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New regularity coefficients

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Abstract. We give two new characterizations of the notion of Lyapunov regularity in terms of the lower and upper exponential growth rates of the singular values. These characterizations motivate the introduction of new regularity coefficients. In particular, we establish relations between these regularity coefficients and the Lyapunov regularity coefficient. Moreover, we construct explicitly bounded sequences of matrices attaining specific values of the new regularity coefficients.

Keywords: Lyapunov regularity, regularity coefficients.

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1 Introduction

The purpose of this work is twofold: to introduce new regularity coefficients and to give new characterizations of Lyapunov regularity. The notion of regularity was introduced by Lyapunov and plays an important role in the stability theory of differential equations and dynamical systems. It is particularly ubiquitous in the context of ergodic theory. The new characterizations of Lyapunov regularity are expressed in terms of the lower and upper exponential growth rates of the singular values.

1.1 The notion of regularity

We start by describing the meaning and some of the implications of Lyapunov regularity. Let $(A_m)_{m\in\mathbb{N}}$ be a sequence of invertible $q \times q$ matrices with real entries. We assume that both sequences A_m and A_m^{-1} are bounded. For each $m \in \mathbb{N}$ let

$$\mathcal{A}_m = \begin{cases} A_{m-1}A_{m-2}\cdots A_1 & \text{if } m > 1, \\ \mathrm{Id} & \text{if } m = 1. \end{cases}$$

Given a basis $v_1, \dots, v_q$ for $\mathbb{R}^q$, any regularity coefficient measures how much

$$\lambda(v_i) := \limsup_{m\to\infty} \frac{1}{m} \log \|\mathcal{A}_m v_i\|, \tag{1.1}$$

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for $i = 1, \dots, q$, differs from being a limit, and how much

$$\alpha_{ij} := \limsup_{m\to\infty} \frac{1}{m} \log \angle(\mathcal{A}_m v_i, \mathcal{A}_m v_j), \tag{1.2}$$

for $i \neq j$, differs from zero. In particular, the *Lyapunov regularity coefficient* determined by the sequence $A = (A_m)_{m\in\mathbb{N}}$ is defined by

$$\sigma(A) = \min \sum_{i=1}^{q} \lambda(v_i) - \liminf_{m\to\infty} \frac{1}{m} \log|\det \mathcal{A}_m|, \tag{1.3}$$

where the minimum is taken over all bases $v_1, \dots, v_q$ for $\mathbb{R}^q$. One can show that $\sigma(A) \geq 0$ and that $\sigma(A) = 0$ if and only if each lim sup in (1.1) is a limit and each lim sup in (1.2) vanishes for any basis $v_1, \dots, v_q$ (see [2,3]). The sequence A is said to be *(Lyapunov) regular* if $\sigma(A) = 0$.

More generally, a *regularity coefficient* is a nonnegative function on the sequences of matrices $A = (A_m)_{m\in\mathbb{N}}$ vanishing only on the Lyapunov regular systems. Besides the Lyapunov regularity coefficient (see [11]), other regularity coefficients were introduced already at an early stage of the theory by Perron (see [13, 14]) and Grobman (see [6]), although for a dynamics with continuous time obtained from a linear ordinary differential equation. We refer the reader to the books [2,3,6,10] for detailed accounts of various parts of the theory, both for discrete and continuous time.

1.2 Origins and relevance of regularity

The notion of regularity first appeared in the works of Lyapunov (see [11]) and Perron [13,14], in connection with the study of the stability of solutions of perturbations of linear ordinary differential equations. As already described above, one can introduce a similar notion of regularity and corresponding regularity coefficients for a dynamics with discrete time

$$x_{m+1} = A_m x_m \quad \text{for } m \in \mathbb{N}$$

on $\mathbb{R}^q$, obtained from a sequence $(A_m)_{m\in\mathbb{N}}$ of $q \times q$ matrices. Some works that consider the case of discrete time include [15] (see also [4]) with a study of the relation of regularity with the exponential growth rates of the singular values, [5, 9] with descriptions of relations between regularity coefficients, and [8] with the introduction of a new regularity coefficient. For further references we refer the reader to [2] (see also [7]).

It turns out that Lyapunov regularity has various nontrivial applications to the stability theory of differential equations and dynamical systems. The reason for this is that any regularity coefficient measures how much the exponential stability or conditional stability of a given trajectory of a linear dynamics differs from being uniform on the initial time. For example, provided that a regularity coefficient is sufficiently small, one can construct stable and unstable invariant manifolds for any sufficiently small nonlinear perturbation when all Lyapunov exponents are nonzero (see [3] for details). This is particularly effective in the context of smooth ergodic theory, since a certain integrability assumption guarantees that the linearizations along almost all trajectories have zero regularity coefficient, as a consequence of Oseledets' Multiplicative ergodic theorem [12].

1.3 Characterizations of regularity

Now we describe briefly our results. In particular, we give new characterizations of Lyapunov regularity that are expressed in terms of the lower and upper exponential growth rates of the singular values. This also serves as a preparation for introducing new regularity coefficients.

Again let $A = (A_m)_{m\in\mathbb{N}}$ be a sequence of invertible $q \times q$ matrices with real entries. We assume that A_m and A_m^{-1} are bounded in m. Now let

$$\rho_1(m) \leq \cdots \leq \rho_q(m)$$

be the eigenvalues of the positive-semidefinite matrix $(\mathcal{A}_m^*\mathcal{A}_m)^{1/2}$. The *lower and upper exponential growth rates of the singular values* are defined by

$$a_i = \liminf_{m\to\infty} \frac{1}{m}\log\rho_i(m) \quad \text{and} \quad b_i = \limsup_{m\to\infty} \frac{1}{m}\log\rho_i(m).$$

In particular, we obtain new characterizations of Lyapunov regularity in terms of these numbers (see Theorem 2.1).

Theorem 1.1. *The following properties are equivalent:*

1. $(A_m)_{m\in\mathbb{N}}$ *is regular;*
2. $\frac{1}{m}\log|\det\mathcal{A}_m| \to \sum_{i=1}^q b_i$ *when* $m\to\infty$*;*
3. $\frac{1}{m}\log|\det\mathcal{A}_m| \to \sum_{i=1}^q a_i$ *when* $m\to\infty$*.*

Some arguments in the proof are inspired by work of Barabanov in [1] who considered a corresponding problem for the case of continuous time.

Now we consider the values $\lambda_1' \leq \cdots \leq \lambda_q'$ of the Lyapunov exponent determined by the sequence A, counted with their multiplicities. If $v_1,\ldots,v_q$ is a basis for $\mathbb{R}^q$ at which the minimum in (1.3) is attained, then $\lambda_i' = \lambda(v_i)$ for $i = 1,\ldots,q$ up to a reordering of the values. One can show that the sequence A is regular if and only if

$$\frac{1}{m}\log|\det\mathcal{A}_m| \to \sum_{i=1}^q \lambda_i' \quad \text{when } m\to\infty$$

(see [3]). Incidentally, we have $a_i \leq b_i \leq \lambda_i'$ for $i = 1,\ldots,q$ and each of these inequalities can be strict (see [2,4]).

As an outcome of our approach, we also obtain a new proof of a characterization of regularity involving only the lower and upper exponential growth rates of the singular values: namely, A is regular if and only if

$$a_i = b_i \quad \text{for } i = 1,\ldots,q. \tag{1.4}$$

It follows from work of Ruelle in [15] that condition (1.4) yields the regularity of A. Barabanov [1] gave a new proof of this property and also obtained the other direction of the equivalence for a dynamics with continuous time. We consider the case of discrete time and we give a new proof of this equivalence (see [4] for a proof based on the existence of a structure of Oseledets type that is present even for a nonregular dynamics). Again, some arguments are inspired in [1].

1.4 New regularity coefficients

Finally, we introduce three new regularity coefficients motivated by Theorem 1.1 (see also Theorem 2.1). Then we establish some relations between these coefficients and the Lyapunov regularity coefficient. Given a sequence $A = (A_m)_{m\in\mathbb{N}}$ of invertible $q \times q$ matrices with real entries, we define

$$\alpha(A) = \max\{b_i - a_i : i = 1, \dots, q\},$$
$$\underline{\sigma}(A) = \sum_{i=1}^{q} b_i - \liminf_{m\to\infty} \frac{1}{m} \log|\det \mathcal{A}_m|,$$
$$\overline{\sigma}(A) = \limsup_{m\to\infty} \frac{1}{m} \log|\det \mathcal{A}_m| - \sum_{i=1}^{q} a_i.$$

The advantage of having various regularity coefficients is that in each specific situation it is often easier to compute or at least to estimate one of them. We show in Theorem 3.1 that

$$0 \le \alpha(A) \le \underline{\sigma}(A) \le q\alpha(A) \quad \text{and} \quad 0 \le \alpha(A) \le \overline{\sigma}(A) \le q\alpha(A).$$

Finally, we construct bounded sequences of matrices attaining specific values of the regularity coefficients (see Theorem 3.3). The construction builds on former work in [5] although it required several nontrivial modifications.

Theorem 1.2. *Given numbers $p, g \ge 0$ such that $p \le g \le qp$, there exists a bounded sequence A of diagonal $q \times q$ matrices with $\alpha(A) = p$ and $\underline{\sigma}(A) = g$.*

In Section 4 we introduce two additional regularity coefficients. We also establish inequalities between these coefficients and the former ones.

1.5 Relevance of the results

Finally, we discuss the relevance of the results obtained in the paper. As noted above, the notion of regularity plays an important role in the stability theory of a dynamics with continuous or discrete time. In fact, a vanishing or sufficiently small regularity coefficient implies that the asymptotic stability of a trajectory a linear dynamics persists under sufficiently small nonlinear perturbations. This leads in particular to the construction of stable and unstable invariant manifolds, as well as to many other nontrivial properties. On the other hand, in each specific situation it may be easier to obtain bounds for a certain regularity coefficient. Thus, it is convenient to have additional coefficients. In particular, it may be easier in some specific situations to use instead the new regularity coefficients introduced in our paper.

In another direction, when a dynamics is regular (when some regularity coefficient vanishes, in which case all regularity coefficients vanish), there is a richer structure, such as for example the one illustrated by (1.1) and (1.2). Our work provides further additional properties caused by regularity that in fact also provide additional structure.

2 Characterizations of regularity

In this section we give new characterizations of Lyapunov regularity that are expressed in terms of the lower and upper exponential growth rates of the singular values. This serves as a preparation for introducing new regularity coefficients in Section 3, although the characterizations are also of interest by themselves.

Let $(A_m)_{m\in\mathbb{N}}$ be a sequence of invertible $q \times q$ matrices with real entries. We shall always assume that there exists $c \in \mathbb{R}$ such that

$$\|A_m\| \le c \quad \text{and} \quad \|A_m^{-1}\| \le c \tag{2.1}$$

for all $m \in \mathbb{N}$. For each $m \in \mathbb{N}$, let

$$\mathcal{A}_m = \begin{cases} A_{m-1}A_{m-2}\cdots A_1 & \text{if } m > 1, \\ \mathrm{Id} & \text{if } m = 1. \end{cases}$$

The Lyapunov exponent $\lambda\colon \mathbb{R}^q \to \mathbb{R} \cup \{-\infty\}$ associated with the sequence $(A_m)_{m\in\mathbb{N}}$ is defined by

$$\lambda(v) = \limsup_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m v\|,$$

with the convention that $\log 0 = -\infty$. By the abstract theory of Lyapunov exponents, λ takes at most a number $s \le q$ of distinct values on $\mathbb{R}^q \setminus \{0\}$, say

$$\lambda_1 < \lambda_2 < \cdots < \lambda_s.$$

Moreover, for each $i = 1, \ldots, s$ the set

$$E_i = \big\{ v \in \mathbb{R}^q : \lambda(v) \le \lambda_i \big\}$$

is a linear subspace of $\mathbb{R}^q$ and

$$\{0\} \subset E_1 \subset E_2 \subset \cdots \subset E_s = \mathbb{R}^q.$$

We denote by $\lambda'_1 \le \lambda'_2 \le \cdots \le \lambda'_q$ the values of the Lyapunov exponent λ counted with their multiplicities. These are obtained repeating each value λ_i a number of times equal to $\dim E_i - \dim E_{i-1}$, with the convention that $E_0 = \{0\}$. The sequence $(A_m)_{m\in\mathbb{N}}$ is said to be *(Lyapunov) regular* if

$$\lim_{m\to\infty} \frac{1}{m} \log|\det \mathcal{A}_m| = \sum_{i=1}^{q} \lambda'_i \tag{2.2}$$

(this includes the requirement that the limit on the left-hand side exists).

We also consider the singular values. The matrix

$$T_m = (\mathcal{A}_m^* \mathcal{A}_m)^{1/2} \tag{2.3}$$

is symmetric and positive-semidefinite. Hence, its eigenvalues

$$\rho_1(m) \le \cdots \le \rho_q(m)$$

(counted with their multiplicities) are real and nonnegative. They are called the *singular values* of the matrix $\mathcal{A}_m$. For $i = 1, \ldots, q$, we define *the lower and upper exponential growth rates of the singular values*, respectively, by

$$a_i = \liminf_{m\to\infty} \frac{1}{m} \log \rho_i(m) \quad \text{and} \quad b_i = \limsup_{m\to\infty} \frac{1}{m} \log \rho_i(m).$$

We note that

$$a_i \le b_i \le \lambda'_i \quad \text{for } i = 1, \ldots, q \tag{2.4}$$

(see for example Proposition 6.1.2 in [2]).

The following result gives two new characterizations of Lyapunov regularity (properties (ii) and (iii)). As an outcome of our approach, we also obtain a new proof of a characterization involving only the lower and upper exponential growth rates of the singular values (property (iv)).

Theorem 2.1. *Let $(A_m)_{m\in\mathbb{N}}$ be a sequence of invertible $q\times q$ matrices with real entries satisfying (2.1). Then the following properties are equivalent:*

(i) $(A_m)_{m\in\mathbb{N}}$ *is regular;*

(ii) $\frac{1}{m}\log|\det\mathcal{A}_m| \to \sum_{i=1}^q b_i$ *when* $m\to\infty$*;*

(iii) $\frac{1}{m}\log|\det\mathcal{A}_m| \to \sum_{i=1}^q a_i$ *when* $m\to\infty$*;*

(iv) $a_i = b_i$ *for* $i=1,\dots,q$*.*

Proof. $(i)\Rightarrow(ii)$. Since $|\det\mathcal{A}_m| = \det T_m$ (see (2.3)), we obtain

$$\frac{1}{m}\log|\det\mathcal{A}_m| = \frac{1}{m}\log\prod_{i=1}^q \rho_i(m) = \sum_{i=1}^q \frac{1}{m}\log\rho_i(m) \tag{2.5}$$

and so

$$\lim_{m\to\infty}\frac{1}{m}\log|\det\mathcal{A}_m| = \lim_{m\to\infty}\sum_{i=1}^q \frac{1}{m}\log\rho_i(m) \le \sum_{i=1}^q b_i.$$

Therefore, it follows from (2.2) that

$$\sum_{i=1}^q \lambda_i' \le \sum_{i=1}^q b_i.$$

By (2.4) we have $\sum_{i=1}^q \lambda_i' = \sum_{i=1}^q b_i$ and property (ii) follows readily from (2.2).

$(ii)\Rightarrow(iv)$. We proceed by contradiction. Assume that $a_k < b_k$ for some $k\in\{1,\dots,q\}$ and take a sequence $(m_l)_{l\in\mathbb{N}}$ such that

$$a_k = \liminf_{m\to\infty}\frac{1}{m}\log\rho_k(m) = \lim_{l\to\infty}\frac{1}{m_l}\log\rho_k(m_l).$$

By (ii) and (2.5), since $a_k < b_k$ we obtain

$$\begin{aligned}\sum_{i=1}^q b_i &= \lim_{m\to\infty}\frac{1}{m}\log|\det\mathcal{A}_m| = \lim_{m\to\infty}\sum_{i=1}^q\frac{1}{m}\log\rho_i(m)\\ &= \lim_{l\to\infty}\sum_{i=1}^q\frac{1}{m_l}\log\rho_i(m_l) = a_k + \lim_{l\to\infty}\sum_{i\ne k}\frac{1}{m_l}\log\rho_i(m_l)\\ &\le a_k + \sum_{i\ne k} b_i < \sum_{i=1}^q b_i.\end{aligned}$$

This contradiction implies that (iv) holds.

Now we obtain the former implications with (ii) replaced by (iii).

$(i)\Rightarrow(iii)$. Let $B_m = (A_m^*)^{-1}$ and define

$$\mathcal{B}_m = \begin{cases} B_{m-1}B_{m-2}\cdots B_1 & \text{if } m>1,\\ \mathrm{Id} & \text{if } m=1.\end{cases}$$

Note that $\mathcal{B}_m = (\mathcal{A}_m^*)^{-1}$. The Lyapunov exponent $\mu\colon \mathbb{R}^q \to \mathbb{R} \cup \{-\infty\}$ associated with the sequence $(B_m)_{m\in\mathbb{N}}$ is defined by

$$\mu(w) = \limsup_{m\to\infty} \frac{1}{m} \log\|\mathcal{B}_m w\|.$$

By the abstract theory of Lyapunov exponents, μ takes at most a number q of distinct values on $\mathbb{R}^q \setminus \{0\}$ and we denote by $\mu_1' \geq \mu_2' \geq \cdots \geq \mu_q'$ these values counted with their multiplicities. One can show that $(A_m)_{m\in\mathbb{N}}$ is regular if and only if $(B_m)_{m\in\mathbb{N}}$ is regular, in which case we have $\lambda_i' = -\mu_i'$ for $i = 1, \ldots, q$ (see for example Theorem 2.4.5 in [2]). Therefore,

$$\lim_{m\to\infty} \frac{1}{m} \log|\det \mathcal{B}_m| = \sum_{i=1}^{q} \mu_i'.$$

Proceeding as in the proof of the implication $(i) \Rightarrow (ii)$, one can show that

$$\lim_{m\to\infty} \frac{1}{m} \log|\det \mathcal{B}_m| = \sum_{i=1}^{q} \beta_i, \tag{2.6}$$

where

$$\beta_i = \limsup_{m\to\infty} \frac{1}{m} \log \sigma_i(m),$$

denoting by

$$\sigma_1(m) \leq \cdots \leq \sigma_q(m)$$

the eigenvalues of the matrix $S_m = (\mathcal{B}_m^* \mathcal{B}_m)^{1/2}$ (which is symmetric and positive-semidefinite). Note that

$$\sigma_i(m) = \frac{1}{\rho_{q-i+1}(m)} \quad \text{for } i = 1, \ldots, q$$

and so

$$\beta_i = -a_{q-i+1} \quad \text{and} \quad \alpha_i = -b_{q-i+1} \quad \text{for } i = 1, \ldots, q, \tag{2.7}$$

where

$$\alpha_i = \liminf_{m\to\infty} \frac{1}{m} \log \sigma_i(m).$$

In view of (2.6) and (2.7) we get

$$\lim_{m\to\infty} \frac{1}{m} \log|\det \mathcal{A}_m| = -\lim_{m\to\infty} \frac{1}{m} \log|\det \mathcal{B}_m| = -\sum_{i=1}^{q} \beta_i = \sum_{i=1}^{q} a_i,$$

which establishes property (iii).

$(iii) \Rightarrow (iv)$**.** The proof is identical to the proof of the implication $(ii) \Rightarrow (iv)$ using the identities in (2.7).

$(iv) \Rightarrow (i)$. We assume that the upper exponential growth rates of the singular values take r distinct values

$$c_1 < \cdots < c_r. \tag{2.8}$$

Let $n_1, \ldots, n_r$ be their multiplicities. Moreover, let $v_1(m), \ldots, v_q(m)$ be an orthonormal basis for $\mathbb{R}^q$ formed by eigenvectors of the matrix T_m associated, respectively, with the eigenvalues $\rho_1(m), \ldots, \rho_q(m)$. For $i = 1, \ldots, q$ and $m \in \mathbb{N}$, let

$$E_i(m) = \text{span}\{v_{n_1+\cdots+n_{i-1}+1}(m), \ldots, v_{n_1+\cdots+n_i}(m)\}$$

and

$$F_i(m) = \text{span}\{v_1(m), \ldots, v_{n_1+\cdots+n_i}(m)\}.$$

Since the basis $v_1(m), \ldots, v_q(m)$ is orthonormal, we have

$$F_i(m)^\perp = \text{span}\{v_{n_1+\cdots+n_i+1}(m), \ldots, v_q(m)\}. \tag{2.9}$$

Considering the 2-norm for $\mathbb{R}^q$, we obtain

$$\|\mathcal{A}_m v\|^2 = v^* \mathcal{A}_m^* \mathcal{A}_m v = v^* T_m^2 v = v^* T_m^* T_m v = \|T_m v\|^2$$

and so

$$\|\mathcal{A}_m|_{F_i(m)}\| = \sup_{v \in F_i(m) \setminus \{0\}} \frac{\|\mathcal{A}_m v\|}{\|v\|} = \sup_{v \in F_i(m) \setminus \{0\}} \frac{\|T_m v\|}{\|v\|} = \rho_{l_i}(m), \tag{2.10}$$

where $l_i = n_1 + \cdots + n_i$. Moreover, for any subspace $L \subset F_i(m)^\perp$, it follows from (2.9) that

$$\rho_{l_{i+1}}(m) \le \|\mathcal{A}_m|_L\| \le \rho_q(m). \tag{2.11}$$

Properties (2.10) and (2.11) are crucial in the remainder of the proof.

Before proceeding, we recall some notions and results concerning the distance between two linear spaces. Let $\angle(v, w)$ be the angle between two vectors v and w. Given linear subspaces $E, F \subset \mathbb{R}^q$, we define

$$d(E, F) = \sin \angle(E, F),$$

where

$$\angle(E, F) = \max\{\theta(E, F), \theta(F, E)\}$$

and

$$\theta(E, F) = \max_{v \in E \setminus \{0\}} \min_{w \in F \setminus \{0\}} \angle(v, w).$$

Note that

$$\theta(E, F) = \max_{v \in E \setminus \{0\}} \angle(v, \text{proj}_F v),$$

where $\text{proj}_F v$ is the orthogonal projection of v onto F.

The following result is proved in [1].

Lemma 2.2. *The following properties hold:*

1. *If* $\dim E = \dim F$, *then* $\theta(E, F) = \theta(F, E)$ *and* $d(E, F) = d(E^\perp, F^\perp)$.
2. *If a sequence* $(E_k)_{k \in \mathbb{N}}$ *of linear spaces of equal dimensions is a Cauchy sequence, then it converges to a linear space* E *of the same dimension.*

3. *Let $(E_k)_{k\in\mathbb{N}}$ be a sequence of linear spaces converging to a linear space E such that $E_k = F_k \oplus G_k$, where G_k is the orthogonal complement of F_k in E_k. If the sequence $(F_k)_{k\in\mathbb{N}}$ converges to a linear space F, then $F \subset E$ and the sequence $(G_k)_{k\in\mathbb{N}}$ converges to a linear space G that is the orthogonal complement of F in E.*

Now let

$$\alpha_i(m) = \angle(F_i(m), F_i(m+1)) \quad \text{for } i = 1, \ldots, r-1.$$

Lemma 2.3. *We have*

$$\limsup_{m\to\infty} \frac{1}{m} \log \alpha_i(m) \le c_i - c_{i+1} \quad \text{for } i = 1, \ldots, r-1.$$

Proof of the lemma. We proceed by contradiction. Assume the contrary. Then for some $i \in \{1, \ldots, r-1\}$ there exist $\varepsilon > 0$ and a sequence $(m_l)_{l\in\mathbb{N}}$ such that

$$\frac{1}{m_l} \log \alpha_i(m_l) > c_i - c_{i+1} + \varepsilon \tag{2.12}$$

for $l \in \mathbb{N}$. Take $v \in F_i(m+1)$ with $\|v\| = 1$ such that

$$\angle(v, F_i(m)) = \alpha_i(m)$$

and write it in the form $v = v_1 + v_2$ with $v_1 \in F_i(m)$ and $v_2 \in F_i(m)^\perp$. By (2.12) we have $v_2 \neq 0$. It follows from (2.10) that

$$\|\mathcal{A}_m v_1\| \le \rho_{l_i}(m)\|v_1\|$$

and taking $L = \operatorname{span}\{v_2\}$ in (2.11) we obtain

$$\|\mathcal{A}_m v_2\| \ge \rho_{l_{i+1}}(m)\|v_2\|.$$

On the other hand, we have

$$\|v_1\| = \|v\| \cos \alpha_i(m) \le \|v\|$$

and

$$\|v_2\| = \|v\| \sin \alpha_i(m) \ge \frac{2}{\pi}\alpha_i(m)\|v\|$$

because $\sin x \ge \frac{2}{\pi}x$ for $x \in [0, \pi/2]$. Therefore,

$$\begin{aligned}
\|A_m v\| &\ge \|\mathcal{A}_m v_2\| - \|\mathcal{A}_m v_1\| \\
&\ge \rho_{l_{i+1}}(m)\|v_2\| - \rho_{l_i}(m)\|v_1\| \\
&\ge \left(\rho_{l_{i+1}}(m)\frac{2}{\pi}\alpha_{l_i}(m) - \rho_{l_i}(m)\right)\|v\|.
\end{aligned}$$

Note that given $\delta > 0$, by (2.8) there exists $m = m(\delta)$ such that

$$e^{(a_j-\delta)m} \le \rho_j(m) \le e^{(a_j+\delta)m}$$

for all $j = 1, \ldots, q$ and $m \ge m(\delta)$. Hence, for $m = m_l$ we obtain

$$\begin{aligned}
\|\mathcal{A}_{m_l} v\| &\ge \left(e^{(c_{i+1}-\delta)m_l}\frac{2}{\pi}e^{(c_i-c_{i+1}+\varepsilon)m_l} - e^{(c_i+\delta)m_l}\right)\|v\| \\
&= \left(\frac{2}{\pi}e^{(c_i-\delta+\varepsilon)m_l} - e^{(c_i+\delta)m_l}\right)\|v\|.
\end{aligned} \tag{2.13}$$

Taking $\delta < \varepsilon/2$, we have $c_i - \delta + \varepsilon > c_i + \delta$. Since $v \in F_i(m_l + 1)$ and $\|A_m^{-1}\| \le c$, it follows from (2.13) that

$$c_i + \delta < \limsup_{l\to\infty} \frac{1}{m_l} \log\|\mathcal{A}_{m_l} v\| \le \lim_{l\to\infty} \frac{1}{m_l+1} \log \rho_{l_i}(m_l+1) = c_i,$$

which is impossible. This contradiction yields the desired result. □

We proceed with the proof of the theorem. We first show that $(F_i(m))_{m\in\mathbb{N}}$ is a Cauchy sequence. By Lemma 2.3, taking $\varepsilon > 0$ such that $c_i - c_{i+1} + \varepsilon < 0$ we have

$$\begin{aligned} d(F_i(m), F_i(k)) &\le \sum_{j=m}^{k-1} d(F_i(j), F_i(j+1)) \\ &\le \sum_{j=m}^{\infty} d(F_i(j), F_i(j+1)) = \sum_{j=m}^{\infty} \sin\alpha_i(j) \\ &\le \sum_{j=m}^{\infty} e^{(c_i - c_{i+1}+\varepsilon)j} = \frac{e^{(c_i-c_{i+1}+\varepsilon)m}}{1 - e^{c_i - c_{i+1}+\varepsilon}} \end{aligned}$$

for all sufficiently large m and all $k > m$. This shows that $(F_i(m))_{m\in\mathbb{N}}$ is a Cauchy sequence. In view of Lemma 2.2 (second item), we conclude that $(F_i(m))_{m\in\mathbb{N}}$ converges to some linear space F satisfying

$$d(F_i(m), F_i) \le \frac{e^{(c_i-c_{i+1}+\varepsilon)m}}{1 - e^{c_i - c_{i+1}+\varepsilon}}. \tag{2.14}$$

Moreover, also in view of Lemma 2.2 (third item), the sequence $(E_i(m))_{m\in\mathbb{N}}$ also converges to some linear space E. Indeed, since $F_{i+1}(m) \to F_{i+1}$ when $m \to \infty$ and

$$F_{i+1}(m) = F_i(m) \oplus E_{i+1}(m)$$

with

$$E_{i+1}(m) = F_i(m)^\perp \cap F_{i+1}(m),$$

we conclude that $(E_i(m))_{m\in\mathbb{N}}$ converges to some linear space E_i.

Lemma 2.4. *For $k = 1, \ldots, r$ we have*

$$\lim_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m w\| = c_k \quad \text{for } w \in E_k \setminus \{0\}$$

and

$$\limsup_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m w\| \le c_{k-1} \quad \text{for } w \in F_{k-1} \setminus \{0\}.$$

Proof of the lemma. We proceed by backwards induction on k. Take $j \in \{0, 1, \ldots, r-2\}$ and given $w \in \mathbb{R}^q$, write it in the form $w = w_1 + w_2$ with $w_1 \in F_{r-j-1}(m)^\perp$ and $w_2 \in F_{r-j-1}(m)$.

First take $j = 0$ and $w \in E_r \setminus \{0\}$. Then $w = w_1 + w_2$ with $w_1 \in F_{r-1}(m)^\perp$ and $w_2 \in F_{r-1}(m)$. Note that $w_1 \ne 0$ for any sufficiently large m, since $\angle(E_r, E_r(m)) \to 0$ when $m \to \infty$ and $E_r(m) = F_{r-1}(m)^\perp$. In view of (2.11) we have

$$\rho_{q-n_r+1}(m)\|w_1\| \le \|\mathcal{A}_m w_1\| \le \rho_q(m)\|w_1\|,$$

which implies that

$$\lim_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m w_1\| = c_r.$$

Moreover, in view of (2.10) we also have

$$\|\mathcal{A}_m w_2\| \leq \rho_{q-r_n}(m)\|w_2\|,$$

which implies that

$$\limsup_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m w_2\| \leq c_{r-1}.$$

Since $\mathcal{A}_m w = \mathcal{A}_m w_1 + \mathcal{A}_m w_2$ and $c_{r-1} < c_r$, we conclude that

$$\lim_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m w\| = c_r.$$

Now take $w \in F_{r-1} \setminus \{0\}$ and recall that $w = w_1 + w_2$ with $w_1 \in F_{r-1}(m)^\perp$ and $w_2 \in F_{r-1}(m)$. Since $\angle(F_{r-1}, F_{r-1}(m)) \to 0$ when $m \to \infty$, we have $w_2 \neq 0$ for any sufficiently large m. Note that

$$\|w_1\| = \|w\| \sin\angle(w, w_2)$$

and

$$\|w_2\| = \|w\| \cos\angle(w, w_2) \leq \|w\|.$$

Since $\sin\angle(w, w_2) \leq d(F_{r-1}, F_{r-1}(m))$, it follows from (2.14) that

$$\begin{aligned}
\|\mathcal{A}_m w\| &\leq \|\mathcal{A}_m w_1\| + \|\mathcal{A}_m w_2\| \\
&\leq \rho_q(m)\|w_1\| + \rho_{q-n_r}(m)\|w_2\| \\
&\leq \frac{e^{(c_{r-1}-c_r+\varepsilon)m}}{1-e^{c_{r-1}-c_r+\varepsilon}} \rho_q(m)\|w\| + \rho_{q-n_r}(m)\|w\|.
\end{aligned}$$

We have

$$\lim_{m\to\infty} \frac{1}{m} \log\left(\frac{e^{(c_{r-1}-c_r+\varepsilon)m}}{1-e^{c_{r-1}-c_r+\varepsilon}} \rho_q(m)\right) = c_{r-1} - c_r + \varepsilon + c_r = c_{r-1} + \varepsilon$$

and

$$\limsup_{m\to\infty} \frac{1}{m} \log \rho_{q-n_r}(m) = c_{r-1}.$$

Therefore,

$$\limsup_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m w\| \leq c_{r-1} + \varepsilon$$

and since ε is arbitrary, we obtain

$$\limsup_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m w\| \leq c_{r-1}.$$

This establishes the induction hypothesis for $j = 0$.

Now assume that the statement in Lemma 2.4 holds for $k = r, \ldots, r-j+1$ and some $j \geq 1$. We want to show that it also holds for $k = r-j$. Take $w \in E_{r-j} \setminus \{0\}$. Since $E_{r-j} \setminus \{0\} \subset F_{r-j} \setminus \{0\}$, it follows from the induction hypothesis that

$$\limsup_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m w\| \leq c_{r-j}. \tag{2.15}$$

We first show that

$$\liminf_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m w\| \geq c_{r-j} \tag{2.16}$$

for $w \in E_{r-j} \setminus \{0\}$. Then it follows from (2.15) and (2.16) that

$$\lim_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m w\| = c_{r-j},$$

which establishes the first statement in the lemma.

Since $E_{r-j} \setminus \{0\} \subset F_{r-j-1}^{\perp} \setminus \{0\}$, we take $w \in F_{r-j-1}^{\perp} \setminus \{0\}$ and write it in the form $w = w_1 + w_2$ with $w_1 \in F_{r-j-1}(m)^{\perp}$ and $w_2 \in F_{r-j-1}(m)$. Since

$$d(F_{r-j-1}^{\perp}, F_{r-j-1}(m)^{\perp}) \to 0 \quad \text{when } m \to \infty,$$

we have $w_1 \neq 0$ for any sufficiently large m. Moreover,

$$\|w_1\| = \|w\| \cos\angle(w, w_1), \quad \|w_2\| = \|w\| \sin\angle(w, w_2). \tag{2.17}$$

In view of (2.14) we have

$$\begin{aligned} \sin\angle(w, w_1) &\leq d(F_{r-j-1}^{\perp}, F_{r-j-1}(m)^{\perp}) \\ &= d(F_{r-j-1}, F_{r-j-1}(m)) \\ &\leq \frac{e^{(c_{r-j-1}-c_{r-j}+\varepsilon)m}}{1 - e^{c_{r-j-1}-c_{r-j}+\varepsilon}} =: \alpha_j(m). \end{aligned} \tag{2.18}$$

Hence, by (2.10) and (2.11) together with (2.17) and (2.18), we obtain

$$\begin{aligned} \|\mathcal{A}_m w\| &\geq \|\mathcal{A}_m w_1\| - \|\mathcal{A}_w w_2\| \\ &\geq \rho_{l_{r-j}}(m)\|w_1\| - \rho_{l_{r-j-1}}(m)\|w_2\| \\ &\geq \rho_{l_{r-j}}(m)\sqrt{1 - \alpha_j(m)^2}\|w\| - \rho_{l_{r-j-1}}(m)\|w\|. \end{aligned}$$

Therefore, since $\alpha_j(m) \to 0$ when $m \to \infty$, we have

$$\lim_{m\to\infty} \frac{1}{m} \log\Big(\rho_{l_{r-j}}(m)\sqrt{1 - \alpha_j(m)^2}\Big) = c_{r-j}$$

and

$$\lim_{m\to\infty} \frac{1}{m} \log \rho_{l_{r-j-1}}(m) = c_{r-j-1}.$$

Finally, since $c_{r-j} > c_{r-j-1}$, we conclude that

$$\liminf_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m w\| \geq c_{r-j},$$

which establishes (2.16).

Now we prove the second statement in the lemma. Take $w \in F_{r-j-1} \setminus \{0\}$ and write it in the form $w = w_1 + w_2$ with $w_1 \in F_{r-j-1}^{\perp}(m)$ and $w_2 \in F_{r-j-1}(m)$. Since

$$\angle(F_{r-j-1}, F_{r-j-1}(m)) \to 0 \quad \text{when } m \to \infty,$$

we have $w_2 \neq 0$ for any sufficiently large m. Note that

$$\|w_1\| = \|w\| \sin\angle(w, w_2)$$

and

$$\|w_2\| = \|w\| \cos\angle(w, w_2) \leq \|w\|.$$

Since $\sin \angle(w, w_2) \le d(F_{r-j-1}, F_{r-j-1}(m))$ and

$$d(F_{r-j-1}, F_{r-j-1}(m)) \le \alpha_j(m),$$

using (2.10) and (2.11), we conclude as above that

$$\begin{aligned}\|\mathcal{A}_m w\| &\le \|\mathcal{A}_m w_1\| + \|\mathcal{A}_m w_2\| \\ &\le \alpha_j(m)\rho_{l_{r-j}}(m)\|w\| + \rho_{l_{r-j-1}}(m)\|w\|.\end{aligned}$$

We have

$$\lim_{m\to\infty} \frac{1}{m} \log\big(\alpha_j(m)\rho_{l_{r-j}}(m)\big) = c_{r-j-1} - c_{r-j} + \varepsilon + c_{r-j} = c_{r-j-1} + \varepsilon$$

and

$$\lim_{m\to\infty} \frac{1}{m} \log \rho_{l_{r-j-1}}(m) = c_{r-j-1}.$$

Therefore,

$$\limsup_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m w\| \le c_{r-j-1} + \varepsilon$$

and since ε is arbitrary, we obtain

$$\limsup_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}_m w\| \le c_{r-j-1}.$$

This establishes the induction hypothesis for $k = r - j$. □

We proceed with the proof of the theorem. Note that

$$\det T_m = \prod_{i=1}^{q} \rho_i(m).$$

Therefore, using (2.5) we have

$$\lim_{m\to\infty} \frac{1}{m} \log|\det \mathcal{A}_m| = \sum_{i=1}^{q} \lim_{m\to\infty} \frac{1}{m} \log \rho_i(m).$$

Finally, in view of Lemma 2.4 we obtain

$$\lim_{m\to\infty} \frac{1}{m} \log|\det \mathcal{A}_m| = \sum_{i=1}^{q} \lambda_i',$$

which shows that $(iv) \Rightarrow (i)$. This completes the proof of the theorem. □

3 New regularity coefficients

In this section we introduce three new regularity coefficients motivated by the properties (ii), (iii) and (iv) in Theorem 2.1. We also establish some relations between these coefficients and the Lyapunov regularity coefficient.

3.1 Regularity coefficients

Given a sequence $A = (A_m)_{m\in\mathbb{N}}$ of invertible $q \times q$ matrices with real entries, we define

$$\begin{aligned}
\alpha(A) &= \max\{b_i - a_i : i = 1, \ldots, q\}, \\
\underline{\sigma}(A) &= \sum_{i=1}^{q} b_i - \liminf_{m\to\infty} \frac{1}{m} \log|\det \mathcal{A}_m|, \\
\overline{\sigma}(A) &= \limsup_{m\to\infty} \frac{1}{m} \log|\det \mathcal{A}_m| - \sum_{i=1}^{q} a_i.
\end{aligned}$$

The following result gives some relations between these functions. In particular, together with Theorem 2.1, it shows that the three are indeed regularity coefficients.

Theorem 3.1. *For any bounded sequence $A = (A_m)_{m\in\mathbb{N}}$ of invertible $q \times q$ matrices, we have*

$$0 \le \alpha(A) \le \underline{\sigma}(A) \le q\alpha(A) \tag{3.1}$$

and

$$0 \le \alpha(A) \le \overline{\sigma}(A) \le q\alpha(A). \tag{3.2}$$

Proof. Clearly, $\alpha(A) \ge 0$. Note that

$$\begin{aligned}
\underline{\sigma}(A) &= \sum_{i=1}^{q} b_i - \liminf_{m\to\infty} \frac{1}{m} \log \prod_{i=1}^{q} \rho_i(m) \\
&\ge \sum_{i=1}^{q} b_i - \limsup_{m\to\infty} \frac{1}{m} \log \prod_{i\ne j} \rho_i(m) - \liminf_{m\to\infty} \frac{1}{m} \log \rho_j(m) \\
&\ge \sum_{i=1}^{q} b_i - \sum_{i\ne j} b_i - a_j = b_j - a_j
\end{aligned}$$

and so $\underline{\sigma}(A) \ge \alpha(A)$. Moreover,

$$\begin{aligned}
\underline{\sigma}(A) &= \sum_{i=1}^{q} b_i - \liminf_{m\to\infty} \frac{1}{m} \log \prod_{i=1}^{q} \rho_i(m) \\
&\le \sum_{i=1}^{q} b_i - \sum_{i=1}^{q} a_i = \sum_{i=1}^{q} (b_i - a_i) \le q\alpha(A),
\end{aligned}$$

which establishes (3.1).

On the other hand, we have

$$\begin{aligned}
\overline{\sigma}(A) &= \limsup_{m\to\infty} \frac{1}{m} \log \prod_{i=1}^{q} \rho_i(m) - \sum_{i=1}^{q} a_i \\
&\ge \limsup_{m\to\infty} \frac{1}{m} \log \rho_j(m) + \liminf_{m\to\infty} \frac{1}{m} \log \prod_{i\ne j} \rho_i(m) - \sum_{i=1}^{q} a_i \\
&\ge b_j + \sum_{i\ne j} a_i - \sum_{i=1}^{q} a_i = b_j - a_j
\end{aligned}$$

and so $\overline{\sigma}(A) \geq \alpha(A)$. Finally,

$$\begin{aligned}\overline{\sigma}(A) &= \limsup_{m\to\infty} \frac{1}{m} \log \prod_{i=1}^{q} \rho_i(m) - \sum_{i=1}^{q} a_i \\ &\leq \sum_{i=1}^{q} b_i - \sum_{i=1}^{q} a_i = \sum_{i=1}^{q} (b_i - a_i) \leq q\alpha(A),\end{aligned}$$

which establishes (3.2). □

It follows readily from Theorem 3.1 that

$$q^{-1}\overline{\sigma}(A) \leq \underline{\sigma}(A) \leq q\overline{\sigma}(A)$$

and

$$q^{-1}\underline{\sigma}(A) \leq \overline{\sigma}(A) \leq q\underline{\sigma}(A).$$

We also establish some relations between these three coefficients and the Lyapunov regularity coefficient. We recall that the *Lyapunov regularity coefficient* of a sequence $A = (A_m)_{m\in\mathbb{N}}$ is defined by

$$\sigma(A) = \sum_{i=1}^{q} \lambda_i' - \liminf_{m\to\infty} \frac{1}{m} \log|\det \mathcal{A}_m|.$$

Theorem 3.2. *For any bounded sequence $A = (A_m)_{m\in\mathbb{N}}$ of invertible $q \times q$ matrices, we have*

$$\underline{\sigma}(A) \leq \sigma(A) \quad \textit{and} \quad \sigma(A) \leq q^2\alpha(A).$$

Proof. The first inequality follows readily from the fact that $b_i \leq \lambda_i'$ for $i = 1, \ldots, q$ and the definitions of $\underline{\sigma}(A)$ and $\sigma(A)$.

For the second identity, we first observe that it suffices to consider upper-triangular matrices. Indeed, given a sequence $A = (A_m)_{m\in\mathbb{N}}$ of invertible $q \times q$ matrices, there exists a sequence $(U_m)_{m\in\mathbb{N}}$ of orthogonal $q \times q$ matrices with $U_1 = \mathrm{Id}$ such that

$$C_m = U_{m+1}^* A_m U_m$$

is upper-triangular for each $m \in \mathbb{N}$ (see Theorem 3.2.1 in [2]). Clearly, the sequence $C = (C_m)_{m\in\mathbb{N}}$ is also bounded and one can easily verify that

$$\alpha(C) = \alpha(A), \quad \underline{\sigma}(C) = \underline{\sigma}(A), \quad \overline{\sigma}(C) = \overline{\sigma}(A) \quad \text{and} \quad \sigma(C) = \sigma(A).$$

Without loss of generality, we assume from now on that all matrices are upper-triangular. We also consider the *Grobman regularity coefficient* $\gamma(A)$ that is defined by

$$\gamma(A) = \min \max\{\lambda(v_i) + \mu(w_i) : 1 \leq i \leq q\},$$

where the minimum is taken over all dual bases $v_1, \ldots, v_q$ and $w_1, \ldots, w_q$. Denoting by $a_{ij}(l)$ the entries of A_l, we have

$$\gamma(A) \leq \sum_{i=1}^{q} \left(\limsup_{m\to\infty} \frac{1}{m} \log \prod_{l=1}^{m} |a_{ii}(l)| - \liminf_{m\to\infty} \frac{1}{m} \log \prod_{l=1}^{m} |a_{ii}(l)| \right)$$

(see Theorem 3.1.3 in [2]). Since the matrices A_l are upper-triangular, we obtain

$$\frac{1}{m} \log \prod_{l=1}^{m} |a_{ii}(l)| = \frac{1}{m} \log \rho_{k_i}(m)$$

for some integer k_i and so

$$\begin{aligned}\gamma(A) &\le \sum_{i=1}^{q}\left(\limsup_{m\to\infty}\frac{1}{m}\log\rho_{k_i}(m)-\liminf_{m\to\infty}\frac{1}{m}\log\rho_{k_i}(m)\right)\\ &=\sum_{i=1}^{q}(b_{k_i}-a_{k_i})\le q\alpha(A).\end{aligned}$$

Moreover, we have

$$\frac{\sigma(A)}{q}\le\gamma(A)$$

(see Theorem 7.3.2 in [2]) and so $\sigma(A)\le q^2\alpha(A)$. This completes the proof of the theorem. □

3.2 Realization problem I

In this section we construct bounded sequences of matrices $A=(A_m)_{m\in\mathbb{N}}$ attaining specific values of the regularity coefficients $\alpha(A)$ and $\underline{\sigma}(A)$.

Theorem 3.3. *Given numbers $p,g\ge 0$ such that*

$$p\le g\le qp,$$

there exists a bounded sequence $A=(A_m)_{m\in\mathbb{N}}$ of diagonal $q\times q$ matrices with $\alpha(A)=p$ and $\underline{\sigma}(A)=g$.

Proof. Note that $\alpha(A)=\underline{\sigma}(A)=0$ for any regular sequence A. So it suffices to take $p>0$. We divide the proof into steps.

Step 1. Construction of sequences of numbers. Given $r,c,d\in\mathbb{R}$ with $r>1$ and $c\ge d$, for each $m\in\mathbb{N}$ let

$$a(m)=\begin{cases}e^d & \text{if } m\in S_k \text{ for } k\in\mathbb{N},\\ e^c & \text{if } m\in T_k \text{ for } k\in\mathbb{N},\end{cases}$$

where

$$S_k=\{m\in\mathbb{N}: r^{2k-2}\le m<r^{2k-1}\}$$

and

$$T_k=\{m\in\mathbb{N}: r^{2k-1}\le m<r^{2k}\}.$$

The following result is taken from [5].

Lemma 3.4. *For $\rho(m)=\prod_{j=1}^{m-1}a(j)$ we have*

$$\rho(m)=\begin{cases}e^{\frac{d+cr}{r+1}(r^{2k-2}-1)+d(m-r^{2k-2})} & \text{if } m\in S_k \text{ for } k\in\mathbb{N},\\ e^{\frac{d+cr}{r+1}(r^{2k-2}-1)+d(r^{2k-1}-r^{2k-2})+c(m-r^{2k-1})} & \text{if } m\in T_k \text{ for } k\in\mathbb{N}.\end{cases}$$

Moreover,

$$\limsup_{m\to\infty}\frac{1}{m}\log\rho(m)=\frac{d+cr}{r+1}$$

and

$$\liminf_{m\to\infty}\frac{1}{m}\log\rho(m)=\frac{c+dr}{r+1}.$$

Step 2. Construction of sequences of matrices. We say that the sequence of matrices $(A_m)_{m\in\mathbb{N}}$, where $A_m = \operatorname{diag}(a_1(m),\dots,a_q(m))$ for $m\in\mathbb{N}$ with

$$a_i(m) = \begin{cases} e^{d_i} & \text{if } m\in S_k \text{ for } k\in\mathbb{N},\\ e^{c_i} & \text{if } m\in T_k \text{ for } k\in\mathbb{N}, \end{cases} \tag{3.3}$$

is *r-regular* if the following conditions hold:

1. $c_i \ge d_i$ for $i=1,\dots,q$;
2. $c_i \le c_{i+1}$ and $d_i \le d_{i+1}$ for $i=1,\dots,q-1$.

Lemma 3.5. *For any r-regular sequence $A=(A_m)_{m\in\mathbb{N}}$, we have*

$$\alpha(A) = \frac{r-1}{r+1}\max_{1\le i\le q}(c_i - d_i)$$

and

$$\underline{\sigma}(A) = \frac{r-1}{r+1}\sum_{i=1}^{q}(c_i-d_i).$$

Proof of the lemma. It follows from Lemma 3.4 that

$$\rho_i(m) = \begin{cases} e^{\frac{d_i+c_ir}{r+1}(r^{2k-2}-1)+d_i(m-r^{2k-2})} & \text{if } m\in S_k \text{ for } k\in\mathbb{N},\\ e^{\frac{d_i+c_ir}{r+1}(r^{2k-2}-1)+d_i(r^{2k-1}-r^{2k-2})+c_i(m-r^{2k-1})} & \text{if } m\in T_k \text{ for } k\in\mathbb{N}, \end{cases}$$

for $i=1,\dots,q$. Indeed, if $m\in S_k$, then

$$\begin{aligned}\rho_i(m) &= e^{\frac{d_i+c_ir}{r+1}(r^{2k-2}-1)+d_i(m-r^{2k-2})}\\ &\le e^{\frac{d_{i+1}+c_{i+1}r}{r+1}(r^{2k-2}-1)+d_{i+1}(m-r^{2k-2})} = \rho_{i+1}(m)\end{aligned}$$

and if $m\in T_k$, then

$$\begin{aligned}\rho_i(m) &= e^{\frac{d_ir}{r+1}(r^{2k-2}-1)+d_i(r^{2k-1}-r^{2k-2})+c_i(m-r^{2k-1})}\\ &\le e^{\frac{d_{i+1}r}{r+1}(r^{2k-2}-1)+d_{i+1}(r^{2k-1}-r^{2k-2})+c_{i+1}(m-r^{2k-1})} = \rho_{i+1}(m).\end{aligned}$$

It also follows from Lemma 3.4 that

$$\alpha(A) = \frac{r-1}{r+1}\max_{1\le i\le q}(c_i - d_i)$$

and

$$\begin{aligned}\underline{\sigma}(A) &= \sum_{i=1}^{q} b_i - \liminf_{m\to\infty}\frac{1}{m}\log\det\mathcal{A}_m\\ &= \sum_{i=1}^{q}\frac{d_i+c_ir}{r+1} - \liminf_{m\to\infty}\frac{1}{m}\log\det\mathcal{A}_m.\end{aligned} \tag{3.4}$$

We have

$$\liminf_{m\to\infty}\frac{1}{m}\log\det\mathcal{A}_m = \liminf_{m\to\infty}\frac{1}{m}\log\prod_{j=1}^{m-1}\prod_{i=1}^{q}a_i(j),$$

where

$$\prod_{i=1}^{q} a_i(m) = \begin{cases} e^{d_1+\cdots+d_q} & \text{if } m \in S_k \text{ for } k \in \mathbb{N}, \\ e^{c_1+\cdots+c_q} & \text{if } m \in T_k \text{ for } k \in \mathbb{N}. \end{cases}$$

Applying Lemma 3.4 with $c = c_1 + \cdots + c_q$ and $d = d_1 + \cdots + d_q$ we obtain

$$\begin{aligned} \liminf_{m\to\infty} \frac{1}{m} \log \prod_{j=1}^{m-1} \prod_{i=1}^{q} a_i(j) &= \frac{c_1 + \cdots + c_q + (d_1 + \cdots + d_q)r}{r+1} \\ &= \sum_{i=1}^{q} \frac{c_i + d_i r}{r+1} \end{aligned}$$

and it follows from (3.4) that

$$\underline{\sigma}(A) = \sum_{i=1}^{q} \frac{d_i + c_i r}{r+1} - \sum_{i=1}^{q} \frac{c_i + d_i r}{r+1} = \frac{r-1}{r+1} \sum_{i=1}^{q} (c_i - d_i).$$

This completes the proof of the lemma. □

Step 3. Conclusion of the argument. We first construct auxiliary sequences.

Lemma 3.6. *Given $c_q > d_q > 0$, there exist an r-regular sequence $C = (C_m)_{m\in\mathbb{N}}$ with*

$$\underline{\sigma}(C) = \alpha(C) = \frac{(r-1)(c_q - d_q)}{r+1}$$

and an r-regular sequence $D = (D_m)_{m\in\mathbb{N}}$ with

$$\underline{\sigma}(D) = q\alpha(D) = q\frac{(r-1)(c_q - d_q)}{r+1}. \tag{3.5}$$

Proof of the lemma. Let

$$c_1 = \cdots = c_{q-1} = d_1 = \cdots = d_{q-1} = 0$$

and denote the corresponding matrices A_m (see (3.3)) by C_m. Then the sequence C is r-regular and by Lemma 3.5 we have

$$\underline{\sigma}(C) = \alpha(C) = \frac{(r-1)(c_q - d_q)}{r+1}.$$

Now let

$$c_1 = \cdots = c_q, \quad d_1 = \cdots = d_q$$

and denote the corresponding matrices A_m (see (3.3)) by D_m. Then the sequence D is r-regular and by Lemma 3.5 we have

$$\underline{\sigma}(D) = q\frac{(r-1)(c_q - d_q)}{r+1} \quad \text{and} \quad \alpha(D) = \frac{(r-1)(c_q - d_q)}{r+1}$$

which gives identity (3.5). □

We use the sequences C and D to show that for each $\mu \in [1, q]$ there exists an r-regular sequence $E = (E_m)_{m\in\mathbb{N}}$ with

$$\underline{\sigma}(E) = \mu\alpha(E).$$

First observe that replacing C by the sequence $C' = (C'_m)_{m\in\mathbb{N}}$ with $C'_m = C_m^\kappa$ for some $\kappa > 0$ corresponds to replace the numbers c_i and d_i, respectively, by κc_i and κd_i for each i. Therefore,

$$\alpha(C') = \kappa\alpha(C) \quad \text{and} \quad \underline{\sigma}(C') = \kappa\underline{\sigma}(C).$$

Moreover, for each $\nu \in [0, 1]$, the sequence of matrices $E = (E_m)_{m\in\mathbb{N}}$ with $E_m = C_m^\nu D_m^{1-\nu}$ for $m \in \mathbb{N}$ is r-regular, with

$$\alpha(E) = \nu\alpha(C) + (1-\nu)\alpha(D) \quad \text{and} \quad \underline{\sigma}(E) = \nu\underline{\sigma}(C) + (1-\nu)\underline{\sigma}(D). \tag{3.6}$$

Indeed, let $e^{c_i(m)}$ and $e^{d_i(m)}$ be, respectively, the entries on the diagonals of C_m and D_m. Then the entries on the diagonal of E_m are $e^{\nu c_i(m)+(1-\nu)d_i(m)}$ and one can easily verify that the two properties in the notion of r-regularity hold as well as (3.6). By Lemma 3.5 and (3.6) we obtain

$$\begin{aligned}\alpha(E) &= \nu\frac{(c_q - d_q)(r-1)}{r+1} + (1-\nu)\frac{(c_q - d_q)(r-1)}{r+1}\\ &= \frac{(c_q - d_q)(r-1)}{r+1}\end{aligned}$$

and

$$\begin{aligned}\underline{\sigma}(E) &= \nu\frac{(c_q - d_q)(r-1)}{r+1} + (1-\nu)\frac{q(c_q - d_q)(r-1)}{r+1}\\ &= \frac{(\nu + (1-\nu)q)(c_q - d_q)(r-1)}{r+1}.\end{aligned}$$

In particular,

$$\underline{\sigma}(E)/\alpha(E) = \nu + (1-\nu)q.$$

Note that when ν goes from 0 to 1, this expression goes from q to 1 and so it takes any value $\mu \in [1, q]$. Moreover, $\alpha(E)$ can take any prescribed positive value by choosing c_q and d_q. This completes the proof of the theorem. □

3.3 Realization problem II

In this section we construct specific sequences of matrices attaining each possible value of the regularity coefficients $\underline{\sigma}(A)$ and $\overline{\sigma}(A)$.

Theorem 3.7. *Given $s \geq 0$, there exists:*

1. *a bounded sequence of matrices A with $\underline{\sigma}(A) = s$;*
2. *a bounded sequence of matrices A with $\overline{\sigma}(A) = s$.*

Proof. Note that $\underline{\sigma}(A) = \overline{\sigma}(A) = 0$ for any regular sequence A. So it suffices to take $s > 0$.

We first show that given $s > 0$, there exists a bounded sequence of matrices A with $\underline{\sigma}(A) = s$. Consider the sequence of diagonal matrices

$$A_m = \text{diag}(a_1(m), \dots, a_q(m)),$$

where

$$a_i(m) = \begin{cases} e^{\beta_i} & \text{if } k! \le m < (k+1)! \text{ with } k \text{ odd}, \\ 1 & \text{otherwise} \end{cases}$$

for some nonnegative numbers

$$\beta_1 \le \beta_2 \le \cdots \le \beta_q$$

such that $\sum_{i=1}^q \beta_i = s$. Then $\rho_i(m) \le e^{\beta_i m}$ and so

$$b_i = \limsup_{m\to\infty} \frac{1}{m} \log \rho_i(m) \le \beta_i.$$

On the other hand, for k odd we have

$$\rho_i((k+1)!) \ge e^{\beta_i((k+1)!-k!)}$$

and so

$$\begin{aligned} b_i &\ge \limsup_{k\to\infty} \frac{1}{(k+1)!} \log \rho_i((k+1)!) \\ &\ge \limsup_{k\to\infty} \frac{1}{(k+1)!} \beta_i((k+1)! - k!) = \beta_i. \end{aligned}$$

This shows that $b_i = \beta_i$ for $i = 1, \dots, q$. Moreover, since $a_i(j) \ge 1$ we have

$$\liminf_{m\to\infty} \frac{1}{m} \log \det \mathcal{A}_m \ge 0. \tag{3.7}$$

On the other hand, denoting by $e_1, \dots, e_q$ the canonical basis for $\mathbb{R}^q$ and writing $r_n = (2n+1)!$, we obtain

$$\begin{aligned} \liminf_{m\to\infty} \frac{1}{m} \log \det \mathcal{A}_m &= \liminf_{m\to\infty} \frac{1}{m} \log \prod_{i=1}^q \|\mathcal{A}_m e_i\| \\ &\le \liminf_{n\to\infty} \frac{1}{r_n} \log \prod_{i=1}^q \|\mathcal{A}_{r_n} e_i\|. \end{aligned}$$

We have

$$\prod_{i=1}^q \|\mathcal{A}_{r_n} e_i\| \le \prod_{i=1}^q e^{\beta_i (2n)!} = e^{s(2n)!}$$

and so

$$\liminf_{m\to\infty} \frac{1}{m} \log \det \mathcal{A}_m \le \liminf_{n\to\infty} \frac{s(2n)!}{r_n} = 0.$$

Together with (3.7), this implies that

$$\liminf_{m\to\infty} \frac{1}{m} \log \det \mathcal{A}_m = 0$$

and so

$$\underline{\sigma}(A) = \sum_{i=1}^q \beta_i - \liminf_{m\to\infty} \frac{1}{m} \log \det \mathcal{A}_m = \sum_{i=1}^q \beta_i = s.$$

Now we show that given $s > 0$, there exists a bounded sequence of matrices A with $\overline{\sigma}(A) = s$. Consider the sequence of diagonal matrices

$$A_m = \text{diag}(a_1(m), \dots, a_q(m)),$$

where

$$a_i(m) = \begin{cases} e^s & \text{if } k! \le m < (k+1)! \text{ and } k \equiv q \bmod (q+1), \\ 1 & \text{otherwise.} \end{cases}$$

Since $a_i(m) \ge 1$, we have $\rho_i(m) \ge 1$ and so

$$a_i = \liminf_{m\to\infty} \frac{1}{m} \log \rho_i(m) \ge 0.$$

Moreover, letting $r_{n,j} = (q + n(q+1) + j)!$ we have

$$a_i \le \liminf_{n\to\infty} \frac{1}{r_{n,2}} \max_{1\le i\le q} \log \prod_{i=1}^{q} \|\mathcal{A}_{r_{n,2}} e_i\| \le \liminf_{n\to\infty} \frac{r_{n,1}}{r_{n,2}} = 0.$$

Therefore, $a_i = 0$ for $i = 1, \dots, q$. Moreover, $\det \mathcal{A}_m \le e^{ms}$ and so

$$\limsup_{m\to\infty} \frac{1}{m} \log \det \mathcal{A}_m \le s. \tag{3.8}$$

Finally, we have

$$\det \mathcal{A}_{(k+1)!} \ge \prod_{i=1}^{q} \prod_{j=k!}^{(k+1)!-1} a_i(j) = e^{s((k+1)!-k!)},$$

for $k \equiv q \bmod (q+1)$, which gives

$$\limsup_{m\to\infty} \frac{1}{m} \log \det \mathcal{A}_m \ge \limsup_{n\to\infty} \frac{s(r_{n,1} - r_{n,0})}{r_{n,1}} = s.$$

Together with (3.8), this implies that

$$\limsup_{m\to\infty} \frac{1}{m} \log \det \mathcal{A}_m = s$$

and so

$$\overline{\sigma}(A) = \limsup_{m\to\infty} \frac{1}{m} \log \det \mathcal{A}_m - \sum_{i=1}^{q} a_i = s.$$

This completes the proof of the theorem. □

4 Further regularity coefficients

In this section we introduce two additional regularity coefficients based on the matrices $\mathcal{A}_m$. We also establish inequalities between these coefficients and the former ones.

For each $k = 1, \dots, q$, let $(\mathbb{R}^q)^{\wedge k}$ be the set of alternating k-linear forms on $\mathbb{R}^q$. We define an inner product on $(\mathbb{R}^q)^{\wedge k}$ by requiring that

$$\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k \rangle = \det B,$$

where B is the $k \times k$ matrix with entries $b_{ij} = \langle v_i, w_j \rangle$ for $i, j = 1, \dots, k$ and where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^q$. In particular, for $k = 1$ we recover the standard inner product and so the 2-norm on $\mathbb{R}^q$.

Now let $A = (A_m)_{m\in\mathbb{N}}$ be a sequence of $q \times q$ matrices with real entries. For each $k = 1, \ldots, q$, we define

$$c_k(A) = \liminf_{m\to\infty} \frac{1}{m} \log\|(\mathcal{A}_m)^{\wedge k}\|$$

and

$$d_k(A) = \limsup_{m\to\infty} \frac{1}{m} \log\|(\mathcal{A}_m)^{\wedge k}\|,$$

where

$$(\mathcal{A}_m)^{\wedge k}(v_1 \wedge \cdots \wedge v_k) = \mathcal{A}_m v_1 \wedge \cdots \wedge \mathcal{A}_m v_k.$$

Finally, let

$$\varepsilon(A) = \max\{d_k(A) - c_k(A) : k = 1, \ldots, q\}.$$

Theorem 4.1. *The function $\varepsilon(A)$ is a regularity coefficient. Moreover, for each bounded sequence of matrices $A = (A_m)_{m\in\mathbb{N}}$ we have*

$$\frac{1}{2}\alpha(A) \le \varepsilon(A) \le q\alpha(A). \tag{4.1}$$

Proof. Note that it suffices to establish (4.1) since then $\varepsilon(A) \ge \alpha(A)/2 \ge 0$ and $\varepsilon(A) = 0$ if and only if $\alpha(A) = 0$, that is, if and only if A is regular.

Recall that for any $q \times q$ matrix B we have

$$\|B^{\wedge k}\| = \prod_{i=1}^{k} \rho_{q-i+1},$$

where $\rho_1 \le \cdots \le \rho_q$ are the (real nonnegative) eigenvalues of the matrix $(B^*B)^{1/2}$. Taking $B = \mathcal{A}_m$ we obtain

$$\|(\mathcal{A}_m)^{\wedge k}\| = \prod_{i=1}^{k} \rho_{q-i+1}(m). \tag{4.2}$$

Therefore,

$$\begin{aligned} c_k(A) &= \liminf_{m\to\infty} \frac{1}{m} \log \prod_{i=1}^{k} \rho_{q-i+1}(m) \\ &\ge \sum_{i=1}^{k} \liminf_{m\to\infty} \frac{1}{m} \log \rho_{q-i+1}(m) = \sum_{i=1}^{k} a_{q-i+1}(A) \end{aligned}$$

and

$$\begin{aligned} d_k(A) &= \limsup_{m\to\infty} \frac{1}{m} \log \prod_{i=1}^{k} \rho_{q-i+1}(m) \\ &\le \sum_{i=1}^{k} \limsup_{m\to\infty} \frac{1}{m} \log \rho_{q-i+1}(m) = \sum_{i=1}^{k} b_{q-i+1}(A). \end{aligned}$$

This readily implies that

$$d_k(A) - c_k(A) \le \sum_{i=1}^{k} (b_{q-i+1}(A) - a_{q-i+1}(A)) \le k\alpha(A)$$

and so $\varepsilon(A) \leq q\alpha(A)$. On the other hand, by (4.2) we have

$$\rho_i(m) = \frac{\|(\mathcal{A}_m)^{\wedge(q-i+1)}\|}{\|(\mathcal{A}_m)^{\wedge(q-i)}\|}$$

for $i = 1, \ldots, q-1$ and

$$\rho_q(m) = \|(\mathcal{A}_m)^{\wedge 1}\| = \|\mathcal{A}_m\|.$$

Therefore,

$$\begin{aligned} a_i(A) &= \liminf_{m\to\infty} \frac{1}{m} \log \rho_i(m) \\ &\geq \liminf_{m\to\infty} \frac{1}{m} \log\|(\mathcal{A}_m)^{\wedge(q-i+1)}\| + \liminf_{m\to\infty} -\frac{1}{m} \log\|(\mathcal{A}_m)^{\wedge(q-i)}\| \\ &= c_{q-i+1}(A) - d_{q-i}(A) \end{aligned}$$

and

$$\begin{aligned} b_i(A) &= \limsup_{m\to\infty} \frac{1}{m} \log \rho_i(m) \\ &\leq \limsup_{m\to\infty} \frac{1}{m} \log\|(\mathcal{A}_m)^{\wedge(q-i+1)}\| + \limsup_{m\to\infty} -\frac{1}{m} \log\|(\mathcal{A}_m)^{\wedge(q-i)}\| \\ &= d_{q-i+1}(A) - c_{q-i}(A) \end{aligned}$$

for $i = 1, \ldots, q-1$. These inequalities also hold for $i = q$, with the convention that $c_0(A) = d_0(A) = 0$. This implies that

$$b_i(A) - a_i(A) \leq d_{q-i+1}(A) - c_{q-i+1}(A) + d_{q-i}(A) - c_{q-i}(A)$$

and so

$$\begin{aligned} \alpha(A) &\leq \max\{d_{q-i+1}(A) - c_{q-i+1}(A) : i = 1, \ldots, q\} + \max\{d_{q-i}(A) - c_{q-i}(A) : i = 1, \ldots, q\} \\ &\leq 2\varepsilon(A). \end{aligned}$$

This completes the proof of the theorem. □

We also introduce a second regularity coefficient. First recall that a bounded sequence of matrices $A = (A_m)_{m\in\mathbb{N}}$ is regular if and only if the sequence of matrices $(\mathcal{A}_m^*\mathcal{A}_m)^{1/(2m)}$ converges entry by entry when $m \to \infty$ (see [4]). Therefore, the function

$$\mu(A) = \sum_{i=1}^{q}\sum_{j=1}^{q}\left(\limsup_{m\to\infty}(\mathcal{A}_m^*\mathcal{A}_m)_{ij}^{1/(2m)} - \liminf_{m\to\infty}(\mathcal{A}_m^*\mathcal{A}_m)_{ij}^{1/(2m)}\right),$$

where B_{ij} denotes the ij entry of a matrix B, is a regularity coefficient. Moreover, we have the following result.

Theorem 4.2. *There exists a constant $C_q > 0$ depending only on q such that for each bounded sequence $A = (A_m)_{m\in\mathbb{N}}$ of invertible $q \times q$ matrices such that $A^{-1} = (A_m^{-1})_{m\in\mathbb{N}}$ is also bounded we have*

$$C_q^{-1}\|A^{-1}\|_\infty^{-1}\alpha(A) \leq \mu(A) \leq C_q\|A\|_\infty\alpha(A).$$

Proof. Since the matrices $(\mathcal{A}_m^*\mathcal{A}_m)^{1/2}$ are symmetric and positive definite, there exist orthogonal matrices S_m such that

$$S_m^{-1}(\mathcal{A}_m^*\mathcal{A}_m)^{1/(2m)}S_m = \operatorname{diag}(\rho_1(m)^{1/m}, \ldots, \rho_q(m)^{1/m}). \tag{4.3}$$

Moreover, it is shown in [4] that one can always choose the matrices S_m so that they converge entry by entry to some orthogonal matrix S when $m \to \infty$. Hence, it follows from (4.3) that there exists a constant $C_q > 0$ depending only on q such that

$$\begin{aligned}\max_{1\le i\le q} &\left(\limsup_{m\to\infty} \rho_i(m)^{1/m} - \liminf_{m\to\infty} \rho_i(m)^{1/m}\right)\\ &\le C_q \sum_{i=1}^q \sum_{j=1}^q \left(\limsup_{m\to\infty} (\mathcal{A}_m^*\mathcal{A}_m)_{ij}^{1/(2m)} - \liminf_{m\to\infty} (\mathcal{A}_m^*\mathcal{A}_m)_{ij}^{1/(2m)}\right).\end{aligned} \tag{4.4}$$

Again by (4.3) we have

$$(\mathcal{A}_m^*\mathcal{A}_m)^{1/(2m)} = S_m \operatorname{diag}(\rho_1(m)^{1/m}, \ldots, \rho_q(m)^{1/m})S_m^{-1}$$

and so we also obtain

$$\begin{aligned}\sum_{i=1}^q \sum_{j=1}^q &\left(\limsup_{m\to\infty} (\mathcal{A}_m^*\mathcal{A}_m)_{ij}^{1/(2m)} - \liminf_{m\to\infty} (\mathcal{A}_m^*\mathcal{A}_m)_{ij}^{1/(2m)}\right)\\ &\le C_q \max_{1\le i\le q} \left(\limsup_{m\to\infty} \rho_i(m)^{1/m} - \liminf_{m\to\infty} \rho_i(m)^{1/m}\right),\end{aligned} \tag{4.5}$$

taking the same constant C_q without loss of generality.

Now observe that by (4.2) with $k = 1$, we have

$$\|\mathcal{A}_m^{-1}\|^{-1} \le \rho_i(m) \le \|\mathcal{A}_m\|$$

for each $i = 1, \ldots, q$. Therefore,

$$\|A^{-1}\|_\infty^{-1} \le \rho_i(m)^{1/m} \le \|A\|_\infty$$

and it follows from the mean value theorem that

$$\begin{aligned}b_i(A) - a_i(A) &= \log \limsup_{m\to\infty} \rho_i(m)^{1/m} - \log \liminf_{m\to\infty} \rho_i(m)^{1/m}\\ &\le \|A^{-1}\|_\infty \left(\limsup_{m\to\infty} \rho_i(m)^{1/m} - \liminf_{m\to\infty} \rho_i(m)^{1/m}\right).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\limsup_{m\to\infty} \rho_i(m)^{1/m} - \liminf_{m\to\infty} \rho_i(m)^{1/m} &= \exp \limsup_{m\to\infty} \frac{1}{m} \log \rho_i(m) - \exp \liminf_{m\to\infty} \frac{1}{m} \log \rho_i(m)\\ &\le \|A\|_\infty (b_i(A) - a_i(A)).\end{aligned}$$

Together with (4.4) and (4.5) this yields the desired result. □

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On global attractivity of a higher order difference equation and its applications

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Abstract. Consider the following higher order difference equation

$$x(n+1) = ax(n) + bf(x(n)) + cf(x(n-k)), \qquad n = 0, 1, \dots$$

where a, b and c are constants with $0 < a < 1, 0 \leq b < 1, 0 \leq c < 1$ and $a + b + c = 1$, $f \in C[[0, \infty), [0, \infty)]$ with $f(x) > 0$ for $x > 0$, and k is a positive integer. Our aim in this paper is to study the global attractivity of positive solutions of this equation and its applications to some population models.

Keywords: higher order difference equation, positive equilibrium, global attractivity, population models.

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1 Introduction

Consider the following higher order difference equation

$$x(n+1) = ax(n) + bf(x(n)) + cf(x(n-k)), \qquad n = 0, 1, \dots, \tag{1.1}$$

where a, b and c are constants with $0 < a < 1, 0 \leq b < 1, 0 \leq c < 1$ and $a + b + c = 1$, $f \in C[[0, \infty), [0, \infty)]$ with $f(x) > 0$ for $x > 0$ and k is a positive integer. Our aim in this paper is to study the global attractivity of positive solutions of Eq. (1.1) and its applications to some population models.

In a recent paper [1], the authors study the dynamics of the following difference system

$$\begin{cases} x(n+1) = (1-\epsilon)f(x(n)) + \epsilon y(n), \\ y(n+1) = (1-\epsilon)y(n) + \epsilon f(x(n)), \\ x(0) \geq 0,\ y(0) \geq 0,\ x(0) + y(0) > 0, \end{cases} \qquad n = 0, 1, \dots, \tag{1.2}$$

where $0 < \epsilon < 1$ is a positive constant. Sys. (1.2) is a population model proposed by Newman et al. [14] which assumes symmetric dispersal between active population $x(n)$ and refuge

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population $y(n)$. The function f describes density-dependent reproduction of the active population. Newman et al. explore the effects of coupling an otherwise chaotic population to a refuge. Specifically, the logistic function $f(x) = \lambda x(1-x)$ and the exponential function $f(x) = \lambda x \exp(-x)$ are studied in [14]. Using numerical simulations, it is concluded that the presence of a passive refuge can greatly stabilize a population that would otherwise exhibit chaotic dynamics [14]. While in [1], Chow et al. assume that the growth function f is monotonically increasing to rule out the chaotic behavior explored in [14]. In particular, the authors study the following two cases: the growth rate is of Beverton–Holt type, that is, $f(x) = \frac{\lambda x}{1+kx}$, and the population is also subject to Allee effects, that is, $f(x) = \frac{\lambda x^2}{(1+kx)(m+x)}$, where λ is the density-independent growth rate, k relates to the population's carrying capacity, and m is the reciprocal of the searching efficiency of an individual when the population is subject to Allee effects. Various properties of solutions of Sys. (1.2) are studied in [1]. Some other results on Beverton–Holt and related equations can be found, e.g., in [15] and [16].

Motivated by these studies, in the present paper we are interested in obtaining an explicit sufficient condition to guarantee the global stability of positive solutions of Sys. (1.2) no matter if f is monotonic or not.

Noting that (1.2) can be converted into a second order scalar difference equation

$$x(n+1) = (1-\epsilon)x(n) + (1-\epsilon)f(x(n)) + (2\epsilon - 1)f(x(n-1)), \tag{1.3}$$

we are led to the more general equation (1.1). When $b = 0$, Eq. (1.1) reduces to the form

$$x(n+1) = ax(n) + cf(x(n-k)), \tag{1.4}$$

which includes several discrete models derived from mathematical biology. The global attractivity and global stability of positive solutions of Eq. (1.4) and some related forms has been studied by many authors; see, for example, [2–13, 17] and the references cited therein.

In the present paper, we are interested in positive solutions of Eq. (1.1). Clearly, if we let

$$x(-k),\ x(-k+1), \dots, x(0) \tag{1.5}$$

be $k+1$ given nonnegative constants with $x(0) > 0$, then Eq.(1.1) has a unique positive solution $\{x(n)\}$ with the initial values (1.5). In the following, we assume that f has a unique positive fixed point $\bar{x}$. It is not difficult to see that $\bar{x}$ is the unique positive equilibrium of (1.1). In the next section, we establish a sufficient condition for $\bar{x}$ to be a global attractor of all positive solutions of Eq. (1.1). Then, in Section 3, we show that our result may be applied to Sys. (1.2) and some other difference equations derived from mathematical biology.

In the following discussion, for the sake of convenience, we adopt the notation $\prod_{i=m}^{n} s(i) = 1$ and $\sum_{i=m}^{n} s(i) = 0$ whenever $\{s(n)\}$ is a real sequence and $m > n$.

2 Main result

In this section we establish a sufficient condition for the global attractivity of positive solutions of Eq. (1.1). The following lemma is needed.

Lemma 2.1. *Assume that f satisfies the negative feedback condition*

$$(x - \bar{x})(f(x) - x) < 0, \qquad x > 0,\ x \neq \bar{x}. \tag{2.1}$$

Then every positive solution $\{x(n)\}$ of Eq. (1.1) is bounded and persistent.

Proof. First we show that $\{x(n)\}$ is bounded. Otherwise, there is a subsequence $\{x(n_i)\}$ of $\{x(n)\}$ such that

$$x(n_i) = \max\{x(n) : -k \le n \le n_i\}, \qquad i = 1, 2, \dots, \quad \text{and} \quad \lim_{i\to\infty} x(n_i) = \infty. \tag{2.2}$$

Then it follows from Eq. (1.1) that

$$b(f(x(n_i - 1)) - x(n_i)) + c(f(x(n_i - 1 - k)) - x(n_i)) = a(x(n_i) - x(n_i - 1)) \ge 0,$$

which, together with (2.2), implies there is a subsequence $\{x(n_{i_j})\}$ of $\{x(n_i)\}$ such that either

$$f(x(n_{i_j} - 1)) \ge x(n_{i_j}) \ge x(n_{i_j} - 1), \qquad j = 1, 2, \dots \tag{2.3}$$

or

$$f(x(n_{i_j} - 1 - k)) \ge x(n_{i_j}) \ge x(n_{i_j} - 1 - k), \qquad j = 1, 2, \dots \tag{2.4}$$

Assume that (2.3) holds. Then by noting (2.1) we see that $x(n_{i_j} - 1) \le \bar{x}$. Since $f(x)$ is bounded on the closed and finite interval $[0, \bar{x}]$, $\{f(x(n_{i_j} - 1))\}$ is bounded. Clearly, this contradicts (2.2) and (2.3). Hence, (2.3) can not hold. Similarly, we can show that (2.4) can not hold either. Hence, $\{x(n)\}$ must be bounded.

Next we show that $\{x(n)\}$ is persistent. Otherwise, there is a subsequence $\{x(n_i)\}$ of $\{x(n)\}$ such that

$$x(n_i) = \min\{x(n) : -k \le n \le n_i\}, \qquad i = 1, 2, \dots, \quad \text{and} \quad \lim_{i\to\infty} x(n_i) = 0. \tag{2.5}$$

Then it follows from Eq. (1.1) that

$$b(f(x(n_i - 1)) - x(n_i)) + c(f(x(n_i - 1 - k)) - x(n_i)) = a(x(n_i) - x(n_i - 1)) \le 0,$$

which, together with (2.5), implies there is a subsequence $\{x(n_{i_j})\}$ of $\{x(n_i)\}$ such that either

$$f(x(n_{i_j} - 1)) \le x(n_{i_j}) \le x(n_{i_j} - 1), \qquad j = 1, 2, \dots \tag{2.6}$$

or

$$f(x(n_{i_j} - 1) - k)) \le x(n_{i_j}) \le x(n_{i_j} - 1 - k), \qquad j = 1, 2, \dots \tag{2.7}$$

Assume that (2.6) holds. Then by noting (2.1) we see that $x(n_{i_j} - 1) \ge \bar{x}$. Since we have shown that $\{x(n)\}$ is bounded, $\{x(n_{i_j} - 1)\}$ is bounded. Hence there is a subsequence $\{x(n_{i_{jm}} - 1)\}$ of $\{x(n_{i_j} - 1)\}$ and a constant δ such that

$$\lim_{m\to\infty} x(n_{i_{jm}} - 1) = \delta,$$

where $\delta \ge \bar{x}$. However, from (2.6) we see that

$$f(x(n_{i_{jm}} - 1)) \le x(n_{i_{jm}}),$$

which implies that $f(\delta) \le 0$. This contradicts the assumption $f(x) > 0$ for $x > 0$. Hence, (2.6) can not hold. Similarly, we can show that (2.7) can not hold either. Hence, there is no such case (2.5) and so $\{x(n)\}$ is persistent. The proof is complete. □

The following theorem is our main result which establishes a sufficient condition for the global attractivity of positive solutions of Eq. (1.1).

Theorem 2.2. *Assume that $ax + bf(x)$ is increasing, $f(x)$ satisfies the negative feedback condition (2.1), and $f(x)$ is L-Lipschitz with*

$$\frac{c(1-a^{k+1})}{c+a^k b}L < 1. \tag{2.8}$$

Then every positive solution $\{x(n)\}$ of Eq. (1.1) converges to $\bar{x}$ as $n \to \infty$.

Proof. First assume that $\{x(n)\}$ does not oscillate about $\bar{x}$. Then there are two cases : $x(n) - \bar{x}$ is eventually positive or $x(n) - \bar{x}$ is eventually negative. For the case that $x(n) - \bar{x}$ is eventually positive, let

$$\limsup_{n\to\infty} x(n) = l.$$

From Lemma 2.1, we know that $\{x(n)\}$ is bounded. Hence, $\bar{x} \le l < \infty$. We now show that $l = \bar{x}$. First assume that $\{x(n)\}$ is eventually decreasing. Then $\lim_{n\to\infty} x(n) = l$. If $l > \bar{x}$, it follows from Eq. (1.1) that

$$\begin{aligned} a(x(n) - x(n-1)) &= (a-1)x(n) + bf(x(n-1)) + cf(x(n-1-k)) \\ &\to (a-1)l + bf(l) + cf(l) \quad \text{as } n \to \infty. \end{aligned} \tag{2.9}$$

Noting that $a + b + c = 1$, and $f(l) < l$ since $l > \bar{x}$, we see that

$$(a-1)l + bf(l) + cf(l) = (a-1)(l - f(l)) < 0.$$

Hence, there is a positive integer N such that

$$a(x(n) - x(n-1)) \le \frac{1}{2}(a-1)(l - f(l)), \qquad n \ge N. \tag{2.10}$$

Summing (2.10) from $N + 1$ to n, we find that

$$a(x(n) - x(N)) \le \frac{1}{2}(a-1)(l - f(l))(n - N) \to -\infty \quad \text{as } n \to \infty,$$

which is a contradiction. Hence, we must have $l = \bar{x}$.

Next, consider the case that $\{x(n)\}$ is not eventually decreasing. Then, there is a subsequence $\{x(n_i)\}$ of $\{x(n)\}$ such that

$$\lim_{i\to\infty} x(n_i) = l \quad \text{and} \quad x(n_i) > x(n_i - 1), \qquad i = 1, 2, \ldots \tag{2.11}$$

Hence it follows from Eq. (1.1) that

$$b(f(x(n_i - 1)) - x(n_i)) + c(f(x(n_i - 1 - k)) - x(n_i)) = a(x(n_i) - x(n_i - 1)) > 0, \quad i = 1, 2, \ldots$$

and so there is a subsequence $\{x(n_{i_j})\}$ of $\{x(n_i)\}$ such that either

$$f(x(n_{i_j} - 1)) > x(n_{i_j}), \qquad j = 1, 2, \ldots \tag{2.12}$$

or

$$f(x(n_{i_j} - 1 - k)) > x(n_{i_j}), \qquad j = 1, 2, \ldots \tag{2.13}$$

If (2.12) holds, then by noting $x(n_{i_j}-1) \geq \bar{x}$, we have $f(x(n_{i_j}-1)) \leq x(n_{i_j}-1)$ and so it follows that $x(n_{i_j}-1) > x(n_{i_j})$, which contradicts (2.11). Hence, (2.12) can not hold and we must have (2.13). Then, by noting $x(n_{i_j}-1-k) \geq \bar{x}$, we have $f(x(n_{i_j}-1-k)) \leq x(n_{i_j}-1-k)$ which yields $x(n_{i_j}-1-k) \geq x(n_{i_j})$. Hence

$$\lim_{j\to\infty} x(n_{i_j}-1-k) = l.$$

Then, by taking limit on both sides of (2.13), we see that $f(l) \geq l$ which yields $l \leq \bar{x}$. Hence, $l = \bar{x}$.

In the above, we have shown that $\{x(n)\}$ converges to $\bar{x}$ when $x(n) - \bar{x}$ is eventually positive. Next we show that $\{x(n)\}$ converges to $\bar{x}$ also when $x(n) - \bar{x}$ is eventually negative. To this end, let

$$\liminf_{n\to\infty} x(n) = r.$$

From Lemma 2.1, we know that $\{x(n)\}$ is persistent. Hence, $0 < r \leq \bar{x}$. We will show that $r = \bar{x}$. First assume that $\{x(n)\}$ is eventually increasing. Then $\lim_{n\to\infty} x(n) = r$. If $r < \bar{x}$, then it follows from Eq. (1.1) that

$$\begin{aligned} a(x(n) - x(n-1)) &= (a-1)x(n) + bf(x(n-1)) + cf(x(n-1-k)) \\ &\to (a-1)r + bf(r) + cf(r) \text{ as } n \to \infty. \end{aligned} \tag{2.14}$$

Noting that $a + b + c = 1$, and $f(r) > r$ since $r < \bar{x}$, we see that

$$(a-1)r + bf(r) + cf(r) = (a-1)(r - f(r)) > 0.$$

Hence, there is a positive integer N such that

$$a(x(n) - x(n-1)) \geq \frac{1}{2}(a-1)(r - f(r)), \qquad n \geq N. \tag{2.15}$$

Summing (2.15) from $N+1$ to n, we find that

$$a(x(n) - x(N)) \geq \frac{1}{2}(a-1)(r-f(r))(n-N) \to \infty \quad \text{as } n \to \infty,$$

which is a contradiction. Hence, we must have $r = \bar{x}$.

Next, consider the case that $\{x(n)\}$ is not eventually increasing. Then, there is a subsequence $\{x(n_i)\}$ of $\{x(n)\}$ such that

$$\lim_{i\to\infty} x(n_i) = r \text{ and } x(n_i) < x(n_i - 1), \qquad i = 1, 2, \ldots \tag{2.16}$$

Hence, it follows from Eq. (1.1) that

$$b(f(x(n_i-1)) - x(n_i)) + c(f(x(n_i-k)) - x(n_i)) = a(x(n_i) - x(n_i-1)) < 0, \qquad i = 1, 2, \ldots$$

and so there is a subsequence $\{x(n_{i_j})\}$ of $\{x(n_i)\}$ such that either

$$f(x(n_{i_j}-1)) < x(n_{i_j}), \qquad j = 1, 2, \ldots \tag{2.17}$$

or

$$f(x(n_{i_j}-1-k)) < x(n_{i_j}), \qquad j = 1, 2, \ldots \tag{2.18}$$

If (2.17) holds, then by noting $x(n_{i_j}-1) < \bar{x}$, we have $f(x(n_{i_j}-1)) \geq x(n_{i_j}-1)$ and so it follows that $x(n_{i_j}-1) < x(n_{i_j})$, which contradicts (2.16). Hence, (2.17) can not hold and we must have (2.18). Then, noting that $x(n_{i_j}-1-k) < \bar{x}$, we have $f(x(n_{i_j}-1-k)) \geq x(n_{i_j}-1-k)$ which yields $x(n_{i_j}-1-k) \leq x(n_{i_j})$. Hence

$$\lim_{j\to\infty} x(n_{i_j}-1-k) = r.$$

Then by taking limit on both sides of (2.18), we see that $f(r) \leq r$ which yields $r \geq \bar{x}$. Hence, $r = \bar{x}$.

In the above, we have shown that every nonoscillatory (about $\bar{x}$) positive solution of Eq. (1.1) converges to $\bar{x}$ as $n \to \infty$. Next, consider the case that $\{x(n)\}$ is a positive solution of Eq. (1.1) and oscillates about $\bar{x}$, that is, $x(n) - \bar{x}$ is not of eventually constant sign. We show that $x(n)$ converges to $\bar{x}$ also as $x \to \infty$. To this end, let $y(n) = x(n) - \bar{x}$. Then $\{y(n)\}$ satisfies the equation

$$y(n+1) = ay(n) + b(f(y(n)+\bar{x}) - \bar{x}) + c(f(y(n-k)+\bar{x}) - \bar{x}) \tag{2.19}$$

and $\{y(n)\}$ oscillates about zero. Let $y(i) < y(j)$ be two consecutive members of the solution $\{y(n)\}$ such that

$$y(i) \leq 0, \quad y(j+1) \leq 0 \quad \text{and} \quad y(n) > 0 \quad \text{for } i+1 \leq n \leq j. \tag{2.20}$$

Let

$$y(t) = \max\{y(i+1), y(i+2), \ldots, y(j)\}. \tag{2.21}$$

We claim that

$$t - (i+1) \leq k. \tag{2.22}$$

Otherwise $t - (i+1) > k$. Then

$$y(t) \geq y(t-1) > 0 \quad \text{and} \quad y(t) \geq y(t-1-k) > 0.$$

By noting $y(t-1) + \bar{x} > \bar{x}$, $y(t-1-k) + \bar{x} > \bar{x}$ and $f(x) < x$ for $x > \bar{x}$, we see that

$$\begin{aligned} & b(f(y(t-1)+\bar{x}) - y(t) - \bar{x}) + c(f(y(t-1-k)+\bar{x}) - y(t) - \bar{x}) \\ & \quad < b((y(t-1)+\bar{x}) - y(t) - \bar{x}) + c((y(t-1-k)+\bar{x}) - y(t) - \bar{x}) \\ & \quad = b(y(t-1) - y(t)) + c(y(t-1-k) - y(t)) \\ & \quad \leq 0. \end{aligned} \tag{2.23}$$

However, on the other hand, (2.19) yields

$$b(f(y(t-1)+\bar{x}) - y(t) - \bar{x}) + c(f(y(t-1-k)+\bar{x}) - y(t) - \bar{x}) = a(y(t) - y(t-1)) > 0,$$

which contradicts (2.23). Hence, (2.22) must hold.

Now, observe that (2.19) yields

$$\frac{y(n+1)}{a^{n+1}} - \frac{y(n)}{a^n} = \frac{b}{a^{n+1}}[f(y(n)+\bar{x}) - \bar{x}] + \frac{c}{a^{n+1}}[f(y(n)+\bar{x}) - \bar{x}] \tag{2.24}$$

Summing (2.24) from i to $t-1$, we see that

$$\frac{y(t)}{a^t} - \frac{y(i)}{a^i} = \sum_{n=i}^{t-1} \frac{b}{a^{n+1}}[f(y(n)+\bar{x})-\bar{x}] + \sum_{n=i}^{t-1} \frac{c}{a^{n+1}}[f(y(n-k)+\bar{x})-\bar{x}]$$

and so it follows that

$$\begin{aligned} y(t) &= a^t \left(\frac{y(i)}{a^i} + \sum_{n=i}^{t-1} \frac{b}{a^{n+1}}[f(y(n)+\bar{x})-\bar{x})] + \sum_{n=i}^{t-1} \frac{c}{a^{n+1}}[f(y(n-k)+\bar{x})-\bar{x})] \right) \\ &= a^t \left(\frac{1}{a^{i+1}}(ay(i)+bf(y(i)+\bar{x})-b\bar{x}) + \sum_{n=i+1}^{t-1} \frac{b}{a^{n+1}}[f(y(n)+\bar{x})-\bar{x}] \right. \\ &\quad \left. + \sum_{n=i}^{t-1} \frac{c}{a^{n+1}}[f(y(n-k)+\bar{x})-\bar{x}] \right). \end{aligned} \tag{2.25}$$

Noting that $ax+bf(x)$ is increasing, $y(i) \le 0$, and $f(\bar{x}) = \bar{x}$, we see that

$$a(y(i)+\bar{x}) + bf(y(i)+\bar{x}) \le a\bar{x} + bf(\bar{x}),$$

which yields

$$ay(i) + bf(y(i)+\bar{x}) - b\bar{x} \le 0.$$

In addition, by noting (2.21) and the fact that $0 < y(n) \le y(t)$ for $n = i+1, i+2, \dots, t-1$, we see that

$$f(y(n)+\bar{x}) < y(n)+\bar{x} \le y(t)+\bar{x}, \qquad n = i+1, i+2, \dots, t-1.$$

Hence, it follows from (2.25) that

$$\begin{aligned} y(t) &\le a^t \left(\sum_{n=i+1}^{t-1} \frac{b}{a^{n+1}}[y(t)+\bar{x}-\bar{x}] + \sum_{n=i}^{t-1} \frac{c}{a^{n+1}}[f(y(n-k)+\bar{x})-\bar{x}] \right) \\ &= a^t \left(y(t) \sum_{n=i+1}^{t-1} \frac{b}{a^{n+1}} + \sum_{n=i}^{t-1} \frac{c}{a^{n+1}}[f(y(n-k)+\bar{x})-f(\bar{x})] \right) \\ &= \frac{(1-a^{t-i-1})b}{1-a} y(t) + a^t \sum_{n=i}^{t-1} \frac{c}{a^{n+1}}[f(y(n-k)+\bar{x})-f(\bar{x})] \\ &\le \frac{(1-a^k)b}{1-a} y(t) + a^t \sum_{n=i}^{t-1} \frac{c}{a^{n+1}}[f(y(n-k)+\bar{x})-f(\bar{x})], \end{aligned}$$

which yields

$$\left(1 - \frac{(1-a^k)b}{1-a}\right) y(t) \le a^t \sum_{n=i}^{t-1} \frac{c}{a^{n+1}}[f(y(n-k)+\bar{x})-f(\bar{x})]. \tag{2.26}$$

Then, by noting

$$1 - \frac{(1-a^k)b}{1-a} = \frac{c+a^k b}{1-a},$$

it follows from (2.26) that

$$y(t) \le \frac{1-a}{c+a^k b} a^t \sum_{n=i}^{t-1} \frac{c}{a^{n+1}}[f(y(n-k)+\bar{x})-f(\bar{x})]. \tag{2.27}$$

Since $\{x(n)\}$ is bounded, there is a positive constant M such that $|y(n)| = |x(n) - \bar{x}| \le M, n = 0, 1, \ldots$ Then by noting the Lipschitz property of f, we see that

$$|f(y(n-k)+\bar{x}) - f(\bar{x})| \le L|y(n-k)| \le LM, \qquad n \ge k.$$

Hence, (2.27) yields

$$y(t) \le \frac{1-a}{c+a^k b} a^t \sum_{n=i}^{t-1} \frac{c}{a^{n+1}} LM = \frac{1-a}{c+a^k b}\left(\frac{1-a^{t-i}}{1-a}\right) cLM \le \frac{1-a}{c+a^k b}\left(\frac{1-a^{k+1}}{1-a}\right) cLM,$$

that is,

$$y(t) \le \frac{c(1-a^{k+1})}{c+a^k b} LM.$$

It follows that

$$y(n) \le \frac{c(1-a^{k+1})}{c+a^k b} LM \quad \text{for } i \le n \le j.$$

Since $y(i)$ and $y(j)$ are two arbitrary members of the solution with property (2.21), we see that there is a positive integer N_1' such that

$$y(n) \le \frac{c(1-a^{k+1})}{c+a^k b} LM, \qquad n \ge N_1'.$$

Then, by a similar argument, it can be shown that there is a positive integer N_1'' such that

$$y(n) \ge -\frac{c(1-a^{k+1})}{c+a^k b} LM, \qquad n \ge N_1''.$$

Hence, there is a positive integer N_1 such that

$$|y(n)| \le \frac{c(1-a^{k+1})}{c+a^k b} LM, \qquad n \ge N_1. \tag{2.28}$$

Now, by noting (2.28) and the Lipschitz property of $f(x)$ again, we see that

$$|f(y(n-k)+\bar{x}) - f(\bar{x})| \le L|y(n-k)| \le \frac{c(1-a^{k+1})}{c+a^k b} L^2 M, \qquad n \ge N_1 + k.$$

Let $y(i)$ and $y(j)$ be two consecutive members of the solution $\{y(n)\}$ with $N_1 + k \le i < j$ such that (2.20) holds. Let $y(t)$ be defined by (2.21). By a similar argument, we may show that (2.22) holds and

$$y(t) \le \left(\frac{c(1-a^{k+1})}{c+a^k b} L\right)^2 M.$$

Then it follows that

$$y(t) \le \left(\frac{c(1-a^{k+1})}{c+a^k b} L\right)^2 M, \qquad i \le n \le j$$

and so again by noting $y(i)$ and $y(j)$ are two arbitrary members of the solution with property (2.20), there is a positive integer $N_2' \ge N_1 + k$ such tat

$$y(n) \le \left(\frac{c(1-a^{k+1})}{c+a^k b} L\right)^2 M, \qquad n \ge N_2'.$$

Similarly, it can be shown that there is a positive integer $N_2'' \geq N_1 + K$ such that

$$y(n) \geq -\left(\frac{c(1-a^{k+1})}{c+a^k b}L\right)^2 M, \qquad n \geq N_2''.$$

Hence, there is a positive integer $N_2 \geq N_1 + k$ such that

$$|y(n)| \leq \left(\frac{c(1-a^{k+1})}{c+a^k b}L\right)^2 M, \qquad n \geq N_2.$$

Finally, by induction, we find that for any positive integer m, there is a positive integer N_m with $N_m \to \infty$ as $m \to \infty$ such that

$$|y_n| \leq \left(\frac{c(1-a^{k+1})}{c+a^k b}L\right)^m M, \qquad n \geq N_m.$$

Then, by noting the hypotheses $\frac{c(1-a^{k+1})}{c+a^k b}L < 1$, we see that $y(n) \to 0$ as $n \to \infty$, and so it follows that $x(n) \to \bar{x}$ as $n \to \infty$. The proof is complete. □

3 Applications

In this section, we apply our result obtained in the last section to some difference equations derived from mathematical biology.

Consider the system (1.2) which has been mentioned in Section 1. By a simple calculation, Sys. (1.2) can be converted into the second order difference equation (1.3). Let us assume that $f \in C[[0,\infty),[0,\infty)]$ with $f(x) > 0$ for $x > 0$, and f has a unique positive fixed point $\bar{x}$. It is not difficult to see that $(\bar{x}, \bar{x})$ is the unique positive equilibrium of Sys. (1.2). By Theorem 2.2, we may have the following result.

Theorem 3.1. *Assume that $1/2 \leq \epsilon < 1$. Suppose also that $x + f(x)$ is increasing, f satisfies the negative feedback condition*

$$(x-\bar{x})(f(x)-x) < 0, \qquad x > 0,\ x \neq \bar{x} \tag{3.1}$$

and f is L-Lipschitz with

$$(2-\epsilon)(2-1/\epsilon)L < 1. \tag{3.2}$$

Then every positive solution $(x(n), y(n))$ of Sys. (1.2) tends to its positive equilibrium $(\bar{x}, \bar{x})$ as $n \to \infty$. Furthermore, if f is differentiable and

$$(1-\epsilon)|1+f'(\bar{x})| < 1 + (1-2\epsilon)f'(\bar{x}) < 2, \tag{3.3}$$

then $(\bar{x}, \bar{x})$ is globally asymptotically stable.

Proof. Eq. (1.3) is in the form of (1.1) with $a = b = 1-\epsilon, c = 2\epsilon - 1$ and $k = 1$. By the hypotheses, $ax + bf(x) = (1-\epsilon)(x + f(x))$ is increasing. In addition,

$$\frac{c(1-a^{k+1})}{c+a^k b}L = (2-\epsilon)(2-1/\epsilon)L < 1.$$

Hence, by Theorem 2.2 every positive solution $\{x(n)\}$ of Eq. (1.3) converges to $\bar{x}$ as $n \to \infty$. Then from (1.2) we see that

$$\epsilon y(n) = x(n+1) - (1-\epsilon)f(x(n)) \to \bar{x} - (1-\epsilon)f(\bar{x}) \quad \text{as } n \to \infty,$$

which yields

$$y(n) \to \bar{x} \quad \text{as } n \to \infty.$$

Hence it follows that every positive solution $(x(n), y(n))$ of Sys. (1.2) converges to $(\bar{x}, \bar{x})$ as $n \to \infty$.

Clearly, to show that $(\bar{x}, \bar{x})$ is globally asymptotically stable when (3.3) holds, it suffices to show that $(\bar{x}, \bar{x})$ is stable. Note that Sys. (1.2) can be converted into the scalar equation (1.3) and the linearized equation of Eq. (1.3) about $\bar{x}$ is

$$x(n+1) = (1-\epsilon)x(n) + (1-\epsilon)f'(\bar{x})x(n) + (2\epsilon - 1)f'(\bar{x})x(n-1),$$

that is,

$$x(n+1) + (\epsilon - 1)(1 + f'(\bar{x}))x(n) + (1-2\epsilon)f'(\bar{x})x(n-1) = 0. \tag{3.4}$$

It is well-known (see, for example [9]) that for the linear equation

$$z(n+1) + \alpha z(n) + \beta z(n-1) = 0,$$

where α and β are constants, a necessary and sufficient condition for the asymptotic stability is

$$|\alpha| < 1 + \beta < 2. \tag{3.5}$$

Hence, when (3.3) holds, the zero solution of Eq. (3.4) is asymptotically stable. Then by linearized stability theory, the positive equilibrium $\bar{x}$ of Eq. (1.3) is asymptotically stable and so it follows that the positive equilibrium $(\bar{x}, \bar{x})$ of Sys. (1.2) is asymptotically stable. This, together with the global attractivity, we have shown above implies that $(\bar{x}, \bar{x})$ is globally asymptotically stable. The proof is complete. □

Sys. (1.2) is a population model proposed by Newman et al. [14] which assumes symmetrical dispersal between an active population $x(n)$ and a refuge population $y(n)$. The exponential function $f(x) = \lambda x e^{-x}$ is a function studied in [14]. Using numerical simulations, it is concluded that the presence of a passive refuge can greatly stabilize a population that would otherwise exhibit chaotic dynamics. Now, by applying Theorem 3.1, we may get an explicit sufficient condition for the global asymptotic stability of Sys. (1.2) when $f(x) = \lambda x e^{-x}$.

When $\lambda > 1$, $\bar{x} = \ln \lambda$ is the unique positive fixed point of f. Clearly, f satisfies the negative feedback condition (3.1). Noting that

$$(x + f(x))' = 1 + (1-x)\lambda e^{-x} \quad \text{and} \quad (x + f(x))'' = \lambda(x-2)e^{-x},$$

we see that when $\lambda \le e^2$,

$$(x + f(x))' \ge (x + f(x))'|_{x=2} = 1 - \lambda e^{-2} \ge 0,$$

and so $x + f(x)$ is increasing. In addition, noting that

$$f'(x) = \lambda(1-x)e^{-x} \quad \text{and} \quad f''(x) = \lambda(x-2)e^{-x},$$

we see that

$$f'(0) = \lambda \quad \text{and} \quad f'(2) = -\lambda e^{-2},$$

and it follows that

$$|f'(x)| \leq \lambda, \qquad x \geq 0.$$

Hence, f is L-Lipschitz with $L = \lambda$. Now observe that

$$f'(\bar{x}) = \lambda(1 - \ln\lambda)e^{-\ln\lambda} = 1 - \ln\lambda,$$

$$1 + (1-2\epsilon)f'(\bar{x}) = 1 + (1-2\epsilon)(1-\ln\lambda) < 1 + (2\epsilon - 1)(\ln e^2 - 1) = 2\epsilon < 2, \qquad \lambda \leq e^2$$

and

$$\begin{aligned} 1 + (1-2\epsilon)f'(\bar{x}) &= 1 + (1-2\epsilon)(1-\ln\lambda) > 1 + (1-2\epsilon) \\ &= 2(1-\epsilon) > (1-\epsilon)(2-\ln\lambda) = (1-\lambda)|1+f'(\bar{x})|. \end{aligned}$$

Hence, by Theorem 3.1, we have the following conclusion: if $1 < \lambda \leq e^2$ and

$$(2-\epsilon)(2-1/\epsilon)\lambda < 1,$$

then Sys. (1.2) has a unique positive equilibrium $(\ln\lambda, \ln\lambda)$ that is globally asymptotically stable.

In the above, we showed that our result can be applied to the case that f is not monotonic. Clearly, our result can be applied to the case that f is monotonic also. The following is an example in which the function f is decreasing. Consider the equation

$$x(n+1) = ax(n) + b\frac{q}{1+x^p} + c\frac{q}{1+x^p(n-k)}, \tag{3.6}$$

where a, b and c are the same as those assumed in Eq. (1.1), and $p \geq 1,\ q \geq 0$ are constants. If $b = 0$, (3.6) reduces to the equation of the form

$$x(n+1) = ax(n) + \frac{B}{1+x^p(n-k)}, \tag{3.7}$$

where $B = cq$. Eq. (3.7) is a discrete analogue of a model that has been used to study blood cells production [10]. The global attractivity of positive solutions of this equation has been studied by numerous authors, see for example, [2–4,7–9] and the references cited therein. Let

$$f(x) = \frac{q}{1+x^p}.$$

Then f is decreasing and has a unique positive fixed point $\bar{x}$. Clearly, f satisfies the feedback condition (3.1). Now observe that

$$f'(x) = \frac{-qpx^{p-1}}{(1+x^p)^2}$$

and

$$f''(x) = \frac{-qpx^{p-2}((p-1)-(p+1)x^p)}{(1+x^p)^3}.$$

We find that $f'(x)$ has a minimum at $x^* = \left(\frac{p-1}{p+1}\right)^{1/p}$ and

$$f'(x^*) = -\frac{q}{4p}(p-1)^{1-1/p}(1+p)^{1+1/p}.$$

Hence, f is L-Lipschitz with

$$L = \frac{q}{4p}(p-1)^{1-1/p}(p+1)^{1+1/p}.$$

In addition, if

$$\frac{bq}{4p}(p-1)^{1-1/p}(p+1)^{1+1/p} < a, \tag{3.8}$$

then

$$(ax + bf(x))' = a + bf'(x) \geq a + bf'(x^*) = a - \frac{bq}{4p}(p-1)^{1-1/p}(p+1)^{1+1/p} > 0$$

and so the function $ax + bf(x)$ is increasing, if

$$\frac{c(1-a^{k+1})}{c+a^k b}\frac{q}{4p}(p-1)^{1-1/p}(p+1)^{1+1/p} < 1, \tag{3.9}$$

then (2.8) is satisfied. Therefore, by Theorem 2.2, if (3.8) and (3.9) hold, then every positive solution of Eq. (3.6) tends to its positive equilibrium $\bar{x}$.

When $f(x) = \frac{q}{1+x^p}$, Sys. (1.2) can be converted into the second order difference equation

$$x(n+1) = (1-\epsilon)x(n) + (1-\epsilon)\frac{q}{1+x^p(n)} + (2\epsilon - 1)\frac{q}{1+x^p(n-1)}, \tag{3.10}$$

which is in the form of Eq. (1.1) with $a = b = 1-\epsilon,\ c = 2\epsilon - 1$ and $k = 1$. In this case, since

$$\frac{c(1-a^{k+1})}{c+a^k b} = (2-\epsilon)\left(2 - \frac{1}{\epsilon}\right),$$

we see that (3.8) and (3.9) reduce to

$$\frac{q}{4p}(p-1)^{1-1/p}(p+1)^{1+1/p} < 1, \tag{3.11}$$

and

$$(2-\epsilon)(2-\frac{1}{\epsilon})\frac{q}{4p}(p-1)^{1-1/p}(p+1)^{1+1/p} < 1, \tag{3.12}$$

respectively. Then by noting $(2-\epsilon)(2-\frac{1}{\epsilon}) < 1$, we see that (3.11) yields (3.12).

From the above discussion and (3.12), we know that

$$-f'(\bar{x}) \leq -f'(x^*) = \frac{q}{4p}(p-1)^{1-1/p}(p+1)^{1+1/p} < 1.$$

Then it follows that

$$1 + (1-2\epsilon)f'(\bar{x}) = 1 + (2\epsilon - 1)(-f'(\bar{x})) \leq 1 + (2\epsilon - 1) = 2\epsilon < 2$$

and

$$(1-\epsilon)|1+f'(\bar{x})| \leq (1-\epsilon)(1-f'(\bar{x})) \leq 2(1-\epsilon) \leq 1 \leq 1 + (2\epsilon-1)(-f'(\bar{x})).$$

Hence, by Theorem 3.1, we find that if (3.11) holds, then Sys. (1.2) with $f(x) = \frac{q}{1+x^p}$ has a unique positive equilibrium $(\bar{x}, \bar{x})$ which is globally asymptotically stable.

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Global existence and blow-up for semilinear parabolic equation with critical exponent in $\mathbb{R}^N$

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Abstract. In this paper, we use the self-similar transformation and the modified potential well method to study the long time behaviors of solutions to the classical semilinear parabolic equation associated with critical Sobolev exponent in $\mathbb{R}^N$. Global existence and finite time blowup of solutions are proved when the initial energy is in three cases. When the initial energy is low or critical, we not only give a threshold result for the global existence and blowup of solutions, but also obtain the decay rate of the L^2 norm for global solutions. When the initial energy is high, sufficient conditions for the global existence and blowup of solutions are also provided. We extend the recent results which were obtained in [R. Ikehata, M. Ishiwata, T. Suzuki, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **27**(2010), No. 3, 877– 900].

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1 Introduction

This paper deals with the following classical semilinear parabolic equation associated with critical Sobolev exponent in $\mathbb{R}^N$:

$$\begin{cases} u_t - \Delta u = |u|^{p-1}u & \text{in } \mathbb{R}^N \times (0,T), \\ u|_{t=0} = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $N \geqslant 3$ and $p = (N+2)/(N-2)$, the critical exponent associated with the Sobolev imbedding.

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There is a great literature on the existence of global solutions and blow-up for the problem (1.1) on the bounded domain (see e.g. [2,6,9,13,21,22,25,27,28]):

$$\begin{cases} u_t - \Delta u = |u|^{p-2}u & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{on } \partial\Omega \times [0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.2}$$

where $p > 1$. It is well known that there exist choices of u_0 for which the corresponding solutions tend to zero as $t \to \infty$ and other choices for which the solutions blow-up in finite time (see e.g. [9]). Tan [25], R. Ikehata and T. Suzuki [6] considered critical problem (1.2). By means of the potential well method, they established the existence of global solutions and studied the asymptotic behavior of solutions which heavily depend on the initial data. Using the the comparison principle and variational methods, Gazzola and Weth [2] obtained global solutions and finite time blow-up solutions with the initial data at high energy level.

The problem (1.1) in $\mathbb{R}^N$ was considered by Ishiwata and Suzuki [4], Ikehata, Ishiwata, and Suzuki [5], Mizoguchi and Yanagida [14–16]. In [14], a sufficient condition on the decay order of initial data, which may change sign, such that the solution of (1.1) blows up infinite time, was gived. Using self-similar transformation, Mizoguchi and Yanagida [15, 16] established the global existence and blow-up results for problem (1.1) in the $\mathbb{R}^1$. In [4,5], the decay and blow-up of the solution with low energy initial data were studied by means of the potential-well and forward self-similar transformation. For a general scope of this topic, we refer the interested readers to the monographs [23] and references therein.

In this article, we consider the problem (1.1) with low initial energy, critical initial energy and high initial energy. The results in our paper will be obtained by the self-similar transformation and the modified potential well method. Potential well method, which was first put forward to consider semi-linear hyperbolic initial boundary value problem by Payne and Sattinger [20, 24] around 1970s, is a powerful tool in studying the long time behaviors of solutions of some evolution equations. The potential well is defined by the level set of energy functional and the derivative functional. It is generally true that solutions starting inside the well are global in time, solutions starting outside the well and at an unstable point blow up in finite time. After the pioneer work of Sattinger and Payne, some authors [7,9–12,17–19,26] used the method to study the global existence and nonexistence of solutions for various nonlinear evolution equations with initial boundary value problem. In [11,12], Liu et al. modified and improved the method by introducing a family of potential wells which include the known potential well as a special case. The modified potential well method has been used to study semilinear pseudo-parabolic equations [9] and fourth-order parabolic equation [3]. In this paper, we use the modified potential well method to obtain global existence and blow up in finite time of solutions when the initial energy is low, critical and high, respectively. When the initial energy is low, similar results are obtained in [5], but our result is more general, moreover, we prove a more precise decay rate of the L^2 norm of global solution.

This paper is organized as follows. In Section 2, we give some notations, definitions and lemmas concerning the basic properties of the related functionals and sets. Sections 3 and 4 will be devoted to the cases $E_K(v_0) < d$ and $E_K(v_0) = d$, respectively, where $E_K(v)$ will be introduced in Section 2. In Section 5, we consider the case when the initial energy is high, i.e. $E_K(v_0) > d$.

2 Preliminaries and main lemmas

In this section, we shall introduce the self-similar transformation and the modified potential well method and give a series of their properties for problem (1.1). The self-similar transformation is defined as follow:

$$v(y,s) = (1+t)^{1/(p-1)}u(x,t), \quad t = e^s - 1, \quad x = (1+t)^{1/2}y.$$

We can easily know that

$$\begin{cases} v_s + Lv = |v|^{p-1}v + \dfrac{1}{p-1}v & \text{in } \mathbb{R}^N \times (0,S), \\ v|_{s=0} = v_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{2.1}$$

where $S = \log(1+T)$ and

$$Lf = -\Delta f - \frac{1}{2}y \cdot \nabla f.$$

Letting

$$K(y) = e^{|y|^2/4},$$

we have

$$Lf = -\frac{1}{K}\nabla \cdot (K\nabla f).$$

Let

$$\|f\|_{2,K} = \left\{ \int_{\mathbb{R}^N} |f(y)|^2 K(y) dy \right\}^{1/2} < +\infty.$$

We also take

$$H^m(K) = \left\{ f \in L^2(K) \mid D^\alpha f \in L^2(K) \text{ for any multi-index } \alpha \text{ in } |\alpha| \leqslant m \right\},$$

where $m = 1, 2, \ldots$ It is a Hilbert space provided with the norm

$$\|f\|_{H^m(K)} = \left\{ \sum_{|\alpha| \leqslant m} \|D^\alpha f\|_{2,K}^2 \right\}^{1/2}.$$

The linear operator L is realized as a self-adjoint operator in $L^2(K)$ through the relation

$$\mathcal{A}_K(u,v) := \int_{\mathbb{R}^N} \nabla u(y) \cdot \nabla v(y) K(y) dy = (Lu, v)_K, \quad u \in D(L) \subset H^1(K), v \in H^1(K),$$

where

$$(u,v)_K = \int_{\mathbb{R}^N} u(y)v(y)K(y)dy.$$

From Lemma 2.1 of [8], the domain $D(L)$ of this operator L is the set of $v \in L^2(K)$ satisfying $Lv \in L^2(K)$, and we have $D(L) = H^2(K)$, It holds also that L is positive selfadjoint and has the compact inverse, and in particular, the set of normalized eigenfunctions of L forms a complete ortho-normal equation in $L^2(K)$. The first eigenvalue λ_1 of L is given by $\lambda_1 = N/2$, and hence from Proposition 2.3 of [1]. the following Poincaré inequality holds,

$$\lambda_1 \|v\|_{2,K}^2 \leqslant \|\nabla v\|_{2,K}^2, \quad v \in H^1(K). \tag{2.2}$$

We have

$$\lambda_1 = \frac{N}{2} > \lambda \equiv \frac{1}{p-1} = \frac{N-2}{4}.$$

Then, the operator

$$A = L - \frac{1}{p-1}$$

in $L^2(K)$ is also positive self-adjoint with the domain $D(A) = H^2(K)$. The semigroup $\{e^{-tA}\}_t \geqslant 0$ are thus defined in $L^2(K)$. These characteristics guarantee the well-posedness of (1.1) locally in time.

Now let us define the level set

$$E^\alpha = \left\{ v \in H^1(K) : E_K(v) < \alpha \right\}. \tag{2.3}$$

Furthermore, by the definition of $E_K(v), \mathcal{N}, E^\alpha$ and d, we easily know that

$$\mathcal{N}_\alpha = \mathcal{N} \cap E^\alpha \equiv \left\{ v \in \mathcal{N} : \|\nabla v\|_{2,K}^2 - \lambda \|v\|_{2,K}^2 < \sqrt{\frac{2\alpha(p+1)}{p-1}} \right\} \neq \varnothing \quad \text{for all } \alpha > d. \tag{2.4}$$

Let

$$\lambda_\alpha = \inf\{\|v\|_{2,K} : v \in \mathcal{N}_\alpha\}, \quad \Lambda_\alpha = \sup\{\|v\|_{2,K} : u \in \mathcal{N}_\alpha\} \quad \text{for all } \alpha > d. \tag{2.5}$$

It is clear that λ_α is nonincreasing and Λ_α is nondecreasing with respect to α . For $0 < \delta < \frac{p+1}{2}$, let us define the modified functional and Nehari manifold as follows:

$$\begin{aligned}
E_K(v) &= \frac{1}{2}\|\nabla v\|_{2,K}^2 - \frac{\lambda}{2}\|v\|_{2,K}^2 - \frac{1}{p+1}\|v\|_{p+1,K}^{p+1}, \\
D_{K,\delta}(v) &= \delta\|\nabla v\|_{2,K}^2 - \lambda\delta\|v\|_{2,K}^2 - \|v\|_{p+1,K}^{p+1}, \\
\mathcal{N}_\delta &= \left\{ u \in H^1(K) : D_{K,\delta}(v) = 0, \|v\| \neq 0 \right\}, \\
d_\delta &= \inf_{v \in N_\delta} E_K(v), \\
r(\delta) &= \delta^{\frac{N-2}{2}} S_\lambda^{\frac{N}{2}}.
\end{aligned}$$

Then we can define the modified potential wells and their corresponding sets as follows:

$$\begin{aligned}
W_\delta &= \left\{ u \in H^1(K) : D_\delta(u) > 0, E(u) < d(\delta) \right\} \cup \{0\}, \\
V_\delta &= \left\{ u \in H^1(K) : D_\delta(u) < 0, E(u) < d(\delta) \right\}, \\
B_\delta &= \left\{ u \in H^1(K) : \|\nabla u\|_{2,K} < r(\delta) \right\}, \\
B_\delta^c &= \left\{ u \in H^1(K) : \|\nabla u\|_{2,K} > r(\delta) \right\}.
\end{aligned} \tag{2.6}$$

We also introduce the following sets

$$\begin{aligned}
\mathcal{B} &= \left\{ u_0 \in H^1(K) : \text{ the solution } u = u(t) \text{ of } (1.2) \text{ blows up in finite time } \right\}, \\
\mathcal{G} &= \left\{ u_0 \in H^1(K) : \text{ the solution } u = u(t) \text{ of } (1.2) \text{ exists for all } t > 0 \right\}, \\
\mathcal{G}_o &= \left\{ u_0 \in G : u(t) \mapsto 0 \text{ in } H^1(K) \text{ as } t \to \infty \right\}.
\end{aligned} \tag{2.7}$$

For future convenience, we give some useful lemmas which will play an important role in the proof of our main results.

Let $L^q(K)$ denote the Banach space composed of measurable functions $v = v(y)$ defined in $\mathbb{R}^N$ such that

$$\|v\|_{q,K} = \left\{ \int_{\mathbb{R}^N} |v(y)|^q K(y) dy \right\}^{1/q} < +\infty$$

for $q \in [1, \infty)$ and

$$\|v\|_{\infty,K} = \operatorname*{ess\,sup}_{y \in \mathbb{R}^N} |f(y)| < +\infty$$

for $q = \infty$. The space $L^\infty(K) = L^\infty(\mathbb{R}^N)$ is thus compatible to the other spaces, i.e.,

$$\lim_{q \uparrow \infty} \|v\|_{q,K} = \|f\|_{\infty,K}, v \in L^1(K) \cap L^\infty\left(\mathbb{R}^N\right)$$

Although the inclusion

$$L^p(K) \subset L^q(K) \quad (1 \leqslant q < p \leqslant \infty)$$

fails, we have

$$H^1(K) \subset L^{2^*}(K)$$

for $2^* = 2N/(N-2) = p+1$. More precisely, Corollary 4.20 of [1] guarantees the following fact, regarded as a Sobolev–Poincaré inequality.

Lemma 2.1 ([1, Corollary 4.20])**.** *It holds that*

$$S_0 \|v\|^2_{p+1,K} + \lambda_* \|v\|^2_{2,K} \leqslant \|\nabla v\|^2_{2,K}, \quad v \in H^1(K),$$

where $\lambda_* = \max(1, N/4)$ *and* S_0 *stands for the Sobolev constant:*

$$S_0 = \inf\left\{ \|\nabla v\|_2^2 \mid v \in C_0^\infty\left(\mathbb{R}^N\right), \|v\|_{p+1} = 1 \right\}.$$

Lemma 2.2 ([5, p. 882])**.** *Set*

$$S_\lambda = \inf\left\{ \frac{\|\nabla v\|^2_{2,K} - \lambda \|v\|^2_{2,K}}{\|v\|^2_{p+1,K}} \mid v \in H^1(K) \right\},$$

We have $S_\lambda = S_0$.

So, it holds that

$$\|v\|^{p+1}_{p+1,K} \leqslant \left(\frac{1}{S_\lambda} (\|\nabla v\|^2_{2,K} - \lambda \|v\|^2_{2,K}) \right)^{\frac{p+1}{2}}, \quad v \in H^1(K), \tag{2.8}$$

and

$$r(\delta) = \delta^{\frac{N-2}{2}} S_\lambda^{\frac{N}{2}} \geq \|\nabla v\|^2_{2,K} - \lambda \|v\|^2_{2,K}.$$

Lemma 2.3. *Let* $u_0 \in H^1(K)$.

(1) *If* $0 < \|\nabla v\|^2_{2,K} - \lambda \|v\|^2_{2,K} < r(\delta)$, *then* $D_{K,\delta}(u) > 0$. *In particular, if* $0 < \|\nabla v\|^2_{2,K} - \lambda \|v\|^2_{2,K} < r(1)$, *then* $D_K(u) > 0$;

(2) *If* $D_{K,\delta}(u) < 0$, *then* $\|\nabla v\|^2_{2,K} - \lambda \|v\|^2_{2,K} > r(\delta)$. *In particular, if* $D_K(u) < 0$, *then* $\|\nabla v\|^2_{2,K} - \lambda \|v\|^2_{2,K} > r(1)$;

(3) *If* $D_{K,\delta}(v)=0$, *then* $\|\nabla v\|_{2,K}^2-\lambda\|v\|_{2,K}^2\geqslant r(\delta)$ *or* $\|\nabla v\|_{2,K}^2-\lambda\|v\|_{2,K}^2=0$. *In particular, if* $D_K(v)=0$, *then* $\|\nabla v\|_{2,K}^2-\lambda\|v\|_{2,K}^2\geqslant r(1)$ *or* $\|\nabla v\|_{2,K}^2-\lambda\|v\|_{2,K}^2=0$;

(4) *If* $D_{K,\delta}(v)=0$ *and* $\|\nabla v\|_{2,K}^2-\lambda\|v\|_{2,K}^2\neq 0$, *then* $E_K(v)>0$ *for* $0<\delta<\frac{p+1}{2}$, $E_K(v)=0$ *for* $\delta=\frac{p+1}{2}$, $E(v)<0$ *for* $\delta>\frac{p+1}{2}$.

Proof. (1) Since $0<\|\nabla v\|_{2,K}^2-\lambda\|v\|_{2,K}^2<r(\delta)$, by the Lemma 2.2 and (2.8), we have from the assumption $0<\|v\|<r(\delta):=\delta^{\frac{N-2}{2}}S_\lambda^{\frac{N}{2}}$, and we obtain

$$\begin{aligned}D_{K,\delta}(v)&=\delta\|\nabla v\|_{2,K}^2-\lambda\delta\|v\|_{2,K}^2-\|v\|_{p+1,K}^{p+1}\\&\geq\delta\|\nabla v\|_{2,K}^2-\lambda\delta\|v\|_{2,K}^2-\left(\frac{1}{S_\lambda}(\|\nabla v\|_{2,K}^2-\lambda\|v\|_{2,K}^2)\right)^{\frac{p+1}{2}}\\&\geq(\|\nabla v\|_{2,K}^2-\lambda\|v\|_{2,K}^2)\left(\delta-\left(\frac{1}{S_\lambda}\right)^{\frac{p+1}{2}}(\|\nabla v\|_{2,K}^2-\lambda\|v\|_{2,K}^2)^{\frac{2}{N-2}}\right)>0.\end{aligned}\tag{2.9}$$

(2) By the assumption $D_{K,\delta}(v)<0$ and (2.8), we have

$$\begin{aligned}0\geq D_{K,\delta}(v)&=\delta\|\nabla v\|_{2,K}^2-\lambda\delta\|v\|_{2,K}^2-\|v\|_{p+1,K}^{p+1}\\&\geq\delta\|\nabla v\|_{2,K}^2-\lambda\delta\|v\|_{2,K}^2-\left(\frac{1}{S_\lambda}(\|\nabla v\|_{2,K}^2-\lambda\|v\|_{2,K}^2)\right)^{\frac{p+1}{2}}\\&\geq(\|\nabla v\|_{2,K}^2-\lambda\|v\|_{2,K}^2)\left(\delta-\left(\frac{1}{S_\lambda}\right)^{\frac{p+1}{2}}(\|\nabla v\|_{2,K}^2-\lambda\|v\|_{2,K}^2)^{\frac{2}{N-2}}\right).\end{aligned}\tag{2.10}$$

Hence, $\|\nabla v\|_{2,K}^2-\lambda\|v\|_{2,K}^2>r(\delta)$.

(3) By the assumption $D_{K,\delta}(v)=0$ and (2.8), we have

$$\begin{aligned}0=D_{K,\delta}(v)&=\delta\|\nabla v\|_{2,K}^2-\lambda\delta\|v\|_{2,K}^2-\|v\|_{p+1,K}^{p+1}\\&\geq\delta\|\nabla v\|_{2,K}^2-\lambda\delta\|v\|_{2,K}^2-\left(\frac{1}{S_\lambda}(\|\nabla v\|_{2,K}^2-\lambda\|v\|_{2,K}^2)\right)^{\frac{p+1}{2}}\\&\geq(\|\nabla v\|_{2,K}^2-\lambda\|v\|_{2,K}^2)\left(\delta-\left(\frac{1}{S_\lambda}\right)^{\frac{p+1}{2}}(\|\nabla v\|_{2,K}^2-\lambda\|v\|_{2,K}^2)^{\frac{2}{N-2}}\right).\end{aligned}\tag{2.11}$$

Hence, $\|\nabla v\|_{2,K}^2-\lambda\|v\|_{2,K}^2=r(\delta)$. or $v=0$.

(4) We easily know that

$$\begin{aligned}E_K(v)&=\frac{1}{2}\|\nabla v\|_{2,K}^2-\frac{\lambda}{2}\|v\|_{2,K}^2-\frac{1}{p+1}\|v\|_{p+1,K}^{p+1}\\&=\left(\frac{1}{2}-\frac{\delta}{p+1}\right)(\|\nabla v\|_{2,K}^2-\lambda\|v\|_{2,K}^2)+\frac{1}{p+1}D_{K,\delta}(v)\\&=\left(\frac{1}{2}-\frac{\delta}{p+1}\right)(\|\nabla v\|_{2,K}^2-\lambda\|v\|_{2,K}^2).\end{aligned}\tag{2.12}$$

Then we can prove the conclusion. □

Lemma 2.4.

(1) $d(\delta) \geqslant a(\delta)r^2(\delta)$ *for* $a(\delta) = \frac{1}{2} - \frac{\delta}{p+1}, 0 < \delta < \frac{p+1}{2}$,

(2) $\lim_{\delta\to 0} d(\delta) = 0, d\left(\frac{p+1}{2}\right) = 0$ *and* $d(\delta) < 0$ *for* $\delta > \frac{p+1}{2}$,

(3) $d(\delta)$ *is increasing on* $0 < \delta \leqslant 1$, *decreasing on* $1 \leqslant \delta \leqslant \frac{p+1}{2}$ *and takes the maximum* $d = d(1)$ *at* $\delta = 1$.

Proof. (1) If $u \in \mathcal{N}_\delta$, by Lemma 2.3 (3), then $\|u\| \geqslant r(\delta)$. Moreover, we can deduce

$$\begin{aligned} E_K(v) &= \frac{1}{2}\|\nabla v\|_{2,K}^2 - \frac{\lambda}{2}\|v\|_{2,K}^2 - \frac{1}{p+1}\|v\|_{p+1,K}^{p+1} \\ &= \left(\frac{1}{2} - \frac{\delta}{p+1}\right)(\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2) + \frac{1}{p+1}D_{K,\delta}(v) \\ &= \left(\frac{1}{2} - \frac{\delta}{p+1}\right)(\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2) \geq a(\delta)r^2(\delta). \end{aligned} \tag{2.13}$$

Hence, $d(\delta) \geqslant a(\delta)r^2(\delta)$.

(2) We easily know that

$$E_K(sv) = \frac{s^2}{2}\|\nabla v\|_{2,K}^2 - \frac{\lambda s^2}{2}\|v\|_{2,K}^2 - \frac{s^{p+1}}{p+1}\|v\|_{p+1,K}^{p+1}.$$

Hence,

$$\lim_{s\to 0} E_K(sv) = 0. \tag{2.14}$$

And if we let $sv \in \mathcal{N}_\delta$, then sv satisfies

$$0 = D_{K,\delta}(sv) = \delta s^2\|\nabla v\|_{2,K}^2 - \lambda s^2\delta\|v\|_{2,K}^2 - s^{p+1}\|v\|_{p+1,K}^{p+1}.$$

Then, we obtain

$$s(\delta) = \left(\frac{\delta(\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2)}{\|v\|_{p+1,K}^{p+1}}\right)^{\frac{1}{p-1}}, \tag{2.15}$$

which yields

$$\lim_{\delta\to 0} s(\delta) = 0. \tag{2.16}$$

Now (2.14) implies that

$$\lim_{\delta\to 0} E_K(sv) = \lim_{\lambda\to 0} E_K(sv) = 0, \tag{2.17}$$

and

$$\lim_{\delta\to 0} d(\delta) = 0. \tag{2.18}$$

It is easy to see that from (2.13)

$$d\left(\frac{p+1}{2}\right) = 0 \text{ and } d(\delta) < 0 \text{ for } \delta > \frac{p+1}{2}.$$

The proof is complete.

(3) We need to prove that for any $0 < \delta' < \delta'' < 1$ or $1 < \delta'' < \delta' < \frac{p+1}{2}$ and for any $w \in \mathcal{N}_{\delta''}$, there is a $v \in \mathcal{N}_{\delta'}$ and a constant $\varepsilon(\delta', \delta'')$ such that $E_K(v) < E_K(w) - \varepsilon(\delta', \delta'')$. Indeed, by the definition of (2.15), we easily know that $D_{K,\delta}(s(\delta)u) = 0$ and $\lambda(\delta'') = 1$. Let $h(s) = E_K(sw)$, we have

$$\begin{aligned}\frac{d}{ds}h(s) &= \frac{1}{s}\left((1-\delta)(\|\nabla sw\|_{2,K}^2 - \lambda\|sw\|_{2,K}^2) + D_{K,\delta}(sw)\right)\\ &= (1-\delta)s(\|\nabla w\|_{2,K}^2 - \lambda\|w\|_{2,K}^2).\end{aligned} \tag{2.19}$$

Take $v = s(\delta')w$, then $v \in \mathcal{N}_{\delta'}$.

For $0 < \delta' < \delta'' < 1$, we obtain

$$\begin{aligned}E_K(w) - E_K(v) &= h(1) - h\left(s\left(\delta'\right)\right)\\ &> \left(1-\delta''\right) r^2\left(\delta''\right) s\left(\delta'\right)\left(1 - s\left(\delta'\right)\right) \equiv \varepsilon\left(\delta', \delta''\right).\end{aligned} \tag{2.20}$$

For $1 < \delta'' < \delta' < \frac{p+1}{2}$, we obtain

$$\begin{aligned}E_K(w) - E_K(v) &= h(1) - h\left(s\left(\delta'\right)\right)\\ &> \left(\delta''-1\right) r^2\left(\delta''\right) s\left(\delta''\right)\left(s\left(\delta'\right) - 1\right) \equiv \varepsilon\left(\delta', \delta''\right).\end{aligned} \tag{2.21}$$

Hence, the proof is complete. □

Lemma 2.5. *Let $u_0 \in H^1(K)$ and $0 < \delta < \frac{p+1}{2}$. If $E_K(v) \leqslant d(\delta)$, then we have*

(1) *If $D_{K,\delta}(v) > 0$, then $\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2 < \frac{d(\delta)}{a(\delta)}$, where $a(\delta) = \frac{1}{2} - \frac{\delta}{p+1}$. In particular, if $E_K(v) \leqslant d$ and $D_K(v) > 0$, then*

$$\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2 < \frac{2(p+1)}{p-1}d. \tag{2.22}$$

(2) *If $\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2 > \frac{d(\delta)}{a(\delta)}$, then $D_{K,\delta}(u) < 0$. In particular, if $E_K(v) \leqslant d$ and*

$$\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2 > \frac{2(p+1)}{p-1}d, \tag{2.23}$$

then $D_K(v) < 0$.

(3) *If $D_{K,\delta}(v) = 0$, then $\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2 \leqslant \frac{d(\delta)}{a(\delta)}$. In particular, if $E_K(v) \leqslant d$ and $D_K(v) = 0$, then*

$$\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2 \leqslant \frac{2(p+1)}{p-1}d. \tag{2.24}$$

Proof. (1) For $0 < \delta < \frac{p+1}{2}$, we see that

$$\begin{aligned}E_K(v) &= \frac{1}{2}\|\nabla v\|_{2,K}^2 - \frac{\lambda}{2}\|v\|_{2,K}^2 - \frac{1}{p+1}\|v\|_{p+1,K}^{p+1}\\ &= \left(\frac{1}{2} - \frac{\delta}{p+1}\right)(\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2) + \frac{1}{p+1}D_\delta(u)\\ &= a(\delta)\|u\|^2 \leq d(\delta).\end{aligned} \tag{2.25}$$

Therefore,

$$\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2 < \frac{d(\delta)}{a(\delta)}.$$

Finally, (2) and (3) follow from (2.25). □

Lemma 2.6. *Let $v \in H^1(K)$. We have*

(1) *0 is away from both $\mathcal{N}$ and $\mathcal{N}_-$, i.e.* $\operatorname{dist}(0,\mathcal{N}) > 0$, $\operatorname{dist}(0, N_-) > 0$.

(2) *For any $\alpha > 0$, the set $E^\alpha \cap \mathcal{N}_+$ is bounded in $H^1(K)$.*

Proof. (1) If $v \in \mathcal{N}$, then we have

$$\begin{aligned} d \le E_K(v) &= \frac{1}{2}\|\nabla v\|_{2,K}^2 - \frac{\lambda}{2}\|v\|_{2,K}^2 - \frac{1}{p+1}\|v\|_{p+1,K}^{p+1} \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right)(\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2) + \frac{1}{p+1}D(u) \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right)(\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2). \end{aligned}$$

If $v \in \mathcal{N}_-$, then we have

$$\begin{aligned} d \le E_K(v) &= \frac{1}{2}\|\nabla v\|_{2,K}^2 - \frac{\lambda}{2}\|v\|_{2,K}^2 - \frac{1}{p+1}\|v\|_{p+1,K}^{p+1} \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right)(\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2) + \frac{1}{p+1}D(u) \\ &\le \left(\frac{1}{2} - \frac{1}{p+1}\right)(\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2). \end{aligned}$$

Hence, 0 is away from both N and $\mathcal{N}_-$, i.e. $\operatorname{dist}(0,\mathcal{N}) > 0$, $\operatorname{dist}(0, N_-) > 0$.

(2) Since $E_K(v) < \alpha$ and $D_K(v) > 0$, we obtain

$$\begin{aligned} \alpha > E_K(v) &= \frac{1}{2}\|\nabla v\|_{2,K}^2 - \frac{\lambda}{2}\|v\|_{2,K}^2 - \frac{1}{p+1}\|v\|_{p+1,K}^{p+1} \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right)(\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2) + \frac{1}{p+1}D(u) \\ &> \left(\frac{1}{2} - \frac{1}{p+1}\right)(\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2). \end{aligned}$$

Hence, for any $\alpha > 0$, the set $E^\alpha \cap \mathcal{N}_+$ is bounded in $H^1(K)$. □

3 Low initial energy $E_K(v_0) < d$

The goal of this section is to prove Theorems 3.2–3.6. A threshold result for the global solutions and finite time blowup will be given.

Theorem 3.1. *Assume that $v_0 \in H^1(K)$, T is the maximal existence time of u, and $0 < e < d$, $\delta_1 < \delta_2$ are two roots of equation $d(\delta) = e$. We have*

(1) *If $D_K(v_0) > 0$, all weak solutions u of equation (2.1) with $E_K(v_0) = e$ belong to W_δ for $\delta_1 < \delta < \delta_2, 0 \leqslant t < T$.*

(2) *If $D_K(v_0) < 0$, all weak solutions u of equation (2.1) with $E_K(v_0) = e$ belong to V_δ for $\delta_1 < \delta < \delta_2, 0 \leqslant t < T$.*

Theorem 3.2 (Global existence)**.** *Assume that $v_0 \in H^1(K)$, $E_K(u_0) < d$, $D_K(u_0) > 0$. Then equation (2.1) has a global solution $v(t) \in L^\infty(0,\infty; H^1(K))$ and $v(t) \in W$ for $0 \leqslant t < \infty$.*

Remark 3.3. Result similar to Theorem 3.2 is obtained in [5]. But our proof is different to [5]. In fact, using the modified potential well method we can obtain the more general conclusion:

If the assumption $D_K(u_0) > 0$ is replaced by $D_{K,\delta_2}(u_0) > 0$, where $\delta_1 < \delta_2$ are the two roots of equation $d(\delta) = E_K(u_0)$, then equation (2.1) admits a global weak solution.

The following result is obtained in [5]. But our proof is different from the proof in [5]. For the reader's convenience, we will give the detailed proof.

Theorem 3.4. *Assume that $v_0 \in H^1(K)$, $E_K(v_0) < d$ and $D(v_0) < 0$. Then the weak solution $v(t)$ of equation (2.1) blows up in finite time, that is, there exists a $T > 0$ such that*

$$\lim_{t \to T} \int_0^t \|v(\tau)\|_{2,K} d\tau = +\infty.$$

Remark 3.5. Assume that $v_0 \in H^1(K)$, $E_K(v_0) < d$. When $D_K(u_0) > 0$, equation (2.1) has a global solution. When $D_K(v_0) < 0$, equation (2.1) does not admit any global weak solution.

Theorem 3.6. *Assume that $v_0 \in H^1(K)$, $E_K(v_0) < d$ and $D_K(v_0) > 0$, $\delta_1 < \delta_2$ are the two roots of equation $d(\delta) = E_K(v_0)$. Then, for the global weak solution v of equation (2.1), it holds*

$$\|v\|_{2,K}^2 \leqslant \|v_0\|_{2,K}^2 e^{-2S_\lambda(1-\delta_1)t}, \quad 0 \leqslant t < \infty. \tag{3.1}$$

Remark 3.7. In comparison with the decay rate in [5], our result concerning the decay rate of $|u|_2$ in Theorem 3.6 is much more precise.

In order to prove Theorems 4.1–4.4, we need the following lemmas:

Lemma 3.8. *For $0 < T \leq \infty$, assume that $v : \Omega \times [0, T) \to \mathbb{R}^3$ is a weak solution to equation (2.1). Then it holds*

$$\int_{t_1}^{t_2} \|v_t\|_{2,K}^2 dt + E_K(v(t_2)) = E_K(v(t_1)), \quad \forall t_1, t_2 \in (0, T). \tag{3.2}$$

Proof. Multiplying (2.1) by v_t and integrating over $\mathbb{R}^N$ via the integration by parts, we get (3.2). □

Lemma 3.9. *If $0 < E_K(v) < d$ for some $v \in H^1(K)$, and $\delta_1 < 1 < \delta_2$ are the two roots of equation $d(\delta) = E_K(v)$, then the sign of $D_{K,\delta}(v)$ does not change for $\delta_1 < \delta < \delta_2$.*

Proof. Since $E_K(v) > 0$, we have $\|v\|_{2,K} \neq 0$. If the sign of $D_{K,\delta}(v)$ is changeable for $\delta_1 < \delta < \delta_2$, then we choose $\bar{\delta} \in (\delta_1, \delta_2)$ such that $D_{K,\bar{\delta}}(v) = 0$. Hence, by the definition of $d(\bar{\delta})$, we can obtain $E_K(v) \geqslant d(\bar{\delta})$, which contradicts $E_K(v) = d(\delta_1) = d(\delta_2) < d(\bar{\delta})$ (by Lemma 2.4 (3)). □

Definition 3.10 (Maximal existence time)**.** Assume that $v(t)$ is a weak solution of equation (2.1). The maximal existence time T of $v(t)$ is defined as follows:

(1) If $v(t)$ exists for $0 \leqslant t < \infty$, then $T = +\infty$.

(2) If there is a $t_0 \in (0, \infty)$ such that $v(t)$ exists for $0 \leqslant t < t_0$, but doesn't exist at $t = t_0$, then $T = t_0$.

Proof of Theorem 3.1. (1) Let $v(t)$ be any weak solution of equation (2.1) with $E_K(v_0) = e$, $D_K(v_0) > 0$, and T be the maximal existence time of $v(t)$. Using $E_K(v_0) = e, D_K(v_0) > 0$ and Lemma 3.9, we have $D_{K,\delta}(v_0) > 0$ and $E_K(v_0) < d(\delta)$. So $v_0(x) \in W_\delta$ for $\delta_1 < \delta < \delta_2$. We need to prove that $v(t) \in W_\delta$ for $\delta_1 < \delta < \delta_2$ and $0 < t < T$. Indeed, if this is not the conclusion, from time continuity of $D_K(v)$ we assume that there must exist a $\delta_0 \in (\delta_1, \delta_2)$ and $t_0 \in (0, T)$ such that $v(t_0) \in \partial W_{\delta_0}$, and $D_{\delta_0}(v(t_0)) = 0, \|v(t_0)\| \neq 0$ or $E_K(v(t_0)) = d(\delta_0)$. From the energy equality

$$\int_0^t \int_\Omega |v_\tau|^2 + E_K(v(t)) = E_K(v_0) < d(\delta),\ \delta_1 < \delta < \delta_2, \quad 0 \leqslant t < T, \tag{3.3}$$

we easily know that $E_K(v(t_0)) \neq d(\delta_0)$. If $D_{K,\delta_0}(v(t_0)) = 0, \|v(t_0)\| \neq 0$, then by the definition of $d(\delta)$ we obtain $E_K(v(t_0)) \geqslant d(\delta_0)$, which contradicts (3.3).

(2) Let $v(t)$ be any weak solution of equation (2.1) with $E_K(v_0) = e$, $D_K(v_0) < 0$, and T be the maximal existence time of $v(t)$. Using $E_K(v_0) = e, D_K(v_0) < 0$ and Lemma 3.9, we have $D_\delta(u_0) < 0$ and $E_K(v_0) < d(\delta)$. So $u_0 \in V_\delta$ for $\delta_1 < \delta < \delta_2$. We need to prove that $v(t) \in V_\delta$ for $\delta_1 < \delta < \delta_2$ and $0 < t < T$. Indeed, if this is not the conclusion, from time continuity of $D_K(v)$ we assume that there must exist a $\delta_0 \in (\delta_1, \delta_2)$ and $t_0 \in (0, T)$ such that $v(t_0) \in \partial V_{\delta_0}$, and $D_{K,\delta_0}(v(t_0)) = 0$, or $E_K(v(t_0)) = d(\delta_0)$. From the energy equality (3.3), we easily know that $E(v(t_0)) \neq d(\delta_0)$. If $D_{K,\delta_0}(v(t_0)) = 0$, and t_0 is the first time such that $D_{K,\delta_0}(v(t)) = 0$, then $D_{K,\delta_0}(v(t)) < 0$ for $0 \leqslant t < T$. By Lemma (2.3) (2), we have $\|v(t_0)\| > r(\delta_0)$ for $0 \leqslant t < T$. So, $\|v(t_0)\| > r(\delta_0)$ and $E_K(v(t_0)) \neq d(\delta_0)$, which contradicts (3.3). As required. □

Proof of Theorem 3.2. From the standard argument in [5], we can prove the local existence result of (2.1) in a more general case of initial value $v_0 \in H^1(K)$ and $v \in C^0([0, T_0], H^1(K))$.

Using $E_K(v_0) < d, D_K(v_0) > 0$ and Lemma 3.9, we have $D_\delta(v_0) > 0$ and $E_K(v_0) < d(\delta)$. So $v_0(x) \in W_\delta$ for $\delta_1 < \delta < \delta_2$. We need to prove that $v(t) \in W_\delta$ for $\delta_1 < \delta < \delta_2$ and $0 < t < T$. Indeed, if this is not the conclusion, from time continuity of $D_K(v)$ we assume that there must exist a $\delta_0 \in (\delta_1, \delta_2)$ and $t_0 \in (0, T)$ such that $v(t_0) \in \partial W_{\delta_0}$, and $D_{K,\delta_0}(v(t_0)) = 0, \|v(t_0)\| \neq 0$ or $E_K(v(t_0)) = d(\delta_0)$. From the energy equality

$$\int_0^t \int_\Omega |v_\tau|^2 + E_K(v(t)) = E_K(v_0) < d(\delta),\ \delta_1 < \delta < \delta_2, \quad 0 \leqslant t < T, \tag{3.4}$$

we easily know that $E_K(v(t_0)) \neq d(\delta_0)$. If $D_{K,\delta_0}(v(t_0)) = 0, \|v(t_0)\| \neq 0$, then by the definition of $d(\delta)$ we obtain $E_K(v(t_0)) \geqslant d(\delta_0)$, which contradicts (3.3). □

Remark 3.11. If in Theorem 3.2 the condition $D_{\delta_2}(u_0) > 0$ is replaced by $\|u_0\| < r(\delta_2)$, then equation (2.1) has a global weak solution $u(t) \in L^\infty(0, \infty; H^1(K))$ with $u_t(t) \in L^2(0, \infty; H^1(K))$ and the following result holds

$$\|u\| < \frac{d(\delta)}{a(\delta)}, \quad \delta_1 < \delta < \delta_2, 0 \leqslant t < \infty, \tag{3.5}$$

$$\int_0^t |u_\tau|^2 \, d\tau < d(\delta), \quad \delta_1 < \delta < \delta_2, 0 \leqslant t < \infty. \tag{3.6}$$

In particular

$$\|u\|^2 < \frac{d(\delta_1)}{a(\delta_1)}, \tag{3.7}$$

$$\int_0^t |u_\tau|^2 \, d\tau < d(\delta_1), \quad 0 \leqslant t < \infty. \tag{3.8}$$

Proof of Theorem 3.4. We argue by contradiction. Suppose that there would exist a global weak solution $v(t)$. Set

$$f(t) = \int_0^t \|v\|_{2,K}^2 \mathrm{d}\tau, t > 0. \tag{3.9}$$

Multiplying (2.1) by u and integrating over $R^N \times (0,t)$, we get

$$\|v(t)\|_{2,K}^2 - \|v_0\|_{2,K}^2 = -2\int_0^t (\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2 - \|v\|_{p+1,K}^{p+1}). \tag{3.10}$$

According to the definition of $f(t)$, we have $f'(t) = \|v(t)\|_{2,K}^2$ and hence

$$f'(t) = \|v(t)\|_{2,K}^2 = \|v_0\|_{2,K}^2 - 2\int_0^t (\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2 - \|v\|_{p+1,K}^{p+1}), \tag{3.11}$$

and

$$f''(t) = -2(\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2 - \|v\|_{p+1,K}^{p+1}) = -2D_K(v). \tag{3.12}$$

Now using (3.2), (3.12) and

$$\begin{aligned} E_K(v) &= \frac{1}{2}\|\nabla v\|_{2,K}^2 - \frac{\lambda}{2}\|v\|_{2,K}^2 - \frac{1}{p+1}\|v\|_{p+1,K}^{p+1} \\ &= \frac{p-1}{2(p+1)}(\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2) + \frac{1}{p+1}D_K(v), \end{aligned}$$

we can obtain

$$\begin{aligned} f''(t) &\geq 2(p+1)\int_0^t \|v_\tau(\tau)\|_{2,K}^2 \mathrm{d}\tau + S_\lambda(p-1)\|v\|_{2,K}^2 - 2(p+1)E_K(v_0) \\ &= 2(p+1)\int_0^t \|v_\tau(\tau)\|_{2,K}^2 \mathrm{d}\tau + S_\lambda(p-1)f'(t) - 2(p+1)E_K(v_0). \end{aligned} \tag{3.13}$$

Note that

$$\begin{aligned} f(t)f''(t) &= f(t)\left[2(p+1)\int_0^t \|v_\tau(\tau)\|_{2,K}^2 \mathrm{d}\tau + S_\lambda(p-1)f'(t) - 2(p+1)E_K(v_0)\right] \\ &= 2(p+1)\int_0^t \|v\|_{2,K}^2 \mathrm{d}\tau \int_0^t \|v_\tau(\tau)\|_{2,K}^2 \mathrm{d}\tau + S_\lambda(p-1)f(t)f'(t) \\ &\quad -2(p+1)E_K(v_0)\int_0^t \|v(\tau)\|_{2,K}^2 \mathrm{d}\tau. \end{aligned} \tag{3.14}$$

Hence, we have

$$\begin{aligned} f(t)f''(t) - \frac{p+1}{2}(f'(t))^2 &= 2(p+1)\int_0^t \|v\|_{2,K}^2 \mathrm{d}\tau \int_0^t \|v_\tau(\tau)\|_{2,K}^2 \mathrm{d}\tau \\ &\quad -2(p+1)\int_0^t (v_\tau, v)_K \mathrm{d}\tau + S_\lambda(p-1)f(t)f'(t) \\ &\quad -2(p+1)E_K(v_0)\int_0^t \|v(\tau)\|_{2,K}^2 \mathrm{d}\tau - (p+1)f'(t)\|v_0\|_{2,K}^2. \end{aligned} \tag{3.15}$$

Making use of the Schwartz inequality, we have

$$\begin{aligned} f(t)f''(t) - \frac{p+1}{2}(f'(t))^2 &\geq C^*(p-1)f(t)f'(t) - (p+1)f'(t)\|v_0\|_{2,K}^2 \\ &\quad -2(p+1)E_K(v_0)\int_0^t \|v(\tau)\|_{2,K}^2 \mathrm{d}\tau. \end{aligned} \tag{3.16}$$

Next, we distinguish two cases:

(1) If $E_K(u_0) \leqslant 0$, then

$$f(t)f''(t) - \frac{p+1}{2}(f'(t))^2 \geq C^*(p-1)f(t)f'(t) - (p+1)f'(t)\|v_0\|_{2,K}^2. \tag{3.17}$$

Now we prove $D_K(v) < 0$ for $t > 0$. If not, we must be allowed to choose a $t_0 > 0$ such that $D_K(v(t_0)) = 0$ and $D_K(v) < 0$ for $0 \leqslant t < t_0$. From Lemma 2.3 (2), we have $\|u\| > r(1)$ for $0 \leqslant t < t_0, \|v(t_0)\| \geqslant r(1)$ and $E_K(v(t_0)) \geqslant d$, which contradicts (3.3). From (3.12) we have $f'(t) > 0$ for $t \geqslant 0$. From $f'(0) = \|v(0)\|_{2,K}^2 \geqslant 0$, we can know that there exists a $t_0 \geqslant 0$ such that $f'(t_0) > 0$. For $t \geqslant t_0$ we have

$$f(t) \geqslant f'(t_0)(t - t_0) > f'(0)(t - t_0). \tag{3.18}$$

Hence, for sufficiently large t , we obtain

$$f(t) > (p+1)\|v_0\|_{2,K}^2, \tag{3.19}$$

then

$$f(t)f''(t) - \frac{p+1}{2}(f'(t))^2 > 0.$$

(2) If $0 < E_K(v_0) < d$, then by Theorem 3.1 we have $v(t) \in V_\delta$ for $1 < \delta < \delta_2, t \geqslant 0$, and $D_\delta(v) < 0$, $\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2 > r(\delta)$ for $1 < \delta < \delta_2, t \geqslant 0$, where δ_2 is the larger root of equation $d(\delta) = E_K(v_0)$. Hence, $D_{\delta_2}(v) \leqslant 0$ and $\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2 \geqslant r(\delta_2)$ for $t \geqslant 0$. By (3.12), we have

$$\begin{aligned} f''(t) &= -2D_K(v) = 2(\delta_2 - 1)(\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2) - 2D_{\delta_2}(v), \\ &\geqslant 2(\delta_2 - 1)(\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2) \geqslant 2(\delta_2 - 1)r^2(\delta_2), \quad t \geqslant 0, \\ f'(t) &\geqslant 2(\delta_2 - 1)r^2(\delta_2)t + f'(0) \geqslant 2(\delta_2 - 1)r^2(\delta_2)t, \quad t \geqslant 0, \\ f(t) &\geqslant (\delta_2 - 1)r^2(\delta_2)t^2, \quad t \geqslant 0. \end{aligned} \tag{3.20}$$

Therefore, for sufficiently large t, we infer

$$\frac{S_\lambda(p-1)}{2}f(t) > (p+1)\|v_0\|_{2,K}^2, \quad \frac{S_\lambda(p-1)}{2}f'(t) > 2(p+1)E_K(v_0). \tag{3.21}$$

Then, (3.16) implies that

$$\begin{aligned} f(t)f''(t) - \frac{p+1}{2}(f'(t))^2 \geq\ & S_\lambda(p-1)f(t)f'(t) - (p+1)f'(t)\|v_0\|_{2,K}^2 \\ & - 2(p+1)f(t)E_K(v_0)\int_0^t \|v(\tau)\|_{2,K}^2. \\ =\ & \left(\frac{S_\lambda(p-1)}{2}f(t) - (p+1)\|v_0\|_{2,K}^2\right)f'(t) \\ & + \left(\frac{S_\lambda(p-1)}{2}f'(t) - 2(p+1)E_K(v_0)\right)f(t) > 0. \end{aligned}$$

The remainder of the proof is the same as that in [12]. □

Proof of Theorem 3.6. Multiplying (2.1) by w, $w \in L^\infty\left(0,\infty;H^1(K)\right)$, we have

$$(v_t,w)_K + (\nabla v,\nabla w)_K = \left(|v|^{p-1}v + \frac{v}{p-1}, w\right)_K. \tag{3.22}$$

Letting $w = v$, (3.22) implies that

$$\frac{1}{2}\frac{d}{dt}\|v\|_{2,K}^2 + D_K(v) = 0, \quad 0 \leqslant t < \infty. \tag{3.23}$$

From $0 < E_K(v_0) < d, D_K(v_0) > 0$ and Lemma 3.1, we have $v(t) \in W_\delta$ for $\delta_1 < \delta < \delta_2$ and $0 \leqslant t < \infty$, where $\delta_1 < \delta_2$ are the two roots of equation $d(\delta) = E_K(v_0)$. Hence, we obtain $D_{K,\delta}(v) \geqslant 0$ for $\delta_1 < \delta < \delta_2$ and $D_{K,\delta_1}(v) \geqslant 0$ for $0 \leqslant t < \infty$. So, (3.23) gives

$$\frac{1}{2}\frac{d}{dt}\|v\|_{2,K}^2 + (1-\delta_1)(\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2) + D_{K,\delta_1}(v) = 0, \quad 0 \leqslant t < \infty. \tag{3.24}$$

Now (3.23) implies that

$$\frac{1}{2}\frac{d}{dt}\|v\|_{2,K}^2 + S_\lambda(1-\delta_1)\,\|v\|_{2,K}^2 \le 0, \quad 0 \leqslant t < \infty. \tag{3.25}$$

and

$$\|v\|_{2,K}^2 \leqslant \|v_0\|_{2,K}^2 - 2S_\lambda\,(1-\delta_1)\int_0^t |v(\tau)|^2 d\tau, \quad 0 \leqslant t < \infty. \tag{3.26}$$

By Gronwall's inequality, we have

$$|v|_{2,K}^2 \leqslant |v_0|_{2,K}^2\, e^{-2S_\lambda(1-\delta_1)t}, \quad 0 \leqslant t < \infty. \tag{3.27}$$

This completes the proof. □

4 Critical initial energy $E_K(v_0) = d$

The goal of this section is to prove Theorem 4.1–4.4.

Theorem 4.1 (Global existence)**.** *Assume that $v_0 \in H^1(K)$,$E(v_0) = d$ and $D_K(v_0) \geqslant 0$. Then equation (2.1) has a global weak solution $u(t) \in L^\infty(0,\infty;H^1(K))$ and $v(t) \in \overline{W} = W \cup \partial W$ for $0 \leqslant t < \infty$.*

Lemma 4.2. *Assume that $v \in H^1(K)$, $\|\nabla v\|_2^2 \neq 0$, and $D_K(v) \geq 0$. Then:*

(1) $\lim_{\mu\to 0} E_K(\lambda v) = 0, \lim_{\mu\to+\infty} E_K(\mu v) = -\infty$,

(2) *On the interval $0 < \mu < \infty$, there exists a unique $\mu^* = \mu^*(u)$, such that*

$$\frac{d}{d\mu}E_K(\mu v)|_{\mu=\mu^*} = 0, \tag{4.1}$$

(3) *$E_K(\mu v)$ is increasing on $0 \leqslant \mu \leqslant \mu^*$, decreasing on $\mu^* \leqslant \mu < \infty$ and takes the maximum at $\mu = \mu^*$,*

(4) *$D_K(\mu v) > 0$ for $0 < \mu < \mu^*$, $D_K(\mu v) < 0$ for $\mu^* < \mu < \infty$, and $D_K(\mu^* v) = 0$.*

Proof. (1) Firstly, from the definition of $E_K(v)$, i.e.

$$E_K(v) = \frac{1}{2}\|\nabla v\|_{2,K}^2 - \frac{\lambda}{2}\|v\|_{2,K}^2 - \frac{1}{p+1}\|v\|_{p+1,K}^{p+1}$$

and we see that

$$E_K(\mu v) = \frac{1}{2}\|\nabla \mu v\|_{2,K}^2 - \frac{\lambda}{2}\|\mu v\|_{2,K}^2 - \frac{1}{p+1}\|\mu v\|_{p+1,K}^{p+1}.$$

Hence, we have

$$\lim_{\mu\to 0} E_K(\mu v) = 0 \quad \text{and} \quad \lim_{\mu\to+\infty} E_K(\mu u) = -\infty. \tag{4.2}$$

(2) It is easy to show that

$$\frac{d}{d\mu}E_K(\mu v) = \mu\|\nabla v\|_{2,K}^2 - \mu\lambda\|v\|_{2,K}^2 - \mu^p\|v\|_{p+1,K}^{p+1},$$

which leads to the conclusion.

(3) By Lemma 4.2 (2), one has

$$\begin{aligned} &\frac{d}{d\mu}E_K(\mu v) > 0 \quad \text{for } 0 < \mu < \mu^*, \\ &\frac{d}{d\mu}E_K(\mu v) < 0 \quad \text{for } \mu^* < \mu < \infty, \end{aligned} \tag{4.3}$$

which leads to the conclusion.

(4) The conclusion follows from

$$D_K(\mu v) = \frac{d}{d\mu}E_K(\mu v) = \mu\|\nabla v\|_{2,K}^2 - \mu\lambda\|v\|_{2,K}^2 - \mu^p\|v\|_{p+1,K}^{p+1}.$$

As desired. □

Proof of Theorem 4.1. Firstly, $E_K(v_0) = d$ implies that $\|v_0\|_{H^1(K)} \neq 0$. Choose a sequence $\{\mu_m\}$ such that $0 < \mu_m < 1$, $m = 1, 2, \ldots$ and $\mu_m \to 1$ as $m \to \infty$. Let $v_{0m} = \mu_m v_0$. We consider the following initial problem

$$\begin{cases} v_s + Lv = |v|^{p-1}v + \dfrac{1}{p-1}v & \text{in } \mathbb{R}^N \times (0, S), \\ v|_{s=0} = v_{0m} & \text{in } \mathbb{R}^N. \end{cases} \tag{4.4}$$

From $D_K(v_0) \geqslant 0$ and Lemma 4.2, we have $\mu^* = \mu^*(u_0) \geqslant 1$. Thus, we get $D_K(v_{0m}) = D_K(\mu_m v_0) > 0$ and $E_K(v_{0m}) = E_K(\mu_m v_0) < E_K(v_0) = d$. From Theorem 3.2, it follows that for each m problem (4.4) admits a global weak solution $v_m(t) \in L^\infty(0,\infty; H^1(K))$ with $v_{mt}(t) \in L^2(0,\infty; H^1(K))$ and $v_m(t) \in W$ for $0 \leqslant t < \infty$ satisfying

$$(v_{m,t}, w)_K + (\nabla v_{m,t}, \nabla w)_K = \left(|v|^{p-1}v + \frac{v}{p-1}, w\right)_K, \quad \text{for all } w \in H^1(K), t > 0. \tag{4.5}$$

$$\int_0^t \|v_{m,\tau}\|_{2,K}^2 + E_K(v_m(t)) = E_K(v_{0m}) < d, \quad 0 \leqslant t < \infty, \tag{4.6}$$

which implies that

$$E_K(v_m) = \frac{1}{2}\|\nabla v_m\|_{2,K}^2 - \frac{\lambda}{2}\|v_m\|_{2,K}^2 - \frac{1}{p+1}\|v_m\|_{p+1,K}^{p+1}$$
$$= \frac{p-1}{2(p+1)}(\|\nabla v_m\|_{2,K}^2 - \lambda\|v_m\|_{2,K}^2) + \frac{1}{p+1}D_K(v_m). \tag{4.7}$$

Therefore, one has

$$\int_0^T \|v_{m,\tau}\|_{2,K}^2 d\tau + \frac{p-1}{2(p+1)}(\|\nabla v_m\|_{2,K}^2 - \lambda\|v_m\|_{2,K}^2) < d, \quad 0 \leqslant t < \infty. \tag{4.8}$$

The remainder of the proof is similar to the proof of Theorem 3.2. □

Theorem 4.3 (Blow-up). *Assume that $v_0 \in H^1(K)$, $E_K(v_0) = d$ and $D(v_0) > 0$, Then the existence time of weak solution for equation (2.1) is finite.*

Proof of Theorem 4.3. Let $v(t)$ be any weak solution of equation (2.1) with $E_K(v_0) = d$ and $D_K(v_0) < 0$, T be the existence time of $v(t)$. We next prove $T < \infty$. We argue by contradiction. Suppose that there would exist a global weak solution $v(t)$. Set

$$f(t) = \int_0^t \|v\|_{2,K}^2 d\tau, t > 0. \tag{4.9}$$

Multiplying (2.1) by u and integrating over $R^N \times (0,t)$, we get

$$\|v(t)\|_{2,K}^2 - \|v_0\|_{2,K}^2 = -2\int_0^t (\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2 - \|v\|_{p+1,K}^{p+1}). \tag{4.10}$$

According to the definition of $f(t)$, we have $f'(t) = \|v\|_{2,K}^2$ and hence

$$f'(t) = \|v(t)\|_{2,K}^2 = \|v_0\|_{2,K}^2 - 2\int_0^t (\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2 - \|v\|_{p+1,K}^{p+1}), \tag{4.11}$$

and

$$f''(t) = -2(\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2 - \|v\|_{p+1,K}^{p+1}) = -2D_K(v). \tag{4.12}$$

Now using (3.2), (4.12) and

$$E_K(v) = \frac{1}{2}\|\nabla v\|_{2,K}^2 - \frac{\lambda}{2}\|v\|_{2,K}^2 - \frac{1}{p+1}\|v\|_{p+1,K}^{p+1}$$
$$= \frac{p-1}{2(p+1)}(\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2) + \frac{1}{p+1}D_K(v),$$

we can obtain

$$f''(t) \geq 2(p+1)\int_0^t \|v_\tau(\tau)\|_{2,K}^2 d\tau + S_\lambda(p-1)\|v\|_{2,K}^2 - 2(p+1)E_K(v_0)$$
$$= 2(p+1)\int_0^t \|v_\tau(\tau)\|_{2,K}^2 d\tau + S_\lambda(p-1)f'(t) - 2(p+1)E_K(v_0). \tag{4.13}$$

Note that

$$f(t)f''(t) = f(t)\left[2(p+1)\int_0^t \|v_\tau(\tau)\|_{2,K}^2 d\tau + C^*(p-1)f'(t) - 2(p+1)E_K(v_0)\right]$$
$$= 2(p+1)\int_0^t \|v\|_{2,K}^2 d\tau \int_0^t \|v_\tau(\tau)\|_{2,K}^2 d\tau + S_\lambda(p-1)f(t)f'(t)$$
$$-2(p+1)E_K(v_0)\int_0^t \|v(\tau)\|_{2,K}^2 d\tau. \tag{4.14}$$

Hence, we have

$$\begin{aligned} f(t)f''(t)-\frac{p+1}{2}(f'(t))^2 = {} & 2(p+1)\int_0^t \|v\|_{2,K}^2 \mathrm{d}\tau \int_0^t \|v_\tau(\tau)\|_{2,K}^2 \mathrm{d}\tau \\ & -2(p+1)\int_0^t (v_\tau, v)_K \mathrm{d}\tau + S_\lambda(p-1)f(t)f'(t) \\ & -2(p+1)E_K(v_0)\int_0^t \|v(\tau)\|_{2,K}^2 \mathrm{d}\tau - (p+1)f'(t)\|v_0\|_{2,K}^2. \end{aligned} \tag{4.15}$$

Hence, according to (4.15) and the Schwartz inequality, we obtain

$$\begin{aligned} f(t)f''(t)-\frac{p+1}{2}(f'(t))^2 \geq {} & S_\lambda(p-1)f(t)f'(t) - (p+1)f'(t)\|v_0\|_{2,K}^2 \\ & -2(p+1)f(t)E_K(v_0)\int_0^t \|v(\tau)\|_{2,K}^2 \mathrm{d}\tau. \\ = {} & \left(\frac{S_\lambda(p-1)}{2}f(t) - (p+1)\|v_0\|_{2,K}^2\right) f'(t) \\ & + \left(\frac{S_\lambda(p-1)}{2}f'(t) - 2(p+1)E_K\left(v_0\right)\right) f(t). \end{aligned} \tag{4.16}$$

On the other hand, from $E_K\left(v_0\right) = d > 0, D_K\left(v_0\right) < 0$ and the continuity of $E_K(v)$ and $D_K(v)$ with respect to t, it follows that there exists a sufficiently small $t_1 > 0$ such that $E_K\left(v\left(t_1\right)\right) > 0$ and $D_K(v) < 0$ for $0 \leqslant t \leqslant t_1$. Hence $(v_t, v)_K = -D_K(v) > 0, \|v_t\|_2 > 0$ for $0 \leqslant t \leqslant t_1$. So, using the continuity of $\int_0^t \|v_\tau\|_{2,K}^2 \, d\tau$, we can choose a t_1 such that

$$0 < d_1 = d - \int_0^{t_1} \|v_\tau\|_{2,K}^2 \, d\tau < d. \tag{4.17}$$

And by (3.4), we get

$$0 < E_K\left(v\left(t_1\right)\right) = d - \int_0^{t_1} \|v_\tau\|_{2,k}^2 \, d\tau = d_1 < d. \tag{4.18}$$

So we can choose $t = t_1$ as the initial time, then we obtain $v(t) \in V_\delta$ for $\delta \in (\delta_1, \delta_2), t_1 \leqslant t < \infty$, where (δ_1, δ_2) is the maximal interval including $\delta = 1$ such that $d(\delta) > d_1$ for $\delta \in (\delta_1, \delta_2)$. Thus we get $D_{K,\delta}(v) < 0$ and $\|v\| > r(\delta)$ for $\delta \in (1, \delta_2), t_1 \leqslant t < \infty$, and $D_{K,\delta_2}(v) \leqslant 0, \|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2 \geqslant r\left(\delta_2\right)$ for $t_1 \leqslant t < \infty$. Thus (4.12) implies that

$$\begin{aligned} f''(t) &= -2D_K(v) = 2\left(\delta_2-1\right)\left(\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2\right) - 2D_{\delta_{K,2}}(v), \\ &\geqslant 2\left(\delta_2-1\right) r\left(\delta_2\right), \quad t \geqslant t_1, \\ f'(t) &\geqslant 2\left(\delta_2-1\right) r\left(\delta_2\right)\left(t-t_1\right) + f'(t_1) \geqslant 2\left(\delta_2-1\right) r\left(\delta_2\right)\left(t-t_1\right), \quad t \geqslant 0, \\ f(t) &\geqslant \left(\delta_2-1\right) r\left(\delta_2\right)\left(t-t_1\right)^2 + M(t_1) > \left(\delta_2-1\right) r\left(\delta_2\right)\left(t-t_1\right)^2, \quad t \geqslant t_1. \end{aligned} \tag{4.19}$$

Therefore, for sufficiently large t, we infer

$$\frac{S_\lambda(p-1)}{2}f(t) > (p+1)\|v_0\|_{2,K}^2, \quad \frac{S_\lambda(p-1)}{2}f'(t) > 2(p+1)E_K\left(v_0\right). \tag{4.20}$$

Then, (4.16) implies that

$$\begin{aligned} f(t)f''(t) - \frac{p+1}{2}(f'(t))^2 \geq & \, S_\lambda(p-1)f(t)f'(t) - (p+1)f'(t)\|v_0\|_{2,K}^2 \\ & - 2(p+1)f(t)E_K(v_0)\int_0^t \|v(\tau)\|_{2,K}^2 \mathrm{d}\tau. \\ = & \left(\frac{S_\lambda(p-1)}{2}f(t) - (p+1)\|v_0\|_{2,K}^2\right) f'(t) \\ & + \left(\frac{S_\lambda(p-1)}{2}f'(t) - 2(p+1)E_K(v_0)\right) f(t) > 0. \end{aligned}$$

The remainder of the proof is the same as that in [12]. □

Theorem 4.4. *Assume that $u_0 \in H^1(K)$, $E_K(v_0) = d$ and $D_K(v_0) > 0$, $\delta_1 < \delta_2$ are the two roots of equation $d(\delta) = E_K(v_0)$. Then, for the global weak solution v of equation (2.1), it holds*

$$|v|_2^2 \leqslant |v_0|_2^2 e^{-2S_\lambda(1-\delta_1)t}, \quad 0 \leqslant t < \infty. \tag{4.21}$$

Proof of Theorem 4.4. We first know that equation (2.1) has a global weak solution from Theorem 4.3. Furthermore, Using Theorem 3.4, Theorem 4.3 and (3.3), if $v(t)$ is a global weak solution of equation (2.1) with $E_K(v_0) = d$, $D_K(v_0) > 0$, then must have $D_K(v) \geq 0$ for $0 \leq t < +\infty$. Next, we distinguish two cases:

(1) Suppose that $D_K(v) > 0$ for $0 \leqslant t < \infty$. Multiplying (2.1) by v, $v \in L^\infty(0,\infty;H^1(K))$, we have

$$(v_t, w)_K + (\nabla v_t, \nabla w)_K = \left(|v|^{p-1}v + \frac{v}{p-1}, w\right)_K, \quad \text{for all } w \in H^1(K), t > 0. \tag{4.22}$$

Letting $w = v$, (4.22) implies that

$$\frac{1}{2}\frac{d}{dt}\|v\|_{2,K}^2 = -D_K(v) < 0, \quad 0 \leqslant t < \infty. \tag{4.23}$$

Since $\|v_t\|_{2,K} > 0$, we have that $\int_0^t \|v_\tau\|^2 d\tau$ is increasing for $0 \leqslant t < \infty$. By choosing any $t_1 > 0$ such that

$$0 < d_1 = d - \int_0^{t_1} \|v_\tau\|_{2,K}^2 d\tau < d. \tag{4.24}$$

From (3.3), if follows that $0 < E_K(v) \leq d_1 < d$, and $v(t) \in W_\delta$ for $\delta_1 < \delta < \delta_2$ and $0 \leqslant t < \infty$, where $\delta_1 < \delta_2$ are the two roots of equation $d(\delta) = E_K(v_0)$. Hence, we obtain $D_{K,\delta_1}(v) \geqslant 0$ for $\delta_1 < \delta < \delta_2$ and $D_{K,\delta_1}(v) \geqslant 0$ for $t_1 \leqslant t < \infty$. So, (4.23) gives

$$\frac{1}{2}\frac{d}{dt}\|v\|_{2,K}^2 + (1-\delta_1)|v|_2^2 + D_{K,\delta}(v) = 0, \quad t_1 \leqslant t < \infty. \tag{4.25}$$

$$\frac{1}{2}\frac{d}{dt}\|v\|_{2,K}^2 + (1-\delta_1)(\|\nabla v\|_{2,K}^2 - \lambda\|v\|_{2,K}^2) + D_{K,\delta_1}(v) = 0, \quad 0 \leqslant t < \infty. \tag{4.26}$$

Now (4.23) implies that

$$\frac{1}{2}\frac{d}{dt}\|v\|_{2,K}^2 + S_\lambda(1-\delta_1)\|v\|_{2,K}^2 \leq 0, \quad 0 \leqslant t < \infty. \tag{4.27}$$

and

$$\|v\|_{2,K}^2 \leqslant \|v_0\|_{2,K}^2 - 2S_\lambda(1-\delta_1)\int_0^t |v(\tau)|^2 d\tau, \quad 0 \leqslant t < \infty. \tag{4.28}$$

and By Gronwall's inequality, we have

$$|v|_{2,K}^2 \leqslant |v_0|_2^2 e^{-2S_\lambda(1-\delta_1)t}, \quad 0 \leqslant t < \infty. \tag{4.29}$$

(2) Suppose that there exists a $t_1 > 0$ such that $D_K(v(t_1)) = 0$ and $D_K(v) > 0$ for $0 \leqslant t < t_1$. Then, $|u_t|_2 > 0$ and $\int_0^t |v_\tau|_2^2 d\tau$ is increasing for $0 \leqslant t < t_1$. By (4.24) we have

$$E_K(v(t_1)) = d - \int_0^{t_1} |v_\tau|_2^2 d\tau < d, \tag{4.30}$$

and $\|v(t_1)\| = 0$. Then, we have that $v(t) \equiv 0$ for $t_1 \leqslant t < \infty$.

Hence, the proof is complete. □

5 High initial energy $E_K(v_0) > d$

In this section, we investigate the conditions to ensure the existence of global solutions or blow-up solutions to system (2.1) with $E_K(v_0) > d$.

Lemma 5.1. *For any $\alpha > d, \lambda_\alpha$ and Λ_α defined in (2.1) satisfy*

$$0 < \lambda_\alpha \leq \Lambda_\alpha < +\infty. \tag{5.1}$$

Proof. (1) By Hölder's inequality, fundamental inequality and $u \in \mathcal{N}$, we have

$$\|\nabla v\|_{2,K}^2 - \lambda \|v\|_{2,K}^2 = \|v\|_{p+1,K}^{p+1}. \tag{5.2}$$

Then from Lemma 2.6 (1), we have $\lambda_\alpha > 0$.

Using Lemma 2.1 and $u \in \mathcal{N}$, we have

$$\|\nabla v\|_{2,K}^2 - \lambda \|v\|_{2,K}^2 = \|v\|_{p+1,K}^{p+1} \leqslant \left(\frac{1}{S_\lambda} (\|\nabla v\|_{2,K}^2 - \lambda \|v\|_{2,K}^2) \right)^{\frac{p+1}{2}}. \tag{5.3}$$

So we have $\|\nabla v\|_{2,K}^2 - \lambda \|v\|_{2,K}^2 \leq \frac{1}{S_\lambda}$ which leads to the conclusion. □

Theorem 5.2. *Suppose that $E_K(v_0) > d$, then we have*

(1) *If $v_0 \in \mathcal{N}_+$ and $\|v_0\|_{2,k} \leq \lambda_{E_K(v_0)}$, then $v_0 \in \mathcal{G}_0$,*

(2) *If $v_0 \in \mathcal{N}_-$ and $\|v_0\|_{2,k} \geq \Lambda_{E_K(v_0)}$, then $v_0 \in \mathcal{B}$.*

Proof. The maximal existence time of the solutions to system (2.1) with initial value v_0 is denoted by T_0. If the solution is global, i.e. $T(v_0) = +\infty$, the limit set of v_0 is denoted by ω_0.

(1) Suppose that $v_0 \in \mathcal{N}_+$ with $|v_0|_2 \leq \lambda_{E_K(v_0)}$. We firstly prove that $v(t) \in \mathcal{N}_+$ for all $t \in [0, T(v_0))$. Assume, on the contrary, that there exists a $t_0 \in (0, T(v_0))$ such that $v(t) \in \mathcal{N}_+$ for $0 \leq t < t_0$ and $v(t_0) \in \mathcal{N}$. It follows from $D_K(v(t)) = -\int_\Omega v_t(x,t)v(x,t)\mathrm{d}x$ that $v_t(x,t) \neq 0$ for $(x,t) \in \Omega \times (0, t_0)$. Recording to(3.2) we then have $E_K(v(t_0)) < E_K(v_0)$, which implies that $u(t_0) \in E_K^{E_K(v_0)}$. Therefore, $v(t_0) \in \mathcal{N}^{E_K(v_0)}$. Recalling the definition of $\lambda_{E_K(v_0)}$, we get

$$|u(t_0)|_2 \geq \lambda_E(v_0). \tag{5.4}$$

Since $D_K(v(t)) > 0$ for $t \in [0, t_0)$, we obtain from (3.23) that

$$|v(t_0)|_2 < |v_0|_2 \leq \lambda_{E_K(v_0)}, \tag{5.5}$$

which contradicts (5.4). Hence, $v(t) \in \mathcal{N}_+$ which shows that $v(t) \in E_K^{E_K(v_0)}$ for all $t \in [0, T(v_0))$. Now Lemma 3.9 (2) implies that the orbit $\{v(t)\}$ remains bounded in $H^1(K)$ for $t \in [0, T(v_0))$ so that $T(v_0) = \infty$. Assume that ω is an arbitrary element in $\omega(v_0)$. Then by (3.2) and (3.23) we obtain

$$|\omega|_2 > \Lambda_{E_K(v_0)}, \quad E_K(\omega) < E_K(v_0), \tag{5.6}$$

which, according to the definition of $\lambda_{E_K(v_0)}$ again, implies that $\omega(v_0) \cap N = \emptyset$. So, $\omega(v_0) = \{0\}$, i.e. $v_0 \in \mathcal{G}_0$.

(2) Suppose that $v_0 \in \mathcal{N}_-$ with $|v_0|_2 \geq \Lambda_{E_K(v_0)}$. We now prove that $v(t) \in \mathcal{N}_-$ for all $t \in [0, T(v_0))$. Assume, on the contrary, that there exists a $t^0 \in (0, T(v_0))$ such that $v(t) \in \mathcal{N}_-$ for $0 \leq t < t^0$ and $v(t^0) \in \mathcal{N}$. Similarly to case (1), one has $E_K(v(t^0)) < E_K(v_0)$, which implies that $v(t^0) \in E_K^{E_K(v_0)}$. Therefore, $v(t^0) \in \mathcal{N}^{E_K(v_0)}$. Recalling the definition of $\Lambda_{E_K(v_0)}$, we infer

$$|v(t^0)|_2 \leq \Lambda_{E_K(v_0)}. \tag{5.7}$$

On the other hand, from (3.23) and the fact that $D_K(v(t)) < 0$ for $t \in [0, t^0)$, we obtain

$$|v(t^0)|_2 > |v_0|_2 \geq \Lambda_{E_K(v_0)}, \tag{5.8}$$

which contradicts (5.7).

Assume that $T(v_0) = \infty$. Then for each $\omega \in \omega(v_0)$, it follows from by (3.2) and (3.23) that

$$\|\omega\|_2 > \Lambda_{E_K(v_0)}, \quad E_K(\omega) < E_K(v_0). \tag{5.9}$$

Noting the definition of $\Lambda_{E_K(v_0)}$ again, we have $\omega(v_0) \cap N = \emptyset$. Hence, it is holded that $\omega(v_0) = \{0\}$, which contradicts Lemma 3.9 (1). Therefore, $T(v_0) < \infty$. This ends the proof. □

Theorem 5.3. *Assume that $v_0 \in H^1(K)$ satisfies*

$$E_K(v_0) \leq \|v_0\|_{2,K} < \frac{p}{p+1}\|v_0\|_{p+1,K}^{p+1}, \tag{5.10}$$

Then, $v_0 \in \mathcal{N}_- \cap \mathcal{B}$.

Proof. Firstly, we observe

$$\begin{aligned} E_K(v_0) &= \frac{1}{2}\|\nabla v_0\|_{2,K}^2 - \frac{\lambda}{2}\|v_0\|_{2,K}^2 - \frac{1}{p+1}\|v_0\|_{p+1,K}^{p+1} \\ &= \frac{1}{2}D_K(v_0) + \frac{p}{p+1}\|v_0\|_{p+1,K}^{p+1}. \end{aligned} \tag{5.11}$$

Thus, we have

$$E_K(v_0) - \frac{p}{p+1}\|v_0\|_{p+1,K}^{p+1} = \frac{1}{2}D_K(v_0) < 0, \tag{5.12}$$

which shows that $v_0 \in \mathcal{N}_-$. Then for any $v \in \mathcal{N}_{E_K(v_0)}$, one has

$$\|v\|_{p+1,K}^{p+1} = \|\nabla v\|_{2,K}^2 - \lambda \|v\|_{2,K}^2 \le E_K(v_0) \le \sqrt{\frac{2(p+1)}{p-1}} E_K(v_0).$$

Taking supremum over $\mathcal{N}_{E_K(v_0)}$ and (5.10), by Theorem 5.2 we can deduce

$$\|v_0\|_2 \ge \Lambda_{E_K(v_0)}.$$

Thus, $v_0 \in \mathcal{N}_- \cap \mathcal{B}$. This finishes the proof. □

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Isolated periodic wave trains in a generalized Burgers–Huxley equation

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Abstract. We study the isolated periodic wave trains in a class of modified generalized Burgers–Huxley equation. The planar systems with a degenerate equilibrium arising after the traveling transformation are investigated. By finding certain positive definite Lyapunov functions in the neighborhood of the degenerate singular points and the Hopf bifurcation points, the number of possible limit cycles in the corresponding planar systems is determined. The existence of isolated periodic wave trains in the equation is established, which is universal for any positive integer n in this model. Within the process, one interesting example is obtained, namely a series of limit cycles bifurcating from a semi-hyperbolic singular point with one zero eigenvalue and one non-zero eigenvalue for its Jacobi matrix.

Keywords: generalized Burgers–Huxley equation, isolated periodic wave solution, positive definite Lyapunov function, degenerate singular point.

2020 Mathematics Subject Classification: 35B32, 34C07, 34D20, 37J20, 35Q51.

1 Introduction

The Burgers–Huxley equation is a well-known nonlinear partial differential equation simulating nonlinear wave phenomena in physics, biology, economics and ecology. In the relation with the in-depth study of practical problems the following generalized Burgers–Huxley equation

$$u_t + \alpha u^n u_x - u_{xx} = \beta u(1 - u^n)(u^n - \gamma) \tag{1.1}$$

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was introduced in [25]. In this paper we consider the further generalization of (1.1) described by the equation

$$u_t + \left(\alpha_0 + \sum_{i=1}^{n} \alpha_i u^i\right) u_x - u_{xx} = \beta u(1-u^n)(u^n - \gamma) \tag{1.2}$$

where α_i, β, γ are real numbers and $n \in \mathbb{N}$. In equation (1.2) a more general version of the convective effect is introduced. Although this modification may go beyond the actual background, it is still related to some real models, for example, when $n = 1$ it corresponds to the single-species model with density-dependent migrations [24]. Of course, we also have to come up with some early models, for example, Burgers equations [3], Burgers–Huxley equation [30, 32], Fitzhugh–Nagumo equation [8, 11], Newwell–Whitehead equation [19].

Regarding to the exact solutions of equation (1.1) we note that some solitary wave solutions were obtained in [6, 29]. Recently via the (G'/G)-expansion method the authors of [22] also obtained a series of exact solutions. In respect to the approximate analytical solutions and numerical solutions of equation (1.1) we can refer to the results in [4, 10, 21] and the references given there. As to the isolated periodic wave solutions, considering the bifurcations of codimension 1 and 2 in the traveling wave system the authors of [32] determined the existence of some bounded traveling waves for the Burgers–Huxley equation. Latter on computing the singular point quantities of the non-degenerate center-focus type equilibrium the authors of [26] proved the existence of the isolated periodic wave solution in the non-degenerate case for equation (1.1).

In this paper we continue the investigation of the isolated periodic wave solutions (which can also be called the isolated periodic wave trains, see [23]) of more general model (1.2). First, we apply the usual approach assuming that equation (1.2) has a travelling wave solution in the form

$$u(x,t) = v(\xi), \qquad \xi = x - ct \tag{1.3}$$

where c is the propagation speed of the wave. Substituting (1.3) into (1.2) we obtain

$$v''(\xi) = -cv'(\xi) + \left(\alpha_0 + \sum_{i=1}^{n} \alpha_i v^i\right) v'(\xi) - \beta v(1-v^n)(v^n - \gamma). \tag{1.4}$$

Then setting $y = v'(\xi)$ we reduce (1.4) to the planar dynamic system

$$\begin{cases} \frac{\mathrm{d}v}{\mathrm{d}\xi} = y \triangleq X(v,y), \\ \frac{\mathrm{d}y}{\mathrm{d}\xi} = -cy + (\alpha_0 + \sum_{i=1}^{n} \alpha_i v^i)y - \beta v(1-v^n)(v^n-\gamma) \triangleq Y(v,y). \end{cases} \tag{1.5}$$

Applying the bifurcation theory of the planar dynamical system to system (1.5) it is possible to investigate the existence of periodic wave trains which correspond to a family of periodic orbits in a neighborhood of a center, the solitary wave solutions of the peak type which correspond to the smooth homoclinic orbits, and the monotone kink solitary wave solutions which correspond to the heteroclinic orbits. As has been indicated in [14] these cases are usually considered for integrable systems. Integrability conditions of system (1.5) are similar to the ones of its special case which has been investigated in [26].

In this paper we focus on the existence of isolated periodic travelling trains which are caused by the presence of limit cycles. Our main idea is to keep track of the conditions of limit cycle bifurcations which can occur in the vicinity of the equilibriums, in particular near the degenerate singular points of the traveling wave system (1.5).

Bifurcations of isolated periodic wave trains for the reaction-diffusion equation have been extensively studied (see [12,13,23,27,28] and references therein). These bifurcations are caused mainly by Hopf bifurcation or Poincaré bifurcation around one non-degenerate equilibrium of the corresponding planar traveling wave system. However the bifurcations of isolated periodic wave trains due to limit cycles bifurcating from one degenerate equilibrium are not well investigated. In this work a particular attention is focused on the cases of a nilpotent focus, a nilpotent node and a semi-hyperbolic singular point whose Jacobi matrix has two eigenvalues: one is zero, another is non-zero. Our main approach is to determine the quasi-Lyapunov constants of system (1.5) by constructing Lyapunov functions not only for the cases of degenerate equilibriums, but also for the multiple Hopf bifurcations of non-degenerate equilibriums.

The paper is organized as follows. In Section 2 the equilibriums and degenerate cases of system (1.5) are determined. Section 3 is devoted to the study of the quasi-Lyapunov constants for the nilpotent critical point of the degenerate planar system (1.5). In Section 4 we apply the positive definite Lyapunov functions to determine the quasi-Lyapunov constants for multiple Hopf bifurcation of the degenerate system (1.5). In the last section using the above analysis we give a general result for the isolated periodic wave trains for any positive integer n in model (1.2).

2 The equilibriums of system (1.5)

In this section we investigate the equilibriums of planar travelling system (1.5). Due to the practical background, the value of u in model (1.2) is nonnegative, thus we only investigate the dynamical behavior near the equilibrium points with $v \geq 0$ for system (1.5). We will focus on the limit cycle bifurcation in system (1.5) with one equilibrium as the degenerate singular point. It is easy to see that when $\gamma > 0$ system (1.5) has only three nonnegative equilibrium points: $(0,0), (1,0)$ and $(\sqrt[n]{\gamma}, 0)$, whereas when $\gamma \leq 0$, there exist only two nonnegative equilibrium points: $(0,0)$ and $(1,0)$.

For the Jacobian matrix at the origin we have

$$\begin{bmatrix} \frac{\partial X}{\partial v} & \frac{\partial X}{\partial y} \\ \frac{\partial Y}{\partial v} & \frac{\partial Y}{\partial y} \end{bmatrix}_{(0,0)} = \begin{bmatrix} 0 & 1 \\ \beta\gamma & \alpha_0 - c \end{bmatrix}. \tag{2.1}$$

Its two eigenvalues are

$$\lambda_{1,2} = \frac{1}{2}\left(\alpha_0 - c \pm \sqrt{(\alpha_0 - c)^2 + 4\beta\gamma}\right). \tag{2.2}$$

Thus, the origin of (1.5) is either a non-degenerate center or a weak focus if and only if $c = \alpha_0$ and $\beta\gamma < 0$. If $\beta\gamma = 0$ then the origin is a degenerate singular point, and the two eigenvalues are $\lambda_1 = 0, \lambda_2 = \alpha_0 - c$. When $\beta\gamma = 0$, according to the monographs [7, 31], we know that if $c \neq \alpha_0$ then the singular point is an elementary degenerate singular point, also called semi-hyperbolic, and when $c = \alpha_0$, the origin is a nilpotent critical point and the limit cycle bifurcation may happen at $(0,0)$.

For the Jacobian matrix at $(1,0)$,

$$\begin{bmatrix} \frac{\partial X}{\partial v} & \frac{\partial X}{\partial y} \\ \frac{\partial Y}{\partial v} & \frac{\partial Y}{\partial y} \end{bmatrix}_{(1,0)} = \begin{bmatrix} 0 & 1 \\ n\beta(1-\gamma) & \sum_{i=0}^{n} \alpha_i - c \end{bmatrix} \tag{2.3}$$

we obtain

$$\lambda_{1,2} = \frac{1}{2}\left[\sum_{i=0}^{n}\alpha_i - c \pm \sqrt{\left(\sum_{i=0}^{n}\alpha_i - c\right)^2 + 4n\beta(1-\gamma)}\right]. \tag{2.4}$$

It has a pair of conjugate pure imaginary eigenvalues and $(1,0)$ is either a non-degenerate center or a weak focus if and only if $c = \sum_{i=0}^{n}\alpha_i$ and $\beta(1-\gamma) < 0$. If $\beta(1-\gamma) = 0$ the singular point is degenerate since at least one of its two eigenvalues is zero. In this case, when $c \neq \sum_{i=0}^{n}\alpha_i$, the singular point is semi-hyperbolic, and when $c = \sum_{i=0}^{n}\alpha_i$, the equilibrium is a nilpotent critical point and limit cycle bifurcation may happen at $(1,0)$.

At the point $(\sqrt[n]{\gamma}, 0)$

$$\begin{bmatrix} \frac{\partial X}{\partial v} & \frac{\partial X}{\partial y} \\ \frac{\partial Y}{\partial v} & \frac{\partial Y}{\partial y} \end{bmatrix}_{(\gamma^{\frac{1}{n}},0)} = \begin{bmatrix} 0 & 1 \\ n\beta\gamma(\gamma-1) & \sum_{i=0}^{n}\alpha_i\gamma^{\frac{i}{n}} - c \end{bmatrix}. \tag{2.5}$$

Then the two eigenvalues are:

$$\lambda_{1,2} = \frac{1}{2}\left[\sum_{i=0}^{n}\alpha_i\gamma^{\frac{i}{n}} - c \pm \sqrt{\left(\sum_{i=0}^{n}\alpha_i\gamma^{\frac{i}{n}} - c\right)^2 + 4n\beta\gamma(\gamma-1)}\right]. \tag{2.6}$$

The Jacobian has a pair of conjugate pure imaginary eigenvalues and the point is non-degenerate, either a center or a focus if and only if $c = \sum_{i=0}^{n}\alpha_i\gamma^{\frac{i}{n}}$ and $\beta\gamma(\gamma-1) < 0$. Furthermore, if $\beta\gamma(\gamma-1) = 0$ the point is a degenerate singular point, and in this case, when $c \neq \sum_{i=0}^{n}\alpha_i\gamma^{\frac{i}{n}}$, the singular point is semi-hyperbolic, and when $c = \sum_{i=0}^{n}\alpha_i\gamma^{\frac{i}{n}}$, the equilibrium is a nilpotent critical point and the limit cycle bifurcation may occur at it.

Therefore we have the following conclusions.

Lemma 2.1. For system (1.5) there exists a degenerate nonnegative equilibrium point if and only if $\beta = 0$, or $\gamma = 0$ or $\gamma = 1$.

Lemma 2.2. For system (1.5), when $\gamma = 0$, only the origin is degenerate, and the equilibrium $(1,0)$ is non-degenerate. In this case, the origin is a nilpotent critical point if and only if $c = \alpha_0$, the equilibrium $(1,0)$ is a center or a focus if and only if $\beta < 0$.

Lemma 2.3. For system (1.5), when $\gamma = 1$, only the equilibrium $(v,y) = (1,0)$ is degenerate, and the origin is non-degenerate. In this case, the equilibrium $(1,0)$ is a nilpotent critical point if and only if $c = \sum_{i=0}^{n}\alpha_i$, the origin is a center or a focus if and only if $\beta < 0$.

Lemma 2.4. For system (1.5), when $\beta = 0$ the line $y = 0$ is a singular straight line on which each equilibrium point is non-isolated and degenerate, and there exists a first integral

$$H(v,y) = (c+\alpha_0)v + \frac{1}{2}\alpha_1 v^2 + \cdots + \frac{1}{n+1}\alpha_n v^{n+1} - y = h. \tag{2.7}$$

In this situation a limit cycle cannot exist.

3 Limit cycle bifurcations from the degenerate equilibriums

In this section we investigate the limit cycle bifurcations near the degenerate equilibriums of system (1.5). First we consider the real polynomial differential system

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = y + \sum_{k+j=2}^{\infty} a_{kj}x^k y^j = X(x,y), \\ \frac{\mathrm{d}y}{\mathrm{d}t} = \lambda y + \sum_{k+j=2}^{\infty} b_{kj}x^k y^j = Y(x,y) \end{cases} \tag{3.1}$$

where $x, y, t, \lambda, a_{kj}, b_{kj} \in \mathbb{R}\,(k, j \in \mathbb{N})$. Obviously, when $\lambda = 0$ the origin $O(0,0)$ is a nilpotent critical point and when $\lambda \neq 0$ the origin is a semi-hyperbolic singular point, whose Jacobi matrix has two eigenvalues: one is zero, another is non-zero.

In the case $\lambda = 0$, according to the implicit function theorem there is the unique function $y = y(x)$ which satisfies $X(x, y(x)) \equiv 0,\ y(0) = 0$ and

$$\begin{aligned} Y(x, y(x)) &= A_k x^k + o(x^k), \qquad A_k \neq 0, \\ \left[\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right]_{y=y(x)} &= B_j x^j + o(x^j) \end{aligned} \tag{3.2}$$

where $k, j \in \mathbb{N}^+$. From [2,20,31], it is known that only when $k = 2m+1, m \in \mathbb{N}^+$, the origin of $(3.1)_{\lambda=0}$ may be a center or a focus or a node, and moreover we have

Proposition 3.1. Suppose that the origin of $(3.1)_{\lambda=0}$ is a nilpotent singular point with multiplicity $k = 2m+1, m \in \mathbb{N}^+$ and $B_j \neq 0$ in (3.2). Then it is a center or a focus if and only if one of the following two conditions is satisfied:

$$\begin{aligned} &C_1: \quad m < j, \quad A_{2m+1} < 0; \\ &C_2: \quad m = j, \quad A_{2m+1} < 0, \quad B_j^2 + 4(m+1)A_{2m+1} < 0. \end{aligned} \tag{3.3}$$

Proposition 3.2. Suppose that the origin of $(3.1)_{\lambda=0}$ is a nilpotent singular point with multiplicity $k = 2m+1, m \in \mathbb{N}^+$. Then it is a node if and only if $B_j \neq 0$ (with $j = 2N, N \in \mathbb{N}^+$ in (3.2)) and one of the following two conditions is satisfied:

$$\begin{aligned} &C_3: \quad m > j, \quad A_{2m+1} < 0; \\ &C_4: \quad m = j, \quad A_{2m+1} < 0, \quad B_j^2 + 4(m+1)A_{2m+1} \geq 0. \end{aligned} \tag{3.4}$$

For the case of Proposition 3.1 the limit cycle bifurcation can be determined by resolving the center-focus problem for the nilpotent critical point (see e.g. [1,5,15–17]). For the case of Proposition 3.2 the limit cycle bifurcation from a nilpotent node was studied in [18]. However, when $\lambda \neq 0$, the origin of (3.1) is a semi-hyperbolic singular point and it can be either a degenerate node or a degenerate saddle, or a saddle-node [2,20,31].

In our study we use the Lyapunov function method to investigate the existence of limit cycles bifurcating from the nilpotent critical points or semi-hyperbolic singular points. As a particular case of the Lyapunov stability theory we have the following statement.

Lemma 3.3. Suppose that the origin of system (3.1) is an isolated degenerate singular point, and there exists a positive definite Lyapunov function $V(x, y)$ such that

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(3.1)} = \frac{\mathrm{d}V}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}V}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}t} = y^2[\beta_{2N}x^{2N} + o(x^{2N})], \qquad \beta_{2N} \neq 0, \tag{3.5}$$

where $(x, y) \in U(O, \delta)$ and δ is certain positive number. Then the origin is stable when $\beta_{2N} < 0$ and unstable when $\beta_{2N} > 0$.

It is easy to prove the following statement.

Theorem 3.4. Suppose that for system (3.1) there exists a positive definite Lyapunov function $V(x, y)$ such that

$$\begin{aligned} \left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(3.1)} &= \frac{\mathrm{d}V}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}V}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}t} \\ &= y^2\Big[\sum_{i=0}^{N} \beta_{2i}(x^{2i} + O(x^{2i+1})) + o(x^{2N+1})\Big] = M(x, y), \qquad \beta_{2N} \neq 0, \end{aligned} \tag{3.6}$$

where $N \in \mathbb{N}^+$, $(x, y) \in U(O, \delta)$, δ is certain positive number, $X^2 + Y^2$ is positive definite on $U(O, \delta)$ and β_{2i} are independent. Assume also that for a vector field χ^* from family (3.1) with fixed parameters $a_{ij} = a^*_{ij}$, $b_{ij} = b^*_{ij}$ the parameters β_{2i} in (3.6) satisfy $\beta_2 = \beta_4 = \cdots = \beta_{2N-2} = 0, \beta_{2N} \neq 0$. Then it is possible to perturb vector field χ^* in such way, that the perturbed system has N small limit cycles in a neighborhood of the origin.

Proof. Let V be the Lyapunov function corresponding to vector field χ^*. Then for sufficiently small c the equation $V(x, y) = c$ defines a contour Γ located in U and surrounding the origin O. Since $X^2 + Y^2$ is positive definite we can assume that inside Γ there no limit cycles and singular points different from O.

Assume for determinacy that $\beta_{2N} < 0$. Then Γ bounds a positive invariant set Ω and all trajectories in Ω tend to the origin. Note also that $M(x, y)/y^2 < \delta < 0$ on Γ. Therefore under sufficiently small perturbations the vector field of the perturbed system is still directed inside Γ.

Therefore, since β_{2i} are independent we can perturb the vector field χ^* keeping $\beta_2 = \cdots = \beta_{2N-4} = 0$ and choose $\beta_{2N-2} > 0$ and sufficiently small such that the system still has the positively invariant set Ω. Since as the result of the perturbation the origin is now unstable singular point, the perturbed system has in Ω a limit cycle surrounding the origin. Repeating the procedure we obtain N small limit cycles around the origin. □

Definition 3.1. For system (3.1) with condition (3.6) satisfied if $\beta_0 = \beta_2 = \cdots = \beta_{2N-2} = 0$ and $\beta_{2N} \neq 0$, then the quantity β_{2N} is called the N-th quasi-Lyapunov constant at the origin.

(I) When $\gamma = 0$ system (1.5) has the form

$$\begin{cases} \frac{\mathrm{d}v}{\mathrm{d}\xi} = y = X(v, y) \\ \frac{\mathrm{d}y}{\mathrm{d}\xi} = -cy + (\sum_{i=0}^{n} \alpha_i v^i) y - \beta v^{n+1}(1 - v^n) = Y(v, y). \end{cases} \tag{3.7}$$

From Lemma 2.2, only when $c = \alpha_0$, that is, $\lambda = \alpha_0 - c = 0$, the origin of system (3.7) is a nilpotent critical point. From (3.2) we have

$$\begin{aligned} Y(v, y(v)) = Y(v, 0) = -\beta v^{n+1} + \beta v^{2n+1}, \\ \left[\frac{\partial X}{\partial v} + \frac{\partial Y}{\partial y}\right]_{y=0} = \alpha_1 v + \alpha_2 v^2 + \cdots + \alpha_n v^n. \end{aligned} \tag{3.8}$$

Moreover, according to Propositions 3.1 and 3.2, only when $n = 2m$, $m \in \mathbb{N}^+$, that is, $A_{2m+1} = -\beta < 0$, the origin is a center, or a focus or a node. When $c \neq \alpha_0$, that is $\lambda = \alpha_0 - c \neq 0$, the origin of system (3.7) is a node.

When $\beta > 0$ for system (3.7) with $n = 2m$ there exists a positive definite Lyapunov function

$$V(v, y) = \frac{1}{2}\left[y^2 + \beta\left(\frac{1}{m+1} - \frac{v^{2m}}{2m+1}\right) v^{2m+2}\right] \tag{3.9}$$

such that

$$\left.\frac{\mathrm{d}V}{\mathrm{d}\xi}\right|_{(3.7)} = \frac{\mathrm{d}V}{\mathrm{d}v}\frac{\mathrm{d}v}{\mathrm{d}\xi} + \frac{\mathrm{d}V}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}\xi} = y^2[(\alpha_0 - c) + \alpha_1 v + \alpha_2 v^2 + \cdots + \alpha_{2m} v^{2m}] \tag{3.10}$$

where

$$|v| < \left(\frac{2m+1}{m+1}\right)^{\frac{1}{2m}} =: \delta_m.$$

Observing that δ_m is a strictly monotonically decreasing with respect to $m \in \mathbb{N}^+$, we have

$$\lim_{m\to\infty} \delta_m = 1 < \delta_m \le \sqrt{\frac{3}{2}} = \delta_1.$$

Thus $V(v, y)$ is a positive definite Lyapunov function in the neighborhood $U(O, \delta_m)$. Since α_i are independent, we can choose the perturbations in such way that

$$0 < |\alpha_0 - c| \ll |\alpha_2| \ll \cdots \ll |\alpha_{2m}|, \qquad \alpha_{2i}\alpha_{2i+2} < 0, \qquad n = 1, 2, \ldots, m-1. \tag{3.11}$$

Then from Theorem 3.4 we obtain the following conclusion.

Theorem 3.5. For system (3.7) with $\gamma = 0$, if $\beta > 0$, $n = 2m$, $\alpha_{2m} \neq 0, m \in \mathbb{N}^+$ then for a suitable choice of α_i there exist m limit cycles in a neighborhood of origin of system (3.7).

Denote the second function in the product on the right side of (3.10) by $g(v)$,

$$g(v) = (\alpha_0 - c) + \alpha_1 v + \alpha_2 v^2 + \cdots + \alpha_{2m} v^{2m}. \tag{3.12}$$

The following statement shows that m small amplitude limit cycles can appear in the system under small perturbations.

Corollary 3.6. In Theorem 3.5, write $\alpha_n = \alpha_{2m} = K$ and let

$$\begin{aligned} \alpha_{2m-2} &= \frac{K}{(2m-2)!} f^{(2m-2)}(0), \ \ldots, \ \alpha_{2j} = \frac{K}{(2j)!} f^{(2j)}(0), \ \ldots, \\ \alpha_0 &= c - Kf(0), \qquad \alpha_{2j+1} = 0, \qquad j = 0, 1, \ldots, m-1 \end{aligned} \tag{3.13}$$

where

$$f(v) = \prod_{j=1}^{m} (v^2 - r_j^2 \varepsilon^2) = (v^2 - r_1^2\varepsilon^2)(v^2 - r_2^2\varepsilon^2)\cdots(v^2 - r_j^2\varepsilon^2) \tag{3.14}$$

where $0 < r_1 < r_2 < \cdots < r_m$. When $0 < \varepsilon << |K|$, there are m limit cycles in a small enough neighborhood of the origin for system (3.7).

Proof. Substituting α_j $(j = 0, 1, \ldots, 2m)$ into (3.12) we have

$$g(v) = Kf(0) + \frac{K}{2!} f^{(2)}(0)\, v^2 + \frac{K}{4!} f^{(4)}(0)\, v^4 + \cdots + \frac{K}{(2j)!} f^{(2j)}(0)\, v^{2j} + \cdots + Kv^{2m}.$$

Note that each $\frac{1}{(2j)!} f^{(2j)}(0)$ in the above expression corresponds to the coefficient of the term v^{2j} of $f(v)$ in (3.14) and $\frac{1}{(2m)!} f^{(2m)}(0) = 1$. That is, $g(v) = Kf(v)$. Because $f(v)$ has just m simple positive roots, $v = r_j\varepsilon$, $j = 1, \cdots, m$, the coefficients of $f(v)$ have alternating signs, namely $\alpha_{2i}\alpha_{2i+2} < 0$. Since the other conditions of Theorem 3.5 are also satisfied, the proof is completed. □

Remark 3.1. When $\alpha_0 - c = 0$, the origin is a nilpotent critical point and there is one less perturbation coefficient, so in such situation there exist just $m - 1$ limit cycles in a neighborhood of origin of system (3.7).

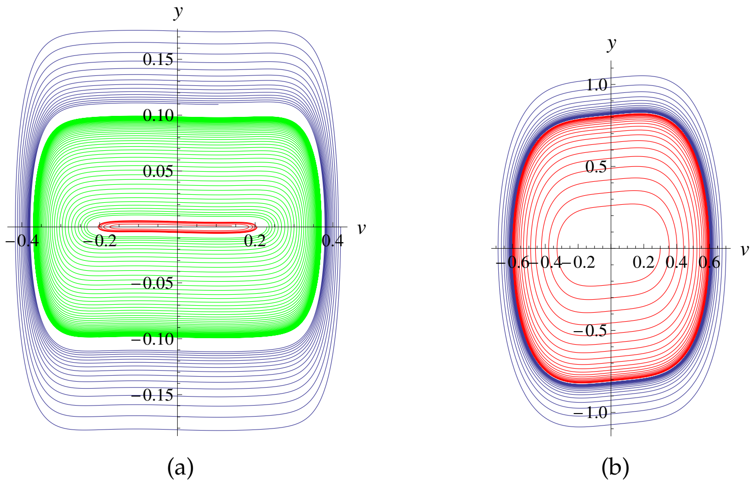

Figure 3.1: (a) The phase portrait showing two limit cycles with the inner one unstable and the outer one stable, where three trajectories have initial points $(0.205, 0), (0.23, 0), (0.42, 0)$ in Case (i). (b) The phase portrait showing one stable limit cycle, where two trajectories have initial points $(0.3, 0)$ and $(0.7, 0)$ in Case (ii).

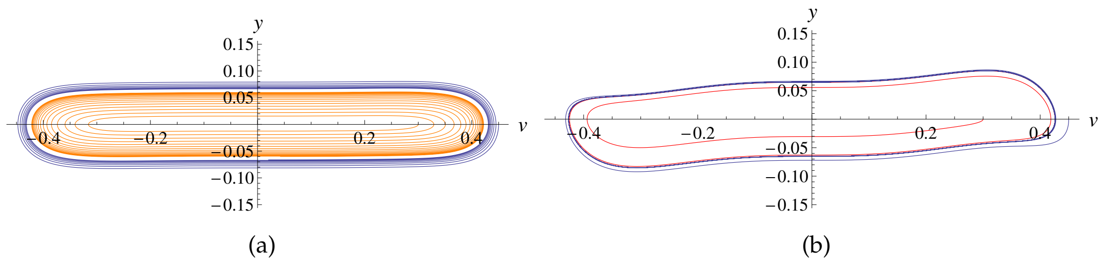

Figure 3.2: (a) The phase portrait showing one stable limit cycle, where two trajectories have initial points $(0.3, 0)$ and $(0.5, 0)$ in Case (iii). (b) The phase portrait showing one stable limit cycle, where two trajectories have initial points $(0.3, 0)$ and $(0.45, 0)$ in Case (iv).

Applying Corollary 3.6, Propositions 3.1 and 3.2 we obtain several examples of system (3.7) with $\alpha_{2i-1} = 0$, $i = 1, 2, \ldots$, as follows:

(i) when $n = 4$, setting $c - \alpha_0 = -0.01$, $\alpha_2 = 1$, $\alpha_4 = K = -10$ and $\beta = 10$, as a semi-hyperbolic singular point, the origin is a stable node and there exist 2 limit cycles, where one is stable, another is unstable, see Fig. 3.1 (a);

(ii) when $n = 2$, setting $c - \alpha_0 = 0.1$, $\alpha_2 = K = -1$ and $\beta = 10$, as a semi-hyperbolic singular point, the origin is an unstable node and there exists one stable limit cycle, see Fig. 3.1 (b);

(iii) when $n = 4$, setting $\alpha_0 - c = 0$, $\alpha_2 = 0.4$, $\alpha_4 = K = -4$ and $\beta = 10$, the origin is a unstable nilpotent focus and there exists one stable limit cycle, see Fig. 3.2 (a);

(iv) when $n = 4$, setting $\alpha_0 - c = 0$, $\alpha_2 = 5$, $\alpha_4 = K = -50$ and $\beta = 2$, the origin is a unstable nilpotent node and there exists one stable limit cycle, see Fig. 3.2 (b).

(II) We consider the degenerate equilibrium $(1, 0)$ of system (1.5) in the case $\gamma = 1$. Applying the translation $v \mapsto v + 1$ and keeping the notation v for the translated system we obtain from (1.5) the system

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}\xi} = y = X(v,y), \\ \frac{\mathrm{d}y}{\mathrm{d}\xi} = -cy + (\sum_{i=0}^{n} \alpha_i (v+1)^i) y + \beta(v+1)[(v+1)^n - 1]^2 = Y(v,y). \end{cases} \tag{3.15}$$

We see that the solution to $X(v, y(v)) = 0, y(0) = 0$ is $y(v) = 0$, thus from (3.2), we have

$$\begin{aligned} Y(v,y(v)) = Y(v,0) = \beta(n^2 v^2 + \cdots + v^{2n+1}), \\ \left[\frac{\partial X}{\partial v} + \frac{\partial Y}{\partial y}\right]_{y=0} = \sum_{i=0}^{n} \alpha_i (v+1)^i - c, \end{aligned} \tag{3.16}$$

and from Lemma 2.2, we obtain that only when $c = \sum_{i=0}^{n} \alpha_i$ the origin of (3.15) is a nilpotent critical point. Since $n > 1$ by Propositions 3.1 and 3.2, when $\beta \neq 0$ and $A_2 = n^2\beta \neq 0$ in (3.2), the origin cannot be a center or a focus, or a node. When $\beta = 0$, from Lemma 2.4 there is no limit cycle bifurcating from the degenerate equilibrium $(1, 0)$ of system (1.5).

Remark 3.2. When $\beta \neq 0$ and $c \neq \sum_{i=0}^{n} \alpha_i$ we verified that there does not exist a limit cycle bifurcating from the origin of (3.15) for some concrete values of n, for which the origin is a degenerate point of saddle-node type. However, for general values of n the problem is open.

4 Hopf bifurcation for the non-degenerate equilibriums

In this section we apply the Lyapunov function method to investigate the Hopf bifurcations in system (1.5), that is, the bifurcations of small amplitude limit cycles from the non-degenerate equilibrium $(1, 0)$ or the origin $(0, 0)$ under the condition $\gamma = 0$ or $\gamma = 1$.

Consider first the real polynomial differential system

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = \lambda x - y + \sum_{k+j=2}^{\infty} a_{kj} x^k y^j = X(x,y), \\ \frac{\mathrm{d}y}{\mathrm{d}t} = x + \lambda y + \sum_{k+j=2}^{\infty} b_{kj} x^k y^j = Y(x,y) \end{cases} \tag{4.1}$$

where $x, y, t, \lambda, a_{kj}, b_{kj} \in \mathbb{R}\,(k, j \in \mathbb{N})$.

With a similar reasoning as in the proof of Theorem 3.4 we obtain the following theorem.

Theorem 4.1. Suppose that for system (4.1) there exists a positive definite function $V(x, y)$ such that

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4.1)} = \frac{\mathrm{d}V}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}V}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}t} = y^2 \left[\sum_{i=0}^{N} \beta_{2i}(x^{2i} + O(x^{2i+1})) + o(x^{2N+1})\right] \tag{4.2}$$

where $N \in \mathbb{N}^+$ and $(x, y) \in U(O, \delta)$, δ is certain positive number and β_{2i} are independent. Assume also that for a vector field χ^* from family (4.1) with fixed parameters $a_{ij} = a^*_{ij}$, $b_{ij} = b^*_{ij}$ the parameters β_{2i} in (4.2) satisfy $\beta_0 = \beta_2 = \beta_4 = \cdots = \beta_{2N-2} = 0, \beta_{2N} \neq 0$. Then it is possible to perturb vector field χ^* in such way, that the perturbed system has N small limit cycles in a neighborhood of the origin.

Definition 4.1. For system (4.1) under condition (4.2), if $\beta_0 = \beta_2 = \cdots = \beta_{2N-2} = 0$ and $\beta_{2N} \neq 0$, then the quantity $\beta_{2N} = V_N$ is called the N-th quasi-Lyapunov constant at the origin $(N = 1, 2, \ldots)$.

(I) For the case $\gamma = 1$ when $\beta < 0$ system (1.5) has a Hopf bifurcation at the singular point $(0,0)$. Indeed, in this case system (1.5) has the form

$$\begin{cases} \frac{dv}{d\xi} = y = X(v,y), \\ \frac{dy}{d\xi} = -cy + (\sum_{i=0}^{n} \alpha_i v^i)y + \beta v(1-v^n)^2 = Y(v,y). \end{cases} \tag{4.3}$$

Thus, when $n \in \mathbb{N}^+$, there exists a positive definite Lyapunov function

$$V(v,y) = \frac{1}{2}\left[y^2 - \beta v^2\left(1 - \frac{4v^n}{2+n} + \frac{v^{2n}}{1+n}\right)\right], \qquad |v| < \left[\frac{2+n}{4}\right]^{\frac{1}{n}} \tag{4.4}$$

such that

$$\left.\frac{dV}{d\xi}\right|_{(4.3)} = \frac{dV}{dv}\frac{dv}{d\xi} + \frac{dV}{dy}\frac{dy}{d\xi} = y^2\left(\sum_{i=0}^{n} \alpha_i v^i - c\right). \tag{4.5}$$

Indeed, for each $n \in \mathbb{N}^+$ we choose $\delta_n = [\frac{2+n}{4}]^{\frac{1}{n}}$ yielding $\frac{3}{4} \le \delta_n \le (\frac{3}{2})^{2/7}$. Then it is easy to verify that $V(v,y)$ is a positive definite Lyapunov function in the neighborhood $U(O,\delta_n)$.

Using Theorem 4.1 we have the following conclusion.

Theorem 4.2. For system (4.3) with $\gamma = 1$, if $\beta < 0$ and $\alpha_{2m} \neq 0$ for either $n = 2m$ or $n = 2m+1$, $m \in \mathbb{N}^+$, we can choose perturbations such that

$$0 < |\alpha_0 - c| \ll |\alpha_2| \ll |\alpha_4| \ll \cdots \ll |\alpha_{2m}|, \qquad \alpha_{2i-2}\alpha_{2i} < 0, \qquad i = 1,2,\ldots,m, \tag{4.6}$$

and there exist m limit cycles in a neighborhood of origin of system (4.3).

(II) For the case of $\gamma = 0$, when $\beta < 0$ system (1.5) has the Hopf bifurcation at the singular point $(1,0)$. After the translation $v \mapsto v+1$ keeping the notation v for the new variable we obtain the system

$$\begin{cases} \frac{dv}{d\xi} = y = X(v,y), \\ \frac{dy}{d\xi} = -cy + (\sum_{i=0}^{n} \alpha_i (v+1)^i)y - \beta(v+1)^{n+1}[1-(v+1)^n] = Y(v,y). \end{cases} \tag{4.7}$$

We show that in the neighborhood $U(O,2)$ of the origin there exists a positive definite Lyapunov function

$$V(v,y) = \frac{1}{2}y^2 - \frac{\beta}{2}\left[\frac{n}{(2+n)(1+n)} - \frac{2(v+1)^{n+2}}{2+n} + \frac{(v+1)^{2n+2}}{1+n})\right] \tag{4.8}$$

such that

$$\left.\frac{dV}{d\xi}\right|_{(4.3)} = \frac{dV}{dv}\frac{dv}{d\xi} + \frac{dV}{dy}\frac{dy}{d\xi} = y^2\left(\sum_{i=0}^{n} \alpha_i (v+1)^i - c\right). \tag{4.9}$$

Clearly, $V(0,0) = 0$. Thus, we need to prove that when $(v,y) \in U(O,\delta_n)$ and $(v,y) \neq (0,0)$ it holds that $V(v,y) > 0$.

Letting $r = v+1$ we have

$$\frac{n}{(2+n)(1+n)} - \frac{2(v+1)^{n+2}}{2+n} + \frac{(v+1)^{2n+2}}{1+n} = \frac{n}{(2+n)(1+n)} - \frac{2r^{n+2}}{2+n} + \frac{r^{2n+2}}{1+n} \triangleq L_r$$

so we will verify that when $r \neq \pm 1$, the inequality $L_r > 0$ holds. From

$$\frac{dL_r}{dr} = 2r^{n+1}(r^n - 1) = 0, \tag{4.10}$$

we find the stationary point. Namely, (i) if n is an odd number, only $r = 0$ and $r = 1$ are the stationary points of L_r; (ii) if n is an even number, then only $r = 0$ and $r = \pm 1$ are the stationary points of L_r.

Performing the monotonicity analysis of the function L_r we see that it has the minimum at $r = 1$, i.e. $L_r(1) = 0$, but there is no extreme at $r = 1$ for the case (i). The function has a minimum at $r = \pm 1$, i.e. $L_r(1) = 0$ and a local maximum at $r = 1$, i.e., $L_r(1) = \frac{n}{(2+n)(1+n)}$ for the case (ii). In summary, for arbitrary $n \in \mathbb{N}$ and $\beta < 0$, when $r \neq \pm 1$, i.e., $v \neq 0, -2$, we have

$$L_r > 0 \quad \text{or} \quad V(v, y) > 0. \tag{4.11}$$

We now can conclude that when the conditions of Theorem 4.1 are satisfied the following statement holds.

Theorem 4.3. For system (4.7) with $\gamma = 0$, if $\beta < 0$, $n = 2m$ or $2m + 1$, $\beta_{2m} \neq 0, m \in \mathbb{N}^+$ we can choose perturbations such that

$$\begin{aligned} &0 < |\beta_0| \ll |\beta_2| \ll |\beta_4| \ll \cdots \ll |\beta_{2m}|, \qquad \beta_{2i-2}\beta_{2i} < 0, \\ &\beta_1 \leq |\beta_0|, |\beta_3| \leq |\beta_2|, \ldots, |\beta_{2i+1}| \leq |\beta_{2i}|, \qquad i = 1, 2, \ldots, m \end{aligned} \tag{4.12}$$

where

$$\beta_0 = \sum_{j=0}^{n} \alpha_j - c,\ \beta_1 = \sum_{j=1}^{n} j\alpha_j,\ \beta_2 = \sum_{j=2}^{n} j(j-1)\alpha_j,\ \ldots, \beta_k = \sum_{j=k}^{n} \binom{k}{j} \alpha_j, \ldots,\ \beta_n = \alpha_n$$

and there exist m limit cycles in a neighborhood of the origin of system (4.7).

5 Illustration of the quasi-Lyapunov constant and isolated periodic wave trains

The readers may be confused by the quasi-Lyapunov constant given in the Definitions 3.1 and 4.1, what is the difference between it and the Lyapunov constant? Here we try to illustrate this. In fact, within the above process certain positive definite Lyapunov functions $V(v, y)$ with finite terms are constructed to investigate limit cycle bifurcations, but these are different from first integral or integrating factor with formal series form to be determined for the Lyapunov constants or focus values, see e.g. [9, 17, 31]. For the later case, in general, the formal series of first integral with infinite terms is also positive definite in the neighborhood of the origin, which terms are derived successively, in a sense, it is a relatively complete sequence of positive definite Lyapunov function. However, for the the former, it is not a complete sequence necessarily. The highest order Lyapunov constant or focus value can be determined under the later case, just at most limit cycles is revealed, while only the quasi-Lyapunov constant are determined under the former case, and the highest order Lyapunov constant can not be determined necessarily.

Obviously, from Theorem 4.3, for the case $n = 1$ in system (4.7) we cannot obtain a limit cycle, and for the case $n = 2$ we can obtain only one limit cycle bifurcating from the origin. However, we have the following complete results by utilizing method of first integral formal series, namely determining the complete sequence of positive definite Lyapunov function.

Proposition 5.1. Under the degenerate condition of $\beta < 0$ and $\gamma = 0$, there exist at least one limit cycle for the case $n = 1$ and two limit cycles for the case $n = 2$, respectively, bifurcating from the origin of system (4.7) as a Hopf bifurcation point.

Proof. (i) When $n = 1$, system (4.7) has the form

$$\dot{v} = y, \qquad \dot{y} = \beta v + (\alpha_0 + \alpha_1 - c)y + h_1(v, y) \tag{5.1}$$

where $h_1(v, y) = \alpha_1 vy + 2\beta v^2 + \beta v^3$. If $c = \alpha_0 + \alpha_1$ holds, the origin is a center or a focus. For system (5.1) there is a formal series

$$H(v, y) = y^2 - \frac{\beta x^4}{2} - \frac{4\beta x^3}{3} - \beta x^2 + \frac{\alpha_1^2 y^4}{2\beta^2} - \frac{2\alpha_1 y^3}{3\beta} + \text{h.o.t} \tag{5.2}$$

such that

$$\frac{dH}{d\xi}\bigg|_{(4.3)} = \frac{dH}{dv}\frac{dv}{d\xi} + \frac{dH}{dy}\frac{dy}{d\xi} = -4\alpha_1 v^2 y^2 + \text{h.o.t.} \tag{5.3}$$

Thus, the first Lyapunov constant is $L_1 = -4\alpha_1$. When $c - \alpha_0 - \alpha_1 \neq 0$ and sufficiently small the origin becomes a weak focus and there exists one limit cycle bifurcated in a neighborhood of it.

(ii) When $n = 2$, system (4.7) becomes

$$\dot{v} = y, \;\; \dot{y} = 2\beta v + (\alpha_0 + \alpha_1 + \alpha_2 - c)y + h_2(v, y) \tag{5.4}$$

where $h_2(v, y) = \beta v^2(7 + 9v + 5v^2 + v^3) + vy(\alpha_1 + 2\alpha_2 + \alpha_2 v)$. Similarly as above there exists a series

$$\begin{aligned}H(v,y) = {} & y^2 - 2\beta v^2 - \tfrac{14\beta}{3}v^3 - \tfrac{(\alpha_1+2\alpha_2)}{3\beta}y^3 + \tfrac{(\alpha_1+2\alpha_2)^2}{8\beta^2}y^4 - \tfrac{9\beta}{2}v^4 \\ & + \tfrac{2}{15}(7\alpha_1^2 + 26\alpha_2\alpha_1 + 24\alpha_2^2 - 15\beta)v^5 + \tfrac{3(\alpha_1+2\alpha_2)}{2\beta}v^2y^3 - \tfrac{(7\alpha_1^2+26\alpha_2\alpha_1+24\alpha_2^2)}{6\beta}v^3y^2 \\ & - \tfrac{1}{20\beta^3}(\alpha_1+2\alpha_2)\left(\alpha_1^2 + 4\alpha_2\alpha_1 + 4\alpha_2^2 + 6\beta\right)y^5 + \tfrac{1}{18}(49\alpha_1^2 + 182\alpha_2\alpha_1 + 168\alpha_2^2 - 6\beta)v^6 \\ & + \tfrac{1}{12\beta^2}(\alpha_1+2\alpha_2)\left(7\alpha_1^2 + 26\alpha_2\alpha_1 + 24\alpha_2^2 + 42\beta\right)v^3y^3 - \tfrac{9(\alpha_1+2\alpha_2)^2}{8\beta^2}v^2y^4 \\ & + \tfrac{1}{48\beta^4}(\alpha_1+2\alpha_2)^2\left(\alpha_1^2 + 4\alpha_2\alpha_1 + 4\alpha_2^2 + 15\beta\right)y^6 + \text{h.o.t}\end{aligned}$$

such that

$$\frac{dH}{d\xi}\bigg|_{(4.3)} = L_1 v^2 y^2 + L_2 v^4 y^4 + \text{h.o.t.} \tag{5.5}$$

where $L_1 = -(7\alpha_1 + 12\alpha_2)$ and $L_2 = -\frac{1}{6\beta}(\alpha_1 + 2\alpha_2)(7\alpha_1^2 + 26\alpha_2\alpha_1 + 24\alpha_2^2 + 285\beta)$ are the first and second Lyapunov constants, respectively. Similarly as above we conclude that there exists two limit cycles bifurcated from the origin. □

Next, we consider the isolated periodic wave trains of the degenerate generalized Burgers–Huxley equation (1.2). As it is known a small amplitude limit cycle corresponds to an isolate bounded periodic solution of system (3.7). Thus from Theorems 3.5, 4.2 and 4.3 we have the following conclusion.

Theorem 5.2. In the Burgers–Huxley equation (1.2), for any $n = 2m$ or $2m + 1$, $m \in \mathbb{N}^+$, at least m isolated periodic wave trains can bifurcate from $u(x, t) = 0$ under the degenerate condition of $\beta > 0, \gamma = 0$ or $\beta < 0, \gamma = 1$; at least m isolated periodic wave trains can bifurcate from $u(x, t) = 1$ under the degenerate condition of $\beta < 0, \gamma = 0$.

Though the exact explicit expressions of the above isolated periodic wave trains cannot be given, some approximation methods can be used. Here we apply numerical computations to get the solutions for the four examples given in Remark 3.1 and shown in Fig. 5.1.

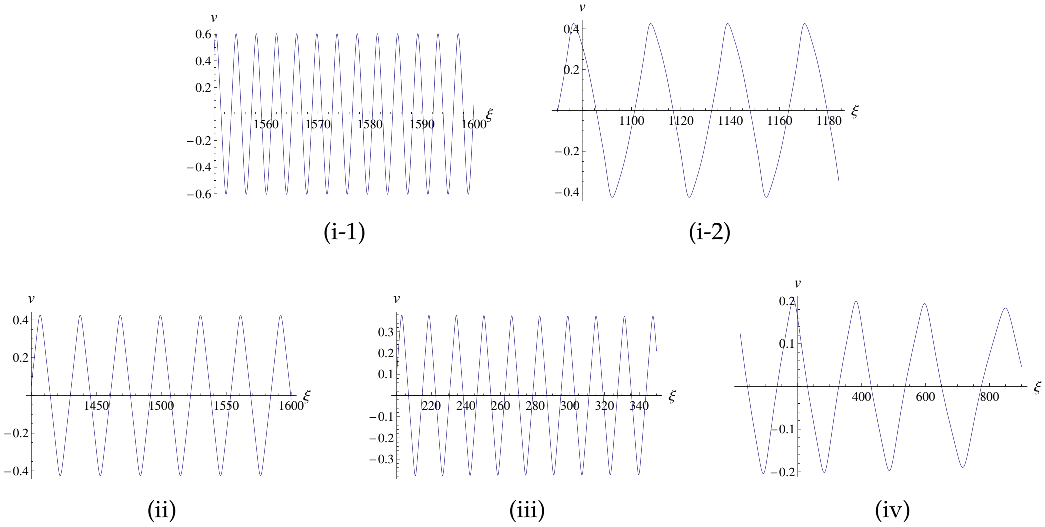

Figure 5.1: The periodic waves in cases (i-1,i-2), (ii), (iii), (iv) corresponding to the isolated periodic wave trains in Cases (i), (ii), (iii), (iv), respectively.

6 Conclusion

We studied the isolated periodic wave trains for a class of modified generalized Burgers–Huxley equation by focusing on the limit cycle bifurcations in a neighborhood of the degenerate equilibrium points. For any positive integer n the number of small amplitude limit cycles bifurcating from the nilpotent point or semi-hyperbolic singular point of system (1.5) is estimated. Finally, the existences of corresponding multiple isolated periodic wave trains in the original model (1.2) is established.

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Existence and multiplicity of eigenvalues for some double-phase problems involving an indefinite sign reaction term

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Abstract.
We study the following class of double-phase nonlinear eigenvalue problems

$$-\operatorname{div}\left[\phi(x, |\nabla u|)\nabla u + \psi(x, |\nabla u|)\nabla u\right] = \lambda f(x, u)$$

in Ω, $u = 0$ on $\partial\Omega$, where Ω is a bounded domain from $\mathbb{R}^N$ and the potential functions ϕ and ψ have $(p_1(x); p_2(x))$ variable growth. The primitive of the reaction term of the problem (the right-hand side) has indefinite sign in the variable u and allows us to study functions with slower growth near $+\infty$, that is, it does not satisfy the Ambrosetti–Rabinowitz condition. Under these hypotheses we prove that for every parameter $\lambda \in \mathbb{R}_+^*$, the problem has an unbounded sequence of weak solutions. The proofs rely on variational arguments based on energy estimates and the use of Fountain Theorem.

Keywords: double-phase differential operator, continuous spectrum, variable exponent, multiplicity of eigenvalues, infinitely many solutions.
2020 Mathematics Subject Classification: 35P30, 49R05.

1 Introduction

The study of variational problems with nonstandard growth conditions has been developed extensively over the last years. Moreover as the technology development in some important areas like robotics, aircraft and airspace and the image restoration was very intensive, and in order to obtain important results, new mathematical models arose.

The $p(x)$-growth conditions can be regarded as a key factor in the modelling of some fluids which have different inhomogeneities, for instance we can mention here the lithium polymetachrylate , which is an electrorheological fluid. The main characteristic of these types of fluids is the fact that their viscosity depends on the electric field in the fluid, that is the viscosity of the fluid is inverse proportional to the strength of the electric field.

As new types of materials arose in the domains that we mentioned before, new problems arose also in the field of variable exponent analysis and partial differential equations which

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involve several variable exponents. Therefore in the last years, double-phase problems which involve several variable exponents and some nonstandard $(p_1(x), p_2(x))$-growth behavior for potential functions have been extensively studied.

In this paper we are concerned with the study of a class of non-autonomous eigenvalue problem with variable $(p_1(x); p_2(x))$-growth rate condition in the left hand side of the problem and a general reaction term (that is in the right-hand side of the problem), which is p_2^+-superlinear at infinity and whose primitive may be sign changing. An important characteristic of the above mentioned problem is the fact that the associated energy density changes its ellipticity according to the point.

The research in this paper in based on some new type of differential operators, which have been introduced by I. H. Kim and Y. H. Kim [8], which enables us to solve some problems which imply the possible lack of uniform convexity. In this paper we extend the results of I. H. Kim and Y. H. Kim by studying a double-phase problem and we use a new type of reaction term which require weaker conditions than the Ambrosetti–Rabinowitz condition (for the sake of simplicity we will denote this condition as the (AR)-condition) and allows us to study functions that have a p_2^+-superlinear growth near infinity but the growth is too slow to satisfy the (AR)-condition. Also,the primitive of the reaction term is allowed to be sign-changing. An example of this type of reaction term will be presented in the last section of this paper together with some important examples and new directions of research. Furthermore, for the best of our knowledge for this type of operators even in the simpler cases, when the differential operator is driven by only one potential function the possibility that the primitive of the reaction function to be sign-changing has not been considered.

This paper also aim to extend some spectral results for some simpler cases studied in the following works: S. Baraket, S. Chebbi, N. Chorfi, V. Rădulescu [2], M. Rodrigues [19], V. F. Uță [22] and K. Q. Wang, M. Zhou [23]. A comparison between these results will be made later in this paper.

Hence, we consider the following double-phase nonlinear eigenvalue problem:

$$\begin{cases} -\operatorname{div}\left[\phi(x,|\nabla u|)\nabla u\right] - \operatorname{div}\left[\psi(x,|\nabla u|)\nabla u\right] = \lambda f(x,u), \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega, \end{cases} \tag{P}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary and $\lambda \in \mathbb{R}$ is a real parameter.

These types of problems generalize a broad variety of models. We will briefly describe the most important ones.

For instance if we may need to model a composite that changes its hardening point exponent according to the point. To this end we refer to the work of M. Colombo, G. Mingione [3], where the associated energies are of type:

$$u \mapsto \int_\Omega |\nabla u|^{p_1(x)} dx + \int_\Omega a(x)|\nabla u|^{p_2(x)} dx \tag{1.1}$$

and

$$u \mapsto \int_\Omega |\nabla u|^{p_1(x)} dx + \int_\Omega a(x)|\nabla u|^{p_2(x)} \log(e + |x|) dx, \tag{1.2}$$

where $p_1(x) \le p_2(x)$, $p_1 \ne p_2$, for all $x \in \Omega$ and $a(x) \ge 0$.

Also a comprehensive study of this variety of models is presented in the following survey paper of G. Mingione, V. Rădulescu [9]. For the regularity of the minimizers functionals for double phase operator we recommend for more details the paper of M. Ragusa, A. Tachikawa [15].

The models presented above describe the behavior of two materials with variable power hardening exponents $p_1(x)$ and $p_2(x)$, with the geometry of a composite for one of the materials described by the coefficient $a(x)$.

As the potentials that drive our nonhomogeneous double-phase operator are very general we will consider the following special cases:

(C_1) The potential functions ϕ and ψ may describe a weighted $p(x)$-Laplacian-like operator

$$-\operatorname{div}[\phi(x,|\nabla u|)\nabla u]-\operatorname{div}[\psi(x,\nabla u)\nabla u]=-\operatorname{div}\left[a(x)|\nabla u|^{p_1(x)-2}\nabla u\right]-\operatorname{div}\left[b(x)|\nabla u|^{p_2(x)-2}\nabla u\right],$$

where the functions $a(x), b(x)\in L^\infty(\Omega)$, and there exist some constant α_0 such that $a(x)\geq\alpha_0$, $b(x)\geq\alpha_0$ for almost all $x\in\Omega$;

(C_2) The potential functions ϕ and ψ may describe the generalized mean curvature operator, thus we obtain the following differential operator:

$$-\operatorname{div}[\phi(x,|\nabla u|)\nabla u]-\operatorname{div}[\psi(x,\nabla u)\nabla u]=-\operatorname{div}\left[\left(1+|\nabla u|^2\right)^{\frac{p_1(x)-2}{2}}\nabla u\right]-\operatorname{div}\left[\left(1+|\nabla u|^2\right)^{\frac{p_2(x)-2}{2}}\nabla u\right]$$

(C_3) The potential functions ϕ and ψ may describe the differential operator that describe the capillary phenomenon:

$$\begin{aligned}&-\operatorname{div}[\phi(x,|\nabla u|)\nabla u]-\operatorname{div}[\psi(x,\nabla u)\nabla u]\\&\quad=-\operatorname{div}\left[\left(|\nabla u|^{p_1(x)-2}+\frac{|\nabla u|^{2p_1(x)-2}}{\left(1+|\nabla u|^{2p_1(x)}\right)^{1/2}}\right)\nabla u\right]\\&\qquad-\operatorname{div}\left[\left(|\nabla u|^{p_2(x)-2}+\frac{|\nabla u|^{2p_2(x)-2}}{\left(1+|\nabla u|^{2p_2(x)}\right)^{1/2}}\right)\nabla u\right].\end{aligned}$$

Remark 1.1. Also there can be considered more complex cases where the potential functions have different behavior, for example potential ϕ may describe the case (C_1), and the potential ψ could describe any of the other cases.

It is obvious that the case (C_1) generalize the relation described by (1.1). In order to obtain the case described by (1.2) we will have to study the following differential operator:

$$-\operatorname{div}[\phi(x,|\nabla u|)\nabla u]-\operatorname{div}[a(x)\psi(x,|\nabla u|)\log(e+|x|)\nabla u]. \tag{1.3}$$

The study of the case (C_3) is motivated by its important applicabilities in various fields varying from the industrial, biomedical and pharmaceutical to the microfluidic systems. In order to describe the capillarity phenomenon we must consider the effects of two opposing forces: adhesion, that is, the attractive (or repulsive) force between the molecules of the liquid and those of the container; and cohesion, that is, the attractive force between the molecules of the liquid.

Problems involving this type of differential operator were intensely studied in the last years. For example we consider the following works: [8,17–19,22]. Also, more closely related results for anisotropic problems with unbalanced growth may be found in [1] and for the double phase operators with lack of compactness we refer to [20].

The main results of this paper consist in two theorems which ensures us that for every $\lambda > 0$, $\lambda \in \mathbb{R}$, the problem (P) admits an unbounded sequence of solutions with higher and higher energies. Both of the proofs are based on variational arguments, energy estimates and the use of the Fountain Theorem.

High energy solutions for similar problems were studied under more restrictive hypotheses in the following works: [19, 22], where the reaction function is supposed to satisfy the so called (AR)-condition, or in [23] where the differential operator enables us to study some simple case, where in order to make connections to our problem the potential function ϕ is supposed to verify just the case (C_2) and the potential function $\psi \equiv 0$, but the nonlinearity in the right-hand side of the problem is more general than the one used in [19] and [22]. This generality comes at a cost, that is, the parameter λ is allowed to take values just in a bounded interval near the origin.

In the last section of this work we give some striking examples and some remarks in order to illustrate the validity of our results. Moreover, we draw a parallel between previous results and the new results presented in this paper as well as some future perspectives of research in this direction.

2 The functional framework

Through this section we will introduce the basic properties of variable exponent spaces, that will constitute necessary the functional framework that we need in the study of problem (P).

These results are described in the following books: J. Musielak [10], L. Diening, P. Hästö, P. Harjulehto, M. Růžička [4], V. Rădulescu and D. Repovš [17]. We also refer to the survey paper by V. Rădulescu [16].

Let Ω be a bounded domain in $\mathbb{R}^N$.

For a measurable function $p : \overline{\Omega} \to \mathbb{R}$ we define:

$$p^+ = \sup_{x\in\Omega} p(x) \quad \text{and} \quad p^- = \inf_{x\in\Omega} p(x).$$

Set:

$$C_+(\Omega) = \left\{ p \in C(\overline{\Omega}) : \ p(x) > 1, \text{ for all } x \in \overline{\Omega} \right\}.$$

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined

$$L^{p(x)}(\Omega) = \left\{ u; \ \ u : \Omega \to \mathbb{R} \text{ a measurable function} : \int_\Omega |u|^{p(x)} dx < \infty \right\},$$

and with the norm:

$$|u|_{p(x)} = \inf \left\{ \mu > 0 : \int_\Omega \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\},$$

$L^{p(x)}(\Omega)$ becomes a Banach space whose dual is the space $L^{p'(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

Remark 2.1. If $1 < p(x) < \infty$, $L^{p(x)}(\Omega)$ is reflexive Banach space. Moreover, if p is measurable and bounded, then $L^{p(x)}(\Omega)$ is also separable.

Remark 2.2. If $0 < |\Omega| < \infty$ and $h(x)$, $r(x)$ with $h(x) < r(x)$ almost everywhere in Ω, are two variable exponents then the following continuous embedding holds

$$L^{r(x)}(\Omega) \hookrightarrow L^{h(x)}(\Omega).$$

Let $L^{p'(x)}(\Omega)$ denotes the dual space of $L^{p(x)}(\Omega)$. For all $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the following Hölder type inequality holds:

$$\left| \int_\Omega uv\, dx \right| \le \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) |u|_{p(x)} |v|_{p'(x)} \le 2|u|_{p(x)}|v|_{p'(x)}. \tag{2.1}$$

A key role in the studies which imply the variable exponent Lebesgue spaces is played by the modular of $L^{p(x)}(\Omega)$, which is $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ and is defined by

$$\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)} dx.$$

Remark 2.3. If $p(x) \not\equiv$ *constant* in Ω, for u, $(u_n) \in L^{p(x)}(\Omega)$, the following relations hold true:

$$|u|_{p(x)} < 1 \Rightarrow \; |u|_{p(x)}^{p^+} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^-}, \tag{2.2}$$

$$|u|_{p(x)} > 1 \Rightarrow \; |u|_{p(x)}^{p^-} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^+}, \tag{2.3}$$

$$|u|_{p(x)} = 1 \Rightarrow \; \rho_{p(x)}(u) = 1, \tag{2.4}$$

$$|u_n - u|_{p(x)} \to 0 \Leftrightarrow \; \rho_{p(x)}(u_n - u) \to 0. \tag{2.5}$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\}.$$

On $W^{1,p(x)}(\Omega)$ we may consider the following equivalent norms:

$$\|u\|_{p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}$$

and

$$\|u\| = \inf \left\{ \mu : \int_\Omega \left(\left| \frac{\nabla u(x)}{\mu} \right|^{p(x)} + \left| \frac{u(x)}{\mu} \right|^{p(x)} \right) dx \le 1 \right\}.$$

We define $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{p(x)}$ or

$$W_0^{1,p(x)}(\Omega) = \left\{ u; u|_{\partial\Omega} = 0, u \in L^{p(x)}(\Omega), |\nabla u| \in L^{p(x)}(\Omega) \right\}.$$

Taking account of [8] for $p \in C_+(\overline{\Omega})$ we have the $p(\cdot)$-*Poincaré type inequality*

$$|u|_{p(x)} \le C|\nabla u|_{p(x)}, \tag{2.6}$$

where $C > 0$ is a constant which depends on p and Ω.

For $\Omega \subset \mathbb{R}^N$ a bounded domain and p a global log-Hölder continuous function, on $W_0^{1,p(x)}(\Omega)$ we can work with the norm $|\nabla u|_{p(x)}$ equivalent with $\|u\|_{p(x)}$.

Remark 2.4. If $p, q : \Omega \to (1, \infty)$ are Lipschitz continuous, $p^+ < N$ and $p(x) \le q(x) \le p^*(x)$, for any $x \in \Omega$, where $p^*(x) = \frac{Np(x)}{N-p(x)}$, the embedding

$$W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$$

is compact and continuous.

Remark 2.5. If $0 < |\Omega| < \infty$, and $p_2(x) < p_1(x)$ in Ω, then there holds the following continuous embedding

$$W_0^{1,p_1(x)}(\Omega) \hookrightarrow W_0^{1,p_2(x)}(\Omega).$$

Remark 2.6 ([5]). Let $p(x)$ and $q(x)$ be measurable functions such that $p(x) \in L^\infty(\Omega)$ and $1 \le p(x)q(x) \le \infty$ almost everywhere in Ω. Let $u \in L^{q(x)}(\Omega)$, $u \ne 0$. Then

$$|u|_{p(x)q(x)} \ge 1 \Rightarrow |u|^{p^-}_{p(x)q(x)} \le \left||u|^{p(x)}\right|_{q(x)} \le |u|^{p^+}_{p(x)q(x)}$$
$$|u|_{p(x)q(x)} \le 1 \Rightarrow |u|^{p^+}_{p(x)q(x)} \le \left||u|^{p(x)}\right|_{q(x)} \le |u|^{p^-}_{p(x)q(x)}.$$

In particular, if $p(x) = p$ is a constant, then $||u|^p|^p_{pq(x)}$.

3 Basic hypotheses and auxiliary results

In this section we will give the basic properties of the potential functions ϕ and ψ which drive us to the differential operator described in the first section. Also we impose the new conditions on the reaction function and the theoretical auxiliary results we need in order to achieve the solutions of problem (P).

Therefore, we assume that the reaction function $f(x, z)$ satisfies the following conditions:

(R_1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, that is:
$\to f(\cdot, z)$ is measurable for all $z \in \mathbb{R}$;
$\to f(x, \cdot)$ is continuous for almost all $x \in \Omega$.

(R_2) There exists $C > 0$, a nonnegative constant such that

$$|f(x,z)| \le C\left(1 + |z|^{q(x)-1}\right)$$

for all $x \in \Omega$ and $z \in \mathbb{R}$, where $q \in C_+(\overline{\Omega})$.

Define

$$F(x,z) = \int_0^z f(x,t)dt. \tag{3.1}$$

(R_3) $\lim_{|z|\to\infty} \frac{|F(x,z)|}{|z|^{p_2^+}} = +\infty$ uniformly in x, and there exists $q_0 > 0$, such that

$$F(x,z) \ge 0 \quad \text{for all } x \in \Omega \text{ and } z \in \mathbb{R},$$

with $|z| > q_0$.

(R_4) Define:

$$\mathcal{R}(x,z) := \frac{1}{p_2^+} f(x,z)z - F(x,z) \geq 0$$

and let $C_1 > 0$, a nonnegative constant and $\mu \in C_+(\Omega)$ with $\mu^- > \max\{1, \frac{N}{p_1^-}\}$ such that

$$|F(x,z)|^{\mu(x)} \leq C_1 |z|^{p_1^- \mu(x)} \mathcal{R}(x,z),$$

for all $x \in \Omega$ and $z \in \mathbb{R}$, with $|z| \geq q_0$.

(R_5) Let $\omega > p_2^+$ and $\eta > 0$ two constants such that

$$\omega F(x,z) \leq f(x,z)z + \eta |z|^{p_1^-},$$

for all $x \in \Omega$, $z \in \mathbb{R}$.

(R_6) $f(x,-z) = -f(x,z)$, for all $x \in \Omega$ and $z \in \mathbb{R}$.

Hypotheses on the potential functions that generates the double-phase differential operator are the following:

(HS_1) $\phi, \psi : \Omega \times [0,\infty) \to [0,\infty)$ and

$\to$ $\phi(\cdot,z)$, $\psi(\cdot,z)$ are measurable on Ω for all $z \geq 0$;

$\to$ $\phi(x,\cdot)$, $\psi(x,\cdot)$ are locally absolutely continuous on $[0,\infty)$ for almost all $x \in \Omega$.

(HS_2) For some functions $\alpha_1 \in L^{p_1'(x)}(\Omega)$ and $\alpha_2 \in L^{p_2'(x)}(\Omega)$ and a nonnegative constant ξ we have that

$\to$ $|\phi(x,|z|)z| \leq \alpha_1(x) + \xi |z|^{p_1(x)-1}$;

$\to$ $|\psi(x,|z|)z| \leq \alpha_2(x) + \xi |z|^{p_2(x)-1}$.

for almost all $x \in \Omega$, and all $z \in \mathbb{R}^N$.

(HS_3) For some constant $C_{\phi,\psi} > 0$, all $x \in \Omega$ and all $z > 0$ we have that:

$\to$ $\phi(x,z) \geq C_{\phi,\psi} z^{p_1(x)-2}$ and $z\frac{\partial \phi}{\partial z} + \phi(x,z) \geq C_{\phi,\psi} z^{p_1(x)-2}$

$\to$ $\psi(x,z) \geq C_{\phi,\psi} z^{p_2(x)-2}$ and $z\frac{\partial \psi}{\partial z} + \psi(x,z) \geq C_{\phi,\psi} z^{p_2(x)-2}$.

Let $S_0(x,z) = \displaystyle\int_0^z \phi(x,t)t\,dt + \int_0^z \psi(x,t)t\,dt$, we define

$$S(u) = \int_\Omega S_0(x,|\nabla u|)dx. \tag{3.2}$$

An important role in our variational approach is played by the fact that the following assumption holds true for the potentials ϕ and ψ:

(HS_4) For all $x \in \overline{\Omega}$, all $z \in \mathbb{R}^N$, the following estimate is true:

$$0 \leq [\phi(x,z) + \psi(x,z)]\,|z|^2 \leq p_2^+ S_0(x,|z|).$$

In order to obtain our results we must state the growth behavior of the variable exponents:

$$\begin{cases} 1 < p_1^- \leq p_1(x) \leq p_1^+ < p_2^- \leq p_2(x) \leq p_2^+ < q^- \leq q(x) \leq q^+ < p_1^*(x); \\ p_1^*(x) = \frac{Np_1(x)}{N-p_1(x)}. \end{cases} \tag{3.3}$$

Remark 3.1. Taking account on the relation (3.3) and the embedding theorems for variable exponent Lebesgue and Sobolev spaces we will choose $W_0^{1,p_2(x)}(\Omega)$ as functional space for the solutions of problem (P), and for the simplicity of the writing by $\|\cdot\|$ we will denote the norm associated to $W_0^{1,p_2(x)}(\Omega)$ ($\|\cdot\|_{p_2(x)}$).

We can now define the weak solution for the problem (P).

Definition 3.2. We say that $u \in W_0^{1,p_2(x)}(\Omega) \setminus \{0\}$ is a nontrivial weak solution of the problem (P) if

$$\int_\Omega [\phi(x,|\nabla u|) + \psi(x,|\nabla u|)] \nabla u \nabla v dx = \lambda \int_\Omega f(x,u) v dx$$

for all $v \in W_0^{1,p_2(x)}(\Omega)$.

In order to point out the existence and multiplicity results for our problem we define the following energy functional associated to the problem (P) as it follows:

$$E_\lambda : W_0^{1,p_2(x)}(\Omega) \to \mathbb{R}$$
$$E_\lambda(z) = S(z) - \lambda T(z),$$

where $S(z)$ is defined by relation (3.2) and $T(z) = \int_\Omega F(x,z)dx$, with $F(x,z)$ defined as in relation (3.1).

Taking account of [8, Lemmas 3.2, 3.4], some details from [2, Section 4] and of [23, Lemma 3.1] it is easily to observe that E_λ is of class $C^1\big(W_0^{1,p_2(x)}(\Omega), \mathbb{R}\big)$.

In order to reveal the existence and multiplicity of eigenvalues associated to our problem, we will point out that the critical points of the energy functional E_λ. We can observe that the critical points of E_λ are weak solutions for the problem (P):

$$\begin{aligned}\langle E_\lambda(u), \varphi\rangle = &\int_\Omega [\phi(x,|\nabla u|) + \psi(x,|\nabla u|)] \nabla u \nabla \varphi dx \\ &- \lambda \int_\Omega f(x,u)\varphi dx, \quad \text{for all } \varphi \in W_0^{1,p_2(x)}(\Omega).\end{aligned}$$

Definition 3.3. We say that $E_\lambda \in C^1\big(W_0^{1,p_2(x)}(\Omega), \mathbb{R}\big)$ fulfills the $(C)_c$-condition if for any sequence $(u_n)_n \subset W_0^{1,p_2(x)}(\Omega)$ the following relation holds true:

$$E_\lambda(u_n) \to c \quad \text{and} \quad \|E'_\lambda(u_n)\|_{W^{-1,p'_2(x)}(\Omega)}(1 + \|u_n\|) \to 0$$

we can find a convergent subsequence.

A central role in the proof of the main results of this paper is played by the Fountain Theorem. As we have seen in the Section 2, the variable exponent Sobolev spaces are reflexive and separable Banach spaces. Therefore, taking account of the Remark 3.1, we consider that for $W_0^{1,p_2(x)}(\Omega)$ we have $(e_j)_j \subset W_0^{1,p_2(x)}(\Omega)$ and $(e_j^*) \subset W^{-1,p'_2(x)}(\Omega)$ such that

$$W_0^{1,p_2(x)}(\Omega) = \overline{\text{span}\{e_j : j = 1,2,\dots\}}$$
$$W^{-1,p'_2(x)}(\Omega) = \overline{\text{span}\{e_j^* : j = 1,2,\dots\}}$$

and

$$\langle e_i, e_j^* \rangle = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j, \end{cases}$$

where $\langle \cdot, \cdot \rangle$ represents the duality product between $W_0^{1,p_2(x)}(\Omega)$ and $W^{-1,p_2'(x)}(\Omega)$. We define

$$\begin{cases} X_j = \operatorname{span}\{e_j\}, \\ Y_k = \overset{k}{\underset{j=1}{\oplus}} X_j, \\ Z_k = \overline{\overset{\infty}{\underset{j=k}{\oplus}} X_j}. \end{cases} \tag{3.4}$$

Theorem 3.4 (Fountain Theorem [18]). *Let $E \in C^1(X)$ be an even functional, where $(X, \|\cdot\|)$ is a separable and reflexive Banach space. Suppose that for every $k \in \mathbb{N}$ large enough, there exists $\rho_k > r_k > 0$ such that*

(i) $\inf\{E(u) : u \in Z_k, \|u\| = r_k\} \to +\infty$ *as* $k \to +\infty$,

(ii) $\max\{E(u) : u \in Y_k, \|u\| = \rho_k\} \leq 0$,

(iii) *E satisfies the Palais–Smale condition for every* $c > 0$.

Then E has a sequence of critical values tending to $+\infty$.

For more details and applications on the Fountain Theorem we refer to X. Fan, Q. Zhang [6], D. Repovš [18] and V. F. Uţă [22]. A comprehensive study for various forms of this theorem and its extensions can be found in the following works of Y. Jabri [7], P. Pucci, V. Rădulescu [11], P. Pucci, J. Serrin [14] and P. Pucci, J. Serrin [13]. Also for double phase problems we recommend the following work of P. Pucci, V. Rădulescu [12], M. Ragusa, A. Tachikawa [15], X. Shi, V. Rădulescu, D. Repovš, Q. Zhang [20].

We proceed now to prove some helpful propositions.

Proposition 3.5. *Suppose that conditions (HS_1)–(HS_4), (R_2)–(R_4) hold true, then every $(C)_c$ sequence associated to the energy functional E_λ is bounded.*

Proof. Let $(u_n)_n \subset W_0^{1,p_2(x)}(\Omega)$ be a $(C)_c$ sequence. In order to prove that it is bounded we argue by contradiction and suppose that

$$\|u_n\| \to +\infty \quad \text{as } n \to \infty. \tag{3.5}$$

Using the above relation and taking n large enough we obtain that:

$$\begin{aligned} c + 1 &\geq E_\lambda(u_n) - \frac{1}{p_2^+}\langle E_\lambda'(u_n), u_n\rangle \\ &= \int_\Omega S_0(x, |\nabla u_n|)dx - \frac{1}{p_2^+}\int_\Omega [\phi(x, |\nabla u_n|) + \psi(x, |\nabla u_n|)]\, |\nabla u_n|^2 dx \\ &\quad - \lambda \int_\Omega F(x, u_n)dx + \frac{\lambda}{p_2^+}\int_\Omega f(x, |\nabla u_n|)u_n dx \end{aligned} \tag{3.6}$$

Now using hypothesis (HS_4) we get that:

$$\begin{aligned} c + 1 &\geq \int_\Omega \left(1 - \frac{p_2^+}{p_2^+}\right) S_0(x, |\nabla u_n|)dx - \lambda \int_\Omega F(x, u_n)dx + \frac{\lambda}{p_2^+}\int_\Omega f(x, u_n)u_n dx \\ &\geq -\lambda \int_\Omega F(x, u_n)dx + \frac{\lambda}{p_2^+}\int_\Omega f(x, u_n)u_n dx. \end{aligned}$$

By assumption (R_4) we obtain that

$$c+1 \geq \lambda \int_\Omega \mathcal{R}(x,u_n)dx.$$

As we supposed, the relation (3.5) holds true, then for n sufficiently large we have that $\|u_n\| > 1$. Hence by the fact that $(u_n)_n$ is a $(C)_c$-sequence we obtain that:

$$\begin{aligned} 0 = \lim_{n\to\infty} \frac{c+o(1)}{\|u_n\|^{p_1^-}} &= \lim_{n\to\infty} \frac{E_\lambda(u_n)}{\|u_n\|^{p_1^-}} \\ &\geq \frac{\int_\Omega S_0(x,|\nabla u_n|)dx - \lambda \int_\Omega F(x,u_n)dx}{\|u_n\|^{p_1^-}}. \end{aligned} \tag{3.7}$$

Now using (HS_4), and (HS_3) we obtain that

$$\begin{aligned} 0 &\geq \frac{\frac{1}{p_2^+}\int_\Omega [\phi(x,|\nabla u_n|)+\psi(x,|\nabla u_n|)]\,|\nabla u_n|^2 dx - \lambda \int_\Omega F(x,u_n)dx}{\|u_n\|^{p_1^-}} \\ &\geq \frac{\frac{1}{p_2^+}\int_\Omega C_{\phi,\psi}\left(|\nabla u_n|^{p_1(x)} + |\nabla u_n|^{p_2(x)}\right)dx - \lambda \int_\Omega F(x,u_n)dx}{\|u_n\|^{p_1^-}}. \end{aligned}$$

Now using the modular properties (2.3), (2.4) we obtain that

$$S(u) \geq \frac{C_{\phi,\psi}}{p_2^+}\left(\|u_n\|^{p_1^-}_{p_1(x)} + \|u_n\|^{p_2^-}\right)$$

Now taking account of the fact that by relation (3.3) $p_1^- < p_2^-$, we have that

$$S(u) \geq \frac{C_{\phi,\psi}}{p_2^+}\|u_n\|^{p_1^-}.$$

Hence from (3.7) we obtain that

$$0 \leq \frac{C_{\phi,\psi}}{p_2^+}\frac{\|u_n\|^{p_1^-}}{\|u_n\|^{p_1^-}} - \frac{\lambda \int_\Omega F(x,u_n)dx}{\|u_n\|^{p_1^-}},$$

which yields to

$$\frac{C_{\phi,\psi}}{p_2^+\lambda} \leq \limsup_{n\to\infty} \int_\Omega \frac{|F(x,u_n)|}{\|u_n\|}dx. \tag{3.8}$$

Let $0 \leq a \leq b$ and $D_n^{a,b} = \{z \in \Omega : a \leq |u_n(z)| < b\}$.

Consider in what follows $w_n = \frac{u_n}{\|u_n\|}$. It is obvious that $\|w_n\| = 1$ and there exist a nonnegative constant C_w such that $|w_n|_{q(x)} \leq C_2\|w_n\| = C_w$.

As a consequence of the above facts (passing eventually to a subsequence), we can find an element w_0 such that $w_n \rightharpoonup w_0$ in $W_0^{1,p_2(x)}(\Omega)$.

Moreover,

$$\begin{aligned} w_n \to w_0 \quad &\text{in } L^{r(x)}(\Omega), \qquad 1 \leq r(x) < p_1^*(x) \\ w_n(x) \to w_0(x) \quad &\text{a.e. on } \Omega. \end{aligned} \tag{3.9}$$

In what follows we have to split the proof in two cases:

(I) $w_0 = 0$;

(II) $w_0 \neq 0$.

Let firstly assume that $w_0 = 0$.
We obtain that

$$\begin{cases} w_n \to 0 \text{ in } L^{r(x)}(\Omega) \\ w_n(x) \to 0 \text{ a.e. on } \Omega \end{cases}$$

and by assumption (R_2) we have that

$$\int_{D_n^{0,\rho}} \frac{|F(x,u_n)|}{\|u_n\|^{p_1^-}} dx \leq \frac{C(\rho + \rho^{\overline{q}})|\Omega|}{\|u_n\|^{p_1^-}} \to 0, \tag{3.10}$$

where $\overline{q} = q^+$ if $\rho \geq 1$ and $\overline{q} = q^-$ if $\rho < 1$.

Let $\mu'(x)$ be the conjugate exponent for $\mu(x)$, i.e., $\mu'(x) = \frac{\mu(x)}{\mu(x)-1}$, by hypothesis (R_4) we have that $\mu^- > \max\left\{1, \frac{N}{p_1^-}\right\}$, hence $1 < p_1^- \mu'(x) < p_1^*(x)$. Therefore we get that $w_n \to 0$ in $L^{p_1^- \mu'(x)}(\Omega)$ as $n \to \infty$.

Using Remark 2.6, assumption (R_4), relation (3.6) and (3.9) one have that

$$\begin{aligned}
\int_{D_n^{\rho,+\infty}} \frac{|F(x,u_n)|}{|u_n|^{p_1^-}} |w_n|^{p_1^-} dx &\leq 2 \left| \frac{|F(x,u_n)|}{|u_n|^{p_1^-}} \right|_{L^{\mu(x)}(D_n^{\rho,+\infty})} \left| |w_n|^{p_1^-} \right|_{L^{\mu'(x)}(D_n^{\rho,+\infty})} \\
&\leq 2 \max\left\{ \left(\int_{D_n^{\rho,+\infty}} \frac{|F(x,u_n)|^{\mu(x)}}{|u_n|^{p_1^- \mu(x)}} dx \right)^{\frac{1}{\mu^+}}, \left(\int_{D_n^{\rho,+\infty}} \frac{|F(x,u_n)|^{\mu(x)}}{|u_n|^{p_1^- \mu(x)}} dx \right)^{\frac{1}{\mu^-}} \right\} \\
&\quad \cdot \max\left\{ \left(\int_{D_n^{\rho,+\infty}} |w_n|^{p_1^- \mu'(x)} dx \right)^{\frac{1}{(\mu')^+}}, \left(\int_{D_n^{\rho,+\infty}} |w_n|^{p_1^- \mu'(x)} dx \right)^{\frac{1}{(\mu')^-}} \right\} \\
&\leq 2 \max\left\{ \left(\int_{D_n^{\rho,+\infty}} \mathcal{R}(x,u_n) dx \right)^{\frac{1}{\mu^+}}, \left(\int_{D_n^{\rho,+\infty}} \mathcal{R}(x,u_n) dx \right)^{\frac{1}{\mu^-}} \right\} \\
&\quad \cdot \max\left\{ \left(\int_{D_n^{\rho,+\infty}} |w_n|^{p_1^- \mu'(x)} dx \right)^{\frac{1}{(\mu')^-}}, \left(\int_{D_n^{\rho,+\infty}} |w_n|^{p_1^- \mu'(x)} dx \right)^{\frac{1}{(\mu')^+}} \right\} \\
&\leq 2 \max\left\{ \left(\frac{C_1}{\lambda} (c+1)^{\frac{1}{\mu^+}} \right), \left(\frac{C_1}{\lambda} (c+1)^{\frac{1}{\mu^-}} \right) \right\} \\
&\quad \cdot \max\left\{ \left(\int_{D_n^{\rho,+\infty}} |w_n|^{p_1^- \mu'(x)} dx \right)^{\frac{1}{(\mu')^-}}, \left(\int_{D_n^{\rho,+\infty}} |w_n|^{p_1^- \mu'(x)} dx \right)^{\frac{1}{(\mu')^-}} \right\} \\
&\to 0 \quad \text{as } n \to +\infty.
\end{aligned} \tag{3.11}$$

By relations (3.10) and (3.11), one have that

$$\begin{aligned}
\int_{\Omega} \frac{|F(x,u_n)|}{\|u_n\|^{p_1^-}} dx &= \int_{D_n^{0,\rho}} \frac{|F(x,u_n)|}{\|u_n\|^{p_1^-}} dx + \int_{D_n^{\rho,+\infty}} \frac{|F(x,u_n)|}{\|u_n\|^{p_1^-}} dx \\
&= \int_{D_n^{0,\rho}} \frac{|F(x,u_n)|}{\|u_n\|^{p_1^-}} dx + \int_{D_n^{\rho,+\infty}} \frac{|F(x,u_n)|}{|u_n|^{p_1^-}} |w_n|^{p_1^-} dx \\
&\to 0 \quad \text{as } n \to +\infty.
\end{aligned} \tag{3.12}$$

which is a contradiction with the fact that $\limsup_{n\to\infty} \int_\Omega \frac{|F(x,u_n)|}{\|u_n\|^{p_1^-}}dx > 0$.

We proceed now to prove the second case, and assume that $w_0 \neq 0$. Therefore there exists D^* such that $D^* := \{z \in \Omega : w_0(z) \neq 0\}$, with $|D^*| > 0$, where $|D^*|$ is the Lebesgue measure of D^*.

So, for almost every $z \in D^*$, we have that

$$\lim_{n\to\infty} |u_n(z)| = +\infty. \tag{3.13}$$

Therefore, by (3.13) we get that $D^* \subset D_n^{\rho,+\infty}$ for $n \in \mathbb{N}$ sufficiently large.

With similar arguments as above (see relation (3.10)) it yields that

$$\int_{D_n^{0,\rho}} \frac{|F(x,u_n)|}{\|u_n\|^{p_2^+}}dx \leq \frac{C(\rho+\rho^{\overline{q}})|\Omega|}{\|u_n\|^{p_2^+}} \to 0 \quad \text{as } n\to\infty \tag{3.14}$$

By hypotheses (R_2), (R_3), relation (3.14) and taking use of the Fatou's Lemma one have that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \frac{c+o(1)}{\|u_n\|^{p_2^+}} = \lim_{n\to\infty} \frac{E_\lambda(u_n)}{\|u_n\|^{p_2^+}} \\ &\leq \lim_{n\to\infty} \left[\frac{\int_\Omega S_0(x,|\nabla u_n|)}{\|u_n\|^{p_2^+}}dx - \frac{\lambda}{\|u_n\|^{p_2^+}} \int_\Omega F(x,u_n)dx \right]. \end{aligned} \tag{3.15}$$

In order to complete our proof and obtain the desired contradiction we will compute the term of the energy functional driven by the double-phase operator and the term driven by the reaction function separately.

We firstly compute the part driven by the differential operator. So, taking use of the fact that $\|u_n\| \to \infty$ as $n \to \infty$, using hypothesis (HS_2), Hölder's inequality and the fact that $W_0^{1,p_2(x)}(\Omega) \hookrightarrow W_0^{1,p_1(x)}(\Omega)$ continuously (and one have $\|u\|_{p_1(x)} \leq C_{p_1}\|u\|$, for some nonnegative constant C_{p_1}) we obtain that

$$\begin{aligned} S(x,|\nabla u_n|) &= \int_\Omega S_0(x,|\nabla u_n|)dx \\ &\leq C_\phi|\alpha_1|_{p_1'(x)}\|u_n\|_{p_1(x)}^{p_1^+} + \frac{\xi}{p_1^-}\|u_n\|_{p_1(x)}^{p_1^+} + C_\psi|\alpha_2|_{p_2'(x)}\|u_n\|^{p_2^+} + \frac{\xi}{p_2^+}\|u_n\|^{p_2^+} \\ &\leq C_M\|u_n\|^{p_2^+} \end{aligned} \tag{3.16}$$

where $C_M = \left(C_\phi|\alpha_1|_{p_1'(x)} \cdot C_{p_1} + \frac{\xi}{p_1^-}C_{p_1}\right) + \left(C_\psi|\alpha_2|_{p_2'(x)} + \frac{\xi}{p_2^-}\right)$, and C_ϕ, C_ψ are two nonnegative constants, $C_\phi, C_\psi > 0$, which depend on the potential functions ϕ, ψ and on the continuous embeddings: $W_0^{1,p_1(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$, $W_0^{1,p_2(x)}(\Omega) \hookrightarrow W_0^{1,p_1(x)}(\Omega)$.

We proceed now to compute the second part of the energy functional driven by our reaction function term of the problem (P).

Combining relations (3.15) and (3.16)

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} \left[\frac{C_M\|u_n\|^{p_2^+}}{\|u_n\|^{p_2^+}} - \lambda \left(\int_{D_n^{0,\rho}} \frac{F(x,u_n)}{\|u_n\|^{p_2^+}}dx + \int_{D_n^{\rho,+\infty}} \frac{F(x,u_n)}{\|u_n\|^{p_2^+}}dx \right) \right] \\ &= \lim_{n\to\infty} \left[C_M - \frac{\lambda}{\|u_n\|^{p_2^+}} \left(\int_{D_n^{0,\rho}} F(x,u_n)dx + \int_{D_n^{\rho,+\infty}} F(x,u_n)dx \right) \right] \end{aligned} \tag{3.17}$$

$$
\begin{aligned}
&\leq \lim_{n\to\infty}\left[C_M - \frac{\lambda}{\|u_n\|^{p_2^+}}\int_{D_n^{\rho,+\infty}} F(x,u_n)dx\right]\\
&\leq \limsup_{n\to\infty}\left[C_M - \frac{\lambda}{\|u_n\|^{p_2^+}}\int_{D_n^{\rho,+\infty}} F(x,u_n)dx\right]\\
&= C_M - \liminf_{n\to\infty}\lambda\int_{D_n^{\rho,+\infty}} \frac{F(x,u_n)}{|u_n|^{p_2^+}}|w_n|^{p_2^+}dx\\
&= C_M - \liminf_{n\to\infty}\lambda\int_{\Omega} \frac{F(x,u_n)}{|u_n|^{p_2^+}}\chi_{D_n^{\rho,+\infty}}(x)|w_n|^{p_2^+}dx\\
&\leq C_M - \lambda\int_{\Omega} \liminf_{n\to\infty}\frac{F(x,u_n)}{|u_n|^{p_2^+}}\chi_{D_n^{\rho,+\infty}}(x)|w_n|^{p_2^+}dx\\
&\to -\infty, \quad \text{as } n\to\infty,
\end{aligned}
$$

which contradicts relation (3.17).

Therefore we obtained the fact that any $(C)_c$-sequence is bounded, so our proof is complete. □

Proposition 3.6. *If assumptions (R_1)–(R_4) hold true, then for every $(C)_c$-sequence of E_λ we can find a convergent subsequence in $W_0^{1,p_2(x)}(\Omega)$.*

Proof. Suppose that $(v_n)_n \subset W_0^{1,p_2(x)}(\Omega)$ is a $(C)_c$-sequence for E_λ. Using Proposition 3.5 we have that $(v_n)_n$ is bounded in $W_0^{1,p_2(x)}(\Omega)$, so, passing eventually to a subsequence we obtain the fact that $v_n \rightharpoonup v_0$ in $W_0^{1,p_2(x)}(\Omega)$. Using Remark 2.6 it yields that $(v_n)_n$ is bounded in $L^{q(x)}(\Omega)$ and by the continuous and compact embedding $W_0^{1,p_2(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, we get that $v_n \to v_0$ in $L^{q(x)}(\Omega)$ as $n\to\infty$.

By straightforward computations we obtain that

$$
\begin{aligned}
&\int_\Omega |f(x,v_n)-f(x,v_0)|\,|v_n-v_0|dx\\
&\quad\leq \int_\Omega (|f(x,v_n)|+|f(x,v_0)|)\,|v_n-v_0|dx\\
&\quad\leq \int_\Omega \left[C(1+|v_n|^{q(x)-1})+C(1+|v_0|^{q(x)-1})\right]|v_n-v_0|dx\\
&\quad\leq 2C\int_\Omega |v_n-v_0|dx + C\int_\Omega |v_n|^{q(x)-1}|v_n-v_0|dx+\int_\Omega |v_0|^{q(x)-1}|v_n-v_0|dx \qquad (3.18)\\
&\quad\leq 2C|v_n-v_0|_{L^1(\Omega)} + 2C\left||v_n|^{q(x)-1}\right|_{q'(x)}\cdot|v_n-v_0|_{q(x)} + 2C\left||v_0|^{q(x)-1}\right|_{q'(x)}|v_n-v_0|_{q(x)}\\
&\quad\leq 2C|v_n-v_0|_{L^1(\Omega)} + 2C\max\left\{|v_n|_{q(x)}^{q^+-1},|v_n|_{q(x)}^{q^--1}\right\}\cdot|v_n-v_0|_{q(x)}\\
&\qquad + 2C\max\left\{|v_0|_{q(x)}^{q^+-1},|v_0|_{q(x)}^{q^--1}\right\}|v_n-v_0|_{q(x)} \to 0 \quad \text{as } n\to\infty,
\end{aligned}
$$

where $q'(x)$ is the conjugate exponent of $q(x)$, i.e. $\frac{1}{q(x)}+\frac{1}{q'(x)}=1$.

Now taking account of [8, Lemma 3.2] one have that:

$$
\begin{aligned}
&\langle S'(v_n)-S'(v_0),v_n-v_0\rangle\\
&\qquad = \langle E'_\lambda(v_n)-E'_\lambda(v_0),v_n-v_0\rangle + \int_\Omega [f(x,v_n)-f(x,v_0)]\,(v_n-v_0)dx \qquad (3.19)
\end{aligned}
$$

and by Definition 3.3, keeping in mind that $(v_n)_n$ is a $(C)_c$-sequence of the energy functional E_λ, we get that:

$$\lim_{n\to\infty} \langle E'_\lambda(v_n) - E'_\lambda(v_0), v_n - v_0\rangle = 0. \tag{3.20}$$

Now by relations (3.18), (3.19), (3.20) and taking account of [8, Lemma 3.4] we obtain the fact that

$$\lim_{n\to\infty} \langle S'(v_n) - S'(v_0), v_n - v_0\rangle = 0,$$

and by the fact that S is of type $(S)_+$ (see also [8, Lemma 3.4]) it yields that $v_n \to v_0$ in $W_0^{1,p_2(x)}(\Omega)$, and so, our proof is complete. □

Proposition 3.7. *If assumptions (R_1)–(R_3) and (R_5) hold true, then for every $(C)_c$-sequence of E_λ, we can find a convergent subsequence in $W_0^{1,p_2(x)}(\Omega)$.*

Proof. Taking use of Proposition 3.5, and keeping in mind the proof of Proposition 3.6 we only have to prove that our sequence is bounded in $W_0^{1,p_2(x)}(\Omega)$.

Let $(v_n)_n \subset W_0^{1,p_2(x)}(\Omega)$ be a $(C)_c$-sequence for E_λ. Arguing by contradiction we suppose that $\|v_n\| \to \infty$ as $n \to \infty$. Now, taking $w_n = \frac{v_n}{\|v_n\|}$, we get that $\|w_n\| = 1$, for all $n \in \mathbb{N}$, futhermore we obtain that $|w_n|_{q(x)} \leq C_w\|w_n\|$, where $C_w > 0$ is a constant.

By the above facts and passing eventually to a subsequence we may find w_0 such that

$$w_n \rightharpoonup w_0 \quad \text{in } W_0^{1,p_2(x)}(\Omega), \tag{3.21}$$

and by the compact embedding $W_0^{1,p_2(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ we obtain that

$$\begin{aligned} w_n &\to w_0 \quad \text{in } L^{q(x)}(\Omega) \\ w_n(x) &\to w_0(x) \quad \text{a.e. on } \Omega. \end{aligned} \tag{3.22}$$

Now by the definition of E_λ it yields that:

$$\begin{aligned} c+1 &\geq E_\lambda(v_n) - \frac{1}{\omega}\langle E'_\lambda(v_n), v_n\rangle \\ &= \int_\Omega S_0(x, |\nabla v_n|) - \frac{1}{\omega}\left[\phi(x,|\nabla v_n|)\nabla v_n + \psi(x,|\nabla v_n|)\nabla v_n\right] dx \\ &\quad + \lambda \int_\Omega \left[\frac{1}{\omega} f(x, v_n)v_n - F(x, v_n)\right] dx. \end{aligned}$$

Now by hypothesis (HS_4) combined with (HS_3) we get that

$$\begin{aligned} c+1 &\geq \int_\Omega \left(1 - \frac{p_2^+}{\omega}\right) S_0(x,|\nabla v_n|)dx + \lambda \int_\Omega \left[\frac{1}{\omega} f(x,v_n)v_n - F(x,v_n)\right] dx \\ &\geq \int_\Omega \left(1 - \frac{p_2^+}{\omega}\right) \cdot \frac{C_{\phi,\psi}}{p_2^+} \left[|\nabla v_n|^{p_1(x)-2} + |\nabla v_n|^{p_2(x)-2}\right] |\nabla v_n|^2 dx \\ &\quad + \lambda \int_\Omega \left[\frac{1}{\omega} f(x,v_n)v_n - F(x,v_n)\right] dx \\ &\geq \left(1 - \frac{p_2^+}{\omega}\right) \cdot \frac{C_{\phi,\psi}}{p_2^+} \int_\Omega |\nabla v_n|^{p_2(x)} dx + \lambda \int_\Omega \left[\frac{1}{\omega} f(x,v_n)v_n - F(x,v_n)\right] dx. \end{aligned}$$

Using assumption (R_5) we obtain that:

$$c+1 \geq \left(1-\frac{p_2^+}{\omega}\right)\cdot\frac{C_{\phi,\psi}}{p_2^+}\|v_n\|^{p_2^-}-\frac{\lambda}{\omega}\eta\int_\Omega |v_n|^{p_1^-}dx$$

for any $n \geq 0$.

Hence passing to $(w_n)_n$ we obtain that:

$$\begin{aligned}\frac{\lambda}{\omega}\eta\int_\Omega |w_n|^{p_1^-}dx &\geq \left(1-\frac{p_2^+}{\omega}\right)\cdot\frac{C_{\phi,\psi}}{p_2^+}\\ &\Rightarrow \frac{\lambda\eta}{\omega}\cdot\frac{\omega p_2^+}{(\omega-p_2^+)C_{\phi,\psi}}\int_\Omega |w_n|^{p_1^-}dx \geq 1\\ &\Rightarrow \frac{\lambda\eta p_2^+}{(\omega-p_2^+)C_{\phi,\psi}}|w_n|_{p_1^-}^{p_1^-} \geq 1\\ &\Rightarrow \frac{\lambda\eta p_2^+}{(\omega-p_2^+)C_{\phi,\psi}}\limsup_{n\to\infty}|w_n|_{p_1^-}^{p_1^-} \geq 1 \qquad (3.23)\end{aligned}$$

Now, keeping in mind relations (3.21) and (3.22) we have that $w_n \to w_0$ in $L^{p_1^-}(\Omega)$, moreover by (3.23) we get that $w_0 \neq 0$.

In order to obtain the desired contradiction we apply the same technique as in the case (II) from the proof of Proposition 3.5 and the contradiction is obtained.

Therefore we have the fact that $(v_n)_n$ is bounded in $W_0^{1,p_2(x)}(\Omega)$. In order to complete the proof we only have to repeat the steps taken in the proof of Proposition 3.6 and the work is accomplished. □

4 Main results

In this section using the Fountain Theorem we will reveal the fact that the problem (P) has an unbounded sequence of weak solutions with higher and higher energies.

We are now ready to enunciate and prove our main results.

Theorem 4.1. *If assumptions (HS_1)–(HS_4), (R_1)–(R_4), (R_6) and (3.3) hold true, then for every $\lambda > 0$ the problem (P) possesses an infinite sequence of nontrivial weak solutions.*

Theorem 4.2. *If assumptions (HS_1)–(HS_4), (R_1)–(R_3), (R_5), (R_6) and (3.3) hold true, then for every $\lambda > 0$ the problem (P) possesses an infinite sequence of nontrivial weak solutions.*

Proof of Theorem 4.1. As we have seen in the previous section, as $W_0^{1,p_2(x)}(\Omega)$ is separable, reflexive Banach space, let us consider Y_k and Z_k denoted by relation (3.4).

Firstly we check if condition (i) from the Theorem 3.4 holds true.

Let $a_k := \sup\left\{|u|_{q(x)} : \|u\| = 1, u \in Z_k\right\}$. It is easily to observe the fact that $a_k \to 0$ as $k \to \infty$. The reasoning behind the above statement is the following. By the definition of $(a_k)_k$ we get that $a_k > a_{k+1} \geq 0$, therefore $a_k \to a \geq 0$, as $k \to \infty$. By the reflexivity of $W_0^{1,p_2(x)}(\Omega)$, and taking $u_k \in Z_k$, $\|u_k\| = 1$ for each $k \in \mathbb{N}$ such that

$$0 \leq a_k - |u_k|_{q(x)} \leq \frac{1}{k},$$

we get that $(u_k)_k$ has a convergent subsequence and suppose $u_k \rightharpoonup u_1$ in $W_0^{1,p_2(x)}(\Omega)$. Keeping in mind the definition of Z_k we obtain that $u_1 = 0$. Taking account of [8, Lemma 3.4] we have that $u_k \to 0$ in $L^{q(x)}(\Omega)$, so it yields that $a = 0$.

Now let $u \in Z_k$ with $\|u\| = \rho_k > 1$, where ρ_k will be specified later.

Using hypotheses (HS_3) and (HS_4) and (2.3) we have that

$$\begin{aligned} E_\lambda(u) &= \int_\Omega S_0(x, |\nabla u|)dx - \lambda \int_\Omega F(x,u)dx \\ &\geq \frac{C_{\phi,\psi}}{p_2^+}\left(\int_\Omega |\nabla u|^{p_1(x)}dx + \|u\|^{p_2^-}\right) - \lambda \int_\Omega F(x,u)dx \\ &\geq \frac{C_{\phi,\psi}}{p_2^+}\|u\|^{p_2^-} - \lambda \int_\Omega F(x,u)dx. \end{aligned} \tag{4.1}$$

Now using assumption (R_2) we get that

$$F(x,z) \leq C(|z| + |z|^{q(x)}) \leq 2C(1 + |z|^{q(x)}) \tag{4.2}$$

for all $(x,z) \in \Omega \times \mathbb{R}$.

Using (4.1) and (4.2) we obtain that

$$\begin{aligned} E_\lambda(u) \geq& \frac{C_{\phi,\psi}}{p_2^+}\|u\|^{p_2^+} - 2\lambda C \int_\Omega \left(1 + |u|^{q(x)}\right) dx \\ \geq& \frac{C_{\phi,\psi}}{p_2^+}\|u\|^{p_2^+} - 2\lambda C \left[|\Omega| + \max\left\{|u|_{q(x)}^{q^-}, |u|_{q(x)}^{q^+}\right\}\right] \end{aligned}$$

(where $|\Omega|$ represents the Lebesgue measure of Ω).

Taking account of the continuous embedding $W_0^{1,p_2(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ we have $|u|_{q(x)} \leq C_3\|u\|$ and then the above inequality becomes:

$$\begin{aligned} E_\lambda(u) &\geq \frac{C_{\phi,\psi}}{p_2^-}\|u\|^{p_2^-} - 2\lambda C \left[|\Omega| + \max\left\{C_3^{q^-}\|u\|^{q^-}, C_3^{q^+}\|u\|^{q^+}\right\}\right] \\ &\geq \frac{C_{\phi,\psi}}{p_2^-}\|u\|^{p_2^-} - 2\lambda C\tilde{C}^q\|u\|^{q^+} - 2\lambda C|\Omega| \\ &\quad \text{(where } \tilde{C}^q = \max\left\{C_3^{q^+}, C_3^{q^-}\right\}) \\ &\geq \frac{C_{\phi,\psi}}{p_2^-}\|u\|^{p_2^-} - 2\lambda C\tilde{C}^q a_k^{q^+}\|u\|^{q^+} - 2\lambda C|\Omega|. \end{aligned}$$

It can be easily checked that if we choose

$$\rho_k = \left(\frac{2\lambda C\tilde{C}^q}{C_{\phi,\psi}} \cdot p_2^- a_k^{p_2^+}\right)^{\frac{1}{p_2^- - q^+}} \tag{4.3}$$

combined with the fact that $p_2^- < q^+$ and $a_k \to 0$ as $k \to +\infty$, we obtain that $\rho_k \to +\infty$ as $k \to +\infty$.

Taking $\|u\| = \rho_k$ with ρ_k as stated in relation (4.3) we obtain that

$$E_\lambda(u) \to +\infty \quad \text{as } k \to +\infty,$$

and so, the validity of condition (i) is proved.

We check now if the condition (ii) from the Fountain Theorem holds true.
Assume that $u \in Y_k$ and $\|u\| = \tau_k > 1$, where τ_k will be specified later.
By hypothesis (HS_2) we have that

$$E_\lambda(u) \leq 2C_4|\alpha_1|_{p_1'(x)} \max\left\{\|u\|_{p_1(x)}^{p_1^-}, \|u\|_{p_1(x)}^{p_1^+}\right\} + \frac{\xi}{p_1^-} \max\left\{\|u\|_{p_1(x)}^{p_1^-}, \|u\|_{p_1(x)}^{p_1^+}\right\}$$
$$+ 2C_5|\alpha_2|_{p_2'(x)}\|u\|^{p_2^+} + \frac{\xi}{p_2^-}\|u\|^{p_2^+} - \lambda \int_\Omega F(x,u)dx,$$

where $C_4,\ C_5$ are some strictly nonnegative constants.

Taking account of the continuous embedding described in Remark 2.5 we obtain that

$$E_\lambda(u) \leq C_6\|u\|^{p_2^+} - \lambda \int_\Omega F(x,u)dx \tag{4.4}$$

where $C_6 = \left(2C_4|\alpha_1|_{p_1'(x)}C_{p_1} + \frac{\xi}{p_1^-}C_{p_1}\right) + \left(2C_5|\alpha_2|_{p_2'(x)} + \frac{\xi}{p_2^-}\right)$, and $C_{p_1} = \max\left\{C_2^{p_1^-}, C_2^{p_2^+}\right\}$.

In order to complete the proof of condition (ii), we argue by contradiction and assume that (ii) is not true for some given n. Hence we can find a sequence $(v_n)_n \subset Y_n$ such that

$$\|v_n\| \to +\infty \quad \text{as } n \to +\infty \quad \text{and} \quad E_\lambda(v_n) \geq 0. \tag{4.5}$$

Suppose now that $w_n = \frac{v_n}{\|v_n\|}$, therefore $\|w_n\| = 1$. As $\dim Y_k < +\infty$, then we can find some $w_0 \in Y_k \setminus \{0\}$ such that, passing eventually to a subsequence we get that

$$\begin{cases} w_n \to w_0, \\ w_n(x) \to w_0(x) \text{ a.e. } x \in \Omega \end{cases} \quad \text{as } n \to +\infty.$$

As $w(x) \neq 0$, we get that $|v_n(x)| \to +\infty$ as $n \to +\infty$. Taking account of hypothesis (R_3) we obtain that

$$\lim_{n\to+\infty} \frac{F(x,|v_n(x)|)}{\|v_n\|^{p_2^+}} = \lim_{n\to+\infty} \frac{F(x,v_n(x))}{|v_n(x)|^{p_2^+}}|w_n(x)|^{p_2^+} = +\infty$$

for all $x \in D_0 := \{x \in \Omega : \ w(x) \neq 0\}$. With the same arguments as in the proof of Proposition 3.6 we get that

$$\int_{D_0} \frac{F(x,v_n)}{\|v_n\|^{p_2^+}}dx \to +\infty \quad \text{as } n \to +\infty.$$

Taking $n \in \mathbb{N}$, large enough we have that $D_0 \subset D_n^{\rho,+\infty}$ (the domain considered in the proof of Proposition 3.5), and so the following estimates hold true:

$$\begin{aligned}
E_\lambda(v_n) &\leq C_6\|v_n\|^{p_2^+} - \lambda\left[\int_{D_n^{0,\rho}} F(x,v_n)dx + \int_{D_n^{\rho,+\infty}} F(x,v_n)dx\right] \\
&\leq C_6\|v_n\|^{p_2^+} + C_7\int_{D_n^{0,\rho}} (\rho + \rho^q)\, dx - \int_{D_n^{\rho,+\infty}} F(x,v_n)dx \\
&\qquad \text{(where } C_7 > 0 \text{ is some nonnegative constant)} \\
&\leq C_6\|v_n\|^{p_2^+} + C_7\,(\rho + \rho^q)\,|\Omega| - \int_{D_n^{\rho,+\infty}\cap D_0} F(x,v_n)dx \\
&\leq \|v_n\|^{p_2^+}\left(C_6 + \frac{C\,(\rho+\rho^q)\,|\Omega|}{\|v_n\|^{p_2^+}} - \int_{D_n^{\rho,+\infty}\cap D_0} \frac{F(x,v_n)}{\|v_n\|^{p_2^+}}dx\right) \\
&\to -\infty \quad \text{as } n \to +\infty,
\end{aligned}$$

which is a contradiction with relation (4.5) and so we have completed the proof that condition (ii) holds true.

As in Proposition 3.6 we have proved that the energy functional E_λ verifies the $(C)_c$-condition and by hypothesis (R_6) the function that gives the reaction term of our equation is odd we can conclude the proof of Theorem 4.1 by simply applying the Fountain Theorem. □

Remark 4.3. Taking account of the above theorem we have proved that for every $\lambda > 0$ we have an unbounded sequence of solutions obtained for higher and higher energies.

Proof of Theorem 4.2. With the same arguments as in the proof of Theorem 4.1, we can point out that condition (i) of the Fountain Theorem is checked (as the assumptions (R_4) and (R_5) plays no role in this part of the proof).

To check the validity of condition (ii) from the Fountain Theorem we combine the arguments from the verification of condition (ii) of the proof to Theorem 4.1 with similar arguments as in the proof of Proposition 3.7 and the condition is checked. Therefore, as in the Proposition 3.7 we have verified the fact that our energy functional satisfies the $(C)_c$-condition and by (R_6) the reaction term of our problem is an odd function, and as $E_\lambda(z) = E_\lambda(-z)$ we only have to apply the Fountain Theorem.

Hence for the energy functional E_λ we have obtained an unbounded sequence of critical values $(u_n)_n \subset W_0^{1,p_2(x)}(\Omega)$ such that $E'_\lambda(u_n) \to 0$ and $E_\lambda(u_n) \to c$ as $n \to +\infty$. □

5 Some examples and final remarks

As the definitions of our double phase-operator and of our reaction term are very general, in what follows we will give some specific examples in order to illustrate the validity of our results.

Example 5.1. Consider the following weight coefficient functions $a, b : \Omega \to \mathbb{R}$, with $a, b \in L^\infty(\Omega)_+$ for all $x \in \Omega$. Suppose there exist a constant $C_{a,b} > 0$ such that $a(x), b(x) \geq C_{a,b}$ for all $x \in \Omega$. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function which satisfy the assumptions $(R_1) - (R_6)$, (3.3) then the results of Theorems 4.1, 4.2 hold true for the following class of Dirichlet problems:

$$\begin{cases} -\operatorname{div}\left[a(x)|\nabla u|^{p_1(x)-2}\nabla u + b(x)|\nabla u|^{p_2(x)-2}\nabla u\right] = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

It is easy to check the fact that our differential operator satisfy hypotheses (HS_1)–(HS_4).

Example 5.2. As we stated in the first section of this paper our potential functions ϕ and ψ generalize the following type of differential operator

$$A(x,z) = \left(1 + \frac{z^{p(x)}}{\sqrt{1+z^{2p(x)}}}\right) z^{p(x)-2} \tag{5.1}$$

corresponding to the differential operator which describes the capillary phenomenon, so we obtain the following class of double-phase problems:

$$\begin{cases} -\operatorname{div}\left[\left(|\nabla u|^{p_1(x)-2}+\frac{|\nabla u|^{2p_1(x)-2}}{\left(1+|\nabla u|^{2p_1(x)}\right)^{1/2}}\right)\nabla u \right. \\ \qquad\qquad \left. +\left(|\nabla u|^{p_2(x)-2}+\frac{|\nabla u|^{2p_2(x)-2}}{\left(1+|\nabla u|^{2p_2(x)}\right)^{1/2}}\right)\nabla u\right]=\lambda f(x,u) & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases}$$

If hypotheses (3.3), (R_1)–(R_6) hold true, then the results of Theorems 4.1 and 4.2 hold true for this class of problems, i.e., this class of problems admits infinitely many nontrivial weak solutions with high and higher energies.

By simple computations we could verify that the potential function of type A from relation (5.1) satisfy the assumptions (HS_1)–(HS_4). For a thorough proof of the validity of our example we can associate the following energy functional to our problem $E_\lambda : W_0^{1,p_2(x)}(\Omega) \to \mathbb{R}$ defined by

$$E_\lambda(u) = \int_\Omega \frac{1}{p_1(x)}\left[|\nabla u|^{p_1(x)} + \left(1+|\nabla u|^{2p_1(x)}\right)^{1/2}\right]dx + \\ + \int_\Omega \frac{1}{p_2(x)}\left[|\nabla u|^{p_2(x)} + \left(1+|\nabla u|^{2p_2(x)}\right)^{1/2}\right]dx - \lambda\int_\Omega F(x,u)dx$$

and recalculate the computations for this functional energy.

In what follows we will give some examples and remarks on the reaction function and the results of this paper.

In order to prove the boundedness of the Palais–Smale sequence it is very popular in the literature to use the (AR)-condition, i.e.,

(AR) There exist some constants $A>0$, $\omega > p_2^+$ such that for $|z|>A$ and for almost every $x \in \Omega$

$$0 < \omega F(x,z) \le z f(x,z)$$

where $F(x,z) = \int_0^z f(x,t)dt$.

Remark 5.3. The (AR)-condition described above implies the fact that our reaction function $f(x,\cdot)$ must have at least $(\omega-1)$-polynomial growth near $+\infty$.

Remark 5.4. There exists an entire class of functions that are superlinear at infinity, but does not satisfy the (AR)-condition for any $\omega > p_2^+$.

An example of this type of function is

$$f(x,z) = p_2^+|z|^{p_2^+-2}z\ln(1+z^2), \tag{5.2}$$

and we obtain that

$$F(x,z) = |z|^{p_2^+}\ln\left(1+z^2\right) - \frac{2|z|^{p_2^+}z}{1+z^2}. \tag{5.3}$$

Remark 5.5. It is easily to observe that the function defined in relation (5.2) does not satisfy the (AR)-condition, but it satisfies conditions (R_3) and (R_4), therefore the results of Theorems 4.1 and 4.2 hold true.

Remark 5.6. (i) Similar results but under the stronger hypothesis (i.e. (AR)-condition is to be satisfied by the reaction term of the problem) where obtained for this problem in [22] and in [8](where furthermore the differential operator is driven only by the potential function ϕ).
(ii) Some spectral results for this type of problem which does not use the (AR)-condition where obtained in [22], but with the price of taking the real parameter λ in a small interval near the origin, and the growth of the reaction function to be more general, i.e., $q^- < p_1^-$, but in this case it is not know the behavior of the quantity $\sup_{x\in\Omega} q(x)$.

Remark 5.7. Also for the coercive case of the problem we refer to [2,21], for the double-phase differential operator and to [8] for the simpler case were the differential operator is driven by only one potential term.

Remark 5.8. According to the terminology used in this paper the study of integral functionals described by relations (1.1), (1.2) correspond to differential operators described by (C_1) and relation (1.3). An interesting extension of the results obtained in this paper can be realized by studying this problems in a more general framework of Musielak–Orlicz spaces. To this end we refer to some results described in [17, Chapter 4].

Remark 5.9. An important role in obtaining our results is played by assumptions (3.3) which indicates the fact that we are in the subcritical framework in the sense of Sobolev variable exponents. No results are known in the critical or supercritical framework. Moreover, no results are known even in the "almost critical" case with lack of compactness where (3.3) is replaced by

$$p_1(x) < p_2(x) < q^- \leq q(x) \leq q^+ \preceq p_1^*(x) \quad \text{for all } x \in \overline{\Omega},$$

where $q(x) \preceq p_1^*(x)$ means that there exists $z \in \Omega$ such that $q(z) = p_1^*(z)$ and $q(x) < p_1^*(x)$ for all $x \in \overline{\Omega} \setminus \{z\}$.

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Uniqueness criteria for ordinary differential equations with a generalized transversality condition at the initial condition

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Abstract. In this paper, we present some uniqueness results for systems of ordinary differential equations. All of them are linked by a weak transversality condition at the initial condition, which generalizes those in the previous literature. Several examples are also provided to illustrate our results.

Keywords: uniqueness, ordinary differential equation.

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1 Introduction

This paper considers local uniqueness of solutions for the initial value problem

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{1.1}$$

where $f : U \subset \mathbb{R}^{n+1} \to \mathbb{R}^n$ is continuous and U is a neighborhood of the point $(t_0, x_0) \in \mathbb{R}^{n+1}$.

Hoag proved in [11] the following result concerning unique solvability of (1.1) in the scalar case ($n = 1$).

Theorem 1.1. *For* $(t_0, x_0) \in \mathbb{R}^2$ *and positive numbers* a *and* b*, define*

$$U = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b].$$

Let $f : U \to \mathbb{R}$ *be a continuous function satisfying the following three conditions:*

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(i) *there are constants* $c > 0$ *and* $r \in (0, 1/2)$ *such that*

$$|f(t,x)| \geq c|x - x_0|^r \quad \textit{for all } (t,x) \in U;$$

(ii) $f(t, x_0)$ *is not identically zero on any interval* $(t_0 - \varepsilon, t_0)$ *or* $(t_0, t_0 + \varepsilon)$ *for* $0 < \varepsilon < a$;

(iii) *there is a number* $K \geq 0$ *such that for all* (t, x) *and* (s, x) *in* U,

$$|f(t,x) - f(s,x)| \leq K|t - s|.$$

Then there is a unique solution to the initial value problem (1.1) *in some interval* $(t_0 - \alpha, t_0 + \alpha)$ *with* $\alpha > 0$.

Basically, Theorem 1.1 replaces the transversality condition $f(t_0, x_0) \neq 0$, employed in previous papers together with the Lipschitz condition with respect to the first variable (iii), see for instance [6,7,13,14,17], by assumptions (i) and (ii). See also [5].

On the other hand, a generalized Lipschitz condition which measures the field differences in different directions to that given by the axis was studied and progressively improved in a series of papers [9,10,16,17]. The following result is the main uniqueness criterion in [16].

Theorem 1.2. *Let* D *be an open neighborhood of the point* $(t_0, x_0) \in \mathbb{R}^{n+1}$ *and* $f : D \to \mathbb{R}^n$ *be a continuous function. Let* $\mathcal{V} \subset \mathbb{R}^{n+1}$ *be a hyperspace and assume that*

(i) $(1, f(t_0, x_0)) \notin \mathcal{V}$,

(ii) f *is Lipschitz continuous along the hyperspace* $\mathcal{V}$ *on* D, *i.e., there exists* $L \geq 0$ *such that for all* $(t, x), (s, y) \in D$,

$$\|f(t,x) - f(s,y)\| \leq L\,\|(t,x) - (s,y)\| \quad \textit{if } (t,x) - (s,y) \in \mathcal{V}.$$

Then problem (1.1) *has a unique local solution.*

The aim of this paper is to extend conditions (i) and (ii) in Theorem 1.1 to the case of systems and to combine them with different hypotheses about f, such as the generalized Lipschitz notion in [16] or the perturbed Lipschitz assumption in [12], in order to obtain uniqueness for (1.1).

The paper is organized as follows: in Section 2, Theorem 1.1 is extended to the case of systems. Our result relies on the concept of *Lipschitz continuous function when fixing a variable*, which was introduced in [4].

The main aim of Section 3 is to adapt the arguments in Theorem 1.1 to functions which are *Lipschitz continuous along a hyperspace* $\mathcal{V}$, as defined in [16, Theorem 2.1 (A2)], and thus allowing the transversality condition $(1, f(t_0, x_0)) \notin \mathcal{V}$ to fail. Therefore, the results in [11] and [16] are simultaneously weakened.

In Section 4 we present a different uniqueness result which was inspired by a particular form of expressing the function f as certain composition of functions due to Bressan and Shen [3]. The weak Lipschitz-type condition required on f is similar to that in the classical uniqueness criterion in [15]. It allows us to construct a set which contains all possible solutions of problem (1.1) and where f is Lipschitz with respect to x.

We point out that all uniqueness results in this paper are connected since they require a relaxed transversality condition at the initial point.

2 Uniqueness via a Lipschitz condition when fixing a variable

Here we prove a relaxed version of Theorem 1.1 in the case of systems. In the sequel we use the notation $\overline{B}_a(x)$ for the closed ball with radius $a > 0$ centered at $x \in \mathbb{R}^p$, with the metric defined by the maximum norm $\|(y_1, y_2, \dots, y_p)\| = \max\{|y_1|, |y_2|, \dots, |y_p|\}$.

Theorem 2.1. *For $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$, $x_0 = (x_{0,1}, x_{0,2}, \dots, x_{0,n})$, and positive numbers a and b, define*

$$U = [t_0 - a, t_0 + a] \times \overline{B}_b(x_0).$$

Let $f : U \to \mathbb{R}^n$ be a continuous function satisfying the following three conditions:

(1) *there is a continuous function $M : [x_{0,n} - b, x_{0,n} + b] \to (0, +\infty)$ such that $1/M^2 \in L^1(x_{0,n} - b, x_{0,n} + b)$ and for all $z \in [x_{0,n} - b, x_{0,n} + b]$, $z \neq x_{0,n}$, we have*

$$|f_n(t, x_1, x_2, \dots, x_{n-1}, z)| \geq M(z) > 0$$

for all $(x_1, \dots, x_{n-1}) \in [x_{0,1} - b, x_{0,1} + b] \times \cdots \times [x_{0,n-1} - b, x_{0,n-1} + b]$ and all $t \in [t_0 - a, t_0 + a]$;

(2) *$f_n(t, y_1(t), \dots, y_{n-1}(t), x_{0,n})$ is not identically zero on any interval $(t_0 - \varepsilon, t_0)$ or $(t_0, t_0 + \varepsilon)$ for $0 < \varepsilon < a$ whenever $(y_1, \dots, y_{n-1}) \in \mathcal{C}(t_0 - \varepsilon, t_0 + \varepsilon)$;*

(3) *f is Lipschitz continuous when fixing the variable x_n, i.e., there exists $K \geq 0$ such that*

$$\|f(t_1, x_1, \dots, x_{n-1}, z) - f(t_2, y_1, \dots, y_{n-1}, z)\| \leq K \|(t_1, x_1, \dots, x_{n-1}) - (t_2, y_1, \dots, y_{n-1})\|$$

for all $(t_1, x_1, \dots, x_{n-1}, z), (t_2, y_1, \dots, y_{n-1}, z) \in U$, $z \neq x_{0,n}$.

Then there is a unique solution to the initial value problem (1.1) in some interval $(t_0 - \alpha, t_0 + \alpha)$ with $\alpha > 0$.

Proof. Firstly, notice that, since f is continuous, there exists $L > 0$ such that $\|f(t, x)\| \leq L$ for all $(t, x) \in U$. Furthermore, condition (1) implies that $f_n(t, x) \neq 0$ for any $(t, x) \in U$ such that $x_n \neq x_{0,n}$. Then f_n has constant sign on the connected sets

$$U^+ := \{(t, x) \in U : x_n > x_{0,n}\} \quad \text{and} \quad U^- := \{(t, x) \in U : x_n < x_{0,n}\}.$$

Now, from (2) it follows that the sign of f_n on both U^+ and U^- must be the same, so in particular f_n does not change sign on U (that is, either $f_n(t, x) \geq 0$ for all $(t, x) \in U$ or $f_n(t, x) \leq 0$ for all $(t, x) \in U$).

Let $x = (x_1, x_2, \dots, x_n)$ be a solution of (1.1) defined on an interval $[t_0 - a_1, t_0 + a_1]$, with $0 < a_1 < a$. We will show that x_n is strictly monotone on a neighborhood of t_0. First, observe that we either have $x_n'(t) \geq 0$ for all $t \in I = [t_0 - a_1, t_0 + a_1]$ or $x_n'(t) \leq 0$ for all $t \in I$, hence x_n is monotone on I.

Let us prove that $x_n'(t) \neq 0$ for all $t \in I$, $t \neq t_0$. Reasoning by contradiction, assume without loss of generality that for some $t^* \in I$, $t^* > t_0$, we have $0 = x_n'(t^*) = f_n(t^*, x(t^*))$. Then we deduce from condition (1) that $x_n(t^*) = x_{0,n}$. Since x_n is monotone and $x_n(t_0) = x_{0,n} = x_n(t^*)$, we deduce that x_n is constant between t_0 and t^*, hence $0 = x_n'(t) = f_n(t, x(t))$ for all $t \in (t_0, t^*)$, but this is impossible due to condition (2).

Summing up, x_n is strictly monotone on I, with nonzero derivative everywhere on $[t_0 - a_1, t_0)$ and on $(t_0, t_0 + a_1]$. Therefore, the function

$$y : J = x_n(I) \to [x_{0,1} - b, x_{0,1} + b] \times \cdots \times [x_{0,n-1} - b, x_{0,n-1} + b] \times I$$

given by $y = (x_1 \circ x_n^{-1}, \dots, x_{n-1} \circ x_n^{-1}, x_n^{-1})$ solves the problem

$$y'(r) = \tilde{f}(r, y(r)) \quad \text{for } r \in J \setminus \{x_{0,n}\},\; y(x_{0,n}) = y_0, \tag{2.1}$$

where $y_0 = (x_{0,1}, \dots, x_{0,n-1}, t_0)$ and $\tilde{f} = (\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_n)$ with

$$\tilde{f}_i(r, y_1, \dots, y_{n-1}, y_n) = \frac{f_i(y_n, y_1, \dots, y_{n-1}, r)}{f_n(y_n, y_1, \dots, y_{n-1}, r)} \quad \text{if } i \in \{1, 2, \dots, n-1\}$$

and

$$\tilde{f}_n(r, y_1, \dots, y_{n-1}, y_n) = \frac{1}{f_n(y_n, y_1, \dots, y_{n-1}, r)}.$$

Indeed, this is a straightforward consequence of the chain rule, the formula for the derivative of the inverse and the fact that x is a solution to (1.1).

Now, it only remains to prove that the problem (2.1) has at most one local solution. Indeed, for all $(r, y_1, \dots, y_n), (r, z_1, \dots, z_n) \in U$, with $r \neq x_{0,n}$, and for all $i \in \{1, 2, \dots, n-1\}$, we have that

$$\begin{aligned} \left|\tilde{f}_i(r, y_1, \dots, y_n) - \tilde{f}_i(r, z_1, \dots, z_n)\right| &= \left| \frac{f_i(y_n, y_1, \dots, y_{n-1}, r)}{f_n(y_n, y_1, \dots, y_{n-1}, r)} - \frac{f_i(z_n, z_1, \dots, z_{n-1}, r)}{f_n(z_n, z_1, \dots, z_{n-1}, r)} \right| \\ &\le \frac{2L\,K}{M^2(r)} \left\| (y_1, \dots, y_n) - (z_1, \dots, z_n) \right\|, \end{aligned}$$

and

$$\begin{aligned} \left|\tilde{f}_n(r, y_1, \dots, y_n) - \tilde{f}_n(r, z_1, \dots, z_n)\right| &= \left| \frac{1}{f_n(y_n, y_1, \dots, y_{n-1}, r)} - \frac{1}{f_n(z_n, z_1, \dots, z_{n-1}, r)} \right| \\ &\le \frac{K}{M^2(r)} \left\| (y_1, \dots, y_n) - (z_1, \dots, z_n) \right\|. \end{aligned}$$

Hence, $\tilde{f}$ satisfies Montel–Tonelli's uniqueness theorem, [1], so it follows the existence of a constant $\alpha > 0$ such that problem (2.1) has at most one solution in the interval $[x_{0,n} - \alpha, x_{0,n} + \alpha]$. □

Remark 2.2. For simplicity, in Theorem 2.1 the function f is assumed to be Lipschitz continuous when fixing the last variable. However, local uniqueness for problem (1.1) is also derived if f is Lipschitz continuous when fixing another variable $i_0 \in \{1, 2, \dots, n-1\}$ and conditions (1) and (2) are given for f_{i_0} instead of f_n.

Remark 2.3. Condition (2) in Theorem 2.1 is satisfied if there exists a function

$$h : [t_0 - a, t_0 + a] \to [0, \infty)$$

such that for all $t \in [t_0 - a, t_0 + a]$ and all $(x_1, \dots, x_{n-1}) \in [x_{0,1} - b, x_{0,1} + b] \times \cdots \times [x_{0,n-1} - b, x_{0,n-1} + b]$, we have

$$|f_n(t, x_1, \dots, x_{n-1}, x_{0,n})| \ge h(t),$$

and $h(t)$ is not identically zero on any interval $(t_0 - \varepsilon, t_0)$ or $(t_0, t_0 + \varepsilon)$ for $0 < \varepsilon < a$.

Moreover, we emphasize that, in the scalar case, this condition (2) means that $f(t, x_0)$ is not identically zero on any interval $(t_0 - \varepsilon, t_0)$ or $(t_0, t_0 + \varepsilon)$, which is exactly condition (ii) in Theorem 1.1.

Remark 2.4. Note that the conclusion of Theorem 2.1 and its proof remain valid if the Lipschitz type condition (3) is replaced by the more general Montel–Tonelli condition, see [1]:

$\overline{(3)}$ There exist functions $\psi : [0,+\infty) \to [0,+\infty)$ continuous and $p : [x_{0,n} - b, x_{0,n} + b] \to [0,+\infty)$ such that for all $(t_1, x_1, \dots, x_{n-1}, z), (t_2, y_1, \dots, y_{n-1}, z) \in U$, $z \neq x_{0,n}$,

$$\begin{aligned}\|f(t_1, x_1, \dots, x_{n-1}, z) - f(t_2, y_1, \dots, y_{n-1}, z)\| \\ \leq p(z)\psi\left(\|(t_1, x_1, \dots, x_{n-1}) - (t_2, y_1, \dots, y_{n-1})\|\right)\end{aligned}$$

where $\psi(\tau) > 0$ when $\tau > 0$, $\int_{0^+} \frac{d\tau}{\psi(\tau)} = +\infty$ and $p/M^2 \in L^1(x_{0,n} - b, x_{0,n} + b)$ being M the function in condition (1) of Theorem 2.1.

Notice that, under $\overline{(3)}$, the condition $1/M^2 \in L^1(x_{0,n} - b, x_{0,n} + b)$ in (1) is not longer required and is replaced by $p/M^2 \in L^1(x_{0,n} - b, x_{0,n} + b)$.

Theorem 2.1 increases the applicability of the main result in [4] in case that $f(t_0, x_0) = 0$, as shown by the following example.

Example 2.5. The function $f : [-1,1] \times \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(t, x_1, x_2) = \left(\sin(x_1 x_2), |t| + x_1^2 + \sqrt[4]{|x_2|}\right)$$

is not Lipschitz continuous with respect to $x = (x_1, x_2)$ in any neighborhood of $(0,0)$ and $f(0,0,0) = (0,0)$.

However, the initial value problem

$$\begin{cases} x_1' = \sin(x_1 x_2), & x_1(0) = 0, \\ x_2' = |t| + x_1^2 + \sqrt[4]{|x_2|}, & x_2(0) = 0, \end{cases} \tag{2.2}$$

has a unique local solution, since f restricted to $U = [-1,1]^3$ is Lipschitz continuous when fixing the last variable, assumption (1) in Theorem 2.1 holds with $M(z) = \sqrt[4]{|z|}$ and, moreover, the inequality

$$f_2(t, x_1, 0) \geq |t| \quad \text{for all } (t, x_1) \in [-1,1] \times \mathbb{R}$$

implies that f_2 satisfies condition (2). Therefore, Theorem 2.1 ensures local uniqueness for (2.2).

3 Uniqueness via a Lipschitz condition along a hyperspace

The following result is a straightforward consequence of the chain rule for functions of several variables.

Lemma 3.1. *Let $U, V \subset \mathbb{R}^{n+1}$ be open sets, $p_0 \in U$, $F : U \subset \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ a continuous function and $\Phi : V \to U$ a diffeomorphism.*

Then $x : (t_0 - \varepsilon, t_0 + \varepsilon) \to U$ is a solution of the autonomous system

$$x'(t) = F(x(t)), \quad x(t_0) = p_0, \tag{3.1}$$

if and only if $y := \Phi^{-1}(x) : (t_0 - \varepsilon, t_0 + \varepsilon) \to V$ is a solution of the problem

$$y'(t) = G(y(t)), \quad y(t_0) = \Phi^{-1}(p_0), \tag{3.2}$$

where

$$G(y) = \Phi'(y)^{-1} F(\Phi(y)).$$

Next, an extension of the main theorem in [16] is established as a consequence of Theorem 2.1 and Lemma 3.1 applied to a linear diffeomorphism. Basically, we assume a weak transversality condition at the initial point and a *Lipschitz condition along a hyperspace* for f.

Let $\mathcal{V} \subset \mathbb{R}^{n+1}$ be a hyperspace and $a_0 \in \mathbb{R}^{n+1}$ be a unit vector such that $\mathcal{V} = a_0^{\perp}$. Note that $\mathbb{R}^{n+1} = \mathcal{V} \oplus \langle a_0 \rangle$, where $\langle a_0 \rangle = \{a_0 s \in \mathbb{R}^{n+1} : s \in \mathbb{R}\}$, and so there exist unique $v_0 \in \mathcal{V}$ and $s_0 \in \mathbb{R}$ such that $(t_0, x_0) = v_0 + a_0 s_0$.

Theorem 3.2. *Let U be a neighborhood of $(t_0, x_0) \in \mathbb{R}^{n+1}$ and $f : U \to \mathbb{R}^n$ be a continuous function satisfying the following conditions:*

(a) *there exist constants $a, b > 0$ and a continuous function $M : [s_0 - a, s_0 + a] \to [0, +\infty)$ such that $1/M^2 \in L^1(s_0 - a, s_0 + a)$ and for all $s \in [s_0 - a, s_0 + a]$, $s \neq s_0$, we have*

$$|a_0 \cdot (1, f(v + a_0 s))| \geq M(s) > 0 \quad \text{for all } v \in \mathcal{V} \cap \overline{B}_b(v_0);$$

(b) *$a_0 \cdot (1, f(v(s) + a_0 s))$ is not identically zero on any interval $(s_0 - \varepsilon, s_0)$ or $(s_0, s_0 + \varepsilon)$ for $0 < \varepsilon < a$ whenever $v \in \mathcal{C}((s_0 - \varepsilon, s_0 + \varepsilon); \mathcal{V})$;*

(c) *f is Lipschitz continuous along the hyperspace $\mathcal{V}$ on U, i.e., there exists $L \geq 0$ such that for all $(t, x), (s, y) \in U$,*

$$\|f(t, x) - f(s, y)\| \leq L \, \|(t, x) - (s, y)\| \quad \text{if } (t, x) - (s, y) \in \mathcal{V}.$$

Then there is a unique local solution to the initial value problem (1.1).

Proof. Since $\mathcal{V}$ is a hyperspace in $\mathbb{R}^{n+1}$, there exists an orthonormal set of vectors $v_1, v_2, \dots, v_n \in \mathbb{R}^{n+1}$ such that $\mathcal{V} = \operatorname{span}\{v_1, \dots, v_n\}$. Let us consider the full rank matrix

$$A := (v_1 \,|\, v_2 \,|\, \cdots \,|\, v_n \,|\, a_0),$$

which gives the change-of-basis matrix from $\{v_1, \dots, v_n, a_0\}$ to the standard Euclidean basis. Notice that A is an orthogonal matrix, so $A^{-1} = A^T$.

Define the linear diffeomorphism

$$\Phi(y) := Ay$$

and consider the map given by $F(x_1, \dots, x_{n+1}) := (1, f(x_1, \dots, x_{n+1}))$.

Let us show that the following autonomous initial value problem

$$y' = G(y), \quad y(t_0) = p_0, \tag{3.3}$$

where $G(y) = A^{-1} F(Ay)$ and $p_0 = A^T (t_0, x_0)^T$, is uniquely locally solvable. Indeed, we shall prove that G satisfies assumptions (1)–(3) in Theorem 2.1. Note that

$$G_{n+1}(y) = a_0 \cdot F(Ay) = a_0 \cdot (1, f(Ay)).$$

Moreover, for each $y \in \mathbb{R}^{n+1}$ we can express $Ay = v + a_0 s$ in a unique way with $v \in \mathcal{V}$ and $s \in \mathbb{R}$, explicitly

$$v = A \begin{pmatrix} y_1 \\ \vdots \\ y_n \\ 0 \end{pmatrix} \quad \text{and} \quad s = y_{n+1}.$$

Hence,

$$G_{n+1}(y) = a_0 \cdot (1, f(v + a_0 s)),$$

and thus conditions (1) and (2) in Theorem 2.1 are directly deduced for G from assumptions (a) and (b). By assumption (c), we can deduce that G is Lipschitz continuous when fixing the last variable: for $(y,z), (\overline{y}, z) \in V$, with V a sufficiently small neighborhood of p_0, we have that

$$\begin{aligned}\|G(y,z) - G(\overline{y},z)\| &= \left\| A^{-1} \left[F(A(y,0)^T + a_0 z) - F(A(\overline{y},0)^T + a_0 z) \right] \right\| \\ &= \left\| f(A(y,0)^T + a_0 z) - f(A(\overline{y},0)^T + a_0 z) \right\| \\ &\leq L \left\| A(y,0)^T - A(\overline{y},0)^T) \right\| = L \left\| A(y - \overline{y}, 0)^T \right\| = L \|y - \overline{y}\|.\end{aligned}$$

Therefore, Theorem 2.1 implies that the initial value problem (3.3) has a unique local solution. Finally, Lemma 3.1 ensures that (1.1) is uniquely locally solvable. □

Remark 3.3. Obviously, conditions (a) and (b) in Theorem 3.2 hold if the transversality condition

$$(1, f(t_0, x_0)) \notin \mathcal{V} \quad (\text{equivalently, } a_0 \cdot (1, f(v_0 + a_0 s_0)) \neq 0)$$

is satisfied, cf. [16, Theorem 2.1].

Notice that a more general version of the previous result can be obtained with a non necessarily linear diffeomorphism. The interested reader is referred to [9] for a general approach based on this idea, which we omit here for the sake of simplicity. However, a nonlinear diffeomorphism will be employed in the scalar case ($n = 1$) in order to provide a relaxed version of the main uniqueness criterion in [10].

Indeed, by using the diffeomorphism

$$\Phi(x_1, x_2) := \begin{pmatrix} x_1 \\ \varphi(x_1) + x_2 \end{pmatrix},$$

for a continuously differentiable function φ, we obtain from Lemma 3.1 the following local equivalence between two scalar initial value problems.

Corollary 3.4. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ a continuously differentiable function. Then, x is a solution of the problem (1.1) if and only if $y(t) = -\varphi(t) + x(t)$ is a solution of problem*

$$y'(t) = g(t, y(t)), \quad y(t_0) = y_0, \tag{3.4}$$

where $y_0 = x_0 - \varphi(t_0)$ and $g : V \to \mathbb{R}$ is defined as

$$g(t,y) = -\varphi'(t) + f(t, \varphi(t) + y)$$

in a neighborhood V of the point (t_0, y_0).

The use of the previous change of variables is standard for instance to translate a given periodic solution $\varphi(t)$ to the origin and then analyze its stability as an equilibrium, [2]. However, its application to derive new uniqueness criteria as in the following result seems to have being unnoticed.

Theorem 3.5. *Let U be a neighborhood of $(t_0, x_0) \in \mathbb{R}^2$, $f : U \to \mathbb{R}$ continuous and $\varphi : \mathbb{R} \to \mathbb{R}$ a continuously differentiable function with Lipschitz derivative. Moreover, assume that f satisfies the following conditions:*

(*a*) *f is Lipschitz along φ, that is, there exists $L \geq 0$ such that if $(t, x + \varphi(t))$, $(t + k, x + \varphi(t + k)) \in U$ for some $k \in \mathbb{R}$, then*

$$|f(t, x + \varphi(t)) - f(t + k, x + \varphi(t + k))| \leq L\,|k|\,.$$

(*b*) *For $y_0 := x_0 - \varphi(t_0)$ and $r > 0$ such that $x - \varphi(t) \in [y_0 - r, y_0 + r]$ for all $(t, x) \in U$, there exists a continuous function $M : [y_0 - r, y_0 + r] \to [0, +\infty)$ such that*

$$\left|-\varphi'(t) + f(t, x)\right| \geq M(x - \varphi(t)) > 0$$

for all $(t, x) \in U \setminus \{(t, x) \in \mathbb{R}^2 : t = t_0 \text{ or } x - \varphi(t) = y_0\}$ and $1/M^2 \in L^1(y_0 - r, y_0 + r)$.

(*c*) *$-\varphi'(t) + f(t, y_0 + \varphi(t))$ is not identically zero on any interval $(t_0 - \varepsilon, t_0)$ or $(t_0, t_0 + \varepsilon)$ for $\varepsilon > 0$.*

Then the scalar problem (1.1) has a unique local solution.

Proof. Let us check that the initial value problem (3.4) is under the assumptions of Theorem 2.1 (with $n = 1$) and, therefore, it has a unique local solution. Then Corollary 3.4 will provide also the uniqueness of local solution for problem (1.1).

First, from assumption (b), we have

$$|g(t, y)| \geq M\left(-\varphi(t) + y + \varphi(t)\right) = M(y) > 0$$

for all $(t, y) \in V \setminus \{(t, y) \in \mathbb{R}^2 : t = t_0 \text{ or } y = y_0\}$.

Similarly, from condition (c), it is immediate to obtain that $g(t, y_0)$ is not identically zero on any interval $(t_0 - \varepsilon, t_0)$ or $(t_0, t_0 + \varepsilon)$.

In addition, g is Lipschitz with respect to the first argument. Indeed, for $(t, y), (t + k, y) \in V$, we have that

$$\begin{aligned}
|g(t, y) - g(t + k, y)| &= \left|-\varphi'(t) + f(t, y + \varphi(t)) + \varphi'(t + k) - f(t + k, y + \varphi(t + k))\right| \\
&\leq \left|\varphi'(t + k) - \varphi'(t)\right| + |f(t, y + \varphi(t)) - f(t + k, y + \varphi(t + k))| \\
&\leq C\,|k| + L\,|k| = (C + L)\,|k|\,,
\end{aligned}$$

as a consequence of the fact that f is Lipschitz along φ and φ' is a Lipschitz continuous function. So condition (3) in Theorem 2.1 is clearly satisfied for $K = C + L$. □

Remark 3.6. Note that if a function φ is under the hypotheses of Theorem 3.5 and $\varphi(t_0) = c \neq 0$, then the function $\tilde{\varphi}(t) = \varphi(t) - c$ is also under the hypotheses of Theorem 3.5. Hence we can just consider functions φ with $\varphi(t_0) = 0$.

Remark 3.7. Observe that for $(v_1, v_2) \in \mathbb{R}^2$, $v_1 \neq 0$, and the function $\varphi(t) = \frac{v_2}{v_1} t$, condition (a) in Theorem 3.5 is equal to the Lipschitz condition in the direction of the vector $v = (1, v_2/v_1)$ or, equivalently, in the direction of (v_1, v_2), see [10]. For this choice of φ, Theorem 3.5 is just the scalar case of Theorem 3.2.

The applicability of Theorem 3.5 is shown by the following example.

Example 3.8. Consider the initial value problem

$$x' = \sqrt[4]{|x - \alpha\, t - \beta\, t^2|} + \alpha + \gamma\, |t|\,, \quad x(0) = 0,$$

where α, β and γ are constants with $\alpha, \beta, \gamma \in \mathbb{R}$ and $\gamma > 2\,|\beta|$.

First, observe that the function $f(t,x) = \sqrt[4]{|x - \alpha\, t - \beta\, t^2|} + \alpha + \gamma\, |t|$ is continuous and Lipschitz along the function $\varphi(t) = \alpha\, t + \beta\, t^2$ for any $\alpha, \beta \in \mathbb{R}$ and $\gamma > 0$. Indeed, for $(t,x) \in \mathbb{R}^2$ and $k \in \mathbb{R}$,

$$|f(t, x + \varphi(t)) - f(t + k, x + \varphi(t + k))| = \gamma\, ||t + k| - |t|| \le \gamma\, |k|\,,$$

and thus condition *(a)* in Theorem 3.5 is satisfied with $L = \gamma$.

Notice also that $\varphi'(0) = f(0,0)$ and so the transversality condition asked in [9] is not satisfied. Nevertheless, assumptions (b) and (c) in Theorem 3.5 hold. Condition (b) can be easily verified by using the continuous function $M(y) = \sqrt[4]{|y|}$, which satisfies that $1/M^2 \in L^1(-\varepsilon, \varepsilon)$, together with the inequality $\gamma > 2\,|\beta|$, and condition (c) follows from the fact that

$$-\alpha - 2\,\beta\, t + f(t, \alpha\, t + \beta\, t^2) = -2\,\beta\, t + \gamma\, |t| > 0 \quad \text{for every } t \in \mathbb{R} \setminus \{0\},$$

which therefore it is not identically zero in any neighborhood of 0.

In conclusion, Theorem 3.5 ensures the uniqueness of a local solution for any $\alpha, \beta \in \mathbb{R}$ and $\gamma > 2\,|\beta|$. Finally, observe that if $\alpha = 1$ and $\beta = \gamma = 0$, then $x_1(t) = t$ and $x_2(t) = \left(\frac{3}{4}t\right)^{\frac{4}{3}} + t$ if $t \ge 0$ and $x_2(t) = -\left(\frac{3}{4}t\right)^{\frac{4}{3}} + t$ if $t < 0$ are two local solutions. In this case, condition (c) is no longer true.

4 Uniqueness via perturbed Lipschitz conditions

Our next result is another local uniqueness criterion for problem (1.1) which was inspired on a specific form of the function f considered in [3]. Basically, we shall assume that the function $f(t,x)$ satisfies a perturbed Lipschitz condition with respect to x outside some hypersurfaces $\tau_i(t,x) = 0$ $(i = 1, 2, \ldots, N)$ which satisfy a weak transversality condition around (t_0, x_0). Our result is closely related to the uniqueness theorem proven in [15] but ours is more general inasmuch our transversality conditions need not be satisfied at the point (t_0, x_0).

Let us state and prove the main result of this section.

Theorem 4.1. *Let U be a neighborhood of $(t_0, x_0) \in \mathbb{R}^{n+1}$, $f : U \to \mathbb{R}^n$ a continuous function and assume that:*

(i) *There exist a constant $K > 0$ and functions $g_i : \mathbb{R} \to \mathbb{R}$ and $\tau_i : U \to \mathbb{R}$, for $i = 1, 2, \ldots, N$, such that*

$$\|f(t,x) - f(t,y)\| \le K\, \|x - y\| + K \max_{1 \le i \le N} |g_i(\tau_i(t,x)) - g_i(\tau_i(t,y))|,$$

for all (t,x), $(t,y) \in U$.

(ii) *Each $\tau_i : U \to \mathbb{R}$ is continuously differentiable and $\tau_i(t_0, x_0) = 0$.*

(iii) *(Transversality) There exists a continuous function $M : \mathbb{R} \to \mathbb{R}$ such that for all $(t,x) \in U$ with $t \neq t_0$ we have*

$$|\nabla \tau_i(t,x) \cdot (1, f(t,x))| \ge M(t) > 0 \quad \textit{for every } i \in \{1, 2, \ldots, N\}.$$

(*iv*) *Each $g_i : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$, differentiable on $\mathbb{R} \setminus \{0\}$, and there exist $\rho_1, \rho_2 > 0$ and $\psi : (0, \rho_1) \to \mathbb{R}$ a decreasing function such that $|g_i'(t)| \le \psi(|t|)$ for all $t \in (-\rho_1, \rho_1) \setminus \{0\}$ and $\psi(|\int_{t_0}^{\cdot} M(s) ds|) \in L^1(t_0 - \rho_2, t_0 + \rho_2)$.*

Then the initial value problem (1.1) *has a unique local solution.*

Proof. Local existence follows from Peano's theorem. So, let us prove local uniqueness to the right, that is in an interval of the form $[t_0, t_0 + \alpha]$ (the proof of the local uniqueness to the left is similar).

Take $b, \delta > 0$ such that $[t_0, t_0 + \delta] \times \overline{B}_b(x_0) \subset U$ and $\|z(t) - x_0\| \le b$ for all $t \in [t_0, t_0 + \delta]$ and for any solution $z(t)$ of (1.1).

Now, by (*iii*), for $i = 1, 2, \dots, N$ and $t \ne t_0$ we have

$$\left| \frac{d}{dt} \tau_i(t, z(t)) \right| = |\nabla \tau_i(t, z(t)) \cdot (1, f(t, z(t)))| \ge M(t),$$

and thus

$$|\tau_i(t, z(t))| \ge \left| \int_{t_0}^{t} M(s)\, ds \right| > 0,$$

for all $t \in [t_0, t_0 + \delta]$, $t \ne t_0$, and $i = 1, 2, \dots, N$.

Define the compact set

$$D = \left\{ (t, x) \in [t_0, t_0 + \delta] \times \overline{B}_b(x_0) \, : \, |\tau_i(t, x)| \ge \left| \int_{t_0}^{t} M(s)\, ds \right| \text{ for } i = 1, 2, \dots, N \right\}.$$

Notice that condition (*iii*) implies that for each $i \in \{1, 2, \dots, N\}$

$$\nabla \tau_i(s, y) \cdot (1, f(s, y)) > 0 \quad \text{for } s \in (t_0, t_0 + \delta) \text{ and } y \in B_b(x_0), \tag{4.1}$$

or

$$\nabla \tau_i(s, y) \cdot (1, f(s, y)) < 0 \quad \text{for } s \in (t_0, t_0 + \delta) \text{ and } y \in B_b(x_0). \tag{4.2}$$

Assume that condition (4.1) holds for all $i \in \{1, 2, \dots, N\}$ (if not, simply replace τ_i with $-\tau_i$ and $g_i(x)$ with $g_i(-x)$ so that (*i*) holds) and let $z(t)$ be a solution of problem (1.1). Then, by the chain rule, we deduce that

$$\frac{d}{dt} \tau_i(t, z(t)) = \nabla \tau_i(t, z(t)) \cdot (1, f(t, z(t))) > 0, \quad \text{for all } t \in (t_0, t_0 + \delta).$$

So $t \mapsto \tau_i(t, z(t))$ is increasing on $(t_0, t_0 + \delta)$ and then

$$\tau_i(t_0, z(t_0)) = 0 < \tau_i(t, z(t)) \quad \text{for all } t \in (t_0, t_0 + \delta).$$

In conclusion, $(t, z(t)) \in \tau_i^{-1}(0, \infty) \cap D$ on $(t_0, t_0 + \delta)$ for all $i \in \{1, 2, \dots, N\}$.

Now, for (t, x), $(t, y) \in \cap_{i=1}^{N} \tau_i^{-1}(0, \infty) \cap D$, we deduce from (*i*), (*iv*) and the Mean Value Theorem that

$$\|f(t, x) - f(t, y)\| \le K \|x - y\| + K \max_{1 \le i \le N} |g_i'(\tau_{i,t})| |\tau_i(t, x) - \tau_i(t, y)|, \tag{4.3}$$

where the last identity is valid for some real numbers $\tau_{i,t}$ located between $\tau_i(t, x)$ and $\tau_i(t, y)$. In particular, $0 < |\int_{t_0}^{t} M(s)\, ds| \le |\tau_{i,t}|$, and then (*iv*) implies that

$$|g_i'(\tau_{i,t})| \le \psi(|\tau_{i,t}|) \le \psi\left(\left| \int_{t_0}^{t} M(s)\, ds \right| \right).$$

Therefore, for $(t,x),\ (t,y) \in \cap_{i=1}^{N} \tau_i^{-1}(0,\infty) \cap D$, we have

$$\begin{aligned} \|f(t,x) - f(t,y)\| &\le K\,\|x-y\| + K\psi\left(\int_{t_0}^{t} M(s)\,ds\right) \max_{1\le i\le N} |\tau_i(t,x) - \tau_i(t,y)| \\ &\le c_1(t)\,\|x-y\|, \end{aligned} \tag{4.4}$$

for some $c_1 \in L^1(t_0, t_0+\alpha)$ with $0 < \alpha \le \delta$, because the τ_i's are Lipschitz continuous with respect to x on the compact set D and assumption (iv).

Finally, let $x(t)$ and $y(t)$ be solutions of (1.1); for $t \in (t_0, t_0+\alpha)$, the previous computations ensure that $(t, x(t)),\ (t, y(t)) \in \cap_{i=1}^{N} \tau_i^{-1}(0,\infty) \cap D$ and thus, by (4.4), we have that

$$\|x(t) - y(t)\| \le \int_{t_0}^{t} \|f(s,x(s)) - f(s,y(s))\|\,ds \le \int_{t_0}^{t} c_1(s)\,\|x(s)-y(s)\|\,ds, \quad t \in (t_0, t_0+\alpha),$$

so we deduce from Gronwall's inequality that $\|x(t) - y(t)\| = 0$ on $(t_0, t_0+\alpha)$. □

Remark 4.2. A particular case of Theorem 4.1, assuming the transversality condition

$$\nabla\tau_i(t_0, x_0)\cdot(1, f(t_0,x_0)) \neq 0, \quad \text{for all } i = 1,2,\ldots,N, \tag{4.5}$$

instead of condition (iii), was proven in [12, Corollary 4.4]. Notice that condition (i) in Theorem 4.1 is satisfied in particular if f can be expressed as the composition

$$f(t,x) = F(t,x,g_1(\tau_1(t,x)), g_2(\tau_2(t,x)),\ldots,g_N(\tau_N(t,x))) \text{ for some } N \in \mathbb{N}, \tag{4.6}$$

where $F : U \times V \subset \mathbb{R}^{n+1} \times \mathbb{R}^N \to \mathbb{R}^n$ satisfies:

$\overline{(i)}$ There exists $K > 0$ such that for every $(t,x,\xi), (t,y,\eta) \in U \times V$ we have

$$\|F(t,x,\xi) - F(t,y,\eta)\| \le K\,\|x-y\| + K\|\xi - \eta\|.$$

The particular form of f given by (4.6) was considered in [3]. On the other hand, the same condition (4.5) is also a key assumption in the uniqueness result obtained in [15] which was motivated by a certain n–body problem of classical electrodynamics.

Example 4.3. The following modification of the Example 2 in [15] illustrates how our Theorem 4.1 can improve the applicability of previous results. Let us consider the initial value problem

$$x' = (|t|^{\alpha} + |x|^{5/3})^{1/3}, \quad x(0) = 0, \quad \alpha \ge 1.$$

Note that $f(t,x) = (|t|^{\alpha} + |x|^{5/3})^{1/3}$ can be expressed as $f(t,x) = g(\tau(t,x))$ with $g(\tau) = \tau^{1/3}$ and $\tau(t,x) = |t|^{\alpha} + |x|^{5/3}$. Hence, it is easy to check that conditions (i)–(iv) in Theorem 4.1 are satisfied for $1 < \alpha < 3/2$.

However, condition (4.5) does not hold for any $1 < \alpha < 3/2$, so neither [12, Corollary 4.4] nor [15] are applicable.

Now we will show that Theorem 4.1 enables us to obtain also alternative uniqueness arguments for the examples provided in [11].

Example 4.4. Consider the initial value problem

$$x'(t) = p(t) + q(t)\,|x(t)|^{r}, \quad x(0) = 0, \tag{4.7}$$

where $0 < r < 1$, p, q are non-negative continuous functions and there exists $\varepsilon > 0$ such that $p(t) > 0$ for all $t \in (-\varepsilon, \varepsilon) \setminus \{0\}$ and

$$\left| \int_0^t p(s)ds \right|^{r-1} \in L^1(-\varepsilon, \varepsilon). \tag{4.8}$$

Note that the function $f(t,x) = p(t) + q(t)|x|^r$ can be expressed in the form $f(t,x) = F(t,x,g(\tau(t,x)))$ with $\tau(t,x) = x$, $g(\tau) = |\tau|^r$ and $F(t,x,\xi) = p(t) + q(t)\xi$. One may easily verify conditions (i)–(ii) in Theorem 4.1 (see also Remark 4.2). In addition,

$$|\nabla\tau(t,x) \cdot (1, f(t,x))| = f(t,x) \geq M(t) = p(t),$$

so conditions (iii) and (iv) hold. Therefore, Theorem 4.1 implies local uniqueness for (4.7).

To end the example, observe that Theorem 4.1 is not applicable to problem (4.7) with $p(t) = t^2$, $q(t) = 1$ and $r = 1/4$ since condition (4.8) does not hold, but uniqueness still can be directly deduced from Theorem 2.1.

Finally, we provide an example for which uniqueness is guaranteed by Theorem 4.1, whereas the criteria in Sections 2 and 3 are not applicable.

Example 4.5. Consider the initial value problem

$$x' = \sqrt{|x-t|} + \sqrt{|t|} + 1, \qquad x(0) = 0,$$

which can be expressed in the form $f(t,x) = F(t,x,g(\tau(t,x)))$, $F(t,x,\xi) = \sqrt{|t|} + \xi + 1$, $g(r) = \sqrt{|r|}$ and $\tau(t,x) = x - t$.

Observe that the functions F, g and τ are under the hypotheses of Theorem 4.1. Moreover, for $t \neq 0$ we have

$$\nabla\tau(t,x) \cdot (1, f(t,x)) = -1 + \sqrt{|x-t|} + \sqrt{|t|} + 1 \geq M(t) = \sqrt{|t|} > 0,$$

so Theorem 4.1 implies the existence and uniqueness of a local solution for the initial value problem.

We highlight that the transversality condition (4.5) is not satisfied at the initial condition $(t_0, x_0) = (0,0)$. In addition, f is not Lipschitz along any function φ with $\varphi(0) = 0$ since

$$|f(0,0) - f(k, \varphi(k))| = \sqrt{|\varphi(k) - k|} + \sqrt{|k|},$$

which is not smaller than $L|k|$ for $k > 0$ small enough. Hence, in virtue of Remark 3.6, Theorem 3.5 cannot be applied here.

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Long-term behavior of nonautonomous neutral compartmental systems

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Abstract. The asymptotic behavior of the trajectories of compartmental systems with a general set of admissible initial data is studied. More precisely, these systems are described by families of monotone nonautonomous neutral functional differential equations with nonautonomous operator. We show that the solutions asymptotically exhibit the same recurrence properties as the transport functions and the coefficients of the neutral operator. Conditions for the cases in which the delays in the neutral and non neutral parts are different, as well as for other cases unaddressed in the previous literature are also obtained.

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1 Introduction

In this work, the long-term behavior of the solutions of neutral compartmental systems is studied. Some interesting results as to the convergence of the solutions of such systems to their omega-limit sets are presented. These results allow for the inclusion of a significantly wider set of possible initial data than those in the previous literature.

Compartmental systems are widely used as successful models in different fields, ranging from physics and biology, to economics or sociology. These models appear naturally when dealing with processes involving a local balance of mass (see e.g. Haddad and Chellaboina [7], Jacquez [9], and Jacquez and Simon [10]).

Compartmental models with finite and infinite delay were initially studied by Győri [4], and Győri and Eller [5]. Later on, Arino and Bourad [1], and Arino and Haourigui [2] found almost periodic solutions for finite delay compartmental systems given by functional differential equations (FDEs for short) or neutral FDEs (NFDEs for short). Győri and Wu [6] also

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studied the dynamics of infinite delay neutral compartmental systems, where the neutral term represents the creation and destruction of material within the compartments. These systems were studied by Wu and Freedman [26], and Wu [25] as well.

Regarding the monotone theory for NFDEs, the fact that the positive cone defined by the exponential ordering introduced by Smith and Thieme [22] has an empty interior, poses a special difficulty. This was overcome by Krisztin and Wu [13] in their paper on scalar NFDEs with finite delay, where they show that the solutions with Lipschitz continuous initial data are asymptotically periodic. More recently, the theory developed by Novo, Obaya, and Sanz [16] gave rise to a series of papers devoted to the study of compartmental systems defined by means of NFDEs with infinite delay, such as Novo, Obaya, and Villarragut [15, 17], and Obaya and Villarragut [19, 20]. In these papers, the neutral part consists of a nonautonomous operator, initial data are assumed to be Lipschitz continuous, and the usual ordering, the exponential ordering, and a new ad hoc exponential ordering defined by means of the neutral operator are considered. Also, these papers generalize the results included in the previous literature.

Finally, a generalization of the results on the asymptotic behavior of NFDEs was given in Novo, Obaya, and Villarragut [18]. In that paper, initial data are only required to have a uniformly bounded variation on compact subintervals of $(-\infty, 0]$, which is a weaker assumption than Lipschitz continuity. The present paper studies compartmental systems defined by NFDEs with infinite delay by applying the results in [18].

Specifically, the family of systems

$$\frac{d}{dt}\left[z_i(t) - \int_{-\infty}^{0} z_i(t+s)\, d\nu_i(\omega{\cdot}t)(s)\right]$$
$$= -\sum_{j=0}^{m} g_{ji}(\omega{\cdot}t, z_i(t)) + \sum_{j=1}^{m} \int_{-\infty}^{0} g_{ij}(\omega{\cdot}(t+s), z_j(t+s))\, d\mu_{ij}(s) + I_i(\omega{\cdot}t)\,,$$

$i = 1, \ldots, m$, where $\mathbb{R} \times \Omega \to \Omega$, $(t, \omega) \mapsto \omega{\cdot}t$ is a minimal flow, $\nu_i(\omega)$ and $\mu_{ij}(\omega)$ are regular Borel measures, and $g_{ij} : \Omega \times \mathbb{R} \to \mathbb{R}$, $I_i : \Omega \to \mathbb{R}$ are real functions, is considered. For each $\omega \in \Omega$ and each $i, j \in \{1, \ldots, m\}$, the measures $\nu_i(\omega)$ represent the creation and destruction of material within each compartment, the functions g_{0i} and I_i represent the flow of material toward and from the environment, respectively, and the functions g_{ij} are the so-called transport functions, modeling the flow of material among the compartments, which is not instantaneous and is regulated by the measures $\mu_{ij}(\omega)$. Some particular cases of these systems were studied by Krisztin (see e.g. [11, 12]) under weaker conditions on the transport functions.

A more specific system of equations is also considered in this work:

$$\frac{d}{dt}[z_i(t) + \widetilde{c}_i(t)\, z_i(t-\alpha_i)] = -\sum_{j=1}^{m} \widetilde{g}_{ji}(t, z_i(t)) + \sum_{j=1}^{m} \widetilde{g}_{ij}(t-\rho_{ij}, z_j(t-\rho_{ij}))\,,$$

$i = 1, \ldots, m$. This system can be included in a family of systems like the previous one by means of a hull construction. Notice that this system has finite delay and it is closed, in that there is no flow of material either toward or from the environment. Specifically, for each $\omega \in \Omega$ and each $i, j \in \{1, \ldots, m\}$, $\nu_i(\omega)$ and $\mu_{ij}(\omega)$ are Dirac measures, and $g_{0i} = I_i \equiv 0$. Besides, a particular example of this system is also considered, and both theoretical and numerical results related to this example are presented.

The structure of the paper is as follows. In Section 2, some preliminaries regarding the usual concepts of topological dynamics are recalled. Section 3 addresses the study of the solutions of a family of compartmental systems with infinite delay with general initial data. An application of the results of Section 3 to the study of compartmental systems with finite delay is included in Section 4. Finally, Section 5 includes the application of the results in Section 3 and Section 4 to a particular compartmental system, together with a numerical simulation of the solutions of that system and their omega-limit sets.

2 Some preliminaries

Given a compact metric space Ω, a *flow* is a continuous mapping $\sigma : \mathbb{R} \times \Omega \to \Omega,\ (t, \omega) \mapsto \sigma_t(\omega)$ which satisfies the following conditions:

(i) $\sigma_0(\omega) = \omega$ for each $\omega \in \Omega$, and

(ii) $\sigma_{t+s} = \sigma_t \circ \sigma_s$ for each $s, t \in \mathbb{R}$.

It is customary to denote $\omega{\cdot}t = \sigma_t(\omega)$ for all $(t, \omega) \in \mathbb{R} \times \Omega$. Given $\omega \in \Omega$, its *orbit* or *trajectory* is the set $\{\sigma_t(\omega) \mid t \in \mathbb{R}\}$. A subset $A \subset \Omega$ is said to be *invariant* if $\sigma_t(A) = A$ for all $t \in \mathbb{R}$, and it is said to be *minimal* if it is compact, invariant, and it contains no proper subsets with those properties apart from the empty set. Equivalently, a subset of Ω is minimal if and only if all the trajectories are dense. Zorn's lemma guarantees that a compact and invariant subset of Ω always contains a minimal subset. If Ω is minimal, it is said that the flow σ is *minimal* or *recurrent*. For example, almost periodic and almost automorphic flows are minimal (see Ellis [3], and Shen and Yi [21] for a thorough description of almost periodic and almost automorphic flows from a topological and ergodic perspective).

Let $\mathbb{R}^+$ be the set of non-negative real numbers. Given a complete metric space X, a *semiflow* is a continuous map $\Phi : \mathbb{R}^+ \times X \to X,\ (t, x) \mapsto \Phi_t(x)$ satisfying

(i) $\Phi_0(x) = x$ for all $x \in X$, and

(ii) $\Phi_{t+s} = \Phi_t \circ \Phi_s$ for all $t, s \in \mathbb{R}^+$.

Given $x \in X$, its *semiorbit* is the set $\{\Phi_t(x) \mid t \geq 0\}$. A subset $A \subset X$ is said to be *positively invariant* if $\Phi_t(A) \subset A$ for all $t \geq 0$. Given $x \in X$ with a relatively compact semiorbit, we define its *omega-limit set*, defined by

$$\mathcal{O}(x) = \bigcap_{s \geq 0} \operatorname{closure}\{\Phi_{t+s}(x) \mid t \geq 0\}\,.$$

It is easy to check that $\mathcal{O}(x)$ is nonempty, compact, connected, and positively invariant. A useful characterization of its elements is as follows: $y \in \mathcal{O}(x)$ if and only if there exists a sequence $t_n \uparrow \infty$ such that $\Phi_{t_n}(x)$ converges to y as $n \uparrow \infty$. A subset $A \subset X$ is said to be *minimal* if it is compact, positively invariant, and it contains no proper subsets with those properties apart from the empty set. If X is minimal, it is said that the semiflow Φ is *minimal*.

We are interested in a particular class of semiflows. Specifically, if Ω is a compact metric space and X is a complete metric space, a *skew-product semiflow* is a semiflow defined on $\Omega \times X$ of the form

$$\begin{aligned} \tau\colon \mathbb{R}^+ \times \Omega \times X &\longrightarrow \Omega \times X \\ (t, \omega, x) &\longmapsto (\omega{\cdot}t, u(t, \omega, x))\,, \end{aligned}$$

where, as before, $\omega{\cdot}t = \sigma_t(\omega)$ for all $(t, \omega) \in \mathbb{R} \times \Omega$, and σ is a flow on Ω referred to as the *base flow*.

3 Transformed exponential ordering for neutral compartmental systems

In this section, we focus on compartmental models. They are primarily used to describe the transport of some material among compartments joined by pipes, which takes some non-negligible time, together with the creation and destruction of material within the compartments.

To this end, we consider the set $X = C((-\infty, 0], \mathbb{R}^m)$ endowed with the compact-open topology, which turns it into a Fréchet space. The space X is metrizable; it suffices to consider the metric

$$\mathsf{d}(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x-y\|_n}{1+\|x-y\|_n}\,, \quad x,y \in X\,,$$

where $\|x\|_n = \sup_{s\in[-n,0]} \|x(s)\|$, and $\|\cdot\|$ is the maximum norm on $\mathbb{R}^m$. We consider the following phase space:

$$BC = \{x \in X \mid x \text{ is bounded}\}\,,$$

together with the supremum norm $\|\cdot\|_\infty$. The space $(BC, \|\cdot\|_\infty)$ is a Banach space. Given $r > 0$, let B_r denote the set $\{x \in BC \mid \|x\|_\infty \le r\}$. Besides, we consider the following subspace of BC:

$$BU = \{x \in BC \mid x \text{ is uniformly continuous}\}\,.$$

Given a, $t \in \mathbb{R}$ with $t \le a$ and a continuous function $x : (-\infty, a] \to \mathbb{R}^m$, we consider $x_t : (-\infty, 0] \to \mathbb{R}^m$, $s \mapsto x(t+s)$. Notice that $x_t \in X$. Finally, we also fix a compact metric space (Ω, d), together with a flow $\sigma : \mathbb{R} \times \Omega \to \Omega$, $(t, \omega) \mapsto \omega{\cdot}t$.

A compartmental system is a device formed by m compartments $C_1, \ldots, C_m$, connected to one another by means of pipes, and the environment. For each $i \in \{1, \ldots, m\}$, let $z_i(t)$ denote the amount of material in compartment C_i at time t. There is a flow of material among compartments. We assume that the material leaves the compartments instantaneously. Then, for each i, $j \in \{1, \ldots, m\}$, the material flows from C_j to C_i according to a transit time distribution regulated by a positive regular Borel measure μ_{ij}. The volume of material being transported is given by the *transport functions* g_{ij}, which depend on time and $z_j(t)$. Besides, there is a bidirectional flow of material from and to the environment; namely, for each $i \in \{1, \ldots, m\}$, some material enters compartment C_i from the environment instantaneously, according to a function I_i depending only on time, and some material leaves compartment C_i for the environment instantaneously, according to a transport function g_{0i}. Finally, for each $i \in \{1, \ldots, m\}$, the rate of creation and destruction of material within compartment C_i is regulated by the past amount of material in that compartment together with some regular Borel measures $\nu_i(\omega)$, $\omega \in \Omega$.

The amount of material in compartment C_i, $1 \le i \le m$, varies according to the difference between the incoming material to C_i and the outgoing material from C_i. This way, our model is given by the following family of NFDEs:

$$\begin{aligned}\frac{d}{dt}&\left[z_i(t) - \int_{-\infty}^{0} z_i(t+s)\, d\nu_i(\omega{\cdot}t)(s)\right] \\ &= -\sum_{j=0}^{m} g_{ji}(\omega{\cdot}t, z_i(t)) + \sum_{j=1}^{m} \int_{-\infty}^{0} g_{ij}(\omega{\cdot}(t+s), z_j(t+s))\, d\mu_{ij}(s) + I_i(\omega{\cdot}t)\,, \qquad (3.1)\end{aligned}$$

$\omega \in \Omega$, $i = 1, \ldots, m$, where $g_{0i} \colon \Omega \times \mathbb{R} \to \mathbb{R}^+$, $g_{ij} \colon \Omega \times \mathbb{R} \to \mathbb{R}$, $I_i \colon \Omega \to \mathbb{R}$ and $\nu_i(\omega)$ are and μ_{ij} are regular Borel measures on $(-\infty, 0]$, $i, j = 1, \ldots, m$, $\omega \in \Omega$.

Let $G\colon \Omega \times BC \to \mathbb{R}^m$ be the map defined by

$$G_i(\omega, x) = -\sum_{j=0}^{m} g_{ji}(\omega, x_i(0)) + \sum_{j=1}^{m} \int_{-\infty}^{0} g_{ij}(\omega{\cdot}s, x_j(s))\, d\mu_{ij}(s) + I_i(\omega)\,, \tag{3.2}$$

$(\omega, x) \in \Omega \times BC$, $i = 1, \dots, m$, and let $D\colon \Omega \times BC \to \mathbb{R}^m$ be the map defined by

$$D_i(\omega, x) = x_i(0) - \int_{-\infty}^{0} x_i(s)\, d\nu_i(\omega)(s)\,, \quad (\omega, x) \in \Omega \times BC\,, \quad i = 1, \dots, m\,.$$

We can now rewrite the family of equations (3.1) as

$$\frac{d}{dt} D(\omega{\cdot}t, z_t) = G(\omega{\cdot}t, z_t)\,, \quad t \geq 0\,,\ \omega \in \Omega\,. \tag{3.3}$$

It is obvious that the sign of the measures $\nu_i(\omega)$ determines different internal mechanisms in the compartment C_i with respect to z_i to produce or swallow material. Since the case in which the measures $\nu_i(\omega)$ are positive and the phase space is BU, the space of uniformly continuous functions in BC, was studied in [15], [17], [19] and [20], we will focus now in the case in which they are regular Borel negative measures. Let us make some assumptions on the family of equations (3.1):

(C1) I_i and g_{ij} are continuous, g_{ij} is nondecreasing in its second variable, $g_{ij}(\cdot, 0) \equiv 0$, and $\Omega \times \mathbb{R} \to \mathbb{R}$, $(\omega, v) \mapsto \frac{\partial g_{ij}}{\partial v}(\omega, v)$ is well-defined and continuous for $i, j \in \{1, \dots, m\}$;

(C2) μ_{ij} is a positive regular Borel measure such that $\mu_{ij}((-\infty, 0]) = 1$ and $\int_{-\infty}^{0} |s|\, d\mu_{ij}(s) < \infty$ for $i, j = 1, \dots m$;

(C3) for each $\omega \in \Omega$ and $i = 1, \dots, m$,

- $\nu_i(\omega)$ is a negative regular Borel measure with $\nu_i(\omega)(\{0\}) = 0$,
- $\sup_{i=1,\dots,m} |\nu_i(\omega)|((-\infty, 0]) < 1$, where $|\nu_i(\omega)|$ denotes the total variation of $\nu_i(\omega)$, and
- $\nu_i\colon \Omega \to \mathcal{M}$, $\omega \mapsto \nu_i(\omega)$ is continuous when the total variation is considered as a norm on the set $\mathcal{M}$ of Borel regular measures on $(-\infty, 0]$;

(C4) for each $i = 1, \dots, m$, there is a negative $a_i < 0$ such that

(i) $-L_i^+(\omega) - a_i > 0$ and

(ii) $1 + \int_{-\infty}^{0} e^{a_i s}\, d\nu_i(\omega)(s) \geq 0\,,$

for each $\omega \in \Omega$, where $L_i^+(\omega) = \sum_{j=0}^{m} \sup_{v \in \mathbb{R}} \frac{\partial g_{ij}}{\partial v}(\omega, v)$.

Let A be the diagonal matrix with the diagonal elements $a_1, \dots, a_m$ given in **(C4)**. Let us consider the partial order relation on BC:

$$x \leq_A y \iff x \leq y \text{ and } y(t) - x(t) \geq e^{A(t-s)}(y(s) - x(s))\,, -\infty < s \leq t \leq 0\,,$$

where $\leq$ denotes the componentwise partial ordering on $\mathbb{R}^m$. The interior of the positive cone

$$BC_A^+ = \{x \in BC \mid x \geq 0 \text{ and } x(t) \geq e^{A(t-s)} x(s) \quad \text{for } -\infty < s \leq t \leq 0\}\,,$$

is empty.

Before stating and proving the main theorem, we need the following result concerning D and its convolution operator

$$\begin{aligned} \widehat{D}: \Omega \times BC &\longrightarrow \Omega \times BC \\ (\omega, x) &\longmapsto (\omega, \widehat{D}_2(\omega, x)), \end{aligned}$$

where $\widehat{D}_2(\omega, x))$ is defined for each $(\omega, x) \in \Omega \times BC$ by

$$\begin{aligned} \widehat{D}_2(\omega, x) : (-\infty, 0] &\longrightarrow \mathbb{R}^m \\ s &\longmapsto D(\omega{\cdot}s, x_s). \end{aligned}$$

Theorem 3.1. *Assume that conditions* (C3) *and* (C4)(ii) *hold. For each $\omega \in \Omega$, let $\widehat{L}_\omega \colon BC \to BC$ be the linear operator defined by*

$$(\widehat{L}_\omega(x))_i(s) = \int_{-\infty}^{0} x_i(s+u)\, d\nu_i(\omega{\cdot}s)(u), \quad x \in BC, \, s \le 0, \, i = 1, \dots, m.$$

Then the following statements hold:

(i) *$\widehat{D}$ is bijective,*

$$(\widehat{D}^{-1})_2(\omega, x) = \sum_{n=0}^{\infty} \widehat{L}_\omega^n(x), \quad (\omega, x) \in \Omega \times BC, \tag{3.4}$$

and $(\widehat{D}^{-1})_2(\omega, x) \ge 0$ for each $(\omega, x) \in \Omega \times BC_A^+$;

(ii) *for each $r > 0$, the map $\Omega \times B_r \to BC$, $(\omega, x) \mapsto \widehat{L}_\omega(x)$ is uniformly continuous for the compact-open topology on BC;*

(iii) *given $(\omega, x) \in \Omega \times BC$ with $D(\omega, x) = 0$, the solution of the difference equation*

$$\begin{cases} D(\omega{\cdot}t, z_t) = 0, & t \ge 0, \\ z_0 = x, \end{cases}$$

satisfies $\|z(t)\| \le c(t)\, \|x\|_\infty$ for all $t \ge 0$, where $c \in C([0,\infty), \mathbb{R})$ and $\lim_{t\to\infty} c(t) = 0$. In this situation, D is said to be stable.

