha,\beta}(f, \cdot)\}$ converges uniformly to f in $[a, b]$, where $0 \leq a < b < \infty$.

Proof. From Lemma 2 we have $T_{n,\rho}^{\alpha,\beta}(1, x) = 1$ for every $n \in \mathbb{N}$,

$$T_{n,\rho}^{\alpha,\beta}(t, x) = \frac{n\rho}{(n\rho - c)}x + \frac{\alpha\rho}{(n\rho - c)(1 + e\beta)}$$

tends to x and

$$T_{n,p}^{\alpha,\beta}(t^2, x) = \frac{n^2 p^2}{(np-c)(np-2c)} x^2 + \frac{np^2 + 2n\alpha p^2 + np^2 e\beta}{(np-c)(np-2c)(1+e\beta)} x + \frac{\alpha^2 p^2 - 2\alpha p^2 e\beta - \alpha p^2 e^2 \beta^2}{(np-c)(np-2c)(1+e\beta)^2}$$

tends to x^2 as $n \rightarrow \infty$, similarly $T_{n,p}^{\alpha,\beta}(t^3, x)$ tends to x^3 uniformly on every compact subset of $[0, \infty)$. Hence, by Bohman-Korovkin theorem the required result holds. $\square$

Consider the K-functional

$$K_2(f, \delta) = \inf_{j \in W^2} \{ \|f - j\| + \delta \|j''\| \}, \quad (3.1)$$

where $\delta > 0$ and $W^2 = \{j \in C_B[0, \infty) : j', j'' \in C_B[0, \infty)\}$. For a constant $C > 0$ such that

$$K_2(f, \delta) \leq C \omega_2(f, \sqrt{\delta}), \quad (3.2)$$

where $f \in C_B[0, \infty)$ and

$$\omega_2(f, \sqrt{\delta}) = \sup_{0 < h \leq \sqrt{\delta}} \sup_{x \in [0, \infty)} |f(x+2h) - 2f(x+h) + f(x)|$$

is the second order modulus of smoothness.

Theorem 2. Let $f \in C_B[0, \infty)$ and $x \in [0, \infty)$. Then we have

$$\left| T_{n,p}^{\alpha,\beta}(f, x) - f(x) \right| \leq C \omega_2\left(f, \sqrt{\delta_n}\right) + \omega(f, |\alpha_2|),$$

where C is a positive constant, $\delta_n = \frac{1}{2}[\alpha_1 + \alpha_2^2]$, $\alpha_1 = T_{n,p}^{\alpha,\beta}((t-x)^2, x)$ and

$$\alpha_2 = \left(\frac{cx}{np-c} + \frac{\alpha p}{(np-c)(1+e\beta)} \right).$$

Proof. Define the auxiliary operators $L_{n,p}^{\alpha,\beta} : C_B[0, \infty) \rightarrow C_B[0, \infty)$ as follows:

$$L_{n,p}^{\alpha,\beta}(f, x) = T_{n,p}^{\alpha,\beta}(f, x) - f\left(\frac{np}{(np-c)}x + \frac{\alpha p}{(np-c)(1+e\beta)}\right) + f(x). \quad (3.3)$$

These operators are linear and $L_{n,p}^{\alpha,\beta}(t-x, x) = 0$.

Let $j \in W^2$ and $x, t \in [0, \infty)$. By Taylor's series expansion

$$j(t) = j(x) + (t-x)j'(x) + \int_x^t (t-u)j''(u)du.$$

Applying the operator $L_{n,p}^{\alpha,\beta}$ on above, we obtain

$$L_{n,p}^{\alpha,\beta}(j; x) = j(x) + j'(x)L_{n,p}^{\alpha,\beta}((t-x); x) + L_{n,p}^{\alpha,\beta}\left(\int_x^t (t-u)j''(u)du; x\right)$$

implying that

$$\begin{aligned}
\left| L_{n,p}^{\alpha,\beta}(j, x) - j(x) \right| &\leq L_{n,p}^{\alpha,\beta} \left(\left| \int_x^t (t-u) j''(u) du \right|, x \right) \\
&\leq T_{n,p}^{\alpha,\beta} \left(\left| \int_x^t (t-u) j''(u) du \right|, x \right) \\
&\quad + \left| \int_x^{\frac{np}{(np-c)x + \frac{\alpha p}{(np-c)(1+e\beta)}}} \left(\frac{np}{(np-c)x + \frac{\alpha p}{(np-c)(1+e\beta)}} - u \right) j''(u) du \right| \\
&\leq \frac{1}{2} T_{n,p}^{\alpha,\beta}((t-x)^2, x) \|j''\| \\
&\quad + \left| \int_x^{\frac{np}{(np-c)x + \frac{\alpha p}{(np-c)(1+e\beta)}}} \left(\frac{np}{(np-c)x + \frac{\alpha p}{(np-c)(1+e\beta)}} - u \right) du \right| \|j''\| \\
\left| L_{n,p}^{\alpha,\beta}(j, x) - j(x) \right| &\leq \frac{1}{2} \left[T_{n,p}^{\alpha,\beta}((t-x)^2, x) + \left(\frac{np}{(np-c)x + \frac{\alpha p}{(np-c)(1+e\beta)}} - x \right)^2 \right] \|j''\| \\
&\leq \frac{1}{2} \left[T_{n,p}^{\alpha,\beta}((t-x)^2, x) + \left(\frac{c}{np-c}x + \frac{\alpha p}{(np-c)(1+e\beta)} \right)^2 \right] \|j''\| \\
&\leq \frac{1}{2} [\alpha_1 + \alpha_2^2] \|j''\| = \delta_n \|j''\|. \tag{3.4}
\end{aligned}$$

Since

$$\begin{aligned}
\left| T_{n,p}^{\alpha,\beta}(f, x) \right| &\leq e^{-nx} \left(\frac{1+e\beta}{2} \right)^\alpha \left(G_0^\alpha(nx; \beta) |f(0)| + \left| \sum_{k=1}^{\infty} \frac{G_k^\alpha(nx; \beta)}{k!} \int_0^\infty \Theta_{n,k}^p(t, c) f(t) dt \right| \right) \\
&\leq \|f\|.
\end{aligned}$$

Now by (3.3), we have

$$\|L_{n,p}^{\alpha,\beta}(f, \cdot)\| \leq \|T_{n,p}^{\alpha,\beta}(f, \cdot)\| + 2\|f\| \leq 3\|f\|, \quad f \in C_B[0, \infty). \tag{3.5}$$

Using (3.3), (3.4) and (3.5), we have

$$\begin{aligned}
\left| T_{n,p}^{\alpha,\beta}(f, x) - f(x) \right| &\leq \left| L_{n,p}^{\alpha,\beta}(f - j, x) - (f - j)(x) \right| + \left| L_{n,p}^{\alpha,\beta}(j, x) - j(x) \right| \\
&\quad + \left| f \left(\frac{np}{(np-c)x + \frac{\alpha p}{(np-c)(1+e\beta)}} \right) - f(x) \right| \\
&\leq 4\|f - j\| + \delta_n \|j''\| + \left| f(x) - f \left(\frac{np}{(np-c)x + \frac{\alpha p}{(np-c)(1+e\beta)}} \right) \right| \\
&\leq 4\|f - j\| + \delta_n \|j''\| + \omega(f, \alpha_2).
\end{aligned}$$

Taking infimum over all $j \in W^2$, and using (3.2), we get the required result. $\square$

Our next result is the Voronovskaja-type asymptotic formula.

Theorem 3. Let f be a bounded and integrable function on $[0, \infty)$ such that $f''(x)$ exists at $0 \leq x < \infty$, then

$$\lim_{n \rightarrow \infty} n \left[T_{n,\rho}^{\alpha,\beta}(f, x) - f(x) \right] = \left[\frac{cx}{\rho} + \frac{\alpha}{(1+e\beta)} \right] f'(x) + \left[\frac{c}{\rho} x^2 + x \right] \frac{f''(x)}{2!}.$$

Proof. Using the well known Taylor's series expansion

$$f(t) = f(x) + (t-x)f'(x) + \frac{(t-x)^2}{2!} f''(x) + \sigma(t, x)(t-x)^2,$$

where $\sigma(t, x) \rightarrow 0$ as $t \rightarrow x$ and the function ρ is bounded on $[0, \infty)$. Now,

$$\begin{aligned} n \left[T_{n,\rho}^{\alpha,\beta}(f, x) - f(x) \right] &= n \left[T_{n,\rho}^{\alpha,\beta}(t-x, x) f'(x) + \frac{T_{n,\rho}^{\alpha,\beta}((t-x)^2, x)}{2!} f''(x) + T_{n,\rho}^{\alpha,\beta}(\sigma(t, x)(t-x)^2, x) \right] \\ &= n \left[\left(\frac{cx}{(n\rho-c)} + \frac{\alpha\rho}{(n\rho-c)(1+e\beta)} \right) f'(x) + \frac{n\rho c + 2c^2}{(n\rho-c)(n\rho-2c)} \frac{x^2}{2!} f''(x) \right. \\ &\quad \left. + \frac{n\rho^2 + n\rho^2 e\beta + 4c\alpha\rho}{(n\rho-c)(n\rho-2c)(1+e\beta)} \frac{x}{2!} f''(x) + \frac{\alpha^2 \rho^2 - 2\alpha\rho^2 e\beta - \alpha\rho^2 e^2 \beta^2}{2!(n\rho-c)(n\rho-2c)(1+e\beta)^2} f''(x) \right] \\ &\quad + \hbar(n, x), \end{aligned}$$

where

$$\begin{aligned} \hbar(n, x) &= e^{-nx} \left(\frac{1+e\beta}{2} \right)^\alpha \\ &\quad \cdot \left(G_0^\alpha(nx; \beta) \sigma(0, x)(-x)^2 + \sum_{k=1}^{\infty} \frac{G_k^\alpha(nx; \beta)}{k!} \int_0^\infty \Theta_{n,k}^\rho(t, c) \sigma(t, x)(t-x)^2 dt \right). \end{aligned}$$

Now it is sufficient to show that $\hbar(n, x) \rightarrow 0$ as large n . Since $\sigma(t, x) \rightarrow 0$ as $t \rightarrow x$, for a given $\varepsilon > 0$, there exists $\delta > 0$ such that $|\sigma(t, x)| < \varepsilon$, whenever $-\delta < t-x < \delta$,

$$\begin{aligned} |\hbar(n, x)| &\leq e^{-nx} \left(\frac{1+e\beta}{2} \right)^\alpha \sum_{k=1}^{\infty} \frac{G_k^\alpha(nx; \beta)}{k!} \\ &\quad \left[\int_{|t-x|<\delta} \Theta_{n,k}^\rho(t, c) |\sigma(t, x)| (t-x)^2 dt + \int_{|t-x|\geq\delta} \Theta_{n,k}^\rho(t, c) |\sigma(t, x)| (t-x)^2 dt \right] \\ &\quad + e^{-nx} \left(\frac{1+e\beta}{2} \right)^\alpha G_0^\alpha(nx; \beta) \sigma(0, x)(-x)^2. \end{aligned}$$

By using Remark 1, we have $I_1 = \sigma O(1)$ and for $m \geq 2$, we obtain

$$\begin{aligned} I_2 &\leq C e^{-nx} \left(\frac{1+e\beta}{2} \right)^\alpha \left(G_0^\alpha(nx; \beta) \sigma(0, x)(-x)^2 + \sum_{k=1}^{\infty} \frac{G_k^\alpha(nx; \beta)}{k!} \int_0^\infty \Theta_{n,k}^\rho(t, c) \frac{(t-x)^{2m}}{\delta^{2m-2}} dt \right) \\ &= O(n^{-m+1}), \end{aligned}$$

where $C = \sup_{t \in [x, \infty)} |\sigma(t, x)|$. Due to arbitrariness of $\varepsilon > 0$, $\hbar(n, x) \rightarrow 0$ for sufficiently large n , the proof is completed. $\square$

In the following theorem, we succeed the rate of convergence of the proposed operators (1.2) for functions in Lipschitz-type spaces.

Suppose that $x \in (0, \infty)$, $t \in [0, \infty)$, and consider the Lipschitz-type space defined as

$$Lip_M^*(\sigma) = \left\{ f \in C[0, \infty) : |f(t) - f(x)| \leq M \frac{|t - x|^\sigma}{(t + x)^{\frac{\sigma}{2}}} \right\},$$

where $M > 0$ is a constant and $0 < \sigma \leq 1$.

Theorem 4. *Let $f \in Lip_M^*(\sigma)$ and $0 < \sigma \leq 1$. Then for each $x \in (0, \infty)$, we have*

$$|T_{n,p}^{\alpha,\beta}(f(t); x) - f(x)| \leq M \left(\frac{T_{n,p}^{\alpha,\beta}((t - x)^2; x)}{x} \right)^{\frac{\sigma}{2}}.$$

Proof. Let $f \in Lip_M^*(\sigma)$ and $x \in (0, \infty)$, $t \in [0, \infty)$, we have

$$\begin{aligned} |T_{n,p}^{\alpha,\beta}(f(t); x) - f(x)| &\leq T_{n,p}^{\alpha,\beta}(|f(t) - f(x)|; x) \\ &\leq M \cdot T_{n,p}^{\alpha,\beta} \left(\frac{|t - x|^\sigma}{(t + x)^{\frac{\sigma}{2}}}; x \right) \\ &\leq \frac{M}{x^{\frac{\sigma}{2}}} T_{n,p}^{\alpha,\beta}(|t - x|^\sigma; x). \end{aligned} \quad (3.6)$$

Taking $p = \frac{2}{\sigma}$ and $q = \frac{2}{2-\sigma}$ and applying Hölder's inequality, we obtain

$$\begin{aligned} T_{n,p}^{\alpha,\beta}(|t - x|^\sigma; x) &\leq \left\{ T_{n,p}^{\alpha,\beta}(|t - x|^2; x) \right\}^{\frac{\sigma}{2}} \cdot \left\{ T_{n,p}^{\alpha,\beta}(1^{\frac{2}{2-\sigma}}; x) \right\}^{\frac{2-\sigma}{2}} \\ &\leq \left\{ T_{n,p}^{\alpha,\beta}(|t - x|^2; x) \right\}^{\frac{\sigma}{2}}. \end{aligned} \quad (3.7)$$

Using (3.6) and (3.7), we get the required result

$$|T_{n,p}^{\alpha,\beta}(f(t); x) - f(x)| \leq M \left(\frac{T_{n,p}^{\alpha,\beta}((t - x)^2; x)}{x} \right)^{\frac{\sigma}{2}}.$$

$\square$

Here, we estimate the convergence using the first order Ditzian-Totik modulus of smoothness $\omega_{\Phi^\gamma}(f; t)$ and The Peetre K -functional $K_{\Phi^\gamma}(f; t)$, $0 \leq \gamma \leq 1$. The Ditzian-Totik modulus of smoothness is defined as

$$\omega_{\Phi^\gamma}(f; t) = \sup_{0 < h \leq t} \left\{ \left| f \left(x + \frac{h\Phi^\gamma(x)}{2} \right) - f \left(x - \frac{h\Phi^\gamma(x)}{2} \right) \right|, x \pm \frac{h\Phi^\gamma(x)}{2} \in [0, \infty) \right\},$$

where $\phi^2(x) = x(cx + \rho)$ and $f \in C[0, \infty)$.

The Peetre K -functional is defined as

$$K_{\phi^\gamma}(f; t) = \inf_{j \in W_\gamma} \{ \|f - j\| + t \|\phi^\gamma j'\| \} \quad (t > 0),$$

where $W_\gamma[0, \infty) = \{j : j \in AC_{loc}[0, \infty), \|\phi^\gamma j'\| < \infty\}$ and $AC_{loc}[0, \infty)$ is the class of the locally absolutely continuous functions. In [2], (Theorem 2.1.1), there is a relation between K -functional and the Ditzian-Totik first order modulus of smoothness

$$C^{-1} \omega_{\phi^\gamma}(f, t) \leq K_{\phi^\gamma}(f; t) \leq C \omega_{\phi^\gamma}(f; t), \quad (3.8)$$

where $C > 0$ is a constant.

Theorem 5. For $f \in C_B[0, \infty)$ and $x \in (0, \infty)$, we have

$$|T_{n,\rho}^{\alpha,\beta}(f(t); x) - f(x)| \leq C \omega_{\phi^\gamma} \left(f; \frac{\phi^{1-\gamma}(x)}{\sqrt{n\rho}} \right),$$

for sufficiently large n and constant $C > 0$, which is independent of f and n .

Proof. For $j \in W_\gamma$, we get

$$j(t) = j(x) + \int_x^t j'(u) du. \quad (3.9)$$

Applying $T_{n,\rho}^{\alpha,\beta}$ in (3.8) and using Hölder's inequality, we obtain

$$\begin{aligned} |T_{n,\rho}^{\alpha,\beta}(j(t); x) - j(x)| &\leq T_{n,\rho}^{\alpha,\beta} \left(\int_x^t |j'| du; x \right) \\ &\leq \|\phi^\gamma j'\| T_{n,\rho}^{\alpha,\beta} \left(\left| \int_x^t \frac{du}{\phi^\gamma(u)} \right|; x \right) \\ &\leq \|\phi^\gamma j'\| T_{n,\rho}^{\alpha,\beta} \left(|t-x|^{1-\gamma} \left| \int_x^t \frac{du}{\phi(u)} \right|^\gamma; x \right). \end{aligned} \quad (3.10)$$

Take $A = \left| \int_x^t \frac{du}{\phi(u)} \right|$, we find

$$\begin{aligned} A &\leq \left| \int_x^t \frac{du}{\sqrt{u}} \right| \left(\frac{1}{\sqrt{cx+\rho}} + \frac{1}{\sqrt{ct+\rho}} \right) \\ &\leq 2|\sqrt{t} - \sqrt{x}| \left(\frac{1}{\sqrt{cx+\rho}} + \frac{1}{\sqrt{ct+\rho}} \right) \\ &\leq 2 \frac{|t-x|}{\sqrt{x} + \sqrt{t}} \left(\frac{1}{\sqrt{cx+\rho}} + \frac{1}{\sqrt{ct+\rho}} \right) \\ &\leq 2 \frac{|t-x|}{\sqrt{x}} \left(\frac{1}{\sqrt{cx+\rho}} + \frac{1}{\sqrt{ct+\rho}} \right). \end{aligned} \quad (3.11)$$

Using the inequality $|a + b|^\gamma \leq |a|^\gamma + |b|^\gamma$, $0 \leq \gamma \leq 1$ and (3.11), we get

$$\left\| \int_x^t \frac{du}{\phi(u)} \right\|^\gamma \leq 2^\beta \frac{|t-x|^\gamma}{x^{\frac{\gamma}{2}}} \left(\frac{1}{(cx+\rho)^{\frac{\gamma}{2}}} + \frac{1}{(ct+\rho)^{\frac{\gamma}{2}}} \right). \quad (3.12)$$

From (3.10), (3.12) and using Cauchy inequality, we get

$$\begin{aligned} |T_{n,\rho}^{\alpha,\beta}(j(t);x) - j(x)| &\leq \frac{2^\gamma \|\phi^\gamma j'\|}{x^{\frac{\gamma}{2}}} T_{n,\rho}^{\alpha,\beta} \left(|t-x| \left(\frac{1}{(cx+\rho)^{\frac{\gamma}{2}}} + \frac{1}{(ct+\rho)^{\frac{\gamma}{2}}} \right); x \right) \\ &\leq \frac{2^\gamma \|\phi^\gamma j'\|}{x^{\frac{\gamma}{2}}} \left(\frac{1}{(ct+\rho)^{\frac{\gamma}{2}}} (T_{n,\rho}^{\alpha,\beta}((t-x)^2; x))^{\frac{1}{2}} \right. \\ &\quad \left. + (T_{n,\rho}^{\alpha,\beta}((t-x)^2; x))^{\frac{1}{2}} \cdot (T_{n,\rho}^{\alpha,\beta}((ct+\rho)^{-\gamma}; x))^{\frac{1}{2}} \right). \end{aligned}$$

If n is adequately large, then we get

$$\left(T_{n,\rho}^{\alpha,\beta}((t-x)^2; x) \right)^{\frac{1}{2}} \leq \sqrt{\frac{2}{n\rho}} \phi(x), \quad (3.13)$$

where $\phi(x) = \sqrt{x(cx+\rho)}$.

For each $x \in (0, \infty)$, $T_{n,\rho}^{\alpha,\beta}((ct+\rho)^{-\gamma}; x) \rightarrow (cx+\rho)^{-\gamma}$ as $n \rightarrow \infty$. For $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$T_{n,\rho}^{\alpha,\beta}((ct+\rho)^{-\gamma}; x) \leq (cx+\rho)^{-\gamma} + \varepsilon, \quad \forall n \geq n_0 = n_0(c, x, \rho, \beta).$$

By choosing $\varepsilon = (cx+\rho)^{-\gamma}$, we obtain

$$T_{n,\rho}^{\alpha,\beta}((ct+\rho)^{-\gamma}; x) \leq 2(cx+\rho)^{-\gamma}, \quad \forall n \geq n_0. \quad (3.14)$$

From (3.10) to (3.14), we get

$$\begin{aligned} |T_{n,\rho}^{\alpha,\beta}(j(t);x) - j(x)| &\leq 2^\gamma \|\phi^\gamma j'\| \sqrt{\frac{2}{n\rho}} \phi(x) \left(\phi^{-\gamma}(x) + \sqrt{2} x^{\frac{-\gamma}{2}} (cx+\rho)^{\frac{-\gamma}{2}} \right) \\ &\leq 2^{\gamma+\frac{1}{2}} (1+\sqrt{2}) \|\phi^\gamma j'\| \sqrt{\frac{2}{n\rho}} \phi^{1-\gamma}(x). \end{aligned} \quad (3.15)$$

We may write

$$\begin{aligned} |T_{n,\rho}^{\alpha,\beta}(f(t);x) - f(x)| &\leq |T_{n,\rho}^{\alpha,\beta}(f(t) - j(t);x)| + |T_{n,\rho}^{\alpha,\beta}(j(t);x) - j(x)| + |j(x) - f(x)| \\ &\leq 2\|f - j\| + |T_{n,\rho}^{\alpha,\beta}(j(t);x) - j(x)|. \end{aligned} \quad (3.16)$$

From (3.15) to (3.16) and for adequately large n , we get

$$|T_{n,\rho}^{\alpha,\beta}(f(t);x) - f(x)| \leq 2\|f - j\| + 2^{\gamma+\frac{1}{2}} (1+\sqrt{2}) \sqrt{\frac{2}{n\rho}} \phi^{1-\gamma} \|\phi^\beta j'\|$$

$$\begin{aligned}
&\leq m_1 \left\{ \|f - j\| + \frac{\phi^{1-\gamma}(x)}{\sqrt{n\rho}} \|\phi^\gamma j'\| \right\} \\
&\leq CK_{\phi^\gamma} \left(f, \frac{\phi^{1-\gamma}(x)}{\sqrt{n\rho}} \right), \tag{3.17}
\end{aligned}$$

where $m_1 = \max(2, 2^{\gamma+\frac{1}{2}}(1+\sqrt{2})\sqrt{2})$, from (3.8) and (3.17) we obtain the result. $\square$

4. WEIGHTED APPROXIMATION

In this section, we will examine the weighted estimation hypothesis. Let

$B_{x^2}[0, \infty) = \{f : \text{for every } x \in [0, \infty), |f(x)| \leq \phi_f(1+x^2), \phi_f \text{ is a constant depending on } f\}$.

Here $C_{x^2}^*[0, \infty)$, denotes the subspace of all continuous functions belonging to $B_{x^2}[0, \infty)$

and satisfying the condition $\lim_{x \rightarrow \infty} \frac{f(x)}{1+x^2}$ is finite. The norm on $C_{x^2}^*[0, \infty)$ is given by

$$\|f\|_{x^2} = \sup_{x \in [0, \infty)} \frac{f(x)}{1+x^2}.$$

Theorem 6. For each $f \in C_{x^2}^*[0, \infty)$ and $n > 2c$, we have

$$\lim_{n \rightarrow \infty} \|T_{n,\rho}^{\alpha,\beta}(f) - f\|_{x^2} = 0.$$

Proof. Using Lemma 2, we see that it is sufficient to verify the conditions

$$\lim_{n \rightarrow \infty} \|T_{n,\rho}^{\alpha,\beta}(t^r, x) - x^r\|_{x^2} = 0, \quad r = 0, 1, 2. \tag{4.1}$$

Since $T_{n,\rho}^{\alpha,\beta}(1, x) = 1$, for $r = 0$, (4.1) holds. For $n\rho > c$, we have

$$\begin{aligned}
\|T_{n,\rho}^{\alpha,\beta}(t, x) - x\|_{x^2} &= \sup_{x \in [0, \infty)} \frac{|T_{n,\rho}^{\alpha,\beta}(t, x) - x|}{1+x^2} \\
&= \sup_{x \in [0, \infty)} \frac{1}{1+x^2} \left| \frac{n\rho x}{(n\rho - c)} + \frac{\alpha\rho}{(n\rho - c)(1+e\beta)} - x \right| \\
&\leq \left[\frac{n\rho}{(n\rho - c)} - 1 \right] \sup_{x \in [0, \infty)} \frac{x}{1+x^2} + \frac{\alpha\rho}{(n\rho - c)(1+e\beta)} \sup_{x \in [0, \infty)} \frac{1}{1+x^2},
\end{aligned}$$

condition (4.1) holds for $r = 1$ as $n \rightarrow \infty$. Again $n\rho > 2c$, we have

$$\begin{aligned}
\|T_{n,\rho}^{\alpha,\beta}(t^2, x) - x^2\|_{x^2} &= \sup_{x \in [0, \infty)} \frac{|T_{n,\rho}^{\alpha,\beta}(t^2, x) - x^2|}{1+x^2} \\
&= \sup_{x \in [0, \infty)} \frac{1}{1+x^2} \left| \frac{n^2\rho^2}{(n\rho - c)(n\rho - 2c)} x^2 \right. \\
&\quad \left. + \frac{n\rho^2(1+2\alpha+e\beta)}{(n\rho - c)(n\rho - 2c)(1+e\beta)} x \right|
\end{aligned}$$

$$\begin{aligned}
& + \frac{\rho^2(\alpha^2 - 2\alpha e\beta - \alpha e^2\beta^2)}{(n\rho - c)(n\rho - 2c)(1 + e\beta)^2} - x^2 \Big| \\
& \leq \left[\frac{n^2\rho^2}{(n\rho - c)(n\rho - 2c)} - 1 \right] \sup_{x \in [0, \infty)} \frac{x^2}{1 + x^2} \\
& + \frac{n\rho^2(1 + 2\alpha + e\beta)}{(n\rho - c)(n\rho - 2c)(1 + e\beta)} \sup_{x \in [0, \infty)} \frac{x}{1 + x^2} \\
& + \frac{\rho^2(\alpha^2 - 2\alpha e\beta - \alpha e^2\beta^2)}{(n\rho - c)(n\rho - 2c)(1 + e\beta)^2} \sup_{x \in [0, \infty)} \frac{1}{1 + x^2}.
\end{aligned}$$

Condition (4.1) holds for $r = 2$ as $n \rightarrow \infty$. This completes the proof of the theorem. $\square$

Corollary 1. For each $f \in C_{x^2}[0, \infty)$, and $\alpha > 0$, we have

$$\lim_{n \rightarrow \infty} \sup_{x \in [0, \infty)} \frac{|T_{n,\rho}^{\alpha,\beta}(f; x) - f(x)|}{(1 + x^2)^\alpha} = 0.$$

Proof. For any fixed $x_0 > 0$,

$$\begin{aligned}
\sup_{x \in [0, \infty)} \frac{|T_{n,\rho}^{\alpha,\beta}(f; x) - f(x)|}{(1 + x^2)^{1+\alpha}} & \leq \sup_{x \leq x_0} \frac{|T_{n,\rho}^{\alpha,\beta}(f; x) - f(x)|}{(1 + x^2)^{1+\alpha}} + \sup_{x \geq x_0} \frac{|T_{n,\rho}^{\alpha,\beta}(f; x) - f(x)|}{(1 + x^2)^{1+\alpha}} \\
& \leq \|T_{n,\rho}^{\alpha,\beta}(f; \cdot) - f\|_{C[0, x_0]} + \|f\|_{x^2} \sup_{x \geq x_0} \frac{|T_{n,\rho}^{\alpha,\beta}(f; x) - f(x)|}{(1 + x^2)^{1+\alpha}} \\
& + \sup_{x \geq x_0} \frac{|f(x)|}{(1 + x^2)^{1+\alpha}}.
\end{aligned}$$

From Theorem 2, in the above inequality first term tends to zero and by Lemma 2 for any fixed x_0 it can be easily seen that

$$\sup_{x \geq x_0} \frac{|T_{n,\rho}^{\alpha,\beta}(1 + t^2; x) - f(x)|}{(1 + x^2)^{1+\alpha}} \leq \frac{\varphi}{(1 + x_0^2)^\alpha}.$$

Constant $\varphi > 0$ is independent of x , and choose adequately large x_0 the right-hand side of the earlier inequality and last part can be made small, we get the required result. $\square$

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ON GENERALIZED PROJECTIVE CURVATURE TENSOR OF PARA-KENMOTSU MANIFOLDS

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Abstract. The object of the present paper is to generalize projective curvature tensor of para-Kenmotsu manifold with the help of a new generalized (0,2) symmetric tensor Z introduced by Mantica and Suh [8]. Various geometric properties of generalized projective curvature tensor of para-Kenmotsu manifold have been studied. It is shown that a generalized projectively ϕ -symmetric para-Kenmotsu manifold is an Einstein manifold.

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1. INTRODUCTION

A relevant tensor from the differential geometric point of view is the projective curvature tensor. Under a geodesic conserving transformation on a semi-Riemannian manifold, the projective curvature tensor P is invariant. In 1982 Szabo [15, 16] studied Riemannian space satisfying $R(X, Y) \cdot R = 0$. In De and Samui [4] studied $P \cdot R = 0$, $R \cdot P = 0$ and $P \cdot S = 0$ in an LP -Sasakian manifold. The projective curvature tensor is therefore a measure of a Riemannian manifold's failure to be of constant curvature. Afterwards several researchers have carried out the study of projective curvature tensor in a variety of directions such as [10, 13, 14].

Several years ago, the notion of paracontact metric structures were introduced in [7]. Since the publication of [19], paracontact metric manifolds have been studied by many authors in recent years. The importance of para-Kenmotsu geometry, have been pointed out especially in the last years by several papers highlighting the exchanges with the theory of para-Kähler manifolds and its role in semi-Riemannian geometry and mathematical physics [3, 5, 6, 11].

In this paper, we consider the generalized projective curvature tensor of para-Kenmotsu manifolds and study some properties of generalized projective curvature

tensor. The organisation of the paper is as follows: After preliminaries on para-Kenmotsu manifold in section 2, we describe briefly the generalized projective curvature tensor on para-Kenmotsu manifold in section 3 and also we study some properties of generalized projective curvature tensor in para-Kenmotsu manifold. In section 4, we study a generalized projectively semi-symmetric para-Kenmotsu manifold is an Einstein manifold. Further in the section 5, we show that a generalized projectively Ricci semi-symmetric para-Kenmotsu manifold is either Einstein manifold or $\psi = 1 - n$ on it. The last section is devoted to the study of the generalized projectively ϕ -symmetric para-Kenmotsu manifold is an Einstein manifold.

2. PRELIMINARIES

The notion of an almost para-contact manifold was introduced by I. Sato [12]. An n -dimensional differentiable manifold M^n is said to have almost para-contact structure (ϕ, ξ, η) , where ϕ is a tensor field of type $(1, 1)$, ξ is a vector field known as characteristic vector field and η is a 1-form satisfying the following relations

$$\phi^2(X) = X - \eta(X)\xi, \quad (2.1)$$

$$\eta(\phi X) = 0, \quad (2.2)$$

$$\phi(\xi) = 0, \quad (2.3)$$

and

$$\eta(\xi) = 1. \quad (2.4)$$

A differentiable manifold with almost para-contact structure (ϕ, ξ, η) is called an almost para-contact manifold. Further, if the manifold M^n has a semi Riemannian metric g satisfying

$$\eta(X) = g(X, \xi) \quad (2.5)$$

and

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y). \quad (2.6)$$

then the structure (ϕ, ξ, η, g) satisfying conditions (2.1) to (2.6) is called an almost para-contact Riemannian structure and the manifold M^n with such a structure is called an almost para-contact Riemannian manifold [1, 12].

On a para-Kenmotsu manifold [2, 11], the following relations hold:

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \quad (2.7)$$

$$\nabla_X \xi = X - \eta(X)\xi, \quad (2.8)$$

$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y), \quad (2.9)$$

$$\eta(R(X, Y, Z)) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \quad (2.10)$$

$$R(X, Y, \xi) = \eta(X)Y - \eta(Y)X, \quad (2.11)$$

$$R(X, \xi, Y) = -R(\xi, X, Y) = g(X, Y)\xi - \eta(Y)X, \quad (2.12)$$

$$S(\phi X, \phi Y) = -(n-1)g(\phi X, \phi Y), \quad (2.13)$$

$$S(X, \xi) = -(n-1)\eta(X), \quad (2.14)$$

$$Q\xi = -(n-1)\xi, \quad (2.15)$$

$$r = -n(n-1), \quad (2.16)$$

for any vector fields X, Y, Z , where Q is the Ricci operator that is $g(QX, Y) = S(X, Y)$, S is the Ricci tensor and r is the scalar curvature.

In [2], Blaga have given an example on para-Kenmotsu manifold:

Example 1. We consider the three dimensional manifold $M^3 = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$, where (x, y, z) are the standard co-ordinates in $\mathbb{R}^3$. The vector fields

$$e_1 := \frac{\partial}{\partial x}, e_2 := \frac{\partial}{\partial y}, e_3 := -\frac{\partial}{\partial z}$$

are linearly independent at each point of the manifold.

Define

$$\begin{aligned} \phi &:= \frac{\partial}{\partial y} \otimes dx + \frac{\partial}{\partial x} \otimes dy, \xi := -\frac{\partial}{\partial z}, \eta := -dz, \\ g &:= dx \otimes dx - dy \otimes dy + dz \otimes dz. \end{aligned}$$

Then it follows that

$$\begin{aligned} \phi e_1 &= e_2, \phi e_2 = e_1, \phi e_3 = 0, \\ \eta(e_1) &= 0, \eta(e_2) = 0, \eta(e_3) = 1. \end{aligned}$$

Let ∇ be the Levi-Civita connection with respect to metric g . Then, we have

$$[e_1, e_2] = 0, [e_2, e_3] = 0, [e_3, e_1] = 0$$

The Riemannian connection ∇ of the metric g is deduced from Koszul's formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ &\quad - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]). \end{aligned}$$

Then Koszul's formula yields

$$\begin{aligned} \nabla_{e_1} e_1 &= -e_3, \nabla_{e_1} e_2 = 0, \nabla_{e_1} e_3 = e_1, \\ \nabla_{e_2} e_1 &= 0, \nabla_{e_2} e_2 = e_3, \nabla_{e_2} e_3 = e_2, \\ \nabla_{e_3} e_1 &= e_1, \nabla_{e_3} e_2 = e_2, \nabla_{e_3} e_3 = 0. \end{aligned}$$

These results shows that the manifold satisfies

$$\nabla_X \xi = X - \eta(X)\xi,$$

for $\xi = e_3$. Hence the manifold under consideration is para-Kenmotsu manifold of dimension three.

3. GENERALIZED PROJECTIVE CURVATURE TENSOR OF PARA-KENMOTSU MANIFOLD

In this section, we give a brief account of generalized projective curvature tensor of para-Kenmotsu manifold and studied various geometric properties of it.

The projective curvature tensor is defined by R.S. Mishra [9]

$$P(X, Y, U) = R(X, Y, U) - \frac{1}{(n-1)}[S(Y, U)X - S(X, U)Y], \quad (3.1)$$

where S is a Ricci tensor, such a tensor field P is known as projective curvature tensor.

Also, the type $(0, 4)$ tensor field $'P$ is given by

$$\begin{aligned} 'P(X, Y, U, V) = 'R(X, Y, U, V) - \frac{1}{(n-1)}[S(Y, U)g(X, V) \\ - S(X, U)g(Y, V)] \end{aligned} \quad (3.2)$$

where

$$'P(X, Y, U, V) = g(P(X, Y, U), V)$$

and

$$'R(X, Y, U, V) = g(R(X, Y, U), V)$$

for the arbitrary vector fields X, Y, U, V .

Differentiating covariantly equation (3.1) with respect to W , we get

$$\begin{aligned} (\nabla_W P)(X, Y, U) = (\nabla_W R)(X, Y, U) - \frac{1}{(n-1)}[(\nabla_W S)(Y, U)X \\ - (\nabla_W S)(X, U)Y]. \end{aligned} \quad (3.3)$$

A new generalized $(0, 2)$ symmetric tensor Z is defined by Mantica and Suh [8]

$$Z(X, Y) = S(X, Y) + \psi g(X, Y), \quad (3.4)$$

where ψ is an arbitrary scalar function.

From equation (3.4), we have

$$Z(\phi X, \phi Y) = S(\phi X, \phi Y) + \psi g(\phi X, \phi Y), \quad (3.5)$$

which on using equations (2.6) and (2.13), gives

$$Z(\phi X, \phi Y) = [\psi - (n-1)][-g(X, Y) + \eta(X)\eta(Y)]. \quad (3.6)$$

From equation (3.2), we have

$$\begin{aligned} 'P(X, Y, U, V) = 'R(X, Y, U, V) - \frac{1}{(n-1)}[S(Y, U)g(X, V) \\ - S(X, U)g(Y, V)]. \end{aligned} \quad (3.7)$$

From equation (3.4) above equation reduces to

$$'P(X, Y, U, V) = 'R(X, Y, U, V) - \frac{1}{(n-1)}[Z(Y, U)g(X, V) \quad (3.8)$$

$$-Z(X, U)g(Y, V)] + \frac{\Psi}{(n-1)}[g(Y, U)g(X, V) - g(Y, V)g(X, U)],$$

Let

$$\begin{aligned} {}'P^*(X, Y, U, V) = {}'R(X, Y, U, V) - \frac{1}{(n-1)}[Z(Y, U)g(X, V) \\ - Z(X, U)g(Y, V)], \end{aligned} \quad (3.9)$$

In the above equation, we get

$$\begin{aligned} {}'P^*(X, Y, U, V) = {}'P(X, Y, U, V) - \frac{\Psi}{(n-1)}[g(X, V)g(Y, U) \\ - g(Y, V)g(X, U)]. \end{aligned} \quad (3.10)$$

Thus $'P^*$ defined in equation (3.9) is called generalized projective curvature tensor of para-Kenmotsu manifold.

If $\Psi=0$, then from equation (3.10), we have

$${}'P^*(X, Y, U, V) = {}'P(X, Y, U, V). \quad (3.11)$$

Lemma 1. *If the scalar function Ψ vanishes on para-Kenmotsu manifold, then the projective curvature tensor and generalized projective curvature tensor are identicle.*

Lemma 2. *Generalized projective curvature tensor of para-Kenmotsu manifold satisfies Bianchi's first identity.*

Remark 1. Generalized projective curvature tensor $'P^*$ of para-Kenmotsu manifold is

- (a) skew symmetric in first two slots.
- (b) skew symmetric in last two slots.
- (c) symmetric in pair of slots.

Proposition 1. *Generalized projective curvature tensor of para-Kenmotsu manifold satisfies the following identities:*

$$\begin{aligned} (a) \quad P^*(\xi, Y, U) = -P^*(Y, \xi, U) = \left[\frac{1-n-\Psi}{n-1} \right] g(Y, U)\xi \\ - \frac{1}{(n-1)}S(Y, U)\xi + \frac{\Psi}{(n-1)}\eta(U)Y, \end{aligned} \quad (3.12)$$

$$(b) \quad P^*(X, Y, \xi) = -\frac{\Psi}{(n-1)}[\eta(Y)X - \eta(X)Y] \quad (3.13)$$

$$\begin{aligned} (c) \quad \eta(P^*(U, V, Y)) = \left[-1 - \frac{\Psi}{(n-1)} \right] [g(V, Y)\eta(U) - g(U, Y)\eta(V)] \\ - \frac{1}{(n-1)}[S(V, Y)\eta(U) - S(U, Y)\eta(V)]. \end{aligned} \quad (3.14)$$

4. GENERALIZED PROJECTIVELY SEMI-SYMMETRIC PARA-KENMOTSU MANIFOLD

Definition 1. Para-Kenmotsu manifold is said to be semi-symmetric if it satisfies the condition

$$R(X, Y) \cdot R = 0, \quad (4.1)$$

where $R(X, Y)$ is considered as the derivative of the tensor algebra at each point of the manifold.

Definition 2. Para-Kenmotsu manifold is said to be generalized projectively semi-symmetric if it satisfies the condition

$$R(X, Y) \cdot P^* = 0, \quad (4.2)$$

where P^* is generalized projective curvature tensor and $R(X, Y)$ is considered as the derivative of the tensor algebra at each point of the manifold.

Theorem 1. A generalized projectively semi-symmetric para-Kenmotsu manifold is an Einstein manifold.

Proof. Consider

$$(R(\xi, X) \cdot P^*)(U, V, Y) = 0,$$

for any $X, Y, U, V \in T_L M$, where P^* is generalized projective curvature tensor.

Then we have

$$\begin{aligned} 0 &= R(\xi, X, P^*(U, V, Y)) - P^*(R(\xi, X, U), V, Y) \\ &\quad - P^*(U, R(\xi, X, V), Y) - P^*(U, V, R(\xi, X, Y)). \end{aligned} \quad (4.3)$$

In view of the equation (2.12) above equation takes the form

$$\begin{aligned} 0 &= \eta(P^*(U, V, Y))X - 'P^*(U, V, Y, X)\xi - \eta(U)P^*(X, V, Y) \\ &\quad + g(X, U)P^*(\xi, V, Y) - \eta(V)P^*(U, X, Y) + g(X, V)P^*(U, \xi, Y) \\ &\quad - \eta(Y)P^*(U, V, X) + g(X, Y)P^*(U, V, \xi). \end{aligned}$$

Taking inner product of above equation with ξ and using equations (2.4), (3.10), (3.12), (3.13), (3.14), we get

$$\begin{aligned} 0 &= -'P(U, V, Y, X) + \frac{\Psi}{(n-1)}[g(X, U)g(Y, V) - g(X, V)g(Y, U)] \\ &\quad + \left[1 + \frac{\Psi}{(n-1)}\right][g(X, V)\eta(U)\eta(Y) - g(X, U)\eta(V)\eta(Y)] \\ &\quad + \left[\frac{1-n-\Psi}{(n-1)}\right]g(X, U)g(Y, V) - \left[\frac{1-n-\Psi}{(n-1)}\right]g(X, V)g(Y, U) \\ &\quad + \frac{\Psi}{(n-1)}g(X, U)\eta(Y)\eta(V) - \frac{\Psi}{(n-1)}g(X, V)\eta(Y)\eta(U) \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{(n-1)}g(X, V)S(Y, U) - \frac{1}{(n-1)}g(X, U)S(Y, V) \\
& + \frac{1}{(n-1)}[S(X, V)\eta(U)\eta(Y) - S(X, U)\eta(V)\eta(Y)].
\end{aligned}$$

By virtue of equation (3.2) above equation reduces to

$$\begin{aligned}
{}'R(U, V, Y, X) &= \frac{\Psi}{(n-1)}[g(X, U)g(Y, V) - g(X, V)g(Y, U)] \\
&+ \left[1 + \frac{\Psi}{(n-1)}\right][g(X, V)\eta(U)\eta(Y) - g(X, U)\eta(V)\eta(Y)] \\
&+ \frac{1}{(n-1)}[S(X, V)\eta(Y)\eta(U) - S(X, U)\eta(Y)\eta(V)] \\
&+ \left[\frac{1-n-\Psi}{(n-1)}\right]g(X, U)g(Y, V) - \left[\frac{1-n-\Psi}{(n-1)}\right]g(X, V)g(Y, U) \\
&+ \frac{\Psi}{(n-1)}g(X, U)\eta(Y)\eta(V) - \frac{\Psi}{(n-1)}g(X, V)\eta(Y)\eta(U).
\end{aligned}$$

Let $\{e_i : i = 1, 2, \dots, n\}$ be an orthonormal basis vector putting $X = U = e_i$ in above equation and taking summation over i , we get

$$S(Y, V) = -(n-1)g(Y, V).$$

This shows that generalized projectively semi-symmetric para-Kenmotsu manifold is an Einstein manifold. $\square$

5. GENERALIZED PROJECTIVELY RICCI SEMI-SYMMETRIC PARA-KENMOTSU MANIFOLD

Definition 3. Para-Kenmotsu manifold M is said to be Ricci semi-symmetric if the condition

$$R(X, Y) \cdot S = 0, \quad (5.1)$$

holds for all $X, Y \in T_L M$.

Definition 4. Para-Kenmotsu manifold is said to be generalized projectively Ricci semi-symmetric if the condition

$$P^*(X, Y) \cdot S = 0, \quad (5.2)$$

holds for all X, Y , where P^* is generalized projective curvature tensor of para-Kenmotsu manifold.

Theorem 2. A generalized projectively Ricci semi-symmetric para-Kenmotsu manifold is either Einstein manifold or $\Psi = 1 - n$ on it.

Proof. Consider

$$(P^*(\xi, X) \cdot S)(U, V) = 0,$$

which gives

$$S(P^*(\xi, X, U), V) + S(U, P^*(\xi, X, V)) = 0.$$

Using equations (2.14) and (3.12) in above equation, we get

$$\begin{aligned} \left[\frac{n-1+\psi}{(n-1)} \right] [S(X, V)\eta(U) + S(X, U)\eta(V)] \\ - (1-n-\psi)[g(X, U)\eta(V) + g(X, V)\eta(U)] = 0. \end{aligned}$$

Putting $U = \xi$ in the above equation and using (2.4), (2.5) and (2.14), we get

$$(n-1+\psi)[S(X, V) + g(X, V)(n-1)] = 0,$$

which gives either $\psi = 1-n$ or

$$S(X, V) = -(n-1)g(X, V).$$

This shows that generalized projectively Ricci semi-symmetric para-Kenmotsu manifold is an Einstein manifold. $\square$

6. GENERALIZED PROJECTIVELY ϕ -SYMMETRIC PARA-KENMOTSU MANIFOLD

Definition 5. A para-Kenmotsu manifold M^n is said to be locally ϕ -symmetric if

$$\phi^2((\nabla_W R)(X, Y, U)) = 0, \quad (6.1)$$

for all vector fields X, Y, U, W orthogonal to ξ .

This notion was introduced by Takahashi for Sasakian manifold [17].

Definition 6. A para-Kenmotsu manifold is said to be ϕ -symmetric if

$$\phi^2((\nabla_W R)(X, Y, U)) = 0, \quad (6.2)$$

for arbitrary vector fields X, Y, U, W .

This notion was also introduced by Takahashi for Sasakian manifold [18]. Also analogous to these definitions, we define

Definition 7. A para-Kenmotsu manifold M^n is said to be generalized projective locally ϕ -symmetric para-Kenmotsu manifold if

$$\phi^2((\nabla_W P^*)(X, Y, U)) = 0, \quad (6.3)$$

for all vector fields X, Y, U, W orthogonal to ξ .

And also

Definition 8. A para-Kenmotsu manifold M^n is said to be generalized projectively ϕ -symmetric para-Kenmotsu manifold if

$$\phi^2((\nabla_W P^*)(X, Y, U)) = 0, \quad (6.4)$$

for arbitrary vector fields X, Y, U, W .

Theorem 3. A generalized projectively ϕ -symmetric para Kenmotsu manifold is an Einstein manifold.

Proof. Taking covariant derivative of equation (3.10) with respect to vector field W , we obtain

$$\begin{aligned} (\nabla_W P^*)(X, Y, U) &= (\nabla_W P)(X, Y, U) - \frac{dr(\psi)}{(n-1)} [g(Y, U)X \\ &\quad - g(X, U)Y], \end{aligned} \quad (6.5)$$

Using equation (3.3) in the above equation, we yields

$$\begin{aligned} (\nabla_W P^*)(X, Y, U) &= (\nabla_W R)(X, Y, U) - \frac{dr(\psi)}{(n-1)} [g(Y, U)X \\ &\quad - g(X, U)Y] - \frac{1}{(n-1)} [(\nabla_W S)(Y, U)X - (\nabla_W S)(X, U)Y], \end{aligned} \quad (6.6)$$

Assume that the manifold is generalized projectively ϕ -symmetric, then from equation (6.4), we have

$$\phi^2((\nabla_W P^*)(X, Y, U)) = 0,$$

which on using equation (2.1), gives

$$(\nabla_W P^*)(X, Y, U) = \eta((\nabla_W P^*)(X, Y, U))\xi. \quad (6.7)$$

Using equation (6.6) in above equation, we get

$$\begin{aligned} &(\nabla_W R)(X, Y, U) - \frac{dr(\psi)}{(n-1)} [g(Y, U)X - g(X, U)Y] \\ &\quad - \frac{1}{(n-1)} [(\nabla_W S)(Y, U)X - (\nabla_W S)(X, U)Y] \\ &= \eta((\nabla_W R)(X, Y, U))\xi - \frac{dr(\psi)}{(n-1)} [g(Y, U)\eta(X) - g(X, U)\eta(Y)]\xi \\ &\quad - \frac{1}{(n-1)} [(\nabla_W S)(Y, U)\eta(X) - (\nabla_W S)(X, U)\eta(Y)]\xi, \end{aligned} \quad (6.8)$$

Taking inner product of the above equation with V , we get

$$\begin{aligned} &g((\nabla_W R)(X, Y, U), V) - \frac{dr(\psi)}{(n-1)} [g(Y, U)g(X, V) - g(X, U)g(Y, V)] \\ &\quad - \frac{1}{(n-1)} [(\nabla_W S)(Y, U)g(X, V) - (\nabla_W S)(X, U)g(Y, V)] \end{aligned}$$

$$\begin{aligned}
&= \eta((\nabla_W R)(X, Y, U))\eta(V) - \frac{dr(\psi)}{(n-1)}[g(Y, U)\eta(X) \\
&\quad - g(X, U)\eta(Y)]\eta(V) - \frac{1}{(n-1)}[(\nabla_W S)(Y, U)\eta(X) - (\nabla_W S)(X, U)\eta(Y)]\eta(V),
\end{aligned} \tag{6.9}$$

Putting $X = V = e_i$ and taking summation over i , we obtain

$$\begin{aligned}
-dr(\psi)g(Y, U) &= \eta((\nabla_W R)(e_i, Y, U))\eta(e_i) - \frac{1}{(n-1)}[(\nabla_W S)(Y, U) \\
&\quad - (\nabla_W S)(e_i, U)\eta(Y)\eta(e_i)] - \frac{dr(\psi)}{(n-1)}[g(Y, U) - \eta(Y)\eta(U)],
\end{aligned} \tag{6.10}$$

Taking $U = \xi$ in the above equation, we have

$$\begin{aligned}
&\eta((\nabla_W R)(e_i, Y, \xi))\eta(e_i) + dr(\psi)\eta(Y) - \frac{1}{(n-1)}[(\nabla_W S)(Y, \xi) \\
&\quad - (\nabla_W S)(e_i, \xi)\eta(e_i)\eta(Y)] = 0.
\end{aligned} \tag{6.11}$$

Now

$$\eta((\nabla_W R)(e_i, Y, \xi))\eta(e_i) = g((\nabla_W R)(e_i, Y, \xi), \xi)g(e_i, \xi). \tag{6.12}$$

Also

$$\begin{aligned}
g((\nabla_W R)(e_i, Y, \xi), \xi) &= g(\nabla_W R(e_i, Y, \xi), \xi) - g(R(\nabla_W e_i, Y, \xi), \xi) \\
&\quad - g(R(e_i, \nabla_W Y, \xi), \xi) - g(R(e_i, Y, \nabla_W \xi), \xi).
\end{aligned} \tag{6.13}$$

Since $\{e_i\}$ is an orthonormal basis, so $\nabla_X e_i = 0$ and using equation (2.11), we get

$$g(R(e_i, \nabla_W Y, \xi), \xi) = 0.$$

As

$$g(R(e_i, Y, \xi), \xi) + g(R(\xi, \xi, Y), e_i) = 0,$$

we have

$$g(\nabla_W R(e_i, Y, \xi), \xi) + g(R(e_i, Y, \xi), \nabla_W \xi) = 0.$$

Using this fact, we get

$$g((\nabla_W R)(e_i, Y, \xi), \xi) = 0. \tag{6.14}$$

Using equation (6.14) in (6.11), we have

$$(\nabla_W S)(Y, \xi) = (n-1)dr(\psi)\eta(Y). \tag{6.15}$$

Taking $Y = \xi$ in above equation and using equations (2.4) and (2.14), we get

$$dr(\psi) = 0, \tag{6.16}$$

which shows that r is constant. Now we have

$$(\nabla_W S)(Y, \xi) = \nabla_W S(Y, \xi) - S(\nabla_W Y, \xi) - S(Y, \nabla_W \xi),$$

Then by using (2.8), (2.9), (2.14) in the above equation, it follows that

$$(\nabla_W S)(Y, \xi) = -S(Y, W) - (n-1)g(Y, W). \tag{6.17}$$

So from equation (6.15), (6.16) and (6.17), This shows that

$$S(Y, W) = -(n-1)g(Y, W), \quad (6.18)$$

which shows that M^n is an Einstein manifold. $\square$

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NON-EXISTENCE OF SOLUTION OF HARAUX-WEISSLER EQUATION ON A STRICTLY STARSHPED DOMAIN

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Abstract. Here, we prove the Haraux-Weissler equation has no solution on a domain which is strictly starshped with respect to 0. Finally, we present some questions.