Proof. Clearly, $\widehat{D}_2(\omega, x) = (I - \widehat{L}_\omega)(x)$ for each $(\omega, x) \in \Omega \times BC$. From condition (C3), we deduce that $\sup_{\omega\in\Omega} \|\widehat{L}_\omega\| < 1$, whence $\widehat{D}$ is invertible and (3.4) holds. In addition, $x \ge_A 0$ implies that $x_i(s) \ge 0$ and $e^{a_i r} x_i(s) \ge x_i(s+r)$ for each r, $s \le 0$, $i = 1, \dots, m$, which, together with the fact that $\nu_i(\omega)$ is a negative measure for each $i = 1, \dots, m$ and condition (C4)(ii), provides

$$\begin{aligned} x_i(s) + (\widehat{L}_\omega(x))_i(s) &= x_i(s) + \int_{-\infty}^{0} x_i(s+r)\, d\nu_i(\omega{\cdot}s)(r) \\ &\ge -\int_{-\infty}^{0} e^{a_i r}\, x_i(s)\, d\nu_i(\omega{\cdot}s)(r) + \int_{-\infty}^{0} x_i(s+r)\, d\nu_i(\omega{\cdot}s)(r) \ge 0. \end{aligned}$$

As a consequence, since $\widehat{L}_\omega^{2n}(x) \ge 0$ for each $x \ge 0$, we deduce that

$$(\widehat{D}^{-1})_2(\omega, x) = \sum_{n=0}^{\infty} \widehat{L}_\omega^{2n}\left(x + \widehat{L}_\omega(x)\right) \ge 0$$

for each $x \ge_A 0$, and the proof of (i) is finished. An adaptation of the arguments in Theorem 3.9(iii) and Theorem 5.2(iv)–(v) of [19] to the phase space BC yield (ii) and (iii). □

Let us define $M : \Omega \times BC \to \mathbb{R}$ as follows:

$$M(\omega, x) = \sum_{i=1}^{m} D_i(\omega, x) + \sum_{i=1}^{m}\sum_{j=1}^{m} \int_{-\infty}^{0} \left(\int_{s}^{0} g_{ji}(\omega{\cdot}u, x_i(u))\, du \right) d\mu_{ji}(s)\,, \tag{3.5}$$

for each $(\omega, x) \in \Omega \times BC$. M is said to be the *total mass* of the family of equations (3.1). Thanks to conditions (C1)–(C2), for all $x \in BC$ and all $i, j \in \{1, \dots, m\}$,

$$\left| \int_{s}^{0} g_{ji}(\omega{\cdot}u, x_i(u))\, du \right| \le c_{ji}\, |s|\,,$$

where $c_{ji} = \sup_{\Omega\times[-\|x\|_\infty,\|x\|_\infty]} g_{ji}$. Hence, M is well-defined.

A key property of the total mass is established by our next result. Similar results can be found in [15], [17], [19], and [26].

Proposition 3.2. *Under assumptions (C1)–(C3), for each $r > 0$, the total mass is uniformly continuous on $\Omega \times B_r$ when the compact-open topology is considered on B_r. In addition, for all $(\omega, x) \in \Omega \times BC$ and all $t \ge 0$ where the solution is defined,*

$$M(\tau(t, \omega, x)) = M(\omega, x) + \sum_{i=1}^{m} \int_{0}^{t} (I_i(\omega{\cdot}s) - g_{0i}(\omega{\cdot}s, z_i(s, \omega, x)))\, ds\,. \tag{3.6}$$

Proof. From (C3) and the definition of D, it is easy to deduce that D is linear and continuous in its second variable for the norm $\|\cdot\|_\infty$, the map $\Omega \to \mathcal{L}(BC, \mathbb{R}^m)$, $\omega \mapsto D(\omega, \cdot)$ is continuous, and the restriction of D to $\Omega \times B_r$ is continuous when we take the restriction of the compact-open topology to B_r for all $r > 0$.

Hence, a natural generalization of Theorem 3.9 in [19] together with properties (C1)–(C2) implies that $\widehat{D}$ is uniformly continuous on $\Omega \times B_r$. The proof of Proposition 5.5 of Muñoz-Villarragut [14] can be adapted to show the uniform continuity of M on $\Omega \times B_r$. Finally, a computation similar to the one given in [26] proves the variation formula (3.6). □

Let us recall a regularity condition introduced in [18] concerning a class of initial data $x \in BC$:

(R) for each $i \in \{1, \dots, n\}$, x_i is of bounded variation on every compact subinterval of $(-\infty, 0]$, and

$$\sup \left\{ V_{[-k,-k+1]}(x_i) \mid i \in \{1, \dots, m\},\, k \ge 1 \right\} < \infty\,,$$

where $V_{[-k,-k+1]}(x_i)$ denotes the total variation of x_i on the interval $[-k, -k+1]$.

Notice that property (R) is satisfied by all the Lipschitz continuous elements of BC, but not all the elements of BC satisfying (R) are Lipschitz continuous. Besides, the subspace $\mathcal{R}$ of BC determined by (R) is a Banach space for the norm

$$\|x\|_\infty + \sup \left\{ V_{[-k,-k+1]}(x_i) \mid i \in \{1, \dots, m\},\, k \ge 1 \right\}, \quad x \in \mathcal{R}\,.$$

Theorem 3.3. *Assume conditions (C1)–(C4). Given $(\omega_0, x_0) \in \Omega \times BC$ with $\widehat{D}_2(\omega_0, x_0)$ satisfying property (R), if $z(\cdot, \omega_0, x_0)$ is a bounded solution of $(3.3)_{\omega_0}$, then the omega-limit set $\mathcal{O}(\omega_0, x_0) = \{(\omega, c(\omega)) \mid \omega \in \Omega\}$ is a copy of the base and*

$$\lim_{t\to\infty} \mathrm{d}(u(t, \omega_0, x_0), c(\omega_0{\cdot}t)) = 0\,,$$

where $c : \Omega \to BU$ is a continuous equilibrium, i.e. $u(t, \omega, c(\omega)) = c(\omega{\cdot}t)$ for each $\omega \in \Omega$ and $t \ge 0$, and it is continuous for the compact-open topology on BU.

Proof. As above, we consider the exponential ordering $\leq_A$. Thanks to Theorem 3.1, we are in a position to consider the *transformed exponential ordering* $\leq_{D,A}$ introduced in [19]. It is defined on each fiber of the product $\Omega \times BC$ as follows: for each (ω, x), $(\omega, y) \in \Omega \times BC$,

$$(\omega, x) \leq_{D,A} (\omega, y) \iff \widehat{D}_2(\omega, x) \leq_A \widehat{D}_2(\omega, y).$$

Our aim is to apply Theorem 4.12 of [18] in order to complete this proof. First, as seen in Theorem 3.1(iii) and in the proof of Proposition 3.2, $D(\omega, \cdot)$ is linear and continuous for the norm for all $\omega \in \Omega$, the mapping $\Omega \to \mathcal{L}(BC, \mathbb{R}^m)$, $\omega \mapsto D(\omega, \cdot)$ is continuous, the restriction of D to $\Omega \times B_r$ is continuous when we consider the compact-open topology on B_r for all $r > 0$, and D is stable, so conditions (D1)–(D4) of [18] hold.

Besides, from assumptions **(C1)**–**(C2)**, it follows that the function G defined by (3.2) is continuous on $\Omega \times BC$. Moreover, its restriction to $\Omega \times B_r$ is Lipschitz continuous in its second variable when the norm is considered on B_r, and it is continuous when the compact-open topology is considered on B_r for each $r > 0$. This implies conditions (N1) and (N2) of [18].

Let (ω, x), $(\omega, y) \in \Omega \times BC$ with $(\omega, x) \leq_{D,A} (\omega, y)$. From Theorem 3.1(i) we deduce that $x \leq y$. Then, from the nondecreasing character of g_{ij}, stated in **(C1)**, the fact that $\mu_{ij}(\omega)$ are positive and $\nu_i(\omega)$ are negative measures, as assumed in **(C2)** and **(C3)** respectively, and $a_i < 0$, we deduce that

$$-\sum_{j=0}^{m} \big(g_{ji}(\omega, y_i(0)) - g_{ji}(\omega, x_i(0))\big) \geq -L_i^+(\omega)\big(y_i(0) - x_i(0)\big)\,,$$

$$\sum_{j=1}^{m} \int_{-\infty}^{0} \big(g_{ij}(\omega{\cdot}s, y_j(s)) - g_{ij}(\omega{\cdot}s, x_j(s))\big)\, d\mu_{ij}(\omega)(s) \geq 0\,,$$

$$c_i(x, y) := a_i \int_{-\infty}^{0} \big(y_i(s) - x_i(s)\big)\, d\nu_i(\omega)(s) \geq 0\,, \tag{3.7}$$

and, from **(C4)**(i), we conclude that

$$\begin{aligned} &G_i(\omega, y) - G_i(\omega, x) - a_i\big(D_i(\omega, y) - D_i(\omega, x)\big) \\ &\qquad \geq \big(-L_i^+(\omega) - a_i\big)\big(y_i(0) - x_i(0)\big) + c_i(x, y) \geq 0\,. \end{aligned} \tag{3.8}$$

This guarantees that condition (N3) of [18] is satisfied.

Fix (ω, x), $(\omega, y) \in \Omega \times BC$ with $(\omega, x) \leq_{D,A} (\omega, y)$. Suppose that (ω, x) and (ω, y) admit a backward orbit extension and there is a subset $J \subset \{1, \ldots, m\}$ such that the following conditions hold

$$\begin{aligned} \widehat{D}_2(\omega, x)_i &= \widehat{D}_2(\omega, y)_i \quad \text{for each } i \notin J\,, \\ \widehat{D}_2(\omega, x)_i(s) &< \widehat{D}_2(\omega, y)_i(s) \quad \text{for each } i \in J \text{ and } s \leq 0\,. \end{aligned}$$

If $i \in J$, then we have $\widehat{D}_2(\omega, x)_i(s) < \widehat{D}_2(\omega, y)_i(s)$, that is, $D_i(\omega{\cdot}s, x_s) < D_i(\omega{\cdot}s, y_s)$ for each $s \leq 0$. In particular, if $s = 0$, we obtain

$$\int_{-\infty}^{0} \big(y_i(s) - x_i(s)\big)\, d\nu_i(\omega)(s) < y_i(0) - x_i(0)\,.$$

As before, from $(\omega, x) \leq_{D,A} (\omega, y)$, we deduce that $x \leq y$ and, since $\nu_i(\omega)$ is a negative measure, we have two options:

$$\int_{-\infty}^{0} \big(y_i(s) - x_i(s)\big)\, d\nu_i(\omega)(s) < 0 \quad \text{or} \quad y_i(0) - x_i(0) > 0\,.$$

In the first case, since $a_i < 0$, we deduce from (3.7) that $c_i(x,y) > 0$ and, in the second case, from **(C4)**(i), we have $\big(-L_i^+(\omega) - a_i\big)\big(y_i(0) - x_i(0)\big) > 0$. Therefore, inequality (3.8) is strict in both cases. As a result, condition (N4) of [18] holds.

Let us check that, if (ω,x), $(\omega,y) \in \Omega \times BC$ with $(\omega,x) \le_{D,A} (\omega,y)$ and $z(t,\omega,x)$, $z(t,\omega,y)$ are defined, then

$$0 \le D_i(\tau(t,\omega,y)) - D_i(\tau(t,\omega,x)) \le M(\omega,y) - M(\omega,x)\,, \quad i = 1,\dots,m\,. \tag{3.9}$$

Thanks to Theorem 4.8 of [18], the skew-product semiflow τ is monotone. As a result, $\tau(t,\omega,x) \le_{D,A} \tau(t,\omega,y)$ whenever they are defined, i.e $\widehat{D}_2(\tau(t,\omega,x)) \le_A \widehat{D}_2(\tau(t,\omega,y))$. Thus, Theorem 3.1(i) provides $z_t(\omega,x) \le z_t(\omega,y)$, and **(C1)** implies

$$g_{ij}(\omega{\cdot}t, z_j(t,\omega,x)) \le g_{ij}(\omega{\cdot}t, z_j(t,\omega,y))\,.$$

Moreover, from $\widehat{D}_2(\tau(t,\omega,x)) \le_A \widehat{D}_2(\tau(t,\omega,y))$, it follows that $\widehat{D}_2(\tau(t,\omega,x)) \le \widehat{D}_2(\tau(t,\omega,y))$, and we deduce that $D_i(\tau(t,\omega,x)) \le D_i(\tau(t,\omega,y))$ for $i = 1,\dots,m$.

As a consequence, **(C2)** and the total mass variation formula (3.6) yield

$$\begin{aligned} 0 &\le D_i(\tau(t,\omega,y)) - D_i(\tau(t,\omega,x)) \le \sum_{i=1}^m [D_i(\tau(t,\omega,y)) - D_i(\tau(t,\omega,x))] \\ &\le M(\tau(t,\omega,y)) - M(\tau(t,\omega,x)) = M(\omega,y) - M(\omega,x) \\ &\quad + \sum_{i=1}^m \int_0^t (g_{0i}(\omega{\cdot}s, z_i(s,\omega,x)) - g_{0i}(\omega{\cdot}s, z_i(s,\omega,y)))\,ds \le M(\omega,y) - M(\omega,x)\,, \end{aligned}$$

as claimed.

Finally, thanks to Proposition 3.2, given $\varepsilon > 0$ and $r > 0$, there exists $\delta > 0$ such that, if x, $y \in B_r$ with $(\omega,x) \le_{D,A} (\omega,y)$ and $\mathsf{d}(x,y) < \delta$, then $0 \le M(\omega,y) - M(\omega,x) < \varepsilon$. Consequently, if x, $y \in B_r$ and $(\omega,x) \le_{D,A} (\omega,y)$, from (3.9), it follows that $0 \le D_i(\omega{\cdot}t, z_t(\omega,y)) - D_i(\omega{\cdot}t, z_t(\omega,x))) < \varepsilon$. Hence, the fact that $\nu_i(\omega)$ is a negative measure implies

$$0 \le z_i(t,\omega,y) - z_i(t,\omega,x) \le \varepsilon + \int_{-\infty}^0 (z_i(t+s,\omega,y) - z_i(t+s,\omega,x))\,d\nu_i(\omega)(s) \le \varepsilon\,,$$

whence, for all x, $y \in B_r$ with $(\omega,x) \le_{D,A} (\omega,y)$ and $\mathsf{d}(x,y) < \delta$, $\|z(t,\omega,y) - z(t,\omega,x)\| \le \varepsilon$ whenever they are defined, so condition (N5) of [18] is satisfied. This property is usually referred to as *uniform stability for the ordering* $\le_{D,A}$ of B_r and it was introduced in [19]. An application of Theorem 4.12 of [18] finishes the proof, as expected. □

Remark 3.4. In compartmental systems, one is interested in initial conditions (ω,x) for which the solutions are bounded and positive, i.e. $u(t,\omega,x) \ge 0$ for each $t \ge 0$. If 0 is a solution, for instance when the compartmental systems (3.1) are closed, the set $\mathcal{P} = \{(\omega,x) \in \Omega \times BC \mid (\omega,x) \ge_{D,A} 0\}$ satisfies this property. Indeed, if $(\omega,x) \in \mathcal{P}$, then $u(t,\omega,x) \ge_{D,A} 0$ for all $t \ge 0$, whence $\widehat{u}(t,\omega,x) \ge_A 0$ for all $t \ge 0$, and Theorem 3.1(i) provides $u(t,\omega,x) \ge 0$ for all $t \ge 0$. The boundedness follows from the uniform stability on each B_r for the ordering $\le_{D,A}$. In addition, note that $(\omega,x) \ge_{D,A} 0$ means that $\widehat{D}_2(\omega,x) \ge_A 0$, which implies, as seen in Proposition 3.8 of [18], that $\widehat{D}_2(\omega,x)$ satisfies **(R)**. Summing up, the conclusions of Theorem 3.3 state that, for closed compartmental systems, $\mathcal{O}(\omega,x)$ is a copy of the base for all initial data in $\mathcal{P}$, those we are interested in.

Under some conditions, in order to verify that $\widehat{D}_2(\omega, x)$ satisfies the property **(R)**, it is sufficient to prove that x does. The following definition is a natural generalization of Definition 6.1 of [19]. As before, $\mathcal{M}$ denotes the set of Borel regular measures on $(-\infty, 0]$.

Definition 3.5. A map $\nu : \Omega \to \mathcal{M}$ is said to be *Lipschitz continuous along the flow* σ if, for each $\omega \in \Omega$, the map $\mathbb{R} \to \mathcal{M}$, $t \mapsto \nu(\omega{\cdot}t)$ is Lipschitz continuous, when the norm given by the total variation is considered on $\mathcal{M}$.

Remark 3.6. Thanks to the minimal character of the base flow on Ω, if the map $\mathbb{R} \to \mathcal{M}$, $t \mapsto \nu(\omega_0{\cdot}t)$ is Lipschitz continuous with Lipschitz constant $L > 0$ for one $\omega_0 \in \Omega$, the same holds for all the maps $\mathbb{R} \to \mathcal{M}$, $t \mapsto \nu(\omega{\cdot}t)$, $\omega \in \Omega$, whence ν is Lipschitz continuous along the flow.

Proposition 3.7. *Assume that ν_i is Lipschitz continuous along the flow and there exists a positive measure $\overline{\nu}$ with finite total variation such that $|\nu_i(\omega)| \le \overline{\nu}$ for all $i \in \{1, \dots, m\}$ and all $\omega \in \Omega$. If $(\omega, x) \in \Omega \times BC$ and x satisfies property* **(R)**, *then $\widehat{D}_2(\omega, x)$ also satisfies property* **(R)**.

Proof. Let $L_\nu > 0$ be a Lipschitz constant valid for ν_i for all $i \in \{1, \dots, m\}$ and

$$V = \sup\left\{ V_{[-k,-k+1]}(x_i) \mid i \in \{1, \dots, m\},\, k \ge 1 \right\}.$$

Fix $i \in \{1, \dots, m\}$, $k \in \mathbb{N}$ and let $-k = t_0 < t_1 < \cdots < t_n = -k+1$ be a partition of the interval $[-k, -k+1]$. Then, for all $s \le 0$, the set $\{t_j + s \mid j \in \{0, \dots, n\}\}$ is included in a partition of an interval of the form $[-l, -l+2]$ for some $l \in \mathbb{N}$ with $l \ge 2$. From this fact, it follows that

$$\begin{aligned}
\sum_{j=1}^{n} \left| \widehat{D}_2(\omega, x)_i(t_j) - \widehat{D}_2(\omega, x)_i(t_{j-1}) \right| &= \sum_{j=1}^{n} \left| D_i(\omega{\cdot}t_j, x_{t_j}) - D_i(\omega{\cdot}t_{j-1}, x_{t_{j-1}}) \right| \\
&\le \sum_{j=1}^{n} |x_i(t_j) - x_i(t_{j-1})| \\
&\quad + \sum_{j=1}^{n} \left| \int_{-\infty}^{0} x_i(t_j + s)\, d\nu_i(\omega{\cdot}t_j)(s) - \int_{-\infty}^{0} x_i(t_{j-1} + s)\, d\nu_i(\omega{\cdot}t_{j-1})(s) \right| \\
&\le V + \sum_{j=1}^{n} \int_{-\infty}^{0} |x_i(t_j + s) - x_i(t_{j-1} + s)|\, d|\nu_i(\omega{\cdot}t_j)|(s) \\
&\quad + \sum_{j=1}^{n} \int_{-\infty}^{0} |x_i(t_{j-1} + s)|\, d|\nu_i(\omega{\cdot}t_j) - \nu_i(\omega{\cdot}t_{j-1})|(s) \\
&\le V + \int_{-\infty}^{0} \sum_{j=1}^{n} |x_i(t_j + s) - x_i(t_{j-1} + s)|\, d\overline{\nu}(s) + \sum_{j=1}^{n} |t_j - t_{j-1}|\, L_\nu \, \|x\|_\infty \\
&\le V + 2\, V\, \overline{\nu}((-\infty, 0]) + L_\nu \, \|x\|_\infty \,,
\end{aligned}$$

which is a bound independent of k and n, and $\widehat{D}_2(\omega, x)$ satisfies **(R)**, as claimed. □

Finally, we remind that [19] contains a dynamical study of the compartmental systems (3.3) on BU when the measures $\nu_i(\omega)$ are positive for $i = 1, \dots, n$ and $\omega \in \Omega$. It provides technical assumptions under which the omega-limit set of every initial datum (ω, x), with x Lipschitz continuous and $z(\cdot, \omega, x)$ bounded, is a copy of the base. From the conclusions of this paper, it follows that this result remains also valid when x satisfies condition **(R)**.

4 Neutral compartmental systems with finite delay

In this section, we include a nonautonomous neutral compartmental system with finite delay satisfying some recurrence conditions on the temporal variation into a family of the form (3.3), and we apply the conclusions of the previous section to the study of the long-term behavior of the solutions of the initial system.

We consider the system of NFDEs with finite delay

$$\frac{d}{dt}[z_i(t)+\widetilde{c}_i(t)\,z_i(t-\alpha_i)] = -\sum_{j=1}^{m}\widetilde{g}_{ji}(t,z_i(t)) + \sum_{j=1}^{m}\widetilde{g}_{ij}(t-\rho_{ij},z_j(t-\rho_{ij})) \tag{4.1}$$

$i=1,\dots,m$. Let us denote by $\widetilde{g}=(\widetilde{g}_{ij})_{i,j}\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}^{m\times m}$, $\widetilde{c}=(c_i)_i\colon\mathbb{R}\to\mathbb{R}^m$ and assume that

- **(c1)** $\widetilde{g}$ is C^1 in its second variable and $\widetilde{g}$, $\frac{\partial}{\partial v}\widetilde{g}$ are uniformly continuous and bounded on $\mathbb{R}\times\{v_0\}$ for all $v_0\in\mathbb{R}$;
- **(c2)** $\widetilde{g}_{ij}(t,0)=0$ and $\widetilde{g}_{ij}(t,\cdot)$ is nondecreasing for all $t\in\mathbb{R}$ and $i,j=1,\dots,m$;
- **(c3)** $\widetilde{c}$ is Lipschitz continuous, nonnegative, and $\sup_{s\in\mathbb{R}}\widetilde{c}_i(s)<1$, $i=1,\dots m$;
- **(c4)** letting $(t,(\mathfrak{c},\mathfrak{g}))\mapsto(\mathfrak{c}_t,\mathfrak{g}_t)$ denote the translation flow, that is, $\mathfrak{c}_t(s)=\mathfrak{c}(t+s)$ and $\mathfrak{g}_t(s,v)=\mathfrak{g}(t+s,v)$ for all $(s,v)\in\mathbb{R}^2$, the closure of the set $\{(\widetilde{c}_t,\widetilde{g}_t)\mid t\in\mathbb{R}\}$ for the compact-open topology, referred to as the *hull* of $(\widetilde{c},\widetilde{g})$, is minimal (in this situation, $(\widetilde{c},\widetilde{g})$ is said to be *recurrent*);
- **(c5)** for each $i=1,\dots,m$, there is a negative $a_i<0$ such that
  - (i) $-L_i^+-a_i>0$ and
  - (ii) $\widetilde{c}_i(t)\le e^{a_i\alpha_i}$, for each $t\in\mathbb{R}$,

  where $L_i^+=\sum_{j=1}^m\sup_{(t,v)\in\mathbb{R}^2}\frac{\partial\widetilde{g}_{ij}}{\partial v}(t,v)$.

Remark 4.1. Notice that no relation among the delays ρ_{ij} and α_i, $i,j=1,\dots,m$, is assumed. Conditions of the same type for the direct exponential ordering and negative coefficients $\widetilde{c}_i$ were obtained in [13], [17] and [20].

In this situation, we may include the system (4.1) in a family of nonautonomous NFDEs. Specifically, let Ω be the hull of $(\widetilde{c},\widetilde{g})$, as defined in **(c4)**. Ω is a compact metric space thanks to **(c1)** and **(c3)** (see Hino, Murakami, and Naito [8]). Let $\sigma:\mathbb{R}\times\Omega\to\Omega$, $(t,\omega)\mapsto\omega{\cdot}t$ be the flow defined on Ω by translation, which is minimal, as specified in **(c4)**. It is noteworthy that the almost periodic and almost automorphic cases are included in this formulation.

Let us consider the continuous map $(c,g):\Omega\times\mathbb{R}\to\mathbb{R}^m\times\mathbb{R}^{m\times m}$, $(\omega,v)\mapsto\omega(0,v)$. Let us denote $c=(c_i)_i$, $g=(g_{ij})_{i,j}$, and define $G:\Omega\times BC\to\mathbb{R}^m$ by

$$G_i(\omega,x) = -\sum_{j=1}^{m}g_{ji}(\omega,x_i(0)) + \sum_{j=1}^{m}g_{ij}(\omega{\cdot}(-\rho_{ij}),x_j(-\rho_{ij}))\,,\quad (\omega,x)\in\Omega\times BC\,,$$

and $D\colon\Omega\times BC\to\mathbb{R}^m$ by

$$D_i(\omega,x)=x_i(0)+c_i(\omega)\,x_i(-\alpha_i)\,,\quad (\omega,x)\in\Omega\times BC\,,\ i=1,\dots,m\,.$$

Hence, the family

$$\frac{d}{dt}D(\omega{\cdot}t,z_t)=G(\omega{\cdot}t,z_t)\,,\quad t\geq 0\,,\;\omega\in\Omega\,,\tag{4.2}$$

is of the form (3.1) for the negative Borel regular measure $\nu_i(\omega)=-c_i(\omega)\,\delta_{-\alpha_i}$ and the positive Borel regular measure $\mu_{ij}=\delta_{-\rho_{ij}}$, $i,\,j=1,\dots,m$, and includes system (4.1) when $\omega=(\widetilde{c},\widetilde{g})$.

Notice that the system (4.2) represents a *closed* compartmental system for each $\omega\in\Omega$, that is, there is no flow of material from or to the environment. Consequently, from **(c2)**, it follows that zero is a bounded solution of all the systems of the family and, thanks to (3.6), the total mass is constant along the trajectories.

Theorem 4.2. *Assume that* **(c1)**–**(c5)** *hold and fix* $(\omega_0,x_0)\in\Omega\times BC$ *such that* x_0 *satisfies property* **(R)**. *Then the solution* $z(\cdot,\omega_0,x_0)$ *of* $(4.2)_{\omega_0}$ *with initial value* x_0 *is bounded, the omega-limit set* $\mathcal{O}(\omega_0,x_0)=\{(\omega,x(\omega))\mid\omega\in\Omega\}$ *is a copy of the base and*

$$\lim_{t\to\infty}\mathsf{d}(u(t,\omega_0,x_0),x(\omega_0{\cdot}t))=0\,,$$

where $x:\Omega\to BU$ *is a continuous equilibrium.*

Proof. From **(c1)**–**(c5)**, $\nu_i(\omega)=-c_i(\omega)\,\delta_{-\alpha_i}$ and $\mu_{ij}=\delta_{-\rho_{ij}}$ for $i,\,j=1,\dots m$, it is easy to check that **(C1)**–**(C4)** hold. Notice in particular that, from Remark 3.6, if $\widetilde{c}$ is Lipschitz continuous, then ν_i is Lipschitz continuous along the flow for all $i\in\{1,\dots,m\}$. In addition, since x_0 satisfies property **(R)** and the conditions of Proposition 3.7 are clearly satisfied, then $\widehat{D}_2(\omega,x_0)$ satisfies property **(R)** as well. Therefore, the result follows from Theorem 3.3, once the boundedness of $z(\cdot,\omega_0,x_0)$ is proved.

As shown in Theorem 3.3, each B_r is uniformly stable for the ordering $\leq_{D,A}$. This and the fact that zero is a solution of (4.2) for all $\omega\in\Omega$ allows us to deduce that each solution $z(\cdot,\omega,x)$ with $(\omega,x)\geq_{D,A}(\omega,0)$ or $(\omega,x)\leq_{D,A}(\omega,0)$ is globally defined and bounded. Since $\widehat{D}_2(\omega,x_0)$ satisfies property **(R)**, thanks to Proposition 3.8 of [18], there exists $h\in BC$ such that $h\geq_A 0$ and $h\geq_A\widehat{D}_2(\omega,x)$, i.e $(\omega,\widehat{h})\geq_{D,A}(\omega,0)$ and $(\omega,\widehat{h})\geq_{D,A}(\omega,x_0)$ for $\widehat{h}=(\widehat{D}^{-1})_2(\omega,h)$. Consequently, from the first inequality we deduce that $z(t,\omega,\widehat{h})$ is globally defined and bounded and, hence, from the second inequality and again the uniform stability of each B_r for the ordering $\leq_{D,A}$, we conclude that $z(\cdot,\omega,x_0)$ is globally defined and bounded, as desired. □

Regarding the solutions of the system (4.1), we obtain the following result, stated in the almost periodic case, but similar conclusions hold changing almost periodicity for constancy, periodicity, almost automorphy or recurrence. All solutions with initial data x_0 satisfying property **(R)**, in particular those with Lipschitz continuous initial data, are asymptotically of the same type as the transport functions and the coefficients of the neutral part.

Theorem 4.3. *Under assumptions* **(c1)**–**(c5)**, *if both* $\widetilde{c}$ *and* $\widetilde{g}$ *are almost periodic, then there exist infinitely many almost periodic solutions of system* (4.1). *Moreover, all the solutions with initial data* $x_0\in BC$ *satisfying property* **(R)** *are asymptotically almost periodic.*

Proof. Fix $\omega_0=(\widetilde{c},\widetilde{g})$ and $x_0\in BC$ satisfying property **(R)**. The omega-limit set $\mathcal{O}(\omega_0,x_0)$ is a copy of the base, whence, $t\mapsto z(t,\omega_0,x(\omega_0))=x(\omega_0{\cdot}t)(0)$ is an almost periodic solution of (4.1) and

$$\lim_{t\to\infty}\|z(t,\omega_0,x_0)-z(t,\omega_0,x(\omega_0))\|=0\,.$$

Let us check that there are infinitely many minimal subsets. Let $x^k \in BC$ denote the constant function $x^k \equiv k\,(1,\dots,1)$. The total mass associated to (ω_0, x^k) is

$$M(\omega_0, x^k) = \sum_{i=1}^{m} k\,(1 + c_i(\omega_0)) + \sum_{i=1}^{m}\sum_{j=1}^{m} \int_{-\infty}^{0} \left(\int_{s}^{0} g_{ji}(\omega_0{\cdot}\tau, k)\, d\tau \right) d\mu_{ji}(s)\,,$$

which diverges to infinity as $k \to \infty$. Therefore, for all $r > 0$, there exists $k_r > 0$ such that $M(\omega_0, x^{k_r}) = r$. Finally, since the total mass is constant along the trajectories, $\mathcal{O}(\omega_0, x^{k_r})$ provides a different minimal set and, hence, a different almost periodic solution for each $r > 0$. □

5 Numerical experiments

In this section, we apply the results included in Section 4 and perform numerical experiments on the neutral compartmental system:

$$\begin{aligned}
&\frac{d}{dt}\big(z_1(t) + 0.1\,(1+\cos(\sqrt{2}\,(t-0.5)))\,z_1(t-0.5)\big)\\
&\qquad = -\,0.1\,(1+\cos(\sqrt{2}\,t))\,\tanh(z_1(t)) - 0.2\,(1+\cos(t))\,\tanh(z_1(t))\\
&\qquad\quad + 0.1\,(1+\cos(\sqrt{2}\,(t-1)))\,\tanh(z_1(t-1))\\
&\qquad\quad + 0.2\,(1+\sin(\sqrt{2}\,(t-2)))\,\tanh(z_2(t-2))\,,\\
&\frac{d}{dt}\big(z_2(t) + 0.05\,(1+\sin((t-1)))\,z_2(t-1)\big)\\
&\qquad = -\,0.2\,(1+\sin(\sqrt{2}\,t))\,\tanh(z_2(t)) - 0.1\,(1+\sin(t))\,\tanh(z_2(t))\\
&\qquad\quad + 0.2\,(1+\cos((t-0.3)))\,\tanh(z_1(t-0.3))\\
&\qquad\quad + 0.1\,(1+\sin((t-0.5)))\,\tanh(z_2(t-0.5))\,,
\end{aligned} \tag{5.1}$$

for $t \ge 0$. Clearly, the system of equations (5.1) is of the form given in (4.1). In this case,

$$\begin{aligned}
&\widetilde{g}_{11}(t,x) = 0.1\,(1+\cos(\sqrt{2}\,t))\,\tanh(x)\,, && \widetilde{g}_{12}(t,x) = 0.2\,(1+\sin(\sqrt{2}\,t))\,\tanh(x)\,,\\
&\widetilde{g}_{21}(t,x) = 0.2\,(1+\cos(t))\,\tanh(x)\,, && \widetilde{g}_{22}(t,x) = 0.1\,(1+\sin(t))\,\tanh(x)\,,\\
&\rho_{11} = 1\,, && \rho_{12} = 2\,,\\
&\rho_{21} = 0.3\,, && \rho_{22} = 0.5\,,\\
&\widetilde{c}_1(t) = 0.1\,(1+\cos(\sqrt{2}\,t))\,, && \widetilde{c}_2(t) = 0.05\,(1+\sin(t))\,,\\
&\alpha_1 = 0.5\,, && \alpha_2 = 1\,.
\end{aligned}$$

The following result shows that the system of equations (5.1) also satisfies the conditions assumed throughout Section 4.

Proposition 5.1. *The system of equations* (5.1) *satisfies conditions* **(c1)**–**(c5)**.

Proof. Conditions **(c1)**–**(c4)** are immediate. As for condition **(c5)**, it is trivial to check that $l_{11}^+ \le 0.2$ and $l_{21}^+ \le 0.4$, whence $L_1^+ \le 0.6$. As a result, by choosing $a_1 = -0.7$, we obtain $e^{a_1\alpha_1} > 0.7$, which is an upper bound of $\widetilde{c}_1$. Analogously, $l_{12}^+ \le 0.4$ and $l_{22}^+ \le 0.2$, whence $L_2^+ \le 0.6$. By choosing $a_2 = -0.7$, we obtain $e^{a_2\alpha_2} > 0.4$, which is an upper bound of $\widetilde{c}_2$, as wanted. □

It is noteworthy that, as in Section 4, we can define Ω as the hull of the coefficients of the system of equations (5.1). It is well-known that the flow defined on Ω by translation is isomorphic to the Kronecker flow defined on the 2-torus $\mathbb{T}^2 = (\mathbb{R}/[0,2\pi])^2$, which is defined by

$$\begin{aligned} \zeta : \mathbb{R}^+ \times \mathbb{T}^2 &\longrightarrow \mathbb{T}^2 \\ (t,\theta_1,\theta_2) &\longmapsto (\theta_1 + \sqrt{2}\,t, \theta_2 + t)\,. \end{aligned}$$

Notice that ζ is a minimal flow (see e.g. Walters [23] for further details). This fact allows us to include the system of equations (5.1) in the family of equations

$$\begin{aligned} &\frac{d}{dt}\big(z_1(t) + 0.1\,(1+\cos(\theta_1+\sqrt{2}\,(t-0.5)))\,z_1(t-0.5)\big) = \\ &\qquad = -\,0.1\,(1+\cos(\theta_1+\sqrt{2}\,t))\,\tanh(z_1(t)) - 0.2\,(1+\cos(\theta_2+t))\,\tanh(z_1(t)) \\ &\qquad\quad + 0.1\,(1+\cos(\theta_1+\sqrt{2}\,(t-1)))\,\tanh(z_1(t-1)) \\ &\qquad\quad + 0.2\,(1+\sin(\theta_1+\sqrt{2}\,(t-2)))\,\tanh(z_2(t-2))\,, \\ &\frac{d}{dt}\big(z_2(t) + 0.05\,(1+\sin(\theta_2+(t-1)))\,z_2(t-1)\big) = \\ &\qquad = -\,0.2\,(1+\sin(\theta_1+\sqrt{2}\,t))\,\tanh(z_2(t)) - 0.1\,(1+\sin(\theta_2+t))\,\tanh(z_2(t)) \\ &\qquad\quad + 0.2\,(1+\cos(\theta_2+(t-0.3)))\,\tanh(z_1(t-0.3)) \\ &\qquad\quad + 0.1\,(1+\sin(\theta_2+(t-0.5)))\,\tanh(z_2(t-0.5))\,, \end{aligned} \tag{5.2}$$

for $t \geq 0$, where $(\theta_1,\theta_2) \in \mathbb{T}^2$.

In this situation, we can apply the results obtained in Section 4 to the system of equations (5.1).

Theorem 5.2. *Fix $(\theta_1,\theta_2,x_0) \in \mathbb{T}^2 \times BC$ such that x_0 satisfies property* **(R)**. *Then the solution $z(\cdot,\theta_1,\theta_2,x_0)$ of (5.2)$_{(\theta_1,\theta_2)}$ with initial value x_0 is bounded, the omega-limit set $\mathcal{O}(\theta_1,\theta_2,x_0) = \{(\varphi_1,\varphi_2,x(\varphi_1,\varphi_2)) \mid (\varphi_1,\varphi_2) \in \mathbb{T}^2\}$ is a copy of the base and*

$$\lim_{t\to\infty} \mathsf{d}(u(t,\theta_1,\theta_2,x_0), x(\zeta_t(\theta_1,\theta_2))) = 0\,,$$

where $x : \mathbb{T}^2 \to BU$ is a continuous equilibrium. Moreover, there are infinitely many almost periodic solutions of system (5.1) and all the solutions with initial data x_0 satisfying property **(R)** *are asymptotically almost periodic.*

Proof. It follows immediately from Proposition 5.1, Theorem 4.2, and Theorem 4.3. □

In what follows, some numerical simulations of the neutral compartmental system (5.1) will be presented. Numerical methods for neutral differential equations with delay are well-known (see e.g. Wen and Yu [24] and the references therein). All our computations were carried out with the Matlab function `ddensd`, with a relative tolerance of 10^{-5} and an absolute tolerance of 10^{-10}.

Figure 5.1 shows the solutions of system (5.1) with the following initial data $x_0^i : (-\infty,0] \to \mathbb{R}^2$, $i = 1, 2, 3$, which satisfy property **(R)**:

(i) $x_0^1(t) = (-\sin(t), \max\{2, \sqrt{-t}\})$ for all $t \in (-\infty,0]$;

(ii) $x_0^2(t) = (0.5, 1)$ for all $t \in (-\infty,0]$;

(iii) $x_0^3(t) = (0.3\,e^t, 0.3\,(1+\cos(2\,t)))$ for all $t \in (-\infty,0]$.

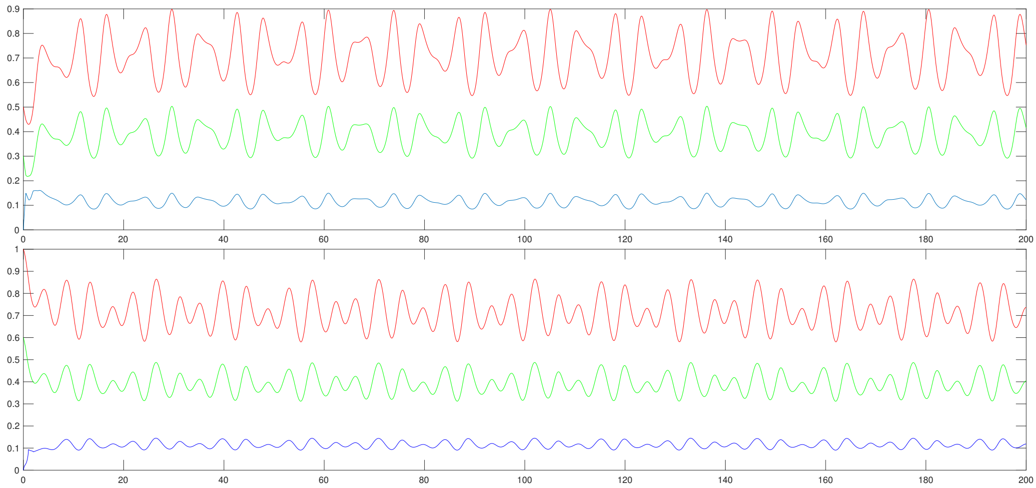


Figure 5.1: First and second components of the solutions of system (5.1) with initial data x_0^1 (blue), x_0^2 (red), and x_0^3 (green) for $t \in [0, 200]$.

Note that x_0^1 is *not* Lipschitz continuous.

In order to approximate the omega-limit of the solution of system (5.1) with initial datum x_0^1, $z : \mathbb{R} \to \mathbb{R}^2$, the system of equations (5.1) was integrated for $t \in [0, 5000]$. We considered the time subintervals $J_k = [500\,k, 500\,(k+1)]$, $k = 0, \ldots, 9$, and restricted the solution z to those subintervals. Then, the graphs over the 2-torus of both components of $z|_{J_k}$,

$$\{(\sqrt{2}\,t \pmod{2\,\pi}, t \pmod{2\,\pi}, z_i(t)) \mid t \in J_k\}, \quad i = 1, 2,$$

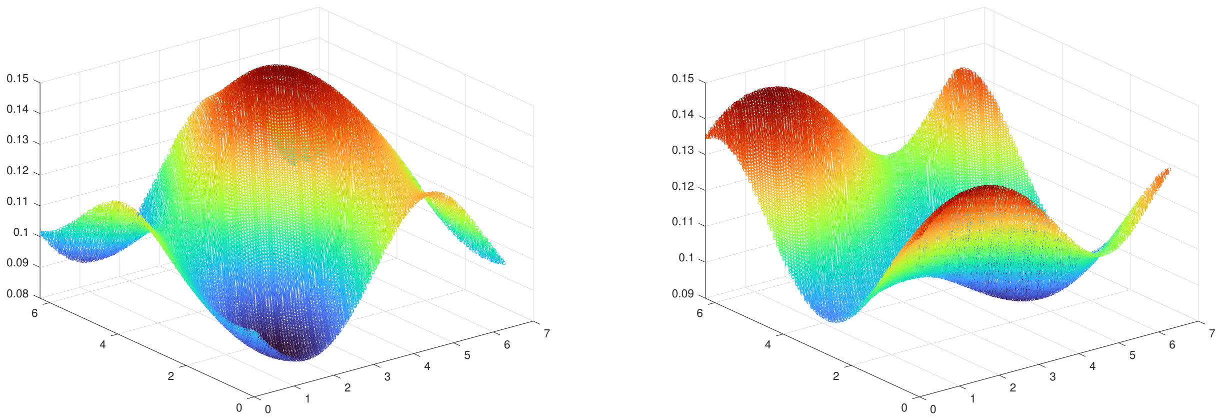
can be approximated by two surfaces S_k^1 and S_k^2, respectively, for $k = 0, \ldots, 9$. These approximated surfaces were computed by means of a cubic interpolation of the scattered data provided by the numerical integration of z on J_k, for $k = 0, \ldots, 9$. Finally, the distance between any two consecutive surfaces in the finite sequence $S_0^i, \ldots, S_9^i$, $i = 1, 2$, was calculated by comparing them on a fixed uniform grid of 32×32 points on the 2-torus $\mathbb{T}^2$, and by taking the supremum over the whole grid. The results for both components of the solution are included in Table 5.1.

Surfaces	Distance ($i = 1$)	Distance ($i = 2$)
S_0^i, S_1^i	$8.76 \cdot 10^{-2}$	$1.21 \cdot 10^{-1}$
S_1^i, S_2^i	$4.32 \cdot 10^{-6}$	$4.81 \cdot 10^{-6}$
S_2^i, S_3^i	$4.36 \cdot 10^{-6}$	$5.02 \cdot 10^{-6}$
S_3^i, S_4^i	$4.40 \cdot 10^{-6}$	$6.21 \cdot 10^{-6}$
S_4^i, S_5^i	$4.52 \cdot 10^{-6}$	$4.93 \cdot 10^{-6}$
S_5^i, S_6^i	$4.32 \cdot 10^{-6}$	$5.03 \cdot 10^{-6}$
S_6^i, S_7^i	$4.28 \cdot 10^{-6}$	$7.92 \cdot 10^{-6}$
S_7^i, S_8^i	$4.32 \cdot 10^{-6}$	$4.58 \cdot 10^{-6}$
S_8^i, S_9^i	$4.38 \cdot 10^{-6}$	$5.17 \cdot 10^{-6}$

Table 5.1: Distance between consecutive surfaces in the finite sequence $S_0^i, \ldots, S_9^i$, $i = 1, 2$.

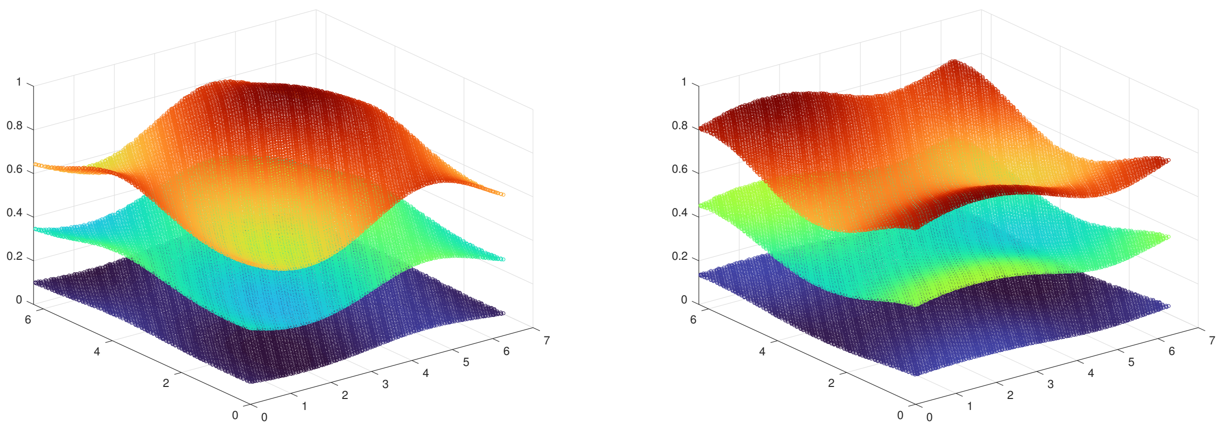
Clearly, the distance between two consecutive surfaces S_k^i and S_{k+1}^i, $i = 1, 2$, $k = 0, \ldots, 8$, gets close to zero very rapidly, but then stabilizes. This fact is not surprising due to the mere uniform stability that was checked in the proof of Theorem 3.3, which does not necessarily imply a fast convergence.

Figure 5.2 contains the omega-limit set of the solution of system (5.1) with initial datum x_0^1. Its components have been approximated by the surfaces S_9^1 and S_9^2, respectively. This omega-limit set is clearly the graph of a continuous function, as proved in Theorem 5.2. Moreover, Figure 5.3 shows the omega-limit sets of the solutions of system (5.1) with initial data x_0^1, x_0^2, and x_0^3. Thanks to Proposition 3.2, a similar argument to that in the proof of Theorem 4.3 implies that there is a different omega-limit set for each value of the total mass of system (5.1) (see e.g. [15] for further results as to the comparison of different omega-limit sets in closed compartmental systems).



(a) First component. (b) Second component.

Figure 5.2: Omega-limit of the solution of system (5.1) with initial data x_0^1.



(a) First components. (b) Second components.

Figure 5.3: First and second components of the omega-limits of the solutions of system (5.1) with initial data x_0^1, x_0^2, and x_0^3.

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Neumann problems of superlinear elliptic systems at resonance

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Abstract. We prove existence of weak solutions of Neumann problem of nonhomogeneous elliptic system with asymmetric nonlinearities that may resonant at $-\infty$ and superlinear at $+\infty$. The proof is based on Mawhin's coincidence theory and the product formula of Brouwer degree.

Keywords: elliptic equation, Neumann problem, weak solution, continuation methods.

2020 Mathematics Subject Classification: 35J25, 35J60, 47H11.

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a smooth bounded connected domain in real N-dimensional Euclidean space. We are concerned with the existence of weak solutions of the following Neumann problem of semilinear elliptic systems

$$\begin{aligned} &\Delta u + f(v) = h_1(x), && \text{in } \Omega, \\ &\Delta v + g(u) = h_2(x), && \text{in } \Omega, \\ &\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions, $\frac{\partial}{\partial \nu}$ denotes the outward normal derivative on $\partial\Omega$, the boundary of Ω, and $h_1, h_2 \in L^1(\Omega)$.

The motivation for this work is the paper F. O. de Paiva, W. Rosa [12], in which the authors showed the following resonant Neumann problems

$$\begin{aligned} &-\Delta u = (v^+)^p + h_1(x), && \text{in } \Omega, \\ &-\Delta v = (u^+)^q + h_2(x), && \text{in } \Omega, \\ &\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, && \text{on } \partial\Omega \end{aligned} \tag{1.2}$$

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has at least one solution (u,v) in $H^1(\Omega)\times H^1(\Omega)$ under the assumptions $h_1,h_2\in L^r(\Omega)$, $r>\frac{N}{2}$, $1<p,q<\frac{N}{N-2}$ and

$$\int_\Omega h_i(x)dx<0, \qquad i=1,2. \tag{1.3}$$

We first define the bilinear form associated with the Laplacian operator. For $u,v\in W^{1,1}(\Omega)$, $\varphi,\psi\in W^{1,\infty}(\Omega)$, we define $B_1(u,\varphi)$ and $B_2(v,\psi)$ by

$$B_1(u,\varphi)=-\sum_{i=1}^N\int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_i}dx,$$

$$B_2(v,\psi)=-\sum_{i=1}^N\int_\Omega \frac{\partial v}{\partial x_i}\frac{\partial \psi}{\partial x_i}dx,$$

where the derivatives are taken in the distributional sense. By a *weak solution* of (1.1), we mean a pair $(u,v)\in W^{1,1}(\Omega)\times W^{1,1}(\Omega)$, such that $f(v(\cdot))\in L^1(\Omega)$, $g(u(\cdot))\in L^1(\Omega)$ and

$$\begin{aligned} B_1(u,\varphi)+\int_\Omega f(v)\varphi dx&=\int_\Omega h_1(x)\varphi dx, &&\forall\ \varphi\in W^{1,\infty}(\Omega),\\ B_2(v,\psi)+\int_\Omega g(u)\psi dx&=\int_\Omega h_2(x)\psi dx, &&\forall\ \psi\in W^{1,\infty}(\Omega).\end{aligned}$$

Denote

$$\begin{aligned} f^{-\infty}&=\limsup_{s\to-\infty} f(s), &\qquad g^{-\infty}&=\limsup_{s\to-\infty} g(s),\\ f_{+\infty}&=\liminf_{s\to+\infty} f(s), &\qquad g_{+\infty}&=\liminf_{s\to+\infty} g(s).\end{aligned}$$

We will make the following assumptions.

(C0) $h_1,h_2\in L^1(\Omega)$.

(Cl) There are the nonnegative constants $C_1,C_2\in(0,\infty)$ such that

$$f(t)\ge -C_1, \qquad g(t)\ge -C_2, \qquad t\in\mathbb{R}$$

and for all $t\le 0$ we have also $|f(t)|\le C_1, |g(t)|\le C_2$.

(C2) There are the constants $a,b\in\mathbb{R}$ and p with $1\le p<N/(N-2)$ for $N\ge 3$ and $1\le p<\infty$ for $N=2$ such that for all $t\ge 0$ the inequality

$$|f(t)|,|g(t)|\le at^p+b \quad \text{a.e. on } \Omega.$$

(C3) We assume f,g tends to be nondecreasing in that there is a $\gamma\in\mathbb{R}$ and a number $M\ge 0$ such that the inequalities

$$f(t_1)\le f(t_2)+\gamma, \qquad g(t_1)\le g(t_2)+\gamma$$

hold a.e. on Ω whenever $t_2-t_1\ge M$.

(C4)

$$\int_\Omega f^{-\infty}<\int_\Omega h_1(x)dx<\int_\Omega f_{+\infty}, \qquad \int_\Omega g^{-\infty}<\int_\Omega h_2(x)dx<\int_\Omega g_{+\infty}.$$

Our main result is the following

Theorem 1.1. *Under assumptions (C0)–(C4) the Neumann problem* (1.1) *has a weak solution* $(u, v) \in W^{1,1}(\Omega) \times W^{1,1}(\Omega)$. *Moreover the solution* $(u, v) \in W^{1,q}(\Omega) \times W^{1,q}(\Omega)$ *for all* $1 \le q < N/(N-1)$.

Remark 1.2. Obviously, (1.3) in F. O. de Paiva, W. Rosa [12] are the special case of (C0) and (C4).

Remark 1.3. Our proof is based upon ideas found in Ward Jr [16]. He used the well-known Mawhin's continuation theorem to get a weak solution of the scale elliptic equation

$$\begin{aligned} &\Delta u + f(x,u) = k(x), &&\text{in } \Omega, \\ &\frac{\partial u}{\partial \nu} = 0, &&\text{on } \partial\Omega \end{aligned} \tag{1.4}$$

under the conditions $k \in L^1(\Omega)$,

$$|f(x,t)| \le \alpha(x)|t|^p + \beta(x), \qquad x \in \Omega,$$

where $p \in [1, \frac{N}{N-2}), \alpha \in L^\infty(\Omega), \beta \in L^1(\Omega)$, and Landesman–Lazer condition

$$\int_\Omega f^{-\infty} < \int_\Omega k(x)dx < \int_\Omega f_{+\infty}.$$

Remark 1.4. Similar problems, under Dirichlet and Neumann boundary condition, can be found in D. Arcoya and S. Villegas [2], M. Cuesta and C. De Coster [3], F. M. Ferreira, F. O. de Paiva [4], R. Kannan and R. Ortega [6,7], S. Kyritsi and N. S. Papageorgiou [8], D. Motreanu, V. Motreanu, N. S. Papageorgiou [10], K. Perera [14], N. S. Papageorgiou and V. D. Rădulescu [13], F. O. de Paiva and A. E. Presoto [11], L. Recova and A. Rumbos [15], J. R. Ward [16].

2 The preliminaries

Before proving Theorem 1.1 we will need a lemma. In the following we will write L^p for $L^p(\Omega)$ and $W^{1,p}$ for $W^{1,p}(\Omega)$. We denote the norm in L^p by $|\cdot|_p$, that of $W^{1,p}$ by $|\cdot|_{1,p}$. For $h \in L^1$. Let Qh be the projection

$$Qh = |\Omega|^{-1} \int_\Omega h dx.$$

Lemma 2.1 ([16])**.** *For each* $h \in L^1(\Omega)$ *with* $Qh = 0$. *There is a unique* $w \in W^{1,1}(\Omega)$ *with* $Qw = 0$ *such that*

$$B(w, \varphi) = \int_\Omega h(x)\varphi dx,$$

for all $\varphi \in W^{1,\infty}$, *where* $B(w, \varphi) = -\sum_{i=1}^N \int_\Omega \frac{\partial w}{\partial x_i} \frac{\partial \varphi}{\partial x_i} dx$. *Moreover* $w \in W^{1,q}$ *for each* q *satisfying* $1 \le q < N/(N-1)$ *and there is a constant* $C(q)$ *such that*

$$|w|_{1,q} \le C(q)|h|_1.$$

By the Rellich–Kondrachov theorem $W^{1,q}$ is compactly imbedded in L^p for $1 \le p < \frac{Nq}{N-q}$ since $q < N/(N-1) \le N$ for all $N \ge 2$. (e.g. see [1, p. 144]). Assume that the number p in condition (C2) is fixed hereafter, satisfying $1 \le p < N/(N-2)$ if $N \ge 3$ and $1 \le p < \infty$ if $N = 2$.

Choose q so that

$$p < \frac{Nq}{N-q} \quad \text{and} \quad 1 < q < \frac{N}{N-1}.$$

We have that $W^{1,q}$ is compactly imbedded in L^p.

Let X_1 denote the closed subspace of L^1 defined by $h \in X_1$ if and only if

$$Qh = 0.$$

Let T denotes the operator mapping X_1 into $W^{1,q} \cap X_1$ given by $h \to w$ where w is the unique weak solution to

$$\begin{aligned} \Delta w &= h \qquad \text{in } \Omega, \qquad Qu = 0, \\ \frac{\partial w}{\partial \nu} &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

Note that $W^{1,q} = (W^{1,q} \cap X_1) \oplus \mathbb{R}$. T maps X_1 into $W^{1,q}$ and we see that $\Psi \circ T$ maps X_1 compactly into L^p if Ψ is the imbedding of $W^{1,q}$ into L^p. Let

$$K = \Psi \circ T,$$

and define an operator $L : L^p \to L^1$. Because L^1 is not the dual space to L^∞, we do not use the usual method of defining L. Instead, we let

$$D(L) = \text{Range}\, K \oplus \mathbb{R}$$

and

$$L(w_1 + \tilde{\alpha}) = h,$$

where $h \in X_1$ and $Kh = w_1$, for $w_1 \in \text{Range}\, K$ and $\tilde{\alpha} \in \mathbb{R}$. It is easy to see that L is a Fredholm operator: it has closed range X_1 and since $\ker(L) = \mathbb{R}$ and the dimension of $L^1 \setminus X_1$ is clearly 1, the index of L is 0,

$$\text{index}(L) = \dim \ker L - \dim \text{coker}\, L.$$

We now define the substitution operators $N_1, N_2 : L^p \to L^1$ by

$$N_1 v(x) = f(v(x)) - h_1(x), \qquad v \in L^p \text{ and } x \in \Omega.$$

$$N_2 u(x) = g(u(x)) - h_2(x), \qquad u \in L^p \text{ and } x \in \Omega.$$

It is well known that the conditions on f and g imply that N_j maps L^p into L^1 continuously and N_j obviously takes sets bounded in L^p into sets bounded in L^1 for $j = 1, 2$.

A function $(u, v) \in W^{1,1} \times W^{1,1}$ is a weak solution of (1.1) if and only if $(u, v) \in D(L) \times D(L)$ and

$$\begin{aligned} Lu + N_1 v &= 0, \\ Lv + N_2 u &= 0. \end{aligned} \tag{2.1}$$

Recalling that for $u \in L^1$ we have defined Qu to be the mean value of u, we have from our remarks above that $K(I - Q)N_j : L^p \to L^p$ is compact and continuous, clearly QN_j is also compact and continuous for $j = 1, 2$. Thus N_j is L-compact (see [5]) on $\bar{G}$ for any open bounded set $\bar{G}$ in L^p for $j = 1, 2$. We will use a well known continuation theorem of Mawhin (see [5] and [9]).

3 Proof of the main result

We are in the position to prove our main result.

Proof of Theorem 1.1. By one of Mawhin's continuation theorems (see [5, p. 40] or [9, Theorem 7.2]) and our remarks above, if we can show the existence of a bounded open set $G := \bar{G} \times \bar{G}$ in $L^p \times L^p$ such that conditions (i) and (ii) below hold, then (2.1) has a solution. The conditions are:

(i) For each $\lambda \in (0,1)$ and each $(u,v) \in (D(L) \times D(L)) \cap \partial G$,

$$\begin{aligned} &Lu + \lambda N_1 v \neq 0, \\ &Lv + \lambda N_2 u \neq 0. \end{aligned} \tag{3.1}$$

(ii) $QN_j w \neq 0$ for each $j = 1,2$, $w \in \ker L \cap \partial \bar{G}$ and

$$d(\Gamma, G \cap (\ker L \times \ker L), 0) \neq 0,$$

where $\Gamma := (JQN_1, JQN_2)$, $J : \operatorname{Im} Q \to \ker L$ is an isomorphism, and d is the Brouwer topological degree.

We first verify (i). We consider

$$\begin{aligned} &Lu + \lambda N_1 v = 0, \\ &Lv + \lambda N_1 u = 0 \end{aligned} \tag{3.2}$$

for $0 < \lambda < 1$. If $((u,v),\lambda)$ is a solution of (3.2) then

$$\begin{aligned} B_1(u,\varphi) + \lambda \int_\Omega f(v)\varphi = \lambda \int_\Omega h_1 \varphi, \qquad \forall\, \varphi \in W^{1,\infty}, \\ B_2(v,\psi) + \lambda \int_\Omega g(u)\psi = \lambda \int_\Omega h_2 \psi, \qquad \forall\, \psi \in W^{1,\infty}. \end{aligned}$$

In particular by taking $\varphi = \psi = 1$, then

$$\int_\Omega f(v) = \int_\Omega h_1,$$

$$\int_\Omega g(u) = \int_\Omega h_2.$$

It follows from (Cl) that for each $t \in \mathbb{R}$

$$|f(t)| \leq f(t) + 2C_1, \qquad |g(t)| \leq g(t) + 2C_2.$$

Thus

$$\begin{aligned} |N_1 v|_1 &= \int_\Omega |f(v) - h_1(x)| dx \\ &\leq \int_\Omega \Big(f(v) + 2C_1 + |h_1(x)| \Big) dx \\ &\leq \int_\Omega h_1 dx + 2|C_1| \cdot |\Omega| + \int_\Omega |h_1(x)| dx =: d_1, \end{aligned}$$

$$\begin{aligned}|N_2u|_1 &= \int_\Omega |g(u)-h_2(x)|dx\\ &\le \int_\Omega \Big(g(u)+2C_2+|h_2(x)|\Big)dx\\ &\le \int_\Omega h_2dx + 2|C_2|\cdot|\Omega| + \int_\Omega |h_2(x)|dx =: d_2.\end{aligned}$$

Writing $u = u_1+\alpha,\ v = v_1+\beta$ with $u_1, v_1 \in \operatorname{Range} K$ and $\alpha, \beta \in \mathbb{R}$ by Lemma 2.1 we have

$$|u_1|_{1,q} \le C(q)d_1 =: m_1,$$

$$|v_1|_{1,q} \le C(q)d_2 =: m_2,$$

where m_1 and m_2 are independently of $\lambda \in (0,1)$. By the Sobolev imbedding theorem

$$|u_1|_p \le m_3, \qquad |v_1|_p \le m_4$$

for some constants m_3 and m_4.

We now show that for solutions $((u,v),\lambda) = \big((u_1+\alpha, v_1+\beta),\lambda\big)$ that α and β are also bounded independently of $\lambda \in (0,1)$.

Suppose this is not the case. Then there is a sequence $((u_n,v_n),\lambda_n)$ of solutions to (3.2) with

$$u_n = u_{1n}+\alpha_n, \qquad v_n = v_{1n}+\beta_n$$

and

$$|\alpha_n|+|\beta_n| \to \infty, \quad \text{as } n\to\infty.$$

Suppose first that a subsequence of $\{\alpha_n\}$, relabeled as $\{\alpha_n\}$, tends to $+\infty$. Then using $|u_{1n}|_{1,q} \le m_1$ is easy to show that

$$\lim_{n\to\infty} u_n(x) = +\infty \quad \text{a.e.} \tag{3.3}$$

For otherwise there is a constant $k_1 > 0$ and sets $\Omega(n)$ in Ω for infinitely many n (without loss of generality we assume for all n) such that $|\Omega(n)| \ge \delta > 0$ and $u_n(x) \le k_1$ for $x \in \Omega(n)$. We have $u_{1n}+\alpha_n \le k_1$ implies

$$\begin{aligned}k_1|\Omega| &\ge \int_{\Omega(n)} k_1dx \ge \int_{\Omega(n)} u_{1n}+\alpha_ndx\\ &\ge \alpha_n|\Omega(n)| - \int_\Omega |u_{1n}|\\ &\ge \alpha_n\delta - C\end{aligned}$$

for C, a constant, which contradicts $\alpha_n \to \infty$. Thus (3.3) holds and

$$\liminf_{n\to\infty} g(u_n(x)) = g_{+\infty} \quad \text{a.e.}$$

Since $g(u_n(x)) \ge -C_2$ for all n and $C_2 \in \mathbb{R}$ we have by Fatou's lemma

$$\int_\Omega h_2 = \liminf_{n\to\infty}\int_\Omega g(u_n(x))dx \ge \int_\Omega g_{+\infty}dx$$

which contradicts (C4). Thus the $\{\alpha_n\}$ must be bounded above.

Suppose $\alpha_n \to -\infty$. It follows as for (3.3) that

$$\lim_{n\to\infty} u_n(x) = -\infty \quad \text{a.e.}$$

Because $g(t)$ is not everywhere bounded above by an L^1 function, we cannot use the simple Fatou's lemma argument as in the case of $\alpha_n \to -\infty$.

We proceed as follows. Since $|u_{1n}|_{1,q} \leq m_1$, we may without loss of generality assume the existence of $\tilde{u}_1 \in L^p$ such that $u_{1n} \to \tilde{u}_1$ in L^p.

Let $0 < \epsilon < |\Omega|$ be given. Then $\tilde{u}_1 \in L^p$ implies that there exists an integer $n(\epsilon)$ and a measurable set $E \subseteq \Omega$ such that if $F = \Omega - E$ then $|F| < \epsilon$ and

$$u_n(x) \leq 0, \qquad x \in E, \quad n \geq n(\epsilon),$$

hence

$$g(u_n(x)) \leq C_2, \qquad x \in E, \quad n \geq n(\epsilon).$$

Moreover there exists another integer m such that for $n \geq m$ we have $\alpha_n \leq -\overline{M}$, where $\overline{M}$ is a positive constant.

Thus, for $n \geq \max\{n(\epsilon), m\}$,

$$\begin{aligned}\int_\Omega h_2 &= \int_E g(u_{1n} + \alpha_n) + \int_F g(u_{1n} + \alpha_n)\\ &\leq \int_E g(u_{1n} + \alpha_n) + \int_F g(u_{1n}) + \int_F \gamma\end{aligned}$$

and

$$\begin{aligned}\int_\Omega h_2 &\leq \limsup_{n\to\infty} \left[\int_E g(u_n) + \int_F g(u_{1n})\right] + \int_F \gamma\\ &\leq \int_E g^{-\infty} dx + \int_F g(\tilde{u}_1) dx + \int_F \gamma\end{aligned} \tag{3.4}$$

by Fatou's lemma for the integral over E and by convergence in L^1 for the integral over F.

Now choose $\eta > 0$ such that

$$\int_\Omega g^{-\infty} dx + \eta < \int_\Omega h_2 dx. \tag{3.5}$$

We may choose $\epsilon > 0$ so small that, since $|F| < \epsilon$,

$$\left|\int_F g^{-\infty} dx\right| < \frac{\eta}{3}, \qquad \left|\int_F g(\tilde{u}_1) dx\right| < \frac{\eta}{3}, \qquad \left|\int_F \gamma\right| < \frac{\eta}{3}.$$

For such as ϵ we have from (3.4) and (3.5)

$$\int_\Omega h_2 \leq \int_\Omega g^{-\infty} dx - \int_F g^{-\infty} dx + \int_F g(\tilde{u}_1) dx + \int_F \gamma \leq \int_\Omega g^{-\infty} dx + \eta < \int_\Omega h_2. \tag{3.6}$$

Therefore we cannot have $\alpha_n \to +\infty$ or $\alpha_n \to -\infty$ and this, combined with $|u_1|_p \leq m_3$ shows that if $((u,v),\lambda)$ is a solution of (3.2) then $|u|_p = |u_1 + \alpha|_p \leq m_3 + C_3$ for some constant C_3. Similarly, We can obtain $|v|_p = |v_1 + \alpha|_p \leq m_4 + C_4$ for some constant C_4.

This verifies condition (i) above for any ball G in $L^1 \times L^1$, centered at the origin and with radius larger than $\rho_1 = \max\{m_3 + C_3, m_4 + C_4\}$.

The verification of condition (ii) is now straightforward. Both the range of Q and the kernel of L may be identified with $\mathbb{R}$, so that the isomorphism J in (ii) we may take to be the identity on $\mathbb{R}$. Now for $\alpha, \beta \in \mathbb{R}$,

$$QN_1\beta = |\Omega|^{-1}\int_\Omega \Big[f(\beta) - h_1(x)\Big]dx, \qquad QN_2\alpha = |\Omega|^{-1}\int_\Omega \Big[g(\alpha) - h_2(x)\Big]dx.$$

We may now make two simple applications of Fatou's lemma using (Cl) to show, using (C4), that there exists an $r > 0$ such that

$$QN_1(\beta) > 0, \qquad QN_1(-\beta) < 0, \quad \text{for } \alpha > r,$$

$$QN_2(\alpha) > 0, \qquad QN_2(-\alpha) < 0, \quad \text{for } \beta > r.$$

Thus for $\bar{r} \geq r\max\{1, |\Omega|\}$,

$$d(QN_j, [-\bar{r}, \bar{r}] \cap \ker L, 0) \neq 0, \qquad j = 1, 2.$$

By the product formula of Brouwer degree, we obtain

$$d(\Gamma, [-\bar{r}, \bar{r}]^2 \cap (\ker L \times \ker L), 0) \neq 0.$$

Now let $\rho := \max\{\rho_1, r \cdot \max\{1, |\Omega|\}\}$. Then we have that both (i) and (ii) are satisfied on $[B_\rho]^2$, where B_ρ is the ball in L^p with radius ρ centered at the origin. Thus (2.1) has a solution $(u, v) \in D(L) \times D(L)$ with

$$|u|_p \leq \rho, \qquad |v|_p \leq \rho,$$

and $(u, v) \in W^{1,p} \times W^{1,p}$ and is a weak solution of (1.1). This completes the proof of Theorem 1.1. □

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Asymptotic behavior of solutions to the multidimensional semidiscrete diffusion equation

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Abstract. We study the asymptotic behavior of solutions to the multidimensional diffusion (heat) equation with continuous time and discrete space. We focus on initial-value problems with bounded initial data, and provide sufficient conditions for the existence of pointwise and uniform limits of solutions.

Keywords: semidiscrete diffusion equation, lattice diffusion equation, modified Bessel function.

2020 Mathematics Subject Classification: 34A33, 35K05.

1 Introduction

In the present paper, we are concerned with the n-dimensional diffusion (heat) equation with continuous time and discrete space, i.e., with the equation

$$\frac{\partial u}{\partial t}(x,t) = a\left(\sum_{i=1}^{n} u(x+e_i,t) - 2nu(x,t) + \sum_{i=1}^{n} u(x-e_i,t)\right), \quad x \in \mathbb{Z}^n, \quad t \geq 0, \tag{1.1}$$

where $e_1, \dots, e_n$ is the canonical basis of $\mathbb{R}^n$, the constant $a > 0$ is the diffusion strength, and the terms inside the parentheses represent the n-dimensional discrete Laplace operator. The study of Eq. (1.1) is meaningful not only from the viewpoint of numerical mathematics, but the equation is of independent interest; for example, it describes the continuous-time symmetric random walk on $\mathbb{Z}^n$, with a being the intensity of transitions between two neighboring lattice points in $\mathbb{Z}^n$. In this case, the value $u(x,t)$ is the probability that the random walk visits point $x \in \mathbb{Z}^n$ at time $t \geq 0$.

We impose the initial condition

$$u(x,0) = c_x, \quad x \in \mathbb{Z}^n, \tag{1.2}$$

where $\{c_x\}_{x\in\mathbb{Z}^n}$ is a collection of real numbers such that $|c_x| \leq M$ for a certain $M \geq 0$ and all $x \in \mathbb{Z}^n$, i.e., $\{c_x\}_{x\in\mathbb{Z}^n} \in \ell^\infty(\mathbb{Z}^n)$. We refer to $\{c_x\}_{x\in\mathbb{Z}^n}$ as a bounded array of real numbers, and

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occasionally write $c_{x_1,\dots,x_n}$ instead of c_x if we need to refer to the components of x. According to [13, Section 5.2], the problem (1.1)–(1.2) has a unique bounded solution, which can be expressed in terms of the modified Bessel functions as follows (see also [9, Section 6] for closely related results):

$$u(x,t) = e^{-2ant} \sum_{k\in\mathbb{Z}^n} c_k I_{x_1-k_1}(2at)\cdots I_{x_n-k_n}(2at), \quad x\in\mathbb{Z}^n, \quad t\geq 0. \tag{1.3}$$

Note that the series is absolutely convergent, because the modified Bessel functions are nonnegative, and if we replace c_k with $|c_k|$, the series represents the solution of the problem with initial values $\{|c_k|\}_{k\in\mathbb{Z}^n}$.

Our goal is to investigate the asymptotic behavior of solutions, focusing on the pointwise limits $\lim_{t\to\infty} u(x,t)$ and the question whether they are uniform with respect to $x\in\mathbb{Z}^n$.

The asymptotic behavior of the classical diffusion equation with continuous time and space was studied in numerous papers, see e.g. [2,7,11,15] and the references therein. The one-dimensional semidiscrete case, i.e., the equation

$$\frac{\partial u}{\partial t}(x,t) = a\left(u(x+1,t) - 2u(x,t) + u(x-1,t)\right), \quad x\in\mathbb{Z}, \quad t\geq 0,$$

was treated in [12], where it was shown that the solution converges to the average of the initial values, provided that the average exists. Generalization of the results from [12] to the multidimensional case is not completely straightforward, and we believe it will be of interest to the readers, also in view of the recent popularity of semidiscrete evolution equations (including those with fractional derivatives), see e.g. [1,3–6,8,14], and the references therein.

2 Main results

Recall that I_k, $k\in\mathbb{Z}$, denotes the modified Bessel function of the first kind of order k. Throughout the paper, we use only a few basic properties of modified Bessel functions, all of which can be found e.g. in the online handbook [10]. Thus, the exposition is accessible also to readers with no prior knowledge of Bessel functions.

Our first goal is to transform the formula (1.3) into an alternative formula, which shows the dependence of the solution on sums (or averages) of initial values. The following statement corresponds to Lemma 2.1 from [12], where it was derived using summation by parts.

Lemma 2.1. *Let $\{c_k\}_{k\in\mathbb{Z}}$ be an arbitrary real sequence. Then for each $N\in\mathbb{N}$ and $t\geq 0$, we have*

$$\sum_{k=-N+1}^{N} c_k I_k(t) = \sum_{k=0}^{N-1} \left(I_k(t) - I_{k+1}(t)\right) \sum_{l=-k}^{k} c_l + I_N(t) \sum_{k=-N+1}^{N} c_k.$$

We need the multidimensional version of Lemma 2.1, which reads as follows.

Lemma 2.2. *Let $n\in\mathbb{N}$ and $\{c_k\}_{k\in\mathbb{Z}^n}$ be an array of real numbers. Then for each $N\in\mathbb{N}$ and $t\geq 0$, we have*

$$\sum_{k_1,\dots,k_n=-N+1}^{N} c_{k_1,\dots,k_n} I_{k_1}(t)\cdots I_{k_n}(t) = \sum_{k_1,\dots,k_n=0}^{N-1} \prod_{j=1}^{n} (I_{k_j}(t) - I_{k_j+1}(t)) \sum_{l_1=-k_1}^{k_1}\cdots\sum_{l_n=-k_n}^{k_n} c_{l_1,\dots,l_n}$$
$$+ I_N(t) \sum_{j=1}^{n} \sum_{k_{j+1},\dots,k_n=0}^{N-1} \sum_{k_1,\dots,k_j=-N+1}^{N} \prod_{i=1}^{j-1} I_{k_i}(t) \prod_{i=j+1}^{n} \left(I_{k_i}(t) - I_{k_i+1}(t)\right) \sum_{l_{j+1}=-k_{j+1}}^{k_{j+1}}\cdots\sum_{l_n=-k_n}^{k_n} c_{k_1,\dots,k_j,l_{j+1},\dots,l_n}.$$

Proof. We use induction with respect to n. For $n = 1$, the statement reduces to Lemma 2.1. Suppose next that the statement holds for $n \in \mathbb{N}$ and let us show that it holds for $n+1$. Using Lemma 2.1, we get

$$\sum_{k_1,\dots,k_{n+1}=-N+1}^{N} c_{k_1,\dots,k_{n+1}} I_{k_1}(t)\cdots I_{k_{n+1}}(t) = \sum_{k_1,\dots,k_n=-N+1}^{N} I_{k_1}(t)\cdots I_{k_n}(t) \sum_{k_{n+1}=-N+1}^{N} I_{k_{n+1}}(t) c_{k_1,\dots,k_{n+1}}$$
$$= \sum_{k_1,\dots,k_n=-N+1}^{N} I_{k_1}(t)\cdots I_{k_n}(t)\left(\sum_{k_{n+1}=0}^{N-1}(I_{k_{n+1}}(t)-I_{k_{n+1}+1}(t))\sum_{l_{n+1}=-k_{n+1}}^{k_{n+1}} c_{k_1,\dots,k_n,l_{n+1}} + I_N(t)\sum_{k_{n+1}=-N+1}^{N} c_{k_1,\dots,k_{n+1}}\right)$$
$$= \sum_{k_{n+1}=0}^{N}\sum_{l_{n+1}=-k_{n+1}}^{k_{n+1}}(I_{k_{n+1}}(t)-I_{k_{n+1}+1}(t))\sum_{k_1,\dots,k_n=-N+1}^{N} I_{k_1}(t)\cdots I_{k_n}(t) c_{k_1,\dots,k_n,l_{n+1}}$$
$$+I_N(t)\sum_{k_1,\dots,k_{n+1}=-N+1}^{N} I_{k_1}(t)\cdots I_{k_n}(t) c_{k_1,\dots,k_{n+1}}.$$

Using the induction hypothesis to rewrite the inner sum in the first term on the right-hand side, we get

$$\sum_{k_{n+1}=0}^{N}\sum_{l_{n+1}=-k_{n+1}}^{k_{n+1}}(I_{k_{n+1}}(t)-I_{k_{n+1}+1}(t))\left(\sum_{k_1,\dots,k_n=0}^{N-1}\prod_{j=1}^{n}(I_{k_j}(t)-I_{k_j+1}(t))\sum_{l_1=-k_1}^{k_1}\cdots\sum_{l_n=-k_n}^{k_n} c_{l_1,\dots,l_{n+1}}\right.$$
$$\left.+I_N(t)\sum_{j=1}^{n}\sum_{k_{j+1},\dots,k_n=0}^{N-1}\sum_{k_1,\dots,k_j=-N+1}^{N}\prod_{i=1}^{j-1} I_{k_i}(t)\prod_{i=j+1}^{n}(I_{k_i}(t)-I_{k_i+1}(t))\sum_{l_{j+1}=-k_{j+1}}^{k_{j+1}}\cdots\sum_{l_n=-k_n}^{k_n} c_{k_1,\dots,k_j,l_{j+1},\dots,l_{n+1}}\right)$$
$$+I_N(t)\sum_{k_1,\dots,k_{n+1}=-N+1}^{N} I_{k_1}(t)\cdots I_{k_n}(t) c_{k_1,\dots,k_{n+1}}.$$

Expanding the product inside the first term and performing some elementary manipulations, we get:

$$\sum_{k_1,\dots,k_{n+1}=0}^{N-1}\prod_{j=1}^{n+1}(I_{k_j}(t)-I_{k_j+1}(t))\sum_{l_1=-k_1}^{k_1}\cdots\sum_{l_{n+1}=-k_{n+1}}^{k_{n+1}} c_{l_1,\dots,l_{n+1}}$$
$$+I_N(t)\sum_{j=1}^{n}\sum_{k_{j+1},\dots,k_{n+1}=0}^{N-1}\sum_{k_1,\dots,k_j=-N+1}^{N}\prod_{i=1}^{j-1} I_{k_i}(t)\prod_{i=j+1}^{n+1}(I_{k_i}(t)-I_{k_i+1}(t))\sum_{l_{j+1}=-k_{j+1}}^{k_{j+1}}\cdots\sum_{l_{n+1}=-k_{n+1}}^{k_{n+1}} c_{k_1,\dots,k_j,l_{j+1},\dots,l_{n+1}}$$
$$+I_N(t)\sum_{k_1,\dots,k_{n+1}=-N+1}^{N} I_{k_1}(t)\cdots I_{k_n}(t) c_{k_1,\dots,k_{n+1}}.$$

This completes the proof, because the third term can be incorporated into the second term as the summand corresponding to $j = n+1$. □

Proposition 2.3. *Let $n \in \mathbb{N}$ and $\{c_k\}_{k\in\mathbb{Z}^n}$ be a bounded array of real numbers. Then the unique bounded solution of the problem (1.1)–(1.2) is given by the formula*

$$u(x,t) = e^{-2ant}\sum_{k_1,\dots,k_n=0}^{\infty}\left(\prod_{j=1}^{n}(I_{k_j}(2at)-I_{k_j+1}(2at))\right)\left(\sum_{l_1=x_1-k_1}^{x_1+k_1}\cdots\sum_{l_n=x_n-k_n}^{x_n+k_n} c_{l_1,\dots,l_n}\right)$$

for all $x \in \mathbb{Z}^n$, $t \geq 0$.

Proof. It suffices to prove the statement for $x = 0$, since for a nonzero $x \in \mathbb{Z}^n$, one can consider the shifted solution satisfying shifted initial conditions (cf. the proof of Lemma 2.2 in [12]).

Using formula (1.3) and the fact that $I_{-k}(t) = I_k(t)$ for all $k \in \mathbb{Z}$ and $t \geq 0$, we have

$$u(0,t) = e^{-2ant} \sum_{k \in \mathbb{Z}^n} c_k I_{k_1}(2at) \cdots I_{k_n}(2at) = e^{-2ant} \lim_{N \to \infty} \sum_{k_1,\dots,k_n=-N+1}^{N} c_{k_1,\dots,k_n} I_{k_1}(2at) \cdots I_{k_n}(2at).$$

The sum can be rewritten using the formula from Lemma 2.2. But let us first observe that the second term on the right-hand of that formula tends to zero as $N \to \infty$. To see this, we perform some estimates. Let $M \geq 0$ be such that $|c_{k_1,\dots,k_n}| \leq M$ for all $k \in \mathbb{Z}^n$. Using the fact that the modified Bessel functions are nonnegative and nonincreasing with respect to the order, we get

$$\left| \prod_{i=1}^{j-1} I_{k_i}(t) \prod_{i=j+1}^{n} (I_{k_i}(t) - I_{k_i+1}(t)) \right| \leq I_0(t)^{n-1}.$$

For $k_{j+1}, \dots, k_n \in \{0, \dots, N-1\}$, this implies that

$$\left| \sum_{l_{j+1}=-k_{j+1}}^{k_{j+1}} \cdots \sum_{l_n=-k_n}^{k_n} \prod_{i=1}^{j-1} I_{k_i}(t) \prod_{i=j+1}^{n} (I_{k_i}(t) - I_{k_i+1}(t)) c_{k_1,\dots,k_j,l_{j+1},\dots,l_n} \right| \leq (2N-1)^{n-j} I_0(t)^{n-1} M.$$

Consequently,

$$\left| I_N(t) \sum_{j=1}^{n} \sum_{k_{j+1},\dots,k_n=0}^{N-1} \sum_{k_1,\dots,k_j=-N+1}^{N} \sum_{l_{j+1}=-k_{j+1}}^{k_{j+1}} \cdots \sum_{l_n=-k_n}^{k_n} \prod_{i=1}^{j-1} I_{k_i}(t) \prod_{i=j+1}^{n} (I_{k_i}(t) - I_{k_i+1}(t)) c_{k_1,\dots,k_j,l_{j+1},\dots,l_n} \right|$$
$$\leq I_N(t) n N^{n-j} (2N)^j (2N-1)^{n-j} I_0(t)^{n-1} M \leq I_N(t) n N^n (2N)^n I_0(t)^{n-1} M,$$

which tends to zero as $N \to \infty$, because $I_N(t) \sim \frac{1}{\sqrt{2\pi N}} \left(\frac{et}{2N}\right)^N$ for $N \to \infty$ (see formula 10.41.1 in [10]). Returning to the beginning of the proof and applying Lemma 2.2, we now see that

$$u(0,t) = e^{-2ant} \lim_{N \to \infty} \left(\sum_{k_1,\dots,k_n=0}^{N-1} \prod_{j=1}^{n} (I_{k_j}(2at) - I_{k_j+1}(2at)) \sum_{l_1=-k_1}^{k_1} \cdots \sum_{l_n=-k_n}^{k_n} c_{l_1,\dots,l_n} \right)$$
$$= e^{-2ant} \sum_{k_1,\dots,k_n=0}^{\infty} \prod_{j=1}^{n} (I_{k_j}(2at) - I_{k_j+1}(2at)) \sum_{l_1=-k_1}^{k_1} \cdots \sum_{l_n=-k_n}^{k_n} c_{l_1,\dots,l_n},$$

and the statement for $x = 0$ is proved. □

We need two more auxiliary lemmas to be able to prove our main result.

Lemma 2.4. *For every $n \in \mathbb{N}$ and $t \geq 0$, we have*

$$e^{-nt} \sum_{k_1,\dots,k_n=0}^{\infty} \prod_{j=1}^{n} [(I_{k_j}(t) - I_{k_j+1}(t))(2k_j + 1)] = 1.$$

Proof. According to Proposition 2.3, the formula on the left-hand side corresponds to the unique bounded solution of the initial-value problem (1.1)–(1.2) with $a = 1/2$ and $c_x = 1$ for all $x \in \mathbb{Z}$. But this problem admits the constant solution $u(x,t) = 1$ for all $x \in \mathbb{Z}$ and $t \geq 0$, and therefore the equality is proved. □

Lemma 2.5. *Let $n, k_0 \in \mathbb{N}$, $l \in \{1, \dots, n\}$. If $i_1, \dots, i_l, j_1, \dots, j_{n-l} \in \mathbb{N}$ are distinct integers such that $\{i_1, \dots, i_l, j_1, \dots, j_{n-l}\} = \{1, \dots, n\}$, then*

$$\lim_{t\to\infty} \sum_{k_{i_1},\dots,k_{i_l}=0}^{k_0-1} \sum_{k_{j_1},\dots,k_{j_{n-l}}=k_0}^{\infty} e^{-nt} \prod_{j=1}^{n} [(I_{k_j}(t) - I_{k_j+1}(t))(2k_j+1)] = 0.$$

Proof. We have

$$0 \le \sum_{k_{i_1},\dots,k_{i_l}=0}^{k_0-1} \sum_{k_{j_1},\dots,k_{j_{n-l}}=k_0}^{\infty} e^{-nt} \prod_{j=1}^{n} [(I_{k_j}(t) - I_{k_j+1}(t))(2k_j+1)]$$

$$\le \sum_{k_{i_1},\dots,k_{i_l}=0}^{k_0-1} e^{-lt} \prod_{m\in\{i_1,\dots,i_l\}} [(I_{k_m}(t) - I_{k_m+1}(t))(2k_m+1)] \sum_{k_{j_1},\dots,k_{j_{n-l}}=0}^{\infty} e^{-(n-l)t} \prod_{m\in\{j_1,\dots,j_{n-l}\}} [(I_{k_m}(t) - I_{k_m+1}(t))(2k_m+1)]$$

$$= \sum_{k_{i_1},\dots,k_{i_l}=0}^{k_0-1} e^{-lt} \prod_{m\in\{i_1,\dots,i_l\}} [(I_{k_m}(t) - I_{k_m+1}(t))(2k_m+1)],$$

where the last equality follows from Lemma 2.4 (with n replaced by $n-l$). Because $I_k(t) \sim \frac{e^t}{\sqrt{2\pi t}}$ for $t \to \infty$ (see formula 10.30.4 in [10]), we get $\lim_{t\to\infty} e^{-t} I_k(t) = 0$ for each $k \in \mathbb{Z}$. Thus, $\lim_{t\to\infty} e^{-t}(I_k(t) - I_{k+1}(t))(2k+1) = 0$ for each fixed $k \in \mathbb{N}_0$, which completes the proof. □

Here is the main result dealing with the asymptotic behavior of solutions to the problem (1.1)–(1.2).

Theorem 2.6. *Let $n \in \mathbb{N}$ and $\{c_k\}_{k\in\mathbb{Z}^n}$ be a bounded array of real numbers. Denote*

$$A_{k_1,\dots,k_n}(x) = \frac{1}{\prod_{j=1}^{n}(2k_j+1)} \sum_{l_1=x_1-k_1}^{x_1+k_1} \cdots \sum_{l_n=x_n-k_n}^{x_n+k_n} c_{l_1,\dots,l_n}, \quad x \in \mathbb{Z}^n, \quad k_1, \dots, k_n \in \mathbb{N}_0. \tag{2.1}$$

Then the unique bounded solution of the problem (1.1)–(1.2) has the following properties:

1. *For every $x \in \mathbb{Z}^n$,*

$$\liminf_{k_1,\dots,k_n\to\infty} A_{k_1,\dots,k_n}(x) \le \liminf_{t\to\infty} u(x,t) \le \limsup_{t\to\infty} u(x,t) \le \limsup_{k_1,\dots,k_n\to\infty} A_{k_1,\dots,k_n}(x).$$

2. *If $x \in \mathbb{Z}^n$ and $\lim_{k_1,\dots,k_n\to\infty} A_{k_1,\dots,k_n}(x) = d$, then $\lim_{t\to\infty} u(x,t) = d$.*

3. *If $\lim_{k_1,\dots,k_n\to\infty} A_{k_1,\dots,k_n}(x) = d$ uniformly for all $x \in \mathbb{Z}^n$, then $\lim_{t\to\infty} u(x,t) = d$ uniformly with respect to $x \in \mathbb{Z}^n$.*

Proof. Fix an arbitrary $x \in \mathbb{Z}^n$ and denote

$$\underline{A} = \liminf_{k_1,\dots,k_n\to\infty} A_{k_1,\dots,k_n}(x), \quad \overline{A} = \limsup_{k_1,\dots,k_n\to\infty} A_{k_1,\dots,k_n}(x).$$

Using Proposition 2.3, we get

$$u(x,t) = e^{-2ant} \sum_{k_1,\dots,k_n=0}^{\infty} \prod_{j=1}^{n} [(I_{k_j}(2at) - I_{k_j+1}(2at))(2k_j+1)] A_{k_1,\dots,k_n}(x). \tag{2.2}$$

Let $M > 0$ be such that $|c_l| \le M$ for all $l \in \mathbb{Z}^n$. Then $|A_{k_1,\dots,k_n}(x)| \le M$ for all $k_1,\dots,k_n \in \mathbb{N}_0$. Given an $\varepsilon > 0$, there exists a $k_0 \in \mathbb{N}$ such that for all $k_1,\dots,k_n \ge k_0$, we have $\underline{A} - \varepsilon < A_{k_1,\dots,k_n}(x) < \overline{A} + \varepsilon$.

From Lemma 2.4, we know that for each $t \ge 0$,

$$1 = e^{-2ant} \sum_{k_1,\dots,k_n=0}^{\infty} \prod_{j=1}^{n} [(I_{k_j}(2at) - I_{k_j+1}(2at))(2k_j+1)].$$

We split the sum in two parts, one containing all terms with $k_1,\dots,k_n \ge k_0$, and the second one containing all remaining terms, i.e., those where $l \in \{1,\dots,n\}$ indices, say $k_{i_1},\dots,k_{i_l}$, are smaller than k_0:

$$1 = e^{-2ant} \sum_{k_1,\dots,k_n=k_0}^{\infty} \prod_{j=1}^{n} [(I_{k_j}(2at) - I_{k_j+1}(2at))(2k_j+1)]$$
$$+e^{-2ant} \sum_{l=1}^{n} \sum_{\substack{i_1,\dots,i_l,j_1,\dots,j_{n-l}\in\{1,\dots,n\} \\ \{i_1,\dots,i_l,j_1,\dots,j_{n-l}\}=\{1,\dots,n\}}} \sum_{k_{i_1},\dots,k_{i_l}=0}^{k_0-1} \sum_{k_{j_1},\dots,k_{j_{n-l}}=k_0}^{\infty} \prod_{j=1}^{n} [(I_{k_j}(2at) - I_{k_j+1}(2at))(2k_j+1)].$$

By Lemma 2.5, the second term tends to zero as $t \to \infty$. Thus, there exists a $t_0 \ge 0$ such that for all $t \ge t_0$, we have

$$0 < e^{-2ant} \sum_{l=1}^{n} \sum_{\substack{i_1,\dots,i_l,j_1,\dots,j_{n-l}\in\{1,\dots,n\} \\ \{i_1,\dots,i_l,j_1,\dots,j_{n-l}\}=\{1,\dots,n\}}} \sum_{k_{i_1},\dots,k_{i_l}=0}^{k_0-1} \sum_{k_{j_1},\dots,k_{j_{n-l}}=k_0}^{\infty} \prod_{j=1}^{n} [(I_{k_j}(2at) - I_{k_j+1}(2at))(2k_j+1)] < \varepsilon,$$
$$1-\varepsilon < e^{-2ant} \sum_{k_1,\dots,k_n=k_0}^{\infty} \prod_{j=1}^{n} [(I_{k_j}(2at) - I_{k_j+1}(2at))(2k_j+1)] < 1.$$

We now use these estimates together with $|A_{k_1,\dots,k_n}(x)| \le M$ and (2.2) to obtain

$$u(x,t) = e^{-2ant} \sum_{k_1,\dots,k_n=k_0}^{\infty} \prod_{j=1}^{n} [(I_{k_j}(2at) - I_{k_j+1}(2at))(2k_j+1)] A_{k_1,\dots,k_n}(x)$$
$$+e^{-2ant} \sum_{l=1}^{n} \sum_{\substack{i_1,\dots,i_l,j_1,\dots,j_{n-l}\in\{1,\dots,n\} \\ \{i_1,\dots,i_l,j_1,\dots,j_{n-l}\}=\{1,\dots,n\}}} \sum_{k_{i_1},\dots,k_{i_l}=0}^{k_0-1} \sum_{k_{j_1},\dots,k_{j_{n-l}}=k_0}^{\infty} \prod_{j=1}^{n} [(I_{k_j}(2at) - I_{k_j+1}(2at))(2k_j+1)] A_{k_1,\dots,k_n}(x)$$
$$< (\overline{A}+\varepsilon) e^{-2ant} \sum_{k_1,\dots,k_n=k_0}^{\infty} \prod_{j=1}^{n} [(I_{k_j}(2at) - I_{k_j+1}(2at))(2k_j+1)] + M\varepsilon.$$

If $\overline{A}+\varepsilon$ is nonnegative, the first term on the right-hand side is majorized by $\overline{A}+\varepsilon$. Otherwise, if $\overline{A}+\varepsilon$ is nonpositive, the term is majorized by $(\overline{A}+\varepsilon)(1-\varepsilon) = \overline{A}+\varepsilon-\varepsilon\overline{A}-\varepsilon^2$. In any case, we get the estimate

$$u(x,t) < \max(\overline{A}+\varepsilon, \overline{A}+\varepsilon-\varepsilon\overline{A}-\varepsilon^2) + \varepsilon M = \overline{A}+\varepsilon M+\varepsilon+\varepsilon\max(0,-\overline{A}-\varepsilon), \quad t \ge t_0.$$

This proves that $\limsup_{t\to\infty} u(x,t) \le \overline{A}$. Similarly, we have

$$u(x,t) > (\underline{A}-\varepsilon) e^{-2ant} \sum_{k_1,\dots,k_n=k_0}^{\infty} \prod_{j=1}^{n} (I_{k_j}(2at) - I_{k_j+1}(2at))(2k_j+1) - M\varepsilon.$$

If $\underline{A} - \varepsilon$ is nonnegative, the first term on the right-hand side is minorized by $(\underline{A} - \varepsilon)(1 - \varepsilon) = \underline{A} - \varepsilon - \varepsilon\underline{A} + \varepsilon^2$. Otherwise, if $\underline{A} - \varepsilon$ is nonpositive, the term is minorized by $\underline{A} - \varepsilon$. In any case, we get the estimate

$$u(x,t) > -\varepsilon M + \min(\underline{A} - \varepsilon - \varepsilon\underline{A} + \varepsilon^2, \underline{A} - \varepsilon) = \underline{A} - \varepsilon M - \varepsilon + \varepsilon \min(-\underline{A} + \varepsilon, 0), \quad t \geq t_0.$$

This proves that $\liminf_{t\to\infty} u(x,t) \geq \underline{A}$.

The second statement of the theorem follows from the first one.

If $\lim_{k_1,\dots,k_n\to\infty} A_{k_1,\dots,k_n}(x) = d$ uniformly for all $x \in \mathbb{Z}$, then the previous estimates are independent of x, which proves the third statement. □

The second part of Theorem 2.6 says that $u(x,t)$ tends to the limit of averages of initial conditions over hyperrectangles centered at x, provided that the limit exists. The next result implies that in fact, one can consider hyperrectangles centered at an arbitrary point. The reason is that if we take two sufficiently large hyperrectangles, then their intersection is large, while their symmetric difference is small. We use the notation introduced in (2.1).

Proposition 2.7. *Let $n \in \mathbb{N}$ and $\{c_k\}_{k\in\mathbb{Z}^n}$ be a bounded array of real numbers. For every $x \in \mathbb{Z}^n$, we have $\lim_{k_1,\dots,k_n\to\infty} A_{k_1,\dots,k_n}(x) = \lim_{k_1,\dots,k_n\to\infty} A_{k_1,\dots,k_n}(0)$ whenever at least one of the limits exists.*

Proof. For each $k = (k_1,\dots,k_n) \in (\mathbb{N}_0)^n$, let

$$S_k = \{x_1 - k_1, \dots, x_1 + k_1\} \times \cdots \times \{x_n - k_n, \dots, x_n + k_n\},$$
$$R_k = \{-k_1, \dots, k_1\} \times \cdots \times \{-k_n, \dots, k_n\}.$$

We need to show that

$$\lim_{k_1,\dots,k_n\to\infty} \frac{1}{\prod_{j=1}^n (2k_j+1)} \left(\sum_{(l_1,\dots,l_n)\in S_k} c_{l_1,\dots,l_n} - \sum_{(l_1,\dots,l_n)\in R_k} c_{l_1,\dots,l_n} \right) = 0.$$

Let $M > 0$ be such that $|c_l| \leq M$ for all $l \in \mathbb{Z}^n$. Then

$$\frac{1}{\prod_{j=1}^n (2k_j+1)} \left| \sum_{(l_1,\dots,l_n)\in S_k} c_{l_1,\dots,l_n} - \sum_{(l_1,\dots,l_n)\in R_k} c_{l_1,\dots,l_n} \right| \leq \frac{M \cdot |R_k \Delta S_k|}{\prod_{j=1}^n (2k_j+1)},$$

where $R_k \Delta S_k = (R_k \setminus S_k) \cup (S_k \setminus R_k)$ is the symmetric difference of the two hyperrectangles. Since both have the same dimensions, it follows from symmetry that $|R_k \Delta S_k| = 2|R_k \setminus S_k| = 2(|R_k| - |R_k \cap S_k|)$. The intersection $R_k \cap S_k$ is again a hyperrectangle. For each $j \in \{1,\dots,n\}$, consider its orthogonal projection on the j-th coordinate axis. If $x_j \geq 0$ and k_j is sufficiently large, then the projection is $\{x_j - k_j, \dots, k_j\}$. If $x_j \leq 0$ and k_j is sufficiently large, then the projection is $\{-k_j, \dots, x_j + k_j\}$. In both cases, the projection contains $2k_j + 1 - |x_j|$ points. Thus, for sufficiently large $k_1,\dots,k_n \in \mathbb{N}$, we have $|R_k \cap S_k| = \prod_{j=1}^n (2k_j + 1 - |x_j|)$, and

$$\frac{|R_k \Delta S_k|}{\prod_{j=1}^n (2k_j+1)} = 2\frac{\prod_{j=1}^n (2k_j+1) - \prod_{j=1}^n (2k_j+1-|x_j|)}{\prod_{j=1}^n (2k_j+1)}.$$

In the numerator of the last fraction, note that $\prod_{j=1}^n (2k_j+1-|x_j|)$ equals $\prod_{j=1}^n (2k_j+1)$ plus $2^n - 1$ additional terms, each of which is a constant multiple of at most $n-1$ terms of the form $2k_j + 1$. Hence, the whole fraction tends to zero when $k_1,\dots,k_n \to \infty$, and the proof is complete. □

Proposition 2.8. *If $\{c_l\}_{l\in\mathbb{Z}^n}$ is an array of real numbers such that*

$$\lim_{\max(|l_1|,\dots,|l_n|)\to\infty} c_{l_1,\dots,l_n} = d \in \mathbb{R},$$

then the unique bounded solution of the problem (1.1)–(1.2) satisfies $\lim_{t\to\infty} u(x,t) = d$ uniformly with respect to x. In particular, if $\{c_l\}_{l\in\mathbb{Z}^n} \in \ell^p(\mathbb{Z}^n)$ for a certain $p \in [1,\infty)$, then $\lim_{t\to\infty} u(x,t) = 0$ uniformly with respect to x.

Proof. It follows from the assumption that there is an $M \geq 0$ such that $|c_l| \leq M$ for all $l \in \mathbb{Z}^n$. Given an $\varepsilon > 0$, there exists a $k_0 \in \mathbb{N}$ such if $\max(|l_1|,\dots,|l_n|) > k_0$, then $|c_{l_1,\dots,l_n} - d| < \varepsilon$.

For each $x \in \mathbb{Z}^n$ and all $k_1,\dots,k_n \in \mathbb{N}_0$, consider the average $A_{k_1,\dots,k_n}(x)$ given by (2.1). In the n-fold sum, there are at most $(2k_0+1)^n$ terms with $\max(|l_1|,\dots,|l_n|) \leq k_0$; their values lie between $-M$ and M. The values of the remaining $\prod_{j=1}^n(2k_j+1) - (2k_0+1)^n$ terms lie between $d-\varepsilon$ and $d+\varepsilon$. Thus, if at least one of $k_1,\dots,k_n$ is sufficiently large, then $A_{k_1,\dots,k_n}(x)$ will lie between $d-2\varepsilon$ and $d+2\varepsilon$. This shows that $\lim_{k_1,\dots,k_n\to\infty} A_{k_1,\dots,k_n}(x) = d$. The convergence is uniform with respect to x, because the previous estimate does not depend on x. The third part of Theorem 2.6 implies that $\lim_{t\to\infty} u(x,t) = d$ uniformly with respect to x. □

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Oscillation of half-linear differential equations with mixed type of argument

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Abstract. This paper is devoted to the study of the oscillatory behavior of half-linear functional differential equations with deviating argument of the form

$$\left(r(t)(y'(t))^{\alpha}\right)' = p(t)y^{\alpha}(\tau(t)). \tag{E}$$

We introduce new technique based on monotonic properties of nonoscillatory solutions to offer new oscillatory criteria for (E). We will show that presented results essentially improve existing ones even for linear differential equations.

Keywords: second order differential equations, delay, advanced, monotonic properties, oscillation.

2020 Mathematics Subject Classification: 34K11, 34C10.

1 Introduction

We consider half-linear second order differential equations with deviating argument

$$\left(r(t)(y'(t))^{\alpha}\right)' = p(t)y^{\alpha}(\tau(t)). \tag{E}$$

Throughout the paper it is assumed that

(H_1) $p, r \in C([t_0,\infty))$, $p(t) > 0$, $r(t) > 0$, α is the ratio of two positive odd integers,

(H_2) $\tau(t) \in C^1([t_0,\infty))$, $\tau'(t) \geq 0$, $\lim_{t\to\infty} \tau(t) = \infty$.

By a solution of Eq. (E) we mean a function $y(t) \in C^1([T_y,\infty))$, $T_y \geq t_0$, such that $r(t)(y'(t))^{\alpha} \in C^1([T_y,\infty))$ and $y(t)$ satisfies Eq. (E) on $[T_y,\infty)$. We consider only those solutions $y(t)$ of (E) which satisfy $\sup\{|y(t)| : t \geq T\} > 0$ for all $T \geq T_y$. We assume that (E) possesses such a solution. A solution of (E) is called oscillatory if it has arbitrarily large zeros on $[T_y,\infty)$ and otherwise it is called to be nonoscillatory. An equation itself is said to be oscillatory if all its solutions are oscillatory.

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Throughout the paper we consider (E) in canonical form, that is,

$$R(t) = \int_{t_0}^{t} \frac{1}{r^{1/\alpha}(s)}\,\mathrm{d}s \to \infty \quad \text{as } t \to \infty.$$

The problem of establishing oscillatory criteria for various types of differential equations has been a very active research area over the past decades (see [1–11]).

The half-linear ordinary differential equation

$$\left((y'(t))^{\alpha}\right)' = p(t)y^{\alpha}(t)$$

to which (E) reduces when $\tau(t) \equiv t$ and $r(t) \equiv 1$ is nonoscillatory in the sense that all of its solutions are nonoscillatory; see Elbert [6]. However, the presence of deviating argument $\tau(t) \not\equiv t$ may generate oscillation of some or all of its solutions.

It is known that (E) may possess only two types of nonoscillatory solutions. So, if $y(t)$ is a nonoscillatory solution of (E) it is easy to see that $y'(t)$ is eventually of constant sign, so that either

$$y(t)y'(t) < 0 \tag{1.1}$$

or

$$y(t)y'(t) > 0, \tag{1.2}$$

eventually. Moreover, if $y(t)$ is an eventually positive solution satisfying inequality (1.2), then $r(t)(y'(t))^{\alpha} > k > 0$, and an integration of $y'(t) > \frac{k^{1/\alpha}}{r^{1/\alpha}(t)}$ yields

$$y(t) \geq k^{1/\alpha} \int_{t_1}^{t} \frac{1}{r^{1/\alpha}(s)}\,\mathrm{d}s \to \infty \quad \text{as } t \to \infty.$$

Consequently, if (E) is in canonical form, then $y(t)$ is bounded or unbounded according to whether (1.1) or (1.2) holds. Effort of mathematicians was aimed to show that (E) admits no bounded or unbounded nonoscillatory solutions in the case where $\tau(t)$ is retarded ($\tau(t) \leq t$) or advanced argument ($\tau(t) \geq t$), respectively. To illustrate this we recall classical result of Kusano and Lalli [10].

Theorem A. Suppose that

(i) $\tau(t) < t$ and

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} \left(\frac{1}{r(u)} \int_{s}^{t} p(s)\,\mathrm{d}s\right)^{1/\alpha} \mathrm{d}u > 1.$$

Then (E) has no bounded nonoscillatory solutions.

(ii) If $\tau(t) > t$ and

$$\limsup_{t\to\infty} \int_{t}^{\tau(t)} \left(\frac{1}{r(u)} \int_{t}^{s} p(s)\,\mathrm{d}s\right)^{1/\alpha} \mathrm{d}u > 1.$$

Then (E) has no unbounded nonoscillatory solutions.

In this paper we are interested in the situation when $\tau(t)$ is of mixed type which means that its retarded part

$$\mathcal{R}_{\tau} = \{t \in (t_0, \infty) : \tau(t) < t\}$$

and its advanced part

$$\mathcal{A}_{\tau} = \{t \in (t_0, \infty) : \tau(t) > t\}$$

are both unbounded subset of (t_0, ∞). The presence of mixed argument may cause that (*E*) has neither bounded nor unbounded nonoscillatory solutions which means oscillation of (*E*). This fact has been observed by Kusano [9], who showed that the second order differential equation

$$y''(t) = p_0 y(t + \sin t) \tag{E_x}$$

is oscillatory provided that

$$p_0 \geq \frac{1}{\sin 1 - 0.5} \approx 2.9285. \tag{1.3}$$

In this paper we present new technique for investigation of (*E*) with mixed argument and the progress achieved will be demonstrated via equation (E_x) and its oscillatory criterion (1.3).

2 Main results

We are about to establish new criteria for (*E*) to do not possess neither bounded nor unbounded solutions. We start with some useful lemma concerning monotonic properties of nonoscillatory solutions for studied equations.

Lemma 2.1. *Let that there exist a sequence $\{t_k\}$ such that $t_k \in \mathcal{R}_\tau$, $t_k \to \infty$ as $k \to \infty$. Assume that $y(t)$ is a positive bounded solution of (E). If there exists some positive constant β such that for all $k \in \{1, 2, \dots\}$*

$$\left[\int_{\tau(t)}^{t} p(s)\,\mathrm{d}s\right]^{1/\alpha} \geq \beta \quad \text{on } [\tau(t_k), t_k], \tag{2.1}$$

then $y(\tau(t))\mathrm{e}^{\beta R(\tau(t))}$ is decreasing on all $[\tau(t_k), t_k]$.

Proof. Assume that $y(t)$ is a positive decreasing solution of (*E*) and $t \in [\tau(t_k), t_k]$. An integration of (*E*) from $\tau(t)$ to t yields

$$r(t)\left(y'(t)\right)^\alpha - r(\tau(t))\left(y'(\tau(t))\right)^\alpha \geq y^\alpha(\tau(t)) \int_{\tau(t)}^{t} p(s)\,\mathrm{d}s \geq \beta^\alpha y^\alpha(\tau(t)).$$

That is

$$-r^{1/\alpha}(\tau(t))y'(\tau(t)) \geq \beta y(\tau(t)).$$

Therefore

$$\left[y(\tau(t))\mathrm{e}^{\beta R(\tau(t))}\right]' = \frac{\mathrm{e}^{\beta R(\tau(t))}\tau'(t)}{r^{1/\alpha}(\tau(t))}\left[\beta y(\tau(t)) + r^{1/\alpha}(\tau(t))y'(\tau(t))\right] \leq 0$$

and we conclude that function $y(\tau(t))\mathrm{e}^{\beta R(\tau(t))}$ is decreasing. The proof is complete. □

Now we apply the above monotonicity to establish criterion for absence of decreasing solutions.

Theorem 2.2. *Let that there exist a sequence $\{t_k\}$ such that $t_k \in \mathcal{R}_\tau$, $t_k \to \infty$ as $k \to \infty$ and (2.1) hold. If*

$$\limsup_{k\to\infty} \mathrm{e}^{\beta R(\tau(t_k))} \int_{\tau(t_k)}^{t_k} \frac{1}{r^{1/\alpha}(u)} \left[\int_{u}^{t_k} p(s)\,\mathrm{e}^{-\alpha\beta R(\tau(s))}\,\mathrm{d}s\right]^{1/\alpha} \mathrm{d}u > 1, \tag{2.2}$$

then (E) has no bounded nonoscillatory solutions.

Proof. Assume on the contrary, that (E) possesses an eventually positive decreasing solution $y(t)$. We assume that $u \in [\tau(t_k), t_k]$ Integrating (E) from u to t_k and using monotonic property of $\mathrm{e}^{\beta R(\tau(t))} y(\tau(t))$, we obtain

$$\begin{aligned} -r(u)(y'(u))^{\alpha} &\geq \int_u^{t_k} p(s) y^{\alpha}(\tau(s)) \mathrm{e}^{\alpha\beta R(\tau(s))} \mathrm{e}^{-\alpha\beta R(\tau(s))} \,\mathrm{d}s \\ &\geq y^{\alpha}(\tau(t_k)) \mathrm{e}^{\alpha\beta R(\tau(t_k))} \int_u^{t_k} p(s) \mathrm{e}^{-\alpha\beta R(\tau(s))} \,\mathrm{d}s. \end{aligned}$$

Extracting the α root and integrating once more from $\tau(t_k)$ to t_k, we get

$$y(\tau(t_k)) \geq \mathrm{e}^{\beta R(\tau(t_k))} y(\tau(t_k)) \int_{\tau(t_k)}^{t_k} \frac{1}{r^{1/\alpha}(v)} \left[\int_v^{t_k} p(s) \mathrm{e}^{-\alpha\beta R(\tau(s))} \,\mathrm{d}s \right]^{1/\alpha} \mathrm{d}v$$

which contradicts to condition (2.2) and we conclude, that (E) does not possess decreasing solutions. □

Now we turn our attention to monotonic properties for possible unbounded solutions of (E).

Lemma 2.3. *Let that there exist a sequence $\{s_k\}$ such that $s_k \in \mathcal{A}_\tau$, $s_k \to \infty$ as $k \to \infty$. Assume that $y(t)$ is a positive unbounded solution of (E). If there exists some positive constant γ such that for all $k \in \{1, 2, \dots\}$*

$$\left[\int_t^{\tau(t)} p(s) \,\mathrm{d}s \right]^{1/\alpha} \geq \gamma \quad \text{on } [s_k, \tau(s_k)], \tag{2.3}$$

then $y(\tau(t))\mathrm{e}^{-\gamma R(\tau(t))}$ is increasing on all $[s_k, \tau(s_k)]$.

Proof. Assume that $y(t)$ is a positive increasing solution of (E) and $t \in [s_k, \tau(s_k)]$. An integration of (E) from t to $\tau(t)$ yields

$$r(\tau(t)) \left(y'(\tau(t)) \right)^{\alpha} \geq y^{\alpha}(\tau(t)) \int_t^{\tau(t)} p(s) \,\mathrm{d}s \geq \gamma^{\alpha} y^{\alpha}(\tau(t)).$$

This means

$$r^{1/\alpha}(\tau(t)) y'(\tau(t)) \geq \gamma\, y(\tau(t)).$$

it is easy to see that

$$\left[y(\tau(t))\mathrm{e}^{-\gamma R(\tau(t))} \right]' = \frac{\mathrm{e}^{-\gamma R(\tau(t))} \tau'(t)}{r^{1/\alpha}(\tau(t))} \left[-\gamma y(\tau(t)) + r^{1/\alpha}(\tau(t)) y'(\tau(t)) \right] \geq 0$$

and we verified that function $y(\tau(t))\mathrm{e}^{-\gamma R(\tau(t))}$ is increasing. The proof is complete. □

Theorem 2.4. *Let that there exist a sequence $\{s_k\}$ such that $s_k \in \mathcal{A}_\tau$, $s_k \to \infty$ as $k \to \infty$ and (2.3) hold. If*

$$\limsup_{k \to \infty} \mathrm{e}^{-\gamma R(\tau(s_k))} \int_{s_k}^{\tau(s_k)} \frac{1}{r^{1/\alpha}(u)} \left[\int_{s_k}^{u} p(s)\, \mathrm{e}^{\alpha\gamma R(\tau(s))} \,\mathrm{d}s \right]^{1/\alpha} \mathrm{d}u > 1, \tag{2.4}$$

then (E) has no unbounded nonoscillatory solutions.

Proof. Assume on the contrary, that (E) has an eventually positive increasing solution $y(t)$. We consider $u \in [s_k, \tau(s_k)]$. Integrating (E) from s_k to u and employing monotonic property of $\mathrm{e}^{-\gamma R(\tau(t))} y(\tau(t))$, one gets

$$\begin{aligned} r(u)(y'(u))^{\alpha} &\geq \int_{s_k}^{u} p(s) y^{\alpha}(\tau(s)) \mathrm{e}^{-\alpha\gamma R(\tau(s))} \mathrm{e}^{\alpha\gamma R(\tau(s))} \,\mathrm{d}s \\ &\geq y^{\alpha}(\tau(s_k)) \mathrm{e}^{-\alpha\gamma R(\tau(s_k))} \int_{s_k}^{u} p(s) \mathrm{e}^{\alpha\gamma R(\tau(s))} \,\mathrm{d}s. \end{aligned}$$

Simplifying and then integrating once more from s_k to u we obtain

$$y(u) \geq \mathrm{e}^{-\gamma R(\tau(s_k))} y(\tau(s_k)) \int_{s_k}^{u} \frac{1}{r^{1/\alpha}(v)} \left[\int_{s_k}^{v} p(s) \mathrm{e}^{\alpha\gamma R(\tau(s))} \,\mathrm{d}s \right]^{1/\alpha} \mathrm{d}v.$$

Putting $u = \tau(s_k)$, we have

$$y(\tau(s_k)) \geq \mathrm{e}^{-\gamma R(\tau(s_k))} y(\tau(s_k)) \int_{s_k}^{\tau(s_k)} \frac{1}{r^{1/\alpha}(v)} \left[\int_{s_k}^{v} p(s) \mathrm{e}^{\alpha\gamma R(\tau(s))} \,\mathrm{d}s \right]^{1/\alpha} \mathrm{d}v$$

which contradicts to condition (2.4) and we conclude, that (E) does not possess decreasing solutions. □

Picking up the previous results we can formulate the following oscillatory criterion.

Theorem 2.5. *Assume that there exist two sequences $\{t_k\}$ and $\{s_k\}$ such that $t_k \in \mathcal{R}_\tau$, $s_k \in \mathcal{A}_\tau$ $t_k, s_k \to \infty$ as $k \to \infty$. Let β and γ be defined by (2.1) and (2.3), respectively. If (2.2) and (2.4) are satisfied, then (E) is oscillatory.*

3 Examples

Example 3.1. We consider the differential equation

$$y''(t) = p y(t + \sin t), \qquad p > 0. \tag{E_x}$$

We shall show that (E_x) is oscillatory provided that $p \geq p_0 = 1.5955$.

To verify that (E_x) has no bounded nonoscillatory solutions we set $t_k = \frac{3}{2}\pi + 2k\pi$. Then it is easy to see that $\tau(t_k) = \frac{3}{2}\pi - 1 + 2k\pi$. So condition (2.1) reduces to

$$-p \sin(t) \geq -p_0 \sin(t) \geq \beta \quad \text{on } [\tau(t_k), t_k], \quad k = 1, 2, \dots$$

Since $-p_0 \sin(t)$ is increasing function on $[\tau(t_k), t_k]$, we can choose

$$\beta = -p_0 \sin(\tau(t_k)) = p_0 \cos 1,$$

On the other hand, condition (2.2) for (E_x) takes the form

$$\limsup_{k\to\infty} \mathrm{e}^{\beta\tau(t_k)} \int_{\tau(t_k)}^{t_k} \int_{u}^{t_k} p_0 \,\mathrm{e}^{-\beta\tau(s)} \,\mathrm{d}s \,\mathrm{d}u > 1. \tag{3.1}$$

Changing order of integration in (3.1) we get simpler form

$$\limsup_{k\to\infty} p_0 \mathrm{e}^{\beta\tau(t_k)} \int_{\tau(t_k)}^{t_k} \mathrm{e}^{-\beta\tau(s)} (s - \tau(t_k)) \,\mathrm{d}s > 1. \tag{3.2}$$

Setting the corresponding values into (3.2) one gets

$$p_0 e^{\beta\tau(t_k)} \int_{\tau(t_k)}^{t_k} e^{-\beta\tau(s)}(s-\tau(t_k))\,ds = p_0 e^{\beta\left(\frac{3}{2}\pi-1+2k\pi\right)} \int_{\tau(t_k)}^{t_k} e^{-\beta(s+\sin s)}\left(s-\left(\frac{3}{2}\pi-1+2k\pi\right)\right)ds.$$

Substitution $s-\left(\frac{3}{2}\pi-1+2k\pi\right)=t$ yields

$$p_0 e^{\beta\tau(t_k)} \int_{\tau(t_k)}^{t_k} e^{-\beta\tau(s)}(s-\tau(t_k))\,ds = p_0 \int_0^1 te^{-\beta(t-\cos(1-t))}\,dt = 0.6268p_0 = 1.000004 > 1,$$

where for evaluation of the above integral we employed Matlab. We have verified that (2.2) holds true and by Theorem 2.2 (E_x) has no bounded solutions.

On the other hand, to ensure that (E_x) has no unbounded nonoscillatory solutions we chose $s_k=\frac{\pi}{2}+2k\pi$. Then $\tau(s_k)=\frac{\pi}{2}+1+2k\pi$. Now,condition (2.4) takes the form

$$p\sin(t) \geq p_0\sin(t) \geq \gamma \quad \text{on } [s_k,\tau(s_k)], \quad k=1,2,\dots$$

Using the fact that $p_0\sin(t)$ is decreasing function on $[s_k,\tau(s_k)]$, we set

$$\gamma = p_0\sin(\tau(s_k)) = p_0\cos 1 = \beta,$$

Condition (2.4) reduces to

$$\limsup_{k\to\infty} e^{-\gamma\tau(t_k)} \int_{s_k}^{\tau(s_k)} \int_{s_k}^{u} p_0\, e^{\gamma\tau(s)}\,ds\,du > 1$$

which is equivalent to

$$\limsup_{k\to\infty} p_0 e^{-\gamma\tau(s_k)} \int_{s_k}^{\tau(s_k)} e^{\gamma\tau(s)}(\tau(s_k)-s)\,ds > 1,$$

which for parameters of (E_x) means

$$p_0 e^{-\gamma\tau(s_k)} \int_{s_k}^{\tau(s_k)} e^{\gamma\tau(s)}(\tau(s_k)-s)\,ds = p_0 e^{-\gamma\left(\frac{\pi}{2}+1+2k\pi\right)} \int_{s_k}^{\tau(s_k)} e^{\gamma(s+\sin s)}\left(\frac{\pi}{2}+1+2k\pi-s\right)ds.$$

Substitution $\frac{\pi}{2}+1+2k\pi-s=t$ provides

$$p_0 e^{-\gamma\tau(s_k)} \int_{s_k}^{\tau(s_k)} e^{\gamma\tau(s)}(\tau(s_k)-s)\,ds = p_0 \int_0^1 te^{-\beta(t-\cos(1-t))}\,dt = 0.6268p_0 = 1.000004 > 1,$$

Consequently, condition (2.4) is satisfied and by Theorem 2.4 Eq. (E_x) has no unbounded nonoscillatory solutions. By comparing with Kusano's result mentioned in the motivation part, our oscillatory constant is significantly better.

4 Summary

In this paper we improved Kusano's technique for investigation of differential equations with mixed arguments. The progress achieved has been presented via Kusano's differential equation.

As a matter of fact the results presented in this paper can be rewritten also for differential equation of the form

$$\big(r(t)|y'(t)|^{\alpha}\operatorname{sgn} y'(t)\big)' = p(t)|y(\tau(t))|^{\alpha}\operatorname{sgn} y(\tau(t)).$$

The details are left to the reader.

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Positive solutions to three classes of non-local fourth-order problems with derivative-dependent nonlinearities

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Abstract. In the article, we investigate three classes of fourth-order boundary value problems with dependence on all derivatives in nonlinearities under the boundary conditions involving Stieltjes integrals. A Gronwall-type inequality is employed to get an a priori bound on the third-order derivative term, and the theory of fixed-point index is used on suitable open sets to obtain the existence of positive solutions. The nonlinearities have quadratic growth in the third-order derivative term. Previous results in the literature are not applicable in our case, as shown by our examples.

Keywords: positive solution, fixed point index, cone, Gronwall inequality.

2020 Mathematics Subject Classification: 34B18, 34B10, 34B15.

1 Introduction

In the article, we investigate the existence of positive solutions to the following three classes of fourth-order boundary value problems (BVPs) with dependence on all derivatives in nonlinearities under the boundary conditions involving Stieltjes integrals

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t), u''(t), u'''(t)), \quad t \in [0,1], \\ u'(0) + \alpha_1[u] = 0, \ u''(0) + \alpha_2[u] = 0, \ u(1) = \alpha_3[u], \ u'''(1) = 0, \end{cases} \tag{1.1}$$

$$\begin{cases} -u^{(4)}(t) = \widetilde{f}(t, u(t), u'(t), u''(t), u'''(t)), \quad t \in [0,1], \\ u(0) = \beta_1[u], \ u'(0) = \beta_2[u], \ u''(0) = \beta_3[u], \ u'''(1) = 0, \end{cases} \tag{1.2}$$

and

$$\begin{cases} u^{(4)}(t) = \overline{f}(t, u(t), u'(t), u''(t), u'''(t)), \quad t \in [0,1], \\ u(0) = u(1) = \eta_1[u], \ u''(0) + \eta_2[u] = 0, \ u''(1) + \eta_2[u] = 0, \end{cases} \tag{1.3}$$

where

$$\alpha_i[u] = \int_0^1 u(t)\,dA_i(t) \quad (i = 1,2,3), \qquad \beta_i[u] = \int_0^1 u(t)\,dB_i(t) \quad (i = 1,2,3),$$

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$$\eta_i[u] = \int_0^1 u(t)dH_i(t) \quad (i = 1, 2)$$

are Stieltjes integrals with A_i, B_i, H_i of bounded variation.

The BVPs (1.1) and (1.2) share the common features that the derivatives of Green's functions, from first to third order in t, do not change sign, however the first-and third-order derivatives of Green's function for the BVP (1.3) are sign-changing. The existence of positive solutions for the BVPs (1.1), (1.2) and (1.3) have been studied respectively in [9] and [5] with $\overline{f}(t, u(t), u''(t))$. The BVP (1.3) with $\eta_1[u] = \eta_2[u] = 0$ is also considered by [16] in which the fourth-order equation is transformed into a second-order problem by order reduction method. The authors in [10] discuss the second-order BVP with non-local boundary conditions

$$\begin{cases} -u''(t) = f(t, u(t), u'(t)), \quad t \in [0, 1], \\ au(0) - bu'(0) = \alpha[u], \ cu(1) + du'(1) = \beta[u], \end{cases} \tag{1.4}$$

where a, b, c and d are nonnegative constants with $\rho = ac + ad + bc > 0$. It is supposed in [10] that the nonlinear term has linear growth in both u and u' and some conditions related to the spectral radius of a related linear operator are used, moreover, the Nagumo condition is applied in one of their results. The BVP (1.4) with $a = d = 1, b = c = 0$ is studied by Zhang et al. [17], but the conditions of theorems in [10] can not contain the ones in [17], see [10, Remark 3.10, Remark 3.11].

Recently, Webb in [12] employs a Gronwall-type inequality proved in [13] to deal with a second order equation with nonlinearity having quadratic dependence in derivative terms, but no growth restriction in the function term. This new Gronwall inequality is used instead of a Nagumo condition to get an a priori bound on the derivative term. The theory of fixed-point index on suitable open sets is applied to obtain the existence of positive solutions to second-order non-local problems.

Motivated by the works mentioned above, in the present paper we adopt the idea and the techniques provided in [12] to consider the positive solutions of the fourth-order BVPs (1.1), (1.2) and (1.3). The nonlinearities contain all terms of the derivatives, and there is quadratic growth in u''' but no growth restriction in u, u' and u''. Li and Chen in [8] investigate the nontrivial solutions to fourth-order BVP with quadratic growth subject to local boundary conditions. In [9], the nonlinearity has linear growth in u and all derivatives with some conditions related to the spectral radius of a related linear operator, the results are not valid for the problems presented in this paper although the Nagumo condition also allows quadratic growth (see Example 2.7 and Remark 2.8). Making use of several different methods, the authors in [3, 5, 11] discuss the existence of positive solutions to some fourth-order BVPs, however not all of the derivatives is included in the nonlinearities since some derivatives of the Green's functions are sign-changing. Some relevant works may refer to [1] for fourth-order BVP with local boundary conditions via an application of contraction mapping theorem, [6] for certain perturbed Hammerstein integral equations with first-order derivative dependence, [7] for fourth-order BVP with local boundary conditions.

We recall the basic properties of fixed point index that we use.

Lemma 1.1 ([2,4]). *Let Ω be a bounded open set relative to a cone P in Banach space X with $0 \in \Omega$. If $A : \overline{\Omega} \to P$ is a completely continuous operator, and $Au \neq \lambda u$ for $u \in \partial_P \Omega$, $\lambda \geq 1$, then the fixed point index $i(A, \Omega, P) = 1$, where $\overline{\Omega}$ and $\partial_P \Omega$ are respectively the closure and boundary of Ω relative to P.*

Lemma 1.2 ([2,4]). *Let Ω be a bounded open set relative to a cone P in Banach space X. If $A : \overline{\Omega} \to P$ is a completely continuous operator, and there exists $v_0 \in P \setminus \{0\}$ such that $u - Au \neq \sigma v_0$ for $u \in \partial_P \Omega$ and $\sigma \geq 0$, then the fixed point index $i(A, \Omega, P) = 0$.*

2 Positive solutions to the BVP (1.1)

Let $[\alpha, \beta] \subset [0, 1]$, we write $L^p_+[\alpha, \beta]$ $(1 \leq p \leq \infty)$ to denote functions that are non-negative almost everywhere (a.e.) and belong to $L^p[\alpha, \beta]$. The proof of the following lemma is completely similar to the method in [12].

Lemma 2.1. *Suppose that there are a constant $d_0 > 0$ and functions $d_1, d_2 \in L^1_+[\alpha, \beta]$ such that $u \in L^\infty_+[\alpha, \beta]$ satisfies*

$$u(t) \leq d_0 + \int_t^\beta d_1(s)u(s)ds + \int_t^\beta d_2(s)u^2(s)ds \quad \text{for a.e. } t \in [\alpha, \beta].$$

If there is a constant $R > 0$ such that $\int_\alpha^\beta d_2(s)u(s)ds \leq R$, then $u(t) \leq d_0 \exp(R) \exp(D_1(t))$ for a.e. $t \in [\alpha, \beta]$, where $D_1(t) := \int_t^\beta d_1(s)ds$.

Proof. Let $v(t) := d_0 + \int_t^\beta d_1(s)u(s)ds + \int_t^\beta d_2(s)u^2(s)ds$. Then v is absolutely continuous, $v(\beta) = d_0, v(t) \geq d_0 > 0$ for all $t \in [\alpha, \beta]$, and $u(t) \leq v(t)$ for a.e. $t \in [\alpha, \beta]$. Moreover, we have

$$v'(t) = -d_1(t)u(t) - d_2(t)u^2(t) \geq -d_1(t)v(t) - d_2(t)u(t)v(t) \quad \text{for a.e. } t \in [\alpha, \beta].$$

Then $v'(t)/v(t) \geq -d_1(t) - d_2(t)u(t)$ which can be integrated to give

$$\ln\left(\frac{v(\beta)}{v(t)}\right) \geq -D_1(t) - \int_t^\beta d_2(s)u(s)ds,$$

hence $u(t) \leq v(t) \leq d_0 \exp(R) \exp(D_1(t))$ for a.e. $t \in [\alpha, \beta]$. □

For BVP (1.1)

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t), u''(t), u'''(t)), \quad t \in [0, 1], \\ u'(0) + \alpha_1[u] = 0, \ u''(0) + \alpha_2[u] = 0, \ u(1) = \alpha_3[u], \ u'''(1) = 0, \end{cases}$$

we make the following assumptions:

(C_1) $f : [0, 1] \times [0, \infty) \times (-\infty, 0]^3 \to [0, \infty)$ is continuous;

(C_2) A_i is of bounded variation, moreover

$$\mathcal{K}_i(s) := \int_0^1 G_0(t, s)dA_i(t) \geq 0, \qquad \forall s \in [0, 1] \quad (i = 1, 2, 3),$$

where

$$G_0(t, s) = \begin{cases} \frac{1}{2}s(1 - s) + \frac{1}{6}(s^3 - t^3), & 0 \leq t \leq s \leq 1, \\ \frac{1}{2}s(1 - s) - \frac{1}{2}ts(t - s), & 0 \leq s \leq t \leq 1; \end{cases}$$

(C_3) The 3×3 matrix $[A]$ is positive whose (i,j)th entry is $\alpha_i[\gamma_j]$, i.e., it has nonnegative entries, where $\gamma_1(t) = 1 - t$, $\gamma_2(t) = \frac{1}{2}(1-t^2)$ and $\gamma_3(t) = 1$ are the solutions of $u^{(4)} = 0$ respectively subject to boundary conditions:

$$u'(0) + 1 = 0, \quad u''(0) = 0, \quad u(1) = 0, \quad u'''(1) = 0;$$

$$u'(0) = 0, \quad u''(0) + 1 = 0, \quad u(1) = 0, \quad u'''(1) = 0;$$

$$u'(0) = 0, \quad u''(0) = 0, \quad u(1) = 1, \quad u'''(1) = 0.$$

Furthermore assume that its spectral radius $r([A]) < 1$.

Webb and Infante [14] in a general framework convert the BVP

$$\begin{cases} u^{(4)}(t) = f(t, u(t)), \quad t \in [0,1], \\ u'(0) + \alpha_1[u] = 0, \ u''(0) + \alpha_2[u] = 0, \ u(1) = \alpha_3[u], \ u'''(1) = 0 \end{cases} \tag{2.1}$$

into the perturbed Hammerstein integral equation of the type

$$u(t) = \sum_{i=1}^{3} \gamma_i(t)\alpha_i[u] + \int_0^1 G_0(t,s) f(s, u(s)) ds,$$

where $G_0(t,s)$ is the Green's function associated with

$$\begin{cases} u^{(4)}(t) = f(t, u(t)), \quad t \in [0,1], \\ u'(0) = u''(0) = u(1) = u'''(1) = 0. \end{cases}$$

Immediately after this they prove that if (C_1)–(C_3) are satisfied, (2.1) is equivalent to

$$u(t) = \int_0^1 G_1(t,s) f(s, u(s)) ds,$$

where

$$G_1(t,s) = \langle (I - [A])^{-1}\mathcal{K}(s), \gamma(t) \rangle + G_0(t,s) = \sum_{i=1}^{3} \kappa_i(s)\gamma_i(t) + G_0(t,s),$$

$\langle (I - [A])^{-1}\mathcal{K}(s), \gamma(t) \rangle$ is the inner product in $\mathbb{R}^3$, $\kappa_i(s)$ is the ith component of $(I - [A])^{-1}\mathcal{K}(s)$.

Similar to the method of Webb–Infante, we define the operator S as

$$(Su)(t) = \int_0^1 G_1(t,s) f(s, u(s), u'(s), u''(s), u'''(s)) ds.$$

Lemma 2.2. *If (C_2) and (C_3) hold, then $\kappa_i(s) \geq 0$ $(i = 1,2,3)$ and for $t, s \in [0,1]$,*

$$c_0(t)\Phi_0(s) \leq G_1(t,s) \leq \Phi_0(s), \tag{2.2}$$

where

$$\Phi_0(s) = \sum_{i=1}^{3} \kappa_i(s) + \frac{1}{2}s(1-s) + \frac{1}{6}s^3, \qquad c_0(t) = \frac{1}{2}(1-t^2),$$

and

$$c_1(t)\Phi_1(s) \leq -\frac{\partial G_1(t,s)}{\partial t} \leq \Phi_1(s), \qquad c_2(t)\Phi_2(s) \leq -\frac{\partial^2 G_1(t,s)}{\partial t^2} \leq \Phi_2(s), \tag{2.3}$$

where

$$\frac{\partial G_1(t,s)}{\partial t} = -\kappa_1(s) - t\kappa_2(s) - \frac{1}{2} \begin{cases} t^2, & 0 \leq t \leq s \leq 1, \\ s(2t - s), & 0 \leq s \leq t \leq 1, \end{cases}$$

$$\frac{\partial^2 G_1(t,s)}{\partial t^2} = -\kappa_2(s) - \begin{cases} t, & 0 \le t \le s \le 1, \\ s, & 0 \le s \le t \le 1, \end{cases}$$

$$\Phi_1(s) = \sum_{i=1}^{2} \kappa_i(s) + \frac{1}{2}s(2-s), \qquad c_1(t) = t^2, \; \Phi_2(s) = \kappa_2(s) + s, \qquad c_2(t) = t.$$

Proof. $\kappa_i(s) \ge 0 (i = 1,2,3)$ by hypotheses (C_2) and (C_3). For $0 \le s \le t \le 1$, $\frac{\partial}{\partial t} G_0(t,s) = \frac{1}{2}s(s-2t) \le 0$ which implies that

$$G_0(t,s) \le G_0(s,s) = \frac{1}{2}s(1-s);$$

For $0 \le t < s \le 1$, $\frac{\partial}{\partial t} G_0(t,s) = -\frac{1}{2}t^2 \le 0$ which implies that

$$G_0(t,s) \le G_0(0,s) = \frac{1}{2}s(1-s) + \frac{1}{6}s^3.$$

Then $G_0(t,s) \le \frac{1}{2}s(1-s) + \frac{1}{6}s^3$, $\forall (t,s) \in [0,1] \times [0,1]$.

Now we find the best function $C_0(t)$ such that $G_0(t,s) \ge C_0(t) \left(\frac{1}{2}s(1-s) + \frac{1}{6}s^3\right)$, $\forall (t,s) \in [0,1] \times [0,1]$.

For $0 \le s \le t \le 1$, this is

$$\frac{1}{2}s(1-s) - \frac{1}{2}ts(t-s) \ge C_0(t) \left(\frac{1}{2}s(1-s) + \frac{1}{6}s^3 \right),$$

thus

$$C_0(t) \le \frac{3(1-t)(1+t-s)}{3-3s+s^2}.$$

Denote

$$g_1(t,s) = \frac{(1-t)(1+t-s)}{3-3s+s^2},$$

from

$$\frac{\partial}{\partial s} g_1(t,s) = \frac{(1-t)(s^2-2s(1+t)+3t)}{(3-3s+s^2)^2} \ge 0$$

it follows that $C_0(t) \le 3g_1(t,0) = 1-t^2$.

For $0 \le t < s \le 1$, this is

$$\frac{1}{2}s(1-s) + \frac{1}{6}(s^3-t^3) \ge C_0(t) \left(\frac{1}{2}s(1-s) + \frac{1}{6}s^3 \right),$$

thus

$$C_0(t) \le \frac{3s-3s^2+s^3-t^3}{s(3-3s+s^2)}.$$

Denote

$$g_2(t,s) = \frac{3s-3s^2+s^3-t^3}{s(3-3s+s^2)},$$

from

$$\frac{\partial}{\partial s} g_2(t,s) = \frac{3(1-s)^2 t^3}{s^2(3-3s+s^2)^2} \ge 0$$

it follows that $C_0(t) \le g_2(t,t) = \frac{3(1-t)}{3-3t+t^2}$.

Therefore

$$C_0(t) = \min\left\{1-t^2,\ \frac{3(1-t)}{3-3t+t^2}\right\} = 1-t^2.$$

Since

$$\frac{1}{2}(1-t^2)\sum_{i=1}^{3}\kappa_i(s) \le \sum_{i=1}^{3}\kappa_i(s)\gamma_i(t) \le \sum_{i=1}^{3}\kappa_i(s),$$

$$\frac{1}{2}(1-t^2)\left(\frac{1}{2}s(1-s)+\frac{1}{6}s^3\right) \le (1-t^2)\left(\frac{1}{2}s(1-s)+\frac{1}{6}s^3\right) \le G_0(t,s) \le \frac{1}{2}s(1-s)+\frac{1}{6}s^3,$$

we know that (2.2) holds. As for (2.3), it comes directly from the inequalities

$$t^2\sum_{i=1}^{2}\kappa_i(s) \le t\sum_{i=1}^{2}\kappa_i(s) \le -\sum_{i=1}^{3}\kappa_i(s)\gamma_i'(t) \le \sum_{i=1}^{2}\kappa_i(s),$$

$$\frac{1}{2}t^2 s(2-s) \le -\frac{\partial G_0(t,s)}{\partial t} \le \frac{1}{2}s(2-s),$$

$$t\kappa_2(s) \le \kappa_2(s) = -\sum_{i=1}^{3}\kappa_i(s)\gamma_i''(t),\ \ ts \le -\frac{\partial^2 G_0(t,s)}{\partial t^2} \le s$$

for $t,s\in[0,1]$. □

Let $C^3[0,1]$ be the Banach space which consists of all third-order continuously differentiable functions on $[0,1]$ with the norm $\|u\|_{C^3} = \max\{\|u\|_C, \|u'\|_C, \|u''\|_C, \|u'''\|_C\}$. In $C^3[0,1]$ we define the cone

$$\begin{aligned} K = \big\{u\in C^3[0,1] : u(t) \ge c_0(t)\|u\|_C,\ -u'(t) \ge c_1(t)\|u'\|_C,\\ -u''(t) \ge c_2(t)\|u''\|_C,\ \forall t\in[0,1];\ u'''(1)=0\big\}. \end{aligned} \tag{2.4}$$

Lemma 2.3. *If (C_1)–(C_3) hold, then $S : K \to K$ is completely continuous and the positive solutions to BVP (1.1) are equivalent to the fixed points of S in K.*

Proof. Because $G_1(t,s)$, and the first- and second-order derivatives are continuous, the third order derivative is integrable in s, from Lemma 2.2 it is easy to prove that $S : K \to K$ is continuous. Let F be a bounded set in K, then there exists $M > 0$ such that $\|u\|_{C^3} \le M$ for all $u \in K$. Denote

$$C = \max_{(t,x_0,x_1,x_2,x_3)\in[0,1]\times[0,M]\times[-M,0]^3} f(t,x_0,x_1,x_2,x_3).$$

By (C_1) and Lemma 2.2 we have that $\forall u \in F$ and $t\in[0,1]$,

$$|(Su)(t)| \le C\int_0^1 \Phi_0(s)ds, \qquad |(Su)'(t)| \le C\int_0^1 \left|\frac{\partial G_1(t,s)}{\partial t}\right|ds \le C\int_0^1 \Phi_1(s)ds,$$

$$|(Su)''(t)| \le C\int_0^1 \left|\frac{\partial^2 G_1(t,s)}{\partial t^2}\right|ds \le C\int_0^1 \Phi_2(s)ds, \qquad |(Su)'''(t)| \le C\int_0^1 \left|\frac{\partial^3 G_1(t,s)}{\partial t^3}\right|ds \le C,$$

then $S(F)$ is uniformly bounded in $C^3[0,1]$. Moreover $\forall u\in F$ and $t_1,t_2\in[0,1]$ with $t_1<t_2$,

$$\begin{aligned} |(Su)(t_1)-(Su)(t_2)| &\le \int_0^1 |G_1(t_1,s)-G_1(t_2,s)| f(s,u(s),u'(s),u''(s),u'''(s))ds\\ &\le C\int_0^1 |G_1(t_1,s)-G_1(t_2,s)|ds, \end{aligned}$$

$$|(Su)'(t_1)-(Su)'(t_2)| \le \int_0^1 \left|\frac{\partial G_1}{\partial t}(t_1,s)-\frac{\partial G_1}{\partial t}(t_2,s)\right| f(s,u(s),u'(s),u''(s),u'''(s))ds$$
$$\le C\int_0^1 \left|\frac{\partial G_1}{\partial t}(t_1,s)-\frac{\partial G_1}{\partial t}(t_2,s)\right| ds,$$

$$|(Su)''(t_1)-(Su)''(t_2)| \le \int_0^1 \left|\frac{\partial^2 G_1}{\partial t^2}(t_1,s)-\frac{\partial^2 G_1}{\partial t^2}(t_2,s)\right| f(s,u(s),u'(s),u''(s),u'''(s))ds$$
$$\le C\int_0^1 \left|\frac{\partial^2 G_1}{\partial t^2}(t_1,s)-\frac{\partial^2 G_1}{\partial t^2}(t_2,s)\right| ds,$$

$$|(Su)'''(t_1)-(Su)'''(t_2)|$$
$$=\left|\int_{t_1}^1 f(s,u(s),u'(s),u''(s),u'''(s))ds-\int_{t_2}^1 f(s,u(s),u'(s),u''(s),u'''(s))ds\right| \le C(t_2-t_1),$$

thus $S(F)$ and $S^{(i)}(F) =: \{v^{(i)} : v^{(i)}(t)=(Su)^{(i)}(t), u\in F\}$ $(i=1,2,3)$ are equicontinuous.

Therefore $S: K\to K$ is completely continuous by the Arzelà–Ascoli theorem. Similar to [14], the positive solutions to BVP (1.1) are equivalent to the fixed points of S in K. □

Lemma 2.4. *Suppose that* (C_1)–(C_3) *hold, there exist constants* $p_0>0, p_3\ge 0$ *and functions* $p_1,p_2\in L^1_+[0,1]$ *such that for all* $(t,x_0,x_1,x_2,x_3)\in[0,1]\times[0,\infty)\times(-\infty,0]^3$,

$$f(t,x_0,x_1,x_2,x_3)\le p_0+p_1(t)g(x_0,x_1,x_2)+p_2(t)|x_3|+p_3|x_3|^2, \tag{2.5}$$

where $g:[0,\infty)\times(-\infty,0]^2\to[0,\infty)$ *is continuous, non-decreasing in the first variable, and non-increasing in the second and third variables. Let* $\lambda\ge 1, \sigma\ge 0, r>0$*, define* $D_i := \int_0^1 p_i(s)ds$ $(i=1,2)$ *and*

$$Q(r) := (p_0+g(r,-r,-r)D_1)\exp(D_2)\exp(p_3 r). \tag{2.6}$$

If $u\in K$ *with* $\|u\|_{C^2}\le r$ *such that* $\lambda u(t)=(Su)(t)+\sigma$*, then* $\|u'''\|_C\le Q(r)$*.*

Proof. Since $u\in K$ and $\lambda u(t)=(Su)(t)+\sigma$, we have that $\lambda u^{(4)}(t)=f(t,u(t),u'(t),u''(t),u'''(t))$ and $\lambda u'''(t)=-\int_t^1 f(s,u(s),u'(s),u''(s),u'''(s))ds\le 0$. From $\|u\|_{C^2}\le r$ it follows that

$$|u'''(t)|\le\lambda|u'''(t)|=\int_t^1 f(s,u(s),u'(s),u''(s),u'''(s))ds$$
$$\le\int_t^1\left(p_0+p_1(s)g(u(s),u'(s),u''(s))+p_2(s)|u'''(s)|+p_3|u'''(s)|^2\right)ds$$
$$\le(p_0+g(r,-r,-r)D_1)+\int_t^1\left(p_2(s)|u'''(s)|+p_3|u'''(s)|^2\right)ds$$

and $\int_0^1 p_3|u'''(s)|ds=-\int_0^1 p_3u'''(s)ds=p_3(u''(0)-u''(1))\le p_3 r$. By Lemma 2.1, we deduce that

$$|u'''(t)|\le(p_0+g(r,-r,-r)D_1)\exp(D_2)\exp(p_3r)=Q(r),$$

the proof is complete. □

Let $[a,b]$ be a subset of $(0,1)$ and denote

$$\gamma:=\min\left\{\min_{t\in[a,b]}c_0(t),\min_{t\in[a,b]}c_1(t),\min_{t\in[a,b]}c_2(t)\right\}=\min\left\{\frac{1}{2}(1-b^2),a^2\right\},$$

$$\frac{1}{m} := \max\left\{\int_0^1 \Phi_0(s)ds, \int_0^1 \Phi_1(s)ds, \int_0^1 \Phi_2(s)ds\right\},$$
$$\frac{1}{M} := \min\left\{\int_a^b \Phi_0(s)ds, \int_a^b \Phi_1(s)ds, \int_a^b \Phi_2(s)ds\right\},$$

where $c_i(t)$ and $\Phi_i(s) (i = 0, 1, 2)$ are provided in Lemma 2.2. Obviously, $\gamma \in (0, 1/2)$ and $m < M$.

Theorem 2.5. *Suppose that* (C_1)-(C_3) *hold and* f *satisfies the growth assumption* (2.5)*. The BVP* (1.1) *has at least one positive solution* $u \in K$ *if either of the following conditions* (F_1), (F_2) *holds, where* Q *is given by* (2.6)*.*

(F_1) *There exist* $0 < r_1 < r_2$ *with* $r_1 < r_2\gamma$*, such that*

$(F_1 a)$ *for* $(t, x_0, x_1, x_2, x_3) \in [0, 1] \times [0, r_1] \times [-r_1, 0]^2 \times [-Q(r_1), 0]$,

$$f(t, x_0, x_1, x_2, x_3) < mr_1; \tag{2.7}$$

$(F_1 b)$ *for* $(t, x_0, x_1, x_2, x_3) \in W_1 := W_{1,0} \cup W_{1,1} \cup W_{1,2}$,

$$f(t, x_0, x_1, x_2, x_3) > Mr_2, \tag{2.8}$$

where

$$W_{1,0} = [a, b] \times [r_2\gamma, r_2] \times [-r_2, 0]^2 \times [-Q(r_2), 0],$$
$$W_{1,1} = [a, b] \times [0, r_2] \times [-r_2, -r_2\gamma] \times [-r_2, 0] \times [-Q(r_2), 0],$$
$$W_{1,2} = [a, b] \times [0, r_2] \times [-r_2, 0] \times [-r_2, -r_2\gamma] \times [-Q(r_2), 0].$$

(F_2) *There exist* $0 < r_1 < r_2$ *with* $Mr_1 < mr_2$*, such that*

$(F_2 a)$ *for* $(t, x_0, x_1, x_2, x_3) \in [0, 1] \times [0, r_2] \times [-r_2, 0]^2 \times [-Q(r_2), 0]$,

$$f(t, x_0, x_1, x_2, x_3) < mr_2; \tag{2.9}$$

$(F_2 b)$ *for* $(t, x_0, x_1, x_2, x_3) \in W_2 := W_{2,0} \cup W_{2,1} \cup W_{2,2}$,

$$f(t, x_0, x_1, x_2, x_3) > Mr_1, \tag{2.10}$$

where

$$W_{2,0} = [a, b] \times [r_1\gamma, r_1] \times [-r_1, 0]^2 \times [-Q(r_1), 0],$$
$$W_{2,1} = [a, b] \times [0, r_1] \times [-r_1, -r_1\gamma] \times [-r_1, 0] \times [-Q(r_1), 0],$$
$$W_{2,2} = [a, b] \times [0, r_1] \times [-r_1, 0] \times [-r_1, -r_1\gamma] \times [-Q(r_1), 0].$$

Proof. Suppose that (F_1) holds. Define an open (relative to K) bounded set

$$U_{r_1} := \{u \in K : \|u\|_{C^2} < r_1, \|u'''\|_C < Q(r_1) + 1\}.$$

Then the boundary $\partial_K U_{r_1}$ of U_{r_1} (relative to K) satisfies $\partial_K U_{r_1} \subset U_{r_1,0} \cup U_{r_1,1} \cup U_{r_1,2}$, where

$$U_{r_1,0} := \{u \in K : \|u\|_C = r_1, \|u'\|_C \le r_1, \|u''\|_C \le r_1, \|u'''\|_C \le Q(r_1) + 1\},$$

$$U_{r_1,1} := \{u \in K : \|u\|_C \le r_1, \|u'\|_C = r_1, \|u''\|_C \le r_1, \|u'''\|_C \le Q(r_1) + 1\},$$
$$U_{r_1,2} := \{u \in K : \|u\|_C \le r_1, \|u'\|_C \le r_1, \|u''\|_C = r_1, \|u'''\|_C \le Q(r_1) + 1\}.$$

We will show that $Su \neq \lambda u$ for all $u \in \partial_K U_{r_1}$ and all $\lambda \ge 1$. If not, there exist $u \in \partial_K U_{r_1}$ and $\lambda \ge 1$ such that $\lambda u(t) = (Su)(t)$. It is clear that $\|u'''\|_C \le Q(r_1)$ by Lemma 2.4.

From Lemma 2.2 and (2.7) it follows that when $u \in U_{r_1,0}$,

$$\lambda u(t) = \int_0^1 G_1(t,s) f(s, u(s), u'(s), u''(s), u'''(s))ds < \int_0^1 \Phi_0(s) m r_1 ds \le r_1;$$

when $u \in U_{r_1,1}$,

$$-\lambda u'(t) = -\int_0^1 \frac{\partial G_1(t,s)}{\partial t} f(s, u(s), u'(s), u''(s), u'''(s))ds < \int_0^1 \Phi_1(s) m r_1 ds \le r_1;$$

when $u \in U_{r_1,2}$,

$$-\lambda u''(t) = -\int_0^1 \frac{\partial^2 G_1(t,s)}{\partial t^2} f(s, u(s), u'(s), u''(s), u'''(s))ds < \int_0^1 \Phi_2(s) m r_1 ds \le r_1.$$

Taking the maximum over $[0,1]$ we give a contradiction $\lambda r_1 < r_1$.

By Lemma 1.1 the fixed point index $i(S, U_{r_1}, K) = 1$.

Define an open (relative to K) set

$$V_{r_2} := \left\{ u \in K : \min_{t\in[a,b]} u(t) < r_2\gamma, \min_{t\in[a,b]} (-u'(t)) < r_2\gamma, \min_{t\in[a,b]} (-u''(t)) < r_2\gamma, \|u'''\|_C < Q(r_2) + 1 \right\}.$$

It is clear that $\overline{U}_{r_1} \subset V_{r_2}$ by $r_1 < r_2\gamma$ and $Q(r_1) < Q(r_2)$. Since $\|u\|_{C^2} \le r_2$ for $u \in V_{r_2}$ by (2.4), V_{r_2} is bounded. The boundary $\partial_K V_{r_2}$ of V_{r_2} (relative to K) satisfies $\partial_K V_{r_2} \subset V_{r_2,0} \cup V_{r_2,1} \cup V_{r_2,2}$, where

$$V_{r_2,0} := \left\{ u \in K : \min_{t\in[a,b]} u(t) = r_2\gamma, \min_{t\in[a,b]} (-u'(t)) \le r_2\gamma, \min_{t\in[a,b]} (-u''(t)) \le r_2\gamma, \|u'''\|_C \le Q(r_2) + 1 \right\},$$

$$V_{r_2,1} := \left\{ u \in K : \min_{t\in[a,b]} u(t) \le r_2\gamma, \min_{t\in[a,b]} (-u'(t)) = r_2\gamma, \min_{t\in[a,b]} (-u''(t)) \le r_2\gamma, \|u'''\|_C \le Q(r_2) + 1 \right\},$$

$$V_{r_2,2} := \left\{ u \in K : \min_{t\in[a,b]} u(t) \le r_2\gamma, \min_{t\in[a,b]} (-u'(t)) \le r_2\gamma, \min_{t\in[a,b]} (-u''(t)) = r_2\gamma, \|u'''\|_C \le Q(r_2) + 1 \right\}.$$

Let $v_0(t) \equiv 1$ and note that $v_0 \in K$. We claim that $u \neq Su + \sigma v_0$ for all $u \in \partial_K V_{r_2}$ and all $\sigma \ge 0$. If the claim is false, there exist $u \in \partial_K V_{r_2}$ and $\sigma \ge 0$ such that $u = Su + \sigma v_0$. Thus $\|u'''\|_C \le Q(r_2)$ for $u \in V_{r_2}$ by Lemma 2.4. From Lemma 2.2 and (2.8) we have the following contradictions. When $u \in V_{r_2,0}$,

$$u(t) = \int_0^1 G_1(t,s) f(s, u(s), u'(s), u''(s), u'''(s))ds + \sigma > \int_a^b c_0(t)\Phi_0(s) M r_2 ds + \sigma \ge r_2\gamma + \sigma,$$

taking the minimum for $t \in [a,b]$ gives the contradiction $r_2\gamma > r_2\gamma + \sigma$. When $u \in V_{r_2,1}$,

$$-u'(t) = -\int_0^1 \frac{\partial G_1(t,s)}{\partial t} f(s, u(s), u'(s), u''(s), u'''(s))ds > \int_a^b c_1(t)\Phi_1(s) M r_2 ds \ge r_2\gamma,$$

taking the minimum for $t \in [a,b]$ gives the contradiction $r_2\gamma > r_2\gamma$. When $u \in V_{r_2,2}$,

$$-u''(t) = -\int_0^1 \frac{\partial^2 G_1(t,s)}{\partial t^2} f(s,u(s),u'(s),u''(s),u'''(s))ds > \int_a^b c_2(t)\Phi_2(s)Mr_2 ds \geq r_2\gamma,$$

taking the minimum for $t \in [a,b]$ also gives the contradiction $r_2\gamma > r_2\gamma$.

By Lemma 1.2 the fixed point index $i(S, V_{r_2}, K) = 0$.

From the additivity property of fixed point index we have $i(S, V_{r_2}\backslash\overline{U}_{r_1}, K) = -1$. So there is a fixed point of S in the set $V_{r_2}\backslash\overline{U}_{r_1}$ which is clearly nonzero and the positive solutions to BVP (1.1) by Lemma 2.3.

Suppose that (F_2) holds, notice that f is well defined since $Mr_1 < mr_2$. Define open (relative to K) bounded sets $U_{r_2} := \{u \in K : \|u\|_{C^2} < r_2, \|u'''\|_C < Q(r_2)+1\}$ and

$$V_{r_1} := \left\{u \in K : \min_{t\in[a,b]} u(t) < r_1\gamma, \min_{t\in[a,b]}(-u'(t)) < r_1\gamma, \min_{t\in[a,b]}(-u''(t)) < r_1\gamma, \|u'''\|_C < Q(r_1)+1\right\}.$$

It is clear that $\overline{V}_{r_1} \subset U_{r_2}$. The rest of proof is similar to the above. □

Example 2.6. Consider the following fourth-order boundary problems under mixed multi-point and integral boundary conditions with sign-changing coefficients and kernel functions.

$$\begin{cases} u^{(4)}(t) = f(t,u(t),u'(t),u''(t),u'''(t)), \quad t \in [0,1], \\ u'(0) + \frac{1}{4}u(\frac{1}{4}) - \frac{1}{12}u(\frac{3}{4}) = 0, \quad u''(0) + \int_0^1 u(t)\cos(\pi t)dt = 0, \\ u(1) = \frac{1}{2}u(\frac{1}{2}) - \frac{1}{4}u(\frac{3}{4}), \quad u'''(1) = 0, \end{cases} \tag{2.11}$$

thus $\alpha_1[u] = \frac{1}{4}u(\frac{1}{4}) - \frac{1}{12}u(\frac{3}{4})$, $\alpha_2[u] = \int_0^1 u(t)\cos(\pi t)dt$, $\alpha_3[u] = \frac{1}{2}u(\frac{1}{2}) - \frac{1}{4}u(\frac{3}{4})$. Then

$$\begin{aligned} 0 \leq \mathcal{K}_1(s) &= \tfrac{1}{4}G_0\left(\tfrac{1}{4},s\right) - \tfrac{1}{12}G_0\left(\tfrac{3}{4},s\right) \\ &= \begin{cases} -\frac{1}{12}s^2 + \frac{19}{192}s, & 0 \leq s \leq \frac{1}{4}, \\ \frac{1}{24}s^3 - \frac{11}{96}s^2 + \frac{41}{384}s - \frac{1}{1536}, & \frac{1}{4} < s \leq \frac{3}{4}, \\ \frac{1}{36}s^3 - \frac{1}{12}s^2 + \frac{1}{12}s + \frac{1}{192}, & \frac{3}{4} < s \leq 1, \end{cases} \end{aligned}$$

$$\mathcal{K}_2(s) = \int_0^1 G_0(t,s)\cos(\pi t)dt = \frac{2s-s^2}{2\pi^2} + \frac{\cos\pi s}{\pi^4} - \frac{1}{\pi^4} \geq 0 \qquad (0 \leq s \leq 1),$$

$$\begin{aligned} 0 \leq \mathcal{K}_3(s) &= \tfrac{1}{2}G_0\left(\tfrac{1}{2},s\right) - \tfrac{1}{4}G_0\left(\tfrac{3}{4},s\right) \\ &= \begin{cases} -\frac{3}{32}s^2 + \frac{17}{128}s, & 0 \leq s \leq \frac{1}{2}, \\ \frac{1}{12}s^3 - \frac{7}{32}s^2 + \frac{25}{128}s - \frac{1}{96}, & \frac{1}{2} < s \leq \frac{3}{4}, \\ \frac{1}{24}s^3 - \frac{1}{8}s^2 + \frac{1}{8}s + \frac{11}{1536}, & \frac{3}{4} < s \leq 1, \end{cases} \end{aligned}$$

the 3×3 matrix

$$[A] = \begin{pmatrix} \alpha_1[\gamma_1] & \alpha_1[\gamma_2] & \alpha_1[\gamma_3] \\ \alpha_2[\gamma_1] & \alpha_2[\gamma_2] & \alpha_2[\gamma_3] \\ \alpha_3[\gamma_1] & \alpha_3[\gamma_2] & \alpha_3[\gamma_3] \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & \frac{19}{192} & \frac{1}{6} \\ \frac{2}{\pi^2} & \frac{1}{\pi^2} & 0 \\ \frac{3}{16} & \frac{17}{128} & \frac{1}{4} \end{pmatrix}$$

and its spectral radius $r([A]) \approx 0.4479 < 1$ (Some values here and later are calculated using the mathematical software *Mathematica*). Therefore, (C_2) and (C_3) are satisfied. We choose $[a, b] = [1/4, 3/4]$ and note that $\gamma = 1/16$,

$$\kappa_1(s) = \begin{cases} \frac{-74+2\pi^4(37-30s)s+23\pi^2 s^2+74\cos(\pi s)}{4\pi^2(-151+114\pi^2)}, & 0 \le s \le \frac{1}{4}, \\ \frac{-592+\pi^2(3-36s+328s^2-192s^3)+\pi^4(-3+628s-624s^2+192s^3)+592\cos(\pi s)}{32\pi^2(-151+114\pi^2)}, & \frac{1}{4} < s \le \frac{1}{2}, \\ \frac{-1776+\pi^2(41-300s+1368s^2-832s^3)+\pi^4(-41+2076s-2256s^2+832s^3)+1776\cos(\pi s)}{96\pi^2(-151+114\pi^2)}, & \frac{1}{2} < s \le \frac{3}{4}, \\ \frac{-888+\pi^2(-47+120s+324s^2-256s^3)+\pi^4(47+768s-768s^2+256s^3)+888\cos(\pi s)}{48\pi^2(-151+114\pi^2)}, & \frac{3}{4} < s \le 1, \end{cases}$$

$$\kappa_2(s) = \begin{cases} \frac{-114+\pi^2(151-87s)s+114\cos(\pi s)}{\pi^2(-151+114\pi^2)}, & 0 \le s \le \frac{1}{4}, \\ \frac{-1824+\pi^2(-3+2452s-1536s^2+192s^3)+1824\cos(\pi s)}{16\pi^2(-151+114\pi^2)}, & \frac{1}{4} < s \le \frac{1}{2}, \\ \frac{-5472+\pi^2(-41+7548s-4992s^2+832s^3)+5472\cos(\pi s)}{48\pi^2(-151+114\pi^2)}, & \frac{1}{2} < s \le \frac{3}{4}, \\ \frac{-2736+\pi^2(47+3504s-2136s^2+256s^3)+2736\cos(\pi s)}{24\pi^2(-151+114\pi^2)}, & \frac{3}{4} < s \le 1, \end{cases}$$

$$\kappa_3(s) =$$
$$\begin{cases} \frac{-794+157\pi^2 s^2-2\pi^4 s(-397+288s)+794\cos(\pi s)}{32\pi^2(-151+114\pi^2)}, & 0 \le s \le \frac{1}{4}, \\ \frac{-12704+4\pi^4(-3+3212s-2448s^2+192s^3)+\pi^2(-5+60s+2272s^2+320s^3)+12704\cos(\pi s)}{512\pi^2(-151+114\pi^2)}, & \frac{1}{4} < s \le \frac{1}{2}, \\ \frac{-38112+\pi^2(3153-18828s+44832s^2-24384s^3)+4\pi^4(-649+13476s-15024s^2+5696s^3)+38112\cos(\pi s)}{1536\pi^2(-151+114\pi^2)}, & \frac{1}{2} < s \le \frac{3}{4}, \\ \frac{-19056+\pi^2(-1029+1008s+8520s^2-6016s^3)+256\pi^4(4+69s-69s^2+23s^3)+19056\cos(\pi s)}{768\pi^2(-151+114\pi^2)}, & \frac{3}{4} < s \le 1, \end{cases}$$

and hence

$$\int_0^1 \Phi_0(s)ds = \frac{-483264+96736\pi^2+79071\pi^4}{57984\pi^2-43776\pi^4},$$

$$\int_0^1 \Phi_1(s)ds = \frac{50880+539\pi^2-16421\pi^4}{3072\pi^2(-151+114\pi^2)},$$

$$\int_0^1 \Phi_2(s)ds = \frac{21888+5371\pi^2-10944\pi^4}{28992\pi^2-21888\pi^4)},$$

$$\frac{1}{m} = \max\left\{\int_0^1 \Phi_0(s)ds, \int_0^1 \Phi_1(s)ds, \int_0^1 \Phi_2(s)ds\right\} = \frac{21888+5371\pi^2-10944\pi^4}{28992\pi^2-21888\pi^4)},$$

$$\int_{1/4}^{3/4} \Phi_0(s)ds = \frac{-483264+103739\pi^2+89136\pi^4}{6144\pi^2(-151+114\pi^2)},$$

$$\int_{1/4}^{3/4} \Phi_1(s)ds = \frac{25440+262\pi^2-9013\pi^4}{57984\pi^2-43776\pi^4},$$

$$\int_{1/4}^{3/4} \Phi_2(s)ds = \frac{10944+2225\pi^2-5472\pi^4}{28992\pi^2-21888\pi^4)},$$

$$\frac{1}{M} = \min\left\{\int_{1/4}^{3/4} \Phi_0(s)ds, \int_{1/4}^{3/4} \Phi_1(s)ds, \int_{1/4}^{3/4} \Phi_2(s)ds\right\} = \frac{-483264+103739\pi^2+89136\pi^4}{6144\pi^2(-151+114\pi^2)},$$

$m \approx 1.8624, M \approx 6.4045.$

Let $f(t,x_0,x_1,x_2,x_3) = d(x_0^{k_0} + (-x_1)^{k_1} + (-x_2)^{k_2} + x_3^2)$ for $(t,x_0,x_1,x_2,x_3) \in [0,1] \times [0,\infty) \times (-\infty,0]^3$, here $k_i > 1$ $(i = 0,1,2)$, and $d > 0$ is a constant which is determined by the next step. Clearly (C_1) holds. For a given $r_1 > 0$, choosing $d_0 > 0$ and d sufficiently small such that

$$d\left(r_1^{k_0} + r_1^{k_1} + r_1^{k_2} + \left(\left(d_0 + \left(r_1^{k_0} + r_1^{k_1} + r_1^{k_2}\right) d\right) \exp(dr_1)\right)^2\right) < mr_1,$$

we have that (2.5) and (2.7) are satisfied with $g(x_0,x_1,x_2) = x_0^{k_0} + (-x_1)^{k_1} + (-x_2)^{k_2}$. Choosing r_2 large enough such that $r_2 > r_1/\gamma$ and $r_2^{k_i-1} > Md^{-1}\gamma^{-k_i}$ $(i = 0,1,2)$, we have that for $(t,x_0,x_1,x_2,x_3) \in W_{1,i}$ (see Theorem 2.5),

$$f(t,x_0,x_1,x_2,x_3) \geq d\,(r_2\gamma)^{k_i} > Mr_2 \qquad (i = 0,1,2),$$

i.e., (2.8) is satisfied. By Theorem 2.5 the BVP (2.11) has at least one positive solution. Of course 0 is also a solution of this problem. Especially, if $r_1 = 0.01, d_0 = 0.01$ and $k_0 = k_1 = k_2 = 2$, we may take $d = 20$.

Example 2.7. Consider BVP (2.11) with $f(t,x_0,x_1,x_2,x_3) = d\left(x_0^{k_0} + (-x_1)^{k_1} + (-x_2)^{k_2} + x_3^2\right)$ for $(t,x_0,x_1,x_2,x_3) \in [0,1] \times [0,\infty) \times (-\infty,0]^3$, here $k_i \in (0,1)$ $(i = 0,1,2)$, and $d > 0$ is a constant which is determined by the next step. Clearly (C_1) holds. For a given $r_2 > 0$, choosing $d_0 > 0$ and d sufficiently small such that

$$d\left(r_2^{k_0} + r_2^{k_1} + r_2^{k_2} + \left(\left(d_0 + \left(r_2^{k_0} + r_2^{k_1} + r_2^{k_2}\right) d\right) \exp(dr_2)\right)^2\right) < mr_2,$$

we have that (2.5) and (2.9) are satisfied with $g(x_0,x_1,x_2) = x_0^{k_0} + (-x_1)^{k_1} + (-x_2)^{k_2}$. Choosing r_1 small enough such that $r_1 < mr_2M^{-1}$ and $r_1^{1-k_i} < d\gamma^{k_i}M^{-1}$ $(i = 0,1,2)$, we have that for $(t,x_0,x_1,x_2,x_3) \in W_{2,i}$ (see Theorem 2.5),

$$f(t,x_0,x_1,x_2,x_3) \geq d\,(r_1\gamma)^{k_i} > Mr_1 \qquad (i = 0,1,2),$$

i.e., (2.10) is satisfied. By Theorem 2.5 the BVP (2.11) has at least one positive solution. Of course 0 is also a solution of this problem. Especially, if $r_2 = 1, d_0 = 0.01$ and $k_0 = k_1 = k_2 = 1/2$, we may take $d = 7/20$.

Remark 2.8. For f as in Example 2.7, if $x_0 = x_1 = x_2 = 0, x_3 \to -\infty$,

$$f(t,x_0,x_1,x_2,x_3) \leq a_0x_0 - a_1x_1 - a_2x_2 - a_3x_3 + C_0$$

does not hold; if $x_0 \to 0^+, x_1 = x_2 = x_3 = 0$,

$$f(t,x_0,x_1,x_2,x_3) \leq b_0x_0 - b_1x_1 - b_2x_2 - b_3x_3$$

does not hold, where $a_i, b_i (i = 0,1,2,3)$ and C_0 are positive constants. Therefore, the conditions in [9, Theorem 2.1, Theorem 2.2] are not satisfied and the results in [9] can not be applied.

3 Positive solutions to the BVP (1.2)

For BVP (1.2)

$$\begin{cases} -u^{(4)}(t) = \widetilde{f}(t, u(t), u'(t), u''(t), u'''(t)), \quad t \in [0,1], \\ u(0) = \beta_1[u], \ u'(0) = \beta_2[u], \ u''(0) = \beta_3[u], \ u'''(1) = 0, \end{cases}$$

we make the following assumptions:

$(\widetilde{C}_1)$ $\widetilde{f} : [0,1] \times [0,\infty)^4 \to [0,\infty)$ is continuous;

$(\widetilde{C}_2)$ B_i is of bounded variation, moreover

$$\widetilde{\mathcal{K}}_i(s) := \int_0^1 \widetilde{G}_0(t,s) dB_i(t) \geq 0, \qquad \forall s \in [0,1] \quad (i = 1,2,3),$$

where

$$\widetilde{G}_0(t,s) = \begin{cases} \frac{1}{6}t^3, & 0 \leq t \leq s \leq 1, \\ \frac{1}{6}s(3t^2 - 3ts + s^2), & 0 \leq s \leq t \leq 1; \end{cases}$$

$(\widetilde{C}_3)$ The 3×3 matrix $[B]$ is positive whose (i,j)th entry is $\beta_i[\delta_j]$, where $\delta_1(t) = 1$, $\delta_2(t) = t$ and $\delta_3(t) = \frac{1}{2}t^2$ are the solutions of $u^{(4)} = 0$ respectively subject to boundary conditions:

$$u'(0) = 1, \quad u''(0) = 0, \quad u(1) = 0, \quad u'''(1) = 0;$$

$$u'(0) = 0, \quad u''(0) = 1, \quad u(1) = 0, \quad u'''(1) = 0;$$

$$u'(0) = 0, \quad u''(0) = 0, \quad u(1) = 1, \quad u'''(1) = 0.$$

Furthermore assume that its spectral radius $r([B]) < 1$.

Define the operator $\widetilde{S}$ as

$$(\widetilde{S}u)(t) = \int_0^1 G_2(t,s) \widetilde{f}(s, u(s), u'(s), u''(s), u'''(s)) ds,$$

where

$$G_2(t,s) = \langle (I - [B])^{-1} \widetilde{\mathcal{K}}(s), \delta(t) \rangle + \widetilde{G}_0(t,s) = \sum_{i=1}^{3} \widetilde{\kappa}_i(s) \delta_i(t) + \widetilde{G}_0(t,s),$$

$\langle (I - [B])^{-1} \widetilde{\mathcal{K}}(s), \delta(t) \rangle$ is the inner product in $\mathbb{R}^3$, $\widetilde{\kappa}_i(s)$ is the ith component of $(I - [B])^{-1} \widetilde{\mathcal{K}}(s)$.

Lemma 3.1. *If $(\widetilde{C}_2)$ and $(\widetilde{C}_3)$ hold, then $\widetilde{\kappa}_i(s) \geq 0$ $(i = 1,2,3)$ and, for $t, s \in [0,1]$,*

$$\widetilde{c}_0(t) \widetilde{\Phi}_0(s) \leq G_2(t,s) \leq \widetilde{\Phi}_0(s), \tag{3.1}$$

where

$$\widetilde{\Phi}_0(s) = \sum_{i=1}^{3} \widetilde{\kappa}_i(s) + \frac{1}{6}s^3 + \frac{1}{2}s(1-s), \qquad \widetilde{c}_0(t) = \frac{1}{2}t^3,$$

and

$$\widetilde{c}_1(t) \widetilde{\Phi}_1(s) \leq \frac{\partial G_2(t,s)}{\partial t} \leq \widetilde{\Phi}_1(s), \qquad \widetilde{c}_2(t) \widetilde{\Phi}_2(s) \leq \frac{\partial^2 G_2(t,s)}{\partial t^2} \leq \widetilde{\Phi}_2(s), \tag{3.2}$$

where

$$\frac{\partial G_2(t,s)}{\partial t} = \widetilde{\kappa}_2(s) + t\widetilde{\kappa}_3(s) + \frac{1}{2}\begin{cases} t^2, & 0 \le t \le s \le 1, \\ s(2t-s), & 0 \le s \le t \le 1, \end{cases}$$

$$\frac{\partial^2 G_2(t,s)}{\partial t^2} = \widetilde{\kappa}_3(s) + \begin{cases} t, & 0 \le t \le s \le 1, \\ s, & 0 \le s \le t \le 1, \end{cases}$$

$$\widetilde{\Phi}_1(s) = \sum_{i=2}^{3} \widetilde{\kappa}_i(s) + \frac{1}{2}s(2-s), \qquad \widetilde{c}_1(t) = t^2, \qquad \widetilde{\Phi}_2(s) = \widetilde{\kappa}_3(s) + s, \qquad \widetilde{c}_2(t) = t.$$

Proof. $\widetilde{\kappa}_i(s) \ge 0$ $(i = 1,2,3)$ by hypotheses $(\widetilde{C}_2)$ and $(\widetilde{C}_3)$. For $0 \le s \le t \le 1$, $\frac{\partial}{\partial t}\widetilde{G}_0(t,s) = \frac{1}{2}s(2t-s) \ge 0$ which implies that

$$\widetilde{G}_0(t,s) \le \widetilde{G}_0(1,s) = \frac{1}{6}s^3 + \frac{1}{2}s(1-s);$$

For $0 \le t < s \le 1$, $\frac{\partial}{\partial t}\widetilde{G}_0(t,s) = \frac{1}{2}t^2 \ge 0$ which implies that

$$\widetilde{G}_0(t,s) \le \widetilde{G}_0(s,s) = \frac{1}{6}s^3.$$

Then $\widetilde{G}_0(t,s) \le \frac{1}{6}s^3 + \frac{1}{2}s(1-s)$, $\forall (t,s) \in [0,1] \times [0,1]$.

Now we find the best function $\widetilde{C}_0(t)$ such that $\widetilde{G}_0(t,s) \ge \widetilde{C}_0(t)\left(\frac{1}{6}s^3 + \frac{1}{2}s(1-s)\right)$, $\forall (t,s) \in [0,1] \times [0,1]$.

For $0 \le s \le t \le 1$, this is

$$\frac{1}{6}s(3t^2 - 3ts + s^2) \ge \widetilde{C}_0(t)\left(\frac{1}{6}s^3 + \frac{1}{2}s(1-s)\right),$$

thus

$$\widetilde{C}_0(t) \le \frac{3t^2 - 3ts + s^2}{3 - 3s + s^2}.$$

Denote

$$\widetilde{g}_1(t,s) = \frac{3t^2 - 3ts + s^2}{3 - 3s + s^2},$$

from

$$\frac{\partial}{\partial s}\widetilde{g}_1(t,s) = \frac{3(t-1)(s^2 - 2s(1+t) + 3t)}{(3 - 3s + s^2)^2} \le 0$$

it follows that $\widetilde{C}_0(t) \le \widetilde{g}_1(t,t) = \frac{t^2}{3-3t+t^2}$.

For $0 \le t < s \le 1$, this is

$$\frac{1}{6}t^3 \ge \widetilde{C}_0(t)\left(\frac{1}{6}s^3 + \frac{1}{2}s(1-s)\right),$$

thus

$$\widetilde{C}_0(t) \le \frac{t^3}{s(3 - 3s + s^2)}.$$

Denote

$$\widetilde{g}_2(t,s) = \frac{1}{s(3 - 3s + s^2)},$$

from

$$\frac{\partial}{\partial s}\widetilde{g}_2(t,s) = -\frac{3(1-s)^2}{s^2(3 - 3s + s^2)^2} \le 0$$

it follows that $\widetilde{C}_0(t) \le t^3 \widetilde{g}_2(t,1) = t^3$.

Therefore
$$\widetilde{C}_0(t) = \min\left\{\frac{t^2}{3-3t+t^2},\ t^3\right\} = t^3.$$

Since
$$\frac{1}{2}t^3\sum_{i=1}^{3}\widetilde{\kappa}_i(s) \le \frac{1}{2}t^2\sum_{i=1}^{3}\widetilde{\kappa}_i(s) \le \sum_{i=1}^{3}\widetilde{\kappa}_i(s)\delta_i(t) \le \sum_{i=1}^{3}\widetilde{\kappa}_i(s),$$
$$\frac{1}{2}t^3\left(\frac{1}{6}s^3+\frac{1}{2}s(1-s)\right) \le t^3\left(\frac{1}{6}s^3+\frac{1}{2}s(1-s)\right) \le \widetilde{G}_0(t,s) \le \frac{1}{6}s^3+\frac{1}{2}s(1-s),$$
we know that (3.1) holds. As for (3.2), it comes directly from the inequalities
$$t^2\sum_{i=2}^{3}\widetilde{\kappa}_i(s) \le t\sum_{i=2}^{3}\widetilde{\kappa}_i(s) \le \sum_{i=1}^{3}\widetilde{\kappa}_i(s)\delta_i'(t) \le \sum_{i=2}^{3}\widetilde{\kappa}_i(s),$$
$$\frac{1}{2}t^2 s(2-s) \le \frac{\partial \widetilde{G}_0(t,s)}{\partial t} \le \frac{1}{2}s(2-s),$$
$$t\widetilde{\kappa}_3(s) \le \widetilde{\kappa}_3(s) = \sum_{i=1}^{3}\widetilde{\kappa}_i(s)\delta_i''(t), \qquad ts \le \frac{\partial^2 \widetilde{G}_0(t,s)}{\partial t^2} \le s$$
for $t,s \in [0,1]$. □

In $C^3[0,1]$ we define the cone
$$\widetilde{K} = \big\{u \in C^3[0,1] : u(t) \ge \widetilde{c}_0(t)\|u\|_C,\ u'(t) \ge \widetilde{c}_1(t)\|u'\|_C,\\ u''(t) \ge \widetilde{c}_2(t)\|u''\|_C,\ \forall t \in [0,1];\ u'''(1) = 0\big\}. \tag{3.3}$$

Lemma 3.2. *If $(\widetilde{C}_1)$–$(\widetilde{C}_3)$ hold, then $\widetilde{S} : \widetilde{K} \to \widetilde{K}$ is completely continuous and the positive solutions to BVP (1.2) are equivalent to the fixed points of $\widetilde{S}$ in $\widetilde{K}$.*

Proof. Because $G_2(t,s)$, and the first- and second-order derivatives are continuous, the third-order derivative is integrable in s, from Lemma 3.1 it is easy to prove that $\widetilde{S} : \widetilde{K} \to \widetilde{K}$ is continuous. Let F be a bounded set in $\widetilde{K}$, then there exists $M > 0$ such that $\|u\|_{C^3} \le M$ for all $u \in \widetilde{K}$. Denote
$$C = \max_{(t,x_0,x_1,x_2,x_3)\in[0,1]\times[0,M]^4} \widetilde{f}(t,x_0,x_1,x_2,x_3).$$

By $(\widetilde{C}_1)$ and Lemma 3.1 we have that $\forall u \in F$ and $t \in [0,1]$,
$$|(\widetilde{S}u)(t)| \le C\int_0^1 \widetilde{\Phi}_0(s)ds, \qquad |(\widetilde{S}u)'(t)| \le C\int_0^1 \left|\frac{\partial G_2(t,s)}{\partial t}\right|ds \le C\int_0^1 \widetilde{\Phi}_1(s)ds,$$
$$|(\widetilde{S}u)''(t)| \le C\int_0^1 \left|\frac{\partial^2 G_2(t,s)}{\partial t^2}\right|ds \le C\int_0^1 \widetilde{\Phi}_2(s)ds, \qquad |(\widetilde{S}u)'''(t)| \le C\int_0^1 \left|\frac{\partial^3 G_2(t,s)}{\partial t^3}\right|ds \le C,$$
then $\widetilde{S}(F)$ is uniformly bounded in $C^3[0,1]$. Moreover $\forall u \in F$ and $t_1, t_2 \in [0,1]$ with $t_1 < t_2$,
$$\begin{aligned}|(\widetilde{S}u)(t_1) - (\widetilde{S}u)(t_2)| &\le \int_0^1 |G_2(t_1,s) - G_2(t_2,s)|\widetilde{f}(s,u(s),u'(s),u''(s),u'''(s))ds\\ &\le C\int_0^1 |G_2(t_1,s) - G_2(t_2,s)|ds,\end{aligned}$$

$$|(\widetilde{S}u)'(t_1)-(\widetilde{S}u)'(t_2)| \leq \int_0^1 \left|\frac{\partial G_2}{\partial t}(t_1,s)-\frac{\partial G_2}{\partial t}(t_2,s)\right| \widetilde{f}(s,u(s),u'(s),u''(s),u'''(s))ds$$
$$\leq C\int_0^1 \left|\frac{\partial G_2}{\partial t}(t_1,s)-\frac{\partial G_2}{\partial t}(t_2,s)\right| ds,$$

$$|(\widetilde{S}u)''(t_1)-(\widetilde{S}u)''(t_2)| \leq \int_0^1 \left|\frac{\partial^2 G_2}{\partial t^2}(t_1,s)-\frac{\partial^2 G_2}{\partial t^2}(t_2,s)\right| \widetilde{f}(s,u(s),u'(s),u''(s),u'''(s))ds$$
$$\leq C\int_0^1 \left|\frac{\partial^2 G_2}{\partial t^2}(t_1,s)-\frac{\partial^2 G_2}{\partial t^2}(t_2,s)\right| ds,$$

$$|(\widetilde{S}u)'''(t_1)-(Su)'''(t_2)|$$
$$=\left|\int_{t_1}^1 \widetilde{f}(s,u(s),u'(s),u''(s),u'''(s))ds-\int_{t_2}^1 \widetilde{f}(s,u(s),u'(s),u''(s),u'''(s))ds\right| \leq C(t_2-t_1),$$

thus $\widetilde{S}(F)$ and $\widetilde{S}^{(i)}(F) =: \{v^{(i)} : v^{(i)}(t) = (\widetilde{S}u)^{(i)}(t), u \in F\}$ $(i = 1,2,3)$ are equicontinuous.

Therefore $\widetilde{S} : \widetilde{K} \to \widetilde{K}$ is completely continuous by the Arzelà–Ascoli theorem. Similar to [14], the positive solutions to BVP (1.2) are equivalent to the fixed points of $\widetilde{S}$ in $\widetilde{K}$. □

Lemma 3.3. *Suppose that $(\widetilde{C}_1)$–$(\widetilde{C}_3)$ hold, there exist constants $\widetilde{p}_0 > 0$, $\widetilde{p}_3 \geq 0$ and functions $\widetilde{p}_1, \widetilde{p}_2 \in L^1_+[0,1]$ such that for all $(t,x_0,x_1,x_2,x_3) \in [0,1]\times[0,\infty)^4$,*

$$\widetilde{f}(t,x_0,x_1,x_2,x_3) \leq \widetilde{p}_0 + \widetilde{p}_1(t)\widetilde{g}(x_0,x_1,x_2) + \widetilde{p}_2(t)x_3 + \widetilde{p}_3 x_3^2, \tag{3.4}$$

where $\widetilde{g} : [0,\infty)^3 \to [0,\infty)$ is continuous, non-decreasing in every variable. Let $\lambda \geq 1, \sigma \geq 0, r > 0$, define $\widetilde{D}_i := \int_0^1 \widetilde{p}_i(s)ds$ $(i = 1,2)$ and

$$\widetilde{Q}(r) := (\widetilde{p}_0 + \widetilde{g}(r,r,r)\widetilde{D}_1)\exp(\widetilde{D}_2)\exp(\widetilde{p}_3 r). \tag{3.5}$$

If $u \in \widetilde{K}$ with $\|u\|_{C^2} \leq r$ such that $\lambda u(t) = (\widetilde{S}u)(t) + \sigma$, then $\|u'''\|_C \leq \widetilde{Q}(r)$.

Let $[a,b]$ be a subset of $(0,1)$ and denote

$$\widetilde{\gamma} := \min\left\{\min_{t\in[a,b]}\widetilde{c}_0(t), \min_{t\in[a,b]}\widetilde{c}_1(t), \min_{t\in[a,b]}\widetilde{c}_2(t)\right\} = \frac{1}{2}a^3,$$
$$\frac{1}{\widetilde{m}} := \max\left\{\int_0^1 \widetilde{\Phi}_0(s)ds, \int_0^1 \widetilde{\Phi}_1(s)ds, \int_0^1 \widetilde{\Phi}_2(s)ds\right\},$$
$$\frac{1}{\widetilde{M}} := \min\left\{\int_a^b \widetilde{\Phi}_0(s)ds, \int_a^b \widetilde{\Phi}_1(s)ds, \int_a^b \widetilde{\Phi}_2(s)ds\right\},$$

where $\widetilde{c}_i(t)$ and $\widetilde{\Phi}_i(s)(i = 0,1,2)$ are provided in Lemma 3.1. Obviously, $\widetilde{\gamma} \in (0,1/2)$ and $\widetilde{m} < \widetilde{M}$.

Similar to the proof of Theorem 2.5, we have the next theorem.

Theorem 3.4. *Suppose that $(\widetilde{C}_1)$-$(\widetilde{C}_3)$ hold and $\widetilde{f}$ satisfies the growth assumption (3.4). The BVP (1.2) has at least one positive solution $u \in \widetilde{K}$ either of the following conditions $(\widetilde{F}_1)$, $(\widetilde{F}_2)$ holds, where $\widetilde{Q}$ is given by (3.5).*

$(\widetilde{F}_1)$ *There exist $0 < r_1 < r_2$ with $r_1 < r_2\widetilde{\gamma}$, such that*

$(\widetilde{F}_1 a)$ *for* $(t, x_0, x_1, x_2, x_3) \in [0,1] \times [0, r_1]^3 \times [0, \widetilde{Q}(r_1)]$,

$$\widetilde{f}(t, x_0, x_1, x_2, x_3) < \widetilde{m} r_1; \tag{3.6}$$

$(\widetilde{F}_1 b)$ *for* $(t, x_0, x_1, x_2, x_3) \in \widetilde{W}_1 := \widetilde{W}_{1,0} \cup \widetilde{W}_{1,1} \cup \widetilde{W}_{1,2}$,

$$\widetilde{f}(t, x_0, x_1, x_2, x_3) > \widetilde{M} r_2, \tag{3.7}$$

where

$$\begin{aligned}
\widetilde{W}_{1,0} &= [a,b] \times [r_2\widetilde{\gamma}, r_2] \times [0, r_2]^2 \times [0, \widetilde{Q}(r_2)],\\
\widetilde{W}_{1,1} &= [a,b] \times [0, r_2] \times [r_2\widetilde{\gamma}, r_2] \times [0, r_2] \times [0, \widetilde{Q}(r_2)],\\
\widetilde{W}_{1,2} &= [a,b] \times [0, r_2]^2 \times [r_2\widetilde{\gamma}, r_2] \times [0, \widetilde{Q}(r_2)].
\end{aligned}$$

$(\widetilde{F}_2)$ *There exist* $0 < r_1 < r_2$ *with* $\widetilde{M} r_1 < \widetilde{m} r_2$, *such that*

$(\widetilde{F}_2 a)$ *for* $(t, x_0, x_1, x_2, x_3) \in [0,1] \times [0, r_2]^3 \times [0, \widetilde{Q}(r_2)]$,

$$\widetilde{f}(t, x_0, x_1, x_2, x_3) < \widetilde{m} r_2; \tag{3.8}$$

$(\widetilde{F}_2 b)$ *for* $(t, x_0, x_1, x_2, x_3) \in \widetilde{W}_2 := \widetilde{W}_{2,0} \cup \widetilde{W}_{2,1} \cup \widetilde{W}_{2,2}$,

$$\widetilde{f}(t, x_0, x_1, x_2, x_3) > \widetilde{M} r_1, \tag{3.9}$$

where

$$\begin{aligned}
\widetilde{W}_{2,0} &= [a,b] \times [r_1\widetilde{\gamma}, r_1] \times [0, r_1]^2 \times [0, \widetilde{Q}(r_1)],\\
\widetilde{W}_{2,1} &= [a,b] \times [0, r_1] \times [r_1\widetilde{\gamma}, r_1] \times [0, r_1] \times [0, \widetilde{Q}(r_1)],\\
\widetilde{W}_{2,2} &= [a,b] \times [0, r_1]^2 \times [r_1\widetilde{\gamma}, r_1] \times [0, \widetilde{Q}(r_1)].
\end{aligned}$$

Example 3.5. Consider

$$\begin{cases}
-u^{(4)}(t) = \widetilde{f}(t, u(t), u'(t), u''(t), u'''(t)), \quad t \in [0,1],\\
u(0) = \frac{1}{2}u(\frac{1}{4}) - \frac{1}{160}u(\frac{3}{4}), \quad u'(0) = \int_0^1 (t - \frac{1}{8})u(t)dt,\\
u''(0) = \frac{1}{2}u(\frac{1}{2}) - \frac{1}{14}u(\frac{3}{4}), \quad u'''(1) = 0,
\end{cases} \tag{3.10}$$

thus $\beta_1[u] = \frac{1}{2}u(\frac{1}{4}) - \frac{1}{160}u(\frac{3}{4})$, $\beta_2[u] = \int_0^1 (t - \frac{1}{8})u(t)dt$, $\beta_3[u] = \frac{1}{2}u(\frac{1}{2}) - \frac{1}{14}u(\frac{3}{4})$. Then

$$\begin{aligned}
0 \le \widetilde{\mathcal{K}}_1(s) &= \tfrac{1}{2}\widetilde{G}_0\left(\tfrac{1}{4}, s\right) - \tfrac{1}{160}\widetilde{G}_0\left(\tfrac{3}{4}, s\right)\\
&= \begin{cases}
\frac{79}{960}s^3 - \frac{77}{1280}s^2 + \frac{71}{5120}s, & 0 \le s \le \frac{1}{4},\\
\frac{1}{768} - \frac{9}{5120}s + \frac{3}{1280}s^2 - \frac{1}{960}s^3, & \frac{1}{4} < s \le \frac{3}{4},\\
\frac{53}{61440}, & \frac{3}{4} < s \le 1,
\end{cases}
\end{aligned}$$

$$\begin{aligned}
\widetilde{\mathcal{K}}_2(s) &= \frac{1}{6}\int_0^s \left(t - \frac{1}{8}\right) t^3 dt + \frac{1}{6}\int_s^1 \left(t - \frac{1}{8}\right) s(3t^2 - 3ts + s^2)dt\\
&= \frac{1}{960}s\big(100 - 130s + 60s^2 + 5s^3 - 8s^4\big) \ge 0 \qquad (0 \le s \le 1),
\end{aligned}$$

$$
\begin{aligned}
0 \le \widetilde{\mathcal{K}}_3(s) &= \tfrac{1}{2}\widetilde{G}_0\left(\tfrac{1}{2}, s\right) - \tfrac{1}{14}\widetilde{G}_0\left(\tfrac{3}{4}, s\right) \\
&= \begin{cases} \frac{1}{14}s^3 - \frac{11}{112}s^2 + \frac{19}{448}s, & 0 \le s \le \frac{1}{2}, \\ \frac{1}{96} - \frac{9}{448}s + \frac{3}{112}s^2 - \frac{1}{84}s^3, & \frac{1}{2} < s \le \frac{3}{4}, \\ \frac{29}{5376}, & \frac{3}{4} < s \le 1, \end{cases}
\end{aligned}
$$

the 3×3 matrix

$$
[B] = \begin{pmatrix} \beta_1[\delta_1] & \beta_1[\delta_2] & \beta_1[\delta_3] \\ \beta_2[\delta_1] & \beta_2[\delta_2] & \beta_2[\delta_3] \\ \beta_3[\delta_1] & \beta_3[\delta_2] & \beta_3[\delta_3] \end{pmatrix} = \begin{pmatrix} \frac{79}{160} & \frac{77}{640} & \frac{71}{5120} \\ \frac{3}{8} & \frac{13}{48} & \frac{5}{48} \\ \frac{3}{7} & \frac{11}{56} & \frac{19}{448} \end{pmatrix}
$$

and its spectral radius $r([B]) \approx 0.6600 < 1$. Therefore, $(\widetilde{C}_2)$ and $(\widetilde{C}_3)$ hold. We choose $[a, b] = [1/4, 3/4]$ and note that $\widetilde{\gamma} = 1/128$,

$$
\widetilde{\kappa}_1(s) = \begin{cases} \frac{s(176400-459360s+504520s^2+4785s^3-7656s^4)}{2252880}, & 0 \le s \le \frac{1}{4}, \\ \frac{6875+93900s-129360s^2+64520s^3+4785s^4-7656s^5}{2252880}, & \frac{1}{4} < s \le \frac{1}{2}, \\ \frac{17425+165750s-214620s^2+99640s^3+9570s^4-15312s^5}{4505760}, & \frac{1}{2} < s \le \frac{3}{4}, \\ \frac{22025+3828s(100-130s+60s^2+5s^3-8s^4)}{9011520}, & \frac{3}{4} < s \le 1, \end{cases}
$$

$$
\widetilde{\kappa}_2(s) = \begin{cases} \frac{s(3986640-6524820s+4630400s^2+171615s^3-274584s^4)}{19900440}, & 0 \le s \le \frac{1}{4}, \\ \frac{36175+3552540s-4788420s^2+2315200s^3+171615s^4-274584s^5}{19900440}, & \frac{1}{4} < s \le \frac{1}{2}, \\ \frac{155405+6606750s-8580180s^2+3965960s^3+343230s^4-549168s^5}{39800880}, & \frac{1}{2} < s \le \frac{3}{4}, \\ \frac{181885+137292s(100-130s+60s^2+5s^3-8s^4)}{79601760}, & \frac{3}{4} < s \le 1, \end{cases}
$$

$$
\widetilde{\kappa}_3(s) = \begin{cases} \frac{s(299565-649440s+553600s^2+6765s^3-10824s^4)}{2487555}, & 0 \le s \le \frac{1}{4}, \\ \frac{4325+247665s-441840s^2+276800s^3+6765s^4-10824s^5}{2487555}, & \frac{1}{4} < s \le \frac{1}{2}, \\ \frac{66715+146940s-186900s^2+89080s^3+13530s^4-21648s^5}{4975110}, & \frac{1}{2} < s \le \frac{3}{4}, \\ \frac{17900+1353s(100-130s+60s^2+5s^3-8s^4)}{2487555}, & \frac{3}{4} < s \le 1, \end{cases}
$$

and hence

$$
\int_0^1 \widetilde{\Phi}_0(s)ds = \frac{1481629721}{7641768960}, \quad \int_0^1 \widetilde{\Phi}_1(s)ds = \frac{3339971}{8547840}, \quad \int_0^1 \widetilde{\Phi}_2(s)ds = \frac{41265293}{79601760},
$$

$$
\frac{1}{\widetilde{m}} = \max\left\{\int_0^1 \widetilde{\Phi}_0(s)ds, \int_0^1 \widetilde{\Phi}_1(s)ds, \int_0^1 \widetilde{\Phi}_2(s)ds\right\} = \frac{41265293}{79601760},
$$

$$
\int_{1/4}^{3/4} \widetilde{\Phi}_0(s)ds = \frac{6666545149}{61134151680}, \quad \int_{1/4}^{3/4} \widetilde{\Phi}_1(s)ds = \frac{14676709}{68382720}, \quad \int_{1/4}^{3/4} \widetilde{\Phi}_2(s)ds = \frac{331536539}{1273628160},
$$

$$
\frac{1}{\widetilde{M}} = \min\left\{\int_{1/4}^{3/4} \widetilde{\Phi}_0(s)ds, \int_{1/4}^{3/4} \widetilde{\Phi}_1(s)ds, \int_{1/4}^{3/4} \widetilde{\Phi}_2(s)ds\right\} = \frac{331536539}{1273628160},
$$

$\widetilde{m} \approx 1.9290$, $\widetilde{M} \approx 9.1703$.

Let $\widetilde{f}(t,x_0,x_1,x_2,x_3) = \widetilde{d}\left(x_0^{k_0}+x_1^{k_1}+x_2^{k_2}+x_3^2\right)$, $(t,x_0,x_1,x_2,x_3) \in [0,1]\times[0,\infty)^4$, here $k_i>1$ $(i=0,1,2)$, and $\widetilde{d}>0$ is a constant which is determined by the next step. Clearly $(\widetilde{C}_1)$ holds. For a given $r_1>0$, choosing $\widetilde{d}_0>0$ and $\widetilde{d}$ sufficiently small such that

$$\widetilde{d}\left(r_1^{k_0}+r_1^{k_1}+r_1^{k_2}+\left(\left(\widetilde{d}_0+\left(r_1^{k_0}+r_1^{k_1}+r_1^{k_2}\right)\widetilde{d}\right)\exp\left(\widetilde{d}r_1\right)\right)^2\right)<\widetilde{m}r_1,$$

we have that (3.4) and (3.6) are satisfied with $\widetilde{g}(x_0,x_1,x_2)=x_0^{k_0}+x_1^{k_1}+x_2^{k_2}$. Choosing r_2 large enough such that $r_2>r_1/\widetilde{\gamma}$ and $r_2^{k_i-1}>\widetilde{M}\widetilde{d}^{-1}\widetilde{\gamma}^{-k_i}$ $(i=0,1,2)$, we have that for $(t,x_0,x_1,x_2,x_3)\in\widetilde{W}_{1,i}$ (see Theorem 3.4),

$$f(t,x_0,x_1,x_2,x_3)\geq\widetilde{d}\,(r_2\widetilde{\gamma})^{k_i}>\widetilde{M}r_2 \qquad (i=0,1,2),$$

i.e., (3.7) is satisfied. By Theorem 3.4 the BVP (3.10) has at least one positive solution. Of course 0 is also a solution of this problem. Especially, if $r_1=0.01,\widetilde{d}_0=0.01$ and $k_0=k_1=k_2=2$, we may take $\widetilde{d}=23$.

Example 3.6. Consider (3.10) with $\widetilde{f}(t,x_0,x_1,x_2,x_3)=\widetilde{d}(x_0^{k_0}+x_1^{k_1}+x_2^{k_2}+x_3^2)$, $(t,x_0,x_1,x_2,x_3)\in[0,1]\times[0,\infty)^4$, here $k_i\in(0,1)$ $(i=0,1,2)$, and $\widetilde{d}>0$ is a constant which is determined by the next step. Clearly $(\widetilde{C}_1)$ holds. For a given $r_2>0$, choosing $\widetilde{d}_0>0$ and $\widetilde{d}$ sufficiently small such that

$$\widetilde{d}\left(r_2^{k_0}+r_2^{k_1}+r_2^{k_2}+\left(\left(\widetilde{d}_0+\left(r_2^{k_0}+r_2^{k_1}+r_2^{k_2}\right)\widetilde{d}\right)\exp\left(\widetilde{d}r_2\right)\right)^2\right)<\widetilde{m}r_2,$$

we have that (3.4) and (3.8) are satisfied with $\widetilde{g}(x_0,x_1,x_2)=x_0^{k_0}+x_1^{k_1}+x_2^{k_2}$. Choosing r_1 small enough such that $r_1<\widetilde{m}r_2\widetilde{M}^{-1}$ and $r_1^{1-k_i}<\widetilde{d}\widetilde{\gamma}^{k_i}\widetilde{M}^{-1}$ $(i=0,1,2)$, we have that for $(t,x_0,x_1,x_2,x_3)\in\widetilde{W}_{2,i}$ (see Theorem 3.4),

$$f(t,x_0,x_1,x_2,x_3)\geq\widetilde{d}\,(r_1\widetilde{\gamma})^{k_i}>\widetilde{M}r_1 \qquad (i=0,1,2),$$

i.e., (3.9) is satisfied. By Theorem 3.4 the BVP (3.10) has at least one positive solution. Of course 0 is also a solution of this problem. Especially, if $r_2=1,\widetilde{d}_0=0.01$ and $k_0=k_1=k_2=1/2$, we may take $\widetilde{d}=7/20$.

Remark 3.7. For $\widetilde{f}$ as in Example 3.6, if $x_0=x_1=x_2=0, x_3\to+\infty$,

$$\widetilde{f}(t,x_0,x_1,x_2,x_3)\leq\widetilde{a}_0x_0+\widetilde{a}_1x_1+\widetilde{a}_2x_2+\widetilde{a}_3x_3+\widetilde{C}_0$$

does not hold; if $x_0\to0^+, x_1=x_2=x_3=0$,

$$\widetilde{f}(t,x_0,x_1,x_2,x_3)\leq\widetilde{b}_0x_0+\widetilde{b}_1x_1+\widetilde{b}_2x_2+\widetilde{b}_3x_3$$

does not hold, where $\widetilde{a}_i,\widetilde{b}_i(i=0,1,2,3)$ and $\widetilde{C}_0$ are positive constants. Therefore, the conditions in [9, Theorem 3.1, Theorem 3.2] are not satisfied and the results in [9] can not be applied.

4 Positive Solutions to the BVP (1.3)

For BVP (1.3)

$$\begin{cases} u^{(4)}(t) = \overline{f}(t, u(t), u'(t), u''(t), u'''(t)), \quad t \in [0,1], \\ u(0) = u(1) = \eta_1[u], \ u''(0) + \eta_2[u] = 0, \ u''(1) + \eta_2[u] = 0, \end{cases}$$

we make the following assumptions:

($\overline{C}_1$) $\overline{f} : [0,1] \times [0,\infty) \times (-\infty,\infty) \times (-\infty,0] \times (-\infty,\infty) \to [0,\infty)$ is continuous;

($\overline{C}_2$) H_i is of bounded variation, moreover

$$\overline{\mathcal{K}}_i(s) := \int_0^1 \overline{G}_0(t,s) dH_i(t) \geq 0, \ \forall s \in [0,1] (i = 1,2),$$

where

$$\overline{G}_0(t,s) = \begin{cases} \frac{1}{6} t(1-s)(2s - t^2 - s^2), & 0 \leq t \leq s \leq 1, \\ \frac{1}{6} s(1-t)(2t - s^2 - t^2), & 0 \leq s \leq t \leq 1; \end{cases}$$

($\overline{C}_3$) The 2×2 matrix $[H]$ is positive whose (i,j)th entry is $\eta_i[\xi_j]$, where $\xi_1(t) = 1$ and $\xi_2(t) = \frac{1}{2} t(1-t)$ are the solutions of $u^{(4)} = 0$ respectively subject to boundary conditions:

$$u(0) = u(1) = 1, \quad u''(0) = u''(1) = 0;$$
$$u(0) = u(1) = 0, \quad u''(0) = u''(1) = -1.$$

Furthermore assume that its spectral radius $r([H]) < 1$.

Define the operator $\overline{S}$ as

$$(\overline{S}u)(t) = \int_0^1 G_3(t,s) \overline{f}(s, u(s), u'(s), u''(s), u'''(s)) ds,$$

where

$$G_3(t,s) = \langle (I - [H])^{-1} \overline{\mathcal{K}}(s), \xi(t) \rangle + \overline{G}_0(t,s) = \sum_{i=1}^{2} \overline{\kappa}_i(s) \xi_i(t) + \overline{G}_0(t,s),$$

$\langle (I - [H])^{-1} \overline{\mathcal{K}}(s), \xi(t) \rangle$ is the inner product in $\mathbb{R}^2$, $\overline{\kappa}_i(s)$ is the ith component of $(I - [H])^{-1} \overline{\mathcal{K}}(s)$.

Lemma 4.1. *If ($\overline{C}_2$) and ($\overline{C}_3$) hold, then $\overline{\kappa}_i(s) \geq 0$ $(i = 1,2)$,*

$$G_3(0,s) = G_3(1,s) = \overline{\kappa}_1(s), \qquad \frac{\partial^2 G_3(0,s)}{\partial t^2} = \frac{\partial^2 G_3(1,s)}{\partial t^2} = -\overline{\kappa}_2(s)$$

and for $t, s \in [0,1]$,

$$\overline{c}_0(t) \overline{\Phi}_0(s) \leq G_3(t,s) \leq \overline{\Phi}_0(s), \tag{4.1}$$

where

$$\overline{\Phi}_0(s) = \overline{\kappa}_1(s) + \frac{1}{8} \overline{\kappa}_2(s) + \widehat{\Phi}_0(s),$$

$$\overline{c}_0(t) = \begin{cases} \frac{3\sqrt{3}}{2} t(1 - t^2), & 0 \leq t \leq \frac{1}{2}, \\ \frac{3\sqrt{3}}{2} t(1-t)(2-t), & \frac{1}{2} < t \leq 1, \end{cases}$$

$$\widehat{\Phi}_0(s) = \begin{cases} \frac{\sqrt{3}}{27} s(1-s^2)^{3/2}, & 0 \le s \le \frac{1}{2}, \\ \frac{\sqrt{3}}{27}(1-s)s^{3/2}(2-s)^{3/2}, & \frac{1}{2} < s \le 1; \end{cases}$$

and

$$\overline{c}_1(t)\overline{\Phi}_1(s) \le -\frac{\partial^2 G_3(t,s)}{\partial t^2} \le \overline{\Phi}_1(s) \tag{4.2}$$

where

$$\frac{\partial^2 G_3(t,s)}{\partial t^2} = -\overline{\kappa}_2(s) - \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1, \end{cases}$$

$$\overline{\Phi}_1(s) = \overline{\kappa}_2(s) + s(1-s), \qquad \overline{c}_1(t) = \min\{t, 1-t\}.$$

Proof. $\overline{\kappa}_i(s) \ge 0$ by hypotheses $(\overline{C}_2)$ and $(\overline{C}_3)$, and the following inequality is proved in [15]

$$\overline{c}_0(t)\widehat{\Phi}_0(s) \le \overline{G}_0(t,s) \le \widehat{\Phi}_0(s).$$

From

$$G_3(t,s) = \sum_{i=1}^{2} \overline{\kappa}_i(s)\xi_i(t) + \overline{G}_0(t,s) \le \overline{\kappa}_1(s) + \frac{1}{8}\overline{\kappa}_2(s) + \widehat{\Phi}_0(s) = \overline{\Phi}_0(s)$$

and

$$\begin{aligned} G_3(t,s) &= \overline{\kappa}_1(s) + \frac{1}{8} \times 4t(1-t)\overline{\kappa}_2(s) + \overline{G}_0(t,s) \\ &\ge 4t(1-t)\left(\overline{\kappa}_1(s) + \frac{1}{8}\overline{\kappa}_2(s)\right) + \overline{c}_0(t)\widehat{\Phi}_0(s) \\ &\ge \frac{9\sqrt{3}}{4} t(1-t)\left(\overline{\kappa}_1(s) + \frac{1}{8}\overline{\kappa}_2(s)\right) + \overline{c}_0(t)\widehat{\Phi}_0(s) \ge \overline{c}_0(t)\overline{\Phi}_0(s), \end{aligned}$$

it follows that (4.1) hold. As for (4.2), it can be checked easily. □

In $C^3[0,1]$ we define the cone

$$\overline{K} = \{u \in C^3[0,1] : u(t) \ge \overline{c}_0(t)\|u\|_C,\ -u''(t) \ge \overline{c}_1(t)\|u''\|_C,\ \forall t \in [0,1]; \\ u(0) = u(1),\ u''(0) = u''(1)\}. \tag{4.3}$$

Lemma 4.2. *If $(\overline{C}_1)$–$(\overline{C}_3)$ hold, then $\overline{S} : \overline{K} \to \overline{K}$ is completely continuous and the positive solutions to BVP (1.3) are equivalent to the fixed points of $\overline{S}$ in $\overline{K}$.*

Lemma 4.3. *Suppose that $(\overline{C}_1)$-$(\overline{C}_3)$ hold, there exist constants $\overline{p}_0 > 0, \overline{p}_3 \ge 0$ and functions $\overline{p}_1, \overline{p}_2 \in L^1_+[0,1]$ such that for all $(t, x_0, x_1, x_2, x_3) \in [0,1] \times [0,\infty) \times (-\infty,\infty) \times (-\infty,0] \times (-\infty,\infty)$,*

$$\overline{f}(t, x_0, x_1, x_2, x_3) \le \overline{p}_0 + \overline{p}_1(t)\overline{g}(x_0, x_1, x_2) + \overline{p}_2(t)|x_3| + \overline{p}_3|x_3|^2, \tag{4.4}$$

where $\overline{g} : [0,\infty) \times (-\infty,\infty) \times (-\infty,0] \to [0,\infty)$ is continuous, non-decreasing in the first variable, even and non-decreasing in $[0,\infty)$ in the second variable, non-increasing in the third variable. Let $\lambda \ge 1, \sigma \ge 0, r > 0$, define $\overline{D}_i := \int_0^1 \overline{p}_i(s)ds$ $(i = 1,2)$ and

$$\overline{Q}(r) := (\overline{p}_0 + \overline{g}(r, r, -r)\overline{D}_1)\exp(\overline{D}_2)\exp(\overline{p}_3 r). \tag{4.5}$$

If $u \in \overline{K}$ with $\|u\|_{C^2} \le r$ such that $\lambda u(t) = (\overline{S}u)(t) + \sigma$, then $\|u'''\|_C \le \overline{Q}(r)$.

Proof. Since $u \in \overline{K}$, there exists $t_0 \in (0,1)$ such that $u'''(t_0) = 0$. From $\lambda u(t) = (\overline{S}u)(t) + \sigma$, we have that $\lambda u^{(4)}(t) = \overline{f}(t, u(t), u'(t), u''(t), u'''(t)) \geq 0$. Therefore, $u'''(t) \leq 0$ $(t \in [0, t_0])$, $u'''(t) \geq 0$ $(t \in [t_0, 1])$ and

$$\lambda u'''(t) = \int_{t_0}^{t} \overline{f}(s, u(s), u'(s), u''(s), u'''(s))ds \qquad (t \in [0,1]). \tag{4.6}$$

If $t \leq t_0$, from $\|u\|_{C^2} \leq r$ and (4.6) it follows that

$$\begin{aligned}|u'''(t)| \leq \lambda|u'''(t)| &= \left|\int_{t_0}^{t} \overline{f}(s, u(s), u'(s), u''(s), u'''(s))ds\right| \\ &\leq \int_{t}^{t_0} \left(\overline{p}_0 + \overline{p}_1(s)\overline{g}(u(s), u'(s), u''(s)) + \overline{p}_2(s)|u'''(s)| + \overline{p}_3|u'''(s)|^2\right) ds \\ &\leq \left(\overline{p}_0 + \overline{g}(r, r, -r)\overline{D}_1\right) + \int_{t}^{t_0} \left(\overline{p}_2(s)|u'''(s)| + \overline{p}_3|u'''(s)|^2\right) ds.\end{aligned}$$

Since

$$\int_0^{t_0} \overline{p}_3|u'''(s)|ds = -\int_0^{t_0} \overline{p}_3 u'''(s)ds = \overline{p}_3(u''(0) - u''(t_0)) \leq \overline{p}_3 r,$$

by Lemma 2.1 we deduce that

$$|u'''(t)| \leq (\overline{p}_0 + \overline{g}(r, r, -r)\overline{D}_1)\exp(\overline{D}_2)\exp(\overline{p}_3 r) = \overline{Q}(r), \qquad t \in [0, t_0].$$

If $t \geq t_0$, we change the variable from s to $\sigma = t_0 + 1 - s$. Denote $w(\sigma) = u(t_0 + 1 - \sigma)$ and then $w'(\sigma) = -u'(s)$, $w''(\sigma) = u''(s)$, $w'''(\sigma) = -u'''(s)$. Setting $\tau = t_0 + 1 - t$, from $\|u\|_{C^2} \leq r$ and (4.6) we have that

$$\begin{aligned}|w'''(\tau)| &= |-u'''(t)| \leq \lambda|u'''(t)| = \left|\int_{t_0}^{t} \overline{f}(s, u(s), u'(s), u''(s), u'''(s))ds\right| \\ &= \left|-\int_{1}^{\tau} \overline{f}(t_0 + 1 - \sigma, w(\sigma), -w'(\sigma), w''(\sigma), -w'''(\sigma))d\sigma\right| \\ &\leq \int_{\tau}^{1} (\overline{p}_0 + \overline{p}_1(t_0 + 1 - \sigma)\overline{g}(w(\sigma), -w'(\sigma), w''(\sigma)) + \overline{p}_2(t_0 + 1 - \sigma)|w'''(\sigma)| + \overline{p}_3|w'''(\sigma)|^2)d\sigma \\ &\leq (\overline{p}_0 + \overline{g}(r, r, -r)\overline{D}_1) + \int_{\tau}^{1} (\overline{p}_2(t_0 + 1 - \sigma)|w'''(\sigma)| + \overline{p}_3|w'''(\sigma)|^2)d\sigma.\end{aligned}$$

Since

$$\int_{t_0}^{1} \overline{p}_3|w'''(\sigma)|d\sigma = -\int_{t_0}^{1} \overline{p}_3 u'''(s)ds = \overline{p}_3(u''(t_0) - u''(1)) \leq \overline{p}_3 r,$$

by Lemma 2.1 we deduce that

$$|w'''(\tau)| \leq (\overline{p}_0 + \overline{g}(r, r, -r)\overline{D}_1)\exp(\overline{D}_2)\exp(\overline{p}_3 r) = \overline{Q}(r), \qquad \tau \in [t_0, 1],$$

i.e. $|u'''(t)| \leq \overline{Q}(r)$, $t \in [t_0, 1]$.

So the proof is complete. □

Let $[a, b]$ be a subset of $(0,1)$ and denote

$$\overline{\gamma} := \min\left\{\min_{t\in[a,b]} \overline{c}_0(t), \min_{t\in[a,b]} \overline{c}_1(t)\right\} = \min\left\{\frac{3\sqrt{3}}{2}a(1 - a^2), \frac{3\sqrt{3}}{2}b(1 - b)(2 - b), a, 1 - b\right\},$$

$$\frac{1}{\overline{m}} := \max\left\{\int_0^1 \overline{\Phi}_0(s)ds, \int_0^1 \overline{\Phi}_1(s)ds\right\}, \qquad \frac{1}{\overline{M}} := \min\left\{\int_a^b \overline{\Phi}_0(s)ds, \int_a^b \overline{\Phi}_1(s)ds\right\},$$

where $\overline{c}_i(t)$ and $\overline{\Phi}_i(s) (i = 0, 1)$ are provided in Lemma 4.1. Obviously, $\overline{\gamma} \in (0, 1/2)$ and $\overline{m} < \overline{M}$.

Theorem 4.4. *Suppose that* $(\overline{C}_1)$–$(\overline{C}_3)$ *hold and* $\overline{f}$ *satisfies the growth assumption* (4.4)*. The BVP* (1.3) *has at least one positive solution* $u \in \overline{K}$ *if either of the following conditions* $(\overline{F}_1)$, $(\overline{F}_2)$ *holds, where* $\overline{Q}$ *is given by* (4.5)*.*

$(\overline{F}_1)$ *There exist* $0 < r_1 < r_2$ *with* $r_1 < r_2\overline{\gamma}$*, such that*

$(\overline{F}_1 a)$ *for* $(t, x_0, x_1, x_2, x_3) \in [0,1] \times [0, r_1] \times [-r_1, r_1] \times [-r_1, 0] \times [-\overline{Q}(r_1), \overline{Q}(r_1)]$,

$$\overline{f}(t, x_0, x_1, x_2, x_3) < \overline{m} r_1; \tag{4.7}$$

$(\overline{F}_1 b)$ *for* $(t, x_0, x_1, x_2, x_3) \in \overline{W}_1 := \overline{W}_{1,0} \cup \overline{W}_{1,1}$,

$$\overline{f}(t, x_0, x_1, x_2, x_3) > \overline{M} r_2, \tag{4.8}$$

where

$$\overline{W}_{1,0} = [a,b] \times [r_2\overline{\gamma}, r_2] \times [-r_2, r_2] \times [-r_2, 0] \times [-\overline{Q}(r_2), \overline{Q}(r_2)],$$
$$\overline{W}_{1,1} = [a,b] \times [0, r_2] \times [-r_2, r_2] \times [-r_2, -r_2\overline{\gamma}] \times [-\overline{Q}(r_2), \overline{Q}(r_2)].$$

$(\overline{F}_2)$ *There exist* $0 < r_1 < r_2$ *with* $\overline{M} r_1 < \overline{m} r_2$*, such that*

$(\overline{F}_2 a)$ *for* $(t, x_0, x_1, x_2, x_3) \in [0,1] \times [0, r_2] \times [-r_2, r_2] \times [-r_2, 0] \times [-\overline{Q}(r_2), \overline{Q}(r_2)]$,

$$\overline{f}(t, x_0, x_1, x_2, x_3) < \overline{m} r_2; \tag{4.9}$$

$(\overline{F}_2 b)$ *for* $(t, x_0, x_1, x_2, x_3) \in \overline{W}_2 := \overline{W}_{2,0} \cup \overline{W}_{2,1}$,

$$\overline{f}(t, x_0, x_1, x_2, x_3) > \overline{M} r_1, \tag{4.10}$$

where

$$\overline{W}_{2,0} = [a,b] \times [r_1\overline{\gamma}, r_1] \times [-r_1, r_1] \times [-r_1, 0] \times [-\overline{Q}(r_1), \overline{Q}(r_1)],$$
$$\overline{W}_{2,1} = [a,b] \times [0, r_1] \times [-r_1, r_1] \times [-r_1, -r_1\overline{\gamma}] \times [-\overline{Q}(r_1), \overline{Q}(r_1)].$$

Proof. Suppose that $(\overline{F}_1)$ holds.

Define an open (relative to $\overline{K}$) set

$$U_{r_1} := \left\{ u \in \overline{K} : \|u\|_C < r_1, \|u''\|_C < r_1, \|u'''\|_C < \overline{Q}(r_1) + 1 \right\}.$$

If $u \in U_{r_1}$, it follows from $u(0) = u(1)$ that there is $\zeta \in (0,1)$ such that $u'(\zeta) = 0$, and $|u'(t)| = \left| \int_\zeta^t u''(s)ds \right| \le \|u''\|_C$ for all $t \in [0,1]$ which implies that $\|u'\|_C < r_1$. Thus U_{r_1} is bounded. Similar to the proof of Theorem 2.5, we have that the fixed point index $i(\overline{S}, U_{r_1}, \overline{K}) = 1$ by Lemma 1.1.

Define an open (relative to $\overline{K}$) set

$$V_{r_2} := \left\{ u \in \overline{K} : \min_{t \in [a,b]} u(t) < r_2\overline{\gamma},\ \min_{t \in [a,b]} (-u''(t)) < r_2\overline{\gamma},\ \|u'''\|_C < Q(r_2) + 1 \right\}.$$

If $u \in V_{r_2}$, it follows from (4.3) that $\|u\|_C < r_2$ and $\|u''\|_C < r_2$. Since $u(0) = u(1)$, there is $\tau \in (0,1)$ such that $u'(\tau) = 0$, and $|u'(t)| = \left| \int_\tau^t u''(s)ds \right| \le \|u''\|_C$ for all $t \in [0,1]$ which implies that $\|u'\|_C < r_2$. Thus V_{r_2} is bounded. Again similar to the proof of Theorem 2.5, we have that the fixed point index $i(\overline{S}, V_{r_2}, \overline{K}) = 0$ by Lemma 1.2.

It is obvious from $r_1 < r_2\overline{\gamma}$ that $\overline{U}_{r_1} \subset V_{r_2}$. So there is a fixed point of $\overline{S}$ in the set $V_{r_2} \setminus \overline{U}_{r_1}$ which is clearly nonzero and the positive solutions to BVP (1.3) by Lemma 4.2.

The other case is proved similarly. □

Example 4.5. Consider

$$\begin{cases} u^{(4)}(t) = \overline{f}(t, u(t), u'(t), u''(t), u'''(t)), \quad t \in [0,1], \\ u(0) = u(1) = \frac{1}{2}u(\frac{1}{4}) - \frac{1}{8}u(\frac{1}{2}), \\ u''(0) + \int_0^1 u(t)(t-\frac{1}{4})dt = 0, \ u''(1) + \int_0^1 u(t)(t-\frac{1}{4})dt = 0, \end{cases} \tag{4.11}$$

thus $\eta_1[u] = \frac{1}{2}u(\frac{1}{4}) - \frac{1}{8}u(\frac{1}{2})$, $\eta_2[u] = \int_0^1 u(t)(t-\frac{1}{4})dt$. Then

$$\begin{aligned} 0 \le \overline{\mathcal{K}}_1(s) &= \tfrac{1}{2}\overline{G}_0\left(\tfrac{1}{4}, s\right) - \tfrac{1}{8}\overline{G}_0\left(\tfrac{1}{2}, s\right) \\ &= \begin{cases} -\frac{5}{96}s^3 + \frac{5}{256}s, & 0 \le s \le \frac{1}{4}, \\ \frac{1}{32}s^3 - \frac{1}{16}s^2 + \frac{9}{256}s - \frac{1}{768}, & \frac{1}{4} < s \le \frac{1}{2}, \\ \frac{1}{96}s^3 - \frac{1}{32}s^2 + \frac{5}{256}s + \frac{1}{768}, & \frac{1}{2} < s \le 1, \end{cases} \end{aligned}$$

$$\overline{\mathcal{K}}_2(s) = \int_0^1 \overline{G}_0(t,s)\left(t - \frac{1}{4}\right)dt = \frac{1}{120}s^5 - \frac{1}{96}s^4 - \frac{1}{144}s^3 + \frac{13}{1440}s \ge 0 \qquad (0 \le s \le 1),$$

the 2×2 matrix

$$[H] = \begin{pmatrix} \eta_1[\xi_1] & \eta_1[\xi_2] \\ \eta_2[\xi_1] & \eta_2[\xi_2] \end{pmatrix} = \begin{pmatrix} \frac{3}{8} & \frac{1}{32} \\ \frac{1}{4} & \frac{1}{48} \end{pmatrix}$$

and its spectral radius $r([H]) = \frac{19}{48} < 1$. Therefore, $(\overline{C}_2)$ and $(\overline{C}_3)$ hold. We choose $[a,b] = [1/4, 3/4]$ and note that $\overline{\gamma} = 1/128$,

$$\overline{\kappa}_1(s) = \begin{cases} \frac{s(3577-9440s^2-60s^3+48s^4)}{111360}, & 0 \le s \le \frac{1}{4}, \\ \frac{-235+6397s-11280s^2+5600s^3-60s^4+48s^5}{111360}, & \frac{1}{4} < s \le \frac{1}{2}, \\ \frac{235+3577s-5640s^2+1840s^3-60s^4+48s^5}{111360}, & \frac{1}{2} < s \le 1, \end{cases}$$

$$\overline{\kappa}_2(s) = \begin{cases} \frac{s(97-160s^2-60s^3+48s^4)}{5568}, & 0 \le s \le \frac{1}{4}, \\ \frac{-3+133s-144s^2+32s^3-60s^4+48s^5}{5568}, & \frac{1}{4} < s \le \frac{1}{2}, \\ \frac{3+97s-72s^2-16s^3-60s^4+48s^5}{5568}, & \frac{1}{2} < s \le 1, \end{cases}$$

and hence

$$\int_0^1 \overline{\Phi}_0(s)ds = -\frac{5051}{712704} + \frac{2}{45\sqrt{3}}, \qquad \int_0^1 \overline{\Phi}_1(s)ds = \frac{15151}{89088},$$

$$\frac{1}{\overline{m}} = \max\left\{\int_0^1 \overline{\Phi}_0(s)ds, \int_0^1 \overline{\Phi}_1(s)ds\right\} = \frac{15151}{89088},$$

$$\int_{1/4}^{3/4} \overline{\Phi}_0(s)ds = -\frac{248861}{28508160} + \frac{5\sqrt{5}}{512}, \qquad \int_{1/4}^{3/4} \overline{\Phi}_1(s)ds = \frac{83365}{712704},$$

$$\frac{1}{\overline{M}} = \min\left\{\int_{1/4}^{3/4} \overline{\Phi}_0(s)ds, \int_{1/4}^{3/4} \overline{\Phi}_1(s)ds\right\} = -\frac{248861}{28508160} + \frac{5\sqrt{5}}{512},$$

$\overline{m} \approx 5.8800, \overline{M} \approx 76.2943$.

Let $\overline{f}(t, x_0, x_1, x_2, x_3) = \overline{d}\big(x_0^{k_0} + x_1^4 + (-x_2)^{k_1} + x_3^2\big)$, $(t, x_0, x_1, x_2, x_3) \in [0,1] \times [0,\infty) \times (-\infty,\infty) \times (-\infty, 0] \times (-\infty,\infty)$, here $k_i > 1$ $(i = 0,1)$, and $\overline{d} > 0$ is a constant which is determined by the next step. Clearly $(\overline{C}_1)$ holds. For a given $r_1 > 0$, choosing $\overline{d}_0 > 0$ and $\overline{d}$ sufficiently small such that

$$\overline{d}\left(r_1^{k_0} + r_1^4 + r_1^{k_1} + \left(\left(\overline{d}_0 + \left(r_1^{k_0} + r_1^4 + r_1^{k_1}\right)\overline{d}\right)\exp(\overline{d}r_1)\right)^2\right) < \overline{m}r_1,$$

we have that (4.4) and (4.7) are satisfied with $\overline{g}(x_0, x_1, x_2) = x_0^{k_0} + x_1^4 + (-x_2)^{k_1}$. Choosing r_2 large enough such that $r_2 > r_1/\overline{\gamma}$ and $r_2^{k_i-1} > \overline{M}\overline{d}^{-1}\overline{\gamma}^{-k_i}$ $(i = 0, 1)$, we have that for $(t, x_0, x_1, x_2, x_3) \in \overline{W}_{1,i}$,

$$\overline{f}(t, x_0, x_1, x_2, x_3) \geq \overline{d}\,(r_2\overline{\gamma})^{k_i} > \overline{M}r_2,$$

i.e., (4.8) is satisfied. By Theorem 4.4 the BVP (4.11) has at least one positive solution. Of course 0 is also a solution of this problem. Especially, if $r_1 = 0.01, \overline{d}_0 = 0.01$ and $k_0 = k_1 = 2$, we may take $\overline{d} = 48$.

Example 4.6. Consider BVP (4.11) with $\overline{f}(t, x_0, x_1, x_2, x_3) = \overline{d}\big(x_0^{k_0} + x_1^4 + (-x_2)^{k_1} + x_3^2\big)$ for $(t, x_0, x_1, x_2, x_3) \in [0, 1] \times [0, \infty) \times (-\infty, \infty) \times (-\infty, 0] \times (-\infty, \infty)$, here $k_i \in (0, 1)$ $(i = 0, 1)$, and $\overline{d} > 0$ is a constant which is determined by the next step. Clearly $(\overline{C}_1)$ holds. For a given $r_2 > 0$, choosing $\overline{d}_0 > 0$ and $\overline{d}$ sufficiently small such that

$$\overline{d}\left(r_2^{k_0} + r_2^4 + r_2^{k_1} + \left(\left(\overline{d}_0 + \left(r_2^{k_0} + r_2^4 + r_2^{k_1}\right)\overline{d}\right)\exp(\overline{d}r_2)\right)^2\right) < \overline{m}r_2,$$

we have that (4.4) and (4.9) are satisfied with $g(x_0, x_1, x_2) = x_0^{k_0} + x_1^4 + (-x_2)^{k_1}$. Choosing r_1 small enough such that $r_1 < \overline{m}r_2\overline{M}^{-1}$ and $r_1^{1-k_i} < \overline{d}\overline{\gamma}^{k_i}\overline{M}^{-1}$ $(i = 0, 1)$, we have that for $(t, x_0, x_1, x_2, x_3) \in \overline{W}_{2,i}$,

$$f(t, x_0, x_1, x_2, x_3) \geq \overline{d}\,(r_1\overline{\gamma})^{k_i} > \overline{M}r_1 \qquad (i = 0, 1),$$

i.e., (4.10) is satisfied. By Theorem 4.4 the BVP (4.11) has at least one positive solution. Of course 0 is also a solution of this problem. Especially, if $r_2 = 1, \overline{d}_0 = 0.01$ and $k_0 = k_1 = 1/2$, we may take $\overline{d} = 1/2$

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Global smooth linearization of nonautonomous contractions on Banach spaces

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Abstract. The main purpose of this paper is to establish a global smooth linearization result for two classes of nonautonomous dynamics with discrete time. More precisely, we consider a nonlinear and nonautonomous dynamics given by a two-sided sequence of maps as well as variational systems whose linear part is contractive, and under suitable assumptions we construct C^1 conjugacies between the original dynamics and its linear part. We stress that our dynamics acts on a arbitrary Banach space. Our arguments rely on related results dealing with autonomous dynamics.

Keywords: nonautonomous contractions, linearization.

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1 Introduction

One of the basic strategies when analysing a complex nonlinear system near its equilibrium is to linearize it, i.e. to study its linear part. Such a procedure is natural since linear systems are much easier to study. However, this strategy is useful only if one knows that a system and the associated linear part have similar behaviour near the equilibrium since only in that case we can conclude something meaningful about the original system by studying its linearization.

Many works have been devoted to the problem of formulating conditions under which the system and its linear part are C^r-conjugated (or equivalent). The first contributions deal with complex dynamics. Indeed, Poincaré [23] proved that an analytic diffeomorphism can be analytically conjugated to its linear part near a fixed point if all eigenvalues of the linear part lie inside the unit circle S^1 (or outside S^1) and satisfy the nonresonant condition. Later, Siegel [32], Brjuno [6] and Yoccoz [37] made contributions to the case of eigenvalues on S^1, in which the small divisor problem is involved.

In the context of real dynamics, the most important result is the famous Hartman–Grobman Theorem [17], which asserts that a C^1-diffeomorphism on $\mathbb{R}^n$ can be C^0-linearized near the hyperbolic fixed point. Later this result was generalized (with simplified proofs) to Banach spaces independently by Palis [21] and Pugh [24].

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It is well-known that in general the conjugacy in the Grobman–Hartman theorem is only locally Hölder continuous and that it may fail to be locally Lipschitz even for C^∞ dynamic (see [3] and references therein). However, C^0-linearization is often not sufficient since for example it can fail to distinguish a node from a focus as pointed out by van Strien [35].

Sternberg [33, 34] proved that C^k ($k \geq 1$) diffeomorphisms can be C^r linearized near the hyperbolic fixed points, where the integer $r > k$ depends on k and the nonresonant conditions. Hence, in order to obtain C^r-linearization, we need to require that our dynamics exhibits higher regularity. Later Belitskii [4, 5] gave conditions for C^k linearization of $C^{k,1}$ ($k \geq 1$) diffeomorphisms under appropriate nonresonant conditions. His results was partially generalized to infinite-dimensional setting in [16, 27, 40].

The C^1 linearization in the case when the linear part is a contraction (its spectrum is contained in the unit circle) was discussed in [15, 26, 27] in the infinite-dimensional setting as well as in [38, 39] in the finite-dimensional setting. We in particularly mention the recent paper by H. M. Rodrigues and J. Solà-Morales [29], in which the global C^1-linearization result for contractions on Banach spaces has been established.

We emphasize that all the above mentioned results deal with *autonomous* dynamics. The first contributions dealing with the linearization of nonautonomous dynamics with continuous time are due to Palmer [22] and to Aulbach and Wanner [2] for dynamics with discrete time. For more recent results we refer to the works of Jiang [18] and Lopez-Fenner and Pinto [19]. The first results related to C^r-linearization ($r \geq 1$) in the framework of nonautonomous dynamics were obtained only quite recently. More precisely, a Sternberg type theorem for nonautonomous dynamics with continuous time was established in [10]. The C^1-linearization for nonautonomous dynamics with discrete time was discussed in [13] (see also [14] for related results for continuous time). Moreover, results related to differentiable linearization of nonautonomous contractions were obtained in interesting papers [7, 8].

The main purpose of this paper is to formulate new conditions for C^1-linearization of nonautonomous contractions on an arbitrary Banach space. Our strategy follows very closely the arguments developed in [13] and consist of passing from nonautonomous to the associated autonomous dynamics acting on a larger space. Then, for the autonomous dynamics we apply results from [29] and after that we return back to the framework of our original nonautonomous dynamics. However, we emphasize that the results from [13] don't imply the results in the present paper. Indeed, the conditions for the linear part that ensure C^1-linearization are given in terms of the spectrum of the associated Mather operator which are difficult to verify in practice, while in the present paper the conditions are given directly in terms of the constants in the notion of an exponential contraction (we refer to Remark 2.15 for a detailed explanation). Furthermore, our results differ from those in [7, 8]. Indeed, besides considering discrete (and not continuous) dynamics on an arbitrary Banach space we also don't require the boundedness for the nonlinearities. Furthermore, we use completely different techniques from those developed in [7, 8].

Following similar ideas (but with substantial changes), we also discuss C^1-linearization of variational contractive dynamical systems with discrete time. We refer to [9, 30, 31] and references therein for a detailed explanation of the relevance of variational systems in the investigation of qualitative properties of nonautonomous dynamics (and to [1] for the exposition of the theory of closely related random dynamical systems).

We note that the main results of the present paper can be viewed as a generalization of Hartman's work [17] to nonautonomous contractions acting on Banach spaces.

2 Nonuniform exponential contractions

Throughout this paper, $X = (X, \|\cdot\|)$ will be an arbitrary Banach space and $B(X)$ will denote the space of all bounded linear operators on X. For a sequence $(A_n)_{n\in\mathbb{Z}} \subset B(X)$ of invertible operators, we set

$$\mathcal{A}(m,n) = \begin{cases} A_{m-1}\cdots A_n & \text{for } m > n; \\ \text{Id} & \text{for } m = n; \\ A_m^{-1}\cdots A_{n-1}^{-1} & \text{for } m < n. \end{cases}$$

Let us introduce a notion of a nonuniform exponential contraction.

Definition 2.1. We say that $(A_n)_{n\in\mathbb{Z}}$ admits a *nonuniform exponential contraction* if there exist $0 < \lambda \le \mu$ and a map $\mathcal{D}\colon \mathbb{Z} \to (0,\infty)$ such that

$$\|\mathcal{A}(m,n)\| \le \mathcal{D}(n)e^{-\lambda(m-n)}, \quad \text{for } m \ge n, \tag{2.1}$$

and

$$\|\mathcal{A}(m,n)\| \le \mathcal{D}(n)e^{\mu(n-m)}, \quad \text{for } m \le n. \tag{2.2}$$

The following example is taken from [11].

Example 2.2. Let $X = \mathbb{R}$ and consider a sequence $(A_n)_{n\in\mathbb{Z}}$ given by

$$A_n = e^{\omega+\epsilon[(-1)^n-\frac{1}{2}]} \quad n \in \mathbb{Z},$$

where $\omega < 0$ and $\epsilon \ge 0$ are some fixed numbers. Then, $(A_n)_{n\in\mathbb{Z}}$ admits a nonuniform exponential contraction with $\mathcal{D}$ being a scalar multiple of the map $n \mapsto e^{\epsilon|n|}$.

Moreover, the notion of a nonuniform exponential contraction is ubiquitous from the ergodic theory point of view (see Remark 3.13 for details).

A sequence $(A_n)_{n\in\mathbb{Z}}$ gives rise to a linear nonautonomous dynamics given by

$$x_{n+1} = A_n x_n, \quad n \in \mathbb{Z}. \tag{2.3}$$

Assume that $(f_n)_{n\in\mathbb{Z}}$ is a sequence of (nonlinear) maps $f_n\colon X \to X$, $n \in \mathbb{Z}$. We consider also the associated nonautonomous dynamics

$$x_{n+1} = A_n x_n + f_n(x_n), \quad n \in \mathbb{Z}. \tag{2.4}$$

The following is our first result. It gives conditions under which nonlinear dynamics (2.4) can be C^1-linearized. We stress that the proof of Theorem 2.3 will follow closely the proof of [13, Theorem 2.], but we include all details for the sake of completeness.

Theorem 2.3. *Assume that $(A_n)_{n\in\mathbb{Z}} \subset B(X)$ is a sequence of invertible operators that admits a nonuniform exponential contraction and assume that $0 < \lambda \le \mu$ and $\mathcal{D}\colon \mathbb{Z} \to (0,\infty)$ are such that* (2.1) *and* (2.2) *hold. Furthermore, suppose that*

$$\mu < 2\lambda. \tag{2.5}$$

In addition, assume that:

- *f_n is differentiable for each $n \in \mathbb{Z}$;*

- *for every $n \in \mathbb{Z}$,*
$$f_n(0) = 0 \quad \text{and} \quad Df_n(0) = 0; \tag{2.6}$$
- *there exists $\gamma > 0$ such that*
$$\|Df_n(x) - Df_n(y)\| \le \frac{\gamma}{\mathcal{D}(n+1)}\|x-y\|, \quad \text{for } n \in \mathbb{Z} \text{ and } x, y \in X; \tag{2.7}$$
- *there exists $\eta > 0$ such that*
$$\|Df_n(x)\| \le \frac{\eta}{\mathcal{D}(n+1)}, \quad \text{for } n \in \mathbb{Z} \text{ and } x \in X. \tag{2.8}$$

Then, if η is sufficiently small, there exists a sequence $(h_n)_{n\in\mathbb{Z}}$ of C^1 diffeomorphisms on X such that
$$h_{n+1} \circ (A_n + f_n) = A_n \circ h_n, \quad \text{for } n \in \mathbb{Z}. \tag{2.9}$$

Proof. Choose $\epsilon > 0$ such that
$$\epsilon < \lambda \quad \text{and} \quad \mu - 2\lambda + 3\epsilon < 0. \tag{2.10}$$
Observe that such ϵ can be chosen since (2.5) holds. For $n \in \mathbb{Z}$ and $x \in X$, we define
$$\|x\|_n = \sum_{m=n}^{\infty} \|\mathcal{A}(m,n)x\| e^{(\lambda-\epsilon)(m-n)} + \sum_{m=-\infty}^{n-1} \|\mathcal{A}(m,n)x\| e^{-(\mu+\epsilon)(n-m)}.$$
Observe that $\|x\| \le \|x\|_n$. Moreover, (2.1) and (2.2) imply that
$$\|x\|_n \le \mathcal{D}(n)\bigg(\sum_{m=n}^{\infty} e^{-\epsilon(m-n)} + \sum_{m=-\infty}^{n-1} e^{-\epsilon(n-m)}\bigg)\|x\|.$$
We conclude that
$$\|x\| \le \|x\|_n \le c\mathcal{D}(n)\|x\| \quad \text{for } x \in X \text{ and } n \in \mathbb{Z}, \tag{2.11}$$
where
$$c = \frac{1+e^{-\epsilon}}{1-e^{-\epsilon}} > 0. \tag{2.12}$$

Lemma 2.4. *We have that*
$$\|A_n x\|_{n+1} \le e^{-(\lambda-\epsilon)}\|x\|_n \quad \text{and} \quad \|A_n^{-1}x\|_n \le e^{\mu+\epsilon}\|x\|_{n+1},$$
for $x \in X$ and $n \in \mathbb{Z}$.

Proof of the lemma. We have that
$$\begin{aligned}
\|A_n x\|_{n+1} &= \sum_{m=n+1}^{\infty} \|\mathcal{A}(m,n+1)A_n x\| e^{(\lambda-\epsilon)(m-n-1)} + \sum_{m=-\infty}^{n} \|\mathcal{A}(m,n+1)A_n x\| e^{-(\mu+\epsilon)(n+1-m)} \\
&= \sum_{m=n+1}^{\infty} \|\mathcal{A}(m,n)x\| e^{(\lambda-\epsilon)(m-n-1)} + \sum_{m=-\infty}^{n} \|\mathcal{A}(m,n)x\| e^{-(\mu+\epsilon)(n+1-m)} \\
&= e^{-(\lambda-\epsilon)} \sum_{m=n}^{\infty} \|\mathcal{A}(m,n)x\| e^{(\lambda-\epsilon)(m-n)} - e^{-(\lambda-\epsilon)}\|x\| \\
&\quad + e^{-(\mu+\epsilon)}\|x\| + e^{-(\mu+\epsilon)} \sum_{m=-\infty}^{n-1} \|\mathcal{A}(m,n)x\| e^{-(\mu+\epsilon)(n-m)} \\
&= e^{-(\lambda-\epsilon)}\|x\|_n + (e^{-(\mu+\epsilon)} - e^{-(\lambda-\epsilon)}) \cdot \bigg(\|x\| + \sum_{m=-\infty}^{n-1} \|\mathcal{A}(m,n)x\| e^{-(\mu+\epsilon)(n-m)}\bigg).
\end{aligned}$$

Since $\lambda - \epsilon < \mu + \epsilon$, we have that $e^{-(\mu+\epsilon)} < e^{-(\lambda-\epsilon)}$ and thus we obtain the first inequality in the statement of the lemma. Moreover, since

$$\|x\| + \sum_{m=-\infty}^{n-1} \|\mathcal{A}(m,n)x\| e^{-(\mu+\epsilon)(n-m)} \le \|x\|_n,$$

we have that

$$\|A_n x\|_{n+1} \ge e^{-(\lambda-\epsilon)}\|x\|_n + (e^{-(\mu+\epsilon)} - e^{-(\lambda-\epsilon)})\|x\|_n = e^{-(\mu+\epsilon)}\|x\|_n,$$

which readily implies the second inequality in the statement of the lemma. □

Set

$$Y_\infty := \left\{ \mathbf{x} = (x_n)_{n\in\mathbb{Z}} \subset X : \|\mathbf{x}\|_\infty := \sup_{n\in\mathbb{Z}} \|x_n\|_n < \infty \right\}.$$

Then, it is easy to verify that $(Y_\infty, \|\cdot\|_\infty)$ is a Banach space. We define a linear operator $\mathbb{A}\colon Y_\infty \to Y_\infty$ by

$$(\mathbb{A}\mathbf{x})_n = A_{n-1}x_{n-1}, \quad \text{for } n \in \mathbb{Z} \text{ and } \mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y_\infty.$$

Lemma 2.5. *$\mathbb{A}$ is a bounded operator and*

$$\|\mathbb{A}^m\| \le e^{-(\lambda-\epsilon)m}, \quad \textit{for } m \in \mathbb{N}.$$

In particular, we have that $r(\mathbb{A}) < 1$, where $r(\mathbb{A})$ denotes the spectral radius of $\mathbb{A}$.

Proof of the lemma. It follows from Lemma 2.4 that

$$\begin{aligned} \|\mathbb{A}^m\mathbf{x}\|_\infty = \sup_{n\in\mathbb{Z}} \|(\mathbb{A}^m\mathbf{x})_n\|_n &= \sup_{n\in\mathbb{Z}} \|\mathcal{A}(n,n-m)x_{n-m}\|_n \\ &\le e^{-(\lambda-\epsilon)m} \sup_{n\in\mathbb{Z}} \|x_{n-m}\|_{n-m} \\ &= e^{-(\lambda-\epsilon)m}\|\mathbf{x}\|_\infty, \end{aligned}$$

for $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y_\infty$, which yields the desired conclusion. □

Lemma 2.6. *$\mathbb{A}$ is invertible and $\|\mathbb{A}^{-1}\| \le e^{\mu+\epsilon}$.*

Proof of the lemma. It is easy to verify that $\mathbb{A}$ is invertible and that its inverse is given by

$$(\mathbb{A}^{-1}\mathbf{x})_n = A_n^{-1}x_{n+1}, \quad \text{for } n \in \mathbb{Z} \text{ and } \mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y_\infty.$$

Moreover, it follows from Lemma 2.5 that

$$\begin{aligned} \|\mathbb{A}^{-1}\mathbf{x}\|_\infty = \sup_{n\in\mathbb{Z}} \|(\mathbb{A}^{-1}\mathbf{x})_n\|_n &= \sup_{n\in\mathbb{Z}} \|A_n^{-1}x_{n+1}\|_n \\ &\le e^{\mu+\epsilon} \sup_{n\in\mathbb{Z}} \|x_n\|_n \\ &= e^{\mu+\epsilon}\|\mathbf{x}\|_\infty, \end{aligned}$$

for each $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y_\infty$, which yields the desired result. □

Lemma 2.7. *We have that*

$$\|\mathbb{A}\|^2 \cdot \|\mathbb{A}^{-1}\| < 1.$$

Proof of the lemma. Observe that it follows from Lemmas 2.5 and 2.6 that

$$\|\mathbb{A}\|^2 \cdot \|\mathbb{A}^{-1}\| \le e^{-2(\lambda-\epsilon)} \cdot e^{\mu+\epsilon}.$$

Hence, the conclusion of the lemma follows from the second inequality in (2.10). □

We now define $F\colon Y_\infty \to Y_\infty$ by

$$(F(\mathbf{x}))_n = f_{n-1}(x_{n-1}), \quad \text{for } n \in \mathbb{Z} \text{ and } \mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y_\infty.$$

Lemma 2.8. *F is well-defined.*

Proof of the lemma. Observe that (2.6) and (2.7) imply that

$$\|f_n(x)\| \le \frac{\gamma}{\mathcal{D}(n+1)}\|x\|^2, \quad \text{for } n \in \mathbb{Z} \text{ and } x \in X. \tag{2.13}$$

By (2.11) and (2.13), we have that

$$\begin{aligned}\|(F(\mathbf{x}))_n\|_n = \|f_{n-1}(x_{n-1})\|_n &\le c\mathcal{D}(n)\|f_{n-1}(x_{n-1})\| \\ &\le c\mathcal{D}(n)\frac{\gamma}{\mathcal{D}(n)}\|x_{n-1}\|^2 \\ &\le c\gamma\|x_{n-1}\|_{n-1}^2,\end{aligned}$$

for $n \in \mathbb{Z}$ and therefore

$$\|F(\mathbf{x})\|_\infty \le c\gamma\|\mathbf{x}\|_\infty^2,$$

for $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y_\infty$. We conclude that F is well-defined. □

Lemma 2.9. *F is differentiable and*

$$(DF(\mathbf{x})\mathbf{y})_n = Df_{n-1}(x_{n-1})y_{n-1},$$

for $n \in \mathbb{Z}$, $\mathbf{x} = (x_n)_{n\in\mathbb{Z}}$, $\mathbf{y} = (y_n)_{n\in\mathbb{Z}} \in Y_\infty$.

Proof of the lemma. Let us fix $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y_\infty$. We define an operator $L\colon Y_\infty \to Y_\infty$ by

$$(L\mathbf{y})_n = Df_{n-1}(x_{n-1})y_{n-1}, \quad \text{for } n \in \mathbb{Z} \text{ and } \mathbf{y} = (y_n)_{n\in\mathbb{Z}} \in Y_\infty.$$

Observe that (2.8) and (2.11) imply that

$$\begin{aligned}\|(L\mathbf{y})_n\|_n = \|Df_{n-1}(x_{n-1})y_{n-1}\|_n &\le c\mathcal{D}(n)\|Df_{n-1}(x_{n-1})y_{n-1}\| \\ &\le c\mathcal{D}(n)\frac{\eta}{\mathcal{D}(n)}\|y_{n-1}\| \\ &\le c\eta\|y_{n-1}\|_{n-1},\end{aligned}$$

for $n \in \mathbb{Z}$ and $\mathbf{y} = (y_n)_{n\in\mathbb{Z}} \in Y_\infty$. Hence,

$$\|L\mathbf{y}\|_\infty \le c\eta\|\mathbf{y}\|_\infty,$$

and we conclude that L is a bounded linear operator. Furthermore, for each $\mathbf{h} = (h_n)_{n\in\mathbb{Z}} \in Y_\infty$, we have that

$$\begin{aligned}(F(\mathbf{x}+\mathbf{h}) - F(\mathbf{x}) - L\mathbf{h})_n &= f_{n-1}(x_{n-1}+h_{n-1}) - f_{n-1}(x_{n-1}) - Df_{n-1}(x_{n-1})h_{n-1} \\ &= \int_0^1 Df_{n-1}(x_{n-1}+th_{n-1})h_{n-1}\,dt - Df_{n-1}(x_{n-1})h_{n-1} \\ &= \int_0^1 \big(Df_{n-1}(x_{n-1}+th_{n-1})h_{n-1} - Df_{n-1}(x_{n-1})h_{n-1}\big)\,dt.\end{aligned}$$

Then, (2.7) and (2.11) imply that

$$\begin{aligned}\|(F(\mathbf{x}+\mathbf{h})-F(\mathbf{x})-L\mathbf{h})_n\|_n &\le \int_0^1 \|Df_{n-1}(x_{n-1}+th_{n-1})h_{n-1}-Df_{n-1}(x_{n-1})h_{n-1}\|_n\,dt\\ &\le c\mathcal{D}(n)\int_0^1 \|Df_{n-1}(x_{n-1}+th_{n-1})h_{n-1}-Df_{n-1}(x_{n-1})h_{n-1}\|\,dt\\ &\le c\gamma\|h_{n-1}\|^2 \le c\gamma\|h_{n-1}\|_{n-1}^2,\end{aligned}$$

for $n\in\mathbb{Z}$ and $\mathbf{h}=(h_n)_{n\in\mathbb{Z}}\in Y_\infty$, and consequently

$$\|F(\mathbf{x}+\mathbf{h})-F(\mathbf{x})-L\mathbf{h}\|_\infty \le c\gamma\|\mathbf{h}\|_\infty^2.$$

We conclude that

$$\lim_{\mathbf{h}\to 0}\frac{\|F(\mathbf{x}+\mathbf{h})-F(\mathbf{x})-L\mathbf{h}\|_\infty}{\|\mathbf{h}\|_\infty}=0,$$

which implies the desired conclusion. □

Lemma 2.10. *We have that DF is uniformly continuous. Moreover,*

$$\sup_{\mathbf{x}\in Y_\infty\setminus\{0\}}\frac{\|DF(\mathbf{x})\|}{\|\mathbf{x}\|_\infty}<\infty.$$

Proof of the lemma. For $\mathbf{x}^i=(x_n^i)_{n\in\mathbb{Z}}$, $i=1,2$ and $\mathbf{y}=(y_n)_{n\in\mathbb{Z}}\in Y_\infty$, it follows from (2.7), (2.11) and Lemma 2.9 that

$$\begin{aligned}\|(DF(\mathbf{x}^1)\mathbf{y})_n-(DF(\mathbf{x}^2)\mathbf{y})_n\|_n &= \|Df_{n-1}(x_{n-1}^1)y_{n-1}-Df_{n-1}(x_{n-1}^2)y_{n-1}\|_n\\ &\le c\mathcal{D}(n)\|Df_{n-1}(x_{n-1}^1)y_{n-1}-Df_{n-1}(x_{n-1}^2)y_{n-1}\|\\ &\le c\gamma\|x_{n-1}^1-x_{n-1}^2\|\cdot\|y_{n-1}\|\\ &\le c\gamma\|x_{n-1}^1-x_{n-1}^2\|_{n-1}\cdot\|y_{n-1}\|_{n-1},\end{aligned}$$

for each $n\in\mathbb{Z}$. Thus,

$$\|DF(\mathbf{x}^1)-DF(\mathbf{x}^2)\|_\infty\le c\gamma\|\mathbf{x}^1-\mathbf{x}^2\|_\infty,$$

which implies that DF is uniformly continuous.

In addition, by (2.6), (2.7) and (2.11) we have that

$$\begin{aligned}\|(DF(\mathbf{x})\mathbf{y})_n\|_n &= \|Df_{n-1}(x_{n-1})y_{n-1}\|_n\\ &\le c\mathcal{D}(n)\|Df_{n-1}(x_{n-1})y_{n-1}\|\\ &\le c\gamma\|x_{n-1}\|\cdot\|y_{n-1}\|\\ &\le c\gamma\|x_{n-1}\|_{n-1}\cdot\|y_{n-1}\|_{n-1},\end{aligned}$$

for $n\in\mathbb{Z}$ and thus

$$\|DF(\mathbf{x})\|\le c\gamma\|\mathbf{x}\|_\infty.$$

Consequently,

$$\sup_{\mathbf{x}\in Y_\infty\setminus\{0\}}\frac{\|DF(\mathbf{x})\|}{\|\mathbf{x}\|_\infty}\le c\gamma<\infty.$$

The proof of the lemma is completed. □

Lemma 2.11. *We have that*
$$\sup_{\mathbf{x}\in Y_\infty}\|DF(\mathbf{x})\| \le c\eta.$$

Proof of the lemma. By (2.8), (2.11) and Lemma 2.9, we have that

$$\|(DF(\mathbf{x})\mathbf{y})_n\|_n = \|Df_{n-1}(x_{n-1})y_{n-1}\|_n \le c\eta\|y_{n-1}\|_{n-1},$$

for $n\in\mathbb{Z}$, $\mathbf{x}=(x_n)_{n\in\mathbb{Z}}$ and $\mathbf{y}=(y_n)_{n\in\mathbb{Z}}\in Y_\infty$. Hence,

$$\sup_{\mathbf{x}\in Y_\infty}\|DF(\mathbf{x})\| \le c\eta. \qquad \square$$

It follows from Lemmas 2.5, 2.7, 2.10, 2.11 and [29, Theorem 3] that for η is sufficiently small, there exists a C^1-diffeomorphism $H\colon Y_\infty\to Y_\infty$ such that

$$H\circ(\mathbb{A}+F)=\mathbb{A}\circ H.$$

For $n\in\mathbb{Z}$ and $v\in X$, we define
$$h_n(v)=(H(\mathbf{v}^n))_n,$$
where $\mathbf{v}^n=(v^n_m)_{m\in\mathbb{Z}}\in Y_\infty$ is given by

$$v^n_m=\begin{cases} v & \text{if } m=n;\\ 0 & \text{of } m\neq n.\end{cases}$$

Proceeding as in the proof of [13, Theorem 2.], one can conclude that h_n is a C^1-diffeomorphism for each $n\in\mathbb{Z}$ and that (2.9) holds. $\square$

Remark 2.12. Note that (2.9) implies the following: if $(x_n)_{n\in\mathbb{Z}}$ solves (2.3), then $(y_n)_{n\in\mathbb{Z}}$ given by $y_n=h_n^{-1}(x_n)$, $n\in\mathbb{Z}$ solves (2.4). Hence, the condition (2.9) can be interpreted as a linearization of the nonlinear dynamics (2.4).

Let us now give an interpretation of Theorem 2.3 in the particular case of uniform exponential contractions.

Definition 2.13. We say that a sequence $(A_n)_{n\in\mathbb{Z}}\subset B(X)$ of invertible operators admits a *uniform exponential contraction* if it admits a nonuniform exponential contraction with a constant function $\mathcal{D}\colon\mathbb{Z}\to(0,\infty)$.

The following result is a direct consequence of Theorem 2.3.

Corollary 2.14. *Assume that $(A_n)_{n\in\mathbb{Z}}$ is a sequence of bounded and invertible linear operators that admits a uniform exponential contraction and assume that $0<\lambda\le\mu$ are such that (2.1) and (2.2) hold (with $\mathcal{D}$ being a constant function). Furthermore, suppose that (2.5) holds. In addition, assume that:*

- *f_n is differentiable for each $n\in\mathbb{Z}$;*
- *for every $n\in\mathbb{Z}$, (2.6) holds;*
- *there exists $\gamma>0$ such that*

$$\|Df_n(x)-Df_n(y)\|\le\gamma\|x-y\|, \quad \text{for } n\in\mathbb{Z} \text{ and } x,y\in X;$$

- *there exists $\eta > 0$ such that*

$$\|Df_n(x)\| \le \eta, \quad \text{for } n \in \mathbb{Z} \text{ and } x \in X.$$

Then, if η is sufficiently small, there exists a sequence $(h_n)_{n\in\mathbb{Z}}$ of C^1 diffeomorphisms on X such that (2.9) holds.

Remark 2.15. We are now in a position to elaborate on how Theorem 2.3 and Corollary 2.14 differ from [13, Theorem 2.]. Firstly, we emphasize that [13, Theorem 2.] works under a more general assumption that (2.3) admits an exponential dichotomy and thus is particularly applicable to our setting when (2.3) is contractive. However, the conditions for the smooth linearization in [13] are given in terms of the spectrum $\sigma(\mathbb{A})$ of the operator $\mathbb{A}$. More precisely, if $X = \mathbb{R}^d$ one can show that

$$\sigma(\mathbb{A}) \cap (0, \infty) = [a_1, b_1] \cup \cdots \cup [a_k, b_k],$$

with $0 < a_1 \le b_1 < a_2 \le b_2 < \cdots < a_k \le b_k$. When (2.3) is contractive, we have that $b_k < 1$. Then [13, Theorem 2.] is applicable under suitable conditions for the quotients b_j/a_j, $1 \le j \le k$. However, it is very difficult to verify these conditions in practice since numbers a_j, b_j are difficult to compute (or even estimate). On the other hand, Theorem 2.3 and Corollary 2.14 do not require this as (2.5) is concerned only with the relationship between λ and μ.

Remark 2.16. We note that the condition (2.7) implies that Df_n is a Lipschitz map for each $n \in \mathbb{Z}$. For certain smooth linearization results which do not require that the derivative of the nonlinear part is Lipschitz, we refer to [25, 36].

3 Nonuniform exponential contractions for variational systems

The purpose of this section is to established result analogous to Theorem 2.3 for variational contractive systems with discrete time. We will begin by recalling some necessary terminology.

Assume that Θ be a metric space and let $\sigma\colon \Theta \to \Theta$ be a homeomorphism.

Definition 3.1. A map $\mathcal{A}\colon \Theta \times \mathbb{Z} \to B(X)$ is said to be a *linear cocycle* over σ if:

- $\mathcal{A}(q, 0) = \text{Id}$ for $q \in \Theta$;
- $\mathcal{A}(q, n+m) = \mathcal{A}(\sigma^n q, m)\mathcal{A}(q, n)$ for $q \in \Theta$ and $n, m \in \mathbb{Z}$.

The map $A\colon \Theta \to B(X)$ defined by $A(q) = \mathcal{A}(q, 1)$, $q \in \Theta$ is said to be a *generator* of $\mathcal{A}$.

Remark 3.2. Let $\mathcal{A}$ be a linear cocycle over σ with generator A. We consider the *discrete variational system* given by

$$x_q(n+1) = A(\sigma^n q)x_q(n), \quad (q, n) \in \Theta \times \mathbb{Z}.$$

Observe that its solution satisfies

$$x_q(m) = \mathcal{A}(\sigma^n q, m-n)x_q(n), \quad \text{for } q \in \Theta \text{ and } m \ge n.$$

Definition 3.3. Let $\Theta_0 \subset \Theta$ be σ-invariant, i.e. that $\sigma(\Theta_0) = \Theta_0$. A linear cocycle $\mathcal{A}$ is said to be *nonuniformly exponentially contractive* on Θ_0 if there exist a map $K\colon \Theta_0 \to (0, \infty)$ and $0 < \lambda \le \mu$ such that:

- $$\|\mathcal{A}(q,n)\| \le K(q)e^{-\lambda n} \quad \text{for } q \in \Theta_0 \text{ and } n \ge 0; \tag{3.1}$$
- $$\|\mathcal{A}(q,-n)\| \le K(q)e^{\mu n} \quad \text{for } q \in \Theta_0 \text{ and } n \ge 0. \tag{3.2}$$

The following is a version of Theorem 2.3 for discrete variational systems.

Theorem 3.4. *Assume that $\mathcal{A}$ is a nonuniformly exponentially contractive linear cocycle on a σ-invariant set $\Theta_0 \subset \Theta$. Furthermore, let $K\colon \Theta_0 \to (0,\infty)$ and $0 < \lambda \le \mu$ be such that (3.1) and (3.2) hold. In addition, suppose that (2.5) holds. Finally, assume that $(f_q)_{q\in\Theta_0}$ is a family of maps $f_q\colon X \to X$ such that:*

- *f_q is differentiable for each $q \in \Theta_0$;*
- *for every $q \in \Theta_0$,*
$$f_q(0) = 0 \quad \text{and} \quad Df_q(0) = 0; \tag{3.3}$$
- *there exists $\gamma > 0$ such that*
$$\|Df_q(x) - Df_q(y)\| \le \frac{\gamma}{K(\sigma q)}\|x-y\|, \quad \text{for } q \in \Theta_0 \text{ and } x,y \in X; \tag{3.4}$$
- *there exists $\eta > 0$ such that*
$$\|Df_q(x)\| \le \frac{\eta}{K(\sigma q)}, \quad \text{for } q \in \Theta_0 \text{ and } x \in X. \tag{3.5}$$

Then, if η is sufficiently small, there exists a family $(h_q)_{q\in\Theta_0}$ of C^1 diffeomorphisms on X such that
$$h_{\sigma q} \circ (A(q) + f_q) = A(q) \circ h_q, \quad \text{for } q \in \Theta_0. \tag{3.6}$$

Proof. We choose $\epsilon > 0$ such that (2.10) holds. For $q \in \Theta_0$ and $x \in X$, set
$$\|x\|_q := \sum_{n=0}^{\infty} \|\mathcal{A}(q,n)x\| e^{(\lambda-\epsilon)n} + \sum_{n=1}^{\infty} \|\mathcal{A}(q,-n)x\| e^{-(\mu+\epsilon)n}.$$
It follows easily from (3.1) and (3.2) that
$$\|x\| \le \|x\|_q \le cK(q)\|x\| \quad \text{for } q \in \Theta_0 \text{ and } x \in X, \tag{3.7}$$
with c as in (2.12).

Lemma 3.5. *We have that*
$$\|A(q)x\|_{\sigma q} \le e^{-(\lambda-\epsilon)}\|x\|_q \quad \text{and} \quad \|A(q)^{-1}x\|_q \le e^{\mu+\epsilon}\|x\|_{\sigma q},$$
for $x \in X$ and $q \in \Theta_0$.

Proof of the lemma. We have that
$$\begin{aligned}
\|A(q)x\|_{\sigma q} &= \sum_{n=0}^{\infty} \|\mathcal{A}(\sigma q,n)A(q)x\| e^{(\lambda-\epsilon)n} + \sum_{n=1}^{\infty} \|\mathcal{A}(\sigma q,-n)A(q)x\| e^{-(\mu+\epsilon)n} \\
&= \sum_{n=0}^{\infty} \|\mathcal{A}(q,n+1)x\| e^{(\lambda-\epsilon)n} + \sum_{n=1}^{\infty} \|\mathcal{A}(q,-(n-1))x\| e^{-(\mu+\epsilon)n} \\
&= e^{-(\lambda-\epsilon)} \sum_{n=0}^{\infty} \|\mathcal{A}(q,n)x\| e^{(\lambda-\epsilon)n} - e^{-(\lambda-\epsilon)}\|x\| \\
&\quad + e^{-(\mu+\epsilon)}\|x\| + e^{-(\mu+\epsilon)} \sum_{n=1}^{\infty} \|\mathcal{A}(q,-n)x\| e^{-(\mu+\epsilon)n} \\
&= e^{-(\lambda-\epsilon)}\|x\|_q + (e^{-(\mu+\epsilon)} - e^{-(\lambda-\epsilon)}) \cdot \left(\|x\| + \sum_{n=1}^{\infty} \|\mathcal{A}(q,-n)x\| e^{-(\mu+\epsilon)n} \right).
\end{aligned}$$

Since $\lambda - \epsilon < \mu + \epsilon$, we have that $e^{-(\mu+\epsilon)} < e^{-(\lambda-\epsilon)}$ and thus we obtain the first inequality in the statement of the lemma. Moreover, since

$$\|x\| + \sum_{n=1}^{\infty} \|\mathcal{A}(q,-n)x\| e^{-(\mu+\epsilon)n} \leq \|x\|_q$$

we have that

$$\|A(q)x\|_{\sigma q} \geq e^{-(\lambda-\epsilon)}\|x\|_q + (e^{-(\mu+\epsilon)} - e^{-(\lambda-\epsilon)})\|x\|_q = e^{-(\mu+\epsilon)}\|x\|_q,$$

which readily implies the second inequality in the statement of the lemma. □

Set

$$Z_\infty := \Big\{ v\colon \Theta_0 \to X : \|v\|_\infty := \sup_{q\in\Theta_0} \|v(q)\|_q < \infty \Big\}.$$

Then, $(Z_\infty, \|\cdot\|_\infty)$ is a Banach space. We define a linear operator $\mathbb{B}\colon Z_\infty \to Z_\infty$ by

$$(\mathbb{B}v)(q) = A(\sigma^{-1}q)v(\sigma^{-1}q), \quad \text{for } q \in \Theta_0 \text{ and } v \in Z_\infty.$$

Lemma 3.6. $\mathbb{B}$ *is a bounded operator and*

$$\|\mathbb{B}^m\| \leq e^{-(\lambda-\epsilon)m}, \quad \textit{for } m \in \mathbb{N}.$$

In particular, we have that $r(\mathbb{B}) < 1$.

Proof of the lemma. It follows from Lemma 3.5 that

$$\begin{aligned}\|\mathbb{B}^m v\|_\infty = \sup_{q\in\Theta_0} \|(\mathbb{B}^m v)(q)\|_q &= \sup_{q\in\Theta_0} \|\mathcal{A}(\sigma^{-m}q, m)v(\sigma^{-m}q)\|_q \\ &\leq e^{-(\lambda-\epsilon)m} \sup_{q\in\Theta_0} \|v(\sigma^{-m}q)\|_{\sigma^{-m}q} \\ &= e^{-(\lambda-\epsilon)m}\|v\|_\infty,\end{aligned}$$

for $v \in Z_\infty$, which yields the desired conclusion. □

Lemma 3.7. $\mathbb{B}$ *is invertible and* $\|\mathbb{B}^{-1}\| \leq e^{\mu+\epsilon}$.

Proof of the lemma. It is easy to verify that $\mathbb{B}$ is invertible and that its inverse is given by

$$(\mathbb{B}^{-1}v)(q) = A(q)^{-1}v(\sigma q), \quad \text{for } q \in \Theta_0 \text{ and } v \in Z_\infty.$$

Moreover, it follows from Lemma 3.5 that

$$\begin{aligned}\|\mathbb{B}^{-1}v\|_\infty = \sup_{q\in\Theta_0} \|(\mathbb{B}^{-1}v)(q)\|_q &= \sup_{q\in\Theta_0} \|A(q)^{-1}v(\sigma q)\|_q \\ &\leq e^{\mu+\epsilon} \sup_{q\in\Theta_0} \|v(\sigma q)\|_{\sigma q} \\ &= e^{\mu+\epsilon}\|v\|_\infty,\end{aligned}$$

for each $v \in Z_\infty$, which yields the desired result. □

As in the proof of Lemma 2.7, it follows from (2.10) and Lemmas 3.6 and 3.7 that

$$\|\mathbb{B}\|^2 \cdot \|\mathbb{B}^{-1}\| < 1. \tag{3.8}$$

We define $G\colon Z_\infty \to Z_\infty$ by

$$(G(v))(q) = f_{\sigma^{-1}q}(v(\sigma^{-1}q)), \quad \text{for } q \in \Theta_0 \text{ and } v \in Z_\infty.$$

Lemma 3.8. *G is well-defined.*

Proof of the lemma. Observe that (3.3) and (3.4) imply that

$$\|f_q(x)\| \le \frac{\gamma}{K(\sigma q)}\|x\|^2, \quad \text{for } q \in \Theta_0 \text{ and } x \in X. \tag{3.9}$$

By (3.7) and (3.9), we have that

$$\begin{aligned}\|(G(v))(q)\|_q = \|f_{\sigma^{-1}q}(v(\sigma^{-1}q))\|_q &\le cK(q)\|f_{\sigma^{-1}q}(v(\sigma^{-1}q))\| \\ &\le cK(q)\frac{\gamma}{K(q)}\|v(\sigma^{-1}q)\|^2 \\ &\le c\gamma\|v(\sigma^{-1}q)\|^2_{\sigma^{-1}q},\end{aligned}$$

for $q \in \Theta_0$ and $v \in Z_\infty$. Hence,

$$\|G(v)\|_\infty \le c\gamma\|v\|^2_\infty \quad \text{for every } v \in Z_\infty,$$

and therefore G is well-defined. □

Lemma 3.9. *G is differentiable and*

$$(DG(v)w)(q) = Df_{\sigma^{-1}q}(v(\sigma^{-1}q))w(\sigma^{-1}q),$$

for $q \in \Theta_0$ and $v, w \in Z_\infty$.

Proof of the lemma. Let us fix $v \in Z_\infty$. We define an operator $L\colon Z_\infty \to Z_\infty$ by

$$(Lw)(q) = Df_{\sigma^{-1}q}(v(\sigma^{-1}q))w(\sigma^{-1}q), \quad \text{for } q \in \Theta_0 \text{ and } w \in Z_\infty.$$

Observe that (3.5) and (3.7) imply that

$$\begin{aligned}\|(Lw)(q)\|_q &= \|Df_{\sigma^{-1}q}(v(\sigma^{-1}q))w(\sigma^{-1}q)\|_q \\ &\le cK(q)\|Df_{\sigma^{-1}q}(v(\sigma^{-1}q))w(\sigma^{-1}q)\| \\ &\le cK(q)\frac{\eta}{K(q)}\|w(\sigma^{-1}q)\| \\ &\le c\eta\|w(\sigma^{-1}q)\|_{\sigma^{-1}q},\end{aligned}$$

for $q \in \Theta_0$ and thus

$$\|Lw\|_\infty \le c\eta\|w\|_\infty,$$

for every $w \in Z_\infty$. Hence, L is a bounded linear operator.

Furthermore, for each $h \in Z_\infty$, we have that

$$
\begin{aligned}
&(G(v+h) - G(v) - Lh)(q) \\
&\quad = f_{\sigma^{-1}q}(v(\sigma^{-1}q) + h(\sigma^{-1}q)) - f_{\sigma^{-1}q}(v(\sigma^{-1}q)) - Df_{\sigma^{-1}q}(v(\sigma^{-1}q))h(\sigma^{-1}q) \\
&\quad = \int_0^1 Df_{\sigma^{-1}q}(v(\sigma^{-1}q) + th(\sigma^{-1}q))h(\sigma^{-1}q)\, dt - Df_{\sigma^{-1}q}(v(\sigma^{-1}q))h(\sigma^{-1}q) \\
&\quad = \int_0^1 \big(Df_{\sigma^{-1}q}(v(\sigma^{-1}q) + th(\sigma^{-1}q))h(\sigma^{-1}q) - Df_{\sigma^{-1}q}(v(\sigma^{-1}q))h(\sigma^{-1}q)\big)\, dt.
\end{aligned}
$$

Then, (3.4) and (3.7) imply that

$$
\begin{aligned}
&\|(G(v+h) - G(v) - Lh)(\omega)\|_q \\
&\quad \le \int_0^1 \|Df_{\sigma^{-1}q}(v(\sigma^{-1}q) + th(\sigma^{-1}q))h(\sigma^{-1}q) - Df_{\sigma^{-1}q}(v(\sigma^{-1}q))h(\sigma^{-1}q)\|_q\, dt \\
&\quad \le cK(q) \int_0^1 \|Df_{\sigma^{-1}q}(v(\sigma^{-1}q) + th(\sigma^{-1}q))h(\sigma^{-1}q) - Df_{\sigma^{-1}q}(v(\sigma^{-1}q))h(\sigma^{-1}q)\|\, dt \\
&\quad \le c\gamma \|h(\sigma^{-1}q)\|^2 \le c\gamma \|h(\sigma^{-1}q)\|^2_{\sigma^{-1}q},
\end{aligned}
$$

for $q \in \Theta_0$ and $h \in Z_\infty$, and consequently

$$\|G(v+h) - G(v) - Lh\|_\infty \le c\gamma \|h\|_\infty^2,$$

which implies the desired conclusion. □

Lemma 3.10. *DG is uniformly continuous. Moreover,*

$$\sup_{v \in Z_\infty \setminus \{0\}} \frac{\|DG(v)\|}{\|v\|_\infty} < \infty.$$

Proof of the lemma. For v_i, $i = 1, 2$ and $h \in Z_\infty$, it follows from (3.4), (3.7) and Lemma 3.9 that

$$
\begin{aligned}
&\|(DG(v_1)h)(q) - (DG(v_2)h)(q)\|_q \\
&\quad = \|Df_{\sigma^{-1}q}(v_1(\sigma^{-1}q))h(\sigma^{-1}q) - Df_{\sigma^{-1}q}(v_2(\sigma^{-1}q))h(\sigma^{-1}q)\|_q \\
&\quad \le cK(q)\|Df_{\sigma^{-1}q}(v_1(\sigma^{-1}q))h(\sigma^{-1}q) - Df_{\sigma^{-1}q}(v_2(\sigma^{-1}q))h(\sigma^{-1}q)\| \\
&\quad \le c\gamma \|v_1(\sigma^{-1}q) - v_2(\sigma^{-1}q)\| \cdot \|h(\sigma^{-1}q)\| \\
&\quad \le c\gamma \|v_1(\sigma^{-1}q) - v_2(\sigma^{-1}q)\|_{\sigma^{-1}q} \cdot \|h(\sigma^{-1}q)\|_{\sigma^{-1}q},
\end{aligned}
$$

for each $q \in \Theta_0$. Therefore,

$$\|DG(v_1) - DG(v_2)\|_\infty \le c\gamma \|v_1 - v_2\|_\infty.$$

In addition, by (3.3), (3.4) and (3.7) we have that

$$
\begin{aligned}
\|(DG(v)h)(q)\|_q &= \|Df_{\sigma^{-1}q}(v(\sigma^{-1}q))h(\sigma^{-1}q)\|_q \\
&\le cK(q)\|Df_{\sigma^{-1}q}(v(\sigma^{-1}q))h(\sigma^{-1}q)\| \\
&\le c\gamma \|v(\sigma^{-1}q)\| \cdot \|h(\sigma^{-1}q)\| \\
&\le c\gamma \|v(\sigma^{-1}q)\|_{\sigma^{-1}q} \cdot \|h(\sigma^{-1}q)\|_{\sigma^{-1}q},
\end{aligned}
$$

for $q \in \Theta_0$ and thus

$$\|DG(v)\| \le c\gamma \|v\|_\infty,$$

which completes the proof of the lemma. □

Lemma 3.11. *We have that*
$$\sup_{v\in Z_\infty}\|DG(v)\|\le c\eta.$$

Proof of the lemma. By (3.5), (3.7) and Lemma 3.9, we have that
$$\|(DG(v)w)(q)\|_q=\|Df_{\sigma^{-1}q}(v(\sigma^{-1}q))w(\sigma^{-1}q)\|_q\le c\eta\|w(\sigma^{-1}q)\|_{\sigma^{-1}q},$$
for $q\in\Theta_0$, $v,w\in Z_\infty$. Hence,
$$\sup_{v\in Z_\infty}\|DG(v)\|\le c\eta.$$
□

It follows from (3.8), Lemmas 3.6, 3.10, 3.11 and [29, Theorem 3] that for η is sufficiently small, there exists a C^1-diffeomorphism $H\colon Z_\infty\to Z_\infty$ such that
$$H\circ(\mathbb{B}+G)=\mathbb{B}\circ H. \tag{3.10}$$
Take now $q_0\in\Theta_0$ and $x_0\in X$. We define $v_{q_0,x_0}\in Z_\infty$ by
$$v_{q_0,x_0}(q)=\begin{cases}x_0 & \text{if } q=q_0;\\ 0 & \text{if } q\neq q_0.\end{cases}$$
Finally, we define $h_{q_0}\colon X\to X$ by $h_{q_0}(x_0)=H(v_{q_0,x_0})(q_0)$. Observe that
$$(\mathbb{B}+G)(v_{q_0,x_0})(q)=\begin{cases}A(q_0)x_0+f_{q_0}(x_0) & \text{if } q=\sigma q_0;\\ 0 & \text{if } q\neq \sigma q_0,\end{cases}$$
and thus
$$(\mathbb{B}+G)(v_{q_0,x_0})=v_{\sigma q_0,A(q_0)x_0+f_{q_0}(x_0)}.$$
Now we observe that it follows from (3.10) that
$$((H\circ(\mathbb{B}+G))(v_{q,x_0}))(\sigma q_0)=((\mathbb{B}\circ H)(v_{q_0,x_0}))(\sigma q_0),$$
and thus
$$(h_{\sigma q_0}\circ(A(q_0)+f_{q_0}))(x_0)=(A(q_0)\circ h_{q_0})(x_0).$$
Since $q_0\in\Theta_0$ and $x_0\in X$ were arbitrary, we conclude that (3.6) holds.

Furthermore, we claim that h_{q_0} is differentiable and that
$$Dh_{q_0}(x_0)y=(DH(v_{q_0,x_0})v_{q_0,y})(q_0). \tag{3.11}$$
Indeed, we have that
$$\begin{aligned}&\frac{\|h_{q_0}(x_0+y)-h_{q_0}(x_0)-(DH(v_{q_0,x_0})v_{q_0,y})(q_0)\|}{\|y\|}\\ &\le cK(q_0)\frac{\|H(v_{q_0,x_0}+v_{q_0,y})(q_0)-H(v_{q_0,x_0})(q_0)-(DH(v_{q_0,x_0})v_{q_0,y})(q_0)\|_{q_0}}{\|y\|_{q_0}}\\ &\le cK(q_0)\frac{\|H(v_{q_0,x_0}+v_{q_0,y})-H(v_{q_0,x_0})-(DH(v_{q_0,x_0})v_{q_0,y})\|_\infty}{\|v_{q_0,y}\|_\infty}.\end{aligned}$$
Letting $\|y\|\to 0$, we have that $\|v_{q_0,y}\|_\infty\to 0$ and we conclude that (3.11) holds.

Let us show that Dh_{q_0} is continuous. For x_0 and $\tilde{x}_0 \in X$, we have that

$$\begin{aligned}
\|Dh_{q_0}(x_0) - Dh_{\omega_0}(\tilde{x}_0)\| &= \sup_{\|y\|\le 1} \|Dh_{q_0}(x_0)y - Dh_{q_0}(\tilde{x}_0)y\| \\
&= \sup_{\|y\|\le 1} \|(DH(v_{q_0,x_0})v_{q_0,y})(q_0) - (DH(v_{\omega_0,\tilde{x}_0})v_{q_0,y})(q_0)\| \\
&\le \sup_{\|y\|\le 1} \|(DH(v_{q_0,x_0})v_{q_0,y})(q_0) - (DH(v_{q_0,\tilde{x}_0})v_{q_0,y})(q_0)\|_{q_0} \\
&\le \sup_{\|y\|\le 1} \|DH(v_{q_0,x_0})v_{q_0,y} - DH(v_{q_0,\tilde{x}_0})v_{q_0,y}\|_\infty \\
&\le cK(q_0)\|DH(v_{q_0,x_0}) - DH(v_{q_0,\tilde{x}_0})\|.
\end{aligned}$$

Letting $\tilde{x}_0 \to x_0$, we have that $v_{q_0,\tilde{x}_0} \to v_{q_0,x_0}$ in Z_∞ and thus since H is of class C^1 we conclude that $Dh_{q_0}(\tilde{x}_0) \to Dh_{q_0}(x_0)$.

Finally, it is easy to show that

$$h_{q_0}^{-1}(x_0) = H^{-1}(v_{q_0,x_0})(q_0),$$

and proceeding as above, one can show that $h_{q_0}^{-1}$ is of class C^1. □

Let us now discuss the applicability of Theorem 3.4 in the setting when we can apply the version of the Oseledets multiplicative ergodic theorem [20] for the cocycle $\mathcal{A}$.

Remark 3.12. Assume that $X = \mathbb{R}^d$ and that on Θ we have a Borel probability measure $\mathbb{P}$ such that σ preserves $\mathbb{P}$. Moreover, suppose that $\mathbb{P}$ is ergodic and that

$$\int_\Theta \log^+ \|A(q)\| \, d\mathbb{P}(q) < \infty.$$

Hence, we can apply the Oseledets multiplicative ergodic theorem [20] to conclude that there exist Lyapunov exponents

$$-\infty < \lambda_r < \cdots < \lambda_2 < \lambda_1 < +\infty, \quad 1 \le r \le d,$$

σ-invariant Borel set $\Theta_0 \subset \Theta$, $\mathbb{P}(\Theta_0) = 1$ and for $q \in \Theta_0$, the corresponding Oseledets splitting

$$\mathbb{R}^d = \bigoplus_{i=1}^r E_i(q)$$

such that:

- $A(q)E_i(q) = E_i(\sigma q)$ for $q \in \Theta_0$ and $1 \le i \le r$;
- for $q \in \Theta_0$, $v \in E_i(q) \setminus \{0\}$ and $1 \le i \le r$,

$$\lim_{n\to\infty} \frac{1}{n} \log\|\mathcal{A}(q,n)v\| = \lambda_i.$$

Assume now that $\lambda_1 < 0$. It follows from [12, Theorem 2] that $\mathcal{A}$ is nonuniformly exponentially contractive on Θ_0. Furthermore, (3.1) and (3.2) hold with

$$\lambda = -\lambda_1 - \epsilon \quad \text{and} \quad \mu = -\lambda_r + 2\epsilon,$$

with any sufficiently small $\epsilon > 0$. Hence, we conclude that in this case (2.5) holds if $2\lambda_1 < \lambda_r$.

As we promised, we now explain why nonuniform contractions introduced in previous section are ubiquitous from the ergodic theory point of view.

Remark 3.13. Let X and $\mathcal{A}$ be as in Remark 3.12. Then, it follows from [12, Theorem 3] that for $q \in \Theta_0$, the sequence $(A_n)_{n\in\mathbb{Z}}$ given by

$$A_n = A(\sigma^n q) \quad n \in \mathbb{Z},$$

admits a nonuniform exponential contraction, where $\mathcal{D}$ is the scalar multiple of $n \mapsto e^{\epsilon|n|}$ and $\epsilon > 0$ is arbitrary.

As in previous section, we will now formulate a direct consequence of Theorem 3.4 dealing with uniformly exponentially contractive variational systems.

Definition 3.14. Let $\Theta_0 \subset \Theta$ be σ-invariant. A linear cocycle $\mathcal{A}$ over σ is said to be *uniformly exponentially contractive* on Θ_0 if there exist $K > 0$ and $0 < \lambda \le \mu$ such that:

- for $q \in \Theta_0$ and $n \ge 0$,
$$\|\mathcal{A}(q,n)\| \le Ke^{-\lambda n}; \tag{3.12}$$
- for $q \in \Theta_0$ and $n \ge 0$,
$$\|\mathcal{A}(q,-n)\| \le Ke^{\mu n}. \tag{3.13}$$

The following is a consequence of Theorem 3.4.

Corollary 3.15. *Assume that $\mathcal{A}$ is a uniformly exponentially contractive linear cocycle on a σ-invariant set $\Theta_0 \subset \Theta$ and suppose that $K > 0$ and $0 < \lambda \le \mu$ are such that (3.12) and (3.13) hold. Furthermore, suppose that (2.5) holds. Finally, assume that $(f_q)_{q\in\Theta_0}$ is a family of maps $f_\omega \colon X \to X$ such that:*

- *f_q is differentiable for each $q \in \Theta_0$;*
- *for every $q \in \Theta_0$,*
$$f_q(0) = 0 \quad \text{and} \quad Df_q(0) = 0;$$
- *there exists $\gamma > 0$ such that*
$$\|Df_q(x) - Df_q(y)\| \le \gamma\|x - y\|, \quad \text{for } q \in \Theta_0 \text{ and } x, y \in X;$$
- *there exists $\eta > 0$ such that*
$$\|Df_q(x)\| \le \eta, \quad \text{for } q \in \Theta_0 \text{ and } x \in X.$$

Then, if η is sufficiently small, there exists a family $(h_q)_{q\in\Theta_0}$ of C^1 diffeomorphisms on X such that

$$h_{\sigma q} \circ (A(q) + f_q) = A(q) \circ h_q, \quad \text{for } q \in \Theta_0.$$

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Existence, nonexistence and multiplicity of positive solutions for singular quasilinear problems

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Abstract. In the present paper we deal with a quasilinear problem involving a singular term and a parametric superlinear perturbation. We are interested in the existence, nonexistence and multiplicity of positive solutions as the parameter $\lambda > 0$ varies. In our first result, the superlinear perturbation has an arbitrary growth and we obtain the existence of a solution for the problem by using the sub-supersolution method. For the second result, the superlinear perturbation has subcritical growth and we employ the Mountain Pass Theorem to show the existence of a second solution.

Keywords: extended functional, sub-supersolution method, singular problem, variational methods.

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1 Introduction

This paper is concerned with the existence, nonexistence and multiplicity of solutions for the family of quasilinear problems with singular nonlinearity

$$\begin{cases} -\Delta u - \Delta(u^2)u = a(x)u^{-\gamma} + \lambda u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where $0 < \gamma, 3 \leq p < \infty, 0 \leq \lambda$ is a parameter, $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded smooth domain and $a(x)$ is a positive measurable function.

We say that a function $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a weak solution (solution, for short) of (P_λ) if $u > 0$ a.e. in Ω, and, for every $\psi \in H_0^1(\Omega)$,

$$au^{-\gamma}\psi, u^p\psi \in L^1(\Omega)$$

and

$$\int_\Omega [(1 + 2u^2)\nabla u \nabla \psi + 2u|\nabla u|^2\psi] = \int_\Omega a(x)u^{-\gamma}\psi + \lambda \int_\Omega u^p \psi.$$

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Solutions of this type are related to the existence of standing wave solutions for quasilinear Schrödinger equations of the form

$$i\partial_t z = -\Delta z + V(x)z + \eta(|z|^2)z - \kappa\Delta\rho(|z|^2)\rho'(|z|^2)z, \tag{1.1}$$

where $z : \mathbb{R}\times\Omega \to \mathbb{C}$, $V(x)$ is a given potential, $\kappa > 0$ is a constant and η, ρ are real functions. Quasilinear equations of form (1.1) appear more naturally in mathematical physics and have been derived as models of several physical phenomena corresponding to various types of ρ. The case of $\rho(s) = s$ was used for the superfluid film equation in plasma physics by Kurihara [20] (cf. [21]). In the case $\rho(s) = (1+s)^{1/2}$, equation (1.1) models the self-channeling of a high-power ultrashort laser in matter, see [7, 9, 11, 28] and the references in [8].

Consider the following quasilinear Schrödinger equation

$$-\Delta u - \Delta(u^2)u = g(x,u) \quad \text{in } \Omega, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$. When g is a singular nonlinearity, problems of type (1.2) was studied by Do Ó–Moameni [25], Liu–Liu–Zhao [23], Wang [32], Dos Santos–Figueiredo–Severo [29], Alves–Reis [2] and Bal–Garain–Mandal–Sreenadh [6]. In particular, the authors in [23] considered the problem

$$\begin{cases} -\Delta_s u - \frac{s}{2^{s-1}}\Delta(u^2)u = a(x)u^{-\gamma} + \lambda u^p & \text{in } \Omega,\\ u > 0 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $N \geq 3$, Δ_s is the s-Laplacian operator, $2 < 2s < p+1 < \infty$, $0 < \gamma$ and $a \geq 0$ is a nontrivial measurable function satisfying the following condition:

(H) There are $\varphi \in C_0^1(\overline{\Omega})$ and $q > N$ such that $\varphi > 0$ on Ω and $a\varphi^{-\gamma} \in L^q(\Omega)$.

The authors used sub-supersolution method, truncation arguments and variational methods to prove the existence of solutions for (1.3) provided $\lambda > 0$ is small enough.

In [29], Dos Santos–Figueiredo–Severo studied the problem

$$\begin{cases} -\Delta u - \Delta(u^2)u = a(x)u^{-\gamma} + z(x,u) & \text{in } \Omega,\\ u > 0 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $N \geq 3$, the function a satisfies the hypothesis (H) and the nonlinearity $z : \Omega\times\mathbb{R} \longrightarrow \mathbb{R}$ is continuous and satisfies (among other conditions):
There exist $C > 0, r \geq 1$ and $b \in L^\infty(\Omega), b \geq 0$ almost everywhere in Ω, such that

$$|z(x,t)| \leq C(1 + b(x)|t|^{r-1}), \qquad \forall t \in \mathbb{R} \text{ and a.e. in } \Omega.$$

By using sub-supersolution method, truncation arguments and the Mountain Pass Theorem they showed the existence of solutions provided $\|b\|_\infty$ is small enough. When $z(x,t) = \lambda|t|^{r-2}t$ this is equivalent to the existence of solutions for $\lambda > 0$ small enough.

In this paper, our first goal is to show the existence and nonexistence of solutions for (P_λ) without restriction on the parameter λ and exponent $p \geq 3$ (see Remark 1.4). We would like to emphasize that for $0 < p < 3$ the arguments carried out in [1, 2] can be adapted to prove that problem (P_λ) has at least one solution for all $\lambda \in \mathbb{R}$ (see Remark 4.1).

It is worth pointing out that to prove our main results, we use the method of changing variables developed in Colin–Jeanjean [13]. Thus, we reformulate problem (P_λ) into a new one, denoted with (Q_λ) (cfr. Section 2), which finds its natural setting in the Sobolev space $H_0^1(\Omega)$.

Our first result is the following.

Theorem 1.1. *Under the assumptions (H) and $p \geq 3$ there exists $0 < \lambda_* < \infty$ such that problem (P_λ) has at least one solution v_λ for $0 < \lambda < \lambda_*$ and no solution for $\lambda > \lambda_*$. Moreover, λ_* is characterized variationally by (3.1) and $v_\lambda \in C_0^1(\overline{\Omega})$.*

The proof of Theorem 1.1 is based on the method of sub-supersolutions. However, by virtue of the arbitrary growth of the singular and superlinear terms that appear in problem (Q_λ) we cannot use directly the method of sub-supersolutions here. An additional difficulty comes from the fact that these singular and superlinear terms are nonhomogeneous. To overcome this difficulty we develop new arguments and a regularity result that allows us to obtain a subsolution to problem (Q_λ) for all $\lambda > 0$. In particular, we establish some preliminary results and we prove a sub-supersolution theorem (see Theorem 2.8).

To prove the multiplicity of solutions for (P_λ), with $\lambda \in (0, \lambda_*)$, we need a refinement of hypotheses (H). We introduce the following assumption:

$(H)_\infty$ There exists $\varphi \in C_0^1(\overline{\Omega})$ such that $\varphi > 0$ on Ω and $a\varphi^{-1-\gamma} \in L^\infty(\Omega)$.

If the function φ satisfies $(H)_\infty$ then it satisfies (H), too (see Section 4).

We denote by $2^* = 2N/(N-2)$ the critical Sobolev exponent. Now we state our second result.

Theorem 1.2. *Under the assumptions $(H)_\infty$ and $3 < p < 22^* - 1$, problem (P_λ) has at least two solutions for $0 < \lambda < \lambda_*$ and no solution for $\lambda > \lambda_*$.*

Example 1.3. When Ω is the unit ball, the functions $a(x) = (1 - |x|^2)^\sigma, \sigma \geq \gamma + 1$ and $\varphi(x) = 1 - |x|^2$ satisfy assumption $(H)_\infty$.

Remark 1.4. The results obtained in Theorems 1.1 and 1.2 are almost global, that is, they do not show the existence and multiplicity of solutions only for $\lambda = \lambda_*$, with parameter λ_* having the property that problem (P_λ) has at least one solution for $\lambda \in (0, \lambda_*)$ and no solutions for $\lambda > \lambda_*$. Thus, in our main results we do not assume the restriction that λ is small enough to guarantee the existence of solutions, because we prove the existence of a solution for all $\lambda \in (0, \lambda_*)$. Furthermore, when $0 < \gamma < 1$ (weak singularity), combining Theorems 1.1, 1.2 and Proposition 4.8 we have a global result:

a) Problem (P_λ) has a solution if and only if $\lambda \in (0, \lambda_*]$. Namely, $\mathcal{L} = (0, \lambda_*]$ (see Section 3 for definition of $\mathcal{L}$).

b) Problem (P_λ) has at least two solutions for $\lambda \in (0, \lambda_*)$ and at least one solution for $\lambda = \lambda_*$ and no solution for $\lambda > \lambda_*$.

Let us highlight that the hypotheses $(H)_\infty$ plays a crucial role in the proof of Theorem 1.2. Indeed, it allows us to show that v_λ is a local minimum of the functional J_λ in the topology of $C_0^1(\overline{\Omega})$ and that the modified functional $\mathcal{J}_\lambda$ belongs to $C^1(H_0^1(\Omega), \mathbb{R})$ and satisfies the assumptions of Theorem 1 in Brezis–Nirenberg [10] (see (4.2) and (4.7) in Section 4 for definition of J_λ and $\mathcal{J}_\lambda$, respectively). In particular, we get that $v = 0$ is a local minimum of the functional $\mathcal{J}_\lambda$ in the $H_0^1(\Omega)$ topology. Then, after fine arguments we apply the Mountain Pass Theorem to obtain a second solution of (P_λ). It is worth pointing out that under the assumption (H) we are not able to show Lemma 4.2 and that $\mathcal{J}_\lambda$ satisfies the assumptions of Theorem 1 in [10].

We emphasize that Theorem 1.1 improve the works [23,29] in the sense that we show the existence and nonexistence of solutions for (P_λ) without restriction on the parameter λ (that is, our results are almost global and we do not assume that λ is small enough to obtain a solution, see Remark 1.4). They also did not prove a result of nonexistence of solutions. As far as we know, Theorem 1.2 is the first result of multiplicity of $H_0^1(\Omega)$-solutions for singular problems with strong singularity $\gamma > 1$ and without restriction on the parameter λ, that is, we do not assume λ small enough. Notice that no restriction on the $\gamma > 0$ is assumed.

Let us compare our parameter λ_* and results with the parameters and results obtained in [6,29].

- Let ϵ_0 and ϵ_1 be the parameters obtained in Theorem 1.2 and 1.3, respectively, in [29] when $h(x,t) = \lambda|t|^{r-2}t$. Then, we will prove in Remark 3.3 that $\epsilon_1 \leq \epsilon_0 < \lambda_*$.

- When $0 < \gamma < 1$ and $3 < p < 22^* - 1$ problem (P_λ) was also studied in [6]. As mentioned by the authors of that work, using the Nehari manifold method they proved the existence of two solutions for λ sufficiently small. More precisely, they proved that there is a parameter $\nu > 0$ such that problem (P_λ) has two solutions for $0 < \lambda < \nu$ and $\mathcal{N}_\lambda^0 = \emptyset$ for $0 < \lambda < \nu$ (here we use ν to avoid confusion with our Λ of Theorem 3.2 and see page 4 of [6] for the definition of $\mathcal{N}_\lambda^0$). In our work we are assuming arbitrary $\gamma > 0$, unlike [6] which assumes $0 < \gamma < 1$. Furthermore the technique used in [6] cannot be used when $\gamma \geq 1$, because they need the continuity of the energy functional associated with the problem and use that $0 < 1-\gamma < 1$ to get estimates (at this point they need Sobolev embeddings and therefore it is very important that $0 < 1-\gamma < 1$). For $\gamma \geq 1$ these facts are not true and therefore for $\gamma \geq 1$ the results obtained in [6] cannot be compared with our results obtained here (in particular with Theorem 1.1 where we assume that p can be supercritical, that is, $22^* - 1 < p$).

- When $0 < \gamma < 1$ and $p < 22^* - 1$, combining Theorem 1.1 and Proposition 4.8 of our work we have that problem (P_λ) has a solution if and only if $\lambda \in (0, \lambda_*]$. Thus, our result is global in this case. As a consequence $\nu \leq \lambda_*$ (see previous paragraph for the definition of ν). In [6] they did not prove the existence of a solution for $\lambda = \nu$, that is, they obtained a local result. Therefore, even in this case our work improves the result of [6] in the sense that we prove that problem (P_λ) has a solution if and only if $\lambda \in (0, \lambda_*]$ (in particular for $\lambda = \lambda_*$), and therefore we do not have the restriction that λ is small enough as in [6]. Finally, we emphasize that a similar problem, but without the term $\Delta(u^2)u$, was studied in [3] using the Nehari manifold method and in that work the authors obtained solutions for parameters $\lambda > 0$ such that $\mathcal{N}_\lambda^0 \neq \emptyset$. This result suggests that parameter ν obtained in [6] is small enough and satisfies $\nu < \lambda_*$.

There is a wide literature dealing with existence and multiplicity results for problems involving both the p-Laplacian operator and singular nonlinearities. The reader who wishes to broaden his/her knowledge on these topics is referred to [3,15–18,26,27], and to the references therein.

The paper is structured as follows: In Section 2, we reformulate problem (P_λ) into a new one which finds its natural setting in the Sobolev space $H_0^1(\Omega)$ and we present some results that will be important for our work. In particular, we prove a nonexistence result and a sub-supersolution theorem. In Section 3, we prove Theorem 1.1 and Section 4 is devoted to prove Theorem 1.2.

Notation. Throughout this paper, we make use of the following notations:

- $L^q(\Omega)$, for $1 \leq q \leq \infty$, denotes the Lebesgue space with usual norm denoted by $\|u\|_q$.
- $H_0^1(\Omega)$ denotes the Sobolev space endowed with inner product
$$(u,v) = \int_\Omega \nabla u \nabla v, \qquad \forall u,v \in H_0^1(\Omega).$$
The norm associated with this inner product will be denoted by $\|\ \|$.
- $W_0^{2,q}(\Omega)$ denotes the Sobolev space with norm
$$\|u\| = \left(\sum_{|\alpha| \leq 2} \|D^\alpha u\|_q^q \right)^{1/q}.$$
- Let us consider the space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}$ equipped with the norm $\|u\|_{C^1} = \max_{x\in\Omega} |u(x)| + \max_{x\in\Omega} |\nabla u(x)|$. If on $C_0^1(\overline{\Omega})$ we consider the pointwise partial ordering (i.e., $u \leq v$ if and only if $u(x) \leq v(x)$ for all $x \in \overline{\Omega}$), which is induced by the positive cone
$$C_0^1(\overline{\Omega})_+ = \left\{ u \in C_0^1(\overline{\Omega}) : u \geq 0 \text{ for all } x \in \Omega \right\},$$
then this cone has a nonempty interior given by
$$\text{int}(C_0^1(\overline{\Omega})_+) = \left\{ u \in C_0^1(\overline{\Omega}) : u > 0 \text{ for all } x \in \Omega \text{ and } \frac{\partial u}{\partial \nu}(x) < 0 \text{ for all } x \in \partial\Omega \right\},$$
where ν is the outward unit normal vector to $\partial\Omega$ at the point $x \in \partial\Omega$.
- $B_r(v)$ denotes the ball centered at $v \in C_0^1(\overline{\Omega})$ with radius $r > 0$ (with respect to the topology of $C_0^1(\overline{\Omega})$).
- The function $d(x) = d(x, \partial\Omega)$ denotes the distance from a point $x \in \overline{\Omega}$ to the boundary $\partial\Omega$, where $\overline{\Omega} = \Omega \cup \partial\Omega$ is the closure of $\Omega \subset \mathbb{R}^N$.
- We denote by ϕ_1 the $L^\infty(\Omega)$-normalized (that is, $\|\phi_1\|_\infty = 1$) positive eigenfunction for the smallest eigenvalue $\lambda_1 > 0$ of $\left(-\Delta, H_0^1(\Omega)\right)$.
- If u is a measurable function, we denote the positive and negative parts by $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$, respectively.
- If A is a measurable set in $\mathbb{R}^N$, we denote by $|A|$ the Lebesgue measure of A.
- $k, c, c_1, c_2, \ldots$ and C denote (possibly different from line to line) positive constants.
- The arrow $\rightharpoonup$ (respectively, $\to$) denotes weak (respectively strong) convergence.

2 Preliminaries

In this section, we will establish some preliminaries which will be important for our work. We reduce the study of the existence of positive solutions for (P_λ) to the existence of positive solutions of a singular elliptic problem. In particular, we will prove a nonexistence result and a sub-supersolution theorem.

We denote by ϕ_1 the $L^\infty(\Omega)$-normalized positive eigenfunction for the smallest eigenvalue $\lambda_1 > 0$ of $\left(-\Delta, H_0^1(\Omega)\right)$. We start by proving that ϕ_1 satisfies the assumption (H). We consider the following assumption.

(H') There is $q > N$ such that $a\phi_1^{-\gamma} \in L^q(\Omega)$.

Lemma 2.1. *Assumptions (H) and (H') are equivalent.*

Proof. Suppose that (H) holds. One has $\phi_1 \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ and $\varphi \in C_0^1(\overline{\Omega})_+$. Then, from Proposition 1 in [24] there exists $k > 0$ such that $\phi_1 \geq k\varphi$ in Ω and hence $a\phi_1^{-\gamma} \leq k^{-\gamma} a\varphi^{-\gamma} \in L^q(\Omega)$, proving (H').

If (H') holds, then the function $\varphi = \phi_1$ and q satisfy (H). This concludes the proof. □

Remark 2.2.

a) The arguments in the proof of Lemma 2.1 can be used to prove that if (H) holds, then any function $u \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ satisfies the assumption (H), too.

b) If φ satisfies the assumption (H) then $a\varphi^{1-\gamma}, a \in L^q(\Omega)$. Indeed, $a = a\varphi^{-\gamma}\varphi^{\gamma} \leq \|\varphi\|_\infty^\gamma a\varphi^{-\gamma} \in L^q(\Omega)$ and $a\varphi^{1-\gamma} \leq \|\varphi\|_\infty a\varphi^{-\gamma} \in L^q(\Omega)$.

c) It is well known that $\phi_1 \in C^1(\overline{\Omega})$ and satisfies $cd(x) \leq \phi_1(x) \leq Cd(x)$, $x \in \Omega$, for some constants $c, C > 0$ (see [31]).

Now, we observe that the natural energy functional corresponding to the problem (P_λ) is the following:

$$Q(u) = \frac{1}{2}\int_\Omega (1 + 2u^2)|\nabla u|^2 + \frac{1}{\gamma - 1}\int_\Omega a(x)F(|u|) - \frac{\lambda}{p+1}\int_\Omega |u|^{p+1}, \qquad u \in A(Q),$$

where

$$A(Q) = \left\{u \in H_0^1(\Omega) : \int_\Omega a(x)F(|u|) < \infty \text{ and } \int_\Omega |u|^{p+1} < \infty\right\}$$

and the function $F : [0, \infty) \to [0, \infty]$ satisfies $F'(s) = s^{-\gamma}$ for $s > 0$ (see [1] for a complete definition of F).

However, this functional is not well defined, because $\int_\Omega u^2|\nabla u|^2 dx$ is not finite for all $u \in H_0^1(\Omega)$, hence it is difficult to apply variational methods directly. In order to overcome this difficulty, we use the following change of variables introduced by [13], namely, $v := g^{-1}(u)$, where g is defined by

$$\begin{cases} g'(t) = \frac{1}{(1+2|g(t)|^2)^{\frac{1}{2}}} & \text{in } [0, \infty), \\ g(t) = -g(-t) & \text{in } (-\infty, 0]. \end{cases} \tag{2.1}$$

We list some properties of g, whose proofs can be found in [2, 13, 22, 30].

Lemma 2.3. *The function g satisfies the following properties:*

(1) *g is uniquely defined, C^∞ and invertible;*

(2) $g(0) = 0$;

(3) $0 < g'(t) \leq 1$ *for all* $t \in \mathbb{R}$;

(4) $\frac{1}{2}g(t) \leq tg'(t) \leq g(t)$ *for all* $t > 0$;

(5) $|g(t)| \leq |t|$ *for all* $t \in \mathbb{R}$;

(6) $|g(t)| \leq 2^{1/4}|t|^{1/2}$ *for all* $t \in \mathbb{R}$;

(7) $(g(t))^2 - g(t)g'(t)t \geq 0$ *for all* $t \in \mathbb{R}$;

(8) *There exists a positive constant C such that* $|g(t)| \geq C|t|$ *for* $|t| \leq 1$ *and* $|g(t)| \geq C|t|^{1/2}$ *for* $|t| \geq 1$;

(9) $g''(t) < 0$ *when* $t > 0$ *and* $g''(t) > 0$ *when* $t < 0$;

(10) *the function* $(g(t))^{1-\gamma}$ *for* $\gamma > 1$ *is decreasing for all* $t > 0$;

(11) *the function* $(g(t))^{-\gamma}$ *is decreasing for all* $t > 0$;

(12) $|g(t)g'(t)| < 1/\sqrt{2}$ *for all* $t \in \mathbb{R}$;

(13) $g^2(ts) \geq tg^2(s)$ *for all* $t \geq 1$ *and* $s \geq 0$.

Corollary 2.4. *For each $s > 0$ there exists a constant $K > 0$ such that $|t^\gamma \ln(g(t))| \leq K$ for all $0 < t \leq s$.*

Proof. Since $h(t) = t^\gamma \ln(g(t))$, $t > 0$, is a continuous function it is sufficient to show that $\lim_{t\to 0} t^\gamma \ln(g(t)) = 0$. From Lemma 2.3 (8) one has

$$|t^\gamma \ln(g(t))| \leq |C^{-\gamma} g^\gamma(t) \ln(g(t))|,$$

for all $0 < t \leq 1$, which implies that $\lim_{t\to 0} t^\gamma \ln(g(t)) = 0$, because $\lim_{t\to 0} t^\gamma \ln(t) = 0$ and $\lim_{t\to 0} g(t) = 0$. □

After a change of variable $v = g^{-1}(u)$, we define an associated problem

$$\begin{cases} -\Delta v = [a(x)(g(v))^{-\gamma} + \lambda (g(v))^p]\, g'(v) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{Q_λ}$$

We say that a function $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a weak solution (solution, for short) of (Q_λ) if $v > 0$ a.e. in Ω, and, for every $\psi \in H_0^1(\Omega)$,

$$a(x)(g(v))^{-\gamma} g'(v)\psi, (g(v))^p g'(v)\psi \in L^1(\Omega)$$

and

$$\int_\Omega \nabla v \nabla \psi = \int_\Omega a(x)(g(v))^{-\gamma} g'(v)\psi + \lambda \int_\Omega (g(v))^p g'(v)\psi.$$

It is easy to see that problem (Q_λ) is equivalent to our problem (P_λ), which takes $u = g(v)$ as its solutions. Thus, our goal is reduced to proving the existence, nonexistence and multiplicity of solutions for the family of problems (Q_λ).

In order to study problem (Q_λ), one introduces the assumption:

$(H)_d$ There are $\varphi \in C_0^1(\overline{\Omega})$ and $q > N$ such that $\varphi > 0$ on Ω and $ag^{-\gamma}(\varphi)g'(\varphi) \in L^q(\Omega)$.

The following lemma show the relation between the assumptions (H) and $(H)_d$.

Lemma 2.5. *Suppose that the function φ satisfies (H). Then φ satisfies $(H)_d$. Moreover, $ag^{1-\gamma}(\varphi) \in L^q(\Omega)$ if $\gamma \neq 1$ and $a(x)\ln(g(\varphi)) \in L^q(\Omega)$ if $\gamma = 1$.*

Proof. Let $0 < \epsilon < 1$ such that $\epsilon\|\varphi\|_\infty < 1$ holds. By using (3), (8), (9) and (11) of Lemma 2.3 and Corollary 2.4 (if $\gamma = 1$) we find

$$ag^{-\gamma}(\varphi)g'(\varphi) \le ag^{-\gamma}(\epsilon\varphi)g'(\epsilon\varphi) \le C^{-\gamma}\epsilon^{-\gamma}a\varphi^{-\gamma} \in L^q(\Omega),$$

$$ag^{1-\gamma}(\varphi) \le g(\|\varphi\|_\infty)ag^{-\gamma}(\epsilon\varphi) \le g(\|\varphi\|_\infty)\epsilon^{-\gamma}C^{-\gamma}a\varphi^{-\gamma} \in L^q(\Omega)$$

and

$$|a(x)\ln(g(\varphi))| = |a(x)\varphi^{-\gamma}\varphi^{\gamma}\ln(g(\varphi))| \le Ka(x)\varphi^{-\gamma} \in L^q(\Omega),$$

namely, $ag^{-\gamma}(\varphi)g'(\varphi) \in L^q(\Omega)$ and $ag^{1-\gamma}(\varphi) \in L^q(\Omega)$ and $a(x)\ln(g(\varphi)) \in L^q(\Omega)$ if $\gamma = 1$. □

To prove the nonexistence of solutions for (Q_λ) we define the function $m(x) = \min\{a(x), 1\} \in L^\infty(\Omega)$ and we will denote by $\lambda_1[m]$ the principal eigenvalue of

$$\begin{cases} -\Delta u = \lambda m(x)u & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{A}$$

It is known that $\lambda_1[m]$ is simples, $\lambda_1[m] > 0$, and the associated eigenfunction $\tilde{\phi}_1$ can be chosen such that $\tilde{\phi}_1 > 0$ in Ω (see [14, Theorem 6.2.9]).

Next, we prove the nonexistence of positive solutions for (Q_λ).

Lemma 2.6. *There exists a constant $\lambda^* > 0$ such that problem (Q_λ) has no solution for all $\lambda \in (\lambda^*, \infty)$.*

Proof. Let us start by defining the function $j_\lambda(t) = (g^{-\gamma}(t)g'(t) + \lambda g^p(t)g'(t))/t$ for $t > 0$. Using (4) of Lemma 2.3 we have that

$$j_\lambda(t) \ge \frac{g^{1-\gamma}(t)}{2t^2} + \lambda\frac{g^{p+1}(t)}{2t^2}, \qquad t > 0.$$

We now distinguish two cases:

Case $\gamma > 1$. From (5) and (8) of Lemma 2.3 we get

$$j_\lambda(t) \ge \begin{cases} \dfrac{t^{-1-\gamma}}{2} + \lambda\dfrac{C^{p+1}t^{p-1}}{2} & \text{if } 0 < t \le 1, \\ \dfrac{t^{-1-\gamma}}{2} + \lambda\dfrac{C^{p+1}t^{(p-3)/2}}{2} & \text{if } t \ge 1. \end{cases} \tag{2.2}$$

In order to find a lower bound for the function j_λ we observe that the function

$$\tilde{f}(t) = \frac{t^{-1-\gamma}}{2} + \lambda\frac{C^{p+1}t^{p-1}}{2}, \qquad t > 0,$$

has a global minimizer

$$t_\lambda = \left[\frac{(1+\gamma)}{\lambda(p-1)C^{p+1}}\right]^{\dfrac{1}{p+\gamma}},$$

such that $t_\lambda < 1$ for λ large enough and

$$\min_{t>0} \tilde{f}(t) = \tilde{f}(t_\lambda) = \frac{1}{2}\left[\frac{\lambda(p-1)C^{p+1}}{1+\gamma}\right]^{\frac{1+\gamma}{p+\gamma}} \left(\frac{p+\gamma}{p-1}\right). \tag{2.3}$$

Hereafter, we fix λ such that $t_\lambda < 1$. Then, by using (2.2) and (2.3), we infer that

$$\min_{t>0} j_\lambda(t) \geq \min\left\{\frac{1}{2}\left[\frac{\lambda(p-1)C^{p+1}}{1+\gamma}\right]^{\frac{1+\gamma}{p+\gamma}} \left(\frac{p+\gamma}{p-1}\right),\ \lambda\frac{C^{p+1}}{2}\right\}$$

and as a consequence there exists λ^* such that

$$j_{\lambda^*}(t_{\lambda^*}) := \min_{t>0} j_{\lambda^*}(t) \geq \lambda_1[m]. \tag{2.4}$$

Case $\gamma \leq 1$. From (8) of Lemma 2.3 we get

$$j_\lambda(t) \geq \begin{cases} \dfrac{C^{1-\gamma}t^{-1-\gamma}}{2} + \lambda\dfrac{C^{p+1}t^{p-1}}{2} & \text{if } 0 < t \leq 1, \\ \dfrac{C^{1-\gamma}t^{(-3-\gamma)/2}}{2} + \lambda\dfrac{C^{p+1}t^{(p-3)/2}}{2} & \text{if } t \geq 1. \end{cases} \tag{2.5}$$

In order to find a lower bound for the function j_λ we observe that the function

$$\tilde{h}(t) = \frac{C^{1-\gamma}t^{-1-\gamma}}{2} + \lambda\frac{C^{p+1}t^{p-1}}{2}, \qquad t > 0,$$

has a global minimizer

$$t_\lambda = \left[\frac{(1+\gamma)}{\lambda(p-1)C^{p+\gamma}}\right]^{\frac{1}{p+\gamma}},$$

such that $t_\lambda < 1$ for λ large enough and

$$\min_{t>0} \tilde{h}(t) = \tilde{h}(t_\lambda) = \frac{C^2}{2}\left[\frac{\lambda(p-1)}{1+\gamma}\right]^{\frac{1+\gamma}{p+\gamma}} \left[\frac{p+\gamma}{p-1}\right]. \tag{2.6}$$

Hereafter, we fix λ such that $t_\lambda < 1$. Then, by using (2.5) and (2.6), we infer that

$$\min_{t>0} j_\lambda(t) \geq \min\left\{\frac{C^2}{2}\left[\frac{\lambda(p-1)}{1+\gamma}\right]^{\frac{1+\gamma}{p+\gamma}} \left(\frac{p+\gamma}{p-1}\right),\ \lambda\frac{C^{p+1}}{2}\right\}$$

and as a consequence there exists λ^* such that

$$j_{\lambda^*}(t_{\lambda^*}) := \min_{t>0} j_{\lambda^*}(t) \geq \lambda_1[m]. \tag{2.7}$$

Now, arguing by contradiction, we suppose that for some $\lambda > \lambda^*$ problem (Q_λ) has a solution v_λ, where λ^* is defined in (2.4) (if $\gamma > 1$) and (2.7) (if $\gamma \leq 1$). By taking $\tilde{\phi}_1$ as a test function in the equation satisfied by v_λ and v_λ in the equation satisfied by $\tilde{\phi}_1$ we obtain

$$\begin{aligned}\int (a(x)g^{-\gamma}(v_\lambda)+\lambda^* g^p(v_\lambda))g'(v_\lambda)\tilde{\phi}_1 &\geq \int m(x)(g^{-\gamma}(v_\lambda)+\lambda^* g^p(v_\lambda))g'(v_\lambda)\tilde{\phi}_1\\ &\geq \int m(x) j_{\lambda^*}(t_{\lambda^*})v_\lambda\tilde{\phi}_1\\ &\geq \int \lambda_1[m]m(x)v_\lambda\tilde{\phi}_1\\ &= \int \nabla\tilde{\phi}_1\nabla v_\lambda\\ &= \int (a(x)g^{-\gamma}(v_\lambda)+\lambda g^p(v_\lambda))g'(v_\lambda)\tilde{\phi}_1\end{aligned}$$

and hence $\lambda^* \geq \lambda$, which is impossible by the choice of λ. By virtue of the relation between (P_λ) and (Q_λ) we deduce that problem (P_λ) has no solution for $\lambda > \lambda^*$. □

Now, we define the notions of subsolution and supersolution and prove a sub-supersolution theorem.

Definition 2.7. We say that v is a subsolution of problem (Q_λ) if $v \in H_0^1(\Omega)\cap L^\infty(\Omega)$, $v > 0$ in Ω, $a(x)(g(v))^{-\gamma}g'(v)\psi, (g(v))^p g'(v)\psi \in L^1(\Omega)$ and

$$\int_\Omega \nabla v\nabla\psi \leq \int_\Omega a(x)(g(v))^{-\gamma}g'(v)\psi + \lambda\int_\Omega (g(v))^p g'(v)\psi,$$

for all $\psi \in H_0^1(\Omega), \psi \geq 0$ in Ω. Similarly, $v \in H_0^1(\Omega)\cap L^\infty(\Omega)$, $v > 0$ in Ω, is a supersolution of (Q_λ) if $a(x)(g(v))^{-\gamma}g'(v)\psi, (g(v))^p g'(v)\psi \in L^1(\Omega)$ and

$$\int_\Omega \nabla v\nabla\psi \geq \int_\Omega a(x)(g(v))^{-\gamma}g'(v)\psi + \lambda\int_\Omega (g(v))^p g'(v)\psi,$$

for all $\psi \in H_0^1(\Omega), \psi \geq 0$ in Ω.

Theorem 2.8. *Let $\underline{v}$ and $\overline{v}$ be a subsolution respectively a supersolution of problem (Q_λ) such that $\underline{v} \leq \overline{v}$ in Ω. Then there exists a solution $v \in H_0^1(\Omega)\cap L^\infty(\Omega)$ of (Q_λ) such that $\underline{v} \leq v \leq \overline{v}$ in Ω.*

Proof. We define a truncated function $\tilde{g} : \Omega\times\mathbb{R} \to \mathbb{R}$ by letting,

$$\tilde{g}(x,t) = \begin{cases} g^p(\underline{v}(x))g'(\underline{v}(x)) & \text{if } t \leq \underline{v}(x),\\ g^p(t)g'(t) & \text{if } \underline{v}(x) \leq t \leq \overline{v}(x),\\ g^p(\overline{v}(x))g'(\overline{v}(x)) & \text{if } \overline{v}(x) \leq t.\end{cases}$$

Clearly, $\tilde{g}$ is a Carathéodory function. Moreover, (3) and (5) of Lemma 2.3 imply that

$$|\tilde{g}(x,t)| \leq |\overline{v}(x)|^p \leq \|\overline{v}\|_\infty^p =: c, \tag{2.8}$$

for all $(x,t) \in \Omega\times\mathbb{R}$. We denote by $\tilde{G}(x,t) = \int_0^t \tilde{g}(x,s)ds$ the primitive of $\tilde{g}$ such that $\tilde{G}(x,0) = 0$.

Now, we consider the auxiliary singular elliptic problem

$$\begin{cases} -\Delta v = a(x)(g(v))^{-\gamma}g'(v) + \lambda\tilde{g}(x,v) & \text{in } \Omega,\\ v > 0 & \text{in } \Omega,\\ v(x) = 0 & \text{on } \partial\Omega.\end{cases} \tag{A_λ}$$

We will show that problem (A_λ) has a solution v such that $\underline{v} \leq v \leq \overline{v}$ in Ω. Thus, from definition of $\tilde{g}$ we obtain that v is a solution of (Q_λ). Define the function G as it follows:
if $0 < \gamma < 1$, $G(t) = \frac{g^{1-\gamma}(|t|)}{1-\gamma}$ and $t \in \mathbb{R}$,
if $\gamma = 1$,

$$G(t) = \begin{cases} \ln g(t), & \text{if } t > 0, \\ +\infty, & \text{if } t = 0, \end{cases}$$

if $\gamma > 1$,

$$G(t) = \begin{cases} \frac{g^{1-\gamma}(t)}{1-\gamma}, & \text{if } t > 0, \\ +\infty, & \text{if } t = 0. \end{cases}$$

We can associate to problem (A_λ) the following energy functional

$$I_\lambda(v) = \frac{1}{2}\|v\|^2 - \int_\Omega a(x)G(|v|) - \lambda \int_\Omega \tilde{G}(x,v), \tag{2.9}$$

for every $v \in D$, where

$$D = \left\{ v \in H_0^1(\Omega) : \int_\Omega a(x)G(|v|) \in \mathbb{R} \right\} \tag{2.10}$$

is the effective domain of I_λ. As we known, the functional I_λ fails to be Gâteaux differentiable because of the singular term, then we can not apply the critical point theory for functionals of class C^1.

The assumption (H) and Lemmas 2.1 and 2.5 imply that $aG(\phi_1) \in L^q(\Omega)$. In particular, one has $\phi_1 \in D$ and hence $D \neq \emptyset$. Then, using (2.8) and arguing as in the proof of Theorems 1.1 and 1.2 of [1] we can show that there exists a solution v of (A_λ) and it satisfies

$$I_\lambda(v) = \inf_{z \in D} I_\lambda(z).$$

It remains to check that $\underline{v} \leq v \leq \overline{v}$ in Ω. We set $(v - \underline{v})^- = \max\{-(v - \underline{v}), 0\}$. Using that $\underline{v}$ is a subsolution and v is a solution, we have

$$\int_\Omega \nabla \underline{v} \nabla (v - \underline{v})^- \leq \int_\Omega a(x)(g(\underline{v}))^{-\gamma} g'(\underline{v})(v - \underline{v})^- + \lambda \int_\Omega (g(\underline{v}))^p g'(\underline{v})(v - \underline{v})^-,$$

$$\int_\Omega \nabla v \nabla (v - \underline{v})^- = \int_\Omega a(x)(g(v))^{-\gamma} g'(v)(v - \underline{v})^- + \lambda \int_\Omega \tilde{g}(x,v)(v - \underline{v})^-,$$

and applying (9), (10) and (11) of Lemma 2.3, we find

$$\begin{aligned} -\int_\Omega |\nabla (v - \underline{v})^-|^2 &\geq \int_{\{v<\underline{v}\}} a(x)((g(v))^{-\gamma} g'(v) - (g(\underline{v}))^{-\gamma} g'(\underline{v}))(v - \underline{v})^- \\ &\quad + \lambda \int_{\{v<\underline{v}\}} (\tilde{g}(x,v) - (g(\underline{v}))^p g'(\underline{v}))(v - \underline{v})^- \\ &\geq \lambda \int_{\{v<\underline{v}\}} (\tilde{g}(x,v) - (g(\underline{v}))^p g'(\underline{v}))(v - \underline{v})^- \\ &= \lambda \int_{\{v<\underline{v}\}} ((g(\underline{v}))^p g'(\underline{v}) - (g(\underline{v}))^p g'(\underline{v}))(v - \underline{v})^- \\ &= 0, \end{aligned}$$

namely $\|(v - \underline{v})^-\| = 0$, which means that $\underline{v} \leq v$ in Ω.

Similarly, setting $(v-\overline{v})^+ = \max\{v-\overline{v}, 0\}$ and using that $\overline{v}$ is a supersolution and v is a solution, jointly with (9), (10) and (11) of Lemma 2.3, we get

$$\begin{aligned}\int_\Omega |\nabla(v-\overline{v})^+|^2 &\le \int_{\{\overline{v}<v\}} a(x)((g(v))^{-\gamma}g'(v) - (g(\overline{v}))^{-\gamma}g'(\overline{v}))(v-\overline{v})^+ \\ &\quad + \lambda \int_{\{\overline{v}<v\}} (\tilde{g}(x,v) - (g(\overline{v}))^p g'(\overline{v}))(v-\overline{v})^+ \\ &\le \lambda \int_{\{\overline{v}<v\}} (\tilde{g}(x,v) - (g(\overline{v}))^p g'(\overline{v}))(v-\overline{v})^+ \\ &= \lambda \int_{\{\overline{v}<v\}} ((g(\overline{v}))^p g'(\overline{v}) - (g(\overline{v}))^p g'(\overline{v}))(v-\overline{v})^+ \\ &= 0,\end{aligned}$$

namely $\|(v-\overline{v})^+\| = 0$, which means that $v \le \overline{v}$ in Ω. This completes the proof of the theorem. □

Remark 2.9.

a) Arguing as in the proof of Lemmas 2.1 and 2.5 we can show that $\mathrm{int}(C_0^1(\overline{\Omega})_+) \subset D$ (see (2.10)). Hence it makes sense to consider the local minimum obtained in Lemma 4.2, because $v_\lambda \in \mathrm{int}(C_0^1(\overline{\Omega})_+) \subset D$.

b) If $0 < \gamma < 1$ holds, then $I_\lambda(v) < 0$. Indeed, applying Lemma 2.6 (8) we obtain

$$I_\lambda(v) \le I_\lambda(t\phi_1) \le \frac{t^2}{2}\|\phi_1\|^2 - \frac{C^{1-\gamma}t^{1-\gamma}}{1-\gamma}\int_\Omega a(x)\phi_1^{1-\gamma} < 0,$$

provided $0 < t < 1$ is small enough.

The following lemma shows the existence of a subsolution of (Q_λ) for all $\lambda > 0$.

Lemma 2.10. *If $v_0 \in H_0^1(\Omega)$ is the unique weak solution of (Q_0), then $v_0 \in C_0^1(\overline{\Omega})$ and $v_0(x) \ge Cd(x)$ in Ω for some constant $C > 0$. Moreover, $a(x)(g(v_0))^{-\gamma}g'(v_0) \in L^q(\Omega)$ and v_0 is a subsolution of (Q_λ) for all $\lambda > 0$.*

Proof. From Lemma 2.1 and Remark 2.2 $b)$ one has $a(x)\phi_1^{1-\gamma} \in L^q(\Omega), q > 1$, and hence, the existence of a unique weak solution $v_0 \in H_0^1(\Omega)$ of (Q_0) follows from Theorem 1.3 in [2]. Now we want to show that $v_0 \in C_0^1(\overline{\Omega})$. Using Theorem 3 of Brezis–Nirenberg [10] there exist constants $c_1, c_2 > 0$ such that $v_0(x) \ge c_2 d(x) \ge c_1\phi_1(x)$ in Ω and $c_1\phi_1(x) < 1$ in Ω. By Lemma 2.3 (3), (8), (11) and Lemma 2.1,

$$a(x)(g(v_0))^{-\gamma}g'(v_0) \le C^{-\gamma}c_1^{-\gamma}a(x)\phi_1^{-\gamma} \in L^q(\Omega),$$

that is, $a(x)(g(v_0))^{-\gamma}g'(v_0) \in L^q(\Omega)$ with $q > N$. Thus, by elliptic regularity, $v_0 \in W_0^{2,q}(\Omega)$, and then by the Sobolev embedding theorem we have $v_0 \in C_0^1(\overline{\Omega})$. Finally, from the fact that v_0 is a solution of (Q_0) and $v_0 \in C_0^1(\overline{\Omega})$ one deduces that v_0 is a subsolution of (Q_λ) for all $\lambda > 0$. This completes the proof. □

We end this section with the following lemma.

Lemma 2.11. *Let $v \in H_0^1(\Omega), v > 0$ in Ω, and suppose that*

$$\int_\Omega \nabla v \nabla \psi \geq \int_\Omega a(x)(g(v))^{-\gamma} g'(v)\psi + \lambda \int_\Omega (g(v))^p g'(v)\psi,$$

for all $\psi \in C_0^1(\overline{\Omega}), \psi \geq 0$, holds. Then

$$\int_\Omega \nabla v \nabla \psi \geq \int_\Omega a(x)(g(v))^{-\gamma} g'(v)\psi + \lambda \int_\Omega (g(v))^p g'(v)\psi,$$

for all $\psi \in H_0^1(\Omega), \psi \geq 0$ in Ω, holds. In particular, $v \geq v_0$ in Ω, where v_0 is the unique solution of (Q_0).

Proof. Let $\psi \in H_0^1(\Omega), \psi \geq 0$ in Ω, then from the proof of Theorem 4.4 of [12] there exists $\psi_n \in C_0^\infty(\overline{\Omega}), \psi_n \geq 0$ such that $\psi_n \to \psi$ in $H_0^1(\Omega)$ and $\psi_n \to \psi$ a.e. in Ω. Hence,

$$\int_\Omega \nabla v \nabla \psi_n \geq \int_\Omega a(x)(g(v))^{-\gamma} g'(v)\psi_n + \lambda \int_\Omega (g(v))^p g'(v)\psi_n,$$

and using the Fatou lemma we deduce that

$$\int_\Omega \nabla v \nabla \psi \geq \int_\Omega a(x)(g(v))^{-\gamma} g'(v)\psi + \lambda \int_\Omega (g(v))^p g'(v)\psi,$$

proving the first statement of the lemma.

It remains to show that $v \geq v_0$ in Ω. For this, we take $(v - v_0)^-$ as a test function in the equation satisfied by v_0 and in the inequality satisfied by v, and arguing as in Theorem 2.8 one finds $v \geq v_0$ in Ω. The proof is complete. □

3 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. In the rest of this paper we will use the same notation introduced in the previous section.

Let us define

$$\begin{aligned}\mathcal{L} &= \{\lambda > 0 : \text{problem } (Q_\lambda) \text{ has at least one solution}\}\\ &= \{\lambda > 0 : \text{problem } (P_\lambda) \text{ has at least one solution}\}\end{aligned}$$

and set

$$\Lambda = \sup \mathcal{L}.$$

We start by proving the following lemma.

Lemma 3.1. *The set $\mathcal{L}$ is nonempty and Λ is finite.*

Proof. Let $\underline{v} = v_0$ and consider the problem

$$\begin{cases} -\Delta v = a(x)(g(\underline{v}))^{-\gamma} g'(\underline{v}) + 1 & \text{in } \Omega,\\ v > 0 & \text{in } \Omega,\\ v(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{T}$$

By using Lemma 2.10 we infer that $a(x)(g(\underline{v}))^{-\gamma} g'(\underline{v}) + 1 \in L^q(\Omega)$. Therefore problem (T) has a solution $\overline{v} \in W^{2,q}(\Omega)$ and by the Sobolev embedding theorem, $\overline{v} \in C_0^1(\overline{\Omega})$. Moreover,

$$-\Delta \overline{v} \geq a(x)(g(\underline{v}))^{-\gamma} g'(\underline{v}) = -\Delta \underline{v} \quad \text{in } \Omega,$$

which implies that $\overline{v} \geq \underline{v}$ in Ω. From this and Lemmas 2.3 (9), (11) and 2.10 we get that

$$\int_\Omega \nabla\overline{v}\nabla\psi \geq \int_\Omega a(x)(g(\overline{v}))^{-\gamma}g'(\overline{v})\psi + \lambda\int_\Omega (g(\overline{v}))^p g'(\overline{v})\psi,$$

for all $\psi \in H_0^1(\Omega), \psi \geq 0$ in Ω, and for $\lambda > 0$ satisfying $\lambda\|(g(\overline{v}))^p g'(\overline{v})\|_\infty \leq 1$. For such values of λ, we can apply Theorem 2.8 to deduce the existence of a solution v of (Q_λ) such that $\underline{v} \leq v \leq \overline{v}$ in Ω (and consequently $v \in L^\infty(\Omega)$). Therefore $\mathcal{L} \neq \emptyset$.

By Lemma 2.6 we obtain that Λ is finite. The proof is complete. □

Following [19] we introduce

$$\lambda_* = \sup_{v\in S}\inf_{\psi\in\Phi}\{L(v,\psi)\} \tag{3.1}$$

where

$$L(v,\psi) := \frac{\int_\Omega \nabla v\nabla\psi - \int_\Omega a(x)(g(v))^{-\gamma}g'(v)\psi}{\int_\Omega (g(v))^p g'(v)\psi}$$

is the extended functional and

$$\Phi = \left\{\psi \in C_0^1(\overline{\Omega})\backslash\{0\} : \psi \geq 0 \text{ in } \Omega\right\},$$

$$S = \left\{v \in H_0^1(\Omega) \cap L^\infty(\Omega) : v \geq C(v)d(x) \text{ in } \Omega\right\},$$

where $0 < C(v) < \infty$ is a positive constant which can depend on v. If $v \in S$ then $v \geq k\phi_1$ in Ω for some $k > 0$ (see Remark 2.2 c)), and from Lemmas 2.1, 2.3 and 2.5 it follows that L is well defined.

Some properties of λ_* are stated in the following theorem.

Theorem 3.2. *The following properties hold true:*

a) $0 < \lambda_* < \infty$.

b) $\lambda_* = \Lambda$.

Proof. *a*) From Lemma 3.1 there exist $\lambda > 0$ and $v \in H_0^1(\Omega) \cap L^\infty(\Omega), v > 0$ in Ω, such that

$$\int_\Omega \nabla v\nabla\psi = \int_\Omega a(x)(g(v))^{-\gamma}g'(v)\psi + \lambda\int_\Omega (g(v))^p g'(v)\psi,$$

for all $\psi \in H_0^1(\Omega)$, which together with Theorem 3 of Brezis–Nirenberg [10] implies that $v \in S$ and $0 < \lambda = L(v,\psi)$ for all $\psi \in \Phi$. As a consequence we get

$$0 < \lambda = \inf_{\psi\in\Phi}\{L(v,\psi)\} \leq \lambda_*.$$

To prove that $\lambda_* < \infty$, we argue by contradiction. Assume that $\lambda_* = \infty$. Then, by the definition of λ_* there exists $v \in S$ such that $\Lambda < \lambda := \inf_{\psi\in\Phi}\{L(v,\psi)\}$, that is,

$$\int_\Omega \nabla v\nabla\psi \geq \int_\Omega a(x)(g(v))^{-\gamma}g'(v)\psi + \lambda\int_\Omega (g(v))^p g'(v)\psi,$$

for all $\psi \in \Phi$. By using Lemma 2.11 we deduce that v is a supersolution of (Q_λ) and $v \geq v_0$ in Ω. Moreover, from Lemma 2.10 one has that v_0 is a subsolution of (Q_λ). As a consequence we can apply Theorem 2.8, with $\underline{v} = v_0$ and $\overline{v} = v$, to deduce the existence of a solution of (Q_λ), which implies $\lambda \le \Lambda$, contradicting the fact that $\lambda > \Lambda$. Therefore $\lambda_* < \infty$.

b) Let $v \in S$ such that $0 < \lambda = \inf_{\psi \in \Phi} \{L(v,\psi)\}$. Arguing as in *a*) we can prove that problem (Q_λ) has a solution, namely, $\lambda \in \mathcal{L}$ and since λ is arbitrary, we have $\lambda_* = \sup_{v\in S} \inf_{\psi\in\Phi} \{L(v,\psi)\} \le \Lambda$. We claim that $\lambda_* = \Lambda$. Otherwise, $\lambda_* < \Lambda$ and by the definition of Λ there exists $\lambda > \lambda_*$ such that problem (Q_λ) has a solution v. Again, arguing as in *a*) we find that $v \in S$ and $\lambda = \inf_{\psi\in\Phi} \{L(v,\psi)\} \le \lambda_*$, contradicting the fact that $\lambda > \lambda_*$. Therefore $\lambda_* = \Lambda$. This finishes the proof. □

Remark 3.3. We will compare parameter λ_* with parameters ϵ_0 and ϵ_1 obtained in Theorems 1.2 and 1.3 of [29]. First, note that when $h(x,t) = \lambda|t|^{r-1}t$ the hypothesis (h_2) in [29] is satisfied with $b(x) := \lambda, C = 1$ and in our notation $r - 1 = p$. Hence $b(x) := \lambda$ is the parameter and problem (Q_λ) (or equivalently problem (P_λ)) has a solution for all $0 < \lambda \le \epsilon_0$. As a consequence, by the definition of $\Lambda = \lambda_*$ (see Theorem 3.2), one has $\epsilon_0 \le \lambda_*$. Let us remark that $\epsilon_1 \le \epsilon_0$, because one of the solutions obtained in Theorem 1.3 is the same as in Theorem 1.2 (both theorems mentioned here are from [29]).

We claim that $\epsilon_0 < \lambda_*$. To show this, we will use some notations and results obtained in Lemma 2.3 of [29]. Let $\underline{v}, \overline{v} \in C_0^1(\overline{\Omega})$ be the sub and supersolution, respectively, obtained in Lemma 2.3 of [29]. Then, $0 < \underline{v} \le \overline{v}$ in Ω and $\overline{v}$ satisfies

$$\begin{cases} -\Delta\overline{v} = a(x)(g(\underline{v}))^{-\gamma}g'(\underline{v}) + 2C & \text{in } \Omega, \\ \overline{v} > 0 & \text{in } \Omega, \\ \overline{v}(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{F}$$

which together with Theorem 3 of Brezis–Nirenberg [10] implies that $\overline{v} \in S$.

Moreover, ϵ_0 satisfies (see end of proof Lemma 2.3 of [29])

$$1 - \epsilon_0(g(\overline{v}))^p \ge 0$$

and since $0 < g'(t) \le 1$ for all $t > 0$, this implies

$$\epsilon_0 \|(g(\overline{v}))^p g'(\overline{v})\|_\infty \le 1. \tag{3.2}$$

Let us evaluate $L(\overline{v},\psi), \psi \in \Phi$. From ($F$), $\underline{v} \le \overline{v}$ in Ω and Lemma 2.3 (9), (11) we get

$$\begin{aligned} L(\overline{v},\psi) &= \frac{\int_\Omega \nabla\overline{v}\nabla\psi - \int_\Omega a(x)(g(\overline{v}))^{-\gamma}g'(\overline{v})\psi}{\int_\Omega (g(\overline{v}))^p g'(\overline{v})\psi} \\ &= \frac{\int_\Omega a(x)(g(\underline{v}))^{-\gamma}g'(\underline{v})\psi - \int_\Omega a(x)(g(\overline{v}))^{-\gamma}g'(\overline{v})\psi + 2C\int_\Omega \psi}{\int_\Omega (g(\overline{v}))^p g'(\overline{v})\psi} \\ &\ge \frac{2C\int_\Omega \psi}{\|(g(\overline{v}))^p g'(\overline{v})\|_\infty \int_\Omega \psi}, \end{aligned}$$

whence

$$L(\overline{v},\psi) \ge \frac{2C}{\|(g(\overline{v}))^p g'(\overline{v})\|_\infty}.$$

Since $C = 1$ and $\overline{v} \in S$, this implies

$$\lambda_* = \sup_{v\in S} \inf_{\psi\in\Phi} \{L(v,\psi)\} \ge \inf_{\psi\in\Phi} \{L(\overline{v},\psi)\} \ge \frac{2}{\|(g(\overline{v}))^p g'(\overline{v})\|_\infty}$$

and therefore

$$\lambda_* \|(g(\overline{v}))^p g'(\overline{v})\|_\infty > 1. \tag{3.3}$$

From $\lambda_* \geq \epsilon_0$, (3.2) and (3.3) one deduces that $\epsilon_1 \leq \epsilon_0 < \lambda_*$.

We are now in position to prove Theorem 1.1.

Proof of Theorem 1.1. Let us show that problem (Q_λ) has a solution for $\lambda \in (0, \lambda_*)$ and no solution for $\lambda \in (\lambda_*, \infty)$, where λ_* is defined in (3.1). Let $\lambda \in (0, \lambda_*)$. Then, by the definition of λ_*, there exists $z \in S$ such that $\lambda \leq L(z, \psi)$ for all $\psi \in \Phi$. We deduce from this inequality and Lemma 2.11 that z is a supersolution of (Q_λ) with $z \geq v_0$ in Ω. Applying Theorem 2.8 with $\underline{v} = v_0$ and $\overline{v} = z$ we obtain that problem (Q_λ) has a solution v_λ with $\underline{v} \leq v_\lambda \leq \overline{v}$ in Ω. To show that $v_\lambda \in C_0^1(\overline{\Omega})$ we follow [2]. By Lemma 2.3 $(3), (5), (9), (11)$ and Lemma 2.10 we infer

$$a(x) g^{-\gamma}(v_\lambda) g'(v) \leq a(x) g^{-\gamma}(\underline{v}) g'(\underline{v}) \in L^q(\Omega)$$

and

$$g^p(v_\lambda) g'(v) \leq |\overline{v}|^p \leq \|\overline{v}\|_\infty^p \in L^\infty(\Omega)$$

and as a consequence there exist $z_1, z_2 \in C_0^{1,\alpha}(\overline{\Omega})$, for some $\alpha \in (0,1)$, satisfying

$$\int_\Omega \nabla z_1 \nabla \psi = \int_\Omega a(x)(g(v_\lambda))^{-\gamma} g'(v_\lambda)\psi \quad \text{and} \quad \int_\Omega \nabla z_2 \nabla \psi = \lambda \int_\Omega (g(v_\lambda))^p g'(v_\lambda)\psi,$$

for all $\psi \in H_0^1(\Omega)$. From this we get

$$\int_\Omega \nabla v_\lambda \nabla \psi = \int_\Omega \nabla z_1 \nabla \psi + \int_\Omega \nabla z_2 \nabla \psi,$$

for all $\psi \in H_0^1(\Omega)$, which implies $v_\lambda = z_1 + z_2$, and hence $v_\lambda \in C_0^{1,\alpha}(\overline{\Omega})$. Furthermore, by the strong maximum principle and the Hopf lemma we find that $v_\lambda \in \text{int}(C_0^1(\overline{\Omega})_+)$.

Finally, from Theorem 3.2 we have $\lambda_* = \Lambda$ and by the definition of Λ problem (Q_λ) has no solution for $\lambda > \lambda_* = \Lambda$. This completes the proof of the theorem. □

4 Proof of Theorem 1.2

In this section we are going to prove Theorem 1.2. In order to do this, we adapt the arguments carried out in [4]. From now on, we will assume $(H)_\infty$ and $3 < p < 22^* - 1$ hold. Proceeding as in Section 1 we can prove that:

- $a\phi_1^{-1-\gamma}, a\phi_1^{-\gamma} \in L^\infty(\Omega)$.
- $ag^{-1-\gamma}(\varphi)g'(\varphi), ag^{-1-\gamma}(\phi_1)g'(\phi_1) \in L^\infty(\Omega)$ and $ag^{1-\gamma}(\varphi) \in L^\infty(\Omega)$ if $\gamma \neq 1$, and $a(x)\ln(g(\varphi)) \in L^\infty(\Omega)$ if $\gamma = 1$.
- if v_λ is the solution obtained in Theorem 1.1, then
$$a(x) g^{-\gamma}(v_\lambda) g'(v_\lambda) \in L^\infty(\Omega). \tag{4.1}$$

We start by defining the functional

$$J_\lambda(v) = \frac{1}{2}\|v\|^2 - \int_\Omega a(x) G(|v|) - \frac{\lambda}{p+1}\int_\Omega g^{p+1}(v), \qquad v \in D. \tag{4.2}$$

It is worth recalling that $\text{int}(C_0^1(\overline{\Omega})_+) \subset D$ (see (2.10) and Remark 2.9). The functional J_λ fails to be Fréchet differentiable in $H_0^1(\Omega)$ because of the singular term, then critical point theory could not be applied to obtain the existence of solutions directly.

Remark 4.1. Let (H) and $0 < p < 3$ hold. Denote by $D^+ = \{u \in D : u \geq 0 \text{ a.e. in } \Omega\}$. Then, by Lemma 2.5 one has $D^+ \neq \emptyset$. The arguments carried out in [1,2] can be adapted to prove that problem (P_λ) has at least one solution for all $\lambda \in \mathbb{R}$.

a) Assume that $\lambda \geq 0$. By Lemma 2.3 (6)

$$J_\lambda(v) \geq \frac{1}{2}\|v\|^2 - \int_\Omega a(x)G(|v|) - \frac{\lambda 2^{\frac{p+1}{4}}}{p+1}\int_\Omega |v|^{\frac{p+1}{2}}$$
$$\geq \frac{1}{2}\|v\|^2 - \int_\Omega a(x)G(|v|) - C\|v\|^{\frac{p+1}{2}},$$

for all $v \in D^+$ and for some constant $C > 0$. Hence, since $1/2 < (p+1)/2 < 2$, we argue in a similar way to the proof of Lemma 2.1 of [1] to show that J_λ is coercive on D^+ and there exists $v_\lambda \in D^+$ such that

$$J_\lambda(v_\lambda) = \inf_{v \in D^+} J_\lambda(v).$$

Finally, considering the cases $\gamma \geq 1$ and $0 < \gamma < 1$ respectively, we argue in a similar way to the first part of the proof of Theorems 1.1 and 1.2 of [1] to show that v_λ is a solution of (Q_λ) (and consequently $u_\lambda = g(v_\lambda)$ is a solution of (P_λ)).

For $\lambda \leq 0$ we get

$$J_\lambda(v) \geq \frac{1}{2}\|v\|^2 - \int_\Omega a(x)G(|v|)$$

for all $v \in D^+$. We argue in the same way as in the case $\lambda \geq 0$ to show that there exists a solution v_λ of (Q_λ) (and consequently $u_\lambda = g(v_\lambda)$ is a solution of (P_λ)).

b) We define the following constraint sets

$$\mathcal{N}_1 = \left\{ v \in D^+ : \|v\|^2 - \int_\Omega (g(v))^p g'(v)v \geq \int_\Omega a(x)(g(v))^{-\gamma} g'(v)v \right\}$$

and

$$\mathcal{N}_2 = \left\{ v \in D^+ : \|v\|^2 - \int_\Omega (g(v))^p g'(v)v = \int_\Omega a(x)(g(v))^{-\gamma} g'(v)v \right\}.$$

Since $1 < p+1 < 4$, by Lemma 2.3 (6) we have $\lim_{t\to\infty} J_\lambda(tv) = \infty$ for all $v \in D^+$. Moreover, $\lim_{t\to 0^+} J_\lambda(tv) = \infty$ if $\gamma \geq 1$ and $\lim_{t\to 0^+} J_\lambda(tv) = 0$ if $0 < \gamma < 1$. Therefore, for all $v \in D^+$ there exists a $t(v) > 0$ such that $J_\lambda(t(v)v) = \inf_{t>0} J_\lambda(tv)$ and $t(v)v \in \mathcal{N}_2$. Using this fact, Lemma 2.3 (6) and that $1 < p+1 < 4$ we show that J_λ is coercive on $\mathcal{N}_1$ and there exists $v_\lambda \in \mathcal{N}_1$ such that $J_\lambda(v_\lambda) = \inf_{v\in\mathcal{N}_1} J_\lambda(v) = \inf_{v\in\mathcal{N}_2} J_\lambda(v)$. Finally, we argue in a similar way to the first part of the proof of Theorem 1.1 of [2] to show that v_λ is a solution of (Q_λ) (and consequently $u_\lambda = g(v_\lambda)$ is a solution of (P_λ)).

In this section, we denote by I_λ the functional defined in (2.9) of Theorem 2.8.
An important property of the solution obtained in Theorem 1.1 is the following.

Lemma 4.2. *Let $0 < \lambda < \lambda_*$. If v_λ is the solution of (Q_λ) obtained in Theorem 1.1, then v_λ is a local minimum of J_λ in the $C_0^1(\overline{\Omega})$ topology.*

Proof. Without loss of generality, we can assume that $\overline{v}$ is a solution of (Q_μ) for some $\mu \in (\lambda, \lambda^*)$. Hence, arguing as in Theorem 1.1 one has $\overline{v} \in C_0^1(\overline{\Omega})$ and by the strong maximum principle and the Hopf lemma we infer that $\overline{v} \in \text{int}(C_0^1(\overline{\Omega})_+)$. Now, the proof is based on the following claims.

Claim 1. $\overline{v} - v_\lambda \in \text{int}(C_0^1(\overline{\Omega})_+)$. We have

$$-\Delta(\overline{v} - v_\lambda) \geq a(x)((g(\overline{v}))^{-\gamma} g'(\overline{v}) - (g(v_\lambda))^{-\gamma} g'(v_\lambda)) + \lambda(g(\overline{v}))^p g'(\overline{v}) - (g(v_\lambda))^p g'(v_\lambda))$$

and by the mean value theorem there exist measurable functions $\theta_1(x)$ and $\theta_2(x)$ such that $v_\lambda(x) \leq \theta_1(x), \theta_2(x) \leq \overline{v}(x)$ for all $x \in \Omega$ and

$$\begin{aligned} -\Delta(\overline{v} - v_\lambda) \geq a(x)&((g(\theta_1(x)))^{-\gamma} g'(\theta_1(x)))'(\overline{v}(x) - v_\lambda(x)) \\ &+ \lambda((g(\theta_2(x))))^p g'(\theta_2(x))))'(\overline{v}(x) - v_\lambda(x)). \end{aligned} \tag{4.3}$$

From the definition of g' and Lemma 2.3 (3), it follows that

$$\begin{aligned} &(g^{-\gamma}(t) g'(t))' \geq -g^{-1-\gamma}(t)(\gamma + 2g^2(t)), && t > 0, \\ &|(g^p(t) g'(t))'| \leq p g^{p-1}(t) + 2 g^{p+1}(t), && t > 0, \end{aligned} \tag{4.4}$$

hold. Then, again by Lemma 2.3 (3), (11) one has

$$\begin{aligned} &(g^{-\gamma}(\theta_1(x)) g'(\theta_1(x)))' \geq -g^{-1-\gamma}(v_\lambda(x))(\gamma + 2g^2(\|\overline{v}\|_\infty)), \\ &|(g^p(\theta_2(x)) g'(\theta_2(x)))'| \leq p g^{p-1}(\|\overline{v}\|_\infty) + 2 g^{p+1}(\|\overline{v}\|_\infty), \end{aligned}$$

for all $x \in \Omega$. We set

$$c_1 = \|a g^{-1-\gamma}(v_\lambda)\|_\infty (\gamma + 2g^2(\|\overline{v}\|_\infty)), \qquad c_2 = p\lambda g^{p-1}(\|\overline{v}\|_\infty) + 2\lambda g^{p+1}(\|\overline{v}\|_\infty)$$

and $c = c_1 + c_2$. With these estimates and definitions, in view of (4.3), we get

$$-\Delta(\overline{v} - v_\lambda) \geq (-c_1 - c_2)(\overline{v} - v_\lambda) = -c(\overline{v} - v_\lambda)$$

that is

$$-\Delta(\overline{v} - v_\lambda) + c(\overline{v} - v_\lambda) \geq 0 \text{ in } \Omega,$$

and since $\overline{v} - v_\lambda \neq 0$, we can apply Theorem 3 of [10] to deduce the existence of constants $c_3, c_4 > 0$ such that

$$\overline{v} - v_\lambda \geq c_3 d(x) \geq c_4 \phi_1(x) \quad \text{in } \Omega.$$

As a consequence we obtain

$$\frac{\partial(\overline{v} - v_\lambda)}{\partial \nu} \leq c_4 \frac{\partial \phi_1}{\partial \nu} < 0 \quad \text{on } \partial\Omega,$$

which jointly with $\overline{v} - v_\lambda > 0$ in Ω means that $\overline{v} - v_\lambda \in \text{int}(C_0^1(\overline{\Omega})_+)$, and this proves Claim 1.

Claim 2. $v_\lambda - \underline{v} \in \text{int}(C_0^1(\overline{\Omega})_+)$. The proof is essentially equal to the one of Claim 1. Indeed, we set

$$c_1 = \|a g^{-1-\gamma}(\underline{v})\|_\infty (\gamma + 2g^2(\|\overline{v}\|_\infty)),$$

and from (4.4) and mean value theorem one has

$$\begin{aligned} -\Delta(v_\lambda - \underline{v}) &\geq a(x)((g(\theta_1(x)))^{-\gamma} g'(\theta_1(x)))'(v_\lambda - \underline{v}) + \lambda(g(v_\lambda))^p g'(v_\lambda) \\ &\geq -c_1(v_\lambda - \underline{v}) \end{aligned}$$

in Ω, because $\underline{v}(x) \leq \theta_1(x) \leq v_\lambda(x)$, and since $v_\lambda - \underline{v} \neq 0$, we can apply Theorem 3 of [10] to deduce the existence of constants $c_3, c_4 > 0$ such that

$$v_\lambda - \underline{v} \geq c_3 d(x) \geq c_4 \phi_1(x) \quad \text{in } \Omega.$$

As a consequence we obtain

$$\frac{\partial(v_\lambda - \underline{v})}{\partial \nu} \leq c_4 \frac{\partial \phi_1}{\partial \nu} < 0 \quad \text{on } \partial\Omega,$$

which jointly with $v_\lambda - \underline{v} > 0$ in Ω means that $v_\lambda - \underline{v} \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$, and this proves Claim 2.

Claim 3. There exists a ball $B = B_\epsilon(v_\lambda)$ in the $C_0^1(\overline{\Omega})$ topology satisfying

$$B \subset [\underline{v}, \overline{v}] := \left\{ v \in C_0^1(\overline{\Omega}) : \underline{v} \leq v \leq \overline{v} \text{ in } \Omega \right\}.$$

From Claims 1 and 2 there exists $\epsilon > 0$ such that the balls $B_1 = B_\epsilon(\overline{v} - v_\lambda), B_2 = B_\epsilon(v_\lambda - \underline{v}) \subset \operatorname{int}(C_0^1(\overline{\Omega})_+)$. We define $B = B_\epsilon(v_\lambda)$. Let $v \in B$. Notice that

$$\overline{v} - B_1 = B_\epsilon(v_\lambda) \quad \text{and} \quad \underline{v} + B_2 = B_\epsilon(v_\lambda),$$

and as a consequence there exist $z \in B_1, w \in B_2$ with

$$\underline{v} + w = v = \overline{v} - z,$$

which implies $\underline{v} < v < \overline{v}$ in Ω, that is, $v \in [\underline{v}, \overline{v}]$. Hence $B \subset [\underline{v}, \overline{v}]$.

We can finally complete the proof of the lemma. Let B as in Claim 3 and consider $v \in B$. Then,

$$\begin{aligned}
J_\lambda(v) - I_\lambda(v) &= -\frac{\lambda}{p+1} \int_\Omega g^{p+1}(v) + \lambda \int_\Omega \tilde{G}(x, v) \\
&= -\frac{\lambda}{p+1} \int_\Omega g^{p+1}(v) + \lambda \int_\Omega \int_0^{\underline{v}(x)} \tilde{g}(x,t)dtdx + \lambda \int_\Omega \int_{\underline{v}(x)}^{v(x)} \tilde{g}(x,t)dtdx \\
&= -\frac{\lambda}{p+1} \int_\Omega g^{p+1}(v) + \lambda \int_\Omega \int_0^{\underline{v}(x)} g^p(\underline{v}(x)) g'(\underline{v}(x)) dtdx \\
&\quad + \lambda \int_\Omega \int_{\underline{v}(x)}^{v(x)} g^p(t) g'(t) dtdx \\
&= \lambda \int_\Omega g^p(\underline{v}(x)) g'(\underline{v}(x)) \underline{v}(x) dx - \frac{\lambda}{p+1} \int_\Omega g^{p+1}(\underline{v}(x)) dx =: c
\end{aligned}$$

where c is a constant.

By virtue of the above equality, we obtain that v_λ is a $C_0^1(\overline{\Omega})$-local minimizer of J_λ. This finishes the proof. □

Remark 4.3. Since $\underline{v}, \overline{v} \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$, it follows that $[\underline{v}, \overline{v}] \subset \operatorname{int}(C_0^1(\overline{\Omega})_+)$ and then, by Remark 2.9, $J_\lambda(v), I_\lambda(v) \in \mathbb{R}$ for all $v \in [\underline{v}, \overline{v}]$. Furthermore, arguing as in Lemma 2.5 we infer

$$\left\{ v \in H_0^1(\Omega) : \underline{v} \leq v \leq \overline{v} \text{ in } \Omega \right\} \subset D.$$

Corollary 4.4. *Let $B = B_\epsilon(0) + v_\lambda$ be as in the proof of Lemma 4.2. Then for all $v \in B_\epsilon(0)$ we have*

$$J_\lambda(v_\lambda + v^+) - J_\lambda(v_\lambda) \geq 0,$$

holds.