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1. INTRODUCTION

A Porous Medium Equation (in short PME) is the equation

$$u_t = \Delta(u^m),$$

where $u = u(x, t)$ is a scalar function and m is a constant larger than 1. The space variable x takes values in R^d , $d \geq 1$, while $t \in R$. Physical considerations lead to the restriction $u \geq 0$, which is mathematically convenient and currently followed, but not essential.

The PME is an example of a nonlinear evolution equation, formally of parabolic type and there are a number application in boundary layer theory mathematical biology, lubrication, water infiltration, heat radiation in plasmas, processes involving diffusion or heat transfer (such as description of the flow of an isentropic gas through a porous medium) and other fields (see [1, 2, 5, 6, 14–17, 19, 20]).

Consider a quasilinear PME with a source term

$$u_t - \Delta(|u|^{m-1}u) = |u|^{p-1}u \text{ in } \mathbb{R}^N \times (0, T),$$

where the parameters are taken as $m > 0$ and $p > 1$. In studying this equation, the self-similar solution $u(x, t) = t^{-\alpha}U(r)$, where $r = |x|t^{-\beta}$, are of interest, as usually describe the large time behaviour of solutions of the Cauchy problem with general initial data (see [12]). Concerning this structure, Haraxu and Weissler [10] proposed

$$(|U|^{m-1}U)'' + \frac{N-1}{r}(|U|^{m-1}U)' + \beta rU' + \alpha U + (|U|^{p-1}U) = 0, \text{ in } \mathbb{R}^+, \quad (1.1)$$

for the semilinear case $m = 1$, where $\alpha, \beta > 0, \alpha(m-1) + 2\beta = 1, (|U|^{m-1}U)(0) = 0, U(0) = a > 0$. In fact, if the equation originates from (PME), the parameters α, β have to fulfill the additional condition $\alpha p = p + 1$ and are thus given by

$$\alpha = \frac{1}{p-1}, \beta = \frac{p-m}{2(p-1)}.$$

Another aspect of problem (1.1) is that it can be regarded as the equation satisfied by radial solutions of a quasilinear elliptic equation with a gradient term

$$\Delta(|u|^{m-1}u) + \beta x \cdot \nabla u + \alpha u + |u|^{p-1}u = 0 \text{ in } \mathbb{R}^N.$$

This equation was studied by Hirose [11], Chipot et al. [4] and Serrin et al. [18].

Haraxu and Weissler [10] studied the equation

$$\Delta u + \frac{1}{2}x \cdot \nabla u + \frac{1}{p-1}u + |u|^{p-1}u = 0 \text{ in } \mathbb{R}^N, \quad (1.2)$$

where $m = 1$ for the self-similar solutions of the semilinear heat equation

$$\partial_t w = \Delta w + w^p. \quad (1.3)$$

Equation (1.3) has a special scaling invariance in the sense that w is a solution if and only if w_λ , defined by

$$w_\lambda(x, t) = \lambda^{\frac{2}{p-1}} w(\lambda x, \lambda^2 t),$$

is a solution for some (equivalently, all) $\lambda > 0$.

Definition 1. A solution w is said to self-similar if $w_\lambda = w$ for all $\lambda > 0$.

Remark 1. Notice that w is a self-similar solution to (1.3) iff $w(x, t) = t^{-\frac{1}{p-1}} u(\frac{x}{\sqrt{t}})$, where u satisfies (1.2).

Also, Kavian et al. [13] study the self-similar solutions to the nonlinear Schrödinger equation

$$iu_t + \Delta u + \varepsilon |u|^\alpha u = 0, \quad (1.4)$$

where $u = u(t, x)$ is a complex-valued function of $t \in \mathbb{R}$ (or a subset of $\mathbb{R}$) and $x \in \mathbb{R}^N$, α is a positive real number, and $\varepsilon := \pm 1$. Self-similar solution of (1.4) have played an important role in the study of the blow-up behavior of nonglobal solutions to (1.4) with $\varepsilon = 1$.

Also, the nonlinear Schrödinger equation (1.4) with the critical power $\alpha = \frac{4}{N}$, that is,

$$iu_t + \Delta u + \varepsilon |u|^{\frac{4}{N}} u = 0, \quad (1.5)$$

admits a special conservation law, called the pseudo-conformal conservation law, and is invariant under a corresponding transformation.

Finally, in [21], the existence of rapidly decaying radial solution of equation (1.2) when $(N-2)p \leq N$ is proved. Also, Fukuizumi et al. [9] gave a sufficient condition

that non-radial H^1 -solutions to the Haraux-Weissler equation should belong to the weighted Sobolev space $H_{\rho}^1(\mathbb{R}^N)$, where ρ is the weight function $\exp(\frac{|x|^2}{4})$.

The main statement of this paper is to prove the existence of positive singular solutions of Haraux-Weissler equation

$$\begin{cases} \Delta u + \frac{2-N}{2}x \cdot \nabla u = 0 & \text{in } \Omega, \\ x \cdot \nabla u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.6)$$

where Ω is a unite ball in $\mathbb{R}^N$ which is strictly starshaped with respect to zero, $N \geq 3$

2. NONEXISTENCE RESULT

Due to prove the nonexistence result, we need to recall some definitions and facts ([3, 8]) as follows.

Definition 2. We say that an open subset Ω of $\mathbb{R}^N$ is starshaped with respect to a point $y \in \Omega$ if for every $x \in \overline{\Omega}$, the segment joining y to x , namely the set

$$\{\lambda x + (1 - \lambda)y : \lambda \in [0, 1]\}$$

is entirely contained in $\overline{\Omega}$.

Definition 3. We say that an open subset Ω of $\mathbb{R}^N$ is strictly starshaped with respect to a point $y \in \Omega$ if for every $x \in \Omega$, the segment joining y to x , namely the set

$$\{\lambda x + (1 - \lambda)y : \lambda \in [0, 1]\}$$

is entirely contained in $\overline{\Omega}$.

Remark 2. If $x \in \overline{\Omega}$ then $\lambda x \in \Omega$ for $\lambda \in [0, 1)$, when $0 \in \Omega$.

Lemma 1. ([3, 8]) Let $\Omega \subset \mathbb{R}^N$ be smooth and strictly starshaped with respect to 0 and $\nu(x)$ denote the outward normal to $\partial\Omega$ at x . Then $\nu(x) \cdot x > 0$ for $x \in \partial\Omega$.

Now, we can present the main result by the the following theorem.

Theorem 1. Let $\Omega \subset \mathbb{R}^N$ be unit ball which is smooth and strictly starshaped with respect to 0 and $N \geq 3$. Then the problem (1.6) has no solutions in $H_0^1(\Omega)$.

Proof. We prove the theorem by contradiction. Assume the problem (1.6) has a solution u in $H_0^1(\Omega)$. This mean that $x \cdot \nabla u > 0$ in Ω . Now we multiply the equation in (1.6) by $x \cdot \nabla u$.

$$-\int_{\Omega} \Delta u x \cdot \nabla u dx = \frac{2-N}{2} \int_{\Omega} |x|^2 |\nabla u|^2 dx. \quad (2.1)$$

Notice that the left hand side by the Green's formula is

$$-\int_{\Omega} \Delta u x \cdot \nabla u dx = \int_{\Omega} \nabla u \cdot (x \cdot \nabla u) dx - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \nabla u \cdot x d\sigma. \quad (2.2)$$

Since

$$\begin{aligned}\frac{\partial}{\partial x_j}(\nabla u \cdot x) &= \frac{\partial}{\partial x_j} \left(\sum_{i=1}^N \frac{\partial u}{\partial x_i} x_i \right) = \sum_{i=1}^N \left(\frac{\partial^2 u}{\partial x_i \partial x_j} x_i + \frac{\partial u}{\partial x_i} \delta_{ij} \right) \\ &= \sum_{i=1}^N \frac{\partial^2 u}{\partial x_i \partial x_j} x_i + \frac{\partial u}{\partial x_j}.\end{aligned}$$

This means that

$$\begin{aligned}\nabla u \cdot \nabla(x \cdot \nabla u) &= \frac{1}{2} \nabla(|\nabla u|^2) \cdot x + |\nabla u|^2 \\ &= \frac{1}{4} \nabla(|\nabla u|^2) \cdot \nabla(|x|^2) + |\nabla u|^2.\end{aligned}$$

By the Green's formula we get

$$\begin{aligned}\int_{\Omega} \nabla u \cdot \nabla(x \cdot \nabla u) dx &= \int_{\Omega} \frac{1}{4} \nabla(|\nabla u|^2) \cdot \nabla(|x|^2) dx + \int_{\Omega} |\nabla u|^2 dx \\ &= \frac{1}{4} \int_{\Omega} |\nabla u|^2 \frac{\partial}{\partial \mathbf{v}}(|x|^2) d\sigma - \frac{N}{2} \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} |\nabla u|^2 dx \\ &= \frac{2-N}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 \mathbf{v}(x) \cdot x d\sigma.\end{aligned}$$

Thus by (2.2) we get

$$\begin{aligned}- \int_{\Omega} \Delta u x \cdot \nabla u dx &= \frac{2-N}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 \mathbf{v}(x) \cdot x d\sigma - \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{v}} \nabla u(x) \cdot x d\sigma \quad (2.3) \\ &= \frac{2-N}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{2} \int_{\partial\Omega} \left(\frac{\partial u}{\partial \mathbf{v}} \right)^2 \mathbf{v}(x) \cdot x d\sigma.\end{aligned}$$

Now considering (2.3) and (2.1) we have

$$\frac{2-N}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{2} \int_{\partial\Omega} \left(\frac{\partial u}{\partial \mathbf{v}} \right)^2 \mathbf{v}(x) \cdot x d\sigma = \frac{2-N}{2} \int_{\Omega} |x|^2 |\nabla u|^2 dx$$

or

$$\frac{2-N}{2} \int_{\Omega} (1 - |x|^2) |\nabla u|^2 dx = \frac{1}{2} \int_{\partial\Omega} \left(\frac{\partial u}{\partial \mathbf{v}} \right)^2 \mathbf{v}(x) \cdot x d\sigma.$$

Since $x \in \Omega$ which is the unit ball, thus the left hand side of the above equation is negative for $N \geq 3$ and the right hand side is positive (Since Ω is strictly starshaped

with respect to 0, we have $v(x) \cdot x > 0$ everywhere on $\partial\Omega$. This means that

$$\frac{1}{2} \int_{\partial\Omega} \left(\frac{\partial u}{\partial v} \right)^2 v(x) \cdot x d\sigma = 0$$

which implies that

$$\frac{2-N}{2} \int_{\Omega} x \cdot \nabla u dx = - \int_{\Omega} \Delta u dx = - \int_{\partial\Omega} \frac{\partial u}{\partial v} d\sigma = 0.$$

Since $x \cdot \nabla u$ is positive in Ω , this is impossible. $\square$

Remark 3. If Ω is a unit ball in $\mathbb{R}^N$ which is strictly convex, Theorem 1 remains true. Because every strictly convex set is strictly starshaped set.

3. SOME QUESTIONS

Here, we present some questions as follow.

Question 1. Is it possible to prove that the equation

$$\begin{cases} \Delta u + \frac{1}{2}x \cdot \nabla u + \frac{1}{p-1}u + u^p = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.1)$$

has no solution, where $\Omega \subset \mathbb{R}^N$ is a strictly starshaped with respect to 0?

Question 2. Is it possible to prove that the equation

$$\begin{cases} \Delta u + a(x) \cdot \nabla u + V(x)u + u^p = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.2)$$

has no solution, where $\Omega \subset \mathbb{R}^N$ is a strictly starshaped with respect to 0, $a(x)$ is a smooth vector field and $V(x)$ is a smooth potential?

Question 3. Is it possible to prove that the equation

$$\begin{cases} \Delta u + a(x) \cdot \nabla u + V(x)u + u^{p(x)} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.3)$$

has no solution, where $\Omega \subset \mathbb{R}^N$ is a strictly starshaped with respect to 0, $a(x)$ is a smooth vector field, $V(x)$ is a smooth potential and $p(x)$ is a variable exponent?

Question 4. Is it possible to prove that the equation

$$\begin{cases} \Delta u + \frac{2-N}{2}x \cdot \nabla v = 0 & \text{in } \Omega, \\ \Delta v + \frac{2-N}{2}x \cdot \nabla u = 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.4)$$

has no solution, where $\Omega \subset \mathbb{R}^N$ is a strictly starshaped with respect to 0?

The Heisenberg group $\mathbb{H}^n = (\mathbb{R}^{2n+1}, \circ)$ is the space $\mathbb{R}^{2n+1}$ with the noncommutative law of product

$$(x, y, z) \circ (x', y', z') = (x + x', y + y', z + z' + 2(\langle y, x' \rangle - \langle x, y' \rangle)),$$

where $x_1, x_2, y_1, y_2 \in \mathbb{R}^n$, $z_1, z_2 \in \mathbb{R}$ and $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^n$. This operation endows $\mathbb{H}^n$ with the structure of a Lie group. The Lie algebra of $\mathbb{H}^n$ is generated by the left-invariant vector fields

$$Z = \frac{\partial}{\partial z}, \quad X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial z}, \quad Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial z}, \quad i = 1, 2, 3, \dots, n.$$

These generators satisfy the noncommutative formula

$$[X_i, Y_j] = -4\delta_{ij}Z, \quad [X_i, X_j] = [Y_i, Y_j] = [X_i, Z] = [Y_i, Z] = 0.$$

The Heisenberg gradient and the Kohn-Laplacian (the Heisenberg Laplacian) operator on $\mathbb{H}^n$ are given by

$$\nabla_{\mathbb{H}^n} = (X_1, X_2, \dots, X_n, Y_1, Y_2, \dots, Y_n),$$

and

$$\Delta_{\mathbb{H}^n} = \sum_{i=1}^n X_i^2 + Y_i^2,$$

respectively (see [7] for more details).

Question 5. The quasilinear PME with a source term in the Heisenberg group $\mathbb{H}^n$ is

$$u_t - \Delta_{\mathbb{H}^n}(|u|^{m-1}u) = |u|^{p-1}u \quad \text{in } \mathbb{R}^{2n+1} \times (0, T), \quad (3.5)$$

where the parameters are taken as $m > 0$ and $p > 1$.

Does the equation (3.5) has a self-similar solution in the Heisenberg group $\mathbb{H}^n$?

In the case $m = 1$, i.e. the equation

$$u_t - \Delta_{\mathbb{H}^n}u = |u|^{p-1}u \quad \text{in } \mathbb{R}^{2n+1} \times (0, T), \quad (3.6)$$

may be regarded as a Haraxu-Weissler equation in the Heisenberg group setting. The existence of self-similar solutions of the parabolic equation (3.6) can be studied.

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ON JORDAN HOMO-DERIVATION OF TRIANGULAR ALGEBRAS

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Abstract. We define an $\mathbb{R}$ -linear map ϕ from $\mathbb{A}$ to an $\mathbb{A}$ -bimodule $\mathbb{M}$ is said to be Jordan homo-derivation if $\phi(x^2) = \phi(x)x + x\phi(x) + \phi(x)^2$ for each $x \in \mathbb{A}$. In this article, we proved that every Jordan homo-derivation to be homo-derivation on triangular algebras.

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1. INTRODUCTION

Let $\mathbb{R}$ be a commutative ring, and $\mathbb{A}$ be an algebra over $\mathbb{R}$. An $\mathbb{R}$ -linear map ψ from $\mathbb{A}$ into an $\mathbb{A}$ -bimodule $\mathbb{M}$ is said to be a derivation if $\psi(xy) = \psi(x)y + x\psi(y)$ hold $\forall x, y \in \mathbb{A}$. It is called a Jordan derivation if $\psi(a^2) = \psi(a)a + a\psi(a)$ for all $a \in \mathbb{A}$. Any derivation is clearly a Jordan derivation. In general, the converse is not true (see [1]). Any Jordan derivation on a 2-torsion free prime ring is a derivation, according to a well-known Herstein [8] result. [4] contains a brief proof of this finding. Bresar [5], demonstrated in 1988 that Herstein's result holds in 2-torsion free semiprime rings. Jordan derivations of unital algebras with idempotents were investigated by Benkovic and Sirovnik [3] in 2012.

Let an $\mathbb{R}$ -linear map ϕ from $\mathbb{A}$ to an $\mathbb{A}$ -bimodule $\mathbb{M}$ is said to be a homo-derivation if $\phi(xy) = \phi(x)y + x\phi(y) + \phi(x)\phi(y)$ for any $x, y \in \mathbb{A}$, (see [7]). We call ϕ a Jordan homo-derivation if it is $\mathbb{R}$ -linear and $\phi(x^2) = \phi(x)x + x\phi(x) + \phi(x)^2$ for each $x \in \mathbb{A}$, (see [9]). Obviously, every homo-derivation is a Jordan homo-derivation. But the converse, in general, is not true, for example see Example 1. If $\mathfrak{S} \subseteq \mathbb{R}$, then a mapping $\phi: \mathbb{R} \rightarrow \mathbb{R}$ preserves $\mathfrak{S}$ if $\phi(\mathfrak{S}) \subseteq \mathfrak{S}$. A mapping $\phi: \mathbb{R} \rightarrow \mathbb{R}$ is said to be zero-power valued on $\mathfrak{S}$ if ϕ preserves $\mathfrak{S}$ and if for each $a \in \mathfrak{S}$, there exists a positive integer $n(a) > 1$ such that $\phi^{n(a)} = 0$, [7].

The $\mathbb{R}$ -algebra $\mathfrak{T} = \text{Tri}(\mathbb{A}, \mathbb{M}, \mathbb{B})$ is an algebra of the form

$$\text{Tri}(\mathbb{A}, \mathbb{M}, \mathbb{B}) = \left\{ \begin{pmatrix} a & m \\ & b \end{pmatrix} : a \in \mathbb{A}, b \in \mathbb{B}, m \in \mathbb{M} \right\}$$

under the usual matrix operations will be called a triangular algebra [2], where $\mathbb{A}$ and $\mathbb{B}$ are two algebras over the commutative ring $\mathbb{R}$, and $\mathbb{M}$ is a $(\mathbb{A}, \mathbb{B})$ -bimodule that is faithful as both a left $\mathbb{A}$ -module and a right $\mathbb{B}$ -module. Recently, some problems on triangular algebra have been determined by many authors. Any Jordan derivation from the algebra of all upper triangular matrices into its arbitrary bimodule, as demonstrated by Benkovic [1], is the sum of a derivation and an antiderivation on triangular matrices over commutative rings. Jordan derivations of triangular algebras are derivations, according to Zhang and Yu (2006) ([10]). It's normal to ask if a Jordan homo-derivation from a triangular algebra to itself is really a derivation. The primary goal of this article is to investigate this problem.

2. JORDAN HOMO-DERIVATIONS ON TRIANGULAR MATRIX RINGS

For the sake of notational clarity, let $\mathfrak{T} = \text{Tri}(\mathbb{A}, \mathbb{M}, \mathbb{B})$ be a triangular algebra. Set

$$\mathfrak{T}_{11} = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in \mathbb{A} \right\}, \quad \mathfrak{T}_{12} = \left\{ \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} : m \in \mathbb{M} \right\}$$

and

$$\mathfrak{T}_{22} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} : b \in \mathbb{B} \right\}.$$

Then we can write $\mathfrak{T} = \mathfrak{T}_{11} \oplus \mathfrak{T}_{12} \oplus \mathfrak{T}_{22}$. Henceforth the element A_{ij} belongs $\mathfrak{T}_{ij}$ and the corresponding elements are in $\mathbb{A}, \mathbb{B}$ or $\mathbb{M}$. By a direct calculation $A_{ij}A_{kl} = 0$ if $j \neq k$ where $i, j, k \in \{1, 2\}$.

Throughout our discussion, $\mathbb{A}$ and $\mathbb{B}$ will be two algebras over 2-torsion free ring $\mathbb{R}$ with the property:

(♣) Let $r \in \mathbb{A}$ (resp $\mathbb{B}$). If $prq + qrp = 0$ holds $\forall p, q \in \mathbb{A}$ (rep. $\mathbb{A}$), then $r = 0$.

Moreover, ϕ will be a Jordan homo-derivation from triangular algebra $\mathfrak{T}$ into itself and zero-power valued on $\mathfrak{T}$, and preserves on $\mathfrak{T}_{11}$ and $\mathfrak{T}_{22}$, where $\mathbb{M}$ is a faithful $(\mathbb{A}, \mathbb{B})$ -bimodule.

We begin with Cheng and Jing's [6, Lemma 2.1] result, which is essential for developing the proof of our main theorem.

Lemma 1. *Let r be in $\mathbb{A}$ (resp. $\mathbb{B}$).*

- (i) *If $prp = 0$ for any $p \in \mathbb{A}$ (resp. $\mathbb{B}$), then $r = 0$;*
- (ii) *If $rp + pr = 0$ for any $p \in \mathbb{A}$ (resp. $\mathbb{B}$), then $r = 0$.*

The main result of the paper is as follows:

Theorem 1. *Let $\mathbb{A}$ and $\mathbb{B}$ be two algebras over a 2-torsion free commutative ring $\mathbb{R}$ with the property (♣). Let $\mathbb{M}$ be a faithful $(\mathbb{A}, \mathbb{B})$ -bimodule and $\mathfrak{T}$ be the triangular algebra $\text{Tri}(\mathbb{A}, \mathbb{M}, \mathbb{B})$. Let a mapping $\phi: \mathfrak{T} \rightarrow \mathfrak{T}$ be zero-power valued on $\mathfrak{T}$, and preserves on $\mathfrak{T}_{11}$ and $\mathfrak{T}_{22}$. Then every Jordan homo-derivation ϕ on $\mathfrak{T}$ into itself is a homo-derivation.*

In Example 1, we are putting $\mathbb{M} = \mathbb{Z}$ and $\mathbb{A} = \mathbb{B} = \mathbb{K}$ and $\mathbb{R} = \mathbb{Z}$. It is easy to see that $\mathfrak{T}$ does not satisfy the property ($\clubsuit$), and ϕ is a Jordan homo-derivation, but it is not homo-derivation.

Example 1. Consider S be a ring such that $s^2 = 0$ for all $s \in S$, but the product of some elements of S is nonzero. Since $s^2 = 0$, so $(s+t)^2 = 0$ for all $s, t \in S$ this implies that $s \circ t = 0$ for all $s, t \in S$. Suppose $\mathbb{K} = \left\{ \begin{pmatrix} 0 & p & q \\ 0 & 0 & r \\ 0 & 0 & 0 \end{pmatrix} : p, q, r \in S \right\}$. Let $\phi: \mathbb{K} \rightarrow \mathbb{K}$ such that $\phi \begin{pmatrix} 0 & p & q \\ 0 & 0 & r \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & r & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Note that ϕ is a Jordan homo-derivation, but it is not homo-derivation.

We start with the following Lemma due to Herstein [8]

Lemma 2. *Let $\mathbb{A}$ be an algebra over a 2-torsion free ring commutative ring, and Let ϕ be a Jordan homo-derivation from $\mathbb{A}$ into its bimodule. Then*

- (i) $\phi(A \circ B) = \phi(A) \circ B + A \circ \phi(B) + \phi(A) \circ \phi(B)$ for all $A, B \in \mathbb{A}$;
- (ii)

$$\begin{aligned} \phi(ABA) &= \phi(A)BA + A\phi(B)A + AB\phi(A) + \phi(A)\phi(B)A \\ &\quad + A\phi(B)\phi(A) + \phi(A)B\phi(A) + \phi(A)\phi(B)\phi(A) \end{aligned}$$

for all $A, B \in \mathbb{A}$.

Proof.

(i) We compute $(A+B)^2$ in two ways, we get the required result.

(ii) Note that $A \circ (A \circ B) = A^2 \circ B + 2ABA$. We compute $\phi(A \circ (A \circ B))$ and $\phi(A^2 \circ B + 2ABA)$ by using Lemma 2 (i) and definition of Jordan homo-derivation.

$$\begin{aligned} \phi(A^2 \circ B + 2ABA) &= \phi(A^2 \circ B) + 2\phi(ABA) \\ &= \phi(A^2) \circ B + A^2 \circ \phi(B) + \phi(A^2) \circ \phi(B) \\ &\quad + 2\phi(ABA) \\ &= \phi(A)AB + A\phi(A)B + \phi(A)^2B \\ &\quad + B\phi(A)A + BA\phi(A) + B\phi(A)^2 \\ &\quad + A^2\phi(B) + \phi(B)A^2 + \phi(A)A\phi(B) \\ &\quad + A\phi(A)\phi(B) + \phi(A)^2\phi(B) + \phi(B)\phi(A)A \\ &\quad + \phi(B)A\phi(A) + \phi(B)\phi(A)^2 + 2\phi(ABA). \end{aligned} \tag{2.1}$$

$$\phi(A \circ (A \circ B)) = \phi(A) \circ (A \circ B) + A \circ \phi(A \circ B)$$

$$\begin{aligned}
& + \phi(A) \circ \phi(A \circ B) \\
= & \phi(A)AB + \phi(A)BA + AB\phi(A) \\
& + BA\phi(A) + A\phi(A)B + AB\phi(A) + A^2\phi(B) \\
& + A\phi(B)A + A\phi(A)\phi(B) + A\phi(B)\phi(A) \\
& + \phi(A)BA + B\phi(A)A + A\phi(B)A \\
& + \phi(B)A^2 + \phi(A)\phi(B)A + \phi(B)\phi(A)A \\
& + \phi(A)^2B + \phi(A)B\phi(A) + \phi(A)A\phi(B) \\
& + \phi(A)\phi(B)A + \phi(A)^2\phi(B) + \phi(A)\phi(B)\phi(A) \\
& + \phi(A)B\phi(A) + B\phi(A)^2 + A\phi(B)\phi(A) \\
& + \phi(B)A\phi(A) + \phi(A)\phi(B)\phi(A) + \phi(B)\phi(A)^2.
\end{aligned} \tag{2.2}$$

Since $\mathbb{R}$ is 2-torsion free and compare (2.1) and (2.2), we get (ii). $\square$

We note that condition (i) is analogous to the Jordan homo-derivation condition.

Lemma 3. $\phi(0) = 0$.

Proof. Putting $B = 0$ and $A = A_{11}$ in Lemma 2 (i), we have $\phi(0) = A_{11} \circ \phi(0) + \phi(A_{11}) \circ \phi(0)$. Since there exists a positive integer $n(A_{11}) > 1$ such that $\phi^n(A_{11}) = 0$, taking A_{11} by $\phi^{n-1}(A_{11})$ in the last relation, we get $\phi(0) = \phi^{n-1}(A_{11}) \circ \phi(0)$. Replacing A_{11} by $\phi(A_{11})$ in the last relation, we obtain $\phi(0) = 0$. $\square$

Lemma 4. For arbitrary $A_{11} \in \mathfrak{T}_{11}$ and $A_{22} \in \mathfrak{T}_{22}$, we have

- (i) $\phi(A_{11}A_{22}) = \phi(A_{11})A_{22} + A_{11}\phi(A_{22}) + \phi(A_{11})\phi(A_{22})$;
- (ii) $\phi(A_{22}A_{11}) = \phi(A_{22})A_{11} + A_{22}\phi(A_{11}) + \phi(A_{22})\phi(A_{11})$.

Proof.

- (i) From conditions of Theorem 1, we have $\phi(A_{22})_{11} = 0$ and $\phi(A_{11})_{22} = 0$. Now, note that

$$\begin{aligned}
& \phi(A_{22})A_{11} + A_{22}\phi(A_{11}) + \phi(A_{22})\phi(A_{11}) \\
= & [\phi(A_{22})_{11} + \phi(A_{22})_{12} + \phi(A_{22})_{22}]A_{11} \\
& + A_{22}[\phi(A_{11})_{11} + \phi(A_{11})_{12} + \phi(A_{11})_{22}] \\
& + [\phi(A_{22})_{11} + \phi(A_{22})_{12} + \phi(A_{22})_{22}][\phi(A_{11})_{11} + \phi(A_{11})_{12} + \phi(A_{11})_{22}] = 0.
\end{aligned} \tag{2.3}$$

We compute

$$\begin{aligned}
\phi(A_{11}A_{22}) & = \phi(A_{11}A_{22} + A_{22}A_{11}) \\
& = \phi(A_{11})A_{22} + A_{11}\phi(A_{22}) + \phi(A_{11})\phi(A_{22}) \\
& \quad + \phi(A_{22})A_{11} + A_{22}\phi(A_{11}) + \phi(A_{22})\phi(A_{11}) \\
& = \phi(A_{11})A_{22} + A_{11}\phi(A_{22}) + \phi(A_{11})\phi(A_{22}), \text{ by using (2.3).}
\end{aligned}$$

(ii) Also, by using (2.3), we get (ii). □

Lemma 5. *For any $A_{12} \in \mathfrak{T}_{12}$, the following are true.*

- (i) $\phi(A_{12})_{11} = 0$;
- (ii) $\phi(A_{12})_{22} = 0$.

Proof. (i) We compute

$$\begin{aligned} 0 &= \phi(\phi^{n-1}(A_{11})A_{12}A_{11}) \\ &= \phi^n(A_{11})A_{12}A_{11} + \phi^{n-1}(A_{11})\phi(A_{12})A_{11} + \phi^{n-1}(A_{11})A_{12}\phi(A_{11}) \\ &\quad + \phi^n(A_{11})\phi(A_{12})A_{11} + \phi^{n-1}(A_{11})\phi(A_{12})\phi(A_{11}) + \phi^n(A_{11})A_{12}\phi(A_{11}) \\ &\quad + \phi^n(A_{11})\phi(A_{12})\phi(A_{11}). \end{aligned}$$

That is

$$(\phi^{n-1}(A_{11}) + \phi^n(A_{11}))[\phi(A_{12})A_{11} + A_{12}\phi(A_{11}) + \phi(A_{12})\phi(A_{11})] = 0. \quad (2.4)$$

Since $\phi^n(A_{11}) = 0$, we have

$$\phi^{n-1}(A_{11})[\phi(A_{12})A_{11} + A_{12}\phi(A_{11}) + \phi(A_{12})\phi(A_{11})] = 0.$$

Similarly, we can compute $\phi(\phi^{n-2}(A_{11})A_{12}A_{11}) = 0$, we get

$$(\phi^{n-2}(A_{11}) + \phi^{n-1}(A_{11}))[\phi(A_{12})A_{11} + A_{12}\phi(A_{11}) + \phi(A_{12})\phi(A_{11})] = 0.$$

By using (2.4) in last relation, we have

$$\phi^{n-2}(A_{11})[\phi(A_{12})A_{11} + A_{12}\phi(A_{11}) + \phi(A_{12})\phi(A_{11})] = 0.$$

Also, we can compute $\phi(\phi^k(A_{11})A_{12}A_{11}) = 0$ for $k = n-3, n-4, \dots, 1$ and $\phi(A_{11}A_{12}A_{11}) = 0$ and so we get

$$A_{11}[\phi(A_{12})A_{11} + A_{12}\phi(A_{11}) + \phi(A_{12})\phi(A_{11})] = 0.$$

Hence

$$A_{11}[\phi(A_{12})A_{11} + A_{12}\phi(A_{11}) + \phi(A_{12})\phi(A_{11})]A_{11} = 0,$$

thus

$$A_{11}[\phi(A_{12})A_{11} + A_{12}\phi(A_{11}) + \phi(A_{12})\phi(A_{11})]_{11}A_{11} = 0$$

and by using Lemma 1 (i)

$$[\phi(A_{12})A_{11} + A_{12}\phi(A_{11}) + \phi(A_{12})\phi(A_{11})]_{11} = 0. \quad (2.5)$$

Also, we have

$$A_{11}[\phi(A_{12})A_{11} + A_{12}\phi(A_{11}) + \phi(A_{12})\phi(A_{11})]_{12} = 0$$

and since $\mathbb{M}$ is a faithful left $\mathbb{A}$ -module, we have

$$[\phi(A_{12})A_{11} + A_{12}\phi(A_{11}) + \phi(A_{12})\phi(A_{11})]_{12} = 0. \quad (2.6)$$

Similarly, we can compute $\phi(\phi^k(A_{22})A_{12}A_{11}) = 0$ for $k = n-3, n-4, \dots, 1$ and $\phi(A_{22}A_{12}A_{11}) = 0$ and so we get

$$A_{22}[\phi(A_{12})A_{11} + A_{12}\phi(A_{11}) + \phi(A_{12})\phi(A_{11})] = 0.$$

Hence

$$A_{22}[\phi(A_{12})A_{11} + A_{12}\phi(A_{11}) + \phi(A_{12})\phi(A_{11})]A_{22} = 0$$

thus

$$A_{22}[\phi(A_{12})A_{11} + A_{12}\phi(A_{11}) + \phi(A_{12})\phi(A_{11})]_{22}A_{22} = 0$$

and by using Lemma 1 (i), we get

$$[\phi(A_{12})A_{11} + A_{12}\phi(A_{11}) + \phi(A_{12})\phi(A_{11})]_{22} = 0. \quad (2.7)$$

From (2.5), (2.6) and (2.7), we have

$$[\phi(A_{12})A_{11} + A_{12}\phi(A_{11}) + \phi(A_{12})\phi(A_{11})] = 0.$$

That is

$$\phi(A_{12})_{11}(A_{11} + \phi(A_{11})) = 0. \quad (2.8)$$

Replacing A_{11} by $\phi^{n-1}(A_{11})$, in (2.8), we get $\phi(A_{12})_{11}\phi^{n-1}(A_{11}) = 0$. Now, replacing A_{11} by $\phi^{n-2}(A_{11})$, in (2.8) and using last relation, we have

$$\phi(A_{12})_{11}\phi^{n-2}(A_{11}) = 0,$$

by continuing, we obtain

$$\phi(A_{12})_{11}A_{11} = 0 \quad \text{that is} \quad A_{11}\phi(A_{12})_{11}A_{11} = 0$$

and by Lemma 1 (i), we get $\phi(A_{12})_{11} = 0$. Similarly we can get (ii) by considering

$$\phi(A_{22}A_{12}\phi^k(A_{22})) = 0$$

for $k = n-1, n-2, \dots, 1$ and $\phi(A_{22}A_{12}A_{22}) = 0$. □

Lemma 6. For any $A_{11} \in \mathfrak{T}_{11}$ and $A_{12} \in \mathfrak{T}_{12}$,

- (i) $\phi(A_{12}A_{11}) = \phi(A_{12})A_{11} + A_{12}\phi(A_{11}) + \phi(A_{12})\phi(A_{11})$;
- (ii) $\phi(A_{11}A_{12}) = \phi(A_{11})A_{12} + A_{11}\phi(A_{12}) + \phi(A_{11})\phi(A_{12})$.

Proof.

(i) We compute

$$\begin{aligned} & \phi(A_{12})A_{11} + A_{12}\phi(A_{11}) + \phi(A_{12})\phi(A_{11}) \\ &= \phi(A_{12})_{11}A_{11} + A_{12}\phi(A_{11})_{22} \\ & \quad + [\phi(A_{12})_{11} + \phi(A_{12})_{12} + \phi(A_{12})_{22}][\phi(A_{11})_{11} + \phi(A_{11})_{12} + \phi(A_{11})_{22}] \\ &= 0 = \phi(A_{12}A_{11}). \end{aligned} \quad (2.9)$$

(ii) We compute

$$\begin{aligned}
 \phi(A_{11}A_{12}) &= \phi(A_{11}A_{12} + A_{12}A_{11}) \\
 &= \phi(A_{11})A_{12} + A_{11}\phi(A_{12}) + \phi(A_{11})\phi(A_{12}) \\
 &\quad + \phi(A_{12})A_{11} + A_{12}\phi(A_{11}) + \phi(A_{12})\phi(A_{11}) \\
 &= \phi(A_{11})A_{12} + A_{11}\phi(A_{12}) + \phi(A_{11})\phi(A_{12}) \text{ by using (2.9).}
 \end{aligned}$$

□

Using similar techniques as used in the proof of Lemma 6, we get the following.

Lemma 7. For any $A_{12} \in \mathfrak{T}_{12}$ and $A_{22} \in \mathfrak{T}_{22}$,

- (i) $\phi(A_{12}A_{22}) = \phi(A_{12})A_{22} + A_{12}\phi(A_{22}) + \phi(A_{12})\phi(A_{22})$;
- (ii) $\phi(A_{22}A_{12}) = \phi(A_{22})A_{12} + A_{22}\phi(A_{12}) + \phi(A_{22})\phi(A_{12})$.

Lemma 8. ϕ is a homo-derivation on $\mathfrak{T}_{12}$.

Proof. For any $A_{12}, B_{12} \in \mathfrak{T}_{12}$, by Lemma 5

$$\begin{aligned}
 &\phi(A_{12})B_{12} + A_{12}\phi(B_{12}) + \phi(A_{12})\phi(B_{12}) \\
 &= \phi(A_{12})_{11}B_{12} + A_{12}\phi(B_{12})_{22} \\
 &\quad + [\phi(A_{12})_{11} + \phi(A_{12})_{12} + \phi(A_{12})_{22}][\phi(B_{12})_{11} + \phi(B_{12})_{12} + \phi(B_{12})_{22}] \\
 &= 0 = \phi(A_{12}B_{12}).
 \end{aligned}$$

□

Lemma 9. ϕ is a homo-derivation on $\mathfrak{T}_{11}$.

Proof. Let $A_{11}, B_{11} \in \mathfrak{T}_{11}$ and $C_{12} \in \mathfrak{T}_{12}$ be arbitrary. On one side, we have

$$\begin{aligned}
 \phi(A_{11}B_{11}C_{12}) &= \phi(A_{11})B_{11}C_{12} + A_{11}\phi(B_{11})C_{12} + A_{11}B_{11}\phi(C_{12}) \\
 &\quad + \phi(A_{11})\phi(B_{11})C_{12} + A_{11}\phi(B_{11})\phi(C_{12}) + \phi(A_{11})B_{11}\phi(C_{12}) \\
 &\quad + \phi(A_{11})\phi(B_{11})\phi(C_{12}).
 \end{aligned}$$

On the other side, we get

$$\phi(A_{11}B_{11}C_{12}) = \phi(A_{11}B_{11})C_{12} + A_{11}B_{11}\phi(C_{12}) + \phi(A_{11}B_{11})\phi(C_{12}).$$

Then we can see that

$$[\phi(A_{11}B_{11}) - \phi(A_{11})B_{11} - A_{11}\phi(B_{11}) - \phi(A_{11})\phi(B_{11})](C_{12} + \phi(C_{12})) = 0. \quad (2.10)$$

Replacing C_{12} by $\phi^{n-1}(C_{12})$ in (2.10) and since $\phi^n(C_{12}) = 0$, we have

$$[\phi(A_{11}B_{11}) - \phi(A_{11})B_{11} - A_{11}\phi(B_{11}) - \phi(A_{11})\phi(B_{11})]\phi^{n-1}(C_{12}) = 0. \quad (2.11)$$

Replacing C_{12} by $\phi^{n-2}(C_{12})$ in (2.10) and using (2.11), we get

$$[\phi(A_{11}B_{11}) - \phi(A_{11})B_{11} - A_{11}\phi(B_{11}) - \phi(A_{11})\phi(B_{11})]\phi^{n-2}(C_{12}) = 0.$$

By continuing, we obtain

$$[\phi(A_{11}B_{11}) - \phi(A_{11})B_{11} - A_{11}\phi(B_{11}) - \phi(A_{11})\phi(B_{11})]C_{12} = 0.$$

This yields that

$$\phi(A_{11}B_{11}) = \phi(A_{11})B_{11} + A_{11}\phi(B_{11}) + \phi(A_{11})\phi(B_{11})$$

since $\mathbb{M}$ is a faithful left $\mathbb{A}$ -module. $\square$

Using the similar arguments as used in the Lemma 9, one can get the following.

Lemma 10. ϕ is a homo-derivation on $\mathfrak{T}_{22}$.

Proof of Theorem 1. The proof of is follows from Lemmas 4-10. $\square$

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A NEW APPROACH TO HOMOTOPY THEORY VIA BEST PROXIMITY POINT

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Abstract. In this paper, we extend the result of Romaguera [21] with the aid of best proximity point theory on partial metric spaces by considering the approach of Haghi et al. [9], and so celebrated Boyd-Wong fixed point theorem [7]. We first introduce two concepts called generalized proximal *BW*-contraction and generalized best *BW*-contraction. Then, we obtain some best proximity point theorems for such mappings. To illustrate the effectiveness of our results, we provide some nontrivial and interesting examples. Finally, unlike homotopy applications existing in the literature, we present for the first time an application of the best proximity result to the homotopy theory.

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1. INTRODUCTION AND PRELIMINARIES

It is well known that $\varkappa = \eta$ if and only if $d(\varkappa, \eta) = 0$ for each points $\varkappa, \eta$ in a metric space (Λ, d) . However, motivated by the experience of computer science, there was a tendency to relax this equivalence. In this context, Matthews [14] introduced a new concept so called partial metric by weakening the mentioned condition as follows:

Definition 1 ([14]). Let Λ be a nonempty set and $\gamma: \Lambda \times \Lambda \rightarrow [0, \infty)$ be a function. Then, γ is said to be a partial metric on Λ if the following conditions hold:

- p1) $\varkappa = \eta$ iff $\gamma(\varkappa, \varkappa) = \gamma(\varkappa, \eta) = \gamma(\eta, \eta)$,
- p2) $\gamma(\varkappa, \varkappa) \leq \gamma(\varkappa, \eta)$,
- p3) $\gamma(\varkappa, \eta) = \gamma(\eta, \varkappa)$,
- p4) $\gamma(\varkappa, \upsilon) \leq \gamma(\varkappa, \eta) + \gamma(\eta, \upsilon) - \gamma(\eta, \eta)$

for all $\varkappa, \eta, \upsilon \in \Lambda$. In this case, the pair (Λ, γ) is called partial metric space.

It is clear that every metric space is a partial metric space but conversely may not be true. Indeed, $(\mathbb{R}^+ = [0, \infty), \gamma)$ is a nonmetric partial metric space, where $\gamma(\varkappa, \eta) = \max\{\varkappa, \eta\}$. For the sake of completeness, we recall the necessary and useful properties of partial metric spaces.

Let (Λ, γ) be a partial metric space. Then, γ generates a T_0 -topology τ_γ on Λ which has as a base the family of open balls $\{B(\varkappa, \varepsilon) : \varkappa \in \Lambda \text{ and } \varepsilon > 0\}$ where

$$B(\varkappa, \varepsilon) = \{\eta \in \Lambda : \gamma(\varkappa, \eta) < \gamma(\varkappa, \varkappa) + \varepsilon\}.$$

Let $\{\varkappa_n\}$ be a sequence in Λ and $\varkappa \in \Lambda$. It is easy to see that the sequence $\{\varkappa_n\}$ converges to $\varkappa$ with respect to τ_γ if and only if

$$\lim_{n \rightarrow \infty} \gamma(\varkappa_n, \varkappa) = \gamma(\varkappa, \varkappa).$$

If $\lim_{n, m \rightarrow \infty} \gamma(\varkappa_n, \varkappa_m)$ exists and is finite, then $\{\varkappa_n\}$ is said to be a Cauchy sequence. If every Cauchy sequence $\{\varkappa_n\}$ converges to a point $\varkappa$ in Λ such that

$$\lim_{n, m \rightarrow \infty} \gamma(\varkappa_n, \varkappa_m) = \gamma(\varkappa, \varkappa)$$

then, (Λ, γ) is said to be a complete partial metric space.

If γ is a partial metric on Λ , then the mapping $d_\gamma : \Lambda \times \Lambda \rightarrow [0, \infty)$ defined by

$$d_\gamma(\varkappa, \eta) = 2\gamma(\varkappa, \eta) - \gamma(\varkappa, \varkappa) - \gamma(\eta, \eta)$$

for all $\varkappa, \eta \in \Lambda$, is a metric on Λ .

The following lemma shows the relation between a partial metric γ and the induced ordinary metric d_γ .

Lemma 1. *Let (Λ, γ) be a partial metric space, $\{\varkappa_n\}$ be a sequence in Λ and $\varkappa \in \Lambda$. Then, the following ones hold:*

- (i) $\{\varkappa_n\}$ is a Cauchy sequence (Λ, γ) iff it is a Cauchy sequence (Λ, d_γ) .
- (ii) (Λ, γ) is a complete partial metric space iff (Λ, d_γ) is a complete metric space.

Further,

$$\lim_{n \rightarrow \infty} d_\gamma(\varkappa_n, \varkappa) = 0 \iff \gamma(\varkappa, \varkappa) = \lim_{n \rightarrow \infty} \gamma(\varkappa_n, \varkappa) = \lim_{n, m \rightarrow \infty} \gamma(\varkappa_n, \varkappa_m).$$

Because of the usefulness of partial metric spaces especially in computer science many authors have studied fixed point theory on these spaces. In this sense, Aydi et al. initiated study of fixed point theory for multivalued mappings on partial metric spaces by introducing partial Hausdorff metric [4]. Considering a binary relation $\mathcal{R}$ on a partial metric space Perveen et al. introduced a new contraction, and then obtained a fixed point theorem for such mappings [19]. Bugajewski et al. also discussed the vital role of bottom sets in fixed point theory on these spaces [8]. Recently, Romaguera [21] obtained the following nice and interesting result on these settings which extends the famous Boyd-Wong fixed point theorem [7].

Theorem 1. *Let (Λ, γ) be a complete partial metric space and let $T : \Lambda \rightarrow \Lambda$ be a mapping such that*

$$\gamma(T\varkappa, T\eta) \leq \varphi(C_T(\varkappa, \eta))$$

for all $\varkappa, \eta \in \Lambda$, where

$$C_T(\varkappa, \eta) = \max \left\{ \gamma(\varkappa, \eta), \gamma(\varkappa, T\varkappa), \gamma(\eta, T\eta), \frac{1}{2} [\gamma(\varkappa, T\eta) + \gamma(\eta, T\varkappa)] \right\}$$

and $\phi : [0, \infty) \rightarrow [0, \infty)$ is upper semicontinuous from the right such that $\phi(\lambda) < \lambda$ for all $\lambda > 0$. Then T has a unique fixed point $\varkappa \in \Lambda$. Moreover, $\gamma(\varkappa, \varkappa) = 0$.

We denote the family of all functions ϕ satisfying conditions in Theorem 1 by Φ .

On the other hand, very recently, fixed point theory has been improved different aspect from the results existing in the literature by using best proximity point theory. Let (Λ, d) be a metric space and $\emptyset \neq P, Q \subseteq \Lambda$. Then, the mapping $T : P \rightarrow Q$ cannot have a fixed point in case of $P \cap Q = \emptyset$. In this case, it is sensible to search the existence a point $\varkappa \in P$ such that $d(\varkappa, T\varkappa) = d(P, Q)$ which is called best proximity point of T . Note that, if $P = Q = \Lambda$, then a best proximity point becomes a fixed point. Therefore, every fixed point results are special cases of the corresponding best proximity point results. For this reason, this topic has been studied in various ways [2, 3, 5, 6, 20, 22]. Now, we state the related fundamental concepts and notations of best proximity point theory in realm of partial metric spaces.

Let (Λ, γ) be a partial metric space and $\emptyset \neq P, Q \subseteq \Lambda$. We will consider the subsets P_0 and Q_0 defined as

$$P_0 = \{\varkappa \in P : \gamma(\varkappa, \eta) = \gamma(P, Q) \text{ for some } \eta \in Q\}$$

and

$$Q_0 = \{\eta \in Q : \gamma(\varkappa, \eta) = \gamma(P, Q) \text{ for some } \varkappa \in P\},$$

where

$$\gamma(P, Q) = \inf\{\gamma(\varkappa, \eta) : \varkappa \in P \text{ and } \eta \in Q\}.$$

We will call Q is approximately compact with respect to P if and only if every sequence $\{\eta_n\}$ in Q satisfying $\gamma(\varkappa, \eta_n) \rightarrow \gamma(\varkappa, Q)$ for some $\varkappa \in P$ has a subsequence $\{\eta_{n_k}\}$ such that $d_\gamma(\eta_{n_k}, \eta) \rightarrow 0$ as $k \rightarrow \infty$ for some $\eta \in Q$.

Definition 2. Let (Λ, γ) be a partial metric space and $\emptyset \neq P, Q \subseteq \Lambda$. Then, the pair (P, Q) is said to have the P -Property if and only if it is satisfied

$$\left. \begin{array}{l} \gamma(\varkappa_1, \eta_1) = \gamma(P, Q) \\ \gamma(\varkappa_2, \eta_2) = \gamma(P, Q) \end{array} \right\} \implies \gamma(\varkappa_1, \varkappa_2) = \gamma(\eta_1, \eta_2)$$

for all $\varkappa_1, \varkappa_2 \in P_0$ and $\eta_1, \eta_2 \in Q_0$.

Remark 1. Recently, taking into account the metric d_γ defined by $d_\gamma(\varkappa, \eta) = 0$ if $\varkappa = \eta$ and $d_\gamma(\varkappa, \eta) = \gamma(\varkappa, \eta)$ if $\varkappa \neq \eta$ with the help of partial metric γ in [10], Haghi et al. [9] showed that some fixed point results for the certain operators on partial metric spaces can be obtained directly from their standard metric counterparts. However, even if it can be shown that a mapping T has a fixed point $\varkappa$ by using this approach, it cannot be obtained that $\gamma(\varkappa, \varkappa) = 0$ which is very useful for computer

sciences (for more details, see [12, 13, 15, 16]). Moreover, the results obtained in this article are even new in the settings of metric spaces.

In this paper, we first introduce two concepts called generalized proximal BW -contraction and generalized best BW -contraction. Then, we obtain some best proximity point theorems for such mappings on partial metric spaces. Hence, many results in the literature are extended and improved. To illustrate the effectiveness of our results, we provide some nontrivial and interesting examples. Finally, unlike homotopy applications existing in the literature, we present for the first time an application of the best proximity result to the homotopy theory.

2. GENERALIZED PROXIMAL BW -CONTRACTIONS

In this section, we first introduce the following new concept, the so called generalized proximal BW -contraction and obtain some best proximity point results for such mappings.

Definition 3. Let (Λ, γ) be a partial metric spaces, $\emptyset \neq P, Q \subseteq \Lambda$ and $T : P \rightarrow Q$ be a mapping. If there exists a $\phi \in \Phi$ such that

$$\left. \begin{array}{l} \gamma(u_1, T\kappa_1) = \gamma(P, Q) \\ \gamma(u_2, T\kappa_2) = \gamma(P, Q) \end{array} \right\} \implies \gamma(u_1, u_2) \leq \phi(R_T(\kappa_1, \kappa_2)) \quad (2.1)$$

for all $u_1, u_2, \kappa_1, \kappa_2 \in P$, where

$$R_T(\kappa_1, \kappa_2) = \max \{ \gamma(\kappa_1, \kappa_2), S_T(\kappa_1, \kappa_2) - \gamma(P, Q) \}$$

and

$$S_T(\kappa_1, \kappa_2) = \max \left\{ \gamma(\kappa_1, T\kappa_1), \gamma(\kappa_2, T\kappa_2), \frac{\gamma(\kappa_1, T\kappa_2) + \gamma(\kappa_2, T\kappa_1)}{2} \right\},$$

then T is called a generalized proximal BW -contraction mapping.