Proof. As we have seen in the proof of Lemma 4.2,

$$\underline{v} < v_\lambda + v < \overline{v} \quad \text{in } \Omega, \tag{4.5}$$

for all $v \in B_\epsilon(0)$. We claim that

$$\underline{v} < v_\lambda + v^+ < \overline{v} \quad \text{in } \Omega,$$

for all $v \in B_\epsilon(0)$. Indeed, by using (4.5) one has

$$\underline{v} < v_\lambda + v = v_\lambda + v^+ - v^- \le v_\lambda + v^+ \quad \text{in } \Omega.$$

Now, let us show that $v_\lambda + v^+ < \overline{v}$ in Ω. Arguing by contradiction, suppose that there exists $x \in \Omega$ such that $v_\lambda(x) + v^+(x) \ge \overline{v}(x)$. Then, from $v_\lambda(x) < \overline{v}(x)$ we infer that $v(x) > 0$, and therefore $v^-(x) = 0$. Thus, the inequality (4.5) implies

$$\overline{v}(x) \le v_\lambda(x) + v^+(x) = v_\lambda(x) + v^+(x) - v^-(x) = v_\lambda(x) + v(x) < \overline{v}(x),$$

a contradiction.

Finally, we can argue as in the proof of Lemma 4.2 to get

$$J_\lambda(v_\lambda + v^+) - I_\lambda(v_\lambda + v^+) = c,$$

where c is a constant, and since $v_\lambda + v^+ \in H_0^1(\Omega)$, by Theorem 2.8, we deduce that

$$J_\lambda(v_\lambda + v^+) - J_\lambda(v_\lambda) = I_\lambda(v_\lambda + v^+) - I_\lambda(v_\lambda) \ge 0,$$

proving the corollary. □

For fixed $\lambda \in (0, \lambda_*)$, we look for a second solution in the form $z = w + v$, where $v \gneqq 0$ and $w = v_\lambda$ is the solution found in the preceding lemma. A straight calculation shows that v satisfies

$$\begin{aligned} -\Delta v = {} & a(x)((g(w+v))^{-\gamma} g'(w+v) - (g(w))^{-\gamma} g'(w)) \\ & + \lambda((g(w+v))^p g'(w+v) - (g(w))^p g'(w)). \end{aligned} \tag{4.6}$$

Denote by $g_\lambda(x,t)$ the right hand side of the preceding equation (with $g_\lambda(x,t) = 0$ for $t \le 0$) and set

$$\mathcal{J}_\lambda(v) = \frac{1}{2}\|v\|^2 - \int_\Omega G_\lambda(x,v), \tag{4.7}$$

where

$$G_\lambda(x,t) = \int_0^t g_\lambda(x,s)ds = \begin{cases} 0 & \text{if } t \le 0, \\ H_1(x,t) + H_2(x,t) + H_3(x,t) & \text{if } t \ge 0, \end{cases}$$

and

$$\begin{aligned} H_1(x,t) &= a(x)(G(w+t) - G(w)), \\ H_2(x,t) &= \frac{\lambda}{p+1}(g^{p+1}(w+t) - g^{p+1}(w)), \\ H_3(x,t) &= -a(x)g^{-\gamma}(w)g'(w)t - \lambda(g(w))^p g'(w)t, \end{aligned}$$

for $t \geq 0$.

We observe that by $(H)_\infty$, Lemma 2.3 $(3), (6), (11), (12)$ and (4.1) one has

$$|g_\lambda(x,t)| \leq c_1 + 2^{(p-3)/4}\lambda c_2 |t|^{(p-1)/2}, \tag{4.8}$$

where $c_1, c_2 > 0$ are constants which depends of $\|ag^{-\gamma}(w)g'(w)\|_\infty$, $\|w\|_\infty$ and p. From this it follows that $\mathcal{J}_\lambda \in C^1(H_0^1(\Omega), \mathbb{R})$.

We shall use the Mountain Pass Theorem to prove the existence of a nontrivial solution to (4.6). In order to do this, we need some preliminary lemmas.

Lemma 4.5. *$v = 0$ is a local minimum of $\mathcal{J}_\lambda$ in $H_0^1(\Omega)$.*

Proof. We write $v = v^+ - v^-$. Using the fact that w is a solution of (Q_λ) and $G(x,t) = 0$ for $t \leq 0$ we get

$$\begin{aligned}
\mathcal{J}_\lambda(v) &= \frac{1}{2}\|v^+\|^2 + \frac{1}{2}\|v^-\|^2 - \int_\Omega G_\lambda(x, v^+) + \frac{1}{2}\|w + v^+\|^2 - \frac{1}{2}\|w + v^+\|^2 \\
&= \frac{1}{2}\|v^-\|^2 - \int_\Omega \nabla w \nabla v^+ + \int_\Omega a(x)(g(w))^{-\gamma} g'(w) v^+ + \lambda \int_\Omega (g(w))^p g'(w) v^+ \\
&\quad + \frac{1}{2}\|w + v^+\|^2 - \int_\Omega a(x) G(w + v^+) - \frac{\lambda}{p+1}\int_\Omega g^{p+1}(w + v^+) \\
&\quad - \frac{1}{2}\|w\|^2 + \int_\Omega a(x) G(w) + \frac{\lambda}{p+1}\int_\Omega g^{p+1}(w) \\
&= \frac{1}{2}\|v^-\|^2 + J_\lambda(w + v^+) - J_\lambda(w).
\end{aligned}$$

This and Corollary 4.4 imply that $\mathcal{J}_\lambda(v) \geq 0$ for all $v \in B_\epsilon(0)$, where $B_\epsilon(0)$ is as in Corollary 4.4. This proves that $v = 0$ is a local minimum in the $C_0^1(\overline{\Omega})$ topology. Therefore, in view of (4.8), Theorem 1 in [10] applies and $v = 0$ is a local minimum of $\mathcal{J}_\lambda$ in the $H_0^1(\Omega)$ topology. This finishes the proof. □

Lemma 4.6. *If $v, w \in L^\infty(\Omega) \cap D$ are positive functions, then*

$$\lim_{t\to\infty} \int_\Omega \frac{a(x)G(v + tw)}{t^{(p+1)/2}} = 0$$

and

$$\int_\Omega g^{p+1}(v + tw) \geq t^{(p+1)/2} \int_\Omega g^{p+1}\left(\frac{v}{t} + w\right),$$

for all $t > 1$.

Proof. First we prove the limit. We divide the proof into three cases.

Case 1. $\gamma < 1$. In this case, by Lemma 2.3 (5) one has

$$0 < \frac{a(x)G(v + tw)}{t^{(p+1)/2}} = \frac{a(x)g^{1-\gamma}(v + tw)}{(1-\gamma)t^{(p+1)/2}} \leq \frac{a(x)(\frac{v}{t} + w)^{1-\gamma}}{(1-\gamma)t^{((p+1)/2)+\gamma-1}} \leq \frac{a(x)(v + w)^{1-\gamma}}{1-\gamma},$$

for all $t \geq 1$. Then taking the limit as $t \to \infty$ we get

$$\frac{a(x)G(v + tw)}{t^{(p+1)/2}} \to 0,$$

and from the Lebesgue dominated convergence theorem we find

$$\lim_{t\to\infty}\int_\Omega \frac{a(x)G(v+tw)}{t^{(p+1)/2}}=0.$$

This proves the case 1.

Case 2. $\gamma=1$. By Lemma 2.3 (3), (5)

$$\frac{a(x)\ln(g(v))}{t^{(p+1)/2}}\le\frac{a(x)G(v+tw)}{t^{(p+1)/2}}=\frac{a(x)\ln(g(v+tw))}{t^{(p+1)/2}}\le\frac{a(x)\left(\frac{v}{t}+w\right)}{t^{((p+1)/2)-1}}\le a(x)(v+w)$$

for all $t\ge1$, and thus

$$\left|\frac{a(x)G(v+tw)}{t^{(p+1)/2}}\right|\le\max\left\{|a(x)\ln(g(v))|,a(x)(v+w)\right\}.$$

Again, by the Lebesgue dominated convergence theorem we have

$$\lim_{t\to\infty}\int_\Omega \frac{a(x)G(v+tw)}{t^{(p+1)/2}}=0.$$

Case 3. $\gamma>1$. By Lemma 2.3 (3), (10) one has

$$0<\left|\frac{a(x)G(v+tw)}{t^{(p+1)/2}}\right|=\frac{a(x)g^{1-\gamma}(v+tw)}{|1-\gamma|t^{(p+1)/2}}\le\frac{a(x)g^{1-\gamma}(v)}{|1-\gamma|t^{(p+1)/2}}\le\frac{a(x)g^{1-\gamma}(v)}{|1-\gamma|},$$

for all $t\ge1$. By the Lebesgue dominated convergence theorem one finds

$$\lim_{t\to\infty}\int_\Omega \frac{a(x)G(v+tw)}{t^{(p+1)/2}}=0.$$

We now fix $t>1$. Then, from Lemma 2.3 (13) we have

$$g^{p+1}(v+tw)=\left[g^2\left(t\left(\frac{v}{t}+w\right)\right)\right]^{(p+1)/2}\ge\left[tg^2\left(\frac{v}{t}+w\right)\right]^{(p+1)/2}=t^{(p+1)/2}g^{p+1}\left(\frac{v}{t}+w\right),$$

and this implies that

$$\int_\Omega g^{p+1}(v+tw)\ge t^{(p+1)/2}\int_\Omega g^{p+1}\left(\frac{v}{t}+w\right),$$

for all $t>1$. The lemma is proved. □

Lemma 4.7. *Let $2<\theta<p+1$. Then, for all $t\ge0$,*

a) $-G_\lambda(x,t)+\frac{\theta}{p+1}g_\lambda(x,t)t\ge c-\frac{a(x)}{1-\gamma}t^{1-\gamma}$ *for some constant $c\in\mathbb{R}$, if $0<\gamma<1$;*

b) $-G_\lambda(x,t)+\frac{\theta}{p+1}g_\lambda(x,t)t\ge c-a(x)t+a(x)\ln(g(w))$ *for some constant $c\in\mathbb{R}$, if $\gamma=1$;*

c) $-G_\lambda(x,t)+\frac{\theta}{p+1}g_\lambda(x,t)t\ge c+\frac{a(x)}{1-\gamma}g^{1-\gamma}(w)$ *for some constant $c\in\mathbb{R}$, if $\gamma>1$.*

Proof. For convenience of notation we write

$$\begin{aligned} h_1(x,t) &= a(x)(g(w+t))^{-\gamma}g'(w+t), \\ h_2(x,t) &= \lambda(g(w+t))^p g'(w+t), \\ h_3(x,t) &= -a(x)(g(w))^{-\gamma}g'(w) - \lambda(g(w))^p g'(w), \end{aligned}$$

for $t \geq 0$. Thus,

$$\begin{aligned} -G_\lambda(x,t) + \frac{\theta}{p+1}g_\lambda(x,t)t = & -H_1(x,t) + \frac{\theta}{p+1}h_1(x,t)t \\ & -H_2(x,t) + \frac{\theta}{p+1}h_2(x,t)t \\ & -H_3(x,t) + \frac{\theta}{p+1}h_3(x,t)t \end{aligned}$$

for $t \geq 0$.

a) In this case, from Lemma 2.3 (5) we have

$$\begin{aligned} -H_1(x,t) + \frac{\theta}{p+1}h_1(x,t)t \geq -H_1(x,t) &\geq -\frac{a(x)}{1-\gamma}g^{1-\gamma}(w+t) \\ &\geq -\frac{a(x)}{1-\gamma}(w+t)^{1-\gamma} \\ &\geq -\frac{a(x)}{1-\gamma}(w^{1-\gamma}+t^{1-\gamma}) \\ &\geq -\frac{\|aw^{1-\gamma}\|_\infty}{1-\gamma} - \frac{a(x)}{1-\gamma}t^{1-\gamma} \end{aligned}$$

and since $p+1 > \theta$,

$$-H_3(x,t) + \frac{\theta}{p+1}h_3(x,t)t = \left(1 - \frac{\theta}{p+1}\right)(a(x)(g(w))^{-\gamma}g'(w) + \lambda(g(w))^p g'(w))t \geq 0. \quad (4.9)$$

Let us observe that this inequality is valid for all $\gamma > 0$.

Now, let us estimate $-H_2(x,t) + \frac{\theta}{p+1}h_2(x,t)t$. From Lemma 2.3 (4) one has

$$\begin{aligned} -H_2(x,t) + \frac{\theta}{p+1}h_2(x,t)t &= \frac{\lambda}{p+1}(-g^{p+1}(w+t) + g^{p+1}(w)) \\ &\quad + \frac{\theta\lambda}{p+1}(g(w+t))^p g'(w+t)t \\ &\geq \frac{\lambda}{p+1}\left[-g^{p+1}(w+t) + \frac{\theta}{2}\frac{g^{p+1}(w+t)t}{w+t} + g^{p+1}(w)\right] \\ &\geq \frac{\lambda}{p+1}\left[g^{p+1}(w+t)\left(-1 + \frac{\theta}{2}\frac{t}{\|w\|_\infty + t}\right) + g^{p+1}(w)\right], \end{aligned}$$

and therefore

$$-H_2(x,t) + \frac{\theta}{p+1}h_2(x,t)t > 0,$$

for all $t > \bar{t} := (2\|w\|_\infty)/(\theta - 2)$.

Moreover, for $0 \leq t \leq \bar{t}$, by using Lemma 2.3 (5) we get

$$-H_2(x,t) + \frac{\theta}{p+1} h_2(x,t)t \geq -\frac{\lambda}{p+1} g^{p+1}(w+t)$$
$$\geq -\frac{\lambda}{p+1}(w+t)^{p+1} \geq -\frac{\lambda}{p+1}(\|w\|_\infty + \bar{t})^{p+1}.$$

By setting $c_1 = -\frac{\lambda}{p+1}(\|w\|_\infty + \bar{t})^{p+1}$, we have proved that

$$-H_2(x,t) + \frac{\theta}{p+1} h_2(x,t)t \geq c_1, \quad \text{for all } t \geq 0. \tag{4.10}$$

Let us observe that this inequality is valid independent of $\gamma > 0$.

In view of the above inequalities we deduce that

$$-G_\lambda(x,t) + \frac{\theta}{p+1} g_\lambda(x,t)t \geq c - \frac{a(x)}{1-\gamma} t^{1-\gamma} \quad \text{for all } t \geq 0,$$

where $c = -\frac{\|aw^{1-\gamma}\|_\infty}{1-\gamma} + c_1$.

b) When $\gamma = 1$, by Lemma 2.3 (5), one has the inequality

$$-H_1(x,t) + \frac{\theta}{p+1} h_1(x,t)t \geq -H_1(x,t) = -a(x)\ln(g(w+t)) + a(x)\ln(g(w))$$
$$\geq -a(x)(w+t) + a(x)\ln(g(w))$$
$$\geq -\|aw\|_\infty - a(x)t + a(x)\ln(g(w)),$$

which combined with (4.9) and (4.10) yield

$$-G_\lambda(x,t) + \frac{\theta}{p+1} g_\lambda(x,t)t \geq c - a(x)t + a(x)\ln(g(w))$$

for some constant $c \in \mathbb{R}$.

c) Indeed, the inequality

$$-H_1(x,t) + \frac{\theta}{p+1} h_1(x,t)t \geq -H_1(x,t) = -\frac{a(x)}{1-\gamma} g^{1-\gamma}(w+t) + \frac{a(x)}{1-\gamma} g^{1-\gamma}(w)$$
$$\geq \frac{a(x)}{1-\gamma} g^{1-\gamma}(w),$$

combined with (4.9) and (4.10) yield

$$-G_\lambda(x,t) + \frac{\theta}{p+1} g_\lambda(x,t)t \geq c + \frac{a(x)}{1-\gamma} g^{1-\gamma}(w)$$

for some constant $c \in \mathbb{R}$. This concludes the proof. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. By Lemma 4.5 $u_0 = 0$ is a local minimizer of $\mathcal{J}_\lambda$ with respect to the topology of $H_0^1(\Omega)$. In the case where u_0 is not a strict local minimizer of $\mathcal{J}_\lambda$, we deduce the existence of further critical points of $\mathcal{J}_\lambda$, and then we are done. In this way, we may assume that

$$u_0 = 0 \text{ is a strict local minimizer of } \mathcal{J}_\lambda. \tag{4.11}$$

For all $t > 1$ we have

$$\mathcal{J}_\lambda(t\phi_1) = J_\lambda(w + t\phi_1) - J_\lambda(w)$$

and by Lemma 4.6 it follows that

$$\mathcal{J}_\lambda(t\phi_1) \le \frac{1}{2}\|w + t\phi_1\|^2 - \int_\Omega a(x)G(w + t\phi_1) - \frac{t^{(p+1)/2}\lambda}{p+1}\int_\Omega g^{p+1}\left(\frac{w}{t} + \phi_1\right) - J_\lambda(w)$$

and using again Lemma 4.6 and the Lebesgue dominated convergence theorem we yield $\lim_{t\to\infty} \mathcal{J}(t\phi_1) = -\infty$. From this and (4.11), we conclude that $\mathcal{J}_\lambda$ has the mountain pass geometry (see [5, Theorem 2.1]). It remains to prove the Palais–Smale condition. Let $4 < 2\theta < p+1$. Let $v_n \in H_0^1(\Omega)$ be such that $\mathcal{J}_\lambda(v_n) \to c$ ($c \in \mathbb{R}$) and $\mathcal{J}'_\lambda(v_n) \to 0$. From the former, respectively the latter multiplied by $\theta v_n/(p+1)$, we get

$$\frac{1}{2}\|v_n\|^2 - \int_\Omega G_\lambda(x, v_n) = c + o(1),$$

$$o(1)\|v_n\| \ge |\frac{\theta}{p+1}\|v_n\|^2 - \frac{\theta}{p+1}\int_\Omega g_\lambda(x, v_n)v_n| \ge \frac{-\theta}{p+1}\|v_n\|^2 + \frac{\theta}{p+1}\int_\Omega g_\lambda(x, v_n)v_n,$$

and therefore (remember that $G_\lambda(x,t) = g_\lambda(x,t)t = 0$ for $t \le 0$),

$$c + o(1) + o(1)\|v_n\| \ge \left(\frac{1}{2} - \frac{\theta}{p+1}\right)\|v_n\|^2 + \int_\Omega (-G_\lambda(x, v_n^+) + \frac{\theta}{p+1} g_\lambda(x, v_n^+)v_n^+).$$

From this and Lemma 4.7 we deduce that

$$c + o(1) + o(1)\|v_n\| \ge \begin{cases} \left(\frac{1}{2} - \frac{\theta}{p+1}\right)\|v_n\|^2 + c|\Omega| - \int_\Omega \frac{a(x)}{1-\gamma}(v_n^+)^{1-\gamma} & \text{if } \gamma < 1, \\ \left(\frac{1}{2} - \frac{\theta}{p+1}\right)\|v_n\|^2 + c|\Omega| - \int_\Omega a(x)v_n^+ + \int_\Omega a(x)\ln(g(w)) & \text{if } \gamma = 1, \\ \left(\frac{1}{2} - \frac{\theta}{p+1}\right)\|v_n\|^2 + c|\Omega| + \int_\Omega \frac{a(x)}{1-\gamma} g^{1-\gamma}(w) & \text{if } \gamma > 1. \end{cases}$$

Thus, in any case, by the Sobolev embedding theorem we have that the sequence $\{v_n\}$ is bounded in $H_0^1(\Omega)$ and a standard argument shows that, up to a subsequence, there exists $v \in H_0^1(\Omega)$ such that $v_n \to v$ in $H_0^1(\Omega)$. Therefore, the Palais–Smale condition has been verified.

Finally, an application of the mountain pass theorem yields a nontrivial critical point v of $\mathcal{J}_\lambda$ (see [5, heorem 2.1]) and by elliptic regularity $v \in C_0^1(\overline{\Omega})$. Moreover, since $g_\lambda(x,t) = 0$ for $t \le 0$ one has $-\|v^-\|^2 = 0$, which implies that $v \gneqq 0$ and $z = w + v \in C_0^1(\overline{\Omega})$ is a second solution of (Q_λ). This finishes the proof of Theorem 1.2. □

We end this section with the following proposition.

Proposition 4.8. *Suppose that $(H)_\infty$ and $3 < p < 22^* - 1$ hold. If $0 < \gamma < 1$, then $\lambda_* \in \mathcal{L}$.*

Proof. In order to prove the proposition one uses the following properties:

- if v_λ is the solution obtained in Theorem 1.1, then $J_\lambda(v_\lambda) < c$ for some constant $c > 0$ independent of $\lambda \in (0, \lambda_*)$. Indeed, as we have seen in the proof of Lemma 4.2,
$$J_\lambda(v_\lambda) = I_\lambda(v_\lambda) + \lambda \int_\Omega g^p(\underline{v}(x))g'(\underline{v}(x))\underline{v}(x)dx - \frac{\lambda}{p+1}\int_\Omega g^{p+1}(\underline{v}(x))dx,$$
which, jointly with Remark 2.9 b), gives
$$J_\lambda(v_\lambda) \le \lambda_* \int_\Omega g^p(\underline{v}(x))g'(\underline{v}(x))\underline{v}(x)dx =: c.$$

- $\frac{-g^{p+1}(t)}{p+1} + \frac{\theta}{p+1}g^p(t)g'(t)t \ge 0$ for all $t > 0$ and $4 < 2\theta < p+1$. Indeed, from Lemma 2.3 (4) we get
$$\frac{-g^{p+1}(t)}{p+1} + \frac{\theta}{p+1}g^p(t)g'(t)t \ge \frac{g^{p+1}(t)}{p+1}\left(-1+\frac{\theta}{2}\right) > 0, \tag{4.12}$$
for all $t > 0$.

Now, let $\lambda_n \in (0, \lambda_*)$ be an increasing sequence such that $\lambda_n \to \lambda_*$ as $n \to \infty$ and let $v_n := v_{\lambda_n}$ be a solution of (Q_λ) obtained in Theorem 1.1 for $\lambda = \lambda_n$. Then
$$J_{\lambda_n}(v_n) = \frac{1}{2}\|v_n\|^2 - \int_\Omega a(x)G(v_n) - \frac{\lambda_n}{p+1}\int_\Omega g^{p+1}(v_n) < c,$$
for some constant $c > 0$ independent of λ_n and
$$\|v_n\|^2 - \int_\Omega a(x)(g(v_n))^{-\gamma}g'(v_n)v_n - \lambda_n\int_\Omega (g(v_n))^p g'(v_n)v_n = 0.$$

Thus, by using (4.12), one deduces
$$\left(\frac{1}{2} - \frac{\theta}{p+1}\right)\|v_n\|^2 - \frac{1}{1-\gamma}\int_\Omega a(x)g^{1-\gamma}(v_n) + \frac{\theta}{p+1}\int_\Omega a(x)(g(v_n))^{-\gamma}g'(v_n)v_n < c,$$
whence, by Lemma 2.3 (3),
$$\left(\frac{1}{2} - \frac{\theta}{p+1}\right)\|v_n\|^2 < \frac{1}{1-\gamma}\int_\Omega a(x)g^{1-\gamma}(v_n) + c \le \frac{\|a\|_\infty}{1-\gamma}\int_\Omega v_n^{1-\gamma} + c.$$
From the previous relation it is easy to see that $\{v_n\}$ is bounded in $H_0^1(\Omega)$. Thus, there exists $v^* \in H_0^1(\Omega)$ such that, up to a subsequence, we have as $n \to \infty$
$$v_n \rightharpoonup v^* \text{ in } H_0^1(\Omega),$$
$$v_n \to v^* \text{ a.e. in } \Omega.$$

Remember that $v_n \ge \underline{v} = v_0$ in Ω and thus, by Lemma 2.3 (9), (11),
$$|a(x)(g(v_n))^{-\gamma}g'(v_n)\psi| \le |a(x)(g(v_0))^{-\gamma}g'(v_0)\psi| \quad \text{in } \Omega.$$

Because v_n is a solution of (Q_{λ_n}), we have
$$\int_\Omega \nabla v_n \nabla \psi = \int_\Omega a(x)(g(v_n))^{-\gamma}g'(v_n)\psi + \lambda_n\int_\Omega (g(v_n))^p g'(v_n)\psi,$$
for all $\psi \in H_0^1(\Omega)$. Passing to the limit in the previous equality and using Lebesgue's theorem, we deduce that v^* is a weak solution of (Q_{λ_*}). Finally, we can adapt the arguments in the proof of Theorem 1 *c*) in [1] to obtain $v^* \in C_0^1(\overline{\Omega})$. This ends the proof of the proposition. □

Proposition 4.8 suggests that $\lambda_* \in \mathcal{L}$ for arbitrary $\gamma > 0$. However, for $\gamma > 1$ and $\lambda \in (0, \lambda_*)$ one has $J_\lambda(v) > 0$ for any solution v of (Q_λ), and thus the proof of Proposition 4.8 cannot be applied to deduce that $\lambda_* \in \mathcal{L}$.

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Global asymptotic stability of a scalar delay Nicholson's blowflies equation in periodic environment

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Abstract. This paper is considered with a scalar delay Nicholson's blowflies equation in periodic environment. By taking advantage of some novel differential inequality techniques and the fluctuation lemma, we set up the sharp condition to characterize the global asymptotic stability of positive periodic solutions on the addressed equation. The obtained results improve and supplement some existing ones in recent literature, and then give a more perfect answer to an open problem proposed by Berezansky et al. in [*Appl. Math. Model.* **34**(2010), 1405–1417]. In particular, two numerical examples are provided to verify the reliability and feasibility of the theoretical findings.

Keywords: positive periodic solution, global asymptotic stability, scalar delay Nicholson's blowflies equation, sharp condition.

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1 Introduction

Classical population dynamics model

$$x'(t) = -\delta x(t) + \beta x(t-\tau)e^{-x(t-\tau)}, \qquad \delta, \beta, \tau \in (0, +\infty), \tag{1.1}$$

is known as Nicholson's blowflies equation [1, 8, 14]. Here $x(t)$ stands for the population of blowflies at time t, δ represents the average daily mortality of adult blowflies, β describes the maximum average daily egg laying rate, and τ denotes mature time delay. Over the past 40 years, plenty of research results have been obtained on the qualitative behaviour and stability of (1.1) (see [1, 3, 4, 10, 11] and their references). In particular, it has been successively shown in [3, 4, 10, 11] that the zero equilibrium point of equation (1.1) possesses global asymptotic stability when $\frac{\beta}{\delta} \leq 1$ and its positive equilibrium point has global asymptotic stability under $1 < \frac{\beta}{\delta} < e^2$. Meanwhile, it was proved in [15] that the positive equilibrium point of (1.1)

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possesses global attractivity when the delay τ is small and $1 < \frac{\beta}{\delta}$. Recently, references [2] and [19] substantiate that the positive equilibrium point of (1.1) is globally asymptotically stable when the condition

$$1 < \frac{\beta}{\delta} \leq e^2, \tag{1.2}$$

holds. It is worth pointing out that Yang and So [18] demonstrate the instability of the positive equilibrium and the existence of a Hopf bifurcation when $\frac{\beta}{\delta} > e^2$ and the delay τ is large. This implies that (1.2) is the sharp stability condition on the positive equilibrium points of the autonomous delay Nicholson's blowflies model (1.1).

In general, the external environment of actual organisms often vary periodically with seasonal changes and climate. Therefore, (1.1) can be normally generalized to the following non-autonomous equation:

$$x'(t) = -\delta(t)x(t) + \beta(t)x(t - \tau(t))e^{-x(t-\tau(t))}, \tag{1.3}$$

where $t \geq t_0$, $\delta(t) > 0$, $\beta(t) > 0$ and $\tau(t) \geq 0$ are continuous ω-periodic functions ($\omega > 0$). As we all know, the periodic population dynamics model often generates a globally stable positive periodic solution. Based on this, the authors of [1] proposed an open problem: Establish global asymptotic stability findings on positive periodic solutions of non-autonomous delay Nicholson's blowflies equation. Subsequently, the global attractivity of the positive periodic solutions of (1.3) is established in [12] when the following condition

$$\kappa \approx 0.7215355, \quad \frac{1-\kappa}{e^{\kappa}} = \frac{1}{e^2} \quad \text{and} \quad e^{\kappa} < \min_{t\in[t_0,\ t_0+\omega]} \frac{\beta(t)}{\delta(t)} \leq \max_{t\in[t_0,\ t_0+\omega]} \frac{\beta(t)}{\delta(t)} < e^2 \tag{1.4}$$

is obeyed, which gives an answer for the above open problem. Recently, [6] studied the periodicity of the delay Nicholson's blowflies system accompanying patch structure, where the main results involving the periodic scalar Nicholson's blowflies case can be described as follows.

Theorem 1.1. *Suppose m is a nonnegative integer, and*

$$\tau(t) \equiv m\omega, \quad \textit{and} \quad 1 < \min_{t\in[t_0,\ t_0+\omega]} \frac{\beta(t)}{\delta(t)} \leq \max_{t\in[t_0,\ t_0+\omega]} \frac{\beta(t)}{\delta(t)} < e^2, \tag{1.5}$$

then the positive periodic solution of the scalar Nicholson's blowflies equation (1.3) is globally attractive.

As in [12], the author of [6] have neither analysed the local stability of positive periodic solutions, nor have they given opinions about the sharp conditions which ensure the global asymptotic stability of positive periodic solutions of (1.3). Therefore, a notable problem naturally arises: What is the sharp condition guaranteeing the globally asymptotic stability of the positive periodic solutions of (1.3)? Because (1.2) is the sharp stability condition on the positive equilibrium points of the autonomous delay Nicholson's blowflies model (1.1), it is reasonable to assert:

$$1 < \min_{t\in[t_0,\ t_0+\omega]} \frac{\beta(t)}{\delta(t)} \leq \max_{t\in[t_0,\ t_0+\omega]} \frac{\beta(t)}{\delta(t)} \leq e^2, \tag{1.6}$$

is the sharp condition ensuring the globally asymptotic stability on the periodic solutions of (1.3). To prove this assertion, the ultimate intention of this work is to develop a new strategy to gain the existence and globally asymptotic stability of positive periodic solutions of equation (1.3) under the assumption (1.6) without any other conditions. Meanwhile, we will establish some completely new results on periodic stability of (1.3) without assuming $\tau = m\omega$, and then a more complete answer is given to the open problem on the global periodic stability conditions of Nicholson's blowflies equation in [1].

2 Some lemmas

For convenience, denote

$$\bar{\tau} = \max_{t\in\in[t_0,\ t_0+\omega]} \tau(t), \qquad B = C([-\bar{\tau}, 0],\ \mathbb{R}), \qquad B_+ = \{\varphi \in B | \varphi(\theta) \geq 0, \forall \theta \in [-\bar{\tau},\ 0]\},$$

and let $x_t(t_0, \varphi)(x(t; t_0, \varphi))$ be the solution of (1.3) satisfying the admissible initial conditions:

$$x_{t_0} = \varphi, \qquad \varphi \in B_+ \quad \text{with} \quad \varphi(0) > 0. \tag{2.1}$$

According to the conclusions of Example 2.8 in [7], we have

Lemma 2.1. *If* $1 < \min_{t\in[t_0, t_0+\omega]} \frac{\beta(t)}{\delta(t)}$*, then* $x(t; t_0, \varphi)$ *exists and possesses uniqueness on* $[t_0, +\infty)$*. Moreover,* $x(t; t_0, \varphi)$ *has positiveness and persistence.*

Lemma 2.2 ([5, Lemma 2.3])**.** *Assume* $a \in (0, 2]$*, then*

$$\left| be^{-b} - ae^{-a} \right| < e^{-a} |b - a| \quad \textit{for all} \;\; b > 0 \textit{ and } b \neq a.$$

Lemma 2.3 ([6, Corollary 3.1])**.** *If* $1 < \min_{t\in[t_0,\ t_0+\omega]} \frac{\beta(t)}{\delta(t)}$ *, then equation* (1.3) *has a positive* ω*-periodic solution* $x^*(t)$*.*

Lemma 2.4. *Suppose that* (1.6) *holds, and equation* (1.3) *has a positive* ω*-periodic solution* $x^*(t)$ *satisfying* $\max_{t\in[t_0,\ t_0+\omega]} x^*(t) \leq 2$*, then* $x^*(t)$ *is globally asymptotically stable.*

Proof. Obviously,

$$0 < k^* := \min_{t\in[t_0,\ t_0+\omega]} x^*(t) \leq \max_{t\in[t_0,\ t_0+\omega]} x^*(t) \leq 2. \tag{2.2}$$

For all $t \in [t_0 - \bar{\tau},\ +\infty)$, let us introduce the notations

$$x(t) = x(t; t_0, \varphi) \quad \text{and} \quad w(t) = \frac{x(t)}{x^*(t)} - 1.$$

Then, for all $t \geq t_0$,

$$\begin{aligned} w'(t) = \frac{\beta(t)}{x^*(t)} \Big\{ &- x^*(t - \tau(t)) e^{-x^*(t-\tau(t))} w(t) \\ &+ \big[x^*(t - \tau(t))(w(t - \tau(t)) + 1) e^{-x^*(t-\tau(t))(w(t-\tau(t))+1)} \\ &- x^*(t - \tau(t)) e^{-x^*(t-\tau(t))} \big] \Big\}. \end{aligned} \tag{2.3}$$

Now, we prove the local stability of $x^*(t)$.

For arbitrary $\varepsilon > 0$, let $H = \frac{k^*\varepsilon}{2}$ and $\|\varphi - x^*\| < H$ with $\|\cdot\|$ denoting the supremum norm, we shall reveal $|x(t) - x^*(t)| < \varepsilon$ for all $t \in [t_0 - \bar{\tau}, +\infty)$. Noting that

$$|w(t)| = \left| \frac{\varphi(t) - x^*(t)}{x^*(t)} \right| < \frac{H}{x^*(t)} \leq \frac{H}{k^*} \quad \text{for arbitrary } t \in [t_0 - \bar{\tau},\ t_0],$$

we assert that

$$|w(t)| < \frac{H}{k^*} \quad \text{for arbitrary } t > t_0. \tag{2.4}$$

Otherwise, there exists $S_1 > t_0$ such that either

$$w(S_1) = \frac{H}{k^*} \quad \text{and} \quad |w(t)| < \frac{H}{k^*} \quad \text{for arbitrary } t \in [t_0 - \bar{\tau},\ S_1) \tag{2.5}$$

or

$$w(S_1) = -\frac{H}{k^*} \quad \text{and} \quad |w(t)| < \frac{H}{k^*} \quad \text{for arbitrary } t \in [t_0 - \bar{\tau},\ S_1) \tag{2.6}$$

holds.

Assuming that (2.5) holds, from Lemma 2.2, we acquire that for $w(S_1 - \tau(S_1)) \neq 0$,

$$\begin{aligned}
0 &\le w'(S_1) \\
&\le \frac{\beta(S_1)}{x^*(S_1)}\Big\{ - x^*(S_1 - \tau(S_1))e^{-x^*(S_1-\tau(S_1))}w(S_1) + |x^*(S_1 - \tau(S_1))(w(S_1 - \tau(S_1)) + 1) \\
&\qquad\qquad \times e^{-x^*(S_1-\tau(S_1))(w(S_1-\tau(S_1))+1)} - x^*(S_1 - \tau(S_1))e^{-x^*(S_1-\tau(S_1))}|\Big\} \\
&< \frac{\beta(S_1)}{x^*(S_1)}\left\{ - x^*(S_1 - \tau(S_1))e^{-x^*(S_1-\tau(S_1))}\frac{H}{k^*} + x^*(S_1 - \tau(S_1))\,|w(S_1 - \tau(S_1))|\, e^{-x^*(S_1-\tau(S_1))}\right\} \\
&= \frac{\beta(S_1)}{x^*(S_1)}x^*(S_1 - \tau(S_1))e^{-x^*(S_1-\tau(S_1))}\left[-\frac{H}{k^*} + |w(S_1 - \tau(S_1))|\right] \\
&\le 0,
\end{aligned}$$

which is an obvious contradiction. Similarly, one can derive a contradiction from the situation (2.6). Moreover, when $w(S_1 - \tau(S_1)) = 0$, one can also derive the above contradiction. Thus, the assertion (2.4) is true and

$$|x(t) - x^*(t)| < x^*(t)\frac{H}{k^*} \le \varepsilon \quad \text{for all} \ \ t \in [t_0,\ +\infty),$$

which follows that $x^*(t)$ is locally stable.

Next, we demonstrate the global attractivity of $x^*(t)$. Let

$$\mu = \limsup_{t\to+\infty} \omega(t) \quad \text{and} \quad \lambda = \liminf_{t\to+\infty} \omega(t).$$

Clearly, the global attractivity of $x^*(t)$ is equivalent to show $\max\{|\mu|, |\lambda|\} = 0$. In order to obtain a contradiction, we just assume $\max\{|\mu|, |\lambda|\} = \mu > 0$ (the situation of $\max\{|\mu|, |\lambda|\} = -\lambda > 0$ is similar). According to the fluctuation lemma [16, Lemma A.1.], one can find a sequence $\{s_k\}_{k=1}^{+\infty}$ obeying

$$\lim_{k\to+\infty} s_k = +\infty, \qquad \lim_{k\to+\infty} w(s_k) = \mu, \quad \text{and} \quad \lim_{k\to+\infty} w'(s_k) = 0.$$

Without any loss of generality, we may also assume that $\lim_{k\to+\infty} \beta(s_k)$, $\lim_{k\to+\infty} \delta(s_k)$, $\lim_{k\to+\infty} w(s_k - \tau(s_k))$, $\lim_{k\to+\infty} x^*(s_k - \tau(s_k))$ and $\lim_{k\to+\infty} x^*(s_k)$ exist. It follows from (2.3) and Lemma 2.2 that for $\lim_{k\to+\infty} w(s_k - \tau(s_k)) \neq 0$,

$$\begin{aligned}
0 &= \lim_{k\to+\infty} w'(s_k) \\
&= \frac{\lim_{k\to+\infty} \beta(s_k)}{\lim_{k\to+\infty} x^*(s_k)}\Big\{ - e^{-\lim_{k\to+\infty} x^*(s_k-\tau(s_k))} \lim_{k\to+\infty} x^*(s_k - \tau(s_k)) \lim_{k\to+\infty} w(s_k) \\
&\quad + \Big[\lim_{k\to+\infty} x^*(s_k - \tau(s_k))(1 + \lim_{k\to+\infty} w(s_k - \tau(s_k)))e^{-\lim_{k\to+\infty} x^*(s_k-\tau(s_k))(1+\lim_{k\to+\infty} w(s_k-\tau(s_k)))} \\
&\quad - \lim_{k\to+\infty} x^*(s_k - \tau(s_k))e^{-\lim_{k\to+\infty} x^*(s_k-\tau(s_k))}\Big]\Big\}
\end{aligned}$$

$$
\begin{aligned}
&< \frac{\lim_{k\to+\infty}\beta(s_k)}{\lim_{k\to+\infty}x^*(s_k)}\Big\{-e^{-\lim_{k\to+\infty}x^*(s_k-\tau(s_k))}\lim_{k\to+\infty}x^*(s_k-\tau(s_k))\lim_{k\to+\infty}w(s_k)\\
&\quad + e^{-\lim_{k\to+\infty}x^*(s_k-\tau(s_k))}\lim_{k\to+\infty}x^*(s_k-\tau(s_k))\lim_{k\to+\infty}|w(s_k-\tau(s_k))|\Big\}\\
&\le 0,
\end{aligned}
$$

which leads to a contradiction. Especially, if $\lim_{k\to+\infty} w(s_k-\tau(s_k)) = 0$, the above contradiction is obvious. This yields $\max\{|\mu|, |\lambda|\} = 0$, and the proof of Lemma 2.4 is finished. □

3 Globally asymptotic stability of positive periodic solutions

Theorem 3.1. *Let (1.6) be satisfied, and m be a nonnegative integer obeying $\tau(t) \equiv m\omega$. Then, equation (1.3) possesses a globally asymptotically stable positive periodic solution.*

Proof. On account of Lemma 2.3, one can discover that equation (1.3) possesses a positive ω-periodic solution $x^*(t)$. In view of Lemma 2.4, to finish the proof of Theorem 3.1, we only need to reveal that $\max_{t\in[t_0,\, t_0+\omega]} x^*(t) \le 2$. For this purpose, let

$$\sigma \in (0,1), \quad \sigma e^{-\sigma} = 2e^{-2}, \quad \text{and} \quad \eta = \sup\left\{\rho \mid \beta(t)e^{-\rho} > \delta(t),\ t\in[0,\omega],\ \rho>0\right\}. \tag{3.1}$$

We claim that

$$k^{**} := \min\{\sigma,\eta\} \le x^*(t) \le 2 \quad \text{for arbitrary } t\in\mathbb{R}.$$

In fact, let $t_1, t_2 \in [\omega, 2\omega]$ such that

$$x^*(t_1) = \max_{t\in\mathbb{R}} x^*(t) \quad \text{and} \quad x^*(t_2) = \min_{t\in\mathbb{R}} x^*(t),$$

then

$$0 = -\delta(t_i)\,x^*(t_i) + \beta(t_i)\,x^*(t_i)\,e^{-x^*(t_i)} \qquad (i = 1,2).$$

Hence, from (1.6) and (3.1), we acquire

$$
\begin{aligned}
&e^{x^*(t_1)} = \frac{\beta(t_1)}{\delta(t_1)} \le e^2 \quad \text{with} \quad x^*(t_1)\le 2, \quad \text{and}\\
&\delta(t_2) = \beta(t_2)\,e^{-x^*(t_2)} \quad \text{with} \quad x^*(t_2) \ge \eta \ge k^{**},
\end{aligned} \tag{3.2}
$$

which finishes the proof. □

Remark 3.2. Evidently, Theorem 1.1 as a main conclusion in [6] is a direct corollary of Theorem 3.1 in this present paper, and the proof of our conclusion is only established under sharp condition (1.6). Meanwhile, we present a detailed proof of the local stability of positive periodic solutions, which is not involved in the existing literature [6,12]. Therefore, the conclusion of this paper improves and generalizes the corresponding ones of the above literature, which provides a more perfect answer to the open problem in [1] which has been mentioned in the Introduction section of this article.

Theorem 3.3. *Assume $\beta^+ = \max_{t\in[t_0,\, t_0+\omega]} \beta(t)$ and*

$$1 < \min_{t\in[t_0,\,t_0+\omega]} \frac{\beta(t)}{\delta(t)} \le \max_{t\in[t_0,\,t_0+\omega]} \frac{\beta(t)}{\delta(t)} \le \max_{t\in[t_0,\,t_0+\omega]} \frac{\beta(t)}{\delta(t)-\tau(t)\beta(t)\beta^+} \le e^2. \tag{3.3}$$

Then, equation (1.3) possesses a globally asymptotically stable positive periodic solution.

Proof. As is seen from Lemma 2.3 and Lemma 2.4, we only need to verify that the positive ω-periodic solution $x^*(t)$ of (1.3) satisfies $\max_{t\in[t_0,\ t_0+\omega]} x^*(t) \le 2$. In order to do this, denote $x(t) = x(t; t_0, \varphi)$ for arbitrary $t \in [t_0 - \bar{\tau},\ +\infty)$, and

$$L = \limsup_{t\to+\infty} x(t), \qquad l = \liminf_{t\to+\infty} x(t). \tag{3.4}$$

Apparently, Lemma 2.1 yields $l > 0$. Now, we verify $L \le 2$. Again from the fluctuation lemma [16, Lemma A.1.], one can pick $\{t_k\}_{k=1}^{+\infty}$ such that

$$\lim_{k\to+\infty} t_k = +\infty, \qquad \lim_{k\to+\infty} x'(t_k) = 0, \qquad \lim_{k\to+\infty} x(t_k) = L. \tag{3.5}$$

Also, we suppose that $\lim_{k\to+\infty} \delta(t_k), \lim_{k\to+\infty} \beta(t_k)$ and $\lim_{k\to+\infty} \tau(t_k)$ exist.

For any $\varepsilon > 0$, it is easy to find $N > 0$ satisfying that $x(t) < L + \varepsilon$ for arbitrary $t > N$, and hence for arbitrary $t \in (N + \bar{\tau},\ +\infty)$,

$$-\delta(t)(L+\varepsilon) < -\delta(t)x(t) + \beta(t)x(t-\tau(t))e^{-x(t-\tau(t))} < \beta(t)(L+\varepsilon).$$

Furthermore,

$$|x'(t)| < \beta(t)(L+\varepsilon), \qquad t \in (N+\bar{\tau},\ +\infty),$$

and

$$\begin{aligned} x'(t_k) &= -\delta(t_k)x(t_k) + \beta(t_k)x(t_k)e^{-x(t_k)} + \beta(t_k)[x(t_k-\tau(t_k))e^{-x(t_k-\tau(t_k))} - x(t_k)e^{-x(t_k)}] \\ &\le -\delta(t_k)x(t_k) + \beta(t_k)x(t_k)e^{-x(t_k)} + \beta(t_k)|(1-\theta)e^{-\theta}||x(t_k) - x(t_k-\tau(t_k))| \\ &\le -\delta(t_k)x(t_k) + \beta(t_k)x(t_k)e^{-x(t_k)} + \beta(t_k)\int_{t_k-\tau(t_k)}^{t_k} |x'(s)|ds \\ &\le -\delta(t_k)x(t_k) + \beta(t_k)x(t_k)e^{-x(t_k)} + \beta^+\beta(t_k)\tau(t_k)(L+\varepsilon), \qquad t_k > N+\bar{\tau}, \end{aligned} \tag{3.6}$$

where θ is the mean value in the Differential Mean Value Theorem. From (3.5), taking the limits on both sides of (3.6) leads to

$$e^L \le \frac{\lim\limits_{k\to+\infty} \beta(t_k)}{\lim\limits_{k\to+\infty} \delta(t_k) - \lim\limits_{k\to+\infty} \beta^+\beta(t_k)\tau(t_k)\frac{L+\varepsilon}{L}}.$$

Let $\varepsilon \to 0$, from (3.3), we derive

$$e^L \le \lim_{k\to+\infty} \frac{\beta(t_k)}{\delta(t_k) - \beta^+\beta(t_k)\tau(t_k)} \le \max_{t\in[t_0,\ t_0+\omega]} \frac{\beta(t)}{\delta(t) - \tau(t)\beta(t)\beta^+}, \quad \text{and} \quad L \le 2.$$

Thus, the positive ω-periodic solution $x^*(t)$ of system (1.3) obeys $\max_{t\in[t_0,\ t_0+\omega]} x^*(t) \le 2$. The verification of Theorem 3.3 is completed. □

Remark 3.4. Theorem 3.3 is established without the assumption of $\tau(t) \equiv m\omega$, and it is easy to verify the feasibility of the conditions (3.3) when the delay is small. Meanwhile, the condition (3.3) is equivalent to the sharp condition (1.6) when the delay vanishes to zero.

4 Numerical simulations

Regard the following scalar delay Nicholson's blowflies equation:

$$x'(t) = -(1+|\sin t|)x(t) + (1+|\sin t|)(1.01+(e^2-1.01)|\cos t|)x(t-2\pi)e^{-x(t-2\pi)}, \quad (4.1)$$

and

$$x'(t) = -(1+|\sin t|)x(t) + (1+|\sin t|)(1.05+(e^2-1.1)|\cos t|)x\big(t-\tfrac{1}{50e^4}|\cos t|\big)e^{-x(t-\frac{1}{50e^4}|\cos t|)}, \quad (4.2)$$

where $t \geq t_0 = 0$. It is easy to verify that (4.1) and (4.2) satisfy

$$\tau(t) \equiv 2\pi, \qquad 1.01 = \min_{t\in[t_0,\ t_0+\pi]} \frac{\beta(t)}{\delta(t)} \leq \max_{t\in[t_0,\ t_0+\pi]} \frac{\beta(t)}{\delta(t)} = e^2, \quad (4.3)$$

and

$$1.05 = \min_{t\in[t_0,\ t_0+\pi]} \frac{\beta(t)}{\delta(t)} \leq \max_{t\in[t_0,\ t_0+\pi]} \frac{\beta(t)}{\delta(t)} \leq \max_{t\in[t_0,\ t_0+\pi]} \frac{\beta(t)}{\delta(t)-\tau(t)\beta(t)\beta^+} \approx e^2-0.02, \quad (4.4)$$

respectively. Therefore, from Theorems 3.1 and 3.3, we know that the above two scalar Nicholson's blowflies models possess global asymptotic stable positive π-periodic solutions. The numerical simulation results of the two examples are shown in Figures 4.1–4.2, and the trajectories of the solutions strongly confirm the correctness and validity of the results in this paper.

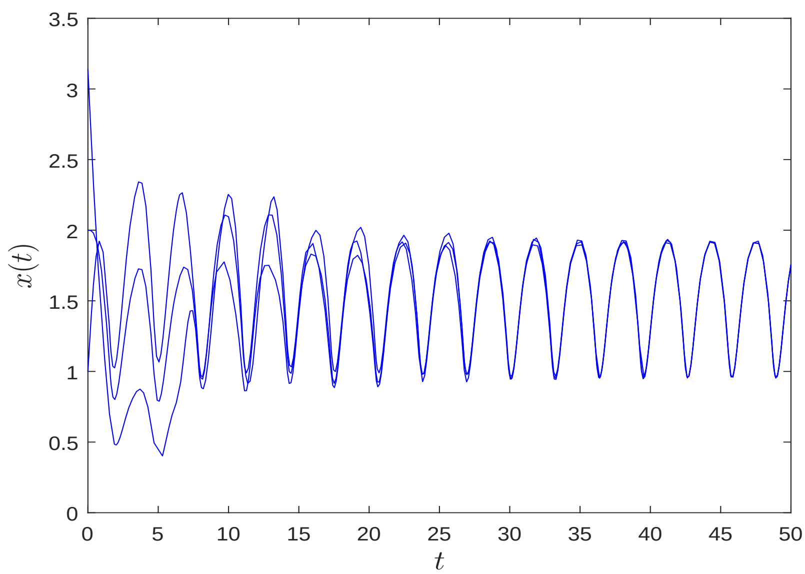

Figure 4.1: Numerical state trajectories of model (4.1) involving the initial values: $1, 2, \pi$, respectively.

Remark 4.1. Nicholson's blowflies equation (4.1) does not satisfy the condition (1.4), equation (4.2) does not obey the conditions (1.4) and (1.5), which have been adopted as fundamental assumptions for the considered periodicity of (1.3) in [6, 12]. Consequently, the conclusions in [6, 12] can not be directly employed to illustrate the globally asymptotic stability of the

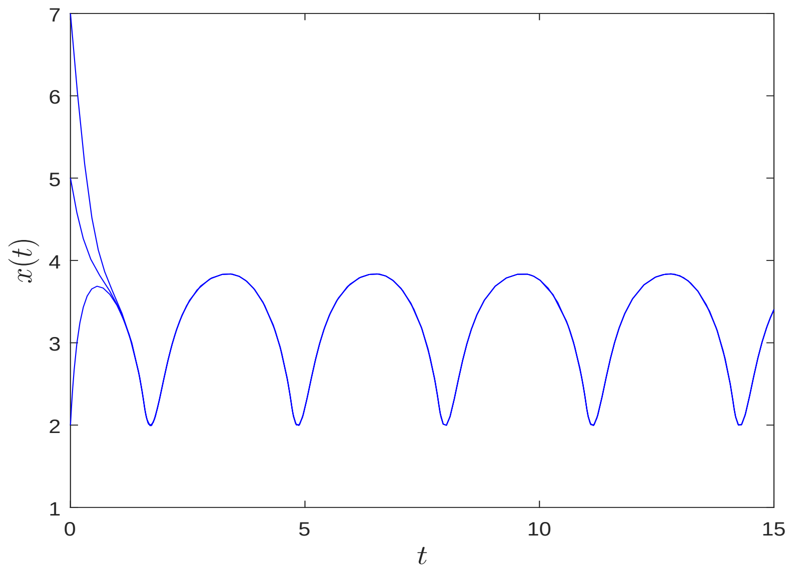

Figure 4.2: Numerical state trajectories of model (4.2) involving the initial values: $2, 5, 7$, respectively.

positive periodic solutions for (4.1) and (4.2), which indicates that the results of this paper improve and extend the corresponding ones of [6,9,12,13,17] and the references cited therein. It is noteworthy that, the method presented in this article can be used to explore the sharp condition of the existence and global asymptotic stability of positive periodic solutions to the scalar Nicholson's blowflies models involving multiple time-varying delays in [12] and the delay Nicholson's blowflies systems accompanying patch structure in [6].

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Expansion of positivity to a class of doubly nonlinear parabolic equations

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Abstract. We establish the *expansion of positivity* of the nonnegative, local, weak solutions to the class of doubly nonlinear parabolic equations

$$\partial_t(u^q) - \operatorname{div}\left(|Du|^{p-2}Du\right) = 0, \qquad p > 1 \text{ and } q > 0$$

considering separately the two possible cases $q+1-p > 0$ and $q+1-p < 0$. The proof relies on the procedure presented by DiBenedetto, Gianazza and Vespri for both the degenerate and the singular parabolic p-Laplacian equation.

Keywords: doubly nonlinear parabolic equations, expansion of positivity, singular PDE, degenerate PDE, intrinsic scaling.

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1 Introduction

In this work we consider the class of doubly nonlinear parabolic equations

$$\partial_t(u^q) - \operatorname{div}\left(|Du|^{p-2}Du\right) = 0, \quad \text{in } \Omega_T, \quad p > 1 \text{ and } q > 0, \tag{1.1}$$

where $\Omega_T = \Omega \times (0, T]$, being Ω a bounded domain in $\mathbb{R}^N$ and T a real positive number; which models, for instance, the turbulent filtration of non-Newtonian fluids through a porous media (see [4]).

Along the past years many authors have studied this class of evolutionary equations: the simpler case $q = 1$ was widely study (see for instance [2, 3] and the references therein); the Trudinger's equation, corresponding to $q = p-1$, is still object of intensive study (cf. [16,18,19] and more recently [5]; and, for the general case, there are already some results (see [12–14]). In a certain extend, this class of doubly nonlinear equations can be seen as

$$\partial_t u - \operatorname{div}\left(|u|^{m-1}|Du|^{p-2}Du\right) = 0, \quad \text{in } \Omega_T, \quad p > 1$$

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and many are the works concerning its weak solutions (just to mane a few, we refer to [6–10, 15, 20–22]).

The doubly nonlinear equation (1.1) presents several difficulties: for $q > 1$ ($0 < q < 1$) equation (1.1) has a degeneracy (singularity) in time, since u^{q-1} is zero (explodes) at the points where $u = 0$; while for $1 < p < 2$ ($p > 2$) (1.1) exhibits a singularity (degeneracy) in space, since the modulus of ellipticity $|Du|^{p-2}$ explodes (is zero) at the points where $Du = 0$. One aspect to always have into consideration is that in order to compensate the degradation of equation's parabolic structure one needs to consider proper cylinders within which the equation behaves as the heat equation – this is known as intrinsic scaling.

The main goal of this work is to give one more contribution to the study of the properties of the weak solutions to this class of doubly nonlinear evolutionary equations (1.1), for $p > 1$ and $q > 0$, namely to present the *expansion of positivity* for its nonnegative bounded weak solutions, which roughly speaking means that the information on the measure of the *positivity set* of u, at a certain time level s over a cuber $K_\rho(y)$, can be expanded to the measure of the *positivity set* of u both in space (say from $K_\rho(y)$ to $K_{2\rho}(y)$) and in time (from the time level s to all further time levels $s + \theta\rho^p$). The proof relies on energy estimates and DeGiorgi-type lemmas and comprehends two steps. The first step consists on the propagation of the positivity information known at a cube located in some time level, say $K_\rho(y) \times \{s\}$, to upper times levels. Not only this is the easiest step but also it holds for all values of $p > 1$ and $q > 0$. As for the second (more demanding and crucial) one, the spacial propagation of positivity is derived from the cube $K_\rho(y)$ to the bigger cube $K_{2\rho}(y)$: the proof is more evolving and requires the cases $q + 1 - p > 0$ and $q + 1 - p < 0$ to be studied separately.

The *expansion of positivity* is the key ingredient to derive Harnack estimates and it can also be an important tool to prove local regularity of the weak solutions. If not only for the mathematical interest per si, these two arguments give extra and relevant reasons for the study at hands.

The paper is organized as follows. In Section 2, we present the notation and some known results needed along the sections to come. In Section 3, we deduce the energy estimates and prove two DeGiorgi-type lemmas which are the core results for the *expansion of positivity*. The proofs of the main results, that is of the *expansion of positivity*, both for $q + 1 - p > 0$ and $q + 1 - p < 0$, are presented in Section 4.

2 Setting the framework

Notation and known results

We start by presenting some notation and some already known results just to keep the text as self contained as possible.

Due to the parabolic nature of our evolutionary equation, we will work with parabolic cylinders and parabolic Sobolev spaces. For that purpose let (x_0, t_0) be an interior point in the space-time domain $\Omega \times (0, T]$. For a cylinder with vertex at (x_0, t_0), of radius $R > 0$ and height τ we can define: the backward cylinder

$$(x_0, t_0) + Q^-(\tau, R) := K_R(x_0) \times (t_0 - \tau, t_0)$$

the forward cylinder

$$(x_0, t_0) + Q^+(\tau, R) := K_R(x_0) \times (t_0, t_0 + \tau),$$

where

$$K_R(x_0) = \{x \in \Omega : \max |x - x_0| < R\}.$$

Let $p \geq 1$. The Sobolev space $H^{1,p}(\Omega)$ is defined to be the completion of $C^\infty(\Omega)$ with respect to the Sobolev norm

$$\|u\|_{1,p,\Omega} = \left(\int_\Omega (|u|^p + |Du|^p) \right)^{1/p}.$$

A function u belongs to the local Sobolev space $H^{1,p}_{loc}(\Omega)$ if it belongs to $H^{1,p}(K)$ for every compactly contained subset K of Ω. Moreover, the Sobolev space with zero boundary values $H^{1,p}_0(\Omega)$ is defined as the completion of $C^\infty_0(\Omega)$ with respect to the Sobolev norm.

The parabolic Sobolev space $L^p(t_1, t_2; H^{1,p}(\Omega))$, with $t_1 < t_2$, is the space of functions $u(x,t)$ such that, for almost every $t \in (t_1, t_2)$ the function $u(\cdot, t)$ belongs to $H^{1,p}(\Omega)$ and

$$\int_{t_1}^{t_2} \int_\Omega (|u|^p + |Du|^p) < \infty.$$

The following result establishes an estimate for the gradient of a certain regular function v at the points where $k < v < l$, $k, l \in \mathbb{R}$.

Lemma 2.1. *Let $v \in H^{1,1}(K) \cap C(K)$ and $k, l \in \mathbb{R}$, $k < l$. There exists a positive constant γ, depending only on N and p, such that*

$$(l - k)\, |K \cap [v > l]\,| \leq \gamma \frac{|K|}{|K \cap [v < k]|} \int_{K \cap [k < v < l]} |\nabla v|. \tag{2.1}$$

The result to come establishes a Sobolev embedding.

Proposition 2.2. *There exists a positive constant γ, depending on N, p, and m, such that*

$$\iint_{\Omega_T} |v|^{p\frac{N+m}{N}} \leq \gamma^{p\frac{N+m}{N}} \left\{ \iint_{\Omega_T} |Dv|^p \right\} \times \left\{ \operatorname*{ess\,sup}_{0<t<T} \int_\Omega |v|^m \right\}^{\frac{p}{N}}, \tag{2.2}$$

for $v \in L^\infty(0, T; L^m(\Omega)) \cap L^p(0, T; W^{1,p}_0(\Omega))$.

The next result concerns algebraic geometric convergence.

Lemma 2.3. *Let $(Y_n)_n$ be a sequence of positive numbers satisfying*

$$Y_{n+1} \leq C b^n Y_n^{1+\alpha},$$

where $C, b > 1$ and $\alpha > 0$. Then $(Y_n)_n$ converges to zero as $n \to \infty$, provided

$$Y_0 \leq C^{-1/\alpha} b^{-1/\alpha^2}.$$

All these results can be found in [2]. The following algebraic result can be found in [1] and in [11], for $0 < q < 1$ and $q > 1$, respectively.

Lemma 2.4. *For any $q > 0$, there exists a positive constant γ, depending on q, such that for all $a, b \in \mathbb{R}$*

$$\frac{1}{\gamma}\left|\mathbf{b}^q - \mathbf{a}^q\right| \leq (|a| + |b|)^{q-1}\,|b-a| \leq \gamma\left|\mathbf{b}^q - \mathbf{a}^q\right|, \tag{2.3}$$

where

$$\mathbf{b}^q = \begin{cases} |b|^{q-1}b, & b \neq 0, \\ 0, & b = 0. \end{cases}$$

As it will be made clearer in the section to come, in order to deduce appropriate energy estimates we'll have to work with the functions

$$g_\pm(u,k) = \pm q\int_k^u |s|^{q-1}(s-k)_\pm\, ds,$$

for which we need lower and upper bounds. This is the content of the next result (although the proof follows quite closely the one presented in [5] we decided to presented it for the sake of completeness).

Lemma 2.5. *There exists a positive constant γ, depending on q, such that for all $u, k \in \mathbb{R}$, the following holds*

$$\frac{1}{\gamma}\left(|u| + |k|\right)^{q-1}(u-k)_\pm^2 \leq g_\pm(u,k) \leq \gamma\left(|u| + |k|\right)^{q-1}(u-k)_\pm^2. \tag{2.4}$$

Proof. We will present the proof for $g_+(u,k)$, the other case can be treated in an analogous way.

Observe that it is enough to considere $u, k \in \mathbb{R}$, with $u > k$, since otherwise $g_+(u,k) = 0$. So, considering $u > k$, on the one hand

$$\begin{aligned} g_+(u,k) &= q\int_k^u |s|^{q-1}(s-k)_+\, ds \\ &\geq q\int_{\frac{k+u}{2}}^u |s|^{q-1}(s-k)\, ds \\ &\geq q\,\frac{u-k}{2}\int_{\frac{k+u}{2}}^u |s|^{q-1}\, ds \\ &= \frac{u-k}{2}\left(|u|^{q-1}u - \left|\frac{k+u}{2}\right|^{q-1}\frac{k+u}{2}\right) \\ &\geq \frac{(u-k)^2}{\gamma}\left(\left|\frac{k+u}{2}\right| + |u|\right)^{q-1} \\ &\geq \frac{1}{\gamma}(u-k)_+^2\left(|u| + |k|\right)^{q-1}; \end{aligned}$$

on the other hand

$$\begin{aligned} g_+(u,k) &\leq (u-k)\, q\int_k^u |s|^{q-1}\, ds \\ &\leq \gamma(u-k)_+^2(|u| + |k|)^{q-1}. \end{aligned}$$

The last inequalities in both lower and upper estimates were obtained realizing that

$$\frac{|u| + |k|}{2} \leq \left|\frac{k+u}{2}\right| + |u| \leq 2(|u| + |k|)$$

and using (2.3). □

Definition of weak solution and Notion of parabolicity

In what follows we state what we mean by a local weak solution to (1.1).

Definition 2.6. A measurable function

$$u \in C(0,T; L^{q+1}_{loc}(\Omega)) \cap L^p_{loc}(0,T; H^{1,p}_{loc}(\Omega))$$

is a weak sub(super)solution to equation (1.1) in $\Omega \times (0,T]$ if, for any compact $K \subset \Omega$ and for almost every $0 < t_1 < t_2 < T$, it satisfies

$$\int_K [\mathbf{u}^q \varphi\ (x,t)]_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_K \left(|Du|^{p-2} Du \cdot D\varphi - \mathbf{u}^q \varphi_t \right) \ \le (\ge)\, 0, \tag{2.5}$$

for every nonnegative test function

$$\varphi \in H^{1,q+1}_{loc}(0,T; L^{q+1}(K)) \cap L^p_{loc}(0,T; H^{1,p}_0(K)).$$

A weak solution to (1.1) is a function that is both a weak subsolution and a weak supersolution to (1.1).

Remark 2.7. Observe that the regularity assumption on u and on test functions η guarantee the convergence of the integrals appearing in (2.5).

In the case $0 < q < 1$, one can consider, and thereby recover, the regularity assumption on u presented for the p-Laplacian, that is, $u \in C(0,T; L^2_{loc}(\Omega)) \cap L^p_{loc}(0,T; H^{1,p}_{loc}(\Omega))$.

Equation (1.1) presents two interesting and relevant features: one is that the nonlinearity exhibit by the time derivative part does not allow us to add constants to the solution and still have a solution; the other one regards the notion of parabolicity (which does not come directly from the differential equation). Taking this into account, we say that equation (1.1) is *parabolic* if

> for all $k \in \mathbb{R}$, whenever u is a weak sub(super)solution, the function $k \pm (u-k)_\pm$ is a local weak sub(super)solution,

where

$$(u-k)_- = \max\{0, k-u\}, \quad (u-k)_+ = \max\{0, u-k\},$$

and

$$k - (u-k)_- = \min\{u,k\}, \quad \text{and} \quad k + (u-k)_+ = \max\{u,k\}.$$

The following result asserts that equation (1.1) is a parabolic equation. The proof follows closely the one presented in [2], for the p-Laplacian equation, and also the one presented in [5] for the Trudinger's equation.

Lemma 2.8. *Let u be a local weak sub(super)solution to (1.1). Then for all $k \in \mathbb{R}$, the truncated functions $k \pm (u-k)_\pm$ are local weak sub(super)solutions to (1.1).*

Proof. Let (x_0, t_0) be an interior point of Ω_T, which by translation we will assume $(x_0, t_0) = (0,0)$. Let u be a subsolution to (1.1) and consider a real number $k \in \mathbb{R}$ (the case of a supersolution can be treat analogously).

It is well known that the time derivative $\partial_t u$ has to be avoided (its notion may not even exist in a Sobolev sense) and so we use the regularization, proposed by Kinnunen and Lindqvist [17],

$$u^\star(x,t) = \frac{1}{\sigma}\int_0^t e^{\frac{s-t}{\sigma}} u(x,s)\, ds, \quad \sigma > 0, \tag{2.6}$$

to overcome this difficulty.

The mollified version of (2.5) is then given by

$$\iint_{\Omega_T} \partial_t\left((\mathbf{u}^q)^\star\right)\varphi + \left(|Du|^{p-2}Du\right)^\star \cdot D\varphi \le \int_\Omega \mathbf{u}^q(x,0)\left(\frac{1}{\sigma}\int_0^T e^{-\frac{s}{\sigma}}\varphi(x,s)\, ds\right) \tag{2.7}$$

for all $0 \le \eta \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^{q+1}(\Omega_T)$.

Consider the test function

$$\varphi(x,t) = \xi^p(x,t)\psi_\epsilon(t)\frac{(u-k)_+}{(u-k)_+ + h}, \qquad h,\epsilon > 0,$$

being $\xi \in C^1(Q(\tau,R))$, verifying $0 \le \xi \le 1$ and vanishing on the lateral boundary of $Q(\tau,R)$; and $\psi_\epsilon(t)$ a piecewise smooth cutoff function defined by

$$\psi_\epsilon(t) = \begin{cases} 0, & -\tau \le t \le t_1 - \epsilon, \\ 1 + \frac{t-t_1}{\epsilon}, & t_1 - \epsilon \le t \le t_1, \\ 1, & t_1 \le t \le t_2, \\ 1 - \frac{t-t_2}{\epsilon}, & t_2 \le t \le t_2 + \epsilon, \\ 0, & t_2 + \epsilon \le t \le 0. \end{cases}$$

Let v_σ be such that $(\mathbf{v}_{\boldsymbol{\sigma}})^q = (\mathbf{u}^q)^\star$.

The parabolic and elliptic terms appearing in (2.7) will be treated separately. As for the parabolic term

$$\begin{aligned}\iint_{\Omega_T} \partial_t\left((\mathbf{u}^q)^\star\right)\varphi = & \iint_{Q(\tau,R)} \partial_t\left((\mathbf{v}_{\boldsymbol{\sigma}})^q\right)\xi^p\psi_\epsilon\left(\frac{(v_\sigma-k)_+}{(v_\sigma-k)_+ + h}\right) \\ & + \iint_{Q(\tau,R)} \partial_t\left((\mathbf{v}_{\boldsymbol{\sigma}})^q\right)\xi^p\psi_\epsilon\left(\frac{(u-k)_+}{(u-k)_+ + h} - \frac{(v_\sigma-k)_+}{(v_\sigma-k)_+ + h}\right).\end{aligned}$$

By observing that

$$\partial_t\left(\int_{\mathbf{k}^q}^{(\mathbf{v}_{\boldsymbol{\sigma}})^q} \frac{(\mathbf{s}^{\frac{1}{q}}-k)_+}{(\mathbf{s}^{\frac{1}{q}}-k)_+ + h}\, ds\right) = \partial_t\left((\mathbf{v}_{\boldsymbol{\sigma}})^q\right)\frac{(v_\sigma-k)_+}{(v_\sigma-k)_+ + h}$$

and

$$\partial_t\left((\mathbf{v}_{\boldsymbol{\sigma}})^q\right) = \partial_t\left((\mathbf{u}^q)^\star\right) = \frac{\mathbf{u}^q - (\mathbf{v}_{\boldsymbol{\sigma}})^q}{\sigma},$$

the second integral appearing in the right hand side of the previous integral identity is nonnegative, since both factors have the same signal due the fact that $f(s) = \frac{(\mathbf{s}^{1/q}-k)_+}{(\mathbf{s}^{1/q}-k)_+ + h}$ is a monotone nondecreasing function. As for the first integral, note that

$$\begin{aligned}\partial_t\left((\mathbf{v}_{\boldsymbol{\sigma}})^q\right)\left(\frac{(v_\sigma-k)_+}{(v_\sigma-k)_+ + h}\right) &= \partial_t\left(\int_{\mathbf{k}^q}^{(\mathbf{v}_{\boldsymbol{\sigma}})^q} \frac{(\mathbf{s}^{\frac{1}{q}}-k)_+}{(\mathbf{s}^{\frac{1}{q}}-k)_+ + h}\, ds\right) \\ &= \partial_t\left(\mathbf{k}^q + q\int_k^{v_\sigma} \frac{(s-k)_+}{(s-k)_+ + h}|s|^{q-1}\, ds\right) \\ &\stackrel{\text{def}}{=} \partial_t\left(\mathcal{I}(v_\sigma,k,h,+)\right).\end{aligned}$$

Gathering these informations we arrive at

$$\iint_{\Omega_T} \partial_t\left((u^q)^\star\right)\varphi \geq \iint_{Q(\tau,R)} \partial_t\left(\mathcal{I}(\mathbf{v}_{\text{œ}},k,h,+)\right)\xi^p\psi_\epsilon$$
$$= -\iint_{Q(\tau,R)} \mathcal{I}(\mathbf{v}_{\text{œ}},k,h,+)\left\{\xi^p\psi_\epsilon' + \partial_t\left(\xi^p\right)\psi_\epsilon\right\}.$$

The regularity assumptions considered allow us to, first pass to limit as $\sigma \to 0$, and then let $\epsilon \to 0$, getting thereby the inferior bound

$$-\iint_{Q(\tau,R)} \mathcal{I}(u,k,h,+)\partial_t\left(\xi^p\right) - \int_{K_R\times\{t_2\}} \mathcal{I}(u,k,h,+)\xi^p + \int_{K_R\times\{t_1\}} \mathcal{I}(u,k,h,+)\xi^p.$$

As for the elliptic term, we start by letting $\sigma \to 0$ and then $\epsilon \to 0$ to arrive at

$$\int_{t_1}^{t_2}\int_{K_R} |Du|^{p-2}Du\cdot\left(\xi^p\frac{(u-k)_+}{(u-k)_+ + h}\right) = \int_{t_1}^{t_2}\int_{K_R} |Du|^{p-2}Du\cdot D(\xi^p)\frac{(u-k)_+}{(u-k)_+ + h}$$
$$+\int_{t_1}^{t_2}\int_{K_R} |D(u-k)_+|^p\frac{h}{((u-k)_+ + h)^2}\xi^p$$
$$\geq \int_{t_1}^{t_2}\int_{K_R} |Du|^{p-2}Du\cdot D(\xi^p)\frac{(u-k)_+}{(u-k)_+ + h},$$

since the last integral appearing in the integral identity is nonnegative. The proof is complete once we let $h \to 0$; just take notice that

$$\mathcal{I}(v_\sigma,k,h,+) \to \left(k^q + q\int_k^u s^{q-1}\,ds\right)\chi_{[u>k]} = (k+(u-k)_+)^q, \quad \text{as } h \to 0 \qquad \square$$

Remark 2.9. The purpose of this work is to present (and prove) the *expansion of positivity* for the nonnegative, local, weak solutions to (1.1). The results to come will be stated in this context (note that for $u \geq 0$, $\mathbf{u}^q = u^q$).

3 Energy estimates and DeGiorgi-type lemmas

The following result comprehends local integral estimates that are the starting point to the proof of the expansion of positivity, the so called energy estimates.

Proposition 3.1. *Let u be a nonnegative, local, weak sub(super)solution to (1.1) in Ω_T in the sense of (2.5). There exists a positive constant C, depending on N, p and q, such that for every cylinder $(x_o,t_o)+Q^-(\tau,R) \subset \Omega_T$, every real number $k \in \mathbb{R}$ and every piecewise smooth nonnegative cutoff function ξ vanishing on the the lateral boundary of $(x_0,t_0)+Q(\tau,R)$ one has*

$$\operatorname*{ess\,sup}_{t_0-\tau<t<t_0}\int_{K_R(x_0)} g_\pm(u,k)\xi^p + \iint_{(x_0,t_0)+Q(\tau,R)} |D(u-k)_\pm|^p\xi^p$$
$$\leq \int_{K_R(x_0)\times\{t_0-\tau\}} g_\pm(u,k)\xi^p + C\iint_{(x_0,t_0)+Q(\tau,R)} \left\{(u-k)_\pm^p|D\xi|^p + g_\pm(u,k)\left|\partial_t\left(\xi^p\right)\right|\right\}. \tag{3.1}$$

Proof. The proof follows quite closely the one presented in Lemma 2.8. In fact, we start by considering a nonnegative, local, weak subsolution u to (1.1) and then work with the mollified version (2.7), taking as test function $\varphi(x,t) = \xi^p(x,t)\psi_\epsilon(t)(u-k)_+$, where ξ and

ψ_ϵ are precisely the same as before. Observe that φ is an admissible test function due to the regularity assumptions on u.

By considering v_σ to be such that $(v_\sigma)^q = (u^q)^\star$ we get

$$\iint_{\Omega_T} \partial_t\left((v_\sigma)^q\right)\varphi + \left(|Du|^{p-2}Du\right)^\star \cdot D\varphi \le \int_\Omega u^q(x,0)\left(\frac{1}{\sigma}\int_0^T e^{-\frac{s}{\sigma}}\varphi(x,s)\,ds\right).$$

The left-hand side is estimated as follows. The term evolving the time derivative

$$\begin{aligned}
\iint_{\Omega_T} \partial_t\left((v_\sigma)^q\right)\varphi = & \iint_{Q(\tau,R)} \partial_t\left((v_\sigma)^q\right)\xi^p\psi_\epsilon(v_\sigma - k)_+ \\
& + \iint_{\Omega_T} \partial_t\left((v_\sigma)^q\right)\left((u-k)_+ - (v_\sigma - k)_+\right)\xi^p\psi_\epsilon \\
= & \iint_{\Omega_T} \partial_t\left(g_+(v_\sigma,k)\right)\xi^p\psi_\epsilon \\
& + \iint_{\Omega_T} \frac{u^q - (v_\sigma)^q}{\sigma}\left((u-k)_+ - (v_\sigma - k)_+\right)\xi^p\psi_\epsilon \\
\ge & \iint_{\Omega_T} \partial_t\left(g_+(v_\sigma,k)\right)\xi^p\psi_\epsilon \\
= & -\iint_{\Omega_T} g_+(v_\sigma,k)\partial_t\left(\xi^p\right)\psi_\epsilon - \iint_{\Omega_T} g_+(v_\sigma,k)\xi^p\psi'_\epsilon
\end{aligned}$$

The inequality relies on the fact that $f_1(s) = (s-k)_+$ and $f_2(s) = s^q$, $q > 0$, are monotone increasing functions. We now let $\sigma \to 0$ and thereby get

$$-\int_{t_1-\epsilon}^{t_2+\epsilon}\!\!\int_{K_R(x_0)} g_+(u,k)\partial_t\left(\xi^p\right)\psi_\epsilon - \frac{1}{\epsilon}\int_{t_1-\epsilon}^{t_1}\!\!\int_{K_R(x_0)} g_+(u,k)\xi^p + \frac{1}{\epsilon}\int_{t_2}^{t_2+\epsilon}\!\!\int_{K_R(x_0)} g_+(u,k)\xi^p,$$

since $u \in L^{q+1} \supset L^q$ we have $(v_\sigma) \to u$ in L^q. We then pass to the limit as ϵ goes to zero, obtaining

$$-\int_{t_1}^{t_2}\!\!\int_{K_R(x_0)} g_+(u,k)\partial_t\left(\xi^p\right) - \int_{K_R(x_0)\times\{t_1\}} g_+(u,k)\xi^p + \int_{K_R(x_0)\times\{t_2\}} g_+(u,k)\xi^p$$

and this completes the estimate of the parabolic term. As for the integral evolving the space derivatives, we start by letting $\sigma \to 0$, then we apply Young's inequality to get the inferior estimate

$$\frac{1}{2}\iint_{\Omega_T} |D(u-k)_+|^p\xi^p\psi_\epsilon - (2(p-1))^{p-1}\iint_{\Omega_T} (u-k)_+^p|D\xi|^p\psi_\epsilon$$

and finally, by letting $\epsilon \to 0$, we obtain

$$\frac{1}{2}\int_{t_1}^{t_2}\!\!\int_{K_R(x_0)} |D(u-k)_+|^p\xi^p - (2(p-1))^{p-1}\int_{t_1}^{t_2}\!\!\int_{K_R(x_0)} (u-k)_+^p|D\xi|^p.$$

As for the right-hand side,

$$\int_\Omega u^q(x,0)\left(\frac{1}{\sigma}\int_0^T e^{-\frac{s}{\sigma}}\varphi(x,s)\,ds\right) \to \int_\Omega u^q(x,0)\varphi(x,0) = 0, \quad \text{as } \sigma \to 0.$$

Combining all the previous estimates we have

$$\begin{aligned}
&\int_{K_R(x_0)\times\{t_2\}} g_+(u,k)\xi^p + \frac{1}{2}\int_{t_1}^{t_2}\!\!\int_{K_R(x_0)} |D(u-k)_+|^p \xi^p \\
&\quad\le \int_{K_R(x_0)\times\{t_1\}} g_+(u,k)\xi^p + \int_{t_1}^{t_2}\!\!\int_{K_R(x_0)} g_+(u,k)\partial_t\left(\xi^p\right) \\
&\qquad + (2(p-1))^{p-1}\int_{t_1}^{t_2}\!\!\int_{K_R(x_0)} (u-k)_+^p |D\xi|^p \\
&\quad\le \int_{K_R(x_0)\times\{t_1\}} g_+(u,k)\xi^p + \int_{t_0-\tau}^{t_0}\int_{K_R(x_0)} g_+(u,k)|\partial_t\left(\xi^p\right)| \\
&\qquad + (2(p-1))^{p-1}\int_{t_0-\tau}^{t_0}\int_{K_R(x_0)} (u-k)_+^p |D\xi|^p.
\end{aligned}$$

By letting $t_1 \to t_0-\tau$ and recalling $u \in C(L^{q+1})$, we have

$$\int_{K_R(x_0)\times\{t_1\}} g_+(u,k)\xi^p \to \int_{K_R(x_0)\times\{t_0-\tau\}} g_+(u,k)\xi^p;$$

as for the left-hand side of the previous inequality we reason as follows: on the one hand, for $t_0-\tau < t_2 < t_0$,

$$\int_{K_R(x_0)\times\{t_2\}} g_+(u,k)\xi^p + \frac{1}{2}\int_{t_0-\tau}^{t_2}\int_{K_R(x_0)} |D(u-k)_+|^p\xi^p \ge \int_{K_R(x_0)\times\{t_2\}} g_+(u,k)\xi^p,$$

and we then take the essential supremum over the set $t_0-\tau < t_2 < t_0$; on the other hand,

$$\begin{aligned}
&\int_{K_R(x_0)\times\{t_2\}} g_+(u,k)\xi^p + \frac{1}{2}\int_{t_0-\tau}^{t_2}\int_{K_R(x_0)} |D(u-k)_+|^p\xi^p \\
&\quad\ge \frac{1}{2}\int_{t_0-\tau}^{t_2}\int_{K_R(x_0)} |D(u-k)_+|^p\xi^p \to \frac{1}{2}\int_{t_0-\tau}^{t_0}\int_{K_R(x_0)} |D(u-k)_+|^p\xi^p,
\end{aligned}$$

as $t_2 \to t_0$.

A final remark: to prove the estimate for supersolutions it suffices to take $\varphi(x,t) = \xi^p(x,t)\psi_\epsilon(t)(u-k)_-$ and proceed in a similar way. □

The next two results are DeGiorgi-type lemmas, being the second one a variant involving information concerning initial data. They are presented for nonnegative, locally bounded, weak supersolutions to (1.1), however, one can state (and prove) similar results for nonnegative, locally bounded, weak subsolutions to (1.1). We recall that the local boundedness of the nonnegative, local, weak solutions u to (1.1) was obtained in [13] and [14].

To simplify the writing consider

$$(y,s) + Q^-(\lambda(2\rho)^p, 2\rho) = (y,s) + Q^-_{2\rho}(\lambda)$$

$$(y,s) + Q^+(\lambda(2\rho)^p, 2\rho) = (y,s) + Q^+_{2\rho}(\lambda)$$

Lemma 3.2. *Let u be a nonnegative, locally bounded, weak supersolution to (1.1) in Ω_T. Let $\tilde{M}$ be a positive number. There exists a constant $\tilde{\nu}$ depending on the N, p, q and on $\tilde{M}$ and λ, such that, if*

$$\left|(y,s) + Q^-_{2\rho}(\lambda) \cap \left[u < \tilde{M}\right]\right| \le \tilde{\nu}\left|Q^-_{2\rho}(\lambda)\right|$$

then

$$u \ge \frac{\tilde{M}}{2} \quad \text{a.e. in } (y,s) + Q^-_\rho(\lambda).$$

Proof. Without loss of generality take $(y,s)=(0,0)$. Construct decreasing sequences of numbers (radii and levels, respectively), for $n=0,1,\dots,$

$$\rho_n=\rho\left(1+\frac{1}{2^n}\right),\qquad k_n=\frac{\tilde M}{2}\left(1+\frac{1}{2^n}\right)$$

and define $Q_n=Q^-_{\rho_n}(\lambda)$ and $K_n=K_{\rho_n}$. Consider the cutoff function $0\le\xi(x,t)=\xi_1(x)\xi_2(t)\le 1$ defined in Q_n and such that

$$\xi_1(x)=1\quad\text{in } K_{n+1};\qquad \xi_1(x)=0\quad\text{in } \mathbb{R}^N\setminus K_n;\qquad |D\xi|\le\frac{2^{n+1}}{\rho};$$

$$\xi_2(t)=1\quad\text{for } t\ge-\lambda\rho^p_{n+1};\qquad \xi_2(t)=0\quad\text{for } t\le-\lambda\rho^p_n;\qquad 0\le\partial_t\xi_2\le\frac{2^{p(n+1)}}{\lambda\rho^p}.$$

From the energy estimates (3.1) written over Q_n for the truncated functions $(u-k_n)_-$, and recalling the estimates for $g_-(u,k_n)$ given in (2.4), we obtain

$$\begin{aligned}
\frac{\tilde M^{q-1}}{\gamma(q)}\ &\operatorname*{ess\,sup}_{-\lambda\rho_n^p<t\le 0}\int_{K_n\times\{t\}}(u-k_n)_-^2\xi^p+\iint_{Q_n}|D(u-k_n)_-|^p\xi^p\\
&\le\operatorname*{ess\,sup}_{-\lambda\rho_n^p<t\le 0}\int_{K_n\times\{t\}}g_-(u,k_n)\xi^p+\iint_{Q_n}|D(u-k_n)_-|^p\xi^p\\
&\le C(p,q)\ 2^{p(n+1)}\ \frac{k_n^p}{\rho^p}\left\{1+\frac{k_n^{q+1-p}}{\lambda}\right\}|A_n|\\
&\le C(p,q)\ 2^{p(n+1)}\ \frac{\tilde M^p}{\rho^p}\left\{1+\frac{\tilde M^{q+1-p}}{\lambda}\right\}|A_n|,
\end{aligned}$$

for $|A_n|=|Q_n\cap[u<k_n]|$.

Observe that, by means of Hölder's inequality together with the Sobolev embedding (2.2),

$$\begin{aligned}
\frac{\tilde M}{2^{n+2}}|A_{n+1}|&=(k_n-k_{n+1})|A_{n+1}|\le\iint_{Q_{n+1}}(u-k_n)_-\\
&\le C(N,p)\left(\iint_{Q_n}|D(u-k_n)_-\xi|^p\right)^{\frac{N}{p(N+2)}}\\
&\quad\times\left(\operatorname*{ess\,sup}_{-\lambda\rho_n^p<t\le 0}\int_{K_n\times\{t\}}(u-k_n)_-^2\xi^p\right)^{\frac{1}{N+2}}\ |A_n|^{1-\frac{N}{p(N+2)}}
\end{aligned}$$

and then, recalling the previous estimates and taking $Y_n=\frac{|A_n|}{|Q_n|}$, we get the recursive algebraic estimate

$$Y_{n+1}\le C(N,p,q)\left(\tilde M^{p-(q+1)}\lambda\right)^{\frac{1}{N+2}}\left(1+\frac{\tilde M^{q+1-p}}{\lambda}\right)^{\frac{N+p}{p(N+2)}}b^n\,Y_n^{1+\frac{1}{N+2}},\quad\text{for } b=2^{\frac{2N+p+2}{N+2}}.$$

The conclusion follows from the fast convergence Lemma 2.3 once we consider

$$\tilde\nu=C(N,p,q)^{-(N+2)}b^{-(N+2)^2}\frac{\frac{\tilde M^{q+1-p}}{\lambda}}{\left(1+\frac{\tilde M^{q+1-p}}{\lambda}\right)^{\frac{N+p}{p}}}.$$

□

Lemma 3.3. *Let u be a nonnegative, locally bounded, weak supersolution to (1.1) in Ω_T. Let $\tilde{M}$ be a positive number such that*

$$u(x,s) \geq \tilde{M} \quad \text{for a.e. } x \in K_{2\rho}(y) \tag{3.2}$$

and

$$\left|(y,s) + Q^+_{2\rho}(\lambda) \cap [u < \tilde{M}]\right| \leq \nu_0 \, \frac{\tilde{M}^{q+1-p}}{\lambda} \, \left|Q^+_{2\rho}(\lambda)\right| \tag{3.3}$$

for ν_0 depending only upon N, p and q. Then

$$u \geq \frac{\tilde{M}}{2} \quad \text{a.e. in } K_\rho(y) + (s, s + \lambda(2\rho)^p].$$

Proof. Take $(y,s) = (0,0)$, construct decreasing sequences of numbers (radii and levels, respectively),

$$\rho_n = \rho\left(1 + \frac{1}{2^n}\right), \quad k_n = \frac{\tilde{M}}{2}\left(1 + \frac{1}{2^n}\right), \quad n = 0, 1, \ldots$$

and take a time independent cutoff function $0 \leq \xi(x) \leq 1$ defined in K_{ρ_n} and, such that, $\xi = 1$ in $K_{\rho_{n+1}}$ and $|D\xi| \leq 2^{n+1}/\rho$.

Keeping in mind that u verifies (3.2), from the energy estimates (3.1) written over $Q_n = K_{\rho_n} \times (0, \lambda(2\rho)^p]$, for the truncated functions $(u-k_n)_-$, we obtain, for all $t \in (0, \lambda(2\rho)^p]$

$$\begin{aligned}
&\frac{\tilde{M}^{q-1}}{\gamma(q)} \int_{K_{\rho_n} \times \{t\}} (u-k_n)_-^2 \xi^p + \iint_{Q_n} |D(u-k_n)_-|^p \xi^p \\
&\quad \leq \int_{K_{\rho_n} \times \{t\}} g_-(u,k_n)\xi^p + \iint_{Q_n} |D(u-k_n)_-|^p \xi^p \\
&\quad \leq C(p) \iint_{Q_n} (u-k_n)_-^p |D\xi|^p \leq C(p)\, 2^{np}\, \frac{\tilde{M}^p}{\rho^p} |A_n|,
\end{aligned}$$

for $|A_n| = |Q_n \cap [u < k_n]|$. Arguing in a similar way as in the proof of Lemma 3.2, we deduce

$$\begin{aligned}
\frac{\tilde{M}}{2^{n+2}} |A_{n+1}| &\leq C(N,p) \left(\iint_{Q_n} |D(u-k_n)_- \xi|^p \right)^{\frac{N}{p(N+2)}} \\
&\quad \times \left(\operatorname*{ess\,sup}_{0<t<\lambda(2\rho)^p} \int_{K_{\rho_n} \times \{t\}} (u-k_n)_-^2 \xi^p \right)^{\frac{1}{N+2}} \; |A_n|^{1-\frac{N}{p(N+2)}}
\end{aligned}$$

and then, recalling the previous estimates and considering $Y_n = \frac{|A_n|}{|Q_n|}$, we arrive at

$$Y_{n+1} \leq C(N,p,q)\, \tilde{M}^{\frac{p-(q+1)}{N+2}}\, \lambda^{\frac{1}{N+2}}\, b^n\, Y_n^{1+\frac{1}{N+2}}, \qquad b = 2^{\frac{2N+p+2}{N+2}}.$$

The proof is completed once we take $\nu_0 = C(N,p,q)^{-(N+2)} b^{-(N+2)^2}$. □

4 Expansion of positivity

As it is now well known the *expansion of positivity* is a property of nonnegative supersolutions, to both stationary and evolutionary equations, that allows one to get Harnack inequalities and to prove regularity results. Heuristically speaking it asserts that the information on the measure of the *positivity set* of u, at a certain time level s over a cuber $K_\rho(y)$, can be expanded to the measure of the *positivity set* of u both in space (say from $K_\rho(y)$ to $K_{2\rho}(y)$) and in time (from the time level s to all further time levels till $s+\theta\rho^p$).

The *expansion of positivity* is based on energy estimates and DeGiorgi-type lemmas and comprehends two steps. The first step consists on the propagation of the positivity information on a cube $K_\rho(y)\times\{s\}$ to upper times levels. Not only this is the easiest step but also it holds for all values of $p>1$ and $q>0$. On the second step, one derives spacial propagation of positivity from the cube $K_\rho(y)$ to the bigger cube $K_{2\rho}(y)$. This is much more demanding and the proof has to be performed separately for the cases $q+1-p>0$ and $q+1-p<0$.

In what follows we adopt the technical scheme devised by DiBenedetto, Gianazza and Vespri for degenerate and singular p-Laplacian parabolic equations – the results can be found in [3]: Chapter 4, Sections 4 and 5, respectively.

Along this section we will assume that u is a nonnegative, locally bounded, weak supersolution to (1.1) in Ω_T, for $p>1$ and $q>0$.

Lemma 4.1. *Assume that, for some $(y,s)\in\Omega_T$ and some $\rho>0$,*

$$\left|K_\rho(y)\cap[u(\cdot,s)>M]\right|\geq\alpha\left|K_\rho(y)\right|, \tag{4.1}$$

for some $M>0$ and some $0<\alpha<1$. Then there exist $\epsilon,\delta\in(0,1)$, depending on α and on N,p and q, such that

$$\left|K_\rho(y)\cap[u(\cdot,t)>\epsilon M]\right|\geq\frac{\alpha}{2}\left|K_\rho(y)\right|, \tag{4.2}$$

for all $t\in(s,s+\delta M^{q+1-p}\rho^p]$.

Proof. Without loss of generality we may take $(y,s)=(0,0)$. Consider the cylinder $Q=K_\rho\times(0,\delta M^{q+1-p}\rho^p]$ and assume $Q\subset\Omega_T$ (take ρ as smaller as needed). Write the energy estimates (3.1) for the cylinder Q, the level $k=M$ and the smooth time independent cutoff function $0\leq\xi=\xi(x)\leq 1$ defined in K_ρ, vanishing on the boundary of K_ρ and verifying, for some $\sigma\in(0,1)$,

$$\xi=1\quad\text{in } K_{(1-\sigma)\rho}\quad\text{and}\quad|D\xi|\leq\frac{1}{\sigma\rho}.$$

We then have, for all $t\in(0,\delta M^{q+p-1}\rho^p]$,

$$\begin{aligned}\int_{K_{(1-\sigma)\rho}\times\{t\}}g_-(u,k)&\leq\int_{K_\rho\times\{t\}}g_-(u,k)\xi^p\\&\leq\int_{K_\rho\times\{0\}}g_-(u,k)\xi^p+C(p)\iint_Q(u-k)_-^p|D\xi|^p\\&\leq\int_{K_\rho\times\{0\}}\left(q\int_u^M s^{q-1}(M-s)\,ds\right)\xi^p\chi_{[u<M]}+C(p)\frac{M^p}{\sigma^p\rho^p}|Q|.\end{aligned}$$

The left-hand side is estimated by considering the integration over the smaller cube $K_{(1-\sigma)\rho}\cap[u(\cdot,t)<\epsilon M]$

$$\begin{aligned}\int_{K_{(1-\sigma)\rho}\times\{t\}}g_-(u,k)&=\int_{K_{(1-\sigma)\rho}\times\{t\}}\left(q\int_u^M s^{q-1}(M-s)\,ds\right)\chi_{[u<M]}\\&\geq\int_{K_{(1-\sigma)\rho}\times\{t\}}\left(q\int_u^M s^{q-1}(M-s)\,ds\right)\chi_{[u<\epsilon M]}\end{aligned}$$

$$
\begin{aligned}
&\geq \int_{K_{(1-\sigma)\rho}\times\{t\}} \left(q\int_{\epsilon M}^{M} s^{q-1}(M-s)\; ds \right) \chi_{[u<\epsilon M]} \\
&= \left| K_{(1-\sigma)\rho} \cap [u(\cdot,t) < \epsilon M] \right| \times \left(q\int_{\epsilon M}^{M} s^{q-1}(M-s)\; ds \right)
\end{aligned}
$$

and then, for all $t \in (0, \delta M^{q+p-1}\rho^p]$,

$$
\begin{aligned}
\left| K_{(1-\sigma)\rho} \cap [u(\cdot,t) < \epsilon M] \right| &\leq \frac{\int_{K_\rho\times\{0\}} \left(q\int_{u}^{M} s^{q-1}(M-s)\; ds \right) \xi^p \chi_{[u<M]}}{q\int_{\epsilon M}^{M} s^{q-1}(M-s)\; ds} \\
&\quad + C(p)\, \frac{\delta M^{q+1}}{\sigma^p}\, \frac{1}{q\int_{\epsilon M}^{M} s^{q-1}(M-s)\; ds}\, |K_\rho| \\
&\leq \left\{ (\gamma(q)\epsilon^q + 1)(1-\alpha) + C(p,q)\frac{\delta}{\sigma^p} \right\} |K_\rho|.
\end{aligned}
$$

These inequalities were obtained arguing as follows: due to (2.4) and considering $0 < \epsilon < \frac{1}{2}$

$$
q\int_{\epsilon M}^{M} s^{q-1}(M-s)\; ds \geq \frac{1}{2\gamma(q)} M^{q+1}(1-\epsilon)^2(1+\epsilon)^{q-1} \geq \frac{M^{q+1}}{\gamma(q)}\;;
$$

and by making use of the same inequality (2.4) and recalling (4.1)

$$
\begin{aligned}
&\frac{\int_{K_\rho\times\{0\}} \left(q\int_{u}^{M} s^{q-1}(M-s)\; ds \right) \xi^p \chi_{[u<M]}}{q\int_{\epsilon M}^{M} s^{q-1}(M-s)\; ds} \\
&\quad = \frac{\int_{K_\rho\times\{0\}} \left(q\int_{u}^{\epsilon M} s^{q-1}(M-s)\; ds \right) \xi^p \chi_{[u<M]}}{q\int_{\epsilon M}^{M} s^{q-1}(M-s)\; ds} + \int_{K_\rho\times\{0\}} \chi_{[u<M]} \\
&\quad \leq \left\{ \frac{\int_0^{\epsilon M} s^{q-1} M\; ds}{\int_{\epsilon M}^{M} s^{q-1}(M-s)\; ds} + 1 \right\} (1-\alpha)|K_\rho| \\
&\quad \leq (\gamma(q)\epsilon^q + 1)(1-\alpha)|K_\rho|.
\end{aligned}
$$

Therefore, for all $t \in (0, \delta M^{q+p-1}\rho^p]$,

$$
\left| K_\rho \cap [u(\cdot,t) < \epsilon M] \right| \leq \left\{ (\gamma(q)\epsilon^q + 1)(1-\alpha) + C(p,q)\frac{\delta}{\sigma^p} + \sigma N \right\} |K_\rho|.
$$

The proof is complete once we choose $\sigma \in (0,1)$ such that $N\sigma \leq \frac{\alpha}{8}$; then choose

$$
\delta \in (0,1) \quad \text{such that} \quad C(p,q)\frac{\delta}{\sigma^p} \leq \frac{\alpha}{8}
$$

and finally choose

$$
\epsilon \in \left(0, \frac{1}{2}\right) \quad \text{such that} \quad (\gamma(q)\epsilon^q + 1)(1-\alpha) \leq 1 - \frac{3}{4}\alpha.
$$

Observe that, with theses choices, the parameters δ and ϵ depend only on α and on N, p, q. $\square$

Expansion of positivity when $q+1-p>0$

Consider a point $(y,s)\in\Omega_T$ and let $\rho>0$ be such that

$$K_{16\rho}(y)\times(s,s+\delta M^{q+1-p}\rho^p]\subset\Omega_T,$$

where δ and M are the same positive real numbers presented in Lemma 4.1.

Consider that (4.1) holds. In order to obtain the expansion of positivity, we start to consider the change of variables: in space, given by $z=\frac{x-y}{\rho}$ and in time, given by

$$-e^{-\tau}=\frac{t-(s+\delta M^{q+1-p}\rho^p)}{\delta M^{q+1-p}\rho^p},$$

which maps the original cylinder $K_{16\rho}(y)\times(s,s+\delta M^{q+1-p}\rho^p]$ into $K_{16}\times(0,+\infty)$.

Introduce the new function

$$v(z,\tau)=\frac{u(x,t)}{M}\,e^{\frac{\tau}{q+1-p}}$$

which verifies

$$\partial_\tau(v^q)-\mathrm{div}\left(\delta|Dv|^{p-2}Dv\right)=\frac{q}{q+1-p}\,v^q \tag{4.3}$$

where D denotes de space derivates of v with respect to z.

Keeping in mind that δ and ϵ are already determined and depend only on N,p,q and on α, the conclusion of Lemma 4.1 now reads

$$\left|K_1\cap[v(\cdot,\tau)>\epsilon\,e^{\frac{\tau}{q+1-p}}]\right|\geq\frac{\alpha}{2}\,|K_1|\,,\quad\forall\tau>0$$

and therefore, once we take $\tau_0>0$ and consider the level $k_0=\epsilon\,e^{\frac{\tau_0}{q+1-p}}$, we have

$$|K_1\cap[v(\cdot,\tau)\geq k_0]|\geq\frac{\alpha}{2}\,|K_1|\,,\quad\forall\tau\geq\tau_0$$

and then, for all $k\leq k_0$,

$$|K_8\cap[v(\cdot,\tau)\geq k]|\geq|K_8\cap[v(\cdot,\tau)\geq k_0]|\geq\frac{\alpha}{2}\,|K_1|=\frac{\alpha}{2^{1+3N}}\,|K_8|\,,\quad\forall\tau\geq\tau_0. \tag{4.4}$$

With the time level τ_0 and the level k_0 we construct the cylinders

$$Q_{\tau_0}=K_8\times\left(\tau_0+k_0^{q+1-p},\tau_0+2k_0^{q+1-p}\right]\subset\tilde{Q}_{\tau_0}=K_8\times\left(\tau_0,\tau_0+2k_0^{q+1-p}\right]$$

and introduce smaller levels

$$k_j=\frac{k_0}{2^j},\quad\text{for } j=0,1,\ldots,s_*,$$

where s_* is to be chosen.

Consider a piecewise smooth cutoff $0\leq\xi(x,t)=\xi_1(x)\xi_2(t)\leq1$ defined in $\tilde{Q}_{\tau_0}$ and such that

$$\xi_1(x)=\begin{cases}1 & \text{in } K_8,\\ 0 & \text{in } \mathbb{R}^N\setminus K_8,\end{cases}\quad\text{and}\quad|D\xi_1|\leq\frac{1}{8}$$

and

$$\xi_2(x)=\begin{cases}0 & \tau<\tau_0,\\ 1 & \tau\geq\tau_0+k_0^{q+1-p},\end{cases}\quad\text{and}\quad0\leq\partial_\tau\xi_2\leq\frac{1}{k_0^{q+1-p}}.$$

At this stage we perform formally (the accurate way to proceed follows the procedure presented before when deducing the energy estimates): we start by multiplying (4.3) by $(v-k_j)_-\xi^p$ and then integrate over $\tilde{Q}_{\tau_0}$. Note that the right-hand side is nonnegative, since $v \geq 0$ and $q+1-p>0$, and therefore we have

$$\iint_{\tilde{Q}_{\tau_0}} \partial_\tau (v^q)(v-k_j)_-\xi^p + \delta \iint_{\tilde{Q}_{\tau_0}} |Dv|^{p-2}Dv \cdot D\left((v-k_j)_-\xi^p\right) \geq 0.$$

This integral inequality is equivalent to

$$\iint_{\tilde{Q}_{\tau_0}} \partial_\tau \left(g_-(v,k_j)\right)\xi^p + \delta \iint_{\tilde{Q}_{\tau_0}} |D(v-k_j)_-|^{p-2}D(v-k_j)_- \cdot D\left((v-k_j)_-\xi^p\right) \leq 0$$

from which we get, for all $\tau \in (0, \tau_0 + 2k_0^{q+1-p}]$,

$$\begin{aligned}\iint_{\tilde{Q}_{\tau_0}} |D(v-k_j)_-|^p\xi^p &\leq \frac{1}{\delta}\int_{K_8\times\{\tau_0\}} g_-(v,k_j)\xi^p + \frac{C(p)}{\delta}\iint_{\tilde{Q}_{\tau_0}} \left\{(v-k_j)_-^p|D\xi|^p + g_-(v,k_j)\partial_\tau(\xi^p)\right\}\\ &\leq \frac{C(p,q)}{\delta}\, k_j^p\, |\tilde{Q}_{\tau_0}|.\end{aligned}$$

We have used (2.4) to obtain an upper bound to $g_-(v,k_j)$

$$\begin{aligned}g_-(v,k_j) &\leq \gamma(q)(v-k_j)_-^2(k_j+v)^{q-1} \leq \gamma(q)(k_j+v)^{q+1} \leq \gamma(q)k_j^{q+1}\\ &\leq \gamma(q)\, k_j^p\, (2k_0)^{q+1-p}.\end{aligned}$$

We now apply inequality (2.1), for the levels k_j and k_{j+1} to arrive at

$$\frac{k_j}{2}\left|K_8 \cap [v(\cdot,\tau < k_{j+1}]\right| \quad\leq\quad \frac{\gamma(N)}{\left|K_8\cap[v(\cdot,\tau)>k_j]\right|}\int_{K_8\cap[k_{j+1}<v(\cdot,\tau)<k_j]}|Dv|.$$

By integrating the previous inequality in time over $(\tau_0 + k_0^{q+1-p}, \tau_0 + 2k_0^{q+1-p}]$, then making use of Hölder's inequality and the estimate obtained previously to $\iint_{\tilde{Q}_{\tau_0}} |D(v-k_j)_-|^p\xi^p$, we get

$$\left|Q_{\tau_0}\cap[v<k_{j+1}]\right| \leq \frac{\gamma(N)}{\alpha}\left(\frac{\gamma(p,q)}{\delta}\right)^{\frac{1}{q}} |Q_{\tau_0}|^{\frac{1}{p}}\ \left|Q_{\tau_0}\cap[k_{j+1}<v<k_j]\right|^{\frac{p-1}{p}}.$$

Take the power $\frac{p}{p-1}$ and add this inequality for $j = 0, 1, \ldots, s_* - 1$

$$\left|Q_{\tau_0}\cap\left[v<\frac{k_0}{2^{s_*}}\right]\right| \leq \frac{\gamma(N,p,q)}{\alpha\,\delta^p}\,\frac{1}{s_*^{\frac{p-1}{p}}}\,|Q_{\tau_0}|. \tag{4.5}$$

The "natural" thought now would be to argue in a DeGiorgi fashion (something like what was done in Lemma 3.2) to conclude that, for an appropriate choice of s_*, this measure theoretical information (4.5) implies that

$$v \geq \frac{k_0}{2^{s_*+1}} \qquad \text{in a smaller cylinder,}$$

say $K_4 \times (\tau_0 + \lambda k_0^{q+1-p}, \tau_0 + 2k_0^{q+1-p}]$, for some $\lambda \in (0,1)$. However, the cylinder's length k_0^{q+1-p} is too big for the level $\frac{k_0}{2^{s_*}}$ at hands; and that is the reason why one needs to work within thinner cylinders with length $\left(\frac{k_0}{2^{s_*}}\right)^{q+1-p}$. That is precisely our purpose in what is to come.

Assume for now that s_* is determined. Consider the cylinder Q_{τ_0} sliced into $(2^{s_*})^{q+1-p}$ subcylinders of length $\left(\frac{k_0}{2^{s_*}}\right)^{q+1-p}$ (if necessary take a bigger s_* so that $(2^{s_*})^{q+1-p}$ is an integer)

$$Q_{\tau_0} = \bigcup_{i=0}^{(2^{s_*})^{q+1-p}-1} Q_i$$

where

$$Q_i = K_8 \times \left(T_0^i, T_0^i + \left(\frac{k_0}{2^{s_*}}\right)^{q+1-p} \right]$$

for

$$T_0^i = \tau_0 + k_0^{q+1-p} + i\left(\frac{k_0}{2^{s_*}}\right)^{q+1-p} .$$

In at least one of these subcylinders, say Q_i, the previous measure theoretical information (4.5) holds, that is

$$\left| Q_i \cap \left[v < \frac{k_0}{2^{s_*}} \right] \right| \leq \frac{\gamma(N,p,q)}{\alpha\ \delta^p} \frac{1}{s_*^{\frac{p-1}{p}}} |Q_i|,$$

for some $i = 0, 1, \ldots, (2^{s_*})^{q+1-p} - 1$. Our goal now is to get a lower bound for v in a cylinder contained in the upper half of the cylinder Q_i, via a DeGiorgi argument. For that purpose, we consider decreasing sequences of radii, cylinders and levels given by, for $n = 0, 1, \ldots$

$$4 < R_n = 4\left(1 + \frac{1}{2^n}\right) \leq 8,$$

$$Q_n = K_{R_n} \times \left(T_0^i + \frac{1}{2}\left(1 - \frac{1}{2^n}\right)\left(\frac{k_0}{2^{s_*}}\right)^{q+1-p}, T_0^i + \left(\frac{k_0}{2^{s_*}}\right)^{q+1-p} \right] \subset Q_i,$$

and

$$\frac{k_0}{2^{s_*+1}} < k_n = \frac{k_0}{2^{s_*+1}}\left(1 + \frac{1}{2^n}\right) \leq \frac{k_0}{2^{s_*}}$$

and cutoff function ξ defined in Q_n and such that: $0 \leq \xi \leq 1$, in Q_n, $\xi = 0$ on parabolic boundary of Q_n, $\xi = 1$ in Q_{n+1} and

$$|D\xi| \leq 2^n \quad \text{and} \quad |\partial_\tau \xi| \leq 2^{n+2}\left(\frac{2^{s_*}}{k_0}\right)^{q+1-p} .$$

We then write the energy estimates (3.1) for v, over Q_n, with $k = k_n$

$$\begin{aligned}
&\operatorname*{ess\,sup}_{T_0^i < \tau < T_0^i + \left(\frac{k_0}{2^{s_*}}\right)^{q+1-p}} \int_{K_{R_n}\{\tau\}} g_-(v,k_n)\xi^p + \iint_{Q_n} |D(v-k_n)_-|^p \xi^p \\
&\leq C(p,q) \iint_{Q_n} \left\{ (v-k_n)_-^p |D\xi|^p + g_-(v,k_n)\, |\partial_t (\xi^p)| \right\} \\
&\leq C(p,q)\, 2^{np}\, k_n^p\, |A_n|,
\end{aligned}$$

where $|A_n| = |Q_n \cap [v < k_n]$. The last inequality was obtained recalling estimate (2.4). To estimate from below the integral term presenting $g_-(v, k_n)$, we use again (2.4) and the fact that $\frac{k_0}{2^{s_*+1}} < k_n \le \frac{k_0}{2^{s_*}}$ to arrive at

$$\int_{K_{R_n}\{\tau\}} g_-(v,k_n)\xi^p \ge \frac{1}{\gamma(q)}\left(\frac{k_0}{2^{s_*}}\right)^{q-1}\int_{K_{R_n}\{\tau\}}(v-k_n)_-^2\xi^p.$$

The previous estimate together with the energy estimates, Hölder's inequality and the Sobolev embedding (2.2), with $m = 2$, allows us to get

$$\begin{aligned}(k_n - k_{n+1})|A_{n+1}| &\le \iint_{Q_{n+1}}(v-k_n)_- \le \iint_{Q_n}(v-k_n)_-\xi \\ &\le \left\{\iint_{Q_n}((v-k_n)_-\xi)^{p\frac{N+2}{N}}\right\}^{\frac{N}{p(N+2)}}|A_n|^{1-\frac{N}{p(N+2)}} \\ &\le C(N,p,q)\, 2^{\frac{N+p}{(N+2)}n}\, k_n^{\frac{N+p}{N+2}}\left(\frac{2^{s_*}}{k_0}\right)^{\frac{q-1}{N+2}}|A_n|^{1+\frac{1}{N+2}}\end{aligned}$$

and from here, by considering $Y_n = \dfrac{|A_n|}{|Q_n|}$, we deduce the algebraic estimate

$$Y_{n+1} \le C(N,p,q)\, b^n\, Y_n^{1+\frac{1}{N+2}}, \quad \text{for } b = 2^{\frac{2N+p+2}{N+2}} > 1.$$

The algebraic convergence Lemma 2.3 says that

if $Y_0 \le C(N.p,q)^{-(N+2)}b^{-(N+2)^2}$, then $Y_n \to 0$, as $n \to +\infty$;

so we just need to choose s_* such that

$$\frac{\gamma(N,p,q)}{\alpha\ \delta^p}\ \frac{1}{s_*^{\frac{p}{p-1}}} = C(N,p,q)^{-(N+2)}b^{-(N+2)^2}.$$

Remark 4.2. Observe that with this choice, s_* only depends on N, p, q and α (since δ is already determined and depends on these parameters as well).

The length of the cylinder Q_i is exactly the one needed so that, when arguing in a DeGiorgi fashion, given by Lemma 3.2, there is an independence of ν_0 on the levels $\tilde{M}$ and the cylinder's length λ. In fact, in our case $\tilde{M} = \frac{k_0}{2^{s_*}}$ and $\lambda = \left(\frac{k_0}{2^{s_*}}\right)^{q+1-p}$.

We thereby obtain the lower bound

$$v \ge \frac{k_0}{2^{s_*+1}} \quad \text{a.e. in } K_4 \times \left(T_0^i + \frac{1}{2}\left(\frac{k_0}{2^{s_*}}\right)^{q+1-p}, T_0^i + \left(\frac{k_0}{2^{s_*}}\right)^{q+1-p}\right];$$

in particular

$$v(,\tau_1) \ge \frac{k_0}{2^{s_*+1}} \quad \text{a.e. in } K_4,$$

for

$$\tau_0 + k_0^{q+1-p} < T_0^i + \frac{1}{2}\left(\frac{k_0}{2^{s_*}}\right)^{q+1-p} < \tau_1 \le T_0^i + \left(\frac{k_0}{2^{s_*}}\right)^{q+1-p} < \tau_0 + 2k_0^{q+1-p}.$$

Returning to the original coordinates and function, we may conclude that

$$u(,t_1) \ge e^{-\frac{\tau_1}{q+1-p}}\frac{k_0}{2^{s_*+1}}\, M = \frac{\epsilon}{2^{s_*+1}}e^{\frac{\tau_0-\tau_1}{q+1-p}}\, M \quad \text{a.e. in } K_{4\rho}(y)$$

where t_1 is defined by

$$-e^{-\tau_1} = \frac{t_1 - (s + \delta M^{q+1-p}\rho^p)}{\delta M^{q+1-p}\rho^p} \Longleftrightarrow t_1 = s + (1 - e^{-\tau_1})\delta M^{q+1-p}\rho^p.$$

We are two steps away to conclude the *expansion of positivity*. First, by considering $\tilde{M} = \frac{\epsilon}{2^{s_*+1}} e^{\frac{\tau_0-\tau_1}{q+1-p}} M$ and choosing $\lambda = \nu_0 \tilde{M}^{q+1-p}$, where $\nu_0 = \nu_0(N,p,q)$, the assumptions of the variant DeGiorgi-type Lemma 3.3 are verified and we may conclude

$$u \geq \frac{\tilde{M}}{2} = \frac{\epsilon}{2^{s_*+2}} e^{\frac{\tau_0-\tau_1}{q+1-p}} M \quad \text{a.e. in } K_{2\rho}(y)$$

for all times

$$t_1 \leq t \leq t_1 + \lambda(2\rho)^p.$$

Finally, we choose τ_0 such that

$$t_1 + \lambda(2\rho)^p = s + \delta M^{q+1-p}\rho^p,$$

that is, keeping in mind the expressions of t_1 and of λ,

$$e^{\tau_0} = \delta C^{N+2} 2^{(N+2)^2-p} \left(\frac{2^{s_*+1}}{\epsilon}\right)^{q+1-p}$$

and from the range of τ_1 we have

$$t_1 < s + \left(1 - e^{-\tau_0} - 2\epsilon^{q+1-p} e^{\tau_0}\right) \delta M^{q+1-p}\rho^p \leq s + \frac{\delta}{2} M^{q+1-p}\rho^p.$$

Gathering these last estimates, we get

$$u(\cdot,t) \geq \frac{\epsilon}{2^{s_*+2}} e^{-\frac{2k_0^{q+1-p}}{q+1-p}} M \quad \text{a.e. in } K_{2\rho}(y),$$

for all

$$t \in \left(s + (1-\lambda)\delta M^{q+1-p}\rho^p, s + \delta M^{q+1-p}\rho^p\right].$$

We have proved

Proposition 4.3. *Let u is a nonnegative, local, weak supersolution to (1.1) in Ω_T. Assume that, for some $(y,s) \in \Omega_T$ and some $\rho > 0$,*

$$\left|K_\rho(y) \cap [u(\cdot,s) \geq M]\right| \geq \alpha \left|K_\rho(y)\right|,$$

for some $M > 0$ and some $\alpha \in (0,1)$. Then there exist $\delta, \lambda, \eta \in (0,1)$, depending on N, p, q and α, such that

$$u(\cdot,t) \geq \eta M \quad \text{a.e. in } K_{2\rho}(y)$$

and for all $t \in \left(s + (1-\lambda)\delta M^{q+1-p}\rho^p, s + \delta M^{q+1-p}\rho^p\right]$.

Expansion of positivity when $q+1-p<0$

Consider a point $(y,s)\in\Omega_T$ and let $\rho>0$ be such that

$$K_{8\rho}(y)\times\left(s,s+\delta\,\frac{b^{p-(q+1)}}{(\eta M)^{p-(q+1)}}\,\rho^p\right]\subset\Omega_T,$$

where M is a given positive number and δ,η,b are positive numbers to be determined.

Proposition 4.4. *Let u is a nonnegative, local, weak supersolution to (1.1) in Ω_T. Assume that (4.1) holds, for some $(y,s)\in\Omega_T$, $\rho>0$ and $\alpha\in(0,1)$. Then there exist $\delta,\eta,b\in(0,1)$, depending on N,p,q and α, such that*

$$u(\cdot,t)\geq\eta M\quad \text{a.e. in } K_{2\rho}(y)$$

and for all $t\in(s+\frac{\delta}{2}\frac{b^{p-(q+1)}}{(\eta M)^{p-(q+1)}}\rho^p,s+\delta\frac{b^{p-(q+1)}}{(\eta M)^{p-(q+1)}}\rho^p]$.

Proof. Assume that (4.1) is verified. Then, for all $0<\sigma_0\leq 1$, one also has

$$\left|K_\rho(y)\cap[u(\cdot,s)>\sigma_0 M]\right|\geq\alpha\left|K_\rho(y)\right|.$$

Consider the energy estimates written over

$$K_\rho(y)\times(s,s+\delta(\sigma_0 M)^{q+1-p}\rho^p]$$

for the levels $k=\sigma_0 M$. By proceeding as in the proof of Lemma 4.1, we obtain the same parameters ϵ and δ, depending on N,p,q and α, for which

$$\left|K_\rho(y)\cap[u(\cdot,t)>\epsilon\,\sigma_0 M]\right|\geq\frac{\alpha}{2}\left|K_\rho(y)\right|,\tag{4.6}$$

for all $t\in(s,s+\delta\,(\sigma_0 M)^{q+1-p}\rho^p]$.

For $\tau\geq 0$, consider the number

$$\sigma_\tau=e^{-\frac{\tau}{p-(q+1)}}\leq 1.$$

Since (4.6) holds for all $0<\sigma_0\leq 1$, it also holds for σ_τ

$$\left|K_\rho(y)\cap[u(\cdot,t)>\epsilon\,\sigma_\tau M]\right|\geq\frac{\alpha}{2}\left|K_\rho(y)\right|,\quad\forall t\in(s,s+\delta\,(\sigma_\tau M)^{q+1-p}\rho^p]$$

and, in particular,

$$\left|K_\rho(y)\cap[u(\cdot,s+\delta\,(\sigma_\tau M)^{q+1-p}\rho^p)>\epsilon\,\sigma_\tau M]\right|\geq\frac{\alpha}{2}\left|K_\rho(y)\right|.$$

Introduce the change of variable

$$e^\tau=(t-s)\,\frac{M^{p-(q+1)}}{\delta\rho^p}$$

and the define the new function

$$v(x,\tau)=\frac{e^{\frac{\tau}{p-(q+1)}}}{M}\,(\delta\rho^p)^{\frac{1}{p-(q+1)}}\,u(x,t).$$

In this new setting, v is a solution to

$$\partial_\tau\left(v^q\right)-\operatorname{div}\left(|Dv|^{p-2}Dv\right)=\frac{q}{p-(q+1)}\,v^q\geq 0$$

and the measure theoretical information on u is translated into the following measure theoretical information on v

$$\left|K_\rho(y)\cap\left[v(\cdot,\tau)>\epsilon\left(\delta\rho^p\right)^{\frac{1}{p-(q+1)}}\right]\right|\geq\frac{\alpha}{2}\left|K_\rho(y)\right|,\qquad\tau\geq 0.$$

Therefore one gets, for all $\tau\geq 0$

$$\left|K_{4\rho}(y)\cap[v(\cdot,\tau)>k_0]\right|\geq\frac{\alpha}{2\,4^N}\left|K_{4\rho}(y)\right|,$$

for

$$k_0=\epsilon\left(\delta\rho^p\right)^{\frac{1}{p-(q+1)}}\qquad\text{(completely determined).}$$

Consider the smaller levels

$$k_j=\frac{k_0}{2^j},\qquad\text{for } j=0,1,\ldots s^*\quad(s^*\text{ to be chosen}),$$

take the stretching factor θ as

$$\theta=\left(\frac{k_0}{2^{s^*}}\right)^{q+1-p}\geq k_j^{q+1-p},\qquad\text{for all } j=0,1,\ldots,s^*,\tag{4.7}$$

construct the cylinders

$$Q=(y,0)+Q^+_{8\rho}(\theta)\quad\text{and}\quad\tilde{Q}=K_{4\rho}(y)\times(\theta(4\rho)^p,\theta(8\rho)^p],\quad\tilde{Q}\subset Q$$

and take $\varphi=(v-k)_-\xi^p$ as a test function, where $\xi\in[0,1]$ is a smooth cutoff function defined in Q, vanishing on its parabolic boundary and verifying

$$\xi=1\quad\text{in }\tilde{Q},\quad|D\xi|\leq\frac{1}{4\rho}\quad\text{and}\quad|\partial_\tau\xi|\leq\frac{1}{\theta(4\rho)^p}.$$

For these choices the arrive at

$$\begin{aligned}\iint_{\tilde{Q}}|D(v-k_j)_-|^p&\leq\iint_Q|D(v-k_j)_-|^p\xi^p\\&\leq C(p)\frac{k_j^p}{(4\rho)^p}\left\{1+\frac{k_j^{q+1-p}}{\theta}\right\}|A_j|\leq C(p)\frac{k_j^p}{(4\rho)^p}|A_j|\end{aligned}$$

due to the definition of θ and taking $|A_j|=|Q\cap[v<k_j]|$. We then proceed in a similar way as in the case $q+1-p>0$, to find out that

$$\left|\tilde{Q}\cap\left[v<\frac{k_0}{2^{s^*}}\right]\right|\leq\frac{C(N,p,q)}{\alpha}\,\frac{1}{(s^*)^{\frac{p-1}{p}}}\,|\tilde{Q}|.$$

This estimate on the measure of the set where v is below the level $\frac{k_0}{2^{s^*}}$ will be the starting point to argument in a DeGiorgi fashion in a backward cylinder, like in Lemma 3.2. Along the way, the length θ of the cylinder will be determined. More precisely, consider the cylinder

$$(y,\tau^*)+Q^-_{4\rho}(\theta)=K_{4\rho}(y)\times(\tau^*-\theta(4\rho)^p,\tau^*]\subset\tilde{Q}\quad\text{for }\tau^*=\theta(8\rho)^p$$

and the sequences of numbers

$$2\rho < \rho_n = 2\rho\left(1+\frac{1}{2^n}\right) \le 4\rho, \quad \frac{k_0}{2^{s^*+1}} < k_n = \frac{k_0}{2^{s^*+1}}\left(1+\frac{1}{2^n}\right) \le \frac{k_0}{2^{s^*}}$$

and of nested and shrinking cylinders

$$Q_n^- = (y,\tau^*) + Q_{\rho_n}^-(\theta),$$

for $n = 0, 1, \ldots$ Take a cutoff function $0 \le \xi \le 1$ defined in Q_n^- and such that: $\xi = 0$ on the parabolic boundary of Q_n^-, $\xi = 1$ in Q_{n+1}^- and

$$|D\xi| \le \frac{2^{n+1}}{\rho} \quad \text{and} \quad |\partial_\tau \xi| \le C\,\frac{2^{np}}{\theta\rho^p}$$

and write the energy estimates (3.1) for v, over Q_n^-, with $k = k_n$. Recalling the estimates (2.4) on $g_-(v,k_n)$ and the definition (4.7) of θ, we obtain

$$\begin{aligned}
&\operatorname*{ess\,sup}_{\tau^*-\theta(\rho_n)^p<\tau<\tau^*} \int_{K_{\rho_n}\times\{\tau\}} g_-(v,k_n)\xi^p + \iint_{Q_n^-} |D(v-k_n)_-|^p \xi^p \\
&\quad \le C(p,q) \iint_{Q_n^-} \left\{ (v-k_n)_-^p |D\xi|^p + g_-(v,k_n)\, |\partial_\tau(\xi^p)| \right\} \\
&\quad \le C(p,q)\, 2^{np}\, \frac{k_n^p}{\rho^p} \left\{ 1 + \frac{k_n^{q+1-p}}{\theta} \right\} |A_n| \le C(p,q)\, 2^{np}\, \frac{k_n^p}{\rho^p}\, |A_n|,
\end{aligned}$$

where, as usually, $|A_n| = |Q_n^- \cap [v < k_n]|$. Observe that, on the one hand

$$\iint_{Q_n^-} (v-k_n)_-\xi \ge \iint_{Q_{n+1}^-} (v-k_n)_- \ge (k_n - k_{n+1})|A_{n+1}| = \frac{k_0}{2^{s^*}}\,\frac{1}{2^{n+2}}|A_{n+1}|$$

and, on the other hand, by applying Hölder's inequality with exponent $p\frac{N+2}{N}$, together with Sobolev's embedding, we get

$$\iint_{Q_n^-} (v-k_n)_-\xi \le C(N,p,q)\left(\frac{2^n}{\rho}\right)^{\frac{N+p}{N+2}} k_n^{\frac{N+p+1-q}{N+2}}\, |A_n|^{1+\frac{1}{N+2}}.$$

Consider the numbers $Y_n = \frac{|A_n|}{|Q_n^-|}$. From the previous estimates we deduce

$$Y_{n+1} \le C(N,p,q)\, b^n\, Y_n^{1+\frac{1}{N+2}}, \quad \text{for } b = 2^{\frac{2N+p+2}{N+2}} > 1$$

and we may conclude that Y_n goes to zero as $n \to +\infty$ once we have

$$\frac{\left|(y,\tau^*) + Q_{4\rho}^-(\theta) \cap \left[v < \frac{k_0}{2^{s^*}}\right]\right|}{\left|(y,\tau^*) + Q_{4\rho}^-(\theta)\right|} = Y_0 \le C(N,p,q)^{-(N+2)} 2^{-(2N+p+2)(N+2)}.$$

Recall that, under our hypothesis, we have $(y,\tau^*) + Q_{4\rho}^-(\theta) \subset \tilde{Q}$ and

$$\left|\tilde{Q} \cap \left[v < \frac{k_0}{2^{s^*}}\right]\right| \le \frac{C(N,p,q)}{\alpha}\,\frac{1}{(s^*)^{\frac{p-1}{p}}}\,|\tilde{Q}|$$

and thereby

$$Y_0 \le \frac{\left|\tilde{Q} \cap \left[v < \frac{k_0}{2^{s^*}}\right]\right|}{|\tilde{Q}|} \frac{|\tilde{Q}|}{\left|(y,\tau^*) + Q^-_{4\rho}(\theta)\right|} \le \frac{\gamma(N,p,q)}{\alpha} \frac{1}{(s^*)^{\frac{p-1}{p}}}.$$

We determine the parameter s^*, and therefore the length of the cylinder, so that

$$\frac{\gamma(N,p,q)}{\alpha} \frac{1}{(s^*)^{\frac{p-1}{p}}} = C(N,p,q)^{-(N+2)} 2^{-(2N+p+2)(N+2)}.$$

This implies

$$v(\cdot,\tau) \ge \frac{k_0}{2^{s^*}+1} \quad \text{a.e. in } K_{2\rho}$$

for all $\tau \in (\tau^* - \theta(2\rho)^p, \tau^*]$.

Returning to the original time variable t and function $u(x,t)$ we get

$$u(x,t) \ge \eta\ M \quad \text{a.e. in } K_{2\rho}(y)$$

for all $t \in (s + \frac{\delta}{2}\ \frac{b^{p-(q+1)}}{(\eta M)^{p-(q+1)}}\ \rho^p, s + \delta \frac{b^{p-(q+1)}}{(\eta M)^{p-(q+1)}}\ \rho^p]$, where

$$\eta = \frac{\epsilon}{2^{s^*+1}} e^{-\left(\frac{\epsilon}{2^{s^*+1}}\right)^{q+1-p} \frac{8^p}{\delta\ (p-q-1)}}.$$

This time interval was obtained from the previous range of τ and realizing that, on such a range,

$$b_1 = e^{-\left(\frac{\epsilon}{2^{s^*+1}}\right)^{q+1-p} \frac{8^p}{\delta\ (p-q-1)}} \le e^{-\frac{\tau}{p-q-1}} < e^{-\left(\frac{\epsilon}{2^{s^*+1}}\right)^{q+1-p} \frac{6^p}{\delta\ (p-q-1)}} = b_2$$

and taking

$$b = \frac{\epsilon}{2^{s^*+1}}.$$

□

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Global phase portraits of a predator–prey system

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Abstract. We classify the global dynamics of a family of Kolmogorov systems depending on three parameters which has ecological meaning as it modelizes a predator–prey system. We obtain all their topologically distinct global phase portraits in the positive quadrant of the Poincaré disc, so we provide all the possible distinct dynamics of these systems.

Keywords: predator–prey system, Kolmogorov system, global phase portrait, Poincaré disc.

2020 Mathematics Subject Classification: 34C05, 37C15.

1 Introduction

Rosenzweig and MacArthur introduced in [25] the following predator–prey model

$$\begin{aligned}\dot{x} &= rx\left(1-\frac{x}{K}\right) - y\frac{mx}{b+x},\\ \dot{y} &= y\left(-\delta + c\frac{mx}{b+x}\right),\end{aligned}$$

where the dot as usual denotes derivative with respect to the time t, $x \geq 0$ denotes the prey density (#/unit of area) and $y \geq 0$ denotes the predator density (#/unit of area), the parameter $\delta > 0$ is the death rate of the predator, the function $mx/(b+x)$ is the # prey caught per predator per unit time, the function $x \to rx(1-x/K)$ is the growth of the prey in the absence of predator, and $c > 0$ is the rate of conversion of prey to predator.

The Rosenzweig and MacArthur system is a particular system of the general predator–prey systems with a Holling type II, see [12, 13].

In [14] Huzak reduced the study of the Rosenzweig and MacArthur system to study a polynomial differential system. In order to do that the first step is to do the rescaling $(\overline{x}, \overline{y}, \overline{b}, \overline{c}, \overline{\delta}) = (x/K, (m/rK)y, b/K, cm/r, \delta/r)$. After denoting again $(\overline{x}, \overline{y}, \overline{b}, \overline{c}, \overline{\delta})$ by (x, y, b, c, δ) and doing

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a time rescaling multiplying by $b + x$, the obtained polynomial differential system of degree three is

$$\begin{aligned} \dot{x} &= x(-x^2 + (1-b)x - y + b), \\ \dot{y} &= y((c-\delta)x - \delta b), \end{aligned} \tag{1.1}$$

where b, c and δ are positive parameters. This system is studied in the positive quadrant of the plane $\mathbb{R}^2$ where it has ecological meaning. See systems (1.1) and (2.2) of [14].

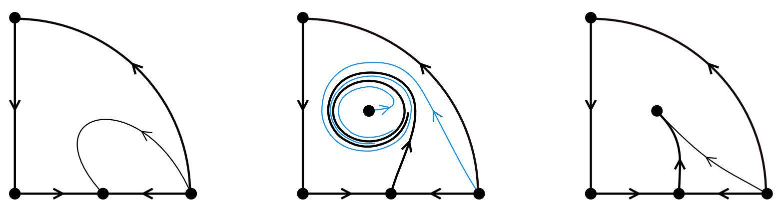
Huzak [14] focuses his work in the study of the periodic sets that can produce the canard relaxation oscillations after perturbations. He finds three types of limit periodic sets and studies their cyclicity by using the geometric singular perturbation theory and the family blow-up at $(x, y, \delta) = (0, br/m, 0)$, where δ is the singular perturbation parameter. He proves that the upper bound on the number of limit cycles of the system is 1 or 2 depending on the parameters.

Systems (1.1) are particular Kolmogorov systems. These systems were proposed in 1936, see [15], as an extension of the Lotka–Volterra systems to arbitrary dimension and arbitrary degree.

We want to complete the study of the dynamics of systems (1.1) and classify all their phase portraits on the closed positive quadrant of the Poincaré disc, in this way we also can control the dynamics of the system near the infinity. This classification is given in the following result, except for the case with the parameters satisfying $0 < b\delta < c - \delta$, $\delta(\delta(b+1) + c(b-1))^2 - 4c(c-\delta)^2(c - \delta(b-1)) < 0$ and $1 + c - \delta - b - b\delta > 0$, in which we make a conjecture about the expected global phase portrait.

Theorem 1.1. *The global phase portrait of system (1.1) in the closed positive quadrant of the Poincaré disc is topologically equivalent to one of the 3 phase portraits of Figure 1.1 in the following way:*

- *If $b\delta \geq c - \delta$ the phase portrait is equivalent to phase portrait (A).*
- *If $0 < b\delta < c - \delta$ and $\delta(\delta(b+1) + c(b-1))^2 - 4c(c-\delta)^2(c - \delta(b-1)) \geq 0$ the phase portrait is equivalent to phase portrait (B).*
- *If $0 < b\delta < c - \delta$ and $\delta(\delta(b+1) + c(b-1))^2 - 4c(c-\delta)^2(c - \delta(b-1)) < 0$ and $1 + c - \delta - b - b\delta < 0$ the phase portrait is equivalent to phase portrait (C).*



(A) (B) (C)

Figure 1.1: Phase portraits of system (1.1) in the positive quadrant of the Poincaré disc.

Conjecture. *The global phase portrait of system (1.1) in the closed positive quadrant of the Poincaré disc if $0 < b\delta < c - \delta$ and $\delta(\delta(b+1) + c(b-1))^2 - 4c(c-\delta)^2(c - \delta(b-1)) < 0$ and $1 + c - \delta - b - b\delta > 0$ is also topologically equivalent to the one in Figure 1.1(C).*

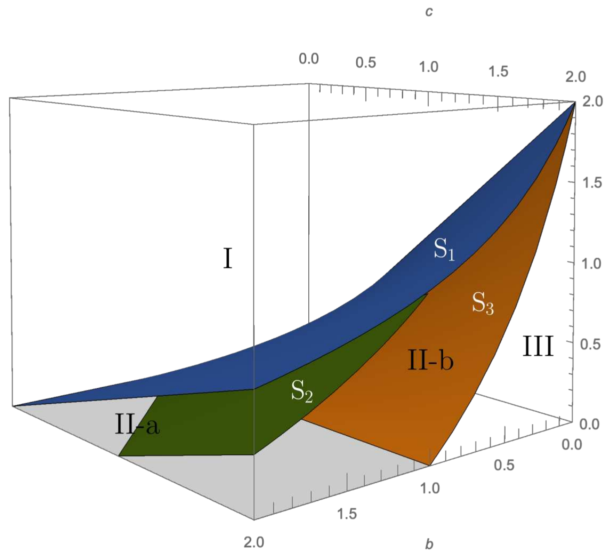


Figure 1.2: The regions I, II-a, II-b, III and the surfaces separating the different phase portraits: $S_1 : \{\delta = c/(b+1) \mid b, c \geq 0\}$, $S_2 : \{\delta = (1+c-b)/(b+1) \mid b, c \geq 0, (1+c-b)/(b+1) < c/(b+1)\}$ and $S_3 : \{\delta = c(1-b)/(1+b) \mid b, c \geq 0\}$.

In Figure 1.2 are represented the regions and surfaces in the parameters space in which each one of the phase portraits are realised. In the region I and over the surface S_1 the phase portrait is the one in Figure 1.1(A) and in the region III the phase portrait is the one in Figure 1.1(B). In region II there are two subregions, II-a and II-b. It is proved that in the region II-a the phase portrait is the one in Figure 1.1(C) and we conjecture that the phase portrait is the same in the region II-b and over the surfaces S_2 and S_3.

2 Preliminaries

Here we introduce the Poincaré compactification, as it allows to control the dynamics of a polynomial differential system near the infinity.

Consider a polynomial system in $\mathbb{R}^2$

$$\dot{x_1} = P(x_1, x_2),$$
$$\dot{x_2} = Q(x_1, x_2),$$

of degree d; the sphere $\mathbb{S}^2 = \{y \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$, which we will call the *Poincaré sphere*, and its tangent plane at the point $(0,0,1)$ which we identify with $\mathbb{R}^2$.

We can obtain an induced vector field in $\mathbb{S}^2 \setminus \mathbb{S}^1$ by means of central projections $f^+ : \mathbb{R}^2 \to \mathbb{S}^2$ and $f^- : \mathbb{R}^2 \to \mathbb{S}^2$, which are defined as

$$f^+(x) = \left(\frac{x_1}{\Delta(x)}, \frac{x_2}{\Delta(x)}, \frac{1}{\Delta(x)}\right) \quad \text{and} \quad f^-(x) = \left(\frac{-x_1}{\Delta(x)}, \frac{-x_2}{\Delta(x)}, \frac{-1}{\Delta(x)}\right),$$

where $\Delta(x) = \sqrt{x_1^2 + x_2^2 + 1}$. The differential Df^+ and Df^- provide a vector field in the northern and southern hemisphere respectively. The points of the equator $\mathbb{S}^1$ of $\mathbb{S}^2$ correspond with the points at infinity of $\mathbb{R}^2$, and we can extend analytically the vector field to these points of the equator multiplying the field by y_3^d. This extended field is called the *Poincaré* *compactification* of the original vector field. Then we must study the dynamics of the Poincaré compactification near $\mathbb{S}^1$, for studying the dynamics of the original field in the neighborhood of the infinity.

We will work in the local charts (U_i, ϕ_i) and (V_i, ψ_i) of the sphere $\mathbb{S}^2$, where $U_i = \{y \in \mathbb{S}^2 : y_i > 0\}$, $V_i = \{y \in \mathbb{S}^2 : y_i < 0\}$, $\phi_i : U_i \longrightarrow \mathbb{R}^2$ and $\psi_i : V_i \longrightarrow \mathbb{R}^2$ for $i = 1,2,3$ with $\phi_i(y) = \psi_i(y) = (y_m/y_i, y_n/y_i)$ for $m < n$ and $m, n \neq i$.

The expression of the Poincaré compactification in the local chart (U_1, ϕ_1) is

$$\dot{u} = v^d \left[-u\, P\left(\frac{1}{v}, \frac{u}{v}\right) + Q\left(\frac{1}{v}, \frac{u}{v}\right) \right], \qquad \dot{v} = -v^{d+1}\, P\left(\frac{1}{v}, \frac{u}{v}\right), \tag{2.1}$$

in the local chart (U_2, ϕ_2) is

$$\dot{u} = v^d \left[P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right) \right], \qquad \dot{v} = -v^{d+1}\, Q\left(\frac{u}{v}, \frac{1}{v}\right), \tag{2.2}$$

and in the local chart (U_3, ϕ_3) the expression is

$$\dot{u} = P(u, v), \qquad \dot{v} = Q(u, v). \tag{2.3}$$

The expression for the Poincaré compactification in the local charts (V_i, ψ_i), with $i = 1,2,3$ is the same as in the charts (U_i, ϕ_i) multiplied by $(-1)^{d-1}$.

As we want to study the behaviour near the infinity, we must study the *infinite singular points*, i.e., the singular points of the Poincaré compactification which lie in the equator $\mathbb{S}^1$. Note that it will be enough to study the infinite points on the local chart U_1 and the origin of the local chart U_2, because if $y \in \mathbb{S}^1$ is an infinite singular point, then $-y$ is also an infinite singular point and they have the same or opposite stability depending on whether the system has odd or even degree.

We shall present the phase portraits of the polynomial differential systems (1.1) in the *Poincaré disc*, i.e. the orthogonal projection of the closed northern hemisphere of $\mathbb{S}^2$ onto the plane $y_3 = 0$. This will be enough since the orbits of the Poincaré compactification on $\mathbb{S}^2$ are symmetric with respect to the origin of $\mathbb{R}^3$ so we only need to consider the flow in the closed northern hemisphere.

See chapter 5 of [8] for more details about the Poincaré compactification.

3 Finite singular points

First we study the finite singular points of system (1.1) in the closed positive quadrant. The origin $P_0 = (0,0)$ and the point $P_1 = (1,0)$ are singular points for any values of the parameters, and $P_2 = \big(b\delta/(c-\delta), (-bc(\delta + b\delta - c))/(c-\delta)^2\big)$ is a positive singular point if $c \neq \delta$ and $0 < b\delta < c - \delta$. Note that if $b\delta = c - \delta$ then $P_1 = P_2$.

Now we study the local phase portraits at these singular points. The origin is a saddle point, as the eigenvalues of the Jacobian matrix at this point are b and $-\delta b$. At the point P_1 the eigenvalues are $-b-1$ and $-\delta b + c - \delta$. The first eigenvalue is always negative, but we distinguish three cases depending on the second one. If $c - \delta < b\delta$ then P_1 is a stable node; if $c - \delta > b\delta$ then P_1 is a saddle (this was the case in [14] because there $\delta > 0$ was kept very small). If $c - \delta = b\delta$, then P_1 is a semi-hyperbolic singular point, so from [8, Theorem 2.19] we obtain that $P_1 = P_2$ is a saddle-node.

At the singular point P_2 the eigenvalues of the Jacobian matrix are

$$\lambda_{1,2} = \frac{2}{(c-\delta)^2}(A \pm \sqrt{\delta B}),$$

where

$$A = \delta(c-\delta) - b\delta(c+\delta) \quad \text{and} \quad B = \delta(\delta(b+1)+c(b-1))^2 - 4c(c-\delta)^2(c-\delta(b-1)).$$

If $B < 0$ then the eigenvalues are complex. In this case for $A > 0$ the singular point P_2 is an unstable focus, and for $A < 0$ it is a stable focus. We deal with this case $B < 0$ in Section 6, where we study the Hopf bifurcation which takes place at P_2.

If $B = 0$ we have $\lambda_1 = \lambda_2 = A/(c-\delta)^2$ and in this case A cannot be zero, because if $A = 0$ then $b = (c-\delta)/(c+\delta)$, and replacing this expression $B = -4c^2(c-\delta)^3/(c+\delta)$, so one of the conditions $c = 0$ or $c - \delta = 0$ must hold, but this is a contradiction as $c > 0$ from the hypotheses, and if $c = \delta$ then $b = 0$ again in contradiction with the hypotheses. Then $A \neq 0$ and its sign determines if the singular point is either a stable or an unstable node.

If $B > 0$ both eigenvalues are real. The determinant of the Jacobian matrix is

$$-\frac{b^2 c\delta}{(c-\delta)^2}(b\delta + \delta - c),$$

which is positive because the singular point P_2 exists only if condition $b\delta < c - \delta$ holds. Then both eigenvalues are nonzero and have the same sign, particularly, if $A > 0$ both are positive and P_2 is an unstable node, and if $A < 0$ both are negative and P_2 is a stable node.

The local phase portrait of the singular point P_2 in the case with $A = 0$ will be proved in Subsection 6.1.

In summary, we describe in Table 3.1 the finite singular points according the values of the parameters b, c and δ.

Case	Conditions	Finite singular points
1	$b\delta > c - \delta$.	P_0 saddle, P_1 stable node.
2	$b\delta = c - \delta$.	P_0 saddle, P_1 saddle-node.
3	$0 < b\delta < c - \delta$, $B \geq 0$, $A > 0$.	P_0 saddle, P_1 saddle, P_2 unstable node.
4	$0 < b\delta < c - \delta$, $B \geq 0$, $A < 0$.	P_0 saddle, P_1 saddle, P_2 stable node.
5	$0 < b\delta < c - \delta$, $B < 0$, $A > 0$.	P_0 saddle, P_1 saddle, P_2 unstable focus.
6	$0 < b\delta < c - \delta$, $B < 0$, $A < 0$.	P_0 saddle, P_1 saddle, P_2 stable focus.
7	$0 < b\delta < c - \delta$, $B < 0$, $A = 0$.	P_0 saddle, P_1 saddle, P_2 weak stable focus.

Table 3.1: The finite singular points in the closed positive quadrant.

4 Infinite singular points

In this section we will consider the Poincaré compactification of system (1.1) as it allows to study the behavior of the trajectories near infinity.

In the chart U_1 system (1.1) writes

$$\begin{aligned} \dot{u} &= uv^2 - b(\delta+1)uv^2 + (b+c-\delta-1)uv + u, \\ \dot{v} &= uv^2 - bv^3 + (b-1)v^2 + v. \end{aligned} \tag{4.1}$$

The only singular point over $v = 0$ is the origin of U_1, which we denote by O_1. The linear part of system (4.1) at the origin is the identity matrix, so O_1 is an unstable node.

In the chart U_2 system (1.1) writes

$$\begin{aligned} \dot{u} &= -u^3 + (\delta + 1 - b - c)u^2 v + b(\delta+1)uv^2 - uv, \\ \dot{v} &= (\delta - c)uv^2 + b\delta v^3. \end{aligned} \tag{4.2}$$

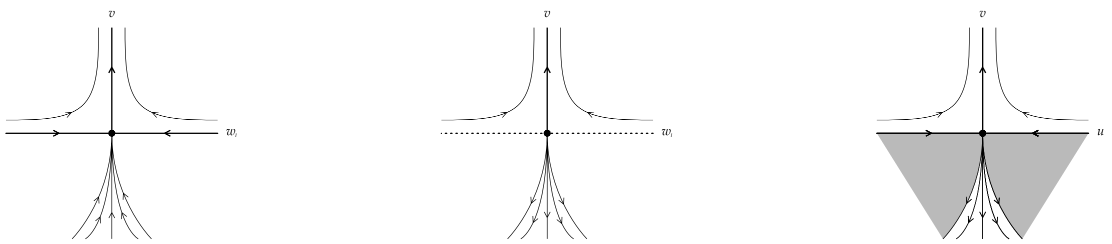
The origin of U_2 is a singular point, O_2, and the linear part of system (4.2) at O_2 is identically zero, so we must use the blow-up technique to study it. We do a horizontal blow up introducing the new variable w_1 by means of the variable change $vw_1 = u$, and get the system

$$\begin{aligned} \dot{w}_1 &= v^2 w_1^3 + (1-b)v^2 w_1^2 + bw_1 v^2 - w_1 v, \\ \dot{v} &= (\delta - c)w_1 v^3 + b\delta v^3. \end{aligned} \tag{4.3}$$

Now rescaling the time variable we cancel the common factor v, getting the system

$$\begin{aligned} \dot{w}_1 &= vw_1^3 + (1-b)vw_1^2 + bw_1 v - w_1, \\ \dot{v} &= (\delta - c)w_1 v^2 + b\delta v^2. \end{aligned} \tag{4.4}$$

The only singular point on $v = 0$ is the origin, which is semi-hyperbolic. Applying [8, Theorem 2.19] we conclude that it is a saddle-node. Studying the sense of the flow over the axis we determine that the phase portrait around the origin of system (4.4) is the one on Figure 4.1(a). If we multiply by v the sense of the orbits on the third and fourth quadrants changes and all the points of the w_1-axis become singular points. With these modifications we obtain the phase portrait for system (4.3), given in Figure 4.1(b). Then we undo the blow up going back to the (u, v)-plane. We must swap the third and fourth quadrants and shrink the exceptional divisor to the origin. The phase portrait obtained for system (4.2) is not totally determined in the shaded regions of the third and fourth quadrants, see Figure 4.1(c). This can be solved by doing a vertical blow up but, in our case, it is not necessary because we only need to know the phase portrait of O_2 in the positive quadrant of the Poincaré disc, which corresponds with the positive quadrant in the plane (u, v), in which the phase portrait is well determined.



(a) Local phase portrait at the origin of system (4.4)

(b) Local phase portrait at the origin of system (4.3)

(c) Local phase portrait at the origin of system (4.2)

Figure 4.1: Desingularization of the origin of system (4.2).

As a conclusion the local phase portrait at the infinite singular points is the same independently of the values of the parameters, so in all cases of Table 3.1 the origin of the chart U_1, i.e. the singular point O_1, is an unstable node and the origin of the chart U_2, i.e. the singular point O_2 has only one hyperbolic sector on the positive quadrant of the Poincaré disc being one separatrix at infinity and the other on $x = 0$.

5 Cases with no singular points in the positive quadrant

In the two first cases of Table 3.1 there is no singular points in the positive quadrant. The finite singular points are the origin P_0 and P_1 which are both over the axes. The axes are invariant lines so there cannot exist a limit cycle surrounding these singular points. Therefore as we have determined the local phase portrait at the finite and infinite singularities, and we know there are no limit cycles, we can study the global portrait in the first quadrant of the Poincaré disc.

In both cases we obtain the same result since in the case in which P_1 is a saddle-node, studying the sense of the flow we determine that the parabolic sector of the saddle-node is always on the positive quadrant of the Poincaré disc. Analysing all the possible alpha and omega-limits, the only possibility is that all the orbits leave the infinite singular point O_1 and go to the finite singular point P_1. This phase portrait is given in Figure 1.1(A).

6 Cases with singular points in the positive quadrant

6.1 Existence of limit cycles

Theorem 6.1. *If $0 < b\delta < c - \delta$ and $A > 0$, then there exists at least one limit cycle surrounding singular point P_2.*

Proof. If conditions $0 < b\delta < c - \delta$ and $A > 0$ hold, then we have case 3 or 5 of Table 3.1. In both cases singular point P_1 is a saddle which has an unstable separatrix on the positive quadrant, P_2 is either an unstable node or an unstable focus, and O_1 is an unstable node. By Poincaré–Bendixson theorem, there must exists at least one limit cycle which is the ω-limit of the orbits leaving O_1, the orbits leaving P_2 and the separatrix of P_1, as there are no other singular points that can be the ω-limit of all these orbits. □

In cases 5, 6 and 7 of Table 3.1 the Jacobian matrix at the point P_2 has complex eigenvalues because $B < 0$. In these cases we study the existence of Hopf bifurcation, leading to the following result.

Theorem 6.2. *The equilibrium P_2 of system (1.1) undergoes a supercritical Hopf bifurcation at $b_0 = (c - \delta)/(c + \delta)$. For $b > b_0$ the system has a unique stable limit cycle bifurcating from the equilibrium point P_2.*

Proof. The Jacobian matrix at this equilibrium is

$$A(b) = \begin{pmatrix} -\dfrac{b\delta(c(b-1) + \delta(b+1))}{(c-\delta)^2} & -\dfrac{b\delta}{c-\delta} \\ -\dfrac{bc(b\delta + \delta - c)}{c-\delta} & 0 \end{pmatrix},$$

and it has eigenvalues $\mu(b) \pm \omega(b)i$, where

$$\mu(b) = \frac{b}{2(c-\delta)^2} A \quad \text{and} \quad \omega(b) = \frac{b}{2(c-\delta)^2}\sqrt{-\delta B}. \tag{6.1}$$

We get $\mu(b_0) = 0$ for

$$b_0 = \frac{c-\delta}{c+\delta}. \tag{6.2}$$

We are working under condition $B < 0$ and from this condition it can be deduced that $c - \delta > 0$, so the expression of b_0 obtained is positive. Therefore at $b = b_0$ the equilibrium point P_2 has a pair of pure imaginary eigenvalues $\pm i\omega(b)$ and the system will have a Hopf bifurcation if some Lyapunov constant is nonzero and $(d\mu/db)(b_0) \neq 0$.

The equilibrium is stable for $b > b_0$ (i.e. for $A < 0$) and unstable for $b < b_0$ (i.e. for $A > 0$). In order to analyze this Hopf bifurcation we will apply [16, Theorem 3.3], so we must prove if the genericity conditions are satisfied. We check that the transversality condition is satisfied as

$$\frac{d\mu}{db}(b_0) = -\frac{\delta}{2(c-\delta)} < 0, \tag{6.3}$$

and the sign is determined because $c - \delta > 0$.

To check the second condition we must compute the first Lyapunov constant. We fix the value $b = b_0$ and then the equilibrium P_2 has the expression

$$P_2 = \left(\frac{\delta}{c+\delta}, \frac{c^2}{(c+\delta)^2} \right). \tag{6.4}$$

We translate P_2 to the origin of coordinates obtaining the system

$$\begin{aligned} \dot{\varepsilon}_1 &= -\varepsilon_1^3 - \frac{\delta}{c+\delta}\varepsilon_1^2 - \varepsilon_1\varepsilon_2 - \frac{\delta}{c+\delta}\varepsilon_2, \\ \dot{\varepsilon_2} &= (c-\delta)\varepsilon_1\varepsilon_2 + \frac{c^2(c-\delta)}{(c+\delta)^2}, \end{aligned} \tag{6.5}$$

which can be represented as

$$\dot{\varepsilon} = A\varepsilon + \frac{1}{2}B(\varepsilon,\varepsilon) + \frac{1}{6}C(\varepsilon,\varepsilon,\varepsilon), \tag{6.6}$$

where $A = A(b_0)$ and the multilinear functions B and C are given by

$$B(\varepsilon,\eta) = \begin{pmatrix} -\dfrac{2\delta}{c+\delta}\varepsilon_1\eta_1 - \varepsilon_1\eta_2 - \varepsilon_2\eta_1 \\ (c-\delta)\varepsilon_1\eta_2 + (c-\delta)\varepsilon_2\eta_1 \end{pmatrix},$$

$$C(\varepsilon,\eta,\zeta) = \begin{pmatrix} 6\varepsilon_1\eta_1\zeta_1 \\ 0 \end{pmatrix}.$$

We need to find two eigenvectors p, q of the matrix A verifying

$$Aq = i\omega q, \qquad A^T p = -i\omega p, \quad \text{and} \quad \langle p, q\rangle = 1,$$

as for example

$$q = \begin{pmatrix} -\dfrac{\delta}{c+\delta} \\ i\omega \end{pmatrix} \quad \text{and} \quad p = \begin{pmatrix} -\dfrac{c+\delta}{2\delta} \\ i\omega\dfrac{(c+\delta)^3}{2c^2\delta(c-\delta)} \end{pmatrix}. \tag{6.7}$$

Now we compute

$$g_{20} = \langle p, B(q,q)\rangle = \frac{\omega^2(c+\delta)^5 - c^2\delta^2(c+\delta)}{2\delta c^4(c-\delta)} + \frac{\omega(c+\delta)^3}{2c^2\delta(c-\delta)}\, i,$$

$$g_{11} = \langle p, B(q, \bar{q}) \rangle = -\frac{\delta(c+\delta)}{2c^2(c-\delta)}, \qquad g_{21} = \langle p, C(q, q, \bar{q}) \rangle = -\frac{3(c+\delta)^4}{4c^4(c-\delta)^2},$$

and the first Lyapunov coefficient

$$\ell_1 = \frac{1}{2\omega^2} \operatorname{Re}(i g_{20} g_{11} + \omega g_{21}) = -\frac{(c+\delta)^4}{4c^4\omega(c-\delta)^2},$$

which is negative for any values of the parameters, and so the second condition of the theorem we are applying is satisfied and we can conclude that a unique stable limit cycle bifurcates from the equilibrium point P_2 through a Hopf bifurcation for $b < b_0$ with $b_0 - b$ sufficiently small. □

Proposition 6.3. *If* $0 < b\delta < c - \delta$ *and* $A > 0$*, the limit cycle surrounding singular point* P_2 *is unique.*

Proof. This result follows from [18] by proving that system (1.1) with $0 < b\delta < c - \delta$ and $A > 0$ satisfies conditions (i)–(iv) in Section 2 of [18].

Condition (i) holds taking $g(x) = (c-\delta)x$ which verifies $g(0) = 0$ and $g'(x) > 0$ for all $x \geq 0$ as we have assumed $c - \delta > 0$.

Condition (ii) holds for $f(x) = -x^2 + (1-b)x + b$, $K = 1$ and $a = (1-b)/2$. From condition $A > 0$ we deduce that

$$\delta(c-\delta) - b\delta(c+\delta) > 0 \Rightarrow \frac{c-\delta}{c+\delta} > \frac{b\delta}{\delta} \Rightarrow 1 > \frac{c-\delta}{c+\delta} > b,$$

and condition $b < 1$ guarantees that $a > 0$.

Condition (iii) holds for $\lambda = b\delta$ and $x^* = \delta b/(c-\delta)$. It can be proved that with the expressions chosen for a and x^* the condition $x^* < a$, is equivalent to the condition $A > 0$:

$$x^* < a \Leftrightarrow \frac{\delta b}{c-\delta} < \frac{1-b}{2} \Leftrightarrow 2\delta b < (1-b)(c-\delta) \Leftrightarrow \delta b + bc < c - \delta \Leftrightarrow b < \frac{c-\delta}{c+\delta} \Leftrightarrow A > 0.$$

Condition (iv) is satisfied with

$$x^* = \frac{\delta b}{c-\delta} \quad \text{and} \quad \bar{x}^* = 1 - \frac{bc}{c-\delta}.$$

We have

$$\frac{d}{dx}\frac{xf'(x)}{g(x)-\lambda} = \frac{-2x^2(c-\delta) + 4x\delta b(b-1)}{((c-\delta)x - \delta b)^2}, \tag{6.8}$$

which is always negative as the polynomial in the numerator is negative in $x = 0$ and has no real roots.

Then, as conditions (i)–(iv) hold for our systems, we can conclude that the limit cycle is unique. □

Remark 6.4. Theorem 6.2 proves that the unique limit cycle of system (1.1) appears from the equilibrium point P_2 in a Hopf bifurcation. From the proof of Theorem 6.2 the singular point P_2 when $B < 0$ and $A = 0$ is a weak stable focus.

So far we have not proved if in cases 4, 6, and 7 of Table 3.1 there are or not limit cycles. The following result proves that in some subcases there are not limit cycles.

Theorem 6.5. *If $0 < b\delta < c - \delta$, $A < 0$ and $1 + c < \delta + b + b\delta$, then system (1.1) does not have periodic orbits in the set $\{(x, y) \in \mathbb{R}^2 : x, z \geq 0\}$.*

Proof. Let

$$f(x, y) = x(-x^2 + (1 - b)x - y + b) \quad \text{and} \quad g(x, y) = y((c - \delta)x - \delta b).$$

In order to prove the non existence of periodic orbits we use the Bendixson–Dulac Theorem that states that if there exists a function $\varphi(x, y)$ such that the term

$$\Delta(x, y) = \frac{\partial(\varphi f)}{\partial x} + \frac{\partial(\varphi g)}{\partial y}$$

does not change sign in a simply connected set $\mathcal{S}$, then there are no periodic orbits on $\mathcal{S}$.

We consider the function $\varphi(x, y) = 1/x$, then:

$$\Delta(x, y) = 1 + c - \delta - 2x - \frac{b(\delta + x)}{x}.$$

We observe that there are no periodic orbits in the set

$$\{(x, y) \in \mathbb{R}^2_+ : \; x \geq 1\},$$

because $\dot{x} < 0$ for all the points in this set and for the same reason there are no periodic orbits crossing the line $\{x = 1, y \geq 0\}$. As a consequence we can restrict to the case $x < 1$ for which we obtain

$$\Delta(x, y) < 1 + c - \delta - \frac{b\delta}{x} - b < 1 + c - \delta - b\delta - b.$$

Then $\Delta(x, y) < 0$ in $\left\{(x, y) \in \mathbb{R}^2 : \; 0 \leq x \leq 1, y \geq 0\right\}$ if $1 + c - \delta - b - b\delta < 0$ and we conclude that there are no periodic orbits in the whole set $\left\{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\right\}$. □

Conjecture. *If $0 < b\delta < c - \delta$, $A < 0$ (i.e., we are in cases 4, 6, or 7 of Table 3.1) and $1 + c > \delta + b + b\delta$, there are not limit cycles.*

We have numerical evidences that the conjecture holds.

6.2 Phase portraits on the positive quadrant of the Poincaré disc

Now we study the global phase portraits of system (1.1) on the positive quadrant of the Poincaré disc when there is a singular point in the positive quadrant, assuming the previous conjecture.

In case 3 of Table 3.1, by Theorem 6.1 there exist a unique limit cycle which is the ω-limit of all orbits leaving O_1 and P_2, and also the ω-limit of the unstable separatrix leaving P_1 in the positive quadrant. Then the global phase portraits is the one on Figure 1.1(B).

In case 5 of Table 3.1 we have again that there exists a unique limit cycle attracting all orbits in the positive quadrant. The global phase portrait is the same as the one in case 3 but here the singular point in the positive quadrant is an unstable focus instead of an unstable node. As the local phase portraits of these two singular points are topologically equivalent we have again phase portrait (B) of Figure 1.1.

In cases 4, 6 and 7 of Table 3.1, if $1 + c < \delta + b + b\delta$ we have proved that there are no limit cycles. In case 4 the only possibility is that the stable node P_2 is a global attractor for all orbits in the positive quadrant, and we have the global phase portrait given in Figure 1.1(C). In cases 6 and 7 of Table 3.1, P_2 is a stable focus and attracts all the orbits of the positive quadrant. As the local phase portrait of a stable focus is topologically equivalent to a stable node, we also have here the phase portrait of Figure 1.1(C).

In the cases 4, 6 and 7 of Table 3.1, if the conditions $1 + c < \delta + b + b\delta$ does not hold, we have assumed that there are not limit cycles, so the conjectured phase portraits will be the same.

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Weighted L^p-type regularity estimates for nonlinear parabolic equations with Orlicz growth

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Abstract. In this paper we obtain the following weighted L^p-type regularity estimates

$$B(|\mathbf{f}|) \in L^q\left(\nu, \nu+T; L^q_w(\Omega)\right) \text{ locally} \Rightarrow B(|\nabla u|) \in L^q\left(\nu, \nu+T; L^q_w(\Omega)\right) \text{ locally}$$

for any $q > 1$ of weak solutions for non-homogeneous nonlinear parabolic equations with Orlicz growth

$$u_t - \operatorname{div}\left(a\left((A\nabla u \cdot \nabla u)^{\frac{1}{2}}\right) A\nabla u\right) = \operatorname{div}\left(a(|\mathbf{f}|)\,\mathbf{f}\right)$$

under some proper assumptions on the functions a, w, A and $\mathbf{f}$, where $B(t) = \int_0^t \tau a(\tau)\, d\tau$ for $t \geq 0$. Moreover, we remark that two natural examples of functions $a(t)$ are

$$a(t) = t^{p-2} \quad (p\text{-Laplace equation}) \quad \text{and} \quad a(t) = t^{p-2}\log^\alpha(1+t) \quad \text{for } \alpha > 0.$$

Moreover, our results improve the known results for such equations.

Keywords: weighted, L^p-type, regularity, gradient, quasilinear, parabolic, non-homogeneous.

2020 Mathematics Subject Classification: 35K55, 35K65.

1 Introduction

This paper is concerned with the local weighted L^p-type gradient estimates for weak solutions of the following non-homogeneous nonlinear parabolic equations with Orlicz growth

$$u_t - \operatorname{div}\left(a\left((A\nabla u \cdot \nabla u)^{\frac{1}{2}}\right) A\nabla u\right) = \operatorname{div}\left(a(|\mathbf{f}|)\,\mathbf{f}\right) \quad \text{in } \Omega_T := \Omega \times (\nu, \nu+T), \tag{1.1}$$

where $\nu \in \mathbb{R}$, Ω is an open bounded domain in $\mathbb{R}^n$, the vector valued function $\mathbf{f} = (f_1, \dots, f_n)$ and $a : (0, +\infty) \to (0, +\infty) \in C^1(0, +\infty)$ satisfies

$$0 \leq i_a := \inf_{t>0} \frac{ta'(t)}{a(t)} \leq \sup_{t>0} \frac{ta'(t)}{a(t)} =: s_a < +\infty. \tag{1.2}$$

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Here, $A(x,t) = \{a_{ij}(x,t)\}_{n\times n}$ is a symmetric matrix with measurable coefficients satisfying the uniformly parabolic condition

$$\Lambda^{-1}|\xi|^2 \le A(x,t)\xi\cdot\xi \le \Lambda|\xi|^2 \tag{1.3}$$

for all $\xi \in \mathbb{R}^n$, almost every $(x,t)\in\mathbb{R}^n\times\mathbb{R}$ and some positive constant Λ. Especially when $a(t) = t^{p-2}$, then $2+i_a = 2+s_a = p$ and (1.1) is reduced to the classical parabolic p-Laplace equations

$$u_t - \operatorname{div}\left((A\nabla u\cdot\nabla u)^{\frac{p-2}{2}} A\nabla u\right) = \operatorname{div}\left(|\mathbf{f}|^{p-2}\mathbf{f}\right). \tag{1.4}$$

Define

$$g(t) = ta(t) \quad \text{and} \quad B(t) = \int_0^t g(\tau)\,d\tau = \int_0^t \tau a(\tau)\,d\tau \quad \text{for } t\ge 0. \tag{1.5}$$

Then (1.2) implies that

$$g(t) \text{ is strictly increasing and continuous over } [0,+\infty), \tag{1.6}$$

and

$$B(t) \text{ is increasing over } [0,+\infty) \text{ and strictly convex with } B(0)=0. \tag{1.7}$$

There are two simple examples satisfying the given condition (1.2)

$$a(t) = t^{p-2} \quad \text{and} \quad a(t) = t^{p-2}\log^{\alpha}(1+t) \quad \text{for any } p\ge 2 \text{ and any } \alpha>0. \tag{1.8}$$

Additionally, another general and interesting example satisfying (1.2) is related to (p,q)-growth condition which is given by appropriate gluing of the monomials (see page 600 in [7] and page 314 in [37]).

Different from the elliptic p-Laplace equation

$$\operatorname{div}\left((A\nabla u\cdot\nabla u)^{\frac{p-2}{2}} A\nabla u\right) = \operatorname{div}\left(|\mathbf{f}|^{p-2}\mathbf{f}\right) \quad \text{in } \Omega,$$

the solution in the p-parabolic setting (1.4) is no longer invariant under multiplication by a constant, which is one of the most difficulties (see [12]). More precisely, it is slightly difficult to use maximal operators, which are typically used in the elliptic cases (see [22]). First of all, Kinnunen and Lewis [36] established the following Gehring's reverse Hölder inequality

$$\nabla u \in L^{p+\epsilon}_{loc}(\Omega_T) \quad \text{for some small } \epsilon>0$$

for weak solutions of (1.4), which implies the local higher integrability of the gradient. In their article they overcome the difficulties in using normalization and scaling methods by choosing the irregular cylinders whose side lengths depend on the function. Meanwhile, Misawa [42] obtained L^q $(q\ge p)$ estimates for gradients for evolutional p-Laplacian equations/systems (1.4) with discontinuous coefficients and external force given by the divergence of BMO-functions. Subsequently, Acerbi & Mingione [1] invented a new covering/iteration argument to prove the sharp local L^q $(q\ge p)$ estimates

$$|\mathbf{f}|^p \in L^q_{loc}(\Omega_T) \Rightarrow |\nabla u|^p \in L^q_{loc}(\Omega_T) \quad \text{for any } q\ge 1$$

with

$$\int_{Q_r} |\nabla u|^{pq} dz \leq C \left[\left(\int_{Q_{2r}} |\nabla u|^p dz \right)^q + \int_{Q_{2r}} |\mathbf{f}|^{pq} + 1 dz \right]^{p/2}, \tag{1.9}$$

where the parabolic cylinder $Q_{2r} = B_{2r} \times (-4r^2, 4r^2] \subset \Omega_T$, for weak solutions of the parabolic p-Laplace equations/systems (1.4) with small BMO coefficients. Furthermore, Byun, Ok & Ryu [17] obtained the global L^q $(q \geq p)$ estimates

$$\mathbf{f} \in L^q(\Omega_T) \Rightarrow \nabla u \in L^q(\Omega_T) \quad \text{for any} \quad q \geq p$$

for weak solutions of the following general parabolic p-Laplace equations

$$u_t - \operatorname{div} \mathrm{a}\,(Du, x, t) = \operatorname{div}\left(|\mathbf{f}|^{p-2}\mathbf{f}\right). \tag{1.10}$$

The corresponding Hölder estimates for (1.4) and (1.10) can be found in the book [29]. On the other hand, the corresponding L^p-type estimates and Hölder estimates for the parabolic $p(x,t)$-Laplacian equations are also developed by [6,10]. In addition, some authors [5,13] have researched the Calderón–Zygmund estimates in the setting of Lorentz spaces

$$|\mathbf{f}|^p \in L^{\gamma,q} \Rightarrow |\nabla u|^p \in L^{\gamma,q} \quad \text{for any} \quad 1 < \gamma < \infty \quad \text{and} \quad 0 < q \leq \infty \tag{1.11}$$

for degenerate parabolic equations/systems (1.4). Most recently, there are also many research results [9,11,16,19,20,25,33,35] on the study of various kinds of regularity estimates for weak solutions of the parabolic equations of p-Laplacian type.

The following non-homogeneous nonlinear elliptic equations with Orlicz growth which is first introduced by Lieberman [37]

$$\operatorname{div}\left(a\left(|\nabla u|\right) \nabla u\right) = f \quad \text{in } \Omega \tag{1.12}$$

can be seen as the most natural generalization of the elliptic p-Laplace equations. Afterward, Cianchi & Maz'ya [26–28] have investigated the global Lipschitz regularity and sharp estimates for (1.12) with the condition (1.2). Moreover, Diening, Stroffolini & Verde [32] obtained the ϕ-harmonic approximation lemma for the gradient of solutions to (1.12) with the condition (1.2) and $f = 0$. Lately, we [48] established the following local L^q estimates

$$B\left(|\mathbf{f}|\right) \in L^q_{loc} \Rightarrow B\left(|\nabla u|\right) \in L^q_{loc} \quad \text{for any} \quad q > 1$$

for weak solutions of

$$\operatorname{div}\left(a\left(|\nabla u|\right) \nabla u\right) = \operatorname{div}\left(a\left(|\mathbf{f}|\right) \mathbf{f}\right) \quad \text{in } \Omega. \tag{1.13}$$

Additionally, the global gradient estimates in Orlicz spaces for weak solutions of (1.13) in a Reifenberg domain have been developed by Byun and Cho [15]. Recently, Beck & Mingione [8] also proved the local Lipschitz regularity of weak solutions for nonuniformly elliptic variational problems, which satisfies the condition (1.2) or the fast, exponential-type growth conditions. Meanwhile, Baasandorj, Byun & Oh [4] discussed the Calderón–Zygmund type estimates for non-uniformly elliptic equations of generalized double phase type in divergence form. A more detailed research progress on (1.12) can be found in the paper [41]. Just like the difference between the elliptic and parabolic quasilinear p-Laplace equations, it is much more difficult to deal with the parabolic case (1.1) than the corresponding elliptic case (1.13).

Recently, Diening, Scharle and Schwarzacher [31] obtained the interior Lipschitz regularity for weak solutions of

$$u_t - \operatorname{div}\left(a\left(|\nabla u|\right)\nabla u\right) = \operatorname{div}\left(a\left(|\mathbf{f}|\right)\mathbf{f}\right). \tag{1.14}$$

Subsequently, we [46, 47] established the local Calderón–Zygmund estimates in the setting of Sobolev spaces and Lorentz spaces for weak solutions of (1.14). Moreover, many authors [14, 38] also studied the regularity estimates of weak solutions for the parabolic case (1.14). In recent years, there are many research activities on the weighted L^p-type estimates for the nonlinear elliptic equations [21, 23, 39, 40, 45] and the nonlinear parabolic cases [3, 18, 49]. The purpose of this paper is to investigate the local weighted L^p-type regularity estimates for weak solutions of (1.1).

Let $Q(\theta,\rho) = B_\rho \times (-\theta,\theta]$ be a centered parabolic cylinder. Especially when $\theta = \rho^2$, we denote $Q_\rho \equiv Q(\rho^2,\rho)$. Throughout this paper we assume that the coefficients of $A = \{a_{ij}\}$ are in parabolic BMO spaces and their parabolic semi-norm are small enough. More precisely, we have the following definition.

Definition 1.1 (Small BMO coefficients)**.** We say that the matrix A of coefficients is (δ, R)-vanishing if

$$\sup_{Q_z(\theta,\rho)\subset\mathbb{R}^n\times\mathbb{R}} \fint_{Q_z(\theta,\rho)} |A(\zeta) - \overline{A}_{Q_z(\theta,\rho)}|\,d\zeta \leq \delta$$

for $\rho \leq R$ and $\theta \leq R^2$, where $z = (x,t)$ and $\zeta = (y,s) \in \mathbb{R}^n \times \mathbb{R}$.

As usual, the solutions of (1.1) are taken in a weak sense. We now state the definition of weak solutions.

Definition 1.2. Assume that $\mathbf{f} \in L^B_{loc}(\Omega_T)$ (see Definition 2.4). A function $u \in L^\infty_{loc}(\nu,\nu+T; L^2_{loc}(\Omega)) \cap L^B_{loc}(\nu,\nu+T; W^{1,B}_{loc}(\Omega))$ is a local weak solution of (1.1) if for any compact subset $\mathcal{K}$ of Ω and for any subinterval $[t_1,t_2]$ of $(\nu,\nu+T)$ we have

$$\int_{\mathcal{K}} u\varphi\,dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_{\mathcal{K}} \left\{ -u\varphi_t + a\left((A\nabla u\cdot\nabla u)^{\frac{1}{2}}\right) A\nabla u\cdot\nabla\varphi \right\} dxdt = -\int_{t_1}^{t_2}\int_{\mathcal{K}} a\left(|\mathbf{f}|\right)\mathbf{f}\cdot\nabla\varphi\,dxdt$$

for any $\varphi \in L^\infty_{loc}(\nu,\nu+T; L^2(\mathcal{K})) \cap L^B_{loc}(\nu,\nu+T; W^{1,B}_0(\mathcal{K}))$.

For convenience of the readers, we shall now give some definitions and properties on the weighted Lebesgue spaces (see [43, 44]).

Definition 1.3. We call the positive function $w(x) \in L^1_{loc}(\mathbb{R}^n)$ belongs to the class of the reverse Hölder weights A_q for some $q > 1$ if

$$\left(\fint_{B_r} w(x)\,dx\right)\left(\fint_{B_r} w(x)^{\frac{-1}{q-1}}\,dx\right)^{q-1} \leq C$$

for any ball $B_r \subset \mathbb{R}^n$ and some constant $C > 0$. Moreover, we denote

$$A_\infty := \bigcup_{1<q<\infty} A_q \quad \text{and} \quad w(B_r) := \int_{B_r} w(x)\,dx.$$

Furthermore, the corresponding weighted Lebesgue space $L^p_w(B_r)$ for any $p \geq 1$ consists of all functions h which satisfy

$$\|h\|_{L^p_w(B_r)} := \left(\int_{B_r} |h|^p\,w(x)\,dx\right)^{1/p} < \infty.$$

Now we are in a position to state the main result of this paper. Here we remark that just like in [49], the occurrence of the assumption $B(|\nabla u|) \in L^{1+\epsilon}\left(\nu, \nu+T; L_w^{1+\epsilon}(\Omega)\right)$ locally is essentially due to the presence of the weight function w.

Theorem 1.4. *Assume that $w(x) \in A_q$ for some $q > 1$. Let u be a local weak solution of (1.1) in Ω_T with $Q_2 \subset \Omega_T$. Then for every $\epsilon \in (0, q-1)$ there exists a small $\delta = \delta(n, w, i_a, s_a, \epsilon, R, \Lambda) > 0$ so that for each uniformly parabolic and (δ, R)-vanishing A, and for all $\mathbf{f}$ with $B(|\mathbf{f}|) \in L^q(\nu, \nu+T; L_w^q(\Omega))$ locally, if $B(|\nabla u|) \in L^{1+\epsilon}(\nu, \nu+T; L_w^{1+\epsilon}(\Omega))$ locally, then we have $B(|\nabla u|) \in L^q(\nu, \nu+T; L_w^q(\Omega))$ locally with the following estimate*

$$\left(\int_{Q_1} [B(|\nabla u|)]^q w(x)\, dz\right)^{\frac{1}{q}}$$
$$\leq C\left[\left\{\int_{Q_2} [B(|\nabla u|)]^{1+\epsilon} w(x) dz\right\}^{\frac{1}{1+\epsilon}} + \left\{\int_{Q_2} [B(|\mathbf{f}|)]^q w(x)\, dz\right\}^{\frac{1}{q}} + 1\right]^{\frac{q(2+s_a)}{2}},$$

where the constant C is independent of u and $\mathbf{f}$.

2 Proof of the main result

This section is devoted to the proof of the main result stated in Theorem 1.4. For convenience of the readers, we recall some definitions and conclusions on the properties of the general Orlicz spaces.

Definition 2.1. A function $B : [0, +\infty) \to [0, +\infty)$ is said to be a Young function if it is convex and $B(0) = 0$. Then a Young function B is called an N-function if

$$0 < B(t) < \infty \quad \text{for} \quad t > 0 \quad \text{and} \quad \lim_{t\to+\infty} \frac{B(t)}{t} = \lim_{t\to 0+} \frac{t}{B(t)} = +\infty. \tag{2.1}$$

Moreover, we call a Young function B satisfies the global Δ_2 condition, denoted by $B \in \Delta_2$, if there exists a positive constant K such that

$$B(2t) \leq KB(t) \quad \text{for every} \quad t > 0. \tag{2.2}$$

Furthermore, an N-function B is said to satisfy the global ∇_2 condition, denoted by $B \in \nabla_2$, if there exists a number $\theta > 1$ such that

$$B(t) \leq \frac{B(\theta t)}{2\theta} \quad \text{for every} \quad t > 0. \tag{2.3}$$

Remark 2.2. For example,

1. $G_1(t) = (1+t)\log(1+t) - t \in \Delta_2$, but $G_1(t) \notin \nabla_2$.
2. $G_2(t) = e^t - t - 1 \in \nabla_2$, but $G_2(t) \notin \Delta_2$.
3. $G_3(t) = t^p \log(1+t) \in \Delta_2 \cap \nabla_2$ for $p > 1$.

Actually, it is easy to check that a Young function $B \in \Delta_2 \cap \nabla_2$ if and only if

$$A_1\left(\frac{s}{t}\right)^{\beta_2} \leq \frac{B(s)}{B(t)} \leq A_2\left(\frac{s}{t}\right)^{\beta_1} \tag{2.4}$$

for some constants $A_2 \geq A_1 > 0$, $\beta_1 \geq \beta_2 > 1$ and any $0 < t \leq s$.

Lemma 2.3. *If B is an N-function, then B satisfies the following Young's inequality*

$$st \leq \widetilde{B}(s) + B(t) \quad \text{for any} \quad s,t \geq 0. \tag{2.5}$$

Additionally, if $B \in \nabla_2 \cap \Delta_2$, then we obtain the following Young's inequality with ϵ

$$st \leq \epsilon \widetilde{B}(s) + C(\epsilon)B(t) \quad \text{for any } s,t \geq 0 \quad \text{and} \quad \epsilon > 0. \tag{2.6}$$

Definition 2.4. If B is an N-function, then the Orlicz class $K^B(\Omega)$ consists of all measurable functions $g : \Omega \to \mathbb{R}$ satisfying

$$\int_\Omega B(|g|)dx < \infty.$$

Also, the Orlicz space $L^B(\Omega)$ is the linear hull of $K^B(\Omega)$ endowed with the Luxemburg norm

$$\|g\|_{L^B(\Omega)} := \inf \left\{ k > 0 : \int_\Omega B\left(\frac{|g(x)|}{k}\right) dx \leq 1 \right\}.$$

On the other hand, the Orlicz–Sobolev space $W^{1,B}(\Omega) := \{g \in L^B(\Omega) \mid \nabla g \in L^B(\Omega)\}$, endowed with the norm $\|g\|_{W^{1,B}(\Omega)} := \|g\|_{L^B(\Omega)} + \|\nabla g\|_{L^B(\Omega)}$.

Remark 2.5. *In general, $K^B(\Omega) \subset L^B(\Omega)$ (see [2, Chapter 8]). But when $B \in \Delta_2$, $K^B(\Omega) = L^B(\Omega)$.*

We first state the following properties on the functions $a(t)$ and $B(t)$ described above.

Lemma 2.6 (see [26, Proposition 2.9] and [48, Lemma 1.9]). *If $a(t)$ satisfies (1.2) and $B(t)$ is defined in (1.5), then we have*

1. *$B(t)$ is strictly convex N-function and*

$$\widetilde{B}(b(t)) \leq C_0 B(t) \quad \text{for} \quad t \geq 0 \ \text{ and some constant } \ C_0 > 0.$$

2. *$B(t) \in \Delta_2 \cap \nabla_2$ with the estimate*

$$A_1 \left(\frac{s}{t}\right)^{2+i_a} \leq \frac{B(s)}{B(t)} \leq A_2 \left(\frac{s}{t}\right)^{2+s_a} \quad \text{for any} \quad 0 < t \leq s. \tag{2.7}$$

3. *$a(t)\theta^{i_a} \leq a(\theta t) \leq a(t)\theta^{s_a}$ for any $t > 0$ and $\theta \geq 1$.*

We continue to require certain properties of the functions $a(t)$ and $B(t)$, whose proofs can be found in Lemma 3 of [30] and Lemma 2.4 and Remark 2.5 of [48].

Lemma 2.7. *If $a(t)$ satisfies (1.2), $A(x,t)$ satisfies (1.3) and $B(t)$ is defined in (1.5), then for any $\xi, \eta \in \mathbb{R}^n$ we have*

$$a\left((A\xi \cdot \xi)^{\frac{1}{2}}\right) A\xi \cdot \xi \geq C(i_a, s_a, \Lambda) B(|\xi|) \tag{2.8}$$

and

$$\left[a\left((A\xi \cdot \xi)^{\frac{1}{2}}\right) A\xi - a\left((A\eta \cdot \eta)^{\frac{1}{2}}\right) A\eta\right] \cdot (\xi - \eta) \geq C(i_a, s_a, \Lambda) B(|\xi - \eta|). \tag{2.9}$$

Moreover, we first give the following L^1 estimate of weak solutions.

Lemma 2.8. *Assume that u is a local weak solution of (1.1) in Ω_T with $Q_2 \subset \Omega_T$ and (1.2). Then we have*

$$\int_{Q_1} B\left(|\nabla u|\right)\, dz \le C \int_{Q_2} B(|u|) + B(|\mathbf{f}|) + 1 dz. \tag{2.10}$$

Proof. We may as well select the test function $\varphi = \zeta u$, which may be justified via Steklov average just like Remark 2.2 in [17], where $\zeta \in C_0^\infty(\mathbb{R}^{n+1})$ is a cut-off function satisfying

$$0 \le \zeta \le 1, \quad \zeta \equiv 1 \text{ in } Q_1 \quad \text{and} \quad \zeta \equiv 0 \text{ in } \mathbb{R}^{n+1}/Q_2.$$

Then by Definition 1.2, after a direct calculation we show the resulting expression as

$$I_1 + I_2 = I_3 + I_4,$$

where

$$\begin{aligned}
I_1 &= \frac{1}{2}\int_{B_2} |u(x,t)|^2\, \zeta(x,t)\, dx\Big|_{t=-4}^{t=4} = 0,\\
I_2 &= \int_{Q_2} \zeta a\left((A\nabla u\cdot\nabla u)^{\frac{1}{2}}\right) A\nabla u\cdot\nabla u\, dz,\\
I_3 &= \frac{1}{2}\int_{Q_2} \zeta_t u^2 dz - \int_{Q_2} ua\left((A\nabla u\cdot\nabla u)^{\frac{1}{2}}\right) A\nabla u\cdot\nabla\zeta dz,\\
I_4 &= -\int_{Q_2} \zeta a\left(|\mathbf{f}|\right)\mathbf{f}\cdot\nabla u + ua\left(|\mathbf{f}|\right)\mathbf{f}\cdot\nabla\zeta dz.
\end{aligned}$$

Estimate of I_2. It follows from (2.8) and the definition of ζ that

$$I_2 \ge C\int_{Q_2} \zeta B\left(|\nabla u|\right)\, dz \ge C\int_{Q_1} B\left(|\nabla u|\right)\, dz.$$

Estimate of I_3. According to Lemma 2.3 and Lemma 2.6 (1), we conclude that

$$\begin{aligned}
|I_3| &\le C\int_{Q_2} a\left(|\nabla u|\right)|\nabla u|\,|u| + |u|^2 dz\\
&\le \frac{\tau}{C_0}\int_{Q_2} \widetilde{B}\left(a\left(|\nabla u|\right)|\nabla u|\right) dz + C(\tau)\int_{Q_2} B\left(|u|\right) + |u|^2 dz\\
&\le \tau\int_{Q_2} B\left(|\nabla u|\right) dz + C(\tau)\int_{Q_2} B\left(|u|\right) + 1 dz \quad \text{for any } \tau > 0,
\end{aligned}$$

where we have used the following inequality

$$B\left(\lambda\right) \ge A_1\lambda^{2+i_a} B\left(1\right) \ge A_1 B\left(1\right)\lambda^2 \quad \text{for } \lambda \ge 1 \tag{2.11}$$

by Lemma 2.6 (2) and the fact that $i_a \ge 0$.

Estimate of I_4. Similarly to the estimate of I_3, we have

$$\begin{aligned}
|I_4| &\le \int_{Q_2} a\left(|\mathbf{f}|\right)|\mathbf{f}|\,|\nabla u| + a\left(|\mathbf{f}|\right)|\mathbf{f}|\,|u|\, dz\\
&\le \tau\int_{Q_2} B\left(|\nabla u|\right) dz + C(\tau)\int_{Q_2} B\left(|\mathbf{f}|\right) + B\left(|u|\right) dz \quad \text{for any } \tau > 0.
\end{aligned}$$

Now we combine all the estimates of I_i $(1 \le i \le 4)$ to deduce that

$$C\int_{Q_1} B\left(|\nabla u|\right)\, dz \le 2\tau\int_{Q_2} B\left(|\nabla u|\right) dz + C(\tau)\int_{Q_2} B\left(|\mathbf{f}|\right) + B\left(|u|\right) + 1 dz.$$

Selecting τ small enough and removing the above right-hand side first integral by a covering and iteration argument (see Lemma 4.1 of Chapter 2 in [24] or Lemma 2.1 of Chapter 3 in [34]), we finish the proof. □

Next, we shall give some lemmas on the properties of A_q weight.

Lemma 2.9 (see [18,39,44]). *Assume that $w \in A_q$ for some $q > 1$. Then there exists a small positive constant $\alpha \in (0,1)$ such that*

$$C_1 \left(\frac{|B_r|}{|B_R|}\right)^q \leq \frac{w(B_r)}{w(B_R)} \leq C_2 \left(\frac{|B_r|}{|B_R|}\right)^\alpha$$

for any balls $B_r \subset B_R \subset \mathbb{R}^n$ and some constants $C_1, C_2 > 0$.

Remark 2.10. *We remark that $A_{p_1} \subset A_p$ for any $1 < p_1 \leq p < \infty$ (see page 195 in [43]).*

Now we recall the following self-improved property and reverse Hölder's inequality.

Lemma 2.11 (see [43,44]). *Assume $w \in A_q$ for some $q > 1$. Then there exist two constants $q_1 \in (1,q)$ and $\epsilon_0 > 0$, depending only on n, q and w such that*

$$w \in A_{q_1} \quad \text{and} \quad \left\{ \fint_{B_r} w^{1+\epsilon_0}(x)\,dx \right\}^{\frac{1}{1+\epsilon_0}} \leq C \fint_{B_r} w(x)\,dx.$$

The following is the measure theory of the weighted Lebesgue spaces.

Lemma 2.12. *If $w(x) \in A_q$ for some $q > 1$ and $g \in L^q_w(\mathbb{R}^{n+1})$, then for any $q > \beta > 1$ we have*

$$\begin{aligned}\int_{\mathbb{R}^{n+1}} |g(z)|^q\, w(x)\,dxdt &= q\int_0^{+\infty} \lambda^{q-1} \int_{\{z\in\mathbb{R}^{n+1}:|g|>\lambda\}} w(x)\,dxdtd\lambda \\ &= (q-\beta)\int_0^{\infty} \lambda^{q-\beta-1} \int_{\{z\in\mathbb{R}^{n+1}:|g|>\lambda\}} |g|^\beta w(x)\,dxdtd\lambda.\end{aligned}$$

Proof. Using Fubini's lemma, we conclude that

$$\begin{aligned}\int_{\mathbb{R}^{n+1}} |g(z)|^q\, w(x)\,dxdt &= \int_{\mathbb{R}^{n+1}} \left[q\int_0^{|g(z)|} \lambda^{q-1} d\lambda \right] w(x)\,dxdt \\ &= \int_{\mathbb{R}^{n+1}} \left[q\int_0^{+\infty} \lambda^{q-1} \chi_{\{z\in\mathbb{R}^{n+1}:|g|>\lambda\}} d\lambda \right] w(x)\,dxdt \\ &= q\int_0^{+\infty} \lambda^{q-1} \int_{\mathbb{R}^{n+1}} \chi_{\{z\in\mathbb{R}^{n+1}:|g|>\lambda\}} w(x)\,dxdtd\lambda \\ &= q\int_0^{+\infty} \lambda^{q-1} \int_{\{z\in\mathbb{R}^{n+1}:|g|>\lambda\}} w(x)\,dxdtd\lambda.\end{aligned}$$

On the other hand, we apply Fubini's lemma to obtain that

$$\begin{aligned}&(q-\beta)\int_0^{+\infty} \lambda^{q-\beta-1} \int_{\{z\in\mathbb{R}^{n+1}:|g|>\lambda\}} |g|^\beta w(x)\,dxdtd\lambda \\ &= (q-\beta)\int_0^{+\infty} \lambda^{q-\beta-1} \int_{\mathbb{R}^{n+1}} |g|^\beta \chi_{\{z\in\mathbb{R}^{n+1}:|g|>\lambda\}} w(x)\,dxdtd\lambda \\ &= (q-\beta)\int_{\mathbb{R}^{n+1}} \int_0^{+\infty} \lambda^{q-\beta-1} |g|^\beta \chi_{\{z\in\mathbb{R}^{n+1}:|g|>\lambda\}} w(x)\,d\lambda dxdt \\ &= \int_{\mathbb{R}^{n+1}} |g|^\beta \left[(q-\beta)\int_0^{|g|} \lambda^{q-\beta-1} d\lambda \right] w(x)\,dxdt \\ &= \int_{\mathbb{R}^{n+1}} |g|^\beta |g|^{q-\beta} w(x)\,dxdt = \int_{\mathbb{R}^{n+1}} |g|^q w(x)\,dxdt.\end{aligned}$$

Thus, we finish the proof. □

Next, we shall need the following important iteration-covering lemma, in which we will divide the domain into several small parts.

Lemma 2.13. *Given* $\lambda \geq \lambda_* := \frac{5^{nq+2}}{\min\left\{\sqrt{A_1 B(1)},1\right\}}\lambda_0$, *where*

$$\lambda_0^2 := \left\{\frac{1}{w\left(B_2\right)}\int_{Q_2}\left[B\left(|\nabla u|\right)\right]^{q_0} w(x)dz\right\}^{\frac{1}{q_0}} + \frac{1}{\delta}\left\{\frac{1}{w\left(B_2\right)}\int_{Q_2}\left[B\left(|\mathbf{f}|\right)\right]^{q_0} w(x)\,dz\right\}^{\frac{1}{q_0}} + 1 \quad (2.12)$$

and $q_0 = 1+\epsilon$ *with* $\epsilon \in (0, q-1)$, *there exist a family of disjoint cylinders* $\{Q_i^0\}_{i\in\mathbb{N}}$ *with*

$$Q_i^0 := Q_{z_i}\left(\frac{\lambda^2}{B(\lambda)}\rho_i^2, \rho_i\right),$$

$z_i = (x_i, t_i) \in E(Q_1, \lambda) := \{z \in Q_1 : B\left(|\nabla u|\right) > B\left(\lambda\right)\}$ *and* $0 < \rho_i = \rho(z_i) \leq 1/10$ *such that*

1.
$$J\left[Q_i^0\right] = J\left[Q_{z_i}\left(\frac{\lambda^2}{B(\lambda)}\rho_i^2, \rho_i\right)\right] = B(\lambda),$$

where

$$J\left[Q(\theta,\rho)\right] := \left\{\frac{1}{2\theta w\left(B_\rho\right)}\int_{Q(\theta,\rho)}\left[B\left(|\nabla u|\right)\right]^{q_0} w(x)dz\right\}^{\frac{1}{q_0}} + \frac{1}{\delta}\left\{\frac{1}{2\theta w\left(B_\rho\right)}\int_{Q(\theta,\rho)}\left[B\left(|\mathbf{f}|\right)\right]^{q_0} w(x)\,dz\right\}^{\frac{1}{q_0}}.$$

2.
$$J\left[Q_i^j\right] = J\left[5jQ_i^0\right] = J\left[Q_{z_i}\left(25j^2\frac{\lambda^2}{B(\lambda)}\rho_i^2, 5j\rho_i\right)\right] < B(\lambda) \quad \textit{for} \quad j = 1, 2,$$

where $Q_i^j = 5jQ_i^0 = Q_{z_i}\left(25j^2\frac{\lambda^2}{B(\lambda)}\rho_i^2, 5j\rho_i\right)$.

3. $E(Q_1,\lambda) \subset \bigcup_{i\in\mathbb{N}} Q_i^1 \cup$ *negligible set.*

Proof. Let $\lambda \geq \lambda_*$. From Lemma 2.9 we have

$$\left\{\frac{\int_{Q_2}\left[B\left(|\nabla u|\right)\right]^{q_0} w(x)dz}{\int_{Q_{z_0}\left(\frac{\lambda^2}{B(\lambda)}\rho^2,\rho\right)} w(x)\,dxdt}\right\}^{\frac{1}{q_0}} = \left\{\frac{\int_{Q_2}\left[B\left(|\nabla u|\right)\right]^{q_0} w(x)dz}{2w\left(B_\rho\left(x_0\right)\right)\frac{\lambda^2}{B(\lambda)}\rho^2}\right\}^{\frac{1}{q_0}}$$
$$\leq \left\{\frac{w(B_2)}{2w\left(B_\rho\left(x_0\right)\right)\frac{\lambda^2}{B(\lambda)}\rho^2}\frac{1}{w(B_2)}\int_{Q_2}\left[B\left(|\nabla u|\right)\right]^{q_0} w(x)\,dz\right\}^{\frac{1}{q_0}}$$
$$\leq \left\{50\cdot 20^{nq}\frac{B(\lambda)}{\lambda^2}\frac{1}{w(B_2)}\int_{Q_2}\left[B\left(|\nabla u|\right)\right]^{q_0} w(x)\,dz\right\}^{\frac{1}{q_0}}$$

for any $z_0 = (x_0, t_0) \in Q_1$ and $1/10 \leq \rho \leq 1$, and then

$$\left\{\frac{\int_{Q_2}\left[B\left(|\mathbf{f}|\right)\right]^{q_0} w(x)dz}{\int_{Q_{z_0}\left(\frac{\lambda^2}{B(\lambda)}\rho^2,\rho\right)} w(x)\,dxdt}\right\}^{\frac{1}{q_0}} \leq \left\{50\cdot 20^{nq}\frac{B(\lambda)}{\lambda^2}\frac{1}{w(B_2)}\int_{Q_2}\left[B\left(|\mathbf{f}|\right)\right]^{q_0} w(x)\,dz\right\}^{\frac{1}{q_0}}.$$

So, (2.11) implies that

$$J\left[Q_{z_0}\left(\frac{\lambda^2}{B(\lambda)}\rho^2,\rho\right)\right] \le \left[50\cdot 20^{nq}A_1B(1)\frac{B(\lambda)}{A_1B(1)\lambda^2}\right]^{\frac{1}{q_0}}\lambda_0^2$$
$$\le \left[25^{nq+2}\cdot A_1B(1)\right]^{\frac{1}{q_0}}\frac{B(\lambda)}{A_1B(1)\lambda^2}\lambda_0^2$$
$$\le 25^{nq+2}\max\{1,A_1B(1)\}\frac{B(\lambda)}{A_1B(1)\lambda^2}\lambda_0^2$$
$$=\lambda_*^2\frac{B(\lambda)}{\lambda^2}\le B(\lambda)$$

for any $z_0\in Q_1$ and $\frac{1}{10}\le\rho\le 1$. Therefore, we deduce that

$$\sup_{z=(x,t)\in Q_1}\sup_{\frac{1}{10}\le\rho\le 1}J\left[Q_z\left(\frac{\lambda^2}{B(\lambda)}\rho^2,\rho\right)\right]\le B(\lambda). \tag{2.13}$$

Using Lebesgue's differentiation theorem, for a.e. $z\in E(Q_1,\lambda)$ we conclude that

$$\lim_{\rho\to 0}J\left[Q_z\left(\frac{\lambda^2}{B(\lambda)}\rho^2,\rho\right)\right]>B(\lambda),$$

which implies that there exists some $\rho_0\in(0,1]$ satisfying

$$J\left[Q_z\left(\frac{\lambda^2}{B(\lambda)}\rho_0^2,\rho_0\right)\right]>B(\lambda). \tag{2.14}$$

Therefore, from (2.13) and (2.14) we can select a radius $\rho_z\in(0,1/10]$ such that

$$\rho_z=:\max\left\{\rho\;\middle|\;J\left[Q_z\left(\frac{\lambda^2}{B(\lambda)}\rho^2,\rho\right)\right]=B(\lambda),\;\;0<\rho\le 1/10\right\},$$

which implies that

$$J\left[Q_z\left(\frac{\lambda^2}{B(\lambda)}\rho_z^2,\rho_z\right)\right]=B(\lambda)\quad\text{and}\quad J\left[Q_z\left(\frac{\lambda^2}{B(\lambda)}\rho^2,\rho\right)\right]<B(\lambda)\quad\text{for }\rho_z<\rho\le 1.$$

Thus, by using Vitali's covering lemma, we can find a family of disjoint cylinders $\{Q_i^0\}_{i\in\mathbb{N}}=\{Q_{z_i}(\frac{\lambda^2}{B(\lambda)}\rho_i^2,\rho_i)\}_{i\in\mathbb{N}}$ with $z_i=(x_i,t_i)\in E(Q_1,\lambda)$ and $\rho_i=\rho(z_i)\le 1/10$ such that (1)–(3) are true. □

Now we employ the above lemma to derive careful estimates on the small parabolic cylinders $\{Q_i^0\}_{i\in\mathbb{N}}$, whose precise structure will be needed below.

Lemma 2.14. *Under the same hypothetical conditions as the lemma above, for $\lambda\ge\lambda_*$ we obtain*

$$w\left(B_{\rho_i}(x_i)\right)\frac{\lambda^2}{B(\lambda)}\rho_i^2<\frac{C}{[B(\lambda)]^{q_0}}\int_{\left\{z\in Q_i^0:B(|\nabla u|)>\frac{B(\lambda)}{4}\right\}}[B(|\nabla u|)]^{q_0}w(x)dz$$
$$+\frac{C}{\delta^{q_0}[B(\lambda)]^{q_0}}\int_{\left\{z\in Q_i^0:B(|\mathbf{f}|)>\frac{B(\lambda)}{4\delta}\right\}}[B(|\mathbf{f}|)]^{q_0}w(x)dz.$$

Proof. From Lemma 2.13 (1), we find that

$$\left\{ \frac{1}{2w\left(B_{\rho_i}(x_i)\right) \frac{\lambda^2}{B(\lambda)}\rho_i^2} \int_{Q_i^0} \left[B\left(|\nabla u|\right)\right]^{q_0} w(x)dz \right\}^{\frac{1}{q_0}} + \frac{1}{\delta} \left\{ \frac{1}{2w\left(B_{\rho_i}(x_i)\right) \frac{\lambda^2}{B(\lambda)}\rho_i^2} \int_{Q_i^0} \left[B\left(|\mathbf{f}|\right)\right]^{q_0} w(x)\,dz \right\}^{\frac{1}{q_0}} = B(\lambda).$$

Therefore, one of the following inequalities must be true

$$\left\{ \frac{1}{2w\left(B_{\rho_i}(x_i)\right) \frac{\lambda^2}{B(\lambda)}\rho_i^2} \int_{Q_i^0} \left[B\left(|\nabla u|\right)\right]^{q_0} w(x)dz \right\}^{\frac{1}{q_0}} > \frac{B(\lambda)}{2} \tag{2.15}$$

and

$$\frac{1}{\delta} \left\{ \frac{1}{2w\left(B_{\rho_i}(x_i)\right) \frac{\lambda^2}{B(\lambda)}\rho_i^2} \int_{Q_i^0} \left[B\left(|\mathbf{f}|\right)\right]^{q_0} w(x)\,dz \right\}^{\frac{1}{q_0}} > \frac{B(\lambda)}{2}. \tag{2.16}$$

If (2.15) is true, then we have

$$\begin{aligned} 2w\left(B_{\rho_i}(x_i)\right) \frac{\lambda^2}{B(\lambda)}\rho_i^2 &< \left[\frac{2}{B(\lambda)}\right]^{q_0} \int_{Q_i^0} \left[B\left(|\nabla u|\right)\right]^{q_0} w(x)dz \\ &\leq \left[\frac{2}{B(\lambda)}\right]^{q_0} \int_{\left\{z\in Q_i^0 : B(|\nabla u|) > \frac{B(\lambda)}{4}\right\}} \left[B\left(|\nabla u|\right)\right]^{q_0} w(x)dz \\ &\quad + \frac{1}{2^{q_0-1}} w\left(B_{\rho_i}(x_i)\right) \frac{\lambda^2}{B(\lambda)}\rho_i^2 \quad \text{for } \lambda \geq \lambda_*, \end{aligned}$$

which implies that

$$w\left(B_{\rho_i}(x_i)\right) \frac{\lambda^2}{B(\lambda)}\rho_i^2 < \frac{C}{[B(\lambda)]^{q_0}} \int_{\left\{z\in Q_i^0 : B(|\nabla u|) > \frac{B(\lambda)}{4}\right\}} \left[B\left(|\nabla u|\right)\right]^{q_0} w(x)dz \quad \text{for } \lambda \geq \lambda_*.$$

Similarly, if (2.16) is true, then we have

$$w\left(B_{\rho_i}(x_i)\right) \frac{\lambda^2}{B(\lambda)}\rho_i^2 < \frac{C}{\delta^{q_0}\,[B(\lambda)]^{q_0}} \int_{\left\{z\in Q_i^0 : B(|\mathbf{f}|) > \frac{B(\lambda)}{4\delta}\right\}} \left[B\left(|\mathbf{f}|\right)\right]^{q_0} w(x)dz \quad \text{for } \lambda \geq \lambda_*.$$

Finally, we combine the two estimates above to find that

$$\begin{aligned} w\left(B_{\rho_i}(x_i)\right) \frac{\lambda^2}{B(\lambda)}\rho_i^2 &< \frac{C}{[B(\lambda)]^{q_0}} \int_{\left\{z\in Q_i^0 : B(|\nabla u|) > \frac{B(\lambda)}{4}\right\}} \left[B\left(|\nabla u|\right)\right]^{q_0} w(x)dz \\ &\quad + \frac{C}{\delta^{q_0}\,[B(\lambda)]^{q_0}} \int_{\left\{z\in Q_i^0 : B(|\mathbf{f}|) > \frac{B(\lambda)}{4\delta}\right\}} \left[B\left(|\mathbf{f}|\right)\right]^{q_0} w(x)dz \end{aligned}$$

for $\lambda \geq \lambda_*$. Thus, this finishes our proof. □

Lemma 2.15. *Under the same hypotheses and results as those in Lemma 2.13, we have*

$$\left(\fint_{Q_i^j} \left[B\left(|\nabla u|\right)\right]^{q_2} dz\right)^{\frac{1}{q_2}} < CB(\lambda) \quad and \quad \left(\fint_{Q_i^j} \left[B\left(|\mathbf{f}|\right)\right]^{q_2} dz\right)^{\frac{1}{q_2}} < C\delta B(\lambda) \tag{2.17}$$

for $j = 1, 2$, $\lambda \geq \lambda_*$ and some constant $q_2 \in (1, q_0)$.

Proof. It follows from Lemma 2.13 (2) that

$$\left(\frac{1}{w\left(B_{5j\rho_i}(x_i)\right) \frac{\lambda^2}{B(\lambda)} 50 j^2 \rho_i^2} \int_{Q_i^j} \left[B\left(|\nabla u|\right)\right]^{q_0} w(x)\, dxdt\right)^{\frac{1}{q_0}} < B(\lambda)$$

and

$$\left(\frac{1}{w\left(B_{5j\rho_i}(x_i)\right) \frac{\lambda^2}{B(\lambda)} 50 j^2 \rho_i^2} \int_{Q_i^j} \left[B\left(|\mathbf{f}|\right)\right]^{q_0} w(x)\, dxdt\right)^{\frac{1}{q_0}} < \delta B(\lambda)$$

for $j = 1, 2$. Since $w \in A_{q_0}$ by Remark 2.10, we find that $w \in A_{q_1}$ for some $q_1 \in (1, q_0)$ in view of Lemma 2.11. Let $q_2 = \frac{q_0}{q_1} \in (1, q_0)$. Then by using Hölder's inequality and the two inequalities above, we see

$$\begin{aligned}
&\left(\fint_{Q_i^j} \left[B\left(|\nabla u|\right)\right]^{q_2} dz\right)^{\frac{1}{q_2}} \\
&= \left(\fint_{Q_i^j} \left[B\left(|\nabla u|\right)\right]^{q_2} w(x)^{\frac{1}{q_1}} w(x)^{-\frac{1}{q_1}} dz\right)^{\frac{1}{q_2}} \\
&\leq \left(\frac{w\left(B_{5j\rho_i}(x_i)\right)}{\left|B_{5j\rho_i}(x_i)\right|} \frac{1}{w\left(B_{5j\rho_i}(x_i)\right) \frac{\lambda^2}{B(\lambda)} 50 j^2 \rho_i^2} \int_{Q_i^j} \left[B\left(|\nabla u|\right)\right]^{q_0} w(x)\, dz\right)^{\frac{1}{q_0}} \left(\fint_{Q_i^j} w(x)^{-\frac{1}{q_1-1}} dz\right)^{\frac{q_1-1}{q_0}} \\
&< B(\lambda) \left[\fint_{B_{5j\rho_i}(x_i)} w(x)\, dx \left(\fint_{B_{5j\rho_i}(x_i)} w(x)^{-\frac{1}{q_1-1}} dx\right)^{q_1-1}\right]^{\frac{1}{q_0}} \\
&< CB(\lambda) \quad \text{for} \quad j = 1, 2.
\end{aligned}$$

Similarly, we have

$$\left(\fint_{Q_i^j} \left[B\left(|\mathbf{f}|\right)\right]^{q_2} dz\right)^{\frac{1}{q_2}} < C\delta B(\lambda) \quad \text{for } j = 1, 2,$$

which finishes our proof. □

Moreover, we can obtain the following comparison result and interior Lipschitz regularity.

Lemma 2.16. *Assume that u is a local weak solution of (1.1) in Ω_T with (1.2). If v is the weak solution of*

$$v_t - \operatorname{div}\left(a\left(\left(\overline{A}_{Q_2} \nabla v \cdot \nabla v\right)^{\frac{1}{2}}\right) \overline{A}_{Q_2} \nabla v\right) = 0 \quad in\ Q_2 \subset \Omega_T \tag{2.18}$$

with $v = u$ on $\partial_p Q_2$, then we have

$$\int_{Q_2} B\left(|\nabla v|\right) dz \leq C \int_{Q_2} B(|\nabla u|) + B(|\mathbf{f}|)\, dz \tag{2.19}$$

and

$$\sup_{Q_1} B\left(|\nabla v|\right) \le C \left[\fint_{Q_2} B(|\nabla u|) + B(|\mathbf{f}|) + 1 dz \right]^{\frac{2+s_a}{2}}, \tag{2.20}$$

where the constant $C = C(i_a, s_a, \Lambda) > 0$.

Proof. Noting that u and v are the weak solutions of (1.1) and (2.18) respectively, we may as well select the test function $\varphi = v - u$ since $v = u$ on $\partial_p Q_2$, which is possible modulo Steklov average. Then a direct calculation shows the resulting expression as

$$I_1 + I_2 = I_3 + I_4 + I_5,$$

where

$$\begin{aligned}
I_1 &= \frac{1}{2} \int_{B_2} |v(x,4) - u(x,4)|^2 dx \ge 0, \\
I_2 &= \int_{Q_2} a\left(\left(\overline{A}_{Q_2} \nabla v \cdot \nabla v\right)^{\frac{1}{2}} \right) \overline{A}_{Q_2} \nabla v \cdot \nabla v \, dz, \\
I_3 &= \int_{Q_2} a\left(\left(\overline{A}_{Q_2} \nabla v \cdot \nabla v\right)^{\frac{1}{2}} \right) \overline{A}_{Q_2} \nabla v \cdot \nabla u \, dz, \\
I_4 &= \int_{Q_2} \left[a\left((A\nabla u \cdot \nabla u)^{\frac{1}{2}} \right) A\nabla u \cdot \nabla v - a\left((A\nabla u \cdot \nabla u)^{\frac{1}{2}} \right) A\nabla u \cdot \nabla u \right] dz, \\
I_5 &= \int_{Q_2} a(|\mathbf{f}|)\, \mathbf{f} \cdot \nabla v - a(|\mathbf{f}|)\, \mathbf{f} \cdot \nabla u \, dz.
\end{aligned}$$

Estimate of I_2. Owing to Lemma 2.7 we thereby discover

$$I_2 \ge C \int_{Q_2} B\left(|\nabla v|\right) dz.$$

Estimate of I_3. According to Lemma 2.3 and Lemma 2.6 (1), we see

$$\begin{aligned}
|I_3| &\le \frac{\tau}{C_0} \int_{Q_2} \widetilde{B}\left(a\left(|\nabla v|\right)|\nabla v|\right) dz + C(\tau) \int_{Q_2} B\left(|\nabla u|\right) dz \\
&\le \tau \int_{Q_2} B\left(|\nabla v|\right) dz + C(\tau) \int_{Q_2} B\left(|\nabla u|\right) dz \quad \text{for any } \tau > 0.
\end{aligned}$$

Estimates of I_i ($4 \le i \le 5$). As in the proof of the estimate of I_3, we compute

$$\begin{aligned}
|I_4| &\le \tau \int_{Q_2} B\left(|\nabla v|\right) dz + C(\tau) \int_{Q_2} B\left(|\nabla u|\right) dz, \\
|I_5| &\le \tau \int_{Q_2} B\left(|\nabla v|\right) dz + C(\tau) \int_{Q_2} B\left(|\nabla u|\right) + B\left(|\mathbf{f}|\right) dz \quad \text{for any } \tau > 0.
\end{aligned}$$

Therefore, by selecting $\tau > 0$ small enough we obtain the desired result (2.19) from the estimates of I_i ($1 \le i \le 5$). Rather, from Theorem 2.2 in [31] we have

$$\min\left\{ \sup_{Q_1} \rho(|\nabla v|), \sup_{Q_1} |\nabla v|^2 \right\} \le C \fint_{Q_2} |\nabla v|^2 + B(|\nabla v|) \, dz,$$

where $C = (i_a, s_a, \Lambda)$ and

$$\rho(t) = (B(t))^{\frac{n}{2}} t^{2-n} \ge C t^n t^{2-n} = C t^2 \quad \text{for any } t \ge 1$$

by (2.11), which implies that

$$\sup_{Q_1} |\nabla v|^2 \leq C \min\left\{\sup_{Q_1} \rho(|\nabla v|), \sup_{Q_1} |\nabla v|^2\right\} + C \leq C \fint_{Q_2} B(|\nabla v|) + 1 dz. \tag{2.21}$$

Furthermore, we deduce from (2.7) that

$$\sup_{Q_1} B\left(|\nabla v|\right) \leq C \left[\fint_{Q_2} B(|\nabla v|) + 1 dz\right]^{\frac{2+s_a}{2}},$$

which implies that (2.20) is valid by (2.19). Thus, we finish the proof. □

Moreover, we shall give the following essential estimate on the level set.

Lemma 2.17. *For any $\epsilon > 0$, there exists a small $\delta = \delta(\epsilon) > 0$ such that if u is a local weak solution of (1.1) in Ω_T with (1.2) and $Q_2 \subset \Omega_T$,*

$$\fint_{Q_2} |A - \overline{A}_{Q_2}| dz \leq \delta, \tag{2.22}$$

$$\left\{\fint_{Q_2} \left[B\left(|\nabla u|\right)\right]^{q_2} dz\right\}^{\frac{1}{q_2}} \leq 1 \quad \text{and} \quad \left\{\fint_{Q_2} \left[B\left(|\mathbf{f}|\right)\right]^{q_2} dz\right\}^{\frac{1}{q_2}} \leq \delta, \tag{2.23}$$

then there exists a constant $N_1 > 1$ such that

$$|\{z \in Q_1 : B(|\nabla u|) > N_1\}| \leq C\epsilon |Q_1|.$$

Proof. If v is the weak solution of (2.18) in Q_2 with $v = u$ on $\partial_p Q_2$, then by selecting the test function $\varphi = u - v$, which is possible modulo Steklov average, after a direct calculation we show the resulting expression as

$$I_1 + I_2 = I_3 + I_4,$$

where

$$\begin{aligned}
I_1 &= \frac{1}{2} \int_{B_2} |u(x,4) - v(x,4)|^2 dx \geq 0,\\
I_2 &= \int_{Q_2} \left[a\left(\left(\overline{A}_{Q_2} \nabla u \cdot \nabla u\right)^{\frac{1}{2}}\right) \overline{A}_{Q_2} \nabla u - a\left(\left(\overline{A}_{Q_2} \nabla v \cdot \nabla v\right)^{\frac{1}{2}}\right) \overline{A}_{Q_2} \nabla v\right] \cdot (\nabla u - \nabla v)\, dz,\\
I_3 &= -\int_{Q_2} \left[a\left((A \nabla u \cdot \nabla u)^{\frac{1}{2}}\right) A \nabla u - a\left(\left(\overline{A}_{Q_2} \nabla u \cdot \nabla u\right)^{\frac{1}{2}}\right) \overline{A}_{Q_2} \nabla u\right] \cdot (\nabla u - \nabla v)\, dz,\\
I_4 &= \int_{Q_2} a\left(|\mathbf{f}|\right) \mathbf{f} \cdot \nabla v dz - \int_{Q_2} a\left(|\mathbf{f}|\right) \mathbf{f} \cdot \nabla u dz.
\end{aligned}$$

Estimate of I_2. Using Lemma 2.7, we observe that

$$I_2 \geq C \int_{Q_2} B\left(|\nabla u - \nabla v|\right) dz.$$

Estimate of I_3. First of all, we discover

$$\begin{aligned}
|I_3| &\leq \int_{Q_2} a\left((A \nabla u \cdot \nabla u)^{\frac{1}{2}}\right) |A - \overline{A}_{Q_2}|\, |\nabla u|\, |\nabla u - \nabla v|\, dz\\
&\quad + \int_{Q_2} \left|a\left((A \nabla u \cdot \nabla u)^{\frac{1}{2}}\right) - a\left(\left(\overline{A}_{Q_2} \nabla u \cdot \nabla u\right)^{\frac{1}{2}}\right)\right| |\overline{A}_{Q_2} \nabla u|\, |\nabla u - \nabla v|\, dz\\
&=: I_{31} + I_{32}.
\end{aligned}$$

Estimate of I_{31}. From (1.3), Lemma 2.6, Young's inequality and Hölder's inequality we find that

$$\begin{aligned}
|I_{31}| &\le C\int_{Q_2} a(|\nabla u|)\,|\nabla u|\,\left|A-\overline{A}_{Q_2}\right|\,|\nabla u-\nabla v|\,dz \\
&\le \frac{\epsilon}{2\Lambda}\int_{Q_2} B(|\nabla u-\nabla v|)\left|A-\overline{A}_{Q_2}\right|dz + C(\epsilon)\int_{Q_2}\widetilde{B}\left(a(|\nabla u|)\,|\nabla u|\right)\left|A-\overline{A}_{Q_2}\right|dz \\
&\le \epsilon\int_{Q_2} B(|\nabla u-\nabla v|)\,dz + C(\epsilon)\int_{Q_2} B(|\nabla u|)\left|A-\overline{A}_{Q_2}\right|dz \\
&\le \epsilon\int_{Q_2} B(|\nabla u-\nabla v|)\,dz + C(\epsilon)\left\{\int_{Q_2}[B(|\nabla u|)]^{q_2}\,dz\right\}^{\frac{1}{q_2}}\left[\int_{Q_2}\left|A-\overline{A}_{Q_2}\right|^{\frac{q_2}{q_2-1}}dz\right]^{\frac{q_2-1}{q_2}}
\end{aligned}$$

for any $\epsilon > 0$, which implies that

$$\begin{aligned}
|I_{31}| &\le \epsilon\int_{Q_2} B(|\nabla u-\nabla v|)\,dz + C(\epsilon)\left[\int_{Q_2}\left|A-\overline{A}_{Q_2}\right|dz\right]^{\frac{q_2-1}{q_2}} \\
&\le \epsilon\int_{Q_2} B(|\nabla u-\nabla v|)\,dz + C(\epsilon)\delta^{\frac{q_2-1}{q_2}},
\end{aligned}$$

where we used the given conditions (2.22)–(2.23).

Estimate of I_{32}. (1.2), (1.3), Lemma 2.6 and Lagrange's mean value theorem yield the bound

$$|I_{32}| \le C\int_{Q_2} a(|\nabla u|)\,|\nabla u|\,\left|A-\overline{A}_{Q_2}\right|\,|\nabla u-\nabla v|\,dz$$

and so

$$|I_{32}| \le \epsilon\int_{Q_2} B(|\nabla u-\nabla v|)\,dz + C(\epsilon)\delta^{\frac{q_2-1}{q_2}} \quad \text{for any } \epsilon > 0,$$

whose proof is totally similar to that of I_{31}.

Estimate of I_4. Lemma 2.3, Lemma 2.6 (1), Hölder's inequality and (2.23) assert

$$\begin{aligned}
|I_4| &\le \int_{Q_2} a(|\mathbf{f}|)\,|\mathbf{f}|\,|\nabla u-\nabla v|\,dz \\
&\le \epsilon\int_{Q_2} B(|\nabla u-\nabla v|)\,dz + C(\epsilon)\int_{Q_2}\widetilde{B}\left(a(|\mathbf{f}|)\,|\mathbf{f}|\right)dz \\
&\le \epsilon\int_{Q_2} B(|\nabla u-\nabla v|)\,dz + C(\epsilon)\int_{Q_2} B(|\mathbf{f}|)\,dz \\
&\le \epsilon\int_{Q_2} B(|\nabla u-\nabla v|)\,dz + C(\epsilon)\left[\fint_{Q_2}(B(|\mathbf{f}|))^{q_2}\,dz\right]^{\frac{1}{q_2}} \\
&\le \epsilon\int_{Q_2} B(|\nabla u-\nabla v|)\,dz + C(\epsilon)\delta
\end{aligned}$$

for any $\epsilon > 0$. So, by selecting $\epsilon > 0$ small enough and combining the estimates of I_i $(1 \le i \le 4)$ we conclude that

$$\int_{Q_2} B(|\nabla u-\nabla v|)\,dz \le C(\epsilon)\delta + C(\epsilon)\delta^{\frac{q_2-1}{q_2}} \le \epsilon, \tag{2.24}$$

where we have chosen a small constant $\delta > 0$ satisfying the above last inequality. Thus, it follows from Lemma 2.16, Hölder's inequality and the assumed condition (2.23) that

$$\begin{aligned}
\sup_{Q_1} B(|\nabla v|) &\leq C\left[\fint_{Q_2} B(|\nabla u|) + B(|\mathbf{f}|) + 1 dz\right]^{\frac{2+s_a}{2}} \\
&\leq C\left\{\left[\fint_{Q_2} (B(|\nabla u|))^{q_2}\, dz\right]^{\frac{1}{q_2}} + \left[\fint_{Q_2} (B(|\mathbf{f}|))^{q_2}\, dz\right]^{\frac{1}{q_2}} + 1\right\}^{\frac{2+s_a}{2}} \\
&\leq N_0 \qquad (2.25)
\end{aligned}$$

for some constant $N_0 > 1$. Finally, from (2.24), (2.25) and the fact that $B \in \Delta_2 \cap \nabla_2$ is convex we have

$$\begin{aligned}
&|\{z \in Q_1 : B(|\nabla u|) > 2C_* N_0\}| \\
&\quad \leq |\{z \in Q_1 : B(|\nabla(u-v)|) > N_0\}| + |\{z \in Q_1 : B(|\nabla v|) > N_0\}| \\
&\quad = |\{z \in Q_1 : B(|\nabla(u-v)|) > N_0\}| \\
&\quad \leq \frac{1}{N_0}\int_{Q_2} B(|\nabla(u-v)|)\, dz \leq C\epsilon|Q_2| \leq C\epsilon|Q_1|,
\end{aligned}$$

where we have used the following inequality

$$B(a+b) \leq \frac{1}{2}B(2a) + \frac{1}{2}B(2b) \leq C_* B(a) + C_* B(b) \quad \text{for any } a, b \geq 0.$$

This completes our proof by choosing $N_1 = 2C_* N_0 > 1$. □

Furthermore, we shall give the following result.

Lemma 2.18. *Assume that $\lambda \geq \lambda_*$. For any $\epsilon > 0$, there exists a small $\delta = \delta(\epsilon) > 0$ such that if u is a local weak solution of (1.1) in Ω_T with $Q_2 \subset \Omega_T$, then we have*

$$\begin{aligned}
\int_{\{z \in Q_1 : B(|\nabla u|) > N_1 B(\lambda)\}} w(x)\, dxdt \leq{}& \frac{C\epsilon^{\alpha}}{[B(\lambda)]^{q_0}} \int_{\left\{z \in Q_2 : B(|\nabla u|) > \frac{B(\lambda)}{4}\right\}} [B(|\nabla u|)]^{q_0}\, w(x) dxdt \\
&+ \frac{C\epsilon^{\alpha}}{\delta^{q_0}\,[B(\lambda)]^{q_0}} \int_{\left\{z \in Q_2 : B(|\mathbf{f}|) > \frac{B(\lambda)}{4\delta}\right\}} [B(|\mathbf{f}|)]^{q_0}\, w(x) dxdt.
\end{aligned}$$

Proof. 1. We first claim that

$$\left|\left\{z \in Q_i^1 : B(|\nabla u|) > N_1 B(\lambda)\right\}\right| \leq C\epsilon|Q_i^1|. \qquad (2.26)$$

To prove this, for each $\lambda \geq 1$ we use the normalization and scaling methods by defining

$$u_\lambda^i(x,t) := \frac{u\left(5\rho_i(x+x_i), \frac{\lambda^2}{B(\lambda)}(5\rho_i)^2(t+t_i)\right)}{\lambda 5\rho_i},$$

$$\mathbf{f}_\lambda^i(x,t) := \frac{\mathbf{f}\left(5\rho_i(x+x_i), \frac{\lambda^2}{B(\lambda)}(5\rho_i)^2(t+t_i)\right)}{\lambda},$$

$$A^i_\lambda(x,t) := A\left(5\rho_i(x+x_i), \frac{\lambda^2}{B(\lambda)}(5\rho_i)^2(t+t_i)\right),$$

$$a_\lambda(t) := \frac{a(\lambda t)}{\frac{B(\lambda)}{\lambda^2}}, \quad b_\lambda(t) := t a_\lambda(t) \quad \text{and} \quad B_\lambda(t) := \int_0^t b_\lambda(\tau)\, d\tau = \frac{B(\lambda t)}{B(\lambda)},$$

which implies that

$$B_\lambda(1) = \frac{B(\lambda)}{B(\lambda)} = 1 \quad \text{and} \quad B_\lambda(t) \quad \text{satisfies (2.7).}$$

From the definitions of u^i_λ, $\mathbf{f}^i_\lambda$, A^i_λ and a_λ, we find that u^i_λ is a local weak solution of

$$(u^i_\lambda)_t - \operatorname{div}\left(a_\lambda\left(\left(A^i_\lambda(x,t)\nabla u^i_\lambda \cdot \nabla u^i_\lambda\right)^{\frac{1}{2}}\right) A^i_\lambda(x,t)\nabla u^i_\lambda\right) = \operatorname{div}\left(a_\lambda\left(|\mathbf{f}^i_\lambda|\right)\mathbf{f}^i_\lambda\right) \quad \text{in } Q_2.$$

Without loss of generality we may as well assume that $R = 2$ in Definition 1.1 by a scaling argument. Consequently, from Definition 1.1 and Lemma 2.15 we conclude that

$$\left\{\fint_{Q_2}\left[B_\lambda(|\nabla u^i_\lambda|)\right]^{q_2} dz\right\}^{\frac{1}{q_2}} \le C, \qquad \left\{\fint_{Q_2}\left[B_\lambda(|\mathbf{f}^i_\lambda|)\right]^{q_2} dz\right\}^{\frac{1}{q_2}} \le C\delta$$

and

$$\fint_{Q_2}\left|A^i_\lambda(x,t) - \overline{A^i_\lambda}_{Q_2}\right| dz \le \delta$$

for any $j = 1, 2$ and $\lambda \ge \lambda_*$. Then according to Lemma 2.17, we conclude that

$$\left|\left\{z \in Q_1 : B_\lambda(|\nabla u^i_\lambda|) > N_1\right\}\right| \le C\epsilon|Q_1|.$$

Then by way of changing variables, we recover the claim.

2. Now we find that

$$\begin{aligned}&\int_{\left\{z\in Q^1_i : B(|\nabla u|) > N_1 B(\lambda)\right\}} w(x)\,dxdt \\ &\le \int_{t_i - 25\frac{\lambda^2}{B(\lambda)}\rho_i^2}^{t_i + 25\frac{\lambda^2}{B(\lambda)}\rho_i^2} \int_{E\left(B_{5\rho_i}(x_i), t\right)} w(x)\,dxdt = \int_{t_i - 25\frac{\lambda^2}{B(\lambda)}\rho_i^2}^{t_i + 25\frac{\lambda^2}{B(\lambda)}\rho_i^2} w\left(E\left(B_{5\rho_i}(x_i), t\right)\right) dt,\end{aligned}$$

where $E\left(B_{5\rho_i}(x_i), t\right) := \left\{x \in B_{5\rho_i}(x_i) : B(|\nabla u(x,t)|) > N_1 B(\lambda)\right\}$, which implies that

$$\begin{aligned}&\frac{\int_{\left\{z\in Q^1_i : B(|\nabla u|) > N_1 B(\lambda)\right\}} w(x)\,dxdt}{50 w\left(B_{5\rho_i}(x_i)\right)\frac{\lambda^2}{B(\lambda)}\rho_i^2} \\ &\le \frac{1}{50\frac{\lambda^2}{B(\lambda)}\rho_i^2}\int_{t_i - 25\frac{\lambda^2}{B(\lambda)}\rho_i^2}^{t_i + 25\frac{\lambda^2}{B(\lambda)}\rho_i^2} \frac{w\left(E\left(B_{5\rho_i}(x_i), t\right)\right)}{w\left(B_{5\rho_i}(x_i)\right)} dt \\ &\le \frac{1}{50\frac{\lambda^2}{B(\lambda)}\rho_i^2}\int_{t_i - 25\frac{\lambda^2}{B(\lambda)}\rho_i^2}^{t_i + 25\frac{\lambda^2}{B(\lambda)}\rho_i^2} \left(\frac{E\left(B_{5\rho_i}(x_i), t\right)}{|B_{5\rho_i}(x_i)|}\right)^\alpha dt \\ &\le \frac{1}{\left[50\frac{\lambda^2}{B(\lambda)}\rho_i^2\right]^\alpha}\left[\int_{t_i - 25\frac{\lambda^2}{B(\lambda)}\rho_i^2}^{t_i + 25\frac{\lambda^2}{B(\lambda)}\rho_i^2} \frac{E\left(B_{5\rho_i}(x_i), t\right)}{|B_{5\rho_i}(x_i)|} dt\right]^\alpha = \left[\frac{\left|\left\{z \in Q^1_i : B(|\nabla u|) > N_1 B(\lambda)\right\}\right|}{|Q^1_i|}\right]^\alpha \le C\epsilon^\alpha,\end{aligned}$$

where we have used Lemma 2.9, Hölder's inequality and (2.26). Therefore, we conclude that

$$\int_{\left\{z\in Q_i^1: B(|\nabla u|)>N_1 B(\lambda)\right\}} w(x)\,dxdt \le C\epsilon^\alpha w\left(B_{5\rho_i}(x_i)\right)\frac{\lambda^2}{B(\lambda)}\rho_i^2 \le C\epsilon^\alpha w\left(B_{\rho_i}(x_i)\right)\frac{\lambda^2}{B(\lambda)}\rho_i^2.$$

So, we can deduce the following result from Lemma 2.13 (3), the fact that the cylinders $\{Q_i^0\}$ are disjoint and Lemma 2.14

$$\begin{aligned}
&\int_{\{z\in Q_1: B(|\nabla u|)>N_1 B(\lambda)\}} w(x)\,dxdt \\
&\quad\le \sum_{i\in\mathbb{N}} \int_{\left\{z\in Q_i^1: B(|\nabla u|)>N_1 B(\lambda)\right\}} w(x)\,dxdt \\
&\quad\le C\epsilon^\alpha \sum_{i\in\mathbb{N}} w\left(B_{\rho_i}(x_i)\right)\frac{\lambda^2}{B(\lambda)}\rho_i^2 \\
&\quad\le \sum_{i\in\mathbb{N}} \frac{C\epsilon^\alpha}{[B(\lambda)]^{q_0}} \int_{\left\{z\in Q_i^0: B(|\nabla u|)>\frac{B(\lambda)}{4}\right\}} \left[B\left(|\nabla u|\right)\right]^{q_0} w(x)dxdt \\
&\qquad + \sum_{i\in\mathbb{N}} \frac{C\epsilon^\alpha}{\delta^{q_0}\left[B(\lambda)\right]^{q_0}} \int_{\left\{z\in Q_i^0: B(|\mathbf{f}|)>\frac{B(\lambda)}{4\delta}\right\}} \left[B\left(|\mathbf{f}|\right)\right]^{q_0} w(x)dxdt \\
&\quad\le \frac{C\epsilon^\alpha}{[B(\lambda)]^{q_0}} \int_{\left\{z\in Q_2: B(|\nabla u|)>\frac{B(\lambda)}{4}\right\}} \left[B\left(|\nabla u|\right)\right]^{q_0} w(x)dxdt \\
&\qquad + \frac{C\epsilon^\alpha}{\delta^{q_0}\left[B(\lambda)\right]^{q_0}} \int_{\left\{z\in Q_2: B(|\mathbf{f}|)>\frac{B(\lambda)}{4\delta}\right\}} \left[B\left(|\mathbf{f}|\right)\right]^{q_0} w(x)dxdt.
\end{aligned}$$

Thus, we finish the proof. □

In the following it is sufficient to consider the proof of Theorem 1.4 as an a priori estimate, therefore assuming a priori that $B\left(|\nabla u|\right) \in L^q(0,T;L^q_w(\Omega))$ locally. This assumption can be removed by an approximation argument like the one in [1, 21]. Now we are ready to prove the main result, Theorem 1.4.

Proof. In light of Lemma 2.12, we find

$$\begin{aligned}
&\int_{Q_1} \left[B\left(|\nabla u|\right)\right]^q w(x)\,dxdt \\
&\quad= qN_1^q \int_0^{+\infty} [B(\lambda)]^{q-1} \int_{\{z\in Q_1: B(|\nabla u|)>N_1 B(\lambda)\}} w(x)\,dxdtdB(\lambda) \\
&\quad= qN_1^q \int_0^{\lambda_*} [B(\lambda)]^{q-1} \int_{\{z\in Q_1: B(|\nabla u|)>N_1 B(\lambda)\}} w(x)\,dxdtdB(\lambda) \\
&\qquad + qN_1^q \int_{\lambda_*}^{+\infty} [B(\lambda)]^{q-1} \int_{\{z\in Q_1: B(|\nabla u|)>N_1 B(\lambda)\}} w(x)\,dxdtdB(\lambda) \\
&\quad=: J_1 + J_2.
\end{aligned}$$

Estimate of J_1. Recalling the definitions of λ_* and λ_0, we estimate

$$J_1 \le C\left\{B\left(\left[\left\{\int_{Q_2} \left[B\left(|\nabla u|\right)\right]^{q_0} w(x)dz\right\}^{\frac{1}{q_0}} + \frac{1}{\delta}\left\{\int_{Q_2} \left[B\left(|\mathbf{f}|\right)\right]^{q_0} w(x)\,dz\right\}^{\frac{1}{q_0}} + 1\right]^{\frac{1}{2}}\right)\right\}^q$$

$$\leq C\left[\left\{\int_{Q_2}[B(|\nabla u|)]^{q_0}w(x)dz\right\}^{\frac{1}{q_0}}+\frac{1}{\delta}\left\{\int_{Q_2}[B(|\mathbf{f}|)]^{q_0}w(x)\,dz\right\}^{\frac{1}{q_0}}+1\right]^{\frac{q(2+s_a)}{2}}$$

$$\leq C\left[\left\{\int_{Q_2}[B(|\nabla u|)]^{q_0}w(x)dz\right\}^{\frac{1}{q_0}}+\left\{\int_{Q_2}[B(|\mathbf{f}|)]^{q}w(x)\,dz\right\}^{\frac{1}{q}}+1\right]^{\frac{q(2+s_a)}{2}},$$

where $C=C(n,i_a,s_a,q,\delta,w)$, since

$$\int_{Q_2}[B(|\mathbf{f}|)]^{q_0}w(x)\,dz=\int_{Q_2}[B(|\mathbf{f}|)]^{1+\epsilon}w(x)^{\frac{1+\epsilon}{q}}w(x)^{\frac{q-1-\epsilon}{q}}dz\leq C\left(\int_{Q_2}[B(|\mathbf{f}|)]^{q}w(x)\,dz\right)^{\frac{1+\varepsilon}{q}}$$

by using Hölder's inequality and the fact that $q_0=1+\epsilon$.

Estimate of J_2. Now we apply Lemma 2.12 and Lemma 2.18 to find that

$$\begin{aligned}J_2\leq C\epsilon^{\alpha}&\left\{\int_0^\infty[B(\lambda)]^{q-q_0-1}\int_{\left\{z\in Q_2:B(|\nabla u|)>\frac{B(\lambda)}{4}\right\}}[B(|\nabla u|)]^{q_0}w(x)dxdtdB(\lambda)\right.\\&\left.+\frac{1}{\delta^{q_0}}\int_0^\infty[B(\lambda)]^{q-q_0-1}\int_{\left\{z\in Q_2:B(|\mathbf{f}|)>\frac{B(\lambda)}{4\delta}\right\}}[B(|\mathbf{f}|)]^{q_0}w(x)\,dxdtdB(\lambda)\right\}\\\leq C_1\epsilon^{\alpha}&\int_{Q_2}[B(|\nabla u|)]^{q}w(x)dz+C_2\int_{Q_2}[B(|\mathbf{f}|)]^{q}w(x)dz,\end{aligned}$$

where $C_1=C_1(n,i_a,s_a,q,\Lambda)$ and $C_2=C_2(n,i_a,s_a,q,\Lambda,\epsilon,\delta)$. Therefore, we combine the estimates of J_1 and J_2 to obtain

$$\begin{aligned}&\int_{Q_1}[B(|\nabla u|)]^{q}w(x)\,dz\\&\quad\leq C_3\left[\left\{\int_{Q_2}[B(|\nabla u|)]^{q_0}w(x)dz\right\}^{\frac{1}{q_0}}+\left\{\int_{Q_2}[B(|\mathbf{f}|)]^{q}w(x)\,dz\right\}^{\frac{1}{q}}+1\right]^{\frac{q(2+s_a)}{2}}\\&\qquad+C_1\epsilon^{\alpha}\int_{Q_2}[B(|\nabla u|)]^{q}w(x)\,dz,\end{aligned}$$

where $C_3=C_3(n,i_a,s_a,q,\Lambda,\epsilon,\delta,w)$. Selecting proper $\epsilon>0$ small enough and using a covering and iteration argument, we can finish the proof of Theorem 1.4. □

Statements

All data generated or analysed during this study are included in this article. Moreover, the author states that there is no conflict of interest.

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Cauchy problem for nonlocal diffusion equations modelling Lévy flights

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Abstract. In the present paper, we study the time-space fractional diffusion equation involving the Caputo differential operator and the fractional Laplacian. This equation describes the Lévy flight with the Brownian motion component and the drift component. First, the asymptotic behavior of the fundamental solution of the fractional diffusion equation is considered. Then, we use the fundamental solution to obtain the representation formula of solutions of the Cauchy problem. In the last, the L^2-decay estimates for solutions are proved by employing the Fourier analysis technique.

Keywords: Caputo differential operator, fractional Laplacian, Cauchy problem, nonlocal diffusion equation, representation of solution, asymptotic behavior, Lévy flight.

2020 Mathematics Subject Classification: 35R11, 35A08, 35B40, 35C15, 45K05, 47G20.

1 Introduction

1.1 Statement of the problem

In this paper, we consider the Cauchy problem for the fractional diffusion equation

$$\partial_t^\alpha u(t,x) = -(-\triangle)^\beta u(t,x) + b \cdot \nabla u(t,x) + h\triangle u(t,x), \qquad t>0,\ x \in \mathbb{R}^n, \tag{1.1}$$

$$u(0,x) = u_0(x), \qquad x \in \mathbb{R}^n, \tag{1.2}$$

where $n \in \mathbb{N}, \alpha \in (0,1], \beta \in (0,1), h>0$ and $b \in \mathbb{R}^n$. Here ∂_t^α denotes the Caputo fractional differential operator defined by [16] ∂_t^1 being the classical differential operator and

$$\partial_t^\alpha v(t) = \frac{d}{dt} J_t^{1-\alpha}(v - v(0))(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} (v(s) - v(0)) ds, \qquad t>0$$

for $\alpha \in (0,1)$, where J_t^a is the Riemann–Liouville fractional integral operator of order $a \geq 0$ defined by [16] J_t^0 being the identity operator and

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$$J_t^a v(t) = \frac{1}{\Gamma(a)} \int_0^t (t-s)^{a-1} v(s) ds$$

for $a > 0$. Also, $(-\triangle)^\beta$ is the fractional Laplacian defined by

$$(-\triangle)^\beta v(x) = F^{-1}(|\xi|^{2\beta} F v(\xi))(x), \qquad x \in \mathbb{R}^n$$

for $\beta \in (0,1)$. Here $\mathscr{F}$ and $\mathscr{F}^{-1}$ are respectively Fourier transform and inverse Fourier transform defined by [16,27]

$$\mathscr{F} v(\xi) = \tilde{v}(\xi) = \int_{\mathbb{R}^n} v(x) e^{-ix\xi} dx, \qquad \xi \in \mathbb{R}^n,$$
$$\mathscr{F}^{-1} v(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} v(\xi) e^{ix\xi} d\xi, \qquad x \in \mathbb{R}^n.$$

1.2 Physical background

The equation (1.1) is derived from the continuous time random walk (CTRW in short) theory, which is characterized by the waiting time probability density function (PDF in short) $\psi(t)$ and the jump length PDF $\omega(x)$. The famous Montroll–Weiss equation is given in the Fourier–Laplace space by [21]

$$\hat{\tilde{u}}(s,\xi) = \frac{1-\hat{\psi}(s)}{s} \frac{1}{1-\hat{\psi}(s)\tilde{\omega}(\xi)}, \tag{1.3}$$

where $\hat{\psi}(s)$ means the Laplace transform of $\psi(t)$ defined by [16]

$$\hat{\psi}(s) = \mathscr{L}\psi(s) = \int_0^\infty e^{-ts}\psi(t) dt$$

and $\hat{\tilde{u}}(s,\xi)$ stands for the Fourier–Laplace transform of the PDF $u(t,x)$ of being at position x at time t.

Take

$$\hat{\psi}(s) = \frac{1}{1+s^\alpha}, \qquad \alpha \in (0,1], \tag{1.4}$$
$$\tilde{\omega}(\xi) = e^{-\zeta_\beta(\xi)}, \tag{1.5}$$

where ζ_β is defined by

$$\zeta_\beta(\xi) = |\xi|^{2\beta} + h|\xi|^2 - ib \cdot \xi. \tag{1.6}$$

Since $\frac{1}{1+s^\alpha}$ is a Stieltjes function, $\psi(t)$ becomes a PDF. In fact, $\psi(t)$ is the Mittag-Leffler PDF given by [11]

$$\psi(t) = \mathscr{L}^{-1}\left(\frac{1}{1+s^\alpha}\right)(t) = t^{\alpha-1} E_{\alpha,\alpha}(-t^\alpha). \tag{1.7}$$

When $\alpha = 1$, $\psi(t)$ becomes a Poisson PDF. By Lemma 6.9 in [17], the function ζ_β is negative definite. It follows from the negative definiteness of ζ_β that $e^{-\zeta_\beta(\xi)}$ is a positive definite function. For details of the Bernstein function theory, see [28]. Then, by the Bochner theorem, $\omega(x)$ becomes a PDF. Also, we have

$$e^{-\zeta_\beta(\xi)} \to 1 - \zeta_\beta(\xi), \qquad \xi \to 0. \tag{1.8}$$

Combining (1.8) with (1.3) and (1.4), we deduce

$$\hat{u}(s,\xi) = \frac{s^{\alpha-1}}{s^\alpha + \zeta_\beta(\xi)}. \tag{1.9}$$

We can rewrite (1.9) as

$$s^\alpha \hat{u}(s,\xi) - s^{\alpha-1} = -\zeta_\beta(\xi)\hat{u}(s,\xi).$$

Taking the inverse Fourier–Laplace transform, we obtain the equation (1.1).

In [3,7], Cartea and del-Castillo-Negrete use the jump length PDF in the Lévy–Khintchine representation to derive a general time-space nonlocal diffusion equation including (1.1). Also, employing the general nonlocal diffusion equation, they described the tempered Lévy flight which looks like a Lévy process in a small time and behaves like a Brownian random walk in a large time. Meerschaert et al. also considered general nonlocal diffusion equations including (1.1) in [20]. In [6], del-Castillo-Negrete and Cartea modelled the resistive pressure-gradient-driven plasma turbulence by employing the time-space fractional diffusion equation.

The equation (1.1) captures the Lévy flight with the Brownian motion component and the drift component. In (1.1), the fractional Laplacian and the classical Laplacian mean the jump component and the Brownian motion component respectively. Also, the gradient in (1.1) stands for the drift component. For details of the Lévy process, see [27].

1.3 State of the art

In [2], Blumenthal and Getoor established the following estimate for the transition density of the 2β-stable Lévy process

$$u(t,x) \sim \min\{t^{-\frac{n}{2\beta}}, t|x|^{-n-2\beta}\}. \tag{1.10}$$

The transition density corresponds to the fundamental solution of the equation (1.1) of the case: $\alpha = 1, \beta \in (0,1)$, $b = h = 0$. In [12], Ignat and Rossi used the energy method to obtain the decay estimate results for solutions of the space fractional diffusion equation. Kaleta and Sztonyk [13] studied the asymptotic behavior of transition density and its derivatives for the tempered Lévy flight.

Eidelman and Kochubei [8] obtained the various estimates for fundamental solutions of the time fractional diffusion equation. In [26], the existence of solutions and the large time behavior for the initial boundary value problem of the Caputo time fractional diffusion-wave equation were investigated. In [14, 31, 32], the optimal decay estimates for solutions of time nonlocal diffusion equations were established.

In [18], Mainardi, Luchko and Pagnini studied the analytical properties of fundamental solutions of the time-space fractional diffusion-wave equation involving the Caputo differential operator and the Riesz–Feller operator. Chen, Meerschaert and Nane [4] established the probabilistic representations for solutions of the equation (1.1) of the case: $\alpha \in (0,1], \beta \in (0,1]$, $b = h = 0$. In [15], Kemppainen, Siljander and Zacher used the properties of the Fox H-function and the Fourier analysis technique to prove the results for the asymptotic behavior of solutions of the equation

$$\partial_t^\alpha u(t,x) = -(-\triangle)^\beta u(t,x) + f(t,x), \qquad t > 0,\ x \in \mathbb{R}^n. \tag{1.11}$$

Allen, Caffarelli and Vasseur [1] studied the Hölder regularity for the nonlocal diffusion equation with the Caputo fractional derivative and a generalization of the fractional Laplacian. By employing the Laplace transform, Cheng, Li and Yamamoto [5] obtained the large time behavior result for initial value problem and initial boundary value problem of the time-space fractional diffusion-reaction equation. In [30], the author proved the existence of solutions of the time-space nonlocal diffusion equation involving the generalized Caputo-type differential operator and the generalized fractional Laplacian introduced in [29]. We mention also [10, 22–25], where analytical solutions of several time-space fractional diffusion-wave equations were established.

The goal of this paper is to obtain the asymptotic behavior of solutions of the Cauchy problem (1.1)–(1.2).

1.4 Outline

This paper is organized as follows.

In Section 2, we give necessary concepts and lemmas for obtaining the main results of the paper.

In Section 3, we study the asymptotic behavior of fundamental solutions of the equation (1.1). The asymptotic behavior result for the fundamental solution shows that the equation (1.1) captures the Lévy flight which looks like a Brownian random walk in a short time and behaves like a Lévy process in a long time.

In Section 4, we prove the representation formula for solutions of the Cauchy problem (1.1)–(1.2) by using the fundamental solution and the properties of the Wright function.

In Section 5, we obtain the L^2-decay estimates for solutions of the Cauchy problem (1.1)–(1.2) by employing the Fourier analysis method.

2 Preliminaries

First of all, we introduce some basic notations. Throughout this paper, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ will mean the sets of natural, real and complex numbers respectively. $C > 0$ stands for a universal positive constant which can be different at different places. Also, $a \lesssim b$ means $a \leq Cb$ for some constant $C > 0$ and $a \gtrsim b$ denotes $a \geq Cb$ for some constant $C > 0$. In addition, we write $a \sim b$ if $a \lesssim b \lesssim a$.

Let $a, b \in \mathbb{C}$ and $Re(a) > 0$. The two parameter Mittag-Leffler function is defined by [16]

$$E_{a,b}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(aj+b)}, \qquad z \in \mathbb{C}.$$

Lemma 2.1 ([16,32])**.** *Let $k \in \mathbb{C}, a > 0$ and $j \in \mathbb{N}$. Then*

$$\partial_t^a E_{a,1}(kt^a) = kE_{a,1}(kt^a), \qquad t > 0. \tag{2.1}$$

Proof. The relation (2.1) was proved in [16]. □

Lemma 2.2. *Let $a \in (0,2)$ and $b \in \mathbb{R}$. Suppose that μ is such that $\pi a/2 < \mu < \min\{\pi, \pi a\}$. Then there exists a constant $C = C(a,b,\mu) > 0$ such that*

$$|E_{a,b}(z)| \leq \frac{C}{1+|z|}, \qquad \mu \leq |\arg(z)| \leq \pi. \tag{2.2}$$

Let $a > -1$ and $b \in \mathbb{C}$. The Wright function $W_{a,b}$ is defined by [16]

$$W_{a,b}(t) = \sum_{j=0}^{\infty} \frac{z^j}{j!\Gamma(aj+b)}, \qquad z \in \mathbb{C}. \tag{2.3}$$

Let $r \in (0,1)$. The functions F_r and M_r are special cases of the Wright function defined by [19]

$$F_r(z) = W_{-r,0}(-z) = \sum_{j=1}^{\infty} \frac{(-z)^j}{j!\Gamma(-rj)}, \qquad z \in \mathbb{C}, \tag{2.4}$$

$$M_r(z) = W_{-r,1-r}(-z) = \sum_{j=0}^{\infty} \frac{(-z)^j}{j!\Gamma(-rj+1-r)}, \qquad z \in \mathbb{C}. \tag{2.5}$$

The functions F_r and M_r are related through

$$F_r(z) = rzM_r(z), \qquad z \in \mathbb{C}. \tag{2.6}$$

By the relation $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$, the following equality holds.

$$F_r(z) = \frac{1}{\pi}\sum_{j=1}^{\infty}(-1)^{j-1}z^j \frac{\Gamma(j\alpha+1)}{j!}\sin(j\pi\alpha), \qquad z \in \mathbb{C}. \tag{2.7}$$

Also, the following relations hold [19].

$$M_r\left(\frac{t}{r}\right) \approx \frac{1}{\sqrt{2\pi(1-r)}} t^{\frac{r-\frac{1}{2}}{1-r}} e^{-\frac{1-r}{r}t^{\frac{1}{1-r}}}, \qquad t \to +\infty. \tag{2.8}$$

$$\int_0^{\infty} M_r(t)dt = 1. \tag{2.9}$$

$$\frac{dW_{a,b}(t)}{dt} = W_{a,a+b}(t). \tag{2.10}$$

By (2.8) and (2.6), we can easily see the asymptotic behavior of the function $F_r(t)$.

$$e^{-\tau s^{\alpha}} = \int_0^{\infty} e^{-st}\theta_{\alpha}(t,\tau)dt, \qquad \tau, s > 0, \tag{2.11}$$

where

$$\theta_{\alpha}(t,\tau) = \frac{1}{\pi}\sum_{j=1}^{\infty}(-1)^{j-1}\tau^j t^{-\alpha j-1}\frac{\Gamma(j\alpha+1)}{j!}\sin(j\pi\alpha), \qquad t,\tau > 0. \tag{2.12}$$

By (2.7) and (2.4),

$$\theta_{\alpha}(t,\tau) = \frac{1}{t}F_{\alpha}\left(\frac{\tau}{t^{\alpha}}\right) = \frac{1}{t}W_{-\alpha,0}\left(-\frac{\tau}{t^{\alpha}}\right), \qquad t,\tau > 0. \tag{2.13}$$

From (2.11), we obtain

$$\lim_{t\to 0}\theta_{\alpha}(t,\tau) = \lim_{s\to\infty} se^{-\tau s^{\alpha}} = 0, \qquad \tau > 0. \tag{2.14}$$

Let $Z_{\alpha,\beta}$ stand for the fundamental solution of the following time-space fractional diffusion equation

$$\partial_t^{\alpha}u(t,x) = -(-\triangle)^{\beta}u(t,x), \qquad t > 0,\ x \in \mathbb{R}^n. \tag{2.15}$$

Let $n \in \mathbb{N}$ and $b > 0$. Denote

$$\bar{p}(n,b) := \begin{cases} \frac{n}{n-2b}, & n > 2b, \\ \infty, & \text{otherwise}. \end{cases}$$

3 Fundamental solution of fractional diffusion equation

In this section, we consider the fundamental solution of the fractional diffusion equation (1.1).

3.1 Fundamental solution of space fractional diffusion equation

In this subsection, we discuss the space fractional diffusion equation of the form

$$\frac{\partial u(t,x)}{\partial t} = -(-\triangle)^{\beta} u(t,x) + b \cdot \nabla u(t,x) + h\triangle u(t,x), \qquad t > 0,\ x \in \mathbb{R}^n, \tag{3.1}$$

which is corresponding to the equation (1.1) when $\alpha = 1$ and $\beta \in (0,1)$. Applying the Fourier transform to (3.1) with respect to the space variable x, we obtain

$$\frac{\partial \tilde{u}(t,\xi)}{\partial t} = -(|\xi|^{2\beta} + h|\xi|^2 - ib \cdot \xi)\tilde{u}(t,\xi), \qquad t > 0,\ \xi \in \mathbb{R}^n. \tag{3.2}$$

The solution of the equation (3.2) with the condition $\tilde{u}(0,\xi) = 1$ has the form:

$$\tilde{u}(t,\xi) = e^{-(|\xi|^{2\beta} + h|\xi|^2 - ib\cdot\xi)t}.$$

For convenience, we write $\zeta_\beta(\xi) = |\xi|^{2\beta} + h|\xi|^2 - ib\cdot\xi$. The fundamental solution $A_{1,\beta}$ of the equation (3.1) is represented by

$$A_{1,\beta}(t,x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-\zeta_\beta(\xi)t}\cos(x\xi)d\xi = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-(|\xi|^{2\beta}+h|\xi|^2)t}\cos((x+bt)\xi)d\xi. \tag{3.3}$$

Since $e^{-(|\xi|^{2\beta}+h|\xi|^2-ib\cdot\xi)t}$ is positive definite, it follows from the Bochner theorem that $A_{1,\beta}(t,x) \geq 0$. Also, the following relation holds.

$$e^{-(|\xi|^{2\beta}+h|\xi|^2-ib\cdot\xi)t} = \int_{\mathbb{R}^n} A_{1,\beta}(t,x)\cos(x\xi)dx = \tilde{A}_{1,\beta}(t,\xi). \tag{3.4}$$

Moreover,

$$\int_{\mathbb{R}^n} A_{1,\beta}(t,x)dx = 1, \qquad t > 0.$$

If $h = 0$, then

$$A_{1,\beta}(t,x) = Z_{1,\beta}(t, x+bt). \tag{3.5}$$

If $h \neq 0$, then

$$A_{1,\beta}(t,x) = \frac{1}{(4\pi ht)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-\frac{|x+bt-y|^2}{4ht}} Z_{1,\beta}(t,y)dy. \tag{3.6}$$

Theorem 3.1. *Let $n \in \mathbb{N}, \beta \in (0,1), h > 0$ and $b \in \mathbb{R}^n$. Then the following relations hold.*

$$A_{1,\beta}(t,x) \lesssim \begin{cases} t^{-\frac{n}{2}}, & \text{if } x \in \mathbb{R}^n \text{ and } t \in (0,1], \\ t^{-\frac{n}{2\beta}}, & \text{if } x \in \mathbb{R}^n \text{ and } t \in (1,\infty). \end{cases}$$

$$A_{1,\beta}(t,x-bt) \lesssim \begin{cases} t\left(|x| - t^{\frac{1}{2\beta}}\right)^{-n-2}, & \text{if } |x| > t^{\frac{1}{2\beta}} \text{ and } t \in (0,1], \\ t\left(|x| - t^{\frac{1}{2\beta}}\right)^{-n-2\beta}, & \text{if } |x| > t^{\frac{1}{2\beta}} \text{ and } t \in (1,\infty). \end{cases}$$

Proof. The asymptotic behavior (1.10) of $Z_{1,\beta}(t,x)$ is the following.

$$Z_{1,\beta}(t,x) \sim \min\{t^{-\frac{n}{2\beta}}, t|x|^{-n-2\beta}\}. \tag{3.7}$$

By (3.7), we have

$$\begin{aligned} A_{1,\beta}(t,x-bt) &= \frac{1}{(4\pi ht)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-\frac{(|x-y|)^2}{4ht}} Z_{1,\beta}(t,y)dy \\ &\sim \frac{1}{(4\pi ht)^{\frac{n}{2}}}\int_{|y|<t^{\frac{1}{2\beta}}} e^{-\frac{|x-y|^2}{4ht}} t^{-\frac{n}{2\beta}}dy + \frac{1}{(4\pi ht)^{\frac{n}{2}}}\int_{|y|>t^{\frac{1}{2\beta}}} e^{-\frac{|x-y|^2}{4ht}} t|y|^{-n-2\beta}dy \\ &\sim t^{-\frac{n}{2\beta}-\frac{n}{2}}\int_{|y|<t^{\frac{1}{2\beta}}} e^{-\frac{|x-y|^2}{4ht}}dy + t^{1-\frac{n}{2}}\int_{|y|>t^{\frac{1}{2\beta}}} e^{-\frac{|x-y|^2}{4ht}}|y|^{-n-2\beta}dy. \end{aligned} \tag{3.8}$$

First, we estimate the integral

$$\int_{|y|<t^{\frac{1}{2\beta}}} e^{-\frac{|x-y|^2}{4ht}}dy.$$

For $x \in \mathbb{R}^n$, we have

$$\int_{|y|<t^{\frac{1}{2\beta}}} e^{-\frac{|x-y|^2}{4ht}}dy \lesssim t^{\frac{n}{2\beta}}.$$

For $x \in \mathbb{R}^n$, we obtain

$$\int_{|y|<t^{\frac{1}{2\beta}}} e^{-\frac{|x-y|^2}{4ht}}dy \lesssim \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4ht}}dy \lesssim (ht)^{\frac{n}{2}} \lesssim t^{\frac{n}{2}}.$$

For $|x| > t^{\frac{1}{2\beta}}$, we estimate

$$\int_{|x-z|<t^{\frac{1}{2\beta}}} e^{-\frac{|z|^2}{4ht}}dz \le e^{-\frac{\left(|x|-t^{\frac{1}{2\beta}}\right)^2}{4ht}}\int_{|x-z|<t^{\frac{1}{2\beta}}} dz \lesssim t^{\frac{n}{2\beta}} e^{-\frac{\left(|x|-t^{\frac{1}{2\beta}}\right)^2}{4ht}}.$$

For $|x| > t^{\frac{1}{2\beta}}$ and $m > 0$, we deduce

$$\begin{aligned} \int_{|y|<t^{\frac{1}{2\beta}}} e^{-\frac{|x-y|^2}{4ht}}dy &\lesssim \int_{|y|<t^{\frac{1}{2\beta}}} \frac{t^{\frac{m}{2}}}{|x-y|^m}dy \lesssim \int_0^{t^{\frac{1}{2\beta}}} \frac{t^{\frac{m}{2}}}{(|x|-r)^m}r^{n-1}dr \\ &\lesssim t^{\frac{m}{2}}\left(|x|-t^{\frac{1}{2\beta}}\right)^{-m}\int_0^{t^{\frac{1}{2\beta}}} r^{n-1}dr \lesssim t^{\frac{m}{2}}t^{\frac{n}{2\beta}}\left(|x|-t^{\frac{1}{2\beta}}\right)^{-m}. \end{aligned}$$

Setting $m = n+2$,

$$\int_{|y|<t^{\frac{1}{2\beta}}} e^{-\frac{|x-y|^2}{4ht}}dy \lesssim t^{\frac{n}{2\beta}+\frac{n}{2}+1}\left(|x|-t^{\frac{1}{2\beta}}\right)^{-n-2}.$$

Setting $m = n+2\beta$,

$$\int_{|y|<t^{\frac{1}{2\beta}}} e^{-\frac{|x-y|^2}{4ht}}dy \lesssim t^{\frac{n}{2\beta}+\frac{n}{2}+\beta}\left(|x|-t^{\frac{1}{2\beta}}\right)^{-n-2\beta}.$$

Next, we estimate the integral

$$\int_{|y|>t^{\frac{1}{2\beta}}} e^{-\frac{|x-y|^2}{4ht}}|y|^{-n-2\beta}dy.$$

For $x \in \mathbb{R}^n$, we obtain

$$\int_{|y|>t^{\frac{1}{2\beta}}} e^{-\frac{|x-y|^2}{4ht}} |y|^{-n-2\beta} dy \lesssim t^{\frac{-n-2\beta}{2\beta}} \int_{|y|>t^{\frac{1}{2\beta}}} e^{-\frac{|x-y|^2}{4ht}} dy \lesssim t^{\frac{-n-2\beta}{2\beta}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4ht}} dy \lesssim t^{-\frac{n}{2\beta}+\frac{n}{2}-1}.$$

For $x \in \mathbb{R}^n$, we have

$$\int_{|y|>t^{\frac{1}{2\beta}}} e^{-\frac{|x-y|^2}{4ht}} |y|^{-n-2\beta} dy \leq \int_{|y|>t^{\frac{1}{2\beta}}} |y|^{-n-2\beta} dy \lesssim t^{-1}.$$

If $|x| > t^{\frac{1}{2\beta}}$, then

$$\begin{aligned}\int_{|y|>t^{\frac{1}{2\beta}}} e^{-\frac{|x-y|^2}{4ht}} |y|^{-n-2\beta} dy &= \int_{|y-x|<\frac{|x|-t^{\frac{1}{2\beta}}}{2}} e^{-\frac{|x-y|^2}{4ht}} |y|^{-n-2\beta} dy \\ &\quad + \int_{|y-x|>\frac{|x|-t^{\frac{1}{2\beta}}}{2},\, |y|>t^{\frac{1}{2\beta}}} e^{-\frac{|x-y|^2}{4ht}} |y|^{-n-2\beta} dy.\end{aligned}$$

Then, for $|x| > t^{\frac{1}{2\beta}}$,

$$\begin{aligned}\int_{|y|>t^{\frac{1}{2\beta}}} e^{-\frac{|x-y|^2}{4ht}} |y|^{-n-2\beta} dy &\leq \left(\frac{|x|+t^{\frac{1}{2\beta}}}{2}\right)^{-n-2\beta} \int_0^{\frac{|x|-t^{\frac{1}{2\beta}}}{2}} e^{-\frac{r^2}{4ht}} r^{n-1} dr \\ &\quad + e^{-\frac{\left(|x|-t^{\frac{1}{2\beta}}\right)^2}{16ht}} \int_{|y|>t^{\frac{1}{2\beta}}} |y|^{-n-2\beta} dy \\ &\lesssim (4ht)^{\frac{n}{2}} \left(\frac{|x|+t^{\frac{1}{2\beta}}}{2}\right)^{-n-2\beta} + \frac{1}{2\beta t} e^{-\frac{\left(|x|-t^{\frac{1}{2\beta}}\right)^2}{16ht}} \\ &\lesssim t^{\frac{n}{2}} \left(|x|+t^{\frac{1}{2\beta}}\right)^{-n-2\beta} + t^{-1} e^{-\frac{\left(|x|-t^{\frac{1}{2\beta}}\right)^2}{16ht}}.\end{aligned}$$

Also, for $|x| > t^{\frac{1}{2\beta}}$,

$$\begin{aligned}\int_{|y|>t^{\frac{1}{2\beta}}} e^{-\frac{|x-y|^2}{4ht}} |y|^{-n-2\beta} dy &\leq \left(\frac{|x|+t^{\frac{1}{2\beta}}}{2}\right)^{-n-2} \int_0^{\frac{|x|-t^{\frac{1}{2\beta}}}{2}} e^{-\frac{r^2}{4ht}} r^{2-2\beta+n-1} dr \\ &\quad + e^{-\frac{\left(|x|-t^{\frac{1}{2\beta}}\right)^2}{16ht}} \int_{|y|>t^{\frac{1}{2\beta}}} |y|^{-n-2\beta} dy \\ &\lesssim (4ht)^{\frac{n}{2}+1-\beta} \left(\frac{|x|+t^{\frac{1}{2\beta}}}{2}\right)^{-n-2} + \frac{1}{2\beta t} e^{-\frac{\left(|x|-t^{\frac{1}{2\beta}}\right)^2}{16ht}} \\ &\lesssim t^{\frac{n}{2}+1-\beta} \left(|x|+t^{\frac{1}{2\beta}}\right)^{-n-2} + t^{-1} e^{-\frac{\left(|x|-t^{\frac{1}{2\beta}}\right)^2}{16ht}}.\end{aligned}$$

Combining the previous estimates, we obtain the following results.
If $|x| > t^{\frac{1}{2\beta}}$ and $t \in (0,1]$, then, by (3.8), for $m > 0$,

$$\begin{aligned}A_{1,\beta}(t, x-bt) &\lesssim t^{-\frac{n}{2\beta}-\frac{n}{2}} t^{\frac{n}{2\beta}} e^{-\frac{\left(|x|-t^{\frac{1}{2\beta}}\right)^2}{4ht}} + t^{1-\frac{n}{2}} \left(t^{\frac{n}{2}+1-\beta} \left(|x|+t^{\frac{1}{2\beta}}\right)^{-n-2} + t^{-1} e^{-\frac{\left(|x|-t^{\frac{1}{2\beta}}\right)^2}{16ht}} \right) \\ &\lesssim t^{-\frac{n}{2}} t^{\frac{m}{2}} \left(|x|-t^{\frac{1}{2\beta}}\right)^{-m} + t^{2-\beta} \left(|x|+t^{\frac{1}{2\beta}}\right)^{-n-2} + t^{-\frac{n}{2}} t^{\frac{m}{2}} \left(|x|-t^{\frac{1}{2\beta}}\right)^{-m}.\end{aligned}$$

Setting $m = n + 2$, we deduce

$$A_{1,\beta}(t, x - bt) \lesssim t\big(|x| - t^{\frac{1}{2\beta}}\big)^{-n-2}.$$

If $x \in \mathbb{R}^n$ and $t \in (0, 1]$, then

$$A_{1,\beta}(t, x - bt) \lesssim t^{-\frac{n}{2\beta} - \frac{n}{2}} t^{\frac{n}{2\beta}} + t^{1-\frac{n}{2}} t^{-1} \lesssim t^{-\frac{n}{2}}.$$

If $|x| > t^{\frac{1}{2\beta}}$ and $t > 1$, then, by (3.8), for $m > 0$,

$$\begin{aligned} A_{1,\beta}(t, x - bt) &\lesssim t^{-\frac{n}{2\beta} - \frac{n}{2}} t^{\frac{n}{2\beta}} e^{-\frac{\left(|x| - t^{\frac{1}{2\beta}}\right)^2}{4ht}} + t^{1-\frac{n}{2}} \left(t^{\frac{n}{2}} \big(|x| + t^{\frac{1}{2\beta}}\big)^{-n-2\beta} + t^{-1} e^{-\frac{\left(|x| - t^{\frac{1}{2\beta}}\right)^2}{16ht}} \right) \\ &\lesssim t^{-\frac{n}{2}} t^{\frac{m}{2}} \big(|x| - t^{\frac{1}{2\beta}}\big)^{-m} + t\big(|x| + t^{\frac{1}{2\beta}}\big)^{-n-2\beta} + t^{-\frac{n}{2}} t^{\frac{m}{2}} \big(|x| - t^{\frac{1}{2\beta}}\big)^{-m}. \end{aligned}$$

Setting $m = n + 2\beta$, we deduce

$$A_{1,\beta}(t, x - bt) \lesssim t^{\beta}\big(|x| - t^{\frac{1}{2\beta}}\big)^{-n-2\beta} + t\big(|x| - t^{\frac{1}{2\beta}}\big)^{-n-2\beta} \lesssim t\big(|x| - t^{\frac{1}{2\beta}}\big)^{-n-2\beta}.$$

If $x \in \mathbb{R}^n$ and $t > 1$, then

$$A_{1,\beta}(t, x - bt) \lesssim t^{-\frac{n}{2\beta} - \frac{n}{2}} t^{\frac{n}{2}} + t^{1-\frac{n}{2}} t^{-1-\frac{n}{2\beta}+\frac{n}{2}} \lesssim t^{-\frac{n}{2\beta}}. \qquad \square$$

Remark 3.2. From the relation (3.7), we can easily see

$$\frac{Z_{1,\beta}(t, x)}{Z_{1,\beta}(t, x + bt)} \lesssim \max\{1, t^{n+2\beta-\frac{n}{2\beta}-1}\}, \quad \text{if } t > 0 \text{ and } x \in \mathbb{R}^n.$$

Then, from (3.6) and Theorem 3.1, we obtain the following result.

If $\beta \in [\frac{1}{2}, 1)$, then

$$A_{1,\beta}(t, x) \lesssim \begin{cases} t\big(|x| - t^{\frac{1}{2\beta}}\big)^{-n-2}, & \text{if } |x| > t^{\frac{1}{2\beta}} \text{ and } t \in (0, 1], \\ t^{n+2\beta-\frac{n}{2\beta}}\big(|x| - t^{\frac{1}{2\beta}}\big)^{-n-2\beta}, & \text{if } |x| > t^{\frac{1}{2\beta}} \text{ and } t \in (1, \infty). \end{cases}$$

If $\beta \in (0, \frac{1}{2})$, then

$$A_{1,\beta}(t, x) \lesssim \begin{cases} t^{n+2\beta-\frac{n}{2\beta}}\big(|x| - t^{\frac{1}{2\beta}}\big)^{-n-2}, & \text{if } |x| > t^{\frac{1}{2\beta}} \text{ and } t \in (0, 1], \\ t\big(|x| - t^{\frac{1}{2\beta}}\big)^{-n-2\beta}, & \text{if } |x| > t^{\frac{1}{2\beta}} \text{ and } t \in (1, \infty). \end{cases}$$

We consider the L^p-estimate of the fundamental solution $A_{1,\beta}$.

Theorem 3.3. *Let $n \in \mathbb{N}, \beta \in (0, 1), h > 0, b \in \mathbb{R}^n$ and $p \in [1, \infty]$. Then the following relation holds.*

$$\|A_{1,\beta}(t, \cdot)\|_{L^p(\mathbb{R}^n)} \lesssim \begin{cases} t^{-\frac{n}{2}(1-\frac{1}{p})}, & t \in (0, 1], \\ t^{-\frac{n}{2\beta}(1-\frac{1}{p})}, & t \in (1, \infty). \end{cases}$$

Proof. Using Theorem 3.1, we can easily obtain the result.
For $t \in (0,1]$, we obtain

$$\begin{aligned}\|A_{1,\beta}(t,\cdot)\|^p_{L^p(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} A_{1,\beta}(t,x)^p dx = \int_{|x|<t^{\frac{1}{2}}} A_{1,\beta}(t,x)^p dx + \int_{|x|>t^{\frac{1}{2}}} A_{1,\beta}(t,x)^p dx \\ &\lesssim t^{-\frac{np}{2}+\frac{n}{2}} + t^p \int_{t^{\frac{1}{2}}}^{\infty} \frac{r^{n-1}}{(r-t^{\frac{1}{2\beta}})^{np+2p}} dr \\ &\lesssim t^{-\frac{np}{2}+\frac{n}{2}} + t^p t^{\frac{n}{2}-\frac{np}{2}-\frac{2p}{2}} \int_1^{\infty} \frac{s^{n-1}}{(s-t^{\frac{1}{2\beta}-\frac{1}{2}})^{np+2p}} ds \lesssim t^{-\frac{np}{2}+\frac{n}{2}}.\end{aligned}$$

For $t > 1$, we have

$$\begin{aligned}\|A_{1,\beta}(t,\cdot)\|^p_{L^p(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} A_{1,\beta}(t,x)^p dx = \int_{|x|<2t^{\frac{1}{2\beta}}} A_{1,\beta}(t,x)^p dx + \int_{|x|>2t^{\frac{1}{2\beta}}} A_{1,\beta}(t,x)^p dx \\ &\lesssim t^{-\frac{np}{2\beta}+\frac{n}{2\beta}} + t^p \int_{2t^{\frac{1}{2\beta}}}^{\infty} \frac{r^{n-1}}{(r-t^{\frac{1}{2\beta}})^{np+2\beta p}} dr \\ &\lesssim t^{-\frac{np}{2\beta}+\frac{n}{2\beta}} + t^p t^{\frac{n}{2\beta}-\frac{np}{2\beta}-\frac{2\beta p}{2\beta}} \int_2^{\infty} \frac{s^{n-1}}{(s-1)^{np+2\beta p}} ds \lesssim t^{-\frac{np}{2\beta}+\frac{n}{2\beta}}.\end{aligned}$$

□

3.2 Fundamental solution of time-space fractional diffusion equation

In this subsection, we consider the time-space fractional diffusion equation (1.1) of the case: $\alpha \in (0,1)$ and $\beta \in (0,1)$. Let $A_{\alpha,\beta}$ denote the fundamental solution of the equation (1.1). Applying the Fourier transform to (1.1) with respect to the space variable x, we obtain

$$\partial_t^\alpha \tilde{u}(t,\xi) = -\zeta_\beta(\xi)\tilde{u}(t,\xi), \qquad t>0,\ \xi \in \mathbb{R}^n. \tag{3.9}$$

The solution of the equation (3.9) with the condition $\tilde{u}(0,\xi) = 1$ is of the form

$$\tilde{u}(t,\xi) = E_{\alpha,1}(-\zeta_\beta(\xi)t^\alpha).$$

By the Laplace transform, we obtain

$$\int_0^\infty E_{\alpha,1}(-\zeta_\beta(\xi)t^\alpha)e^{-st}dt = \frac{s^{\alpha-1}}{s^\alpha + \zeta_\beta(\xi)}. \tag{3.10}$$

Now we define the function $\phi(t,\tau)$ by

$$\phi(t,\tau) = J_t^{1-\alpha}\theta_\alpha(t,\tau) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\theta_\alpha(s,\tau)ds, \qquad t,\tau>0. \tag{3.11}$$

Then

$$\int_0^\infty \phi(t,\tau)e^{-ts}dt = s^{\alpha-1}e^{-\tau s^\alpha}, \qquad s,\tau>0$$

and

$$\lim_{t\to 0}\phi(t,\tau) = \lim_{s\to\infty} s^\alpha e^{-\tau s^\alpha} = 0, \qquad \tau>0.$$

From (4.26) in [18], we obtain

$$\phi(t,\tau) = \frac{1}{t^\alpha}M_\alpha\Big(\frac{\tau}{t^\alpha}\Big) = \frac{1}{t^\alpha}W_{-\alpha,1-\alpha}\Big(-\frac{\tau}{t^\alpha}\Big), \qquad t,\tau>0. \tag{3.12}$$

By the asymptotic behavior of M_α, we have

$$\lim_{\tau\to 0}\phi(t,\tau)=\lim_{\tau\to\infty}\phi(t,\tau)=0,\qquad t>0.$$

Also,

$$\begin{aligned}\int_0^\infty\int_0^\infty \phi(t,\tau)e^{-\zeta_\beta(\xi)\tau}d\tau e^{-st}dt &= \int_0^\infty\int_0^\infty \phi(t,\tau)e^{-st}dte^{-\zeta_\beta(\xi)\tau}d\tau\\ &=\int_0^\infty s^{\alpha-1}e^{-\tau s^\alpha}e^{-\zeta_\beta(\xi)\tau}d\tau=\frac{s^{\alpha-1}}{s^\alpha+\zeta_\beta(\xi)}.\end{aligned}\tag{3.13}$$

It follows from the uniqueness of the Laplace transform and (3.10) that

$$\tilde{u}(t,\xi)=E_{\alpha,1}(-\zeta_\beta(\xi)t^\alpha)=\int_0^\infty \phi(t,\tau)e^{-\zeta_\beta(\xi)\tau}d\tau,\qquad t>0,\ \xi\in\mathbb{R}^n.\tag{3.14}$$

From (3.7) and (3.12), we deduce

$$\begin{aligned}u(t,x)&=\frac{1}{(2\pi)^n}\int_0^\infty\phi(t,\tau)\int_{\mathbb{R}^n}e^{-\zeta_\beta(\xi)\tau}\cos(\xi x)d\xi d\tau=\int_0^\infty\phi(t,\tau)A_{1,\beta}(\tau,x)d\tau\\ &=\frac{1}{t^\alpha}\int_0^\infty M_\alpha\Big(\frac{\tau}{t^\alpha}\Big)A_{1,\beta}(\tau,x)d\tau=\int_0^\infty M_\alpha(s)A_{1,\beta}(st^\alpha,x)ds,\qquad t>0,\ x\in\mathbb{R}^n.\end{aligned}$$

Therefore the fundamental solution of the equation (1.1) is represented by

$$A_{\alpha,\beta}(t,x)=\int_0^\infty M_\alpha(s)A_{1,\beta}(st^\alpha,x)ds,\qquad t>0,\ x\in\mathbb{R}^n.\tag{3.15}$$

It follows from $A_{1,\beta}(s,x)\ge 0$ that $A_{\alpha,\beta}(t,x)\ge 0$. By the relation (2.9), we obtain

$$\int_{\mathbb{R}^n}A_{\alpha,\beta}(t,x)dx=\int_0^\infty M_\alpha(s)\int_{\mathbb{R}^n}A_{1,\beta}(st^\alpha,x)dxds=\int_0^\infty M_\alpha(s)ds=1,\qquad t>0.$$

Now we consider the asymptotic behavior of $A_{\alpha,\beta}(t,x)$ when $|x|>t^{\frac{\alpha}{2\beta}}$.

Lemma 3.4. *Let $n\in\mathbb{N},\alpha\in(0,1),\beta\in(0,1),h>0,b\in\mathbb{R}^n,t\in(0,1)$ and $|x|>t^{\frac{\alpha}{2\beta}}$. Then the following relation holds.*

$$A_{\alpha,\beta}(t,x)\lesssim\begin{cases}t^\alpha|x|^{-n-2}, & |x|\le 2,\\ t^\alpha|x|^{-n-2\beta}, & |x|>2.\end{cases}\tag{3.16}$$

Proof. Now we will prove the relation (3.16) when of $\beta\in[\frac12,1)$. If $\beta\in(0,\frac12)$, then the relation can be proved similarly.

First, we consider the case of $|x|\le 2$. It follows from Theorem 3.1 and Remark 3.2 that

$$\begin{aligned}A_{\alpha,\beta}(t,x)=&\int_0^{\frac{|x|^{2\beta}}{2^{2\beta}t^\alpha}}M_\alpha(s)A_{1,\beta}(st^\alpha,x)ds+\int_{\frac{|x|^{2\beta}}{2^{2\beta}t^\alpha}}^{\frac{1}{t^\alpha}}M_\alpha(s)A_{1,\beta}(st^\alpha,x)ds\\ &+\int_{\frac{1}{t^\alpha}}^\infty M_\alpha(s)A_{1,\beta}(st^\alpha,x)ds\\ \le&\int_0^{\frac{|x|^{2\beta}}{2^{2\beta}t^\alpha}}M_\alpha(s)st^\alpha\big(|x|-(st^\alpha)^{\frac{1}{2\beta}}\big)^{-n-2}ds+\int_{\frac{|x|^{2\beta}}{2^{2\beta}t^\alpha}}^{\frac{1}{t^\alpha}}M_\alpha(s)(st^\alpha)^{-\frac{n}{2}}ds\\ &+\int_{\frac{1}{t^\alpha}}^\infty M_\alpha(s)(st^\alpha)^{-\frac{n}{2\beta}}ds.\end{aligned}$$

We obtain the following relations:

$$\int_{\frac{|x|^{2\beta}}{2^{2\beta}t^\alpha}}^{\frac{1}{t^\alpha}} M_\alpha(s)s^{-\frac{n}{2}}ds = \int_{\frac{|x|^{2\beta}}{2^{2\beta}t^\alpha}}^{\frac{1}{t^\alpha}} sM_\alpha(s)s^{-\frac{n}{2}-1}ds \le \left(\frac{|x|^{2\beta}}{2^{2\beta}t^\alpha}\right)^{-\frac{n}{2}-1}\int_1^\infty M_\alpha(s)sds,$$

$$\int_{\frac{1}{t^\alpha}}^\infty M_\alpha(s)(st^\alpha)^{-\frac{n}{2\beta}}ds \le \int_{\frac{1}{t^\alpha}}^\infty M_\alpha(s)\left(\frac{1}{t^\alpha}t^\alpha\right)^{-\frac{n}{2\beta}}ds \le \int_{\frac{1}{t^\alpha}}^\infty M_\alpha(s)ds.$$

By the asymptotic behavior of M_α, the function

$$r\int_r^\infty M_\alpha(s)ds$$

has a maximum value in $[1,\infty)$. Then we have

$$A_{\alpha,\beta}(t,x) \lesssim t^\alpha|x|^{-n-2} + t^{\alpha+\frac{n\alpha}{2}}|x|^{-n\beta-2\beta} + t^\alpha \lesssim t^\alpha|x|^{-n-2}.$$

Next, we consider the case of $|x|>2$. By Theorem 3.1 and Remark 3.2, we have

$$\begin{aligned}
A_{\alpha,\beta}(t,x) = & \int_0^{\frac{1}{t^\alpha}} M_\alpha(s)A_{1,\beta}(st^\alpha,x)ds + \int_{\frac{1}{t^\alpha}}^{\frac{|x|^{2\beta}}{2^{2\beta}t^\alpha}} M_\alpha(s)A_{1,\beta}(st^\alpha,x)ds \\
& + \int_{\frac{|x|^{2\beta}}{2^{2\beta}t^\alpha}}^\infty M_\alpha(s)A_{1,\beta}(st^\alpha,x-bt)ds \\
\lesssim & \int_0^{\frac{1}{t^\alpha}} M_\alpha(s)st^\alpha\left(|x|-(st^\alpha)^{\frac{1}{2\beta}}\right)^{-n-2}ds \\
& + \int_{\frac{1}{t^\alpha}}^{\frac{|x|^{2\beta}}{2^{2\beta}t^\alpha}} M_\alpha(s)(st^\alpha)^{n+2\beta-\frac{n}{2\beta}}\left(|x|-(st^\alpha)^{\frac{1}{2\beta}}\right)^{-n-2\beta}ds \\
& + \int_{\frac{|x|^{2\beta}}{2^{2\beta}t^\alpha}}^\infty M_\alpha(s)(st^\alpha)^{-\frac{n}{2\beta}}ds.
\end{aligned}$$

Since

$$\int_{\frac{|x|^{2\beta}}{2^{2\beta}t^\alpha}}^\infty M_\alpha(s)(st^\alpha)^{-\frac{n}{2\beta}}ds \le \int_{\frac{|x|^{2\beta}}{2^{2\beta}t^\alpha}}^\infty M_\alpha(s)\left(\frac{|x|^{2\beta}t^\alpha}{2^{2\beta}t^\alpha}\right)^{-\frac{n}{2\beta}}ds \lesssim \frac{|x|^{-n}t^\alpha}{|x|^{2\beta}}\frac{|x|^{2\beta}}{t^\alpha}\int_{\frac{|x|^{2\beta}}{2^{2\beta}t^\alpha}}^\infty M_\alpha(s)ds \lesssim t^\alpha|x|^{-n-2\beta},$$

we have

$$A_{\alpha,\beta}(t,x) \lesssim t^\alpha(|x|-1)^{-n-2} + t^{\alpha(n+2\beta-\frac{n}{2\beta})}|x|^{-n-2\beta} + t^\alpha|x|^{-n-2\beta} \lesssim t^\alpha|x|^{-n-2\beta}. \qquad \square$$

Now we obtain the L^p-decay estimate for the fundamental solution $A_{\alpha,\beta}(t,x)$.

Theorem 3.5. *Let $n\in\mathbb{N}$, $\alpha\in(0,1)$, $\beta\in(0,1)$, $h>0$ and $b\in\mathbb{R}^n$. Then,*

$$\|A_{\alpha,\beta}(t,\cdot)\|_{L^p(\mathbb{R}^n)} \lesssim t^{-\frac{\alpha n}{2}(1-\frac{1}{p})}, \qquad t\in(0,1] \tag{3.17}$$

for $p\in[1,\bar{p}(n,1))$. Also,

$$\|A_{\alpha,\beta}(t,\cdot)\|_{L^p(\mathbb{R}^n)} \lesssim t^{-\frac{\alpha n}{2\beta}(1-\frac{1}{p})}, \qquad t\in(1,\infty) \tag{3.18}$$

for $p\in[1,\bar{p}(n,\beta))$.

Proof. If $p \in [1, \bar{p}(n,1))$, then the integral

$$\int_0^\infty M_\alpha(s) s^{-\frac{n}{2}(1-\frac{1}{p})} ds$$

is finite. Then, for $t > 0$ and $p \in [1, \bar{p}(n,1))$, we have

$$\begin{aligned}
\|A_{\alpha,\beta}(t,\cdot)\|_{L^p(\mathbb{R}^n)} &\leq \int_0^{\frac{1}{t^\alpha}} M_\alpha(s) \|A_{1,\beta}(st^\alpha,\cdot)\|_{L^p(\mathbb{R}^n)} ds + \int_{\frac{1}{t^\alpha}}^\infty M_\alpha(s) \|A_{1,\beta}(st^\alpha,\cdot)\|_{L^p(\mathbb{R}^n)} ds \\
&\lesssim \int_0^{\frac{1}{t^\alpha}} M_\alpha(s)(st^\alpha)^{-\frac{n}{2}(1-\frac{1}{p})} ds + \int_{\frac{1}{t^\alpha}}^\infty M_\alpha(s)(st^\alpha)^{-\frac{n}{2\beta}(1-\frac{1}{p})} ds \\
&\lesssim \int_0^{\frac{1}{t^\alpha}} M_\alpha(s)(st^\alpha)^{-\frac{n}{2}(1-\frac{1}{p})} ds + \int_{\frac{1}{t^\alpha}}^\infty M_\alpha(s)(st^\alpha)^{-\frac{n}{2}(1-\frac{1}{p})} ds \\
&\lesssim t^{-\frac{\alpha n}{2}(1-\frac{1}{p})} \int_0^{\frac{1}{t^\alpha}} M_\alpha(s) s^{-\frac{n}{2}(1-\frac{1}{p})} ds + t^{-\frac{\alpha n}{2}(1-\frac{1}{p})} \int_{\frac{1}{t^\alpha}}^\infty M_\alpha(s) s^{-\frac{n}{2}(1-\frac{1}{p})} ds \\
&\lesssim t^{-\frac{\alpha n}{2}(1-\frac{1}{p})}.
\end{aligned}$$

If $p \in [1, \bar{p}(n,\beta))$, then the integral

$$\int_0^\infty M_\alpha(s) s^{-\frac{n}{2\beta}(1-\frac{1}{p})} ds$$

is finite. Then, for $t > 0$ and $p \in [1, \bar{p}(n,\beta))$, we have

$$\begin{aligned}
\|A_{\alpha,\beta}(t,\cdot)\|_{L^p(\mathbb{R}^n)} &\leq \int_0^{\frac{1}{t^\alpha}} M_\alpha(s) \|A_{1,\beta}(st^\alpha,\cdot)\|_{L^p(\mathbb{R}^n)} ds + \int_{\frac{1}{t^\alpha}}^\infty M_\alpha(s) \|A_{1,\beta}(st^\alpha,\cdot)\|_{L^p(\mathbb{R}^n)} ds \\
&\lesssim \int_0^{\frac{1}{t^\alpha}} M_\alpha(s)(st^\alpha)^{-\frac{n}{2}(1-\frac{1}{p})} ds + \int_{\frac{1}{t^\alpha}}^\infty M_\alpha(s)(st^\alpha)^{-\frac{n}{2\beta}(1-\frac{1}{p})} ds \\
&\lesssim \int_0^{\frac{1}{t^\alpha}} M_\alpha(s)(st^\alpha)^{-\frac{n}{2\beta}(1-\frac{1}{p})} ds + \int_{\frac{1}{t^\alpha}}^\infty M_\alpha(s)(st^\alpha)^{-\frac{n}{2\beta}(1-\frac{1}{p})} ds \\
&\lesssim t^{-\frac{\alpha n}{2\beta}(1-\frac{1}{p})} \int_0^{\frac{1}{t^\alpha}} M_\alpha(s) s^{-\frac{n}{2\beta}(1-\frac{1}{p})} ds + t^{-\frac{\alpha n}{2\beta}(1-\frac{1}{p})} \int_{\frac{1}{t^\alpha}}^\infty M_\alpha(s) s^{-\frac{n}{2\beta}(1-\frac{1}{p})} ds \\
&\lesssim t^{-\frac{\alpha n}{2\beta}(1-\frac{1}{p})}.
\end{aligned}$$

□

Remark 3.6. Comparing Theorem 3.3 and Theorem 3.5 with Lemma 5.1 in [15], we can see that the equation (1.1) describes the Lévy flight which looks like a Brownian random walk in a small time and behaves like a Lévy process in a large time.

4 Representation formula of solutions

In this section, we establish a representation formula of the fractional diffusion equation (1.1) with the initial condition (1.2).

4.1 Classical solution

In this subsection, we discuss a classical solution of the problem (1.1)–(1.2).

Definition 4.1. We call $u \in C([0,\infty)\times\mathbb{R}^n)$ a classical solution of the Cauchy problem (1.1)–(1.2) if

(P1) $\mathscr{F}^{-1}((\zeta_\beta(\cdot)\tilde{u}(t,\cdot))(x)$ is a continuous function of x for any $t>0$,

(P2) for any $x\in\mathbb{R}^n$, $J_t^{1-\alpha}u(t,x)$ is continuously differentiable with respect to $t>0$,

(P3) $u(t,x)$ satisfies the equation (1.1) for any $(t,x)\in(0,\infty)\times\mathbb{R}^n$ and the initial condition (1.2) for any $x\in\mathbb{R}^n$.

Theorem 4.2. *Let $n\in\mathbb{N}, \alpha=1, \beta\in(0,1), h>0$ and $b\in\mathbb{R}^n$. Let $u_0\in C(\mathbb{R}^n)\bigcap L^1(\mathbb{R}^n)$ be a function such that $\tilde{u}_0\in L^1(\mathbb{R}^n)$. Then the Cauchy problem (1.1)–(1.2) has a classical solution represented by*

$$u(t,x)=\int_{\mathbb{R}^n}A_{1,\beta}(t,x-y)u_0(y)dy. \tag{4.1}$$

Proof. First, we prove that the function (4.1) satisfies the condition (P1). Using (3.4), we have

$$\zeta_\beta(\xi)\tilde{u}(t,\xi)=\zeta_\beta(\xi)\tilde{A}_{1,\beta}(t,\xi)\tilde{u}_0(\xi)=\zeta_\beta(\xi)e^{-\zeta_\beta(\xi)t}\tilde{u}_0(\xi).$$

Then it follows from the condition $\tilde{u}_0\in L^1(\mathbb{R}^n)$ that $\zeta_\beta(\cdot)\tilde{u}(t,\cdot)\in L^1(\mathbb{R}^n)$ for any $t>0$. By the Riemann–Lebesgue lemma, $\mathscr{F}^{-1}((\zeta_\beta(\cdot)\tilde{u}(t,\cdot))(x)$ is a continuous function of x for any $t>0$.

Next, we show that the function (4.1) satisfies the condition (P2). We have

$$\frac{\partial A_{1,\beta}(t,x)}{\partial t}=-\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\zeta_\beta(\xi)e^{-\zeta_\beta(\xi)t}\cos(x\xi)d\xi, \tag{4.2}$$

which implies that $\frac{\partial A_{1,\beta}(t,x)}{\partial t}$ is continuous with respect to t and x. Since the function

$$\frac{\partial A_{1,\beta}(t,x)}{\partial t}$$

is a bounded continuous function of x for any $t>0$ and $u_0\in L^1(\mathbb{R}^n)$,

$$\frac{\partial A_{1,\beta}(t,x-\cdot)}{\partial t}u_0(\cdot)\in L^1(\mathbb{R}^n),$$

which implies that the function

$$\int_{\mathbb{R}^n}\frac{\partial A_{1,\beta}(t,x-y)}{\partial t}u_0(y)dy$$

is a continuous function of x for any $t>0$. Then, we have

$$\frac{\partial u(t,x)}{\partial t}=\int_{\mathbb{R}^n}\frac{\partial A_{1,\beta}(t,x-y)}{\partial t}u_0(y)dy. \tag{4.3}$$

In the last, we deduce that the function (4.1) satisfies the condition (P3). For $(t,x)\in(0,\infty)\times\mathbb{R}^n$, we deduce

$$\mathscr{F}^{-1}(\zeta_\beta(\xi)\tilde{u}(t,\xi))(x)=\mathscr{F}^{-1}\Big(\zeta_\beta(\xi)e^{-\zeta_\beta(\xi)t}\tilde{u}_0(\xi)\Big)(x)=\int_{\mathbb{R}^n}\frac{\partial A_{1,\beta}(t,x-y)}{\partial t}u_0(y)dy=\frac{\partial u(t,x)}{\partial t}.$$

For any $\epsilon > 0$, there exists a $\delta > 0$ such that $|u_0(y) - u_0(x)| < \epsilon$ for $x, y \in \mathbb{R}^n$ satisfying the relation $|x - y| < 2\delta$. By the asymptotic behavior of $A_{1,\beta}(t,x)$, for any $x \in \mathbb{R}^n$ and $t \in (0, \min\{\delta/|b|, \delta^{2\beta}/2^{2\beta}\})$, we have

$$\begin{aligned} |u(t,x) - u_0(x)| &= \left| \int_{\mathbb{R}^n} A_{1,\beta}(t, x-y-bt)(u_0(y+bt) - u_0(x))dy \right| \\ &\leq \int_{|x-y|<\delta} A_{1,\beta}(t, x-y-bt)|u_0(y+bt) - u_0(x)|dy \\ &\quad + \int_{|x-y|>\delta} A_{1,\beta}(t, x-y-bt)|u_0(y+bt) - u_0(x)|dy \\ &\lesssim \epsilon \int_{|x-y|<\delta} A_{1,\beta}(t, x-y-bt)dy + 2\|u_0\|_{L^\infty(\mathbb{R}^n)} \int_{|x-y|>\delta} t(|x-y| - t^{\frac{1}{2\beta}})^{-n-2}dy \\ &\lesssim \epsilon + 2t\|u_0\|_{L^\infty(\mathbb{R}^n)} \int_\delta^\infty (r - t^{\frac{1}{2\beta}})^{-n-2} r^{n-1} dr \\ &\lesssim \epsilon + 2t\|u_0\|_{L^\infty(\mathbb{R}^n)} \int_\delta^\infty \left(r - \frac{\delta}{2}\right)^{-n-2} r^{n-1} dr. \end{aligned}$$

If t is sufficiently small, $|u(t,x) - u_0(x)| < 2\epsilon$ for $x \in \mathbb{R}^n$. Since ϵ is arbitrary, for any $x \in \mathbb{R}^n$,

$$\lim_{t \to 0} |u(t,x) - u_0(x)| = 0. \qquad \square$$

Theorem 4.3. *Let $n \in \mathbb{N}, \alpha \in (0,1), \beta \in (0,1), h > 0$ and $b \in \mathbb{R}^n$. Let $u_0 \in C(\mathbb{R}^n) \bigcap L^1(\mathbb{R}^n)$ be a function such that $\tilde{u}_0 \in L^1(\mathbb{R}^n)$. Then the Cauchy problem (1.1)–(1.2) has a classical solution represented by*

$$u(t,x) = \int_{\mathbb{R}^n} A_{\alpha,\beta}(t, x-y) u_0(y) dy. \tag{4.4}$$

Proof. First, we prove that the function (4.1) satisfies the condition (P1). Using (3.14) and (2.2), we have

$$\zeta_\beta(\xi)\tilde{u}(t,\xi) = \zeta_\beta(\xi)\tilde{A}_{\alpha,\beta}(t,\xi)\tilde{u}_0(\xi) = \zeta_\beta(\xi) E_{\alpha,1}(-\zeta_\beta(\xi)t^\alpha)\tilde{u}_0(\xi) \lesssim \frac{1}{t^\alpha}\tilde{u}_0(\xi), \qquad t > 0,\ \xi \in \mathbb{R}^n.$$

Then it follows from the condition $\tilde{u}_0 \in L^1(\mathbb{R}^n)$ that $\zeta_\beta(\cdot)\tilde{u}(t,\cdot) \in L^1(\mathbb{R}^n)$ for any $t > 0$. By the Riemann–Lebesgue lemma, $\mathscr{F}^{-1}((\zeta_\beta(\cdot)\tilde{u}(t,\cdot))(x)$ is a continuous function of x for any $t > 0$.

Next, we show that the function (4.1) satisfies the condition (P2). From the relation

$$\int_0^\infty J_t^{1-\alpha}\phi(t,\tau)e^{-st}dt = s^{2\alpha-2}e^{-\tau s^\alpha}, \qquad s, \tau > 0,$$

we obtain

$$\lim_{t\to 0} J_t^{1-\alpha}\phi(t,\tau) = \lim_{s\to\infty} s^{2\alpha-1}e^{-\tau s^\alpha} = 0, \qquad \tau > 0.$$

From the relations (2.13) and (2.10), we deduce

$$\frac{\partial \theta_\alpha(t,\tau)}{\partial t} = -\frac{1}{t^2}F_\alpha\left(\frac{\tau}{t^\alpha}\right) - \frac{1}{t^{1+\alpha}}F'_\alpha\left(\frac{\tau}{t^\alpha}\right) = -\frac{1}{t^2}F_\alpha\left(\frac{\tau}{t^\alpha}\right) - \frac{1}{t^{1+\alpha}}W_{-\alpha,-\alpha}\left(\frac{\tau}{t^\alpha}\right). \tag{4.5}$$

By the formula (2.3) and the asymptotic behavior of the Wright function given by (1.11.8) in [16], we have

$$\frac{\partial \theta_\alpha(t,\tau)}{\partial t} \to 0, \qquad \tau \to 0 \text{ or } \tau \to \infty. \tag{4.6}$$

Since

$$\int_0^\infty \frac{\partial \theta_\alpha(t,\tau)}{\partial t} e^{-ts} dt = s e^{-\tau s^\alpha}, \qquad s, \tau > 0,$$

we obtain

$$\lim_{t \to 0} \frac{\partial \theta_\alpha(t,\tau)}{\partial t} = \lim_{s \to \infty} s^2 e^{-\tau s^\alpha} = 0, \qquad \tau > 0, \tag{4.7}$$

$$\lim_{t \to \infty} \frac{\partial \theta_\alpha(t,\tau)}{\partial t} = \lim_{s \to 0} s^2 e^{-\tau s^\alpha} = 0, \qquad \tau > 0. \tag{4.8}$$

Then

$$\partial_t^\alpha \phi(t,\tau) = \frac{\partial}{\partial t} J_t^{1-\alpha} \phi(t,\tau) = \frac{\partial}{\partial t} J_t^{2-2\alpha} \theta_\alpha(t,\tau) = J_t^{2-2\alpha} \frac{\partial \theta_\alpha(t,\tau)}{\partial t}, \qquad t, \tau > 0. \tag{4.9}$$

Meanwhile, we have

$$J_t^{1-\alpha} A_{\alpha,\beta}(t,x) = \int_0^\infty J_t^{1-\alpha} \phi(t,\tau) A_{1,\beta}(\tau,x) d\tau = \frac{1}{(2\pi)^n} \int_0^\infty J_t^{1-\alpha} \phi(t,\tau) \int_{\mathbb{R}^n} e^{-\zeta_\beta(\xi)\tau} \cos(\xi x) d\xi d\tau.$$

By (4.6) and (4.9), $J_t^{1-\alpha} A_{\alpha,\beta}(t,x)$ is continuously differentiable with respect to $t > 0$. From Theorem 3.1 and (3.3), for $t > 0$ and $x \in \mathbb{R}^n$, we obtain

$$\partial_t^\alpha A_{\alpha,\beta}(t,x) = \int_0^\infty \partial_t^\alpha \phi(t,\tau) A_{1,\beta}(\tau,x) d\tau = \frac{1}{(2\pi)^n} \int_0^\infty \partial_t^\alpha \phi(t,\tau) \int_{\mathbb{R}^n} e^{-\zeta_\beta(\xi)\tau} \cos(x\xi) d\xi d\tau. \tag{4.10}$$

Then the function $\partial_t^\alpha A_{\alpha,\beta}(t,x)$ is a continuous function of x for any $t > 0$. It follows from Theorem 3.3 that $\partial_t^\alpha A_{\alpha,\beta}(t,\cdot) \in L^1(\mathbb{R}^n)$ for any $t > 0$. Also, the function

$$\int_{\mathbb{R}^n} \partial_t^\alpha A_{\alpha,\beta}(t, x-y) u_0(y) dy$$

is a continuous function of x for any $t > 0$. Therefore, we have

$$\partial_t^\alpha u(t,x) = \int_{\mathbb{R}^n} \partial_t^\alpha A_{\alpha,\beta}(t, x-y) u_0(y) dy. \tag{4.11}$$

In the last, we deduce that the function (4.4) satisfies the condition (P3). For $t > 0$ and $x \in \mathbb{R}^n$, we deduce

$$\begin{aligned}
\mathscr{F}^{-1}((\zeta_\beta(\xi)\tilde{u}(t,\xi))(x) &= \mathscr{F}^{-1}\big(\zeta_\beta(\xi) E_{\alpha,1}(-\zeta_\beta(\xi)t^\alpha)\tilde{u}_0(\xi)\big)(x) \\
&= \mathscr{F}^{-1}\big(\partial_t^\alpha (E_{\alpha,1}(-\zeta_\beta(\xi)t^\alpha))\tilde{u}_0(\xi)\big)(x) \\
&= \mathscr{F}^{-1}\left(\int_0^\infty \partial_t^\alpha \phi(t,\tau) e^{-\zeta_\beta(\xi)\tau^\alpha} \tilde{u}_0(\xi) d\tau\right)(x) \\
&= \int_0^\infty \partial_t^\alpha \phi(t,\tau) \mathscr{F}^{-1}\big(e^{-\zeta_\beta(\xi)\tau^\alpha} \tilde{u}_0(\xi)\big)(x) d\tau \\
&= \int_0^\infty \partial_t^\alpha \phi(t,\tau) \int_{\mathbb{R}^n} A_{1,\beta}(\tau, x-y) u_0(y) dy d\tau \\
&= \int_{\mathbb{R}^n} \partial_t^\alpha A_{\alpha,\beta}(t, x-y) u_0(y) dy = \partial_t^\alpha u(t,x).
\end{aligned}$$

As in the proof of Theorem 4.2, using the asymptotic behavior of $A_{\alpha,\beta}(t,x)$ obtained in Lemma 3.4, we can prove the initial condition $\lim_{t \to 0} |u(t,x) - u_0(x)| = 0$ for any $x \in \mathbb{R}^n$. □

4.2 Mild solution

In this subsection, we consider a mild solution of the Cauchy problem (1.1)–(1.2). Now we give a rigorous definition of the solution of the equation (1.1)–(1.2).

Definition 4.4. We call u a mild solution to (1.1)–(1.2) if (1.1) holds in $L^2(\mathbb{R}^n)$ and $u(t,\cdot) \in H^2(\mathbb{R}^n)$ for $t>0$ and $u \in C([0,\infty); L^2(\mathbb{R}^n))$, $\lim_{t\to 0} \|u(t,\cdot) - u_0\|_{L^2(\mathbb{R}^n)} = 0$.

Theorem 4.5. *Let $\alpha \in (0,1], \beta \in (0,1), h > 0, b \in \mathbb{R}^n$ and $u_0 \in L^2(\mathbb{R}^n)$. Then the Cauchy problem (1.1)–(1.2) has a unique mild solution u represented by (4.4). Moreover, $u \in C((0,\infty); H^2(\mathbb{R}^n))$ and the following relations hold.*

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \leq \|u_0\|_{L^2(\mathbb{R}^n)}, \qquad t \geq 0, \tag{4.12}$$

$$\|u(t,\cdot)\|_{H^2(\mathbb{R}^n)} \lesssim \|u_0\|_{L^2(\mathbb{R}^n)}(1 + t^{-\alpha}), \qquad t > 0, \tag{4.13}$$

$$\|u(t,\cdot)\|_{H^1(\mathbb{R}^n)} \lesssim \|u_0\|_{L^2(\mathbb{R}^n)}(1 + t^{-\frac{\alpha}{2}}), \qquad t > 0. \tag{4.14}$$

If $n < 4\beta$, then

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \lesssim \begin{cases} t^{-\frac{n\alpha}{4}}, & t \in (0,1), \\ t^{-\frac{n\alpha}{4\beta}}, & t \in [1,\infty). \end{cases}$$

Proof. Using Lemma 2.2, we have

$$\begin{aligned} ||\xi|^2 \tilde{u}(t,\xi)| &\leq |\tilde{u}_0(\xi)||\xi|^2 \left|E_{\alpha,1}(-(|\xi|^{2\beta} + h|\xi|^2 - ib\cdot\xi)t^\alpha)\right| \\ &\lesssim |\tilde{u}_0(\xi)||\xi|^2 \frac{1}{1+|\xi|^2 t^\alpha} \lesssim \frac{|\tilde{u}_0(\xi)|}{t^\alpha}, \qquad t > 0, \xi \in \mathbb{R}^n, \\ ||\xi|\tilde{u}(t,\xi)| &\leq |\tilde{u}_0(\xi)||\xi| \left|E_{\alpha,1}(-(|\xi|^{2\beta} + h|\xi|^2 - ib\cdot\xi)t^\alpha)\right| \\ &\lesssim |\tilde{u}_0(\xi)||\xi| \frac{1}{1+|\xi|^2 t^\alpha} \lesssim \frac{|\tilde{u}_0(\xi)|}{t^{\frac{\alpha}{2}}}, \qquad t > 0, \xi \in \mathbb{R}^n. \end{aligned}$$

By using the Plancherel theorem, for $t > 0$, we deduce

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} = \|\tilde{u}(t,\cdot)\|_{L^2(\mathbb{R}^n)} = \|\tilde{u}_0 \tilde{A}_{\alpha,\beta}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \leq \|\tilde{u}_0\|_{L^2(\mathbb{R}^n)} = \|u_0\|_{L^2(\mathbb{R}^n)}.$$

Also, for $t > 0$, we have

$$\begin{aligned} \|u(t,\cdot)\|_{H^2(\mathbb{R}^n)} &= \|(1 + |\cdot|^2)\tilde{u}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \left(1 + \frac{1}{t^\alpha}\right)\|u_0\|_{L^2(\mathbb{R}^n)}, \\ \|u(t,\cdot)\|_{H^1(\mathbb{R}^n)} &= \|(1 + |\cdot|)\tilde{u}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \left(1 + \frac{1}{t^{\frac{\alpha}{2}}}\right)\|u_0\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

For $t > 0$, we estimate

$$\|u(t,\cdot) - u_0\|_{L^2(\mathbb{R}^n)} = \|\tilde{u}(t,\cdot) - \tilde{u}_0\|_{L^2(\mathbb{R}^n)} = \|\tilde{u}_0(1 - \tilde{A}_{\alpha,\beta}(t,\cdot))\|_{L^2(\mathbb{R}^n)} \leq 2\|u_0\|_{L^2(\mathbb{R}^n)}.$$

From the Lebesgue dominated convergence theorem, we obtain

$$\lim_{t\to 0} \|u(t,\cdot) - u_0\|_{L^2(\mathbb{R}^n)} = \lim_{t\to 0} \|\tilde{u}(t,\cdot) - \tilde{u}_0\|_{L^2(\mathbb{R}^n)} = \|\tilde{u}_0 \lim_{t\to 0}(1 - \tilde{A}_{\alpha,\beta}(t,\cdot))\|_{L^2(\mathbb{R}^n)} = 0.$$

For $t_1, t_2 > 0$, we deduce

$$\begin{aligned}\|u(t_1,\cdot)-u(t_2,\cdot)\|_{H^2(\mathbb{R}^n)} &= \|(1+|\cdot|^2)(\tilde{u}(t_1,\cdot)-\tilde{u}(t_2,\cdot)\|_{L^2(\mathbb{R}^n)}\\ &= \|(1+|\cdot|^2)\tilde{u}_0(\tilde{A}_{\alpha,\beta}(t_1,\cdot)-\tilde{A}_{\alpha,\beta}(t_2,\cdot))\|_{L^2(\mathbb{R}^n)}\\ &\lesssim \frac{1}{t_1^\alpha}\|u_0\|_{L^2(\mathbb{R}^n)}+\frac{1}{t_2^\alpha}\|u_0\|_{L^2(\mathbb{R}^n)}+2\|u_0\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

By the Lebesgue dominated convergence theorem, we estimate

$$\begin{aligned}\lim_{t_1\to t_2}\|u(t_1,\cdot)-u(t_2,\cdot)\|_{H^2(\mathbb{R}^n)} &= \lim_{t\to 0}\|(1+|\cdot|^2)(\tilde{u}(t_1,\cdot)-\tilde{u}(t_2,\cdot))\|_{L^2(\mathbb{R}^n)}\\ &= \|(1+|\cdot|^2)\tilde{u}_0(\cdot)\lim_{t_1\to t_2}(\tilde{A}_{\alpha,\beta}(t_1,\cdot)-\tilde{A}_{\alpha,\beta}(t_2,\cdot))\|_{L^2(\mathbb{R}^n)}=0.\end{aligned}$$

Similarly, we can prove $u \in C((0,\infty); H^1(\mathbb{R}^n))$.

Using the Plancherel theorem, (2.1) and (3.9), for $t > 0$, we have

$$\begin{aligned}&\left\|\frac{\partial^\alpha u(t,\cdot)}{\partial t^\alpha}+(-\triangle)^\beta u(t,x)-b\cdot\nabla u(t,x)-h\triangle u(t,x)\right\|_{L^2(\mathbb{R}^n)}\\ &=\left\|\frac{\partial^\alpha \tilde{u}(t,\cdot)}{\partial t^\alpha}-\tilde{u}_0(\xi)(|\xi|^{2\beta}+h|\xi|^2-ib\cdot\xi)E_{\alpha,1}(-(|\xi|^{2\beta}+h|\xi|^2-ib\cdot\xi)t^\alpha)\right\|_{L^2(\mathbb{R}^n)}=0.\end{aligned}$$

In the case of $n < 4\beta$, using Young inequality for convolution and Theorem 3.5, we obtain

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \le \|u_0\|_{L^2(\mathbb{R}^n)}\|A_{\alpha,\beta}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \begin{cases} t^{-\frac{n\alpha}{4}}, & t\in(0,1),\\ t^{-\frac{n\alpha}{4\beta}}, & t\in[1,\infty).\end{cases}$$

□

5 Decay behavior of solutions

In this section, we consider the L^2-decay of solutions of the nonlocal diffusion equation (1.1) with the initial condition (1.2).

Theorem 5.1. *Let $n\in\mathbb{N}$, $\alpha=1, \beta\in(0,1), h>0, b\in\mathbb{R}^n$ and $u_0\in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$. Then the mild solution u of the Cauchy problem* (1.1)–(1.2) *satisfies the following relation.*

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \begin{cases} t^{-\frac{n}{4}}\|u_0\|_{L^1(\mathbb{R}^n)}, & t\in(0,1],\\ t^{-\frac{n}{4\beta}}\|u_0\|_{L^1(\mathbb{R}^n)}, & t\in(1,\infty).\end{cases}$$

Proof. Using Theorem 3.3, Young's inequality for convolution and the Plancherel theorem, for $t>0$, we obtain

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \le \|A_{1,\beta}(t,\cdot)\|_{L^2(\mathbb{R}^n)}\|u_0\|_{L^1(\mathbb{R}^n)} \lesssim \begin{cases} t^{-\frac{n}{4}}\|u_0\|_{L^1(\mathbb{R}^n)}, & t\in(0,1],\\ t^{-\frac{n}{4\beta}}\|u_0\|_{L^1(\mathbb{R}^n)}, & t\in(1,\infty).\end{cases}$$

□

Theorem 5.2. *Let $n \in \mathbb{N}$, $\alpha \in (0,1), \beta \in (0,1), h > 0, b \in \mathbb{R}^n$ and $u_0 \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then the mild solution u of the Cauchy problem (1.1)–(1.2) satisfies the following relations.*

If $n \neq 4\beta$, then

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \begin{cases} t^{-\frac{\alpha}{2}\min\{\frac{n}{2},1\}}, & t \in (0,1], \\ t^{-\alpha \min\{\frac{n}{4\beta},1\}}, & t \in (1,\infty). \end{cases}$$

If $n = 4\beta$, then

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \begin{cases} t^{-\frac{\alpha}{2}\min\{\frac{n}{2},1\}}, & t \in (0,1], \\ t^{-\frac{\alpha}{2}\max\{\frac{n}{2},1\}}, & t \in (1,\infty). \end{cases}$$

Proof. Using Theorem 3.5 and Young inequality for convolution, we have

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \le \|A_{\alpha,\beta}(t,\cdot)\|_{L^2(\mathbb{R}^n)}\|u_0\|_{L^1(\mathbb{R}^n)} \lesssim \begin{cases} t^{-\frac{\alpha n}{4}}\|u_0\|_{L^1(\mathbb{R}^n)}, & \text{for } n < 4 \text{ and } t \in (0,1], \\ t^{-\frac{\alpha n}{4\beta}}\|u_0\|_{L^1(\mathbb{R}^n)}, & \text{for } n < 4\beta \text{ and } t \in (1,\infty). \end{cases}$$

If $n > 4\beta$, then, from the Plancherel theorem, Lemma 2.2 and the Hardy–Littlewood–Sobolev theorem [9], we deduce

$$\begin{aligned}(2\pi)^n\|u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} &= \|\tilde{u}(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\tilde{A}_{\alpha,\beta}(t,\xi)|^2|\tilde{u}_0(\xi)|^2 d\xi \\ &= \int_{\mathbb{R}^n} |E_{\alpha,1}(-\zeta_\beta(\xi)t^\alpha)|^2|\tilde{u}_0(\xi)|^2 d\xi \lesssim \int_{\mathbb{R}^n} \frac{|\tilde{u}_0(\xi)|^2}{(1+|\zeta_\beta(\xi)|t^\alpha)^2}d\xi \\ &\lesssim t^{-2\alpha}\int_{\mathbb{R}^n} \frac{|\xi|^{4\beta}t^{2\alpha}}{(1+|\xi|^{2\beta}t^\alpha)^2}||\xi|^{-2\beta}\tilde{u}_0(\xi)|^2 d\xi \lesssim t^{-2\alpha}\|(-\triangle)^{-\beta}u_0\|^2_{L^2(\mathbb{R}^n)} \lesssim t^{-2\alpha}\|u_0\|^2_{L^{\frac{2n}{n+4\beta}}(\mathbb{R}^n)}.\end{aligned}$$

If $n > 2\beta$, then we obtain

$$\begin{aligned}\|u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} &\lesssim \int_{\mathbb{R}^n} \frac{|\tilde{u}_0(\xi)|^2}{(1+|\zeta_\beta(\xi)|t^\alpha)^2}d\xi \lesssim t^{-\alpha}\int_{\mathbb{R}^n} \frac{|\xi|^{2\beta}t^\alpha}{(1+|\xi|^{2\beta}t^\alpha)^2}||\xi|^{-\beta}\tilde{u}_0(\xi)|^2 d\xi \\ &\lesssim t^{-\alpha}\int_{\mathbb{R}^n} \frac{||\xi|^{-\beta}\tilde{u}_0(\xi)|^2}{(|\xi|^{-\beta}t^{-\frac{\alpha}{2}}+|\xi|^\beta t^{\frac{\alpha}{2}})^2}d\xi \lesssim t^{-\alpha}\|(-\triangle)^{-\frac{\beta}{2}}u_0\|^2_{L^2(\mathbb{R}^n)} \lesssim t^{-\alpha}\|u_0\|^2_{L^{\frac{2n}{n+2\beta}}(\mathbb{R}^n)}.\end{aligned}$$

If $n > 4$, then we have

$$\begin{aligned}\|u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} &\lesssim \int_{\mathbb{R}^n} \frac{|\tilde{u}_0(\xi)|^2}{(1+|\zeta_\beta(\xi)|t^\alpha)^2}d\xi \lesssim t^{-2\alpha}\int_{\mathbb{R}^n} \frac{|\xi|^4t^{2\alpha}}{(1+|\xi|^2t^\alpha)^2}||\xi|^{-2}\tilde{u}_0(\xi)|^2 d\xi \\ &\lesssim t^{-2\alpha}\|(-\triangle)^{-1}u_0\|^2_{L^2(\mathbb{R}^n)} \lesssim t^{-2\alpha}\|u_0\|^2_{L^{\frac{2n}{n+4}}(\mathbb{R}^n)}.\end{aligned}$$

If $n > 2$, then we estimate

$$\begin{aligned}\|u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} &\lesssim \int_{\mathbb{R}^n} \frac{|\tilde{u}_0(\xi)|^2}{(1+|\zeta_\beta(\xi)|t^\alpha)^2}d\xi \lesssim t^{-\alpha}\int_{\mathbb{R}^n} \frac{|\xi|^2t^\alpha}{(1+|\xi|^2t^\alpha)^2}||\xi|^{-1}\tilde{u}_0(\xi)|^2 d\xi \\ &\lesssim t^{-\alpha}\int_{\mathbb{R}^n} \frac{||\xi|^{-1}\tilde{u}_0(\xi)|^2}{(|\xi|^{-1}t^{-\frac{\alpha}{2}}+|\xi|t^{\frac{\alpha}{2}})^2}d\xi \lesssim t^{-\alpha}\|(-\triangle)^{-\frac{1}{2}}u_0\|^2_{L^2(\mathbb{R}^n)} \lesssim t^{-\alpha}\|u_0\|^2_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}.\end{aligned}$$

If $n < 4$, then we deduce

$$\begin{aligned}\|u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} &\lesssim \int_{\mathbb{R}^n} \frac{|\tilde{u}_0(\xi)|^2}{(1+|\xi|^2t^\alpha)^2}d\xi \lesssim \|\tilde{u}_0\|^2_{L^\infty(\mathbb{R}^n)}\int_0^\infty \frac{r^{n-1}}{(1+r^2t^\alpha)^2}dr \\ &= t^{-\frac{\alpha n}{2}}\|u_0\|^2_{L^1(\mathbb{R}^n)}\int_0^\infty \frac{w^{n-1}}{(1+w^2)^2}dw.\end{aligned}$$

Combining the previous estimates, we obtain the desired result. □

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Stability results for the functional differential equations associated to water hammer in hydraulics

Dedicated to László Hatvani, outstanding scholar and long life friend

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Abstract. We consider a system of two sets of partial differential equations describing the water hammer in a hydroelectric power plant containing the dynamics of the tunnel, turbine penstock, surge tank and hydraulic turbine. Under standard simplifying assumptions (negligible Darcy–Weisbach losses and dynamic head variations), a system of functional differential equations of neutral type, with two delays, can be associated to the aforementioned partial differential equations and existence, uniqueness and continuous data dependence can be established. Stability is then discussed using a Lyapunov functional deduced from the energy identity. The Lyapunov functional is "weak" i.e. its derivative function is only non-positive definite. Therefore only Lyapunov stability is obtained while for asymptotic stability application of the Barbashin–Krasovskii–LaSalle invariance principle is required. A necessary condition for its validity is the asymptotic stability of the difference operator associated to the neutral system. However, its properties in the given case make the asymptotic stability *non-robust* (fragile) in function of some arithmetic properties of the delay ratio.

Keywords: differential equations, neutral functional differential equations, energy Lyapunov functional, asymptotic stability, water hammer.

2020 Mathematics Subject Classification: 34D20, 34K40, 34K20, 35L50.

1 The engineering model and problem statement

The transient processes of the hydraulic power plants are quite important since, if uncontrolled, they can produce technical and/or environment catastrophes. Consequently their theoretical analysis is of major concern: we can cite but a few references on this subject [3, 21, 24, 27].

We shall consider here, as in other papers of ours, the (relatively) standard structure of a hydroelectric power plant consisting of a water reservoir ("lake"), tunnel, penstock and hydraulic turbine (Fig. 1.1)

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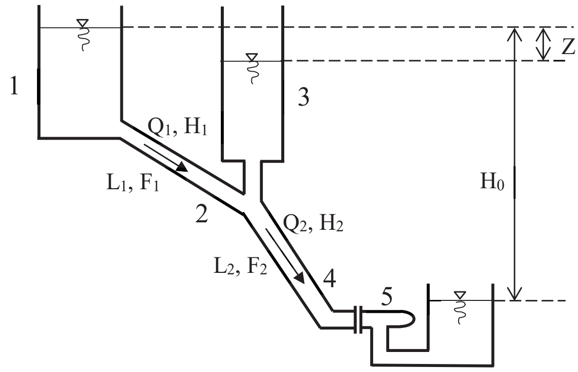

Figure 1.1: Hydroelectric plant structure. 1. Lake. 2. Tunnel. 3. Surge tank. 4. Penstock. 5. Hydraulic turbine.

This structure is common for the hydroelectric power plants throughout the world: such examples as *"Bicaz"* and *"Someș Mărișelu"* in Romania [27] or *"Tanzmühle"* in Germany [25,26] illustrate this assertion.

If distributed parameters are considered along the two water conduits, the adapted Saint Venant partial differential equations have to be used and the following mathematical model is obtained – we reproduce it after [10]

$$
\begin{aligned}
&\partial_{x_i}\left(H_i+\frac{V_i^2}{2g}\right)+\frac{1}{g}\partial_t V_i+\frac{\lambda_i}{2D_i}\frac{1}{g}V_i|V_i|=0, \quad \partial_t H_i+\frac{a_i^2}{g}\partial_{x_i}V_i=0, \quad i=1,2,\\
&H_1(0,t)=H_0; \quad H_1(L_1,t)+\frac{V_1^2(L_1,t)}{2g}=Z(t)+R_s\frac{\mathrm{d}Z}{\mathrm{d}t}=H_2(0,t)+\frac{V_2^2(0,t)}{2g},\\
&Q_i=F_iV_i, \quad i=1,2; \quad \bar{Q}=\alpha_q F_{\theta\,\max}\sqrt{H_0}; \quad F_s\frac{\mathrm{d}Z}{\mathrm{d}t}=Q_1(L_1,t)-Q_2(0,t),\\
&Q_2(L_2,t)=(1-k)\alpha_q F_\theta(t)\sqrt{H_2(L_2,t)}+k\bar{Q}\Omega(t)/\Omega_c,\\
&J\Omega_c\frac{\mathrm{d}\Omega}{\mathrm{d}t}=\eta_\theta\frac{\gamma}{2g}Q_2(L_2,t)H_2(L_2,t)-N_g,
\end{aligned}
\tag{1.1}
$$

where the notations are the usual ones in the field and are enumerated in the Appendix (also reproduced after the Appendix of [10]). In (1.1) the flow crossing the wicket gates of the turbine, namely $Q_2(L_2,t)$ (the subscript 2 accounts for the penstock state variables and parameters) is expressed according to an improved formula of [3], thus being dependent of the turbine rotating speed. The terms depending on V_i^2 account for the dynamic heads and those in $V_i|V_i|$ for the Darcy–Weisbach losses. It is worth mentioning that all conduits are assumed to be described by distributed parameters. The throttling of the surge tank is represented by its parameter R_s; letting $R_s=0$ means assuming a surge tank without throttling. Also the flow $\bar{Q}$ as defined in (1.1) represents the maximally available flow at the wicket gates and serves to flow rating (at this point we do not yet discuss the state variables ratings). It is worth mentioning however that this model reproduces the usual models of the hydraulic plants incorporating the dynamic (velocity) heads $(Q_i/F_i)^2$ and the distributed Darcy–Weisbach losses $(\lambda_i/(2D_ig))(Q_i/F_i))(|Q_i|/F_i)$. The boundary condition for the water flow $Q_2(L_2,t)$ is borrowed from [3] and incorporates the turbine rotating speed effect on the flow: it is stated that $0<k<0.3$ but in general there is taken $k=0$; k is thus a numeric coefficient having a corrective character from the engineering point of view. As it will appear in the following, k is irrelevant in water hammer analysis. As it can be seen in the Appendix containing the notation list, F_θ is the cross section area of the hydraulic turbine wicket gates. In the equations of the model (1.1) it acts as an input (forcing) signal, being defined by the speed controller of the turbine; its amplitude is limited physically: $0 \leq F_\theta \leq F_{\theta\max}$. During water hammer the hydraulic turbine is decoupled from the hydraulic system upstream and the forcing signal is blocked at some constant value, being thus irrelevant for the water hammer dynamics.

In hydraulic engineering two are the types of transients which are discussed: the normal and the abnormal ones. The normal exploitation regimes of the hydraulic power plants are concerned firstly with the so called *frequency/megawatt control* of the Electric Grid. The frequency/megawatt control is achieved by the control of the turbine rotating speed through water flow admission – controlled by the cross section area $F_\theta(t)$ of the wicket gates. The turbine controllers can be mechanical, hydraulic or electro-hydraulic (as technical implementation); the most recent control approach is based on predictive control [24]. The turbine controller is not included in (1.1) since normal regimes are outside the aim of this paper.

The abnormal regimes are concerned with sudden large power changes including turbine shut down. Especially in the last case, the turbine with the rotating speed controller are "cut" from the upstream dynamics; the only stabilizing device for the upstream dynamics remains the surge tank.

The present paper is concerned with the second case – the dynamics of the abnormal regimes. Again, two will be the problems analyzed. The first one will be the inherent stability of the surge tank as stabilizing device. The problem occurs from the engineering conviction that a stabilizing device incorporated in a feedback structure must be stable itself. Moreover, the surge tank is not a miniaturized electronic device but a construction which cannot be rebuilt in case of a design error. The second problem, already mentioned, is the stability of the upstream dynamics of the turbine (tunnel, surge tank, penstock) under water hammer.

Several simplifying assumptions are introduced, considered as covering from the engineering point of view (this aspect will be explained in what follows). The newly obtained model will allow a rigorous mathematical study by associating certain functional differential equations of neutral type.

2 Rated variables and parameters – the basic working model

A specific feature of the analysis of the real world mathematical models is the use of the rated (scaled) variables: the real physical variables are rated to certain reference values, the aim being at least twofold: to use relative i.e. comparable values and to reduce numerical ill conditioning. In our case the flows will be rated to the maximally available water flow at the wicket gates of the hydraulic turbine $\bar{Q} = \alpha_q F_{\theta\max}\sqrt{H_0}$; the piezometric heads are rated to the maximal head H_0 of the reservoir; the rotating speed of the turbine is rated to the synchronous speed Ω_c. These are the scalings of the state variables. The next scalings are those of the conduit coordinates $x_i (i = 1,2)$ to the conduit lengths L_i namely $\xi_i = x_i/L_i$.

We introduce further the following time constants of the conduits

- the starting time constant $T_{wi} = (L_i\bar{Q})(F_iH_0g)^{-1}$ $(i = 1,2)$;
- the fill up time constant $T_i = (L_iF_i)/\bar{Q}$ $(i = 1,2)$;
- the wave propagation time $T_{pi} = L_i/a_i$ $(i = 1,2)$,

and also

- the fill up time constant of the surge tank $T_s = F_s H_0/\bar{Q}$;
- the starting time constant of the turbine

$$T_a = \frac{J\Omega_c^2}{\eta_\theta \frac{\gamma}{2g}\bar{Q}H_0}.$$

The time constants notations are also listed in the Appendix.

After some simple and straightforward manipulation the following equations are obtained

$$\begin{aligned}
&\partial_{\xi_i}\left(h_i + \frac{1}{2}\frac{T_{wi}}{T_i}q_i^2\right) + T_{wi}\partial_t q_i + \frac{\lambda_i L_i}{D_i}\frac{1}{2}\frac{T_{wi}}{T_i}q_i|q_i| = 0, \quad \frac{T_{pi}^2}{T_{wi}}\partial_t h_i + \partial_{\xi_i} q_i = 0,\\
&h_1(0,t) \equiv 1; \quad h_1(1,t) = 1 + z(t) + R_s\frac{\mathrm{d}z}{\mathrm{d}t} = h_2(0,t),\\
&T_s\frac{\mathrm{d}z}{\mathrm{d}t} = q_1(1,t) - q_2(0,t); \quad q_2(1,t) = (1-k)f_\theta(t)\sqrt{h_2(1,t)} + k\varphi(t),\\
&T_a\frac{\mathrm{d}\varphi}{\mathrm{d}t} = q_2(1,t)h_2(1,t) - \nu_g.
\end{aligned} \tag{2.1}$$

By lower case letters q_i, h_i, z we denoted the rated state variables – flows, piezometric heads and water level in the surge tank respectively. We introduced also the rated rotating speed $\varphi = \Omega/\Omega_c$, the rated load mechanical power

$$\nu_g = \frac{N_g}{\eta_\theta \frac{\gamma}{2g}\bar{Q}H_0} \tag{2.2}$$

and the rated cross section area of the wicket gates $f_\theta = F_\theta/F_{\theta\max}$. The aforementioned rated variables are also listed in the Appendix. Observe also in (2.1) that the local dynamic heads $V_1^2(L_1,t)/(2g)$ and $V_2^2(0,t)/(2g)$ have been neglected, as it is customary in hydropower engineering. Another remark concerns the water level in the surge tank: again, as it is customary in hydraulic engineering, the rated level is "counted" from the maximal head H_0 of the lake i.e. $z := (Z - H_0)/H_0$ and this explains the presence of 1 in the boundary conditions at $\xi_1 = 1$ and $\xi_2 = 0$ where the surge tank is located – equations (2.1).

The next model transformation is connected with the rating of the time to the largest time constant T_1 (this assertion – T_1 being the largest – holds for most hydroelectric power plants). We shall have $\tau = t/T_1$; only the equations containing time derivatives will be modified since the corresponding time constant will be now rated to T_1 – from the chain rule differentiation.

Before writing down the modified model, an explanation for this time rating appears as necessary. Model (1.1) is considered in hydraulics as *fundamental* in the sense that various particular models for various analysis are deduced from it according to corresponding assumptions (as it will appear throughout this paper also). Among other features of the model – which correspond to a physical reality – is the property of *several time scales*. This property follows by comparison of the time constants introduced previously: if we refer again to the aforementioned hydroelectric power plants of Romania (for which numerical data are available) we can see e.g. that $T_1 = 1005$ sec., $T_s = 502.25$ sec., $T_{w1} = 14.71$ sec., $T_{p1} = 3.81$ sec., $T_2 = 44.33$ sec., $T_{w2} = 0.38$ sec., $T_a \approx 8$ sec. etc. Several time scales are usually tackled within the framework of the *singular perturbations*. Therefore it is useful for a basic model to have the "small parameters" as ratios of time constants ensuring their dimensionless.

Denoting by $\theta_{wi} = T_{wi}/T_1$, $\theta_i = T_i/T_1$ ($\theta_1 = 1$) etc. the rated to T_1 time constants, the following work model is obtained

$$
\begin{aligned}
&\partial_{\xi_i}\left(h_i + \frac{1}{2}\frac{\theta_{wi}}{\theta_i}q_i^2\right) + \theta_{wi}\partial_\tau q_i + \frac{\lambda_i L_i}{D_i}\frac{1}{2}\frac{\theta_{wi}}{\theta_i}q_i|q_i| = 0, \quad \frac{\theta_{pi}^2}{\theta_{wi}}\partial_\tau h_i + \partial_{\xi_i} q_i = 0,\\
&h_1(0,\tau) \equiv 1; \quad h_1(1,\tau) = 1 + z(\tau) + \lambda_s\frac{\mathrm{d}z}{\mathrm{d}\tau} = h_2(0,\tau),\\
&\theta_s\frac{\mathrm{d}z}{\mathrm{d}\tau} = q_1(1,\tau) - q_2(0,\tau); \quad q_2(1,\tau) = (1-k)f_\theta(\tau)\sqrt{h_2(1,\tau)} + k\varphi(\tau),\\
&\theta_a\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = q_2(1,\tau)h_2(1,\tau) - \nu_g
\end{aligned}
\tag{2.3}
$$

with $\lambda_s := R_s/T_1$; the term $\lambda_s \mathrm{d}z/\mathrm{d}\tau$ in the boundary conditions at the surge tank accounts for a surge tank with throttling [15,27,39].

The rated time constants are also listed in the Appendix. The fact that $\theta_1 = 1$ appears in the equations is due to the similarity of the Saint Venant partial differential equations for the two conduits, suggesting the more compact writing of the equations.

Model (2.3) is, generally speaking, completed with certain equations of the speed controller for the hydraulic turbine. We already mentioned at Section 1 that this controller is decoupled (even blocked) during the abnormal regime of the water hammer hence that its dynamics will be irrelevant throughout this paper. Its equations are nevertheless given in order to make clearer the passage from normal to abnormal exploitation.

The speed controller has various engineering implementations (mechanical, mechanic-hydraulic, electro-hydraulic). For the aforementioned purpose we can write down a general form

$$
\begin{aligned}
\dot{x}_c &= A_c x_c + b_c(\varphi_0 - \varphi),\\
f_\theta &= f_c^T x_c + \gamma_c(\varphi_0 - \varphi),
\end{aligned}
\tag{2.4}
$$

where $x_c \in \mathbb{R}^n$ is the state vector of the controller dynamics and $\varphi_0 = 1 = \Omega/\Omega_c$ – the rated synchronous speed of the hydraulic turbine, imposed by the Power Grid. The controller's coefficients A_c, b_c, f_c, γ_c have appropriate dimensions.

Obviously the speed controller acts by modifying the cross section area f_θ of the turbine wicket gates. Its role is firstly to ensure a stable steady state for system (2.3)–(2.4). Let us compute it, by letting the "time" derivatives from (2.3)–(2.4) go to zero. We obtain firstly for the steady state flows $\bar{q}_i(\xi_i)$ that they are constant and equal i.e. $\bar{q}_1(\xi_1) \equiv \bar{q}_2(\xi_2) \equiv \bar{q}$. Let $\bar{h}_i(\xi_i)$ be the steady state values for the piezometric heads. The steady state boundary condition at $\xi_2 = 1$ that is

$$\bar{q} = (1-k)\bar{f}_\theta\sqrt{\bar{h}_2(1)} + k\bar{\varphi}$$

shows that $\bar{h}_2(1) > 0$. Therefore the steady state load condition $\bar{q}\bar{h}_2(1) = \nu_g > 0$ shows that $\bar{q} > 0$ what is only natural since no normal exploitation would require an upstream flow. Therefore we deduce the differential steady state equations for the piezometric heads

$$\frac{\mathrm{d}\bar{h}_i}{\mathrm{d}\xi_i} + \frac{1}{2}\frac{\lambda_i L_i}{D_i}\frac{\theta_{wi}}{\theta_i}\bar{q}^2 = 0.$$

From here it follows

$$
\begin{aligned}
\bar{h}_1(\xi_1) &= 1 - \frac{1}{2}\frac{\lambda_1 L_1}{D_1}\frac{\theta_{w1}}{\theta_1}\bar{q}^2\xi_1, \quad 0 \le \xi_1 \le 1,\\
\bar{h}_2(\xi_2) &= 1 - \frac{1}{2}\frac{\lambda_1 L_1}{D_1}\frac{\theta_{w1}}{\theta_1}\bar{q}^2 - \frac{1}{2}\frac{\lambda_2 L_2}{D_2}\frac{\theta_{w2}}{\theta_2}\bar{q}^2\xi_2, \quad 0 \le \xi_2 \le 1
\end{aligned}
$$

and, therefore

$$\bar{h}_2(1) = 1 - \frac{1}{2}\left(\frac{\theta_{w1}}{\theta_1}\frac{\lambda_1 L_1}{D_1} + \frac{\theta_{w2}}{\theta_2}\frac{\lambda_2 L_2}{D_2}\right)\bar{q}^2.$$

The load condition $\bar{q}\bar{h}_2(1) = \nu_g$ will then send to the following equation of third degree which allows determination of the flow as function of the mechanical load ν_g

$$\frac{1}{2}\left(\frac{\theta_{w1}}{\theta_1}\frac{\lambda_1 L_1}{D_1} + \frac{\theta_{w2}}{\theta_2}\frac{\lambda_2 L_2}{D_2}\right)\bar{q}^3 - \bar{q} + \nu_g = 0. \tag{2.5}$$

It is worth mentioning that the design is such that the coefficients of (2.5) allow existence of a solution $\bar{q} > 0$. Observe that this solution results from the steady state equations of the water supply of the turbine (upstream it), being imposed by the steady state mechanical load ν_g of the turbine. Afterwards the piezometric heads, which are linearly decreasing, will follow, also $\bar{z} = \bar{h}_1(1) = \bar{h}_2(0)$. Now it becomes possible to obtain the steady state of the hydraulic turbine and of its controller by solving the equations

$$\begin{aligned} &A_c\bar{x}_c + b_c(\varphi_0 - \bar{\varphi}) = 0; \quad \bar{f}_\theta = f_c^T\bar{x}_c + \gamma_c(\varphi_0 - \bar{\varphi}), \\ &(1-k)\bar{f}_\theta\sqrt{\nu_g/\bar{q}} + k\bar{\varphi} = \bar{q} \end{aligned}$$

allowing to find $\bar{\varphi}$, $\bar{f}_\theta$, $\bar{x}_c$, the reference φ_0 being given.

However, the steady state of the normal exploitation, just computed, is not of interest in this paper. We just mention that stability of this normal exploitation steady state is ensured by both the surge tank – which regulates the upstream water flow $q_2(\xi_2, \tau)$ *via* the water level oscillations $z(\tau)$ of the tank – and the speed controller – which regulates the water flow admitted in the turbine to realize the frequency/megawatt (φ *versus* ν_g) control of the Power Grid.

As already mentioned in Section 1, during the abnormal regimes generating water hammer – the sudden turbine load discharge – the turbine with its speed controller are "cut" from the upstream dynamics and f_θ – the controlled cross-section area of the wicket gates – is assumed "blocked" at a constant value.

3 Inherent stability of the surge tank

It has been just shown that the surge tank has regulatory role for the water flow upstream the turbine. This regulatory role is more obvious during water hammer, when the turbine and its speed controller are "cut" from the upstream. A standard engineering philosophy states that a stabilizing device should display inherent stability itself. The stability analysis for the surge tank is done under some unanimously accepted assumptions going back to the early period of hydraulic power engineering [3,20,24,27].

3.1 The inferred engineering model

According the the aforementioned literature (and not only), the stability model for the surge tank relies on three equations: the dynamics equation, the continuity equation and the load control equation.

The dynamics equation is the so called *inelastic water column, upstream the surge tank, equation* - in fact the water column in the tunnel; the term inelastic defines the lumped flow parameters.

It is adopted as such and it reads

$$\frac{L_1}{g}\frac{\mathrm{d}V_1}{\mathrm{d}t} + (Z - H_0) + P_1|V_1|V_1 + R_s|V_s|V_s = 0 \tag{3.1}$$

where $P_1|V_1|V_1$ accounts for the hydraulic losses at the input of the surge tank and $R_s|V_s|V_s$ are the losses through the tank throttling: we have $V_s = \mathrm{d}Z/\mathrm{d}t$ and in some cases this term is linearized. The introduction of the modulus in the losses terms is done for the case of the reverse (upstream) flow which might appear during transients.

The continuity equation is nothing more but the mass balance equation for the surge tank

$$F_s\frac{\mathrm{d}Z}{\mathrm{d}t} = F_1V_1 - Q_T \tag{3.2}$$

where Q_T is the "load" water flow reaching the hydraulic turbine. This flow, which should ensure delivery of the required mechanical power, is defined by the *load control equation*, which is static – of the form

$$Q_T = f_T(N_g, Z). \tag{3.3}$$

The static load control function is an inference, deduced from several facts: firstly, the hydroelectric plants had relatively small powers and, as a consequence, the penstocks were short and the turbines located near the surge tanks. At its turn this fact allowed neglecting the dynamics of the penstock and of the turbine, also of the hydraulic losses. Following the load instantaneously induces also a more difficult dynamic condition for the surge tank and may therefore be considered as covering ("worst case") from the engineering point of view.

Starting from the hydraulic power definition, namely $N_g = \eta_\theta(\gamma/(2g))H_TQ_T$, H_T being the piezometric head at the wicket gates of the turbine, taking into account that head losses between the surge tank and the hydraulic turbine are negligible and neglected, it follows that $H_T = Z$ and

$$f_T(N_g, Z) = \frac{N_g}{\eta_\theta\frac{\gamma}{2g}Z}. \tag{3.4}$$

The model is thus given by (3.1), (3.2), (3.4) and, as already specified, represents an inference – at the engineering level of rigor – from certain equations of the Hydraulic engineering. From this moment, however, no additional physical or engineering assumptions can be introduced and the analysis will deal with the differential equations

$$\begin{aligned} &\frac{L_1}{g}\frac{\mathrm{d}V_1}{\mathrm{d}t} + (Z - H_0) + P_1|V_1|V_1 + R_s\left|\frac{\mathrm{d}Z}{\mathrm{d}t}\right|\frac{\mathrm{d}Z}{\mathrm{d}t} = 0,\\ &F_s\frac{\mathrm{d}Z}{\mathrm{d}t} = F_1V_1 - \frac{N_g}{\eta_\theta\frac{\gamma}{2g}Z}. \end{aligned} \tag{3.5}$$

In order to use a unitary framework, we rate the flows at $\bar{Q}$ (as in Section 2), the piezometric heads to H_0 and denote

$$q_1 := F_1V_1/\bar{Q}\,, \qquad z := (Z - H_0)/H_0$$

Therefore equations (3.5) become

$$\begin{aligned} &T_{w1}\frac{\mathrm{d}q_1}{\mathrm{d}t} + z + \frac{P_1\bar{Q}^2}{F_1^2H_0}|q_1|q_1 + \frac{R_s}{H_0^2}\left|\frac{\mathrm{d}z}{\mathrm{d}t}\right|\frac{\mathrm{d}z}{\mathrm{d}t} = 0\\ &T_s\frac{\mathrm{d}z}{\mathrm{d}t} = q_1 - \frac{\nu_g}{1+z} \end{aligned} \tag{3.6}$$

with T_{w1} and T_s as defined in Section 2. While the fill up time constant T_1 does not appear in the present inference, it is however possible to introduce the rated time $\tau = t/T_1$ to transform (3.6) as below

$$\begin{aligned} &\theta_{w1}\frac{dq_1}{d\tau} + z + P_1'|q_1|q_1 + \lambda_s\left|\frac{dz}{d\tau}\right|\frac{dz}{d\tau} = 0, \\ &\theta_s\frac{dz}{d\tau} = q_1 - \frac{\nu_g}{1+z}, \end{aligned} \tag{3.7}$$

where the rated coefficients of the losses and of the throttling are as follows

$$P_1' := \frac{P_1\bar{Q}^2}{F_1^2 H_0}, \qquad \lambda_s := \frac{R_s}{H_0^2 T_1^2} = \frac{R_s\bar{Q}^2}{F_1^2 L_1^2 H_0}. \tag{3.8}$$

We proceed now to analyze stability of the surge tank based on (3.7), following [14, 16]. The steady state (equilibrium) imposed by following the load ν_g is given by

$$\bar{z} + P_1'|\bar{q}_1|\bar{q}_1 = 0, \qquad \bar{q}_1 = \frac{\nu_g}{1+\bar{z}}$$

The physically significant steady states correspond to positive flows (flowing downstream), what implies $1+\bar{z} > 0$ i.e. the water level in the surge tank, usually lower than the lake water level ($\bar{z} < 0$ since $\nu_g < 1$), cannot be under the basic reference level. Therefore $\bar{z}$ is a real solution of the third degree equation

$$\bar{z}(1+\bar{z})^2 + P_1'\nu_g^2 = 0. \tag{3.9}$$

If $P_1'\nu_g^2 > 4/27$, equation (3.9) has a single real root which is lower than -1 hence this case is not acceptable from the engineering point of view. In practice the parameters are chosen to have the reverse i.e. $P_1'\nu_g^2 < 4/27$ – when (3.9) has three real roots: $\bar{z}_1 \in (-1/3, 0)$, $\bar{z}_2 \in (-1, -1/3)$, $\bar{z}_3 \in (-\infty, -1)$. The third has no engineering significance, as already mentioned, while $\bar{z}_1$ is the acceptable one. We shall discuss the stability of the equilibrium defined by it. We introduce firstly the deviations

$$\zeta := z - \bar{z}_1, \qquad v := \frac{d\zeta}{d\tau} \tag{3.10}$$

which are subject to the following differential equations

$$\begin{aligned} \frac{d\zeta}{d\tau} &= v, \\ \theta_s\frac{dv}{d\tau} &= \frac{d}{d\tau}\left(\theta_s\frac{d\zeta}{d\tau}\right) \\ &= \frac{d}{d\tau}\left(q_1 - \frac{\nu_g}{1+\bar{z}_1+\zeta}\right) \\ &= \frac{\nu_g}{(1+\bar{z}_1+\zeta)^2}v - \frac{1}{\theta_{w1}}[\bar{z}_1 + \zeta + P_1'|q_1|q_1 + \lambda_s|v|v] \\ &= \frac{\nu_g}{(1+\bar{z}_1+\zeta)^2}v \\ &\quad - \frac{1}{\theta_{w1}}\left[\bar{z}_1 + \zeta + P_1'\left|\theta_s v + \frac{\nu_g}{1+\bar{z}_1+\zeta}\right|\left(\theta_s v + \frac{\nu_g}{1+\bar{z}_1+\zeta}\right) + \lambda_s|v|v\right]. \end{aligned} \tag{3.11}$$

To system (3.11) it is attached the following Lyapunov function

$$\mathcal{V}(\zeta, v) = \frac{1}{2}\theta_s v^2 + \frac{1}{2\theta_{w1}}\left[1 - 2P_1'\frac{\nu_g^2}{(1+\bar{z}_1)^2(1+\bar{z}_1+\zeta)}\right]\zeta^2. \tag{3.12}$$

This function is positive definite in the following domain of the phase plane (ζ, v)

$$1 - 2P_1' \frac{\nu_g^2}{(1+\bar{z}_1)^2(1+\bar{z}_1+\zeta)} = 1 + \frac{2\bar{z}_1}{1+\bar{z}_1+\zeta} = \frac{1+3\bar{z}_1+\zeta}{1+\bar{z}_1+\zeta} > 0. \tag{3.13}$$

The acceptable condition is the strip $\zeta > -(1+3\bar{z}_1)$ which contains $(0,0)$ – the equilibrium of (3.11) corresponding to the equilibrium of (3.9) $(\bar{z}_1, \nu_g(1+\bar{z}_1)^{-1})$.

The next condition is given by positiveness of the water flow q_1 (flowing downstream – what happens in most time even during water hammer transients). The condition

$$q_1 > 0 \;\Leftrightarrow\; \theta_s v + \frac{\nu_g}{1+\bar{z}_1+\zeta} > 0 \;\Leftrightarrow\; v(1+\bar{z}_1+\zeta) > -\frac{\nu_g}{\theta_s} \tag{3.14}$$

defines a domain also containing the origin $(0,0)$.

Consider firstly the simpler case of following a zero load – the completely discharged turbine. Therefore $\nu_g = 0$, $\bar{z}_1 = 0$. System (3.11) becomes

$$\frac{\mathrm{d}\zeta}{\mathrm{d}\tau} = v \;;\; \theta_s \frac{\mathrm{d}v}{\mathrm{d}\tau} - \frac{1}{\theta_{w1}}[\zeta + (P_1'\theta_s^2 + \lambda_s)|v|v] \tag{3.15}$$

with the Lyapunov function

$$\mathcal{V}(\zeta, v) = \frac{1}{2}(\theta_s v^2 + \frac{1}{\theta_{w1}}\zeta^2) > 0. \tag{3.16}$$

The derivative function will be

$$\mathcal{W}(\zeta, v) = -\frac{1}{\theta_{w1}}(P_1'\theta_s^2 + \lambda_s)|v|v^2 \le 0.$$

The asymptotic stability follows immediately by applying the Barbashin–Krasovskii–LaSalle invariance principle. The same result is however straightforward from (3.7), where $\nu_g = 0$ will show $(0,0)$ to be the unique steady state. Further, system (3.7) can be written as a single second order differential equation

$$\frac{\theta_s^2}{\theta_{w1}} \frac{\mathrm{d}^2 z}{\mathrm{d}\tau^2} + (P_1'\theta_s^2 + \lambda_s)\left|\frac{\mathrm{d}z}{\mathrm{d}\tau}\right|\frac{\mathrm{d}z}{\mathrm{d}\tau} + z = 0 \tag{3.17}$$

which describes an oscillator with nonlinear damping. The Lyapunov function (3.16) is just oscillator's total energy.

Let now $\nu_g > 0$ i.e. the load discharge is not full. The stability domain is delimited by (3.13) and (3.14). The derivative function of (3.12) will be now, under conditions (3.13) and (3.14)

$$\begin{aligned}
\mathcal{W}(\zeta, v) = &\left\{ \frac{\nu_g}{(1+\bar{z}_1+\zeta)^2} v - \frac{1}{\theta_{w1}} \left[\bar{z}_1 + \zeta + P_1' \left(\theta_s v + \frac{\nu_g}{1+\bar{z}_1+\zeta} \right)^2 + \lambda_s |v| v \right] \right\} v \\
&+ \frac{1}{\theta_{w1}} \left\{ \left[1 - 2P_1' \frac{\nu_g^2}{(1+\bar{z}_1)^2(1+\bar{z}_1+\zeta)} \right] \zeta + P_1' \frac{\nu_g^2}{(1+\bar{z}_1)^2(1+\bar{z}_1+\zeta)^2} \zeta^2 \right\} v \\
= &\frac{\nu_g}{(1+\bar{z}_1+\zeta)^2} v^2 - \frac{1}{\theta_{w1}} \left\{ -\frac{P_1'\nu_g^2}{(1+\bar{z}_1)^2} + P_1' \left(\theta_s v + \frac{\nu_g}{1+\bar{z}_1+\zeta} \right)^2 + \right. \\
&\left. + 2P_1' \frac{\nu_g^2}{(1+\bar{z}_1)^2(1+\bar{z}_1+\zeta)} \zeta - P_1' \frac{\nu_g^2}{(1+\bar{z}_1)^2(1+\bar{z}_1+\zeta)^2} \zeta^2 + \lambda_s |v| v \right\} v
\end{aligned}$$

We compute

$$-\frac{P_1'\nu_g^2}{(1+\bar{z}_1)^2}\left[1-\frac{2\zeta}{1+\bar{z}_1+\zeta}+\frac{\zeta^2}{(1+\bar{z}_1+\zeta)^2}\right]=-\frac{P_1'\nu_g^2}{(1+\bar{z}_1+\zeta)^2}$$

to obtain further

$$\begin{aligned}\mathcal{W}(\zeta,v)&=\frac{\nu_g}{(1+\bar{z}_1+\zeta)^2}v^2-\frac{1}{\theta_{w1}}\left[P_1'\left(\theta_s v+\frac{\nu_g}{1+\bar{z}_1+\zeta}\right)^2-\frac{P_1'\nu_g^2}{(1+\bar{z}_1+\zeta)^2}+\lambda_s|v|v\right]v\\&=-\left[\frac{\theta_s}{\theta_{w1}}P_1'\left(\theta_s v+\frac{2\nu_g}{1+\bar{z}_1+\zeta}\right)-\frac{\nu_g}{(1+\bar{z}_1+\zeta)^2}+\frac{\lambda_s}{\theta_{w1}}|v|\right]v^2.\end{aligned}\tag{3.18}$$

We seek conditions for $\mathcal{W}(\zeta,v)\le 0$ under (3.13) and (3.14). A *necessary* (not sufficient) condition would be fulfilment of $\mathcal{W}(\zeta,v)\le 0$ in a *small neighborhood* of the origin $(0,0)$. This condition reads

$$2\frac{\theta_s}{\theta_{w1}}P_1'\frac{\nu_g}{1+\bar{z}_1}-\frac{\nu_g}{(1+\bar{z}_1)^2}>0\ \Leftrightarrow\ 2\frac{\theta_s}{\theta_{w1}}P_1'(1+\bar{z}_1)>1\tag{3.19}$$

which imposes a lower limit for the time constant θ_s of the surge tank, in fact for the cross-section area of the surge tank. Taking into account the definitions of θ_s, θ_{w1}, P_1' and $\bar{z}_1$ we shall have

$$\begin{aligned}2\frac{\theta_s}{\theta_{w1}}P_1'(1+\bar{z}_1)&=2\frac{T_s}{T_{w1}}P_1'(1+\bar{z}_1)=2\frac{F_sH_0}{\bar{Q}}\frac{F_1H_0g}{L_1\bar{Q}}\frac{P_1\bar{Q}^2}{F_1^2H_0}\frac{H_0+\bar{Z}_1}{H_0}\\&=2\frac{F_s}{F_1}\frac{g}{L_1}P_1(H_0+\bar{Z}_1)>1\end{aligned}$$

or

$$F_s>\frac{1}{2}\frac{L_1}{g}\frac{1}{P_1(H_0+\bar{Z}_1)}F_1=F_{Th}\tag{3.20}$$

where F_{Th} is the so called *Thoma cross-section area* introduced by D. Thoma in his doctoral thesis [38]. Since, as mentioned, (3.19) is necessary, not sufficient for $\mathcal{W}(\zeta,v)\le 0$, we turn again to (3.18) and re-write it as follows

$$\mathcal{W}(\zeta,v)=\left[\frac{\theta_s^2}{\theta_{w1}}P_1'(\alpha|v|+v)+2\frac{\theta_s}{\theta_{w1}}P_1'\frac{\nu_g}{1+\bar{z}_1+\zeta}-\frac{\nu_g}{(1+\bar{z}_1+\zeta)^2}\right]v^2$$

where $\alpha=\lambda_s(P_1'\theta_s^2)^{-1}$. Suppose $\alpha>1$, then $\mathcal{W}(\zeta,v)\le 0$ provided

$$2\frac{\theta_s}{\theta_{w1}}P_1'(1+\bar{z}_1+\zeta)>1.\tag{3.21}$$

It can be seen that (3.21) implies (3.19) which thus appears genuinely as necessary but not sufficient. Even if $\alpha<1$, then (3.21) becomes a necessary condition for $\mathcal{W}(\zeta,v)\le 0$ with (3.19) as necessary for the fulfilment of (3.21).

Consider now the expressions of the aforementioned parameters. We find

$$\alpha=\frac{R_sH_0}{T_1^2}\frac{F_1^2H_0}{P_1\bar{Q}^2}\frac{T_1^2}{T_s^2}=\frac{R_s}{P_1}\left(\frac{F_1}{F_s}\right)^2$$

and, for (3.21)

$$2\frac{F_s}{F_1}\frac{g}{L_1}P_1(\bar{Z}_1+Y)>1\,,\qquad Y:=Z-\bar{Z}_1.\tag{3.22}$$

The last inequality is easily re-written as

$$\bar{Z}_1 + Y > \bar{Z}_1 \frac{F_{Th}}{F_s} \tag{3.23}$$

and it implies (3.20) which again appears as a necessary condition.

Unfortunately condition $\alpha > 1$ turns to be completely *unrealistic* since normally $R_s < P_1$ and $F_1 \ll F_s$. We have thus to consider the case $\alpha < 1$. Following [16], we consider the following function

$$\Phi(\zeta, v) = v - \frac{\theta_{w1}}{\theta_s^2 P_1'} \frac{\nu_g}{1 + \bar{z}_1 + \zeta} \left[\frac{1}{1 + \bar{z}_1 + \zeta} - \frac{2}{\theta_s} \right] \tag{3.24}$$

and if $\Phi(\zeta, v) > 0$, it follows that $\mathcal{W}(\zeta, v) \leq 0$. Together with the Barbashin–Krasovskii–LaSalle invariance principle, this will give asymptotic stability.

Concerning the estimate of the attraction domain of the equilibrium $(0,0)$, we have to consider the interior of the domain defined by (3.13), (3.14) and $\Phi(\zeta, v) > 0$, together with the family of curves $\Psi_c(\zeta, v) = \{(\zeta, v) \mid \mathcal{V}(\zeta, v) = c > 0\}$ which are closed for $c > 0$ small enough. An estimate of the attraction domain is the domain inside Ψ_c completely included in the domain defined by (3.13), (3.14) and $\Phi(\zeta, v) > 0$, $c > 0$ being maximal from this point of view. Summarizing, the mathematical result is as follows.

Theorem 3.1. *Consider the system* (3.11), *the associated Lyapunov function* (3.12) *and its derivative function along* (3.11) *– its Lie derivative* $\mathcal{W}(\zeta, v)$ *– given by* (3.18). *The equilibrium* $(0,0)$ *of (3.11) is asymptotically stable with the attraction domain contained in the set of the phase plane* (ζ, v) *defined by* (3.13), (3.14) *and* $\Phi(\zeta, v) > 0$. *A standard estimate of this domain is given by inequalities of the form* $\mathcal{V}(\zeta, v) < c$ *with* $c > 0$ *maximally possible in order to have* $\mathcal{V}(\zeta, v) = c$ *closed curves and the domain inside fully contained in the aforementioned set defined by* (3.13), (3.14) *and* $\{(\zeta, v) | \Phi(\zeta, v) > 0\}$.

Finally, let us remark that, since the attraction domain does not encompass the entire phase plane, the analysis should be completed with additional studies dealing with limit cycles and hidden attractors [2,22,32,33]. This extended analysis is outside the aims of this paper.

3.2 Modeling the surge tank in the context of several time scales

In this subsection we shall consider modeling of the surge tank stability dynamics as resulting from the model (2.3)–(2.4). This model displays distributed parameters, being defined by partial differential equations: it is valid for larger hydroelectric power plants unlike those for which the model considered in the previous subsection was inferred. Nevertheless, in the contemporary water hammer analysis, a difference is made between fast water mass oscillations – where partial differential equations are used in modeling – and slow water mass oscillations. This last case is more suitable for surge tank stability analysis. The explanation is that model (2.3)–(2.4) has several time scales, as follows e.g. from the size analysis of the occurring time constants. Taking as examples the two hydroelectric plants of Romania, mentioned at the beginning of Section 1, we can see that

a) for the "Bicaz" plant: $\theta_{w1} = 14.71 \times 10^{-3}$, $\theta_{p1} = 3.81 \times 10^{-3}$, $\theta_{w2} = 0.38 \times 10^{-3}$, $\theta_{p2} = 0.14 \times 10^{-3}$, $\theta_s = 0.502$, $\theta_a = 5.1 \times 10^{-3}$;

b) for the "Someș-Mărișelu" plant: $\theta_{w1} = 2.34 \times 10^{-3}$, $\theta_{p1} = 2.7 \times 10^{-3}$, $\theta_{w2} = 0.36 \times 10^{-3}$, $\theta_{p2} = 0.27 \times 10^{-3}$, $\theta_s = 0.108$, $\theta_a = 1.3 \times 10^{-3}$.

Consequently, the surge tank stability has to be studied at its time scale – given by the time constant θ_s. We take therefore the approach of (formal) singular perturbations, following also the standard engineering assumptions enumerated in the previous subsection.

By taking $\theta_{p1}^2/\theta_{w1} = \delta_1^2\theta_{w1} \approx 0$ ($\delta_1 = \theta_{p1}/\theta_{w1} = 0.26$), it follows that $\partial_{\xi_1} q_1 = 0$ hence $q_1(\xi_1, \tau) \equiv q_1(\tau)$; also $\partial_{\xi_1} q_1^2 = 0$. The equation for $h_1(\xi_1, \tau)$ becomes

$$\partial_{\xi_1} h_1 + \theta_{w1}\frac{dq_1}{d\tau} + \frac{1}{2}\frac{\theta_{w1}}{\theta_1}\frac{\lambda_1 L_1}{D_1}|q_1|q_1 = 0$$

and can be integrated with respect to ξ_1 from 0 to 1 to obtain

$$h_1(1,\tau) - h_1(0,\tau) + \theta_{w1}\frac{dq_1}{d\tau} + \frac{1}{2}\frac{\theta_{w1}}{\theta_1}\frac{\lambda_1 L_1}{D_1}|q_1|q_1 = 0,$$

$$h_1(1,\tau) = 1 + z(\tau) + \lambda_s\frac{dz}{d\tau}\,; \qquad h_1(0,\tau) \equiv 1.$$

Therefore

$$\theta_{w1}\frac{dq_1}{d\tau} + \frac{1}{2}\frac{\theta_{w1}}{\theta_1}\frac{\lambda_1 L_1}{D_1}|q_1|q_1 + z(\tau) + \lambda_s\frac{dz}{d\tau} = 0. \tag{3.25}$$

A comparison with the first equation of (3.7) is useful. This first equation of (3.7) and (3.25) are almost identical – the difference appears in the model of the surge tank throttling. In the model (2.3)–(2.4) it was considered linear [15,27], thus migrating in (3.25) while in other studies is taken quadratic [19,27,41]. In fact the modeling of the local hydraulic losses is made of engineering inferences: starting from the general laws of the Fluid Mechanics, a formula is inferred and then verified experimentally. Many constructive elements in engineering are modeled in this way, based on steady state behavior and measurements, then put together in a comprehensive dynamical model thus extending the steady state properties to dynamics. This explains the necessary validation of the mathematical model [28].

The second equation of the surge tank model is the continuity one i.e.

$$\theta_s\frac{dz}{d\tau} = q_1(\tau) - q_2(0,\tau) \tag{3.26}$$

(we already took $q_1(\tau)$ from the equation (3.25)). The remaining modeling problem is to represent the load flow. The engineering requirement is that the load flow should follow a *static* external mechanical load. Since in statics (but rated variables) we have the formula $\nu_g = q_2 h_2$ with all terms – constant, it follows that $q_2 = \nu_g/h_2 = \nu_g(1+z)^{-1}$. This is an engineering inference which *has not been deduced* from (2.3). The model

$$\begin{aligned} &\theta_{w1}\frac{dq_1}{d\tau} + \frac{1}{2}\frac{\theta_{w1}}{\theta_1}\frac{\lambda_1 L_1}{D_1}|q_1|q_1 + z(\tau) + \lambda_s\frac{dz}{d\tau} = 0,\\ &\theta_s\frac{dz}{d\tau} = q_1 - \frac{\nu_g}{1+z} \end{aligned} \tag{3.27}$$

is very much alike to (3.7) but it is obtained (partly) from (2.3)–(2.4). We can try however to point out possible assumptions leading to the second equation of (3.27) and/or (3.7).

Following a static load means firstly letting to zero all time constants multiplying the time derivatives downstream the surge tank: $\theta_{p2}^2/\theta_{w2} = \delta_2^2\theta_{w2} = 0$, $\theta_{w2} = 0$, $\theta_a = 0$. It follows that $q_2(\xi_2, \tau) \equiv q_2(\tau)$ and

$$\frac{dh_2}{d\xi_2} + \frac{1}{2}\frac{\theta_{w2}}{\theta_2}|q_2|q_2 = 0.$$

Since the penstock is much shorter than the tunnel, the engineering assumption is that the losses along the penstock are negligible; this inference is consistent with $\theta_{w2} \approx 0$ (according to the numerical data $\theta_{w2}/\theta_2 < \theta_{w2}$). Therefore $h_2(\xi_2, \tau) \equiv h_2(\tau)$. This implies

$$h_2(\tau) = 1 + z(\tau) + \lambda_s\frac{dz}{d\tau}\,; \qquad q_2 = \nu_g\left(1 + z(\tau) + \lambda_s\frac{dz}{d\tau}\right)^{-1}.$$

The resulting model will be

$$\begin{aligned} &\theta_{w1}\frac{\mathrm{d}q_1}{\mathrm{d}\tau}+\frac{1}{2}\frac{\theta_{w1}}{\theta_1}\frac{\lambda_1 L_1}{D_1}|q_1|q_1+z(\tau)+\lambda_s\frac{\mathrm{d}z}{\mathrm{d}\tau}=0,\\ &\theta_s\frac{\mathrm{d}z}{\mathrm{d}\tau}=q_1-\nu_g\left(1+z(\tau)+\lambda_s\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)^{-1} \end{aligned} \tag{3.28}$$

being different from (3.27) by the term $\lambda_s(\mathrm{d}z/\mathrm{d}\tau)$ in the second equation. Its significance is that now *the surge tank load flow must follow a dynamic load*.

Some additional comments are necessary. Model (3.28) *is not equivalent* to (2.3)–(2.4), but *it is obtained from it* by letting some small time constants to zero and neglecting the losses along the penstock (the water conduit corresponding to $i = 2$). The connection of the two models can be viewed at the level of their solutions by: a) neglecting the losses along the penstock in (2.3) also; b) comparing the solutions of the two mathematical models for the "small" time constants sufficiently small.

Other details are also to be specified. They are related to the hydraulic turbine and its speed controller and we have to consider again the engineering assumptions and inferences.

Surge tank stability is related to water hammer – an abnormal transient occurring as a result of a sudden, rather large load discharge. This load discharge initiates a safety maneuver of decoupling the controller (2.4), stopping the turbine ($\varphi = 0$) and blocking the wicket gates crossing area f_θ at a constant value $\bar{f}_\theta$. The boundary condition

$$q_2=(1-k)f_\theta\sqrt{h_2(\tau)}+k\varphi$$

combined with $q_2(\tau)h_2(\tau) = \nu_g$ and $\varphi = 0$ will give

$$\nu_g=(1-k)f_\theta(h_2(\tau))^{3/2}\ \Rightarrow\ f_\theta=\frac{\nu_g}{1-k}(h_2(\tau))^{3/2}=\frac{\nu_g}{1-k}\left(1+z(\tau)+\lambda_s\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)^{-1}$$

and

$$\bar{f}_\theta=\lim_{\tau\to\infty}\frac{\nu_g}{1-k}(h_2(\tau))^{3/2}=(1+\bar{z}_1)^{3/2}\frac{\nu_g}{1-k}.$$

It follows that the blocking value $\bar{f}_\theta$ is reached after a transient process due to the surge tank which, again, must be stable.

To end these considerations, let us mention that they contain several engineering inferences resulting from practice, some of them being assumed here for the sake of completeness, because civil hydraulic engineers, hydroelectric power engineers and automatic control engineers have rather few interactions: each of them is following the prescriptions and the experience of the corresponding domain of expertise.

3.3 Asymptotic stability and total stability

We shall consider here stability for the models (3.27) and (3.28). Both models have the same steady state given by

$$\frac{1}{2}\frac{\theta_{w1}}{\theta_1}\frac{\lambda_1 L_1}{D_1}\bar{q}_1^2+\bar{z}=0\,,\qquad \bar{q}_1=\frac{\nu_g}{1+\bar{z}} \tag{3.29}$$

which reduces to the third degree equation

$$\bar{z}(1+\bar{z})^2+A_q\nu_g^2=0\,;\qquad A_q:=\frac{1}{2}\frac{\theta_{w1}}{\theta_1}\frac{\lambda_1 L_1}{D_1} \tag{3.30}$$

like in the case discussed in Subsection 3.1 In fact (3.7) and (3.27) have the same structure. For $A_q\nu_g^2 < 4/27$ equation (3.30) has three real roots, among which $\bar{z}_1 \in (-1/3,0)$ is the acceptable one in applications. It is interesting to give computed data for the aforementioned inequality. Using the same data for the two already mentioned hydroelectric power plants of Romania, we shall have $A_q \approx 0.073$ for "Bicaz" and $A_q \approx 0.032$ for "Someș-Mărișelu". Since $0 < \nu_g < 1$ and $4/27 \approx 0.148$, the fulfilment of $A_q\nu_g^2 < 4/27$ is obvious.

Model (3.27) having the same structure as (3.7), we can introduce again the deviations (3.10) which are subject to the system in deviations – much alike to (3.11)

$$\begin{aligned} \frac{\mathrm{d}\zeta}{\mathrm{d}\tau} &= v, \\ \theta_s\frac{\mathrm{d}v}{\mathrm{d}\tau} &= \frac{\nu_g}{(1+\bar{z}_1+\zeta)^2}v \\ &\quad - \frac{1}{\theta_{w1}}\left[\bar{z}_1+\zeta+A_q\left|\theta_s v+\frac{\nu_g}{1+\bar{z}_1+\zeta}\right|\left(\theta_s v+\frac{\nu_g}{1+\bar{z}_1+\zeta}\right)+\lambda_s v\right]. \end{aligned} \tag{3.31}$$

The only differences in comparison to (3.11) are to have A_q instead of P_1' and $\lambda_s v$ instead of $\lambda_s|v|v$; the last difference is introduced as a result of having a linear throttling model. Associate the same (in fact) Lyapunov function

$$\mathcal{V}(\zeta,v) = \frac{1}{2}\theta_s v^2 + \frac{1}{2\theta_{w1}}\left[1-2A_q\frac{\nu_g^2}{(1+\bar{z}_1)^2(1+\bar{z}_1+\zeta)}\right]\zeta^2 \tag{3.32}$$

which is strictly positive definite in the strip $\zeta > -(1+3\bar{z}_1)$ containing the equilibrium $(0,0)$. Computing the derivative function

$$\mathcal{W}(\zeta,v) = -\left[\frac{\theta_s}{\theta_{w1}}A_q\left(\theta_s v+\frac{2\nu_g}{1+\bar{z}_1+\zeta}\right)-\frac{\nu_g}{(1+\bar{z}_1+\zeta)^2}+\frac{\lambda_s}{\theta_{w1}}\right]v^2$$

valid in the phase plane domain $q_1 > 0$ i.e. $\theta_s v + \nu_g(1+\bar{z}_1+\zeta)^{-1} > 0$, we remark that the "helpful" term $\lambda_s/\theta_{w1} > 0$ is now constant everywhere in the phase plane. The necessary condition of (3.19) type is now

$$2\frac{\theta_s}{\theta_{w1}}A_q\frac{\nu_g}{1+\bar{z}_1} - \frac{\nu_g}{(1+\bar{z}_1)^2} + \frac{\lambda_s}{\theta_{w1}} > 0 \tag{3.33}$$

and it is relaxed in comparison to (3.19), because of the term $\lambda_s/\theta_{w1} > 0$. Condition (3.24) also can be relaxed since now we shall have

$$\Phi(\zeta,v) = v - \frac{\theta_{w1}}{\theta_s^2A_q}\frac{\nu_g}{1+\bar{z}_1+\zeta}\left[\frac{1}{1+\bar{z}_1+\zeta}-\frac{2}{\theta_s}\right] + \frac{\lambda_s}{\theta_s^2A_q} > 0 \tag{3.34}$$

the term $\lambda_s(\theta_s^2A_q)^{-1}$ being again helpful.

Consider now the model (3.28). While the model (3.27) relies on the load curve $q_2 = \nu_g(1+z)^{-1}$ which is inferred while accepted by the hydraulic engineering community, model (3.28) is obtained partly from (2.3)–(2.4) and this imposes a dynamic load curve defined by $q_2 = \nu_g(1+z+\lambda_s\mathrm{d}z/\mathrm{d}\tau)^{-1}$. This model is not *homologated* within the hydraulic engineering community, possibly because for $\lambda_s = 0$ (surge tank without throttling) the two models coincide, also because for most surge tanks the throttling effect is neglected ($\lambda_s \approx 0$ in real data).

Constructing a Lyapunov function for (3.28) is a new and distinct problem which is outside the mainstream of the present paper, dealing with a model which has to be adopted as such.

As a preliminary analysis, we can however consider (3.28) written as

$$
\begin{aligned}
&\theta_{w1}\frac{dq_1}{d\tau}+\frac{1}{2}\frac{\theta_{w1}}{\theta_1}\frac{\lambda_1 L_1}{D_1}|q_1|q_1+z(\tau)+\lambda_s\frac{dz}{d\tau}=0,\\
&\theta_s\frac{dz}{d\tau}=q_1-\frac{\nu_g}{1+z}+\frac{\nu_g\lambda_s}{(1+z)(1+z+\lambda_s dz/d\tau)}\frac{dz}{d\tau}.
\end{aligned}
\tag{3.35}
$$

The last term in the second equation of (3.35) is a *persistent perturbation*. It is thus possible to search for the total stability [30,40] i.e. stability with respect to persistent perturbations [13,23]. The basic result on stability under persistent perturbations (total stability) is from 1944 and is due to Malkin – see [23], pages 301 and next, also [30] (Theorems II.4.4 and II.4.5), [40], pages 118 and next.

The assumptions of these basic results of Malkin are fulfilled by (3.35) and its associated Lyapunov function (3.32) – re-written in the state variables of (3.27), (3.28), (3.35), provided they are considered on bounded domains of the state space (e.g. of the form $\mathcal{V}(\zeta,v)\le c$). Obviously the attraction domain of the origin under persistent perturbations is rather small. Improvements can be sought using more refined results on perturbed dynamical systems [4–7,34–37]

4 Water hammer stability analysis

For this analysis we shall start from the mathematical model (2.3) under the assumptions of [15,27]. The basic one is to neglect the Darcy–Weisbach losses along the two water conduits (the tunnel and the penstock). From the engineering point of view, with an argument at the physical level of rigor, this assumption is covering: water hammer oscillation quenching is connected to energy dissipation and the analysis is done precisely without certain energy dissipation terms. Next, neglecting the dynamic (velocity) heads variation $\partial_{\xi_i}q_i^2$ is standard in hydraulic engineering – see all the cited "hydraulic" literature – and we shall not elaborate on this assumptions. Therefore the starting model will be now the following

$$
\begin{aligned}
&\theta_{wi}\partial_\tau q_i+\partial_{\xi_i}h_i=0,\quad \frac{\theta_{pi}^2}{\theta_{wi}}\partial_\tau h_i+\partial_{\xi_i}q_i=0;\quad i=1,2,\\
&h_1(0,\tau)\equiv 1,\quad h_1(1,\tau)=1+z(\tau)+\lambda_s\frac{dz}{d\tau}=h_2(0,\tau),\\
&q_2(1,\tau)=(1-k)f_\theta(\tau)\sqrt{h_2(1,\tau)}+k\varphi(\tau),\\
&\theta_s\frac{dz}{d\tau}=q_1(1,\tau)-q_2(0,\tau),\quad \theta_a\frac{d\varphi}{d\tau}=q_2(1,\tau)h_2(1,\tau)-\nu_g.
\end{aligned}
\tag{4.1}
$$

This form of the equations will turn to be helpful for the basic theory for (4.1).

The ignition of the water hammer takes place as follows: system (4.1) starts from a normal steady state defined by

$$
\begin{aligned}
&\bar{h}_i(\xi_i)\equiv\text{const},\quad \bar{q}_i(\xi_i)\equiv\text{const}\\
&\bar{h}_1(0)=1,\quad \bar{h}_1(1)=1+\bar{z}=\bar{h}_2(0)\quad\Rightarrow\quad \bar{z}=0,\quad \bar{h}_1\equiv 1,\quad \bar{h}_2\equiv 1\\
&\bar{q}_1=\bar{q}_2=\bar{q},\quad \bar{q}=(1-k)\bar{f}_\theta+k\bar{\varphi};\quad \bar{q}=\nu_g
\end{aligned}
\tag{4.2}
$$

The steady state value of $\bar{\varphi}$ – the rotating speed of the hydraulic turbine – is imposed by the frequency/megawatt control of the Grid (together with the power level ν_g) and is ensured by the speed controller of the turbine.

From this steady state the system is moved to an abnormal operation by turbine shutdown $f_\theta \equiv 0$. Usually the case $k = 0$ is considered; the case $k \neq 0$ [3] is somehow unusual and we do not know if k does change during turbine shutdown. If k is kept at its previous value and the turbine is not unloaded instantaneously then the tendency will be to have the steady state (4.2) but with $\bar{\varphi} = \nu_g/k$. The turbine is probably unloaded before this steady state is reached (provided it is asymptotically stable); nevertheless the stability of the steady state defined by $\bar{f}_\theta = 0$, $\bar{\varphi} = \nu_g/k$ – starting from (4.2) is interesting in itself and its study was not undertaken (prior to our knowledge).

We shall however deal with basic theory for (4.1) with $\bar{f}_\theta(\tau) \equiv 0$ in order to deal with a linear Boundary value problem of nonstandard type. We call it nonstandard because its boundary conditions (of Dirichlet type) are controlled by ordinary differential equations which at their turn are controlled by the boundary conditions (an internal feedback).

4.1 Basic theory

We shall take the approach arising from the papers of A.D. Myshkis [1] and K.L. Cooke [8] (this one summarized and completely proven in [28]). This approach consists in associating to (4.1) a system of functional differential equations with deviated argument and establishing a one to one correspondence between the solutions of the two mathematical objects. As a consequence, *any property proven for one mathematical object is thus projected back on the other.*

We shall thus turn to (4.1) with $f_\theta(\tau) \equiv 0$. Introduce first the Riemann invariants $r_i^\pm(\xi_i,\tau)$ by

$$r_i^\pm(\xi_i,\tau) = \frac{1}{2}\left[\frac{\theta_{pi}}{\theta_{wi}} h_i(\xi_i,\tau) \pm q_i(\xi_i,\tau)\right] \tag{4.3}$$

with their inverses

$$h_i(\xi_i,\tau) = \frac{\theta_{wi}}{\theta_{pi}}\left[r_i^+(\xi_i,\tau) + r_i^-(\xi_i,\tau)\right], \quad q_i(\xi_i,\tau) = r_i^+(\xi_i,\tau) - r_i^-(\xi_i,\tau). \tag{4.4}$$

Consequently the boundary value problem (4.1) – with $f_\theta(\tau) \equiv 0$ – will be written with respect to the Riemann invariants as follows

$$\begin{aligned}
&\theta_{pi}\partial_\tau r_i^\pm \pm \partial_{\xi_i} r_i^\pm = 0, \quad i = 1,2,\\
&r_1^+(0,\tau) + r_1^-(0,\tau) = \frac{\theta_{p1}}{\theta_{w1}},\\
&\frac{\theta_{w1}}{\theta_{p1}}[r_1^+(1,\tau) + r_1^-(1,\tau)] = 1 + z(\tau) + \lambda_s \frac{\mathrm{d}z}{\mathrm{d}\tau} = \frac{\theta_{w2}}{\theta_{p2}}[r_2^+(0,\tau) + r_2^-(0,\tau)],\\
&\theta_s \frac{\mathrm{d}z}{\mathrm{d}\tau} = r_1^+(1,\tau) - r_1^-(1,\tau) - r_2^+(0,\tau) + r_2^-(0,\tau),\\
&r_2^+(1,\tau) - r_2^-(1,\tau) = k\varphi(\tau),\\
&\theta_a \frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = k\frac{\theta_{w2}}{\theta_{p2}}[r_2^+(1,\tau) + r_2^-(1,\tau)]\varphi - \nu_g.
\end{aligned} \tag{4.5}$$

In the strip $[0,1] \times \mathbb{R}$ we define the characteristic lines crossing $(\xi_i,\tau) \in [0,1] \times \mathbb{R}^+$, $i = 1,2$

$$\tau_i^\pm(\sigma;\xi_i,\tau) = \tau \pm \theta_{pi}(\sigma - \xi_i), \quad i = 1,2. \tag{4.6}$$

We now make use of the following properties of the characteristic lines and of the Riemann invariants along them: a) any increasing characteristic τ_i^+ can be extended "to the right" up to $\xi_i = 1$ and any decreasing characteristic τ_i^- can be extended "to the left" up to $\xi_i = 0$; b) the Riemann invariant r_i^+ (the forward wave) is constant along the increasing characteristic τ_i^+ and the Riemann invariant r_i^- (the backward wave) is constant along the decreasing characteristic τ_i^-. Consequently, the following representation formulae for the Riemann invariants, based on their boundary values, are obtained

$$r_i^+(\xi_i,\tau) = r_i^+(1,\tau+\theta_{pi}(1-\xi_i)), \quad r_i^-(\xi_i,\tau) = r_i^-(0,\tau+\theta_{pi}\xi) \ , \quad i=1,2. \tag{4.7}$$

Consider now those characteristics which can be extended on $(0,1)$ – r_i^+ "to the left" and r_i^- "to the right" – to obtain, after denoting $y_i^+(\tau) := r_i^+(1,\tau)$, $y_i^-(\tau) := r_i^-(0,\tau)$

$$\begin{aligned} r_i^+(0,\tau) &= r_i^+(1,\tau+\theta_{pi}) = y_i^+(\tau+\theta_{pi}), \\ r_i^-(1,\tau) &= r_i^-(0,\tau+\theta_{pi}) = y_i^+(\tau+\theta_{pi}). \end{aligned} \tag{4.8}$$

The functions $y_i^\pm(\tau)$ are then substituted in the boundary conditions of (4.5) to obtain

$$\begin{aligned} & y_1^+(\tau+\theta_{p1}) + y_1^-(\tau) = \frac{\theta_{p1}}{\theta_{w1}}, \\ & \frac{\theta_{w1}}{\theta_{p1}}(y_1^+(\tau) + y_1^-(\tau+\theta_{p1})) = 1 + z(\tau) + \lambda_s\frac{\mathrm{d}z}{\mathrm{d}\tau} = \frac{\theta_{w2}}{\theta_{p2}}(y_2^+(\tau+\theta_{p2}) + y_2^-(\tau)), \\ & \theta_s\frac{\mathrm{d}z}{\mathrm{d}\tau} = y_1^+(\tau) - y_1^-(\tau+\theta_{p1}) - y_2^+(\tau+\theta_{p2}) + y_2^-(\tau), \\ & y_2^+(\tau) - y_2^-(\tau+\theta_{p2}) = k\varphi(\tau), \\ & \theta_a\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = k\frac{\theta_{w2}}{\theta_{p2}}[y_2^+(\tau) + y_2^-(\tau+\theta_{p2})]\varphi - \nu_g. \end{aligned} \tag{4.9}$$

We introduce now the new functions $w_i^\pm(\tau) := y_i^\pm(\tau+\theta_{pi})$ to give (4.9) a form which is more "at hand" in the study of the systems with deviated argument

$$\begin{aligned} & w_1^+(\tau) + w_1^-(\tau-\theta_{p1}) = \frac{\theta_{p1}}{\theta_{w1}}, \\ & \frac{\theta_{w1}}{\theta_{p1}}(w_1^-(\tau) + w_1^+(\tau-\theta_{p1})) = 1 + z(\tau) + \lambda_s\frac{\mathrm{d}z}{\mathrm{d}\tau} = \frac{\theta_{w2}}{\theta_{p2}}(w_2^+(\tau) + w_2^-(\tau-\theta_{p2})), \\ & \theta_s\frac{\mathrm{d}z}{\mathrm{d}\tau} = w_1^+(\tau-\theta_{p1}) - w_1^-(\tau) - w_2^+(\tau) + w_2^-(\tau-\theta_{p2}), \\ & w_2^-(\tau) - w_2^+(\tau-\theta_{p2}) = -k\varphi(\tau), \\ & \theta_a\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = k\frac{\theta_{w2}}{\theta_{p2}}[w_2^-(\tau) + w_2^+(\tau-\theta_{p2})]\varphi - \nu_g. \end{aligned} \tag{4.10}$$

The differential and difference system (4.10) should be expressed in a form allowing the construction by steps of its solution. Our main concern is the two boundary conditions corresponding to those of the surge tank, namely

$$\begin{aligned}
\frac{\theta_{w1}}{\theta_{p1}}(w_1^-(\tau)+w_1^+(\tau-\theta_{p1})) &= z(\tau)+\frac{\lambda_s}{\theta_s}(w_1^+(\tau-\theta_{p1})-w_1^-(\tau) \\
&\quad - w_2^+(\tau)+w_2^-(\tau-\theta_{p2})), \\
\frac{\theta_{w2}}{\theta_{p2}}(w_2^+(\tau)+w_2^-(\tau-\theta_{p2})) &= z(\tau)+\frac{\lambda_s}{\theta_s}(w_1^+(\tau-\theta_{p1})-w_1^-(\tau) \\
&\quad - w_2^+(\tau)+w_2^-(\tau-\theta_{p2})).
\end{aligned} \tag{4.11}$$

Denoting for the simplicity of the writing $\delta_i := \theta_{pi}/\theta_{wi}$, $\lambda_s' := \lambda_s/\theta_s$, we obtain, after a straightforward manipulation including the inversion of a 2×2 matrix, the following system of coupled delay differential and difference equations

$$\begin{aligned}
\theta_s\frac{\mathrm{d}z}{\mathrm{d}\tau} &= \frac{1}{1+(\delta_1+\delta_2)\lambda_s'}[-(\delta_1+\delta_2)z(\tau)+2w_1^+(\tau-\theta_{p1})+2w_2^-(\tau-\theta_{p2})], \\
\delta_2\theta_a\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} &= k(2w_2^+(\tau-\theta_{p2})-k\varphi)\varphi-\nu_g, \\
w_1^+(\tau) &= \delta_1-w_1^-(\tau-\theta_{p1});\quad w_2^-(\tau)=w_2^+(\tau-\theta_p2)-k\varphi(\tau), \\
w_1^-(\tau) &= \frac{1}{1+(\delta_1+\delta_2)\lambda_s'}[\delta_1 z(\tau)-(1+(\delta_2-\delta_1)\lambda_s')w_1^+(\tau-\theta_{p1}) \\
&\quad +2\delta_1\lambda_s' w_2^-(\tau-\theta_{p2})], \\
w_2^+(\tau) &= \frac{1}{1+(\delta_1+\delta_2)\lambda_s'}[\delta_2 z(\tau)+2\delta_2\lambda_s' w_1^+(\tau-\theta_{p1}) \\
&\quad -(1+(\delta_1-\delta_2)\lambda_s')w_2^-(\tau-\theta_{p2})]
\end{aligned} \tag{4.12}$$

Observe that the equation of φ is a Riccati equation and this might ignite finite time escape. Now, the solution of (4.12) can be constructed by steps provided initial conditions are given on $(-\theta_{pi},0)$ for $w_i^\pm(\tau)$, $i=1,2$. If $\varphi(0),z(0)$ are given, as well as $w_i^\pm(\tau)$ on $(-\theta_{pi},0)$, then $(\varphi(\tau),z(\tau))$ can be obtained on $(0,\theta_{pi})$. Next, using the initial data and $(\varphi(\tau),z(\tau)$ on $(0,\theta_{pi})$, $w_i^\pm(\tau)$ can be obtained on $(0,\theta_{pi})$ from the difference equations. The process is then iterated on the following interval. The resulting solution appears to be continuous and piecewise differentiable – the state variables z and φ – while $w_i^\pm$ have the smoothness of their initial conditions and, in general, have finite discontinuities ("jumps") in $\tau = m_1\theta_{p1}+m_2\theta_{p2}$, where m_i are integers. It is also quite clear that the solution can be constructed also backwards.

All this construction is conditioned by the knowledge of the initial conditions $w_{io}^\pm(\tau)$, $-\theta_{pi}\le\tau<0$, $i=1,2$. These initial conditions can be obtained starting from the initial conditions of (4.3): starting from the initial conditions of (4.1) namely $(q_i^o(\xi_i),h_i^o(\xi_i))$ given on $(0,1)$, we use (4.3) to obtain $r_{io}^\pm(\xi_i)$ on $(0,1)$.

Consider those points (ξ_i,τ) which are such that the characteristic $\tau_i^+(\sigma;\xi_i,\tau)$ cannot be extended "to the left" up to $\xi_i=0$ but only to the point where $\tau+\theta_{pi}(\sigma-\xi_i)=0$ i.e. up to $\sigma=\xi_i-\tau/\theta_{pi}$. It follows that

$$r_i^+(\xi_i-\tau/\theta_{pi},0)=r_i^+(1,\tau+\theta_{pi}(1-\xi_i))=w_i^+(\tau-\theta_{pi}\xi_i).$$

Since $0\le\xi_i-\tau/\theta_{pi}\le 1$, it follows that $w_{io}^+(\theta)=r_{io}^+(-\theta/\theta_{pi})$ with $-\theta_{pi}\le\theta\le 0$. In the same way, using those characteristic lines $\tau_i^-(\sigma;\xi_i,\tau)$ which cannot be extended to $\sigma=1$ but only to the point where $\tau-\theta_{pi}(\sigma-\xi_i)=0$, i.e. to $\sigma=\xi_i+\tau/\theta_{pi}$, the following initial condition is obtained

$$r_i^-(\xi_i+\tau/\theta_{pi},0)=r_i^-(0,\tau+\theta_{pi}\xi_i)=w_i^-(\tau+\theta_{pi}(\xi_i-1)),$$

hence $w_{io}^-(\theta) = r_{io}^-(1+\theta/\theta_{pi})$ with $-\theta_{pi} \le \theta \le 0$.

Consider now the converse: let $\{z(0),\varphi(0),w_{io}^\pm(\theta),-\theta_{pi} \le \theta \le 0\}$ be a set of initial conditions for (4.12). The procedure by steps allows construction of the corresponding solution for (4.12). Define

$$\begin{aligned} &r_i^+(\xi_i,\tau) = w_i^+(\tau-\theta_{pi}\xi_i)\ ,\ r_i^-(\xi_i,\tau) = w_i^-(\tau+\theta_{pi}(\xi_i-1)),\\ &h_i(\xi_i,\tau) = \frac{1}{\delta_i}[w_i^+(\tau-\theta_{pi}\xi_i)+w_i^-(\tau+\theta_{pi}(\xi_i-1))],\\ &q_i(\xi_i,\tau) = w_i^+(\tau-\theta_{pi}\xi_i)-w_i^-(\tau+\theta_{pi}(\xi_i-1)). \end{aligned} \tag{4.13}$$

Then, if $w_{io}^\pm(\theta)$ are sufficiently smooth, the set of functions $\{z(\tau),\varphi(\tau);h_i(\xi_i,\tau),q_i(\xi_i,\tau)\}$ is a (possibly discontinuous) classical solution of (4.1) with the initial conditions $\{z(0),\varphi(0);$ $h_i(\xi_i,0),q_i(\xi_i,0)\}$. Summarizing, the following result has been obtained and proven.

Theorem 4.1. *Consider the boundary value problem defined by (4.1) with $f_\theta(\tau)\equiv 0$ and a set of initial conditions $\{z(0),\varphi(0);h_i^o(\xi_i),q_i^o(\xi_i),0\le\xi_i\le 1,i=1,2\}$ with $\{h_i^o,q_i^o\}$ sufficiently smooth to define a classical solution for (4.1). Let $r_i^\pm(\xi_i,\tau)$ – defined by (4.3) – be the corresponding Riemann invariants of this solution. Let $y_i^\pm(\tau)$ defined by (4.8) and $w_i^\pm(\tau) := y_i^\pm(\tau+\theta_{pi})$. Then $\{z(\tau),\varphi(\tau);w_i^\pm(\tau)\}$ is a solution of (4.12) with the initial conditions $\{z(0),\varphi(0);w_{io}^\pm(\tau),-\theta_{pi}\le\tau\le 0\}$ where $w_{io}^\pm(\tau)$ are obtained as defined above.*

Conversely, let $\{z(\tau),\varphi(\tau);w_i^\pm(\tau)\}$ be a solution of (4.12) defined by the initial conditions $\{z(0),$ $\varphi(0);w_{io}^\pm(\tau),-\theta_{pi}\le\tau\le 0\}$ with $w_{io}^\pm(\tau)$ sufficiently smooth e.g. of class C^1. Then the set of functions $\{z(\tau),\varphi(\tau);h_i(\xi_i,\tau),q_i(\xi_i,\tau)\}$ with $\{h_i(\xi_i,\tau),q_i(\xi_i,\tau)\}$ defined by (4.13) is a (possibly discontinuous) solution of (4.1) with $f_\theta(\tau)\equiv 0$ and the initial conditions following by taking $\tau=0$ in the aforementioned set of functions.

4.2 Steady state for the turbine shutdown

A. Consider $f_\theta(\tau)\equiv 0$ in (4.1) and let the time derivatives be zero: this will give the steady state at shutdown after water hammer (if the turbine is not unloaded and this steady state is stable). Its equations are given by (4.2) with $\bar f_\theta = 0$. Consider now (4.1) and, after taking $f_\theta(\tau)\equiv 0$, we introduce the deviations of the state variables with respect to the steady state

$$\begin{aligned} &\chi_i(\xi_i,\tau) = h_i(\xi_i,\tau)-1, \quad \varpi_i(\xi_i,\tau) = q_i(\xi_i,\tau)-\bar q = q_i(\xi_i,\tau)-\nu_g;\\ &\zeta(\tau)\equiv z(\tau), \quad s(\tau) = \varphi(\tau)-\bar\varphi = \varphi(\tau)-\nu_g/k. \end{aligned} \tag{4.14}$$

Re-write now (4.1) with respect to the deviations

$$\begin{aligned} &\theta_{wi}\partial_\tau\varpi_i + \partial_{\xi_i}\chi_i = 0, \quad \frac{\theta_{pi}^2}{\theta_{wi}}\partial_\tau\chi_i + \partial_{\xi_i}\varpi_i = 0; \quad i=1,2,\\ &\chi_1(0,\tau) = 0\ ,\ \chi_1(1,\tau) = z(\tau)+\lambda_s\frac{\mathrm{d}z}{\mathrm{d}\tau} = \chi_2(0,\tau),\\ &\theta_s\frac{\mathrm{d}z}{\mathrm{d}\tau} = \varpi_1(1,\tau)-\varpi_2(0,\tau); \quad \varpi_2(1,\tau) = ks(\tau),\\ &\theta_a\frac{\mathrm{d}s}{\mathrm{d}\tau} = \varpi_2(1,\tau)(1+\chi_2(1,\tau))+\nu_g\chi_2(1,\tau). \end{aligned} \tag{4.15}$$

Except the last equation (of the hydraulic turbine), equations (4.15) are linear. If the standard dependence flow – piezometric head is considered ($k=0$), system (4.15) becomes fully linear since the boundary condition at $\xi_2 = 1$ becomes $\varpi_2(1,\tau) = 0$ and the equation for s is decoupled being thus independent. Due to linearity, the steady state is 0 but the basic steady state with respect to which the deviations (4.14) has no importance.

B. Suppose we continue with $k \neq 0$ but take into account that the turbine is unloaded simultaneously with shutdown ($f_\theta(\tau) \equiv 0$, $\nu_g = 0$). In this case the steady state of (4.1) is defined by

$$\bar{z} = 0, \quad \bar{h}_1 = 1\ ,\ \bar{h}_2 = 1; \quad \bar{q}_1 = \bar{q}_2 = \bar{q}; \quad \bar{q} = k\bar{\varphi}, \quad \bar{q} = \nu_g = 0 \tag{4.16}$$

(see also (4.2)). Also the deviations are given by (4.14) with $\nu_g = 0$, $\bar{\varphi} = 0$. The system in deviations will be (4.15) but with $\varpi_2(1,\tau) = 0$ and without the equation for s which is decoupled and, therefore, not involved in the stability analysis under water hammer.

In what follows we shall consider only the case $k = 0$ i.e. of the decoupled turbine under water hammer. This option is motivated by the fact that the expression of $q_2(1,\tau)$ in (2.3) has to be connected to an adequate expression for the active torque of the turbine – see [3]. In fact the expression in (2.3) i.e. $q_2(1,\tau)h_2(1,\tau)$ is used with the boundary condition with $k = 0$. Moreover, the aforementioned expression for the torque in the case $k \neq 0$ is determined in steady state and its use during transients might be questionable.

4.3 The energy identity for stability analysis

We start from the energy identity as deduced following e.g. [12]

$$\frac{1}{2} \cdot \frac{\mathrm{d}}{\mathrm{d}\tau} \int_0^1 \left(\theta_{wi}\varpi_i(\xi_i,\tau)^2 + \frac{\theta_{pi}^2}{\theta_{wi}}\chi_i(\xi_i,\tau)^2 \right) \mathrm{d}\xi_i + \varpi_i(\xi_i,\tau)\chi_i(\xi_i,\tau)|_0^1 \equiv 0. \tag{4.17}$$

The energy identity suggests the following Lyapunov functional, written along the solutions of (4.15) with $k = 0$

$$\mathcal{V}(z(\tau),\varpi_i(\cdot,\tau),\chi_i(\cdot,\tau)) = \frac{1}{2}\left\{ \theta_s z(\tau)^2 + \sum_1^2 \theta_{wi} \int_0^1 \left[\varpi_i(\xi_i,\tau)^2 + \delta_i^2 \chi_i(\xi_i,\tau)^2\right] \mathrm{d}\xi_i \right\}. \tag{4.18}$$

We differentiate (4.18) with respect to (4.15), taking into account the boundary conditions. After a straightforward manipulation we obtain

$$\mathcal{W}(\varpi_1(\cdot,\tau),\varpi_2(\cdot,\tau)) = -\frac{\lambda_s}{\theta_s}(\varpi_1(1,\tau) - \varpi_2(0,\tau))^2 = -\lambda_s' \left(\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)^2 \leq 0. \tag{4.19}$$

It is clear that (4.18) and (4.19) imply Lyapunov stability of the zero solution of (4.15) in the sense of the metrics induced by the Lyapunov function (4.18):

$$\mathcal{V}(z,\varpi_i(\cdot,\tau),\chi_i(\cdot,\tau)) \leq \mathcal{V}(z,\varpi_i^o(\cdot),\chi_i^o(\cdot)). \tag{4.20}$$

It remains now to discuss asymptotic stability. Since $\mathcal{W}$ is only non-positive definite, application of the invariance principle Barbashin–Krasovskii–LaSalle is required. This principle is known to be valid for neutral functional differential equations [18]; therefore we make use of Theorem 4.1 and consider the system of functional differential equations (4.12) – but associated to the system in deviations (4.15) with $k = 0$

$$\begin{aligned}
\theta_s \frac{\mathrm{d}z}{\mathrm{d}\tau} &= \frac{1}{1+(\delta_1+\delta_2)\lambda_s'}[-(\delta_1+\delta_2)z(\tau) - 2\eta_1^-(\tau-2\theta_{p1}) + 2\eta_2^+(\tau-2\theta_{p2})],\\
\eta_1^-(\tau) &= \frac{1}{1+(\delta_1+\delta_2)\lambda_s'}[\delta_1 z(\tau) + (1+(\delta_2-\delta_1)\lambda_s')\eta_1^-(\tau-2\theta_{p1}) + 2\delta_1\lambda_s'\eta_2^+(\tau-2\theta_{p2})],\\
\eta_2^+(\tau) &= \frac{1}{1+(\delta_1+\delta_2)\lambda_s'}[\delta_2 z(\tau) - 2\delta_2\lambda_s'\eta_1^-(\tau-2\theta_{p1}) - (1+(\delta_1-\delta_2)\lambda_s')\eta_2^+(\tau-2\theta_{p2})]
\end{aligned} \tag{4.21}$$

(for stability analysis, where large τ are concerned, one can consider $\tau > 2\max\{\theta_{p1},\theta_{p2}\}$ and the variables η_1^+, η_2^- can be substituted from their equations in the remaining ones).

Consider now the representation formulae (4.13) written for the system in deviations

$$\begin{aligned} \chi_i(\xi_i,\tau) &= \frac{1}{\delta_i}[\eta_i^+(\tau-\theta_{pi}\xi_i)+\eta_i^-(\tau+\theta_{pi}(\xi_i-1))], \\ \omega_i(\xi_i,\tau) &= \eta_i^+(\tau-\theta_{pi}\xi_i)-\eta_i^-(\tau+\theta_{pi}(\xi_i-1))] \end{aligned} \tag{4.22}$$

to re-write $\mathcal{V}$ and $\mathcal{W}$

$$\begin{aligned} \chi_1(\xi_1,\tau) &= \frac{1}{\delta_1}[\eta_1^-(\tau+\theta_{p1}(\xi_1-1))-\eta_1^-(\tau-\theta_{p1}(\xi_1+1))], \\ \omega_1(\xi_1,\tau) &= -\eta_1^-(\tau+\theta_{p1}(\xi_1-1))-\eta_1^-(\tau-\theta_{p1}(\xi_1+1)), \\ \chi_2(\xi_2,\tau) &= \frac{1}{\delta_2}[\eta_2^+(\tau-\theta_{p2}\xi_2)+\eta_2^+(\tau+\theta_{p2}(\xi_2-2))], \\ \omega_2(\xi_2,\tau) &= \eta_2^+(\tau-\theta_{p2}\xi_2)-\eta_2^+(\tau+\theta_{p2}(\xi_2-2)). \end{aligned} \tag{4.23}$$

Therefore

$$\begin{aligned} &\mathcal{V}(z(\tau),\eta_1^-(\cdot,\tau),\eta_2^+(\cdot,\tau)) \\ &\quad = \frac{1}{2}\left\{\theta_s z(\tau)^2+\theta_{w1}\int_{-2\theta_{p1}}^0 \eta_1^-(\tau+\lambda)^2\mathrm{d}\lambda+\theta_{w2}\int_{-2\theta_{p2}}^0 \eta_2^+(\tau+\lambda)^2\mathrm{d}\lambda\right\}, \\ &\mathcal{W}(\eta_1^-(\cdot,\tau),\eta_2^+(\cdot,\tau)), \\ &\quad = -\lambda_s'[-\eta_1^-(\tau)-\eta_1^-(\tau-2\theta_{p1})-\eta_2^+(\tau)+\eta_2^+(\tau-2\theta_{p2})]^2 = -\lambda_s'\left(\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)^2 \leq 0. \end{aligned} \tag{4.24}$$

Since $\mathcal{W} \leq 0$ we shall try to apply the invariance Barbashin–Krasovskii–LaSalle principle hence we seek first for the largest invariant set with respect to the solutions of (4.21), contained in the set where $\mathcal{W} = 0$. Since $\mathrm{d}z/\mathrm{d}\tau = 0$ we deduce from (4.21)

$$z(\tau) = \frac{2}{\delta_1+\delta_2}[-\eta_1^-(\tau-2\theta_{p1})+\eta_2^+(\tau-2\theta_{p2})]$$

and substitute $z(\tau)$ in the remaining difference equations; a simple manipulation will show that on the set where $\mathcal{W} = 0$ the system is restricted to

$$\begin{aligned} \eta_1^-(\tau) &= \frac{\delta_2-\delta_1}{\delta_1+\delta_2}\eta_1^-(\tau-2\theta_{p1})+\frac{2\delta_1}{\delta_1+\delta_2}\eta_2^+(\tau-2\theta_{p2}), \\ \eta_2^+(\tau) &= -\frac{2\delta_2}{\delta_1+\delta_2}\eta_1^-(\tau-2\theta_{p1})-\frac{\delta_1-\delta_2}{\delta_1+\delta_2}\eta_2^+(\tau-2\theta_{p2}). \end{aligned} \tag{4.25}$$

The invariant solutions with respect to τ are the constant solutions. The non-zero determinant

$$\begin{vmatrix} 1-\frac{\delta_2-\delta_1}{\delta_1+\delta_2} & -\frac{2\delta_1}{\delta_1+\delta_2} \\ \frac{2\delta_2}{\delta_1+\delta_2} & 1+\frac{\delta_1-\delta_2}{\delta_1+\delta_2} \end{vmatrix} = \frac{4\delta_1}{(\delta_1+\delta_2)} \neq 0$$

shows that the only invariant set located in the set where $\mathcal{W} = 0$ is the origin $\{0;0,0\}$. Application of the Barbashin–Krasovskii–LaSalle invariance principle [18], Theorem 9.8.2 will give asymptotic stability.

There exists however a restriction to the application of the Barbashin–Krasovskii–LaSalle invariance theorem – *the stability if not even the strong stability of the difference subsystem* of (4.21). We shall consider this problem as applied to the stability of (4.21) and, *via* Theorem 4.1, to (4.15) – both with $k = 0$. We make first the following notations

$$\rho_1 = \frac{1+(\delta_2-\delta_1)\lambda_s'}{1+(\delta_1+\delta_2)\lambda_s'}\,, \qquad \rho_2 = \frac{1+(\delta_1-\delta_2)\lambda_s'}{1+(\delta_1+\delta_2)\lambda_s'} \tag{4.26}$$

to give (4.21) the following form

$$\begin{aligned} \theta_s\frac{\mathrm{d}z}{\mathrm{d}\tau} &= \frac{\rho_1+\rho_2}{2}[-(\delta_1+\delta_2)z - 2\eta_1^-(\tau-2\theta_{p1}) + 2\eta_2^+(\tau-2\theta_{p2})],\\ \eta_1^-(\tau) &= \frac{\rho_1+\rho_2}{2}\delta_1 z(\tau) + \rho_1\eta_1^-(\tau-2\theta_{p1}) + (1-\rho_1)\eta_2^+(\tau-2\theta_{p2}),\\ \eta_2^+(\tau) &= \frac{\rho_1+\rho_2}{2}\delta_2 z(\tau) - (1-\rho_2)\eta_1^-(\tau-2\theta_{p1}) - \rho_2\eta_2^+(\tau-2\theta_{p2}). \end{aligned} \tag{4.27}$$

The restriction in applying Theorem 9.8.2 of [18], page 293 thus refers to (asymptotic) stability of a difference system having the form

$$y(t) = \sum_1^p A_k y(t-r_k), \qquad t \geq 0. \tag{4.28}$$

In order to make the development which follows more clear, we shall recall in brief certain development of [18], Section 9.3, the part tackling difference equations and operators – pp. 274–276. With the notations of *op. cit.*, we consider the homogeneous and non-homogeneous difference equations

$$\mathrm{D}y_t = 0\,, \quad t \geq 0; \qquad \mathrm{D}y_t = h(t), \quad t \geq 0, \tag{4.29}$$

where $h \in \mathcal{C}([0,\infty);\mathbb{R}^n)$ and the difference operator $\mathrm{D} : \mathcal{C}(-r,0;\mathbb{R}^n) \mapsto \mathbb{R}^n$ is continuous and atomic at 0 being thus defined as

$$\mathrm{D}\phi = \phi(0) - \int_{-r}^0 \mathrm{d}[\mu(\theta)]\phi(\theta). \tag{4.30}$$

In (4.30) the kernel $\mu : \mathbb{R} \mapsto \mathbb{R}^{n\times n}$ is measurable, normalized such that $\mu(\theta) \equiv 0$ for $\theta \geq 0$ and $\mu(\theta) \equiv \mu(-r)$ for $\theta \leq -r$; the kernel is continuous from the left and of bounded variation. The following assumption is supposed to hold for the kernel μ.

Assumption 4.2 (Assumption (J) of [18], page 271). *The entries μ_{ij} of μ have an atom before they become constant i.e. there is a t_{ij} such that $\mu_{ij}(t) \equiv \mu_{ij}(t_{ij}+0)$ for $t \geq t_{ij}$ and $\mu_{ij}(t_{ij}-0) \neq \mu_{ij}(t_{ij}+0)$.*

This assumption is particularly true for (4.28) where $\mu(\theta)$ is reduced to a stepwise function. Let $\Delta_0(\lambda)$ defined below be the characteristic function of (4.30)

$$\Delta_0(\lambda) = \det\left(I - \int_{-r}^0 \mathrm{e}^{\lambda\theta}\mathrm{d}[\mu(\theta)]\right) \tag{4.31}$$

which in the case of (4.28) reads

$$\Delta_0(\lambda) = \det\left(I - \sum_1^p A_k \mathrm{e}^{-\lambda r_k}\right) \tag{4.32}$$

Let $a_D := \sup\{\Re e(\lambda) \mid \Delta_0(\lambda) = 0\}$. We state firstly

Definition 4.3 (Definition 9.3.1 of [18], page 275)**.** Suppose D is linear, continuous and atomic at 0 – see (4.30). The operator D is said to be stable if the zero solution of the homogeneous equation of (4.29) with the initial condition $\psi \in \mathcal{C}(-r,0;\mathbb{R}^n)$ subject to $\mathrm{D}\psi = 0$ is asymptotically stable

The following result [18], p. 275, concerns the aforementioned stability property.

Theorem 4.4 (Theorem 9.3.5 of [18], p. 275)**.** *The following statements are equivalent*

(i) D *is (asymptotically) stable in the sense of Definition* 4.3*.*

(ii) $a_D < 0$*.*

(iii) There exists constants $\alpha > 0$ *and* $\gamma(\alpha) > 0$ *such that for any* $h \in \mathcal{C}([0,\infty);\mathbb{R}^n)$ *any solution of the non-homogeneous equation of* (4.29) *satisfies*

$$|y(\psi,h)(t)| \leq \gamma(\alpha)[|\psi|\mathrm{e}^{-\alpha t} + \sup_{0\leq s\leq t} |h(s)|].$$

(iv) If D *is given by* (4.31) *with* $\lim_{s\to 0} \mathrm{Var}_{[-s,0]}\mu = 0$ *and* μ *is also subject to Assumption* 4.2 *– Assumption (J), then there exists a* $\delta > 0$ *such that all roots of the characteristic equation*

$$\Delta_0(\lambda) := \det\left(I - \int_{-r}^{0} \mathrm{e}^{\lambda\theta}\mathrm{d}[\mu(\theta)]\right) = 0$$

satisfy $\Re e(\lambda) \leq -\delta < 0$*.*

Observe that (iii) shows that (asymptotic) stability of D in the sense of Definition 4.3 is equivalent to exponential stability (the principle of K. P. Persidskii). Therefore stability of (4.28) means in fact *exponential stability*. Moreover, (iv) shows – along the same line – that $a_D < 0$ ensures that the roots of the characteristic equation (4.32) *have their real parts well delimited from* 0. In fact Theorem 4.1 states *equivalence of (apparently) weak properties with other, stronger ones*.

Turning to (4.29), its (asymptotic, exponential) stability is equivalent to the location of the roots of the characteristic equation $\Delta_0(\lambda) = 0$ with $\Delta_0(\lambda)$ given by (4.32) in the open left half plane $\mathbb{C}^-$. But for difference operators there exists another property called *strong stability*. This property is introduced also in [18], Section 9.6, for difference operators occurring in (4.28) i.e. defined by

$$\mathrm{D}(r,A)\phi = \phi(0) - \sum_{1}^{p} A_k\phi(-r_k). \tag{4.33}$$

Observe that the difference operator (4.33) is a special case of (4.30) with μ containing only the stepwise component with a finite number of steps.

Let $r = \mathrm{col}(r_1,\ldots,r_p)$ be the vector of the delays $r_k > 0$, $\forall k$.

Definition 4.5 (Definitions 9.6.1 and 9.6.2 of [18], p. 285)**.** The operator $\mathrm{D}(r,A)$ is said to be stable locally in the delays if there is an open neighborhood $I(r) \subset \mathbb{R}_+^p$ of r such that $\mathrm{D}(v,A)$ is stable in the sense of Definition 4.3 for each $v \in I(r)$.

The operator $\mathrm{D}(r,A)$ is said to be stable globally in the delays (strongly stable) if it is stable for each $r \in \mathbb{R}_+^p$.

For *strong stability* the following result is true.

Theorem 4.6 (Theorem 9.6.1 of [18], p. 286)**.** *The following statements are equivalent.*

(i) *For some* $r \in \mathbb{R}_+^p$, $r = \mathrm{col}(r_1, \dots, r_p)$ *with* $r_k > 0$ *rationally independent,* $\mathrm{D}(r, A)$ *is stable in the sense of Definition 4.3.*

(ii) *If* $\gamma(B)$ *is the spectral radius of a matrix* B*, then* $\gamma_0(A) < 1$ *where*

$$\gamma_0(A) := \sup\left\{ \gamma\left(\sum_1^p A_k e^{\imath\theta_k} \right) \,\middle|\, \theta_k \in [0, 2\pi),\ k = 1, 2, \dots, p \right\} \tag{4.34}$$

(iii) $\mathrm{D}(r, A)$ *is stable locally in the delays in the sense of Definition 4.5.*

(iv) $\mathrm{D}(r, A)$ *is stable globally in the delays (strongly stable) in the sense of Definition 4.5.*

We are now in position to consider the stability properties of the difference subsystem of (4.27) namely the linear difference subsystem

$$\begin{aligned} \eta_1^-(\tau) &= \rho_1 \eta_1^-(\tau - 2\theta_{p1}) + (1 - \rho_1)\eta_2^+(\tau - 2\theta_{p2}), \\ \eta_2^+(\tau) &= -(1 - \rho_2)\eta_1^-(\tau - 2\theta_{p1}) - \rho_2 \eta_2^+(\tau - 2\theta_{p2}). \end{aligned} \tag{4.35}$$

Therefore we shall have

$$A_1 = \begin{pmatrix} \rho_1 & 0 \\ -(1 - \rho_2) & 0 \end{pmatrix}; \qquad A_2 = \begin{pmatrix} 0 & (1 - \rho_1) \\ 0 & -\rho_2 \end{pmatrix}. \tag{4.36}$$

The characteristic equation of (4.35) results in

$$\left(1 - \rho_1 e^{-2\lambda\theta_{p1}}\right)\left(1 + \rho_2 e^{-2\lambda\theta_{p2}}\right) + (1 - \rho_1)(1 - \rho_2) e^{-2\lambda(\theta_{p1} + \theta_{p2})} = 0 \tag{4.37}$$

the two delays being, generally speaking, rationally independent; this equation ought to have its roots in a left half plane $\Re e(\lambda) \le -\alpha_0 < 0$. Denoting

$$e^{2\lambda\theta_{p2}} =: s\,, \qquad \nu = \theta_{p1}/\theta_{p2}, \tag{4.38}$$

the aforementioned condition reduces to the condition for the equation

$$(s^\nu - \rho_1)(s + \rho_2) + (1 - \rho_1)(1 - \rho_2) = 0 \tag{4.39}$$

to have its roots with $|s| < 1$ (inside the unit disk of $\mathbb{C}$).

Let firstly $s = re^{\imath\varphi}$, $r > 1$. We deduce

$$\begin{aligned} |r^\nu e^{\imath\nu\varphi} - \rho_1|^2 \cdot |re^{\imath\varphi} + \rho_2|^2 &= \left(r^{2\nu} + \rho_1^2 - 2r^\nu \rho_1 \cos\nu\varphi\right)\left(r^2 + \rho_2^2 + 2\rho_2 \cos\varphi\right) \\ &> (r^\nu - \rho_1)^2 (r - \rho_2)^2. \end{aligned}$$

Further

$$\begin{aligned} &(r^\nu - \rho_1)(r - \rho_2) > (1 - \rho_1)(1 - \rho_2) \\ &(r^\nu - \rho_1)(r + \rho_2) > (1 - \rho_1)(1 - \rho_2) + 2\rho_2(1 - \rho_1) > (1 - \rho_1)(1 - \rho_2). \end{aligned}$$

It follows that (4.39) cannot have roots with $|s| > 1$.

Let now $\nu = p/q \in \mathbb{Q}$ – an irreducible ratio i.e. assume the delays to be rationally dependent. Equation (4.39) becomes by taking again $s := re^{\imath\varphi}$

$$(r^{p/q}e^{\imath p\varphi/q} - \rho_1)(re^{\imath\varphi} + \rho_2) + (1-\rho_1)(1-\rho_2) = 0. \tag{4.40}$$

Denote $(re^{\imath\varphi})^{1/q} =: z$. Consequently equation (4.40) reads

$$(z^p - \rho_1)(z^q + \rho_2) + (1-\rho_1)(1-\rho_2) = 0. \tag{4.41}$$

Observe that if $z := \rho e^{\imath\theta}$ then $\varphi = q\theta$ and since $\varphi \in [0, 2\pi)$ it follows that $\theta \in [0, 2\pi/q)$. Also if $r \leq 1$ then $\rho \leq 1$.

Let $\rho = 1$. Equation (4.41) becomes

$$(e^{\imath p\theta} - \rho_1)(e^{\imath q\theta} + \rho_2) + (1-\rho_1)(1-\rho_2) = 0; \qquad \theta \in [0, 2\pi/q). \tag{4.42}$$

This equation is equivalent to

$$\begin{aligned} &(1 - 2\rho_1\cos p\theta + \rho_1^2)(1 + 2\rho_2\cos q\theta + \rho_2^2) = (1-\rho_1)^2(1-\rho_2)^2, \\ &\frac{\sin p\theta}{\cos p\theta - \rho_1} + \frac{\sin q\theta}{\cos q\theta + \rho_2} = 0. \end{aligned} \tag{4.43}$$

(The modulus and phase equations.) The first (modulus) equation is re-written as

$$\left[1 + \frac{2\rho_1}{(1-\rho_1)^2}(1 - \cos p\theta)\right] \cdot \left[1 + \frac{2\rho_2}{(1-\rho_2)^2}(1 + \cos q\theta)\right] = 1 \tag{4.44}$$

and holds for the unique combination

$$\cos p\theta = 1 \ (p\theta = 2m\pi), \qquad \cos q\theta = -1 \ (q\theta = (2n+1)\pi); \qquad m, n \in \mathbb{N}. \tag{4.45}$$

From (4.45) it follows that $\theta = 2m\pi/p = (2n+1)\pi/q$. The first conclusion is that $p/q = (2m)(2n+1)^{-1}$. Therefore the modulus equation of (4.43) has a solution iff p/q is such that p is even and q is odd. Now, since $0 \leq \theta \leq 2\pi/q$, it follows that $m \leq p/q$ and $2n+1 \leq 2$ i.e. $n = 0$. *Therefore $\theta = \pi/q$ is the only possible solution of* (4.43). Observe that for this value – corresponding to $\varphi = \pi$ – the phase equation in (4.43) is automatically fulfilled. An elementary computation shows that *this root is simple*. The other roots have their modulus less than 1 and their number is finite and among them there is one whose modulus is maximal hence $\rho_k \leq \rho_0 < 1$ for all roots inside the unit disk.

Otherwise i.e. iff both p and q are *odd* (4.43) has no solution that is (4.41) has no roots with modulus 1; therefore in this case *all* roots of (4.41) satisfy $\rho_k \leq \rho_0 < 1$. The aforementioned properties are thus valid for equation (4.40) hence for (4.39) in the case of $\nu \in \mathbb{Q}$.

Consider now the case of irrational ν that is of rationally independent delays in (4.35). This case can be tackled *via* Theorem 4.4 dealing with strong stability of (4.35). Since there are two delays the computation of the spectral radius reduces to its computation for $A_1 + A_2e^{\imath\theta}$ where $\theta \in [0, 2\pi)$. The matrix being of dimension 2×2, it has two eigenvalues which both have to be inside the unit disk for *all* $\theta \in [0, 2\pi)$. The characteristic equation of $A_1 + A_2e^{\imath\theta}$ with A_i given by (4.36) results as

$$(z - \rho_1)(z + \rho_2e^{\imath\theta}) + (1-\rho_1)(1-\rho_2)e^{\imath\theta} = 0. \tag{4.46}$$

Location of the roots of (4.46) – a second degree equation *with complex coefficients* – inside the unit disk can be checked with the Schur–Cohn criterion. However, since (4.46) is very much alike to (4.41), we can take the same approach. Let $z = re^{\imath\varphi}$, $r > 1$, $\varphi \in [0,2\pi)$. We shall have after a straightforward computation

$$\begin{aligned}
|z-\rho_1|^2 \cdot |z+\rho_2 e^{\imath\theta}|^2 &= |re^{\imath\varphi}-\rho_1|^2 \cdot |re^{\imath\varphi}+\rho_2 e^{\imath\theta}|^2 \\
&= (r^2 - 2r\rho_1\cos\varphi + \rho_1)^2 (r^2 + 2r\rho_2\cos\varphi\cos\theta + \rho_2)^2 \\
&= [(r-\rho_1)^2 + 2\rho_1 r(1-\cos\varphi)][(r-\rho_2)^2 + 2\rho_2 r(1-\cos\varphi)\cos\theta] \\
&> (r-\rho_1)^2(r-\rho_2)^2 > (1-\rho_1)^2(1-\rho_2)^2.
\end{aligned}$$

Therefore equation (4.46) has no roots of modulus larger than 1. Let now $r = 1$. We have to check the equality

$$(e^{\imath\varphi}-\rho_1)(e^{\imath\varphi}+\rho_2 e^{\imath\theta}) + (1-\rho_1)(1-\rho_2)e^{\imath\theta} = 0. \tag{4.47}$$

We can proceed as in the case of (4.42) but here the problem is simpler. Let $\varphi = 0$, $\theta = \pi$: obviously (4.47) is fulfilled. Therefore $z = 1$ is one of the two roots of (4.46) for $\theta = \pi$, the other one being $(\rho_1 + \rho_2 - 1) \in (-1,1)$. This result is sufficient to obtain $\gamma_0(A) = 1$ and, therefore, that statement (ii) of Theorem 4.4 is not fulfilled. Since $(ii) \Leftrightarrow (i)$, we have also $\text{non}(i) \Leftrightarrow \text{non}(ii)$. It follows that that there will be *no (asymptotic) stability of the difference system* (4.35) *for irrational* ν.

We are now in position to summarize the results concerning (asymptotic) stability of the difference system (4.35). This system is stable *in the sense of Lyapunov*: the result was obtained using the Lyapunov functional (4.24) whose derivative, also given in (4.24) is non-positive definite; stability should be viewed in the sense of the Lyapunov functional itself.

For the sake of completeness, it should be mentioned that the difference system (4.35) is obtained from (4.21) for $z(\tau) \equiv 0$. In this case the derivative functional of (4.24) is subject to $\mathcal{W}(\eta_1^-(\cdot,\tau),\eta_1^+(\cdot,\tau)) \equiv 0$, system (4.35) resulting conservative – in the metrics of the Lyapunov functional $\mathcal{V}$ of (4.24), also restricted to $z(\tau) \equiv 0$. Therefore system (4.35) is stable in the sense of the metrics of $\mathcal{V}$.

If this aspect is viewed from the point of view of the characteristic equation (4.39), an elementary computation of the derivative of its right hand side will show that for any real ν the possible roots of modulus 1 will be *simple*. We reiterated here that system (4.35) as well as system (4.21) are Lyapunov stable in the sense of the metrics induced by the Lyapunov functional (4.24). Based on Theorem 4.1 and on the representation formulae (4.13), Lyapunov stability is ensured for system (4.15) – in the sense of the metrics defined by the Lyapunov functional (4.18).

Return now to the problem of asymptotic stability. We showed in the previous development, based on Theorems 4.4 and 4.6 that the asymptotic stability of system (4.35) is true only in a single case of rationally dependent delays – when their ratio $\nu = p/q \in \mathbb{Q}$ in the particular case when both p and q are odd numbers. *Only in this case* the invariance principle Barbashin–Krasovskii–LaSalle (Theorem 9.8.2 of [18], page 293) can be applied to system (4.27) to obtain its asymptotic stability and, *via* Theorem 4.1 and the representation formulae (4.13), of system (4.15).

Summarizing, Lyapunov stability is ensured for (4.21) hence for (4.15) but the *asymptotic stability is fragile*: it holds for a countable set of rational ratios of propagation time constants – those rational ν having both odd numerator and denominator. The fragility appears from the fact that the set of irrationals is dense and a small uncertainty in the delays will modify ν from rational to irrational.

5 Some conclusions

The examined stability analysis concerns a model arising from hydraulic engineering, largely accepted among engineers, as it might be seen from the cited references. As it appeared from the stability analysis of the water hammer, the only stabilizing device of this phenomenon is the surge tank. Since the engineering philosophy states that a stabilizing device should be stable itself (display inherent stability), the paper contains a standard stability analysis of the surge tank. Its result is the dimension of the equivalent cross-section area of the surge tank which must be larger than the so called *Thoma cross-section area*. The analysis was made using a suitable Lyapunov function giving an asymptotic stability result combined with an estimate of the attraction domain. The aforementioned analysis is valid for the physically accepted equilibrium. Other steady states may be foreseen, corresponding to rather abnormal situations (from the engineering point of view). The analysis might point out instabilities, limit cycles, hidden attractors.

On the other hand, the model for the water hammer itself is described by a nonstandard (i.e. with derivative boundary conditions) boundary value problem for hyperbolic $1D$ equations. We applied here a well established method, coming from the paper of A. D. Myshkis [1], to associate a system of functional differential equations (in most cases, of neutral type) to a nonstandard initial boundary value problem for hyperbolic partial differential equations. A one to one correspondence between the solutions of the two mathematical objects being established e.g. [28], all results obtained for one mathematical object are thus projected back on the other one.

Consider here stability obtained *via* "weak" (in the sense of N. G. Četaev) Lyapunov function(al)s i.e. having the derivative function(al) only non-positive (the best known are the energy type function(al)s). In this case the main instrument for the asymptotic stability is the Barbashin–Krasovskii–LaSalle invariance principle. For neutral functional differential equations this principle is established as a theorem for equations with stable difference operator (Theorem 9.8.2 of [18]). However this stability is *robust (non-fragile)* with respect to delay uncertainties only if the difference operator is strongly stable. As an example, for a single delay case – $p = 1$ in (4.28) – the strong stability follows from the location of the eigenvalues of A_1 inside the unit disk of $\mathbb{C}$; moreover, in this case stability and strong stability are the same thing.

In our opinion, this assumption, occurring for the first time in the paper [9], turned to be capital for stability studies. This was also due to the fact that many applications leading to neutral functional differential equations displayed conditions for the fulfilment of the (strong) stability assumptions of the difference operator.

In the last years there were however exposed applications (mainly from Mechanics and Mechanical Engineering) with matrix A_1 – again the case $p = 1$ in (4.28) – having its eigenvalues on the unit circle i.e. in a critical case (a list of such applications is available in [28]; other applications, dealing with synchronization of mechanical oscillators, can be found in [29]); these cases were not yet seriously tackled.

The case described in this paper looks different: displaying two delays, it displays also a *fragile asymptotic stability* – valid for rationally dependent delays, but only in one case of two possible. The fragility of the asymptotic stability with respect to the delays is confirmed by practical measurement (in-site), displaying some oscillatory modes. Such aspects arising from practice should stimulate revival of some "old" studies which have been obscured by the Cruz-Hale assumptions: the book [11] and its reference list are a good starting point to meet this challenge.

There exists however another challenge, arising from the fact that stability of the difference operator is a premise to apply the Barbashin–Krasovskii–LaSalle invariance principle. As pointed out in [31], the assumption of stability for the difference operator is necessary to obtain pre-compactness of the positive orbits whenever the solution is bounded (Chapter VI, p. 341 and next). The cited reference gives an alternative in its Chapter V (Section 4). Interesting enough, the case considered in Chapter V is a boundary value problem for hyperbolic partial differential equations. With the aforementioned one to one correspondence between the solutions of the boundary value problem for hyperbolic partial differential equations and those of the associated system of neutral functional differential equations, the things become clearer. Our point of view is that all this is a question of choosing the state space for neutral functional differential equations (other than $\mathcal{C}$) – see [17]. In any case there is plenty of motivation to follow this line of research.

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It is more than 40 years since our paths crossed with László Hatvani on the "field" of the mathematical research in differential equations and dynamical systems. His deep mathematical knowledge doubled by the highest human qualities made me feel proud and lucky of meeting him and being almost side by side in our scientific interest. At this cross road moment I wish him good health and open mind to continue – for the good of our science and for the joy of his friends.

Appendix

We shall give in this Appendix (reproduced after [10]) the principal notations of the paper – in fact the notations which are usual in the field of hydroelectric engineering and can be met in field's references, in particular in those of the present paper.

The notations of the state variables are as follows

- $V_i(x,t)$, $Q_i(x,t)$, $H_i(x,t)$, $i = 1,2$ – water flow velocity, water flow and piezometric head at $(x,t) \in \{(x,t) \mid 0 \le x \le L_i, t \in \mathbb{R}\}$, x being the coordinates along the conduits ($i = 1$ accounts for the tunnel and $i = 2$ for the penstock);
- H_0 – piezometric head of the lake;
- $Z(t)$ – water level in the surge tank;
- $\Omega(t)$ – turbine rotating speed; Ω_c – the synchronous speed.

Also the notations for system's parameters are as follows

- F_i, D_i, L_i ($i = 1,2$) – the cross section areas, the hydraulic diameters and the lengths of the conduits, respectively;
- F_s, F_θ – equivalent cross section areas of the surge tank and regulated flow area of the turbine wicket gates, respectively;
- J, η_θ – moment of inertia and efficiency of the hydraulic turbine, respectively;

- γ, g – specific weight of the water and gravity acceleration, respectively;
- $N_g = \Omega_c M_g$ – the mechanical power supplied to the hydrogenerator, where M_g is the torque;
- λ_s – coefficient of losses of the throttling of the surge tank;
- λ_i, a_i $(i = 1,2)$ – coefficients of the Darcy–Weisbach losses and the propagation speeds of the water hammer along the conduits respectively;
- α_q – a flow coefficient;
- k – a corrective coefficient for the flow through the wicket gates of the turbine.

We list further the following time constants of the conduits

- T_{wi} – the starting time constant: $T_{wi} = (L_i\bar{Q})(F_i H_0 g)^{-1}$ $(i = 1,2)$;
- T_i – the fill up time constant: $T_i = (L_i F_i)/\bar{Q}$ $(i = 1,2)$;
- T_{pi} – the wave propagation time: $T_{pi} = L_i/a_i$ $(i = 1,2)$,

and also

- T_s – the fill up time constant of the surge tank: $T_s = F_s H_0/\bar{Q}$;
- T_a – the starting time constant of the turbine:

$$T_a = \frac{J\Omega_c^2}{\eta_\theta \frac{\gamma}{2g}\bar{Q}H_0}.$$

The following rated state variables and parameters are listed below

- q_i – rated water flow along the conduit i defined by $q_i = Q_i/\bar{Q}$, $i = 1,2$;
- h_i – rated piezometric head along the conduit i defined by $h_i = H_i/H_0$, $i = 1,2$;
- z – rated piezometric head at the surge tank defined by $z = Z/H_0$;
- φ – rated rotating speed of the hydraulic turbine, defined by $\varphi = \Omega/\Omega_c$;
- ν_g – rated load mechanical power of the hydraulic turbine, defined by

$$\nu_g = \frac{N_g}{\eta_\theta \frac{\gamma}{2g}\bar{Q}H_0}.$$

We list finally the rated (to T_1) time constants as follows

- θ_{wi} – the rated starting time constant $\theta_{wi} = T_{wi}/T_1$ $(i = 1,2)$;
- θ_i – the rated fill up time constant $\theta_i = T_i/T_1$ $(i = 1,2, \theta_1 = 1)$;
- θ_{pi} – the rated wave propagation time $\theta_{pi} = T_{pi}/T_1$ $(i = 1,2)$;
- θ_s – the rated fill up time constant of the surge tank $\theta_s = T_s/T_1$;
- θ_a – the rated starting time constant of the turbine $\theta_a = T_a/T_1$

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Nonconstant positive steady states and pattern formation of a diffusive epidemic model

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Abstract. It is our purpose in this paper to make a detailed description for the structure of the set of the nonconstant steady states for the two-dimensional epidemic S-I model with diffusion incorporating demographic and epidemiological processes with zero-flux boundary conditions. We first study the conditions of diffusion-driven instability occurrence, which induces spatial inhomogeneous patterns. The results will extend to the derivative of prey's functional response with prey is positive. Moreover, we establish the local and global structure of nonconstant positive steady state solutions. A priori estimates for steady state solutions will play a key role in the proof. Our results indicate that the diffusion has a great influence on the spread of the epidemic and extend well the finding of spatiotemporal dynamics in the epidemic model.

Keywords: epidemic model, Turing instability, local and global structure, pattern.

2020 Mathematics Subject Classification: 35B35, 35K57, 92D25.

1 Introduction

Since the pioneer work of King [15], Kermack and McKendrick [14], mathematical models have been contributing to improve our understanding of infectious disease dynamics and help us develop preventive measures to control infection spread. Over a period of time, researchers in theoretical and mathematical epidemiology have proposed many epidemic models, and the temporal dynamics of infectious disease transmission described with differential equations has been investigated in either qualitative or numerical analysis [1, 2, 6, 20].

In epidemic models, the incidence rate plays a key role in the spread of an infection [3, 6, 8, 17, 19, 21, 24]. Traditionally, two different types of incidence rate are been frequently used in well-known epidemic models [4, 9]: The density-dependent transmission is the case in which the contact rate between susceptible and infective individuals increases linearly with population size; the frequency-dependent transmission is the case in which the number of contacts is independent of population size [13].

In [5], the susceptible S is a capable of reproducing with logistic law and strong Allee effect and the infected individuals I do not reproduce but they still contribute with S to population

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growth toward the carrying capacity. This assumption is based on [28–30]. If we assume I is capable of reproducing without strong Allee effect, and assume that the disease is not to be transmitted to offspring, newborns of the infected are in the susceptible class. The infected I is removed only by death at rate μ, there is no recovery from the disease. The disease transmission is assumed to be standard incidence term $\frac{\beta SI}{S+I}$, and no vertical transmission, i.e., the number of contacts between infected and susceptible individuals is constant [11]. The transmission coefficient is $\beta > 0$.

From the above assumption, we can establish the following model

$$\begin{aligned} \frac{dS}{dt} &= r(S+\rho I)[1-a(S+I)] - \frac{\beta SI}{S+I}, \\ \frac{dI}{dt} &= \frac{\beta SI}{S+I} - \mu I. \end{aligned} \tag{1.1}$$

Here S and I denote the density of the uninfected (susceptible) and infected hosts, respectively. All parameters are nonnegative. Parameter r denotes the maximum birth rate of the hosts; and $0 \le \rho \le 1$ describes the reducing reproduction ability of infected hosts: $\rho = 0$ means that infected hosts lose their reproducing ability while $\rho = 1$ indicates that they experience no reduction in reproduction fitness; a measures the per capita density-dependent reduction in birth rate. If $a \neq 0$, then $1/a$ is also called the carrying capacity; if $a = 0$, then the model not consider horizontally transmitted that not reduces fecundity and survival of its host, which in turn is not regulated by density-dependent birth. However, considering the impact of various aspects such as resources and environment on population growth and the model has more practical significance, this paper mainly considers $a \neq 0$.

Suppose that the susceptible (S) and the infections individuals (I) move randomly in the space-described as Brownian random motion [10], and then we propose a simple spatial model corresponding to (1.1) as follows

$$\begin{cases} \frac{\partial S}{\partial t} - d_1 \Delta S = r(S+\rho I)[1-a(S+I)] - \frac{\beta SI}{S+I}, & x \in \Omega,\ t > 0, \\ \frac{\partial I}{\partial t} - d_2 \Delta I = \frac{\beta SI}{S+I} - \mu I, & x \in \Omega,\ t > 0, \\ \frac{\partial S}{\partial \mathbf{n}} = \frac{\partial I}{\partial \mathbf{n}} = 0, & x \in \partial\Omega,\ t > 0, \\ S(x,0) = S_0(x), I(x,0) = I_0(x), & x \in \Omega. \end{cases} \tag{1.2}$$

Here Ω is a bounded domain in $\mathbb{R}^N (N \ge 1)$ with smooth boundary $\partial\Omega$, $\mathbf{n}$ is the outward unit normal vector of the boundary $\partial\Omega$ and the homogeneous Neumann boundary condition is being considered. The diffusion coefficients d_1 and d_2 are positive constants, and the initial data $S_0(x), I_0(x)$ are continuous functions. $\Delta = \frac{\partial^2}{\partial x^2}$ is the Laplacian operator in two-dimensional space, which describes the Brownian random motion. The diffusion model provides a useful framework to evaluate some spatially related control measures.

The *Turing instability* refers to "diffusion driven instability", i.e., the stability of the endemic equilibrium changing from stable for the ordinary differential equations (ODE) dynamics (1.1), to unstable, for the partial differential equations (PDE) dynamics (1.2). And the reason of the occurrence of Turing pattern is the existence of nonconstant positive steady states of model (1.2) as a result of diffusion. And there naturally comes two questions:

(1) How about the existence of nonconstant positive steady states of model (1.2)?

(2) What is the structure of nonconstant positive steady states of model (1.2)?

The main goal of this paper is to solve the two questions above completely. So, we will concentrate on the following steady state problem corresponding to (1.2) is given by

$$\begin{cases} -d_1\Delta S = r(S+\rho I)[1-a(S+I)] - \dfrac{\beta SI}{S+I}, & x \in \Omega, \\ -d_2\Delta I = \dfrac{\beta SI}{S+I} - \mu I, & x \in \Omega, \\ \dfrac{\partial S}{\partial \mathbf{n}} = \dfrac{\partial I}{\partial \mathbf{n}} = 0, & x \in \partial\Omega. \end{cases} \tag{1.3}$$

The rest of the paper is organized as follows. In section 2, we perform a priori estimates of positive steady state solutions of (1.3). In section 3, the stability of constant steady state solution and the conditions of Turing instability of model (1.2) are discussed. In section 4, the existence, local and global structure of nonconstant positive solutions of (1.3) are investigated. In the last section, we make some comments on our studies and propose some interesting problems for future studies.

2 A priori estimates

In this section, we investigate the basic estimates of the reaction-diffusion model (1.3) use the following lemma.

Lemma 2.1 ([23]). *Suppose that $g \in C(\overline{\Omega} \times \mathbb{R})$.*

(1) Assume that $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and satisfies $\Delta w(x) + g(x, w(x)) \geq 0 \quad x \in \Omega, \partial_\nu w \leq 0,\ x \in \partial\Omega$, if $w(x_0) = \max_{\overline{\Omega}} w$, then $g(x_0, w(x_0)) \geq 0$;

(2) Assume that $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$, and satisfies $\Delta w(x) + g(x, w(x)) \leq 0, x \in \Omega, \partial_\nu w \geq 0,\ x \in \partial\Omega$, if $w(x_0) = \min_{\overline{\Omega}} w$, then $g(x_0, w(x_0)) \leq 0$.

Lemma 2.2 ([25]). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, and let $g \in C(\overline{\Omega} \times \mathbb{R})$.*

(1) If $z \in W^{1,2}(\Omega)$ is a weak solution of the inequalities

$$\Delta z + g(x,z) \geq 0 \quad \text{in } \Omega, \qquad \partial_n z \leq 0 \quad \text{on } \partial\Omega.$$

and if there is a constant K such that $g(x,z) < 0$ for $z > K$, then $z \leq K$ a.e. in Ω.

(2) If $z \in W^{1,2}(\Omega)$ is a weak solution of the inequalities

$$\Delta z + g(x,z) \leq 0 \quad \text{in } \Omega, \qquad \partial_n z \geq 0 \quad \text{on } \partial\Omega.$$

and if there is a constant K such that $g(x,z) > 0$ for $z < K$, then $z \geq K$ a.e. in Ω.

In order to obtain the existence of nonconstant positive steady states, a priori estimates will play a key role. Our main result in this section is the following.

Theorem 2.3. *If $d_1 < d_2$, or $d_1 > d_2$ and $d_1(\beta - \mu) < d_2\beta$, then all the non-negative solutions of model (1.3) that start in Ω are bounded with ultimate bound $\Gamma = \frac{1}{a}$ independent of the initial conditions.*

Proof. Model (1.3) can reduces to

$$\begin{aligned} -d_1\Delta S &= r(S+\rho I)[1-a(S+I)] - \frac{\beta SI}{S+I}, \\ -d_1\Delta I &= \frac{d_1}{d_2}\frac{\beta SI}{S+I} - \frac{d_1}{d_2}\mu I. \end{aligned} \tag{2.1}$$

Summing up the two equations of (2.1), we have

$$-d_1\Delta(S+I) = r(S+\rho I)[1-a(S+I)] - \frac{\beta SI}{S+I} + \frac{d_1}{d_2}\frac{\beta SI}{S+I} - \frac{d_1}{d_2}\mu I. \tag{2.2}$$

(i) If $d_1 < d_2$, then from (2.2) it follows that

$$\begin{aligned} -d_1\Delta(S+I) &\le r(S+\rho I)[1-a(S+I)] - \left(1-\frac{d_1}{d_2}\right)\frac{\beta SI}{S+I} \\ &\le r(S+\rho I)[1-a(S+I)]. \end{aligned}$$

(ii) If $d_1 > d_2$ and $d_1(\beta-\mu) < d_2\beta$, then from (2.2) lead to

$$\begin{aligned} -d_1\Delta(S+I) &\le r(S+\rho I)[1-a(S+I)] + \left(\left(\frac{d_1}{d_2}-1\right)\beta - \frac{d_1}{d_2}\mu\right)I \\ &\le r(S+\rho I)[1-a(S+I)]. \end{aligned}$$

In addition, by Lemma 2.2, we have $0 < S+I \le \frac{1}{a}$, and easy to see that $\Gamma = \frac{1}{a}$ independent of the initial conditions, then we can conclude the proof. □

Theorem 2.4. *If $(S(x), I(x))$ is any positive solution of (1.3) and $\beta > \mu$ holds, then*

$$0 < S(x) < \frac{1}{a}, \qquad 0 < I(x) < \frac{\beta-\mu}{\mu a}, \qquad x \in \overline{\Omega}.$$

Furthermore, if $M := \frac{r - ra\alpha(1+\rho)-\beta}{ra} > 0$ holds, then $(S(x), I(x))$ satisfies

$$M < S(x) < \frac{1}{a}, \qquad \frac{\beta-\mu}{\mu}M < I(x) < \frac{\beta-\mu}{\mu a}, \qquad x \in \overline{\Omega}, \tag{2.3}$$

where $\alpha = \frac{\beta-\mu}{\mu a}$.

Proof. Let (S, I) be a given positive solution of (1.3). First of all, by Theorem 2.3, it is clear that $S(x) < \frac{1}{a}$, for all $x \in \overline{\Omega}$. To obtain the upper bound for I, we let for some $z_0 \in \overline{\Omega}$ such that $I(z_0) = \max I(x)$. By virtue of Lemma 2.1, we have

$$\frac{\beta S(z_0)I(z_0)}{S(z_0)+I(z_0)} \ge \mu I(z_0).$$

Thus

$$I(z_0) \le \frac{\beta-\mu}{\mu}S(z_0) < \frac{\beta-\mu}{\mu a}.$$

In the following, we proof the lower bound of $(S(x), I(x))$, and $\alpha = \frac{\beta-\mu}{\mu a}$. Since

$$\begin{aligned} -d_1\Delta S &= r(S+\rho I)[1-a(S+I)] - \frac{\beta SI}{S+I} \\ &\ge r(S+\rho I)[1-a(S+I)] - \beta S \\ &\ge S(r - raS - ra\alpha(1+\rho) - \beta) + r\rho I(1-aI). \end{aligned}$$

By Theorem 2.3, we have

$$-d_1\Delta S \geq S(r - raS - ra\alpha(1+\rho) - \beta).$$

Hence, by Lemma 2.2 and strong maximum principle, we can obtain

$$S(x) > \frac{r - ra\alpha(1+\rho) - \beta}{ra} := M > 0.$$

Similarly, we have

$$I(x) > \frac{\beta - \mu}{\mu} M.$$

This completes our proof. □

3 Constant steady states and Turing instability

In this section, we mainly discuss the stability of constant steady state solution. For convenience, we denote

$$g_1(S,I) = r(S+\rho I)[1 - a(S+I)] - \frac{\beta SI}{S+I}, \qquad g_2(S,I) = \frac{\beta SI}{S+I} - \mu I. \tag{3.1}$$

Clearly, ODE model (1.1) or PDE model (1.2) has a unique constant steady state $E^* = (S^*, I^*)$ with positive coordinates

$$S^* = \left(1 - \frac{\mu(\beta-\mu)}{r(\mu+\rho(\beta-\mu))}\right)\frac{\mu}{a\beta}, \qquad I^* = S^*\frac{\beta-\mu}{\mu}$$

if and only if

(**P**) $\mu < \beta < \frac{r\mu}{\mu - r\rho} + \mu$ and $\mu > r\rho$ hold.

In addition, model (1.1) or model (1.2) has a trivial equilibrium $U_0 = \left(\frac{1}{a}, 0\right)$. By the standard linearization method, we can easily prove the following result.

Theorem 3.1. *The trivial equilibrium U_0 is locally asymptotically stable if $\mu > \beta$ and is unstable if $\mu < \beta$.*

Next, we will focus on the stability of E^* for model (1.1) and model (1.2), respectively. By simple calculation, the Jacobian matrix of (1.1) evaluated at E^* is given by

$$J(E^*) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tag{3.2}$$

where

$$\begin{aligned} a_{11} &= r[1 - a(S^*+I^*)] - ar(S^*+\rho I^*) - \frac{\beta I^{*2}}{(S^*+I^*)^2}, \\ a_{12} &= r\rho[1 - a(S^*+I^*)] - ar(S^*+\rho I^*) - \frac{\beta S^{*2}}{(S^*+I^*)^2}, \\ a_{21} &= \frac{\beta I^{*2}}{(S^*+I^*)^2} > 0, \qquad a_{22} = -\frac{\beta S^* I^*}{(S^*+I^*)^2} < 0. \end{aligned} \tag{3.3}$$

The characteristic equation of $J(E^*)$ is

$$\eta^2 - T\eta + Q = 0,$$

where

$$T = a_{11} + a_{22}, \qquad Q = a_{11}a_{22} - a_{21}a_{12}. \tag{3.4}$$

By direct calculation, under the condition $(\mathbf{P})$, we have $T < 0$, $Q > 0$. Thus, we can obtain the following theorem.

Theorem 3.2. *Assume condition* $(\mathbf{P})$ *holds, then the constant steady state solution* E^* *of model* (1.1) *is locally asymptotically stable.*

Next, we analyse the stability of the endemic equilibrium E^* for the reaction-diffusion model (1.2). Form now on, let

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_i < \cdots$$

be the sequence of eigenvalues for the operator $-\Delta$ subject to the Neumann boundary condition on Ω [7]. By $E(\lambda_i)$, we denote the space of eigenfunctions corresponding to λ_i in $H^1(\Omega)$. Set $\{\phi_{ij} : j = 1, 2, \cdots, \dim E(\lambda_i)\}$ be the orthonormal basis of $E(\lambda_i)$, $\mathbf{X} = [H^1(\Omega)]^2, \mathbf{X_{ij}} = \{\mathbf{c}\phi_{ij} : \mathbf{c} \in \mathbb{R}^2\}$. Then

$$\mathbf{X} = \bigoplus_{i=1}^{+\infty} \mathbf{X_i} \quad \text{and} \quad \mathbf{X_i} = \bigoplus_{j=1}^{\dim E(\lambda_i)} \mathbf{X_{ij}}.$$

Assume that $a_{11} > 0$ and $d_1\lambda_1 < a_{11}$, then we may define $N^0 = N^0(r, a, \rho, \beta, \mu, \Omega)$ to be the largest positive integer such that

$$d_1\lambda_i < a_{11}, \quad \text{for } i \leq N^0.$$

Obviously, if $d_1\lambda_1 < a_{11}$ is satisfied, then $1 \leq N^0 < \infty$. In this situation, define

$$\widetilde{d_2} := \min_{1 \leq i \leq N^0} d_{2,i}, \qquad d_{2,i} = \frac{a_{11}a_{22} - a_{12}a_{21} - a_{22}d_1\lambda_i}{\lambda_i(a_{11} - d_1\lambda_i)}. \tag{3.5}$$

And naturally we can give the stability of E^* of model (1.2).

Theorem 3.3. *Assume condition* $(\mathbf{P})$ *holds.*

(i) If $a_{11} < 0$, then E^* is locally asymptotically stable.

(ii) If $a_{11} > 0$, then

(ii-1) if $d_1\lambda_1 < a_{11}$ and $0 < d_2 < \widetilde{d_2}$, then E^* is locally asymptotically stable;

(ii-2) if $d_1\lambda_1 < a_{11}$ and $d_2 > \widetilde{d_2}$, then E^* is Turing unstable.

Proof. Consider the linearization operator evaluated at E^* of model (1.2)

$$L = \begin{pmatrix} d_1\Delta + a_{11} & a_{12} \\ a_{21} & d_2\Delta + a_{22} \end{pmatrix}.$$

It is easy to see that the eigenvalues of L are given by those of the following operator L_i

$$L_i = \begin{pmatrix} -d_1\lambda_i + a_{11} & a_{12} \\ a_{21} & -d_2\lambda_i + a_{22} \end{pmatrix},$$

whose characteristic equation is

$$\xi^2 - \xi T_i + Q_i = 0, \qquad i = 0, 1, 2, \ldots, \tag{3.6}$$

where

$$\begin{aligned} T_i &= -(d_1 + d_2)\lambda_i + a_{11} + a_{22}, \\ Q_i &= \lambda_i(d_1\lambda_i - a_{11}) \left\{ d_2 - \frac{d_1 a_{22}\lambda_i - a_{11}a_{22} + a_{12}a_{21}}{\lambda_i(d_1\lambda_i - a_{11})} \right\}. \end{aligned} \tag{3.7}$$

(i) If $a_{11} < 0$, then $T_i < 0$ and $Q_i > 0$, which implies that $\mathrm{Re}\{\xi_i\} < 0$ for all eigenvalues ξ. Therefore, the constant solution E^* is locally asymptotically stable.

(ii) Since $T < 0, Q > 0$, then $T_i < 0$ and $d_1 a_{22}\lambda_i - a_{11}a_{22} + a_{12}a_{21} < 0$.

(ii-1) If $a_{11} > 0$, $d_1\lambda_1 < a_{11}$ and $0 < d_2 < \widetilde{d}_2$, then $d_1\lambda_i < a_{11}$ and $d_2 < d_{2,i}$ for all $i \in [1, N^0]$. Thus

$$Q_i = \lambda_i(d_1\lambda_i - a_{11}) \left\{ d_2 - \frac{d_1 a_{22}\lambda_i - a_{11}a_{22} + a_{12}a_{21}}{\lambda_i(d_1\lambda_i - a_{11})} \right\} > 0.$$

One the other hand, if $i > N^0$, then $d_1\lambda_i > a_{11}$. Thus, we have $Q_i > 0$. The analysis yields the local asymptotic stability of E^*.

(ii-2) If $a_{11} > 0$, $d_1\lambda_1 < a_{11}$ and $d_2 > \widetilde{d}_2$, then we may assume the minimum is attained at $j \in [1, N^0]$. Thus $d_2 > d_{2,j}$, which implies

$$Q_j = \lambda_j(d_1\lambda_j - a_{11}) \left\{ d_2 - \frac{d_1 a_{22}\lambda_j - a_{11}a_{22} + a_{12}a_{21}}{\lambda_j(d_1\lambda_j - a_{11})} \right\} < 0.$$

Hence, E^* is unstable in this case. □

Remark 3.4. From Theorem 3.2 and 3.3, we can know that if $a_{11} > 0$, under mild extra conditions, the stability of the constant equilibrium E^* may change from stable, for the (ODE) dynamics (1.1), to unstable, for the (PDE) dynamics (1.2), whereas those of other constant equilibria are invariant.

Remark 3.5. When we regard Q_i as a quadratic polynomial with respect to λ_i, i.e., $Q_i = d_1 d_2 \lambda_i^2 - (d_1 a_{22} + d_2 a_{11})\lambda_i + a_{11}a_{22} - a_{12}a_{21}$, using the method of [26], we can also get that the condition of Turing instability: Assume that ($\mathbf{P}$) and $a_{11} > 0$ hold. If

$$\frac{d_2}{d_1} > \frac{-(2a_{12}a_{21} - a_{11}a_{22}) + 2\sqrt{a_{12}a_{21}(a_{12}a_{21} - a_{11}a_{22})}}{a_{11}^2},$$

then Turing instability occurs.

Example 3.6. As an example, we take the parameters in model (1.2) as

$$a = 1, \qquad \rho = 0.1, \qquad \beta = 1, \qquad \mu = 0.35, \qquad r = 0.61, \qquad d_1 = 0.01.$$

There is a unique positive equilibrium $E^* \approx (0.03546, 0.06586)$, and $a_{11} = 0.1 > 0, \widetilde{d}_2 = 0.1073$.

For the ODE model (1.1), easy to verify that $T = -0.1275 < 0, Q = 0.0166 > 0$, then E^* is locally asymptotically stable from Theorem 3.2.

For the PDE model (1.2) on one-dimensional space domain $(0, \pi)$, $d_1\lambda_1 - a_{11} = -0.09 < 0$. If $0.1 = d_2 < \widetilde{d}_2$, then E^* is locally asymptotically stable (see Fig. 3.1), and if $0.25 = d_2 > \widetilde{d}_2$, E^* is Turing instability from Theorem 3.3. The model (1.2) exhibits *Turing pattern* (see Fig. 3.2).

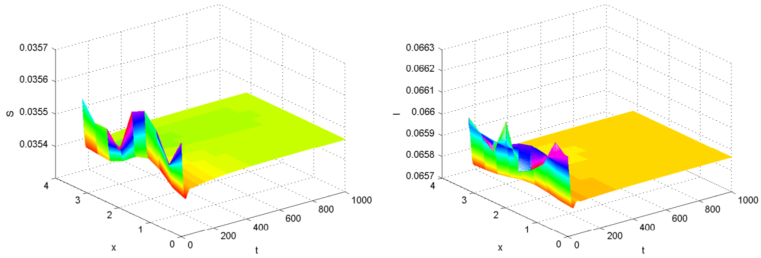


Figure 3.1: Stable behavior with $d_2 = 0.1$ for the model (1.2).

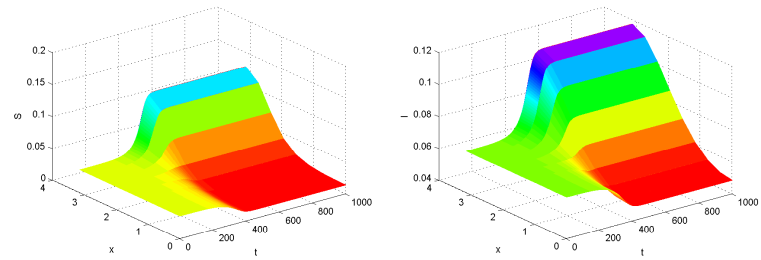
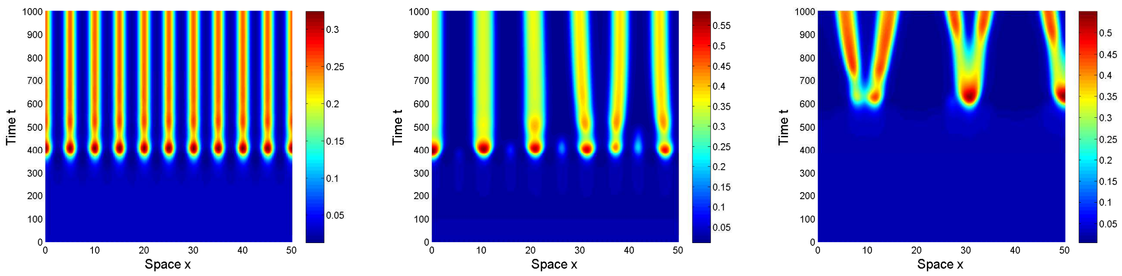
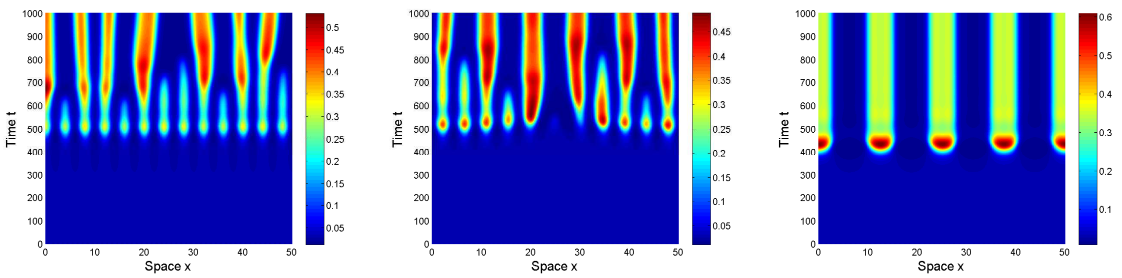


Figure 3.2: Turing instability behavior with $d_2 = 0.25$ for the model (1.2).

For the sake of learning the effect of the diffusion on the Turing pattern of model (1.2) more, as an example, in Fig. 3.3, we demonstrate that the spatial-temporal dynamics to (1.2) are complicated and the pattern formation is extremely sensitive to the variation in diffusion rate d_2 around 0.1073. The transitions between regular and irregular patterning have been well observed in model (1.2).



(a) $d_2 = 0.4$ (b) $d_2 = 1$ (c) $d_2 = 1.35$



(d) $d_2 = 1.5$ (e) $d_2 = 1.54$ (f) $d_2 = 1.8$

Figure 3.3: Transitions between regular and irregular patterning in model (1.2) with different values of d_2.

4 Nonconstant positive steady states

In this section, we will focus on the existence and the structure of nonconstant positive solution for the system (1.3).

4.1 Existence of nonconstant positive steady states

In this subsection, we apply priori estimates to yield the existence and nonexistence results of positive nonconstant solutions to (1.3). First, we can easily obtain the nonexistence of nonconstant positive solutions by using the energy method [27], which is relatively simple and we omit the proof here. Also, for notational convenience, we write $\Theta = (r, a, \rho, \beta, \mu)$ in the sequel.

Theorem 4.1. *Under the assumption* $(\mathbf{P})$*, let* D_2 *be a fixed positive constant satisfying* $D_2 > \frac{\mu}{\lambda_1}$*. Then there exists a positive constant* $D_1 = D_1(\Theta, D_2)$ *such that model* (1.3) *has no nonconstant positive solution provided that* $d_1 \geq D_1$ *and* $d_2 \geq D_2$*.*

With the help of Theorem 4.1, by using the Leray–Schauder degree theory, we discuss the existence of positive nonconstant solutions to (1.3) when the diffusion coefficients d_1 and d_2 vary while the parameters r, a, ρ, β, μ keep fixed.

Rewrite model (1.3) in the form:

$$\begin{cases} -\Delta E = D^{-1}F(E), & x \in \Omega, \\ \dfrac{\partial E}{\partial \mathbf{n}} = 0, & x \in \partial\Omega, \end{cases} \tag{4.1}$$

where $D = \text{diag}(d_1, d_2), E = (S, I), F(E) = (g_1(S, I), g_2(S, I))^T$. Therefore, E solves (4.1) if and only if it satisfies

$$\widehat{f}(d_1, d_2, E) := E - (\mathbf{I} - \Delta)^{-1}\{D^{-1}F(E) + E\} = 0 \quad \text{on } \mathbf{X}, \tag{4.2}$$

where $\mathbf{I}$ is the identity matrix, $(\mathbf{I} - \Delta)^{-1}$ represents the inverse of $\mathbf{I} - \Delta$ with homogeneous Neumann boundary condition.

A straightforward computation reveals

$$D_E\widehat{f}(d_1, d_2, E^*) = \mathbf{I} - (\mathbf{I} - \Delta)^{-1}(D^{-1}J + \mathbf{I}).$$

For each $\mathbf{X_i}$, ξ is an eigenvalue of $D_E\widehat{f}(d_1, d_2, E^*)$ on $\mathbf{X_i}$ if and only if $\xi(1 + \lambda_i)$ is an eigenvalue of the matrix

$$M_i := \lambda_i I - D^{-1}J = \begin{pmatrix} \lambda_i - d_1^{-1}a_{11} & -d_1^{-1}a_{12} \\ -d_2^{-1}a_{21} & \lambda_i - d_2^{-1}a_{22} \end{pmatrix}.$$

Clearly,

$$\text{det}M_i = d_1^{-1}d_2^{-1}[d_1d_2\lambda_i^2 + (-d_1a_{22} - d_2a_{11})\lambda_i + a_{11}a_{22} - a_{12}a_{21}],$$

and

$$\text{tr}M_i = 2\lambda_i - d_1^{-1}a_{11} - d_2^{-1}a_{22}.$$

Define

$$\widehat{g}(d_1, d_2, \lambda) := d_1d_2\lambda^2 + (-d_1a_{22} - d_2a_{11})\lambda + a_{11}a_{22} - a_{12}a_{21}.$$

Thus, $\widehat{g}(d_1, d_2, \lambda_i) = d_1d_2 \det M_i$. If

$$d_1a_{22} + d_2a_{11} > 2\sqrt{d_1d_2(a_{11}a_{22} - a_{12}a_{21})}, \tag{4.3}$$

then $\widehat{g}(d_1,d_2,\lambda)=0$ has two real roots, that is

$$\lambda_+(d_1,d_2)=\frac{d_1a_{22}+d_2a_{11}+\sqrt{(d_1a_{22}+d_2a_{11})^2-4d_1d_2(a_{11}a_{22}-a_{12}a_{21})}}{2d_1d_2},$$

$$\lambda_-(d_1,d_2)=\frac{d_1a_{22}+d_2a_{11}-\sqrt{(d_1a_{22}+d_2a_{11})^2-4d_1d_2(a_{11}a_{22}-a_{12}a_{21})}}{2d_1d_2}.$$

Let

$$\begin{aligned}A&=A(d_1,d_2)=\{\lambda:\lambda\geq 0,\lambda_-(d_1,d_2)<\lambda<\lambda_+(d_1,d_2)\},\\ S_p&=\{\lambda_0,\lambda_1,\lambda_2,\dots\},\end{aligned}$$

and let $m(\lambda_i)$ be multiplicity of λ_i. In order to calculate the index of $\widehat{f}(d_1,d_2,\cdot)$ at E^*, we need the following lemma.

Lemma 4.2. *Suppose $\widehat{g}(d_1,d_2,\lambda_i)\neq 0$ for all $\lambda_i\in S_p$. Then*

$$\operatorname{index}(\widehat{f}(d_1,d_2,\cdot),E^*)=(-1)^\sigma,$$

where

$$\sigma=\begin{cases}\sum_{\lambda_i\in A\cap S_p}m(\lambda_i), & A\cap S_p\neq\varnothing,\\ 0, & A\cap S_p=\varnothing.\end{cases}$$

In particular, $\sigma=0$ if $\widehat{g}(d_1,d_2,\lambda_i)>0$ for all $\lambda_i\geq 0$.

From Lemma 4.2, in order to calculate the index of $\widehat{f}(d_1,d_2,\cdot)$ at E^*, we need to determine the range of λ for which $\widehat{g}(d_1,d_2,\lambda)<0$.

Theorem 4.3. *Under the conditions of Theorem 2.4 and* (**P**), *$a_{11}>0$ hold. If $\frac{a_{11}}{d_1}\in(\lambda_k,\lambda_{k+1})$ for some $k\geq 1$, and $\sigma_k=\sum_{i=1}^k m(\lambda_i)$ is odd, then there exists a positive constant D^* such that for all $d_2\geq D^*$, model (1.3) has at least one nonconstant positive solution.*

Proof. Since $a_{11}>0$, it follows that if d_2 is large enough, then (4.3) holds and $\lambda_+(d_1,d_2)>\lambda_-(d_1,d_2)>0$. Furthermore,

$$\lim_{d_2\to\infty}\lambda_+(d_1,d_2)=\frac{a_{11}}{d_1},\qquad \lim_{d_2\to\infty}\lambda_-(d_1,d_2)=0.$$

As $\frac{a_{11}}{d_1}\in(\lambda_k,\lambda_{k+1})$, there exists $d_0\gg 1$ such that

$$\lambda_+(d_1,d_2)\in(\lambda_k,\lambda_{k+1}),\qquad 0<\lambda_-(d_1,d_2)<\lambda_1\qquad \forall d_2\geq d_0. \tag{4.4}$$

From Theorem 4.1, we know that there exists $d>d_0$ such that (1.3) with $d_1=d$ and $d_2\geq d$ has no nonconstant positive solution. Let $d>0$ be large enough such that $\frac{a_{11}}{d_1}<\lambda_1$. Then there exists $D^*>d$ such that

$$0<\lambda_-(d_1,d_2)<\lambda_+(d_1,d_2)<\lambda_1\quad \text{for all } d_2\geq D^*. \tag{4.5}$$

Now we prove that, for any $d_2\geq D^*$, (1.3) has at least one nonconstant positive solution. By way of contradiction, assume that the assertion is not true for some $D_2^*\geq D^*$. By using the homotopy argument, we can derive a contradiction in the sequel. Fixing $d_2 = D_2^*$, for $\tau \in [0,1]$, we define

$$D(\tau) = \begin{pmatrix} \tau d_1 + (1-\tau)d & 0 \\ 0 & \tau d_2 + (1-\tau)D^* \end{pmatrix},$$

and consider the following problem

$$\begin{cases} -\Delta E = D^{-1}(\tau)F(E), & x \in \Omega, \\ \dfrac{\partial E}{\partial \mathbf{n}} = 0, & x \in \partial\Omega. \end{cases} \tag{4.6}$$

Thus, E is a positive nonconstant solution of (1.3) if and only if it solves (4.6) with $\tau = 1$. Evidently, E^* is the unique positive constant solution of (4.6). For any $\tau \in [0,1]$, E is a positive nonconstant solution of (4.6) if and only if

$$h(E,\tau) = E - (\mathbf{I} - \Delta)^{-1}\{D^{-1}(\tau)F(E) + E\} = 0 \quad \text{on } \mathbf{X}. \tag{4.7}$$

From the discussion above, we know that (4.7) has no positive nonconstant solution when $\tau = 0$, and we have assumed that there is no such solution for $\tau = 1$ at $d_2 = D_2^*$. Clearly, $h(E,1) = \widehat{f}(d_1, d_2, E), h(E,0) = \widehat{f}(d, D^*, E)$ and

$$D_E\widehat{f}(d_1, d_2, E^*) = \mathbf{I} - (\mathbf{I} - \Delta)^{-1}(D^{-1}J + \mathbf{I}),$$
$$D_E\widehat{f}(d, D^*, E^*) = \mathbf{I} - (\mathbf{I} - \Delta)^{-1}(\widetilde{D}^{-1}J + \mathbf{I}),$$

where $\widehat{f}(\cdot,\cdot,\cdot)$ is as given in (4.2) and $\widetilde{D} = \operatorname{diag}(d, D^*)$. From (4.4) and (4.5), we have $A(d_1, d_2) \cap S_p = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ and $A(d, D^*) \cap S_p = \varnothing$. Since σ_k is odd, Lemma 4.2 yields

$$\operatorname{index}(h(\cdot,1), E^*) = \operatorname{index}(\widehat{f}(d_1, d_2, \cdot), E^*) = (-1)^{\sigma_k} = -1,$$
$$\operatorname{index}(h(\cdot,0), E^*) = \operatorname{index}(\widehat{f}(d, D^*, \cdot), E^*) = (-1)^0 = 1.$$

From Theorem 2.4, there exist positive constants $\underline{C} = \underline{C}(d, d_1, D^*, D_2^*, \Theta)$ and $\overline{C} = \overline{C}(d, D^*, \Theta)$ such that the positive solutions of (4.7) satisfy $\underline{C} < S(x), I(x) < \overline{C}$ on $\overline{\Omega}$ for all $\tau \in [0,1]$.

Define $\Sigma = \{(S,I)^T \in C^1(\overline{\Omega}, \mathbb{R}^2) : \underline{C} < S(x), I(x) < \overline{C}, x \in \overline{\Omega}\}$. Then $h(E,\tau) \neq 0$ for all $E \in \partial\Sigma$ and $\tau \in [0,1]$. By virtue of the homotopy invariance of the Leray–Schauder degree, we have

$$\deg(h(\cdot,0), \Sigma, 0) = \deg(h(\cdot,1), \Sigma, 0). \tag{4.8}$$

Notice that both equations $h(E,0) = 0$ and $h(E,1) = 0$ have a unique positive solution E^* in Σ, and we obtain

$$\deg(h(\cdot,0), \Sigma, 0) = \operatorname{index}(h(\cdot,0), E^*) = 1,$$
$$\deg(h(\cdot,1), \Sigma, 0) = \operatorname{index}(h(\cdot,1), E^*) = -1,$$

which contradicts (4.8). The proof is complete. □

Remark 4.4. Theorem 4.3 shows that, if the parameters are properly chosen, the existence of nonconstant steady states, i.e., Turing pattern can arise as a result of diffusion.

Next we investigate the structure of nonconstant positive solution for the system (1.3) in the one-dimensional space domain $\Omega = (0, \pi)$. Thus, (1.3) become

$$\begin{aligned} &d_1 \Delta S + r(S+\rho I)[1 - a(S+I)] - \frac{\beta S I}{S+I} = 0, && x \in (0,\pi), \\ &d_2 \Delta I + \frac{\beta S I}{S+I} - \mu I = 0, && x \in (0,\pi), \\ &S' = I' = 0, && x = 0, \pi. \end{aligned} \tag{4.9}$$

For the sake of simplicity, we denote $d_1 = 1$ and $d_2 = d$.

It is well known that the operator $u \to -\Delta u$ with no-flux boundary conditions has eigenvalues

$$\lambda_0 = 0, \qquad \lambda_j = j^2, \qquad j = 1, 2, 3, \ldots$$

and eigenfunctions

$$\phi_0(x) = \sqrt{\frac{1}{\pi}}, \qquad \phi_j(x) = \sqrt{\frac{2}{\pi}} \cos jx, \quad j = 1, 2, 3, \ldots$$

We translate (S^*, I^*) to the origin by the translation $(\widehat{S}, \widehat{I}) = (S - S^*, I - I^*)$. For convenience, we still denote $\widehat{S}, \widehat{I}$ by S, I respectively, then we can obtain the following system

$$\begin{aligned} &\Delta S + r(S + S^* + \rho(I + I^*))[1 - a((S+S^*) + (I+I^*))] - \frac{\beta(S+S^*)(I+I^*)}{(S+S^*)+(I+I^*)} = 0, && x \in (0,\pi), \\ &d\Delta I + \frac{\beta(S+S^*)(I+I^*)}{(S+S^*)+(I+I^*)} - \mu(I+I^*) = 0, && x \in (0,\pi), \\ &S' = I' = 0, && x = 0, \pi. \end{aligned} \tag{4.10}$$

4.2 Local structure of nonconstant positive steady states

In this subsection, we study the local structure of nonconstant positive solutions for the new system (4.10). In brief, by regarding d as the bifurcation parameter, we verify the existence of positive solutions bifurcating form $(d, (0,0))$. The Crandall–Rabinowitz bifurcation theorem from the simple eigenvalue in [18] will be applied to obtain bifurcations. For the case of double eigenvalues, we shall apply some techniques in [16] and [22] to deal with it.

Let $X = \{(S, I) \in W^{2,p}(0,\pi) \times W^{2,p}(0,\pi) : S' = I' = 0, x = 0, \pi\}$ and $Y = L^p(0,\pi) \times L^p(0,\pi)$. We define the map $F : \mathbb{R}^+ \times X \to Y$ by

$$F(d,(S,I)) = \begin{pmatrix} \Delta S + r(S+S^*+\rho(I+I^*))[1-a((S+S^*)+(I+I^*))] - \frac{\beta(S+S^*)(I+I^*)}{(S+S^*)+(I+I^*)} \\ d\Delta I + \frac{\beta(S+S^*)(I+I^*)}{(S+S^*)+(I+I^*)} - \mu(I+I^*) \end{pmatrix}.$$

Thus, the solutions of (4.10) are exactly zeros of this map $F(d,(S,I))$. Note that $(0,0)$ is the unique constant solution of (4.10), then we have $F(d,(0,0)) = 0$. The Fréchet derivative of $F(d,(S,I))$ with respect to (S,I) at $(0,0)$ can be given by

$$L(d) = F_{(S,I)}(d,(0,0)) = \begin{pmatrix} \Delta + a_{11} & a_{12} \\ a_{21} & d\Delta + a_{22} \end{pmatrix}.$$

The characteristic equation of $L(d)$ is given by

$$\xi^2 - \xi T_i + Q_i = 0, \qquad i = 0, 1, 2, \ldots, \tag{4.11}$$

where, $T_i = -(1+d)\lambda_i + a_{11} + a_{22}$ and $Q_i = d\lambda_i^2 + (-a_{22} - da_{11})\lambda_i + a_{11}a_{22} - a_{12}a_{21}$.

Throughout this section, we always assume that $\lambda_1 < a_{11}$. Then there exists the largest positive integer $N^0 \geq 1$ such that $\lambda_j < a_{11}$ for $1 \leq j \leq N^0$. Letting $\xi = 0$ in (4.11), we have

$$d = d_j = \frac{a_{11}a_{22} - a_{12}a_{21} - a_{22}\lambda_j}{\lambda_j(a_{11} - \lambda_j)}, \quad \text{for } 1 \leq j \leq N^0.$$

We shall prove that there exists a nonconstant positive solution of $F(d,(S,I)) = 0$ near $(d_j,(0,0))$.

Theorem 4.5. *Let $d = d_j$, $\lambda_j = j^2$, for $1 \leq j \leq N^0$. Assume that*

$$r \neq \left\{ \frac{(\beta-\mu)(2\mu^2 - \rho(\beta-\mu)(\beta-2\mu))}{\rho^2(\beta-\mu)^2 + 2\mu\rho(\beta-\mu) + \mu^2}, \frac{2\mu\beta(\beta-\mu)^2 + \mu^2(\beta-2\mu)}{\rho^2(\beta-\mu)^2 + 2\mu\rho(\beta-\mu) + \mu^2} \right\}.$$

Suppose that $d_i \neq d_j$ whenever $i \neq j, 1 \leq i,j \leq N^0$. Then $(d_j,(0,0))$ is a bifurcation point of $F(d,(S,I)) = 0$. Moreover, there is a curve of nonconstant solutions $(d(s),(S(s),I(s)))$ of $F(d,(S,I)) = 0$ for $|s|$ sufficiently small, satisfying $d(0) = d_j, (S(0)\ I(0)), S(s) = s\phi_j + O(s^2)$, $I(s) = se_j\phi_j + O(s^2)$, where $d(s), S(s), I(s)$ are continuously differentiable function with respect to s and $e_j = \frac{\lambda_j - a_{11}}{a_{12}}$.

Proof. By the Crandall–Rabinowitz bifurcation theorem about simple eigenvalues in [18], we see that $(d,(0,0))$ is a bifurcation point if the following conditions are satisfied:

(a) the partial derivatives $F_d, F_{(S,I)}$, and $F_{(d,(S,I))}$ exist and are continuous.

(b) $\dim\ker F_{(S,I)}(d,(0,0)) = \text{codim}\mathrm{R}(F_{(S,I)}(d,(0,0))) = 1$.

(c) Let $\ker F_{(S,I)}(d,(0,0)) = \text{span}\{\Phi\}$, then $F_{(d,(S,I))}(d,(0,0))\Phi \notin \mathrm{R}(F_{(S,I)}(d,(0,0)))$.

Note that

$$L(d_j) = F_{(S,I)}(d,(0,0)) = \begin{pmatrix} \Delta + a_{11} & a_{12} \\ a_{21} & d_j\Delta + a_{22} \end{pmatrix},$$

and

$$F_d(d,(0,0)) = \begin{pmatrix} 0 \\ \Delta \end{pmatrix}, \qquad F_{(d,(S,I))}(d,(0,0)) = \begin{pmatrix} 0 & 0 \\ 0 & \Delta \end{pmatrix}.$$

It is obvious that the linear operators $F_d, F_{(S,I)}, F_{(d,(S,I))}$ are continuous. So assertion (a) holds.

Suppose $\Phi_j = (\phi,\psi) \in \ker L(d_j)$, and write $\phi = \Sigma a_j\phi_j, \psi = \Sigma b_j\phi_j$. Then

$$\sum_{j=0}^{\infty} B_j \begin{pmatrix} a_j \\ b_j \end{pmatrix} \phi_j = 0, \quad \text{where} \quad B_j = \begin{pmatrix} a_{11} - \lambda_j & a_{12} \\ a_{21} & a_{22} - d_j\lambda_j \end{pmatrix}. \tag{4.12}$$

And

$$\det B_j = 0 \quad \Leftrightarrow \quad d = d_j = \frac{a_{11}a_{22} - a_{12}a_{21} - a_{22}\lambda_j}{\lambda_j(a_{11} - \lambda_j)}$$

implies that

$$\ker L(d_j) = \text{span}\{\Phi_j\}, \qquad \Phi_j = \begin{pmatrix} 1 \\ e_j \end{pmatrix} \phi_j,$$

where $e_j = \frac{\lambda_j - a_{11}}{a_{12}}$. The adjoint operator is defined by

$$L^*(d_j) = \begin{pmatrix} \Delta + a_{11} & a_{21} \\ a_{12} & d_j\Delta + a_{22} \end{pmatrix}.$$

In the same way as above we obtain

$$\ker L^*(d_j) = \text{span}\{\Phi_j^*\}, \qquad \Phi_j^* = \begin{pmatrix} 1 \\ e_j^* \end{pmatrix} \phi_j,$$

where $e_j^* = \frac{\lambda_j - a_{11}}{a_{21}}$.

Since $\mathrm{R}(L) = \ker(L^*)^{\perp}$ (R: image; $^{\perp}$: complementary set), thus

$$\text{codim}(\mathrm{R}(L(d_j))) = \dim(\ker(L^*(d_j))) = 1.$$

So assertion (b) holds.

Next, we verify assertion (c) holds. Since

$$F_{(d,(S,I))}(d_j, (0,0))\Phi_j = \begin{pmatrix} 0 & 0 \\ 0 & \Delta \end{pmatrix} \Phi_j = \begin{pmatrix} 0 \\ -\lambda_j e_j \phi_j \end{pmatrix}.$$

and

$$\langle F_{(d,(S,I))}(d_j, (0,0))\Phi_j, \Phi_j^* \rangle_Y = \langle -\lambda_j e_j \phi_j, e_j^* \phi_j \rangle = -\lambda_j e_j e_j^* \neq 0.$$

we see that $F_{(d,(S,I))}(d_j, (0,0))\Phi_j \notin \mathrm{R}(L(d_j))$. Hence, the proof is completed. □

Remark 4.6. Under the assumption of Theorem 4.5, each $(d_j, (S^*, I^*))$ is a bifurcation point with respect to the trivial branch $(d, (S^*, I^*))$. The number of such bifurcation points is infinite.

Remark 4.7. Theorem 4.5 implies that each bifurcation curve Γ_j around $(d_j, (S^*, I^*))$ is of pitchfork type.

4.3 Global structure of nonconstant positive steady states

In this subsection, we study the global structure of the bifurcation solutions form simple eigenvalues. Let J denote the closure of the nonconstant solution set of (4.9), and Γ_j the connected component of $J \cup (d_j, (S^*, I^*))$ to which $(d_j, (S^*, I^*))$ belongs. Theorem 4.5 provides no information on the bifurcating curve Γ_j far form the equilibrium (S^*, I^*). In order to understand its global structure, a further study is necessary.

Theorem 4.8. *Under the same assumption of Theorem 4.5, the projection of the bifurcation curve Γ_j on the d-axis contains (d_j, ∞).*

Proof. Rewrite (4.9) as

$$\begin{cases} -\Delta S = a_{11}S + a_{12}I + h_1(S,I), \\ -d\Delta I = a_{21}S + a_{22}I + h_2(S,I), \end{cases}$$

where $h_1(S,I), h_2(S,I)$ are higher-order terms of S and I. The constant steady state (S^*, I^*) of (1.3) is shifted to $(0,0)$ of this new system. Let

$$G = (-\Delta + a_{11})^{-1}, \qquad G_d = (-d\Delta - a_{22})^{-1}, \qquad E = (S,I).$$

We next rewrite (4.9) in a form that the standard global bifurcation theory can be more conveniently used. Then

$$K(d)E = (2a_{11}G(S) + a_{12}G(I), a_{21}G_d(S))$$

and

$$H(E) = (G(h_1(S, I)), G_d(h_2(S, I))).$$

Then (4.9) can be interpreted as the equation

$$E = K(d)E + H(E). \tag{4.13}$$

For any fixed $d > 0$, it is noted that $K(d)$ is a compact liner operator on X. $H(E) = o(|E|)$ for E near zero uniformly on closed d sub-intervals of $(0, \infty)$, and is also a compact operator on X.

To apply Rabinowitz's global bifurcation theorem, we first verify that 1 is an eigenvalue of $K(d_j)$ of algebraic multiplicity one. From the argument in the proof of Theorem 4.5 it is seen that $\ker(K(d_j) - \mathrm{Id}) = \ker L = \mathrm{span}\{\Phi_j\}$ (Id: identity operator), so 1 is indeed an eigenvalue of $K(d_j)$, and $\dim \ker(K(d_j) - \mathrm{Id}) = 1$. According to [12], we know that the algebraic multiplicity of the eigenvalue 1 is the dimension of the generalized null space $\cup_{i=1}^{\infty} \ker(K(d_j) - \mathrm{Id})^i$. For our purpose, we need to verify that

$$\ker(K(d_j) - \mathrm{Id}) = \ker(K(d_j) - \mathrm{Id})^2, \quad \text{or} \quad \ker(K(d_j) - \mathrm{Id}) \cap \mathrm{R}(K(d_j) - \mathrm{Id}) = \{0\}.$$

Now, We compute $\ker(K^*(d_j) - \mathrm{Id})$, where $K^*(d_j)$ is the adjoint of $K(d_j)$. Let $(\varphi, \chi) \in \ker(K^*(d_j) - \mathrm{Id})$. Then we have

$$2a_{11}G(\varphi) + a_{21}G_d(\chi) = \varphi, \qquad a_{12}G(\varphi) = \chi.$$

By the definition of G and G_d, we obtain

$$-d_j a_{12}\Delta\varphi = f_\varphi \varphi + f_\chi \chi, \qquad -\Delta\chi = a_{12}\varphi - a_{11}\chi,$$

where

$$f_\varphi = 2d_j a_{11}a_{12} + a_{12}a_{22}, \qquad f_\chi = a_{12}a_{21} - 2a_{11}a_{22} - 2d_j a_{11}^2.$$

Let $\varphi = \Sigma a_i\phi_i, \chi = \Sigma b_i\phi_i$. Then

$$\sum_{i=0}^{\infty} B_i^* \begin{pmatrix} a_i \\ b_i \end{pmatrix} \phi_i = 0, \quad \text{where} \quad B_i^* = \begin{pmatrix} -d_j a_{12}\lambda_i + f_\varphi & f_\chi \\ a_{12} & -\lambda_i - a_{11} \end{pmatrix}.$$

By a straightforward calculation one can check that $\det B_i^* = a_{12} \det B_i$, where B_i is given in (4.12). Thus $\det B_i^* = 0$ only for $i = j$, and

$$\ker(K^*(d_j) - \mathrm{Id}) = \mathrm{span}\{\Phi_j'\} \quad \text{where } \Phi_j' = \left(\frac{\lambda_j + a_{11}}{a_{12}}, 1\right)^{\top} \phi_j.$$

In addition, we can check that $\int_0^\pi \Phi_j'\Phi_j dx = \frac{2\lambda_j}{a_{12}} \neq 0$, which implies that

$$\Phi_j \notin (\ker(K^*(d_j) - \mathrm{Id}))^{\perp} = \mathrm{R}((K(d_j) - \mathrm{Id})).$$

Hence, we show $\ker(K(d_j) - \mathrm{Id}) \cap \mathrm{R}(K(d_j) - \mathrm{Id}) = \{0\}$ and the eigenvalue 1 has algebraic multiplicity one.

Suppose that $0 < d \neq d_j$ is in a small neighborhood of d_j, then, for this given d, the liner operator $\mathrm{Id} - K(d) : X \to X$ is a bijection and 0 is an isolated solution of (4.13). The index of this isolated zero of $\mathrm{Id} - K(d) - H$ is given by

$$\mathrm{index}(\mathrm{Id} - K(d) - H, (d, 0)) = \deg(\mathrm{Id} - K(d), B, 0) = (-1)^P,$$

where B is a sufficiently small ball center at 0, and P is the sum of the algebraic multiplicities of the eigenvalues of $K(d)$ that are greater than one.

For our purpose, it is also necessary to show that this index changes when d crosses d_j, that is, for $\varepsilon > 0$ sufficiently small, we need verify

$$\text{index}(\text{Id} - K(d_j - \varepsilon) - H, (d_j - \varepsilon, 0)) \neq \text{index}(\text{Id} - K(d_j + \varepsilon) - H, (d_j + \varepsilon, 0)). \tag{4.14}$$

Indeed, suppose that ζ is an eigenvalue of $K(d)$ with an eigenfunction $(\tilde{\phi}, \tilde{\psi})$, then

$$\begin{aligned} -\zeta \Delta \tilde{\phi} &= (2-\zeta) a_{11} \tilde{\phi} + a_{12} \tilde{\psi}, \\ -d\zeta \Delta \tilde{\psi} &= a_{21} \tilde{\phi} + a_{22} \zeta \tilde{\psi}. \end{aligned}$$

Using the Fourier cosine series $\tilde{\phi} = \Sigma \tilde{a}_i \phi_i$ and $\tilde{\psi} = \Sigma \tilde{b}_i \phi_i$ leads to

$$\sum_{i=0}^{\infty} \tilde{B}_i \begin{pmatrix} \tilde{a}_i \\ \tilde{b}_i \end{pmatrix} \phi_i = 0, \quad \text{where} \quad \tilde{B}_i = \begin{pmatrix} (2-\zeta) a_{11} - \lambda_i \zeta & a_{12} \\ a_{21} & (a_{22} - d\lambda_i)\zeta \end{pmatrix}.$$

Thus, the set of eigenvalues of $K(d)$ consists of all $\zeta' s$, which solve the characteristic equation

$$\zeta^2 - \frac{2a_{11}}{a_{11} + \lambda_i} \zeta - \frac{a_{12} a_{21}}{(a_{11} + \lambda_i)(d\lambda_i - a_{22})} = 0, \tag{4.15}$$

where the integer i runs from zero to ∞. In particular, for $d = d_j$, if $\zeta = 1$ is a root of (4.15), then a simple calculation leads to $d_j = d_i$, and so $j = i$ by the assumption. Therefore, without counting the eigenvalues corresponding to $i \neq j$ in (4.15), $K(d)$ has the same number of eigenvalues greater than 1 for all d close to d_j, and they have the same multiplicities. On the other hand, for $i = j$ in (4.15), we let $\zeta(d), \tilde{\zeta}(d)$ denote the two roots. By a straightforward calculation, we find that

$$\zeta(d_j) = 1 \quad \text{and} \quad \tilde{\zeta}(d_j) = \frac{a_{11} - \lambda_j}{a_{11} + \lambda_j} < 1.$$

When d close to d_j, we obtain $\tilde{\zeta}(d) < 1$. As the constant term $-a_{12}a_{21}/(d\lambda_i - a_{22})$ in (4.15) is a decreasing function of d when $a_{12} < 0$, we know that

$$\zeta(d_j + \varepsilon) > 1, \quad \zeta(d_j - \varepsilon) < 1.$$

Consequently, $K(d_j + \varepsilon)$ has exactly one more eigenvalues that are larger than 1 than $K(d_j - \varepsilon)$ does. Furthermore, by a similar argument above, we can show this eigenvalue has algebraic multiplicity one. So (4.14) holds. And the proof is complete. □

Remark 4.9. Theorem 4.8 shows that there is a smooth curve Γ_j bifurcating from $(d_j, (S^*, I^*))$, with Γ_j contained in a global branch of the positive solutions of (4.9).

5 Conclusions

In this paper, we study the dynamics of a reaction-diffusion model in the susceptible population. In particular, we are interested in the positive steady states. Diffusion-induced instability of the positive equilibrium E^* is investigated, which produces spatial inhomogeneous patterns (see Theorem 3.3). Since a priori estimates for steady states are necessary in obtaining the existence of nonconstant positive steady states by applying the global bifurcation theory, establishing a priori bounds for steady states is the key point.

The condition $d_i \neq d_j$ for any integer $i \neq j$ in Theorem 4.5 guarantees $\dim \mathrm{Ker} L(d_j) = 1$, that is 0 is a simple eigenvalue of $L(d_j)$. Hence, we can apply the global bifurcation theory from a simple eigenvalue in this paper. In fact, $j \mapsto d_j$ is not a one-to-one correspondence. On the other hand, d_j is not monotonous function for λ_j. If $d_i = d_j$ for some integer $i \neq j$, then $\dim \mathrm{Ker}\, L(d_j) > 1$. We hope to discuss this case in the near future.

We also remark that we do not know if it is possible that Γ_j obtained in Theorem 4.8 meets some bifurcation points and then reaches infinity; note that our argument only rules out the possibility that Γ_j meets some bifurcation points without finally reaching infinity. If this case occurs, then some bifurcation branches "collide" each other and the solution undergo a symmetry breaking. Understanding this phenomenon is very important in studying the pattern formation in living organisms.

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Multi-bump solutions for the magnetic Schrödinger–Poisson system with critical growth

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Abstract. In this paper, we are concerned with the following magnetic Schrödinger–Poisson system

$$\begin{cases} -(\nabla + iA(x))^2 u + (\lambda V(x) + 1)u + \phi u = \alpha f(|u|^2)u + |u|^4 u, & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2, & \text{in } \mathbb{R}^3, \end{cases}$$

where $\lambda > 0$ is a parameter, f is a subcritical nonlinearity, the potential $V : \mathbb{R}^3 \to \mathbb{R}$ is a continuous function verifying some conditions, the magnetic potential $A \in L^2_{loc}(\mathbb{R}^3, \mathbb{R}^3)$. Assuming that the zero set of $V(x)$ has several isolated connected components $\Omega_1, \dots, \Omega_k$ such that the interior of Ω_j is non-empty and $\partial\Omega_j$ is smooth, where $j \in \{1, \dots, k\}$, then for $\lambda > 0$ large enough, we use the variational methods to show that the above system has at least $2^k - 1$ multi-bump solutions.

Keywords: Schrödinger–Poisson system, multi-bump solutions, magnetic field, critical growth, variational methods.

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1 Introduction

In the past few decades, there is a vast literature concerning the nonlinear Schrödinger–Poisson system

$$\begin{cases} -i\frac{\partial\psi}{\partial t} = -\Delta\psi + V(x)\psi + \phi(x)\psi - |\psi|^{p-1}\psi, & \text{in } \mathbb{R}^3, \\ -\Delta\phi = \psi^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $V : \mathbb{R}^3 \to \mathbb{R}$ is a nonnegative continuous function with $\inf_{x\in\mathbb{R}^3} V(x) > 0$, $1 < p < 5$ and $\psi : \mathbb{R}^3 \to \mathbb{C}$ and $\phi : \mathbb{R}^3 \to \mathbb{R}$ are two unknown functions. In fact, the first equation in the above system describes quantum (non-relativistic) particles interacting with the electromagnetic field generated by the motion. And $\phi(x)$ satisfies the second equation (Poisson

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equation) in the system, because the potential $\phi(x)$ is determined by the charge of wave function itself. Therefore, system (1.1) can be regarded as the coupling of the Schrödinger equation and Poisson equation.

If one looks for stationary solutions $\psi(x,t) := e^{-it}u(x)$ of system (1.1), the system can be reduced by

$$\begin{cases} -\Delta u + V(x)u + \phi(x)u = |u|^{p-1}u, & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2, & \text{in } \mathbb{R}^3. \end{cases} \tag{1.2}$$

In [4], Azzollini and Pomponio considered system (1.2). More precisely, if V is a positive constant, they proved the existence of a ground state solution (u,ϕ) for $2 < p < 5$. If V is a nonconstant potential that is measurable and (possibly) not bounded from below, they obtained a similar existence result for $3 < p < 5$. Existence and nonexistence results were also proved when the nonlinearity exhibits a critical growth.

In a celebrated paper [13], by using the variational methods, Ding and Tanaka established multiplicity of multi-bump solutions for a semilinear elliptic equation with deepening potential well. Recently, in [2], Alves and Yang considered system (1.2) which having a general nonlinear term f and assumed the potential $V(x)$ has the form $V(x) = \lambda a(x) + 1$, where λ is a positive parameter and $a : \mathbb{R}^3 \to \mathbb{R}^3$ is a nonnegative continuous function. In the interesting paper, the authors proved the existence of positive multi-bump solutions for the system

$$\begin{cases} -\Delta u + (\lambda a(x) + 1)u + \phi(x)u = f(u), & \text{in } \mathbb{R}^3, \\ -\Delta\phi = 4\pi u^2, & \text{in } \mathbb{R}^3. \end{cases}$$

For more results on the Schrödinger–Poisson system, we refer the reader to [3, 5, 7, 10, 11, 18, 19, 23–26, 28, 31–34, 36, 38, 40, 41] and the references therein.

In recent years, the magnetic nonlinear Schrödinger equation has also received considerable attention

$$i\hbar\frac{\partial\psi}{\partial t} = \left(\frac{\hbar}{i}\nabla - A(x)\right)^2\psi + U(x)\psi - f(|\psi|^2)\psi, \quad \text{in } \mathbb{R}^N \times \mathbb{R},$$

where i is the imaginary unit, $\hbar$ is the Planck constant, and $A : \mathbb{R}^N \to \mathbb{R}^N$ is the magnetic potential. When one looks for standing wave solutions $\psi(x,t) := e^{-iEt/\hbar}u(x)$, with $E \in \mathbb{R}$, of the above equation, the problem can be reduced by

$$\left(\frac{\hbar}{i}\nabla - A(x)\right)^2 u + V(x)u = f\left(|u|^2\right)u, \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

From a physical point of view, the existence of such solutions and the study of their shape in the semiclassical limit, namely, as $\hbar \to 0^+$ is of the greatest importance, since the transition from Quantum Mechanics to Classical Mechanics can be formally performed by sending the Planck constant $\hbar$ to zero.

As far as we know, the first result involving the magnetic field was obtained by Esteban and Lions [15]. In [15], for $\hbar > 0$ fixed and special classes of magnetic fields, the authors found the existence of standing waves to problem (1.3) by solving an appropriate minimization problem for the corresponding energy functional in the cases of $N = 2$ and 3. Afterwards, in [27], Kurata assumed a technical condition relating $V(x)$ and $A(x)$. Under these assumptions, he proved that the associated functional satisfies the Palais–Smale condition at any level and further obtained a least energy solution of the problem for any $\epsilon > 0$. Also, Alves *et al.*

[1] studied the multiple solutions by combining a local assumption on V, the penalization techniques of del Pino and Felmer [12] and the Ljusternic–Schnirelmann theory.

Recently, Tang [35] considered multi-bump solutions of the following nonlinear magnetic Schrödinger equation with critical frequency

$$-(\nabla + iA(x))^2 u + (\lambda V(x) + E)u = f\left(|u|^2\right) u, \quad \text{in } \mathbb{R}^2,$$

where $\lambda > 0$, $E \in \mathbb{R}$ is a constant, $\inf_{x\in\mathbb{R}^N} V(x) = E$ and f satisfies subcritical growth. Later, by using the variational methods, Ji and Rădulescu [22] established the existence and multiplicity of multi-bump solutions for the following nonlinear magnetic Schrödinger equation

$$-(\nabla + iA(x))^2 u + (\lambda V(x) + Z(x))u = f\left(|u|^2\right) u, \quad \text{in } \mathbb{R}^2,$$

where $\lambda > 0$, $f(t)$ is a continuous function with exponential critical growth, the magnetic potential $A : \mathbb{R}^2 \to \mathbb{R}^2$ is in $L^2_{loc}\left(\mathbb{R}^2, \mathbb{R}^2\right)$ and the potentials $V, Z : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions verifying some conditions. Recently, Ma and Ji [30] studied the existence and multiplicity of multi-bump solutions for the magnetic Schrödinger–Poisson system with subcritical growth. It is natural to consider multiplicity of multi-bump solutions for the magnetic Schrödinger–Poisson system with *critical* growth. To the best of our knowledge, this problem has not ever been studied. For more results related to the nonlinear partial differential equations with magnetic field, we refer to [6,8,9,14,17,20,21,39,42] and references therein.

Inspired by the previous works of [22,30,35], the aim of this paper is to study existence of multi-bump solutions for the magnetic Schrödinger–Poisson system with critical growth

$$\begin{cases} -(\nabla + iA(x))^2 u + (\lambda V(x) + 1)u + \phi u = \alpha f(|u|^2)u + |u|^4 u, & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.4}$$

where $\lambda > 0$ is a parameter, the magnetic potential A is in $L^2_{loc}(\mathbb{R}^3, \mathbb{R}^3)$, f has subcritical growth and the potential $V : \mathbb{R}^3 \to \mathbb{R}$ is continuous. Due to the appearance of magnetic field $A(x)$, problem (1.4) can not be changed into a pure real-valued problem, hence we should deal with a complex-valued directly. Also, since the electrostatic potential $\phi(x)$ depends on the wave function, $\phi(x)u$ is nonlocal which will make some estimates more difficult and complicated. Moreover, since the problem we deal with has critical growth, we need more refined estimates to overcome the lack of compactness.

Now we present the general assumptions on the potentials in this paper:

(A) $A : \mathbb{R}^3 \to \mathbb{R}^3$ be in $L^2_{loc}(\mathbb{R}^3, \mathbb{R}^3)$;

(V_1) $V(x) \in C\left(\mathbb{R}^3, \mathbb{R}\right)$ and $V(x) \geq 0$, for all $x \in \mathbb{R}^3$;

(V_2) $\Omega = \operatorname{int} V^{-1}(0)$ is a nonempty bounded open subset with smooth boundary and $\overline{\Omega} = V^{-1}(0)$ where $\operatorname{int} V^{-1}(0)$ denotes the set of the interior points of $V^{-1}(0)$, Ω consists of k components:
$$\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_k,$$
and $\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset$ for all $i \neq j$.

Furthermore, the nonlinearity f is a continuous function satisfying the following conditions:

(f_1) $f(t) = 0, \forall\, t \leq 0$, and $\lim_{t\to 0^+} \frac{f(t)}{t} = 0$;

(f_2) There exists $q, \iota \in (4,6)$ and $\varsigma > 0$ such that

$$\lim_{t\to+\infty} \frac{f(t)}{t^{\frac{q-2}{2}}} = 0, \text{ and } f(t) \geq \varsigma t^{(\iota-2)/2} \text{ for any } t > 0;$$

(f_3) There exists $\theta \in (4,6)$ such that

$$0 < \frac{\theta}{2} F(t) \leq t f(t), \text{ for any } t > 0$$

where $F(t) = \int_0^t f(s)ds$;

(f_4) $f(t)$ is an increasing function in $t > 0$.

The main result of this paper to be proved is the theorem below:

Theorem 1.1. *Assume that (A), $(V_1)-(V_2)$ and $(f_1)-(f_4)$ hold. Then, for any non-empty subset Γ of $\{1,2,\ldots,k\}$, there exist constants $\alpha^* > 0$ and $\lambda^* = \lambda^*(\alpha^*)$ such that, for all $\alpha \geq \alpha^*$ and $\lambda \geq \lambda^*$, problem (1.4) has a nontrivial solution u_λ. Moreover, the family $\{u_\lambda\}_{\lambda\geq\lambda^*}$ has the following properties: for any sequence $\lambda_n \to \infty$, we can extract a subsequence λ_{n_i} such that $u_{\lambda_{n_i}}$ converges in $H^1_A(\mathbb{R}^3,\mathbb{C})$ to a function u, which satisfies $u = 0$ for $x \notin \Omega_\Gamma = \cup_{j\in\Gamma}\Omega_j$, and the restriction $u\,|_{\Omega_j}$ is a least energy solution of*

$$\begin{cases} -(\nabla + iA(x))^2 u + u + \left(\frac{1}{4\pi}\int_{\Omega_j} \frac{|u(y)|^2}{|x-y|} dy\right) u = f(|u|^2)u + |u|^4 u, \ x \in \Omega_j, \\ u \in H^{0,1}_A(\Omega_j), \end{cases}$$

where $j \in \Gamma$.

Corollary 1.2. *Under the assumptions of Theorem 1.1, there exist $\alpha_* > 0$ and $\lambda_* = \lambda_*(\alpha_*)$ such that, for all $\alpha \geq \alpha_*$ and $\lambda \geq \lambda_*$, problem (1.4) has at least $2^k - 1$ nontrivial solutions.*

The paper is organized as follows. In Section 2, we shall introduce the variational setting and give some necessary preliminaries. In Section 3, we study an modified problem, and prove the Palais–Smale condition for the modified problem and study the behavior of $(PS)_\infty$ sequence. Moreover, we establish L^∞ estimate of the solution of the modified problem. In Section 4, by adapting the deformation flow method, we show that the existence of a special critical point and prove the main theorem.

2 Preliminaries

In this section, we shall present the variational framework for problem (1.4) and some useful preliminary lemmas.

For $u : \mathbb{R}^3 \to \mathbb{C}$, let us denote by

$$\nabla_A u = (\nabla + iA)\, u,$$

and

$$H^1_A\left(\mathbb{R}^3,\mathbb{C}\right) = \left\{u \in L^2\left(\mathbb{R}^3,\mathbb{C}\right) : |\nabla_A u| \in L^2\left(\mathbb{R}^3,\mathbb{R}\right)\right\}.$$

The space $H^1_A\left(\mathbb{R}^3,\mathbb{C}\right)$ is an Hilbert space under the scalar product

$$\langle u, v\rangle = \mathrm{Re}\int_{\mathbb{R}^3} \left(\nabla_A u \overline{\nabla_A v} + u\bar{v}\right) dx, \qquad \forall u, v \in H^1_A\left(\mathbb{R}^3,\mathbb{C}\right),$$

where Re and the bar denote the real part of a complex number and the complex conjugation, respectively. Moreover, the norm induced by the product $\langle\cdot,\cdot\rangle$ is $\|u\|_A = \left(\int_{\mathbb{R}^3} |\nabla_A u|^2 + |u|^2 dx\right)^{\frac{1}{2}}$.

By (A), on $H_A^1(\mathbb{R}^3,\mathbb{C})$, we have the important diamagnetic inequality (see [29], Theorem 7.21) which is frequently used in this paper:

$$|\nabla_A u(x)| \geq |\nabla |u(x)||. \tag{2.1}$$

Let

$$E_\lambda = \left\{ u \in H_A^1(\mathbb{R}^3,\mathbb{C}) : \int_{\mathbb{R}^3} \lambda V(x) |u|^2 dx < \infty \right\},$$

with the norm

$$\|u\|_\lambda^2 = \int_{\mathbb{R}^3} \left(|\nabla_A u|^2 + (\lambda V(x) + 1)|u|^2 \right) dx.$$

For $\lambda \geq 0$, a direct computation gives that $(E_\lambda, \|\cdot\|_\lambda)$ is an Hilbert space and $E_\lambda \subset H_A^1(\mathbb{R}^3,\mathbb{C})$.

Also, for an open set $K \subset \mathbb{R}^3$,

$$\begin{aligned}
H_A^1(K,\mathbb{C}) &:= \left\{ u \in L^2(K,\mathbb{C}) : |\nabla_A u| \in L^2(K,\mathbb{R}) \right\}, \\
\|u\|_{H_A^1(K,\mathbb{C})} &= \left(\int_K \left(|\nabla_A u|^2 + |u|^2 \right) dx \right)^{\frac{1}{2}}, \\
E_\lambda(K,\mathbb{C}) &:= \left\{ u \in H_A^1(K,\mathbb{C}) : \int_K \lambda V(x)|u|^2 dx < \infty \right\}, \\
\|u\|_{\lambda,K}^2 &= \int_K \left(|\nabla_A u|^2 + (\lambda V(x) + 1)|u|^2 \right) dx.
\end{aligned}$$

Let $H_A^{0,1}(K,\mathbb{C})$ be the Hilbert space obtained as the closure of $C_0^\infty(K,\mathbb{C})$ under the norm $\|u\|_{H_A^1(K,\mathbb{C})}$.

The diamagnetic inequality (2.1) implies that, if $u \in H_A^1(\mathbb{R}^3,\mathbb{C})$, then $|u| \in H^1(\mathbb{R}^3,\mathbb{R})$ and $\|u\| \leq \|u\|_A$. Therefore, the embedding $H_A^1(\mathbb{R}^3,\mathbb{C}) \hookrightarrow L^r(\mathbb{R}^3,\mathbb{C})$ is continuous for $2 \leq r \leq 6$ and the embedding $H_A^1(\mathbb{R}^3,\mathbb{C}) \hookrightarrow L_{loc}^r(\mathbb{R}^3,\mathbb{C})$ is compact for $1 \leq r < 6$.

By the continuous embedding $H^1(\mathbb{R}^3,\mathbb{R}) \hookrightarrow L^r(\mathbb{R}^3,\mathbb{R})$ for $2 \leq r \leq 6$, we have

$$H^1(\mathbb{R}^3,\mathbb{R}) \hookrightarrow L^{\frac{12}{5}}(\mathbb{R}^3,\mathbb{R}).$$

For any $u \in H_A^1(\mathbb{R}^3,\mathbb{C})$, we obtain that $|u| \in H^1(\mathbb{R}^3,\mathbb{R})$, and the linear functional $\mathcal{L}_{|u|} : D^{1,2}(\mathbb{R}^3,\mathbb{R}) \to \mathbb{R}$ given by

$$\mathcal{L}_{|u|}(v) = \int_{\mathbb{R}^3} |u|^2 v dx$$

is well defined and continuous in view of the Hölder inequality and (2.2). Indeed, we can see that

$$|\mathcal{L}_{|u|}(v)| \leq \left(\int_{\mathbb{R}^3} |u|^{\frac{12}{5}} dx \right)^{\frac{5}{6}} \left(\int_{\mathbb{R}^3} |v|^6 dx \right)^{\frac{1}{6}} \leq C\|u\|_A^2 \|v\|_{D^{1,2}}. \tag{2.2}$$

Then, given $u \in H_A^1(\mathbb{R}^3,\mathbb{C})$, $|u| \in H^1(\mathbb{R}^3,\mathbb{R})$, by the Lax–Milgram Theorem, there exists an unique $\phi = \phi_{|u|} \in D^{1,2}(\mathbb{R}^3,\mathbb{R})$ such that

$$-\Delta\phi = u^2.$$

Moreover, $\phi_{|u|}$ can be expressed as

$$\phi_{|u|}(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{|u(y)|^2}{|x-y|} dy.$$

Next, we provide the following properties about $\phi_{|u|}$ in the following lemma whose proof is similar to one in [11,32,41], so we omit it.

Lemma 2.1. *For any $u \in H_A^1(\mathbb{R}^3, \mathbb{C})$, we have*

(i) there exists $C > 0$ such that

$$\int_{\mathbb{R}^3} |\nabla \phi_{|u|}|^2 dx = \int_{\mathbb{R}^3} \phi_{|u|} |u|^2 dx \le C \|u\|_A^4, \qquad \forall u \in H_A^1(\mathbb{R}^3, \mathbb{C});$$

(ii) $\phi_{|u|} \ge 0$, $\forall u \in H_A^1(\mathbb{R}^3, \mathbb{C})$;

(iii) $\phi_{|tu|} = t^2 \phi_{|u|}$, $\forall t \in \mathbb{R}$ and $u \in H_A^1(\mathbb{R}^3, \mathbb{C})$;

(iv) if $u_n \rightharpoonup u$ in $H_A^1(\mathbb{R}^3, \mathbb{C})$, then $\phi_{|u_n|} \rightharpoonup \phi_{|u|}$ in $D^{1,2}(\mathbb{R}^3, \mathbb{R})$ and

$$\lim_{n\to+\infty} \int_{\mathbb{R}^3} \phi_{|u_n|} |u_n|^2 dx \ge \int_{\mathbb{R}^3} \phi_{|u|} |u|^2 dx;$$

(v) if $u_n \to u$ in $H_A^1(\mathbb{R}^3, \mathbb{C})$, then $\phi_{|u_n|} \to \phi_{|u|}$ in $D^{1,2}(\mathbb{R}^3, \mathbb{R})$. Hence,

$$\lim_{n\to+\infty} \int_{\mathbb{R}^3} \phi_{|u_n|} |u_n|^2 dx = \int_{\mathbb{R}^3} \phi_{|u|} |u|^2 dx.$$

Now, we define the energy functional I_λ associated with problem (1.4) given by

$$\begin{aligned} I_\lambda(u) = &\frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla_A u|^2 + (\lambda V(x) + 1) |u|^2 \right) dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_{|u|}(x) |u|^2 dx \\ &- \frac{\alpha}{2} \int_{\mathbb{R}^3} F\left(|u|^2\right) dx - \frac{1}{6} \int_{\mathbb{R}^3} |u_n|^6 dx, \end{aligned}$$

it is standard to prove that $I_\lambda(u) \in C^1(E_\lambda, \mathbb{R})$, and for any $\varphi \in E_\lambda$, we have

$$\begin{aligned} \langle I_\lambda'(u), \varphi \rangle = &\mathrm{Re} \int_{\mathbb{R}^3} (\nabla_A u \overline{\nabla_A \varphi} + (\lambda V(x) + 1) u \overline{\varphi}) dx + \mathrm{Re} \int_{\mathbb{R}^3} \phi_{|u|}(x) u \overline{\varphi} dx \\ &- \mathrm{Re} \int_{\mathbb{R}^3} \alpha f\left(|u|^2\right) u \overline{\varphi} dx - \mathrm{Re} \int_{\mathbb{R}^3} |u|^4 u \overline{\varphi} dx. \end{aligned}$$

Definition 2.2. A pair $(u, \phi) \in E_\lambda \times D^{1,2}(\mathbb{R}^3, \mathbb{R})$ is said to be a weak solution of problem (1.4), if $I_\lambda'(u)\varphi = 0$, $\forall \varphi \in E_\lambda$, where $\phi_{|u|} = \phi$.

By (V_3), we can derive that for any open set $K \subset \mathbb{R}^3$,

$$M_0 \|u\|_{2,K}^2 \le \int_K \left(|\nabla_A u|^2 + (\lambda V(x) + 1)|u|^2 \right) dx,$$

for all $u \in E_\lambda(K)$, and $\lambda > 0$, where $\|u\|_{2,K}^2 = \int_K |u|^2 dx$. So, from this relation, we have the following result:

Lemma 2.3. *There exist $\delta_0, \nu_0 > 0$ with $\delta_0 \approx 1$ and $\nu_0 \approx 0$ such that for any open set $K \subset \mathbb{R}^3$,*

$$\delta_0 \|u\|_{\lambda,K}^2 \le \|u\|_{\lambda,K}^2 - \nu_0 \|u\|_{2,K}^2, \quad \textit{for all } u \in E_\lambda(K, \mathbb{C}), \textit{ and } \lambda > 0.$$

3 A modified problem

Since $\mathbb{R}^3$ is unbounded and nonlinear term has the critical growth, we know that the Sobolev embeddings are not compact, as so I_λ can not verify the Palais–Smale condition. In order to overcome this difficulty, we adapt the argument of the penalization method introduced by del Pino and Felmer [12] and Ding and Tanaka [13], and consider a modified problem satisfying the Palais–Smale condition.

Let $\nu_0 > 0$ be a constant given in Lemma 2.3, $\kappa > \frac{\theta}{\theta-2}$ and $a > 0$ verifying $\alpha f(a) + a^2 = \frac{\nu_0}{\kappa}$ and $\widetilde{f}, \widetilde{F} : \mathbb{R} \to \mathbb{R}$ given by

$$\tilde{f}(t) = \begin{cases} \alpha f(t) + t^2, & t \le a, \\ \frac{\nu_0}{\kappa}, & t \ge a, \end{cases}$$

thus

$$\widetilde{f}(t) \le \alpha f(t) + t^2, \qquad t \ge 0. \tag{3.1}$$

Also,

$$\widetilde{F}(t) = \int_0^t \widetilde{f}(s)\, ds.$$

Now, since the potential well $\Omega = \operatorname{int} V^{-1}(0)$ can be decomposed into k connected components $\Omega_1, \dots, \Omega_k$ with $\operatorname{dist}(\Omega_i, \Omega_j) > 0$, $i \ne j$, then for each $j \in \{1, 2, \dots, k\}$, we fix a smooth bounded domain Ω'_j such that

(i) $\overline{\Omega_j} \subset \Omega'_j$;

(ii) $\overline{\Omega'_i} \cap \overline{\Omega'_j} = \emptyset$ for all $i \ne j$.

Next, we fix a non-empty subset $\Gamma \subset \{1, \dots, k\}$ and

$$\Omega_\Gamma = \bigcup_{j\in\Gamma} \Omega_j, \qquad \Omega'_\Gamma = \bigcup_{j\in\Gamma} \Omega'_j,$$

$$\chi_\Gamma(x) := \begin{cases} 1 & \text{for } x \in \Omega'_\Gamma, \\ 0 & \text{for } x \notin \Omega'_\Gamma. \end{cases}$$

Using the above notations, we set the functions

$$\begin{aligned} g(x,t) &= \chi_\Gamma(x)(\alpha f(t) + t^2) + (1 - \chi_\Gamma(x))\, \tilde{f}(t), \\ G(x,t) &= \int_0^t g(x,s) ds = \chi_\Gamma(x) \alpha F(t) + (1 - \chi_\Gamma(x))\, \tilde{F}(t). \end{aligned} \tag{3.2}$$

In view of (f_1)–(f_4), we have that g is a Carathéodory function satisfying the following properties:

(g_1) $g(x,t) = 0$ for each $t \le 0$;

(g_2) $\lim_{t\to 0^+} \frac{g(x,t)}{t} = 0$ uniformly in $x \in \mathbb{R}^3$;

(g_3) $g(x,t) \le \alpha f(t) + t^2$ for all $t \ge 0$ and any $x \in \mathbb{R}^3$;

(g_4) $0 < \theta G(x,t) \le 2 g(x,t) t$ for each $x \in \Omega'_\Gamma$ and $t > 0$;

(g_5) $0 < G(x,t) \leq g(x,t)t \leq \nu_0 t/\kappa$, for each $x \in \mathbb{R}^3 \setminus \Omega'_\Gamma$, $t > 0$;

(g_6) for each $x \in \Omega'_\Gamma$, the function $t \mapsto \frac{g(x,t)}{t}$ is strictly increasing in $t \in (0, +\infty)$ and for each $x \in \mathbb{R}^3 \setminus \Omega'_\Gamma$, the function $t \mapsto \frac{g(x,t)}{t}$ is strictly increasing in $(0, a)$.

Moreover, we have the modified problem

$$-(\nabla + iA(x))^2 u + (\lambda V(x) + 1)u + \phi_{|u|} u = g\left(x, |u|^2\right) u, \qquad x \in \mathbb{R}^3, \tag{3.3}$$

and the energy functional $\Phi_\lambda(u) : E_\lambda\left(\mathbb{R}^3, \mathbb{C}\right) \to \mathbb{R}$ given by

$$\Phi_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla_A u|^2 + (\lambda V(x)+1)|u|^2\right)dx + \frac{1}{4}\int_{\mathbb{R}^3}\phi_{|u|}|u|^2 dx - \frac{1}{2}\int_{\mathbb{R}^3} G\left(x, |u|^2\right)dx.$$

We want to get some nontrivial solutions of (3.3) are ones of the original problem (1.4), more precisely, if u_λ is a nontrivial solution of (3.3) verifying $|u_\lambda(x)|^2 \leq a$ in $\mathbb{R}^3 \setminus \Omega'_\Gamma$, then it is a nontrivial solution to (1.4).

Next, we prove that the energy functional $\Phi_\lambda(u)$ satisfies the (PS) condition.

Lemma 3.1. *All $(PS)_c$ sequences for Φ_λ are bounded in E_λ.*

Proof. Let (u_n) be a $(PS)_c$ sequence for Φ_λ. Thus, we have

$$\Phi_\lambda(u_n) - \frac{1}{\theta}\Phi'_\lambda(u_n)u_n = c + o_n(1) + o_n(1)\|u_n\|_\lambda .$$

On the other hand, by (g_4), (g_5), $\kappa > \frac{\theta}{\theta-2}$, and Lemma 2.3, we derive

$$\begin{aligned}
\Phi_\lambda(u_n) - \frac{1}{\theta}\Phi'_\lambda(u_n)u_n &= \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|_\lambda^2 + \left(\frac{1}{4} - \frac{1}{\theta}\right)\int_{\mathbb{R}^3}\phi_{|u_n|}(x)|u_n|^2 dx \\
&\quad + \int_{\mathbb{R}^3}\left(\frac{1}{\theta}g\left(x,|u_n|^2\right)|u_n|^2 - \frac{1}{2}G\left(x,|u_n|^2\right)\right)dx \\
&\geq \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|_\lambda^2 + \frac{2-\theta}{2\theta}\int_{\mathbb{R}^3\setminus\Omega'_\Gamma}\tilde{F}(|u_n|^2)dx \\
&\geq \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|_\lambda^2 + \frac{(\theta-2)\nu_0}{2\theta\kappa}\int_{\mathbb{R}^3\setminus\Omega'_\Gamma}|u_n|^2 dx \\
&\geq \left(\frac{1}{2} - \frac{1}{\theta}\right)(1 - \frac{1}{\kappa})\|u_n\|_\lambda^2 .
\end{aligned}$$

So,

$$\left(\frac{1}{2} - \frac{1}{\theta}\right)(1 - \frac{1}{\kappa})\|u_n\|_\lambda^2 \leq c + o_n(1) + o_n(1)\|u_n\|_\lambda .$$

This shows that (u_n) is bounded in E_λ. □

For each fixed $j \in \Gamma$, let us denote by c_j the minimax level of the functional $I_j \colon H_A^{0,1}(\Omega_j, \mathbb{C}) \to \mathbb{R}$ given by

$$I_j(u) = \frac{1}{2}\int_{\Omega_j}\left(|\nabla_A u|^2 + |u|^2\right)dx + \frac{1}{4}\int_{\Omega_j}\phi_{|u|}|u|^2 dx - \frac{\alpha}{2}\int_{\Omega_j}F(|u|^2)dx - \frac{1}{6}\int_{\Omega_j}|u|^6 dx,$$

and

$$c_j = \inf_{\gamma\in\Lambda_j}\max_{t\in[0,1]} I_j(\gamma(t)),$$

where

$$\Lambda_j = \left\{ \gamma \in C([0,1], H_A^{0,1}(\Omega_j, \mathbb{C})) : \gamma(0) = 0, I_j(\gamma(1)) < 0 \right\}.$$

It is well-known that the critical points of are the weak solutions of the problem

$$\begin{cases} -(\nabla + iA(x))^2 u + u + \phi_{|u|} u = \alpha f\left(|u|^2\right) u + |u|^4 u, & \text{in } \Omega_j, \\ u = 0, & \text{on } \partial\Omega_j. \end{cases} \tag{3.4}$$

Moreover, we have the following important result.

Lemma 3.2. *There exists $\alpha^* > 0$ such that, for all $\alpha \geq \alpha^*$, we have*

$$c_j \in \left(0, \frac{1}{3(k+1)} S^{3/2}\right), \quad \textit{for all } j \in \{1, \cdots, k\} \textit{ and all } \alpha \in [\alpha^*, +\infty).$$

Proof. We choose a function $\varphi_j \in H_A^{0,1}(\Omega_j, \mathbb{C})) \setminus \{0\}$ for each $j \in \{1, \cdots, k\}$. There exists $t_{\alpha,j} \in (0, +\infty)$ such that

$$c_j \leq I_j(t_{\alpha,j}\varphi_j) = \max_{t \geq 0} I_j(t\varphi_j)$$

and hence, by (f_4), one has

$$\begin{aligned} t_{\alpha,j}^2 \int_{\mathbb{R}^3} \left(|\nabla_A \varphi_j|^2 + |\varphi_j|^2\right) dx + t_{\alpha,j}^4 \int_{\mathbb{R}^3} \phi_{|\varphi_j|} |\varphi_j|^2 dx \\ = \alpha \int_{\mathbb{R}^3} f\left(|t_{\alpha,j}\varphi_j|^2\right) |t_{\alpha,j}\varphi_j|^2 dx + t_{\alpha,j}^6 \int_{\mathbb{R}^3} |\varphi_j|^6 dx \\ \geq \alpha \int_{\mathbb{R}^3} f\left(|t_{\alpha,j}\varphi_j|^2\right) |t_{\alpha,j}\varphi_j|^2 dx \geq \alpha \varsigma t_{\alpha,j}^{\iota} \int_{\mathbb{R}^3} |\varphi_j|^{\iota} dx. \end{aligned} \tag{3.5}$$

If $|t_{\alpha,j}| \leq 1$, by (3.5), we have

$$t_{\alpha,j}^2 \int_{\mathbb{R}^3} \left(|\nabla_A \varphi_j|^2 + |\varphi_j|^2\right) dx + t_{\alpha,j}^2 \int_{\mathbb{R}^3} \phi_{|\varphi_j|} |\varphi_j|^2 dx \geq \alpha \varsigma t_{\alpha,j}^{\iota} \int_{\mathbb{R}^3} |\varphi_j|^{\iota} dx.$$

The above inequality implies that

$$t_{\alpha,j} \leq \left[\frac{\int_{\mathbb{R}^3} \left(|\nabla_A \varphi_j|^2 + |\varphi_j|^2\right) dx + \int_{\mathbb{R}^3} \phi_{|\varphi_j|} |\varphi_j|^2 dx}{\alpha \varsigma \int_{\mathbb{R}^3} |\varphi_j|^{\iota} dx} \right]^{1/(\iota-2)}.$$

If $|t_{\alpha,j}| \geq 1$, by (3.5), one has

$$t_{\alpha,j}^4 \int_{\mathbb{R}^3} \left(|\nabla_A \varphi_j|^2 + |\varphi_j|^2\right) dx + t_{\alpha,j}^4 \int_{\mathbb{R}^3} \phi_{|\varphi_j|} |\varphi_j|^2 dx \geq \alpha \varsigma t_{\alpha,j}^{\iota} \int_{\mathbb{R}^3} |\varphi_j|^{\iota} dx.$$

The above inequality implies that

$$t_{\alpha,j} \leq \left[\frac{\int_{\mathbb{R}^3} \left(|\nabla_A \varphi_j|^2 + |\varphi_j|^2\right) dx + \int_{\mathbb{R}^3} \phi_{|\varphi_j|} |\varphi_j|^2 dx}{\alpha \varsigma \int_{\mathbb{R}^3} |\varphi_j|^{\iota} dx} \right]^{1/(\iota-4)}.$$

Using the above limits, we have $t_{\alpha,j} \to 0$ as $\alpha \to +\infty$. This fact yields that $I_j(t_{\alpha,j}\varphi_j) \to 0$ as $\alpha \to +\infty$. Thus, there exists $\alpha^* > 0$ such that

$$c_j \in \left(0, \frac{1}{3(k+1)} S^{3/2}\right), \quad \text{for all } j \in \{1, \cdots, k\}. \qquad \square$$

Remark 3.3. In particular, the above lemma implies for $\alpha > 0$ large that

$$\sum_{j=1}^{k} c_j \in \left(0, \frac{1}{3}S^{\frac{3}{2}}\right). \tag{3.6}$$

Proposition 3.4. *For any $\lambda > 0$, the functional Φ_λ satisfies the $(PS)_c$ condition at the level $c < \frac{1}{3}S^{\frac{3}{2}}$.*

Proof. Let $(u_n) \subset E_\lambda$ be a $(PS)_c$ sequence for $\Phi_\lambda$at the level $c < \frac{1}{3}S^{\frac{3}{2}}$, that is

$$\Phi_\lambda(u_n) \to c < \frac{1}{3}S^{\frac{3}{2}} \quad \text{and} \quad \Phi_\lambda'(u_n) \to 0.$$

From Lemma 3.1, we know that the sequence (u_n) is bounded in E_λ. Thus, there exists $u \in E_\lambda$ such that $u_n \rightharpoonup u$ in E_λ, up to a subsequence if necessary. Then it is standard to check that for any $C_0^\infty\left(\mathbb{R}^3, \mathbb{C}\right) \subset E_\lambda$,

$$\operatorname{Re}\int_{\mathbb{R}^3} \nabla_A u_n \overline{\nabla_A \varphi} dx \to \operatorname{Re}\int_{\mathbb{R}^3} \nabla_A u \overline{\nabla_A \varphi} dx,$$

$$\operatorname{Re}\int_{\mathbb{R}^3} (\lambda V(x)+1) u_n \overline{\varphi} dx \to \operatorname{Re}\int_{\mathbb{R}^3} (\lambda V(x)+1) u \overline{\varphi} dx,$$

and

$$\operatorname{Re}\int_{\mathbb{R}^3} g\left(x, |u_n|^2\right) u_n \overline{\varphi} dx \to \operatorname{Re}\int_{\mathbb{R}^3} g\left(x, |u|^2\right) u \overline{\varphi} dx. \tag{3.7}$$

Form (3.7), the density of $C_0^\infty\left(\mathbb{R}^3, \mathbb{C}\right)$ in E_λ, and $\Phi_\lambda'(u_n) \to 0$, we can obtain that the weak limit u is a critical point of Φ_λ and so

$$\|u\|_\lambda^2 + \int_{\mathbb{R}^3} \phi_{|u|} |u|^2 dx = \int_{\mathbb{R}^3} g(x, |u|^2)|u|^2 dx. \tag{3.8}$$

On the other hand, we know that $< \Phi_\lambda'(u_n), u_n >= o_n(1)$ which implies that

$$\|u_n\|_\lambda^2 + \int_{\mathbb{R}^3} \phi_{|u_n|} |u_n|^2 dx = \int_{\mathbb{R}^3} g(x, |u_n|^2)|u_n|^2 dx + o_n(1). \tag{3.9}$$

Step 1: We show that for any given $\zeta > 0$, there exists $R > 0$ large enough such that $\Omega_\Gamma' \subset B_{R/2}(0)$ and

$$\limsup_n \int_{B_R^c(0)} (|\nabla_A u_n|^2 + (\lambda V(x)+1)|u_n|^2) dx \leq \zeta. \tag{3.10}$$

Now, we take $R > 0$ large such that $\Omega_\Gamma' \subset B_{\frac{R}{2}}(0)$ and $\eta_R \in C^\infty\left(\mathbb{R}^3, \mathbb{R}\right)$ satisfying

$$\eta_R = 0 \quad x \in B_{\frac{R}{2}}(0), \quad \eta_R = 1 \quad x \in B_R^c(0), \quad 0 \leq \eta_R \leq 1, \quad \text{and} \quad |\nabla \eta_R| \leq \frac{C}{R},$$

where $C > 0$ is a constant independent of R.

By a direct computation, we have

$$\begin{aligned} o_n(1) = \langle \Phi_\lambda'(u_n), u_n \eta_R \rangle = & \int_{\mathbb{R}^3} \left(|\nabla_A u_n|^2 + (\lambda V(x)+1)|u_n|^2\right) \eta_R dx \\ & + \int_{\mathbb{R}^3} \phi_{|u_n|}(x) |u_n|^2 \eta_R dx + \operatorname{Re}\left(\int_{\mathbb{R}^3} \overline{u_n} \nabla_A u_n \nabla \eta_R dx\right) \\ & - \int_{\mathbb{R}^3} \tilde{f}\left(|u_n|^2\right) |u_n|^2 \eta_R dx. \end{aligned}$$

Notice that

$$|\mathrm{Re}\,(\overline{u_n}\nabla_A u_n)| = |\mathrm{Re}\,((\nabla u_n + iAu_n)\,\overline{u_n})| = |\mathrm{Re}\,(\overline{u_n}\nabla u_n)| = |u_n|\,|\nabla|u_n||.$$

Using the Hölder inequality and the above equality, we derive

$$\limsup_{n\to\infty}\left|\mathrm{Re}\left(\int_{\mathbb{R}^3}\overline{u_n}\nabla_A u_n \nabla\eta_R dx\right)\right| \le \frac{C}{R}.$$

So, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^3}\left(|\nabla_A u_n|^2 + (\lambda V(x)+1)\,|u_n|^2\right)\eta_R dx\\ &\le \int_{\mathbb{R}^3}\tilde{f}\left(|u_n|^2\right)|u_n|^2\,\eta_R dx + \frac{C}{R} + o_n(1)\\ &\le \frac{\nu_0}{\kappa}\int_{\mathbb{R}^3}|u_n|^2\,\eta_R dx + \frac{C}{R} + o_n(1),\end{aligned}$$

which implies that for any $\zeta > 0$, choosing a $R > 0$ larger if necessary, we have

$$\limsup_{n\to\infty}\int_{B_R^c(0)}\left(|\nabla_A u_n|^2 + (\lambda V(x)+1)\,|u_n|^2\right)dx \le \zeta.$$

Step 2: We show that

$$\lim_n \int_{\mathbb{R}^3}\phi_{|u_n|}|u_n|^2 dx = \int_{\mathbb{R}^3}\phi_{|u|}|u|^2 dx. \tag{3.11}$$

By (3.10) and the Sobolev embedding, for any $\zeta > 0$, there exists $R > 0$ such that for n large enough and $q \in [2, 6)$

$$\begin{aligned}\|u_n - u\|_{L^q(\mathbb{R}^3)} &= \|u_n - u\|_{L^q(B_R(0))} + \|u_n - u\|_{L^q(B_R^c(0))}\\ &\le \|u_n - u\|_{L^q(B_R(0))} + \|u_n\|_{L^q(B_R^c(0))} + \|u\|_{L^q(B_R^c(0))}\\ &\le C\zeta,\end{aligned}$$

which implies

$$u_n \to u \quad \text{in } L^q(\mathbb{R}^3, \mathbb{C}), \qquad \forall q \in [2, 6).$$

Since $||u_n| - |u|| \le |u_n - u||$ and $\frac{12}{5} \in (2, 6)$, one has

$$|u_n| \to |u| \quad \text{in } L^{12/5}(\mathbb{R}^3, \mathbb{R}). \tag{3.12}$$

Let

$$\mathbb{D}(u) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^2\,|u(y)|^2}{|x-y|}dxdy,$$

we have

$$\begin{aligned}|\mathbb{D}(u_n) - \mathbb{D}(u)| &= \left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_n(x)|^2\,|u_n(y)|^2}{|x-y|}dxdy - \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^2\,|u(y)|^2}{|x-y|}dxdy\right|\\ &= \left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(|u_n(x)|^2 - |u(x)|^2)(|u_n(y)|^2 + |u(y)|^2)}{|x-y|}dxdy\right|\\ &\le \left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|\,|u_n(x)|^2 - |u(x)|^2\,|(|u_n(y)|^2 + |u(y)|^2)}{|x-y|}dxdy\right|\\ &\le C\sqrt{\mathbb{D}(||u_n|^2 - |u|^2|^{1/2})}\sqrt{\mathbb{D}(||u_n|^2 + |u|^2|^{1/2})}\end{aligned}$$

Then, by the Hardy–Littlewood–Sobolev inequality, the Hölder inequality and (3.12), it follows that

$$|\mathbb{D}(u_n)-\mathbb{D}(u)|^2 = C\||\,|u_n|^2-|u|^2|^{1/2}\|^4_{L^{12/5}(\mathbb{R}^3)}\||\,|u_n|^2+|u|^2|^{1/2}\|^4_{L^{12/5}(\mathbb{R}^3)}$$
$$\le C\||\,|u_n|^2-|u|^2|^{1/2}\|^4_{L^{12/5}(\mathbb{R}^3)} \to 0.$$

Step 3:

$$\lim_n \int_{\mathbb{R}^3} g(x,|u_n|^2)|u_n|^2 dx = \int_{\mathbb{R}^3} g(x,|u|^2)|u|^2 dx. \tag{3.13}$$

By (g_3), (f_1) and (f_2), (3.10), for n large enough,

$$\int_{B^c_R(0)} \left|g(x,|u_n|^2)\,|u_n|^2\right| dx \le C_1 \int_{B^c_R(0)} (|u_n|^2+|u_n|^q+|u_n|^6)dx$$
$$\le C_2(\zeta+\zeta^{\frac{q}{2}}+\zeta^3) \tag{3.14}$$

On the other hand, choosing $R>0$ large if necessary, we may assume that

$$\int_{B^c_R(0)} \left|g(x,|u|^2)\,|u|^2\right| dx \le \zeta.$$

Hence, from the last inequality and (3.14), we have that

$$\lim_n \int_{B^c_R(0)} g(x,|u_n|^2)\,|u_n|^2\,dx = \int_{B^c_R(0)} g(x,|u|^2)\,|u|^2\,dx. \tag{3.15}$$

By the definition of g, one has

$$g(x,|u_n|^2)\,|u_n|^2 \le \alpha f(|u_n|^2)\,|u_n|^2 + a^3 + \frac{\nu_0}{\kappa}|u_n|^2, \quad \text{for any } x\in\mathbb{R}^3\setminus\Omega'_\Gamma.$$

Since the set $B_R(0)\bigcap(\mathbb{R}^3\setminus\Omega'_\Gamma)$ is bounded, we can use the above estimates, (f_1), (f_2) and Lebesgue dominated convergence theorem to obtain that

$$\lim_n \int_{B_R(0)\bigcap(\mathbb{R}^3\setminus\Omega'_\Gamma)} g(x,|u_n|^2)\,|u_n|^2\,dx = \int_{B_R(0)\bigcap(\mathbb{R}^3\setminus\Omega'_\Gamma)} g(x,|u|^2)\,|u|^2\,dx. \tag{3.16}$$

We show now

$$\lim_n \int_{\Omega'_\Gamma} |u_n|^6 dx = \int_{\Omega'_\Gamma} |u|^6 dx. \tag{3.17}$$

If (3.17) holds, by (g_3), (f_1), (f_2) and Lebesgue dominated convergence theorem, we have

$$\lim_n \int_{B_R(0)\cap\Omega'_\Gamma} g(x,|u_n|^2)\,|u_n|^2\,dx = \int_{B_R(0)\cap\Omega'_\Gamma} g(x,|u|^2)\,|u|^2\,dx. \tag{3.18}$$

Hence, by (3.16) and (3.18), $\lim_n \int_{\mathbb{R}^3} g(x,|u_n|^2)|u_n|^2 dx = \int_{\mathbb{R}^3} g(x,|u|^2)|u|^2 dx$. Using (3.10) and the diamagnetic inequality (2.1), the sequence $(|u_n|)$ is tight in, we may assume that

$$|\nabla|u_n||^2 \rightharpoonup \mu \quad \text{and} \quad |u_n|^6 \rightharpoonup \nu \tag{3.19}$$

in the sense of measures. By the concentration-compactness principle in [37], we can find an at most countable index I, sequences $(x_i)\subset\mathbb{R}^3$, $(\mu_i),(\nu_i)\subset(0,\infty)$ such that

$$\mu \ge |\nabla|u||^2 dx + \sum_{i\in I}\mu_i\delta_{x_i},$$
$$\nu = |u|^6 + \sum_{i\in I}\nu_i\delta_{x_i} \quad \text{and} \quad S\nu_i^{1/3} \le \mu_i \tag{3.20}$$

for any $i \in I$, where δ_{x_i} is the Dirac mass at the point x_i. Let us show that $(x_i)_{i\in I} \cap \Omega'_\Gamma = \emptyset$. Assume, by contradiction, that $x_i \in \Omega'_\Gamma$ for some $i \in I$. For any $\rho > 0$, we define $\psi_\rho(x) = \psi(\frac{x-x_i}{\rho})$ where $\psi \in C_0^\infty(\mathbb{R}^3, [0,1])$ is such that $\psi = 1$ in B_1, $\psi = 0$ in $\mathbb{R}^3 \setminus B_2$ and $\|\nabla\psi\|_{L^\infty(\mathbb{R}^3,\mathbb{R})} \leq 2$. We suppose that $\rho > 0$ is such that $\operatorname{supp}(\psi_\rho) \subset \Omega'_\Gamma$. Since $(\psi_\rho u_n)$ is bounded in E_λ, we can see that $\Phi'_\lambda(u_n)[\psi_\rho u_n] = o_n(1)$, that is

$$\begin{aligned}&\int_{\mathbb{R}^3} |\nabla_A u_n|^2 \psi_\rho dx + \operatorname{Re}\int_{\mathbb{R}^3} i\overline{u_n}\nabla_A u_n \nabla\psi_\rho dx + \int_{\mathbb{R}^3}(\lambda V(x)+1)|u_n|^2\psi_\rho dx\\ &\quad= \int_{\mathbb{R}^3} g(x,|u_n|^2)|u_n|^2\psi_\rho dx + o_n(1)\\ &\quad= \alpha\int_{\mathbb{R}^3} f(|u_n|^2)|u_n|^2\psi_\rho dx + \int_{\mathbb{R}^3}|u_n|^6\psi_\rho dx + o_n(1).\end{aligned}$$

Using the diamagnetic inequality (2.1) again, it follows that

$$\begin{aligned}&\int_{\mathbb{R}^3} |\nabla|u_n||^2 \psi_\rho dx + \operatorname{Re}\int_{\mathbb{R}^3} i\overline{u_n}\nabla_A u_n \nabla\psi_\rho dx\\ &\qquad\leq \alpha\int_{\mathbb{R}^3} f(|u_n|^2)|u_n|^2\psi_\rho dx + \int_{\mathbb{R}^3}|u_n|^6\psi_\rho dx + o_n(1).\end{aligned}\tag{3.21}$$

Due to the fact that f has the subcritical growth and ψ_ρ has the compact support, we have that

$$\lim_{\rho\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^3} f(|u_n|^2)|u_n|^2\psi_\rho dx = \lim_{\rho\to 0}\int_{\mathbb{R}^3} f(|u|^2)|u|^2\psi_\rho dx = 0. \tag{3.22}$$

Now, we show that

$$\lim_{\rho\to 0}\limsup_{n\to\infty}\left|\operatorname{Re}\int_{\mathbb{R}^3} i\overline{u_n}\nabla_A u_n\nabla\psi_\rho dx\right| = 0. \tag{3.23}$$

Because of the boundedness of (u_n) in E_λ, using the Hölder inequality, the strong convergence of $(|u_n|)$ in $L^2_{loc}(\mathbb{R}^3,\mathbb{R})$, $|u| \in L^6(\mathbb{R}^3,\mathbb{R})$, $|\nabla\psi_\rho| \leq C\rho^{-1}$ and $|B_{2\rho}(x_i)| \sim \rho^3$, we have that

$$\begin{aligned}0 &\leq \lim_{\rho\to 0}\limsup_{n\to\infty}\left|\operatorname{Re}\int_{\mathbb{R}^3} i\overline{u_n}\nabla_A u_n\nabla\psi_\rho dx\right|\\ &\leq \lim_{\rho\to 0}\limsup_{n\to\infty}\int_{\mathbb{R}^3}|\overline{u_n}\nabla\psi_\rho||\nabla_A u_n| dx\\ &\leq \lim_{\rho\to 0}\lim_{n\to\infty}\left(\int_{B_{2\rho}(x_i)}|\overline{u_n}\nabla\psi_\rho|^2 dx\right)^{1/2}\|u_n\|_\lambda\\ &\leq C\lim_{\rho\to 0}\left(\int_{B_{2\rho}(x_i)}|u|^2 dx\right)^{1/2} = 0\end{aligned}$$

which shows that (3.23) holds.

Then, taking into account (3.19), (3.21), (3.22) and (3.23), we can conclude that $\nu_i \geq \mu_i$ for all $i \in I$. Together with the inequality $S\nu_i^{1/3} \leq \mu_i$ in (3.20), we have

$$\nu_i \geq S^{\frac{3}{2}}. \tag{3.24}$$

Now, from (f_3), (g_4) and (g_5), we have

$$\begin{aligned}
c &= \Phi_\lambda(u_n) - \frac{1}{4} < \Phi'_\lambda(u_n), u_n > +o_n(1) \\
&= \frac{1}{4}\|u_n\|_\lambda^2 + \int_{\mathbb{R}^3}\Big(\frac{1}{4}g(x,|u_n|^2)|u_n|^2 - \frac{1}{2}G(x,|u_n|^2)\Big)dx + o_n(1) \\
&\geq \frac{1}{4}\|u_n\|_\varepsilon^2 + \int_{\mathbb{R}^3\setminus\Omega'_\Gamma}\Big(\frac{1}{4}g(x,|u_n|^2)|u_n|^2 - \frac{1}{2}G(x,|u_n|^2)\Big)dx \\
&\quad + \frac{1}{12}\int_{\Omega'_\Gamma}|u_n|^6dx + o_n(1) \\
&\geq \frac{1}{4}\Big(\int_{\Omega'_\Gamma}\psi_\rho|\nabla|u_n||^2dx + \int_{\mathbb{R}^3\setminus\Omega'_\Gamma}(\lambda V(x)+1)|u_n|^2\Big) - \frac{1}{4}\int_{\mathbb{R}^3\setminus\Omega'_\Gamma}G(x,|u_n|^2)dx \\
&\quad + \frac{1}{12}\int_{\Omega'_\Gamma}|u_n|^6dx + o_n(1) \\
&\geq \frac{1}{4}\int_{\Lambda_\varepsilon}\psi_\rho|\nabla|u_n||^2dx + (\frac{1}{4}-\frac{1}{4\kappa})\int_{\mathbb{R}^3\setminus\Omega'_\Gamma}(\lambda V(x)+1)|u_n|^2dx + \frac{1}{12}\int_{\Lambda_\varepsilon}\psi_\rho|u_n|^6dx + o_n(1) \\
&\geq \frac{1}{4}\int_{\Omega'_\Gamma}\psi_\rho|\nabla|u_n||^2dx + \frac{1}{12}\int_{\Omega'_\Gamma}\psi_\rho|u_n|^6dx + o_n(1).
\end{aligned}$$

From the above arguments, (3.20) and (3.24), we have

$$\begin{aligned}
c &\geq \frac{1}{4}\sum_{\{i\in I: x_i\in\Omega'_\Gamma\}}\psi_\rho(x_i)\mu_i + \frac{1}{12}\sum_{\{i\in I: x_i\in\Omega'_\Gamma\}}\psi_\rho(x_i)\nu_i \\
&\geq \frac{1}{4}\sum_{\{i\in I: x_i\in\Omega'_\Gamma\}}\psi_\rho(x_i)S\nu_i^{1/3} + \frac{1}{12}\sum_{\{i\in I: x_i\in\Omega'_\Gamma\}}\psi_\rho(x_i)\nu_i \\
&\geq \frac{1}{4}S^{\frac{3}{2}} + \frac{1}{12}S^{\frac{3}{2}} = \frac{1}{3}S^{\frac{3}{2}}
\end{aligned}$$

which gives a contradiction. This means that (3.17) holds.

From (3.8), (3), (3.12) and (3.13), we may obtain that $\|u_n\|_\lambda^2 \to \|u\|_\lambda^2$ which means that $u_n \to u$ in E_λ. □

Next we study the behavior of a $(PS)_\infty$ sequence, that is, a sequence $(u_n) \subset H_A^1(\mathbb{R}^3, \mathbb{C})$ satisfying

$$\begin{aligned}
&u_n \in E_{\lambda_n} \quad \text{and} \quad \lambda_n \to \infty, \\
&\Phi_{\lambda_n}(u_n) \to c, \\
&\left\|\Phi'_{\lambda_n}(u_n)\right\|_{E^*_{\lambda_n}} \to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

Proposition 3.5. *Let $(u_n) \subset H_A^1(\mathbb{R}^3, \mathbb{C})$ be a $(PS)_\infty$ sequence with $c \in (0, \frac{1}{3}S^{\frac{3}{2}})$. Then, up to a subsequence, there exists $u \in H_A^1(\mathbb{R}^3, \mathbb{C})$ such that $u_n \rightharpoonup u$ in $H_A^1(\mathbb{R}^3, \mathbb{C})$. Moreover,*

(i) $u = 0$ in $\mathbb{R}^3 \setminus \Omega_\Gamma$, and for all $j \in \Gamma$, $u|_{\Omega_j}$ is a solution for

$$\begin{cases} -(\nabla + iA(x))^2u + u + \phi_{|u|}u = \alpha f(|u|^2)u + |u|^4u, & \text{in } \Omega_j, \\ u = 0, & \text{on } \partial\Omega_j; \end{cases} \tag{3.25}$$

(ii) $u_n \to u$ in E_{λ_n}. Hence

$$u_n \to u \quad \text{in } H_A^1(\mathbb{R}^3, \mathbb{C}); \tag{3.26}$$

(iii) $\lambda_n \int_{\mathbb{R}^3} V(x)\,|u_n|^2\,dx \to 0.$

(iv) $\|u_n\|^2_{\lambda_n,\Omega'_j} \to \int_{\Omega_j}(|\nabla_A u|^2 + |u|^2)dx,$ *for* $j \in \Gamma$;

(v) $\|u_n\|^2_{\lambda_n,\mathbb{R}^3\setminus\Omega_\Gamma} \to 0;$

(vi) $\Phi_{\lambda_n}(u_n) \to \frac{1}{2}\int_{\Omega_\Gamma}\left(|\nabla_A u|^2 + |u|^2\right)dx + \frac{1}{4}\int_{\Omega_\Gamma}\phi_{|u|}|u|^2dx - \alpha\int_{\Omega_\Gamma}F(|u|^2)dx - \frac{1}{6}\int_{\Omega_\Gamma}|u|^6dx.$

Proof. As in Lemma 3.1, we know that (u_n) is bounded in $H^1_A(\mathbb{R}^3,\mathbb{C})$. Thus we may assume that for some $u \in H^1_A(\mathbb{R}^3,\mathbb{C})$, up to a subsequence, if necessary

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } H^1_A\left(\mathbb{R}^3,\mathbb{C}\right), \\ u_n &\to u \quad \text{in } L^r_{loc}\left(\mathbb{R}^3,\mathbb{C}\right), \quad \forall r \geq 1, \\ |u_n| &\to |u| \quad \text{a.e. in } \mathbb{R}^3. \end{aligned}$$

(i) We fix the set $C_m = \left\{x \in \mathbb{R}^3; V(x) \geq \frac{1}{m}\right\}$, for each $m \in \mathbb{N}$. Then, we have

$$\begin{aligned} \int_{C_m}|u_n|^2\,dx &\leq \frac{m}{\lambda_n}\int_{\mathbb{R}^3}\lambda_n V(x)\,|u_n|^2\,dx \\ &\leq \frac{2m}{\lambda_n}\int_{\mathbb{R}^3}\left(|\nabla_A u_n|^2 + (\lambda_n V(x)+1)\,|u_n|^2\right)dx \\ &= \frac{2m}{\lambda_n}\,\|u_n\|^2_{\lambda_n}. \end{aligned}$$

By the Fatou's lemma, we derive

$$\int_{C_m}|u|^2\,dx = 0.$$

So, $u = 0$ in $\cup_{m=1}^{+\infty}C_m = \mathbb{R}^3 \setminus \Omega$, from which we can assert that $u|_{\Omega_j} \in H^{0,1}_A\left(\Omega_j,\mathbb{C}\right)$ for any $j \in \{1,2,\ldots,k\}$.

From (f_1), (f_2), for any $\zeta > 0$, there exists $C_\zeta > 0$ such that

$$|f(t)| \leq \zeta\,|t| + C_\zeta\,|t|^{\frac{q-2}{2}}.$$

So, we derive

$$\left|\text{Re}\int_{\mathbb{R}^3}g\left(x,|u_n|^2\right)u_n\bar{v}dx\right| \leq \zeta\alpha\int_{\mathbb{R}^3}|u_n|^3\,|\bar{v}|dx + C_\zeta\alpha\int_{\mathbb{R}^3}|u_n|^{q-1}\,|\bar{v}|dx + \int_{\mathbb{R}^3}|u_n|^5\,|\bar{v}|dx.$$

Therefore,

$$\text{Re}\int_{\mathbb{R}^3}g\left(x,|u_n|^2\right)u_n\bar{v}dx \to \text{Re}\int_{\mathbb{R}^3}g\left(x,|u|^2\right)u\bar{v}dx.$$

Since for each $v \in C_0^\infty\left(\Omega_j,\mathbb{C}\right)$, $\Phi'_{\lambda_n}(u_n)\,v \to 0$ as $n \to \infty$, from the above information and the argument explored in Proposition 3.4, we have

$$\text{Re}\left(\int_{\Omega_j}\left(\nabla_A u\overline{\nabla_A v} + u\bar{v}\right)dx + \int_{\Omega_j}\phi_{|u|}u\bar{v}dx - \int_{\Omega_j}g\left(x,|u|^2\right)u\bar{v}dx\right) = 0,$$

which implies that $u|_{\Omega_j}$ is a solution of problem (3.25) for each $j \in \Gamma$.

On the other hand, if $j \in \{1,2,\ldots,k\} \setminus \Gamma$, setting $v = u|_{\Omega_j}$,

$$\int_{\Omega_j}\left(|\nabla_A u|^2 + |u|^2\right)dx + \int_{\Omega_j}\phi_{|u|}\,|u|^2\,dx - \int_{\Omega_j}\widetilde{f}\left(|u|^2\right)|u|^2\,dx = 0.$$

By Lemma 2.3 and the definition of $\widetilde{f}$, we have

$$0 \leq \delta_0 \|u\|^2_{\lambda,\Omega_j} \leq \|u\|^2_{\lambda,\Omega_j} - \frac{\nu_0}{k}\|u\|^2_{2,\Omega_j}$$
$$\leq \int_{\Omega_j} \left(|\nabla_A u|^2 + |u|^2\right) dx - \int_{\Omega_j} \tilde{f}\left(|u|^2\right)|u|^2 dx \leq 0.$$

Thus $u|_{\Omega_j} = 0$ for $j \in \{1, 2, \dots, k\} \setminus \Gamma$. This proves that $u = 0$ in $\mathbb{R}^3 \setminus \Omega_\Gamma$.

(ii) From the similar arguments in the proof of Proposition 3.4,

$$\int_{\mathbb{R}^3} g\left(x, |u_n|^2\right)|u_n|^2 dx \to \int_{\mathbb{R}^3} g\left(x, |u|^2\right)|u|^2 dx$$
$$= \alpha \int_{\Omega_\Gamma} f\left(|u|^2\right)|u|^2 dx + \int_{\Omega_\Gamma} |u|^6 dx \quad \text{as } n \to +\infty.$$

By (i), we have

$$o_n(1) = \Phi'_{\lambda_n}(u_n)(u_n)$$
$$= \|u_n\|^2_{\lambda_n} + \int_{\mathbb{R}^3} \phi_{|u_n|}(x)|u_n|^2 dx - \int_{\mathbb{R}^3} g\left(x, |u_n|^2\right)|u_n|^2 dx$$
$$= \|u_n\|^2_{\lambda_n} - \|u\|^2_{\lambda_n} + o_n(1),$$

which implies $u_n \to u$ in E_{λ_n}. Hence $u_n \to u$ in $H^1_A(\mathbb{R}^3, \mathbb{C})$.

(iii) By (ii),

$$\lambda_n \int_{\mathbb{R}^3} V(x)|u_n|^2 dx = \lambda_n \int_{\mathbb{R}^3} V(x)|u_n - u|^2 dx$$
$$\leq C\|u_n - u\|^2_{\lambda_n} \to 0 \quad \text{as } n \to \infty.$$

(iv) Let $j \in \Gamma$. By (ii),

$$|u_n - u|^2_{2,\Omega'_j} \to 0, \qquad |\nabla_A u_n - \nabla_A u|^2_{2,\Omega'_j} \to 0,$$

therefore,

$$\int_{\Omega'_\Gamma} \left(|\nabla_A u_n|^2 - |\nabla_A u|^2\right) dx \to 0 \quad \text{and} \quad \int_{\Omega'_\Gamma} \left(|u_n|^2 - |u|^2\right) dx \to 0.$$

Also, by (iii),

$$\int_{\Omega'_\Gamma} \lambda_n V(x)|u_n|^2 dx \to 0.$$

Thus,

$$\|u_n\|^2_{\lambda_n,\Omega'_\Gamma} \to \int_{\Omega_\Gamma} \left(|\nabla_A u|^2 + u^2\right) dx.$$

(v) By (ii), it is easy to obtain that

$$\|u_n\|^2_{\lambda,\mathbb{R}^3\setminus\Omega_\Gamma} \to 0.$$

(vi) Since

$$\Phi_{\lambda_n}(u_n) = \sum_{j\in\Gamma}\left[\frac{1}{2}\int_{\Omega'_j} \left(|\nabla_A u_n|^2 + (\lambda_n V(x) + 1)|u_n|^2\right) dx + \frac{1}{4}\int_{\Omega'_j} \phi_{|u_n|}|u_n|^2 dx\right]$$
$$+ \frac{1}{2}\int_{\mathbb{R}^3\setminus\Omega'_\Gamma} \left(|\nabla_A u_n|^2 + (\lambda_n V(x) + 1)|u_n|^2\right) dx + \frac{1}{4}\int_{\mathbb{R}^3\setminus\Omega'_\Gamma} \phi_{|u_n|}|u_n|^2 dx$$
$$- \int_{\mathbb{R}^3} G(x, u_n)\, dx,$$

by (i)–(v), we can derive

$$\Phi_{\lambda_n}(u_n) \to \frac{1}{2}\int_{\Omega_\Gamma}\left(|\nabla_A u|^2 + |u|^2\right)dx + \frac{1}{4}\int_{\Omega_\Gamma}\phi_{|u|}u^2 dx - \alpha\int_{\Omega_\Gamma}F(|u|^2)dx - \frac{1}{6}\int_{\Omega_\Gamma}|u|^6 dx. \quad \square$$

Now, we study L^∞ estimate of the solution of problem (3.3).

Proposition 3.6. *Let (u_λ) be a family of nontrivial solutions of (3.3). Then, there exists $\lambda^* > 0$ such that*

$$\|u_\lambda\|^2_{L^\infty(\mathbb{R}^3\setminus\Omega'_\Gamma)} \le a, \quad \forall \lambda \ge \lambda^*.$$

In particular, u_λ is a solution of the original problem (1.4) for any $\lambda \ge \lambda^*$.

Proof. We give notation $B_r(x) = \{y \in \mathbb{R}^3 : |x-y| < r\}$. Since $u_\lambda \in E_\lambda$ is a critical point of $\Phi_\lambda(u)$, that is, u_λ satisfies the following equation

$$-\Delta_A u_\lambda + (\lambda V(x)+1)u_\lambda + \phi_{|u_\lambda|}u_\lambda = g\left(x, |u_\lambda|^2\right)u_\lambda, \quad x \in \mathbb{R}^3.$$

By the Kato's inequality

$$\Delta|u_\lambda| \ge \mathrm{Re}\left(\frac{\overline{u_\lambda}}{|u_\lambda|}(\nabla + iA(x))^2 u_\lambda(x)\right),$$

there holds

$$\Delta|u_\lambda(x)| - (\lambda V(x)+1)|u_\lambda(x)| - \phi_{|u_\lambda|}|u_\lambda(x)| - g(x, |u_\lambda|^2)|u_\lambda(x)| \ge 0, \qquad x \in \mathbb{R}^3,$$

since $|u_\lambda| \ge 0$, $\phi_{|u_\lambda|} \ge 0$ and $(\lambda V(x)+1) \ge M_0 > 0$ if $\lambda \ge 1$, we have

$$\Delta|u_\lambda(x)| - g\left(x, |u_\lambda|^2\right)|u_\lambda(x)| \ge 0, \qquad x \in \mathbb{R}^3.$$

We use the subsolution estimate (see [16], Theorem 8.17) and obtain that there exists a constant $C(r)$ such that for $1 < q < 2$

$$\sup_{y\in B_r(x)}|u_\lambda(y)| \le C(r)\Big(\int_{B_{2r}(x)}|u_\lambda|^q dy\Big)^{1/q}.$$

By Proposition 3.5, for any sequence $\lambda_n \to \infty$, we can extract a subsequence λ_{n_i} such that

$$u_{\lambda_{n_i}} \to u \in H_A^{0,1}(\Omega_\Gamma, \mathbb{C}) \quad \text{strongly in } H_A^1(\mathbb{R}^N, \mathbb{C}).$$

In particular,

$$u_{\lambda_{n_i}} \to 0 \quad \text{in } L^2(\mathbb{R}^N \setminus \overline{\Omega_\Gamma}, \mathbb{C}).$$

Since $\lambda_n \to \infty$ is arbitrary, we have

$$u_\lambda \to 0 \quad \text{in } L^2(\mathbb{R}^N \setminus \overline{\Omega_\Gamma}, \mathbb{C}) \quad \text{as } \lambda \to \infty.$$

Thus, choosing $r \in (0, \mathrm{dist}(\Omega_\Gamma, \mathbb{R}^N \setminus \Omega'_\Gamma))$, we have uniformly in $x \in \mathbb{R}^N \setminus \Omega'_\Gamma$ that

$$\begin{aligned}|u_\lambda(y)| &\le C(r)\|u_\lambda\|_{L^q(B_{2r}(x))}\\ &\le C(r)|B_{2r}(x)|^{\frac{2-q}{2q}}\|u_\lambda\|_{L^2(\mathbb{R}^N\setminus\overline{\Omega_\Gamma})}\\ &\to 0.\end{aligned}$$

This finishes the proof. $\square$

4 Existence of multi-bump solutions

In this section, we start to prove the existence of multi-bump solutions. First of all, for each fixed $j \in \Gamma$, let us denote by c_j the minimax level of the functional $I_j : H_A^{0,1}(\Omega_j, \mathbb{C}) \to \mathbb{R}$ given by

$$I_j(u) = \frac{1}{2}\int_{\Omega_j} \left(|\nabla_A u|^2 + |u|^2\right) dx + \frac{1}{4}\int_{\Omega_j} \phi_{|u|}|u|^2 dx - \frac{\alpha}{2}\int_{\Omega_j} F(|u|^2)dx - \frac{1}{6}\int_{\Omega_j} |u|^6 dx,$$

and

$$c_j = \inf_{\gamma \in \Lambda_j} \max_{t \in [0,1]} I_j(\gamma(t)),$$

where

$$\Lambda_j = \left\{\gamma \in C([0,1], H_A^{0,1}(\Omega_j, \mathbb{C})) : \gamma(0) = 0, I_j(\gamma(1)) < 0\right\}.$$

For each $j \in \Gamma$, we denote by $\Phi_{\lambda,j} : H_A^1\left(\Omega_j', \mathbb{C}\right) \to \mathbb{R}$ the functional given by

$$\begin{aligned}\Phi_{\lambda,j}(u) = &\frac{1}{2}\int_{\Omega_j'} \left(|\nabla_A u|^2 + (\lambda V(x) + 1)|u|^2\right) dx \\ &+ \frac{1}{4}\int_{\Omega_j'} \left(\frac{1}{4\pi}\int_{\Omega_j'} \frac{|\widetilde{u}|^2}{|x-y|} dy\right) u^2 dx - \frac{\alpha}{2}\int_{\Omega_j'} F(|u|^2)dx - \frac{1}{6}\int_{\Omega_j'} |u|^6 dx,\end{aligned}$$

and the above functional is associated to the following problem

$$\begin{cases} -\Delta_A u + (\lambda V(x) + 1)u + \left(\frac{1}{4\pi}\int_{\Omega_j'} \frac{|\widetilde{u}|^2}{|x-y|} dy\right) u = \alpha f(|u|^2)u + |u|^4 u, & \text{in } \Omega_j', \\ \frac{\partial u}{\partial \eta} = 0, & \text{on } \partial\Omega_j', \end{cases}$$

where

$$\widetilde{u}(x) = \begin{cases} u(x), & \text{in } \Omega_j', \\ 0, & \text{in } \mathbb{R}^3 \setminus \Omega_j'. \end{cases}$$

In what follows, we denote by $c_{\lambda,j}$ the minimax level of the above functional given by

$$c_{\lambda,j} = \inf_{\gamma \in \Lambda_{\lambda,j}} \max_{t \in [0,1]} \Phi_{\lambda,j}(\gamma(t)),$$

where

$$\Lambda_{\lambda,j} = \left\{\gamma \in C\left([0,1], H_A^1\left(\Omega_j', \mathbb{C}\right)\right) : \gamma(0) = 0, \Phi_{\lambda,j}(\gamma(1)) < 0\right\}.$$

Repeating the same method used in the previous section, we are able to prove that there exist $\omega_j \in H_A^{0,1}(\Omega_j, \mathbb{C})$ and $\omega_{\lambda,j} \in H_A^1\left(\Omega_j', \mathbb{C}\right)$ such that

$$I_j(\omega_j) = c_j \quad \text{and} \quad I_j'(\omega_j) = 0,$$

and

$$\Phi_{\lambda,j}(\omega_{\lambda,j}) = c_{\lambda,j} \quad \text{and} \quad \Phi_{\lambda,j}'(\omega_{\lambda,j}) = 0.$$

Furthermore, we have the following important lemma.

Lemma 4.1. *The following statements hold:*

(i) $0 < c_{\lambda,j} \leq c_j$, *for* $\lambda \geq 1$ *and* $j \in \Gamma$.

(ii) c_j *(*$c_{\lambda,j}$ *respectively) is a least energy level for* $I_j(u)$ *(*$\Phi_{\lambda,j}(u)$ *respectively), that is*

$$c_j = \inf\left\{ I_j(u) : u \in H_A^{0,1}(\Omega_j, \mathbb{C}) \setminus \{0\}, I_j'(u)u = 0 \right\},$$

and

$$c_{\lambda,j} = \inf\left\{ \Phi_{\lambda,j}(u) : u \in H_A^1\left(\Omega_j', \mathbb{C}\right) \setminus \{0\}, \Phi_{\lambda,j}'(u)u = 0 \right\}.$$

(iii) $c_{\lambda,j} \to c_j$, *as* $\lambda \to \infty$ *for any* $j \in \Gamma$.

Proof. (i) From (f_3), we have $c_j > 0$ and $c_{\lambda,j} > 0$ for any $j \in \Gamma$ and $\lambda \geq 1$. For any $u \in H_A^{0,1}(\Omega_j, \mathbb{C})$, we may extend u to $\widehat{u} \in H_A^1\left(\Omega_j', \mathbb{C}\right)$ by

$$\widehat{u}(x) = \begin{cases} u(x), & \text{in } \Omega_j, \\ 0, & \text{in } \Omega_j' \setminus \overline{\Omega}_j. \end{cases}$$

Using the fact that $H_A^{0,1}(\Omega_j, \mathbb{C}) \subset H_A^1\left(\Omega_j', \mathbb{C}\right)$, we have

$$\begin{aligned} c_{\lambda,j} &= \inf_{\gamma \in \Lambda_{\lambda,j}} \max_{t \in [0,1]} \Phi_{\lambda,j}(\gamma(t)) \\ &\leq \inf_{\gamma \in \Lambda_j} \max_{t \in [0,1]} \Phi_{\lambda,j}(\gamma(t)) \\ &= \inf_{\gamma \in \Lambda_j} \max_{t \in [0,1]} I_j(\gamma(t)) = c_j. \end{aligned}$$

(ii) By the monotonicity of the term $f(t)$ with respect to t for $t > 0$, we are able to prove this.

(iii) Using Proposition 3.5, for sequences (λ_n) with $\lambda_n \to \infty$, as $n \to \infty$, there exists $\omega \in H_A^{0,1}(\Omega_j, \mathbb{C})$ is a solution of (3.25) such that

$$\omega_{\lambda_n,j} \to \omega \quad \text{in } H_A^1(\Omega_j', \mathbb{C}),$$

and

$$\Phi_{\lambda_n,j}(\omega_{\lambda_n,j}) \to I_j(\omega).$$

By the definition of c_j, we have

$$\limsup_{\lambda \to \infty} c_{\lambda,j} = \limsup_{\lambda \to \infty} \Phi_{\lambda,j}\left(\omega_{\lambda,j}\right) \geq I_j(\omega) \geq c_j.$$

Together with (i), we get the asserted result. □

In what follows, we fix $R > 1$ verifying

$$\left| I_j\left(\frac{1}{R}\omega_j\right) \right| < \frac{1}{2}c_j, \qquad \forall j \in \Gamma, \tag{4.1}$$

and

$$\left| I_j\left(R\omega_j\right) - c_j \right| \geq 1, \qquad \forall j \in \Gamma. \tag{4.2}$$

By the definition of c_j, we are able to obtain

$$\max_{s_j \in [1/R^2, 1]} I_j\left(s_j R\omega_j\right) = c_j, \qquad \forall j \in \Gamma.$$

Then, for $\Gamma = \{1, 2, \ldots, l\}$ $(l \le k)$, we define

$$\gamma_0(\boldsymbol{s})(x) = \sum_{j=1}^{l} s_j R\omega_j(x) \quad \forall \boldsymbol{s} = (s_1, s_2, \ldots, s_l) \in \left[1/R^2, l\right]^l,$$

$$\Lambda_* = \left\{\gamma \in C\left(\left[1/R^2, 1\right]^l, E_\lambda \setminus \{0\}\right) : \gamma = \gamma_0 \text{ on } \partial\left(\left[1/R^2, 1\right]^l\right)\right\},$$

and

$$b_{\lambda,\Gamma} = \inf_{\gamma \in \Lambda_*} \max_{\boldsymbol{s} \in [1/R^2, 1]^l} \Phi_\lambda\left(\gamma\left(\boldsymbol{s}\right)\right).$$

Next, let us denote by $c_\Gamma = \sum_{j=1}^{l} c_j$ and $c_{\lambda,\Gamma} = \sum_{j=1}^{l} c_{\lambda,j}$. Moreover, from Remark 3.3, we know that $c_\Gamma \in (0, \frac{1}{3}S^{\frac{3}{2}})$. To prove an important relation among $b_{\lambda,\Gamma}$, c_Λ and $c_{\lambda,\Gamma}$, we need to the following lemma.

Lemma 4.2. *For any $\gamma \in \Lambda_*$, there exists $(t_1, t_2, \ldots, t_l) \in \left[1/R^2, 1\right]^l$ such that*

$$\Phi'_{\lambda,j}\left(\gamma\left(t_1, t_2, \ldots, t_l\right)\right)\left(\gamma\left(t_1, t_2, \ldots, t_l\right)\right) = 0 \quad \textit{for all } j \in \{1, 2, \ldots, l\}.$$

Proof. Given $\gamma \in \Lambda_*$, consider $\widetilde{\gamma} : \left[1/R^2, 1\right]^l \to \mathbb{C}^l$ defined by

$$\widetilde{\gamma}\left(s_1, s_2, \ldots, s_l\right) = \left(\Phi'_{\lambda,1}(\gamma)(\gamma), \Phi'_{\lambda,2}(\gamma)(\gamma), \ldots, \Phi'_{\lambda,l}(\gamma)(\gamma)\right),$$

where

$$\Phi'_{\lambda,j}(\gamma)(\gamma) = \Phi'_{\lambda,j}\left(\gamma\left(s_1, s_2, \ldots, s_l\right)\right)\left(\gamma\left(s_1, s_2, \ldots, s_l\right)\right) \qquad \text{for all } j \in \Gamma.$$

By (f_4) and $I'_j(\omega_j) = 0$, we have

$$I'_j\left(R\omega_j\right)\left(R\omega_j\right) < 0 \quad \text{and} \quad I'_j\left(\frac{1}{R}\omega_j\right)\left(\frac{1}{R}\omega_j\right) > 0.$$

For $\boldsymbol{s} \in \partial(\left[1/R^2, 1\right]^l)$, it holds $\gamma(\boldsymbol{s}) = \gamma_0(\boldsymbol{s})$, and

$$\Phi'_{\lambda,j}\left(\gamma_0\left(\boldsymbol{s}\right)\right)\left(\gamma_0\left(\boldsymbol{s}\right)\right) = 0 \Rightarrow s_j \notin \left\{1/R^2, 1\right\}, \qquad \forall j \in \Gamma.$$

Thus,

$$(0, 0, \ldots, 0) \notin \widetilde{\gamma}\left(\partial\left(\left[1/R^2, 1\right]^l\right)\right).$$

Since

$$\deg\left(\widetilde{\gamma}, \left(1/R^2, 1\right)^l, (0, \ldots, 0)\right) = \deg\left(\widetilde{\gamma}_0, \left(1/R^2, 1\right)^l, (0, \ldots, 0)\right)$$

and, for $\boldsymbol{s} \in \left(1/R^2, 1\right)^l$,

$$\widetilde{\gamma}_0(\boldsymbol{s}) = 0 \Longleftrightarrow \boldsymbol{s} = \left(\frac{1}{R}, \ldots, \frac{1}{R}\right),$$

we have

$$\deg\left(\widetilde{\gamma}, \left(1/R^2, 1\right)^l, (0, \ldots, 0)\right) \neq 0.$$

This shows what was stated. □

Proposition 4.3. *The following facts hold:*

(i) $c_{\lambda,\Gamma} \leq b_{\lambda,\Gamma} \leq c_\Gamma, \forall \lambda \geq 1$;

(ii) $b_{\lambda,\Gamma} \to c_\Gamma$, *as* $\lambda \to \infty$;

(iii) $\Phi_\lambda(\gamma(\boldsymbol{s})) < c_\Gamma, \forall \lambda \geq 1, \gamma \in \Lambda_*$ *and* $\boldsymbol{s} = (s_1, \ldots, t_l) \in \partial([1/R^2, 1]^l)$;

(iv) $b_{\lambda,\Gamma}$ *is a critical point of* Φ_λ *for large* λ.

Proof. (i) Since $\gamma_0 \in \Lambda_*$,

$$\begin{aligned} b_{\lambda,\Gamma} &\leq \max_{(s_1,s_2,\ldots,s_l)\in[1/R^2,1]} \Phi_\lambda\left(\gamma_0\left(s_1,s_2,\ldots,s_l\right)\right) \\ &= \max_{(s_1,s_2,\ldots,s_l)\in[1/R^2,1]} \sum_{j=1}^{l} I_j\left(s_j R\omega_j\right) \\ &= \sum_{j=1}^{l} c_j = c_\Gamma. \end{aligned}$$

Fixing $(t_1, t_2, \ldots, t_l) \in \left[1/R^2, 1\right]^l$ given in Lemma 4.2 and recalling that $c_{\lambda,j}$ can be characterized by

$$c_{\lambda,j} = \inf\left\{\Phi_{\lambda,j}(u) : u \in H_A^1\left(\Omega_j', \mathbb{C}\right) \setminus \{0\}, \Phi_{\lambda,j}'(u)u = 0\right\},$$

it follows that

$$\Phi_{\lambda,j}\left(\gamma\left(t_1,t_2,\ldots,t_l\right)\right) \geq c_{\lambda,j} \qquad \forall j \in \Gamma.$$

Since $\forall u \in H_A^1\left(\mathbb{R}^3 \setminus \Omega_\Gamma', \mathbb{C}\right)$, $\Phi_{\lambda,\mathbb{R}^2\setminus\Omega_\Gamma'}(u) \geq 0$, we have

$$\Phi_\lambda\left(\gamma\left(s_1,s_2,\ldots,s_l\right)\right) \geq \sum_{j=1}^{l} \Phi_{\lambda,j}\left(\gamma\left(s_1,s_2,\ldots,s_l\right)\right).$$

Hence

$$\max_{(s_1,s_2,\ldots,s_l)\in[1/R^2,1]^l} \Phi_\lambda\left(\gamma\left(s_1,s_2,\ldots,s_l\right)\right) \geq \Phi_\lambda\left(\gamma\left(t_1,t_2,\ldots,t_l\right)\right) \geq \sum_{j=1}^{l} c_{\lambda,j}$$

showing that

$$b_{\lambda,\Gamma} \geq \sum_{j=1}^{l} c_{\lambda,j} = c_{\lambda,\Gamma}.$$

(ii) Since $c_{\lambda,j} \to c_j$, as $\lambda \to \infty$, by the previous item,

$$b_{\lambda,\Gamma} \to c_\Gamma, \quad as\ \lambda \to \infty.$$

(iii) For $\boldsymbol{s} \in \partial([1/R^2, 1]^l)$, it holds $\gamma(\boldsymbol{s}) = \gamma_0(\boldsymbol{s})$. Hence,

$$\Phi_\lambda\left(\gamma_0\left(s_1,s_2,\ldots,s_l\right)\right) = \sum_{j=1}^{l} I_j\left(s_j R\omega_j\right).$$

From (4.1) and (4.2), we have

$$\Phi_\lambda\left(\gamma_0\left(s_1,s_2,\ldots,s_l\right)\right) \leq c_\Gamma - \epsilon.$$

for some $\epsilon > 0$, so (iii) holds.
(iv) By (ii), we can choose λ large enough such that $b_{\lambda,\Gamma}, c_\Gamma \in (0, \frac{1}{3}S^{\frac{3}{2}})$. From Proposition 3.4 and (3.6), we know that any $(PS)_{b_{\lambda,\Gamma}}$ sequence of Φ_λ has a convergence subsequence in E_λ. Moreover, from the deformation lemma, we can conclude that $b_{\lambda,\Gamma}$ is a critical level of Φ_λ for λ large. □

To prove Theorem 1.1, we need to find a nontrivial solution u_λ for the large λ which approaches a least energy solution in each Ω_j ($j \in \Gamma$) and to 0 elsewhere as $\lambda \to \infty$. Therefore, we shall show two propositions which imply together with the estimates made in the previous section that Theorem 1.1 holds.

Henceforth, let

$$\Phi_\lambda^{c_\Gamma} = \{u \in E_\lambda : \Phi_\lambda(u) \le c_\Gamma\}.$$

For small $\mu > 0$, we denote by

$$A_\mu^\lambda = \left\{u \in E_\lambda : \|u\|_{\lambda,\mathbb{R}^3\setminus\Omega_j'} \le \mu, |\Phi_{\lambda,j}(u) - c_j| \le \mu, \forall j \in \Gamma\right\},$$

and observe that $\omega = \sum_{j=1}^{l} \omega_j \in A_\mu^\lambda \cap \Phi_\lambda^{c_\Gamma}$, showing that $A_\mu^\lambda \cap \Phi_\lambda^{c_\Gamma} \ne \emptyset$. Fixing

$$0 < \mu < \frac{1}{3}\min\{c_j, j \in \Gamma\}. \tag{4.3}$$

We obtain the following uniform estimate of $\|\Phi_\lambda'(u)\|_\lambda$ on the annulus $\left(A_{2\mu}^\lambda \setminus A_\mu^\lambda\right) \cap \Phi_\lambda^{c_\Gamma}$.

Proposition 4.4. *Let $\mu > 0$ satisfying (4.3). Then there exist $\sigma_0 > 0$ and $\lambda^* \ge 1$ independent of λ such that*

$$\|\Phi_\lambda'(u)\|_\lambda \ge \sigma_0 \quad \text{for } \lambda \ge \lambda^* \quad \text{for all } u \in \left(A_{2\mu}^\lambda \setminus A_\mu^\lambda\right) \cap \Phi_\lambda^{c_\Gamma}.$$

Proof. Arguing by contradiction, we assume that there exist $\lambda_n \to \infty$ and $u_n \in \left(A_{2\mu}^{\lambda_n} \setminus A_\mu^{\lambda_n}\right) \cap \Phi_{\lambda_n}^{c_\Gamma}$ such that $\|\Phi_{\lambda_n}'(u)\|_{\lambda_n} \to 0$.

Since $u_n \in A_{2\mu}^{\lambda_n}$, we can obtain that $\{\|u_n\|_{\lambda_n}\}$ is a bounded in $E_{\lambda_n}(\mathbb{R}^3, \mathbb{C})$ and $H_A^1(\mathbb{R}^3, \mathbb{C})$, and $\{\Phi_{\lambda_n}(u_n)\}$ is also bounded. Thus, passing a subsequence if necessary, we may assume that

$$\Phi_{\lambda_n}(u_n) \to c \in (-\infty, c_\Gamma].$$

From Proposition 3.5, there exists $u \in H_A^{0,1}(\Omega_\Gamma, \mathbb{C})$ such that u is a solution of (3.25),

$$\begin{aligned}
&u_n \to u \quad \text{in } H_A^1(\mathbb{R}^3, \mathbb{C}),\\
&\lim_{n\to\infty} \Phi_{\lambda_n}(u_n) = \sum_{j=1}^{l} I_j(u) \le c_\Gamma,\\
&\|u_n\|_{\lambda_n,\Omega_j'}^2 \to \int_{\Omega_j} \left(|\nabla_A u|^2 + |u|^2\right) dx, \qquad \forall j \in \Gamma,\\
&\lambda_n \int_{\mathbb{R}^3} V(x)\,|u_n|^2\, dx \to 0,\\
&\|u_n\|_{\lambda_n,\mathbb{R}^3\setminus\Omega_\Gamma}^2 \to 0.
\end{aligned}$$

Since $c_\Gamma = \sum_{j=1}^{l} c_j$ and c_j is the least energy level for I_j, we have two possibilities:

(*i*) $I_j(u|_{\Omega_j}) = c_j$ $\forall j \in \Gamma$;

(*ii*) $I_{j_0}(u|_{\Omega_{j_0}}) = 0$, that is $u|_{\Omega_{j_0}} \equiv 0$ for some $j_0 \in \Gamma$.

If (i) occurs, we have

$$\frac{1}{2}\int_{\Omega_j}\left(|\nabla_A u|^2+|u|^2\right)dx+\frac{1}{4}\int_{\Omega_j}\phi_{|u|}\,|u|^2\,dx-\frac{\alpha}{2}\int_{\Omega_j}F(|u|^2)dx-\frac{1}{6}\int_{\Omega_j}|u|^6dx=c_j,\qquad \forall j\in\Gamma.$$

Thus, $\left|\Phi_{\lambda,j}(u)-c_j\right|\le\mu$, $\forall j\in\Gamma$, that is, $u_n\in A_\mu^{\lambda_n}$ for large n, which is a contradiction to the assumption $u_n\in A_{2\mu}^{\lambda_n}\setminus A_\mu^{\lambda_n}$.

If (ii) occurs, we have

$$\left|\Phi_{\lambda_n,j_0}(u_n)-c_{j_0}\right|\to c_{j_0}\ge 3\mu,$$

which is a contradiction with the fact that $u_n\in A_{2\mu}^{\lambda_n}\setminus A_\mu^{\lambda_n}$. Thus neither (i) nor (ii) can hold, and the proof is completed. □

Proposition 4.5. *Let $\mu>0$ satisfying (4.3) and $\lambda^*\ge 0$ be a constant given in Proposition 4.4. Then, for any $\lambda\ge\lambda^*$, there exists a nontrivial solution u_λ of (3.3) satisfying $u_\lambda\in A_\mu^\lambda\cap\Phi_\lambda^{c_\Gamma}$.*

Proof. Arguing by contradiction, we assume that there are no critical points in $A_\mu^\lambda\cap\Phi_\lambda^{c_\Gamma}$. Since Φ_λ verifies the (PS) condition in the level $(0,\frac{1}{3}S^{\frac{3}{2}})$, there exists a constant $d_\lambda>0$ such that

$$\left\|\Phi_\lambda'(u)\right\|\ge d_\lambda\quad\text{for all}\quad u\in A_\mu^\lambda\cap\Phi_\lambda^{c_\Gamma}.$$

From Proposition 4.4, we have

$$\left\|\Phi_\lambda'(u)\right\|\ge\sigma_0\quad\text{for all } u\in\left(A_{2\mu}^\lambda\setminus A_\mu^\lambda\right)\cap\Phi_\lambda^{c_\Gamma},$$

where $\sigma_0>0$ does not depend on λ. In what follows, $\Psi:E_\lambda\to\mathbb{R}$ is a continuous functional verifying

$$\begin{array}{ll}\Psi(u)=1 & \text{for } u\in A_{3\mu/2}^\lambda,\\ \Psi(u)=0 & \text{for } u\notin A_{2\mu}^\lambda,\\ 0\le\Psi(u)\le 1 & \text{for } u\in E_\lambda\left(\mathbb{R}^3,\mathbb{C}\right).\end{array}$$

We consider $H:\Phi_\lambda^{c_\Gamma}\to E_\lambda$ given by

$$H(u)=\begin{cases}-\Psi(u)\frac{\Phi_\lambda'(u)}{\left\|\Phi_\lambda'(u)\right\|_\lambda}, & u\in A_{2\mu}^\lambda,\\ 0, & u\notin A_{2\mu}^\lambda.\end{cases}$$

Hence, we have the inequality

$$\|H(u)\|_\lambda\le 1\quad\forall\lambda\ge\Lambda_*\quad\text{and}\quad u\in\Phi_\lambda^{c_\Gamma}.$$

Considering the deformation flow $\eta:[0,\infty)\times\Phi_\lambda^{c_\Gamma}\to\Phi_\lambda^{c_\Gamma}$ defined by

$$\frac{d\eta}{dt}=H(\eta)\quad\text{and}\quad\eta(0,u)=u\in\Phi_\lambda^{c_\Gamma}.$$

Thus η has the following properties

$$\frac{d}{dt}\Phi_\lambda(\eta(t,u))=-\Psi(\eta(t,u))\left\|\Phi_\lambda'(\eta(t,u))\right\|_\lambda\le 0,\tag{4.4}$$

$$\eta(t,u)=u\quad\text{for all } t\ge 0 \text{ and } u\in\Phi_\lambda^{c_\Gamma}\setminus A_{2\mu}^\lambda,\tag{4.5}$$

$$\left\|\frac{d\eta}{dt}\right\|_\lambda \le 1 \quad \text{for all } t,u. \tag{4.6}$$

Now let $\gamma_0(\boldsymbol{s}) \in \Lambda_*$ and we consider $\eta(t,\gamma_0(\boldsymbol{s}))$ for large t. If μ satisfies (4.3), we have that

$$\gamma_0(\boldsymbol{s}) \notin \mathcal{A}^\lambda_{2\mu}, \qquad \forall \boldsymbol{s} \in \partial\left([1/R^2,1]^l\right).$$

Since

$$\Phi_\lambda\left(\gamma_0(\boldsymbol{s})\right) < c_\Gamma, \qquad \forall \boldsymbol{s} \in \partial\left([1/R^2,1]^l\right),$$

from (4.5), it follows that

$$\eta\left(t,\gamma_0(\boldsymbol{s})\right) = \gamma_0(\boldsymbol{s}), \quad \forall \boldsymbol{s} \in \partial\left([1/R^2,1]^l\right).$$

So, $\eta\left(t,\gamma_0(\boldsymbol{s})\right) \in \Lambda_*$, for each $t \ge 0$.

On the other hand, $\operatorname{supp}\gamma_0(\boldsymbol{s})(x) \subset \overline{\Omega}_\Gamma$ for all $\boldsymbol{s} \in \partial([1/R^2,1]^l)$, then $\Phi_\lambda(\gamma_0(\boldsymbol{s}))$ does not depend on $\lambda \ge 0$. Moreover,

$$\Phi_\lambda\left(\gamma_0(\boldsymbol{s})\right) \le c_\Gamma, \quad \forall \boldsymbol{s} \in [1/R^2,1]^l$$

and $\Phi_\lambda\left(\gamma_0(\boldsymbol{s})\right) = c_\Gamma$ if and only if $s_j = \frac{1}{R}, \forall j \in \Gamma$.

Therefore, we have that

$$m_0 = \max\left\{\Phi_\lambda(u) : u \in \gamma_0([1/R^2,1]^l) \setminus A^\lambda_\mu\right\}$$

is independent of λ and $m_0 \le c_\Gamma$. From (4.6), it is easy to see that for any $t > 0$,

$$\left\|\eta\left(0,\gamma_0\left(s_1,s_2,\dots,s_l\right)\right) - \eta\left(t,\gamma_0\left(s_1,s_2,\dots,s_l\right)\right)\right\|_\lambda \le t.$$

Since $\Phi_{\lambda,j}(u) \in C^1(E_\lambda,\mathbb{R})$ for all $j = 1,2,\dots,l$, and the assumptions $(f_1)-(f_4)$, it is easy to see that for large number $T > 0$, there exists a positive number $\rho_0 > 0$ which is independent of λ such that for all $j = 1,2,\dots,l$ and $t \in [0,T]$,

$$\left\|\Phi'_{\lambda,j}(\eta(t,\gamma_0(s_1,s_2,\dots,s_l)))\right\|_\lambda \le \rho_0. \tag{4.7}$$

We claim that for large T,

$$\max_{\boldsymbol{s}\in[1/R^2,1]^l} \Phi_\lambda\left(\eta\left(T,\gamma_0(\boldsymbol{s})\right)\right) \le \max\left\{m_0, c_\Gamma - \frac{1}{2}\tau_0\mu\right\},$$

where $\tau_0 = \max\{\sigma_0, \frac{\sigma_0}{\rho_0}\}$.

In fact, if $\gamma_0(\boldsymbol{s}) \notin A^\lambda_\mu$, from (4.4),

$$\Phi_\lambda(\eta(t,\boldsymbol{s})) \le \Phi_\lambda(\boldsymbol{s}) \le m_0, \qquad \forall t \ge 0.$$

If $\gamma_0(\boldsymbol{s}) \in A^\lambda_\mu$, we set

$$\widetilde{\eta}(t) = \eta(t,\boldsymbol{s}), \qquad \widetilde{d_\lambda} = \min\left\{d_\lambda,\sigma_0\right\} \quad \text{and} \quad T = \frac{\sigma_0\mu}{2\widetilde{d_\lambda}}.$$

Next we differentiate two cases:

(1) $\widetilde{\eta}(t) \in A^\lambda_{3\mu/2}$ for all $t \in [0,T]$.

(2) $\widetilde{\eta}\left(t_{0}\right) \in \partial A_{3\mu/2}^{\lambda}$ for some $t_{0} \in[0, T]$.

If (1) holds, we have $\Psi(\widetilde{\eta}(t)) \equiv 1$ and $\left\|\Phi_{\lambda}^{\prime}(\widetilde{\eta}(t))\right\|_{\lambda} \geq \widetilde{d_{\lambda}}$ for all $t \in[0, T]$. Hence, from (4.4), we get

$$
\begin{aligned}
\Phi_{\lambda}(\widetilde{\eta}(T)) & =\Phi_{\lambda}\left(\gamma_{0}(s)\right)+\int_{0}^{T} \frac{d}{d s} \Phi_{\lambda}(\widetilde{\eta}(s)) d s \\
& =\Phi_{\lambda}\left(\gamma_{0}(s)\right)-\int_{0}^{T} \Psi(\widetilde{\eta}(s))\left\|\Phi_{\lambda}^{\prime}(\widetilde{\eta}(s))\right\|_{\lambda} d s \\
& \leq c_{\Gamma}-\int_{0}^{T} \widetilde{d}_{\lambda} d s \\
& =c_{\Gamma}-\widetilde{d}_{\lambda} T \\
& =c_{\Gamma}-\frac{1}{2} \sigma_{0} \mu \\
& \leq c_{\Gamma}-\frac{1}{2} \tau_{0} \mu .
\end{aligned}
$$

If (2) holds, there exists $0 \leq t_{1} \leq t_{2} \leq T$ such that

$$
\widetilde{\eta}\left(t_{1}\right) \in \partial A_{u}^{\lambda}, \tag{4.8}
$$

$$
\widetilde{\eta}\left(t_{2}\right) \in \partial A_{3\mu/2}^{\lambda}, \tag{4.9}
$$

$$
\widetilde{\eta}(t) \in A_{3\mu/2}^{\lambda} \setminus A_{u}^{\lambda}, \quad \text{for all } t \in\left[t_{1}, t_{2}\right].
$$

It follows from (4.9)

$$
\left\|\widetilde{\eta}\left(t_{2}\right)\right\|_{\lambda, \mathbb{R}^{3} \setminus \Omega_{\Gamma}^{\prime}}=\frac{3 \mu}{2},
$$

or

$$
\left|\Phi_{\lambda, \Omega_{j_{0}}^{\prime}}\left(\widetilde{\eta}\left(t_{2}\right)\right)-c_{j_{0}}\right|=\frac{3 \mu}{2},
$$

for some $j_{0} \in \Gamma$.

Now we consider the later case, the former case can be obtained in a similar way. By (4.8),

$$
\left|\Phi_{\lambda, \Omega_{j_{0}}^{\prime}}\left(\widetilde{\eta}\left(t_{1}\right)\right)-c_{j_{0}}\right| \leq \mu,
$$

thus, we obtain

$$
\left|\Phi_{\lambda, \Omega_{j_{0}}^{\prime}}(\widetilde{\eta}(t_{2}))-\Phi_{\lambda, \Omega_{j_{0}}^{\prime}}(\widetilde{\eta}(t_{1}))\right| \geq\left|\Phi_{\lambda, \Omega_{j_{0}}^{\prime}}\left(\widetilde{\eta}\left(t_{2}\right)\right)-c_{j_{0}}\right|-\left|\Phi_{\lambda, \Omega_{j_{0}}^{\prime}}\left(\widetilde{\eta}\left(t_{1}\right)\right)-c_{j_{0}}\right| \geq \frac{1}{2} \mu .
$$

On the other hand, by the mean value theorem, there exists $t_{3} \in\left(t_{1}, t_{2}\right)$ such that

$$
\left|\Phi_{\lambda, \Omega_{j_{0}}^{\prime}}(\widetilde{\eta}(t_{2}))-\Phi_{\lambda, \Omega_{j_{0}}^{\prime}}(\widetilde{\eta}(t_{1}))\right|=\left|\Phi_{\lambda, \Omega_{j_{0}}^{\prime}}^{\prime} \cdot \frac{d \widetilde{\eta}}{d t}\left(t_{3}\right)\right|\left(t_{2}-t_{1}\right).
$$

Moreover, from (4.6) and (4.7), we have

$$
t_{2}-t_{1} \geq \frac{\mu}{2 \rho_{0}}.
$$

Hence, we obtain

$$\begin{aligned}\Phi_\lambda(\widetilde{\eta}(T)) &= \Phi_\lambda(\gamma_0(\mathbf{s})) + \int_0^T \frac{d}{ds}\Phi_\lambda(\widetilde{\eta}(s))ds \\ &= \Phi_\lambda(\gamma_0(\mathbf{s})) - \int_0^T \Psi(\widetilde{\eta}(s))\left\|\Phi'_\lambda(\widetilde{\eta}(s))\right\|_\lambda ds \\ &\le c_\Gamma - \int_{t_1}^{t_2} \Psi(\widetilde{\eta}(s))\left\|\Phi'_\lambda(\widetilde{\eta}(s))\right\|_\lambda ds \\ &= c_\Gamma - \sigma_0(t_2 - t_1) \\ &\le c_\Gamma - \frac{1}{2}\tau_0\mu,\end{aligned}$$

and so (4.7) is proved. Now we recall that $\widetilde{\eta}(T) = \eta(T, \gamma_0(\mathbf{0})) \in \Lambda_*$, thus

$$b_{\lambda,\Gamma} \le \Phi_\lambda(\widetilde{\eta}(T)) \le \max\left\{m_0, c_\Gamma - \frac{1}{2}\tau_0\mu\right\},$$

which contradicts the fact that $b_{\lambda,\Gamma} \to c_\Gamma$ as $\lambda \to \infty$. □

Proof of Theorem 1.1. From Proposition 4.5, there exists a nontrivial solutions u_λ to problem (3.3) such that $u_\lambda \in A^\lambda_\mu \cap \Phi^{c_\Gamma}_\lambda$, for all $\lambda \ge \lambda^*$. So, using the proof of Proposition 3.6, we can derive that

$$\|u_\lambda\|^2_{L^\infty(\mathbb{R}^3\setminus\Omega'_\Gamma)} \le a, \qquad \forall \lambda \ge \lambda^*,$$

which shows that u_λ is a nontrivial solution to the original problem (1.4).

Moreover, for any given sequence (λ_n) with $\lambda_n \to +\infty$, up to a subsequence if necessary, it is easy to show that (u_{λ_n}) is a $(PS)_\infty$ sequence. Hence, by Proposition 3.5, we obtain

$$u_{\lambda_n} \to u \quad \text{in } H^1_A\left(\mathbb{R}^3, \mathbb{C}\right) \text{ with } u \in H^{0,1}_A\left(\Omega_\Gamma, \mathbb{C}\right), \quad u \equiv 0 \quad \text{in } \mathbb{R}^3 \setminus \Omega_\Gamma,$$

and the restriction $u\,|_{\Omega_j}$ is a least energy solution of

$$\begin{cases} -(\nabla + iA(x))^2 u + u + \left(\frac{1}{4\pi}\int_{\Omega_j}\frac{|u(y)|^2}{|x-y|}dy\right)u = \alpha f(|u|^2)u + |u|^4 u, \ x \in \Omega_j, \\ u \in H^{0,1}_A\left(\Omega_j\right), \end{cases}$$

where $j \in \Gamma$. We complete the proof of Theorem 1.1. □

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Weak center for a class of Λ–Ω differential systems

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Abstract. In this paper, we give the necessary and sufficient conditions for a class of higher degree polynomial systems to have a weak center. As corollaries, we prove the correctness of the two conjectures about the weak center problem for the Λ–Ω differential systems.

Keywords: weak center, Λ–Ω system, composition center, center conditions.

2020 Mathematics Subject Classification: 34C07, 34C05, 34C25, 37G15.

1 Introduction

Consider differential system of the form

$$\begin{cases} x' = -y + P, \\ y' = x + Q, \end{cases} \tag{1.1}$$

where $P = \sum_{k=2}^{m} P_k(x, y)$ and $Q = \sum_{k=2}^{m} Q_k(x, y)$, P_k and Q_k are homogeneous polynomials in x and y of degree k . The equilibrium point $O(0, 0)$ is a center if there exists an open neighborhood U of O where all the orbits contained in U/O are periodic. The center-focus problem asks about the conditions on the coefficients of P and Q under which the origin of system (1.1) is a center. The study of the centers of analytical or polynomial differential system (1.1) has a long history. The first works are due to Poincaré [13] and Dulac [8], and continued by Liapunov [9] and many others. Unfortunately, the center-focus problem has been solved only for quadratic system and some special cubic system and others [2, 6, 7, 12]. Up to now, very little is known about the center conditions for polynomial differential system with arbitrary degree m $(m > 2)$.

A center of (1.1) is called a **weak center** if the Poincaré–Liapunov first integral can be written as $H = \frac{1}{2}(x^2 + y^2)(1 + h.o.t.)$. By literature [10, 11] we know that a center of a polynomial differential system (1.1) is a weak center if and only if it can be written as

$$\begin{cases} x' = -y(1 + \Lambda) + x\Omega, \\ y' = x(1 + \Lambda) + y\Omega, \end{cases} \tag{1.2}$$

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where $\Lambda = \Lambda(x,y)$ and $\Omega = \Omega(x,y)$ are polynomials of degree at most $m-1$ such that $\Lambda(0,0) = \Omega(0,0) = 0$. The weak centers contain the uniform isochronous centers and the holomorphic isochronous centers [10], they also contain the class of centers studied by Alwash and Lloyd [5], but they do not coincide with all classes of isochronous centers [10].

The class of differential system (1.2) is called the **Λ–Ω system**. The reason of called such system in this way is due to the fact that a subclass of these systems already appears in physics [11].

In [11] the authors put forward such conjectures:

Conjecture 1.1. *The polynomial differential system of degree m*

$$\begin{cases} x' = -y(1+\mu(a_2x - a_1y)) + x((a_1x + a_2y) + \Phi_{m-1}), \\ y' = x(1+\mu(a_2x - a_1y)) + y((a_1x + a_2y) + \Phi_{m-1}), \end{cases} \tag{1.3}$$

where $(\mu + m - 2)(a_1^2 + a_2^2) \neq 0$ and $\Phi_{m-1} = \Phi_{m-1}(x,y)$ is a homogeneous polynomial of degree $m-1$, has a weak center at the origin if and only if system (1.3) after a linear change of variables $(x,y) \to (X,Y)$ is invariant under the transformations $(X,Y,t) \to (-X,Y,-t)$.

Conjecture 1.2. *The polynomial differential system of degree m*

$$\begin{cases} x' = -y(1 + a_1x + a_2y) + x\Phi_{m-1}, \\ y' = x(1 + a_1x + a_2y) + y\Phi_{m-1} \end{cases} \tag{1.4}$$

has a weak center at the origin if and only if system (1.4) after a linear change of variables $(x,y) \to (X,Y)$ is invariant under the transformations $(X,Y,t) \to (-X,Y,-t)$.

The authors of [11] have used Poincaré–Liapunov first integral and Reeb inverse integrating factor to prove that Conjecture 1.1 and Conjecture 1.2 are correct when $m = 2,3,4,5,6$. They remarked that the only difficulty for proving Conjectures 1.1 and 1.2 for the Λ–Ω system of degree m with $m > 6$ is the huge number of computations for obtaining the conditions that characterize the centers.

In this paper we will research the weak center problem of the Λ–Ω system

$$\begin{cases} x' = -y(1+\mu(a_2x - a_1y)) + x(\nu(a_1x + a_2y) + \Lambda_{m-1} + \Omega_{2m-1}), \\ y' = x(1+\mu(a_2x - a_1y)) + y(\nu(a_1x + a_2y) + \Lambda_{m-1} + \Omega_{2m-1}), \end{cases} \tag{1.5}$$

in which $m > 2$ and $(\mu^2 + \nu^2)(\mu + \nu(m-2))(a_1^2 + a_2^2) \neq 0$, $\Lambda_{m-1} = \Lambda_{m-1}(x,y), \Omega_{2m-1} = \Omega_{2m-1}(x,y)$ are respectively homogeneous polynomials of degree $m-1$ and $2m-1$. In the section 3 we will see that by suitable transformation this system can be transformed into

$$\begin{cases} x' = -y(1 - \mu y) + x(\nu x + \Phi_{m-1} + \Psi_{2m-1}), \\ y' = x(1 - \mu y) + y(\nu x + \Phi_{m-1} + \Psi_{2m-1}). \end{cases} \tag{1.6}$$

In the following we use a method different from Llibre [11] and more simply, without huge number of computation, to prove that for system (1.6), under several restrictive conditions, it has a weak center at the origin if and only if

$$\int_0^{2\pi} \sin^i\theta\, \Phi_{m-1}(\cos\theta, \sin\theta) d\theta = 0 \qquad (i = 0,1,2,\ldots,m-1) \tag{1.7}$$

and

$$\int_0^{2\pi} \sin^j\theta\,\Psi_{2m-1}(\cos\theta,\sin\theta)d\theta = 0 \qquad (j = 0,1,2,\dots,2m-1). \tag{1.8}$$

As corollaries, we also show that for arbitrary $m\,(>2)$, Conjecture 1.1 with $\mu = 1$ and Conjecture 1.2 are correct; When $\mu \neq 1$ under several restrictive conditions Conjecture 1.1 is correct, too.

2 Several lemmas

In polar coordinates, the system (1.1) becomes

$$\frac{dr}{d\theta} = \frac{\sum_{k=2}^{m} A_k(\theta)r^k}{1+\sum_{k=2}^{m} B_k(\theta)r^{k-1}},$$

where

$$A_k(\theta) = \cos\theta P_k(\cos\theta,\sin\theta) + \sin\theta Q_k(\cos\theta,\sin\theta),$$
$$B_k(\theta) = \cos\theta Q_k(\cos\theta,\sin\theta) - \sin\theta P_k(\cos\theta,\sin\theta).$$

By [3,4], the **composition condition** is satisfied if there exists a trigonometric polynomial $u(\theta)$ such that

$$A_k(\theta) = u'(\theta)\sum a_{kj}u^j(\theta), \qquad B_k(\theta) = \sum b_{kj}u^j(\theta) \qquad (k = 2,3,\dots,m), \tag{2.1}$$

where a_{kj}, b_{kj} are real numbers.

Lemma 2.1 ([4]). *If the conditions* (2.1) *are satisfied, then the origin point of* (1.1) *is a center and this center is called* **composition center**.

Lemma 2.2 ([14]). *If*

$$P_n = \sum_{i+j=n} p_{ij}\cos^i\theta\sin^j\theta, \qquad p_{ij} \in R,$$

$$\hat{P}_1 = p_{10}\sin\theta - p_{01}\cos\theta, \qquad p_{10}^2 + p_{01}^2 \neq 0$$

and

$$\int_0^{2\pi} \hat{P}_1^k P_n d\theta = 0 \qquad (k = 0,1,2,\dots,n),$$

then

$$P_n = P_1\sum_{i=1}^{n}\lambda_i\hat{P}_1^{i-1},$$

where λ_i $(i = 1,2,\dots,n)$ are real numbers.

Lemma 2.3. *Let $\Phi_{m-1}(x,y) = \sum_{i+j=m-1}\phi_{ij}x^iy^j$ $(\phi_{ij} \in R)$. If relation* (1.7) *holds, then*

$$\Phi_{m-1}(\cos\theta,\sin\theta) = \cos\theta\sum_{i=1}^{m-1}\lambda_i\sin^{i-1}\theta,$$

where λ_i $(i = 1,2,\dots,m-2)$ are real numbers and

$$\lambda_{m-1} = \sum_{i=0}^{[\frac{m-2}{2}]}(-1)^i\phi_{2i+1\;m-2-2i}. \tag{2.2}$$

Proof. In Lemma 2.2, taking $P_1 = \cos\theta, \hat{P}_1 = \sin\theta$ we get

$$\Phi_{m-1}(\cos\theta, \sin\theta) = \cos\theta \sum_{i=1}^{m-1} \lambda_i \sin^{i-1}\theta,$$

thus

$$\Phi_{m-1}(x,y) = \sum_{i+j=2n} \phi_{ij} x^i y^j = x \sum_{i=1}^{n} \lambda_{2i} y^{2i-1} (x^2+y^2)^{n-i}, \qquad m-1 = 2n;$$

$$\Phi_{m-1}(x,y) = \sum_{i+j=2n+1} \phi_{ij} x^i y^j = x \sum_{i=0}^{n} \lambda_{2i+1} y^{2i} (x^2+y^2)^{n-i}, \qquad m-1 = 2n+1.$$

Equating the corresponding coefficients of the same power of x, y, we obtain

$$\lambda_{m-1} = \sum_{i=0}^{n-1} (-1)^i \phi_{2i+1\,2(n-i)-1}, \qquad m-1 = 2n;$$

$$\lambda_{m-1} = \sum_{i=0}^{n} (-1)^i \phi_{2i+1\,2(n-i)}, \qquad m-1 = 2n+1.$$

Therefore, the conclusion of the present lemma is valid. □

By this lemma, it is easy to deduce the following conclusion.

Lemma 2.4. *Let $\Phi_{m-1}(x,y)$ be a homogeneous polynomial of degree $m-1$. Then it can be written as*

$$\Phi_{m-1}(x,y) = x\check{\Phi}(x^2+y^2, y)$$

if and only if the relation (1.7) holds. Where $\check{\Phi}$ is a polynomial on x^2+y^2 and y.

3 Main results

As $a_1^2 + a_2^2 \neq 0$, taking the linear change:

$$X = a_1 x + a_2 y, \qquad Y = -a_2 x + a_1 y, \tag{3.1}$$

the system (1.5) becomes

$$\begin{cases} X' = -Y(1-\mu Y) + X(\nu X + \Phi_{m-1} + \Psi_{2m-1}), \\ Y' = X(1-\mu Y) + Y(\nu X + \Phi_{m-1} + \Psi_{2m-1}), \end{cases}$$

where $\Phi_{m-1} = \Lambda_{m-1}\left(\frac{a_1X - a_2Y}{a_1^2+a_2^2}, \frac{a_1Y+a_2X}{a_1^2+a_2^2}\right)$, $\Psi_{2m-1} = \Omega_{2m-1}\left(\frac{a_1X-a_2Y}{a_1^2+a_2^2}, \frac{a_1Y+a_2X}{a_1^2+a_2^2}\right)$, and they are respectively homogeneous polynomials of degree $m-1$ and $2m-1$.

Obviously, if $\Phi_{m-1} = X\check{\Phi}_{m-1}(X^2+Y^2, Y)$, $\Psi_{2m-1} = X\check{\Psi}_{2m-1}(X^2+Y^2, Y)$, then the Λ–Ω system (1.5) after a linear change of variables $(x,y) \to (X,Y)$ is invariant under the transformations $(X,Y,t) \to (-X,Y,-t)$. By Lemma 2.4, in order to find the necessary and sufficient conditions for (1.5) to have a weak center, only need to seek the conditions under which the identities (1.7) and (1.8) are valid.

Case A. If $\nu \neq 0$, applying the transformation $X = \frac{1}{\nu}x$, $Y = \frac{1}{\nu}y$, we get

$$\begin{cases} x' = -y(1-\hat{\mu}y) + x(x + \hat{\Phi}_{m-1} + \hat{\Psi}_{2m-1}), \\ y' = x(1-\hat{\mu}y) + y(x + \hat{\Phi}_{m-1} + \hat{\Psi}_{2m-1}), \end{cases}$$

where $\hat{\mu} = \frac{\mu}{\nu}$, $\hat{\Phi}_{m-1} = \frac{1}{\nu^{m-1}}\Phi_{m-1}(x,y)$, $\hat{\Psi}_{2m-1} = \frac{1}{\nu^{2m-1}}\Psi_{2m-1}(x,y)$. Thus, if the identities (1.7) and (1.8) are valid, then replacing Φ_{m-1} and Ψ_{2m-1} by $\hat{\Phi}_{m-1}$ and $\hat{\Psi}_{2m-1}$ respectively, these identities also hold.

Case 1. $\nu \neq 0$, $\hat{\mu} = 1$.

Consider the Λ–Ω system

$$\begin{cases} x' = -y(1-y) + x(x + \Phi_{m-1} + \Psi_{2m-1}), \\ y' = x(1-y) + y(x + \Phi_{m-1} + \Psi_{2m-1}). \end{cases} \tag{3.2}$$

Theorem 3.1. *Suppose that*

$$\prod_{m-1 \leq k \leq 2m-3} L_k \neq 0; \tag{3.3}$$

$$L_{2m-2} + \frac{m(2m-1)}{2(m-1)^2}\lambda_{m-1}^2 \neq 0; \tag{3.4}$$

$$L_{2m-1} + \left(2d_1 + e_1\frac{m(m+1)}{(m-1)^2}\right)\lambda_{m-1}^2 \neq 0, \tag{3.5}$$

where λ_{m-1} is expressed by (2.2),

$$\begin{gathered} L_k := e_k + \sum_{i=0}^{k-m+1} \frac{k+1-2i}{m-1+i} d_i e_{k-m+1-i} \lambda_{m-1} \qquad (k = m-1, m, \dots, 2m-1), \\ d_k = (m-1)\frac{(m+k-1)^{k-1}}{k!}, \qquad e_k = (2m-1)\frac{(2m+k-1)^{k-1}}{k!} \\ (k = 1,2,3,\dots), \quad d_0 = 1,\ e_0 = 1. \end{gathered} \tag{3.6}$$

Then the origin point of (3.2) is a center if and only if (1.7) and (1.8) hold.

Moreover, this center is a composition center and weak center.

Proof. In polar coordinates, the system (3.2) can be written as

$$\frac{dr}{d\theta} = \frac{r^2\cos\theta + \Phi_{m-1}r^m + \Psi_{2m-1}r^{2m}}{1 - r\sin\theta},$$

where $\Phi_{m-1} = \Phi_{m-1}(\cos\theta, \sin\theta), \Psi_{2m-1} = \Psi_{2m-1}(\cos\theta, \sin\theta)$.

Taking $\rho = \frac{r}{e^{r\sin\theta}}$, the above equation becomes

$$\frac{d\rho}{d\theta} = \rho^m e^{(m-1)r\sin\theta}\Phi_{m-1} + \rho^{2m}e^{(2m-1)r\sin\theta}\Psi_{2m-1}. \tag{3.7}$$

Now we recall the Langrange–Bürman formula [1]. If real or complex w and z satisfy that $w = \frac{z}{\phi(z)}$, where $\phi(0) = 1$ and $\phi(z)$ is analytic at $z = 0$, then in a neighborhood of $w = 0$, the analytic function $F(z)$ can be expressed as a power series:

$$F(z) = F(0) + \sum_{n=1}^{\infty} \frac{w^n}{n!} \frac{d^{n-1}(F'(x)\phi^n(x))}{dx^{n-1}}\bigg|_{x=0},$$

which is analytic at $w = 0$.

Applying the Langrange–Bürman formula we have

$$e^{(m-1)r\sin\theta} = 1 + (m-1)\sum_{n=1}^{\infty}\frac{(m+n-1)^{n-1}}{n!}\rho^n \sin^n\theta,$$
$$e^{(2m-1)r\sin\theta} = 1 + (2m-1)\sum_{n=1}^{\infty}\frac{(2m+n-1)^{n-1}}{n!}\rho^n \sin^n\theta.$$

Thus the equation (3.7) can be written as

$$\frac{d\rho}{d\theta} = \Phi_{m-1}\sum_{n=0}^{\infty} d_n\rho^{m+n}\sin^n\theta + \Psi_{2m-1}\sum_{n=0}^{\infty} e_n\rho^{2m+n}\sin^n\theta, \tag{3.8}$$

where d_n, e_n $(n = 0, 1, 2, \dots)$ are expressed by (3.6).

Therefore, the system (3.2) has a center at $(0, 0)$ if and only if all the solutions $\rho(\theta)$ of equation (3.8) near $\rho = 0$ are periodic [2].

Let $\rho(\theta, c)$ be the solution of (3.8) such that $\rho(0, c) = c$ $(0 < c \ll 1)$. We write

$$\rho(\theta, c) = c\sum_{n=0}^{\infty} a_n(\theta)c^n,$$

where $a_0(0) = 1$ and $a_n(0) = 0$ for $n \geq 1$. The origin point of (3.2) is a center if and only if $\rho(\theta + 2\pi, c) = \rho(\theta, c)$, i.e., $a_0(2\pi) = 1$, $a_n(2\pi) = 0$ $(n = 1, 2, 3, \dots)$ [5].

Substituting $\rho(\theta, c)$ into (3.8) we obtain

$$c\sum_{i=0}^{\infty} a_i'(\theta)c^n = \Phi_{m-1}\sum_{n=0}^{\infty} d_n\sin^n\theta\left(c\sum_{i=0}^{\infty} a_i(\theta)c^i\right)^{m+n} + \Psi_{2m-1}\sum_{n=0}^{\infty} e_n\sin^n\theta\left(c\sum_{i=0}^{\infty} a_i(\theta)c^i\right)^{2m+n}. \tag{3.9}$$

Equating the corresponding coefficients of c^n of (3.9) yields

$$a_0(\theta) = 1,\ a_i(\theta) = 0, \qquad (i = 1, 2, \dots, m-2).$$

Rewriting

$$\rho = c(1 + c^{m-1}h), \qquad h = \sum_{i=0}^{\infty} h_i(\theta)c^i, h_i(0) = 0, \qquad (i = 0, 1, 2 \dots).$$

Substituting it into (3.8) we get

$$\begin{aligned}\sum_{k=0}^{\infty} h_k'(\theta)c^k &= \Phi_{m-1}\sum_{k=0}^{\infty} d_k c^k\sin^k\theta\sum_{j=0}^{m+k} C_{m+k}^j h^j c^{(m-1)j} \\ &\quad + \Psi_{2m-1}\sum_{k=0}^{\infty} e_k c^{m+k}\sin^k\theta\sum_{j=0}^{2m+k} C_{2m+k}^j h^j c^{(m-1)j}, \quad h_k(0) = 0 \quad (k = 0, 1, 2, \dots).\end{aligned} \tag{3.10}$$

In the following we denote

$$g_k = d_k\overline{\sin^k\theta\,\Phi_{m-1}}, \qquad \beta_k = e_k\overline{\sin^k\theta\,\Psi_{2m-1}}, \qquad (k = 0, 1, 2, \dots), \tag{3.11}$$

where

$$\overline{\sin^k\theta\,\Phi_{m-1}} = \int_0^\theta \sin^k\theta\Phi_{m-1}d\theta, \qquad \overline{\sin^k\theta\,\Psi_{2m-1}} = \int_0^\theta \sin^k\theta\Psi_{2m-1}d\theta.$$

Equating the corresponding coefficients of c^k of the equation (3.10) we obtain

$$h_k' = d_k \sin^k\theta\,\Phi_{m-1}, \qquad h_k(0) = 0 \qquad (k = 0, 1, 2, \ldots, m-2),$$

$$h_{m-1}' = \Phi_{m-1} C_m^1 h_0 + \Phi_{m-1} d_{m-1} \sin^{m-1}\theta, \qquad h_{m-1}(0) = 0,$$

solving these equations we get

$$h_k(\theta) = g_k, \qquad (k = 0, 1, 2, \ldots, m-2),$$

$$h_{m-1}(\theta) = g_{m-1} + \alpha_0, \qquad \alpha_0 = \frac{m}{2}\bar{\Phi}_{m-1}^2.$$

As $d_k \neq 0$ $(k = 0, 1, 2 \ldots.)$, from $h_k(2\pi) = 0$ $(k = 0, 1, 2, \ldots, m-1)$ follow that

$$\int_0^{2\pi} \sin^k\theta\,\Phi_{m-1} d\theta = 0 \qquad (k = 0, 1, 2, \ldots, m-1),$$

i.e., the condition (1.7) is a necessary condition for $\rho = 0$ to be a center. By Lemma 2.3 which implies that

$$\Phi_{m-1} = \cos\theta \sum_{k=1}^{m-1} \lambda_k \sin^{k-1}\theta, \qquad \bar{\Phi}_{m-1} = \int_0^{\theta} \Phi_{m-1} d\theta = \sum_{k=1}^{m-1} \frac{\lambda_k}{k} \sin^k\theta, \tag{3.12}$$

where λ_k $(k = 1, 2, \ldots, m-1)$ are real numbers and λ_{m-1} is expressed by (2.2).

Applying (3.12) we get

$$\int_0^{2\pi} \sin^k\theta\,\Phi_{m-1} d\theta = 0, \qquad g_k = g_k(\sin\theta), \qquad g_k(2\pi) = 0 \qquad (k = 0, 1, 2, \ldots.). \tag{3.13}$$

Equating the corresponding coefficients of c^{m-1+k} of the equation (3.10) we obtain

$$h_{m-1+k}' = \Phi_{m-1} \sum_{i=0}^{k} d_i \sin^i\theta C_{m+i}^1 h_{k-i} + d_{m-1+k} \sin^{m-1+k}\theta\Phi_{m-1} + e_{k-1} \sin^{k-1}\theta\Psi_{2m-1},$$

$$h_{m-1+k}(0) = 0 \qquad (k = 1, 2, \ldots, m-2),$$

solving these equations we get

$$h_{m-1+k}(\theta) = g_{m-1+k} + \alpha_k + \beta_{k-1} \qquad (k = 1, 2, \ldots, m-2),$$

where g_{m-1+k} and β_{k-1} are expressed by (3.11), α_k is the solution of the following equation

$$\alpha_k' = \Phi_{m-1} \sum_{i=0}^{k} d_i d_{k-i} \sin^i\theta C_{m+i}^1 \overline{\sin^{k-i}\theta\Phi_{m-1}},\ \alpha_k(0) = 0. \tag{3.14}$$

By this we get: when $k = 2n$,

$$\begin{aligned}\alpha_k = \sum_{i=0}^{n-1} d_i d_{k-i} \Big(& C_{m+i}^1 \overline{\sin^i\theta\Phi_{m-1}}\,\overline{\sin^{k-i}\theta\Phi_{m-1}} + (C_{m+k-i}^1 - C_{m+i}^1)\overline{\overline{\sin^i\theta\Phi_{m-1}}\sin^{k-i}\theta\Phi_{m-1}} \Big) \\ & + \frac{1}{2} d_n^2 C_{m+n}^1 \overline{\sin^n\theta\Phi_{m-1}}^2;\end{aligned} \tag{3.15}$$

when $k = 2n+1$,

$$\alpha_k = \sum_{i=0}^{n} d_i d_{k-i} \left(C^1_{m+i} \overline{\sin^i \theta \Phi_{m-1}}\, \overline{\sin^{k-i} \theta \Phi_{m-1}} + (C^1_{m+k-i} - C^1_{m+i}) \overline{\overline{\sin^i \theta \Phi_{m-1}} \sin^{k-i} \theta \Phi_{m-1}} \right). \tag{3.16}$$

By (3.13) we see that $\alpha_k = \alpha_k(\sin\theta)$, $\alpha_k(2\pi) = 0$ $(k = 0,1,2,3,\dots)$. Then from

$$h_{m-1+k}(2\pi) = g_{m-1+k}(2\pi) + \alpha_k(2\pi) + \beta_{k-1}(2\pi) = 0 \qquad (k = 1,2,\dots,m-2)$$

imply that

$$\beta_k(2\pi) = 0 \qquad (k = 0,1,2,\dots,m-3),$$

in view of $e_k \neq 0$ $(k = 0,1,2\dots.)$, so

$$\int_0^{2\pi} \sin^k \theta \Psi_{2m-1} d\theta = 0 \qquad (k = 0,1,2\dots,m-3). \tag{3.17}$$

Equating the corresponding coefficients of c^{2m-2} of the equation (3.10) we get

$$h'_{2m-2} = \Phi_{m-1} \sum_{i=0}^{m-1} d_i \sin^i \theta C^1_{m+i} h_{m-1-i} + \Phi_{m-1}(C^1_m \alpha_0 + C^2_m h_0^2)$$
$$+ d_{2m-2} \sin^{2m-2} \theta \Phi_{m-1} + e_{m-2} \sin^{m-2} \theta \Psi_{2m-1}, \qquad h_{2m-2}(0) = 0,$$

by this we get

$$h_{2m-2}(\theta) = g_{2m-2} + \alpha_{m-1} + \beta_{m-2} + \delta_0,$$

where

$$\delta_0 = \frac{m(2m-1)}{6} \bar{\Phi}^3_{m-1}.$$

α_{m-1} is a solution of (3.14) with $k = m-1$ and $\alpha_{m-1} = \alpha_{m-1}(\sin\theta)$. Thus, using (3.12) and (3.13), from $h_{2m-2}(2\pi) = 0$ follows that $\beta_{m-2}(2\pi) = 0$, i.e.,

$$\int_0^{2\pi} \sin^{m-2} \theta\, \Psi_{2m-1} d\theta = 0. \tag{3.18}$$

Equating the corresponding coefficients of c^{2m-2+k} of the equation (3.10) we obtain

$$h'_{2m-2+k} = \Phi_{m-1} \sum_{i=0}^{m-1+k} d_i \sin^i \theta C^1_{m+i} h_{m-1+k-i} + \Phi_{m-1} \sum_{i=0}^{k} d_i \sin^i \theta C^2_{m+i} \sum_{j+l=k-i} h_j h_l$$
$$+ d_{2m-2+k} \sin^{2m-2+k} \theta \Phi_{m-1} + e_{m-2+k} \sin^{m-2+k} \theta \Psi_{2m-1}$$
$$+ \Psi_{2m-1} \sum_{i=0}^{k-1} e_i \sin^i \theta C^1_{2m+i} h_{k-1-i},$$
$$h_{2m-2+k}(0) = 0 \qquad (k = 1,2,\dots,m-2),$$

solving these equations we get

$$h_{2m-2+k} = g_{2m-2+k} + \alpha_{k+m-1} + \beta_{k+m-2} + \delta_k + \varepsilon_{k-1} \qquad (k = 1,2,\dots,m-2),$$

where α_{k+m-1} is a solution of (3.14), δ_k and ε_{k-1} are the solutions of the following equations, respectively,

$$\delta'_k = \Phi_{m-1} \left(\sum_{i=0}^{k} d_i \sin^i \theta C^1_{m+i} \alpha_{k-i} + \sum_{i=0}^{k} C^2_{m+i} d_i \sin^i \theta \sum_{j+l=k-i} h_l h_j \right),$$

$$\varepsilon'_{k-1} = \Phi_{m-1}\sum_{i=0}^{k-1} C^1_{m+i} d_i \sin^i\theta\beta_{k-1-i} + \Psi_{2m-1}\sum_{i=0}^{k-1} e_i \sin^i\theta C^1_{2m+i} g_{k-1-i}. \tag{3.19}$$

By (3.12) and (3.13) we see that $\delta_k = \delta_k(\sin\theta)$ and $\delta_k(2\pi) = 0$.

Solving (3.19) we get

$$\varepsilon_{k-1} = \sum_{i=0}^{k-1} d_i e_{k-1-i}\Big(C^1_{m+i}\overline{\sin^i\theta\Phi_{m-1}}\,\overline{\sin^{k-1-i}\theta\Psi_{2m-1}} + (C^1_{2m+k-1-i} - C^1_{m+i})\overline{\overline{\sin^i\theta\Phi_{m-1}}\sin^{k-1-i}\theta\Psi_{2m-1}}\Big). \tag{3.20}$$

Therefore, from $h_{2m-2+k}(2\pi) = 0$ $(k = 1, 2, \ldots, m-2)$ implies that

$$\beta_{k+m-2}(2\pi) + \varepsilon_{k-1}(2\pi) = 0 \qquad (k = 1, 2, \ldots, m-2),$$

simplifying this relation by using (3.17) and (3.18), (3.20) and (3.12) we get

$$\left(e_k + \sum_{i=0}^{k-m+1}\frac{k+1-2i}{m-1+i} d_i e_{k-m+1-i}\lambda_{m-1}\right)\int_0^{2\pi}\sin^k\theta\Psi_{2m-1}d\theta = L_k\int_0^{2\pi}\sin^k\theta\Psi_{2m-1}d\theta = 0,$$

$$(k = m-1, m, \ldots, 2m-4).$$

By the hypothesis (3.3), $L_k \neq 0$, so

$$\int_0^{2\pi}\sin^k\theta\Psi_{2m-1}d\theta = 0 \qquad (k = m-1, m, \ldots, 2m-4). \tag{3.21}$$

Equating the corresponding coefficients of c^{3m-3} of the equation (3.10) we obtain

$$h_{3m-3} = g_{3m-3} + \alpha_{2m-2} + \beta_{2m-3} + \delta_{m-1} + \varepsilon_{m-2},$$

where α_{2m-2} is a solution of (3.14) with $k = 2m-2$ and $\alpha_{2m-2} = \alpha_{2m-2}(\sin\theta)$, ε_{m-2} is expressed by (3.20) with $k = m-1$, δ_{m-1} is a solution of the following equation

$$\delta'_{m-1} = \Phi_{m-1}\left(\sum_{i=0}^{m-1} d_i\sin^i\theta C^1_{m+i}\alpha_{m-1-i} + \sum_{i=0}^{m-1} C^2_{m+i} d_i\sin^i\theta\sum_{j+l=m-1-i} g_l g_j + C^1_m\delta_0 + 2C^2_m h_0\alpha_0 + C^3_m h_0^3\right).$$

By (3.12) and (3.13) we see that $g_k = g_k(\sin\theta)$ $(k = 0, 1, 2, \ldots, m-1)$ and $\delta_{m-1} = \delta_{m-1}(\sin\theta)$. Thus, from $h_{3m-3}(2\pi) = 0$ follows that

$$\beta_{2m-3}(2\pi) + \varepsilon_{m-2}(2\pi) = 0,$$

simplifying this relation by using (3.17) and (3.18) and (3.21), (3.20) and (3.12) we get

$$L_{2m-3}\int_0^{2\pi}\sin^{2m-3}\theta\Psi_{2m-1}d\theta = 0,$$

as $L_{2m-3} \neq 0$,

$$\int_0^{2\pi}\sin^{2m-3}\theta\Psi_{2m-1}d\theta = 0. \tag{3.22}$$

Equating the corresponding coefficients of c^{3m-2} of the equation (3.10) we obtain

$$h_{3m-2}(\theta) = g_{3m-2} + \alpha_{2m-1} + \beta_{2m-2} + \delta_m + \varepsilon_{m-1} + \eta_0,$$

where α_{2m-1} is a solution of (3.14) with $k = 2m-1$, ε_{m-1} is a solution of (3.19) with $k = m$, δ_m is a solution of the following equation

$$\delta_m' = \Phi_{m-1}\left(\sum_{i=0}^{m} d_i \sin^i \theta C_{m+i}^1 \alpha_{m-i} + \sum_{i=0}^{m} d_i \sin^i \theta C_{m+i}^2 \sum_{j+l=m-i} g_j g_l\right) \\ + \Phi_{m-1}(C_m^1 \delta_1 + d_1 \sin\theta C_{m+1}^1 \delta_0 + C_m^2(2h_0\alpha_1 + 2h_1\alpha_0) + 3C_m^3 h_0^2 h_1 + d_1 \sin\theta C_{m+1}^3 h_0^3),$$

by (3.12) and (3.13) we see that $\delta_m = \delta_m(\sin\theta)$. η_0 is a solution of equation

$$\eta_0' = \Phi_{m-1}(C_m^1 \varepsilon_0 + 2C_m^2 h_0 \beta_0) + \Psi_{2m-1}(C_{2m}^2 h_0^2 + C_{2m}^1 \alpha_0),$$

solving this we get

$$\eta_0 = \frac{1}{2} m(m-1)\bar{\Phi}_{m-1}^2 \bar{\Psi}_{2m-1} + m^2 \bar{\Phi}_{m-1} \overline{\bar{\Phi}_{m-1}\Psi_{2m-1}} + \frac{2m^2-m}{2}\overline{\bar{\Phi}_{m-1}^2 \Psi_{2m-1}}. \tag{3.23}$$

Thus, from $h_{3m-2}(2\pi) = 0$ follows that

$$\beta_{2m-2}(2\pi) + \varepsilon_{m-1}(2\pi) + \eta_0(2\pi) = 0,$$

calculating this relation by using (3.17) and (3.18) and (3.20)–(3.23) and (3.12) we get

$$\left(L_{2m-2} + \frac{2m^2-m}{2(m-1)^2}\lambda_{m-1}^2\right)\int_0^{2\pi} \sin^{2m-2} \Psi_{2m-1} d\theta = 0,$$

in view of the condition (3.4) we have

$$\int_0^{2\pi} \sin^{2m-2}\theta \Psi_{2m-1} d\theta = 0. \tag{3.24}$$

Equating the corresponding coefficients of c^{3m-1} of the equation (3.10) we obtain

$$h_{3m-1}(\theta) = g_{3m-1} + \alpha_{2m} + \beta_{2m-1} + \delta_{m+1} + \varepsilon_m + \eta_1, \tag{3.25}$$

where g_{3m-1}, α_{2m}, β_{2m-1} and ε_m are the same as above, δ_{m+1} is a solution of the equation

$$\delta_{m+1}' = \Phi_{m-1}\Bigg(\sum_{i=0}^{m+1} d_i \sin^i \theta C_{m+i}^1 \alpha_{m+1-i} + \sum_{i=0}^{m+1} d_i \sin^i \theta C_{m+i}^2 \sum_{j+l=m+1-i} g_j g_l \\ + \sum_{i=0}^{2} d_i \sin^i \theta \Big(C_{m+i}^1 \delta_{2-i} + C_{m+i}^2 \sum_{l+j=2-i} g_j \alpha_i + C_{m+i}^3 \sum_{l+j+k=2-i} g_l g_j g_k\Big)\Bigg).$$

By (3.12) and (3.13) and $\alpha_k = \alpha_k(\sin\theta)$ $(k = 0, 1, 2\dots)$, $\delta_i = \delta_i(\sin\theta)$ $(i = 0, 1, 2)$, which imply that $\delta_{m+1} = \delta_{m+1}(\sin\theta)$. η_1 is a solution of the following equation

$$\eta_1' = \Phi_{m-1}(C_m^1 \varepsilon_1 + d_1 \sin\theta C_{m+1}^1 \varepsilon_0 + C_m^2(2h_0\beta_1 + 2h_1\beta_0) + d_1 \sin\theta C_{m+1}^2 2h_0\beta_0) \\ + \Psi_{2m+1}(C_{2m}^2 2h_0 h_1 + e_1 \sin\theta C_{2m+1}^2 h_0^2 + C_{2m}^1(\alpha_1 + \beta_0) + e_1 \sin\theta C_{2m+1}^1 \alpha_0),$$

solving this equation we get

$$\begin{aligned}\eta_1 = m\bar{\Psi}^2_{2m-1} + e_1\Big(\Big(m^2 - \frac{m}{2}\Big)\bar{\Phi}^2_{m-1}\overline{\sin\theta\Psi_{2m-1}} + m(m+1)\bar{\Phi}_{m-1}\overline{\bar{\Phi}_{m-1}\sin\theta\Psi_{2m-1}}\\ + m(m+1)\overline{\sin\theta\Psi_{2m-1}\bar{\Phi}^2_{m-1}}\Big)\\ + d_1\Big(2m^2\bar{\Phi}_{m-1}\bar{\Psi}_{2m-1}\overline{\sin\theta\Phi_{m-1}} + m(m-1)\bar{\Phi}_{m-1}\overline{\overline{\sin\theta\Phi_{m-1}}\Psi_{2m-1}}\\ + m(m+1)\overline{\sin\theta\Phi_{m-1}}\,\overline{\bar{\Phi}_{m-1}\Psi_{2m-1}}\\ + 2m\bar{\Psi}_{2m-1}\overline{\bar{\Phi}_{m-1}\Psi_{m-1}\sin\theta} + 2(m^2-m)\overline{\bar{\Phi}_{m-1}\Psi_{2m-1}\overline{\sin\theta\Phi_{m-1}}}\Big).\end{aligned} \tag{3.26}$$

By (3.25) we see that if $h_{3m-1}(2\pi) = 0$, then

$$\beta_{2m-1}(2\pi) + \varepsilon_m(2\pi) + \eta_1(2\pi) = 0,$$

simplifying this equation by using (3.17) and (3.18) and (3.20)–(3.24), (3.26) and (3.12) we get

$$\left(L_{2m-1} + \left(2d_1 + e_1\frac{m(m+1)}{(m-1)^2}\right)\lambda^2_{m-1}\right)\int_0^{2\pi}\sin^{2m-1}\theta\Psi_{2m-1}d\theta = 0,$$

by the hypothesis (3.5) we obtain

$$\int_0^{2\pi}\sin^{2m-1}\theta\Psi_{2m-1}d\theta = 0.$$

In summary, under the conditions (3.3)–(3.5), the (1.7) and (1.8) are the necessary conditions for $\rho = 0$ to be a center of (3.2). Therefore, the necessity has been proved. On the other hand, by Lemma 2.1 and Lemma 2.3, if the conditions (1.7) and (1.8) are satisfied, then $\rho = 0$ is a center of equation (3.2), this means that the sufficiency is proved. By Lemma 2.3 this center is a composition center, by Lemma 2.4 this center is a weak center. □

Corollary 3.2. *For arbitrary $m\ (>2)$, if $\mu = 1$, then the origin point of (1.3) is a center if and only if (1.7) is satisfied.*

Proof. Under the linear change of variables (3.1) the system (1.3) becomes

$$\begin{cases} x' = -y(1-y) + x(x + \Phi_{m-1}),\\ y' = x(1-y) + y(x+\Phi_{m-1}),\end{cases}$$

which in polar coordinates becomes

$$\frac{dr}{d\theta} = \frac{r^2\cos\theta + \Phi_{m-1}r^m}{1 - r\sin\theta}.$$

Taking $\rho = \frac{r}{e^{r\sin\theta}}$ we get

$$\frac{d\rho}{d\theta} = \Phi_{m-1}\rho^m\sum_{n=0}^{\infty} d_n\rho^n\sin^n\theta,$$

where $d_0 = 1$, $d_n = \frac{1}{n!}(m-1)(m+n-1)^{n-1}$, $(n = 1,2,3,\dots)$. Similar to Theorem 3.1, it can be deduced that the solution ρ of this equation such that $\rho(0) = c\ (0 < |c| \ll 1)$ is

$$\rho = c + c^m\sum_{k=0}^{m-2} c^k d_k\overline{\sin^k\theta\Phi_{m-1}} + c^{2m-1}\left(d_{m-1}\overline{\sin^{m-1}\theta\Phi_{m-1}} + \frac{m}{2}\bar{\Phi}^2_{m-1}\right) + o(c^{2m-1}).$$

As $d_n \neq 0\ (n = 0,1,2\dots)$, from $\rho(2\pi) = c$ it follows that the condition (1.7) is satisfied. Using Lemma 2.3 and Lemma 2.4, the conclusion of the present corollary is valid. □

Remark 3.3. By Corollary 3.2, when $\mu = 1$, Conjecture 1.1 is correct for arbitrary $m > 2$.

Case 2. $\nu \neq 0$, $\hat{\mu} \neq 1$.

Consider Λ–Ω system

$$\begin{cases} x' = -y(1-\hat{\mu}y) + x(x + \Phi_{m-1} + \Psi_{2m-1}), \\ y' = x(1-\hat{\mu}y) + y(x + \Phi_{m-1} + \Psi_{2m-1}). \end{cases} \tag{3.27}$$

Theorem 3.4. *Suppose that*

$$\prod_{1\le n\le m-1} \tilde{d}_n \neq 0; \qquad \prod_{m-1\le k\le 2m-3} \tilde{L}_k \neq 0;$$

$$\tilde{L}_{2m-2} + \frac{m(2m-1)}{2(m-1)^2}\lambda_{m-1}^2 \neq 0;$$

$$\tilde{L}_{2m-1} + \left(2\tilde{d}_1 + \tilde{e}_1\frac{m(m+1)}{(m-1)^2}\right)\lambda_{m-1}^2 \neq 0,$$

where λ_{m-1} *is expressed by* (2.2),

$$\tilde{L}_k := \tilde{e}_k + \sum_{i=0}^{k-m+1} \frac{k+1-2i}{m-1+i}\tilde{d}_i\tilde{e}_{k-m+1-i}\lambda_{m-1}, \qquad (k = m-1, m, \ldots, 2m-1),$$

$$\tilde{d}_n = \frac{\tilde{d}_1}{n!}\prod_{0\le r\le n-2}(\sigma - r(1-\hat{\mu}))$$
$$(n = 2,3,\ldots),\ \tilde{d}_0 = 1,\ \tilde{d}_1 = m + \hat{\mu} - 2,\ \sigma = n + m + 2\hat{\mu} - 3; \tag{3.28}$$

$$\tilde{e}_n = \frac{\tilde{e}_1}{n!}\prod_{0\le r\le n-2}(\epsilon - r(1-\hat{\mu}))$$
$$(n = 2,3,\ldots),\ \tilde{e}_0 = 1,\ \tilde{e}_1 = 2m + \hat{\mu} - 2,\ \epsilon = n + 2m + 2\hat{\mu} - 3. \tag{3.29}$$

Then the origin point of (3.27) *is a center if and only if* (1.7) *and* (1.8) *hold.*

Moreover, this center is a composition center and weak center.

Proof. In polar coordinates, the system (3.27) becomes

$$\frac{dr}{d\theta} = \frac{r^2\cos\theta + \Phi_{m-1}r^m + \Psi_{2m-1}r^{2m}}{1 - \hat{\mu}r\sin\theta}, \tag{3.30}$$

where $\Phi_{m-1} = \Phi_{m-1}(\cos\theta, \sin\theta), \Psi_{2m-1} = \Psi_{2m-1}(\cos\theta, \sin\theta)$.

Taking

$$\rho = \frac{r}{(1 + (1-\hat{\mu})r\sin\theta)^{\frac{1}{1-\hat{\mu}}}},$$

the equation (3.30) can be written as

$$\frac{d\rho}{d\theta} = \rho^m\Phi_{m-1}(1 + (1-\hat{\mu})r\sin\theta)^{\frac{m+\hat{\mu}-2}{1-\hat{\mu}}} + \rho^{2m}\Psi_{2m-1}(1 + (1-\hat{\mu})r\sin\theta)^{\frac{2m+\hat{\mu}-2}{1-\hat{\mu}}}. \tag{3.31}$$

Applying the Langrange–Bürman formula we have

$$(1+(1-\hat{\mu})r\sin\theta)^{\frac{m+\hat{\mu}-2}{1-\hat{\mu}}} = \sum_{n=0}^{\infty} \tilde{d}_n\rho^n \sin^n\theta;$$
$$(1+(1-\hat{\mu})r\sin\theta)^{\frac{2m+\hat{\mu}-2}{1-\hat{\mu}}} = \sum_{n=0}^{\infty} \tilde{e}_n\rho^n \sin^n\theta,$$

where $\tilde{d}_n$, $\tilde{e}_n$ are expressed by (3.28), (3.29), respectively.

Substituting them into (3.31) we get

$$\frac{d\rho}{d\theta} = \Phi_{m-1}\rho^m \sum_{n=0}^{\infty} \tilde{d}_n\rho^n \sin^n\theta + \rho^{2m}\Psi_{2m-1}\sum_{n=0}^{\infty} \tilde{e}_n\rho^n \sin^n\theta. \tag{3.32}$$

Comparing the equations (3.8) and (3.32), we see that they have the same form, only with different coefficients. Similar to Theorem 3.1, the present theorem can be derived. □

Remark 3.5. When $\hat{\mu}=0$, from Theorem 3.4 implies the Theorem 3.1 of [15].

Corollary 3.6. *If $\mu \neq 1$ and $\hat{d}_n = \tilde{d}_n|_{\hat{\mu}=\mu} \neq 0$ $(n=1,2,\dots,m-1)$ $(m>2)$, where $\tilde{d}_n$ $(n=1,2,\dots,m-1)$ is expressed by (3.28). Then the origin point of (1.3) is a center if and only if (1.7) is satisfied.*

Proof. Similar to Theorem 3.4, when $\Psi_{2m-1}=0$, the equation (1.3) can be transformed as following

$$\frac{d\rho}{d\theta} = \Phi_{m-1}\rho^m \sum_{n=0}^{\infty} \hat{d}_n\rho^n \sin^n\theta. \tag{3.33}$$

Similar to Theorem 3.1, we get that the solution of (3.33) such that $\rho(0)=c$ $(0<|c|\ll 1)$ is

$$\rho = c + c^m \sum_{k=0}^{m-2} c^k\hat{d}_k\overline{\sin^k\theta\Phi_{m-1}} + c^{2m-1}\left(\hat{d}_{m-1}\overline{\sin^{m-1}\theta\Phi_{m-1}} + \frac{m}{2}\bar{\Phi}^2_{m-1}\right) + o(c^{2m-1}).$$

As $\hat{d}_n = \tilde{d}_n|_{\hat{\mu}=\mu} \neq 0\,(n=1,2\dots,m-1)$, $\tilde{d}_0=1$, from $\rho(2\pi)=c$ follows that the condition (1.7) is satisfied. Using Lemma 2.4, the conclusion of the present corollary is valid. □

Remark 3.7. By Corollary 3.6, if $\mu\neq 1$, Conjecture 1.1 is valid when $\prod_{1\le n\le m-1}\hat{d}_n \neq 0$, $(m>2)$.

Case B. $\nu=0$, $\mu\neq 0$.

Consider Λ–Ω system

$$\begin{cases} x' = -y(1-\mu y) + x(\Phi_{m-1}+\Psi_{2m-1}), \\ y' = x(1-\mu y) + y(\Phi_{m-1}+\Psi_{2m-1}). \end{cases} \tag{3.34}$$

Theorem 3.8. *Suppose that*

$$\prod_{m-1\le k\le 2m-3} \hat{L}_k \neq 0;$$
$$\hat{L}_{2m-2} + \frac{m(2m-1)}{2(m-1)^2}\lambda^2_{m-1} \neq 0;$$
$$\hat{L}_{2m-1} + \mu\left(2+\frac{m(m+1)}{(m-1)^2}\right)\lambda^2_{m-1} \neq 0,$$

where λ_{m-1} is expressed by (2.2), $\hat{L}_k := \mu^k + \mu^{1-m+k}\sum_{i=0}^{k-m+1}\frac{k+1-2i}{m-1+i}\lambda_{m-1}$, $(k=m-1,m,\dots,2m-1)$. Then the origin point of (3.34) is a center if and only if (1.7) and (1.8) hold.

Moreover, this center is a composition center and weak center.

Proof. In polar coordinates, the system (3.34) becomes

$$\frac{dr}{d\theta} = \Phi_{m-1} \sum_{n=0}^{\infty} \mu^n r^{m+n} \sin^n \theta + \Psi_{2m-1} \sum_{n=0}^{\infty} \mu^n r^{2m+n} \sin^n \theta, \tag{3.35}$$

where $\Phi_{m-1} = \Phi_{m-1}(\cos\theta, \sin\theta), \Psi_{2m-1} = \Psi_{2m-1}(\cos\theta, \sin\theta)$.

Obviously, the equation (3.35) has the same form as (3.8), in Theorem 3.1 taking $d_k = e_k = \mu^k$ $(k = 0, 1, 2, \dots)$, the present theorem can be derived directly. □

Corollary 3.9. *For arbitrary $m > 2$, the origin point of (1.4) is a center if and only if (1.7) is satisfied.*

Proof. Under the linear changes of variables (3.1) the system (1.4) becomes

$$\begin{cases} x' = -y(1-y) + x\Phi_{m-1}, \\ y' = x(1-y) + y\Phi_{m-1}. \end{cases} \tag{3.36}$$

In polar coordinates (3.36) can be written as

$$\frac{dr}{d\theta} = \Phi_{m-1} \sum_{n=0}^{\infty} r^{m+n} \sin^n \theta. \tag{3.37}$$

Similar to Theorem 3.1, we get that the solution of (3.37) such that $r(0) = c$ $(0 < |c| \ll 1)$ is

$$r = c + c^m \sum_{i=0}^{m-2} \overline{\sin^k \theta \Phi_{m-1}} + c^{2m-1} \left(\overline{\sin^{m-1} \theta \Phi_{m-1}} + \frac{m}{2} \bar{\Phi}_{m-1}^2 \right) + o(c^{2m-1}).$$

Obviously, from $r(2\pi) = c$ follows that the condition (1.7) is satisfied. Using Lemma 2.4 the conclusion of the present corollary is correct. □

Remark 3.10. By Corollary 3.9, Conjecture 1.2 is valid for $m > 2$.

Remark 3.11. In the case of $\mu = \nu = 0$, $m = 2$ the center problem of system (1.5) has been discussed by [14].

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On a differential equation involving a new kind of variable exponents

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Abstract. In this paper, we are concerned with some new first order differential equation defined on the whole real axis $\mathbb{R}$. The principal part of the equation involves an operator with variable exponent p depending on the variable $x \in \mathbb{R}$ through the unknown solution while the nonlinear part involves the classical variable exponent $p(x)$. Such kind of situation is very related to the presence of the variable exponent and has not been treated before. Our existence result of nontrivial solution cannot be reached using standard variational or topological methods of nonlinear analysis and some sophisticated arguments have to be employed.

Keywords: $p(u)$-Laplacian, variable exponents, Schauder's fixed point theorem, approximation scheme, weighted Sobolev space, existence result.

2020 Mathematics Subject Classification: 26A24, 34A34, 47H05, 47H10.

1 Introduction and Statement of Main results

Nonlinear partial differential equations involving variable exponents have many applications in physics. In fact, such equations are used as models to describe many phenomena arising in applied sciences. For instance, we can mention the study of materials with strong inhomogeneities such as electrorheological fluids or thermo-rheological, image restoration, phenomenon of elasticity or the continuum mechanics. See [5,10,15,16,22].

Actually, the observation of the image restoration process through some numerical techniques has proved that considering the case of variable exponents depending on the solution u (or its derivatives) reduces the noise of the restored image u. See [8,9,17]. The same situation is observed when treating the problem of thermistor which describes the electric current in a conductor that may change its properties in dependence of temperature (see [4]).

When we try to deal with a problem involving an exponent depending on the solution, we are quickly faced with many obstacles which are essentially related to the theoretical well-posedness of the problem itself. Indeed, such a problem is not standard because its weak

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formulation cannot be written as an equation in terms of duality in a fixed Banach space. This is why, in the mathematical literature, one can find only few works devoted to the study of elliptic and parabolic equations involving an exponent of the type $p(u)$ with local and nonlocal dependence of p on u. The first one is due to B. Andreianov, M. Bendahmane and S. Ouaro who have considered in [1] the problem

$$\begin{cases} u - \operatorname{div}\left(|\nabla u|^{p(u)-2}\nabla u\right) = f, & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is some bounded domain of $\mathbb{R}^N$, $N \geq 2$, $f \in L^1(\Omega)$ and $p : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous such that $p^- = \inf_{s\in\mathbb{R}} p(s) > N$. Under the key restriction $p^- > N$, the authors proved that the problem (1.1) is well-posed in $L^1(\Omega)$. By this way, using some approximation method, they can establish the existence of so-called narrow and broad weak solution (definitions related to the fact that the source f is integrable). The version of the problem (1.1) with homogeneous Neumann boundary conditions has been treated in [14].

Recently, M. Chipot and H. B. de Oliveira proposed in [11] a new simple approach to deal with a problem very similar to (1.1). More precisely, M. Chipot and H. B. de Oliveira studied the problem

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p(u)-2}\nabla u\right) = f, & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 2$ with smooth boundary, $p : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function such that $p^- > N$, and $f \in W^{-1,(p^-)'}(\Omega)$. The approach in [11] is mainly based on a perturbation of the problem (1.2) and the use of Schauder's fixed point theorem to solve the approximated problem. Finally, a process of passage to the limit in the spirit of [23] is carried out to prove the existence of a weak solution u of the problem (1.2) in the sense that $u \in W_0^{1,p(u)}(\Omega)$ and satisfies

$$\int_\Omega |\nabla u|^{p(u)-2}\nabla u \nabla v dx = \langle f, v\rangle, \qquad \forall\, v \in W_0^{1,p(u)}(\Omega).$$

The nonlocal version of (1.2) has been also considered in [11]. More precisely, the authors studied the problem

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p(b(u))-2}\nabla u\right) = f, & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where p is merely bounded continuous and satisfies that $1 < p^- < p(s)$, $\forall\, s \in \mathbb{R}$, and $b : W_0^{1,p^-}(\Omega) \to \mathbb{R}$ sends bounded sets of $W_0^{1,p^-}(\Omega)$ into bounded sets of $\mathbb{R}$. Using the Brower's fixed point theorem applied to some compact interval of $\mathbb{R}$, M. Chipot and H. B. de Oliveira proved that (1.3) has at least one weak solution u in the sense that $u \in W_0^{1,p(b(u))}(\Omega)$ and satisfies

$$\int_\Omega |\nabla u|^{p(b(u))-2}\nabla u \nabla v dx = \langle f, v\rangle, \qquad \forall\, v \in W_0^{1,p(b(u))}(\Omega).$$

This work has been completed in [20] where the authors treated the case when $f \in L^1(\Omega)$ for which they prove the existence of an entropy solution. The work of M. Chipot and H. B. de Oliveira has given a new impulse to the study of problems involving exponents depending on the unknown solution. In [2], S. Antontsev and S. Shmarev studied the homogeneous Dirichlet problem for the parabolic equation

$$u_t - \operatorname{div}\left(|\nabla u|^{p[u]-2}\nabla u\right) = f, \quad \text{in } Q_T = \Omega\times]0,T[,$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a smooth domain, $p[u] = p(l(u))$, p is a given differentiable function such that $\frac{2N}{N+2} < p^- \leq p^+ < 2$, and $\sup_{s\in\mathbb{R}} |p'(s)| < +\infty$; $l(u) = \int_\Omega |u(x,t)|^\alpha\, dx$, $\alpha \in [1,2]$, and $f \in L^{(p^-)'}(Q_T)$. A result of existence and uniqueness of a solution $u \in C^0\left([0,T]; L^2(\Omega)\right)$, $|\nabla u|^{p[u]} \in L^\infty\left(0,T; L^1(\Omega)\right)$, $u_t \in L^2(Q_T)$ has been proved. This result has been extended in [3] to the case when the source f is replaced by the nonlinear term $f((x,t),u,l(u))$, and in [4] where the authors (together with I. Kuznetsov) treated the case when the exponent p is depending on the gradient of u, i.e. when $p[u]$ is replaced by $p[\nabla u] = p(l(|\nabla u|))$. The case of unbounded domain has been considered in [7] where S. Aouaoui and A. E. Bahrouni studied the equation

$$-\operatorname{div}(w_1(x)\,|\nabla u|^{p(u)-2}\,\nabla u) + w_0(x)\,|u|^{p(u)-2}\,u = f(x,u), \qquad x \in \mathbb{R}^N,\ N \geq 2,$$

where $p : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function such that $N < p^- < p^+ < +\infty$; $w_0, w_1 \in L^1(\mathbb{R}^N)$ and f is a Carathéodory function having a polynomial growth with exponent lower than $p^- - 1$. A result of the existence of a nontrivial solution has been established.

The present work is a contribution in the same direction. Indeed, in this paper, we are concerned with the following nonlinear differential equation:

$$-\left(w_1(x)\,|u'|^{p(u)-2}\,u'\right)' + w_0(x)\,|u|^{p(u)-2}\,u = g(x)\,|u|^{p(x)-2}\,u, \qquad x \in \mathbb{R}, \tag{1.4}$$

where $p : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function such that

$$1 < p^- = \inf_{s\in\mathbb{R}} p(s) < p^+ = \sup_{s\in\mathbb{R}} p(s) < +\infty.$$

The equation (1.4) is taken under the following assumptions:

(H_1) We assume that there exists $0 < \alpha < 1$ such that $p(\alpha) = p^+$. Moreover, we assume that the function $x \longmapsto x^{p(x)-1}$ is increasing on the interval $[0,\alpha]$.

(H_2) $w_0, w_1 \in L^1(\mathbb{R})$ are such that

$$0 < \sup_{|x|\leq R} w_0(x) < +\infty, \qquad 0 < \inf_{|x|\leq R} w_1(x) < +\infty, \qquad \forall\, R > 0.$$

We also assume that there exists a positive constant $C_0 > 0$ such that

$$w_1(x) \leq C_0 w_0(x), \qquad \forall\, x \in \mathbb{R}.$$

(H_3) $g \in L^1(\mathbb{R})$, $g(x) > 0$, $\forall\, x \in \mathbb{R}$. We assume that

$$g(x) \leq w_0(x) \leq g(x)\alpha^{p(x)-p^+}, \qquad \forall\, x \in \mathbb{R},$$

where α is defined in (H_1).

A similar differential equation to (1.4) has been treated in [6] where the author dealt with the nonlinear equation

$$-\left(\left|u'\right|^{p(x)-2}u'\right)+|u|^{p(x)-2}u=\lambda\varphi(x)\,|u|^{p(u)-2}u,\qquad x\in\mathbb{R},\tag{1.5}$$

where $p\in C^1(\mathbb{R})$ is such that $2<p^-<p^+<+\infty$, λ is a positive parameter and $\varphi\in C(\mathbb{R})\cap L^\infty(\mathbb{R})$, $\varphi(x)>0$, $\forall\ x\in\mathbb{R}$. Under some suitable additional conditions on p and φ, the author used a variational method to prove the existence of a nontrivial solution to (1.5). Comparing to (1.4), the problem (1.5) is easier because the exponent appearing in the principal part depends directly on the variable $x\in\mathbb{R}$ and by consequence the solution has been searched in the fixed classical variable exponent Sobolev space $W^{1,p(x)}(\mathbb{R})$.

Definition 1.1. A function $u:\mathbb{R}\to\mathbb{R}$ is said to be a weak solution to the equation (1.4) if it satisfies that $u\in L^1_{loc}(\mathbb{R})$,

$$\int_{\mathbb{R}}w_0(x)\,|u|^{p(u)}\,dx<+\infty,\qquad\int_{\mathbb{R}}w_1(x)\left|u'\right|^{p(u)}dx<+\infty,$$

and

$$\int_{\mathbb{R}}w_1(x)\left|u'\right|^{p(u)-2}u'v'dx+\int_{\mathbb{R}}w_0(x)\,|u|^{p(u)-2}uvdx=\int_{\mathbb{R}}g(x)\,|u|^{p(x)-2}uvdx,\qquad\forall\ v\in E_u,$$

where

$$E_u=\left\{v\in L^1_{loc}(\mathbb{R}),\ \int_{\mathbb{R}}w_0(x)\,|v|^{p(u)}\,dx<+\infty,\ \int_{\mathbb{R}}w_1(x)\left|v'\right|^{p(u)}dx<+\infty\right\}.$$

The main result in this work is given by the following theorem.

Theorem 1.2. *Assume that* (H_1), (H_2) *and* (H_3) *hold. Then, there exists at least one weak nontrivial and positive solution to the equation* (1.4) *in the sense of Definition 1.1.*

Example 1.3. As an example of functions p, w_0, w_1 and g satisfying the hypotheses of Theorem 1.2, one can choose

$$p(x)=k+e^{-(x-\frac{1}{2})^2},\quad k\geq 2,\quad w_0(x)=w_1(x)=g(x)=e^{-x^2},\quad x\in\mathbb{R}.$$

2 Preliminaries

In this section, we study the functional space E_u. For $u:\mathbb{R}\to\mathbb{R}$ a fixed measurable function, set $q=p(u)$. In view of this notation, one can easily see that E_u is the weighted Sobolev space with variable exponent

$$E_u=\left\{v\in L^1_{loc}(\mathbb{R}),\ \int_{\mathbb{R}}w_0(x)\,|v|^{q(x)}\,dx<+\infty,\ \int_{\mathbb{R}}w_1(x)\left|v'\right|^{q(x)}dx<+\infty\right\}.$$

This space is equipped with the well known Luxemburg norm

$$\|u\|_{E_u}=\inf\left\{\lambda>0,\ \int_{\mathbb{R}}\left(\frac{w_1(x)\,|u'|^{q(x)}+w_0(x)\,|u|^{q(x)}}{\lambda^{q(x)}}\right)dx\leq 1\right\}.$$

Since $w_0, w_1 \in L^1_{loc}(\mathbb{R})$ and $w_0^{-\frac{1}{q(x)-1}}, w_1^{-\frac{1}{q(x)-1}} \in L^1_{loc}(\mathbb{R})$, then $(E_u, \|\cdot\|_{E_u})$ is a Banach, reflexive and separable space (see [21]).

If $v \in E_u$, $(v_n)_n \subset E_u$, then the following relations hold true.

$$\|v\|_{E_u} < 1 \Rightarrow \|v\|_{E_u}^{q^+} \leq \int_{\mathbb{R}} \left(w_1(x) \left|v'\right|^{q(x)} + w_0(x) \left|v\right|^{q(x)} \right) dx \leq \|v\|_{E_u}^{q^-},$$

$$\|v\|_{E_u} > 1 \Rightarrow \|v\|_{E_u}^{q^-} \leq \int_{\mathbb{R}} \left(w_1(x) \left|v'\right|^{q(x)} + w_0(x) \left|v\right|^{q(x)} \right) dx \leq \|v\|_{E_u}^{q^+},$$

$$\|v_n - v\|_{E_u} \to 0 \Leftrightarrow \int_{\mathbb{R}} \left(w_1(x) \left|v_n' - v'\right|^{q(x)} + w_0(x) \left|v_n - v\right|^{q(x)} \right) dx \to 0, \qquad n \to +\infty.$$

One of the most important properties of the space E_u is the density of the space of smooth functions $C_0^\infty(\mathbb{R})$ in it with respect to the norm $\|\cdot\|_{E_u}$.

Proposition 2.1. *Assume that $u \in L^1_{loc}(\mathbb{R})$, and satisfies*

$$\int_{\mathbb{R}} w_0(x) \left|u\right|^{p^-} dx < +\infty, \quad \text{and} \quad \int_{\mathbb{R}} w_1(x) \left|u'\right|^{p^-} dx < +\infty.$$

Then, $C_0^\infty(\mathbb{R})$ is dense in E_u.

Proof. The proof relies essentially on a truncation procedure. Let $v \in E_u$, $\psi \in C_0^\infty(\mathbb{R})$ with $0 \leq \psi \leq 1$, $\psi(x) = 1$, if $|x| \leq 1$, $\psi(x) = 0$, if $|x| \geq 2$, and, for $n \geq 1$ an integer, set $\psi_n(x) = \psi(\frac{x}{n})$ and $v_n = v\psi_n$. We claim that $v_n \to v$ strongly in E_u. We have,

$$\int_{\mathbb{R}} w_0(x) \left|v_n - v\right|^{q(x)} dx = \int_{\mathbb{R}} w_0(x) \left|1 - \psi_n(x)\right|^{q(x)} \left|v\right|^{q(x)} dx.$$

Since $|1 - \psi_n(x)| \to 0$, $\forall\, x \in \mathbb{R}$ and $|1 - \psi_n(x)| \leq 2$, $\forall\, x \in \mathbb{R}$, $\forall\, n \geq 1$, then one can use the Lebesgue's dominated convergence theorem to deduce that

$$\lim_{n \to +\infty} \int_{\mathbb{R}} w_0(x) \left|v_n - v\right|^{q(x)} dx = 0. \tag{2.1}$$

On the other hand,

$$\begin{aligned}
&\int_{\mathbb{R}} w_1(x) \left|v_n' - v'\right|^{q(x)} dx \\
&\quad = \int_{\mathbb{R}} w_1(x) \left|(1-\psi_n)v' - v\psi_n'\right|^{q(x)} dx \\
&\quad \leq c_0 \int_{\mathbb{R}} w_1(x) \left|1-\psi_n\right|^{q(x)} \left|v'\right|^{q(x)} dx + c_0 \int_{\mathbb{R}} w_1(x) \left|v\right|^{q(x)} \left(\frac{1}{n}\right)^{q(x)} \left|\psi'\left(\frac{x}{n}\right)\right|^{q(x)} dx \\
&\quad \leq c_0 \int_{\mathbb{R}} w_1(x) \left|1-\psi_n\right|^{q(x)} \left|v'\right|^{q(x)} dx + c_0 C_0 \int_{\mathbb{R}} w_0(x) \left|v\right|^{q(x)} \left(\frac{1}{n}\right)^{q(x)} \left|\psi'\left(\frac{x}{n}\right)\right|^{q(x)} dx,
\end{aligned} \tag{2.2}$$

where we used the fact that $w_1(x) \leq C_0 w_0(x)$, $\forall\, x \in \mathbb{R}$. Plainly,

$$\int_{\mathbb{R}} w_0(x) \left|v\right|^{q(x)} \left(\frac{1}{n}\right)^{q(x)} \left|\psi'\left(\frac{x}{n}\right)\right|^{q(x)} dx \to 0, \qquad n \to +\infty. \tag{2.3}$$

Again by the virtue of the Lebesgue's dominated convergence theorem, we get

$$\lim_{n \to +\infty} \int_{\mathbb{R}} w_1(x) \left|1-\psi_n\right|^{q(x)} \left|v'\right|^{q(x)} dx = 0. \tag{2.4}$$

By (2.4) and (2.3), we infer

$$\lim_{n\to+\infty}\int_{\mathbb{R}} w_1(x)\left|v_n'-v'\right|^{q(x)}dx=0. \tag{2.5}$$

Combining (2.5) and (2.1), it follows that $v_n \to v$ strongly in E_u. Hence, for every $\epsilon > 0$, there exists $n_0 = n_0(\epsilon) \geq 1$ such that $\|v_{n_0}-v\|_{E_u} \leq \frac{\epsilon}{2}$. Now, taking into account that

$$0<\inf_{|x|<2n_0} w_0(x)\leq \sup_{|x|<2n_0} w_0(x)<+\infty, \quad \text{and} \quad 0<\inf_{|x|<2n_0} w_1(x)\leq \sup_{|x|<2n_0} w_1(x)<+\infty,$$

one can easily see that

$$\left\{w\in L^1(]-2n_0,2n_0[),\ \int_{-2n_0}^{2n_0} w_0(x)\,|w|^{q(x)}dx<+\infty,\ \int_{-2n_0}^{2n_0} w_1(x)\left|w'\right|^{q(x)}dx<+\infty\right\} = W^{1,q(x)}(]-2n_0,2n_0[).$$

We also see that $u\in W^{1,p^-}(]-2n_0,2n_0[)$. Hence, $u\in C([-2n_0,2n_0])$ and there exists a constant C depending on p and ϵ such that

$$|u(x)-u(y)|\leq C\,\|u\|_{W^{1,p^-}(]-2n_0,2n_0[)}\,|x-y|^{1-\frac{1}{p^-}}, \qquad \forall\, x,y\in]-2n_0,2n_0[.$$

By hypothesis, there is a constant $L>0$ such that

$$|p(u(x))-p(u(y))|\leq L\,|u(x)-u(y)|, \qquad \forall\, x,y\in\mathbb{R},$$

which implies that

$$|q(x)-q(y)|\leq LC\,\|u\|_{W^{1,p^-}(]-2n_0,2n_0[)}\,|x-y|^{1-\frac{1}{p^-}}, \qquad \forall\, x,y\in]-2n_0,2n_0[.$$

Therefore, q is log-Hölder continuous, that is, there is a constant $C'>0$ such that

$$|q(x)-q(y)|\leq \frac{-C'}{\log|x-y|}, \qquad \forall\, x,y\in]-2n_0,2n_0[,\ |x-y|<\frac{1}{2}.$$

That result guarantees that $C_0^\infty(]-2n_0,2n_0[)$ is dense in $W^{1,q(x)}(]-2n_0,2n_0[)\cap W_0^{1,1}(]-2n_0,2n_0[)$ (see [13, 21]). Having in mind that $v_{n_0}\in W^{1,q(x)}(]-2n_0,2n_0[)\cap W_0^{1,1}(]-2n_0,2n_0[)$, we can conclude the proof of Proposition 2.1. □

3 Proof of Theorem 1.2

For $(x,s)\in\mathbb{R}^2$, set

$$f(x,s)=\begin{cases} g(x), & \text{if } s\geq 1,\\ g(x)s^{p(x)-1}, & \text{if } \alpha\leq s\leq 1,\\ g(x)\alpha^{p(x)-1}, & \text{if } s\leq\alpha.\end{cases}$$

Consider the weighted Sobolev space

$$W^{1,p^+}_{w_0,w_1}(\mathbb{R})=\left\{u\in L^1_{loc}(\mathbb{R}),\ \int_{\mathbb{R}} w_0(x)\,|u|^{p^+}dx<+\infty,\ \int_{\mathbb{R}} w_1(x)\left|u'\right|^{p^+}dx<+\infty\right\}.$$

This space is naturally equipped with the norm

$$\|u\|_{W^{1,p^+}_{w_0,w_1}(\mathbb{R})}=\left(\int_{\mathbb{R}}\left(w_1(x)\left|u'\right|^{p^+}+w_0(x)\,|u|^{p^+}\right)dx\right)^{\frac{1}{p^+}}.$$

Lemma 3.1. *For each $\epsilon > 0$, there exists a function $u_\epsilon \in W^{1,p^+}_{w_0,w_1}(\mathbb{R})$ such that*

$$\begin{aligned}\int_{\mathbb{R}} w_1(x)\left|u'_\epsilon\right|^{p(u_\epsilon)-2} u'_\epsilon v' dx &+ \int_{\mathbb{R}} w_0(x)\left|u_\epsilon\right|^{p(u_\epsilon)-2} u_\epsilon v dx \\ &+ \epsilon\left(\int_{\mathbb{R}} w_1(x)\left|u'_\epsilon\right|^{p^+-2} u'_\epsilon v' dx + \int_{\mathbb{R}} w_0(x)\left|u_\epsilon\right|^{p^+-2} u_\epsilon v dx\right) \\ &= \int_{\mathbb{R}} f(x,u_\epsilon) v dx, \ \forall\, v \in W^{1,p^+}_{w_0,w_1}(\mathbb{R}).\end{aligned}$$

Proof. Let $\epsilon > 0$ fixed. For $w : \mathbb{R} \to \mathbb{R}$ a measurable function, define the operator $A_w : W^{1,p^+}_{w_0,w_1}(\mathbb{R}) \to (W^{1,p^+}_{w_0,w_1}(\mathbb{R}))^*$ by

$$\begin{aligned}\langle A_w u, v\rangle = &\int_{\mathbb{R}} w_1\left|u'\right|^{p(w)-2} u'v' dx + \int_{\mathbb{R}} w_0\left|u\right|^{p(w)-2} uv dx \\ &+ \epsilon\left(\int_{\mathbb{R}} w_1\left|u'\right|^{p^+-2} u'v' dx + \int_{\mathbb{R}} w_0\left|u\right|^{p^+-2} uv dx\right), \qquad u,v \in W^{1,p^+}_{w_0,w_1}(\mathbb{R}).\end{aligned}$$

Observe that A_w is well defined. In fact, for $u, v \in W^{1,p^+}_{w_0,w_1}(\mathbb{R})$, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}} w_1\left|u'\right|^{p(w)-2} u'v' dx + \int_{\mathbb{R}} w_0\left|u\right|^{p(w)-2} uv dx\right| \\ &\quad\le \int_{\mathbb{R}} w_1\left|u'\right|^{p(w)-1}\left|v'\right| dx + \int_{\mathbb{R}} w_0\left|u\right|^{p(w)-1}\left|v\right| dx \\ &\quad\le \int_{\mathbb{R}} w_1\left|v'\right| dx + \int_{\mathbb{R}} w_1\left|u'\right|^{p^+-1}\left|v'\right| dx + \int_{\mathbb{R}} w_0\left|v\right| dx + \int_{\mathbb{R}} w_0\left|u\right|^{p^+-1}\left|v\right| dx \\ &\quad\le |w_1|^{\frac{p^+-1}{p^+}}_{L^1(\mathbb{R})}\left(\int_{\mathbb{R}} w_1\left|v'\right|^{p^+} dx\right)^{\frac{1}{p^+}} + \left(\int_{\mathbb{R}} w_1\left|u'\right|^{p^+} dx\right)^{\frac{p^+-1}{p^+}}\left(\int_{\mathbb{R}} w_1\left|v'\right|^{p^+} dx\right)^{\frac{1}{p^+}} \\ &\qquad + |w_0|^{\frac{p^+-1}{p^+}}_{L^1(\mathbb{R})}\left(\int_{\mathbb{R}} w_0\left|v\right|^{p^+} dx\right)^{\frac{1}{p^+}} + \left(\int_{\mathbb{R}} w_0\left|u\right|^{p^+} dx\right)^{\frac{p^+-1}{p^+}}\left(\int_{\mathbb{R}} w_0\left|v\right|^{p^+} dx\right)^{\frac{1}{p^+}}.\end{aligned}$$

Hence, for u fixed in $W^{1,p^+}_{w_0,w_1}(\mathbb{R})$, the linear mapping $v \longmapsto \langle A_w u, v\rangle$ lies in the dual $(W^{1,p+}_{w_0,w_1}(\mathbb{R}))^*$. Clearly, A_w is coercive and continuous. Moreover, A_w is strictly monotone, i.e.

$$\langle A_w u_1 - A_w u_2, u_1 - u_2\rangle > 0, \qquad \forall\, u_1, u_2 \in W^{1,p^+}_{w_0,w_1}(\mathbb{R}),\ u_1 \neq u_2.$$

On the other hand, for $w : \mathbb{R} \to \mathbb{R}$ measurable and $v \in W^{1,p^+}_{w_0,w_1}(\mathbb{R})$, we have

$$\begin{aligned}\left|\int_{\mathbb{R}} f(x,w) v dx\right| &\le \int_{w\ge 1} g(x)\left|v\right| dx + \int_{w\le\alpha} g(x)\alpha^{p(x)-1}\left|v\right| dx + \int_{\alpha\le w\le 1} g(x)\left|w\right|^{p(x)-1}\left|v\right| dx \\ &\le \int_{\mathbb{R}} g(x)\left|v\right| dx \\ &= \int_{\mathbb{R}} \frac{g(x)}{w_0^{\frac{1}{p^+}}} w_0^{\frac{1}{p^+}}\left|v\right| dx \\ &\le \left(\int_{\mathbb{R}} \frac{(g(x))^{\frac{p^+}{p^+-1}}}{w_0^{\frac{1}{p^+-1}}} dx\right)^{\frac{p^+-1}{p^+}}\left(\int_{\mathbb{R}} w_0\left|v\right|^{p^+} dx\right)^{\frac{1}{p^+}}.\end{aligned}$$

Thus, $(f(\cdot,w)) \in (W^{1,p^+}_{w_0,w_1}(\mathbb{R}))^*$. By the virtue of the Minty–Browder theorem (see [19, Theorem 26.A]), we deduce that there exists a unique element $u_w \in W^{1,p^+}_{w_0,w_1}(\mathbb{R})$ such that

$$A_w(u_w) = f(\cdot,w) \quad \text{in } (W^{1,p^+}_{w_0,w_1}(\mathbb{R}))^*.$$

That is

$$\langle A_w u_w, v\rangle = \int_{\mathbb{R}} f(x,w)v\,dx, \qquad \forall\, v \in W^{1,p^+}_{w_0,w_1}(\mathbb{R}). \tag{3.1}$$

Taking $v = u_w$ in (3.1), we infer

$$\int_{\mathbb{R}} w_1(x)\,|u_w'|^{p(w)}\,dx + \int_{\mathbb{R}} w_0(x)\,|u_w|^{p(w)}\,dx + \epsilon\left(\int_{\mathbb{R}} w_1(x)\,|u_w'|^{p^+}\,dx + \int_{\mathbb{R}} w_0(x)\,|u_w|^{p^+}\,dx\right)$$
$$= \int_{\mathbb{R}} f(x,w)u_w dx \le \int_{\mathbb{R}} g(x)\,|u_w|\,dx.$$

Thus,

$$\epsilon\,\|u_w\|^{p^+}_{W^{1,p^+}_{w_0,w_1}(\mathbb{R})} \le \left(\int_{\mathbb{R}} \frac{g(x)^{\frac{p^+}{p^+-1}}}{w_0^{\frac{1}{p^+-1}}}dx\right)^{\frac{p^+-1}{p^+}} \|u_w\|_{W^{1,p^+}_{w_0,w_1}(\mathbb{R})}.$$

Consequently, there exists a constant C_ϵ depending on ϵ but independent of w such that

$$\|u_w\|_{W^{1,p^+}_{w_0,w_1}(\mathbb{R})} \le C_\epsilon. \tag{3.2}$$

Now, we claim that $W^{1,p^+}_{w_0,w_1}(\mathbb{R})$ is compactly embedded in the weighted Lebesgue space

$$L^{p^-}_{w_0}(\mathbb{R}) = \left\{u : \mathbb{R} \to \mathbb{R} \text{ measurable}, \int_{\mathbb{R}} w_0(x)\,|u|^{p^-}\,dx < +\infty\right\}$$

equipped with the norm $u \longmapsto |u|_{L^{p^-}_{w_0}(\mathbb{R})} = \left(\int_{\mathbb{R}} w_0(x)\,|u|^{p^-}\,dx\right)^{\frac{1}{p^-}}$. For that aim, take a sequence $(u_n)_n \subset W^{1,p^+}_{w_0,w_1}(\mathbb{R})$ such that $u_n \rightharpoonup 0$ weakly in $W^{1,p^+}_{w_0,w_1}(\mathbb{R})$. We claim that, up to a subsequence $u_n \to 0$ strongly in $L^{p^-}_{w_0}(\mathbb{R})$. We have,

$$\int_{\mathbb{R}} w_0(x)\,|u_n|^{p^-}\,dx = \int_{\mathbb{R}} (w_0(x))^{1-\frac{p^-}{p^+}}(w_0(x))^{\frac{p^-}{p^+}}\,|u_n|^{p^-}\,dx. \tag{3.3}$$

Observing that the sequence $(w_0^{\frac{p^-}{p^+}}\,|u_n|^{p^-})_n$ is bounded in $L^{\frac{p^+}{p^-}}(\mathbb{R})$ and, up to a subsequence, is weakly convergent to 0 in $L^{\frac{p^+}{p^-}}(\mathbb{R})$, and that $w_0^{1-\frac{p^-}{p^+}} \in L^{\frac{p^+}{p^+-p^-}}(\mathbb{R})$, we can immediately see from (3.3) that

$$\int_{\mathbb{R}} w_0(x)\,|u_n|^{p^-}\,dx \to 0, \qquad n \to +\infty.$$

Let $C_1 > 0$ be a positive constant such that

$$|u|_{L^{p^-}_{w_0}(\mathbb{R})} \le C_1\,\|u\|_{W^{1,p^+}_{w_0,w_1}(\mathbb{R})}, \qquad \forall\, u \in W^{1,p^+}_{w_0,w_1}(\mathbb{R}). \tag{3.4}$$

Set $\widetilde{C_\epsilon} = C_1 C_\epsilon$, and $\mathcal{K} = \{w \in L^{p^-}_{w_0}(\mathbb{R}),\ |w|_{L^{p^-}_{w_0}(\mathbb{R})} \le \widetilde{C_\epsilon}\}$ the closed ball of $L^{p^-}_{w_0}(\mathbb{R})$ centered at the origin and of radius $\widetilde{C_\epsilon}$. Define the mapping $T : \mathcal{K} \to \mathcal{K}$ by $Tw = u_w$ given by (3.1). In view of (3.2) and (3.4), it yields that $T(\mathcal{K}) \subset \mathcal{K}$. Moreover, since $W^{1,p^-}_{w_0,w_1}(\mathbb{R})$ is compactly embedded in $L^{p^-}_{w_0}(\mathbb{R})$, we can easily show that $T(\mathcal{K})$ is relatively compact. Observing that T is continuous, then one can use the Schauder's fixed point Theorem (see [18, Theorem 2.A]) to deduce the existence of $\widetilde{w} \in \mathcal{K}$ such that $u_{\widetilde{w}} = \widetilde{w}$. Consequently,

$$\begin{aligned}\int_{\mathbb{R}} w_1(x)\left|u'_{\widetilde{w}}\right|^{p(u_{\widetilde{w}})-2} u'_{\widetilde{w}} v' dx + \int_{\mathbb{R}} w_0(x)\left|u_{\widetilde{w}}\right|^{p(u_{\widetilde{w}})-2} u_{\widetilde{w}} v dx \\ + \epsilon\left(\int_{\mathbb{R}} w_1(x)\left|u'_{\widetilde{w}}\right|^{p^+-2} u'_{\widetilde{w}} v' dx + \int_{\mathbb{R}} w_0(x)\left|u_{\widetilde{w}}\right|^{p^+-2} u_{\widetilde{w}} v dx\right) \\ = \int_{\mathbb{R}} f(x, u_{\widetilde{w}}) v dx, \qquad \forall\, v \in W^{1,p^+}_{w_0,w_1}(\mathbb{R}).\end{aligned}$$

This concludes the proof of Lemma 3.1. □

The completion of the proof of Theorem 1.2

Choosing $\epsilon = \frac{1}{n}$, $n \geq 1$, in Lemma 3.1, we deduce that there exists $u_n \in W^{1,p^+}_{w_0,w_1}(\mathbb{R})$ such that

$$\begin{aligned}\int_{\mathbb{R}} w_1(x)\left|u'_n\right|^{p(u_n)-2} u'_n v' dx + \int_{\mathbb{R}} w_0(x)\left|u_n\right|^{p(u_n)-2} u_n v dx \\ + \frac{1}{n}\left(\int_{\mathbb{R}} w_1(x)\left|u'_n\right|^{p^+-2} u'_n v' dx + \int_{\mathbb{R}} w_0(x)\left|u_n\right|^{p^+-2} u_n v dx\right) \\ = \int_{\mathbb{R}} f(x, u_n) v dx, \qquad \forall\, v \in W^{1,p^+}_{w_0,w_1}(\mathbb{R}).\end{aligned} \tag{3.5}$$

Taking $v = u_n$ as test function in (3.5), we get

$$\begin{aligned}\int_{\mathbb{R}} w_1(x)\left|u'_n\right|^{p(u_n)} dx + \int_{\mathbb{R}} w_0(x)\left|u_n\right|^{p(u_n)} dx + \frac{1}{n}\left\|u_n\right\|^{p^+}_{W^{1,p^+}_{w_0,w_1}(\mathbb{R})} \\ = \int_{\mathbb{R}} f(x, u_n) u_n dx, \qquad \forall\, n \geq 1.\end{aligned} \tag{3.6}$$

We have

$$\begin{aligned}\left|\int_{\mathbb{R}} f(x,u_n)u_n dx\right| &\leq \int_{\mathbb{R}^N} g(x)\left|u_n\right| dx \\ &\leq \left(\int_{\mathbb{R}}\left(\frac{g(x)}{(w_0(x))^{\frac{1}{p^-}}}\right)^{\frac{p^-}{p^--1}} dx\right)^{\frac{p^--1}{p^-}} \left(\int_{\mathbb{R}} w_0(x)\left|u_n\right|^{p^-} dx\right)^{\frac{1}{p^-}} \\ &\leq c_2\left(\int_{|u_n|\leq 1} w_0(x)\left|u_n\right|^{p^-} dx + \int_{|u_n|\geq 1} w_0(x)\left|u_n\right|^{p^-} dx\right)^{\frac{1}{p^-}} \\ &\leq c_2\left(\int_{\mathbb{R}} w_0(x)\left|u_n\right|^{p(u_n)} dx + \left|w_0\right|_{L^1(\mathbb{R})}\right)^{\frac{1}{p^-}}.\end{aligned}$$

By (3.6), we infer

$$\int_{\mathbb{R}} w_1(x)\left|u'_n\right|^{p(u_n)} dx + \int_{\mathbb{R}} w_0(x)\left|u_n\right|^{p(u_n)} dx + \frac{1}{n}\left\|u_n\right\|^{p^+}_{W^{1,p^+}_{w_0,w_1}(\mathbb{R})} \leq c_3, \qquad \forall\, n \geq 1. \tag{3.7}$$

In particular, there exists a positive constant $c_4 > 0$ (independent of n) such that

$$\int_{\mathbb{R}} w_1(x)\left|u'_n\right|^{p^-} dx + \int_{\mathbb{R}} w_0(x)\left|u_n\right|^{p^-} dx \leq c_4, \qquad \forall\, n \geq 1. \tag{3.8}$$

Indeed,

$$\begin{aligned}\int_{\mathbb{R}} w_1(x)\left|u_n'\right|^{p^-} dx &= \int_{|u_n'|\geq 1} w_1(x)\left|u_n'\right|^{p^-} dx + \int_{|u_n'|<1} w_1(x)\left|u_n'\right|^{p^-} dx \\ &\leq \int_{|u_n'|\geq 1} w_1(x)\left|u_n'\right|^{p(u_n)} dx + |w_1|_{L^1(\mathbb{R})} \\ &\leq \int_{\mathbb{R}} w_1(x)\left|u_n'\right|^{p(u_n)} dx + |w_1|_{L^1(\mathbb{R})}, \qquad \forall\, n \geq 1.\end{aligned}$$

Similarly,

$$\int_{\mathbb{R}} w_0(x)\left|u_n\right|^{p^-} dx \leq \int_{\mathbb{R}} w_0(x)\left|u_n\right|^{p(u_n)} dx + |w_0|_{L^1(\mathbb{R})}, \qquad \forall\, n \geq 1.$$

Hence, (3.8) immediately follows from (3.7). By the reflexivity of the weighted Sobolev space $W^{1,p^-}_{w_0,w_1}(\mathbb{R})$, there exists $u \in W^{1,p^-}_{w_0,w_1}(\mathbb{R})$ such that, up to a subsequence, $u_n \rightharpoonup u$ weakly in $W^{1,p^-}_{w_0,w_1}(\mathbb{R})$ and $u_n(x) \to u(x)$ a.e. $x \in \mathbb{R}$. Now, we claim that

$$\int_{\mathbb{R}} w_1(x)\left|u'\right|^{p(u)} dx + \int_{\mathbb{R}} w_0(x)\left|u\right|^{p(u)} dx < +\infty. \tag{3.9}$$

For that aim, for $x \in \mathbb{R}$ and $n \geq 1$, set $q_n(x) = p(u_n(x))$ and $q(x) = p(u(x))$. For $k > 0$, set

$$v_k = \begin{cases} qu\left|u\right|^{q-2}, & \text{if } |u| \leq k, \\ qk^{q-1}\frac{u}{|u|}, & \text{if } |u| > k. \end{cases}$$

By Young's inequality, it yields

$$u_n v_k \leq |u_n|^{q_n} + \frac{q_n - 1}{q_n^{q_n'}}\left|v_k\right|^{q_n'}, \qquad \forall\, k > 0,\ \forall\, n \geq 1,$$

where $q_n' = \frac{q_n}{q_n - 1}$. Thus,

$$\int_{\mathbb{R}} w_0(x)u_n v_k dx \leq \int_{\mathbb{R}} w_0(x)\left|u_n\right|^{q_n} dx + \int_{\mathbb{R}} w_0(x)\frac{q_n - 1}{q_n^{q_n'}}\left|v_k\right|^{q_n'} dx, \qquad \forall\, k > 0,\ \forall\, n \geq 1.$$

Tending n to $+\infty$ and having (3.7) in mind, we get

$$\int_{\mathbb{R}} w_0(x)uv_k dx \leq c_3 + \int_{\mathbb{R}} w_0(x)\frac{q-1}{q^{q'}}\left|v_k\right|^{q'} dx.$$

Consequently,

$$\begin{aligned}&\int_{|u|\leq k} w_0(x)q\left|u\right|^q dx + \int_{|u|>k} w_0(x)qk^{q-1}\left|u\right| dx \\ &\quad\leq c_3 + \int_{|u|\leq k} w_0(x)(q-1)\left|u\right|^q dx + \int_{|u|>k} w_0(x)(q-1)k^q dx.\end{aligned}$$

Thus,

$$\int_{|u|\leq k} w_0(x)\left|u\right|^q dx + \int_{|u|>k} w_0(x)k^q dx \leq c_3.$$

Passing to the limit as k tends to $+\infty$ in that last inequality, we obtain

$$\int_{\mathbb{R}} w_0(x)\left|u\right|^q dx \leq c_3.$$

Similarly, for $k > 0$, one can choose the function

$$\widetilde{v_k} = \begin{cases} qu' \left|u'\right|^{q-2}, & \text{if } |u'| \leq k, \\ qk^{q-1} \frac{u'}{|u'|}, & \text{if } |u'| > k, \end{cases}$$

Using Young's inequality and proceeding exactly as previously, we can easily show that

$$\int_{|u'| \leq k} w_1(x) \left|u'\right|^q dx \leq c_3, \qquad \forall\, k > 0,$$

and after passing to the limit as k tends to $+\infty$, we finally obtain

$$\int_{\mathbb{R}} w_1(x) \left|u'\right|^q dx \leq c_3.$$

Hence, the claim (3.9) holds. Let $v \in W^{1,p^+}_{w_0,w_1}(\mathbb{R})$. We have

$$\begin{aligned} \int_{\mathbb{R}} w_1(x) & \left(\left|u_n'\right|^{p(u_n)-2} u_n' - \left|v'\right|^{p(u_n)-2} v' \right) (u_n' - v') dx \\ & + \int_{\mathbb{R}} w_0(x) \left(|u_n|^{p(u_n)-2} u_n - |v|^{p(u_n)-2} v \right) (u_n - v) dx \\ & + \frac{1}{n} \int_{\mathbb{R}} w_1(x) \left(\left|u_n'\right|^{p^+-2} u_n' - \left|v'\right|^{p^+-2} v' \right) (u_n' - v') dx \\ & + \frac{1}{n} \int_{\mathbb{R}} w_0(x) \left(|u_n|^{p^+-2} u_n - |v|^{p^+-2} v \right) (u_n - v) dx \\ \geq 0, & \qquad \forall\, n \geq 1. \end{aligned}$$

Thus,

$$\begin{aligned} \int_{\mathbb{R}} w_1(x) & \left(\left|u_n'\right|^{p(u_n)-2} u_n' + \frac{1}{n} \left|u_n'\right|^{p^+-2} u_n' \right) (u_n' - v') dx \\ & + \int_{\mathbb{R}} w_0(x) \left(|u_n|^{p(u_n)-2} u_n + \frac{1}{n} |u_n|^{p^+-2} u_n \right) (u_n - v) dx \\ \geq & \int_{\mathbb{R}} w_1(x) \left(\left|v'\right|^{p(u_n)-2} v' + \frac{1}{n} \left|v'\right|^{p^+-2} v' \right) (u_n' - v') dx \\ & + \int_{\mathbb{R}} w_0(x) \left(|v|^{p(u_n)-2} v + \frac{1}{n} |v|^{p^+-2} v \right) (u_n - v) dx. \end{aligned}$$

By (3.5), it follows

$$\begin{aligned} \int_{\mathbb{R}} f(x, u_n)(u_n - v) dx \geq & \int_{\mathbb{R}} w_1(x) \left(\left|v'\right|^{p(u_n)-2} v' + \frac{1}{n} \left|v'\right|^{p^+-2} v' \right) (u_n' - v') dx \\ & + \int_{\mathbb{R}} w_0(x) \left(|v|^{p(u_n)-2} v + \frac{1}{n} |v|^{p^+-2} v \right) (u_n - v) dx. \end{aligned} \tag{3.10}$$

We have

$$\begin{aligned}
&\left|\frac{1}{n}\int_{\mathbb{R}} w_1(x)\left|v'\right|^{p^+-2}v'(u_n'-v')dx\right| \\
&\quad\leq \frac{1}{n}\int_{\mathbb{R}} w_1(x)\left|v'\right|^{p^+-1}\left|(u_n'-v')\right|dx \\
&\quad\leq \frac{1}{n}\left(\int_{\mathbb{R}} w_1(x)\left|v'\right|^{p^+}dx\right)^{\frac{p^+-1}{p^+}}\left(\int_{\mathbb{R}} w_1(x)\left|(u_n'-v')\right|^{p^+}dx\right)^{\frac{1}{p^+}} \\
&\quad= \left(\frac{1}{n}\right)^{\frac{p^+-1}{p^+}}\left(\frac{1}{n}\right)^{\frac{1}{p^+}}\left(\int_{\mathbb{R}} w_1(x)\left|v'\right|^{p^+}dx\right)^{\frac{p^+-1}{p^+}}\left(\int_{\mathbb{R}} w_1(x)\left|(u_n'-v')\right|^{p^+}dx\right)^{\frac{1}{p^+}} \\
&\quad\leq \left(\frac{1}{n}\right)^{\frac{p^+-1}{p^+}}\left(\frac{1}{n}\left\|u_n-v\right\|^{p^+}_{W^{1,p^+}_{w_0,w_1}(\mathbb{R})}\right)^{\frac{1}{p^+}}\left\|v\right\|^{p^+-1}_{W^{1,p^+}_{w_0,w_1}(\mathbb{R})}.
\end{aligned}\tag{3.11}$$

By (3.7), we know that

$$\sup_{n\geq 1}\left(\frac{1}{n}\left\|u_n\right\|^{p^+}_{W^{1,p^+}_{w_0,w_1}(\mathbb{R})}\right) < +\infty.$$

Then, from (3.11), we deduce that

$$\frac{1}{n}\int_{\mathbb{R}} w_1(x)\left|v'\right|^{p^+-2}v'(u_n'-v')dx \to 0, \qquad n\to+\infty. \tag{3.12}$$

Similarly,

$$\frac{1}{n}\int_{\mathbb{R}} w_0(x)\left|v\right|^{p^+-2}v(u_n-v)dx \to 0, \qquad n\to+\infty. \tag{3.13}$$

We claim that

$$\int_{\mathbb{R}} f(x,u_n)(u_n-v)dx \to \int_{\mathbb{R}} f(x,u)(u-v)dx, \qquad n\to+\infty. \tag{3.14}$$

First, note that $f(x,u_n(x))(u_n(x)-v(x)) \to f(x,u(x))(u(x)-v(x))$, a.e. $x\in\mathbb{R}$. Second, by (H_2), it yields

$$\left|f(x,u_n)(u_n-v)\right| \leq g(x)\left|u_n-v\right| = \frac{g(x)}{(w_0(x))^{\frac{1}{p^-}}}(w_0(x))^{\frac{1}{p^-}}\left|u_n-v\right|, \qquad \forall\, n\geq 1.$$

Having in mind that $(w_0(x))^{\frac{1}{p^-}}\left|u_n-v\right| \rightharpoonup (w_0(x))^{\frac{1}{p^-}}\left|u-v\right|$ weakly in $L^{p^-}(\mathbb{R})$ and that $g/w_0^{\frac{1}{p^-}}$ belongs to the dual of $L^{p^-}(\mathbb{R})$, it follows that

$$\int_{\mathbb{R}} g(x)\left|u_n-v\right|dx \to \int_{\mathbb{R}} g(x)\left|u-v\right|dx, \qquad n\to+\infty,$$

which implies that $g(x)\left|u_n-v\right| \to g(x)\left|u-v\right|$ strongly in $L^1(\mathbb{R})$ and by consequence, there exists $g_1\in L^1(\mathbb{R})$ such that, up to a subsequence,

$$g(x)\left|u_n-v\right| \leq g_1(x), \quad \text{a.e. } x\in\mathbb{R},\ \forall\, n\geq 1. \tag{3.15}$$

Using (3.15), we can easily apply the Lebesgue's dominated convergence theorem to deduce (3.14). In view of (3.12), (3.13), and (3.14), from (3.10) we get that

$$\begin{aligned}
\int_{\mathbb{R}} f(x,u)(u-v)dx \geq{} & \lim_{n\to+\infty}\int_{\mathbb{R}} w_1(x)\left|v'\right|^{p(u_n)-2}v'(u_n'-v')dx \\
& + \lim_{n\to+\infty}\int_{\mathbb{R}} w_0(x)\left|v\right|^{p(u_n)-2}v(u_n-v)dx.
\end{aligned}\tag{3.16}$$

Now, observe that

$$\int_{\mathbb{R}} w_1(x)\left|v'\right|^{p(u_n)-2} v'(u_n'-v')dx = \int_{\mathbb{R}} w_1(x)\left|v'\right|^{p(u)-2} v'(u_n'-v')dx + \int_{\mathbb{R}} w_1(x)\left(\left|v'\right|^{p(u_n)-2} v' - \left|v'\right|^{p(u)-2} v'\right)(u_n'-v')dx, \tag{3.17}$$

and

$$\int_{\mathbb{R}} w_0(x)\left|v\right|^{p(u_n)-2} v(u_n-v)dx = \int_{\mathbb{R}} w_0(x)\left|v\right|^{p(u)-2} v(u_n-v)dx + \int_{\mathbb{R}} w_0(x)\left(\left|v\right|^{p(u_n)-2} v - \left|v\right|^{p(u)-2} v\right)(u_n-v)dx. \tag{3.18}$$

Next, we introduce the functional subspace Z of $W^{1,p^+}_{w_0,w_1}(\mathbb{R})$ defined by

$$Z = \left\{ v \in W^{1,p^+}_{w_0,w_1}(\mathbb{R}),\ \int_{\mathbb{R}} w_1(x)\left|v'\right|^{\frac{p^-(p^+-1)}{p^--1}} dx < +\infty,\ \int_{\mathbb{R}} w_0(x)\left|v\right|^{\frac{p^-(p^+-1)}{p^--1}} dx < +\infty \right\}.$$

For $v \in Z$ and $w \in W^{1,p^-}_{w_0,w_1}(\mathbb{R})$, we have

$$\begin{aligned} &\left|\int_{\mathbb{R}} w_1(x)\left|v'\right|^{p(u)-2} v'w'dx\right| \\ &\quad\leq \int_{\mathbb{R}} w_1(x)\left|v'\right|^{p(u)-1}\left|w'\right| dx \\ &\quad\leq \left(\int_{\mathbb{R}} w_1(x)\left|v'\right|^{\frac{p^-(p(u)-1)}{p^--1}} dx\right)^{\frac{p^--1}{p^-}} \left(\int_{\mathbb{R}} w_1(x)\left|w'\right|^{p^-} dx\right)^{\frac{1}{p^-}} \\ &\quad\leq \left(|w_1|_{L^1(\mathbb{R})} + \int_{|v'|\geq 1} w_1(x)\left|v'\right|^{\frac{p^-(p^+-1)}{p^--1}} dx\right)^{\frac{p^--1}{p^-}} \left(\int_{\mathbb{R}} w_1(x)\left|w'\right|^{p^-} dx\right)^{\frac{1}{p^-}} \\ &\quad\leq \left(|w_1|_{L^1(\mathbb{R})} + \int_{\mathbb{R}} w_1(x)\left|v'\right|^{\frac{p^-(p^+-1)}{p^--1}} dx\right)^{\frac{p^--1}{p^-}} \|w\|_{W^{1,p^-}_{w_0,w_1}(\mathbb{R})}, \end{aligned}$$

where

$$W^{1,p^-}_{w_0,w_1}(\mathbb{R}) = \left\{ u \in L^1_{loc}(\mathbb{R}),\ \int_{\mathbb{R}} w_0(x)\left|u\right|^{p^-} dx < +\infty,\ \int_{\mathbb{R}} w_1(x)\left|u'\right|^{p^-} dx < +\infty \right\},$$

equipped with the norm

$$\|u\|_{W^{1,p^-}_{w_0,w_1}(\mathbb{R})} = \left(\int_{\mathbb{R}} \left(w_1(x)\left|u'\right|^{p^-} + w_0(x)\left|u\right|^{p^-}\right) dx\right)^{\frac{1}{p^-}}.$$

Thus, the functional

$$w \longmapsto \int_{\mathbb{R}} w_1(x)\left|v'\right|^{p(u)-2} v'w'dx$$

belongs to the dual of $W^{1,p^-}_{w_0,w_1}(\mathbb{R})$. The same result holds for the functional

$$w \longmapsto \int_{\mathbb{R}} w_0(x)\left|v\right|^{p(u)-2} vwdx.$$

Since $(u_n - v) \rightharpoonup (u-v)$ weakly in $W^{1,p^-}_{w_0,w_1}(\mathbb{R})$, then

$$\begin{aligned} &\int_{\mathbb{R}} w_1(x) \left|v'\right|^{p(u)-2} v'(u_n' - v')dx + \int_{\mathbb{R}} w_0(x) \left|v\right|^{p(u)-2} v(u_n - v)dx \\ &\qquad \to \int_{\mathbb{R}} w_1(x) \left|v'\right|^{p(u)-2} v'(u' - v')dx + \int_{\mathbb{R}} w_0(x) \left|v\right|^{p(u)-2} v(u - v)dx. \end{aligned} \tag{3.19}$$

Furthermore, we have

$$\begin{aligned} &\left|\int_{\mathbb{R}} w_1(x) \left(\left|v'\right|^{p(u_n)-2} v' - \left|v'\right|^{p(u)-2} v'\right)(u_n' - v')dx\right| \\ &\quad \le \left(\int_{\mathbb{R}} w_1(x) \left| \left|v'\right|^{p(u_n)-2} v' - \left|v'\right|^{p(u)-2} v'\right|^{\frac{p^-}{p^- -1}} dx\right)^{\frac{p^- -1}{p^-}} \\ &\quad \times \left(\int_{\mathbb{R}} w_1(x) \left|(u_n' - v')\right|^{p^-} dx\right)^{\frac{1}{p^-}} \\ &\quad \le c_5 \left(\int_{\mathbb{R}} w_1(x) \left| \left|v'\right|^{p(u_n)-2} v' - \left|v'\right|^{p(u)-2} v'\right|^{\frac{p^-}{p^- -1}} dx\right)^{\frac{p^- -1}{p^-}}. \end{aligned} \tag{3.20}$$

Observe that

$$\begin{aligned} &w_1(x) \left| \left|v'\right|^{p(u_n)-2} v' - \left|v'\right|^{p(u)-2} v'\right|^{\frac{p^-}{p^- -1}} \\ &\quad \le w_1(x) 2^{\frac{p^-}{p^- -1}} \left|v'\right|^{\frac{p^-(p^+-1)}{p^- -1}} \mathbb{1}_{\{|v'|\ge 1\}} + w_1(x) 2^{\frac{p^-}{p^- -1}} \mathbb{1}_{\{|v'|\le 1\}} \\ &\quad \le w_1(x) 2^{\frac{p^-}{p^- -1}} \left(1 + \left|v'\right|^{\frac{p^-(p^+-1)}{p^- -1}}\right). \end{aligned}$$

Taking into account that, for a.e. $x \in \mathbb{R}$, $p(u_n(x)) \to p(u(x))$ as $n \to +\infty$, then we can apply the Lebesgue's dominated convergence theorem to get

$$\int_{\mathbb{R}} w_1(x) \left| \left|v'\right|^{p(u_n)-2} v' - \left|v'\right|^{p(u)-2} v'\right|^{\frac{p^-}{p^- -1}} dx \to 0, \qquad n \to +\infty.$$

By (3.20), it follows

$$\int_{\mathbb{R}} w_1(x) \left(\left|v'\right|^{p(u_n)-2} v' - \left|v'\right|^{p(u)-2} v'\right)(u_n' - v')dx \to 0, \qquad n \to +\infty. \tag{3.21}$$

In a similar way, we have

$$\int_{\mathbb{R}} w_0(x) \left(\left|v\right|^{p(u_n)-2} v - \left|v\right|^{p(u)-2} v\right)(u_n - v)dx \to 0, \qquad n \to +\infty. \tag{3.22}$$

Combining (3.21), (3.22), (3.17) and (3.18), we deduce that

$$\begin{aligned} &\int_{\mathbb{R}} w_1(x) \left|v'\right|^{p(u_n)-2} v'(u_n' - v')dx + \int_{\mathbb{R}} w_0(x) \left|v\right|^{p(u_n)-2} v(u_n - v)dx \\ &\qquad \to \int_{\mathbb{R}} w_1(x) \left|v'\right|^{p(u)-2} v'(u' - v')dx + \int_{\mathbb{R}} w_0(x) \left|v\right|^{p(u)-2} v(u - v)dx. \end{aligned} \tag{3.23}$$

Inserting (3.23) in (3.16), we infer

$$\begin{aligned} \int_{\mathbb{R}} f(x,u)(u-v)dx \ge &\int_{\mathbb{R}} w_1(x) \left|v'\right|^{p(u)-2} v'(u' - v')dx \\ &+ \int_{\mathbb{R}} w_0(x) \left|v\right|^{p(u)-2} v(u - v)dx, \qquad \forall\, v \in Z. \end{aligned} \tag{3.24}$$

In particular,

$$\int_{\mathbb{R}} f(x,u)(u-v)dx \geq \int_{\mathbb{R}} w_1(x)\left|v'\right|^{p(u)-2} v'(u-v)'dx + \int_{\mathbb{R}} w_0(x)\left|v\right|^{p(u)-2} v(u-v)dx, \qquad \forall\, v \in C_0^{\infty}(\mathbb{R}). \tag{3.25}$$

Next, observe that the function $w \longmapsto \int_{\mathbb{R}} w_0(x)\left|u-w\right|^{p^-}$ is continuous on $(E_u, \|\cdot\|_{E_u})$. Taking into account that

$$\int_{\mathbb{R}} g(x)\left|u-w\right| dx \leq \left(\int_{\mathbb{R}} \left(\frac{g(x)}{(w_0(x))^{\frac{1}{p^-}}}\right)^{\frac{p^-}{p^--1}} dx\right)^{\frac{p^--1}{p^-}} \left(\int_{\mathbb{R}} w_0(x)\left|u-w\right|^{p^-} dx\right)^{\frac{1}{p^-}}, \quad \forall\, w \in E_u,$$

then by (H_2), we can deduce that the function

$$w \longmapsto \int_{\mathbb{R}} f(x,u)(u-w)dx$$

is continuous on $(E_u, \|\cdot\|_{E_u})$. Using that fact together with Proposition 2.1, we can immediately see that the inequality (3.25) can be extended to the whole space E_u, i.e.

$$\int_{\mathbb{R}} f(x,u)(u-v)dx \geq \int_{\mathbb{R}} w_1(x)\left|v'\right|^{p(u)-2} v'(u'-v')dx + \int_{\mathbb{R}} w_0(x)\left|v\right|^{p(u)-2} v(u-v)dx, \qquad \forall\, v \in E_u. \tag{3.26}$$

For $s > 0$ and $w \in E_u$, choosing $v = u - sw$ as test function in (3.26), it yields

$$s\int_{\mathbb{R}} f(x,u)wdx \geq s\int_{\mathbb{R}} w_1(x)\left|u'-sw'\right|^{p(u)-2}(u'-sw')w'dx + s\int_{\mathbb{R}} w_0(x)\left|u-sw\right|^{p(u)-2}(u-sw)wdx.$$

Thus,

$$\int_{\mathbb{R}} f(x,u)wdx - \int_{\mathbb{R}} w_1(x)\left|u'-sw'\right|^{p(u)-2}(u'-sw')w'dx - \int_{\mathbb{R}} w_0(x)\left|u-sw\right|^{p(u)-2}(u-sw)wdx \geq 0.$$

Tending s to 0^+ in that last inequality, we obtain

$$\int_{\mathbb{R}} f(x,u)wdx \geq \int_{\mathbb{R}} w_1(x)\left|u'\right|^{p(u)-2} u'w'dx + \int_{\mathbb{R}} w_0(x)\left|u\right|^{p(u)-2} uwdx.$$

Clearly, the same inequality holds for $(-w)$. Therefore,

$$\int_{\mathbb{R}} w_1(x)\left|u'\right|^{p(u)-2} u'w'dx + \int_{\mathbb{R}} w_0(x)\left|u\right|^{p(u)-2} uwdx = \int_{\mathbb{R}} f(x,u)wdx, \qquad \forall\, w \in E_u. \tag{3.27}$$

In order to conclude the proof of Theorem 1.2, we need to prove that $f(x,u(x)) = g(x)\left|u\right|^{p(x)-2} u(x)$ a.e. $x \in \mathbb{R}$. For that aim, we start by taking $w = (u-1)^+$ as test function in (3.27):

$$\int_{\mathbb{R}} w_1(x)\left|u'\right|^{p(u)-2} u'((u-1)^+)'dx + \int_{\mathbb{R}} w_0(x)\left|u\right|^{p(u)-2} u(u-1)^+dx = \int_{\mathbb{R}} f(x,u)(u-1)^+dx.$$

Since

$$\int_{\mathbb{R}} w_1(x)\left|u'\right|^{p(u)-2}u'((u-1)^+)'dx = \int_{u\geq 1} w_1(x)\left|u'\right|^{p(u)}dx \geq 0,$$

by (H_3) we get

$$\begin{aligned}\int_{\mathbb{R}} w_0(x)\,|u|^{p(u)-2}u(u-1)^+dx &\leq \int_{\mathbb{R}} f(x,u)(u-1)^+dx \\ &= \int_{u\geq 1} g(x)(u-1)^+dx \leq \int_{u\geq 1} w_0(x)(u-1)^+dx.\end{aligned}$$

Thus,

$$\int_{u\geq 1} w_0(x)\left(|u|^{p(u)-2}u-1\right)(u-1)^+dx \leq 0.$$

We immediately deduce that $u(x)\leq 1$ a.e. $x\in\mathbb{R}$. On the other hand, it is easy to see that $u\geq 0$. In fact, taking $w=u^-=\min(u,0)$ as test function in (3.27), we have

$$\int_{\mathbb{R}} w_1(x)\left|(u^-)'\right|^{p(u)}dx + \int_{\mathbb{R}} w_0(x)\left|u^-\right|^{p(u)}dx = \int_{\mathbb{R}} f(x,u)u^-dx \leq 0,$$

which immediately implies that $u^-=0$ and by consequence $u(x)\geq 0$ a.e. $x\in\mathbb{R}$. Now, taking $w=(\alpha-u)^+$ as test function in (3.27) and having in mind that

$$\int_{\mathbb{R}} w_1(x)\left|u'\right|^{p(u)-2}u((\alpha-u)^+)'dx = -\int_{\alpha\geq u} w_1(x)\left|u'\right|^{p(u)}dx \leq 0,$$

by (H_3) it yields

$$\begin{aligned}\int_{\mathbb{R}} w_0(x)u^{p(u)-1}(\alpha-u)^+dx &\geq \int_{\mathbb{R}} f(x,u)(\alpha-u)^+dx \\ &= \int_{\alpha\geq u} g(x)\alpha^{p(x)-1}(\alpha-u)^+dx \\ &\geq \int_{\alpha\geq u} w_0(x)\alpha^{p(\alpha)-1}(\alpha-u)^+dx.\end{aligned}$$

Hence,

$$\int_{\alpha\geq u} w_0(x)\left(\alpha^{p(\alpha)-1}-u^{p(u)-1}\right)(\alpha-u)^+dx \leq 0. \tag{3.28}$$

In view of (H_1), we know that $\alpha^{p(\alpha)-1}\geq u^{p(u)-1}$ on the set $\{x\in\mathbb{R},\ \alpha\geq u(x)\}$. From (3.28), we deduce that $(\alpha-u)^+=0$ and by consequence $u(x)\geq\alpha$ a.e. $x\in\mathbb{R}$. Finally, we conclude that $u\neq 0$ and $f(x,u(x))=g(x)u^{p(x)-1}$ a.e. $x\in\mathbb{R}$. This ends the proof of Theorem 1.2. □

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Sobolev inequality with non-uniformly degenerating gradient

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Abstract. In this paper we prove the following weighted Sobolev inequality in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$, of a homogeneous space $(\mathbb{R}^n, \rho, w dx)$, under suitable compatibility conditions on the positive weight functions $(v, w, \omega_1, \omega_2, \dots, \omega_n)$ and on the quasi-metric ρ,

$$\left(\int_\Omega |f|^q v\, w dz \right)^{\frac{1}{q}} \leq C \sum_{i=1}^{N} \left(\int_\Omega |f_{z_i}|^p \omega_i M_S w\, dz \right)^{\frac{1}{p}}, \quad f \in \mathrm{Lip}_0(\overline{\Omega}),$$

where $q \geq p > 1$ and M_S denotes the strong maximal operator. Some corollaries on non-uniformly degenerating gradient inequalities are derived.

Keywords: Sobolev's inequality, homogeneous space, non-uniformly degenerating gradient.

2020 Mathematics Subject Classification: 26D10, 35B45, 42B25.

1 Introduction

In this paper we aim to prove the following weighted Sobolev type inequality in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$, of a homogeneous space $(\mathbb{R}^n, \rho, w dx)$

$$\left(\int_\Omega |f|^q v\, w dz \right)^{\frac{1}{q}} \leq C \sum_{i=1}^{N} \left(\int_\Omega |f_{z_i}|^p \omega_i M_S w\, dz \right)^{\frac{1}{p}}, \quad f \in \mathrm{Lip}_0(\overline{\Omega}), \tag{1.1}$$

where $q \geq p > 1$ and M_S denotes the strong maximal operator. This can be done under suitable compatibility conditions on the positive weight functions $(v, w, \omega_1, \omega_2, \dots, \omega_n)$ and on the quasi-metric ρ.

We say that (1.1) is a non-uniform weighted Sobolev inequality since the functions $\omega_i \omega_j^{-1}$, $i, j = 1, \dots, n$, are not assumed to be neither bounded nor bounded away from zero in any compact subset of Ω.

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Poincaré–Sobolev type inequalities are essential in many contexts of the theory of elliptic and parabolic partial differential equations such as the Harnack's inequality, the regularity of solutions, the continuation of differential inequalities, the absence of positive eigenvalues, the estimation of negative eigenvalues, the spectrum discreetness of Schrödinger operator etc. (see, e.g., [1,3,5,11,12,23–27,30,33,36,38,40]).

The study of the above mentioned qualitative properties of second order elliptic equations in absence of uniform ellipticity condition and in lack of uniform degeneration relies on Poincaré–Sobolev type weighted inequalities having non-uniformly degenerating gradient. Meanwhile, the theory has also been extended to more general contexts, such as that of Carnot–Carathéodory metrics associated with families of vector fields (see, e.g., [8–10,13]).

To deduce the inequality (1.1) one could first derive a suitable representation formula in terms of integral operators of potential type, and then use some continuity results for these operators in proper metric spaces, endowed with doubling measures (see, for example, [18]). In this paper, we show a new different approach to obtain the inequality (1.1). The arguments of our proofs are inspired by those of [31] (see, also the recent papers [29,32]), where the Euclidean metric was considered. However, the ideas of [31,32] cannot be simply adapted to homogeneous spaces and to non-uniformly degenerating gradients, since not all the homogeneous spaces posses the Besicovitch covering property (see, e.g., [35]). To overcome this difficulty, we use the "$5B$" covering lemma that holds in any homogeneous space, see e.g., [7,39]. We refer to [37], and to the references therein where the Euclidean metric and equal weights ω_i, , $i = 1, \dots, n$, are considered.

In general, when dealing with multi-weighted Sobolev inequalities the task is to find sufficient (and hopefully necessary) conditions on the measures $\omega_i(x)dx$, $i = 1, \dots, n$, and $v(x)dx$ which give

$$\Big(\int_\Omega |f|^q v\,dz\Big)^{\frac{1}{q}} \le C \sum_{i=1}^{N} \Big(\int_\Omega |f_{z_i}|^p \omega_i\,dz\Big)^{\frac{1}{p}}, \quad f \in \mathrm{Lip}_0(\overline{\Omega}), \tag{1.2}$$

where $1 \le p \le q < \infty$ and the constant C does not depend on f and Ω. For equal weights ω_i, $i = 1, \dots, n$, sharp sufficient conditions can be found in [4,15] and in the papers [20,32,34]. Though this subject has been extensively studied in the last years it is still far from its full characterization (see, [6,14–22]). Some progresses in deriving sufficient conditions for the Sobolev–Poincaré type inequalities with Grushin type weights were made in the works [15,29]. In this article we give sufficient conditions for the inequality (1.2) to hold and we show some generalizations.

2 Notation and main results

We say that $(\mathbb{R}^n, \rho)$ is a quasi-metric space if the function $\rho : \mathbb{R}^n \times \mathbb{R}^n \to (0, \infty)$ satisfies the following properties:

1) $\rho(x, y) \ge 0$ for all $x, y \in \mathbb{R}^n$; $\rho(x, y) = 0$ if and only if $x = y$;

2) $\rho(x, y) \le K_0\big(\rho(x, z) + \rho(y, z)\big)$ for all $x, y, z \in \mathbb{R}^n$, with K_0 positive constant;

3) $\rho(x, y) = \rho(y, x)$ for all $x, y \in \mathbb{R}^n$.

A useful result by Macías and Segovia (see [28]) asserts that, every quasi-metric space is metrizable, i.e. there exist a distance d and a positive number $\alpha > 0$ such that ρ^α is equivalent to d.

Now, let us denote by $B(x,r) = \{y \in \mathbb{R}^n : \rho(y,x) < r\}$ the ρ-metric ball with center in $x \in \mathbb{R}^n$ and radius $r > 0$, and let μ be a nonnegative Borel measure on $\mathbb{R}^n$ satisfying the doubling condition. We say the measure μ is a doubling measure if there exists C_1 such that

$$\mu\left(B(x,2r)\right) \leq C_1 \mu\left(B(x,r)\right) \quad \text{for all } x \in \mathbb{R}^n,\ r > 0.$$

The quasi-metric space $(\mathbb{R}^n, \rho)$ equipped with a doubling measure μ is called a homogeneous space and it is denoted by $(\mathbb{R}^n, \rho, d\mu)$ (see [7]). In Section 3 we will give an example of homogeneous space.

In sequel, the notation $Q_n(x,r)$ (or simply $Q(x,r)$) denotes the n-dimensional Euclidean ball $Q(x,r) = \{y \in \mathbb{R}^n : |y-x| < r\}$ centered in x and of radius r. For $i = 1, \ldots, n$, we denote by $\ell_i(B(x,r)) = \sup\left\{|z_i - y_i| : z = (z_1, \ldots, z_n), y = (y_1, \ldots, y_n) \in B(x,r)\right\}$ and by $d(\Omega) = \sup\{\rho(x,y) : x, y \in \Omega\}$ the ρ-diameter of the domain Ω. We also let Σ be the collection of ρ-metric balls with center in Ω and radius less then $d(\Omega)$.

Given an integrable function f and a measurable set $E \subset \mathbb{R}^n$ we denote by $f(E) = \int_E f(x)dx$ the weighted measure of E, while $|E|$ denotes the Lebesgue measure of E. Denote by p' the conjugate number of $1 < p < \infty$ such that $\frac{1}{p} + \frac{1}{p'} = 1$

A measurable function taking a.e. finite positive values is called a weight. A weight function $f : \mathbb{R}^n \to (0, \infty)$ belongs to the A_p-Muckenhoupt weight class, $1 < p < \infty$, with respect to the quasi-metric ρ, if for any ρ-metric ball $B = B(x,r) \subset \mathbb{R}^n$, one has

$$\left(\frac{1}{|B|}\int_B f(z)dz\right)\left(\frac{1}{|B|}\int_B f^{-\frac{1}{p-1}}(z)dz\right)^{p-1} \leq C, \tag{2.1}$$

while it belongs to the A_1-class if

$$\frac{1}{|B|}\int_B f(z)dz \leq C \inf_B f(z),$$

where the constants $C > 0$ do not depend on $x \in \mathbb{R}^n$ and $r > 0$.

A weight function $f : \mathbb{R}^n \to (0, \infty)$ belongs to the A_∞-Muckenhoupt weight class $A_\infty(dx)$ if there exist two constants $C, \delta > 0$ such that for any ρ-metric ball $B = B(x,r)$ and any measurable subset $E \subset B$ it holds that

$$\frac{f(E)}{f(B)} \leq C\left(\frac{|E|}{|B|}\right)^{\delta}. \tag{2.2}$$

Let $g : \mathbb{R}^n \to (0, \infty)$ be a weight function and μ be a doubling measure. We say g belongs to the $A_\infty(\mu)$ weight class if there exist two constants $C, \delta > 0$ such that for any ρ quasi-metric ball $B = B(x,r)$ and any measurable subset $E \subset B$ one has

$$\frac{\int_E g d\mu}{\int_B g d\mu} \leq C\left(\frac{\mu(E)}{\mu(B)}\right)^{\delta}, \tag{2.3}$$

For the main properties of the A_p-Muckenhoupt's weight classes, we refer the reader, for instance, to [7]. It is well-known that $A_p \subset A_\infty$ for any fixed $1 \leq p \leq \infty$ and moreover $A_\infty = \cup_{1 \leq p < \infty} A_p$. Furthermore, $A_p \subset A_{p-\varepsilon}$, for some $\varepsilon > 0$ depending on the constant C in the A_p class definition.