Before the main result of this section, we give the following lemma which is useful in our results.

Lemma 2. Let (Λ, γ) be a partial metric space, $\emptyset \neq P, Q \subseteq \Lambda$ and $T : P \rightarrow Q$ be a mapping. Then,

$$R_T(\kappa, \eta) = \max \{ \gamma(\kappa, \eta), \gamma(\eta, T\eta) - \gamma(P, Q) \}$$

for all $\kappa, \eta \in P$ satisfying $\gamma(\eta, T\kappa) = \gamma(P, Q)$.

Proof. Let $\kappa, \eta \in \Lambda$ satisfying $\gamma(\eta, T\kappa) = \gamma(P, Q)$. Then, we have

$$\gamma(\kappa, T\kappa) \leq \gamma(\kappa, \eta) + \gamma(\eta, T\kappa),$$

and so,

$$\gamma(\kappa, T\kappa) - \gamma(P, Q) \leq \gamma(\kappa, \eta). \quad (2.2)$$

Further, using triangle inequality and (2.4), we obtain

$$\begin{aligned}
 \frac{\gamma(\varkappa, T\eta) + \gamma(\eta, T\varkappa)}{2} - \gamma(P, Q) &\leq \frac{\gamma(\varkappa, \eta) + \gamma(\eta, T\eta)}{2} + \frac{\gamma(P, Q)}{2} - \gamma(P, Q) \\
 &= \frac{\gamma(\varkappa, \eta) + \gamma(\eta, T\eta)}{2} - \frac{\gamma(P, Q)}{2} \\
 &= \frac{\gamma(\varkappa, \eta)}{2} + \frac{\gamma(\eta, T\eta) - \gamma(P, Q)}{2} \\
 &\leq \max \{ \gamma(\varkappa, \eta), \gamma(\eta, T\eta) - \gamma(P, Q) \}. \quad (2.3)
 \end{aligned}$$

Thus, from (2.2) and (2.3), we get

$$\begin{aligned}
 R_T(\varkappa, \eta) &= \max \{ \gamma(\varkappa, \eta), S_T(\varkappa, \eta) - \gamma(P, Q) \} \\
 &= \max \left\{ \gamma(\varkappa, \eta), \max \left\{ \gamma(\varkappa, T\varkappa), \gamma(\eta, T\eta), \frac{\gamma(\varkappa, T\eta) + \gamma(\eta, T\varkappa)}{2} \right\} - \gamma(P, Q) \right\} \\
 &\leq \max \{ \gamma(\varkappa, \eta), \gamma(\eta, T\eta) - \gamma(P, Q) \} \leq R_T(\varkappa, \eta)
 \end{aligned}$$

and so

$$R_T(\varkappa, \eta) = \max \{ \gamma(\varkappa, \eta), \gamma(\eta, T\eta) - \gamma(P, Q) \}.$$

□

Theorem 2. Let (Λ, γ) be a partial metric space, $\emptyset \neq P, Q \subseteq \Lambda$ with $P_0 \neq \emptyset$ and $T : P \rightarrow Q$ be a generalized proximal BW-contraction mapping satisfying $T(P_0) \subseteq Q_0$. If (P_0, γ) is complete, then the mapping T has a best proximity point $\varkappa^*$ in P_0 . Moreover, $\gamma(\varkappa^*, \varkappa^*) = 0$.

Proof. Let $\varkappa_0 \in P_0$ be an arbitrary point. Since $T\varkappa_0 \in T(P_0) \subseteq Q_0$, there exists $\varkappa_1 \in P_0$ such that

$$\gamma(\varkappa_1, T\varkappa_0) = \gamma(P, Q).$$

Similarly, since $T\varkappa_1 \in T(P_0) \subseteq Q_0$, there exists $\varkappa_2 \in P_0$ such that

$$\gamma(\varkappa_2, T\varkappa_1) = \gamma(P, Q).$$

Hence, by the generalized proximal BW-contractivity of T , we have

$$\gamma(\varkappa_1, \varkappa_2) \leq \Phi(R_T(\varkappa_0, \varkappa_1)).$$

Continuing this process, we can construct a sequence $\{\varkappa_n\}$ in P_0 such that

$$\gamma(\varkappa_{n+1}, T\varkappa_n) = \gamma(P, Q) \quad (2.4)$$

and

$$\gamma(\varkappa_n, \varkappa_{n+1}) \leq \Phi(R_T(\varkappa_{n-1}, \varkappa_n)) \quad (2.5)$$

for all $n \geq 1$. If there exists $n_0 \geq 1$ such that $\gamma(\varkappa_{n_0}, \varkappa_{n_0+1}) = 0$, then $\varkappa_{n_0}$ is a best proximity point of T . Moreover $\gamma(\varkappa_{n_0}, \varkappa_{n_0}) = 0$. Assume that $\gamma(\varkappa_n, \varkappa_{n+1}) > 0$ for all $n \geq 1$. Thus, from (2.4) and Lemma 2, we have

$$R_T(\varkappa_{n-1}, \varkappa_n) = \max \{ \gamma(\varkappa_{n-1}, \varkappa_n), \gamma(\varkappa_n, T\varkappa_{n-1}) - \gamma(P, Q) \}$$

for all $n \geq 1$. Now from (2.5), we get

$$\gamma(\mathcal{X}_n, \mathcal{X}_{n+1}) \leq \Phi(\max\{\gamma(\mathcal{X}_{n-1}, \mathcal{X}_n), \gamma(\mathcal{X}_n, T\mathcal{X}_n) - \gamma(P, Q)\}) \quad (2.6)$$

for all $n \geq 1$. If there exists $n_0 \geq 1$ such that

$$\gamma(\mathcal{X}_{n_0}, T\mathcal{X}_{n_0}) - \gamma(P, Q) \geq \gamma(\mathcal{X}_{n_0-1}, \mathcal{X}_{n_0})$$

then, from (2.4) and (2.6), we have

$$\begin{aligned} \gamma(\mathcal{X}_{n_0}, \mathcal{X}_{n_0+1}) &\leq \Phi(\gamma(\mathcal{X}_{n_0}, T\mathcal{X}_{n_0}) - \gamma(P, Q)) \\ &< \gamma(\mathcal{X}_{n_0}, T\mathcal{X}_{n_0}) - \gamma(P, Q) \\ &\leq \gamma(\mathcal{X}_{n_0}, \mathcal{X}_{n_0+1}) + \gamma(\mathcal{X}_{n_0+1}, T\mathcal{X}_{n_0}) - \gamma(P, Q) \\ &= \gamma(\mathcal{X}_{n_0}, \mathcal{X}_{n_0+1}), \end{aligned}$$

which is a contradiction. Therefore, we have

$$\gamma(\mathcal{X}_n, T\mathcal{X}_n) - \gamma(P, Q) < \gamma(\mathcal{X}_{n-1}, \mathcal{X}_n)$$

for all $n \geq 1$ and so, we get

$$\gamma(\mathcal{X}_n, \mathcal{X}_{n+1}) \leq \Phi(\gamma(\mathcal{X}_{n-1}, \mathcal{X}_n)) < \gamma(\mathcal{X}_{n-1}, \mathcal{X}_n) \quad (2.7)$$

for all $n \geq 1$. Hence $\{\gamma(\mathcal{X}_n, \mathcal{X}_{n+1})\}$ is a decreasing sequences in $[0, \infty)$ and so it is convergent. Then, there exists $u \geq 0$ such that

$$\lim_{n \rightarrow \infty} \gamma(\mathcal{X}_n, \mathcal{X}_{n+1}) = u.$$

We claim that $u = 0$. Assume that $u > 0$. Then, using equation (2.7), we have

$$u = \lim_{n \rightarrow \infty} \gamma(\mathcal{X}_n, \mathcal{X}_{n+1}) \leq \lim_{n \rightarrow \infty} \Phi(\gamma(\mathcal{X}_{n-1}, \mathcal{X}_n)) = \lim_{n \rightarrow \infty} \sup \Phi(\gamma(\mathcal{X}_{n-1}, \mathcal{X}_n)) \leq \Phi(u) < u,$$

which is a contradiction. Therefore, $\lim_{n \rightarrow \infty} \gamma(\mathcal{X}_n, \mathcal{X}_{n+1}) = 0$. Now, we shall show that $\lim_{n, m \rightarrow \infty} \gamma(\mathcal{X}_n, \mathcal{X}_m) = 0$. Assume the contrary. Then, there exist $\varepsilon > 0$ and two sequences $\{\mathcal{X}_{n_k}\}$ and $\{\mathcal{X}_{m_k}\}$ of $\{\mathcal{X}_n\}$ such that

$$\gamma(\mathcal{X}_{n_k}, \mathcal{X}_{m_k}) \geq \varepsilon \text{ and } \gamma(\mathcal{X}_{n_k}, \mathcal{X}_{m_k-1}) < \varepsilon \quad (2.8)$$

for all $m_k > n_k \geq k$, where m_k is the smallest natural number satisfying (2.8) corresponding to n_k . Then, we have

$$\varepsilon \leq \gamma(\mathcal{X}_{n_k}, \mathcal{X}_{m_k}) \leq \gamma(\mathcal{X}_{n_k}, \mathcal{X}_{m_k-1}) + \gamma(\mathcal{X}_{m_k-1}, \mathcal{X}_{m_k}) < \varepsilon + \gamma(\mathcal{X}_{m_k-1}, \mathcal{X}_{m_k}). \quad (2.9)$$

Letting $k \rightarrow \infty$ in inequality (2.9), we get $\lim_{k \rightarrow \infty} \gamma(\mathcal{X}_{n_k}, \mathcal{X}_{m_k}) = \varepsilon$. Also, we have

$$\begin{aligned} &\frac{\gamma(\mathcal{X}_{n_k}, T\mathcal{X}_{m_k}) + \gamma(\mathcal{X}_{m_k}, T\mathcal{X}_{n_k})}{2} - \gamma(P, Q) \\ &\leq \frac{\gamma(\mathcal{X}_{n_k}, \mathcal{X}_{m_k}) + \gamma(\mathcal{X}_{m_k}, T\mathcal{X}_{m_k}) - \gamma(P, Q)}{2} + \frac{\gamma(\mathcal{X}_{n_k}, \mathcal{X}_{m_k}) + \gamma(\mathcal{X}_{n_k}, T\mathcal{X}_{n_k}) - \gamma(P, Q)}{2} \\ &= \gamma(\mathcal{X}_{n_k}, \mathcal{X}_{m_k}) + \frac{\gamma(\mathcal{X}_{n_k}, T\mathcal{X}_{n_k}) - \gamma(P, Q)}{2} + \frac{\gamma(\mathcal{X}_{m_k}, T\mathcal{X}_{m_k}) - \gamma(P, Q)}{2} \end{aligned} \quad (2.10)$$

for all $k \geq 1$. Then, taking into account inequalities (2.4) and (2.10), we obtain

$$\begin{aligned}\gamma(\mathcal{X}_{n_k}, \mathcal{X}_{m_k}) &\leq R_T(\mathcal{X}_{n_k}, \mathcal{X}_{m_k}) \\ &= \max \{ \gamma(\mathcal{X}_{n_k}, \mathcal{X}_{m_k}), S_T(\mathcal{X}_{n_k}, \mathcal{X}_{m_k}) - \gamma(P, Q) \} \\ &\leq \gamma(\mathcal{X}_{n_k}, \mathcal{X}_{m_k}) + \gamma(\mathcal{X}_{n_k}, T\mathcal{X}_{n_k}) - \gamma(P, Q) + \gamma(\mathcal{X}_{m_k}, T\mathcal{X}_{m_k}) - \gamma(P, Q) \\ &\leq \gamma(\mathcal{X}_{n_k}, \mathcal{X}_{m_k}) + \gamma(\mathcal{X}_{n_k}, \mathcal{X}_{n_k+1}) + \gamma(\mathcal{X}_{m_k}, \mathcal{X}_{m_k+1})\end{aligned}$$

which implies that

$$\lim_{k \rightarrow \infty} R_T(\mathcal{X}_{n_k}, \mathcal{X}_{m_k}) = \varepsilon.$$

Since $R_T(\mathcal{X}_{n_k}, \mathcal{X}_{m_k}) \geq \gamma(\mathcal{X}_{n_k}, \mathcal{X}_{m_k}) \geq \varepsilon$ for all $k \geq 1$, we have

$$\limsup_{k \rightarrow \infty} \varphi(R_T(\mathcal{X}_{n_k}, \mathcal{X}_{m_k})) \leq \varphi(\varepsilon).$$

On the other hand, for all $k \geq 1$, we obtain

$$\begin{aligned}\varepsilon &\leq \gamma(\mathcal{X}_{n_k}, \mathcal{X}_{m_k}) \\ &\leq \gamma(\mathcal{X}_{n_k}, \mathcal{X}_{n_k+1}) + \gamma(\mathcal{X}_{n_k+1}, \mathcal{X}_{m_k+1}) + \gamma(\mathcal{X}_{m_k}, \mathcal{X}_{m_k+1}) \\ &\leq \gamma(\mathcal{X}_{n_k}, \mathcal{X}_{n_k+1}) + \varphi(R_T(\mathcal{X}_{n_k}, \mathcal{X}_{m_k})) + \gamma(\mathcal{X}_{m_k}, \mathcal{X}_{m_k+1})\end{aligned}$$

and so, taking limit supremum in last inequality, we have

$$\varepsilon \leq \varphi(\varepsilon) < \varepsilon,$$

which is a contradiction. Hence, $\{\mathcal{X}_n\}$ is a Cauchy sequence in P_0 . Since P_0 is a complete partial metric space, there exist $\mathcal{X}^* \in P_0$ such that

$$\lim_{n \rightarrow \infty} \gamma(\mathcal{X}_n, \mathcal{X}^*) = \gamma(\mathcal{X}^*, \mathcal{X}^*) = 0 = \lim_{n, m \rightarrow \infty} \gamma(\mathcal{X}_n, \mathcal{X}_m).$$

In this case, since $T\mathcal{X}^* \in T(P_0) \subseteq Q_0$, there exists $\mathfrak{v} \in P_0$ such that

$$\gamma(\mathfrak{v}, T\mathcal{X}^*) = \gamma(P, Q). \quad (2.11)$$

Now, we have

$$\begin{aligned}\gamma(\mathcal{X}^*, T\mathcal{X}^*) - \gamma(P, Q) &\leq R_T(\mathcal{X}_n, \mathcal{X}^*) \\ &\leq \gamma(\mathcal{X}_n, \mathcal{X}^*) + [\gamma(\mathcal{X}_n, T\mathcal{X}_n) - \gamma(P, Q)] + [\gamma(\mathcal{X}^*, T\mathcal{X}^*) - \gamma(P, Q)] \\ &\leq \gamma(\mathcal{X}_n, \mathcal{X}^*) + \gamma(\mathcal{X}_n, \mathcal{X}_{n+1}) + [\gamma(\mathcal{X}^*, T\mathcal{X}^*) - \gamma(P, Q)]\end{aligned}$$

and so

$$\lim_{n \rightarrow \infty} R_T(\mathcal{X}_n, \mathcal{X}^*) = \gamma(\mathcal{X}^*, T\mathcal{X}^*) - \gamma(P, Q).$$

Then, from (2.4), (2.11) and the generalized proximal BW -contractivity of T , we get

$$\gamma(\mathcal{X}^*, \mathfrak{v}) = \lim_{n \rightarrow \infty} \gamma(\mathcal{X}_{n+1}, \mathfrak{v}) \leq \lim_{n \rightarrow \infty} \varphi(R_T(\mathcal{X}_n, \mathcal{X}^*)) \leq \varphi(\gamma(\mathcal{X}^*, T\mathcal{X}^*) - \gamma(P, Q)).$$

Now, assume $\gamma(\mathcal{X}^*, T\mathcal{X}^*) > \gamma(P, Q)$. In this case, from the last inequality, we have

$$\gamma(\mathcal{X}^*, \mathfrak{v}) < \gamma(\mathcal{X}^*, T\mathcal{X}^*) - \gamma(P, Q) \leq \gamma(\mathcal{X}^*, \mathfrak{v}) + \gamma(\mathfrak{v}, T\mathcal{X}^*) - \gamma(P, Q) = \gamma(\mathcal{X}^*, \mathfrak{v}),$$

which is a contradiction. Therefore, $\gamma(\mathcal{X}^*, T\mathcal{X}^*) = \gamma(P, Q)$, that is, $\mathcal{X}^*$ is a best proximity point of T . $\square$

Remark 2. Let (Λ, γ) be a complete partial metric space, $\emptyset \neq P, Q \subseteq \Lambda$ where P is closed and Q is approximately compact with respect to P . Then, (P_0, γ) is complete. Indeed, let $\{\mathcal{X}_n\}$ be a Cauchy sequence in P_0 . Then, $\{\mathcal{X}_n\}$ is a Cauchy sequence in Λ . Since (Λ, γ) is complete, there exists $\mathcal{X}^* \in \Lambda$ such that

$$\lim_{n \rightarrow \infty} \gamma(\mathcal{X}_n, \mathcal{X}^*) = \gamma(\mathcal{X}^*, \mathcal{X}^*) = \lim_{n, m \rightarrow \infty} \gamma(\mathcal{X}_n, \mathcal{X}_m)$$

Because of Lemma 1, $\{\mathcal{X}_n\}$ converges to $\mathcal{X}^*$ with respect to ordinary metric d_γ , that is,

$$\lim_{n \rightarrow \infty} d_\gamma(\mathcal{X}_n, \mathcal{X}^*) = 0. \quad (2.12)$$

Moreover, since $\mathcal{X}_n \in P_0$ for all $n \geq 1$, there exists a sequence $\{\eta_n\}$ in Q_0 such that

$$\gamma(\mathcal{X}_n, \eta_n) = \gamma(P, Q) \quad (2.13)$$

for all $n \geq 1$. On the other hand, since

$$\begin{aligned} \gamma(\mathcal{X}^*, Q) &\leq \gamma(\mathcal{X}^*, \eta_n) \\ &\leq \gamma(\mathcal{X}^*, \mathcal{X}_n) + \gamma(\mathcal{X}_n, \eta_n) - \gamma(\mathcal{X}_n, \mathcal{X}_n) \\ &= \gamma(\mathcal{X}^*, \mathcal{X}_n) - \gamma(\mathcal{X}_n, \mathcal{X}_n) + \gamma(P, Q) \\ &\leq \gamma(\mathcal{X}^*, \mathcal{X}_n) - \gamma(\mathcal{X}_n, \mathcal{X}_n) + \gamma(\mathcal{X}^*, Q), \end{aligned}$$

we have $\gamma(\mathcal{X}^*, \eta_n) \rightarrow \gamma(\mathcal{X}^*, Q)$ as $n \rightarrow \infty$. Since Q is an approximately compact with respect to P , we get that there exists a subsequence $\{\eta_{n_k}\}$ of $\{\eta_n\}$ such that $\lim_{k \rightarrow \infty} d_\gamma(\eta_{n_k}, \eta^*) = 0$ for some $\eta^* \in Q$. Hence, from (2.12) and (2.13), we have $\gamma(\mathcal{X}^*, \eta^*) = \gamma(P, Q)$ and so, $\mathcal{X}^* \in P_0$.

Taking into account Remark 2, we can obtain the following corollary.

Corollary 1. Let (Λ, γ) be a complete partial metric space, $\emptyset \neq P, Q \subseteq \Lambda$ where P is closed and Q is approximately compact with respect to P . Assume that $T : P \rightarrow Q$ is a generalized proximal BW-contraction with $T(P_0) \subseteq Q_0$ and $P_0 \neq \emptyset$. Then T has a best proximity point $\mathcal{X}^*$ in Λ . Moreover, $\gamma(\mathcal{X}^*, \mathcal{X}^*) = 0$.

Now, we present an example to illustrate both the effectiveness of Theorem 2 and the meaningfulness of Remark 2.

Example 1. Let $\Lambda = (\{0\} \cup [1, \infty)) \times [0, \infty)$ endowed with the partial metric γ defined by

$$\gamma(\mathcal{X}, \eta) = \begin{cases} \frac{\mathcal{X}_1}{2}, & \mathcal{X} = \eta \\ \mathcal{X}_1 + \eta_1 + |\mathcal{X}_2 - \eta_2|, & \mathcal{X} \neq \eta \end{cases}$$

for all $\mathcal{X} = (\mathcal{X}_1, \mathcal{X}_2), \eta = (\eta_1, \eta_2) \in \Lambda$. Consider the subsets P and Q as

$$P = (\{0\} \cup [2, \infty)) \times [0, \infty)$$

and

$$Q = [1, 2) \times [0, \infty).$$

Then, $\gamma(P, Q) = 1$, $P_0 = \{0\} \times [0, \infty)$ and $Q_0 = \{1\} \times [0, \infty)$. Further, we have (P_0, γ) is complete. Now, we define mappings $T : P \rightarrow Q$ and $\phi : [0, \infty) \rightarrow [0, \infty)$ by

$$T(\varkappa_1, \varkappa_2) = \begin{cases} (1, \frac{\varkappa_2}{1+\varkappa_2}) & , \quad \varkappa_1 = 0 \\ (2 - \frac{1}{\varkappa_1}, \varkappa_2) & , \quad \text{otherwise} \end{cases}$$

and

$$\phi(\lambda) = \begin{cases} \frac{1}{2} & , \quad \lambda = 0 \\ \frac{\lambda}{1+\lambda} & , \quad \lambda \neq 0 \end{cases}.$$

It can be seen that $T(P_0) \subseteq Q_0$ and T is a generalized proximal BW -contraction. Then, all hypotheses of Theorem 2 are satisfied. Therefore, the mapping T has a best proximity point $\varkappa^*$ in P . Moreover $\gamma(\varkappa^*, \varkappa^*) = 0$.

Note that Q is not approximately compact with respect to P . Indeed, let us consider the sequence $\eta_n = (1 + \frac{1}{2n}, 0)$ for all $n \geq 1$. In this case, $\gamma((0, 0), \eta_n) \rightarrow \gamma((0, 0), Q)$ as $n \rightarrow \infty$. However, the sequence $\{\eta_n\}$ does not have a d_γ -convergent subsequence in Q . Therefore Corollary 1 can not be applied to this example.

3. GENERALIZED BEST BW -CONTRACTIONS

We start to this section by giving the definition of generalized best BW -contraction.

Definition 4. Let (Λ, γ) be a partial metric space, $\emptyset \neq P, Q \subseteq \Lambda$ and $T : P \rightarrow Q$ be a mapping. Then, T is called generalized best BW -contraction if there exists $\phi \in \Phi$ such that

$$\gamma(T\varkappa, T\eta) \leq \phi(R_T(\varkappa, \eta)) \quad (3.1)$$

for all $\varkappa, \eta \in P$.

Theorem 3. Let (Λ, γ) be a complete partial metric space, P, Q be nonempty closed subsets of Λ and $T : P \rightarrow Q$ be a generalized best BW -contraction. Assume that the pair (P, Q) has the P -Property, $P_0 \neq \emptyset$ and $T(P_0) \subseteq Q_0$. Then, T has a best proximity point $\varkappa^*$ in P . Moreover, $\gamma(\varkappa^*, \varkappa^*) = 0$.

Proof. Let $\varkappa_0 \in P_0$ be an arbitrary point. Since $T\varkappa_0 \in T(P_0) \subseteq Q_0$, there exists $\varkappa_1 \in P_0$ such that

$$\gamma(\varkappa_1, T\varkappa_0) = \gamma(P, Q). \quad (3.2)$$

Similarly, since $T\varkappa_1 \in T(P_0) \subseteq Q_0$ there exists $\varkappa_2 \in P_0$ such that

$$\gamma(\varkappa_2, T\varkappa_1) = \gamma(P, Q). \quad (3.3)$$

Considering (3.2), (3.3) and P -Property, we have

$$\gamma(\varkappa_1, \varkappa_2) = \gamma(T\varkappa_0, T\varkappa_1).$$

Repeating this process, we construct a sequence $\{\varkappa_n\}$ such that

$$\gamma(\varkappa_{n+1}, T\varkappa_n) = \gamma(P, Q) \quad (3.4)$$

and

$$\gamma(\mathcal{X}_n, \mathcal{X}_{n+1}) = \gamma(T\mathcal{X}_{n-1}, T\mathcal{X}_n) \quad (3.5)$$

for all $n \geq 1$. If there exists $n_0 \geq 1$ such that $\gamma(\mathcal{X}_{n_0}, \mathcal{X}_{n_0+1}) = 0$, then $\mathcal{X}_{n_0}$ is a best proximity point of T . Moreover, $\gamma(\mathcal{X}_{n_0}, \mathcal{X}_{n_0}) = 0$. Therefore, assume $\gamma(\mathcal{X}_n, \mathcal{X}_{n+1}) > 0$ for all $n \geq 1$. In equation (3.1), taking $\mathcal{X} = \mathcal{X}_n$ and $\eta = \mathcal{X}_{n+1}$, we have

$$\gamma(\mathcal{X}_{n+1}, \mathcal{X}_{n+2}) = \gamma(T\mathcal{X}_n, T\mathcal{X}_{n+1}) \leq \Phi(R_T(\mathcal{X}_n, \mathcal{X}_{n+1})). \quad (3.6)$$

Thus, from (3.4) and Lemma 2, we get

$$\gamma(\mathcal{X}_{n+1}, \mathcal{X}_{n+2}) \leq \Phi(\max\{\gamma(\mathcal{X}_n, \mathcal{X}_{n+1}), \gamma(\mathcal{X}_{n+1}, T\mathcal{X}_{n+1}) - \gamma(P, Q)\}) \quad (3.7)$$

for all $n \geq 1$. Now, if there exists $n_0 \geq 1$ such that

$$\gamma(\mathcal{X}_{n_0}, \mathcal{X}_{n_0+1}) \leq \gamma(\mathcal{X}_{n_0+1}, T\mathcal{X}_{n_0+1}) - \gamma(P, Q)$$

then, we have

$$\begin{aligned} \gamma(\mathcal{X}_{n_0+1}, \mathcal{X}_{n_0+2}) &\leq \Phi(\gamma(\mathcal{X}_{n_0+1}, T\mathcal{X}_{n_0+1}) - \gamma(P, Q)) \\ &< \gamma(\mathcal{X}_{n_0+1}, T\mathcal{X}_{n_0+1}) - \gamma(P, Q) \\ &\leq \gamma(\mathcal{X}_{n_0+1}, \mathcal{X}_{n_0+2}) + \gamma(\mathcal{X}_{n_0+2}, T\mathcal{X}_{n_0+1}) - \gamma(P, Q) \\ &= \gamma(\mathcal{X}_{n_0+1}, \mathcal{X}_{n_0+2}) \end{aligned}$$

which is a contradiction. Therefore, we obtain

$$\gamma(\mathcal{X}_n, \mathcal{X}_{n+1}) > \gamma(\mathcal{X}_{n+1}, T\mathcal{X}_{n+1}) - \gamma(P, Q)$$

for all $n \geq 1$ and so, from (3.7) we get

$$\gamma(\mathcal{X}_{n+1}, \mathcal{X}_{n+2}) = \gamma(T\mathcal{X}_n, T\mathcal{X}_{n+1}) \leq \Phi(\gamma(\mathcal{X}_n, \mathcal{X}_{n+1})) < \gamma(\mathcal{X}_n, \mathcal{X}_{n+1}) \quad (3.8)$$

for all $n \geq 1$. Hence $\{\gamma(\mathcal{X}_n, \mathcal{X}_{n+1})\}$ is a decreasing sequences in $[0, \infty)$ and so it is convergent. Then, there exists $u \geq 0$ such that

$$\lim_{n \rightarrow \infty} \gamma(\mathcal{X}_n, \mathcal{X}_{n+1}) = u.$$

We claim that $u = 0$. Assume that $u > 0$. Then, using equations (3.5) and (3.8), we have

$$\begin{aligned} u &= \lim_{n \rightarrow \infty} \gamma(\mathcal{X}_n, \mathcal{X}_{n+1}) = \lim_{n \rightarrow \infty} \gamma(T\mathcal{X}_{n-1}, T\mathcal{X}_n) \leq \lim_{n \rightarrow \infty} \Phi(\gamma(\mathcal{X}_{n-1}, \mathcal{X}_n)) \\ &= \lim_{n \rightarrow \infty} \sup \Phi(\gamma(\mathcal{X}_{n-1}, \mathcal{X}_n)) \leq \Phi(u) < u, \end{aligned}$$

which is a contradiction. Therefore, $\lim_{n \rightarrow \infty} \gamma(\mathcal{X}_n, \mathcal{X}_{n+1}) = 0$. Now, we shall show that $\lim_{n, m \rightarrow \infty} \gamma(\mathcal{X}_n, \mathcal{X}_m) = 0$. Assume the contrary, that is, there exist $\varepsilon > 0$ and two sequences $\{\mathcal{X}_{n_k}\}$ and $\{\mathcal{X}_{m_k}\}$ of $\{\mathcal{X}_n\}$ such that

$$\gamma(\mathcal{X}_{n_k}, \mathcal{X}_{m_k}) \geq \varepsilon \text{ and } \gamma(\mathcal{X}_{n_k}, \mathcal{X}_{m_k-1}) < \varepsilon \quad (3.9)$$

for all $m_k > n_k \geq k$ where m_k is the smallest natural number satisfying (3.9) corresponding to n_k . Then, we have

$$\varepsilon \leq \gamma(\mathcal{X}_{n_k}, \mathcal{X}_{m_k}) \leq \gamma(\mathcal{X}_{n_k}, \mathcal{X}_{m_k-1}) + \gamma(\mathcal{X}_{m_k-1}, \mathcal{X}_{m_k}) < \varepsilon + \gamma(\mathcal{X}_{m_k-1}, \mathcal{X}_{m_k}) \quad (3.10)$$

Letting $k \rightarrow \infty$ in inequality (3.10), we get

$$\lim_{k \rightarrow \infty} \gamma(\mathcal{X}_{n_k}, \mathcal{X}_{m_k}) = \varepsilon.$$

Then, we obtain

$$\begin{aligned} \gamma(\mathcal{X}_{n_k}, \mathcal{X}_{m_k}) &\leq R_T(\mathcal{X}_{n_k}, \mathcal{X}_{m_k}) \\ &= \max \{ \gamma(\mathcal{X}_{n_k}, \mathcal{X}_{m_k}), S_T(\mathcal{X}_{n_k}, \mathcal{X}_{m_k}) - \gamma(P, Q) \} \\ &\leq \gamma(\mathcal{X}_{n_k}, \mathcal{X}_{m_k}) + \gamma(\mathcal{X}_{n_k}, \mathcal{X}_{n_k+1}) + \gamma(\mathcal{X}_{m_k}, \mathcal{X}_{m_k+1}) + \frac{\gamma(\mathcal{X}_{n_k}, \mathcal{X}_{n_k+1})}{2} + \frac{\gamma(\mathcal{X}_{m_k}, \mathcal{X}_{m_k+1})}{2} \end{aligned}$$

which implies that

$$\lim_{k \rightarrow \infty} R_T(\mathcal{X}_{n_k}, \mathcal{X}_{m_k}) = \varepsilon$$

Since $R_T(\mathcal{X}_{n_k}, \mathcal{X}_{m_k}) \geq \varepsilon$ for all $k \geq 1$, we have

$$\limsup_{k \rightarrow \infty} \varphi(R_T(\mathcal{X}_{n_k}, \mathcal{X}_{m_k})) \leq \varphi(\varepsilon).$$

On the other hand, for all $k \geq 1$, we obtain

$$\begin{aligned} \varepsilon &\leq \gamma(\mathcal{X}_{n_k}, \mathcal{X}_{m_k}) \\ &\leq \gamma(\mathcal{X}_{n_k}, \mathcal{X}_{n_k+1}) + \gamma(\mathcal{X}_{n_k+1}, \mathcal{X}_{m_k+1}) + \gamma(\mathcal{X}_{m_k}, \mathcal{X}_{m_k+1}) \\ &\leq \gamma(\mathcal{X}_{n_k}, \mathcal{X}_{n_k+1}) + \varphi(R_T(\mathcal{X}_{n_k}, \mathcal{X}_{m_k})) + \gamma(\mathcal{X}_{m_k}, \mathcal{X}_{m_k+1}) \end{aligned}$$

and taking limit supremum we have

$$\varepsilon \leq \varphi(\varepsilon) < \varepsilon,$$

which is a contradiction. Hence, $\{\mathcal{X}_n\}$ is a Cauchy sequence in P . From (3.5), we have $\{T\mathcal{X}_n\}$ is a Cauchy sequence in Q . Since (Λ, γ) is complete partial metric space and P, Q are closed subsets of Λ , there exist $\mathcal{X}^* \in P$ and $\eta^* \in Q$ such that

$$\lim_{n \rightarrow \infty} \gamma(\mathcal{X}_n, \mathcal{X}^*) = \gamma(\mathcal{X}^*, \mathcal{X}^*) = 0 = \lim_{n, m \rightarrow \infty} \gamma(\mathcal{X}_n, \mathcal{X}_m)$$

and

$$\lim_{n \rightarrow \infty} \gamma(T\mathcal{X}_n, \eta^*) = \gamma(\eta^*, \eta^*) = 0 = \lim_{n, m \rightarrow \infty} \gamma(T\mathcal{X}_n, T\mathcal{X}_m).$$

Letting $n \rightarrow \infty$ in (3.4), we have

$$\gamma(\mathcal{X}^*, \eta^*) = \gamma(P, Q).$$

From (3.1), we have

$$\begin{aligned} \gamma(\mathcal{X}^*, T\mathcal{X}^*) - \gamma(P, Q) &\leq R_T(\mathcal{X}_n, \mathcal{X}^*) \\ &\leq \gamma(\mathcal{X}_n, \mathcal{X}^*) + \max \left\{ \frac{\gamma(\mathcal{X}_n, T\mathcal{X}_n), \gamma(\mathcal{X}^*, T\mathcal{X}^*)}{\frac{\gamma(\mathcal{X}^*, T\mathcal{X}_n) + \gamma(\mathcal{X}_n, T\mathcal{X}^*)}{2}} \right\} - \gamma(P, Q) \end{aligned}$$

and so taking limit $n \rightarrow \infty$ we have

$$\gamma(\mathcal{X}^*, T\mathcal{X}^*) - \gamma(P, Q) \leq \lim_{n \rightarrow \infty} R_T(\mathcal{X}_n, \mathcal{X}^*) \leq \gamma(\mathcal{X}^*, T\mathcal{X}^*) - \gamma(P, Q).$$

Therefore we have

$$\lim_{n \rightarrow \infty} R_T(\mathcal{X}_n, \mathcal{X}^*) = \gamma(\mathcal{X}^*, T\mathcal{X}^*) - \gamma(P, Q)$$

and then

$$\begin{aligned} \gamma(\eta^*, T\mathcal{X}^*) &= \lim_{n \rightarrow \infty} \gamma(T\mathcal{X}_n, T\mathcal{X}^*) \\ &\leq \lim_{n \rightarrow \infty} \sup \phi(R_T(\mathcal{X}_n, \mathcal{X}^*)) \\ &\leq \phi(\gamma(\mathcal{X}^*, T\mathcal{X}^*) - \gamma(P, Q)). \end{aligned}$$

Now, assume $\gamma(\mathcal{X}^*, T\mathcal{X}^*) > \gamma(P, Q)$. In this case, from the last inequality, we have

$$\gamma(\eta^*, T\mathcal{X}^*) < \gamma(\mathcal{X}^*, T\mathcal{X}^*) - \gamma(P, Q).$$

Therefore we have

$$\begin{aligned} \gamma(\mathcal{X}^*, T\mathcal{X}^*) - \gamma(P, Q) &\leq \gamma(\mathcal{X}^*, \eta^*) + \gamma(\eta^*, T\mathcal{X}^*) - \gamma(P, Q) \\ &= \gamma(\eta^*, T\mathcal{X}^*) \\ &< \gamma(\mathcal{X}^*, T\mathcal{X}^*) - \gamma(P, Q), \end{aligned}$$

which is a contradiction. Therefore, $\gamma(\mathcal{X}^*, T\mathcal{X}^*) = \gamma(P, Q)$, that is, $\mathcal{X}^*$ is a best proximity point of T . $\square$

We aim to show the importance of Theorem 3 with the nontrivial following example.

Example 2. Let $\Lambda = [0, \infty) \times [0, \infty)$ and $\gamma: \Lambda \times \Lambda \rightarrow [0, \infty)$ be a function defined by

$$\gamma(\mathcal{X}, \eta) = \max\{\mathcal{X}_1, \eta_1\} + |\mathcal{X}_2 - \eta_2|$$

for all $\mathcal{X} = (\mathcal{X}_1, \mathcal{X}_2), \eta = (\eta_1, \eta_2) \in \Lambda$. Then, (Λ, γ) is a complete partial metric space. Consider the closed subsets P, Q of Λ as follows:

$$P = [0, \infty) \times \{0\}$$

and

$$Q = [0, \infty) \times \{1\}.$$

Then, $\gamma(P, Q) = 1$, $P_0 = \{(0, 0)\}$ and $Q_0 = \{(0, 1)\}$. Also, it can be seen that (P, Q) has P -property. Define the mappings $T: P \rightarrow Q$ and $\phi: [0, \infty) \rightarrow [0, \infty)$ by

$$T(\mathcal{X}, 0) = \begin{cases} (\frac{\mathcal{X}}{6}, 1) & , \mathcal{X} \in [0, e^2) \\ (\arctan \mathcal{X}, 1) & , \mathcal{X} \in [e^2, \infty) \end{cases}$$

for all $\mathcal{X} \in [0, \infty)$ and

$$\phi(\lambda) = \begin{cases} \frac{2\lambda}{e^2} & , \lambda \in [0, e^2) \\ \ln \lambda & , \lambda \in [e^2, \infty) \end{cases}$$

for all $\lambda \in [0, \infty)$. Then, we have $T(P_0) \subseteq Q_0$ and $\varphi \in \Phi$. Finally, considering the following cases, we shall show that T is a generalized best BW -contraction:

Case 1. Let $\varkappa = (\varkappa_1, 0)$ and $\eta = (\eta_1, 0)$ with $\varkappa_1, \eta_1 \in [0, e^2)$. Then, we have

$$\gamma(T\varkappa, T\eta) = \max\left\{\frac{\varkappa_1}{6}, \frac{\eta_1}{6}\right\} \leq \frac{2\max\{\varkappa_1, \eta_1\}}{e^2} = \varphi(\gamma(\varkappa, \eta)) \leq \varphi(R_T(\varkappa, \eta)).$$

Case 2. Let $\varkappa = (\varkappa_1, 0)$ and $\eta = (\eta_1, 0)$ with $\varkappa_1 \in [0, e^2)$ and $\eta_1 \in [e^2, \infty)$. Then, we have

$$\begin{aligned} \gamma(T\varkappa, T\eta) &= \max\left\{\frac{\varkappa_1}{6}, \arctan \eta_1\right\} = \arctan \eta_1 \\ &\leq \ln \eta_1 = \varphi(\gamma(\varkappa, \eta)) = 9\varphi(R_T(\varkappa, \eta)). \end{aligned}$$

Case 3: Let $\varkappa = (\varkappa_1, 0)$ and $\eta = (\eta_1, 0)$ with $\varkappa_1, \eta_1 \in [e^2, \infty)$. Then, we have

$$\begin{aligned} \gamma(T\varkappa, T\eta) &= \max\{\arctan \varkappa_1, \arctan \eta_1\} = \arctan(\max\{\varkappa_1, \eta_1\}) \\ &\leq \ln(\max\{\varkappa_1, \eta_1\}) = \varphi(\gamma(\varkappa, \eta)) = \varphi(R_T(\varkappa, \eta)) \end{aligned}$$

The related inequalities can be seen in Figure 1 and Figure 2. Therefore, all hypotheses of Theorem 3 are satisfied and so the mapping T has a best proximity point $\varkappa^*$ in P . Moreover $\gamma(\varkappa^*, \varkappa^*) = 0$.

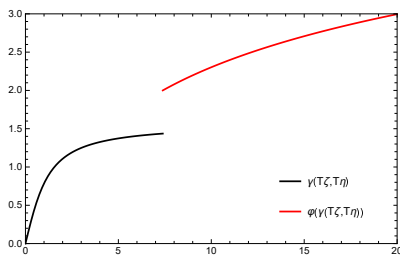


FIGURE 1. $\gamma(T\varkappa, T\eta)$ and $\varphi(\gamma(\varkappa, \eta))$ in Case 2.

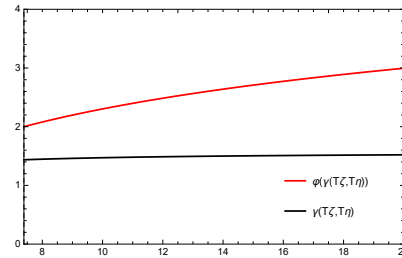


FIGURE 2. $\gamma(T\varkappa, T\eta)$ and $\varphi(\gamma(\varkappa, \eta))$ in Case 3.

From Theorem 3, we get the following corollary.

Corollary 2. Let (Λ, γ) be a complete partial metric space, P, Q be nonempty closed subsets of Λ and $T : P \rightarrow Q$ be a mapping satisfying

$$\gamma(T\varkappa, T\eta) \leq \varphi(\gamma(\varkappa, \eta))$$

for all $\varkappa, \eta \in P$, where $\varphi \in \Phi$. Assume that the pair (P, Q) has the P -Property, $P_0 \neq \emptyset$ and $T(P_0) \subseteq Q_0$. Then T has a best proximity point $\varkappa^*$ in P . Moreover, $\gamma(\varkappa^*, \varkappa^*) = 0$.

Taking $P = Q = \Lambda$ in both Theorem 2 and Theorem 3, we obtain Theorem 1 which is the main result of [21]. Moreover, in a similar way, from Corollary 2 we deduce the following fixed point result.

Corollary 3. *Let (Λ, γ) be a complete partial metric space and $T : \Lambda \rightarrow \Lambda$ be a mapping. Assume that the mapping T satisfies*

$$\gamma(T\kappa, T\eta) \leq \varphi(\gamma(\kappa, \eta))$$

for all $\kappa, \eta \in \Lambda$, where $\varphi \in \Phi$. Then, T has a fixed point κ^ in Λ . Moreover, $\gamma(\kappa^*, \kappa^*) = 0$.*

4. HOMOTOPY RESULT

Homotopy theory is one of the important parts of algebraic topology. Recently, the relationship of this topic with other branches of mathematics has revealed and has attracted the attention of many authors. Although there are many applications of fixed point results to homotopy theory, until now there is no any application of best proximity point results to homotopy theory [1, 11, 18]. In this section, we present an application of our new best proximity point result to homotopy theory. So, we investigate that if a mapping T satisfies all hypotheses of Corollary 2, then we show that all mappings which are homotopic to T also have a best proximity point. We begin this section by recalling the definition of homotopy.

Definition 5. Let (Λ_1, τ_1) and (Λ_2, τ_2) be topological spaces, $T, F : \Lambda_1 \rightarrow \Lambda_2$ be continuous mappings. If there exists continuous function $H : \Lambda_1 \times [0, 1] \rightarrow \Lambda_2$ such that $H(\kappa, 0) = T\kappa$ and $H(\kappa, 1) = F\kappa$ for all $\kappa \in \Lambda_1$, then it is said to be that T and F are homotopic mappings. Also, the mapping H is called homotopy.

In the rest of the paper, we denote the family of all functions in Φ satisfying the following implication by Φ_H as in [17]:

$$\lim_{n \rightarrow \infty} \{s_n - \varphi(s_n)\} = 0 \Rightarrow \lim_{n \rightarrow \infty} s_n = 0 \quad (4.1)$$

for all sequence in $\{s_n\} \subseteq [0, \infty)$. The following example is important to show that the family Φ_H is nonempty set. Let's consider the sequence $(s_n) = (\frac{1}{n})_{n \geq 1}$ and the mapping $\varphi : [0, \infty) \rightarrow [0, \infty)$ defined by

$$\varphi(\lambda) = \begin{cases} 0 & , \lambda < 1 \\ \frac{1}{2\lambda} & , \lambda \geq 1 \end{cases}.$$

Then, it can be easily seen that $\varphi \in \Phi$ and the implication (4.1) is satisfied. Hence, we have $\varphi \in \Phi_H$.

Now, we can present the main result of this section.

Theorem 4. *Let (Λ, γ) be a complete partial metric space, P, Q be nonempty closed subsets of Λ and $\emptyset \neq U \subseteq P$. Assume that the pair (P, Q) has P -Property and $H : P \times [0, 1] \rightarrow Q$ is a mapping satisfying*

$$(i) \quad \gamma(\kappa, H(\kappa, \lambda)) > \gamma(P, Q) \text{ for all } \kappa \in P \setminus U \text{ and } \lambda \in [0, 1],$$

(ii) there exists $\varphi \in \Phi_H$ such that

$$\gamma(H(\varkappa, \lambda), H(\eta, \lambda)) \leq \varphi(\gamma(\varkappa, \eta)) \quad (4.2)$$

for all $\varkappa, \eta \in P$ and $\lambda \in [0, 1]$,

(iii) there exists a continuous function $\eta : [0, 1] \rightarrow [0, \infty)$ such that

$$\gamma(H(\varkappa, \lambda), H(\varkappa, s)) \leq |\eta(\lambda) - \eta(s)|$$

for all $\varkappa \in P$ and $\lambda, s \in [0, 1]$,

(iv) for all $\lambda \in [0, 1]$ satisfying $\gamma(\varkappa, H(\varkappa, \lambda)) = \gamma(P, Q)$ for some $\varkappa \in U$, there exists $\varepsilon_\lambda > 0$ such that $H(P_0, \lambda^*) \subseteq Q_0$ for all $\lambda^* \in (\lambda - \varepsilon_\lambda, \lambda + \varepsilon_\lambda)$.

If $H(\cdot, 0)$ has a best proximity point in P , then $H(\cdot, 1)$ has a best proximity point in P .

Proof. Define a set

$$K = \{\lambda \in [0, 1] : \gamma(\varkappa, H(\varkappa, \lambda)) = \gamma(P, Q) \text{ for some } \varkappa \in U\}.$$

Since $H(\cdot, 0)$ has a best proximity point in P and (i) holds, then $0 \in K$. Hence, K is a nonempty set. We shall show that K is both open and closed in $[0, 1]$ and hence by connectedness of $[0, 1]$, we have that $K = [0, 1]$. We first show that K is closed. For this, let $\{\lambda_n\}$ be a sequence in K with $\lambda_n \rightarrow \lambda^* \in [0, 1]$ as $n \rightarrow \infty$. Using definition of K , there exists $\varkappa_n \in U$ such that

$$\gamma(\varkappa_n, H(\varkappa_n, \lambda_n)) = \gamma(P, Q) \quad (4.3)$$

for all $n \geq 1$. Then, from P -Property and (ii), we have

$$\begin{aligned} \gamma(\varkappa_n, \varkappa_m) &= \gamma(H(\varkappa_n, \lambda_n), H(\varkappa_m, \lambda_m)) \\ &\leq \gamma(H(\varkappa_n, \lambda_n), H(\varkappa_n, \lambda_m)) + \gamma(H(\varkappa_n, \lambda_m), H(\varkappa_m, \lambda_m)) \\ &\leq |\eta(\lambda_n) - \eta(\lambda_m)| + \varphi(\gamma(\varkappa_n, \varkappa_m)) \end{aligned}$$

for all $n, m \geq 1$. Since η is a continuous function and the sequence $\{\lambda_n\}$ is convergent, we have

$$\lim_{n, m \rightarrow \infty} \{\gamma(\varkappa_n, \varkappa_m) - \varphi(\gamma(\varkappa_n, \varkappa_m))\} = 0 \text{ as } n, m \rightarrow \infty.$$

From (4.1), we have $\lim_{n, m \rightarrow \infty} \gamma(\varkappa_n, \varkappa_m) = 0$. Therefore, $\{\varkappa_n\}$ is a Cauchy sequence. Since (Λ, γ) is complete and P is a closed subset of Λ , there exists $\varkappa^* \in P$ such that

$$\lim_{n, m \rightarrow \infty} \gamma(\varkappa_n, \varkappa_m) = \lim_{n \rightarrow \infty} \gamma(\varkappa_n, \varkappa^*) = \gamma(\varkappa^*, \varkappa^*) = 0.$$

Now, we have to consider the following two cases:

(a) Assume that there exists $n_0 \geq 1$ such that $\varkappa_n = \varkappa^*$ for all $n \geq n_0$. In this case, from (4.3), we have

$$\gamma(\varkappa^*, H(\varkappa^*, \lambda_n)) = \gamma(P, Q) \quad (4.4)$$

for all $n \geq n_0$. Also, from (iii), we obtain

$$\gamma(H(\varkappa^*, \lambda_n), H(\varkappa^*, \lambda^*)) \leq |\eta(\lambda_n) - \eta(\lambda^*)|$$

for all $n \geq n_0$. Then, we have $\lim_{n \rightarrow \infty} \gamma(H(\varkappa^*, \lambda_n), H(\varkappa^*, \lambda^*)) = 0$ and so, from (4.4), we get

$$\gamma(\varkappa^*, H(\varkappa^*, \lambda^*)) = \lim_{n \rightarrow \infty} \gamma(\varkappa^*, H(\varkappa^*, \lambda_n)) = \gamma(P, Q).$$

(b) Now, assume the contrary to (a), that is, there exists a subsequence $\{\varkappa_{n_k}\}$ of $\{\varkappa_n\}$ such that $\varkappa_{n_k} \neq \varkappa^*$ for all $k \geq 1$. Then, from (4.3), we have

$$\begin{aligned} \gamma(P, Q) &\leq \gamma(\varkappa_{n_k}, H(\varkappa^*, \lambda^*)) \\ &\leq \gamma(\varkappa_{n_k}, H(\varkappa_{n_k}, \lambda_{n_k})) + \gamma(H(\varkappa_{n_k}, \lambda_{n_k}), H(\varkappa_{n_k}, \lambda^*)) + \gamma(H(\varkappa_{n_k}, \lambda^*), H(\varkappa^*, \lambda^*)) \\ &\leq \gamma(P, Q) + |\eta(\lambda_{n_k}) - \eta(\lambda^*)| + \phi(\gamma(\varkappa_{n_k}, \varkappa^*)) \\ &< \gamma(P, Q) + |\eta(\lambda_{n_k}) - \eta(\lambda^*)| + \gamma(\varkappa_{n_k}, \varkappa^*) \end{aligned}$$

and so

$$\gamma(\varkappa^*, H(\varkappa^*, \lambda^*)) = \lim_{k \rightarrow \infty} \gamma(\varkappa_{n_k}, H(\varkappa^*, \lambda^*)) = \gamma(P, Q).$$

Thus, in both cases, we obtain $\lambda^* \in K$ and so K is closed in $[0, 1]$.

Now, we shall show that K is open. Let $\lambda_0 \in K$. Then, there exists $\varkappa_0 \in U$ such that $\gamma(\varkappa_0, H(\varkappa_0, \lambda_0)) = \gamma(P, Q)$. From (iv), for $\lambda_0 \in [0, 1]$, there exists $\varepsilon_{\lambda_0} > 0$ such that $H(P_0, \lambda^*) \subseteq Q_0$ for all $\lambda^* \in (\lambda_0 - \varepsilon_{\lambda_0}, \lambda_0 + \varepsilon_{\lambda_0})$. If we consider the mappings $H(\cdot, \lambda^*) : P \rightarrow Q$ for all $\lambda^* \in (\lambda_0 - \varepsilon_{\lambda_0}, \lambda_0 + \varepsilon_{\lambda_0})$, then, from (ii), the mappings $H(\cdot, \lambda^*)$ are *BW*-contraction. Therefore, all hypotheses of Corollary 2 are satisfied. Hence, for all $\lambda^* \in (\lambda_0 - \varepsilon_{\lambda_0}, \lambda_0 + \varepsilon_{\lambda_0})$, $H(\cdot, \lambda^*)$ has a best proximity point $\varkappa_{\lambda^*}^*$ in P . Moreover, $\gamma(\varkappa_{\lambda^*}^*, \varkappa_{\lambda^*}^*) = 0$. From (i), $\varkappa_{\lambda^*}^* \in U$ for all $\lambda^* \in (\lambda_0 - \varepsilon_{\lambda_0}, \lambda_0 + \varepsilon_{\lambda_0})$. Hence, we have $(\lambda_0 - \varepsilon_{\lambda_0}, \lambda_0 + \varepsilon_{\lambda_0}) \subseteq K$, that is, K is open in $[0, 1]$. $\square$

Taking $Q = \Lambda$ in Theorem 4, we obtain the following corollary.