In the statement and proof of Theorem 2.5 below, we use the strong maximal function $M_S w$. For sake of completeness, let us recall its definition. Let $\mathcal{R}$ denote the collection of rectangles R in $\mathbb{R}^n$ with sides parallel to the coordinate axes, we define the strong maximal function M_S as

$$M_S f(x) = \sup_{R \ni x} \frac{1}{|R|} \int_R |f(y)|\, dy, \quad f \in L^{1,loc}$$

In Theorem 2.6, we make use of the classical fractional maximal operator $\mathcal{M}_\varepsilon$ defined as

$$\mathcal{M}_\varepsilon f(x) = \sup_{Q \ni x} \frac{1}{|Q|^{(n-\varepsilon)/n}} \int_Q |f(y)| dy,$$

where the supremum is taken all over the Euclidean balls $\{Q\}$ containing the point x.

In the proofs of our main results we avail ourselves of the so called "$5B$" covering lemma below. This lemma, unlike Besicovich covering property, is valid in any homogeneous space.

Lemma 2.1 ([1, Covering Lemma, p. 270])**.** *Let (X, ρ, μ) be a homogeneous space. Let $B = \{B_\alpha = B(x_\alpha, r_\alpha) : \alpha \in \Gamma\}$ be a family of balls in X such that $\cup_{\alpha \in \Gamma} B_\alpha$ is bounded. Then there exists a sequence of disjoint balls $\{B(x_i, r_i)\}_{i \in \mathbb{N}} \subset B$ such that for every $\alpha \in \Gamma$ there exists i satisfying $r_\alpha \leq 2r_i$ and $B_\alpha \subset B(x_i, 5K_0^2 r_i)$.*

Definition 2.2. Throughout this paper, we consider the quasi-metrics ρ satisfying the following "S"-condition

$$|B^\#| \leq C|B|, \tag{2.4}$$

for all the ρ quasi-metric balls B (see next section for their definition). Here $B^\#$ is the smallest parallelepiped with edges parallel to coordinate axes containing the ρ quasi-metric ball B.

Definition 2.3. Moreover, we assume that there exists a constant $c > 0$ such that for every B and every $x, y \in B$, $t \in (0,1)$ one has

$$x + t(y - x) \in cB. \tag{2.5}$$

Now, we are ready to state our main results.

Theorem 2.4. *Let $q \geq p \geq 1$, $(\mathbb{R}^n, \rho, dx)$ be a homogeneous space and $\Omega \subset \mathbb{R}^N$ be a bounded domain. Assume that the ρ quasi-metric balls $B \in \Sigma$ satisfy the S-condition (2.4) and (2.5). Let $v \in A_\infty(dx)$ and $\omega_i^{1-p'}$, $i = 1, 2, \ldots, n$, be doubling functions on Σ. If*

$$(\ell_i(B)/|B|) \Big(\int_{B \cap \Omega} v\, dx \Big)^{\frac{1}{q}} \Big(\int_{B \cap \Omega} \omega_i^{1-p'}\, dx \Big)^{\frac{1}{p'}} \leq \tilde{A}, \tag{2.6}$$

$i = 1, 2, \ldots, n$, on any $B \in \Sigma$, then

$$\Big(\int_\Omega |f|^q v\, dx \Big)^{\frac{1}{q}} \leq C_0 \tilde{A} \sum_{i=1}^n \Big(\int_\Omega |f_{z_i}|^p \omega_i\, dx \Big)^{\frac{1}{p}}, \tag{2.7}$$

for all Lipschitz continuous functions $f : \overline{\Omega} \to \mathbb{R}$ vanishing on $\partial\Omega$, and with a constant C_0 depending only p, q, n and on C, δ in (2.2).

Theorem 2.4 is an easy consequence of the next assertion.

Theorem 2.5. *Let $q \geq p \geq 1$, $(\mathbb{R}^n, \rho, dx)$ be a homogeneous space and $\Omega \subset \mathbb{R}^N$ be a bounded domain. Assume that the ρ quasi-metric balls $B \in \Sigma$ satisfy the S-condition (2.4) and (2.5). Let*

$v : \mathbb{R}^N \to (0, \infty)$ be an $A_\infty(wdx)$ function and $\omega_i^{1-p'} M_S w$, $i = 1, 2, \ldots, n$, be doubling functions on Σ. If

$$\ell_i(B) \left(\int_B vw\, dy \right)^{1/q} \left(\int_B \omega_i^{1-p'} M_S w(y)\, dy \right)^{1/p'} \le A \int_B w(y)\, dy, \tag{2.8}$$

$i = 1, 2, \ldots, n$, on any $B \in \Sigma$, then

$$\Big(\int_\Omega |f|^q v\, w(z)\, dz \Big)^{\frac{1}{q}} \le C_0 A \sum_{i=1}^N \Big(\int_\Omega |f_{z_i}|^p \omega_i\, M_S w(z)\, dz \Big)^{\frac{1}{p}}, \tag{2.9}$$

for all Lipschitz continuous functions $f : \overline{\Omega} \to \mathbb{R}$ vanishing on $\partial\Omega$, and with a constant C_0 depending only on p, q, n and on C, δ in (2.3).

We remark that, in Theorem 2.5, the Sobolev type weight inequality (2.9) is proven with different weights for the partial derivatives. This is due to the fact that the weights and the metric must be in a balance with the geometry of the quasi-metric balls. Taking $v \equiv \omega_i \equiv 1$ in (2.9) we get the measure $w(x)dx$ to be a doubling function on Σ, hence we obtain the inequality

$$\Big(\int_\Omega |f|^q w(x)\, dx \Big)^{\frac{1}{q}} \le C_0 \sum_{i=1}^N \Big(\int_\Omega |f_{z_i}|^p\, M_S w(x) dx \Big)^{\frac{1}{p}}.$$

Moreover, let us mention that the doubling condition on the weights $\omega_i^{1-p'} M_S w$ in Theorem 2.5 is motivated by the use Lemma 4 of [39, Chapter 8].

In the next Theorem 2.6 we give a better estimate. In order to do that, the sufficiency condition (2.8) needs to be suitably strengthened (see (2.11)). Theorem 2.6 below gives, locally, a finer inequality since

$$d(\Omega)^\varepsilon \sup_{B \in \Sigma, B \ni x} w(B)/|B| \ge \sup_{B \in \Sigma, B \ni x} w(B)/|B|^{1-\varepsilon/n}.$$

Theorem 2.6. *Let $q \ge p \ge 1$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain. Let $(\mathbb{R}^N, \rho, wdx)$ be a homogeneous space and assume that there exists a positive constant C_1 such that*

$$C_1 |x - y| \le \rho(x, y) \tag{2.10}$$

for all $x, y \in \Omega$. Let $v : \mathbb{R}^N \to (0, \infty)$ be an $A_\infty(wdx)$ weight function and $\omega_i^{1-p'} \mathcal{M}_\varepsilon w$, $i = 1, 2, \ldots, n$, be doubling functions on Σ. Assume that the ρ quasi-metric balls $B \in \Sigma$ satisfy the S-condition (2.4) and (2.5). If

$$\ell_i(B) \left(\frac{r(B)^{n-\varepsilon}}{|B|} \right) \left(\int_B vw\, dy \right)^{1/q} \left(\int_B \omega_i^{1-p'} \mathcal{M}_\varepsilon w\, dy \right)^{1/p'} \le \bar{A} \int_B w(y)\, dy \tag{2.11}$$

$i = 1, 2, \ldots, n$, with $\varepsilon \in [0, 1)$ uniformly with respect to $B \in \Sigma$, then

$$\Big(\int_\Omega |f|^q v\, w(z)\, dz \Big)^{\frac{1}{q}} \le C_0 \bar{A} \sum_{i=1}^N \Big(\int_\Omega |f_{z_i}|^p \omega_i\, \mathcal{M}_\varepsilon w(z)\, dz \Big)^{\frac{1}{p}}, \tag{2.12}$$

for all Lipschitz continuous functions $f : \overline{\Omega} \to \mathbb{R}$ vanishing on $\partial\Omega$, and with a constant C_0 depending on p, q, n and on C, δ in (2.3).

3 An example of homogeneous space

Let $\omega : \mathbb{R}^n \to (0,\infty)$ be a positive measurable function, such that $\sigma(x) = \frac{1}{\omega(x)}$ is in the Muckenhoupt A_2-weight class all over the n-dimensional Euclidean balls. This condition used in proofs of the corollaries below. Observe that this gives that also ω is in the Muckenhoupt's A_2-class all over the n-dimensional Euclidean balls.

For $x \in \mathbb{R}^n$, define a function $h_x : t \in [0,\infty) \to h_x(t) \in [0,\infty)$ as

$$h_x(t) = t\left(\frac{1}{|Q(x,t)|}\int_{Q(x,t)} \sigma(s)\,ds\right)^{\frac{1}{2}}, \quad t > 0$$

and assume that $h_x(0) = 0$, $\lim_{t\to+\infty} h_x(t) = +\infty$ for a fixed $x \in \mathbb{R}^n$. Then we may consider an inverse function $h_x^{-1} : s \in [0,\infty) \to h_x^{-1}(s) \in [0,\infty)$ defined as

$$h_x^{-1}(s) = \inf\Big\{t > 0 : \ h_x(t) \geq s\Big\}, \quad s > 0.$$

and $h_x^{-1}(0) = 0$. We can define a quasi-metric ρ on $\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m = \{z = (x,y)\,|x \in \mathbb{R}^n\,, y \in \mathbb{R}^m\}$ as follows: for any $z_1 = (x_1,y_1), z_2 = (x_2,y_2) \in \mathbb{R}^N$ we put

$$\rho(z_1,z_2) = \max\Big\{|x_1 - x_2|,\ h_{x_1}^{-1}(|y_2 - y_1|),\ h_{x_2}^{-1}(|y_2 - y_1|)\Big\}. \tag{3.1}$$

The function $\rho : \mathbb{R}^N \times \mathbb{R}^N \to [0,\infty)$ is a quasi-metric satisfying the triangle inequality

$$\rho(z_1,z_2) \leq K_0\Big(\rho(z_1,z_3) + \rho(z_2,z_3)\Big) \tag{3.2}$$

with a constant $K_0 \geq 1$ independent of $z_1, z_2, z_3 \in \mathbb{R}^N$, (see, e.g., [1,15]). Therefore, the above defined quasi-metric space $(\mathbb{R}^N,\rho)$ endowed with the Lebesgue measure is a homogeneous space.

In general, the balls of a homogeneous space are not convex, therefore the conditions (2.4), (2.5) may be failed. The condition (2.4) means that the Lebesgue measure of a metric ball comparable with Lebesgue measure of its circumscribed parallelepiped. Also as we have noted the balls of a metric space are not convex the line segment connecting any two points of a ball may get out of that ball. The meaning of condition (2.5) is that, although the points on a line segment get out of the ball its points are contained on the comparable ball. It easily seen that the balls of metric (3.1) are convex and conditions (2.4), (2.5) are satisfied for that.

4 Applications

In this section, we give two examples of applications of Theorem 2.4. To this aim, let ρ be the quasi-metric defined in (3.1). It is not difficult to see that the ball $B(z_0,R)$ with center in $z_0 = (x_0,y_0) \in \mathbb{R}^N$ and radius $R > 0$ of this quasi-metric is given by

$$B(z_0,R) = \Bigg\{z = (x,y) \in \mathbb{R}^n \times \mathbb{R}^m : \ |x - x_0| < R, \\ |y - y_0| < R\left(\frac{1}{|Q(x_0,R)|}\int_{Q(x_0,R)} \sigma(t)dt\right)^{\frac{1}{2}}\Bigg\} \tag{4.1}$$

Let ω as in the beginning of Section 3 and $f : (x,y) \in \mathbb{R}^n \times \mathbb{R}^m \to f(x,y) \in \mathbb{R}$ be a Lipschitz continuous function. The degenerated gradient of f is given by

$$|\nabla_\omega f|^2 = \omega(x)|\nabla_x f|^2 + |\nabla_y f|^2.$$

For $m \geq 2$ we can prove the following result:

Corollary 4.1. *Let $n+m \geq 3$, $q = \frac{2(n+m)}{n+m-2}$, $t = \frac{n}{n+m-2}$ and let $\omega \in A_2$-Muckenhoupt class function on $\mathbb{R}^n$. Then,*

$$\Big(\int_{B(z_0,R)} \omega^t \, |f|^q \, dz \Big)^{\frac{1}{q}} \leq C_0 \Big(\int_{B(z_0,R)} |\nabla_\omega f|^2 \, dz \Big)^{\frac{1}{2}} \tag{4.2}$$

for any function f, Lipschitz continuous in the ball $B(z_0,R) \subset \mathbb{R}^N$, vanishing on $\partial B(z_0,R)$. The positive constant C_0 in (4.2) depends on n,m and on the constants in the A_2-condition from (2.1).

For $m = 1$, we have:

Corollary 4.2. *Let $n > 1$, $q = \frac{2(n+1)}{n-1}$, and let ω^{-1} be a classical $A_{1+\frac{1}{n'}}$-Muckenhoupt class function on $\mathbb{R}^n$. Then,*

$$\Big(\int_{B(z_0,R)} \omega^{\frac{n}{n-1}} \, |f|^q dz \Big)^{\frac{1}{q}} \leq C_0 \Big(\int_{B(z_0,R)} |\nabla_\omega f|^2 \, dz \Big)^{\frac{1}{2}} \tag{4.3}$$

for any function f, Lipschitz continuous in the ball $B(z_0,R) \subset \mathbb{R}^N$ and vanishing on $\partial B(z_0,R)$. The positive constant C_0 in (4.2) depends on n and on the constants in the $A_{1+\frac{1}{n'}}$-condition from (2.1).

Corollary 4.3. *Let $q \in [2, 2N/(N-2)]$ and let v, $\omega : \mathbb{R}^n \to (0,\infty)$ be functions of the variable x only of classes A_∞ and A_2, respectively. Let*

$$\left(\frac{r}{R}\right)^{1-\frac{(N-n)(n+2)}{2}\left(\frac{1}{2}-\frac{1}{q}\right)} \left(\frac{v\left(Q_r^x\right)}{v\left(Q_R^x\right)}\right)^{\frac{1}{q}} \leq C \left(\frac{\omega\left(Q_r^x\right)}{\omega\left(Q_R^x\right)}\right)^{\frac{1}{2}-\frac{N-n}{2}\left(\frac{1}{2}-\frac{1}{q}\right)} \tag{4.4}$$

for any $x \in \mathbb{R}^n$ and $r > 0$. Then for all $f \in Lip_0\left(B_R^{z_0}\right)$

$$\Big(\int_{B(z_0,R)} v \, |f|^q \, dz \Big)^{1/q} \leq C_0 A(x_0,R) \, R \left(\int_{B(z_0,R)} |\nabla_\omega f|^2 \, dz \right)^{1/2} \tag{4.5}$$

holds with

$$A(x_0,R) = R^{-\frac{(N-n)(n+2)}{2}\left(\frac{1}{2}-\frac{1}{q}\right)} v\left(Q_R^{x_0}\right)^{\frac{1}{q}} \Big/ \omega\left(Q_R^{x_0}\right)^{\frac{1}{2}-\frac{N-n}{2}\left(\frac{1}{2}-\frac{1}{q}\right)},$$

C_0 depends on the A_∞, A_2 conditions for v,ω and n,q.

The given above corollaries generalize the two-weight Sobolev inequalities to the case of non-uniformly degenerate gradient $\nabla_\omega f$. Therefore, those inequalities are of the well-known inequalities type by Chanillo–Wheeden, Fabes–Kenig–Serapioni with $\omega \equiv 1$. Such inequalities may be applied to the study of equations with Grushin type operator $\partial_{x_i}\big(\omega(x)\partial_{x_i}\big) + \partial^2_{y_j}$ or its generalizations $\partial_{x_i}\big(\omega(x)w\partial_{x_i}\big) + \partial_{y_j}\big(w\partial_{y_j}\big)$ when $w(x,y)$ is a function of two variables x,y obligated to satisfy some conditions.

Note that, the condition (4.4) is a balance condition of Chanillo–Wheeden type [4] for the case of non-uniformly degenerate gradient inequality of the Sobolev type. Note again, the function v depends only the variable x while the function f is dependent of two variables $z = (x,y)$.

5 Proofs of the main results

Let us start proving Theorem 2.5.

5.1 Proof of Theorem 2.5

Assume that f is not equal to zero almost everywhere in Ω, otherwise the result of Theorem 2.5 is trivial. For $\alpha > 0$ set $\Omega_\alpha = \{x \in \Omega : |f(x)| > \alpha\}$. Since f is continuous the set Ω_α is open. Let a fixed α be such that the set $\Omega_{3\alpha}$ is nonempty. Choose a countable covering of Ω_α made up of connected components $\Omega_{\alpha,j} \subset \Omega_\alpha$, $j \in \mathbb{N}$. Denote the parts of $\Omega_{3\alpha}$ and $\Omega_{2\alpha}$ contained in $\Omega_{\alpha,j}$ by $\Omega_{3\alpha,j}$ and $\Omega_{2\alpha,j}$, respectively (note that the sets $\Omega_{3\alpha,j}$ and $\Omega_{2\alpha,j}$ need not to be connected).

For the reader's convenience, let us recall that the weight function w of the homogeneous space $(\mathbb{R}^n, \rho, w dx)$ satisfies the doubling condition on the ρ quasi-metric balls. Let $b \in \Omega_{3\alpha,j}$ be a fixed point. Let us show that there exists a ρ-quasi metric ball $B = B(b, r(b))$ such that

$$w\left(B \setminus \Omega_{\alpha,j}\right) = \gamma\, w(B), \tag{5.1}$$

where γ is a small positive number that will be chosen later on. To this aim, let $\gamma > 0$ and define the function

$$F(t) = \frac{1}{\gamma} w\left(B(b,t) \setminus \Omega_{\alpha,j}\right) - w\left(B(b,t)\right),$$

which is continuous and negative for sufficiently small $t > 0$ since b is an interior point of $\Omega_{3\alpha,j}$.

From the doubling property of w on the ρ-quasimetric balls it follows that there exists a positive real number τ such that

$$w\left(B\left(b, d(\Omega)\right) \setminus \Omega\right) \geq \tau w\left(B\left(b, d(\Omega)\right)\right).$$

Let us choose the constant $\gamma > 0$ so that the function $F(t)$ is positive for $t = d(\Omega)$. Observe, that is always possible since

$$\begin{aligned} F(d(\Omega)) &= \frac{1}{\gamma} w\left(B\left(b, d(\Omega)\right) \setminus \Omega_{\alpha,j}\right) - w\left(B(b, d(\Omega))\right) \\ &\geq \frac{1}{\gamma} w\left(B\left(b, d(\Omega)\right) \setminus \Omega\right) - w\left(B(b, d(\Omega))\right) \\ &\geq \left(\frac{\tau}{\gamma} - 1\right) w\left(B\left(b, d(\Omega)\right)\right), \end{aligned}$$

thus it suffices to choose γ such that $\frac{\tau}{\gamma} - 1 > 0$ in order to get $F(d(\Omega)) \geq 0$. Hence, by the Bolzano–Cauchy theorem for continuous functions we get that there exists a $t^* \in (0, d(\Omega))$ such that $F(t^*) = 0$. Therefore, if we take $r(b) = t^*$ we achieve equality (5.1).

Now, there are two possibilities:

Case 1)

$$w\left(B^* \cap \Omega_{3\alpha,j}\right) \leq \gamma w(B^*), \tag{5.2}$$

Case 2)

$$w\left(B^* \cap \Omega_{3\alpha,j}\right) > \gamma w(B^*), \tag{5.3}$$

where $B^* = B(b, 5K_0^2 r(b))$.

In case 1), denoted by $\lambda = vw$, using the doubling property of the function $v \in A_\infty(wdx)$, it follows

$$\lambda\left(B^* \cap \Omega_{3\alpha,j}\right) \le C\gamma^\delta \lambda(B^*) \le CC_1\gamma^\delta\lambda(B). \tag{5.4}$$

By (5.1) and since $v \in A_\infty(wdx)$ we have again

$$\lambda(B) = \lambda\left(B \cap \Omega_{\alpha,j}\right) + \lambda\left(B \setminus \Omega_{\alpha,j}\right) \le \lambda\left(B \cap \Omega_{\alpha,j}\right) + C\gamma^\delta\lambda(B),$$

therefore, eventually reducing γ

$$\lambda(B) \le \frac{1}{1 - C\gamma^\delta}\lambda\left(B \cap \Omega_{\alpha,j}\right).$$

Thus, by (5.4) we get

$$\lambda\left(B^* \cap \Omega_{3\alpha,j}\right) \le \frac{CC_1\gamma^\delta}{1 - C\gamma^\delta}\lambda\left(B \cap \Omega_{\alpha,j}\right) \tag{5.5}$$

In case 2), we have two possibilities:

2a)

$$\left|B^* \setminus \Omega_{2\alpha,j}\right| \ge \frac{1}{2}|B^*| \tag{5.6}$$

and

2b)

$$\left|B^* \cap \Omega_{2\alpha,j}\right| > \frac{1}{2}|B^*|. \tag{5.7}$$

If 2a) takes place, let us show that

$$1 \le \frac{2}{\gamma\alpha}\sum_{i=1}^n \frac{\ell_i(B^*)|(B^*)^\#|}{|B^*|w(B^*)}\int_{B^{**}\cap\left(\Omega_{2\alpha,j}\setminus\Omega_{3\alpha,j}\right)} |f_{x_i}(z)|M_S w(z)\,dz, \tag{5.8}$$

where $M_S w$ denotes the strong maximal function of w, B^{**} is the ρ-metric ball $B^{**} = cB^*$ and $(B^*)^\#$ denotes the smallest rectangular with edges parallel to coordinate axes containing B^*.

To prove inequality (5.8), we follow an idea of [31], formula (3.7). Denote $\hat{A} = B^* \setminus \Omega_{2\alpha,j}$ and $Z = B^* \cap \Omega_{3\alpha,j}$. Let the points $x \in \hat{A}$ and $y \in Z$ be arbitrary fixed. Since the quasi-metric balls are not assumed to be convex, the line segment $\overline{xy} = \{x + t(y - x) : t \in [0,1]\}$ connecting x, y may get out of the ball B^* as t varies in $(0,1)$. But, due to hypothesis (2.5) it will stay in the congruent ball $B^{**} = cB^*$.

Also, the line segment $\overline{xy}$ intersects the surfaces $\{z' \in \Omega_{\alpha,j} : |f(z')| = \alpha\}$ and $\{z'' \in \Omega_\alpha^j : |f(z'')| = 2\alpha\}$ in some points $z' = x + t_1(y - x)$ and $z'' = x + t_2(y - x)$ where $t_1, t_2 \in [0,1]$, with $t_2 > t_1$ depend on x, y. Here, t_2 corresponds to the value of t for which $\overline{xy}$ meets for the first time the surface $\partial\Omega_{2\alpha,j}$ after leaving $\partial\Omega_{\alpha,j}$ while t_1 corresponds to the value of t when $\overline{xy}$ intersects the surface $\partial\Omega_{\alpha,j}$.

Having this in mind and using (5.1), (5.6) it follows that

$$\frac{1}{2}\gamma w(B^*)|B^*| \le \frac{1}{\alpha}\int_{\hat{A}}\Big(\int_Z |f(z'') - f(z')|dy\Big)w(x)dx. \tag{5.9}$$

Whence,

$$\frac{1}{2}\gamma w(B^*)|B^*| \le \frac{1}{\alpha}\int_{\hat{A}}\Big(\int_Z\Big(\int_{t_1(z,y)}^{t_2(z,y)}\Big|\frac{\partial f}{\partial t}(x + t(y - x))\Big|dt\Big)dy\Big)w(x)dx.$$

By Fubini's theorem,

$$\frac{1}{2}\gamma w(B^*)|B^*| \le \sum_{i=1}^{n} \frac{\ell_i(B^*)}{\alpha} \int_{\hat{A}} \Big(\int_0^1 \Big(\int_{\{y\in B^*:\, x+t(y-x)\in G\}} \Big|\frac{\partial f}{\partial z_i}\big(x+t(y-x)\big)\Big| dy\Big) dt\Big) w(x)dx,$$

where $G = B^{**} \cap (\Omega_{2\alpha,j} \setminus \Omega_{3\alpha,j})$.

Let us now make the change of variable $z = x + t(y - x)$ in the interior integral to pass from y to z. Since $dy = t^{-n}dz$, one has

$$\frac{1}{2}\gamma w(B^*)|B^*| \le \sum_{i=1}^{n} \frac{\ell_i(B^*)}{\alpha} \int_{\hat{A}} \Big(\int_0^1 \Big(\int_{\{z\in G:\, \frac{z-x}{t}+x\in Z\}} \Big(\Big|\frac{\partial f}{\partial z_i}(z)\Big| dz\Big) \frac{dt}{t^n}\Big) w(x)dx. \tag{5.10}$$

For $t \in (0,1)$ and $z \in G$ it follows $|x_s - z_s| < tl_s(B^*)$, $s = 1, 2, \ldots, n$, therefore applying Fubini's formula again, we get

$$\frac{1}{2}\gamma w(B^*)|B^*| \le \sum_{i=1}^{N} \frac{\ell_i(B^*)}{\alpha} \int_0^1 \Big(\int_G \Big|\frac{\partial f}{\partial z_i}(z)\Big| \Big(\int_{\{z:\, |z_s-x_s|<tl_s(B^*),\, s=1,2,\ldots,N\}} w(x)dx\Big) dz\Big) \frac{dt}{t^n}, \tag{5.11}$$

where $G = B^{**} \cap \left(\Omega_{2\alpha,j} \setminus \Omega_{3\alpha,j}\right)$.

Then

$$1 \le \frac{2}{\gamma\alpha} \sum_{i=1}^{n} \frac{\ell_i(B^*)|\,(B^*)^{\#}\,|}{|B^*|w(B^*)} \int_{B^{**}\cap(\Omega_{2\alpha,j}\setminus\Omega_{3\alpha,j})} |f_{z_i}(z)| M_S w(z)\, dz,$$

where M_S is the strong maximal operator. Therefore,

$$\lambda\left(\Omega_{3\alpha,j} \cap B^*\right) \le \frac{2}{\gamma\alpha} \sum_{i=1}^{n} \frac{\ell_i(B^*)\lambda(B^*)|\,(B^*)^{\#}\,|}{|B^*|w(B^*)} \int_{B^{**}\cap(\Omega_{2\alpha,j}\setminus\Omega_{3\alpha,j})} |f_{z_i}(z)| M_S w(z)\, dz. \tag{5.12}$$

In the case 2b) we can argue as in case 2a) by putting $\hat{A} = B^* \setminus \Omega_{\alpha,j}$ and $Z = \Omega_{2\alpha,j} \cap B^*$. Thus, we have

$$\frac{1}{2}\gamma w(B^*)|B^*| \le \frac{1}{\alpha} \int_{B^*\setminus\Omega_{\alpha,j}} \Big(\int_{\Omega_{2\alpha,j}\cap B^*} |f(z'') - f(z')| dy\Big) w(x)dx.$$

In this case the line segment $\overline{xy}$ intersects the surfaces $\{z' \in \Omega_{\alpha,j} : |f(z')| = \alpha\}$ and $\{z'' \in \Omega_\alpha^j : |f(z'')| = 2\alpha\}$ in points that can be expressed as $z' = x + t_1(y - x)$ and $z'' = x + t_2(y - x)$ where $t_1, t_2 \in [0,1]$, with $t_2 > t_1$ depend on x, y. Here, t_2 corresponds to the value of t for which $\overline{xy}$ meets for the first time the surface $\partial\Omega_{2\alpha,j}$ after leaving $\partial\Omega_{\alpha,j}$ while t_1 corresponds to the value of t when $\overline{xy}$ intersects the surface $\partial\Omega_{\alpha,j}$.

In this case, in place of (5.11), we get the following inequality

$$\frac{1}{2}\gamma w(B^*)|B^*| \le \sum_{i=1}^{N} \frac{\ell_i(B^*)}{\alpha} \int_0^1 \Big(\int_G \Big|\frac{\partial f}{\partial z_i}(z)\Big| \Big(\int_{\{z:\, |z_s-x_s|<tl_s(B),\, s=1,2,\ldots,N\}} w(x)dx\Big) dz\Big) \frac{dt}{t^n},$$

where $G = B^{**} \cap \left(\Omega_{\alpha,j} \setminus \Omega_{2\alpha,j}\right)$.

Therefore,

$$1 \le \frac{2}{\gamma\alpha} \sum_{i=1}^{n} \frac{\ell_i(B^*)|\,(B^*)^{\#}\,|}{|B^*|w(B^*)} \int_{B^{**}\cap(\Omega_{\alpha,j}\setminus\Omega_{2\alpha,j})} |f_{z_i}(z)| M_S w(z)\, dz$$

and then

$$\lambda\left(\Omega_{2\alpha,j}\cap B^*\right)\leq\frac{2}{\gamma\alpha}\sum_{i=1}^{n}\frac{\ell_i(B^*)\lambda(B^*)|\left(B^*\right)^{\#}|}{|B^*|w(B^*)}\int_{B^{**}\cap(\Omega_{\alpha,j}\setminus\Omega_{2\alpha,j})}|f_{z_i}(z)|M_S w(z)\,dz. \tag{5.13}$$

Now, since $\Omega_{3\alpha,j}\subset\Omega_{2\alpha,j}$, combining (5.5), (5.12), and (5.13) we have

$$\begin{aligned}\lambda\left(\Omega_{3\alpha,j}\cap B^*\right)\leq{}&\frac{CC_1\gamma^\delta}{1-C\gamma^\delta}\lambda\left(B\cap\Omega_\alpha^j\right)\\&+\frac{2}{\gamma\alpha}\sum_{i=1}^{n}\frac{\ell_i(B^*)\lambda(B^*)|\left(B^*\right)^{\#}|}{|B^*|w(B^*)}\int_{B^{**}\cap(\Omega_{2\alpha,j}\setminus\Omega_{3\alpha,j})}|f_{z_i}(z)|M_S w(z)\,dz\\&+\frac{2}{\gamma\alpha}\sum_{i=1}^{n}\frac{\ell_i(B^*)\lambda(B^*)|\left(B^*\right)^{\#}|}{|B^*|w(B^*)}\int_{B^{**}\cap(\Omega_{\alpha,j}\setminus\Omega_{2\alpha,j})}|f_{z_i}(z)|M_S w(z)\,dz.\end{aligned} \tag{5.14}$$

Summing up over $j=1,2,\ldots,$ we obtain

$$\begin{aligned}\lambda\left(\Omega_{3\alpha}\cap B^*\right)\leq{}&\frac{CC_1\gamma^\delta}{1-C\gamma^\delta}\lambda\left(B^*\cap\Omega_\alpha\right)\\&+\frac{2}{\gamma\alpha}\sum_{i=1}^{n}\frac{\ell_i(B^*)\lambda(B^*)|\left(B^*\right)^{\#}|}{|B^*|w(B^*)}\int_{B^{**}\cap(\Omega_{2\alpha}\setminus\Omega_{3\alpha})}|f_{z_i}(z)|M_S w(z)\,dz\\&+\frac{2}{\gamma\alpha}\sum_{i=1}^{n}\frac{\ell_i(B^*)\lambda(B^*)|\left(B^*\right)^{\#}|}{|B^*|w(B^*)}\int_{B^{**}\cap(\Omega_{\alpha}\setminus\Omega_{2\alpha})}|f_{z_i}(z)|M_S w(z)\,dz.\end{aligned} \tag{5.15}$$

Recall that the balls system $\left\{B^*=B(b,5K_0^2r(b))\right\}_{b\in\Omega_{3\alpha}}$ covers $\Omega_{3\alpha}$. Using Lemma 2.1, from those balls one can choose a countable subcover $\left\{B_m^*=B(x_m,5K_0^2r(x_m))\right\}_{m\in\mathbb{N}}$ such that

$$\Omega_{3\alpha}\subset\bigcup_m B_m^*. \tag{5.16}$$

Moreover, the balls $\left\{B_m=B(x_m,r(x_m))\right\}_{m\in\mathbb{N}}$ are disjoint, i.e.

$$\bigcap_m B_m=\emptyset. \tag{5.17}$$

Writing (5.15) for the system of balls B_m^*, we get

$$\begin{aligned}\lambda\left(\Omega_{3\alpha}\cap B_m^*\right)\leq{}&\frac{CC_1\gamma^\delta}{1-C\gamma^\delta}\lambda\left(B_m\cap\Omega_\alpha\right)\\&+\frac{2}{\gamma\alpha}\sum_{i=1}^{n}\frac{\ell_i(B_m^*)\lambda(B_m^*)|\left(B_m^*\right)^{\#}|}{|B_m^*|w(B_m^*)}\int_{B_m^{**}\cap(\Omega_{2\alpha}\setminus\Omega_{3\alpha})}|f_{z_i}(z)|M_S w(z)\,dz\\&+\frac{2}{\gamma\alpha}\sum_{i=1}^{n}\frac{\ell_i(B_m^*)\lambda(B_m^*)|\left(B_m^*\right)^{\#}|}{|B_m^*|w(B_m^*)}\int_{B_m^{**}\cap(\Omega_{\alpha}\setminus\Omega_{2\alpha})}|f_{z_i}(z)|M_S w(z)\,dz.\end{aligned} \tag{5.18}$$

Summing up over $m=1,2,\ldots,$ we get

$$\begin{aligned}\lambda\left(\Omega_{3\alpha}\right)\leq{}&\frac{CC_1\gamma^\delta}{1-C\gamma^\delta}\lambda\left(\Omega_\alpha\right)\\&+\frac{2}{\gamma\alpha}\sum_{i=1}^{n}\sum_m\frac{\ell_i(B_m^*)\lambda(B_m^*)|\left(B_m^*\right)^{\#}|}{|B_m^*|w(B_m^*)}\int_{\Omega_{2\alpha}\setminus\Omega_{3\alpha}}\chi_{B_m^{**}}(z)\,|f_{z_i}(z)|M_S w(z)\,dz\\&+\frac{2}{\gamma\alpha}\sum_{i=1}^{n}\sum_m\frac{\ell_i(B_m^*)\lambda(B_m^*)|\left(B_m^*\right)^{\#}|}{|B_m^*|w(B_m^*)}\int_{\Omega_{\alpha}\setminus\Omega_{2\alpha}}|\chi_{B_m^{**}}(z)\,f_{z_i}(z)|M_S w(z)\,dz.\end{aligned} \tag{5.19}$$

Denote

$$c_m = \frac{\ell_i(B_m^*)\lambda(B_m^*)|\,(B_m^*)^{\#}\,|}{|B_m^*|w(B_m^*)},$$

then

$$\begin{aligned} \lambda\left(\Omega_{3\alpha}\right) \leq \frac{CC_1\gamma^\delta}{1-C\gamma^\delta}\lambda\left(\Omega_\alpha\right) &+ \frac{2}{\gamma\alpha}\sum_{i=1}^n \int_{\Omega_{2\alpha}\setminus\Omega_{3\alpha}} \left(\sum_m c_m\chi_{B_m^{**}}(z)\right) |f_{z_i}(z)|M_S w(z)\,dz \\ &+ \frac{2}{\gamma\alpha}\sum_{i=1}^n \int_{\Omega_\alpha\setminus\Omega_{2\alpha}} \left(\sum_m c_m\chi_{B_m^{**}}(z)\right) |f_{z_i}(z)|M_S w(z)\,dz. \end{aligned} \tag{5.20}$$

Using Hölder's inequality, this implies

$$\begin{aligned} \lambda\left(\Omega_{3\alpha}\right) \leq \frac{CC_1\gamma^\delta}{1-C\gamma^\delta}\lambda\left(\Omega_\alpha\right) &+ \frac{2}{\gamma\alpha}\sum_{i=1}^n \left(\int_{\Omega_{2\alpha}\setminus\Omega_{3\alpha}} \omega_i(z)\,|f_{z_i}(z)|^p M_S w(z)\,dz\right)^{1/p} \\ &\times \left(\int_{\Omega_{2\alpha}\setminus\Omega_{3\alpha}} \left(\sum_m c_m\chi_{B_m^{**}}(z)\right)^{p'} \sigma_i(z)\,M_S w(z)\,dz\right)^{1/p'} \\ &+ \frac{2}{\gamma\alpha}\sum_{i=1}^n \left(\int_{\Omega_\alpha\setminus\Omega_{2\alpha}} \omega_i(z)\,|f_{z_i}(z)|^p M_S w(z)\,dz\right)^{1/p} \\ &\times \left(\int_{\Omega_\alpha\setminus\Omega_{2\alpha}} \left(\sum_m c_m\chi_{B_m^{**}}(z)\right)^{p'} \sigma_i(z)\,M_S w(z)\,dz\right)^{1/p'}, \end{aligned}$$

where $\sigma_i = \omega_i^{1-p'}$. Now, using Lemma of 4 in [39, Chapter 8], we have

$$\begin{aligned} \lambda\left(\Omega_{3\alpha}\right) \leq \frac{CC_1\gamma^\delta}{1-C\gamma^\delta}\lambda\left(\Omega_\alpha\right) &+ \frac{2C_2}{\gamma\alpha}\sum_{i=1}^n \left(\int_{\Omega_{2\alpha}\setminus\Omega_{3\alpha}} \omega_i(z)\,|f_{x_i}(z)|^p M_S w(z)\,dz\right)^{1/p} \\ &\times \left(\int_{\Omega_{2\alpha}\setminus\Omega_{3\alpha}} \left(\sum_m c_m\chi_{B_m}(z)\right)^{p'} \sigma_i(z)\,M_S w(z)\,dz\right)^{1/p'} \\ &+ \frac{2C_2}{\gamma\alpha}\sum_{i=1}^n \left(\int_{\Omega_\alpha\setminus\Omega_{2\alpha}} \omega_i(z)\,|f_{z_i}(z)|^p M_S w(z)\,dz\right)^{1/p} \\ &\times \left(\int_{\Omega_\alpha\setminus\Omega_{2\alpha}} \left(\sum_m c_m\chi_{B_m}(z)\right)^{p'} \sigma_i(z)\,M_S w(z)\,dz\right)^{1/p'}. \end{aligned} \tag{5.21}$$

By the property (5.17) of the covering $\{B_m\}$ and by the doubling assumption on $\sigma_i M_S w$ on the ρ quasi-metric balls, we get

$$\begin{aligned} \left(\int_{\Omega_{2\alpha}\setminus\Omega_{3\alpha}} \left(\sum_m c_m\chi_{B_m}(z)\right)^{p'} \sigma_i(z)\,M_S w(z)\,dz\right)^{1/p'} &= \left(\sum_m c_m^{p'}\kappa_i(B_m)\right)^{1/p'} \\ &\leq CA\left(\sum_m \lambda(B_m)^{p'/q'}\right)^{1/p'}, \end{aligned} \tag{5.22}$$

where $\kappa_i = \sigma_i M_S w$. Note that in (5.22) we have used that the condition (2.4) and (2.8) and the doubling assumption on the measures yield

$$c_m^{p'} \kappa_i(B_m) \leq C_3 A^{p'} \left(\lambda(B_m)\right)^{p'/q'}.$$

Now, by (5.1) and since $p'/q' \geq 1$,

$$\begin{aligned}
&\left(\int_{\Omega_{2\alpha}\setminus\Omega_{3\alpha}} \left(\sum_m c_m \chi_{B_m}(z)\right)^{p'} \sigma_i(z)\, M_S w(z)\, dz\right)^{1/p'} \\
&\quad \leq C/(1-\gamma)^{1/q'} A \left(\sum_m \lambda\left(B_m \cap \Omega_\alpha\right)^{p'/q'}\right)^{1/p'} \\
&\quad \leq CA/(1-\gamma)^{1/q'} \lambda(\Omega_\alpha)^{1/q'}.
\end{aligned} \tag{5.23}$$

Observe that the same inequality can be obtained also for integrals over the sets $\Omega_\alpha \setminus \Omega_{2\alpha}$. Thus, by (5.21), we get

$$\begin{aligned}
\lambda\left(\Omega_{3\alpha}\right) &\leq \frac{CC_1\gamma^\delta}{1-C\gamma^\delta} \lambda\left(\Omega_\alpha\right) \\
&\quad + \frac{2C_3 A}{(1-\gamma)^{1/q'}\gamma\alpha} \lambda(\Omega_\alpha)^{1/q'} \sum_{i=1}^n \left(\int_{\Omega_{2\alpha}\setminus\Omega_{3\alpha}} \omega_i(z)\, |f_{z_i}(z)|^p M_S w(z)\, dz\right)^{1/p} \\
&\quad + \frac{2C_3 A}{(1-\gamma)^{1/q'}\gamma\alpha} \lambda(\Omega_\alpha)^{1/q'} \sum_{i=1}^n \left(\int_{\Omega_\alpha\setminus\Omega_{2\alpha}} \omega_i(z)\, |f_{z_i}(z)|^p M_S w(z)\, dz\right)^{1/p}, \quad \alpha > 0
\end{aligned}$$

and

$$\begin{aligned}
\int_0^\infty \lambda(\Omega_{3\alpha}) d\alpha^q &\leq \frac{CC_1\gamma^\delta}{1-C\gamma^\delta} \int_0^\infty \lambda(\Omega_\alpha) d\alpha^q \\
&\quad + \frac{2C_3 q}{(1-\gamma)^{1/q'}\gamma} \sum_{j=1}^N A \int_0^\infty \lambda(\Omega_\alpha)^{1/q'} \left(\int_{\Omega_{2\alpha}\setminus\Omega_{3\alpha}} \omega_i(z)\, |f_{z_i}(z)|^p M_S w(z)\, dz\right)^{1/p} \frac{\alpha^{q-1} d\alpha}{\alpha} \\
&\quad + \frac{2C_3 q}{(1-\gamma)^{1/q'}\gamma} \sum_{j=1}^N A \int_0^\infty \lambda(\Omega_\alpha)^{1/q'} \left(\int_{\Omega_\alpha\setminus\Omega_{2\alpha}} \omega_i(z)\, |f_{z_i}(z)|^p M_S w(z)\, dz\right)^{1/p} \frac{\alpha^{q-1} d\alpha}{\alpha}.
\end{aligned} \tag{5.24}$$

Notice that

$$\int_0^\infty \lambda(\Omega_{3\alpha}) d\alpha^q = \frac{1}{3^q} \int_\Omega |f|^q\, v\, w dx \quad \text{and} \quad \int_0^\infty \lambda(\Omega_\alpha) d\alpha^q = \int_\Omega |f|^q v\, w dx. \tag{5.25}$$

Therefore, from (5.25) and Hölder's inequality, we get

$$\begin{aligned}
\frac{1}{3^q} \int_\Omega |f|^q\, v\, w dx &\leq \frac{CC_1\gamma^\delta}{1-C\gamma^\delta} \int_\Omega |f|^q v\, w dx \\
&\quad + \frac{2C_3 q}{(1-\gamma)^{1/q'}\gamma} \sum_{i=1}^n A \left(\int_0^\infty \left(\int_{\Omega_{2\alpha}\setminus\Omega_{3\alpha}} \omega_i(z)\, |f_{z_i}(z)|^p M_S w(z)\, dz\right) \frac{d\alpha}{\alpha}\right)^{1/p} \\
&\quad \times \left(\int_0^\infty \lambda(\Omega_\alpha)^{p'/q'} \alpha^{(q-1)p'-1} d\alpha\right)^{1/p'} \\
&\quad + \frac{2C_3 q}{(1-\gamma)^{1/q'}\gamma} \sum_{i=1}^n A \left(\int_0^\infty \left(\int_{\Omega_\alpha\setminus\Omega_{2\alpha}} \omega_i(z)\, |f_{z_i}(z)|^p M_S w(z)\, dz\right) \frac{d\alpha}{\alpha}\right)^{1/p} \\
&\quad \times \left(\int_0^\infty \lambda(\Omega_\alpha)^{p'/q'} \alpha^{(q-1)p'-1} d\alpha\right)^{1/p'}.
\end{aligned} \tag{5.26}$$

Now, by Fubini's theorem,

$$\left(\int_0^\infty \left(\int_{\Omega_{2\alpha}\setminus\Omega_{3\alpha}} \omega_i(z)\,|f_{z_i}(z)|^p M_S w(z)\,dz\right)\frac{d\alpha}{\alpha}\right)^{1/p} = \left(\ln\frac{3}{2}\right)^{1/p} \|f_{z_i}(\cdot)\|_{p,\omega_i M_S w,\Omega},$$

$$\left(\int_0^\infty \left(\int_{\Omega_{2\alpha}\setminus\Omega_{3\alpha}} \omega_i(z)\,|f_{z_i}(z)|^p \omega_i M_S w(z)\,dz\right)\frac{d\alpha}{\alpha}\right)^{1/p} = (\ln 2)^{1/p}\, \|f_{z_i}(\cdot)\|_{p,\omega_i M_S w,\Omega}.$$

On the other hand, Minkowski's inequality gives

$$\begin{aligned}\left(\int_0^\infty \lambda(\Omega_\alpha)^{p'/q'}\alpha^{(q-1)p'-1}d\alpha\right)^{1/p'} &\le \left(\frac{1}{(q-1)p'}\right)^{1/p'} \left\|\int_{\Omega_{(\cdot)}} v\,wdx\right\|^{1/q'}_{p'/q',d\alpha^{(q-1)p'}}\\ &\le \left(\frac{1}{(q-1)p'}\right)^{1/p'} \|f\|^{1/q'}_{q,vw,\Omega}.\end{aligned}$$

Using the last inequalities and choosing

$$\frac{1}{3^q} - \frac{CC_1\gamma^\delta}{1-C\gamma^\delta} > 0, \tag{5.27}$$

from (5.26) we get

$$\|f\|_{q,vw,\Omega} \le \left(\frac{1}{(q-1)p'}\right)^{1/p'} \frac{2C_3 q 2^{1/p'}(\ln 3)^{1/p}}{(1-\gamma)^{1/q'}\gamma} A \sum_{i=1}^n \|f_{z_i}(\cdot)\|_{p,\omega_i M_S w,\Omega}. \tag{5.28}$$

This completes the proof of Theorem 2.5

5.2 Proof of Theorem 2.6

To prove Theorem 2.6 we may argue following along the lines the proof of Theorem 2.5 until formula (5.10). Then, from hypothesis (2.10), for $t \in (0,1)$ and $z = x + t(y-x) \in G$, using the condition (2.10) it follows

$$|x-z| < t|x-y| \le \rho(x,y) \le 2K_0 r(B^*)\,t,$$

therefore applying Fubini's formula again,

$$\begin{aligned}&\frac{1}{2}\gamma w(B^*)|B^*|\\ &\quad\le \sum_{i=1}^N \frac{\ell_i(B^*)}{\alpha}\int_G \left|\frac{\partial f}{\partial x_i}(z)\right| \left(\int_0^1 \left(\frac{1}{t^{n-\varepsilon}}\int_{\{x\in B:\,|z-x|<2K_0 r(B^*)\,t\}} w(x)dx\right)\frac{dt}{t^\varepsilon}\right)dz.\end{aligned} \tag{5.29}$$

Now, by the definition of the fractional order Hardy–Littlewood maximal operator over Euclidean balls and since $B(x, 2K_0 r(B^*)t) \ni z$ it follows

$$\int_{\{x\in B:\,|z-x|<2K_0 r(B^*)\,t\}} w(x)dx \le M_\varepsilon w(z)\,(2K_0 r(B^*)\,t)^{n-\varepsilon}.$$

By (5.29), one has

$$1 \le \frac{2^{n+1-\varepsilon}K_0^{n-\varepsilon}}{(1-\varepsilon)\gamma\alpha}\sum_{i=1}^n \frac{\ell_i(B^*)r(B^*)^{n-\varepsilon}}{|B^*|w(B^*)}\int_{B^{**}\cap(\Omega_{2\alpha,j}\setminus\Omega_{3\alpha,j})} |f_{z_i}(z)|M_\varepsilon w(z)\,dz.$$

Arguing further as in Theorem 2.5 we obtain estimate (5.28) with $\bar{A}$ in place of A. The proof of Theorem 2.6 is then complete. □

5.3 Proof of Theorem 2.4

Theorem 2.4 is a corollary of Theorem 2.5 for $w \equiv 1$.

5.4 Proof of Corollary 4.1

The result follows from Theorem 2.4. It is enough to choose $(x, y) \in \mathbb{R}^N$, with $N = n + m$, $v(x, y) = \omega(x)^t$, $t = \frac{n}{n+m-2}$, and $\omega_1 = \cdots = \omega_n = \omega(x)$, $\omega_i \equiv 1$, $i = n+1, n+2, \ldots, n+m$ in the statement of Theorem 2.4. Observe that the A_∞-condition on the ρ-quasimetric balls on $\omega(x)^{\frac{n}{n+m-2}}$ as well as the A_2-condition on the ρ-quasimetric balls for ω are satisfied, in view of (3.1) and (4.1). Indeed, it is well-known that the A_p condition for some $p \geq 1$ implies the A_∞ condition. Therefore, in order to show that ω^t belongs to A_∞ let us show that it belongs to A_p, for some $p \geq 1$. To this aim, observe that, by our assumptions, $\sigma \in A_2$ hence

$$\sigma(Q) \int_Q \omega \, dx \leq C|Q|^2. \tag{5.30}$$

Using the Hölder inequality with powers $\frac{n+m-2}{n}$ and $\frac{n+m-2}{m-2}$,

$$\int_Q \omega^{\frac{n}{n+m-2}} \, dx \leq \left(\int_Q \omega \, dx \right)^{\frac{n}{n+m-2}} |Q|^{\frac{m-2}{n+m-2}},$$

thus, by (5.30), we get

$$\sigma(Q) \left(\int_Q \omega^{\frac{n}{n+m-2}} \, dx \right)^{\frac{n+m-2}{n}} \leq C|Q|^{2+\frac{m-2}{n}}.$$

The last inequality implies $\omega^t \in A_p$ with $p = 1 + \frac{n}{n+m-2}$.

For what concerns hypothesis (2.6), by the definition of the quasimetric ρ given in Section 3, in this case it can be derived by the following inequality

$$Cr|B(z,r)|^{-\left(\frac{1}{2}-\frac{1}{q}\right)} \left(\frac{1}{|Q(x,r)|} \int_{Q(x,r)} \omega^t \, ds \right)^{\frac{1}{q}} \left(\frac{1}{|Q(x,r)|} \int_{Q(x,r)} \sigma(s) ds \right)^{\frac{1}{2}} \leq A, \tag{5.31}$$

where $B(z,r)$ is a ρ-quasimetric ball of center z and radius r, $0 < r < R$, while $Q(x,r)$ is the projection of $B(z,r)$ on $\mathbb{R}^n$. Thus, in order to satisfy condition (2.6) we need to estimate the left hand side of (5.31) from above. To this aim, observe that

$$\begin{aligned}
& r|B(z,r)|^{-\left(\frac{1}{2}-\frac{1}{q}\right)} \left(\frac{1}{|Q(x,r)|} \int_{Q(x,r)} \omega^t \, ds \right)^{\frac{1}{q}} \left(\frac{1}{|Q(x,r)|} \int_{Q(x,r)} \sigma(s) ds \right)^{\frac{1}{2}} \\
& \quad \leq \left(\frac{1}{|Q(x,r)|} \int_{Q(x,r)} \omega(s)^t ds \right)^{\frac{1}{q}} \left(\frac{1}{|Q(x,r)|} \int_{Q(x,r)} \sigma(s) ds \right)^{\frac{1}{2}-\frac{m}{2}\left(\frac{1}{2}-\frac{1}{q}\right)} r^{1-(n+m)\left(\frac{1}{2}-\frac{1}{q}\right)} \\
& \quad = C \left(\frac{1}{|Q(x,r)|} \int_{Q(x,r)} \omega(s)^t ds \right)^{\frac{1}{q}} \left(\frac{1}{|Q(x,r)|} \int_{Q(x,r)} \sigma(s) ds \right)^{\frac{1}{2}-\frac{m}{2}\left(\frac{1}{2}-\frac{1}{q}\right)} \\
& \quad \leq C_1 \left(\frac{1}{|Q(x,r)|} \int_{Q(x,r)} \omega(s) ds \right)^{\frac{t}{q}} \left(\frac{1}{|Q(x,r)|} \int_{Q(x,r)} \sigma(s) ds \right)^{\frac{1}{2}-\frac{m}{2}\left(\frac{1}{2}-\frac{1}{q}\right)},
\end{aligned}$$

where we used the fact that $q = \frac{2(n+m)}{n+m-2}$ gives $1 - (n+m)\left(\frac{1}{2} - \frac{1}{q}\right) = 0$ and Hölder's inequality.

Now, since $\frac{t}{q} = \frac{1}{2} - \frac{m}{2}\left(\frac{1}{2} - \frac{1}{q}\right) = \frac{n}{2(n+m)}$, taking into account assumption $\omega \in A_2$ we get

$$C_1 \left(\frac{1}{|Q(x,r)|} \int_{Q(x,r)} \omega(s)ds \right)^{\frac{t}{q}} \left(\frac{1}{|Q(x,r)|} \int_{Q(x,r)} \sigma(s)ds \right)^{\frac{1}{2}-\frac{m}{2}\left(\frac{1}{2}-\frac{1}{q}\right)}$$
$$= C_1 \left[\left(\frac{1}{|Q(x,r)|} \int_{Q(x,r)} \omega(s)ds \right) \left(\frac{1}{|Q(x,r)|} \int_{Q(x,r)} \sigma(s)ds \right) \right]^{\frac{n}{2(n+m)}} \le C_2$$

Hence condition (2.6) of Theorem 2.4 satisfied. This completes the proof of Corollary 4.1. □

5.5 Proof of Corollary 4.2

To prove this result, one can follow along the lines the proof of Corollary 4.1, for $t = \frac{n}{n-1}$, with suitable modifications.

5.6 Proof of Corollary 4.3

The proof of Corollary 4.3 is obtained from Theorem 2.4 similarly to that of Corollary 4.1, so we leave the proof to the Reader.

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Dynamical behavior of a parametrized family of one-dimensional maps

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Abstract. The connection of these maps to homoclinic loops acts like an amplifier of the map behavior, and makes it interesting also in the case where all map orbits approach zero (but in many possible ways). We introduce so-called 'flat' intervals containing exactly one maximum or minimum, and so-called 'steep' intervals containing exactly one zero point of $f_{\mu,\omega}$ and no zero of $f'_{\mu,\omega}$. For specific parameters μ and ω, we construct an open set of points with orbits staying entirely in the 'flat' intervals in section three. In section four, we describe orbits staying in the 'steep' intervals (for open parameter sets), and in section five (for specific parameters) orbits regularly changing between 'steep' and 'flat' intervals. Both orbit types are described by symbol sequences, and it is shown that their Lebesgue measure is zero.

Keywords: homoclinic behavior, one-dimensional maps, symbolic dynamics, measure.

2020 Mathematics Subject Classification: Primary 37E05, Secondary 34C37, 37B10, 37D45.

1 Introduction

Our aim in this paper is to analyze the dynamics of certain parametrized families $f_{\mu,\omega}$ of one-dimensional maps. These arise in the dynamics of flows in three dimensions of saddle-focus homoclinic connections which were studied by Šil'nikov [6] and Holmes [2]. Holmes considered maps f similar to

$$f_{\mu,\omega} : x \to x^{\mu} \sin(\omega \ln(x))$$

for $\mu > 1, \omega > 0$ (and odd continuation). The property $\mu > 1$ implies that all points $x \in (-1, 1)$ approach 0 under f^n as $n \to \infty$. The connection of the map f to a doubly homoclinic loop (as explained below) implies that the small difference between $f^n(x)$ being positive or negative corresponds to the 'macroscopic' difference that the $n+1$st return will take place along the upper or lower branch of the homoclinic loop, and is therefore of interest. Holmes claimed that the set Z of points x for which there exists an $n_x \in \mathbb{N}$ such that $f^{n_x}(x) = 0$ can be a dense subset of $[0, 1]$, but it seems that this proof is not conclusive. (We briefly write f for $f_{\mu,\omega}$ now.) In section four, we are interested in the orbit $x, f(x), f(f(x)), \ldots$ We first assign

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to x a symbolic trajectory $s_0, s_1, s_2, \ldots$ where $s_n = \text{sign}(f^n(x))$. Then we construct sets Ω_n^c (depending on a parameter c and $n \in \mathbb{N}$) of points with the first n iterates contained in certain 'steep' intervals and following arbitrary symbol sequences. We show that Ω_n^c is contained in the closure of the set Z, but $\Omega_\infty^c = \bigcap_{n\in\mathbb{N}} \Omega_n^c$ has measure zero. The remark on the bottom of the page 395 of [2] conjectures that open sets of points with orbit only in the 'flat' intervals can exist for certain parameters. (These 'flat' intervals are disjoint to Z.) We prove this in Section 3.

In the last section, we focus on constructing another type of orbit whose points travel regularly from a 'flat' interval to a 'steep' interval, then again from the 'steep' interval to a 'flat' interval. These points form a Cantor type set and are described by sequences of the type $(L, R, R, L, \ldots)$, indicating whether iterates of the initial points are to the left or to the right of corresponding maxima of f. Taking counter images $f^{-1}(J)$ of intervals J with $f^{-1}(J)$ close to a quadratic maximum of f involves inversion of the second order Taylor expansion and thus taking square roots. We also show that, despite the expanding effect of the square root, the measure of the points with such orbits (and thus the measure of the Cantor set) is also zero.

1.1 Motivation of the map

We consider the differential equation

$$\begin{aligned} \dot{x} &= sx - \nu y + F_1(x,y,z) \\ \dot{y} &= \nu x + sy + F_2(x,y,z) \quad \text{or} \quad \dot{X} = F(X), \\ \dot{z} &= \lambda z + F_3(x,y,z) \end{aligned} \tag{1.1}$$

where $X = (x,y,z)$, with smooth functions F_1, F_2, F_3 which vanish at the origin together with their derivatives and assume that there exists a doubly homoclinic connection associated to a saddle-focus singularity at the origin $(0,0,0)$ with eigenvalues $s \pm i\nu$, $s < 0$, $\nu \neq 0$, $\lambda > 0$. We also assume that the saddle value satisfies $s + \lambda < 0$ and F possesses symmetry under the change of sign, $F(X) = -F(-X)$. Here, note that while the stable manifold $W^s(0)$ is two-dimensional, the unstable manifold $W^u(0)$ is one-dimensional. The global unstable manifold $W^u(0)$ consists of the homoclinic loops and is contained in $W^s(0)$ (see Figure 1.1). Note also that in case $s + \lambda < 0$ stable periodic orbits bifurcate from the homoclinic loop as described by L. P. Šil'nikov in reference [5], even in case of only one homoclinic loop.

Furthermore, to obtain expressions for a Poincaré first return map defined by the trajectories close to the homoclinic loop Λ, we assume that the vector field is linear (i.e. $F_1 = F_2 = F_3 = 0$) in a neighborhood of $(0,0,0)$. First, in a neighborhood of $(0,0,0)$ we introduce a cross section Σ_0 that is transversal to Λ and has a nonzero projection to the unstable direction. The second property is an automatic consequence of the first in three dimensions. The stable manifold W^s_{loc} splits Σ_0 into the upper and lower components Σ_0^+ and Σ_0^- respectively, and the homoclinic loop intersects Σ_0 at some point $p = (\xi, 0, 0) \in \Lambda \cap \Sigma_0$ on W^s_{loc}. We next introduce two cross-sections $\Sigma_1^\mp$ transversal to W^u_{loc}. Using the trajectories which travel from Σ_0^+ to Σ_1^+ we aim at computing local maps $G_0^+ : \Sigma_0^+ \to \Sigma_1^+$ and $G_0^- : \Sigma_0^- \to \Sigma_1^-$. These local maps associate to each point $p \in \Sigma_0$ the first intersection with Σ_1 of the trajectory which starts at p. Thus, a local map G_0 is defined by the flow on subsets $\Sigma_0^\mp$ of Σ_0. Note that since the upper and lower homoclinic orbit of the system have analogous behavior, we shall continue with one (the upper homoclinic loop) of them. For simplification we assume that there exist $\xi > 0$, $\zeta > 0$ such that $\Sigma_0^+ \subset \{(\xi, y, z) : (y,z) \in \mathbb{R}^2\}$ and $\Sigma_1^+ \subset \{(x, y, \zeta) : (x,y) \in \mathbb{R}^2\}$.

The solution $(x(t), y(t), z(t))$ of (1.1), which starts from a point $(x_0 = \xi, y_0, z_0) \in \Sigma_0^+$ close to the origin at the time $t = 0$ and ends up at the point $(x_1, y_1, z_1 = \zeta) \in \Sigma_1^+$ at the time $t = \tau$, is written (taking into account only the linear terms in (1.1)) as follows:

$$x(t) + iy(t) = e^{(s+iv)t}(x_0 + iy_0) = e^{(s+iv)t}(\xi + iy_0) \tag{1.2}$$
$$z(t) = z_0 e^{\lambda t}.$$

The flight time τ that the trajectory takes from Σ_0^+ to Σ_1^+ is given by $\tau = \frac{1}{\lambda}\ln(\frac{\zeta}{z_0})$. Substituting τ and ξ into formula (1.2), we get the following expression for the local map G_0^+, in complex notation:

$$x_1 + iy_1 = e^{(s+iv)\tau(z_0)}(x_0 + iy_0) = e^{(s+iv)\tau(z_0)}(\xi + iy_0). \tag{1.3}$$

On the other hand, due to the existence of the homoclinic connection and its transversal intersection with Σ_0 and Σ_1^+, we also have a Poincaré type map

$$G_1^+ : \Sigma_1^+ \to \Sigma_0$$

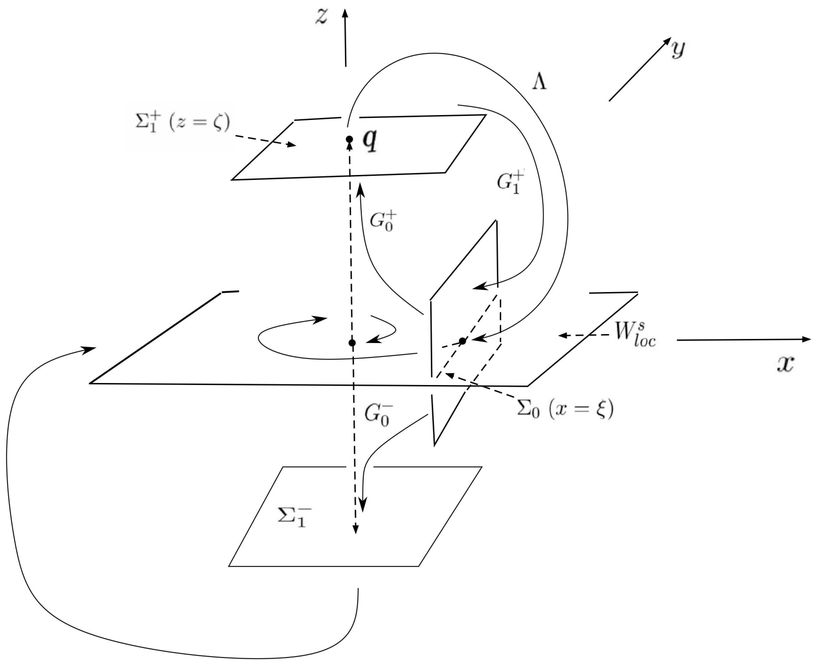

Figure 1.1: Cross sections Σ_0, Σ_1 and homoclinic orbit Λ.

Hence, for $(x_1, y_1, z_1 = \zeta) \in \Sigma_1^+$ we have $G_1^+(x_1, y_1, z_1 = \zeta) = (\xi, y_2, z_2) \in \Sigma_0$. With $DG_1^+(0, 0, \zeta)$ represented by the matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial y_1} \\ \frac{\partial z_2}{\partial x_1} & \frac{\partial z_2}{\partial y_1} \end{pmatrix}(0, 0),$$

we have for the composite map

$$(G_1^+ \circ G_0^+) : (\xi, y_0, z_0) \to (\xi, y_2, z_2),$$

$$\begin{pmatrix} y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 + \beta y_1 \\ \gamma x_1 + \delta y_1 \end{pmatrix} + h.o.t. \text{ (higher order terms)},$$

and finally we get

$$z_2 = \gamma x_1 + \delta y_1. \tag{1.4}$$

Substituting the value of $\tau(z_0)$, x_1 and y_1 in (1.4), in particular for $y_0 = 0$, one obtains

$$z_2 = \xi e^{s\tau(z_0)} [\gamma \cos + \delta \sin] (\nu \tau(z_0)).$$

Hence, with $c := \xi\sqrt{\gamma^2 + \delta^2}$ and choosing φ with $e^{i\varphi}\sqrt{\gamma^2+\delta^2} = \delta + i\gamma$, the z-component after one return is approximately given by

$$z_0 \to z_2 = c\left(\frac{\zeta}{z_0}\right)^{\frac{s}{\lambda}} \left[\sin\left(\frac{\nu}{\lambda}\left(\ln\frac{\zeta}{z_0}\right) + \varphi\right)\right].$$

Note that $\frac{s}{\lambda} < -1$ so $\mu := -\frac{s}{\lambda} > 1$, with $x := \frac{z_0}{\zeta}$, and $\omega = \frac{\nu}{\lambda}$ we can rewrite the last equation as

$$z_2 = cx^{\mu}[\sin(-\omega(\ln x) + \varphi)].$$

This motivates the study of the one-dimensional map $f_{\omega,\mu} : [-1,1] \to [-1,1]$ given by the following simpler expression

$$f_{\mu,\omega}(x) = \begin{cases} x^{\mu}\sin(\omega\ln(x)), & x > 0, \\ 0, & x = 0, \\ -f_{\mu,\omega}(-x), & x < 0, \end{cases}$$

where we use x instead of z from now on. Here, note that odd continuation in the definition of $f_{\mu,\omega}$ is motivated by the corresponding symmetry of vector field. The above process shows how to arrive at this map starting from homoclinic orbits; similar considerations are given in Šil'nikov, L. P. [6], P. J. Holmes [2], or J. Guckenheimer/P. Holmes [1, pp. 320–321]. Analogous infinite-dimensional examples with attracting homoclinic behavior (not necessarily with a double loop) were studied by Walther in [7] and by Ignatenko in [3]. The maps of this kind (see Figure 1.2) were also studied by M. J. Pacifico, A. Rovella and M. Viana [4], but for $\mu < 1$, which has expansion properties of $f_{\mu,\omega}$ as a consequence. Briefly, they proved that a family of one dimensional maps with infinitely many critical points exhibit global chaotic behavior in a persistent way: For a positive Lebesgue measure set of values μ, the map f has positive Lyapunov exponent at every critical value and at Lebesgue almost all points in its domain; moreover, f is topologically transitive, i.e. has dense orbits [4].

After giving some preparatory calculations for the following chapters, we are going to study the orbit $f^n_{\omega,\mu}(x) = f^n(x)$; $n = 1,2,3,\dots$ of a typical point $x \in (0,1)$. If $f^n(x) = 0$ for some $n < \infty$, then it is clear that all $\left(f^j(x)\right)_{j\geq n}$ will equal to 0. To orbits of f we can associate symbol sequences $(s_j) = \left(\operatorname{sign} f^j(x)\right)_{j\geq 0} = (+1,+1,-1,\dots)$. $f^n(x) = 0$ implies that $s_n = 0$, then $s_k = 0$ for all $k \geq n$. Here $+1, -1$ and 0 correspond to the upper, to the lower homoclinic branch or to the stable manifold $W^s(0)$ in terms of the original motivation. Consequently, the following questions arise:

(i) Are all symbol sequences possible or not?

(ii) Does the symbol sequence change in every interval? (Is there chaotic motion?)

(iii) Is it possible to construct open intervals where the symbol sequence does not change?

In the fifth chapter, we shall also consider symbol sequences different from (sign $f^j(x)$), describing whether $f^n(x)$ is to the left or to the right hand side of maximum points of f.

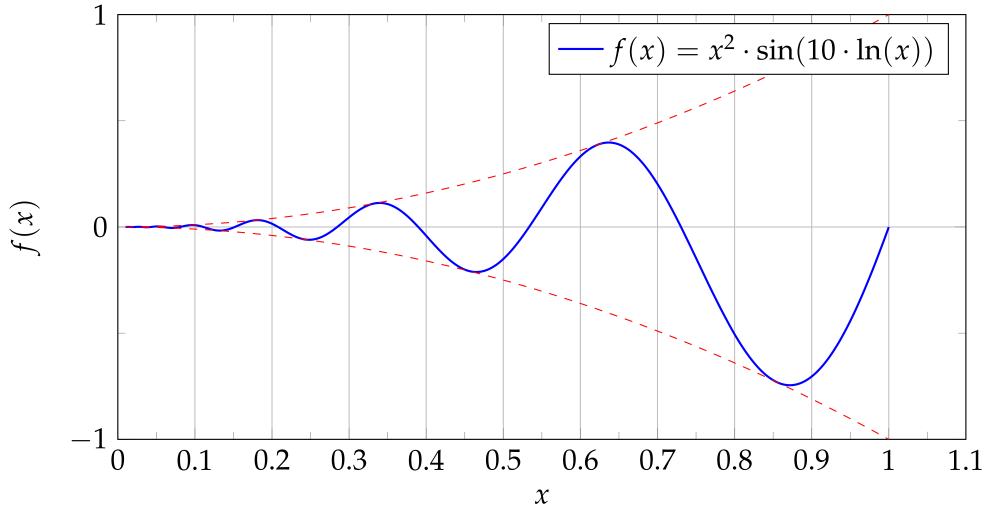

Figure 1.2: Graph of f for $\mu = 2, \omega = 10$.

2 Formulas for the derivatives of $f_{\mu,\omega}$

Lemma 2.1. *Define for $\mu, \omega > 0$ the map*

$$f_{\mu,\omega}(x) = \begin{cases} x^\mu \sin(\omega \ln(x)), & x > 0, \\ 0, & x = 0, \\ -f_{\mu,\omega}(-x), & x < 0. \end{cases}$$

Assume now $\mu \in (2, \infty)$, $\omega > 0$. Set $\varphi_j := \arctan(\frac{\omega}{\mu+1-j}) \in (0, \frac{\pi}{2})$ and

$$g_{\omega,\mu+1-j} := \sqrt{(\mu+1-j)^2 + \omega^2}$$

for $j \in \{1, 2, 3\}$. It is convenient to also define the more general class of functions

$$f_{\mu,\omega,\varphi}(x) := x^\mu \sin(\omega \ln(x) + \varphi).$$

Then, the following formulas hold for $x \in \mathbb{R}$, if $\mu > 3$:

(i)

$$f'_{\mu,\omega}(x) = g_{\omega,\mu} \cdot f_{\mu-1,\omega,\varphi_1}(x), \tag{2.1}$$

$$\cos(\varphi_1) = \frac{\mu}{\sqrt{\mu^2 + \omega^2}} = \frac{\mu}{g_{\omega,\mu}}, \tag{2.2}$$

$$\sin(\varphi_1) = \frac{\omega}{\sqrt{\mu^2 + \omega^2}} = \frac{\omega}{g_{\omega,\mu}}. \tag{2.3}$$

(ii)

$$f''_{\mu,\omega}(x) = g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot f_{\mu-2,\omega,\varphi_1+\varphi_2}(x). \tag{2.4}$$

(iii)

$$f'''_{\mu,\omega}(x) = g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot g_{\omega,\mu-2} \cdot f_{\mu-3,\omega,\varphi_1+\varphi_2+\varphi_3}(x). \tag{2.5}$$

Proof. (i) From the definition of φ_j, we have $\varphi_1 = \arctan(\frac{\omega}{\mu})$, and also from the definition of $g_{\omega,\mu+1-j}$, we have $g_{\omega,\mu} = \sqrt{\mu^2+\omega^2}$. It follows that

$$\cos(\varphi_1) = \frac{\mu}{g_{\omega,\mu}} \text{ and } \sin(\varphi_1) = \frac{\omega}{g_{\omega,\mu}}.$$

This proves (2.2) and (2.3). For $x > 0$, we have

$$\begin{aligned} f'_{\mu,\omega,\varphi}(x) &= x^{\mu}\cos(\omega\ln(x)+\varphi)\left(\frac{1}{x}\omega\right) + x^{\mu-1}\mu\sin(\omega\ln(x)+\varphi) \\ &= x^{\mu-1}\left(\mu\sin(\omega\ln(x)+\varphi)+\omega\cos(\omega\ln(x)+\varphi)\right). \end{aligned}$$

By multiplying and dividing the last equation with $g_{\omega,\mu}$, we have

$$f'_{\mu,\omega,\varphi}(x) = g_{\omega,\mu}\cdot x^{\mu-1}\left(\frac{\mu}{g_{\omega,\mu}}\sin(\omega\ln(x)+\varphi) + \frac{\omega}{g_{\omega,\mu}}\cos(\omega\ln(x)+\varphi)\right). \tag{2.6}$$

Putting (2.2) and (2.3) in (2.6), we finally obtain

$$\begin{aligned} f'_{\mu,\omega,\varphi}(x) &= g_{\omega,\mu}\cdot x^{\mu-1}\left(\cos(\varphi_1)\cdot\sin(\omega\ln(x)+\varphi)+\sin(\varphi_1)\cdot\cos(\omega\ln(x)+\varphi)\right) \\ &= g_{\omega,\mu}\cdot x^{\mu-1}\sin(\omega\ln(x)+\varphi+\varphi_1) \\ &= g_{\omega,\mu}\cdot f_{\mu-1,\omega,\varphi+\varphi_1}(x). \end{aligned} \tag{2.7}$$

(ii) Further, using (2.7) with $\varphi+\varphi_1$ instead of φ, and $\mu-1$ instead of μ, we see that

$$\begin{aligned} f''_{\mu,\omega}(x) = f''_{\mu,\omega,0}(x) &= \left(g_{\omega,\mu}\cdot f_{\mu-1,\omega,\varphi_1}\right)'(x) \\ &= g_{\omega,\mu}\cdot g_{\omega,\mu-1}\cdot f_{\mu-2,\omega,\varphi_1+\varphi_2}(x), \end{aligned}$$

which proves (2.4).

(iii) Using (2.7) we obtain (2.5) analogously. □

Lemma 2.2. *Let $\mu > 3$ and $\omega > 0$ be given. Define $q := e^{-\frac{\pi}{\omega}}$ and φ_j as in Lemma 2.1. Then, the following properties are satisfied in $(0,1]$:*

(i) $f_{\mu,\omega}$ has the zero points

$$q^k = e^{-\frac{k\pi}{\omega}}, \tag{2.8}$$

$(k\in\mathbb{N})$ and

$$f'_{\mu,\omega}(q^k) = (-1)^k\,\omega q^{k(\mu-1)}. \tag{2.9}$$

(ii) $f_{\mu,\omega}$ has the extremal points

$$m_k = q^k e^{\frac{-\varphi_1}{\omega}} \tag{2.10}$$

and

$$f_{\mu,\omega}(m_k) = (-1)^{k+1}\cdot\exp\left(-\frac{k\pi\mu+\varphi_1\mu}{\omega}\right)\cdot\sin(\varphi_1). \tag{2.11}$$

(iii) If μ is an even integer, and $\beta \in \mathbb{N}$ is odd and $l(k) := k\mu + \beta$, then $f_{\mu,\omega}$ has a maximum at

$$m_{l(k)} = q^{l(k)} e^{\frac{-\varphi_1}{\omega}}. \tag{2.12}$$

Proof. (i) We first find the zeros of $f_{\mu,\omega}$. For $x \in (0,1)$ one has

$$\sin(\omega \ln(x)) = 0 \quad \Leftrightarrow \quad \exists k \in \mathbb{N} : \omega \ln(x) = -k\pi \quad \Leftrightarrow \quad \exists k \in \mathbb{N} \ \ln(x) = \frac{-k\pi}{\omega},$$

and hence $x = e^{\frac{-k\pi}{\omega}}$. With $q = e^{-\frac{\pi}{\omega}}$, the zeros of $f_{\mu,\omega}$ in $(0,q]$ are given by $x = e^{\frac{-k\pi}{\omega}} = q^k$. Therefore, by inserting q^k in (2.1), we have

$$\begin{aligned} f'_{\mu,\omega}(q^k) &= q^{k(\mu-1)} \cdot g_{\omega,\mu} \cdot \sin(\omega \left(\ln q^k\right) + \varphi_1) \\ &= q^{k(\mu-1)} \cdot g_{\omega,\mu} \cdot \sin(\omega \left(\ln e^{\frac{-k\pi}{\omega}}\right) + \varphi_1) \\ &= (-1)^k q^{k(\mu-1)} \cdot g_{\omega,\mu} \cdot \sin(\varphi_1). \end{aligned}$$

Using (2.3) we obtain

$$\begin{aligned} f'_{\mu,\omega}(q^k) &= (-1)^k q^{k(\mu-1)} \cdot g_{\omega,\mu} \cdot \frac{\omega}{g_{\omega,\mu}} \\ &= (-1)^k \omega q^{k(\mu-1)}. \end{aligned}$$

Hence, assertion **(i)** is proved.

(ii) Let $k \in \mathbb{N}$. We find the extremum points of $f_{\mu,\omega}$ in the interval $I_k = \left[q^{k+1}, q^k\right]$ by solving $f'_{\mu,\omega}(x) = 0$ for $x \in I_k$. Since $x > 0$, $x^{\mu-1} \neq 0$. So, we have

$$\sin(\omega (\ln x) + \varphi_1) = 0,$$

and hence $x = e^{\frac{-k\pi-\varphi_1}{\omega}}$. The last expression equals to $q^k e^{\frac{-\varphi_1}{\omega}} = m_k$, which proves (2.10). Furthermore, for the extremum point m_k of $f_{\mu,\omega}$ in the interval $\left(q^{k+1}, q^k\right)$ we have

$$\begin{aligned} f_{\mu,\omega}(m_k) &= m_k^\mu \sin(\omega \ln(m_k)) \\ &= \left(q^k e^{-\frac{\varphi_1}{\omega}}\right)^\mu \sin(\omega \ln \left(q^k e^{-\frac{\varphi_1}{\omega}}\right)) \\ &= \left(e^{-\frac{k\pi}{\omega}} e^{-\frac{\varphi_1}{\omega}}\right)^\mu \sin(\omega \ln \left(e^{-\frac{k\pi}{\omega}} e^{-\frac{\varphi_1}{\omega}}\right)) \\ &= \exp\left(-\frac{k\pi\mu + \varphi_1\mu}{\omega}\right) \sin(\omega \frac{-k\pi - \varphi_1}{\omega}) \\ &= (-1)^{k+1} \exp\left(-\frac{k\pi\mu + \varphi_1\mu}{\omega}\right) \sin(\varphi_1). \end{aligned}$$

(iii) Substituting $l(k)$ instead of k in (2.11), we have

$$\begin{aligned} f_{\mu,\omega}\left(m_{l(k)}\right) &= (-1)^{l(k)+1} \exp\left(-\frac{l(k)\pi\mu + \varphi_1\mu}{\omega}\right) \sin(\varphi_1) \\ &= \exp\left(-\frac{l(k)\pi\mu + \varphi_1\mu}{\omega}\right) (-1)^{k\mu+\beta+1} \sin(\varphi_1) \end{aligned}$$

Therefore, it is clear that $f_{\mu,\omega}(m_{l(k)}) > 0$ (and hence $f_{\mu,\omega}$ has a maximum at $m_{l(k)}$), if μ is even and β is odd. □

We shall frequently use the simple lemma below.

Lemma 2.3. *Assume $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). If $|f'| \geq c$, or $|f'| \leq d$ (c and d are constant), then we have*

$$c\,|b-a| \leq |f(b) - f(a)| \leq d\,|b-a|\,. \tag{2.13}$$

Proof. (Follows from the mean value theorem.) □

3 The behavior of orbits remaining in some 'flat' intervals

In this part we find some parameters μ and ω such that $f_{\mu,\omega}$ maps some extremal points m_k to some other extremal points $m_{\ell(k)}$ (see Figure 3.1). Then, we construct some open intervals U_k around m_k and orbits of $f_{\mu,\omega} = f$ which are entirely contained in $\bigcup_{k\in\mathbb{N}} U_k$.

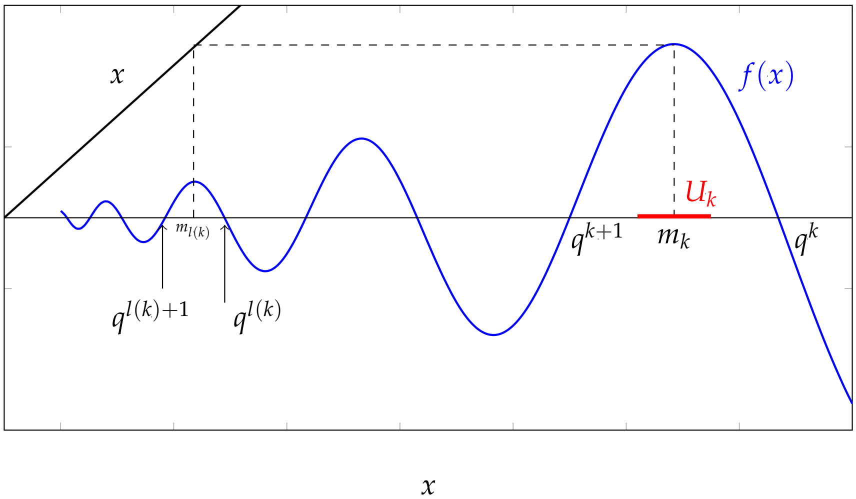

Figure 3.1: $f(m_k) = m_{\ell(k)}$ for special parameters (picture not produced with realistic parameters, for better visibility).

Theorem 3.1. *For $k \in \mathbb{N}$, $\omega > 0$, and even integer $\mu > 5$, define*

$$\eta := \min\left\{ \frac{q}{g_{\omega,\mu}\cdot g_{\omega,\mu-1}}, \frac{e^{-\frac{\varphi_1}{\omega}} - q}{2}, \frac{1-e^{-\frac{\varphi_1}{\omega}}}{2} \right\}, \tag{3.1}$$

and set $\ell(k) := k\mu + 1$ (which corresponds to $\beta = 1$ in assertion (iii) of Lemma 2.2), $\delta_k := \eta q^k$, $\delta_{\ell(k)} := \eta q^{\ell(k)}$. Then, for every large enough even integer μ there exists a corresponding ω such that the following properties are satisfied:

(i) With the intervals $U_k = (m_k - \delta_k, m_k + \delta_k)$ one has $f(U_k) \subset U_{\ell(k)}$ and

$$\forall k \in \mathbb{N} : f^{-1}(\{0\}) \cap U_k = \emptyset.$$

(ii) If k is odd, then for $x \in U_k$, the orbits $\left(f^j(x)\right)_{j\in\mathbb{N}}$ all have the symbol sequence

$$(s_j) = \left(\operatorname{sign} f^j(x)\right)_{j\in\mathbb{N}} = (+1, +1, +1, \dots)\,.$$

(iii) *The set*

$$Z = \{x \mid \exists n \in \mathbb{N} : f^n(x) = 0\} \tag{3.2}$$

is disjoint to $\bigcup_k U_k$ *and, in particular, is not dense in* $[-1,1]$.

The proof is divided into several lemmas.

Lemma 3.2. *Let* $k \in \mathbb{N}$ *and define* φ_1 *as in Lemma 2.1. Define* η *and* δ_k *as in Theorem 3.1, and*

$$\overline{\eta} := \min\left\{\frac{e^{-\frac{\varphi_1}{\omega}} - q}{2}, \frac{1 - e^{-\frac{\varphi_1}{\omega}}}{2}\right\}.$$

Then we have

$$(m_k - \delta_k, m_k + \delta_k) \subset [m_k - \delta_k, m_k + \delta_k] \subset \left[m_k - \overline{\eta}q^k, m_k + \overline{\eta}q^k\right] \subset \left(q^{k+1}, q^k\right).$$

Proof. From (3.1) we have $\eta \leq \overline{\eta}$. Multiplying both sides with q^k, and using (2.10) we have

$$\delta_k \leq \overline{\eta}q^k = \min\left\{\frac{q^k e^{-\frac{\varphi_1}{\omega}} - q^{k+1}}{2}, \frac{q^k - q^k e^{-\frac{\varphi_1}{\omega}}}{2}\right\} = \min\left\{\frac{m_k - q^{k+1}}{2}, \frac{q^k - m_k}{2}\right\},$$

it follows that $(m_k - \delta_k, m_k + \delta_k) \subset [m_k - \delta_k, m_k + \delta_k] \subset \left[m_k - \overline{\eta}q^k, m_k + \overline{\eta}q^k\right] \subset \left(q^{k+1}, q^k\right)$. □

Lemma 3.3. *Define* φ_1 *as in Lemma 2.1, and define* $\ell(k)$ *as in Theorem 3.1. Then the following statements are true.*

(i) *For every even integer* $\mu \geq 32$, *there exists* $\omega \in (0,1)$ *such that for all* $k \in \mathbb{N}$ f *has the property*

$$f(m_k) = m_{\ell(k)}.$$

(ii) *For any choice of* ω *as in assertion* ***(i)***, *one has* $\omega \to 0$ *as* $\mu \to \infty$.

Proof. **(i)** From (2.11) we have for all $k \in \mathbb{N}$

$$|f(m_k)| = \exp\left(-\frac{k\pi\mu + \varphi_1\mu}{\omega}\right)\sin(\varphi_1). \tag{3.3}$$

On the other hand, from the third assertion of Lemma 2.2 we know that for even μ, f has a maximum at the point

$$m_{\ell(k)} = \exp\left(-\frac{\pi\ell(k) + \varphi_1}{\omega}\right). \tag{3.4}$$

Using (2.3), (3.3) and (3.4), we obtain the following equivalences:

$$\begin{aligned}
m_{\ell(k)} = f(m_k) \quad &\Leftrightarrow \quad \exp\left(-\frac{\pi\ell(k) + \varphi_1}{\omega}\right) = \exp\left(-\frac{k\pi\mu + \varphi_1\mu}{\omega}\right)\cdot\sin(\varphi_1) \\
&\Leftrightarrow \quad \exp\left(-\frac{\pi\ell(k) + \varphi_1}{\omega}\right) = \exp\left(-\frac{k\pi\mu + \varphi_1\mu}{\omega}\right)\cdot\frac{1}{\sqrt{1 + \frac{\mu^2}{\omega^2}}} \\
&\Leftrightarrow \quad \exp\left(\frac{-\pi}{\omega}[k\mu - \ell(k)] + \frac{\varphi_1(1-\mu)}{\omega}\right) = \sqrt{1 + \frac{\mu^2}{\omega^2}}.
\end{aligned} \tag{3.5}$$

Substituting $\ell(k) = k\mu + 1$ in (3.5), we have

$$\exp\left(\frac{\pi + \varphi_1(1-\mu)}{\omega}\right) = \sqrt{1 + \frac{\mu^2}{\omega^2}}$$

or, using the definition of φ_1,

$$\exp\left(\frac{\pi - (\mu - 1)\arctan\left(\frac{\omega}{\mu}\right)}{\omega}\right) = \sqrt{1 + \frac{\mu^2}{\omega^2}}\,. \tag{3.6}$$

In view of (3.6), we define

$$F(\omega,\mu) = \exp\left(\frac{\pi - (\mu - 1)\arctan\left(\frac{\omega}{\mu}\right)}{\omega}\right) - \sqrt{1 + \frac{\mu^2}{\omega^2}} \tag{3.7}$$

for all $\omega > 0$ and $\mu > 1$. We try to find (ω, μ) with $F(\omega,\mu) = 0$ (see Figure 3.2). Noting that for fixed μ, $\lim_{\omega\to 0} F(\omega,\mu) = +\infty$, it is enough to find at least one pair (ω,μ) with $F(\omega,\mu) < 0$. For $\omega = 1$, we have

$$\begin{aligned} F(1,\mu) &= \exp\left(\pi - (\mu-1)\arctan\left(\frac{1}{\mu}\right)\right) - \sqrt{1+\mu^2} \\ &= \exp\left(\pi - \mu\arctan\left(\frac{1}{\mu}\right) + \arctan\left(\frac{1}{\mu}\right)\right) - \sqrt{1+\mu^2}. \end{aligned} \tag{3.8}$$

Since $\arctan'(x) = \frac{1}{1+x^2}$, we have $\arctan'(x) \geq \frac{1}{2}$ for $|x| \leq 1$. Hence, (2.13) shows $\arctan(x) \geq \frac{1}{2}x$ for $x \in [0,1]$. It follows that for $\mu > 1$,

$$\mu\arctan\left(\frac{1}{\mu}\right) \geq \frac{1}{2}. \tag{3.9}$$

Using (3.9) and $\arctan\left(\frac{1}{\mu}\right) < \frac{\pi}{4}$ for $\mu > 1$ in (3.8), we have

$$F(1,\mu) \leq \exp\left(\pi - \frac{1}{2} + \frac{\pi}{4}\right) - \sqrt{1+\mu^2} = \exp\left(\frac{5\pi}{4} - \frac{1}{2}\right) - \sqrt{1+\mu^2}.$$

From the fact that $\exp\left(\frac{5\pi}{4} - \frac{1}{2}\right) < 32$, we have $F(\omega,\mu) < 0$, if we set $\omega = 1$ and $\mu \geq 32$. With the intermediate value theorem, it is trivial that $F(\omega,\mu)$ has at least one zero point $\omega \in (0,1)$. It follows that (3.7) is satisfied with this ω depending on the even integer $\mu \geq 32$. Hence, the proof of assertion (i) is completed.

(ii) Consider a sequence μ_k, $\mu_k \to \infty$ with corresponding $\omega_k \in (0,1)$ such that $F(\mu_k,\omega_k) = 0$. Then $\sqrt{1 + \frac{\mu_k^2}{\omega_k^2}} \to \infty$. Further, $(\mu_k - 1)\arctan(\frac{\omega_k}{\mu_k})$ is bounded. The exponential term in (3.7) must go to $+\infty$, so $\omega_k \to 0$ necessarily. This completes the proof of (ii) and the proof of Lemma 3.3. □

Remark 3.4. Consider the equation (3.5). Because $\mu > 1$, so $\frac{\varphi_1(1-\mu)}{\omega} < 0$, and $\sqrt{1+\frac{\mu^2}{\omega^2}} > 1$, the term $\frac{-\pi}{\omega}[k\mu - \ell(k)]$ must be positive, if we have a solution. Accordingly, $\ell(k) > k\mu$ must be satisfied. It means (3.6) has no solution for $\ell(k) \leq k\mu$. Thus $\ell(k) \geq k\mu + 1$ necessarily; we made the choice $\ell(k) = k\mu + 1$.

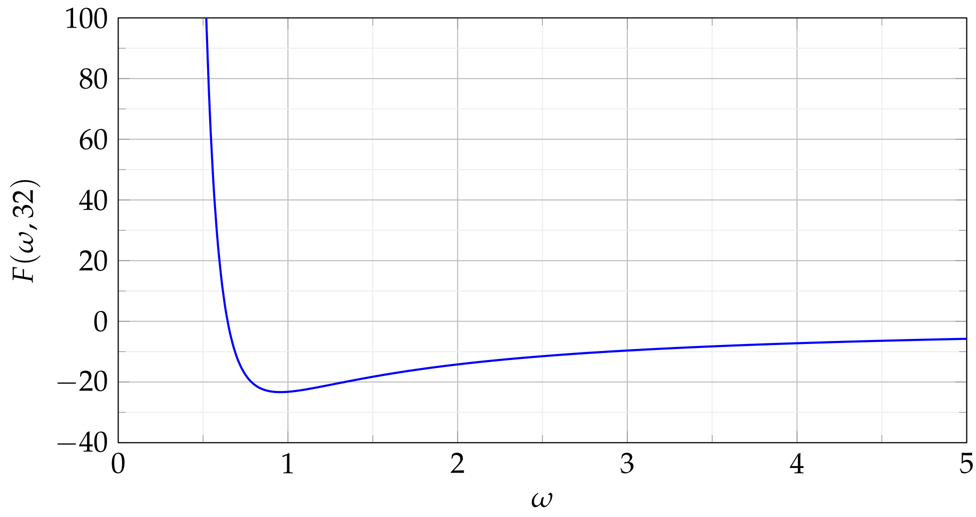

Figure 3.2: Graph of $F(\cdot, \mu)$ for $\mu = 32$.

Numerical observations. In order to find a numerical solution we use two starting points where $F(\cdot, \mu)$ has opposite signs and at the 9 th step of a bisection method we obtained $\omega = 0.69895$ and $\mu = 24$ as an appropriate $F(\omega, \mu) = 0$. Although one can obtain some other solution points ω, for some other the parameters μ, we numerically found out that there is no solution for $\mu < 3.1$.

Lemma 3.5. *Choose an even integer $\mu \geq 32$ and $\omega \in (0, 1)$ with the properties as in Lemma 3.3. Define $\ell(k)$, η, δ_k and $\delta_{\ell(k)}$ as in Theorem 3.1. Then with the intervals $U_k = (m_k - \delta_k, m_k + \delta_k)$, we have $f(U_k) \subset U_{\ell(k)}$.*

Proof. Let μ and ω be as in the assumption of the lemma, and $x \in U_k$. With $\ell(k) = k\mu + 1$ we claim that

$$\left| f(x) - m_{\ell(k)} \right| < \delta_{\ell(k)} = \eta q^{\ell(k)} . \tag{3.10}$$

From the second order Taylor expansion, we have

$$f(x) = f(m_k) + f'(m_k)(x - m_k) + \frac{f''(\xi)}{2}(x - m_k)^2 \tag{3.11}$$

with $\xi \in (m_k - \delta_k, m_k + \delta_k)$. Since $\mu > 2$, note that we also have

$$q^{(k+1)(\mu-2)} \leq |\xi|^{\mu-2} \leq q^{k(\mu-2)}. \tag{3.12}$$

Substituting the equality (3.11) in the left hand side of (3.10), we get

$$\left| f(x) - m_{\ell(k)} \right| = \left| f(m_k) + f'(m_k)(x - m_k) + f''(\xi)\frac{(x - m_k)^2}{2} - m_{\ell(k)} \right|.$$

From the fact that we now have fixed parameters μ, ω with the property $f(m_k) = m_{\ell(k)}$ as in Lemma 3.3 and using $f'(m_k) = 0$ and $(x - m_k) < \delta_k$, the last equality gives

$$\left| f(x) - m_{\ell(k)} \right| \leq \left| f''(\xi)\frac{\delta_k^2}{2} \right|.$$

Using (2.4) in the last equality, we obtain

$$\left|f(x)-m_{\ell(k)}\right| = \left|g_{\omega,\mu}\cdot g_{\omega,\mu-1}\cdot \sin\left(\omega \ln(\xi)+\varphi_1+\varphi_2\right)|\xi|^{\mu-2}\frac{\delta_k^2}{2}\right|. \tag{3.13}$$

Using the upper estimate of (3.12) and substituting the value of δ_k in (3.13), we get

$$\begin{aligned}\left|f(x)-m_{\ell(k)}\right| &\leq \left|g_{\omega,\mu}\cdot g_{\omega,\mu-1}\cdot \left|q^k\right|^{\mu-2}\frac{\eta^2 q^{2k}}{2}\right| \\ &= \left|g_{\omega,\mu}\cdot g_{\omega,\mu-1}\cdot q^{k\mu}\frac{\eta\eta}{2}\right|. \end{aligned} \tag{3.14}$$

Finally, using the definition of η from (3.1) in (3.14), we have

$$\begin{aligned}\left|f(x)-m_{\ell(k)}\right| &\leq \left|g_{\omega,\mu}\cdot g_{\omega,\mu-1}\cdot q^{k\mu}\frac{\eta}{2}\frac{q}{g_{\omega,\mu}\cdot g_{\omega,\mu-1}}\right| \\ &= q^{k\mu+1}\frac{\eta}{2} < \eta q^{k\mu+1} = \eta q^{\ell(k)} = \delta_{\ell(k)}. \end{aligned}$$

□

Proof of Theorem 3.1. Choose μ, ω as in Lemma 3.3, and let $\ell(k)$ be as in Theorem 3.1.

(i) Lemma 3.5 shows $f(U_k) \subset U_{\ell(k)}$ and the definition of $U_{\ell(k)}$ implies $0 \notin U_{\ell(k)}$, so

$$f^{-1}(\{0\}) \cap U_k = \emptyset.$$

(ii) If k is odd and μ is as above (therefore even), then all $\ell^j(k)$ $(j \geq 0)$ are odd and all $U_{\ell^j(k)}$ are intervals around maxima of f, where f is positive. Hence the assertion is proved.

(iii) For $k_0 \in \mathbb{N}$, $x \in U_{k_0}$ and $n \in \mathbb{N}_0$, $f^n(x) \in \bigcup_{k\in\mathbb{N}} U_k$, in particular $f^n(x) \neq 0$, which proves assertion (iii). □

Note that the possible existence of the orbits which remain close to critical points, i.e implying non-density has been mentioned as a remark by P. J. Holmes in the bottom of the page 395 of [2] with only a vague indication of proof. With this section we gave a rigorous proof of that idea.

4 Behavior of the map $f_{\mu,\omega}$ in some 'steep' intervals

In this section we first construct some orbits whose points stay entirely in so-called 'steep' intervals, and then analyze the measure of the set of points which have such orbits. In contrast to Sections 3 and 5, where the parameters μ and ω are connected by the conditions given in assertion (i) of Lemma 3.3 and in (5.1), in this section both of them can be varied independently.

Consider the interval $(-m_k, -m_{k+1})$ or (m_{k+1}, m_k). From Lemma 2.2 we have

$$\left|f'_{\mu,\omega}(q^{k+1})\right| = \omega\left(q^{k+1}\right)^{(\mu-1)}.$$

Since $f'_{\mu,\omega}(\mp m_k) = f'_{\mu,\omega}(\mp m_{k+1}) = 0$, continuity of $f'_{\mu,\omega}$ implies that we can choose a 'steep' interval S_k, either as a subset of (m_{k+1}, m_k) or as a subset of $(-m_k, -m_{k+1})$, on which $|f'_{\mu,\omega}|$ satisfies a lower estimate. We begin by specifying the boundaries of the 'steep' interval S_k and by giving some new notations.

We use the notation $|I|$ for the length of an interval I.

Definition 4.1. Let $k \in \mathbb{N}$ and $c \in (0,1)$. Define

$$a_k := \min \left\{ x \in \left(m_{k+1}, q^{k+1} \right] : \left| f'_{\mu,\omega}(x) \right| \geq c\omega \left(q^{k+1} \right)^{(\mu-1)} \text{ on } \left[x, q^{k+1} \right] \right\}$$

and

$$b_k := \max \left\{ x \in \left[q^{k+1}, m_k \right) : \left| f'_{\mu,\omega}(x) \right| \geq c\omega \left(q^{k+1} \right)^{(\mu-1)} \text{ on } \left[q^{k+1}, x \right] \right\}.$$

Note that $q^{k+2} < a_k < q^{k+1} < b_k < q^k$ (see Figure 4.1). Given a symbol sequence of the form

$$\mathbf{s} = (s_0, s_1, s_2, \dots) \in \{+1, -1\}^{\mathbb{N}_0},$$

where symbols represent the signs of $f^j_{\mu,\omega}(x)$ for some starting value x, we construct corresponding orbits of $f_{\mu,\omega}$. Note that in terms of the motivation by the three dimensional vector field, such orbits correspond to solutions converging to the doubly homoclinic loop, and taking turns along the upper and lower homoclinic orbit according to the symbol sequence. For $0 \leq a \leq b$, define

$$[a,b]_{+1} := [a,b],$$
$$[a,b]_{-1} := [-b,-a],$$

and define 'steep' intervals by

$$S^c_{k,s} := [a_k, b_k]_s = \begin{cases} [a_k, b_k], & \text{if } s = +1, \\ [-b_k, -a_k], & \text{if } s = -1. \end{cases}$$

So, we have

$$\left| f'_{\mu,\omega}(x) \right| \geq c\omega \left(q^{k+1} \right)^{(\mu-1)} \quad \text{for } x \in S^c_{k,s},\ s \in \{\pm 1\},\ k \in \mathbb{N}. \tag{4.1}$$

We also define $S_{k,\pm 1} := S^c_{k,+1} \cup S^c_{k,-1}$ and define the union of all 'steep' intervals by

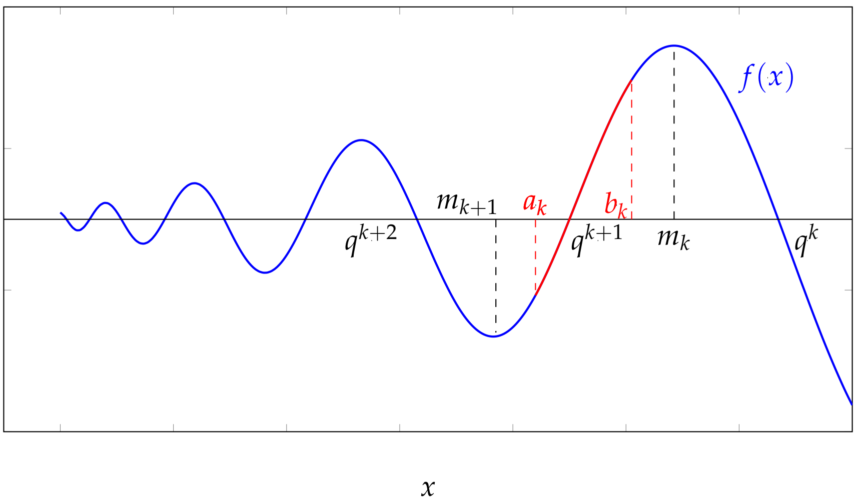

Figure 4.1: One interval (q^{k+2}, q^k), with corresponding 'steep' interval $S^c_{k,+1} = [a_k, b_k]$.

$$\Psi^c = \bigcup_{k \in \mathbb{N}} S_{k,\pm 1}.$$

Note that for $s \in \{\pm 1\}$, $S^c_{k,s} \subset (m_{k+1}, m_k)_s$, and hence

$$\left| S^c_{k,s} \right| \leq m_k - m_{k+1} = q^k e^{-\frac{\varphi_1}{\omega}} (1 - q) \tag{4.2}$$

Setting $f := f_{\mu,\omega}$, we define sets of points with forward orbits which are contained in these 'steep' intervals (see Figure 4.2). Namely,

$$\Omega^c_n = \bigcap_{j=0}^{n} f^{-j}(\Psi^c); \; \Omega^c_\infty = \bigcap_{j=0}^{\infty} f^{-j}(\Psi^c).$$

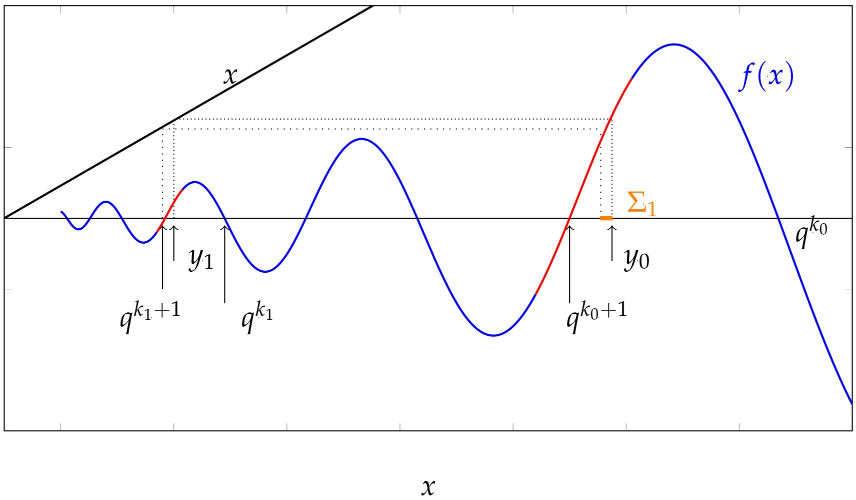

Figure 4.2: Graphical construction of Σ_1 from $\Sigma_0 = [q^{k_0+1}, y_0]$ (in case $s_0 = s_1 = +1$).

Theorem 4.2. *Let $c \in (0, 1)$. Assume $\mu > 1$ and define $S^c_{k,\pm 1}$ and Ω^c_∞ as above. Then for $k_0 \in \mathbb{N}$ the following statements are true:*

(i) For every symbol sequence $\mathbf{s} = (s_0, s_1, s_2, \ldots)$ there exists a point $y_0 \in \left(S^c_{k_0,s_0} \cap \Omega^c_\infty \right)$ with the property that sign $f^j(y_0)$ *is given by $s_j \in \{\pm 1\}$, where $j \in \mathbb{N}_0$.*

(ii) Let $\omega > \frac{1}{c} + \pi(\mu + 1)$. Then with the set Z from (3.2) we have $\left(S^c_{k_0,s_0} \cap \Omega^c_\infty \right) \subset \overline{Z}$.

(iii) Let $c \in \left(\frac{2}{\pi}, 1 \right)$ and $\omega > \frac{c\pi^2(2\mu+3)}{2(c\pi-2)}$. Then $\Omega^c_\infty \subset \overline{Z}$, and Ω^c_∞ has Lebesgue measure zero.

Remark 4.3. A similar argument is sketched in the page 395 of [2], with the purpose to show that Z can be dense, but it seems that the method gives density only in a set of measure zero (see part **(iii)** of the above theorem).

The proof starts with the following lemma.

Lemma 4.4. *Let $k_0 \in \mathbb{N}$, $c \in (0,1)$ and $\mathbf{s} = (s_0, s_1, s_2, \dots) \in \{+1, -1\}^{\mathbb{N}_0}$ be given. Define $S^c_{k_0,s_j}$ as in the passage given before Theorem 4.2. Then the following statements are true:*

(i) There exists a point $y_0 \in S^c_{k_0,s_0}$ and a sequence $k_0 < k_1 < k_2 < \cdots$ such that $\forall j \in \mathbb{N}_0$ $f^j(y_0) \in S^c_{k_j,s_j}$, in particular, $y_0 \in \Omega^c_\infty$.

(ii) Let $y_0 \in \left(S^c_{k_0,s_0} \cap \Omega^c_\infty\right)$ be given and define the sequence $k_0 < k_1 < k_2 < \cdots$ by $f^j(y_0) = y_j \in S^c_{k_j,s_j}$ $(j \in \mathbb{N}_0)$. Then there exists a sequence (Σ_j) of intervals in $S^c_{k_0,s_0}$ with $\Sigma_j \supset \Sigma_{j+1} \ni y_0$, $\left(f^j\right)' \neq 0$ on Σ_j and

$$f^j(\Sigma_j) = \left[q^{k_j+1}, y_j\right]_{s_j} = \begin{cases} \left[q^{k_j+1}, y_j\right], & \text{if } s_j = +1, \\ \left[y_j, -q^{k_j+1}\right], & \text{if } s_j = -1 \end{cases} \subset S^c_{k_j,s_j} \quad \text{for } j \in \mathbb{N}_0, \tag{4.3}$$

in particular, $Z \cap \Sigma_j \neq \emptyset$ for all $j \in \mathbb{N}_0$.

(iii) For $y_0 \in \left(S^c_{k_0,s_0} \cap \Omega^c_\infty\right)$ and $k_0, k_1, k_2, \dots$ as in assertion (ii) and all $j \in \mathbb{N}$ we have

$$\left|\left(f^j\right)'(y_0)\right| \geq (c\omega)^j \left(\prod_{n=0}^{j-1} q^{k_n+1}\right)^{\mu-1}. \tag{4.4}$$

(iv) Let y_0 and the sequence $k_0 < k_1 < k_2 < \cdots$ be as in (ii). Then

$$\forall j \in \mathbb{N}: \; q^{k_j\mu} \geq q^{k_{j+1}+2}. \tag{4.5}$$

(v) Let $\omega > \frac{1}{c} + \pi(\mu+1)$. Let y_0 and the associated Σ_j be as in assertion (ii) and φ_1 be as in Lemma 2.1. Then $|\Sigma_j| \leq \frac{q^{k_0} e^{-\frac{\varphi_1}{\omega}}(1-q)}{\left(c\omega q^{\mu+1}\right)^j}$ and $c\omega q^{\mu+1} > 1$; in particular, $|\Sigma_j| \to 0$, as $j \to \infty$.

Proof. **(i)** Let $k_0 \in \mathbb{N}$ and $\mathbf{s} = (s_0, s_1, s_2, \dots)$ be given. For $S^c_{k_0,s_0} = [a_{k_0}, b_{k_0}]_{s_0}$ it is clear that $f(S^c_{k_0,s_0})$ is an interval which contains 0 in its interior, and since $a_k \to 0$, $b_k \to 0$ as $k \to \infty$, there exists $k_1 > k_0$ with $S^c_{k_1,s_1} \subset f(S^c_{k_0,s_0})$. Further $f|_{S^c_{k_0,s_0}}$ is injective, and we set

$$J_1 := \left(f|_{S^c_{k_0,s_0}}\right)^{-1}\left(S^c_{k_1,s_1}\right).$$

(f maps J_1 bijectively onto $S^c_{k_1,s_1}$.) Similarly, there exists $k_2 > k_1$ with $S^c_{k_2,s_2} \subset f(S^c_{k_1,s_1})$, and a closed subinterval $J_2 \subset J_1$ such that $f^2|_{J_2} : J_2 \to S^c_{k_2,s_2}$ is bijective. Thus, we obtain a nested sequence

$$J_1 \supset J_2 \supset J_3 \supset \cdots$$

of closed intervals and sequence of numbers

$$k_0 < k_1 < k_2 < \cdots$$

with the property that $f^j(J_j) = S^c_{k_j,s_j}$, $j = 1, 2, 3, \dots$ Furthermore, the intersection of nested closed intervals $\bigcap_{j\in\mathbb{N}} J_j$ is not empty. It means that there exists a point $y_0 \in \bigcap_{j\in\mathbb{N}} J_j$ which follows the symbol sequence $\mathbf{s}$, and this result completes the proof of assertion (i).

(ii) For the proof of this assertion we use a recursive construction. Define

$$\Sigma_0 := \left[q^{k_0+1}, y_0\right]_{s_0} = \begin{cases} \left[q^{k_0+1}, y_0\right], & \text{if } s_0 = +1 \\ \left[y_0, -q^{k_0+1}\right], & \text{if } s_0 = -1 \end{cases} \subset S^c_{k_0,s_0}.$$

Then $y_0 \in \Sigma_0$, and the definition of $S^c_{k_0,s_0}$ implies $f' \neq 0$ on Σ_0, so (4.3) holds for $j = 0$. Assume Σ_j with the properties in (4.3) is constructed and we want to construct $\Sigma_{j+1} \subset \Sigma_j$ such that (4.3) is also satisfied for $j + 1$. We have, observing that sign $(y_j) = s_j$,

$$f\left(\left[q^{k_j+1}, y_j\right]_{s_j}\right) = \left[0, f\left(y_j\right)\right]_{s_{j+1}} = \left[0, y_{j+1}\right]_{s_{j+1}},$$

and $f^j|_{\Sigma_j}$ as well as $f|_{\left[q^{k_j+1}, y_j\right]_{s_j}}$ are invertible. Hence, we can define

$$\Sigma_{j+1} = \left(f^{-j}|_{\Sigma_j}\right)^{-1} \left(f|_{\left[q^{k_j+1}, y_j\right]_{s_j}}\right)^{-1} \left(\left[q^{k_{j+1}+1}, y_{j+1}\right]_{s_{j+1}}\right).$$

Then $y_0 \in \Sigma_{j+1} \subset \Sigma_j$, the chain rule shows $\left(f^{j+1}\right)' \neq 0$ on Σ_{j+1}, and

$$\left(f^{j+1}\right)\left(\Sigma_{j+1}\right) = \left[q^{k_{j+1}+1}, y_{j+1}\right]_{s_{j+1}} \subset S^c_{k_{j+1},s_{j+1}}.$$

Hence, the recursive construction is completed. Note also that for $j \in \mathbb{N}$, Σ_j contains a point x_j with $f^j\left(x_j\right) = q^{k_j+1}$, so $f^{j+1}(x_j) = f\left(q^{k_j+1}\right) = 0$, hence $x_j \in \Sigma_j \cap Z$.

(iii) By the chain rule the derivative $\left(f^j\right)'$ at $y_0 \in \bigcap_{j\in\mathbb{N}} \Sigma_j$ can be calculated as the product of the derivatives of f along the orbit

$$\left|\left(f^j\right)'(y_0)\right| = \left|f'(y_0) \cdot f'(y_1) \cdot \ldots \cdot f'(y_{j-2}) \cdot f'(y_{j-1})\right| = \prod_{n=0}^{j-1} \left|f'(y_n)\right|.$$

Using (4.1) for each derivative in the last equality, we have

$$\begin{aligned} \left|\left(f^j\right)'(y_0)\right| &= \prod_{n=0}^{j-1} \left|f'(y_n)\right| \geq (c\omega)^j \prod_{n=0}^{j-1} \left(q^{k_n+1}\right)^{\mu-1} \\ &= (c\omega)^j \left(\prod_{n=0}^{j-1} q^{k_n+1}\right)^{\mu-1}. \end{aligned}$$

This gives the proof of (4.4).

(iv) Let now $y_0 \in S^c_{k_0,s_0}$ and sequence $k_0 < k_1 < k_2 < \cdots$ as in (ii) be given. With Σ_j from (4.3) we have $f^j\left(\Sigma_j\right) \subset S^c_{k_j,s_j}$, and so

$$f^{j+1}(y_0) \in S^c_{k_{j+1},s_{j+1}} \cap f^{j+1}\left(\Sigma_j\right) \subset f\left(f^j\left(\Sigma_j\right)\right) \subset f\left(S^c_{k_j,s_j}\right), \quad \text{for } j \in \mathbb{N}_0$$

which implies $f(S^c_{k_j,s_j}) \cap S^c_{k_{j+1},s_{j+1}} \neq \emptyset$. Moreover, since $|f| \leq q^{k_j\mu}$ on $S^c_{k_j,s_j}$, we obviously have $q^{k_j\mu} \geq \max\{|f(x)| : x \in S^c_{k_j,s_j}\}$. Together with

$$\max\left\{|f(x)| : x \in S^c_{k_j,s_j}\right\} \geq \min\left\{|y| : y \in S^c_{k_{j+1},s_{j+1}}\right\},$$

we conclude

$$q^{k_j\mu} \geq \max\left\{|f(x)| : x \in S^c_{k_j,s_j}\right\} \geq \min\left\{|y| : y \in S^c_{k_{j+1},s_{j+1}}\right\} = a_{k_{j+1}} \geq q^{k_{j+1}+2}.$$

Hence, the proof of (iv) is also completed.

(v) Finally, from (2.13) we know that on Σ_j we have

$$|\Sigma_j| \leq \frac{\left|\left(f^j\right)(\Sigma_j)\right|}{\min_{\Sigma_j}\left|\left(f^j\right)'\right|}. \tag{4.6}$$

From (4.3) we have $\left|\left(f^j\right)(\Sigma_j)\right| \leq \left|S^c_{k_j,s_j}\right|$, and from (4.2) we have $\left|S^c_{k_j,s_j}\right| \leq q^{k_j}e^{-\frac{\varphi_1}{\omega}}(1-q)$. Combining both inequalities, we get

$$\left|\left(f^j\right)(\Sigma_j)\right| \leq \left|S^c_{k_j,s_j}\right| \leq q^{k_j}e^{-\frac{\varphi_1}{\omega}}(1-q). \tag{4.7}$$

Using (4.7) and (4.4) in (4.6), we obtain

$$|\Sigma_j| \leq \frac{q^{k_j}e^{-\frac{\varphi_1}{\omega}}(1-q)}{(c\omega)^j\left(\prod_{n=0}^{j-1} q^{k_n+1}\right)^{\mu-1}}. \tag{4.8}$$

By using (4.5) we can estimate the denominator of (4.8) as follows:

$$\begin{aligned}
(c\omega)^j\left(\prod_{n=0}^{j-1} q^{k_n+1}\right)^{\mu-1} &= (c\omega)^j\cdot\left(\prod_{n=0}^{j-1} q\right)^{\mu-1}\left(\prod_{n=0}^{j-1} q^{k_n}\right)^{\mu-1}\\
&= (c\omega)^j\cdot\frac{q^{j(\mu-1)}\prod_{n=0}^{j-1} q^{k_n\mu}}{\prod_{n=0}^{j-1} q^{k_n}} \geq (c\omega)^j\cdot q^{j(\mu-1)}\cdot\frac{\prod_{n=0}^{j-1} q^{k_{n+1}+2}}{\prod_{n=0}^{j-1} q^{k_n}}\\
&= (c\omega)^j\cdot q^{(\mu-1)j}\cdot\frac{q^{2j}\prod_{n=0}^{j-1} q^{k_{n+1}}}{\prod_{n=0}^{j-1} q^{k_n}} = \left(c\omega q^{\mu+1}\right)^j\cdot\frac{q^{k_j}}{q^{k_0}}.
\end{aligned}$$

Substituting this estimate in (4.8), we finally have

$$|\Sigma_j| \leq \frac{q^{k_j}e^{-\frac{\varphi_1}{\omega}}(1-q)\,q^{k_0}}{(c\omega q^{\mu+1})^j\,q^{k_j}} = \frac{q^{k_0}e^{-\frac{\varphi_1}{\omega}}(1-q)}{(c\omega q^{\mu+1})^j}.$$

To show that $|\Sigma_j| \to 0$ as $j \to \infty$, it is enough to show $\left(c\omega q^{\mu+1}\right) > 1$. Note that the first order Taylor expansion of $q^{\mu+1} = \exp\left(-\frac{\pi}{\omega}(\mu+1)\right)$ is

$$\exp\left(-\frac{\pi}{\omega}(\mu+1)\right) = 1 - \frac{\pi(\mu+1)}{\omega} + R_1(\xi),$$

where $R_1(\xi) = \frac{\exp''(\xi)}{2}\left(\frac{\pi(\mu+1)}{\omega}\right)^2 > 0$, and $\xi \in \left(-\frac{\pi(\mu+1)}{\omega}, 0\right)$. The assumption of (e) gives us $\frac{1}{c} + \pi(\mu+1) < \omega$, and hence

$$1 < c\omega - c\pi(\mu+1) = c\omega\left(1 - \frac{\pi(\mu+1)}{\omega}\right).$$

Since $R_1(\xi) > 0$, we obtain

$$1 < c\omega \left(1 - \frac{\pi(\mu+1)}{\omega}\right) < c\omega \left(1 - \frac{\pi(\mu+1)}{\omega} + R_1(\xi)\right) = c\omega \exp\left(-\frac{\pi}{\omega}(\mu+1)\right) = c\omega q^{\mu+1},$$

and this completes the proof of (v). $\square$

The next lemma estimates the measure of the points in the 'steep' interval $S^c_{k_0,+1}$ which have the first n iterates in the union of all 'steep' intervals.

Lemma 4.5. *Let $k_0 \in \mathbb{N}$, $c \in \left(\frac{1}{\pi}, 1\right)$. Let Ψ^c and $S^c_{k_0,\pm 1}$ be as in the passage before Theorem 4.2. Define φ_1 as in Lemma 2.1. Then for $k_0 \in \mathbb{N}$ we have*

$$\left| S^c_{k_0,+1} \cap \bigcap_{i=1}^{n} f^{-i}(\Psi^c) \right| \le \frac{2^n q^{k_0} e^{-\frac{\varphi_1}{\omega}}(1-q)}{\left(c\omega q^{\mu+1}(1-q)\right)^n}. \tag{4.9}$$

(The same estimate holds for $S^c_{k_0,-1}$.)

Proof. Let $k_0 \in \mathbb{N}$ and $c \in \left(\frac{1}{\pi}, 1\right)$ be given. It is clear that $f\left(S^c_{k_0,+1}\right)$ contains infinitely many 'steep' intervals, because it is a neighborhood of zero. Assume $\ell, i \in \mathbb{N}$ are such that $S^c_{k_0,+1} \cap f^{-i}\left(S^c_{\ell,\pm 1}\right) \neq \emptyset$. Since $\left|f^i(x)\right| \le |x|^{\mu^i}$ on $S^c_{k_0,\pm 1}$, one must have $q^{k_0\mu^i} \ge \min\{|y| : y \in S^c_{\ell,\pm 1}\} \ge q^{\ell+2}$. It follows that $\ell \ge k_0\mu^i - 2 \ge k_0\mu - 2$. Thus

$$f^{-i}(\Psi^c) = f^{-i}\left(\bigcup_{\ell \in \mathbb{N}} S^c_{\ell,\pm 1}\right) = f^{-i}\left(\bigcup_{\substack{\ell \in \mathbb{N} \\ f^{-i}\left(S^c_{\ell,\pm 1}\right) \neq \emptyset}} S^c_{\ell,\pm 1}\right) = f^{-i}\left(\bigcup_{\ell \ge k_0\mu - 2} S^c_{\ell,\pm 1}\right).$$

Hence, the intersection in (4.9) equals $S^c_{k_0,+1} \cap \bigcap_{i=1}^{n} f^{-i}\left(\bigcup_{\ell \ge k_0\mu-2} S^c_{\ell,\pm 1}\right)$. We now prove (4.9) by induction over n. For $n = 1$,

$$\begin{aligned} \left| S^c_{k_0,+1} \cap f^{-1}(\Psi^c) \right| &= \left| S^c_{k_0,+1} \cap f^{-1}\left(\bigcup_{\ell \ge k_0\mu-2} S^c_{\ell,\pm 1}\right) \right| \\ &= \sum_{\ell \ge k_0\mu-2} \left| S^c_{k_0,+1} \cap f^{-1}\left(S^c_{\ell,\pm 1}\right) \right|. \end{aligned} \tag{4.10}$$

From (4.2) we have

$$\left| S^c_{\ell,\pm 1} \right| \le 2q^{\ell} e^{-\frac{\varphi_1}{\omega}}(1-q). \tag{4.11}$$

Using (2.13), (4.1) and (4.11) in (4.10), we have

$$\begin{aligned} \left| S^c_{k_0,+1} \cap f^{-1}(\Psi^c) \right| &= \sum_{\ell \ge k_0\mu-2} \left| S^c_{k_0,+1} \cap f^{-1}\left(S^c_{\ell,\pm 1}\right) \right| \le \sum_{\ell \ge k_0\mu-2} \frac{1}{c\omega q^{(k_0+1)(\mu-1)}} \left| S^c_{\ell,\pm 1} \right| \\ &\le \frac{2e^{-\frac{\varphi_1}{\omega}}(1-q)}{c\omega q^{(k_0+1)(\mu-1)}} \sum_{\ell \ge k_0\mu-2} q^{\ell}. \end{aligned} \tag{4.12}$$

Here, note that

$$\sum_{\ell \ge k_0\mu-2} q^{\ell} = \sum_{\ell \ge \lceil k_0\mu-2 \rceil} q^{\ell} = q^{\lceil k_0\mu-2 \rceil} \frac{1}{1-q}, \tag{4.13}$$

where $\lceil\cdot\rceil$ denotes the ceiling function. Setting $\varepsilon(k_0) := \lceil k_0\mu - 2\rceil - (k_0\mu - 1) \in [-1, 0)$ and using (4.13) in (4.12), we obtain

$$\left|S^c_{k_0,+1} \cap f^{-1}(\Psi^c)\right| \leq \frac{2e^{-\frac{\varphi_1}{\omega}}}{c\omega q^{(k_0+1)(\mu-1)}} q^{\lceil k_0\mu-2\rceil} = \frac{2q^{k_0}e^{-\frac{\varphi_1}{\omega}}}{c\omega} \cdot \frac{q^{\lceil k_0\mu-2\rceil}}{q^{k_0\mu-1}} \cdot \frac{1}{q^\mu}$$
$$= 2q^{\varepsilon(k_0)} \cdot \frac{q^{k_0}e^{-\frac{\varphi_1}{\omega}}}{c\omega q^\mu} \leq \frac{2q^{-1}q^{k_0}e^{-\frac{\varphi_1}{\omega}}}{c\omega q^\mu} = \frac{2q^{k_0}e^{-\frac{\varphi_1}{\omega}}(1-q)}{c\omega q^{\mu+1}(1-q)}$$

which proves the case $n = 1$.

Assume the assertion is true for n, i.e, for all $k_0 \in \mathbb{N}$ we have

$$\left|S^c_{k_0,+1} \cap \bigcap_{i=1}^{n} f^{-i}(\Psi^c)\right| \leq q^{k_0}e^{-\frac{\varphi_1}{\omega}}(1-q)\left(\frac{2}{c\omega q^{\mu+1}(1-q)}\right)^n, \tag{4.14}$$

then the same estimate is true for $S^c_{k_0,-1}$. Now we show that it is true for $n+1$. Using (4.10) for the third equality we obtain

$$\left|S^c_{k_0,+1} \cap \bigcap_{i=1}^{n+1} f^{-i}(\Psi^c)\right| = \left|S^c_{k_0,+1} \cap f^{-1}(\boldsymbol{\Psi}^c) \cap \cdots \cap f^{-n-1}(\boldsymbol{\Psi}^c)\right|$$
$$= \left|S^c_{k_0,+1} \cap f^{-1}\left(\bigcap_{i=0}^{n} f^{-i}(\boldsymbol{\Psi}^c)\right)\right|$$
$$= \left|S^c_{k_0,+1} \cap f^{-1}\left(\left(\bigcup_{\ell\geq k_0\mu-2} S^c_{\ell,\pm1}\right) \cap \bigcap_{i=0}^{n} f^{-i}(\boldsymbol{\Psi}^c)\right)\right|.$$

Note that $S^c_{\ell,\pm1} \subset \boldsymbol{\Psi}^c$ implies

$$S^c_{\ell,\pm1} \cap \bigcap_{i=0}^{n} f^{-i}(\boldsymbol{\Psi}^c) = S^c_{\ell,\pm1} \cap \bigcap_{i=1}^{n} f^{-i}(\boldsymbol{\Psi}^c).$$

So, we obtain

$$\left|S^c_{k_0,+1} \cap \bigcap_{i=1}^{n+1} f^{-i}(\Psi^c)\right| = \left|S^c_{k_0,+1} \cap f^{-1}\left(\bigcup_{\ell\geq k_0\mu-2}\left(S^c_{\ell,\pm1} \cap \bigcap_{i=1}^{n} f^{-i}(\boldsymbol{\Psi}^c)\right)\right)\right| \tag{4.15}$$

Using (2.13), (4.1), (4.11), (4.13) and (4.14) for $S^c_{k_0,+1}$ and $S^c_{k_0,-1}$ in (4.15), we have

$$\left|S^c_{k_0,+1} \cap \bigcap_{i=1}^{n+1} f^{-i}(\Psi^c)\right| \leq \frac{1}{(c\omega)\, q^{(k_0+1)(\mu-1)}} \sum_{\ell\geq k_0\mu-2} 2\left(\frac{2}{c\omega q^{\mu+1}(1-q)}\right)^n q^\ell e^{-\frac{\varphi_1}{\omega}}(1-q)$$
$$= \frac{2^{n+1}q^{k_0}e^{-\frac{\varphi_1}{\omega}}}{(c\omega)^{n+1}(q^{\mu+1})^n q^\mu q^{k_0\mu-1}}\left(\frac{1}{1-q}\right)^{n-1} \sum_{\ell\geq k_0\mu-2} q^\ell$$
$$= \frac{2^{n+1}q^{k_0}e^{-\frac{\varphi_1}{\omega}}}{(c\omega)^{n+1}(q^{\mu+1})^n q^\mu}\left(\frac{1}{1-q}\right)^{n-1} \frac{q^{\lceil k_0\mu-2\rceil}}{q^{k_0\mu-1}}\frac{1}{1-q}$$

With $\varepsilon(k_0)$ as above, we obtain

$$\begin{aligned}\left|S^c_{k_0,+1}\cap\bigcap_{i=1}^{n+1}f^{-i}(\Psi^c)\right| &\le \frac{2^{n+1}q^{k_0}e^{-\frac{\varphi_1}{\omega}}q^{\varepsilon(k_0)}}{(c\omega)^{n+1}(q^{\mu+1})^n q^{\mu}}\left(\frac{1}{1-q}\right)^n\\ &\le \frac{2^{n+1}q^{k_0}e^{-\frac{\varphi_1}{\omega}}q^{-1}}{(c\omega)^{n+1}(q^{\mu+1})^n q^{\mu}}\left(\frac{1}{1-q}\right)^n\\ &= q^{k_0}e^{-\frac{\varphi_1}{\omega}}(1-q)\left(\frac{2}{c\omega q^{\mu+1}(1-q)}\right)^{n+1},\end{aligned}$$

so the assertion is true for $n+1$ and hence, the proof of Lemma 4.5 is completed. □

Remark 4.6. Let $c\in\left(\frac{2}{\pi},1\right)$ and $\mu>1$. Then $\frac{1}{c}+\pi(\mu+1)\le\frac{c\pi^2(2\mu+3)}{2(c\pi-2)}$.

Proof. Let $c\in\left(\frac{2}{\pi},1\right)$. Then

$$\frac{1}{c}+\pi(\mu+1)=\frac{1+c\pi\mu+c\pi}{c}=\frac{\pi+c\pi^2\mu+c\pi^2}{c\pi}\le\frac{2c\pi^2\mu+2c\pi^2+2\pi}{2(c\pi-2)}.$$

Since $c\pi>2$, we have $2\pi<c\pi^2$ and hence

$$\frac{1}{c}+\pi(\mu+1)\le\frac{2c\pi^2\mu+3c\pi^2}{2(c\pi-2)}=\frac{c\pi^2(2\mu+3)}{2(c\pi-2)}.$$ □

***Proof of Theorem* 4.2.** **(i)** From assertion (i) in Lemma 4.4 we see that there exists a point $y_0\in(S^c_{k_0,s_0}\cap\Omega^c_\infty)$ with sign $f^j(y_0)=s_j$, because $f^j(y_0)\in S^c_{k_j,s_j}$.

(ii) Assume $y_0\in(S^c_{k_0,s_0}\cap\Omega^c_\infty)$. Assertion (ii) of Lemma 4.4 shows that $\Sigma_j\ni y_0$ and $Z\cap\Sigma_j\ne\emptyset$. Further, assertion (v) of Lemma 4.4 shows that $|\Sigma_j|\to 0$ as $j\to\infty$. This means that there exists a sequence $(z_j)\subset Z$ with $z_j\to y_0$, and this completes the proof.

(iii) Let $c\in\left(\frac{2}{\pi},1\right)$ be given. Remark 4.6 shows that the condition $\omega>\frac{c\pi^2(2\mu+3)}{2(c\pi-2)}$ from assertion (iii) of Theorem 4.2 implies the condition $\omega>\frac{1}{c}+\pi(\mu+1)$ of assertion (ii). Hence, $(S^c_{k_0,\pm1}\cap\Omega^c_\infty)\subset\overline{Z}$ for all $k_0\in\mathbb{N}$. It follows that $\Omega^c_\infty=\bigcup_{k_0\in\mathbb{N}}(S^c_{k_0,\pm1}\cap\Omega^c_\infty)\subset\overline{Z}$, so $\Omega^c_\infty\subset\overline{Z}$. To prove that Ω^c_∞ has measure zero, we show $\lim_{n\to\infty}|\Omega^c_n\cap S^c_{k_0,\pm1}|=0$ for every $k_0\in\mathbb{N}$. For this purpose it is enough to show that under the conditions of assertion (iii) of Theorem 4.2, $c\omega q^{\mu+1}(1-q)>2$ in (4.9). We use the second order Taylor expansion of e^{-y} around 0 for $y>0$,

$$e^{-y}=1-y+\frac{y^2}{2}+R_3,$$

with $R_3=\frac{\exp'''(\xi)}{3!}(-y)^3<0$ for some $\xi\in(-y,0)$. Hence, since

$$q=e^{-\frac{\pi}{\omega}}=1-\frac{\pi}{\omega}+\frac{(\pi)^2}{2\omega^2}+R_3\left(\frac{\pi}{\omega}\right)<1-\frac{\pi}{\omega}+\frac{(\pi)^2}{2\omega^2},$$

we have

$$1-q=1-e^{-\frac{\pi}{\omega}}=\frac{\pi}{\omega}-\frac{\pi^2}{2\omega^2}-R_3\left(\frac{\pi}{\omega}\right)>\frac{\pi}{\omega}-\frac{\pi^2}{2\omega^2}. \tag{4.16}$$

On the other hand, with appropriate ξ,

$$q^{\mu+1}=e^{-\frac{\pi}{\omega}(\mu+1)}=1-\frac{\pi(\mu+1)}{\omega}+\frac{\exp''(\xi)}{2!}\left(-\frac{\pi(\mu+1)}{\omega}\right)^2>1-\frac{\pi(\mu+1)}{\omega}. \tag{4.17}$$

Using (4.16) and (4.17), we get

$$\begin{aligned} c\omega q^{\mu+1}(1-q) &> c\omega\left(1-\frac{\pi(\mu+1)}{\omega}\right)\left(\frac{\pi}{\omega}-\frac{\pi^2}{2\omega^2}\right) = c\pi\left(1-\frac{\pi\mu+\pi}{\omega}\right)\left(1-\frac{\pi}{2\omega}\right) \\ &= c\pi\left(1-\frac{(3\pi+2\pi\mu)}{2\omega}+\frac{\pi^2(\mu+1)}{2\omega^2}\right) \\ &> c\pi\left(1-\frac{\pi(3+2\mu)}{2\omega}\right). \end{aligned} \tag{4.18}$$

In view of Remark 4.6, and using the assumption which is given in the assertion (iii) of Theorem 4.2 in (4.18), we finally obtain

$$c\omega q^{\mu+1}(1-q) > c\pi\left(1-\frac{\pi(3+2\mu)}{2\cdot\frac{c\pi^2(2\mu+3)}{2(c\pi-2)}}\right) = c\pi\left(1-\frac{c\pi-2}{c\pi}\right) = 2. \qquad \square$$

5 The behavior of the points whose orbits follow 'flat-steep-flat' intervals

In chapter three we analyzed the behavior of the points which are mapped from 'flat' intervals to some other 'flat' intervals, and in chapter four we studied the behavior of the points which are mapped from 'steep' intervals to some other 'steep' intervals. Finally in this chapter, as we briefly mentioned in the summary of this thesis, we first construct a specific type of orbit whose points travel from 'flat' intervals to 'steep' intervals, then from 'steep' intervals again to 'flat' intervals under the iteration (see Figure 5.1).

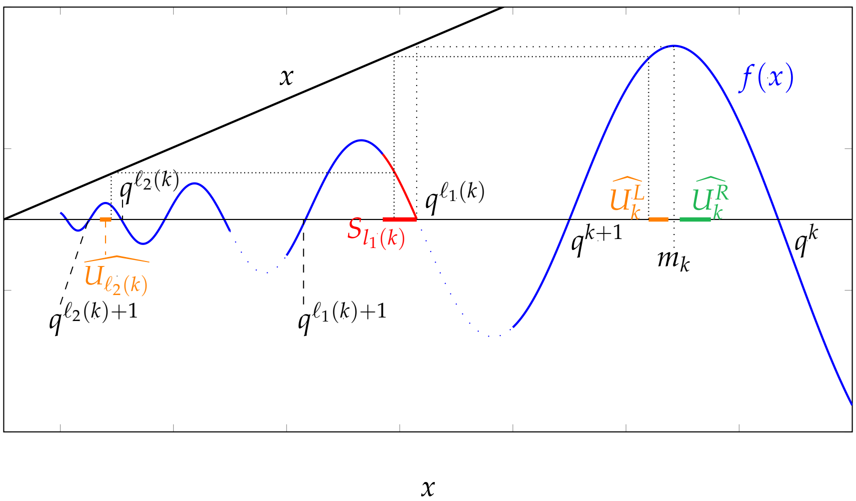

Figure 5.1: The parameters adjusted so that $f(m_k) = q^{\ell_1(k)}$, and the sets $\widehat{U_k^L}$ and $\widehat{U_k^R}$ constructed as counterimages under f^2 of the interval $\widehat{U_{\ell_2(k)}} \subset U_{\ell_2(k)}$ (indicated only for the lower endpoint of $\widehat{U_k^L}$). The dotted parts of the graph indicate possible maxima/minima in between which are not shown.

Besides, to avoid repeating the same expression, we shall use $g_{\omega,\mu+1-j}$ as in Lemma 2.1 and $c \in (0,1)$ for the rest of the paper. For a specific choice of μ, $\omega > 0$, maxima m_k get mapped to zeros $q^{\ell_1(k)}$ of $f_{\mu,\omega} = f$. We shall first introduce 'flat' intervals of the form $U_k = [m_k - \delta_k, m_k + \delta_k]$ for odd k and use the notations $U_k^R = [m_k, m_k + \delta_k]$ and $U_k^L = [m_k - \delta_k, m_k]$ for the right and left part of U_k respectively. We also introduce 'steep' intervals $S_{\ell_1(k)}$, where $\ell_1(k) = k\mu + 1$, of the form $\left[q^{\ell_1(k)} - r_{\ell_1(k)}, q^{\ell_1(k)}\right]$, with a suitable $r_{\ell_1(k)}$. Then we define $U = \bigcup_{\substack{k\in\mathbb{N}\\ k \text{ odd}}} U_k$, $S = \bigcup_{\substack{k\in\mathbb{N}\\ k \text{ odd}}} S_{\ell_1(k)}$, and we construct orbits $\left(f^j(x)\right)_{j\in\mathbb{N}}$, with the properties

$$f^j(x) \in \begin{cases} U, & j \text{ is even,} \\ S, & j \text{ is odd.} \end{cases}$$

Furthermore, for $k, \mu \in \mathbb{N}$, $\omega > 0$ and with φ_1 as in Lemma 2.1, we define

$$\ell_2(k) := \min\left\{\ell \in \mathbb{N} : q^\ell \le q^{\ell_1(k)\mu} \cdot q^{\frac{\varphi_1(\mu-2)}{\pi}} \cdot \frac{c(1-c)\,\omega^3}{4g_{\omega,\mu} \cdot g_{\omega,\mu-1}^2}\right\}.$$

We denote by $\ell_2^j(k)$ the j th iterate of the function ℓ_2 applied to k. Then, given a symbol sequence of the form $\{L, R\}^{n+1}$, where symbols represent the left 'L' or right 'R' hand part of U_k (that is U_k^L, U_k^R), we construct corresponding orbits of f. Given a finite sequence

$$\mathbf{s} = (s_0, s_1, s_2, \ldots, s_n) \in \{L, R\}^{n+1}$$

and $k \in \mathbb{N}$, we first construct the subset of points x in U_k which follow this symbol sequence in the sense that $f^{2j}(x) \in U^L_{\ell_2^j(k)}$ or $f^{2j}(x) \in U^R_{\ell_2^j(k)}$, $j = 0, 1, 2, \ldots, n$ depending on whether $s_j = L$ or $s_j = R$. Hence, we construct the set $I^n_{k,\mathbf{s}} = \bigcap_{j=0}^n f^{-2j}\left(U^{s_j}_{\ell_2^j(k)}\right)$ and the set $\Gamma_k^n = \bigcup_{s\in\{L,R\}^{n+1}} I^n_{k,s}$ which is the set of points following symbol sequences in the set $\{L, R\}^{\{0,1,2,\ldots,n\}}$. Note that strict monotonicity of f^2 on each interval U_k^L and U_k^R implies that the sets $I^n_{k,\mathbf{s}}$ are closed intervals. The corresponding set for infinite symbol sequences is $I^\infty_{k,s} = \bigcap_{j=0}^\infty f^{-2j}\left(U^{s_j}_{\ell_2^j(k)}\right)$. Finally, we analyze the Lebesgue measure of the set Γ_k^n, and consider the limit as $n \to \infty$.

Note that the 'steep' intervals S_k that we use in our calculations in this chapter are some subintervals of $\left(m_k, q^k\right]$, whereas the 'steep' intervals which were used in the fourth chapter are some subintervals of (m_{k+1}, m_k). In the theorem below we restrict ourselves to $\mu \in \mathbb{N}$ for simplicity.

Theorem 5.1. *Let k be a positive odd integer. Let $c \in (0,1)$, and $\mu \in \mathbb{N}$, $\mu \ge \max\left\{\left(\frac{30e}{7\pi}\right)^2\left(\frac{1-c}{2c}\right), 15\right\}$ be given. Then there exist an $\omega > 0$ (depending on μ) such that $f = f_{\mu,\omega}$ has the following properties:*

(i) Let a sequence of the form $\mathbf{s} \in \{L, R\}^{\mathbb{N}_0}$ be given. Then, there exists exactly one point $x_{k,s} \in U_k$ with the property:

For all $n \in \mathbb{N}_0$, $f^{2n}(x_{k,s}) \in U_{\ell_2^n(k)}$, and $f^{2n}(x_{k,s})$ is to the left of $m_{\ell_2^n(k)}$ or to the right of $m_{\ell_2^n(k)}$, depending on whether $s_n = L$ or $s_n = R$. That is, $I^\infty_{k,s} = \{x_{k,s}\}$.

(ii) The measure of Γ_k^n as defined above goes to zero, as $n \to \infty$.

The proof requires several lemmas and propositions. The proof of the following lemma is analogous to the proof of Lemma 3.3, but is included for completeness.

Lemma 5.2. *Define φ_1 as in Lemma 2.1. Then the following statements are true.*

(i) Assume $\mu \in \mathbb{N}$, $\mu \geq 15$ and define $\ell_1(k)$ as in the passage before Theorem 5.1. Then there exists an $\omega \in (0,1)$ such that for all $k \in \mathbb{N}$, f has the property

$$|f(m_k)| = q^{\ell_1(k)}, \tag{5.1}$$

which is equivalent to

$$\exp\left(\frac{\pi - \mu \arctan\left(\frac{\omega}{\mu}\right)}{\omega}\right) = \sqrt{1 + \frac{\mu^2}{\omega^2}}\,. \tag{5.2}$$

(ii) For any choice of ω as in assertion (i), we have $\omega \to 0$ as $\mu \to \infty$.

Proof. **(i)** Let $k \in \mathbb{N}$ be given. With m_k from (2.10), we have from (2.11)

$$|f(m_k)| = \exp\left(-\frac{k\pi\mu + \varphi_1\mu}{\omega}\right) \cdot \sin(\varphi_1).$$

Using (2.3) we obtain

$$|f(m_k)| = \exp\left(-\frac{k\pi\mu + \varphi_1\mu}{\omega}\right) \cdot \frac{1}{\sqrt{1 + \frac{\mu^2}{\omega^2}}}. \tag{5.3}$$

On the other hand, from (2.8) we have

$$q^{\ell_1(k)} = \exp\left(-\frac{\pi\ell_1(k)}{\omega}\right). \tag{5.4}$$

With (5.3) and (5.4) together, we see that (5.1) is equivalent to

$$\exp\left(-\frac{\pi\ell_1(k)}{\omega}\right) = \exp\left(-\frac{k\pi\mu + \varphi_1\mu}{\omega}\right) \frac{1}{\sqrt{1 + \frac{\mu^2}{\omega^2}}}$$

and hence to

$$\exp\left(\frac{\pi(\ell_1(k) - k\mu) - \varphi_1\mu}{\omega}\right) = \sqrt{1 + \frac{\mu^2}{\omega^2}}. \tag{5.5}$$

So, if we substitute $\ell_1(k) = k\mu + 1$ and the value of φ_1 given by Lemma 2.1 in (5.5), we finally get that (5.1) is equivalent to

$$\exp\left(\frac{\pi - \mu \arctan\left(\frac{\omega}{\mu}\right)}{\omega}\right) = \sqrt{1 + \frac{\mu^2}{\omega^2}},$$

which proves the equivalence of (5.1) and (5.2). Now, we want to find ω and μ such that $|f(m_k)| = q^{\ell_1(k)}$. Define

$$F(\omega, \mu) = \exp\left(\frac{\pi - \mu \arctan\left(\frac{\omega}{\mu}\right)}{\omega}\right) - \sqrt{1 + \frac{\mu^2}{\omega^2}}$$

and try to find $F(\omega,\mu)=0$ at least for a special pair of (ω,μ) (see Figure 5.2). On the one hand, for a fixed $\mu>2$, $\arctan\left(\frac{\omega}{\mu}\right)\to 0$ as $\omega\to 0$. Hence, due to the exponential growth, $F(\omega,\mu)\to\infty$ as $\omega\to 0$. On the other hand, for $\omega=1$ we have

$$F(1,\mu)=\exp\left(\pi-\mu\arctan\left(\frac{1}{\mu}\right)\right)-\sqrt{1+\mu^2}. \tag{5.6}$$

From 3.9 we have $\mu\arctan(\frac{1}{\mu})\geq\frac{1}{2}$ for $\mu>2$, and using this estimate in (5.6), we finally have $F(1,\mu)<e^{\pi-\frac{1}{2}}-\sqrt{1+\mu^2}$. From the fact that $e^{\pi-\frac{1}{2}}<15$, we finally have $F(1,\mu)<0$, if we choose $\mu\geq 15$. With the intermediate value theorem, it is clear that there exists at least one $\omega\in(0,1)$ which satisfies $F(\omega,\mu)=0$ for fixed μ. This gives the proof of assertion (i).

(ii) The proof is analogous to the proof of the assertion (ii) of Lemma 3.3. □

In order to find a numerical solution, one can use the bisection method, and we found numerically that there is no solution for $\mu<2.3$. The numerical investigation suggests that ω in Lemma 5.2 is unique. We made no effort to prove that, because part (ii) is true for any possible choice of ω.

The next three propositions (5.3–5.5) give some preparatory calculations.

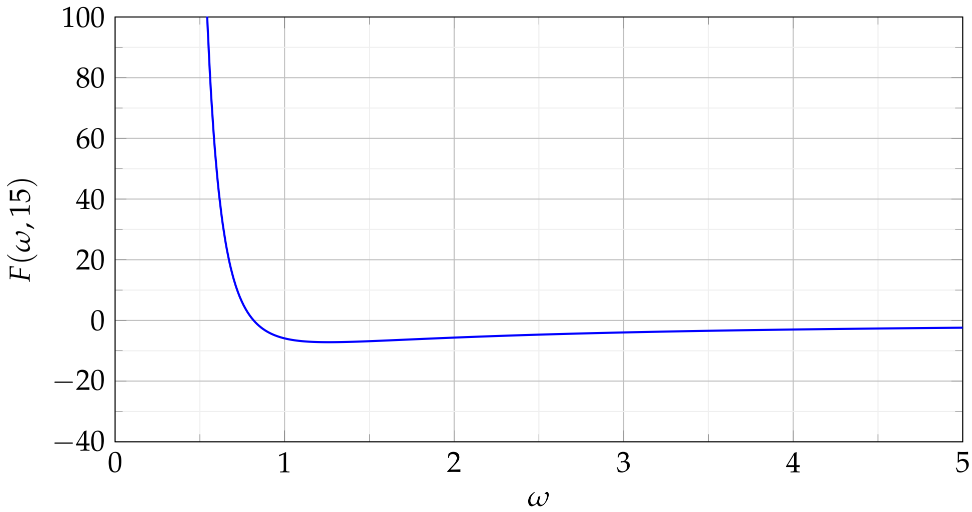

Figure 5.2: Graph of $F(\cdot,\mu)$ for $\mu=15$.

Proposition 5.3. *Let φ_1 be as in Lemma 2.1. Set $\alpha(\omega,\mu,c):=\frac{\exp\left(\frac{(\mu-2)\varphi_1}{2\omega}\right)}{g_{\omega,\mu-1}}\sqrt{\frac{1-c}{2c\omega}}$. If $\mu\in\mathbb{N}$,*

$$\mu\geq\max\left\{\left(\frac{30e}{7\pi}\right)^2\left(\frac{1-c}{2c}\right),15\right\},$$

and ω is a corresponding value obtained as in Lemma 5.2, then we have $\alpha(\omega,\mu,c)<\frac{1}{2}$.

Proof. Let μ and $\omega\in(0,1)$ be as in the assumption. Then, it is clear that $\frac{3\mu^2}{\omega^2}\geq 1$, and in view of (5.2) we have

$$\frac{2\mu}{\omega}=\sqrt{\frac{3\mu^2}{\omega^2}+\frac{\mu^2}{\omega^2}}\geq\sqrt{1+\frac{\mu^2}{\omega^2}}=\exp\left(\frac{\pi-\mu\arctan\left(\frac{\omega}{\mu}\right)}{\omega}\right).$$

Since $\arctan\left(\frac{\omega}{\mu}\right) \leq \frac{\omega}{\mu}$, we get

$$\frac{2\mu}{\omega} \geq \exp\left(\frac{\pi - \mu \cdot \frac{\omega}{\mu}}{\omega}\right) = \exp\left(\frac{\pi}{\omega} - 1\right),$$

and hence we have $2\mu e \geq \omega e^{\frac{\pi}{\omega}}$. Using the second order Taylor expansion of $e^{\frac{\pi}{\omega}}$ in the last inequality, we obtain

$$2\mu e \geq \omega\left(1 + \frac{\pi}{\omega} + \frac{1}{2}\frac{\pi^2}{\omega^2}\right) \geq \frac{1}{2}\frac{\pi^2}{\omega}$$

or

$$4e\mu \geq \frac{\pi^2}{\omega}. \tag{5.7}$$

On the other hand, we know that $\mu \geq \left(\frac{30e}{7\pi}\right)^2\left(\frac{1-c}{2c}\right)$, so $\sqrt{\mu} \geq \frac{30e}{7\pi}\sqrt{\frac{1-c}{2c}}$ which implies

$$\frac{1}{2} \geq e^{\frac{1}{2}}\sqrt{\frac{1-c}{2c}}\frac{30e^{\frac{1}{2}}}{14\pi\sqrt{\mu}}.$$

Since $e^{\frac{1}{2}} > e^{\frac{1}{2}-\frac{1}{\mu}}$, it follows that

$$\frac{1}{2} > \exp\left(\frac{1}{2} - \frac{1}{\mu}\right) \cdot \sqrt{\frac{1-c}{2c}} \cdot \frac{15\sqrt{4e}}{14\pi\sqrt{\mu}} = \exp\left(\frac{\omega}{2\mu\omega}(\mu-2)\right) \cdot \sqrt{\frac{1-c}{2c}} \cdot \frac{15\sqrt{4e\mu}}{14\pi\mu}.$$

On the other hand, since $\mu \geq 15$, we have $\frac{1}{\mu-1} \leq \frac{15}{14\mu}$, and with the fact that $\arctan\left(\frac{\omega}{\mu}\right) \leq \frac{\omega}{\mu}$ we get

$$\frac{1}{2} > \exp\left(\frac{(\mu-2)\arctan\left(\frac{\omega}{\mu}\right)}{2\omega}\right) \cdot \sqrt{\frac{1-c}{2c}} \cdot \frac{\sqrt{4e\mu}}{\pi(\mu-1)}.$$

Finally, using (5.7) and the definition of φ_1, we obtain

$$\begin{aligned}\frac{1}{2} &> \exp\left(\frac{(\mu-2)\varphi_1}{2\omega}\right) \cdot \sqrt{\frac{1-c}{2c\omega}} \cdot \sqrt{\frac{1}{(\mu-1)^2}} \\ &\geq \exp\left(\frac{(\mu-2)\varphi_1}{2\omega}\right) \cdot \sqrt{\frac{1-c}{2c\omega}} \cdot \sqrt{\frac{1}{\omega^2 + (\mu-1)^2}} \\ &= \frac{\exp\left(\frac{(\mu-2)\varphi_1}{2\omega}\right)}{g_{\omega,\mu-1}} \cdot \sqrt{\frac{1-c}{2c\omega}} = \alpha(\omega,\mu,c)\end{aligned}$$

and this completes the proof. □

Proposition 5.4. *Let φ_1 be as in Lemma 2.1 and $c \in (0,1)$ be given. Set $\widetilde{\eta}_1(\omega,\mu) := \frac{\omega e^{-\frac{\varphi_1(\mu-2)}{\omega}}}{2g_{\omega,\mu-1} \cdot g_{\omega,\mu-2}}$, $\widetilde{\eta}_2(\omega) := \frac{e^{-\frac{\varphi_1}{\omega}} - q}{2}$, $\widetilde{\eta}_3(\omega) := \frac{1-e^{-\frac{\varphi_1}{\omega}}}{2}$ and $\widetilde{\eta}_4(\omega,\mu) := \frac{\sqrt{(1-c)q\omega}}{g_{\omega,\mu} \cdot g_{\omega,\mu-1}}$. There exist $\omega_0 > 0$ and $\mu_0 > 3$ such that for $\omega \leq \omega_0$ and $\mu \in \mathbb{N}$, $\mu \geq \mu_0$, the number*

$$\widetilde{\eta} := \min\left\{\widetilde{\eta}_1(\omega,\mu),\ \widetilde{\eta}_2(\omega),\ \widetilde{\eta}_3(\omega),\ \widetilde{\eta}_4(\omega,\mu)\right\}$$

satisfies

$$\widetilde{\eta} = \widetilde{\eta}_4(\omega,\mu). \tag{5.8}$$

Proof. We prove that $\widetilde{\eta}_4(\omega,\mu) \leq \widetilde{\eta}_1(\omega,\mu) \leq \min\{\widetilde{\eta}_2(\omega), \widetilde{\eta}_3(\omega)\}$ for μ large enough, ω small enough. For $\omega > 0$, we have $g_{\omega,\mu-1} \geq \sqrt{(\mu-1)^2}$, $g_{\omega,\mu-2} \geq \sqrt{(\mu-2)^2}$ and using these simplifications, we obtain

$$\widetilde{\eta}_1(\omega,\mu) \leq \frac{\omega e^{-\frac{\varphi_1(\mu-2)}{\omega}}}{2\sqrt{(\mu-1)^2(\mu-2)^2}} = \frac{\omega e^{-\frac{\varphi_1(\mu-2)}{\omega}}}{2(\mu-1)(\mu-2)}. \tag{5.9}$$

We have already defined

$$\widetilde{\eta}_2(\omega) = \frac{e^{-\frac{\varphi_1}{\omega}} - q}{2} = \frac{1}{2}e^{-\frac{\varphi_1}{\omega}}\left(1 - e^{\frac{\varphi_1-\pi}{\omega}}\right),$$

and there exists $\widetilde{\omega_1} > 0$ such that for $\omega \in \omega_0 \in (0, \widetilde{\omega_1}]$, the property $\varphi_1 - \pi < 0$ implies $\left(1 - e^{\frac{\varphi_1-\pi}{\omega}}\right) > \frac{1}{2}$. Hence, for such ω we have

$$\widetilde{\eta}_2(\omega) \geq \frac{1}{4}e^{-\frac{\varphi_1}{\omega}}. \tag{5.10}$$

From (5.9) and (5.10) it is obvious that $\widetilde{\eta}_1(\omega,\mu) \leq \frac{\omega e^{-\frac{\varphi_1}{\omega}}}{2} \leq \widetilde{\eta}_2(\omega)$ for $\mu \geq 3$ and $\omega \leq \omega_{12}$, where $\omega_{12} =: \min\{\frac{1}{2}, \widetilde{\omega_1}\}$. Analogously there exists $\omega_{13} > 0$ such that for $\mu \geq 3$ and $\omega \leq \omega_{13}$, one has $\widetilde{\eta}_1(\omega,\mu) \leq \widetilde{\eta}_3(\omega)$; observe $\widetilde{\eta}_3(\omega) \to \frac{1}{2}$ as $\omega \to 0$. There exist $c_1, c_2 > 0$, and $\widetilde{\mu_0} > 0$ such that for $\mu \geq \widetilde{\mu_0}$ we have $\widetilde{\eta}_1(\omega,\mu) \geq c_1\frac{\omega e^{-\frac{\varphi_1(\mu-2)}{\omega}}}{\mu^2}$ and $\widetilde{\eta}_4(\omega,\mu) \leq c_2\frac{\sqrt{\omega e^{-\frac{\pi}{\omega}}}}{\mu^2}$. Hence, we have for $\mu \geq \widetilde{\mu_0}$ and $\omega > 0$

$$\frac{\widetilde{\eta}_4(\omega,\mu)}{\widetilde{\eta}_1(\omega,\mu)} \leq \frac{c_2\sqrt{\omega e^{-\frac{\pi}{\omega}}}}{c_1\omega e^{-\frac{\varphi_1(\mu-2)}{\omega}}} = \frac{c_2}{c_1}\frac{1}{\sqrt{\omega}}\exp\left(\frac{\varphi_1(\mu-2) - \frac{\pi}{2}}{\omega}\right).$$

Substituting the explicit form of φ_1 as in Lemma 2.1, the last equality turns to

$$\frac{\widetilde{\eta}_4(\omega,\mu)}{\widetilde{\eta}_1(\omega,\mu)} \leq \frac{c_2}{c_1}\frac{1}{\sqrt{\omega}}\exp\left(\frac{(\mu-2)\arctan\left(\frac{\omega}{\mu}\right) - \frac{\pi}{2}}{\omega}\right). \tag{5.11}$$

Using the fact that

$$\lim_{\mu\to\infty,\ \omega\to 0}\left(\frac{(\mu-2)\arctan\left(\frac{\omega}{\mu}\right)}{\omega}\right) = \lim_{\mu\to\infty,\ \omega\to 0}\left(\frac{(\mu-2)\frac{\omega}{\mu}}{\omega}\right) = 1, \text{ and}$$

$\lim_{\omega\to 0}\frac{1}{\sqrt{\omega}}\exp\left(\frac{-\pi}{2\omega}\right) = 0$ in (5.11), we finally have

$$\lim_{\mu\to\infty,\ \omega\to 0}\left(\frac{\widetilde{\eta}_4(\omega,\mu)}{\widetilde{\eta}_1(\omega,\mu)}\right) \leq \lim_{\mu\to\infty,\ \omega\to 0}\frac{c_2}{c_1}\frac{1}{\sqrt{\omega}}\exp\left(1 - \frac{\pi}{2\omega}\right) = 0$$

and that shows that there exists $\mu_0 \geq \widetilde{\mu_0}$ and $\omega_0 \in (0, \min\{\omega_{12}, \omega_{13}\}]$ such that for $\omega \leq \omega_0$ and $\mu \in \mathbb{N}$, $\mu \geq \mu_0$ one has $\widetilde{\eta}_4(\omega,\mu) \leq \widetilde{\eta}_1(\omega,\mu) \leq \min\{\widetilde{\eta}_2(\omega), \widetilde{\eta}_3(\omega)\}$. □

Now, we aim at finding an interval $U_k := [m_k - \delta_k, m_k + \delta_k]$ as indicated in the passage before Theorem 5.1, which gets mapped to a 'steep' interval $S_{\ell_1(k)}$, but we first provide upper and lower estimates for the second derivative $f''_{\mu,\omega}$ of f.

Proposition 5.5. *Let $k \in \mathbb{N}$. Assume that with μ_0 and ω_0 as in Proposition 5.4, one has $\mu \geq \mu_0$ and $\omega \leq \omega_0$. Define $\widetilde{\eta}$ as in Proposition 5.4 and set $\delta_k := \widetilde{\eta}\delta_k$, $J_k := \left[q^{k+1}, q^k\right]$. Then*

$$U_k := [m_k - \delta_k, m_k + \delta_k] \subset \left[q^{k+1}, q^k\right] = J_k$$

and the following estimates hold:

$$\forall x \in [m_k - \delta_k, m_k + \delta_k] : g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot q^{k(\mu-2)} \geq \left|f''(x)\right| \geq \frac{\left(q^k e^{-\frac{\varphi}{\omega}}\right)^{\mu-2} \cdot \omega \cdot g_{\omega,\mu}}{2}. \tag{5.12}$$

Proof. Let $k \in \mathbb{N}$. With $\overline{\eta}$ from Lemma 3.2, the definition of $\widetilde{\eta}$ given in Proposition 5.4 shows $\widetilde{\eta} \leq \min\{\widetilde{\eta}_2, \widetilde{\eta}_3\} = \overline{\eta}$. Hence, in view of Lemma 3.2, we see that

$$U_k = [m_k - \delta_k, m_k + \delta_k] \subset \left[m_k - \overline{\eta}q^k, m_k + \overline{\eta}q^k\right] \subset \left[q^{k+1}, q^k\right] = J_k.$$

Further, inserting m_k from (2.10) in (2.4) we have

$$\begin{aligned}\left|f''(m_k)\right| &= g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot |m_k|^{\mu-2} \left|\sin\left((\omega \ln(m_k) + \varphi_1) + \varphi_2\right)\right| \\ &= g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot |m_k|^{\mu-2} \left|\sin\left(-k\pi + \varphi_2\right)\right|.\end{aligned}$$

Using $\varphi_2 = \arctan(\frac{\omega}{\mu-1})$ from Lemma 2.1, we have $\sin(\varphi_2) = \frac{\omega}{g_{\omega,\mu-1}}$ and, inserting this value in the last equality, we have

$$\left|f''(m_k)\right| = g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot |m_k|^{\mu-2} \frac{\omega}{g_{\omega,\mu-1}} = |m_k|^{\mu-2} \omega \cdot g_{\omega,\mu}. \tag{5.13}$$

From (2.5) we have on $\left[q^{k+1}, q^k\right]$

$$\begin{aligned}\left|f'''(x)\right| &= \left|g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot g_{\omega,\mu-2} \cdot x^{\mu-3} \sin\left(\omega \ln(x) + (\varphi_1 + \varphi_2 + \varphi_3)\right)\right| \\ &\leq g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot g_{\omega,\mu-2} \cdot q^{k(\mu-3)}.\end{aligned} \tag{5.14}$$

From (2.13) for $x \in [m_k - \delta_k, m_k + \delta_k]$ and with the definition of δ_k, we also have

$$\begin{aligned}\left|f''(x)\right| &\geq \left|f''(m_k)\right| - \delta_k \cdot \max_{[m_k-\delta_k, m_k+\delta_k]} \left|f'''\right| \\ &= \left|f''(m_k)\right| - \widetilde{\eta}q^k \cdot \max_{[m_k-\delta_k, m_k+\delta_k]} \left|f'''\right| \\ &\geq \left|f''(m_k)\right| - \widetilde{\eta}_1 q^k \cdot \max_{[m_k-\delta_k, m_k+\delta_k]} \left|f'''\right|.\end{aligned} \tag{5.15}$$

With the definition of $\widetilde{\eta}_1$, using (2.10), (5.13) and (5.14) in (5.15), we finally have

$$\begin{aligned}\left|f''(x)\right| &\geq m_k^{\mu-2}\omega \cdot g_{\omega,\mu} - \frac{q^k \omega e^{-\frac{\varphi_1(\mu-2)}{\omega}}}{2g_{\omega,\mu-1} \cdot g_{\omega,\mu-2}} g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot g_{\omega,\mu-2} \cdot q^{k(\mu-3)} \\ &= \left(q^k e^{-\frac{\varphi_1}{\omega}}\right)^{\mu-2} \cdot \omega \cdot g_{\omega,\mu} - \frac{q^{k(\mu-2)} \omega e^{-\frac{\varphi_1(\mu-2)}{\omega}}}{2} g_{\omega,\mu} \\ &= \left(q^k e^{-\frac{\varphi_1}{\omega}}\right)^{\mu-2} \cdot \omega \cdot g_{\omega,\mu} - \frac{\left(q^k e^{-\frac{\varphi_1}{\omega}}\right)^{\mu-2} \cdot \omega \cdot g_{\omega,\mu}}{2} \\ &= \frac{\left(q^k e^{-\frac{\varphi_1}{\omega}}\right)^{\mu-2} \cdot \omega \cdot g_{\omega,\mu}}{2}.\end{aligned}$$

This is the lower estimate for $|f''(x)|$; the upper estimate even on the interval $\left[q^{k+1}, q^k\right]$ follows with the formula for f'' in (2.4). □

For $k \in \mathbb{N}$, we specify the boundaries of an associated 'steep' interval $S_{\ell_1(k)}$ with the next proposition.

Proposition 5.6. *Let $k \in \mathbb{N}$. Assume μ and ω are as in Proposition 5.4 and define $\ell_1(k) = k\mu + 1$ as in the passage before the Theorem 5.1. Set $r_{\ell_1(k)} := \frac{(1-c)\omega}{g_{\omega,\mu} \cdot g_{\omega,\mu-1}} q^{\ell_1(k)}$ and $S_{\ell_1(k)} := \left[q^{\ell_1(k)} - r_{\ell_1(k)}, q^{\ell_1(k)}\right]$. Then, $S_{\ell_1(k)} \subset \left(m_{\ell_1(k)}, q^{\ell_1(k)}\right]$ and on $S_{\ell_1(k)}$ we have*

$$\left|f'\right| \geq c\omega q^{\ell_1(k)(\mu-1)}. \tag{5.16}$$

Proof. Let $k \in \mathbb{N}$. From the upper estimate of (5.12), on $S_{\ell_1(k)}$ we have

$$\left\|f''\right\|_{\infty, S_{\ell_1(k)}} \leq g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot q^{\ell_1(k)(\mu-2)}. \tag{5.17}$$

From (2.13) we have

$$\forall x \in S_{\ell_1(k)} : \left|f'(x)\right| \geq \left|f'\left(q^{\ell_1(k)}\right)\right| - \left\|f''\right\|_{\infty, S_{\ell_1(k)}} \cdot r_{\ell_1(k)}, \tag{5.18}$$

and from (2.9) we also have $\left|f'\left(q^{\ell_1(k)}\right)\right| = \omega q^{\ell_1(k)(\mu-1)}$. Using (5.17) and substituting the explicit values of both $\left|f'\left(q^{\ell_1(k)}\right)\right|$ and $r_{\ell_1(k)}$ in (5.18), we get

$$\begin{aligned}
\forall x \in S_{\ell_1(k)} : \left|f'(x)\right| &\geq \left|f'\left(q^{\ell_1(k)}\right)\right| - g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot q^{\ell_1(k)(\mu-2)} \cdot r_{\ell_1(k)} \\
&= \omega q^{\ell_1(k)(\mu-1)} - g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot q^{\ell_1(k)(\mu-2)} \cdot \frac{(1-c)\,\omega}{g_{\omega,\mu-1} \cdot g_{\omega,\mu}} q^{\ell_1(k)} \\
&= \omega q^{\ell_1(k)(\mu-1)} - (1-c)\,\omega q^{\ell_1(k)(\mu-1)} = c\omega q^{\ell_1(k)(\mu-1)}.
\end{aligned}$$

It follows now from $f'(m_{\ell_1(k)}) = 0$ that $m_{\ell_1(k)} < q^{\ell_1(k)} - r_{\ell_1(k)}$. □

From the graph of the map one can understand that the image of $S_{\ell_1(k)}$ under $f_{\mu,\omega}$ includes many 'steep' and 'flat' intervals, but we continue our calculations with a subinterval $\widetilde{S_{\ell_1(k)}}$ of $S_{\ell_1(k)}$ which is contained in $f(U_k)$. The next lemma gives an estimate for the size of $f(U_k)$ with a relation between $\widetilde{S_{\ell_1(k)}}$ and $S_{\ell_1(k)}$.

Note that for the sake of simplicity we shall use k as a positive odd integer number for the rest of the paper. Note also that in addition to the notations U_k^L, U_k^R which represent to the left 'L' and right 'R' hand part of U_k respectively, we also use the notation $U_k^{L\backslash R}$ in statements which are valid for both U_k^L and U_k^R.

Lemma 5.7. *Let k be a positive odd integer number. Let ω and even integer μ be as in Proposition 5.4 and satisfying (5.2). Define $\widetilde{\eta}$ as in Proposition 5.4, δ_k and U_k as in Proposition 5.5, and $U_k^{L\backslash R}$ as in the passage before Theorem 5.1. Then the following statements are true.*

(i) Define $r_{\ell_1(k)}$ and $S_{\ell_1(k)}$ as in Proposition 5.6. Then we have $f(U_k) \subset S_{\ell_1(k)}$;

(ii) Set

$$\widetilde{r_{\ell_1(k)}} := q^{\ell_1(k)} \cdot q^{\frac{\varphi_1(\mu-2)}{\pi}} \cdot \frac{(1-c) \cdot \omega^2}{4 g_{\omega,\mu} \cdot g_{\omega,\mu-1}^2} \tag{5.19}$$

and $\widetilde{S_{\ell_1(k)}} = \left[q^{\ell_1(k)} - \widetilde{r_{\ell_1(k)}}, q^{\ell_1(k)}\right]$. Then we have $f\left(U_k^{L\backslash R}\right) \supset \widetilde{S_{\ell_1(k)}}$.

Proof. Note that due to (5.1), and since k is odd (see (2.11)), $\max\{f(U_k)\} = f(m_k) = q^{\ell_1(k)}$. For the interval U_k we have

$$\begin{aligned}\min&\left\{\left|f(m_k-\delta_k)-q^{\ell_1(k)}\right|,\left|f(m_k+\delta_k)-q^{\ell_1(k)}\right|\right\}\\&\leq |f(U_k)|\\&\leq \max\left\{\left|f(m_k-\delta_k)-q^{\ell_1(k)}\right|,\left|f(m_k+\delta_k)-q^{\ell_1(k)}\right|\right\}.\end{aligned}$$

It follows from second order Taylor expansion of f around the extremum m_k and from (5.1) that

$$\min_{\xi\in[m_k-\delta_k,m_k+\delta_k]}\left|f''(\xi)\right|\frac{\delta_k^2}{2}\leq |f(U_k)|\leq \max_{\xi\in[m_k-\delta_k,m_k+\delta_k]}\left|f''(\xi)\right|\frac{\delta_k^2}{2}. \tag{5.20}$$

Consequently, using (5.8) and inserting the upper estimate of $|f''|$ given in (5.12) and the value of δ_k in the upper estimate of (5.20), we finally get

$$\begin{aligned}|f(U_k)| &\leq \max_{\xi\in[m_k-\delta_k,m_k+\delta_k]}\left|f''(\xi)\right|\frac{\delta_k^2}{2}\leq g_{\omega,\mu}\cdot g_{\omega,\mu-1}\cdot q^{k(\mu-2)}\frac{\delta_k^2}{2}\\&= g_{\omega,\mu}\cdot g_{\omega,\mu-1}\cdot q^{k(\mu-2)}\widetilde{\eta}^2 q^{2k}\\&\leq g_{\omega,\mu}\cdot g_{\omega,\mu-1}\cdot q^{k(\mu-2)}\left(\widetilde{\eta}_4\right)^2 q^{2k}\\&\leq g_{\omega,\mu}\cdot g_{\omega,\mu-1}\cdot q^{k(\mu-2)}\left(\frac{\sqrt{(1-c)\,q\omega}}{g_{\omega,\mu}\cdot g_{\omega,\mu-1}}\right)^2 q^{2k}\\&= q^{k\mu+1}\frac{(1-c)\,\omega}{g_{\omega,\mu}\cdot g_{\omega,\mu-1}}=\frac{(1-c)\,\omega}{g_{\omega,\mu}\cdot g_{\omega,\mu-1}}q^{\ell_1(k)}=r_{\ell_1(k)}=\left|S_{\ell_1(k)}\right|.\end{aligned}$$

From (5.1) we know that $f(m_k)=q^{\ell_1(k)}$. So, $f(U_k)=\left[\min f(U_k), q^{\ell_1(k)}\right]$ and the estimate

$$|f(U_k)|\leq r_{\ell_1(k)} \text{ shows } f(U_k)\subset\left[q^{\ell_1(k)}-r_{\ell_1(k)},q^{\ell_1(k)}\right]=S_{\ell_1(k)}$$

and this completes the proof of assertion (i).

Note that, although there is no symmetry between the graph of $f_{\mu,\omega}$ to the left and right hand side of U_k, we can estimate the size of $f\left(U_k^L\right)$ and $f\left(U_k^R\right)$ in a similar way. Substituting the lower bound of $|f''|$ given by (5.12), the value of δ_k in the analogue of the lower estimate of (5.20) for $U_k^{L\backslash R}$, and using (5.8) we obtain

$$\begin{aligned}\left|f\left(U_k^{L\backslash R}\right)\right| &\geq \min_{\xi\in U_k}\left|f''(\xi)\right|\frac{\delta_k^2}{2}\geq\frac{\left(q^k e^{-\frac{\varphi_1}{\omega}}\right)^{\mu-2}\cdot\omega\cdot g_{\omega,\mu}}{2}\frac{\delta_k^2}{2}\\&=\frac{\left(q^k e^{-\frac{\varphi_1}{\omega}}\right)^{\mu-2}\cdot\omega\cdot g_{\omega,\mu}}{4}\widetilde{\eta}^2 q^{2k}=\frac{q^{k\mu-2k}\cdot e^{-\frac{\varphi_1(\mu-2)}{\omega}}\omega\cdot g_{\omega,\mu}}{4}\left(\widetilde{\eta}_4\right)^2 q^{2k}\\&=\frac{q^{k\mu-2k}\cdot e^{-\frac{\varphi_1(\mu-2)}{\omega}}\omega\cdot g_{\omega,\mu}}{4}\left(\frac{\sqrt{(1-c)\,q\omega}}{g_{\omega,\mu}\cdot g_{\omega,\mu-1}}\right)^2 q^{2k}\\&=\frac{q^{k\mu+1}\cdot e^{-\frac{\varphi_1(\mu-2)}{\omega}}\cdot\omega^2}{4}\frac{(1-c)}{g_{\omega,\mu}\cdot g_{\omega,\mu-1}^2}\\&=q^{\ell_1(k)}\cdot q^{\frac{\varphi_1(\mu-2)}{\pi}}\cdot\frac{(1-c)\cdot\omega^2}{4g_{\omega,\mu}\cdot g_{\omega,\mu-1}^2}=\widetilde{r_{\ell_1(k)}}=\left|\widetilde{S_{\ell_1(k)}}\right|,\end{aligned}$$

and this completes the proof of assertion (ii) and the proof of the lemma. □

We continue analyzing the next ‘flat’ interval obtained by the second iteration of f.

Lemma 5.8. *Let k be a positive odd integer number. Let ω, μ be as in Lemma 5.7. Define $\ell_1(k)$ and $\ell_2(k)$ as in the passage before Theorem 5.1. Then for $\widetilde{S_{\ell_1(k)}}$ as in Lemma 5.7, we have*

$$f\left(\widetilde{S_{\ell_1(k)}}\right) \supset \left[0, q^{\ell_2(k)}\right].$$

Proof. Using (2.13) on $\widetilde{S_{\ell_1(k)}}$, we obtain

$$\left|f\left(\widetilde{S_{\ell_1(k)}}\right)\right| \geq \widetilde{r_{\ell_1(k)}} \cdot \min_{x\in\widetilde{S_{\ell_1(k)}}} \left|f'(x)\right|. \tag{5.21}$$

Using (5.16) and (5.19) in (5.21), and also the definition of $\ell_2(k)$ at the beginning of this section, we have

$$\begin{aligned}\left|f\left(\widetilde{S_{\ell_1(k)}}\right)\right| &\geq \widetilde{r_{\ell_1(k)}} \cdot \min_{x\in\widetilde{S_{\ell_1(k)}}} \left|f'(x)\right| \\ &= q^{\ell_1(k)} \cdot q^{\frac{\varphi_1(\mu-2)}{\pi}} \cdot \frac{(1-c)\cdot\omega^2}{4g_{\omega,\mu}\cdot g^2_{\omega,\mu-1}} \cdot c\omega q^{\ell_1(k)(\mu-1)} \\ &= q^{\ell_1(k)\mu} \cdot q^{\frac{\varphi_1(\mu-2)}{\pi}} \cdot \frac{c(1-c)\,\omega^3}{4g_{\omega,\mu}\cdot g^2_{\omega,\mu-1}} \geq q^{\ell_2(k)}.\end{aligned}$$

Note also that $\ell_1(k) = k\mu+1$ is odd, since μ is even. Hence, $f \geq 0$ on $\widetilde{S_{\ell_1(k)}}$ and since $f\left(q^{\ell_1(k)}\right) = 0$, $f\left(\widetilde{S_{\ell_1(k)}}\right) = \left[0, \max f\left(\widetilde{S_{\ell_1(k)}}\right)\right]$. The estimate $\left|f\left(\widetilde{S_{\ell_1(k)}}\right)\right| \geq q^{\ell_2(k)}$ implies that $f\left(\widetilde{S_{\ell_1(k)}}\right) \supset \left[0, q^{\ell_2(k)}\right]$.

From Lemma 5.7 we know $f\left(U_k^{L\setminus R}\right) \supset \widetilde{S_{\ell_1(k)}}$. In Lemma 5.8 we showed that $f\left(\widetilde{S_{\ell_1(k)}}\right) \supset \left[0, q^{\ell_2(k)}\right]$. In particular, $U_{\ell_2(k)} \subset \left[q^{\ell_2(k)+1}, q^{\ell_2(k)}\right] \subset f\left(\widetilde{S_{\ell_1(k)}}\right)$. Now, in the next lemma we estimate the counterimage of subsets of $U_{\ell_2(k)}$ under $\left(f^2|_{U_k^{L\setminus R}}\right)$. □

Lemma 5.9. *Let k be a positive odd integer. Assume μ is an even integer, $\mu \geq \max\left\{\left(\frac{30e}{7\pi}\right)^2\left(\frac{1-c}{2c}\right), 15\right\}$ and $\omega \in (0,1)$ is a corresponding value satisfying (5.2) and such that the assertion of Proposition 5.4 is true (this is possible due to assertion (ii) of Lemma 5.2). Define $\alpha(\omega, \mu, c)$ as in Proposition 5.3 and J_k as in Proposition 5.5. Then, for $p \in (0,1]$ and any subinterval $\widehat{U_{\ell_2(k)}}$ of $U_{\ell_2(k)}$ with $\ell_2(k)$ as in the passage before Theorem 5.1, if*

$$\left|\widehat{U_{\ell_2(k)}}\right| = p\left|J_{\ell_2(k)}\right|$$

then $\left(f|_{U_k}\right)^{-2}\left(\widehat{U_{\ell_2(k)}}\right)$ has two parts of the form

$$\widehat{U_k^L} = \left[m_k - \delta^L_{k,2}, m_k - \delta^L_{k,1}\right] \subset U_k^L \quad \text{and} \quad \widehat{U_k^R} = \left[m_k + \delta^R_{k,1}, m_k + \delta^R_{k,2}\right] \subset U_k^R,$$

where $\delta^L_{k,1}, \delta^L_{k,2} \in \left(0, m_k - q^{k+1}\right)$ and $\delta^R_{k,1}, \delta^R_{k,2} \in \left(0, q^k - m_k\right)$, and each of them has the size

$$\left|\widehat{U_k^{L\setminus R}}\right| \leq \alpha \cdot p \cdot |J_k|. \tag{5.22}$$

Proof. Set $\widehat{S_{\ell_1(k)}} := \left(f|_{\widetilde{S_{\ell_1(k)}}}\right)^{-1}\left(\widehat{U_{\ell_2(k)}}\right)$. Note that injectivity of $f|_{S_{\ell_1(k)}}$ and Lemma 5.8 imply that $\left(f|_{S_{\ell_1(k)}}\right)^{-1}\left(\widehat{U_{\ell_2(k)}}\right) = \left(f|_{\widetilde{S_{\ell_1(k)}}}\right)^{-1}\left(\widehat{U_{\ell_2(k)}}\right)$. Using (2.13) on $\widehat{S_{\ell_1(k)}}$, we have

$$\left|\left(f|_{\widetilde{S_{\ell_1(k)}}}\right)^{-1}\left(\widehat{U_{\ell_2(k)}}\right)\right| = \left|\widehat{S_{\ell_1(k)}}\right| \leq \frac{\left|\widehat{U_{\ell_2(k)}}\right|}{\min_{\widehat{S_{\ell_1(k)}}} |f'|}.$$

On the other hand, from Proposition 5.6 we already know that on $S_{\ell_1(k)}$, $|f'| \geq c\omega q^{\ell_1(k)(\mu-1)}$. Because of $\widehat{S_{\ell_1(k)}} \subset \widetilde{S_{\ell_1(k)}} \subset S_{\ell_1(k)}$, this property also satisfied on $\widehat{S_{\ell_1(k)}}$. Hence, inserting both $\left|\widehat{U_{\ell_2(k)}}\right| = p\left|J_{\ell_2(k)}\right|$ and the estimate of $\min_{S_{\ell_1(k)}} |f'|$ in the last expression, we have

$$\left|\widehat{S_{\ell_1(k)}}\right| \leq \frac{\left|\widehat{U_{\ell_2(k)}}\right|}{\min_{\widehat{S_{\ell_1(k)}}} |f'|} \leq \frac{p\left|J_{\ell_2(k)}\right|}{\min_{S_{\ell_1(k)}} |f'|} \leq \frac{p \cdot q^{\ell_2(k)}(1-q)}{c\omega q^{\ell_1(k)(\mu-1)}}. \tag{5.23}$$

Now, we calculate subintervals of $(m_k - \delta_k, m_k + \delta_k)$ which get mapped bijectively to $\widehat{S_{\ell_1(k)}}$. Note that the counterimage of $\widehat{S_{\ell_1(k)}}$ has two parts in the form $\widehat{U_k^L} \subset U_k^L$, and $\widehat{U_k^R} \subset U_k^R$. It follows from strict monotonicty of f on $[m_k - \delta_k, m_k]$ and $[m_k, m_k + \delta_k]$ and from the fact that $f\left(U_k^{L\backslash R}\right) \supset \widetilde{S_{\ell_1(k)}}$ that there exist $\delta_{k,1}^{L\backslash R}, \delta_{k,2}^{L\backslash R}$ with

$$\begin{aligned}\left|f\left(\widehat{U_k^R}\right)\right| &= \left|f\left(\left[m_k + \delta_{k,1}^R, m_k + \delta_{k,2}^R\right]\right)\right| \\ &= \left|f\left(\left[m_k - \delta_{k,2}^L, m_k - \delta_{k,1}^L\right]\right)\right| = \left|f\left(\widehat{U_k^L}\right)\right| = \left|\widehat{S_{\ell_1(k)}}\right|.\end{aligned} \tag{5.24}$$

We continue our calculations by using the boundaries of $\widehat{U_k^R}$. Note that for the interval $\left[m_k, m_k + \delta_{k,1}^R\right]$ we know that $f\left(m_k + \delta_{k,1}^R\right) = \max \widehat{S_{\ell_1(k)}}$ and $f(m_k) = q^{\ell_1(k)}$. Again from the monotonicity of the map it follows that $f\left(\left[m_k, m_k + \delta_{k,1}^R\right]\right) = \left[\max \widehat{S_{\ell_1(k)}}, q^{\ell_1(k)}\right]$. Consequently, since $f\left(q^{\ell_1(k)}\right) = 0$ and $f\left(\max \widehat{S_{\ell_1(k)}}\right) \in \left[q^{\ell_2(k)+1}, q^{\ell_2(k)}\right]$, from (2.13) we have

$$\left|\max \widehat{S_{\ell_1(k)}} - q^{\ell_1(k)}\right| \geq \frac{q^{\ell_2(k)+1}}{\|f'\|_{\infty, S_{\ell_1(k)}}}. \tag{5.25}$$

From (2.1) we also have that $\|f'\|_{\infty, S_{\ell_1(k)}} \leq g_{\omega,\mu} \cdot q^{\ell_1(k)(\mu-1)}$. Inserting this estimate in (5.25), we obtain

$$\left|\max \widehat{S_{\ell_1(k)}} - q^{\ell_1(k)}\right| \geq \frac{q^{\ell_2(k)+1}}{g_{\omega,\mu} \cdot q^{\ell_1(k)(\mu-1)}}. \tag{5.26}$$

In addition, from (2.4) we know that

$$\|f''\|_{\infty, U_k} \leq g_{\omega,\mu} \cdot g_{\omega,\mu-1} \cdot q^{k(\mu-2)}. \tag{5.27}$$

Now, using the second order Taylor expansion of $f\left(m_k + \delta_{k,1}^R\right)$, we have

$$\left|f\left(m_k + \delta_{k,1}^R\right) - f(m_k)\right| \leq \left|f''(\xi) \frac{\left(\delta_{k,1}^R\right)^2}{2}\right|, \tag{5.28}$$

where $\xi \in \left(m_k, m_k + \delta_{k,1}^R\right)$. Substituting the values of $f\left(m_k + \delta_{k,1}^R\right)$ and $f\left(m_k\right)$ in (5.28), we have

$$\left|f\left(m_k + \delta_{k,1}^R\right) - f\left(m_k\right)\right| = \left|\max \widehat{S_{\ell_1(k)}} - q^{\ell_1(k)}\right| \leq \|f''\|_{\infty,U_k} \frac{\left(\delta_{k,1}^R\right)^2}{2},$$

which implies

$$\delta_{k,1}^R \geq \sqrt{2\frac{\left|\max \widehat{S_{\ell_1(k)}} - q^{\ell_1(k)}\right|}{\|f''\|_{\infty,U_k}}}. \tag{5.29}$$

Using both estimates (5.26) and (5.27) in (5.29), we finally get

$$\delta_{k,1}^R \geq \sqrt{2\frac{q^{\ell_2(k)+1}}{g_{\omega,\mu}^2 \cdot g_{\omega,\mu-1} \cdot q^{k(\mu-2)} \cdot q^{\ell_1(k)(\mu-1)}}}. \tag{5.30}$$

On the other hand, from Taylor's formula with the integral remainder term we have

$$\begin{aligned} f\left(m_k + \delta\right) &= f\left(m_k\right) + \int_{m_k}^{m_k+\delta} \left(m_k + \delta - t\right) f''\left(t\right) dt \\ &= f\left(m_k\right) + \int_0^{\delta} \left(\delta - t\right) f''\left(m_k + t\right) dt. \end{aligned} \tag{5.31}$$

Consequently, applying (5.31) for the boundaries of $\widehat{U_k^R}$, we have

$$\begin{aligned} \left|\widehat{S_{\ell_1(k)}}\right| &= \left|f\left(\widehat{U_k^R}\right)\right| = \left|f\left(m_k + \delta_{k,2}^R\right) - f\left(m_k + \delta_{k,1}^R\right)\right| \\ &= \left|\int_0^{\delta_{k,2}^R} \left(\delta_{k,2}^R - t\right) f''\left(m_k + t\right) dt - \int_0^{\delta_{k,1}^R} \left(\delta_{k,1}^R - t\right) f''\left(m_k + t\right) dt\right|. \end{aligned}$$

From (5.12) we already know that $M := \min_{x \in U_k} \left|f''\left(x\right)\right| \geq \frac{q^{k(\mu-2)} q^{\frac{\varphi_1(\mu-2)}{\pi}} \cdot \omega \cdot g_{\omega,\mu}}{2}$. In particular, f'' has constant sign on U_k. Using the fact that $\delta_{k,1}^R < \delta_{k,2}^R$ in the last equality, we obtain

$$\begin{aligned} \left|\widehat{S_{\ell_1(k)}}\right| &= \left|\int_0^{\delta_{k,1}^R} \left(\delta_{k,2}^R - \delta_{k,1}^R\right) f''\left(m_k + t\right) dt + \int_{\delta_{k,1}^R}^{\delta_{k,2}^R} \left(\delta_{k_2}^R - t\right) f''\left(m_k + t\right) dt\right| \\ &\geq \left|\int_0^{\delta_{k,1}^R} \left(\delta_{k,2}^R - \delta_{k,1}^R\right) f''\left(m_k + t\right) dt\right| \geq \left|\delta_{k,2}^R - \delta_{k,1}^R\right| \cdot M \cdot \delta_{k,1}^R, \end{aligned}$$

so

$$\left|\delta_{k,2}^R - \delta_{k,1}^R\right| \leq \frac{\left|\widehat{S_{\ell_1(k)}}\right|}{M \cdot \delta_{k,1}^R}. \tag{5.32}$$

Substituting the estimate of M and the estimate $\delta_{k,1}^R$ given by (5.30) in (5.32), we obtain

$$\left|\delta_{k,2}^R - \delta_{k,1}^R\right| \leq \frac{\left|\widehat{S_{\ell_1(k)}}\right|}{\frac{q^{k(\mu-2)} q^{\frac{\varphi_1(\mu-2)}{\pi}} \cdot \omega \cdot g_{\omega,\mu}}{2} \cdot \sqrt{2\frac{q^{\ell_2(k)+1}}{q^{k(\mu-2)} \cdot g_{\omega,\mu-1} \cdot g_{\omega,\mu}^2 \cdot q^{\ell_1(k)(\mu-1)}}}}. \tag{5.33}$$

Combining the estimate of $\widehat{S_{\ell_1(k)}}$ given by (5.23) with (5.33), we finally have

$$\begin{aligned}
\left|\widehat{U_k^R}\right| &= \left|\delta_{k,2}^R - \delta_{k,1}^R\right| \\
&\le \frac{\sqrt{2}p \cdot q^{\ell_2(k)}\,(1-q)}{c\omega q^{\ell_1(k)(\mu-1)}} \frac{\sqrt{q^{k(\mu-2)} \cdot g_{\omega,\mu-1} \cdot g_{\omega,\mu}^2 \cdot q^{\ell_1(k)(\mu-1)}}}{q^{k(\mu-2)} q^{\frac{\varphi_1(\mu-2)}{\pi}} \cdot \omega \cdot g_{\omega,\mu} \cdot \sqrt{q^{\ell_2(k)+1}}} \\
&= \sqrt{2}p \frac{q^{\ell_2(k)}\,(1-q)}{c\omega q^{\ell_1(k)(\mu-1)}} \cdot \frac{\sqrt{q^{k(\mu-2)} g_{\omega,\mu-1}}}{q^{k(\mu-2)} q^{\frac{\varphi_1(\mu-2)}{\pi}} \cdot \omega} \cdot \frac{\sqrt{q^{\ell_1(k)(\mu-1)}}}{\sqrt{q^{\ell_2(k)+1}}} \\
&= \sqrt{2}p \frac{(1-q)\sqrt{g_{\omega,\mu-1}}}{c\omega^2 \cdot q^{\frac{\varphi_1(\mu-2)}{\pi}}} \cdot \frac{q^{\ell_2(k)}}{q^{\frac{\ell_2(k)}{2}}} \cdot \frac{\sqrt{q^{k(\mu-2)}}}{q^{k(\mu-2)+\frac{1}{2}}} \cdot \frac{\sqrt{q^{\ell_1(k)(\mu-1)}}}{q^{\ell_1(k)(\mu-1)}} \\
&= \frac{\sqrt{2}p \cdot q^k\,(1-q) \cdot q^{\frac{\ell_2(k)}{2}}}{q^{\frac{\ell_1(k)(\mu-1)}{2}} \cdot q^{\frac{k\mu+1}{2}} \cdot q^{\frac{\varphi_1(\mu-2)}{\pi}}} \cdot \frac{\sqrt{g_{\omega,\mu-1}}}{c\omega^2}.
\end{aligned}$$

Here, using the estimate of $q^{\ell_2(k)}$ given in the passage before Theorem 5.1 and $|J_k| = q^k\,(1-q)$, we obtain

$$\begin{aligned}
\left|\widehat{U_k^R}\right| &\le \frac{\sqrt{2}p \cdot |J_k| \cdot \sqrt{q^{\ell_1(k)\mu} \cdot q^{\frac{\varphi_1(\mu-2)}{\pi}} \cdot \frac{c(1-c)\omega^3}{4g_{\omega,\mu} \cdot g_{\omega,\mu-1}^2}}}{q^{\frac{\ell_1(k)(\mu-1)}{2}} \cdot q^{\frac{k\mu+1}{2}} \cdot q^{\frac{\varphi_1(\mu-2)}{\pi}}} \cdot \frac{\sqrt{g_{\omega,\mu-1}}}{c\omega^2} \\
&= \frac{\sqrt{2}}{2} p \cdot |J_k| \cdot \frac{\sqrt{q^{\ell_1(k)\mu}}}{q^{\frac{\ell_1(k)\mu-\ell_1(k)}{2}} \cdot q^{\frac{k\mu+1}{2}}} \cdot \frac{\sqrt{q^{\frac{\varphi_1(\mu-2)}{\pi}}}}{q^{\frac{\varphi_1(\mu-2)}{\pi}}} \cdot \sqrt{\frac{(1-c)}{c\omega g_{\omega,\mu} \cdot g_{\omega,\mu-1}}}.
\end{aligned}$$

Inserting $\ell_1(k) = k\mu + 1$ one gets

$$\begin{aligned}
\left|\widehat{U_k^R}\right| &\le \frac{\sqrt{2}}{2} p \cdot |J_k| \cdot \frac{\sqrt{q^{(k\mu+1)\mu}}}{q^{\frac{(k\mu+1)\mu-(k\mu+1)}{2}} \cdot q^{\frac{k\mu+1}{2}}} \cdot \frac{\sqrt{q^{\frac{\varphi_1(\mu-2)}{\pi}}}}{q^{\frac{\varphi_1(\mu-2)}{\pi}}} \cdot \sqrt{\frac{(1-c)}{c\omega g_{\omega,\mu} \cdot g_{\omega,\mu-1}}} \\
&= p \cdot |J_k| \cdot q^{-\frac{\varphi_1(\mu-2)}{2\pi}} \cdot \sqrt{\frac{(1-c)}{2c\omega g_{\omega,\mu} \cdot g_{\omega,\mu-1}}}.
\end{aligned}$$

Since $g_{\omega,\mu-1} < g_{\omega,\mu}$, we can simplify the last inequality as follows:

$$\begin{aligned}
\left|\widehat{U_k^R}\right| &\le p \cdot |J_k| \cdot q^{-\frac{\varphi_1(\mu-2)}{2\pi}} \cdot \sqrt{\frac{(1-c)}{2c\omega g_{\omega,\mu-1}^2}} \\
&= p \cdot |J_k| \cdot \frac{q^{-\frac{\varphi_1(\mu-2)}{2\pi}}}{g_{\omega,\mu-1}} \cdot \sqrt{\frac{1-c}{2c\omega}}.
\end{aligned}$$

Inserting $q = e^{-\frac{\pi}{\omega}}$ we have

$$\begin{aligned}
\left|\widehat{U_k^R}\right| &\le p \cdot |J_k| \cdot \frac{\exp\left(\frac{\pi}{\omega} \cdot \frac{(\mu-2)\varphi_1}{2\pi}\right)}{g_{\omega,\mu-1}} \cdot \sqrt{\frac{1-c}{2c\omega}} \\
&= p \cdot |J_k| \cdot \frac{\exp\left(\frac{(\mu-2)\varphi_1}{2\omega}\right)}{g_{\omega,\mu-1}} \cdot \sqrt{\frac{1-c}{2c\omega}}.
\end{aligned}$$

Finally, using the definition of $\alpha(\mu,\omega,c)$, we get

$$\left|\widehat{U_k^R}\right| \leq \alpha \cdot p \cdot |J_k|,$$

and this completes the proof for $\widehat{U_k^R}$. The proof for $\widehat{U_k^L}$ is analogous. □

Corollary 5.10. *If the set $\widehat{U_{\ell_2(k)}}$ in Lemma 5.9 is not only one interval, but a disjoint union of subintervals of $U_{\ell_2(k)}$, and $|\widehat{U_{\ell_2(k)}}|$ (the measure of $\widehat{U_{\ell_2(k)}}$) satisfies $|\widehat{U_{\ell_2(k)}}| = p|J_{\ell_2(k)}|$, then $(f|_{U_k})^{-2}(\widehat{U_{\ell_2(k)}})$ has two parts (one in U_k^L and the other in U_k^R) and each of them has measure less or equal $\alpha p\,|J_k|$.*

Proof. (By summation over the subintervals.) □

Now, we consider symbol sequences of the form $\{L,R\}^{n+1}$ and construct corresponding orbits of f. For given a finite sequence

$$\mathbf{s} = (s_0, s_1, s_2, \ldots, s_n) \in \{L,R\}^{n+1}$$

and odd $k \in \mathbb{N}$, we now construct the subset of points x in U_k which follow this symbol sequence. Recall the set $I^n_{k,\mathbf{s}} = \bigcap_{j=0}^n f^{-2j}(U^{s_j}_{\ell_2^j(k)})$ defined in the passage before Theorem 5.1. We estimate the size of $|I^n_{k,\mathbf{s}}|$.

Corollary 5.11. *Let $\mathbf{s} = (s_0, s_1, s_2, \ldots, s_n)$ and an odd $k \in \mathbb{N}$ be given. Then, with ω, μ as in Lemma 5.9 and $\alpha(\omega,\mu,c)$ as in Proposition 5.3 we have $\emptyset \neq I^n_{k,\mathbf{s}}$ and*

$$\left|I^n_{k,\mathbf{s}}\right| \leq \alpha^n\,|J_k|.$$

Proof. We prove the corollary by induction over n. For $n=0$, $I^0_{k,\mathbf{s}} = U_k^{s_0} \neq \emptyset$, and

$$\left|I^0_{k,\mathbf{s}}\right| = \left|U_k^{s_0}\right| \leq |J_k|.$$

Now, we assume the result is true for n, and we verify it for $n+1$. Let $\mathbf{s} = (s_0, s_1, s_2, \ldots, s_{n+1})$ be given. Define $\widetilde{\mathbf{s}} = (s_1, s_2, \ldots, s_{n+1})$. From the induction hypothesis we have $I^n_{\ell_2(k),\widetilde{\mathbf{s}}} \neq \emptyset$, $I^n_{\ell_2(k),\widetilde{\mathbf{s}}} \subset U_{\ell_2(k)}$, and

$$\begin{aligned}\left|I^n_{\ell_2(k),\widetilde{\mathbf{s}}}\right| &= \left|\bigcap_{j=0}^{n} f^{-2j}\left(U^{s_{j+1}}_{\ell_2^j(\ell_2(k))}\right)\right| \\ &\leq \alpha^n \left|J_{\ell_2(k)}\right|.\end{aligned}$$

Note that $I^{n+1}_{k,\mathbf{s}} = f^{-2}\left(I^n_{\ell_2(k),\widetilde{\mathbf{s}}}\right) \cap U_k^{s_0}$. Hence, we have

$$\left|I^{n+1}_{k,\mathbf{s}}\right| = \left|f^{-2}\left(I^n_{\ell_2(k),\widetilde{\mathbf{s}}}\right) \cap U_k^{s_0}\right|. \tag{5.34}$$

Applying Corollary 5.10 with $p := \alpha^n$ and $I^n_{\ell_2(k),\widetilde{\mathbf{s}}}$ instead of $\widehat{U_{\ell_2(k)}}$ in (5.34), and using this p together with 5.22, we finally obtain

$$\left|I^{n+1}_{k,\mathbf{s}}\right| = \left|f^{-2}\left(I^n_{\ell_2(k),\widetilde{\mathbf{s}}}\right) \cap U_k^{s_0}\right| \leq \alpha \cdot p \cdot |J_k| = \alpha^{n+1}\,|J_k|.$$

This completes the induction and the proof of Corollary 5.11. □

Proof of Theorem 5.1. Assume k, c and μ are as in the assumptions of the Theorem 5.1 and, $\alpha = \alpha(\omega, \mu, c)$ be as in Proposition 5.3, so that $\alpha < \frac{1}{2}$. Choose $\omega \in (0,1)$ as in Lemma 5.2.

(i) Let a symbol sequence $s = (s_0, s_1, s_2, \dots) \in \{L, R\}^{\mathbb{N}_0}$ be given. From Corollary 5.11 one can see that for $n \in \mathbb{N}_0$ the closed interval $I^n_{k,s}$ consists of the points $x \in U_k$ which follow the finite symbol sequence $s = (s_0, s_1, s_2, \dots, s_n) \in \{L, R\}^{n+1}$. Further we have $I^{n+1}_{k,s} \subset I^n_{k,s}$. It follows that $\bigcap_{n \in \mathbb{N}_0} I^n_{k,s} \neq \emptyset$. Since, in view of Corollary 5.11 and $\alpha < \frac{1}{2}$, we have $\left|I^n_{k,s}\right| \to 0$ for $n \to \infty$, the intersection $\bigcap_{n \in \mathbb{N}_0} I^n_{k,s}$ contains exactly one point $x_{k,s}$. This point $x_{k,s}$ has the asserted properties. Any point in U_k with these properties would also be contained in this intersection and thus equal $x_{k,s}$.

(ii) The set $\{L, R\}^{\{0,1,\dots,n\}}$ has 2^{n+1} elements and from Corollary 5.11 we know that each set corresponding to one $s \in \{L, R\}^{\{0,1,2,\dots,n\}}$ satisfies the estimate $\left|I^n_{k,\mathbf{s}}\right| \le \alpha^n |J_k|$. It follows that $\left|\Gamma^n_k\right| \le 2^{n+1} \alpha^n |J_k|$, and it turns out that the measure

$$\lim_{n \to \infty} |\Gamma^n_k| = \lim_{n \to \infty} \left| \bigcup_{s \in \{L,R\}^{\{0,1,2,\dots,n\}}} I^n_{k,\mathbf{s}} \right| \le 2 \lim_{n \to \infty} 2^n \alpha^n |J_k| = 0$$

and this completes the proof. □

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Solvability of thirty-six three-dimensional systems of difference equations of hyperbolic-cotangent type

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Abstract. We present thirty-six classes of three-dimensional systems of difference equations of the hyperbolic-cotangent type which are solvable in closed form.

Keywords: system of difference equations, solvable systems, closed-form formulas.

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1 Introduction

Let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{C}$ be the sets of natural, integer and complex numbers, respectively, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Difference equations and systems have been studied for a long time. Some information on old results can be found in the classical books such as [4, 8, 9, 15–18]. Each of the books contain a part devoted to solvability. Solvability seems the first topic which has been seriously studied. The following papers and books contain some of the most important classical results on solvability [3,5–7,12–14]. Quite old presentations of these and other old results can be found in [10,11]. Many difference equations and systems have appeared in some applications. Some classical applications can be found in [8, 9, 17, 29, 44]. A great majority of the equations and systems is very difficult or impossible to solve, because of which it is of some interest to look for their invariants, which might also help in studying of long-term behaviour of their solutions as it was the case, for example, in [20–22,25,30,31,35]. The following papers: [26,28,32–34,36–43] contain some recent results on solvability.

During the '90s Papaschinopoulos and Schinas started studying systems which frequently possessed some kind of symmetry (see, e.g., [19–25, 27, 30, 31]), which was one of the motivations for our investigation of solvability of such systems (see, e.g., [32–34, 36–39, 42, 43]). Product-type difference equations and systems are closely related to linear ones, some of which are solvable. This fact motivated some recent investigations of their solvability (see, e.g., [32]).

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The following class of systems (the hyperbolic-cotangent-type class)

$$x_{n+1} = \frac{p_{n-k}q_{n-l} + a}{p_{n-k} + q_{n-l}}, \quad y_{n+1} = \frac{r_{n-k}s_{n-l} + a}{r_{n-k} + s_{n-l}}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

with complex initial values, where $k, l \in \mathbb{N}_0$, $a \in \mathbb{C}$, and p_n, q_n, r_n, s_n are x_n or y_n for all n, has been studied considerably during the last several years. The corresponding scalar equation has been studied for the last two decades (see [28,40] and the related references therein). The fact that the equation can be easily reduced to the case $a = 1$, which has the form of the hyperbolic-cotangent sum formula, has suggested the name of the class of systems.

System (1.1) is closely related to some product-type ones. Depending on the characteristic polynomial associated with a linear difference equation appearing during finding closed-form formulas for solutions to such a system, some of them are theoretically, but some are practically solvable, i.e., solvable in closed form, due to the Abel theorem [1] (each linear homogeneous difference equation with constant coefficients of order less than or equal to four is practically solvable, unlike the case when the order is bigger than four when in some cases the equation is only theoretically solvable). This fact suggests that for practical solvability delays k and l have to take small values. In a series of papers we have studied solvability of the systems of the form in (1.1) with small k and l. In [34,42,43] was investigated the case $k = 0$ and $l = 1$, in [33] the case $k = 1$ and $l = 2$, in [37] the case $k = 0$ and $l = 2$, in [38] the case $k = 2$ and $l = 3$, in [39] the case $k = 0$ and $l = 3$, in [36] the case $k = l \in \mathbb{N}_0$. The case $k = 1$ and $l = 3$ reduces to the case $k = 0$ and $l = 1$, since in the case the system is with interlacing indices (for the notion and some basic fact see, e.g., [41]). In these papers was shown that the corresponding systems are practically solvable, which in some cases is a bit surprising result, e.g., when $k = 2$ and $l = 3$. Namely, it turns out that all the associated polynomials appearing during finding solutions to the corresponding systems are solvable by radicals.

It is a natural problem to study solvability of the corresponding thee-dimensional systems of difference equations. Hence, in this paper we study solvability of some of the three-dimensional systems of difference equations of the form

$$x_{n+1} = \frac{p_n q_n + a}{p_n + q_n}, \quad y_{n+1} = \frac{r_n s_n + a}{r_n + s_n}, \quad z_{n+1} = \frac{t_n g_n + a}{t_n + g_n}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

where $a, p_0, q_0, r_0, s_0, t_0, g_0 \in \mathbb{C}$, and p_n, q_n, r_n, s_n, t_n, g_n are one of the sequences x_n, y_n, z_n.

2 Systems studied in the paper

In this section we transform the system (1.2) into another and give a list of its special cases which are studied in the paper. Before we do this we first note that if $a = 0$, then the system (1.2) is essentially linear with constant coefficients (see [34]; special cases frequently appear in problem books [2]), so the case is not much interesting. Hence, the case will not be treated here and we may assume that $a \neq 0$.

First note that from (1.2) we easily obtain

$$x_{n+1} \pm \sqrt{a} = \frac{(p_n \pm \sqrt{a})(q_n \pm \sqrt{a})}{p_n + q_n},$$
$$y_{n+1} \pm \sqrt{a} = \frac{(r_n \pm \sqrt{a})(s_n \pm \sqrt{a})}{r_n + s_n},$$
$$z_{n+1} \pm \sqrt{a} = \frac{(t_n \pm \sqrt{a})(g_n \pm \sqrt{a})}{t_n + g_n},$$

for $n \in \mathbb{N}_0$, and consequently

$$\begin{aligned} \frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} &= \frac{p_n+\sqrt{a}}{p_n-\sqrt{a}} \cdot \frac{q_n+\sqrt{a}}{q_n-\sqrt{a}}, \\ \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} &= \frac{r_n+\sqrt{a}}{r_n-\sqrt{a}} \cdot \frac{s_n+\sqrt{a}}{s_n-\sqrt{a}}, \\ \frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} &= \frac{t_n+\sqrt{a}}{t_n-\sqrt{a}} \cdot \frac{g_n+\sqrt{a}}{g_n-\sqrt{a}}, \end{aligned} \tag{2.1}$$

for $n \in \mathbb{N}_0$.

Since each of the sequences $p_n, q_n, r_n, s_n, t_n, g_n$ is one of the sequences x_n, y_n, z_n, there are a lot of systems of difference equations of the form in (2.1). They all are not different, since some of them are equivalent to each other.

By using the change of variables

$$u_n = \frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}, \quad v_n = \frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}, \quad w_n = \frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}, \quad n \in \mathbb{N}_0, \tag{2.2}$$

the systems of difference equations in (2.1) are transformed to some product type systems of difference equations. Bearing in mind that the product type systems of difference equations are theoretically solvable and that some of them are practically solvable, it is a natural problem to study practical solvability of (1.2).

Note that from (2.2) we have the following relations

$$x_n = \sqrt{a}\,\frac{u_n+1}{u_n-1}, \quad y_n = \sqrt{a}\,\frac{v_n+1}{v_n-1}, \quad z_n = \sqrt{a}\,\frac{w_n+1}{w_n-1}, \quad n \in \mathbb{N}_0, \tag{2.3}$$

which will be used in the proofs of all the theorems in the paper.

Our aim here is to show practical solvability of the following 36 systems of difference equations by presenting closed-form formulas for their well-defined solutions.

System 1. Case $p_n = q_n = r_n = s_n = t_n = g_n = x_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2. \end{aligned} \tag{2.4}$$

System 2. Case $p_n = q_n = r_n = s_n = t_n = x_n$, $g_n = y_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right). \end{aligned} \tag{2.5}$$

System 3. Case $p_n = q_n = r_n = s_n = x_n$, $t_n = g_n = y_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)^2. \end{aligned} \tag{2.6}$$

System 4. $p_n = q_n = r_n = s_n = t_n = x_n$, $g_n = z_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right). \end{aligned} \tag{2.7}$$

System 5. Case $p_n = q_n = r_n = s_n = x_n$, $t_n = g_n = z_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right)^2. \end{aligned} \tag{2.8}$$

System 6. Case $p_n = q_n = r_n = s_n = x_n$, $t_n = y_n$, $g_n = z_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right). \end{aligned} \tag{2.9}$$

System 7. Case $p_n = q_n = r_n = t_n = g_n = x_n$, $s_n = y_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right), \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2. \end{aligned} \tag{2.10}$$

System 8. Case $p_n = q_n = r_n = t_n = x_n$, $s_n = g_n = y_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right), \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right). \end{aligned} \tag{2.11}$$

System 9. Case $p_n = q_n = r_n = x_n$, $s_n = t_n = g_n = y_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right), \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)^2. \end{aligned} \tag{2.12}$$

System 10. Case $p_n = q_n = r_n = t_n = x_n$, $s_n = y_n$, $g_n = z_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right), \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right). \end{aligned} \tag{2.13}$$

System 11. Case $p_n = q_n = r_n = x_n$, $s_n = y_n$, $t_n = g_n = z_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right), \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right)^2. \end{aligned} \tag{2.14}$$

System 12. Case $p_n = q_n = r_n = x_n$, $s_n = t_n = y_n$, $g_n = z_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right), \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right). \end{aligned} \tag{2.15}$$

System 13. Case $p_n = q_n = t_n = g_n = x_n$, $r_n = s_n = y_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)^2, \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2. \end{aligned} \tag{2.16}$$

System 14. Case $p_n = q_n = t_n = x_n$, $r_n = s_n = g_n = y_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)^2, \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right). \end{aligned} \tag{2.17}$$

System 15. Case $p_n = q_n = x_n$, $r_n = s_n = t_n = g_n = y_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)^2, \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)^2. \end{aligned} \tag{2.18}$$

System 16. Case $p_n = q_n = t_n = x_n$, $r_n = s_n = y_n$, $g_n = z_n$.

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)^2, \quad \frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right). \tag{2.19}$$

System 17. Case $p_n = q_n = x_n$, $r_n = s_n = y_n$, $t_n = g_n = z_n$.

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)^2, \quad \frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right)^2. \tag{2.20}$$

System 18. Case $p_n = q_n = x_n$, $r_n = s_n = t_n = y_n$, $g_n = z_n$.

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)^2, \quad \frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right). \tag{2.21}$$

System 19. Case $p_n = q_n = r_n = t_n = g_n = x_n$, $s_n = z_n$.

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right), \quad \frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2. \tag{2.22}$$

System 20. Case $p_n = q_n = r_n = t_n = x_n$, $s_n = z_n$, $g_n = y_n$.

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right), \quad \frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right). \tag{2.23}$$

System 21. Case $p_n = q_n = r_n = x_n$, $s_n = z_n$, $t_n = g_n = y_n$.

$$\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right), \quad \frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)^2. \tag{2.24}$$

System 22. Case $p_n = q_n = r_n = t_n = x_n$, $s_n = g_n = z_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right), \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right). \end{aligned} \tag{2.25}$$

System 23. Case $p_n = q_n = r_n = x_n$, $s_n = t_n = g_n = z_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right), \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right)^2. \end{aligned} \tag{2.26}$$

System 24. Case $p_n = q_n = r_n = x_n$, $s_n = g_n = z_n$, $t_n = y_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right), \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right). \end{aligned} \tag{2.27}$$

System 25. Case $p_n = q_n = t_n = g_n = x_n$, $r_n = s_n = z_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right)^2, \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2. \end{aligned} \tag{2.28}$$

System 26. Case $p_n = q_n = t_n = x_n$, $r_n = s_n = z_n$, $g_n = y_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right)^2, \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right). \end{aligned} \tag{2.29}$$

System 27. Case $p_n = q_n = x_n$, $r_n = s_n = z_n$, $t_n = g_n = y_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right)^2, \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)^2. \end{aligned} \tag{2.30}$$

System 28. Case $p_n = q_n = t_n = x_n$, $r_n = s_n = g_n = z_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right)^2, \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right). \end{aligned} \tag{2.31}$$

System 29. Case $p_n = q_n = x_n$, $r_n = s_n = t_n = g_n = z_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right)^2, \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right)^2. \end{aligned} \tag{2.32}$$

System 30. Case $p_n = q_n = x_n$, $r_n = s_n = g_n = z_n$, $t_n = y_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right)^2, \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right). \end{aligned} \tag{2.33}$$

System 31. Case $p_n = q_n = t_n = g_n = x_n$, $s_n = z_n$, $r_n = y_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right), \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2. \end{aligned} \tag{2.34}$$

System 32. Case $p_n = q_n = t_n = x_n$, $s_n = z_n$, $r_n = g_n = y_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right), \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right). \end{aligned} \tag{2.35}$$

System 33. Case $p_n = q_n = x_n$, $s_n = z_n$, $r_n = t_n = g_n = y_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right), \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)^2. \end{aligned} \tag{2.36}$$

System 34. Case $p_n = q_n = t_n = x_n$, $s_n = g_n = z_n$, $r_n = y_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right), \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right). \end{aligned} \tag{2.37}$$

System 35. Case $p_n = q_n = x_n$, $s_n = t_n = g_n = z_n$, $r_n = y_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right), \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right)^2. \end{aligned} \tag{2.38}$$

System 36. Case $p_n = q_n = x_n$, $s_n = g_n = z_n$, $r_n = t_n = y_n$.

$$\begin{aligned} &\frac{x_{n+1}+\sqrt{a}}{x_{n+1}-\sqrt{a}} = \left(\frac{x_n+\sqrt{a}}{x_n-\sqrt{a}}\right)^2, \quad \frac{y_{n+1}+\sqrt{a}}{y_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right), \\ &\frac{z_{n+1}+\sqrt{a}}{z_{n+1}-\sqrt{a}} = \left(\frac{y_n+\sqrt{a}}{y_n-\sqrt{a}}\right)\left(\frac{z_n+\sqrt{a}}{z_n-\sqrt{a}}\right). \end{aligned} \tag{2.39}$$

3 Main results

Here we analyse solvability of each of the systems (2.4)–(2.39), and as a consequence of the analysis, for each of them, we state the corresponding result on its solvability. For each system we also use (2.3).

System 1. By using the change of variables (2.2) system (2.4) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = u_n^2, \quad w_{n+1} = u_n^2, \quad n \in \mathbb{N}_0. \tag{3.1}$$

From the first equation in (3.1) we easily obtain

$$u_n = u_0^{2^n}, \quad n \in \mathbb{N}_0. \tag{3.2}$$

By using (3.2) in the second and third equation in (3.1) we get

$$v_n = u_0^{2^n}, \quad w_n = u_0^{2^n}, \quad n \in \mathbb{N}. \tag{3.3}$$

From (2.3), (3.2) and (3.3) we have that the following theorem holds.

Theorem 3.1. *If $a \neq 0$, then the general solution to system (2.4) is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N},$$

$$z_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}.$$

Remark 3.2. Note that in the formulas in Theorem 3.1 do not participate the initial values y_0 and z_0, which is caused by the form of system (2.4). The situation that some of the initial values x_0, y_0, z_0, do not participate in the corresponding formulas appears also in several other systems considered in this paper. Such systems seem less interesting than the other ones. Nevertheless, we will also consider them.

System 2. By using the change of variables (2.2) system (2.5) becomes

$$u_{n+1} = u_n^2, \quad v_{n+1} = u_n^2, \quad w_{n+1} = u_n v_n, \quad n \in \mathbb{N}_0. \tag{3.4}$$

We have that (3.2) and the first equality in (3.3) hold. By using these relations in the third equation in (3.4) we have

$$w_n = u_{n-1}v_{n-1} = u_0^{2^{n-1}} u_0^{2^{n-1}} = u_0^{2^n}, \quad n \geq 2. \tag{3.5}$$

From (2.3), (3.2), (3.3) and (3.5) we have that the following theorem holds.

Theorem 3.3. *If $a \neq 0$, then the general solution to system (2.5) is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N},$$

$$z_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \geq 2.$$

Remark 3.4. Note that the solutions to systems (2.4) and (2.5) are not the same. Namely, the formula for z_n holds for $n \in \mathbb{N}$, that is, $n \geq 2$, respectively, whereas the values for z_1 can be different.

System 3. By using the change of variables (2.2) system (2.6) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = u_n^2, \quad w_{n+1} = v_n^2, \quad n \in \mathbb{N}_0. \tag{3.6}$$

We have that (3.2) and the first equality in (3.3) hold. By using (3.3) in the third equation in (3.6) we have

$$w_n = v_{n-1}^2 = (u_0^{2^{n-1}})^2 = u_0^{2^n}, \quad n \geq 2. \tag{3.7}$$

From (2.3), (3.2), (3.3) and (3.7) we have that the following theorem holds.

Theorem 3.5. *If $a \neq 0$, then the general solution to system* (2.6) *is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N},$$

$$z_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \geq 2.$$

Remark 3.6. Note that the solutions to systems (2.5) and (2.6) are not the same. Namely, the corresponding values for z_1 can be different.

System 4. By using the change of variables (2.2) system (2.7) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = u_n^2, \quad w_{n+1} = u_n w_n, \quad n \in \mathbb{N}_0. \tag{3.8}$$

We have that (3.2) and the first equality in (3.3) hold. By using (3.2) in the third equation in (3.8) we have

$$w_n = u_{n-1}w_{n-1} = u_0^{2^{n-1}}w_{n-1} = w_0\prod_{j=0}^{n-1}u_0^{2^j} = w_0u_0^{\sum_{j=0}^{n-1}2^j} = w_0u_0^{2^n-1}, \quad n \in \mathbb{N}_0. \tag{3.9}$$

From (2.3), (3.2), (3.3) and (3.9) we have that the following theorem holds.

Theorem 3.7. *If $a \neq 0$, then the general solution to system* (2.7) *is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N},$$

$$z_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N}_0.$$

System 5. By using the change of variables (2.2) system (2.8) becomes

$$u_{n+1} = u_n^2, \quad v_{n+1} = u_n^2, \quad w_{n+1} = w_n^2, \quad n \in \mathbb{N}_0. \tag{3.10}$$

We have that (3.2) and the first equality in (3.3) hold, whereas from the third equation in (3.10) we have

$$w_n = w_0^{2^n}, \quad n \in \mathbb{N}_0. \tag{3.11}$$

From (2.3), (3.2), (3.3) and (3.11) we have that the following theorem holds.

Theorem 3.8. *If $a \neq 0$, then the general solution to system (2.8) is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N},$$

$$z_n = \sqrt{a}\frac{\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0.$$

System 6. By using the change of variables (2.2) system (2.9) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = u_n^2, \quad w_{n+1} = v_n w_n, \quad n \in \mathbb{N}_0. \tag{3.12}$$

We have that (3.2) and the first equality in (3.3) hold. By using (3.3) in the third equation in (3.12) we have

$$w_n = v_{n-1}w_{n-1} = u_0^{2^{n-1}}w_{n-1} = w_1\prod_{j=1}^{n-1}u_0^{2^j} = v_0w_0u_0^{\sum_{j=1}^{n-1}2^j} = v_0w_0u_0^{2^n-2}, \tag{3.13}$$

for $n \in \mathbb{N}$.

From (2.3), (3.2), (3.3) and (3.13) we have that the following theorem holds.

Theorem 3.9. *If $a \neq 0$, then the general solution to system (2.9) is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N},$$

$$z_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-2}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-2}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N}.$$

System 7. By using the change of variables (2.2) system (2.10) becomes

$$u_{n+1} = u_n^2, \quad v_{n+1} = u_n v_n, \quad w_{n+1} = u_n^2, \quad n \in \mathbb{N}_0. \tag{3.14}$$

We have that (3.2) and the second equality in (3.3) hold. By using (3.2) in the second equation in (3.14) we have

$$v_n = u_{n-1}v_{n-1} = u_0^{2^{n-1}}v_{n-1} = v_0\prod_{j=0}^{n-1}u_0^{2^j} = v_0u_0^{\sum_{j=0}^{n-1}2^j} = v_0u_0^{2^n-1}, \quad n \in \mathbb{N}_0. \tag{3.15}$$

From (2.3), (3.2), (3.3) and (3.15) we have that the following theorem holds.

Theorem 3.10. *If $a \neq 0$, then the general solution to system* (2.10) *is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N}_0,$$

$$z_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}.$$

System 8. By using the change of variables (2.2) system (2.11) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = u_nv_n, \quad w_{n+1} = u_nv_n, \quad n \in \mathbb{N}_0. \tag{3.16}$$

We have that (3.2) and (3.15) hold. From this and since $v_n = w_n$, $n \in \mathbb{N}$, we have

$$w_n = v_0u_0^{2^n-1}, \quad n \in \mathbb{N}. \tag{3.17}$$

From (2.3), (3.2), (3.15) and (3.17) we have that the following theorem holds.

Theorem 3.11. *If $a \neq 0$, then the general solution to system* (2.11) *is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N}_0,$$

$$z_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N}.$$

System 9. By using the change of variables (2.2) system (2.12) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = u_nv_n, \quad w_{n+1} = v_n^2, \quad n \in \mathbb{N}_0. \tag{3.18}$$

We have that (3.2) and (3.15) hold. From this and the third equation in (3.18), we have

$$w_n = v_{n-1}^2 = v_0^2u_0^{2^n-2}, \quad n \in \mathbb{N}. \tag{3.19}$$

From (2.3), (3.2), (3.15) and (3.19) we have that the following theorem holds.

Theorem 3.12. *If $a \neq 0$, then the general solution to system (2.12) is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N}_0,$$

$$z_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-2}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^2+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-2}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^2-1}, \quad n \in \mathbb{N}.$$

System 10. By using the change of variables (2.2) system (2.13) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = u_n v_n, \quad w_{n+1} = u_n w_n, \quad n \in \mathbb{N}_0. \tag{3.20}$$

We have that (3.2) and (3.15) hold. From this and the third equation in (3.20), we have

$$w_n = u_{n-1}w_{n-1} = w_0 \prod_{j=0}^{n-1} u_0^{2^j} = w_0 u_0^{\sum_{j=0}^{n-1} 2^j} = w_0 u_0^{2^n-1}, \quad n \in \mathbb{N}_0. \tag{3.21}$$

From (2.3), (3.2), (3.15) and (3.21) we have that the following theorem holds.

Theorem 3.13. *If $a \neq 0$, then the general solution to system (2.13) is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N}_0,$$

$$z_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N}_0.$$

System 11. By using the change of variables (2.2) system (2.14) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = u_n v_n, \quad w_{n+1} = w_n^2, \quad n \in \mathbb{N}_0. \tag{3.22}$$

We have that (3.2), (3.11), (3.15) hold, form which along with (2.3) it follows that the following theorem holds.

Theorem 3.14. *If $a \neq 0$, then the general solution to system* (2.14) *is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N}_0,$$

$$z_n = \sqrt{a}\frac{\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0.$$

System 12. By using the change of variables (2.2) system (2.15) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = u_n v_n, \quad w_{n+1} = v_n w_n, \quad n \in \mathbb{N}_0. \tag{3.23}$$

We have that (3.2) and (3.15) hold. From this and the third equation in (3.23), we have

$$\begin{aligned} w_n &= v_{n-1}w_{n-1} = w_0\prod_{j=0}^{n-1} v_j = w_0\prod_{j=0}^{n-1} v_0 u_0^{2^j-1} \\ &= w_0 v_0^n u_0^{\sum_{j=0}^{n-1}(2^j-1)} = w_0 v_0^n u_0^{2^n-n-1}, \quad n \in \mathbb{N}_0. \end{aligned} \tag{3.24}$$

From (2.3), (3.2), (3.15) and (3.24) we have that the following theorem holds.

Theorem 3.15. *If $a \neq 0$, then the general solution to system* (2.15) *is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N}_0,$$

$$z_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-n-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^n\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-n-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^n\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N}_0.$$

System 13. By using the change of variables (2.2) system (2.16) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = v_n^2, \quad w_{n+1} = u_n^2, \quad n \in \mathbb{N}_0. \tag{3.25}$$

We have that (3.2) and the second equality in (3.3) hold, whereas from the second equation in (3.25) we have

$$v_n = v_0^{2^n}, \quad n \in \mathbb{N}_0. \tag{3.26}$$

From (2.3), (3.2), (3.3) and (3.26) we have that the following theorem holds.

Theorem 3.16. *If $a \neq 0$, then the general solution to system* (2.16) *is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$z_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}.$$

System 14. By using the change of variables (2.2) system (2.17) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = v_n^2, \quad w_{n+1} = u_n v_n, \quad n \in \mathbb{N}_0. \tag{3.27}$$

We have that (3.2) and (3.26) hold, whereas from the third equation in (3.27) we have

$$w_n = u_{n-1}v_{n-1} = u_0^{2^{n-1}} v_0^{2^{n-1}}, \quad n \in \mathbb{N}. \tag{3.28}$$

From (2.3), (3.2), (3.26) and (3.28) we have that the following theorem holds.

Theorem 3.17. *If $a \neq 0$, then the general solution to system* (2.17) *is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$z_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^{n-1}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^{n-1}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^{n-1}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^{n-1}}-1}, \quad n \in \mathbb{N}.$$

System 15. By using the change of variables (2.2) system (2.18) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = v_n^2, \quad w_{n+1} = v_n^2, \quad n \in \mathbb{N}_0. \tag{3.29}$$

We have that (3.2) and (3.26) hold, whereas from the third equation in (3.29) we have

$$w_n = v_{n-1}^2 = v_0^{2^n}, \quad n \in \mathbb{N}. \tag{3.30}$$

From (2.3), (3.2), (3.26) and (3.30) we have that the following theorem holds.

Theorem 3.18. *If $a \neq 0$, then the general solution to system (2.18) is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$z_n = \sqrt{a}\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}.$$

System 16. By using the change of variables (2.2) system (2.19) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = v_n^2, \quad w_{n+1} = u_n w_n, \quad n \in \mathbb{N}_0. \tag{3.31}$$

We have that (3.2) and (3.26) hold, whereas from the third equation in (3.31) we have

$$w_n = u_{n-1}w_{n-1} = w_0\prod_{j=0}^{n-1} u_j = w_0\prod_{j=0}^{n-1} u_0^{2^j} = w_0 u_0^{2^n-1}, \quad n \in \mathbb{N}_0. \tag{3.32}$$

From (2.3), (3.2), (3.26) and (3.32) we have that the following theorem holds.

Theorem 3.19. *If $a \neq 0$, then the general solution to system (2.19) is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$z_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N}_0.$$

System 17. By using the change of variables (2.2) system (2.20) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = v_n^2, \quad w_{n+1} = w_n^2, \quad n \in \mathbb{N}_0. \tag{3.33}$$

We have that (3.2), (3.11) and (3.26) hold, from which the following theorem follows.

Theorem 3.20. *If $a \neq 0$, then the general solution to system* (2.20) *is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$z_n = \sqrt{a}\frac{\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0.$$

System 18. By using the change of variables (2.2) system (2.21) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = v_n^2, \quad w_{n+1} = v_n w_n, \quad n \in \mathbb{N}_0. \tag{3.34}$$

We have that (3.2) and (3.26) hold, whereas from the third equation in (3.34) we have

$$w_n = v_{n-1}w_{n-1} = w_0 \prod_{j=0}^{n-1} v_0^{2^j} = w_0 v_0^{\sum_{j=0}^{n-1} 2^j} = w_0 v_0^{2^n-1}, \quad n \in \mathbb{N}_0. \tag{3.35}$$

From (2.3), (3.2), (3.26) and (3.35) we have that the following theorem holds.

Theorem 3.21. *If $a \neq 0$, then the general solution to system* (2.21) *is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$z_n = \sqrt{a}\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N}_0.$$

System 19. By using the change of variables (2.2) system (2.22) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = u_n w_n, \quad w_{n+1} = u_n^2, \quad n \in \mathbb{N}_0. \tag{3.36}$$

We have that (3.2) and the second relation in (3.3) hold, whereas from the second equation in (3.36) we have

$$v_n = u_{n-1}w_{n-1} = u_0^{2^{n-1}} u_0^{2^{n-1}} = u_0^{2^n}, \quad n \geq 2. \tag{3.37}$$

From (2.3), (3.2), (3.3) and (3.37) we have that the following theorem holds.

Theorem 3.22. *If $a \neq 0$, then the general solution to system (2.22) is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$
$$y_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \geq 2,$$
$$z_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}.$$

System 20. By using the change of variables (2.2) system (2.23) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = u_n w_n, \quad w_{n+1} = u_n v_n, \quad n \in \mathbb{N}_0. \tag{3.38}$$

We have that (3.2) holds. From the second and third relation in (3.38) we have

$$v_n = u_{n-1}w_{n-1} = u_{n-1}u_{n-2}v_{n-2} = u_0^{3\cdot 2^{n-2}}v_{n-2}, \quad n \geq 2,$$

from which we obtain

$$v_{2n} = u_0^{3\cdot 2^{2n-2}}v_{2n-2} = v_0\prod_{j=1}^{n}u_0^{3\cdot 2^{2j-2}} = v_0 u_0^{3\sum_{j=1}^{n}4^{j-1}} = v_0 u_0^{4^n-1}, \quad n \in \mathbb{N}_0, \tag{3.39}$$

and

$$v_{2n+1} = u_0^{3\cdot 2^{2n-1}}v_{2n-1} = v_1\prod_{j=1}^{n}u_0^{3\cdot 2^{2j-1}} = u_0 w_0 u_0^{6\sum_{j=1}^{n}4^{j-1}} = w_0 u_0^{2^{2n+1}-1}, \tag{3.40}$$

for $n \in \mathbb{N}_0$.

Further, by (3.2), (3.39) and (3.40), we have

$$w_{2n} = u_{2n-1}v_{2n-1} = u_0^{2^{2n-1}}w_0 u_0^{2^{2n-1}-1} = w_0 u_0^{4^n-1}, \quad n \in \mathbb{N}_0, \tag{3.41}$$

and

$$w_{2n+1} = u_{2n}v_{2n} = u_0^{2^{2n}}v_0 u_0^{4^n-1} = v_0 u_0^{2^{2n+1}-1}, \quad n \in \mathbb{N}_0. \tag{3.42}$$

From (2.3), (3.2), (3.39)–(3.42) we have that the following theorem holds.

Theorem 3.23. *If $a \neq 0$, then the general solution to system (2.23) is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_{2n} = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^{2n}-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^{2n}-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N}_0,$$

$$y_{2n+1} = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^{2n+1}-1}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^{2n+1}-1}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N}_0,$$

$$z_{2n} = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^{2n}-1}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^{2n}-1}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N}_0,$$

$$z_{2n+1} = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^{2n+1}-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^{2n+1}-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N}_0.$$

System 21. By using the change of variables (2.2) system (2.24) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = u_n w_n, \quad w_{n+1} = v_n^2, \quad n \in \mathbb{N}_0. \tag{3.43}$$

We have that (3.2) holds. From the second and third relation in (3.43) we have

$$v_n = u_{n-1}w_{n-1} = u_0^{2^{n-1}} v_{n-2}^2, \quad n \geq 2, \tag{3.44}$$

from which we obtain

$$\begin{aligned} v_{2n} &= u_0^{2^{2n-1}} v_{2n-2}^2 = u_0^{2^{2n-1}} (u_0^{2^{2n-3}} v_{2n-4}^2)^2 = u_0^{2^{2n-1}+2^{2n-2}} v_{2n-4}^{2^2} \\ &= u_0^{2^{2n-1}+2^{2n-2}} (u_0^{2^{2n-5}} v_{2n-6}^2)^{2^2} = u_0^{2^{2n-1}+2^{2n-2}+2^{2n-3}} v_{2n-6}^{2^3}. \end{aligned}$$

Assume that we have proved

$$v_{2n} = u_0^{2^{2n-1}+2^{2n-2}+\cdots+2^{n+1}+2^n} v_0^{2^n} = u_0^{2^n(2^n-1)} v_0^{2^n}, \tag{3.45}$$

for an $n \in \mathbb{N}$.

Then by using (3.44) and (3.45) we have

$$v_{2n+2} = u_0^{2^{2n+1}} v_{2n}^2 = u_0^{2^{2n+1}} (u_0^{2^n(2^n-1)} v_0^{2^n})^2 = u_0^{2^{n+1}(2^{n+1}-1)} v_0^{2^{n+1}},$$

from which along with the method of induction it follows that formula (3.45) holds for every $n \in \mathbb{N}$. In fact, a simple calculation shows that it also holds for $n = 0$.

Further, we have

$$v_{2n+1} = u_0^{2^{2n}} v_{2n-1}^2 = u_0^{2^{2n}} (u_0^{2^{2n-2}} v_{2n-3}^2)^2 = u_0^{2^{2n}+2^{2n-1}} v_{2n-3}^{2^2}, \quad n \geq 2.$$

Assume that we have proved

$$\begin{aligned} v_{2n+1} &= u_0^{2^{2n}+2^{2n-1}+\cdots+2^{n+2}+2^{n+1}} v_1^{2^n} = u_0^{2^{n+1}(2^n-1)}(u_0w_0)^{2^n} \\ &= u_0^{2^n(2^{n+1}-1)} w_0^{2^n}, \end{aligned} \tag{3.46}$$

for an $n \in \mathbb{N}_0$.

Then by using (3.44) and (3.46) we have

$$v_{2n+3} = u_0^{2^{2n+2}} v_{2n+1}^2 = u_0^{2^{2n+2}} (u_0^{2^n(2^{n+1}-1)} w_0^{2^n})^2 = u_0^{2^{n+1}(2^{n+2}-1)} w_0^{2^{n+1}},$$

from which along with the method of induction it follows that formula (3.46) holds for every $n \in \mathbb{N}_0$.

By using (3.45) and (3.46) into the third equation in (3.43) we get

$$w_{2n} = v_{2n-1}^2 = (u_0^{2^{n-1}(2^n-1)} w_0^{2^{n-1}})^2 = u_0^{2^n(2^n-1)} w_0^{2^n}, \quad n \in \mathbb{N}_0, \tag{3.47}$$

and

$$w_{2n+1} = v_{2n}^2 = (u_0^{2^n(2^n-1)} v_0^{2^n})^2 = u_0^{2^{n+1}(2^n-1)} v_0^{2^{n+1}}, \quad n \in \mathbb{N}_0. \tag{3.48}$$

From (2.3), (3.2), (3.45)–(3.48) we have that the following theorem holds.

Theorem 3.24. *If $a \neq 0$, then the general solution to system (2.24) is*

$$\begin{aligned} x_n &= \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0, \\ y_{2n} &= \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n(2^n-1)}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n(2^n-1)}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0, \\ y_{2n+1} &= \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n(2^{n+1}-1)}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n(2^{n+1}-1)}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0, \\ z_{2n} &= \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n(2^n-1)}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n(2^n-1)}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0, \\ z_{2n+1} &= \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^{n+1}(2^n-1)}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^{n+1}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^{n+1}(2^n-1)}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^{n+1}}-1}, \quad n \in \mathbb{N}_0. \end{aligned}$$

System 22. By using the change of variables (2.2) system (2.25) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = u_n w_n, \quad w_{n+1} = u_n w_n, \quad n \in \mathbb{N}_0. \tag{3.49}$$

We have that (3.2) holds and that $v_n = w_n$, $n \in \mathbb{N}$. Hence

$$v_n = u_{n-1}w_{n-1} = u_0^{2^{n-1}} v_{n-1} = v_1 \prod_{j=1}^{n-1} u_0^{2^j} = w_0 u_0^{\sum_{j=0}^{n-1} 2^j} = w_0 u_0^{2^n-1} \tag{3.50}$$

for $n \in \mathbb{N}$, and consequently

$$w_n = w_0 u_0^{2^n - 1}, \quad n \in \mathbb{N}. \tag{3.51}$$

In fact, a simple calculation shows that (3.51) also holds for $n = 0$.

From (2.3), (3.2), (3.50) and (3.51) we have that the following theorem holds.

Theorem 3.25. *If $a \neq 0$, then the general solution to system (2.25) is*

$$x_n = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n} + 1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n} - 1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right) + 1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right) - 1}, \quad n \in \mathbb{N},$$

$$z_n = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right) + 1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right) - 1}, \quad n \in \mathbb{N}_0.$$

System 23. By using the change of variables (2.2) system (2.26) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = u_n w_n, \quad w_{n+1} = w_n^2, \quad n \in \mathbb{N}_0. \tag{3.52}$$

We have that (3.2) and (3.11) hold, from which it follows that

$$v_n = u_{n-1} w_{n-1} = u_0^{2^{n-1}} w_0^{2^{n-1}} = (u_0 w_0)^{2^{n-1}}, \quad n \in \mathbb{N}. \tag{3.53}$$

From (2.3), (3.2), (3.11) and (3.53) we have that the following theorem holds.

Theorem 3.26. *If $a \neq 0$, then the general solution to system (2.26) is*

$$x_n = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n} + 1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n} - 1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^{n-1}}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^{n-1}} + 1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^{n-1}}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^{n-1}} - 1}, \quad n \in \mathbb{N},$$

$$z_n = \sqrt{a}\,\frac{\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n} + 1}{\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n} - 1}, \quad n \in \mathbb{N}_0.$$

System 24. By using the change of variables (2.2) system (2.27) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = u_n w_n, \quad w_{n+1} = v_n w_n, \quad n \in \mathbb{N}_0. \tag{3.54}$$

We have that (3.2) holds. From this, the second and third equation in (3.54) we get

$$w_n = v_{n-1} w_{n-1} = w_{n-1} w_{n-2} u_{n-2} = w_{n-1} w_{n-2} u_0^{2^{n-2}}, \quad n \geq 2.$$

By using the change of variables

$$w_n = \zeta_n u_0^{\alpha_n}, \quad n \in \mathbb{N}_0, \tag{3.55}$$

the last equation becomes

$$\zeta_n = \zeta_{n-1}\zeta_{n-2}u_0^{\alpha_{n-1}+\alpha_{n-2}+2^{n-2}-\alpha_n}, \quad n \geq 2. \tag{3.56}$$

Since $w_0 = w_0$ and $w_1 = v_0 w_0$, and they do not contain u_0, we may take

$$\alpha_0 = 0 \quad \text{and} \quad \alpha_1 = 0. \tag{3.57}$$

Let $(\alpha_n)_{n\in\mathbb{N}_0}$ be the solution to the difference equation

$$\alpha_n = \alpha_{n-1} + \alpha_{n-2} + 2^{n-2}, \quad n \geq 2, \tag{3.58}$$

satisfying the initial conditions in (3.57).

We find a particular solution to equation (3.58) in the form $\alpha_n^p = c2^n$, $n \in \mathbb{N}_0$, where c is a constant ([17]). By employing it in (3.58) we have that it must be

$$c2^n = c(2^{n-1} + 2^{n-2}) + 2^{n-2}, \quad n \in \mathbb{N}$$

from which it follows that $c = 1$. Hence, the general solution to (3.58) has the form

$$\alpha_n = c_1\lambda_1^n + c_2\lambda_2^n + 2^n, \quad n \in \mathbb{N}, \tag{3.59}$$

where λ_1 and λ_2 are the roots of the polynomial $P_2(\lambda) = \lambda^2 - \lambda - 1$.

From (3.57) and (3.59) we have

$$c_1 + c_2 = -1 \quad \text{and} \quad c_1\lambda_1 + c_2\lambda_2 = -2$$

from which it follows that

$$c_1 = \frac{1}{\lambda_2 - \lambda_1}\begin{vmatrix} -1 & 1 \\ -2 & \lambda_2 \end{vmatrix} = \frac{2-\lambda_2}{\lambda_2 - \lambda_1} \quad \text{and} \quad c_2 = \frac{1}{\lambda_2 - \lambda_1}\begin{vmatrix} 1 & -1 \\ \lambda_1 & -2 \end{vmatrix} = \frac{\lambda_1 - 2}{\lambda_2 - \lambda_1},$$

from which along with (3.59) we have

$$\alpha_n = \frac{(2-\lambda_2)\lambda_1^n + (\lambda_1 - 2)\lambda_2^n}{\lambda_2 - \lambda_1} + 2^n, \quad n \in \mathbb{N}_0. \tag{3.60}$$

For such a chosen sequence α_n, we have that $(\zeta_n)_{n\in\mathbb{N}_0}$ satisfies the equation

$$\zeta_n = \zeta_{n-1}\zeta_{n-2}, \quad n \geq 2, \tag{3.61}$$

with the initial conditions

$$\zeta_0 = w_0 \quad \text{and} \quad \zeta_1 = v_0 w_0. \tag{3.62}$$

Let $a_1 = b_1 = 1$. Then we have

$$\zeta_n = \zeta_{n-1}^{a_1}\zeta_{n-2}^{b_1} = (\zeta_{n-2}\zeta_{n-3})^{a_1}\zeta_{n-2}^{b_1} = \zeta_{n-2}^{a_1+b_1}\zeta_{n-3}^{a_1} = \zeta_{n-2}^{a_2}\zeta_{n-3}^{b_2},$$

where $a_2 = a_1 + b_1$ and $b_2 = a_1$. By using a simple inductive argument we obtain

$$\zeta_n = \zeta_{n-k}^{a_k}\zeta_{n-k-1}^{b_k}$$

for $1 \le k \le n-1$, and

$$a_k = a_{k-1} + b_{k-1}, \quad b_k = a_{k-1}. \tag{3.63}$$

The relations in (3.63) hold for every $k \in \mathbb{Z}$.

Hence for $k = n-1$ is obtained

$$\zeta_n = \zeta_1^{a_{n-1}} \zeta_0^{b_{n-1}} = \zeta_1^{a_{n-1}} \zeta_0^{a_{n-2}}, \quad n \in \mathbb{N}, \tag{3.64}$$

and also

$$a_n = a_{n-1} + a_{n-2}, \quad n \ge 3.$$

From this and since $a_1 = 1$ and $a_2 = 2$, we have $a_n = f_{n+1}$ and $b_n = f_n$, where f_n is the Fibonacci sequence ([44]).

From (3.62), (3.63) and (3.64) we have

$$\zeta_n = (v_0 w_0)^{f_n} w_0^{f_{n-1}} = v_0^{f_n} w_0^{f_n + f_{n-1}} = v_0^{f_n} w_0^{f_{n+1}},$$

from which together with (3.55) we obtain

$$w_n = v_0^{f_n} w_0^{f_{n+1}} u_0^{\alpha_n}, \quad n \in \mathbb{N}_0. \tag{3.65}$$

By using (3.2) and (3.65) in the second equation in (3.54) we get

$$v_n = u_{n-1} w_{n-1} = u_0^{\alpha_{n-1} + 2^{n-1}} v_0^{f_{n-1}} w_0^{f_n}, \quad n \in \mathbb{N}. \tag{3.66}$$

A simple calculation shows that (3.66) holds also for $n = 0$.

From (2.3), (3.2), (3.65), (3.66) we have that the following theorem holds.

Theorem 3.27. *If $a \neq 0$, then the general solution to system (2.27) is*

$$x_n = \sqrt{a} \frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n} + 1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n} - 1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a} \frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{\alpha_{n-1}+2^{n-1}} \left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{f_{n-1}} \left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{f_n} + 1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{\alpha_{n-1}+2^{n-1}} \left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{f_{n-1}} \left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{f_n} - 1}, \quad n \in \mathbb{N}_0,$$

$$z_n = \sqrt{a} \frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{\alpha_n} \left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{f_n} \left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{f_{n+1}} + 1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{\alpha_n} \left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{f_n} \left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{f_{n+1}} - 1}, \quad n \in \mathbb{N}_0,$$

where the sequence $(\alpha_n)_{n\in\mathbb{N}_0}$ is given by (3.60).

System 25. By using the change of variables (2.2) system (2.28) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = w_n^2, \quad w_{n+1} = u_n^2, \quad n \in \mathbb{N}_0. \tag{3.67}$$

We have that (3.2) and the second relation in (3.3) hold. Hence

$$v_n = w_{n-1}^2 = u_0^{2^n}, \quad n \ge 2. \tag{3.68}$$

From (2.3), (3.2), (3.3), (3.68) we have that the following theorem holds.

Theorem 3.28. *If $a \neq 0$, then the general solution to system (2.28) is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \geq 2,$$

$$z_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}.$$

System 26. By using the change of variables (2.2) system (2.29) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = w_n^2, \quad w_{n+1} = u_n v_n, \quad n \in \mathbb{N}_0. \tag{3.69}$$

This system is obtained from system (3.43) by interchanging letters v and w. Hence, from (3.45)–(3.48) we have

$$v_{2n} = u_0^{2^n(2^n-1)} v_0^{2^n}, \quad n \in \mathbb{N}_0, \tag{3.70}$$

$$v_{2n+1} = u_0^{2^{n+1}(2^n-1)} w_0^{2^{n+1}}, \quad n \in \mathbb{N}_0, \tag{3.71}$$

$$w_{2n} = u_0^{2^n(2^n-1)} w_0^{2^n}, \quad n \in \mathbb{N}_0, \tag{3.72}$$

$$w_{2n+1} = u_0^{2^n(2^{n+1}-1)} v_0^{2^n}, \quad n \in \mathbb{N}_0. \tag{3.73}$$

From (2.3), (3.2), (3.70)–(3.73) we have that the following theorem holds.

Theorem 3.29. *If $a \neq 0$, then the general solution to system (2.29) is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_{2n} = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n(2^n-1)}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n(2^n-1)}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_{2n+1} = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^{n+1}(2^n-1)}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^{n+1}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^{n+1}(2^n-1)}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^{n+1}}-1}, \quad n \in \mathbb{N}_0,$$

$$z_{2n} = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n(2^n-1)}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n(2^n-1)}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$z_{2n+1} = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n(2^{n+1}-1)}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n(2^{n+1}-1)}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0.$$

System 27. By using the change of variables (2.2) system (2.30) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = w_n^2, \quad w_{n+1} = v_n^2, \quad n \in \mathbb{N}_0. \tag{3.74}$$

We have that (3.2) holds. From the second and third equation in (3.74) we have

$$v_n = w_{n-1}^2 = v_{n-2}^4, \quad n \geq 2,$$

from which it follows that

$$v_{2n} = v_{2n-2}^4 = v_0^{4^n}, \quad n \in \mathbb{N}_0, \tag{3.75}$$

and

$$v_{2n+1} = v_{2n-1}^4 = v_1^{4^n} = w_0^{2\cdot 4^n}, \quad n \in \mathbb{N}_0, \tag{3.76}$$

By using (3.75) and (3.76) in the third equation in (3.74) we get

$$w_{2n} = v_{2n-1}^2 = (w_0^{2\cdot 4^{n-1}})^2 = w_0^{4^n}, \quad n \in \mathbb{N}_0, \tag{3.77}$$

and

$$w_{2n+1} = v_{2n}^2 = (v_0^{4^n})^2 = v_0^{2\cdot 4^n}, \quad n \in \mathbb{N}_0. \tag{3.78}$$

From (2.3), (3.2), (3.75)–(3.78) we have that the following theorem holds.

Theorem 3.30. *If $a \neq 0$, then the general solution to system (2.30) is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_{2n} = \sqrt{a}\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{4^n}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{4^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_{2n+1} = \sqrt{a}\frac{\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2\cdot 4^n}+1}{\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2\cdot 4^n}-1}, \quad n \in \mathbb{N}_0,$$

$$z_{2n} = \sqrt{a}\frac{\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{4^n}+1}{\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{4^n}-1}, \quad n \in \mathbb{N}_0,$$

$$z_{2n+1} = \sqrt{a}\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2\cdot 4^n}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2\cdot 4^n}-1}, \quad n \in \mathbb{N}_0.$$

System 28. By using the change of variables (2.2) system (2.31) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = w_n^2, \quad w_{n+1} = u_n w_n, \quad n \in \mathbb{N}_0. \tag{3.79}$$

We have that (3.2) holds. From this and the third equation in (3.79) we have

$$w_n = u_{n-1}w_{n-1} = u_0^{2^{n-1}}w_{n-1} = w_0\prod_{j=0}^{n-1}u_0^{2^j} = w_0 u_0^{\sum_{j=0}^{n-1}2^j} = w_0 u_0^{2^n-1}, \tag{3.80}$$

for $n \in \mathbb{N}_0$, from which and the second equation in (3.79) it follows that

$$v_n = w_{n-1}^2 = (w_0 u_0^{2^{n-1}-1})^2 = w_0^2 u_0^{2^n-2}, \quad n \in \mathbb{N}. \tag{3.81}$$

From (2.3), (3.2), (3.80), (3.81) we have that the following theorem holds.

Theorem 3.31. *If $a \neq 0$, then the general solution to system (2.31) is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-2}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-2}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2}-1}, \quad n \in \mathbb{N},$$

$$z_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N}_0.$$

System 29. By using the change of variables (2.2) system (2.32) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = w_n^2, \quad w_{n+1} = w_n^2, \quad n \in \mathbb{N}_0. \tag{3.82}$$

We have that (3.2) and (3.11) hold, and consequently

$$v_n = w_{n-1}^2 = w_0^{2^n}, \quad n \in \mathbb{N}. \tag{3.83}$$

From (2.3), (3.2), (3.11), (3.83) we have that the following theorem holds.

Theorem 3.32. *If $a \neq 0$, then the general solution to system (2.32) is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N},$$

$$z_n = \sqrt{a}\frac{\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0.$$

System 30. By using the change of variables (2.2) system (2.33) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = w_n^2, \quad w_{n+1} = v_n w_n, \quad n \in \mathbb{N}_0. \tag{3.84}$$

We have that (3.2) holds. From the second and third equation in (3.84) we have

$$w_n = v_{n-1}w_{n-1} = w_{n-1}w_{n-2}^2, \quad n \geq 2. \tag{3.85}$$

Let $a_1 = 1$ and $b_1 = 2$. Then we have

$$w_n = w_{n-1}^{a_1}w_{n-2}^{b_1} = (w_{n-2}w_{n-3}^2)^{a_1}w_{n-2}^{b_1} = w_{n-2}^{a_1+b_1}w_{n-3}^{2a_1} = w_{n-2}^{a_2}w_{n-3}^{b_2},$$

where $a_2 = a_1 + b_1$ and $b_2 = 2a_1$. By using a simple inductive argument we obtain

$$w_n = w_{n-k}^{a_k}w_{n-k-1}^{b_k}$$

for $1 \le k \le n-1$, and

$$a_k = a_{k-1} + b_{k-1}, \quad b_k = 2a_{k-1}. \tag{3.86}$$

Hence for $k = n-1$ we get

$$w_n = w_1^{a_{n-1}} w_0^{b_{n-1}} = (v_0 w_0)^{a_{n-1}} w_0^{2a_{n-2}} = v_0^{a_{n-1}} w_0^{a_n}, \quad n \in \mathbb{N}_0, \tag{3.87}$$

and

$$a_n = a_{n-1} + 2a_{n-1}, \quad n \ge 2. \tag{3.88}$$

In fact, (3.88) holds for each $n \in \mathbb{Z}$.

The characteristic polynomial associated to equation (3.88) is $\widehat{P}_2(\lambda) = \lambda^2 - \lambda - 2 = (\lambda+1)(\lambda-2)$. Hence, general solution to the equation is

$$a_n = c_1(-1)^n + c_2 2^n.$$

From this and since $a_0 = a_1 = 1$, we have

$$c_1 + c_2 = 1, \quad -c_1 + 2c_2 = 1$$

from which it follows that $c_1 = 1/3$ and $c_2 = 2/3$. Hence

$$a_n = \frac{2^{n+1} + (-1)^n}{3}, \quad n \in \mathbb{N}_0,$$

from which together with (3.87) it follows that

$$w_n = v_0^{\frac{2^n - (-1)^n}{3}} w_0^{\frac{2^{n+1} + (-1)^n}{3}}, \quad n \in \mathbb{N}_0. \tag{3.89}$$

By using (3.89) in the second equation in (3.84) we get

$$v_n = w_{n-1}^2 = v_0^{\frac{2^n + 2(-1)^n}{3}} w_0^{\frac{2^{n+1} - 2(-1)^n}{3}}, \quad n \in \mathbb{N}. \tag{3.90}$$

Direct calculation shows that this formula also holds for $n = 0$.

From (2.3), (3.2), (3.89), (3.90) we have that the following theorem holds.

Theorem 3.33. *If $a \neq 0$, then the general solution to system (2.33) is*

$$x_n = \sqrt{a}\,\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n} + 1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n} - 1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{\frac{2^n+2(-1)^n}{3}} \left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{\frac{2^{n+1}-2(-1)^n}{3}} + 1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{\frac{2^n+2(-1)^n}{3}} \left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{\frac{2^{n+1}-2(-1)^n}{3}} - 1}, \quad n \in \mathbb{N}_0,$$

$$z_n = \sqrt{a}\,\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{\frac{2^n-(-1)^n}{3}} \left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{\frac{2^{n+1}+(-1)^n}{3}} + 1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{\frac{2^n-(-1)^n}{3}} \left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{\frac{2^{n+1}+(-1)^n}{3}} - 1}, \quad n \in \mathbb{N}_0.$$

System 31. By using the change of variables (2.2) system (2.34) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = v_n w_n, \quad w_{n+1} = u_n^2, \quad n \in \mathbb{N}_0. \tag{3.91}$$

We have that (3.2) and the second equality in (3.3) hold. By using these formulas we have

$$v_n = w_{n-1}v_{n-1} = v_1 \prod_{j=1}^{n-1} u_0^{2^j} = v_0 w_0 u_0^{\sum_{j=1}^{n-1} 2^j} = v_0 w_0 u_0^{2^n-2}, \quad n \in \mathbb{N}. \tag{3.92}$$

From (2.3), (3.2), (3.3), (3.92) we have that the following theorem holds.

Theorem 3.34. *If $a \neq 0$, then the general solution to system (2.34) is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-2}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-2}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N},$$

$$z_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}.$$

System 32. By using the change of variables (2.2) system (2.35) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = v_n w_n, \quad w_{n+1} = u_n v_n, \quad n \in \mathbb{N}_0. \tag{3.93}$$

This system is obtained from system (3.54) by interchanging letters v and w. Hence, from (3.65) and (3.66) we have

$$v_n = v_0^{f_{n+1}} w_0^{f_n} u_0^{\alpha_n}, \quad n \in \mathbb{N}_0, \tag{3.94}$$

$$w_n = u_0^{\alpha_{n-1}+2^{n-1}} v_0^{f_n} w_0^{f_{n-1}}, \quad n \in \mathbb{N}_0. \tag{3.95}$$

From (2.3), (3.2), (3.94), (3.95) we have that the following theorem holds.

Theorem 3.35. *If $a \neq 0$, then the general solution to system (2.35) is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{\alpha_n}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{f_{n+1}}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{f_n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{\alpha_n}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{f_{n+1}}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{f_n}-1}, \quad n \in \mathbb{N}_0,$$

$$z_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{\alpha_{n-1}+2^{n-1}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{f_n}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{f_{n-1}}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{\alpha_{n-1}+2^{n-1}}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{f_n}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{f_{n-1}}-1}, \quad n \in \mathbb{N}_0.$$

where the sequence $(\alpha_n)_{n\in\mathbb{N}_0}$ is given by (3.60).

System 33. By using the change of variables (2.2) system (2.36) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = v_n w_n, \quad w_{n+1} = v_n^2, \quad n \in \mathbb{N}_0. \tag{3.96}$$

This system is obtained from system (3.84) by interchanging letters v and w. Hence, from (3.89) and (3.90) we have

$$v_n = v_0^{\frac{2^{n+1}+(-1)^n}{3}} w_0^{\frac{2^n-(-1)^n}{3}}, \quad n \in \mathbb{N}_0, \tag{3.97}$$

$$w_n = v_0^{\frac{2^{n+1}-2(-1)^n}{3}} w_0^{\frac{2^n+2(-1)^n}{3}}, \quad n \in \mathbb{N}_0. \tag{3.98}$$

From (2.3), (3.2), (3.97), (3.98) we have that the following theorem holds.

Theorem 3.36. *If $a \neq 0$, then the general solution to system (2.36) is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{\frac{2^{n+1}+(-1)^n}{3}}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{\frac{2^n-(-1)^n}{3}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{\frac{2^{n+1}+(-1)^n}{3}}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{\frac{2^n-(-1)^n}{3}}-1}, \quad n \in \mathbb{N}_0,$$

$$z_n = \sqrt{a}\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{\frac{2^{n+1}-2(-1)^n}{3}}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{\frac{2^n+2(-1)^n}{3}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{\frac{2^{n+1}-2(-1)^n}{3}}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{\frac{2^n+2(-1)^n}{3}}-1}, \quad n \in \mathbb{N}_0.$$

System 34. By using the change of variables (2.2) system (2.37) is transformed to

$$u_{n+1} = u_n^2, \quad v_{n+1} = v_n w_n, \quad w_{n+1} = u_n w_n, \quad n \in \mathbb{N}_0. \tag{3.99}$$

This system is obtained from system (3.23) by interchanging letters v and w. Hence, from (3.15) and (3.24) we have

$$v_n = v_0 w_0^n u_0^{2^n-n-1}, \quad n \in \mathbb{N}_0, \tag{3.100}$$

$$w_n = w_0 u_0^{2^n-1}, \quad n \in \mathbb{N}_0. \tag{3.101}$$

From (2.3), (3.2), (3.100), (3.101) we have that the following theorem holds.

Theorem 3.37. *If $a \neq 0$, then the general solution to system (2.37) is*

$$x_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1}, \quad n \in \mathbb{N}_0,$$

$$y_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-n-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-n-1}\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{n}-1}, \quad n \in \mathbb{N}_0,$$

$$z_n = \sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n-1}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)-1}, \quad n \in \mathbb{N}_0.$$

System 35. By using the change of variables (2.2) system (2.38) is transformed to

$$u_{n+1}=u_n^2,\quad v_{n+1}=v_nw_n,\quad w_{n+1}=w_n^2,\quad n\in\mathbb{N}_0. \tag{3.102}$$

We have that (3.2) and (3.11) hold. By using (3.11) in the second equation in (3.102) we have

$$v_n=v_{n-1}w_{n-1}=v_0\prod_{j=0}^{n-1}w_0^{2^j}=v_0w_0^{2^n-1},\quad n\in\mathbb{N}_0. \tag{3.103}$$

From (2.3), (3.2), (3.11), (3.103) we have that the following theorem holds.

Theorem 3.38. *If $a\neq 0$, then the general solution to system (2.38) is*

$$x_n=\sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1},\quad n\in\mathbb{N}_0,$$

$$y_n=\sqrt{a}\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n-1}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n-1}-1},\quad n\in\mathbb{N}_0,$$

$$z_n=\sqrt{a}\frac{\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^n}-1},\quad n\in\mathbb{N}_0.$$

System 36. By using the change of variables (2.2) system (2.39) is transformed to

$$u_{n+1}=u_n^2,\quad v_{n+1}=v_nw_n,\quad w_{n+1}=v_nw_n,\quad n\in\mathbb{N}_0. \tag{3.104}$$

We have that (3.2) holds and that $v_n=w_n$, $n\in\mathbb{N}$. By using the last relation in the second equation in (3.104) we have

$$v_n=v_{n-1}w_{n-1}=v_{n-1}^2,\quad n\geq 2. \tag{3.105}$$

Hence

$$v_n=v_1^{2^{n-1}}=(v_0w_0)^{2^{n-1}},\quad n\in\mathbb{N}, \tag{3.106}$$

and consequently

$$w_n=(v_0w_0)^{2^{n-1}},\quad n\in\mathbb{N}. \tag{3.107}$$

From (2.3), (3.2), (3.106), (3.107) we have that the following theorem holds.

Theorem 3.39. *If $a\neq 0$, then the general solution to system (2.39) is*

$$x_n=\sqrt{a}\frac{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}+1}{\left(\frac{x_0+\sqrt{a}}{x_0-\sqrt{a}}\right)^{2^n}-1},\quad n\in\mathbb{N}_0,$$

$$y_n=\sqrt{a}\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^{n-1}}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^{n-1}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^{n-1}}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^{n-1}}-1},\quad n\in\mathbb{N},$$

$$z_n=\sqrt{a}\frac{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^{n-1}}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^{n-1}}+1}{\left(\frac{y_0+\sqrt{a}}{y_0-\sqrt{a}}\right)^{2^{n-1}}\left(\frac{z_0+\sqrt{a}}{z_0-\sqrt{a}}\right)^{2^{n-1}}-1},\quad n\in\mathbb{N}.$$

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Limit cycles of planar piecewise linear Hamiltonian differential systems with two or three zones

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Abstract. In this paper, we study the existence of limit cycles in continuous and discontinuous planar piecewise linear Hamiltonian differential system with two or three zones separated by straight lines and such that the linear systems that define the piecewise one have isolated singular points, i.e. centers or saddles. In this case, we show that if the planar piecewise linear Hamiltonian differential system is either continuous or discontinuous with two zones, then it has no limit cycles. Now, if the planar piecewise linear Hamiltonian differential system is discontinuous with three zones, then it has at most one limit cycle, and there are examples with one limit cycle. More precisely, without taking into account the position of the singular points in the zones, we present examples with the unique limit cycle for all possible combinations of saddles and centers.

Keywords: limit cycles, piecewise linear differential system, Hamiltonian systems.

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1 Introduction

The first works on piecewise differential systems appeared in the 1930s, see [1]. This class of systems is very important due to numerous applications, for example in control theory, mechanics, electrical circuits, neurobiology, etc (see for instance the book [7]). Recently, this subject has piqued the attention of researchers in qualitative theory of differential equations and numerous studies about this topic have arisen in the literature (see [6, 15, 19, 20, 30]).

Piecewise linear differential systems are an interesting class of piecewise differential systems and, unlike the smooth case, have a rich dynamic that is far from being fully understood. In addition to numerous applications in various areas of knowledge. In 1990, Lum and Chua [28] conjectured that a continuous piecewise differential systems in the plane with two zones has at most one limit cycle. In 1998 this conjecture was proved by Freire, Ponce, Rodrigo and Torres in [9]. The problem becomes more complicated when we have three zones. Conditions for non existence and existence of one, two or three limit cycles have been obtained,

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see [10,23,32]. However, the maximum number of limit cycles, as far as we know, is not yet known.

In the discontinuous case, the maximum number of limit cycles is not known even in the simplest case, i.e. for piecewise linear differential systems with two zones separated by a straight line. However, important partial results about this problem have been obtained. In summary, the results about the number of limit cycles of discontinuous piecewise linear differential systems with two zones separated by a straight line are given in Table 1.1. The symbol "—" indicates that those cases appear repeated in the table and the empty entries on it correspond to cases not studied in the literature, at least as far as we know.

	F_r	F_v	F_b	S_r	S_r^0	N_v	iN_v	C	C_b
F_r		3	2*	3	2*	3	3	2*	2*
F_v	—	2	2*	2	1*		2	1*	1*
F_b	—	—	1*	2*	1*	2*	2*	1*	1*
S_r	—	—	—	2*	1*	2	2	1*	1*
S_r^0	—	—	—	—	0*	1*	1*	0*	0*
N_v	—	—	—	—	—			1*	1*
iN_v	—	—	—	—	—	—	2	1*	1*
C	—	—	—	—	—	—	—	0*	0*
C_b	—	—	—	—	—	—	—	—	0*

Table 1.1: Lower bounds (Upper bounds*) of the maximum number of limit cycles of discontinuous piecewise linear differential systems with two zones separated by a straight line. Here F_r, F_v, F_b, S_r, S_r^0, N_v, iN_v, C and C_b denote real focus, virtual focus, boundary focus, real saddle, real saddle with zero trace, virtual node, improper node, center and boundary center, respectively.

We denote the lower bounds of the entrances from Table 1.1 by the symbols that indicate its position on the table. For example, the lower bound for the case with a real focus F_r and a virtual focus F_v is detonated by F_rF_v, i.e. $F_rF_v = 3$. A proof for the lower bound F_rF_v can be found in [22]. A proof for the lower bound F_rS_r can be found in [18]. A proof for the lower bounds F_rN_v and F_riN_v can be found in [12]. A proof for the lower bound F_vF_v can be found in [11]. A proof for the lower bound F_vS_r can be found in [36]. A proof for the lower bound F_viN_v can be found in [35]. A proof for the upper bound S_rS_r can be found in [2]. A proof for the lower bounds S_rN_v and S_riN_v can be found in [26]. A proof for the lower bound iN_viN_v can be found in [17]. The other cases listed in Table 1.1 can be found in [21]. In the papers [3,5,13,27,34] we can also find proofs for some lower bounds of Table 1.1.

If the curve between two linear zones is not a straight line it is possible to obtain as many cycles as you want. This fact has been conjectured by Braga and Mello in [4] and firstly proved by Novaes and Ponce in [31]. Exact number of limit cycles, for discontinuous piecewise linear systems with two zones separated by a straight line, were obtained in particular cases. Llibre and Teixeira [24] proved that if the linear systems, that define the piecewise one, has no singular point, then it has at most one limit cycle. Medrado and Torregrossa [29] proved that if the straight line has only crossing sewing points and the piecewise linear system has only a monodronic singular point on it, then the system has at most one limit cycle.

There are a few papers on discontinuous piecewise linear systems with three zones separated by two straight lines (see [8,25,38,39]). In [25], Llibre and Teixeira study the existence of limit cycles for continuous and discontinuous planar piecewise linear differential system with three zones separated by two parallel straight lines and such that the linear systems involved have a unique singular point which are centers. More precisely, in the continuous case, they prove that the piecewise system has no limit cycles. Now, in the discontinuous case, the piecewise system has at most one limit cycle and there are examples with one. Mello, Llibre and Fonseca, in [8], propose a mix of [24] and [25]. They proved that a piecewise linear Hamiltonian systems with three zones separated by two parallel straight lines without singular points have at most one crossing limit cycle.

In this paper, we contribute along these lines, that is, we are interested in studying the existence and the number of limit cycles of piecewise linear differential systems with two or three zones in the plane with the following hypotheses:

(H1) The separations curves are straight lines, and parallel if there are more than one.

(H2) The vector fields which define the piecewise one are linear.

(H3) The vector fields which define the piecewise one are Hamiltonian.

(H4) The vector fields which define the piecewise one have isolated singularities.

Note that, hypotheses (H2), (H3) and (H4) imply that the singular points of the linear systems that define the piecewise differential systems are saddles or centers.

We can classify the systems that satisfy the above hypotheses according to the configuration of their singular points. Thus, denoting the centers by the capital letter C and by S the saddles, in the case of two zones we have systems of the type CC, SC and SS. This is, CC indicates that the singular points of the linear systems that define the piecewise differential system are centers and so on. Following this idea, for three zones, we have the following six class of piecewise linear Hamiltonian systems: CCC, SCC, SCS, CSC, SSS and SSC.

The case with two zones has been study in the literature, i.e. the next theorem is already proved.

Theorem 1.1. *A continuous or discontinuous planar piecewise linear Hamiltonian differential system with two zones separated by a straight line and such that the linear systems that define it have isolated singular points, i.e. centers or saddles, has no limit cycles.*

A proof for Theorem 1.1 is contained in the proofs of Theorem 2 and 4 from [21]. Alternative proofs can also be found in other papers. See the proof of Theorem 1 from [27] for the case where one of the linear systems has a center and the other has a center or saddle, and see the proof of Theorem 3.4 from [16] for the case where the linear systems has saddles.

We include a proof of Theorem 1.1 in Section 3 just for the sake of completeness.

Assuming hypotheses (H1)–(H4), the main results in this paper are the follows:

Theorem 1.2. *A continuous planar piecewise linear Hamiltonian differential system with three zones separated by two parallel straight lines and such that the linear systems that define it have isolated singular points, i.e. centers or saddles, has no limit cycles.*

Theorem 1.3. *A discontinuous planar piecewise linear Hamiltonian differential system with three zones separated by two parallel straight lines and such that the linear systems that define it have isolated singular points, i.e. centers or saddles, has at most one limit cycle.*

Theorems 1.2 and 1.3 have been proved for the particular case in which the linear systems that define the piecewise one has only isolated centers, see [25]. Theorem 1.2 has also been proved for the particular case SCS, see the proof of Lemma 11 from [33]. For the other possibilities, as far as we know, the results of Theorems 1.2 and 1.3 are new.

The paper is organized as follows. In Section 2 we introduce the basic definitions and results. In Section 3 we prove Theorems 1.2–1.3. Examples of discontinuous planar piecewise linear Hamiltonian differential system with three zones separated by two parallel straight lines such that the linear systems that define it have isolated singular points are analyzed in Section 4. That is, we give examples of piecewise linear Hamiltonian systems of type CCC, SCC, SCS, CSC, SSS and SSC with exactly one limit cycle.

2 Preliminary results

In this section, we will present the basic concepts that we need to prove the main results of this paper.

Let $h_i : \mathbb{R}^2 \to \mathbb{R}$, $i = C, L, R$, be the function $h_C(x, y) = x$, $h_L(x, y) = x + 1$ and $h_R(x, y) = x - 1$. By means of rotations, translations and homotheties we can assume without loss of generality that the *switching curve* Σ_C of a piecewise linear system with two zones in the plane is defined as

$$\Sigma_C = h_C^{-1}(0) = \{(x, y) \in \mathbb{R}^2 : x = 0\}.$$

This straight line decomposes the plane in two regions

$$R_L = \{(x, y) \in \mathbb{R}^2 : x < 0\} \quad \text{and} \quad R_R = \{(x, y) \in \mathbb{R}^2 : x > 0\}.$$

Assuming the hypotheses (H2) and (H3), the piecewise linear Hamiltonian vector field with two zones is given by

$$\begin{aligned} X_L(x, y) &= (a_L x + b_L y + \alpha_L, c_L x - a_L y + \beta_L), \qquad x \le 0, \\ X_R(x, y) &= (a_R x + b_R y + \alpha_R, c_R x - a_R y + \beta_R), \qquad x > 0. \end{aligned} \tag{2.1}$$

Note that the Hamiltonian functions that determine the vector field (2.1) are

$$\begin{aligned} H_L(x, y) &= \frac{b_L}{2} y^2 - \frac{c_L}{2} x^2 + a_L xy + \alpha_L y - \beta_L x, \qquad x \le 0, \\ H_R(x, y) &= \frac{b_R}{2} y^2 - \frac{c_R}{2} x^2 + a_R xy + \alpha_R y - \beta_R x, \qquad x > 0. \end{aligned} \tag{2.2}$$

Assuming the hypothesis (H1), by means of rotations, translations and homotheties we can assume without loss of generality, for the case with three zones, that the switching curves Σ_L and Σ_R are given by

$$\Sigma_L = h_L^{-1}(0) = \{(x, y) \in \mathbb{R}^2 : x = -1\},$$

and

$$\Sigma_R = h_R^{-1}(0) = \{(x, y) \in \mathbb{R}^2 : x = 1\}.$$

This straight lines decomposes the plane in three regions

$$R_L = \{(x, y) \in \mathbb{R}^2 : x < -1\}, \quad R_C = \{(x, y) \in \mathbb{R}^2 : -1 < x < 1\},$$

and

$$R_R = \{(x, y) \in \mathbb{R}^2 : x > 1\}.$$

Assuming the hypotheses (H2) and (H3), the piecewise linear Hamiltonian vector field with three zones is give by

$$\begin{aligned} X_L(x,y) &= (a_L x + b_L y + \alpha_L, c_L x - a_L y + \beta_L), && x \le -1, \\ X_C(x,y) &= (a_C x + b_C y + \alpha_C, c_C x - a_C y + \beta_C), && -1 \le x \le 1, \\ X_R(x,y) &= (a_R x + b_R y + \alpha_R, c_R x - a_R y + \beta_R), && x \ge 1. \end{aligned} \tag{2.3}$$

The Hamiltonian functions that determine the vector field (2.3) are

$$\begin{aligned} H_L(x,y) &= \frac{b_L}{2}y^2 - \frac{c_L}{2}x^2 + a_L xy + \alpha_L y - \beta_L x, && x \le -1, \\ H_C(x,y) &= \frac{b_C}{2}y^2 - \frac{c_C}{2}x^2 + a_C xy + \alpha_C y - \beta_C x, && -1 \le x \le 1, \\ H_R(x,y) &= \frac{b_R}{2}y^2 - \frac{c_R}{2}x^2 + a_R xy + \alpha_R y - \beta_R x, && x \ge 1. \end{aligned} \tag{2.4}$$

We will use the vector field X_L and the switching curve Σ_L in the next definitions. However, we can easily adapt the definitions to the vector fields X_C and X_R and the switching curves Σ_C and Σ_R.

The derivative of function h_L in the direction of the vector field X_L, i.e., the expression

$$X_L h_L(p) = \langle X_L(p), \nabla h_L(p) \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{R}^2$, characterize the contact between the vector field X_L and the switching curve Σ_L.

We distinguish the following subsets of Σ_L (the same for Σ_C and Σ_R)

- Crossing set:
$$\Sigma_L^c = \{p \in \Sigma_L : X_L h_L(p) \cdot X_C h_L(p) > 0\};$$
- Sliding set:
$$\Sigma_L^s = \{p \in \Sigma_L : X_L h_L(p) > 0, X_C h_L(p) < 0\};$$
- Escaping set:
$$\Sigma_L^e = \{p \in \Sigma_L : X_L h_L(p) < 0, X_C h_L(p) > 0\}.$$

In a piecewise vector field with two or three zones in the plane, the limit cycles can be of two types: sliding limit cycles or crossing limit cycles; the first one contain some segment of sliding or escaping sets, and the second one does not contain any segments of sliding or escaping sets. In this paper, we only study the crossing limit cycles. In what follows, when we talk about limit cycles, we are talking about crossing limit cycles.

Piecewise vector field (2.1) is called continuous if

$$X_L(p) = X_R(p), \quad \forall p \in \Sigma_C.$$

Otherwise, it is called discontinuous. Similarly, piecewise vector field (2.3) is called continuous if

$$\begin{aligned} X_L(p) &= X_C(p), \quad \forall p \in \Sigma_L \quad \text{and} \\ X_C(q) &= X_R(q), \quad \forall q \in \Sigma_R. \end{aligned}$$

Otherwise, it is called discontinuous.

3 Proof of Theorems 1.1–1.3

This section is devoted to present the proof of main results.

Proof of Theorem 1.1. Consider a discontinuous piecewise linear Hamiltonian vector field with two zones separated by a straight line, such that the linear vector fields, that define it, have isolated singular points. That is, we have piecewise vector field (2.1), with $a_i^2 + b_i c_i \neq 0$, for $i = L, R$. If the piecewise linear vector field has a periodic orbit, then it intersects the straight line $x = 0$ at two points, $(0, y_0)$ and $(0, y_1)$, with $y_1 < y_0$, satisfying

$$H_R(0, y_1) = H_R(0, y_0),$$
$$H_L(0, y_0) = H_L(0, y_1),$$

where H_L and H_R are given by (2.2). More precisely, we have the equations

$$-\frac{1}{2}(y_0 - y_1)(b_R(y_0 + y_1) + 2\alpha_R) = 0,$$
$$\frac{1}{2}(y_0 - y_1)(b_L(y_0 + y_1) + 2\alpha_L) = 0.$$

As $y_1 < y_0$, if $b_R = 0$ and $\alpha_R \neq 0$ or $b_L = 0$ and $\alpha_L \neq 0$ the above system has no solutions. If $b_R = \alpha_R = 0$ and $b_L \neq 0$ the solution (y_0, y_1) of the above system with $y_1 < y_0$ satisfies $y_0 = -(b_L y_1 + 2\alpha_L)/b_L$, with arbitrary y_1. If $b_L = \alpha_L = 0$ and $b_R \neq 0$ the solution (y_0, y_1) of the above system with $y_1 < y_0$ satisfies $y_0 = -(b_R y_1 + 2\alpha_R)/b_R$, with arbitrary y_1. If $b_L b_R \neq 0$, then the above system has a solution (y_0, y_1) with $y_1 < y_0$ only when $b_L = b_R = b$ and $\alpha_L = \alpha_R = \alpha$. Moreover, $y_0 = (-b y_1 + 2\alpha)/b$ with arbitrary y_1. If $b_R = b_L = \alpha_R = \alpha_L = 0$, then the system has infinitely many solutions. Therefore, the piecewise linear vector field (2.1) has no periodic orbits or has a continuum of periodic orbits, and consequently, it has no limit cycle.

Note that the continuous case is a constraint of the discontinuous one. In fact, the continuous condition is given by

$$X_R(0, y) = X_L(0, y), \quad \forall y \in \mathbb{R}.$$

which implies

$$a_R = a_L = a, \quad b_R = b_L = b, \quad \alpha_R = \alpha_L = \alpha \quad \text{and} \quad \beta_R = \beta_L = \beta.$$ □

Proof of Theorem 1.2. Consider a continuous piecewise linear Hamiltonian vector field with three zones separated by two parallel straight lines, such that the linear vector fields, that define it, have isolated singular points. That is, we have piecewise vector fields (2.3), with $a_i^2 + b_i c_i \neq 0$, for $i = L, C, R$, and due to continuity

$$X_R(1, y) = X_C(1, y) \quad \text{and} \quad X_C(-1, y) = X_L(-1, y), \quad \forall y \in \mathbb{R}.$$

These equalities imply that

$$a_R = a_C = a_L = a, \quad b_R = b_C = b_L = b, \quad \alpha_R = \alpha_C = \alpha_L = \alpha$$

and

$$\beta_R - \beta_C - c_C + c_R = \beta_L - \beta_C - c_L + c_C = 0.$$

By Theorem 1.1, the piecewise linear vector field has no limit cycles contained in two zones. Thus, if the piecewise linear vector field has a periodic orbit, then it intersects the straight lines $x = \pm 1$ at four points, $(1, y_0)$, $(1, y_1)$, with $y_1 < y_0$, and $(-1, y_2)$, $(-1, y_3)$, with $y_2 < y_3$, respectively, satisfying

$$\begin{aligned} H_R(1,y_1) &= H_R(1,y_0), \\ H_C(1,y_0) &= H_C(-1,y_3), \\ H_L(-1,y_3) &= H_L(-1,y_2), \\ H_C(-1,y_2) &= H_C(1,y_1), \end{aligned} \tag{3.1}$$

where H_L, H_C and H_R are given by (2.4). More precisely, we have the equations

$$-\frac{1}{2}(y_0 - y_1)(b(y_0 + y_1) + 2(a + \alpha)) = 0, \tag{3.2}$$

$$a(y_0 + y_3) + \frac{1}{2}(y_0 - y_3)(b(y_0 + y_3) + 2\alpha) - 2\beta_C = 0, \tag{3.3}$$

$$-\frac{1}{2}(y_2 - y_3)(b(y_2 + y_3) - 2(a - \alpha)) = 0, \tag{3.4}$$

$$-a(y_1 + y_2) - \frac{1}{2}(y_1 - y_2)(b(y_1 + y_2) + 2\alpha) + 2\beta_C = 0. \tag{3.5}$$

As $y_1 < y_0$, $y_2 < y_3$ and $a^2 + bc_i \neq 0$, for $i = L, C, R$, if either $b = 0$ and $a + \alpha \neq 0$ or $b = 0$ and $a + \alpha = 0$ the above system has no solutions. If $b \neq 0$, as $y_1 < y_0$ and $y_2 < y_3$, from equation (3.2) we can obtain y_0 as a function of y_1, i.e.

$$y_0 = \frac{-by_1 - 2(a + \alpha)}{b}. \tag{3.6}$$

Now, from equation (3.4) we can obtain y_2 as a function of y_3, i.e.

$$y_2 = \frac{-by_3 - 2(\alpha - a)}{b}. \tag{3.7}$$

Substituting (3.6) and (3.7) in equations (3.3) and (3.5), respectively, we obtain a solution (y_0, y_1, y_2, y_3) of the system (3.1) satisfying $y_1 < y_0$ and $y_2 < y_3$, given by $(\varphi_1(y_1), y_1, \varphi_2(y_1), \varphi_3(y_1))$, where

$$\begin{aligned} \varphi_1(y_1) &= \frac{-by_1 - 2(a + \alpha)}{b}, \\ \varphi_2(y_1) &= \frac{a - \alpha + \sqrt{b^2 y_1^2 + 2b(a + \alpha)y_1 + (a - \alpha)^2 - 4b\beta_C}}{b}, \\ \varphi_3(y_1) &= \frac{a - \alpha + \sqrt{b^2 y_1^2 + 2b(a + \alpha)y_1 + (a - \alpha)^2 - 4b\beta_C}}{b}, \end{aligned}$$

with arbitrary y_1. Note that the inequality $b^2 y_1^2 + 2b(a + \alpha)y_1 + (a - \alpha)^2 - 4b\beta_C \leq 0$ for all $y_1 \in \mathbb{R}$ is not possible. Therefore, the piecewise linear vector field (2.3) has no periodic orbits or has a continuum of periodic orbits, and consequently, it has no limit cycle. □

Proof of Theorem 1.3. Consider a discontinuous piecewise linear Hamiltonian vector field with three zones separated by two parallel straight lines, such that the linear vector fields, that define it, have isolated singular points. That is, we have piecewise vector fields (2.3), with $-a_i^2 - b_i c_i \neq 0$, for $i = L, C, R$. By Theorem 1.1, the piecewise linear vector field has no limit cycles contained in two zones. Thus, if the piecewise linear vector field has a periodic orbit, then it intersects the straight lines $x = \pm 1$ at four points, $(1, y_0)$, $(1, y_1)$, with $y_1 < y_0$, and $(-1, y_2)$, $(-1, y_3)$, with $y_2 < y_3$, respectively, satisfying

$$\begin{aligned} H_R(1, y_1) &= H_R(1, y_0), \\ H_C(1, y_0) &= H_C(-1, y_3), \\ H_L(-1, y_3) &= H_L(-1, y_2), \\ H_C(-1, y_2) &= H_C(1, y_1), \end{aligned} \tag{3.8}$$

where H_L, H_C and H_R are given by (2.4). More precisely, we have the equations

$$\frac{1}{2}(y_1 - y_0)(b_R(y_0 + y_1) + 2(a_R + \alpha_R)) = 0, \tag{3.9}$$

$$\frac{1}{2}(y_0 - y_3)(b_C(y_0 + y_3) + 2\alpha_C) - 2\beta_C + a_C(y_0 + y_3) = 0, \tag{3.10}$$

$$\frac{1}{2}(y_3 - y_2)(b_L(y_2 + y_3) - 2(a_L - \alpha_L)) = 0, \tag{3.11}$$

$$\frac{1}{2}(y_2 - y_1)(b_C(y_1 + y_2) + 2\alpha_C) + 2\beta_C - a_C(y_1 + y_2) = 0. \tag{3.12}$$

To determine all the solutions of the above systems, restricted to the conditions $y_1 < y_0$, $y_2 < y_3$ and $a_i^2 + b_i c_i \neq 0$, for $i = L, C, R$, we distinguish two cases. In the first case we assume that $b_R b_L b_C = 0$. For this cases, system (3.9)–(3.12) has no solutions when

- $b_R = 0$ and $a_R + \alpha_R \neq 0$;
- $b_L = 0$ and $a_L - \alpha_L \neq 0$;
- $b_R = a_R + \alpha_R = b_L = a_L - \alpha_L = b_C = \alpha_C - a_C = 0$;
- $b_R = a_R + \alpha_R = b_C = \alpha_C - a_C = 0$ and $b_L \neq 0$;
- $b_L = a_L - \alpha_L = b_C = \alpha_C - a_C = 0$ and $b_R \neq 0$;
- $b_C = 0$, $b_R b_L \neq 0$ and $b_R \alpha_C (a_L - \alpha_L) + a_C b_R (\alpha_L - a_L) + b_L (a_R + \alpha_R)(a_C + \alpha_C) + 2 b_L b_R \beta_C \neq 0$;

and it has infinitely many solutions when

- $b_R = a_R + \alpha_R = b_L = a_L - \alpha_L = b_C = 0$ and $\alpha_C - a_C \neq 0$;
- $b_R = a_R + \alpha_R = b_L = a_L - \alpha_L = 0$ and $b_C \neq 0$;
- $b_R = a_R + \alpha_R = b_C = 0$, $\alpha_C - a_C \neq 0$ and $b_L \neq 0$;
- $b_R = a_R + \alpha_R = 0$ and $b_L b_C \neq 0$;
- $b_L = a_L - \alpha_L = b_C = 0$, $b_R \neq 0$ and $\alpha_C - a_C \neq 0$;
- $b_L = a_L - \alpha_L = 0$ and $b_R b_C \neq 0$;
- $b_C = 0$, $b_R b_L \neq 0$ and $b_R \alpha_C (a_L - \alpha_L) + a_C b_R (\alpha_L - a_L) + b_L (a_R + \alpha_R)(a_C + \alpha_C) + 2 b_L b_R \beta_C = 0$.

In the second case, we assume that $b_L b_C b_R \neq 0$. From equation (3.9), we can obtain y_0 as a function of y_1, i.e.

$$y_0 = \frac{-b_R y_1 - 2(a_R + \alpha_R)}{b_R}. \tag{3.13}$$

Now, from equation (3.11), we can obtain y_2 as a function of y_3, i.e.

$$y_2 = \frac{-b_L y_3 - 2(\alpha_L - a_L)}{b_L}. \tag{3.14}$$

Substituting (3.13) and (3.14) in equations (3.10) and (3.12), respectively, we obtain the equations of two hyperbolas in the $y_1 y_3$ plane, given by

$$\begin{aligned} \frac{(y_1 - A)^2}{K} - \frac{(y_3 - B)^2}{K} - C = 0, \\ \frac{(y_1 - D)^2}{K} - \frac{(y_3 - E)^2}{K} - C = 0, \end{aligned} \tag{3.15}$$

with

$$K = \frac{2}{b_C}, \quad A = \frac{b_R(a_C + \alpha_C) - 2b_C(a_R + \alpha_R)}{b_C b_R},$$

$$B = \frac{a_C - \alpha_C}{b_C}, \quad C = \frac{2(a_C\alpha_C + b_C\beta_C)}{b_C},$$

$$D = -\frac{(a_C + \alpha_C)}{b_C} \quad \text{and} \quad E = \frac{b_L(\alpha_C - a_C) - 2b_C(\alpha_L - a_L)}{b_C b_L}.$$

Note that the system (3.15) is equivalent to the system

$$\begin{aligned} & y_1^2 - 2Ay_1 + A^2 - y_3^2 + 2By_3 - B^2 - KC = 0, \\ & 2(A - D)y_1 + 2(E - B)y_3 + D^2 - E^2 + B^2 - A^2 = 0. \end{aligned} \tag{3.16}$$

The system above eventually could have infinitely many solutions (y_1, y_3), for instance when $A = D$ and $B = E$. In this case, the piecewise linear vector field (2.3) has a continuum of periodic orbits, and consequently, it has no limit cycle. Suppose that system (3.16) has finitely many solutions. According to Bezout's Theorem, if a system of polynomial equations has finitely many solutions, then the number of its solutions is at most the product of the degrees of the polynomials, that for system (3.16) is two. Therefore, the two hyperbolas above intersect at most two points. Note that, by (3.9)–(3.12), if (y_0, y_1, y_2, y_3) is solution of the system (3.8) then (y_1, y_0, y_3, y_2) is also a solution. However, for $y_1 < y_0$ and $y_2 < y_3$ we have at most a single solution. Therefore, the piecewise linear vector field (2.3) can have at most one limit cycle. □

4 Examples

In this section, we will give some examples of discontinuous planar piecewise linear Hamiltonian differential system with three zones separated by two parallel straight lines with one limit cycle, such that the linear systems that define it have isolated singular points. That is, we given examples of piecewise linear Hamiltonian systems of type CCC, SCC, SCS, CSC, SSS and SSC with exactly one limit cycle. In [25], the authors presented an example of a discontinuous piecewise linear differential system of type CCC with exactly one limit cycle. Here we will show another example for this case.

Example 4.1 (Case CCC)**.** Consider the discontinuous planar piecewise linear Hamiltonian vector field (2.3) with $a_L = 4$, $b_L = 8$, $\alpha_L = 3/2$, $c_L = -5/2$, $\beta_L = 11/4$, $a_C = 0$, $b_C = 2$, $\alpha_C = \beta_C = 2/3$, $c_C = -2$, $a_R = 4$, $b_R = 2$, $c_R = -10$ and $\alpha_R = \beta_R = -4$. The eigenvalues of the linear part of X_i, $i = L, C, R$, from (2.3) for this case, are $\pm 2i$, $\pm 2i$ and $\pm 2i$, respectively, i.e. we have three centers. Therefore, a candidate to limit cycle of vector field (2.3), in this case, correspond to the solution of system (3.8), i.e.

$$\begin{aligned}
(y_1 - y_0)(y_1 + y_0) &= 0,\\
\frac{1}{3}(y_0(2+3y_0) - y_3(2+3y_3) - 4) &= 0,\\
\frac{1}{2}(y_3 - y_2)(8(y_2+y_3) - 5) &= 0,\\
\frac{1}{3}(4 - y_1(2+3y_1) + y_2(2+3y_2)) &= 0.
\end{aligned}$$

After some computations, the unique solution (y_0, y_1, y_2, y_3) of the above system, satisfying the condition $y_1 < y_0$ and $y_2 < y_3$, is given by

$$\left(\frac{31}{48}\sqrt{\frac{1259}{235}},\ -\frac{31}{48}\sqrt{\frac{1259}{235}},\ \frac{5}{16} - \frac{1}{3}\sqrt{\frac{1259}{235}},\ \frac{5}{16} + \frac{1}{3}\sqrt{\frac{1259}{235}}\right).$$

The points $(-1, y_2), (-1, y_3) \in \Sigma_L$ and $(1, y_0), (1, y_1) \in \Sigma_R$ are crossing points because

$$\begin{aligned}
\langle X_L(-1,y_2),(1,0)\rangle \cdot \langle X_C(-1,y_2),(1,0)\rangle &\approx 1.5518 > 0,\\
\langle X_L(-1,y_3),(1,0)\rangle \cdot \langle X_C(-1,y_3),(1,0)\rangle &\approx 17.4969 > 0,\\
\langle X_C(1,y_0),(1,0)\rangle \cdot \langle X_R(1,y_0),(1,0)\rangle &\approx 10.9315 > 0,\\
\langle X_C(1,y_1),(1,0)\rangle \cdot \langle X_R(1,y_1),(1,0)\rangle &\approx 6.9452 > 0.
\end{aligned}$$

The orbit $(x_R(t), y_R(t))$ of X_R, such that $(x_R(0), y_R(0)) = (1, y_0)$, is given by

$$\begin{aligned}
x_R(t) &= 7\cos(2t) + \frac{31}{48}\sqrt{\frac{1259}{235}}\sin(2t) - 6,\\
y_R(t) &= \left(\frac{31}{48}\sqrt{\frac{1259}{235}} - 14\right)\cos(2t) - \left(7 + \frac{31}{24}\sqrt{\frac{1259}{235}}\right)\sin(2t) + 14.
\end{aligned}$$

The orbit $(x_{C_1}(t), y_{C_1}(t))$ of X_C, such that $(x_{C_1}(0), y_{C_1}(0)) = (1, y_1)$, is given by

$$\begin{aligned}
x_{C_1}(t) &= \frac{2}{3}\cos(2t) + \left(\frac{1}{3} - \frac{31}{48}\sqrt{\frac{1259}{235}}\right)\sin(2t) + \frac{1}{3},\\
y_{C_1}(t) &= \left(\frac{1}{3} - \frac{31}{48}\sqrt{\frac{1259}{235}}\right)\cos(2t) - \frac{1}{3}(1 + 4\cos(t)\sin(t)).
\end{aligned}$$

The orbit $(x_L(t), y_L(t))$ of X_L, such that $(x_L(0), y_L(0)) = (-1, y_2)$, is given by

$$\begin{aligned}
x_L(t) &= -8\cos(2t) - \frac{4}{3}\sqrt{\frac{1259}{235}}\sin(2t) + 7,\\
y_L(t) &= \left(4 - \frac{1}{3}\sqrt{\frac{1259}{235}}\right)\cos(2t) + \left(4 + \frac{4}{3}\sqrt{\frac{1259}{235}}\right)\cos(t)\sin(t) - \frac{59}{16}.
\end{aligned}$$

The orbit $(x_{C_2}(t), y_{C_2}(t))$ of X_C, such that $(x_{C_2}(0), y_{C_2}(0)) = (-1, y_3)$, is given by

$$x_{C_2}(t) = -\frac{4}{3}\cos(2t) + \left(\frac{31}{48} + \frac{1}{3}\sqrt{\frac{1259}{235}}\right)\sin(2t) + \frac{1}{3},$$
$$y_{C_2}(t) = \left(\frac{31}{48} + \frac{1}{3}\sqrt{\frac{1259}{235}}\right)\cos(2t) + \frac{1}{3}(8\cos(t)\sin(t) - 1).$$

The fly time of the orbit $(x_R(t), y_R(t))$, from $(1, y_0) \in \Sigma_R$ to $(1, y_1) \in \Sigma_R$, is

$$t_R = \frac{1}{2}\arctan\left(\frac{20832\sqrt{295865}}{25320661}\right).$$

The fly time of the orbit $(x_{C_1}(t), y_{C_1}(t))$, from $(1, y_1) \in \Sigma_R$ to $(-1, y_2) \in \Sigma_L$, is

$$t_{C_1} = \frac{\pi}{2} - \frac{1}{2}\arctan\left(\frac{96(10810 + \sqrt{295865})}{496001}\right).$$

The fly time of the orbit $(x_L(t), y_L(t))$, from $(-1, y_2) \in \Sigma_L$ to $(-1, y_3) \in \Sigma_L$, is

$$t_L = \frac{1}{2}\arctan\left(\frac{12\sqrt{295865}}{7201}\right).$$

Finally, the fly time of the orbit $(x_{C_2}(t), y_{C_2}(t))$, from $(-1, y_3) \in \Sigma_L$ to $(1, y_0) \in \Sigma_R$, is

$$t_{C_2} = -\frac{1}{2}\arctan\left(\frac{96(\sqrt{295865} - 10810)}{496001}\right).$$

Using the Mathematica software (see [37]), we can draw the orbits $(x_i(t), y_i(t))$ for the time $t \in [0, t_i]$, $i = R, L, C_1, C_2$, i.e. we obtain the limit cycle given in Figure 4.1 (a). Figure 4.1 (b) was made with the help of P5 software (see [14]), and provides the phase portrait of vector field (2.3) in this case (the symbol $\circ$ indicates an invisible singular point).

Example 4.2 (Case SCC)**.** Consider the discontinuous planar piecewise linear Hamiltonian vector field (2.3) with $a_L = b_L = 1$, $\alpha_L = 2/3$, $c_L = 35$, $\beta_L = 214/3$, $a_C = 0$, $b_C = 2$, $\alpha_C = \beta_C = 2/3$, $c_C = -2$, $a_R = 4$, $b_R = 2$, $\alpha_R = \beta_R = -4$ and $c_R = -10$. The eigenvalues of the linear part of X_i, $i = L, C, R$, from (2.3) for this case, are ± 6, $\pm 2i$ and $\pm 2i$, respectively, i.e. we have one saddle and two centers. In this case, as in Example 4.1, the unique solution (y_0, y_1, y_2, y_3) of system (3.8) satisfying the condition $y_1 < y_0$ and $y_2 < y_3$, which is given by

$$\left(\frac{2\sqrt{5}}{3}, -\frac{2\sqrt{5}}{3}, \frac{1-\sqrt{5}}{3}, \frac{1+\sqrt{5}}{3}\right),$$

correspond to the unique limit cycle of vector field (2.3).

Note that the points $(-1, y_2), (-1, y_3) \in \Sigma_L$ and $(1, y_0), (1, y_1) \in \Sigma_R$ are crossing points.

Now, we can compute: the orbit $(x_R(t), y_R(t))$ of X_R with $(x_R(0), y_R(0)) = (1, y_0)$; the orbit $(x_{C_1}(t), y_{C_1}(t))$ of X_C with $(x_{C_1}(0), y_{C_1}(0)) = (1, y_1)$; the orbit $(x_L(t), y_L(t))$ of X_L with $(x_L(0), y_L(0)) = (-1, y_2)$; and the orbit $(x_{C_2}(t), y_{C_2}(t))$ of X_C, with $(x_{C_2}(0), y_{C_2}(0)) = (-1, y_3)$. We can also compute the fly times of the orbits: $(x_R(t), y_R(t))$ from $(1, y_0) \in \Sigma_R$ to $(1, y_1) \in \Sigma_R$;

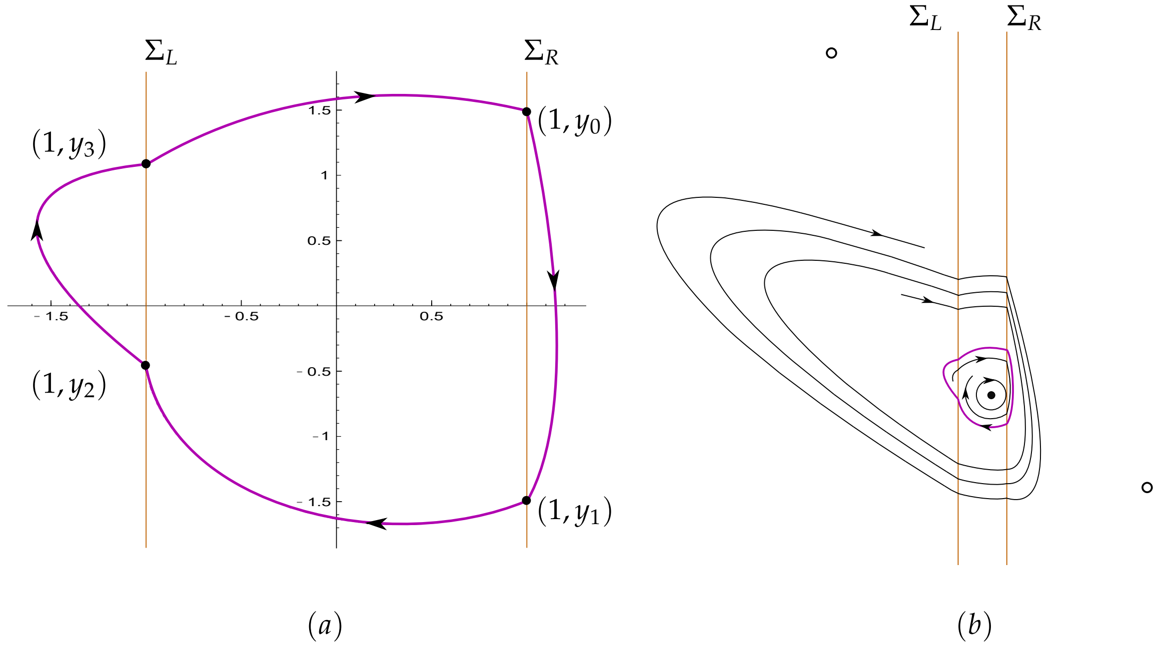


Figure 4.1: The limit cycle of vector field (2.3) with $a_L = 4$, $b_L = 8$, $\alpha_L = 3/2$, $c_L = -5/2$, $\beta_L = 11/4$, $a_C = 0$, $b_C = 2$, $\alpha_C = \beta_C = 2/3$, $c_C = -2$, $a_R = 4$, $b_R = 2$, $c_R = -10$ and $\alpha_R = \beta_R = -4$.

$(x_{C_1}(t), y_{C_1}(t))$ from $(1, y_1) \in \Sigma_R$ to $(-1, y_2) \in \Sigma_L$; $(x_L(t), y_L(t))$ from $(-1, y_2) \in \Sigma_L$ to $(-1, y_3) \in \Sigma_L$; and $(x_{C_2}(t), y_{C_2}(t))$ from $(-1, y_3) \in \Sigma_L$ to $(1, y_0) \in \Sigma_R$. Hence, using the *Mathematica* software, we can draw the orbits $(x_i(t), y_i(t))$ for the time $t \in [0, t_i]$, $i = R, L, C_1, C_2$, i.e. we obtain the limit cycle given in Figure 4.2 (a). The Figure 4.2 (b) has been made with the help of P5 software, and provides the phase portrait of vector field (2.3) in this case.

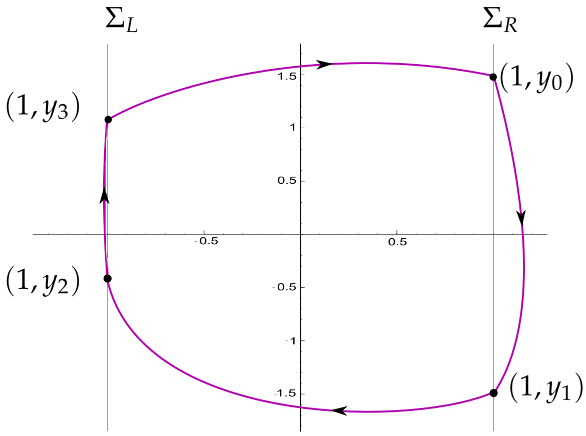
Example 4.3 (Case SCS)**.** Consider the discontinuous planar piecewise linear Hamiltonian vector field (2.3) with $a_L = b_L = 1$, $\alpha_L = 3/5$, $c_L = 35$, $\beta_L = 357/5$, $a_C = 0$, $b_C = 2$, $\alpha_C = \beta_C = 1$, $c_C = -2$, $a_R = b_R = 1$, $\alpha_R = -1$, $c_R = 15$ and $\beta_R = -31$. The eigenvalues of the linear part of X_i, $i = L, C, R$, from (2.3) for this case, are ± 6, $\pm 2i$ and ± 4, respectively, i.e. we have two saddles and one center. In this case, as in Example 4.1, the unique solution (y_0, y_1, y_2, y_3) of system (3.8) satisfying the condition $y_1 < y_0$ and $y_2 < y_3$, which is given by

$$\left(\frac{18}{5}\sqrt{\frac{2}{7}}, -\frac{18}{5}\sqrt{\frac{2}{7}}, \frac{2}{5} - 2\sqrt{\frac{2}{7}}, \frac{2}{5} + 2\sqrt{\frac{2}{7}}\right),$$

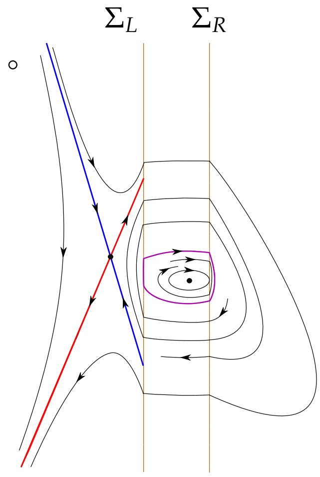
correspond to the unique limit cycle of vector field (2.3).

Note that the points $(-1, y_2), (-1, y_3) \in \Sigma_L$ and $(1, y_0), (1, y_1) \in \Sigma_R$ are crossing points.

Now, we can compute: the orbit $(x_R(t), y_R(t))$ of X_R with $(x_R(0), y_R(0)) = (1, y_0)$; the orbit $(x_{C_1}(t), y_{C_1}(t))$ of X_C with $(x_{C_1}(0), y_{C_1}(0)) = (1, y_1)$; the orbit $(x_L(t), y_L(t))$ of X_L with $(x_L(0), y_L(0)) = (-1, y_2)$; and the orbit $(x_{C_2}(t), y_{C_2}(t))$ of X_C, with $(x_{C_2}(0), y_{C_2}(0)) = (-1, y_3)$. We can also compute the fly times of the orbits: $(x_R(t), y_R(t))$ from $(1, y_0) \in \Sigma_R$ to $(1, y_1) \in \Sigma_R$; $(x_{C_1}(t), y_{C_1}(t))$ from $(1, y_1) \in \Sigma_R$ to $(-1, y_2) \in \Sigma_L$; $(x_L(t), y_L(t))$ from $(-1, y_2) \in \Sigma_L$ to $(-1, y_3) \in \Sigma_L$; and $(x_{C_2}(t), y_{C_2}(t))$ from $(-1, y_3) \in \Sigma_L$ to $(1, y_0) \in \Sigma_R$. Hence, using the mathematica software, we can draw the orbits $(x_i(t), y_i(t))$ for the time $t \in [0, t_i]$, $i = R, L, C_1, C_2$, i.e. we obtain the limit cycle given in Figure 4.3 (a). The Figure 4.3 (b) has been made with the help of P5 software, and provides the phase portrait of vector field (2.3) in this case.



(a)



(b)

Figure 4.2: The limit cycle of vector field (2.3) with $a_L = b_L = 1$, $\alpha_L = 2/3$, $c_L = 35$, $\beta_L = 214/3$, $a_C = 0$, $b_C = 2$, $\alpha_C = \beta_C = 2/3$, $c_C = -2$, $a_R = 4$, $b_R = 2$, $\alpha_R = \beta_R = -4$ and $c_R = -10$.

Example 4.4 (Case CSC)**.** Consider the discontinuous planar piecewise linear Hamiltonian vector field (2.3) with $a_L = 4$, $b_L = 8$, $\alpha_L = 2$, $c_L = -5/2$, $\beta_L = 5/2$, $a_C = 2/5$, $b_C = 24/5$, $\alpha_C = -9/5$, $c_C = 4/5$, $\beta_C = -4/15$, $a_R = 8$, $b_R = 10$ and $\alpha_R = c_R = \beta_R = -8$. The eigenvalues of the linear part of X_i, $i = L, C, R$, from (2.3) for this case, are $\pm 2i$, ± 2 and $\pm 4i$, respectively, i.e. we have one saddle and two centers. In this case, as in Example 4.1, the unique solution (y_0, y_1, y_2, y_3) of system (3.8) satisfying the condition $y_1 < y_0$ and $y_2 < y_3$, which is given by

$$\left(\frac{5}{12}\sqrt{\frac{7}{3}}, -\frac{5}{12}\sqrt{\frac{7}{3}}, \frac{1}{4} - \frac{7\sqrt{21}}{36}, \frac{1}{4} + \frac{7\sqrt{21}}{36}\right),$$

correspond to the unique limit cycle of vector field (2.3).

Note that the points $(-1, y_2), (-1, y_3) \in \Sigma_L$ and $(1, y_0), (1, y_1) \in \Sigma_R$ are crossing points.

Now, we can compute: the orbit $(x_R(t), y_R(t))$ of X_R with $(x_R(0), y_R(0)) = (1, y_0)$; the orbit $(x_{C_1}(t), y_{C_1}(t))$ of X_C with $(x_{C_1}(0), y_{C_1}(0)) = (1, y_1)$; the orbit $(x_L(t), y_L(t))$ of X_L with $(x_L(0), y_L(0)) = (-1, y_2)$; and the orbit $(x_{C_2}(t), y_{C_2}(t))$ of X_C, with $(x_{C_2}(0), y_{C_2}(0)) = (-1, y_3)$. We can also compute the fly times of the orbits: $(x_R(t), y_R(t))$ from $(1, y_0) \in \Sigma_R$ to $(1, y_1) \in \Sigma_R$; $(x_{C_1}(t), y_{C_1}(t))$ from $(1, y_1) \in \Sigma_R$ to $(-1, y_2) \in \Sigma_L$; $(x_L(t), y_L(t))$ from $(-1, y_2) \in \Sigma_L$ to $(-1, y_3) \in \Sigma_L$; and $(x_{C_2}(t), y_{C_2}(t))$ from $(-1, y_3) \in \Sigma_L$ to $(1, y_0) \in \Sigma_R$. Hence, using the *Mathematica* software, we can draw the orbits $(x_i(t), y_i(t))$ for the time $t \in [0, t_i]$, $i = R, L, C_1, C_2$, i.e. we obtain the limit cycle given in Figure 4.4 (a). The Figure 4.4 (b) has been made with the help of P5 software, and provides the phase portrait of vector field (2.3) in this case.

Example 4.5 (Case SSS)**.** Consider the discontinuous planar piecewise linear Hamiltonian vector field (2.3) with $a_L = \alpha_L = -2/3$, $b_L = 4/3$, $c_L = 8/3$, $\beta_L = 35/3$, $a_C = 2/11$, $b_C = 120/11$, $\alpha_C = -41/11$, $c_C = 4/11$, $\beta_C = -4/33$, $a_R = -2/11$, $b_R = 4/11$, $\alpha_R = 1/5$, $c_R = 120/11$ and $\beta_R = -749/55$. The eigenvalues of the linear part of X_i, $i = L, C, R$, from (2.3) for this case, are ± 2, ± 2 and ± 2, respectively, i.e. we have three saddles. In this case, as in Example 4.1, the unique solution (y_0, y_1, y_2, y_3) of system (3.8) satisfying the condition $y_1 < y_0$ and $y_2 < y_3$,

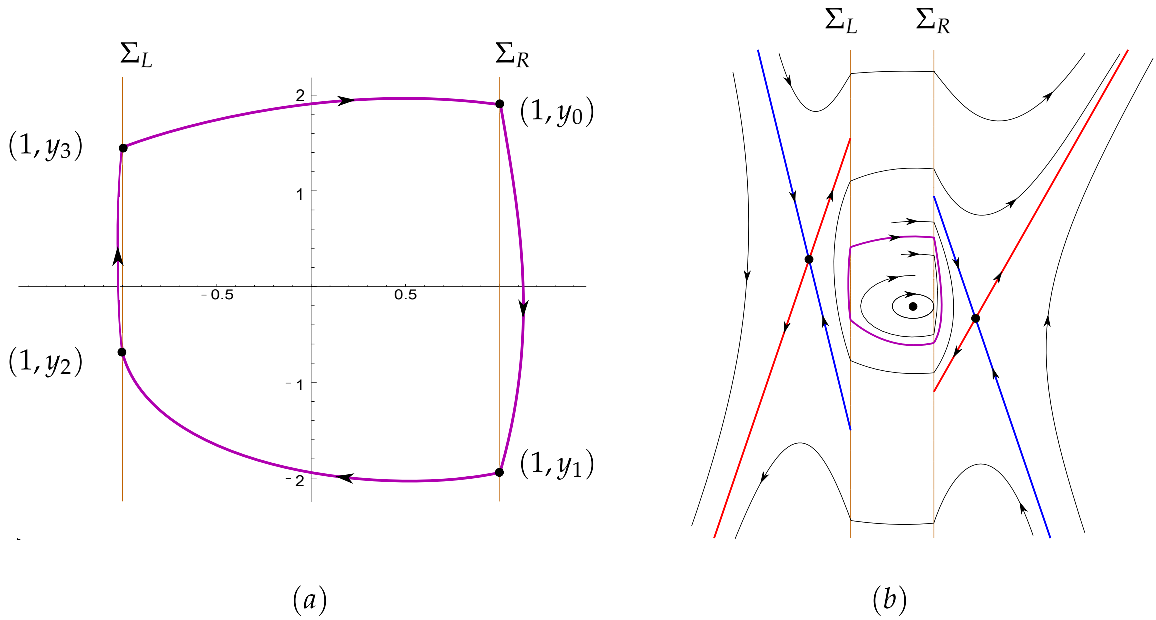


Figure 4.3: The limit cycle of vector field (2.3) with $a_L = b_L = 1$, $\alpha_L = 3/5$, $c_L = 35$, $\beta_L = 357/5$, $a_C = 0$, $b_C = 2$, $\alpha_C = \beta_C = 1$, $c_C = -2$, $a_R = b_R = 1$, $\alpha_R = -1$, $c_R = 15$ and $\beta_R = -31$.

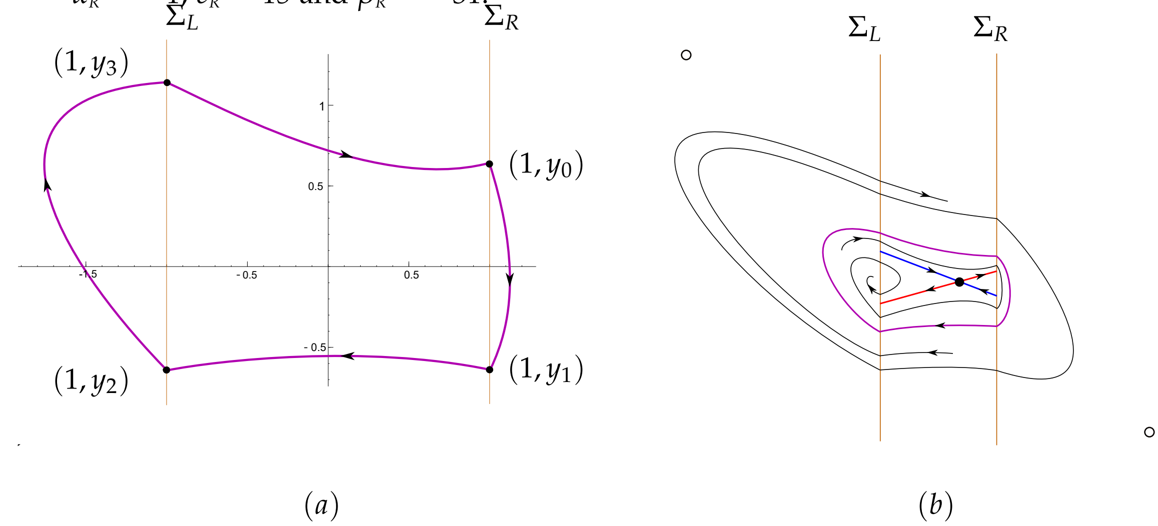


Figure 4.4: The limit cycle of vector field (2.3) with $a_L = 4$, $b_L = 8$, $\alpha_L = 2$, $c_L = -5/2$, $\beta_L = 5/2$, $a_C = 2/5$, $b_C = 24/5$, $\alpha_C = -9/5$, $c_C = 4/5$, $\beta_C = -4/15$, $a_R = 8$, $b_R = 10$ and $\alpha_R = c_R = \beta_R = -8$.

which is given by

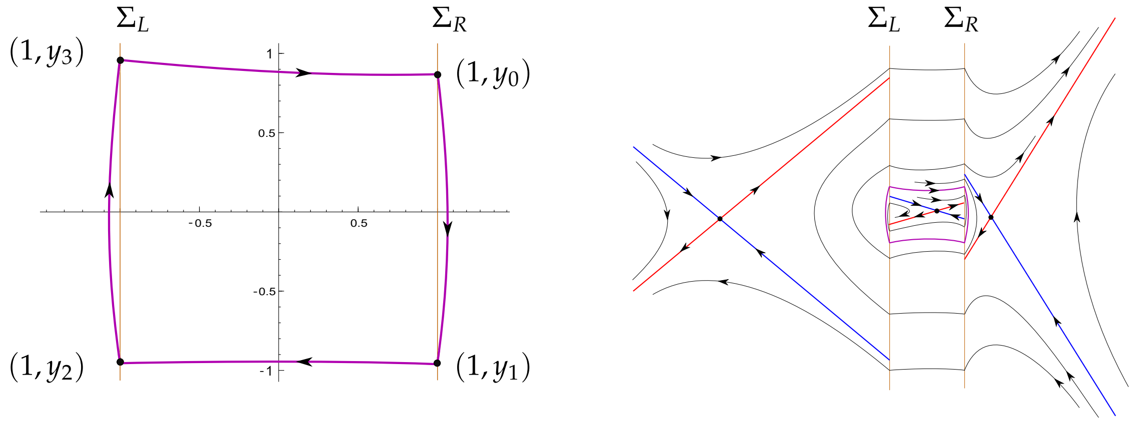
$$\left(\frac{43\sqrt{26}-12}{240}, -\frac{43\sqrt{26}+12}{240}, -\frac{3}{8}\sqrt{\frac{13}{2}}, \frac{3}{8}\sqrt{\frac{13}{2}}\right),$$

correspond to the unique limit cycle of vector field (2.3).

Note that the points $(-1, y_2), (-1, y_3) \in \Sigma_L$ and $(1, y_0), (1, y_1) \in \Sigma_R$ are crossing points.

Now, we can compute: the orbit $(x_R(t), y_R(t))$ of X_R with $(x_R(0), y_R(0)) = (1, y_0)$; the orbit $(x_{C_1}(t), y_{C_1}(t))$ of X_C with $(x_{C_1}(0), y_{C_1}(0)) = (1, y_1)$; the orbit $(x_L(t), y_L(t))$ of X_L with $(x_L(0), y_L(0)) = (-1, y_2)$; and the orbit $(x_{C_2}(t), y_{C_2}(t))$ of X_C, with $(x_{C_2}(0), y_{C_2}(0)) = (-1, y_3)$. We can also compute the fly times of the orbits: $(x_R(t), y_R(t))$ from $(1, y_0) \in \Sigma_R$ to $(1, y_1) \in \Sigma_R$;

$(x_{C_1}(t), y_{C_1}(t))$ from $(1, y_1) \in \Sigma_R$ to $(-1, y_2) \in \Sigma_L$; $(x_L(t), y_L(t))$ from $(-1, y_2) \in \Sigma_L$ to $(-1, y_3) \in \Sigma_L$; and $(x_{C_2}(t), y_{C_2}(t))$ from $(-1, y_3) \in \Sigma_L$ to $(1, y_0) \in \Sigma_R$. Hence, using the *Mathematica* software, we can draw the orbits $(x_i(t), y_i(t))$ for the time $t \in [0, t_i]$, $i = R, L, C_1, C_2$, i.e. we obtain the limit cycle given in Figure 4.5 (a). The Figure 4.5 (b) has been made with the help of P5 software, and provides the phase portrait of vector field (2.3) in this case..



(a) (b)

Figure 4.5: The limit cycle of vector field (2.3) with $a_L = \alpha_L = -2/3$, $b_L = 4/3$, $c_L = 8/3$, $\beta_L = 35/3$, $a_C = 2/11$, $b_C = 120/11$, $\alpha_C = -41/11$, $c_C = 4/11$, $\beta_C = -4/33$, $a_R = -2/11$, $b_R = 4/11$, $\alpha_R = 1/5$, $c_R = 120/11$ and $\beta_R = -749/55$.

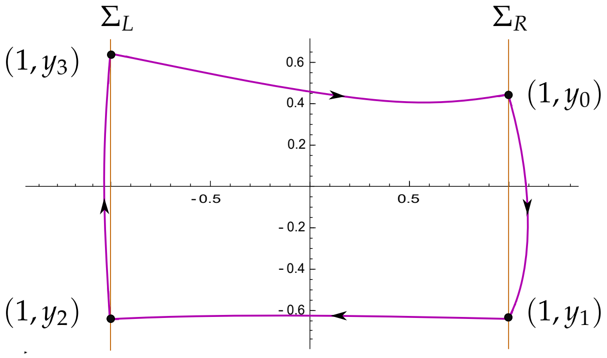
Example 4.6 (Case SSC)**.** Consider the discontinuous planar piecewise linear Hamiltonian vector field (2.3) with $a_L = \alpha_L = -2/3$, $b_L = 4/3$, $c_L = 8/3$, $\beta_L = 35/3$, $a_C = 2/11$, $b_C = 120/11$, $\alpha_C = -41/11$, $c_C = 4/11$, $\beta_C = -4/33$, $a_R = 8$, $b_R = 10$, $\alpha_R = -7$ and $c_R = \beta_R = -8$. The eigenvalues of the linear part of X_i, $i = L, C, R$, from (2.3) for this case, are ± 2, ± 2 and $\pm 4i$, respectively, i.e. we have two saddles and one center. In this case, as in Example 4.1, the unique solution (y_0, y_1, y_2, y_3) of system (3.8) satisfying the condition $y_1 < y_0$ and $y_2 < y_3$, which is given by

$$\left(\frac{43}{24}\sqrt{\frac{43}{470}} - \frac{1}{10}, -\frac{43}{24}\sqrt{\frac{43}{470}} - \frac{1}{10}, -\frac{17}{8}\sqrt{\frac{43}{470}}, \frac{17}{8}\sqrt{\frac{43}{470}}\right),$$

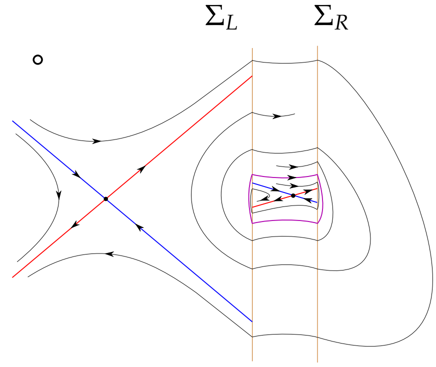
correspond to the unique limit cycle of vector field (2.3).

Note that the points $(-1, y_2), (-1, y_3) \in \Sigma_L$ and $(1, y_0), (1, y_1) \in \Sigma_R$ are crossing points.

Now, we can compute: the orbit $(x_R(t), y_R(t))$ of X_R with $(x_R(0), y_R(0)) = (1, y_0)$; the orbit $(x_{C_1}(t), y_{C_1}(t))$ of X_C with $(x_{C_1}(0), y_{C_1}(0)) = (1, y_1)$; the orbit $(x_L(t), y_L(t))$ of X_L with $(x_L(0), y_L(0)) = (-1, y_2)$; and the orbit $(x_{C_2}(t), y_{C_2}(t))$ of X_C, with $(x_{C_2}(0), y_{C_2}(0)) = (-1, y_3)$. We can also compute the fly times of the orbits: $(x_R(t), y_R(t))$ from $(1, y_0) \in \Sigma_R$ to $(1, y_1) \in \Sigma_R$; $(x_{C_1}(t), y_{C_1}(t))$ from $(1, y_1) \in \Sigma_R$ to $(-1, y_2) \in \Sigma_L$; $(x_L(t), y_L(t))$ from $(-1, y_2) \in \Sigma_L$ to $(-1, y_3) \in \Sigma_L$; and $(x_{C_2}(t), y_{C_2}(t))$ from $(-1, y_3) \in \Sigma_L$ to $(1, y_0) \in \Sigma_R$. Hence, using the *Mathematica* software, we can draw the orbits $(x_i(t), y_i(t))$ for the time $t \in [0, t_i]$, $i = R, L, C_1, C_2$, i.e. we obtain the limit cycle given in Figure 4.6 (a). The Figure 4.6 (b) has been made with the help of P5 software, and provides the phase portrait of vector field (2.3) in this case.



(a)



(b)

Figure 4.6: The limit cycle of vector field (2.3) with $a_L = \alpha_L = -2/3$, $b_L = 4/3$, $c_L = 8/3$, $\beta_L = 35/3$, $a_C = 2/11$, $b_C = 120/11$, $\alpha_C = -41/11$, $c_C = 4/11$, $\beta_C = -4/33$, $a_R = 8$, $b_R = 10$, $\alpha_R = -7$ and $c_R = \beta_R = -8$.

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Fixed-time and state-dependent time discontinuities in the theory of Stieltjes differential equations

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Abstract. In the present paper, we are concerned with a very general problem, namely the Stieltjes differential Cauchy problem involving state-dependent discontinuities.

Given that the theory of Stieltjes differential equations covers the framework of impulsive problems with fixed-time impulses, in the present work we generalize this setting by allowing the occurrence of fixed-time impulses, as well as the occurrence of state-dependent impulses.

Along with an existence result obtained under an overarching set of assumptions involving Stieltjes integrals, it is showed that a least and a greatest solution can be found.

Keywords: Stieltjes differential equation, state-dependent impulsive equation, Stieltjes integral, extremal solution.

2020 Mathematics Subject Classification: 34A37, 34A06, 26A24, 26A42.

1 Introduction

The important role played by the theory of initial value impulsive differential problems in describing the evolution of many processes in the real life is well-known [1, 15, 27]. The most encountered framework in literature is that of impulsive equations with impulses occurring at fixed times [1, 5].

The more general setting of state-dependent time discrete perturbations is (despite its wide applicability, e.g. [6, 12, 24]) far less studied, due to its complexity – see [2, 4, 10] or [25] and the references therein. To give just an idea, fixed point results are not applicable since the continuity of Nemytskii operator cannot be checked, while the control of the number and position of the state-dependent impulse moments requires strong specific assumptions.

At the same time, the theory of differential equations with Stieltjes derivative – see [19] (called Stieltjes differential equations, e.g. [11, 17]), which has been shown to be generally equivalent to the theory of measure differential equations (see [8, 9, 21]) covers a wide variety

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of real life processes. For instance, it allows the occurrence of stationary intervals (where the derivator g is constant) coupled with moments with abrupt changes in the state (where g has discontinuities).

We have in mind the possibility to allow both behaviours: stationary intervals coupled with pre-established moments with abrupt changes and also with state-dependent time impulses.

We thus focus on Stieltjes first-order Cauchy differential problem with impulses depending on the state

$$\begin{cases} x'_g(t) = f(t, x(t)), \ \mu_g\text{-a.e. } t \in [0,1] \setminus (A_x \setminus A) \\ \Delta^+ x(t) = x(t+) - x(t) = I_i(x(t)), \text{ if } t \in A_x^i \setminus A, \text{ for } i = 1, \dots, k \\ x(0) = x_0 \end{cases} \tag{1.1}$$

where $g : [0,1] \to \mathbf{R}$ is a left-continuous nondecreasing function which induces the Stieltjes measure μ_g, $B \subset \mathbf{R}$ is a closed set containing x_0, $f : [0,1] \times B \to \mathbf{R}$ is the function describing the rate of change of the unknown function, while $I_i : \mathbf{R} \to \mathbf{R}$, $i = 1, \dots, k$ give the jumps at the points where the barriers $\gamma_i : \mathbf{R} \to [0,1]$, $i = 1, \dots, k$ are reached.

By A, A_x^i and A_x (x being a real valued function on $[0,1]$) one denotes the sets

$$A = \text{the set of points of discontinuity of } g,$$

$$A_x^i = \{t \in [0,1] : t = \gamma_i(x(t))\} \text{ for every } i = 1, \dots, k,$$

respectively

$$A_x = \bigcup_{i=1}^{k} A_x^i.$$

To avoid ambiguity at the common points of A_x^i and A_x^j (with $i \neq j$), respectively of A_x^i and A, we impose the conditions $H4).iii)$, respectively $H4).iv)$ below.

Using of the Stieltjes derivative x'_g with respect to a left-continuous nondecreasing map g enables the presence of dead times (intervals where the process is stationary – corresponding to intervals where g is constant) as well as of fixed-time discrete perturbations (at the discontinuities of g).

In the particular case where $g(t) = t$ for every $t \in [0,1]$, the existence of solutions for this problem has been provided e.g. in [2, 10, 13] or [25]. However, even in this specific case, basic properties of the set of solutions are difficult to be proved (we refer to [13] or [33] for a detailed discussion).

The very wide framework of Stieltjes differential problems (which already covers many classical cases, such as ordinary differential and difference equations, impulsive equations, time-scales dynamic equations) with state-dependent discontinuities is studied here for the first time, as far as the author knows.

More precisely, we first present an existence result inspired by [22] (available for measure differential equations without allowing state dependent discontinuities, in particular for impulsive problems with fixed time impulse moments) by taking the advantage of the method used in [10] for state-dependent impulsive equations with $g(t) = t$.

Finally, we prove, using a nice result for measure differential problems without variable time impulses in [22], that a least and a greatest solution can be found. Note that, by a different method and different hypotheses, the existence of extremal solutions has been obtained in [13] when $g(t) = t$ under assumptions involving that each barrier is hit only once.

2 Notions and preliminary facts

A function $u : [0,1] \to \mathbf{R}$ is said to be regulated if for every $t \in [0,1)$ there exists the limit $u(t+)$ and for every $s \in (0,1]$ there exists the limit $u(s-)$. The set of discontinuity points of a regulated function is at most countable and the bounded variation or continuous functions are, without any doubt, regulated. The space $G([0,1],\mathbf{R})$ of regulated functions $u : [0,1] \to \mathbf{R}$ is a Banach space with respect to the sup-norm. By $G_{-}([0,1],\mathbf{R})$ we denote its subspace consisting in left-continuous functions.

Given a left-continuous nondecreasing function $g : [0,1] \to \mathbf{R}$, the measurability with respect to (in short, w.r.t.) the σ-algebra defined by g will be called g-measurability, μ_g denotes the Stieltjes measure generated by g and the Lebesgue–Stieltjes (shortly, LS-) integrability w.r.t. g means the abstract Lebesgue integrability w.r.t. the Stieltjes measure μ_g. It is well known that if f is LS-integrable w.r.t. g, the primitive $\int_0^{\cdot} f(s)dg(s) = \int_{[0,\cdot)} f(s)dg(s)$ is a g-absolutely continuous function in the following sense (see [31], [11] or [19]): a function $u : [0,1] \to \mathbf{R}$ is g-absolutely continuous if for every $\varepsilon > 0$ there is $\delta_\varepsilon > 0$ such that

$$\sum_{j=1}^{m} |u(t_j'') - u(t_j')| < \varepsilon$$

for any set $\{(t_j', t_j'')\}_{i=1}^m$ of non-overlapping subintervals of $[0,1]$ with $\sum_{j=1}^m (g(t_j'') - g(t_j')) < \delta_\varepsilon$.

We shall also use the theory of Kurzweil–Stieltjes integral (we refer the reader to [14,23,30], see also [28,29]) motivated by the fact that it is easy to handle (by integral sums), it fits well with the setting of regulated functions (i.e. it covers the situation where both the integrand and the integrator possess discontinuities) and, moreover, it can integrate functions that are not absolutely integrable.

Below are listed the basic properties of KS-integrals.

Definition 2.1. A function $f : [0,1] \to \mathbf{R}$ is Kurzweil–Stieltjes integrable with respect to $g : [0,1] \to \mathbf{R}$ (or KS-integrable w.r.t. g) if there exists $\int_0^1 f(s)dg(s) \in \mathbf{R}$ such that, for every $\varepsilon > 0$, there is a positive function $\delta_\varepsilon : [0,1] \to \mathbf{R}$ with

$$\left| \sum_{i=1}^{p} f(\xi_i)(g(t_i) - g(t_{i-1})) - \int_0^1 f(s)dg(s) \right| < \varepsilon$$

for every δ_ε-fine partition $\{([t_{i-1}, t_i], \xi_i) : \ i = 1, \dots, p\}$ of $[0,1]$. This means that $[t_{i-1}, t_i] \subset \,]\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)[$, for all $i = 1, \dots, p$.

The function $t \mapsto \int_0^t f(s)dg(s)$ is called the KS-primitive of f w.r.t. g.

Proposition 2.2 ([23])**.** *Let $f : [0,1] \to \mathbf{R}$ be Kurzweil–Stieltjes integrable w.r.t. $g : [0,1] \to \mathbf{R}$. If g is regulated, then so is the primitive $h : [0,1] \to \mathbf{R}$, $h(t) = \int_0^t f(s)dg(s)$ and for every $t \in [0,1]$,*

$$h(t+) - h(t) = f(t)\,[g(t+) - g(t)] \quad \text{and} \quad h(t) - h(t-) = f(t)\,[g(t) - g(t-)]\,.$$

Therefore, h is left-continuous, respectively right-continuous at the points where g has the same property.

Note that the Lebesgue–Stieltjes integrability of a function f implies the Kurzweil–Stieltjes integrability and in the framework of a left-continuous nondecreasing function g, as a consequence of [23, Theorem 6.11.3] (see also [26, Theorem 8.1]), if $t \in [0,1]$ then

$$\int_0^t f(s)dg(s) = \int_{[0,t]} f(s)d\mu_g(s) - f(t)(g(t+) - g(t)) = \int_{[0,t)} f(s)d\mu_g(s).$$

In order to recall more properties of the primitive, we need the notion of (Stieltjes) derivative of a function with respect to another function, given in [19] (see also [31]).

Definition 2.3. Let $g : [0,1] \to \mathbf{R}$ be nondecreasing and left-continuous. The derivative of $f : [0,1] \to \mathbf{R}$ with respect to g (or the g-derivative) at the point $t \in [0,1]$ is

$$f'_g(t) = \lim_{t' \to t} \frac{f(t') - f(t)}{g(t') - g(t)} \quad \text{if } g \text{ is continuous at } t,$$

$$f'_g(t) = \lim_{t' \to t+} \frac{f(t') - f(t)}{g(t') - g(t)} \quad \text{if } g \text{ is discontinuous at } t,$$

if the limit exists.

The g-derivative has found meaningful applications in solving real-world problems where periods of time where no activity occurs and instants with abrupt changes are both involved, such as [11], [18] or [20].

Remark that if t is a discontinuity point of g, then

$$f'_g(t) = \frac{f(t+) - f(t)}{g(t+) - g(t)}.$$

There is a set where Definition 2.3 does not work, more precisely,

$$C_g = \{t \in [0,1] : g \text{ is constant on } (t - \varepsilon, t + \varepsilon) \text{ for some } \varepsilon > 0\}$$

but we must take into account that $\mu_g(C_g) = 0$ [19] and, when studying differential equations, the equation has to be satisfied μ_g-almost everywhere.

The connection between Stieltjes integrals and the Stieltjes derivative is given by Fundamental Theorems of Calculus [19, Theorems 5.4, 6.2, 6.5].

For Lebesgue–Stieltjes integrals, it is contained in [19, Theorem 5.4], we give the entire statement below.

Theorem 2.4. *Let $g : [0,1] \to \mathbf{R}$ be a nondecreasing left-continuous function. Then $f : [0,1] \to \mathbf{R}$ is g-absolutely continuous if and only if it is g-differentiable μ_g-a.e., f'_g is Lebesgue–Stieltjes integrable w.r.t g and*

$$f(t') = f(t'') + \int_{[t'',t')} f'_g(s)d\mu_g(s), \quad \text{for every } 0 \le t'' < t' \le 1.$$

3 Main results

We are concerned with the Stieltjes initial value differential problem with state-dependent discontinuities

$$\begin{cases} x'_g(t) = f(t, x(t)), \ \mu_g - a.e. \ t \in [0,1] \setminus (A_x \setminus A) \\ \Delta^+ x(t) = x(t+) - x(t) = I_i(x(t)), \text{ if } t \in A_x^i \setminus A, \text{ for } i = 1, \dots, k \end{cases}$$

where $B \subset \mathbf{R}$ is closed, $x_0 \in B$, $f : [0,1] \times B \to \mathbf{R}$ and for each $i = 1, \dots, k$, $I_i : B \to \mathbf{R}$ describes the jumps at the points where the barrier $\gamma_i : \mathbf{R} \to [0,1]$ is reached. Recall that A is the set of discontinuity points of the left-continuous nondecreasing function $g : [0,1] \to \mathbf{R}$ continuous at 0, A_x^i is the set of points where the function $x : [0,1] \to \mathbf{R}$ hits the barrier γ_i, i.e. $\tau \in A_x^i$ if $t = \gamma_i(x(t))$ and A_x is the union of these A_x^i.

3.1 Existence result

Definition 3.1.

i) A function $x : [0, a] \to \mathbf{R}$ ($a \in (0, 1]$) is called an integral solution of the state-dependent impulsive Stieltjes differential problem (1.1) on $[0, a]$ if it is a solution of the impulsive integral equation

$$x(t) = x_0 + \int_0^t f(s, x(s))dg(s) + \sum_{i=1}^{k} \sum_{\tau \in (A_x^i \setminus A) \cap [0,t)} I_i(x(\tau)), \ t \in [0, a]. \tag{3.1}$$

ii) (e.g. [10]) We say that a function $x : [0, a] \to \mathbf{R}$ ($a \in (0, 1]$) is a g-Carathéodory solution of the state-dependent impulsive Stieltjes differential problem (1.1) on $[0, a]$ if:

- it is g-absolutely continuous and $x'_g(t) = f(t, x(t))$, μ_g-a.e. on $[0, a] \setminus (A_x \setminus A)$;
- for each $i = 1, \ldots, k$, at every $t \in (A_x^i \setminus A) \cap [0, a]$, x is left-continuous, it has finite right limit and $x(t+) = x(t) + I_i(x(t))$;
- $x(0) = x_0$.

Consider $B \subset \mathbf{R}$ a compact set.
We shall impose the following hypotheses on $f : [0, 1] \times B \to \mathbf{R}$:

$H1$) For each $x \in B$, the map $f(\cdot, x)$ is Kurzweil–Stieltjes integrable w.r.t. g on $[0, 1]$;

$H2$) One can find a non-decreasing function $h : [0, 1] \to \mathbf{R}$ and a function $M : [0, 1] \to \mathbf{R}$ KS-integrable w.r.t. h such that for every $x \in G_-([0, 1], B)$,

$$\left| \int_u^v f(t, x(t))dg(t) \right| \le \int_u^v M(t)dh(t), \quad \text{for all } 0 \le u \le v \le 1;$$

$H3$) For any $t \in [0, 1]$, $f(t, \cdot)$ is continuous on B;

Remark 3.2. Using [22, Lemma 3.1], from the preceding assumptions it follows that for each $x \in G_-([0, 1], B)$, the map $f(\cdot, x(\cdot))$ is Kurzweil–Stieltjes integrable w.r.t. g.

The assumptions on the barriers $\gamma_i : \mathbf{R} \to [0, 1]$ (known as transversality assumptions) and on the jumps $I_i : \mathbf{R} \to \mathbf{R}$, $i = 1, \ldots, k$ are described below:

$H4$) i) The maps γ_i, $i = 1, \ldots, k$ are strictly monotone and continuous;

ii) $\gamma_i^{-1}(0) \neq x_0$ for all i and $\gamma_i^{-1}(t) \neq \gamma_j^{-1}(t) + I_j(\gamma_j^{-1}(t))$ for all $i, j = 1, \ldots, k$, $t \in [0, 1]$;

iii) if $\gamma_i(x) = \gamma_j(x)$ for some $x \in B$ and $i \neq j$ then $I_i(x) = I_j(x)$;

iv) whenever $\tau \in A_x^i \cap A$ for some $x \in G_-([0, 1], B)$,

$$I_i(x(\tau)) = f(\tau, x(\tau)) \cdot \Delta^+ g(\tau);$$

$H5$) There is a positive integer $\widetilde{M}$ such that each integral solution of (1.1) on any subinterval of $[0, 1]$ hits the barriers at at most $\widetilde{M}$ points.

We make the convention that, whenever a solution hits the intersection of two barriers, the moment is counted only once.

Remark 3.3. The last part of Condition $H4$) means that for every $\tau \in A$ satisfying $x(\tau) = \gamma_i^{-1}(\tau)$ for some $x \in G_-([0,1], B)$ and some $i \in \{1, \dots, k\}$,

$$I_i(\gamma_i^{-1}(\tau)) = f(\tau, \gamma_i^{-1}(\tau)) \cdot \Delta^+ g(\tau).$$

Condition $H5$) is presented in a very general form, but we stress that it is ensured by the hypotheses imposed in other works on state-dependent impulsive differential problems when $g(t) = t$.

For instance, in [10] it is assumed that the distance between any two consecutive points where a solution hits the barriers is bigger than some constant, see (3.4) in Theorem 3.1. Also, in [2] there are a fixed number of barriers which are hit at most once by any solution, while in [25] there is only one barrier hit exactly once by any solution, see [25, Lemma 5.1].

By combining the hypotheses imposed for integral measure driven equations in [22] with the method used in the framework of state dependent impulsive equations in [10], we can prove an existence result for the state dependent impulsive Stieltjes differential problem (1.1):

Theorem 3.4. *Let $f : [0,1] \times B \to \mathbf{R}$ satisfy the hypotheses $H1$)–$H3$) and the barriers and jumps satisfy $H4$), $H5$). Suppose that*

$$\left\{ x \in \mathbf{R}; |x - x_0| \le \int_0^1 M(s)dh(s) + K_1 + \cdots + K_{\widetilde{M}} \right\} \subset B, \tag{$*$}$$

where

$$K_1 = \max_{i=1}^{k} \sup_{|x-x_0| \le \int_0^1 M(s)dh(s)} |I_i(x)|,$$

$$K_{n+1} = \max_{i=1}^{k} \sup_{|x-x_0| \le \int_0^1 M(s)dh(s)+K_1+\cdots+K_n} |I_i(x)|, \qquad \forall n \ge 1.$$

Then the problem (1.1) admits integral solutions on $[0,1]$.

Proof. Consider at the beginning the measure-driven integral equation

$$x(t) = x_0 + \int_0^t f(s, x(s))dg(s), \quad t \in [0,1].$$

Since our assumption on B implies that

$$\left\{ x \in \mathbf{R} : |x - x_0| \le \int_0^1 M(t)dh(t) \right\} \subset B,$$

by [22, Theorem 3.2], one can find an integral solution x_1 on $[0,1]$. By usual properties of Kurzweil–Stieltjes integrals, x_1 is left-continuous on $[0,1]$ and continuous at any point where g is continuous (thus, it is continuous at 0).

Define then $r_{i,1} : [0,1] \to \mathbf{R}$ by

$$r_{i,1}(t) = \gamma_i(x_1(t)) - t.$$

Due to $H4).ii$), $r_{i,1}(0) \ne 0$ for all $i = 1, \dots, k$ and, since $r_{i,1}$ is continuous at 0, we might encounter the following situations:

- if $r_{i,1}(t) \ne 0$ for all $i = 1, \dots, k$ and $t \in (0,1] \setminus A$, then x_1 is a solution of (1.1) on $[0,1]$.

- if $r_{i,1}(t) = 0$ for some $i \in \{1, \dots, k\}$ and $t \in (0,1] \setminus A$, then let t_1 be a continuity point of g (i.e. $t \notin A$) such that $r_{i_1,1}(t_1) = 0$ for some i_1 and $r_{i,1}(t) \neq 0$ on $[0, t_1) \setminus A$ for all i.

Consider in what follows the measure-driven integral problem

$$x(t) = x_1(t_1) + I_{i_1}(x_1(t_1)) + \int_{t_1}^{t} f(s, x(s))dg(s), \quad t \in [t_1, 1].$$

The assumption made on B brings us to

$$\left\{x \in \mathbf{R} : |x - (x_1(t_1) + I_{i_1}(x_1(t_1)))| \leq \int_{t_1}^{1} M(s)dh(s)\right\} \subset B$$

since for each such x

$$\begin{aligned} |x - x_0| &\leq |x - (x_1(t_1) + I_{i_1}(x_1(t_1)))| + |x_1(t_1) - x_0| + |I_{i_1}(x_1(t_1)))| \\ &\leq \int_{t_1}^{1} M(s)dh(s) + \int_{0}^{t_1} M(s)dh(s) + \sup_{|x - x_0| \leq \int_0^1 M(s)dh(s)} |I_{i_1}(x)| \\ &\leq \int_{0}^{1} M(s)dh(s) + K_1 \end{aligned}$$

and so, $x \in B$ and the inclusion is proved.

We can thus apply [22, Theorem 3.2] once again and one can find an integral solution x_2 on $[t_1, 1]$; it is left-continuous on $[t_1, 1]$ and continuous at any point where g is continuous (in particular, at t_1). As above, define $r_{i,2} : [t_1, 1] \to \mathbf{R}$ by $r_{i,2}(t) = \gamma_i(x_2(t)) - t$ which is continuous at t_1.

Besides, by $H4).ii)$, for all i

$$\gamma_i^{-1}(t_1) \neq \gamma_{i_1}^{-1}(t_1) + I_{i_1}(\gamma_{i_1}^{-1}(t_1)) = x_2(t_1),$$

so $r_{i,2}(t_1) \neq 0$, whence we might have the following situations:

- if $r_{i,2}(t) \neq 0$ for all $i = 1, \dots, k$ and $t \in (t_1, 1] \setminus A$, then a solution of (1.1) on $[0, 1]$ can be found if we take x_1 on $[0, t_1]$ and x_2 on $(t_1, 1]$;
- if $r_{i,2}(t) = 0$ for some $i \in \{1, \dots, k\}$ and $t \in (t_1, 1] \setminus A$, then let $t_2 > t_1$ be a continuity point of g chosen such that $r_{i_2,2}(t_2) = 0$ for some i_2 and $r_{i,2}(t) \neq 0$ on $[t_1, t_2) \setminus A$ for all i.

Let us next look at the measure-driven Cauchy problem

$$x(t) = x_2(t_2) + I_{i_2}(x_2(t_2)) + \int_{t_2}^{t} f(s, x(s))dg(s), \quad t \in [t_2, 1].$$

It is not difficult to see that

$$\left\{x \in \mathbf{R} : |x - (x_2(t_2) + I_{i_2}(x_2(t_2)))| \leq \int_{t_2}^{1} M(s)dh(s)\right\} \subset B$$

as for each x in this set, as before,

$$\begin{aligned} |x - x_0| &\leq |x - (x_2(t_2) + I_{i_2}(x_2(t_2)))| + |x_2(t_2) - x_1(t_1)| + |x_1(t_1) - x_0| + |I_{i_2}(x_2(t_2)))| \\ &\leq \int_{t_2}^{1} M(s)dh(s) + \int_{t_1}^{t_2} M(s)dh(s) + |I_{i_1}(x_1(t_1)))| + \int_{0}^{t_1} M(s)dh(s) + |I_{i_2}(x_2(t_2)))| \\ &\leq \int_{0}^{t_1} M(s)dh(s) + \sup_{|x - x_0| \leq \int_0^1 M(s)dh(s)} |I_{i_1}(x)| + \sup_{|x - x_0| \leq \int_0^1 M(s)dh(s) + K_1} |I_{i_2}(x)| \\ &\leq \int_{0}^{1} M(s)dh(s) + K_1 + K_2 \end{aligned}$$

and so, $x \in B$.

We can thus apply [22, Theorem 3.2] and one can continue the process and we claim that it will be finished after less than $\widetilde{M}+1$ steps (otherwise, hypothesis $H5)$ would be contradicted).

□

Under stronger assumptions on f and keeping the hypothesis on the barriers, one can obtain the existence of g-Carathéodory solutions for the impulsive measure differential problem (1.1).

Theorem 3.5. *Let $f : [0,1] \times B \to \mathbf{R}$ satisfy the hypotheses*

$H1')$ *For each $x \in G_-([0,1], B)$, the map $f(\cdot, x(\cdot))$ is g-measurable on $[0,1]$;*

$H2')$ *One can find a function $M : [0,1] \to \mathbf{R}$ Lebesgue–Stieltjes-integrable w.r.t. g such that for every $x \in B$,*

$$|f(t,x)| \leq M(t), \quad \text{for } \mu_g\text{-a.e. } t \in [0,1];$$

together with $H3)$ and the barriers and jumps satisfy $H4), H5)$.

Suppose that

$$\left\{ x \in \mathbf{R}; |x - x_0| \leq \int_0^1 M(s)dg(s) + K_1 + \cdots + K_{\widetilde{M}} \right\} \subset B, \tag{$**$}$$

where

$$K_1 = \max_{i=1}^{k} \sup_{|x-x_0| \leq \int_0^1 M(s)dg(s)} |I_i(x)|,$$

$$K_{n+1} = \max_{i=1}^{k} \sup_{|x-x_0| \leq \int_0^1 M(s)dg(s)+K_1+\cdots+K_n} |I_i(x)|, \qquad \forall n \geq 1.$$

Then the problem (1.1) admits g-Carathéodory solutions on $[0,1]$.

Proof. We follow the same lines as in the previous result. Consider first the measure-driven Cauchy problem

$$\begin{cases} x_g'(t) = f(t, x(t)), \ \mu_g\text{-a.e. } t \in [0,1], \\ x(0) = x_0. \end{cases}$$

By the Peano existence result [11, Theorem 7.5], one can find a g-Carathéodory solution x_1 on $[0,1]$.

Define then $r_{i,1} : [0,1] \to \mathbf{R}$ as before and we can fall into one of the following situations:

- if $r_{i,1}(t) \neq 0$ for all $i = 1, \ldots, k$ and $t \in (0,1] \setminus A$, then x_1 is a g-Carathéodory solution of (1.1) on $[0,1]$;
- if $r_{i,1}(t) = 0$ for some $i \in \{1, \ldots, k\}$ and $t \in (0,1] \setminus A$, then let $t_1 \in (0,1] \setminus A$ be chosen such that $r_{i_1,1}(t_1) = 0$ for some i_1 and $r_{i,1}(t) \neq 0$ on $[0, t_1) \setminus A$ for all i.

Consider then the measure-driven Cauchy problem

$$\begin{cases} x_g'(t) = f(t, x(t)), \ \mu_g\text{-a.e. } t \in [t_1, 1], \\ x(t_1) = x_1(t_1) + I_{i_1}(x_1(t_1)). \end{cases}$$

We can again apply [11, Theorem 7.5] in order to get a g-Carathéodory solution on $[t_1, 1]$ and so on.

□

Remark 3.6. We could have obtained the previous result by applying Theorem 3.4 and remarking that the assumptions $H1'$), $H2'$) together with the Fundamental Theorem of Calculus imply that any integral solution of our problem is a g-Carathéodory solution.

3.2 Existence of extremal solutions

Using the existence of extremal solutions for measure differential equations ([22, Theorem 4.4]), we get the existence of extremal solutions for measure differential equations with state-dependent impulses.

We need several additional assumptions.

$H6$) One of the following sets of conditions holds:

a) $x_0 > \gamma_i^{-1}(0)$ for each i, together with
   i) $\gamma_i^{-1}(t) < x + f(t,x)\Delta^+ g(t)$ for every $i = 1, \dots, k$, $t \in A$ whenever $\gamma_i^{-1}(t) < x$;
   ii) $\gamma_i^{-1}(t) < \gamma_j^{-1}(t) + I_j(\gamma_j^{-1}(t))$ for all $i, j = 1, \dots, k$, $t \in [0,1]$

or

b) $x_0 < \gamma_i^{-1}(0)$ for each i, together with
   i) $\gamma_i^{-1}(t) > x + f(t,x)\Delta^+ g(t)$ for every $i = 1, \dots, k$, $t \in A$ whenever $\gamma_i^{-1}(t) > x$;
   ii) $\gamma_i^{-1}(t) > \gamma_j^{-1}(t) + I_j(\gamma_j^{-1}(t))$ for all $i, j = 1, \dots, k$, $t \in [0,1]$.

Remark 3.7. In the first case, when $i = j$ one gets $I_i(\gamma_i^{-1}(t)) > 0$ and, obviously, in the second case, $I_i(\gamma_i^{-1}(t)) < 0$.

$H7$) For every $x, y \in B$ with $x \le y$,

$$x + f(t,x) \cdot \Delta^+ g(t) \le y + f(t,y) \cdot \Delta^+ g(t), \quad \forall t \in A$$

together with

$$\gamma_i^{-1}(t) + I_i(\gamma_i^{-1}(t)) \le \gamma_j^{-1}(t) + I_j(\gamma_j^{-1}(t)) \quad \text{whenever } \gamma_i^{-1}(t) \le \gamma_j^{-1}(t)$$

for some $t \in [0,1]$, $i, j \in \{1, \dots, k\}$.

Definition 3.8. A solution $y : [0,1] \to \mathbf{R}$ is said to be the least (resp. greatest) solution of (1.1) if for any other solution $x : [0,1] \to \mathbf{R}$,

$$y(t) \le x(t) \quad \text{for every } t \in [0,1],$$

respectively

$$y(t) \ge x(t) \quad \text{for every } t \in [0,1].$$

Theorem 3.9. *Let the hypotheses $H1$)–$H7$) and ($*$) be satisfied. Then the problem (1.1) admits a greatest integral solution and a least integral solution on $[0,1]$.*

Proof. We proceed as in the proof of Theorem 3.4, with convenient adjustments, in order to get the existence of a least solution.

Thus, consider in the first place the measure-driven integral equation

$$x(t) = x_0 + \int_0^t f(s, x(s))dg(s), \quad t \in [0,1].$$

Since all the hypotheses of [22, Theorem 4.4] are satisfied, one can find a least solution y_1 on $[0,1]$ (left-continuous everywhere and continuous at the continuity points of g, such as 0).

Let $r_{i,1} : [0,1] \to \mathbf{R}$ be defined by

$$r_{i,1}(t) = \gamma_i(y_1(t)) - t.$$

Due to $H4).ii)$, $r_{i,1}(0) \neq 0$ for all $i = 1, \dots, k$ and since $r_{i,1}$ is continuous at 0, the following situations are possible:

- if $r_{i,1}(t) \neq 0$ for all $i = 1, \dots, k$ and all $t \in (0,1] \setminus A$, let y_- be y_1 on $[0,1]$.
- if $r_{i,1}(t) = 0$ for some $i \in \{1, \dots, k\}$ and $t \in (0,1] \setminus A$, then let t_1 be a continuity point of g such that $r_{i_1,1}(t_1) = 0$ (i.e. $y_1(t_1) = \gamma_{i_1}^{-1}(t_1)$) for some i_1 and $r_{i,1}(t) \neq 0$ on $[0, t_1) \setminus A$ for all i.

Consider in what follows the measure-driven integral problem:

$$x(t) = \gamma_{i_1}^{-1}(t_1) + I_{i_1}(\gamma_{i_1}^{-1}(t_1)) + \int_{t_1}^{t} f(s, x(s))dg(s), \quad t \in [t_1, 1].$$

The assumption we made on B implies that we can apply [22, Theorem 4.4] once again in order to find a least solution on $[t_1, 1]$, denoted by y_2.

As above, define $r_{i,2} : [t_1, 1] \to \mathbf{R}$ by $r_{i,2}(t) = \gamma_i(y_2(t)) - t$ and since it is continuous at t_1 and $r_{i,2}(t_1) \neq 0$, we might have the following situations:

- if $r_{i,2}(t) \neq 0$ for all $i = 1, \dots, k$ and $t \in (t_1, 1] \setminus A$, then we construct the solution y_- of (1.1) on $[0,1]$ taking y_1 on $[0, t_1]$ and y_2 on $(t_1, 1]$.
- if $r_{i,2}(t) = 0$ for some $i \in \{1, \dots, k\}$ and $t \in (t_1, 1] \setminus A$, then let $t_2 > t_1$ be a continuity point of g chosen such that $r_{i_2,2}(t_2) = 0$ for some i_2 and $r_{i,2}(t) \neq 0$ on $[t_1, t_2) \setminus A$ for all i.

Let us next look at the problem:

$$x(t) = \gamma_{i_2}^{-1}(t_2) + I_{i_2}(\gamma_{i_2}^{-1}(t_2)) + \int_{t_2}^{t} f(s, x(s))dg(s), \quad t \in [t_2, 1]$$

for each $i \in \{1, \dots, k\}$.

The hypothesis on B implies that, by [22, Theorem 4.4], we can find, for each of these problems, a least solution on $[t_2, 1]$, denoted by y_3 and one can continue the process, which will be finished after less than $\widetilde{M} + 1$ steps (otherwise, hypothesis $H5$) would be contradicted).

Let us see that the solution constructed in this way, namely y_-, is a least solution of (1.1) on $[0,1]$. Suppose that $H6).a)$ is satisfied (the case $b)$ can be analyzed in a similar way).