Theorem 5. Let (Λ, γ) be a complete partial metric space, P be a nonempty closed subset of Λ and $\emptyset \neq U \subseteq P$. Assume that $H : P \times [0, 1] \rightarrow \Lambda$ is a mapping satisfying

- (i) $\gamma(\varkappa, H(\varkappa, \lambda)) > 0$ for all $\varkappa \in P \setminus U$ and $\lambda \in [0, 1]$,
- (ii) there exists a $\phi \in \Phi_H$ such that

$$\gamma(H(\varkappa, \lambda), H(\eta, \lambda)) \leq \phi(\gamma(\varkappa, \eta))$$

for all $\varkappa, \eta \in P$ and $\lambda \in [0, 1]$,

- (iii) there exists a continuous function $\eta : [0, 1] \rightarrow [0, \infty)$ such that

$$\gamma(H(\varkappa, \lambda), H(\varkappa, s)) \leq |\eta(\lambda) - \eta(s)|$$

for all $\varkappa \in P$ and $\lambda, s \in [0, 1]$,

- (iv) for all $\lambda \in [0, 1]$ satisfying $\varkappa = H(\varkappa, \lambda)$ for some $\varkappa \in U$, there exists $\varepsilon_\lambda > 0$ such that $H(P, \lambda^*) \subseteq P$ for all $\lambda^* \in (\lambda - \varepsilon_\lambda, \lambda + \varepsilon_\lambda)$.

If $H(\cdot, 0)$ has a fixed point in P , then $H(\cdot, 1)$ has a fixed point P .

Proof. Assume that $H(\cdot, 0)$ has a fixed point $\varkappa$ in P . In this case, since $\varkappa = H(\varkappa, 0)$, we have

$$\gamma(\varkappa, \varkappa) \leq \gamma(H(\varkappa, 0), H(\varkappa, 0)) = 0.$$

Hence, $\gamma(\varkappa, \varkappa) = 0$ and so $\gamma(P, \Lambda) = 0$. Hence the conditions (i), (ii) and (iii) of Theorem 4 are hold. Further, we have $P_0 = Q_0 = P$ whenever $Q = \Lambda$, and so condition (iv) is also hold. Therefore, there exists $\varkappa^* \in P$ such that

$$\gamma(\varkappa^*, H(\varkappa^*, 1)) = \gamma(P, \Lambda) = 0.$$

This shows that $\varkappa^*$ is a fixed point of $H(\cdot, 1)$. □

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INTERVAL-VALUED Iq^b -CALCULUS AND APPLICATIONS

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Abstract. This study investigates the Iq^b -differentiability and Iq^b -integrability for interval-valued functions defined on the q -geometric set. We also establish some Iq^b -Hermite-Hadamard type inequalities. Furthermore, some examples are presented to illustrate our results.

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Keywords: interval-valued functions, Iq^b -calculus, Hermite-Hadamard type inequalities

1. INTRODUCTION

Quantum calculus was proposed by the famous mathematician Euler in the 18th century. It uses the difference operator instead of the classical derivative to study the set of non-differentiable functions. In 1910, Jackson introduced the concept of quantum definite integral and extended the concept of quantum calculus. Quantum calculus is a bridge between mathematics and physics. It has been widely used in number theory, quantum theory, mechanics and other fields. In recent years, with the higher and higher requirements of analog quantum computing on mathematics, it has attracting considerable scholarly attention and triggered a huge amount of innovative scientific research [6, 7, 9, 14].

As a branch of mathematics, interval analysis is an effective tool for dealing with data inaccurate models caused by some types of measurements. Interval analysis has been attracting considerable interest since it was firstly applied to automatic error analysis by Moore. We now see applications in fuzzy set and possibility theory [5], uncertain quantification and propagation procedure in the case of the small sample measurement data [17], stochastic analysis of structures with uncertain-but-bounded parameters [13], the terminal error bound of the Stewart platform [18], for more profound results and applications, we refer to the papers [3, 10]. Particularly, Younus [19] investigated the fractional q -differentiability and fractional q -integrability for

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interval-valued functions (IVFs) defined on the q -geometric set of real numbers. Lou [11] proved the Iq -Hermite-Hadamard inequalities for IVFs. Recently, several studies have focused on the quantum differentials, integrals and inequalities of IVFs.

Motivated by the works mentioned, in this paper, we discuss the interval-valued quantum calculus for (shortly, Iq^b -calculus). Firstly, we give the concepts of Iq^b -calculus and define the Iq^b -derivative and Iq^b -integral. We also give some basic properties and present some examples to illustrate our theorems. Moreover, using the notion of Iq^b -derivative and Iq^b -integral, some new inequalities like Hermite-Hadamard are offered. The results of this paper can serve as a base for future studies. These results of this paper can be used as a powerful tool in fuzzy analysis, interval optimization, and interval-valued differential equations.

2. PRELIMINARIES

2.1. Calculus for IVFs

Let $\mathbb{R}_I$ be the set of all non-empty compact intervals on the real line $\mathbb{R}$. For all $[\underline{a}, \bar{a}], [\underline{b}, \bar{b}] \in \mathbb{R}_I$, $\lambda \in \mathbb{R}$ we have

$$[\underline{a}, \bar{a}] + [\underline{b}, \bar{b}] = [\underline{a} + \underline{b}, \bar{a} + \bar{b}],$$

and

$$\lambda [\underline{a}, \bar{a}] = \begin{cases} [\lambda \underline{a}, \lambda \bar{a}], & \lambda > 0, \\ 0, & \lambda = 0, \\ [\lambda \bar{a}, \lambda \underline{a}], & \lambda < 0, \end{cases}$$

respectively.

The generalized Hukuhara (gH for brevity) difference of two intervals $[\underline{a}, \bar{a}], [\underline{b}, \bar{b}] \in \mathbb{R}_I$ is defined by Stefanini [15]:

$$[\underline{a}, \bar{a}] \ominus_g [\underline{b}, \bar{b}] = [\min\{\underline{a} - \underline{b}, \bar{a} - \bar{b}\}, \max\{\underline{a} - \underline{b}, \bar{a} - \bar{b}\}].$$

Let $A = [\underline{a}, \bar{a}] \in \mathbb{R}_I$, $w(A) = \bar{a} - \underline{a}$ is called the length of the interval A . Then, for all $A = [\underline{a}, \bar{a}], B = [\underline{b}, \bar{b}] \in \mathbb{R}_I$, we have

$$[\underline{a}, \bar{a}] \ominus_g [\underline{b}, \bar{b}] = \begin{cases} [\underline{a} - \underline{b}, \bar{a} - \bar{b}], & \text{if } w(A) \geq w(B), \\ [\bar{a} - \bar{b}, \underline{a} - \underline{b}], & \text{if } w(A) \leq w(B). \end{cases}$$

The gH -difference $\ominus_g$ have the following properties [12]:

- (1) $A \ominus_g A = \{0\}$, $A \ominus_g \{0\} = A$, $\{0\} \ominus_g A = -A$,
- (2) $A \ominus_g B = (-B) \ominus_g (-A) = -(B \ominus_g A)$,
- (3) $A \ominus_g (-B) = B \ominus_g (-A) = -(B \ominus_g A)$,
- (4) $(A + B) \ominus_g B = A$,
- (5) $\lambda A \ominus_g \lambda B = \lambda (A \ominus_g B)$.

For $A, B, C, D \in \mathbb{R}_I$, consider $t_1 = w(A) - w(C)$, $t_2 = w(B) - w(D)$, $t_3 = w(A) - w(B)$, $t_4 = w(C) - w(D)$. The following properties are satisfied [12]:

(1)

$$(A + B) \ominus_g (C + D) = \begin{cases} (A \ominus_g C) + (B \ominus_g D), & t_1 t_2 \geq 0, \\ (A \ominus_g C) \ominus_g (-B \ominus_g D), & t_1 t_2 < 0. \end{cases}$$

(2)

$$(A \ominus_g B) + (C \ominus_g D) = \begin{cases} (A \ominus_g (-C)) \ominus_g (B \ominus_g (-D)), & t_1 t_2 \geq 0, t_3 t_4 < 0, \\ (A \ominus_g (-C)) + (-B \ominus_g D), & t_1 t_2 < 0, t_3 t_4 < 0, \\ (A + C) \ominus_g (B + D), & t_3 t_4 \geq 0. \end{cases}$$

(3)

$$(A \ominus_g B) \ominus_g (C \ominus_g D) = \begin{cases} (A \ominus_g C) \ominus_g (B \ominus_g D), & t_1 t_2 \geq 0, t_3 t_4 \geq 0, \\ (A \ominus_g C) + (-B \ominus_g D), & t_1 t_2 < 0, t_3 t_4 \geq 0, \\ (A + (-C)) \ominus_g (B + (-D)), & t_3 t_4 < 0. \end{cases}$$

A function $f: [a, b] \rightarrow \mathbb{R}_I$ is said to be IVF, if $f(x) = [\underline{f}(x), \bar{f}(x)]$ such that $\underline{f}(x) \leq \bar{f}(x)$ for all $x \in [a, b]$. It is well known that $\lim_{x \rightarrow x_0} f(x)$ exist if and only if $\lim_{x \rightarrow x_0} \underline{f}(x)$ and $\lim_{x \rightarrow x_0} \bar{f}(x)$ exist, and is given by

$$\lim_{x \rightarrow x_0} f(x) = \left[\lim_{x \rightarrow x_0} \underline{f}(x), \lim_{x \rightarrow x_0} \bar{f}(x) \right].$$

Particularly, an IVF f is continuous if and only if $\underline{f}, \bar{f}$ are continuous. For two IVFs $f, g: [a, b] \rightarrow \mathbb{R}_I$, we define the IVF $f \ominus_g g: [a, b] \rightarrow \mathbb{R}_I$ by

$$(f \ominus_g g)(x) = f(x) \ominus_g g(x).$$

If $\lim_{x \rightarrow x_0} f(x) = A$ and $\lim_{x \rightarrow x_0} g(x) = B$, then

$$\lim_{x \rightarrow x_0} (f \ominus_g g)(x) = A \ominus_g B.$$

Further more, if $f, g: [a, b] \rightarrow \mathbb{R}_I$ are both continuous, then $f \ominus_g g$ is continuous.

Definition 1 ([1, Definition 20]). Let $f: [a, b] \rightarrow \mathbb{R}_I$ be an IVF and let $x_0 \in [a, b]$. We define $f'(x_0)$ if it exist by

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) \ominus_g f(x_0)}{h},$$

and call it the gH -derivative of f at x_0 . We say that f is gH -differentiable on (a, b) if it is differentiable at every point of (a, b) . The IVF $f': [a, b] \rightarrow \mathbb{R}_I$ is called a gH -derivative of f on $[a, b]$.

Theorem 1 ([16, Theorem 17]). An IVF $f: [a, b] \rightarrow \mathbb{R}_I$ such that $f(x) = [\underline{f}(x), \overline{f}(x)]$ is gH -differentiable at $x \in [a, b]$ if $\underline{f}, \overline{f}$ are differentiable at $x \in [a, b]$ and

$$f'(x) = \left[\min \left\{ \underline{f}'(x), \overline{f}'(x) \right\}, \max \left\{ \underline{f}'(x), \overline{f}'(x) \right\} \right].$$

Let's recall the definition of μ -monotone IVFs given by Markov in [12]:

An IVF $f: [a, b] \rightarrow \mathbb{R}_I$ is μ -increasing (μ -decreasing) on $[a, b]$ if the function $x \mapsto w(f(x))$ is increasing (decreasing) on $[a, b]$. Therefore, f is called μ -monotone on $[a, b]$ if f is μ -decreasing or μ -increasing on $[a, b]$.

Proposition 1 ([4, Proposition 2]). Let $f: [a, b] \rightarrow \mathbb{R}_I$ such that $f(x) = [\underline{f}(x), \overline{f}(x)]$. If f is μ -monotone and gH -differentiable on $[a, b]$, then $\frac{d}{dx}\underline{f}(x)$ and $\frac{d}{dx}\overline{f}(x)$ exist for all $x \in [a, b]$. Moreover, we have

- (i) $f'(x) = [\underline{f}'(x), \overline{f}'(x)]$ for all $t \in [a, b]$, if f is μ -increasing;
- (ii) $f'(x) = [\overline{f}'(x), \underline{f}'(x)]$ for all $t \in [a, b]$, if f is μ -decreasing.

Definition 2 ([12, Definition 4]). The integral of an IVF $f: [a, b] \rightarrow \mathbb{R}_I$ such that $f(x) = [\underline{f}(x), \overline{f}(x)]$ is defined by

$$\int_a^b f(x)dx = \left[\int_a^b \underline{f}(x)dx, \int_a^b \overline{f}(x)dx \right].$$

2.2. q -calculus for real-valued functions

We first present some known Definitions and related inequalities in q -calculus. Set the following notation [8]:

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \cdots + q^{n-1}.$$

Let A be a subset of $\mathbb{R}$ and $0 < q < 1$ be a fixed number. The set A is said to be q -geometric if $qz \in A$ whenever $z \in A$.

For a real-valued function f defined on a q -geometric set A , the q -difference operator D_q is defined by Bermudo in [2]:

$$D_q f(x) = \frac{f(x) - f(qx)}{x - qx}, \text{ for } x \in A \setminus \{0\}.$$

Definition 3 ([2, Definition 4]). For a continuous $f: [a, b] \rightarrow \mathbb{R}$ and $q \in (0, 1)$, the q^b -derivative of f at $x \in [a, b]$ is characterized by the expression

$${}^bD_q f(x) = \frac{f(qx + (1 - q)b) - f(x)}{(1 - q)(b - x)}, \quad x \neq b.$$

For $x = b$, we define ${}^bD_q f(b) = \lim_{x \rightarrow b} D_q f(x)$ if it exist and it is finite.

Definition 4 ([2, Definition 6]). For a continuous $f: [a, b] \rightarrow \mathbb{R}$ and $q \in (0, 1)$, the q^b -integral of f at $x \in [a, b]$ is characterized by the expression

$${}^bI_q f(t) = \int_t^b f(x) {}^b d_q x = (1-q)(b-t) \sum_{n=0}^{\infty} q^n f(q^n t + (1-q^n)b).$$

Theorem 2 ([2, Theorem 12]). Let $f: [a, b] \rightarrow \mathbb{R}$ be a convex function and $q \in (0, 1)$. Then we have

$$f\left(\frac{a+qb}{[2]_q}\right) \leq \frac{1}{b-a} \int_a^b f(x) {}^b d_q x \leq \frac{f(a)+qf(b)}{[2]_q}. \quad (2.1)$$

3. Iq^b -DERIVATIVE FOR IVFs

Now we introduce Iq^b -derivative and corresponding properties.

Definition 5 (Iq^b -derivative). For a continuous IVF $f: [a, b] \rightarrow \mathbb{R}_I^+$, the Iq^b -derivative of f at $x \in [a, b]$ is given as:

$${}^bD_q f(x) = \begin{cases} \frac{f(qx+(1-q)b) \ominus_g f(x)}{(1-q)(b-x)}, & x \neq b, \\ \lim_{x \rightarrow b^-} {}^bD_q f(x), & x = b, \end{cases}$$

where $q \in (0, 1)$ and $D_q f(x) = \frac{f(x) \ominus_g f(qx)}{(1-q)x}$.

Example 1. Consider $f: [0, 1] \rightarrow \mathbb{R}_I$ given by $f(x) = [-x, x]$. It follows that f is Iq^b -differentiable. By Definition 5, for all $x \in [0, 1)$, we have

$$\begin{aligned} {}^bD_q f(x) &= \frac{f(qx+(1-q)b) \ominus_g f(x)}{(1-q)(b-x)} \\ &= \frac{[-qx-(1-q), qx+(1-q)] \ominus_g [-x, x]}{(1-q)(1-x)} \\ &= \frac{[(1-q)(x-1), (1-q)(1-x)] \ominus_g [-x, x]}{(1-q)(1-x)} = [-1, 1], \end{aligned}$$

and for $x = 1$, we get

$$\begin{aligned} {}^bD_q f(1) &= \lim_{x \rightarrow 1^-} {}^bD_q f(x) = \lim_{x \rightarrow 1^-} \frac{f(x) \ominus_g f(qx)}{(1-q)x} \\ &= \lim_{x \rightarrow 1^-} \frac{[-x, x] \ominus_g [-qx, qx]}{(1-q)x} = [-1, 1]. \end{aligned}$$

Theorem 3. An IVF $f: [a, b] \rightarrow \mathbb{R}_I^+$ such that $f(x) = [\underline{f}(x), \bar{f}(x)]$. Then f is Iq^b -differentiable function if and only if $\underline{f}, \bar{f}$ are q^b -differentiable function, and

$${}^bD_q f(x) = \left[\min \left\{ {}^bD_q \underline{f}(x), {}^bD_q \bar{f}(x) \right\}, \max \left\{ {}^bD_q \underline{f}(x), {}^bD_q \bar{f}(x) \right\} \right]. \quad (3.1)$$

Proof. Suppose f is a Iq^b -differentiable function, then there exist $\underline{g}, \bar{g}$ such that

$${}^bD_q f(x) = [\underline{g}(x), \bar{g}(x)].$$

According to Definition 5,

$$\underline{g}(x) = \min \left\{ \frac{\underline{f}(qx + (1-q)b) - \underline{f}(x)}{(1-q)(b-x)}, \frac{\bar{f}(qx + (1-q)b) - \bar{f}(x)}{(1-q)(b-x)} \right\}$$

and

$$\bar{g}(x) = \max \left\{ \frac{\underline{f}(qx + (1-q)b) - \underline{f}(x)}{(1-q)(b-x)}, \frac{\bar{f}(qx + (1-q)b) - \bar{f}(x)}{(1-q)(b-x)} \right\}$$

exist. Then ${}^bD_q \underline{f}(x)$ and ${}^bD_q \bar{f}(x)$ exist, and (3.1) is feasible.

Conversely, if $\underline{f}, \bar{f}$ are q^b -differentiable at x and suppose ${}^bD_q \underline{f}(x) \leq {}^bD_q \bar{f}(x)$, then

$$\begin{aligned} [{}^bD_q \underline{f}(x), {}^bD_q \bar{f}(x)] &= \left[\frac{\underline{f}(qx + (1-q)b) - \underline{f}(x)}{(1-q)(b-x)}, \frac{\bar{f}(qx + (1-q)b) - \bar{f}(x)}{(1-q)(b-x)} \right] \\ &= \frac{f(qx + (1-q)b) \ominus_g f(x)}{(1-q)(b-x)} = {}^bD_q f(x). \end{aligned}$$

So, f is a Iq^b -differentiable IVF.

Similarly, If ${}^bD_q \underline{f}(x) \geq {}^bD_q \bar{f}(x)$, then ${}^bD_q f(x) = [{}^bD_q \bar{f}(x), {}^bD_q \underline{f}(x)]$. $\square$

To illustrate the nature of the derivatives more clearly, we give the following results.

Theorem 4. Let $f: [a, b] \rightarrow \mathbb{R}_I^+$. If f is Iq^b -differentiable on $[a, b]$, then we have

- (1) ${}^bD_q f(x) = [{}^bD_q \underline{f}(x), {}^bD_q \bar{f}(x)]$ for all $x \in [a, b]$ if f is μ -increasing;
- (2) ${}^bD_q f(x) = [{}^bD_q \bar{f}(x), {}^bD_q \underline{f}(x)]$ for all $x \in [a, b]$ if f is μ -decreasing.

Proof. First, we supposed f is μ -increasing and Iq^b -differentiable on $[a, b]$. For all $x \in [a, b]$, we have

$$\begin{aligned} [\bar{f}(qx + (1-q)b) - \underline{f}(qx + (1-q)b)] - [\bar{f}(x) - \underline{f}(x)] &> 0, \\ \bar{f}(qx + (1-q)b) - \bar{f}(x) &> \underline{f}(qx + (1-q)b) - \underline{f}(x). \end{aligned}$$

Therefore,

$$\begin{aligned} {}^bD_q f(x) &= \frac{[\bar{f}(qx + (1-q)b) - \underline{f}(qx + (1-q)b)] \ominus_g [\bar{f}(x) - \underline{f}(x)]}{(1-q)(b-x)} \\ &= \left[\frac{\underline{f}(qx + (1-q)b) - \underline{f}(x)}{(1-q)(b-x)}, \frac{\bar{f}(qx + (1-q)b) - \bar{f}(x)}{(1-q)(b-x)} \right] \\ &= [{}^bD_q \underline{f}(x), {}^bD_q \bar{f}(x)]. \end{aligned}$$

The other condition can be similarly proved. $\square$

Example 2. Furthermore, by Example 1, we have

$$w(f(x)) = 2x, \quad x \in [0, 1].$$

So, f is a μ -increasing IVF.

On the other hand,

$$\begin{aligned} {}^bD_q \underline{f}(x) &= \frac{\underline{f}(qx + (1-q)b) - \underline{f}(x)}{(1-q)(b-x)} = \frac{(1-q)(x-1)}{(1-q)(1-x)} = -1, \\ {}^bD_q \bar{f}(x) &= \frac{\bar{f}(qx + (1-q)b) - \bar{f}(x)}{(1-q)(b-x)} = \frac{(1-q)(1-x)}{(1-q)(1-x)} = 1. \end{aligned}$$

Then, we obtained

$${}^bD_q f(x) = [-1, 1] = [{}^bD_q \underline{f}(x), {}^bD_q \bar{f}(x)].$$

Theorem 5. Let $f, g: [a, b] \rightarrow \mathbb{R}_I^+$ be Iq^b -differentiable IVFs. For all $x \in [a, b]$, $q \in (0, 1)$, consider

$$\begin{aligned} t_1 &= w(f(qx + (1-q)b)) - w(f(x)), \\ t_2 &= w(g(qx + (1-q)b)) - w(g(x)), \\ t_3 &= w(f(qx + (1-q)b)) - w(g(qx + (1-q)b)), \\ t_4 &= w(f(x)) - w(g(x)). \end{aligned}$$

(1) The sum $f + g: [a, b] \rightarrow \mathbb{R}_I^+$ is a Iq^b -differentiable IVF with

$${}^bD_q (f + g)(x) = \begin{cases} {}^bD_q f(x) + {}^bD_q g(x), & t_1 t_2 \geq 0, \\ {}^bD_q f(x) \ominus_g (-{}^bD_q g(x)), & t_1 t_2 < 0. \end{cases}$$

(2) For any constant $\lambda \in \mathbb{R}$, $\lambda f: [a, b] \rightarrow \mathbb{R}_I^+$ is a Iq^b -differentiable IVF with

$${}^bD_q \lambda f(x) = \lambda {}^bD_q f(x).$$

(3) The difference $f \ominus_g g: [a, b] \rightarrow \mathbb{R}_I^+$ is a Iq^b -differentiable IVF with

$${}^bD_q (f \ominus_g g)(x) = \begin{cases} {}^bD_q f(x) \ominus_g {}^bD_q g(x), & t_1 t_2 \geq 0, t_3 t_4 \geq 0, \\ {}^bD_q f(x) + (-{}^bD_q g(x)), & t_1 t_2 < 0, t_3 t_4 \geq 0. \end{cases}$$

Proof.

(1) By Definition 5, we have

$$\begin{aligned} {}^bD_q (f + g)(x) &= \frac{(f + g)(qx + (1-q)b) \ominus_g (f + g)(x)}{(1-q)(b-x)} \\ &= \frac{(f(qx + (1-q)b) + g(qx + (1-q)b)) \ominus_g (f(x) + g(x))}{(1-q)(b-x)} \end{aligned}$$

$$\begin{aligned}
&= \begin{cases} \frac{(f(qx+(1-q)b) \ominus_g f(x)) + (g(qx+(1-q)b) \ominus_g g(x))}{(1-q)(b-x)}, & t_1 t_2 \geq 0, \\ \frac{(f(qx+(1-q)b) \ominus_g f(x)) \ominus_g (-g(qx+(1-q)b) \ominus_g g(x))}{(1-q)(b-x)}, & t_1 t_2 < 0, \end{cases} \\
&= \begin{cases} \frac{(f(qx+(1-q)b) \ominus_g f(x))}{(1-q)(b-x)} + \frac{(g(qx+(1-q)b) \ominus_g g(x))}{(1-q)(b-x)}, & t_1 t_2 \geq 0, \\ \frac{(f(qx+(1-q)b) \ominus_g f(x))}{(1-q)(b-x)} \ominus_g \frac{(-g(qx+(1-q)b) \ominus_g g(x))}{(1-q)(b-x)}, & t_1 t_2 < 0, \end{cases} \\
&= \begin{cases} {}^b D_q f(x) + {}^b D_q g(x), & t_1 t_2 \geq 0, \\ {}^b D_q f(x) \ominus_g (-{}^b D_q g(x)), & t_1 t_2 < 0. \end{cases}
\end{aligned}$$

(2) By Definition 5, we have

$$\begin{aligned}
{}^b D_q (\lambda f)(x) &= \frac{(\lambda f)(qx + (1-q)b) \ominus_g (\lambda f)(x)}{(1-q)(b-x)} \\
&= \lambda \frac{f(qx + (1-q)b) \ominus_g f(x)}{(1-q)(b-x)} \\
&= \lambda {}^b D_q f(x).
\end{aligned}$$

(3) By Definition 5, we have

$$\begin{aligned}
{}^b D_q (f \ominus_g g)(x) &= \frac{(f \ominus_g g)(qx + (1-q)b) \ominus_g (f \ominus_g g)(x)}{(1-q)(b-x)} \\
&= \begin{cases} \frac{(f(qx+(1-q)b) \ominus_g f(x)) \ominus_g (g(qx+(1-q)b) \ominus_g g(x))}{(1-q)(b-x)}, & t_1 t_2 \geq 0, t_3 t_4 \geq 0, \\ \frac{(f(qx+(1-q)b) \ominus_g f(x)) + (-g(qx+(1-q)b) \ominus_g g(x))}{(1-q)(b-x)}, & t_1 t_2 < 0, t_3 t_4 \geq 0, \end{cases} \\
&= \begin{cases} {}^b D_q f(x) \ominus_g {}^b D_q g(x), & t_1 t_2 \geq 0, t_3 t_4 \geq 0, \\ {}^b D_q f(x) + (-{}^b D_q g(x)), & t_1 t_2 < 0, t_3 t_4 \geq 0. \end{cases}
\end{aligned}$$

□

4. Iq^b -INTEGRAL FOR INTERVAL-VALUED FUNCTIONS

In this section, we present the concepts of Iq^b -integral for IVFs and give some properties.

Definition 6 (Iq^b -integral). Let $f: [a, b] \rightarrow \mathbb{R}_I$ be a continuous IVF, the definite integral Iq^b -integral of f on $[a, b]$ is given by

$${}^b I_q f(x) = \int_x^b f(t) {}^b d_q^I t = (1-q)(b-x) \sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)b).$$

Theorem 6. Let $f, g: [a, b] \rightarrow \mathbb{R}_I$ are continuous IVFs and $\lambda \in \mathbb{R}$, we have the following properties:

- (1) ${}^b I_q(f+g)(x) = {}^b I_q f(x) + {}^b I_q g(x)$,
- (2) ${}^b I_q(\lambda f)(x) = \lambda {}^b I_q f(x)$.

Proof. By Definition 6, we have

$$\begin{aligned} {}^b I_q(f+g)(x) &= (1-q)(b-x) \sum_{n=0}^{\infty} q^n (f+g)(q^n x + (1-q^n)b) \\ &= (1-q)(b-x) \sum_{n=0}^{\infty} q^n (f(q^n x + (1-q^n)b) + g(q^n x + (1-q^n)b)) \\ &= (1-q)(b-x) \sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)b) \\ &\quad + (1-q)(b-x) \sum_{n=0}^{\infty} q^n g(q^n x + (1-q^n)b) \\ &= {}^b I_q f(x) + {}^b I_q g(x), \end{aligned}$$

and

$$\begin{aligned} {}^b I_q(\lambda f)(x) &= (1-q)(b-x) \sum_{n=0}^{\infty} q^n (\lambda f)(q^n x + (1-q^n)b) \\ &= \lambda (1-q)(b-x) \sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)b) = \lambda {}^b I_q f(x). \end{aligned}$$

□

Theorem 7. Let $f: [a, b] \rightarrow \mathbb{R}_I$ be a continuous IVF, then f is Iq^b -integral on $[a, b]$ if and only if $\underline{f}$ and $\bar{f}$ is q^b -integral on $[a, b]$, and

$${}^b I_q f(x) = \left[{}^b I_q \underline{f}(x), {}^b I_q \bar{f}(x) \right].$$

Proof. The proof can be obtained by combining Definitions 4 and 6 and hence is omitted. □

Example 3. Let $f: [0, 1] \rightarrow \mathbb{R}_I$ be given by

$$f(x) = [x^2, x].$$

For $0 < q < 1$, we have

$$\begin{aligned} \int_0^1 f(x) {}^1 d_q^I x &= \left[\int_0^1 x^2 {}^1 d_q^I x, \int_0^1 x {}^1 d_q^I x \right] \\ &= \left[(1-q) \sum_{n=0}^{\infty} q^n (1-q^n)^2, (1-q) \sum_{n=0}^{\infty} q^n (1-q^n) \right] = \left[\frac{q(1+q^2)}{[3]_q}, \frac{q}{[2]_q} \right]. \end{aligned}$$

Theorem 8. Let $f, g: [a, b] \rightarrow \mathbb{R}_I$ are continuous IVFs, then

$$\int_a^b f(x) {}^b d_q^I x \ominus_g \int_a^b g(x) {}^b d_q^I x \subseteq \int_a^b (f(x) \ominus_g g(x)) {}^b d_q^I x.$$

Moreover, if $w(f(x)) - w(g(x))$ has a constant sign on $[a, b]$, then

$$\int_a^b f(x) {}^b d_q^I x \ominus_g \int_a^b g(x) {}^b d_q^I x = \int_a^b (f(x) \ominus_g g(x)) {}^b d_q^I x.$$

Proof. First, we have

$$\begin{aligned} & \int_a^b \min [\underline{f}(x) - \underline{g}(x), \bar{f}(x) - \bar{g}(x)] {}^b d_q^I x \\ & \leq \min \left\{ \int_a^b (\underline{f}(x) - \underline{g}(x)) {}^b d_q^I x, \int_a^b (\bar{f}(x) - \bar{g}(x)) {}^b d_q^I x \right\} \\ & \leq \max \left\{ \int_a^b (\underline{f}(x) - \underline{g}(x)) {}^b d_q^I x, \int_a^b (\bar{f}(x) - \bar{g}(x)) {}^b d_q^I x \right\} \\ & \leq \int_a^b \max [\underline{f}(x) - \underline{g}(x), \bar{f}(x) - \bar{g}(x)] {}^b d_q^I x. \end{aligned}$$

This implies that

$$\begin{aligned} & \int_a^b f(x) {}^b d_q^I x \ominus_g \int_a^b g(x) {}^b d_q^I x \\ & = \left[\min \left\{ \int_a^b (\underline{f}(x) - \underline{g}(x)) {}^b d_q^I x, \int_a^b (\bar{f}(x) - \bar{g}(x)) {}^b d_q^I x \right\}, \right. \\ & \quad \left. \max \left\{ \int_a^b (\underline{f}(x) - \underline{g}(x)) {}^b d_q^I x, \int_a^b (\bar{f}(x) - \bar{g}(x)) {}^b d_q^I x \right\} \right] \\ & \subseteq \left[\int_a^b \min \{ \underline{f}(x) - \underline{g}(x), \bar{f}(x) - \bar{g}(x) \} {}^b d_q^I x, \right. \\ & \quad \left. \int_a^b \max \{ \underline{f}(x) - \underline{g}(x), \bar{f}(x) - \bar{g}(x) \} {}^b d_q^I x \right] \\ & = \int_a^b (f(x) \ominus_g g(x)) {}^b d_q^I x. \end{aligned}$$

Moreover, we assume that $w(f(x)) - w(g(x)) \geq 0$, $x \in [a, b]$, then

$$f(x) \ominus_g g(x) = [\underline{f}(x) - \underline{g}(x), \bar{f}(x) - \bar{g}(x)].$$

So, we have

$$\int_a^b (\underline{f}(x) - \underline{g}(x)) {}^b d_q^I x \leq \int_a^b (\bar{f}(x) - \bar{g}(x)) {}^b d_q^I x.$$

This implies that

$$\int_a^b f(x) {}^b d_q^I x \ominus_g \int_a^b g(x) {}^b d_q^I x$$

$$\begin{aligned}
&= \left[\min \left\{ \int_a^b (\underline{f}(x) - \underline{g}(x)) {}^b d_q^I x, \int_a^b (\overline{f}(x) - \overline{g}(x)) {}^b d_q^I x \right\}, \right. \\
&\quad \left. \max \left\{ \int_a^b (\underline{f}(x) - \underline{g}(x)) {}^b d_q^I x, \int_a^b (\overline{f}(x) - \overline{g}(x)) {}^b d_q^I x \right\} \right] \\
&= \left[\int_a^b \underline{f}(x) {}^b d_q^I x, \int_a^b \overline{f}(x) {}^b d_q^I x \right] \ominus_g \left[\int_a^b \underline{g}(x) {}^b d_q^I x, \int_a^b \overline{g}(x) {}^b d_q^I x \right] \\
&= \int_a^b (f(x) \ominus_g g(x)) {}^b d_q^I x.
\end{aligned}$$

□

Example 4. Let $f, g, h: [0, 1] \rightarrow \mathbb{R}_I$ are given by

$$f(x) = [-x, x], \quad g(x) = [-x^2, x^2], \quad h(x) = [-2x^2, 2x^2].$$

We have $w(f(x)) - w(g(x)) = 2x(1-x) \geq 0$, $x \in [0, 1]$ and

$$\begin{aligned}
f(x) \ominus_g g(x) &= [-x + x^2, x - x^2], \\
f(x) \ominus_g h(x) &= \begin{cases} [2x^2 - x, x - 2x^2], & x \in [0, \frac{1}{2}], \\ [x - 2x^2, 2x^2 - x], & x \in [\frac{1}{2}, 1]. \end{cases}
\end{aligned}$$

Then, we obtained that

$$\begin{aligned}
\int_0^1 f(x) {}^b d_q^I x \ominus_g \int_0^1 g(x) {}^b d_q^I x &= \left[-\frac{q^2}{[2]_q[3]_q}, \frac{q^2}{[2]_q[3]_q} \right], \\
\int_0^1 f(x) {}^b d_q^I x \ominus_g \int_0^1 h(x) {}^b d_q^I x &= \left[-\frac{q(1-q+q^2)}{[2]_q[3]_q}, \frac{q(1-q+q^2)}{[2]_q[3]_q} \right], \\
\int_0^1 (f(x) \ominus_g g(x)) {}^b d_q^I x &= \left[-\frac{q^2}{[2]_q[3]_q}, \frac{q^2}{[2]_q[3]_q} \right], \\
\int_0^1 (f(x) \ominus_g h(x)) {}^b d_q^I x &= \frac{1}{2} \left[-\frac{q}{[2]_q}, \frac{q}{[2]_q} \right].
\end{aligned}$$

Hence,

$$\int_0^1 f(x) {}^b d_q^I x \ominus_g \int_0^1 g(x) {}^b d_q^I x = \int_0^1 (f(x) \ominus_g g(x)) {}^b d_q^I x$$

and

$$\int_0^1 f(x) {}^b d_q^I x \ominus_g \int_0^1 h(x) {}^b d_q^I x \subseteq \int_0^1 (f(x) \ominus_g h(x)) {}^b d_q^I x.$$

Theorem 9. Let $f: [a, b] \rightarrow \mathbb{R}_I$. If f is Iq^b -differentiable on $[a, b]$, then ${}^b D_q f(x)$ is Iq^b -integrable. Moreover, if f is μ -monotone on $[a, b]$, then

$$f(c) \ominus_g f(x) = \int_x^c {}^b D_q f(s) {}^b d_q^I s \quad \text{for all } s \in [x, b]. \quad (4.1)$$

Proof. If f is Iq^b -integrable on $[a, b]$, then from Theorem 7 it follows that $\underline{f}$ and $\bar{f}$ are q^b -integrable. Hence, ${}^bD_q \underline{f}(x)$ and ${}^bD_q \bar{f}(x)$ are q^b -integrable. Therefore, Theorem 3 imply that ${}^bD_q f(x)$ is Iq^b -integrable.

If f is μ -increasing on $[a, b]$, then

$${}^bD_q f(x) = \left[{}^bD_q \underline{f}(x), {}^bD_q \bar{f}(x) \right],$$

for all $x \in [a, b]$. Then we have that

$$\underline{f}(c) - \underline{f}(x) = \int_x^c {}^bD_q \underline{f}(s) {}^b d_q s, \quad \bar{f}(c) - \bar{f}(x) = \int_x^c {}^bD_q \bar{f}(s) {}^b d_q s.$$

It follows that

$$f(c) = f(x) + \int_x^c {}^bD_q f(s) {}^b d_q^I s.$$

Since f is μ -increasing on $[a, b]$, we have

$$f(c) \ominus_g f(x) = \int_x^c {}^bD_q f(s) {}^b d_q^I s.$$

If f is μ -decreasing on $[a, b]$, then

$${}^bD_q f(x) = \left[{}^bD_q \bar{f}(x), {}^bD_q \underline{f}(x) \right],$$

for all $x \in [a, b]$. Then we get that

$$\begin{aligned} \int_x^c {}^bD_q f(s) {}^b d_q^I s &= \left[\int_x^c {}^bD_q \bar{f}(s) {}^b d_q s, \int_x^c {}^bD_q \underline{f}(s) {}^b d_q s \right] \\ &= [\underline{f}(c) - \underline{f}(x), \bar{f}(c) - \bar{f}(x)] \\ &= [\underline{f}(c), \bar{f}(c)] \ominus_g [\underline{f}(x), \bar{f}(x)] = f(c) \ominus_g f(x). \end{aligned}$$

□

Remark 1. If f is μ -increasing on $[a, b]$, then (4.1) is equivalent with

$$f(c) = f(x) + \int_x^c {}^bD_q f(s) {}^b d_q^I s,$$

and if f is μ -decreasing on $[a, b]$, then (4.1) is equivalent with

$$f(x) = f(c) + \left(- \int_x^c {}^bD_q f(s) {}^b d_q^I s \right).$$

5. Iq^b -HERMITE-HADAMARD INEQUALITIES

Now we review the definition of interval convex function.

Definition 7. [20, Definition 5] Let $f: [a, b] \rightarrow \mathbb{R}_I^+$ be an IVF. We say that f is interval convex function or that $f \in SX([a, b], \mathbb{R}_I^+)$, if for all $x, y \in [a, b]$ and $t \in [0, 1]$, we have

$$f(tx + (1-t)y) \supseteq tf(x) + (1-t)f(y).$$

Next, we prove the Hermite-Hadamard type inequalities for Iq^b -integrable IVFs.

Theorem 10. Let $f: [a, b] \rightarrow \mathbb{R}_I^+$ be a gH -differentiable convex IVF over $[a, b]$, then the following inequalities hold for Iq^b -integral:

$$f\left(\frac{a+qb}{[2]_q}\right) \supseteq \frac{1}{b-a} \int_a^b f(x) {}^b d_q^I x \supseteq \frac{f(a) + qf(b)}{[2]_q}, \quad (5.1)$$

where $q \in (0, 1)$.

Proof. We observe that

$$\frac{a+qb}{[2]_q} = \sum_{n=0}^{\infty} (1-q)q^n (q^n a + (1-q^n)b),$$

where $\sum_{n=0}^{\infty} (1-q)q^n = 1$. Thus, by the convexity of f , Jensen's inequality implies

$$f\left(\frac{a+qb}{[2]_q}\right) \supseteq \sum_{n=0}^{\infty} (1-q)q^n f(q^n a + (1-q^n)b) = \frac{1}{b-a} \int_a^b f(x) {}^b d_q^I x$$

and the first inequality in (5.1) is proved.

Now, for the second inequality in (5.1), we suppose that

$$s(x) = f(b) + \frac{f(b) - f(a)}{b-a}(x-b) = \frac{f(b) - f(a)}{b-a}x + \frac{bf(a) - af(b)}{b-a}$$

and for the convexity of f , we have

$$f(x) \supseteq s(x).$$

Hence by using Iq^b -integration, we have

$$\int_a^b f(x) {}^b d_q^I x \supseteq \int_a^b s(x) {}^b d_q^I x = (b-a) \left[\frac{f(a) + qf(b)}{[2]_q} \right]$$

and the second inequality in (5.1) is also proved. The proof is completed. $\square$

Lemma 1. If $f(x) = [\underline{f}(x), \bar{f}(x)]$ and $\underline{f}(x) = \bar{f}(x)$ for all $x \in [a, b]$, then we get the (2.1). If in (5.1), we get the classical Hermite-Hadamard inequality.

Example 5. Let $f: [0, 1] \rightarrow \mathbb{R}_I^+$ is given by $f(x) = [x^2, x]$, then f is interval convex function, and

$$\begin{aligned} f\left(\frac{a+qb}{[2]_q}\right) &= \left[\frac{q^2}{[2]_q^2}, \frac{q}{[2]_q}\right], \\ \frac{1}{b-a} \int_a^b f(x) {}^b d_q^I x &= \left[\frac{q(1+q^2)}{[2]_q[3]_q}, \frac{q}{[2]_q}\right], \\ \frac{f(a)+qf(b)}{[2]_q} &= \left[\frac{q}{[2]_q}, \frac{q}{[2]_q}\right]. \end{aligned}$$

Then, we obtain that

$$\left[\frac{q^2}{[2]_q^2}, \frac{q}{[2]_q}\right] \supseteq \left[\frac{q(1+q^2)}{[2]_q[3]_q}, \frac{q}{[2]_q}\right] \supseteq \left[\frac{q}{[2]_q}, \frac{q}{[2]_q}\right].$$

Consequently, Theorem 10 is verified.

Summing up the results in Theorem 10 and Theorem 5.3 of [11] yields the next corollary.

Corollary 1. Let $f: [a, b] \rightarrow \mathbb{R}_I^+$ be a gH -differentiable convex IVF over $[a, b]$, then we have

$$\begin{aligned} f\left(\frac{qa+b}{[2]_q}\right) + f\left(\frac{a+qb}{[2]_q}\right) &\supseteq \frac{1}{b-a} \int_a^b f(x) {}^b d_q^I x + \frac{1}{b-a} \int_a^b f(x) {}^a d_q^I x \\ &\supseteq f(a) + f(b) \end{aligned} \quad (5.2)$$

where $q \in (0, 1)$.

Corollary 2. Let $f: [a, b] \rightarrow \mathbb{R}_I^+$ be a gH -differentiable convex IVF over $[a, b]$, then we have

$$2f\left(\frac{a+b}{2}\right) \supseteq \frac{1}{b-a} \left\{ \int_a^b f(x) {}^b d_q^I x + \int_a^b f(x) {}^a d_q^I x \right\} \supseteq f(a) + f(b), \quad (5.3)$$

where $q \in (0, 1)$.

Proof. By Corollary 1, it is enough to see that by the convexity of f ,

$$f\left(\frac{a+b}{2}\right) = f\left(\frac{1}{2} \frac{qa+b}{[2]_q} + \frac{1}{2} \frac{a+qb}{[2]_q}\right) \supseteq \frac{1}{2} f\left(\frac{qa+b}{[2]_q}\right) + \frac{1}{2} f\left(\frac{a+qb}{[2]_q}\right).$$

□

Remark 2. If in (5.2) or (5.3), we make $q \rightarrow 1$, we get the classical Hermite-Hadamard inequality.

Theorem 11. Let $f: [a, b] \rightarrow \mathbb{R}_I^+$ be a gH -differentiable convex IVF over $[a, b]$, then the following inequalities hold for Iq^b -integral:

$$\begin{aligned} f\left(\frac{qa+b}{[2]_q}\right) + \frac{(1-q)(b-a)}{[2]_q} {}^bD_q f\left(\frac{qa+b}{[2]_q}\right) &\supseteq \frac{1}{b-a} \int_a^b f(x) {}^b d_q^I x \\ &\supseteq \frac{f(a) + qf(b)}{[2]_q}, \end{aligned}$$

where $q \in (0, 1)$.

Proof. According to the Iq^b -differentiability of f on $[a, b]$, there are two tangents at the point $\frac{qa+b}{1+q} \in (a, b)$, and their equations are

$$h_2(x) = \underline{f}\left(\frac{qa+b}{[2]_q}\right) + \frac{(1-q)(b-a)}{1+q} {}^bD_q \underline{f}\left(\frac{qa+b}{[2]_q}\right)$$

and

$$\bar{h}_2(x) = \bar{f}\left(\frac{qa+b}{[2]_q}\right) + \frac{(1-q)(b-a)}{1+q} {}^bD_q \bar{f}\left(\frac{qa+b}{[2]_q}\right).$$

Since $f \in SX([a, b], \mathbb{R}_I)$, we have

$$h_2(x) \supseteq f(x),$$

for all $x \in [a, b]$. By Iq^b -integrating this inequality with respect to x on $[a, b]$ we have

$$\begin{aligned} &\int_a^b h_2(x) {}^b d_q^I x \\ &= \int_a^b \left[f\left(\frac{qa+b}{[2]_q}\right) + {}^bD_q f\left(\frac{qa+b}{[2]_q}\right) \left(x - \frac{qa+b}{[2]_q}\right) \right] {}^b d_q^I x \\ &= (b-a) f\left(\frac{qa+b}{[2]_q}\right) + {}^bD_q f\left(\frac{qa+b}{[2]_q}\right) \left(\int_a^b x {}^b d_q^I x - (b-a) \frac{qa+b}{[2]_q} \right) \\ &= (b-a) f\left(\frac{qa+b}{[2]_q}\right) + {}^bD_q f\left(\frac{qa+b}{[2]_q}\right) \\ &\quad \times \left((1-q)(b-a) \sum_{n=0}^{\infty} q^n (q^n a + (1-q^n)b) - (b-a) \frac{qa+b}{[2]_q} \right) \\ &= (b-a) f\left(\frac{qa+b}{[2]_q}\right) + {}^bD_q f\left(\frac{qa+b}{[2]_q}\right) \\ &\quad \times \left((1-q)(b-a) \left[\left(\frac{1}{1-q} - \frac{1}{1-q^2} \right) a + \frac{1}{1-q^2} b \right] - (b-a) \frac{a+qb}{[2]_q} \right) \\ &= (b-a) f\left(\frac{qa+b}{[2]_q}\right) + {}^bD_q f\left(\frac{qa+b}{[2]_q}\right) \left((b-a) \frac{a+qb}{[2]_q} - (b-a) \frac{qa+b}{[2]_q} \right) \end{aligned}$$

$$\begin{aligned}
&= (b-a)f\left(\frac{a+qb}{[2]_q}\right) + {}^bD_q f\left(\frac{qa+b}{[2]_q}\right) \frac{(b-a)^2(1-q)}{1+q} \\
&\supseteq \int_a^b f(x) {}^b d_q^I x.
\end{aligned}$$

Combining the above formula with (5), we come to the conclusion. $\square$

CONCLUSIONS

We introduced the concept of Iq^b -calculus of interval-valued functions and studied their important properties. Furthermore, we gave Hermite-Hadamard-type inequalities by using these results. Our results generalized some existing theories of quantum calculus. Next, we intend to further study some applications of quantum calculus and quantum calculus of fuzzy interval-valued functions.

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CAYLEY LINE GRAPHS OF TRANSITIVE GROUPOIDS

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Abstract. A groupoid is a small category with inverses. Adding appropriate colors to the edges of the line graph of a transitive groupoid creates a Cayley line graph of the groupoid. The groupoid of partial automorphisms of the Cayley line graph is isomorphic to a semidirect product of the original groupoid. Using the trivial coloring to build the Cayley line graph makes the semidirect product trivial, hence the groupoid of partial automorphisms of this Cayley line graph is isomorphic to the original groupoid.

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1. INTRODUCTION

The Cayley color graph of a group is an important object both in group theory and in graph theory. A comprehensive reference can be found for example in [6]. The line graph of the Cayley color graph is studied in [2–4]. In particular, it is shown in [4] that the edges of the line graph of a Cayley color graph of a group can be colored so that the automorphism group of the resulting color digraph is isomorphic to a semidirect product of the group.

A groupoid is a small category with inverses. The notion of Cayley color graph of a group is extended to inverse semigroups and groupoids in [9], where it is shown that the groupoid of partial automorphisms of the Cayley color graph of a transitive groupoid is isomorphic to the original groupoid. In this paper we study the line graph of the Cayley color graph of a transitive groupoid and extend the above mentioned result of [4] to transitive groupoids.

In Section 2 we recall definitions and results from [9] concerning the Cayley color graph of a groupoid and give a few examples. In Section 3 we define a Cayley line graph as a special edge coloring of the line graph of the Cayley graph. The Cayley line graph might not be unique, we can color the edges more than one way. In Section 4 we show that the groupoid of partial automorphisms of the Cayley line

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graph is isomorphic to a semidirect product of the original groupoid. The semidirect product depends on the choice of the coloring, but the coloring always can be chosen trivially to make this semidirect product isomorphic to the original groupoid.

2. PRELIMINARIES

A *groupoid* $\mathcal{G}$ is a set with a subset $\mathcal{G}^{(2)}$ of $\mathcal{G} \times \mathcal{G}$, a product map $(x, y) \mapsto xy: \mathcal{G}^{(2)} \rightarrow \mathcal{G}$ and an inverse map $x \mapsto x^{-1}: \mathcal{G} \rightarrow \mathcal{G}$ such that:

- (a) $(xy)z = x(yz)$ for all $(x, y), (y, z) \in \mathcal{G}^{(2)}$;
- (b) $(x, x^{-1}) \in \mathcal{G}^{(2)}$ for all $x \in \mathcal{G}$;
- (c) $x^{-1}(xy) = y, (xy)y^{-1} = x$ for all $(x, y) \in \mathcal{G}^{(2)}$.

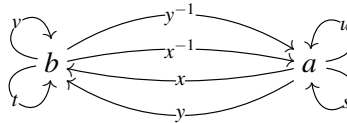
The set $\mathcal{G}^{(2)}$ is called the set of *composable pairs*. The *domain* and *range* maps $d, r: \mathcal{G} \rightarrow U$ are defined by $d(x) = x^{-1}x$ and $r(x) = xx^{-1}$, where $U = \{xx^{-1} \mid x \in \mathcal{G}\}$ is the set of *units* of $\mathcal{G}$.

A groupoid $\mathcal{G}$ is called *transitive* if for all $u, v \in U$ there is an element $g \in \mathcal{G}$ satisfying $d(g) = u$ and $r(g) = v$. This means $\mathcal{G}$ is connected if viewed as a small category with inverses. Every groupoid is the disjoint union of transitive groupoids, and every transitive groupoid is the direct product of a group G and a trivial groupoid $A \times A$. More precisely, a transitive groupoid $\mathcal{G}$ is isomorphic to a groupoid $A \times G \times A$ where (d, h, c) and (b, g, a) are composable whenever $b = c$, in which case their product is (d, hg, a) . The inverse of (b, g, a) is (a, g^{-1}, b) . The set A can be identified with the unit space $\{(a, e, a) \mid a \in A\}$ of $\mathcal{G}$ where e is the identity of G . The group G is isomorphic to the isotropy subgroup $\mathcal{G}_u^u = \{x \mid d(x) = u = r(x)\}$ for any unit u of $\mathcal{G}$. We only work with finite groupoids. Our references for groupoids are [1, 7, 8].

Example 1. If $A = \{a, b\}$ then the transitive groupoid $\mathcal{G} = A \times \mathbb{Z}_2 \times A$ has eight elements

$$\mathcal{G} = \{x, y, x^{-1}, y^{-1}, u = x^{-1}x, v = xx^{-1}, s = y^{-1}y, t = yy^{-1}\}$$

where $x = (b, 0, a)$ and $y = (b, 1, a)$. The groupoid can be visualized by an arrow diagram.



A subset Δ of the groupoid $\mathcal{G}$ *generates* $\mathcal{G}$ if every element of $\mathcal{G}$ can be written as a finite product of elements of Δ .

A *color digraph* is a directed graph with possible multiple edges and loops, together with a color function defined on the set of edges.

The *tail* of a vertex v in a color digraph is the set $\text{tail}(v)$ of vertices that can be reached by a finite directed walk starting at v . We say that v is a *head* of its tail. Note that a tail may have more than one head, and a tail contains each of its heads.

If Δ is a set of generators of the groupoid $\mathcal{G}$, then the *Cayley color graph* [9] $D = D_\Delta(\mathcal{G})$ of $\mathcal{G}$ with respect to the generating set Δ is the color digraph with vertices $\mathcal{G}$ and edges

$$\{(x, z) \mid x \in \mathcal{G}, z \in \Delta, (x, z) \in \mathcal{G}^{(2)}\}$$

such that the edge (x, z) connects x to xz and has color z . See [5] for another version of groupoid Cayley graphs.

Example 2. If $\mathcal{G} = A \times \mathbb{Z}_2 \times A$ is the transitive groupoid of Example 1, then $\Delta = \{x, x^{-1}, s\}$ is a generating set and the Cayley color graph has two tails:

$$x^{-1} \xrightarrow[x^{-1}]{x} u \xrightarrow[s]{s} s \xrightarrow[x]{x^{-1}} y^{-1} \qquad v \xrightarrow[x^{-1}]{x} x \xrightarrow[s]{s} y \xrightarrow[x]{x^{-1}} t$$

If x is a vertex of $D_\Delta(\mathcal{G})$, then $\text{tail}(x) = \{xy \mid y \in \mathcal{G}\}$ and $r(x)$ is the unique unit u of $\mathcal{G}$ for which $\text{tail}(x) = \text{tail}(u)$. Every tail is strongly connected, and every element of a tail is a head of the tail. A tail contains all the elements of the groupoid whose range is a given unit, so the number of tails is equal to the number of units.

A *partial automorphism* of a Cayley color graph $D = D_\Delta(\mathcal{G})$ is a bijection between two tails of the graph that preserves the colors of the edges. Every partial automorphism of D is implemented by a left multiplication by an element of $\mathcal{G}$ and this representation gives an isomorphism between the partial automorphisms of D and $\mathcal{G}$ (see [9]).

If $\mathcal{G} = A \times G \times A$ and T is a tail of D , then the partial automorphisms of D whose domain and range is T form a group isomorphic to G .

A *color permuting partial automorphism* of a Cayley color graph $D_\Delta(\mathcal{G})$ is a bijection α between two tails of $D_\Delta(\mathcal{G})$ and a permutation ρ of Δ such that $\alpha(xz) = \alpha(x)\rho(z)$ for all $x \in \text{dom}(\alpha)$, $(x, z) \in \mathcal{G}^{(2)}$ and $z \in \Delta$.

Let $H = \{\pi \in \text{Aut}(\mathcal{G}) \mid \pi(\Delta) = \Delta\}$ be the group containing the automorphisms of $\mathcal{G}$ preserving Δ . Let $\iota: H \rightarrow \text{Aut}(\mathcal{G})$ be the canonical embedding. Recall [8] that the semidirect product $\mathcal{G} \times_1 H$ is the groupoid with operations

$$(x_1, \pi_1)(x_2, \pi_2) = (x_1\pi_1(x_2), \pi_1\pi_2) \quad \text{and} \quad (x, \pi)^{-1} = (\pi^{-1}(x^{-1}), \pi^{-1})$$

whenever x_1 and $\pi_1(x_2)$ are composable. The map $(y, \rho) \mapsto \beta_{(y, \rho)}$ where $\beta_{(y, \rho)}(x) = y\rho(x)$ is an isomorphism between the semidirect product $\mathcal{G} \times_1 H$ and the groupoid $\text{PAut}^*(D)$ of color permuting partial automorphisms of $D_\Delta(\mathcal{G})$.

3. CAYLEY LINE GRAPHS

Definition 1. Let $D = D_\Delta(\mathcal{G})$ be a Cayley color graph and π_z be a permutation of Δ for all $z \in \Delta$ satisfying $\pi_z \circ \pi_w = \pi_{\pi_z(w)}$ for all $w, z \in \Delta$. The *Cayley line graph* $L = L_\pi(D)$ with respect to π is the color digraph with vertices $\{(x, z) \mid x \in \mathcal{G}, z \in \Delta, (x, z) \in \mathcal{G}^{(2)}\}$ and edges

$$\{((x, z), w) \mid x \in \mathcal{G} \text{ and } z, w \in \Delta \text{ and } (x, z), (z, \pi_z(w)) \in \mathcal{G}^{(2)}\}$$

such that the edge $((x, z), w)$ connects vertex (x, z) to vertex $(xz, \pi_z(w))$ and has color w .

The correspondence can be visualized by the following diagram where the color is written next to the edge, and the name is written on the edge:

$$\begin{array}{lcl}
 \text{groupoid } \mathcal{G} : & \bullet \xleftarrow{x} \bullet & \xleftarrow{z_1} \bullet \xleftarrow{z_2} \bullet \\
 \text{Cayley graph } D : & x & \xrightarrow{z_1} xz_1 \xrightarrow{z_2} xz_1z_2 \\
 \text{Cayley line graph } L : & (x, z_1) & \xrightarrow[\substack{w=\pi_{z_1}^{-1}(z_2)}]{((x, z_1), w)} (xz_1, z_2)
 \end{array}$$

Example 3. If $\Delta = \{z_0, \dots, z_{n-1}\}$ then both of the definitions

- (a) $\pi_{z_i} = \text{id}$;
- (b) $\pi_{z_i}(z_j) = z_{i+j \bmod n}$;

satisfy $\pi_z \circ \pi_w = \pi_{\pi_z(w)}$ for all $w, z \in \Delta$.

Example 4. Let $\mathcal{G} = \{x, y, u, v\}$ be the trivial groupoid, that is, $y = x^{-1}$, $v = xy$, and $u = yx$. If $\Delta = \{x, y\}$ then the Cayley graph is:

$$u \xrightleftharpoons[x]{y} y \quad v \xrightleftharpoons[y]{x} x$$

If $\pi_x = \text{id}$ and $\pi_y = (x \ y)$ is the cycle permutation as in Example 3(b), then the Cayley line graph is:

$$(y, x) \xrightleftharpoons[y]{y} (u, y) \quad (x, y) \xrightleftharpoons[y]{y} (v, x)$$

Example 5. Let D be the Cayley color graph of Example 2, $\pi_x = \text{id}$, $\pi_{x^{-1}} = (x \ x^{-1} \ s)$, and $\pi_s = (x \ s \ x^{-1})$. The Cayley line graph is:

$$\begin{array}{ccccc}
 (x^{-1}, x) & \xrightarrow{s} & (u, s) & \xrightarrow{s} & (s, x^{-1}) & (v, x) & \xrightarrow{s} & (x, s) & \xrightarrow{s} & (y, x^{-1}) \\
 \uparrow \downarrow x^{-1} & & \uparrow \downarrow x & & \uparrow \downarrow s & \uparrow \downarrow x^{-1} & & \uparrow \downarrow x & & \uparrow \downarrow s \\
 (u, x^{-1}) & \xleftarrow{s} & (s, s) & \xleftarrow{s} & (y^{-1}, x) & (x, x^{-1}) & \xleftarrow{s} & (y, s) & \xleftarrow{s} & (t, x)
 \end{array}$$

Throughout this paper let $L = L_\pi(D)$ denote the Cayley line graph of the Cayley color graph $D = D_\Delta(\mathcal{G})$ of the transitive groupoid $\mathcal{G}$.

Lemma 1. For all $w, z \in \Delta$ we have $\pi_z^{-1} \circ \pi_w = \pi_{\pi_z^{-1}(w)}$.

Proof. The result follows from the calculation:

$$\pi_z^{-1} \circ \pi_w = \pi_z^{-1} \circ \pi_{\pi_z^{-1}(w)} = \pi_z^{-1} \circ \pi_z \circ \pi_{\pi_z^{-1}(w)} = \pi_{\pi_z^{-1}(w)}. \quad \square$$

Lemma 2. If the vertex (x_0, δ_0) is connected to the vertex (y, z) in L through n edges with colors $\delta_1, \dots, \delta_n$, then $z = \pi_{\delta_0} \circ \pi_{\delta_1} \circ \dots \circ \pi_{\delta_{n-1}}(\delta_n)$.

Proof. Suppose vertex (x_0, δ_0) is connected to vertex (y, z) through the vertices

$$(x_1, z_1), \dots, (x_{n-1}, z_{n-1}),$$

so we have the following walk:

$$(x_0, \delta_0) \xrightarrow{\delta_1} (x_1, z_1) \xrightarrow{\delta_2} \cdots \longrightarrow (x_{n-2}, z_{n-2}) \xrightarrow{\delta_{n-1}} (x_{n-1}, z_{n-1}) \xrightarrow{\delta_n} (y, z)$$

Then

$$\begin{aligned} z &= \pi_{z_{n-1}}(\delta_n) = \pi_{\pi_{z_{n-2}}(\delta_{n-1})}(\delta_n) = \pi_{z_{n-2}} \circ \pi_{\delta_{n-1}}(\delta_n) \\ &= \cdots = \pi_{\delta_0} \circ \pi_{\delta_1} \circ \cdots \circ \pi_{\delta_{n-1}}(\delta_n). \end{aligned}$$

□

Definition 2. Define

$$H_L := \{\rho \in \text{Aut}(\mathcal{G}) \mid \rho(\Delta) = \Delta \text{ and } \rho \circ \pi_z = \pi_{\rho(z)} \text{ for all } z \in \Delta\}.$$

Note that H_L is not empty since it contains the identity.

Example 6.

- (a) If π_z is the identity for all $z \in \Delta$ as in Example 3(a), then $H_L = \{\text{id}\}$.
- (b) In Example 4, $H_L = \{\text{id}, \rho\} \cong \mathbb{Z}_2$ where $\rho = \begin{pmatrix} x & y \\ u & v \end{pmatrix}$.
- (c) In Example 5, $H_L = \{\text{id}\}$.

Lemma 3. H_L is a subgroup of $\text{Aut}(\mathcal{G})$.

Proof. If $\sigma, \rho \in H_L$ then for all $z \in \Delta$ we have $\sigma\rho \circ \pi_z = \sigma \circ \pi_{\rho(z)} = \pi_{\sigma(\rho(z))}$, so $\sigma\rho \in H_L$. Also,

$$\rho^{-1} \circ \pi_z = \rho^{-1} \circ \pi_{\rho(\rho^{-1}(z))} = \rho^{-1} \rho \circ \pi_{\rho^{-1}(z)} = \pi_{\rho^{-1}(z)}$$

for all $z \in \Delta$. Thus, $\rho^{-1} \in H_L$. □

4. PARTIAL AUTOMORPHISMS

Lemma 4. If (y, w) and (y, z) are vertices of L , then $\text{tail}(y, w) = \text{tail}(y, z)$.

Proof. By symmetry, it suffices to show that $(y, z) \in \text{tail}(y, w)$. Write $w^{-1} = \delta_1 \cdots \delta_n$ as a product of elements of Δ . Then (y, z) can be reached from (y, w) along the following walk:

$$(y, w) \rightarrow (yw, \delta_1) \rightarrow \cdots \rightarrow (yw\delta_1 \cdots \delta_{n-1}, \delta_n) \rightarrow (y, z)$$

where the color of the last edge is $\pi_{\delta_n}^{-1}(z)$. □

Similar proof shows that $\text{tail}(y, z) = \text{tail}(r(y), \delta)$ for some $\delta \in \Delta$. It is easy to see that if u and v are different elements of U , then $\text{tail}(u, \delta)$ and $\text{tail}(v, \mu)$ are not the same. Thus, the tails of L are of the form $\text{tail}(u, \delta)$, where u is a unit of $\mathcal{G}$ and δ is an arbitrary element of Δ whose range is u .

Definition 3. A *partial automorphism* of L is a bijection α between two tails of L that preserves the colors of the edges, that is,

$$\pi_{\alpha(x,z)_2}(w) = \alpha(xz, \pi_z(w))_2$$

for all $((x, z), w)$ satisfying $(x, z) \in \text{dom}(\alpha)$. The set of partial automorphisms of L is denoted by $\text{PAut}(L)$.

The mapping can be visualized by:

$$(x, z) \xrightarrow{w} (xz, \pi_z(w)) \quad \alpha(x, z) \xrightarrow{w} \alpha(xz, \pi_z(w))$$

Lemma 5. For all $(y, \rho) \in \mathcal{G} \times {}_1H_L$, the map $\alpha_{(y, \rho)}: \text{tail}(\rho^{-1}(y^{-1}), \delta) \rightarrow \text{tail}(y, \delta_0)$ defined by

$$\alpha_{(y, \rho)}(x, z) = (y\rho(x), \rho(z))$$

is a partial automorphism of L . Furthermore $\alpha_{(y, \rho)}^{-1} = \alpha_{(y, \rho)}$.

Proof. If $((x, z), w)$ is an edge with $(x, z) \in \text{dom}(\alpha_{(y, \rho)})$, then

$$\pi_{\alpha_{(y, \rho)}(x, z)_2}(w) = \pi_{\rho(z)}(w) = \rho \circ \pi_z(w) = \alpha_{(y, \rho)}(xz, \pi_z(w))_2.$$

So $\alpha_{(y, \rho)}$ preserves the colors of the edges.

If $(x, z) \in \text{tail}(y, \delta_0)$ then $(x, z) = (y\delta_0 \cdots \delta_n, z)$ for some $\delta_0, \dots, \delta_n \in \Delta$. Since $r(y)$ and δ_0 are composable, $(\rho^{-1}(\delta_0 \cdots \delta_n), \rho^{-1}(z))$ is an element of $\text{tail}(\rho^{-1}(y^{-1}), \delta)$ that is mapped to (x, z) by $\alpha_{(y, \rho)}$. Thus, $\alpha_{(y, \rho)}$ is surjective. It is easy to check that $\alpha_{(y, \rho)}$ is injective.

If $(x, z) \in \text{dom}(\alpha_{(y, \rho)}^{-1})$ then

$$\begin{aligned} \alpha_{(y, \rho)}^{-1}(x, z) &= \alpha_{(\rho^{-1}(y^{-1}), \rho^{-1})}(x, z) = (\rho^{-1}(y^{-1})\rho^{-1}(x), \rho^{-1}(z)) \\ &= (\rho^{-1}(y^{-1}x), \rho^{-1}(z)) = \alpha_{(y, \rho)}^{-1}(x, z). \end{aligned}$$

□

Lemma 6. If $\alpha \in \text{PAut}(L)$, $\text{dom}(\alpha) = \text{tail}(u, \delta)$ with $u \in U$, $\delta \in \Delta$, and $(t, z) \in \text{dom}(\alpha)$, then $\alpha(t, z)_2 = \pi_{\alpha(u, \delta)_2}(\pi_\delta^{-1}(z))$.

Proof. Since (u, δ) is connected to (t, z) through edges say with colors $\delta_1, \dots, \delta_n$. By Lemma 2, $z = \pi_\delta \circ \pi_{\delta_1} \circ \cdots \circ \pi_{\delta_{n-1}}(\delta_n)$, and so $\pi_\delta^{-1}(z) = \pi_{\delta_1} \circ \cdots \circ \pi_{\delta_{n-1}}(\delta_n)$. Since α preserves the colors of the edges, $\alpha(u, \delta)$ is connected to $\alpha(t, z)$ through edges with colors $\delta_1, \dots, \delta_n$, so

$$\alpha(t, z)_2 = \pi_{\alpha(u, \delta)_2} \circ \pi_{\delta_1} \circ \cdots \circ \pi_{\delta_{n-1}}(\delta_n) = \pi_{\alpha(u, \delta)_2}(\pi_\delta^{-1}(z)).$$

□

Lemma 7. If $\alpha \in \text{PAut}(L)$ and $(x, z), (x, w) \in \text{dom}(\alpha)$, then $\alpha(x, z)_1 = \alpha(x, w)_1$.

Proof. Since L has no sources, there is a vertex (y, δ) of L that is connected to both (x, z) and (x, w) by single edges. So $\alpha(x, z)_1 = \alpha(y, \delta)_1 \alpha(y, \delta)_2 = \alpha(x, w)_1$. □

Proposition 1. Every partial automorphism α of L is $\alpha_{(y,\rho)}$ for some $(y,\rho) \in \mathcal{G} \times_1 H_L$.

Proof. Let u be the unique unit of $\mathcal{G}$ such that $\text{tail}(u, \delta) = \text{dom}(\alpha)$ for some $\delta \in \Delta$. By the previous lemma,

$$y := \alpha(u, \delta)_1$$

is independent of the choice of δ .

Let $z \in \Delta$. Since $\mathcal{G}$ is transitive, we can find $t \in \mathcal{G}$ such that $u = r(tz)$. The definition

$$\rho(z) := \alpha(t, z)_2$$

is independent of the choice of t since by Lemma 6, $\alpha(t, z)_2 = \pi_{\alpha(u, \delta)_2}(\pi_\delta^{-1}(z))$.

We now extend ρ to $\mathcal{G}$. If $x \in \mathcal{G}$ then $x = z_1 \cdots z_n$ for some $z_1, \dots, z_n \in \Delta$. Define

$$\rho(x) = \rho(z_1) \cdots \rho(z_n).$$

We need to check that this definition is independent of the choice of the z_i 's. Suppose we also have $x = w_1 \cdots w_m$ for some $w_1, \dots, w_m \in \Delta$. Choose $t \in \mathcal{G}$ and $\mu \in \Delta$ such that $u = r(tx)$ and $(x, \mu) \in \mathcal{G}^{(2)}$. Then we have the following walks in L

$$\begin{array}{ccccccc} (y, \delta) & \longrightarrow & \bullet \cdots \bullet & \longrightarrow & (t, z_1) & \longrightarrow & \bullet \cdots \bullet & \longrightarrow & (tz_1 \cdots z_{n-1}, z_n) & \searrow \\ & \searrow & & & & & & & & \\ & & \bullet \cdots \bullet & \longrightarrow & (t, w_1) & \longrightarrow & \bullet \cdots \bullet & \longrightarrow & (tw_1 \cdots w_{m-1}, w_m) & \longrightarrow & (tx, \mu) \end{array}$$

which are mapped by α to the walks

$$\begin{array}{ccccccc} \alpha(y, \delta) & \longrightarrow & \bullet \cdots \bullet & \longrightarrow & (\mu_1, \rho(z_1)) & \longrightarrow & \bullet \cdots \bullet & \longrightarrow & (\mu_n, \rho(z_n)) & \searrow \\ & \searrow & & & & & & & & \\ & & \bullet \cdots \bullet & \longrightarrow & (v_1, \rho(w_1)) & \longrightarrow & \bullet \cdots \bullet & \longrightarrow & (v_m, \rho(w_m)) & \longrightarrow & \alpha(tx, \mu) \end{array}$$

for some $\mu_1, \dots, \mu_n, v_1, \dots, v_m \in \mathcal{G}$. Hence, we have

$$\mu_1 \rho(z_1) \cdots \rho(z_n) = \alpha(tx, \mu)_1 = v_1 \rho(w_1) \cdots \rho(w_m).$$

By Lemma 7, $\mu_1 = v_1$. So, $\rho(z_1) \cdots \rho(z_n) = \rho(w_1) \cdots \rho(w_m)$.

We show that ρ is an automorphism of $\mathcal{G}$. If $(w, z) \in \mathcal{G}^{(2)}$ then we can find a $t \in \mathcal{G}$ such that $u = r(tw) = r(twz)$. The edge $((t, w), \pi_w^{-1}(z))$ connects vertex (t, w) to (tw, z) . Taking the image under α shows that $\alpha(t, w) = (\alpha(t, w)_1, \delta(w))$ is connected to $(\alpha(tw, z)_1, \delta(z))$ by an edge with color $\pi_w^{-1}(z)$, so $(\delta(w), \delta(z)) \in \mathcal{G}^{(2)}$. It follows easily now that ρ is multiplicative. Since $\rho = \pi_{\alpha(u, \delta)_2} \circ \pi_\delta^{-1}$ is a permutation of Δ , $\rho: \mathcal{G} \rightarrow \mathcal{G}$ is surjective. Since $\mathcal{G}$ is finite, ρ must be injective.

We show that $\rho \in H_L$. Since $\rho(\Delta) = \Delta$, we only need to check that $\rho \circ \pi_z = \pi_{\rho(z)}$ for all $z \in \Delta$. If $u = r(tz)$ for some $t \in \mathcal{G}$, then $\rho(z) = \pi_{\alpha(u, \delta)_2}(\pi_\delta^{-1}(z))$. So,

$$\pi_{\rho(z)} = \pi_{\alpha(u, \delta)_2} \circ \pi_{\pi_\delta^{-1}(z)} = \pi_{\alpha(u, \delta)_2} \circ \pi_\delta^{-1} \circ \pi_z = \rho \circ \pi_z.$$

It remains to show that $\alpha = \alpha_{(y,\rho)}$. To check that α and $\alpha_{(y,\rho)}$ have the same domain, write $u = u_1 \cdots u_n$ as a product of elements of Δ . Then

$$\begin{aligned} y\rho(u) &= \alpha(u, u_1)_1 \rho(u_1) \cdots \rho(u_n) \\ &= \alpha(u, u_1)_1 \alpha(u, u_1)_2 \alpha(uu_1, u_2)_2 \cdots \alpha(uu_1 \cdots u_{n-1}, u_n)_2 \\ &= \alpha(u, u_1)_1 = y \end{aligned}$$

so

$$\begin{aligned} \text{dom}(\alpha_{(y,\rho)}) &= \text{tail}(\rho^{-1}(y^{-1}), \delta) = \text{tail}(r(\rho^{-1}(y^{-1})), \delta) \\ &= \text{tail}(\rho^{-1}(y^{-1}y), \delta) = \text{tail}(u, \delta) = \text{dom}(\alpha). \end{aligned}$$

If $(x, z) \in \text{dom}(\alpha)$ then $ux = x = x_1 \cdots x_n$ for some $x_1, \dots, x_n \in \Delta$, so we have

$$\begin{aligned} \alpha_{(y,\rho)}(x, z) &= (y\rho(x), \rho(z)) = (\alpha(u, x_1)_1 \rho(x_1) \cdots \rho(x_n), \alpha(x, z)_2) \\ &= (\alpha(u, x_1)_1 \alpha(u, x_1)_2 \alpha(ux_1, x_2)_2 \cdots \alpha(ux_1 \cdots x_{n-1}, x_n)_2, \alpha(x, z)_2) \\ &= (\alpha(ux_1, x_2)_1 \alpha(ux_1, x_2)_2 \cdots \alpha(ux_1 \cdots x_{n-1}, x_n)_2, \alpha(x, z)_2) \\ &= \cdots \\ &= (\alpha(ux_1 \cdots x_n, z)_1, \alpha(x, z)_2) = (\alpha(x, z)_1, \alpha(x, z)_2) \\ &= \alpha(x, z). \end{aligned}$$

□

Theorem 1. *If $L = L_\pi(D_\Delta(\mathcal{G}))$ then $\mathcal{G} \times_{\mathfrak{t}} H_L \cong \text{PAut}(L)$.*

Proof. We saw that the map $(y, \rho) \mapsto \alpha_{(y,\rho)}$ is onto and $\alpha_{(y,\rho)^{-1}} = \alpha_{(y,\rho)}^{-1}$. It is multiplicative since if $(x, \sigma), (y, \rho) \in \mathcal{G} \times_{\mathfrak{t}} H_L$ are composable and $(x, z) \in \text{dom}(\alpha_{(y,\rho)})$, then

$$\begin{aligned} \alpha_{(x,\sigma)(y,\rho)}(x, z) &= \alpha_{(x\sigma(y), \sigma\rho)}(x, z) = (x\sigma(y)\sigma(\rho(x)), \sigma(\rho(z))) \\ &= (x\sigma(y\rho(x)), \sigma(\rho(z))) = \alpha_{(x,\sigma)}(y\rho(x), \rho(z)) \\ &= \alpha_{(x,\sigma)}\alpha_{(y,\rho)}(x, z). \end{aligned}$$

It remains to show that $(y, \rho) \mapsto \alpha_{(y,\rho)}$ is injective. Suppose $\alpha_{(y,\rho)} = \alpha_{(x,\sigma)}$. There is a unique unit $u \in \mathcal{G}$ such that $(u, \delta) \in \text{dom}(\alpha_{(y,\rho)})$. Hence

$$\begin{aligned} (y, \rho(\delta)) &= (y\rho(u), \rho(\delta)) = \alpha_{(y,\rho)}(u, \delta) \\ &= \alpha_{(x,\sigma)}(u, \delta) = (x\sigma(u), \sigma(\delta)) \\ &= (x, \sigma(\delta)), \end{aligned}$$

so $y = x$. If $z \in \Delta$ and $u = r(tz)$, then

$$\rho(z) = \alpha_{(y,\rho)}(t, z)_2 = \alpha_{(x,\sigma)}(t, z)_2 = \sigma(z).$$

Thus, $(y, \rho) = (x, \sigma)$. □

Corollary 1. *If π_z is the identity for all $z \in \Delta$ as in Example 6(a), then $\text{PAut}(L) \cong \mathcal{G}$.*

Note that H_L is a subgroup of $H = \{\pi \in \text{Aut}(\mathcal{G}) \mid \pi(\Delta) = \Delta\}$ and so $\text{PAut}(L) \cong \mathcal{G} \times_1 H_L$ is isomorphic to a subgroupoid of the groupoid $\text{PAut}^*(D) \cong \mathcal{G} \times_1 H$ of color permuting partial automorphisms of D .

Example 7. In Example 4, $H_L = H$ and $\mathcal{G} \times_1 H_L \cong \text{PAut}(L) \cong \{a, b\} \times \mathbb{Z}_2 \times \{a, b\}$. Note that the Cayley line graph L is not a Cayley color graph since the number of vertices of L is not the same as the number of elements of $\text{PAut}(L)$.

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A MODIFIED NONMONOTONE FILTER QP-FREE METHOD

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Abstract. In this paper, an infeasible QP-free method without penalty function is proposed for inequality constrained optimization. We first compute a fundamental direction and then bend the search direction based on the constraint function and the Lagrange multiplier. Based on the modified nonmonotone filter technique, the acceptable criterion of trial points is relaxed and Maratos effects are avoided to a certain degree. At each iteration, only two or three systems of linear equations with the same coefficient are needed to solve to obtain the search direction. Under suitable conditions, the global convergence of the algorithm is proved without the strict complementarity conditions. In the end, some numerical results are reported.

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Keywords: inequality constrained optimization, QP-free method, active set, global convergence, filter

1. INTRODUCTION

Consider the nonlinear optimization with general inequality constraints:

$$\begin{aligned} & \min f(x) \\ & \text{subject to } g_i(x) \leq 0, \quad i \in I = \{1, 2, \dots, m\}, \end{aligned} \quad (1.1)$$

where $f: R^n \rightarrow R$ and $g_i: R^n \rightarrow R^m$, $i \in I$ are twice continuously differentiable functions.

It is well known that sequential quadratic programming (SQP) method is one of the effective methods for solving nonlinearly constrained optimization problem and has been widely investigated by many authors [8, 10]. However, the search direction of SQP method is obtained by solving a quadratic programming subproblem in each iteration, which greatly increases the computational scale. To avoid the drawback, various QP-free approaches, also called sequential systems of linear equations (SSLE) methods, are proposed for (1.1).

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In [15], Panier et al proposed a feasible sequential quadratic programming algorithm, the algorithm is shown to possess global convergence as well as two-step superlinear convergence under the relevant conditions, and Maratos effect can be overcome to some extent. But it requires that the number of isolated stationary points is finite. The algorithm is later improved by Gao et al. [7]. In 2000, Qi [16] gave a new feasible QP-free algorithm based on Fischer-Burmeister function [4], which does not require strict complementarity conditions and does not need to assume the isolation of stable points. Later, Qi et al. [17] proposed an infeasible QP-free algorithm based on a continuous differentiable exact penalty function and an efficient recursive QP algorithm model, which are presented by Lucidi [14] and Facchinei [3] respectively. Tits and Bakhtiari [2] presented a simple primal-dual feasible interior point algorithm for the problem (1.1). Based on the idea in [2], Jian et al. [11] presented a primal-dual quasi interior-point algorithm, the associated system of linear equations possesses a smaller scale and requires less computational cost than that in [2]. Then Li et al. [1] and Huang et al. [12] developed QP-free methods based on the smoothing techniques and the working set techniques, but this method assumed that the Hessian estimate was positive definite. In 2017, Wang et al. [21] proposed an infeasible active-set QP-free algorithm based on filter technique without the positive definite assumption on the Hessian estimate.

Penalty function is usually used as merit function to decide whether the trial point is accepted at the new iteration, but it is well known that the chosen of penalty parameter is difficult. If the penalty parameter is too large, then any monotonic method would be forced to follow the nonlinear constraint manifold very closely, resulting in shortened Newton steps and slow convergence. On the other hand, too small a choice of the penalty parameter may result in an infeasible point, or even an unbounded increase in the penalty. Therefore, Fletcher and Leyffer [5] proposed filter technique and gave a large number of numerical experiments to prove the validity of filter method. After that, many different filter methods were proposed. For example, filter interior point approach [6], line search filter method [20], a modified trust-region filter [19], a nonmonotone filter method which used a global g-filter for global convergence [18], a nonmonotone line search multidimensional filter-SQP method [9].

Motivated by the above ideas, we propose a QP-free method with filter technique which solve two or three linear equation systems with the same coefficient matrix. Compared with the existing methods, our method has several advantages:

- (1) We first obtain a fundamental direction and then bend the search direction based on the constraint function and Lagrange multiplier.
- (2) The initial point is not needed to be feasible.
- (3) There is no penalty function so that the penalty parameter is avoided.

- (4) The strict complementarity conditions are not required, computation scale is decreased by working set, moreover, restoration phase is not required, which is needed in most of the traditional filter methods.

By the numerical results, we show that the proposed method is effective.

The remainder of this paper is organized as follows: In Section 2, we introduce our QP-free algorithm based on filter technique. The global convergence of this algorithm is established in Section 3. Some numerical experiments are shown in Section 4.

2. THE ALGORITHM

In this section, we present some related concepts and symbols. A given point $x \in R^n$ is said to be a Karush-Kuhn-Tucker (KKT) point of problem (1.1) if there exists a vector $\lambda \in R^m$ such that

$$\nabla_x L(x, \lambda) = 0, \quad \lambda_i g_i(x) = 0, \quad \lambda_i \geq 0, \quad g_i(x) \leq 0 \quad \forall i \in I \quad (2.1)$$

where the vector $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)^T$ is the corresponding lagrangian multiplier, $L(x, \lambda) = f(x) + \sum_{i=1}^m \lambda_i g_i(x)$ is the Lagrangian function of problem (1.1).

Let (x^*, λ^*) denote a KKT point of problem (1.1). Define $\Phi: R^{n+m} \rightarrow R^{n+m}$:

$$\Phi(x, \lambda) = \begin{pmatrix} \nabla_x L(x, \lambda) \\ \min\{-G(x), \lambda\} \end{pmatrix},$$

where $G(x) = (g_1(x), g_2(x), \dots, g_m(x))^T$, $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)^T$. It is obviously that (2.1) and $\Phi(x, \lambda) = 0$ are equivalent. Then, we define another function $\varphi: R^{n+m} \rightarrow R$

$$\varphi(x, \lambda) = \sqrt{\|\Phi(x, \lambda)\|},$$

where $\|\cdot\|$ denotes the Euclidean norm. The function φ is non-negative and continuous. And it means that (x^*, λ^*) is a KKT point of problem (1.1) if and only if $\varphi(x^*, \lambda^*) = 0$. Let the active set

$$I(x) = \{i \in I \mid g_i(x) = 0\},$$

and two working sets

$$J_\varepsilon(x, \lambda) = \{i \in I \mid g_i(x) \geq -\varepsilon \min\{\varphi(x, \lambda), \varphi_{\max}\}\},$$

$$\bar{J}_\varepsilon(x, \lambda) = \{i \in J_\varepsilon \mid \lambda_i \geq \varepsilon \min\{\varphi(x, \lambda), \varphi_{\max}\}\},$$

where $\varphi_{\max} > 0$, $\varepsilon > 0$, $J_\varepsilon(x, \lambda)$ is an estimate of the final active set $I(x)$, and $\bar{J}_\varepsilon(x, \lambda)$ is stronger than working set $J_\varepsilon(x, \lambda)$. To simplify the presentation, we set $J_{\varepsilon_k}(x^k, \lambda^{k-1,0})$ be W_k and $\bar{J}_{\varepsilon_k}(x^k, \lambda^{k-1,0})$ be $\bar{W}_k$, where ε_k and (x^k, λ^k) are at the k th iteration.

In order to avoid computing linearly independent constraint gradients, inspired by [13], the coefficient matrix V_k of our Newton equations involve only constraints in the working set W_k ,

$$V_k = \begin{pmatrix} H_k & \nabla g_{W_k}(x^k) \\ U_k \nabla g_{W_k}(x^k)^T & G_{W_k}(x^k) \end{pmatrix}, \quad (2.2)$$

where $H_k \in R^{n \times n}$ is an estimate of the Lagrangian Hessian, $U_k = \text{diag}(\mu_i^k)$, $i \in W_k$ and $G_{W_k}(x^k) = \text{diag}(g_i(x^k))$, $i \in W_k$. Note that μ_i^k which are defined as

$$\mu_i^k = \begin{cases} \theta_k + \max\{\lambda_i^{k-1,0}\}, & i \in W_k; \\ \theta_k, & i \in I \setminus W_k; \end{cases}$$

are controlled to be componentwise bounded, where $\theta_k = v \min\{\lambda_i^{k-1,0}, i \in \overline{W}_k\}$ if $\overline{W}_k \neq \emptyset$ and $\varphi(x^k, \lambda^{k-1,0}) > 0$; otherwise $\theta_k = \theta$, θ and v are positive constants.

In this paper, we use a nonmonotone filter to determine a trial point is accepted or not. In order to prove the following lemma, we give some definitions as follows:

Definition 1. Define the constrained violation function $h: R^n \rightarrow R$ by

$$h(x) = \sum_{i \in I} \max\{g_i(x), 0\}.$$

It is easy to see that $h(x) = 0$ if and only if x is a feasible point.

Definition 2. A point x^i is dominated by x^j if and only if $h(x^i) \leq h(x^j)$ and $f(x^i) \leq f(x^j)$ for each $i \neq j$.

We give the concept of filter subsets based on the above definitions.

Definition 3. A filter set $\mathcal{F}$ is a set of pairs (h, f) such that no pair dominates any other.

So, we have a definition of whether the trial point x^i is accepted by the filter.

Definition 4. A trial point x^i is called acceptable to the filter if and only if either

$$h(x^i) \leq h(x^j) \text{ or } f(x^i) \leq f(x^j) \quad \forall (h(x^j), f(x^j)) \in \mathcal{F}, i \neq j.$$

In the actual calculation, some filter point pairs may fall on the boundary, resulting in convergence to the infeasible limit points where $h > 0$. In order to avoid this situation, a modified filter method is presented by adding an envelope to the current filter.

Definition 5. A trial point x^k is called acceptable to the filter if and only if either

$$h(x^k) \leq (1 - \gamma)h(x^j) \text{ or } f(x^k) \leq f(x^j) - \gamma h(x^k) \\ \forall (h(x^j), f(x^j)) \in \mathcal{F}$$

where γ is a constant that close to 0.

So we give a nonmonotone modified filter that substitute Definition 4 and Definition 5 with the following Definition 6 in our actual algorithm.

Definition 6. A trial point x^k is called acceptable to the filter if and only if either

$$h(x^k) \leq \max_{0 < j < m(k)} (1 - \gamma)h(x^j) \text{ or } f(x^k) \leq \max_{0 < j < m(k)} f(x^j) - \gamma h(x^k)$$

$$\forall (h(x^j), f(x^j)) \in \mathcal{F}$$

where γ is a constant that close to 0, $m(0) = 0$, $0 \leq m(k) \leq \min[m(k-1) + 1, M]$ for $k \geq 1$, $M \geq 1$ is a given positive constant.

We add the point x^k to the filter, and update the filter set which is $\mathcal{F}_+ = \mathcal{F} \cup (h^k, f^k)$, then remove point x^j that satisfied

$$h(x^j) \geq h(x^k) \text{ and } f(x^j) - \gamma h(x^j) \geq f(x^k) - \gamma h(x^k),$$

so, the new filter set is

$$\mathcal{F}_+ = \mathcal{F} \cup (h^k, f^k) \setminus \{(h^j, f^j) | h^j \geq h^k \text{ and } f^j - \gamma h^j \geq f^k - \gamma h^k\},$$

we also refer to this operation as 'adding x^k to the filter'.

In the following algorithm, we are going to update θ_k and μ_{k+1} through the working set $\bar{W}_k$, and the algorithm is finished on the working set W_k , which reduces the calculation scale. The algorithm obtains the initial direction $d^{k,0}$ and the corresponding multiplier $\lambda^{k,0}$ by solving the first linear equation system, and then bends the search direction based on $d^{k,0}$, $\lambda^{k,0}$ and constraint functions.

In order to obtain the convergence of the algorithm, we use the nonmonotone filter technique to replace the traditional filter, thus, the acceptance criteria of trial points x^k are relaxed. We are now ready to state the algorithm.

Algorithm 1.

Step 1. Give an initial point $x^1 \in R^n$. $t, \gamma \in (0, 1)$, $h_{\max} > 1$, $\chi_1 \gg \varphi_{\max}$, $\varepsilon_1 > 0$, $\varphi_{\max} > 0$, $\lambda^{0,0} > 0$, $\omega \in (2, 3)$, $\mathcal{F}_1 = \{(h_{\max}, -\infty)\}$ with $h(x^1) \ll h_{\max}$, $\theta > 0$, $v \in (0, 1)$, $\rho \in [0, 1]$. If $\bar{W}_1 \neq \emptyset$ and $\varphi(x^1, \lambda^{0,0}) > 0$, $\theta_1 = v \min\{\lambda_i^{0,0}, i \in \bar{W}_1\}$; otherwise, $\theta_1 = \theta$. $\mu_1 = \lambda^{0,0} + \theta_1 e$, $H_1 = \nabla_{xx}^2 L(x^1, \lambda^{0,0})$. Set $k = 1$.

Step 2. Compute $d^{k,0}$ and $\lambda^{k,0}$ by solving the linear system in (d, λ) :

$$V_k \begin{pmatrix} d \\ \lambda \end{pmatrix} = - \begin{pmatrix} \nabla f(x^k) \\ 0 \end{pmatrix} \quad (2.3)$$

Set $\lambda_i^{k,0} = 0$, $i \in I \setminus W_k$.

Step 3. Compute $d^{k,1}$ and $\lambda^{k,1}$ by solving the linear system in (d, λ) :

$$V_k \begin{pmatrix} d \\ \lambda \end{pmatrix} = - \begin{pmatrix} \nabla f(x^k) \\ (1 - \rho)\mu \|d^{k,0}\|^\omega + \rho \theta_k v_{W_k}^k \end{pmatrix} \quad (2.4)$$

where $v_{W_k}^k = (v_i^k, i \in W_k)$,

$$v_i^k = \begin{cases} \min\{-g_i(x^k), \lambda_i^{k,0}\}, & \lambda_i^{k,0} < 0; \\ -g_i(x^k), & \text{otherwise.} \end{cases}$$

Set $\lambda_i^{k,0} = 0$, $i \in I \setminus W_k$. If $\nabla f(x^k)^T d^{k,1} = 0$ and $h(x^k) = 0$, stop.

Step 4. Set $l = 0$, $\alpha_{k,l} = 1$.

Step 5. If $x^k + \alpha_{k,l}d^{k,1}$ is acceptable for the filter, let $\alpha_k = \alpha_{k,l}$, $p^k = \alpha_k d^{k,1}$, go to step 7.

Step 6. Compute $d^{k,2}$ and $\lambda^{k,2}$ by solving the linear system in (d, λ) :

$$V_k \begin{pmatrix} d \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ -G_{W_k}(x^k + d^{k,1}) \end{pmatrix}$$

Set $\lambda_i^{k,2} = 0$, $i \in I \setminus W_k$. If $\|d^{k,2}\| > \|d^{k,1}\|$, set $\|d^{k,2}\| = 0$. If $x^k + d^{k,1} + d^{k,2}$ is acceptable for the filter, set $p^k = d^{k,1} + d^{k,2}$, go to step 7. Set $\alpha_{k,l+1} = t\alpha_{k,l}$, $l = l + 1$, go back to step 5.

Step 7. Set $x^{k+1} = x^k + p^k$ and add x^{k+1} to the filter.

Step 8. Update. If $\|\lambda^{k,0}\|_\infty > \chi_k$, set $\varepsilon_{k+1} = \frac{1}{2}\varepsilon_k$ and $\chi_{k+1} = 2\chi_k$; else set $(\varepsilon_{k+1}, \chi_{k+1}) = (\varepsilon_k, \chi_k)$. If $\overline{W}_{k+1} \neq \emptyset$ and $\varphi(x^{k+1}, \lambda^{k,0}) > 0$, set $\theta_{k+1} = \vee \min\{\lambda_i^{k,0}, i \in \overline{W}_{k+1}\}$; otherwise, $\theta_{k+1} = \theta$. Set

$$\mu_i^{k+1} = \begin{cases} \theta_{k+1} + \max\{\lambda_i^{k,0}, 0\}, & i \in W_k; \\ \theta_{k+1}, & i \in I \setminus W_k. \end{cases}$$

Update H_{k+1} , and set $k = k + 1$ and go to step 2.

Remark 1. We obtain a fundamental direction by solving equation (2.3), and then bend the search direction according to the constraint function or the Lagrange multiplier by equation (2.4). Moreover, we make the modified nonmonotone filter technique to avoid the Maratos effect effectively.

3. GLOBAL CONVERGENCE

In this section, we show that Algorithm 1 is global convergent to KKT points of problem (1.1). To prove, we have the following assumptions:

Assumption 1. The sequence $\{x^k\}$ and $\{x^k + p^k\}$ which are generated by Algorithm 1 are contained in a bounded set $\Omega \subseteq R^n$.

Assumption 2. The functions $f(x)$ and $g_i(x)$, $i \in I$ are twice continuously differentiable, and their function values are bounded over $\Omega \subseteq R^n$.

Assumption 3. The vectors $\{\nabla g_i(x), i \in W_k\}$ are linearly independent for each point $x \in \Omega \subseteq R^n$.

Assumption 4. There exist $\beta_1, \beta_2 > 0$ such that for all k , $\|H_k\| \leq \beta_2$, and

$$d^T \widehat{H}_k d \geq \beta_1 \|d\|^2 \quad \forall d \in \mathfrak{K}(x^k),$$

where

$$\begin{aligned}\widehat{H}_k &= H_k - \sum_{i \in W_k \setminus I(x^k)} \frac{\mu_i^k}{g_i(x^k)} \nabla g_i(x^k) \nabla g_i(x^k)^T, \\ \mathfrak{K}(x) &= \{d \in R^n \mid \nabla g_i(x^k)^T d = 0, i \in W_k\}.\end{aligned}$$

Lemma 1. *Sequences $\{\chi_k\}$ and ε_k in Algorithm 1 is changed in finite number of times.*

Proof. Suppose that χ_k and ε_k are changed infinitely many times, that is, there exists an infinite index set K such that $\chi_{k+1} = 2\chi_k$ and $\varepsilon_{k+1} = \varepsilon_k/2$ for all $k \in K$. Then we have $\{\chi_k\} \rightarrow +\infty$ and $\varepsilon_k \rightarrow 0^+$ as $k \rightarrow \infty$.

Due to the finiteness of set I , we assume $W_K = W_k$ for all $k \in K$. Assume $x^k \rightarrow \bar{x}$ as $k \rightarrow \infty$ and $k \in K$. We get from the definition of W_k that $W_K \subseteq I(\bar{x})$, since $\varepsilon_k \min\{\varphi(x^k, \lambda^{k-1,0})\} \rightarrow 0$ as $k \rightarrow \infty$. It is known in the algorithm that $\|\lambda^{k,0}\|_\infty > \chi_k$, then $\|\lambda^{k,0}\|_\infty \rightarrow \infty$. Therefore, we have sequence $\{t_k\}$, with

$$t_k = \max\{\|d^{k,0}\|, \|\lambda_{W_K}^{k,0}\|_\infty, 1\}$$

tends to infinity on K . Define $\widehat{d}_k = d^{k,0}/t_k$ and $\widehat{\lambda}_{W_K}^k = \lambda_{W_K}^{k,0}/t_k$ for $k \in K$.

Therefore, $\max\{\|\widehat{d}_k\|, \|\widehat{\lambda}_{W_K}^k\|_\infty\} = 1$ for all $k \in K$ large enough. Then we have a non-zero vector $(\widehat{d}_k, \widehat{\lambda}_{W_K}^k) \rightarrow (\widehat{d}, \widehat{\lambda}_{W_K})$ as $k \in K_1 \subseteq K \rightarrow \infty$, where K_1 is an infinite index set.

Since $I(x^k) \subseteq W_k$, we have that $g_i(x^k) = 0, i \in I(x^k)$, it follows from equation (2.3) that

$$\mu_i^k \nabla g_i(x^k)^T d^{k,0} = -\lambda_i^{k,0} g_i(x^k) = 0,$$

therefore $d^{k,0} \in \mathfrak{K}(x^k)$ as $\mu^k > 0$, that is $\nabla g_i(x^k)^T d = 0$, so from (2.3) of Algorithm 1 and Assumption 4, we have

$$\begin{aligned}\nabla f(x^k)^T d^{k,0} &= -(d^{k,0})^T H_k d^{k,0} - \sum_{i \in I(x^k)} \lambda_i^{k,0} \nabla g_i(x^k)^T d^{k,0} \\ &= -(d^{k,0})^T (\widehat{H}_k - \sum_{i \in W_k \setminus I(x^k)} \frac{\mu_i^k}{g_i(x^k)} \nabla g_i(x^k) \nabla g_i(x^k)^T) d^{k,0} \\ &\quad - \sum_{i \in I(x^k)} \lambda_i^{k,0} \nabla g_i(x^k)^T d^{k,0} \\ &= -(d^{k,0})^T \widehat{H}_k d^{k,0} \leq -\beta_1 \|d^{k,0}\|^2.\end{aligned}$$

Let $k \in K_1 \rightarrow \infty$ yields $\widehat{d} = 0$, and then $\widehat{\lambda}_{W_K}$ is nonzero. Besides, from equation (2.3), If $k > 1$, we have

$$H_k d^{k,0} + \nabla g_{W_K}(x^k) \lambda_{W_K}^{k,0} = -\nabla f(x^k). \quad (3.1)$$

Therefore, dividing both sides of equation (3.1) by t_k and letting $k \in K_1 \rightarrow \infty$, we have $\nabla g_{W_k}(\bar{x}) \hat{\lambda}_{W_k} = 0$

This is inconsistent with Assumption 3. The conclusion follows. $\square$

Lemma 2. Suppose Assumption 3 holds, given any vector $x \in X$, and any nonnegative vector $\mu^k \in R^m$ such that $\mu_i^k > 0$ if $g_i(x^k) = 0$ for all $i \in W_k$, and let $H_k \in R^{n \times n}$ be a symmetric matrix satisfying (3.1), then the matrix V_k defined by equation (2.2) is non-singular.

Proof. Suppose (d, λ) is a solution of the following equation

$$V_k \begin{pmatrix} d \\ \lambda \end{pmatrix} = 0.$$

So we have that $\lambda_i = -(\mu_i/g_i(x^k)) \nabla g_i(x^k)^T d$, $i \in W_k \setminus I(x^k)$ and $\nabla g_i(x^k)^T d = 0$, $i \in I(x^k)$ therefore,

$$d^T \left(H - \sum_{W_k \setminus I(x^k)} \frac{\mu_i}{g_i(x^k)} \nabla g_i(x^k) \nabla g_i(x^k)^T \right) d = 0$$

which implies that $d = 0$. Moreover, $\nabla g_{W_k}(x^k) \lambda = 0$, $G_{W_k} \lambda = 0$. So $\nabla g_{I(x^k)}(x^k) \lambda_{I(x^k)} = 0$ and $\lambda_{W_k \setminus I(x^k)} = 0$. Since Assumption 3 implies $\lambda_{I(x^k)} = 0$, zero is the unique solution, that is V_k is non-singular. $\square$

Lemma 3. Under Assumption 1-4, sequences $\{\lambda^{k,0}\}$ and $\{\mu^k\}$ in Algorithm 1 are bounded.

Proof. It follows that $\{\chi_k\}$ has an upper bound from Lemma 1, and thus $\{\lambda^{k,0}\}$ is bounded by Algorithm 1. The boundedness of $\{\mu^k\}$ follows directly from their definitions and the boundedness of $\{\lambda^{k,0}\}$. $\square$

Lemma 4. Under the condition of Lemma 2, denote

$$V_k^{-1} = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix}$$

then $C_k = U_k B_k^T$.

Proof. Similar to the proof of Lemma 3.5 in [21]. $\square$

Lemma 5. If $\{x^{k_j}\}$ is a subset of iterations for which $\Gamma_s^{k_j} \geq \epsilon$, $s = 1, 2, 3$ with a constant ϵ_1 and ϵ_2 , independent of j , such that if $h(x^{k_j}) \leq \epsilon_1$ and $\|d^{k,0}\|^\omega \leq \epsilon_2$, then $\nabla f(x^{k_j})^T d^{k_j,1} \leq -\epsilon/2$ for all j .

Proof. Substituting the conclusion of Lemma 4 into equation (2.3) and (2.4) yields that

$$\begin{aligned} d^{k,0} &= -A_k \nabla f(x^k), \quad \lambda^{k,0} = -C_k \nabla f(x^k); \\ d^{k,1} &= d^{k,0} - B_k[(1-\rho)\mu^k \|d^{k,0}\|^\omega + \rho\theta_k v^k], \\ \lambda^{k,1} &= \lambda^{k,0} - D_k[(1-\rho)\mu^k \|d^{k,0}\|^\omega + \rho\theta_k v^k]. \end{aligned} \quad (3.2)$$

From equations (2.3) and (3.2), we have

$$\begin{aligned} \nabla f(x^{k_j})^T d^{k_j,1} &= \nabla f(x^{k_j})^T \{d^{k_j,0} - B_{k_j}[(1-\rho)\mu^{k_j} \|d^{k_j,0}\|^\omega + \rho\theta_{k_j} v^{k_j}]\} \\ &= -(d^{k_j,0})^T \widehat{H}_{k_j} d^{k_j,0} \\ &\quad - \nabla f(x^{k_j})^T B_{k_j}[(1-\rho)\mu^{k_j} \|d^{k_j,0}\|^\omega + \rho\theta_{k_j} v^{k_j}] \\ &= -(d^{k_j,0})^T \widehat{H}_{k_j} d^{k_j,0} \\ &\quad + (\lambda^{k_j,0})_{W_{k_j}}^T U_{k_j}^{-1} [(1-\rho)\mu^{k_j} \|d^{k_j,0}\|^\omega + \rho\theta_{k_j} v^{k_j}], \end{aligned} \quad (3.3)$$

Case I: $\rho = 1$.

$$\begin{aligned} \nabla f(x^{k_j})^T d^{k_j,1} &= -(d^{k_j,0})^T \widehat{H}_{k_j} d^{k_j,0} + (\lambda^{k_j,0})_{W_{k_j}}^T U_{k_j}^{-1} \theta_{k_j} v^{k_j} \\ &= -(d^{k_j,0})^T \widehat{H}_{k_j} d^{k_j,0} - \sum_{i \in \overline{W}_k^-} \frac{\lambda_i^{k_j,0} \theta_{k_j}}{\mu_i^{k_j}} g_i(x^{k_j}) - \sum_{i \in \overline{W}_k^+} \frac{\lambda_i^{k_j,0} \theta_{k_j}}{\mu_i^{k_j}} g_i(x^{k_j}) \\ &\quad + \sum_{i \in W_k \setminus \overline{W}_k} \frac{\lambda_i^{k_j,0} \theta_{k_j}}{\mu_i^{k_j}} \min\{-g_i(x^{k_j}), \lambda_i^{k_j,0}\} \\ &= -\Gamma_1^{k_j} - \sum_{i \in \overline{W}_k^-} \frac{\lambda_i^{k_j,0} \theta_{k_j}}{\mu_i^{k_j}} g_i(x^{k_j}). \end{aligned}$$

Therefore, it follows with $\Gamma_1^{k_j} \geq \varepsilon$ and for all $c > 0$, such that

$$\nabla f(x^{k_j})^T d^{k_j,1} = -\Gamma_1^{k_j} - \sum_{i \in \overline{W}_k^-} \frac{\lambda_i^{k_j,0} \theta_{k_j}}{\mu_i^{k_j}} g_i(x^{k_j}) \leq -\varepsilon + ch(x^{k_j}).$$

If $h(x^{k_j}) \leq \varepsilon_1 = \varepsilon/(2c)$, then $\nabla f(x^{k_j})^T d^{k_j,1} \leq -\varepsilon_2 = -\varepsilon/2$.