Let x be an arbitrary solution of (1.1) on $[0,1]$. We first show that $y_-(t) \leq x(t)$ for every $t \in [0, t_1]$.

i) If $(A_x \setminus A) \cap [0, t_1) = \emptyset$, then $y_-(t) \leq x(t)$ for every $t \in [0, t_1]$.

ii) If there are points in $(A_x \setminus A) \cap [0, t_1)$, let us focus on the first one since their number is finite and for all such points the discussion can be led in the same way; let $\tau_1 \in (A_x \setminus A) \cap [0, t_1)$ be the first point where x hits some barrier γ_{i_0}. Then the following situations can be encountered:

ii.a) none of the discontinuity points of g lies in between 0 and τ_1; in this case, since $y_-(0) = x_0 > \gamma_{i_0}^{-1}(0)$ and $y_-(\tau_1) \leq x(\tau_1) = \gamma_{i_0}^{-1}(\tau_1)$ (as y_- is the least solution of the measure integral equation on $[0, t_1)$), by the continuity of $\gamma_{i_0}^{-1}$ and y_- on $(0, \tau_1)$, it would follow that y_- hits the barrier γ_{i_0} on $(0, \tau_1]$, contradiction with the choice of t_1.

ii.b) if there are discontinuity points of g lying in between 0 and τ_1, we can fall into one of the three cases below:

• this is a finite subset of A, $\{\tilde{\tilde{t}}_i, i = 1, \ldots, k\}$; then for each i, $x(\tilde{\tilde{t}}_i), y_-(\tilde{\tilde{t}}_i) > \gamma_{i_0}^{-1}(\tilde{\tilde{t}}_i)$ since otherwise the graphs of x, y_- would hit the barrier γ_{i_0} before τ_1 and this is not possible.

By *H*6).*a*).*i*), for each $i = 1, \ldots, k$,

$$\gamma_{i_0}^{-1}(\tilde{\tilde{t}}_i) < x(\tilde{\tilde{t}}_i+) \quad \text{and} \quad \gamma_{i_0}^{-1}(\tilde{\tilde{t}}_i) < y_-(\tilde{\tilde{t}}_i+),$$

whence, due to the fact that $y_-(\tau_1) \leq x(\tau_1) = \gamma_{i_0}^{-1}(\tau_1)$ (since y_- is the least solution of the measure integral equation on $[0, t_1)$), y_- would hit the barrier γ_{i_0} on $(t_k, \tau_1]$ which again is impossible.

• this is a countable set $\{\tilde{\tilde{t}}_i, i \in \mathbb{N}\}$ accumulating towards $\bar{t} < \tau_1$; then, as before, at each such point

$$\gamma_{i_0}^{-1}(\tilde{\tilde{t}}_i) < x(\tilde{\tilde{t}}_i) \quad \text{and} \quad \gamma_{i_0}^{-1}(\tilde{\tilde{t}}_i) < y_-(\tilde{\tilde{t}}_i),$$

which, taking into account the left continuity of x, y_- and the continuity of $\gamma_{i_0}^{-1}$, imply

$$\gamma_{i_0}^{-1}(\bar{t}) \leq x(\bar{t}) \quad \text{and} \quad \gamma_{i_0}^{-1}(\bar{t}) \leq y_-(\bar{t}).$$

But equality is impossible as this would mean that x, respectively y_- would hit some barrier at $\bar{t}$, therefore

$$\gamma_{i_0}^{-1}(\bar{t}) < x(\bar{t}) \quad \text{and} \quad \gamma_{i_0}^{-1}(\bar{t}) < y_-(\bar{t})$$

and thus, by *H*6).*a*).*i*),

$$\gamma_{i_0}^{-1}(\bar{t}) < x(\bar{t}+) \quad \text{and} \quad \gamma_{i_0}^{-1}(\bar{t}) < y_-(\bar{t}+).$$

Again it would imply that y_- hits the barrier γ_{i_0} on $(\bar{t}, \tau_1]$ which cannot happen.

• this is a countable set $\{\tilde{\tilde{t}}_i, i \in \mathbb{N}\}$ accumulating towards τ_1, in which case, as before,

$$\gamma_{i_0}^{-1}(\tau_1) \leq x(\tau_1) \quad \text{and} \quad \gamma_{i_0}^{-1}(\tau_1) \leq y_-(\tau_1)$$

and since $y_-(\tau_1) \leq x(\tau_1) = \gamma_{i_0}^{-1}(\tau_1)$, y_- would hit the barrier before t_1 and this is a contradiction.

Let us now check that $y_-(t) \leq x(t)$ for every $t \in [t_1, t_2]$ (on the next intervals the same discussion is to be carried out).

Hypothesis *H*6).*a*).*ii*) implies that

$$y_-(t_1+) > \gamma_i^{-1}(t_1) \quad \text{for each } i = 1, \ldots, k \tag{3.2}$$

and from the preceding step we know that $y_-(t_1) \leq x(t_1)$ which, by H7), implies that

$$y_-(t_1+) \leq x(t_1+). \tag{3.3}$$

If in (3.3) one has equality, then the proof on $[0, t_1]$ has to be repeated in order to get the assertion on $[t_1, t_2]$. If strict inequality holds, then some modifications are necessary but we take the same steps as on $[0, t_1]$ in order to prove that $y_-(t) \leq x(t)$ for every $t \in [t_1, t_2]$. Thus:

i) If $(A_x \setminus A) \cap [t_1, t_2) = \emptyset$, then suppose there is $\bar{t} \in (t_1, t_2]$ such that $y_-(\bar{t}) > x(\bar{t})$. We could be in the following situations:

i).a) $[t_1, \bar{t}] \cap A = \emptyset$, in which case x and y_- are continuous on $(t_1, \bar{t}]$, $y_-(t_1+) < x(t_1+)$ is valid and so there is a point $\bar{\bar{t}}$ in this interval where the two trajectories intersect; then the solution defined by

$$\begin{cases} y_-(t), & \text{for } t \in (t_1, \bar{\bar{t}}], \\ x(t), & \text{for } t \in (\bar{\bar{t}}, \bar{t}] \end{cases}$$

would contradict the definition of y_- on $(t_1, t_2]$ as being the least solution of the measure integral equation.

i).b) $[t_1, \bar{t}] \cap A \neq \emptyset$, in which case we might have:

• this is a finite subset of A, $\{\tilde{\tilde{t}}_i, i = 1, \ldots, k\}$; then, since $y_-(t_1+) < x(t_1+)$, for each i, $y_-(\tilde{\tilde{t}}_i) \leq x(\tilde{\tilde{t}}_i)$ since otherwise, as in $i).a)$, the fact that y_- is a least solution of the measure integral equation would be disobeyed.

So, by $H7)$, $y_-(\tilde{\tilde{t}}_k+) \leq x(\tilde{\tilde{t}}_k+)$ and, as $y_-(\bar{t}) > x(\bar{t})$, as in $i).a)$, the fact that y_- is the least solution of the measure integral equation on $[t_1, t_2)$ is contradicted.

• this is a countable set $\{\tilde{\tilde{t}}_i, i \in \mathbb{N}\}$ accumulating towards $\bar{\bar{t}} < \bar{t}$; then, as before, at each such point $y_-(\tilde{\tilde{t}}_i) \leq x(\tilde{\tilde{t}}_i)$ which, taking into account the left continuity of x, y_-, imply

$$y_-(\bar{\bar{t}}) \leq x(\bar{\bar{t}})$$

and thus

$$y_-(\bar{\bar{t}}+) \leq x(\bar{\bar{t}}+).$$

Again it would follow that y_- is not a least solution on $(\bar{\bar{t}}, \bar{t}]$.

• this is a countable set $\{\tilde{\tilde{t}}_i, i \in \mathbb{N}\}$ accumulating towards $\bar{t}$, in which case, as before, $y_-(\bar{t}) \leq x(\bar{t})$ - contradiction.

ii) If there are points in $(A_x \setminus A) \cap [t_1, t_2)$, let us focus only on the first one $\tau_1 \in (A_x \setminus A) \cap [t_1, t_2)$ where x hits some barrier γ_{i_0}. Then the following situations can be encountered:

ii.a) none of the discontinuity points of g lies in between t_1 and τ_1; in this case, let us put together (3.2) and the fact that $y_-(\tau_1) \leq x(\tau_1) = \gamma_{i_0}^{-1}(\tau_1)$ (since otherwise, together with (3.3) and the continuity of y_- and x it would be contradicted, as before, the choice of y_- as the least solution of the measure integral equation on $[t_1, t_2)$). By the continuity of $\gamma_{i_0}^{-1}$ and y_- on (t_1, τ_1), it would then follow that y_- hits the barrier γ_{i_0} on $(t_1, \tau_1]$, contradiction with the choice of t_2.

ii.b) if there are discontinuity points of g lying in between t_1 and τ_1, we can fall again into one of the three cases below:

• this is a finite subset of A, $\{\tilde{\tilde{t}}_i, i = 1, \ldots, k\}$; then for each i, $x(\tilde{\tilde{t}}_i), y_-(\tilde{\tilde{t}}_i) > \gamma_{i_0}^{-1}(\tilde{\tilde{t}}_i)$ since otherwise the graphs of x, y_- would hit the barrier γ_{i_0} on (t_1, τ_1) and this is not possible.

By $H6).a).i)$, for each $i = 1, \ldots, k$,

$$\gamma_{i_0}^{-1}(\widetilde{\widetilde{t}}_i) < x(\widetilde{\widetilde{t}}_i+) \quad \text{and} \quad \gamma_{i_0}^{-1}(\widetilde{\widetilde{t}}_i) < y_-(\widetilde{\widetilde{t}}_i+),$$

whence, due to the fact that $y_-(\tau_1) \leq x(\tau_1) = \gamma_{i_0}^{-1}(\tau_1)$ (otherwise, as before, the fact that y_- is the least solution of the measure integral equation on $[t_1, t_2)$ would be contradicted), y_- would hit the barrier γ_{i_0} on $(t_k, \tau_1]$ which again is impossible.

• this is a countable set $\{\widetilde{\widetilde{t}}_i, i \in \mathbb{N}\}$ accumulating towards $\bar{t} < \tau_1$; then, as before, at each such point

$$\gamma_{i_0}^{-1}(\widetilde{\widetilde{t}}_i) < x(\widetilde{\widetilde{t}}_i) \quad \text{and} \quad \gamma_{i_0}^{-1}(\widetilde{\widetilde{t}}_i) < y_-(\widetilde{\widetilde{t}}_i),$$

which, taking into account the left continuity of x, y_-, imply

$$\gamma_{i_0}^{-1}(\bar{t}) \leq x(\bar{t}) \quad \text{and} \quad \gamma_{i_0}^{-1}(\bar{t}) \leq y_-(\bar{t}).$$

Equality is not possible (because it would mean that x, y_- hit the barrier at $\bar{t}$), so

$$\gamma_{i_0}^{-1}(\bar{t}) < x(\bar{t}) \quad \text{and} \quad \gamma_{i_0}^{-1}(\bar{t}) < y_-(\bar{t})$$

and thus

$$\gamma_{i_0}^{-1}(\bar{t}) < x(\bar{t}+) \quad \text{and} \quad \gamma_{i_0}^{-1}(\bar{t}) < y_-(\bar{t}+).$$

Again it would follow that y_- would hit the barrier γ_{i_0} on $(\bar{t}, \tau_1]$ which cannot happen.

• this is a countable set $\{\widetilde{\widetilde{t}}_i, i \in \mathbb{N}\}$ accumulating towards τ_1, in which case, as before,

$$\gamma_{i_0}^{-1}(\tau_1) \leq x(\tau_1) \quad \text{and} \quad \gamma_{i_0}^{-1}(\tau_1) \leq y_-(\tau_1)$$

and since $y_-(\tau_1) \leq x(\tau_1) = \gamma_{i_0}^{-1}(\tau_1)$, y_- would hit the barrier before t_2 and this is a contradiction. □

Corollary 3.10. *If $H1'), H2')$ are imposed, then there exist a least and a greatest g-Carathéodory solutions of* (1.1).

Remark 3.11. The present work provides the existence of extremal solutions for a large class of differential problems, namely Stieltjes differential equations involving fixed time and state-dependent time impulses. Let us note once again that, even in the particular case where the Stieltjes derivative is the usual derivative, the allowance of state-dependent time impulses leads to really complex situations.

The very general discussion developed here could be applied to study real life problems where the results available in literature for measure (Stieltjes) differential equations or for classical impulsive ODEs fail.

The wide applicability of our results can be seen by looking at the following example, which represents a generalization of a problem in [11], describing the evaporating water in an open top cylindrical tank.

Example 3.12. Suppose that the initial level of the water in the tank is x_0 and that the water level decreases due to evaporation. If $x(t)$ denotes the water height at time $t > 0$, then the model adopted in [11] (which takes into account that the level remains constant during the nights, while during the days the evaporation speed is maximum at middays) states that the evolution of x can be described by the Stieltjes differential problem

$$x'_g(t) = f(t, x(t)), \quad t \in [0, T] \quad \text{and} \quad x(0) = x_0.$$

Note that in [11] the map f is supposed to be linear in x, but the nonlinear framework is realistic as well. The nondecreasing left-continuous function g can be chosen conveniently [11, page 20], for instance if we want to refill the tank every morning with an amount of water depending on to the level before refilling, then one may set

$$g(t) = \int_0^t \max\left(\sin(\pi s), 0\right) ds + \max\{k \in \mathbb{N} : 2k \le t\}$$

and

$$f(2k, x(2k)) = \Delta^+ x(2k) = \lambda_k x(2k), \quad \lambda_k > 0$$

(the intervals $[2k, 2k+1), k \in \mathbb{N}$ correspond to day times and, obviously, the intervals $[2k+1, 2k+2), k \in \mathbb{N}$ to night times).

In other words, the moments $2k, k \in \mathbb{N}$ are fixed-time impulsive moments with $\Delta^+ g(2k) = 1, \forall k$ and so far, the problem can be solved through the theory of Stieltjes differential equations.

Suppose now that we want to add an amount of water (equal to $I(x(t))$) whenever a state-dependent condition is satisfied, such as $x(t) = \beta(t)$, where β is a decreasing function measuring the water level in a huge second tank where the level water decreases due to evaporation, without adding or removing any quantity and without stationary intervals.

In this case, the theory in [11] cannot be applied due to the occurrence of state-dependent impulses. At the same time, nor the studies developed for state-dependent impulsive problems ([10], [2] or [25]) apply since the involved derivative is the Stieltjes derivative (not the usual derivative).

The announced problem can be investigated by applying our results for

$$\begin{cases} x'_g(t) = f(t, x(t)), \ \mu_g\text{-a.e. } t \in [0, T] \setminus (A_x \setminus A) \\ \Delta^+ x(t) = x(t+) - x(t) = I(x(t)), \ \text{if } t \in A_x \setminus A \\ x(0) = x_0 \end{cases}$$

where $A = \{2k : k \in \mathbb{N}\} \cap [0, T]$ and, for some function $x \in G_-([0, T], \mathbf{R})$, $A_x = \{t \in [0, T] : x(t) = \beta(t)\}$; we thus face the occurrence of only one barrier $\gamma_1 = \beta^{-1}$.

Theorem 3.9 yields the existence of a least g-Carathéodory solution and of a greatest g-Carathéodory solution provided f and I satisfy the following conditions:

a) for each $x \in G_-([0, T], B)$, the map $f(\cdot, x(\cdot))$ is g-measurable;

b) one can find a function $M : [0, T] \to \mathbf{R}$ Lebesgue–Stieltjes-integrable w.r.t. g such that for every $x \in B$,
$$|f(t, x)| \le M(t), \ \text{for } \mu_g\text{-a.e. } t \in [0, T]$$
such that ($**$) is valid (with T instead of 1);

c) f is continuous with respect to its second argument;

d) $\beta : [0, 1] \to \mathbf{R}$ is strictly monotone and continuous and whenever $\tau \in A_x \cap A$ for some $x \in G_-([0, T], B)$,
$$I(\beta(\tau)) = f(\tau, \beta(\tau));$$

e) there is a positive integer $\widetilde{M}$ such that each integral solution of (1.1) on any subinterval of $[0, T]$ hits the barrier at at most $\widetilde{M}$ points;

f) $x_0 > \beta(0)$, $I(\beta(t)) > 0$ for every $t \in [0, T]$ and $\beta(t) < x + f(t, x)$ for every $t \in A$, $x \in B$ with $\beta(t) < x$;

g) for every $x, y \in B$ with $x \leq y$,

$$x + f(t, x) \leq y + f(t, y), \quad \forall t \in A.$$

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$C^{1,\gamma}$ regularity for fully nonlinear elliptic equations on a convex polyhedron

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Abstract. In this note, we prove the boundary and global $C^{1,\gamma}$ regularity for viscosity solutions of fully nonlinear uniformly elliptic equations on a convex polyhedron by perturbation and iteration techniques.

Keywords: $C^{1,\gamma}$ regularity, fully nonlinear elliptic equation, viscosity solution.

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1 Introduction

The purpose of this note is to investigate the $C^{1,\gamma}$ regularity up to the boundary for viscosity solutions of the following fully nonlinear elliptic equation

$$\begin{cases} F(D^2u, x) = f & \text{in } \Omega; \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset R^n$ is a convex polyhedron, F is assumed to be uniformly elliptic (see (1.4)).

With respect to the boundary regularity of solutions of linear elliptic equations, Li and Wang [7,8] proved the boundary differentiability on convex domains and demonstrated that only under the assumption that Ω is convex, no continuity of the gradient of solutions along the boundary can be expected (see the counterexamples in [7]). On the other hand, Lian and Zhang [11, Theorem 1.6] proved the boundary $C^{1,\alpha}$ regularity of solutions of fully nonlinear elliptic equations under the assumption that the boundary $\partial\Omega$ is $C^{1,\alpha}$. In this note, we show the $C^{1,\gamma}$ regularity for fully nonlinear elliptic equations (linear elliptic equations as a special case) by strengthening convex domain into convex polyhedron. And we do not need such high smoothness condition on the boundary as in [11].

Before stating our main results, we give several definitions.

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Definition 1.1. Let $\Omega \subset R^n$ be a bounded set and f be a function defined on Ω. We say that f is $C^{1,\alpha}$ at $x_0 \in \Omega$ denoted by $f \in C^{1,\alpha}(x_0)$ if there exist a linear polynomial L, constants C and $r_0 > 0$ such that

$$|f(x) - L(x)| \le C|x - x_0|^{1+\alpha}, \quad \forall\, x \in \Omega \cap B_{r_0}(x_0). \tag{1.2}$$

Note that there may exist many L and C (e.g. $\Omega = B_1 \cap R^{n-1}$). We take L_0 with

$$\|L_0\| = \min \{\|L\| \mid \exists C \text{ such that (1.2) holds with } L \text{ and } C\},$$

where $\|L\| = |L(x_0)| + |DL(x_0)|$. Define

$$Df(x_0) = DL_0(x_0),$$

$$\|f\|_{C^1(x_0)} = \|L_0\|,$$

$$[f]_{C^{1,\alpha}(x_0)} = \min \{C \mid \text{(1.2) holds with } L_0 \text{ and } C\}$$

and

$$\|f\|_{C^{1,\alpha}(x_0)} = \|f\|_{C^1(x_0)} + [f]_{C^{1,\alpha}(x_0)}.$$

If $f \in C^{1,\alpha}(x)$ for any $x \in \Omega$ with the same r_0 and

$$\|f\|_{C^{1,\alpha}(\bar{\Omega})} := \sup_{x\in\Omega} \|f\|_{C^1(x)} + \sup_{x\in\Omega} [f]_{C^{1,\alpha}(x)} < +\infty,$$

we say $f \in C^{1,\alpha}(\bar{\Omega})$.

Definition 1.2. Let Ω and f be as in Definition 1.1. We call that f is $C^{-1,\alpha}$ at $x_0 \in \Omega$ denoted by $f \in C^{-1,\alpha}(x_0)$ if there exist constants C and $r_0 > 0$ such that

$$\|f\|_{L^n(\bar{\Omega}\cap B_r(x_0))} \le Cr^{\alpha}, \quad \forall\, 0 < r < r_0, \tag{1.3}$$

and denote

$$\|f\|_{C^{-1,\alpha}(x_0)} = \min \{C \mid \text{(1.3) holds with } C\}.$$

If $f \in C^{-1,\alpha}(x)$ for any $x \in \Omega$ with the same r_0 and

$$\|f\|_{C^{-1,\alpha}(\bar{\Omega})} := \sup_{x\in\Omega} \|f\|_{C^{-1,\alpha}(x)} < +\infty,$$

we say $f \in C^{-1,\alpha}(\bar{\Omega})$.

Remark 1.3. Without loss of generality, we can assume $r_0 = 1$ throughout this paper.

Remark 1.4. If Ω is a Lipschitz domain, the definition of $C^{1,\alpha}(\bar{\Omega})$ in Definition 1.1 is equivalent to the usual classical definition of $C^{1,\alpha}(\bar{\Omega})$ (see [9]).

Definition 1.5 ([13])**.** A bounded set Ω is called a convex polyhedron if it is the intersection of a finite number of closed half-spaces.

For an n-dimensional convex polyhedron Ω, let F_k $(k = 0, 1, \ldots, n-1)$ be its k-dimensional faces. Specially, 0-dimensional faces are vertices and 1-dimensional faces are edges. Then we classify the boundary points of Ω into two categories. For any $x_0 \in \partial\Omega$, if $x_0 \in F_{n-1}$, we call it the first class boundary point and denote $x_0 \in S_1$. If $x_0 \notin F_{n-1}$, we call it the second class boundary point and denote $x_0 \in S_2$.

We call that $F : S^n \times \Omega \to R$ is a fully nonlinear uniformly elliptic operator with ellipticity constants $0 < \lambda \leq \Lambda$ if

$$\lambda\|N\| \leq F(M+N,x) - F(M,x) \leq \Lambda\|N\|, \qquad \forall\, M,N \in S^n,\ N \geq 0, \tag{1.4}$$

where S^n denotes the set of $n \times n$ symmetric matrices; $\|N\|$ is the spectral radius of N and $N \geq 0$ means the nonnegativeness. The standard notions and notations such as Pucci operators $M^+(M,\lambda,\Lambda)$, $M^-(M,\lambda,\Lambda)$ and Pucci class $\bar{S}(\lambda,\Lambda,f)$, $\underline{S}(\lambda,\Lambda,f)$, $S^*(\lambda,\Lambda,f)$ will be used. For the details, one can refer to [1–3].

Now we state our main results.

Theorem 1.6 (boundary $C^{1,\gamma}$ regularity). *Let $0 < \alpha < \alpha_1$ where α_1 is a universal constant (see Lemma 2.1). Suppose that Ω is a convex polyhedron, $x_0 \in \partial\Omega$ and u is a viscosity solution of*

$$\begin{cases} u \in S^*(\lambda,\Lambda,f) & \text{in } \Omega; \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $f \in C^{-1,\alpha}(x_0)$ and $g \in C^{1,\alpha}(x_0)$. Then u is $C^{1,\gamma}$ at x_0, i.e., for any $x_0 \in \partial\Omega$, there exists a linear polynomial L_{x_0} such that

$$|u(x) - L_{x_0}(x)| \leq C|x-x_0|^{1+\gamma}\left(\|u\|_{L^\infty(\Omega)} + \|f\|_{C^{-1,\alpha}(x_0)} + \|g\|_{C^{1,\alpha}(x_0)}\right), |\ \forall\, x \in \bar{\Omega} \tag{1.6}$$

and

$$|Du(x_0)| \leq C\left(\|u\|_{L^\infty(\Omega)} + \|f\|_{C^{-1,\alpha}(x_0)} + \|g\|_{C^{1,\alpha}(x_0)}\right), \tag{1.7}$$

where $0 < \gamma < \alpha$ and C depend only on n, λ, Λ, α and Ω.

Remark 1.7. The viscosity solutions of (1.1) are in the classes $S^*(\lambda,\Lambda,f)$ (see [1, Proposition 2.13]). So all results for functions in the classes $S^*(\lambda,\Lambda,f)$ are valid for solutions of (1.1).

Combining the interior $C^{1,\gamma}$ estimate [1, Theorem 8.3], we have

Theorem 1.8 (global $C^{1,\gamma}$ regularity). *Let α and Ω be as in Theorem 1.6. Suppose that u is a viscosity solution of (1.1) with $f \in C^{-1,\alpha}(\bar{\Omega})$ and $g \in C^{1,\alpha}(\partial\Omega)$. Then there exists $\theta > 0$ depending only on n, λ, Λ and α such that if*

$$\beta_F(x) = \sup_{M \in S\setminus\{0\}} \frac{|F(M,x) - F(M,0)|}{\|M\|} \leq \theta, \quad \forall\, x \in \Omega,$$

then $u \in C^{1,\gamma}(\bar{\Omega})$ and

$$\|u\|_{C^{1,\gamma}(\bar{\Omega})} \leq C\left(\|u\|_{L^\infty(\Omega)} + \|f\|_{C^{-1,\alpha}(\bar{\Omega})} + \|g\|_{C^{1,\alpha}(\partial\Omega)}\right),$$

where $0 < \gamma < \alpha$ and C depend only on n, λ, Λ, α and Ω.

The following corollary of Theorem 1.8 is a new result for linear elliptic equations.

Corollary 1.9. *Let u be a viscosity solution of*

$$\begin{cases} -a^{ij}(x)\dfrac{\partial^2 u(x)}{\partial x_i x_j} = f & \text{in } \Omega; \\ u = g & \text{on } \partial\Omega, \end{cases}$$

where α, Ω, f and g are as in Theorem 1.8. Then there exists $\theta > 0$ depending only on n, λ, Λ and α such that if

$$\|a^{ij} - \delta_{ij}\|_{L^\infty(\Omega)} \leq \theta,$$

then $u \in C^{1,\gamma}(\bar{\Omega})$ and

$$\|u\|_{C^{1,\gamma}(\bar{\Omega})} \leq C\left(\|u\|_{L^\infty(\Omega)} + \|f\|_{C^{-1,\alpha}(\bar{\Omega})} + \|g\|_{C^{1,\alpha}(\partial\Omega)}\right), \tag{1.8}$$

where $0 < \gamma < \alpha$ and C depend only on n, λ, Λ, α and Ω.

Remark 1.10. The $C^{1,\gamma}$ estimate (1.8) is also called Cordes–Nirenberg estimate.

Remark 1.11. In this paper, C depending on n, λ, Λ, α and Ω will denote constants which may differ at different occurrences.

The main route of proving Theorem 1.6 is the following. For $x_0 \in S_1$, the $C^{1,\gamma}$ regularity can be obtained as a simple corollary of [11]. For $x_0 \in S_2$, there exist a half ball $B_r(x_0)$ and a cone K such that $\Omega \subset B_r(x_0)$ and $K \subset B_r(x_0) \cap \Omega^c$. This will lead to a higher regularity of u. In addition, if $f, g \equiv 0$, the solutions of (1.5) on the half ball have sufficient regularity (see Lemma 2.1). Noting that cone has the scaling invariance, the boundary $C^{1,\gamma}$ regularity for $x_0 \in S_2$ can be derived by perturbation and iteration techniques which are inspired by [10]. Then the boundary $C^{1,\gamma}$ regularity can be obtained by the technique of patching. Finally, the global $C^{1,\gamma}$ regularity will be deduced by combining the interior $C^{1,\gamma}$ estimate.

In Section 2, we will prove an important estimate (about the $C^{1,\gamma}$ regularity for $x_0 \in S_2$). Theorem 1.6 and Theorem 1.8 will be proved in Section 3. In this note, we use the following notations.

Notation

1. $R^n_+ = \{x \in R^n \mid x_n > 0\}$.
2. $B_r(x_0) = \{x \in R^n \mid ||x - x_0| < r\}$, $B_r = B_r(0)$, $B_r^+(x_0) = B_r(x_0) \cap R^n_+$ and $B_r^+ = B_r^+(0)$.
3. $T_r(x_0) = \{(x', 0) \in R^n \mid |x' - x_0'| < r\}$ and $T_r = T_r(0)$.
4. Ω^c: the complement of Ω and $\bar{\Omega}$: the closure of Ω, $\forall \Omega \subset R^n$.
5. $\Omega_r = \Omega \cap B_r$ and $(\partial\Omega)_r = \partial\Omega \cap B_r$.

2 An important estimate

In this section, we introduce some known lemmas. The first concerns the boundary $C^{1,\alpha}$ regularity for solutions with flat boundaries. It was first proved by Krylov [6] for classical solutions and further simplified by Caffarelli (see [4, Theorem 9.31] and [5, Theorem 4.28]), which is applicable to viscosity solutions (see [12]).

Lemma 2.1. *Let u be a viscosity solution of*

$$\begin{cases} u \in S(\lambda, \Lambda, 0) & \text{in } B_1^+; \\ u = 0 & \text{on } T_1. \end{cases}$$

Then u *is* C^{1,α_1} *at* 0*, i.e., there exists a constant* a *such that*

$$|u(x) - ax_n| \le C_1 |x|^{1+\alpha_1} \|u\|_{L^\infty(B_1^+)}, \qquad \forall\, x \in B_{1/2}^+$$

and

$$|a| \le C_1 \|u\|_{L^\infty(B_1^+)},$$

where α_1 *and* C_1 *depend only on* n*,* λ *and* Λ*.*

The next Lemma presents the boundary $C^{1,\alpha}$ estimate for solutions of fully nonlinear elliptic equations with the suitable right hand function f and the boundary value g on the curved boundary (see [11, Theorem 1.6]).

Lemma 2.2. *Let* $0 < \alpha_2 < \alpha_1$ *where* α_1 *is a universal constant (see Lemma 2.1). Suppose that* $\partial\Omega$ *is* C^{1,α_2} *at* 0 *and* u *is a viscosity solution of*

$$\begin{cases} u \in S(\lambda, \Lambda, f) & \text{in } \Omega \cap B_1; \\ u = g & \text{on } \partial\Omega \cap B_1, \end{cases}$$

where $f \in C^{-1,\alpha_2}(0)$ *and* $g \in C^{1,\alpha_2}(0)$*. Then* u *is* C^{1,α_2} *at* 0*, i.e., there exists a linear polynomial* $\tilde{L}_0$ *such that*

$$|u(x) - \tilde{L}_0(x)| \le \tilde{C} |x|^{1+\alpha_2} \left(\|u\|_{L^\infty(\Omega\cap B_1)} + \|f\|_{C^{-1,\alpha_2}(0)} + \|g\|_{C^{1,\alpha_2}(0)} \right), \qquad \forall\, x \in \Omega \cap B_{1/2}$$

and

$$|Du(0)| \le \tilde{C} \left(\|u\|_{L^\infty(\Omega\cap B_1)} + \|f\|_{C^{-1,\alpha_2}(0)} + \|g\|_{C^{1,\alpha_2}(0)} \right),$$

where $\tilde{C}$ *depends only on* n, λ, Λ*,* α_2 *and* Ω*.*

Remark 2.3. The $C^{1,\gamma}$ regularity for the viscosity solutions of (1.5) at $x_0 \in S_1$ is true as a special case of Lemma 2.2.

The next is a Hopf type lemma (see [14, Lemma 2.15]).

Lemma 2.4. *Let* $\Gamma \subset \partial B_1^+ \backslash T_1$*, and* u *be a viscosity solution of*

$$\begin{cases} M^-(D^2 u, \lambda, \Lambda) = 0 & \text{in } B_1^+; \\ u = x_n & \text{on } \Gamma; \\ u = 0 & \text{on } \partial B_1^+ \backslash \Gamma. \end{cases}$$

Then

$$u(x) \ge c_1 x_n \quad \text{in } B_{1/2}^+,$$

where $c_1 > 0$ *depends only on* n, λ*,* Λ *and* Γ*.*

It has been known that if Ω occupies a smaller portion in a ball centered at 0 (e.g. $|\Omega \cap B_r|/|B_r|$ is smaller), the regularity of u is higher (roughly speaking). Inspired by this, we have the following result.

Theorem 2.5. *Let* α *and* Ω *be as in Theorem 1.6. Suppose that* $x_0 \in S_2$ *and* u *is a viscosity solution of (1.5) with* $f \in C^{-1,\alpha}(x_0)$ *and* $g \in C^{1,\alpha}(x_0)$*. Then* u *is* $C^{1,\gamma}$ *at* x_0*, i.e., for any* $x_0 \in S_2$*, there exists a linear polynomial* $\bar{L}_{x_0}$ *such that*

$$|u(x) - \bar{L}_{x_0}(x)| \le C|x - x_0|^{1+\gamma} \left(\|u\|_{L^\infty(\Omega\cap B_1)} + \|f\|_{C^{-1,\alpha}(x_0)} + \|g\|_{C^{1,\alpha}(x_0)} \right), \qquad \forall\, x \in \bar{\Omega} \quad (2.1)$$

with

$$Du(x_0) = Dg(x_0), \tag{2.2}$$

where $0 < \gamma < \alpha$ *and* C *depend only on* n, λ, Λ*,* α *and* Ω*.*

Proof. For $x_0 \in S_2$, we can assume that $x_0 = 0$, $\Omega \subset R^n_+$ and there exists a cone $K \subset \Omega^c \cap R^n_+$ with 0 being the vertex (by translating and rotating the coordinate system). Further, we assume that $g(0) = 0$ and $Dg(0) = 0$. Otherwise, we can consider $v(x) = u(x) - g(0) - Dg(0) \cdot x$, then the regularity of u follows easily from v. Let $C_g = [g]_{C^{1,\alpha}(0)}$, then

$$|g(x)| \le C_g |x|^{1+\alpha}, \qquad \forall\, x \in (\partial\Omega)_1. \tag{2.3}$$

Let $M = \|u\|_{L^\infty(\Omega\cap B_1)} + \|f\|_{C^{-1,\alpha}(0)} + \|g\|_{C^{1,\alpha}(0)}$. To prove Theorem 2.5, we only need to show that there exists a nonnegative sequence $\{a_k\}$ $(k \ge -1)$ with $a_0 = 0$ such that for all $k \ge 0$,

$$\sup_{\Omega_{\eta^k}} (u - a_k x_n) \le \hat{C} M \eta^{k(1+\alpha)}, \tag{2.4}$$

$$\inf_{\Omega_{\eta^k}} (u + a_k x_n) \ge -\hat{C} M \eta^{k(1+\alpha)} \tag{2.5}$$

and

$$a_k \le (1 - c_1) a_{k-1} + \bar{C}\hat{C} M \eta^{(k-1)\alpha}, \tag{2.6}$$

where $\bar{C}$ depends only on n, λ and Λ; $0 < c_1 < 1$ depends only on n, λ, Λ and Ω; $\hat{C}$ and $0 < \eta < 1/4$ depend only on n, λ, Λ and α.

Now we show that (2.4)-(2.6) imply that u is $C^{1,\gamma}$ at 0. Indeed, from (2.6), we have

$$a_k \le \bar{C}\hat{C} M \sum_{i=0}^{k-1} (1 - c_1)^{k-1-i} \eta^{i\alpha} \le \bar{C}\hat{C} M \eta^{(k-1)\gamma} \sum_{i=0}^{k-1} \eta^{i(\alpha-\gamma)} \le C M \eta^{k\gamma},$$

provided

$$1 - c_1 \le \eta^\gamma, \quad 0 < \gamma < \alpha.$$

For any $x \in \Omega_1$, there exists $k \ge 0$ such that $\eta^{k+1} \le |x| < \eta^k$. From (2.4), we have

$$u(x) \le \sup_{\Omega_{\eta^k}} (u - a_k x_n) + a_k x_n \le C M \eta^{k(1+\gamma)} \le C M |x|^{1+\gamma}.$$

Similarly, (2.5) and (2.6) imply

$$u(x) \ge -C M |x|^{1+\gamma}.$$

Therefore, u is $C^{1,\gamma}$ at 0 with $Du(0) = Dg(0)$.

We only give the proofs of (2.4) and (2.6); the proof of (2.5) is similar with (2.4) and we omit it. We prove (2.4) and (2.6) by induction. For $k = 0$, by setting $a_{-1} = 0$, they hold clearly. Supposing that they hold for k, we need to prove that they hold for $k + 1$.

Let $r = \eta^k/2$ and v_1 solve

$$\begin{cases} M^+(D^2 v_1, \lambda, \Lambda) = 0 & \text{in } B_r^+; \\ v_1 = 0 & \text{on } T_r; \\ v_1 = \hat{C} M \eta^{k(1+\alpha)} & \text{on } \partial B_r^+ \backslash T_r. \end{cases}$$

By the boundary $C^{1,\alpha}$ estimate for v_1 (see Lemma 2.1) and the maximum principle, there exists $\bar{a} \ge 0$ such that

$$\begin{aligned} \|v_1 - \bar{a} x_n\|_{L^\infty(\Omega_{\eta^{k+1}})} &= \|v_1 - \bar{a} x_n\|_{L^\infty(\Omega_{2\eta r})} \\ &\le C_1 \frac{|x|^{1+\alpha_1}}{r^{1+\alpha_1}} \|v_1\|_{L^\infty(B_r^+)} \\ &\le C_1 \eta^{\alpha_1 - \alpha} \cdot \hat{C} M \eta^{(k+1)(1+\alpha)} \end{aligned} \tag{2.7}$$

and

$$\bar{a} \le C_1 \hat{C} M \eta^{k\alpha},$$

where α_1 and C_1 depend only on n, λ and Λ.

Let v_2 solve

$$\begin{cases} M^-(D^2 v_2, \lambda, \Lambda) = 0 & \text{in } B_r^+; \\ v_2 = a_k x_n & \text{on } \partial B_r^+ \cap K; \\ v_2 = 0 & \text{on } \partial B_r^+ \backslash K. \end{cases}$$

By Lemma 2.4, there exists $0 < c_1 < 1$ depending only on n, λ, Λ and K such that

$$v_2 \ge c_1 a_k x_n \quad \text{in } B_{2\eta r}^+. \tag{2.8}$$

In addition, by the comparison principle,

$$v_2 \le a_k x_n \quad \text{in } B_r^+.$$

Letting $w = u - a_k x_n - v_1 + v_2$, it follows that (note that $v_1, v_2 \ge 0$)

$$\begin{cases} w \in \underline{S}(\lambda, \Lambda, -|f|) & \text{in } \Omega \cap B_r^+; \\ w \le g & \text{on } \partial\Omega \cap B_r^+; \\ w \le 0 & \text{on } \partial B_r^+ \cap \bar{\Omega}. \end{cases}$$

By the Alexandrov–Bakel'man–Pucci maximum principle, we have

$$\sup_{\Omega_{\eta^{k+1}}} w \le \sup_{\Omega_r} w \le C_g \eta^{k(1+\alpha)} + C_2 r \|f\|_{L^n(\Omega_r)} \le \frac{1 + C_2}{\hat{C}\eta^{1+\alpha}} \cdot \hat{C} M \eta^{(k+1)(1+\alpha)}, \tag{2.9}$$

where C_2 depend only on n, λ and Λ.

Let $\bar{C} := C_1$. Take η small enough such that

$$C_1 \eta^{\alpha_1 - \alpha} \le \frac{1}{2}.$$

Next, take $\hat{C}$ large enough such that

$$\frac{1 + C_2}{\hat{C}\eta^{1+\alpha}} \le \frac{1}{2}.$$

Let $a_{k+1} = (1 - c_1) a_k + \bar{a}$. Then (2.6) holds for $k + 1$. Recalling (2.7), (2.8) and (2.9), we have

$$\begin{aligned} u - a_{k+1} x_n &= u - a_k x_n - v_1 + v_2 + v_1 - a x_n + c_1 a_k x_n - v_2 \\ &= w + v_1 - a x_n + c_1 a_k x_n - v_2 \\ &\le w + v_1 - a x_n \\ &\le \hat{C} M \eta^{(k+1)(1+\alpha)} \quad \text{in } \Omega_{\eta^{k+1}}. \end{aligned}$$

By induction, the proofs of (2.4) and (2.6) are completed. □

3 Proofs of the main results

Combining Theorem 2.5 and Lemma 2.2, we give the

Proof of Theorem 1.6. We only need to prove that for any $x_0 \in S_1$, there exists a linear polynomial L_{x_0} such that

$$|u(x) - L_{x_0}(x)| \leq C|x - x_0|^{1+\gamma}, \qquad \forall\, x \in \bar{\Omega}. \tag{3.1}$$

In fact, for any $x_0 \in S_1$, there exists $y \in S_2$ such that $|y - x_0| = d_{x_0} = d(x_0, S_2)$. We know from Theorem 2.5 that there exist a linear polynomial $\bar{L}_y$ and a constant C such that

$$|u(x) - \bar{L}_y(x)| \leq C|x - y|^{1+\gamma}, \qquad \forall\, x \in \bar{\Omega}. \tag{3.2}$$

Let $v(x) = u(x) - \bar{L}_y(x)$. There exists a constant $0 < \tau \leq 1$ (depending only on Ω) such that $\Omega \cap B_{\tau d_{x_0}}(x_0)$ is a half ball. That is, $\Omega \cap B_{\tau d_{x_0}}(x_0) = \{x \in R^n | \vec{n} \cdot (x - x_0) > 0\} \cap B_{\tau d_{x_0}}(x_0)$, where $\vec{n}$ is the unit inward normal of Ω at x_0. Applying Lemma 2.2 in $\Omega \cap B_{\tau d_{x_0}}(x_0)$ and recalling (3.2), there exists a linear polynomial

$$R_{x_0}(x) = R(x_0) + DR(x_0) \cdot (x - x_0)$$

such that

$$|R(x_0)| = |v(x_0)| \leq C|d_{x_0}|^{1+\gamma},$$
$$|DR(x_0)| \leq C|\tau d_{x_0}|^{\gamma} \leq C|d_{x_0}|^{\gamma}$$

and

$$\begin{aligned} |v(x) - R_{x_0}(x)| &\leq C\frac{|x - x_0|^{1+\gamma}}{|\tau d_{x_0}|^{1+\gamma}} \left(\|v\|_{L^\infty(\Omega \cap B_{\tau d_{x_0}}(x_0))} + |\tau d_{x_0}|^{1+\gamma} (\|f\|_{C^{-1,\alpha}(x_0)} + \|g\|_{C^{1,\alpha}(x_0)}) \right) \\ &\leq C|x - x_0|^{1+\gamma}, \qquad \forall\, x \in \Omega \cap B_{\tau d_{x_0}/2}(x_0). \end{aligned} \tag{3.3}$$

Define

$$L_{x_0}(x) = \bar{L}_y(x) + R_{x_0}(x).$$

If $|x - x_0| < \tau d_{x_0}/2$, by (3.3), we have

$$|u(x) - L_{x_0}(x)| = |v(x) - R_{x_0}(x)| \leq C|x - x_0|^{1+\gamma}.$$

If $|x - x_0| \geq \tau d_{x_0}/2$, by (3.2), we have

$$\begin{aligned} |u(x) - L_{x_0}(x)| &\leq |u(x) - \bar{L}_y(x)| + |R_{x_0}(x)| \\ &\leq C|x - y|^{1+\gamma} + |R(x_0)| + |DR(x_0)||x - x_0| \\ &\leq C|x - x_0|^{1+\gamma}. \end{aligned}$$

Combining the two cases, we get (3.1). □

The proof of the global $C^{1,\gamma}$ regularity is ended by Theorem 1.6 and the interior $C^{1,\gamma}$ estimate. Now we give the details.

Proof of Theorem 1.8. For any $x_0 \in \Omega$, there exists $y \in \partial\Omega$ such that $|y - x_0| = d_{x_0} = d(x_0, \partial\Omega)$. Then from Theorem 1.6 and Remark 1.7, there exist a linear polynomial L_y and a constant C such that

$$|u(x) - L_y(x)| \leq C|x - y|^{1+\gamma}, \quad \forall\, x \in \bar{\Omega}. \tag{3.4}$$

Let $v(x) = u(x) - L_y(x)$. By the interior $C^{1,\alpha}$ estimate in $B_{d_{x_0}}(x_0)$ and (3.4), there exists a linear polynomial

$$Q_{x_0}(x) = Q(x_0) + DQ(x_0) \cdot (x - x_0)$$

such that

$$\begin{aligned} |Q(x_0)| = |v(x_0)| &\leq C|d_{x_0}|^{1+\gamma}, \\ |DQ(x_0)| &\leq C|d_{x_0}|^{\gamma} \end{aligned}$$

and

$$\begin{aligned} |v(x) - Q_{x_0}(x)| &\leq C\frac{|x - x_0|^{1+\gamma}}{|d_{x_0}|^{1+\gamma}} \left(\|v\|_{L^\infty(B_{d_{x_0}}(x_0))} + |d_{x_0}|^{1+\gamma}(\|f\|_{C^{-1,\alpha}(x_0)} + \|g\|_{C^{1,\alpha}(x_0)})\right) \\ &\leq C|x - x_0|^{1+\gamma}, \qquad \forall\, x \in B_{d_{x_0}/2}(x_0). \end{aligned} \tag{3.5}$$

Define

$$P_{x_0}(x) = L_y(x) + Q_{x_0}(x).$$

If $|x - x_0| < d_{x_0}/2$, by (3.5), we have

$$|u(x) - P_{x_0}(x)| = |v(x) - Q_{x_0}(x)| \leq C|x - x_0|^{1+\gamma}.$$

If $|x - x_0| \geq d_{x_0}/2$, by (3.4), we have

$$\begin{aligned} |u(x) - P_{x_0}(x)| &\leq |u(x) - L_y(x)| + |Q_{x_0}(x)| \\ &\leq C|x - y|^{1+\gamma} + |Q(x_0)| + |DQ(x_0)||x_0 - x| \\ &\leq C|x - x_0|^{1+\gamma}. \end{aligned}$$

Combining the two cases, it follows that for any $x_0 \in \Omega$, there exists a linear polynomial P_{x_0} such that

$$|u(x) - P_{x_0}(x)| \leq C|x - x_0|^{1+\gamma}, \qquad \forall\, x \in \bar{\Omega}.$$

The proof of Theorem 1.8 is finished. □

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The existence of ground state solutions for semi-linear degenerate Schrödinger equations with steep potential well

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Abstract. In this article, we study the following degenerated Schrödinger equations:

$$\begin{cases} -\Delta_\gamma u + \lambda V(x) u = f(x,u) & \text{in } \mathbb{R}^N, \\ u \in E_\lambda \ , \end{cases}$$

where $\lambda > 0$ is a parameter, Δ_γ is a degenerate elliptic operator, the potential $V(x)$ has a potential well with bottom and the nonlinearity $f(x,u)$ is either super-linear or sub-linear at infinity in u. The existence of ground state solution be obtained by using the variational methods.

Keywords: steep well potential, mountain pass theorem, strongly degenerate elliptic operator.

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1 Introduction

This article is concerned with a class of nonlinear Schrödinger equations:

$$\begin{cases} -\Delta_\gamma u + \lambda V(x) u = f(x,u) & \text{in } \mathbb{R}^N, \\ u \in E_\lambda \ , \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a parameter, Δ_γ is a degenerate elliptic operator of the form

$$\Delta_\gamma := \sum_{j=1}^{N} \partial_{x_j}(\gamma_j^2 \partial_{x_j}), \qquad \partial_{x_j} = \frac{\partial}{\partial x_j}, \qquad \gamma = (\gamma_1(x), \gamma_2(x), \dots, \gamma_N(x)).$$

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Here, the functions $\gamma_j : \mathbb{R}^N \to \mathbb{R}$ are assumed to be continuous, different from zero and of class C^1 in $\mathbb{R}^N \setminus \Pi$, where

$$\Pi := \left\{ x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : \prod_{j=1}^{N} x_j = 0 \right\}.$$

Moreover, the function γ_j satisfy the following properties:

(i) There exists a semigroup of dilations $\{\delta_t\}_{t>0}$ such that

$$\delta_t : \mathbb{R}^N \to \mathbb{R}, \qquad \delta_t(x_1, \ldots, x_N) = (t^{\varepsilon_1} x_1, \ldots, t^{\varepsilon_N} x_N),$$

where $1 = \varepsilon_1 \leq \varepsilon_2 \leq \cdots \leq \varepsilon_N$, such that γ_j is δ_t-homogeneous of degree $\varepsilon_j - 1$, i.e.

$$\gamma_j(\delta_t(x)) = t^{\varepsilon_j - 1}\gamma_j(x), \qquad \forall x \in \mathbb{R}^N,\ \forall t > 0,\ j = 1, \ldots, N.$$

The number

$$\tilde{N} := \sum_{j=1}^{N} \varepsilon_j$$

is called the homogeneous dimension of $\mathbb{R}^N$ with respect to $\{\delta_t\}_{t>0}$.

(ii) $\gamma_1 = 1,\ \gamma_j(x) = \gamma_j(x_1, x_2, \ldots, x_{j-1}),\ j = 2, \ldots, N.$

(iii) There exists a constant $\rho \geq 0$ such that

$$0 \leq x_k \partial_{x_k} \gamma_j(x) \leq \rho \gamma_j(x), \qquad \forall k \in \{1, 2, \ldots, j-1\},\ \forall j = 2, \ldots, N,$$

and for every $x \in \overline{\mathbb{R}}^N_+ = \{(x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : x_j \geq 0,\ \forall j = 1, 2, \ldots, N\}$.

(iv) Equalities $\gamma_j(x) = \gamma_j(x^*) (j = 1, 2, \ldots, N)$ are satisfied for every $x \in \mathbb{R}^N$, where

$$x^* = (|x_1|, \ldots, |x_N|), \quad \text{if } x = (x_1, x_2, \ldots, x_N).$$

The Δ_γ-operator contains the following operator of Grušin-type

$$G_\alpha := \Delta_x + |x|^{2\alpha}\Delta_y, \qquad \alpha \geq 0,$$

where (x, y) denotes the point of $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$. This operator was studied by Grušin in [8] when α is an integer, and by Franchi and Lanconeli in [6,7], Loiudice in [11], Monti and Morbidelli in [13] when α is not an integer. The Δ_γ-operator also contains following semi-linear strongly degenerate operator

$$P_{\alpha,\beta} = \Delta_x + \Delta_y + |x|^{2\alpha}|y|^{2\beta}\Delta_z, \qquad (x, y, z) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3},$$

where α, β are nonnegative real numbers. The $P_{\alpha,\beta}$-operator was studied in [1]. For more information about the operator Δ_γ, please see [10].

In this paper, we study the existence of ground state solutions for the equation (1.1) under the assumptions that V is neither radially symmetric nor coercive. Precisely, we make the following assumptions.

$(V1)$ $V(x) \in C(\mathbb{R}^N, \mathbb{R})$ satisfying $\inf_{x\in\mathbb{R}^N} V(x) > 0$.

$(V2)$ There exists $b > 0$ such that the set $\{x \in \mathbb{R}^N : V(x) < b\}$ is nonempty and has finite measure.

The conditions $(V1) \sim (V2)$ are special cases of steep potential well which were first introduced by Bartsch and Wang in [2]. In recent years, steep potential well are widely used in various equation, such as Schrödinger equations, Schrödinger–Poisson equations and Klein–Gordon–Maxwell system and so on (see [2–4, 9, 14, 15]).

Nextly, wee will require that the nonlinear term satisfies either the assumptions:

$(f1)'$ $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and there are constants $0 < a_1 < a_2 < a_3 \cdots < a_m < 1$ and functions $b_i(x) \in L^{\frac{2}{1-a_i}}(\mathbb{R}^N, (0, +\infty))$ such that

$$|f(x,z)| \le \sum_{i=1}^{m} (a_i + 1) b_i(x) |z|^{a_i}, \qquad \forall (x,z) \in \mathbb{R}^N \times \mathbb{R};$$

$(f2)'$ There exist constants $\eta, \delta > 0, a_0 \in (1,2), \Omega \subset \mathbb{R}^N$ such that $\text{meas}(\Omega) \neq 0$ and

$$F(x,z) = \int_0^z f(x,t)dt \ge \eta |z|^{a_0}, \qquad \forall x \in \Omega \text{ and } \forall\ |z| \le \delta,$$

or the assumptions:

$(f1)$ $\lim_{|z|\to 0} \frac{f(x,z)}{|z|} = 0$ uniformly for $x \in \mathbb{R}^N$.

$(f2)$ For some $2 < p < 2^*_\gamma$, $C_0 > 0$,

$$|f(x,z)| \le C_0(|z| + |z|^{p-1}), \qquad \forall (x,z) \in \mathbb{R}^N \times \mathbb{R},$$

where $2^*_\gamma := \frac{2\tilde{N}}{\tilde{N}-2}$ is the critical Sobolev exponent;

$(f3)$ $F(x,z) := \int_0^z f(x,t)dt \ge 0$ for all $x \in \mathbb{R}^N$, and

$$\lim_{|z|\to+\infty} \frac{F(x,z)}{|z|^2} = +\infty, \qquad \forall (x,z) \in \mathbb{R}^N \times \mathbb{R};$$

$(f4)$ There exist $a_1 > 0, L_0 > 0$ and $\tau > \frac{\tilde{N}}{2}$, such that

$$|f(x,z)|^\tau \le a_1 \mathcal{F}(x,z)|z|^\tau, \quad \text{for all } x \in \mathbb{R}^N \text{ and } |z| \ge L_0,$$

where

$$\mathcal{F}(x,z) := \frac{1}{2} f(x,z)z - F(x,z) \ge 0, \qquad \forall (x,z) \in \mathbb{R}^N \times \mathbb{R};$$

$(f5)$ $\frac{f(x,z)}{|z|}$ is an increasing function of z on $\mathbb{R} \setminus \{0\}$ for every $x \in \mathbb{R}^N$.

Before stating our main results, we give several notations. For $\lambda > 0$, let

$$S^2_\gamma(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) : \gamma_j \partial_{x_j} u \in L^2(\mathbb{R}^N),\ j = 1, \ldots, N \right\},$$
$$E_\lambda := \left\{ u \in S^2_\gamma(\mathbb{R}^N) : \int_{\mathbb{R}^N} \lambda V(x) u^2 dx < +\infty \right\}.$$

Then, by assumption $(V1)$, E_λ is a Hilbert space with the inner product and norm respectively

$$(u,v)_\lambda = \int_{\mathbb{R}^N} (\nabla_\gamma u \nabla_\gamma v + \lambda V(x) uv) dx, \qquad \|u\|_\lambda = (u,u)_\lambda^{\frac{1}{2}},$$

where

$$\nabla_\gamma u = (\gamma_1 \partial_{x_1} u, \gamma_2 \partial_{x_2} u, \dots, \gamma_N \partial_{x_N} u).$$

Obviously, the embedding $E_\lambda \hookrightarrow S_\gamma^2(\mathbb{R}^N)$ is continuous. It follows that $E_\lambda \hookrightarrow L^s(\mathbb{R}^N)$ is continuous for each $s \in [2, 2_\gamma^*]$ (see [12]). Thus for each $2 \le s \le 2_\gamma^*$, there exists $d_s > 0$ such that

$$|u|_s \le d_s \|u\|_\lambda, \qquad \forall u \in E_\lambda, \tag{1.2}$$

where $L^s(\mathbb{R}^N)$ denote a Lebesgue space, the norm in $L^s(\mathbb{R}^N)$ is denoted by $|\cdot|_s$.

We point out that there are Rellich-type compact embeddings hold on bounded domains for subcritical exponents. By $S_\gamma^2(\Omega)$ we denote the set of all functions $u \in L^2(\Omega)$ such that $\gamma_j \partial_{x_j} u \in L^2(\Omega)$ for all $j = 1, \dots, N$, where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$. The space $S_{\gamma,0}^2(\Omega)$ is defined as the closure of $C_0^1(\Omega)$ in the space $S_\gamma^2(\Omega)$. We define the norm on this space as

$$\int_\Omega (|\nabla_\gamma u|^2 + \lambda V(x) u^2) dx,$$

which is equivalent to $\int_\Omega |\nabla_\gamma u|^2 dx$, by $(V1)$. Then, we have that the embedding $S_{\gamma,0}^2(\Omega) \hookrightarrow L^s(\Omega)$ is compact for every $s \in [1, 2_\gamma^*)$ (see Proposition 3.2. in [10]).

We can now state the main result:

Theorem 1.1. *Assume $(V1)$ and $(f1)' \sim (f2)'$ are satisfied. Then $\forall \lambda > 0$, problem (1.1) admits at least a ground state solution in E_λ.*

Remark 1.2. To the best of our knowledge, it seems that Theorem 1.1 is the first result about the existence of ground state solutions for the semi-linear Δ_γ differential equation in $\mathbb{R}^N$. By the way, we would like to point out that in [12] the authors study existence of infinitely many solutions for semi-linear degenerate Schrödinger equations with the potential $V(x)$ satisfying the coercivity condition which implies $E_\lambda \hookrightarrow\hookrightarrow L^s(\mathbb{R}^N)$ for any $s \in [2, 2_\gamma^*)$.

Theorem 1.3. *Assume $(V1)$, $(V2)$ and $(f1) \sim (f5)$ are satisfied. Then there exists $\Lambda > 0$ such that problem (1.1) has at least a ground state solution in E_λ, for all $\lambda > \Lambda$.*

Remark 1.4. We point out that the Schrödinger equation with general steep potential well is considered in reference [3, 4], but they consider a special nonlinear term, where $f(x,z) = |z|^{p-2}z (2 < p < 2^*)$. At the same time, we also point out that although the Schrödinger equation with general steep potential well and the general nonlinear term are considered in reference [2, 9], the nonlinear term there satisfies the following Ambrosetti–Rabinowitz type condition:

(AR) There exist $\mu > 2$ and $L > 0$, such that

$$\mu F(x,z) \le z f(x,z), \qquad \forall x \in \mathbb{R}^N,\ \forall |z| \ge L.$$

The nonlinear term we consider here is not required to satisfy the Ambrosetti–Rabinowitz type condition, for example we allow nonlinearities of the type

$$f(x,z) = 2z \ln(1+z^2) + \frac{2z^3}{1+z^2}, \qquad \forall (x,z) \in \mathbb{R}^3 \times \mathbb{R}.$$

By a simple calculation, we have

$$F(x,z) = \int_0^z f(x,t)dt = z^2\ln(1+z^2), \qquad \mathcal{F}(x,z) = \frac{2z^4}{1+z^2},$$

and

$$zf(x,z) - \mu F(x,z) = z^2\left((2-\mu)\ln(1+z^2) + \frac{2z^2}{1+z^2}\right).$$

Now, it is easy to verify that the function f satisfies our assumptions and does not satisfy the Ambrosetti–Rabinowitz type condition.

To obtain our main results, we have to overcome some difficulties in our proof. The main difficulty consists in the lack of compactness of the $E_\lambda \hookrightarrow L^s(\mathbb{R}^N)$ with $s \in [2, 2^*_\gamma]$. Since we assume that the potential is not radially symmetric, we cannot use the usual way to recover compactness, for example, restricting in the subspace of radial functions of E_λ. We also cannot borrow some ideas in [12] to recover compactness because the potential do not satisfied the coercivity condition. To recover the compactness, we establish the parameter dependent compactness conditions.

Now, we define the following energy functional

$$J_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla_\gamma u|^2 + \lambda V(x)u^2)dx - \int_{\mathbb{R}^N} F(x,u)dx, \tag{1.3}$$

for any $u \in E_\lambda$. It is well known that J_λ is a C^1 functional with derivative given by

$$\langle J'_\lambda(u), v\rangle = \int_{\mathbb{R}^N}(\nabla_\gamma u \nabla_\gamma v + \lambda V(x)uv)dx - \int_{\mathbb{R}^N} f(x,u)vdx, \tag{1.4}$$

for any $u, v \in E_\lambda$. We have that u is a weak solution of equation (1.1) if only if it is a critical point of $J_\lambda(u)$ in E_λ.

2 The proof of main results for f sub-linear at infinity in u

Lemma 2.1 (see [17]). *Let E be a real Banach space and $J \in C^1(E,\mathbb{R})$ satisfy the (PS) condition. If J is bounded from below, then $c = \inf_E J$ is critical value of J.*

Lemma 2.2. *Assume that $(V1)$ and $(f1)'$ hold, then J_λ is bounded from below.*

Proof. It follows from $(f1)'$ that we can get

$$|F(x,z)| \le \sum_{i=1}^m b_i(x)|z|^{a_i+1}, \qquad \forall (x,z) \in \mathbb{R}^N \times \mathbb{R}. \tag{2.1}$$

The above inequality combined with the Hölder inequality and (1.2) shows that

$$\begin{aligned}\int_{\mathbb{R}^N}|F(x,z)|dx &\le \int_{\mathbb{R}^N}\sum_{i=1}^m b_i(x)|z|^{a_i+1}dx \\ &\le \sum_{i=1}^m\left(\int_{\mathbb{R}^N}|b_i(x)|^{\frac{2}{1-a_i}}dx\right)^{\frac{1-a_i}{2}}\left(\int_{\mathbb{R}^N}|z|^2dx\right)^{\frac{1+a_i}{2}} \\ &\le \sum_{i=1}^m d_2^{1+a_i}|b_i(x)|_{\frac{2}{1-a_i}}\|z\|_\lambda^{1+a_i}.\end{aligned} \tag{2.2}$$

Thus

$$\begin{aligned} J_\lambda(u) &= \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla_\gamma u|^2 + \lambda V(x)u^2)dx - \int_{\mathbb{R}^N} F(x,u)dx \\ &\geq \frac{1}{2}\|u\|_\lambda^2 - \sum_{i=1}^m d_2^{1+a_i}|b_i(x)|_{\frac{2}{1-a_i}}\|u\|_\lambda^{1+a_i}. \end{aligned}$$

In view of $0 < a_1 < a_2 < a_3 < \cdots < a_m < 1$ and $b_i(x) \in L^{\frac{2}{1-a_i}}(\mathbb{R}^N,(0,+\infty))$, it is clearly shows that J_λ is coercive, then J_λ is bounded from below. □

Lemma 2.3. *Assume that $(V1)$ and $(f1)'$ are satisfied, then J_λ satisfies the (PS) condition for each $\lambda > 0$.*

Proof. We suppose that $\{u_n\}$ is a Palais–Smale sequence of J_λ, that is for some $c_\lambda \in \mathbb{R}$, $J_\lambda(u_n) \to c_\lambda$, $J'_\lambda(u_n) \to 0$, as $n \to \infty$. According to lemma 2.2, $\{u_n\}$ is bounded in E_λ. Therefore, up to a subsequence, there are $u \in E_\lambda$, we have

$$\begin{aligned} &u_n \rightharpoonup u, \quad \text{in } E_\lambda; \\ &u_n \to u, \quad \text{in } L^s_{loc}(\mathbb{R}^N),\ 2 \leq s < 2^*_\gamma. \end{aligned} \tag{2.3}$$

By $(f1)'$, for any fixed $\varepsilon > 0$, we can choose $R_\varepsilon > 0$ such that

$$\left(\int_{\mathbb{R}^N - B_{R_\varepsilon}} |b_i(x)|^{\frac{2}{1-a_i}}dx\right)^{\frac{1-a_i}{2}} < \varepsilon, \qquad i = 1,2,\ldots,m. \tag{2.4}$$

It follows that (2.3), we obtain that

$$\lim_{n\to\infty}\int_{B_{R_\varepsilon}} |u_n - u|^2 dx = 0.$$

Hence, there exists $N_0 \in \mathbb{N}$ such that we have

$$\int_{B_{R_\varepsilon}} |u_n - u|^2 dx < \varepsilon^2, \qquad \forall n \geq N_0. \tag{2.5}$$

Combing this with the Hölder inequality and $(f1)'$, for any $n \geq N_0$ we have that

$$\begin{aligned} &\int_{B_{R_\varepsilon}} |f(x,u_n) - f(x,u)||u_n - u|dx \\ &\leq \left(\int_{B_{R_\varepsilon}} |f(x,u_n) - f(x,u)|^2 dx\right)^{\frac{1}{2}} \left(\int_{B_{R_\varepsilon}} |u_n - u|^2 dx\right)^{\frac{1}{2}} \\ &\leq \left(\int_{B_{R_\varepsilon}} |f(x,u_n) - f(x,u)|^2 dx\right)^{\frac{1}{2}} \cdot \varepsilon \\ &\leq \left\{\int_{B_{R_\varepsilon}} 2m\left[\sum_{i=1}^m (a_i+1)^2 b_i^2(x)|u_n|^{2a_i} + \sum_{i=1}^m (a_i+1)^2 b_i^2(x)|u|^{2a_i}\right]dx\right\}^{\frac{1}{2}} \cdot \varepsilon \\ &\leq \sqrt{2m}\left[\sum_{i=1}^m (a_i+1)^2 |b_i(x)|^2_{\frac{2}{1-a_i}}(|u_n|_2^{2a_i} + |u|_2^{2a_i})\right]^{\frac{1}{2}} \cdot \varepsilon. \end{aligned} \tag{2.6}$$

Again by $(f1)'$, the Hölder inequality and (2.4), we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^N - B_{R_\varepsilon}} |f(x,u_n) - f(x,u)||u_n - u|dx \\
&\quad \le \int_{\mathbb{R}^N - B_{R_\varepsilon}} \sum_{i=1}^{m} (a_i+1) b_i(x) (|u_n|^{a_i+1} + |u|^{a_i}|u_n| + |u_n|^{a_i}|u| + |u|^{a_i+1}) dx \\
&\quad \le \sum_{i=1}^{m} (a_i+1) \left(\int_{\mathbb{R}^N - B_{R_\varepsilon}} |b_i|^{\frac{2}{1-a_i}} dx \right)^{\frac{1-a_i}{2}} \left(|u_n|_2^{a_i+1} + |u|_2^{a_i}|u_n|_2 + |u_n|_2^{a_i}|u|_2 + |u|_2^{a_i+1} \right) \\
&\quad \le \varepsilon \sum_{i=1}^{m} (a_i+1) \left(|u_n|_2^{a_i+1} + |u|_2^{a_i}|u_n|_2 + |u_n|_2^{a_i}|u|_2 + |u|_2^{a_i+1} \right).
\end{aligned} \tag{2.7}$$

Since ε is arbitrary, by (2.6) and (2.7), we known that

$$\int_{\mathbb{R}^N} |f(x,u_n) - f(x,u)||u_n - u|dx \to 0, \quad \text{as } n \to \infty. \tag{2.8}$$

Thus, from (1.4) and (2.3), it holds

$$\|u_n - u\|_\lambda^2 = \langle J'_\lambda(u_n) - J'_\lambda(u), u_n - u \rangle + \int_{\mathbb{R}^N} |f(x,u_n) - f(x,u)||u_n - u|dx \to 0, \quad \text{as } n \to \infty.$$

So, $u_n \to u$ in E_λ. □

Proof of Theorem 1.1. By Lemmas 2.1, 2.2 and 2.3, we known that $c_\lambda = \inf_{E_\lambda} J_\lambda(u)$ is critical value of J_λ. Next, we will prove $c_\lambda \neq 0$. Let $u \in E_\lambda$ and $\|u\|_\lambda = 1$, by $(f2)'$, we can get

$$\begin{aligned}
J_\lambda(tu) &= \frac{t^2}{2}\|u\|_\lambda^2 - \int_{\mathbb{R}^N} F(x,tu)dx \\
&\le \frac{t^2}{2} - \eta |t|^{a_0} \int_\Omega |u|^{a_0} dx.
\end{aligned}$$

Since $1 < a_0 < 2$, as $t > 0$ small enough, $J_\lambda(tu) < 0$. Hence $c_\lambda = \inf_{E_\lambda} J_\lambda(u) < 0$, equation (1.1) possesses at least a nontrivial ground state solution u_λ for every $\lambda > 0$. Then the proof of Theorem 1.1 is completed. □

3 The proof of main results for f super-linear at infinity in u

To complete the proof of our theorem, we need the following definition of Cerami condition and critical point theorem(see [16]).

If any sequence $\{u_n\} \subset H$ such that $J(u_n) \to c$ and $J'(u_n)(1 + \|u_n\|) \to 0$, then this sequence is called a $(C)_c$ sequence. If any $(C)_c$ sequence $\{u_n\} \subset H$ of J has a convergent subsequence, then this C^1 functional J satisfies $(C)_c$ condition.

Theorem 3.1 (Mountain Pass Theorem). *Let H be a real Banach space and $J \in C^1(H, \mathbb{R})$. Assume that there exist $v_0 \in H, v_1 \in H$, and a bounded open neighborhood Ω of v_0 such that $v_1 \notin \Omega$ and*

$$\inf_{u \in \partial\Omega} J(u) > \max\{J(v_0), J(v_1)\}.$$

Let

$$\Gamma := \{\gamma \in C([0,1]), H) : \gamma(0) = v_0, \gamma(1) = v_1\}$$

and

$$c=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}J(\gamma(t)).$$

If J satisfies the $(C)_c$ condition, then c is a critical value of J and $c>\max\{J(v_0),J(v_1)\}$.

We choose $H=E_\lambda, J=J_\lambda, v_0=0$ and define $\Omega=B(0,\rho)$ is a ball with radius ρ and origin at $0\in H$, where radius ρ is given in following lemma.

Lemma 3.2. *Assume $(V1)$ and $(f1),(f2)$ are satisfied, then for each $\lambda>0$, there exist $\rho>0$ such that*

$$\inf_{\|u\|_\lambda=\rho}J_\lambda(u)>0.$$

Proof. According to $(f1)$, for any $\varepsilon>0$, there exist $\delta=\delta(\varepsilon)>0$, such that

$$|f(x,z)|\le\varepsilon|z|,\qquad\forall x\in\mathbb{R}^N\text{ and }|z|\le\delta.\tag{3.1}$$

By $(f2)$ we can obtain that

$$|f(x,z)|\le C_0(|z|+|z|^{p-1})\le|z|^{p-1}\left(C_0\frac{1}{\delta^{p-2}}+1\right):=C_\varepsilon|z|^{p-1},\qquad\forall x\in\mathbb{R}^N,\ |z|\ge\delta.\tag{3.2}$$

Combining this with (3.1), (3.2) and $F(x,z)=\int_0^1 f(x,tz)zdt$, we get

$$|f(x,z)|\le C_\varepsilon|z|^{p-1}+\varepsilon|z|,\qquad\forall(x,z)\in\mathbb{R}^N\times\mathbb{R},\tag{3.3}$$

and

$$|F(x,z)|\le\frac{C_\varepsilon}{p}|z|^p+\frac{\varepsilon}{2}|z|^2,\qquad\forall(x,z)\in\mathbb{R}^N\times\mathbb{R}.\tag{3.4}$$

Then, from (3.4) and (1.2), we have that

$$\begin{aligned}J_\lambda(u)&=\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla_\gamma u|^2+\lambda V(x)u^2)dx-\int_{\mathbb{R}^N}F(x,u)dx\\&\ge\frac{1}{2}\|u\|_\lambda^2-\int_{\mathbb{R}^N}\frac{\varepsilon}{2}|u|^2dx-\int_{\mathbb{R}^N}\frac{C_\varepsilon}{p}|u|^pdx\\&\ge\frac{1}{2}\|u\|_\lambda^2-\frac{\varepsilon}{2}d_2^2\|u\|_\lambda^2-\frac{C_\varepsilon}{p}d_p^p\|u\|_\lambda^p\\&\ge\frac{1}{4}\|u\|_\lambda^2-\frac{C_\varepsilon}{p}d_p^p\|u\|_\lambda^p,\end{aligned}$$

where $2<p<2_\gamma^*$ and $0<\varepsilon<\frac{1}{2d_2^2}$. Choosing $\rho=\|u\|_\lambda$ small enough concludes the proof. □

Lemma 3.3. *Under assumption $(V1)$ and $(f3)$, there exist $v_1\in E_\lambda$, such that $\|v_1\|_\lambda>\rho$ and $J_\lambda(v_1)<0$.*

Proof. Let $u\in E_\lambda$ satisfied $u\ne0$, then $\operatorname{meas}(\{x\in\mathbb{R}^N:u(x)\ne0\})>0$. If there exists $M_0>0$ such that $J_\lambda(tu)>-M_0$, then by $(f3)$ and the Fatou lemma, we have that

$$\begin{aligned}0&=\lim_{t\to+\infty}\frac{-M_0}{t^2}\le\limsup_{t\to+\infty}\frac{J_\lambda(tu)}{t^2}\\&=\limsup_{t\to+\infty}\left(\frac{\frac{t^2}{2}\|u\|_\lambda^2}{t^2}-\int_{\mathbb{R}^N}\frac{F(x,tu)}{t^2}dx\right)\\&\le\frac{1}{2}\|u\|_\lambda^2-\liminf_{t\to+\infty}\int_{u(x)\ne0}\frac{F(x,tu)}{(tu)^2}u^2dx\\&=-\infty.\end{aligned}$$

Obviously, this is a contradiction. So $J_\lambda(tu) \to -\infty$, as $t \to +\infty$. Let $v_1 = tu$, for large enough t, we have $\|v_1\|_\lambda > \rho$ and $J_\lambda(v_1) < 0$. The proof is complete. □

It is clear that

$$\inf_{u\in\partial\Omega} J_\lambda(u) = \inf_{\|u\|_\lambda=\rho} J_\lambda(u) > 0 = \max\{J_\lambda(0), J_\lambda(v_1)\} = \max\{J_\lambda(v_0), J_\lambda(v_1)\}.$$

That is, the geometric conditions of mountain pass theorem are satisfied. Thus, the mountain pass value

$$c_\lambda = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_\lambda(\gamma(t)).$$

exists.

Lemma 3.4. *Let $(V1), (V2)$ and $(f1) \sim (f4)$ be satisfied. For any $M > c_\lambda$, the $(C)_{c_\lambda}$ sequence of J_λ is bounded in E_λ for enough large λ.*

Proof. Let $\{u_n\} \subset E_\lambda$ be a $(C)_{c_\lambda}$ sequence of J_λ, that is

$$J_\lambda(u_n) \to c_\lambda, \qquad J'_\lambda(u_n)(1+\|u_n\|_\lambda) \to 0, \quad \text{as } n\to\infty. \tag{3.5}$$

Arguing by contradiction, up to subsequence, we assume that $\|u_n\|_\lambda \to \infty$ as $n \to \infty$. Let $w_n = \frac{u_n}{\|u_n\|_\lambda}$, then $\|w_n\|_\lambda = 1$, $\{w_n\}$ is bounded. Going if necessary to a subsequence, there exists a $w \in E_\lambda$ such that we have

$$\begin{aligned} w_n &\to w, \qquad \text{in } L^s_{loc}(\mathbb{R}^N), \text{ for } 2 \le s < 2^*_\gamma; \\ w_n(x) &\to w(x), \quad \text{a.e. } x \in \mathbb{R}^N. \end{aligned} \tag{3.6}$$

Firstly, we consider the case $w = 0$. By (1.4) and (3.5), we obtain that

$$\int_{\mathbb{R}^N} \frac{f(x,u_n)u_n}{\|u_n\|^2_\lambda}dx = 1 - \frac{\langle J'_\lambda(u_n), u_n\rangle}{\|u_n\|^2_\lambda} \to 1, \quad \text{as } n\to\infty. \tag{3.7}$$

From $(f1)$, there exist $\delta > 0$, such that

$$\left|\frac{f(x,z)z}{z^2}\right| = \left|\frac{f(x,z)}{z}\right| \le 1, \qquad \forall x\in\mathbb{R}^N,\ 0<|z|<\delta. \tag{3.8}$$

By $(f2)$, there exist $C > 0$ satisfy

$$\left|\frac{f(x,z)z}{z^2}\right| \le \left|\frac{C_0(|z|^2+|z|^p)}{z^2}\right| \le C, \qquad \forall x\in\mathbb{R}^N,\ \delta\le|z|\le L_0. \tag{3.9}$$

Hence, from (3.8) and (3.9), we have that

$$|f(x,z)z| \le (C+1)z^2, \qquad \forall x\in\mathbb{R}^N,\ 0<|z|\le L_0. \tag{3.10}$$

By $(V2)$, (3.6) and $\|w_n\|_\lambda = 1$, we get that

$$\begin{aligned} \int_{\mathbb{R}^N} w_n^2dx &= \int_{V(x)\ge b} w_n^2dx + \int_{V(x)<b} w_n^2dx \\ &\le \frac{1}{\lambda b}\int_{V(x)\ge b} \lambda V(x)w_n^2dx + \int_{V(x)<b} w_n^2dx \\ &\le \frac{1}{\lambda b}\int_{\mathbb{R}^N} \lambda V(x)w_n^2dx + \int_{V(x)<b} w_n^2dx \\ &\le \frac{1}{\lambda b} + \int_{V(x)<b} w_n^2dx \to 0, \quad \text{as } n\to\infty,\ \lambda\to+\infty. \end{aligned} \tag{3.11}$$

In view of (3.10) and (3.11), we obtain that

$$\begin{aligned}\int_{|u_n|\le L_0}\frac{|f(x,u_n)u_n|}{\|u_n\|_\lambda^2}dx &\le (C+1)\int_{|u_n|\le L_0}\frac{u_n^2}{\|u_n\|_\lambda^2}dx\\ &=(C+1)\int_{|u_n|\le L_0}w_n^2dx\\ &\le (C+1)\int_{\mathbb{R}^N}w_n^2dx\to 0,\quad \text{as } n\to\infty,\ \lambda\to+\infty.\end{aligned}\tag{3.12}$$

Combing the Hölder inequality, (1.2), $\|w_n\|_\lambda=1$ and (3.11), for any $s\in(2,2_\gamma^*)$ we have that

$$\begin{aligned}\left(\int_{\mathbb{R}^N}|w_n|^sdx\right)^{\frac{1}{s}} &= \left(\int_{\mathbb{R}^N}|w_n|^{\theta s}|w_n|^{(1-\theta)s}dx\right)^{\frac{1}{s}}\\ &\le \left(\int_{\mathbb{R}^N}|w_n|^{\theta s\cdot\frac{2}{\theta s}}dx\right)^{\frac{\theta s}{2}\cdot\frac{1}{s}}\left(\int_{\mathbb{R}^N}|w_n|^{(1-\theta)s\cdot\frac{2_\gamma^*}{(1-\theta)s}}dx\right)^{\frac{(1-\theta)s}{2_\gamma^*}\cdot\frac{1}{s}}\\ &=\left(\int_{\mathbb{R}^N}|w_n|^2dx\right)^{\frac{\theta}{2}}\left(\int_{\mathbb{R}^N}|w_n|^{2_\gamma^*}dx\right)^{\frac{1-\theta}{2_\gamma^*}}\\ &\le d_{2_\gamma^*}^{1-\theta}\left(\int_{\mathbb{R}^N}|w_n|^2dx\right)^{\frac{\theta}{2}}\to 0,\quad \text{as } n\to\infty,\ \lambda\to+\infty,\end{aligned}\tag{3.13}$$

where $\theta=\frac{2(2_\gamma^*-s)}{s(2_\gamma^*-2)}$. By (3.5) and $(f4)$, we get that for n large enough

$$M>J_\lambda(u_n)-\frac{1}{2}\langle J_\lambda'(u_n),u_n\rangle=\int_{\mathbb{R}^N}\mathcal{F}(x,u_n)dx\ge 0.\tag{3.14}$$

From $\tau>\frac{\tilde{N}}{2}$, we easily obtain $\frac{2\tau}{\tau-1}\in(2,2_\gamma^*)$. So, by the Hölder inequality, $(f4)$, (3.14) and (3.13) with $s=\frac{2\tau}{\tau-1}$, we get that

$$\begin{aligned}\int_{|u_n|\ge L_0}\frac{|f(x,u_n)u_n|}{\|u_n\|_\lambda^2}dx &= \int_{|u_n|\ge L_0}\left|\frac{f(x,u_n)}{u_n}\right|w_n^2dx\\ &\le\left(\int_{|u_n|\ge L_0}\left|\frac{f(x,u_n)}{u_n}\right|^\tau dx\right)^{\frac{1}{\tau}}\left(\int_{|u_n|\ge L_0}|w_n|^{2\cdot\frac{\tau}{\tau-1}}dx\right)^{\frac{\tau-1}{\tau}}\\ &\le\left(\int_{|u_n|\ge L_0}a_1\mathcal{F}(x,u_n)dx\right)^{\frac{1}{\tau}}\left(\int_{\mathbb{R}^N}|w_n|^{\frac{2\tau}{\tau-1}}dx\right)^{\frac{\tau-1}{\tau}}\\ &\le a_1^{\frac{1}{\tau}}\left(\int_{\mathbb{R}^N}\mathcal{F}(x,u_n)dx\right)^{\frac{1}{\tau}}\left(\left(\int_{\mathbb{R}^N}|w_n|^{\frac{2\tau}{\tau-1}}dx\right)^{\frac{\tau-1}{2\tau}}\right)^2\\ &\le(a_1M)^{\frac{1}{\tau}}\left(\left(\int_{\mathbb{R}^N}|w_n|^{\frac{2\tau}{\tau-1}}dx\right)^{\frac{\tau-1}{2\tau}}\right)^2\to 0,\quad\text{as } n\to+\infty,\ \lambda\to+\infty.\end{aligned}$$

Thus, combining with (3.12), we obtain that

$$\int_{\mathbb{R}^N}\frac{f(x,u_n)u_n}{\|u_n\|_\lambda^2}dx=\int_{|u_n|\le L_0}\frac{f(x,u_n)u_n}{\|u_n\|_\lambda^2}dx+\int_{|u_n|\ge L_0}\frac{f(x,u_n)u_n}{\|u_n\|_\lambda^2}dx\to 0,\quad\text{as } n\to\infty,\ \lambda\to+\infty,$$

which is a contradiction with (3.7).

Secondly, we consider the case $w \neq 0$. Evidently, meas$(\{x \in \mathbb{R}^N : w(x) \neq 0\}) > 0$ and $|u_n(x)| \to \infty$ as $n \to \infty$, for a.e. $x \in \{x \in \mathbb{R}^N : w(x) \neq 0\}$. Thus, from $(f3)$ and Fatou's lemma, we can get

$$\begin{aligned} \liminf_{n\to\infty} \frac{\int_{\mathbb{R}^N} F(x,u_n)dx}{\|u_n\|_\lambda^2} &\geq \liminf_{n\to\infty} \int_{w(x)\neq 0} \frac{F(x,u_n)}{u_n^2} w_n^2 dx \\ &\geq \int_{w(x)\neq 0} \liminf_{n\to\infty} \frac{F(x,u_n)}{u_n^2} w_n^2 dx \\ &= +\infty. \end{aligned} \tag{3.15}$$

By (3.5), we have

$$\begin{aligned} \liminf_{n\to\infty} \frac{\int_{\mathbb{R}^N} F(x,u_n)dx}{\|u_n\|_\lambda^2} &\leq \limsup_{n\to\infty} \frac{\int_{\mathbb{R}^N} F(x,u_n)dx}{\|u_n\|_\lambda^2} \\ &= \limsup_{n\to\infty} \left(\frac{1}{2} - \frac{J_\lambda(u_n)}{\|u_n\|_\lambda^2} \right) \\ &= \frac{1}{2}, \end{aligned}$$

which is contradiction with (3.15).

So $\{u_n\}$ is bounded. □

Lemma 3.5. *Assume $(V1), (V2)$ and $(f1) \sim (f4)$ be satisfied, then for any $M > c_\lambda$, there exist $\Lambda = \Lambda(M) > 0$ such that J_λ satisfies $(C)_{c_\lambda}$ condition for all $\lambda > \Lambda$.*

Proof. Let $\{u_n\} \subset E_\lambda$ satisfies (3.5). By Lemma 3.4, we known that $\{u_n\}$ is bounded in E_λ. Thus, up to a subsequence, we have that

$$u_n \rightharpoonup u, \quad \text{in } E_\lambda; \tag{3.16}$$

$$u_n \to u, \quad \text{in } L^s_{loc}(\mathbb{R}^N), \text{ for } 2 \leq s < 2^*_\gamma; \tag{3.17}$$

$$u_n(x) \to u(x), \quad \text{a.e. } x \in \mathbb{R}^N. \tag{3.18}$$

Let $v_n := u_n - u$, then $v_n \rightharpoonup 0$ in E_λ by (3.16), which implies that

$$\|u_n\|_\lambda^2 = (v_n + u, v_n + u)_\lambda = \|v_n\|_\lambda^2 + \|u\|_\lambda^2 + o(1). \tag{3.19}$$

Next, by using the similar proof method of Proposition A.1 in the literature [5], we can get that

$$\int_{\mathbb{R}^N} F(x,u_n)dx = \int_{\mathbb{R}^N} F(x,v_n)dx + \int_{\mathbb{R}^N} F(x,u)dx + o(1), \tag{3.20}$$

and

$$\int_{\mathbb{R}^N} f(x,u_n)\varphi dx = \int_{\mathbb{R}^N} f(x,v_n)\varphi dx + \int_{\mathbb{R}^N} f(x,u)\varphi dx + o(1), \tag{3.21}$$

for any $\varphi \in E_\lambda$. By (3.19) and (3.20), we can obtain that

$$J_\lambda(u_n) = J_\lambda(v_n) + J_\lambda(u) + o(1). \tag{3.22}$$

Combing with (3.21) and $u_n = v_n + u$, for any $\varphi \in E_\lambda$ we have that

$$\langle J'_\lambda(u_n), \varphi \rangle = \langle J'_\lambda(v_n), \varphi \rangle + \langle J'_\lambda(u), \varphi \rangle + o(1). \tag{3.23}$$

From (3.3), (3.18) and the dominated convergence theorem, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, we obtain that

$$\int_{\mathbb{R}^N} (f(x,u_n) - f(x,u))\varphi dx = \int_{\Omega_\varphi} (f(x,u_n) - f(x,u))\varphi dx \to 0, \quad \text{as } n \to \infty, \tag{3.24}$$

here Ω_φ is the support set of φ. For each $\varphi \in C_0^\infty(\mathbb{R}^N)$, by (3.16) we have

$$(u_n - u, \varphi)_\lambda \to 0, \quad \text{as } n \to \infty. \tag{3.25}$$

By (3.25), (3.24), (3.5) and the dense of $C_0^\infty(\mathbb{R}^N)$ in E_λ, it shows that

$$\lim_{n\to\infty} \langle J'_\lambda(u_n), \varphi\rangle = \langle J'_\lambda(u), \varphi\rangle = 0, \qquad \forall \varphi \in E_\lambda. \tag{3.26}$$

Hence, $J'_\lambda(u) = 0$ and from $(f4)$ we can obtain that

$$J_\lambda(u) = J_\lambda(u) - \frac{1}{2}\langle J'_\lambda(u), u\rangle = \int_{\mathbb{R}^N} \mathcal{F}(x,u)dx \geq 0.$$

So, by (3.22), (3.23), (3.26) and the boundedness of $\{v_n\}$, we get that

$$\begin{aligned}\int_{\mathbb{R}^N} \mathcal{F}(x,v_n)dx &= J_\lambda(v_n) - \frac{1}{2}\langle J'_\lambda(v_n), v_n\rangle \\ &= J_\lambda(u_n) - J_\lambda(u) - \frac{1}{2}\langle J'_\lambda(u_n) - J'_\lambda(u), v_n\rangle + o(1) \\ &\leq J_\lambda(u_n) + o(1).\end{aligned}$$

Thus, for enough large n, we have that

$$\int_{\mathbb{R}^N} \mathcal{F}(x,v_n)dx < M. \tag{3.27}$$

Now, we will show that $v_n \to 0$ in E_λ. By $(V2)$ and (3.17) that

$$\int_{\mathbb{R}^N} v_n^2 dx = \int_{V(x)\geq b} v_n^2 dx + \int_{V(x)<b} v_n^2 dx \leq \frac{1}{\lambda b}\|v_n\|_\lambda^2 + o(1). \tag{3.28}$$

Thus, combing with the Hölder inequality and (1.2), for any $s \in (2, 2^*_\gamma)$ we have

$$\begin{aligned}\left(\int_{\mathbb{R}^N} |v_n|^s dx\right)^{\frac{1}{s}} &= \left(\int_{\mathbb{R}^N} |v_n|^{\theta s}|v_n|^{(1-\theta)s} dx\right)^{\frac{1}{s}} \\ &\leq \left(\int_{\mathbb{R}^N} |v_n|^{\theta s\cdot\frac{2}{\theta s}} dx\right)^{\frac{\theta s}{2}\cdot\frac{1}{s}} \left(\int_{\mathbb{R}^N} |v_n|^{(1-\theta)s\cdot\frac{2^*_\gamma}{(1-\theta)s}} dx\right)^{\frac{(1-\theta)s}{2^*_\gamma}\cdot\frac{1}{s}} \\ &= \left(\int_{\mathbb{R}^N} |v_n|^2 dx\right)^{\frac{\theta}{2}} \left(\int_{\mathbb{R}^N} |v_n|^{2^*_\gamma} dx\right)^{\frac{1-\theta}{2^*_\gamma}} \\ &\leq d_{2^*_\gamma}^{1-\theta}(\lambda b)^{-\frac{\theta}{2}}\|v_n\|_\lambda + o(1),\end{aligned} \tag{3.29}$$

where $\theta = \frac{2(2^*_\gamma - s)}{s(2^*_\gamma - 2)}$. According to (3.28) and (3.10), we obtain that

$$\begin{aligned}\int_{v_n\leq L_0} f(x,v_n)v_n dx &\leq (C+1)\int_{v_n\leq L_0} v_n^2 dx \\ &\leq (C+1)\int_{\mathbb{R}^N} v_n^2 dx \\ &\leq \frac{C+1}{\lambda b}\|v_n\|_\lambda^2 + o(1).\end{aligned} \tag{3.30}$$

By $\tau > \frac{\tilde{N}}{2}$, it is easy obtained that $\frac{2\tau}{\tau-1} \in (2, 2^*_\gamma)$. Thus, from the Hölder inequality, (3.27), (3.29) with $s = \frac{2\tau}{\tau-1}$ and the boundedness of $\{v_n\}$, we can see that

$$\begin{aligned}
\int_{v_n \geq L_0} f(x, v_n) v_n dx &\leq \int_{|u_n| \geq L_0} \left| \frac{f(x, v_n)}{v_n} \right| v_n^2 dx \\
&\leq \left(\int_{v_n \geq L_0} \left| \frac{f(x, v_n)}{v_n} \right|^\tau dx \right)^{\frac{1}{\tau}} \left(\int_{v_n \geq L_0} |v_n|^{2 \cdot \frac{\tau}{\tau-1}} dx \right)^{\frac{\tau-1}{\tau}} \\
&\leq \left(\int_{v_n \geq L_0} a_1 \mathcal{F}(x, v_n) dx \right)^{\frac{1}{\tau}} \left(\int_{\mathbb{R}^N} |v_n|^{\frac{2\tau}{\tau-1}} dx \right)^{\frac{\tau-1}{\tau}} \\
&\leq a_1^{\frac{1}{\tau}} \left(\int_{\mathbb{R}^N} \mathcal{F}(x, v_n) dx \right)^{\frac{1}{\tau}} \left(\left(\int_{\mathbb{R}^N} |v_n|^{\frac{2\tau}{\tau-1}} dx \right)^{\frac{\tau-1}{2\tau}} \right)^2 \\
&\leq (a_1 M)^{\frac{1}{\tau}} d_{2^*_\gamma}^{2(1-\theta)} (\lambda b)^{-\theta} \|v_n\|_\lambda^2 + o(1).
\end{aligned} \tag{3.31}$$

Therefore, by (3.30) and (3.31), we have

$$\begin{aligned}
o(1) &= \langle J'_\lambda(v_n), v_n \rangle \\
&= \|v_n\|_\lambda^2 - \int_{\mathbb{R}^N} f(x, v_n) v_n dx \\
&= \|v_n\|_\lambda^2 - \int_{v_n \leq L_0} f(x, v_n) v_n dx - \int_{v_n \geq L_0} f(x, v_n) v_n dx \\
&\geq \left[1 - \frac{C+1}{\lambda b} - (a_1 M)^{\frac{1}{\tau}} d_{2^*_\gamma}^{2(1-\theta)} (\lambda b)^{-\theta} \right] \|v_n\|_\lambda^2 + o(1).
\end{aligned}$$

So, there exist $\Lambda = \Lambda(M) > 0$ such that $v_n \to 0$ in E_λ as $n \to \infty$ for any $\lambda > \Lambda$. The proof is complete. □

Proof of Theorem 1.3. By Lemma 3.2, 3.3, 3.4 and 3.5, all condition of Theorem 3.1 are satisfied. Thus equation (1.1) possesses at least a nontrivial solution $u_\lambda \in E_\lambda$ and $J_\lambda(u_\lambda) = c_\lambda$ is a critical value, as $\lambda > \Lambda$. Set $S = \{u \in E_\lambda - \{0\} : J'_\lambda(u) = 0\}$. Evidently, by $u_\lambda \in S$ we have that

$$\inf_{u \in S} J_\lambda(u) \leq J_\lambda(u_\lambda) = c_\lambda.$$

For any $u \in S$, let $\gamma_u(t) = tt_0 u, t \in [0,1]$, then $\gamma \in \Gamma$ for enough large t_0 by Lemma 3.3. Thus, according to the definition of c_λ for any $u \in S$ we have

$$c_\lambda \leq \max_{t \in [0,1]} J_\lambda(\gamma_u(t)) = \max_{t \in [0,1]} J_\lambda(tt_0 u) = \max_{t \in [0,t_0]} J_\lambda(tu) = \max_{t \geq 0} J_\lambda(tu).$$

It is easy obtained that $J_\lambda(u) = \max_{t \geq 0} J_\lambda(tu)$ by $(f5)$ for any $u \in S$. So, from the arbitrariness of u, we obtain

$$\inf_{u \in S} J_\lambda(u) \geq c_\lambda.$$

Thus,

$$c_\lambda = \inf_{u \in S} J_\lambda(u),$$

and we can conclude that u_λ is the ground state solution, then the proof of Theorem 1.3 is completed. □

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On the solution manifold of a differential equation with a delay which has a zero

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Abstract. For a differential equation with a state-dependent delay we show that the associated solution manifold X_f of codimension 1 in the space $C^1([-r,0],\mathbb{R})$ is an almost graph over a hyperplane, which implies that X_f is diffeomorphic to the hyperplane. For the case considered previous results only provide a covering by 2 almost graphs.

Keywords: delay differential equation, state-dependent delay, solution manifold, almost graph.

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1 Introduction

Let $r > 0$ be given, choose a norm on $\mathbb{R}^n$, and let $C_n = C([-r,0],\mathbb{R}^n)$ and $C_n^1 = C^1([-r,0],\mathbb{R}^n)$ denote the Banach spaces of continuous and continuously differentiable functions $[-r,0] \to \mathbb{R}^n$, respectively, with the norms given by $|\phi|_C = \max_{-r\le t\le 0}|\phi(t)|$ and $|\phi| = |\phi|_C + |\phi'|_C$. For a delay differential equation

$$x'(t) = f(x_t)$$

with a vector-valued functional $f : C_n^1 \supset U \to \mathbb{R}^n$ and with the solution segment $x_t \in U$ defined as $x_t(s) = x(t+s)$, the associated *solution manifold* is the set

$$X_f = \{\phi \in U : \phi'(0) = f(\phi)\}.$$

Assume that f is continuously differentiable and

(e) *each derivative $Df(\phi) : C_n^1 \to \mathbb{R}^n$, $\phi \in U$, has a linear extension $D_e f(\phi) : C_n \to \mathbb{R}^n$ so that the map*

$$U \times C_n \ni (\phi,\chi) \mapsto D_e f(\phi)\chi \in \mathbb{R}^n$$

is continuous.

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The extension property (e) is a relative of the notion of being *almost Fréchet differentiable* from [4] and can be verified for a variety of differential equations with state-dependent delay. – Under the said, mild conditions a non-empty set X_f is a continuously differentiable submanifold of codimension n in the Banach space C^1_n, and it is on this manifold that the initial value problem associated with $x'(t) = f(x_t)$ is well-posed, with solutions which are continuously differentiable with respect to initial data [1,5].

The present note continues the description of solution manifolds initiated in [6,7]. In [6] we saw that in case f satisfies a condition which in examples with explicit delays corresponds to all of these delays being bounded away from zero the solution manifold is a graph over the closed subspace

$$X_0 = \{\phi \in C^1_n : \phi'(0) = 0\},$$

which also is the solution manifold for $f = 0$. An example of a scalar equation with a single state-dependent delay which is positive but not bounded away from zero shows that in general solution manifolds do not admit any graph representation [6, Section 3]. However, the main result from [6] guarantees that for a reasonably large class of systems with explicit discrete state-dependent delays which are all positive the solution manifolds are nearly as simple as graphs: They are *almost graphs* over X_0, in the terminology introduced in [6,7].

Let us recall from [7] that given a continuously differentiable submanifold X of a Banach space E and a closed subspace H with a closed complementary space,

(i) X is called a graph (over H) if there are a closed complementary space Q for H and a continuously differentiable map $\gamma : H \supset \text{dom} \to Q$ with

$$X = \{\zeta + \gamma(\zeta) \in E : \zeta \in \text{dom}\},$$

and that

(ii) X is called an *almost graph* (over H) if there is a continuously differentiable map $\alpha : H \supset \text{dom} \to E$ with

$$\alpha(\zeta) = 0 \quad \text{on } \text{dom} \cap X,$$
$$\alpha(\zeta) \in E \setminus H \quad \text{on } \text{dom} \setminus X,$$

so that the map

$$\text{dom} \ni \zeta \mapsto \zeta + \alpha(\zeta) \in E$$

defines a diffeomorphism onto X.

Furthermore,

(iii) a diffeomorphism A from an open neighbourhood $\mathcal{O}$ of X in E onto an open subset of E is called an *almost graph diffeomorphism* (associated with X and H) if

$$A(X) \subset H$$

and

$$A(\zeta) = \zeta \quad \text{on } X \cap H.$$

In [7, Section 1] it is verified that in case there is an almost graph diffeomorphism A associated with X and H the submanifold X is an almost graph over H.

An example of an almost graph in finite dimension is the unit circle in the plane without the point on top. The inverse of the stereographic projection onto the real line serves as the map $\zeta \mapsto \zeta + \alpha(\zeta)$ in Part (ii) of the previous definition.

We return to the results in [6] and [7]. It is not difficult to see that the approach used in [6] for an almost graph representation of a solution manifold fails in case some of the delays in the system considered have zeros. For the solution manifold of such a system, with k delays some of which have zeros, the approach can be used, however, in order to obtain a finite atlas of manifold charts whose domains are almost graphs over X_0. This is achieved in [7], with the size of the atlas independent of the number of equations in the system considered, solely determined by the zerosets of the delays, and not exceeding 2^k.

The immediate question with regard to the topological properties of these solution manifolds is whether the size of the atlas found in [7] is minimal. The result of the present note shows that for a prototype of the systems studied in [6, 7] this is not the case: The entire solution manifolds of the prototype equation is in fact an almost graph over X_0. The proof relies on a major modification of the approach taken in [7].

The prototype equation belongs to the simplest cases of the systems studied in [6,7] which are scalar equations with a single delay (k=1) and have the form

$$x'(t) = g(x(t - d(Lx_t))) \tag{1.1}$$

with a continuously differentiable map $g : \mathbb{R} \to \mathbb{R}$, a continuously differentiable *delay function* $d : F \to [0, r]$ defined on a finite-dimensional topological vectorspace F, and a surjective continuous linear map $L : C \to F$. We abbreviate $C = C_1$ and $C^1 = C^1_1$. For $f : C^1 \to \mathbb{R}$ given by

$$f(\phi) = g(\phi(-d(L\phi)))$$

Eq. (1.1) takes the form $x'(t) = f(x_t)$. Proposition 2.1 from [7] applies and shows that f is continuously differentiable with property (e). Proposition 2.3 from [7] yields that the associated solution manifold

$$X_f = \{\phi \in C^1 : \phi'(0) = g(\phi(-d(L\phi)))\}$$

is non-empty, and it follows that it is a continuously differentiable submanifold of codimension 1 in C^1.

If $d(\xi) > 0$ everywhere than we know from [6] that X_f is an almost graph over $X_0 = \{\chi \in C^1 : \chi'(0) = 0\}$. If d has zeros then the result of [7] yields an atlas of $2 = 2^1$ manifold charts whose domains are almost graphs over X_0.

In [8] an adaptation of the approach from [6, 7] to (1.1) with a linear map L for which, loosely spoken, $L\phi$ does not depend on $\phi(0)$ is used to prove that the associated solution manifold is an almost graph over X_0, no matter whether d has zeros or not.

In the sequel we consider a prototype for the remaining, critical cases, namely, Eq. (1.1) for

$$F = \mathbb{R}, \quad L\phi = \phi(0) \quad \text{for all } \phi \in C,$$

and for

$$d \quad \text{with a single zero} \quad \eta_0 \in \mathbb{R}.$$

We find an almost graph diffeomorphism $A : C^1 \to C^1$ which maps X_f onto X_0.

A part of the construction of the diffeomorphism A uses the technique developed in [6,7]. For another part it was helpful to have in mind an idea of Krisztin [2] which yields graph representations of solution manifolds from bounds on extended derivatives as in property (e), compare the proof of Lemma 1 in [3].

The assumption that d has a single zero is for simplicity and may be relaxed in future work.

2 Preliminaries

Differentiable maps are always defined on open subsets of Banach spaces or Banach manifolds.

Differentiation $\partial : C^1 \to C$, $\partial\phi = \phi'$, is linear and continuous, and the evaluation map $ev : C \times [-r,0] \ni (\phi,t) \mapsto \phi(t) \in \mathbb{R}$ is continuous but not locally Lipschitz continuous. The composition

$$ev(\cdot,0) \circ \partial : C^1 \ni \phi \mapsto ev(\partial\phi,0) \in \mathbb{R}$$

is linear and continuous.

The restriction ev_1 of ev to $C^1 \times (-r,0)$ is continuously differentiable with

$$D\,ev_1(\phi,t)(\chi,s) = \chi(t) + \phi'(t)\,s,$$

and the composition

$$h : C^1 \ni \phi \mapsto ev(\phi, -d(L\phi)) \in \mathbb{R}$$

is continuously differentiable with

$$Dh(\phi)\chi = \chi(-d(L\phi)) - \phi'(-d(L\phi))d'(L\phi)L\chi,$$

see Part 2.1 of the proof of [7, Proposition 2.1].

In Section 1 we quoted [7, Proposition 2.3] for $X_f \neq \emptyset$. Using the choice of L this also follows directly from the fact that for every $\xi \in \mathbb{R}$ each $\phi \in C^1$ with $L(\phi) = \phi(0) = \xi$, $\phi(-d(\xi)) = \xi$, and $\phi'(0) = g(\xi)$ belongs to X_f.

The tangent space $T_\phi X_f$ of the solution manifold X_f at $\phi \in X_f$ consists of the vectors $c'(0) = Dc(0)1$ of differentiable curves $c : I \to C^1$ with $0 \in I \subset \mathbb{R}$, $c(0) = \phi$, $c(I) \subset X_f$. We have

$$\begin{aligned} T_\phi X_f &= \{\chi \in C^1 : \chi'(0) = Df(\phi)\chi\} \\ &= \{\chi \in C^1 : \chi'(0) = g'(\phi(-d(L\phi)))[\chi(-d(L\phi)) - \phi'(-d(L\phi))d'(L\phi)L\chi]\}. \end{aligned}$$

3 A map A taking X_f into X_0

As d is minimal at η_0 we have $d'(\eta_0) = 0$.

Notice that for $\phi \in C$ with $L\phi = \eta_0$,

$$\phi(-d(L\phi)) = \phi(-d(\eta_0)) = \phi(0) = L\phi = \eta_0. \tag{3.1}$$

For reals η we introduce the continuous linear maps

$$L_\eta : C \to \mathbb{R}$$

given by $L_\eta\phi = \phi(-d(\eta))$.

In order to develop a bit of intuition about the shape of X_f observe that the sets

$$\begin{aligned} X_{f\eta} &= X_f \cap L^{-1}(\eta) \\ &= \{\phi \in C^1 : L\phi = \eta \text{ and } \phi'(0) = g(\phi(-d(L\phi)))\} \\ &= \{\phi \in C^1 : L\phi = \eta \text{ and } \phi'(0) = g(\phi(-d(\eta)))\} \end{aligned}$$

are mutually disjoint and decompose X_f, and that

$$X_{f\eta_0} = \{\phi \in C^1 : L\phi = \eta_0 \text{ and } \phi'(0) = g(\eta_0)\}$$

(with Eq. (3.1) for $L\phi = \eta_0$) is a closed affine subspace of codimension 2 in C^1.

Choose $\rho > 0$ and set

$$c = \max_{|\xi-\eta_0|\le\rho} |g(\xi)| + \max_{|\xi-\eta_0|\le\rho} |g'(\xi)|,$$
$$c_* = \frac{\rho}{4(c+1)(\rho+3)}.$$

The map which we are going to construct relies on vectors $\psi_\eta \in C^1$ which are transversal to the solution manifold at points in $X_{f\eta}$. We begin with the case $\eta = \eta_0$ and choose $\psi_{\eta_0} \in C^1$ with the properties

$$\psi_{\eta_0}(0) = 0, \quad \psi'_{\eta_0}(0) = 1, \quad |\psi_{\eta_0}|_C \le c_*.$$

For each $\phi \in X_f$ with $L\phi = \eta_0$ (or, for each $\phi \in X_{f\eta_0}$) we have

$$\begin{aligned}\psi'_{\eta_0}(0) = 1 > c\,c_* &\ge |g'(\eta_0)||\psi_{\eta_0}|_C\\ &= |g'(\phi(-d(L\phi)))||\psi_{\eta_0}|_C \ge |g'(\phi(-d(L\phi)))\psi_{\eta_0}(-d(L\phi))|\\ &= |g'(\phi(-d(L\phi)))[\psi_{\eta_0}(-d(L\phi)) - \phi'(-d(L\phi))d'(L\phi)L\psi_{\eta_0}]|\\ &\quad (\text{with} \quad d'(L\phi) = d'(\eta_0) = 0)\end{aligned}$$

which means

$$\psi_{\eta_0} \in C^1 \setminus T_\phi X_f.$$

Proposition 3.1. *There exists a continuously differentiable map*

$$\mathbb{R} \setminus \{\eta_0\} \ni \eta \mapsto \psi_\eta \in C^1$$

so that for every $\eta \in \mathbb{R} \setminus \{\eta_0\}$ we have

$$\psi_\eta(t) = 0 \quad \text{on } [-r, -d(\eta)] \cup \{0\}, \quad \psi'_\eta(0) = 1, \quad |\psi_\eta|_C \le c_*,$$

and for all $\phi \in X_f$ with $L\phi = \eta$,

$$\psi_\eta \in C^1 \setminus T_\phi X_f.$$

Proof. 1. For each $z \in [-r, 0)$ choose $\psi_z \in C^1$ with $\psi_z(t) = 0$ on $[-r, z] \cup \{0\}$, $\psi'(0) = 1$, and $|\psi_z|_C \le c_*$. Then proceed as in the proof of [7, Proposition 4.1], with $\mathcal{F} = \mathbb{R}, \mathcal{W} = \mathbb{R} \setminus \{\eta_0\}, \lambda = L$, and construct the desired map $\mathbb{R} \setminus \{\eta_0\} \to C^1$ from a sequence of maps ψ_{z_m}, $m \in \mathbb{N}$, with $z_m \to \min d = 0$ as $m \to \infty$. Observe that $|\psi_\eta|_C \le c_*$ is achieved.

2. For $\eta \in \mathbb{R} \setminus \{\eta_0\}$ and $\phi \in X_f$ with $L\phi = \eta$ the function ψ_η does not satisfy the equation characterizing the tangent space $T_\phi X_f$, due to $\psi'_\eta(0) = 1$ and $\psi_\eta(-d(L\phi)) = \psi_\eta(-d(\eta)) = 0 = \psi_\eta(0) = L\psi_\eta$. □

The map from Proposition 3.1 has no continuous extension to $\mathbb{R}$. Nevertheless, for all $\eta \in \mathbb{R}$,

$$L\psi_\eta = \psi_\eta(0) = 0. \tag{3.2}$$

Also, for each $\eta \in \mathbb{R}$,

$$L_\eta \psi_\eta = 0, \tag{3.3}$$

which in case $\eta = \eta_0$ holds with $L_{\eta_0} = L$, and,

$$\psi'_\eta(0) = 1. \tag{3.4}$$

It follows from Eq. (3.4) that for all $\eta \in \mathbb{R}$,

$$C^1 = X_0 \oplus \mathbb{R}\psi_\eta,$$

and the continuous linear projection $P_\eta : C^1 \to C^1$ along $\mathbb{R}\psi_\eta$ onto X_0 is given by

$$P_\eta \phi = \phi - \phi'(0)\psi_\eta.$$

Now we are ready for the definition of the map $A : C^1 \to C^1$ which in the next section will be shown to be an almost graph diffeomorphism associated with X_f and X_0. Let

$$a : \mathbb{R} \to [0,1]$$

be a continuously differentiable map with $a(\xi) = 1$ for $|\xi - \eta_0| \leq \frac{\rho}{2}$, $a(\xi) = 0$ for $|\xi - \eta_0| \geq \rho$, and $|a'(\xi)| \leq \frac{3}{\rho}$ for $\frac{\rho}{2} \leq |\xi - \eta_0| \leq \rho$.

The maps $A_{\rho/2} : C^1_{\rho/2} \to C^1$ and $A_+ : C^1_+ \to C^1$ given by

$$\begin{aligned} C^1_{\rho/2} &= \{\phi \in C^1 : |\phi(-d(L\phi)) - \eta_0| < \rho/2\}, \\ A_{\rho/2}(\phi) &= \phi - g(\tau)\psi_{\eta_0}, \\ C^1_+ &= \{\phi \in C^1 : d(L\phi) > 0\}, \\ A_+(\phi) &= \phi - g(\tau)[a(\tau)\psi_{\eta_0} + (1 - a(\tau))\psi_\eta], \end{aligned}$$

with

$$\tau = L_\eta \phi = ev(\phi, -d(L\phi)), \quad \eta = L\phi,$$

are continuously differentiable. On the intersection of their domains they coincide: For $\phi \in C^1_{\rho/2} \cap C^1_+$ we have

$$|\tau - \eta_0| = |\phi(-d(L\phi)) - \eta_0| < \frac{\rho}{2},$$

which yields $a(\tau) = 1$, and thereby, $A_{\rho/2}(\phi) = A_+(\phi)$.

Also,

$$C^1 = C^1_{\rho/2} \cup C^1_+,$$

since for $\phi \in C^1 \setminus C^1_+$, $d(L\phi) = 0$, hence $L\phi = \eta_0$, and due to Eq. (3.1), $\phi(-d(L\phi)) = \eta_0$, which means $|\phi(-d(L\phi)) - \eta_0| = 0$, or $\phi \in C^1_{\rho/2}$.

It follows that $A_{\rho/2}$ and A_+ define a continuously differentiable map $A : C^1 \to C^1$.

Using Eq. (3.4) we infer that for every $\phi \in X_f$,

$$(A(\phi))'(0) = \phi'(0) - g(\tau) \cdot 1 = \phi'(0) - g(\phi(-d(L\phi))) = 0,$$

or, $A(\phi) \in X_0$. Hence

$$A(X_f) \subset X_0.$$

Notice also that due to Eq. (3.2),

$$LA(\phi) = L\phi \quad \text{for all } \phi \in C^1. \tag{3.5}$$

4 A is an almost graph diffeomorphism

In order to find the inverse of A we first consider $\phi \in C^1_+ \cap X_f$ and $\chi = A(\phi) = A_+(\phi) \in X_0$, and compare $\tau = L_\eta \phi$ where $\eta = L\phi$ to $\sigma = L_{\hat\eta}\chi$ where $\hat\eta = L\chi$. By Eq. (3.5), $\hat\eta = \eta$, hence

$$\begin{aligned}\sigma &= L_\eta \chi = L_\eta A_+(\phi)\\ &= L_\eta \phi - g(\tau)[a(\tau)L_\eta \phi_{\eta_0} + (1-a(\tau))L_\eta \phi_\eta]\\ &= \tau - g(\tau)a(\tau)L_\eta\phi_{\eta_0} \quad \text{(with Eq. (3.3))}.\end{aligned} \tag{4.1}$$

For every $\eta \in \mathbb{R}$ the map

$$h_\eta : \mathbb{R} \ni \tau \mapsto \tau - (ga)(\tau)L_\eta \psi_{\eta_0} \in \mathbb{R}$$

is continuously differentiable.

Proposition 4.1. *Every map h_η, $\eta \in \mathbb{R}$, is a bijection, the map $T : \mathbb{R}^2 \to \mathbb{R}$ given by $T(\eta,\sigma) = (h_\eta)^{-1}(\sigma)$ is continuously differentiable, and for all $(\eta,\sigma,\tau) \in \mathbb{R}^3$ Eq. (4.1) is equivalent to $\tau = T(\eta,\sigma)$.*

Proof. 1. For every $\eta \in \mathbb{R}$ the map h_η satisfies $h_\eta(\tau) = \tau$ for $|\tau - \eta_0| \geq \rho$, and for $|\tau - \eta_0| \leq \rho$,

$$\begin{aligned}h'_\eta(\tau) &= 1 - (g'(\tau)a(\tau) + g(\tau)a'(\tau))L_\eta\psi_{\eta_0}\\ &\geq 1 - (c + c(3/\rho))c_* > 0.\end{aligned}$$

It follows that h_η is bijective.

2. The map

$$F : \mathbb{R}^3 \ni (\eta,\sigma,\tau) \mapsto \sigma - h_\eta(\tau) \in \mathbb{R}$$

is continuously differentiable (with $L_\eta\psi_{\eta_0} = \psi_{\eta_0}(-d(\eta))$). For each $(\eta,\sigma,\tau) \in \mathbb{R}^3$ Eq. (4.1) and the relations $F(\eta,\sigma,\tau) = 0$ and $\tau = (h_\eta)^{-1}(\sigma) = T(\eta,\sigma)$ are equivalent. For $|\tau - \eta_0| \leq \rho$ we have

$$\partial_3 F(\eta,\sigma,\tau) = -1 + (ga)'(\tau)L_\eta\psi_{\eta_0} \leq -1 + (c + c(3/\rho))c_* < 0,$$

and

$$\partial_3 F(\eta,\sigma,\tau) = -1 \neq 0$$

for $|\tau - \eta_0| \geq \rho$. Applications of the Implicit Function Theorem to the zeroset of F show that the map T is locally given by continuously differentiable maps. □

For the open subsets $C^1_{\rho/4} = \{\chi \in C^1 : |\chi(-d(L\chi)) - \eta_0| < \rho/4\}$ and C^1_+ of the space C^1 we have

$$C^1 = C^1_{\rho/4} \cup C^1_+$$

since for $\chi \in C^1 \setminus C^1_+$, $d(L\chi) = 0$, hence $L\chi = \eta_0$, and due to Eq. (3.1) $\chi(-d(L\chi)) = \eta_0$, which means $|\chi(-d(L\chi)) - \eta_0| = 0$. The maps $B_{\rho/4} : C^1_{\rho/4} \to C^1$ and $B_+ : C^1_+ \to C^1$ given by

$$\begin{aligned}B_{\rho/4}(\chi) &= \chi + g(\tau)\psi_{\eta_0},\\ B_+(\chi) &= \chi + g(\tau)[a(\tau)\psi_{\eta_0} + (1-a(\tau))\psi_\eta],\end{aligned}$$

with

$$\tau = T(\eta,\sigma), \quad \eta = L\chi, \quad \sigma = L_\eta\chi = ev(\chi, -d(L\chi))$$

are continuously differentiable. On the intersection $C^1_{\rho/4} \cap C^1_+$ both maps $B_{\rho/4}$ and B_+ coincide. In order to verify this we need to know $a(\tau) = 1$ for

$$\tau = T(\eta, \sigma), \quad \eta = L\chi, \quad \chi \in C^1, \quad \sigma = L_\eta \chi$$

with $|\chi(-d(L\chi)) - \eta_0| < \rho/4$ and $d(L\chi) > 0$. The equation $a(\tau) = 1$ holds provided $|\tau - \eta_0| \leq \rho/2$, which follows from

$$|\tau - \eta_0| = |\sigma + (ga)(\tau) L_\eta \psi_{\eta_0} - \eta_0| \leq \frac{\rho}{4} + \max_{\xi \in \mathbb{R}} |(ga)(\xi)| \cdot c_*$$

in combination with

$$\begin{aligned} (ga)(\tau) = 0 \quad & \text{for } |\tau - \eta_0| \geq \rho \\ |(ga)(\tau)| \leq c \quad & \text{for } |\tau - \eta_0| \leq \rho, \\ c\, c_* < \rho/4. & \end{aligned}$$

So $B_{\rho/4}$ and B_+ define a continuously differentiable map $B : C^1 \to C^1$.
Observe that due to Eq. (3.2),

$$LB(\chi) = L\chi \quad \text{for all } \chi \in C^1. \tag{4.2}$$

We have

$$B(X_0) \subset X_f.$$

Proof of this: For $\chi \in X_0$ let $\phi = B(\chi)$ and $\eta = L\chi$, $\sigma = L_\eta \chi$, and $\tau = T(\eta, \sigma) = h_\eta^{-1}(\sigma)$. Then

$$\sigma = h_\eta(\tau) = \tau - (ga)(\tau) L_\eta \psi_{\eta_0}.$$

Using $\chi'(0) = 0$, Eq. (3.4), and the preceding equation we get

$$\phi'(0) = (B(\chi))'(0) = 0 + g(\tau) \cdot 1 = g(\tau)$$

with

$$\begin{aligned} \tau &= \sigma + (ga)(\tau) L_\eta \psi_{\eta_0} \\ &= L_\eta \chi + g(\tau)[a(\tau) L_\eta \psi_{\eta_0} + (1 - a(\tau)) L_\eta \psi_\eta] \quad \text{(with Eq. (3.3))} \\ &= L_\eta B(\chi) = L_\eta \phi. \end{aligned}$$

Eq. (4.2) yields $\eta = L\chi = L\phi$, and we obtain

$$\phi'(0) = g(\tau) = g(L_\eta \phi) = g(\phi(-d(\eta))) = g(\phi(-d(L\phi))),$$

or, $B(\chi) = \phi \in X_f$.

Proposition 4.2. *$B(A(\phi)) = \phi$ for all $\phi \in C^1$.*

Proof. 1. The case $d(L\phi) > 0$. Consider

$$\chi = A(\phi) = A_+(\phi) = \phi - g(\tau)[a(\tau)\psi_{\eta_0} + (1 - a(\tau))\psi_{L\phi}]$$

with

$$\tau = L_\eta \phi, \quad \eta = L\phi.$$

From Eq. (3.5) we infer

$$L\chi = L\phi, \tag{4.3}$$

hence $d(L\chi) = d(L\phi) > 0$, and thereby

$$B(\chi) = B_+(\chi) = \chi + g(\hat{\tau})[a(\hat{\tau})\psi_{\eta_0} + (1 - a(\hat{\tau}))\psi_{L\chi}]$$

with

$$\hat{\tau} = T(\hat{\eta}, \hat{\sigma}) = h_{\hat{\eta}}^{-1}(\hat{\sigma}), \quad \hat{\eta} = L\chi, \quad \hat{\sigma} = L_{\hat{\eta}}\chi.$$

It follows that

$$B(\chi) = B(A(\phi)) = \{\phi - g(\tau)[a(\tau)\psi_{\eta_0} + (1 - a(\tau))\psi_{L\phi}]\} + g(\hat{\tau})[a(\hat{\tau})\psi_{\eta_0} + (1 - a(\hat{\tau}))\psi_{L\chi}].$$

Using the preceding equation in combination with Eq. (4.3) we obtain $B(A(\phi)) = \phi$ provided we have $\hat{\tau} = \tau$. Proof of this: By Eq. (4.3), $\hat{\eta} = L\chi = L\phi = \eta$. We get

$$\begin{aligned}
h_\eta(\hat{\tau}) = h_{\hat{\eta}}(\hat{\tau}) = \hat{\sigma} = L_{\hat{\eta}}\chi = L_\eta\chi = L_\eta A(\phi) \\
&= L_\eta\phi - g(\tau)[a(\tau)L_\eta\psi_{\eta_0} + (1 - a(\tau))L_\eta\psi_\eta] \\
&= L_\eta\phi - g(\tau)a(\tau)L_\eta\psi_{\eta_0} \quad \text{(with Eq. (3.3))} \\
&= \tau - (ga)(\tau)L_\eta\psi_{\eta_0} \\
&= h_\eta(\tau),
\end{aligned}$$

and the injectivity of h_η yields $\hat{\tau} = \tau$.

2. The case $d(L\phi) = 0$. Then $\phi(0) = L\phi = \eta_0$. Choose a sequence of points $\phi_j \in C^1$, $j \in \mathbb{N}$, with $\phi_j(0) \neq \eta_0$ for all $j \in \mathbb{N}$ and $\phi_j \to \phi$ in C^1 as $j \to \infty$. For all $j \in \mathbb{N}$, $B(A(\phi_j)) = \phi_j$, due to Part 1 of the proof, and continuity yields $B(A(\phi)) = \phi$. □

Proposition 4.3. *$A(B(\chi)) = \chi$ for all $\chi \in C^1$.*

Proof. 1. The case $d(L\chi) > 0$. Then $\chi \in C^1_+$ and

$$B(\chi) = B_+(\chi) = \chi + g(\tau)[a(\tau)\psi_{\eta_0} + (1 - a(\tau))\psi_\eta]$$

with $\tau = T(\eta, \sigma) = h_\eta^{-1}(\sigma)$, $\eta = L\chi$, $\sigma = L_\eta\chi$. Set $\phi = B(\chi)$. By Eq. (4.2), $L\phi = L\chi = \eta$. Hence $d(L\phi) = d(L\chi) > 0$, or $\phi \in C^1_+$, and

$$A(\phi) = A_+(\phi) = \phi - g(\hat{\tau})[a(\hat{\tau})\psi_{\eta_0} + (1 - a(\hat{\tau}))\psi_{\hat{\eta}}]$$

with

$$\hat{\tau} = L_{\hat{\eta}}\phi, \quad \hat{\eta} = L\phi \quad (= \eta).$$

It follows that

$$\begin{aligned}
A(B(\chi)) = A(\phi) &= \{\chi + g(\tau)[a(\tau)\psi_{\eta_0} + (1 - a(\tau))\psi_\eta]\} - g(\hat{\tau})[a(\hat{\tau})\psi_{\eta_0} + (1 - a(\hat{\tau}))\psi_{\hat{\eta}}] \\
&= \{\chi + g(\tau)[a(\tau)\psi_{\eta_0} + (1 - a(\tau))\psi_\eta]\} - g(\hat{\tau})[a(\hat{\tau})\psi_{\eta_0} + (1 - a(\hat{\tau}))\psi_\eta] \quad (\text{with } \hat{\eta} = \eta),
\end{aligned}$$

and for $A(B(\chi)) = \chi$ it remains to show $\hat{\tau} = \tau$. Proof of this:

$$\begin{aligned}
\hat{\tau} = L_{\hat{\eta}}\phi = L_\eta\phi &= L_\eta(\chi + g(\tau)[a(\tau)\psi_{\eta_0} + (1 - a(\tau))\psi_\eta]) \\
&= L_\eta\chi + g(\tau)[a(\tau)L_\eta\psi_{\eta_0} + (1 - a(\tau))L_\eta\psi_\eta] \\
&= \sigma + (ga)(\tau)L_\eta\psi_{\eta_0} \quad \text{(with Eq. (3.3))} \\
&= h_\eta(\tau) + (ga)(\tau)L_\eta\psi_{\eta_0} = \tau.
\end{aligned}$$

2. In case $d(L\chi) = 0$ use the result of Part 1 above and continuity as in Part 2 of the proof of Proposition 4.2. □

Corollary 4.4. *The map A is an almost graph diffeomorphism associated with X_f and X_0.*

Proof. Propositions 4.2 and 4.3 yield that A is a diffeomorphism onto C^1 with inverse B. Using $A(X_f) \subset X_0$ and $B(X_0) \subset X_f$ one finds $A(X_f) = X_0$. For $\phi \in X_f \cap X_0$, $\tau = L_\eta \phi$, and $\eta = L\phi$, we have

$$g(\tau) = g(L_\eta \phi) = g(\phi(-d(\eta))) = g(\phi(-d(L\phi))) = \phi'(0) = 0.$$

This yields $A(\phi) = \phi$. □

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Strong maximum principle for a sublinear elliptic problem at resonance

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Abstract. We examine the semilinear resonant problem

$$-\Delta u = \lambda_1 u + \lambda g(u) \text{ in } \Omega, \quad u \geq 0 \text{ in } \Omega, \ \ u_{|\partial\Omega} = 0,$$

where $\Omega \subset \mathbb{R}^N$ is a smooth, bounded domain, λ_1 is the first eigenvalue of $-\Delta$ in Ω, $\lambda > 0$. Inspired by a previous result in literature involving power-type nonlinearities, we consider here a generic sublinear term g and single out conditions to ensure: the existence of solutions for all $\lambda > 0$; the validity of the strong maximum principle for sufficiently small λ. The proof rests upon variational arguments.

Keywords: resonant problem, existence, maximum principle.

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded domain of class C^2, and let λ_1 be the first eigenvalue of $-\Delta$ in Ω with Dirichlet boundary conditions. The issue of the existence of solutions of the problem

$$\begin{cases} -\Delta u = \lambda_1 u + u^{s-1} - \mu u^{r-1} & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

$s \in (1, 2)$, $r \in (1, s)$, and $\mu > 0$, has been the subject of study of the recent [3]. As a distinctive feature, the right-hand side term $f(t) := \lambda_1 t + t^{s-1} - \mu t^{r-1}$ in (1.1) is not locally Lipschitz near 0, and moreover satisfies the sign property

$$f^{-1}((-\infty, 0]) \supseteq (0, a], \quad \text{for some } a > 0.$$

As a result, from the celebrated paper [13] (see also [8]), it is known that the strong maximum principle may fail to be valid in this context. By adopting minimax and perturbation

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techniques, the author of [3] showed instead that such a principle does hold as long as the perturbation parameter is chosen sufficiently large. More precisely, the main results in [3] state that problem (1.1) has non-zero solutions for the entire positive range of μ; positive solutions for μ large enough.

The fact that, after a rescaling, (1.1) can be turned into the problem

$$\begin{cases} -\Delta u = \lambda_1 u + \lambda(u^{s-1} - u^{r-1}) & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

for a suitable $\lambda > 0$, raises the natural question whether, as explicitly expressed in [3, Remark 2.4], the same results mentioned above continue to hold when the powers in (1.2) are replaced by a generic nonlinear term g. And, if it is so, it would be interesting of course to identify some "minimal" structure conditions on g for the validity of such results. In the present paper we address these questions and consider the problem

$$\begin{cases} -\Delta u = \lambda_1 u + \lambda g(u) & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where $g : [0, +\infty) \to \mathbb{R}$ is continuous, $g(0) = 0$, and obeys the following conditions:

(g_1) there exists $q \in (1, 2)$ such that $k_1 := \sup_{t>0} \frac{|g(t)|}{1 + t^{q-1}} < +\infty$;

(g_2) $\lim_{t \to 0^+} \frac{g(t)}{t} = -\infty$;

(g_3) $\liminf_{t \to +\infty} G(t) > 0$;

(g_4) $\lim_{t \to +\infty} (g(t)t - 2G(t)) = -\infty$,

where, as usual,

$$G(t) := \int_0^t g(s)ds, \quad \text{for all } t \geq 0.$$

Problems like (P_λ) are being investigated since Landesman and Lazer's pioneering work [9], in which sufficient conditions, based on the interaction between the nonlinearity and the spectrum of the linear operator, were given for them to have a solution. Noteworthy contributions following that work can be found in [2, 5, 12] and also in [6, 7, 10, 11, 14] (see the related references as well) in which several classes of elliptic problems at resonance are investigated via variational and topological methods.

Coming back to (P_λ), our approach develops along the same line of reasoning as [3]. We prove initially that (P_λ) has at least a non-zero solution for all $\lambda > 0$. This is accomplished by considering a sequence of problems near resonance whose solutions are shown to converge to a solution of the original problem. In this regard, assumption (g_4) comes into play to prove the boundedness of the sequence of approximating solutions. Then, by exploiting the classical decomposition of $H_0^1(\Omega)$ into the first eigenspace and its orthogonal complement, we show that, for sufficiently small λ, the set of solutions to (P_λ) is contained in the interior of the positive cone of $C^1_0(\overline{\Omega})$. It still remains an open question to investigate the uniqueness of positive solutions to (P_λ) (in the one-dimensional case and for power-nonlinearities it has instead been established in [4]), as well as the existence of non-zero solutions compactly supported in Ω, in the spirit of [8].

Our main results, Theorems 2.3 and 2.4, are stated and proved in the coming section. Before going on, we arrange some notation and the variational framework for (P_λ). We set

$$\|u\| := \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}}, \quad \text{for all } u \in H^1_0(\Omega),$$

and denote by $\|\cdot\|_p$, $p \in [1, +\infty]$, the classical L^p-norm on Ω. We also set

$$c_p := \sup_{u \in H^1_0(\Omega)\setminus\{0\}} \frac{\|u\|_p}{\|u\|}$$

for each $p \geq 1$, with $p \leq \frac{2N}{N-2}$ if $N \geq 3$, and denote by ϕ_1 the positive eigenfunction associated with λ_1 and normalized with respect to $\|\cdot\|_\infty$. We recall that the first two eigenvalues λ_1, λ_2 of $-\Delta$ in Ω admit the variational characterization

$$\lambda_1 = \inf_{u \in H^1_0(\Omega)\setminus\{0\}} \frac{\|u\|^2}{\|u\|_2^2}, \quad \lambda_2 = \inf_{u \in \operatorname{span}\{\phi_1\}^\perp\setminus\{0\}} \frac{\|u\|^2}{\|u\|_2^2}.$$

Given a set $E \subset \mathbb{R}^N$, its Lebesgue measure will be denoted by the symbol $|E|$. Throughout this paper, the symbols $C, C_1, C_2, \ldots$ represent generic positive constants whose exact value may change from occurence to occurrence.

For all $\lambda > 0$, we denote by $I_\lambda : H^1_0(\Omega) \to \mathbb{R}$ the energy functional associated with (P_λ),

$$I_\lambda(u) := \frac{1}{2}\|u\|^2 - \frac{\lambda_1}{2}\|u_+\|_2^2 - \lambda \int_\Omega G(u_+)dx, \quad \text{for all } u \in H^1_0(\Omega),$$

where $u_+ = \max\{u, 0\}$. By a weak solution to (P_λ) we mean any $u \in C^0(\overline{\Omega}) \cap H^1_0(\Omega)$ verifying

$$\int_\Omega (\nabla u \nabla v - \lambda_1 uv - \lambda g(u)v)\, dx = 0, \quad \text{for all } v \in H^1_0(\Omega).$$

2 Results

As already mentioned, we start by considering a sequence of approximating problems.

Lemma 2.1. *For each $\lambda > 0$, there exists $\bar{n} \in \mathbb{N}$ such that the problem*

$$\begin{cases} -\Delta u = \left(\lambda_1 - \dfrac{1}{n}\right) u + \lambda g(u) & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_n}$$

admits a non-zero weak solution u_n, with positive energy, for all $n \geq \bar{n}$.

Proof. Fix $\lambda > 0$ and let $n \in \mathbb{N}$ with $n > \frac{1}{\lambda_1}$. Let us first show that the energy functional $I_n : H_0^1(\Omega) \to \mathbb{R}$ corresponding to (P_n),

$$I_n(u) := I_\lambda(u) + \frac{1}{2n}\|u_+\|_2^2 = \frac{1}{2}\|u\|^2 - \frac{1}{2}\left(\lambda_1 - \frac{1}{n}\right)\|u_+\|_2^2 - \lambda\int_\Omega G(u_+)dx, \tag{2.1}$$

for all $u \in H_0^1(\Omega)$, has the mountain pass geometry for sufficiently large $n \in \mathbb{N}$.

Fix $k \in (2, 2^*)$ and set

$$M := \frac{k}{2}\sup_{t>0}\frac{\lambda_1 t^2 + 2\lambda G(t)}{t^k}.$$

By (g_1) and (g_2) one has $0 < M < +\infty$ and $\frac{\lambda_1}{2}t^2 + \lambda G(t) \leq \frac{M}{k}t^k$, for all $t \geq 0$. Then, defining

$$R := (Mc_k^k)^{\frac{1}{2-k}},$$

we easily obtain

$$\begin{aligned}\inf_{u\in S_R} I_n(u) &\geq \inf_{\|u\|=R}\left(\frac{1}{2}\|u\|^2 - \frac{M}{k}\|u\|_k^k\right)\\ &\geq \inf_{u\in S_R}\left(\frac{1}{2}\|u\|^2 - \frac{Mc_k^k}{k}\|u\|^k\right)\\ &= \left(\frac{1}{2} - \frac{1}{k}\right)R^2 > 0,\end{aligned} \tag{2.2}$$

for any $n \in \mathbb{N}$, where $S_R := \{u \in H_0^1(\Omega) : \|u\| = R\}$.

Now, let us show that there exist $u_1 \in H_0^1(\Omega)$, with $\|u_1\| > R$, and $\bar{n} \in \mathbb{N}$, such that $I_n(u_1) < 0$ for all $n \geq \bar{n}$. Owing to (g_3), there exist $L, b > 0$ such that

$$G(t) > L, \quad \text{for all } t \geq b.$$

If we denote by

$$E_\gamma := \{x \in \Omega : \phi_1(x) < \gamma\},$$

with $\gamma > 0$, then there exists $\gamma_1 > 0$ such that

$$L > \frac{k_1(bq + b^q)|E_\gamma|}{q(|\Omega| - |E_\gamma|)}, \quad \text{for all } \gamma \in (0, \gamma_1). \tag{2.3}$$

Fix $\bar{\gamma} \in \mathbb{R}$ satisfying

$$0 < \bar{\gamma} < \min\left\{\gamma_1, \frac{b}{R}\right\}.$$

Since the function $\psi(t) := q\bar{\gamma}t + \bar{\gamma}^q t^q$ is continuous in $(0, +\infty)$ and $\psi\left(\frac{b}{\bar{\gamma}}\right) = bq + b^q$, thanks to (2.3), there exists $\bar{t} > \frac{b}{\bar{\gamma}}$ such that

$$L > \frac{k_1(q\bar{\gamma}\bar{t} + \bar{\gamma}^q\bar{t}^q)|E_{\bar{\gamma}}|}{q(|\Omega| - |E_{\bar{\gamma}}|)}. \tag{2.4}$$

With the aid of (g_1) and (2.4) we then obtain

$$\begin{aligned}\int_\Omega G(\bar{t}\phi_1)dx &= \int_{E_{\bar{\gamma}}} G(\bar{t}\phi_1)dx + \int_{\{\phi_1 \geq \bar{\gamma}\}} G(\bar{t}\phi_1)dx \\ &\geq -k_1 \int_{E_{\bar{\gamma}}} \left(\bar{t}\phi_1 + \frac{(\bar{t}\phi_1)^q}{q} \right) dx + \int_{\{\phi_1 \geq \bar{\gamma}\}} G(\bar{t}\phi_1)dx \\ &\geq -k_1 \left(\bar{t}\bar{\gamma} + \frac{\bar{t}^q \bar{\gamma}^q}{q} \right) |E_{\bar{\gamma}}| + L(|\Omega| - |E_{\bar{\gamma}}|) \\ &> 0.\end{aligned}$$

As a result, there exists $\bar{n} \in \mathbb{N}$, with $\bar{n} > \frac{1}{\lambda_1}$, such that

$$I_n(\bar{t}\phi_1) = \frac{\bar{t}^2}{2n} \|\phi_1\|_2^2 - \lambda \int_\Omega G(\bar{t}\phi_1)dx < 0$$

for all $n \geq \bar{n}$. Therefore, the functional I_n satisfies the geometric conditions required by the mountain pass theorem for all $n \geq \bar{n}$.

Moreover, by (g_1) and Sobolev embeddings, one has

$$\begin{aligned} I_n(u) &\geq \frac{1}{2n\lambda_1} \|u\|^2 - \lambda k_1 \left(\int_\Omega |u| dx + \frac{1}{q} \int_\Omega |u|^q dx \right) \\ &\geq \frac{1}{2n\lambda_1} \|u\|^2 - \lambda c_1 k_1 \|u\| - \frac{\lambda c_q k_1}{q} \|u\|^q, \end{aligned}$$

and thus $I_n(u) \to +\infty$ as $\|u\| \to +\infty$. This fact, in addition to standard arguments (see for instance Example 38.25 of [15]), ensures that I_n satisfies the Palais–Smale condition. Then, by invoking the classical mountain pass theorem, I_n admits a critical point $u_n \in H_0^1(\Omega) \setminus \{0\}$ for all $n \geq \bar{n}$, and, by (2.2), one also has

$$I_n(u_n) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_n(\gamma(t)) \geq \left(\frac{1}{2} - \frac{1}{k} \right) R^2, \tag{2.5}$$

where $\Gamma := \{\gamma \in C^0([0,1], H_0^1(\Omega)) : \gamma(0) = 0,\ \gamma(1) = u_1\}$. This concludes the proof. □

Lemma 2.2. *Let $\lambda > 0$, $\bar{n} \in \mathbb{N}$ and let u_n, with $n \geq \bar{n}$, be as in Lemma 2.1. Then, the sequence $\{u_n\}_{n \geq \bar{n}}$ is bounded in $H_0^1(\Omega)$.*

Proof. Let $n \in \mathbb{N}$, $n \geq \bar{n}$. By standard regularity theory, $u_n \in C^{1,\alpha}(\overline{\Omega})$, for some $\alpha \in (0,1)$. For any $n \in \mathbb{N}$, $n \geq \bar{n}$ there exist, uniquely determined, $t_n \in \mathbb{R}$ and $w_n \in \operatorname{span}\{\phi_1\}^\perp$ such that

$$u_n = t_n \phi_1 + w_n.$$

It is straightforward to verify that $w_n \in C^{1,\alpha}(\overline{\Omega})$ is a weak solution to

$$\begin{cases} -\Delta u = \left(\lambda_1 - \frac{1}{n} \right) u + \lambda g(t_n \phi_1 + u) - \frac{t_n}{n} \phi_1 & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.6}$$

and therefore, also by (g_1), one has

$$\begin{aligned}\|w_n\|^2 &\leq \left(\frac{\lambda_1 - \frac{1}{n}}{\lambda_2}\right)\|w_n\|^2 + \lambda \int_\Omega g(t_n\phi_1 + w_n)w_n dx \\ &\leq \left(\frac{\lambda_1 - \frac{1}{n}}{\lambda_2}\right)\|w_n\|^2 + \lambda k_1 \|w_n\|_1 + \lambda k_1 t_n^{q-1}\|\phi_1\|_\infty^{q-1}\|w_n\|_1 + \lambda k_1 \|w_n\|_q^q.\end{aligned} \tag{2.7}$$

From (2.7), it follows that

$$\|w_n\| \leq C\left((1 + t_n^{q-1}) + \|w_n\|^{q-1}\right), \tag{2.8}$$

for some $C > 0$. We claim that the sequence $\{t_n\}_{n\geq\bar{n}}$ is bounded in $\mathbb{R}$. Arguing by contradiction, assume that, up to a subsequence, $t_n \to +\infty$ as $n \to +\infty$. Without loss of generality, we can assume that $t_n \geq 1$ for all $n \geq \bar{n}$ and, since

$$y^{q-1} \leq C_1 + \frac{1}{2C}y \leq C_1 t_n^{q-1} + \frac{1}{2C}y, \quad \text{for all } y > 0,$$

from (2.8) we deduce

$$\|w_n\| \leq 2Ct_n^{q-1} + C\|w_n\|^{q-1} \leq 2Ct_n^{q-1} + CC_1t_n^{q-1} + \frac{1}{2}\|w_n\|,$$

and then

$$\|w_n\| \leq C_2 t_n^{q-1}.$$

Therefore, fixing $p > \max\left\{\frac{N}{2}, \frac{q}{q-1}\right\}$, we obtain

$$\begin{aligned}\|w_n\|_\infty &\leq C_3\left(\|w_n\|_p + \|g(t_n\phi_1 + w_n)\|_p + \frac{t_n}{n}\|\phi_1\|_p\right) \\ &\leq C_4\left(\|w_n\|_\infty^{\frac{p-1}{p}}\|w_n\|_1^{\frac{1}{p}} + 1 + t_n^{q-1} + \|w_n\|_\infty^{q-1-\frac{q}{p}}\|w_n\|_q^{\frac{q}{p}} + \frac{t_n}{n}\right) \\ &\leq C_5\left(\|w_n\|_\infty^{\frac{p-1}{p}} t_n^{\frac{q-1}{p}} + t_n^{q-1} + \|w_n\|_\infty^{q-1-\frac{q}{p}} t_n^{\frac{q(q-1)}{p}} + \frac{t_n}{n}\right).\end{aligned}$$

Dividing the first and the last side of the previous inequality by t_n and bearing in mind that $y^m \leq 1 + y$, for all $m \in [0,1]$ and $y > 0$, we get

$$\begin{aligned}\left\|\frac{w_n}{t_n}\right\|_\infty &\leq C_5\left(\left\|\frac{w_n}{t_n}\right\|_\infty^{\frac{p-1}{p}} t_n^{\frac{q-2}{p}} + t_n^{q-2} + \left\|\frac{w_n}{t_n}\right\|_\infty^{q-1-\frac{q}{p}} t_n^{(q-2)\left(1+\frac{q}{p}\right)} + \frac{1}{n}\right) \\ &\leq C_5\left(t_n^{q-2} + \left(t_n^{\frac{q-2}{p}} + t_n^{(q-2)\left(1+\frac{q}{p}\right)}\right)\left(1 + \left\|\frac{w_n}{t_n}\right\|_\infty\right) + \frac{1}{n}\right) \\ &\leq C_5\left(t_n^{\frac{q-2}{p}} + 2t_n^{\frac{q-2}{p}}\left(1 + \left\|\frac{w_n}{t_n}\right\|_\infty\right) + \frac{1}{n}\right).\end{aligned}$$

It follows that

$$\left(1 - 2C_5 t_n^{\frac{q-2}{p}}\right)\left\|\frac{w_n}{t_n}\right\|_\infty \leq 3C_5 t_n^{\frac{q-2}{p}} + \frac{C_5}{n},$$

and, as a consequence,

$$\lim_{n\to+\infty}\left\|\frac{w_n}{t_n}\right\|_\infty = 0,$$

i.e.,

$$\frac{u_n}{t_n} \to \phi_1 \quad \text{uniformly in } \overline{\Omega}.$$

So, fixing $\gamma \in (0, \|\phi_1\|_\infty)$, we can find $E \subset \Omega$, with $|E| > 0$, and $\tilde{n} \in \mathbb{N}$, $\tilde{n} \geq \bar{n}$, such that

$$u_n(x) \geq \gamma t_n, \quad \text{for all } n \geq \tilde{n} \text{ and } x \in E.$$

At this point, set

$$\delta := \sup_{t>0}(g(t)t - 2G(t)) \in [0, +\infty),$$

and let $\bar{t} > 0$ such that

$$g(t)t - 2G(t) \leq -\frac{(\delta+1)|\Omega|}{|E|}, \quad \text{for all } t \geq \bar{t},$$

and $n^* \geq \tilde{n}$ such that $t_n \geq \frac{\bar{t}}{\gamma}$ for all $n \geq n^*$. Then, for all $n \geq n^*$, taking also (2.5) into account, we obtain

$$\begin{aligned}
0 &< \int_\Omega (g(u_n)u_n - 2G(u_n))dx \\
&= \int_{\Omega \setminus E} (g(u_n)u_n - 2G(u_n))dx + \int_E (g(u_n)u_n - 2G(u_n))dx \\
&\leq \delta|\Omega| - (\delta+1)|\Omega| < 0,
\end{aligned}$$

a contradiction. Therefore, the sequence $\{t_n\}_{n\geq\bar{n}}$ is bounded in $\mathbb{R}$ and (2.8) yields the boundedness of $\{w_n\}_{n\geq\bar{n}}$ in $H_0^1(\Omega)$, as well. As a consequence, we get the boundedness of $\{u_n\}_{n\geq\bar{n}}$ in $H_0^1(\Omega)$, as desired. □

Collecting the results of the previous lemmas, it is now easy to derive our first existence result.

Theorem 2.3. *For all $\lambda > 0$, problem (P_λ) has at least one non-zero solution.*

Proof. Let $\{u_n\}$ be the sequence of solutions to (P_n) in Lemma 2.1. By Lemma 2.2 there exists $u^* \in H_0^1(\Omega)$ such that, up to a subsequence,

$$u_n \rightharpoonup u^* \text{ in } H_0^1(\Omega), \quad u_n \to u^* \text{ in } L^p(\Omega), \text{ for all } p \in [1, 2^*).$$

Fixing $v \in H_0^1(\Omega)$ and taking the limit as $n \to +\infty$ in the identity $I_n'(u_n)(v) = 0$, we get $I_\lambda'(u^*)(v) = 0$, i.e. u^* is a weak solution to (P_λ). To justify that $u^* \neq 0$, observe that, by (2.5) one has

$$\begin{aligned}
0 &< \left(\frac{1}{2} - \frac{1}{k}\right) R^2 \\
&\leq \lambda \int_\Omega (g(u_n)u_n dx - 2G(u_n))\, dx \\
&\leq \lambda k_1 \left(\|u_n\|_1 + \|u_n\|_q^q\right) + 2\lambda k_1 \left(\|u_n\|_1 + \frac{1}{q}\|u_n\|_q^q\right),
\end{aligned}$$

and so, letting $n \to +\infty$, the conclusion is achieved. □

We now show that, when λ approaches zero, every non-zero solution to (P_λ) is actually positive. To this aim, for all $\lambda > 0$, set

$$S_\lambda := \{u \in H_0^1(\Omega) \setminus \{0\} : u \text{ is a solution to } (P_\lambda)\},$$

and denote by $\mathcal{P}$ the interior of the positive cone of $C_0^1(\overline{\Omega})$, i.e.

$$\mathcal{P} := \left\{ u \in C_0^1(\overline{\Omega}) : u > 0 \text{ in } \Omega,\ \frac{\partial u}{\partial \nu} < 0 \text{ on } \partial\Omega \right\},$$

ν being the unit outer normal to $\partial\Omega$. Our second result reads as follows:

Theorem 2.4. *There exists $\Lambda^* > 0$ such that for each $\lambda \in (0, \Lambda^*)$, $S_\lambda \subset \mathcal{P}$.*

Proof. We first observe that, by the regularity theory of elliptic equations, for all $\lambda > 0$ and $u_\lambda \in S_\lambda$, one has $u_\lambda \in C^{1,\alpha}(\overline{\Omega})$, for some $\alpha \in (0, 1)$.

If $u_\lambda \in S_\lambda$, it is straightforward to check that $v_\lambda := \lambda^{-1} u_\lambda$ is a solution to the problem

$$\begin{cases} -\Delta u = \lambda_1 u + g(\lambda u) & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\tilde{P}_\lambda$}$$

clearly equivalent to (P_λ). Note that (g_2) ensures the existence of some $a > 0$ such that $g(t) < 0$ for all $t \in (0, a)$, and moreover it must hold

$$\|v_\lambda\|_\infty \geq \frac{a}{\lambda}, \tag{2.9}$$

otherwise we would get $g(u_\lambda) < 0$ in $\Omega \setminus u_\lambda^{-1}(0)$, and so

$$\|u_\lambda\|^2 - \lambda_1 \|u_\lambda\|_2^2 = \lambda \int_\Omega g(u_\lambda) u_\lambda dx < 0,$$

against the definition of λ_1. From now on, we will then focus on ($\tilde{P}_\lambda$). We split the proof in several steps.

Step 1. We show that there exist two constants $C^*, \Lambda_0 > 0$ such that, for any $\lambda \in (0, \Lambda_0]$ and for any $v_\lambda \in S_\lambda$,

$$\|v_\lambda\| \geq \frac{C^*}{\lambda}. \tag{2.10}$$

Fix $\beta > \max\{\frac{N}{2}, \frac{1}{q-1}\}$. By [1, Theorem 8.2] and the embedding $W^{2,\beta}(\Omega) \hookrightarrow C^1(\overline{\Omega})$, one has $v_\lambda \in W^{2,\beta}(\Omega)$ and there exists a constant $C_0 > 0$, independent of λ, such that

$$\|v_\lambda\|_{C^1(\overline{\Omega})} \leq C_0 \left((\lambda_1 + 1) \|v_\lambda\|_\beta + \|g(\lambda v_\lambda)\|_\beta \right). \tag{2.11}$$

So, by (g_1) and Hölder's inequality, we get

$$\begin{aligned} \int_\Omega |g(\lambda v_\lambda)|^\beta dx &\leq k_1^\beta \int_\Omega \left(1 + (\lambda v_\lambda)^{q-1}\right)^\beta dx \\ &\leq 2^{\beta-1} k_1^\beta \left(|\Omega| + \lambda^{\beta(q-1)} \|v_\lambda\|_\infty^{\beta(q-1)-1} \|v_\lambda\|_1 \right), \end{aligned}$$

and therefore

$$\|v_\lambda\|_\infty \le C_0 \Bigg((\lambda_1+1)\,\|v_\lambda\|_\infty^{\frac{\beta-1}{\beta}}\,\|v_\lambda\|_1^{\frac{1}{\beta}} + 2^{\frac{\beta-1}{\beta}} k_1 \left(|\Omega|^{\frac{1}{\beta}} + \lambda^{q-1}\,\|v_\lambda\|_\infty^{q-1-\frac{1}{\beta}}\,\|v_\lambda\|_1^{\frac{1}{\beta}} \right) \Bigg).$$

Now, dividing by $\|v_\lambda\|_\infty^{\frac{\beta-1}{\beta}}$ both sides of the previous inequality and taking (2.9) into account, we obtain,

$$\begin{aligned} \left(\frac{a}{\lambda}\right)^{\frac{1}{\beta}} \le \|v_\lambda\|_\infty^{\frac{1}{\beta}} &\le C_1 \left(\|v_\lambda\|_1^{\frac{1}{\beta}} + \|v_\lambda\|_\infty^{\frac{1-\beta}{\beta}} + \lambda^{q-1}\,\|v_\lambda\|_\infty^{q-2}\,\|v_\lambda\|_1^{\frac{1}{\beta}} \right) \\ &\le C_1 \left(\|v_\lambda\|_1^{\frac{1}{\beta}} + a^{\frac{1-\beta}{\beta}}\lambda^{\frac{\beta-1}{\beta}} + a^{q-2}\lambda\,\|v_\lambda\|_1^{\frac{1}{\beta}} \right) \\ &\le C_2 \left((1+\lambda)\,\|v_\lambda\|^{\frac{1}{\beta}} + \lambda^{\frac{\beta-1}{\beta}} \right). \end{aligned} \tag{2.12}$$

Now, if $0 < \lambda \le \min\{1, a(2C_2)^{-\beta}\} := \Lambda_0$, one has

$$\|v_\lambda\|^{\frac{1}{\beta}} \ge \frac{1}{2C_2}\left(\frac{a}{\lambda}\right)^{\frac{1}{\beta}} - \frac{1}{2} \ge \frac{1}{4C_2}\left(\frac{a}{\lambda}\right)^{\frac{1}{\beta}}$$

and hence (2.10) is fulfilled with $C^* = a(4C_2)^{-\beta}$. Since of course $\|v_\lambda\| \to +\infty$ as $\lambda \to 0^+$, by (2.12) we can determine $C_3 > 0$ and $\Lambda_1 \in (0, \Lambda_0]$ such that $\|v_\lambda\| \ge 1$ and

$$\|v_\lambda\|_\infty \le C_3\,\|v_\lambda\| \tag{2.13}$$

for any $\lambda \in (0, \Lambda_1]$. For the rest of the proof, we assume $\lambda \in (0, \Lambda_1]$.

Step 2. We now show that, writing v_λ as

$$v_\lambda = t_\lambda \phi_1 + w_\lambda,$$

with $t_\lambda \in \mathbb{R}$ and $w_\lambda \in \operatorname{span}\{\phi_1\}^\perp$, then it holds

$$\|w_\lambda\|_{C^1(\overline{\Omega})} \le \tilde{C}\,\|v_\lambda\|^{\frac{q}{2}}, \tag{2.14}$$

for some $\tilde{C} > 0$. By the same arguments as [3], it is easily seen that $t_\lambda > 0$ and that w_λ is a weak solution to

$$\begin{cases} -\Delta u = \lambda_1 u + g(\lambda v_\lambda) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.15}$$

The relation $I'_\lambda(v_\lambda)(\phi_1) = 0$ and the definition of ϕ_1 imply that

$$\int_\Omega \nabla v_\lambda \nabla \phi_1 dx - \lambda_1 \int_\Omega v_\lambda \phi_1 dx - \int_\Omega g(\lambda v_\lambda)\phi_1 dx = -\int_\Omega g(\lambda v_\lambda)\phi_1 dx = 0,$$

and therefore

$$\int_\Omega g(\lambda v_\lambda) w_\lambda dx = \int_\Omega g(\lambda v_\lambda)(v_\lambda - t_\lambda \phi_1) dx = \int_\Omega g(\lambda v_\lambda) v_\lambda dx.$$

So, we get

$$\begin{aligned}\|w_\lambda\|^2 &= \lambda_1 \|w_\lambda\|_2^2 + \int_\Omega g(\lambda v_\lambda) w_\lambda dx\\ &\le \frac{\lambda_1}{\lambda_2}\|w_\lambda\|^2 + \int_\Omega g(\lambda v_\lambda) v_\lambda dx\\ &\le \frac{\lambda_1}{\lambda_2}\|w_\lambda\|^2 + k_1\left(\|v_\lambda\|_1 + \lambda^{q-1}\|v_\lambda\|_q^q\right)\\ &\le \frac{\lambda_1}{\lambda_2}\|w_\lambda\|^2 + C_4\|v_\lambda\|^q,\end{aligned}$$

from which we deduce the estimate

$$\|w_\lambda\|^2 \le C_5 \|v_\lambda\|^q, \tag{2.16}$$

being $C_5 = \frac{\lambda_2 C_4}{\lambda_2 - \lambda_1}$. By applying the same arguments as before to the function w_λ and bearing in mind also (2.13) and (2.16), we obtain

$$\begin{aligned}\|w_\lambda\|_{C^1(\overline{\Omega})} &\le C_6\left((\lambda_1+1)\|w_\lambda\|_\beta + \|g(\lambda v_\lambda)\|_\beta\right)\\ &\le C_6\left((\lambda_1+1)\|w_\lambda\|_\infty^{\frac{\beta-1}{\beta}}\|w_\lambda\|_1^{\frac{1}{\beta}} + 2^{\frac{\beta-1}{\beta}}k_1\left(|\Omega|^{\frac{1}{\beta}} + \lambda^{q-1}\|v_\lambda\|_\infty^{q-1-\frac{1}{\beta}}\|v_\lambda\|_1^{\frac{1}{\beta}}\right)\right)\\ &\le C_7\left(\|w_\lambda\|_{C^1(\overline{\Omega})}^{\frac{\beta-1}{\beta}}\|v_\lambda\|^{\frac{q}{2\beta}} + 1 + \lambda^{q-1}\|v_\lambda\|^{q-1}\right)\\ &\le C_7\left(\|w_\lambda\|_{C^1(\overline{\Omega})}^{\frac{\beta-1}{\beta}}\|v_\lambda\|^{\frac{q}{2\beta}} + 2\|v_\lambda\|^{q-1}\right).\end{aligned}$$

So, either

$$\|w_\lambda\|_{C^1(\overline{\Omega})} \le 2C_7\|w_\lambda\|_{C^1(\overline{\Omega})}^{\frac{\beta-1}{\beta}}\|v_\lambda\|^{\frac{q}{2\beta}}$$

or

$$\|w_\lambda\|_{C^1(\overline{\Omega})} \le 4C_7\|v_\lambda\|^{q-1}.$$

In any case, we get

$$\|w_\lambda\|_{C^1(\overline{\Omega})} \le \tilde{C}\|v_\lambda\|^{\frac{q}{2}}, \tag{2.17}$$

where $\tilde{C} = 4C_7$, as desired.

Step 3 (conclusion). Taking (2.10) and (2.16) into account, for $0 < \lambda \le \min\{1, \Lambda_0, \Lambda_1, \Lambda_2\}$, where $\Lambda_2 := \left(\frac{1}{2C_5}\right)^{\frac{1}{2-q}} C^*$, we obtain

$$t_\lambda^2 \ge \frac{\|v_\lambda\|^2 - C_5\|v_\lambda\|^q}{\|\phi_1\|^2} \ge \frac{\|v_\lambda\|^2}{\|\phi_1\|^2}\left(1 - \frac{C_5 {C^*}^{q-2}}{\lambda^{q-2}}\right) \ge \frac{\|v_\lambda\|^2}{2\|\phi_1\|^2} = C_8\|v_\lambda\|^2, \tag{2.18}$$

where $C_8 = \frac{1}{2\|\phi_1\|^2}$. For this range of λ, in view of (2.17), we then obtain

$$\left\|t_\lambda^{-1}v_\lambda - \phi_1\right\|_{C^1(\overline{\Omega})} = t_\lambda^{-1}\|w_\lambda\|_{C^1(\overline{\Omega})} \le \tilde{C}C_8^{-\frac{1}{2}}\|v_\lambda\|^{\frac{q}{2}-1} \le C_9\lambda^{1-\frac{q}{2}}$$

with $C_9 = \tilde{C}C_8^{-\frac{1}{2}}{C^*}^{\frac{q}{2}-1}$. Since $\phi_1 \in \mathcal{P}$ and $\mathcal{P}$ is an open subset of $C^1(\overline{\Omega})$, there exists $\delta > 0$ such that

$$\{u \in C^1(\overline{\Omega}) : \|u - \phi_1\|_{C^1(\overline{\Omega})} < \delta\} \subset \mathcal{P}.$$

So, setting $\Lambda_3 := \left(\frac{\delta}{C_9}\right)^{\frac{2}{2-q}}$, for all $0 < \lambda \le \min\{1, \Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3\} := \Lambda^*$, one has $t_\lambda^{-1}v_\lambda \in \mathcal{P}$ and hence $v_\lambda \in \mathcal{P}$. This concludes the proof. □

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Periodic and bounded solutions of functional differential equations with small delays

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Abstract. We study existence and local uniqueness of periodic solutions of nonlinear functional differential equations of first order with small delays. Bifurcations of periodic and bounded solutions of particular periodically forced second-order equations with small delays are investigated as well.

Keywords: periodic and bounded solutions, small delay, topological degree, bifurcation.

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1 Introduction

In this paper, we study existence and local uniqueness of periodic solutions of nonlinear delay first-order equation

$$\dot{x}(t) = f(x(t-\varepsilon), t), \quad t \in \mathbb{R} \tag{1.1}$$

where ε is a positive parameter, $f : \mathbb{R}^2 \to \mathbb{R}$ and f satisfies assumptions fully specified in Theorems 3.1 and 3.2 below. We use classical methods such as Leray–Schauder degree and a priori estimates to prove that for sufficiently small parameters ε and under certain assumptions to right-hand-side function f, there is a locally unique periodic solution that depends continuously on ε. Similar methods were used e.g., in [7,8] where more complicated neutral differential equations were studied. Bifurcation theory is applied for perturbed second order case of (1.1) to get existence and non-existence results for periodic and bounded solutions with examples in Section 4. Related results to this paper are derived in [1]. We refer the reader to [2] for more papers dealing with the effects of small delays on the dynamical behaviors of systems compared with differential equations without delays.

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2 Preliminaries

As we already mentioned our paper deals with periodic solutions of equation (1.1). Such equations are usually equipped with initial condition $x(t) = \varphi(t)$ for $t \in [-\varepsilon, 0]$ where function φ is given. To avoid defining an initial condition for periodic solutions we introduce the following new problem. We see that a function x is a T-periodic solution of (1.1) and $\varepsilon \in (0, T)$ if and only if it is a solution of the problem

$$\begin{aligned} \dot{x}(t) &= f\left(x\left(t-\varepsilon\right), t\right), \quad && t \in [0, T], \\ x(t) &= x(T+t), && t \in [-\varepsilon, 0]. \end{aligned} \tag{2.1}$$

The corresponding problem (when $\varepsilon = 0$) is then

$$\begin{aligned} \dot{x}(t) &= f\left(x\left(t\right), t\right), \quad && t \in [0, T], \\ x(0) &= x(T). \end{aligned} \tag{2.2}$$

Here we introduce some notation we will use in the rest of our paper. Let X be the space of continuous, T-periodic functions defined on $\mathbb{R}$ equipped with the maximum norm $\|x\|_\infty := \max_{t\in\mathbb{R}} |x(t)|$. We define the closed ball in X as

$$B_r(y) =: \{x \in X;\ \|x - y\|_\infty \le r\}$$

and let $I : X \to X$ be the identical operator. For the Leray–Schauder degree of function f on domain Ω at point 0, we will use the standard notation $\deg(f, \Omega, 0)$. Properties of Leray–Schauder degree can be found in e.g., in [4].

3 Existence results

Theorem 3.1. *Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f = f(x, t)$ be a uniformly Lipschitz continuous function with respect to x, T-periodic in variable t. Let there exist $\delta, \eta, K, L > 0$ such that $-L + \eta < K - \eta$ and either*

$$f(x,t) \ge \delta \text{ for } (x,t) \in [-L-\eta, -L+\eta] \times [0,T] \quad \text{and} \quad f(x,t) \le -\delta \text{ for } (x,t) \in [K-\eta, K+\eta] \times [0,T], \tag{3.1}$$

or

$$f(x,t) \le -\delta \text{ for } (x,t) \in [-L-\eta, -L+\eta] \times [0,T] \quad \text{and} \quad f(x,t) \ge \delta \text{ for } (x,t) \in [K-\eta, K+\eta] \times [0,T] \tag{3.2}$$

is satisfied. Then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$, there exists a solution of problem (2.1) that is bounded by K and $-L$.

Proof. Denote

$$\Omega = \{x \in X;\ x([0,T]) \subset (-L, K)\}.$$

First, we assume that the condition (3.1) is valid. We define the following operator

$$F : [0,T] \times X \to X, \qquad F(\varepsilon, x)(t) = x(T) + \int_0^t f\left(\widetilde{x}\left(s-\varepsilon\right), s\right)\, \mathrm{d}s \tag{3.3}$$

where $\tilde{x}(t) = x(t)$ if $t \in [0,T]$, and $\tilde{x}(t) = x(T+t)$ if $t \in [-\varepsilon, 0)$. Clearly, the operator F is well defined. Note that F is also continuous and compact due to the local boundedness and continuity of f. In the following, we will use the notation $F_\varepsilon := F(\varepsilon, .)$ whenever $\varepsilon \geq 0$ is fixed.

Our goal is to prove that if $\varepsilon_0 > 0$ is sufficiently small then for the Leray–Schauder degree, there holds

$$\deg(I - F_\varepsilon, \Omega, 0) = \deg(I - F_0, \Omega, 0) = 1$$

for $\varepsilon \in (0, \varepsilon_0]$.

We will prove that $(I - \theta F_0)x \neq 0$ for every $x \in \partial\Omega$ and $\theta \in [0,1]$. This is clearly true for $\theta = 0$. Assume that there exists some $\theta \in (0,1)$ and a solution $x \in \partial\Omega$ of problem

$$\begin{aligned} \dot{x}(t) &= \theta f\left(x\left(t\right), t\right) \quad \text{for } t \in [0,T], \\ x(0) &= \theta x(T). \end{aligned} \tag{3.4}$$

This means that x is a fixed point of the operator θF_0. Since $x \in \partial\Omega$, there exists either $t_0 \in [0,T]$ and $x(t_0) = K$, or $t_1 \in [0,T]$ and $x(t_1) = -L$. We will deal with the first case that x attains maximum $K > 0$ at some t_0, since the proof is similar in the second case.

Due to the assumption (3.1), we see that $\dot{x}_0(t_0) \leq -\delta$ and hence x is decreasing in some neighbourhood of t_0. Then necessarily $t_0 = 0$, otherwise x would attain higher values for some $t < t_0$. The solution x satisfies the problem (3.4), hence $x(T) \geq K$. This is not possible, since x is decreasing in some neighbourhood of 0 and decreases whenever x reaches value K due to (3.1). Thus, we proved $\deg(I - \theta F_0, \Omega, 0) = \deg(I, \Omega, 0) = 1$.

Finally, we will prove that $(I - \theta F_0 - (1-\theta)F_\varepsilon)x \neq 0$ for every $x \in \partial\Omega$ and $\theta \in [0,1]$. This means we have to show that there are no solutions of problem

$$\begin{aligned} \dot{x}(t) &= \theta f\left(x\left(t\right), t\right) + (1-\theta) f\left(x\left(t-\varepsilon\right), t\right), \quad t \in [0,T], \\ x(t) &= x(T+t), \quad t \in [-\varepsilon, 0] \end{aligned} \tag{3.5}$$

that lie on the boundary of Ω. Let there exist a solution x of (3.5) that attains its maximum K at some $t_0 \in [0,T]$ (the case when x attains minimum $-L$ would be treated similarly). This solution can be periodically extended to the whole real line. Moreover, x is Lipschitz continuous with some Lipschitz constant M that is a bound for f on set $\overline{\Omega}$. For some $\varepsilon_0 > 0$ sufficiently small and dependent on x, there holds $x(t_0 - \varepsilon) \in [K - \eta, K]$ for $\varepsilon \in (0, \varepsilon_0]$ and so $\dot{x}(t_0) \leq -\delta$. This is a contradiction, since x is periodic and attains maximum at t_0.

The next step is to remove the dependence of ε_0 on solution x. For every $x \in \overline{\Omega}$, there holds

$$|x(t_0) - x(t_0 - \varepsilon)| \leq M\varepsilon_0 \quad \text{for all } \varepsilon \in [0, \varepsilon_0].$$

If we choose $\varepsilon_0 = \frac{\eta}{M}$ then both values $x(t_0)$ and $x(t_0 - \varepsilon)$ stay in the interval $[K - \eta, K]$ for $\varepsilon \in [0, \varepsilon_0]$. Therefore since x satisies the problem (3.5), we get $\dot{x}(t_0) \leq -\delta$ what contradicts the fact that x attains its maximum at t_0. Thus we proved $\deg(I - F_\varepsilon, \Omega, 0) = \deg(I - F_0, \Omega, 0) = 1$ for $\varepsilon \in (0, \varepsilon_0]$.

Next, we assume that the condition (3.2) is valid. In this case, our goal is to prove the existence of T-periodic solution of problem

$$\begin{aligned} \dot{y}(t) &= -f\left(y\left(t+\varepsilon\right), -t\right), \quad t \in [0,T], \\ y(t) &= y(T+t), \qquad\qquad\quad t \in \left[0, \varepsilon\right]. \end{aligned} \tag{3.6}$$

for all $\varepsilon > 0$ sufficiently small, since then we just set $s = -t$ and $x(s) := y(-s)$. Then x will be a T-periodic solution of the original problem (2.1).

We define the following operator

$$G : [0,T] \times X \to X, \qquad G(\varepsilon, x)(t) = x(T) - \int_0^t f\left(\widetilde{x}\left(s+\varepsilon\right), -s\right)\, \mathrm{d}s$$

where $\widetilde{x}(t) = x(t)$ if $t \in [0,T]$, and $\widetilde{x}(t) = x(t-T)$ if $t \in (T, T+\varepsilon]$. Note that the right-hand-side function in problem (3.6) satisfies the assumption (3.1). Using the notation $G_\varepsilon := G(\varepsilon, .)$ for $\varepsilon \geq 0$ fixed and using similar arguments as in the previous part of the proof, we come to conclusion

$$\deg(I - G_\varepsilon, \Omega, 0) = \deg(I - G_0, \Omega, 0) = 1$$

for $\varepsilon \in (0, \varepsilon_0]$ with $\varepsilon_0 > 0$ sufficiently small. □

Theorem 3.2. *Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f = f(x,t)$ be a bounded continuous function, T-periodic in variable t and there exist $K, L, \delta > 0$ such that either*

$$\int_0^T f(x,t)\, \mathrm{d}t \geq \delta \quad \textit{for } x \geq K \quad \textit{and} \quad \int_0^T f(x,t)\, \mathrm{d}t \leq -\delta \quad \textit{for } x \leq -L \tag{3.7}$$

or

$$\int_0^T f(x,t)\, \mathrm{d}t \leq -\delta \quad \textit{for } x \geq K \quad \textit{and} \quad \int_0^T f(x,t)\, \mathrm{d}t \geq \delta \quad \textit{for } x \leq -L. \tag{3.8}$$

Then for every $\varepsilon > 0$ there exists a solution of problem (2.1).

Proof. In our proof, we will proceed under the case (3.7) only since the case (3.8) would be dealt with similarly as in the proof of Theorem 3.1 under the assumption (3.2). We define the following operators

$$\begin{aligned} H : X \to X, \quad & T : \mathbb{R} \times X \to \mathbb{R} \times X, \\ H(x)(t) &= \int_0^t f(x(s-\varepsilon), s)\, \mathrm{d}s - \frac{t}{T} \int_0^T f(x(s-\varepsilon), s)\, \mathrm{d}s, \\ T(r,x)(t) &= \left(\int_0^T f(x(s-\varepsilon), s)\, \mathrm{d}s, x(t) - r - H(x)(t) \right). \end{aligned}$$

Observe that $T(r,x) = 0$ for some $r \in \mathbb{R}$ and $x \in X$ if and only if x is a solution of (2.1) and $x(0) = r$.

As in the proof of Theorem 3.1, we will use the Leray–Schauder degree to prove the assertion. Let $M > 0$ be a global bound of right-hand-side function f and let $\alpha = \max\{K, L\} + 2MT + 1$, $\beta = \alpha + 2MT + 1$. We define the domain

$$\Omega := \{(r,x) \in \mathbb{R} \times X;\ |r| < \alpha,\ \|x\|_\infty < \beta\}$$

and the homotopy

$$T_\theta(r,x) = \left(\int_0^T f(r + \theta H(x)(s-\varepsilon), s)\, \mathrm{d}s, x(t) - \theta(r + H(x)(t)) \right)$$

where $\theta \in [0,1]$. Our goal is to prove that

$$\deg(T_1, \Omega, 0) = \deg(T_0, \Omega, 0) = 1$$

which means that there exist $(r,x) \in \Omega$ such that $T_1(r,x) = 0$. One can easily prove that $T_1(r,x) = 0$ if and only if $T(r,x) = 0$.

Now, we prove that $T_\theta(r,x) \neq 0$ for $(r,x) \in \partial\Omega$ and $\theta \in [0,1]$. Note that $H(x)(0) = 0$ and $H(x) \in X$ for every $x \in X$. Since

$$|\dot{H}(x)(t)| = \left| f(x(t-\varepsilon),t) - \frac{1}{T}\int_0^T f(x(s-\varepsilon),s)\,\mathrm{d}s \right| \leq 2M$$

due to the boundedness of f, we have $|H(x)(t)| \leq 2MT$ for every $x \in X$ and $t \in \mathbb{R}$. Next, assume by contradiction that there is some $(r,x) \in \partial\Omega$ and $\theta \in [0,1]$ such that $T_\theta(r,x) = 0$. Then it holds

$$|x(t)| = \theta|r + H(x)(t)| \leq \alpha + 2MT < \beta, \quad t \in \mathbb{R}$$

for every $r \in [-\alpha,\alpha]$. Then necessarily $r = \pm\alpha$, otherwise $(r,x) \notin \partial\Omega$. For the case $r = \alpha$, we obtain

$$r + \theta H(x)(s-\varepsilon) \geq \alpha - 2MT \geq K,$$

so due to the assumption (3.7), it holds

$$\int_0^T f(r + \theta H(x)(s-\varepsilon),s)\,\mathrm{d}s \geq \delta.$$

This means that $T_\theta(\alpha,x) \neq 0$ and this is a contradiction. For the case $r = -\alpha$, we obtain a similar estimate

$$r + \theta H(x)(s-\varepsilon) \leq -\alpha + 2MT \leq -L$$

and using (3.7) leads to a contradiction. Thus $\deg(T_1,\Omega,0) = \deg(T_0,\Omega,0)$.

The identity $\deg(T_0,\Omega,0) = 1$ follows from the basic properties of the Leray–Schauder degree. In fact, the domain Ω can be represented as a Cartesian product of interval and a ball in the maximum norm. Hence

$$\deg(T_0,\Omega,0) = \deg\left((g,I),\Omega,0\right) = \deg(g,(-\alpha,\alpha),0)$$

where $g = g(r) = \int_0^T f(r,s)\,\mathrm{d}s$. Since $g(-\alpha) < 0 < g(\alpha)$ due to the assumption (3.7), we can define homotopy $g_\theta(r) = \theta g(r) + (1-\theta)r$ and we conclude that $\deg(g,(-\alpha,\alpha),0) = 1$. □

Lemma 3.3. *Let $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $f = f(x,t)$ be a uniformly Lipschitz continuous function with respect to x, T-periodic in variable t. Denote by M a Lipschitz constant for function f and let $L > M$ be a given constant. Then for any $\varepsilon > 0$ such that*

$$\varepsilon < \frac{L-M}{L}e^{-LT}, \tag{3.9}$$

(periodic) solutions of problem (2.1) *do not intersect each other.*

Proof. Let x, y be two (periodic) solutions of (2.1) that intersect at some $t_0 \in [0,T]$. Introduce the norm $\|x\|_L := \max_{t\in[t_0,t_0+T]} e^{-L(t-t_0)}|x(t)|$. Let $z =: x - y$ and $t \in [t_0,t_0+T]$. Using standard estimates, the Lipschitz continuity of f (and M denotes the Lipschitz constant), the periodicity of function z and the equality

$$z(t) = \int_{t_0}^t (\dot{x}-\dot{y})(s)\,\mathrm{d}s = \int_{t_0}^t f(x(s-\varepsilon),s) - f(y(s-\varepsilon),s)\,\mathrm{d}s,$$

we obtain the following estimation

$$\begin{aligned}
e^{-L(t-t_0)}|z(t)| &\le M\int_{t_0}^{t} e^{-L(t-s+\varepsilon)}e^{-L(s-\varepsilon-t_0)}|z(s-\varepsilon)|\,\mathrm{d}s\\
&\le M\left(\int_{t_0+\varepsilon}^{t} e^{-L(-s+\varepsilon+t)}e^{-L(s-\varepsilon-t_0)}|z(s-\varepsilon)|\,\mathrm{d}s\right.\\
&\quad\left.+\int_{t_0}^{t_0+\varepsilon} e^{-L(-s+\varepsilon+t)}e^{-L(s-\varepsilon-t_0)}|z(s-\varepsilon)|\,\mathrm{d}s\right)\\
&\le \frac{M}{L}\left(e^{-L\varepsilon}-e^{-L(t-t_0+\varepsilon)}\right)\|z\|_L\\
&\quad+\int_{t_0}^{t_0+\varepsilon} e^{L(s+T-t-\varepsilon)}e^{-L(s-\varepsilon+T-t_0)}|z(s-\varepsilon+T)|\,\mathrm{d}s\\
&\le \left(\frac{M}{L}+\varepsilon e^{LT}\right)\|z\|_L\,.
\end{aligned}$$

Hence $z\equiv 0$ due to the assumption (3.9) and this concludes the proof of the lemma. □

Now, we are ready to prove the following Theorem 3.4. The proof relies on Theorem 3.1, however, Theorem 3.4 can be proven also using Theorem 3.2.

Theorem 3.4. *Let $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $f=f(x,t)$ be a uniformly Lipschitz continuous function with respect to x, T-periodic in variable t and let x_0 be a (T-periodic) solution of problem (2.2). Assume that there exists a constant $\eta>0$ such that function f is either increasing, or decreasing in variable x for every $t\in[0,T]$ and $x\in[x_0(t)-\eta,x_0(t)+\eta]$. Then there exists $\varepsilon_0>0$ such that for every $\varepsilon\in(0,\varepsilon_0]$, there exists a locally unique solution x_ε of problem (2.1) and x_ε depends continuously on ε.*

Proof. Assume that f is increasing in variable x for every $t\in[0,T]$ and $x\in[x_0(t)-\eta,x_0(t)+\eta]$. We define a new right-hand-side function

$$g(x,t):=\begin{cases} f(x_0(t)-\eta,t)+x-x_0(t)+\eta, & (x,t)\in(-\infty,x_0(t)-\eta)\times[0,T]\,,\\ f(x,t), & (x,t)\in[x_0(t)-\eta,x_0(t)+\eta]\times[0,T]\,,\\ f(x_0(t)+\eta,t)+x-x_0(t)-\eta, & (x,t)\in(x_0(t)+\eta,\infty)\times[0,T]\,.\end{cases}$$

Since g is increasing in variable x, the function $x_0\in X$ is the only periodic solution of equation

$$\dot{x}(t)=g\left(x,t\right),\quad t\in[0,T]. \tag{3.10}$$

In fact, since x_0 is the periodic solution of (3.10) then necessarily $\int_0^T g(x_0(t),t)\,\mathrm{d}t=0$. Due to the Lipschitz continuity of g, we know that any other solution y does not cross x_0, hence either $y(t)>x_0(t)$ or $y(t)<x_0(t)$ for all $t\in[0,T]$. In both cases due to the strict monotonicity of g, we get $\int_0^T g(y(t),t)\,\mathrm{d}t\neq 0$ so y cannot be periodic.

The new right-hand-side function g satisfies the assumptions of Theorem 3.1 and thus there exists $\varepsilon_0>0$ such that for every $\varepsilon\in(0,\varepsilon_0]$, there exists at least one (periodic) solution x_ε of problem

$$\begin{aligned}\dot{x}(t)&=g\left(x\left(t-\varepsilon\right),t\right), && t\in[0,T],\\ x(t)&=x(T+t), && t\in[-\varepsilon,0]\,.\end{aligned} \tag{3.11}$$

Moreover, all such solutions are uniformly bounded independently of ε. We need to verify that for some $\varepsilon_0>0$ sufficiently small, the solution x_ε of (3.11) is also a solution of original problem (2.2). More precisely, we prove that there exists some $\varepsilon_0>0$ such that for every $\varepsilon\in(0,\varepsilon_0]$, there holds $x_\varepsilon \in B_\eta(x_0)$. Assume that this is not true, i.e. there exists a sequence of positive parameters $\{\varepsilon_n\}_{n\in\mathbb{N}}$ such that $\varepsilon_n \to 0$ and for every $n \in \mathbb{N}$, it holds $x_{\varepsilon_n} \notin B_\eta(x_0)$. Recall that the solutions x_{ε_n} are fixed points of the operator F defined by (3.3) and due to the compactness of F and the uniform boundedness of functions x_{ε_n}, the set $\{x_{\varepsilon_n}\}_{n\in\mathbb{N}}$ is relatively compact in X. Hence some subsequence converges uniformly to a periodic solution of equation (3.10) (since $\varepsilon_n \to 0$) that is not equal to x_0. This contradicts the uniqueness of periodic solution x_0.

Next, we will prove the local uniqueness of periodic solutions of original problem (2.2). More precisely, the solution x_ε is unique in the ball $B_\eta(x_0)$ In fact, let y be a periodic solution that lies in $B_\eta(x_0)$ and is not equal to x_ε. Then necessarily, there exists some $t_0 \in [0, T]$ such that $y(t_0) = x_\varepsilon(t_0)$, otherwise we would come to $\int_0^T f(y(t), t)\,\mathrm{d}t \neq 0$ due to the strict monotonicity of f. We choose some $L > M$ and $\varepsilon_0 > 0$ sufficiently small so the inequality (3.9) is valid for all $\varepsilon \in [0, \varepsilon_0]$. Hence the uniqueness follows from Lemma 3.3. This completes the proof of the local uniqueness of periodic solutions of problem (2.1) for small $\varepsilon > 0$.

The continuous dependence of solution x_ε on parameter ε is a consequence of uniform boundedness of these solutions and compactness of the operator F defined by (3.3). □

Remark 3.5. Lemma 3.3 need not to be true if the inequality (3.9) is not valid. In fact, consider the equation

$$\dot{x}(t) = x\left(t - \frac{3\pi}{2}\right).$$

This equation possesses infinitely many 2π-periodic solutions of form $a\sin(x+b)$ for $a \in \mathbb{R}$ and $b \in [0, \pi)$ and every two of these solutions intersect each other.

4 Bifurcations

We consider the perturbed equation

$$\ddot{x}(t) + g(x(t - \varepsilon\mu_1)) + \varepsilon\mu_2 h(t) = 0 \tag{4.1}$$

where $g, h \in C^3(\mathbb{R}, \mathbb{R})$, $\mu_1, \mu_2 \in \mathbb{R}$ and $h(t)$ is T-periodic.

Theorem 4.1. *Assume that there is a T-periodic solution $u(t)$ of equation*

$$\ddot{u} + g(u) = 0 \tag{4.2}$$

such that $v(t) = \dot{u}(t)$ is the only T-periodic solution up to a scalar multiple of

$$\ddot{v} + g'(u(t))v = 0.$$

If the function

$$M(\alpha) = \mu_2 \int_0^T h(t+\alpha)\dot{u}(t)dt - \mu_1 \int_0^T g'(u(t))\dot{u}^2(t)dt$$

has a simple zero α_0, i.e., $M(\alpha_0) = 0$ and $M'(\alpha_0) \neq 0$, then for any $\varepsilon \neq 0$ small, the equation (4.1) has the unique T-periodic solution $x_\varepsilon(t)$ that satisfies

$$\sup_{t\in\mathbb{R}} |x_\varepsilon(t) - u(t-\alpha_0)| + |\dot{x}_\varepsilon(t) - \dot{u}(t-\alpha_0)| = O(\varepsilon). \tag{4.3}$$

Proof. Note that the equation (4.1) can be written in the form

$$\ddot{x}(t) + g(x(t)) - \varepsilon\mu_1 g'(x(t))\dot{x}(t) + \varepsilon\mu_2 h(t) = O(\varepsilon^2).$$

Hence we can apply the well-known Melnikov theory (see [3,5]) to obtain the result. □

Remark 4.2. 1. Note that $M(\alpha)$ is T-periodic and

$$M(\alpha) = -\mu_1 \int_0^T \ddot{u}^2(t)dt - \mu_2 \int_0^T \dot{h}(t+\alpha)u(t)dt. \tag{4.4}$$

2. The function $u(t)$ is embedded into a 1-parametric family of periodic solutions $u_a(t)$, $a \in (a_1, a_2) \subset \mathbb{R}$ of (4.2) with minimal periods $T(a)$. So $u_{a_0}(t) = u(t)$. If $T'(a_0) \neq 0$ then the assumption of Theorem 4.1 holds.

3. The existence part of Theorem 4.1 holds if

$$\max_{\alpha\in\mathbb{R}} M(\alpha) \min_{\alpha\in\mathbb{R}} M(\alpha) < 0.$$

4. If $M(\alpha) \neq 0, \forall \alpha \in \mathbb{R}$ then there is no bifurcation.

5. If $x(t)$ is a solution of (4.1) then $x(t+kT)$, $k \in \mathbb{Z}$ is also a solution. So we consider in Theorem 4.1 just $\alpha_0 \in [0, T]$.

To illustrate the theory, we consider the following example

$$\ddot{x}(t) + x(t-\varepsilon\mu_1) + x^3(t-\varepsilon\mu_1) + \varepsilon\mu_2 \cos 2t = 0. \tag{4.5}$$

So (4.2) is the Duffing equation

$$U''(t) + U(t) + U^3(t) = 0$$

possessing a family of periodic solutions

$$u_a(t) = a\,\mathrm{cn}(\sqrt{1+a^2}t)$$

for $a > 0$ with periods $T(a) = \frac{4K(k)}{\sqrt{1+a^2}}$, $k = \frac{a}{\sqrt{2+2a^2}}$. Note $u_a(0) = a$ and $u_a'(0) = 0$. Here cn is the Jacobi elliptic function, $K(k)$ is the complete elliptic function of the first kind and k is the elliptic modulus, see [6]. Moreover, we have

$$T'(a) = \frac{8(E(k)-K(k)) - 4a^2K(k)}{a\sqrt{1+a^2}\,(2+a^2)} < 0,$$

since $E(k) \leq K(k)$, where $E(k)$ is the complete elliptic function of the second kind. So $T(a)$ is decreasing from $T(0) = 2\pi$ to 0, and hence Remark 4.2 2 can be applied. Now $T = \pi$, so we numerically solve $T(a) = \pi$ to get $a_0 \cong 2.03284$ and then (4.4) has the form

$$M(\alpha) = -105.817\mu_1 + 6.17466\mu_2 \sin 2\alpha. \tag{4.6}$$

Applying Theorem 4.1 we get the following result.

Theorem 4.3. *If $|\mu_1| < 0.058352|\mu_2|$ and $\varepsilon \neq 0$ is small, then (4.5) has precisely two π-periodic solutions orbitally near $u_{a_0}(t) = 2.03284\,\mathrm{cn}(2.26549t)$, i.e., (4.3) holds just for two $\alpha_{0,1}, \alpha_{0,2} \in [0, \pi)$, namely for roots of (4.6). If $|\mu_1| > 17.1374|\mu_2|$ then (4.5) has no π-periodic solutions orbitally near $u_{a_0}(t)$ for any $\varepsilon \neq 0$ small, i.e., (4.3) does not hold for any $\alpha \in \mathbb{R}$.*

We end this paper with extending the above bifurcation results of periodic solutions to bounded ones.

Theorem 4.4. *Assume that there are x_0 and x_1 such that $g(x_0) = g(x_1) = 0$ and $g'(x_0) < 0$, $g'(x_1) < 0$. Suppose there is a solution $u(t)$ of (4.2) such that $\lim_{t\to-\infty} u(t) = x_0$ and $\lim_{t\to\infty} u(t) = x_1$. If the function*

$$M(\alpha) = -\mu_1 \int_{-\infty}^{\infty} \dot{u}^2(t)dt + \mu_2 \int_{-\infty}^{\infty} h(t+\alpha)\dot{u}(t)dt \tag{4.7}$$

has a simple zero α_0 then for any $\varepsilon \neq 0$ small, the equation (4.1) has the unique solution $x_\varepsilon(t)$ that satisfies (4.3).

Remark 4.5. The points 3, 4 and 5 of Remark 4.2 remain valid for this case.

To illustrate the theory, we consider

$$\ddot{x}(t) - x(t-\varepsilon\mu_1) + x^3(t-\varepsilon\mu_1) + \varepsilon\mu_2 \cos 2t = 0. \tag{4.8}$$

So (4.2) is the Duffing equation

$$U''(t) - U(t) + U^3(t) = 0$$

possessing a homoclinic solution

$$u(t) = \sqrt{2}\,\mathrm{sech}\, t$$

to $x_0 = x_1 = 0$. Again $h(t) = \cos 2t$. Then the Melnikov function (4.7) is now

$$M(\alpha) = -\frac{28}{15}\mu_1 + 2\sqrt{2}\pi\,\mathrm{sech}\,\pi\mu_2 \sin 2\alpha. \tag{4.9}$$

Applying Theorem 4.1 we get the following result.

Theorem 4.6. *If $|\mu_1| < \frac{15\pi\,\mathrm{sech}\,\pi}{7\sqrt{2}}|\mu_2|$ and $\varepsilon \neq 0$ is small, then (4.8) has precisely two bounded solutions orbitally near $\sqrt{2}\,\mathrm{sech}\,t$, i.e., (4.3) holds just for two $\alpha_{0,1}, \alpha_{0,2} \in [0,\pi)$, namely for roots of (4.9). If $|\mu_1| > \frac{15\pi\,\mathrm{sech}\,\pi}{7\sqrt{2}}|\mu_2|$ then (4.8) has no bounded solutions orbitally near $\sqrt{2}\,\mathrm{sech}\,t$ for any $\varepsilon \neq 0$ small, i.e., (4.3) does not hold for any $\alpha \in \mathbb{R}$. Note $\frac{15\pi\,\mathrm{sech}\,\pi}{14\sqrt{2}} \cong 0.41065$.*

Finally, we consider

$$\ddot{x}(t) + x(t-\varepsilon\mu_1) - x^3(t-\varepsilon\mu_1) + \varepsilon\mu_2 \cos 2t = 0. \tag{4.10}$$

So (4.2) is the Duffing equation

$$U''(t) + U(t) - U^3(t) = 0$$

possessing a heteroclinic solution

$$u(t) = \tanh(t/\sqrt{2}).$$

to $x_0 = -1$ and $x_1 = 1$. Again $h(t) = \cos 2t$. Then the Melnikov function (4.7) is now

$$M(\alpha) = -\frac{4\sqrt{2}}{15}\mu_1 + 2\sqrt{2}\pi\,\mathrm{csch}\,\sqrt{2}\pi\mu_2 \cos 2\alpha. \tag{4.11}$$

Applying Theorem 4.1 we get the following result.

Theorem 4.7. *If $|\mu_1| < \frac{15}{2}\pi\,\mathrm{csch}\,\sqrt{2}\pi|\mu_2|$ and $\varepsilon \neq 0$ is small, then (4.10) has precisely two bounded solutions orbitally near $\tanh(t/\sqrt{2})$, i.e., (4.3) holds just for two $\alpha_{0,1}, \alpha_{0,2} \in [0,\pi)$, namely for roots of (4.11). If $|\mu_1| > \frac{15}{2}\pi\,\mathrm{csch}\,\sqrt{2}\pi|\mu_2|$ then (4.10) has no bolunded solutions orbitally near $\tanh(t/\sqrt{2})$ for any $\varepsilon \neq 0$ small, i.e., (4.3) does not hold for any $\alpha \in \mathbb{R}$. Note $\frac{15}{2}\pi\,\mathrm{csch}\,\sqrt{2}\pi \cong 0.554347$.*

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Iterative solution of elliptic equations

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Abstract. We reduce solution of the Dirichlet problem ($x \in D \subset R^m$)

$$\Delta u(x) + a(x)u(x) = f(x) \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D$$

to iterative solution of a simpler problem

$$\Delta u = f(x) \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D\,,$$

for which one can use either Fourier series or Green's function method. The method is suitable for numerical computations, particularly when one uses Newton's method for semilinear problems

$$\Delta u + g(x,u) = 0 \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D\,,$$

in dimensions $m \geq 3$.

Keywords: iterative method, Lyapunov–Schmidt reduction.

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1 Introduction

If Green's function $G(x,y)$ is available for a domain $D \subset R^m$, it is easy to solve numerically the Dirichlet problem for Laplace's equation

$$-\Delta u = f(x) \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D\,. \tag{1.1}$$

The solution is $u(x) = \int_D G(x,y)f(y)\,dy$. *Mathematica* software can compute such integrals quickly and accurately even in dimensions $m > 2$, say for $m = 5$. When solving semilinear problems

$$\Delta u + g(x,u) = 0 \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D$$

one usually uses Newton's method

$$\Delta u_{p+1} + g(x,u_p) + g_u(x,u_p)\left(u_{p+1} - u_p\right) = 0 \quad \text{in } D, \qquad u_{p+1} = 0 \quad \text{on } \partial D\,,$$

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which requires repeated solution of the linear problems

$$\Delta u + a(x)u = f(x) \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D\,, \tag{1.2}$$

with given functions $a(x)$ and $f(x)$. It is very unlikely to have eigenfunctions (or Green's function) available for the problem (1.2). *Question: can one reduce solving* (1.2) *to iterative solution of* (1.1)*?* It turns out that the answer is affirmative for any bounded $a(x)$. We show that either the iterations

$$-\Delta u_{n+1} = a(x)u_n - f(x) \quad \text{in } D, \qquad u_{n+1} = 0 \quad \text{on } \partial D \tag{1.3}$$

converge to the solution of (1.2), or else there is a modified iterative process that converges to the solution of (1.2). Eigenfunctions of the Laplacian, or Green's functions, are available for some domains. For other domains their computation is a one time effort, while solving nonlinear problems requires repeated solutions of the problem (1.2), particularly in connection to curve following.

Turning to the description of the method, let $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$ be the eigenvalues of $-\Delta$ with zero boundary conditions on D (λ_1 is simple, while some other eigenvalues may be repeated), and $\varphi_1 > 0, \varphi_2, \varphi_3, \ldots$ be the corresponding eigenfunctions of $-\Delta$, forming an orthonormal set in $L^2(D)$, so that $\int_D \varphi_k^2\, dx = 1$. Represent $f(x) = \sum_{k=1}^{\infty} f_k \varphi_k(x)$, with $f_k = \int_D f(x)\varphi_k(x)\, dx$. Recall that $\|f\|^2_{L^2(D)} = \int_D f^2(x)\, dx = \sum_{k=1}^{\infty} f_k^2$ (Parseval's identity), see e.g., W. Craig [1] or P. Korman [2]. The solution of (1.1) is

$$u(x) = \sum_{k=1}^{\infty} \frac{f_k}{\lambda_k} \varphi_k(x) \equiv (-\Delta)^{-1}\left(f(x)\right),$$

where $(-\Delta)^{-1}$ is the common notation for the solution operator of (1.1). By Parseval's identity

$$\|(-\Delta)^{-1} f\|^2_{L^2(D)} = \sum_{k=1}^{\infty} \frac{f_k^2}{\lambda_k^2} \le \frac{1}{\lambda_1^2} \sum_{k=1}^{\infty} f_k^2 = \frac{1}{\lambda_1^2} \|f\|^2_{L^2(D)}\,. \tag{1.4}$$

In case $f_1 = 0$, or $f \perp \varphi_1$ in L^2, the same argument shows that

$$\|(-\Delta)^{-1} f\|_{L^2} \le \frac{1}{\lambda_2} \|f\|_{L^2}\,, \tag{1.5}$$

and if $f_1 = f_2 = \cdots = f_j = 0$, then

$$\|(-\Delta)^{-1} f\|_{L^2} \le \frac{1}{\lambda_{j+1}} \|f\|_{L^2}\,. \tag{1.6}$$

Proposition 1.1. *Assume that $a(x) \in C(\bar{D})$ satisfies*

$$\max_{\bar{D}} |a(x)| < \lambda_1\,. \tag{1.7}$$

Then the iterates given by (1.3) *converge in $L^2(D)$, to a solution $u(x) \in H^2(D)$ of* (1.2)*, for any $f(x) \in L^2(D)$.*

Proof. Write (1.3) in the form

$$u_{n+1} = (-\Delta)^{-1}\left[a(x)u_n - f(x)\right].$$

Subtracting a similar formula for u_n, and then using (1.4), we obtain

$$u_{n+1} - u_n = (-\Delta)^{-1}\left[a(x)\,(u_n - u_{n-1})\right],$$
$$\|u_{n+1} - u_n\|_{L^2} \le \frac{1}{\lambda_1}\|a(x)\,(u_n - u_{n-1})\,\|_{L^2} \le \theta\|u_n - u_{n-1}\|_{L^2},$$

where $\theta \equiv \frac{\max_{\bar{D}}|a(x)|}{\lambda_1} < 1$, which implies that $\{u_n(x)\}$ is a Cauchy sequence in $L^2(D)$, and the proof follows in view of completeness of $L^2(D)$. □

If the condition (1.7) is violated then the iterations (1.3) diverge in $L^2(D)$, in general, as the following example shows.

Example 1.2. Let $a(x) = a$, a constant, with $\lambda_1 < a < \lambda_2$. For the iterations

$$-\Delta u_{n+1} = au_n - f(x) \quad \text{in } D, \qquad u_{n+1} = 0 \quad \text{on } \partial D, \tag{1.8}$$

write $f(x) = \sum_{k=1}^{\infty} f_k\varphi_k$, and $u_n = \sum_{k=1}^{\infty} u_n^k\varphi_k$, to obtain

$$\lambda_k u_{n+1}^k = a u_n^k - f_k.$$

Denoting $\delta = \frac{a}{\lambda_1} > 1$, obtain for the $k = 1$ component

$$u_{n+1}^1 - u_n^1 = \delta\left(u_n^1 - u_{n-1}^1\right),$$

so that the iterations (1.8) diverge (because the first component diverges).

Now suppose that the condition (1.7) does not hold, but we have

$$\max_{\bar{D}}|a(x)| < \lambda_2 \tag{1.9}$$

instead. Decompose

$$u(x) = \xi_1\varphi_1(x) + U(x), \tag{1.10}$$

with $\int_D U(x)\varphi_1(x)\,dx = 0$, i.e., $u(x)$ is the sum of the first harmonic of $u(x)$, and the projection of $u(x)$ on $\varphi_1^{\perp}$, the orthogonal complement of φ_1 in $L^2(D)$. Now the iterates given by (1.3) diverge, in general, but we shall show that both ξ_1 and the U part can be obtained by using two converging iteration processes. (Unless $a(x)$ is a constant, the harmonics do not decouple, making the problem nontrivial.) Then we extend the method for any $a(x)$ bounded on $\bar{D}$.

2 The case $\max_{\bar{D}}|a(x)| < \lambda_2$

Let P denote the projection operator on $\varphi_1^{\perp}$ in $L^2(D)$ ($Pv = v - \left(\int_D v\varphi_1\,dx\right)\varphi_1$). Then one can write $U(x) = Pu(x)$ in the decomposition (1.10). Similarly, decompose $f(x) = \mu_1\varphi_1 + e(x)$, with $e(x) = Pf(x)$. Applying the operator P to the equation (1.2) gives

$$\Delta U + P\left[a(x)\,(\xi_1\varphi_1(x) + U(x))\right] = e(x) \quad \text{in } D, \qquad U = 0 \quad \text{on } \partial D. \tag{2.1}$$

Projection of (1.2) onto φ_1 gives

$$\int_D (\Delta u + a(x)u)\,\varphi_1\,dx = \int_D f(x)\varphi_1\,dx = \mu_1. \tag{2.2}$$

Clearly, $u(x) = \xi_1\varphi_1(x) + U(x)$ is a solution of (1.2) if and only if (2.1) and (2.2) hold. The decomposition (2.1), (2.2) is similar to the Lyapunov–Schmidt reduction, see e.g., L. Nirenberg [5].

We now modify the problem (2.1): find $V(x) \in \varphi_1^\perp \cap H^2(D)$ solving

$$\Delta V + P\left[a(x)V(x)\right] = e(x) \quad \text{in } D, \qquad V = 0 \quad \text{on } \partial D\,. \tag{2.3}$$

Proposition 2.1. *Assume that the condition* (1.9) *holds. Then the problem* (2.3) *can be solved by the converging iterations* $V_n(x) \in \varphi_1^\perp \cap H^2(D)$

$$-\Delta V_{n+1} = P\left[a(x)V_n(x)\right] - e(x) \quad \text{in } D, \qquad V_{n+1} = 0 \quad \text{on } \partial D\,, \tag{2.4}$$

beginning with $V_0 = 0$.

Proof. The iterates belong to $\varphi_1^\perp$, since the right hand sides of (2.4) do. Subtracting the equations for two consecutive iterates, and then using (1.5) and $\|Pv\|_{L^2} \le \|v\|_{L^2}$, we obtain from (2.4):

$$V_{n+1} - V_n = (-\Delta)^{-1} P\left[a(x)\left(V_n - V_{n-1}\right)\right],$$
$$\|V_{n+1} - V_n\|_{L^2} \le \frac{1}{\lambda_2}\|a(x)\left(V_n - V_{n-1}\right)\|_{L^2} \le \theta\|V_n - V_{n-1}\|_{L^2}\,,$$

where $\theta \equiv \frac{\max_D |a(x)|}{\lambda_2} < 1$ by (1.9), and the proof follows. □

The difference $W(x) = U(x) - V(x)$ satisfies

$$\Delta W + P\left[a(x)W(x)\right] = -\xi_1 P\left[a(x)\varphi_1\right] \quad \text{in } D, \qquad W = 0 \quad \text{on } \partial D\,.$$

It follows that $W = \xi_1\bar{W}$, where $\bar{W}$ is the unique solution of

$$\Delta W + P\left[a(x)W(x)\right] = -P\left[a(x)\varphi_1\right] \quad \text{in } D, \qquad W = 0 \quad \text{on } \partial D\,, \tag{2.5}$$

which in view of Proposition 2.1 is the limit of the iterations

$$-\Delta W_{n+1} = P\left[a(x)W_n(x)\right] + P\left[a(x)\varphi_1\right] \quad \text{in } D, \qquad W_{n+1} = 0 \quad \text{on } \partial D\,, \tag{2.6}$$

starting with $W_0 = 0$.

We conclude that $U = V + \xi_1\bar{W}$, so that $u = \xi_1\varphi_1 + U = \xi_1\varphi_1 + V + \xi_1\bar{W}$, and it remains to determine the value of ξ_1. Substitute this $u(x)$ into (1.2)

$$-\lambda_1\xi_1\varphi_1 + \Delta V + \xi_1\Delta\bar{W} + a(x)\left(\xi_1\varphi_1 + V + \xi_1\bar{W}\right) = f(x)\,.$$

Multiplication by φ_1 and integration over D gives a linear equation for ξ_1, with the solution (observe that both ΔV and $\Delta\bar{W}$ are in $\varphi_1^\perp$)

$$\bar{\xi}_1 = \frac{\int_D f\varphi_1\,dx - \int_D a(x)V\varphi_1\,dx}{-\lambda_1 + \int_D a(x)\varphi_1^2\,dx + \int_D a(x)\bar{W}\varphi_1\,dx}\,. \tag{2.7}$$

Then the solution of (1.2) is

$$u(x) = \bar{\xi}_1\varphi_1 + V + \bar{\xi}_1\bar{W}\,. \tag{2.8}$$

Remark 2.2. In case $\max_D a(x) > \lambda_1$, it is possible to have resonance, when the problem

$$\Delta u + a(x)u = 0 \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D,$$

has a nontrivial solution. In such a case the denominator in (2.7) is zero, and the problem (1.2) is not solvable for general $f(x)$.

Example 2.3. As a feasibility check we solved the problem

$$u''(x) + \left(2 + \frac{1}{3}x\right) u(x) = x^2 \quad \text{for } 0 < x < \pi, \qquad u(0) = u(\pi) = 0. \tag{2.9}$$

Here $\lambda_1 = 1$, $\lambda_2 = 4$, so that $a(x) = 2 + \frac{1}{3}x$ satisfies $\lambda_1 < a(x) < \lambda_2$ on $(0, \pi)$. Calculate $\varphi_1(x) = \sqrt{\frac{2}{\pi}} \sin x$, $e(x) = f(x) - \left(\int_0^\pi f(x)\varphi_1(x)\, dx\right) \varphi_1(x)$, with $f(x) = x^2$. We achieved good accuracy performing twelve iterations for both (2.4) and (2.6). The graph of the solution of (2.9) was identical to the one produced by *Mathematica's* NDSolve command.

3 The general $a(x)$

We now prove directly that the formulas (2.3), (2.5), (2.7), (2.8) give the solution of (1.2), and then generalize for any bounded $a(x)$.

Theorem 3.1. *Assume that the condition* (1.9) *holds. Then the formulas* (2.3), (2.5), (2.7), (2.8) *give the solution of* (1.2).

Proof. We will show that $u(x) = \bar{\xi}_1\varphi_1(x) + U(x)$, with $U(x) = V + \bar{\xi}_1\bar{W}$ satisfies (2.1) and (2.2) (where V and $\bar{W}$ are the unique solutions (2.3) and (2.5) respectively, and $\bar{\xi}$ is determined by (2.7)). Indeed,

$$\begin{aligned} \Delta U + P\left[a(x)\left(\bar{\xi}_1\varphi_1(x) + U(x)\right)\right] &= \Delta V + \bar{\xi}_1\Delta\bar{W} + P\left[a(x)\left(\bar{\xi}_1\varphi_1(x) + V + \bar{\xi}_1\bar{W}\right)\right] \\ &= \Delta V + P\left[a(x)V\right] + \bar{\xi}_1\left\{\Delta\bar{W} + P\left[a(x)\bar{W}\right] + P\left[a(x)\varphi_1\right]\right\} \\ &= e(x), \end{aligned}$$

verifying (2.1). Using (2.7) we obtain

$$\begin{aligned} &\int_D (\Delta u + a(x)u)\,\varphi_1\, dx = -\lambda_1\bar{\xi}_1 + \int_D a(x)\left[\bar{\xi}_1\varphi_1 + V + \bar{\xi}_1\bar{W}\right]\varphi_1\, dx \\ &\quad = \bar{\xi}_1\left[-\lambda_1 + \int_D a(x)\varphi_1^2\, dx + \int_D \bar{a}(x)W\varphi_1\, dx\right] + \int_D a(x)V\varphi_1\, dx = \int_D f\varphi_1\, dx, \end{aligned}$$

justifying (2.2). ∎

Turning to any $a(x) \in C(\bar{D})$, we can find the first index j so that

$$\max_{\bar{D}} |a(x)| < \lambda_{j+1}. \tag{3.1}$$

Decompose

$$u(x) = \sum_{i=1}^{j} \xi_i\varphi_i(x) + U(x), \tag{3.2}$$

with $\int_D U(x)\varphi_i(x)\,dx = 0$ for all $i = 1, \dots, j$. Let P denote the projection operator on the orthogonal complement of the first j eigenfunctions, i.e., the projection on Span $\{\varphi_1, \varphi_2, \dots, \varphi_j\}^\perp$ in $L^2(D)$. Decompose $f(x) = \sum_{i=1}^j \mu_i \varphi_i + e(x)$, with $e(x) = Pf(x)$. Applying P to the equation (1.2) gives

$$\Delta U + P\left[a(x)\left(\sum_{i=1}^j \xi_i \varphi_i(x) + U(x)\right)\right] = e(x) \quad \text{in } D, \qquad U = 0 \quad \text{on } \partial D\,. \tag{3.3}$$

We now modify the problem (3.3): find $V(x) \in \text{Span}\,\{\varphi_1, \varphi_2, \dots, \varphi_j\}^\perp$ solving

$$\Delta V + P\left[a(x)V(x)\right] = e(x) \quad \text{in } D, \qquad V = 0 \quad \text{on } \partial D\,. \tag{3.4}$$

The following proposition is proved the same way as Proposition 2.1.

Proposition 3.2. *Under the condition (3.1) the problem (3.4) can be solved by the converging iterations* $V_n(x) \in \text{Span}\,\{\varphi_1, \varphi_2, \dots, \varphi_j\}^\perp$

$$-\Delta V_{n+1} = P\left[a(x)V_n(x)\right] - e(x) \quad \textit{in } D, \qquad V_{n+1} = 0 \quad \textit{on } \partial D\,, \tag{3.5}$$

beginning with $V_0 = 0$.

The difference $W(x) = U(x) - V(x) \in \text{Span}\,\{\varphi_1, \varphi_2, \dots, \varphi_j\}^\perp$ satisfies

$$\Delta W + P\left[a(x)W(x)\right] = -\sum_{i=1}^j \xi_i P\left[a(x)\varphi_i\right] \quad \text{in } D, \quad W = 0 \quad \text{on } \partial D\,.$$

By linearity $W = \sum_{i=1}^j \xi_i \bar{W}_i$, where $\bar{W}_i$ is the unique solution of

$$\Delta W + P\left[a(x)W(x)\right] = -P\left[a(x)\varphi_i\right] \quad \text{in } D, \qquad W = 0 \quad \text{on } \partial D\,, \tag{3.6}$$

which in view of Proposition 3.2 is the limit of the iterations

$$-\Delta W_{n+1} = P\left[a(x)W_n(x)\right] + P\left[a(x)\varphi_i\right] \quad \text{in } D, \qquad W_{n+1} = 0 \quad \text{on } \partial D\,, \tag{3.7}$$

starting with $W_0 = 0$. It follows that

$$u = \sum_{i=1}^j \xi_i \varphi_i + U = \sum_{i=1}^j \xi_i \varphi_i + V + \sum_{i=1}^j \xi_i \bar{W}_i\,, \tag{3.8}$$

and it remains to determine the values of ξ_i. Substitute this $u(x)$ into (1.2)

$$-\sum_{i=1}^j \lambda_i \xi_i \varphi_i + \Delta V + \sum_{i=1}^j \xi_i \Delta \bar{W}_i + a(x)\left(\sum_{i=1}^j \xi_i \varphi_i + V + \sum_{i=1}^j \xi_i \bar{W}_i\right) = f(x)\,.$$

Multiplication by φ_k and integration over D gives a $j \times j$ system of linear equations for ξ_i's ($k = 1, 2, \dots, j$)

$$-\lambda_k \xi_k + \sum_{i=1}^j \xi_i \left[\int_D a(x)\left(\varphi_i + \bar{W}_i\right)\varphi_k\,dx\right] = \int_D \left(f(x) - a(x)V\right)\varphi_k\,dx\,. \tag{3.9}$$

This system has a unique solution, provided that (1.2) is solvable. Using the solution of (3.9) in (3.8) provides the solution of (1.2).

So that in case the condition (3.1) holds, the algorithm for solving (1.2) is as follows.

1. Solve the problem (3.4) by using the iterates (3.5).
2. Solve j problems (3.6) by using the iterates (3.7) for each problem.
3. Solve the $j \times j$ linear algebraic system (3.9) to find $\bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\xi}_j$.
4. The solution is $u(x) = \sum_{i=1}^{j} \bar{\xi}_i \varphi_i + V + \sum_{i=1}^{j} \bar{\xi}_i \bar{W}_i$.

4 Semilinear Poisson equation in higher dimensions

Elliptic PDE's, like the problem (1.2), are rarely solved numerically in dimensions $m > 2$. Using finite differences in the dimension $m = 4$ with 20 subdivision points along each axis (which is not many), requires solving a system of $20^4 = 160000$ linear equations. Richard Bellman coined a phrase "the curse of dimensionality" to describe the computational challenges in higher dimensions. Since then there has been a tremendous advance in computer power and software (e.g., parallel computations). In particular, a system of $20^4 = 160000$ linear equations nowadays is not considered to be very large. However the accuracy will be low with only 20 subdivision points along each axis, so that challenges remain. Another problem in higher dimensions is representation of solutions. Once a solution in dimension $m = 4$ is computed, should the result be presented as a graph in 5 dimensions, or as a 4-dimensional table? The iterative method developed above addresses both issues. Represent $f(x) = \sum_{k=1}^{\infty} f_k \varphi_k(x)$, with the coefficients $f_k = \int_D f(y)\varphi_k(y)\, dy$. The solution of (1.1) is

$$u(x) = \sum_{k=1}^{\infty} \frac{f_k}{\lambda_k} \varphi_k(x) , \quad \text{with } f_k = \int_D f(y)\varphi_k(y)\, dy . \tag{4.1}$$

Replacing f_k's in the sum by their expressions as integrals, one can express the solution of (1.1) as

$$u(x) = \int_D G(x,y) f(y)\, dy , \tag{4.2}$$

with Green's function

$$G(x,y) = \sum_{k=1}^{\infty} \frac{\varphi_k(x)\varphi_k(y)}{\lambda_k} . \tag{4.3}$$

However, it is easier to use the form (4.1) rather than (4.2) because *Mathematica* cannot handle numerical integration in y variables, when x variables are present (even with the delayed assignment). This is perfectly understandable, because addition of thousands of functions (obtained by interpolation) is an enormous task. One can introduce a mesh, and compute (4.2) in parallel at each point, using as many processors as there are points on the mesh, but this "industrial strength" computational effort is beyond our scope. However, the usefulness of our method probably lies in this direction.

We did try the eigenfunction expansion in both two and three dimensions, using the first 50 eigenfunctions. Conclusion: the method is slow. The method requires either the knowledge or calculation of the eigenvalues and the eigenfuctions of the Laplacian. On a rectangle $R = [0, a] \times [0, b] \times [0, c]$ in three dimensions, the eigenfunctions (vanishing on ∂R) are

$$\varphi = c_0 \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \sin \frac{p\pi}{c} z , \qquad c_0 = \sqrt{\frac{8}{abc}} ,$$

with $\int_R \varphi^2\, dxdydz = 1$. The corresponding eigenvalues are

$$\lambda = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} + \frac{p^2}{c^2} \right).$$

The order and multiplicity of eigenvalues depends on a particular choice of a, b, c. Let us take $a = b = 1$, $c = \sqrt{2}$. Then $\lambda = \pi^2 \left(m^2 + n^2 + \frac{p^2}{2}\right)$. The order of eigenvalues is determined by $(m, n, p) \equiv m^2 + n^2 + \frac{p^2}{2}$.

a. $(1,1,1)$ gives $\lambda_1 = \frac{5}{2}\pi^2$, $\varphi_1 = c_0 \sin \pi x \sin \pi y \sin \frac{\pi}{\sqrt{2}} z$.

b. $(1,1,2)$ gives $\lambda_2 = 4\pi^2$, $\varphi_2 = c_0 \sin \pi x \sin \pi y \sin \frac{2\pi}{\sqrt{2}} z$.

c. $(2,1,1)$ and $(1,2,1)$ give a repeated eigenvalue $\lambda_3 = \lambda_4 = \frac{11}{2}\pi^2$, with the eigenfunctions $\varphi_3 = c_0 \sin 2\pi x \sin \pi y \sin \frac{\pi}{\sqrt{2}} z$ and $\varphi_4 = \sin \pi x \sin 2\pi y \sin \frac{\pi}{\sqrt{2}} z$, and so on.

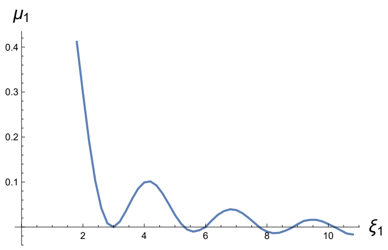

Figure 4.1: The solution curve $\mu_1 = \mu_1(\xi_1)$ of the problem (4.4), oscillating around the ξ_1-axis.

We wrote a code, allowing us to calculate a large number of eigenfunctions automatically. We solved a number of examples for the problem (1.2), obtaining the expected results, but the computations were slow.

Example 4.1. We performed curve-following for the following semilinear problem on a parallelepiped $\Omega = (0,1) \times (0,1) \times (0, \sqrt{2})$ in three dimensions

$$\Delta u + \lambda_1 u + \sin u = \mu_1 \varphi_1(x,y,z) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{4.4}$$

see Figure 4.1. Here $\lambda_1 = \frac{5}{2}\pi^2$ is the principal eigenvalue of the Laplacian on Ω with zero boundary condition, so that the problem is at resonance. Decompose the solution as $u(x) = \xi_1 \varphi_1(r) + U(x,y,z)$, with $U(x,y,z) \in \varphi_1^\perp$ in $L^2(\Omega)$, where $\varphi_1 = \sqrt{\frac{8}{\sqrt{2}}} \sin \pi x \sin \pi y \sin \frac{\pi}{\sqrt{2}} z$. The following facts follow from the results proved in [3] and [4]. The solution set of (4.4) is exhausted by a single continuous curve $(u(x,y,z), \mu_1)(\xi_1)$. Moreover $\mu_1(\xi_1) \to 0$ as $\xi_1 \to \infty$, while $\mu_1(\xi_1)$ changes sign infinitely many times. In particular, the problem (4.4) has infinitely many solutions at $\mu_1 = 0$. Performing the curve following required solving linear problems of the type (1.2) repeatedly. We used eigenfunction expansions, and it took long time to compute the solution curve in Figure 4.1.

Mathematica's NDSolve command can also handle the problem (1.2) in two and three dimensions. It appears that the accuracy is excellent in two dimensions, but not in dimension three.

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On the cyclicity of Kolmogorov polycycles

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Abstract. In this paper we study planar polynomial Kolmogorov's differential systems

$$X_\mu \quad \begin{cases} \dot{x} = xf(x,y;\mu), \\ \dot{y} = yg(x,y;\mu), \end{cases}$$

with the parameter μ varying in an open subset $\Lambda \subset \mathbb{R}^N$. Compactifying X_μ to the Poincaré disc, the boundary of the first quadrant is an invariant triangle Γ, that we assume to be a hyperbolic polycycle with exactly three saddle points at its vertices for all $\mu \in \Lambda$. We are interested in the cyclicity of Γ inside the family $\{X_\mu\}_{\mu\in\Lambda}$, i.e., the number of limit cycles that bifurcate from Γ as we perturb μ. In our main result we define three functions that play the same role for the cyclicity of the polycycle as the first three Lyapunov quantities for the cyclicity of a focus. As an application we study two cubic Kolmogorov families, with $N = 3$ and $N = 5$, and in both cases we are able to determine the cyclicity of the polycycle for all $\mu \in \Lambda$, including those parameters for which the return map along Γ is the identity.

Keywords: limit cycle, polycycle, cyclicity, asymptotic expansion.

2020 Mathematics Subject Classification: 34C07, 34C20, 34C23.

1 Introduction and main results

The present paper is motivated by the results obtained by Gasull, Mañosa and Mañosas [8] with regard to the *stability* of an unbounded polycycle Γ in the Kolmogorov's polynomial differential systems

$$\begin{cases} \dot{x} = xf(x,y), \\ \dot{y} = yg(x,y). \end{cases}$$

These systems are widely used in ecology to describe the interaction between two populations, see [18] for instance. That being said, the stability of the polycycle is not the main issue to

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which this paper is addressed. Indeed, assuming that the coefficients of the polynomials f and g depend analytically on a parameter μ, we are interested in the *cyclicity* of the polycycle (see Definition 1.2 below), which roughly speaking is the number of limit cycles that can bifurcate from Γ as we perturb μ. In our main result (Theorem A) we define three functions, $d_1(\mu)$, $d_2(\mu)$ and $d_3(\mu)$, that play the same role for the cyclicity of the polycycle as the first three *Lyapunov quantities* for the cyclicity of a focus. Recall that the displacement map can be analytically extended to a focus and that the Lyapunov quantities are the coefficients of its Taylor's series. On the contrary the displacement map has no smooth extension to a polycycle. At best one can hope that it has some asymptotic expansion. This is indeed the case for the polycycle that we study in the present paper and in order to obtain it we strongly rely in our previous results [14–16] about the asymptotic expansion of the Dulac map of an unfolding of hyperbolic saddles. The principal part of the asymptotic expansion of the displacement map is given in a monomial scale containing a deformation of the logarithm, the so-called Ecalle–Roussarie compensator, and the remainder is uniformly flat with respect to the parameters. The functions $d_i(\mu)$ in Theorem A are essentially the coefficients of the first three monomials in the principal part, which explains their relation with the cyclicity. For other results regarding the cyclicity of polycycles and more general limit periodic sets the reader is referred to [5,6,11,22] and references therein.

Most of the work on planar polynomial differential systems, including this paper, is related to the questions surrounding Hilbert's 16th problem (see for instance [10,21,23] and references therein) and its various weakened versions. In this setting it is worth to mention that, using a compactness argument, Roussarie [20] showed that to prove the existential part of Hilbert's 16th problem in the family $\mathcal{P}_n$ of all polynomial vector fields of degree $\leqslant n$ it is sufficient to show that each *limit periodic set* in $\mathcal{P}_n$ has finite cyclicity.

Definition 1.1. Let X be a vector field on $\mathbb{R}^2$ (or $\mathbb{S}^2$). A *graphic* Γ for X is a compact, non-empty invariant subset which is a continuous image of $\mathbb{S}^1$ and consists of a finite number of isolated singular points $\{p_1,\ldots,p_m,p_{m+1}=p_1\}$ (not necessarily distinct) and compatibly oriented separatrices $\{s_1,\ldots,s_m\}$ connecting them (i.e., such that the α-limit set of s_j is p_j and the ω-limit set of s_j is p_{j+1}). A graphic is said to be *hyperbolic* if all its singular points are hyperbolic saddles. A *polycyle* is a graphic with a return map defined on one of its sides.

The polycycle that we aim to study is unbounded. In order to investigate the behaviour of the trajectories of a polynomial vector field Y near infinity we can consider its Poincaré compactification $p(Y)$, see [2, §5] for details, which is an analytically equivalent vector field defined on the sphere $\mathbb{S}^2$. The points at infinity of $\mathbb{R}^2$ are in bijective correspondence with the points of the equator of $\mathbb{S}^2$, that we denote by ℓ_∞. Furthermore, the trajectories of $p(Y)$ in $\mathbb{S}^2$ are symmetric with respect to the origin and so it suffices to draw its flow in the closed northern hemisphere only, the so called Poincaré disc.

Definition 1.2. Let $\{X_\mu\}_{\mu\in\Lambda}$ be a family of vector fields on $\mathbb{S}^2$ and suppose that Γ is a polycyle for X_{μ_0}. We say that Γ has finite cyclicity in the family $\{X_\mu\}_{\mu\in\Lambda}$ if there exist $\kappa\in\mathbb{N}$, $\varepsilon>0$ and $\delta>0$ such that any X_μ with $\|\mu-\mu_0\|<\delta$ has at most κ limit cycles γ_i with $\text{dist}_H(\Gamma,\gamma_i)<\varepsilon$. The minimum of such κ when ε and δ go to zero is called the *cyclicity* of Γ in $\{X_\mu\}_{\mu\in\Lambda}$ and denoted by $\text{Cycl}\big((\Gamma,X_{\mu_0}),X_\mu\big)$.

In this paper we consider the family of vector fields $\{X_\mu\}_{\mu\in\Lambda}$ given by

$$X_\mu := f(x,y;\mu)x\partial_x + g(x,y;\mu)y\partial_y \tag{1.1}$$

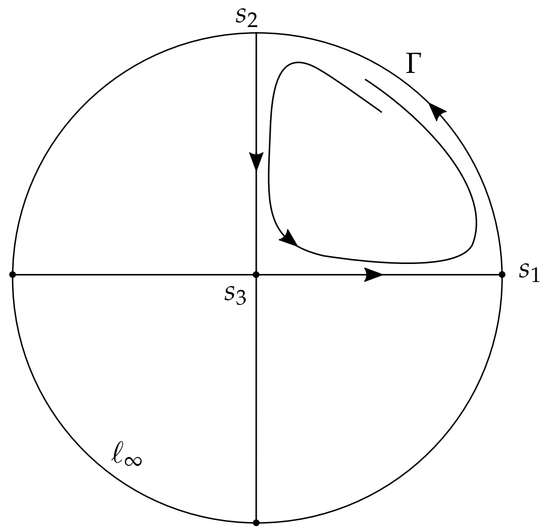

Figure 1.1: Placement of the hyperbolic saddles and the polycycle Γ in the Poincaré disc.

where Λ is an open subset of $\mathbb{R}^N$ and f and g are polynomials in x and y of degree $n \in \mathbb{N}$ with the coefficients depending analytically on μ. The standing hypothesis on the family $\{X_\mu\}_{\mu\in\Lambda}$ are the following:

H1 $f(z,0;\mu) > 0$, $g(0,z;\mu) < 0$ and $\big(f_n - g_n\big)(1,z;\mu) < 0$ for all $z > 0$ and $\mu \in \Lambda$.

H2 $\lambda_1(\mu) := \big(\frac{f_n}{g_n - f_n}\big)(1,0;\mu)$, $\lambda_2(\mu) := \big(\frac{f_n - g_n}{g_n}\big)(0,1;\mu)$ and $\lambda_3(\mu) := -\big(\frac{g}{f}\big)(0,0;\mu)$, are well defined and strictly positive for all $\mu \in \Lambda$.

Here, and in what follows, f_n and g_n denote, respectively, the homogeneous part of degree n of f and g. Conditions **H1** and **H2** guarantee that, after compactifying the polynomial vector field X_μ to the Poincaré disc, the boundary of the first quadrant is a polycyle with three hyperbolic saddles, see Figure 1.1,

$$s_1 := \{y=0\} \cap \ell_\infty, \qquad s_2 := \{x=0\} \cap \ell_\infty \quad \text{and} \quad s_3 := (0,0).$$

From now on we shall denote this polycycle by Γ, that we remark is a compact subset of the Poincaré disc. The hyperbolicity ratios of the saddles at its vertices are precisely the ones given in **H2**. We also define:

$$L_{11}(u) = \exp\left(\int_0^u \left(\left(\tfrac{f-g}{f}\right)(1/z,0) + \tfrac{1}{\lambda_1}\right)\tfrac{dz}{z}\right), \qquad L_{12}(u) = \exp\left(\int_0^u \left(\left(\tfrac{f_n}{f_n-g_n}\right)(1,z) + \lambda_1\right)\tfrac{dz}{z}\right),$$
$$L_{21}(u) = \exp\left(\int_0^u \left(\left(\tfrac{g_n}{g_n-f_n}\right)(z,1) + \tfrac{1}{\lambda_2}\right)\tfrac{dz}{z}\right), \qquad L_{22}(u) = \exp\left(\int_0^u \left(\left(\tfrac{g-f}{g}\right)(0,1/z) + \lambda_2\right)\tfrac{dz}{z}\right),$$
$$L_{31}(u) = \exp\left(\int_0^u \left(\left(\tfrac{g}{f}\right)(z,0) + \lambda_3\right)\tfrac{dz}{z}\right), \qquad L_{32}(u) = \exp\left(\int_0^u \left(\left(\tfrac{f}{g}\right)(0,z) + \frac{1}{\lambda_3}\right)\tfrac{dz}{z}\right)$$

together with

$$M_1(u) = -\frac{L_{11}(u)}{u}\,\partial_2\left(\frac{g}{f}\right)(1/u,0) \quad \text{and} \quad M_3(u) = L_{31}(u)\,\partial_2\left(\frac{g}{f}\right)(u,0). \tag{1.2}$$

We point out that all these functions depend on the parameter μ. This dependence is omitted for the sake of shortness when there is no risk of confusion.

We can now state our main result, which is addressed to the cyclicity of the polycycle Γ inside the polynomial family $\{X_\mu\}_{\mu\in\Lambda}$. More formally, we should refer to the compactified family $\{p(X_\mu)\}_{\mu\in\Lambda}$ of vector fields on $\mathbb{S}^2$ but for the simplicity in the exposition we commit an abuse of language by identifying both families. It is clear that the number of limit cycles of $p(X_\mu)$ and X_μ is the same. In the statement $\mathscr{R}(\,\cdot\,;\mu)$ stands for the return map of the vector field X_μ around the polycycle Γ (see Figure 1.1) and we use the notion of functional independence that is given in Definition 2.8.

Theorem A. *Consider the family of Kolmogorov polynomial vector fields $\{X_\mu\}_{\mu\in\Lambda}$ given in (1.1) and verifying the assumptions* **H1** *and* **H2**. *Then, for any $\mu_0\in\Lambda$, the following assertions hold with regard to the cyclicity of the polycycle Γ inside the family:*

(a) $\operatorname{Cycl}\big((\Gamma,X_{\mu_0}),X_\mu\big)=0$ *if $d_1(\mu):=1-\lambda_1\lambda_2\lambda_3$ does not vanish at μ_0.*

(b) $\operatorname{Cycl}\big((\Gamma,X_{\mu_0}),X_\mu\big)\geqslant 1$ *if d_1 vanishes and is independent at μ_0 and $\mathscr{R}(\,\cdot\,;\mu_0)\not\equiv Id$.*

(c) $\operatorname{Cycl}\big((\Gamma,X_{\mu_0}),X_\mu\big)\leqslant 1$ *if $d_2(\mu):=\log\left(\left(\frac{L_{12}}{L_{21}}\right)^{\lambda_2}\left(\frac{L_{31}}{L_{11}}\right)^{\lambda_1\lambda_2}\frac{L_{22}}{L_{32}}\right)(1)$ does not vanish at μ_0.*

(d) $\operatorname{Cycl}\big((\Gamma,X_{\mu_0}),X_\mu\big)\geqslant 2$ *if d_1 and d_2 vanish and are independent at μ_0 and $\mathscr{R}(\,\cdot\,;\mu_0)\not\equiv Id$.*

In case that $\lambda_1(\mu_0)<1$, $\lambda_2(\mu_0)>1$ and $\lambda_3(\mu_0)>1$ then the following assertions hold as well:

(e) $\operatorname{Cycl}\big((\Gamma,X_{\mu_0}),X_\mu\big)\leqslant 2$ *if $d_3(\mu):=\hat{M}_3(\lambda_3,1)L_{11}(1)-\hat{M}_1(\frac{1}{\lambda_1},1)L_{31}(1)$ does not vanish at μ_0.*

(f) $\operatorname{Cycl}\big((\Gamma,X_{\mu_0}),X_\mu\big)\geqslant 3$ *if d_1, d_2 and d_3 vanish and are independent at μ_0 and $\mathscr{R}(\,\cdot\,;\mu_0)\not\equiv Id$.*

Let us make some remarks with regard to the regularity of the functions d_1, d_2 and d_3 defined in the statement. On account of the hypothesis **H1** and **H2** it is evident that d_1 is analytic on the whole parameter space Λ. On the other hand, d_2 is defined in terms of the functions $\mu\mapsto L_{ij}(1)$, which in turn are given by some (apparently) improper integrals. By applying the Weierstrass Division Theorem one can easily show that each $L_{ij}(1)$ is an analytic strictly positive function, so that d_2 is also analytic on Λ. Finally, d_3 is given by means of a sort of incomplete Mellin transform (which is defined in Proposition 2.5) of the functions M_1 and M_3 in (1.2). One can show that the hypothesis **H1** and **H2** imply that each $M_i(u;\mu)$ is analytic on $(-\varepsilon,+\infty)\times\Lambda$ for some $\varepsilon>0$. Taking this into account, by applying (d) in Proposition 2.5 it follows that d_3 is a meromorphic function on Λ having poles only at those μ_0 such that $1/\lambda_1(\mu_0)\in\mathbb{N}$ or $\lambda_3(\mu_0)\in\mathbb{N}$.

Also with regard to the statement of Theorem A, the assertions (e) and (f) hold under the assumptions $\lambda_1(\mu_0)<1$, $\lambda_2(\mu_0)>1$ and $\lambda_3(\mu_0)>1$. However, one can always reduce to this case provided that $\lambda_i(\mu_0)\neq 1$ for $i=1,2,3$ by means of a rescaling of time and a projective change of coordinates that permute conveniently the three singular points of the polycycle.

The paper [8] constitutes an important previous contribution to the study of Kolmogorov polycycles that should be referred. Indeed, following our notations and definitions, the authors prove (see [8, Theorem 1]) that if $d_1(\mu_0)=0$ then the return map of X_{μ_0} around the polycycle Γ is of the form

$$\mathscr{R}(s;\mu_0)=\Delta s+\mathrm{o}(s), \tag{1.3}$$

cf. (b) in Theorem 2.6, and they also provide the explicit expression of the coefficient Δ. This coefficient is given as the limit of a sum of three improper integrals, which computed separately diverge. An easy manipulation of the integrals shows that these divergences cancel each other, yielding to the expression of d_2 given in Theorem A. It is important to remark that the expansion in (1.3) can not be used to obtain an upper bound for $\mathrm{Cycl}\big((\Gamma, X_{\mu_0}), X_\mu\big)$ because the remainder is not uniform with respect to the parameters. It is possible, however, to use it to obtain lower bounds. In this direction the authors prove in [8, Corollary 5] that if d_1 vanishes and is independent at μ_0 and $d_2(\mu_0) \neq 0$ then $\mathrm{Cycl}\big((\Gamma, X_{\mu_0}), X_\mu\big) \geqslant 1$. Since $d_2(\mu_0) \neq 0$ implies $\mathscr{R}(\,\cdot\,;\mu_0) \not\equiv Id$ by Theorem 2.6, this lower bound follows by applying (b) in Theorem A.

The paper is organised in the following way. Section 2 is entirely devoted to prove Theorem A and for that purpose we rely in our previous results about the asymptotic expansion of the Dulac map of a hyperbolic saddle that we obtain in [14–16]. For this reason, before starting the proof of Theorem A we first state these results and introduce the necessary definitions. The asymptotic expansion of the displacement map near the polycycle is given in Theorem 2.6 and constitutes the fundamental tool in order to prove Theorem A. As a by-product of this expansion we obtain a method to study the stability of the polycycle, see Remark 2.7. Section 3 is addressed to the applications. The first one is Theorem 3.1, where we consider a Kolmogorov's cubic system depending on three parameters that was previously studied in [8]. The authors in that paper show that there exist parameters for which the cyclicity of the polycycle is at least 1. In the present paper we obtain the exact cyclicity of all the parameters in the family (that can be 0 or 1), including the case in which the return map along the polycycle is the identity. We also show that there exists exactly one singularity in the first quadrant, which can be a focus or a center, and we compute its cyclicity. Finally we prove that it is not possible a simultaneous bifurcation of limit cycles from the polycycle and that singularity. We give our second application in Theorem 3.2, where we consider a Kolmogorov's cubic system depending on five parameters. In this case we also provide the exact cyclicity of Γ for all the parameters in the family, which again can be 0 or 1.

2 Proof of Theorem A

In order to tackle the proof of Theorem A we will appeal to some previous results from [14–16] about the asymptotic expansion of the Dulac map. For reader's convenience we gather these results in Proposition 2.4. To this end it is first necessary to introduce some new notation and definitions.

Setting $\hat{\nu} := (\lambda, \nu) \in \hat{W} := (0, +\infty) \times W$ with W an open set of $\mathbb{R}^N$, we consider the family of vector fields $\{X_{\hat{\nu}}\}_{\hat{\nu} \in \hat{W}}$ with

$$X_{\hat{\nu}}(x_1, x_2) = x_1 P_1(x_1, x_2; \hat{\nu})\partial_{x_1} + x_2 P_2(x_1, x_2; \hat{\nu})\partial_{x_2} \tag{2.1}$$

where

- P_1 and P_2 belong to $\mathscr{C}^\omega(\mathscr{U} \times \hat{W})$ for some open set $\mathscr{U}$ of $\mathbb{R}^2$ containing the origin,
- $P_1(x_1, 0; \hat{\nu}) > 0$ and $P_2(0, x_2; \hat{\nu}) < 0$ for all $(x_1, 0), (0, x_2) \in \mathscr{U}$ and $\hat{\nu} \in \hat{W}$,
- $\lambda = -\frac{P_2(0,0;\nu)}{P_1(0,0;\nu)}$.

Thus, for all $\hat{\nu} \in \hat{W}$, the origin is a hyperbolic saddle of $X_{\hat{\nu}}$ with the separatrices lying in the axis. We point out that here the hyperbolicity ratio of the saddle is an independent parameter, although in the proof of Theorem A we will have $\lambda = \lambda(\nu)$. The reason for this is that the

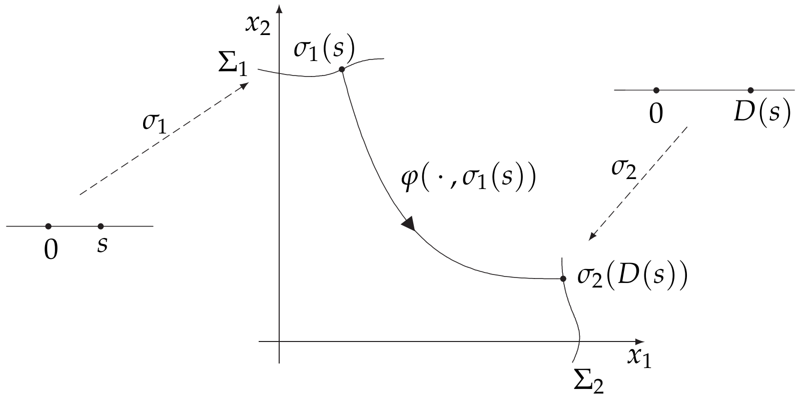

Figure 2.1: Definition of the Dulac map $D(\,\cdot\,;\hat{v})$, where $\varphi(t,p;\hat{v})$ is the solution of $X_{\hat{v}}$ passing through the point $p \in \mathscr{U}$ at time $t=0$.

hyperbolicity ratio turns out to be the ruling parameter in our results and, besides, having it uncoupled from the rest of parameters simplifies the notation in the statements. Moreover, for $i=1,2$, we consider a $\mathscr{C}^\omega$ transverse section $\sigma_i\colon(-\varepsilon,\varepsilon)\times\hat{W}\longrightarrow\Sigma_i$ to $X_{\hat{v}}$ at $x_i=0$ defined by

$$\sigma_i(s;\hat{v}) = \big(\sigma_{i1}(s;\hat{v}),\sigma_{i2}(s;\hat{v})\big)$$

such that $\sigma_1(0,\hat{v})\in\{(0,x_2);x_2>0\}$ and $\sigma_2(0,\hat{v})\in\{(x_1,0);x_1>0\}$ for all $\hat{v}\in\hat{W}$. We denote the Dulac map of $X_{\hat{v}}$ from Σ_1 to Σ_2 by $D(\,\cdot\,;\hat{v})$, see Figure 2.1. The asymptotic expansion of $D(s;\hat{v})$ at $s=0$ consists of a remainder and a principal part. The principal part is given in a monomial scale that contains a deformation of the logarithm, the so-called Ecalle-Roussarie compensator, whereas the remainder has good flatness properties with respect to the parameters. We next give precise definitions of these key notions.

Definition 2.1. The function defined for $s>0$ and $\alpha\in\mathbb{R}$ by means of

$$\omega(s;\alpha) = \begin{cases} \frac{s^{-\alpha}-1}{\alpha} & \text{if } \alpha\neq 0,\\ -\log s & \text{if } \alpha=0,\end{cases}$$

is called the *Ecalle–Roussarie compensator*.

Definition 2.2. Consider an open subset $U\subset\hat{W}\subset\mathbb{R}^{N+1}$. We say that a function $\psi(s;\hat{v})$ belongs to the class $\mathscr{C}^\infty_{s>0}(U)$ if there exist an open neighbourhood Ω of

$$\{(s,\hat{v})\in\mathbb{R}^{N+2};s=0,\hat{v}\in U\} = \{0\}\times U$$

in $\mathbb{R}^{N+2}$ such that $(s,\hat{v})\mapsto\psi(s;\hat{v})$ is $\mathscr{C}^\infty$ on $\Omega\cap\big((0,+\infty)\times U\big)$.

More formally, the definition of $\mathscr{C}^\infty_{s>0}(U)$ must be thought in terms of germs with respect to relative neighbourhoods of $\{0\}\times U$ in $(0,+\infty)\times U$. In doing so $\mathscr{C}^\infty_{s>0}(U)$ becomes a ring.

We can now introduce the notion of flatness that we shall use in the sequel.

Definition 2.3. Consider an open subset $U\subset\hat{W}\subset\mathbb{R}^{N+1}$. Given $L\in\mathbb{R}$ and $\hat{v}_0\in U$, we say that a function $\psi(s;\hat{v})\in\mathscr{C}^\infty_{s>0}(U)$ is *L-flat with respect to s at* $\hat{v}_0$, and we write $\psi\in\mathcal{F}^\infty_L(\hat{v}_0)$, if for each $\ell=(\ell_0,\ell_1,\ldots,\ell_{N+1})\in\mathbb{Z}^{N+2}_{\geq 0}$ there exist a neighbourhood V of $\hat{v}_0$ and $C,s_0>0$ such that

$$\left|\frac{\partial^{|\ell|}\psi(s;\hat{v})}{\partial s^{\ell_0}\partial\hat{v}_1^{\ell_1}\cdots\partial\hat{v}_{N+1}^{\ell_{N+1}}}\right| \leqslant Cs^{L-\ell_0} \text{ for all } s\in(0,s_0) \text{ and } \hat{v}\in V.$$

If W is a (not necessarily open) subset of U then define $\mathcal{F}^\infty_L(W) := \bigcap_{\hat{v}_0\in W}\mathcal{F}^\infty_L(\hat{v}_0)$.

Apart from the remainder and the monomial order, the most important ingredient for our purposes is the explicit expression of the coefficients in the asymptotic expansion. In order to give them we introduce next some additional notation, where for the sake of shortness the dependence on $\hat{\nu} = (\lambda, \nu)$ is omitted. We define the functions:

$$
\begin{aligned}
L_1(u) &:= \exp \int_0^u \left(\frac{P_1(0,z)}{P_2(0,z)} + \frac{1}{\lambda} \right) \frac{dz}{z} \qquad & L_2(u) &:= \exp \int_0^u \left(\frac{P_2(z,0)}{P_1(z,0)} + \lambda \right) \frac{dz}{z} \\
M_1(u) &:= L_1(u) \partial_1 \left(\frac{P_1}{P_2} \right) (0,u) & M_2(u) &:= L_2(u) \partial_2 \left(\frac{P_2}{P_1} \right) (u,0)
\end{aligned}
\tag{2.2}
$$

On the other hand, for shortness as well, we use the compact notation σ_{ijk} for the kth derivative at $s = 0$ of the jth component of $\sigma_i(s; \hat{\nu})$, i.e.,

$$
\sigma_{ijk}(\hat{\nu}) := \partial_s^k \sigma_{ij}(0; \hat{\nu}).
$$

Taking this notation into account we also introduce the following real values, where once again we omit the dependence on $\hat{\nu}$:

$$
\begin{aligned}
S_1 &:= \frac{\sigma_{112}}{2\sigma_{111}} - \frac{\sigma_{121}}{\sigma_{120}} \left(\frac{P_1}{P_2} \right) (0, \sigma_{120}) - \frac{\sigma_{111}}{L_1(\sigma_{120})} \hat{M}_1(1/\lambda, \sigma_{120}) \\
S_2 &:= \frac{\sigma_{222}}{2\sigma_{221}} - \frac{\sigma_{211}}{\sigma_{210}} \left(\frac{P_2}{P_1} \right) (\sigma_{210}, 0) - \frac{\sigma_{221}}{L_2(\sigma_{210})} \hat{M}_2(\lambda, \sigma_{210}).
\end{aligned}
\tag{2.3}
$$

Here $\hat{M}_i$ stands for a sort of incomplete Mellin transform of M_i that will be defined by Proposition 2.5 below. We can now state the following result, which gathers Theorem A and Theorem 4.1 in [16] and that it will constitute the key tool in order to prove the main result in the present paper.

Proposition 2.4. *Let $D(s; \hat{\nu})$ be the Dulac map of the hyperbolic saddle (2.1) from Σ_1 and Σ_2 and define*

$$
\Delta_0(\hat{\nu}) = \frac{\sigma_{111}^{\lambda} \sigma_{120}}{L_1^{\lambda}(\sigma_{120})} \frac{L_2(\sigma_{210})}{\sigma_{221} \sigma_{210}^{\lambda}}, \quad \Delta_1(\hat{\nu}) = \Delta_0 \lambda S_1 \quad \text{and} \quad \Delta_2(\hat{\nu}) = -\Delta_0^2 S_2,
$$

where $\hat{\nu} = (\lambda, \nu) \in \hat{W} = (0, +\infty) \times W$. Then Δ_0 is analytic and strictly positive on $\hat{W}$, Δ_1 is meromorphic on $\hat{W}$ with poles only at $\lambda \in \frac{1}{\mathbb{N}}$ and Δ_2 is meromorphic on $\hat{W}$ with poles only at $\lambda \in \mathbb{N}$. Moreover the following assertions hold:

(1) *If $\lambda_0 < 1$ then $D(s; \hat{\nu}) = s^{\lambda} \big(\Delta_0(\hat{\nu}) + \Delta_2(\hat{\nu}) s^{\lambda} + \mathcal{F}_{\ell}^{\infty}(\{\lambda_0\} \times W) \big)$ for any $\ell \in \big[\lambda_0, \min(2\lambda_0, 1)\big)$.*

(2) *If $\lambda_0 = 1$ then $D(s; \hat{\nu}) = s^{\lambda} \big(\Delta_0(\hat{\nu}) + \mathbf{\Delta}^{\lambda_0}(\omega; \hat{\nu}) s + \mathcal{F}_{\ell}^{\infty}(\{\lambda_0\} \times W) \big)$ for any $\ell \in [1, 2)$, where*

$$
\mathbf{\Delta}^{\lambda_0}(\omega; \hat{\nu}) = \Delta_1(\hat{\nu}) + \Delta_2(\hat{\nu})(1 + \alpha\omega),
$$

$\alpha = 1 - \lambda$ and $\omega = \omega(s; \alpha)$.

(3) *If $\lambda_0 > 1$ then $D(s; \hat{\nu}) = s^{\lambda} \big(\Delta_0(\hat{\nu}) + \Delta_1(\hat{\nu}) s + \mathcal{F}_{\ell}^{\infty}(\{\lambda_0\} \times W) \big)$ for any $\ell \in \big[1, \min(\lambda_0, 2)\big)$.*

In particular, $D(s; \hat{\nu}) = s^{\lambda} \big(\Delta_0(\hat{\nu}) + \mathcal{F}_{\ell}^{\infty}(\{\lambda_0\} \times W) \big)$ for any $\ell \in \big(0, \min(\lambda_0, 1)\big)$.

The flatness ℓ of the remainder can range in a certain interval depending on λ_0. The left endpoint of this interval is only given for completeness to guarantee that all the monomials in the principal part are relevant (i.e., they cannot be included in the remainder). The important information about the flatness is given by the right endpoint. A key tool in order to give a closed expression of the coefficients Δ_i is the use of a sort of incomplete Mellin transform, which is accurately defined in the next result. For a proof of this result the reader is referred to [16, Appendix B].

Proposition 2.5. *Consider an open interval I of $\mathbb{R}$ containing $x=0$ and an open subset U of $\mathbb{R}^M$.*

(a) Given $f(x;v)\in\mathscr{C}^\infty(I\times U)$, there exits a unique $\hat f(\alpha,x;v)\in\mathscr{C}^\infty((\mathbb{R}\setminus\mathbb{Z}_{\geq 0})\times I\times U)$ such that

$$x\partial_x\hat f(\alpha,x;v)-\alpha\hat f(\alpha,x;v)=f(x;v).$$

(b) If $x\in I\setminus\{0\}$ then $\partial_x(\hat f(\alpha,x;v)|x|^{-\alpha})=f(x;v)\frac{|x|^{-\alpha}}{x}$ and, taking any $k\in\mathbb{Z}_{\geq 0}$ with $k>\alpha$,

$$\hat f(\alpha,x;v)=\sum_{i=0}^{k-1}\frac{\partial_x^i f(0;v)}{i!(i-\alpha)}x^i+|x|^\alpha\int_0^x\Big(f(s;v)-T_0^{k-1}f(s;v)\Big)|s|^{-\alpha}\frac{ds}{s},$$

where $T_0^k f(x;v)=\sum_{i=0}^k\frac{1}{i!}\partial_x^i f(0;v)x^i$ is the k-th degree Taylor polynomial of $f(x;v)$ at $x=0$.

(c) For each $(i_0,x_0,v_0)\in\mathbb{Z}_{\geq 0}\times I\times W$ the function $(\alpha,x,v)\mapsto(i_0-\alpha)\hat f(\alpha,x;v)$ extends $\mathscr{C}^\infty$ at (i_0,x_0,v_0) and, moreover, it tends to $\frac{1}{i_0!}\partial_x^{i_0}f(0;v_0)x_0^{i_0}$ as $(\alpha,x,v)\to(i_0,x_0,v_0)$.

(d) If $f(x;v)$ is analytic on $I\times U$ then $\hat f(\alpha,x;v)$ is analytic on $(\mathbb{R}\setminus\mathbb{Z}_{\geq 0})\times I\times U$. Finally, for each $(\alpha_0,x_0,v_0)\in\mathbb{Z}_{\geq 0}\times I\times U$ the function $(\alpha,x,v)\mapsto(\alpha_0-\alpha)\hat f(\alpha,x;v)$ extends analytically to (α_0,x_0,v_0).

On account of this result for each $M_i(u;\hat v)$ in (2.2) we have that $(\alpha,u;\hat v)\mapsto\hat M_i(\alpha,u;\hat v)$ is a well defined meromorphic function with poles only at $\alpha\in\mathbb{Z}_{\geq 0}$. Accordingly, see (2.3), $\hat M_1(1/\lambda,\sigma_{120})$ and $\hat M_2(\lambda,\sigma_{210})$ are the values (depending on $\hat v$) that we obtain by taking $\hat M_1(\alpha,u;\hat v)$ with $\alpha=1/\lambda$ and $u=\sigma_{120}(\hat v)$ and by taking $\hat M_2(\alpha,u;\hat v)$ with $\alpha=\lambda$ and $u=\sigma_{210}(\hat v)$, respectively.

At this point we get back to the setting treated in the present paper and from now on we recover the original notation for the parameters in the family under consideration, see (1.1). In order to study the Dulac maps of the hyperbolic saddles at the vertices of the polycycle Γ we take three local transverse sections Σ_1, Σ_2, and Σ_3 parametrised, respectively, by $s\mapsto(1,s)$, $s\mapsto(1/s,1/s)$ and $s\mapsto(s,1)$ with $s>0$. We define $D_1(s;\mu)$ to be the Dulac map of X_μ from Σ_1 to Σ_2, $D_2(s;\mu)$ to be the Dulac map of X_μ from Σ_2 to Σ_3 and, finally, $D_3(s;\mu)$ to be the Dulac map of $-X_\mu$ from Σ_1 to Σ_3, see Figure 2.2 x It is then clear that the limit cycles of X_μ near Γ are in one to one correspondence with the isolated positive zeroes of

$$\mathscr{D}(s;\mu):=\big(D_2\circ D_1-D_3\big)(s;\mu)$$

near $s=0$. The proof of Theorem A strongly relies in our next result, where we get the asymptotic expansion of $\mathscr{D}(s;\mu)$ at $s=0$ and we compute its coefficients. In its statement $d_i(\mu)$, for $i=1,2,3$, are the functions defined in Theorem A and $\mathbb{R}\{\mu\}_{\mu_0}$ stands for the local ring of convergent power series at μ_0.

Theorem 2.6. *Let us fix any $\mu_0\in\Lambda$ and set $\lambda_i^0:=\lambda_i(\mu_0)$ for $i=1,2,3$.*

(a) If $\lambda_1^0\lambda_2^0\lambda_3^0\neq 1$ then, for any $\ell_1\in\big(\min(\lambda_1^0\lambda_2^0,1/\lambda_3^0),\min(\lambda_1^0+\lambda_1^0\lambda_2^0,1+\lambda_1^0\lambda_2^0,2\lambda_1^0\lambda_2^0,1+1/\lambda_3^0,2/\lambda_3^0)\big)$,

$$\mathscr{D}(s;\mu)=a_1(\mu)s^{\lambda_1\lambda_2}-a_2(\mu)s^{1/\lambda_3}+\mathcal{F}_{\ell_1}^\infty(\mu_0),$$

where a_1 and a_2 are analytic and strictly positive functions on Λ.

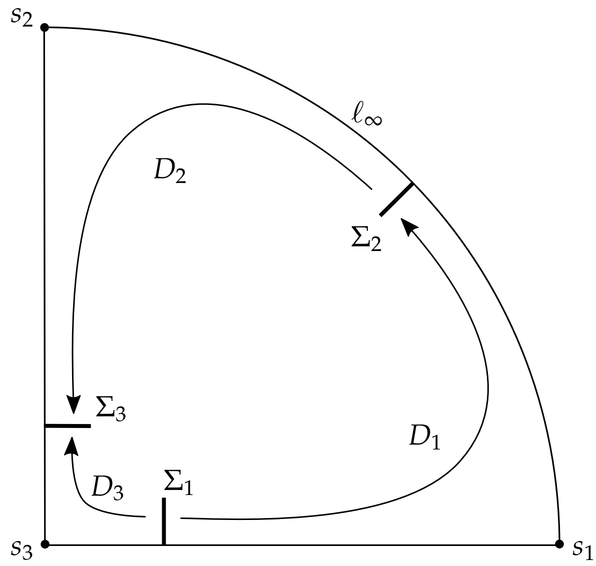

Figure 2.2: Auxiliary Dulac maps for the definition of $\mathscr{D} = D_2 \circ D_1 - D_3$ in Theorem 2.6. The return map in Theorem A, with respect to the transverse section Σ_1, would be $\mathscr{R} = D_3^{-1} \circ D_2 \circ D_1 = D_3^{-1} \circ \mathscr{D} + \mathrm{Id}$.

(*b*) *If* $\lambda_1^0\lambda_2^0\lambda_3^0 = 1$ *then, for any* $\ell_2 \in (0, \min(1, \lambda_1^0, \lambda_1^0\lambda_2^0))$,

$$\mathscr{D}(s;\mu) = \big(b_1(\mu)\,\omega\big(s;\alpha(\mu)\big) + b_2(\mu) + \mathcal{F}_{\ell_2}^\infty(\mu_0)\big)s^{1/\lambda_3},$$

where $\alpha = 1/\lambda_3 - \lambda_1\lambda_2$, $b_1 = \alpha a_1$ *and* $b_2 = a_1 - a_2$. *Moreover the equalities*

$$(b_1) = (d_1) \text{ and } (b_1, b_2) = (d_1, d_2)$$

between ideals over the ring $\mathbb{R}\{\mu\}_{\mu_0}$ *are verified. Assuming* $\lambda_1^0 > 1$, $\lambda_2^0 > 1$ *and* $\lambda_3^0 < 1$ *additionally then, for any* $\ell_3 \in \big(1, \min(2, \lambda_1^0, 1/\lambda_3^0)\big)$,

$$\mathscr{D}(s;\mu) = \big(b_1(\mu)\,\omega\big(s;\alpha(\mu)\big) + b_2(\mu) + b_3(\mu)s + \mathcal{F}_{\ell_3}^\infty(\mu_0)\big)s^{1/\lambda_3}U(s;\mu),$$

where b_3 *is an analytic function at* μ_0 *verifying that*

$$(b_1, b_2, b_3) = (d_1, d_2, d_3)$$

over the ring $\mathbb{R}\{\mu\}_{\mu_0}$ *and* U *is an analytic function such that* $U(0;\mu_0) = 1$.

Proof. In order to study the Dulac map D_1 from Σ_1 to Σ_2 we compactify X_μ by means of the coordinate change $\{x_1 = \frac{y}{x}, x_2 = \frac{1}{x}\}$. One can easily verify that the new vector field is orbitally conjugated to (2.1) particularised with $P_1(x_1,x_2) = x_2^n(f-g)(\frac{1}{x_2}, \frac{x_1}{x_2})$ and $P_2(x_1,x_2) = x_2^n f(\frac{1}{x_2}, \frac{x_1}{x_2})$, whereas in these coordinates the transverse sections Σ_1 and Σ_2 are parametrised by $\sigma_1(s) = (s,1)$ and $\sigma_2(s) = (1,s)$, respectively. The hyperbolicity ratio of the saddle at the origin is $\lambda_1 = -(\frac{P_2}{P_1})(0,0) = (\frac{f_n}{g_n - f_n})(1,0)$. Therefore, by applying Proposition 2.4 we can assert that

$$D_1(s) = \Delta_{10}s^{\lambda_1}\big(1 + \mathcal{F}_{\ell_1}^\infty(\mu_0)\big), \quad \text{with } \Delta_{10} := \big(L_{12}L_{11}^{-\lambda_1}\big)(1) \tag{2.4}$$

and any $0 < \ell_1 < \min(1, \lambda_1^0)$. Recall here that $\lambda_i^0 = \lambda_i(\mu_0)$ for $i = 1,2,3$ by definition.

Next, to analyse the Dulac map D_2 from Σ_2 to Σ_3 we compactify X_μ performing the change of coordinates given by $\{x_1 = \frac{1}{y}, x_2 = \frac{x}{y}\}$. One can check that the new vector field is orbitally conjugated to (2.1) with $P_1(x_1, x_2) = x_1^n g(\frac{x_2}{x_1}, \frac{1}{x_1})$ and $P_2(x_1, x_2) = x_1^n (g - f)(\frac{x_2}{x_1}, \frac{1}{x_1})$ and that in these coordinates the transverse sections Σ_2 and Σ_3 are parametrised by $\sigma_1(s) = (s, 1)$ and $\sigma_2(s) = (1, s)$, respectively. The hyperbolicity ratio of the saddle at the origin is $\lambda_2 = -(\frac{P_2}{P_1})(0,0) = (\frac{f_n - g_n}{g_n})(0,1)$. Thus, by Proposition 2.4 again,

$$D_2(s) = \Delta_{20} s^{\lambda_2} \big(1 + \mathcal{F}^\infty_{\ell_2}(\mu_0)\big), \quad \text{with } \Delta_{20} := \big(L_{22} L_{21}^{-\lambda_2}\big)(1) \tag{2.5}$$

and any $0 < \ell_2 < \min(1, \lambda_2^0)$.

Finally, to study the Dulac map D_3 of from Σ_1 to Σ_3 we make the reflection $\{x_1 = y, x_2 = x\}$, which brings $-X_\mu$ to (2.1) with $P_1(x_1, x_2) = -g(x_2, x_1)$ and $P_2(x_1, x_2) = -f(x_2, x_1)$. In these coordinates the transverse sections Σ_1 and Σ_3 are parametrised by $\sigma_1(s) = (s, 1)$ and $\sigma_2(s) = (1, s)$, respectively, and the hyperbolicity ratio of the saddle is $\frac{1}{\lambda_3} = -(\frac{P_2}{P_1})(0,0) = -(\frac{f}{g})(0,0)$. Hence by Proposition 2.4 once again,

$$D_3(s) = \Delta_{30} s^{1/\lambda_3} \big(1 + \mathcal{F}^\infty_{\ell_3}(\mu_0)\big), \quad \text{with } \Delta_{30} := \big(L_{32} L_{31}^{-1/\lambda_3}\big)(1) \tag{2.6}$$

and any $0 < \ell_3 < \min(1, 1/\lambda_3^0)$. Consequently, from (2.4), (2.5) and (2.6) we get that

$$\begin{aligned}
\mathscr{D}(s) = \big(D_2 \circ D_1 - D_3\big)(s) &= \Delta_{20} \Delta_{10}^{\lambda_2} s^{\lambda_1 \lambda_2} \big(1 + \mathcal{F}^\infty_{\ell_1}(\mu_0)\big)^{\lambda_2} \big(1 + \mathcal{F}^\infty_{\ell_4}(\mu_0)\big) - \Delta_{30} s^{1/\lambda_3} \big(1 + \mathcal{F}^\infty_{\ell_3}(\mu_0)\big) \\
&= \Delta_{20} \Delta_{10}^{\lambda_2} s^{\lambda_1 \lambda_2} \big(1 + \mathcal{F}^\infty_{\ell_1}(\mu_0)\big) \big(1 + \mathcal{F}^\infty_{\ell_4}(\mu_0)\big) - \Delta_{30} s^{1/\lambda_3} \big(1 + \mathcal{F}^\infty_{\ell_3}(\mu_0)\big) \\
&= \Delta_{20} \Delta_{10}^{\lambda_2} s^{\lambda_1 \lambda_2} - \Delta_{30} s^{1/\lambda_3} + \mathcal{F}^\infty_{\ell_5}(\mu_0),
\end{aligned}$$

where, by [15, Lemma A.2], we can take ℓ_4 and ℓ_5 to be any numbers such that $0 < \ell_4 < \min(\lambda_1^0, \lambda_1^0 \lambda_2^0)$ and $\min(\lambda_1^0 \lambda_2^0, 1/\lambda_3^0)) < \ell_5 < \min(\lambda_1^0 + \lambda_1^0 \lambda_2^0, 1 + \lambda_1^0 \lambda_2^0, 2\lambda_1^0 \lambda_2^0, 1 + 1/\lambda_3^0, 2/\lambda_3^0)$, respectively. Thus, setting $a_1 := \Delta_{20} \Delta_{10}^{\lambda_2}$ and $a_2 := \Delta_{30}$, we obtain

$$\mathscr{D}(s) = a_1 s^{\lambda_1 \lambda_2} - a_2 s^{1/\lambda_3} + \mathcal{F}^\infty_{\ell_5}(\mu_0) \tag{2.7}$$

and this proves (a) because each Δ_{i0} is a strictly positive analytic function by Proposition 2.4.

Let us proceed next with the proof of the assertions in (b), thus from now on we assume that $\lambda_1^0 \lambda_2^0 \lambda_3^0 = 1$. Then, setting $\alpha = 1/\lambda_3 - \lambda_1 \lambda_2$ and taking any $\ell_6 \in (0, \min(1, \lambda_1^0, \lambda_1^0 \lambda_2^0))$, from the above expression we get

$$\begin{aligned}
\mathscr{D}(s) &= s^{1/\lambda_3} \big(a_1 s^{\lambda_1 \lambda_2 - 1/\lambda_3} - a_2 + \mathcal{F}^\infty_{\ell_6}(\mu_0)\big) \\
&= s^{1/\lambda_3} \big(a_1 (1 + \alpha \omega(s; \alpha)) - a_2 + \mathcal{F}^\infty_{\ell_6}(\mu_0)\big) \\
&= s^{1/\lambda_3} \big(b_1 \omega(s; \alpha) + b_2 + \mathcal{F}^\infty_{\ell_6}(\mu_0)\big),
\end{aligned}$$

where $b_1 := \alpha a_1$ and $b_2 := a_1 - a_2$. Here we also take Definition 2.1 into account. Since a_1 is an analytic non-vanishing function at μ_0 and $d_1 = \lambda_3 \alpha$, we obtain the equality $(b_1) = (d_1)$ between ideals over the local ring $\mathbb{R}\{\mu\}_{\mu_0}$. In order to show that $(b_1, b_2) = (d_1, d_2)$ holds as well we note that, from (2.4), (2.5) and (2.6) again,

$$
\begin{aligned}
b_2(\mu) &= a_1 - a_2 = \Delta_{20}\Delta_{10}^{\lambda_2} - \Delta_{30} \\
&= \left(L_{22}L_{21}^{-\lambda_2}L_{12}^{\lambda_2}L_{11}^{-\lambda_1\lambda_2} - L_{32}L_{31}^{-1/\lambda_3}\right)\Big|_{u=1} \\
&= L_{32}L_{31}^{-1/\lambda_3}\left(\frac{L_{22}}{L_{32}}\frac{L_{31}^{1/\lambda_3}}{L_{11}^{\lambda_1\lambda_2}}\frac{L_{12}^{\lambda_2}}{L_{21}^{\lambda_2}} - 1\right)\Bigg|_{u=1} \\
&= L_{32}L_{31}^{-1/\lambda_3}\left(L_{31}^{\alpha}\left(\frac{L_{22}}{L_{32}}\frac{L_{31}^{\lambda_1\lambda_2}}{L_{11}^{\lambda_1\lambda_2}}\frac{L_{12}^{\lambda_2}}{L_{21}^{\lambda_2}} - 1\right) + (L_{31}^{\alpha} - 1)\right)\Bigg|_{u=1} \\
&= \kappa_1(\mu)d_2(\mu) + \alpha(\mu)\kappa_2(\mu),
\end{aligned}
$$

where $\kappa_1 = \frac{L_{32}}{L_{31}^{1/\lambda_3}}\big|_{u=1}\frac{e^{d_2}-1}{d_2}$ and $\kappa_2 = \frac{L_{32}}{L_{31}^{1/\lambda_3}}\frac{L_{31}^{\alpha}-1}{\alpha}\big|_{u=1}$ are analytic functions at μ_0 because so it is each L_{ij} by [16, Lemma 2.3]. Thus, on account of $d_1 = \lambda_3\alpha$, we get that $b_2 = \kappa_1 d_2 + \kappa_3 d_1$. Hence $(b_1, b_2) = (d_1, d_2)$ over the ring $\mathbb{R}\{\mu\}_{\mu_0}$ since $\kappa_1(\mu_0) > 0$ and we have already shown that $(b_1) = (d_1)$.

So far we have proved the first assertion in (b). To show the second one, besides $\lambda_1^0\lambda_2^0\lambda_3^0 = 1$, we assume $\lambda_1^0 > 1$, $\lambda_2^0 > 1$ and $\lambda_3^0 < 1$. On account of this we can apply point (3) in Proposition 2.4 to conclude that

$$D_1(s) = s^{\lambda_1}\left(\Delta_{10} + \Delta_{11}s + \mathcal{F}_{\ell_7}^{\infty}(\mu_0)\right) \quad \text{for any } \ell_7 \in \left[1, \min(\lambda_1^0, 2)\right), \tag{2.8}$$

$$D_2(s) = s^{\lambda_2}\left(\Delta_{20} + \Delta_{21}s + \mathcal{F}_{\ell_8}^{\infty}(\mu_0)\right) \quad \text{for any } \ell_8 \in \left[1, \min(\lambda_2^0, 2)\right), \tag{2.9}$$

and

$$D_3(s) = s^{1/\lambda_3}\left(\Delta_{30} + \Delta_{31}s + \mathcal{F}_{\ell_9}^{\infty}(\mu_0)\right) \quad \text{for any } \ell_9 \in \left[1, \min(1/\lambda_3^0, 2)\right). \tag{2.10}$$

Here the first order coefficients Δ_{10}, Δ_{20} and Δ_{30} are the ones already defined in (2.4), (2.5) and (2.6), respectively. With regard to the second order coefficients, only the ones of D_1 and D_3 are relevant for our purposes, which are given by

$$\Delta_{11} := -\frac{\lambda_1\Delta_{10}\hat{M}_1(1/\lambda_1, 1)}{L_{11}(1)} \quad \text{and} \quad \Delta_{31} := -\frac{\Delta_{30}\hat{M}_3(\lambda_3, 1)}{\lambda_3 L_{31}(1)}, \tag{2.11}$$

respectively. In each case, on account of (1.2), this follows easily from the formula $\Delta_1 = \lambda\Delta_0 S_1$ given in Proposition 2.4 and taking S_1 in (2.3) particularised to $\sigma_1(s) = (s, 1)$.

From (2.8) and (2.9), by applying [15, Lemma A.2] we can assert that

$$
\begin{aligned}
(D_2 \circ D_1)(s) &= s^{\lambda_1\lambda_2}\left(\Delta_{10} + \Delta_{11}s + \mathcal{F}_{\ell_7}^{\infty}\right)^{\lambda_2}\left(\Delta_{20} + \Delta_{21}s^{\lambda_1}\left(\Delta_{10} + \Delta_{11}s + \mathcal{F}_{\ell_7}^{\infty}\right) + \mathcal{F}_{\ell_{10}}^{\infty}\right) \\
&= s^{\lambda_1\lambda_2}\left(\Delta_{10}^{\lambda_2} + \lambda_2\Delta_{10}^{\lambda_2-1}\Delta_{11}s + \mathcal{F}_{\ell_7}^{\infty}\right)\left(\Delta_{20} + \Delta_{21}s^{\lambda_1}\left(\Delta_{10} + \Delta_{11}s\right) + \mathcal{F}_{\ell_{11}}^{\infty} + \mathcal{F}_{\ell_{10}}^{\infty}\right) \\
&= s^{\lambda_1\lambda_2}\left(\Delta_{10}^{\lambda_2} + \lambda_2\Delta_{10}^{\lambda_2-1}\Delta_{11}s + \mathcal{F}_{\ell_7}^{\infty}\right)\left(\Delta_{20} + \Delta_{10}\Delta_{21}s^{\lambda_1} + \Delta_{11}\Delta_{21}s^{\lambda_1+1} + \mathcal{F}_{\ell_{12}}^{\infty}\right)
\end{aligned}
$$

for any $\ell_{10} \in [\lambda_1^0, \lambda_1^0\min(\lambda_2^0, 2))$ in the first equality, any $\ell_{11} \in \left[\lambda_1^0 + 1, \lambda_1^0 + \min(\lambda_1^0, 2)\right)$ in the second one and any $\ell_{12} \in \left[\lambda_1^0 + 1, \min(2\lambda_1^0, \lambda_1^0 + 2, \lambda_1^0\lambda_2^0)\right)$ in the third one. Furthermore, in the second equality we use that, for any $\eta = \eta(\mu)$,

$$(1 + as + \mathcal{F}_\ell)^\eta = (1 + as)^\eta + \mathcal{F}_\ell = 1 + a\eta s + \mathcal{F}_{2-\varepsilon} + \mathcal{F}_\ell, \tag{2.12}$$

for any $\varepsilon > 0$, which in turn follows noting that

$$(1+as+\mathcal{F}_\ell)^\eta - (1+as)^\eta = (1+as)^\eta\big((1+\tfrac{\mathcal{F}_\ell}{1+as})^\eta - 1\big) = (1+as)^\eta\big((1+\mathcal{F}_\ell)^\eta - 1\big) = (1+as)^\eta\mathcal{F}_\ell = \mathcal{F}_\ell.$$

Consequently

$$\big(D_2\circ D_1\big)(s) = s^{\lambda_1\lambda_2}\big(\Delta_{20}\Delta_{10}^{\lambda_2} + \lambda_2\Delta_{11}\Delta_{20}\Delta_{10}^{\lambda_2-1}s + \mathcal{F}_{\ell_7}^\infty\big),$$

where we use again [15, Lemma A.2] taking $\lambda_1^0 > 1$ and $\lambda_2^0 > 1$ into account. Hence, from (2.10) and plug in $s^{-\alpha} = 1+\alpha\omega(s;\alpha)$ as before, we get

$$\begin{aligned}\mathscr{D}(s) &= \big(D_2\circ D_1 - D_3\big)(s)\\ &= s^{1/\lambda_3}\Big(s^{-\alpha}\big(\Delta_{20}\Delta_{10}^{\lambda_2} + \lambda_2\Delta_{11}\Delta_{20}\Delta_{10}^{\lambda_2-1}s + \mathcal{F}_{\ell_7}^\infty\big) - \Delta_{30} - \Delta_{31}s - \mathcal{F}_{\ell_9}^\infty\Big)\\ &= s^{1/\lambda_3}\Big((1+\alpha\omega(s;\alpha))\Delta_{20}\Delta_{10}^{\lambda_2}U(s) - \Delta_{30} - \Delta_{31}s - \mathcal{F}_{\ell_9}^\infty\Big),\end{aligned}$$

where we define $U(s) = 1+\lambda_2\Delta_{11}\Delta_{10}^{-1}s + \mathcal{F}_{\ell_7}^\infty$. The application of the formula given in (2.12) with $\eta = -1$ shows that $U(s)^{-1} = 1-\lambda_2\Delta_{11}\Delta_{10}^{-1}s + \mathcal{F}_{\ell_7}$. Thus one can easily verify that the above expression yields to

$$\mathscr{D}(s) = s^{1/\lambda_3}U(s)\big(b_1\omega(s;\alpha) + b_2 + b_3 s + \mathcal{F}_{\ell_{13}}^\infty\big)$$

with $\ell_{13}\in[1,\min(\lambda_1^0, 1/\lambda_3^0, 2))$ and $b_3 := \lambda_2\Delta_{11}\Delta_{30}\Delta_{10}^{-1} - \Delta_{31}$. Let us recall here that $b_1 = \alpha a_1 = \alpha\Delta_{20}\Delta_{10}^{\lambda_2}$ and $b_2 = a_1 - a_2 = \Delta_{20}\Delta_{10}^{\lambda_2} - \Delta_{30}$, where a_1 and a_2 are the analytic and strictly positive functions in (2.7). On account of the assumptions $\lambda_1^0 > 1$ and $\lambda_3^0 < 1$ we have $\lambda_1(\mu)\notin\frac{1}{\mathbb{N}}$ and $\lambda_3(\mu)\notin\mathbb{N}$ for $\mu\approx\mu_0$, which imply respectively that Δ_{11} and Δ_{31} are analytic at μ_0 by Proposition 2.4. Consequently b_3 is an analytic function at μ_0. That being said we claim that the equality $(b_1,b_2,b_3) = (d_1,d_2,d_3)$ between ideals over the local ring $\mathbb{R}\{\mu\}_{\mu_0}$ is true. In order to prove this, for the sake of shortness in the next computation we follow the convention that κ stands for an analytic function at μ_0 and $\hat\kappa$ stands for an analytic strictly positive function at μ_0. Some easy computations following this convention yield

$$\begin{aligned} b_3 &= \Delta_{10}^{-1}(\lambda_2\Delta_{11}\Delta_{30} - \Delta_{31}\Delta_{10}) = \Delta_{10}^{-1}(\lambda_2\Delta_{11}\Delta_{30} - \Delta_{31}\Delta_{10})\\ &= -\Delta_{30}\Big(\lambda_2\lambda_1\frac{\hat M_1(1/\lambda_1,1)}{L_{11}(1)} - \frac{1}{\lambda_3}\frac{\hat M_3(\lambda_3,1)}{L_{31}(1)}\Big) = -\hat\kappa\Big(\lambda_1\lambda_2\lambda_3\hat M_1\big(\tfrac{1}{\lambda_1},1\big)L_{31}(1) - \hat M_3(\lambda_3,1)L_{11}(1)\Big)\\ &= -\hat\kappa\Big((1-d_1)\hat M_1\big(\tfrac{1}{\lambda_1},1\big)L_{31}(1) - \hat M_3(\lambda_3,1)L_{11}(1)\Big)\\ &= \hat\kappa\Big(\hat M_3(\lambda_3,1)L_{11}(1) - \hat M_1\big(\tfrac{1}{\lambda_1},1\big)L_{31}(1)\Big) + \kappa d_1. \end{aligned}$$

where in the third and fifth equalities we use (2.11) and $d_1 := 1-\lambda_1\lambda_2\lambda_3$, respectively. Hence $b_3 = \hat\kappa d_3 + \kappa d_1$ since $d_3 := \hat M_3(\lambda_3,1)L_{11}(1) - \hat M_1(\frac{1}{\lambda_1},1)L_{31}(1)$. On account of $(b_1) = (d_1)$ and $(b_1,b_2) = (d_1,d_2)$, this shows the validity of the claim and completes the proof of the result. □

Remark 2.7. There are two important observations to be made about Theorem 2.6:

(*a*) The statement claims that the equalities $(b_1) = (d_1)$, $(b_1,b_2) = (d_1,d_2)$ and $(b_1,b_2,b_3) = (d_1,d_2,d_3)$ between ideals over the local ring $\mathbb{R}\{\mu\}_{\mu_\star}$ are satisfied. As a matter of fact, in the proof we show a stronger property, namely that the following holds:

$$\begin{pmatrix} d_1\\ d_2\\ d_3\end{pmatrix} = \begin{pmatrix}\kappa_1 & 0 & 0\\ * & \kappa_2 & 0\\ * & * & \kappa_3\end{pmatrix}\begin{pmatrix} b_1\\ b_2\\ b_3\end{pmatrix},$$

where all the entries in the matrix are analytic functions on Λ and each κ_i is strictly positive.

(b) From the dynamical point of view it is interesting to point out that the asymptotic expansion of the displacement map $\mathscr{D}(s;\mu_0)$ given in Theorem 2.6 provides a method to study the stability of the polycycle Γ. Indeed, taking also the previous observation into account, it shows that

1. if $d_1(\mu_0) < 0$ (respectively, > 0) then Γ is asymptotically stable (respectively, unstable),
2. if $d_1(\mu_0) = 0$ and $d_2(\mu_0) < 0$ (respectively, > 0) then Γ is asymptotically stable (respectively, unstable), and
3. if $d_1(\mu_0) = d_2(\mu_0) = 0$ and $d_3(\mu_0) < 0$ (respectively, > 0) then Γ is asymptotically stable (respectively, unstable).

Of course this is relevant because we have an explicit expression of these functions by Theorem A. In this regard let us note that the first assertion is well known since $d_1(\mu_0) < 0$ is equivalent to require that $\lambda_1^0\lambda_2^0\lambda_3^0 > 1$, while the second assertion was already proved by Gasull *et al.* in [8], see Theorem 1. On the contrary the third assertion constitutes a new result to the best of our knowledge. □

We give at this point the precise definition of independence of functions that we use in the present paper.

Definition 2.8. Let us consider the functions $g_i\colon \Lambda \longrightarrow \mathbb{R}$ for $i = 1,2,\ldots,k$. The *real variety* $V(g_1, g_2, \ldots, g_k)$ is defined to be the set of $\mu \in \Lambda$ such that $g_i(\mu) = 0$ for $i = 1,2,\ldots,k$. We say that $g_1, g_2, \ldots, g_k$ are *independent* at $\mu_\star \in V(g_1, g_2, \ldots, g_k)$ if the following conditions are fulfilled:

(1) Every neighbourhood of $\mu_\star$ contains two points $\mu_1, \mu_2 \in V(g_1, \ldots, g_{k-1})$ verifying that $g_k(\mu_1)g_k(\mu_2) < 0$ (if $k = 1$ then we set $V(g_1, \ldots, g_{k-1}) = V(0) = \Lambda$ for this to hold).

(2) The varieties $V(g_1, \ldots, g_i)$, $2 \leqslant i \leqslant k-1$, are such that if $\mu_0 \in V(g_1, \ldots, g_i)$ then every neighbourhood of μ_0 contains two points $\mu_1, \mu_2 \in V(g_1, \ldots, g_{i-1})$ so that $g_i(\mu_1)g_i(\mu_2) < 0$.

(3) If $\mu_0 \in V(g_1)$ then every neighbourhood of μ_0 contains two points μ_1, μ_2 such that $g_1(\mu_1)g_1(\mu_2) < 0$.

It is clear that if $g_i \in \mathscr{C}^1(\Lambda)$ for $i = 1,2,\ldots,k$ and the gradients $\nabla g_1(\mu_\star), \nabla g_2(\mu_\star) \ldots, \nabla g_k(\mu_\star)$ are linearly independent vectors of $\mathbb{R}^{N+1}$ then there exists a neighbourhood $U_\star$ of $\mu_\star$ such that the restrictions of $g_1, g_2, \ldots, g_k$ to $U_\star$ are independent at $\mu_\star$.

Lemma 2.9. *Suppose that the equalities $(c_1, \ldots, c_k) = (d_1, \ldots, d_k)$ between ideals over the local ring $\mathbb{R}\{\mu\}_{\mu_\star}$ hold for $k = 1,2,\ldots,n$, where $\mu_\star \in V(c_1, \ldots, c_n) = V(d_1, \ldots, d_n)$. Then $c_1, \ldots, c_n$ are independent at $\mu_\star$ if, and only if, $d_1, \ldots, d_n$ are independent at $\mu_\star$.*

Proof. Let us assume for instance that $c_1, \ldots, c_n$ are independent at $\mu_\star$ and prove that then $d_1, \ldots, d_n$ are also independent. To this aim we note that the equalities $(c_1, \ldots, c_k) = (d_1, \ldots, d_k)$ for $k = 1,2,\ldots,n$ imply the existence of two triangular matrices $A = (a_{ij})$ and $B = (b_{ij})$ with coefficients in $\mathbb{R}\{\mu\}_{\mu_\star}$ such that

$$\begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ * & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & \cdots & * & a_{nn} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

and

$$\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} b_{11} & 0 & \cdots & 0 \\ * & b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & \cdots & * & b_{nn} \end{pmatrix} \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}.$$

Clearly $R = (r_{ij}) := BA$ is also a triangular matrix with coefficients in the local ring $\mathbb{R}\{\mu\}_{\mu_\star}$ and

$$\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} r_{11} & 0 & \cdots & 0 \\ * & r_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & \cdots & * & r_{nn} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

We claim that, since $c_1, \ldots, c_n$ are independent at $\mu_\star$, then $r_{kk}(\mu_\star) = 1$ for all $k = 1, 2, \ldots, n$. The fact that this is true for $k = 1$ follows easily by continuity. Let us prove by contradiction that this is also true for $k \geqslant 2$. So assume that $r_{kk}(\mu_\star) \neq 1$ for some $k \in \{2, \ldots, n\}$. Then the equality $c_k = r_{k1}c_1 + \ldots + r_{kk}c_k$ implies that $c_k = \alpha_1 c_1 + \ldots + \alpha_{k-1}c_{k-1}$ where each $\alpha_i := \frac{r_{ki}}{1-r_{kk}}$ is an analytic function at $\mu_\star$. This clearly contradicts the assumption that $c_1, \ldots, c_n$ are independent at $\mu_\star$ (see Definition 2.8). Hence the claim is true and, consequently, $\det(R) = \det(A)\det(B) = 1$ at $\mu = \mu_\star$. This shows, in particular, that A is an invertible matrix in the local ring $\mathbb{R}\{\mu\}_{\mu_\star}$ and so there exists a neighbourhood U of $\mu_\star$ such that $a_{kk}(\mu) \neq 0$ for all $\mu \in U$ and $k = 1, 2, \ldots, n$. On account of this, the fact that $d_1, \ldots, d_n$ are independent at $\mu_\star$ follows easily noting that if we take any two points $\mu_1, \mu_2 \in U \cap V(c_1, \ldots, c_{i-1}) = U \cap V(d_1, \ldots, d_{i-1})$ verifying $c_i(\mu_1)c_i(\mu_2) < 0$ then we have that $d_i(\mu_1)d_i(\mu_2) = a_{ii}(\mu_1)a_{ii}(\mu_2)c_i(\mu_1)c_i(\mu_2) < 0$. This completes the proof of the result. □

Proof of Theorem A. Let us fix any $\mu_0 \in \Lambda$ and set $\lambda_i^0 := \lambda_i(\mu_0)$ for $i = 1, 2, 3$. Recall that the limit cycles of X_μ near Γ are in one to one correspondence with the isolated positive zeros of

$$\mathscr{D}(s;\mu) = \big(D_2 \circ D_1 - D_3\big)(s;\mu)$$

near $s = 0$. If $d_1(\mu_0) = 1 - \lambda_1^0\lambda_2^0\lambda_3^0$ is not zero then by applying (a) in Theorem 2.6 we have that, for any $\ell_1 \in \big(\min(\lambda_1^0\lambda_2^0, 1/\lambda_3^0), \min(\lambda_1^0 + \lambda_1^0\lambda_2^0, 1 + \lambda_1^0\lambda_2^0, 2\lambda_1^0\lambda_2^0, 1 + 1/\lambda_3^0, 2/\lambda_3^0)\big)$,

$$\mathscr{D}(s;\mu) = a_1(\mu)s^{\lambda_1\lambda_2} + a_2(\mu)s^{1/\lambda_3} + \mathcal{F}_{\ell_1}^\infty(\mu_0),$$

where a_1 and a_2 are analytic and strictly positive functions on Λ. Thus

$$\lim_{(s,\mu)\to(0,\mu_0)} s^{-\lambda_1\lambda_2}\mathscr{D}(s;\mu) = a_1(\mu_0) \text{ in case that } \lambda_1^0\lambda_2^0 < \lambda_3^0$$

and

$$\lim_{(s,\mu)\to(0,\mu_0)} s^{-1/\lambda_3}\mathscr{D}(s;\mu) = a_2(\mu_0) \quad \text{in case that } \lambda_1^0\lambda_2^0 > \lambda_3^0.$$

Since $a_i(\mu_0) \neq 0$ for $i = 1, 2$, this implies the existence of an open neighbourhood U of μ_0 and $\varepsilon > 0$ small enough such that $\mathscr{D}(s;\mu) \neq 0$ for all $\mu \in U$ and $s \in (0,\varepsilon)$ when $\lambda_1^0\lambda_2^0\lambda_3^0 \neq 1$. Hence $\mathrm{Cycl}\big((\Gamma, X_{\mu_0}), X_\mu\big) = 0$ and the assertion in (a) is true.

In order to prove (b) we note that if $d_1(\mu_0) = 1 - \lambda_1^0\lambda_2^0\lambda_3^0$ is equal to zero then, by (b) in Theorem 2.6, we can assert that, for any $\ell_2 \in (0, \min(1, \lambda_1^0, \lambda_1^0\lambda_2^0))$,

$$s^{-1/\lambda_3}\mathscr{D}(s;\mu) = b_1(\mu)\,\omega\big(s;\alpha(\mu)\big) + b_2(\mu) + \mathcal{F}^\infty_{\ell_2}(\mu_0), \tag{2.13}$$

where $\alpha = 1/\lambda_3 - \lambda_1\lambda_2$ and b_1 and b_2 are analytic functions at μ_0 such that $(b_1) = (d_1)$ and $(b_1, b_2) = (d_1, d_2)$ over the ring $\mathbb{R}\{\mu\}_{\mu_0}$. The assumptions in this case imply that $\mathscr{D}(s;\mu_0) \not\equiv 0$, $b_1(\mu_0) = 0$ and, thanks to Lemma 2.9, that b_1 is independent at μ_0. Thus, given any $\varepsilon > 0$, there exists $s_1 \in (0,\varepsilon)$ such that $\mathscr{D}(s_1;\mu_0) \neq 0$. Let us assume for instance that $\mathscr{D}(s_1;\mu_0) > 0$, the other case follows verbatim. Then, thanks to (1) in Definition 2.8, there exists $\mu_1 \approx \mu_0$ with $b_1(\mu_1) < 0$ and, by continuity, such that $\mathscr{D}(s_1;\mu_1) > 0$. Moreover by applying Lemmas A.3 and A.4 in [14],

$$Z_1(s;\mu) := \frac{s^{-1/\lambda_3}\mathscr{D}(s;\mu)}{\omega\big(s;\alpha(\mu)\big)} = b_1(\mu) + \frac{b_2(\mu)}{\omega\big(s;\alpha(\mu)\big)} + \mathcal{F}^\infty_{\ell_2-\delta}(\mu_0) \to \kappa(\mu) \quad \text{as } s \to 0, \tag{2.14}$$

where

$$\kappa(\mu) := b_1(\mu) - b_2(\mu)\min(\alpha(\mu), 0)).$$

Here we use that $1/\omega(s;\alpha(\mu)) \in \mathcal{F}^\infty_{-\delta}(\mu_0)$ for any $\delta > 0$ and that $\lim_{s\to 0} 1/\omega(s;\alpha) = \max(-\alpha, 0)$ by assertions (a) and (b) in [14, Lemma A.4], respectively. Note on the other hand that, by (b) in Theorem 2.6, $b_1 = \alpha a_1$ and $b_2 = a_1 - a_2$, where each a_i is an analytic strictly positive function. Thus $\kappa(\mu) = \alpha(\mu)a_2(\mu)$ if $\alpha(\mu) < 0$ and $\kappa(\mu) = \alpha(\mu)a_1(\mu)$ if $\alpha(\mu) \geqslant 0$. Therefore, since $b_1 = \alpha a_1$, we can write

$$\kappa = b_1\eta \quad \text{with } \eta > 0. \tag{2.15}$$

Hence, on account of $b_1(\mu_1) < 0$, we can assert that $\kappa(\mu_1) < 0$, which in turn, from (2.14), guarantees the existence of some $s_2 \in (0, s_1)$ such that $Z_1(s_2;\mu_1) < 0$. Thus $\mathscr{D}(s_1;\mu_1)\mathscr{D}(s_2;\mu_1) < 0$ and, by continuity, $\mathscr{D}(\hat{s};\mu_1) = 0$ for some $\hat{s} \in (s_2, s_1) \subset (0,\varepsilon)$. This implies $\mathrm{Cycl}\big((\Gamma, X_{\mu_0}), X_\mu\big) \geqslant 1$ as desired.

Let us prove (c) next, i.e., that if $d_2(\mu_0) \neq 0$ then $\mathrm{Cycl}\big((\Gamma, X_{\mu_0}), X_\mu\big) \leqslant 1$. Note first that to this end we can also assume that $d_1(\mu_0) = 0$, otherwise $\mathrm{Cycl}\big((\Gamma, X_{\mu_0}), X_\mu\big) = 0$. Consequently the expression in (2.13) is valid. That being said, the proof will follow by applying the well-known derivation-division algorithm. In doing so, from (2.14) and the fact that $\partial_s\omega(s;\alpha) = -s^{-\alpha-1}$,

$$\partial_s Z_1(s;\mu) = \frac{b_2(\mu)}{s^{\alpha(\mu)+1}\omega^2\big(s;\alpha(\mu)\big)} + \mathcal{F}^\infty_{\ell_2-1-\delta}(\mu_0),$$

where the flatness of the remainder follows from (f) in Lemma A.3 of [14]. Therefore

$$s^{\alpha(\mu)+1}\omega^2\big(s;\alpha(\mu)\big)\partial_s Z_1(s;\mu) = b_2(\mu) + \mathcal{F}^\infty_{\ell_2-4\delta}(\mu_0) \to b_2(\mu_0) \quad \text{as } (s,\mu) \to (0,\mu_0).$$

Recall at this point that $(b_1) = (d_1)$ and $(b_1, b_2) = (d_1, d_2)$ over the local ring $\mathbb{R}\{\mu\}_{\mu_0}$ by Theorem 2.6. Thus, the assumptions $d_1(\mu_0) = 0$ and $d_2(\mu_0) \neq 0$ imply that $b_2(\mu_0) \neq 0$. Accordingly, on account of the above limit and by Bolzano's Theorem, we obtain $\varepsilon > 0$ such that if $\|\mu - \mu_0\| < \varepsilon$ then $Z_1(\,\cdot\,;\mu)$, and so $\mathscr{D}(\,\cdot\,;\mu)$, has at most one zero for $s \in (0,\varepsilon)$, multiplicities taking into account. Hence $\mathrm{Cycl}\big((\Gamma, X_{\mu_0}), X_\mu\big) \leqslant 1$ and (c) follows.

Let us turn next to the proof of (d), in which the assumptions are $\mathscr{R}(\,\cdot\,;\mu_0)\not\equiv \mathrm{Id}$ and that d_1 and d_2 vanish and are independent at μ_0. Then $\mathscr{D}(\,\cdot\,;\mu_0)\not\equiv 0$ and, due to $(b_1)=(d_1)$ and $(b_1,b_2)=(d_1,d_2)$ once again, b_1 and b_2 vanish and are independent at μ_0 by Lemma 2.9. Thus, given any $\varepsilon>0$, there exists $s_1\in(0,\varepsilon)$ such that, for instance, $\mathscr{D}(s_1;\mu_0)<0$. Then, by continuity and condition (1) in Definition 2.8, there exists $\mu_1\approx\mu_0$ such that $b_2(\mu_1)>0$, $b_1(\mu_1)=0$ and $\mathscr{D}(s_1;\mu_1)<0$. Hence, from (2.13),

$$s^{-1/\lambda_3}\mathscr{D}(s;\mu_1)=b_2(\mu_1)+\mathcal{F}^{\infty}_{\ell_2}(\mu_0)\to b_2(\mu_1)\quad\text{as } s\to 0,$$

which shows the existence of $s_2\in(0,s_1)$ such that $\mathscr{D}(s_2;\mu_1)>0$. For the same reasons we can choose $\mu_2\approx\mu_1$ satisfying $\mathscr{D}(s_1;\mu_2)<0$ and $\mathscr{D}(s_2;\mu_2)>0$ together with $b_1(\mu_2)<0$. Then, from (2.14) and (2.15), $\lim_{s\to 0}Z_1(s;\mu_2)=b_1(\mu_2)\eta(\mu_2)<0$ and so there exists $s_3\in(0,s_2)$ verifying that $\mathscr{D}(s_3;\mu_2)<0$. By continuity there exist $\hat{s}_1,\hat{s}_2\in(0,\varepsilon)$ with $D(\hat{s}_1;\mu_2)=D(\hat{s}_2;\mu_2)=0$. Accordingly $\mathrm{Cycl}\big((\Gamma,X_{\mu_0}),X_\mu\big)\geqslant 2$.

From now on, in order to prove (e) and (f), we assume $\lambda_1^0<1$, $\lambda_2^0>1$ and $\lambda_3^0>1$. Then by applying Theorem 2.6, for any $\ell_3\in\big(1,\min(2,\lambda_1^0,1/\lambda_3^0)\big)$,

$$\mathscr{D}(s;\mu)=\big(b_1(\mu)\,\omega(s;\alpha(\mu))+b_2(\mu)+b_3(\mu)s+\mathcal{F}^{\infty}_{\ell_3}(\mu_0)\big)s^{1/\lambda_3}U(s;\mu),\tag{2.16}$$

where b_3 is an analytic function at μ_0 verifying that $(b_1,b_2,b_3)=(d_1,d_2,d_3)$ over the ring $\mathbb{R}\{\mu\}_{\mu_0}$ and U is an analytic function such that $U(0;\mu_0)=1$. Hence

$$\begin{aligned}Z_2(s;\mu)&:=\frac{\mathscr{D}(s;\mu)}{s^{1/\lambda_3}\omega(s;\alpha(\mu))U(s)}\\&=b_1(\mu)+b_2(\mu)\frac{1}{\omega(s;\alpha(\mu))}+b_3(\mu)\frac{s}{\omega(s;\alpha(\mu))}+\mathcal{F}^{\infty}_{\ell_3-\delta}(\mu_0),\end{aligned}\tag{2.17}$$

where we use once again that $1/\omega(s;\alpha(\mu))\in\mathcal{F}^{\infty}_{-\delta}(\mu_0)$ for any $\delta>0$. Note furthermore that, for $\mu\approx\mu_0$, the positive zeros of $\mathscr{D}(\,\cdot\,;\mu)$ and $Z_2(\,\cdot\,;\mu)$ near $s=0$ are in one to one correspondence because $\frac{1}{s^{1/\lambda_3}\omega(s;\alpha(\mu))}$ tends to $+\infty$ as $(s,\mu)\to(0,\mu_0)$. That being stablished we begin first with the proof of assertion (e) and to this aim, besides $d_3(\mu_0)\neq 0$, we can also suppose $d_1(\mu_0)=d_2(\mu_0)=0$, otherwise $\mathrm{Cycl}\big((\Gamma,X_{\mu_0}),X_\mu\big)\leqslant 1$ by (a) or (b), which already have been proved. In this case, since $(b_1)=(d_1)$, $(b_1,b_2)=(d_1,d_2)$ and $(b_1,b_2,b_3)=(d_1,d_2,d_3)$, it turns out that $b_3(\mu_0)\neq 0$. As in the proof of (c), we will apply to steps of the derivation-division algorithm in (2.17). In doing so we obtain that

$$Z_3(s;\mu):=s^{\alpha(\mu)+1}\omega^2(s;\alpha(\mu))\partial_s Z_2(s;\mu)=b_2(\mu)+b_3(\mu)\big(s+s^{\alpha(\mu)+1}\omega(s;\alpha(\mu))\big)+\mathcal{F}^{\infty}_{\ell_3-4\delta}(\mu_0),$$

where the flatness of the remainder follows by applying Lemmas A.3 and A.4 in [14] as before and we use that $\partial_s\omega(s;\alpha)=-s^{-\alpha-1}$. Note also that the positive zeros of $Z_3(\,\cdot\,;\mu)$ and $\partial_s Z_2(\,\cdot\,;\mu)$ near $s=0$ are in one to one correspondence for $\mu\approx\mu_0$ because $\omega(s;\alpha(\mu))$ tends to $+\infty$ as $(s,\mu)\to(0,\mu_0)$. Finally

$$\frac{\partial_s Z_3(s;\mu)}{s^{\alpha(\mu)}\omega(s;\alpha(\mu))}=(\alpha(\mu)+1)b_3(\mu)+\mathcal{F}^{\infty}_{\ell_3-1-6\delta}(\mu_0)\to b_3(\mu_0)\neq 0\quad\text{as } (s,\mu)\to(0,\mu_0).$$

By applying twice Bolzano's Theorem, we can assert the existence of some $\varepsilon>0$ such that if $\|\mu-\mu_0\|<\varepsilon$ then $Z_2(\,\cdot\,;\mu)$, and so $\mathscr{D}(\,\cdot\,;\mu)$, has at most two zeros for $s\in(0,\varepsilon)$, multiplicities taking into account. Hence $\mathrm{Cycl}\big((\Gamma,X_{\mu_0}),X_\mu\big)\leqslant 2$, which proves (e).

Finally, in order to prove (f) we suppose that $\mathscr{R}(\,\cdot\,;\mu_0)\not\equiv\mathrm{Id}$ and that d_1, d_2 and d_3 vanish and are independent at μ_0. Consequently $\mathscr{D}(\,\cdot\,;\mu_0)\not\equiv 0$ and, due to the equality between the corresponding ideals over the local ring, b_1, b_2 and b_3 vanish and are independent at μ_0 by Lemma 2.9. Thus, given any $\varepsilon > 0$, there exists $s_1 \in (0, \varepsilon)$ such that, for instance, $\mathscr{D}(s_1; \mu_0) < 0$. Then, by continuity and condition (1) in Definition 2.8, there exists $\mu_1 \approx \mu_0$ such that $b_3(\mu_1) > 0$, $b_1(\mu_1) = b_2(\mu_1) = 0$ and $\mathscr{D}(s_1; \mu_1) < 0$. Hence, from (2.16),

$$\frac{\mathscr{D}(s;\mu_1)}{s^{1+1/\lambda_3}U(s;\mu)} = b_3(\mu_1) + \mathcal{F}^{\infty}_{\ell_3-1}(\mu_0) \to b_3(\mu_1) > 0 \quad \text{as } s \to 0,$$

which shows the existence of $s_2 \in (0, s_1)$ such that $\mathscr{D}(s_2; \mu_1) > 0$. For the same reasons we can choose $\mu_2 \approx \mu_1$ satisfying $\mathscr{D}(s_1; \mu_2) < 0$ and $\mathscr{D}(s_2; \mu_2) > 0$ together with $b_1(\mu_2) = 0$ and $b_2(\mu_2) < 0$. Accordingly, from (2.16) again,

$$\frac{\mathscr{D}(s;\mu_2)}{s^{1/\lambda_3}U(s;\mu)} = b_2(\mu_2) + b_3(\mu_2)s + \mathcal{F}^{\infty}_{\ell_3-1}(\mu_0) \to b_2(\mu_2) < 0 \quad \text{as } s \to 0,$$

which shows the existence of $s_3 \in (0, s_2)$ such that $\mathscr{D}(s_3; \mu_2) < 0$. In the final step we take $\mu_3 \approx \mu_2$ satisfying $\mathscr{D}(s_1; \mu_3) < 0$ and $\mathscr{D}(s_2; \mu_3) > 0$ and $\mathscr{D}(s_3; \mu_3) < 0$ together with $b_1(\mu_3) > 0$. Then, from (2.14) and (2.15), $\lim_{s\to 0} s^{-1/\lambda_3} Z_1(s; \mu_3) = b_1(\mu_3)\eta(\mu_3) > 0$ and so there exists $s_4 \in (0, s_3)$ such that $\mathscr{D}(s_3; \mu_3) > 0$. By continuity there exist $\hat{s}_1, \hat{s}_2, \hat{s}_3 \in (0, \varepsilon)$ with $D(\hat{s}_i; \mu_3) = 0$ for $i = 1, 2, 3$. Hence $\operatorname{Cycl}\big((\Gamma, X_{\mu_0}), X_\mu\big) \geqslant 3$ and this completes the proof of the result. □

3 Applications

We begin this section by revisiting in Theorem 3.1 a family of Kolmogorov differential systems that was first studied in [8], where the authors (following the notation in our statement) prove that if $\mu_0 = (a_0, p_0, q_0)$ verifies $p_0 + q_0 = 0$ and $a_0 \neq 0$ then $\operatorname{Cycl}\big((\Gamma, X_{\mu_0}), X_\mu\big) \geqslant 1$, cf. assertion (b).

Theorem 3.1. *Consider the family of Kolmogorov differential systems*

$$X_\mu \quad \begin{cases} \dot{x} = x(1 + x + x^2 + axy + py^2), \\ \dot{y} = y(-1 - y + qx^2 + axy - y^2), \end{cases}$$

where $\mu = (a, p, q) \in \mathbb{R}^3$ with $p < -1$ and $q > 1$ and let us fix any $\mu_0 = (a_0, p_0, q_0)$. Then, compactifying X_μ to the Poincaré disc, the boundary of the first quadrant is a polycycle Γ such that:

(a) $\operatorname{Cycl}\big((\Gamma, X_{\mu_0}), X_\mu\big) = 0$ *if $p_0 + q_0 \neq 0$.*

(b) $\operatorname{Cycl}\big((\Gamma, X_{\mu_0}), X_\mu\big) = 1$ *if $p_0 + q_0 = 0$ and $a_0 \neq 0$.*

(c) *The return map of X_{μ_0} along Γ is the identity if, and only if, $a_0 = p_0 + q_0 = 0$. In this case Γ is the outer boundary of the period annulus of a center at (x_0, y_0) with $x_0 = y_0 = -\frac{1+\sqrt{-3-4p_0}}{2(1+p_0)}$ that foliates the first quadrant and, moreover,* $\operatorname{Cycl}\big((\Gamma, X_{\mu_0}), X_\mu\big) = 1$.

On the other hand the vector field X_μ has a unique singularity $Q_\mu = (v_1, v_2)$ in the first quadrant, which is either a focus or a center, and has trace equal to $\tau(\mu) = v_1 + 2v_1^2 + 2av_1v_2 - v_2 - 2v_2^2$. Furthermore, the following holds:

(d) *If $\tau(\mu_0) \neq 0$ then* $\operatorname{Cycl}\big((Q_{\mu_0}, X_{\mu_0}), X_\mu\big) = 0$ *and a sufficient condition for $\tau(\mu_0) \neq 0$ to hold is that $p_0 + q_0 = 0$ and $a_0 \neq 0$.*

(*e*) *If* $\tau(\mu_0)=0$ *and* $p_0+q_0\neq 0$ *then* Q_{μ_0} *is a weak focus of order 1 and* $\mathrm{Cycl}\big((Q_{\mu_0},X_{\mu_0}),X_\mu\big)=1$.

(*f*) *If* $\tau(\mu_0)=0$ *and* $p_0+q_0=0$ *then* $a_0=0$. *In addition* Q_{μ_0} *is a center if, and only if,* $p_0+q_0=a_0=0$ *and, in this case,* $\mathrm{Cycl}\big((Q_{\mu_0},X_{\mu_0}),X_\mu\big)=1$.

Finally it is not possible a simultaneous bifurcation of limit cycles from Γ *and* Q_μ.

Proof. The assertions in (a) and (b) follow directly by applying Theorem A. Indeed, in this case, following the notation in (1.1), $f(x,y)=1+x+x^2+axy+py^2$ and $g(x,y)=-1-y+qx^2+axy-y^2$, so that

$$f_2(x,y)=x^2+axy+py^2 \text{ and } g_2(x,y)=qx^2+axy-y^2.$$

Taking this into account, together with $p<-1$ and $q>1$, one can easily check that the assumptions **H1** and **H2** are verified. As a matter of fact the first assumption holds not only for $z>0$ but for all $z\in\mathbb{R}$, and this implies that the boundary of each quadrant is a monodromic polycycle for the compactified vector field. Hence, by the Poincaré–Bendixson theorem (see [21] for instance), there exists at least one singularity of X_μ inside each one of the four quadrants. Due to $\deg(f)=\deg(g)=2$, by Bézout's theorem there exists exactly one in each quadrant. From now on we denote the singularity of X_μ in the first quadrant by Q_μ. That being said, the hyperbolicity ratios of the saddles at Γ are $\lambda_1=\frac{1}{q-1}$, $\lambda_2=-(p+1)$ and $\lambda_3=1$. Consequently the first assertion follows from (a) in Theorem A because

$$d_1(\mu)=1-\lambda_1\lambda_2\lambda_3=\frac{p+q}{q-1}. \tag{3.1}$$

The second assertion will follow by applying (b) and (c) in Theorem A. To show this we first recall that

$$d_2(\mu)=\lambda_2\log\left(\frac{L_{12}}{L_{21}}\right)(1)+\log\left(\frac{L_{22}}{L_{32}}\right)(1)+\lambda_1\lambda_2\log\left(\frac{L_{31}}{L_{11}}\right)(1)$$

and this leads us to the computation of the following improper integrals:

$$\Delta_1(\mu):=\log\left(\frac{L_{12}}{L_{21}}\right)(1)=\int_0^1\left(\left(\frac{f_2}{f_2-g_2}\right)(1,z)+\lambda_1-\left(\frac{g_2}{g_2-f_2}\right)(z,1)-\frac{1}{\lambda_2}\right)\frac{dz}{z}$$

$$\Delta_2(\mu):=\log\left(\frac{L_{22}}{L_{32}}\right)(1)=\int_0^1\left(\left(\frac{g-f}{g}\right)(0,1/z)+\lambda_2-\left(\frac{f}{g}\right)(0,z)-\frac{1}{\lambda_3}\right)\frac{dz}{z}$$

$$\Delta_3(\mu):=\log\left(\frac{L_{31}}{L_{11}}\right)(1)=\int_0^1\left(\left(\frac{g}{f}\right)(z,0)+\lambda_3-\left(\frac{f-g}{f}\right)(1/z,0)-\frac{1}{\lambda_1}\right)\frac{dz}{z}$$

These expressions have to be computed assuming that $p+q=0$, i.e., $\lambda_1\lambda_2=1$. In doing so we obtain that

$$\Delta_1(a,p,-p)=\frac{2a}{1-q}\int_0^1\frac{zdz}{z^2+1}=\frac{a\pi}{2(p+1)}$$

and

$$\Delta_2(a,p,-p)=-\Delta_3(a,p,-p)=-(p+1)\int_0^1\frac{zdz}{z^2+z+1}=-\frac{(p+1)\pi}{3\sqrt{3}}.$$

Therefore $d_2(a,p,-p)=-\frac{a\pi}{2}$ is zero if, and only if, $a=0$. Taking this into account, the combination of (b) and (c) in Theorem A shows that $\mathrm{Cycl}\big((\Gamma,X_{\mu_0}),X_\mu\big)=1$ for any $\mu_0=$

$(a_0, p_0, -p_0)$ with $a_0 \neq 0$, as desired. It is important to remark for the forthcoming analysis that by applying the Weierstrass Division Theorem (see for instance [9,12]) we can assert that

$$d_2(\mu) = -\frac{a\pi}{2} + (p+q)h(\mu) \tag{3.2}$$

for some analytic function h.

Next we proceed with the proof of (c).To this aim we fix any $\mu_0 = (a_0, p_0, q_0)$ and apply Theorem 2.6, which gives the asymptotic expansion of $\mathscr{D}(s;\mu)$ at $s = 0$ for $\mu \approx \mu_0$. This result, taking (3.1) and (3.2) into account, shows that if $\mathscr{D}(s;\mu_0) \equiv 0$ then $a_0 = p_0 + q_0 = 0$. In order to prove the converse observe that if $\mu_0 = (0, p_0, -p_0)$ then the vector field X_{μ_0} writes as

$$\begin{cases} \dot{x} = x(1 + x + x^2 + p_0 y^2), \\ \dot{y} = -y(1 + y + p_0 x^2 + y^2). \end{cases}$$

One can easily check that Q_{μ_0}, the only singularity of X_{μ_0} in the first quadrant, is a weak focus at the point (x_0, y_0) with $x_0 = y_0 = -\frac{1+\sqrt{-3-4p_0}}{2(1+p_0)}$. Furthermore, setting $\sigma(x,y) = (y,x)$, it turns out that $\sigma^\star X_{\mu_0} = -X_{\mu_0}$ and so the vector field is reversible with respect to the straight line $y = x$. Hence Q_{μ_0} is a center and a straightforward application of the Poincaré–Bendixson theorem shows that its period annulus fills the first quadrant, which in particular implies that $\mathscr{D}(s;\mu_0) \equiv 0$.

So far we have proved that the return map of X_{μ_0} along Γ is the identity if, and only if, $\mu_0 = (0, p_0, -p_0)$. Our next task is to show that, in this case, $\mathrm{Cycl}\big((\Gamma, X_{\mu_0}), X_\mu\big) = 1$. With this aim in view we apply (b) in Theorem 2.6, which shows that if $\mu \approx \mu_0$ then

$$\mathscr{D}(s;\mu) = \big(b_1(\mu)\,\omega\big(s;\alpha(\mu)\big) + b_2(\mu) + r(s;\mu)\big)s^{1/\lambda_3}, \tag{3.3}$$

where $\alpha = 1/\lambda_3 - \lambda_1\lambda_2$, $r \in \mathcal{F}_\ell^\infty(\mu_0)$ with $\ell \in \big(0, \min(1, \frac{-1}{p+1})\big)$ and, in addition,

$$(b_1) = (d_1) \quad \text{and} \quad (b_1, b_2) = (d_1, d_2)$$

over the local ring $\mathbb{R}\{\mu\}_{\mu_0}$. Consequently if $\mu = (a,p,q)$ satisfies $a = p + q = 0$ then $b_1(\mu) = b_2(\mu) = 0$ and $r(s;\mu) \equiv 0$. Furthermore, since the vectors $\nabla d_1(\mu_0)$ and $\nabla d_2(\mu_0)$ are linearly independent, see (3.1) and (3.2), the above equalities between ideals show that this is also the case of $\nabla b_1(\mu_0)$ and $\nabla b_2(\mu_0)$. We can thus take $(\eta_1, \eta_2, \eta_3) \notin \langle \nabla b_1(\mu_0), \nabla b_2(\mu_0) \rangle$ and define $b_3(\mu) := \eta_1 a + \eta_2(p - p_0) + \eta_3(q + p_0)$ so that $\nu = \Psi(\mu) := \big(b_1(\mu), b_2(\mu), b_3(\mu)\big)$ is a local analytical change of coordinates in a neighbourhood of $\mu = \mu_0$. Note that Ψ maps μ_0 to $0_3 := (0,0,0)$ and $\{a = p + q = 0\}$ to $\{\nu_1 = \nu_2 = 0\}$ and in addition

$$\mathscr{R}_1(s;\nu) := s^{-1/\lambda_3}\mathscr{D}(s;\mu)\Big|_{\mu=\Psi^{-1}(\nu)} = \nu_1\,\omega(s;\hat{\alpha}) + \nu_2 + \hat{r}(s;\nu),$$

where $\hat{\alpha} = \hat{\alpha}(\nu) := \alpha\big(\Psi^{-1}(\nu)\big)$ and $\hat{r}(s;\nu) := r\big(s;\Psi^{-1}(\nu)\big) \in \mathcal{F}_\ell^\infty(0_3)$. The key point is that $\hat{r}(s;0,0,\nu_3) \equiv 0$ implies, thanks to [17, Lemma 4.1], that $\hat{r}(s;\nu) = \nu_1 h_1(s;\nu) + \nu_2 h_2(s;\nu)$ with $h_i \in \mathcal{F}_\ell^\infty(0_3)$. Accordingly

$$\mathscr{R}_1(s;\nu) = \nu_1\big(\omega(s;\hat{\alpha}) + h_1(s;\nu)\big) + \nu_2\big(1 + h_2(s;\nu)\big). \tag{3.4}$$

Observe that if $s \to 0^+$ and $\nu \to (0,0,0)$ then the factor multiplying ν_1 tends to $+\infty$, whereas the factor multiplying ν_2 tends to 1. Here we use Definition 2.3 and that $\lim_{(s,\alpha)\to(0,0)} \omega(s;\alpha) = +\infty$. We claim that there exists $s_0 > 0$ and an open neighbourhood U of $\nu = (0,0,0)$ such that

$$\mathscr{R}_2(s;\nu) := \frac{\mathscr{R}_1(s;\nu)}{\omega(s;\hat{\alpha}) + h_1(s;\nu)} = \nu_1 + \nu_2\frac{1 + h_2(s;\nu)}{\omega(s;\hat{\alpha}) + h_1(s;\nu)}$$

has at most one zero on $(0,s_0)$, counted with multiplicities, for all $\nu=(\nu_1,\nu_2,\nu_3)\in U$ with $\nu_1^2+\nu_2^2\neq 0$. This will imply that $\mathrm{Cycl}\big((\Gamma,X_{\mu_0}),X_\mu\big)\leqslant 1$ because $\mathscr{R}_2(s;0,0,\nu_3)\equiv 0$, so that it has not any isolated zero. The claim is clear in case that $\nu_2=0$. To tackle the case $\nu_2\neq 0$ we compute the derivative with respect to s to obtain that

$$
\begin{aligned}
\mathscr{R}_2'(s;\nu)&=\nu_2\partial_s\left(\frac{1+\mathcal{F}_\ell^\infty}{\omega(s;\hat\alpha)+\mathcal{F}_\ell^\infty}\right)=\nu_2\partial_s\left(\frac{1+\mathcal{F}_\ell^\infty}{\omega(s;\hat\alpha)(1+\mathcal{F}_{\ell-\varepsilon}^\infty)}\right)\\
&=\nu_2\partial_s\left(\frac{1+\mathcal{F}_{\ell-\varepsilon}^\infty}{\omega(s;\hat\alpha)}\right)=\frac{\nu_2}{s^{\hat\alpha+1}\omega^2(s;\hat\alpha)}(1+\mathcal{F}_{\ell-\varepsilon}^\infty)+\frac{\nu_2}{\omega(s;\hat\alpha)}\mathcal{F}_{\ell-\varepsilon-1}^\infty\\
&=\frac{\nu_2}{s^{\hat\alpha+1}\omega^2(s;\hat\alpha)}\big(1+\mathcal{F}_{\ell-\varepsilon}^\infty+s^{\hat\alpha+1}\omega(s;\hat\alpha)\mathcal{F}_{\ell-\varepsilon-1}^\infty\big)=\frac{\nu_2}{s^{\hat\alpha+1}\omega^2(s;\hat\alpha)}\big(1+\mathcal{F}_{\ell-3\varepsilon}^\infty\big).
\end{aligned}
$$

Here, in the second equality we apply first assertion (c) of Lemma A.4 in [14] to get that $1/\omega(s;\hat\alpha)\in\mathcal{F}_{-\varepsilon}^\infty(0_3)$ for all $\varepsilon>0$ small enough, due to $\hat\alpha(0_3)=0$, and use next that $\mathcal{F}_{-\varepsilon}^\infty\mathcal{F}_\ell^\infty\subset\mathcal{F}_{\ell-\varepsilon}^\infty$ from (g) of Lemma A.3 in [14] . In the third equality, on account of $\frac{1}{1+s}-1\in\mathcal{F}_1^\infty$ and by (h) of Lemma A.3 in [14] , we use first the inclusion $\frac{1}{1+\mathcal{F}_{\ell-\varepsilon}^\infty}\subset 1+\mathcal{F}_{\ell-\varepsilon}^\infty$. Then, by using (d) and (g) of Lemma A.3 in [14] , we expand the numerator to get that $(1+\mathcal{F}_\ell^\infty)(1+\mathcal{F}_{\ell-\varepsilon}^\infty)\subset 1+\mathcal{F}_{\ell-\varepsilon}^\infty$. Next, in the fourth equality we use that $\partial_s\omega(s;\alpha)=s^{-\alpha-1}$ and assertion (f) of Lemma A.3 in [14] to deduce that $\partial_s\mathcal{F}_{\ell-\varepsilon}^\infty\subset\mathcal{F}_{\ell-\varepsilon-1}^\infty$. Finally in the last equality we apply (c) of Lemma A.4 in [14] to get that $s^{\hat\alpha+1}\omega(s;\hat\alpha)\in\mathcal{F}_{1-2\varepsilon}^\infty$ and we use again that $\mathcal{F}_{1-2\varepsilon}^\infty\mathcal{F}_{\ell-\varepsilon-1}^\infty\subset\mathcal{F}_{\ell-3\varepsilon}^\infty$. On account of Definition 2.3 we can assert the existence of some $s_0\in(0,1)$ and a neighbourhood U of $\nu=(0,0,0)$ such that $\mathscr{R}_2'(s;\nu)\neq 0$ for all $s\in(0,s_0)$ and $\nu\in U$ with $\nu_2\neq 0$. Hence the application of Rolle's theorem shows that the claim is true for $\nu_2\neq 0$ as well. So far we have proved that $\mathrm{Cycl}\big((\Gamma,X_{\mu_0}),X_\mu\big)\leqslant 1$. The fact that this upper bound is attained follows by applying the assertion in (b) taking $\mu_0=(a_0,p_0,-p_0)$ with $a_0\approx 0$ but different from zero. This completes the proof of (c).

Let us turn now to the proof of the assertions regarding the singularity of X_μ at Q_μ. The approach here is rather standard and the technical difficulty is that we do not dispose of a feasible expression of the coordinates of Q_μ. To overcome this problem we shall parametrise the family of vector fields more conveniently. For reader's convenience we summarise the chain of reparametrisations that we shall perform:

$$\mu=(a,p,q)\to(a,p,\varepsilon)\to(v_1,v_2,\varepsilon)\to(v_1,v_2,\tau)\to\hat\mu=(\rho,\sigma,\tau).$$

For the first one we simply introduce $\varepsilon=p+q$. In the second one we take the coordinates of the singular point $Q_\mu=(v_1,v_2)$ as new parameters, i.e., we isolate a and p from

$$\begin{cases}1+v_1+v_1^2+av_1v_2+pv_2^2=0,\\ 1+v_2+(p-\varepsilon)v_1^2-av_1v_2+v_2^2=0,\end{cases}\tag{3.5}$$

to obtain

$$a=-\frac{v_1^2v_2^2\varepsilon+v_1^4-v_2^4+v_1^3-v_2^3+v_1^2-v_2^2}{v_1v_2\left(v_1^2+v_2^2\right)}\quad\text{and}\quad p=\frac{v_1^2\varepsilon-v_1^2-v_2^2-v_1-v_2-2}{v_1^2+v_2^2}.$$

In this respect we point out that v_1 and v_2 are strictly positive because Q_μ is inside the first quadrant for all admissible μ. More important, the map $\varphi:(a,p,\varepsilon)\mapsto(v_1,v_2,\varepsilon)$ is smooth and, taking (3.5) into account, injective. The smoothness follows by the Inverse Function

Theorem since one can check that the determinant of the Jacobian of $(v_1, v_2, \varepsilon) \mapsto (a, p, \varepsilon)$ is non-zero at the image by φ of any admissible parameter. Then one can check that the trace of the Jacobian of the vector field at $(x, y) = (v_1, v_2)$ is

$$\begin{aligned} x f_x(x,y) + y g_y(x,y)\big|_{(x,y)=(v_1,v_2)} &= v_1 + 2v_1^2 + 2a v_1 v_2 - v_2 - 2v_2^2\Big|_{a=a(v_1,v_2,\varepsilon)} \\ &= -\frac{2\varepsilon v_1^2 v_2^2 + (v_1 - v_2)(v_1 + 2 + v_2)(v_1 + v_2)}{v_2^2 + v_1^2}, \end{aligned}$$

and we introduce $\tau = \tau(v_1, v_2, \varepsilon)$ isolating ε from

$$2\varepsilon v_1^2 v_2^2 + (v_1 - v_2)(v_1 + 2 + v_2)(v_1 + v_2) = \tau. \tag{3.6}$$

In other words, τ is (up to a non-vanishing factor) the trace of the vector field. Finally, for convenience, we define $\rho = \frac{v_1 - v_2}{2}$ and $\sigma = \frac{v_1 + v_2}{2}$. Observe then that $\{p + q = a = 0\}$ becomes $\{\rho = \tau = 0\}$. In what follows, setting $\hat{\mu} = (\rho, \sigma, \tau)$ for shortness, we denote the vector field by $X_{\hat{\mu}}$. Let us also remark that the map $\mu \mapsto \hat{\mu}$ is smooth and injective as a consequence of the previous discussion.

At this point we claim that Q_μ is either a focus or a center. To show this we will check that the discriminant D_μ of the characteristic polynomial of the Jacobian matrix of X_μ at Q_μ is strictly negative for all admissible parameter. Indeed, one can verify that D_μ expressed in terms of (v_1, v_2, ε) can be written as

$$D_\mu = \frac{4(v_1 v_2)^4 \varepsilon^2 + A_1(v_1, v_2)\varepsilon + A_0(v_1, v_2)}{(v_1^2 + v_2^2)^2},$$

where A_i are polynomials of degree 7. Thus $D_\mu = 0$ gives two roots $\varepsilon = \hat{\varepsilon}_i(v_1, v_2)$, for $i = 1, 2$, that one can check to be well-defined continuous functions on $V := \{(v_1, v_2) \in \mathbb{R}^2 : v_1 > 0, v_2 > 0\}$. To see the claim we first prove that, for $i = 1, 2$,

$$(p+1)(q-1)\big|_{\varepsilon=\hat{\varepsilon}_i} > 0 \quad \text{for all } (v_1, v_2) \in V. \tag{3.7}$$

This implies that D_μ can not vanish at an admissible parameter due to the assumptions $p < -1$ and $q > 1$. In this regard we note that the product $(p+1)(q-1)$ expressed in terms of (v_1, v_2, ε) is given by

$$(p+1)(q-1) = \frac{(v_1 v_2)^2 \varepsilon^2 + B_1(v_1, v_2)\varepsilon + B_0(v_1, v_2)}{(v_1^2 + v_2^2)^2},$$

where $\deg(B_1) = 3$ and $\deg(B_0) = 2$. A computation shows that the *resultant* (see [4, Chapter 3] for instance) between the numerators of D_μ and $(p+1)(q-1)$ with respect to ε is a polynomial in v_1 and v_2 with all the coefficients being natural numbers. Consequently the resultant does not vanish on V and, accordingly, $(p+1)(q-1)|_{\varepsilon=\hat{\varepsilon}_i} \neq 0$ for all $(v_1, v_2) \in V$. Thus, since V is arc-connected and the function $(v_1, v_2) \mapsto (p+1)(q-1)|_{\varepsilon=\hat{\varepsilon}_i}$ is continuous on V, it suffices to verify (3.7) at some particular choice of parameter. For instance, taking $v_1 = v_2 = 1$ we obtain that $(p+1)(q-1)|_{\varepsilon=\hat{\varepsilon}_i} = 92$ for $i = 1, 2$. Therefore $D_\mu \neq 0$ at any admissible parameter. Thus, exactly as before, since $\mu \mapsto D_\mu$ is continuous and the set of admissible parameters is arc-connected, the claim will follow once we verify its validity at some particular parameter. For instance the choice $\mu = (0, -2, 2)$ yields $D_\mu = -\frac{67+25\sqrt{5}}{2} < 0$.

We proceed now with the study of the cyclicity of Q_μ. The fact that $\mathrm{Cycl}\big((Q_{\mu_0}, X_{\mu_0}), X_\mu\big) = 0$ when $\tau(\mu_0) \neq 0$ is well-known. On the other hand, if $p+q=0$ then $\varepsilon = 0$ and

$$\tau(v_1, v_2, 0) = (v_1 - v_2)(v_1 + 2 + v_2)(v_1 + v_2),$$

which vanishes at an admissible parameter if, and only if, $v_1 = v_2$. For this to happen, see (3.5) with $\varepsilon = 0$, it is necessary that $a = 0$. This proves the validity of (d).

We shall next solve the center-focus problem in the family. With this aim in view, taking a local transversal section at Q_μ we consider the displacement map $\mathscr{D}(s;\hat{\mu})$, which extends analytically to $s = 0$, so that we can compute its Taylor's expansion

$$\mathscr{D}(s;\hat{\mu}) = \eta_1(\hat{\mu})s + \eta_2(\hat{\mu})s^2 + \eta_3(\hat{\mu})s^3 + sR(s;\hat{\mu}),$$

where the remainder R is $o(s^2)$. Recall that the trace of $X_{\hat{\mu}}$ at $Q_{\hat{\mu}}$ is equal to $\tau u_1(\hat{\mu})$, where u_1 is a unity. The coefficients η_i are called the *Lyapunov quantities* of the focus. We have in particular (see for instance [19, p. 94]) that $\eta_1(\hat{\mu}) = e^{\tau u_1(\hat{\mu})} - 1 = \tau u_2(\hat{\mu})$, where u_2 is again a unity. Since the first nonzero coefficient of the expansion is the coefficient of an odd power of s, see [19, p. 94] again, we get that $\eta_2(\hat{\mu}) = \tau \ell_1(\hat{\mu})$ for some analytic function ℓ_1. In order to obtain η_3 we shall appeal to the well-known relation between the Lyapunov and focus quantities which, following the notation in [19, Theorem 6.2.3], we denote by g_{ii}. The first ones are the coefficients in the Taylor's expansion of the displacement map that we already introduced, while the second ones are the obstructions for the existence of a first integral. It occurs that $\eta_{2i+1} - \pi g_{ii} \in (g_{11}, \ldots, g_{i-1,i-1})$ and, more important for our purposes, that $\eta_3 = \pi g_{11}$. On account of this we can compute g_{11} instead of η_3, which is easier to obtain, and in doing so (see [3, p. 29]) we get that

$$\eta_3(\hat{\mu})\big|_{\tau=0} = \pi \frac{2\rho\,(\rho-\sigma)\,(\sigma+1)\,\big(4 + 34\sigma + 29\sigma^2 + 8\sigma^3 - 3\rho^2\big)}{3\,(\rho+\sigma)^3\,(\rho + 2\sigma + 2\sigma^2 + 2\rho^2 + 2)^2}.$$

In this respect we claim that $\eta_3(\hat{\mu})\big|_{\tau=0} = \rho h(\rho,\sigma)$ with $h(\rho,\sigma) \neq 0$ in case that $|\rho| < \sigma$, which corresponds to the admissible values $v_1, v_2 > 0$ due to $\rho = \frac{v_1 - v_2}{2}$ and $\sigma = \frac{v_1 + v_2}{2}$. Indeed, it is clear that the factor $(\rho - \sigma)(\sigma + 1)$ does not vanish inside the admissible set, while the other one does not vanish neither because

$$4 + 34\sigma + 29\sigma^2 + 8\sigma^3 - 3\rho^2 > 4 + 34\sigma + 26\sigma^2 + 8\sigma^3 > 0,$$

where the first inequality follows using that $|\rho| < \sigma$ and the second one the fact that $\sigma > 0$. Hence the claim is true. Therefore, if Q_μ is a center then $\tau = \rho = 0$, and the assertion in (c) shows that these two conditions are also sufficient because $\{p + q = a = 0\} = \{\tau = \rho = 0\}$. Observe moreover that we can write $\eta_3(\hat{\mu}) = \tau \ell_2(\hat{\mu}) + \rho h(\rho,\sigma)$ for some analytic function ℓ_2. On the other hand, due to $R(s;\hat{\mu})|_{\rho=\tau=0} \equiv 0$, we can also write $R = \tau R_1 + \rho R_2$ with $R_i \in o(s^2)$ and, accordingly,

$$\mathscr{D}(s;\mu) = \tau s\big(u_2 + \ell_1 s + \ell_2 s^2 + R_1\big) + \rho s\big(h s^2 + R_2\big). \tag{3.8}$$

Note that if $\tau(\mu) = 0$ and $\varepsilon = p + q \neq 0$ then $\rho = \frac{v_1 - v_2}{2}$ must be different from zero because otherwise, from (3.6), we would get that $\varepsilon = 0$. Consequently, due to $h(\rho,\sigma) \neq 0$ for all admissible ρ and σ, the equality in (3.8) implies $\mathrm{Cycl}\big((Q_{\mu_0}, X_{\mu_0}), X_\mu\big) \leqslant 1$ in case that $\tau(\mu_0) = 0$ and $p_0 + q_0 \neq 0$. The fact that this upper bound is attained follows by means of an easy perturbative argument using that $\partial_\varepsilon \tau(\mu) = -2v_1^2 v_2^2 \neq 0$. This proves the validity of the assertion in (e).

In order to prove (f) note that if $\tau(\mu)=0$ and $\varepsilon=p+q=0$ then, from (3.6), $\rho=\frac{v_1-v_2}{2}=0$. Hence, from (3.5), $2av_1v_2=0$, which implies $a=0$ and shows the first assertion. That being stablished, we have already proved that Q_{μ_0} is a center if, and only if $p_0+q_0=a_0=0$. We show next that, in this case, $\mathrm{Cycl}\big((Q_{\mu_0},X_{\mu_0}),X_\mu\big)\leqslant 1$. Indeed, since u_2 is a unity we can consider

$$\mathscr{D}_1(s;\hat{\mu}):=\frac{\mathscr{D}(s;\hat{\mu})}{s(u_2+\ell_1 s+\ell_2 s^2+R_1)}=\tau+\rho\frac{hs^2+R_2}{u_2+\ell_1 s+\ell_2 s^2+R_1}.$$

The upper bound for the cyclicity of Q_{μ_0} in the center case will follow once we prove that there exist $s_0>0$ and an open neighbourhood U of $(\rho,\sigma,\tau)=(0,\hat{\sigma},0)$ such that $\mathscr{D}_1(s;\hat{\mu})$ has at most one zero on $(0,s_0)$, counted with multiplicities, for all $\hat{\mu}\in U$ with $\rho^2+\tau^2\neq 0$. Recall in this regard that $\mathscr{D}_1(s;\hat{\mu})|_{\rho=\tau=0}\equiv 0$ and that, on account of (c), $\hat{\sigma}=-\frac{1+\sqrt{-3-4p_0}}{2(1+p_0)}>0$ is the first component of Q_{μ_0}. The idea to show this is exactly the same as in the proof of (c) but with less technicalities because the involved functions are analytic at $s=0$. The desired property is evident when $\rho=0$. In case that $\rho\neq 0$ we compute the derivative of $\mathscr{D}_1$ with respect to s to obtain

$$\partial_s\mathscr{D}_1(s;\hat{\mu})=\rho s\big(2h/u_2+\mathrm{o}(1)\big).$$

Since $h(\rho,\sigma)\neq 0$ in case that $|\rho|<\sigma$ and $u_2=u_2(\hat{\mu})$ is a unity, the existence of the desired $s_0>0$ and the open neighbourhood U follows by Rolle's theorem. So far we have proved that $\mathrm{Cycl}\big((Q_{\mu_0},X_{\mu_0}),X_\mu\big)\leqslant 1$ if $p_0+q_0=a_0=0$. The fact that this upper bound is attained follows noting that we can take $\mu=(a,p,q)$ with $\tau(\mu)=0$ and $p+q\neq 0$ arbitrarily close to $\mu_0=(0,p_0,-p_0)$ and apply then the assertion in (e). This proves (f).

Let us turn now to the proof of the last assertion in the statement. Observe in this respect that the combination of (a) and (b) together with (d) and (e) shows that a simultaneous bifurcation of limit cycles from Γ and Q_μ can only occur if we perturb some $\mu_\star=(a_\star,p_\star,q_\star)$ with $a_\star=p_\star+q_\star=0$. We shall prove by contradiction that this is neither possible. So assume that for each $n\in\mathbb{N}$ there exist $\mu_n=(a_n,p_n,q_n)$ and two limit cycles γ_n and γ_n' of the vector field X_{μ_n} in the first quadrant such that the Hausdorff distances $d_H(\gamma_n,\Gamma)$ and $d_H(\gamma_n',Q_{\mu_n})$ tend to zero and μ_n tends to $\mu_\star$ as $n\to+\infty$. Let us consider the asymptotic expansion of the displacement map of X_μ at the polycycle Γ that we compute in (3.3) and denote it by $\mathscr{D}_p(s;\mu)$. We also consider its Taylor's expansion near the focus Q_μ given in (3.8) and denote it by $\mathscr{D}_c(s';\mu)$. Then the assumption implies the existence of two sequences $s_n\to 0^+$ and $s_n'\to 0^+$ such that $\mathscr{D}_p(s_n;\mu_n)=0$ and $\mathscr{D}_c(s_n';\mu_n)=0$ for all $n\in\mathbb{N}$. We claim that the first equality implies that

$$\lim_{n\to+\infty}\frac{p_n+q_n}{a_n}=0. \tag{3.9}$$

Indeed, from (3.4) we have that

$$\mathscr{R}_1(s_n;\nu)\big|_{\nu=\Psi(\mu_n)}=b_1(\mu_n)\left(\omega(s_n;\alpha(\mu_n))+h_1(s_n;\nu)\right)+b_2(\mu_n)\big(1+h_2(s;\nu)\big)\big|_{\nu=\Psi(\mu_n)}=0$$

for all $n\in\mathbb{N}$. Thus, due to $\lim_{n\to+\infty}\omega(s_n;\alpha(\mu_n))=+\infty$ and $h_i\in\mathcal{F}_\ell^\infty(0_3)$, we obtain that $\lim_{n\to+\infty}\frac{b_2(\mu_n)}{b_1(\mu_n)}=-\infty$. Moreover, since $(b_1)=(d_1)$ and $(b_1,b_2)=(d_1,d_2)$ with ∇d_1 and ∇d_2 independent at $\mu=\mu_\star$, we can write

$$\frac{b_2}{b_1}=\frac{\kappa_1 d_1+\hat{\kappa}_2 d_2}{\hat{\kappa}_3 d_1}$$

with $\hat{\kappa}_i(\mu_\star)\neq 0$ and, consequently, $\lim_{n\to+\infty}\frac{d_2(\mu_n)}{d_1(\mu_n)}=\infty$. This, on account of (3.1) and (3.2), gives the limit in (3.9) and so the claim is true. Recall on the other hand that in order to study the displacement map near the focus Q_μ we use a more convenient parametrisation given by $\hat{\mu} := (\rho, \sigma, \tau) = \phi(\mu)$. That being said, setting $(\rho_n, \sigma_n, \tau_n) := \phi(a_n, p_n, q_n)$, similarly as we argue before, the fact that $\mathscr{D}_c(s_n'; \mu_n) = 0$ for all $n \in \mathbb{N}$ implies from (3.8) that

$$\lim_{n\to+\infty} \frac{\tau_n}{\rho_n} = 0. \tag{3.10}$$

Let us remark that here we also take into account that u_2 is a unity. We next arrive to contradiction showing that (3.9) and (3.10) cannot hold simultaneously. Indeed, one can verify that, setting $\sigma_\star = -\frac{1+\sqrt{-3-4p_\star}}{2(1+p_\star)}$,

$$\begin{aligned} \left.\frac{p_n+q_n}{a_n}\right|_{\mu_n=\phi^{-1}(\hat{\mu}_n)} &= 4\,\frac{\rho_n^2+\sigma_n^2}{\rho_n^2-\sigma_n^2}\,\frac{\tau_n+\rho_n\sigma_n(\sigma_n+2)}{2\tau_n-\rho_n(2\sigma_n+1)(\rho_n^2+\sigma_n^2)} \\ &= 4\,\frac{\rho_n^2+\sigma_n^2}{\rho_n^2-\sigma_n^2}\,\frac{\tau_n/\rho_n+\sigma_n(\sigma_n+2)}{2\tau_n/\rho_n-(2\sigma_n+1)(\rho_n^2+\sigma_n^2)} \to \frac{4(\sigma_\star+2)}{\sigma_\star(2\sigma_\star+1)} \neq 0 \text{ as } n\to+\infty. \end{aligned}$$

Here, in addition to (3.10), we use that if $p+q$ and a tend to zero then $\rho \to 0$ and $\sigma \to \sigma_\star$, where $\sigma_\star$ is precisely the first component of the center at $Q_{\mu_\star}$ (which is in the diagonal of the first quadrant). This shows that (3.9) and (3.10) cannot occur simultaneously, which yields to the desired contradiction and finishes the proof of the result. □

The following is our second example of application of Theorem A. In this case the family of Kolmogorov's systems is five-parametric and to the best of our knowledge it has not been studied previously.

Theorem 3.2. *Consider the family of Kolmogorov differential systems*

$$X_\mu \quad \begin{cases} \dot{x} = x(c + x^2 + axy - (p+1)y^2), \\ \dot{y} = y(-1 + (q+1)x^2 + (a-b)xy - y^2), \end{cases}$$

where $\mu = (a,b,c,p,q) \in \mathbb{R}^5$ with $c > 0$, $p > 0$, $q > 0$ and $b < 2\sqrt{pq}$ and let us fix any $\mu_0 = (a_0, b_0, c_0, p_0, q_0)$. Then there exists a unique singular point Q_μ in the first quadrant, which is either a center, a focus or a node. Moreover, compactifying X_μ to the Poincaré disc, the boundary of the first quadrant is a polycycle Γ such that:

(*a*) $\text{Cycl}\big((\Gamma, X_{\mu_0}), X_\mu\big) = 0$ *if* $p_0 - c_0q_0 \neq 0$.

(*b*) $\text{Cycl}\big((\Gamma, X_{\mu_0}), X_\mu\big) = 1$ *if* $p_0 - c_0q_0 = 0$ *and* $2c_0q_0a_0 - (c_0q_0 - c_0 + 1)b_0 \neq 0$.

(*c*) *The return map of X_{μ_0} along Γ is the identity if, and only if, $p_0 - c_0q_0 = 2c_0q_0a_0 - (c_0q_0 - c_0 + 1)b_0 = 0$. In this case Q_{μ_0} is a center with first integral*

$$H(x,y) = \frac{q_0(x^2 + c_0(y^2+1)) - b_0xy}{(xy^{c_0})^{\frac{2}{c_0q_0+c_0+1}}},$$

which foliates the first quadrant. Moreover Γ is the outer boundary of its period annulus and, in addition, $\text{Cycl}\big((\Gamma, X_{\mu_0}), X_\mu\big) = 1$.

Remark 3.3. In contrast to the family of Kolmogorov's cubic systems studied in Theorem 3.1, for the family in Theorem 3.2 there exist parameters μ_0 with $d_1(\mu_0) = 0$ and $d_2(\mu_0) \neq 0$, so that $\text{Cycl}\big((\Gamma, X_{\mu_0}), X_\mu\big) = 1$, and satisfying additionally that the unique singular point Q_{μ_0} in the first quadrant is a non-degenerate node. Hence, for appropriate $\mu \approx \mu_0$ we will have a limit cycle γ_μ with a non-monodromic singular point Q_μ as unique singularity in its interior. For instance, the choice $\mu_0 = (-800.01, -900.99999, 1000, 1, 0.001)$ leads to this phenomenon with $Q_{\mu_0} = (0.1, 10)$. A similar occurrence is observed in [2, p. 203] to take place in the family of cubic Liénard systems studied in [7]. □

Proof of Theorem 3.2. In this case, following the notation in (1.1), we have that

$$f(x,y;\mu) := c + x^2 + axy - (p+1)y^2 \quad \text{and} \quad g(x,y;\mu) := -1 + (q+1)x^2 + (a-b)xy - y^2.$$

Since $f(z,0;\mu) = c + z^2$, $g(0,z;\mu) = -1 - z^2$ and $(f_2 - g_2)(1,z;\mu) = -q + bz - pz^2$, one can check that the hypothesis **H1** and **H2** are satisfied for the admissible parameters, i.e., $c > 0$, $p > 0$, $q > 0$ and $b < 2\sqrt{pq}$. Moreover the hyperbolicity ratios are

$$\lambda_1 = 1/q, \qquad \lambda_2 = p \quad \text{and} \quad \lambda_3 = 1/c. \tag{3.11}$$

Then Γ is a polycycle and by applying the Poincaré–Bendixson theorem we deduce the existence of at least one singular point of X_μ in the first quadrant. We claim that there exists exactly one. In order to show this we suppose that (v_1, v_2) is a singular point of X_μ in the first quadrant and solve $f(v_1, v_2; \mu) = 0$ and $g(v_1, v_2; \mu) = 0$ for a and b as a function of c, p, q, v_1 and v_2. In doing so we obtain that

$$a = \frac{pv_2^2 - v_1^2 + v_2^2 - c}{v_1 v_2} \quad \text{and} \quad b = \frac{pv_2^2 + qv_1^2 - c - 1}{v_1 v_2}.$$

The substitution of these values in $f + cg$, which is homogeneous of degree 2 in x and y, yields

$$(f + cg)(x,y;\mu)\big|_{y=rx} = \frac{x^2(v_2 - rv_1)\big(rv_2(c+p+1) + v_1(cq+c+1)\big)}{v_1 v_2}.$$

It is clear then that the vanishing of the above numerator provides the possible values of r such that X_μ has a singular point at the straight-line $y = rx$, namely,

$$r_1 = \frac{v_2}{v_1} > 0 \quad \text{and} \quad r_2 = -\frac{v_1(cq+c+1)}{v_2(c+p+1)} < 0.$$

Since $x \mapsto f(x, r_1 x) = c + x^2(1 + ar_1 - (p+1)r_1^2)$ vanishes at $x = v_1 > 0$, it must have another real zero, which has to be negative. Therefore X_μ has exactly one singular point in the first quadrant and exactly one singular point in the third quadrant, showing in particular the validity of the claim. An easy computation shows that the determinant of the Jacobian of X_μ at $Q_\mu = (v_1, v_2)$ is equal to $2v_1^2(cq+c+1) + 2v_2^2(p+c+1) > 0$, so that it can be a center, a focus or a node.

So far we have proved that the first assertion in the statement is true. Let us turn to the proof of the assertions in (a), (b) and (c). The first one follows from (a) in Theorem A because

$$d_1(\mu) = 1 - \lambda_1\lambda_2\lambda_3 = \frac{cq - p}{cq}. \tag{3.12}$$

The second assertion will follow by applying (b) and (c) in Theorem A. In this regard let us recall that

$$d_2(\mu) = \lambda_2 \log\left(\frac{L_{12}}{L_{21}}\right)(1) + \log\left(\frac{L_{22}}{L_{32}}\right)(1) + \lambda_1\lambda_2 \log\left(\frac{L_{31}}{L_{11}}\right)(1). \tag{3.13}$$

On account of the definition of each L_{ij}, see (1.2), we easily obtain that

$$\begin{aligned} \log L_{11}(1) &= \frac{1+c(q+1)}{2c}\log(c+1), \\ \log L_{31}(1) &= \frac{1+c(q+1)}{2c}\log\left(1/c+1\right), \\ \log L_{22}(1) = \log L_{32}(1) &= \frac{\log 2}{2}(p+c+1). \end{aligned} \tag{3.14}$$

Moreover

$$\log L_{12}(1) = \frac{1}{q}\int_0^1 \frac{mz+n}{-pz^2+bz-q}dz \quad \text{and} \quad \log L_{21}(1) = \frac{1}{p}\int_0^1 \frac{mz+n'}{-qz^2+bz-p}dz,$$

where

$$m := -(pq+p+q), \qquad n := qa+b \quad \text{and} \quad n' := p(b-a)+b. \tag{3.15}$$

The explicit integration of these functions leads to several cases depending on the parameters. To avoid this we note that

$$\frac{mz+n}{-pz^2+bz-q} = -\frac{m}{2p}\frac{-2pz+b}{-pz^2+bz-q} + \frac{1}{2p}\frac{mb+2np}{-pz^2+bz-q},$$

so that

$$\log L_{12}(1) = -\frac{m}{2pq}\log\left(\frac{p+q-b}{q}\right) + \frac{mb+2np}{2pq}\int_0^1 \frac{dz}{-pz^2+bz-q}.$$

It is clear that the same formula holds for $\log L_{21}(1)$ replacing p, q and n by q, p and n', respectively. On account of this and the fact that, from (3.15), $mb+2n'q = -mb-2np$, we get

$$\log\left(\frac{L_{12}}{L_{21}}\right)(1) = \log L_{12}(1) - \log L_{21}(1) = -\frac{m}{2pq}\log\left(\frac{p}{q}\right) + (mb+2np)\Phi(\mu), \tag{3.16}$$

where

$$\Phi(\mu) := \frac{1}{2pq}\int_0^1 \left(\frac{1}{-pz^2+bz-q} + \frac{1}{-qz^2+bz-p}\right)dz.$$

Notice, and this is the key point in the forthcoming arguments, that Φ is a non-vanishing function because, thanks to property **H1**, $\Phi(\mu) < 0$ at any admissible parameter μ. On the other hand, from (3.14),

$$\log\left(\frac{L_{31}}{L_{11}}\right)(1) = -\frac{(1+c+cq)}{2c}\log c \quad \text{and} \quad \log\left(\frac{L_{22}}{L_{32}}\right)(1) = 0. \tag{3.17}$$

Accordingly the substitution of (3.16) and (3.17) in (3.13) yields

$$\begin{aligned} d_2(\mu) &= -\frac{m}{2q}\log\left(\frac{p}{q}\right) + p(mb+2np)\Phi(\mu) - \frac{p(1+c+cq)}{2qc}\log c \\ &= \frac{pq+p+q}{2q}\log\left(\frac{p}{q}\right) + p\big(2pqa - (pq-p+q)b\big)\Phi(\mu) - \frac{p(1+c+cq)}{2qc}\log c, \end{aligned}$$

where in the first equality we set the values of the hyperbolicity ratios given in (3.11) and in the second one the expressions of m and n defined in (3.15). Observe at this point, see (3.12), that $d_1(\mu) = 0$ if, and only if, $p = cq$. Moreover the two logarithmic summands in the above expression of $d_2(\mu)$ cancel each other after the substitution $p = cq$, so that

$$d_2(\mu)\big|_{p=cq} = cq^2\big(2cqa - (cq - c + 1)b\big)\Phi(a, b, c, cq, q).$$

Thus, by the Weierstrass Division Theorem (see [9, 12]), there exists an analytic function κ_1 such that

$$d_2(\mu) = d_1(\mu)\kappa_1(\mu) + \big(2cqa - (cq - c + 1)b\big)\kappa_2(\mu), \tag{3.18}$$

where $\kappa_2(\mu) := cq^2\Phi(\mu)$ is a unity in the admissible set. This expression shows that if we take an admissible parameter $\mu_0 = (a_0, b_0, c_0, p_0, q_0)$ such that $p_0 - c_0q_0 = 0$ and $2c_0q_0a_0 - (c_0q_0 - c_0 + 1)b_0 \neq 0$ then $d_2(\mu_0) \neq 0$, which by (c) in Theorem A implies that $\mathrm{Cycl}\big((\Gamma, X_{\mu_0}), X_\mu\big) \leqslant 1$. The fact that $\mathrm{Cycl}\big((\Gamma, X_{\mu_0}), X_\mu\big) = 1$ follows by applying (b) in Theorem A because $\nabla d_1(\mu_0)$ is not the zero vector, see (3.12). This proves the validity of the assertion in (b).

To show (c) we take any $\mu_0 = (a_0, b_0, c_0, p_0, q_0)$ satisfying $p_0 - c_0q_0 = 2c_0q_0a_0 - (c_0q_0 - c_0 + 1)b_0 = 0$. Then one can verify that the function

$$H(x, y) = \frac{q_0(x^2 + c_0(y^2 + 1)) - b_0xy}{(xy^{c_0})^{\frac{2}{c_0q_0+c_0+1}}}$$

is a first integral of X_{μ_0}, which is clearly analytic on the whole first quadrant. (For reader's convenience let us mention that we found this first integral looking for an integrating factor of the form x^ry^s with $r, s \in \mathbb{R}$.) Thus, since the determinant of the Jacobian of X_{μ_0} at Q_{μ_0} is strictly positive, we can assert that it is a center. A straightforward application of the Poincaré-Bendixson theorem shows that Γ is the outer boundary of its period annulus, which fills the first quadrant. This proves that if $p_0 - c_0q_0 = 2c_0q_0a_0 - (c_0q_0 - c_0 + 1)b_0 = 0$ then the displacement map $\mathscr{D}(\,\cdot\,; \mu_0)$ of X_{μ_0} along Γ is identically zero. The converse follows by Theorem 2.6 noting that $d_1(\mu) = d_2(\mu) = 0$ if, and only if, $p - cq = 2cqa - (cq - c + 1)b = 0$.

It only remains to be proved that $\mathrm{Cycl}\big((\Gamma, X_{\mu_0}), X_\mu\big) = 1$. This follows verbatim the proof of the same fact in assertion (c) of Theorem 3.1 and so we shall omit the details for the sake of shortness. Indeed, by (b) in Theorem 2.6 we have that if $\mu \approx \mu_0$ then

$$\mathscr{D}(s; \mu) = (b_1(\mu)\omega(s; \alpha(\mu)) + b_2(\mu) + r(s; \mu))s^c,$$

where $\alpha = c - p/q$ and $r \in \mathcal{F}^\infty_\ell(\mu_0)$ with $\ell \in (0, \min(1, p))$. Here we use that the three hyperbolicity ratios are $\lambda_1 = 1/q$, $\lambda_2 = p$ and $\lambda_3 = 1/c$. We also have that $(b_1) = (d_1)$ and $(b_1, b_2) = (d_1, d_2)$ over the local ring $\mathbb{R}\{\mu\}_{\mu_0}$. Therefore, if $\mu = (a, b, c, p, q)$ verifies $p - cq = 2cqa - (cq - c + 1)b = 0$ then $b_1(\mu) = b_2(\mu) = 0$ and $r(s; \mu) \equiv 0$. Moreover, since $\nabla d_1(\mu_0)$ and $\nabla d_2(\mu_0)$ are linearly independent, see (3.12) and (3.18), this is also the case of $\nabla b_1(\mu_0)$ and $\nabla b_2(\mu_0)$. We can thus take three linear functions, say $b_3(\mu)$, $b_4(\mu)$ and $b_5(\mu)$, such that $\nu = \Psi(\mu) := \big(b_1(\mu), b_2(\mu), b_3(\mu), b_4(\mu), b_5(\mu)\big)$ is a local analytic change of coordinates in a neighbourhood of $\mu = \mu_0$ with $\Psi(\mu_0) = 0_5 := (0, 0, 0, 0, 0)$. Notice then that Ψ maps $\{p - cq = 2cqa - (cq - c + 1)b = 0\}$ to $\{\nu_1 = \nu_2 = 0\}$ and, moreover,

$$\mathscr{R}_1(s; \nu) := s^{-c}\mathscr{D}(s; \mu)\Big|_{\mu=\Psi^{-1}(\nu)} = \nu_1\omega(s; \hat{\alpha}) + \nu_2 + \hat{r}(s; \nu),$$

where $\hat{\alpha} = \hat{\alpha}(\nu) := \alpha(\Psi^{-1}(\nu))$ and $\hat{r}(s; \nu) := r(s; \hat{\Psi}^{-1}(\nu)) \in \mathcal{F}^\infty_\ell(0_5)$. Due to $\hat{r}(s; 0, 0, \nu_3, \nu_4, \nu_5) \equiv 0$, by applying [17, Lemma 4.1] we can write the remainder as $\hat{r}(s; \nu) = \nu_1h_1(s; \nu) + \nu_2h_2(s; \nu)$

with $h_i \in \mathcal{F}_\ell^\infty(0_5)$ and, consequently,

$$\mathscr{R}_1(s;\nu) = \nu_1\big(\omega(s;\hat{\alpha}) + h_1(s;\nu)\big) + \nu_2\big(1 + h_2(s;\nu)\big).$$

From this expression we conclude that there exists $s_0 > 0$ and an open neighbourhood U of $\nu = 0_5$ such that $\mathscr{R}_1(s;\nu)$ that has at most one zero on $(0, s_0)$, counted with multiplicities, for all $\nu = (\nu_1, \nu_2, \nu_3, \nu_4, \nu_5) \in U$ with $\nu_1^2 + \nu_2^2 \neq 0$, which implies that $\mathrm{Cycl}\big((\Gamma, X_{\mu_0}), X_\mu\big) \leqslant 1$. The proof of this follows exactly as we argue to show the same fact in Theorem 3.1, cf. (3.4), and it is omitted for brevity. Finally the fact that $\mathrm{Cycl}\big((\Gamma, X_{\mu_0}), X_\mu\big) \geqslant 1$ follows taking $\mu_1 \approx \mu_0$ with $p_1 - c_1 q_1 = 0$ and $2c_1 q_1 a_1 - (c_1 q_1 - c_1 + 1)b_1 \neq 0$, and applying the assertion in (b). This completes the proof of the result. □

Remark 3.4. In order to prove Theorem 3.2 it is only necessary to compute the functions d_1 and d_2 in Theorem A, which give the conditions for cyclicity 0 and 1, respectively. Let us explain that, as a matter of fact, we computed the function d_3 as well, realizing that it vanishes when $d_1 = d_2 = 0$. It was this fact that lead us to investigate if the return map along the polycycle is the identity in that case. For completeness let us explain succinctly the computations that involve the obtention of d_3 for the Kolmogorov's family considered in Theorem 3.2. Recall, see (e) in Theorem A, that

$$d_3(\mu) := \hat{M}_3(\lambda_3, 1)L_{11}(1) - \hat{M}_1(1/\lambda_1, 1)L_{31}(1).$$

In this case, cf. (3.14), we have that

$$L_{11}(u) = (1 + cu^2)^{\frac{1+(q+1)c}{2c}} \quad \text{and} \quad L_{31}(u) = \big(1 + u^2/c\big)^{\frac{1+(q+1)c}{2c}}$$

and then, from the definition in (1.2),

$$M_1(u) = (1 + cu^2)^{\frac{1+(q-3)c}{2c}}\big(aq + b + (cb - (c+1)a)u^2\big)$$

and

$$M_3(u) = -u\,\big(1 + u^2/c\big)^{\frac{1+(q-3)c}{2c}}\,\big((aq + b)u^2 + cb - (c+1)a\big)/c^2.$$

In order to proceed with the computation of $\hat{M}_1(\frac{1}{\lambda_1}, 1)$ and $\hat{M}_3(\lambda_3, 1)$ we note that if $J(x;\eta, r) := (1 + \eta x^2)^r$ with $\eta > 0$ and $r \in \mathbb{R}$ then

$$\hat{J}(\alpha, 1;\eta, r) = \int_0^1 (1 + \eta x^2)^r x^{-\alpha-1} dx = -\frac{1}{\alpha}\,{}_2F_1(-r, -\alpha/2; 1 - \alpha/2; -\eta) \text{ for all } \alpha < 0,$$

where in the first equality we apply (b) in Proposition 2.5 with $k = 0$ and in the second one we use the equality in [1, 15.3.1] to express the definite integral as a hypergeometric function. In principle the above equality is only true provided that $\alpha < 0$. However, its validity can be extended to any $\alpha \notin \mathbb{N}$ thanks to the meromorphic properties of the functions ${}_2F_1$ and $\hat{J}$ established, respectively, by [17, Lemma B.2] and (d) in Proposition 2.5. Consequently, thanks to this observation and applying twice the above formula, we get

$$\hat{M}_1(1/\lambda_1, 1) = -\frac{aq + b}{q}\,\varphi_1(c, q) + \frac{cb - (c+1)a}{2 - q}\,\varphi_2(c, q),$$

where

$$\varphi_1(c, q) := {}_2F_1\big((3 - q)/2 - 1/(2c), -q/2; 1 - q/2; -c\big),$$

$$\varphi_2(c, q) := {}_2F_1\big((3 - q)/2 - 1/(2c), 1 - q/2; 2 - q/2; -c\big).$$

Here we also use that if $h = f + g$ then $\hat{h}_\alpha = \hat{f}_\alpha + \hat{g}_\alpha$ and that if $f(x) = x^n g(x)$ then $\hat{f}_\alpha(x) = x^n \hat{g}_{\alpha-n}(x)$, see [16, Corollary B3]. Similarly

$$\hat{M}_3(\lambda_3, 1) = \frac{aq+b}{c(1-3c)}\,\varphi_3(c,q) + \frac{cb-(c+1)a}{c(1-c)}\,\varphi_4(c,q),$$

where

$$\varphi_3(c,q) := {}_2F_1\big((3-q)/2 - 1/(2c), 3/2 - 1/(2c); 5/2 - 1/(2c); -1/c\big),$$
$$\varphi_4(c,q) := {}_2F_1\big((3-q)/2 - 1/(2c), 1/2 - 1/(2c); 3/2 - 1/(2c); -1/c\big).$$

In the proof of Theorem 3.2 we show that $d_1(\mu) = d_2(\mu) = 0$ if, and only if, $\mu = (a, b, c, p, q)$ verifies $p = cq$ and $a = \frac{b(1-c+cq)}{2cq}$. Long but easy computations show that, under these two conditions, $d_3(\mu) = 0$ if, and only if,

$$\frac{q}{1-3c}\varphi_3(c,q) - \varphi_4(c,q) + c^{-\frac{1+c+cq}{2c}}\left(\varphi_1(c,q) + \frac{c-1}{q-2}\varphi_2(c,q)\right) = 0.$$

This is an equation for b, c and q that involves four hypergeometric functions. Surprisingly enough it turns out, by applying the formula in [1, 15.3.7], that the function on the left hand side of the above equation is identically zero. In other words, $d_1(\mu_0) = d_2(\mu_0) = 0$ implies $d_3(\mu_0) = 0$. □

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On the existence of periodic solutions to second order Hamiltonian systems

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Abstract. In this paper, the existence of periodic solutions to the second order Hamiltonian systems is investigated. By introducing a new growth condition which generalizes the Ambrosetti–Rabinowitz condition, we prove a existence result of nontrivial T-periodic solution via the variational methods. Our result is new because it can deal with not only the superquadratic case, but also the anisotropic case which allows the potential to be superquadratic growth in only one direction and asymptotically quadratic growth in other directions.