Case II: $\rho = 0$.

$$\begin{aligned} \nabla f(x^{k_j})^T d^{k_j,1} &= -(d^{k_j,0})^T \widehat{H}_{k_j} d^{k_j,0} + (\lambda^{k_j,0})_{W_{k_j}}^T U_{k_j}^{-1} \mu^k \|d^{k,0}\|^\omega \\ &= -(d^{k_j,0})^T \widehat{H}_{k_j} d^{k_j,0} + \sum_{i \in \overline{W}_k^-} \lambda_i^{k_j,0} \|d^{k,0}\|^\omega + \sum_{i \in \overline{W}_k^+} \lambda_i^{k_j,0} \|d^{k,0}\|^\omega \\ &\quad + \sum_{i \in W_k \setminus \overline{W}_k} \lambda_i^{k_j,0} \|d^{k,0}\|^\omega \end{aligned}$$

$$= -\Gamma_2^{k_j} + \sum_{i \in \overline{W}_k} \lambda_i^{k_j,0} \|d^{k,0}\|^\omega.$$

There exists a scalar $c > 0$ such that $\sum_{i \in \overline{W}_k^+} \lambda_i^{k_j,0} \|d^{k,0}\|^\omega < c \|d^{k,0}\|^\omega$, combining with $\Gamma_2^{k_j} \geq \varepsilon$, we have

$$\nabla f(x^{k_j})^T d^{k_j,1} = -\Gamma_2^{k_j} + \sum_{i \in \overline{W}_k} \lambda_i^{k_j,0} \|d^{k,0}\|^\omega \leq -\varepsilon + c \|d^{k,0}\|^\omega.$$

If $\|d^{k,0}\|^\omega \leq \varepsilon_1 = \varepsilon/(2c)$, then $\nabla f(x^{k_j})^T d^{k_j,1} \leq -\varepsilon_2 = -\varepsilon/2$.

Case III: $0 < \rho < 1$.

$$\begin{aligned} \nabla f(x^{k_j})^T d^{k_j,1} &= -(d^{k_j,0})^T \widehat{H}_{k_j} d^{k_j,0} + (\lambda^{k_j,0})_{\overline{W}_{k_j}}^T U_{k_j}^{-1} [(1-\rho)\mu^{k_j} \|d^{k,0}\|^\omega + \rho \theta_{k_j} v^{k_j}] \\ &= -(d^{k_j,0})^T \widehat{H}_{k_j} d^{k_j,0} + \sum_{i \in \overline{W}_k} \frac{\lambda_i^{k_j,0}}{\mu_i^{k_j}} [(1-\rho)\mu_i^{k_j} \|d^{k,0}\|^\omega - \rho \theta_{k_j} g_i(x^{k_j})] \\ &\quad + \sum_{i \in \overline{W}_k^+} \frac{\lambda_i^{k_j,0}}{\mu_i^{k_j}} [(1-\rho)\mu_i^{k_j} \|d^{k,0}\|^\omega - \rho \theta_{k_j} g_i(x^{k_j})] \\ &\quad + \sum_{i \in W_k \setminus \overline{W}_k} \frac{\lambda_i^{k_j,0}}{\mu_i^{k_j}} [(1-\rho)\mu_i^{k_j} \|d^{k,0}\|^\omega - \rho \theta_{k_j} \min\{-g_i(x^{k_j}), \lambda_i^{k_j,0}\}] \\ &= -\Gamma_3^{k_j} + \sum_{i \in \overline{W}_k^+} \lambda_i^{k_j,0} (1-\rho) \|d^{k,0}\|^\omega - \sum_{i \in \overline{W}_k^+} \frac{\lambda_i^{k_j,0}}{\mu_i^{k_j}} \rho \theta_{k_j} g_i(x^{k_j}). \end{aligned}$$

There exists a scalar $c_1 > 0$ such that $\sum_{i \in \overline{W}_k^+} \lambda_i^{k_j,0} (1-\rho) \|d^{k,0}\|^\omega < c_1 \|d^{k,0}\|^\omega$ and combining with $\Gamma_3^{k_j} \geq \varepsilon$, for all $c_2 > 0$, such that

$$\begin{aligned} \nabla f(x^{k_j})^T d^{k_j,1} &\leq -\Gamma_3^{k_j} + \sum_{i \in \overline{W}_k^+} \lambda_i^{k_j,0} (1-\rho) \|d^{k,0}\|^\omega - \sum_{i \in \overline{W}_k^+} \frac{\lambda_i^{k_j,0} \theta_{k_j}}{\mu_i^{k_j}} \rho g_i(x^{k_j}) \\ &\leq -\varepsilon + c_1 \|d^{k,0}\|^\omega + c_2 h(x^{k_j}). \end{aligned}$$

If $\|d^{k,0}\|^\omega \leq \varepsilon_2 = \varepsilon/(4c)$, $h(x^{k_j}) \leq \varepsilon_1 = \varepsilon/(4c)$, then $\nabla f(x^{k_j})^T d^{k_j,1} \leq -\varepsilon_3 = -\varepsilon/2$. Therefore, the conclusion holds. $\square$

Lemma 6. *The inner loop terminates in finite iterations.*

Proof. Suppose that the inner loop run infinitely, then the filter rejects the trial point $x^k + \alpha_{k,l} d^{k,1}$ and $\lim_{l \rightarrow \infty} \alpha_{k,l} = 0$. If $h(x^k) = 0$, from the the definition of $h(x^k)$, we

have

$$\begin{aligned} h(x^k + \alpha_{k,l}d^{k,1}) &= \sum_{i \in I} \max\{g_i(x^k + \alpha_{k,l}d^{k,1}), 0\} \\ &\leq \sum_{i \in I} \max\{g_i(x^k) + \alpha_{k,l} \nabla g_i(x^k)^T d^{k,1} + o(\|\alpha_{k,l}d^{k,1}\|^2), 0\}, \end{aligned}$$

it is obvious that $\nabla g_i(x^k)^T d^{k,1} < 0$ for all $i \in W_k$, so there exists a constant γ , such that

$$\begin{aligned} h(x^k + \alpha_{k,l}d^{k,1}) &\leq \sum_{i \in I} \max\{g_i(x^k)\} \leq \sum_{i \in I} (1 - \gamma) \max\{g_i(x^k)\} \\ &= (1 - \gamma)h(x^k) \leq \max_{0 \leq k \leq m(k)} (1 - \gamma)h(x^k) \end{aligned}$$

From Lemma 5, we have

$$\begin{aligned} f(x^k + \alpha_{k,l}d^{k,1}) &\leq f(x^k) + \alpha_{k,l} \nabla f(x^k)^T d^{k,1} + O(\|\alpha_{k,l}d^{k,1}\|^2) \\ &\leq f(x^k) \leq \max_{0 \leq k \leq m(k)} f(x^k) \end{aligned}$$

In view of equations (2.3) and (2.4), we know that $x^k + \alpha_{k,l}d^{k,1}$ must be acceptable for the filter and x^k , which is a contradiction. $\square$

Lemma 7. Suppose Algorithm 1 dose not terminate finitely and the assumptions hold, then $\lim_{k \rightarrow \infty} h(x^k) = 0$.

Proof. Suppose there exists a subsequence $\{x^{k_j}\}$ such that $\lim_{j \rightarrow \infty} h(x^{k_j}) = \varepsilon$ for some constant $\varepsilon > 0$. Without loss of generality, for all j , we assume that $((1 - \gamma)/2)^{1/2} \varepsilon \leq h(x^{k_j}) \leq ((1 - \gamma)/2)^{-1/2} \varepsilon$.

From Lemma 6, we know that there exists a constant K such that $x^{k_{j+1}}$ must be accepted by $\mathcal{F}_{k_{j+1}}$ for all $k_{j+1} \geq K$.

Therefore,

$$h(x^{k_{j+1}}) \leq \max_{0 \leq k_j \leq m(k)} (1 - \gamma)h(x^k) \text{ or } f(x^{k_{j+1}}) \leq \max_{0 \leq k_j \leq m(k)} f(x^{k_j}) - \gamma h(x^{k_{j+1}}).$$

So, we have

$$f(x^{k_{j+1}}) \leq \max_{0 \leq k_j \leq m(k)} f(x^{k_j}) - \gamma h(x^{k_{j+1}}) \leq \max_{0 \leq k_j \leq m(k)} f(x^{k_j}) - \gamma((1 - \gamma)/2)^{1/2} \varepsilon.$$

Let $j \rightarrow \infty$ and the above inequality implies $f(x^{k_j}) \rightarrow -\infty$ in contradiction to the assumption that f is bounded below. $\square$

Lemma 8. If Algorithm 1 dose not terminate finitely, then, there exists an index set K , such that $\lim_{k \rightarrow \infty, k \in K} \nabla f(x^k)^T d^{k,1} = 0$.

Proof. Since $\nabla f(x^k)^T d^{k,1} \leq 0$, there exists a constant $\xi_k > 0$, which $\{\xi_k\}$ is uniformly bounded and $\liminf_k \xi_k > 0$, such that $f(x^k + \alpha_k d^{k,1}) - \max_{0 \leq k \leq m(k)} f(x^k) \leq$

$f(x^k + \alpha_k d^{k,1}) - f(x^k) \leq \xi_k \alpha_k \nabla f(x^k)^T d^{k,1} \leq 0$. It follows from Assumption 1 that $\{x^k\}_{k \in K} \rightarrow x^*$ for some index set K . Combing with continuity of f yields $f(x^k) \rightarrow f(x^*)$, $k \in K, k \rightarrow +\infty$.

Therefore,

$$0 = \lim_{k \rightarrow \infty, k \in K} [f(x^k + \alpha_k d^{k,1}) - \max_{0 \leq k \leq m(k)} f(x^k)] \leq \lim_{k \rightarrow \infty, k \in K} \xi_k \alpha_k \nabla f(x^k)^T d^{k,1} \leq 0.$$

Since $\liminf_k \xi_k \alpha_k > 0$, so $\lim_{k \rightarrow \infty, k \in K} \nabla f(x^k)^T d^{k,1} = 0$. $\square$

Lemma 9. *If $h(x^k) = 0$ and $\nabla f(x^k)^T d^{k,1} = 0$ hold, then x^k is a KKT point of problem (1.1).*

Proof. Since $h(x^k) = 0$, then $g_i(x^k) \leq 0, i \in I$, and then $\bar{W}_k^+ = \emptyset$. And since $\nabla f(x^k)^T d^{k,1} = 0$, it follows from equation (3.3),

$$\begin{aligned} \nabla f(x^k)^T d^{k,1} &= -(d^{k,0})^T \hat{H}_k d^{k,0} - (\lambda^{k,0})_{\bar{W}_k}^T U_k^{-1} [(1-\rho)\mu^k \|d^{k,0}\|^\omega + \rho\theta_k v^k] \\ &= -(d^{k,0})^T \hat{H}_k d^{k,0} - \sum_{i \in \bar{W}_k} \frac{\lambda_i^{k,0}}{\mu^k} [(1-\rho)\mu^k \|d^{k,0}\|^\omega - \rho\theta_k g_i(x^k)] \\ &\quad - \sum_{i \in W_k \setminus \bar{W}_k} \frac{\lambda_i^{k,0}}{\mu^k} [(1-\rho)\mu^k \|d^{k,0}\|^\omega - \rho\theta_k \min\{-g_i(x^k), \lambda_i^{k,0}\}] = 0. \end{aligned}$$

therefore,

$$\begin{aligned} (d^{k,0})^T \hat{H}_k d^{k,0} &= 0, \\ \sum_{i \in \bar{W}_k} \lambda_i^{k,0} [(1-\rho)\mu^k \|d^{k,0}\|^\omega - \frac{\rho\theta_k}{\mu^k} g_i(x^k)] &= 0, \\ \sum_{i \in W_k \setminus \bar{W}_k} \lambda_i^{k,0} [(1-\rho)\mu^k \|d^{k,0}\|^\omega - \frac{\rho\theta_k}{\mu^k} \min\{-g_i(x^k), \lambda_i^{k,0}\}] &= 0. \end{aligned}$$

Since $I(x^k) \subseteq W_k$, hence it follows from equation (2.3) that,

$$\mu_i^k \nabla g_i(x^k)^T d^{k,0} = -g_i(x^k) \lambda_i^{k,0} = 0 \forall i \in I(x^k).$$

We have $\mu^k > 0$ by step 8 of Algorithm 1, so from Assumption 4, we know that $d^{k,0} = 0$, $\lambda_i^{k,0} g_i(x^k) = 0, i \in I$. And because $(\rho\theta_k)/\mu^k > 0$, so combining these with equation (2.3), the KKT condition is established. $\square$

Theorem 1. *Suppose the Assumption 1-4 hold, and the sequence $\{(x^k, \lambda^k)\}$ which is generated by Algorithm 1 is infinite, then every accumulation point of the sequence $\{(x^k, \lambda^k)\}$ is a KKT pair of problem (1.1).*

4. NUMERICAL RESULTS

Algorithm 1 is implemented in the environment of MATLAB R2016a. We give our preliminary results on the some test problems from Hock and Schittkowski [22], and compare with the algorithm in [21] and that in Matlab.

The results are summarized in Table 1. The details about the implementation are described as follows.

- (a) The parameter values as chosen as follows: $\gamma = 10^{-4}$, $h_{\max} = 10^6$, $v = 0.5$, $\rho = 0.5$, $\chi_1 = 10$, $\phi_{\max} = 0.5$, $\varepsilon = 5$, $\omega = 2.5$.
- (b) The meanings of some notations in Table 1 are described as follows:
- No: the problem number given in Hock and Schittkowski [22];
 - n: the number of variables;
 - m: the number of constraints;
 - NIT: the number of iterations;
 - NF: the number of evaluations for $f(x)$;
 - NG: the number of evaluations for $g(x)$.
- (c) A stops if $|\nabla f(x^k)^T d^{k,1}|/(\|f(x^k)\| + 1) \leq 10^{-6}$ and $h(x^k) \leq 10^{-6}$.
- (d) H_k is updated by the damped BFGS formula.

Table 1 Numerical results

No	n	m	Algorithm 1			Algorithm in [21]			Matlab
			NIT	NF	NG	NIT	NF	NG	NIT - NF
HS1	2	1	7	13	9	23	48	54	27 – 95
HS3	2	1	5	9	9	13	19	23	4 – 15
HS4	2	2	5	9	9	2	4	2	2 – 6
HS5	2	4	12	75	75	10	12	31	10 – 34
HS6	2	1	3	5	3	7	16	17	6 – 28
HS11	2	1	3	5	3	23	35	35	7 – 25
HS12	2	1	16	54	53	10	12	10	8 – 25
HS15	2	3	8	37	33	14	28	16	3 – 9
HS16	2	5	7	57	53	10	27	11	4 – 12
HS17	2	5	8	15	9	4	7	4	14 – 43
HS18	2	6	9	17	12	7	14	9	9 – 28
HS21	2	5	7	13	7	7	11	11	3 – 9
HS22	2	2	8	15	10	21	45	49	4 – 15
HS26	3	1	6	11	6	10	36	41	6 – 27
HS27	3	1	6	11	8	13	46	17	44 – 303
HS28	3	1	8	15	9	17	33	40	7 – 29
HS30	3	7	9	18	18	6	11	20	11 – 44
HS33	3	6	5	9	9	3	13	22	5 – 20
HS35	3	4	8	15	15	14	43	59	6 – 24
HS43	4	3	6	11	6	12	22	17	12 – 63

$HS4652$	13	32	25	13	57	83	12 – 75
$HS4852$	10	21	19	14	24	14	8 – 50
$HS4952$	27	69	51	103	198	213	19 – 117

In the table, we first give the result of algorithm 1. For comparison, we have included the corresponding results obtained by Wang et al. [21] and the optimization code in Matlab (column 'MATLAB'). Compared with [21] and the code in Matlab, algorithm 1 has a relatively small iteration number both in NIT and in NF/NG. Therefore, our algorithm is effective and has numerical promising.

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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR FRACTIONAL Q -DIFFERENCE EQUATIONS

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Abstract. In this work, two different boundary value problems of fractional q -difference equation are discussed. The first one is fractional q -difference equation with integral boundary conditions and the other one is three point boundary value problem of fractional q -difference equation with p -Laplacian. By using fixed point theorems, some sufficient conditions that guarantee the existence and uniqueness of solutions are given.

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Keywords: fractional q -difference equations, integral boundary value problems, the fixed point theorem, existence, uniqueness

1. INTRODUCTION

The quantum calculus deals with quantum derivatives and integrals, and has proven to be relevant for quantum mechanics. It allows to deal with continuous functions, which do not need to be smooth. There are several types of quantum calculus: h -calculus (also known as the calculus of finite differences), the q -calculus and Hahn's calculus. In this paper we are concerned with the q -calculus. The q -derivative and the q -integral (also known as Jackson derivative and Jackson integral, respectively) were first defined by Jackson [12, 13] and had proven to have important applications on many subjects, like in hypergeometric series, complex analysis, particle physics and quantum mechanics. For a general introduction to the q -calculus we refer the reader to the books [10, 14]. Fractional calculus, on the other hand, generalizes integer-order analysis, by considering derivatives and integrals of non-integer order, and found many applications in several fields of applied sciences and engineering such as physics, chemistry, biology, economics, signal and image processing, calculus of variations, control theory, electrochemistry, electrical networks and many other fields for example [3, 7, 9, 21]. Several definitions of fractional derivatives are available in the literature, including the Riemann-Liouville, Grunwald-Letnikov, Caputo, Riesz, Riesz-Caputo, Weyl, Hadamard, and Chen derivatives. In this paper we are concerned with the Riemann-Liouville type fractional derivative. For an excellent book on the theory of fractional calculus we refer the reader to [17].

The natural extension, which we investigate here, is to consider a quantum fractional calculus, which unifies these two theories by considering quantum derivatives of non-integer order. Nowadays, fractional q -difference calculus has been given in wide applications of different science areas, which include basic hyper-geometric functions, mechanics, the theory of relativity, combinatorics and discrete mathematics. Therefore, fractional q -difference calculus has been of great interest and many good results can be found in [2, 4–6] and references therein. For different problems of fractional q -difference equations, the existence and the uniqueness of solutions have been always considered in literature. To solve these boundary value problems, some techniques have been applied, such as the monotone iterative technique, the lower-upper solution method, the Schauder fixed point theorem and the Krasnoselskii fixed point theorem. For details, one can see [19, 20, 23, 24].

By using Schauder fixed point theorem and the Banach fixed point theorem, Yang [22] discussed a fractional q -difference equation with three-point boundary conditions:

$$D_q^\alpha x(t) + f(t, x(t), x(t)) + g(t, x(t)) = 0, \quad 0 < t < 1, \quad 2 < \alpha < 3,$$

$$x(0) = D_q x(0) = 0, \quad D_q x(1) = \beta D_q x(\eta),$$

where $0 < \beta \eta^{\alpha-2} < 1$, $0 < q < 1$ and D_q^α is the Riemann-Liouville fractional q -derivative of order α .

In [25], Zhao studied the existence of positive solutions for the following q -fractional boundary value problems with p -Laplacian

$$D_q^\beta (\phi_p(D_q^\alpha u(t))) = f(t, u(t)), \quad t \in (0, 1),$$

$$u(0) = 0, \quad u(1) = \int_0^1 h(t) u(t) d_q t, \quad D_q^\alpha u(0) = 0, \quad D_q^\alpha u(1) = b D_q^\alpha u(\eta),$$

where D_q^α , D_q^β are the fractional q -derivative of Riemann-Liouville type with $1 < \alpha$, $\beta \leq 2$, $0 \leq b < 1$, $0 < \eta < 1$, $\phi_p(s) = |s|^{p-2}s$, $\phi_p^{-1} = \phi_r$, $p^{-1} + r^{-1} = 1$, $p > 1$, $r > 1$ and $f \in C([0, 1] \times \mathbb{R}^+, \mathbb{R}^+)$, $h \in C([0, 1], \mathbb{R}^+)$.

In the light of the studies mentioned, in this paper firstly we investigate the existence and uniqueness of positive solutions for the following q -fractional boundary value problem

$$D_q^\alpha x(t) + f(t, x(t)) = 0, \quad t \in (0, 1) \quad (1.1)$$

$$x(0) = a \int_0^1 f_1(t) x(t) d_q t, \quad x(1) = b \int_0^1 f_2(t) x(t) d_q t, \quad (1.2)$$

where D_q^α is the fractional q -derivative of Riemann-Liouville type with $1 < \alpha \leq 2$, $a, b \in \mathbb{R}^+$ and $f \in C([0, 1] \times \mathbb{R}^+, \mathbb{R}^+)$, $f_1, f_2 \in C([0, 1], \mathbb{R}^+)$.

Secondly, we will give the existence result for the following q -fractional boundary value problems with p -Laplacian

$$D_q^\beta (\phi_p(D_q^\alpha x(t))) = f(t, x(t)) \quad (1.3)$$

$$\begin{aligned} x(0) &= a \int_0^1 f_1(t) x(t) d_q t, \quad x(1) = b \int_0^1 f_2(t) x(t) d_q t, \\ D_q^\alpha x(0) &= 0, \quad D_q^\alpha x(1) = c D_q^\alpha x(\eta), \end{aligned} \quad (1.4)$$

where D_q^α, D_q^β are the fractional q -derivative of Riemann-Liouville type with $1 < \alpha, \beta \leq 2$, $a, b \in \mathbb{R}^+$, $0 \leq c < 1$, $0 < \eta < 1$, $\varphi_p(s) = |s|^{p-2}s$ such that $\varphi_p^{-1} = \varphi_r$ with $p^{-1} + r^{-1} = 1$, $p > 1$, $r > 1$ and $f \in C([0, 1] \times \mathbb{R}^+, \mathbb{R}^+)$, $f_1, f_2 \in C([0, 1], \mathbb{R}^+)$.

2. PRELIMINARIES

In this section, we list some useful definitions and preliminaries, which will be used in the proofs of the main results.

Let $q \in (0, 1)$ and define

$$[a]_q = \frac{1 - q^a}{1 - q}, a \in \mathbb{R}.$$

The q -analogue of the power function $(a - b)^k$, $k \in N_0 = \{0, 1, 2, \dots\}$ is

$$(a - b)^0 = 1, \quad (a - b)^{(k)} = \prod_{i=0}^{k-1} (a - bq^i), \quad k \in N, a, b \in \mathbb{R}.$$

More generally, if $\alpha \in \mathbb{R}$, then

$$(a - b)^{(\alpha)} = a^\alpha \prod_{n=0}^{\infty} \frac{a - bq^n}{a - bq^{\alpha+n}}.$$

Note that, if $b = 0$ then $a^{(\alpha)} = a^\alpha$.

The q -gamma function is defined by

$$\Gamma_q(x) = \frac{(1 - q)^{(x-1)}}{(1 - q)^{x-1}}, \quad x \in \mathbb{R} \setminus \{0, -1, -2, \dots\}$$

then

$$\Gamma_q(x + 1) = [x] \Gamma_q(x).$$

The q -derivative of a function f is here defined by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1 - q)^x}, (D_q f)(0) = \lim_{x \rightarrow 0} (D_q f)(x) \quad \text{for } x \neq 0$$

and q -derivatives of higher order by

$$(D_q^0 f)(x) = f(x) \quad \text{and} \quad (D_q^n f)(x) = D_q(D_q^{n-1} f)(x), \quad n \in N.$$

The q -integral of a function f defined in the interval $[0, b]$ is given by

$$(I_q f)(x) = \int_0^x f(t) d_q t = x(1 - q) \sum_{n=0}^{\infty} f(xq^n) q^n, \quad x \in [0, b].$$

If $a \in [0, b]$ and f is defined in the interval $[0, b]$, its integral from a to b is defined by

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t.$$

Similarly as done for derivatives, an operator I_q^n can be defined, i.e.,

$$(I_q^0 f)(x) = f(x) \quad \text{and} \quad (I_q^n f)(x) = I_q(I_q^{n-1} f)(x), \quad n \in \mathbb{N}.$$

The fundamental theorem of calculus applies to these operators I_q and D_q , i.e.,

$$(D_q I_q f)(x) = f(x)$$

and if f is continuous at $x = 0$, then

$$(I_q D_q f)(x) = f(x) - f(0).$$

We now point out two formulas that will be used later (${}_t D_q$ denotes the derivative with respect to variable t)

$${}_t D_q(t-s)^{(\alpha)} = [\alpha]_q (t-s)^{(\alpha-1)},$$

$$\left({}_x D_q \int_0^x f(x, t) d_q t \right)(x) = \int_0^x {}_x D_q f(x, t) d_q t + f(qx, x).$$

Remark 1. If $\alpha > 0$ and $a \leq b \leq t$, then $(t-a)^{(\alpha)} \geq (t-b)^{(\alpha)}$.

Definition 1 ([1]). Let $\alpha \geq 0$ and f be a function defined on $[0, 1]$. The fractional q -integral of the Riemann-Liouville type is

$$(I_q^0 f)(x) = f(x)$$

and

$$(I_q^\alpha f)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x-qt)^{(\alpha-1)} f(t) d_q t, \quad x \in [0, 1].$$

Definition 2 ([16]). The fractional q -derivative of the Riemann-Liouville type of order $\alpha \geq 0$ is defined by

$$(D_q^\alpha f)(x) = f(x)$$

and

$$(D_q^\alpha f)(x) = (D_q^p I_q^{p-\alpha} f)(x), \quad \alpha > 0,$$

where p is the smallest integer greater than or equal to α .

Next, we list some properties about q -derivative and q -integral that are already known in the literature, which are helpful in proofs of our main results.

Lemma 1 ([15]). (1) If f and g are q -integral on the interval $[a, b]$, $\alpha \in \mathbb{R}$, $c \in [a, b]$, then

- (1) $\int_a^b (f(t) + g(t)) d_q t = \int_a^b f(t) d_q t + \int_a^b g(t) d_q t$
- (2) $\int_a^b \alpha f(t) d_q t = \alpha \int_a^b f(t) d_q t$

$$(3) \int_a^b f(t) d_q t = \int_a^c f(t) d_q t + \int_c^b f(t) d_q t$$

$$(4) \int x^\alpha d_q s = \frac{x^{\alpha+1}}{[\alpha+1]}, \quad (\alpha \neq -1);$$

(2) If $|f|$ is q -integral on the interval $[0, x]$, then

$$\left| \int_0^x f(t) d_q t \right| \leq \int_0^x |f(t)| d_q t;$$

(3) If f and g are q -integral on the interval $[0, x]$, $f(t) \leq g(t)$, $\forall t \in [0, x]$, then

$$\int_0^x f(t) d_q t \leq \int_0^x g(t) d_q t.$$

Lemma 2 ([8]). Let $\alpha > 0$ and p be a positive integer. Then, the following equality holds:

$$(I_q^\alpha D_q^p f)(x) = (D_q^p I_q^\alpha f)(x) - \sum_{k=0}^{p-1} \frac{x^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)} (D_q^k f)(0).$$

Now, we will give the existence theorems used in our main results.

Theorem 1 ([11]). Let $T: X \rightarrow X$ be a map on a complete non-empty metric space. If some iterate T^n of T is a contraction, then T has a unique fixed point.

Theorem 2 ([18]). (Schauder Fixed Point Theorem) Let $(X, \|\cdot\|)$ be a Banach space and $S \subset X$ is compact, convex and nonempty. Then any continuous operator $A: S \rightarrow S$ has at least one fixed point.

Theorem 3 ([16]). (Nonlinear alternative for single valued maps) Let E be a Banach space, C a closed, convex subset of E , U an open subset of C and $0 \in U$. Suppose that $F: \overline{U} \rightarrow C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either

- (1) F has a fixed point in $\overline{U}$, or
- (2) There is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0, 1)$ with $u = \lambda F(u)$.

Suppose that the following condition is satisfied.

$$(H_1): 1 - b \int_{t=0}^1 f_2(t) t^{\alpha-2} d_q t - a \int_{t=0}^1 f_1(t) t^{\alpha-2} d_q t > 0.$$

Lemma 3. Let $h(t) \in C[0, 1]$ and $1 < \alpha \leq 2$, then the boundary value problem

$$(D_q^\alpha x)(t) + h(t) = 0, \quad 0 < t < 1, \quad (2.1)$$

$$x(0) = a \int_0^1 f_1(t) x(t) d_q t, \quad x(1) = b \int_0^1 f_2(t) x(t) d_q t \quad (2.2)$$

has a unique solution

$$x(t) = \int_{s=0}^1 G(t, qs) h(s) d_q s$$

such that

$$\begin{aligned} G(t, s) = & g(t, s) + \frac{at^{\alpha-2}(1-t)}{\Delta} \left\{ \left[1 - b \int_0^1 t^{\alpha-1} f_2(t) d_q t \right] \int_0^1 g(t, s) f_1(t) d_q t \right. \\ & + b \int_0^1 t^{\alpha-1} f_1(t) d_q t \int_0^1 g(t, s) f_2(t) d_q t \left. \right\} \\ & + \frac{bt^{\alpha-1}}{\Delta} \left\{ \left[1 - a \int_0^1 t^{\alpha-2}(1-t) f_1(t) d_q t \right] \int_0^1 g(t, s) f_2(t) d_q t \right. \\ & + a \int_0^1 t^{\alpha-2}(1-t) f_2(t) d_q t \int_0^1 g(t, s) f_1(t) d_q t \left. \right\} \end{aligned}$$

and

$$g(t, s) = \frac{1}{\Gamma_q(\alpha)} \begin{cases} (t(1-s))^{\alpha-1} - (t-s)^{\alpha-1} & , \quad s \leq t \\ (t(1-s))^{\alpha-1} & , \quad s \geq t. \end{cases}$$

Proof. From Lemma 2, since $D_q^\alpha x(t) = -h(t)$ we have

$$\begin{aligned} x(t) &= -I_q^\alpha h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, \quad c_1, c_2 \in \mathbb{R} \\ &= -\frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} h(s) d_q s + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}. \end{aligned}$$

Using the first boundary condition

$$x(0) = a \int_0^1 f_1(t) x(t) d_q t$$

we get

$$c_2 = a \int_0^1 f_1(t) x(t) d_q t.$$

So we have

$$x(t) = -\frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} h(s) d_q s + c_1 t^{\alpha-1} + at^{\alpha-2} \int_0^1 f_1(t) x(t) d_q t.$$

Then using the second boundary condition

$$\begin{aligned} x(1) &= b \int_0^1 f_2(t) x(t) d_q t \\ &= -\frac{1}{\Gamma_q(\alpha)} \int_0^t (1-qs)^{(\alpha-1)} h(s) d_q s + c_1 + a \int_0^1 f_1(t) x(t) d_q t \end{aligned}$$

we get

$$c_1 = \frac{1}{\Gamma_q(\alpha)} \int_0^t (1-qs)^{(\alpha-1)} h(s) d_q s - a \int_0^1 f_1(t) x(t) d_q t + b \int_0^1 f_2(t) x(t) d_q t.$$

Therefore, we have

$$\begin{aligned}
 x(t) &= -\frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} h(s) d_qs + \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^t (1-qs)^{(\alpha-1)} h(s) d_qs \\
 &\quad - at^{\alpha-1} \int_0^1 f_1(t) x(t) d_qt + bt^{\alpha-1} \int_0^1 f_2(t) x(t) d_qt + at^{\alpha-2} \int_0^1 f_1(t) x(t) d_qt \\
 &= -\frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} h(s) d_qs + \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^t (1-qs)^{(\alpha-1)} h(s) d_qs \\
 &\quad - at^{\alpha-2}(t-1) \int_0^1 f_1(t) x(t) d_qt + bt^{\alpha-1} \int_0^1 f_2(t) x(t) d_qt \\
 &= \int_0^1 g(t, qs) h(s) d_qs + at^{\alpha-2}(1-t) \int_0^1 f_1(t) x(t) d_qt + bt^{\alpha-1} \int_0^1 f_2(t) x(t) d_qt.
 \end{aligned} \tag{2.3}$$

At first, multiply both sides of (2.3) by $f_1(t)$, then take the integral from 0 to 1, we get

$$\begin{aligned}
 \int_0^1 f_1(t) x(t) d_qt &= \int_{t=0}^1 f_1(t) \int_{s=0}^1 g(t, qs) h(s) d_qs d_qt \\
 &\quad + a \int_0^1 \left(f_1(t) t^{\alpha-2}(1-t) \int_0^1 f_1(t) x(t) d_qt \right) d_qt \\
 &\quad + b \int_0^1 \left(f_1(t) t^{\alpha-1} \int_0^1 f_2(t) x(t) d_qt \right) d_qt
 \end{aligned}$$

and so

$$\begin{aligned}
 \int_0^1 f_1(t) x(t) d_qt &\left[1 - a \int_0^1 f_1(t) t^{\alpha-2}(1-t) d_qt \right] \\
 &- b \int_0^1 f_2(t) x(t) d_qt \int_0^1 f_1(t) t^{\alpha-1} d_qt = \int_{t=0}^1 f_1(t) \int_{s=0}^1 g(t, qs) h(s) d_qs d_qt.
 \end{aligned}$$

Similarly, multiply both sides of (2.3) by $f_2(t)$, then take the integral from 0 to 1, we have

$$\begin{aligned}
 \int_0^1 f_2(t) x(t) d_qt &= \int_{t=0}^1 f_2(t) \int_{s=0}^1 g(t, qs) h(s) d_qs d_qt \\
 &\quad + a \int_0^1 \left(f_2(t) t^{\alpha-2}(1-t) \int_0^1 f_1(t) x(t) d_qt \right) d_qt \\
 &\quad + b \int_0^1 \left(f_2(t) t^{\alpha-1} \int_0^1 f_2(t) x(t) d_qt \right) d_qt
 \end{aligned}$$

and so

$$\int_0^1 f_1(t) x(t) d_qt \left[-a \int_0^1 f_2(t) t^{\alpha-2}(1-t) d_qt \right]$$

$$+ \int_0^1 f_2(t) x(t) d_q t \left[1 - b \int_0^1 f_2(t) t^{\alpha-1} d_q t \right] = \int_{t=0}^1 f_2(t) \int_{s=0}^1 g(t, qs) h(s) d_q s d_q t.$$

If we define

$$\begin{aligned} \Delta &= \begin{vmatrix} 1 - a \int_0^1 f_1(t) t^{\alpha-2} (1-t) d_q t & -b \int_0^1 f_1(t) t^{\alpha-1} d_q t \\ -a \int_0^1 f_2(t) t^{\alpha-2} (1-t) d_q t & 1 - b \int_0^1 f_2(t) t^{\alpha-1} d_q t \end{vmatrix} \\ &= (1 - a \int_0^1 f_1(t) t^{\alpha-2} (1-t) d_q t) (1 - b \int_0^1 f_2(t) t^{\alpha-1} d_q t) \\ &\quad - ab \int_0^1 f_2(t) t^{\alpha-2} (1-t) d_q t \int_0^1 f_1(t) t^{\alpha-1} d_q t, \end{aligned}$$

we have

$$\begin{aligned} &\int_0^1 f_1(t) x(t) d_q t \\ &= \frac{1}{\Delta} \begin{vmatrix} \int_{t=0}^1 f_1(t) \int_{s=0}^1 g(t, qs) h(s) d_q s d_q t & -b \int_0^1 f_1(t) t^{\alpha-1} d_q t \\ \int_{t=0}^1 f_2(t) \int_{s=0}^1 g(t, qs) h(s) d_q s d_q t & 1 - b \int_0^1 f_2(t) t^{\alpha-1} d_q t \end{vmatrix} \\ &= \frac{1}{\Delta} \left\{ \left(1 - b \int_0^1 f_2(t) t^{\alpha-1} d_q t \right) \int_{s=0}^1 \int_{t=0}^1 g(t, qs) h(s) f_1(t) d_q t d_q s \right. \\ &\quad \left. + b \int_0^1 f_1(t) t^{\alpha-1} d_q t \int_{s=0}^1 \int_{t=0}^1 g(t, qs) h(s) f_2(t) d_q t d_q s \right\} \end{aligned}$$

and

$$\begin{aligned} &\int_0^1 f_2(t) x(t) d_q t \\ &= \frac{1}{\Delta} \begin{vmatrix} 1 - a \int_0^1 f_1(t) t^{\alpha-2} (1-t) d_q t & \int_{t=0}^1 f_1(t) \int_{s=0}^1 g(t, qs) h(s) d_q s d_q t \\ -a \int_0^1 f_2(t) t^{\alpha-2} (1-t) d_q t & \int_{t=0}^1 f_2(t) \int_{s=0}^1 g(t, qs) h(s) d_q s d_q t \end{vmatrix} \\ &= \frac{1}{\Delta} \left\{ \left(1 - a \int_0^1 f_1(t) t^{\alpha-2} (1-t) d_q t \right) \int_{s=0}^1 \int_{t=0}^1 g(t, qs) h(s) f_2(t) d_q t d_q s \right. \\ &\quad \left. + a \int_0^1 f_2(t) t^{\alpha-2} (1-t) d_q t \int_{s=0}^1 \int_{t=0}^1 g(t, qs) h(s) f_1(t) d_q t d_q s \right\}. \end{aligned}$$

Because of the condition (H_1) , we can easily see that $\Delta > 0$. Therefore, we have

$$\begin{aligned} x(t) &= \int_0^1 g(t, qs) h(s) d_q s \\ &\quad + at^{\alpha-2} (1-t) \frac{1}{\Delta} \left\{ \left(1 - b \int_0^1 f_2(t) t^{\alpha-1} d_q t \right) \int_{s=0}^1 \int_{t=0}^1 g(t, qs) h(s) f_1(t) d_q t d_q s \right. \\ &\quad \left. + b \int_0^1 f_1(t) t^{\alpha-1} d_q t \int_{s=0}^1 \int_{t=0}^1 g(t, qs) h(s) f_2(t) d_q t d_q s \right\} \end{aligned}$$

$$\begin{aligned}
 & + bt^{\alpha-1} \frac{1}{\Delta} \left\{ \left(1 - a \int_0^1 f_1(t) t^{\alpha-2} (1-t) d_q t \right) \int_{s=0}^1 \int_{t=0}^1 g(t, qs) h(s) f_2(t) d_q t d_q s \right. \\
 & \left. + a \int_0^1 f_2(t) t^{\alpha-2} (1-t) d_q t \int_{s=0}^1 \int_{t=0}^1 g(t, qs) h(s) f_1(t) d_q t d_q s \right\}
 \end{aligned}$$

or

$$\begin{aligned}
 x(t) &= \int_0^1 g(t, qs) h(s) d_q s \\
 &+ at^{\alpha-2} (1-t) \frac{1}{\Delta} \int_{s=0}^1 \left[\int_{t=0}^1 g(t, qs) \left\{ f_1(t) \left(1 - b \int_0^1 f_2(t) t^{\alpha-1} d_q t \right) \right. \right. \\
 &\quad \left. \left. + b f_2(t) \int_0^1 f_1(t) t^{\alpha-1} d_q t \right\} d_q t \right] h(s) d_q s \\
 &+ bt^{\alpha-1} \frac{1}{\Delta} \int_{s=0}^1 \left[\int_{t=0}^1 g(t, qs) \left\{ f_2(t) \left(1 - a \int_0^1 f_1(t) t^{\alpha-2} (1-t) d_q t \right) \right. \right. \\
 &\quad \left. \left. + a f_1(t) \int_0^1 f_2(t) t^{\alpha-2} (1-t) d_q t \right\} d_q t \right] h(s) d_q s \\
 &= \int_0^1 G(t, qs) h(s) d_q s.
 \end{aligned}$$

Consequently, we can write

$$\begin{aligned}
 G(t, s) &= g(t, s) + \frac{at^{\alpha-2} (1-t)}{\Delta} \left\{ \left[1 - b \int_0^1 f_2(t) t^{\alpha-1} d_q t \right] \int_0^1 g(t, s) f_1(t) d_q t \right. \\
 &\quad \left. + b \int_0^1 f_1(t) t^{\alpha-1} d_q t \int_0^1 g(t, s) f_2(t) d_q t \right\} \\
 &+ \frac{bt^{\alpha-1}}{\Delta} \left\{ \left[1 - a \int_0^1 f_1(t) t^{\alpha-2} (1-t) d_q t \right] \int_0^1 g(t, s) f_2(t) d_q t \right. \\
 &\quad \left. + a \int_0^1 f_2(t) t^{\alpha-2} (1-t) d_q t \int_0^1 g(t, s) f_1(t) d_q t \right\}.
 \end{aligned}$$

□

Using the definition of the Green's function for our problem, we get the following inequality

$$\begin{aligned}
 G(t, s) &\leq \frac{1}{\Gamma_q(\alpha)} \left\{ 1 + \frac{as^{\alpha-2}}{\Delta} \left[B_1 \int_0^1 f_1(t) d_q t + B_2 \int_0^1 f_2(t) d_q t \right] \right. \\
 &\quad \left. + \frac{bs^{\alpha-1}}{\Delta} \left[A_1 \int_0^1 f_2(t) d_q t + A_2 \int_0^1 f_1(t) d_q t \right] \right\} \left(\gamma(1-s)^{\alpha-1} \right).
 \end{aligned}$$

So, if we define

$$\frac{1}{\Gamma_q(\alpha)} \left\{ 1 + \frac{a}{\Delta} \left[B_1 \int_0^1 f_1(t) d_q t + B_2 \int_0^1 f_2(t) d_q t \right] + \frac{b}{\Delta} \left[A_1 \int_0^1 f_2(t) d_q t + A_2 \int_0^1 f_1(t) d_q t \right] \right\} = M, \quad (2.4)$$

we get $G(t, s) \leq M$.

3. MAIN RESULTS

In this section, primarily, we will give the existence and uniqueness results for the problem (1.1)-(1.2). Let $B = C([0, 1], \mathbb{R})$ is the Banach space with the norm $\|x\| = \max_{t \in [0, 1]} |x(t)|$.

We define the operator $T: B \rightarrow B$ by

$$Tx(t) = \int_0^1 G(t, qs) f(s, x(s)) d_q s.$$

Let $x \in B$, in view of the continuity of $G(t, s)$ and $f(t, x(t))$, we have $T: B \rightarrow B$ is continuous. Let $\Omega \subset B$ be bounded, it is easy to see that $T(\Omega)$ is uniformly bounded. On the other hand, since $G(t, s)$ is uniformly continuous on $[0, 1] \times [0, 1]$ we can get that $T(\Omega)$ is equicontinuous. By the Arzela-Ascoli theorem, we have $T: B \rightarrow B$ is completely continuous.

Theorem 4. Assume that $f(t, 0) \neq 0$ and there exist $k, h \in L^1[0, 1]$ such that

$$|f(t, x)| \leq k(t) |x| + h(t), \quad (t, x) \in [0, 1] \times \mathbb{R}$$

and

$$M \int_0^1 k(s) d_q s < 1,$$

where M is defined by (2.4). Then the boundary value problem (1.1)-(1.2) has at least one solution $x^* \in C[0, 1]$ which is different from 0.

Proof. Let we define the numbers

$$\int_0^1 k(s) d_q s = K$$

and

$$\int_0^1 h(s) d_q s = L.$$

Thus we get $MK < 1$, since $f(t, 0) \neq 0$ there exist such $[a, b] \subseteq [0, 1]$ that $\min_{a \leq t \leq b} |f(t, 0)| > 0$. Moreover $h(t) > |f(t, 0)|$ then $L > 0$.

Assume that $A = ML(1 - MK)^{-1}$ and $\Omega = \{x \in E: \|x\| < A\}$. Let $x \in \partial\Omega$ and $\lambda > 1$ with $T(x) = \lambda x$.

In that case, we have

$$\lambda A = \lambda \|x\| = \|Tx\| = \max_{t \in [0, 1]} |Tx(t)|$$

$$\begin{aligned}
 &\leq \max_{t \in [0,1]} \left| \int_0^1 G(t, qs) f(s, x(s)) d_qs \right| \\
 &\leq \max_{t \in [0,1]} \int_0^1 |G(t, qs)| |f(s, x(s))| d_qs \\
 &\leq \max_{t \in [0,1]} \int_0^1 M \cdot \{k(s) |x(s)| + h(s)\} d_qs \\
 &= MK \cdot \|x\| + ML.
 \end{aligned}$$

Therefore, we get

$$\lambda \leq MK + \frac{ML}{A} = MK + \frac{ML}{ML(1 - MK)^{-1}} = 1.$$

This conflicts with $\lambda > 1$.

Consequently, by the nonlinear alternative of Leray Schauder type, we deduce that T has a fixed point $x^* \in \bar{\Omega}$ which is a solution of ((1.1)-(1.2)). $\square$

Now, we will give the uniqueness of solution for our problem, using the following well-known contraction mapping theorem named also as the Banach fixed point theorem. Let B be a Banach space and S a nonempty closed subset of B . Assume $A: S \rightarrow S$ is a contraction, i.e., there is a λ ($0 < \lambda < 1$) such that $\|Ax - Ay\| \leq \lambda \|x - y\|$ for all x, y in S . Then A has a unique fixed point in S .

Theorem 5. Suppose that (H_1) holds. Also, we assume that (H_2) : Let the function $f(t, x)$ satisfy the following Lipschitz condition: there is a constant $L > 0$ such that $|f(t, x) - f(t, y)| \leq L|x - y|$, for $t \in [0, 1]$ for all $x, y \in C[0, 1]$. Moreover, $LM < 1$ where M is defined in (2.4). Then the problem (1.1)-(1.2) has a unique solution in $C[0, 1]$.

Proof. For $x, y \in C[0, 1]$ and $t \in [0, 1]$, we have

$$\begin{aligned}
 |Tx(t) - Ty(t)| &= \left| \int_0^1 G(t, qs) (f(s, x(s)) - f(s, y(s))) d_qs \right| \\
 &\leq \int_0^1 |G(t, qs)| |f(s, x(s)) - f(s, y(s))| d_qs \\
 &\leq M \int_0^1 L \cdot |x(s) - y(s)| d_qs \\
 &\leq ML \cdot \|x - y\|.
 \end{aligned}$$

Then, we obtain

$$\|Tx - Ty\| = \max_{t \in [0,1]} |Tx(t) - Ty(t)| \leq ML \cdot \|x - y\| = \lambda \|x - y\|,$$

where $\lambda = ML \in (0, 1)$. Hence, T is a contraction mapping and the theorem is proved. $\square$

We are now in a position to state and prove the existence result for the problem (1.3)-(1.4).

Lemma 4. Suppose that (H_1) holds. Let $g \in C[0, 1]$, $1 < \alpha$, $\beta \leq 2$, $0 \leq c < 1$, $0 < \eta < 1$. Then

$$\begin{aligned} D_q^\beta (\varphi_p(D_q^\alpha x(t))) &= g(t) \\ x(0) &= a \int_0^1 f_1(t)x(t) d_q t, \quad x(1) = b \int_0^1 f_2(t)x(t) d_q t, \\ D_q^\alpha x(0) &= 0, \quad D_q^\alpha x(1) = c D_q^\alpha x(\eta), \end{aligned}$$

is equivalent to

$$x(t) = \int_0^1 G(t, qs) \varphi_r \left(\int_0^1 H(s, q\tau) g(\tau) d_q \tau \right) d_q s,$$

where

$$\begin{aligned} H(t, s) &= m(t, s) + \frac{c^{p-1} t^{\beta-1}}{1 - c^{p-1} \eta^{\beta-1}} m(\eta, s), \\ m(t, s) &= \frac{1}{\Gamma_q(\beta)} \begin{cases} (t(1-s))^{\beta-1} - (t-s)^{\beta-1}, & 0 \leq s \leq t \leq 1 \\ (t(1-s))^{\beta-1}, & 0 \leq t \leq s \leq 1. \end{cases} \end{aligned}$$

and $G(t, s)$ is as given in Lemma 3.

Proof. This lemma can be proven similarly to Lemma 2.6 in [25]. $\square$

Remark 2. We can easily get

$$0 \leq H(t, qs) \leq \frac{1}{\Gamma_q(\beta)} t^{\beta-1} (1-qs)^{(\beta-1)} \leq \frac{1}{\Gamma_q(\beta)}, \quad t, s \in [0, 1] \times [0, 1].$$

Theorem 6. Assume that the function $f(t, x)$ is continuous with respect to $x \in \mathbb{R}$. If $N > 0$ satisfies $M \cdot Q^{r-1} \leq N \cdot (\Gamma_q(\beta))^{r-1}$, where $Q > 0$ satisfies

$$Q \geq \max_{\|x\| \leq N} |f(t, x)|, \quad \text{for } t \in [0, 1]$$

then the problem (1.3)-(1.4) has a solution $x(t)$.

Proof. Let $K = \{x \in B : \|x\| \leq N\}$. Note that K is closed, bounded and convex subset of B to which Schauder fixed point theorem is applicable. Define $A : K \rightarrow B$ by

$$Ax(t) := \int_0^1 G(t, qs) \varphi_r \left(\int_0^1 H(s, q\tau) f(\tau, x(\tau)) d_q \tau \right) d_q s,$$

for $t \in [0, 1]$. Obviously the solutions of problem (1.3)-(1.4) are the fixed points of operator A . It can be shown that $A : K \rightarrow B$ is continuous.

Claim that $A : K \rightarrow K$. Let $x \in K$. Consider

$$|Ax(t)| = \left| \int_0^1 G(t, qs) \varphi_r \left(\int_0^1 H(s, q\tau) f(\tau, x(\tau)) d_q \tau \right) d_q s \right|$$

$$\begin{aligned}
&\leq \int_0^1 |G(t, qs)| \left| \Phi_r \left(\int_0^1 H(s, q\tau) f(\tau, x(\tau)) d_q \tau \right) \right| d_q s \\
&\leq \int_0^1 M \cdot \left[\int_0^1 H(s, q\tau) d_q \tau \cdot Q \right]^{r-1} d_q s \\
&= M \cdot \left(\frac{Q}{\Gamma_q(\beta)} \right)^{r-1} \leq N,
\end{aligned}$$

for every $t \in [0, 1]$. This implies $\|Ax\| \leq N$.

It can be shown that $A: K \rightarrow K$ is a compact operator by the Arzela-Ascoli theorem. Hence A has a fixed point in K by Schauder fixed point theorem. $\square$

Corollary 1. *If f is continuous and bounded on $[0, 1] \times \mathbb{R}$, then the problem (1.3)-(1.4) has a solution.*

Proof. Since the function $f(t, x)$ is bounded, it has a supremum for $t \in [0, 1]$ and $x \in \mathbb{R}$. Let us choose $P > \sup \{|f(t, x)| : (t, x) \in [0, 1] \times \mathbb{R}\}$. Pick N large enough such that $P < N$. Then there is a number Q such that $P > Q$, where $Q \geq \max \{|f(t, x)| : t \in [0, 1], |x| \leq N\}$. Hence

$$1 < \frac{N}{P} \leq \frac{N}{Q},$$

and thus the problem (1.3)-(1.4) has a solution by Theorem 6. $\square$

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NUMERICAL OSCILLATION OF NONLINEAR GENERALIZED DELAY SINGLE SPECIES POPULATION MODEL

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Abstract. In this paper, we mainly consider the oscillation of numerical solutions for a nonlinear delay differential equation which is generalized from a delay Lotka-Volterra type single species population growth model. By studying the corresponding difference scheme of the equation discretized by θ -method, forward Euler method and backward Euler method, some sufficient conditions under which the numerical solutions oscillate are obtained. Furthermore, we prove that the positive non-oscillatory numerical solutions tend to the equilibrium of the original differential equation. Finally, some numerical experiments are given to verify the theoretical results.

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Keywords: nonlinear differential equation, delay differential equation, θ -method, forward Euler method, backward Euler method, oscillation

1. INTRODUCTION

In recent years, the reason why the properties of solutions of nonlinear delay differential equations (NDDEs) are getting more and more attention is that this type of equation has been widely used to describe phenomena in various fields of science such as life science, biology, ecology and physics [1, 3, 7, 14, 15, 17]. Many researchers hold strong interest in all kinds of behaviors of solutions of NDDEs such as stability [24], periodicity [4], hopf bifurcation [25] and oscillation [11, 13]. Relative to the oscillation of analytic solutions, it is necessary to study the oscillation of the corresponding numerical solutions. Because only those numerical methods that preserve the inherent properties of the continuous model are meaningful and valuable. Nowadays, some results on the numerical oscillation of NDDEs have been published in [18–20]. Different from them, in this paper, we consider the following NDDEs

$$\dot{x}(t) = x(t)(a + bx^p(t - \tau) - cx^q(t - \tau)), \quad t \geq 0, \quad (1.1)$$

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which is called generalized single species population model. The parameters in (1.1) satisfy

$$a, c, p, q, \tau > 0, b \in \mathbb{R}, \quad p < q. \quad (1.2)$$

We consider the solutions of (1.1) with initial condition of the form

$$x(t) = \varphi(t), \quad -\tau \leq t \leq 0,$$

where $\varphi \in C([- \tau, 0], (0, \infty))$ and $\varphi(0) > 0$.

In [12], Ladas and Qian studied globally asymptotically stability for every positive solution of (1.1). After that, some extended forms of (1.1) were considered. The global attractive and the oscillation of a unique positive periodic solution of delay periodic logistic equation were studied in [22], sufficient conditions were obtained for the existence and global attractivity of positive periodic solution of an impulsive DDE with Allee effect [23], the existence of a positive periodic solution of the delayed periodic logistic equation was established in [5], and some corresponding discrete models were studied also [6].

In particular, when $p = 1$ and $q = 2$, then (1.1) gives

$$\dot{x}(t) = x(t)(a + bx(t - \tau) - cx^2(t - \tau)), \quad (1.3)$$

which is a delay Lotka-Volterra type single species population growth model proposed by Gopalsamy and Ladas [9]. As a case in exhibition of the Allee effect [21], (1.3) means the positive feedback effects of aggregation and cooperation are dominated by density-dependent stabilizing negative feedback effects due to intraspecific competition when the density of the population is higher than a critical value. In this model, the per capita growth is expressed as a quadratic function, related either at present or in history, of the density. Specifically, this per-capita growth rate can be interpreted as a "first order" nonlinear approximation of more general types of plausible nonlinear growth rates with single humps. As for the research on the properties of (1.3), the interested readers can refer to [8].

The definitions of oscillatory for solutions of differential equation and difference equation can be found in [2], next we list some helpful results which will be used in the upcoming analysis.

Theorem 1 ([10, Corollary 7.1.1]). *Consider the difference equation*

$$a_{n+1} - a_n + \sum_{j=-k}^l q_j a_{n+j} = 0, \quad (1.4)$$

assume that $k, l \in \mathbb{N}$ and $q_j \in \mathbb{R}$ for $j = -k, \dots, l$. Then the following statements are equivalent:

- (i) Every solution of (1.4) oscillates.
- (ii) The characteristic equation $r - 1 + \sum_{j=-k}^l q_j r^j = 0$ has no positive roots.

Theorem 2 ([10, Theorem 7.2.1]). *For the difference equation*

$$a_{n+1} - a_n + la_{n-k} = 0, \quad (1.5)$$

where $l \in \mathbb{R}$, $k \in \mathbb{Z}$. Then every solution of (1.5) oscillates if and only if one of the following conditions holds:

- (i) $k = -1$ and $l \leq -1$;
- (ii) $k = 0$ and $l \geq 1$;
- (iii) $k \in \{\dots, -3, -2\} \cup \{1, 2, \dots\}$ and $l(k+1)^{k+1}/k^k > 1$.

Lemma 1 ([12, Theorem 1]). *Assume that (1.2) holds, then every solution of (1.1) oscillates about x^* if and only if*

$$(qc(x^*)^q - pb(x^*)^p)\tau > \frac{1}{e}, \quad (1.6)$$

where x^* is the positive equilibrium of (1.1).

2. OSCILLATIONS OF NUMERICAL SOLUTIONS

In this section, we will apply the θ -method to the simplified form of (1.1), then by analyzing the characteristic equation of the corresponding difference scheme we get the conditions under which the numerical solutions oscillate for two different ranges of parameter θ .

At first, making the change of variable $x(t) = x^*e^{y(t)}$, (1.1) is reformulated as

$$\dot{y}(t) + (qc(x^*)^q - pb(x^*)^p)g(y(t - \tau)) = 0, \quad (2.1)$$

where

$$g(v) = \frac{c(x^*)^q(e^{qv} - 1) - b(x^*)^p(e^{pv} - 1)}{qc(x^*)^q - pb(x^*)^p},$$

so the linearized form of (2.1) is

$$\dot{z}(t) + (cq(x^*)^q - bp(x^*)^p)z(t - \tau) = 0. \quad (2.2)$$

In the rest paper, we mainly investigate the conditions under which the numerical solution of (1.1) is oscillatory when (1.6) is satisfied. In this situation, the θ -method preserves the oscillation of the analytic solution of (1.1). Therefore this numerical method is meaningful and available.

Let $h = \tau/m$ be a given stepsize with integer $m \in \mathbb{N}^+$. Application of the θ -method to (2.1) gives

$$\begin{aligned} y_{n+1} = y_n - h\theta(qc(x^*)^q - pb(x^*)^p)g(y_{n+1-m}) \\ - h(1 - \theta)(qc(x^*)^q - pb(x^*)^p)g(y_{n-m}), \end{aligned} \quad (2.3)$$

where $\theta \in [0, 1]$, y_{n+1} and y_{n+1-m} are approximations to $y(t)$ and $y(t - \tau)$ of (2.1) at t_{n+1} , respectively.

Let $y_n = \ln(x_n/x^*)$, we have the difference scheme for (1.1)

$$x_{n+1} = x_n \exp \left\{ h\theta \left[c \left((x^*)^q - x_{n+1-m}^q \right) + b \left(x_{n+1-m}^p - (x^*)^p \right) \right] + h(1-\theta) \left[c \left((x^*)^q - x_{n-m}^q \right) + b \left(x_{n-m}^p - (x^*)^p \right) \right] \right\}. \quad (2.4)$$

Next, we consider whether the θ -method preserves the oscillation of (1.1). Thus, we study the oscillation of (2.3) to get the conditions under which (2.4) oscillates.

On the other hand, using the θ -method to (2.2) yields difference equation

$$z_{n+1} = z_n - h\theta(qc(x^*)^q - pb(x^*)^p)z_{n+1-m} - h(1-\theta)(qc(x^*)^q - pb(x^*)^p)z_{n-m}, \quad (2.5)$$

which is exactly the linearized form of (2.3).

For simplicity, let $\alpha = qc(x^*)^q$ and $\beta = pb(x^*)^p$, then (1.6) and (2.5) reduces to the following simplified inequality

$$(\alpha - \beta)\tau > \frac{1}{e} \quad (2.6)$$

and

$$z_{n+1} = z_n - h\theta(\alpha - \beta)z_{n+1-m} - h(1-\theta)(\alpha - \beta)z_{n-m}, \quad (2.7)$$

respectively.

Lemma 2. *The characteristic equation of (2.5) is given by*

$$r = S(-h(\alpha - \beta)r^{-m}). \quad (2.8)$$

Proof. Substituting $z_n = r^n z_0$ into (2.7), and according to the expression of the stability function of the θ -method, we can get this proof. $\square$

The next lemma is for the first case of θ which is in the interval $[0, 0.5]$.

Lemma 3. *If (1.2) and (2.6) hold, then for $\theta \in [0, 0.5]$, (2.8) has no positive roots.*

Proof. We introduce $D(r) = r - S(-h(\alpha - \beta)r^{-m})$. On the other hand, from Lemma 3 in [16] we have

$$S(-h(\alpha - \beta)r^{-m}) \leq \exp(-h(\alpha - \beta)r^{-m}).$$

In the following, we shall prove $E(r) = r - \exp(-h(\alpha - \beta)r^{-m}) > 0$ for all $r > 0$. We consider it by using the proof by contradiction. Suppose that there exists a $r_0 > 0$ such that $E(r_0) \leq 0$, then

$$r_0 \leq \exp(-h(\alpha - \beta)r_0^{-m}),$$

hence

$$r_0^m \leq \exp(-\tau(\alpha - \beta)r_0^{-m}),$$

which, rearranging terms, writes also as

$$\tau(\alpha - \beta)e \leq \tau(\alpha - \beta)r_0^{-m} \exp(1 - \tau(\alpha - \beta)r_0^{-m}).$$

We examine it in two cases:

- If $1 - \tau(\alpha - \beta)r_0^{-m} = 0$, then $\tau(\alpha - \beta)e \leq 1$, which contradicts to (2.6).
- If $1 - \tau(\alpha - \beta)r_0^{-m} \neq 0$, by the fact $e^x < 1/(1-x)$ we have

$$\exp(1 - \tau(\alpha - \beta)r_0^{-m}) < \frac{1}{\tau(\alpha - \beta)r_0^{-m}},$$

that is $\tau(\alpha - \beta)e \leq 1$, which also contradicts to (2.6).

Therefore, the following inequality is true for all $r > 0$,

$$D(r) = r - S(-h(\alpha - \beta)r^{-m}) \geq r - \exp(-h(\alpha - \beta)r^{-m}) = E(r) > 0,$$

the proof is finished. $\square$

For the second case: $\theta \in (0.5, 1]$, we might as well assume $m > 1$.

Lemma 4. *If (1.2) and (2.6) hold, then for $\theta \in (0.5, 1]$, (2.8) has no positive roots for $h < h_0$, where*

$$h_0 = \begin{cases} \infty, & (\alpha - \beta)\tau \geq 1; \\ \tau(1 + \ln(\alpha - \beta)), & (\alpha - \beta)\tau < 1. \end{cases} \quad (2.9)$$

Proof. We only to prove that $D(r) > 0$. Note that, the function $S(-h(\alpha - \beta)r^{-m})$ is increasing about θ if $r > 0$, that is

$$S(-h(\alpha - \beta)r^{-m}) = \frac{1 - h(1 - \theta)(\alpha - \beta)r^{-m}}{1 + h\theta(\alpha - \beta)r^{-m}} \leq \frac{1}{1 + h(\alpha - \beta)r^{-m}}.$$

Next, we are going to prove

$$r - \frac{1}{1 + h(\alpha - \beta)r^{-m}} > 0 \quad (2.10)$$

holds conditionally. Rearranging terms on the left of (2.10) gives

$$r - \frac{1}{1 + h(\alpha - \beta)r^{-m}} = \frac{r^{1-m}}{1 + h(\alpha - \beta)r^{-m}}(r^m - r^{m-1} + h(\alpha - \beta)).$$

Denoting

$$G(r) = r^m - r^{m-1} + h(\alpha - \beta),$$

then $G(r)$ is the characteristic polynomial of the difference equation

$$v_{n+1} - v_n + h(\alpha - \beta)v_{n+1-m} = 0.$$

Next we aim to prove $G(r) > 0$ for $r > 0$. By means of Theorems 1 and 2, we know that $G(r)$ has no positive roots if and only if

$$h(\alpha - \beta) \frac{m^m}{(m-1)^{m-1}} > 1,$$

which can be restated as

$$\ln(\alpha - \beta)\tau + (m-1) \ln \frac{m}{m-1} > 0. \quad (2.11)$$

Therefore we have the following two cases:

- If $(\alpha - \beta)\tau \geq 1$, then (2.11) holds naturally for $m > 1$;
- If $(\alpha - \beta)\tau < 1$, in view of (2.9) as well as the fact $\ln(1+x) > x/(1+x)$ we know that (2.11) also holds.

Thus (2.10) is true for $h < h_0$. Then we arrive at

$$D(r) = r - S(-h(\alpha - \beta)r^{-m}) > r - \frac{1}{1 + h(\alpha + \beta)r^{-m}} > 0,$$

which indicates that (2.8) has no positive roots. This proof is completed. $\square$

We therefore have the following theorem.

Theorem 3. *If (1.2) and (2.6) hold, then (2.4) is oscillatory for*

$$h < \begin{cases} \infty & \text{if } \theta \in [0, 0.5], \\ h_0 & \text{if } \theta \in (0.5, 1], \end{cases}$$

where h_0 is defined by (2.9).

3. ASYMPTOTIC BEHAVIOR OF NON-OSCILLATORY SOLUTION

In this section, we study the asymptotic behavior of non-oscillatory numerical solution of (1.1). That is, we aim to prove that the non-oscillatory solution of (2.4) approaches the equilibrium point.

Lemma 5 ([12, Lemma 4]). *Let $x(t)$ be a positive solution of (1.1) which does not oscillate about x^* , then $\lim_{t \rightarrow \infty} x(t) = x^*$.*

Lemma 6. *Let y_n be a non-oscillatory solution of (2.3), then $\lim_{n \rightarrow \infty} y_n = 0$.*

Proof. Assume $y_n > 0$ for n sufficiently large. The case of $y_n < 0$ is similar and be omitted. To begin with, we notice that

$$0 = a + b(x^*)^p - c(x^*)^q,$$

together with (1.2) which implies

$$\alpha - \beta = qc(x^*)^q - pb(x^*)^p > q(c(x^*)^q - b(x^*)^p) = qa > 0. \quad (3.1)$$

From (2.3) we have

$$y_{n+1} - y_n = -h\theta(\alpha - \beta)g(y_{n+1-m}) - h(1 - \theta)(\alpha - \beta)g(y_{n-m}) < 0, \quad (3.2)$$

then $\{y_n\}$ is decreasing. Hence there is a $\zeta > 0$ such that

$$\lim_{n \rightarrow \infty} y_n = \zeta. \quad (3.3)$$

In order to prove that $\zeta = 0$, in contrast we assume $\zeta > 0$, then there exists $\tilde{N} \in \mathbb{N}^+$ and any given $\varepsilon > 0$, such that for $n - m > \tilde{N}$, $0 < \zeta - \varepsilon < y_n < \zeta + \varepsilon$. So $y_{n-m} > \zeta - \varepsilon$ and $y_{n+1-m} > \zeta - \varepsilon$. Thus (3.2) gives

$$\begin{aligned} y_{n+1} - y_n &= -h\theta(\alpha - \beta)g(y_{n+1-m}) - h(1 - \theta)(\alpha - \beta)g(y_{n-m}) \\ &< -h\theta(\alpha - \beta)g(\zeta - \varepsilon) - h(1 - \theta)(\alpha - \beta)g(\zeta - \varepsilon) \\ &= -h(\alpha - \beta)g(\zeta - \varepsilon) < 0, \end{aligned}$$

results in $y_{n+1} - y_n < H < 0$, where

$$H = h(b(x^*)^p(e^{p(\zeta - \varepsilon)} - 1) - c(x^*)^q(e^{q(\zeta - \varepsilon)} - 1)).$$

Thus $y_n \rightarrow -\infty$ as $n \rightarrow \infty$, which contradicts to (3.3). The proof is complete. $\square$

Based on Lemma 6, we obtain the following theorem.

Theorem 4. *Let x_n be a positive solution of (2.4), which does not oscillate about x^* , then $\lim_{n \rightarrow \infty} x_n = x^*$.*

4. THE FORWARD EULER METHOD AND THE BACKWARD EULER METHOD

In this section, we shall carry out the analysis of oscillation and non-oscillation of (1.1) in the case of discretization by the forward Euler method and the backward Euler method. The results for the numerical oscillation and the asymptotic behavior will be given in detail.

4.1. The forward Euler method

Let $h = \tau/m$ be a given stepsize with $m \in \mathbb{N}^+$. Apply the forward Euler method to (2.1) and (2.2) gives

$$y_{n+1} = y_n - h(qc(x^*)^q - pb(x^*)^p)g(y_{n-m}) \quad (4.1)$$

and

$$z_{n+1} = z_n - h(qc(x^*)^q - pb(x^*)^p)z_{n-m}, \quad (4.2)$$

respectively.

Let $y_n = \ln(x_n/x^*)$ in (4.1), we have the discrete scheme for (1.1)

$$x_{n+1} = x_n \exp \left\{ h \left[c \left((x^*)^q - x_{n-m}^q \right) + b \left(x_{n-m}^p - (x^*)^p \right) \right] \right\}. \quad (4.3)$$

Lemma 7. *If (1.2) and (2.6) hold, then the characteristic equation of (4.2) has no positive roots.*

Proof. Set $z_n = r^n z_0$ in (4.2), we get the characteristic equation of (4.2) as follows

$$r = 1 - h(\alpha - \beta)r^{-m}. \quad (4.4)$$

Denoting

$$F(r) = r - (1 - h(\alpha - \beta)r^{-m}),$$

next we will discuss in two cases.

- (i) When $r \geq 1$, from (3.1) we know that $F(r) > 0$, so (4.4) has no positive roots in $[1, +\infty)$.
(ii) When $0 < r < 1$, in view of $1 + x < e^x$ for all $x > 0$, we get

$$1 - h(\alpha - \beta)r^{-m} < e^{-h(\alpha - \beta)r^{-m}},$$

then

$$F(r) = r - (1 - h(\alpha - \beta)r^{-m}) > r - e^{-h(\alpha - \beta)r^{-m}},$$

in order to prove $F(r) > 0$, we only need to prove $r > e^{-h(\alpha - \beta)r^{-m}}$, that is

$$\ln r > -h(\alpha - \beta)r^{-m}.$$

Set

$$\Gamma(r) = \ln r + h(\alpha - \beta)r^{-m},$$

from

$$\frac{d\Gamma}{dr} = \frac{r^m - \tau(\alpha - \beta)}{r^{m+1}} = 0$$

we have

$$r = (\tau(\alpha - \beta))^{\frac{1}{m}}.$$

So it is easy to know that

$$\min_{0 < r < 1} \Gamma(r) = \Gamma((\tau(\alpha - \beta))^{\frac{1}{m}}) = \frac{1 + \ln(\tau(\alpha - \beta))}{m},$$

in terms of (2.6) we have $\Gamma(r) > 0$ for $0 < r < 1$, thus $F(r) > 0$ for $0 < r < 1$. So (4.4) has no positive roots in $(0, 1)$. The proof is finished. $\square$

Theorem 5. If (1.2) and (2.6) hold, then (4.3) is oscillatory for any $h > 0$.

Similar to Lemma 6 and Theorem 4 we have the following theorem.

Theorem 6. Let x_n be a positive solution of (4.3), which does not oscillate about x^* , then $\lim_{n \rightarrow \infty} x_n = x^*$.

4.2. The backward Euler method

Using the similar technique in Section 4.1, we obtain the corresponding results for the backward Euler method.

Let $h = \tau/m$ ($m \in \mathbb{N}^+$), apply the backward Euler method to (2.1) and (2.2) gives

$$y_{n+1} = y_n - h(qc(x^*)^q - pb(x^*)^p)g(y_{n+1-m}) \quad (4.5)$$

and

$$z_{n+1} = z_n - h(qc(x^*)^q - pb(x^*)^p)z_{n+1-m}, \quad (4.6)$$

respectively.

Let $y_n = \ln(x_n/x^*)$ in (4.5), we have the discrete scheme for (1.1)

$$x_{n+1} = x_n \exp \left\{ h \left[c \left((x^*)^q - x_{n+1-m}^q \right) + b \left(x_{n+1-m}^p - (x^*)^p \right) \right] \right\}. \quad (4.7)$$

Lemma 8. If (1.2) and (2.6) hold, then the characteristic equation of (4.6) has no positive roots for $h < \bar{h}$, here

$$\bar{h} = \tau - \frac{1}{(\alpha - \beta)e}. \quad (4.8)$$

Theorem 7. If (1.2) and (2.6) hold, then (4.7) is oscillatory for $h < \bar{h}$, here $\bar{h}$ is defined in (4.8).

Theorem 8. Let x_n be a positive solution of (4.7), which does not oscillate about x^* , then $\lim_{n \rightarrow \infty} x_n = x^*$.

5. NUMERICAL EXPERIMENTS

In this section, we present some examples to verify the above theoretical results.

First of all, we consider the equation

$$\dot{x}(t) = x(t) (3 - 10x(t - 0.27) - 10x^2(t - 0.27)), \quad t \geq 0 \quad (5.1)$$

with initial condition $\varphi(t) = 1$ for $-0.27 \leq t \leq 0$. Since $a = 3$, $b = -10$, $c = 10$, $p = 1$, $q = 2$ and $\tau = 0.27$, then we compute that $x^* \approx 0.2416$ and $1/e < (\alpha - \beta)\tau \approx 0.9676 < 1$, so (2.6) is satisfied, which indicates the analytic solution of (5.1) is oscillatory. From Theorem 3 we know that the numerical solution of (5.1) should be oscillatory if we choose $\theta = 0.4 \in [0, 0.5]$, $m = 5$ and $h = \tau/m = 0.0540 < \infty$. At the same time, the numerical solution of (5.1) should also be oscillatory if we choose $\theta = 0.7 \in (0.5, 1]$, $m = 7$ and $h = \tau/m \approx 0.0385 < h_0 = \tau(1 + \ln(\alpha - \beta)) \approx 0.6146$. We draw the figures of the analytic solution and numerical solution of (5.1) in each of Figures 1 and 2 with the parameters mentioned above. From these figures we can

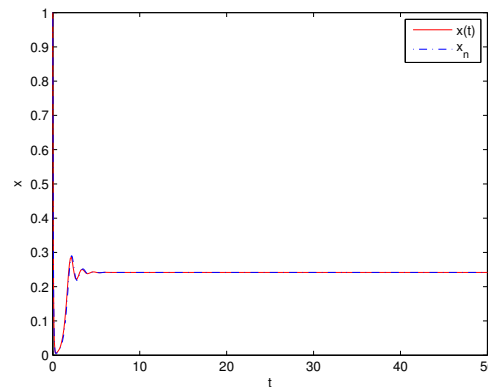


FIGURE 1. The analytic solution and the numerical solution of (5.1) with $m = 5$ and $\theta = 0.4$.

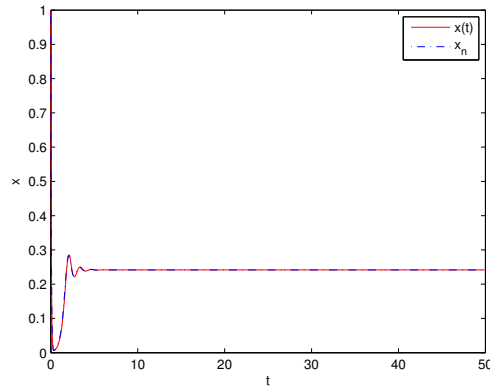


FIGURE 2. The analytic solution and the numerical solution of (5.1) with $m = 7$ and $\theta = 0.7$.

see that the numerical solution of (5.1) oscillates about x^* . That is, the numerical phenomena are consistent with Theorem 3.

Secondly, for another equation

$$\dot{x}(t) = x(t) \left(\frac{1}{11} + 2x(t-2) - 7x^2(t-2) \right), \quad t \geq 0 \quad (5.2)$$

with initial condition $\varphi(t) = 0.5$ for $-2 \leq t \leq 0$. It is easy to get that $x^* \approx 0.3256$ and $1/e < 1 < (\alpha - \beta)\tau \approx 1.6660$, so (2.6) holds, then the analytic solution of (5.2) is oscillatory. We set $\theta = 0.5 \in [0, 0.5]$, $m = 10$, $h = 0.2 < \infty$ in Figure 3 and $\theta = 0.55 \in (0.5, 1]$, $m = 20$, $h = 0.1 < h_0 = \infty$ in Figure 4, respectively. From the two

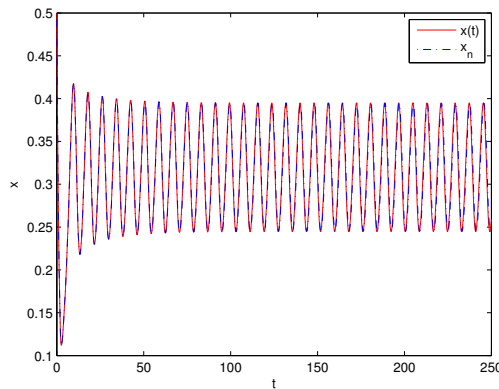


FIGURE 3. The analytic solution and the numerical solution of (5.2) with $m = 10$ and $\theta = 0.5$.

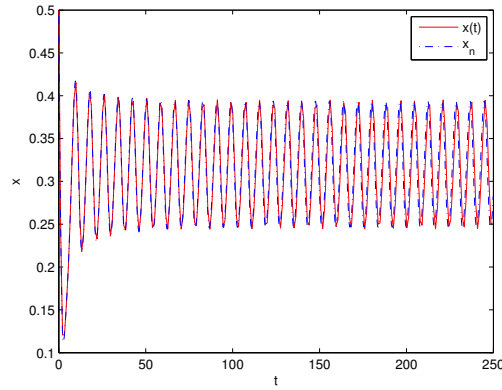


FIGURE 4. The analytic solution and the numerical solution of (5.2) with $m = 20$ and $\theta = 0.55$.

figures we see that the numerical solution of (5.2) oscillates about x^* , which are in agreement with Theorem 3.

Thirdly, we give an example to verify the asymptotic behavior. Consider the following equation

$$\dot{x}(t) = x(t) \left(\frac{1}{13} + \frac{1}{7}x(t-0.5) - \frac{1}{5}x^2(t-0.5) \right), \quad t \geq 0 \quad (5.3)$$

with initial condition $\varphi(t) = 2$ for $-0.5 \leq t \leq 0$. So we have $x^* \approx 1.0728$ and $(\alpha - \beta)\tau \approx 0.1536 < 1/e$, then (2.6) is not fulfilled. Thus the analytic solution of (5.3) is non-oscillatory. In Figure 5 we draw the figures of the analytic solution and

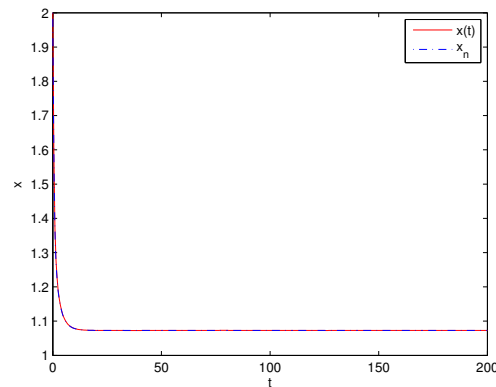


FIGURE 5. The analytic solution and the numerical solution of (5.3) with $m = 5$ and $\theta = 0.4$.

the numerical solution of (5.3) with $\theta = 0.4, m = 5$. We can see that $x(t) \rightarrow x^*$ as $t \rightarrow \infty$ and $x_n \rightarrow x^*$ as $n \rightarrow \infty$. The numerical results are identical with Theorem 4.

Finally, for the following four equations

$$\begin{aligned} \dot{x}(t) &= x(t) (0.5 - 13x^2(t-1.2) - 5x^5(t-1.2)), \quad t \geq 0, \\ \varphi(t) &= \frac{1}{3}, \quad -1.2 \leq t \leq 0, \end{aligned} \quad (5.4)$$

$$\begin{aligned} \dot{x}(t) &= x(t) \left(\frac{4}{21} - x^3(t-0.6) - \frac{1}{13}x^7(t-0.6) \right), \quad t \geq 0, \\ \varphi(t) &= 0.4, \quad -0.6 \leq t \leq 0, \end{aligned} \quad (5.5)$$

$$\begin{aligned} \dot{x}(t) &= x(t) (0.25 - 2x^2(t-2.2) - 13x^9(t-2.2)), \quad t \geq 0, \\ \varphi(t) &= 0.6, \quad -2.2 \leq t \leq 0, \end{aligned} \quad (5.6)$$

$$\begin{aligned} \dot{x}(t) &= x(t) \left(\frac{1}{17} - 4x^5(t-0.7) - 18x^8(t-0.7) \right), \quad t \geq 0, \\ \varphi(t) &= 0.12, \quad -0.7 \leq t \leq 0, \end{aligned} \quad (5.7)$$

we can verify Theorems 5 and 6 for the forward Euler method with (5.4) and (5.5) (see Figures 6, 7), Theorems 7 and 8 for the backward Euler method with (5.6) and (5.7) (see Figures 8, 9) in the similar way.

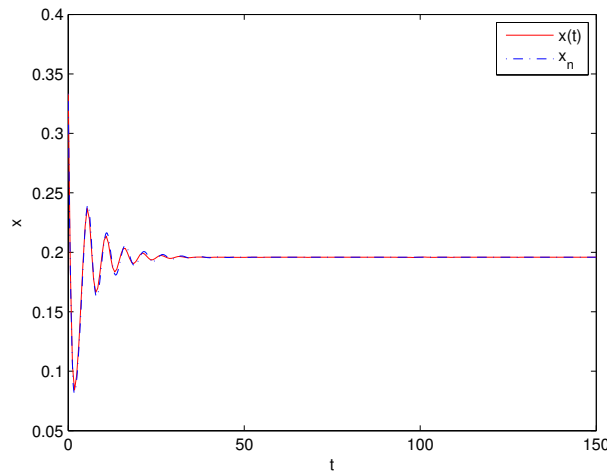


FIGURE 6. The analytic solution and the forward Euler numerical solution of (5.4) with $m = 20$.

From all these figures we can see that the numerical methods inherit the corresponding oscillation and asymptotic property of the above equations.

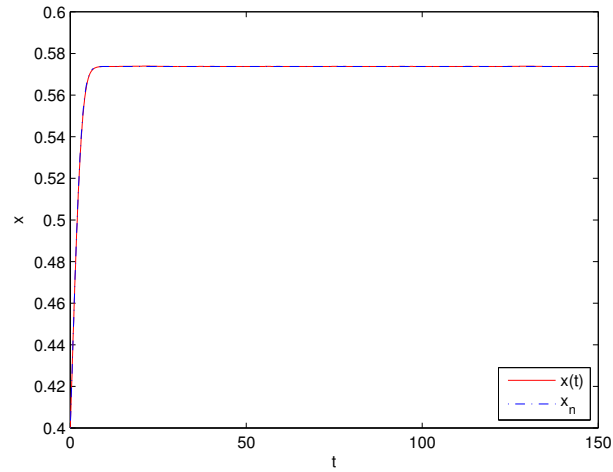


FIGURE 7. The analytic solution and the forward Euler numerical solution of (5.5) with $m = 10$.

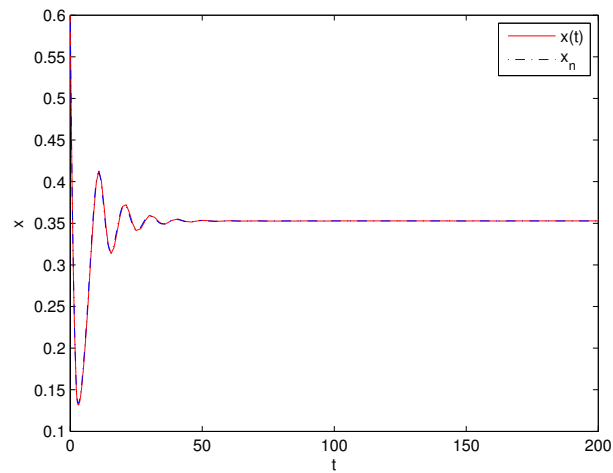


FIGURE 8. The analytic solution and the backward Euler numerical solution of (5.6) with $m = 35$.

6. CONCLUSION

In this paper, we investigate the oscillation and the asymptotic behavior of the numerical solution of nonlinear generalized delay single species population model with three kinds of numerical methods: the θ -method, the forward Euler method and the

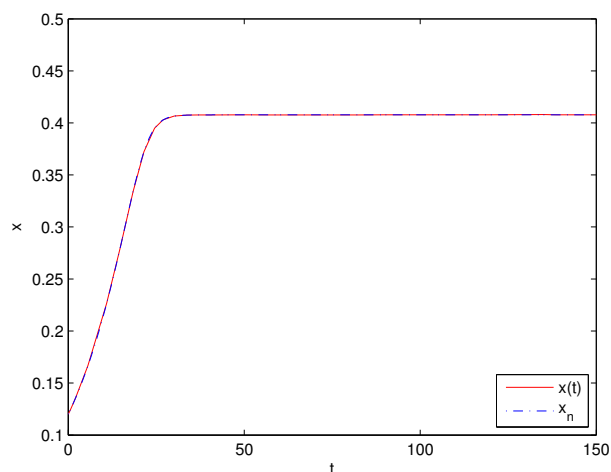


FIGURE 9. The analytic solution and the backward Euler numerical solution of (5.7) with $m = 14$.

backward Euler method. Under the condition that the analytic solution is oscillatory, several conditions under which the numerical solution oscillates are obtained. Furthermore, we confirmed that the non-oscillatory numerical solution approaches to the steady state of the equation. In our future work, we will consider the higher order numerical method, fractional order problem and variable exponents problem including the equation $\dot{x}(t) = x(t)(a + bx^{p(x)}(t - \tau) - cx^{q(x)}(t - \tau))$.

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ON QUATERNION-GAUSSIAN FIBONACCI POLYNOMIALS

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Abstract. In this paper, we define Gaussian Fibonacci quaternion polynomials and Gaussian Lucas quaternion polynomials. We also investigate some properties of these quaternion polynomials.

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Keywords: Gaussian numbers, Fibonacci numbers, Gaussian Fibonacci polynomials, quaternions

1. INTRODUCTION

Gaussian number, investigated by Gauss in 1832, is a complex number with integer coefficients. Horadam introduced the concept of complex Fibonacci numbers (Gaussian Fibonacci numbers) in 1963. Then, Gaussian Fibonacci numbers and Gaussian Lucas numbers are studied by many authors. Some example of these studies can be found in [1, 7, 8, 10, 12], among others.

The n th Gaussian Fibonacci number is defined by the relation

$$GF_n = GF_{n-1} + GF_{n-2}, \quad n \geq 2,$$

with initial conditions $GF_0 = i$ and $GF_1 = 1$.

It is easy to see that $GF_n = F_n + iF_{n-1}$, where F_n is the n th Fibonacci number defined recursively by $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0$, $F_1 = 1$.

Similarly, the n th Gaussian Lucas number is defined by the relation

$$GL_n = GL_{n-1} + GL_{n-2}, \quad n \geq 2,$$

with initial conditions $GL_0 = 2 - i$ and $GL_1 = 1 + 2i$.

It is clear that $GL_n = L_n + iL_{n-1}$, where L_n is the n th Lucas number defined recursively by $L_n = L_{n-1} + L_{n-2}$ with $L_0 = 2$, $L_1 = 1$.

The Fibonacci polynomials studied by Catalan in 1883 are defined by the relation

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \quad n \geq 2,$$

with initial conditions $F_0(x) = 0$ and $F_1(x) = 1$.

The Lucas polynomials studied by Bicknell in 1970 are defined by the relation

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x), \quad n \geq 2,$$

with initial conditions $L_0(x) = 2$ and $L_1(x) = x$.

In [11], Özkan and Taştan studied Gaussian Fibonacci polynomials, Gaussian Lucas polynomials and also their applications. They gave the definitions of the Gaussian Fibonacci and Lucas polynomials, and Binet's formulas for these polynomials as follows:

The Gaussian Fibonacci polynomials are defined by the relation

$$GF_n(x) = xGF_{n-1}(x) + GF_{n-2}(x), \quad n \geq 2, \quad (1.1)$$

with initial conditions $GF_0(x) = i$ and $GF_1(x) = 1$.

Moreover, it is easy to see that $GF_n(x) = F_n(x) + iF_{n-1}(x)$. Setting $x = 1$ in the eq. (1.1), the Gaussian Fibonacci number GF_n can be obtained.

The Binet's formulas for the Gaussian Fibonacci polynomials are given by

$$GF_n(x) = \frac{\alpha^{n-1}(x)(\alpha(x) + i) - \beta^{n-1}(x)(\beta(x) + i)}{\alpha(x) - \beta(x)}, \quad (1.2)$$

where $\alpha(x) = \frac{x+\sqrt{x^2+4}}{2}$ and $\beta(x) = \frac{x-\sqrt{x^2+4}}{2}$ are the roots of the equation $t^2 - xt - 1 = 0$.

The Gaussian Lucas polynomials are defined by the relation

$$GL_n(x) = xGL_{n-1}(x) + GL_{n-2}(x), \quad n \geq 2, \quad (1.3)$$

with initial conditions $GL_0(x) = 2 - ix$ and $GL_1(x) = x + 2i$.

Furthermore, it is clear that $GL_n(x) = L_n(x) + iL_{n-1}(x)$. Setting $x = 1$ in the eq. (1.3), the Gaussian Lucas number GL_n can be obtained.

The Binet's formulas for the Gaussian Lucas polynomials are given by

$$GL_n(x) = \alpha^{n-1}(x)(\alpha(x) + i) + \beta^{n-1}(x)(\beta(x) + i), \quad (1.4)$$

where $\alpha(x)$ and $\beta(x)$ are same as defined in eq. (1.2).

Quaternions, four-dimensional hyper-complex numbers, introduced by Sir William Rowan Hamilton in 1843. These numbers have found widespread application in quantum physics, computer graphics, robotics and signal processing.

A quaternion q is of the form

$$q = q_0 + q_1i + q_2j + q_3k = (q_0, q_1, q_2, q_3),$$

where q_0, q_1, q_2, q_3 are real numbers, and i, j, k are quaternionic units which satisfy the equalities

$$i^2 = j^2 = k^2 = ijk = -1, ij = k = -ji, \quad jk = i = -kj, \quad ki = j = -ik. \quad (1.5)$$

The set of all quaternions denoted by $\mathbb{H}$ is a non-commutative associative algebra over the real numbers. For a survey on quaternions, we refer the reader to [6, 15].

In [8], Horadam defined the Fibonacci and Lucas quaternions as

$$FQ_n = F_n + F_{n+1}i + F_{n+2}j + F_{n+3}k$$

and

$$LQ_n = L_n + L_{n+1}i + L_{n+2}j + L_{n+3}k,$$

respectively, where F_n is the n th Fibonacci number, L_n is the n th Lucas number, and i, j, k are quaternionic units which satisfy the rules (1.5).

There have been many studies in literature on Fibonacci and Lucas quaternions, see for example [2, 3, 9, 13, 14], among others.

The main objective of this paper is to define and study Gaussian Fibonacci and Lucas quaternion polynomials. We shall give recurrence relations, Binet's formulas, generating functions and summation formulas involving these quaternion polynomials.

2. THE GAUSSIAN FIBONACCI QUATERNION POLYNOMIALS

In this section, we first give the definitions of Gaussian Fibonacci quaternion polynomials and Gaussian Lucas quaternion polynomials. We then obtain some results for these quaternion polynomials.

For $n \geq 0$, Gaussian Fibonacci quaternion polynomials $\{GFQ_n(x)\}_{n=0}^\infty$ and Gaussian Lucas quaternion polynomials $\{GLQ_n(x)\}_{n=0}^\infty$ are defined by

$$GFQ_n(x) = (GF_n(x), GF_{n+1}(x), GF_{n+2}(x), GF_{n+3}(x)) \quad (2.1)$$

and

$$GLQ_n(x) = (GL_n(x), GL_{n+1}(x), GL_{n+2}(x), GL_{n+3}(x)), \quad (2.2)$$

respectively, where $GF_n(x)$ is the n th Gaussian Fibonacci polynomial, $GL_n(x)$ is the n th Gaussian Lucas polynomial.

It is easy to see that the n th Gaussian Fibonacci quaternion polynomial is defined recursively by

$$GFQ_n(x) = xGFQ_{n-1}(x) + GFQ_{n-2}(x), \quad n \geq 2, \quad (2.3)$$

with initial conditions

$$GFQ_0(x) = (i, 1, x + i, x^2 + 1 + i) = 2i + (x - 1)j + (x^2 + 2)k \quad (2.4)$$

and

$$\begin{aligned} GFQ_1(x) &= (1, x + i, x^2 + 1 + i, x^3 + 2x + i(x + 1)) \\ &= xi + (x^2 - x)j + (x^3 + 2x + 1)k. \end{aligned} \quad (2.5)$$

Similarly, the n th Gaussian Lucas quaternion polynomial is defined recursively by

$$GLQ_n(x) = xGLQ_{n-1}(x) + GLQ_{n-2}(x), \quad n \geq 2, \quad (2.6)$$

with initial conditions

$$GLQ_0(x) = (x^3 + 4x)k \quad (2.7)$$

and

$$GLQ_1(x) = (x^2 + 4)i + (x^4 + 5x^2 + 4)k. \quad (2.8)$$

It must be noted that if we set $x = 1$ in eqs. (2.1) or (2.3), we obtain Gaussian Fibonacci quaternions (see, [3, 5])

$$GFQ_n = GFQ_{n-1} + GFQ_{n-2}, \quad n \geq 2, \quad (2.9)$$

with initial conditions

$$\begin{aligned} GFQ_0 &= (i, 1, 1+i, 2+i) = 2i + 3k, \\ GFQ_1 &= (1, 1+i, 2+i, 3+2i) = i + 4k. \end{aligned}$$

and if we set $x = 1$ in eqs. (2.2) or (2.6), we obtain Gaussian Lucas quaternions (see, [4])

$$GLQ_n = GLQ_{n-1} + GLQ_{n-2}, \quad n \geq 2, \quad (2.10)$$

with initial conditions

$$\begin{aligned} GLQ_0 &= (2-i, 1+2i, 3+i, 4+3i) = 5k, \\ GLQ_1 &= (1+2i, 3+i, 4+3i, 7+4i) = 5i + 10k. \end{aligned}$$

Let $\alpha(x)$ and $\beta(x)$ be the roots of the characteristic equation $t^2 - xt - 1 = 0$ on the recurrence relation (2.3) of Gaussian Fibonacci quaternion polynomials. Here, $\alpha(x) = \frac{x+\sqrt{x^2+4}}{2}$ and $\beta(x) = \frac{x-\sqrt{x^2+4}}{2}$. These roots satisfy the following rules:

$$\begin{aligned} \alpha(x) + \beta(x) &= x, \quad \alpha(x) - \beta(x) = \sqrt{x^2+4}, \quad \alpha(x)\beta(x) = -1, \\ \frac{\alpha(x)}{\beta(x)} &= -\alpha^2(x), \quad \frac{\beta(x)}{\alpha(x)} = -\beta^2(x). \end{aligned} \quad (2.11)$$

We now give the Binet's formulas for the Gaussian Fibonacci and Lucas quaternion polynomials in the following theorem.

Theorem 1. *The Binet's formulas for the Gaussian Fibonacci and Lucas quaternion polynomials are given by*

$$GFQ_n(x) = \frac{GFQ_1(x)(\alpha^n(x) - \beta^n(x)) + GFQ_0(x)(\alpha^{n-1}(x) - \beta^{n-1}(x))}{\alpha(x) - \beta(x)} \quad (2.12)$$

and

$$GLQ_n(x) = \frac{GLQ_1(x)(\alpha^n(x) - \beta^n(x)) + GLQ_0(x)(\alpha^{n-1}(x) - \beta^{n-1}(x))}{\alpha(x) - \beta(x)} \quad (2.13)$$

respectively.

Proof. From the general solution for the recurrence relation and using initial conditions the desired results can be obtained easily. $\square$

If we set $x = 1$ in eq. (2.12), we obtain the Binet's formula for the Gaussian Fibonacci quaternions (see, [3, 5]) as follow:

$$GFQ_n = \frac{GFQ_1(\alpha^n - \beta^n) + GFQ_0(\alpha^{n-1} - \beta^{n-1})}{\alpha - \beta}.$$

Moreover, if we set $x = 1$ in eq. (2.13), we obtain the Binet's formula for the Gaussian Lucas quaternions (see, [4]) as follow:

$$GLQ_n = \frac{GLQ_1(\alpha^n - \beta^n) + GLQ_0(\alpha^{n-1} - \beta^{n-1})}{\alpha - \beta}.$$

Now, the ordinary generating functions, exponential generating functions and Poisson generating functions for the Gaussian Fibonacci quaternion polynomials and Gaussian Lucas quaternion polynomials are given in the following results, respectively.

Theorem 2. *The ordinary generating functions for the Gaussian Fibonacci quaternion polynomials and Gaussian Lucas quaternion polynomials are given by*

$$\begin{aligned} f(t) &= \frac{GFQ_0(x) + (GFQ_1(x) - xGFQ_0(x))t}{1 - xt - t^2} \\ &= \frac{2i + (x-1)j + (x^2+2)k + (-xi+k)t}{1 - xt - t^2}, \end{aligned} \quad (2.14)$$

and

$$\begin{aligned} g(t) &= \frac{GLQ_0(x) + (GLQ_1(x) - xGLQ_0(x))t}{1 - xt - t^2} \\ &= \frac{(x^3+4x)k + (x^2+4)(i+k)t}{1 - xt - t^2}, \end{aligned} \quad (2.15)$$

respectively.

Proof. Let $f(t)$ be the generating function for the Gaussian Fibonacci quaternion polynomials. Then we write

$$f(t) = \sum_{n=0}^{\infty} GFQ_n(x)t^n = GFQ_0(x) + GFQ_1(x)t + \dots + GFQ_n(x)t^n + \dots \quad (2.16)$$

Multiplying the eq. (2.16) with $-xt$ and $-t^2$, respectively, we get

$$-xtf(t) = -xGFQ_0(x)t - xGFQ_1(x)t^2 - \dots - xGFQ_{n-1}(x)t^n - \dots \quad (2.17)$$

and

$$-t^2f(t) = -GFQ_0(x)t^2 - GFQ_1(x)t^3 - \dots - GFQ_{n-2}(x)t^n - \dots \quad (2.18)$$

Then adding the eqs. (2.16), (2.17) and (2.18), we obtain

$$\begin{aligned} (1 - xt - t^2)f(t) &= GFQ_0(x) + (GFQ_1(x) - xGFQ_0(x))t \\ &\quad + \sum_{n=2}^{\infty} (GFQ_n(x) - xGFQ_{n-1}(x) - GFQ_{n-2}(x))t^n. \end{aligned}$$

From the eq. (2.3), we get

$$(1 - xt - t^2)f(t) = GFQ_0(x) + (GFQ_1(x) - xGFQ_0(x))t.$$

Using the eqs. (2.4) and (2.5), we have

$$(1 - xt - t^2)f(t) = 2i + (x - 1)j + (x^2 + 2)k + (-xi + k)t$$

which completes the proof of the first statement.

The second statement of the theorem can be proved in a similar manner. $\square$

If we set $x = 1$ in eq. (2.14), we obtain the ordinary generating function for the Gaussian Fibonacci quaternions (see, [3, 5]) as follow:

$$f(t) = \frac{2i + 3k + (k - i)t}{1 - t - t^2}.$$

Furthermore, if we set $x = 1$ in eq. (2.15), we obtain the ordinary generating function for the Gaussian Lucas quaternions (see, [4]) as follow:

$$g(t) = \frac{5k + 5(i + k)t}{1 - t - t^2}.$$

Theorem 3. *The exponential generating functions for the Gaussian Fibonacci quaternion polynomials and Gaussian Lucas quaternion polynomials are given by*

$$F(t) = \frac{(GFQ_1(x) - \beta(x)GFQ_0(x))e^{\alpha(x)t} - (GFQ_1(x) - \alpha(x)GFQ_0(x))e^{\beta(x)t}}{\alpha(x) - \beta(x)},$$

and

$$G(t) = \frac{(GLQ_1(x) - \beta(x)GLQ_0(x))e^{\alpha(x)t} - (GLQ_1(x) - \alpha(x)GLQ_0(x))e^{\beta(x)t}}{\alpha(x) - \beta(x)},$$

respectively.

Proof. Using the Binet's formula (2.12) of the Gaussian Fibonacci quaternion polynomials, we have

$$\begin{aligned} F(t) &= \sum_{n=0}^{\infty} GFQ_n(x) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{GFQ_1(x)(\alpha^n(x) - \beta^n(x))}{\alpha(x) - \beta(x)} \frac{t^n}{n!} + \sum_{n=0}^{\infty} \frac{GFQ_0(x)(\alpha^{n-1}(x) - \beta^{n-1}(x))}{\alpha(x) - \beta(x)} \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(\alpha(x)GFQ_1(x) + GFQ_0(x))(\alpha(x)t)^n}{\alpha(x)(\alpha(x) - \beta(x))n!} \\ &\quad - \sum_{n=0}^{\infty} \frac{(\beta(x)GFQ_1(x) + GFQ_0(x))(\beta(x)t)^n}{\beta(x)(\alpha(x) - \beta(x))n!} \\ &= \frac{(\alpha(x)GFQ_1(x) + GFQ_0(x))}{\alpha(x)(\alpha(x) - \beta(x))} \sum_{n=0}^{\infty} \frac{(\alpha(x)t)^n}{n!} \\ &\quad - \frac{(\beta(x)GFQ_1(x) + GFQ_0(x))}{\beta(x)(\alpha(x) - \beta(x))} \sum_{n=0}^{\infty} \frac{(\beta(x)t)^n}{n!} \end{aligned}$$

$$\begin{aligned}
&= \frac{(\alpha(x)GFQ_1(x) + GFQ_0(x))}{1 + \alpha^2(x)} e^{\alpha(x)t} + \frac{(\beta(x)GFQ_1(x) + GFQ_0(x))}{1 + \beta^2(x)} e^{\beta(x)t} \\
&= \frac{(GFQ_1(x) - \beta(x)GFQ_0(x))e^{\alpha(x)t} - (GFQ_1(x) - \alpha(x)GFQ_0(x))e^{\beta(x)t}}{\alpha(x) - \beta(x)}.
\end{aligned}$$

Thus, the proof of the first statement is completed. The second statement can be proved using the Binet's formula (2.13) of the Gaussian Lucas quaternion polynomials in a similar manner. $\square$

Theorem 4. *The Poisson generating functions for the Gaussian Fibonacci quaternion polynomials and Gaussian Lucas quaternion polynomials are given by*

$$\mathcal{F}(t) = \frac{(GFQ_1(x) - \beta(x)GFQ_0(x))e^{\alpha(x)t} - (GFQ_1(x) - \alpha(x)GFQ_0(x))e^{\beta(x)t}}{e^t(\alpha(x) - \beta(x))},$$

and

$$\mathcal{G}(t) = \frac{(GLQ_1(x) - \beta(x)GLQ_0(x))e^{\alpha(x)t} - (GLQ_1(x) - \alpha(x)GLQ_0(x))e^{\beta(x)t}}{e^t(\alpha(x) - \beta(x))},$$

respectively.

Proof. Since $\mathcal{F}(t) = e^{-t}F(t)$ and $\mathcal{G}(t) = e^{-t}G(t)$, the proof of the theorem is clear. $\square$

Theorem 5. *For $n \geq 2$, we have*

$$\sum_{k=1}^n GFQ_k(x) = \frac{1}{x}(GFQ_{n+1}(x) + GFQ_n(x) - GFQ_1(x) - GFQ_0(x)), \quad (2.19)$$

and

$$\sum_{k=1}^n GLQ_k(x) = \frac{1}{x}(GLQ_{n+1}(x) + GLQ_n(x) - GLQ_1(x) - GLQ_0(x)). \quad (2.20)$$

Proof. From eq. (1.1), we can write $GFQ_{n-1}(x) = \frac{1}{x}(GFQ_n(x) - GFQ_{n-2}(x))$. By the telescoping sum, we get

$$\begin{aligned}
\sum_{k=1}^n GFQ_k(x) &= \frac{1}{x} \sum_{k=2}^{n+1} GFQ_k(x) - \frac{1}{x} \sum_{k=0}^{n-1} GFQ_k(x) \\
&= \frac{1}{x}(GFQ_{n+1}(x) + GFQ_n(x) - GFQ_1(x) - GFQ_0(x))
\end{aligned}$$

which completes the proof of the eq. (2.19).