Keywords: second order Hamiltonian systems, periodic solutions, existence, variational method.

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1 Introduction and main result

Consider the following second order Hamiltonian systems

$$\begin{cases} -\ddot{u}(t) + L(t)u(t) = \nabla_x F(t, u(t)), \quad \text{a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases} \tag{1.1}$$

where $u(t) = (u_1(t), u_2(t), \dots, u_N(t))$, $N \geq 1$, $T > 0$, $L(t) := \big(l_{ij}(t)\big) \in C(0, T; \mathbb{R}^{N\times N})$ is a symmetric positive matrix and T-periodic in t, $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ is T-periodic in t and satisfies the following assumptions:

(A) $F(t, x)$ is measurable in t for every $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0, T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1(0, T; \mathbb{R}^+)$ such that

$$|F(t, x)| \leq a(|x|)b(t), \qquad |\nabla_x F(t, x)| \leq a(|x|)b(t)$$

for $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$, where $x := (x_1, \dots, x_N)$, $\nabla_x F(t, x) := \big(\frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_N}\big)$.

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The periodic solutions to non-autonomous system (1.1) has an extensive history in the case of singular systems (cf., e.g., Ambrosetti–Coti Zelati [1]). The first to consider it for nonsingular potentials were Berger and Schechter [3] in 1977. Since then, the existence of periodic solutions to system (1.1) have been deeply studied by a large number of researchers. Many solvability conditions about the potentials have been obtained, we refer the readers to [4, 11, 12, 16, 17, 19–23, 26–28] and their references. In 1978, Rabinowitz [13] established the existence of a non-constant T-periodic solution when $L(t) \equiv 0$ by assuming that the potential F satisfies the following superquadratic condition

(AR) there exist constants $r_0 > 0$ and $\theta > 2$ such that

$$0 < \theta F(t,x) \leq (\nabla_x F(t,x), x)$$

for $|x| \geq r_0$ and a.e. $t \in [0,T]$, where $(\cdot,\cdot)$ is the inner product in $\mathbb{R}^N$.

This is the so-called Ambrosetti–Rabinowitz ((AR) for short) condition which plays a key role in verifying the mountain pass geometry and the compactness for the Euler–Lagrange functional associated to system (1.1). So (AR) condition has been widely used in follow-up research for the superquadratic problem, for example, see [5] and their references. If $F \in C^1([0,T] \times \mathbb{R}^N, \mathbb{R})$, one can easily deduce from (AR) that

$$F(t,x) \geq a|x|^\theta - b$$

for $x \in \mathbb{R}^{\mathbb{N}}$ and a.e. $t \in [0,T]$, where $a, b > 0$. This implies a more intrinsic superquadratic condition

(SQ) $\lim_{|x|\to\infty} \frac{F(t,x)}{|x|^2} = +\infty$ uniformly in a.e. $t \in [0,T]$.

Under condition (SQ), one can also add some other conditions on F to guarantee the existence of T-periodic solutions. For example, Fei [7] assumed the nonquadratic condition

(NQ) $\liminf_{|x|\to\infty} \frac{(\nabla_x F(t,x),x) - 2F(t,x)}{|x|^\beta} > 0$ uniformly in a.e. $t \in [0,T]$,

where $\beta > 1$. Luan–Mao [10] supposed that F satisfied the following condition

(LM) there exist $c > 0$, $r_1 > 0$ and some $\sigma > 1$ such that

$$\frac{|\nabla_x F(t,x)|^\sigma}{|x|^\sigma} \leq cH(x,s)$$

for $|x| \geq r_1$ and a.e. $t \in [0,T]$, where $H(x,s) := (\nabla_x F(t,x), x) - 2F(t,x)$.

Wu and Tang [24] introduced a new superquadratic situation

(WT) there exist $c > 0$, $r_2 > 0$ such that

$$\frac{F(t,x)}{|x|^2} \leq cH(x,s)$$

for $|x| \geq r_2$ and a.e. $t \in [0,T]$.

Ye–Tang [30] and Li–Schechter [9] studied the situation that F satisfied the following monotonic condition

(M) there exist $D \geq 1$ and $C_* \in L^1(0, T; \mathbb{R}^+)$ such that

$$H(t, sx) \leq DH(t, x) + C_*(t), \qquad \forall s \in [0, 1]$$

for $x \in \mathbb{R}^{\mathbb{N}}$ and a.e. $t \in [0, T]$.

Schechter [18] assumed

(S1) $2F(t, x) \geq \lambda_{l-1}|x|^2$ for $x \in \mathbb{R}^{\mathbb{N}}$ and a.e. $t \in [0, T]$, where λ_i is the *ith* eigenvalue of the operator $-\frac{d^2}{dt^2} + L(t)$,

(S2) there are constants $m > 0$ and $\vartheta > 0$ such that

$$2F(t, x) \leq \vartheta |x|^2$$

for $|x| \leq m$ and a.e. $t \in [0, T]$.

The readers are referred to [6,8,29] for more types of conditions under condition (SQ).

In addition, without condition (SQ), Schechter [14] assumed that

(S3) $F(t, x) \geq 0$ for $x \in \mathbb{R}^{\mathbb{N}}$ and a.e. $t \in [0, T]$,

(S4) there are constants $m > 0$, $\alpha \leq 6m^2/T^2$ such that

$$F(t, x) \leq \alpha$$

for $|x| \leq m$ and a.e. $t \in [0, T]$,

(S5) there are $\mu > 2$, $r_3 > 0$ and $W \in L^1([0, T])$ such that

$$\begin{cases} (i) \ \ \frac{H_\mu(t,x)}{|x|^2} \leq W(t) \text{ for } |x| \geq r_3 \text{ and a.e. } t \in [0, T], \\ (ii) \ \ \limsup_{|x| \to +\infty} \frac{H_\mu(t,x)}{|x|^2} \leq 0, \end{cases}$$

where $H_\mu(t, x) := \mu F(t, x) - (\nabla_x F(t, x), x)$,

(S6) there is a subset $\Sigma \subset [0, T]$ of positive measure such that

$$\liminf_{|x| \to +\infty} \frac{F(t, x)}{|x|^2} > 0 \quad \text{uniformly in a.e. } t \in \Sigma.$$

In [15], the potentials F satisfy (S3)–(S5) and

(S7) there are constants $\beta > \frac{2\pi^2}{T^2}$ and $r_3 > 0$ such that

$$F(t, x) \geq \beta |x|^2$$

for $|x| > r_3$ and a.e. $t \in [0, T]$.

Wang–Zhang [25] assumed F satisfies (S3), (S4), (S6) and

(WZ) (i) there exist $M_1 > 0$, $\sigma > 1$ and $f \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\lim_{|x|\to\infty} f(|x|) = +\infty$ and $\frac{f(|x|)}{|x|^\sigma}$ is non-increasing on $\mathbb{R}^+$ such that

$$(\nabla_x F(t,x), x) - 2F(t,x) \geq f(|x|)\frac{|\nabla F(t,x)|^\sigma}{|x|^\sigma}$$

for $|x| \geq M_1$ and a.e. $t \in [0,T]$, or
(ii) there exist $M_2 > 0$ and $g \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\lim_{|x|\to\infty} g(|x|) = +\infty$ and $\frac{g(|x|)}{|x|^2}$ is non-increasing on $\mathbb{R}^+$ such that

$$(\nabla_x F(t,x), x) - 2F(t,x) \geq g(|x|)\frac{F(t,x)}{|x|^2}$$

for $|x| \geq M_2$ and a.e. $t \in [0,T]$.

In [31], Zhang–Tang assumed

(ZT) there exist constants $\mu > 2$, $0 < \beta < 2$, $L > 0$ and a function $a \in L^1(0,T;\mathbb{R}^+)$ such that

$$\mu F(t,x) \leq (\nabla_x F(t,x)x) + a(t)|x|^\beta$$

for $|x| \leq L$ and a.e. $t \in [0,T]$.

In this paper, we will give a new solvable condition. Our main result is the following theorem.

Theorem 1.1. *Assume that F satisfies assumptions* (A) *and*

(F_1) $\lim_{|x|\to 0} \frac{F(t,x)}{|x|^2} = 0$ *uniformly in a.e.* $t \in [0,T]$,

(F_2) *there exist a constant $r_* > 0$ and a function θ such that*

$$0 < (2+\theta(x))F(t,x) \leq (\nabla_x F(t,x), x)$$

for $|x| \geq r_*$ and a.e. $t \in [0,T]$, where $\theta : \{x \in \mathbb{R}^N : |x| \geq r_*\} \to \mathbb{R}$ is continuous and satisfies the following assumption

$$(\bigstar)\begin{cases} (i) \quad \theta(x) > 0,\ \forall |x| \geq r_*, \\ (ii) \quad \lim\limits_{|x|\to+\infty} \theta(x)|x|^2 = +\infty, \\ (iii) \quad \text{there is } x^0 \in \mathbb{R}^N \text{ with } |x^0| = 1 \text{ satisfying } \lim\limits_{r\to+\infty}\displaystyle\int_{r_*}^{r} \frac{\theta(sx^0)}{s}ds = +\infty, \end{cases}$$

then system (1.1) *has a nontrivial periodic solution.*

Remark 1.2. **(1)** Condition (F_2) is strictly weaker than the (AR) condition. In fact, we can derive from condition (F_2) that $\inf_{|x|\geq r_*}\theta(x) \geq 0$, and the (AR) condition is exactly equivalent to condition (F_2) when $\inf_{|x|\geq r_*}\theta(x) > 0$. On the one hand, the (AR) condition implies condition (F_2) with $\theta(x) \equiv \theta - 2 > 0$ and $r_* = r_0$. On the other hand, when $\inf_{|x|\geq r_*}\theta(x) > 0$, condition ($F_2$) implies the (AR) condition with $\theta := 2 + \inf_{|x|\geq r_*}\theta(x) > 2$ and $r_0 = r_*$. In addition, there are functions F satisfying condition (F_2) with $\inf_{|x|\geq r_*}\theta(x) = 0$, for example,

(Superlinear case) let

$$F(t,x)=\begin{cases}|x|^2\ln|x|-\frac{1}{2}e^2-\frac{1}{16}, & |x|\geq e;\\ \frac{1}{2}|x|^2-\frac{1}{16}, & \frac{1}{2}\leq|x|\leq e;\\ |x|^4, & |x|\leq\frac{1}{2}.\end{cases}$$

Then we have

$$\nabla_x F(t,x)=\begin{cases}2x\ln|x|+x, & |x|\geq e;\\ x, & \frac{1}{2}\leq|x|\leq e;\\ 4|x|^2x, & |x|\leq\frac{1}{2}\end{cases}$$

and

$$(\nabla_x F(t,x),x)-2F(t,x)=\begin{cases}|x|^2+e^2+\frac{1}{8}, & |x|\geq e;\\ \frac{1}{16}, & \frac{1}{2}\leq|x|\leq e;\\ 2|x|^4, & |x|\leq\frac{1}{2}.\end{cases}$$

It is easy to verify that F satisfies assumptions (A), (F_1), (F_2) with $\theta(x)=\frac{1}{\ln|x|}$ and $r_*=e$. However, $\inf_{|x|\geq e}\frac{1}{\ln|x|}=0$, so condition (AR) is not satisfied.

(2) It is particularly noteworthy that our Theorem 1.1 can deal with the potentials F without condition (SQ). In fact, there are functions with anisotropic growth satisfying condition (F_2), for example,

(Anisotropic case) let $N=2$, $x:=(x_1,x_2)\in\mathbb{R}^2$, and

$$F(t,x)=\begin{cases}x_1^4+\frac{5}{6}x_2^4, & x_2\leq 1;\\ x_1^4+x_2^2\cdot e^{-x_2^{-\frac{4}{3}}+1}-\frac{1}{6}, & x_2\geq 1.\end{cases}$$

Through simple calculation, we have

$$\nabla_x F(t,x):=\left(\frac{\partial F}{\partial x_1},\frac{\partial F}{\partial x_2}\right)=\begin{cases}4x_1^3+\frac{10}{3}x_2^3, & x_2\leq 1;\\ 4x_1^3+\left(2x_2+\frac{4}{3}x_2^{-\frac{1}{3}}\right)\cdot e^{-x_2^{-\frac{4}{3}}+1}, & x_2\geq 1\end{cases}$$

and

$$(\nabla_x F(t,x),x)-2F(t,x)=\begin{cases}2x_1^4+\frac{5}{3}x_2^4, & x_2\leq 1;\\ 2x_1^4+\frac{4}{3}x_2^{\frac{2}{3}}e^{-x_2^{-\frac{4}{3}}+1}+\frac{1}{3}, & x_2\geq 1.\end{cases}$$

Let

$$\theta(x)=\begin{cases}2, & x_2\leq 1;\\ \frac{2x_1^4+\frac{4}{3}x_2^{\frac{2}{3}}e^{-x_2^{-\frac{4}{3}}+1}+\frac{1}{3}}{x_1^4+x_2^2\cdot e^{-x_2^{-\frac{4}{3}}+1}-\frac{1}{6}}, & x_2\geq 1,\end{cases}$$

then for $|x|=\sqrt{x_1^2+x_2^2}$, $r_*=\sqrt{2}$ and $\mathbf{e}_1=(1,0)\in\mathbb{R}^2$, we can deduce that $\theta(x)>0$ for $|x|\geq\sqrt{2}$, $\lim_{|x|\to\infty}\theta(x)|x|^2=+\infty$, and

$$\lim_{r\to+\infty}\int_{\sqrt{2}}^{r}\frac{\theta(s\mathbf{e}_1)}{s}ds=\lim_{r\to+\infty}\int_{\sqrt{2}}^{r}\frac{2}{s}ds=+\infty,$$

which implies that F is superquadratic growth in direction $\mathbf{e}_1$. In addition, it is easy to verify that F satisfies assumptions (A), (F_1), (F_2). However, for $\mathbf{e}_2 = (0,1) \in \mathbb{R}^2$, we have

$$\lim_{x=s\mathbf{e}_2,\ |x|\to+\infty} \frac{F(t,x)}{|x|^2} = \lim_{|s|\to+\infty} \frac{F(t,s\mathbf{e}_2)}{|s|^2} = e,$$

which shows that F is asymptotically quadratic growth in direction $\mathbf{e}_2$. In conclusion, F satisfy condition (F_2) but not condition (SQ).

(3) Our Theorem 1.1 is different from all the results mentioned above. Firstly, condition (F_2) is strictly weaker than the (AR) condition. Secondly, we do not need the condition (SQ). More precisely, Theorem 1.1 can deal with not only the superquadratic case but also the the anisotropic case. Thirdly, we do not need more stringent and complex growth assumptions on F at 0.

2 Proof of the theorem

Let

$$H_T^1 = \left\{ u \in L^2(0,T;\mathbb{R}^N) \mid u \text{ is weakly differentiable and } \dot{u} \in L^2(0,T;\mathbb{R}^N) \right\}$$

be a Hilbert space with the inner product and the induced norm respectively given by

$$\langle u,v\rangle_{H_T^1} = \int_0^T (\dot{u},\dot{v}) + (u(t),v(t))dt, \qquad \|u\|_{H_T^1} = \left(\int_0^T |\dot{u}(t)|^2 + |u(t)|^2 dt \right)^{\frac{1}{2}}.$$

Denoting by $\lambda_{\min}(t)$ and $\lambda_{\max}(t)$ respectively the smallest and the biggest eigenvalue of $L(t)$, then $\lambda_{\min}(t)$, $\lambda_{\max}(t) \in C(0,T;\mathbb{R}^+)$. Setting

$$\underline{\lambda} := \min_{t\in[0,T]} \lambda_{\min}(t), \qquad \overline{\lambda} := \max_{t\in[0,T]} \lambda_{\max}(t),$$

we have $0 < \underline{\lambda} \le \overline{\lambda}$ and

$$\underline{\lambda}|\xi|^2 \le \big(L(t)\xi,\xi\big) \le \overline{\lambda}|\xi|^2$$

for $\xi \in \mathbb{R}^N$ and $t \in [0,T]$. Thus, the following inner product and the corresponding induced norm on H_T^1 defined by

$$\langle u,v\rangle = \int_0^T (\dot{u},\dot{v}) + \big(L(t)u(t),v(t)\big)dt, \qquad \|u\| = \left(\int_0^T |\dot{u}(t)|^2 + \big(L(t)u(t),u(t)\big)dt \right)^{\frac{1}{2}}$$

are respectively equivalent to $\langle u,v\rangle_{H_T^1}$ and $\|u\|_{H_T^1}$. In fact, it is easy to verify that

$$\sqrt{\min\{1,\underline{\lambda}\}}\|u\|_{H_T^1} \le \|u\| \le \sqrt{\max\{1,\overline{\lambda}\}}\|u\|_{H_T^1}$$

for $u \in H_T^1$. By Sobolev's inequality, there is $M > 0$ such that

$$\|u\|_\infty \le M\|u\|, \qquad \forall u \in H_T^1,$$

where $\|u\|_\infty := \max_{t\in[0,T]}|u(t)|$. In addition, from the assumption (A) it follows that the functional Φ given by

$$\Phi(u) = \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt + \frac{1}{2}\int_0^T \big(L(t)u(t),u(t)\big)dt - \int_0^T F(t,u(t))dt$$

is continuously differentiable on H^1_T, and

$$\langle \Phi'(u), v\rangle = \int_0^T \left[\big(\dot{u}(t), \dot{v}(t)\big) + \big(L(t)u(t), v(t)\big) - \big(\nabla_x F(t, u(t)), v(t)\big)\right] dt.$$

Furthermore, the weak solutions to system (1.1) are exactly the critical points of Φ in H^1_T.

Lemma 2.1. *Assume that $\theta : \{x \in \mathbb{R}^N : |x| \geq r_*\} \to \mathbb{R}$ is continuous and satisfies condition $(\bigstar)(i)$, and suppose that there is a sequence $\{y_n\} \subset \{x \in \mathbb{R}^N : |x| \geq r_*\}$ such that*

$$\theta(y_n) \to 0 \quad \text{as } n \to \infty,$$

then $|y_n| \to +\infty$ as $n \to \infty$.

Proof. By negation, there exists a subsequence, still denoted by $\{y_n\}$, is bounded. After passing to a subsequence, we may assume that there is $y_0 \in \{x \in \mathbb{R}^N : |x| \geq r_*\}$ such that

$$y_n \to y_0 \quad \text{as } n \to \infty,$$

from this, $(\bigstar)(i)$ and the continuity of θ it follows that

$$0 = \lim_{n\to\infty} \theta(y_n) = \theta(y_0) > 0,$$

a contradiction. The proof of Lemma 2.1 is completed. □

Lemma 2.2. *Assume that F satisfies (F_1), then there are $\rho > 0$ and $\alpha > 0$ such that $\Phi(u) \geq \alpha$ for $u \in H^1_T$ with $\|u\| = \rho$.*

Proof. From (F_1), for $\varepsilon \in \big(0, \frac{1}{4TM^2}\big)$, there exists a constant $\delta > 0$ such that

$$|F(t, x)| \leq \varepsilon |x|^2$$

for $|x| < \delta$ and a.e. $t \in [0, T]$. Arbitrarily taking $\rho \in \big(0, \frac{\delta}{M}\big)$, we have

$$\|u\|_\infty \leq M\|u\| \leq M\rho < \delta$$

for $u \in H^1_T$ with $\|u\| = \rho$, this leads to

$$\Phi(u) \geq \frac{1}{2}\|u\|^2 - \varepsilon \int_0^T |u(t)|^2 dt \geq \left(\frac{1}{2} - \varepsilon M^2 T\right)\|u\|^2 \geq \frac{\rho^2}{4}$$

for $u \in H^1_T$ with $\|u\| = \rho$. Setting $\alpha := \frac{\rho^2}{4} > 0$, then the proof Lemma 2.2 is completed. □

Lemma 2.3. *Assume that F satisfies assumptions (A) and (F_2), then there is $u_0 \in H^1_T$ with $\|u_0\| > \rho$ such that $\Phi(u_0) < 0$.*

Proof. From assumptions (A) and (F_2) it follows that

$$F(t, sx^0) \geq \frac{F(t, r_* x^0)}{r_*^2} \cdot e^{\int_{r_*}^s \frac{\theta(\tau x^0)}{\tau} d\tau} \cdot s^2$$

for $s \geq r_*$ and a.e. $t \in [0, T]$, then we have

$$\Phi(sx^0) = \frac{1}{2}\|sx^0\|^2 - \int_0^T F(t, sx^0) dt \leq \left(\frac{\overline{\lambda}^2}{2} - \int_0^T \frac{F(t, r_* x^0)}{r_*^2} dt \cdot e^{\int_{r_*}^s \frac{\theta(\tau x^0)}{\tau} d\tau}\right) s^2$$

for $s \geq r_*$ and a.e. $t \in [0, T]$, which implies $\Phi(u_0) < 0$ with $u_0 = sx^0$ for large s. This completes the proof of Lemma 2.3 □

Lemma 2.4. *Assume that F satisfies assumptions* (A), (F_1) *and* (F_2), *then* Φ *satisfies the* (C) *condition, that is, for any $c \in \mathbb{R}$ and every sequence $\{u_n\}$ such that*

$$\|\Phi'(u_n)\|(1+\|u_n\|) \to 0 \quad \text{and} \quad \Phi(u_n) \to c \quad \text{as } n \to \infty \tag{2.1}$$

has a convergent subsequence.

Proof. It suffices to prove that $\{u_n\}$ is bounded. Moreover, the proof is trivial when $\inf_{|x|\geq r_*}\theta(x) > 0$, so we just need to prove this lemma when $\inf_{|x|\geq r_*}\theta(x) = 0$.

We argue by contradiction. If $\{u_n\}$ is unbounded, then after passing to a subsequence, we may assume that

$$\lambda_n := \|u_n\| \to +\infty \quad \text{as } n \to \infty. \tag{2.2}$$

Setting $w_n = \frac{u_n}{\|u_n\|}$, then $\|w_n\| = 1$ and $u_n = \lambda_n w_n$. Thus, we deduce

$$\|w_n\|_\infty \leq M\|w_n\| \leq M.$$

Fixing $x_{\lambda_n} \in \left\{x \in \mathbb{R}^N : r_* \leq |x| \leq \lambda_n M\right\}$ to be such that

$$\theta(x_{\lambda_n}) = \min_{r_* \leq |x| \leq \lambda_n M} \theta(x), \tag{2.3}$$

then we have $\lambda_n \geq \frac{|x_{\lambda_n}|}{M}$, $0 < \theta(x_{\lambda_n}) \leq \theta^* := \min_{|x|=r_*}\theta(x)$,

$$0 < (2+\theta(x_{\lambda_n}))F(t,x) \leq (2+\theta(x))F(t,x) \leq (\nabla_x F(t,x), x) \tag{2.4}$$

for $r_* \leq |x| \leq \lambda_n M$ and a.e. $t \in [0,T]$. Moreover, from (2.2), (2.3) and $\inf_{|x|\geq r_*}\theta(x) = 0$ it follows that

$$\theta(x_{\lambda_n}) \to 0 \quad \text{as } n \to \infty.$$

Thus, from Lemma 2.1, we obtain

$$|x_{\lambda_n}| \to +\infty \quad \text{as } n \to \infty. \tag{2.5}$$

Setting

$$E_n^- = \{t \in [0,T] : |u_n(t)| < r_*\}, \qquad E_n^+ = \{t \in [0,T] : |u_n(t)| \geq r_*\},$$

then from (2.1) it follows that

$$\begin{aligned} o(1) &= \left|\langle \Phi'(u_n), u_n\rangle\right| \\ &= \left|\lambda_n^2 - \int_{E_n^-}(u_n(t), \nabla_x F(t,u_n(t))dt - \int_{E_n^+}(u_n(t), \nabla_x F(t,u_n(t))dt\right|, \end{aligned}$$

where $o(1) \to 0$ as $n \to \infty$, this implies that

$$\int_{E_n^+}(u_n(t), \nabla_x F(t,u_n(t))dt \leq \lambda_n^2 + \left|\int_{E_n^-}(u_n(t), \nabla_x F(t,u_n(t))dt\right| + o(1). \tag{2.6}$$

In addition, it follows from assumption (A) that there is $\beta > 0$ such that

$$\left|\int_{E_n^-}(u_n(t), \nabla_x F(t,u_n(t))dt\right|, \quad \left|\int_{E_n^-} F(t,u_n(t))dt\right| \leq \beta,$$

which together with (2.1), (2.4) and (2.6) gives

$$\begin{aligned} c + o(1) &= \Phi(u_n) \\ &= \frac{\lambda_n^2}{2} - \int_{E_n^-} F(t, u_n(t))dt - \int_{E_n^+} F(t, u_n(t))dt \\ &\geq \frac{\lambda_n^2}{2} - \beta - \frac{1}{2+\theta(x_{\lambda_n})} \int_{E_n^+} (u_n(t), \nabla_x F(t, u_n(t))dt \\ &\geq \frac{\lambda_n^2}{2} - \beta - \frac{1}{2+\theta(x_{\lambda_n})} \left(\lambda_n^2 + \left| \int_{E_n^-} (u_n(t), \nabla_x F(t, u_n(t))dt \right| + o(1) \right) \\ &\geq \frac{\theta(x_{\lambda_n})\lambda_n^2}{2(2+\theta(x_{\lambda_n}))} - \beta - \frac{\beta + o(1)}{2+\theta(x_{\lambda_n})} \\ &\geq \frac{\theta(x_{\lambda_n})|x_{\lambda_n}|^2}{2(2+\theta^*)M^2} - \frac{3\beta + o(1)}{2}, \end{aligned}$$

which is in contradiction with (2.5) and the assumption $\theta(x)|x|^2 \to +\infty$ as $|x| \to \infty$. Hence, $\{u_n\}$ is bounded, the proof of Lemma 2.4 is completed. □

Now, we give the proof of Theorem 1.1.

Proof of Theorem 1.1. By Lemmas 2.2–2.4, we obtain a nontrivial solution to system (1.1) via the Mountain Pass Theorem under the (C) condition which the readers can refer to [2]. □

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On a solvable class of nonlinear difference equations of fourth order

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Abstract. We consider a class of nonlinear difference equations of the fourth order, which extends some equations in the literature. It is shown that the class of equations is solvable in closed form explaining theoretically, among other things, solvability of some previously considered very special cases. We also present some applications of the main theorem through two examples, which show that some results in the literature are not correct.

Keywords: nonlinear difference equation, solvable difference equation, general solution, closed-form formula for solutions.

2020 Mathematics Subject Classification: 39A20.

1 Introduction

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ stand for the sets of natural, integer and real numbers respectively, and let $\mathbb{N}_k = \{n \in \mathbb{Z} : n \geq k\}$ where $k \in \mathbb{Z}$. If $l \in \mathbb{Z}$, then, as usual, we regard that $\prod_{j=l}^{l-1} c_j = 1$.

To obtain some information on solutions of difference equations and systems of difference equations scientists first tried to find some closed form formulas for their solutions. The first important results can be found, for example, in [7,10,11,17,18], as well as in the books [15,16] where many results up to the end of the eighteenth century can be found.

The linear homogeneous second order difference equation with constant coefficients

$$x_{n+2} + ax_{n+1} + bx_n = 0, \quad n \in \mathbb{N}_0, \tag{1.1}$$

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where $a \in \mathbb{R}$ and $b \in \mathbb{R} \setminus \{0\}$, was solved by de Moivre [10].

If the coefficients a and b satisfy the condition $a^2 \neq 4b$, then the general solution to equation (1.1) is given by the formula

$$x_n = \frac{(x_1 - \lambda_2 x_0)\lambda_1^n - (x_1 - \lambda_1 x_0)\lambda_2^n}{\lambda_1 - \lambda_2}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

where

$$\lambda_1 = \frac{-a + \sqrt{a^2 - 4b}}{2} \quad \text{and} \quad \lambda_2 = \frac{-a - \sqrt{a^2 - 4b}}{2}$$

are the roots of the polynomial equation $p_2(\lambda) := \lambda^2 + a\lambda + b = 0$ (see [10, p.84]).

If $a^2 = 4b$, then the polynomial has two equal roots

$$\lambda_1 = \lambda_2 = -\frac{a}{2},$$

and the general solution to equation (1.1) in this case is given by the closed-form formula

$$x_n = ((x_1 - \lambda_1 x_0)n + \lambda_1 x_0)\lambda_1^{n-1}, \quad n \in \mathbb{N}_0. \tag{1.3}$$

See [7], where, among other things, the method for finding solutions to linear homogeneous difference equations with constant coefficients of arbitrary order in the form

$$x_n = \lambda^n, \quad n \in \mathbb{N}_0,$$

was described.

Closed-form formulas for solutions to linear homogeneous difference equations with constant coefficients of the third order were presented by Euler in [11]. For some later presentations of results in the topic see, for example, the books [8,13,19,20,22].

Beside solvability of difference equations and systems of difference equations, some recent investigations in the topic include also finding invariants of the equations and systems. For some recent results in the topics, as well as their applications see, for example, [4–6,12,23–37], as well as many related references cited therein.

One of the difference equations which, by using some changes of variables, reduces to equation (1.1) is

$$x_{n+1} = \frac{ax_n + b}{cx_n + d}, \quad n \in \mathbb{N}_0, \tag{1.4}$$

the bilinear/fractional linear difference equation. Equation (1.4) and some of related systems of difference equations have been investigated since the time of Laplace and frequently appear in the literature (see, for example, [1,2,6,8,9,14–16,19,21,22,31,32,34,35,37]).

Many other classes of difference equations can be reduced to linear difference equations with constant coefficients. It is of some interest to find such classes, as well as some which reduces to equation (1.4). By using some changes of variables it is easy to form many such classes.

There have been some investigations on solvability and behaviour of solutions to the difference equation

$$x_{n+1} = ax_n + \frac{bx_n x_{n-3}}{cx_{n-2} + dx_{n-3}}, \quad n \in \mathbb{N}_0, \tag{1.5}$$

where $a, b, c, d \in \mathbb{R}$.

Here we show that a more general class of difference equations can be solved in closed form, extending some of the results on equation (1.5) in the literature. We use some methods and ideas related to the ones, e.g., in [12,31,32,34,37]. By using obtained closed-form formulas for solutions to equation (1.5), we present some applications of our main theorem by giving two examples which show that some results in [3] are not correct.

2 Main results

This section presents the main result in the paper. It shows the solvability of a generalization of equation (1.5), by finding closed form formulas for their solutions.

Theorem 2.1. *Assume $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, $\alpha^2 + \beta^2 \neq 0 \neq \gamma^2 + \delta^2$, g is a strictly monotone and continuous function, $g(\mathbb{R}) = \mathbb{R}$ and $g(0) = 0$. Then, the equation*

$$x_{n+1} = g^{-1}\left(g(x_n)\frac{\alpha g(x_{n-2}) + \beta g(x_{n-3})}{\gamma g(x_{n-2}) + \delta g(x_{n-3})}\right), \quad n \in \mathbb{N}_0, \tag{2.1}$$

is solvable in closed form.

Proof. By a well known theorem in real analysis we see that the conditions posed on function g imply the existence of the inverse function g^{-1} which satisfies the same conditions as the function g ([38]).

Assume that $x_{n_0} = 0$ for some $n_0 \in \mathbb{N}_0$. Then from (2.1) we have $x_{n_0+1} = 0$. These facts along with (2.1) imply that x_{n_0+4} is not defined. Thus, of interest are the solutions of equation (2.1) such that $x_n \neq 0$, $n \in \mathbb{N}_0$. We may also assume that $x_{-j} \neq 0$, $j = \overline{1,3}$, otherwise the equation can be considered only on the domain $\mathbb{N}_0$. Hence, we suppose

$$x_n \neq 0, \quad \text{for } n \in \mathbb{N}_{-3}. \tag{2.2}$$

From (2.2) and the conditions of the theorem we have

$$g(x_n) \neq 0, \quad \text{for } n \in \mathbb{N}_{-3}. \tag{2.3}$$

First, assume $\alpha\delta \neq \beta\gamma$ and $\gamma \neq 0$. Let

$$y_n = \frac{g(x_n)}{g(x_{n-1})}, \quad n \in \mathbb{N}_{-2}. \tag{2.4}$$

From (2.1) and monotonicity of g, we have

$$g(x_{n+1}) = g(x_n)\frac{\alpha g(x_{n-2}) + \beta g(x_{n-3})}{\gamma g(x_{n-2}) + \delta g(x_{n-3})}, \quad n \in \mathbb{N}_0. \tag{2.5}$$

Employing the change of variables (2.4) in (2.5) we have

$$y_{n+1} = \frac{\alpha y_{n-2} + \beta}{\gamma y_{n-2} + \delta}, \quad n \in \mathbb{N}_0. \tag{2.6}$$

Let

$$z_m^{(j)} = y_{3m-j}, \quad m \in \mathbb{N}_0,\ j = \overline{0,2}. \tag{2.7}$$

Then, from (2.6) and (2.7) we have

$$z_{m+1}^{(j)} = \frac{\alpha z_m^{(j)} + \beta}{\gamma z_m^{(j)} + \delta}, \tag{2.8}$$

for $m \in \mathbb{N}_0$, $j = \overline{0,2}$, which is a bilinear difference equation.

Let

$$z_m^{(j)} = \frac{u_{m+1}^{(j)}}{u_m^{(j)}} + f_j, \quad m \in \mathbb{N}_0,\ j = \overline{0,2}, \tag{2.9}$$

for some $f_j \in \mathbb{R}$, $j = \overline{0,2}$.

Then from (2.8) and (2.9) we have

$$\left(\frac{u_{m+2}^{(j)}}{u_{m+1}^{(j)}} + f_j\right)\left(\gamma\frac{u_{m+1}^{(j)}}{u_m^{(j)}} + \gamma f_j + \delta\right) - \left(\alpha\frac{u_{m+1}^{(j)}}{u_m^{(j)}} + \alpha f_j + \beta\right) = 0,$$

for $m \in \mathbb{N}_0$, $j = \overline{0,2}$.

Let

$$f_j = -\frac{\delta}{\gamma}, \quad j = \overline{0,2}.$$

Then we have

$$\gamma^2 u_{m+2}^{(j)} - \gamma(\alpha+\delta)u_{m+1}^{(j)} + (\alpha\delta - \beta\gamma)u_m^{(j)} = 0, \tag{2.10}$$

for $m \in \mathbb{N}_0$, $j = \overline{0,2}$.

Suppose $\Delta := (\alpha+\delta)^2 - 4(\alpha\delta - \beta\gamma) \neq 0$. Then by using formula (1.2) we have that

$$u_m^{(j)} = \frac{(u_1^{(j)} - \lambda_2 u_0^{(j)})\lambda_1^m - (u_1^{(j)} - \lambda_1 u_0^{(j)})\lambda_2^m}{\lambda_1 - \lambda_2}, \tag{2.11}$$

for $m \in \mathbb{N}_0$, $j = \overline{0,2}$, where

$$\lambda_1 = \frac{\alpha+\delta+\sqrt{\Delta}}{2\gamma} \quad \text{and} \quad \lambda_2 = \frac{\alpha+\delta-\sqrt{\Delta}}{2\gamma},$$

is the general solution to (2.10).

Formulas (2.9) and (2.11) imply

$$\begin{aligned} z_m^{(j)} &= \frac{(u_1^{(j)} - \lambda_2 u_0^{(j)})\lambda_1^{m+1} - (u_1^{(j)} - \lambda_1 u_0^{(j)})\lambda_2^{m+1}}{(u_1^{(j)} - \lambda_2 u_0^{(j)})\lambda_1^m - (u_1^{(j)} - \lambda_1 u_0^{(j)})\lambda_2^m} - \frac{\delta}{\gamma} \\ &= \frac{(z_0^{(j)} - \lambda_2 + \frac{\delta}{\gamma})\lambda_1^{m+1} - (z_0^{(j)} - \lambda_1 + \frac{\delta}{\gamma})\lambda_2^{m+1}}{(z_0^{(j)} - \lambda_2 + \frac{\delta}{\gamma})\lambda_1^m - (z_0^{(j)} - \lambda_1 + \frac{\delta}{\gamma})\lambda_2^m} - \frac{\delta}{\gamma}, \end{aligned}$$

for $m \in \mathbb{N}_0$, $j = \overline{0,2}$, from which along with (2.7) it follows that

$$y_{3m-j} = \frac{(y_{-j} - \lambda_2 + \frac{\delta}{\gamma})\lambda_1^{m+1} - (y_{-j} - \lambda_1 + \frac{\delta}{\gamma})\lambda_2^{m+1}}{(y_{-j} - \lambda_2 + \frac{\delta}{\gamma})\lambda_1^m - (y_{-j} - \lambda_1 + \frac{\delta}{\gamma})\lambda_2^m} - \frac{\delta}{\gamma}, \tag{2.12}$$

for $m \in \mathbb{N}_0$, $j = \overline{0,2}$.

From (2.4) and (2.12) it follows that

$$g(x_{3m-j}) = \left(\frac{\left(\frac{g(x_{-j})}{g(x_{-j-1})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m+1} - \left(\frac{g(x_{-j})}{g(x_{-j-1})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m+1}}{\left(\frac{g(x_{-j})}{g(x_{-j-1})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m} - \left(\frac{g(x_{-j})}{g(x_{-j-1})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m}} - \frac{\delta}{\gamma} \right) g(x_{3m-j-1}),$$

for $m \in \mathbb{N}_0$, $j = \overline{0,2}$.

From (2.4) we easily get

$$g(x_{3m-j}) = y_{3m-j} y_{3m-j-1} y_{3m-j-2} g(x_{3m-j-3}), \tag{2.13}$$

for $m \in \mathbb{N}$, $j = \overline{1,3}$.

Hence

$$\begin{aligned} g(x_{3m}) &= g(x_{-3}) \prod_{i=0}^{m} y_{3i} y_{3i-1} y_{3i-2}, \\ g(x_{3m+1}) &= g(x_{-2}) \prod_{i=0}^{m} y_{3i+1} y_{3i} y_{3i-1}, \\ g(x_{3m+2}) &= g(x_{-1}) \prod_{i=0}^{m} y_{3i+2} y_{3i+1} y_{3i}, \end{aligned}$$

for $m \in \mathbb{N}_0$, and consequently

$$x_{3m} = g^{-1}\left(g(x_{-3}) \prod_{i=0}^{m} y_{3i} y_{3i-1} y_{3i-2} \right), \tag{2.14}$$

$$x_{3m+1} = g^{-1}\left(g(x_{-2}) \prod_{i=0}^{m} y_{3i+1} y_{3i} y_{3i-1} \right), \tag{2.15}$$

$$x_{3m+2} = g^{-1}\left(g(x_{-1}) \prod_{i=0}^{m} y_{3i+2} y_{3i+1} y_{3i} \right), \tag{2.16}$$

for $m \in \mathbb{N}_0$, where

$$\begin{aligned} y_{3m} y_{3m-1} y_{3m-2} = &\left(\frac{\left(\frac{g(x_0)}{g(x_{-1})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m+1} - \left(\frac{g(x_0)}{g(x_{-1})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m+1}}{\left(\frac{g(x_0)}{g(x_{-1})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m} - \left(\frac{g(x_0)}{g(x_{-1})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m}} - \frac{\delta}{\gamma} \right) \\ &\times \left(\frac{\left(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m+1} - \left(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m+1}}{\left(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m} - \left(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m}} - \frac{\delta}{\gamma} \right) \\ &\times \left(\frac{\left(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m+1} - \left(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m+1}}{\left(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m} - \left(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m}} - \frac{\delta}{\gamma} \right), \end{aligned} \tag{2.17}$$

$$y_{3m+1}y_{3m}y_{3m-1} = \left(\frac{\left(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m+2} - \left(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m+2}}{\left(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m+1} - \left(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m+1}} - \frac{\delta}{\gamma}\right)$$
$$\times \left(\frac{\left(\frac{g(x_0)}{g(x_{-1})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m+1} - \left(\frac{g(x_0)}{g(x_{-1})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m+1}}{\left(\frac{g(x_0)}{g(x_{-1})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m} - \left(\frac{g(x_0)}{g(x_{-1})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m}} - \frac{\delta}{\gamma}\right)$$
$$\times \left(\frac{\left(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m+1} - \left(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m+1}}{\left(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m} - \left(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m}} - \frac{\delta}{\gamma}\right), \quad (2.18)$$

$$y_{3m+2}y_{3m+1}y_{3m} = \left(\frac{\left(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m+2} - \left(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m+2}}{\left(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m+1} - \left(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m+1}} - \frac{\delta}{\gamma}\right)$$
$$\times \left(\frac{\left(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m+2} - \left(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m+2}}{\left(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m+1} - \left(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m+1}} - \frac{\delta}{\gamma}\right)$$
$$\times \left(\frac{\left(\frac{g(x_0)}{g(x_{-1})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m+1} - \left(\frac{g(x_0)}{g(x_{-1})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m+1}}{\left(\frac{g(x_0)}{g(x_{-1})} - \lambda_2 + \frac{\delta}{\gamma}\right)\lambda_1^{m} - \left(\frac{g(x_0)}{g(x_{-1})} - \lambda_1 + \frac{\delta}{\gamma}\right)\lambda_2^{m}} - \frac{\delta}{\gamma}\right), \quad (2.19)$$

for $m \in \mathbb{N}_0$. Formulas (2.14)–(2.19) present general solution to equation (2.1) in this case.

Assume $\Delta = 0$. Then, by using formula (1.3) we see that the general solution to equation (2.10) in this case is given by

$$u_m^{(j)} = ((u_1^{(j)} - \lambda_1 u_0^{(j)})m + \lambda_1 u_0^{(j)})\lambda_1^{m-1}, \quad (2.20)$$

for $m \in \mathbb{N}_0$, $j = \overline{0,2}$, where

$$\lambda_1 = \frac{\alpha + \delta}{2\gamma} \neq 0.$$

From (2.9) and (2.20) we have

$$z_m^{(j)} = \frac{((u_1^{(j)} - \lambda_1 u_0^{(j)})(m+1) + \lambda_1 u_0^{(j)})\lambda_1}{(u_1^{(j)} - \lambda_1 u_0^{(j)})m + \lambda_1 u_0^{(j)}} - \frac{\delta}{\gamma}$$
$$= \frac{((z_0^{(j)} - \lambda_1 + \frac{\delta}{\gamma})(m+1) + \lambda_1)\lambda_1}{(z_0^{(j)} - \lambda_1 + \frac{\delta}{\gamma})m + \lambda_1} - \frac{\delta}{\gamma},$$

for $m \in \mathbb{N}_0$, $j = \overline{0,2}$, from which along with (2.7) it follows that

$$y_{3m-j} = \frac{((y_{-j} - \lambda_1 + \frac{\delta}{\gamma})(m+1) + \lambda_1)\lambda_1}{(y_{-j} - \lambda_1 + \frac{\delta}{\gamma})m + \lambda_1} - \frac{\delta}{\gamma}, \quad (2.21)$$

for $m \in \mathbb{N}_0$, $j = \overline{0,2}$.

From (2.4) and (2.21) we have

$$g(x_{3m-j}) = \left(\frac{(\frac{g(x_{-j})}{g(x_{-j-1})} - \lambda_1 + \frac{\delta}{\gamma})(m+1) + \lambda_1)\lambda_1}{(\frac{g(x_{-j})}{g(x_{-j-1})} - \lambda_1 + \frac{\delta}{\gamma})m + \lambda_1} - \frac{\delta}{\gamma}\right) g(x_{3m-j-1}), \quad (2.22)$$

for $m \in \mathbb{N}_0$, $j = \overline{0,2}$.

We also have

$$y_{3m}y_{3m-1}y_{3m-2} = \left(\frac{(\frac{g(x_0)}{g(x_{-1})} - \lambda_1 + \frac{\delta}{\gamma})(m+1) + \lambda_1)\lambda_1}{(\frac{g(x_0)}{g(x_{-1})} - \lambda_1 + \frac{\delta}{\gamma})m + \lambda_1} - \frac{\delta}{\gamma}\right) \times \left(\frac{(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_1 + \frac{\delta}{\gamma})(m+1) + \lambda_1)\lambda_1}{(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_1 + \frac{\delta}{\gamma})m + \lambda_1} - \frac{\delta}{\gamma}\right) \times \left(\frac{(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_1 + \frac{\delta}{\gamma})(m+1) + \lambda_1)\lambda_1}{(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_1 + \frac{\delta}{\gamma})m + \lambda_1} - \frac{\delta}{\gamma}\right), \tag{2.23}$$

$$y_{3m+1}y_{3m}y_{3m-1} = \left(\frac{(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_1 + \frac{\delta}{\gamma})(m+2) + \lambda_1)\lambda_1}{(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_1 + \frac{\delta}{\gamma})(m+1) + \lambda_1} - \frac{\delta}{\gamma}\right) \times \left(\frac{(\frac{g(x_0)}{g(x_{-1})} - \lambda_1 + \frac{\delta}{\gamma})(m+1) + \lambda_1)\lambda_1}{(\frac{g(x_0)}{g(x_{-1})} - \lambda_1 + \frac{\delta}{\gamma})m + \lambda_1} - \frac{\delta}{\gamma}\right) \times \left(\frac{(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_1 + \frac{\delta}{\gamma})(m+1) + \lambda_1)\lambda_1}{(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_1 + \frac{\delta}{\gamma})m + \lambda_1} - \frac{\delta}{\gamma}\right), \tag{2.24}$$

$$y_{3m+2}y_{3m+1}y_{3m} = \left(\frac{(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_1 + \frac{\delta}{\gamma})(m+2) + \lambda_1)\lambda_1}{(\frac{g(x_{-1})}{g(x_{-2})} - \lambda_1 + \frac{\delta}{\gamma})(m+1) + \lambda_1} - \frac{\delta}{\gamma}\right) \times \left(\frac{(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_1 + \frac{\delta}{\gamma})(m+2) + \lambda_1)\lambda_1}{(\frac{g(x_{-2})}{g(x_{-3})} - \lambda_1 + \frac{\delta}{\gamma})(m+1) + \lambda_1} - \frac{\delta}{\gamma}\right) \times \left(\frac{(\frac{g(x_0)}{g(x_{-1})} - \lambda_1 + \frac{\delta}{\gamma})(m+1) + \lambda_1)\lambda_1}{(\frac{g(x_0)}{g(x_{-1})} - \lambda_1 + \frac{\delta}{\gamma})m + \lambda_1} - \frac{\delta}{\gamma}\right), \tag{2.25}$$

for $m \in \mathbb{N}_0$.

The above consideration, shows that the general solution to equation (2.1) in this case is given by formulas (2.14)–(2.16), (2.23)–(2.25).

Now assume $\gamma = 0$. Then $\delta \neq 0$ and equation (2.6) becomes

$$y_{n+1} = \frac{\alpha}{\delta}y_{n-2} + \frac{\beta}{\delta}, \quad n \in \mathbb{N}_0. \tag{2.26}$$

Hence,

$$z_{m+1}^{(j)} = \frac{\alpha}{\delta}z_m^{(j)} + \frac{\beta}{\delta}, \quad m \in \mathbb{N}_0, \ j = \overline{0,2}. \tag{2.27}$$

If $\alpha = \delta$, then from (2.27) we obtain

$$z_m^{(j)} = \frac{\beta}{\delta}m + z_0^{(j)}, \quad m \in \mathbb{N}_0, \ j = \overline{0,2}.$$

that is

$$y_{3m-j} = \frac{\beta}{\delta}m + y_{-j}, \quad m \in \mathbb{N}_0,\ j = \overline{0,2},$$

from which along with (2.4) and (2.13) it follows that

$$\begin{aligned}
g(x_{3m}) &= \left(\frac{\beta}{\delta}m + \frac{g(x_0)}{g(x_{-1})}\right)\left(\frac{\beta}{\delta}m + \frac{g(x_{-1})}{g(x_{-2})}\right)\left(\frac{\beta}{\delta}m + \frac{g(x_{-2})}{g(x_{-3})}\right)g(x_{3m-3}),\\
g(x_{3m+1}) &= \left(\frac{\beta}{\delta}(m+1) + \frac{g(x_{-2})}{g(x_{-3})}\right)\left(\frac{\beta}{\delta}m + \frac{g(x_0)}{g(x_{-1})}\right)\left(\frac{\beta}{\delta}m + \frac{g(x_{-1})}{g(x_{-2})}\right)g(x_{3m-2}),\\
g(x_{3m+2}) &= \left(\frac{\beta}{\delta}(m+1) + \frac{g(x_{-1})}{g(x_{-2})}\right)\left(\frac{\beta}{\delta}(m+1) + \frac{g(x_{-2})}{g(x_{-3})}\right)\left(\frac{\beta}{\delta}m + \frac{g(x_0)}{g(x_{-1})}\right)g(x_{3m-1}),
\end{aligned}$$

for $m \in \mathbb{N}_0$, from which it follows that

$$\begin{aligned}
g(x_{3m}) &= g(x_{-3})\prod_{j=0}^{m}\left(\frac{\beta}{\delta}j + \frac{g(x_0)}{g(x_{-1})}\right)\left(\frac{\beta}{\delta}j + \frac{g(x_{-1})}{g(x_{-2})}\right)\left(\frac{\beta}{\delta}j + \frac{g(x_{-2})}{g(x_{-3})}\right),\\
g(x_{3m+1}) &= g(x_{-2})\prod_{j=0}^{m}\left(\frac{\beta}{\delta}(j+1) + \frac{g(x_{-2})}{g(x_{-3})}\right)\left(\frac{\beta}{\delta}j + \frac{g(x_0)}{g(x_{-1})}\right)\left(\frac{\beta}{\delta}j + \frac{g(x_{-1})}{g(x_{-2})}\right),\\
g(x_{3m+2}) &= g(x_{-1})\prod_{j=0}^{m}\left(\frac{\beta}{\delta}(j+1) + \frac{g(x_{-1})}{g(x_{-2})}\right)\left(\frac{\beta}{\delta}(j+1) + \frac{g(x_{-2})}{g(x_{-3})}\right)\left(\frac{\beta}{\delta}j + \frac{g(x_0)}{g(x_{-1})}\right),
\end{aligned}$$

for $m \in \mathbb{N}_0$, and consequently

$$x_{3m} = g^{-1}\left(g(x_{-3})\prod_{j=0}^{m}\left(\frac{\beta}{\delta}j + \frac{g(x_0)}{g(x_{-1})}\right)\left(\frac{\beta}{\delta}j + \frac{g(x_{-1})}{g(x_{-2})}\right)\left(\frac{\beta}{\delta}j + \frac{g(x_{-2})}{g(x_{-3})}\right)\right), \tag{2.28}$$

$$x_{3m+1} = g^{-1}\left(g(x_{-2})\prod_{j=0}^{m}\left(\frac{\beta}{\delta}(j+1) + \frac{g(x_{-2})}{g(x_{-3})}\right)\left(\frac{\beta}{\delta}j + \frac{g(x_0)}{g(x_{-1})}\right)\left(\frac{\beta}{\delta}j + \frac{g(x_{-1})}{g(x_{-2})}\right)\right), \tag{2.29}$$

$$x_{3m+2} = g^{-1}\left(g(x_{-1})\prod_{j=0}^{m}\left(\frac{\beta}{\delta}(j+1) + \frac{g(x_{-1})}{g(x_{-2})}\right)\left(\frac{\beta}{\delta}(j+1) + \frac{g(x_{-2})}{g(x_{-3})}\right)\left(\frac{\beta}{\delta}j + \frac{g(x_0)}{g(x_{-1})}\right)\right), \tag{2.30}$$

for $m \in \mathbb{N}_0$. Hence, the general solution to equation (2.1) in this case is given by formulas (2.28)–(2.30).

If $\alpha \neq \delta$, then from (2.27) we have

$$z_m^{(j)} = \frac{\beta}{\alpha-\delta}\left(\left(\frac{\alpha}{\delta}\right)^m - 1\right) + \left(\frac{\alpha}{\delta}\right)^m z_0^{(j)},$$

for $m \in \mathbb{N}_0$, $j = \overline{0,2}$, that is,

$$y_{3m-j} = \beta\frac{(\alpha/\delta)^m - 1}{\alpha-\delta} + \left(\frac{\alpha}{\delta}\right)^m y_{-j}, \tag{2.31}$$

for $m \in \mathbb{N}_0$, $j = \overline{0,2}$.

From (2.4), (2.13) and (2.31) we have

$$\begin{aligned} g(x_{3m}) = &\left(\beta\frac{(\alpha/\delta)^m-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^m\frac{g(x_0)}{g(x_{-1})}\right)\\ &\times\left(\beta\frac{(\alpha/\delta)^m-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^m\frac{g(x_{-1})}{g(x_{-2})}\right)\\ &\times\left(\beta\frac{(\alpha/\delta)^m-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^m\frac{g(x_{-2})}{g(x_{-3})}\right)g(x_{3m-3}),\\ g(x_{3m+1}) = &\left(\beta\frac{(\alpha/\delta)^{m+1}-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^{m+1}\frac{g(x_{-2})}{g(x_{-3})}\right)\\ &\times\left(\beta\frac{(\alpha/\delta)^m-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^m\frac{g(x_0)}{g(x_{-1})}\right)\\ &\times\left(\beta\frac{(\alpha/\delta)^m-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^m\frac{g(x_{-1})}{g(x_{-2})}\right)g(x_{3m-2}),\\ g(x_{3m+2}) = &\left(\beta\frac{(\alpha/\delta)^{m+1}-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^{m+1}\frac{g(x_{-1})}{g(x_{-2})}\right)\\ &\times\left(\beta\frac{(\alpha/\delta)^{m+1}-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^{m+1}\frac{g(x_{-2})}{g(x_{-3})}\right)\\ &\times\left(\beta\frac{(\alpha/\delta)^m-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^m\frac{g(x_0)}{g(x_{-1})}\right)g(x_{3m-1}), \end{aligned}$$

for $m \in \mathbb{N}_0$.

Hence

$$\begin{aligned} g(x_{3m}) = g(x_{-3})\prod_{j=0}^{m}&\left(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^j\frac{g(x_0)}{g(x_{-1})}\right)\\ &\times\left(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^j\frac{g(x_{-1})}{g(x_{-2})}\right)\\ &\times\left(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^j\frac{g(x_{-2})}{g(x_{-3})}\right),\\ g(x_{3m+1}) = g(x_{-2})\prod_{j=0}^{m}&\left(\beta\frac{(\alpha/\delta)^{j+1}-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^{j+1}\frac{g(x_{-2})}{g(x_{-3})}\right)\\ &\times\left(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^j\frac{g(x_0)}{g(x_{-1})}\right)\\ &\times\left(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^j\frac{g(x_{-1})}{g(x_{-2})}\right),\\ g(x_{3m+2}) = g(x_{-1})\prod_{j=0}^{m}&\left(\beta\frac{(\alpha/\delta)^{j+1}-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^{j+1}\frac{g(x_{-1})}{g(x_{-2})}\right)\\ &\times\left(\beta\frac{(\alpha/\delta)^{j+1}-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^{j+1}\frac{g(x_{-2})}{g(x_{-3})}\right)\\ &\times\left(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\left(\frac{\alpha}{\delta}\right)^j\frac{g(x_0)}{g(x_{-1})}\right), \end{aligned}$$

for $m \in \mathbb{N}_0$, and consequently

$$x_{3m} = g^{-1}\Bigg(g(x_{-3})\prod_{j=0}^{m}\Bigg(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^j\frac{g(x_0)}{g(x_{-1})}\Bigg)\times\Bigg(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^j\frac{g(x_{-1})}{g(x_{-2})}\Bigg)\times\Bigg(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^j\frac{g(x_{-2})}{g(x_{-3})}\Bigg)\Bigg), \tag{2.32}$$

$$x_{3m+1} = g^{-1}\Bigg(g(x_{-2})\prod_{j=0}^{m}\Bigg(\beta\frac{(\alpha/\delta)^{j+1}-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^{j+1}\frac{g(x_{-2})}{g(x_{-3})}\Bigg)\times\Bigg(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^j\frac{g(x_0)}{g(x_{-1})}\Bigg)\times\Bigg(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^j\frac{g(x_{-1})}{g(x_{-2})}\Bigg)\Bigg), \tag{2.33}$$

$$x_{3m+2} = g^{-1}\Bigg(g(x_{-1})\prod_{j=0}^{m}\Bigg(\beta\frac{(\alpha/\delta)^{j+1}-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^{j+1}\frac{g(x_{-1})}{g(x_{-2})}\Bigg)\times\Bigg(\beta\frac{(\alpha/\delta)^{j+1}-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^{j+1}\frac{g(x_{-2})}{g(x_{-3})}\Bigg)\times\Bigg(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^j\frac{g(x_0)}{g(x_{-1})}\Bigg)\Bigg), \tag{2.34}$$

for $m \in \mathbb{N}_0$. Hence, the general solution to equation (2.1) in this case is given by formulas (2.32)–(2.34).

Assume $\alpha\delta = \beta\gamma$. If $\alpha = 0$, then $\beta \neq 0$. This implies $\gamma = 0$ and $\delta \neq 0$. Hence

$$x_{n+1} = g^{-1}\Big(\frac{\beta}{\delta}g(x_n)\Big), \quad n \in \mathbb{N}_0. \tag{2.35}$$

From (2.35) we easily get

$$x_n = g^{-1}\Bigg(\Big(\frac{\beta}{\delta}\Big)^n g(x_0)\Bigg), \tag{2.36}$$

for $n \in \mathbb{N}_0$.

If $\alpha \neq 0$ and $\beta = 0$, then $\delta = 0$, from which it follows that $\gamma \neq 0$. Hence

$$x_{n+1} = g^{-1}\Big(\frac{\alpha}{\gamma}g(x_n)\Big), \quad n \in \mathbb{N}_0. \tag{2.37}$$

From (2.37) we obtain

$$x_n = g^{-1}\Bigg(\Big(\frac{\alpha}{\gamma}\Big)^n g(x_0)\Bigg), \quad n \in \mathbb{N}_0. \tag{2.38}$$

If $\delta = 0$, then $\gamma \neq 0$. This implies $\beta = 0$, and consequently $\alpha \neq 0$, so we get equation (2.37) whose solutions are given by formula (2.38). If $\gamma = 0$, then $\delta \neq 0$. Hence $\alpha = 0$ which implies $\beta \neq 0$, so we get equation (2.35) whose solutions are given by formula (2.36).

If $\alpha\beta\gamma\delta \neq 0$, then $\alpha = \beta\gamma/\delta$, so we again get equation (2.35), which in this case coincides with equation (2.37).

From above obtained closed-form formulas for solutions to equation (2.1) the theorem follows. □

3 Some applications and discussions

A part of recent literature on difference equations contains many claims which are not established and/or explained. In some of our papers we discussed some aspects of the phenomena (see, e.g., [32–34, 36]). Here we discuss some incorrect claims on long-term behaviour of solutions to equation (1.5) given in [3].

Note that equation (1.5) can be written in the form

$$x_{n+1} = x_n \frac{acx_{n-2} + (ad+b)x_{n-3}}{cx_{n-2} + dx_{n-3}}, \quad n \in \mathbb{N}_0. \tag{3.1}$$

In [3] was first tried to find the equilibria of the equation. After some simple algebraic manipulations it was concluded that $\bar{x} = 0$ is a unique equilibrium point of equation (1.5), when

$$(1-a)(c+d) \neq b.$$

Assume that $\bar{x}$ is an equilibrium of equation (1.5). Then it must satisfy the algebraic equation

$$\bar{x} = a\bar{x} + \frac{b\bar{x}^2}{(c+d)\bar{x}}. \tag{3.2}$$

From (3.2) we see that it must be

$$\bar{x} \neq 0 \quad \text{and} \quad c+d \neq 0.$$

This eliminates the possibility $\bar{x} = 0$.

If $\bar{x} \neq 0$, then (3.2) implies

$$\bar{x}\left(1 - a - \frac{b}{c+d}\right) = 0,$$

and consequently

$$1 - a - \frac{b}{c+d} = 0.$$

Therefore, under the last condition any $\bar{x} \neq 0$ is an equilibrium of the difference equation.

This means that the claim in [3, Theorem 1] that, under a condition, the zero equilibrium point of equation (1.5) is locally asymptotically stable is not correct, since it is not an equilibrium at all.

Further, Theorem 2 in [3] claims the following:

Theorem 3.1. *The equilibrium point $\bar{x}$ of equation (1.5) is global attractor if $d(1-a) \neq b$.*

Note that equation (3.1) is a special case of equation (2.1) with

$$g(x) = x, \quad \alpha = ac, \quad \beta = ad + b, \quad \gamma = c \quad \text{and} \quad \delta = d.$$

Example 3.2. Consider the equation (1.5) with

$$a = 3, \quad b = -5, \quad c = 1 \quad \text{and} \quad d = 2, \tag{3.3}$$

that is, the equation

$$x_{n+1} = x_n \frac{3x_{n-2} + x_{n-3}}{x_{n-2} + 2x_{n-3}}, \quad n \in \mathbb{N}_0. \tag{3.4}$$

This is the equation (2.1) with $g(x) = x$, $x \in \mathbb{R}$,

$$\alpha = 3, \quad \beta = \gamma = 1, \quad \delta = 2. \tag{3.5}$$

The associated characteristic polynomial to the corresponding linear equation in (2.10) is

$$p_2(\lambda) = \lambda^2 - 5\lambda + 5,$$

and its roots are

$$\lambda_1 = \frac{5+\sqrt{5}}{2} \quad \text{and} \quad \lambda_2 = \frac{5-\sqrt{5}}{2}.$$

Since in this case we have

$$d(1-a) - b = 1 \neq 0,$$

the condition $d(1-a) \neq b$ in Theorem 3.1 is satisfied.

Employing the formulas in (2.14)–(2.19), where $g(x) = x$, $x \in \mathbb{R}$, and the coefficients $\alpha, \beta, \gamma, \delta$ are as in (3.5), we have

$$x_{3m} = x_{-3}\prod_{i=0}^{m} y_{3i}y_{3i-1}y_{3i-2}, \tag{3.6}$$

$$x_{3m+1} = x_{-2}\prod_{i=0}^{m} y_{3i+1}y_{3i}y_{3i-1}, \tag{3.7}$$

$$x_{3m+2} = x_{-1}\prod_{i=0}^{m} y_{3i+2}y_{3i+1}y_{3i}, \tag{3.8}$$

for $m \in \mathbb{N}_0$, where

$$\begin{aligned} y_{3m}y_{3m-1}y_{3m-2} = &\left(\frac{\left(\frac{x_0}{x_{-1}} - \lambda_2 + 2\right)\lambda_1^{m+1} - \left(\frac{x_0}{x_{-1}} - \lambda_1 + 2\right)\lambda_2^{m+1}}{\left(\frac{x_0}{x_{-1}} - \lambda_2 + 2\right)\lambda_1^{m} - \left(\frac{x_0}{x_{-1}} - \lambda_1 + 2\right)\lambda_2^{m}} - 2\right)\\ &\times\left(\frac{\left(\frac{x_{-1}}{x_{-2}} - \lambda_2 + 2\right)\lambda_1^{m+1} - \left(\frac{x_{-1}}{x_{-2}} - \lambda_1 + 2\right)\lambda_2^{m+1}}{\left(\frac{x_{-1}}{x_{-2}} - \lambda_2 + 2\right)\lambda_1^{m} - \left(\frac{x_{-1}}{x_{-2}} - \lambda_1 + 2\right)\lambda_2^{m}} - 2\right)\\ &\times\left(\frac{\left(\frac{x_{-2}}{x_{-3}} - \lambda_2 + 2\right)\lambda_1^{m+1} - \left(\frac{x_{-2}}{x_{-3}} - \lambda_1 + 2\right)\lambda_2^{m+1}}{\left(\frac{x_{-2}}{x_{-3}} - \lambda_2 + 2\right)\lambda_1^{m} - \left(\frac{x_{-2}}{x_{-3}} - \lambda_1 + 2\right)\lambda_2^{m}} - 2\right), \end{aligned} \tag{3.9}$$

$$\begin{aligned} y_{3m+1}y_{3m}y_{3m-1} = &\left(\frac{\left(\frac{x_{-2}}{x_{-3}} - \lambda_2 + 2\right)\lambda_1^{m+2} - \left(\frac{x_{-2}}{x_{-3}} - \lambda_1 + 2\right)\lambda_2^{m+2}}{\left(\frac{x_{-2}}{x_{-3}} - \lambda_2 + 2\right)\lambda_1^{m+1} - \left(\frac{x_{-2}}{x_{-3}} - \lambda_1 + 2\right)\lambda_2^{m+1}} - 2\right)\\ &\times\left(\frac{\left(\frac{x_0}{x_{-1}} - \lambda_2 + 2\right)\lambda_1^{m+1} - \left(\frac{x_0}{x_{-1}} - \lambda_1 + 2\right)\lambda_2^{m+1}}{\left(\frac{x_0}{x_{-1}} - \lambda_2 + 2\right)\lambda_1^{m} - \left(\frac{x_0}{x_{-1}} - \lambda_1 + 2\right)\lambda_2^{m}} - 2\right)\\ &\times\left(\frac{\left(\frac{x_{-1}}{x_{-2}} - \lambda_2 + 2\right)\lambda_1^{m+1} - \left(\frac{x_{-1}}{x_{-2}} - \lambda_1 + 2\right)\lambda_2^{m+1}}{\left(\frac{x_{-1}}{x_{-2}} - \lambda_2 + 2\right)\lambda_1^{m} - \left(\frac{x_{-1}}{x_{-2}} - \lambda_1 + 2\right)\lambda_2^{m}} - 2\right), \end{aligned} \tag{3.10}$$

$$\begin{aligned} y_{3m+2}y_{3m+1}y_{3m} = &\left(\frac{\left(\frac{x_{-1}}{x_{-2}} - \lambda_2 + 2\right)\lambda_1^{m+2} - \left(\frac{x_{-1}}{x_{-2}} - \lambda_1 + 2\right)\lambda_2^{m+2}}{\left(\frac{x_{-1}}{x_{-2}} - \lambda_2 + 2\right)\lambda_1^{m+1} - \left(\frac{x_{-1}}{x_{-2}} - \lambda_1 + 2\right)\lambda_2^{m+1}} - 2\right)\\ &\times\left(\frac{\left(\frac{x_{-2}}{x_{-3}} - \lambda_2 + 2\right)\lambda_1^{m+2} - \left(\frac{x_{-2}}{x_{-3}} - \lambda_1 + 2\right)\lambda_2^{m+2}}{\left(\frac{x_{-2}}{x_{-3}} - \lambda_2 + 2\right)\lambda_1^{m+1} - \left(\frac{x_{-2}}{x_{-3}} - \lambda_1 + 2\right)\lambda_2^{m+1}} - 2\right)\\ &\times\left(\frac{\left(\frac{x_0}{x_{-1}} - \lambda_2 + 2\right)\lambda_1^{m+1} - \left(\frac{x_0}{x_{-1}} - \lambda_1 + 2\right)\lambda_2^{m+1}}{\left(\frac{x_0}{x_{-1}} - \lambda_2 + 2\right)\lambda_1^{m} - \left(\frac{x_0}{x_{-1}} - \lambda_1 + 2\right)\lambda_2^{m}} - 2\right), \end{aligned} \tag{3.11}$$

for $m \in \mathbb{N}_0$.

Now note that

$$\begin{aligned}
&\lim_{m\to+\infty} \frac{\left(\frac{x_0}{x_{-1}} - \lambda_2 + 2\right)\lambda_1^{m+1} - \left(\frac{x_0}{x_{-1}} - \lambda_1 + 2\right)\lambda_2^{m+1}}{\left(\frac{x_0}{x_{-1}} - \lambda_2 + 2\right)\lambda_1^{m} - \left(\frac{x_0}{x_{-1}} - \lambda_1 + 2\right)\lambda_2^{m}} - 2 \\
&= \lim_{m\to+\infty} \frac{\left(\frac{x_{-1}}{x_{-2}} - \lambda_2 + 2\right)\lambda_1^{m+1} - \left(\frac{x_{-1}}{x_{-2}} - \lambda_1 + 2\right)\lambda_2^{m+1}}{\left(\frac{x_{-1}}{x_{-2}} - \lambda_2 + 2\right)\lambda_1^{m} - \left(\frac{x_{-1}}{x_{-2}} - \lambda_1 + 2\right)\lambda_2^{m}} - 2 \\
&= \lim_{m\to+\infty} \frac{\left(\frac{x_{-2}}{x_{-3}} - \lambda_2 + 2\right)\lambda_1^{m+1} - \left(\frac{x_{-2}}{x_{-3}} - \lambda_1 + 2\right)\lambda_2^{m+1}}{\left(\frac{x_{-2}}{x_{-3}} - \lambda_2 + 2\right)\lambda_1^{m} - \left(\frac{x_{-2}}{x_{-3}} - \lambda_1 + 2\right)\lambda_2^{m}} - 2 \\
&= \lambda_1 - 2 = \frac{1+\sqrt{5}}{2} > 1,
\end{aligned}$$

when

$$\frac{x_{-i}}{x_{-(i+1)}} \neq \lambda_2 - 2 = \frac{1-\sqrt{5}}{2}, \quad i = \overline{0,2}. \tag{3.12}$$

By choosing positive initial values satisfying (3.12) and using formulas (3.6)–(3.11) we have

$$\lim_{n\to+\infty} x_n = +\infty.$$

This means that the solution is not convergent, which is a counterexample to the claim in Theorem 3.1.

Bearing in mind that in [3] is stated that it considers equation (1.5) for the case when all the coefficients a, b, c and d are positive, and that one of the coefficients in Example 3.2 is negative (see (3.3)), in the following example we also give a counterexample to the statement in Theorem 3.1 for the case of positive coefficients.

Example 3.3. Consider the equation (1.5) with

$$a = b = c = d = 1, \tag{3.13}$$

that is, the equation

$$x_{n+1} = x_n \frac{x_{n-2} + 2x_{n-3}}{x_{n-2} + x_{n-3}}, \quad n \in \mathbb{N}_0. \tag{3.14}$$

This is the equation (2.1) with $g(x) = x$, $x \in \mathbb{R}$,

$$\alpha = \gamma = \delta = 1, \quad \beta = 2. \tag{3.15}$$

The associated characteristic polynomial to the corresponding linear equation in (2.10) is

$$p_2(\lambda) = \lambda^2 - 2\lambda - 1,$$

and its roots are

$$\lambda_1 = 1 + \sqrt{2} \quad \text{and} \quad \lambda_2 = 1 - \sqrt{2}.$$

Since in this case we have

$$d(1-a) - b = -1 \neq 0,$$

the condition $d(1-a) \neq b$ in Theorem 3.1 is satisfied.

Employing (2.14)–(2.19), where $g(x) = x$, $x \in \mathbb{R}$, and the coefficients $\alpha, \beta, \gamma, \delta$ are as in (3.15) we have that the relations in (3.6)–(3.8) hold for $m \in \mathbb{N}_0$, where

$$\begin{aligned} y_{3m}y_{3m-1}y_{3m-2} = &\left(\frac{\left(\frac{x_0}{x_{-1}}-\lambda_2+1\right)\lambda_1^{m+1}-\left(\frac{x_0}{x_{-1}}-\lambda_1+1\right)\lambda_2^{m+1}}{\left(\frac{x_0}{x_{-1}}-\lambda_2+1\right)\lambda_1^{m}-\left(\frac{x_0}{x_{-1}}-\lambda_1+1\right)\lambda_2^{m}}-1\right)\\ &\times\left(\frac{\left(\frac{x_{-1}}{x_{-2}}-\lambda_2+1\right)\lambda_1^{m+1}-\left(\frac{x_{-1}}{x_{-2}}-\lambda_1+1\right)\lambda_2^{m+1}}{\left(\frac{x_{-1}}{x_{-2}}-\lambda_2+1\right)\lambda_1^{m}-\left(\frac{x_{-1}}{x_{-2}}-\lambda_1+1\right)\lambda_2^{m}}-1\right)\\ &\times\left(\frac{\left(\frac{x_{-2}}{x_{-3}}-\lambda_2+1\right)\lambda_1^{m+1}-\left(\frac{x_{-2}}{x_{-3}}-\lambda_1+1\right)\lambda_2^{m+1}}{\left(\frac{x_{-2}}{x_{-3}}-\lambda_2+1\right)\lambda_1^{m}-\left(\frac{x_{-2}}{x_{-3}}-\lambda_1+1\right)\lambda_2^{m}}-1\right), \end{aligned} \tag{3.16}$$

$$\begin{aligned} y_{3m+1}y_{3m}y_{3m-1} = &\left(\frac{\left(\frac{x_{-2}}{x_{-3}}-\lambda_2+1\right)\lambda_1^{m+2}-\left(\frac{x_{-2}}{x_{-3}}-\lambda_1+1\right)\lambda_2^{m+2}}{\left(\frac{x_{-2}}{x_{-3}}-\lambda_2+1\right)\lambda_1^{m+1}-\left(\frac{x_{-2}}{x_{-3}}-\lambda_1+1\right)\lambda_2^{m+1}}-1\right)\\ &\times\left(\frac{\left(\frac{x_0}{x_{-1}}-\lambda_2+1\right)\lambda_1^{m+1}-\left(\frac{x_0}{x_{-1}}-\lambda_1+1\right)\lambda_2^{m+1}}{\left(\frac{x_0}{x_{-1}}-\lambda_2+1\right)\lambda_1^{m}-\left(\frac{x_0}{x_{-1}}-\lambda_2+1\right)\lambda_2^{m}}-1\right)\\ &\times\left(\frac{\left(\frac{x_{-1}}{x_{-2}}-\lambda_2+1\right)\lambda_1^{m+1}-\left(\frac{x_{-1}}{x_{-2}}-\lambda_1+1\right)\lambda_2^{m+1}}{\left(\frac{x_{-1}}{x_{-2}}-\lambda_2+1\right)\lambda_1^{m}-\left(\frac{x_{-1}}{x_{-2}}-\lambda_1+1\right)\lambda_2^{m}}-1\right), \end{aligned} \tag{3.17}$$

$$\begin{aligned} y_{3m+2}y_{3m+1}y_{3m} = &\left(\frac{\left(\frac{x_{-1}}{x_{-2}}-\lambda_2+1\right)\lambda_1^{m+2}-\left(\frac{x_{-1}}{x_{-2}}-\lambda_1+1\right)\lambda_2^{m+2}}{\left(\frac{x_{-1}}{x_{-2}}-\lambda_2+1\right)\lambda_1^{m+1}-\left(\frac{x_{-1}}{x_{-2}}-\lambda_1+1\right)\lambda_2^{m+1}}-1\right)\\ &\times\left(\frac{\left(\frac{x_{-2}}{x_{-3}}-\lambda_2+1\right)\lambda_1^{m+2}-\left(\frac{x_{-2}}{x_{-3}}-\lambda_1+1\right)\lambda_2^{m+2}}{\left(\frac{x_{-2}}{x_{-3}}-\lambda_2+1\right)\lambda_1^{m+1}-\left(\frac{x_{-2}}{x_{-3}}-\lambda_1+1\right)\lambda_2^{m+1}}-1\right)\\ &\times\left(\frac{\left(\frac{x_0}{x_{-1}}-\lambda_2+1\right)\lambda_1^{m+1}-\left(\frac{x_0}{x_{-1}}-\lambda_1+1\right)\lambda_2^{m+1}}{\left(\frac{x_0}{x_{-1}}-\lambda_2+1\right)\lambda_1^{m}-\left(\frac{x_0}{x_{-1}}-\lambda_1+1\right)\lambda_2^{m}}-1\right), \end{aligned} \tag{3.18}$$

for $m \in \mathbb{N}_0$.

Now note that

$$\begin{aligned} &\lim_{m\to+\infty}\frac{\left(\frac{x_0}{x_{-1}}-\lambda_2+1\right)\lambda_1^{m+1}-\left(\frac{x_0}{x_{-1}}-\lambda_1+1\right)\lambda_2^{m+1}}{\left(\frac{x_0}{x_{-1}}-\lambda_2+1\right)\lambda_1^{m}-\left(\frac{x_0}{x_{-1}}-\lambda_1+1\right)\lambda_2^{m}}-1\\ &\quad=\lim_{m\to+\infty}\frac{\left(\frac{x_{-1}}{x_{-2}}-\lambda_2+1\right)\lambda_1^{m+1}-\left(\frac{x_{-1}}{x_{-2}}-\lambda_1+1\right)\lambda_2^{m+1}}{\left(\frac{x_{-1}}{x_{-2}}-\lambda_2+1\right)\lambda_1^{m}-\left(\frac{x_{-1}}{x_{-2}}-\lambda_1+1\right)\lambda_2^{m}}-1\\ &\quad=\lim_{m\to+\infty}\frac{\left(\frac{x_{-2}}{x_{-3}}-\lambda_2+1\right)\lambda_1^{m+1}-\left(\frac{x_{-2}}{x_{-3}}-\lambda_1+1\right)\lambda_2^{m+1}}{\left(\frac{x_{-2}}{x_{-3}}-\lambda_2+1\right)\lambda_1^{m}-\left(\frac{x_{-2}}{x_{-3}}-\lambda_1+1\right)\lambda_2^{m}}-1\\ &\quad=\lambda_1-1=\sqrt{2}>1, \end{aligned}$$

when

$$\frac{x_{-i}}{x_{-(i+1)}} \neq \lambda_2 - 1 = -\sqrt{2}, \quad i = \overline{0,2}. \tag{3.19}$$

By choosing positive initial values satisfying (3.19), and using formulas (3.6)–(3.8), (3.16)–(3.18) we have

$$\lim_{n\to+\infty} x_n = +\infty.$$

Hence, the solutions are not convergent, which is a counterexample to the claim in Theorem 3.1 in the case $\min\{a, b, c, d\} > 0$.

Remark 3.4. The closed-form formulas for some special cases of equation (1.5) presented in [3] easily follow from the ones in Theorem 2.1. We leave the verification of the fact to the interested reader as some simple exercises. Hence, our Theorem 2.1 gives a theoretical explanation for the closed-form formulas therein.

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Exact solution of the Susceptible–Infectious–Recovered–Deceased (SIRD) epidemic model

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Abstract. Exact solution of the Susceptible–Infectious–Recovered–Deceased (SIRD) epidemic model is established, and various properties of solution are derived directly from the exact solution. The exact solution of an initial value problem for SIRD epidemic model is represented in an explicit form, and it is shown that the parametric form of the exact solution is a solution of some linear differential system.

Keywords: exact solution, SIRD epidemic model, initial value problem, linear differential system.

2020 Mathematics Subject Classification: 34A34.

1 Introduction

Recently there is an increasing requirement for mathematical approach to epidemic models. It goes without saying that a vast literature and research papers, dealing with epidemic models has been published so far (see, e.g., [2–4,7]). It seems that little is known about exact solutions of epidemic models. Exact solutions of the Susceptible-Infectious-Recovered (SIR) epidemic model were studied by Bohner, Streipert and Torres [1], Harko, Lobo and Mak [5], Shabbir, Khan and Sadiq [9] and Yoshida [11]. However there appears to be no known results about exact solutions of the Susceptible–Infectious–Recovered–Deceased (SIRD) epidemic models. The objective of this paper is to obtain an exact solution of SIRD differential system, and to derive various properties of the exact solution. Furthermore we show that the parametric form of the exact solution satisfies some linear differential system.

The differential system called Susceptible–Infectious–Recovered–Deceased (SIRD) epidem-

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ic model is the following:

$$\frac{dS(t)}{dt} = -\beta S(t)I(t) + \nu R(t),$$
$$\frac{dI(t)}{dt} = \beta S(t)I(t) - \gamma I(t) - \mu I(t),$$
$$\frac{dR(t)}{dt} = \gamma I(t) - \nu R(t),$$
$$\frac{dD(t)}{dt} = \mu I(t)$$

(see, e.g., [8]). If $\nu = 0$, we obtain the simplified SIRD differential system

$$\frac{dS(t)}{dt} = -\beta S(t)I(t), \tag{1.1}$$
$$\frac{dI(t)}{dt} = \beta S(t)I(t) - \gamma I(t) - \mu I(t), \tag{1.2}$$
$$\frac{dR(t)}{dt} = \gamma I(t), \tag{1.3}$$
$$\frac{dD(t)}{dt} = \mu I(t) \tag{1.4}$$

for $t > 0$, where β, γ and μ are positive constants. We consider the following initial condition:

$$S(0) = \tilde{S}, \qquad I(0) = \tilde{I}, \qquad R(0) = \tilde{R}, \qquad D(0) = \tilde{D}, \tag{1.5}$$

where $\tilde{S} + \tilde{I} + \tilde{R} + \tilde{D} = N$ (positive constant). Since

$$\frac{d}{dt}\left(S(t) + I(t) + R(t) + D(t)\right) = \frac{dS(t)}{dt} + \frac{dI(t)}{dt} + \frac{dR(t)}{dt} + \frac{dD(t)}{dt} = 0$$

by (1.1)–(1.4), it follows that

$$S(t) + I(t) + R(t) + D(t) = k \qquad (t \geq 0)$$

for some constant k. In view of the fact that

$$k = S(0) + I(0) + R(0) + D(0) = \tilde{S} + \tilde{I} + \tilde{R} + \tilde{D} = N,$$

we conclude that

$$S(t) + I(t) + R(t) + D(t) = N \qquad (t \geq 0).$$

It is assumed throughout this paper that:

(A_1) $\tilde{S} > \dfrac{\gamma + \mu}{\beta}$;

(A_2) $\tilde{I} > 0$;

(A_3) $\tilde{R} \geq 0$ satisfies

$$N - \tilde{D} > \tilde{S}e^{(\beta/\gamma)\tilde{R}} + \tilde{R}; \tag{1.6}$$

(A_4) $\tilde{D} \geq 0$.

In Section 2 we show that a positive solution of the SIRD differential system can be represented in a parametric form, and we derive an exact solution of the SIRD differential system (1.1)–(1.4). Section 3 is devoted to the investigation of various properties of the exact solution.

2 Exact solution of SIRD differential system

First we need the following important lemma.

Lemma 2.1. *If $S(t) > 0$ for $t > 0$, then the following holds:*

$$R'(t) = \gamma \left(N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} - \tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)R(t)} - \left(1 + \frac{\mu}{\gamma}\right) R(t) \right) \tag{2.1}$$

for $t > 0$.

Proof. From (1.1) and (1.3) we see that

$$R'(t) = \gamma I(t) = \gamma \left(\frac{S'(t)}{-\beta S(t)} \right) = -\frac{\gamma}{\beta}(\log S(t))',$$

and integrating the above on $[0, t]$ yields

$$R(t) - \tilde{R} = -\frac{\gamma}{\beta}(\log S(t) - \log \tilde{S}).$$

Therefore we obtain

$$\log S(t) = -\frac{\beta}{\gamma}(R(t) - \tilde{R}) + \log \tilde{S}$$

and hence

$$S(t) = \exp\left(\log \tilde{S} - \frac{\beta}{\gamma}R(t) + \frac{\beta}{\gamma}\tilde{R} \right) = \tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)R(t)}. \tag{2.2}$$

It follows from (1.3) and (1.4) that

$$D'(t) = \mu I(t) = \frac{\mu}{\gamma}(\gamma I(t)) = \frac{\mu}{\gamma}R'(t)$$

and hence we get

$$D(t) = \frac{\mu}{\gamma}R(t) + C$$

for some constant C. The initial condition (1.5) implies

$$C = \tilde{D} - \frac{\mu}{\gamma}\tilde{R}.$$

Consequently we obtain

$$D(t) = \frac{\mu}{\gamma}R(t) + \tilde{D} - \frac{\mu}{\gamma}\tilde{R}. \tag{2.3}$$

Taking account of (2.2), (2.3) and $I(t) = N - S(t) - R(t) - D(t)$, we observe that

$$\begin{aligned}
R'(t) &= \gamma I(t) \\
&= \gamma(N - S(t) - R(t) - D(t)) \\
&= \gamma \left(N - \tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)R(t)} - R(t) - \frac{\mu}{\gamma}R(t) - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} \right) \\
&= \gamma \left(N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} - \tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)R(t)} - \left(1 + \frac{\mu}{\gamma}\right) R(t) \right),
\end{aligned}$$

which is the desired identity (2.1). □

By a *solution* of the SIRD differential system (1.1)–(1.4) we mean a vector-valued function $(S(t), I(t), R(t), D(t))$ of class $C^1(0,\infty) \cap C[0,\infty)$ which satisfies (1.1)–(1.4). Associated with every continuous function $f(t)$ on $[0,\infty)$, we define

$$f(\infty) := \lim_{t\to\infty} f(t).$$

Lemma 2.2. *Let $(S(t), I(t), R(t), D(t))$ be a solution of the SIRD differential system (1.1)–(1.4) such that $S(t) > 0$ and $I(t) > 0$ for $t > 0$. Then there exist the limits $S(\infty)$, $I(\infty)$, $R(\infty)$ and $D(\infty)$.*

Proof. Since $S(t) > 0$ and $I(t) > 0$, we see that $S'(t) < 0$, and therefore $S(t)$ is decreasing on $[0,\infty)$. It is trivial that $S(t)$ is bounded from below because $S(t) > 0$. Hence, there exists the limit $S(\infty)$. We observe that $R(t)$ is increasing on $[0,\infty)$ and bounded from above in view of the fact that $R'(t) = \gamma I(t) > 0$ and $R(t) < N$. Therefore there exists $R(\infty)$. Similarly there exists $D(\infty)$. Since $I(t) = N - S(t) - R(t) - D(t)$ and there exist $S(\infty)$, $R(\infty)$ and $D(\infty)$, it follows that there exists $I(\infty)$. □

Theorem 2.3. *Let $(S(t), I(t), R(t), D(t))$ be a solution of the initial value problem (1.1)–(1.5) such that $S(t) > 0$ and $I(t) > 0$ for $t > 0$. Then $(S(t), I(t), R(t), D(t))$ can be represented in the following parametric form:*

$$S(t) = S(\varphi(u)) = \tilde{S}e^{(\beta/\gamma)\tilde{R}}u, \tag{2.4}$$

$$I(t) = I(\varphi(u)) = N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} - \tilde{S}e^{(\beta/\gamma)\tilde{R}}u + \frac{\gamma+\mu}{\beta}\log u, \tag{2.5}$$

$$R(t) = R(\varphi(u)) = -\frac{\gamma}{\beta}\log u, \tag{2.6}$$

$$D(t) = D(\varphi(u)) = -\frac{\mu}{\beta}\log u + \tilde{D} - \frac{\mu}{\gamma}\tilde{R} \tag{2.7}$$

for $e^{-(\beta/\gamma)R(\infty)} < u \le e^{-(\beta/\gamma)\tilde{R}}$, where $t = \varphi(u)$ is given in the proof.

Proof. Define the function $u(t)$ by

$$u(t) := e^{-(\beta/\gamma)R(t)}.$$

We note that there exists the limit $R(\infty)$ by Lemma 2.2. Then $u = u(t)$ is decreasing on $[0,\infty)$, $e^{-(\beta/\gamma)R(\infty)} < u \le e^{-(\beta/\gamma)\tilde{R}}$ and $\lim_{t\to\infty} u(t) = e^{-(\beta/\gamma)R(\infty)}$ since $R(t)$ is increasing on $[0,\infty)$ and $\tilde{R} \le R(t) < R(\infty)$. It is clear that $u(t)$ is of class $C^1(0,\infty)$ in view of $e^{-(\beta/\gamma)R(t)} \in C^1(0,\infty)$. Therefore, there exists the inverse function $\varphi(u)$ of $u = u(t)$ such that

$$\begin{aligned}
&t = \varphi(u) \quad \left(e^{-(\beta/\gamma)R(\infty)} < u \le e^{-(\beta/\gamma)\tilde{R}}\right),\\
&\varphi(u) \in C^1(e^{-(\beta/\gamma)R(\infty)}, e^{-(\beta/\gamma)\tilde{R}}),\\
&\varphi(u) \text{ is decreasing in } (e^{-(\beta/\gamma)R(\infty)}, e^{-(\beta/\gamma)\tilde{R}}],\\
&\varphi(e^{-(\beta/\gamma)\tilde{R}}) = 0,\\
&\lim_{u\to e^{-(\beta/\gamma)R(\infty)}+0} \varphi(u) = \infty.
\end{aligned}$$

Substituting $t = \varphi(u)$ into (2.1) in Lemma 2.1 yields

$$R'(\varphi(u)) = \gamma\left(N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} - \tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)R(\varphi(u))} - \left(1 + \frac{\mu}{\gamma}\right)R(\varphi(u))\right) \tag{2.8}$$

for $e^{-(\beta/\gamma)R(\infty)} < u \le e^{-(\beta/\gamma)\tilde{R}}$. Differentiating both sides of $u = e^{-(\beta/\gamma)R(\varphi(u))}$ with respect to u, we get

$$\begin{aligned} 1 &= -\frac{\beta}{\gamma}R'(\varphi(u))\varphi'(u)e^{-(\beta/\gamma)R(\varphi(u))} \\ &= -\frac{\beta}{\gamma}R'(\varphi(u))\varphi'(u)u, \end{aligned}$$

and therefore

$$R'(\varphi(u)) = -\frac{\gamma}{\beta}\frac{1}{\varphi'(u)u}. \tag{2.9}$$

It is obvious that

$$R(\varphi(u)) = -\frac{\gamma}{\beta}\log u \tag{2.10}$$

in view of $u = e^{-(\beta/\gamma)R(\varphi(u))}$. Combining (2.8)–(2.10), we have

$$-\frac{\gamma}{\beta}\frac{1}{\varphi'(u)u} = \gamma N - \gamma\tilde{D} + \mu\tilde{R} - \gamma\tilde{S}e^{(\beta/\gamma)\tilde{R}}u + \frac{\gamma}{\beta}(\gamma+\mu)\log u$$

and therefore

$$\begin{aligned} -\varphi'(u) &= \frac{\gamma}{\beta}\frac{1}{u\left(\gamma N - \gamma\tilde{D} + \mu\tilde{R} - \gamma\tilde{S}e^{(\beta/\gamma)\tilde{R}}u + (\gamma/\beta)(\gamma+\mu)\log u\right)} \\ &= \frac{1}{u\left(\beta N - \beta\tilde{D} + (\beta\mu/\gamma)\tilde{R} - \beta\tilde{S}e^{(\beta/\gamma)\tilde{R}}u + (\gamma+\mu)\log u\right)}. \end{aligned} \tag{2.11}$$

Integrating (2.11) over $[u, e^{-(\beta/\gamma)\tilde{R}}]$ and taking account of $\varphi(e^{-(\beta/\gamma)\tilde{R}}) = 0$, we get

$$\varphi(u) = \int_u^{e^{-(\beta/\gamma)\tilde{R}}} \frac{d\xi}{\xi\psi(\xi)},$$

where

$$\psi(\xi) = \beta N - \beta\tilde{D} + \frac{\beta\mu}{\gamma}\tilde{R} - \beta\tilde{S}e^{(\beta/\gamma)\tilde{R}}\xi + (\gamma+\mu)\log\xi. \tag{2.12}$$

It follows from (2.2), (2.3) and (2.10) that

$$\begin{aligned} S(t) &= S(\varphi(u)) = \tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)R(\varphi(u))} = \tilde{S}e^{(\beta/\gamma)\tilde{R}}u, \\ R(t) &= R(\varphi(u)) = -\frac{\gamma}{\beta}\log u, \\ D(t) &= D(\varphi(u)) = \frac{\mu}{\gamma}R(\varphi(u)) + \tilde{D} - \frac{\mu}{\gamma}\tilde{R} = -\frac{\mu}{\beta}\log u + \tilde{D} - \frac{\mu}{\gamma}\tilde{R}, \\ I(t) &= I(\varphi(u)) = N - S(\varphi(u)) - R(\varphi(u)) - D(\varphi(u)) \\ &= N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} - \tilde{S}e^{(\beta/\gamma)\tilde{R}}u + \frac{\gamma+\mu}{\beta}\log u, \end{aligned}$$

which is the desired solution (2.4)–(2.7). Since $\lim_{u\to e^{-(\beta/\gamma)R(\infty)}+0}\varphi(u) = \infty$, it is necessary that

$$\begin{aligned} \lim_{\xi\to e^{-(\beta/\gamma)R(\infty)}+0}\psi(\xi) &= \lim_{\xi\to e^{-(\beta/\gamma)R(\infty)}+0}\left(\beta N - \beta\tilde{D} + \frac{\beta\mu}{\gamma}\tilde{R} - \beta\tilde{S}e^{(\beta/\gamma)\tilde{R}}\xi + (\gamma+\mu)\log\xi\right) \\ &= \lim_{x\to R(\infty)-0}\beta\left(N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} - \tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)x} - \frac{\gamma+\mu}{\gamma}x\right) \\ &= \beta\left(N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} - \tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)R(\infty)} - \frac{\gamma+\mu}{\gamma}R(\infty)\right) \\ &= 0, \end{aligned}$$

which implies

$$R(\infty) = \frac{\gamma}{\gamma+\mu}N - \frac{\gamma}{\gamma+\mu}\tilde{D} + \frac{\mu}{\gamma+\mu}\tilde{R} - \frac{\gamma}{\gamma+\mu}\tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)R(\infty)}. \tag{2.13}$$

We find that $\psi'(\xi) = 0$ for $\xi = \overline{\xi} = \left((\gamma+\mu)/(\beta\tilde{S})\right)e^{-(\beta/\gamma)\tilde{R}}$, and that $e^{-(\beta/\gamma)R(\infty)} < \overline{\xi} < e^{-(\beta/\gamma)\tilde{R}}$ if $(\gamma+\mu)/\beta < \tilde{S} < ((\gamma+\mu)/\beta)e^{(\beta/\gamma)(R(\infty)-\tilde{R})}$. Since $\psi'(\xi) > 0$ for $e^{-(\beta/\gamma)R(\infty)} < \xi < \overline{\xi}$ and $\psi'(\xi) < 0$ for $\overline{\xi} < \xi < e^{-(\beta/\gamma)\tilde{R}}$, we observe that $\psi(\xi)$ is increasing in $(e^{-(\beta/\gamma)R(\infty)}, \overline{\xi})$ and is decreasing in $(\overline{\xi}, e^{-(\beta/\gamma)\tilde{R}})$. In view of the fact that $\psi\left(e^{-(\beta/\gamma)\tilde{R}}\right) = \beta\left(N - \tilde{S} - \tilde{R} - \tilde{D}\right) = \beta\tilde{I} > 0$ and $\lim_{\xi \to e^{-(\beta/\gamma)R(\infty)}+0} \psi(\xi) = 0$, we see that

$$\psi(\xi) > 0 \quad \text{in } (e^{-(\beta/\gamma)R(\infty)}, e^{-(\beta/\gamma)\tilde{R}}]$$

under the condition $(\gamma+\mu)/\beta < \tilde{S} < ((\gamma+\mu)/\beta)e^{(\beta/\gamma)(R(\infty)-\tilde{R})}$. Moreover, we get

$$\begin{aligned}
\lim_{t\to\infty} \frac{\beta}{\gamma}R'(t) &= \lim_{t\to\infty} \beta\left(N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} - \tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)R(t)} - \left(1+\frac{\mu}{\gamma}\right)R(t)\right) \\
&= \beta\left(N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} - \tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)R(\infty)} - \left(1+\frac{\mu}{\gamma}\right)R(\infty)\right) \\
&= 0,
\end{aligned}$$

which implies $I(\infty) = 0$ in light of (1.3). □

Lemma 2.4. *Under the hypothesis* (A$_3$)*, the transcendental equation*

$$x = F(N, \tilde{D}, \tilde{R}, \gamma, \mu) - \frac{\gamma}{\gamma+\mu}\tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)x}$$

has a unique solution $x = \alpha$ such that

$$\tilde{R} < \alpha < N,$$

where

$$F(N, \tilde{D}, \tilde{R}, \gamma, \mu) := \frac{\gamma}{\gamma+\mu}N - \frac{\gamma}{\gamma+\mu}\tilde{D} + \frac{\mu}{\gamma+\mu}\tilde{R}$$

(*cf.* Figure 2.1).

Proof. First we note that

$$F(N, \tilde{D}, \tilde{R}, \gamma, \mu) - \frac{\gamma}{\gamma+\mu}\tilde{S}e^{(\beta/\gamma)\tilde{R}} = \frac{\gamma}{\gamma+\mu}\left(N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} - \tilde{S}e^{(\beta/\gamma)\tilde{R}}\right) > 0 \tag{2.14}$$

in view of (1.6). We define the sequence $\{a_n\}_{n=1}^{\infty}$ by

$$\begin{aligned}
a_1 &= \tilde{a} \quad \left(0 < \tilde{a} \le F(N, \tilde{D}, \tilde{R}, \gamma, \mu) - \frac{\gamma}{\gamma+\mu}\tilde{S}e^{(\beta/\gamma)\tilde{R}}\right), \\
a_{n+1} &= F(N, \tilde{D}, \tilde{R}, \gamma, \mu) - \frac{\gamma}{\gamma+\mu}\tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)a_n} \qquad (n = 1, 2, \dots).
\end{aligned} \tag{2.15}$$

It is easily seen that

$$\begin{aligned}
a_1 = \tilde{a} &\le F(N, \tilde{D}, \tilde{R}, \gamma, \mu) - \frac{\gamma}{\gamma+\mu}\tilde{S}e^{(\beta/\gamma)\tilde{R}} \\
&\le F(N, \tilde{D}, \tilde{R}, \gamma, \mu) - \frac{\gamma}{\gamma+\mu}\tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)a_1} \\
&= a_2.
\end{aligned}$$

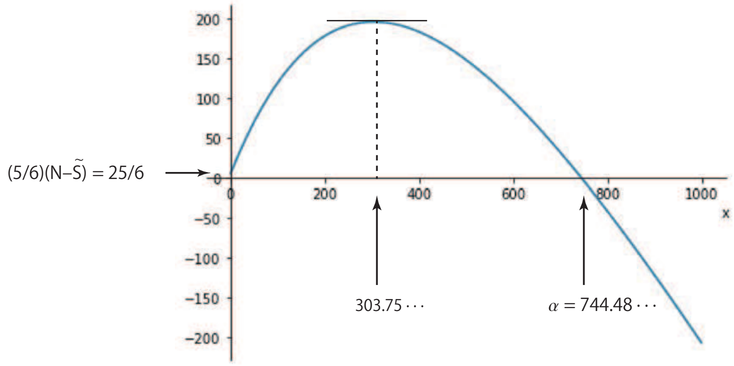

Figure 2.1: Variation of $F(N,\tilde{D},\tilde{R},\gamma,\mu) - (\gamma/(\gamma+\mu))\tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)x} - x$ for $N = 1000, \tilde{S} = 995, \tilde{R} = 0, \tilde{D} = 0, \beta = 0.15/1000$, $\gamma = 0.05$ and $\mu = 0.01$. In this case we see that $F(N,\tilde{D},\tilde{R},\gamma,\mu) - (\gamma/(\gamma+\mu))\tilde{S} = (5/6)(N-\tilde{S}) = 25/6$ and $0 < \alpha = 744.48\cdots < 1000$.

If $a_{n+1} \geq a_n$, then

$$\begin{aligned} a_{n+2} - a_{n+1} &= \frac{\gamma}{\gamma+\mu}\tilde{S}e^{(\beta/\gamma)\tilde{R}}\left(e^{-(\beta/\gamma)a_n} - e^{-(\beta/\gamma)a_{n+1}}\right) \\ &\geq 0. \end{aligned}$$

Therefore we find that $a_{n+2} \geq a_{n+1}$, and hence the sequence $\{a_n\}$ is nondecreasing by the mathematical induction. We observe that the sequence $\{a_n\}$ is bounded because

$$|a_{n+1}| \leq F(N,\tilde{D},\tilde{R},\gamma,\mu) + \frac{\gamma}{\gamma+\mu}\tilde{S}e^{(\beta/\gamma)\tilde{R}}.$$

Since $\{a_n\}$ is nondecreasing and bounded, there exists $\lim_{n\to\infty} a_n = \alpha$. Taking the limit as $n \to \infty$ in (2.15), we have

$$\alpha = F(N,\tilde{D},\tilde{R},\gamma,\mu) - \frac{\gamma}{\gamma+\mu}\tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)\alpha}. \tag{2.16}$$

The straight line $y = F(N,\tilde{D},\tilde{R},\gamma,\mu) - x$ and the exponential curve $y = (\gamma/(\gamma+\mu))\tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)x}$ has only one intersecting point in $0 < x < N$ by virtue of (2.14), and so the uniqueness of α follows. We claim that $\tilde{R} < \alpha < N$. Since

$$\begin{aligned} F(N,\tilde{D},\tilde{R},\gamma,\mu) &= \frac{\gamma}{\gamma+\mu}N - \frac{\gamma}{\gamma+\mu}\tilde{D} + \frac{\mu}{\gamma+\mu}\tilde{R} \\ &= N - \frac{\mu}{\gamma+\mu}(N-\tilde{R}) - \frac{\gamma}{\gamma+\mu}\tilde{D} \\ &\leq N, \end{aligned}$$

we obtain

$$\begin{aligned} \alpha &= F(N,\tilde{D},\tilde{R},\gamma,\mu) - \frac{\gamma}{\gamma+\mu}\tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)\alpha} \\ &< F(N,\tilde{D},\tilde{R},\gamma,\mu) \leq N. \end{aligned}$$

The inequality $\alpha > \tilde{R}$ follows from the following inequality

$$\begin{aligned}\alpha &\geq F(N,\tilde{D},\tilde{R},\gamma,\mu) - \frac{\gamma}{\gamma+\mu}\tilde{S}e^{(\beta/\gamma)\tilde{R}} \\ &= \frac{\gamma}{\gamma+\mu}\left(N-\tilde{D}-\tilde{S}e^{(\beta/\gamma)\tilde{R}}\right) + \frac{\mu}{\gamma+\mu}\tilde{R} \\ &> \frac{\gamma}{\gamma+\mu}\tilde{R} + \frac{\mu}{\gamma+\mu}\tilde{R} = \tilde{R}\end{aligned}$$

in view of (1.6). □

We assume that the following hypothesis

(A$_5$) $\tilde{S} < \dfrac{\gamma+\mu}{\beta}e^{(\beta/\gamma)(\alpha-\tilde{R})}$

holds in the rest of this paper. We note that (A$_5$) is equivalent to the following

(A$_5'$) $\dfrac{\gamma+\mu}{\beta} > N - \tilde{D} + \dfrac{\mu}{\gamma}\tilde{R} - \dfrac{\gamma+\mu}{\gamma}\alpha$

in view of $\tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)\alpha} = N - \tilde{D} + (\mu/\gamma)\tilde{R} - ((\gamma+\mu)/\gamma)\,\alpha$.

Theorem 2.5. *The initial value problem* (1.1)–(1.5) *has the solution*

$$S(t) = \tilde{S}e^{(\beta/\gamma)\tilde{R}}\varphi^{-1}(t), \tag{2.17}$$

$$I(t) = N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} - \tilde{S}e^{(\beta/\gamma)\tilde{R}}\varphi^{-1}(t) + \frac{\gamma+\mu}{\beta}\log\varphi^{-1}(t), \tag{2.18}$$

$$R(t) = -\frac{\gamma}{\beta}\log\varphi^{-1}(t), \tag{2.19}$$

$$D(t) = -\frac{\mu}{\beta}\log\varphi^{-1}(t) + \tilde{D} - \frac{\mu}{\gamma}\tilde{R}, \tag{2.20}$$

where $\varphi^{-1}(t)$ denotes the inverse function of $\varphi : (e^{-(\beta/\gamma)\alpha}, e^{-(\beta/\gamma)\tilde{R}}] \to [0,\infty)$ such that

$$t = \varphi(u) := \int_u^{e^{-(\beta/\gamma)\tilde{R}}} \frac{d\xi}{\xi\psi(\xi)},$$

where $\psi(\xi)$ is given by (2.12).

Proof. We note that $\varphi(u) \in C^1(e^{-(\beta/\gamma)\alpha}, e^{-(\beta/\gamma)\tilde{R}})$, $\varphi(u)$ is decreasing in $(e^{-(\beta/\gamma)\alpha}, e^{-(\beta/\gamma)\tilde{R}}]$, $\varphi\big(e^{-(\beta/\gamma)\tilde{R}}\big) = 0$. We claim that $\lim_{u\to e^{-(\beta/\gamma)\alpha}+0}\varphi(u) = \infty$. A little calculation yields

$$\begin{aligned}\lim_{\xi\to e^{-(\beta/\gamma)\alpha}+0}\psi(\xi) &= \lim_{\xi\to e^{-(\beta/\gamma)\alpha}+0}\left(\beta N - \beta\tilde{D} + \frac{\beta\mu}{\gamma}\tilde{R} - \beta\tilde{S}e^{(\beta/\gamma)\tilde{R}}\xi + (\gamma+\mu)\log\xi\right) \\ &= \lim_{x\to\alpha-0}\beta\left(N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} - \tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)x} - \frac{\gamma+\mu}{\gamma}x\right) \\ &= \beta\left(N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} - \tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)\alpha} - \frac{\gamma+\mu}{\gamma}\alpha\right) \\ &= 0\end{aligned} \tag{2.21}$$

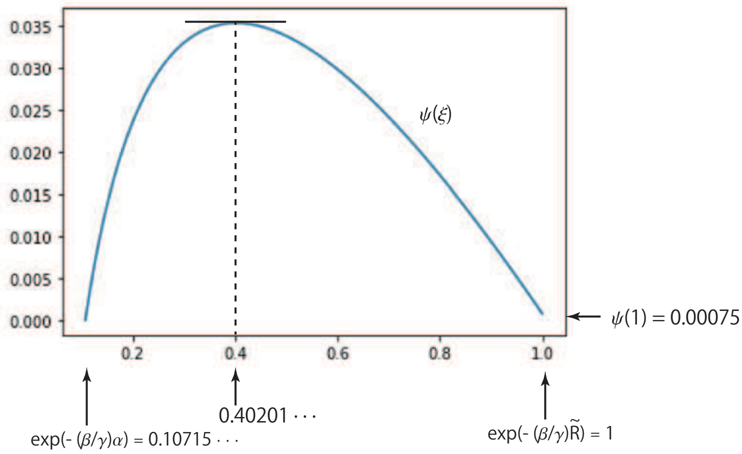

Figure 2.2: Variation of $\psi(\xi) = \beta N - \beta\tilde{D} + (\beta\mu/\gamma)\tilde{R} - \beta\tilde{S}e^{(\beta/\gamma)\tilde{R}}\xi + (\gamma+\mu)\log\xi$ for $N = 1000, \tilde{S} = 995, \tilde{I} = 5, \tilde{R} = 0, \tilde{D} = 0, \beta = 0.15/1000$, $\gamma = 0.05$ and $\mu = 0.01$. In this case we observe that $e^{-(\beta/\gamma)\alpha} = 0.10715\ldots$, $e^{-(\beta/\gamma)\tilde{R}} = 1$, $\psi(1) = \beta N - \beta\tilde{S} = \beta\tilde{I} = 0.00075$, $\psi'(\xi) = 0$ for $\xi = 0.40201\ldots$, $(\gamma+\mu)/\beta = 40$, $((\gamma+\mu)/\beta)e^{(\beta/\gamma)\alpha} > 2000$, and $(\gamma+\mu)/\beta < \tilde{S} < ((\gamma+\mu)/\beta)e^{(\beta/\gamma)\alpha}$.

in view of (2.16). Taking account of the hypotheses (A_1) and (A_5), we find that $\psi(\xi) > 0$ in $(e^{-(\beta/\gamma)\alpha}, e^{-(\beta/\gamma)\tilde{R}}]$ by the same arguments as in the proof of Theorem 2.3. Since

$$\frac{1}{\xi\psi(\xi)} = \frac{\beta}{\gamma+\mu}\tilde{S}e^{(\beta/\gamma)\tilde{R}}\frac{1}{\psi(\xi)} + \frac{1}{\gamma+\mu}\frac{-\beta\tilde{S}e^{(\beta/\gamma)\tilde{R}} + (\gamma+\mu)/\xi}{\psi(\xi)},$$

we get

$$\begin{aligned}
\varphi(u) &= \int_u^{e^{-(\beta/\gamma)\tilde{R}}} \frac{d\xi}{\xi\psi(\xi)} \\
&= \frac{\beta}{\gamma+\mu}\tilde{S}e^{(\beta/\gamma)\tilde{R}} \int_u^{e^{-(\beta/\gamma)\tilde{R}}} \frac{d\xi}{\psi(\xi)} + \frac{1}{\gamma+\mu}\int_u^{e^{-(\beta/\gamma)\tilde{R}}} \frac{\psi'(\xi)}{\psi(\xi)}d\xi \\
&= \frac{\beta}{\gamma+\mu}\tilde{S}e^{(\beta/\gamma)\tilde{R}} \int_u^{e^{-(\beta/\gamma)\tilde{R}}} \frac{d\xi}{\psi(\xi)} + \frac{1}{\gamma+\mu}\left(\log\, \psi(e^{-(\beta/\gamma)\tilde{R}}) - \log\, \psi(u)\right) \\
&= \frac{\beta}{\gamma+\mu}\tilde{S}e^{(\beta/\gamma)\tilde{R}} \int_u^{e^{-(\beta/\gamma)\tilde{R}}} \frac{d\xi}{\psi(\xi)} + \frac{1}{\gamma+\mu}\left(\log\,(\beta\tilde{I}) - \log\, \psi(u)\right). \qquad (2.22)
\end{aligned}$$

Therefore, we see from (2.21) and (2.22) that

$$\lim_{u\to e^{-(\beta/\gamma)\alpha}+0} \varphi(u) = \lim_{u\to e^{-(\beta/\gamma)\alpha}+0} \int_u^{e^{-(\beta/\gamma)\tilde{R}}} \frac{d\xi}{\xi\psi(\xi)} = \infty.$$

Then we conclude that $\varphi^{-1}(t) \in C^1(0,\infty)$, $\varphi^{-1}(t)$ is decreasing on $[0,\infty)$, and that

$$\varphi^{-1}(0) = e^{-(\beta/\gamma)\tilde{R}},$$
$$\lim_{t\to\infty}\varphi^{-1}(t) = e^{-(\beta/\gamma)\alpha}.$$

It is easily verified that

$$\begin{aligned}
\left(\varphi^{-1}(t)\right)' &= \frac{1}{\varphi'(u)}\Big|_{u=\varphi^{-1}(t)} = -u\psi(u)\Big|_{u=\varphi^{-1}(t)} \\
&= -\varphi^{-1}(t)\psi(\varphi^{-1}(t)) = -\beta I(t)\varphi^{-1}(t), \qquad (2.23)
\end{aligned}$$

in light of

$$\psi(\varphi^{-1}(t)) = \beta I(t). \tag{2.24}$$

We observe, using (2.17)–(2.20) and (2.23), that

$$\begin{aligned}
S'(t) &= \tilde{S}e^{(\beta/\gamma)\tilde{R}} \left(\varphi^{-1}(t)\right)' = \tilde{S}e^{(\beta/\gamma)\tilde{R}} \left(-\beta I(t)\varphi^{-1}(t)\right) \\
&= -\beta\tilde{S}e^{(\beta/\gamma)\tilde{R}}\varphi^{-1}(t)I(t) = -\beta S(t)I(t), \\
I'(t) &= -\tilde{S}e^{(\beta/\gamma)\tilde{R}} \left(\varphi^{-1}(t)\right)' + \frac{\gamma+\mu}{\beta}\frac{(\varphi^{-1}(t))'}{\varphi^{-1}(t)} \\
&= -(-\beta S(t)I(t)) + \frac{\gamma+\mu}{\beta}(-\beta I(t)) = \beta S(t)I(t) - \gamma I(t) - \mu I(t), \\
R'(t) &= -\frac{\gamma}{\beta}\frac{(\varphi^{-1}(t))'}{\varphi^{-1}(t)} = -\frac{\gamma}{\beta}(-\beta I(t)) = \gamma I(t), \\
D'(t) &= -\frac{\mu}{\beta}\frac{(\varphi^{-1}(t))'}{\varphi^{-1}(t)} = -\frac{\mu}{\beta}(-\beta I(t)) = \mu I(t),
\end{aligned}$$

and consequently (2.17)–(2.20) satisfy (1.1)–(1.4), respectively. It is easy to check that

$$\begin{aligned}
S(0) &= \tilde{S}e^{(\beta/\gamma)\tilde{R}}\varphi^{-1}(0) = \tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)\tilde{R}} = \tilde{S}, \\
R(0) &= -\frac{\gamma}{\beta}\log\varphi^{-1}(0) = -\frac{\gamma}{\beta}\log e^{-(\beta/\gamma)\tilde{R}} = \tilde{R}, \\
D(0) &= -\frac{\mu}{\beta}\log\varphi^{-1}(0) + \tilde{D} - \frac{\mu}{\gamma}\tilde{R} \\
&= -\frac{\mu}{\beta}\log e^{-(\beta/\gamma)\tilde{R}} + \tilde{D} - \frac{\mu}{\gamma}\tilde{R} \\
&= -\frac{\mu}{\beta}\left(-\frac{\beta}{\gamma}\tilde{R}\right) + \tilde{D} - \frac{\mu}{\gamma}\tilde{R} = \tilde{D}, \\
I(0) &= N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} - \tilde{S}e^{(\beta/\gamma)\tilde{R}}\varphi^{-1}(0) + \frac{\gamma+\mu}{\beta}\log\varphi^{-1}(0) \\
&= N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} - \tilde{S} + \frac{\gamma+\mu}{\beta}\left(-\frac{\beta}{\gamma}\tilde{R}\right) \\
&= N - \tilde{D} - \tilde{S} - \tilde{R} = \tilde{I}.
\end{aligned}$$

□

3 Various properties of the exact solution

We can derive various properties of solutions of SIRD epidemic model via the differential system qualitatively, however we obtain more detailed properties directly from the exact solution of the SIRD differential system.

Theorem 3.1. *We observe that $I(\infty) = 0$ and $I(t) > 0$ on $[0,\infty)$, and that $I(t)$ has the maximum*

$$\max_{t\geq 0} I(t) = N - \tilde{D} - \tilde{R} - \frac{\gamma+\mu}{\beta}\left(1 + \log\tilde{S} - \log\frac{\gamma+\mu}{\beta}\right)$$

at

$$t = T := \varphi\left(\frac{\gamma+\mu}{\beta\tilde{S}e^{(\beta/\gamma)\tilde{R}}}\right) = S^{-1}\left(\frac{\gamma+\mu}{\beta}\right).$$

Furthermore, $I(t)$ is increasing in $[0,T)$ and is decreasing in (T,∞).

Proof. Taking account of (2.16), we easily check that

$$\begin{aligned}
I(\infty) &= \lim_{t\to\infty} I(t) \\
&= \lim_{u\to e^{-(\beta/\gamma)\alpha}+0} \left(N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} - \tilde{S}e^{(\beta/\gamma)\tilde{R}}u + \frac{\gamma+\mu}{\beta}\log u\right) \\
&= \frac{\gamma+\mu}{\gamma}\left(F(N,\tilde{D},\tilde{R},\gamma,\mu) - \frac{\gamma}{\gamma+\mu}\tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)\alpha} - \alpha\right) \\
&= 0.
\end{aligned}$$

Since $e^{-(\beta/\gamma)\alpha} < \varphi^{-1}(t) \le e^{-(\beta/\gamma)\tilde{R}}$ for $t \ge 0$ and $\psi(\xi) > 0$ for $e^{-(\beta/\gamma)\alpha} < \xi \le e^{-(\beta/\gamma)\tilde{R}}$, we find that $I(t) = (1/\beta)\psi\big(\varphi^{-1}(t)\big) > 0$ on $[0,\infty)$. Since $e^{-(\beta/\gamma)\alpha} < \varphi^{-1}(t) \le e^{-(\beta/\gamma)\tilde{R}}$ for $t > 0$ and $\psi(\xi) > 0$ for $e^{-(\beta/\gamma)\alpha} < \xi \le e^{-(\beta/\gamma)\tilde{R}}$, we see that $I(t) = (1/\beta)\psi\big(\varphi^{-1}(t)\big)$ given by (2.24) is positive for $t > 0$. Differentiating both sides of (2.24), we arrive at

$$\begin{aligned}
I'(t) &= \frac{1}{\beta}\psi'\big(\varphi^{-1}(t)\big)\big(\varphi^{-1}(t)\big)' \\
&= \frac{1}{\beta}\left(-\beta\tilde{S}e^{(\beta/\gamma)\tilde{R}} + \frac{\gamma+\mu}{\varphi^{-1}(t)}\right)\big(\varphi^{-1}(t)\big)' \\
&= \frac{1}{\beta}\left(-\beta S(t) + \gamma + \mu\right)\frac{\big(\varphi^{-1}(t)\big)'}{\varphi^{-1}(t)} \\
&= \frac{1}{\beta}\left(-\beta S(t) + \gamma + \mu\right)\left(-\beta I(t)\right) \\
&= \big(\beta S(t) - (\gamma+\mu)\big)I(t)
\end{aligned} \tag{3.1}$$

in view of (2.23). It is obvious that $I'(t) = 0$ holds if and only if

$$\varphi^{-1}(t) = \frac{\gamma+\mu}{\beta\tilde{S}e^{(\beta/\gamma)\tilde{R}}}$$

or

$$S(t) = \frac{\gamma+\mu}{\beta},$$

which yield

$$t = T = \varphi\left(\frac{\gamma+\mu}{\beta\tilde{S}e^{(\beta/\gamma)\tilde{R}}}\right) = S^{-1}\left(\frac{\gamma+\mu}{\beta}\right).$$

We note that

$$e^{-(\beta/\gamma)\alpha} < \frac{\gamma+\mu}{\beta\tilde{S}e^{(\beta/\gamma)\tilde{R}}} = \frac{\gamma+\mu}{\beta\tilde{S}}e^{-(\beta/\gamma)\tilde{R}} < e^{-(\beta/\gamma)\tilde{R}}$$

in light of the hypotheses (A_1) and (A_5). Since $\big(\varphi^{-1}(t)\big)' < 0$ and $\varphi^{-1}(t)$ is decreasing on $[0,\infty)$, we observe that $I'(t) > 0$ [resp. < 0] if and only if $t < T$ [resp. $> T$]. Hence, $I(t)$ is increasing in $[0,T)$ and is decreasing in (T,∞). We find that the maximum of $I(t)$ on $[0,\infty)$ is given by

$$\begin{aligned}
\frac{1}{\beta}\psi\left(\frac{\gamma+\mu}{\beta\tilde{S}e^{(\beta/\gamma)\tilde{R}}}\right) &= N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} - \tilde{S}e^{(\beta/\gamma)\tilde{R}}\frac{\gamma+\mu}{\beta\tilde{S}e^{(\beta/\gamma)\tilde{R}}} + \frac{\gamma+\mu}{\beta}\log\left(\frac{\gamma+\mu}{\beta\tilde{S}e^{(\beta/\gamma)\tilde{R}}}\right) \\
&= N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} - \frac{\gamma+\mu}{\beta} + \frac{\gamma+\mu}{\beta}\left(\log\frac{\gamma+\mu}{\beta} - \log\tilde{S} - \frac{\beta}{\gamma}\tilde{R}\right) \\
&= N - \tilde{D} - \tilde{R} - \frac{\gamma+\mu}{\beta}\left(1 + \log\tilde{S} - \log\frac{\gamma+\mu}{\beta}\right). \qquad \square
\end{aligned}$$

Corollary 3.2. *The function $I(t)$ has the maximum at*

$$
\begin{aligned}
T &= \varphi\left(\frac{\gamma+\mu}{\beta\tilde{S}e^{(\beta/\gamma)\tilde{R}}}\right) \\
&= \frac{\beta}{\gamma+\mu}\tilde{S}e^{(\beta/\gamma)\tilde{R}}\int_{(\gamma+\mu)/(\beta\tilde{S}e^{(\beta/\gamma)\tilde{R}})}^{e^{-(\beta/\gamma)\tilde{R}}}\frac{d\xi}{\psi(\xi)} \\
&\quad+\frac{1}{\gamma+\mu}\left(\log(\beta\tilde{I})-\log\left(\beta H(N,\tilde{S},\tilde{R},\tilde{D},\beta,\gamma,\mu)\right)\right),
\end{aligned}
$$

and T satisfies the following inequality

$$\tau_1 \le T \le \tau_2,$$

where

$$\tau_1 = \frac{(\beta/(\gamma+\mu))\tilde{S}-1}{\beta H(N,\tilde{S},\tilde{R},\tilde{D},\beta,\gamma)} + \frac{1}{\gamma+\mu}\left(\log(\beta\tilde{I})-\log\left(\beta H(N,\tilde{S},\tilde{R},\tilde{D},\beta,\gamma,\mu)\right)\right)$$

and

$$\tau_2 = \frac{(\beta/(\gamma+\mu))\tilde{S}-1}{\beta\tilde{I}} + \frac{1}{\gamma+\mu}\left(\log(\beta\tilde{I})-\log\left(\beta H(N,\tilde{S},\tilde{R},\tilde{D},\beta,\gamma,\mu)\right)\right)$$

with $H(N,\tilde{S},\tilde{R},\tilde{D},\beta,\gamma,\mu)$ being

$$H(N,\tilde{S},\tilde{R},\tilde{D},\beta,\gamma,\mu) := N-\tilde{D}-\tilde{R}-\frac{\gamma+\mu}{\beta}\left(1+\log\tilde{S}-\log\frac{\gamma+\mu}{\beta}\right).$$

Proof. It follows from (2.22) that

$$
\begin{aligned}
T &= \varphi\left(\frac{\gamma+\mu}{\beta\tilde{S}e^{(\beta/\gamma)\tilde{R}}}\right) \\
&= \frac{\beta}{\gamma+\mu}\tilde{S}e^{(\beta/\gamma)\tilde{R}}\int_{(\gamma+\mu)/(\beta\tilde{S}e^{(\beta/\gamma)\tilde{R}})}^{e^{-(\beta/\gamma)\tilde{R}}}\frac{d\xi}{\psi(\xi)} \\
&\quad+\frac{1}{\gamma+\mu}\left(\log(\beta\tilde{I})-\log\left(\beta H(N,\tilde{S},\tilde{R},\tilde{D},\beta,\gamma,\mu)\right)\right)
\end{aligned}
$$

because of

$$\psi\left(\frac{\gamma+\mu}{\beta\tilde{S}e^{(\beta/\gamma)\tilde{R}}}\right) = \beta H(N,\tilde{S},\tilde{R},\tilde{D},\beta,\gamma,\mu).$$

From (2.12) we see that

$$\psi'(\xi) = -\beta\tilde{S}e^{(\beta/\gamma)\tilde{R}} + \frac{\gamma+\mu}{\xi},$$

and that $\psi'\left((\gamma+\mu)/(\beta\tilde{S}e^{(\beta/\gamma)\tilde{R}})\right) = 0$, $\psi(e^{-(\beta/\gamma)\tilde{R}}) = \beta\tilde{I}$, and $\psi(\xi)$ is decreasing on $[(\gamma+\mu)/(\beta\tilde{S}e^{(\beta/\gamma)\tilde{R}}), e^{-(\beta/\gamma)\tilde{R}}]$. Then we get

$$\beta\tilde{I} \le \psi(\xi) \le \beta H(N,\tilde{S},\tilde{R},\tilde{D},\beta,\gamma,\mu),$$

and hence

$$\frac{1}{\beta H(N,\tilde{S},\tilde{R},\tilde{D},\beta,\gamma,\mu)} \le \frac{1}{\psi(\xi)} \le \frac{1}{\beta\tilde{I}}.$$

Integrating the above inequality over $[(\gamma+\mu)/(\beta\tilde{S}e^{(\beta/\gamma)\tilde{R}}), e^{-(\beta/\gamma)\tilde{R}}]$, and then multiplying by $(\beta/(\gamma+\mu))\tilde{S}e^{(\beta/\gamma)\tilde{R}}$, we are led to

$$\begin{aligned}\frac{(\beta/(\gamma+\mu))\tilde{S}-1}{\beta H(N,\tilde{S},\tilde{R},\tilde{D},\beta,\gamma,\mu)} &\le \frac{\beta}{\gamma+\mu}\tilde{S}e^{(\beta/\gamma)\tilde{R}}\int_{(\gamma+\mu)/(\beta\tilde{S}e^{(\beta/\gamma)\tilde{R}})}^{e^{-(\beta/\gamma)\tilde{R}}}\frac{d\xi}{\psi(\xi)}\\ &\le \frac{(\beta/(\gamma+\mu))\tilde{S}-1}{\beta\tilde{I}},\end{aligned}$$

which yields the desired inequality. □

Theorem 3.3. *We find that* $R(\infty)=\alpha$*,*

$$R(\infty)=N-\tilde{D}+\frac{\mu}{\gamma}\tilde{R}-\tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)R(\infty)}-\frac{\mu}{\gamma}R(\infty), \tag{3.2}$$

and that $R(t)$ *is an increasing function on* $[0,\infty)$ *such that*

$$\tilde{R}\le R(t)<\alpha=R(\infty).$$

Proof. It follows from (2.19) that

$$\begin{aligned}R(\infty)=\lim_{t\to\infty}R(t)&=\lim_{t\to\infty}-\frac{\gamma}{\beta}\log\varphi^{-1}(t)\\ &=\lim_{u\to e^{-(\beta/\gamma)\alpha}+0}-\frac{\gamma}{\beta}\log u\\ &=\alpha.\end{aligned}$$

Since $\alpha=R(\infty)$, the identity (3.2) follows from (2.16). Since $e^{-(\beta/\gamma)\alpha}<\varphi^{-1}(t)\le e^{-(\beta/\gamma)\tilde{R}}$, we obtain

$$-\frac{\gamma}{\beta}\log e^{-(\beta/\gamma)\tilde{R}}\le R(t)<-\frac{\gamma}{\beta}\log e^{-(\beta/\gamma)\alpha},$$

or

$$\tilde{R}\le R(t)<\alpha=R(\infty).$$

It is obvious that $R(t)$ is increasing on $[0,\infty)$ in view of the fact that $\varphi^{-1}(t)$ is decreasing $[0,\infty)$. □

Theorem 3.4. *We see that*

$$S(\infty)=\tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)R(\infty)}, \tag{3.3}$$

and that $S(t)$ *is a decreasing function on* $[0,\infty)$ *such that*

$$\tilde{S}\ge S(t)>\tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)\alpha}=S(\infty).$$

Proof. The identity (3.3) follows from

$$\begin{aligned}S(\infty)=\lim_{t\to\infty}S(t)&=\lim_{t\to\infty}\tilde{S}e^{(\beta/\gamma)\tilde{R}}\varphi^{-1}(t)\\ &=\lim_{u\to e^{-(\beta/\gamma)\alpha}+0}\tilde{S}e^{(\beta/\gamma)\tilde{R}}u\\ &=\tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)\alpha}\\ &=\tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)R(\infty)}.\end{aligned}$$

Since $e^{-(\beta/\gamma)\alpha} < \varphi^{-1}(t) \le e^{-(\beta/\gamma)\tilde{R}}$, we have

$$\tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)\alpha} < \tilde{S}e^{(\beta/\gamma)\tilde{R}}\varphi^{-1}(t) \le \tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)\tilde{R}}.$$

Hence we obtain

$$\tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)\alpha} < S(t) \le \tilde{S}.$$

Since $\varphi^{-1}(t)$ is decreasing on $[0,\infty)$, we deduce that $S(t)$ is also decreasing on $[0,\infty)$. □

Theorem 3.5. *The following holds:*

$$D(\infty) = \frac{\mu}{\gamma}R(\infty) + \tilde{D} - \frac{\mu}{\gamma}\tilde{R}, \tag{3.4}$$

and $D(t)$ is an increasing function on $[0,\infty)$ such that

$$\tilde{D} \le D(t) < D(\infty).$$

Proof. Taking account of (2.20), we obtain

$$\begin{aligned}\lim_{t\to\infty} D(t) &= \lim_{t\to\infty}\left(-\frac{\mu}{\beta}\log\varphi^{-1}(t)\right) + \tilde{D} - \frac{\mu}{\gamma}\tilde{R} \\ &= \lim_{u\to e^{-(\beta/\gamma)\alpha}+0}\left(-\frac{\mu}{\beta}\log u\right) + \tilde{D} - \frac{\mu}{\gamma}\tilde{R} \\ &= \frac{\mu}{\gamma}\alpha + \tilde{D} - \frac{\mu}{\gamma}\tilde{R} \\ &= \frac{\mu}{\gamma}R(\infty) + \tilde{D} - \frac{\mu}{\gamma}\tilde{R},\end{aligned}$$

which is the desired identity (3.4). Since $e^{-(\beta/\gamma)\alpha} < \varphi^{-1}(t) \le e^{-(\beta/\gamma)\tilde{R}}$, we get

$$-\frac{\beta}{\gamma}\alpha < \log\varphi^{-1}(t) \le -\frac{\beta}{\gamma}\tilde{R},$$

and hence

$$\frac{\mu}{\gamma}\tilde{R} \le -\frac{\mu}{\beta}\log\varphi^{-1}(t) < \frac{\mu}{\gamma}\alpha,$$

which implies

$$\tilde{D} \le D(t) < \frac{\mu}{\gamma}\alpha + \tilde{D} - \frac{\mu}{\gamma}\tilde{R} = \frac{\mu}{\gamma}R(\infty) + \tilde{D} - \frac{\mu}{\gamma}\tilde{R}.$$

We conclude that $D(t)$ is increasing on $[0,\infty)$ since $\varphi^{-1}(t)$ is decreasing on $[0,\infty)$. □

Theorem 3.6. *If*

$$\tilde{S} \le \frac{\gamma+\mu}{\beta} + \frac{1}{2}\left(\tilde{I} + \sqrt{\frac{4(\gamma+\mu)}{\beta}\tilde{I} + \tilde{I}^2}\right), \tag{3.5}$$

then there exists a number T_1 $(T < T_1)$ such that $I(t)$ is concave in $(0,T_1)$, and is convex in (T_1,∞). If

$$\tilde{S} > \frac{\gamma+\mu}{\beta} + \frac{1}{2}\left(\tilde{I} + \sqrt{\frac{4(\gamma+\mu)}{\beta}\tilde{I} + \tilde{I}^2}\right), \tag{3.6}$$

then there exist two numbers T_2 and T_3 $(0 < T_2 < T < T_3)$ such that $I(t)$ is convex in $(0,T_1) \cup (T_3,\infty)$, and is concave in (T_2,T_3) (cf. Figures 3.1, 3.2).

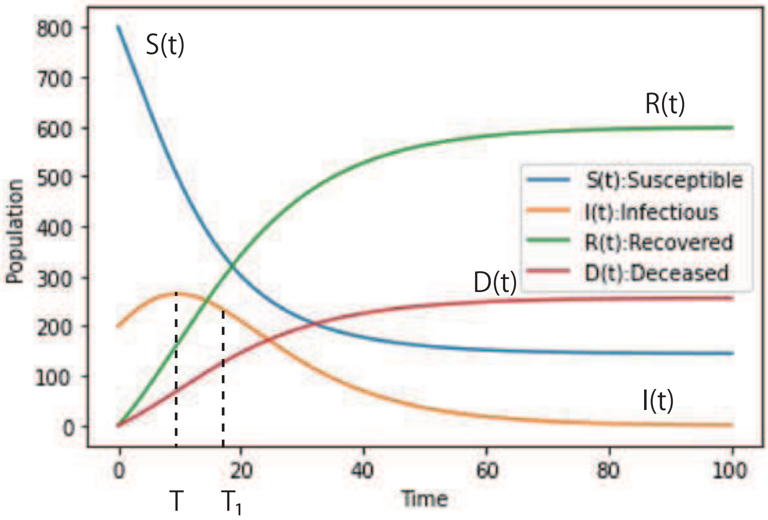

Figure 3.1: Variation of $S(t)$, $I(t)$, $R(t)$ and $D(t)$ obtained by the numerical integration of the initial value problem (1.1)–(1.5) for $N = 1000, \tilde{S} = 800, \tilde{I} = 200, \tilde{R} = \tilde{D} = 0, \beta = 0.2/1000, \gamma = 0.07$ and $\mu = 0.03$. In this case $\tilde{S}(= 800) > (\gamma+\mu)/\beta(= 500) > S(\infty)(= 144.57\ldots)$, and the condition (3.5) is satisfied because $(\gamma+\mu)/\beta = 500$ and $\tilde{S}(= 800) < (\gamma+\mu)/\beta + (1/2)\left(\tilde{I} + \sqrt{(4(\gamma+\mu)/\beta)\tilde{I} + \tilde{I}^2}\right) (= 931.66\ldots)$.

Proof. First we note that the hypotheses (A_1) and (A'_5) imply that

$$\begin{aligned}\tilde{S} > \frac{\gamma+\mu}{\beta} &> N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} - \frac{\gamma+\mu}{\gamma}\alpha \\ &= N - R(\infty) - \left(\frac{\mu}{\gamma}R(\infty) + \tilde{D} - \frac{\mu}{\gamma}\tilde{R}\right) \\ &= N - R(\infty) - D(\infty) = S(\infty)\end{aligned}$$

in light of $\alpha = R(\infty)$ and $I(\infty) = 0$. Differentiating (3.1) with respect to t and taking (1.1), (1.2) into account, we obtain

$$\begin{aligned}I''(t) &= \big(\beta S'(t)\big)I(t) + \big(\beta S(t) - (\gamma+\mu)\big)I'(t) \\ &= \beta\big(-\beta S(t)I(t)\big)I(t) + \big(\beta S(t) - (\gamma+\mu)\big)\big(\beta S(t)I(t) - \gamma I(t) - \mu I(t)\big) \\ &= \Big(-\beta^2 S(t)I(t) + \beta^2 S(t)^2 - 2\beta(\gamma+\mu)S(t) + (\gamma+\mu)^2\Big)I(t) \\ &= \beta^2\left(S(t)^2 - S(t)I(t) - \frac{2(\gamma+\mu)}{\beta}S(t) + \frac{(\gamma+\mu)^2}{\beta^2}\right)I(t) \\ &= \beta^2\left(S(t) - \frac{2(\gamma+\mu)}{\beta} + \frac{(\gamma+\mu)^2}{\beta^2}\frac{1}{S(t)} - I(t)\right)S(t)I(t).\end{aligned}$$

Now we investigate the sign of $I''(t)$. We define

$$G(t) := S(t) - \frac{2(\gamma+\mu)}{\beta} + \frac{(\gamma+\mu)^2}{\beta^2}\frac{1}{S(t)} - I(t)$$

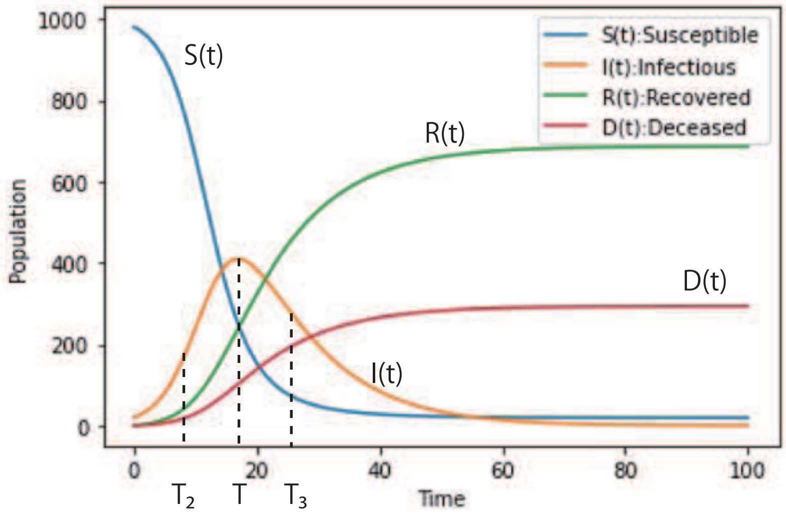

Figure 3.2: Variation of $S(t)$, $I(t)$, $R(t)$ and $D(t)$ obtained by the numerical integration of the initial value problem (1.1)–(1.5) for $N = 1000, \tilde{S} = 980, \tilde{I} = 20, \tilde{R} = \tilde{D} = 0, \beta = 0.4/1000, \gamma = 0.07$ and $\mu = 0.03$. In this case $\tilde{S}(= 980) > (\gamma+\mu)/\beta(= 250) > S(\infty)(= 19.39\ldots)$, and the condition (3.6) is satisfied because $\tilde{S}(= 980) > (\gamma+\mu)/\beta + (1/2)\big(\tilde{I} + \sqrt{(4(\gamma+\mu)/\beta)\tilde{I} + \tilde{I}^2}\big)(= 331.41\ldots)$.

and differentiate both sides of the above with respect to t to obtain

$$\begin{aligned}
G'(t) &= S'(t) - \frac{(\gamma+\mu)^2}{\beta^2}\frac{S'(t)}{S(t)^2} - I'(t) \\
&= S'(t) - \frac{(\gamma+\mu)^2}{\beta^2}\frac{-\beta S(t)I(t)}{S(t)^2} - I'(t) \\
&= -\beta S(t)I(t) + \frac{(\gamma+\mu)^2}{\beta}\frac{I(t)}{S(t)} - \big(\beta S(t)I(t) - \gamma I(t) - \mu I(t)\big) \\
&= -2\beta S(t)I(t) + \frac{(\gamma+\mu)^2}{\beta}\frac{I(t)}{S(t)} + (\gamma+\mu)I(t) \\
&= -2\beta\left(S(t)^2 - \frac{(\gamma+\mu)}{2\beta}S(t) - \frac{(\gamma+\mu)^2}{2\beta^2}\right)\frac{I(t)}{S(t)} \\
&= -2\beta\left(S(t) - \frac{\gamma+\mu}{\beta}\right)\left(S(t) + \frac{\gamma+\mu}{2\beta}\right)\frac{I(t)}{S(t)}.
\end{aligned}$$

Since $S(t) + (\gamma+\mu)/(2\beta) > 0$, it follows that $G'(t) = 0$ for $t = T = S^{-1}((\gamma+\mu)/\beta)$, and that $G'(t) < 0$ [resp. > 0] if $t < T$ [resp. $> T$]. Therefore, $G(t)$ is decreasing in $[0, T)$ and increasing in (T, ∞). It is readily seen that

$$\begin{aligned}
G(0) &= \tilde{S} - \frac{2(\gamma+\mu)}{\beta} + \frac{(\gamma+\mu)^2}{\beta^2}\frac{1}{\tilde{S}} - \tilde{I} \\
&= \frac{1}{\tilde{S}}\left(\tilde{S}^2 - \left(\frac{2(\gamma+\mu)}{\beta} + \tilde{I}\right)\tilde{S} + \frac{(\gamma+\mu)^2}{\beta^2}\right) \\
&= \frac{1}{\tilde{S}}(\tilde{S} - s_1)(\tilde{S} - s_2),
\end{aligned}$$

where

$$s_1 = \frac{\gamma+\mu}{\beta} + \frac{1}{2}\left(\tilde{I} - \sqrt{\frac{4(\gamma+\mu)}{\beta}\tilde{I} + \tilde{I}^2}\right) \left(< \frac{\gamma+\mu}{\beta}\right),$$
$$s_2 = \frac{\gamma+\mu}{\beta} + \frac{1}{2}\left(\tilde{I} + \sqrt{\frac{4(\gamma+\mu)}{\beta}\tilde{I} + \tilde{I}^2}\right) \left(> \frac{\gamma+\mu}{\beta}\right).$$

Moreover we observe that

$$\begin{aligned} G(T) &= S(T) - \frac{2(\gamma+\mu)}{\beta} + \frac{(\gamma+\mu)^2}{\beta^2}\frac{1}{S(T)} - I(T) \\ &= \frac{\gamma+\mu}{\beta} - \frac{2(\gamma+\mu)}{\beta} + \frac{(\gamma+\mu)^2}{\beta^2}\frac{\beta}{\gamma+\mu} - \max_{t\geq 0} I(t) \\ &= -\max_{t\geq 0} I(t) < 0, \end{aligned}$$

and that

$$\begin{aligned} \lim_{t\to\infty} G(t) &= S(\infty) - \frac{2(\gamma+\mu)}{\beta} + \frac{(\gamma+\mu)^2}{\beta^2}\frac{1}{S(\infty)} - I(\infty) \\ &= \frac{1}{S(\infty)}\left(\frac{\gamma+\mu}{\beta} - S(\infty)\right)^2 > 0. \end{aligned}$$

If (3.5) is satisfied, then $G(0) \leq 0$, and therefore there exists a number $T_1 > T$ such that $G(T_1) = 0$, $G(t)$ is negative in $(0, T_1)$, and $G(t)$ is positive in (T_1, ∞). Since $I''(t) = \beta^2 G(t)S(t)I(t)$, we deduce that $I(t)$ is concave in $(0, T_1)$, and is convex in (T_1, ∞). If (3.6) is satisfied, then $G(0) > 0$, and hence there exist two numbers T_2 and T_3 $(0 < T_2 < T < T_3)$ such that $G(T_2) = G(T_3) = 0$, $G(t)$ is positive in $(0, T_2) \cup (T_3, \infty)$, and $G(t)$ is negative in (T_2, T_3). Consequently we conclude that $I(t)$ is convex in $(0, T_2) \cup (T_3, \infty)$, and is concave in (T_2, T_3). □

Theorem 3.7. *The following identity holds:*

$$S(\infty) = \tilde{S} + \tilde{I} + \frac{\gamma+\mu}{\beta}\log\frac{S(\infty)}{\tilde{S}}.$$

Proof. Since $I(\infty) = 0$, we observe, using (3.4), that

$$\begin{aligned} S(\infty) &= N - R(\infty) - D(\infty) \\ &= N - R(\infty) - \frac{\mu}{\gamma}R(\infty) - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} \\ &= N - \tilde{D} - \tilde{R} + \left(1 + \frac{\mu}{\gamma}\right)\tilde{R} - \left(1 + \frac{\mu}{\gamma}\right)R(\infty) \\ &= \tilde{S} + \tilde{I} + \left(1 + \frac{\mu}{\gamma}\right)\frac{\gamma}{\beta}\left(\frac{\beta}{\gamma}\tilde{R} - \frac{\beta}{\gamma}R(\infty)\right) \\ &= \tilde{S} + \tilde{I} + \frac{\gamma+\mu}{\beta}\log\left(e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)R(\infty)}\right) \\ &= \tilde{S} + \tilde{I} + \frac{\gamma+\mu}{\beta}\log\frac{S(\infty)}{\tilde{S}} \end{aligned}$$

by virtue of (3.3). □

Theorem 3.8. *It follows that*

$$S'(\infty) = I'(\infty) = R'(\infty) = D'(\infty) = 0.$$

Proof. Since $I(\infty) = 0$, we conclude that

$$\begin{aligned} S'(\infty) &= -\beta S(\infty)I(\infty) = 0, \\ I'(\infty) &= \beta S(\infty)I(\infty) - \gamma I(\infty) - \mu I(\infty) = 0, \\ R'(\infty) &= \gamma I(\infty) = 0, \\ D'(\infty) &= \mu I(\infty) = 0, \end{aligned}$$

by taking account of (1.1)–(1.4). □

Theorem 3.9. *Let* $(S(t), I(t), R(t), D(t))$ *be the exact solution* (2.17)–(2.20) *of the initial value problem* (1.1)–(1.5)*, and let*

$$\left(\hat{S}(u), \hat{I}(u), \hat{R}(u), \hat{D}(u)\right) := \left(S(\varphi(u)), I(\varphi(u)), R(\varphi(u)), D(\varphi(u))\right).$$

Then $\left(\hat{S}(u), \hat{I}(u), \hat{R}(u), \hat{D}(u)\right)$ *is a solution of the initial value problem for the linear differential system*

$$\frac{d\hat{S}(u)}{du} = \frac{\hat{S}(u)}{u}, \tag{3.7}$$

$$\frac{d\hat{I}(u)}{du} = -\frac{\hat{S}(u)}{u} + \frac{\gamma}{\beta}\frac{1}{u} + \frac{\mu}{\beta}\frac{1}{u}, \tag{3.8}$$

$$\frac{d\hat{R}(u)}{du} = -\frac{\gamma}{\beta}\frac{1}{u}, \tag{3.9}$$

$$\frac{d\hat{D}(u)}{du} = -\frac{\mu}{\beta}\frac{1}{u} \tag{3.10}$$

for $u \in (e^{-(\beta/\gamma)\alpha}, e^{-(\beta/\gamma)\tilde{R}})$*, with the initial condition*

$$\hat{S}\left(e^{-(\beta/\gamma)\tilde{R}}\right) = \tilde{S}, \tag{3.11}$$

$$\hat{I}\left(e^{-(\beta/\gamma)\tilde{R}}\right) = \tilde{I}, \tag{3.12}$$

$$\hat{R}\left(e^{-(\beta/\gamma)\tilde{R}}\right) = \tilde{R}, \tag{3.13}$$

$$\hat{D}\left(e^{-(\beta/\gamma)\tilde{R}}\right) = \tilde{D}. \tag{3.14}$$

Proof. First we remark that

$$\hat{I}(u) = I(\varphi(u)) = \frac{1}{\beta}\psi(u) \tag{3.15}$$

in light of (2.24). Noting

$$S'(\varphi(u)) = -\beta S(\varphi(u))I(\varphi(u)) = -\beta\hat{S}(u)\hat{I}(u),$$

we are led to

$$\begin{aligned} \frac{d\hat{S}(u)}{du} &= \left.\frac{dS(t)}{dt}\right|_{t=\varphi(u)} \varphi'(u) = S'(\varphi(u))\left(-\frac{1}{u\psi(u)}\right) \\ &= (-\beta\hat{S}(u)\hat{I}(u))\left(-\frac{1}{u\psi(u)}\right) = \beta\hat{S}(u)\hat{I}(u)\left(\frac{1}{u\psi(u)}\right) \\ &= \frac{\hat{S}(u)}{u} \end{aligned}$$

in view of (3.15). Similarly we obtain

$$\begin{aligned}\frac{d\hat{R}(u)}{du} &= \frac{dR(t)}{dt}\Big|_{t=\varphi(u)}\varphi'(u) = R'(\varphi(u))\left(-\frac{1}{u\psi(u)}\right)\\ &= \left(\gamma\hat{I}(u)\right)\left(-\frac{1}{u\psi(u)}\right)\\ &= -\frac{\gamma}{\beta}\frac{1}{u},\end{aligned}$$

$$\begin{aligned}\frac{d\hat{D}(u)}{du} &= \frac{dD(t)}{dt}\Big|_{t=\varphi(u)}\varphi'(u) = D'(\varphi(u))\left(-\frac{1}{u\psi(u)}\right)\\ &= \left(\mu\hat{I}(u)\right)\left(-\frac{1}{u\psi(u)}\right)\\ &= -\frac{\mu}{\beta}\frac{1}{u},\end{aligned}$$

and

$$\begin{aligned}\frac{d\hat{I}(u)}{du} &= \frac{dI(t)}{dt}\Big|_{t=\varphi(u)}\varphi'(u) = I'(\varphi(u))\left(-\frac{1}{u\psi(u)}\right)\\ &= \left(\beta\hat{S}(u)\hat{I}(u) - \gamma\hat{I}(u) - \mu\hat{I}(u)\right)\left(-\frac{1}{u\psi(u)}\right)\\ &= -\frac{\hat{S}(u)}{u} + \frac{\gamma}{\beta}\frac{1}{u} + \frac{\mu}{\beta}\frac{1}{u}.\end{aligned}$$

It is clear that

$$\begin{aligned}\hat{S}\left(e^{-(\beta/\gamma)\tilde{R}}\right) &= S\left(\varphi\left(e^{-(\beta/\gamma)\tilde{R}}\right)\right) = S(0) = \tilde{S},\\ \hat{I}\left(e^{-(\beta/\gamma)\tilde{R}}\right) &= I\left(\varphi\left(e^{-(\beta/\gamma)\tilde{R}}\right)\right) = I(0) = \tilde{I},\\ \hat{R}\left(e^{-(\beta/\gamma)\tilde{R}}\right) &= R\left(\varphi\left(e^{-(\beta/\gamma)\tilde{R}}\right)\right) = R(0) = \tilde{R},\\ \hat{D}\left(e^{-(\beta/\gamma)\tilde{R}}\right) &= D\left(\varphi\left(e^{-(\beta/\gamma)\tilde{R}}\right)\right) = D(0) = \tilde{D}.\end{aligned}$$

Hence, $\left(\hat{S}(u), \hat{I}(u), \hat{R}(u), \hat{D}(u)\right)$ is a solution of the initial value problem (3.7)–(3.14). □

Theorem 3.10. *Solving the initial value problem* (3.7)–(3.14), *we obtain the solution* (2.4)–(2.7) *for* $u \in (e^{-(\beta/\gamma)\alpha}, e^{-(\beta/\gamma)\tilde{R}}]$.

Proof. Since (3.7) is equivalent to

$$\frac{d}{du}\left(\frac{1}{u}\hat{S}(u)\right) = 0,$$

we derive

$$\hat{S}(u) = ku$$

for some constant k. It follows from (3.11) that

$$\hat{S}\left(e^{-(\beta/\gamma)\tilde{R}}\right) = ke^{-(\beta/\gamma)\tilde{R}} = \tilde{S}$$

and hence

$$k = \tilde{S}e^{(\beta/\gamma)\tilde{R}},$$

which yields

$$\hat{S}(u) = \tilde{S}e^{(\beta/\gamma)\tilde{R}}u.$$

Solving (3.9) yields

$$\hat{R}(u) = -\frac{\gamma}{\beta}\log u + k$$

for some constant k. The initial condition (3.13) implies

$$\hat{R}\left(e^{-(\beta/\gamma)\tilde{R}}\right) = -\frac{\gamma}{\beta}\log e^{-(\beta/\gamma)\tilde{R}} + k = \tilde{R} + k = \tilde{R}$$

and hence $k = 0$. Consequently we have

$$\hat{R}(u) = -\frac{\gamma}{\beta}\log u.$$

We solve (3.10) to obtain

$$\hat{D}(u) = -\frac{\mu}{\beta}\log u + k$$

for some constant k. The initial condition (3.14) implies

$$\hat{D}\left(e^{-(\beta/\gamma)\tilde{R}}\right) = -\frac{\mu}{\beta}\log e^{-(\beta/\gamma)\tilde{R}} + k = \frac{\mu}{\gamma}\tilde{R} + k = \tilde{D}$$

and hence $k = \tilde{D} - (\mu/\gamma)\tilde{R}$. Consequently we have

$$\hat{D}(u) = -\frac{\mu}{\beta}\log u + \tilde{D} - \frac{\mu}{\gamma}\tilde{R}.$$

Since

$$\frac{\hat{S}(u)}{u} = \tilde{S}e^{(\beta/\gamma)\tilde{R}},$$

we obtain

$$\frac{d\hat{I}(u)}{du} = -\tilde{S}e^{(\beta/\gamma)\tilde{R}} + \frac{\gamma}{\beta}\frac{1}{u} + \frac{\mu}{\beta}\frac{1}{u}.$$

Hence we get

$$\hat{I}(u) = -\tilde{S}e^{(\beta/\gamma)\tilde{R}}u + \frac{\gamma}{\beta}\log u + \frac{\mu}{\beta}\log u + k$$

for some constant k. From the initial condition (3.12) it follows that

$$\hat{I}\left(e^{-(\beta/\gamma)\tilde{R}}\right) = -\tilde{S} + \frac{\gamma}{\beta}\log e^{-(\beta/\gamma)\tilde{R}} + \frac{\mu}{\beta}\log e^{-(\beta/\gamma)\tilde{R}} + k = -\tilde{S} - \tilde{R} - \frac{\mu}{\gamma}\tilde{R} + k = \tilde{I},$$

which implies

$$k = \tilde{S} + \tilde{I} + \tilde{R} + \frac{\mu}{\gamma}\tilde{R} = N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R}.$$

Therefore we deduce that

$$\hat{I}(u) = -\tilde{S}e^{(\beta/\gamma)\tilde{R}}u + \frac{\gamma+\mu}{\beta}\log u + N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R}. \qquad \square$$

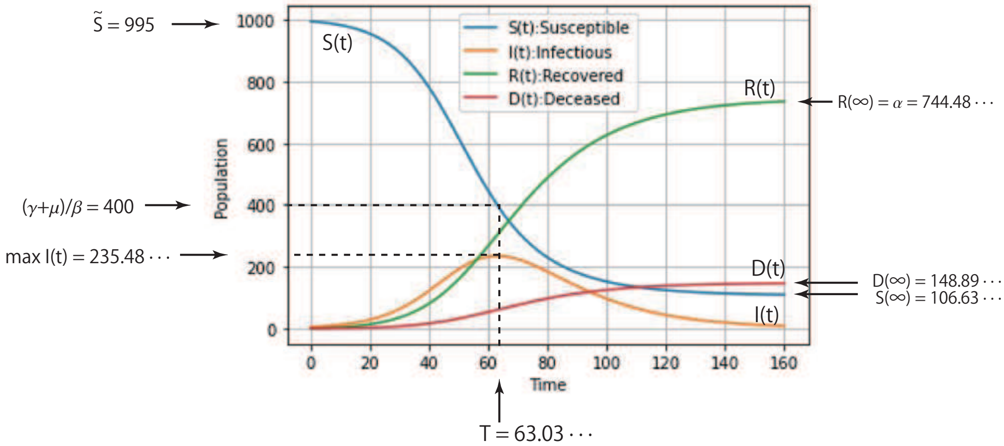

Figure 3.3: Variation of $S(t), I(t), R(t)$ and $D(t)$ obtained by the numerical integration of the initial value problem (1.1)–(1.5) for $N = 1000, \tilde{S} = 995, \tilde{I} = 5, \tilde{R} = \tilde{D} = 0, \beta = 0.15/1000$, $\gamma = 0.05$ and $\mu = 0.01$. In this case we obtain $R(\infty) = \alpha = 744.48\ldots$, $I(\infty) = 0$, $D(\infty) = 148.89\ldots$, $S(\infty) = N - R(\infty) - D(\infty) = 106.63\ldots$, $(\gamma+\mu)/\beta = 400$, $\tilde{S}(= 995) > (\gamma+\mu)/\beta(= 400) > S(\infty)(= 106.63\ldots)$, $\max_{t\geq 0} I(t) = 235.48\ldots$ and $T = 63.03\ldots$, where T is calculated by

$$\begin{aligned} T = \varphi\big((\gamma+\mu)/(\beta\tilde{S})\big) &= \int_{400/995}^{1} \frac{d\xi}{\xi\psi(\xi)} \\ &= \int_{400/995}^{1} \frac{d\xi}{\xi(0.15 - (0.15/1000)\times 995\,\xi + 0.06\log\xi)} = 63.03\ldots \end{aligned}$$

Remark 3.11. The hypothesis (A3) is satisfied if $\tilde{R} = 0$, since $N > \tilde{S} + \tilde{D}$.

Remark 3.12. The right differential coefficient $I'_+(0)$ is positive because

$$\begin{aligned} I'_+(0) &= \lim_{t\to+0} I'(t) = \lim_{t\to+0} \big(\beta S(t)I(t) - \gamma I(t) - \mu I(t)\big) \\ &= \beta\tilde{S}\,\tilde{I} - \gamma\tilde{I} - \mu\tilde{I} = \big(\beta\tilde{S} - \gamma - \mu\big)\tilde{I} > 0 \end{aligned}$$

in view of the hypotheses (A_1) and (A_2).

Remark 3.13. In the case where $(S(\infty) <)\,\tilde{S} \leq (\gamma+\mu)/\beta$ (i.e., $I'_+(0) \leq 0$) we deduce that $\psi(\xi)$ is increasing in $\big(e^{-(\beta/\gamma)\alpha}, e^{-(\beta/\gamma)\tilde{R}}\big]$, $\lim_{\xi\to e^{-(\beta/\gamma)\alpha}+0} \psi(\xi) = 0$ and $\psi\big(e^{-(\beta/\gamma)\tilde{R}}\big) = \beta\tilde{I}$. Since $\varphi^{-1}(t)$ is decreasing on $[0,\infty)$, $\varphi^{-1}(0) = e^{-(\beta/\gamma)\tilde{R}}$ and $\lim_{t\to\infty} \varphi^{-1}(t) = e^{-(\beta/\gamma)\alpha}$, it follows that $I(t) = (1/\beta)\psi\big(\varphi^{-1}(t)\big)$ is decreasing on $[0,\infty)$, and that $I(0) = (1/\beta)\psi\big(\varphi^{-1}(0)\big) = \tilde{I}$ and $I(\infty) = \lim_{t\to\infty} I(t) = \lim_{\xi\to e^{-(\beta/\gamma)\alpha}+0}(1/\beta)\psi(\xi) = 0$ (*cf.* Figure 3.4).

Remark 3.14. The constant $H(N, \tilde{S}, \tilde{R}, \tilde{D}, \beta, \gamma, \mu)$ defined in Corollary 3.2 is equal to $\max_{t\geq 0} I(t)$ given in Theorem 3.1.

Remark 3.15. It follows from Theorems 3.1, 3.3–3.5 that $S(t) > 0, I(t) > 0$ for $t \geq 0$ and $R(t) > 0, D(t) > 0$ for $t > 0$.

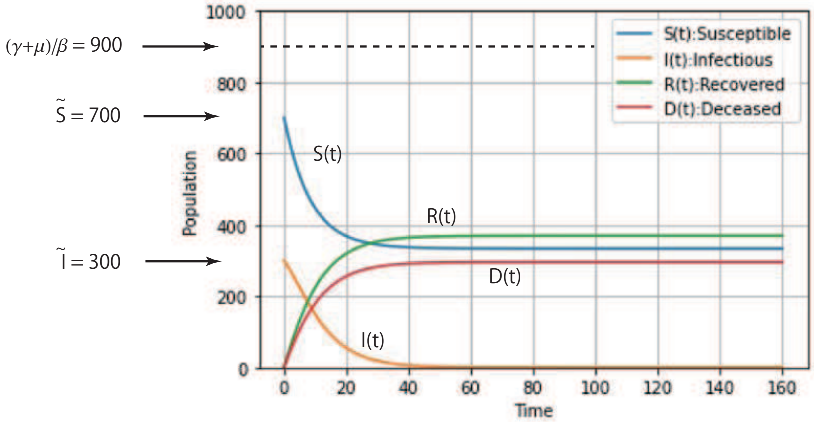

Figure 3.4: Variation of $S(t), I(t), R(t), D(t)$ obtained by the numerical integration of the initial value problem (1.1)–(1.5) for $N = 1000, \tilde{S} = 700, \tilde{I} = 300, \tilde{R} = \tilde{D} = 0, \beta = 0.2/1000$, $\gamma = 0.1$ and $\mu = 0.08$. In this case we see that $(\gamma + \mu)/\beta (= 900) > \tilde{S}(= 700)$, $I(\infty) = 0$ and $I(t)$ is decreasing on $[0, \infty)$.

Remark 3.16. Under the hypothesis

(A_3') $\tilde{D} \geq 0$ satisfies

$$N - \tilde{R} > \tilde{S}e^{(\beta/\mu)\tilde{D}} + \tilde{D},$$

the transcendental equation

$$y = \frac{\mu}{\mu+\gamma}N - \frac{\mu}{\mu+\gamma}\tilde{R} + \frac{\gamma}{\mu+\gamma}\tilde{D} - \frac{\mu}{\mu+\gamma}\tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)y} \tag{3.16}$$

has a unique solution $y = \alpha_*$ such that

$$\tilde{D} < \alpha_* < N$$

by the same arguments as in Lemma 2.4. Since the equation (3.16) reduces to the transcendental equation in Lemma 2.4 by the transformation $y = \tilde{D} - (\mu/\gamma)(\tilde{R} - x)$, we see that $\alpha_* = \tilde{D} - (\mu/\gamma)(\tilde{R} - \alpha)$. We define

$$\varphi_*(w) := \int_w^{e^{-(\beta/\mu)\tilde{D}}} \frac{d\xi}{\xi\psi_*(\xi)}$$

for $e^{-(\beta/\mu)\alpha_*} < w \leq e^{-(\beta/\mu)\tilde{D}}$, where

$$\psi_*(\xi) = \beta N - \beta\tilde{R} + \frac{\beta\gamma}{\mu}\tilde{D} - \beta\tilde{S}e^{(\beta/\mu)\tilde{D}}\xi + (\mu+\gamma)\log\xi.$$

It follows from the transformation

$$\xi = e^{-(\beta/\mu)\tilde{D}}e^{(\beta/\gamma)\tilde{R}}\eta$$

that

$$\varphi_*(w) = \int_{e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\gamma)\tilde{R}}w}^{e^{-(\beta/\gamma)\tilde{R}}} \frac{d\eta}{\eta\psi(\eta)},$$

where $e^{-(\beta/\gamma)\alpha} < e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\gamma)\tilde{R}}w \le e^{-(\beta/\gamma)\tilde{R}}$. Then there exist the inverse functions $\varphi_*^{-1}(t)$ and $\varphi^{-1}(t)$ of the functions

$$t = \varphi_*(w), \quad t = \varphi\big(e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\gamma)\tilde{R}}w\big),$$

respectively, and the following hold:

$$w = \varphi_*^{-1}(t), \quad e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\gamma)\tilde{R}}w = \varphi^{-1}(t) \quad (0 \le t < \infty).$$

Hence we obtain

$$\varphi_*^{-1}(t) = e^{-(\beta/\mu)\tilde{D}}e^{(\beta/\gamma)\tilde{R}}\varphi^{-1}(t) \quad (0 \le t < \infty).$$

Let $\big(S_*(t), I_*(t), R_*(t), D_*(t)\big)$ be the exact solution of the initial value problem (1.1)–(1.5) by starting our arguments utilizing (1.4) instead of (1.3). Then we observe that

$$\begin{aligned}
S_*(t) &= \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi_*^{-1}(t) \\
&= \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)\tilde{D}}e^{(\beta/\gamma)\tilde{R}}\varphi^{-1}(t) \\
&= \tilde{S}e^{(\beta/\gamma)\tilde{R}}\varphi^{-1}(t) = S(t), \\
I_*(t) &= N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi_*^{-1}(t) + \frac{\mu+\gamma}{\beta}\log\varphi_*^{-1}(t) \\
&= N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} - \tilde{S}e^{(\beta/\gamma)\tilde{R}}\varphi^{-1}(t) + \frac{\gamma+\mu}{\beta}\log\varphi^{-1}(t) = I(t), \\
R_*(t) &= -\frac{\gamma}{\beta}\log\varphi_*^{-1}(t) + \tilde{R} - \frac{\gamma}{\mu}\tilde{D} \\
&= -\frac{\gamma}{\beta}\log\big(e^{-(\beta/\mu)\tilde{D}}e^{(\beta/\gamma)\tilde{R}}\varphi^{-1}(t)\big) + \tilde{R} - \frac{\gamma}{\mu}\tilde{D} \\
&= -\frac{\gamma}{\beta}\log\varphi^{-1}(t) = R(t), \\
D_*(t) &= -\frac{\mu}{\beta}\log\varphi_*^{-1}(t) \\
&= -\frac{\mu}{\beta}\log\big(e^{-(\beta/\mu)\tilde{D}}e^{(\beta/\gamma)\tilde{R}}\varphi^{-1}(t)\big) \\
&= \tilde{D} - \frac{\mu}{\gamma}\tilde{R} - \frac{\mu}{\beta}\log\varphi^{-1}(t) = D(t)
\end{aligned}$$

for $0 \le t < \infty$. Consequently we conclude that

$$\big(S_*(t), I_*(t), R_*(t), D_*(t)\big) \equiv \big(S(t), I(t), R(t), D(t)\big) \quad \text{on } [0, \infty).$$

Remark 3.17. In this paper we derived the explicit formula for the exact solution of the SIRD epidemic model, and obtained various properties of the exact solution including the maximum of $I(t)$, the concavity and convexity of $I(t)$, time T which attains $\max_{t\ge 0} I(t)$ and the linear differential system which is satisfied by the parametric form of the exact solution. If $\mu = 0$ and $D(t) \equiv 0$, then the SIRD epidemic model reduces to the SIR epidemic model. We note that our results can be applied to the SIR epidemic model if we set $\mu = 0$ and $D(t) \equiv 0$.

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The existence of solutions for the modified $(p(x), q(x))$-Kirchhoff equation

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Abstract. We consider the Dirichlet problem

$$-\Delta_{p(x)}^{K_p} u(x) - \Delta_{q(x)}^{K_q} u(x) = f(x, u(x), \nabla u(x)) \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = 0,$$

driven by the sum of a $p(x)$-Laplacian operator and of a $q(x)$-Laplacian operator, both of them weighted by indefinite (sign-changing) Kirchhoff type terms. We establish the existence of weak solution and strong generalized solution, using topological tools (properties of Galerkin basis and of Nemitsky map). In the particular case of a positive Kirchhoff term, we obtain the existence of weak solution (= strong generalized solution), using the properties of pseudomonotone operators.

Keywords: Brouwer fixed point theorem, Galerkin basis, Kirchhoff term, Nemitsky map, pseudomonotone operator.

2020 Mathematics Subject Classification: 35J60, 35J92, 35J55.

1 Introduction

In this manuscript we consider equations driven by Kirchhoff type operators of the form $u \to -K(r,u)\Delta_{r(x)}u$ for functions u, defined on a bounded domain $\Omega \subseteq \mathbb{R}^N$ with smooth boundary $\partial\Omega$. The analysis is carried out in a suitable anisotropic Dirichlet Sobolev space $W_0^{1,r(x)}(\Omega)$, with variable exponent $r \in C(\overline{\Omega})$ satisfying certain regularity and bound conditions. The operator $\Delta_{r(x)}$ is the $r(x)$-Laplacian operator, which for every $u \in W_0^{1,r(x)}(\Omega)$ is defined by $\Delta_{r(x)}u = \operatorname{div}(|\nabla u|^{r(x)-2}\nabla u)$. Additionally, the nonlocal Kirchhoff type term $K(r,u)$ is assumed indefinite (sign changing) and given as

$$K(r,u) = a_r - b_r \int_\Omega \frac{1}{r(x)} |\nabla u|^{r(x)} dx, \quad \text{with } a_r, b_r > 0. \tag{K}$$

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Precisely, the Dirichlet problem we study is

$$-\Delta_{p(x)}^{K_p} u(x) - \Delta_{q(x)}^{K_q} u(x) = f(x, u(x), \nabla u(x)) \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = 0. \tag{P}$$

Here, we have the sum of two such Kirchhoff type operators $-\Delta_{p(x)}^{K_p} u := -K(p,u)\Delta_{p(x)} u$ and $-\Delta_{q(x)}^{K_q} u := -K(q,u)\Delta_{q(x)} u$, with variable exponents $p, q \in C(\overline{\Omega})$ such that

$$\begin{aligned} 1 < q^- &= \inf_{x\in\overline{\Omega}} q(x) \le q(x) \le q^+ = \sup_{x\in\overline{\Omega}} q(x) \\ &< p^- = \inf_{x\in\overline{\Omega}} p(x) \le p(x) \le p^+ = \sup_{x\in\overline{\Omega}} p(x) < +\infty. \end{aligned}$$

The reaction (right hand side of (P)) is a Carathéodory function $f(x,z,y)$ (that is, for all $(z,y) \in \mathbb{R} \times \mathbb{R}^N$, $x \to f(x,z,y)$ is measurable and for almost all $x \in \Omega$, $(z,y) \to f(x,z,y)$ is continuous). The presence of the gradient ∇u is crucial to be considered when the convection in fluid dynamical processes cannot be neglected (that is, when an energy transfer is accomplished by moving particles). Turning to the Kirchhoff type term (K), it is related to physical modeling of the changes in length of a string subject to transverse vibrations. In [13], Kirchhoff generalized the classical D'Alembert wave equation

$$\rho \frac{\partial^2}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2}{\partial x^2} = 0,$$

with ρ, P_0, h, E, L denoting physical parameters (respectively, mass density, initial tension, area of the cross-section, Young modulus of the material, length of the string) and describing the change of string's length during free vibration.

The existence results we establish use topological techniques (fixed-point arguments together with the theory of pseudomonotone operators) in order to overcome the loss of variational structure, due to the presence of gradient term in the reaction. The classical strategies are also adapted to deal with the nonlocal nature of the Kirchhoff term. Following the similar approach as in Vetro [26], we prove the existence of strong generalized solutions as well as weak solutions to (P). To have a more complete picture of the relevant literature, we mention that standard $-\Delta_p - \Delta_q$ operator was considered by Faria et al. [5] and Zeng & Papageorgiou [28], in the case of positive solutions. For single $-\Delta_p$ operator we mention Papageorgiou et al. [19], dealing also with positive solutions. Precisely, in [5] the authors adopt an approximating process involving a Schauder basis of $W_0^{1,p(x)}(\Omega)$, then apply a generalized strong maximum principle. In [28], the authors use the Leray–Schauder alternative principle in combination with the frozen variable method (to freeze the effects of the gradient term). In [19], the authors use also Leray–Schauder alternative principle, together with truncation and comparison techniques. Additionally, the case of double phase problems (that is, $-\Delta_p - \mu(x)\Delta_q$ operator, with suitable weight function $\mu(\cdot)$) was studied by Gasiński & Winkert [10], using surjectivity result of pseudomonotone operators. Finally, we mention the work of Motreanu [18] dealing with $-\Delta_p + \Delta_q$ operator. In that paper, the author uses a consequence of the Brouwer fixed point theorem, in respect of a Galerkin basis of $W_0^{1,p}(\Omega)$. A main feature of the present manuscript and of the works [18, 26] is the consideration of two different types of solutions of (P), that is, the authors employ both classical weak solutions and new concepts of strong generalized solutions. Additionally, [26] deals with the variable exponents Lebesgue and Sobolev spaces, in the case of a single $p(x)$-Kirchhoff type operator. The similar problem was previously studied by Wang et al. [27], in absence of Kirchhoff type term. Moreover, the Kirchhoff type term herein was considered by Hamdani et al. [11], whose reaction is not gradient dependent. Therefore, [11] employs a (classical) variational approach. It is worth mentioning that the Lions' work [16] originated a revival interest for equations involving a Kirchhoff term, but a large amount of manuscripts imposes a positive restriction to the values of the Kirchhoff term (that is, they consider a sign "$+$" instead of "$-$" in (K), deriving from the classical theory). The interested reader can also refer to Molica Bisci & Pizzimenti [17] (looking infinitely many solutions), Figueiredo & Nascimento [6] (looking for nodal (sign-changing) solutions), Santos Júnior & Siciliano [24] and Gasiński & Santos Júnior [8, 9] (both of them introducing non positivity conditions on the Kirchhoff term). In the last three papers, the authors assume that Kirchhoff terms can vanish in many different points. Additionally, their strategy of proofs also involve fixed point results, and aims to establish both existence and nonexistence theorems. Before concluding this introduction, it is very important to say that recently in the literature, we find many papers where the authors study the existence and multiplicity of solutions to problems involving the Kirchhoff operator, Choquard-Pekar equations and functionals of double phase with variable exponents. As a partial list we mention the works by Albalawi [1], He et al. [12], Liang et al. [15], Qin et al. [21], Ragusa & Tachikawa [22], Hi et al. [25], and references therein.

2 Preliminaries

Referring to the books of Diening et al. [2] and of Rădulescu & Repovš [23], we provide the mathematical background of the present study. The natural setting where finding solutions to (P) is the anisotropic Dirichlet Sobolev space $W_0^{1,p(x)}(\Omega)$, which means the completion of $C_0^\infty(\Omega)$ with respect to the $W^{1,p(x)}$-norm defined below. Starting with

$$L^{p(x)}(\Omega) = \left\{ u \in M(\Omega) : \int_\Omega |u(x)|^{p(x)} dx < +\infty \right\},$$

which is the variable exponent Lebesgue space, we consider the norm

$$\|u\|_{L^{p(x)}(\Omega)} := \inf \left\{ \lambda > 0 : \rho_p\left(\frac{u}{\lambda}\right) \le 1 \right\}.$$

Here $M(\Omega)$ means the space of all measurable functions $u : \Omega \to \mathbb{R}$, and

$$\rho_p(u) := \int_\Omega |u(x)|^{p(x)} dx \quad \text{for all } u \in L^{p(x)}(\Omega)$$

denotes the modular. As it is well known, $(L^{p(x)}(\Omega), \|\cdot\|_{L^{p(x)}(\Omega)})$ is a separable, reflexive and uniformly convex Banach space. The norm $\|\cdot\|_{L^{p(x)}(\Omega)}$ and the modular $\rho_p(\cdot)$ are related each other by the following statements.

Theorem 2.1 ([4, Theorem 1.3]). *Let $u \in L^{p(x)}(\Omega)$, then we have:*

(i) $\|u\|_{L^{p(x)}(\Omega)} < 1 \ (=1, \ >1) \Leftrightarrow \rho_p(u) < 1 \ (=1, \ >1)$;

(ii) *if* $\|u\|_{L^{p(x)}(\Omega)} > 1$, *then* $\|u\|_{L^{p(x)}(\Omega)}^{p^-} \le \rho_p(u) \le \|u\|_{L^{p(x)}(\Omega)}^{p^+}$;

(iii) *if* $\|u\|_{L^{p(x)}(\Omega)} < 1$, *then* $\|u\|_{L^{p(x)}(\Omega)}^{p^+} \le \rho_p(u) \le \|u\|_{L^{p(x)}(\Omega)}^{p^-}$.

In view of Theorem 2.1, we obtain the relation:

$$\|u\|^{p^+}_{L^{p(x)}(\Omega)} + 1 \geq \rho_p(u) \geq \|u\|^{p^-}_{L^{p(x)}(\Omega)} - 1. \tag{2.1}$$

We are able to introduce the conjugate variable exponent to p, namely $p' \in C(\overline{\Omega})$ satisfying

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1 \quad \text{for all } x \in \overline{\Omega}.$$

As it is well known $L^{p(x)}(\Omega)^* = L^{p'(x)}(\Omega)$ and if $p^- > 1$ we have

$$\int_\Omega uw dx \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right)\|u\|_{L^{p(x)}(\Omega)}\|w\|_{L^{p'(x)}(\Omega)} \leq 2\|u\|_{L^{p(x)}(\Omega)}\|w\|_{L^{p'(x)}(\Omega)},$$

for $u \in L^{p(x)}(\Omega)$, $w \in L^{p'(x)}(\Omega)$. This Hölder's inequality plays a crucial role in establishing suitable embedding results. We refer to [4, Theorem 1.11] for the continuity of the embedding $L^{p_1(x)}(\Omega) \hookrightarrow L^{p_2(x)}(\Omega)$, provided that $p_1, p_2 \in C(\overline{\Omega})$ with $p_1(x) \geq p_2(x) > 1$ for all $x \in \overline{\Omega}$. Using the variable exponent Lebesgue space, we can define the variable exponent Sobolev space

$$W^{1,p(x)}(\Omega) := \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}, \quad p \in C(\overline{\Omega}).$$

Starting with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)} \quad (\text{where } \|\nabla u\|_{L^{p(x)}(\Omega)} = \||\nabla u|\|_{L^{p(x)}(\Omega)}),$$

we recall that

$$\|u\|_{L^{p(x)}(\Omega)} \leq c_1\|\nabla u\|_{L^{p(x)}(\Omega)} \quad \text{for all } u \in W^{1,p(x)}_0(\Omega), \text{ some } c_1 > 0, \tag{2.2}$$

see [2, Theorem 8.2.18]. Thus, as it is well known, $\|u\|_{W^{1,p(x)}(\Omega)}$ and $\|\nabla u\|_{L^{p(x)}(\Omega)}$ are equivalent norms on $W^{1,p(x)}_0(\Omega)$. This implies that we can use $\|\nabla u\|_{L^{p(x)}(\Omega)}$ instead of $\|u\|_{W^{1,p(x)}(\Omega)}$, and set

$$\|u\| = \|\nabla u\|_{L^{p(x)}(\Omega)} \quad \text{in } W^{1,p(x)}_0(\Omega) \quad (\text{by (2.2)}).$$

We mention that judicious choices of norms and norm inequalities are needed for establishing bounds and a priori estimates. Additionally, Fan & Zhao [4] established that with these norms the spaces $W^{1,p(x)}(\Omega)$ and $W^{1,p(x)}_0(\Omega)$, become Banach spaces which are separable and uniformly convex (hence reflexive). Now, for $p \in C(\overline{\Omega})$ we are able to define the critical Sobolev exponent p^* by

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } N \leq p(x), \end{cases} \quad \text{for all } x \in \overline{\Omega}.$$

About the continuity and compactness of Sobolev embeddings, we recall the following well-known result.

Proposition 2.2. *Suppose $p \in C(\overline{\Omega})$ with $p(x) > 1$ for all $x \in \overline{\Omega}$. If $\alpha \in C(\overline{\Omega})$ and $1 < \alpha(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then $W^{1,p(x)}(\Omega) \hookrightarrow L^{\alpha(x)}(\Omega)$ is continuous and compact.*

As already mentioned in the Introduction, our approach here makes use of properties of pseudomonotone operators. So, we collect some definitions and results, as follows.

Definition 2.3. For a reflexive Banach space X, let X^* the dual space of X and $\langle \cdot, \cdot \rangle$ the duality pairing. Let $A : X \to X^*$, then A is called

(i) to satisfy the (S_+)-property if $u_n \xrightarrow{w} u$ in X and $\limsup_{n\to+\infty}\langle A(u_n), u_n - u\rangle \leq 0$ imply $u_n \to u$ in X;

(ii) pseudomonotone if $u_n \xrightarrow{w} u$ in X and $\limsup_{n\to+\infty}\langle A(u_n), u_n - u\rangle \leq 0$ imply

$$\liminf_{n\to+\infty}\langle A(u_n), u_n - v\rangle \geq \langle A(u), u - v\rangle \text{ for all } v \in X;$$

(iii) coercive if

$$\lim_{\|u\|_X\to+\infty} \frac{\langle A(u), u\rangle}{\|u\|_X} = +\infty.$$

Remark 2.4. We point out that if the operator $A : X \to X^*$ is bounded, then pseudomonotonicity in Definition 2.3 (ii) is equivalent to $u_n \xrightarrow{w} u$ in X and $\limsup_{n\to+\infty}\langle A(u_n), u_n - u\rangle \leq 0$ imply $A(u_n) \xrightarrow{w} A(u)$ and $\langle A(u_n), u_n\rangle \to \langle A(u), u\rangle$. In the following we are going to use this fact since our involved operators are bounded.

Pseudomonotone operators exhibit remarkable surjectivity properties. In particular, we have the following result, see, for example, Papageorgiou & Winkert [20, Theorem 6.1.57].

Theorem 2.5. *Let X be a real and reflexive Banach space. Let $A: X \to X^*$ be a pseudomonotone, bounded, and coercive operator, and let $b \in X^*$. Then, the equation $A(u) = b$ admits a solution.*

From Gasiński & Papageorgiou [7, Lemma 2.2.27], we get the following result of continuous embedding and density.

Theorem 2.6. *Let X, Y be Banach spaces such that $X \subseteq Y$. If X is dense in Y and the embedding is continuous, then the embedding $Y^* \subseteq X^*$ is continuous too. Moreover, if X is reflexive then Y^* is dense in X^*.*

Our arguments of proofs are also based on Brouwer's fixed point theorem, which leads to the existence of solutions to certain operator equations as stated in the following proposition.

Proposition 2.7. *For a normed finite-dimensional space $(X, \|\cdot\|_X)$ and a continuous map $A : X \to X^*$, we have that:*

If there exists some $R > 0$ such that

$$\langle A(w), w\rangle \geq 0 \quad \textit{for all } w \in X \textit{ with } \|w\|_X = R,$$

then $A(w) = 0$ has a solution $\widehat{w} \in X$ such that $R \geq \|\widehat{w}\|_X$.

3 Hypotheses and results

In this section, we introduce the hypotheses on the data and collect the statements of our results. First, we put some restrictions on the exponent p, useful to give us the Rayleigh quotient

$$\widehat{\lambda} := \inf_{u\in W_0^{1,p(x)}(\Omega), u\neq 0} \frac{\int_\Omega |\nabla u|^{p(x)}dx}{\int_\Omega |u|^{p(x)}dx} > 0. \tag{3.1}$$

$H(p)$: There exists $\xi_0 \in \mathbb{R}^N \setminus \{0\}$ such that for all $x \in \Omega$ the function $p_x : \Omega_x \to \mathbb{R}$ defined by $p_x(z) = p(x + z\xi_0)$ is monotone, where $\Omega_x := \{z \in \mathbb{R} : x + z\xi_0 \in \Omega\}$.

We get (3.1) by [3, Theorem 3.3]. Alternatively, one can adopt a different condition, see for example [3, Theorem 3.4]. Here, we will also impose the condition:

$H'(p)$: $p \in C(\overline{\Omega})$ is finite with $p^+ < 2p^-$.

A similar condition was used in [11, 26]. Additionally, we impose growth conditions on the right hand side of (P). Precisely, our hypotheses will be the following:

$H(f)$: $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function such that

(i) there exist $\sigma \in L^{\alpha'(x)}(\Omega)$, $\alpha \in C(\overline{\Omega})$ with $1 < \alpha(x) < p^*(x)$ for all $x \in \overline{\Omega}$ and $c > 0$ such that

$$|f(x,z,y)| \leq c(\sigma(x) + |z|^{\alpha(x)-1} + |y|^{\frac{p(x)}{\alpha'(x)}}) \quad \text{for a.a. } x \in \Omega, \text{ all } z \in \mathbb{R}, \text{ all } y \in \mathbb{R}^N;$$

(ii) there exist $\sigma_0 \in L^1(\Omega)$ and $b_1, b_2 \geq 0$ such that

$$|f(x,z,y)z| \leq \sigma_0(x) + b_1|z|^{p(x)} + b_2|y|^{p(x)} \quad \text{for a.a. } x \in \Omega, \text{ all } z \in \mathbb{R}, \text{ all } y \in \mathbb{R}^N.$$

Remark 3.1. Let $\lambda^* = b_1\widehat{\lambda}^{-1} + b_2$. By $H(f)$(ii) and $H(p)$, we get the following estimate:

$$\int_\Omega |f(x,u,\nabla u)u|dx \leq \lambda^* \rho_p(\nabla u) + \|\sigma_0\|_{L^1(\Omega)} \quad \text{for all } u \in W_0^{1,p(x)}(\Omega). \tag{3.2}$$

Here we establish the existence of solutions both in the usual weak form and in a specific (for Dirichlet problem (P)) form. As it is well known, $u \in W_0^{1,p(x)}(\Omega)$ is weak solution whenever

$$\left\langle -\Delta^{K_p}_{p(x)} u, w \right\rangle + \left\langle -\Delta^{K_q}_{q(x)} u, w \right\rangle = \int_\Omega f(x, u(x), \nabla u(x)) w(x) dx \tag{3.3}$$

for all $w \in W_0^{1,p(x)}(\Omega)$.

On the other hand, we introduce a new definition of strong generalized solution to (P), as follows (see, the corresponding notion of [26]).

Definition 3.2. $u \in W_0^{1,p(x)}(\Omega)$ is a strong generalized solution to (P), if we can find a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p(x)}(\Omega)$ verifying the convergences:

(i) $u_n \xrightarrow{w} u$ in $W_0^{1,p(x)}(\Omega)$, as $n \to +\infty$;

(ii) $-\Delta^{K_p}_{p(x)} u_n - \Delta^{K_q}_{q(x)} u_n - f(\cdot, u_n(\cdot), \nabla u_n(\cdot)) \xrightarrow{w} 0$ in $W^{-1,p'(x)}(\Omega)$, as $n \to +\infty$;

(iii) $\lim_{n\to+\infty} \left\langle -\Delta^{K_p}_{p(x)} u_n - \Delta^{K_q}_{q(x)} u_n, u_n - u \right\rangle = 0$.

Remark 3.3. We point out that every weak solution to (P) satisfies the conditions in Definition 3.2. It is sufficient to use as test sequence, $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p(x)}(\Omega)$ defined by $u_n = u$ for all $n \in \mathbb{N}$.

In view of the above remark, we provide an answer to the question:

When does a strong generalized solution to (P) lead to a weak solution?

Note that the source of difficulty in answering this question, is related to the indefinite behavior of (K). Thus we assume the following non-negative bound conditions:

$$\liminf_{n\to+\infty} |K(p, u_n)| > 0 \quad \text{and} \quad K(p, u_n)\, K(q, u_n) \geq 0 \quad \text{for all } n \in \mathbb{N}. \tag{K_+}$$

Proposition 3.4. *Consider a strong generalized solution of* (P), *namely $u \in W_0^{1,p(x)}(\Omega)$, in respect to the sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p(x)}(\Omega)$. Then $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of* (P), *provided that hypotheses $H(f)$ and* (K_+) *hold.*

We shall prove two auxiliary propositions needed for the proof of the main result. These propositions require both the hypotheses $H(p)$ and $H(f)$, as given in the statements. Since $W_0^{1,p(x)}(\Omega)$ is a separable Banach space, then we consider a Galerkin basis of $W_0^{1,p(x)}(\Omega)$, which means that there exists a sequence $\{X_n\}_{n\in\mathbb{N}}$ of vector subspaces of $W_0^{1,p(x)}(\Omega)$ satisfying

(j) $\dim(X_n) < +\infty$ for all $n \in \mathbb{N}$;

(jj) $X_n \subseteq X_{n+1}$ for all $n \in \mathbb{N}$;

(jjj) $\overline{\cup_{n=1}^{\infty} X_n} = W_0^{1,p(x)}(\Omega)$.

Proposition 3.5. *Consider a Galerkin basis of $W_0^{1,p(x)}(\Omega)$, namely $\{X_n\}_{n\in\mathbb{N}}$. Then for all $n \in \mathbb{N}$ we can find $u_n \in X_n$ with*

$$\Big\langle -\Delta_{p(x)}^{K_p} u_n, w \Big\rangle + \Big\langle -\Delta_{q(x)}^{K_q} u_n, w \Big\rangle = \int_\Omega f(x, u_n(x), \nabla u_n(x)) w(x) dx \tag{3.4}$$

for all $w \in X_n$, provided that hypotheses $H(p)$ and $H(f)$ hold.

Remark 3.6. From Theorem 2.1 we deduce that $S \subseteq W_0^{1,p(x)}(\Omega)$ is bounded in its norm if the set $\{\rho_p(\nabla u) : u \in S\}$ is bounded.

Focusing on the sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \cup_{n=1}^{\infty} X_n$ mentioned in Proposition 3.5 (see also the corresponding proof, in next section), we will show that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p(x)}(\Omega)$.

Proposition 3.7. *Consider the sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \cup_{n=1}^{\infty} X_n$ generated in Proposition 3.5. Then $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p(x)}(\Omega)$, provided that hypotheses $H(p)$ and $H(f)$ hold.*

Consequently, we prove our first existence result.

Theorem 3.8. *Problem* (P) *admits a strong generalized solution $u \in W_0^{1,p(x)}(\Omega)$, provided that hypotheses $H(p)$ and $H(f)$ hold.*

The analogous of Propositions 3.5 and 3.7, and Theorem 3.8 can be obtained imposing $H'(p)$ instead of $H(p)$ (see also Remark 4.1 at the end of Section 4).

4 Proofs of results

In this section we collect the technical proofs of the results stated previously.

Proof of Proposition 3.4. We only prove the case $\liminf_{n\to+\infty} K(p,u_n) > 0$, the other cases can be proved by a similar argument. The previous assumption ensures that we can suppose that

$$K(p,u_n) \geq \beta > 0 \quad \text{and} \quad K(q,u_n) \geq 0 \quad \text{for all } n \in \mathbb{N}, \tag{4.1}$$

are true at least for a relabeled subsequence of $\{u_n\}_{n\in\mathbb{N}}$. Next, we recall that the $-\Delta_{q(x)}$ operator is monotone and hence

$$\langle -\Delta_{q(x)}u_n, u_n - u\rangle \geq \langle -\Delta_{q(x)}u, u_n - u\rangle \quad \text{for all } n \in \mathbb{N}.$$

Multiplying both sides of last inequality by $K(q,u_n)$, then we get

$$K(q,u_n)\langle -\Delta_{q(x)}u_n, u_n - u\rangle \geq K(q,u_n)\langle -\Delta_{q(x)}u, u_n - u\rangle \quad \text{for all } n \in \mathbb{N} \text{ (by (4.1))},$$

that is, adopting the notation introduced at the beginning of this manuscript,

$$\Big\langle -\Delta^{K_q}_{q(x)}u_n, u_n - u\Big\rangle \geq K(q,u_n)\langle -\Delta_{q(x)}u, u_n - u\rangle \quad \text{for all } n \in \mathbb{N}.$$

It is clear that using condition (iii) of Definition 3.2, we get

$$\begin{aligned}\limsup_{n\to+\infty}\Big\langle -\Delta^{K_p}_{p(x)}u_n, u_n - u\Big\rangle &= \limsup_{n\to+\infty}\Big[\Big\langle -\Delta^{K_p}_{p(x)}u_n, u_n - u\Big\rangle - K(q,u_n)\langle \Delta_{q(x)}u, u_n - u\rangle\Big] \\ &\leq \lim_{n\to+\infty}\Big\langle -\Delta^{K_p}_{p(x)}u_n - \Delta^{K_q}_{q(x)}u_n, u_n - u\Big\rangle = 0.\end{aligned}$$

From the previous inequality and (4.1), we deduce

$$\limsup_{n\to+\infty}\langle -\Delta_{p(x)}u_n, u_n - u\rangle \leq 0,$$

and hence we retrieve the $(S)_+$-property of the $p(x)$-Laplacian operator, provided that $u_n \to u$ in $W^{1,p(x)}_0(\Omega)$, as $n \to +\infty$. Using condition (ii) of Definition 3.2, we deduce that

$$-\Delta^{K_p}_{p(x)}u_n - \Delta^{K_q}_{q(x)}u_n - f(\cdot, u_n(\cdot), \nabla u_n(\cdot)) \xrightarrow{w} 0 \quad \text{in } W^{-1,p'(x)}(\Omega),$$

which implies

$$-\Delta^{K_p}_{p(x)}u - \Delta^{K_q}_{q(x)}u - f(\cdot, u(\cdot), \nabla u(\cdot)) = 0,$$

and hence we conclude that $u \in W^{1,p(x)}_0(\Omega)$ is a weak solution to problem (P) (recall the definition of weak solution in (3.3)). □

Proof of Proposition 3.5. Fixed $n \in \mathbb{N}$, let $A_n : X_n \to X_n^*$ be the operator defined by

$$\langle A_n(u), w\rangle = \Big\langle -\Delta^{K_p}_{p(x)}u, w\Big\rangle + \Big\langle -\Delta^{K_q}_{q(x)}u, w\Big\rangle - \int_\Omega f(x, u(x), \nabla u(x))w(x)dx$$

for all $u, w \in X_n$.

Now, by the estimate (3.2), we get

$$\begin{aligned}\langle -A_n(w), w\rangle &= \left(b_p \int_\Omega \frac{1}{p(x)}|\nabla w|^{p(x)}dx - a_p\right)\int_\Omega |\nabla w|^{p(x)}dx \\ &\quad + \left(b_q \int_\Omega \frac{1}{q(x)}|\nabla w|^{q(x)}dx - a_q\right)\int_\Omega |\nabla w|^{q(x)}dx - \int_\Omega f(x,w,\nabla w)w dx \\ &\geq \left(b_p \int_\Omega \frac{1}{p(x)}|\nabla w|^{p(x)}dx - a_p\right)\int_\Omega |\nabla w|^{p(x)}dx \\ &\quad - a_q\int_\Omega |\nabla w|^{q(x)}dx - \int_\Omega |f(x,w,\nabla w)w|dx \\ &\geq \frac{b_p}{p^+}\rho_p^2(\nabla w) - a_p\rho_p(\nabla w) - a_q\int_\Omega (1+|\nabla w|^{p(x)})dx \\ &\quad - \lambda^*\rho_p(\nabla w) - \|\sigma_0\|_{L^1(\Omega)} \quad \text{(by (3.2))} \\ &\geq \frac{b_p}{p^+}\rho_p^2(\nabla w) - (a_p + a_q + \lambda^*)\rho_p(\nabla w) - a_q|\Omega| - \|\sigma_0\|_{L^1(\Omega)},\end{aligned}$$

where $|\Omega|$ is the Lebesgue measure of the set Ω. So, we have

$$\langle -A_n(w), w\rangle \geq \frac{b_p}{p^+}\rho_p^2(\nabla w) - (a_p + a_q + \lambda^*)\rho_p(\nabla w) - C \quad \text{for all } w \in X_n,$$

where $C = a_q|\Omega| + \|\sigma_0\|_{L^1(\Omega)}$. Now, if $\rho_p(\nabla w) > 1$ we get

$$\begin{aligned}\langle -A_n(w), w\rangle &\geq \frac{b_p}{p^+}\rho_p^2(\nabla w) - (a_p + a_q + \lambda^* + C)\rho_p(\nabla w) \\ &= \left[\frac{b_p}{p^+}\rho_p(\nabla w) - \left(a_p + a_q + \lambda^* + C\right)\right]\rho_p(\nabla w),\end{aligned}$$

which gives us the condition

$$\langle -A_n(w), w\rangle \geq 0 \quad \text{if } \rho_p(\nabla w) \geq \frac{p^+}{b_p}(a_p + a_q + \lambda^* + C).$$

Let

$$R > \max\left\{\left[\frac{p^+}{b_p}(a_p + a_q + \lambda^* + C)\right]^{1/p^-}, 1\right\}$$

be fixed. For each $w \in X_n$ with $\|w\| = R$ we obtain

$$\langle -A_n(w), w\rangle \geq 0 \quad \text{(recall we have } \|w\| = \|\nabla w\|_{L^{p(x)}(\Omega)} \leq \rho_p^{\frac{1}{p^-}}(\nabla w)).$$

A simple application of Proposition 2.7 ensures that $-A_n(w) = 0$ (and hence, $A_n(w) = 0$) possesses a solution $u_n \in X_n$. This is sufficient to conclude that the equation (3.4) is proved. □

Proof of Proposition 3.7. The crucial point of the proof consists in showing that

$$\rho_p(\nabla u_n) \leq \max\left\{\frac{p^+}{b_p}\left(a_p + a_q + \lambda^* + a_q|\Omega| + \|\sigma_0\|_{L^1(\Omega)}\right), 1\right\} \quad \text{for all } n \in \mathbb{N}. \tag{4.2}$$

Hence, we start obtaining the inequality

$$\rho_p(\nabla u_n) \le \frac{p^+}{b_p}\left(a_p + a_q + \lambda^* + a_q|\Omega| + \|\sigma_0\|_{L^1(\Omega)}\right),$$

provided that $\rho_p(\nabla u_n) > 1$. By (3.4), putting $w = u_n$ we get

$$\begin{aligned}\frac{b_p}{p^+}\rho_p^2(\nabla u_n) &\le a_p\rho_p(\nabla u_n) + a_q\rho_q(\nabla u_n) - \frac{b_q}{q^+}\rho_q^2(\nabla u_n) - \int_\Omega f(x,u_n,\nabla u_n)u_n dx\\ &\le (a_p+a_q)\rho_p(\nabla u_n) + a_q|\Omega| + \int_\Omega |f(x,u_n,\nabla u_n)u_n|dx\\ &\le (a_p+a_q)\rho_p(\nabla u_n) + a_q|\Omega| + \lambda^*\rho_p(\nabla u_n) + \|\sigma_0\|_{L^1(\Omega)} \quad \text{(by (3.2))}.\end{aligned}$$

Keeping in mind that $\rho_p(\nabla u_n) > 1$, it follows that

$$\begin{aligned}&\frac{b_p}{p^+}\rho_p^2(\nabla u_n) \le \left(a_p + a_q + \lambda^* + a_q|\Omega| + \|\sigma_0\|_{L^1(\Omega)}\right)\rho_p(\nabla u_n),\\ \Rightarrow\quad &\frac{b_p}{p^+}\rho_p(\nabla u_n) \le a_p + a_q + \lambda^* + a_q|\Omega| + \|\sigma_0\|_{L^1(\Omega)},\end{aligned}$$

and multiplying both sides by $\frac{p^+}{b_p}$, we get

$$\rho_p(\nabla u_n) \le \frac{p^+}{b_p}\left(a_p + a_q + \lambda^* + a_q|\Omega| + \|\sigma_0\|_{L^1(\Omega)}\right).$$

This concludes the proof of inequality (4.2). Consequently, we get that $\{u_n\}_{n\in\mathbb{N}} \subseteq \cup_{n=1}^\infty X_n$ is a bounded sequence in $W_0^{1,p(x)}(\Omega)$. □

Proof of Theorem 3.8. First, we introduce the Nemitsky map corresponding to the Carathéodory function f. Namely, $N_f^* : W_0^{1,p(x)}(\Omega) \subset L^{\alpha(x)}(\Omega) \to L^{\alpha'(x)}(\Omega)$ defined by

$$N_f^*(u)(\cdot) = f(\cdot, u(\cdot), \nabla u(\cdot)) \quad \text{for all } u \in W_0^{1,p(x)}(\Omega).$$

Hypothesis $H(f)$(i) implies that $N_f^*(\cdot)$ is well-defined, bounded and continuous, see Fan & Zhao [4] and Kováčik & Rákosník [14]. By Theorem 2.6, the embedding $i^* : L^{\alpha'(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ is continuous and hence the operator $N_f : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ defined by $N_f = i^* \circ N_f^*$ is bounded and continuous.

Now, we have established in Proposition 3.7, that the sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \cup_{n=1}^\infty X_n$ (generated in Proposition 3.5) is bounded in the anisotropic Dirichlet Sobolev space $W_0^{1,p(x)}(\Omega)$. Additionally, this Sobolev space is reflexive, and hence for some $u \in W_0^{1,p(x)}(\Omega)$, we suppose that

$$u_n \xrightarrow{w} u \quad \text{in } W_0^{1,p(x)}(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in } L^{\alpha(x)}(\Omega). \tag{4.3}$$

Since the Nemitsky map is bounded, then we deduce that

$$\{N_f(u_n)\}_{n\in\mathbb{N}} \quad \text{is bounded in } W^{-1,p'(x)}(\Omega).$$

We already know that $-\Delta_{p(x)}^{K_p}, -\Delta_{q(x)}^{K_q} : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ are bounded, and hence

$$\left\{-\Delta_{p(x)}^{K_p}u_n - \Delta_{q(x)}^{K_q}u_n - N_f(u_n)\right\}_{n\in\mathbb{N}} \quad \text{is bounded in } W^{-1,p'(x)}(\Omega). \tag{4.4}$$

Consequently, for a relabeled subsequence of (4.4) we get

$$-\Delta^{K_p}_{p(x)} u_n - \Delta^{K_q}_{q(x)} u_n - N_f(u_n) \xrightarrow{w} g \quad \text{in } W^{-1,p'(x)}(\Omega), \text{ for some } g \in W^{-1,p'(x)}(\Omega), \tag{4.5}$$

as the dual space $W^{-1,p'(x)}(\Omega)$ is reflexive too.

Choosing w in $\cup_{n=1}^{\infty} X_n$, there will be $n(w) \in \mathbb{N}$ such that w belongs to $X_{n(w)}$. By Proposition 3.5, we deduce that (3.4) holds true for every $n \geq n(w)$. Letting n to infinity in (3.4), we obtain

$$\langle g, w \rangle = 0 \quad \text{for all } w \in \cup_{n=1}^{\infty} X_n.$$

The density of $\cup_{n=1}^{\infty} X_n$ in $W_0^{1,p(x)}(\Omega)$ (as $\{X_n\}_{n\in\mathbb{N}}$ is a Galerkin basis), leads to the conclusion $g = 0$, and using (4.5) we get

$$-\Delta^{K_p}_{p(x)} u_n - \Delta^{K_q}_{q(x)} u_n - N_f(u_n) \xrightarrow{w} 0 \quad \text{in } W^{-1,p'(x)}(\Omega). \tag{4.6}$$

Turning to equation (3.4), we consider $w = u_n$ and obtain

$$\Big\langle -\Delta^{K_p}_{p(x)} u_n - \Delta^{K_q}_{q(x)} u_n - N_f(u_n), u_n \Big\rangle = 0 \quad \text{for all } n \in \mathbb{N}. \tag{4.7}$$

By (4.6) we have

$$\Big\langle -\Delta^{K_p}_{p(x)} u_n - \Delta^{K_q}_{q(x)} u_n - N_f(u_n), u \Big\rangle \to 0, \quad \text{as } n \to +\infty,$$

and using (4.7) we get

$$\lim_{n\to+\infty} \Big\langle -\Delta^{K_p}_{p(x)} u_n - \Delta^{K_q}_{q(x)} u_n - N_f(u_n), u_n - u \Big\rangle = 0. \tag{4.8}$$

Since $\{u_n\}_{n\in\mathbb{N}}$ converges weakly in $W_0^{1,p(x)}(\Omega)$, it is bounded and so $\{N_f^*(u_n)\}_{n\in\mathbb{N}}$ is bounded. Using this fact along with Hölder's inequality and the compact embedding $W_0^{1,p(x)} \hookrightarrow L^{\alpha(x)}(\Omega)$ (see Proposition 2.2), we get

$$\begin{aligned} \left| \int_\Omega f(x, u_n, \nabla u_n)(u_n - u)\, dx \right| &\leq 2 \|N_f^*(u_n)\|_{L^{\alpha'(x)}(\Omega)} \|u - u_n\|_{L^{\alpha(x)}(\Omega)} \\ &\leq 2 \left(\sup_{n\in\mathbb{N}} \|N_f^*(u_n)\|_{L^{\alpha'(x)}(\Omega)} \right) \|u - u_n\|_{L^{\alpha(x)}(\Omega)} \to 0, \end{aligned}$$

as $n \to +\infty$. It follows that

$$\lim_{n\to+\infty} \Big\langle -\Delta^{K_p}_{p(x)} u_n - \Delta^{K_q}_{q(x)} u_n, u_n - u \Big\rangle = 0 \quad (\text{recall (4.8)}). \tag{4.9}$$

Combining (4.3), (4.6) and (4.9) we conclude that $u \in W_0^{1,p(x)}(\Omega)$ is a strong generalized solution to (P). This completes the proof. □

Remark 4.1. Changing $H(p)$ by $H'(p)$, the proofs of Propositions 3.5 and 3.7 above need minor adaptations. Thus, to avoid repetitions, we omit the details. We leave to the reader the easy computations, see also the similar lines in Section 4, pp. 12–13, of [26].

5 Case of positive Kirchhoff term

In this section, we briefly discuss the existence of weak solutions to (P), in the case the Kirchhoff type term (K) is substituted by the classical positive Kirchhoff term in the literature, that is

$$\widetilde{K}(r,u) = a_r + b_r \int_\Omega \frac{1}{r(x)} |\nabla u|^{r(x)} dx, \quad \text{with } a_r, b_r > 0. \tag{5.1}$$

This means that our hypothesis (K_+) this time is trivially satisfied as from (5.1) we have

$$\widetilde{K}(r,u) \geq a_r > 0 \quad \text{for all } u \in W_0^{1,r(x)}(\Omega),$$

and consequently we focus only on the notion of weak solution. Indeed, every weak solution obtained in this case, is a strong generalized solution too (recall (K_+)).

The main problem (P) becomes as follows

$$-\Delta_{p(x)}^{\widetilde{K}_p} u(x) - \Delta_{q(x)}^{\widetilde{K}_q} u(x) = f(x, u(x), \nabla u(x)) \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = 0. \tag{P_+}$$

This time, $-\Delta_{p(x)}^{\widetilde{K}_p}, -\Delta_{q(x)}^{\widetilde{K}_p} : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ are the operators defined by

$$\begin{aligned}\Big\langle -\Delta_{p(x)}^{\widetilde{K}_p} u, w \Big\rangle &= \widetilde{K}(p,u) \langle -\Delta_{p(x)} u, w \rangle \\ &= \widetilde{K}(p,u) \int_\Omega |\nabla u|^{p(x)-2} (\nabla u, \nabla w)_{\mathbb{R}^N} dx \quad \text{for all } u, w \in W_0^{1,p(x)}(\Omega),\end{aligned}$$

$$\begin{aligned}\Big\langle -\Delta_{q(x)}^{\widetilde{K}_q} u, w \Big\rangle &= \widetilde{K}(q,u) \langle -\Delta_{q(x)} u, w \rangle \\ &= \widetilde{K}(q,u) \int_\Omega |\nabla u|^{q(x)-2} (\nabla u, \nabla w)_{\mathbb{R}^N} dx \quad \text{for all } u, w \in W_0^{1,p(x)}(\Omega).\end{aligned}$$

Simplifying, $-\Delta_{r(x)}^{\widetilde{K}_r} : W_0^{1,r(x)}(\Omega) \to W^{-1,r'(x)}(\Omega)$ can be seen as positive-weight version of the operator $-\Delta_{r(x)} : W_0^{1,r(x)}(\Omega) \to W^{-1,r'(x)}(\Omega)$, in respect to the theory of pseudomonotone operators. Since $-\Delta_{r(x)}$ is continuous, bounded, strictly monotone convex and of type $(S)_+$, we deduce trivially that $-\Delta_{r(x)}^{\widetilde{K}_r}$ is continuous, bounded and of type $(S)_+$.

Our approach remains purely topological (because of the presence of convection), so we involve the Nemitsky map $N_f : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$, and introduce the operator $A : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ defined by

$$A(u) = -\Delta_{p(x)}^{\widetilde{K}_p} u - \Delta_{q(x)}^{\widetilde{K}_q} u - N_f(u) \quad \text{for all } u \in W_0^{1,p(x)}(\Omega). \tag{5.2}$$

Clearly, this operator is bounded and continuous. We establish the following existence theorem.

Theorem 5.1. *If hypotheses $H(p)$ and $H(f)$ hold, then problem (P_+) admits at least a weak solution.*

A similar theorem can be established using hypothesis $H'(p)$ instead of $H(p)$. In both the cases, the new strategy develops through two steps: the proof of pseudo-monotonicity of $A(\cdot)$ and the proof of coercivity of $A(\cdot)$.

Proof of Theorem 5.1. In the first step of the proof, we establish the pseudo-monotonicity of $A(\cdot)$ defined by (5.2), in the sense of Remark 2.4. To this end, let $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p(x)}$ be a sequence such that

$$u_n \xrightarrow{w} u \quad \text{in } W_0^{1,p(x)} \quad \text{and} \quad \limsup_{n\to+\infty}\langle A(u_n), u_n - u\rangle \le 0. \tag{5.3}$$

Using (5.3) we deduce that

$$\limsup_{n\to+\infty}\left[\left\langle -\Delta_{p(x)}^{\widetilde{K}_p} u_n - \Delta_{q(x)}^{\widetilde{K}_q} u_n, u_n - u\right\rangle - \int_\Omega f(x,u_n,\nabla u_n)(u_n-u)dx\right] \le 0. \tag{5.4}$$

Since $\{u_n\}_{n\in\mathbb{N}}$ converges weakly in $W_0^{1,p(x)}(\Omega)$, it is bounded and so $\{N_f^*(u_n)\}_{n\in\mathbb{N}}$ is bounded. Using this fact along with Hölder's inequality and the compact embedding $W_0^{1,p(x)} \hookrightarrow L^{\alpha(x)}(\Omega)$ (see Proposition 2.2), we get

$$\int_\Omega f(x,u_n,\nabla u_n)(u_n-u)dx \to 0, \quad \text{as } n\to+\infty. \tag{5.5}$$

Therefore (5.4) leads to the following chain of implications

$$\begin{aligned}
&\limsup_{n\to+\infty}\left\langle -\Delta_{p(x)}^{\widetilde{K}_p} u_n - \Delta_{q(x)}^{\widetilde{K}_q} u_n, u_n - u\right\rangle \le 0,\\
\Rightarrow\ &\limsup_{n\to+\infty}\left[\left\langle -\Delta_{p(x)}^{\widetilde{K}_p} u_n, u_n-u\right\rangle + \widetilde{K}(q,u_n)\langle -\Delta_{q(x)}u, u_n-u\rangle\right] \le 0,\\
\Rightarrow\ &\limsup_{n\to+\infty}\left\langle -\Delta_{p(x)}^{\widetilde{K}_p} u_n, u_n-u\right\rangle \le 0\\
\Rightarrow\ &u_n \to u \text{ in } W_0^{1,p(x)}(\Omega) \quad (\text{since } -\Delta_{p(x)}^{\widetilde{K}_p} \text{ has the } (S)_+\text{-property}).
\end{aligned} \tag{5.6}$$

Since $A(\cdot)$ is continuous, using (5.6) we get the convergences $A(u_n)\to A(u)$ and $\langle A(u_n),u_n\rangle \to \langle A(u),u\rangle$. So, we conclude that $A(\cdot)$ is a pseudomonotone operator.

It remains to prove the coercivity of $A(\cdot)$. Using hypothesis $H(f)$(ii), we deduce that

$$\begin{aligned}
\langle A(u),u\rangle &= \left(a_p + b_p\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\int_\Omega |\nabla u|^{p(x)}dx\\
&\quad + \left(a_q + b_q\int_\Omega \frac{1}{q(x)}|\nabla u|^{q(x)}dx\right)\int_\Omega |\nabla u|^{q(x)}dx - \int_\Omega f(x,u,\nabla u)u dx\\
&\ge \frac{b_p}{p^+}\rho_p^2(\nabla u) + a_p\rho_p(\nabla u) + \frac{b_q}{q^+}\rho_q^2(\nabla u) + a_q\rho_q(\nabla u) - \int_\Omega |f(x,u,\nabla u)u|dx\\
&\ge \left[\frac{b_p}{p^+}\rho_p(\nabla u) + a_p - \lambda^*\right]\rho_p(\nabla u) - \|\sigma_0\|_{L^1(\Omega)} \quad (\text{by } (3.2)),
\end{aligned}$$

and hence we get

$$\langle A(u),u\rangle \ge \left[\frac{b_p}{p^+}(\|u\|^{p^-}-1) + a_p - \lambda^*\right](\|u\|^{p^-}-1) - \|\sigma_0\|_{L^1(\Omega)} \quad (\text{by } (2.1)).$$

Therefore the coercivity of $A(\cdot)$ follows immediately since $1 < p^-$. Now, we can apply Theorem 2.5 to the operator $A(\cdot)$, and hence we deduce that there exists $\hat{u} \in W_0^{1,p(x)}(\Omega)$ such that $A(\hat{u}) = 0$. Obviously, such $\hat{u} \in W_0^{1,p(x)}(\Omega)$ is a weak solution to (P_+). □

Remark 5.2. When we use hypothesis $H(f)$(ii) and $H'(p)$ to prove the coercivity of $A(\cdot)$, the precise calculations are as follows

$$\begin{aligned}\langle A(u),u\rangle &= \left(a_p + b_p\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\int_\Omega |\nabla u|^{p(x)}dx \\ &\quad + \left(a_q + b_q\int_\Omega \frac{1}{q(x)}|\nabla u|^{q(x)}dx\right)\int_\Omega |\nabla u|^{q(x)}dx - \int_\Omega f(x,u,\nabla u)u dx \\ &\geq \frac{b_p}{p^+}\rho_p^2(\nabla u) - |a_p - b_2|\rho_p(\nabla u) - b_1\int_\Omega |u|^{p(x)}dx - \|\sigma_0\|_{L^1(\Omega)} \\ &\geq \frac{b_p}{p^+}\|u\|^{2p^-} - C\|u\|^{p^+} \quad \text{for some } C>0 \text{ if } \|u\|>1,\end{aligned}$$

and hence the coercivity of $A(\cdot)$ is proved.

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Strong solutions for singular Dirichlet elliptic problems

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Abstract. We prove an existence result for strong solutions $u \in W^{2,q}(\Omega)$ of singular semilinear elliptic problems of the form $-\Delta u = g(\cdot, u)$ in Ω, $u = \tau$ on $\partial\Omega$, where $1 < q < \infty$, Ω is a bounded domain in $\mathbb{R}^n$ with C^2 boundary, $0 \leq \tau \in W^{2-\frac{1}{q},q}(\partial\Omega)$, and with $g : \Omega \times (0, \infty) \to [0, \infty)$ belonging to a class of nonnegative Carathéodory functions, which may be singular at $s = 0$ and also at $x \in S$ for some suitable subsets $S \subset \overline{\Omega}$. In addition, we give results concerning the uniqueness and regularity of the solutions. A related problem on punctured domains is also considered.

Keywords: singular elliptic problems, strong solutions, Schauder's fixed point theorem, approximation method.

2020 Mathematics Subject Classification: 35J75, 35J25, 35J61.

1 Introduction and statement of the main results

Our aim in this paper is to state existence and uniqueness results for strong solutions $u \in W^{2,q}(\Omega)$ of singular elliptic problems of the form

$$\begin{cases} -\Delta u = g(\cdot, u) & \text{in } \Omega, \\ u = \tau & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $1 < q < \infty$, Ω is a bounded domain in $\mathbb{R}^n$ with C^2 boundary, $0 \leq \tau \in W^{2-\frac{1}{q},q}(\partial\Omega)$, with the boundary condition understood in the sense of the trace, and where $g : \Omega \times (0, \infty) \to [0, \infty)$ is a suitable nonnegative Carathéodory function which may be singular at $s = 0$ and at $x \in S$ for some suitable subsets $S \subset \overline{\Omega}$.

Singular elliptic problems appear in the study of nonlinear phenomena such as non-Newtonian fluids, the temperature of some electrical conductors, thin films, micro electro-mechanicals devices, and chemical catalysts process, (see e.g., [6, 15, 19, 20, 28] and the references therein).

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Existence of classical solutions $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of problem (1.1) were obtained, in the pioneering works [11,41] (in both cases for a general second order linear operator instead of the Laplacian, but in [11] with homogeneous boundary condition), and in [9,15,20]. Cases where g has the form $g(x,s) = a(x)s^{-\alpha}$, $\alpha \in (0,\infty)$, and $\tau = 0$ were studied in [26] and [14], and more recently, in [16,40], and [29]. Let us mention also that in [15], problem (1.1) was studied when $\tau = 0$ and $g(x,s) = -\frac{1}{s^\gamma} + f(x)$ for some $\gamma > 0$ and $f \in L^1(\Omega)$.

Existence results for classical solutions of Lane–Emden–Fowler equations with convection and singular potential were obtained in [17], and related problems were studied in [8] and [22]. Problem (1.1) was studied, again in a classical sense, in [1,27,31,34,35,42], and [43], in some cases where $g = g(x,s)$ is singular at $s = 0$, and with some kind of singularity at $x \in \partial\Omega$. Related problems can be found also in [37], [38], and [39].

In [30] it was studied the existence, uniqueness, and regularity properties of the weak solutions of problems of the form $-\operatorname{div}(A(x)\nabla u) = \frac{f(x)}{u^\gamma} + \mu$ in Ω, $u > 0$ in Ω, $u = 0$ on $\partial\Omega$, in the case when $A(x)$ is a uniformly elliptic and bounded matrix, $\gamma > 0$, $0 \leq f \in L^1(\Omega)$ in Ω, and μ is a nonnegative bounded Radon measure.

Existence and nonexistence of solutions of problems of the form $-\operatorname{div}(A(x)\nabla u) = fu^{-\gamma}$ in Ω, $u > 0$ in Ω, $u = 0$ on $\partial\Omega$, was studied in [4], in the case where A is a bounded elliptic matrix and f is, either a nonnegative function in a suitable $L^p(\Omega)$ or a nonnegative and bounded Radon measure. The existence and uniqueness of solutions of problem of the form $-\operatorname{div}(A(x)\nabla u) = H(u)\mu$ in Ω, $u > 0$ in Ω, $u = 0$ on $\partial\Omega$, was studied in [31] in the case when μ a bounded Radon measure, $A(x)$ is a uniformly elliptic and bounded matrix with Lipschitz continuous coefficients, and $H : (0,\infty) \to (0,\infty)$ satisfies some suitable conditions which allow that $\lim_{s\to 0^+} H(s) = \infty$.

Problems of the form $-\Delta u = H(u)\mu$ in Ω, $u > 0$ in Ω, $u = 0$ on $\partial\Omega$, with $H : (0,\infty) \to (0,\infty)$ allowed to be singular at the origin, in the sense that $\lim_{s\to 0^+} H(s) = \infty$, and where μ is a bounded Radon measure were studied, under different assumptions, in [13] and [32], and the analogous problem $-\Delta_p u = H(u)\mu$ in Ω, $u > 0$ in Ω, $u = 0$ on $\partial\Omega$ (where Δ_p is the usual p-Laplacian operator $\Delta_p(u) := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$), was studied in [12].

In [18] it was proved, via a comparison principle, the uniqueness of the weak solutions of problems of the form $-\Delta_p u = F(\cdot,u)$ in Ω, $u > 0$ in Ω, $u = 0$ on $\partial\Omega$, in the case when F is a nonnegative Carathéodory function on $\Omega \times (0,\infty)$ such that $s \to s^{1-p}F(x,s)$ is decreasing on $(0,\infty)$ for *a.e.* $x \in \Omega$. In addition, again in [18], it was proved the existence of weak solutions of problems of the form $-\Delta_p u = fu^{-\gamma} + gu^q$ in Ω, $u > 0$ in Ω, $u = 0$ on $\partial\Omega$, in the case when $\gamma \geq 0$, $0 \leq q \leq p-1$; f and g are nonnegative functions belonging to suitable Lebesgue spaces.

The existence of weak solutions in $W_0^{1,q}(\Omega)$ of problem (1.1) was studied in [7] in some cases where $\tau = 0$, $g(x,s) = a(x)s^{-\alpha(x)}$. In [24] it was studied the existence of weak solutions, in $H_0^1(\Omega)$, for problems of the form $-\Delta u = g(\cdot,u)$ in Ω, $u = 0$ on $\partial\Omega$, $u > 0$ in Ω, including some cases where $g(x,s)$ is singular at $s = 0$, and also at $x \in \partial\Omega$.

Singular problems on punctured domains were studied in [3]. There it was proved that, if $x_0 \in \Omega$ and if $a : \Omega \to \mathbb{R}$ satisfies certain condition related to the Karamata class, then the problem $-\Delta u = au^{-\alpha}$ in $\Omega \setminus \{x_0\}$, $u > 0$ in $\Omega \setminus \{x_0\}$, $u = 0$ on $\partial\Omega$ has at least one solution such that , $\lim_{x\to x_0} |x - x_0|^{n-2} u(x) = 0$.

The interested reader will find an updated account, concerning the topic of singular elliptic problems, as well as additional references, in the research books [36], and [21].

We assume, from now on, that $n \geq 2$ and that Ω is a bounded domain in $\mathbb{R}^n$ with C^2 boundary. Let $q \in (1,\infty)$, which we fix from now on. We recall that (see, e.g., [25, Theorem 2.4.2.5]), for $f \in L^q(\Omega)$ and $\tau \in W^{2-\frac{1}{q},q}(\partial\Omega)$, there exists a unique strong solution $u \in W^{2,q}(\Omega)$ of the problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = \tau & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

with the boundary condition understood in the sense of the trace, and that u satisfies $\|u\|_{W^{2,q}(\Omega)} \leq c\big(\|f\|_{L^q(\Omega)} + \|\tau\|_{W^{2-\frac{1}{q},q}(\partial\Omega)} \big)$, where c is a positive constant independent of u. We will write $(-\Delta)^{-1}$ for the solution operator $(-\Delta)^{-1} : L^q(\Omega) \to W^{2,q}(\Omega)$ of the homogeneous Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

i.e., for the operator defined by $(-\Delta)^{-1} f := u$, where $u \in W^{2,q}(\Omega)$ is the unique strong solution u of problem (1.3).

We will write d_Ω for the function $d_\Omega : \Omega \to \mathbb{R}$, defined by $d_\Omega(x) := \operatorname{dist}(x, \partial\Omega)$. With these notations, our first result reads as follows:

Theorem 1.1. *Let $n \geq 2$, let Ω be a bounded domain in $\mathbb{R}^n$ with C^2 boundary, and let τ be a nonnegative function in $W^{2-\frac{1}{q},q}(\partial\Omega) \cap C(\partial\Omega)$. Let $g : \Omega \times (0,\infty) \to \mathbb{R}$ satisfying the following three conditions H1)–H3):*

- *H1) g is a Carathéodory function (that is $g(\cdot, s)$ is measurable for any $s > 0$ and $g(x, \cdot)$ is continuous on $(0,\infty)$ for any $x \in \Omega$) and such that, for any $x \in \Omega$, $g(x, \cdot)$ is nonnegative and nonincreasing on $(0,\infty)$.*
- *H2) There exists $A \subset \Omega$ such that $|A| > 0$ and $g(x,s) > 0$ for all $(x,s) \in A \times (0,\infty)$.*
- *H3) $g(\cdot, cd_\Omega) \in L^q(\Omega)$ for all $c \in (0,\infty)$.*

Then problem (1.1) has a strong solution $u \in W^{2,q}(\Omega)$ which satisfies $\tau^* + cd_\Omega \leq u \leq \tau^* + (-\Delta)^{-1}(g(\cdot, cd_\Omega))$ a.e. in Ω, where c is a positive constant and $\tau^* \in W^{2,q}(\Omega)$ is the (unique) strong solution of the problem

$$\begin{cases} -\Delta z = 0 & \text{in } \Omega, \\ z = \tau & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

Remark 1.2. For τ as in the statement of Theorem 1.1, since $\tau \in C(\partial\Omega)$, problem (1.4) has a classical solution $\zeta \in C^2(\Omega) \cap C(\overline{\Omega})$ (see e.g., [23, Theorem 2.14]) which, by the classical maximum principle (as stated e.g., in [23, Theorem 3.1]), satisfies $\zeta \geq 0$ in Ω. On the other hand, since $\tau \in W^{2-\frac{1}{q},q}(\partial\Omega)$, ([2, Theorem 15.2]) gives that $\zeta \in W^{2,q}(\Omega)$ and that ζ is the strong solution of (1.4). Then $\tau^* \geq 0$ in Ω and $\tau^* \in C(\overline{\Omega})$. Moreover, τ^* is harmonic in Ω, then $\tau^* \in C^\infty(\Omega)$, and so $\tau^* \in W^{2,p}_{loc}(\Omega)$ for any $p \in [1,\infty)$.

The next result states that, if H1)–H3) hold, and if some additional assumptions on g are fulfilled, then the solution u of problem (1.1) is unique and has additional regularity properties:

Theorem 1.3. *Assume the hypothesis of Theorem 1.1 and that, in addition, the following conditions H4)–H5) hold:*

- *H4) g is continuous on $\Omega \times (0,\infty)$,* *H5)* $(-\Delta)^{-1}(g(\cdot, cd_\Omega)) \in C(\overline{\Omega})$ *for any* $c > 0$.

Then problem (1.1) *has a unique strong solution* $u \in W^{2,q}(\Omega)$, *and it belongs to* $W^{2,n}_{loc}(\Omega) \cap C(\overline{\Omega})$. *In particular,* $u \in C^1(\Omega)$.

Our third result refers to the punctured domain $U := \Omega \setminus \{x_0\}$, where $x_0 \in \Omega$, and reads as follows:

Theorem 1.4. *Let* $x_0 \in \Omega$, $U := \Omega \setminus \{x_0\}$ *and, for* $\delta > 0$, *let*

$$A_\delta := \left\{x \in \Omega : \frac{\delta}{2} \leq |x - x_0| \leq \delta\right\}. \tag{1.5}$$

Let $h : \Omega \times (0, \infty) \to \mathbb{R}$ *and let* $w \in W^{2,q}(U)$. *Assume that* w *is a strong solution of the problem*

$$\begin{cases} -\Delta w = h(\cdot, w) & \text{in } U, \\ w = \tau & \text{on } \partial\Omega \end{cases} \tag{1.6}$$

(with the boundary condition understood in the sense of the trace). If either $w \in C(\Omega)$ *or* $\limsup_{\delta \to 0^+} \frac{1}{\delta^2} \int_{A_\delta} |w| = 0$, *then* $w \in W^{2,q}(\Omega)$ *and* w *is a strong solution of the problem*

$$\begin{cases} -\Delta w = h(\cdot, w) & \text{in } \Omega, \\ w = \tau & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

We have also the following:

Theorem 1.5. *Assume the hypothesis of Theorem* 1.3. *Let* $x_0 \in \Omega$, $U := \Omega \setminus \{x_0\}$, *and let* $w \in W^{2,q}(U)$. *If* w *is a strong solution of the problem*

$$\begin{cases} -\Delta w = g(\cdot, w) & \text{in } U, \\ w = \tau & \text{on } \partial\Omega, \\ w > 0 & \text{in } U. \end{cases}$$

Then:

i) If $\limsup_{x \to x_0} |x - x_0|^{n-2} w(x) = 0$ *then, after redefining* w *in a set with zero measure, it hold that* $w \in W^{2,q}(\Omega) \cap C(\overline{\Omega}) \cap C^1(\Omega)$ *and* w *is the unique solution of problem* (1.1)

ii) If $\|w\|_{L^\infty(U)} = \infty$, *then* $\limsup_{x \to x_0} |x - x_0|^{n-2} w(x) > 0$.

The paper is organized as follows: in Section 2 we study, for $M \geq 1$ and $\varepsilon \in (0, 1]$, the approximated problems

$$\begin{cases} -\Delta u = g_M(\cdot, \varepsilon + u) & \text{in } \Omega, \\ u = \tau & \text{on } \partial\Omega, \end{cases}$$

where $g_M(x, s) := \min\{M, g(x, s)\}$. By using Schauder's fixed point theorem, we show that this problem has a unique solution $u_{M,\varepsilon} \in \cap_{1<p<\infty} W^{2,p}(\Omega)$ (see Lemmas 2.2 and 2.4). Lemma 2.6 states that $\varepsilon \to u_{M,\varepsilon}$ is nonincreasing, $M \to u_{M,\varepsilon}$ is nondecreasing, and that $\tau^* + c_0 d_\Omega \leq u_{M,\varepsilon} \leq \tau^* + (-\Delta)^{-1}(\cdot, c_0 d_\Omega)$ in Ω, with c_0 a positive constant independent of M

and ε, and where τ^* is the strong solution of (1.4). Lemma 2.7 shows that if $u_M := \lim_{\varepsilon\to 0^+} u_{M,\varepsilon}$, then $u_M \in W^{2,q}(\Omega)$ and u_M is a strong solution of the problem

$$\begin{cases} -\Delta u_M = g_M(\cdot, u_M) & \text{in } \Omega, \\ u_M = \tau & \text{on } \partial\Omega. \end{cases}$$

The main results are proved in Section 3. To prove Theorem 1.1 we define $\boldsymbol{u} := \lim_{M\to\infty} u_M$ and we show that $\boldsymbol{u}$ is a strong solution of problem (1.1) with the desired properties. This is achieved from thanks to Lemma 2.7 by showing that $g(\cdot, \boldsymbol{u}) := \lim_{M\to\infty} g_M(\cdot, u_M)$ with convergence in $L^q(\Omega)$. To prove Theorem 1.3 we show that, for any strong solution u of problem (1.1), there exists a positive constant c such that $\tau^* + cd_\Omega \le u \le \tau^* + (-\Delta)^{-1}(\cdot, cd_\Omega)$ in Ω, which will give the continuity of u at $\partial\Omega$, next we show, by a suitable bootstrap argument, that $u \in W^{2,n}_{loc}(\Omega)$, which gives that $u \in C^1(\Omega)$. Proved that $\boldsymbol{u} \in W^{2,n}_{loc}(\Omega) \cap C(\overline{\Omega})$, the uniqueness assertion of Theorem 1.3 will follow from the fact that $s \to g(x,s)$ is nonincreasing, combined with the application of an appropriate maximum principle. Finally, Theorem 1.4 is proved by showing that, if $w \in W^{2,q}(\Omega \setminus \{x_0\})$ satisfies the conditions of Theorem 1.4, then w, viewed as a distribution on Ω, belongs to $W^{2,q}(\Omega)$.

2 Preliminaries

Let $g : \Omega \times (0,\infty) \to \mathbb{R}$ be a function satisfying the conditions H1)–H3) of Theorem 1.1 and, for $M \in [1,\infty)$, $\varepsilon \in (0,1]$, let $g_M : \Omega \times (0,\infty) \to \mathbb{R}$ be defined by

$$g_M(x,s) := \min\{M, g(x,s)\}.$$

let $K_M := \|\tau^*\|_{L^q(\Omega)} + M\|(-\Delta)^{-1}(\mathbf{1})\|_{L^q(\Omega)}$, where τ^* is the strong solution of problem (1.4), and let

$$C_M := \{v \in L^q(\Omega) : 0 \le v \le K_M\}.$$

For $v \in C_M$, since g is a Carathéodory function, $g_M(\cdot, \varepsilon + v)$ is a measurable function. Let η be a positive and small enough number such that $\eta d_\Omega \le \varepsilon$ in Ω. Then, since g is nonincreasing in the second variable and $v \ge 0$ in Ω, we have $0 \le g(\cdot, \varepsilon + v) \le g(\cdot, \varepsilon) \le g(\cdot, \eta d_\Omega)$ in Ω. By H3), $g(\cdot, \eta d_\Omega) \in L^q(\Omega)$, then $0 \le g_M(\cdot, \varepsilon + v) \le g_M(\cdot, \eta d_\Omega) \in L^q(\Omega)$ and thus $g_M(\cdot, \varepsilon + v) \in L^q(\Omega)$. Then $(-\Delta)^{-1}(g_M(\cdot, \varepsilon + v))$ is a well defined element in $W^{2,q}(\Omega)$. Let $T_{M,\varepsilon} : C_M \to W^{2,q}(\Omega)$ be the operator defined by

$$T_{M,\varepsilon}(v) := \tau^* + (-\Delta)^{-1}(g_M(\cdot, \varepsilon + v)).$$

Remark 2.1.

i) Let us recall the following form of the Aleksandrov maximum principle (which is a particular case of [23], Theorem 9.1): If U is a bounded domain in $\mathbb{R}^n$ and if $u \in W^{2,n}_{loc}(U) \cap C(\overline{U})$ satisfies $-\Delta u \ge 0$ in U (respectively $-\Delta u \le 0$ in U) and $u \ge 0$ on ∂U (resp. $u \le 0$ on ∂U), then $u \ge 0$ in U (resp. $u \le 0$ in U).

ii) If $0 \le f \in L^q(\Omega)$ then $(-\Delta)^{-1} f \ge 0$ in Ω (note that we do not assume $q \ge n$). Indeed, let $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}$ be the extension by zero of f. Then $0 \le \widetilde{f} \in L^q(\mathbb{R}^n)$ and so $\widetilde{f}$ can be approximated, in the $L^q(\mathbb{R}^n)$ norm, by a sequence $\{\widetilde{f}_j\}_{j\in N} \subset C^\infty(\mathbb{R}^n)$ obtained by convolving $\widetilde{f}$ with suitable mollifiers (see [33, Proposition 1.1.3]). Thus, for each j, $0 \leq \widetilde{f}_{j|\Omega} \in L^{\infty}(\Omega)$, and so the solution u_j of the problem

$$\begin{cases} -\Delta u_j = \widetilde{f}_{j|\Omega} & \text{in } \Omega, \\ u_j = 0 & \text{on } \partial\Omega \end{cases}$$

belongs to $W^{2,p}(\Omega)$ for any $p \in [1, \infty)$ and, since $\{\widetilde{f}_{j|\Omega}\}_{j \in N}$ converges to f in $L^q(\mathbb{R}^n)$, it follows that $\{u_j\}_{j \in N}$ converges to u in $W^{2,q}(\Omega)$. Now, by i), $u_j \geq 0$ in Ω, and then $u \geq 0$ in Ω.

iii) From ii), it follows immediately that if f and h belong to $L^q(\Omega)$ and $f \leq h$ in Ω, then $(-\Delta)^{-1} f \leq (-\Delta)^{-1} h$ in Ω.

Lemma 2.2. *Assume the conditions H1)–H3) of Theorem 1.1, let τ be a nonnegative function in $W^{2-\frac{1}{q},q}(\partial\Omega)$, and let $\tau^* \in W^{2,q}(\Omega)$ be the strong solution of problem (1.4). Then, for $M \in [1, \infty)$ and $\varepsilon \in (0, 1]$,*

i) C_M is a closed and convex subset of $L^q(\Omega)$.

ii) $T_{M,\varepsilon}(C_M) \subset C_M$.

iii) $T_{M,\varepsilon} : C_M \to C_M$ is continuous.

iv) $T_{M,\varepsilon} : C_M \to C_M$ is a compact operator.

Proof. i) is immediate. To prove ii), observe that, for $v \in C_M$, since $g(\cdot, \varepsilon + v)$ is nonnegative, Remark 2.1 iii) gives that $(-\Delta)^{-1}(g_M(\cdot, \varepsilon + v)) \geq 0$ and so, since $\tau^* \geq 0$ in Ω, we have $T_{M,\varepsilon}(v) \geq 0$ in Ω. Also,

$$\begin{aligned} \|T_{M,\varepsilon}(v)\|_q &\leq \|\tau^*\|_q + \left\|(-\Delta)^{-1}(g_M(\cdot, \varepsilon + v))\right\|_q \\ &\leq \|\tau^*\|_q + M\left\|(-\Delta)^{-1}(\mathbf{1})\right\|_q = K_M. \end{aligned}$$

Then $T_{M,\varepsilon}(v) \in C_M$.

To show iii), it is enough to see that if $v \in C_M$ and if $\{v_j\}_{j \in \mathbb{N}}$ is a sequence in C_M such that $\{v_j\}_{j \in \mathbb{N}}$ converges to v in $L^q(\Omega)$, then there exists a subsequence $\{v_{j_k}\}_{k \in \mathbb{N}}$ such that $\{T_{M,\varepsilon}(v_{j_k})\}_{k \in \mathbb{N}}$ converges to $T_{M,\varepsilon}(v)$ in $L^q(\Omega)$.

Let $v \in C_M$ and let $\{v_j\}_{j \in \mathbb{N}} \subset C_M$ be such that $\{v_j\}_{j \in \mathbb{N}}$ converges to v in $L^q(\Omega)$. Then there exists a subsequence $\{v_{j_k}\}_{k \in \mathbb{N}}$ such that $\{v_{j_k}\}_{k \in \mathbb{N}}$ converges to v *a.e.* in Ω. Then, since g_M is a Carathéodory function, $\{g_M(\cdot, \varepsilon + v_{j_k})\}_{k \in \mathbb{N}}$ converges to $g_M(\cdot, \varepsilon + v)$ *a.e.* in Ω. Thus $\lim_{k \to \infty} |g_M(\cdot, \varepsilon + v_{j_k}) - g_M(\cdot, \varepsilon + v)|^q = 0$ *a.e.* in Ω. Also, $|g_M(\cdot, \varepsilon + v_{j_k}) - g_M(\cdot, \varepsilon + v)|^q \leq (2M)^q$ and then, by Lebesgue's dominated convergence theorem, $\{g_M(\cdot, \varepsilon + v_{j_k})\}_{k \in \mathbb{N}}$ converges to $g_M(\cdot, \varepsilon + v)$ in $L^q(\Omega)$. Thus $\{(-\Delta)^{-1}(g_M(\cdot, \varepsilon + v_{j_k}))\}_{k \in \mathbb{N}}$ converges to $(-\Delta)^{-1}(g_M(\cdot, \varepsilon + v))$ in $W^{2,q}(\Omega)$. Then iii) holds.

To prove iv), consider a sequence $\{v_j\}_{j \in \mathbb{N}} \subset C_M$. Then $\{v_j\}_{j \in \mathbb{N}}$ is bounded in $L^q(\Omega)$, and thus $\{(-\Delta)^{-1}(g_M(\cdot, \varepsilon + v_{j_k}))\}_{k \in \mathbb{N}}$ is bounded in $W^{2,q}(\Omega)$. Then there exists a subsequence $\{v_{j_k}\}_{k \in \mathbb{N}}$ such that $\{(-\Delta)^{-1}(g_M(\cdot, \varepsilon + v_{j_k}))\}_{k \in \mathbb{N}}$ converges in $L^q(\Omega)$, and so iv) holds. □

Lemma 2.3. *Let $h : \Omega \times (0, \infty) \to \mathbb{R}$ be a function such that $h(x, \cdot)$ is nonincreasing on $(0, \infty)$ for any $x \in \Omega$, and let u, v be two functions in $W^{2,n}_{loc}(\Omega) \cap C(\overline{\Omega})$. If u, v satisfy $-\Delta u = h(\cdot, u)$ in Ω, $-\Delta v = h(\cdot, v)$ in Ω, and $u = v$ on $\partial\Omega$, then $u = v$ in Ω.*

Proof. Let $U := \{x \in \Omega : u(x) > v(x)\}$ and let $V := \{x \in \Omega : u(x) < v(x)\}$. Then U and V are open subsets of Ω. Suppose that $U \neq \varnothing$, Then

$$-\Delta(u - v) = h(\cdot, u) - h(\cdot, v) \leq 0 \quad \text{in } U. \tag{2.1}$$

Also,

$$u - v = 0 \quad \text{on } \partial U. \tag{2.2}$$

Indeed, if $x \in \partial U \cap \partial\Omega$ then $u(x) - v(x) = 0$, and if $x \in \partial U \cap \Omega$ then $u(x) - v(x) \geq 0$ (because $u - v > 0$ in U and $u - v$ is continuous in $\overline{\Omega}$), but if $u(x) - v(x) > 0$ we would have $u - v > 0$ in a neighborhood of x, in contradiction with the fact that $x \in U$. Then $u(x) - v(x) = 0$ also in the case when $x \in \partial U \cap \Omega$. Thus (2.2) holds. Now, from (2.1), (2.2) and Remark 2.1, we obtain $u - v \leq 0$ in U. which is impossible. Thus $U = \varnothing$. Similarly, $V = \varnothing$, and so $u = v$ in Ω. □

Lemma 2.4. *Assume the hypothesis of Theorem 1.1. Then, for $M \in [1, \infty)$ and $\varepsilon \in (0, 1]$*

i) The problem

$$\begin{cases} -\Delta u = g_M(\cdot, \varepsilon + u) & \text{in } \Omega, \\ u = \tau & \text{on } \partial\Omega \end{cases} \tag{2.3}$$

has a unique strong solution $u_{M,\varepsilon} \in W^{2,q}(\Omega) \cap C_M$.

ii) The problem

$$\begin{cases} -\Delta v = g_M(\cdot, \varepsilon + \tau^* + v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

has a unique strong solution $v_{M,\varepsilon} \in \cap_{1<p<\infty} W^{2,p}(\Omega)$ and $u_{M,\varepsilon} = \tau^* + v_{M,\varepsilon}$.

Proof. Taking into account Lemma 2.2 and Schauder's fixed point theorem (as stated, e.g., in [23, Corollary 11.2]), $T_{M,\varepsilon}$ has a fixed point $u_{M,\varepsilon} \in C_M$, which, by the definition of $T_{M,\varepsilon}$, belongs also to $W^{2,q}(\Omega)$ and that is a strong solution of problem (2.3). Clearly a function $u \in W^{2,q}(\Omega)$ is solution of (2.3) if and only if $v := u - \tau^*$ is a solution of (2.4), and so (2.4) has, at least, a solution $v_{M,\varepsilon} \in W^{2,q}(\Omega)$. Moreover, if v is a solution of (2.4), since $g_M(\cdot, \varepsilon + \tau^* + v) \in L^\infty(\Omega)$ and $v = 0$ on $\partial\Omega$, it follows that $v \in \cap_{1 \leq p < \infty} W^{2,p}(\Omega)$. In particular $v \in C(\overline{\Omega}) \cap W^{2,n}_{loc}(\Omega)$. Suppose now that v and w are two solutions of (2.4). Then v and w belong to $C(\overline{\Omega}) \cap W^{2,n}_{loc}(\Omega)$ and $v = w = 0$ on $\partial\Omega$. Since $s \to g(x, \varepsilon + \tau^*(x) + s)$ is nonincreasing for any $x \in \Omega$, the function $h(x, s) := g_M(x, \varepsilon + \tau^*(x) + s)$ is also nonincreasing for any $x \in \Omega$. Then, by Lemma 2.3, $v = w$ in Ω and so the solution of (2.4) is unique. Now, from the equivalence of problems (2.3) and (2.4), the solution of (2.3) is also unique. □

For $M \in [1, \infty)$ and $\varepsilon \in (0, 1]$ we will denote by $u_{M,\varepsilon}$ and $v_{M,\varepsilon}$ the solutions of problems (2.3) and (2.4) given by Lemma 2.4.

Remark 2.5.

i) Let us recall the following form of the Hopf maximum principle (see [5], Lemma 3.2): Suppose that $\rho \geq 0$ belongs to $L^{\infty}(\Omega)$. Let v be the solution of $-\Delta v = \rho$ in Ω, $v = 0$ on $\partial\Omega$. Then
$$v(x) \geq c d_{\Omega}(x) \int_{\Omega} \rho d_{\Omega} \quad a.e. \text{in } \Omega, \tag{2.5}$$
where c is a positive constant depending only on Ω.

ii) Suppose that $\rho \geq 0$ belongs to $L^{\infty}(\Omega)$. If $h \in L^{q}(\Omega)$ and $h \geq \rho$ in Ω, then, from Remark 2.1 iii) and (2.5) it follows immediately that $(-\Delta)^{-1} h \geq c d_{\Omega}(x) \int_{\Omega} \rho d_{\Omega}$ *a.e.* in Ω, where c is the constant given in (2.5).

iii) We recall also Hardy's inequality (see e.g., [33], Theorem 1.10.15): There exists a positive constant c such that $\left\| \frac{\varphi}{d_{\Omega}} \right\|_2 \leq c \left\| \nabla \varphi \right\|_2$ for any $\varphi \in H_0^1(\Omega)$.

Lemma 2.6. *Assume the hypothesis of Theorem 1.1. Then*

i) For each $M \in [1, \infty)$ the map $\varepsilon \to u_{M,\varepsilon}$ is nonincreasing on $(0, 1]$.

ii) For each $\varepsilon \in (0, 1]$ the map $M \to u_{M,\varepsilon}$ is nondecreasing on $[1, \infty)$.

iii) There exists a positive constant c_0 such that, for any $\varepsilon \in (0, 1]$ and $M \in [1, \infty)$, $\tau^* + c_0 d_{\Omega} \leq u_{M,\varepsilon} \leq \tau^* + (-\Delta)^{-1}(\cdot, c_0 d_{\Omega})$ in Ω.

Proof. To see i), suppose that $0 < \varepsilon \leq \eta \leq 1$. Let $U := \left\{ x \in \Omega : v_{M,\varepsilon}(x) < v_{M,\eta}(x) \right\}$ and suppose that $U \neq \varnothing$. Since g is nonincreasing in the second variable, the same is true for g_M and so,

$$\begin{aligned} -\Delta(v_{M,\varepsilon}) &= g_M(\cdot, \varepsilon + \tau^* + v_{M,\varepsilon}) \geq g_M(\cdot, \eta + \tau^* + v_{M,\varepsilon}) \\ &\geq g_M(\cdot, \eta + \tau^* + v_{M,\eta}) = -\Delta(v_{M,\eta}) \text{ in } U. \end{aligned}$$

Also, as in the proof of Lemma 2.3, we have $v_{M,\varepsilon} = v_{M,\eta}$ on ∂U. Then, by Remark 2.1 iii), $v_{M,\varepsilon} \geq v_{M,\eta}$ in U, which is impossible. Then $U = \varnothing$ and so $v_{M,\varepsilon} \geq v_{M,\eta}$ in Ω, which implies $u_{M,\varepsilon} \geq u_{M,\eta}$ in Ω. Thus i) holds.

To see ii), suppose $1 \leq M_1 \leq M_2$ and $\varepsilon \in (0, 1]$. Let $U := \{v_{M_1,\varepsilon} > v_{M_2,\varepsilon}\}$. If $U \neq \varnothing$, then

$$\begin{aligned} -\Delta(v_{M_2,\varepsilon}) &= g_{M_2}(\cdot, \varepsilon + \tau^* + v_{M_2,\varepsilon}) \geq g_{M_1}(\cdot, \varepsilon + \tau^* + v_{M_2,\varepsilon}) \\ &\geq g_{M_1}(\cdot, \varepsilon + \tau^* + v_{M_1,\varepsilon}) = -\Delta(v_{M_1,\varepsilon}) \quad \text{in } U. \end{aligned}$$

Also, $v_{M_1,\varepsilon} = v_{M_2,\varepsilon}$ on ∂U. Then, by Remark 2.1 iii), $v_{M_1,\varepsilon} \leq v_{M_2,\varepsilon}$ in U, which is impossible. Therefore $U = \varnothing$ and so $v_{M_1,\varepsilon} \leq v_{M_2,\varepsilon}$ in Ω, which implies $u_{M_1,\varepsilon} \leq u_{M_2,\varepsilon}$ in Ω. Thus ii) holds.

To prove iii), observe that by i) and ii) we have, for $M \in [1, \infty)$ and $\varepsilon \in (0, 1]$,
$$v_{M,\varepsilon} \geq v_{M,1} \geq v_{1,1} \quad \text{in } \Omega. \tag{2.6}$$

Now,
$$\begin{cases} -\Delta v_{1,1} = g_1(\cdot, 1 + \tau^* + v_{1,1}) & \text{in } \Omega, \\ v_{1,1} = 0 & \text{on } \partial\Omega \end{cases}$$

and $0 \leq g_1(\cdot, 1+\tau^*+v_{1,1}) \in L^\infty(\Omega)$. Note that $g_1(\cdot, 1+\tau^*+v_{1,1}) \not\equiv 0$ in Ω (that is: $|\{x \in \Omega : g_1(x, 1+\tau^*(x)+v_{1,1}(x)) > 0\}| > 0$) because if $g_1(\cdot, 1+\tau^*+v_{1,1}) \equiv 0$ in Ω then $g(\cdot, 1+\tau^*+v_{1,1}) \equiv 0$ in Ω, which contradicts H2). Then

$$\int_\Omega d_\Omega g_1(\cdot, 1+\tau^*+v_{1,1}) > 0,$$

and so, taking into account Remark 2.5, there exists a positive constant c', depending only on Ω, such that

$$v_{1,1} \geq c' d_\Omega \int_\Omega g_1(\cdot, 1+\tau^*+v_{1,1})\, d_\Omega \quad \textit{a.e.} \text{ in } \Omega.$$

Then, from (2.6), $v_{M,\varepsilon} \geq c_0 d_\Omega$ with

$$c_0 := c' \int_\Omega g_1(\cdot, 1+\tau^*+v_{1,1})\, d_\Omega > 0. \tag{2.7}$$

and so, since $u_{M,\varepsilon} = \tau^* + v_{M,\varepsilon}$, we get that $u_{M,\varepsilon} \geq \tau^* + c_0 d_\Omega$ in Ω.

On the other hand, $v_{M,\varepsilon} = (-\Delta)^{-1}(g_M(\cdot, \varepsilon+\tau^*+v_{M,\varepsilon}))$. Now, $v_{M,\varepsilon} \geq v_{1,\varepsilon} \geq v_{1,1}$ in Ω, and so $g_M(\cdot, \varepsilon+\tau^*+v_{M,\varepsilon}) \leq g_M(\cdot, v_{1,1}) \leq g_M(\cdot, c_0 d_\Omega) \leq g(\cdot, c_0 d_\Omega)$, with c_0 given by (2.7). Then, by Remark 2.1 iii), $(-\Delta)^{-1} g_M(\cdot, \varepsilon+\tau^*+v_{M,\varepsilon}) \leq (-\Delta)^{-1}(g(\cdot, c_0 d_\Omega))$ in Ω, that is, $v_{M,\varepsilon} \leq (-\Delta)^{-1}(g(\cdot, c_0 d_\Omega))$ in Ω. Thus $u_{M,\varepsilon} = \tau^* + v_{M,\varepsilon} \leq \tau^* + (-\Delta)^{-1}(g(\cdot, c_0 d_\Omega))$ in Ω, which completes the proof of the lemma. □

For $M \in [1, \infty)$, let u_M and v_M be the functions, defined on Ω by

$$u_M(x) := \lim_{\varepsilon \to 0^+} u_{M,\varepsilon}(x), \qquad v_M(x) := \lim_{\varepsilon \to 0^+} v_{M,\varepsilon}(x). \tag{2.8}$$

Note that, by Lemma 2.6, $u_M(x)$ is well defined and finite for *a.e.* $x \in \Omega$ and so, since $u_{M,\varepsilon} = \tau^* + v_{M,\varepsilon}$, the same assertion holds also for v_M.

Lemma 2.7. *Assume the hypothesis of Theorem 1.1 and let c_0 be the constant given by Lemma 2.6 iii). Then:*

i) The map $M \to u_M$ is nondecreasing on $[1, \infty)$.

ii) $\tau^* + c_0 d_\Omega \leq u_M \leq \tau^* + (-\Delta)^{-1}(g(\cdot, c_0 d_\Omega))$ in Ω,for any $M \geq 1$ (in particular $u_M > 0$ in Ω)

iii) For each $M > 0$, $u_M \in W^{2,q}(\Omega)$ and u_M is a strong solution of the problem

$$\begin{cases} -\Delta u_M = g_M(\cdot, u_M) & \textit{in } \Omega, \\ u_M = \tau & \textit{on } \partial\Omega. \end{cases}$$

Proof. If $1 \leq M_1 \leq M_2$ and $\varepsilon \in (0, 1]$ then, by Lemma 2.6, $u_{M_1,\varepsilon} \leq u_{M_2,\varepsilon}$, and so, by taking $\lim_{\varepsilon \to 0^+}$, we get $u_{M_1} \leq u_{M_2}$. Thus i) holds. Also, taking $\lim_{\varepsilon \to 0^+}$ in the inequalities of Lemma 2.6 iii) we get ii).

To prove iii) note that, by Lemma 2.4 ii), we have, for $\varepsilon \in (0, 1]$ and $M \in [1, \infty)$,

$$u_{M,\varepsilon} = \tau^* + v_{M,\varepsilon}, \tag{2.9}$$

where $v_{M,\varepsilon} = (-\Delta)^{-1}(g_M(\cdot, \varepsilon+\tau^*+v_{M,\varepsilon}))$. From (2.9),

$$\lim_{\varepsilon \to 0^+} (\tau^* + v_{M,\varepsilon}) = u_M \quad \textit{a.e.} \text{ in } \Omega,$$

and so, since g_M is a Carathéodory function,

$$\lim_{\varepsilon\to 0^+} g_M(\cdot,\varepsilon+\tau^*+v_{M,\varepsilon}) = g_M(\cdot,u_M) \quad a.e. \text{ in } \Omega.$$

Then $\lim_{\varepsilon\to 0^+} |g_M(\cdot,\varepsilon+\tau^*+v_{M,\varepsilon}) - g_M(\cdot,u_M)|^q = 0$ *a.e.* in Ω. Also,

$$|g_M(\cdot,\varepsilon+\tau^*+v_{M,\varepsilon}) - g_M(\cdot,u_M)|^q \leq (2M)^q$$

for any $\varepsilon \in (0,1]$. Then, by the Lebesgue's dominated convergence theorem,

$$\lim_{\varepsilon\to 0^+} g_M(\cdot,\varepsilon+\tau^*+v_{M,\varepsilon}) = g_M(\cdot,u_M)$$

with convergence in $L^q(\Omega)$. Then

$$\lim_{\varepsilon\to 0^+} (-\Delta)^{-1}(g_M(\cdot,\varepsilon+\tau^*+v_{M,\varepsilon})) = (-\Delta)^{-1}(g_M(\cdot,u_M))$$

with convergence in $W^{2,q}(\Omega)$, and so, in particular, $(-\Delta)^{-1}(g_M(\cdot,u_M)) \in W^{2,q}(\Omega)$. Therefore $\lim_{\varepsilon\to 0^+} v_{M,\varepsilon} = (-\Delta)^{-1}(g_M(\cdot,u_M))$ with convergence in $W^{2,q}(\Omega)$, and thus $u_M = \lim_{\varepsilon\to 0^+} u_{M,\varepsilon} = \lim_{\varepsilon\to 0^+}(\tau^*+v_{M,\varepsilon}) = \tau^* + (-\Delta)^{-1}(g_M(\cdot,u_M))$, with convergence in $W^{2,q}(\Omega)$. Then $-\Delta u_M = g_M(\cdot,u_M)$ in Ω and $u_M = \tau$ on $\partial\Omega$. □

3 Proof of Theorems 1.1 and 1.3

Proof of Theorem 1.1. Define $\boldsymbol{u} : \Omega \to \mathbb{R}$ by

$$\boldsymbol{u} := \lim_{M\to\infty} u_M. \tag{3.1}$$

Note that, by Lemma 2.7 i), the map $M \to u_M$ is nondecreasing on $(0,\infty)$, and so $\boldsymbol{u}$ is well defined. By Lemma 2.7 we have, for any $M \geq 1$,

$$\tau^* + c_0 d_\Omega \leq u_M \leq \tau^* + (-\Delta)^{-1}(g(\cdot,c_0 d_\Omega)) \quad \text{in } \Omega, \tag{3.2}$$

with τ^* given by by (1.4). Then

$$\tau^* + c_0 d_\Omega \leq \boldsymbol{u} \leq \tau^* + (-\Delta)^{-1}(g(\cdot,c_0 d_\Omega)) \quad \text{in } \Omega \tag{3.3}$$

Also, by Lemma 2.7,

$$\begin{cases} -\Delta u_M = g_M(\cdot,u_M) & \text{in } \Omega, \\ u_M = \tau & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

Note that

$$\lim_{M\to\infty} g_M(\cdot,u_M) = g(\cdot,\boldsymbol{u}) \quad a.e. \text{ in } \Omega. \tag{3.5}$$

Indeed, for $k \in \mathbb{N}$ let

$$\Omega_k := \left\{ x \in \Omega : \frac{1}{k} d_\Omega(x) < \boldsymbol{u}(x) < k \right\},$$

and let $E := \Omega \setminus \cup_{k\in\mathbb{N}}\Omega_k$. Thus $E = \{x \in \Omega : \boldsymbol{u}(x) = 0\} \cup \{x \in \Omega : \boldsymbol{u}(x) = \infty\}$ and so, from (3.3) and taking into account that $\tau^* \geq 0$ in Ω and that $(-\Delta)^{-1}(g(\cdot,c_0 d_\Omega)) < \infty$ *a.e.* in Ω, we get that $|E| = 0$. Then

$$\Omega = \cup_{k\in\mathbb{N}}\Omega_k \cup E. \tag{3.6}$$

with $|E| = 0$. Now, for each $k \in \mathbb{N}$ and $x \in \Omega_k$, we have $\boldsymbol{u}(x) > \frac{1}{k} d_\Omega(x)$ and so, since $\boldsymbol{u}(x) = \lim_{M\to\infty} u_M(x)$, there exists $N_{k,x}$ such that $u_M(x) > \frac{1}{k} d_\Omega(x)$ for any $M > N_{k,x}$. Let $M_{k,x} := \max\left\{N_{k,x}, g\left(x, \frac{1}{k} d_\Omega(x)\right)\right\}$. Since $g(x,\cdot)$ is nonincreasing we have, for $M > M_{k,x}$,

$$g(x, u_M(x)) \leq g\left(x, \frac{1}{k} d_\Omega(x)\right) \leq M_{k,x} < M,$$

and so $g_M(x, u_M(x)) = g(x, u_M(x))$ whenever $M > M_{k,x}$. Thus, for any $x \in \Omega_k$,

$$\lim_{M\to\infty} g_M(x, u_M(x)) = \lim_{M\to\infty} g(x, u_M(x)) = g(x, \boldsymbol{u}(x)),$$

the last equality because g is a Carathéodory function. Then, for each k,

$$\lim_{M\to\infty} g_M(\cdot, u_M) = g(\cdot, \boldsymbol{u}) \quad a.e. \text{ in } \Omega_k,$$

and so, taking into account (3.6) and that $|E| = 0$, we get (3.5).

Let us see that $\{g_M(\cdot, u_M)\}_{M\in\mathbb{N}}$ converges to $g(\cdot, \boldsymbol{u})$ with convergence in $L^q(\Omega)$. From (3.5),

$$\lim_{M\to\infty} |g_M(\cdot, u_M) - g(\cdot, \boldsymbol{u})|^q = 0 \quad a.e. \text{ in } \Omega.$$

Also, since $\tau^* \geq 0$, from (3.3) and (3.2) we have that $\boldsymbol{u} \geq c_0 d_\Omega$ in Ω and that $u_M \geq c_0 d_\Omega$ in Ω for any $M \geq 1$. Then, recalling that g and g_M are nonincreasing in the second variable,

$$\begin{aligned} |g_M(\cdot, u_M) - g(\cdot, \boldsymbol{u})|^q &\leq (g_M(\cdot, u_M) + g(\cdot, \boldsymbol{u}))^q \\ &\leq (2g(\cdot, c_0 d_\Omega))^q \quad a.e. \text{ in } \Omega. \end{aligned}$$

By H3), $(2g(\cdot, c_0 d_\Omega))^q \in L^1(\Omega)$. By Lebesgue's dominated convergence theorem,

$$\begin{gathered} g(\cdot, \boldsymbol{u}) \in L^q(\Omega), \\ \text{and } \lim_{M\to\infty} g_M(\cdot, u_M) = g(\cdot, \boldsymbol{u}) \text{ with convergence in } L^q(\Omega). \end{gathered} \tag{3.7}$$

Let $\boldsymbol{v} = \boldsymbol{u} - \tau^*$. Since $v_M = u_M - \tau^*$, Lemma 2.7 gives

$$\begin{cases} -\Delta v_M = -\Delta u_M = g_M(\cdot, u_M) & \text{in } \Omega, \\ v_M = 0 & \text{on } \partial\Omega. \end{cases}$$

i.e., $v_M = (-\Delta)^{-1} g_M(\cdot, u_M)$; and so, by (3.7),

$$\boldsymbol{v} = \lim_{M\to\infty} v_M = (-\Delta)^{-1} g(\cdot, \boldsymbol{u}) \quad \text{with convergence in } W^{2,q}(\Omega). \tag{3.8}$$

Then $\boldsymbol{u} - \tau^* = \boldsymbol{v} = (-\Delta)^{-1} g(\cdot, \boldsymbol{u})$, which gives that $\boldsymbol{u} \in W^{2,q}(\Omega)$ and that

$$\begin{cases} -\Delta \boldsymbol{u} = g(\cdot, \boldsymbol{u}) & \text{in } \Omega, \\ \boldsymbol{u} = \tau & \text{on } \partial\Omega. \end{cases}$$

□

Remark 3.1. It is a well known fact that, for $\eta \in \mathbb{R}$, $d_\Omega^{-\eta} \in L^1(\Omega)$ if, and only if, $\eta < 1$. Moreover, if $S \subset \Omega$ is a closed C^2 and $n-1$ dimensional surface, and if $\rho_S(x) := \operatorname{dist}(x, S)$, then $\rho_S^{-\eta} \in L^1(\Omega)$ whenever $\eta < 1$. From these facts, and taking into account that $\operatorname{dist}(S, \partial\Omega) > 0$, it follows easily that if $\alpha : \Omega \to \mathbb{R}$ and $\beta : \Omega \to \mathbb{R}$ are measurable functions such that $ess\sup_\Omega \alpha < \frac{1}{q}$ and $ess\sup_\Omega \beta < \frac{1}{q}$, then $d_\Omega^{-\alpha} s^{-\beta} \in L^q(\Omega)$.

Example 3.2. The conditions H1–H3) of Theorem 1.1 allow some cases where the function $g(x,s)$ is singular at $s=0$, and also at $x \in \partial\Omega$. For instance, consider the case where $g(x,s) := b(x)\, d_\Omega^{-\alpha} s^{-\beta}$, with $\alpha : \Omega \to \mathbb{R}$, and $\beta : \Omega \to [0,\infty)$ measurable functions such that $ess\sup_\Omega \alpha + ess\sup_\Omega \beta < \frac{1}{q}$, and with $b : \Omega \to \mathbb{R}$ such that

$$0 \le b \in L^\infty(\Omega) \quad \text{and} \quad |\{x \in \Omega : b(x) > 0\}| > 0. \tag{3.9}$$

Clearly g satisfies H1) and H2) and, for $q \in \left(1, \frac{1}{\alpha+\beta}\right)$, the first assertion of Remark (3.1), jointly with (3.9), implies that g satisfies also H3).

Example 3.3. A second example of application of Theorem 1.1 is given by the function $g(x,s) := |x-x_0|^{-\gamma} b(x)\, s^{-\beta}$, where $x_0 \in \Omega$, $0<\gamma<n$, $0<\beta<1$, $1<q<\min\left\{\frac{1}{\beta}, \frac{n}{\gamma}\right\}$ and with $b : \Omega \to \mathbb{R}$ satisfying (3.9).

Example 3.4. A third example can be given by taking $g(x,s) := b(x)\, \rho_S^{-\gamma}(x)\, s^{-\beta}$, where $S \subset \Omega$ is a closed C^2 and $n-1$ dimensional surface, $\rho_S(x) := \operatorname{dist}(x,S)$, $0<\gamma<1$, $0<\beta<1$, $1<q<\min\left\{\frac{1}{\beta}, \frac{1}{\gamma}\right\}$ and with b satisfying (3.9). Indeed, H1) and H2) clearly hold, and H3) follows easily from the last assertion of Remark 3.1.

If U and V' are domains in $\mathbb{R}^n$, we will write $U \subset\subset V$ to mean that $U \subset \overline{U} \subset V$.

Proof of Theorem 1.3. Let u be a solution of (1.1). By H1) and H2), $g(\cdot,u)$ is nonnegative and nonidentically zero on Ω and, since u is a strong solution of problem (1.1), then $g(\cdot,u) \in L^q(\Omega)$. Let $v := u - \tau^*$. Then $-\Delta v = -\Delta u = g(\cdot,u)$ in Ω and $v = 0$ on $\partial\Omega$, i.e., $v = (-\Delta)^{-1} g(\cdot,u)$. Then, by Remark 2.5 ii), there exists a positive constant c' such that $v \ge c' d_\Omega$ in Ω. On the other hand, $\tau^* \ge 0$ in Ω. Thus, since $u = v + \tau^*$,

$$u \ge \tau^* + c' d_\Omega \text{ in } \Omega. \tag{3.10}$$

Also, since $\tau^* \ge 0$ in Ω, and taking into account that g is nonincreasing in the second variable and that $v \ge c' d_\Omega$ in Ω, we have $g(\cdot,\tau^*+v) \le g(\cdot,c' d_\Omega)$ and so $v = (-\Delta)^{-1} g(\cdot,u) = (-\Delta)^{-1} g(\cdot,\tau^*+v) \le (-\Delta)^{-1} g(\cdot,c' d_\Omega)$. Then

$$u \le \tau^* + (-\Delta)^{-1} g(\cdot,c' d_\Omega) \quad \text{in } \Omega. \tag{3.11}$$

Then

$$\tau^* + c' d_\Omega \le u \le \tau^* + (-\Delta)^{-1} g(\cdot,c' d_\Omega) \quad \text{in } \Omega, \tag{3.12}$$

which, taking into account H5) and that $\tau^* \in C(\overline{\Omega})$, implies that u is continuous at $\partial\Omega$.

Now we prove, by a bootstrap argument, that $u \in W^{2,n}_{loc}(\Omega)$. For $1 \le p \le \infty$ define p^* by $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$ if $p < n$ and by $p^* = \infty$ if $p \ge n$; and, for $k \in \mathbb{N} \cup \{0\}$, define inductively q_k, by $q_0 = q$, and by $q_{k+1} = q_k^*$. Thus $\frac{1}{q_k} = \frac{1}{q} - \frac{k}{n}$ when $k < \frac{n}{q}$ and $q_k = \infty$ if $k \ge \frac{n}{q}$. Let $j \in \mathbb{N} \cup \{0\}$ be such that $\frac{j}{n} < \frac{1}{q} \le \frac{j+1}{n}$. Then $0 < \frac{1}{q} - \frac{j}{n} < \frac{1}{n}$, and so $n < q_j < \infty$. Given a domain $\widetilde{\Omega} \subset\subset \Omega$, let $\Omega_0, \Omega_1, \Omega_2, \dots, \Omega_j$ be regular domains such that $\widetilde{\Omega} \subset \Omega_j \subset\subset \Omega_{j-1} \subset\subset \cdots \subset\subset \Omega_1 \subset\subset \Omega_0 = \Omega$. Now, $u \in W^{2,q}(\Omega) = W^{2,q_0}(\Omega_0)$. Suppose that $u \in W^{2,q_k}(\Omega_k)$ for some $k = 0,1,\dots,j-1$ and let $\widetilde{\Omega}_k$ be a domain such that $\Omega_{k+1} \subset\subset \widetilde{\Omega}_k \subset\subset \Omega_k$. Then $u \in W^{2,q_k}(\widetilde{\Omega}_k)$ and so, by the embedding theorems for Sobolev spaces, $u \in L^{q_k^*}\left(\widetilde{\Omega}_k\right) = L^{q_k+1}(\widetilde{\Omega}_k)$. Also, by H4), g is continuous on $\Omega \times (0,\infty)$, and so, since $0 \le g(\cdot,u) = g(\cdot,\tau^*+v) \le g(\cdot,c' d_\Omega)$, we have $g(\cdot,c' d_\Omega) \in L^\infty(\widetilde{\Omega}_k)$. Thus, by the inner elliptic estimates (as stated, e.g., in [23, Theorem 9.11]), $u \in W^{2,q_{k+1}}(\Omega_{k+1})$. Thus, inductively, we get that $u \in W^{2,q_j}(\Omega_j)$ and so, since $\widetilde{\Omega} \subset \Omega_j$

and $j > n$, we have $u \in W^{2,n}(\widetilde{\Omega})$. Thus (since $\widetilde{\Omega}$ was an arbitrary domain such that $\widetilde{\Omega} \subset\subset \Omega$), $u \in W^{2,n}_{loc}(\Omega)$. Then $u \in C(\Omega)$ and so, since we had already seen that u is continuous at $\partial\Omega$, we conclude that $u \in C(\overline{\Omega})$.

Suppose now that u and $\widetilde{u}$ are solutions of problem (1.1). Then u and $\widetilde{u}$ belong to $W^{2,n}_{loc}(\Omega) \cap C(\overline{\Omega})$ and

$$\begin{cases} -\Delta(u - \widetilde{u}) = g(\cdot, u) - g(\cdot, v) & \text{in } \Omega, \\ u - \widetilde{u} = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus, by Lemma 2.3, $u = \widetilde{u}$ in Ω. □

Remark 3.5. Assume the hypothesis of Theorem 1.1 and that $\tau = 0$ in problem (1.1). Assume also that $d_\Omega g(\cdot, cd_\Omega) \in L^2(\Omega)$ for any $c \in (0, \infty)$, and let $u \in W^{2,q}(\Omega)$ be the strong solution of problem (1.1) given by Theorem 1.1. Then $u \in H^1_0(\Omega)$ and u is a weak solution of problem (1.1), i.e., for any $\varphi \in H^1_0(\Omega)$,

$$g(\cdot, u)\varphi \in L^1(\Omega) \quad \text{and} \quad \int_\Omega \langle \nabla u, \nabla \varphi \rangle = \int_\Omega g(\cdot, u)\varphi. \tag{3.13}$$

Indeed, by Theorem 1.1, we have $u \geq c_u d_\Omega$ for some $c_u \in (0, \infty)$ and so $0 \leq g(\cdot, u) \leq g(\cdot, c_u d_\Omega)$. Now, for $\varphi \in H^1_0(\Omega)$, the Holder's inequality and the Hardy's inequality of Remark 2.5 iii) give

$$\begin{aligned} \int_\Omega |g(\cdot, u)\varphi| &= \int_\Omega d_\Omega g(\cdot, u) \left| \frac{\varphi}{d_\Omega} \right| \leq \int_\Omega d_\Omega g(\cdot, c_u d_\Omega) \left| \frac{\varphi}{d_\Omega} \right| \\ &\leq \|d_\Omega g(\cdot, c_u d_\Omega)\|_2 \left\| \frac{\varphi}{d_\Omega} \right\|_2 \leq c \|d_\Omega g(\cdot, c_u d_\Omega)\|_2 \|\nabla \varphi\|_2, \end{aligned}$$

and thus $g(\cdot, u)\varphi \in L^1(\Omega)$. Moreover, the above inequality gives that the map $\varphi \to \int_\Omega g(\cdot, u)\varphi$ is continuous on $H^1_0(\Omega)$. Then, since $H^1_0(\Omega)$ is a Hilbert space with respect to the inner product $(u, v) := \int_\Omega \langle \nabla u, \nabla v \rangle$, it follows that there exists a function $\widetilde{u} \in H^1_0(\Omega)$ such that, for any $\varphi \in H^1_0(\Omega)$,

$$\int_\Omega \langle \nabla \widetilde{u}, \nabla \varphi \rangle = \int_\Omega \langle \nabla u, \nabla \varphi \rangle.$$

Then $\int_\Omega \langle \nabla(\widetilde{u} - u), \nabla \varphi \rangle = 0$ for any $\varphi \in C^\infty_c(\Omega)$ and so $z := \widetilde{u} - u$ satisfies, in the sense of distributions, $-\Delta z = 0$ in Ω. Also, $z \in W^{1,\overline{q}}_0(\Omega)$ with $\overline{q} := \min(q, 2)$ and so, in the sense of the trace, $z = 0$ on $\partial\Omega$. Then $z = 0$ and thus $u = \widetilde{u}$ in Ω. Therefore $u \in H^1_0(\Omega)$. Since u is a strong solution of problem (1.1) we have

$$\int_\Omega \langle \nabla u, \nabla \psi \rangle = \int_\Omega g(\cdot, u)\psi \quad \text{for any } \psi \in C^\infty_c(\Omega). \tag{3.14}$$

and then, by density, (3.14) holds also for any $\varphi \in H^1_0(\Omega)$.

For $f : \Omega \to \mathbb{R}$ and $h : \Omega \to \mathbb{R}$ we will write $f \approx h$ to mean that there exist positive constants c_1 and c_2 such that $c_1 f \leq h \leq c_2 h$ *a.e.* in Ω

Remark 3.6. In order to illustrate the relationship between the existence of classical solutions, strong solutions and weak solutions in $H^1_0(\Omega)$ let us consider the case when Ω is a $C^{2+\alpha}$ domain in $\mathbb{R}^n$ for some $\alpha \in (0, 1)$, $n \geq 3$ and $g(x, s) = a(x)s^{-\gamma}$ with $a \in C^\alpha(\overline{\Omega})$ such that $\min_\Omega \alpha > 0$. Assume also that $\tau = 0$ in problem (1.1). In this situation, [26, Theorem 1] states that problem (1.1) has a unique classical solution $u \in C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$ for any $\gamma > 0$ and that, when $\gamma > 1$, $u \approx d_\Omega^{\frac{2}{1+\gamma}}$ in Ω. In addition, [26, Theorem 2] says that when $\gamma > 3$ no (classical) solution belonging to $H^1(\Omega)$ exists. In the case $\gamma = 1$ [29, Theorem 1] states that $u \approx d_\Omega \big(\ln\big(\frac{\omega}{d_\Omega}\big)\big)^{\frac{1}{2}}$, where ω is any constant such that $\omega > \operatorname{diam}(\Omega)$. On the other hand, as a consequence of [42, Theorems 1 and 2] a weak solution $u \in H_0^1(\Omega)$ exists if and only if $\gamma < 3$. However, for $1 \le \gamma < 3$ these weak solution are not strong solutions. Indeed, when $\gamma = 1$, $au^{-\gamma} = au^{-1} \approx d_\Omega^{-1}\big(\ln\big(\frac{\omega}{d_\Omega}\big)\big)^{-\frac{1}{2}}$ and it is easy to see that, for all $q \ge 1$, $\int_\Omega d_\Omega^{-q}\big(\ln\big(\frac{\omega}{d_\Omega}\big)\big)^{-\frac{q}{2}} = \infty$ (in fact, $\int_\Omega d_\Omega^{-q}\big(\ln\big(\frac{\omega}{d_\Omega}\big)\big)^{-\frac{q}{2}} < \infty$ if and only if $I(\varepsilon) := \int_0^\varepsilon t^{-q}\big(\ln\big(\frac{\omega}{t}\big)\big)^{-\frac{q}{2}} dt < \infty$ for some $\varepsilon > 0$, but the change of variable $s = \ln\frac{\omega}{t}$ immediately shows that $I(\varepsilon) = \infty$ for all $\varepsilon > 0$). When $1 < \gamma < 3$, we have $u \approx d_\Omega^{\frac{2}{1+\gamma}}$ in Ω, and thus $au^{-\gamma} \approx d_\Omega^{-\frac{2\gamma}{1+\gamma}}$. Then, for $q \ge 1$, $au^{-\gamma} \in L^q(\Omega)$ if and only if $\frac{2\gamma q}{1+\gamma} < 1$, that is $\gamma < \frac{1}{2q-1}$. Since $\frac{1}{2q-1} \le 1$ we get that $\gamma < 1$, which contradicts our assumption $1 < \gamma < 3$.

4 A related problem in a punctured domain

Let $x_0 \in \Omega$, let $U := \Omega \setminus \{x_0\}$ and let $w \in L^1(U)$. Then $w \in L^1(\Omega)$, and so w can be viewed as a distribution on U and also as a distribution on Ω. For $1 \le i, j \le n$, we will denote by $\partial_i^U w$ and $\partial_i^U \partial_j^U w$ (respectively by $\partial_i^\Omega w$ and $\partial_i^\Omega \partial_j^\Omega w$) the first and the second derivatives of w considered as a distribution on U (resp. as a distribution on Ω), and, if $\varphi \in C^\infty(\mathbb{R}^n)$, we will write simply $\partial_i \varphi$ and $\partial_i \partial_j \varphi$ for the first and the second derivatives of φ.

If $w \in W^{2,q}(U)$ for some $q \in (1,\infty)$, then $\partial_i^U w$ and $\partial_i^U \partial_j^U w$ belong to $L^q(U)$ and so they also belong to $L^q(\Omega)$. One may ask if $\partial_i^U w = \partial_i^\Omega w$ and $\partial_i^U \partial_j^U w = \partial_i^\Omega \partial_j^\Omega w$, i.e., if the equalities

$$\left\langle \partial_i^U w, \varphi \right\rangle = -\int_\Omega w \partial_i \varphi \text{ and } \left\langle \partial_i^U \partial_j^U w, \varphi \right\rangle = \int_\Omega w \partial_i \partial_j \varphi,$$

which hold for $\varphi \in C_c^\infty(U)$, hold also for $\varphi \in C_c^\infty(\Omega)$. The next lemma provides a partial answer to this question.

Lemma 4.1. *Let $x_0 \in \Omega$, let $U := \Omega \setminus \{x_0\}$, and, for $\delta > 0$, let A_δ be defined by (1.5) and let $w \in W^{2,q}(U)$. If either $\lim_{x \to x_0} w(x)$ exists and is finite, or if*

$$\limsup_{\delta \to 0^+} \frac{1}{\delta^2} \int_{A_\delta} |w| = 0, \tag{4.1}$$

then $\partial_i^U w = \partial_i^\Omega w$ and $\partial_i^U \partial_j^U w = \partial_i^\Omega \partial_j^\Omega w$ for each i and j, and so, in particular, $w \in W^{2,q}(\Omega)$.

Proof. Observe that, in the case when $\lim_{x \to x_0} w(x)$ exists and is finite, it is enough to prove the lemma under the additional assumption that $\lim_{x \to x_0} w(x) = 0$ (because the functions $w - \lim_{x \to x_0} w(x)$ and w have the same derivatives, either in $D'(U)$ or in $D'(\Omega)$). Let $\psi \in C_c^\infty(\mathbb{R}^n)$ be such that $0 \le \psi \le 1$ in $\mathbb{R}^n$, $\psi(x) = 1$ for $|x| \le \frac{1}{2}$ and $\psi(x) = 0$ for $|x| \ge 1$; and for $\delta > 0$, define $\psi_{1,\delta}$ and $\psi_{2,\delta}$ by $\psi_{1,\delta}(x) := \psi\left(\frac{x - x_0}{\delta}\right)$ and $\psi_{2,\delta}(x) := 1 - \psi_{1,\delta}(x)$. For $\varphi \in C_c^\infty(\Omega)$ and $0 < \delta < \min\{1, \operatorname{dist}(x_0, \partial\Omega)\}$ we have

$$\int_\Omega \varphi \partial_i^U \partial_j^U w = \int_\Omega \varphi \psi_{1,\delta} \partial_i^U \partial_j^U w + \int_\Omega \varphi \psi_{2,\delta} \partial_i^U \partial_j^U w.$$

Now, $\left|\varphi\psi_{1,\delta}\partial_i^U\partial_j^U w\right| \leq \|\varphi\|_{L^\infty(\Omega)} \left|\partial_i^U\partial_j^U w\right| \in L^q(U) = L^q(\Omega) \subset L^1(\Omega)$. Also $\lim_{\delta\to 0}\varphi\psi_{1,\delta}\partial_i^U\partial_j^U w = 0$ *a.e.* in Ω. Then, by Lebesgue's dominated convergence theorem,

$$\lim_{\delta\to 0}\int_\Omega \varphi\psi_{1,\delta}\partial_i^U\partial_j^U w = 0. \tag{4.2}$$

Thus, to prove the assertion of the lemma for the second derivatives, it suffices to show that

$$\lim_{\delta\to 0}\int_\Omega \varphi\psi_{2,\delta}\partial_i^U\partial_j^U w = \int_\Omega w\partial_i\partial_j\varphi \quad \text{for any } \varphi \in C_c^\infty(\Omega). \tag{4.3}$$

Notice that $\partial_i\psi_{2,\delta}(x) = \frac{1}{\delta}\frac{\partial\psi}{\partial x_i}\left(\frac{x-x_0}{\delta}\right)$ and $\partial_i\partial_j\psi_{2,\delta}(x) = \frac{1}{\delta^2}\left(\partial_i\partial_j\psi\right)\left(\frac{x-x_0}{\delta}\right)$, and so there exists a positive constant c, independent of δ, such that

$$|\partial_i\psi_{2,\delta}| \leq \frac{c}{\delta} \quad \text{and} \quad \left|\partial_i\partial_j\psi_{2,\delta}\right| \leq \frac{c}{\delta^2} \quad \text{in } \Omega. \tag{4.4}$$

Now,

$$\int_\Omega \varphi\psi_{2,\delta}\partial_i^U\partial_j^U w = \int_U \varphi\psi_{2,\delta}\partial_i^U\partial_j^U w = \int_U w\partial_i\partial_j\left(\varphi\psi_{2,\delta}\right),$$

and a computation gives that

$$\partial_i\partial_j\left(\varphi\psi_{2,\delta}\right) = \partial_i\varphi\partial_j\psi_{2,\delta} + \varphi\partial_i\partial_j\psi_{2,\delta} + \partial_i\psi_{2,\delta}\partial_j\varphi + \psi_{2,\delta}\partial_i\partial_j\varphi,$$

and so, $\psi_{2,\delta}$, $\partial_i\psi_{2,\delta}$ and $\partial_i\partial_j\psi_{2,\delta}$ have their supports contained in A_δ, we have

$$\int_U w\partial_i\partial_j\left(\varphi\psi_{2,\delta}\right) = I_{1,\delta} + I_{2,\delta} + I_{3,\delta} + I_{4,\delta}, \tag{4.5}$$

where

$$I_{1,\delta} := \int_{A_\delta} w\partial_i\varphi\partial_j\psi_{2,\delta}, \qquad I_{2,\delta} := \int_{A_\delta} w\varphi\partial_i\partial_j\psi_{2,\delta},$$
$$I_{3,\delta} := \int_{A_\delta} w\partial_i\psi_{2,\delta}\partial_j\varphi, \qquad I_{4,\delta} := \int_U w\psi_{2,\delta}\partial_i\partial_j\varphi.$$

Thus, by (4.4),

$$|I_{1,\delta}| \leq c\,\|\partial_i\varphi\|_{L^\infty(\Omega)}\frac{1}{\delta}\int_{A_\delta}|w|, \tag{4.6}$$

with c a positive constant independent of δ. If (4.1) holds then clearly

$$\lim_{\delta\to 0^+} I_{1,\delta} = 0. \tag{4.7}$$

and, in the case when $\lim_{x\to x_0} w(x) = 0$, we have $\lim_{\delta\to 0^+}\sup_{A_\delta}|w| = 0$, and so, from (4.6), $|I_{1,\delta}| \leq \frac{c}{\delta}\|\partial_i\varphi\|_{L^\infty(\Omega)}|A_\delta|\sup_{A_\delta}|w|$, where $|A_\delta|$ denotes the Lebesgue measure of A_δ. Since $|A_\delta| = \alpha_n\left(1-\frac{1}{2^n}\right)\delta^n$ where α_n is the volume of the unit ball in $\mathbb{R}^n$ and taking into account that $n \geq 2$, we get (4.7) again in this case. Similarly,

$$\lim_{\delta\to 0^+} I_{3,\delta} = 0. \tag{4.8}$$

To estimate $I_{2,\delta}$ observe that, by (4.4),

$$|I_{2,\delta}| \leq \frac{c}{\delta^2}\|\varphi\|_{L^\infty(\Omega)}\int_{A_\delta}|w|, \tag{4.9}$$

and so, proceeding similarly to the estimative of $I_{3,\delta}$ we get, in both cases of the lemma, that

$$\lim_{\delta\to 0^+} I_{2,\delta} = 0. \tag{4.10}$$

Consider now $I_{4,\delta}$. We have $|w\psi_{2,\delta}\partial_i\partial_j\varphi| \leq |w||\partial_i\partial_j\varphi| \in L^1(\Omega)$, and clearly $\lim_{\delta\to 0^+} w\psi_{2,\delta}\partial_i\partial_j\varphi = w\partial_i\partial_j\varphi$ *a.e.* in Ω. Then, by Lebesgue's dominated convergence theorem,

$$\lim_{\delta\to 0^+} I_{4,\delta} = \int_\Omega w\partial_i\partial_j\varphi = \left\langle \partial_i^\Omega\partial_j^\Omega w, \varphi\right\rangle. \tag{4.11}$$

From (4.7), (4.8), (4.10), and (4.11), we get (4.3), and so the assertion of the lemma for the second derivatives holds. The proof of the assertion of the lemma for the first derivatives follows similar lines and we omit it. □

Proof of Theorem 1.4. Let $w \in W^{2,q}(U)$ be a strong solution of problem (1.6). If either $w \in C(\Omega)$ or $\limsup_{\delta\to 0^+} \frac{1}{\delta^2}\int_{A_\delta}|w| = 0$, then, by Lemma 4.1, $w \in W^{2,q}(\Omega)$. Since the equality $-\Delta w = h(\cdot, w)$ holds *a.e.* in Ω, and, in the sense of the trace, $w = \tau$ on $\partial\Omega$, we have that w is a strong solution u of problem (1.7). □

Proof of Theorem 1.5. To see i), suppose that $\limsup_{x\to x_0} |x-x_0|^{n-2} w(x) = 0$, and let $\varepsilon > 0$. Then there exist $\delta_0 > 0$ such that $|x-x_0|^{n-2} w(x) \leq \varepsilon$ if $0 < |x-x_0| < \delta_0$. Now, for $\delta \in (0, \delta_0)$,

$$\begin{aligned}
\frac{1}{\delta^2}\int_{A_\delta}|w| &= \frac{1}{\delta^2}\int_{A_\delta}\frac{1}{|x-x_0|^{n-2}}|x-x_0|^{n-2} w(x)\,dx\\
&\leq \frac{1}{\delta^2}\int_{A_\delta}\left(\frac{2}{\delta}\right)^{n-2}|x-x_0|^{n-2} w(x)\,dx\\
&\leq 2^{n-2}\varepsilon\delta^{-n}|A_\delta| = 2^{n-2}\left(1-\frac{1}{2^n}\right)\alpha_n\varepsilon,
\end{aligned}$$

where α_n is the volume of the unit ball in $\mathbb{R}^n$. Thus $\lim_{\delta\to 0}\frac{1}{\delta^2}\int_{A_\delta}|w| = 0$, and then i) follows from Theorems 1.4 and 1.3.

ii) follows directly from i). If $\|w\|_{L^\infty(U)} = \infty$ and $\limsup_{x\to x_0}|x-x_0|^{n-2} w(x) = 0$, then, by i), after redefining w in a set with zero measure, we would have $C(\overline{\Omega})$, which is impossible when $\|w\|_{L^\infty(U)} = \infty$. □

Remark 4.2. Theorems 1.4 and 1.5 say that if $x_0 \in \Omega$, $U = \Omega\setminus\{x_0\}$, and if w is a nice enough strong solution of problem (1.7) then w is a strong solution of problem 1.1.

On the other hand, it was proved in ([30], Theorem 3.6) that, if μ is a bounded Radon measure in Ω, $\gamma \leq 1$, and $f \in L^1(\Omega)$, then the problem

$$\begin{cases} -\Delta w = fu^{-\gamma} + \mu & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega,\\ u > 0 & \text{in } \Omega \end{cases}$$

has a solution in the sense that:

i) $u \in W_0^{1,1}(\Omega)$ and for any compact $K \subset \Omega$ there exists a positive constant c such that $u \geq c$ *a.e.* in K,

ii) $\int_\Omega \langle \nabla w, \nabla\varphi\rangle = \int_\Omega fu^{-\gamma}\varphi + \int_\Omega \varphi d\mu$ for any $\varphi \in C_c^1(\Omega)$.

By taking $\mu = \delta_{x_0}$ (the Dirac's measure concentrated at x_0), and, for instance, $f = 1$, in [30, Theorem 3.6] it is clear that the conclusions of Theorems 1.4 and 1.5 could not hold anymore if the notion of solution is changed and the requirement that w is "nice enough" is dropped.

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Dirichlet problems with unbalanced growth and convection

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Abstract. We consider a double phase Dirichlet problem with a gradient dependent reaction term (convection). Using the theory of nonlinear operators of monotone type, we show the existence of a bounded strictly positive solution. Moreover, we show that the set of these solutions is compact in the corresponding generalized Sobolev–Orlicz space.

Keywords: weighted p-Laplacian, eigenvalue, generalized Orlicz space, pseudomonotone operator, double phase.

2020 Mathematics Subject Classification: 35B02, 35B40, 35J15.

1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a Lipschitz boundary $\partial\Omega$. In this paper we study the existence of a positive solution for the following nonlinear Dirichlet problem

$$\begin{cases} -\Delta_p^a u(z) - \Delta_q u(z) = f(z, u(z), Du(z)) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, u > 0, 1 < q < p < N. \end{cases} \tag{1.1}$$

Given $a \in L^\infty(\Omega) \setminus \{0\}$ with $a(z) \geq 0$ for a.a. $z \in \Omega$ and $r \in (1, \infty)$ by Δ_r^a denotes the weighted r-Laplace differential operator defined by

$$\Delta_r^a u = \operatorname{div}(a(z)|Du|^{r-2} Du).$$

When $a(\cdot) \equiv 1$, then we write $\Delta_r^a = \Delta_r$ which is the standard r-Laplace differential operator.

In (1.1) the differential operator is not homogeneous and is related to two-phase integral functional

$$u \to \int_\Omega [a(z)|Du|^p + |Du|^q]\, dz.$$

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The integrand of this functional is the function

$$\mathcal{E}(z,t) = a(z)t^p + t^q \qquad \forall z \in \Omega,\ \forall t \geq 0.$$

We do not assume that the weight $a(\cdot)$ is bounded away from zero (that is, we do not require that $\operatorname{ess\,inf}_\Omega a > 0$) and so $\mathcal{E}(t,\cdot)$ exhibits unbalanced growth, namely we have

$$t^q \leq \mathcal{E}(z,t) \leq c_0[t^p + t^q] \quad \text{a.a. } z \in \Omega, \text{ all } t \geq 0, \text{ some } c_0 > 0.$$

Such integral functionals, were first examined by Marcellini [11] and Zhikov [18] in the context of problems of the calculus of variations and of nonlinear elasticity theory. Until now there is no global regularity theory for unbalanced growth (double phase) boundary value problems analogous to the one for balanced growth problems developed by Lieberman [7]. Only local (interior) regularity results exist, produced primarily by Marcellini [12], Baroni–Colombo–Mingione [1] and Ragusa–Tachikawa [17].

In the reaction (right hand side) of (1.1), we have a Carathéodory function $f(z,x,y)$ (that is, for all $(x,y) \in \mathbb{R} \times \mathbb{R}^N, z \to f(z,x,y)$ is measurable and for a.a. $z \in \Omega, (x,y) \to f(z,x,y)$ is continuous). Since the reaction (source) term is gradient dependent, problem (1.1) is nonvariational. For this reason our approach is topological based on the theory of nonlinear operators of monotone type.

Recently there have been existence and multiplicity results for double phase equations with no gradient dependence (variational problems). We refer to the works of Gasiński–Papageorgiou [3], Gasiński–Winkert [4], Liu–Dai [8], Papageorgiou–Rădulescu–Repovš [14], Papageorgiou–Rădulescu–Zhang [15], Papageorgiou–Vetro–Vetro [16] and the references therein. Double phase problems with gradient dependence (convection), were studied only by Gasiński–Winkert [5] and Liu–Papageorgiou [9] using different conditions on the reaction $f(z,x,y)$.

2 Mathematical background

The unbalanced growth of $\mathcal{E}(z,\cdot)$ leads to a functional framework for problem (1.1) based on generalized Orlicz spaces. A comprehensive account of the theory of these spaces can be found in the book of Harjulehto–Hästö [6].

Let $M(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable}\}$. We identify two such functions which differ only on a Lebesgue-null set. Also by $C^{0,1}(\overline{\Omega})$ we denote the space of all functions $u : \overline{\Omega} \to \mathbb{R}$ which are Lipschitz continuous. For the moment we assume that

$$a \in C^{0,1}(\overline{\Omega}),\ a(z) > 0 \qquad \forall z \in \Omega,\ 1 < q < p < N,\ \frac{p}{q} < 1 + \frac{1}{N}. \tag{2.1}$$

The last inequality in (2.1) implies that $p < q^* = \frac{Nq}{N-q}$ and this then leads to useful compact embeddings for some relevant spaces (see Proposition 2.1 below). Also these conditions guarantee the validity of the Poincaré inequality in the appropriate Sobolev–Orlicz space.

Then the Lebesgue–Orlicz space $L^{\mathcal{E}}(\Omega)$ is defined by

$$L^{\mathcal{E}}(\Omega) = \{u \in M(\Omega) : \rho_{\mathcal{E}}(u) < \infty\},$$

with $\rho_{\mathcal{E}}(\cdot)$ being the modular function defined by

$$\rho_{\mathcal{E}}(u) = \int_{\Omega} \mathcal{E}(z, |u|)\, dz = \int_{\Omega} [a(z)|u|^p + |u|^q] dz.$$

We equip this space with the so-called "Luxemburg norm" defined by

$$\|u\|_{\mathcal{E}} = \inf\Big[\lambda > 0 : \rho_{\mathcal{E}}\Big(\frac{u}{\lambda}\Big) \le 1\Big].$$

Normed this way, $L^{\mathcal{E}}(\Omega)$ becomes a Banach space which is separable and reflexive (in fact uniformly convex). Then using $L^{\mathcal{E}}(\Omega)$ we can define the corresponding Sobolev–Orlicz space $W^{1,\mathcal{E}}(\Omega)$ by

$$W^{1,\mathcal{E}}(\Omega) = \{u \in L^{\mathcal{E}}(\Omega) : |Du| \in L^{\mathcal{E}}(\Omega)\}.$$

Here Du denotes the weak gradient of $u(\cdot)$. This space is given the following norm

$$\|u\|_{1,\mathcal{E}} = \|u\|_{\mathcal{E}} + \|Du\|_{\mathcal{E}} \quad \text{for all } u \in W^{1,\mathcal{E}}(\Omega).$$

Here $\|Du\|_{\mathcal{E}} = \||Du|\|_{\mathcal{E}}$. This too is a Banach space which separable and reflexive (in fact uniformly convex). Also set

$$W_0^{1,\mathcal{E}}(\overline{\Omega}) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{1,\mathcal{E}}},$$

with $C_c^\infty(\Omega) = \{u \in C^\infty(\Omega)$ with compact support$\}$. Conditions (2.1) imply that the Poincaré inequality is valid on $W_0^{1,\mathcal{E}}(\Omega)$ and we can use the following equivalent norm on $W_0^{1,\mathcal{E}}(\Omega)$.

$$\|u\| = \|Du\|_{\mathcal{E}} \quad \text{for all } u \in W_0^{1,\mathcal{E}}(\Omega).$$

For these spaces we have the following useful embeddings.

Proposition 2.1.

(*a*) $L^{\mathcal{E}}(\Omega) \hookrightarrow L^\tau(\Omega)$ *and* $W_0^{1,\mathcal{E}}(\Omega) \hookrightarrow W_0^{1,\tau}(\Omega)$ *continuously for all* $\tau \in [1, q]$.

(*b*) $W_0^{1,\mathcal{E}}(\Omega) \hookrightarrow L^\tau(\Omega)$ *continuously for all* $\tau \in [1, q^*]$ *and compactly for all* $\tau \in [1, q^*)$;

(*c*) $L^p(\Omega) \hookrightarrow L^{\mathcal{E}}(\Omega)$ *continuously.*

There is a close relation between the norm $\|\cdot\|_{\mathcal{E}}$ and the modular function $\rho_{\mathcal{E}}(\cdot)$ on the space $W_0^{1,\mathcal{E}}(\Omega)$.

Proposition 2.2.

(*a*) $\|u\|_{\mathcal{E}} = \lambda \Leftrightarrow \rho_{\mathcal{E}}(\frac{u}{\lambda}) = 1$;

(*b*) $\|u\|_{\mathcal{E}} < 1$ (*resp.* $= 1, > 1$) $\Leftrightarrow \rho_{\mathcal{E}}(u) < 1$ (*resp.* $= 1, > 1$);

(*c*) $\|u\|_{\mathcal{E}} \le 1 \Rightarrow \|u\|_{\mathcal{E}}^p \le \rho_{\mathcal{E}}(u) \le \|u\|_{\mathcal{E}}^q$;

(*d*) $\|u\|_{\mathcal{E}} > 1 \Rightarrow \|u\|_{\mathcal{E}}^q \le \rho_{\mathcal{E}}(u) \le \|u\|_{\mathcal{E}}^p$;

(*e*) $\|u\|_{\mathcal{E}} \to 0$ (*resp.* $\to +\infty$) $\Leftrightarrow \rho_{\mathcal{E}}(u) \to 0q$(*resp.* $\to +\infty$).

Let $V : W_0^{1,\mathcal{E}}(\Omega) \to W_0^{1,\mathcal{E}}(\Omega)^*$ be the nonlinear operator defined by

$$\langle V(u), h\rangle = \int_{\Omega} (a(z)|Du|^{p-2}Du + |Du|^{p-2}Du, Dh)_{\mathbb{R}^N} dz, \quad \text{for all } u, h \in W_0^{1,\mathcal{E}}(\Omega).$$

This operator has the following properties (see [8]).

Proposition 2.3. *The operator $V(\cdot)$ is bounded (maps bounded sets to bounded sets), continuous, strictly monotone (thus maximal monotone too) and of type $(S)_+$, that is, "if $u_n \xrightarrow{w} u$ in $W_0^{1,\mathcal{E}}(\Omega)$ and $\limsup_{n\to\infty}\langle V(u_n), u_n - u\rangle \le 0$, then $u_n \to u$ in $W_0^{1,\mathcal{E}}(\Omega)$."*

For $x \in \mathbb{R}$, we set $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$. Then if $u \in M(\Omega)$, we define $u^+(z) = u(z)^+$ and $u^-(z) = u(z)^-$ for all $z \in \Omega$. We know that $u = u^+ - u^-, |u| = u^+ + u^-$ and if $u \in W_0^{1,\mathcal{E}}(\Omega)$, then $u^\pm \in W_0^{1,\mathcal{E}}(\Omega)$.

3 Some auxiliary results

In this section we prove some auxiliary results concerning the weighted p-Laplacian Δ_p^a, which we will need in the analysis of problem (1.1)).

We strengthen the conditions on the weight $a(\cdot)$. By $\widetilde{A}_p$ we denote the p-Muckenhoupt class (see Harjulehto–Hästö [6, p. 114]). The stronger conditions on the weight $a(\cdot)$ are the following:

H_0: $a \in C^{0,1}(\overline{\Omega}) \cap \widetilde{A}_p$, $a(z) > 0$ for all $z \in \Omega$, $1 < q < p < N$, $\frac{p}{q} < 1 + \frac{1}{N}$.

Let $\mathcal{E}_0(z,t) = a(z)t^p$ for all $z \in \Omega$ for all $t \ge 0$. On account of hypotheses H_0 above we have that $W_0^{1,\mathcal{E}_0}(\Omega) \hookrightarrow L^{\mathcal{E}_0}(\Omega)$ compactly (see Liu–Papageorgiou [10]). We will use this fact to produce a smallest eigenvalue for $(-\Delta_p^a, W_0^{1,\mathcal{E}_0}(\Omega))$. So, we consider the following nonlinear eigenvalue problem

$$-\Delta_p^a u(z) = \hat{\lambda} a(z)|u(z)|^{p-2}u(z) \quad \text{in } \Omega, \quad \mathrm{u}|_{\partial\Omega} = 0. \tag{3.1}$$

We say that $\hat{\lambda} \in \mathbb{R}$ is an "eigenvalue", if the above Dirichlet problem admits a nontrivial solution $\hat{u} \in W_0^{1,\mathcal{E}_0}(\Omega)$ known as an"eigenfunction" corresponding to $\hat{\lambda}$.

Proposition 3.1. *If hypotheses H_0 hold, then problem (3.1) has a smallest eigenvalue $\hat{\lambda}_1^a = \hat{\lambda}_1^a(p) > 0$ and every corresponding eigenfunction $\hat{u} \in W_0^{1,\mathcal{E}_0}(\Omega)$ satisfies $\hat{u}(z) > 0$ or $\hat{u}(z) < 0$ a.a. in Ω (has constant sign).*

Proof. Let $\hat{\lambda}_1^a = \inf\left[\frac{\rho_a(Du)}{\rho_a(u)} : u \in W_0^{1,\mathcal{E}_0}(\Omega), u \neq 0\right]$, where for every $v \in L^{\mathcal{E}_0}(\Omega)$ we define $\rho_a(v) = \int_\Omega a(z)|v|^p dz$. The homogeneity of $\rho_a(\cdot)$ implies that

$$\hat{\lambda}_1^a = \inf\Big[\rho_a(Du) : u \in W_0^{1,\mathcal{E}_0}(\Omega), \rho_a(u) = 1\Big]. \tag{3.2}$$

Consider a sequence $\{u_n\}_{n\ge\mathbb{N}} \subseteq W_0^{1,\mathcal{E}_0}(\Omega)$ such that

$$\rho_a(Du_n) \downarrow \hat{\lambda}_1^a \quad \text{and} \quad \rho_a(u_n) = 1 \quad \text{for all } n \in \mathbb{N}. \tag{3.3}$$

Evidently $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\mathcal{E}_0}(\Omega)$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} \hat{u} \quad \text{in } W_0^{1,\mathcal{E}_0}(\Omega) \quad \text{and} \quad u_n \to \hat{u} \text{ in } L^{\mathcal{E}_0}(\Omega). \tag{3.4}$$

The function $\rho_a(\cdot)$ is continuous, convex, thus sequentially weakly lower semicontinuous. So, from (3.4) we have

$$\begin{aligned} & \rho_a(D\hat{u}) \le \liminf_{n\to+\infty} \rho_a(Du_n), \quad \rho_a(u_n) \to \rho_a(\hat{u}), \\ \Rightarrow \quad & \rho_a(D\hat{u}) \le \hat{\lambda}_1^a, \quad \rho_a(\hat{u}) = 1 \quad (\text{see } (3.3)), \\ \Rightarrow \quad & \rho_a(D\hat{u}) = \hat{\lambda}_1^a > 0. \end{aligned}$$

From (3.2) and the Lagrange multiplier rule (see [13, p. 422]), we have

$$-\Delta_p^a \hat{u} = \hat{\lambda}_1^a a(z)|\hat{u}|^{p-2}\hat{u} \quad \text{in } \Omega, \quad \hat{u}|_{\partial\Omega} = 0. \tag{3.5}$$

Suppose $\hat{u}^+ \neq 0$, since $\hat{u}^+ \in W_0^{1,\mathcal{E}_0}(\Omega)$, acting on (3.4) with $\hat{u}^+$, we obtain

$$\begin{aligned} &\rho_a(D\hat{u}^+) = \hat{\lambda}_1^a \rho_a(\hat{u}^+), \\ \Rightarrow \quad &\hat{u}^+ \text{ is an eigenfunction for } \quad \hat{\lambda}_1^a > 0. \end{aligned}$$

From Colasuonno–Squassina [2, Section 3.3], we have that

$$u^+ \in W_0^{1,\mathcal{E}_0}(\Omega) \cap L^\infty(\Omega).$$

Invoking Proposition 2.4 of Papageorgiou–Vetro–Vetro [16], we infer that

$$\begin{aligned} &\hat{u}^+(z) > 0 \quad \text{for } a.a.\ z \in \Omega, \\ \Rightarrow \quad &\hat{u} = \hat{u}^+. \end{aligned}$$

Similarly if $u^- \neq 0$. □

This proposition leads to the following estimate which is useful in case we have nonuniform nonresonance.

Proposition 3.2. *If hypotheses H_0 hold, $\eta \in L^\infty(\Omega)$, $\eta(z) \leq \hat{\lambda}_1^a$ for a.a. $z \in \Omega$ and*

$$\eta \not\equiv \hat{\lambda}_1^a,$$

then there exists $c_1 > 0$ such that

$$c_1 \|u\|_{1,\mathcal{E}_0}^p \leq \rho_a(Du) - \int_\Omega \eta(z)a(z)|u|^p dz \quad \textit{for all } u \in W_0^{1,\mathcal{E}_0}(\Omega).$$

Proof. We argue by contradiction. So, suppose that the conclusion of the proposition is not true. We can find $\{u_n\}_{n\geq \mathbb{N}} \subseteq W_0^{1,\mathcal{E}_0}(\Omega)$ such that

$$\rho_a(Du_n) - \int_\Omega \eta(z)a(z)|u_n|^p dz < \frac{1}{n}\|u_n\|_{1,\mathcal{E}_0}^p \quad \text{for all } n \in \mathbb{N}.$$

Exploiting the p-homogeneity of this inequality, we can say that

$$\left\{ \begin{aligned} &\rho_a(Du_n) - \int_\Omega \eta(z)a(z)|u_n|^p dz < \frac{1}{n}, \\ &\|u_n\|_{1,\mathcal{E}_0} = 1 \quad \text{for all } n \in \mathbb{N}. \end{aligned} \right\} \tag{3.6}$$

We may assume that

$$u_n \xrightarrow{w} u \quad \text{in } W_0^{1,\mathcal{E}_0}(\Omega) \quad \text{and} \quad u_n \to u \text{ in } L^{\mathcal{E}_0}(\Omega). \tag{3.7}$$

If $u = 0$, then

$$\begin{aligned} &\rho_a(Du_n) \to 0, \\ \Rightarrow \quad &u_n \to 0 \quad \text{in } W_0^{1,\mathcal{E}_0}(\Omega) \quad \text{(see Proposition 2.2)}, \end{aligned}$$

a contradiction, since $\|u_n\|_{1,\mathcal{E}_0} = 1$, for all $n \in \mathbb{N}$ (see (3.6)).

If $u \neq 0$, then from (3.6) and (3.7)

$$\rho_a(Du) \leq \int_\Omega \eta(z)a(z)|u|^p dz, \tag{3.8}$$

$$\begin{aligned} &\Rightarrow \quad \rho_a(Du) = \hat{\lambda}_1^a \rho_a(u), \quad \text{(see (3.1))}, \\ &\Rightarrow \quad |u(z)| > 0 \quad \text{for } a.a.\ z \in \Omega, \quad \text{(see Proposition 3.1)}, \\ &\Rightarrow \quad \rho_a(Du) < \hat{\lambda}_1^a \rho_a(u) \quad \text{(see (3.8))}, \end{aligned}$$

which contradicts (3.1).

Therefore we conclude that there exists $c_1 > 0$ such that

$$c_1 \|u\|_{1,\mathcal{E}_0}^p \leq \rho_a(Du) - \int_\Omega \eta(z)a(z)|u|^p dz \quad \text{for all } u \in W_0^{1,\mathcal{E}_0}(\Omega). \qquad \square$$

4 Positive solution

In this section, using the theory of pseudomonotone operators (see Papageorgiou–Rădulescu–Repovš [13, Section 2.10]), we prove the existence of a positive solution for problem (1.1).

We impose the following conditions on the reaction $f(z,x,y)$. In what follows, by $\hat{\lambda}_1(q) > 0$, we denote the principal eigenvalue of $(-\Delta_q, W_0^{1,q}(\Omega))$ (that is $\hat{\lambda}_1(q) = \hat{\lambda}_1^a(q)$ with $a \equiv 1$).

H_1: $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function such that

(i) $|f(z,x,y)| \leq \hat{a}(z)[1 + |x|^{p-1}] + \mu|y|^{q-1}$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, all $y \in \mathbb{R}^N$ with $\hat{a} \in L^\infty(\Omega)$ and $\mu < \hat{\lambda}_1(q)$;

(ii) there exists a function $\eta \in L^\infty(\Omega)$ such that

$$\eta(z) \leq \hat{\lambda}_1^a \text{ for } a.a.\ z \in \Omega, \eta \not\equiv \hat{\lambda}_1^a,$$

and for every $\varepsilon > 0$, there exists $M_\varepsilon > 0$ such that

$$f(z,x,y) \leq [\eta(z) + \varepsilon]a(z)x^{p-1} + \mu|y|^q \text{ for } a.a.\ z \in \Omega,\ all\ x \geq M_\varepsilon;$$

(iii) there exists $\vartheta \in L^\infty(\Omega)$ and $\delta > 0$ such that

$$\begin{aligned} &\vartheta(z) \geq \hat{\lambda}_1(q) \text{ for } a.a.\ z \in \Omega, \vartheta \not\equiv \hat{\lambda}_1(q), \\ &f(z,x,y) \geq \vartheta(z)x^{q-1} \text{ for } a.a.\ z \in \Omega, \text{ all } 0 \leq x \leq \delta, \text{ all } y \in \mathbb{R}^N \\ &f(z,x,y) \geq -c_2 x^{r-1} \text{ for } a.a.\ z \in \Omega, \text{ all } x \geq \delta, \text{ all } y \in \mathbb{R}^N, \text{some } c_2 > 0,\ p < r < p^*. \end{aligned}$$

Remark 4.1. Hypothesis H_1(ii) implies that

$$\limsup_{n \to +\infty} \frac{f(z,x,y)}{a(z)x^{p-1}} \leq \eta(z)$$

uniformly for a.a.$z \in \Omega$ and all $y \in \mathbb{R}^N$ on a bounded set. Similarly, hypothesis H_1(iii) implies that

$$\liminf_{x \to 0^+} \frac{f(z,x,y)}{x^{q-1}} \geq \vartheta(z)$$

uniformly for a.a.$z \in \Omega$ and all $y \in \mathbb{R}^N$.

Example 4.2. The following function satisfies all the above hypotheses

$$f(z,x,y) = \begin{cases} \vartheta(x^+)^{q-1} + \left[\mu|y|^{q-1} + (\eta a(z) - \vartheta)\right](x^+)^{s-1}, & \text{if } x \le 1, \\ \eta a(z) x^{p-1} + \mu |y|^{q-1}, & \text{if } 1 < x, \end{cases}$$

with $\mu < \hat{\lambda}_1(q) < \vartheta$, $\eta < \hat{\lambda}_1^a$, $1 < q < s$.

On account of hypotheses H_1(i),(ii), we have

$$f(z,x,y) \ge \vartheta(z)x^{q-1} - c_3 x^{r-1} \text{ for a.a. } z \in \Omega, \text{ all } x \ge 0, \text{ all } y \in \mathbb{R}^N, \text{ some } c_3 > 0. \tag{4.1}$$

Based on this unilateral growth condition, we consider the following auxiliary double phase Dirichlet problem

$$\left\{ \begin{array}{l} -\Delta_p^a u(z) - \Delta_q u(z) = \vartheta(z)u(z)^{p-1} - c_3 u(z)^{r-1} \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \; u > 0, \; 1 < q < p < N, \; r > p. \end{array} \right\} \tag{4.2}$$

From Liu–Papageorgiou [9, Proposition 3.1], we have the following result for problem (4.2).

Proposition 4.3. *If hypotheses H_0 hold, then problem (4.2) has a unique positive solution $\bar{u} \in W_0^{1,\mathcal{E}}(\Omega) \cap L^\infty(\Omega)$ and $\bar{u}(z) > 0$ for a.a. $z \in \Omega$.*

Using the solution $\bar{u}$ we introduce the Carathéodory function $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ defined by

$$g(z,x,y) = \begin{cases} f(z,\bar{u}(z),y), & \text{if } x \le \bar{u}(z), \\ f(z,x,y), & \text{if } \bar{u}(z) < x. \end{cases} \tag{4.3}$$

Let $N_g(u)(\cdot) = g(\cdot, u(\cdot), Du(\cdot))$ for all $u \in W_0^{1,\mathcal{E}}(\Omega)$ (the Nemytski map corresponding to g) and consider the nonlinear operator $K : W_0^{1,\mathcal{E}}(\Omega) \to W_0^{1,\mathcal{E}}(\Omega)^*$ defined by

$$K(u) = V(u) - N_g(u) \quad \text{for all } u \in W_0^{1,\mathcal{E}}(\Omega).$$

Proposition 4.4. *If hypotheses H_0, H_1 hold, then the operator $K(\cdot)$ is pseudomonotone.*

Proof. We consider a sequence $\{u_n\}_{n \ge \mathbb{N}} \subseteq W_0^{1,\mathcal{E}}(\Omega)$ such that

$$\left\{ \begin{array}{l} u_n \xrightarrow{w} u \quad \text{in} \quad W_0^{1,\mathcal{E}}(\Omega) \quad K(u_n) \xrightarrow{w} u^* \text{ in } W_0^{1,\mathcal{E}}(\Omega)^*, \\ \limsup_{n\to\infty} \langle K(u_n), u_n - u \rangle \le 0. \end{array} \right\} \tag{4.4}$$

Hypotheses H_0 imply $p < q^*$ and so by Proposition 2.1, we have that

$$\begin{aligned} & W_0^{1,\mathcal{E}}(\Omega) \hookrightarrow L^p(\Omega) \text{ compactly}, \\ \Rightarrow \quad & u_n \to u \quad \text{in } L^p(\Omega) \quad (\text{see } (4.4)). \end{aligned} \tag{4.5}$$

We have

$$\begin{aligned} \int_\Omega g(z,u_n,Du_n)(u_n - u)dz = & \int_{\{u_n \le \bar{u}\}} f(z,\bar{u},Du_n)(u_n-u)dz \\ & + \int_{\{\bar{u} < u_n\}} f(z,u_n,Du_n)(u_n-u)dz \quad (\text{see } (4.4)). \end{aligned} \tag{4.6}$$

On account of hypothesis H_1(i), we have that

$$\{f(\cdot,\bar{u}(\cdot),Du_n(\cdot)\}_{n\geq\mathbb{N}}\subseteq L^{p'}(\Omega)$$
$$\{f(\cdot,u_n(\cdot),Du_n(\cdot)\}_{n\geq\mathbb{N}}\subseteq L^{p'}(\Omega),\quad \left(\frac{1}{p}+\frac{1}{p'}=1\right),$$

are both bounded (recall $q<p$). Therefore, from (4.5) we infer that

$$\int_{\{u_n\leq\bar{u}\}} f(z,\bar{u},Du_n)(u_n-u)dz\to 0,\qquad \int_{\{\bar{u}<u_n\}} f(z,u_n,Du_n)(u_n-u)dz\to 0\quad\text{as } n\to\infty.$$

From (4.6) it follows that

$$\begin{aligned}&\lim_{n\to\infty}\int_\Omega g(z,u_n,Du_n)(u_n-u)dz=0,\\ \Rightarrow\ &\limsup_{n\to\infty}\langle V(u_n),u_n-u\rangle\leq 0\quad\text{(see (4.4))},\\ \Rightarrow\ &u_n\to u\quad\text{in}\quad W_0^{1,\mathcal{E}}(\Omega)\quad\text{(see Proposition 2.3)}.\end{aligned}$$

Exploiting the continuity of $K(\cdot)$, we have

$$\begin{aligned}&K(u_n)\to K(u)\text{ in } W_0^{1,\mathcal{E}}(\Omega)^*,\\ \Rightarrow\ &u^*=K(u)\quad\text{and}\quad\langle K(u_n),u_n\rangle\to\langle K(u),u\rangle\quad\text{(see (4.4))},\\ \Rightarrow\ &K(\cdot)\quad\text{is generalized pseudomonotone}\quad\text{(see [13, p. 150])}.\end{aligned}$$

Invoking Proposition 2.10.3, p. 51, of Papageorgiou–Rădulescu–Repovš [13], we conclude that $K(\cdot)$ is pseudomonotone. □

Next we show that $K(\cdot)$ is strongly coercive, that is,

$$\frac{\langle K(u),u\rangle}{\|u\|}\to+\infty\quad\text{as } \|u\|\to\infty.$$

Proposition 4.5. *If hypotheses H_0, H_1 hold, then the operator $K(\cdot)$ is strongly coercive.*

Proof. For every $u\in W_0^{1,\mathcal{E}}(\Omega)$ with $\|u\|\geq 1$, $\|u\|_{1,\mathcal{E}_0}\geq 1$, we have

$$\begin{aligned}\langle K(u),u\rangle&=\langle V(u),u\rangle-\int_\Omega g(z,u,Du)udz\\ &=\rho_{\mathcal{E}}(Du)-\int_{\{u\leq\bar{u}\}} f(z,\bar{u},Du)udz-\int_{\{\bar{u}<u\}} f(z,u,Du)udz\quad\text{(see (4.3))}.\end{aligned}\tag{4.7}$$

We have

$$\int_{\{u\leq\bar{u}\}} f(z,\bar{u},Du)udz\leq c_4\|u\|+\int_{\{u\leq\bar{u}\}}\mu|Du|^{q-1}udz\tag{4.8}$$
for some $c_4>0$ (see hypothesis H_1(i))

$$\int_{\{\bar{u}<u\}} f(z,u,Du)udz\leq c_5+\int_\Omega[\eta(z)+\varepsilon]a(z)u^pdz+\int_{\{\bar{u}<u\}}\mu|Du|^{q-1}udz\tag{4.9}$$
for some $c_5=c_5(\varepsilon)>0$ (see H_1(ii)).

From (4.8) and (4.9) it follows that

$$\int_\Omega g(z,u,Du)udz\leq c_5+c_4\|u\|+\frac{\mu}{\hat{\lambda}_1(q)}\|Du\|_q^q+\int_\Omega\eta(z)a(z)u^pdz+\varepsilon\rho_a(u)\tag{4.10}$$
(here we have used Hölder's inequality) .

We return to (4.7) and use (4.10). We obtain

$$\begin{aligned}
\langle K(u), u\rangle &\geq \rho_a(Du) - \int_\Omega \eta(z)a(z)|u|^p dz - \frac{\varepsilon}{\hat{\lambda}_1^a}\|u\|_{1,\mathcal{E}_0}^p \\
&\quad + \left(1 - \frac{\mu}{\hat{\lambda}_1(q)}\right)\|Du\|_q^q - c_4\|u\| - c_5 \\
&\qquad \text{(recall that } \|u\|_{1,\mathcal{E}_0} \geq 1 \text{ and see Proposition 2.2)} \\
&\geq \left[c_1 - \frac{\varepsilon}{\hat{\lambda}_1^a}\right]\|u\|_{1,\mathcal{E}_0}^p + c_6\|Du\|_q^q - c_4\|u\| - c_5 \\
&\qquad \text{with } c_6 = 1 - \frac{\mu}{\hat{\lambda}_1(q)} > 0 \quad \text{(see Proposition 3.2).}
\end{aligned}$$

Choosing $\varepsilon \in (0, c_1\hat{\lambda}_1^a)$, we see that

$$\begin{aligned}
\langle K(u), u\rangle &\geq c_7\rho_{\mathcal{E}}(Du) - c_4\|u\| - c_5 \\
&\geq c_7\|u\|^q - c_4\|u\| - c_5 \quad \text{for some } c_7 > 0 \text{ (recall } \|u\| \geq 1 \text{ and see Proposition 2.2)} \\
&\Rightarrow K(\cdot) \quad \text{is strongly coercive.}
\end{aligned}$$

□

Now we are ready to prove the existence of a bounded positive solution for problem (1.1).

Theorem 4.6. *If hypotheses H_0, H_1 hold, then problem* (1.1) *admits a positive solution*

$$\hat{u} \in W_0^{1,\mathcal{E}}(\Omega) \cap L^\infty(\Omega).$$

such that $\hat{u}(z) > 0$ for a.a. $z \in \Omega$.

Proof. From Propositions 4.4 and 4.5 we have that the operator $K(\cdot)$ is pseudomonotone and strongly coercive. So, by Theorem 2.10.10, p. 156, of Papageorgiou–Rădulescu–Repovš [13], $K(\cdot)$ is surjective. Hence we can find $\hat{u} \in W_0^{1,\mathcal{E}}(\Omega)$ such that

$$\begin{aligned}
&K(\hat{u}) = 0 \quad \text{in} \quad W_0^{1,\mathcal{E}}(\Omega)^*, \\
\Rightarrow \quad &\langle K(\hat{u}), (\bar{u} - \hat{u})^+\rangle = 0 \quad \text{(since } (\bar{u} - \hat{u})^+ \text{ in } W_0^{1,\mathcal{E}}(\Omega))
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \langle V(\hat{u}), (\bar{u} - \hat{u})^+\rangle &= \int_\Omega g(z, \hat{u}, D\hat{u})(\bar{u} - \hat{u})^+ dz \\
&= \int_\Omega f(z, \hat{u}, D\hat{u})(\bar{u} - \hat{u})^+ dz \quad \text{(see (4.3))} \\
&\geq \int_\Omega [\vartheta(z)\bar{u}^{q-1} - c_8\bar{u}^{r-1}](\bar{u} - \hat{u})^+ dz \quad \text{(see (4.1))} \\
&= \langle V(\bar{u}), (\bar{u} - \hat{u})^+\rangle \quad \text{(see Proposition 4.3)} \\
&\Rightarrow \bar{u} \leq \hat{u} \quad \text{(see Proposition 2.3).}
\end{aligned}$$

Therefore $\hat{u} \in W_0^{1,\mathcal{E}}(\Omega)$ is a positive solution for problem (1.1). From Theorem 3.1 of Gasiński–Winkert [4], we have that $\hat{u} \in W_0^{1,\mathcal{E}}(\Omega) \cap L^\infty(\Omega)$. Finally Proposition 2.4 of Papageorgiou–Vetro–Vetro [16] implies that $\hat{u}(z) > 0$ for a.a. $z \in \Omega$. □

Let $S_+ \subseteq W_0^{1,\mathcal{E}}(\Omega)$ denote the set of positive solutions of problem (1.1). From Theorem 4.6 we have

$$\emptyset \neq S_+ \subseteq W_0^{1,\mathcal{E}}(\Omega) \cap L^\infty(\Omega). \tag{4.11}$$

Proposition 4.7. *If hypotheses H_0, H_1 hold, then $S_+ \subseteq W_0^{1,\mathcal{E}}(\Omega)$ is nonempty, compact.*

Proof. We already know that $S_+ \neq \emptyset$ (see Theorem 4.6 and (4.11)). Clearly $S_+ \subseteq W_0^{1,\mathcal{E}}(\Omega)$ is closed. Let $\{u_n\}_{n\geq \mathbb{N}} \subseteq S_+$. We have

$$\langle V(u_n), h\rangle = \int_\Omega f(z,u_n,Du_n)hdz \quad \text{for all } h \in W_0^{1,\mathcal{E}}(\Omega) \text{ all } n \in \mathbb{N}. \tag{4.12}$$

On account of hypotheses H_1(i)(ii), we have

$$f(z,x,y)x \leq [\eta(z)+\varepsilon]a(z)|x|^p + c_8 + \mu|y|^{q-1} \quad \text{for } a.a.\ z\in\Omega, \text{ all } x \in \mathbb{R}, \text{ some } c_8>0. \tag{4.13}$$

In (4.12) we use $h = u_n \in W_0^{1,\mathcal{E}}(\Omega)$. Using (4.13) we obtain

$$\begin{aligned}
&\rho_a(Du_n) - \int_\Omega \eta(z)a(z)|u_n|^p dz - \frac{\varepsilon}{\hat\lambda_1^a}\|u_n\|^p_{1,\mathcal{E}_0} + \|Du_n\|_q^q - \mu\|u_n\|_q^q \leq c_8 \quad \text{for all } n\in\mathbb{N},\\
\Rightarrow\ & \left[c_1 - \frac{\varepsilon}{\hat\lambda_1^a}\right]\|u_n\|^p_{1,\mathcal{E}_0} + \left[1-\frac{\mu}{\hat\lambda_1(q)}\right]\|Du\|_q^q \leq c_8,\\
\Rightarrow\ & \|u_n\|^p \leq c_9 \quad \text{for some } c_9>0, \text{ all } n\in\mathbb{N}\\
&\qquad\qquad (\text{choose } \varepsilon\in(0,c_1\hat\lambda_1^a) \text{ and recall that } \mu<\hat\lambda_1(q))\\
\Rightarrow\ & \{u_n\}_{n\geq\mathbb{N}} \subseteq W_0^{1,\mathcal{E}}(\Omega) \quad \text{is bounded.}
\end{aligned}$$

So, we may assume that

$$u_n \xrightarrow{w} u \quad \text{in } W_0^{1,\mathcal{E}}(\Omega) \text{ and } u_n \to u \text{ in } L^p(\Omega) \tag{4.14}$$
(recall that $p<q^*$ and see Proposition 2.1).

Then (4.14) and hypothesis H_1(i) imply that

$$\begin{aligned}
&\int_\Omega f(z,u_n,Du_n)(u_n-u)dz \to 0,\\
\Rightarrow\ & \lim_{n\to\infty}\langle V(u_n), u_n-u\rangle = 0 \quad (\text{see (4.12) with } h = u_n-u)\\
\Rightarrow\ & u_n \to u \quad \text{in } W_0^{1,\mathcal{E}}(\Omega) \quad (\text{see Proposition 4.4})
\end{aligned}$$

Since S_+ is closed, we conclude that it is compact in $W_0^{1,\mathcal{E}}(\Omega)$. □

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Limit cycles in mass-conserving deficiency-one mass-action systems

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Abstract. We present some simple mass-action systems with limit cycles that fall under the scope of the Deficiency-One Theorem. All the constructed examples are mass-conserving and their stoichiometric subspace is two-dimensional. Using the continuation software MATCONT, we depict the limit cycles in all stoichiometric classes at once. The networks are trimolecular and tetramolecular, and some exhibit two or even three limit cycles. Finally, we show that the associated mass-action system of a bimolecular reaction network with two-dimensional stoichiometric subspace does not admit a limit cycle.

Keywords: Andronov–Hopf bifurcation, focal value, limit cycle, parallelogram.

2020 Mathematics Subject Classification: 92E20, 34C23, 34C25, 37G15.

1 Introduction

The intensively studied field of reaction networks investigates, amongst many other questions, the existence, the uniqueness, and the stability of equilibria and limit cycles of mass-action systems. Often, properties of the underlying network alone have some consequences on the dynamics. For example, the associated mass-action system of a deficiency-zero network, regardless of the values of the rate constants, does not admit periodic solutions (the deficiency is a nonnegative integer, to be defined in Section 2).

Recently we have constructed a number of planar deficiency-one mass-action systems that oscillate [5]. The state space of those systems is the positive quadrant, an unbounded set. Often, physically realistic systems have bounded state space: the law of atomic balance means that in a closed environment the numbers of atoms of each element are expected to be conserved [9]. In this paper we provide a couple of examples that admit limit cycles, all with three species (whose concentrations are denoted by x, y, z) and a linear conservation law $d_1\dot{x} + d_2\dot{y} + d_3\dot{z} = 0$ with d_1, d_2, $d_3 > 0$. Thus, $d_1x + d_2y + d_3z = c$ holds for all positive time, where $c = d_1x(0) + d_2y(0) + d_3z(0)$. Consequently, the state space, after fixing the initial condition, is a bounded subset of the positive orthant. The main approach we follow is that

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we regard c a bifurcation parameter. We will be interested in the stability of equilibria and limit cycles as c varies, and in the case of limit cycles their number can also vary by c. We will see examples with multiple limit cycles, in one of the cases we can even prove the existence of three limit cycles that are all born via a degenerate Andronov–Hopf bifurcation. Probably the most interesting phenomenon found in this paper is a mass-action system with the unique positive equilibrium being asymptotically stable for all $c > 0$, but not globally stable for $c_1^* \leq c \leq c_2^*$, because a torus formed of stable and unstable limit cycles surrounds the curve of equilibria. We extensively use the continuation software MATCONT [8] to visualise the limit cycles in the (x, y, z)-space for all $c > 0$ at once, while fixing all the rate constants. The codes are available on GitHub [3].

A common feature of the networks analysed in this paper is that they all have at least one chemical complex that is trimolecular or tetramolecular. This is necessary, as the mass-action system associated to a bimolecular reaction network with two-dimensional stoichiometric subspace does not admit limit cycles. This latter fact is known for 2 and 3 species [18,19], while for arbitrary number of species we prove it in Section 4. In fact, we show that essentially the only bimolecular reaction networks with a two-dimensional stoichiometric subspace whose associated mass-action system oscillates are the Lotka and the Ivanova reactions, where each positive non-equilibrium solution is periodic.

The rest of this paper is organised as follows. In Section 2 we collect the needed terminology and some basic results from chemical reaction network theory. In Section 3 we present a number of mass-conserving networks that admit (multiple) limit cycles. Finally, in Section 4 we show that the associated mass-action system of a bimolecular reaction network with two-dimensional stoichiometric subspace does not admit a limit cycle.

2 Mass-action systems

In this section we briefly introduce mass-action systems and related notions that are necessary for our exposition. An illustrative example is included at the end of this section. For more details about mass-action systems, consult e.g. [10,15]. The symbols $\mathbb{R}_+$ and $\mathbb{Z}_{\geq 0}$ denote the set of positive real numbers and the set of nonnegative integers, respectively.

Definition 2.1. A *Euclidean embedded graph* (or a *reaction network*) is a directed graph (V, E), where V is a nonempty finite subset of $\mathbb{Z}_{\geq 0}^n$.

Denote by $\mathsf{X}_1, \ldots, \mathsf{X}_n$ the n *species* and by $y^1, \ldots, y^m$ the elements of V, called *complexes*. Accordingly, we often refer to y^i as $y_1^i \mathsf{X}_1 + \cdots + y_n^i \mathsf{X}_n$. The entries of y^i are the *stoichiometric coefficients*. The concentrations of the species $\mathsf{X}_1, \ldots, \mathsf{X}_n$ at time τ are collected in the vector $x(\tau) \in \mathbb{R}_+^n$.

Definition 2.2. A *mass-action system* is a triple (V, E, κ), where (V, E) is a reaction network and $\kappa \colon E \to \mathbb{R}_+$ is the collection of the *rate constants*. Its *associated differential equation* on $\mathbb{R}_+^n$ is

$$\dot{x}(\tau) = \sum_{(i,j) \in E} \kappa_{ij} x_1(\tau)^{y_1^i} \cdots x_n(\tau)^{y_n^i} (y^j - y^i). \tag{2.1}$$

The span $\mathcal{S} = \operatorname{span}\{y^j - y^i \colon (i, j) \in E\} \leq \mathbb{R}^n$ is called the *stoichiometric subspace* and the sets $(p + \mathcal{S}) \cap \mathbb{R}_+^n$ for $p \in \mathbb{R}_+^n$ are called the *(positive) stoichiometric classes*. The stoichiometric classes provide a foliation of the positive orthant $\mathbb{R}_+^n$ into forward invariant sets of the mass-action differential equation (2.1). Therefore, dynamical questions (e.g. existence, uniqueness, stability, or number of equilibria or limit cycles) are examined relative to a stoichiometric class. The *rank* of a reaction network (or its associated mass-action system) is defined to be the dimension of its stoichiometric subspace.

In some cases, a network property alone has consequences on the qualitative behaviour of the differential equation (2.1). For instance, if the directed graph (V, E) is strongly connected (i.e., for all $i, j \in V$ there exists a directed path from i to j) then the associated mass-action differential equation is permanent [14, Theorem 1.3], [1, Theorem 5.5], [4, Theorem 4.2]. We now define permanence.

Definition 2.3. A mass-action system is *permanent in a stoichiometric class* $\mathcal{P}$ if there exists a compact set $K \subseteq \mathcal{P}$ with the property that for each solution $\tau \mapsto x(\tau)$ with $x(0) \in \mathcal{P}$ there exists a $\tau_0 \geq 0$ such that $x(\tau) \in K$ holds for all $\tau \geq \tau_0$. A mass-action system is *permanent* if it is permanent in every stoichiometric class.

Theorem 2.4 ([1, 4, 14])**.** *If (V, E) is strongly connected then the mass-action system (V, E, κ) is permanent.*

We now recall a classical theorem on the number of positive equilibria for mass-action systems with low deficiency. The *deficiency* of a reaction network (V, E) is the nonnegative integer $\delta = m - \ell - \dim \mathcal{S}$, where $m = |V|$, ℓ is the number of connected components of the directed graph (V, E), and $\mathcal{S}$ is the stoichiometric subspace.

Theorem 2.5 (Deficiency-One Theorem [11])**.** *Assume that the reaction network (V, E) is strongly connected and its deficiency is zero or one. Then the following statements hold.*

(i) There exists a unique positive equilibrium in every stoichiometric class.

(ii) The set of positive equilibria equals $\{x \in \mathbb{R}^n_+ \colon \log x - \log x^* \in \mathcal{S}^\perp\}$, where x^* is any given positive equilibrium.

(iii) Denoting by $J \in \mathbb{R}^{n\times n}$ the Jacobian matrix at a positive equilibrium,

(a) the linear map $J|_\mathcal{S} \colon \mathcal{S} \to \mathcal{S}$ is nonsingular and

(b) the linear map $-J|_\mathcal{S} \colon \mathcal{S} \to \mathcal{S}$ is orientation-preserving, i.e., its determinant is positive, or equivalently $\operatorname{sgn} \det J|_\mathcal{S} = (-1)^{\dim \mathcal{S}}$.

Proof. Parts (i), (ii), (iii)(a) are proven in [11].

We now prove part (iii)(b). For a fixed stoichiometric class, consider the map $\mathbb{R}^E_+ \to \mathbb{R}$ that assigns to each set of rate constants the determinant of the restricted Jacobian map at the unique positive equilibrium. This map is continuous, it is everywhere nonzero by part (iii)(a), and hence has a constant sign. For the subset of complex balanced systems the equilibrium is linearly stable (see [16, Theorem 4.3.2], [12, Theorem 15.2.2], or [7, Theorem 8]), and hence the sign is $(-1)^{\dim \mathcal{S}}$. □

If $\delta = 0$ in Theorem 2.5, one can even prove the asymptotic stability of the unique positive equilibrium. For a streamlined exposition and a biological application, see [21].

As a consequence of part (iii)(b) in Theorem 2.5, for $\dim \mathcal{S} = 2$ the product of the two nonzero eigenvalues at a positive equilibrium is positive, hence it is enough to look at the trace for deciding stability: if the trace is negative (respectively, positive) then the equilibrium is asymptotically stable (respectively, repelling) within its stoichiometric class. When the trace vanishes, the two nonzero eigenvalues are purely imaginary and the stability can be decided by computing the focal values.

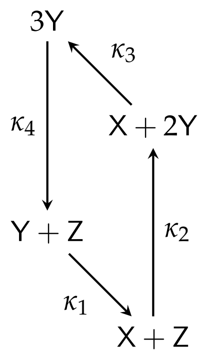
In the special case when $\dim \mathcal{S} = n-1$, part (ii) in Theorem 2.5 has the immediate consequence that the set of positive equilibria can be parametrised as follows:

$$\{(x_1^* t^{d_1}, \ldots, x_n^* t^{d_n}) \colon t > 0\}, \tag{2.2}$$

where $x^* \in \mathbb{R}^n_+$ is any positive equilibrium and d is any nonzero vector in $\mathcal{S}^\perp$.

A reaction network is *mass-conserving* if the stoichiometric classes are bounded, or equivalently there exists a $d \in \mathcal{S}^\perp$ with all coordinates being positive.

To illustrate the notions introduced in this section, consider the following reaction network and its associated mass-action differential equation:



$$\begin{aligned}\dot{x} &= \kappa_1 yz - \kappa_3 xy^2,\\ \dot{y} &= -\kappa_1 yz + \kappa_3 xy^2 + 2(\kappa_2 xz - \kappa_4 y^3),\\ \dot{z} &= -\kappa_2 xz + \kappa_4 y^3.\end{aligned}$$

The stoichiometric subspace of the network is

$$\mathcal{S} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

Since $[1,1,2]^\top \in \mathcal{S}^\perp$, the network is mass-conserving. The stoichiometric classes are $\mathcal{P}_c = \{(x,y,z) \in \mathbb{R}^3_+ \colon x + y + 2z = c\}$ for $c > 0$. By Theorem 2.5, the system is permanent in $\mathcal{P}_c$ for every $c > 0$. Since $\dim \mathcal{S} = 2$, the rank of the network is two and its deficiency is $\delta = 4 - 1 - 2 = 1$. The set of positive equilibria is the curve

$$\left(\left(\frac{\kappa_1\kappa_4}{\kappa_2\kappa_3}\right)^{\frac{1}{2}} t, t, \left(\frac{\kappa_3\kappa_4}{\kappa_1\kappa_2}\right)^{\frac{1}{2}} t^2 \right) \quad \text{for } t > 0$$

which is in line with (2.2). Further, this curve intersects $\mathcal{P}_c$ in exactly one point for each $c > 0$, as predicted by Theorem 2.5.

In Sections 3.1 to 3.3, all the networks fall under the scope of the Theorem 2.5 (with $\delta = 1$) and satisfy $n = 3$ and $\dim \mathcal{S} = 2$. Thus, in all of the examples the set of positive equilibria is parametrised as in (2.2) and their stability is decided by the sign of the trace of the Jacobian matrix (provided it is nonzero).

3 Parallelograms

Our goal is to find some simple mass-conserving reaction networks that fall under the scope of the Deficiency-One Theorem and their associated mass-action systems admit limit cycles. By the recent result [2, Theorem 1], a way to achieve this is the following: first find a planar mass-action system with a limit cycle (or multiple limit cycles) and then add a new species to some of the reactions in such a way that the rank of the new network is still two and the stoichiometric classes become bounded. For instance, one confirms that all the three parallelograms in Figure 3.1 admit limit cycles. This can be proven by showing that there exist rate constants such that the unique positive equilibrium is repelling. The existence of a stable limit cycle then follows from the permanence of the system and the Poincaré–Bendixson Theorem. One finds that each of the second and the third parallelograms admits even two limit cycles (to show this, one calculates the first focal value at the positive equilibrium and finds that it can be positive for some rate constants, allowing an unstable limit cycle to be born via a subcritical Andronov–Hopf bifurcation).

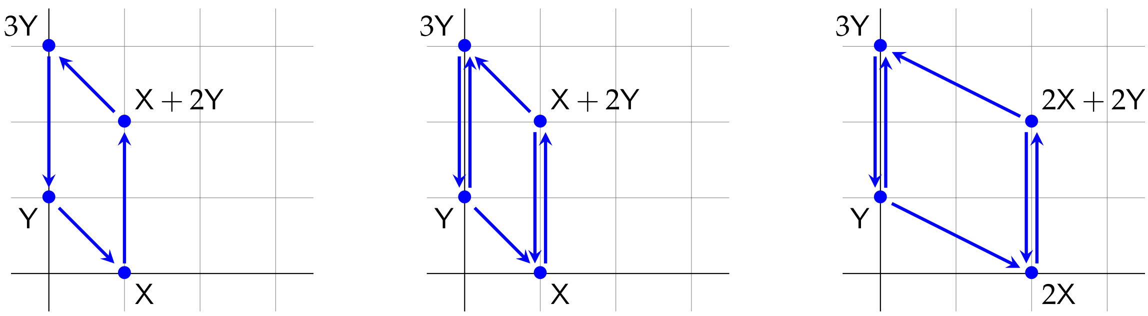
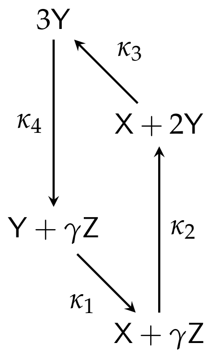


Figure 3.1: Three planar reaction networks that all fall under the scope of the Deficiency-One Theorem and their associated mass-action systems admit limit cycles. Each of the second and the third parallelograms admits even two limit cycles: there exist rate constants such that the unique positive equilibrium is asymptotically stable and is surrounded by an unstable and a stable limit cycle.

In Sections 3.1 to 3.3, we respectively lift the three parallelograms in Figure 3.1 by adding a new species. In all the three cases, we obtain a three species mass-action system that is mass-conserving and that falls under the scope of the Deficiency-One Theorem. Thus, the set of positive equilibria is of the form (2.2), it intersects every stoichiometric class in exactly one point. The aim is to find rate constants such that the positive equilibria are surrounded by limit cycles. In fact, we adopt the approach that the rate constants are fixed, and regard the stoichiometric class as a bifurcation parameter. In a number of cases, we visualise the (stable and unstable) limit cycles across the stoichiometric classes. This is performed using the numerical continuation software MATCONT [8]. The MATLAB codes, along with the symbolic computations performed with Mathematica, are available on GitHub [3].

3.1 Supercritical Andronov–Hopf bifurcation

Let us take the first planar parallelogram in Figure 3.1 and add a new species, Z, with stoichiometric coefficient $\gamma > 0$ as follows:



$$\begin{aligned} \dot{x} &= \kappa_1 y z^\gamma - \kappa_3 x y^2, \\ \dot{y} &= -\kappa_1 y z^\gamma + \kappa_3 x y^2 + 2(\kappa_2 x z^\gamma - \kappa_4 y^3), \\ \dot{z} &= \gamma(-\kappa_2 x z^\gamma + \kappa_4 y^3), \end{aligned} \tag{3.1}$$

where we also displayed the associated mass-action differential equation. Note that $\gamma x + \gamma y + 2z$ is a conserved quantity, and therefore the network is mass-conserving. By a short calculation, the set of positive equilibria is the curve

$$\left(\left(\frac{\kappa_1\kappa_4}{\kappa_2\kappa_3} \right)^{\frac{1}{2}} t^\gamma, t^\gamma, \left(\frac{\kappa_3\kappa_4}{\kappa_1\kappa_2} \right)^{\frac{1}{2\gamma}} t^2 \right) \quad \text{for } t > 0.$$

Since the stability of a positive equilibrium within its stoichiometric class is determined by the sign of the trace of the Jacobian matrix, we compute the trace along the curve of equilibria and find it is

$$\left(\left(\frac{\kappa_1\kappa_3\kappa_4}{\kappa_2} \right)^{\frac{1}{2}} - (\kappa_3 + 6\kappa_4) \right) t^{2\gamma} - \gamma^2 \kappa_4 \left(\frac{\kappa_1\kappa_2}{\kappa_3\kappa_4} \right)^{\frac{1}{2\gamma}} t^{3\gamma-2} \quad \text{for } t > 0. \tag{3.2}$$

As a function of t, it behaves differently for $\gamma < 2$, $\gamma = 2$, and $\gamma > 2$. In Sections 3.1.1 and 3.1.2 we study the cases $\gamma = 2$ and $\gamma \neq 2$, respectively.

3.1.1 Case $\gamma = 2$

Notice that for $\gamma = 2$, every complex in the network is trimolecular. As a consequence, the r.h.s. of the mass-action differential equation is a homogeneous polynomial (every monomial is of degree three). Furthermore, $x + y + z$ is conserved and it is not hard to see that in every stoichiometric class the dynamics is the same (up to scaling).

For $\gamma = 2$, the set of positive equilibria is a half-line and the trace of the Jacobian matrix along that, formula (3.2), equals

$$\left(\left(\frac{\kappa_1\kappa_3\kappa_4}{\kappa_2} \right)^{\frac{1}{2}} - \kappa_3 - 6\kappa_4 - 4\kappa_4 \left(\frac{\kappa_1\kappa_2}{\kappa_3\kappa_4} \right)^{\frac{1}{4}} \right) t^4,$$

an expression whose sign is independent of t. With $a(\kappa)$ denoting the coefficient of t^4, every positive equilibrium is asymptotically stable (respectively, repelling) if $a(\kappa) < 0$ (respectively, $a(\kappa) > 0$). For $a(\kappa) = 0$ one computes the first focal value and finds it is negative, implying that the equilibria are asymptotically stable and the corresponding Andronov–Hopf bifurcation is supercritical. Thus, setting the rate constants such that $a(\kappa) = 0$ and then perturbing them slightly to achieve $a(\kappa) > 0$, results in the emergence of a stable limit cycle in every stoichiometric class. By the homogeneity, the phase portrait is the same in every stoichiometric class, and the limit cycles that are born via a supercritical Andronov–Hopf bifurcation indeed coexist in every stoichiometric class. In fact, since the system is permanent and the equilibrium is repelling for $a(\kappa) > 0$, a stable limit cycle exists in every stoichiometric class for all rate constants with $a(\kappa) > 0$. We depicted these limit cycles in Figure 3.2 with $\kappa_1 = 16$, $\kappa_2 = \frac{1}{16}$, $\kappa_3 = 1$, $\kappa_4 = 1$ (thus, $a(\kappa) = 5 > 0$).

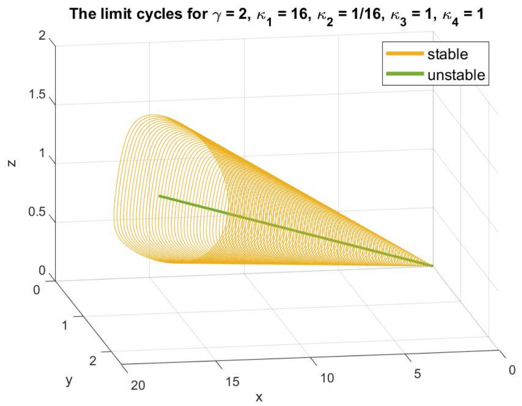


Figure 3.2: The stable limit cycles that are born via a supercritical Andronov–Hopf bifurcation for the mass-action system (3.1) (case $\gamma = 2$).

3.1.2 Case $\gamma \neq 2$

When $\gamma \neq 2$, the exponents 2γ and $3\gamma - 2$ of t in the trace formula (3.2) are unequal. Since the coefficient of $t^{3\gamma-2}$ is negative, the trace vanishes at some $t > 0$ if and only if the coefficient of $t^{2\gamma}$ is positive. Thus, if $\sqrt{\frac{\kappa_1}{\kappa_2}} \leq \sqrt{\frac{\kappa_3}{\kappa_4}} + 6\sqrt{\frac{\kappa_4}{\kappa_3}}$ then every positive equilibrium is asymptotically stable. On the other hand, if $\sqrt{\frac{\kappa_1}{\kappa_2}} > \sqrt{\frac{\kappa_3}{\kappa_4}} + 6\sqrt{\frac{\kappa_4}{\kappa_3}}$ then the trace vanishes at exactly one positive t, call it t^*. One computes the first focal value at $t = t^*$ and finds it is negative (due to the high computational complexity, we could, unfortunately, verify this only for some specific values of γ, e.g. 1, 3, 4, 5, 6). Therefore, the equilibrium at $t = t^*$ is asymptotically stable and the corresponding Andronov–Hopf bifurcation is supercritical (regarding t as a parameter, while all the rate constants are fixed). Hence, we obtain the following qualitative pictures:

Case $\gamma < 2$. For $t \leq t^*$ the equilibrium is asymptotically stable, for $t > t^*$ the equilibrium is repelling, and for t slightly larger than t^* there exists a stable limit cycle that is born via a supercritical Andronov–Hopf bifurcation. In fact, since the system is permanent and the equilibrium is repelling for all $t > t^*$, a stable limit cycle exists for all $t > t^*$. Using MATCONT, we depicted these limit cycles in the left panel of Figure 3.3 with $\gamma = 1$, $\kappa_1 = 8$, $\kappa_2 = \frac{1}{8}$, $\kappa_3 = 1$, $\kappa_4 = 1$.

Case $\gamma > 2$. For $t \geq t^*$ the equilibrium is asymptotically stable, for $t < t^*$ the equilibrium is repelling, and for t slightly smaller than t^* there exists a stable limit cycle that is born via a supercritical Andronov–Hopf bifurcation. In fact, since the system is permanent and the equilibrium is repelling for all $t < t^*$, a stable limit cycle exists for all $t < t^*$. Using MATCONT, we depicted these limit cycles in the right panel of Figure 3.3 with $\gamma = 3$, $\kappa_1 = 16$, $\kappa_2 = \frac{1}{16}$, $\kappa_3 = 1$, $\kappa_4 = 1$.

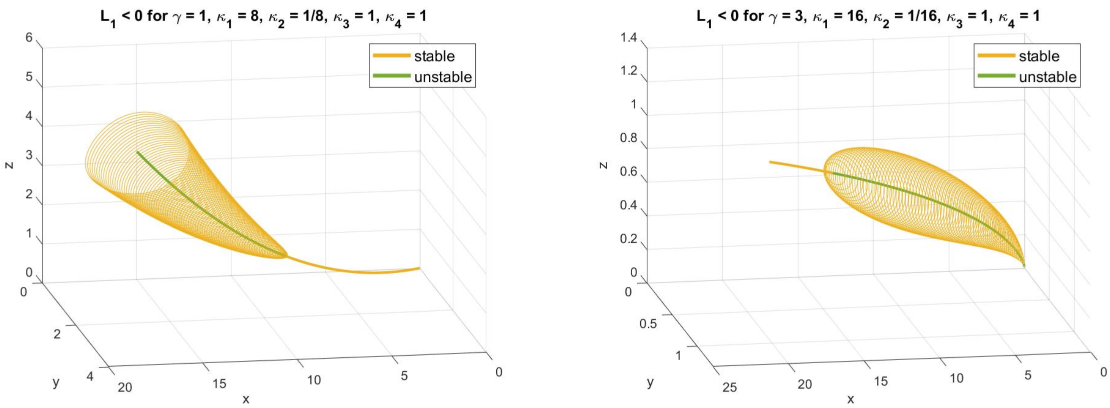
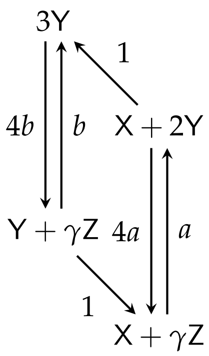


Figure 3.3: The stable limit cycles that are born via a supercritical Andronov–Hopf bifurcation for the mass-action system (3.1) (case $\gamma < 2$ on the left, case $\gamma > 2$ on the right).

3.2 Subcritical Andronov–Hopf bifurcation

Let us take the second planar parallelogram in Figure 3.1 and add a new species, Z, with stoichiometric coefficient $\gamma > 0$ as follows:



$$\begin{aligned} \dot{x} &= yz^\gamma - xy^2, \\ \dot{y} &= -yz^\gamma + xy^2 + 2(axz^\gamma - 4axy^2 + byz^\gamma - 4by^3), \\ \dot{z} &= \gamma(-axz^\gamma + 4axy^2 - byz^\gamma + 4by^3), \end{aligned} \tag{3.3}$$

where we employed special rate constants (for $a, b > 0$) and also displayed the associated mass-action differential equation. This special choice of the rate constants makes the calculations somewhat easier and they still allow us to present some qualitative pictures that were not seen in Section 3.1. Note that $\gamma x + \gamma y + 2z$ is a conserved quantity, and therefore the network is mass-conserving. By a short calculation, the set of positive equilibria is the curve

$$(2t^\gamma, t^\gamma/2, t^2) \quad \text{for } t > 0.$$

Since the stability of a positive equilibrium within its stoichiometric class is determined by the sign of the trace of the Jacobian matrix, we compute the trace along the curve of equilibria and find it is

$$4t^{2\gamma}\left[\left(\frac{3}{16} - 4a - b\right) - \frac{\gamma^2}{8}(4a+b)t^{\gamma-2}\right] \quad \text{for } t > 0. \tag{3.4}$$

As a function of t, it behaves differently for $\gamma < 2$, $\gamma = 2$, and $\gamma > 2$. In Sections 3.2.1 and 3.2.2 we study the cases $\gamma = 2$ and $\gamma \neq 2$, respectively.

3.2.1 Case $\gamma = 2$

First notice that for $\gamma = 2$, the same way as in Section 3.1.1, the phase portrait is the same (up to scaling) in every stoichiometric class.

For $\gamma = 2$, the set of positive equilibria is a half-line and the trace of the Jacobian matrix along that, formula (3.4), equals $6t^4\left(\frac{1}{8} - 4a - b\right)$. Thus, all positive equilibria are asymptotically stable if $4a + b > \frac{1}{8}$, while all of them are repelling if $4a + b < \frac{1}{8}$. On the $4a + b = \frac{1}{8}$ line in parameter space, one computes the first focal value and gets

$$L_1 = 1280b^2 - 16b - 7 \quad \text{for } 0 < b < \frac{1}{8}.$$

Hence, with $b^* = \frac{1+\sqrt{141}}{160} \approx 0.08$, the first focal value is negative for $0 < b < b^*$, vanishes at b^*, and is positive for $b^* < b < \frac{1}{8}$. This allows us to construct two limit cycles in each stoichiometric class in the following way. Take $a = \frac{1}{200}$ and $b = \frac{21}{200}$. Then the trace vanishes and the first focal value is positive, so the equilibrium is repelling. By permanence, there exists a stable limit cycle. By increasing b a tiny bit, the trace becomes negative, and an unstable limit cycle is born via a subcritical Andronov–Hopf bifurcation. Thus, there exist a and b such that two limit cycles coexist. A numerical experiment suggests that these two limit cycles merge and disappear through a fold bifurcation around $b \approx \frac{23.37}{200}$. Using MATCONT, we depicted in Figure 3.4 the two nested cones of limit cycles for $a = \frac{1}{200}$ and $b = \frac{22}{200}$.

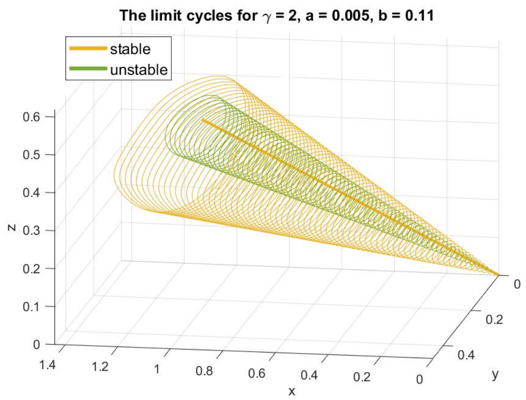


Figure 3.4: The two nested cones of limit cycles, the unstable limit cycles are born via a subcritical Andronov–Hopf bifurcation for the mass-action system (3.3) (case $\gamma = 2$).

3.2.2 Case $\gamma \neq 2$

When $\gamma \neq 2$, for any fixed $a, b > 0$ with $4a + b < \frac{3}{16}$ there exists a unique $t^* > 0$ for which the trace (3.4) vanishes. To decide stability of the equilibrium, one computes the first focal value and finds it can have any sign, see the left (case $\gamma = 1$) and right (case $\gamma = 3$) panels in the top row in Figure 3.5. In particular, the first focal value can be positive and the corresponding Andronov–Hopf bifurcation is then subcritical, allowing an unstable limit cycle to be born. Since the system is permanent, each unstable equilibrium and each unstable limit cycle is surrounded by a stable limit cycle. Using MATCONT, we depicted these limit cycles in the bottom row of Figure 3.5 for some particular choices of the parameters.

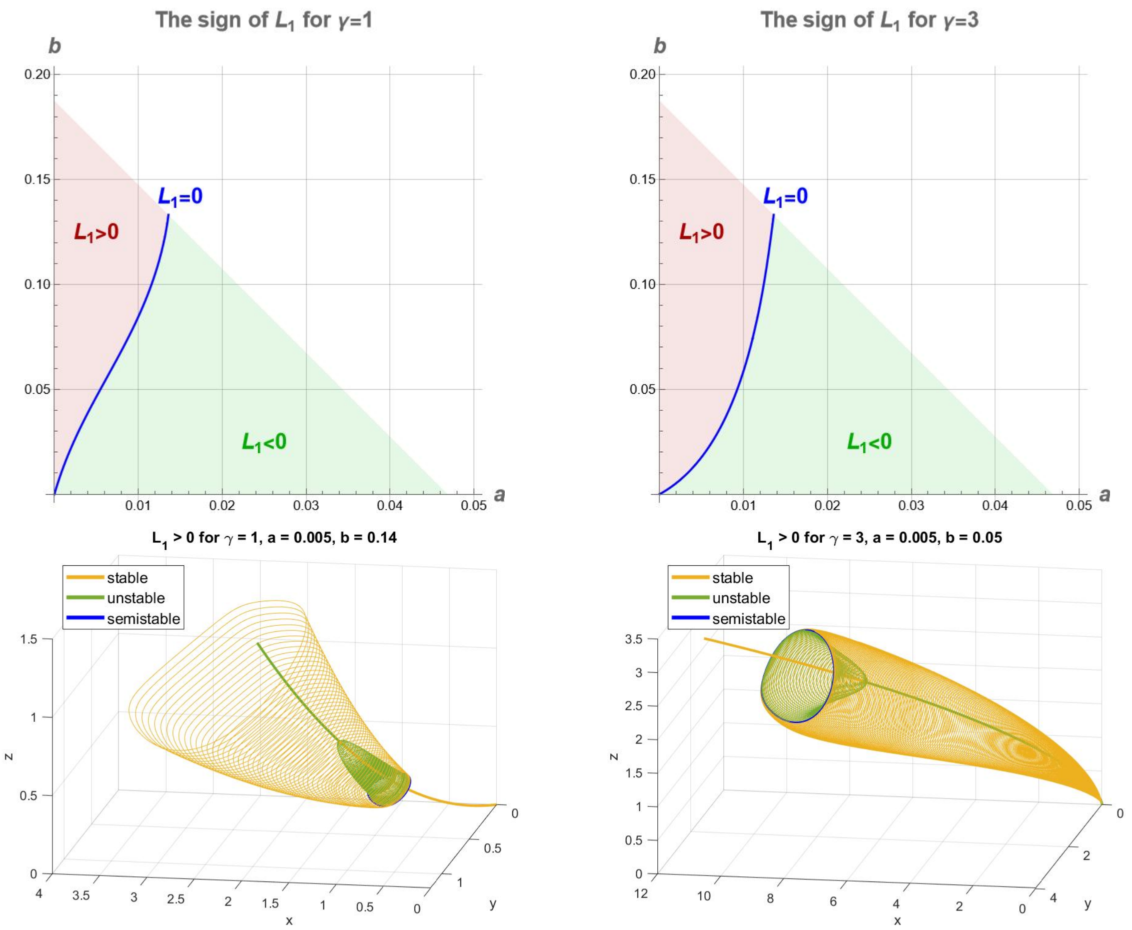
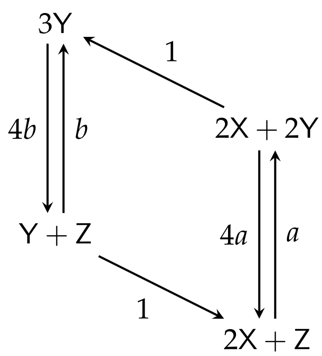


Figure 3.5: Top row: the sign of the first focal value for the mass-action system (3.3) (case $\gamma = 1$ on the left, case $\gamma = 3$ on the right). Bottom row: the unstable limit cycles that are born via a subcritical Andronov–Hopf bifurcation (parameters are taken such that $L_1 > 0$ holds), surrounded by stable limit cycles (case $\gamma = 1$ on the left, case $\gamma = 3$ on the right). The stable and unstable limit cycles merge through a fold bifurcation to a semistable limit cycle (shown in blue).