Since $GLQ_{n-1}(x) = \frac{1}{x}(GLQ_n(x) - GLQ_{n-2}(x))$ from eq. (1.3), eq. (2.20) can be obtained in a similar manner. $\square$

If we set $x = 1$ in eq. (2.20), we have [4]

$$\sum_{k=0}^n GLQ_k = GLQ_{n+2} - 5(i + 2k).$$

Corollary 1. For $n \geq 2$, we have

$$\begin{aligned}\sum_{k=1}^n GFQ_{2k}(x) &= \frac{1}{x}(GFQ_{2n+1}(x) - GFQ_1(x)), \\ \sum_{k=1}^n GFQ_{2k-1}(x) &= \frac{1}{x}(GFQ_{2n}(x) - GFQ_0(x)), \\ \sum_{k=1}^n GLQ_{2k}(x) &= \frac{1}{x}(GLQ_{2n+1}(x) - GLQ_1(x)), \\ \sum_{k=1}^n GLQ_{2k-1}(x) &= \frac{1}{x}(GLQ_{2n}(x) - GLQ_0(x)).\end{aligned}$$

We now define the matrices $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{S}$ as follows:

$$\mathbf{Q} = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} GFQ_2(x) & GFQ_1(x) \\ GFQ_1(x) & GFQ_0(x) \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} GLQ_2(x) & GLQ_1(x) \\ GLQ_1(x) & GLQ_0(x) \end{pmatrix}.$$

Theorem 6. For $n \geq 1$, we have

$$\mathbf{Q}^n \mathbf{R} = \begin{pmatrix} GFQ_{n+2}(x) & GFQ_{n+1}(x) \\ GFQ_{n+1}(x) & GFQ_n(x) \end{pmatrix}$$

and

$$\mathbf{Q}^n \mathbf{S} = \begin{pmatrix} GLQ_{n+2}(x) & GLQ_{n+1}(x) \\ GLQ_{n+1}(x) & GLQ_n(x) \end{pmatrix}.$$

Proof. The results can be obtained easily using the mathematical induction on n . $\square$

Theorem 7 (Cassini's Identity). For $n \geq 1$, we have

$$GFQ_{n+1}(x)GFQ_{n-1}(x) - GFQ_n^2(x) = (-1)^{n-1}(GFQ_2(x)GFQ_0(x) - GFQ_1^2(x)),$$

and

$$GLQ_{n+1}(x)GLQ_{n-1}(x) - GLQ_n^2(x) = (-1)^{n-1}(GLQ_2(x)GLQ_0(x) - GLQ_1^2(x)).$$

Proof. It is clear that $\det \mathbf{Q}^{n-1} = (-1)^{n-1}$, $\det \mathbf{R} = GFQ_2(x)GFQ_0(x) - GFQ_1^2(x)$ and $\det \mathbf{S} = GLQ_2(x)GLQ_0(x) - GLQ_1^2(x)$. Taking $\det \mathbf{Q}^{n-1} \mathbf{R}$ and $\det \mathbf{Q}^{n-1} \mathbf{S}$, we can obtain the desired results, respectively. $\square$

3. CONCLUSION

In this paper, we study the Gaussian Fibonacci and Lucas quaternion polynomials. We give some results including recurrence relations, Binet's formulas, generating functions and summation formulas for these quaternion polynomials.

It must be noted that for $x = 1$, the results for the Gaussian Fibonacci quaternion polynomials and Gaussian Lucas quaternion polynomials given in this study correspond to the Gaussian Fibonacci quaternions [3, 5] and Gaussian Lucas quaternions [4], respectively.

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INTEGRATIONS ON LATTICES

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Abstract. In this paper, we introduce the notion of integration with respect to a given derivation on a lattice. More precisely, we give the definitions of integrable elements of a lattice and their integral sets. We investigate several characterizations and properties of integrations on a lattice. Also, we give a lattice structure to the family of integral sets with respect to a given integration. Further, we provide a representation theorem for the lattice of fixed points of an isotone derivation based on the family of integral sets. As an application of this notion of integration, we use the integrable elements of a Boolean lattice to determine the necessary and sufficient conditions under which a linear differential equation on this Boolean lattice has a solution.

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1. INTRODUCTION

In 1957, the author E.C. Posner introduced the notion of derivations in prime rings [9]. Later on, this notion had many applications (see, e.g. [2]). Szász [10] has extended this notion of derivation to the setting of lattice structures. He has defined a derivation on a given lattice L as a function d satisfying the following two conditions:

$$d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y)) \quad \text{and} \quad d(x \vee y) = d(x) \vee d(y) \quad \text{for any } x, y \in L.$$

Ferrari [4] has investigated some properties of this notion and provided some interesting examples in particular classes of lattices. Xin et al. [14] have ameliorated this notion of derivation on a lattice by considering only the first condition, and they have shown that the second condition obviously holds for the isotone derivations on a distributive lattice. In the same paper, they have characterized distributive and modular lattices in terms of their isotone derivations. Later on, Xin [13] has focused his attention on the structure of the set of the fixed points of a derivation on a lattice, and has shown some relationships between this set and the notion of an ideal of a lattice.

This notion of derivation on a lattice is witnessing increased attention. It was applied in partially ordered sets [1, 17], in distributive lattices [16], in semilattices

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[15], in bounded hyperlattices [11], in quantales and residuated lattices [5, 12] and in several kinds of algebra [6–8].

Inspired by the notion of integrations on ring structures introduced by Banič [3], we extend this notion of integration to any lattice structure. More precisely, we introduce the notions of an integrable element and its integral set with respect to a given derivation on a lattice. We investigate several characterizations and properties of integrations on lattices. Moreover, we pay particular attention to the lattice structure of the family of integral sets with respect to a given integration on a lattice. We provide a representation theorem for the lattice of fixed points of an isotone derivation based on the family of integral sets. As an application, we use integrations on a lattice to determine the necessary and sufficient conditions under which a linear differential equation on an arbitrary Boolean lattice has a solution.

The rest of the paper is organized as follows. In Section 2, we recall some necessary concepts and properties of derivations on lattices. In Section 3, we introduce the notion of integration on a lattice and investigate several characterizations and properties for this concept. In Section 4, we give a lattice structure to the family of integral sets with respect to a given integration on a lattice. Moreover, we provide a representation theorem for the lattice of fixed points of an isotone derivation based on the family of integral sets. In Section 5, we give an application of integrations on lattices. Finally, we present some conclusions and discuss a future research in Section 6.

2. DERIVATIONS ON LATTICES

In this section, we recall some basic concepts and properties of derivations on lattices that will be needed throughout this paper.

Definition 1 ([14, Definition 3.1]). Let $(L, \wedge, \vee)$ be a lattice and $\leq$ be its order relation. A function $d: L \rightarrow L$ is called a *derivation* on L if it satisfies the following condition:

$$d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y)) \quad \text{for any } x, y \in L.$$

In the rest of the paper, we shortly write dx instead of $d(x)$.

Definition 2 ([14, Definition 3.7]). Let $(L, \wedge, \vee)$ be a lattice and d be a derivation on L . d is called *isotone* if it satisfies the following condition:

$$x \leq y \text{ implies } dx \leq dy \quad \text{for any } x, y \in L.$$

Example 1 ([14, Example 3.8]). Let $(L, \wedge, \vee)$ be a lattice, α be an element of L and $d_\alpha: L \rightarrow L$ be a function defined as $d_\alpha(x) = \alpha \wedge x$ for any $x \in L$. The function d_α is called a *principal derivation* on L .

As a remark, any principal derivation d_α on L is isotone.

Remark 1 ([14, Remark 2.]). Let $(L, \wedge, \vee)$ be a lattice and d be a derivation on L . If $\text{Fix}_d(L) = \{x \in L \mid dx = x\}$ the set of fixed points of d is non-empty, then it is a down-set. Moreover, if d is isotone, then $\text{Fix}_d(L)$ is an ideal of L .

The following proposition gives some properties of derivations on a lattice.

Proposition 1 ([14, Proposition 3.6]). *Let $(L, \wedge, \vee)$ be a lattice and d be a derivation on L . Then it holds that*

- (i) $dx \leq x$ for any $x \in L$;
- (ii) $d(dx) = dx$ for any $x \in L$;
- (iii) if L has a least element $0 \in L$, then $d0 = 0$;
- (iv) d is isotone if and only if $d(x \wedge y) = dx \wedge dy$ for any $x, y \in L$;
- (v) if L is distributive and d is isotone, then $d(x \vee y) = dx \vee dy$ for any $x, y \in L$.

For more details concerning derivations on lattices, we refer to [13, 14].

3. INTEGRATIONS ON A LATTICE

The notion of integrations on a ring was introduced by Banič [3]. Inspired by this notion, we introduce the notion of integration with respect to a given derivation on a lattice. Also, we investigate several characterizations and properties of integrations on a lattice.

3.1. Definitions and examples

Definition 3. Let $(L, \wedge, \vee)$ be a lattice and d be a derivation on L . Let $i_d: L \rightarrow \mathcal{P}(L)$ be a function defined as $i_d(x) = d^{-1}(x) = \{z \in L \mid dz = x\}$ for any $x \in L$. The function i_d is called the integration with respect to d (the d -integration, for short) on L . The set $i_d(x)$ is called the integral set of x with respect to d (the d -integral set of x , for short).

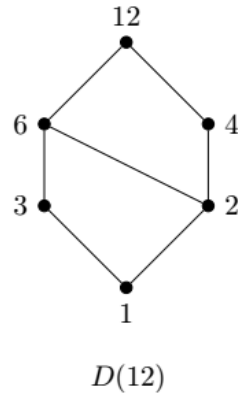
Definition 4. Let $(L, \wedge, \vee)$ be a lattice and i_d be a d -integration on L . An element x of L is called an integrable element with respect to i_d (a d -integrable element, for short) if $i_d(x) \neq \emptyset$.

Remark 2. In any lattice $(L, \wedge, \vee, 0)$ with 0 , the least element 0 is an integrable element with respect to any d -integration on L . Indeed, let i_d be a d -integration on L . Since $d0 = 0$ (see Proposition 1 (iii)) we conclude that $i_d(0) \neq \emptyset$. Hence 0 is a d -integrable element of L .

In the following, we give two illustrative examples of integrations on lattices.

Example 2. Let $(D(12), gcd, lcm)$ be the lattice of positive divisors of 12 ordered by the divisibility order relation $|$ and given by the Hasse diagram in Figure 1. Let d be a principal derivation on $D(12)$ defined as $dx = gcd(6, x)$ for any $x \in D(12)$. The d -integration function i_d on $D(12)$ is defined in the following table:

x	1	2	3	4	6	12
dx	1	2	3	2	6	6
$i_d(x)$	$\{1\}$	$\{2, 4\}$	$\{3\}$	$\emptyset$	$\{6, 12\}$	$\emptyset$

FIGURE 1. The Hasse diagram of the lattice $(D(12), gcd, lcm)$.

Example 3. Let $(\mathbb{N}^*, gcd, lcm)$ be the lattice of positive integers ordered by the divisibility order relation $|$ and d be a derivation on $\mathbb{N}^*$ defined as $dx = gcd(2, x)$ for any $x \in \mathbb{N}^*$. We denote by $\mathbb{P}$ the set of all positive even numbers and by $\mathbb{I}$ the set of all positive odd numbers. One can easily verify that

$$dx = \begin{cases} 1 & \text{if } x \text{ is odd;} \\ 2 & \text{if } x \text{ is even.} \end{cases}$$

The d -integration mapping i_d on $\mathbb{N}^*$ is defined as:

$$i_d(x) = \begin{cases} \mathbb{I} & \text{if } x = 1; \\ \mathbb{P} & \text{if } x = 2; \\ \emptyset & \text{otherwise.} \end{cases}$$

3.2. Properties of the integrable elements of a lattice with respect to an integration

In this subsection, we investigate several properties concerning the integrable elements of a lattice and their integral sets. First, we show the following two characterizations of an integrable element of a lattice.

Lemma 1. *Let $(L, \wedge, \vee)$ be a lattice and i_d be a d -integration on L . Then an element x of L is a d -integrable element if and only if $x \in \text{Fix}_d(L)$.*

Proof. To prove the direct implication, suppose that x is a d -integrable element of L . Then there exists $z \in L$ such that $dz = x$. Applying Proposition 1 (ii), we obtain $dx = d(dz) = dz = x$. Thus, x is a fixed point of d , i.e., $x \in \text{Fix}_d(L)$. The converse implication is immediate. $\square$

The following propositions show several properties of integrable elements of a lattice.

Proposition 2. *Let $(L, \wedge, \vee)$ be a lattice, i_d be a d -integration on L and x be a d -integrable element of L . Then it holds that*

- (i) *x is the least element of $i_d(x)$;*
- (ii) *if $y \leq x$, then y is also a d -integrable element of L for any $y \in L$.*

Proof. Suppose that x is a d -integrable element of L .

- (i) On the one hand, Lemma 1 guarantees that $x \in i_d(x)$. On the other hand, let $z \in i_d(x)$, then $dz = x$. Since d is a derivation on L , it holds from Proposition 1 (i) that $dz \leq z$. Hence, $x \leq z$. Thus, x is the least element of $i_d(x)$.
- (ii) Using Lemma 1 guarantees that $x \in \text{Fix}_d(L)$. Let $y \in L$ such that $y \leq x$. The fact that $\text{Fix}_d(L)$ is a down-set (see Remark 1) implies $y \in \text{Fix}_d(L)$. Therefore, y is also a d -integrable element of L .

□

Here, we mention that if x is a d -integrable element of L , then it holds from Proposition 2 (ii) that $m \wedge x$ is also a d -integrable element of L for any $m \in L$.

Proposition 3. *Let $(L, \wedge, \vee)$ be a lattice and i_{d_1}, i_{d_2} be two integrations on L such that $d_1 \leq d_2$ (i.e., $d_1(x) \leq d_2(x)$ for any $x \in L$). If x is a d_1 -integrable element of L , then x is also a d_2 -integrable element.*

Proof. Let x be a d_1 -integrable element of L . Then from Lemma 1, we have x is a fixed point of d_1 . On the one hand, the fact that $d_1 \leq d_2$ implies $x = d_1(x) \leq d_2(x)$. Hence, $x \leq d_2(x)$. On the other hand, since d_2 is a derivation on L , we obtain from Proposition 1 (i) that $d_2(x) \leq x$. Thus, $d_2(x) = x$, i.e., x is also a fixed point of d_2 . Therefore, Lemma 1 guarantees that x is also a d_2 -integrable element. □

Proposition 4. *Let $(L, \wedge, \vee)$ be a lattice and i_d be a d -integration on L such that d is isotone. The following implications hold:*

- (i) *if $y_1 \in i_d(x_1)$ and $y_2 \in i_d(x_2)$, then $y_1 \wedge y_2 \in i_d(x_1 \wedge x_2)$ for any $x_1, x_2, y_1, y_2 \in L$;*
- (ii) *if $y_1 \in i_d(x_1)$, $y_2 \in i_d(x_2)$ and L is distributive, then $y_1 \vee y_2 \in i_d(x_1 \vee x_2)$ for any $x_1, x_2, y_1, y_2 \in L$.*

Proof. Let $x_1, x_2, y_1, y_2 \in L$ such that $y_1 \in i_d(x_1)$ and $y_2 \in i_d(x_2)$, then $dy_1 = x_1$ and $dy_2 = x_2$.

- (i) The fact that d is an isotone derivation on L implies from Proposition 1 (iv) that $d(y_1 \wedge y_2) = dy_1 \wedge dy_2 = x_1 \wedge x_2$. Thus, $y_1 \wedge y_2 \in i_d(x_1 \wedge x_2)$.
- (ii) Since L is distributive and d is an isotone derivation on L , we conclude from Proposition 1 (v) that $d(y_1 \vee y_2) = dy_1 \vee dy_2 = x_1 \vee x_2$. Therefore, $y_1 \vee y_2 \in i_d(x_1 \vee x_2)$.

□

Next, we give relationships between the integral sets on a lattice with zero.

Proposition 5. *Let $(L, \wedge, \vee, 0)$ be a lattice with a least element 0 and i_d be a d -integration on L . Let x be a d -integrable element of L . Then the following statements hold:*

- (i) $m \wedge i_d(x) \subseteq i_d(m \wedge x) \subseteq i_d(m \wedge dx)$ for any $m \in i_d(0)$, where $m \wedge i_d(x) = \{m \wedge z \mid z \in i_d(x)\}$;
- (ii) if L is distributive and d is isotone, then $m \vee i_d(x) \subseteq i_d(x)$ for any $m \in i_d(0)$, where $m \vee i_d(x) = \{m \vee z \mid z \in i_d(x)\}$.

Proof. Let x be a d -integrable element of L and $m \in i_d(0)$. We know that $m \wedge x$ is also a d -integrable elements of L . Hence, $i_d(m \wedge x) \neq \emptyset$.

- (i) Let $y \in m \wedge i_d(x)$, then $y = m \wedge z$ with $dz = x$. Since d is a derivation on L and $dm = 0$, it follows that

$$dy = d(m \wedge z) = (dm \wedge z) \vee (m \wedge dz) = m \wedge dz = m \wedge x.$$

Hence, $y \in i_d(m \wedge x)$. Thus, $m \wedge i_d(x) \subseteq i_d(m \wedge x)$. Next, let $t \in i_d(m \wedge x)$, then $dt = m \wedge x$. Using the fact that d is a derivation on L and Proposition 1 (ii) we obtain

$$dt = d(dt) = d(m \wedge x) = (dm \wedge x) \vee (m \wedge dx) = m \wedge dx.$$

Hence, $t \in i_d(m \wedge dx)$. Therefore, $i_d(m \wedge x) \subseteq i_d(m \wedge dx)$. Consequently,

$$m \wedge i_d(x) \subseteq i_d(m \wedge x) \subseteq i_d(m \wedge dx).$$

- (ii) Let $y \in m \vee i_d(x)$, then $y = m \vee z$ with $dz = x$. Since L is distributive and d is isotone derivation, it follows from Proposition 1 (v) that $dy = d(m \vee z) = dm \vee dz = dz = x$. Hence, $y \in i_d(x)$. Therefore, $m \vee i_d(x) \subseteq i_d(x)$

□

4. A LATTICE STRUCTURE OF INTEGRAL SETS ON A LATTICE

In this section, we give a lattice structure to the family of integral sets with respect to a given integration on a lattice. Also, we provide a representation theorem for the lattice of fixed points of an isotone derivation based on the family of integral sets.

Notation 1. Let $(L, \wedge, \vee)$ be a lattice and i_d be an integration on L . We denote by

- (i) $\mathbb{I}_d(L) := \{i_d(x) \mid x \in L\}$ the family of d -integral sets of L ;
- (ii) $I_d(L) := \{i_d(x) \mid x \in \text{Fix}_d(L)\}$ the family of d -integral sets of the d -integrable elements of L .

In the following theorem, we give a poset structure to the family of integral sets of a lattice with respect to a given integration.

Theorem 1. *Let $(L, \wedge, \vee)$ be a lattice and i_d be an integration on L with d is isotone. Let $\sqsubseteq$ be a binary relation on $\mathbb{I}_d(L)$ defined for any $i_d(x), i_d(y) \in \mathbb{I}_d(L)$ as:*

$$i_d(x) \sqsubseteq i_d(y) \quad \text{if and only if} \quad i_d(x) = \emptyset \text{ or } \exists t \in i_d(y) \text{ such that } x \leq t.$$

Then the structure $(\mathbb{I}_d(L), \sqsubseteq)$ is a partially ordered set.

Proof. Let $i_d(x) \in \mathbb{I}_d(L)$, if $i_d(x) = \emptyset$, then $i_d(x) \sqsubseteq i_d(x)$. If $i_d(x) \neq \emptyset$, this means x is a d -integrable element of L . Lemma 1 guarantees $x \in i_d(x)$. Since $x \in i_d(x)$ and $x \leq x$, we conclude $i_d(x) \sqsubseteq i_d(x)$. Thus, $\sqsubseteq$ is reflexive. Next, let $i_d(x), i_d(y) \in \mathbb{I}_d(L)$ with $i_d(x) \sqsubseteq i_d(y)$ and $i_d(y) \sqsubseteq i_d(x)$. Then, $[i_d(x) = \emptyset \text{ or } (\exists t_1 \in i_d(y) \text{ such that } x \leq t_1)]$ and $[i_d(y) = \emptyset \text{ or } (\exists t_2 \in i_d(x) \text{ such that } y \leq t_2)]$. Here, we discuss the following four possible cases:

- (i) if $i_d(x) = \emptyset$ and $i_d(y) = \emptyset$, then $i_d(x) = i_d(y)$;
- (ii) if $i_d(x) = \emptyset$ and $\exists t_2 \in i_d(x)$ such that $y \leq t_2$, this is an impossible case;
- (iii) if $i_d(y) = \emptyset$ and $\exists t_1 \in i_d(y)$ such that $x \leq t_1$, this is also an impossible case;
- (iv) if $\exists t_1 \in i_d(y)$ such that $x \leq t_1$ and $\exists t_2 \in i_d(x)$ such that $y \leq t_2$, then x and y are d -integrable elements of L . Lemma 1 assures $x, y \in \text{Fix}_d(L)$, i.e., $dx = x$ and $dy = y$. Since $t_1 \in i_d(y)$ and $t_2 \in i_d(x)$, we get $dt_1 = y$ and $dt_2 = x$. Using the hypotheses of d is an isotone derivation on L , we obtain $x = dx \leq dt_1 = y$ and $y = dy \leq dt_2 = x$. Thus, $x = y$, i.e., $i_d(x) = i_d(y)$.

Therefore, $\sqsubseteq$ is antisymmetric. Now, let $i_d(x), i_d(y), i_d(z) \in \mathbb{I}_d(L)$ with $i_d(x) \sqsubseteq i_d(y)$ and $i_d(y) \sqsubseteq i_d(z)$. Then $[i_d(x) = \emptyset \text{ or } (\exists t_1 \in i_d(y) \text{ such that } x \leq t_1)]$ and $[i_d(y) = \emptyset \text{ or } (\exists t_2 \in i_d(z) \text{ such that } y \leq t_2)]$. Here, we have the following three possible cases:

- (i) if $i_d(x) = \emptyset$, then $i_d(x) \sqsubseteq i_d(z)$;
- (ii) if $\exists t_1 \in i_d(y)$ such that $x \leq t_1$ and $i_d(y) = \emptyset$, this is an impossible case;
- (iii) if $\exists t_1 \in i_d(y)$ such that $x \leq t_1$ and $\exists t_2 \in i_d(z)$ such that $y \leq t_2$, then $dt_1 = y$ and $dt_2 = z$. Since, $i_d(x) \neq \emptyset$, $i_d(y) \neq \emptyset$ and $i_d(z) \neq \emptyset$, it holds from Lemma 1 that $dx = x$, $dy = y$ and $dz = z$. The fact that d is an isotone derivation on L gives $x = dx \leq dt_1 = y$ and $y = dy \leq dt_2 = z$. Hence, $x \leq z$ and $z \in i_d(z)$. Thus, $i_d(x) \sqsubseteq i_d(z)$.

Therefore, $\sqsubseteq$ is transitive. Consequently, $\sqsubseteq$ is an order relation on $\mathbb{I}_d(L)$, i.e., the structure $(\mathbb{I}_d(L), \sqsubseteq)$ is a partially ordered set. $\square$

In the following proposition, we present the least element of $(\mathbb{I}_d(L), \sqsubseteq)$

Proposition 6. *Let $(L, \wedge, \vee)$ be a lattice and d be an isotone derivation on L different from the identity function of L . Then the empty-set $\emptyset$ is the least element of $\mathbb{I}_d(L)$.*

Proof. The fact that d is not the identity function implies that there exists $x \in L$ where x is not a fixed point of d . Lemma 1 guarantees that x is not a d -integrable element of L . Hence, $i_d(x) = \emptyset$, i.e., $\emptyset \in \mathbb{I}_d(L)$. Since $\emptyset \sqsubseteq i_d(y)$ for any $i_d(y) \in \mathbb{I}_d(L)$, it holds that $\emptyset$ is the least element of $\mathbb{I}_d(L)$. $\square$

The following lemma is the key to prove the next main results of this section.

Lemma 2. *Let $(L, \wedge, \vee)$ be a lattice and d be an isotone derivation on L . Then $i_d(x) \sqsubseteq i_d(y)$ if and only if $x \leq y$ for any $x, y \in \text{Fix}_d(L)$.*

Proof. Let $x, y \in \text{Fix}_d(L)$, then from Lemma 1 we have x and y are two d -integrable elements of L , i.e., $i_d(x) \neq \emptyset$ and $i_d(y) \neq \emptyset$. Firstly, suppose that $i_d(x) \sqsubseteq i_d(y)$, then there exists $t \in i_d(y)$ such that $x \leq t$. Since $x \in \text{Fix}_d(L)$ and d is an isotone derivation on L , it holds that $x = dx \leq dt = y$. Thus, $x \leq y$. Conversely, suppose that $x \leq y$. Using Lemma 1 gives that $y \in i_d(y)$. Thus, there exists $t = y$ such that $t \in i_d(y)$ and $x \leq t$. Therefore, $i_d(x) \sqsubseteq i_d(y)$. $\square$

In the following theorem, we give a lattice structure to the family of integral sets of a lattice.

Theorem 2. Let $(L, \wedge, \vee)$ be a lattice and d be an isotone derivation on L . Define on $\mathbb{I}_d(L)$ two binary operations for any $i_d(x), i_d(y) \in \mathbb{I}_d(L)$ as:

$$i_d(x) \sqcap i_d(y) = \begin{cases} i_d(x \wedge y) & \text{if } x, y \in \text{Fix}_d(L), \\ \emptyset & \text{otherwise.} \end{cases}$$

and

$$i_d(x) \sqcup i_d(y) = \begin{cases} i_d(x \vee y) & \text{if } x, y \in \text{Fix}_d(L), \\ i_d(y) & \text{if } x \notin \text{Fix}_d(L), \\ i_d(x) & \text{if } y \notin \text{Fix}_d(L). \end{cases}$$

Then the structure $(\mathbb{I}_d(L), \sqcap, \sqcup)$ is a lattice with respect to the order relation $\sqsubseteq$.

Proof. Let $i_d(x), i_d(y) \in \mathbb{I}_d(L)$. Firstly, we are aiming to prove that $i_d(x) \sqcap i_d(y)$ is the greatest lower bound of $i_d(x)$ and $i_d(y)$. On the one hand, if $x \notin \text{Fix}_d(L)$ or $y \notin \text{Fix}_d(L)$, then $i_d(x) \sqcap i_d(y) = \emptyset$. Hence, $i_d(x) \sqcap i_d(y) \sqsubseteq i_d(x)$ and $i_d(x) \sqcap i_d(y) \sqsubseteq i_d(y)$. If $x, y \in \text{Fix}_d(L)$, then $i_d(x) \sqcap i_d(y) = i_d(x \wedge y)$. Remark 1 assures that $\text{Fix}_d(L)$ is a down-set, then $x \wedge y \in \text{Fix}_d(L)$. Since $x \wedge y \leq x$ and $x \wedge y \leq y$, it holds from Lemma 2 that $i_d(x) \sqcap i_d(y) = i_d(x \wedge y) \sqsubseteq i_d(x)$ and $i_d(x) \sqcap i_d(y) = i_d(x \wedge y) \sqsubseteq i_d(y)$. Thus, $i_d(x) \sqcap i_d(y)$ is a lower bound of $i_d(x)$ and $i_d(y)$. On the other hand, let $i_d(z) \in \mathbb{I}_d(L)$ be a lower bound of $i_d(x)$ and $i_d(y)$. If $i_d(z) = \emptyset$, then $i_d(z) \sqsubseteq i_d(x) \sqcap i_d(y)$. Otherwise, $i_d(z) \sqsubseteq i_d(x)$ and $i_d(z) \sqsubseteq i_d(y)$ such that $x, y, z \in \text{Fix}_d(L)$. So, Lemma 2 guarantees that $z \leq x$ and $z \leq y$. Hence, $z \leq x \wedge y$. Thus, $i_d(z) \sqsubseteq i_d(x \wedge y) = i_d(x) \sqcap i_d(y)$. Therefore, $i_d(x) \sqcap i_d(y)$ is the greatest lower bound of $i_d(x)$ and $i_d(y)$, i.e., the poset $(\mathbb{I}_d(L), \sqcap)$ is a meet-semilattice.

Secondly, we show that $i_d(x) \sqcup i_d(y)$ is the least upper bound of $i_d(x)$ and $i_d(y)$. On the one hand,

- (i) if $x \notin \text{Fix}_d(L)$, then $i_d(x) = \emptyset$ and $i_d(x) \sqcup i_d(y) = i_d(y)$. Hence, $i_d(x) \sqsubseteq i_d(x) \sqcup i_d(y)$ and $i_d(y) \sqsubseteq i_d(x) \sqcup i_d(y)$;
- (ii) if $y \notin \text{Fix}_d(L)$, then $i_d(y) = \emptyset$ and $i_d(x) \sqcup i_d(y) = i_d(x)$. Hence, $i_d(x) \sqsubseteq i_d(x) \sqcup i_d(y)$ and $i_d(y) \sqsubseteq i_d(x) \sqcup i_d(y)$;
- (iii) if $x, y \in \text{Fix}_d(L)$, then $i_d(x) \sqcup i_d(y) = i_d(x \vee y)$. Since d is an isotone derivation on L , it holds from Remark 1 that $\text{Fix}_d(L)$ is an ideal of L , then $x \vee y \in \text{Fix}_d(L)$. The fact that $x \leq x \vee y$ and $y \leq x \vee y$ with $x, y, x \vee y \in \text{Fix}_d(L)$ imply

from Lemma 2 that $i_d(x) \sqsubseteq i_d(x \vee y) = i_d(x) \sqcup i_d(y)$ and $i_d(y) \sqsubseteq i_d(x \vee y) = i_d(x) \sqcup i_d(y)$.

Therefore, $i_d(x) \sqcup i_d(y)$ is an upper bound of $i_d(x)$ and $i_d(y)$. On the other hand, let $i_d(t) \in \mathbb{I}_d(L)$ be an upper bound of $i_d(x)$ and $i_d(y)$. Two possible cases to consider.

- (i) If $i_d(t) = \emptyset$, since $i_d(x) \sqsubseteq i_d(t)$ and $i_d(y) \sqsubseteq i_d(t)$ we obtain $i_d(x) = \emptyset$ and $i_d(y) = \emptyset$. Hence, $i_d(x) \sqcup i_d(y) = \emptyset \sqsubseteq i_d(t)$.
- (ii) If $i_d(t) \neq \emptyset$, then $t \in \text{Fix}_d(L)$. So, if $x \notin \text{Fix}_d(L)$ or $y \notin \text{Fix}_d(L)$, then $i_d(x) \sqcup i_d(y) = i_d(y) \sqsubseteq i_d(t)$ or $i_d(x) \sqcup i_d(y) = i_d(x) \sqsubseteq i_d(t)$. If $x, y, t \in \text{Fix}_d(L)$ with $i_d(x) \sqsubseteq i_d(t)$ and $i_d(y) \sqsubseteq i_d(t)$, then Lemma 2 gives that $x \leq t$ and $y \leq t$. Hence, $x \vee y \leq t$ such that $t, x \vee y \in \text{Fix}_d(L)$. Thus, Lemma 2 guarantees that $i_d(x) \sqcup i_d(y) = i_d(x \vee y) \sqsubseteq i_d(t)$.

Therefore, $i_d(x) \sqcup i_d(y)$ is the least upper bound of $i_d(x)$ and $i_d(y)$, i.e., the poset $(\mathbb{I}_d(L), \sqcup)$ is a *join*-semilattice. Consequently, the structure $(\mathbb{I}_d(L), \sqcap, \sqcup)$ is a lattice. $\square$

The following proposition shows that $I_d(L)$ is a sublattice of $\mathbb{I}_d(L)$.

Proposition 7. *Let $(L, \wedge, \vee)$ be a lattice and d be an isotone derivation on L . Then $(I_d(L), \sqcap, \sqcup)$ is a sublattice of $(\mathbb{I}_d(L), \sqcap, \sqcup)$, where $i_d(x) \sqcap i_d(y) = i_d(x \wedge y)$ and $i_d(x) \sqcup i_d(y) = i_d(x \vee y)$ for any $i_d(x), i_d(y) \in I_d(L)$.*

Proof. Let $i_d(x), i_d(y) \in I_d(L)$, then $x, y \in \text{Fix}_d(L)$. Since $\text{Fix}_d(L)$ is an ideal of L (see Remark 1), we have $x \wedge y \in \text{Fix}_d(L)$ and $x \vee y \in \text{Fix}_d(L)$. Then we conclude that $i_d(x) \sqcap i_d(y) = i_d(x \wedge y) \in I_d(L)$ and $i_d(x) \sqcup i_d(y) = i_d(x \vee y) \in I_d(L)$. Thus, $I_d(L)$ is closed under $(\sqcap)$ and $(\sqcup)$ operations. Therefore, the structure $(I_d(L), \sqcap, \sqcup)$ is a sublattice of $(\mathbb{I}_d(L), \sqcap, \sqcup)$. $\square$

In the following theorem, we provide a representation theorem for the lattice of fixed points of an isotone derivation based on the family of integral sets of the integrable elements.

Theorem 3. *Let $(L, \wedge, \vee)$ be a lattice and d be an isotone derivation on L . Then $(I_d(L), \sqcap, \sqcup)$ is isomorphic to $(\text{Fix}_d(L), \wedge, \vee)$, where $(\text{Fix}_d(L), \wedge, \vee)$ is the lattice of fixed points of d .*

Proof. Since $\text{Fix}_d(L)$ is an ideal of L (see Remark 1), we conclude that $(\text{Fix}_d(L), \wedge, \vee)$ is a lattice. Also, Proposition 7 guarantees that $(I_d(L), \sqcap, \sqcup)$ is a lattice. Next, let $\psi: \text{Fix}_d(L) \rightarrow I_d(L)$ be a mapping defined as $\psi(x) = i_d(x)$ for any $x \in \text{Fix}_d(L)$. One can easily verify that ψ is surjective. Further, Lemma 2 guarantees

$$x \leq y \quad \text{if and only if} \quad \psi(x) \sqsubseteq \psi(y) \quad \text{for any } x, y \in \text{Fix}_d(L).$$

Now, we show that ψ is injective. Let $x_1, x_2 \in \text{Fix}_d(L)$ such that $\psi(x_1) = \psi(x_2)$, so $\psi(x_1) \sqsubseteq \psi(x_2)$ and $\psi(x_2) \sqsubseteq \psi(x_1)$. Then $x_1 \leq x_2$ and $x_2 \leq x_1$. Therefore, $x_1 = x_2$. Hence, ψ is injective.

Thus, ψ is an order isomorphism between the lattices $Fix_d(L)$ and $I_d(L)$. Therefore, ψ is a lattice isomorphism from $Fix_d(L)$ into $I_d(L)$. Consequently, $(I_d(L), \sqcap, \sqcup)$ and $(Fix_d(L), \wedge, \vee)$ are isomorphic. $\square$

Next, we give an illustrative example to present this isomorphism.

Example 4. Let $(D(12), gcd, lcm)$ be the lattice of positive divisors of 12 ordered by the divisibility order relation $|$ and i_d be the d -integration function defined in the table of Example 2.

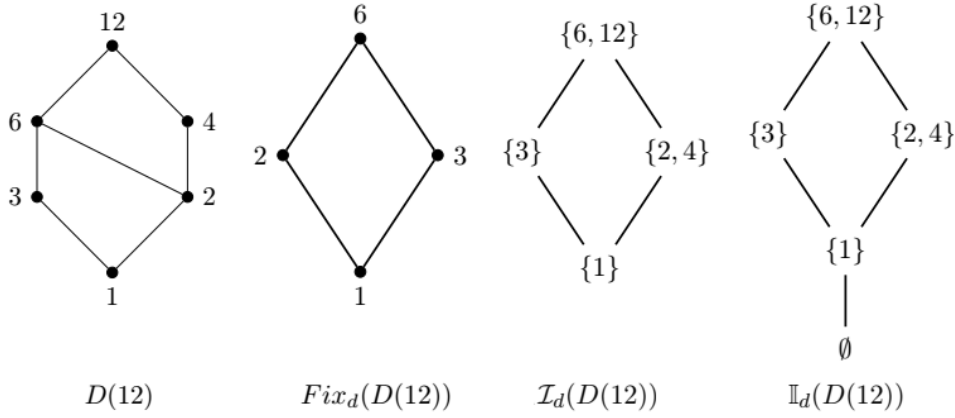


FIGURE 2. The Hasse diagrams of the lattices $(D(12), gcd, lcm)$, $(Fix_d(D(12)), gcd, lcm)$, $(I_d(D(12)), \sqcap, \sqcup)$ and $(\mathbb{I}_d(D(12)), \sqcap, \sqcup)$.

The above Theorem 3 leads to the following corollary.

Corollary 1. *Let $(L, \wedge, \vee)$ be a lattice and d be an isotone derivation on L . If $(L, \wedge, \vee)$ is modular (resp. distributive), then $(I_d(L), \sqcap, \sqcup)$ is also modular (resp. distributive).*

Remark 3. Let $(L, \wedge, \vee)$ be a lattice and d be an isotone derivation on L different from the identity function. Then $\mathbb{I}_d(L) = I_d(L) \cup \{\emptyset\}$. Indeed, $\mathbb{I}_d(L) = \{i_d(x) \mid x \in L\} = \{i_d(x) \mid x \in Fix_d(L)\} \cup \{i_d(x) \mid x \notin Fix_d(L)\} = I_d(L) \cup \{\emptyset\}$.

Combining Proposition 6, Remark 3 and Corollary 1 leads to the following corollary.

Corollary 2. *Let $(L, \wedge, \vee)$ be a lattice and d be an isotone derivation on L different from the identity function. If $(L, \wedge, \vee)$ is modular (resp. distributive), then $(\mathbb{I}_d(L), \sqcap, \sqcup)$ is also modular (resp. distributive).*

5. AN APPLICATION OF INTEGRATIONS ON LATTICES

In this section, we use this notion of integrations to determine the necessary and sufficient conditions under which a linear differential equation on an arbitrary Boolean lattice has a solution.

Definition 5. Let $(B, \wedge, \vee, 0, 1, ')$ be a Boolean lattice and $\leq$ be its order relation. Let i_d be an integration on B and a, b be two elements of B . A *linear differential equation* with respect to d is any equation with the form

$$a \cdot d(X) + b = 0, \quad (5.1)$$

where $a \cdot d(X) = a \wedge d(X)$ and $x + y = (x \wedge y') \vee (x' \wedge y)$ for any $x, y \in B$.

Theorem 4. Let $(B, \wedge, \vee, 0, 1, ')$ be a Boolean lattice and i_d be an integration on B . Then the equation (5.1) has a solution if and only if $b \leq a$ and b is a d -integrable element of B .

Proof. Let x_0 be a solution of the equation (5.1), then $a \cdot d(x_0) + b = 0$, i.e.,

$$a \cdot d(x_0) = b.$$

Thus, $b \leq a$. On an other hand, the fact that d is a derivation on B implies that

$$\begin{aligned} d(b) &= d(a \cdot d(x_0)) \\ &= (d(a) \cdot d(x_0)) \vee (a \cdot d(d(x_0))) \\ &= (d(a) \cdot d(x_0)) \vee (a \cdot d(x_0)) \\ &= (d(a) \cdot d(x_0)) \vee b. \end{aligned}$$

Thus, $b \leq d(b)$. Also, from Proposition 1 (i) we have $d(b) \leq b$. Then $d(b) = b$, i.e., b is a fixed point of d . Therefore, Lemma 1 guarantees that b is a d -integrable element of B . Conversely, suppose that $b \leq a$ and b a d -integrable element of B . Then $a \cdot b = b$ and $d(b) = b$. Hence, $a \cdot d(b) + b = a \cdot b + b = b + b = 0$. Thus, b is a solution of the equation (5.1). $\square$

Example 5. Let $(D(30), gcd, lcm, 1, 30, ')$ be the Boolean lattice of positive divisors of 30 ordered by the divisibility order relation $|$ and given by the Hasse diagram in Figure 3. Let d be a principal derivation on $D(30)$ defined as $dx = gcd(6, x)$ for any $x \in D(30)$. The d -integration function i_d on $D(30)$ is defined in the following table:

x	1	2	3	5	6	10	15	30
$d(x)$	1	2	3	1	6	2	3	6
$i_d(x)$	$\{1, 5\}$	$\{2, 10\}$	$\{3, 15\}$	$\emptyset$	$\{6, 30\}$	$\emptyset$	$\emptyset$	$\emptyset$

Let $10 \cdot d(X) + 2 = 0$ and $10 \cdot d(X) + 5 = 0$ be two linear differential equations on $D(30)$. Since $2 | 10$ and 2 is an integrable element of $D(30)$, then Theorem 4 guarantees that the $10 \cdot d(X) + 2 = 0$ has a solution. The solutions are given by this set:

$$S = \{2, 6, 10, 30\}.$$

Moreover, the fact that $5 \mid 10$ but 5 is not an integrable element of $D(30)$, then Theorem 4 guarantees that the $10 \cdot d(X) + 5 = 0$ has not a solution on the Boolean lattice $D(30)$.

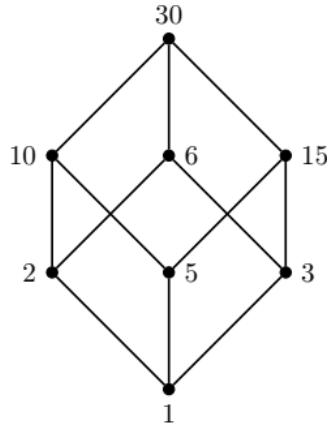


FIGURE 3. The Hasse diagram of the Boolean lattice $(D(30), \gcd, \text{lcm}, 1, 30, ')$.

Theorem 5. Let $(B, \wedge, \vee, 0, 1, ')$ be a Boolean lattice and i_d be an integration on B . If x_0 is a solution of the equation (5.1), then $b \leq d(x_0) \leq a + b + 1$.

Proof. Let $x_0 \in B$ is a solution of the equation (5.1), i.e., $a \cdot d(x_0) + b = 0$. Then $a \cdot d(x_0) = b$, so on the one hand $b \leq d(x_0)$. On the other hand, $(a + b + 1) \cdot d(x_0) = a \cdot d(x_0) + b \cdot d(x_0) + 1 \cdot d(x_0) = b + b + d(x_0) = d(x_0)$. Thus, $d(x_0) \leq (a + b + 1)$. Therefore, $b \leq d(x_0) \leq a + b + 1$. $\square$

6. CONCLUSION AND FUTURE RESEARCH

In this paper, we have introduced the notion of integration with respect to a given derivation on a lattice. More precisely, we have given the definitions of integrable elements of a lattice and their integral sets. We have investigated several characterizations and properties concerning the integrable elements and their integral sets. Moreover, we have given a lattice structure to the family of integral sets with respect to a given integration on a lattice. We have provided a representation theorem for the lattice of fixed points of an isotone derivation based on the family of integral sets. As an application, we have used this notion of integration to determine the necessary and sufficient conditions under which a linear differential equation on an arbitrary Boolean lattice has a solution.

Finally, we anticipate that there exist relationships between isotone derivations and their integrations on a lattice. Also, we intend to extend this notion of integration to several interesting algebraic structures.

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A NOTE ON S -CURVATURE OF RANDERS MEASURE SPACES

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Abstract. In this paper, we give some characterizations on S -curvature in Randers measure spaces, which shows that only the Busemann-Hausdorff volume form can admit a vanishing S -curvature.

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Keywords: Randers measure space; S -curvature; Busemann-Hausdorff volume form

1. INTRODUCTION

In Finsler geometry, the S -curvature, which was introduced by Shen [4], plays a very important role in the classification of Finsler metrics and the global geometric analysis.

Let $(M, F, d\mu)$ be a Finsler measure space, where $d\mu = \sigma(x)dx^1 \wedge \cdots \wedge dx^n$ is an arbitrary volume form. Then the S -curvature is defined by

$$S(x, y) := \frac{d}{dt} [\tau(c(t), \dot{c}(t))] |_{t=0},$$

where $c(t)$ is the geodesic with $c(0) = x$ and $\dot{c}(0) = y$, and τ is the distortion, which is defined by

$$\tau(x, y) := \log \frac{\sqrt{\det(g_{ij}(x, y))}}{\sigma(x)},$$

where $g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 F(x, y)^2}{\partial y^i \partial y^j}$. Clearly, the S -curvature demonstrates the rate of change of the distortion along geodesics. In local coordinates, the S -curvature can be written as

$$S(x, y) = \frac{\partial G^i}{\partial y^i} - y^i \frac{\partial}{\partial x^i} \log \sigma(x), \quad (1.1)$$

where G^i is called the spray coefficients of F .

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Let us take a look at the special case: when F is a Riemannian metric and

$$d\mu = e^{-f(x)}dV, \quad \text{where} \quad dV = \sqrt{\det(g_{ij}(x))}dx^1 \wedge \cdots \wedge dx^n,$$

the S -curvature is

$$S(x, y) = \frac{\partial f(x)}{\partial x^i} y^i, \quad (1.2)$$

where f is called the potential function.

Using Ricci curvature, S -curvature and its derivative $\dot{S}$ along geodesics, one can define the Bakry-Emery Ricci curvature (the so called weighted Ricci curvature in the Finsler setting).

In many literatures, S -curvature is defined with respect to the Busemann-Hausdorff volume form. See [2] [1] [4] [5] and the references therein. Now we give the corresponding definitions by using an arbitrary volume form as follows.

Definition 1. Let $(M, F, d\mu)$ be a Finsler measure space, where $d\mu = \sigma(x)dx^1 \wedge \cdots \wedge dx^n$ is an arbitrary volume form.

(1) F is of weakly isotropic S -curvature if the S -curvature

$$S = (n+1)cF + \eta,$$

where $c = c(x)$ is a scalar function and $\eta = \eta_i(x)y^i$ is a 1-form on M .

(2) F is of almost isotropic S -curvature if $d\eta = 0$.

(3) F is of isotropic S -curvature if $\eta = 0$.

(4) F is of constant S -curvature if c is a constant and $\eta = 0$.

(5) F is of exact S -curvature if $c = 0$ and $\eta = d\phi$ for some smooth function on M .

Notice that in Riemannian manifold (M, g, dV) , the S -curvature is zero, while for the volume form $d\mu = e^{-f(x)}dV$, M has exact S -curvature (see (1.2)). In the Finsler situation, F is of exact S -curvature if and only if there is a volume form $d\mu$ such that $S_{d\mu} = 0$.

In the following, we focus on the Randers case. Let $F = \alpha + \beta$ be a solution of Zermelo's navigation problem, which is expressed by

$$F = \frac{\sqrt{\lambda h^2 + W_0^2}}{\lambda} - \frac{W_0}{\lambda}, \quad (1.3)$$

where $h := \sqrt{h_{ij}(x)y^i y^j}$ is a Riemannian metric, $W_0 := W_i(x)y^i$ and $\lambda := 1 - \|W_x\|_h^2$.

It is shown that, for the Busemann-Hausdorff volume form, F has vanishing S -curvature if and only if W is a Killing vector field with respect to h (see [5]).

A natural problem is to determine all volume forms such that S -curvature is zero.

In this short note, we study this problem and obtain the following main result.

Theorem 1. *Only (constant multiplications of) the Busemann-Hausdorff volume form can satisfy $S = 0$ on Randers spaces.*

Remark 1. In [3], Ohta has discussed this problem and obtained that, a Randers space $(M, \alpha + \beta)$ admits a measure m with $S = 0$ if and only if β is a Killing form of constant length. However, the "only if" part of Theorem 1.1 in [3] is not necessarily true.

2. THE PROOF OF THE MAIN RESULT

We first recall some fundamental facts in Randers geometry. For more details, refer to [2]. Let F be a Randers metric in terms of (h, W) as (1.3). Let $d\mu_{BH} = \sigma_{BH}(x) dx^1 \wedge \cdots \wedge dx^n$ denote the Busemann-Hausdorff volume form. Then

$$\sigma_{BH}(x) = \sigma_h(x),$$

where $dV_h = \sigma_h(x) dx^1 \wedge \cdots \wedge dx^n$ is the volume form of h .

From (p.35-36, [2]), we see

$$\begin{aligned} S_{BH} &= \frac{\partial G^i}{\partial y^i} - y^i \frac{\partial}{\partial x^i} \log \sigma_{BH}(x) \\ &= \frac{\partial \mathcal{G}^i}{\partial y^i} + \frac{n+1}{2F} (2F \mathcal{R}_0 - \mathcal{R}_{00} - F^2 \mathcal{R}) - y^i \frac{\partial}{\partial x^i} \log \sigma_h(x) \\ &= \frac{n+1}{2F} (2F \mathcal{R}_0 - \mathcal{R}_{00} - F^2 \mathcal{R}), \end{aligned}$$

where $\mathcal{G}^i$ is the spray coefficients of h , and

$$\begin{aligned} \mathcal{R}_{ij} &:= \frac{1}{2} (W_{i;j} + W_{j;i}), \quad \mathcal{R}_{00} := \mathcal{R}_{ij} y^i y^j, \\ \mathcal{R}_i &:= \mathcal{R}_{ij} W^j, \quad \mathcal{R}_0 := \mathcal{R}_i y^i, \quad \mathcal{R} := \mathcal{R}_i W^i, \end{aligned}$$

where $W^i := h^{ij} W_j$, and ";" denotes the covariant differentiation with respect to h .

To prove the main result, we give two important lemmas in the following.

Lemma 1. [1] *Let $F = \alpha + \beta$ be a Randers metric on an n -dimensional manifold M with the Busemann-Hausdorff volume form. For any $c = c(x)$ on M , the following are equivalent:*

- (1) $S_{BH} = (n+1)cF$;
- (2) $S_{BH} = (n+1)cF + \eta$, where η is a 1-form.

Lemma 2. [5] *Let $F = \alpha + \beta$ be a Randers metric on a manifold M , which is expressed in terms of a Riemannian metric h and a vector field W by (1.3). Then F is of isotropic S -curvature with respect to the Busemann-Hausdorff volume form, $S_{BH} = (n+1)cF$, if and only if W satisfies*

$$\mathcal{R}_{00} = -2ch^2.$$

Theorem 2. Let $(M, F, d\mu)$ be a Randers measure space, where $d\mu = \sigma(x)dx^1 \wedge \cdots \wedge dx^n$ is an arbitrary volume form, and F is expressed in terms of a Riemannian metric h and a vector field W by (1.3). Then F is of weakly isotropic S -curvature with respect to the volume form $d\mu$ if and only if W satisfies

$$\mathcal{R}_{00} = -2ch^2.$$

Proof. Suppose that $\mathcal{R}_{00} = -2ch^2$. Then by Lemma 2, we have

$$S_{BH} = (n+1)cF. \quad (2.1)$$

Let $d\mu_{BH} = \sigma_{BH}(x)dx^1 \wedge \cdots \wedge dx^n$ be the Busemann-Hausdorff volume form. Then there is a positive C^∞ function φ on M such that

$$\sigma(x) = \varphi(x)\sigma_{BH}(x).$$

Thus, from (1.1), we have

$$\begin{aligned} S &= \frac{\partial G^i}{\partial y^i} - y^i \frac{\partial}{\partial x^i} \log \sigma(x) \\ &= \frac{\partial G^i}{\partial y^i} - y^i \frac{\partial}{\partial x^i} \log \sigma_{BH}(x) - y^i \frac{\partial}{\partial x^i} \log \varphi \\ &= S_{BH} - y^i \frac{\partial}{\partial x^i} \log \varphi. \end{aligned} \quad (2.2)$$

Combining (2.1) with (2.2), it follows that S is weakly isotropic.

Conversely, if S is weakly isotropic, we then can write

$$S = (n+1)cF + \eta, \quad (2.3)$$

where η is a 1-form. Then by (2.2), we obtain

$$S_{BH} = (n+1)cF + \eta + y^i \frac{\partial}{\partial x^i} \log \varphi. \quad (2.4)$$

It is obviously to see from (2.4) that S_{BH} is weakly isotropic. By using Lemma 1, it is equivalent to

$$S_{BH} = (n+1)cF. \quad (2.5)$$

Therefore, it holds from Lemma 2 that $\mathcal{R}_{00} = -2ch^2$. $\square$

Corollary 1. Let $(M, F, d\mu)$ be a Randers measure space, where $d\mu = \sigma(x)dx^1 \wedge \cdots \wedge dx^n$ is an arbitrary volume form, and F is expressed in terms of a Riemannian metric h and a vector field W by (1.3). If F is of weakly isotropic S -curvature with respect to volume form $d\mu$, then $d\mu$ is the Busemann-Hausdorff volume form up to a constant multiplication.

Proof. If F is of weakly isotropic S -curvature with respect to an arbitrary volume form, then from Theorem 2, we have

$$\mathcal{R}_{00} = -2ch^2.$$

Thus, by Lemma 2, we have (2.5). On the other hand, we also have (2.4) from (2.2) and (2.3). Therefore, combining (2.4) with (2.5), we deduce

$$\eta + y^i \frac{\partial}{\partial x^i} \log \varphi = 0,$$

for any η and any y . By the arbitrariness of η and y , we have

$$\frac{\partial \varphi}{\partial x^i} = 0, \quad \forall i,$$

which yields φ is a constant. This ends the proof. $\square$

Proof of Theorem 1. It follows from Corollary 1 directly. $\square$

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