3.3 Two Andronov–Hopf points

Let us take the third planar parallelogram in Figure 3.1 and add a new species, Z, as follows:



$$\begin{aligned} \dot{x} &= 2(yz - x^2y^2), \\ \dot{y} &= -yz + x^2y^2 + 2(ax^2z - 4ax^2y^2 + byz - 4by^3), \\ \dot{z} &= -ax^2z + 4ax^2y^2 - byz + 4by^3, \end{aligned} \tag{3.5}$$

where we employed special rate constants (for $a, b > 0$) and also displayed the associated mass-action differential equation. Similarly to Section 3.2, this special choice of the rate constants makes the calculations somewhat easier and they still allow us to present some qualitative pictures that were not seen in Sections 3.1 and 3.2. Note that $x + 2y + 4z$ is a conserved quantity, and therefore the network is mass-conserving. By a short calculation, the set of positive equilibria is the curve

$$(t, t^2/4, t^4/4) \quad \text{for } t > 0.$$

Since the stability of a positive equilibrium within its stoichiometric class is determined by the sign of the trace of the Jacobian matrix, we compute the trace along the curve of equilibria and find it is

$$\frac{t^2}{4}[-t^3 + (1 - 4(4a+b))t^2 - (4a+b)] \quad \text{for } t > 0.$$

One finds that, as a function of t, the trace has exactly two positive roots if and only if $4a + b < 1/16$. Call the two roots $t^{(1)}$ and $t^{(2)}$ (with $0 < t^{(1)} < t^{(2)}$; the dependence on a and b is omitted for readability). To understand the stability of the equilibria at $t^{(1)}$ and $t^{(2)}$, one computes the respective first focal values $L_1^{(1)}$ and $L_1^{(2)}$. Their sign is shown in the left panel of Figure 3.6, the generic cases are

- both of $L_1^{(1)}$ and $L_1^{(2)}$ are negative (first row in Figure 3.7),
- $L_1^{(1)}$ is negative and $L_1^{(2)}$ is positive (left panel in the second row in Figure 3.7),
- both of $L_1^{(1)}$ and $L_1^{(2)}$ are positive (right panel in the second row in Figure 3.7).

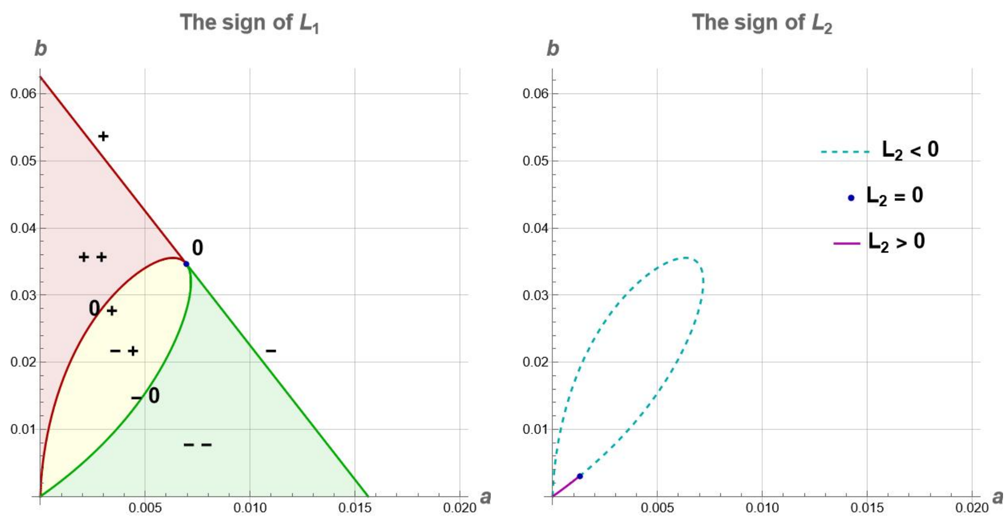


Figure 3.6: Left panel: the sign of the first focal values $L_1^{(1)}$ and $L_1^{(2)}$ for $4a + b \leq 1/16$ for the mass-action system (3.5) (since $t^{(1)}$ and $t^{(2)}$ coincide if $4a + b = 1/16$, we displayed only one sign there). Right panel: the sign of L_2 along the curve $L_1 = 0$. The third focal value is negative at the unique $(a, b, t) \in \mathbb{R}_+^3$, where $\operatorname{tr} J = L_1 = L_2 = 0$.

In the boundary case $4a + b = 1/16$, we have $t^{(1)} = t^{(2)} = 1/2$. If additionally the first focal value is positive (take for instance $a = 1/256$ and $b = 12/256$) then the trace, as a function of

t, vanishes, but does not change sign at $1/2$. The equilibrium at $t = 1/2$ is repelling, since the first focal value is positive. Furthermore, since this is the limiting case of $L_1^{(1)} > 0$ and $L_1^{(2)} > 0$, both for t slightly smaller and slightly larger than $1/2$, the stable equilibrium is surrounded by an unstable limit cycle. Furthermore, by permanence, any unstable equilibrium and any unstable limit cycle is surrounded by a stable limit cycle. Using MATCONT, the picture we get in this case is shown in the left panel in the third row in Figure 3.7.

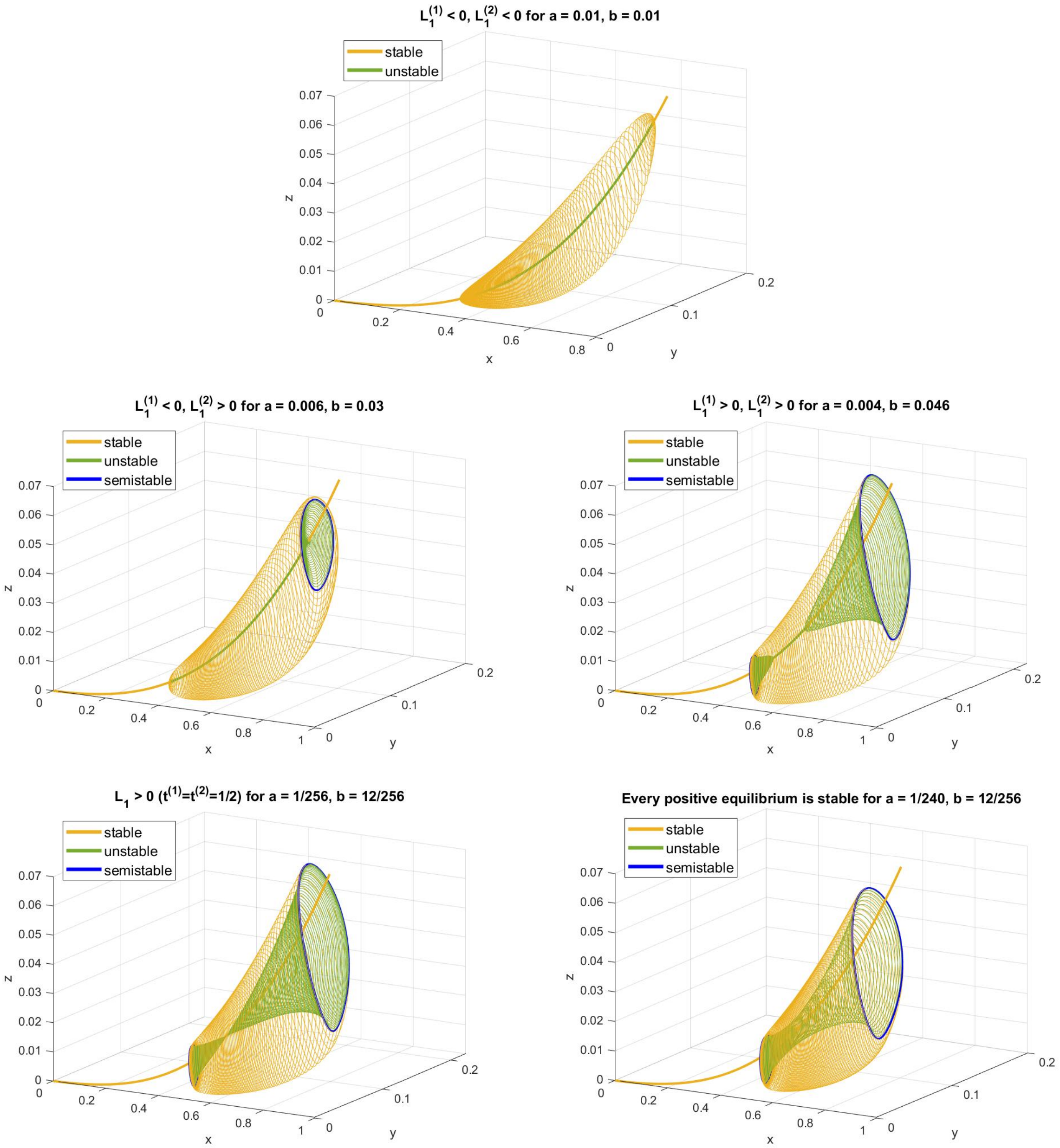


Figure 3.7: The various shapes that the stable and unstable limit cycles could form for the mass-action system (3.5). The most interesting is the last one, where all the positive equilibria are asymptotically stable, but not all of them are globally stable, because a torus of limit cycles surrounds the curve of positive equilibria.

Perhaps the most interesting qualitative picture in this paper is obtained by taking parameter values a and b as in the previous paragraph and then perturbing them slightly to make $4a + b > 1/16$. Then every positive equilibrium is asymptotically stable, but not all of them are globally stable within their stoichiometric classes, because a torus of stable and unstable limit cycles is created around the curve of positive equilibria as shown in the right panel in the third row in Figure 3.7.

We conclude this section by explaining how can one construct even three limit cycles. Viewing now $(a, b, t) \in \mathbb{R}^3_+$ as parameter, the trace of the Jacobian matrix vanishes along a surface $\mathcal{M}$. On this surface, there is a curve γ, where the first focal value vanishes (the projection of this curve to the (a, b)-plane is shown in Figure 3.6). Along this curve, there is a point $(a^*, b^*, t^*) \approx (0.001291, 0.003044, 0.958228)$, where the second focal value changes sign (this is shown in the right panel in Figure 3.6). One computes the third focal value at (a^*, b^*, t^*) and finds it is negative. The equilibrium is therefore asymptotically stable. Then do the following steps.

(i) First perturb (a, b, t) slightly along the curve γ such that the second focal value becomes positive. Then a stable limit cycle Γ_3 is born via a degenerate supercritical Andronov–Hopf bifurcation and the equilibrium becomes repelling.

(ii) Then perturb (a, b, t) slightly along the surface $\mathcal{M}$ such that the first focal value becomes negative. Then an unstable limit cycle Γ_2 is born via a degenerate subcritical Andronov–Hopf bifurcation and the equilibrium becomes asymptotically stable.

(iii) Finally perturb (a, b, t) slightly away from the surface $\mathcal{M}$ such that the trace of the Jacobian matrix becomes positive. Then a stable limit cycle Γ_1 is born via a nondegenerate supercritical Andronov–Hopf bifurcation and the equilibrium becomes repelling.

We have thus proved that there exists an (a, b, t) such that the unique positive equilibrium is repelling and is surrounded by three limit cycles (two stable and one unstable one). It would be interesting to continue these limit cycles in MATCONT as t is varied, however, apparently the three limit cycles coexist only in a very tiny region of the parameter space, and thus, it is numerically not easy to create meaningful pictures.

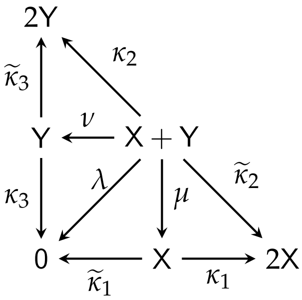
4 Bimolecular networks and limit cycles

The *molecularity* of a complex $y \in \mathbb{Z}^n_{\geq 0}$ (or $y_1\mathsf{X}_1 + \cdots + y_n\mathsf{X}_n$) is the sum $y_1 + \cdots + y_n \in \mathbb{Z}_{\geq 0}$. We say that a reaction network (V, E) (or a mass-action system (V, E, κ)) is *bimolecular* if the molecularity of each element of V is at most two. None of the examples in Section 3 is bimolecular. In this section we prove that for a rank-two bimolecular reaction network the associated mass-action system does not admit a limit cycle. For 2 and 3 species this was proven by Póta in [18] and [19], respectively. Theorem 4.1 below is an extension of these results to arbitrary number of species. On the other hand, there are two famous bimolecular mass-action systems of rank two that do oscillate: the Lotka and the Ivanova networks give rise to centers.

The Lotka network and its associated mass-action system are

$$\begin{aligned} \mathsf{X} &\xrightarrow{\kappa_1} \mathsf{2X} \\ \mathsf{X}+\mathsf{Y} &\xrightarrow{\kappa_2} \mathsf{2Y} \\ \mathsf{Y} &\xrightarrow{\kappa_3} \mathsf{0} \end{aligned} \qquad \begin{aligned} \dot{x} &= \kappa_1 x - \kappa_2 xy, \\ \dot{y} &= \kappa_2 xy - \kappa_3 y. \end{aligned}$$

Notice however that the mass-action system



$$\begin{aligned} \dot{x} &= (\kappa_1 - \widetilde{\kappa}_1)x - (\kappa_2 - \widetilde{\kappa}_2 + \lambda + \nu)xy, \\ \dot{y} &= (\kappa_2 - \widetilde{\kappa}_2 - \lambda - \mu)xy - (\kappa_3 - \widetilde{\kappa}_3)y \end{aligned}$$

also gives rise to a center, provided that

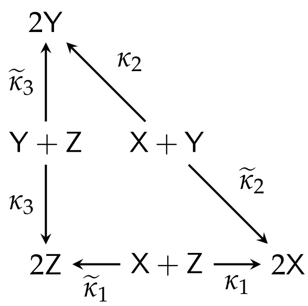
$$\operatorname{sgn}(\kappa_1 - \widetilde{\kappa}_1) = \operatorname{sgn}(\kappa_2 - \widetilde{\kappa}_2 + \lambda + \nu) = \operatorname{sgn}(\kappa_2 - \widetilde{\kappa}_2 - \lambda - \mu) = \operatorname{sgn}(\kappa_3 - \widetilde{\kappa}_3) \neq 0.$$

In the above, we allow some of κ_1, $\widetilde{\kappa}_1$, κ_2, $\widetilde{\kappa}_2$, κ_3, $\widetilde{\kappa}_3$, λ, μ, ν to vanish, in which case the corresponding reaction is not present. Finally, we note that the sum of the quadratic terms in $\dot{x}$ and $\dot{y}$ is nonpositive. This is because for a bimolecular reaction network in any reaction the molecularity of the product complex cannot be larger than the molecularity of the reactant complex if the latter equals two.

The Ivanova network and its associated mass-action system are

$$\begin{aligned} \mathsf{Z}+\mathsf{X} &\xrightarrow{\kappa_1} \mathsf{2X} \\ \mathsf{X}+\mathsf{Y} &\xrightarrow{\kappa_2} \mathsf{2Y} \\ \mathsf{Y}+\mathsf{Z} &\xrightarrow{\kappa_3} \mathsf{2Z} \end{aligned} \qquad \begin{aligned} \dot{x} &= \kappa_1 xz - \kappa_2 xy, \\ \dot{y} &= \kappa_2 xy - \kappa_3 yz, \\ \dot{z} &= \kappa_3 yz - \kappa_1 xz. \end{aligned}$$

Notice however that the mass-action system



$$\begin{aligned} \dot{x} &= (\kappa_1 - \widetilde{\kappa}_1)xz - (\kappa_2 - \widetilde{\kappa}_2)xy, \\ \dot{y} &= (\kappa_2 - \widetilde{\kappa}_2)xy - (\kappa_3 - \widetilde{\kappa}_3)yz, \\ \dot{z} &= (\kappa_3 - \widetilde{\kappa}_3)yz - (\kappa_1 - \widetilde{\kappa}_1)xz \end{aligned}$$

also gives rise to a center, provided that

$$\operatorname{sgn}(\kappa_1 - \widetilde{\kappa}_1) = \operatorname{sgn}(\kappa_2 - \widetilde{\kappa}_2) = \operatorname{sgn}(\kappa_3 - \widetilde{\kappa}_3) \neq 0.$$

In the above, we allow some of κ_1, $\widetilde{\kappa}_1$, κ_2, $\widetilde{\kappa}_2$, κ_3, $\widetilde{\kappa}_3$ to vanish, in which case the corresponding reaction is not present. Interestingly, the Ivanova network can be obtained from the Lotka network by adding a new species, Z, in a way that the molecularity of every complex becomes two.

Observe that there is another way for a rank-two bimolecular mass-action system to admit periodic solutions, as in the following example:

$$\begin{aligned} \mathsf{X} &\xrightarrow{\kappa_1} 2\mathsf{X} \\ \mathsf{X}+\mathsf{Y} &\xrightarrow{\kappa_2} 2\mathsf{Y} \\ \mathsf{Y} &\xrightarrow{\kappa_3} 0 \\ \mathsf{X}+\mathsf{Z} &\xrightarrow{\kappa_4} \mathsf{Z} \end{aligned} \qquad \begin{aligned} \dot{x} &= (\kappa_1 - \kappa_4 z)x - \kappa_2 xy, \\ \dot{y} &= \kappa_2 xy - \kappa_3 y, \\ \dot{z} &= 0. \end{aligned}$$

Here, the positive stoichiometric classes are given by $z = c$ for some $c > 0$. For $c < \frac{\kappa_1}{\kappa_4}$, the unique positive equilibrium is a global center (the dynamics is the same as for the Lotka network), while for $c \geq \frac{\kappa_1}{\kappa_4}$ there is no positive equilibrium.

The following theorem says that essentially the Lotka and the Ivanova differential equations are the only ones with a periodic solution that are derived from a rank-two bimolecular reaction network.

Theorem 4.1. *Suppose that a rank-two bimolecular mass-action system has a periodic solution in a positive stoichiometric class $\mathcal{P}$. Then the dynamics in $\mathcal{P}$ is described either by*

$$\begin{aligned} \dot{x} &= x(a - by), \\ \dot{y} &= y(b'x - c) \end{aligned}$$

for some $a, b, b', c \in \mathbb{R}$ with $\operatorname{sgn} a = \operatorname{sgn} b = \operatorname{sgn} b' = \operatorname{sgn} c \neq 0$ and $b' \leq b$ or by

$$\begin{aligned} \dot{x} &= x(az - by), \\ \dot{y} &= y(bx - cz), \\ \dot{z} &= z(cy - ax) \end{aligned}$$

for some $a, b, c \in \mathbb{R}$ with $\operatorname{sgn} a = \operatorname{sgn} b = \operatorname{sgn} c \neq 0$. In particular, there is a unique positive equilibrium in $\mathcal{P}$, every non-equilibrium solution is periodic, and there is no limit cycle.

Proof. For $k = 1, \ldots, n$ we collect the terms in $\dot{x}_k$ according to their degree in x_k. To ease the notation, let $x_{-k} = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) \in \mathbb{R}^{n-1}$ for $x \in \mathbb{R}^n$. With this,

$$\dot{x}_k = a_k(x_{-k}) + x_k b_k(x_{-k}) + c_k x_k^2 \quad \text{for } k = 1, \ldots, n,$$

where the function a_k collects all the terms without x_k, the middle term collects all the terms, where x_k appears linearly, and finally, the only term which is quadratic in x_k is denoted by $c_k x_k^2$. Note that $a_k(x_{-k}) \geq 0$, because in a reaction with X_k not being a reactant species, X_k can only be gained. Also, $c_k \leq 0$, since X_k can only be consumed in a bimolecular reaction whose reactant complex is $2\mathsf{X}_k$.

After dividing the vectorfield by $\prod_{i=1}^n x_i$, the divergence equals to

$$\frac{1}{\prod_{i=1}^n x_i} \sum_{k=1}^n \left(-\frac{a_k(x_{-k})}{x_k} + c_k x_k \right),$$

which is nonpositive in $\mathbb{R}^n_+$ by the above discussion. By a multidimensional version of the Bendixson–Dulac test, the existence of a periodic solution together with $\dim \mathcal{P} = 2$ implies that the divergence vanishes everywhere, see e.g. [20, Satz 1] for the $n = 3$ case and [17, Theorem 3.3, Remark (3)] for the general case. Therefore, both $a_k(x_{-k})$ and c_k vanish for all $k = 1, \ldots, n$ and

$$\dot{x}_k = x_k b_k(x_{-k}) = x_k \left(r_k + \sum_{i \neq k} b_{ki} x_i \right) \quad \text{for } k = 1, \ldots, n. \tag{4.1}$$

If there exists a k' such that $r_{k'} = 0$ and $b_{k'i} = 0$ for all $i \neq k'$ then $\dot{x}_{k'} = 0$ and thus $x_{k'} \equiv d$ for some $d > 0$. In this case, for each $k \neq k'$ we update r_k to be $r_k + b_{kk'} d$ and omit the equation for $\dot{x}_{k'}$. For the rest of this proof, we assume that the differential equation (4.1) is such that

$$\text{for all } k = 1, \ldots, n \text{ at least one of the } n \text{ numbers } r_k \text{ and } b_{ki} \ (i \neq k) \text{ is nonzero.} \tag{4.2}$$

Case $n = 2$. The differential equation (4.1) takes the form

$$\begin{aligned} \dot{x}_1 &= x_1(r_1 + b_{12} x_2), \\ \dot{x}_2 &= x_2(r_2 + b_{21} x_1). \end{aligned}$$

Since there exists a periodic solution, $\operatorname{sgn} r_1 = -\operatorname{sgn} b_{12} \neq 0$ and $\operatorname{sgn} r_2 = -\operatorname{sgn} b_{21} \neq 0$ follow. Furthermore, $\operatorname{sgn} b_{12} = -\operatorname{sgn} b_{21}$ (otherwise the unique positive equilibrium inside the closed orbit would be a saddle with index -1, a contradiction). Finally, $b_{12} + b_{21} \leq 0$ follows from the assumption that every product complex has molecularity at most two.

Case $n = 3$. The differential equation (4.1) takes the form

$$\begin{aligned} \dot{x}_1 &= x_1(r_1 + b_{12} x_2 + b_{13} x_3), \\ \dot{x}_2 &= x_2(r_2 + b_{21} x_1 + b_{23} x_3), \\ \dot{x}_3 &= x_3(r_3 + b_{31} x_1 + b_{32} x_2). \end{aligned} \tag{4.3}$$

Since the stoichiometric subspace is two-dimensional, there exists a $d \in \mathbb{R}^3 \setminus \{0\}$ such that $d_1 \dot{x}_1 + d_2 \dot{x}_2 + d_3 \dot{x}_3 = 0$. If exactly one coordinate of d is nonzero, say d_1, then $\dot{x}_1 = 0$, contradicting the assumption (4.2). If exactly two coordinates of d are nonzero, say d_1 and d_2, then $r_1 = r_2 = b_{13} = b_{23} = 0$ follows. Taking also into account (4.2), $b_{12} \neq 0$ and $b_{21} \neq 0$ hold. But then both x_1 and x_2 are strictly monotonic over time, contradicting the existence of a periodic solution. Thus, each of d_1, d_2, d_3 is nonzero. Hence, $d_1 \dot{x}_1 + d_2 \dot{x}_2 + d_3 \dot{x}_3 = 0$ implies that $r_1 = r_2 = r_3 = 0$. Furthermore, the existence of a periodic solution and the assumption (4.2) together imply $\operatorname{sgn} b_{12} = -\operatorname{sgn} b_{13} \neq 0$, $\operatorname{sgn} b_{21} = -\operatorname{sgn} b_{23} \neq 0$, and $\operatorname{sgn} b_{31} = -\operatorname{sgn} b_{32} \neq 0$. Since for reactions with reactant complexes $\mathsf{X}_1 + \mathsf{X}_2$, $\mathsf{X}_1 + \mathsf{X}_3$, or $\mathsf{X}_2 + \mathsf{X}_3$, the molecularity cannot increase (by the assumption that there is no product complex with molecularity higher than two), it follows that $\dot{x}_1 + \dot{x}_2 + \dot{x}_3 \leq 0$. Thus, along a periodic solution, $\dot{x}_1 + \dot{x}_2 + \dot{x}_3 = 0$, implying $b_{12} = -b_{21}$, $b_{13} = -b_{31}$, and $b_{23} = -b_{32}$.

Case $n \geq 4$. We now show that this case is actually empty. Since the stoichiometric subspace is 2-dimensional, the linear conservation laws form an $(n-2)$-dimensional subspace of $\mathbb{R}^n$. In such a subspace there always exists a nonzero vector d with at most three nonzero entries. Assume that d has a support that is minimal w.r.t. inclusion. If d has exactly one or two nonzero entries then we arrive at a contradiction in the same way as in the $n = 3$ case above. If d has exactly three nonzero entries, say d_1, d_2, d_3, then (x_1, x_2, x_3) evolves according to the equations (4.3), because $b_{1i} = b_{2i} = b_{3i} = 0$ for all $i = 4, \dots, n$ follows. By the minimality of the support of d, these three variables already occupy two dimensions, and thus $\dot{x}_4 = \cdots = \dot{x}_n = 0$, contradicting the assumption (4.2). $\square$

Theorem 4.1 shows that the rank of a bimolecular mass-action system with a limit cycle is at least three. We analyse some simple bimolecular oscillators of rank three in [6].

We conclude with a remark. For bimolecular networks we required that every complex's molecularity is at most two. If one relaxes this and imposes only that every reactant complex's molecularity is at most two, while it is allowed to have a product complex with molecularity three then limit cycles in rank-two mass-action systems are not excluded in general. Indeed, the mass-action system

$$\begin{array}{l} \mathsf{X} \xrightarrow{\kappa_1} 2\mathsf{X} \xrightarrow{\kappa_4} 3\mathsf{X} \\ \mathsf{X}+\mathsf{Y} \xrightarrow{\kappa_2} 2\mathsf{Y} \\ \mathsf{Y} \underset{\kappa_5}{\overset{\kappa_3}{\rightleftarrows}} 0 \end{array} \qquad \begin{aligned} \dot{x} &= \kappa_1 x - \kappa_2 xy + \kappa_4 x^2, \\ \dot{y} &= \kappa_2 xy - \kappa_3 y + \kappa_5 \end{aligned}$$

by Frank-Kamenetsky and Salnikov [13] admits a stable limit cycle.

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Asymptotic behavior of solutions of quasilinear differential-algebraic equations

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Abstract. This paper is concerned with the asymptotic behavior of solutions of linear differential-algebraic equations (DAEs) under small nonlinear perturbations. Some results on the asymptotic behavior of solutions which are well known for ordinary differential equations are extended to DAEs. The main tools are the projector-based decoupling and the contractive mapping principle. Under certain assumptions on the linear part and the nonlinear term, asymptotic behavior of solutions are characterized. As the main result, a Perron type theorem that establishes the exponential growth rate of solutions is formulated.

Keywords: quasilinear differential-algebraic equation, asymptotic behavior, index, projector, contractive mapping.

2020 Mathematics Subject Classification: 34A09, 34D05, 34D08, 34E10.

1 Introduction

Qualitative theory and numerical analysis of differential-algebraic equations (DAEs) have been extensively studied since the 80's, see for example the monographs [8,9,11] and the references therein. It is well known that DAEs play an important role in mathematical modeling and arise in many real-life applications such as multibody mechanics, electronic circuit design, chemical engineering, etc, see [4,9,10]. Since the derivative cannot be solved explicitly, DAEs are also called singular (or generalized) systems of differential equations. DAEs are generalizations of ordinary differential equations (ODEs) whose qualitative theory is well known, see [6,7]. Roughly speaking, DAEs are mixed systems of implicit differential and algebraic equations, which may involve hidden constraints as well. The facts that the systems are coupled and the dynamics is constrained makes the analysis and numerical treatment of DAEs more complicated. Even the existence and uniqueness of solutions for linear DAEs can be established only under extra restrictive assumptions. Furthermore, solutions of DAEs may be very sensitive to changes in the system data.

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In this paper, we study the asymptotic behavior of solutions of linear DAEs under nonlinear perturbations

$$Ex'(t) = Ax(t) + f(t, x(t), x'(t)), \qquad t \in \mathbb{I} = [0, \infty), \tag{1.1}$$

where $E, A \in \mathbb{C}^{n\times n}$, $f : \mathbb{I} \times \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ is continuous, and E is assumed to be singular. The question is that if a nonlinear perturbation f is supposed to be sufficiently small in some sense, how certain solutions of the quasilinear DAE (1.1) behave asymptotically comparing to those of the unperturbed linear DAE as t tends to infinity. In [12], asymptotic integration of solutions of linear DAEs with coefficients subjected to linear time-varying perturbations was studied. If the perturbations are small enough in some sense, then the exponential growth rate of solutions is established. It is known that the exponential growth rate of solutions of linear systems characterized by Lyapunov exponents plays a very important role in the qualitative study of dynamical systems, see [1]. Characterizations of Lyapunov exponents were extended from linear time-varying ODEs to linear time-varying DAEs in [13–15]. In particular, the stability of Lyapunov exponents is investigated when the coefficients are subjected to structured perturbations in [14, 15]. For some other remarkable results on the asymptotics and stability of solutions for DAEs, see [3, 5, 16, 18].

Our aim is to extend some classical results which are well known for quasilinear ordinary differential equations [6, 7] to quasilinear DAEs. One of the most important results for quasilinear systems is the Perron type theorem which was established for ODEs a long time ago, see [7, Theorem 5, p. 97]. Recently, extensions of this result to functional differential equations [17] and nonautonomous ODEs [2] were done. Unlike the approach in [12], in order to characterize the asymptotic behavior of solutions of (1.1), in this work we use the projector-based approach. Conditions for the pencil (E, A) and perturbation f are given so that the asymptotic behavior of solutions of (1.1) is shown to be related to those of the corresponding linear DAE. The paper is organized as follows. In the next section, we briefly introduce the projector-based analysis of linear DAEs and recall some classical results for quasilinear ODEs. In Section 3, the existence and uniqueness of solutions for the initial value problem for DAE (1.1) are established. A simple example is also given for illustrating the feasibility of the assumptions. Then, in Section 4, the asymptotic behavior of solutions is characterized under certain assumptions. As the main result, a Perron type theorem that establishes the exponential growth rate of solutions is formulated. A discussion and some open questions will close the paper.

2 Preliminaries

2.1 Projector-based analysis for linear DAEs

Consider the linear time-invariant homogeneous DAEs of the form

$$Ex'(t) = Ax(t), \qquad t \in \mathbb{I}, \tag{2.1}$$

where $E, A \in \mathbb{C}^{n\times n}$, E is singular and $x : \mathbb{I} \to \mathbb{C}^n$. As in the classical theory of ODEs, the search for solutions of (2.1) having the form $e^{\lambda t}x_0$ naturally leads to the generalized eigenvalue problem defined by $\det(\lambda E - A) = 0$, and therefore drives the analysis of homogeneous linear time-invariant DAEs to the theory of matrix pencils, see [8–10], where the Kronecker index is used for the analysis of DAEs (2.1).

The matrix pencil $\{E, A\}$ is said to be *regular* if there exists $\lambda \in \mathbb{C}$ such that the determinant $\det(\lambda E - A)$ is nonzero. Otherwise, if $\det(\lambda E - A) = 0$ for all $\lambda \in C$, then we say that $\{E, A\}$ is irregular or non-regular. If $\{E, A\}$ is regular, then $\lambda \in C$ is a (generalized finite) eigenvalue of $\{E, A\}$ and a nonzero vector ζ is the associated eigenvector if $\lambda E\zeta = A\zeta$. It is known that the system (2.1) is solvable if and only if the matrix pencil $\{E, A\}$ is regular [4,8,9]. The following theorem is known as the Kronecker–Weierstraß canonical form, which plays an important role in the analysis of linear constant-coefficient DAEs.

Theorem 2.1. *Suppose that $\{E, A\}$ is a regular pencil. Then, there exist nonsingular matrices G and H such that*

$$GEH = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, \qquad GAH = \begin{bmatrix} J_{n_1} & 0 \\ 0 & I_{n_2} \end{bmatrix}, \tag{2.2}$$

where $n_1 + n_2 = n$, J_{n_1} is a $n_1 \times n_1$ matrix and N is a matrix of nilpotency index k, i.e. $N^k = 0$, but $N^{k-1} \neq 0$. If N is a zero matrix, then we define $k = 1$.

The *Kronecker index* of the pencil $\{E, A\}$ is defined by the nilpotency index of the matrix N in (2.2).

Now, we suppose that the matrix pencil $\{E, A\}$ is of index one (in the Kronecker sense) and $\operatorname{rank} E = d < n$. Let Q be any projector onto $\ker E$. Then, we have the following result, which is also presented as the definition of tractability index one, see [8].

Proposition 2.2. *Let $E \in \mathbb{C}^{n\times n}$ be a singular matrix, and Q be an arbitrary projector onto* $\ker E$*. Then, the matrix pencil $\{E, A\}$ is regular with Kronecker index one if and only if the matrix $E_1 = E - AQ$ is non-singular.*

Let us define $P = I - Q$, which is a projector, too. It is easy to show that

$$E_1^{-1}E = P, \qquad E_1^{-1}AQ = -Q.$$

Multiplying (2.1) with PE_1^{-1}, QE_1^{-1} and using the relation $x = Px + Qx$, we obtain

$$\begin{aligned} (Px)' &= PE_1^{-1}APx, \\ 0 &= QE_1^{-1}APx - Qx. \end{aligned} \tag{2.3}$$

Denoting $u = Px$ and $v = Qx$, then the first equation of the system (2.3) can be rewritten as

$$u' = PE_1^{-1}Au. \tag{2.4}$$

In addition, we can rewrite the second equation of the system (2.3) as

$$v = QE_1^{-1}Au. \tag{2.5}$$

Thus, equations (2.4) and (2.5) yield a decoupling of the DAE (2.1) in terms of the differential component u and the algebraic one v. The equation (2.4) is called an inherent ODE for the DAE (2.1). The linear subspace $\operatorname{im} P$ is invariant with respect to this equation. Indeed, an initial condition $u_0 \in \operatorname{im} P$ implies $Qu_0 = 0$. Since $(Qu)' = Qu' = QPE_1^{-1}Au = 0$, we obtain $Qu(t) = 0$ i.e., $u(t) \in \operatorname{im} P$ for all t. Solutions of the DAE (2.1) will be described in terms of solutions u of (2.4) lying in the invariant subspace $\operatorname{im} P$. A projector onto $\ker E$ along $S = \{x \in \mathbb{C}^n \,|\, Ax \in \operatorname{im} E\}$ is called the canonical projector. For index one DAE (2.4), specially, if we choose Q being the canonical projector, then we have $v = 0$, see [8,10].

Using $E = EP$, let us reformulate (2.1) as

$$E(Px)'(t) = Ax(t). \tag{2.6}$$

This makes sense to look for solutions defined in the space

$$C^1_P(\mathbb{I}, \mathbb{C}^n) = \{x \in C(\mathbb{I}, \mathbb{C}^n) \mid Px \in C^1(\mathbb{I}, \mathbb{C}^n)\} \supset C^1(\mathbb{I}, \mathbb{C}^n),$$

where $C^1(\mathbb{I}, \mathbb{C}^n)$ denotes the space of continuously differentiable functions defined on $\mathbb{I}$. Therefore, $x(\cdot) \in C^1_P(\mathbb{I}, \mathbb{C}^n)$ is a solution of (2.1) if and only if it can be written as

$$x(t) = u(t) + v(t), \tag{2.7}$$

where $u \in C^1(\mathbb{I}, \mathbb{C}^n)$ is a solution of (2.4) in the invariant space $\operatorname{im} P$ and $v \in C(\mathbb{I}, \mathbb{C}^n)$ given by (2.5), see [8, 11]. It is sufficient to assign an initial condition to the differential component u. The initial value for the algebraic component v follows from the algebraic constraint, namely $v(0) = QE_1^{-1}Au(0)$.

Now, let us construct a fundamental solution for DAE (2.1) as follows. In space $\operatorname{im} P$, we consider an orthogonal basis $u_1, u_2, \ldots, u_d$. Clearly, $U = (u_1, u_2, \ldots, u_d)$ is an $n \times d$-size matrix and $U^T U = I_d$.

Suppose that $Y(t)$ is a fundamental matrix of the inherent ODE (2.4) restricted on $\operatorname{im} P$, which is defined by the solution of the matrix-valued IVP

$$\begin{aligned} Y'(t) &= PE_1^{-1}AY(t), \\ Y(0) &= U. \end{aligned} \tag{2.8}$$

It is easy to verify that $Y(t) = e^{tPE_1^{-1}A}U$ and the columns of $Y(t)$ are linearly independent solutions of the equation (2.4) restricted on $\operatorname{im} P$. Then, we define a fundamental matrix X of the DAE (2.1) by

$$X(t) = (I + QE_1^{-1}A)Y(t). \tag{2.9}$$

We note that we can also obtain another important associated ODE, the so-called essentially underlying ODE as follows. Let us introduce the change of variables $u(t) = Uw(t)$. Then, w satisfies

$$w'(t) = U^T PE_1^{-1}AUw(t). \tag{2.10}$$

It is also easy to verify that $Z(t) = U^T Y(t)$ is the normalized fundamental solution of the EUODE (2.10) and we also have the representation $Y(t) = UZ(t)$.

The following lemma, which is an extension of Lemma 3.1 [18], characterizes the spectra of the eigenvalue problems associated with the DAE (2.1), the inherent ODE (2.4), and the essentially underlying ODE (2.10).

Lemma 2.3. *Let a regular index-1 pencil $\{E, A\}$ be given and Q denotes an arbitrary projector onto* $\ker E$. *Further,* $M := PE_1^{-1}A$, $N := U^T PE_1^{-1}AU$, $d :=$ rank $E = n - \dim(\ker E)$. *Then* $\deg(\det(\lambda E - A)) = d$, *i.e.,* $\{E, A\}$ *has d finite eigenvalues, say* $\lambda_1, \ldots, \lambda_d$. *Moreover,* $\lambda_1, \ldots, \lambda_d$ *belong also to the spectrum of M and they are exactly the same as the eigenvalues of N. The remaining eigenvalues of M are zero.*

If Q is chosen being the canonical projector, then the eigenvectors associated with the finite eigenvalues belong to $\operatorname{im} P$ and the eigenvectors associated with the other zero eigenvalues of M span $\ker P$, see [18, Lemma 3.1].

Proof. Let λ_k be an arbitrary finite eigenvalue of matrix pencil $\{E, A\}$ and ξ_k be an associated eigenvector. From the equality $\lambda_k E\xi_k = A\xi_k$ with $\xi_k = P\xi_k + Q\xi_k$, it is easy to see that

$$\lambda_k P\xi_k = PE_1^{-1}AP\xi_k,$$
$$QE_1^{-1}AP\xi_k = Q\xi_k.$$

This means that ξ_k is an eigenvector corresponding to the eigenvalue λ_k of the pencil $\{E, A\}$ if and only if $P\xi_k$ is an eigenvector corresponding to the eigenvalue λ_k of the matrix $PE_1^{-1}A$. Furthermore, let us define the vector ζ_k by $P\xi_k = U\zeta_k$. Then, we obtain $\lambda_k U\zeta_k = PE_1^{-1}APU\zeta_k$. It follows that $\lambda_k\zeta_k = U^T PE_1^{-1}APU\zeta_k$. This means that λ_k is an eigenvalue and ζ_k is a corresponding eigenvector of N. □

Remark 2.4. It is quite obvious to see that all the solutions of DAE (2.1) are bounded if and only if all the solutions of the inherent ODE (2.4) (and also those of the essential underlying ODE (2.10)) are so. It is also well known that this happens if and only if all the finite eigenvalues of pencil $\{E, A\}$ have non-positive real parts and any eigenvalue with zero real part must be semi-simple.

2.2 Preliminary results for quasilinear ODEs

Consider a special case of (2.1), namely the case of well-known quasilinear ODE

$$x'(t) = Ax(t) + h(t, x), \tag{2.11}$$

i.e., $E = I$, $f(t, x, y) \equiv h(t, x)$, where I is the identity matrix. According to the stability theory of ODEs, if the spectrum $\sigma\{I, A\}$ belong to $\mathbb{C}^-$ and the nonlinear term is sufficiently small in some sense, then the trivial solution is asymptotically stable in Lyapunov sense. This result was extended to DAEs in [16].

Next, we recall some other well-known results on the asymptotic behavior of solutions in the theory of ODEs, see [6,7].

Proposition 2.5 ([6, Problem 1, p. 344])**.** *Let all solutions of the linear system with constant coefficients $y' = Ay$ be bounded for $t \geq 0$, that is, let $\|e^{tA}\| \leq M$, $t \geq 0$, for some constant M. Let h be continuous and let there exist a constant k and a function $\alpha(t)$ such that*

$$\|h(t, x)\| \leq \alpha(t)\|x\| \quad \text{for } \|x\| \leq k \text{ and } t \geq 0, \tag{2.12}$$

and let

$$\int_0^\infty \alpha(t)dt < \infty. \tag{2.13}$$

Then, there exists a constant M_1 such that any solution x of the system (2.11) satisfies

$$\|x(t)\| < M_1\|x(0)\| \quad \text{if } \|x(0)\| \leq \frac{k}{M_1}.$$

Proposition 2.6 ([6, Problem 2, p. 345])**.** *Let the assumptions of Proposition 2.5 be satisfied. It is clear that $e^{tA} = X_1(t) + X_2(t)$, where $X_1(t)$ contains elements which are sums of exponential terms $e^{i\lambda_j t}$ for real λ_j and*

$$\|X_1(t)\| \leq K_1, \quad -\infty < t < \infty, \text{ and}$$
$$\|X_2(t)\| \leq K_2 e^{-\sigma t}, \quad 0 \leq t < \infty$$

for some positive constants $\sigma > 0$, K_1 and K_2.

Then, corresponding to any solution x of (2.11), there is a constant vector p such that

$$x(t) - X_1(t)p \to 0 \quad \text{as } t \to \infty.$$

Theorem 2.7 ([7, Theorem 5, p. 97]). *Suppose that $x(t)$ is a bounded solution of (2.11) and*

$$\|h(t, x(t))\| \leq \alpha(t)\|x(t)\|, \tag{2.14}$$

for $t \geq 0$, where $\alpha : [0, \infty) \longrightarrow [0, \infty)$ is a continuous nonnegative function satisfying

$$\int_t^{t+1} \alpha(s)ds \to 0 \quad \text{as } t \to \infty, \tag{2.15}$$

then either $x(t) = 0$ for all large t or

$$\mu = \lim_{t\to\infty} \frac{\log \|x(t)\|}{t} \tag{2.16}$$

exists and is equal to the real part of one of the eigenvalues of the matrix A.

Obviously, if $\alpha(t) \to 0$ as $t \to \infty$, or $\int_\sigma^\infty \alpha^p(s)ds < \infty$ for some $p \in [1, \infty)$, then the condition (2.15) holds. This theorem is known as a Perron type theorem for ODEs.

3 Existence and uniqueness of solutions for quasilinear DAEs

Throughout the remainder of the paper, we consider the quasilinear DAE (1.1) where it is assumed that the matrix pencil $\{E, A\}$ is regular of index-1, f is continuous, and the Jacobian $f_y(t, x, y)$ exists.

Clearly, (1.1) generalizes the well-understood case of ODEs (2.11). Now, we focus on the case of singular E, i.e. the equation (1.1) is a DAE. To make sure that the nonlinear perturbation in (1.1) plays a proper role, we need a technical assumption

$$\ker E \subseteq \ker f_y(t, x, y), \qquad (t, x, y) \in \mathbb{I} \times \mathbb{C}^n \times \mathbb{C}^n. \tag{3.1}$$

It is shown, e.g., in [8], that (3.1) is sufficient for implying the identity

$$f(t, x, y) = f(t, x, Py). \tag{3.2}$$

This suggests a more proper reformulation of equation (1.1) as follows

$$E(Px)'(t) = Ax(t) + f(t, x(t), (Px)'(t)). \tag{3.3}$$

This is just a special case of DAEs with properly stated derivative discussed in [10]. We look for solutions of (3.3) that belong to the class $C_P^1(\mathbb{I}, \mathbb{C}^m)$. It is worth mentioning that this class is independent of the choice of projector P, see [8].

First, we establish the (local) existence and uniqueness of solutions of IVPs for (1.1).

Theorem 3.1. *Let pencil $\{E, A\}$ be of index-1 and let f satisfy*

$$\|PE_1^{-1}f(t, x, y) - PE_1^{-1}f(t, \bar{x}, \bar{y})\| \leq \alpha_1(t)\|x - \bar{x}\| + \beta_1(t)\|y - \bar{y}\|, \tag{3.4}$$

$$\|QE_1^{-1}f(t, x, y) - QE_1^{-1}f(t, \bar{x}, \bar{y})\| \leq \alpha_2(t)\|x - \bar{x}\| + \beta_2(t)\|y - \bar{y}\|, \tag{3.5}$$

for all $t \geq 0$ and $x, \bar{x}, y, \bar{y} \in \mathbb{C}^n$, $\alpha_i(t)$ and $\beta_i(t)$ are non-negative bounded functions ($i = 1, 2$ and $t \geq 0$) such that $\sup_{t\in[0,\infty)} \alpha_2(t) < 1$ and $\sup_{t\in[0,\infty)} \gamma(t) < 1$, where $\gamma(t) = \frac{\alpha_1(t)\beta_2(t)}{1-\alpha_2(t)} + \beta_1(t)$. Then, for any $t_1 \geq 0$ and $x^1 \in \mathbb{C}^n$, there exists a positive a such that the IVP for equation (1.1) with initial condition $P(x(t_1) - x^1) = 0$ has a unique solution defined on $[t_1, t_1 + a)$.

Proof. Since $\{E, A\}$ is regular of index-1, we have $QE_1^{-1}A = -Q, E_1^{-1}E = P$ and $PE_1^{-1}A = PE_1^{-1}AP$. Multiplying both sides of (1.1) with PE_1^{-1}, QE_1^{-1} respectively, using the relation $x = Px + Qx$ and noting $\operatorname{im} Q = \ker E \subset \ker f_y(t, x, y)$ for all $t \in \mathbb{I}$, we obtain

$$\begin{aligned}(Px)' &= PE_1^{-1}A(Px) + PE_1^{-1}f(t, Px + Qx, (Px)'),\\ Qx &= QE_1^{-1}A(Px) + QE_1^{-1}f(t, Px + Qx, (Px)').\end{aligned}$$

Denoting again $u = Px$ and $v = Qx$, then the system can be rewritten as

$$u' = PE_1^{-1}Au + PE_1^{-1}f(t, u + v, u'), \tag{3.6}$$

$$v = QE_1^{-1}Au + QE_1^{-1}f(t, u + v, u'). \tag{3.7}$$

Using on the second equation of (3.7), we will try to represent v by u and u'. Put $F(t, u, u', v) = QE_1^{-1}Au + QE_1^{-1}f(t, u + v, u')$. Due to (3.5), we have

$$\|F(t, x, y, z_1) - F(t, x, y, z_2)\| \le \alpha_2(t)\|z_1 - z_2\|, \quad \text{for all } x,\ y,\ z_1,\ z_2 \in \mathbb{C}^n,\ t \in \mathbb{I}.$$

Since $\sup_{t\in\mathbb{I}} \alpha_2(t) < 1$, $F(t, x, y, z)$ defined as above is a contractive mapping with respect to variable z. Applying the contractive mapping principle, there exists a function $\psi(t, x, y)$ such that $z = \psi(t, x, y)$, i.e.,

$$\psi(t, u, u') = QE_1^{-1}Au + QE_1^{-1}f(t, u + \psi(t, u, u'), u').$$

We can see that ψ is invariant under projector Q, i.e., $Q\psi(t, u, u') = \psi(t, u, u')$. Due to (3.5), we have

$$\begin{aligned}&\|\psi(t, x_1, y_1) - \psi(t, x_2, y_2)\|\\ &\quad\le \|QE_1^{-1}A(x_1 - x_2)\| + \|QE_1^{-1}f(t, x_1 + \psi(t, x_1, y_1), y_1) - QE_1^{-1}f(t, x_2 + \psi(t, x_2, y_2), y_2)\|,\\ &\quad\le C_1\|x_1 - x_2\| + \alpha_2(t)\|x_1 - x_2\| + \alpha_2(t)\|\psi(t, x_1, y_1) - \psi(t, x_2, y_2)\| + \beta_2(t)\|y_1 - y_2\|,\end{aligned}$$

for all $x_1,\ x_2,\ y_1,\ y_2 \in \mathbb{C}^n,\ t \in \mathbb{I}$, where $C_1 = \|QE_1^{-1}A\|$.

Hence, we get

$$\|\psi(t, x_1, y_1) - \psi(t, x_2, y_2)\| \le \frac{C_1 + \alpha_2(t)}{1 - \alpha_2(t)}\|x_1 - x_2\| + \frac{\beta_2(t)}{1 - \alpha_2(t)}\|y_1 - y_2\|. \tag{3.8}$$

Replacing $v = \psi(t, u, u')$ in (3.6), we obtain

$$u' = PE_1^{-1}Au + PE_1^{-1}f(t, u + \psi(t, u, u'), u').$$

Put $K(t, u, u') = PE_1^{-1}Au + PE_1^{-1}f(t, u + \psi(t, u, u'), u')$. We will show that $K(t, x, y)$ is a contractive mapping with respect to variable y. Indeed, for $x,\ y_1,\ y_2 \in \mathbb{C}^n,\ t \in \mathbb{I}$, we have

$$\begin{aligned}\|K(t, x, y_1) - K(t, x, y_2)\| &= \|PE_1^{-1}f(t, x + \psi(t, x, y_1), y_1) - PE_1^{-1}f(t, x + \psi(t, x, y_2), y_2)\|\\ &\le \alpha_1(t)\|\psi(t, x, y_1) - \psi(t, x, y_2)\| + \beta_1(t)\|y_1 - y_2\|\\ &\le \frac{\alpha_1(t)\beta_2(t)}{1 - \alpha_2(t)}\|y_1 - y_2\| + \beta_1(t)\|y_1 - y_2\| \quad \text{(by (3.8))}\\ &\le \Big(\frac{\alpha_1(t)\beta_2(t)}{1 - \alpha_2(t)} + \beta_1(t)\Big)\|y_1 - y_2\| = \gamma(t)\|y_1 - y_2\|.\end{aligned}$$

Since $\sup_{t\in\mathbb{I}}\gamma(t) < 1$, it follows that $K(t,x,y)$ is a contraction with respect to variable y. Applying the contractive mapping principle, there exists a function $g(t,x)$ such that $y = g(t,x)$, i.e.,

$$g(t,u) = PE_1^{-1}Au + PE_1^{-1}f(t,u+\psi(t,u,g(t,u)),g(t,u)).$$

Obviously, g is invariant under projector P, i.e., $Pg(t,u) = g(t,u)$. By (3.4) and (3.8), for $u_1,\ u_2 \in \mathbb{C}^n,\ t\in\mathbb{I}$, we have

$$\begin{aligned}
&\|g(t,u_1)-g(t,u_2)\| \\
&\quad\le \|PE_1^{-1}A\|\|u_1-u_2\| + \alpha_1(t)\|u_1-u_2\| + \alpha_1(t)\|\psi(t,u_1,g(t,u_1)) - \psi(t,u_2,g(t,u_2))\| \\
&\qquad + \beta_1(t)\|g(t,u_1)-g(t,u_2)\| \\
&\quad\le (C_2+\alpha_1(t))\|u_1-u_2\| + \frac{\alpha_1(t)(C_1+\alpha_2(t))}{1-\alpha_2(t)}\|u_1-u_2\| \\
&\qquad + \frac{\alpha_1(t)\beta_2(t)}{1-\alpha_2(t)}\|g(t,u_1)-g(t,u_2)\| + \beta_1(t)\|g(t,u_1)-g(t,u_2)\| \quad (\text{put } C_2 = \|PE_1^{-1}A\|) \\
&\quad\le \left(C_2 + \frac{(C_1+1)\alpha_1(t)}{1-\alpha_2(t)}\right)\|u_1-u_2\| + \gamma(t)\|g(t,u_1)-g(t,u_2)\|.
\end{aligned}$$

Thus, we have

$$(1-\gamma(t))\|g(t,u_1)-g(t,u_2)\| \le \left(C_2 + \frac{(C_1+1)\alpha_1(t)}{1-\alpha_2(t)}\right)\|u_1-u_2\|,$$

i.e.,

$$\|g(t,u_1)-g(t,u_2)\| \le \frac{1}{1-\gamma(t)}\left(C_2 + \frac{(C_1+1)\alpha_1(t)}{1-\alpha_2(t)}\right)\|u_1-u_2\|. \tag{3.9}$$

Put $\widetilde{g}(t,u) = g(t,u) - PE_1^{-1}Au$. Then $\widetilde{g}$ is also invariant under projector P, i.e., $P\widetilde{g}(t,u) = \widetilde{g}(t,u)$. We have

$$\begin{aligned}
\|\widetilde{g}(t,u_1)-\widetilde{g}(t,u_2)\| &\le \alpha_1(t)\|u_1-u_2\| + \alpha_1(t)\|\psi(t,u_1,g(t,u_1)) - \psi(t,u_2,g(t,u_2))\| \\
&\quad + \beta_1(t)\|g(t,u_1)-g(t,u_2)\| \\
&\le \alpha_1(t)\|u_1-u_2\| + \frac{\alpha_1(t)(C_1+\alpha_2(t))}{1-\alpha_2(t)}\|u_1-u_2\| \\
&\quad + \frac{\alpha_1(t)\beta_2(t)}{1-\alpha_2(t)}\|g(t,u_1)-g(t,u_2)\| + \beta_1(t)\|g(t,u_1)-g(t,u_2)\| \\
&\le \frac{(C_1+1)\alpha_1(t)}{1-\alpha_2(t)}\|u_1-u_2\| + \frac{\gamma(t)}{1-\gamma(t)}\left(C_2 + \frac{(C_1+1)\alpha_1(t)}{1-\alpha_2(t)}\right)\|u_1-u_2\| \\
&\le \left[\frac{(C_1+1)\alpha_1(t)}{(1-\alpha_2(t))(1-\gamma(t))} + \frac{C_2\gamma(t)}{1-\gamma(t)}\right]\|u_1-u_2\|.
\end{aligned}$$

Thus, we get

$$\|\widetilde{g}(t,u_1)-\widetilde{g}(t,u_2)\| \le \widetilde{\gamma}(t)\|u_1-u_2\|, \tag{3.10}$$

where

$$\widetilde{\gamma}(t) = \frac{(C_1+1)\alpha_1(t)}{(1-\alpha_2(t))(1-\gamma(t))} + \frac{C_2\gamma(t)}{1-\gamma(t)}. \tag{3.11}$$

On the other hand, $\alpha_1(t)$ is bounded, $\sup_{t\in[0,\infty)}\alpha_2(t) < 1$ and $\sup_{t\in[0,\infty)}\gamma(t) < 1$, then there exists a positive constant L such that $\|\widetilde{g}(t,u_1)-\widetilde{g}(t,u_2)\| \le L\|u_1-u_2\|$, for all $u_1,u_2\in\mathbb{C}^n$, i.e., $\widetilde{g}(t,u)$ is Lipschitz continuous with respect to u.

We conclude that system (1.1) can be reduced to the decoupled form

$$u' = PE_1^{-1}Au + \widetilde{g}(t,u), \tag{3.12}$$
$$v = \psi(t,u,u'). \tag{3.13}$$

and initial condition $P(x(t_1) - x^1) = 0$ is equivalent to $u(t_1) = Px^1$. Since $\widetilde{g}(t,u)$ is a Lipschitz continuous function for u, the IVP for equation (3.12) with initial condition $u(t_1) = Px^1$ has a unique solution $u(t)$ defined on $[t_1, t_1 + a)$ for some positive number a and this solution satisfies $Pu(t) = u(t)$. Then, we obtain $v(t)$ from (3.13). Hence, the unique solution $x(t)$ is defined by $x(t) = u(t) + v(t)$ for all $t \in [t_1, t_1 + a)$. The proof is complete. □

We present a simple example that illustrates the feasibility of the conditions given in Theorem 3.1.

Example 3.2. We consider the equation

$$Ex' = Ax + f(t,x,x') \tag{3.14}$$

with

$$E = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 0 \\ 1 & 2 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}.$$

The nonlinear part $f = (f_1, f_2, f_3)^\top$ will be specified later. Let us choose

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is seen that Q is a projection onto $\ker E$ and the matrix pencil $\{E, A\}$ is of index-1. Furthermore, we have

$$PE_1^{-1} = \begin{bmatrix} 4 & -2 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad QE_1^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

Let us define f such that

$$PE_1^{-1}f(t,x,y) = \begin{bmatrix} a_1(t)\sin(x_2) + b_1(t)\cos(y_1) \\ a_2(t)\sin(x_3) + a_3(t)\cos(x_2) \\ 0 \end{bmatrix}$$

and

$$QE_1^{-1}f(t,x,y) = \begin{bmatrix} 0 \\ 0 \\ -a_4(t)\cos(x_3) - b_2(t)\cos(y_2) \end{bmatrix}.$$

Choose $a_i(t) = \frac{\delta_i}{(1+t)^2}$, $(i = 1,2,3,4)$, $b_1(t) = \frac{\varepsilon_1}{(1+t)^2}$, and $b_2(t) = \frac{1-\delta_4}{(1+t)^2}$, where δ_i and ε_1 are positive constants such that $\delta_4 < 1$.

Using the maximum norm, for any $t \geq 0$ and any x, $\bar{x}$, y, $\bar{y} \in \mathbb{R}^3$, we have

$$\begin{aligned} \|PE_1^{-1}f(t,x,y) - PE_1^{-1}f(t,\bar{x},\bar{y})\| \leq & \max\{a_1(t)|\sin(x_2) - \sin(\bar{x}_2)| + b_1(t)|\cos(y_1) - \cos(\bar{y}_1)|, \\ & a_2(t)|\sin(x_3) - \sin(\bar{x}_3)| + a_3(t)|\cos(x_2) - \cos(\bar{x}_2)|\} \\ \leq & (a_1(t) + a_2(t) + a_3(t))\|x - \bar{x}\| + b_1(t)\|y - \bar{y}\| \end{aligned}$$

and

$$\|QE_1^{-1}f(t,x,y) - QE_1^{-1}f(t,\bar{x},\bar{y})\| \le a_4(t)|\cos(x_3) - \cos(\bar{x}_3)| + b_2(t)|\cos(y_2) - \cos(\bar{y}_2)| \\ \le a_4(t)\|x-\bar{x}\| + b_2(t)\|y-\bar{y}\|.$$

We put

$$\alpha_1(t) = \frac{\delta_1+\delta_2+\delta_3}{(1+t)^2}, \qquad \beta_1(t) = \frac{\varepsilon_1}{(1+t)^2}, \qquad \alpha_2(t) = \frac{\delta_4}{(1+t)^2}, \qquad \beta_2 = \frac{1-\delta_4}{(1+t)^2}.$$

It is trivial to see that, for t in $[0,\infty)$, the following estimates hold:

$$0 < \alpha_1(t) \le \delta_1+\delta_2+\delta_3, \qquad 0 < \alpha_2(t) \le \delta_4,$$
$$0 < \beta_1(t) \le \varepsilon_1, \qquad 0 < \beta_2(t) \le 1-\delta_4,$$
$$0 < \gamma(t) = \frac{\alpha_1(t)\beta_2(t)}{1-\alpha_2(t)} + \beta_1(t) < \delta_1+\delta_2+\delta_3+\varepsilon_1.$$

Therefore, if $\delta_4 < 1$ and $\delta_1+\delta_2+\delta_3+\varepsilon_1 < 1$ simultaneously hold, then all the conditions in Theorem 3.1 are satisfied. We conclude that for any $t_1 \in [0,\infty)$ and $x^1 \in \mathbb{R}^3$, the IVP for equation (3.14) with initial condition $P(x(t_1)-x^1) = 0$ has a unique solution defined in $[t_1, t_1+a)$ with some positive number a.

4 Asymptotic behavior of solutions for quasilinear DAEs

In this section, we extend the results in Section 2.2 to quasilinear DAEs of the form (1.1).

Theorem 4.1. *Let pencil $\{E, A\}$ be regular of index-1 and let f satisfy all the conditions in Theorem 3.1 with*

$$\int_0^\infty \alpha_1(t)dt < \infty, \qquad \int_0^\infty \beta_1(t)dt < \infty, \tag{4.1}$$

and $f(t,0,0) \equiv 0$. Let all the solutions of the linear DAE (2.1) be bounded for $t \ge 0$, i.e., there exists a positive constant M such that the fundamental matrix $X(t)$ of (2.1) satisfies $\|X(t)\| \le M$ for all $t \ge 0$. Then, there exists positive constant M_1 such that any solution $x = x(t)$ of the system (1.1) satisfies $\|x(t)\| \le M_1\|x(0)\|$ for all $t \ge 0$.

Proof. In space $\operatorname{im} P$, let us consider again an orthogonal basis $u_1, u_2, \ldots, u_d$. We denote $U = (u_1, u_2, \ldots, u_d)$ which is a $n \times d$-size matrix and $U^TU = I_d$. Using the change of variables $u(t) = Uw(t)$, it is easy to see that from the equation (3.12) we obtain the EUODE

$$w'(t) = U^TPE_1^{-1}AUw(t) + U^T\widetilde{g}(t, Uw(t)). \tag{4.2}$$

Due to Lemma 2.3, the spectra of pencil $\{E, A\}$ and of $N = U^TPE_1^{-1}AU$ coincide. Put $\overline{g}(t,w(t)) = U^T\widetilde{g}(t,Uw(t))$. Clearly, the equation (4.2) is an ODE for $w(t)$. From (3.10), together with (4.1), $\alpha_1(t)$ and $\beta_2(t)$ are bounded, $\sup_{t\in[0,\infty)}\alpha_2(t) < 1$ and $\sup_{t\in[0,\infty)}\gamma(t) < 1$, $\widetilde{g}(t,0) = 0$, we have

$$\|\widetilde{g}(t,u)\| \le \widetilde{\gamma}(t)\|u\|,$$

where $\widetilde{\gamma}(t)$ defined in (3.11) satisfies $\int_0^\infty \widetilde{\gamma}(t)dt < \infty$. Without loss of generality let us use the Euclidean norm, due to the definition of U and the properties that $u(t) = Pu(t)$, $\widetilde{g}(t,u) = P\widetilde{g}(t,u)$, we obtain $\|u(t)\| = \|w(t)\|$ and

$$\|\overline{g}(t,w(t))\| = \|U^T\widetilde{g}(t,Uw(t))\| = \|\widetilde{g}(t,Uw(t))\| \le \widetilde{\gamma}(t)\,\|Uw(t)\| = \widetilde{\gamma}(t)\,\|w(t)\|.$$

Thus, we get

$$\|\overline{g}(t, w(t))\| \leq \widetilde{\gamma}(t)\|w(t)\|, \quad \text{for all } t \geq 0. \tag{4.3}$$

Due to the properties of $\overline{g}$ and $\widetilde{\gamma}$, by Proposition 2.5, one concludes that if the equation (4.2) has a solution $w(t)$ defined for all $t \in [0, \infty)$, there exists a constant $\widetilde{M}$ such that

$$\|w(t)\| < \widetilde{M}\|w(0)\|.$$

Then, $u(t) = Uw(t)$ is a solution of the equation (3.12) such that

$$\|u(t)\| = \|w(t)\| < \widetilde{M}\|w(0)\| = \widetilde{M}\|u(0)\|.$$

The equation (1.1) has the solution of the form $x(t) = u(t) + v(t)$. Therefore, we obtain $x(t) = u(t) + \psi(t, u(t), g(t, u(t)))$. Note that, since $f(t, 0, 0) \equiv 0$, it is not difficult to show that $\psi(t, 0, 0) \equiv 0$, $g(t, 0) \equiv 0$ and $\widetilde{g}(t, 0) \equiv 0$. Then, we have

$$\begin{aligned}
\|x(t)\| &\leq \|u(t)\| + \|\psi(t, u(t), g(t, u(t)))\| \\
&\leq \|u(t)\| + \frac{(C_1 + \alpha_2(t))}{1 - \alpha_2(t)}\|u(t)\| + \frac{\beta_2(t)}{1 - \alpha_2(t)}\|g(t, u(t))\| \quad \text{(by (3.8))} \\
&\leq \frac{C_1 + 1}{1 - \alpha_2(t)}\|u(t)\| + \frac{\beta_2(t)}{(1 - \alpha_2(t))(1 - \gamma(t))}\left(C_2 + \frac{(C_1 + 1)\alpha_1(t)}{1 - \alpha_2(t)}\right)\|u(t)\| \quad \text{(by (3.9))} \\
&\leq \frac{1}{1 - \alpha_2(t)}\left[C_1 + 1 + \frac{\beta_2(t)}{1 - \gamma(t)}\left(C_2 + \frac{(C_1 + 1)\alpha_1(t)}{1 - \alpha_2(t)}\right)\right]\|u(t)\|.
\end{aligned}$$

In addition, $\alpha_1(t)$ and $\beta_2(t)$ are bounded, $\sup_{t \in [0, \infty)} \alpha_2(t) < 1$ and $\sup_{t \in [0, \infty)} \gamma(t) < 1$, there exists a constant $\bar{K}$ such that

$$\|x(t)\| \leq \bar{K}\|u(t)\|. \tag{4.4}$$

Therefore, we obtain

$$\|x(t)\| \leq \bar{K}\|u(t)\| \leq \bar{K}\widetilde{M}\|u(0)\| \leq \bar{K}\widetilde{M}\|Px(0)\| \leq \bar{K}\widetilde{M}\|P\|\|x(0)\|.$$

Thus, by setting $M_1 = \bar{K}\widetilde{M}\|P\|$, we get

$$\|x(t)\| < M_1\|x(0)\|, \quad \text{for all } t \in [0, \infty).$$

The proof of Theorem 4.1 is complete. □

The boundedness of the solutions of DAE (2.1) implies that

$$Z(t) = e^{tU^T P E_1^{-1} A U} = Z_1(t) + Z_2(t), \tag{4.5}$$

where $Z_1(t)$ contains elements which are sums of exponential terms $e^{i\lambda_j t}$ for real λ_j and

$$\|Z_1(t)\| \leq K_1, \quad -\infty < t < \infty, \tag{4.6}$$

$$\|Z_2(t)\| \leq K_2 e^{-\sigma t}, \quad 0 \leq t < \infty \tag{4.7}$$

for some $\sigma > 0$, where K_1 and K_2 are constants. From (4.5) and (2.10) we have

$$Y(t) = UZ(t) = UZ_1(t) + UZ_2(t). \tag{4.8}$$

Thus, the fundamental matrix $X(t)$ of equation (2.1) can be decomposed as

$$X(t) = X_1(t) + X_2(t), \tag{4.9}$$

where

$$X_1(t) = (I + QE_1^{-1}A)UZ_1(t),\ X_2(t) = (I + QE_1^{-1}A)UZ_2(t).$$

Therefore, the estimates

$$\|X_1(t)\| \le \overline{K}_1, \quad -\infty < t < \infty, \tag{4.10}$$

$$\|X_2(t)\| \le \overline{K}_2 e^{-\sigma t}, \quad 0 \le t < \infty \tag{4.11}$$

hold for some positive constants $\sigma > 0$, $\overline{K}_1$ and $\overline{K}_2$.

Theorem 4.2. *Let the assumptions of Theorem 4.1 be satisfied. Moreover, let $\alpha_2(t) \to 0$ and $\beta_2(t) \to 0$ as $t \to \infty$ hold. Then, for any solution x of (1.1), there is a constant vector $p \in \mathbb{R}^d$ such that*

$$x(t) - X_1(t)p \to 0 \quad \text{as } t \to \infty.$$

Proof. Given a solution x, let us define u, v and w as above. By Proposition 2.6, there exists a constant vector $p \in \mathbb{R}^d$ such that

$$\lim_{t\to\infty} [w(t) - Z_1(t)p] = 0$$

i.e.,

$$\lim_{t\to\infty} [u(t) - UZ_1(t)p] = 0,$$

where $u(t)$ is a solution of (3.12).

On the other hand, the solution of (1.1) of the form $x(t) = u(t) + \psi(t, u(t), g(t, u(t)))$. Thus, we obtain

$$\begin{aligned} x(t) - X_1(t)p &= u(t) + \psi(t, u(t), g(t, u(t))) - (I + QE_1^{-1}A)UZ_1(t)p \\ &= u(t) + QE_1^{-1}Au(t) + QE_1^{-1}f(t, u(t) + \psi(t, u(t), g(t, u(t))), g(t, u(t))) \\ &\quad - (I + QE_1^{-1}A)UU_1(t)p \\ &= (I+QE_1^{-1}A)[u(t)-UZ_1(t)p] + QE_1^{-1}f(t, u(t)+\psi(t, u(t), g(t, u(t))), g(t, u(t))). \end{aligned}$$

Therefore, we have the following inequality

$$\begin{aligned} \|x(t) - X_1(t)p\| \le{}& \|(I + QE_1^{-1}A)(u(t) - UZ_1(t)p)\| \\ &+ \|QE_1^{-1}f(t, u(t) + \psi(t, u(t), g(t, u(t))), g(t, u(t)))\|. \end{aligned}$$

Moreover, we have the following estimate

$$\begin{aligned} &\|QE_1^{-1}f(t, u(t) + \psi(t, u(t), g(t, u(t))), g(t, u(t)))\| \\ &\quad \le \alpha_2(t)\|u(t) + \psi(t, u(t), g(t, u(t)))\| + \beta_2(t)\|g(t, u(t))\| \\ &\quad \le \alpha_2(t)\left(\|u(t)\| + \frac{C_1 + \alpha_2(t)}{1 - \alpha_2(t)}\|u(t)\|\right) + \frac{\alpha_2(t)\beta_2(t)}{1 - \alpha_2(t)}\|g(t, u(t))\| + \beta_2(t)\|g(t, u(t))\| \\ &\quad \le \frac{(C_1 + 1)\alpha_2(t)}{1 - \alpha_2(t)}\|u(t)\| + \frac{\beta_2(t)}{1 - \alpha_2(t)}\|g(t, u(t))\| \\ &\quad \le \frac{(C_1 + 1)\alpha_2(t)}{1 - \alpha_2(t)}\|u(t)\| + \frac{\beta_2(t)}{1 - \alpha_2(t)}\frac{1}{1 - \gamma(t)}\left(C_2 + \frac{(C_1 + 1)\alpha_1(t)}{1 - \alpha_2(t)}\right)\|u(t)\| \\ &\quad \le \left[\frac{(C_1 + 1)\alpha_2(t)}{1 - \alpha_2(t)} + \frac{\beta_2(t)}{1 - \alpha_2(t)}\frac{1}{1 - \gamma(t)}\left(C_2 + \frac{(C_1 + 1)\alpha_1(t)}{1 - \alpha_2(t)}\right)\right]\|u(t)\| \\ &\quad \le K(t)\|u(t)\|, \end{aligned}$$

where

$$K(t) = \frac{(C_1+1)\alpha_2(t)}{1-\alpha_2(t)} + \frac{\beta_2(t)}{1-\alpha_2(t)}\frac{1}{1-\gamma(t)}\left(C_2 + \frac{(C_1+1)\alpha_1(t)}{1-\alpha_2(t)}\right).$$

Since $\lim_{t\to\infty}\alpha_2(t) = 0, \lim_{t\to\infty}\beta_2(t) = 0$, we have $\lim_{t\to\infty}K(t) = 0$. Thus, we obtain

$$\|x(t) - X_1(t)p\| \le \|I + QE_1^{-1}A\|\|u(t) - UZ_1(t)p\| + K(t)\|u(t)\| \to 0$$

as $t \to \infty$, because $u(t)$ is bounded for $t \ge 0$. The proof of Theorem 4.2 is complete. □

Theorem 4.3. *Let the assumptions in Theorem 3.1 be satisfied and let*

$$\int_t^{t+1}\alpha_1(s)ds \to 0, \qquad \int_t^{t+1}\beta_1(s)ds \to 0 \quad \text{as } t \to \infty. \tag{4.12}$$

Furthermore $f(t,0,0) = 0$. Suppose that x is a bounded solution of (1.1). Then, either

i) the limit

$$\lim_{t\to\infty}\frac{\ln\|x(t)\|}{t}$$

exists and is equal to the real part of one of the eigenvalues of the pencil matrix $\{E, A\}$, or

ii) $x(t) = 0$ for all large t.

Proof. Since $f(t,0,0) = 0$, equation (1.1) has the trivial solution. We consider again the EUODE (4.2), where $\|U^T\widetilde{g}(t,Uw(t))\| \le \widetilde{\gamma}(t)\|w(t)\|$ and $\int_t^{t+1}\widetilde{\gamma}(s)ds \to 0$ as $t \to \infty$. By Theorem 2.7, the solution $w(t)$ of (4.2) satisfies either the limit $\lim_{t\to\infty}\frac{\ln\|w(t)\|}{t}$ exists and is equal to the real part of one of the eigenvalues of the matrix $N = U^TPE_1^{-1}AU$, or $w(t) = 0$ for all large t. Therefore, the solution $u(t)$ of (3.12) which satisfies either the limit $\lim_{t\to\infty}\frac{\ln\|u(t)\|}{t}$ exists and is equal to the real part of one of the finite eigenvalues of the matrix pencil $\{E, A\}$ or $u(t) = 0$ for all large t.

On the other hand, from $x(t) = u(t) + \psi(t,u(t),g(t,u(t)))$ it follows that the following estimate holds:

$$\begin{aligned}
\|x(t)\| &\le \|u(t)\| + \|\psi(t,u(t),g(t,u(t)))\| \\
&\le \|u(t)\| + \frac{C_1+\alpha_2(t)}{1-\alpha_2(t)}\|u(t)\| + \frac{\beta_2(t)}{1-\alpha_2(t)}\|g(t,u(t)))\| \\
&\le \frac{C_1+1}{1-\alpha_2(t)}\|u(t)\| + \frac{1}{1-\gamma(t)}\frac{\beta_2(t)}{1-\alpha_2(t)}\left(C_2 + \frac{(C_1+1)\alpha_1(t)}{1-\alpha_2(t)}\right)\|u(t)\| \\
&\le \frac{1}{1-\alpha_2(t)}\left[1 + C_1 + \frac{\beta_2(t)}{1-\gamma(t)}\left(C_2 + \frac{(C_1+1)\alpha_1(t)}{1-\alpha_2(t)}\right)\right]\|u(t)\|.
\end{aligned}$$

Put $\sup_{t\in[0,\infty)}\alpha_i = a_i$, $\sup\limits_{t\in[0,\infty)}\beta_i = b_i$, $i = 1,2$. We obtain the following inequality

$$\|x(t)\| \le \widetilde{K}\|u(t)\|, \quad t \ge 0, \tag{4.13}$$

where

$$\widetilde{K} = \frac{1}{1-a_2}\left[C_1 + 1 + \frac{b_2}{1 - \frac{a_1b_2}{1-a_2} - b_1}\left(C_2 + \frac{(C_1+1)a_1}{1-a_2}\right)\right].$$

Obviously, if $u(t) = 0$ for all large t then $x(t) = 0$ for all $t \ge 0$, too. Otherwise, from the inequality (4.13) it follows that

$$\lim_{t\to\infty}\frac{\ln\|x(t)\|}{t} \le \lim_{t\to\infty}\frac{\ln\|u(t)\|}{t}. \tag{4.14}$$

Moreover, we have

$$\|u(t)\| = \|Px(t)\| \le \|P\|\|x(t)\|, \qquad \forall t \ge 0.$$

Therefore, we obtain

$$\lim_{t\to\infty} \frac{\ln \|u(t)\|}{t} \le \lim_{t\to\infty} \frac{\ln \|x(t)\|}{t}. \tag{4.15}$$

Thus, from (4.14) and (4.15), we have

$$\lim_{t\to\infty} \frac{\ln \|x(t)\|}{t} = \lim_{t\to\infty} \frac{\ln \|u(t)\|}{t}.$$

By Theorem 2.7 and Lemma 2.3, we conclude that the limit $\lim_{t\to\infty} \frac{\ln\|x(t)\|}{t}$ exists and it is equal to the real part of one of the eigenvalues of the matrix pencil $\{E, A\}$. The proof of Theorem 4.3 is complete. □

Remark 4.4. As a special case, if the nonlinear term f does not involve the derivative term x', i.e., DAE (1.1) becomes

$$Ex'(t) = Ax(t) + f(t, x), \qquad t \in \mathbb{I}, \tag{4.16}$$

then the situation is simpler and the assumptions in Theorem 3.1 can be significantly simplified. Namely, instead of (3.4) and (3.5), we assume

$$\begin{aligned} \|PE_1^{-1}f(t,x) - PE_1^{-1}f(t,\bar{x})\| &\le \alpha_1(t)\|x-\bar{x}\|, \\ \|QE_1^{-1}f(t,x) - QE_1^{-1}f(t,\bar{x})\| &\le \alpha_2(t)\|x-\bar{x}\| \end{aligned} \tag{4.17}$$

for all $t \ge 0$ and $x, \bar{x} \in \mathbb{C}^n$, $\alpha_i(t)$ are non-negative bounded functions ($i = 1,2$) such that $\sup_{t\in[0,\infty)} \alpha_2(t) < 1$. Furthermore, all the results in Section 4 can be stated analogously under appropriately reduced assumptions for $f(t, x)$, too.

5 Discussion

In this paper we have studied the asymptotic behavior of solutions for quasilinear DAEs, where the linear part is a DAE of index one and the nonlinearity is assumed to be small in some sense. As the main results, we have shown that any non-vanishing, bounded solution has the strict Lyapunov exponent which coincides with one of the Lyapunov exponents of the linear system. Since the coefficients of the linear system are constant, one might use alternatively the more simple Kronecker–Weierstraß decomposition or the Singular Value Decomposition as in [12] for decoupling. However, these tools will not work for time-varying systems, in general. Here we prefer using the projector-based approach because as a future problem, we want to use this approach to extend the results to quasilinear DAEs whose linear part is time-varying. The derivative should be properly stated as in [10]. This problem expects more technical difficulties since the Lyapunov spectrum of a linear time-varying system may be unstable under infinitesimally small perturbations.

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Existence of positive ground state solutions of critical nonlinear Klein–Gordon–Maxwell systems

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Abstract. In this paper we study the following nonlinear Klein–Gordon–Maxwell system

$$\begin{cases} -\Delta u + [m_0^2 - (\omega+\varphi)^2]u = f(u) & \text{in } \mathbb{R}^3, \\ \Delta\varphi = (\omega+\varphi)u & \text{in } \mathbb{R}^3, \end{cases}$$

where $0 < \omega < m_0$. Based on an abstract critical point theorem established by Jeanjean, the existence of positive ground state solutions is proved, when the nonlinear term $f(u)$ exhibits linear near zero and a general critical growth near infinity. Compared with other recent literature, some different arguments have been introduced and some results are extended.

Keywords: Klein–Gordon–Maxwell system, general critical growth, positive ground state solutions, variational methods.

2020 Mathematics Subject Classification: 35J20, 35J65, 35J60.

1 Introduction

This article is concerned with the following Klein–Gordon–Maxwell equations

$$\begin{cases} -\Delta u + [m_0^2 - (\omega+\varphi)^2]u = f(u) & \text{in } \mathbb{R}^3, \\ \Delta\varphi = (\omega+\varphi)u & \text{in } \mathbb{R}^3, \end{cases} \tag{KGME}$$

where $0 < \omega < m_0$. We assume that the followings hold for f:

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$ is odd;

(f_2) $\lim_{s\to 0} \frac{f(s)}{s} = -m < 0$;

(f_3) $\lim_{|s|\to+\infty} \frac{f(s)}{s^5} = K > 0$;

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(f_4) there exist $D > 0$ and $q \in (2, 6)$ such that $f(s) + ms \geq Ks^5 + Ds^{q-1}$ for all $s > 0$;

(f_5) there exists constant $\gamma > 2$ such that $f(s)s - \gamma F(s) \geq 0$ for all $s \in \mathbb{R}$, where $F(s) = \int_0^s f(t)dt$.

This system is well known as a model describing the interaction between the nonlinear Klein–Gordon field and the electrostatic field. The presence of nonlinear term $f(u)$ simulates the interaction between many particles or external nonlinear perturbations.

In recent years, there is large quality works devoted to the system (KGME), and we would like to recall some of them. In a remarkable work, V. Benci and D. Fortunato [4] are the first to study the following system

$$\begin{cases} -\Delta u + [m_0^2 - (\omega + \varphi)^2]u = |u|^{p-2}u & \text{in } \mathbb{R}^3, \\ \Delta \varphi = (\omega + \varphi)u & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

using the variational method, the authors proved the existence of infinitely many radially symmetric solutions when $m_0 > \omega > 0$ and $4 < q < 6$. In [15,16], D'Aprile and Mugnai considered the case $2 < p \leq 4$ and established some non-existence results for $p > 6$. Afterwards, there are also more literatures focusing on the existence and multiplicity of solutions for the problem (KGME). See [12,13,19] and the references therein.

There are some results related the critical case. In [11], Cassani considered the following system with the critical term:

$$\begin{cases} -\Delta u + [m_0^2 - (\omega + \varphi)^2]u = \mu|u|^{p-2}u + |u|^{2^*-2}u & \text{in } \mathbb{R}^3, \\ \Delta \varphi = (\omega + \varphi)u & \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

where $\mu > 0$. He showed that system (1.2) possesses a radially symmetric solution under one of the following conditions:

(i) $4 < p < 6$ and $|m_0| > |\omega|$;

(ii) $p = 4$, $|m_0| > |\omega|$ and μ large enough.

Soon afterwards, the authors of [9] studied the following critical Klein–Gordon–Maxwell system with external potential:

$$\begin{cases} -\Delta u + \mu V(x)u - (2\omega + \varphi)\varphi^2]u = \lambda f(u) + |u|^5 & \text{in } \mathbb{R}^3, \\ \Delta \varphi = (\omega + \varphi)u & \text{in } \mathbb{R}^3. \end{cases} \tag{1.3}$$

Provided $f(u)$ satisfying assumptions:

(f_2') $\lim_{s \to 0} \frac{f(s)}{s} = 0$;

(f_3') $\lim_{s \to \infty} \frac{f(s)}{s^5} = 0$;

(f_4') $\frac{1}{4} f(u)u - F(u) \geq 0$.

They obtained a nontrivial solution for (1.3). For more related results, we refer the readers to [3,24].

The existence of ground state solutions, that is, couples (u, φ) which solve (KGME) and minimize the action functional associated to (KGME) among all possible nontrivial solutions, has been investigated by many authors. Inspired by the approach of Benci and Fortunato, Azzollini and Pomponio [10] proved that (1.1) admits a ground state solution provided one of the following assumptions:

(i) $3 \le p < 5$ and $m_0 > \omega$;

(ii) $1 < p < 3$, $m_0\sqrt{p-1} > \omega\sqrt{6-p}$.

Soon afterwards, Carrião *et al*. [22] dealt with the critical Klein–Gordon–Maxwell system (1.2) with potentials. Combining the minimization of the corresponding Euler–Lagrange functional on the Nehari manifold, they proved the existence of positive ground state solutions for system (1.2). Very recently, Moura, Miyagaki *et al*. [14] considered quasicritical Klein–Gordon–Maxwell systems with potential, and obtained positive ground state solutions. For other related results about Klein–Gordon–Maxwell systems the authors maybe see [7,17,25].

Here we also mention that the papers [2,6], Berestycki and Lions studied the following elliptic equation

$$-\Delta u = f(u), \qquad u \in H^1(\mathbb{R}^N). \tag{1.4}$$

Under the following conditions on $f(u)$:

(A_1) $f(u) \in C(\mathbb{R}, \mathbb{R})$ is odd;

(A_2) $-\infty < \liminf_{u\to 0^+} \frac{f(u)}{u} \le \limsup_{u\to 0^+} \frac{f(u)}{u} = -m < 0$ for $N \ge 3$ and $\lim_{u\to 0} \frac{f(u)}{u} = -m < 0$ for $N = 2$;

(A_3) when $N \ge 3$, $-\infty < \limsup_{u\to\infty} \frac{f(u)}{u^{\frac{N+2}{N-2}}} \le 0$; when $N = 2$ for any $\alpha > 0$, there exists $C_\alpha > 0$ such that $f(u) \le C_\alpha \exp(\alpha u^2)$ for $u > 0$;

(A_4) there exists $\varsigma > 0$ such that$F(\varsigma) = \int_0^\varsigma f(s)ds > 0$,

Berestycki and Lions [6] proved the existence of a positive least energy solution when $N \ge 3$ and Berestycki *et al*. [2] investigated the existence of infinitely many bound state solutions when $N = 2$. Under the above assumptions, Azzollini, d'Avenia and Pomponio [1] obtained the existence of at least a radial positive solution to a class of Schrödinger–Poisson problems, and Azzollini [28] proved the existence of ground state solutions for Kirchhoff-type problems, and soon after Zhang and Zou [27] investigated the existence of ground state solutions of the problem (1.4) with the critical growth assumption on $f(u)$.

Under the assumptions (f_1)–(f_5), Zhang [21] studied a class of Schrödinger–Poisson problems and established the existence of ground state solutions for $q \in (2,4]$ with D large enough, or $q \in (4,6)$, where $m = 0$; Liu [20] considered a Kirchhoff-type problem and obtained the existence of ground state solutions without (f_5).

Motivated by the above mentioned works, in particular by [9,20,21,27], the main purpose of this paper is to consider the existence of positive least energy solutions of (KGME) with a general nonlinearity in the critical growth. To our best knowledge, under the assumptions (f_1)–(f_5), there is no work on the the existence of positive ground state solutions for problem (KGME). Precisely, we have the following results.

Theorem 1.1. *If (f_1)–(f_5) hold. Assume that either $q \in (2,4]$ with D sufficiently large, or $q \in (4,6)$, then the problem (KGME) possesses a positive radial solution if one of the following conditions is satisfied:*

(i) $2 < \gamma < 3$ *and* $0 < \omega < \sqrt{(\gamma-2)(4-\gamma)}m_0$;

(ii) $3 \le \gamma < \infty$ *and* $0 < \omega < m_0$.

Theorem 1.2. *If (f_1)–(f_5) hold. Assume that either $q \in (2,4]$ with D sufficiently large, or $q \in (4,6)$, then the problem (KGME) possesses a positive ground state solution provided one of the following conditions holds:*

(i) $2 < \gamma < 3$ *and* $0 < \omega < \sqrt{(\gamma-2)(4-\gamma)}m_0$*;*

(ii) $3 \le \gamma < \infty$ *and* $0 < \omega < m_0$.

Theorem 1.3. *If we replace the condition (f_5) by the following condition:*

(f_6) *there exists $\gamma > 2$ such that $t :\to \frac{f(x,t)}{t^{\gamma-1}}$ is increasing on $(-\infty, 0)$ and $(0, +\infty)$.*

Then the conclusions of Theorems 1.1 and 1.2 remain true.

Remark 1.4. Assumptions (f_1)–(f_4) were used by [20,21]. Since the problem in [20] is different from ours, the methods used in [20] do not work here. The similar hypotheses on $f(u)$ as above (f_1)–(f_5) are introduced in [21], where the authors used a cut-off functional to obtain bounded (PS) sequences. However, our device is different from the main arguments of [21]. Moreover, the results of [21] hold under $\gamma > 3$, and in our case, $\gamma > 2$.

Remark 1.5. The condition (f_4) plays a crucial role to ensure the existence of ground state solution to the problem (KGME). And the condition (f_5) is a technical condition to overcome the difficulty caused by the critical exponential growth case.

In our paper, due to the presence of a nonlocal term φ and the effect of the nonlinearity in the critical growth, there exist several difficulties to solve. In the first place, the lack of the following Ambrosetti–Rabinowitz growth hypothesis on f :

$$\exists\, \mu > 4 \text{ s. t. } 0 < \mu F(s) \le s f(s), \quad \forall t \in \mathbb{R}$$

brings a obstacle in proving the boundedness of (PS) sequence. To overcome this difficulty, we will use approaches developed by Jeanjean [23] to obtain the boundedness. In the next place, since we deal with the critical case, the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ is not compact, and the functional I does not satisfy $(PS)_c$ condition at every energy level c. To avoid the difficulty, we try to pull the energy level down below some critical level c_1^* (Section 3). In the end, we apply the Strauss' compactness result [5] to obtain the convergence of $(PS)_c$ sequence.

An outline of the paper is as follows. In Section 2, we give some preliminary lemmas. Section 3 is devoted to the existence of the mountain pass solution and positive ground state solution. Throughout the paper we denote by C the various positive constants. Let $D^{1,2}(\mathbb{R}^3) = \{u \in L^6(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\}$ be the Sobolev space equipped with the norm $\|u\|^2_{D^{1,2}} = \int_{\mathbb{R}^3} |\nabla u|^2 dx$. $S = \inf\limits_{u \in D^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 dx}{(\int_{\mathbb{R}^3} |u|^6 dx)^{\frac{1}{3}}}$ denotes the best Sobolev constant.

2 Preliminaries

In this section we give notations and prove some preliminary lemmas. Let us define an equivalent norm on $H^1(\mathbb{R}^3)$, that is

$$\|u\|^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + mu^2) dx \quad \text{for fixed } m > 0.$$

For any $1 \le s < \infty$, we denote that $L^s(\mathbb{R}^3)$ is the usual Lebesgue space endowed with the norm $\|u\|_{L^s}^s = \int_{\mathbb{R}^3} |u|^s dx$. Then we have that, for $2 \le s \le 2^*$, $H^1(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)$ continuously. Let $H := H_r^1(\mathbb{R}^3) := \{u \in H^1(\mathbb{R}^3) \mid u \text{ is radial functions}\}$. Then $H \hookrightarrow L^s(\mathbb{R}^3)$ is compact for $2 < s < 2^*$.

According to the variational nature of (KGME), we define its the energy functional as follows:

$$\Phi(u,\varphi) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 - |\nabla\varphi|^2 + [m_0^2 - (\omega+\varphi)^2]u^2)dx - \int_{\mathbb{R}^3} F(u)dx. \tag{2.1}$$

Under the assumptions (f_1)–(f_2), by standard arguments, we can prove that $\Phi(u,\varphi)$ is a well defined C^1 function on $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ and that the weak solutions of (KGME) is critical points of the functional Φ. Obviously, the functional Φ is the strongly indefiniteness, which means that it is unbounded both from below and from above on infinite-dimensional subspaces. In order to avoid this indefiniteness, we apply the reduction method developed by Benci and Fortunato [8]. For deducing our results, we introduce the following results whose idea of proof comes from [15,16].

Lemma 2.1. *For any $u \in H^1(\mathbb{R}^3)$, there is a unique $\varphi = \varphi_u \in D^{1,2}(\mathbb{R}^3)$ which satisfies the following equation*

$$-\Delta\varphi + \varphi u^2 = -\omega u^2. \tag{2.2}$$

Furthermore the map $\Psi : u \in H^1(\mathbb{R}^3) \to \varphi_u \in D^{1,2}(\mathbb{R}^3)$ is continuously differentiable and

(i) in the set $\{x : u(x) \neq 0\}$, for $\omega > 0$,

$$-\omega \le \varphi \le 0;$$

(ii) $\|\varphi_u\|_{D^{1,2}} \le C\|u\|^2$ and $\int_{\mathbb{R}^3} |\varphi_u| u^2 dx \le C\|u\|_{\frac{12}{5}}^4$.

Multiplying (2.2) by φ_u and integrating by parts, we obtain

$$\int_{\mathbb{R}^3} |\nabla\varphi_u|^2 dx = -\int_{\mathbb{R}^3} u^2\varphi_u^2 dx - \int_{\mathbb{R}^3} \omega u^2\varphi_u dx. \tag{2.3}$$

Lemma 2.2. *If $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, then, up to subsequence, $\varphi_{u_n} \rightharpoonup \varphi_u$ in $D^{1,2}(\mathbb{R}^3)$. As a consequence, $\Psi'(u_n) \to \Psi'(u)$ in the sense of distributions.*

By the definition of Φ and (2.3) the functional $I(u) = \Phi(u,\varphi)$ may be rewritten as the following form

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + (m_0^2 - \omega^2)u^2 - \omega\varphi_u u^2)dx - \int_{\mathbb{R}^3} F(u)dx. \tag{2.4}$$

In view of Lemmas 2.1 and 2.2, the conditions (f_1)–(f_3) imply $I(u) \in C^1$ and its Gateaux derivative is

$$\langle I'(u), v\rangle = \int_{\mathbb{R}^3}[\nabla u\nabla v + (m_0^2 - \omega^2)uv - (2\omega + \varphi_u)\varphi_u uv]dx - \int_{\mathbb{R}^3} f(u)v dx \tag{2.5}$$

for all $u, v \in H$. Then (u,φ) is a weak solution of (KGME) if and only if $\varphi = \varphi_u$ and u is a critical point of I on H.

For simplicity, in this paper we may assume that $K = 1$. Set $g(t) = f(t) + mt$, so the functional I is reduced as

$$I(u) = \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_{\mathbb{R}^3}(m_0^2 - \omega^2)u^2dx - \frac{1}{2}\int_{\mathbb{R}^3}\omega\varphi_u u^2dx - \int_{\mathbb{R}^3}G(u)dx, \tag{2.6}$$

where $G(s) = \int_0^s g(t)dt$. In the following we give the abstract result established by Jeanjean [23].

Lemma 2.3. *Let $(X, \|\cdot\|)$ be a Banach space and $\mathrm{h} \subset R_+$ an interval. Consider the family of C^1 functionals on X*

$$J_\lambda(u) = A(u) - \lambda B(u), \qquad \lambda \in \mathrm{h}$$

with B nonnegative and either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\| \to \infty$ and such that $J_\lambda(0) = 0$. For any $\lambda \in \mathrm{h}$, we set

$$\Gamma_\lambda = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) < 0\}.$$

If for every $\lambda \in \mathrm{h}$ the set Γ_λ is nonempty and

$$c_\lambda = \inf_{\gamma\in\Gamma_\lambda}\max_{t\in[0,1]} J_\lambda(\gamma(t)) > 0.$$

Then for every almost $\lambda \in \mathrm{h}$ there is a sequence $\{u_n\} \subset X$ such that

(i) *$\{u_n\}$ is bounded;*

(ii) *$J_\lambda(u_n) \to c_\lambda$;*

(iii) *$J'_\lambda(u_n) \to 0$ in the dual X^{-1} of X.*

In our case, $X = H$, $h = [\frac{1}{2}, 1]$,

$$A(u) = \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_{\mathbb{R}^3}(m_0^2 - \omega^2)u^2dx - \frac{1}{2}\int_{\mathbb{R}^3}\omega\varphi_u u^2dx, \qquad B(u) = \int_{\mathbb{R}^3}G(u)dx,$$

and so the family of functionals we study is

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_{\mathbb{R}^3}(m_0^2 - \omega^2)u^2dx - \frac{1}{2}\int_{\mathbb{R}^3}\omega\varphi_u u^2dx - \lambda\int_{\mathbb{R}^3}G(u)dx. \tag{2.7}$$

and for every $u, v \in H$,

$$\begin{aligned}\langle I'_\lambda(u), v\rangle = &\int_{\mathbb{R}^3}(\nabla u\nabla v + muv)dx + \int_{\mathbb{R}^3}(m_0^2 - \omega^2)uvdx \\ &- \int_{\mathbb{R}^3}(2\omega + \varphi_u)\varphi_u uvdx - \lambda\int_{\mathbb{R}^3}g(u)vdx. \end{aligned}\tag{2.8}$$

We shall use the following Pohožaev type identity. Its proof can be done as in [16].

Lemma 2.4. *For $\lambda \in [\frac{1}{2}, 1]$, let $u \in H$ be a critical point of I_λ, then*

$$\begin{aligned}P_\lambda(u) := &\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2dx + \frac{3}{2}\int_{\mathbb{R}^3}mu^2dx + \frac{3}{2}\int_{\mathbb{R}^3}(m_0^2 - \omega^2)u^2dx \\ &- \frac{1}{2}\int_{\mathbb{R}^3}(5\omega + 2\varphi_u)\varphi_u u^2dx - 3\lambda\int_{\mathbb{R}^3}G(u)dx = 0.\end{aligned}$$

If $\lambda = 1$, the above Pohožaev equality turns to be the following

$$\begin{aligned}P(u) := &\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2dx + \frac{3}{2}\int_{\mathbb{R}^3}mu^2dx + \frac{3}{2}\int_{\mathbb{R}^3}(m_0^2 - \omega^2)u^2dx \\ &- \frac{1}{2}\int_{\mathbb{R}^3}(5\omega + 2\varphi_u)\varphi_u u^2dx - 3\int_{\mathbb{R}^3}G(u)dx = 0.\end{aligned}$$

Next we shall cite a variant of the Strauss compactness result [5], which plays a fundamental tool in our arguments:

Lemma 2.5. *Let P and $Q : \mathbb{R} \to \mathbb{R}$ be two continuous functions satisfying*

$$\lim_{s\to\infty} \frac{P(s)}{Q(s)} = 0.$$

Let $\{v_n\}_n$, v and ψ be measurable functions from $\mathbb{R}^N$ to $\mathbb{R}$, with z bounded, such that

$$\sup_n \int_{\mathbb{R}^N} |Q(v_n(x))|\psi dx < \infty,$$
$$P(v_n(x)) \to v(x) \quad a.e.\ in\ \mathbb{R}^N,\ as\ n \to +\infty.$$

Then for any bounded Borel set B one has $\|(P(v_n) - v)\psi\|_{L^1(B)} \to 0$. Moreover, if

$$\lim_{s\to 0} \frac{P(s)}{Q(s)} = 0,$$
$$\lim_{|x|\to+\infty} \sup_n |v_n(x)| = 0,$$

then

$$\|(P(v_n) - v)\psi\|_{L^1(\mathbb{R}^N)} \to 0 \quad as\ n \to \infty.$$

3 Proof of main results

In this section we will look for a positive ground state solutions of (KGME). First, we will prove the existence of a mountain pass solution. Now, we give several lemmas which imply that I_λ satisfies the conditions of Lemma 2.3.

Lemma 3.1. *Assume that (f_1)–(f_4) hold. Then*

(i) $\Gamma_\lambda \neq \emptyset$ for every $\lambda \in h$;

(ii) there exists a constant $\tilde{c}$ such that $c_\lambda \geq \tilde{c} > 0$.

Proof. (i) For any $\lambda \in h$, it follows from Lemma 2.1, (2.7) and (f_4) that

$$I_\lambda(u) \leq \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_{\mathbb{R}^3} (m_0^2 - \omega^2)u^2 dx - \frac{1}{2}\int_{\mathbb{R}^3} \omega\varphi_u u^2 dx - \frac{1}{12}\int_{\mathbb{R}^3} |u|^6 dx - \frac{D}{2q}\int_{\mathbb{R}^3} |u|^q dx.$$

Then

$$I_\lambda(tu) \leq \frac{1}{2}t^2\|u\|^2 + \frac{1}{2}t^2 \int_{\mathbb{R}^3} m_0^2 u^2 dx - \frac{1}{12}t^6 \int_{\mathbb{R}^3} |u|^6 dx - \frac{D}{2q}t^q \int_{\mathbb{R}^3} |u|^q dx.$$

Then we can choose $t_0 > 0$ large and $u \in H\backslash\{0\}$ such that $I_\lambda(t_0 u) < 0$ for every $\lambda \in h$. Define $\gamma_1 : [0,1] \to H$ in the following way

$$\gamma_1(t) = tt_0 u, \qquad 0 \leq t \leq 1.$$

It is easy to see γ_1 a continuous path from $t_0 u$. Moreover, for every $\lambda \in h$, $I_\lambda(\gamma_1(1)) < 0$ and $I_\lambda(\gamma_1(0)) = 0$. The proof is completed.

(ii) Using (f_1)–(f_3), for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that $|g(u)| \le \varepsilon|u| + C_\varepsilon|u|^5$. Then by Sobolev's embedding theorem, one has

$$I_\lambda(u) \ge \frac{1}{2}\|u\|^2 - \frac{\varepsilon}{2}\int_{\mathbb{R}^3}|u|^2dx - \frac{C_\varepsilon}{6}\int_{\mathbb{R}^3}|u|^6dx \ge \frac{m-\varepsilon}{2m}\|u\|^2 - \frac{CC_\varepsilon}{6}\|u\|^6.$$

For fixed $\varepsilon \in (0, m)$, there exists $\tilde{c} > 0$ such that $I_\lambda(u) \ge \tilde{c} > 0$ for any $\lambda \in h$ and $u \in H$ with $\|u\| = \rho$ small enough. Now fix $\lambda \in h$ and $\gamma \in \Gamma_\lambda$. Since $\gamma(0) = 0$ and $I_\lambda(\gamma(1)) < 0$, certainly $\|\gamma(1)\| > \rho$. By continuity, we deduce that there exists $t_\gamma \in (0,1)$ such that $\|\gamma(t_\gamma)\| = \rho$. Therefore for every $\lambda \in h$

$$c_\lambda \ge \inf_{\gamma\in\Gamma_\lambda} I_\lambda(\gamma(t_\gamma)) \ge \tilde{c} > 0,$$

which implies (ii) of Lemma 3.1. □

It follows from Lemma 3.1 that the conclusions of Lemma 2.3 hold.

Lemma 3.2. *Assume that (f_1)–(f_4) hold. If $q \in (4,6)$ or $q \in (2,4]$ with D is large enough, then $c_\lambda < c_\lambda^* := \frac{1}{3}\lambda^{-\frac{1}{2}}S^{\frac{3}{2}}$.*

Proof. For $\varepsilon, r > 0$, define $u_\varepsilon(x) = \frac{\phi(x)\varepsilon^{\frac{1}{4}}}{(\varepsilon+|x|^2)^{\frac{1}{2}}}$, where $\phi \in C_0^\infty(B_{2r}(0)), 0 \le \phi \le 1$ and $\phi|_{B_r(0)} \equiv 1$. And it is well known that the best Sobolev constant S is attained by the functions $\frac{\varepsilon^{\frac{1}{4}}}{(\varepsilon+|x|^2)^{\frac{1}{2}}}$. Direct calculation yields that

$$\int_{\mathbb{R}^3}|\nabla u_\varepsilon|^2dx = K_1 + O(\varepsilon^{\frac{1}{2}}), \qquad \int_{\mathbb{R}^3}|u_\varepsilon|^6dx = K_2 + O(\varepsilon^{\frac{3}{2}}) \tag{3.1}$$

and

$$\int_{\mathbb{R}^3}|u_\varepsilon|^tdx = \begin{cases} K\varepsilon^{\frac{t}{4}}, & t \in [2,3), \\ K\varepsilon^{\frac{3}{4}}|In\varepsilon|, & t = 3, \\ K\varepsilon^{\frac{6-t}{4}}, & t \in (3,6), \end{cases} \tag{3.2}$$

where K_1, K_2, K are positive constants. Moreover, $S = K_1K_2^{-\frac{1}{3}}$. Using (3.1) and (3.2), we have

$$\frac{\int_{\mathbb{R}^3}|\nabla u_\varepsilon|^2dx}{(\int_{\mathbb{R}^3}u_\varepsilon^6dx)^{\frac{1}{3}}} = S + O(\varepsilon^{\frac{1}{2}}).$$

Set

$$g(t) = \frac{1}{2}t^2\|u_\varepsilon\|^2 + \frac{1}{2}t^2(m_0^2-\omega^2)\int_{\mathbb{R}^3}|u_\varepsilon|^2dx - \frac{\lambda}{6}t^6\int_{\mathbb{R}^3}|u_\varepsilon|^6dx.$$

It is easy to see that $g(t)$ attains its maximum at $t_0 = [\frac{\|u_\varepsilon\|^2+(m_0^2-\omega^2)\int_{\mathbb{R}^3}|u_\varepsilon|^2dx}{\frac{\lambda}{6}\int_{\mathbb{R}^3}|u_\varepsilon|^6dx}]^{\frac{1}{4}}$ and then

$$\begin{aligned}\max_{t\ge0} g(t) = &\frac{1}{2}\lambda^{-\frac{1}{2}}\sqrt{[\frac{\|u_\varepsilon\|^2)}{(\int_{\mathbb{R}^3}|u_\varepsilon|^6dx)^{\frac{1}{3}}}]^3 + \frac{(m_0^2-\omega^2)\|u_\varepsilon\|^4\int_{\mathbb{R}^3}|u_\varepsilon|^2dx}{\int_{\mathbb{R}^3}|u_\varepsilon|^6dx}} \\ &+ \frac{1}{2}(m_0^2-\omega^2)\lambda^{-\frac{1}{2}}\sqrt{\frac{\|u_\varepsilon\|^2(\int_{\mathbb{R}^3}|u_\varepsilon|^2dx)^4 + (m_0^2-\omega^2)(\int_{\mathbb{R}^3}|u_\varepsilon|^2dx)^6}{\int_{\mathbb{R}^3}|u_\varepsilon|^6dx}} \\ &- \frac{\lambda^{-\frac{1}{2}}}{6}\sqrt{[\frac{\|u_\varepsilon\|^2 + (m_0^2-\omega^2)\int_{\mathbb{R}^3}|u_\varepsilon|^2dx}{(\int_{\mathbb{R}^3}|u_\varepsilon|^6dx)^{\frac{1}{3}}}]^3} = \frac{1}{3}\lambda^{-\frac{1}{2}}S^{\frac{3}{2}}\end{aligned}$$

for $\varepsilon > 0$ small enough. Obviously, there exists $0 < t' < 1$ such that, for $\varepsilon < 1$, one has

$$\begin{aligned}\max_{t' \geq t \geq 0} I_\lambda(tu_\varepsilon(x)) &\leq \max_{t' \geq t \geq 0} \left(\frac{1}{2}t^2\|u_\varepsilon\|^2 + \frac{1}{2}(m_0^2 - w^2)t^2 \int_{\mathbb{R}^3} |u_\varepsilon|^2 dx + Ct^2 \int_{\mathbb{R}^3} |u_\varepsilon|^2 dx\right) \\ &\leq Ct^2\|u_\varepsilon\|^2 < \frac{1}{3}\lambda^{-\frac{1}{2}} S^{\frac{3}{2}}. \end{aligned} \tag{3.3}$$

Using (f_4), (2.7) and Lemma 2.1, one has

$$\begin{aligned} I_\lambda(tu_\varepsilon(x)) &= \frac{1}{2}t^2\|u_\varepsilon\|^2 + \frac{1}{2}t^2 \int_{\mathbb{R}^3} (m_0^2 - \omega^2) u_\varepsilon^2 dx - \frac{1}{2}t^2 \int_{\mathbb{R}^3} \omega \varphi_{tu_\varepsilon} u_\varepsilon^2 dx - \lambda \int_{\mathbb{R}^3} G(tu_\varepsilon) dx \\ &\leq \frac{1}{2}t^2\|u_\varepsilon\|^2 + \frac{1}{2}t^2 \int_{\mathbb{R}^3} (m_0^2 - \omega^2) u_\varepsilon^2 dx - \frac{1}{2}t^2 \int_{\mathbb{R}^3} \omega \varphi_{tu_\varepsilon} u_\varepsilon^2 dx - \frac{\lambda}{6}t^2 \int_{\mathbb{R}^3} |u_\varepsilon|^6 dx \\ &\quad - \frac{\lambda D}{q} \int_{\mathbb{R}^3} |u_\varepsilon|^q dx \\ &= g(t) - \frac{1}{2}t^2 \int_{\mathbb{R}^3} \omega \varphi_{tu_\varepsilon} u_\varepsilon^2 dx - \frac{\lambda D}{q} \int_{\mathbb{R}^3} |u_\varepsilon|^q dx \\ &\leq g(t) + Ct^2 \int_{\mathbb{R}^3} |u_\varepsilon|^2 dx - CDt^q \int_{\mathbb{R}^3} |u_\varepsilon|^q dx. \end{aligned} \tag{3.4}$$

It follows from (3.4) and Lemma 3.1 that

$$\lim_{t \to \infty} I_\lambda(tu_\varepsilon(x)) = -\infty \tag{3.5}$$

and

$$I_\lambda(tu_\varepsilon(x)) > 0 \tag{3.6}$$

as t is close to 0. Now we prove that there exists $0 < \varepsilon_0 < 1$ such that $\lim_{t\to\infty} I_\lambda(tu_\varepsilon(x)) < 0$ uniformly in $\varepsilon \in (0, \varepsilon_0)$. Set

$$\begin{aligned}\eta(t) = \frac{1}{2}t^2\|u_\varepsilon\|^2 + \frac{1}{2}t^2(m_0^2 - \omega^2) \int_{\mathbb{R}^3} |u_\varepsilon|^2 dx - \frac{1}{2}t^2 \int_{\mathbb{R}^3} \omega \varphi_{tu_\varepsilon} u_\varepsilon^2 dx \\ - \frac{\lambda}{6}t^6 \int_{\mathbb{R}^3} |u_\varepsilon|^6 dx - \frac{\lambda D}{q}t^q \int_{\mathbb{R}^3} |u_\varepsilon|^q dx. \end{aligned} \tag{3.7}$$

Following from (3.5) and (3.6), (3.7) means that there exists $t_\varepsilon > 0$ such that $\eta(t_\varepsilon) = 0$ and for $t > t_\varepsilon$, $\eta(t) < 0$. Then we get

$$\begin{aligned} 0 = \eta(t_\varepsilon) = t_\varepsilon^2 \bigg(\frac{1}{2}\|u_\varepsilon\|^2 + \frac{1}{2}(m_0^2 - \omega^2) \int_{\mathbb{R}^3} |u_\varepsilon|^2 dx - \frac{1}{2} \int_{\mathbb{R}^3} \omega \varphi_{tu_\varepsilon} u_\varepsilon^2 dx \\ - \frac{\lambda}{6}t_\varepsilon^4 \int_{\mathbb{R}^3} |u_\varepsilon|^6 dx - \frac{\lambda D}{q}t_\varepsilon^{q-2} \int_{\mathbb{R}^3} |u_\varepsilon|^q dx\bigg), \end{aligned} \tag{3.8}$$

thus (3.1) and (3.2) mean that for $\varepsilon \in (0, \varepsilon_0)$

$$\begin{aligned} \frac{\lambda}{6}t_\varepsilon^4 &\leq \frac{1}{\int_{\mathbb{R}^3} |u_\varepsilon|^6 dx} \left[\frac{1}{2}\|u_\varepsilon\|^2 + \frac{1}{2}(m_0^2 - \omega^2) \int_{\mathbb{R}^3} |u_\varepsilon|^2 dx - \frac{1}{2} \int_{\mathbb{R}^3} \omega \varphi_{tu_\varepsilon} u_\varepsilon^2 dx\right] \\ &\leq \frac{1}{\int_{\mathbb{R}^3} |u_\varepsilon|^6 dx} \left[\frac{1}{2}\|u_\varepsilon\|^2 + \frac{1}{2}(m_0^2 - \omega^2) \int_{\mathbb{R}^3} |u_\varepsilon|^2 dx + C \int_{\mathbb{R}^3} |u_\varepsilon|^2 dx\right] \\ &\leq \frac{1}{2} \frac{K_1 + O(\varepsilon^{\frac{1}{2}})}{K_2 + O(\varepsilon^{\frac{3}{2}})} + O(\varepsilon^{\frac{1}{2}}) \leq \frac{1}{2} \frac{K_1 + O(\varepsilon_0^{\frac{1}{2}})}{K_2} + O(\varepsilon_0^{\frac{1}{2}}), \end{aligned} \tag{3.9}$$

where ε_0 is small enough. (3.9) implies that for some $t^* > 0$, t_ε is bounded from above uniformly for $\varepsilon \in (0, \varepsilon_0)$, where t^* is independent of ε. Using (3.5) and (3.9) we easily get that there exists $0 < \varepsilon_0 < 1$ such that $\lim_{t\to\infty} I_\lambda(tu_\varepsilon(x)) < 0$ uniformly in $\varepsilon \in (0, \varepsilon_0)$. Thus there exists $t^{''} > t^*$ such that for $\varepsilon \in (0, \varepsilon_0)$,

$$\max_{t \geq t^{''}} I_\lambda(tu_\varepsilon) < \frac{1}{3}\lambda^{-\frac{1}{2}} S^{\frac{3}{2}}. \tag{3.10}$$

It follows from (3.1), (3.2) and (3.4) that

$$\begin{aligned} \max_{t^{''} \geq t \geq t'} I_\lambda(tu_\varepsilon) &\leq g(t_0) + C\int_{\mathbb{R}^3} |u_\varepsilon|^2 dx - CD\int_{\mathbb{R}^3} |u_\varepsilon|^q dx \\ &= \frac{1}{3}\lambda^{-\frac{1}{2}} S^{\frac{3}{2}} + O(\varepsilon^{\frac{1}{2}}) - CD\int_{\mathbb{R}^3} |u_\varepsilon|^q dx. \end{aligned} \tag{3.11}$$

For $q \in (2, 4]$ and D sufficiently large, $\varepsilon \in (0, \varepsilon_0)$ fixed, we derive from (3.11) that

$$\max_{t^{''} \geq t \geq t'} I_\lambda(tu_\varepsilon) < \frac{1}{3}\lambda^{-\frac{1}{2}} S^{\frac{3}{2}}. \tag{3.12}$$

For $q \in (4, 6)$, observe that $\frac{6-q}{4} < \frac{1}{2}$, then it follows from (3.2) and (3.11) that, there exists $0 < \varepsilon_1 < \varepsilon_0$ small enough such that for $\varepsilon \in (0, \varepsilon_1)$,

$$\max_{t^{''} \geq t \geq t'} I_\lambda(tu_\varepsilon) < \frac{1}{3}\lambda^{-\frac{1}{2}} S^{\frac{3}{2}}. \tag{3.13}$$

It follows from (3.3), (3.10), (3.12) and (3.13) that $c_\lambda < c_\lambda^* := \frac{1}{3}\lambda^{-\frac{1}{2}} S^{\frac{3}{2}}$. □

Lemma 3.3. *Assume that* (f_1)–(f_3) *and* (f_5) *hold. Let* $\{u_\lambda\}$ *be a critical point for* $I_\lambda(u_\lambda)$ *at level* c_λ. *Then* $I_\lambda(u_\lambda) \geq 0$.

Proof. If $\gamma \geq 4$ in (f_5), then it follows from (f_5), (2.7) and (2.8) that

$$\begin{aligned} I_\lambda(u_\lambda) &= I_\lambda(u_\lambda) - \frac{1}{\gamma}\langle I'_\lambda(u_\lambda), u_\lambda\rangle \\ &= \left(\frac{1}{2} - \frac{1}{\gamma}\right)\int_{\mathbb{R}^3} \left[|\nabla u_\lambda|^2 + (m_0^2 - \omega^2)u_\lambda^2\right] dx \\ &\quad + \int_{\mathbb{R}^3} \left[\left(\frac{2}{\gamma} - \frac{1}{2}\right)\omega\varphi_{u_\lambda} u_\lambda^2 + \frac{1}{\gamma}\varphi_{u_\lambda}^2 u_\lambda^2\right] dx + \int_{\mathbb{R}^3} \left[\frac{1}{\gamma} f(u_\lambda)u_\lambda - F(u_\lambda)\right] dx \\ &\geq \left(\frac{1}{2} - \frac{1}{\gamma}\right)\int_{\mathbb{R}^3} |\nabla u_\lambda|^2 dx \geq 0. \end{aligned} \tag{3.14}$$

Now, we consider $2 < \gamma < 4$ in (f_5). By (f_5), (2.7) and (2.8) and Lemma 2.4, we obtain that

$$
\begin{aligned}
I_\lambda(u_\lambda) &= I_\lambda(u_\lambda) - \frac{2}{6-\gamma}\langle I'_\lambda(u_\lambda), u_\lambda\rangle - \frac{2-\gamma}{2(6-\gamma)}P_\lambda(u_\lambda) \\
&= I_\lambda(u_\lambda) - \Big\{ \int_{\mathbb{R}^3} \frac{1}{2}|\nabla u_\lambda|^2 dx + \frac{10-3\gamma}{2(6-\gamma)} \int_{\mathbb{R}^3} (m_0^2 - \omega^2) u_\lambda^2 dx \\
&\qquad - \int_{\mathbb{R}^3} \Big[\frac{18-5\gamma}{2(6-\gamma)} \omega \varphi_{u_\lambda} + \frac{8-2\gamma}{2(6-\gamma)} \varphi_{u_\lambda}^2 u_\lambda^2 \Big] dx \\
&\qquad - \lambda \int_{\mathbb{R}^3} \Big[\frac{2}{6-\gamma} f(u_\lambda) u_\lambda + \frac{6(2-\gamma)}{2(6-\gamma)} F(u_\lambda) \Big] dx \Big\} \\
&= \frac{\gamma-2}{6-\gamma} \int_{\mathbb{R}^3} (m_0^2 - \omega^2) u_\lambda^2 dx + \int_{\mathbb{R}^3} \Big[\frac{2(3-\gamma)}{6-\gamma} \omega \varphi_{u_\lambda} + \frac{4-\gamma}{6-\gamma} \varphi_{u_\lambda}^2 \Big] u_\lambda^2 dx \\
&\quad + \frac{2}{6-\gamma} \lambda \int_{\mathbb{R}^3} [f(u_\lambda) u_\lambda - \gamma F(u_\lambda)] dx.
\end{aligned}
\tag{3.15}
$$

Set $h(t) = (4-\gamma)t^2 + 2(3-\gamma)\omega t$. We distinguish two cases:

Case 1. $3 \le \gamma < 4$ and $0 < \omega < m_0$. In this case, one has

$$
h(t) \ge 0, \qquad \forall -\omega \le t \le 0. \tag{3.16}
$$

Note that $-\omega \le \varphi_{u_\lambda} \le 0$. From (f_5), (3.15), (3.16), we have

$$
I_\lambda(u_\lambda) \ge \frac{\gamma-2}{6-\gamma} \int_{\mathbb{R}^3} (m_0^2 - \omega^2) u_\lambda^2 dx \ge 0. \tag{3.17}
$$

Case 2. $2 < \gamma < 3$ and $0 < \omega < \sqrt{(\gamma-2)(4-\gamma)}m_0$. For $\forall -\omega \le t \le 0$, an elementary computation means that

$$
\begin{aligned}
(\gamma-2)(m_0 - \omega^2) + h(t) &= (\gamma-2)(m_0 - \omega^2) + (4-\gamma)(t^2 + \frac{2(3-\gamma)}{4-\gamma}\omega t) \\
&\ge (\gamma-2)(m_0 - \omega^2) - \frac{(\gamma-3)^2}{4-\gamma}\omega^2 \\
&= \frac{(\gamma-2)(4-\gamma)m_0^2 - \omega^2}{4-\gamma} > 0.
\end{aligned}
\tag{3.18}
$$

Then from (f_5), (3.15) and (3.18), we get

$$
I_\lambda(u_\lambda) \ge \frac{1}{(6-\gamma)(4-\gamma)} [(\gamma-2)(4-\gamma)m_0^2 - \omega^2] \int_{\mathbb{R}^3} u_\lambda^2 dx \ge 0. \tag{3.19}
$$

It follows from (3.14), (3.17) and (3.19) that $I_\lambda(u_\lambda) \ge 0$. □

Lemma 3.4. *Assume that (f_1)–(f_5). For almost every $\lambda \in [\frac{1}{2}, 1]$, there is $u_\lambda \in H \setminus \{0\}$ such that $I'_\lambda(u_\lambda) = 0$ and $I_\lambda(u_\lambda) = c_\lambda$.*

Proof. By Lemma 2.3 and Lemma 3.1, for almost every $\lambda \in [\frac{1}{2}, 1]$, there exists a bounded (PS) sequence $\{u_n\} \subset H$ such that

$$
I_\lambda(u_n) \to c_\lambda, \qquad I'_\lambda(u_n) = 0 \quad \text{in } H', \tag{3.20}
$$

where H' is the dual space of H. Using Lemma 2.2, up to a subsequence, we can suppose that there exists $u \in H$ such that

$$\begin{aligned} u_n &\rightharpoonup u_\lambda && \text{weakly in } H, \\ u_n &\to u_\lambda && \text{in } L^p(\mathbb{R}^3),\ 2 < p < 6, \\ u_n &\to u_\lambda && \text{a.e. in } \mathbb{R}^3, \\ \varphi_{u_n} &\rightharpoonup \varphi_{u_\lambda} && \text{weakly in } D^{1,2}(\mathbb{R}^3). \end{aligned} \tag{3.21}$$

If we apply Lemma 2.5 for $P(t) = g(t) - t^5$, $Q(t) = t^5$,$\{v_n\}_n = \{u_n\}_n$, $v = g(u_\lambda) - u_\lambda^5$ and $\psi \in C_0^\infty(\mathbb{R}^3)$. By (f_2)–(f_4) and (3.21), we have

$$\int_{\mathbb{R}^3} (g(u_n) - u_n^5)\psi dx \to \int_{\mathbb{R}^3} (g(u_\lambda) - u_\lambda^5)\psi dx. \tag{3.22}$$

If we apply Lemma 2.5 for $P(t) = F(t) + \frac{1}{2}mt^2 - \frac{1}{6}t^6 = G(t) - \frac{1}{6}t^6$, $Q(t) = t^2 + t^6$, $\{v_n\}_n = \{u_n\}_n$, $v = F(u_\lambda) + \frac{1}{2}mu_\lambda^2 - \frac{1}{6}u_\lambda^6 = G(u_\lambda) - \frac{1}{6}u_\lambda^6$, and $\psi = 1$. By (f_2)–(f_4) and (3.21), we have

$$\int_{\mathbb{R}^3} \left(G(u_n) - \frac{1}{6}u_n^6 \right) dx \to \int_{\mathbb{R}^3} \left(G(u_\lambda) - \frac{1}{6}u_\lambda^6 \right) dx. \tag{3.23}$$

Similarly, we also have

$$\int_{\mathbb{R}^3} (g(u_n)u_n - u_n^6) dx \to \int_{\mathbb{R}^3} (g(u_\lambda)u_\lambda - u_\lambda^6) dx. \tag{3.24}$$

Introduce the notation $\mathrm{Y} = \mathrm{Supp}(\psi)$. Using (3.21) and the Sobolev inequality, one has

$$\begin{aligned} \left| \int_{\mathbb{R}^3} (\varphi_{u_n} u_n^2 \psi - \varphi_{u_\lambda} u_\lambda^2 \psi) dx \right| &\le \int_{\mathbb{R}^3} |\varphi_{u_n}||u_n^2 - u_\lambda^2||\psi| dx + \int_{\mathbb{R}^3} |\varphi_{u_n} - \varphi_{u_\lambda}||u_\lambda||\psi| dx \\ &\le \left(\int_{\mathbb{R}^3} |\varphi_{u_n}|^6 dx \right)^{\frac{1}{6}} \left(\int_{\mathrm{Y}} |u_n^2 - u_\lambda^2|^{\frac{6}{5}} dx \right)^{\frac{5}{6}} \sup|\psi| \\ &\quad + \left(\int_{\mathrm{Y}} |\varphi_{u_n} - \varphi_{u_\lambda}|^{\frac{6}{5}} dx \right)^{\frac{5}{6}} \left(\int_{\mathrm{Y}} |u_\lambda^2|^6 dx \right)^{\frac{1}{6}} \sup|\psi| = o(1), \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} \left| \int_{\mathbb{R}^3} (\varphi_{u_n} u_n \psi - \varphi_{u_\lambda} u_\lambda \psi) dx \right| &\le \int_{\mathbb{R}^3} |\varphi_{u_n}||u_n - u_\lambda||\psi| dx + \int_{\mathbb{R}^3} |\varphi_{u_n} - \varphi_{u_\lambda}||u_\lambda||\psi| dx \\ &\le \left(\int_{\mathbb{R}^3} |\varphi_{u_n}|^6 dx \right)^{\frac{1}{6}} \left(\int_{\mathrm{Y}} |u_n - u_\lambda|^{\frac{6}{5}} dx \right)^{\frac{5}{6}} \sup|\psi| \\ &\quad + \left(\int_{\mathrm{Y}} |\varphi_{u_n} - \varphi_{u_\lambda}|^{\frac{6}{5}} dx \right)^{\frac{5}{6}} \left(\int_{\mathrm{Y}} |u_\lambda|^6 dx \right)^{\frac{1}{6}} \sup|\psi| = o(1), \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} \left| \int_{\mathbb{R}^3} (\varphi_{u_n}^2 u_n \psi - \varphi_{u_\lambda}^2 u_\lambda \psi) dx \right| &\le \int_{\mathbb{R}^3} \varphi_{u_n}^2 |u_n - u_\lambda||\psi| dx + \int_{\mathbb{R}^3} |\varphi_{u_n}^2 - \varphi_{u_\lambda}^2||u_\lambda||\psi| dx \\ &\le \left(\int_{\mathbb{R}^3} |\varphi_{u_n}|^6 dx \right)^{\frac{1}{6}} \left(\int_{\mathrm{Y}} |u_n - u_\lambda|^{\frac{3}{2}} dx \right)^{\frac{2}{3}} \sup|\psi| \\ &\quad + \left(\int_{\mathrm{Y}} |\varphi_{u_n}^2 - \varphi_{u_\lambda}^2|^{\frac{6}{5}} dx \right)^{\frac{5}{6}} \left(\int_{\mathrm{Y}} |u_\lambda|^6 dx \right)^{\frac{1}{6}} \sup|\psi| = o(1). \end{aligned} \tag{3.27}$$

It follows from $\langle I'_\lambda(u_n),\psi\rangle = 0$, (3.21), (3.22), (3.26) and (3.27) that

$$\int_{\mathbb{R}^3}(\nabla u_\lambda \nabla\psi + m u_\lambda\psi + (m_0^2 - w^2)u_\lambda\psi)dx - \int_{\mathbb{R}^3}(2w+\varphi_{u_\lambda})\varphi_{u_\lambda}u_\lambda\psi dx \\ - \lambda\int_{\mathbb{R}^3}(g(u_\lambda)\psi - u_\lambda^5\psi)dx - \lambda\int_{\mathbb{R}^3}u_\lambda^5\psi dx = 0,$$

i.e. $J'_\lambda(u_\lambda) = 0$, where

$$J_\lambda(u) := \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_{\mathbb{R}^3}(m_0^2-\omega^2)u^2dx - \frac{1}{2}\int_{\mathbb{R}^3}\omega\varphi_u u^2dx \\ - \lambda\int_{\mathbb{R}^3}(G(u) - \frac{1}{6}u^6)dx - \frac{\lambda}{6}\int_{\mathbb{R}^3}u^6dx.$$

Set $v_n = u_n - u_\lambda$. Then $v_n \rightharpoonup 0$ in H. Following from the well-known Brezis–Lieb lemma [18], we get

$$\begin{aligned}\|v_n\|_2^2 &= \|u_n\|_2^2 - \|u_\lambda\|_2^2 + o(1),\\ \|\nabla v_n\|_2^2 &= \|\nabla u_n\|_2^2 - \|\nabla u_\lambda\|_2^2 + o(1),\\ \|v_n\|_6^6 &= \|u_n\|_6^6 - \|u_\lambda\|_6^6 + o(1).\end{aligned} \tag{3.28}$$

Then, by Lemma 3.3, (3.24), (3.26), (3.27), (3.28), $J'_\lambda(u_n) = 0$ and $J'_\lambda(u_\lambda) = 0$, we have

$$\begin{aligned}o(1) &= \langle J'_\lambda(u_n), u_n\rangle - \langle J'_\lambda(u_\lambda), u_\lambda\rangle\\ &= \|v_n\|^2 + (m_0^2-\omega^2)\|v_n\|_2^2 - \lambda\int_{\mathbb{R}^3}|v_n|^6dx.\end{aligned} \tag{3.29}$$

Up to a subsequence, we may assume that $\|v_n\|^2 + (m_0^2-\omega^2)\|v_n\|_2^2 \to l \geq 0$. By (3.29), $\lambda\int_{\mathbb{R}^3}|v_n|^6dx \to l$. If $l > 0$, then the Sobolev embedding theorem means that $S \leq \frac{\int_{\mathbb{R}^3}|\nabla v_n|^2dx}{(\int_{\mathbb{R}^3}|v_n|^6dx)^{\frac{1}{3}}} \leq \frac{\|v_n\|^2+(m_0^2-\omega^2)\|v_n\|_2^2}{(\int_{\mathbb{R}^3}|v_n|^6dx)^{\frac{1}{3}}}$, which implies that

$$l \geq \lambda^{-\frac{1}{2}}S^{\frac{3}{2}}. \tag{3.30}$$

By (3.20), (3.21), (3.23), (3.25) and (3.28), we get

$$\begin{aligned}c_\lambda - I_\lambda(u_\lambda) &= I_\lambda(u_n) - I_\lambda(u_\lambda) + o(1)\\ &= \frac{1}{2}\|v_n\|^2 + \frac{1}{2}\int_{\mathbb{R}^3}(m_0^2-\omega^2)|v_n|^2dx - \frac{\lambda}{6}\int_{\mathbb{R}^3}|v_n|^6dx + o(1).\end{aligned} \tag{3.31}$$

Then, by (3.30)–(3.31), we have $c_\lambda - I_\lambda(u_\lambda) = \frac{1}{3}l > \frac{1}{3}\lambda^{-\frac{1}{2}}S^{\frac{3}{2}}$, by Lemma 3.3, which contradicts with $c_\lambda - I_\lambda(u_\lambda) < \frac{1}{3}\lambda^{-\frac{1}{2}}S^{\frac{3}{2}}$ since that $I_\lambda(u_\lambda) \geq 0$. Therefore, $l = 0$, i.e. $\|v_n\|^2 = o(1)$, hence $u_n \to u_\lambda$ in H. This completes Lemma 3.4. □

Proof of Theorem 1.1. According to Lemma 3.4, there exists sequences $\{\lambda_n\} \subset h$ with $\lambda_n \to 1$, $c_{\lambda_n} \in (0, \frac{1}{3}\lambda_n^{-\frac{1}{2}}S^{\frac{3}{2}})$ and a sequence of $\{u_{\lambda_n}\}$, denoted by $\{u_n\}$ such that $I_{\lambda_n}(u_n) = c_{\lambda_n}$ and $I'_{\lambda_n}(u_n) = 0$. Next we show $\{u_n\}$ is bounded. The proof will be developed in several steps: Indeed, by Lemma 2.4, $I_{\lambda_n}(u_n) = c_{\lambda_n}$ and $I'_{\lambda_n}(u_n) = 0$, we have

$$\begin{cases}\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u_n|^2dx + \frac{3}{2}\int_{\mathbb{R}^3}(m_0^2-\omega^2)u_n^2dx - \frac{1}{2}\int_{\mathbb{R}^3}(5\omega+2\varphi_{u_n})\varphi_{u_n}u_n^2dx - 3\lambda_n\int_{\mathbb{R}^3}F(u_n)dx = 0,\\ \frac{1}{2}\int_{\mathbb{R}^3}[|\nabla u_n|^2 + (m_0^2-\omega^2)u_n^2 - \omega\varphi_{u_n}u^2]dx - \lambda_n\int_{\mathbb{R}^3}F(u_n)dx = c_{\lambda_n} \leq c_{\frac{1}{2}},\\ \int_{\mathbb{R}^3}[|\nabla u_n|^2 + (m_0^2-\omega^2)u_n^2 - (2\omega+\varphi_{u_n})\varphi_{u_n}u_n^2]dx - \lambda_n\int_{\mathbb{R}^3}f(u_n)u_ndx = 0.\end{cases}$$

Step 1. If $\gamma \geq 4$ in (f_5), then it follows from (3.14) that

$$c_{\frac{1}{2}} \geq c_{\lambda_n} = I_{\lambda_n}(u_n) = I_{\lambda_n}(u_n) - \frac{1}{\gamma}\langle I'_{\lambda_n}(u_n), u_n\rangle \geq \left(\frac{1}{2} - \frac{1}{\gamma}\right)\int_{\mathbb{R}^3}[|\nabla u_\lambda|^2 + (m_0^2 - \omega^2)u_\lambda^2]dx \geq C\|u_n\|^2. \tag{3.32}$$

Thus, we deduce from (3.32) that $\{u_n\}$ is bounded in H if $\gamma \geq 4$.

Step 2. If $2 < \gamma < 4$ in (f_5), we distinguish two cases:

Case 1. $3 \leq \gamma < 4$ and $0 < \omega < m_0$. Following from (3.17), we have

$$c_{\frac{1}{2}} \geq c_{\lambda_n} = I_{\lambda_n}(u_n) \geq \frac{\gamma - 2}{6 - \gamma}\int_{\mathbb{R}^3}(m_0^2 - \omega^2)u_n^2 dx. \tag{3.33}$$

Case 2. $2 < \gamma < 3$ and $0 < \omega < \sqrt{(\gamma-2)(4-\gamma)}m_0$. From (3.19), we have

$$c_{\frac{1}{2}} \geq c_{\lambda_n} = I_{\lambda_n}(u_\lambda) \geq \frac{1}{(6-\gamma)(4-\gamma)}[(\gamma-2)(4-\gamma)m_0^2 - \omega^2]\int_{\mathbb{R}^3} u_n^2 dx. \tag{3.34}$$

Deriving from (3.33) and (3.34), we get the boundedness of $\|u_n\|_2$. Then by Lemma 2.1, one has

$$0 \leq \int_{\mathbb{R}^3} -\omega\varphi_{u_n}u_n^2 dx \leq \int_{\mathbb{R}^3} \omega^2 u_n^2 dx \leq C. \tag{3.35}$$

Thus from Lemma 2.4 and (3.35) we deduce that

$$c_{\frac{1}{2}} \geq c_{\lambda_n} = I_{\lambda_n}(u_n) - \frac{1}{6}P_{\lambda_n}(u_n) = \frac{1}{3}\int_{\mathbb{R}^3}|\nabla u_n|^2 dx + \frac{1}{3}\int_{\mathbb{R}^3}(\omega + \varphi_{u_n})\varphi_{u_n}u_n^2 dx \geq \frac{1}{3}\int_{\mathbb{R}^3}|\nabla u_n|^2 dx + \frac{1}{3}\int_{\mathbb{R}^3}\omega\varphi_{u_n}u_n^2 dx \geq \frac{1}{3}\int_{\mathbb{R}^3}|\nabla u_n|^2 dx - C,$$

which means the boundedness of $\{\|\nabla u_n|_2\}$. This completes the proof. □

Note that $I(u_n) = I_{\lambda_n}(u_n) - (\lambda_n - 1)\int_{\mathbb{R}^3} F(u_n)dx$ and $I'(u_n) = I'_{\lambda_n}(u_n) - (\lambda_n - 1)\int_{\mathbb{R}^3} f(u_n)u_n dx$. By using the fact that the map $\lambda \to c_\lambda$ is left-continuous (see [23]), $\lambda_n \to 1$, the boundedness of $\{u_n\}$, we can show that

$$\lim_{n\to\infty} I(u_n) = c_1, \qquad \lim_{n\to\infty} I'(u_n) = 0.$$

Lemma 3.4 yields that there exists $u_0 \in H \setminus \{0\}$ being a critical point of I and $I(u_0) = c_1$. Set

$$I^+(u) = \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_{\mathbb{R}^3}(m_0^2 - \omega^2)u^2 dx - \frac{1}{2}\int_{\mathbb{R}^3}\omega\varphi_u u^2 dx - \left(\int_{\mathbb{R}^3} G(u)dx - \frac{1}{6}\int_{\mathbb{R}^3}|u|^6 dx\right) - \frac{1}{6}\int_{\mathbb{R}^3}|u^+|^6 dx,$$

where $u^+ = \max\{u, 0\}$. Repeating all the calculations above word by word, there is nonzero function u_0 solving the equation

$$-\Delta u + (m_0^2 - \omega^2)u + mu - \omega\varphi_u u = (g(u) - u^5) + (u^+)^5. \tag{3.36}$$

Using $u^- = \max\{-u_0, 0\}$ as a text function and integrating (3.36) by parts, we obtain

$$0 = \int_{\mathbb{R}^3} (|\nabla u_0^-|^2 + m|u_0^-|^2 + (m_0^2 - \omega^2)|u_0^-|^2)dx - \int_{\mathbb{R}^3} \omega\varphi_{u_0}|u_0^-|^2)dx - \int_{\mathbb{R}^3} (g(u_0) - u_0^5)u_0^- dx. \tag{3.37}$$

We deduce from (f_1) and (f_4) that $g(t) - t^5$ is an odd function and $g(t) - t^5 > 0$ for $t > 0$. So from (3.37) one has

$$0 = \int_{\mathbb{R}^3} (|\nabla u_0^-|^2 + m|u_0^-|^2 + (m_0^2 - \omega^2)|u_0^-|^2)dx - \int_{\mathbb{R}^3} \omega\varphi_{u_0}|u_0^-|^2)dx.$$

From Lemma 2.1, we obtain that $u_0^- = 0$ and $u_0 \geq 0$. Then u_0 is a nonnegative solution of the problem (KGME). Deducing from Harnack's inequality (see [26]), we can obtain that $u_0 > 0$ for all $x \in \mathbb{R}^3$, and u_0 is a positive critical point of the functional $I(u)$. Then by Lemma 2.1, we have $\varphi = \varphi_{u_0}$. From (2.1), (2.3) and (2.4) that (u_0, φ_{u_0}) is a positive solution of (KGME). The proof is complete. In what follows, we prove the existence of a positive ground state solution for the problem (KGME).

Proof of Theorem 1.2. Set $\tilde{m} := \inf\{I(u) : u \in H \setminus \{0\}, I'(u) = 0\}$. According to the arguments as above, we know that $0 < \tilde{m} \leq c < c_1^* := \frac{1}{3}S^{\frac{3}{2}}$. By the definition of $\tilde{m}$, there exists a sequence $\{u_n\} \subset H$ such that $u_n \neq 0$, $I(u_n) \to \tilde{m}$ and $I'(u_n) = 0$. Similar to the arguments as **Step 1** and **Step 2** in Theorem 1.1, we obtain the boundedness of $\{u_n\}$ in H. Since $I'(u_n) = 0$, we deduce from (2.3), (2.5), (f_1)–(f_3) and the Sobolev embedding inequality that

$$\begin{aligned}\|u_n\|^2 &= \int_{\mathbb{R}^3} (|\nabla u_n|^2 + m|u_n|^2 + (m_0^2 - \omega^2)|u_n|^2)dx + \int_{\mathbb{R}^3} [2|\nabla\varphi_{u_n}|^2 + \varphi_{u_n}^2 u_n^2]dx \\ &= \int_{\mathbb{R}^3} g(u_n)u_n dx \leq \varepsilon \int_{\mathbb{R}^3} (|u_n|^2 + |u_n|^6)dx,\end{aligned}$$

and so, there exists $C > 0$ such that $\|u_n\| \geq C$. Then we can claim that there exists $\sigma > 0$ such that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |u_n|^2 dx \geq \sigma > 0. \tag{3.38}$$

Otherwise, $\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |u_n|^2 dx = 0$. Using Lemma I.1 of [26], it follows that, for $2 < s < 6$, $\int_{\mathbb{R}^3} |u_n|^s dx \to 0$ in $L^s(\mathbb{R}^3)$. Using the same arguments as Lemma 3.4, we can obtain $\tilde{m} \geq c_1^* := \frac{1}{3}S^{\frac{3}{2}}$, which contradicts $\tilde{m} < c_1^* := \frac{1}{3}S^{\frac{3}{2}}$. Then (3.38) holds. Going if necessary to a subsequence, by (3.37), we may assume the existence of $y_n \in \mathbb{R}^3$ such that

$$\int_{B_1(y_n)} |u_n|^2 dx \geq \frac{\sigma}{2} > 0.$$

Set $v_n(x) = u_n(x + y_n)$. Then

$$\|v_n\| = \|u_n\|, \qquad \int_{B_1(0)} |v_n|^2 dx \geq \frac{\sigma}{2} > 0.$$

Since $\varphi_{u_n}(x + y_n) = \varphi_{v_n}(x)$, by (2.4), (2.5), we have

$$I(v_n) \to \tilde{m}, \qquad I'(v_n) = 0.$$

From the boundedness of $\{u_n\}$ in H, $\{v_n\}$ is also bounded. Then there exits $v_0 \neq 0$ such that $v_n \rightharpoonup v_0$ weakly in H. In view of Lemma 3.4, one can conclude that

$$\langle I'(v_0), v_0\rangle = 0, \qquad I(v_0) \geq \tilde{m}. \tag{3.39}$$

On the other hand, we will prove that $\tilde{m} \geq I(v_0)$.

In fact, if $\gamma \geq 4$ in (f_5), by (2.3), (2.5), (3.39) and Fatou's lemma, one has

$$\begin{aligned}
\tilde{m} &= \lim_{n\to\infty}\left\{I(v_n) - \frac{1}{\gamma}\langle I'(v_n), v_n\rangle\right\} \\
&= \lim_{n\to\infty}\left\{\left(\frac{1}{2}-\frac{1}{\gamma}\right)\int_{\mathbb{R}^3}[|\nabla v_n|^2 + (m_0^2-\omega^2)v_n^2]dx \right.\\
&\qquad \left. + \int_{\mathbb{R}^3}\left[\left(\frac{2}{\gamma}-\frac{1}{2}\right)\omega\varphi_{v_n}v_n^2 + \frac{1}{\gamma}\varphi_{v_n}^2 v_n^2\right]dx + \int_{\mathbb{R}^3}\left[\frac{1}{\gamma}f(v_n)v_n - F(v_n)\right]dx\right\} \\
&\geq \left(\frac{1}{2}-\frac{1}{\gamma}\right)\int_{\mathbb{R}^3}[|\nabla v_0|^2 + (m_0^2-\omega^2)v_0^2]dx \\
&\quad + \int_{\mathbb{R}^3}\left[\left(\frac{2}{\gamma}-\frac{1}{2}\right)\omega\varphi_{v_0}v_0^2 + \frac{1}{\gamma}\varphi_{v_0}^2 v_0^2\right]dx + \int_{\mathbb{R}^3}\left[\frac{1}{\gamma}f(v_0)v_0 - F(v_0)\right]dx \\
&= I(v_0) - \frac{1}{\gamma}\langle I'(v_0), v_0\rangle = I(v_0).
\end{aligned} \tag{3.40}$$

If $2 < \gamma < 3$ in (f_5) and $0 < \omega < \sqrt{(\gamma-2)(4-\gamma)}m_0$ or $3 \leq \gamma < 4$ in (f_5) and $0 < \omega < m_0$, by (3.15) and Fatou's lemma, one also has

$$\begin{aligned}
\tilde{m} &= \lim_{n\to\infty}\left\{I(v_n) - \frac{2}{6-\gamma}\langle I'(v_n), v_n\rangle - \frac{2-\gamma}{2(6-\gamma)}P(v_n)\right\} \\
&= \lim_{n\to\infty}\left\{\frac{\gamma-2}{6-\gamma}\int_{\mathbb{R}^3}(m_0^2-\omega^2)v_n^2dx + \int_{\mathbb{R}^3}\left[\frac{2(3-\gamma)}{6-\gamma}\omega\varphi_{v_n} + \frac{4-\gamma}{6-\gamma}\varphi_{v_n}^2\right]v_n^2dx \right.\\
&\qquad \left. + \frac{2}{6-\gamma}\lambda\int_{\mathbb{R}^3}[f(v_n)v_n - \gamma F(v_n)]dx\right\} \\
&\geq \frac{\gamma-2}{6-\gamma}\int_{\mathbb{R}^3}(m_0^2-\omega^2)v_0^2dx + \int_{\mathbb{R}^3}\left[\frac{2(3-\gamma)}{6-\gamma}\omega\varphi_{v_0} + \frac{4-\gamma}{6-\gamma}\varphi_{v_0}^2\right]v_0^2dx \\
&\quad + \frac{2}{6-\gamma}\lambda\int_{\mathbb{R}^3}[f(v_0)v_0 - \gamma F(v_0)]dx \\
&= I(v_0) - \frac{2}{6-\gamma}\langle I'(v_0), v_0\rangle - \frac{2-\gamma}{2(6-\gamma)}P(v_0) = I(v_0).
\end{aligned} \tag{3.41}$$

Combining (3.39), (3.40) with (3.41), we derive that $I(v_0) = \tilde{m} = \inf\{I(u) : u \in H \setminus \{0\}\} > 0$. Arguments as Theorem 1.1, we get $v_0 > 0$. Thus, by Lemma 2.1, (2.1), (2.3) and (2.4), $(v_0, \varphi_{v_0}) \in H \times D^{1,2}(\mathbb{R}^3)$ is a positive ground state solution of problem (KGME). The proof is complete. □

Proof of Theorem 1.3. It is sufficient to prove (f_5). Indeed, by (f_6), whenever $u > 0$,

$$F(x,u) = \int_0^1 f(x,ut)u dt = \int_0^1 \frac{f(x,ut)}{(ut)^{\gamma-1}}u^\gamma t^{\gamma-1}dt \leq \int_0^1 \frac{f(x,u)}{u^{\gamma-1}}u^\gamma t^{\gamma-1}dt = \frac{1}{\gamma}uf(u),$$

and whenever $u < 0$,

$$F(x,u) = \int_0^1 f(x,ut)u dt = -\int_0^1 \frac{f(x,ut)}{(-ut)^{\gamma-1}}(-u)^\gamma t^{\gamma-1} dt$$
$$= -\int_0^1 \frac{f(x,ut)}{|ut|^{\gamma-1}}|u|^\gamma t^{\gamma-1} dt \le -\int_0^1 \frac{f(x,u)}{|u|^{\gamma-1}}|u|^\gamma t^{\gamma-1} dt = \frac{1}{\gamma} u f(u).$$

The above results mean (f_5) holds. The proof is complete. □

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Stability of delay equations

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Abstract. For a large class of nonautonomous linear delay equations with distributed delay, we obtain the equivalence of hyperbolicity, with the existence of an exponential dichotomy, and Ulam–Hyers stability. In particular, for linear equations with constant or periodic coefficients and with a simple spectrum these two properties are equivalent. We also show that any linear delay equation with an exponential dichotomy and its sufficiently small Lipschitz perturbations are Ulam–Hyers stable.

Keywords: Ulam–Hyers stability, delay equations, exponential dichotomies.

2020 Mathematics Subject Classification: 37D99.

1 Introduction

We consider delay equations with distributed delay with the objective of relating hyperbolicity and Ulam–Hyers stability. More precisely, the aim of our work is twofold. In a first part, we show that any linear delay equation with an exponential dichotomy and its sufficiently small Lipschitz perturbations are Ulam–Hyers stable. We emphasize that we consider arbitrary nonautonomous delay equations with distributed delay. In a second part, we obtain a converse for a large class of linear equations by showing that hyperbolicity and Ulam–Hyers stability are equivalent properties. This includes in particular linear delay equations with constant coefficients, always with distributed delay, provided for example that the generator has a simple spectrum. We also consider delay equations with periodic coefficients.

Before proceeding, we recall the notion of Ulam–Hyers stability for an autonomous delay equation (the general nonautonomous case is analogous but is left for the main text). Let $|\cdot|$ be a norm on $\mathbb{C}^n$. Given $r > 0$, we denote by $C = C([-r,0],\mathbb{C}^n)$ the Banach space of all continuous functions $\phi\colon [-r,0] \to \mathbb{C}^n$ equipped with the supremum norm $\|\cdot\|$. Now let $L\colon C \to \mathbb{C}^n$ be a bounded linear operator and let $f\colon C \to \mathbb{C}^n$ be a continuous function. We say that the equation

$$v' = Lv_t + f(v_t), \tag{1.1}$$

where $v_t(\theta) = v(t+\theta)$ for $\theta \in [-r,0]$, is *Ulam–Hyers stable* if there exists $\kappa > 0$ such that for each $\varepsilon > 0$ and each continuous function $v\colon [-r,+\infty) \to \mathbb{C}^n$ of class C^1 on $[0,+\infty)$ (taking the

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right-hand derivative at 0) satisfying

$$\sup_{t\geq 0}\left|v'(t) - Lv_t - f(v_t)\right| < \varepsilon,$$

there exists a solution $w\colon [-r,+\infty) \to \mathbb{C}^n$ of equation (1.1) satisfying

$$\sup_{t\geq 0}\|v_t - w_t\| < \kappa\varepsilon$$

We detail briefly the origins and developments of Ulam–Hyers stability (sometimes the names are reordered in the literature), particularly in the context of differential equations and dynamical systems. Often it is also called Ulam–Hyers–Rassias stability. The concept goes back to a question of Ulam [37] for functional equations. Hyers [17] soon gave a solution for a particular functional equation and much later Rassias [34] made a considerable generalization for a notion of stability that includes the one studied by Hyers as a particular case (we refer the reader to the book [18] for details and for many additional references). The notion essentially requires that if there exists an approximate solution of a differential equation, in the sense that it satisfies the differential equation up to a certain error, then there exists an actual solution that is sufficiently close to the approximate solution. For many developments of the theory we refer the reader to the books [10,20,36] and the references therein.

The developments described above include in particular many works giving conditions leading to Ulam–Hyers stability, both for linear and nonlinear differential equations, or even that are equivalent to Ulam–Hyers stability for some classes of equations. The first to consider Ulam–Hyers stability in the context of differential equations seem to have been Alsina and Ger [1] (see [35] for a generalization). Further developments include for example the works [6,13,14,19,32,33] as well as related work for difference equations, such as [5,11,31]. There are various other variants, including for integral equations, differential integral equations, impulsive differential equations, and partial differential equations. There are also some works for delay equations, such as [16,21,26,27,38], although to our best knowledge never for distributed delays and never considering the problem of whether hyperbolicity is equivalent to Ulam–Hyers stability. These two aspects are precisely the main novelties of our work.

We are mainly interested in the relation between Ulam–Hyers stability and hyperbolicity. The equivalence between these two properties, under some additional assumptions, has been established in a few cases. Namely, this was established in [25] for differential equations with constant coefficients and in [6] for differential equations with periodic coefficients. Related results for discrete time were obtained, respectively, in [5] for constant coefficients and in [12] for periodic coefficients. To the possible extent, and similarly also under some additional assumptions, we want to obtain related results for delay equations. To our best knowledge, no similar problem was considered before for delays equations.

Certainly, our work is related to all these works since we study similar properties, but the techniques (either for delays equations or others) cannot be used in our work. This is due to the fact that we consider distributed delays, for which in particular the variation of constants formula requires extending some operators to a space of discontinuous functions. Moreover, unlike in all former works concerning Ulam–Hyers stability for delay equations, which put their emphasis on Lipschitz properties and then deduce the stability of the equation, our emphasis is instead on the hyperbolicity of the linear part, which allows us in particular to give a complete characterization of Ulam–Hyers stability for linear delay equations.

Incidentally, Ulam–Hyers stability can be described, equivalently, as the shadowing of approximate orbits and specifically as what is called Lipschitz shadowing (we refer the reader to the books [28, 29] for details and references). Nonetheless, the two theories first emerged independently. Shadowing theory was mainly motivated by hyperbolic dynamics. In particular, Anosov's closing lemma [2] shows how to shadow pseudo-orbits by periodic orbits. The general shadowing theorem of Anosov [3] and Bowen [9] leads to the structural stability of hyperbolic sets. On the other hand, it was shown by Pilyugin and Tikhomirov [30] that the Lipschitz shadowing property of a diffeomorphism is equivalent to its structural stability. These closing and shadowing results have important generalizations to nonuniformly hyperbolic systems. In particular, a closing lemma was proved by Katok in [22]. It is also a Lipschitz shadowing result, although some applications require its sharper bounds. We refer the reader to the book [7] for a detailed presentation of these results, but we refrain from giving further references since our work concerns only delay equations (with continuous time).

In the remainder of the introduction we recall briefly the notion of hyperbolicity and we formulate our main results in the particular case of autonomous delay equations. This allows us to avoid some technicalities that are present in the general nonautonomous case and for which we refer to the main text.

We consider an autonomous delay equation

$$v' = Lv_t, \tag{1.2}$$

where $L\colon C \to \mathbb{C}^n$ is a bounded linear operator. For each initial condition $v_0 = \phi \in C$, equation (1.2) has a unique solution v on $[-r, +\infty)$. These solutions determine a semigroup $S(t)\colon C \to C$, for $t \geq 0$, defined by

$$S(t)\phi = v_t(\cdot, 0, \phi) \quad \text{for } \phi \in C.$$

It is a strongly continuous semigroup with generator $A\colon D(A) \to C$ given by

$$A\phi := \lim_{t \searrow 0} \frac{S(t)\phi - \phi}{t} = \phi'$$

in the domain $D(A)$ formed by all $\phi \in C$ such that $\phi' \in C$ and $\phi'(0) = L\phi$. It turns out that the spectrum $\sigma(A)$ is composed entirely of eigenvalues.

Now we can formulate prototypes of our results in the particular case of autonomous equations (we refer to the main text for general results).

Theorem 1.1. *If the spectrum $\sigma(A)$ does not intersect the imaginary axis and the function $f\colon C \to \mathbb{C}^n$ satisfies*

$$|f(\phi) - f(\psi)| \leq K\|\phi - \psi\| \quad \textit{for all } \phi, \psi \in C,$$

then provided that K is sufficiently small the equation $v' = Lv_t + f(v_t)$ is Ulam–Hyers stable.

One can take $f = 0$ to obtain a result for the linear equation $v' = Lv_t$.

Theorem 1.2. *For a linear equation $v' = Lv_t$, if the spectrum $\sigma(A)$ does not intersect the imaginary axis, then the equation is Ulam–Hyers stable.*

We also consider the converse problem for a linear delay equation, among other results in the main text. Again we consider here only autonomous delay equations.

Theorem 1.3. *Assume that any $\lambda \in \sigma(A)$ on the imaginary axis is a simple eigenvalue. Then the equation $v' = Lv_t$ is Ulam–Hyers stable if and only if the spectrum $\sigma(A)$ does not intersect the imaginary axis.*

In addition, we show that for differential difference equations of the form

$$v' = A_0 v + \sum_{i=1}^{k} A_i v(t - \tau_i),$$

for some positive numbers $\tau_1 < \tau_2 < \cdots < \tau_k$ and some $n \times n$ matrices A_i for $i = 0, \ldots, k$, the simplicity condition in Theorem 1.3 is an open condition.

A more general condition than the simplicity of the spectrum is considered in the main text. It corresponds to assume that the Jordan form of each eigenvalue on the imaginary axis is diagonal. We also consider equations with periodic coefficients and, using the version of Floquet theory for delay equations, we obtain an appropriate version of the former theorem.

2 Preliminaries

In this section we recall a few notions and results from the theory of delay equations. This includes the notions of an exponential dichotomy and of an exponential trichotomy. We refer the reader to the books [8, 15] for details as well as proofs of all the results recalled in this section.

2.1 Basic notions

Let $|\cdot|$ be a norm on $\mathbb{C}^n$. Given $r > 0$ (the delay), we denote by $C = C([-r, 0], \mathbb{C}^n)$ the Banach space of all continuous functions $\phi\colon [-r, 0] \to \mathbb{C}^n$ equipped with the supremum norm

$$\|\phi\| = \sup_{-r \le \theta \le 0} |\phi(\theta)|. \tag{2.1}$$

We consider perturbations of a linear delay equation of the form

$$v' = L(t) v_t + g(t), \tag{2.2}$$

writing $v_t(\theta) = v(t + \theta)$ for $\theta \in [-r, 0]$ and where:

1. $L(t)\colon C \to \mathbb{C}^n$, for $t \ge 0$, are bounded linear operators such that the map $t \mapsto L(t)$ is strongly continuous on $[0, +\infty)$ and

$$\sup_{t \ge 0} \int_t^{t+1} \|L(\tau)\| \, d\tau < +\infty; \tag{2.3}$$

2. $g\colon [0, +\infty) \to \mathbb{C}^n$ is a bounded continuous function, that is,

$$\sup_{t \ge 0} |g(t)| < +\infty.$$

We recall that a map $t \mapsto L(t)$ is said to be *strongly continuous* on $[0, +\infty)$ if $t \mapsto L(t)\phi$ is continuous on $[0, +\infty)$ for each $\phi \in C$. It follows from the uniform boundedness principle and the strong continuity of the map $t \mapsto L(t)$ that

$$\sup_{\tau \in [s,t]} \|L(\tau)\| < +\infty \quad \text{for all } t > s. \tag{2.4}$$

Note that condition (2.3) holds for example when the map $t \mapsto L(t)$ is bounded and so in particular when the operators $L(t)$ are independent of t or are periodic in t.

A continuous function $v\colon [s-r,a) \to \mathbb{C}^n$ with $a \le +\infty$ is called a *solution* of equation (2.2) if

$$v(t) = v(s) + \int_s^t (L(\tau)v_\tau + g(\tau))\, d\tau \quad \text{for } t \in [s,a). \tag{2.5}$$

Since v is uniformly continuous on bounded intervals, the map $\tau \mapsto v_\tau$ is continuous. This implies that the function

$$h(\tau) = L(\tau)v_\tau + g(\tau)$$

is also continuous. Indeed,

$$\begin{aligned} |h(\tau) - h(\overline{\tau})| &\le |L(\tau)v_\tau - L(\overline{\tau})v_{\overline{\tau}}| + |g(\tau) - g(\overline{\tau})| \\ &\le |L(\tau)(v_\tau - v_{\overline{\tau}})| + |L(\tau)v_{\overline{\tau}} - L(\overline{\tau})v_{\overline{\tau}}| + |g(\tau) - g(\overline{\tau})| \end{aligned}$$

and the right-hand side converges to 0 when $\tau \to \overline{\tau}$, in view of (2.4) and the continuity of the maps $\tau \mapsto v_\tau$, $\tau \mapsto L(\tau)v_{\overline{\tau}}$ and g. Therefore, by (2.5), any solution of equation (2.2) is of class C^1 on $[s,a)$ and satisfies

$$v'(t) = L(t)v_t + g(t) \quad \text{for } t \in [s,a),$$

taking the right-hand derivative at s. Moreover, it follows from standard results on the existence and uniqueness of solutions of a delay equation that equation (2.2) has a unique solution on $[s-r,+\infty)$ for each initial condition $v_s = \phi \in C$. These solutions can be expressed in terms of the variation of constants formula (which we recall in the following section).

2.2 Linear equations

Now we consider the particular case of a linear equation

$$v' = L(t)v_t, \tag{2.6}$$

with the same hypotheses on the operators $L(t)$ as before. Equation (2.6) determines an evolution family $T(t,s)\colon C \to C$, for $t \ge s \ge 0$, defined by

$$T(t,s)\phi = v_t(\cdot,s,\phi) \quad \text{for } \phi \in C, \tag{2.7}$$

where v is the unique solution of equation (2.6) on $[s-r,+\infty)$ with $v_s = \phi$. One can easily verify that indeed

$$T(t,s) = \mathrm{Id} \quad \text{and} \quad T(t,\tau)T(\tau,s) = T(t,s)$$

for any $t \ge \tau \ge s \ge 0$. Moreover, one can show that

$$\|T(t,s)\| \le \exp\left(\int_s^t \|L(\tau)\|\, d\tau\right)$$

for any $t \ge s \ge 0$ and so it follows from any of the properties (2.3) and (2.4) that each $T(t,s)$ is bounded.

It turns out that the linear operators $T(t,s)$ can be extended to a certain space of discontinuous functions. Let C_0 be the set of all functions $\phi\colon [-r,0] \to \mathbb{C}^n$ that are continuous on $[-r,0)$ and for which the limit

$$\phi(0^-) = \lim_{\theta \to 0^-} \phi(\theta)$$

exists. This is a Banach space when equipped with the supremum norm $\|\cdot\|$ in (2.1). We write the linear operator $L(t)\colon C \to \mathbb{C}^n$ as a Riemann–Stieltjes integral

$$L(t)\phi = \int_{-r}^{0} d\eta(t,\theta)\phi(\theta) \tag{2.8}$$

for some measurable map $\eta\colon [0,+\infty) \times [-r,0] \to M_n$, where M_n is the set of all $n \times n$ matrices with complex entries, such that $\theta \mapsto \eta(t,\theta)$ has bounded variation and is left-continuous for each $t \geq 0$. We extend the linear operator $L(t)$ to C_0 using the integral in (2.8) (we continue to denote the extension by $L(t)$ since there is no danger of confusion). Finally, given $t \geq 0$, we define a linear operator $T_0(t,s)$ on the space C_0 by

$$T_0(t,s)\phi = v_t(\cdot,s,\phi) \quad \text{for } \phi \in C_0,$$

where v is the unique solution of equation (2.6) on $[s-r,+\infty)$ with $v_s = \phi$.

By the variation of constants formula for delay equations, the unique solution v of equation (2.2) on $[s-r,+\infty)$ with $v_s = \phi \in C$ satisfies

$$v_t = T(t,s)\phi + \int_s^t T_0(t,\tau)X_0 g(\tau)\,d\tau \tag{2.9}$$

for all $t \geq s$, where $X_0\colon \mathbb{C}^n \to C_0$ is the linear operator defined by

$$(X_0 p)(\theta) = \begin{cases} 0 & \text{if } -r \leq \theta < 0, \\ p & \text{if } \theta = 0 \end{cases}$$

for each $p \in \mathbb{C}^n$. Identity (2.9) means that

$$v(t+\theta) = (T(t,s)\phi)(\theta) + \int_s^{t+\theta} \big(T_0(t,\tau)X_0 g(\tau)\big)(\theta)\,d\tau \tag{2.10}$$

for all $t \geq s$ and $\theta \in [-r,0]$ with $t+\theta \geq s$. In particular, this formula gives the solution $v(t)$ taking $\theta = 0$.

2.3 Partial hyperbolicity

We say that the linear equation (2.6) has an *exponential trichotomy* if:

1. there exist projections $P(t), Q(t), R(t)\colon C \to C$ for $t \geq 0$ satisfying

   $$P(t) + Q(t) + R(t) = \mathrm{Id}$$

   such that for any $t \geq s \geq 0$ we have

   $$P(t)T(t,s) = T(t,s)P(s), \quad Q(t)T(t,s) = T(t,s)Q(s)$$

   and

   $$R(t)T(t,s) = T(t,s)R(s);$$

2. the linear operator

   $$\overline{T}(t,s) := T(t,s)|_{\ker P(s)}\colon \ker P(s) \to \ker P(t) \tag{2.11}$$

   is onto and invertible for each $t \geq s \geq 0$;

3. there exist $\mu, \nu, D > 0$ with $\mu < \nu$ such that for any $t \geq s \geq 0$ we have
$$\|T(t,s)P(s)\| \leq De^{-\nu(t-s)}, \quad \|T(t,s)Q(s)\| \geq D^{-1}e^{\nu(t-s)}$$
and
$$D^{-1}e^{-\mu(t-s)} \leq \|T(t,s)R(s)\| \leq De^{\mu(t-s)}.$$

An *exponential dichotomy* is an exponential trichotomy with $R(s) = 0$ for some $s \geq 0$ (and so with $R(s) = 0$ for all $s \geq 0$). For each $t \geq s \geq 0$ we denote the inverse of the operator $\overline{T}(t,s)$ in (2.11) by
$$\overline{T}(s,t) := \overline{T}(t,s)^{-1} \colon \ker P(t) \to \ker P(s).$$

The *stable, unstable and center spaces* of an exponential trichotomy (or of an exponential dichotomy) at time t are defined, respectively, by
$$E(t) = P(t)(C), \quad F(t) = Q(t)(C) \quad \text{and} \quad G(t) = R(t)(C).$$

Clearly,
$$C = E(t) \oplus F(t) \oplus G(t).$$

The unstable and center spaces are always finite-dimensional, with dimensions independent of t (see for example [8, Chapter 10]). For each $t \geq 0$ we define linear operators
$$P_0(t), Q_0(t), R_0(t) \colon \mathbb{C}^n \to C_0$$
by
$$Q_0(t) = \overline{T}(t,t+r)Q(t+r)T_0(t+r,t)X_0,$$
$$R_0(t) = \overline{T}(t,t+r)R(t+r)T_0(t+r,t)X_0$$
and
$$P_0(t) = X_0 - Q_0(t) - R_0(t).$$

Then
$$P_0(t)p \in C_0 \setminus C, \quad Q_0(t)p \in F \subset C \quad \text{and} \quad R_0(t)p \in G \subset C$$
for each $p \in \mathbb{C}^n$. The following result extends the exponential bounds of an exponential trichotomy to the space C_0.

Proposition 2.1. *If condition (2.3) holds and equation (2.6) has an exponential trichotomy, then there exist $\mu, \nu, N > 0$ such that for any $t \geq s \geq 0$ we have*
$$\|T_0(t,s)P_0(s)\| \leq Ne^{-\nu(t-s)}, \quad \|T(t,s)Q_0(s)\| \geq N^{-1}e^{\nu(t-s)}$$
and
$$N^{-1}e^{-\mu(t-s)} \leq \|T(t,s)R_0(s)\| \leq Ne^{\mu(t-s)}.$$

Proposition 2.1 also holds for an exponential dichotomy, in which case we have $R(s) = 0$ for all $s \geq 0$ and so also $R_0(s) = 0$ for all $s \geq 0$.

3 From hyperbolicity to Ulam–Hyers stability

In this section we establish the Ulam–Hyers stability of an arbitrary nonautonomous linear delay equation with an exponential dichotomy and of its sufficiently small Lipschitz perturbations.

3.1 Basic notions

We first introduce the notion of Ulam–Hyers stability for a delay equation. We consider general perturbations of a nonautonomous linear delay equation. Namely, we assume that:

1. $L(t)\colon C \to \mathbb{C}^n$, for $t \ge 0$, are bounded linear operators such that the map $t \mapsto L(t)$ is strongly continuous on $[0,+\infty)$ and (2.3) holds;
2. $f\colon [0,+\infty) \times C \to \mathbb{C}^n$ is a continuous function.

We say that the equation

$$v' = L(t)v_t + f(t, v_t) \tag{3.1}$$

is *Ulam–Hyers stable* if there exists $\kappa > 0$ such that for each $\varepsilon > 0$ and each continuous function $v\colon [-r,+\infty) \to \mathbb{C}^n$ of class C^1 on $[0,+\infty)$ (taking the right-hand derivative at 0) satisfying

$$\sup_{t\ge 0} \big| v'(t) - L(t)v_t - f(t,v_t) \big| < \varepsilon, \tag{3.2}$$

there exists a solution $w\colon [-r,+\infty) \to \mathbb{C}^n$ of equation (3.1) satisfying

$$\sup_{t\ge 0} \|v_t - w_t\| < \kappa\varepsilon \tag{3.3}$$

Before proceeding, we make a few comments on this notion of stability. We must assume that each function v has derivative on $[0,+\infty)$ so that the supremum in (3.2) is well defined. But in fact one can show that any solution of equation (3.1) is of class C^1 on the interval $[0,+\infty)$ (taking the right-hand derivative at 0). Indeed, let w be any solution of the equation and consider the continuous function $g(t) = f(t, w_t)$. Then, as detailed in Section 2.1, any solution of equation (2.2) is of class C^1 on the interval $[0,+\infty)$ (taking the right-hand derivative at 0). But the function w is a solution of this equation, which thus gives the desired result. On the other hand, this also motivates assuming that the function v in (3.2) is of class C^1 on $[0,+\infty)$.

3.2 Linear case

The following theorem is our first result relating Ulam–Hyers stability and hyperbolicity. It considers the particular case of a nonautonomous *linear* equation (2.6) and shows that the existence of an exponential dichotomy yields the Ulam–Hyers stability of the equation. The proof has the advantage of being more direct than in the general nonlinear case since we construct explicitly the function w in (3.3).

Theorem 3.1. *If the equation $v' = L(t)v_t$ has an exponential dichotomy, then it is Ulam–Hyers stable.*

Proof. Take $\varepsilon > 0$ and a continuous function $v\colon [-r,+\infty) \to \mathbb{C}^n$ of class C^1 on the interval $[0,+\infty)$ satisfying

$$\sup_{t\ge 0} |v'(t) - L(t)v_t| < \varepsilon.$$

Consider the continuous function $g\colon [0,+\infty) \to \mathbb{C}^n$ given by

$$g(t) = v'(t) - L(t)v_t.$$

Note that $\sup_{t\ge 0}|g(t)| < \varepsilon$. For each $t \ge 0$ let

$$w(t) = v(t) - \int_0^t \big(T_0(t,\tau)P_0 g(\tau)\big)(0)\, d\tau + \int_t^{+\infty} \big(\overline{T}(t,\tau)Q_0 g(\tau)\big)(0)\, d\tau.$$

Then for any $t \geq 0$ and $\theta \in [-r, 0]$ with $t + \theta \geq 0$ we have

$$\begin{aligned} w_t(\theta) &= v_t(\theta) - \int_0^{t+\theta} \big(T_0(t+\theta, \tau) P_0 g(\tau)\big)(0)\, d\tau + \int_{t+\theta}^{+\infty} \big(\overline{T}(t+\theta, \tau) Q_0 g(\tau)\big)(0)\, d\tau \\ &= v_t(\theta) - \int_0^{t+\theta} \big(T_0(t, \tau) P_0 g(\tau)\big)(\theta)\, d\tau + \int_{t+\theta}^{+\infty} \big(\overline{T}(t, \tau) Q_0 g(\tau)\big)(\theta)\, d\tau. \end{aligned}$$

This can be written in the form

$$w_t = v_t - \int_0^t T_0(t, \tau) P_0 g(\tau)\, d\tau + \int_t^{+\infty} \overline{T}(t, \tau) Q_0 g(\tau)\, d\tau, \tag{3.4}$$

in a similar manner to that in (2.10). It follows from Proposition 2.1 that

$$\begin{aligned} \int_0^t \|T_0(t, \tau) P_0 g(\tau)\|\, d\tau &\leq \sup_{s \geq 0} |g(s)| \int_0^t N e^{-\nu(t-\tau)}\, d\tau \\ &= \sup_{s \geq 0} |g(s)| \frac{N(1 - e^{-\nu t})}{\nu} < \frac{N\varepsilon}{\nu} \end{aligned}$$

and, similarly,

$$\int_t^{+\infty} \|\overline{T}(t, \tau) Q_0 g(\tau)\|\, d\tau \leq \sup_{s \geq 0} |g(s)| \frac{N}{\nu} < \frac{N\varepsilon}{\nu},$$

for all $t \geq 0$. Therefore, the function $w\colon [-r, +\infty) \to \mathbb{C}^n$ is well defined. Moreover, for any $t \geq s \geq 0$ we have

$$\begin{aligned} v_t - w_t = & \int_s^t T_0(t, \tau) X_0 g(\tau)\, d\tau - \int_s^t T_0(t, \tau) P_0 g(\tau)\, d\tau - \int_s^t T(t, \tau) Q_0 g(\tau)\, d\tau \\ & + \int_0^t T_0(t, \tau) P_0 g(\tau)\, d\tau - \int_t^{+\infty} \overline{T}(t, \tau) Q_0 g(\tau)\, d\tau \\ = & \int_s^t T_0(t, \tau) X_0 g(\tau)\, d\tau + \int_0^s T_0(t, \tau) P_0 g(\tau)\, d\tau - \int_s^{+\infty} \overline{T}(t, \tau) Q_0 g(\tau)\, d\tau. \end{aligned}$$

On the other hand, it also follows from (3.4) that

$$T(t, s)(v_s - w_s) = \int_0^s T_0(t, \tau) P_0 g(\tau)\, d\tau - \int_s^{+\infty} \overline{T}(t, \tau) Q_0 g(\tau)\, d\tau$$

(see for example [8, Section 3.4]). Therefore,

$$v_t - w_t = T(t, s)(v_s - w_s) + \int_s^t T_0(t, \tau) X_0 g(\tau)\, d\tau$$

for all $t \geq s \geq 0$. It follows from the variation of constants formula that

$$(v - w)' = L(t)(v_t - w_t) + g(t).$$

Since v satisfies the equation

$$v' = L(t) v_t + g(t),$$

we conclude that $w' = L(t) w_t$. Moreover,

$$\begin{aligned} \|v_t - w_t\| &\leq \left\| \int_0^t T_0(t, \tau) P_0 g(\tau)\, d\tau - \int_t^{+\infty} \overline{T}(t, \tau) Q_0 g(\tau)\, d\tau \right\| \\ &\leq \int_0^t \|T_0(t, \tau) P_0 g(\tau)\|\, d\tau + \int_t^{+\infty} \|\overline{T}(t, \tau) Q_0 g(\tau)\|\, d\tau \end{aligned}$$

and so

$$\sup_{t\geq 0}\|v_t - w_t\| < \frac{2N\varepsilon}{\nu}.$$

This shows that equation (2.6) is Ulam–Hyers stable with $\kappa = 2N/\nu$. □

A consequence of Theorem 3.1 is the following result. Let $M(t)\colon C \to \mathbb{C}^n$, for $t \geq 0$, be bounded linear operators such that the map $t \mapsto M(t)$ is strongly continuous on $[0, +\infty)$.

Corollary 3.2. *If the equation $v' = L(t)v_t$ has an exponential dichotomy, then there exists $\delta > 0$ such that if*

$$\sup_{t\geq 0}\int_t^{t+1}\|L(\tau) - M(\tau)\|\,d\tau < \delta, \tag{3.5}$$

then the equation $v' = M(t)v_t$ is Ulam–Hyers stable.

Proof. When $v' = L(t)v_t$ has an exponential dichotomy and $\delta > 0$ is sufficiently small, condition (3.5) implies that the equation $v' = M(t)v_t$ also has an exponential dichotomy (see Theorem 6.1 in [8]). Hence, the desired statement follows readily from Theorem 3.1. □

3.3 Nonlinear case

The following theorem is our main result relating Ulam–Hyers stability and hyperbolicity for a *nonlinear* delay equation obtained from perturbing a linear equation with an exponential dichotomy by a continuous map that is Lipschitz on the space variable.

Theorem 3.3. *Assume that the equation $v' = L(t)v_t$ has an exponential dichotomy and that there exists $K > 0$ such that*

$$|f(t,\phi) - f(t,\psi)| \leq K\|\phi - \psi\| \quad \text{for all } t \geq 0 \text{ and } \phi, \psi \in C. \tag{3.6}$$

If K is sufficiently small, then equation (3.1) is Ulam–Hyers stable.

Proof. Take $\varepsilon > 0$ and a continuous function $v\colon [-r, +\infty) \to \mathbb{C}^n$ of class C^1 on $[0, +\infty)$ satisfying (3.2). We consider also the continuous function $g\colon [0, +\infty) \to \mathbb{C}^n$ defined by

$$g(t) = v'(t) - L(t)v_t - f(t, v_t),$$

which satisfies $\sup_{t\geq 0}|g(t)| < \varepsilon$. We want to show that there exists a continuous function $w\colon [-r, +\infty) \to \mathbb{C}^n$ satisfying

$$\begin{aligned} w_t = v_t &- \int_0^t T_0(t,\tau)P_0(\tau)\big[f(\tau, v_\tau) - f(\tau, w_\tau) + g(\tau)\big]\,d\tau \\ &+ \int_t^{+\infty} \overline{T}(t,\tau)Q_0(\tau)\big[f(\tau, v_\tau) - f(\tau, w_\tau) + g(\tau)\big]\,d\tau \end{aligned} \tag{3.7}$$

for all $t \geq 0$ such that the map $t \mapsto v_t - w_t$ is bounded. Let

$$u(t) = w(t) - v(t) \quad \text{and so} \quad u_t = w_t - v_t. \tag{3.8}$$

Moreover, let

$$h(t,\phi) = f(t, v_t) - f(t, \phi + v_t) + g(t). \tag{3.9}$$

Then identity (3.7) becomes

$$u_t = -\int_0^t T_0(t,\tau)P_0(\tau)h(\tau,u_\tau)\,d\tau + \int_t^{+\infty} \overline{T}(t,\tau)Q_0(\tau)h(\tau,u_\tau)\,d\tau.$$

Now we consider the map $G\colon C_b \to C_b$ defined by

$$G(u)_t = -\int_0^t T_0(t,\tau)P_0(\tau)h(\tau,u_\tau)\,d\tau + \int_t^{+\infty} \overline{T}(t,\tau)Q_0(\tau)h(\tau,u_\tau)\,d\tau,$$

where C_b denotes the Banach space of all bounded continuous functions $u\colon [-r,+\infty) \to \mathbb{C}^n$ equipped with the supremum norm

$$\|u\|_\infty = \sup_{t\geq -r}|u(t)|.$$

For each $t \geq 0$ and $u, \overline{u} \in C_b$ we have

$$\begin{aligned}
\|G(u)_t - G(\overline{u})_t\| &\leq \int_0^t \left\|T_0(t,\tau)P_0(\tau)(f(\tau,\overline{u}_\tau + v_\tau) - f(\tau,u_\tau + v_\tau))\right\| d\tau \\
&\quad + \int_t^{+\infty} \left\|\overline{T}(t,\tau)Q_0(\tau)(f(\tau,\overline{u}_\tau + v_\tau) - f(\tau,u_\tau + v_\tau))\right\| d\tau \\
&\leq K\|u - \overline{u}\|_\infty \left(\int_0^t Ne^{-\nu(t-\tau)}\,d\tau + \int_t^{+\infty} Ne^{-\nu(\tau-t)}\,d\tau\right) \\
&\leq \frac{2KN}{\nu}\|u - \overline{u}\|_\infty.
\end{aligned}$$

Therefore,

$$\|G(u) - G(\overline{u})\|_\infty = \sup_{t\geq 0}\|G(u)_t - G(\overline{u})_t\| \leq \frac{2KN}{\nu}\|u - \overline{u}\|_\infty$$

and so the map G is a contraction provided that K is sufficiently small. Moreover, taking $\overline{u} = 0$ we obtain

$$\|G(u)\|_\infty = \sup_{t\geq 0}\|G(u)_t\| \leq \frac{2KN}{\nu}\|u\|_\infty + \sup_{t\geq 0}\|G(0)_t\|.$$

Since $h(\tau,0) = g(\tau)$, proceeding as before we get

$$\sup_{t\geq 0}\|G(0)_t\| \leq \frac{2N}{\nu}\sup_{t\geq 0}|g(t)| < \varepsilon\frac{2N}{\nu}$$

and so

$$\|G(u)\|_\infty \leq \frac{2KN}{\nu}\|u\|_\infty + \varepsilon\frac{2N}{\nu}.$$

This shows that the map G is well-defined. Hence, by the contraction mapping principle, there exists $u \in C_b$ satisfying (3.7). Moreover, u satisfies

$$\|u\|_\infty \leq \frac{2KN}{\nu}\|u\|_\infty + \varepsilon\frac{2N}{\nu}$$

and for $K < \nu/(2N)$ we obtain

$$\|u\|_\infty \leq \varepsilon\frac{2N}{\nu - 2KN}. \tag{3.10}$$

Finally, proceeding as in the proof of Theorem 3.1 with $g(t)$ replaced by $h(t,u_t)$, we find that

$$u_t = T(t,s)u_s - \int_s^t T_0(t,\tau)X_0h(\tau,u_\tau)\,d\tau$$

for all $t \geq s \geq 0$. It follows from the variation of constants formula that

$$u' = L(t)u_t - h(t, u_t).$$

Since v satisfies the equation

$$v' = L(t)v_t + f(t, v_t) + g(t),$$

it follows from (3.8) and (3.9) that

$$w' = L(t)w_t + f(t, w_t).$$

By (3.10), we finally obtain

$$\sup_{t \geq 0} \|v_t - w_t\| < \varepsilon \frac{2N}{\nu - 2KN},$$

and so equation (3.1) is Ulam–Hyers stable with $\kappa = 2N/(\nu - 2KN)$. □

We were informed by the referee that Theorem 3.3 was obtained independently in [4] and extended to the case of weighted Ulam–Hyers stability.

3.4 Measurable right-hand side

While Theorem 3.3 considers equations with continuous right-hand side, one can consider a class of equations with measurable right-hand side. Similarly, one can also consider a notion of Ulam–Hyers stability for which the approximate solution need not be of class C^1 on $[0, +\infty)$.

Theorem 3.4. *Let $L(t)\colon C \to \mathbb{C}^n$ be linear operators for $t \geq 0$, bounded for almost all t, with $t \mapsto L(t)\phi$ measurable for each $\phi \in C$, and satisfying (2.3), and let $f\colon [0, +\infty) \times C \to \mathbb{C}^n$ be a measurable function satisfying (3.6). Then there exists $\kappa > 0$ such that for each $\varepsilon > 0$ and each continuous function $v\colon [-r, +\infty) \to \mathbb{C}^n$ with measurable derivative on $[0, +\infty)$ satisfying (3.2), there exists a solution $w\colon [-r, +\infty) \to \mathbb{C}^n$ of equation (3.1) satisfying (3.3).*

One can in fact replace condition (2.3) by the more general requirement that there exist constants $C, \omega > 0$ such that

$$\|T_0(t, s)\| \leq Ce^{\omega(t-s)} \quad \text{for } t \geq s.$$

The proof of Theorem 3.4 follows almost verbatim the proof of Theorem 3.3, although now the functions g and h may be only measurable in t.

4 From Ulam–Hyers stability to hyperbolicity

In this section we establish the converse of Theorem 3.1 for a large class of autonomous linear equations $v' = Lv_t$ and, more generally, linear equations $v' = L(t)v_t$ for which the map $t \mapsto L(t)$ is periodic. This class includes for example all equations for which the generator of the semigroup induced by the equation has a simple spectrum on the imaginary axis. We first recall a few basic notions, including the spectral properties of the generator, since these are necessary for the proofs. Again we refer the reader to the books [8, 15] for details as well as proofs of these basic notions.

4.1 Basic notions

In this section we consider perturbations of a linear delay equation of the form

$$v' = Lv_t + g(t), \tag{4.1}$$

where $L\colon C \to \mathbb{C}^n$ is a bounded linear operator and $g\colon \mathbb{R} \to \mathbb{C}^n$ is a bounded continuous function. Note that conditions 1 and 2 in Section 2.1 are automatically satisfied. Therefore, for each initial condition $v_s = \phi \in C$ equation (4.1) has a unique solution on $[s-r, +\infty)$. This solution is of class C^1 on the interval $(s, +\infty)$ and satisfies

$$v'(t) = Lv_t + g(t) \quad \text{for } t \in [s, +\infty),$$

taking the right-hand derivative at s.

Now we consider the particular case of a linear equation

$$v' = Lv_t. \tag{4.2}$$

This includes for example the differential difference equations of the form

$$v' = A_0 v + \sum_{i=1}^{k} A_i v(t - \tau_i) \tag{4.3}$$

for some positive numbers $\tau_1 < \tau_2 < \cdots < \tau_k$ and some $n \times n$ matrices A_i for $i = 0, \ldots, k$. Equation (4.2) determines a semigroup $S(t)\colon C \to C$, for $t \geq 0$, defined by

$$S(t)\phi = v_t(\cdot, 0, \phi) \quad \text{for } \phi \in C,$$

where v is the unique solution of equation (4.2) on $[-r, +\infty)$ with $v_0 = \phi$. In fact,

$$S(t-s) = T(t,s) \quad \text{for any } t \geq s \geq 0,$$

where $T(t,s)$ is the evolution family in (2.7). One can also extend the linear operators $S(t)$ to the space C_0. Namely, we first write the linear operator $L\colon C \to \mathbb{C}^n$ in the form

$$L\phi = \int_{-r}^{0} d\eta(\theta)\phi(\theta)$$

for some left-continuous measurable map $\eta\colon [-r, 0] \to M_n$ of bounded variation and then we use it to extend L to C_0. Given $t \geq 0$, we define a linear operator $S_0(t)$ on the space C_0 by

$$S_0(t)\phi = v_t(\cdot, 0, \phi) \quad \text{for } \phi \in C_0,$$

where v is the unique solution of equation (4.2) on $[-r, +\infty)$ with $v_0 = \phi$.

We note that equation (4.2) has an *exponential trichotomy* if:

1. there exist projections $P, Q, R\colon C \to C$ satisfying $P + Q + R = \mathrm{Id}$ such that for all $t \geq 0$ we have
$$PS(t) = S(t)P, \quad QS(t) = S(t)Q \quad \text{and} \quad RS(t) = S(t)R;$$
2. the linear operator
$$\overline{S}(t) := S(t)|_{\ker P}\colon \ker P \to \ker P$$
is onto and invertible for each $t \geq 0$;

3. there exist $\mu, \nu, D > 0$ with $\mu < \nu$ such that for any $t \geq 0$ we have
$$\|S(t)P\| \leq De^{-\nu t}, \quad \|S(t)Q\| \geq D^{-1}e^{\nu t}$$
and
$$D^{-1}e^{-\mu t} \leq \|S(t)R\| \leq De^{\mu t}. \tag{4.4}$$

An *exponential dichotomy* is an exponential trichotomy with $R = 0$. It turns out that any autonomous linear equation (4.2) has an exponential trichotomy (possibly with $R = 0$), as a consequence of the spectral properties of the generator of $S(t)$ (see Proposition 4.1).

We define linear operators $P_0, Q_0, R_0 \colon \mathbb{C}^n \to C_0$ by
$$Q_0 = \overline{S}(-r)QS_0(r)X_0, \quad R_0 = \overline{S}(-r)RS_0(r)X_0$$
and
$$P_0 = X_0 - Q_0 - R_0.$$
For each $p \in \mathbb{C}^n$ we have
$$P_0 p \in C_0 \setminus C, \quad Q_0 p \in F \subset C \quad \text{and} \quad R_0 p \in G \subset C.$$
In a similar manner to that in Proposition 2.1, one can extend the exponential bounds of an exponential trichotomy to the space C_0 (notice that condition (2.3) is now automatically satisfied).

Finally, we recall some properties of the semigroup $S(t)$ and its generator that will be used later on.

Proposition 4.1. *The following properties hold:*

1. *$S(t)$ is a strongly continuous semigroup with generator $A \colon D(A) \to C$ given by*
$$A\phi := \lim_{t \searrow 0} \frac{S(t)\phi - \phi}{t} = \phi'$$
in the domain
$$D(A) = \{\phi \in C : \phi' \in C, \phi'(0) = L\phi\};$$
2. *the spectrum $\sigma(A)$ is composed of eigenvalues, for each $\gamma \in \mathbb{R}$ there are finitely many numbers $\lambda \in \sigma(A)$ satisfying $\operatorname{Re}\lambda > \gamma$, and*
$$\sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\} < +\infty;$$
3. *the generalized eigenspace space M_λ of each $\lambda \in \sigma(A)$ is finite-dimensional, there exists $k \in \mathbb{N}$ such that $M_\lambda = \ker(A - \lambda \mathrm{Id})^k$ and*
$$C = M_\lambda \oplus N_\lambda \quad \text{with } N_\lambda = \operatorname{Im}(A - \lambda \mathrm{Id})^k;$$
4. *if $\Phi_\lambda = \{\phi_1, \ldots, \phi_d\}$ is a basis for M_λ, then there exists a $d \times d$ matrix B_λ with a single eigenvalue λ such that $A\Phi_\lambda = \Phi_\lambda B_\lambda$,*
$$S(t)\Phi_\lambda = \Phi_\lambda e^{B_\lambda t} \quad \text{for } t \in \mathbb{R}$$
and so also
$$(S(t)\Phi_\lambda)(\theta) = \Phi_\lambda(0)e^{B_\lambda(t+\theta)} \quad \text{for } t \in \mathbb{R} \text{ and } \theta \in [-r, 0]. \tag{4.5}$$

The identity $A\Phi_\lambda = \Phi_\lambda B_\lambda$ in property 4 means that $A\Phi_\lambda a = \Phi_\lambda B_\lambda a$ for any $a \in \mathbb{C}^d$ and a similar observation applies to the remaining identities. A consequence of the former proposition is that any autonomous linear equation (4.2) has an exponential trichotomy (possibly with $R = 0$), in fact with an arbitrarily small constant μ in (4.4).

The stable, unstable and center spaces of an exponential trichotomy are now independent of time and are given, respectively, by

$$E = P(C), \quad F = Q(C) \quad \text{and} \quad G = R(C).$$

In fact, we have

$$E = \bigcap_{\operatorname{Re}\lambda \geq 0} N_\lambda, \quad F = \bigoplus_{\operatorname{Re}\lambda > 0} M_\lambda \quad \text{and} \quad G = \bigoplus_{\operatorname{Re}\lambda = 0} M_\lambda.$$

Clearly, $C = E \oplus F \oplus G$ and the spaces F and G are finite-dimensional.

Moreover, we have the following result.

Proposition 4.2. *Equation* (4.2) *has an exponential dichotomy if and only if the spectrum $\sigma(A)$ does not intersect the imaginary axis.*

4.2 Autonomous case

Now we consider the converse of Theorem 3.1 for an autonomous linear delay equation assuming that for any eigenvalue $\lambda \in \sigma(A)$ with $\operatorname{Re}\lambda = 0$ the corresponding matrix B_λ (see Proposition 4.1) has a diagonal Jordan form. Under this assumption, we present our main result for a *linear* delay equation: the Ulam–Hyers stability of the equation implies that there exists an exponential dichotomy.

Theorem 4.3. *If equation* (4.2) *is Ulam–Hyers stable, then either it has an exponential dichotomy or for some eigenvalue $\lambda \in \sigma(A)$ with* $\operatorname{Re}\lambda = 0$ *the matrix B_λ has a nondiagonal Jordan form.*

Proof. Assume that equation (4.2) does not have an exponential dichotomy and that for any eigenvalue $\lambda \in \sigma(A)$ with $\operatorname{Re}\lambda = 0$ the matrix B_λ has a diagonal Jordan form. Since the equation does not have an exponential dichotomy, by Proposition 4.2 indeed there exists an eigenvalue $\lambda \in \sigma(A)$ with $\operatorname{Re}\lambda = 0$ (for which the matrix B_λ has thus a diagonal Jordan form). Take $\phi_\lambda \in M_\lambda$ normalized so that $|\phi_\lambda(0)| < 1$. Then necessarily $\phi_\lambda(0) \neq 0$ for some $\phi_\lambda \in M_\lambda$ since otherwise it would follow from (4.5) that the generalized eigenspace was $M_\lambda = \{0\}$. More precisely, by property (4.5), the solutions $w(t)$ of equation (4.2) with $w_0 \in M_\lambda$ are

$$w(t) = c(S(t)\phi_\lambda)(0) = ce^{\lambda t}\phi_\lambda(0) \tag{4.6}$$

with $c \in \mathbb{C}$. Given $\varepsilon > 0$, let

$$\psi(t) = \varepsilon e^{\lambda t}\phi_\lambda(0)/(1 + r\|L\|)$$

and consider the function

$$g(t) = \psi(t) - \int_{-r}^{0} d\eta(\theta)\theta\psi(t+\theta). \tag{4.7}$$

Note that g is continuous and that

$$\sup_{t\geq 0}|g(t)| \leq \|\psi\|(1 + r\|L\|) < \varepsilon.$$

The unbounded function $v(t) = t\psi(t)$ satisfies

$$v'(t) = \psi(t) + t\psi'(t) = \psi(t) + tL\psi_t$$

since ψ is a solution of equation (4.2). Moreover,

$$Lv_t = \int_{-r}^{0} d\eta(\theta)(t+\theta)\psi(t+\theta) = tL\psi_t + \int_{-r}^{0} d\eta(\theta)\theta\psi(t+\theta)$$

and so

$$v'(t) - Lv_t = \psi(t) - \int_{-r}^{0} d\eta(\theta)\theta\psi(t+\theta) = g(t).$$

This shows that v is an unbounded function satisfying

$$\sup_{t\ge 0} |v'(t) - Lv_t| < \varepsilon.$$

In order to obtain a contradiction, we consider an arbitrary solution $w(t)$ of the linear equation (4.2). Note that

$$\|Pw_t\| = \|S(t)Pw_0\| \le De^{-\nu t}\|w_0\| \le D\|w_0\|$$

and

$$\|Qw_t\| = \|S(t)Qw_0\| \ge D^{-1}e^{\nu t}\|w_0\|$$

for all $t \ge 0$. Now we observe that there are finitely many eigenvalues λ of A with $\operatorname{Re}\lambda = 0$. Moreover, the matrix B_λ of each of them has a diagonal Jordan form. Finally, denoting by $\Pi_{\lambda'}$ the projection onto $M_{\lambda'}$ it follows readily from (4.6) that

$$\|Rw_t\| = \|S(t)Rw_0\| \le \sum_{\operatorname{Re}\lambda'=0} \|S(t)\Pi_{\lambda'}w_0\| = \sum_{\operatorname{Re}\lambda'=0} \|\Pi_{\lambda'}w_0\|$$

for all $t \ge 0$. Therefore, there exists a constant $N > 0$ such that

$$\|(P+R)w_t\| = \|S(t)(P+R)w_0\| \le N$$

for all $t \ge 0$. This implies that if $Qw_0 \ne 0$, then

$$\begin{aligned}\|v_t - w_t\| &\ge \|Qw_t\| - \|v_t - (P+R)w_t\| \\ &\ge D^{-1}e^{\nu t}\|w_0\| - \sup_{\theta\in[-r,0]} |t+\theta|\varepsilon|\phi_\lambda(0)|/(1+r\|L\|) - N \to +\infty\end{aligned}$$

when $t \to +\infty$, which shows that

$$\sup_{t\ge 0}\|v_t - w_t\| = +\infty \tag{4.8}$$

when $Qw_0 \ne 0$. Now we assume that $Qw_0 = 0$. In this case we have

$$\begin{aligned}\|v_t - w_t\| &\ge |v(t)| - \|(P+R)w_t\| \\ &\ge t\varepsilon|\phi_\lambda(0)|/(1+r\|L\|) - N \to +\infty.\end{aligned}$$

when $t \to +\infty$. This shows that (4.8) also holds when $Qw_0 = 0$, which contradicts the hypothesis that equation (4.2) is Ulam–Hyers stable. Therefore, either the equation has an exponential dichotomy or for some eigenvalue $\lambda \in \sigma(A)$ with $\operatorname{Re}\lambda = 0$ the matrix B_λ has a nondiagonal Jordan form. □

The more general case of a linear delay equation with periodic coefficients is considered later on in Section 4.4. However, the proof requires substantial additional material that also needs to be introduced. For this reason we have preferred to give first the former streamlined proof for autonomous equations. In its turn, it is this proof that motivates the approach for delay equations with periodic coefficients.

4.3 Diagonal Jordan forms

The main difficulty in considering nondiagonal normal forms for the eigenvalues of the generator A is that one may not be able to obtain a bounded function g as in (4.7). Indeed, if the function ψ is obtained from a generalized eigenvector a in the form

$$\psi(t) = \varepsilon(S(t)\Phi_\lambda a)(0),$$

then g may not be bounded, simply because it may involve nonconstant polynomials. This means that this approach need not work for an arbitrary autonomous linear equation $v' = Lv_t$.

A corollary of the former Theorems 3.1 and 4.3 is a complete characterization of the Ulam–Hyers stability of a linear delay equation when the eigenvalues of the generator A on the imaginary axis have diagonal Jordan forms.

Corollary 4.4. *Assume that for any $\lambda \in \sigma(A)$ with $\operatorname{Re}\lambda = 0$ the matrix B_λ has a diagonal Jordan form. Then equation (4.2) is Ulam–Hyers stable if and only if it has an exponential dichotomy.*

It remains an open problem whether a similar characterization holds without the hypothesis on diagonal Jordan forms. We are not aware of counterexamples, even though explicit computations are always somewhat involved.

On the other hand, at least for differential difference equations as in (4.3) we can show that if the spectrum of the generator A is simple on the imaginary axis (for which thus the hypothesis on diagonal Jordan forms holds), then any sufficiently close equation is Ulam–Hyers stable if and only if it has an exponential dichotomy. More precisely, we have the following result.

Corollary 4.5. *For equation (4.3) assume that any eigenvalue $\lambda \in \sigma(A)$ with $\operatorname{Re}\lambda = 0$ is simple. Then there exists $\delta > 0$ such that any equation*

$$v' = A_0'v + \sum_{i=1}^{k} A_i'v(t-\tau_i) \tag{4.9}$$

with

$$\|A_i' - A_i\| < \delta \quad \text{for } i = 0,\dots,k \tag{4.10}$$

is Ulam–Hyers stable if and only if it has an exponential dichotomy.

Proof. We recall that the eigenvalues of the generator A are the roots of the characteristic equation $\det\Delta(\lambda) = 0$, where

$$\Delta(\lambda) = \int_{-r}^{0} e^{\lambda\theta}d\eta(\theta) - \lambda\mathrm{Id}.$$

For equation (4.3) this becomes

$$h(\lambda) := \det\left(A_0 + \sum_{i=1}^{k} A_i e^{-\lambda\tau_i} - \lambda\mathrm{Id}\right) = 0.$$

Since the function h is holomorphic, one can use Rouché's theorem to deduce the continuity of the eigenvalues of A on the matrices A_i. In particular, given an eigenvalue λ of multiplicity m, there exists $\delta > 0$ such that for any equation (4.9) satisfying (4.10) there are exactly m eigenvalues, counted with multiplicities, of the corresponding generator of the induced semigroup (see [23,24] for details and related discussions).

This has the following consequence. If for equation (4.3) any eigenvalue $\lambda \in \sigma(A)$ with $\operatorname{Re}\lambda = 0$ is simple, then any equation (4.9) satisfying (4.10) for some sufficiently small $\delta > 0$ has the property that any eigenvalue of the generator of the corresponding semigroup on the imaginary axis is also simple. It follows from Corollary 4.4 that any such equation is Ulam–Hyers stable if and only if it has an exponential dichotomy. □

4.4 Periodic case

In this section we consider the more general case of a linear equation

$$v' = L(t)v_t, \tag{4.11}$$

where $L(t)\colon C \to \mathbb{C}^n$, for $t \geq 0$, are bounded linear operators such that:

1. the map $t \mapsto L(t)$ is strongly continuous on $[0, +\infty)$;
2. there exists $\omega > 0$ such that $L(t+\omega) = L(t)$ for all $t \geq 0$.

Note that condition (2.3) is automatically satisfied.

We recall a few properties of the solutions of the linear equation (4.11) that are necessary for the arguments. We refer the reader to the book [15] for details. Consider the operator $U\colon C \to C$ defined by

$$U\phi = T(\omega, 0)\phi$$

with the evolution family $T(t,s)$ as in (2.7). The spectrum $\sigma(U)$ is a countable compact subset of $\mathbb{C}$ accumulating at most at 0 and any number $\mu \in \sigma(U) \setminus \{0\}$ is an eigenvalue of U, called a *characteristic multiplier* of equation (4.11). Moreover, any number $\lambda \in \mathbb{C}$ satisfying $\mu = e^{\lambda\omega}$ for some eigenvalue $\mu \neq 0$ is called a *characteristic exponent* of the equation.

Proposition 4.6. *Given a characteristic multiplier μ, the following properties hold:*

1. *for each $s \in \mathbb{R}$ there exists a splitting $C = M_\mu(s) \oplus N_\mu(s)$ into closed subspaces with $M_\mu(s)$ is finite-dimensional such that*

   $$U(s)M_\mu(s) \subset M_\mu(s) \quad \text{and} \quad U(s)N_\mu(s) \subset N_\mu(s),$$

   where $U(s) = T(s+\omega, s)$;

2. *we have*

   $$\sigma(U(s)|_{M_\mu(s)}) = \{\mu\} \quad \text{and} \quad \sigma(U(s))|_{N_\mu(s)}) = \sigma(U) \setminus \{\mu\};$$

3. *if $\Phi_{\mu,s}$ is a basis for $M_\mu(s)$, then $T(t,s)\Phi_{\mu,s}$ is a basis for the space $M_\mu(t)$ for each $t \in \mathbb{R}$;*

4. *if $\dim M_\mu(s) = d$, then there exist a $d \times d$ matrix C_μ and vectors $P(t) \in C^d$ for $t \in \mathbb{R}$ such that $\sigma(e^{C_\mu\omega}) = \{\mu\}$,*

   $$P(t+\omega) = P(t) \quad \text{for } t \in \mathbb{R}$$

   and

   $$T(t,0)\Phi_{\mu,0} = P(t)e^{C_\mu t} \quad \text{for } t \in \mathbb{R}. \tag{4.12}$$

Identity (4.12) means that

$$(T(t,0)\Phi_{\mu,0})(\theta) = P(t)(\theta)e^{C_\mu t}$$

for all $t \in \mathbb{R}$ and $\theta \in [-r, 0]$. In particular, taking $\theta = 0$ we find that any solution $v(t)$ of equation (4.11) with initial condition in the space $M_\mu(0)$ for some characteristic multiplier $\mu = e^{\lambda\omega}$ is obtained multiplying the exponential $e^{\lambda t}$ by a polynomial in t whose coefficients are ω-periodic in t.

A consequence of the former properties is that any linear equation (4.11) with periodic coefficients has an exponential trichotomy (possibly with projections $R(s) = 0$ for each $s \in \mathbb{R}$), whose stable, unstable and center spaces at time s are given, respectively, by

$$E(s) = \bigcap_{|\mu|\geq 1} N_\mu(s), \quad F(s) = \bigoplus_{|\mu|>1} M_\mu(s) \quad \text{and} \quad G(s) = \bigoplus_{|\mu|=1} M_\mu(s).$$

Note that $F(s)$ and $G(s)$ are always finite-dimensional (for example since each space $M_\mu(s)$ is finite-dimensional and since $\sigma(U)$ accumulates at most at 0, although it is always the case that the unstable and center spaces of an exponential trichotomy are finite-dimensional). Moreover, equation (4.11) has an exponential dichotomy if and only if $\sigma(U)$ does not intersect the unit circle S^1.

The following result is a generalization of Theorem 4.3 for linear equations with periodic coefficients.

Theorem 4.7. *If equation (4.11) is Ulam–Hyers stable, then either it has an exponential dichotomy or for some characteristic multiplier $\mu \in \sigma(U)$ with $|\mu| = 1$ the matrix C_μ has a nondiagonal Jordan form.*

Proof. The proof is analogous to that of Theorem 4.7 and so we only give a sketch. Assume that equation (4.11) does not have an exponential dichotomy and that for any characteristic multiplier $\mu \in \sigma(U)$ with $|\mu| = 1$ the matrix C_μ has a diagonal Jordan form. Since by hypothesis the equation does not have an exponential dichotomy, there exists such a characteristic multiplier. Writing $\mu = e^{\lambda\omega}$ with $\operatorname{Re}\lambda = 0$, we consider the solutions $w(t)$ of equation (4.11) with $w_0 \in M_\mu$ of the form $w(t) = ce^{\lambda t}p(t)$ with $c \in \mathbb{C}$ and where $p\colon \mathbb{R} \to \mathbb{C}^n$ is a continuous function such that $p(t+\omega) = p(t)$ for all $t \in \mathbb{R}$. Given $\varepsilon > 0$, let $\psi(t) = \varepsilon e^{\lambda t}p(t)$ and

$$g(t) = \psi(t) - \int_{-r}^{0} d\eta(t,\theta)\theta\psi(t+\theta).$$

Note that

$$\begin{aligned} |g(t)| &\leq |\psi(t)| + \|L(t)\| \sup_{\theta\in[-r,0]} |\theta\psi(t+\theta)| \\ &\leq |\varepsilon e^{\lambda t}p(t)| + \|L(t)\| \sup_{\theta\in[-r,0]} r|\varepsilon e^{\lambda(t+\theta)}p(t+\theta)| \\ &\leq \varepsilon|p(t)| + \|L(t)\| \sup_{\theta\in[-r,0]} r\varepsilon|p(t+\theta)| \leq \varepsilon\alpha, \end{aligned}$$

where

$$\alpha := \sup_{t\in\mathbb{R}} \left(|p(t)| + \|L(t)\| \sup_{\theta\in[-r,0]} r|p(t+\theta)| \right)$$

is finite because it is the supremum of a periodic function. One can easily verify that $v(t) = t\psi(t)$ satisfies $v'(t) - L(t)v_t = g(t)$ and so

$$\sup_{t\geq 0}|v'(t) - L(t)v_t| < \varepsilon\alpha.$$

In a similar manner to that in the proof of Theorem 4.7 we obtain

$$\|P(t)w_t\| = \|T(t,0)P(0)w_0\| \leq De^{-\nu t}\|w_0\| \leq D\|w_0\| \tag{4.13}$$

and

$$\|Q(t)w_t\| = \|T(t,0)Q(0)Qw_0\| \geq D^{-1}e^{\nu t}\|w_0\|$$

for all $t \geq 0$. Moreover, denoting by $\Pi_{\mu'}$ the projection onto $M_{\mu'}(0)$ we have

$$\|R(t)w_t\| = \|T(t,0)R(0)w_0\| \leq \sum_{|\mu'|=1} \|T(t,0)\Pi_{\mu'}w_0\|$$

for all $t \geq 0$. On the other hand, writing $\mu' = e^{\lambda'\omega}$ with $\operatorname{Re}\lambda' = 0$, the solutions $w(t)$ of equation (4.11) with $w_0 \in M_{\mu'}$ are given by $w(t) = ce^{\lambda' t}q(t)$ with $c \in \mathbb{C}$ and where $q\colon \mathbb{R} \to \mathbb{C}^n$ is a continuous function such that $q(t+\omega) = q(t)$ for all $t \in \mathbb{R}$. Therefore,

$$\begin{aligned}
\|T(t,0)\Pi_{\mu'}w_0\| &\leq \sup_{\theta\in[-r,0]} |ce^{\lambda'(t+\theta)}q(t+\theta)| \\
&\leq \sup_{t\in\mathbb{R}} \sup_{\theta\in[-r,0]} |cq(t+\theta)| \\
&= \sup_{t\in\mathbb{R}} |cq(t)| < +\infty,
\end{aligned}$$

for some $c \in \mathbb{R}$. Since there are finitely many characteristic multipliers on S^1, this implies that there exists a constant $K > 0$ such that

$$\|R(t)w_t\| \leq \sum_{|\mu'|=1} \|T(t,0)\Pi_{\mu'}w_0\| \leq K. \tag{4.14}$$

Finally, by (4.13) and (4.14) there exists $N > 0$ such that

$$\|(\mathrm{Id} - Q(t))w_t\| \leq N$$

for all $t \geq 0$. If $Q(0)w_0 \neq 0$, then

$$\begin{aligned}
\|v_t - w_t\| &\geq \|Q(t)w_t\| - \|v_t - (\mathrm{Id} - Q(t))w_t\| \\
&\geq D^{-1}e^{\nu t}\|w_0\| - \sup_{\theta\in[-r,0]} \varepsilon|(t+\theta)e^{\lambda(t+\theta)}p(t+\theta)| - N \to +\infty
\end{aligned}$$

when $t \to +\infty$, because the function $e^{\lambda t}p(t)$ is bounded. On the other hand, if $Q(0)w_0 = 0$, then

$$\|v_t - w_t\| \geq |v(t)| - \|(\mathrm{Id} - Q(t))w_t\| \geq t\varepsilon|e^{\lambda t}p(t)| - N.$$

Note that the function $|e^{\lambda t}p(t)| = |p(t)|$ is ω-periodic and so its maximum is attained at some times $t_k = t_0 + k\omega$ with $k \in \mathbb{N}$. This implies that

$$\|v_{t_k} - w_{t_k}\| \geq t_k\varepsilon \max_{t\in\mathbb{R}} |p(t)| - N \to +\infty$$

when $k \to +\infty$. In both cases property (4.8) holds and so we obtain a contradiction to the hypothesis that equation (4.11) is Ulam–Hyers stable. This yields the desired statement. □

Building on the proof of the former theorem we formulate a result for arbitrary nonautonomous linear delay equations $v' = L(t)v_t$ under certain additional assumptions. We refrain from including the proof since it corresponds to make slight changes in the former argument.

Theorem 4.8. *Assume that:*

1. *$L(t) \colon C \to \mathbb{C}^n$, for $t \geq 0$, are bounded linear operators such that the map $t \mapsto L(t)$ is bounded and strongly continuous on $[0, +\infty)$;*
2. *equation (4.11) has an exponential trichotomy such that all solutions with initial condition in the center space $G(0)$ are bounded;*
3. *there exist $\delta > 0$ and a solution ψ of equation (4.11) with initial condition in $G(0)$ such that $|\psi(t_n)| > \delta$ for some sequence $t_n \to +\infty$.*

If equation (4.11) is Ulam–Hyers stable, then it has an exponential dichotomy.

Note that by property 1 condition (2.3) is automatically satisfied.

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Bifurcation for a class of piecewise cubic systems with two centers

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Abstract. In this paper, a class of symmetric cubic planar piecewise polynomial systems are presented, which have two symmetric centers corresponding to two period annuli. By perturbation and considering piecewise first order Melnikov function, we show the existence of 18 limit cycles (not small-amplitude limit cycles) with the configuration $(9, 9)$ bifurcating from the two period annuli and 22 small-amplitude limit cycles with the configuration $(11, 11)$, respectively.

Keywords: piecewise cubic system, limit cycle, piecewise first order Melnikov function.

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1 Introduction

During the past sixty years, many problems arising from mechanics, electrical engineering and automatic control are described by non-smooth systems in [1, 2, 9]. Piecewise systems, a class of non-smooth systems which have different definitions for the vector fields in different regions divided by lines or curves, have attracted much attention due to their complex dynamic phenomena and wide applications. Usually, a planar piecewise system with two zones has the form

$$(\dot{x}, \dot{y}) = \begin{cases} Z^{+}(x, y), & h(x, y) > 0, \\ Z^{-}(x, y), & h(x, y) < 0, \end{cases}$$

where $Z^{\pm}(x, y)$ are analytic functions in $\{(x, y) : \pm h(x, y) \geq 0\}$ respectively, and $h(x, y)$ is a continuous function.

Similar to the planar smooth systems, a natural and important topic in the qualitative theory of planar piecewise systems is to find the number and configuration of limit cycles. Moreover, the piecewise systems can exhibit more complex dynamic behaviors than the classical smooth systems. For instance, in contrast to non-existence of the limit cycle in

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planar linear systems, the piecewise linear ones can possess limit cycles, one is referred to [4, 10, 14, 15, 17, 18, 22] for details.

Below we give a brief introduction to the piecewise quadratic and cubic systems. For piecewise quadratic system, da Cruz *et al.* in [8] recently constructed a system with at least 16 limit cycles. For more stories, one can see [5, 21] and references therein. For piecewise cubic systems, only a small amount of work has been done recently, see [11, 13, 16, 19, 23] for example. In [16] Llibre *et al.* obtained 12 limit cycles bifurcating from a period annulus of some cubic system under piecewise cubic perturbation. Later, a piecewise cubic system with 15 limit cycles was constructed by Li *et al.* [19]. Recently, Guo *et al.* considered in [13] a class of $\mathbf{Z}_2$-equivariant piecewise cubic systems with two centers at $(-1, 0)$ and $(1, 0)$, and showed by computing the Lyapunov quantities that there exist 18 small-amplitude limit cycles with configuration $(9, 9)$. Here by configuration $(9, 9)$, we mean that 9 limit cycles surround $(1, 0)$ and the remaining ones surround $(-1, 0)$, simultaneously. Yu *et al.* in [23] also obtained the existence of 18 small-amplitude limit cycles by computing the Lyapunov quantities in a planar piecewise cubic polynomial system. Note that the difference between them is that the authors in [23] obtained 18 limit cycles bifurcating from the two symmetric foci, and each of them will present 9 limit cycles. Of course, the calculations in [13, 23] have high techniques, as they involves nonlinear equations. In [11], Gouveia and Torregrosa got, also by computing the Lyapunov quantities through the parallelization algorithm, 24 crossing small-amplitude limit cycles emerging from a piecewise cubic polynomial center at the cost of quite complicated computations. Very recently, an improvement in the number of crossing limit cycles in the cubic family is obtained by Gouveia and Torregrosa in [12], where the calculations are also based on the parallelization algorithm.

Motivated by [13, 23] in which two symmetrical nests are considered simultaneously, this paper is devoted to investigating limit cycles bifurcating from piecewise cubic polynomial system with two symmetric centers, each of which corresponds to a period annulus full of closed orbits. In detail, we focus on the following piecewise cubic polynomial system

$$\begin{cases} H^+(x,y) = \Psi^+(x) + \Phi^+(y), & x > 0, \\ H^-(x,y) = \Psi^-(x) + \Phi^-(y), & x < 0, \end{cases} \tag{1.1}$$

where $\Psi^\pm(x)$ and $\Phi^\pm(y)$ are quartic polynomials such that $\Phi^\pm(y) = \Phi^\pm(-y)$. Concretely, system (1.1) has form

$$(\dot{x}, \dot{y}) = \begin{cases} \left(a_1 y + a_2 y^3,\, a_3 + a_4 x + a_5 x^2 + a_6 x^3\right), & x > 0, \\ \left(b_1 y + b_2 y^3,\, b_3 + b_4 x + b_5 x^2 + b_6 x^3\right), & x < 0, \end{cases} \tag{1.2}$$

where a_i and b_i, $i = 1, \dots, 6$, are real coefficients. Without loss of generality, assume that system (1.2) has two symmetric centers at $(0, \pm 1)$ which yield

$$a_1 = -a_2, \quad b_1 = -b_2, \quad a_3 = b_3 = 0, \quad a_1 a_4 > 0, \quad b_1 b_4 > 0.$$

Next, by the transformations $(x, y, t) \to \left(\sqrt{\frac{a_1}{2a_4}}x, y, \frac{1}{\sqrt{2a_1 a_4}}t\right)$ and $(x, y, t) \to \left(\sqrt{\frac{b_1}{2b_4}}x, y, \frac{1}{\sqrt{2b_1 b_4}}t\right)$ for $x > 0$ and $x < 0$, respectively, the system (1.2) becomes

$$(\dot{x}, \dot{y}) = \begin{cases} \left(y(1-y^2),\, \dfrac{1}{2}x + \bar{a}x^2 + \bar{c}x^3\right), & x > 0, \\ \left(y(1-y^2),\, \dfrac{1}{2}x + \bar{b}x^2 + \bar{d}x^3\right), & x < 0, \end{cases} \tag{1.3}$$

where $\bar{a} = \frac{a_5}{2a_4}\sqrt{\frac{a_1}{2a_4}}$, $\bar{c} = \frac{a_1 a_6}{4a_4^2}$, $\bar{b} = \frac{b_5}{2b_4}\sqrt{\frac{b_1}{2b_4}}$, and $\bar{d} = \frac{b_1 b_6}{4b_4^2}$. Under piecewise cubic polynomial perturbation, we consider

$$(\dot{x}, \dot{y}) = \begin{cases} \left(y(1-y^2) + \epsilon f^+(x,y),\ \dfrac{1}{2}x + \bar{a}x^2 + \bar{c}x^3 + \epsilon g^+(x,y) \right), & x > 0, \\ \left(y(1-y^2) + \epsilon f^-(x,y),\ \dfrac{1}{2}x + \bar{b}x^2 + \bar{d}x^3 + \epsilon g^-(x,y) \right), & x < 0, \end{cases} \tag{1.4}$$

where $f^+(x,y) = \sum_{i+j=0}^{3} a_{ij}x^i y^j$, $f^-(x,y) = \sum_{i+j=0}^{3} c_{ij}x^i y^j$, $g^+(x,y) = \sum_{i+j=0}^{3} b_{ij}x^i y^j$, $g^-(x,y) = \sum_{i+j=0}^{3} d_{ij}x^i y^j$.

To investigate the number of the limit cycles bifurcating from the two period annuli, we will apply the first order Melnikov function, also known as the Abelian integral, rather than the Lyapunov quantities to reduce the computation. One of our two main results is stated as follows.

Theorem 1.1. *For sufficiently small $|\epsilon| > 0$, there exists a system of the form (1.4) possessing at least 18 limit cycles with configuration $(9,9)$.*

Note that 18 limit cycles in Theorem 1.1 obtained by the first order Melnikov function and Lemma 2.2 are no longer small-amplitude, which differs from the conclusions of 18 small-amplitude limit cycles in [13,23].

The following result states the existence of 22 small-amplitude limit cycles (near the centers) with configuration $(11,11)$, which improves the results in [13,23]. Although 22 small-amplitude limit cycles is not as good as previous results in [11,12], this is new and good from the point of view of simultaneity.

Theorem 1.2. *For sufficiently small $|\epsilon| > 0$, there exists a system of the form (1.4) possessing at least 22 small-amplitude limit cycles with configuration $(11,11)$.*

The rest of this paper is organized as follows. In section 2, we will introduce the piecewise first order Melnikov function firstly. Meanwhile, some lemmas which will be applied to prove our main theorems are presented. Section 3 is devoted to the proofs of Theorems 1.1 and 1.2.

2 Preliminary results

In this section, we introduce the piecewise first order Melnikov function. For this we need the following result from the work of Liu and Han [20] in which the authors studied system

$$(\dot{x}, \dot{y}) = \begin{cases} (H_y^+(x,y) + \epsilon f^+(x,y), -H_x^+(x,y) + \epsilon g^+(x,y)) & x > 0, \\ (H_y^-(x,y) + \epsilon f^-(x,y), -H_x^-(x,y) + \epsilon g^-(x,y)) & x < 0, \end{cases} \tag{2.1}$$

where $f^\pm(x,y), g^\pm(x,y), H^\pm(x,y)$ are analytic functions and suppose the following two assumptions **H1** and **H2** hold.

H1. There exists an open interval (α, β), and two points $A(h) = (0, r(h)), C(h) = (0, \tilde{r}(h))$, where $r(h) \neq \tilde{r}(h)$. For $h \in (\alpha, \beta)$, we have $H^+(A(h)) = H^+(C(h)) = h$, $H^-(A(h)) = H^-(C(h))$.

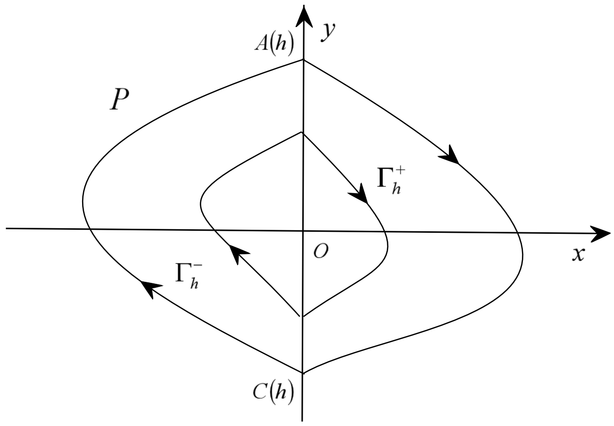

Figure 2.1: The graph shows the structure of **H1** and **H2**.

H2. When $x > 0$, system (2.1)$|_{\epsilon=0}$ has an orbital arc Γ_h^+ starting from $A(h)$ and ending at $C(h)$ defined by $H^+(x,y) = h$. When $x \leq 0$, system (2.1)$|_{\epsilon=0}$ has an orbital arc Γ_h^- starting from $C(h)$ and ending at $A(h)$ defined by $H^-(x,y) = H^-(A(h))$, as illustrated in Figure 2.1.

Then Liu and Han give the piecewise first order Melnikov function, also known as Abelian integral, for (2.1) in [20] as follows.

Lemma 2.1. *Under assumptions* **H1** *and* **H2**, *for sufficiently small* $|\epsilon| > 0$, *then*

(1) *the Abelian integral of system* (2.1) *can be expressed as*

$$I(h) = \frac{H_y^+(A(h))}{H_y^-(A(h))}\left(\frac{H_y^-(C(h))}{H_y^+(C(h))}\int_{\Gamma_h^+} g^+(x,y)\mathrm{d}x - f^+(x,y)\mathrm{d}y + \int_{\Gamma_h^-} g^-(x,y)\mathrm{d}x - f^-(x,y)\mathrm{d}y\right); \quad (2.2)$$

(2) *system* (2.1) *has a limit cycle near* Γ_{h^*}, *if* $I(h)$ *has a simple root* h^* $(I(h^*) = 0, I'(h^*) \neq 0)$;

(3) *system* (2.1) *has at least k limit cycles, if* $I(h)$ *has k roots.*

Applying Lemma 2.1 to consider the problem of limit cycles, a difficult and necessary work is to estimate the number of roots of (2.2). For this purpose, a well-known result, see Lemma 4.5 in [7], is presented as follows.

Lemma 2.2. *Consider* $p+1$ *linearly independent analytical functions* $f_i : U \subset R \to R, i = 0, 1, \ldots, p$.

(1) *Given p arbitrary values* $x_i \in U, i = 1, 2, \ldots, p$, *there exist* $p+1$ *constants* $C_i, i = 0, 1, \ldots, p$, *such that*

$$f(x) := \sum_{i=0}^{p} C_i f_i(x), \quad (2.3)$$

is not the zero function and $f(x_i) = 0$ *for* $i = 1, 2, \ldots, p$.

(2) *Furthermore, if there exists* $j \in \{0, 1, \ldots, p\}$ *such that* $f_j |_U$ *has constant sign, it is possible to get* $f(x)$ *in* (2.3) *such that it has at least p simple roots in U.*

If we consider small-amplitude limit cycles, the following lemma gives sufficient conditions, see Lemma 3.2 in [5] and Theorem 2.1 in [6].

Lemma 2.3. *Suppose* $c = (c_1, c_2, \ldots, c_N)$, $I(h) = \sum_{i=0}^{\infty} A_i(c)h^i$ *where* $A_i(c^*) = 0, i = 0, 1, 2, \ldots, N-1, A_N(c^*) \neq 0$, *and*

$$\text{rank}\left(\frac{\partial(A_0(c), A_1(c), \ldots, A_{N-1}(c))}{\partial(c_1, c_2, \ldots, c_N)}\,|_{c^*}\right) = N,$$

then there exists $(c_1, c_2, \ldots, c_N)$ *such that* $I(h)$ *can have N simple real positive roots near* $h = 0$.

3 Proof of the main results

In this section, we will prove Theorem 1.1 and Theorem 1.2 by Lemma 2.2 and Lemma 2.3 respectively. For system $(1.4)_{\epsilon=0}$, there exists a first integral

$$H(x,y) = \begin{cases} H^+(x,y) = \frac{1}{4}(y^2-1)^2 + \frac{1}{4}x^2(1 + \frac{4}{3}\bar{a}x + \bar{c}x^2), \\ H^-(x,y) = \frac{1}{4}(y^2-1)^2 + \frac{1}{4}x^2(1 + \frac{4}{3}\bar{b}x + \bar{d}x^2). \end{cases}$$

Define $\Gamma_{hi}^{\pm} = \{(x,y) : H^{\pm}(x,y) = h,\ 0 < h < \frac{1}{4}\}, i = 1, 2$, which form two annuli corresponding to two centers $(0,1)$ and $(0,-1)$, respectively (see Figure 3.1). More precisely, $\Gamma_{h1}^+ \cup \Gamma_{h1}^-$ and $\Gamma_{h2}^+ \cup \Gamma_{h2}^-$ are the closed orbits surrounding $(0,1)$ and $(0,-1)$, respectively.

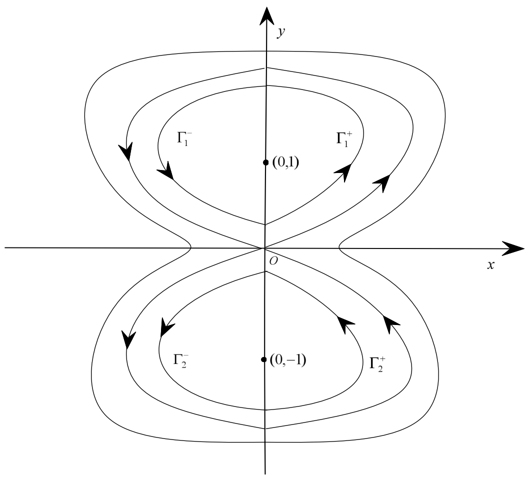

Figure 3.1: The phase graph of system $(1.4)_{\epsilon=0}$.

Since $H_y^+(x,y) = H_y^-(x,y)$, the Abelian integral (2.2) of the system (1.4) corresponding to the two annuli can be written as

$$I_i(h) = \int_{\Gamma_{hi}^+} g^+(x,y)\mathrm{d}x - f^+(x,y)\mathrm{d}y + \int_{\Gamma_{hi}^-} g^-(x,y)\mathrm{d}x - f^-(x,y)\mathrm{d}y, \qquad i = 1, 2, \tag{3.1}$$

respectively. A direct computation for (3.1) yields the following result.

Lemma 3.1.

$$\begin{aligned} I_i(h) = & -(a_{00} - c_{00}) \int_{\Gamma_{hi}^+} \mathrm{d}y - (a_{01} - c_{01}) \int_{\Gamma_{hi}^+} y\mathrm{d}y - (a_{02} - c_{02}) \int_{\Gamma_{hi}^+} y^2\mathrm{d}y - (a_{03} - c_{03}) \int_{\Gamma_{hi}^+} y^3\mathrm{d}y \\ & - \int_{\Gamma_{hi}^+} \Big[(a_{10} + b_{01})x + (a_{11} + 2b_{02})xy + \left(a_{20} + \frac{1}{2}b_{11}\right)x^2 + \left(a_{30} + \frac{1}{3}b_{21}\right)x^3 \\ & + (a_{21} + b_{12})x^2y + \left(a_{12} + 3b_{03}\right)xy^2\Big]\mathrm{d}y - \int_{\Gamma_{hi}^-} \Big[\left(c_{10} + d_{01}\right)x + (c_{11} + 2d_{02})xy \\ & + \left(c_{20} + \frac{1}{2}d_{11}\right)x^2 + \left(c_{30} + \frac{1}{3}d_{21}\right)x^3 + (c_{21} + d_{12})x^2y + (c_{12} + 3d_{03})xy^2\Big]\mathrm{d}y, \quad i = 1, 2. \end{aligned}$$

In the proof of Lemma 3.1, we will impose

$$\int_{\Gamma_{hi}^+} \mathrm{d}y = -\int_{\Gamma_{hi}^-} \mathrm{d}y, \ \int_{\Gamma_{hi}^+} x^m y^n \mathrm{d}x = -\int_{\Gamma_{hi}^+} \frac{n}{m+1} x^{m+1} y^{n-1} \mathrm{d}y.$$

Since the proof is direct, we omit it. Furthermore, if we take the following hypothesis

$$\text{(H)} \qquad a_{01} - c_{01} = a_{03} - c_{03} = a_{11} + 2b_{02} = a_{21} + b_{12} = c_{11} + 2d_{02} = c_{21} + d_{12} = 0,$$

the Abelian integrals $I_1(h)$ and $I_2(h)$ will have the same expression defined as $I(h)$. Define

$$J_1(h) = \int_{\Gamma_{h1}^+} dy, \quad J_2(h) = \int_{\Gamma_{h1}^+} y^2 dy, \quad J_3(h) = \int_{\Gamma_{h1}^+} x dy, \quad J_4(h) = \int_{\Gamma_{h1}^+} x^2 dy, \quad J_5(h) = \int_{\Gamma_{h1}^+} x^3 dy,$$

$$J_6(h) = \int_{\Gamma_{h1}^+} xy^2 dy, \quad J_7(h) = \int_{\Gamma_{h1}^-} x dy, \quad J_8(h) = \int_{\Gamma_{h1}^-} x^2 dy, \quad J_9(h) = \int_{\Gamma_{h1}^-} x^3 dy, \quad J_{10}(h) = \int_{\Gamma_{h1}^-} xy^2 dy.$$

Then we show the expression of $I(h)$ as the following result.

Lemma 3.2. *When the hypothesis* (H) *holds, then*

$$\begin{aligned} I(h) = & -(a_{00} - c_{00}) J_1(h) - (a_{02} - c_{02}) J_2(h) - (a_{10} + b_{01}) J_3(h) - \left(a_{20} + \frac{1}{2} b_{11}\right) J_4(h) \\ & - \left(a_{30} + \frac{1}{3} b_{21}\right) J_5(h) - (a_{12} + 3b_{03}) J_6(h) - (c_{10} + d_{01}) J_7(h) - \left(c_{20} + \frac{1}{2} d_{11}\right) J_8(h) \\ & - \left(c_{30} + \frac{1}{3} d_{21}\right) J_9(h) - (c_{12} + 3d_{03}) J_{10}(h). \end{aligned}$$

Proof. Using symmetry, we have

$$\int_{\Gamma_{h1}^+} x^i y^j \mathrm{d}y = (-1)^j \int_{\Gamma_{h2}^+} x^i y^j \mathrm{d}y, \qquad \int_{\Gamma_{h1}^-} x^i y^j \mathrm{d}y = (-1)^j \int_{\Gamma_{h2}^-} x^i y^j \mathrm{d}y,$$

where $0 \le i + j \le 3$. Combining Lemma 3.1 and hypothesis (H), we directly obtain that $I_1(h)$ and $I_2(h)$ have the same expression $I(h)$. □

Proof of Theorem 1.1. For simplicity, we may take $\bar{a} = \frac{3}{4}, \bar{b} = \frac{6}{4}$, and $\bar{c} = \bar{d} = 1$, then

$$\Gamma_{h1}^+ = \left\{ (x, y) : \frac{1}{4}(y^2 - 1)^2 + \frac{1}{4}x^2(1 + x + x^2) = h \right\},$$

and

$$\Gamma_{h1}^- = \left\{ (x, y) : \frac{1}{4}(y^2 - 1)^2 + \frac{1}{4}x^2(1 + 2x + x^2) = h \right\},$$

where $0 < h < \frac{1}{4}$.

From the proof of Lemma 3.2, it is easy to check that the coefficients of $J_i(h), i = 1, 2, \ldots, 10$, are arbitrary and $J_1(h) > 0$ for all $h \in (0, \frac{1}{4})$. By Lemma 2.2 and Theorem 2.1, we only need to prove that $J_i(h), i = 1, 2, \ldots, 10$, are linearly independent functions.

Let $h = \frac{r^2}{4}$. When $0 < h \ll 1$, on Γ_{1h}^+ we apply the transformations $x^2(1 + x + x^2) = u^2$, $(y^2 - 1)^2 = v^2$ with $x > 0, y > 0$, where $u = r\cos\theta, \ v = r\sin\theta, \ \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. With the aid of algebra-system Maple [3], we obtain

$$\begin{aligned} x &= u - \frac{1}{2}u^2 + \frac{1}{8}u^3 + \frac{1}{2}u^4 - \frac{161}{128}u^5 + \frac{3}{2}u^6 + \frac{33}{1024}u^7 - \frac{9}{2}u^8 + \frac{350779}{32768}u^9 - \frac{23}{2}u^{10} + O(u^{11}), \\ y &= 1 + \frac{1}{2}v - \frac{1}{8}v^2 + \frac{1}{16}v^3 - \frac{5}{128}v^4 + \frac{7}{256}v^5 - \frac{21}{1024}v^6 + \frac{33}{2048}v^7 - \frac{429}{32768}v^8 + \frac{715}{65536}v^9 \\ &\quad - \frac{2431}{262144}v^{10} + O(v^{11}). \end{aligned}$$

Furthermore,

$$J_1(h)=\int_{\Gamma_{h1}^+}\mathrm{d}y=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r\cos\theta\frac{\mathrm{d}y}{\mathrm{d}v}\,|_{v=r\sin\theta}\,\mathrm{d}\theta=r+\frac{1}{8}r^3+\frac{7}{128}r^5+\frac{33}{1024}r^7+\frac{715}{32768}r^9+O(r^{11}),$$

$$J_2(h)=r-\frac{1}{24}r^3-\frac{1}{128}r^5-\frac{3}{1024}r^7-\frac{143}{98304}r^9+O(r^{11}),$$

$$J_3(h)=\frac{1}{4}\pi r^2-\frac{1}{3}r^3+\frac{3}{64}\pi r^4+\frac{29}{120}r^5-\frac{191}{1024}\pi r^6+\frac{443}{640}r^7-\frac{82949}{46080}r^9+\frac{1382241}{1048576}\pi r^{10}+O(r^{11}),$$

$$J_4(h)=\frac{2}{3}r^3-\frac{3}{16}\pi r^4+\frac{19}{60}r^5+\frac{1}{8}\pi r^6-\frac{601}{448}r^7+\frac{4935}{8192}\pi r^8-\frac{3023}{4608}r^9-\frac{78477}{65536}\pi r^{10}+O(r^{11}),$$

$$J_5(h)=\frac{3}{16}\pi r^4-\frac{4}{5}r^5+\frac{3}{16}\pi r^6+\frac{29}{70}r^7-\frac{5655}{8192}\pi r^8+\frac{12317}{3360}r^9-\frac{47691}{65536}\pi r^{10}+O(r^{11}),$$

$$J_6(h)=\frac{1}{4}\pi r^2-\frac{1}{3}r^3+\frac{1}{64}\pi r^4+\frac{11}{40}r^5-\frac{203}{1024}\pi r^6+\frac{9167}{13440}r^7+\frac{115}{16384}\pi r^8-\frac{592951}{322560}r^9$$
$$+\frac{1381233}{1048576}\pi r^{10}+O(r^{11}).$$

On Γ_{h1}^-, let $x^2(1+2x+x^2)=u^2,\ (y^2-1)^2=v^2$ with $x<0$ and $y>0$, where $u=r\cos\theta,\ v=r\sin\theta,\ \theta\in(\frac{\pi}{2},\frac{3\pi}{2})$. Similarly, we have

$$x=u-u^2+2u^3-5u^4+14u^5-42u^6+132u^7-429u^8+1430u^9-4862u^{10}+O(u^{11}),$$

$$y=1+\frac{1}{2}v-\frac{1}{8}v^2+\frac{1}{16}v^3-\frac{5}{128}v^4+\frac{7}{256}v^5-\frac{21}{1024}v^6+\frac{33}{2048}v^7-\frac{429}{32768}v^8+\frac{715}{65536}v^9$$
$$-\frac{2431}{262144}v^{10}+O(v^{11}).$$

Furthermore,

$$J_7(h)=\frac{1}{4}\pi r^2+\frac{2}{3}r^3+\frac{51}{128}\pi r^4+\frac{163}{60}r^5+\frac{9091}{4096}\pi r^6+\frac{43363}{2240}r^7+\frac{4760595}{262144}\pi r^8+\frac{28250723}{161280}r^9$$
$$+\frac{2963888949}{16777216}\pi r^{10}+O(r^{11}),$$

$$J_8(h)=-\frac{2}{3}r^3-\frac{3}{8}\pi r^4-\frac{163}{60}r^5-\frac{283}{128}\pi r^6-\frac{43363}{2240}r^7-\frac{297465}{16384}\pi r^8-\frac{28250723}{161280}r^9$$
$$-\frac{46310061}{262144}\pi r^{10}+O(r^{11}),$$

$$J_9(h)=\frac{3}{16}\pi r^4+\frac{8}{5}r^5+\frac{363}{256}\pi r^6+\frac{451}{35}r^7+\frac{405465}{32768}\pi r^8+\frac{203683}{1680}r^9+\frac{64826685}{524288}\pi r^{10}+O(r^{11}),$$

$$J_{10}(h)=\frac{1}{4}\pi r^2+\frac{2}{3}r^3+\frac{47}{128}\pi r^4+\frac{53}{20}r^5+\frac{8923}{4096}\pi r^6+\frac{128689}{6720}r^7+\frac{4721575}{262144}\pi r^8+\frac{28071031}{161280}r^9$$
$$+\frac{2948223621}{16777216}\pi r^{10}+O(r^{11}).$$

Define $J_i(h)=\sum_{j=1}^{10}C_{i,j}r^j+O(r^{10}),\ i=1,2,\dots,10$, and $C=(C_{i,j})_{10\times 10}$, we obtain $\mathrm{rank}(C)=10$ which means $J_i(h),\ i=1,2,\dots,10$, are linearly independent functions.

The proof is completed. □

Next, we will apply Lemma 2.3 to prove that there exists a system of (1.4) with 22 small-amplitude limit cycles. Here we will take $\bar{a}=\frac{3}{4}a,\ \bar{b}=\frac{3}{4}b$, and $\bar{c}=\bar{d}=1$ for simplicity.

Proof of Theorem 1.2. Let

$$I(h)=\sum_{j=1}^{10}k_jJ_j(h),$$

where k_j, $j=1,2,\dots,10$, are arbitrary real constants.

By similar calculations used in Theorem 1.1, with the relation $h=\frac{r^2}{4}$, we get the Taylor expansions of $J_i(h)$, $i=1,2,\dots,10$, with 12th-order r which yield

$$I(h)=\sum_{i=0}^{12}F_ir^i+O(r^{13})=\sum_{i=0}^{12}2F_ih^{\frac{i}{2}}+O(h^{\frac{13}{2}}),$$

where

$$\begin{aligned}
F_0 &= 0,\\
F_1 &= k_1+k_2,\\
F_2 &= \frac{1}{4}\pi k_3+\frac{1}{4}\pi k_6+\frac{1}{4}\pi k_7+\frac{1}{4}\pi k_{10},\\
F_3 &= \frac{1}{8}k_1-\frac{1}{24}k_2-\frac{1}{3}ak_3+\frac{2}{3}k_4-\frac{1}{3}ak_6+\frac{1}{3}bk_7-\frac{2}{3}k_8+\frac{1}{3}bk_{10},\\
F_4 &= \Big(-\frac{9}{128}\pi+\frac{15}{128}a^2\pi\Big)k_3-\frac{3}{16}\pi ak_4+\frac{3}{16}\pi k_5+\Big(-\frac{13}{128}\pi+\frac{15}{128}a^2\pi\Big)k_6\\
&\quad+\Big(-\frac{9}{128}\pi+\frac{15}{128}b^2\pi\Big)k_7-\frac{3}{16}\pi bk_8+\frac{3}{16}\pi k_9+\Big(-\frac{13}{128}\pi+\frac{15}{128}b^2\pi\Big)k_{10},\\
F_5 &= \frac{7}{128}k_1-\frac{1}{128}k_2+\Big(-\frac{8}{15}a^3+\frac{31}{40}a\Big)k_3+\Big(-\frac{29}{60}+\frac{4}{5}a^2\Big)k_4-\frac{4}{5}ak_5+\Big(-\frac{8}{15}a^3+\frac{97}{120}a\Big)k_6\\
&\quad+\Big(\frac{8}{15}b^3-\frac{31}{40}b\Big)k_7+\Big(-\frac{4}{5}b^2+\frac{29}{60}\Big)k_8+\frac{4}{5}bk_9+\Big(\frac{8}{15}b^3-\frac{97}{120}b\Big)k_{10},\\
F_6 &= \Big(\frac{571}{4096}\pi-\frac{1245}{2048}a^2\pi+\frac{1155}{4096}a^4\pi\Big)k_3+\Big(-\frac{105}{256}\pi a^3+\frac{137}{256}\pi a\Big)k_4\\
&\quad+\Big(-\frac{57}{256}\pi+\frac{105}{256}a^2\pi\Big)k_5+\Big(\frac{563}{4096}\pi-\frac{1265}{2048}a^2\pi+\frac{1155}{4096}a^4\pi\Big)k_6\\
&\quad+\Big(\frac{571}{4096}\pi-\frac{1245}{2048}b^2\pi+\frac{1155}{4096}b^4\pi\Big)k_7+\Big(-\frac{105}{256}\pi b^3+\frac{137}{256}\pi b\Big)k_8\\
&\quad+\Big(-\frac{57}{256}\pi+\frac{105}{256}b^2\pi\Big)k_9+\Big(\frac{563}{4096}\pi-\frac{1265}{2048}b^2\pi+\frac{1155}{4096}b^4\pi\Big)k_{10},
\end{aligned}$$

and F_i, $i=7,8,9,10,11,12$, are polynomials of a, b, and k_j, $j=1,2,\dots,10$, which are omitted here because of the large scale.

Solving $F_i=0$, $i=1,\dots,9$, we obtain $k_1,k_3,k_2,k_4,k_5,k_6,k_7,k_8,k_9$ as follows.

$$\begin{aligned}
k_1 &= -k_2,\\
k_3 &= -k_6-k_7-k_{10},\\
k_2 &= 4k_4+(2a+2b)k_7-4k_8+(2a+2b)k_{10},\\
k_4 &= -\frac{1}{24a}\Big(-24k_5+4k_6+(15a^2-15b^2)k_7+24bk_8-24k_9+(15a^2-15b^2+4)k_{10}\Big),\\
k_5 &= \Big(-\frac{3}{22}a^2+\frac{1}{6}\Big)k_6+\Big(\frac{1}{22}a^4+\frac{15}{22}a^2b^2+\frac{8}{11}b^3a-\frac{53}{88}a^2-\frac{27}{22}ba-\frac{5}{8}b^2\Big)k_7\\
&\quad+\Big(-\frac{12}{11}a^2b-\frac{12}{11}b^2a+a+b\Big)k_8+\Big(\frac{12}{11}a^2+\frac{12}{11}ba-1\Big)k_9\\
&\quad+\Big(\frac{1}{22}a^4+\frac{15}{22}a^2b^2+\frac{8}{11}b^3a-\frac{69}{88}a^2-\frac{14}{11}ba-\frac{5}{8}b^2+\frac{1}{6}\Big)k_{10},
\end{aligned}$$

$$
\begin{aligned}
k_6 = {} & \frac{1}{16(9a^2-22)}\Big((103a^4+390a^2b^2-2048ab^3-2541b^4-768a^2+3456ab+4224b^2)k_7 \\
& +(-624a^2b+3072ab^2+3696b^3-2816a-2816b)k_8+(624a^2-3072ab-3696b^2)k_9 \\
& +(103a^4+390a^2b^2-2048ab^3-2541b^4-872a^2+3584ab+4312b^2+352)k_{10}\Big), \\
k_7 = {} & -\frac{N_7(a,b,k_8,k_9,k_{10})}{3\,M_7(a,b)}, \qquad k_8 = -\frac{N_8(a,b,k_9,k_{10})}{6\,M_8(a,b)}, \qquad k_9 = -\frac{k_{10}\,N_9(a,b)}{24\,(a+b)^2\,M(a,b)},
\end{aligned}
$$

where M_7, N_7, M_8, N_8, N_9, and M are polynomials of degree 7, 6, 10, 10, 14, and 10, respectively. Substituting the above results into F_{10}, F_{11}, and F_{12}, we have

$$
F_{10} = -\frac{21\pi k_{10}}{1048576M}(a^2-b^2)P_{10}(a,b), \qquad F_{11} = \frac{k_{10}}{3465M}(a+b)P_{11}(a,b),
$$
$$
F_{12} = -\frac{21\pi k_{10}}{33554432M}(a^2-b^2)P_{12}(a,b),
$$

where $P_{10}(a,b)$, $P_{11}(a,b)$, and $P_{12}(a,b)$ are polynomials of degree 12, 14, and 14 respectively.

Define

$$
P = \det\left[\frac{\partial(F_1,F_2,F_3,F_4,F_5,F_6,F_7,F_8,F_9,F_{10},F_{11})}{\partial(k_1,k_3,k_2,k_4,k_5,k_6,k_7,k_8,k_9,a,b)}\right].
$$

When $F_i = 0$, $i = 1,\dots,9$, we take $k_1,k_3,k_2,k_4,k_5,k_6,k_7,k_8$, and k_9 into function P one by one. Then we obtain

$$
P = \frac{\pi^5 k_{10}^2 (a+b)^6}{8192610672140353536000}\,\frac{\bar{P}(a,b)}{M},
$$

where $\bar{P}(a,b)$ is a polynomial of degree 36.

Next, we prove the existence of a, b such that $P_{10}(a,b) = P_{11}(a,b) = 0$ and $P_{12}(a,b)\cdot\bar{P}(a,b) \neq 0$ in three steps.

Firstly, we determine the common roots of $P_{10}(a,b)$ and $P_{11}(a,b)$. By the Maple built-in command 'RealRootIsolate' where the width of the interval is less than or equal $\frac{1}{2^{15}}$, we have

$$
\begin{aligned}
R_1 \triangleq \{ & [[1.678741455, 1.678771973], [-0.9492201089, -0.9492201089]], \\
& [[0.9492034912, 0.9492263794], [-1.678745722, -1.678745722]], \\
& [[-0.9492263794, -0.9492034912], [1.678745722, 1.678745722]], \\
& [[-1.678771973, -1.678741455], [0.9492201089, 0.9492201089]], \\
& [[-1.901992798, -1.901976585], [-0.03034826851, -0.03034826851]], \\
& [[-2.100128174, -2.100101471], [-0.8025713876, -0.8025713876]], \\
& [[-2.040435791, -2.040405273], [-1.611435396, -1.611435396]], \\
& [[-0.03036117554, -0.03033447266], [-1.901976879, -1.901976879]], \\
& [[-1.611436844, -1.611415863], [-2.040434847, -2.040434847]], \\
& [[-0.8025817871, -0.8025512695], [-2.100106852, -2.100106852]], \\
& [[0.8025512695, 0.8025817871], [2.100106852, 2.100106852]], \\
& [[1.611415863, 1.611436844], [2.040434847, 2.040434847]], \\
& [[0.03033447266, 0.03036117554], [1.901976879, 1.901976879]], \\
& [[2.040405273, 2.040435791], [1.611435396, 1.611435396]], \\
& [[2.100101471, 2.100128174], [0.8025713876, 0.8025713876]], \\
& [[1.901976585, 1.901992798], [0.03034826851, 0.03034826851]]\},
\end{aligned}
$$

where the common roots are located.

Secondly, we estimate the common roots of $P_{10}(a,b)$, $P_{11}(a,b)$, and $P_{12}(a,b)$, and the common roots of $P_{10}(a,b)$, $P_{11}(a,b)$, and $\bar{P}(a,b)$. By Groebner Basis and the Maple built-in command 'Basis', we get

$$\text{Basis}([P_{10}(a,b),P_{11}(a,b),P_{12}(a,b)],\text{plex}(a,b)) = \text{Basis}([P_{10}(a,b),P_{11}(a,b),\bar{P}(a,b)],\text{plex}(a,b)),$$

and two polynomials $P_1(b)$ and $P_2(a,b)$ with degrees 44 and 43, respectively, which mean the common roots of P_{10}, P_{11}, and P_{12} are the same as the common roots of P_{10}, P_{11}, and $\bar{P}(a,b)$ and they are determined by the common roots of $P_1(b)$ and $P_2(a,b)$. Furthermore, we find the intervals where the roots of $P_1(b)$ are located as follows

$$\begin{aligned} R_2 \triangleq \{&[-1.678746223,-1.678745270],[-0.9492206573,-0.9492197037],\\ &[0.9492197037,0.9492206573],[1.678745270,1.678746223]\}. \end{aligned}$$

Thirdly, we take

$$(a^*,b^*) \in [[-1.901992798,-1.901976585],[-0.03034826851,-0.03034826851]],$$

which means $a^* \in [-1.901992798,-1.901976585]$, $b^* \in [-0.03034826851,-0.03034826851]$ with $b^* \notin R_2$. Then $P_{10}(a^*,b^*) = P_{11}(a^*,b^*) = 0$ and $P_{12}(a^*,b^*) \cdot \bar{P}(a^*,b^*) \neq 0$. By the same method, we can prove (a^*,b^*) such that $M \neq 0$ and $M_i \neq 0, i = 7,8$. Furthermore, these properties imply $F_{10} = F_{11} = 0$ and $F_{12} \cdot P \neq 0$.

Finally, we can solve $k_9^*,\ldots,k_1^*$ one by one, which combined with a^* and b^* imply $F_i = 0$, $i = 1,2,\ldots,11$, and $F_{12} \cdot P \neq 0$. According to Lemma 2.3, $I(h)$ has 11 simple positive roots near $h = 0$. By symmetry, the proof is completed. □

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On the localization and numerical computation of positive radial solutions for ϕ-Laplace equations in the annulus

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Abstract. The paper deals with the existence and localization of positive radial solutions for stationary partial differential equations involving a general ϕ-Laplace operator in the annulus. Three sets of boundary conditions are considered: Dirichlet–Neumann, Neumann–Dirichlet and Dirichlet–Dirichlet. The results are based on the homotopy version of Krasnosel'skiĭ's fixed point theorem and Harnack type inequalities, first established for each one of the boundary conditions. As a consequence, the problem of multiple solutions is solved in a natural way. Numerical experiments confirming the theory, one for each of the three sets of boundary conditions, are performed by using the MATLAB object-oriented package Chebfun.

Keywords: ϕ-Laplace operator, radial solution, positive solution, fixed point index, Harnack type inequality, numerical solution.

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1 Introduction

In the short but clever paper [22], Hayan Wang solved the problem of existence of positive radial solutions for the semilinear elliptic equation

$$\Delta w + g\left(|x|\right) f\left(w\right) = 0, \qquad R_1 < |x| < R_2, \quad x \in \mathbb{R}^N, \quad N \geq 2,$$

with one of the following sets of boundary conditions,

$$w = 0 \quad \text{on } |x| = R_1 \quad \text{and} \quad |x| = R_2, \tag{1.1}$$

$$w = 0 \quad \text{on } |x| = R_1 \quad \text{and} \quad \partial w/\partial r = 0 \quad \text{on } |x| = R_2, \tag{1.2}$$

$$\partial w/\partial r = 0 \quad \text{on } |x| = R_1 \quad \text{and} \quad w = 0 \quad \text{on } |x| = R_2, \tag{1.3}$$

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where $r = |x|$ and $\partial w/\partial r$ denotes differentiation in the radial direction, and $0 < R_1 < R_2 < +\infty$.

The tools were a Krasnosel'skiĭ type fixed point theorem in cones and the property of bilateral boundedness of the corresponding Green functions. The first one is based on the fixed point index theory, while the second, as shown in [16], on Harnack type inequalities. Since then, many authors have considered the problem of radial solutions for equations and systems involving the Laplacian or some of its generalizations, various boundary conditions and domains, by using different topological or variational methods. We refer the interested reader to some of these contributions [1–3,5,6,9] and the references therein.

Most of the works that followed deviated from the spirit of the original ideas. On their line, we mention our recent papers [17], [18] and [19]. It is the scope of the present paper to complement them, as close as possible to paper [22], for the case of equations with a general ϕ-Laplacian. Here in the absence of a Green function we are forced to produce Harnack type inequalities for each set of boundary conditions.

More exactly, in this paper, we deal with the existence, localization and multiplicity of positive radial solutions to equations involving ϕ-Laplacian operators:

$$-\operatorname{div}\left(\psi\left(|\nabla w|\right)\nabla w\right) = g\left(|x|\right) f\left(u\right), \qquad R_1 < |x| < R_2, \quad x \in \mathbb{R}^N, \quad N \geq 2, \tag{1.4}$$

where $0 < R_1 < R_2 < +\infty$, the functions $g : [R_1, R_2] \to \mathbb{R}_+$, $f : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and $\psi : (-a, a) \to \mathbb{R}$ is such that $\phi(s) := s\,\psi(s)$ is an increasing homeomorphism between two intervals $(-a, a)$ and $(-b, b)$ $(0 < a, b \leq +\infty)$.

The following particular cases are of much interest due to their corresponding models arising from physics:

(a) $\phi : \mathbb{R} \to \mathbb{R}, \quad \phi(s) = |s|^{p-2} s$, where $p > 1$ (here $a = b = +\infty$), when the left side $L_0 w$ in (1.4) is

$$L_0 w = -\operatorname{div}\left(|\nabla w|^{p-2}\nabla w\right) \quad (p\text{-Laplace operator}),$$

involved in a nonlinear Darcy law for flows through porous media;

(b) (*singular* homeomorphism) $\phi : (-a, a) \to \mathbb{R}$, $\phi(s) = \frac{s}{\sqrt{a^2 - s^2}}$ (here $0 < a < +\infty$ and $b = +\infty$), when

$$L_0 w = -\operatorname{div}\left(\frac{\nabla w}{\sqrt{a^2 - |\nabla w|^2}}\right) \quad (\textit{Minkowski mean curvature operator}),$$

arose from the relativistic mechanics;

(c) (*bounded* homeomorphism) $\phi : \mathbb{R} \to (-b, b)$, $\phi(s) = \frac{bs}{\sqrt{1+s^2}}$ (here $a = +\infty$ and $0 < b < +\infty$), when

$$L_0 w = -b \operatorname{div}\left(\frac{\nabla w}{\sqrt{1 + |\nabla w|^2}}\right) \quad (\textit{Euclidian mean curvature operator}),$$

associated to capillarity problems.

Looking for radial solutions of (1.4), that is, functions of the form $w(x) = v(r)$ with $r = |x|$, (1.4) reduces to the ordinary differential equation

$$L(v) := -r^{1-N}\left(r^{N-1}\phi\left(v'\right)\right)' = g(r) f(v), \qquad R_1 < r < R_2, \tag{1.5}$$

while boundary conditions (1.1)–(1.3) become

$$v(R_1) = 0 \quad \text{and} \quad v(R_2) = 0, \tag{1.6}$$
$$v(R_1) = 0 \quad \text{and} \quad v'(R_2) = 0, \tag{1.7}$$
$$v'(R_1) = 0 \quad \text{and} \quad v(R_2) = 0. \tag{1.8}$$

2 Harnack type inequalities

Originally, Harnack's inequality was introduced in order to give estimates from above and from below for the ratio $u(x)/u(y)$ of two values of a positive harmonic function. It can be generically put under the form

$$\min_{x\in\omega} u(x) \geq k \max_{x\in\omega} u(x),$$

where k is a positive constant depending on the subdomain ω. Next it was generalized to nonnegative solutions or supersolutions of a wide class of linear elliptic equations. For the origin of the notion and many references, we refer the reader to [14].

More general, we speak about a Harnack type inequality whenever for a given operator L acting on a space of functions defined on a set Ω and endowed with a norm $\|\cdot\|$, there is a subdomain $\omega \subset \Omega$ and a constant $k > 0$ such that

$$\min_{x\in\omega} u(x) \geq k\|u\|$$

for all nonnegative functions u satisfying $L(u) \geq 0$ and eventually some additional behavior properties. In [15], Harnack inequalities have been put in connection with the compression-expansion method of Krasnosel'skiĭ for the localization of positive solutions of nonlinear problems. In case of boundary value problems for ordinary differential equations, when a Green function is known, a Harnack inequality immediately can be derived using the bilateral estimates of the Green function. However, Harnack inequalities can be obtained even for differential operators for which a Green function does not exist. This is the case of the ϕ-Laplace operators. Deduction of such inequalities requires a fine analysis and makes use of priority properties of solutions such as monotony and concavity (see, e.g., [10] and [11]). The analysis is even more difficult in the case of radial solutions. It is the goal of this section to obtain Harnack inequalities for ϕ-Laplace operators subject to each of the three boundary conditions (1.6), (1.7), (1.8).

2.1 Case of the boundary conditions (1.7)

Theorem 2.1. *If $v \in C^1[R_1, R_2]$ is such that (1.7) are satisfied, $r^{N-1}\phi(v')$ is differentiable and $L(v)(r) \in [0, b(R_1/R_2)^{N-1})$ for all $r \in [R_1, R_2]$, then v is nonnegative, increasing and concave. In addition, for any $c \in (R_1, R_2)$, one has*

$$v(c) \geq \frac{c - R_1}{R_2 - R_1} v(R_2). \tag{2.1}$$

Proof. Let $h := L(v)$. Integrating from r to R_2 and taking into account that $v'(R_2) = 0$ yields

$$v'(r) = \phi^{-1}\left(r^{1-N}\int_r^{R_2} \tau^{N-1} h(\tau)\, d\tau\right).$$

A new integration, this time from R_1 to r gives the expression of the solution, namely

$$v(r) = \int_{R_1}^{r} \phi^{-1}\left(s^{1-N}\int_{s}^{R_2}\tau^{N-1}h(\tau)\,d\tau\right)ds$$

and of the associated solution operator

$$S(h)(r) = \int_{R_1}^{r} \phi^{-1}\left(s^{1-N}\int_{s}^{R_2}\tau^{N-1}h(\tau)\,d\tau\right)ds.$$

Since $h \geq 0$, these formulas show that v is nonnegative and increasing. Also v' is decreasing, i.e., v is concave. Finally, the concavity implies that the graph of v is over the line joining the point $(R_1, 0)$ and $(R_2, v(R_2))$, whose equation is $y = \frac{v(R_2)}{R_2 - R_1}(x - R_1)$. Taking $x = c$ gives (2.1). □

Note that under the assumptions of Theorem 2.1, in (2.1), one has $v(c) = \min_{r\in[c,R_2]} v(r)$ and $v(R_2) = \max_{r\in[R_1,R_2]} v(r) = \|v\|$. Hence,

$$\min_{r\in[c,R_2]} v(r) \geq k_1 \|v\|,$$

with $k_1 = (c - R_1)/(R_2 - R_1)$.

2.2 Case of the boundary conditions (1.8)

Theorem 2.2. *If $v \in C^1[R_1, R_2]$ is such that (1.8) are satisfied, $r^{N-1}\phi(v')$ is differentiable, $L(v)(r) \in [0, b)$ and $L(v)$ is increasing on $[R_1, R_2]$, then v is nonnegative, decreasing and concave. In addition, for any $c \in (R_1, R_2)$, one has*

$$v(c) \geq \frac{R_2 - c}{R_2 - R_1} v(R_1). \tag{2.2}$$

Proof. If we let $h = L(v)$, then by integration we obtain

$$v'(r) = \phi^{-1}\left(-r^{1-N}\int_{R_1}^{r}\tau^{N-1}h(\tau)\,d\tau\right)$$

and

$$S(h)(r) = v(r) = -\int_{r}^{R_2}\phi^{-1}\left(-s^{1-N}\int_{R_1}^{s}\tau^{N-1}h(\tau)\,d\tau\right)ds.$$

Since h is nonnegative, these formulas immediately imply that v is nonnegative and decreasing.

To show that v is concave we need to prove that v' is decreasing, equivalently, that the function $\eta(r) = r^{1-N}\int_{R_1}^{r}\tau^{N-1}h(\tau)\,d\tau$ is increasing. Indeed, using the monotonicity of h, one has

$$\begin{aligned}
\eta'(r) &= h(r) - \frac{N-1}{r^N}\int_{R_1}^{r}\tau^{N-1}h(\tau)\,d\tau \\
&\geq h(r) - \frac{N-1}{r^N}h(r)\int_{R_1}^{r}\tau^{N-1}d\tau \\
&= h(r) - \frac{N-1}{N}h(r)\left(1 - \left(\frac{R_1}{r}\right)^N\right) \\
&\geq 0.
\end{aligned}$$

Finally, since the graph of the concave function v is over the line joining the points $(R_1, v(R_1))$ and $(R_2, 0)$, if c is any point in (R_1, R_2), we have (2.2). □

Note that under the assumptions of Theorem 2.2, in (2.2), one has $v(c) = \min_{r\in[R_1,c]} v(r)$ and $v(R_1) = \max_{r\in[R_1,R_2]} v(r) = \|v\|$. Therefore,

$$\min_{r\in[R_1,c]} v(r) \geq k_2 \|v\|,$$

where $k_2 = (R_2 - c)/(R_2 - R_1)$.

2.3 Case of the boundary conditions (1.6)

Theorem 2.3. *For each function $h \in L^1(R_1, R_2)$ not identically zero satisfying $h(r) \geq 0$ a.e. on (R_1, R_2) and $\|h\|_{L^1} < b(R_1/R_2)^{N-1}$, the equation $L(v) = h$ endowed with the boundary conditions (1.6) has a unique nonzero nonnegative solution v which is concave and such that for any $c \in [0, (R_2 - R_1)/2)$, one has:*

$$\min_{r\in[c_1,c_2]} v(r) \geq \frac{1}{R_2 - R_1}\left(\frac{R_2 - R_1}{2} - c\right)\|v\|, \tag{2.3}$$

where $c_1 = R_m - c$, $c_2 = R_m + c$ and $R_m = (R_1 + R_2)/2$.

Proof. Let v be a nonnegative solution. Since h is not identically zero, v is nonzero and since it vanishes at R_1 and R_2, any maximum point R is interior and so $v'(R) = 0$. Integrating from R to r then gives

$$v'(r) = \phi^{-1}\left(-r^{1-N}\int_R^r \tau^{N-1}h(\tau)\,d\tau\right). \tag{2.4}$$

This shows that v' is decreasing on $[R_1, R_2]$. Hence v is concave on $[R_1, R_2]$. Let R be such that $v(R) = \|v\| = \max_{r\in[R_1,R_2]} v(r)$. First assume that $R \leq R_m$. The concavity of v implies that the graph of v restricted to $[R, R_2]$ is over the line joining the points $(R, v(R))$ and $(R_2, 0)$ which at its turn is over the line joining the points $(R_1, v(R))$, $(R_2, 0)$, of equation $y = \frac{v(R)}{R_2-R_1}(R_2 - x)$. Thus, since $c_2 \in [R, R_2]$, we have

$$v(c_2) \geq \frac{v(R)}{R_2 - R_1}(R_2 - c_2) = \frac{1}{R_2 - R_1}\left(\frac{R_2 - R_1}{2} - c\right)v(R).$$

In addition the graph of v on $[R_1, R_m]$ is over the line joining the points $(R_1, 0)$, $(R_2, v(R))$. Then

$$v(c_1) \geq \frac{v(R)}{R_2 - R_1}(c_1 - R_1) = \frac{1}{R_2 - R_1}\left(\frac{R_2 - R_1}{2} - c\right)v(R).$$

As a result

$$\min_{r\in[c_1,c_2]} v(r) = \min\{v(c_1), v(c_2)\} \geq \frac{1}{R_2 - R_1}\left(\frac{R_2 - R_1}{2} - c\right)\|v\|.$$

The proof of the case $R \geq R_m$ is similar.

Next, integration in (2.4) gives the representation formulas (for the solution operator)

$$v(r) = \int_{R_1}^r \phi^{-1}\left(-s^{1-N}\int_R^s \tau^{N-1}h(\tau)\,d\tau\right)ds \quad (r \in [R_1, R_2]), \tag{2.5}$$

$$v(r) = -\int_r^{R_2} \phi^{-1}\left(-s^{1-N}\int_R^s \tau^{N-1}h(\tau)\,d\tau\right)ds \quad (r \in [R_1, R_2]).$$

To prove the existence of a solution, in virtue of (2.5) and (2.4) it is enough to prove the existence of a number $R \in (R_1, R_2)$ such that

$$\int_{R_1}^{R_2} \phi^{-1}\left(-s^{1-N}\int_R^s \tau^{N-1}h(\tau)\,d\tau\right)ds = 0.$$

This immediately follows since the continuous function

$$t \longmapsto \int_{R_1}^{R_2} \phi^{-1}\left(-s^{1-N}\int_t^s \tau^{N-1}h(\tau)\,d\tau\right)ds \quad (t \in [R_1, R_2])$$

takes values of opposite sign at the ends R_1 and R_2.

To prove the uniqueness of the solution, assume that v_1 and v_2 are two nonnegative solutions and let R', R'' be two of their maximum points, respectively. Using the representation formula (2.4) it is easy to see that $(v_2 - v_1)'$ preserves its sign on the whole interval (R_1, R_2), positive or negative depending on the ordering between R' and R''. Thus $v_2 - v_1$ is monotone and being zero at the ends of the interval it must be identically zero. Hence $v_1 = v_2$. □

3 Existence and localization

As mentioned above, the key ingredient together with Harnack inequalities to obtain positive solutions in this paper will be the fixed point index in cones. In particular, we recall the well–known homotopy version of Krasnosel'skiĭ fixed point theorem in cones.

Theorem 3.1 (Krasnosel'skiĭ). *Let X be a Banach space, K a cone of X and Ω_1 and Ω_2 two relatively open and bounded subsets of K with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. Let $N : K \to K$ be a completely continuous operator satisfying one of the following two conditions:*

(i) *$\lambda u \neq N(u)$ for all $u \in \partial_K \Omega_1$ and all $\lambda \geq 1$; and there exists $h \in K \setminus \{0\}$ such that $u \neq N(u) + \lambda h$ for all $u \in \partial_K \Omega_2$ and all $\lambda \geq 0$.*

(ii) *$\lambda u \neq N(u)$ for all $u \in \partial_K \Omega_2$ and all $\lambda \geq 1$; and there exists $h \in K \setminus \{0\}$ such that $u \neq N(u) + \lambda h$ for all $u \in \partial_K \Omega_1$ and all $\lambda \geq 0$.*

Then N has a fixed point $u \in K$ such that $u \in \Omega_2 \setminus \overline{\Omega}_1$.

In the sequel, consider the Banach space of continuous functions $C[R_1, R_2]$ endowed with the usual maximum norm $\|v\| = \max_{r \in [R_1, R_2]} |v(r)|$ and denote by P the cone of nonnegative functions in $C[R_1, R_2]$.

3.1 Case of the boundary conditions (1.7)

By a solution of (1.5)–(1.7) we mean a function $v \in C^1[R_1, R_2]$ with $v(R_1) = 0 = v'(R_2)$ such that $v' \in (-a, a)$, $r^{n-1}\phi(v') \in W^{1,1}[R_1, R_2]$ and satisfies (1.5). We will look for nonnegative nontrivial solutions on $[R_1, R_2]$.

It is clear that v is a nonnegative solution of (1.5)–(1.7) if and only if v is a fixed point of the operator

$$T_1(v)(r) = \int_{R_1}^{r} \phi^{-1}\left(s^{1-N}\int_s^{R_2} \tau^{N-1}g(\tau)\,f(v(\tau))d\tau\right)ds.$$

If f is such that $f(s)\int_{R_1}^{R_2} g(\tau)\,d\tau < \left(\frac{R_1}{R_2}\right)^{N-1} b$ for all $s \in \mathbb{R}_+$, then T_1 is defined on the whole cone P and $T_1(P) \subset P$. Moreover, T_1 is completely continuous as follows from the Arzelà–Ascoli theorem.

Here, for a fixed $c \in (R_1, R_2)$, we will look for fixed points of the operator T_1 in a subcone of P, namely,

$$K_1 = \left\{ v \in P : \min_{r \in [c, R_2]} v(r) \geq k_1 \|v\| \right\},$$

where $k_1 = (c - R_1)/(R_2 - R_1)$. By the Harnack inequality given by Theorem 2.1, it is easy to check that the operator T_1 maps the cone K_1 into itself.

Now, for any numbers $\alpha, \beta > 0$, consider the open (in K_1) sets

$$V_\alpha := \{ v \in K_1 : \|v\| < \alpha \} \tag{3.1}$$

and

$$W_\beta := \left\{ v \in K_1 : \min_{r \in [c, R_2]} v(r) < \beta \right\}. \tag{3.2}$$

Note that $V_\beta \subset W_\beta \subset V_{\beta/k_1}$, so W_β is bounded.

We are in the position to apply Theorem 3.1 in order to obtain existence and localization results for problem (1.5)–(1.7). In this way, we localize a solution in the set $W_\beta \setminus \overline{V}_\alpha$ if $\beta > \alpha$ and in the set $V_\alpha \setminus \overline{W}_\beta$ if $\alpha > \beta/k_1$.

We will use the following notations:

$$A := \int_{R_1}^{R_2} g(\tau) d\tau \quad \text{and} \quad B := \int_c^{R_2} g(\tau) d\tau.$$

Also, for any $\alpha, \beta > 0$, we denote

$$M_\alpha := \max\{ f(s) : s \in [0, \alpha] \} \quad \text{and} \quad m_\beta := \min\{ f(s) : s \in [\beta, \beta/k_1] \}.$$

Theorem 3.2. *Assume that*

$$f(s) < \frac{b}{A} \left(\frac{R_1}{R_2} \right)^{N-1} \quad \text{for all } s \in \mathbb{R}_+. \tag{3.3}$$

In addition assume that there exist $\alpha, \beta > 0$ such that

$$(R_2 - R_1)\phi^{-1}\left((R_2/R_1)^{N-1} A\, M_\alpha \right) < \alpha, \tag{3.4}$$

$$(c - R_1)\phi^{-1}\left(B\, m_\beta \right) > \beta. \tag{3.5}$$

(1^0) *If $\alpha < \beta$, then problem (1.5)–(1.7) has a positive solution v such that $\alpha < \|v\| < \beta/k_1$.*

(2^0) *If $\alpha > \beta/k_1$, then problem (1.5)–(1.7) has a positive solution v such that $\beta < \|v\| < \alpha$.*

Proof. We shall apply Theorem 3.1. First, let us see that

$$\|T_1(v)\| < \alpha \quad \text{for all } v \in K_1 \text{ with } \|v\| = \alpha,$$

which clearly implies that $\lambda v \neq T_1(v)$ for all $v \in \partial_{K_1} V_\alpha$ and all $\lambda \geq 1$. Indeed, for $v \in K_1$ with $\|v\| = \alpha$, we have that $f(v(s)) \leq M_\alpha$ and so from (3.4) it follows that

$$\begin{aligned}
\|T_1(v)\| &= \int_{R_1}^{R_2} \phi^{-1}\left(s^{1-N} \int_s^{R_2} \tau^{N-1} g(\tau) f(v(\tau)) d\tau \right) ds \\
&\leq \int_{R_1}^{R_2} \phi^{-1}\left(M_\alpha\, s^{1-N} \int_{R_1}^{R_2} \tau^{N-1} g(\tau)\, d\tau \right) ds \\
&\leq (R_2 - R_1)\phi^{-1}\left((R_2/R_1)^{N-1} A M_\alpha \right) < \alpha,
\end{aligned}$$

as wished.

On the other hand, let us prove that $v \neq T_1(v) + \lambda h$ for all $v \in \partial_{K_1} W_\beta$ and all $\lambda \geq 0$ with $h \equiv 1$. Notice that for $v \in K_1$ with $\min_{r\in[c,R_2]} v(r) = \beta$, we have that $\beta \leq v(r) \leq \beta/k_1$ for all $r \in [c, R_2]$, and thus $m_\beta \leq f(v(r))$ for all $r \in [c, R_2]$. Hence, for any $r \in [c, R_2]$,

$$\begin{aligned} T_1(v)(r) &\geq \int_{R_1}^{c} \phi^{-1}\left(s^{1-N}\int_s^{R_2} \tau^{N-1} g(\tau) f(v(\tau))d\tau\right) ds \\ &\geq \int_{R_1}^{c} \phi^{-1}\left(s^{1-N}\int_c^{R_2} \tau^{N-1} g(\tau) f(v(\tau))d\tau\right) ds \\ &\geq (c-R_1)\phi^{-1}\left(B\, m_\beta\right). \end{aligned}$$

Now, (3.5) implies that $T_1(v)(r) > \beta = \min_{r\in[c,R_2]} v(r)$ for all $r \in [c, R_2]$, which clearly ensures that $v \neq T_1(v) + \lambda$ for all $v \in \partial_{K_1} W_\beta$ and all $\lambda \geq 0$.

Now, if $\alpha < \beta$, then $\overline{V}_\alpha \subset W_\beta$, so Theorem 3.1 guarantees that the operator T_1 has at least a fixed point in $W_\beta \setminus \overline{V}_\alpha \subset V_{\beta/k_1} \setminus \overline{V}_\alpha$. But if one has $\alpha > \beta/k_1$, then $W_\beta \subset V_{\beta/k_1} \subset \overline{V}_{\beta/k_1} \subset V_\alpha$ and thus Theorem 3.1 implies that the operator T_1 has at least a fixed point located in $V_\alpha \setminus \overline{W}_\beta \subset V_\alpha \setminus \overline{V}_\beta$. □

Note that condition (3.3) trivially holds if $b = +\infty$. Obviously, if ϕ is a classical or a bounded homeomorphism, i.e., if $a = +\infty$, then conditions (3.4) and (3.5) can be rewritten as

$$\frac{M_\alpha}{\phi\,(C_1\,\alpha)} < C_2 \quad \text{and} \quad \frac{m_\beta}{\phi\,(C_3\,\beta)} > C_4,$$

with suitable positive constants C_1, C_2, C_3 and C_4 as come from (3.4)–(3.5).

Hence, if we are only interested on the existence and not on the localization of the solutions, we can establish sufficient conditions for the existence of the numbers α and β satisfying the inequalities above. They are given by asymptotic conditions on the ratio f/ϕ at 0 and at infinity.

Theorem 3.3. *Assume that the following conditions are satisfied: $a = +\infty$,*

$$\limsup_{x\to 0} \frac{\phi(\tau x)}{\phi(x)} < +\infty, \qquad \limsup_{x\to+\infty} \frac{\phi(x)}{\phi(\tau x)} < +\infty \quad \textit{for all } \tau > 0 \tag{3.6}$$

and

$$f_0 := \lim_{x\to 0^+} \frac{f(x)}{\phi(x)} = +\infty, \qquad f_\infty := \lim_{x\to+\infty} \frac{f(x)}{\phi(x)} = 0.$$

Then problem (1.5)–(1.7) has at least one positive solution.

Proof. First we show that there exists $\beta > 0$ such that

$$m_\beta > C_4 \phi\,(C_3\,\beta)\,. \tag{3.7}$$

By (3.6), with $\tau = C_3$, there exist $L > 0$ and $\rho > 0$ such that

$$\frac{L}{C_4}\phi(x) > \phi(C_3\,x) \quad \text{for all } x \in (0,\rho).$$

Now, since $f_0 = +\infty$, there exists $\tau > 0$ (we may assume $\tau < \rho$) such that

$$f(x) \geq L\phi(x) \quad \text{for all } x \in (0,\tau].$$

Hence, the fact that ϕ is increasing implies that

$$\min_{x\in[\tau k_1,\tau]} f(x) \geq L\phi(\tau k_1).$$

Then, taking $\beta = \tau k_1$, one has

$$m_\beta := \min_{x\in[\beta,\beta/k_1]} f(x) \geq L\phi(\beta) > C_4\phi(C_3\,\beta),$$

and so (3.7) holds.

Secondly, we prove that there exists $\alpha > \beta/k_1$ such that

$$M_\alpha < C_2\phi\left(C_1\,\alpha\right). \tag{3.8}$$

By (3.6), with $\tau = C_1$, there exist $\tilde{L} > 0$ and $\tilde{\rho} > 0$ such that

$$C_2\,\tilde{L}\,\phi(C_1\,x) > \phi(x) \quad \text{for all } x \in (\tilde{\rho}, +\infty).$$

Since $f_\infty = 0$, there exists $\sigma > 0$ such that

$$f(x) \leq \sigma + \frac{1}{2\tilde{L}}\phi(x) \quad \text{for all } x \geq 0.$$

Now, it follows from the fact that $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing unbounded homeomorphism that there exists $\alpha > 0$ such that $2\tilde{L}\,\sigma \leq \phi(\alpha)$. Thus,

$$f(x) \leq \frac{1}{\tilde{L}}\phi(\alpha) \quad \text{for all } x \in [0, \alpha],$$

and so

$$M_\alpha := \max_{x\in[0,\alpha]} f(x) \leq \frac{1}{\tilde{L}}\phi(\alpha) < C_2\phi(C_1\,\alpha),$$

that is, (3.8) holds.

Finally, the conclusion follows from Theorem 3.2. □

Similarly, an existence result can be obtained if f is sublinear at 0 and superlinear at infinity with respect to ϕ.

Theorem 3.4. *Assume that ϕ is a classical homeomorphism such that*

$$\limsup_{x\to 0} \frac{\phi(x)}{\phi(\tau x)} < +\infty, \quad \limsup_{x\to +\infty} \frac{\phi(\tau x)}{\phi(x)} < +\infty \quad \textit{for all } \tau > 0 \tag{3.9}$$

and f satisfies

$$f_0 = 0, \qquad f_\infty = +\infty.$$

Then problem (1.5)–(1.7) has at least one positive solution.

Remark 3.5. Note that if ϕ is bounded, then condition $f_\infty = +\infty$ is not possible, since $\lim_{x\to+\infty} \phi(x) = b$ and f must be bounded.

Note that if ϕ is singular (i.e., $a < +\infty$, $b = +\infty$), then condition (3.4) is trivially satisfied for α large enough and so the existence of a positive solution for problem (1.5)–(1.7) is ensured provided that there exists a positive number β satisfying (3.5). This holds if f is superlinear at 0 with respect to ϕ, i.e., $f_0 = +\infty$. Thus we have

Theorem 3.6. *Assume that ϕ is a singular homeomorphism such that*

$$\limsup_{x\to 0} \frac{\phi(\tau x)}{\phi(x)} < +\infty \quad \textit{for all } \tau > 0 \tag{3.10}$$

and f satisfies

$$f_0 = +\infty.$$

Then problem (1.5)–(1.7) has at least one positive solution.

Obviously, the localization of solutions given by Theorem 3.2 allows us to derive multiplicity results provided that there exist several couples of positive numbers (α, β) satisfying assumptions (3.4)–(3.5). Some conclusions in this line are collected in the following

Theorem 3.7. *Assume that condition (3.3) holds.*

(1) Let $(\alpha_i)_{1\le i\le k}$, $(\beta_i)_{1\le i\le k}$ $(k \in \mathbb{N})$ be sets of positive numbers with $\alpha_i < \beta_i \le k_1\alpha_{i+1}$ for each i. If the assumptions of Theorem 3.2 hold for each couple (α_i, β_i), then problem (1.5)–(1.7) has k different solutions v_i such that $\alpha_i < \|v_i\| < \beta_i/k_1$.

(2) Let $(\alpha_i)_{1\le i\le k}$, $(\beta_i)_{1\le i\le k}$ $(k \in \mathbb{N})$ be sets of positive numbers with $\alpha_i < \beta_i < k_1\alpha_{i+1}$ for each i. If the assumptions of Theorem 3.2 hold for each couple (α_i, β_i), then problem (1.5)–(1.7) has $2k-1$ different solutions v_i, w_j $(i = 1, \dots, k,\ j = 1, \dots, k-1)$ such that

$$\alpha_i < \|v_i\|, \qquad \min_{r\in[c,R_2]} v_i(r) < \beta_i \quad \textit{and} \quad \min_{r\in[c,R_2]} w_j(r) > \beta_j, \qquad \|w_j\| < \alpha_{j+1}.$$

(3) Let $(\alpha_i)_{i\in\mathbb{N}}$, $(\beta_i)_{i\in\mathbb{N}}$ be two sequences of positive numbers with $\alpha_i < \beta_i \le k_1\alpha_{i+1}$ for each i. If the assumptions of Theorem 3.2 hold for each couple (α_i, β_i), then problem (1.5)–(1.7) has infinitely many different solutions v_i such that $\alpha_i < \|v_i\| < \beta_i/k_1$.

Proof. Let us prove cases (1) and (2).

(1) For each i, since $\alpha_i < \beta_i$, Theorem 3.2 ensures that problem (1.5)–(1.7) has a positive solution v_i such that $\alpha_i < \|v_i\| < \beta_i/k_1$. Now, it suffices to remark that $\beta_i/k_1 \le \alpha_{i+1}$ implies that $\|v_i\| < \|v_{i+1}\|$, so there exist at least k different such solutions.

(2) For each i, since $\alpha_i < \beta_i$, we can derive from the proof of Theorem 3.2 a better localization result: the solution v_i belongs to the set $W_{\beta_i} \setminus \overline{V}_{\alpha_i}$, that is,

$$\alpha_i < \|v_i\|, \qquad \min_{r\in[c,R_2]} v_i(r) < \beta_i.$$

On the other hand, for each $j \in \{1, \dots, k-1\}$, since $\beta_j < k_1\alpha_{j+1}$, Theorem 3.2 also implies that problem (1.5)–(1.7) has a positive solution w_j located in the set $V_{\alpha_{j+1}} \setminus \overline{W}_{\beta_j}$. Thus,

$$\min_{r\in[c,R_2]} w_j(r) > \beta_j, \qquad \|w_j\| < \alpha_{j+1}.$$

Since $\alpha_{j+1} < \beta_{j+1}$, one has that $\|w_j\| < \|w_{j+1}\|$. Finally, the estimations

$$\min_{r\in[c,R_2]} v_n(r) < \beta_j < \min_{r\in[c,R_2]} w_j(r) \quad \text{and} \quad \|w_j\| < \alpha_{j+1} < \|v_m\|,$$

for $n \in \{1, \dots, j\}$ and $m \in \{j+1, \dots, k\}$, show that w_j is also distinct from any v_i and so problem (1.5)–(1.7) has at least $2k-1$ different solutions.

The proof of case (3) is analogous and thus we omit it. □

3.2 Case of the boundary conditions (1.8)

By a solution of (1.5)–(1.8) we mean a function $v \in C^1[R_1, R_2]$ with $v'(R_1) = 0 = v(R_2)$ such that $v' \in (-a, a)$, $r^{n-1}\phi(v') \in W^{1,1}[R_1, R_2]$ and satisfies (1.5). We will look for nonnegative nontrivial solutions on $[R_1, R_2]$.

It is clear that v is a nonnegative solution of (1.5)–(1.8) if and only if v is a fixed point of the operator $T_2 : P \to P$ defined as

$$T_2(v)(r) = -\int_r^{R_2} \phi^{-1}\left(-s^{1-N}\int_{R_1}^{s} \tau^{N-1} g\left(\tau\right) f(v(\tau))d\tau\right) ds,$$

which is a completely continuous operator.

Let us assume that the functions f and g satisfy the following monotonicity assumptions:

(H_f) f is decreasing on $\mathbb{R}_+$ and $0 < f(0) < b\int_{R_1}^{R_2} g(\tau)d\tau$;

(H_g) $r^{N-1}g(r)$ is increasing on $[R_1, R_2]$.

For a fixed $c \in (R_1, R_2)$, we consider the following subcone of P:

$$K_2 = \left\{v \in P : \ v \text{ is decreasing and } \min_{r\in[R_1,c]} v\left(r\right) \geq k_2 \left\|v\right\|\right\},$$

where $k_2 = (R_2 - c)/(R_2 - R_1)$.

Note that the operator T_2 maps the cone K_2 into itself. Indeed, take $v \in K_2$ and let us show that $w := T_2(v)$ belongs to K_2. Since f and g are nonnegative, then w is nonnegative and decreasing. Moreover, the monotonicity assumptions on f and g given by (H_f) and (H_g) together with the fact that v is decreasing imply that the function $r \mapsto r^{N-1}g(r)f(v(r))$ is increasing. Thus,

$$r \mapsto L(w)(r) = r^{N-1}g(r)f(v(r))$$

is increasing on $[R_1, R_2]$. Then Theorem 2.2 ensures that w satisfies that

$$\min_{r\in[R_1,c]} w\left(r\right) \geq k_2 \left\|w\right\|,$$

so $w \in K_2$.

For any numbers $\alpha, \beta > 0$, define the sets V_α and W_β as in (3.1) and (3.2), with K_2 instead of K_1. Then the following existence and localization result for problem (1.5)–(1.8) can be proved as an application of Theorem 3.1, which guarantees the existence of a fixed point of T_2 in $W_\beta \setminus \overline{V}_\alpha$ or in $V_\alpha \setminus \overline{W}_\beta$.

We will use the following notation:

$$A := \int_{R_1}^{R_2} g(\tau)d\tau \quad \text{and} \quad B := \int_{R_1}^{c} g(\tau)d\tau.$$

Moreover, for any $\alpha, \beta > 0$, denote

$$M_\alpha := \max\{f(s) : s \in [0, \alpha]\} = f(0) \quad \text{and} \quad m_\beta := \min\{f(s) : s \in [\beta, \beta/k_2]\} = f(\beta/k_2).$$

It is obvious that the following result can be proved in a similar way to Theorem 3.2, so we omit the proof here.

Theorem 3.8. *Assume that conditions* (H_f) *and* (H_g) *hold and that there exist* $\alpha, \beta > 0$ *such that*

$$-(R_2 - R_1)\phi^{-1}(-A\, M_\alpha) < \alpha, \tag{3.11}$$

$$-(R_2 - c)\phi^{-1}\left(-(R_1/R_2)^{N-1} B\, m_\beta\right) > \beta. \tag{3.12}$$

(1^0) *If* $\alpha < \beta$, *then problem* (1.5)–(1.8) *has a positive solution* v *such that* $\alpha < \|v\| < \beta/k_2$.

(2^0) *If* $\alpha > \beta/k_2$, *then problem* (1.5)–(1.8) *has a positive solution* v *such that* $\beta < \|v\| < \alpha$.

Remark 3.9. If we take into account that f is decreasing, then conditions (3.11) and (3.12) can be rewritten as

$$-(R_2 - R_1)\phi^{-1}(-A\, f(0)) < \alpha, \tag{3.13}$$

$$-(R_2 - c)\phi^{-1}\left(-(R_1/R_2)^{N-1} B\, f(\beta/k_2)\right) > \beta. \tag{3.14}$$

Note that condition (3.13) is always satisfied for α sufficiently large since the left-hand side in the inequality is independent of α. Furthermore, from the fact that f is continuous with $f(0) > 0$, it follows that condition (3.14) holds for any β close enough to 0.

In view of Theorem 3.8 and Remark 3.9, it is clear that problem (1.5)–(1.8) is always solvable under assumptions (H_f) and (H_g). Thus we have

Corollary 3.10. *If conditions* (H_f) *and* (H_g) *hold, then problem* (1.5)–(1.8) *has at least one positive solution.*

Remark 3.11. Observe that multiplicity results cannot be derived from Theorem *3.8*. Indeed, since $A \geq B$ and f is decreasing, one has

$$A\, f(0) \geq (R_1/R_2)^{N-1} B\, f(\beta/k_2),$$

and so

$$-(R_2 - R_1)\phi^{-1}(-A\, f(0)) > -(R_2 - c)\phi^{-1}\left(-(R_1/R_2)^{N-1} B\, f(\beta/k_2)\right).$$

Therefore, any α satisfying (3.13) must be bigger than any β for which (3.14) holds.

Remark 3.12. Observe that the results contained in Section 3.2 remain valid for $R_1 = 0$, i.e., in the ball.

Note that problem (1.5)–(1.8) with $R_1 = 0$ and $R_2 = 1$, that is, in the unit ball, was considered in *[19]*, but the results are not comparable since there f was assumed to be nondecreasing.

3.3 Case of the boundary conditions (1.6)

By a solution of (1.5)–(1.6) we mean a function $v \in C^1[R_1, R_2]$ with $v(R_1) = 0 = v(R_2)$ such that $v' \in (-a, a)$, $r^{n-1}\phi(v') \in W^{1,1}[R_1, R_2]$ and satisfies (1.5). We will look for nonnegative nontrivial solutions on $[R_1, R_2]$.

To construct the fixed point operator, we need the following technical result, similar to Lemma 1 in [4].

Denote

$$D_b = \left\{ h \in P : \ \|h\|_{L^1} < b \left(\frac{R_1}{R_2}\right)^{N-1} \right\}.$$

Lemma 3.13. *For each function $h \in D_b$, there exists $R \in (R_1, R_2)$ such that*

$$\gamma = \int_{R_1}^{R} \tau^{N-1} h(\tau)\, d\tau$$

is the unique number γ satisfying

$$-s^{1-N}\left[\int_{R_1}^{s} \tau^{N-1} h(\tau)\, d\tau - \gamma\right] \in (-b, b)$$

and

$$\int_{R_1}^{R_2} \phi^{-1}\left(-s^{1-N}\left[\int_{R_1}^{s} \tau^{N-1} h(\tau)\, d\tau - \gamma\right]\right) ds = 0.$$

Moreover, the function $Q_\phi : D_b \to \mathbb{R},\ Q_\phi(h) = \gamma$ is continuous.

Proof. The existence of R with the desired property follows from the proof of Theorem 2.3. Note that for any $h \in D_b$, one has

$$-s^{1-N}\left[\int_{R_1}^{s} \tau^{N-1} h(\tau)\, d\tau - \gamma\right] \in (-b, b) \quad \text{for all } s \in [R_1, R_2]\,.$$

Indeed

$$\left|-s^{1-N}\left[\int_{R_1}^{s} \tau^{N-1} h(\tau)\, d\tau - \gamma\right]\right| = \left|s^{1-N}\int_{R}^{s} \tau^{N-1} h(\tau)\, d\tau\right| \le \left(\frac{R_2}{R_1}\right)^{N-1} \|h\|_{L^1} < b.$$

For uniqueness, assume that there exist $\gamma_i \in \mathbb{R}$ $(i = 1, 2)$ such that

$$\int_{R_1}^{R_2} \phi^{-1}\left(-s^{1-N}\left[\int_{R_1}^{s} \tau^{N-1} h(\tau)\, d\tau - \gamma_i\right]\right) ds = 0.$$

Now, by the mean value theorem for integration, there exists $s_0 \in [R_1, R_2]$ such that

$$\phi^{-1}\left(-{s_0}^{1-N}\left[\int_{R_1}^{s_0} \tau^{N-1} h(\tau)\, d\tau - \gamma_1\right]\right) ds = \phi^{-1}\left(-{s_0}^{1-N}\left[\int_{R_1}^{s_0} \tau^{N-1} h(\tau)\, d\tau - \gamma_2\right]\right) ds.$$

This clearly implies that $\gamma_1 = \gamma_2$.

Finally, for the continuity of Q_ϕ, let $\{h_n\}_{n\in\mathbb{N}} \subset D_b$ such that $h_n \to h_0 \in D_b$ in $C[R_1, R_2]$. We may assume that $Q_\phi(h_n) \to \gamma_0$. Passing to limit we find that

$$\int_{R_1}^{R_2} \phi^{-1}\left(-s^{1-N}\left[\int_{R_1}^{s} \tau^{N-1} h_0(\tau)\, d\tau - \gamma_0\right]\right) ds = 0,$$

and so $\gamma_0 = Q_\phi(h_0)$, as wished. □

In addition, the solution operator

$$S : D_b \to C[R_1, R_2], \qquad S(h)(r) = \int_{R_1}^{r} \phi^{-1}\left(-s^{1-N}\left[\int_{R_1}^{s} \tau^{N-1} h(\tau)\, d\tau - Q_\phi(h)\right]\right) ds,$$

is monotone as shows the next lemma. The proof follows similar ideas to those in [11].

Lemma 3.14. *Let $h_1, h_2 \in D_b$, $h_1 \ge h_2$ a.e. on $[R_1, R_2]$, and let $v_1, v_2 \in C^1[R_1, R_2]$ be such that for $i = 1, 2$, one has $v_i(R_1) = 0 = v_i(R_2)$ and*

$$L(v_i)(r) = h_i(r) \quad \text{for } r \in (R_1, R_2).$$

Then $v_1 \ge v_2$ on $[R_1, R_2]$.

Proof. Assume to the contrary that $v_1 \not\geq v_2$. Then there exists an interval $[t_1, t_2]$, with $R_1 \leq t_1 < t_2 \leq R_2$, such that $v_1 < v_2$ on (t_1, t_2) and $v_1(t_i) = v_2(t_i)$, $i = 1, 2$. Hence, by the mean value theorem, there exists $R \in (t_1, t_2)$ such that $(v_1 - v_2)'(R) = 0$. Then

$$\phi(v_2')(r) - \phi(v_1')(r) = \frac{1}{r^{N-1}} \int_R^r s^{N-1} \left[h_1(s) - h_2(s)\right] ds.$$

Since $h_1 \geq h_2$, we deduce that $\phi(v_2')(r) - \phi(v_1')(r) \geq 0$ on (R, t_2). Thus, $v_2'(r) \geq v_1'(r)$ on (R, t_2) which joint with $v_2(t_2) = v_1(t_2)$ imply $v_1 \geq v_2$ on (R, t_2), a contradiction. □

If f satisfies condition (3.3), then for each $v \in P$, the function $h := gf(v) \in D_b$ and since $h \geq 0$, one has $S(h) \geq S(0) = 0$. Hence the operator

$$T_3 : P \to P, \qquad T_3(v) = S(gf(v))$$

is well-defined. In addition, thanks to the continuity of Q_ϕ and the Arzelà–Ascoli theorem, it is completely continuous.

Notice that v is a nonnegative solution of (1.5)–(1.6) if and only if v is a fixed point of the operator T_3. Here, for a fixed $c \in (0, (R_2 - R_1)/2)$, we shall look for fixed points of the operator T_3 in a subcone of P, namely,

$$K_3 = \left\{ v \in P : \min_{r \in I_c} v(r) \geq k_3 \|v\| \right\},$$

where $k_3 = ((R_2 - R_1)/2 - c) / (R_2 - R_1)$ and $I_c = [R_m - c, R_m + c]$. By the Harnack inequality given by Theorem 2.3, it follows that the operator T_3 maps the cone K_3 into itself.

Now, for any numbers $\alpha, \beta > 0$, consider the relatively open sets

$$V_\alpha := \{v \in K_3 : \|v\| < \alpha\} \quad \text{and} \quad W_\beta := \left\{ v \in K_3 : \min_{r \in I_c} v(r) < \beta \right\}.$$

We will use the following notation:

$$A := \int_{R_1}^{R_2} g(\tau) d\tau \quad \text{and} \quad B := \min\left\{ \int_{R_m - c}^{R_m} g(\tau) d\tau, \int_{R_m}^{R_m + c} g(\tau) d\tau. \right\}.$$

Moreover, for any $\alpha, \beta > 0$, denote

$$M_\alpha := \max\{f(s) : s \in [0, \alpha]\} \quad \text{and} \quad m_\beta := \min\{f(s) : s \in [\beta, \beta/k_3]\}.$$

Theorem 3.15. *Assume that f satisfies (3.3) and there exist $\alpha, \beta > 0$ such that*

$$(R_2 - R_1)\phi^{-1}\left(\left(\frac{R_2}{R_1}\right)^{N-1} A\, M_\alpha \right) < \alpha, \tag{3.15}$$

$$k_3 (R_m - R_1 - c)\, \phi^{-1}(B\, m_\beta) > \beta, \tag{3.16}$$

$$-k_3 (R_2 - R_m - c)\, \phi^{-1}\left(-(R_m/R_2)^{N-1} B\, m_\beta \right) > \beta. \tag{3.17}$$

(1^0) *If $\alpha < \beta$, then problem (1.5)–(1.6) has a positive solution v such that $\alpha < \|v\| < \beta/k_3$.*

(2^0) *If $\alpha > \beta/k_3$, then problem (1.5)–(1.6) has a positive solution v such that $\beta < \|v\| < \alpha$.*

Proof. We shall apply Theorem 3.1. First, let us show that

$$\|T_3(v)\| < \alpha \quad \text{for all } v \in K_3 \text{ with } \|v\| = \alpha,$$

which clearly implies that $\lambda v \neq T_3(v)$ for all $v \in \partial_{K_3} V_\alpha$ and all $\lambda \geq 1$. Indeed, for $v \in K_3$ with $\|v\| = \alpha$, we have that there exists $R \in (R_1, R_2)$ such that $\|T_3(v)\| = T_3(v)(R)$ and $(T_3(v))'(R) = 0$. Thus,

$$\begin{aligned} T_3(v)(r) &= \int_{R_1}^{r} \phi^{-1}\left(-s^{1-N}\int_R^s \tau^{N-1} g(\tau) f(v(\tau))\, d\tau\right) ds \\ &= -\int_r^{R_2} \phi^{-1}\left(-s^{1-N}\int_R^s \tau^{N-1} g(\tau) f(v(\tau))\, d\tau\right) ds \qquad (r \in [R_1, R_2]). \end{aligned}$$

Since $f(v(s)) \leq M_\alpha$ for every $s \in [R_1, R_2]$ and S is monotone, we have

$$\begin{aligned} \|T_3(v)\| = T_3(v)(R) = S\left(gf\left(v\right)\right)(R) \leq S\left(gM_\alpha\right)(R) &= \int_{R_1}^{R} \phi^{-1}\left(-s^{1-N} M_\alpha \int_R^s \tau^{N-1} g(\tau)\, d\tau\right) ds \\ &= \int_{R_1}^{R} \phi^{-1}\left(s^{1-N} M_\alpha \int_s^R \tau^{N-1} g(\tau)\, d\tau\right) ds \\ &\leq (R_2 - R_1)\phi^{-1}\left(\left(\frac{R_2}{R_1}\right)^{N-1} A\, M_\alpha\right) < \alpha, \end{aligned}$$

as wished.

On the other hand, let us prove that $v \neq T_3(v) + \lambda\, h$ for all $v \in \partial_{K_3} W_\beta$ and all $\lambda \geq 0$ with $h \equiv 1$. Notice that for $v \in K_3$ with $\min_{r \in I_c} v(r) = \beta$, we have that $\beta \leq v(r) \leq \beta / k_3$ for all $r \in I_c$, and thus $m_\beta \leq f(v(r))$ for all $r \in I_c$. Hence, $f(v(r)) \geq m_\beta \chi_{I_c}(r)$ for all $r \in [R_1, R_2]$ (where χ_{I_c} denotes the characteristic function of I_c). Then Lemma 3.14 implies that

$$T_3(v)(r) \geq S(m_\beta\, g\, \chi_{I_c})(r), \qquad (r \in [R_1, R_2]).$$

Note that there is $R \in (R_1, R_2)$ such that

$$S(m_\beta\, g\, \chi_{I_c})(r) = \int_{R_1}^{r} \phi^{-1}\left(-s^{1-N} m_\beta \int_R^s \tau^{N-1} g(\tau) \chi_{I_c}(\tau)\, d\tau\right) ds.$$

Now, suppose that $R \geq R_m$. Then

$$\begin{aligned} T_3(v)(R_m - c) &\geq S(m_\beta\, g\, \chi_{I_c})(R_m - c) \\ &= \int_{R_1}^{R_m - c} \phi^{-1}\left(s^{1-N} m_\beta \int_{R_m - c}^{R} \tau^{N-1} g(\tau) \chi_{I_c}(\tau)\, d\tau\right) ds \\ &\geq \int_{R_1}^{R_m - c} \phi^{-1}\left(s^{1-N} m_\beta \int_{R_m - c}^{R_m} \tau^{N-1} g(\tau) \chi_{I_c}(\tau)\, d\tau\right) ds \\ &\geq (R_m - R_1 - c)\phi^{-1}\left(B\, m_\beta\right) \\ &> \beta / k_3, \end{aligned}$$

that is, $T_3(v)(R_m - c) > \beta / k_3 \geq v(r)$ for all $r \in I_c$. In particular, $T_3(v)(R_m - c) > v(R_m - c)$.

Analogously, if $R \le R_m$, then

$$\begin{aligned} T_3(v)(R_m+c) &\ge S(m_\beta\, g\, \chi_{I_c})(R_m+c) \\ &= -\int_{R_m+c}^{R_2} \phi^{-1}\left(-s^{1-N} m_\beta \int_R^s \tau^{N-1} g(\tau)\chi_{I_c}(\tau)\, d\tau\right) ds \\ &= -\int_{R_m+c}^{R_2} \phi^{-1}\left(-s^{1-N} m_\beta \int_{R_m}^{R_m+c} \tau^{N-1} g(\tau)\chi_{I_c}(\tau)\, d\tau\right) ds \\ &\ge -(R_2-R_m-c)\,\phi^{-1}\left(-(R_m/R_2)^{N-1}\, B\, m_\beta\right) \\ &> \beta/k_3. \end{aligned}$$

we may prove that $T_3(v)(R_m+c) > \beta/k_3 \ge v(r)$ for all $r \in I_c$.

Therefore, $v \ne T_3(v) + \lambda$ for all $v \in \partial_{K_3} W_\beta$ and all $\lambda \ge 0$. The conclusion follows from Theorem 3.1. □

Remark 3.16. If ϕ is odd then the two conditions (3.16) and (3.17) on β reduce to the unique inequality

$$k_3\,(R_2-R_m-c)\,\phi^{-1}\left((R_m/R_2)^{N-1}\, Bm_\beta\right) > \beta. \tag{3.18}$$

We emphasize that if ϕ is a classical or bounded odd homeomorphism, then conditions (3.15) and (3.18) can be rewritten as

$$\frac{M_\alpha}{\phi\,(C_1\,\alpha)} < C_2 \quad \text{and} \quad \frac{m_\beta}{\phi\,(C_3\,\beta)} > C_4,$$

for certain positive constants C_1, C_2, C_3 and C_4. Therefore, existence results for sublinear and superlinear nonlinearities can be proven exactly as in Section 3.1.

Theorem 3.17. *Assume that ϕ is odd and that one of the following conditions holds:*

(i) $f_0 = +\infty$, $f_\infty = 0$ *and ϕ is a classical or bounded homeomorphism satisfying* (3.6).

(ii) $f_0 = 0$, $f_\infty = +\infty$ *and ϕ is a classical homeomorphism satisfying* (3.9).

(iii) $f_0 = +\infty$ *and ϕ is a singular homeomorphism satisfying* (3.10).

Then problem (1.5)–(1.6) *has at least one positive solution.*

Remark 3.18. Theorem 3.15 allows us to deduce the existence of multiple positive solutions for problem (1.5)–(1.6) provided that there are several pairs of positive numbers (α, β) satisfying conditions (3.15)–(3.17).

4 Numerical examples

From numerical point of view we will consider three distinct boundary value problems. In order to solve them we make use of the new and powerful MATLAB package Chebfun which is a product of the numerical analysis group at Oxford University led by Professor Trefethen (see for instance [20] and [21] to quote but a few).

The philosophy behind this package is non-standard in numerical analysis and can be summed up in the words of its initiator as “Feel symbolic but run at the speed of numerics”.

In short, the method implemented by Chebfun is a Chebyshev type collocation one. Chebfun tries to solve a BVP by using successively to approximate the solution Chebyshev polynomials on grids of size 17, 33, 65 ... until the spectral convergence is reached. The relative accuracy of each computation carried out by a Chebfun algorithm is usually about 16 digits, and in principle the user need have no knowledge of the underlying algorithms. However, when solving a nonlinear BVP, Chebfun provides useful information on the convergence of the Newtonian method used to solve nonlinear algebraic systems obtained by discretization. In addition, the behavior of the solution coefficients can be visualized (the way in which they decrease to the machine accuracy). We will display these two outputs for each of the three issues considered. In fact, we must emphasize that we have used Chebfun with excellent results in our previous works [7] and [17].

Moreover, in order to observe the behavior of the Chebfun system in solving genuinely nonlinear boundary value problems, we have reported in our recent work [8] the solutions of eight non-linear problems, some of them even singular. In the vast majority of cases, the asymptotic rate of convergence of the Chebyshev collocation implemented by Chebfun is exponential (geometric). Only in the case of the singular problem was this reduced to an algebraic one.

4.1 First example: a Dirichlet–Neumann problem

Consider the Dirichlet–Neumann problem for an equation involving a singular homeomorphism

$$\begin{cases} -\left(r\frac{v'}{\sqrt{1-v'^2}}\right)' = rg(r)\,f(v)\,, \quad r \in (1,2) \\ v(1) = v'(2) = 0, \end{cases} \tag{4.1}$$

where

$$g(r) = \frac{r+1}{2r^2+1}, \qquad f(v) = v^2+1.$$

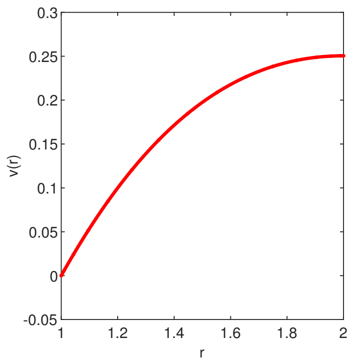

Figure 4.1: Graph of the numerical solution of problem (4.1). The initial guess for the initialization of the Newton procedure is $v_0 := 1$.

The residual Chebfun satisfies the operator is of order 10^{-10} and the boundary conditions are satisfied exactly. From the left panel of Fig. 4.2 it is very clear that Newton method converges with an order of at most 2. From the right panel of the same figure one can observe that a Chebyshev polynomial of order 16, with highly and smoothly decreasing coefficients is the solution of this problem and the asymptotic rate of convergence is exponential.

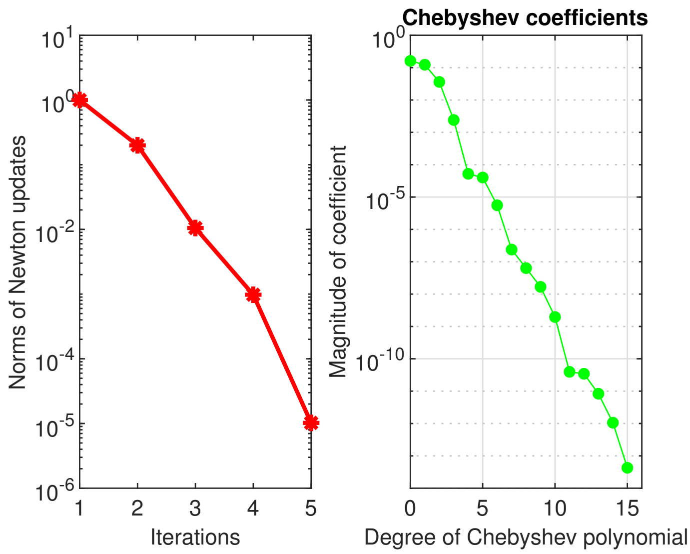

Figure 4.2: Newton iterations (left panel) and the coefficients of Chebyshev solution when Chebfun solves problem (4.1).

4.2 Second example: a Neumann–Dirichlet problem

We now solve numerically the following problem

$$\begin{cases} -\left(r\frac{v'}{\sqrt{1-v'^2}}\right)' = rg\,(r)\, f\,(v)\,, \quad r \in (0,1) \\ v'\,(0) = v\,(1) = 0, \end{cases} \tag{4.2}$$

where

$$g\,(r) = e^{-r} + \frac{1}{2}r, \qquad f\,(v) = \frac{1}{v^2+1}.$$

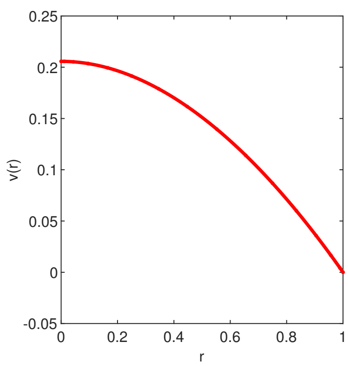

Figure 4.3: Graph of the numerical solution of problem (4.2). The initial guess for the initialization of the Newton procedure is $v_0 := 1$.

The residual Chebfun satisfies the operator is of order 10^{-11} and the boundary conditions are satisfied exactly. From the left panel of Fig. 4.4 it is very clear that Newton method converges with an order of at least 2. From the right panel of the same figure one can observe that a Chebyshev polynomial of order 17, with highly decreasing coefficients is the solution of the problem and the asymptotic rate of convergence is again exponential.

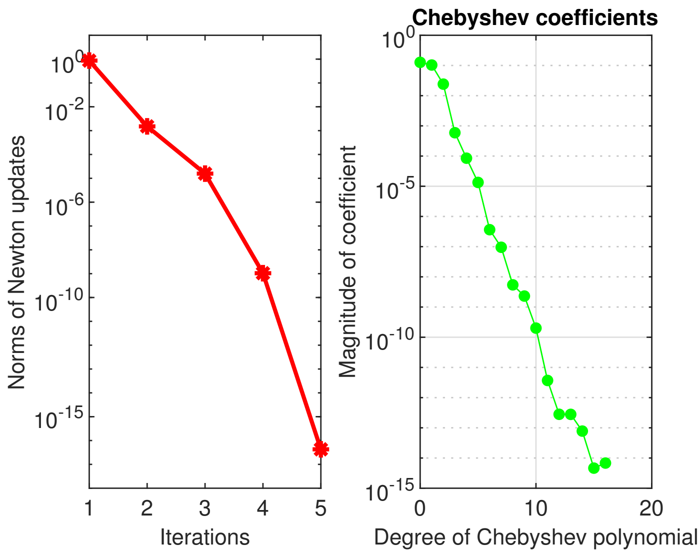

Figure 4.4: Newton iterations (left panel) and the coefficients of Chebyshev solution when Chebfun solves problem (4.2).

4.3 Third example: a Dirichlet problem

The last example is giving by the Dirichlet problem

$$\begin{cases} -\left(r\frac{v'}{\sqrt{1-v'^2}}\right)' = rg\left(r\right)f\left(v\right), & r \in (1,2) \\ v\left(1\right) = v\left(2\right) = 0, \end{cases} \tag{4.3}$$

where

$$g\left(r\right) = 1, \qquad f\left(v\right) = \frac{v+1}{v^2+1}.$$

The residual Chebfun satisfies the operator is of order 10^{-10} and the boundary conditions are

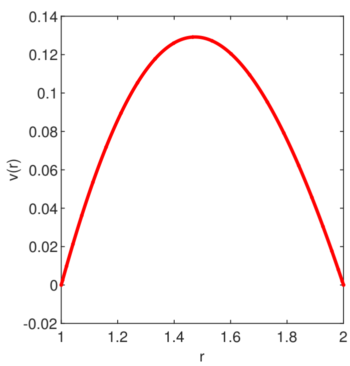

Figure 4.5: Graph of the numerical solution of problem (4.3). The initial guess for the initialization of the Newton procedure is $v_0 := 1$.

satisfied exactly. From the left panel of Fig. 4.6 it is very clear that Newton method converges with an order of at least 2 and from the right panel of the same figure one can observe that a Chebyshev polynomial of order 24, with highly decreasing coefficients is the solution of the problem and the asymptotic rate of convergence continues to be exponential.

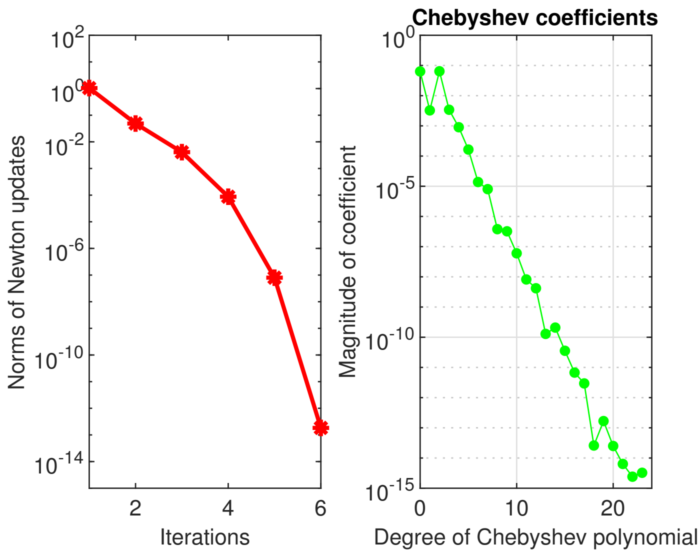

Figure 4.6: Newton iterations (left panel) and the coefficients of Chebyshev solution when Chebfun solves problem (4.3).

We must make an important remark at the end of these three examples. Exponential convergence occurs for solutions represented by Chebyshev polynomials of relatively small order (of the order of a few tens). Moreover and more important, the convergence is so fast that no rounding off plateau appears (see the right panels of the Figures 4.2, 4.4 and 4.6).

From this point of view, the problems in the present work, compared to those in [8], appear to be only slightly nonlinear.

In the papers [12] and [13], the authors solve numerically similar problems. They exclusively use shooting type methods, i.e., they transform a nonlinear boundary value problem into a Cauchy problem and then solve it by finite difference schemes.

Variants of the shooting method have produced remarkable results over time, but we consider that the Chebyshev collocation implemented by Chebfun, through the information it provides, is very reliable. Unfortunately, a direct comparison of our results with the numerical results from the last two cited works is almost impossible.

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Positive ground state of coupled planar systems of nonlinear Schrödinger equations with critical exponential growth

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Abstract. In this paper, we prove the existence of a positive ground state solution to the following coupled system involving nonlinear Schrödinger equations:

$$\begin{cases} -\Delta u + V_1(x)u = f_1(x,u) + \lambda(x)v, & x \in \mathbb{R}^2, \\ -\Delta v + V_2(x)v = f_2(x,v) + \lambda(x)u, & x \in \mathbb{R}^2, \end{cases}$$

where $\lambda, V_1, V_2 \in C(\mathbb{R}^2, (0,+\infty))$ and $f_1, f_2 : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ have critical exponential growth in the sense of Trudinger–Moser inequality. The potentials $V_1(x)$ and $V_2(x)$ satisfy a condition involving the coupling term $\lambda(x)$, namely $0 < \lambda(x) \le \lambda_0 \sqrt{V_1(x)V_2(x)}$. We use non-Nehari manifold, Lions's concentration compactness and strong maximum principle to get a positive ground state solution. Moreover, by using a bootstrap regularity lifting argument and L^q-estimates we get regularity and asymptotic behavior. Our results improve and extend the previous results.

Keywords: coupled system, nonlinear Schrödinger equations, variational methods, Trudinger–Moser inequality, positive ground state solution, regularity.

2020 Mathematics Subject Classification: 35J10, 35J50, 35J61, 35B33, 35Q55.

1 Introduction and main results

This article is devoted to studying standing waves for the following system of nonlinear Schrödinger equations:

$$\begin{cases} -\Delta u + V_1(x)u = f_1(x,u) + \lambda(x)v, & x \in \mathbb{R}^2, \\ -\Delta v + V_2(x)v = f_2(x,v) + \lambda(x)u, & x \in \mathbb{R}^2, \end{cases} \tag{1.1}$$

where $\lambda, V_1, V_2 \in C(\mathbb{R}^2, \mathbb{R})$ and $f_1, f_2 : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ satisfy the following basic assumptions:

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(V) $V_1(x),\ V_2(x),\ \lambda(x) \in C(\mathbb{R}^2, (0, +\infty))$ are all 1-periodic in each of x_1 and x_2. Moreover, there exists $\lambda_0 \in (0,1)$ such that

$$0 < \lambda(x) \le \lambda_0 \sqrt{V_1(x)V_2(x)}, \qquad \forall x \in \mathbb{R}^2;$$

(F1) $f_i \in C(\mathbb{R}^2 \times \mathbb{R}, \mathbb{R})$, $f_i(x,t)$ is 1-periodic in each of x_1 and x_2, and there exists $\alpha_1, \alpha_2 > 0$ such that

$$\lim_{t\to\infty} \frac{|f_i(x,t)|}{e^{\alpha t^2}} = 0, \text{ uniformly on } x \in \mathbb{R}^2 \text{ for all } \alpha > \alpha_i, i = 1,2;$$

and

$$\lim_{t\to\infty} \frac{|f_i(x,t)|}{e^{\alpha t^2}} = +\infty, \text{ uniformly on } x \in \mathbb{R}^2 \text{ for all } \alpha < \alpha_i, i = 1,2;$$

(F2) $f_i(x,t) = o(t)$ as $t \to 0$ uniformly on $x \in \mathbb{R}^2$, for $i = 1,2$. $f_i(x,t) = 0$ for all $x \in \mathbb{R}^2, t \le 0$.

Solutions of system (1.1) are related with standing waves of the following two-component system:

$$\begin{cases} -i\frac{\partial \psi}{\partial t} = \Delta\psi - V_1(x)\psi + f_1(x,\psi) + \lambda(x)\phi, & (x,t) \in \mathbb{R}^2 \times \mathbb{R}, \\ -i\frac{\partial \phi}{\partial t} = \Delta\phi - V_2(x)\phi + f_2(x,\psi) + \lambda(x)\psi, & (x,t) \in \mathbb{R}^2 \times \mathbb{R}, \end{cases} \tag{1.2}$$

where i denotes the imaginary unit. Such class of systems arise in various branches of mathematical physics and nonlinear optics, see [1]. For instance, solutions of (1.1) are related to the existence of solitary wave solutions for nonlinear Schrödinger equations and Klein–Gordon equations, see [4]. For system (1.2), a solution of the form

$$(\psi(x,t), \phi(x,t)) = (e^{-iMt}u(x), e^{-iMt}v(x)),$$

where M is some real constant, called standing wave solution.

In order to motivate our results, we begin by giving a brief survey on this subject. Let us consider the scalar case. Notice that if $\lambda \equiv 0, V_1 \equiv V_2 = V(x), f_1 \equiv f_2 = f$ and $u \equiv v$, system (1.1) reduces to the scalar equation

$$-\Delta u + V(x)u = f(x,u). \tag{1.3}$$

This class of nonlinear Schrödinger equation has been widely studied by many researchers, under various hypotheses on the potential $V(x)$ and nonlinear term $f(x,u)$. Such as coercive potential, axially symmetric potential, positive potential and periodic potential. In particular, Chen and Tang [8] developed a direct approach to get nontrivial solutions and ground state solutions when they considered the equation (1.3) in $\mathbb{R}^2$ where $V(x)$ was a 1-periodic function with respect to x_1 and x_2, 0 lies in the gap of $-\Delta + V$, and the nonlinear term was of Trudinger-Moser critical exponential growth. Using the generalized linking theorem to obtain a Cerami sequence, they showed that the Cerami sequence was bounded and the minimax-level was less than the threshold value by virtue of Moser type functions. Furthermore, they obtained that the Cerami sequence was nonvanishing, which extended and improved the results of [2,17].

For the system of nonlinear Schrödinger equations, there are some results on the linearly coupled system in subcritical and critical case. Chen and Zou [9] studied the following system

$$\begin{cases} -\Delta u + u = f(x,u) + \lambda v, & x \in \mathbb{R}^N, \\ -\Delta v + v = g(x,v) + \lambda u, & x \in \mathbb{R}^N, \end{cases} \tag{1.4}$$

where $0 < \lambda < 1$. They discussed the system for non-autonomous and autonomous nonlinearities of subcritical growth respectively. When $N \geq 2$, $f(x,u) = (1+a(x))|u|^{p-1}u$ and $g(x,v) = (1+b(x))|v|^{p-1}v$, they improved the results of [3] for establishing energy estimates of the ground states. Under some assumptions of potential $a(x)$ and $b(x)$, they obtained not only the existence of positive bound states, but also a precise description of the limit behavior of the bound states as the parameter λ goes to zero. When $N \geq 3$, $f(x,u) = f(u)$, $g(x,v) = g(v)$, and Berestycki–Lions type assumptions were satisfied, they proved system (1.4) had a positive radial ground state, moreover, the behavior and energy estimates of the bound states as $\lambda \to 0$ were also obtained.

Later, Chen and Zou [10] investigated the following coupled systems with critical power-type nonlinearity:

$$\begin{cases} -\Delta u + \mu u = |u|^{p-1}u + \lambda v, & x \in \mathbb{R}^N, \\ -\Delta v + \nu v = |v|^{2^*-2}v + \lambda u, & x \in \mathbb{R}^N, \end{cases} \tag{1.5}$$

where $0 < \lambda < \sqrt{\mu\nu}, 1 < p < 2^* - 1$ and $N \geq 3$. They proved the existence of positive ground states for system (1.5) when $0 < \mu \leq \mu_0$, where $\mu_0 \in (0,1)$ was some critical value. When μ and λ were both large, system (1.5) had a positive ground state also. While, when μ was large but λ was small, the system (1.5) had no ground state solutions. In addition, when $p = 2^* - 1$, system (1.5) had no nontrivial solutions by the Pohozaev identity. Motivated by [10], Li and Tang [14] considered system (1.5) in $\mathbb{R}^N$, $N \geq 3$, when $\mu = a(x) > 0$, $\nu = b(x) > 0$ and $\lambda = \lambda(x)$ were continuous functions, 1-periodic in each of $x_1, x_2, \ldots, x_N$, and satisfied $\lambda(x) < \sqrt{a(x)b(x)}$, they proved system (1.5) had a Nehari-type ground state solution when $0 < a(x) < \mu_0$ for some $\mu_0 \in (0,1)$. Some related linearly coupled systems were also studied in [3,11,12] and the references therein.

In the above references we refer to, it is noticed that the nonlinearities were only considered the polynominal growth of subcritical or critical type in terms of the Sobolev embedding. As we all know, the Trudinger–Moser inequality in $\mathbb{R}^2$ with critical exponential growth instead of the Sobolev inequality in $\mathbb{R}^N$ with critical polynominal growth, which was first established by Cao in [5], reads as follows.

Lemma 1.1 ([5]).

i) If $\alpha > 0$ and $u \in H^1(\mathbb{R}^2)$, then

$$\int_{\mathbb{R}^2} (e^{\alpha u^2} - 1)\mathrm{d}x < \infty;$$

ii) if $u \in H^1(\mathbb{R}^2)$, $\|\nabla u\|_2^2 \leq 1$, $\|u\|_2^2 \leq M < \infty$, and $\alpha < 4\pi$. then there exists a constant $C(M,\alpha)$, which depends only on M and α, such that

$$\int_{\mathbb{R}^2} (e^{\alpha u^2} - 1)\mathrm{d}x \leq C(M,\alpha).$$

By virtue of the Trudinger–Moser inequality, do Ó and de Albuquerque [16] investigated the following linear coupled system with constant potential in $\mathbb{R}^2$,

$$\begin{cases} -\Delta u + u = f_1(u) + \lambda(x)v, & x \in \mathbb{R}^2, \\ -\Delta v + v = f_2(v) + \lambda(x)u, & x \in \mathbb{R}^2. \end{cases} \tag{1.6}$$

By using the minimization technique over the Nehari manifold and strong maximum principle, the existence of positive ground state solution and the corresponding asymptotic behavior were obtained.

In the paper [15], do Ó and de Albuquerque used the same idea as [16] to investigate the existence of positive ground state solution and asymptotic behaviors for the coupled system (1.1) with nonnegative variable potentials. The main problem they faced was to overcome the difficulty originated from the lack of compactness when the nonlinear terms had critical exponential growth in $\mathbb{R}^2$. Based on this, they considered the following weighted Sobolev space defined by

$$H_{V_i}(\mathbb{R}^2) = \Big\{ u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} V_i(x) u^2 \mathrm{d}x < \infty \Big\},$$

endowed with the norm

$$\|u\|_{V_i} = \left(\int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x + \int_{\mathbb{R}^2} V_i(x) u^2 \mathrm{d}x \right)^{\frac{1}{2}}.$$

They assumed the following conditions on the potential $V_i(x),\ i = 1, 2$.

(V1′) $V_i(x) \geq 0$, for all $x \in \mathbb{R}^2$ and $V_i \in L^\infty_{loc}(\mathbb{R}^2)$;

(V2′) The infimum

$$\inf_{u \in H_{V_i}(\mathbb{R}^2)} \left\{ \int_{\mathbb{R}^2} (|\nabla u|^2 + V_i(x) u^2) \mathrm{d}x : \int_{\mathbb{R}^2} u^2 \mathrm{d}x = 1 \right\}$$

is positive;

(V3′) There exists $s \in [2, +\infty)$ such that

$$\lim_{R \to \infty} \nu_s^i(\mathbb{R}^2 \setminus \overline{B_R}) = \infty,$$

here,

$$\nu_s^i(\Omega) = \begin{cases} \inf\limits_{u \in H_0^1(\Omega) \setminus \{0\}} \dfrac{\int_\Omega (|\nabla u|^2 + V_i(x) u^2) \mathrm{d}x}{(\int_\Omega |u|^s \mathrm{d}x)^{\frac{2}{s}}}, & \Omega \neq \emptyset, \\ \infty, & \Omega = \emptyset; \end{cases}$$

(V4′) There exists functions $A_i(x) \in L^\infty_{loc}(\mathbb{R}^2)$, with $A_i(x) \geq 1$, and constants $\beta_i > 1, C_0, R_0 > 0$ such that

$$A_i(x) \leq C_0 [1 + V_i(x)^{\frac{1}{\beta_i}}], \quad \text{for all } |x| \geq R_0.$$

Here, (V1′) and (V2′) is assumed to ensure that $H_{V_i}(\mathbb{R}^2)$ is a Hilbert space, (V3′) and (V4′) play a crucial role in overcoming the lack of compactness.

In terms of nonlinearities, they defined $f_i : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ had α_0^i-critical growth at $+\infty$ involving the term $A_i(x)$, such as

$$\lim_{t \to +\infty} \frac{|f_i(x,t)|}{A_i(x) e^{\alpha t^2}} = 0, \text{ uniformly on } x \in \mathbb{R}^2 \text{ for all } \alpha > \alpha_0^i.$$

$$\lim_{t \to +\infty} \frac{|f_i(x,t)|}{A_i(x) e^{\alpha t^2}} = +\infty, \text{ uniformly on } x \in \mathbb{R}^2 \text{ for all } \alpha < \alpha_0^i.$$

Here, $A_i(x)$ was defined in (V4′). When $A_i(x) = 1$, (F1) holds. In addition, they assumed the following hypotheses:

(F1′) $f_i : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is C^1, $f_i(x,t) = 0$ for all $x \in \mathbb{R}^2, t \le 0$, and

$$\lim_{t\to 0} \frac{|f_i(x,t)|}{A_i(x)|t|} = 0, \quad \text{uniformly on } x \in \mathbb{R}^2;$$

(F2′) $f_i(x,t)$ is locally bounded in t, that is, for any bounded interval $\Lambda \subset \mathbb{R}$, there exists $C > 0$ such that $f_i(x,t) \le C$, for all $(x,t) \in \mathbb{R}^2 \times \Lambda$;

(F3′) There exists $\mu_i > 2$ such that

$$t f_i(x,t) \ge \mu_i F_i(x,t) := \mu_i \int_0^t f_i(x,s)\mathrm{d}s > 0, \qquad \forall (x,t) \in \mathbb{R}^2 \times \mathbb{R}^+;$$

(F4′) For each fixed $x \in \mathbb{R}^2$ the function $t \mapsto \frac{f_i(x,t)}{t}$ is increasing for $t > 0$;

(F5′) There exists $q > 2$ such that

$$F_1(x,s) + F_2(x,t) \ge \vartheta(s^q + t^q)$$

for all $x \in \mathbb{R}^2$ and $s, t \ge 0$, $\vartheta > 0$ is a constant.

In [15], jointly with (V4′), one can find that the growth of f_i are controlled by the growth of $V_i(x)$, $i = 1,2$ form (F1′). Moreover, the condition $f_i \in C^1$ in (F1′) plays a crucial role to obtain the Nehari-type ground state solutions for (1.1) via the Nehari manifold method. (F3′) is the well-known Ambrosetti–Rabinowitz condition ((AR) condition), which ensures that the functional associated with the problem has a mountain pass geometry and guarantees the boundedness of the Palais–Smale sequence. (F4′) is the Nehari monotonic condition. (F5′) needs that the nonlinearities are super-q growth at zero, $q > 2$. It is noticed that sufficiently large ϑ in (F5′) is very crucial in their arguments. In fact, by virtue of this condition, the minimax-level for the energy functional can be choosen sufficiently small, therefore the difficult arising from the critical growth of Trudinger–Moser type is easily overcome. But this result has no relationship with the exponential velocity α_0^i, $i = 1,2$, hence it does not reveal the essential characteristics with the critical growth of Trudinger–Moser type.

Recently, Wei, Lin and Tang [20] used non-Nehari manifold methods (see [19]), Lions's concentration compactness and a direct approach derived from [7] for obtaining the minimax estimate to investigate system (1.6) in the non-autonomous case. They proved that (1.6) still possessed a Nehari-type ground state solution and a nontrivial solution. Their results improved the existence results of [16] by weakening the nonlinearities to be continuous, and only needed to satisfy the weaker Nehari monotonic condition, even without (AR) condition. Additionally, since the generalized linking theorem did not work for the strongly indefinite Hamiltonian elliptic system with critical exponential growth in $\mathbb{R}^2$, Qin, Tang and Zhang [18] developed a new approach to seek Cerami sequences for the energy functional and estimated the minimax levels of these sequences. Furthermore, they used non-Nehari manifold method to obtain the existence of ground state solutions without (AR) condition.

It is interesting to ask if the existence of positive ground state solutions for linearly coupled systems with variable potentials is preserved without (AR) condition. Our aim in this paper is to prove the existence of positive Nehari-type ground state solution of (1.1) and obtain the asymptotic behaviors of ground states with some mild assumptions. This work is motivated by the results of [15, 18, 20]. Our main result below (Theorem 1.2) can handle the case of

$f_i(x,t)$ with less restrictions, which are in the true sense of critical exponential growth, and are independent of (F5′) with some large constant ϑ (see [15, Theroem 1.1]).

To this end, we emphasize that we need refinements in order to treat the different setting from the constant potentials to the variable ones. Indeed, it is easy to get the mountain pass geometry for the problem with the constant potentials, while for variable potentials, some new analysis techniques and imbedding inequalities such as (2.2) are needed. We borrow the ideas from [18, 20] to look for the minimizing Cerami sequence for the energy functional associated with (1.1) by using the non-Nehari manifold approach. By means of slightly weaker monotonic conditions, we show the boundedness of the Cerami sequence. Furthermore, to recover the compactness of the minimizing Cerami sequence, we estimate an accurate threshold for the minimax-level, meanwhile, we use Lions's concentration compactness principle and the invariance of the energy functional by translation to show that the sequence does not vanish. Then by using a standard bootstrap argument and L^q-estimates we get regularity and asymptotic behavior of the ground state solution.

To state our main results, in addition to (F1) and (F2), we also introduce the following assumptions:

(F3) There exists $M_0 > 0$ and $t_0 > 0$ such that for every $x \in \mathbb{R}^2$,
$$F_i(x,t) \le M_0|f_i(x,t)|, \qquad \forall |t| \ge t_0;$$

(F4) For every $x \in \mathbb{R}^2$, $\frac{f_i(x,t)}{t}$ is non-decreasing on $(0,\infty)$;

(F5) $\liminf_{|t|\to\infty} \frac{t^2F_i(x,t)}{e^{\alpha_0 t^2}} \ge \kappa > \frac{V_M}{\alpha_0^2}$ uniformly on $x \in \mathbb{R}^2$, where $\alpha_0 = \max\{\alpha_1, \alpha_2\}$, $V_M = \max_{\mathbb{R}^2}\{V_1, V_2\}$.

In view of Lemma 1.1 i), under assumption (V), (F1) and (F2), the weak solutions of (1.1) correspond to the critical points of the energy functional defined by
$$\Phi(u,v) = \frac{1}{2}\left[\|(u,v)\|^2 - 2\int_{\mathbb{R}^2} \lambda(x)uv\mathrm{d}x\right] - \int_{\mathbb{R}^2}[F_1(x,u) + F_2(x,v)]\mathrm{d}x. \tag{1.7}$$
where $\|\cdot\|$ is defined in Section 2, (2.3).

Now our main results can be stated as follow.

Theorem 1.2. *Let (V), $(F1)$–$(F5)$ be satisfied. Then* (1.1) *has a solution $(\bar{u},\bar{v}) \in \mathcal{N}$ with $\bar{u} > 0$ and $\bar{v} > 0$ such that $\Phi(\bar{u},\bar{v}) = b := \inf_{\mathcal{N}} \Phi$, where*
$$\mathcal{N} := \{u \in E \setminus \{(0,0)\} : \langle \Phi'(u,v),(u,v)\rangle = 0\}, \tag{1.8}$$
where E is defined in Section 2. Moreover, $(\bar{u},\bar{v}) \in C^{1,\beta}_{loc}(\mathbb{R}^2) \times C^{1,\beta}_{loc}(\mathbb{R}^2)$ for some $\beta \in (0,1)$ with the following asymptotic behavior
$$\|\bar{u}\|_{C^{1,\beta}(\overline{B_R})} \to 0 \quad \text{and} \quad \|\bar{v}\|_{C^{1,\beta}(\overline{B_R})} \to 0 \quad \text{as } |x| \to \infty. \tag{1.9}$$

Remark 1.3. Theorem 1.2 improves and extends the results in [15, Theorem 1.1]. In the sense of the conditions of nonlinearities, f_1 and f_2 are continuous and the growth of them are independent on $V_i(x)$. For obtaining the boundedness of Cerami sequence, we only need the condition (F3) used for the exponential growth problems instead of (F3′). When it comes to the minimax level estimates of the energy functional, the authors in [15] made use of a rigorous limitation on the norm of the minimizing sequence by the the polynomial controlled condition (F5′), while we use the direct calculation argument with the exponential controlled condition (F5). Moreover, we use the weaker monotonicity condition (F4) to replace (F4′).

Remark 1.4. There are many functions satisfying the conditions (F1)–(F5) of the nonlinearities in this paper, but not satisfying the conditions (F4′) and (F5′) in [15]. For example, for$a_1, a_2 > 0$,

$$f_1(x,t) = \begin{cases} a_1(e^{2t^2} - 1), & t > 0, \\ 0, & t \le 0, \end{cases}$$

$$f_2(x,t) = \begin{cases} a_2|t|te^{t^2}, & t > 0, \\ 0, & t \le 0. \end{cases}$$

The paper is organized as follows. In Section 2, we give the variational setting and preliminaries. In Section 3, we establish the minimax estimates of the energy functional. The proof of ground state solution will be stated in Section 4. Then in Section 5, we give the proof of regularity and asymptotic behavior.

Throughout the paper, we make use of the following notations:

- $L^s(\mathbb{R}^2)(1 \le s < \infty)$ denotes the Lebesgue space with the norm $\|u\|_s = (\int_{\mathbb{R}^2} |u|^s \mathrm{d}x)^{1/s}$;
- $\forall x \in \mathbb{R}^2$ and $r > 0$, $B_r(x) := \{y \in \mathbb{R}^2 : |y - x| < r\}$;
- $C_1, C_2, C_3, \dots$ denote positive constants possibly different in different places.

2 Variational setting and preliminaries

Consider that the potentials are positive, we define the inner product in $H^1(\mathbb{R}^2)$ and the associated norm as follows,

$$(u,v) := \int_{\mathbb{R}^2} [\nabla u \nabla v + V(x)uv]\mathrm{d}x, \qquad \|u\|^2 := (u,u), \quad \forall u,v \in H^1(\mathbb{R}^2). \tag{2.1}$$

For any $s \in [2, +\infty)$, the Sobolev embedding theorem yields the existence of $\gamma_s \in (0, +\infty)$ such that

$$\|u\|_s \le \gamma_s \|u\|, \qquad \forall u \in H^1(\mathbb{R}^2). \tag{2.2}$$

Under (V), let $H_{V_1}(\mathbb{R}^2)$ and $H_{V_2}(\mathbb{R}^2)$ be endowed with the norm

$$\|u\|_{V_1} = \left(\int_{\mathbb{R}^2} |\nabla u|^2 + V_1(x)u^2 \mathrm{d}x \right)^{\frac{1}{2}}, \qquad \|v\|_{V_2} = \left(\int_{\mathbb{R}^2} |\nabla v|^2 + V_2(x)v^2 \mathrm{d}x \right)^{\frac{1}{2}}.$$

Define $E := H_{V_1}(\mathbb{R}^2) \times H_{V_2}(\mathbb{R}^2)$ and

$$((u,v),(\phi,\psi)) := \int_{\mathbb{R}^2} (\nabla u \nabla \phi + \nabla v \nabla \psi + V_1(x)u\phi + V_2(x)v\psi)\mathrm{d}x, \quad \forall (u,\phi) \in H_{V_1}, (v,\psi) \in H_{V_2}.$$

Then E is a Hilbert space on the above inner product. The induced norm

$$\|(u,v)\|^2 := \int_{\mathbb{R}^2} (|\nabla u|^2 + |\nabla v|^2 + V_1(x)u^2 + V_2(x)v^2)\mathrm{d}x, \qquad \forall (u,v) \in E. \tag{2.3}$$

That is $\|(u,v)\|^2 = \|u\|_{V_1}^2 + \|v\|_{V_2}^2$. By (V), (1.7) and Lemma 1.1, we know that the functional $\Phi(u,v)$ is well defined on E. Moreover, by standard arguments, $\Phi \in C^1(E, \mathbb{R})$ and its derivative is given by

$$\langle \Phi'(u,v),(\phi,\psi) \rangle = ((u,v),(\phi,\psi)) - \int_{\mathbb{R}^2} \lambda(x)(u\psi + v\phi)\mathrm{d}x - \int_{\mathbb{R}^2} [f_1(x,u)\phi + f_2(x,v)\psi]\mathrm{d}x \tag{2.4}$$

and

$$\langle \Phi'(u,v),(u,v)\rangle = \|(u,v)\|^2 - 2\int_{\mathbb{R}^2}\lambda(x)uv\mathrm{d}x - \int_{\mathbb{R}^2}[f_1(x,u)u + f_2(x,v)v]\mathrm{d}x. \tag{2.5}$$

For any $\varepsilon > 0, \alpha > \alpha_0$ and $\bar{q} > 0$, it follows from (F1) and (F2) that there exists $C = C(\varepsilon,\alpha,\bar{q}) > 0$ such that

$$|F_i(x,t)| \le \varepsilon t^2 + C|t|^{\bar{q}} e^{\alpha t^2}. \tag{2.6}$$

Now we choose $(u_0,v_0) \in E \setminus \{(0,0)\}$, it is easy to show that $\lim_{t\to\infty}\Phi(tu_0,tv_0) = -\infty$ due to (V) and (F1).

Lemma 2.1. *Assume that* (V), (F1) *and* (F2) *hold. Then there exists a sequence* $(u_n,v_n) \subset E$ *satisfying*

$$\Phi(u_n,v_n) \to c^*, \quad \|\Phi'(u_n,v_n)\|(1+\|(u_n,v_n)\|) \to 0. \tag{2.7}$$

where c^* *is given by*

$$c^* = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\Phi(\gamma(t)),$$

$$\Gamma = \{\gamma \in C([0,1],E) : \gamma(0) = 0, \Phi(\gamma(1)) < 0\}.$$

Proof. By (2.6), one has for some constants $\alpha > \alpha_0$ and $C_1 > 0$

$$F_i(x,t) \le \frac{1-\lambda_0}{4\gamma_2^2}t^2 + C_1|t|^3(e^{\alpha t^2}-1). \tag{2.8}$$

From (2.8) and Lemma 1.1 ii), we obtain

$$\begin{aligned}\int_{\mathbb{R}^2}F_1(x,u)\mathrm{d}x &\le \frac{1-\lambda_0}{4\gamma_2^2}\|u\|_2^2 + C_1\int_{\mathbb{R}^2}(e^{\alpha u^2}-1)|u|^3\mathrm{d}x \\ &\le \frac{1-\lambda_0}{4\gamma_2^2}\|u\|_2^2 + C_1\left[\int_{\mathbb{R}^2}(e^{2\alpha u^2}-1)\mathrm{d}x\right]^{\frac{1}{2}}\|u\|_6^3 \\ &\le \frac{1-\lambda_0}{4}\|u\|_{V_1}^2 + C_2\|u\|_{V_1}^3, \qquad \forall\|(u,v)\| \le \sqrt{\pi/\alpha}. \end{aligned}\tag{2.9}$$

Similarly, we have

$$\int_{\mathbb{R}^2}F_2(x,v)\mathrm{d}x \le \frac{1-\lambda_0}{4}\|v\|_{V_2}^2 + C_2\|v\|_{V_2}^3, \qquad \forall\|(u,v)\| \le \sqrt{\pi/\alpha}. \tag{2.10}$$

Hence, it follows from (V), (1.7), (2.9) and (2.10) that

$$\begin{aligned}\Phi(u,v) &= \frac{1}{2}\left[\|(u,v)\|^2 - 2\int_{\mathbb{R}^2}\lambda(x)uv\mathrm{d}x\right] - \int_{\mathbb{R}^2}[F_1(x,u)+F_2(x,v)]\mathrm{d}x \\ &\ge \frac{1}{2}\left[\|(u,v)\|^2 - \lambda_0\int_{\mathbb{R}^2}(V_1(x)u^2+V_2(x)v^2)\mathrm{d}x\right] - \frac{1-\lambda_0}{4}(\|u\|_{V_1}^2+\|v\|_{V_2}^2) \\ &\quad - C_2(\|u\|_{V_1}^3+\|v\|_{V_2}^3) \\ &\ge \frac{1-\lambda_0}{4}\|(u,v)\|^2 - C_3\|(u,v)\|^3. \end{aligned}\tag{2.11}$$

Therefore, there exists $\kappa_0 > 0$ and $0 < \rho < \sqrt{\pi/\alpha}$ such that

$$\Phi(u,v) \ge \kappa_0, \qquad \forall (u,v) \in S := \{(u,v)\in E : \|(u,v)\| = \rho\}. \tag{2.12}$$

Since $\lim_{t\to\infty}\Phi(tu_0,tv_0) = -\infty$, we can choose $T > 0$ such that $e = (Tu_0,Tv_0) \in \{(u,v) \in E : \|(u,v)\| \ge \rho\}$ and $\Phi(e) < 0$, then according to the mountain pass lemma, we deduce that there exists $c^* \in [\kappa_0, \sup_{t\ge 0}\Phi(tu_0,tv_0)]$ and a sequence $\{(u_n,v_n)\} \subset E$ satisfying (2.7). □

Lemma 2.2. *Assume that* (V), (F1), (F2) *and* (F4) *hold. Then*

$$\Phi(u,v) \geq \Phi(tu,tv) + \frac{1-t^2}{2} \left\langle \Phi'(u,v),(u,v) \right\rangle, \qquad \forall t > 0. \tag{2.13}$$

Proof. It is obvious that (F4) implies the following inequality:

$$\frac{1-t^2}{2} f_i(x,s)s + F_i(x,ts) - F_i(x,s) = \int_t^1 \left[\frac{f_i(x,s)}{s} - \frac{f_i(x,\tau s)}{\tau s} \right] \tau s^2 \mathrm{d}\tau \geq 0. \tag{2.14}$$

From (1.7), (2.5) and (2.14), we have

$$\begin{aligned}
\Phi(u,v) - \Phi(tu,tv) &= \frac{1}{2}\left[\|(u,v)\|^2 - \int_{\mathbb{R}^2} \lambda(x)uv\mathrm{d}x \right] - \int_{\mathbb{R}^2} [F_1(x,u) + F_2(x,v)]\mathrm{d}x \\
&\quad - \left\{ \frac{t^2}{2}\|(u,v)\|^2 - t^2 \int_{\mathbb{R}^2} \lambda(x)uv\mathrm{d}x - \int_{\mathbb{R}^2} F_1(x,tu) + F_2(x,tv)\mathrm{d}x \right\} \\
&= \frac{1-t^2}{2} \left\langle \Phi'(u,v),(u,v) \right\rangle + \int_{\mathbb{R}^2} \left[\frac{1-t^2}{2} f_1(x,u)u + F_1(x,tu) - F_1(x,u) \right] \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^2} \left[\frac{1-t^2}{2} f_2(x,v)v + F_2(x,tv) - F_2(x,v) \right] \mathrm{d}x \\
&\geq \frac{1-t^2}{2} \left\langle \Phi'(u,v),(u,v) \right\rangle.
\end{aligned}$$

□

From Lemma 2.2, we get the following corollary easily.

Corollary 2.3. *Assume that* (V), (F1), (F2) *and* (F4) *hold. Then*

$$\Phi(u,v) \geq \max_{t \geq 0} \Phi(tu,tv), \qquad \forall (u,v) \in \mathcal{N}. \tag{2.15}$$

Lemma 2.4. *Assume that* (V), (F1), (F2) *and* (F4) *hold. Then for any* $(u,v) \in E \setminus \{(0,0)\}$, *there exists a unique* $t_{(u,v)} > 0$ *such that* $(t_{(u,v)}u, t_{(u,v)}v) \in \mathcal{N}$.

Proof. Let $(u,v) \in E \setminus \{(0,0)\}$ be fixed and define a function $\zeta(t) := \Phi(tu,tv)$ on $[0,\infty)$. Clearly, by (2.5), we have

$$\begin{aligned}
\zeta'(t) = 0 &\Leftrightarrow t^2\|(u,v)\|^2 - 2t^2 \int_{\mathbb{R}^2} \lambda(x)uv\mathrm{d}x - \int_{\mathbb{R}^2} [f_1(x,tu)tu + f_2(x,tv)tv]\mathrm{d}x = 0 \\
&\Leftrightarrow \left\langle \Phi'(tu,tv),(tu,tv) \right\rangle = 0 \Leftrightarrow (tu,tv) \in \mathcal{N}.
\end{aligned}$$

By (2.11) and (F1), one has $\zeta(0) = 0$ and $\zeta(t) > 0$ for $t > 0$ small and $\zeta(t) < 0$ for t large. Therefore, $\max_{t\in(0,\infty)} \zeta(t)$ is achieved at some $t_0 = t_{(u,v)} > 0$, so that $\zeta'(t_0) = 0$ and $t_{(u,v)}(u,v) \in \mathcal{N}$.

Next we claim that $t_{(u,v)}$ is unique for any $(u,v) \in E \setminus \{(0,0)\}$, let $t_1, t_2 > 0$ such that $\zeta'(t_1) = \zeta'(t_2) = 0$. Then $\langle \Phi'(t_1u,t_1v),(t_1u,t_1v) \rangle = \langle \Phi'(t_2u,t_2v),(t_2u,t_2v) \rangle = 0$. Jointly with (2.13) , we have

$$\Phi(t_1u,t_1v) \geq \Phi(t_2u,t_2v) + \frac{1-t^2}{2} \left\langle \Phi'(t_1u,t_1v),(t_1u,t_1v) \right\rangle \tag{2.16}$$

and

$$\Phi(t_2u,t_2v) \geq \Phi(t_1u,t_1v) + \frac{1-t^2}{2} \left\langle \Phi'(t_2u,t_2v),(t_2u,t_2v) \right\rangle. \tag{2.17}$$

By (2.16) and (2.17), it is obvious that $t_1 = t_2$. Therefore $t_{(u,v)} > 0$ is unique for any $(u,v) \in E \setminus \{(0,0)\}$. □

From Corollary 2.3 and Lemma 2.4, we directly have the following lemma about minimax characterization of $\inf_{\mathcal{N}} \Phi$.

Lemma 2.5. *Assume that* (V), (F1), (F2) *and* (F4) *hold. Then*

$$b := \inf_{\mathcal{N}} \Phi = \inf_{(u,v)\in E\setminus\{(0,0)\}} \max_{t\geq 0} \Phi(tu, tv). \tag{2.18}$$

Lemma 2.6. *Assume that* (V), (F1), (F2) *and* (F4) *hold. Then there exist a constant $\bar{c} \in (0, b]$ and a sequence $\{(u_n, v_n)\} \subset E$ satisfying*

$$\Phi(u_n, v_n) \to \bar{c}, \qquad \|\Phi'(u_n, v_n)\|_{E^*}(1 + \|(u_n, v_n)\|) \to 0. \tag{2.19}$$

Similarly with [20, Lemma 2.6], the proof is omitted here.

Lemma 2.7. *Assume that* (V), (F1)–(F4) *hold. Then any sequence $\{(u_n, v_n)\}$ satisfying* (2.19) *is bounded.*

Proof. Arguing by contradiction, suppose that $\|(u_n, v_n)\| \to \infty$ as $n \to \infty$. Let $(\tilde{u}_n, \tilde{v}_n) = (u_n, v_n)/\|(u_n, v_n)\|$. Then $1 = \|(\tilde{u}_n, \tilde{v}_n)\|^2$. By (F2) and (F4), we have

$$\frac{f_i(x, \theta t)\theta t}{\theta^2} \geq f_i(x,t)t \geq 2F_i(x,t) \geq 0, \qquad \forall x \in \mathbb{R}^2, t \in \mathbb{R}, \theta \geq 1, i = 1, 2. \tag{2.20}$$

It follows from (F3) and (2.20) that there exists $R > t_0$ such that

$$f_i(x,t)t \geq 4F_i(x,t), \qquad \forall |t| \geq R. \tag{2.21}$$

From (1.7), (2.5), (2.19), (2.20), and (2.21), we have

$$\begin{aligned}
\bar{c} + o(1) &= \Phi(u_n, v_n) - \frac{1}{2}\langle \Phi'(u_n, v_n), (u_n, v_n)\rangle \\
&= \int_{\mathbb{R}^2}\left[\frac{1}{2}f_1(x,u_n)u_n - F_1(x,u_n)\right]\mathrm{d}x + \int_{\mathbb{R}^2}\left[\frac{1}{2}f_2(x,v_n)v_n - F_2(x,v_n)\right]\mathrm{d}x \\
&\geq \int_{|u_n|\leq R}\left[\frac{1}{2}f_1(x,u_n)u_n - F_1(x,u_n)\right]\mathrm{d}x + \int_{|v_n|\leq R}\left[\frac{1}{2}f_2(x,v_n)v_n - F_2(x,v_n)\right]\mathrm{d}x \\
&\quad + \frac{1}{4}\int_{|u_n|>R} f_1(x,u_n)u_n \mathrm{d}x + \frac{1}{4}\int_{|v_n|>R} f_2(x,v_n)v_n \mathrm{d}x \\
&\geq \frac{1}{4}\int_{|u_n|>R} f_1(x,u_n)u_n \mathrm{d}x + \frac{1}{4}\int_{|v_n|>R} f_2(x,v_n)v_n \mathrm{d}x.
\end{aligned} \tag{2.22}$$

Let $\tau \geq \left(\frac{4(\bar{c}+1)}{1-2\lambda_0}\right)^{\frac{1}{2}}$ and $t_n = \tau/\|(u_n, v_n)\|$. Then $t_n \to 0$ as $n \to \infty$. It follows from (F2), (2.21) and (2.22) that

$$
\begin{aligned}
&\int_{\mathbb{R}^2}[F_1(x,t_nu_n)+F_2(x,t_nu_n)]\mathrm{d}x\\
&\quad=\int_{|u_n|\le R}F_1(x,t_nu_n)\mathrm{d}x+\int_{|u_n|>R}F_1(x,t_nu_n)\mathrm{d}x\\
&\qquad+\int_{|v_n|\le R}F_2(x,t_nv_n)\mathrm{d}x+\int_{|v_n|>R}F_2(x,t_nv_n)\mathrm{d}x\\
&\quad\le\frac{1}{4\gamma_2^2}\int_{|u_n|\le R}|t_nu_n|^2\mathrm{d}x+\frac{1}{4\gamma_2^2}\int_{|v_n|\le R}|t_nv_n|^2\mathrm{d}x\\
&\qquad+\frac{t_n^2}{4}\int_{|u_n|>R}f_1(x,u_n)u_n\mathrm{d}x+\frac{t_n^2}{4}\int_{|v_n|>R}f_2(x,v_n)v_n\mathrm{d}x\\
&\quad\le\frac{t_n^2}{4\gamma_2^2}\int_{|u_n|\le R}|u_n|^2\mathrm{d}x+\frac{t_n^2}{4\gamma_2^2}\int_{|v_n|\le R}|v_n|^2\mathrm{d}x+\frac{\tau^2(\bar{c}+1)}{\|(u_n,v_n)\|^2}\\
&\quad\le\frac{\tau^2}{4}+o(1).
\end{aligned}
\tag{2.23}
$$

Hence, from (2.13), (2.19) and (2.23), we have

$$
\begin{aligned}
\bar{c}+o(1)&=\Phi(u_n,v_n)\\
&\ge\Phi(t_nu_n,t_nv_n)+\frac{1-t_n^2}{2}\left\langle\Phi'(u_n,v_n),(u_n,v_n)\right\rangle\\
&=\frac{t_n^2}{2}\left[\|(u_n,v_n)\|^2-2\int_{\mathbb{R}^2}\lambda(x)u_nv_n\mathrm{d}x\right]-\int_{\mathbb{R}^2}[F_1(x,t_nu_n)+F_2(x,t_nv_n)]\mathrm{d}x+o(1)\\
&\ge\frac{(1-2\lambda_0)\tau^2}{4}+o(1)\\
&\ge\bar{c}+1+o(1).
\end{aligned}
\tag{2.24}
$$

This contradiction shows that $\{(u_n,v_n)\}$ is bounded. □

3 Minimax estimates

In this section, we give a accurate estimation about the minimax level c^* defined by Lemma 2.1.

At first, we define a Moser type function $w_n(x)$ supported in $B_{\sqrt{2/V_M}}:=B_{\sqrt{2/V_M}}(0)$ as follows:

$$
w_n(x)=\frac{1}{\sqrt{2\pi}}\begin{cases}\sqrt{\log n}, & 0\le|x|\le\sqrt{2}/(\sqrt{V_M}n);\\ \dfrac{\log(\sqrt{2}/\sqrt{V_M}|x|)}{\sqrt{\log n}}, & \sqrt{2}/(\sqrt{V_M}n)\le|x|\le\sqrt{2/V_M};\\ 0, & |x|\ge\sqrt{2/V_M}.\end{cases}
\tag{3.1}
$$

By an elementary computation, we have

$$
\|\nabla w_n\|_2^2=\int_{\mathbb{R}^2}|\nabla w_n|^2\mathrm{d}x=1,
\tag{3.2}
$$

and

$$
\|w_n\|_2^2=\int_{\mathbb{R}^2}|w_n|^2\mathrm{d}x=\frac{2\delta_n}{V_M},
\tag{3.3}
$$

where

$$\delta_n := \frac{1}{4\log n} - \frac{1}{4n^2\log n} - \frac{1}{2n^2} > 0. \tag{3.4}$$

Lemma 3.1. *Assume that* (V), (F1), (F2), *and* (F5) *hold. Furthermore, suppose that* $F_i(x,t) \geq 0$ *for all* $t \in \mathbb{R}$. *Then there exists* $\bar{n} \in \mathbb{N}$ *such that*

$$c^* \leq \max_{t\geq 0} \Phi(tw_{\bar{n}}, tw_{\bar{n}}) < \frac{4\pi}{\alpha_0}. \tag{3.5}$$

Proof. By (F5), we can choose $\varepsilon > 0$ and $t_\varepsilon > 0$ such that

$$\log \frac{V_M(1+\epsilon)^2}{(2-\epsilon)(\kappa-\epsilon)\alpha_0^2} < -\epsilon \tag{3.6}$$

and

$$t^2F_i(x,t) \geq (\kappa-\epsilon)e^{\alpha_0 t^2}, \quad \forall x \in \mathbb{R}^2, \quad |t| \geq t_\epsilon, \quad i = 1,2. \tag{3.7}$$

From (1.7), (3.2) and (3.3), we have

$$\begin{aligned}
\Phi(tw_n, tw_n) &= \frac{t^2}{2}\int_{\mathbb{R}^2}[|\nabla w_n|^2 + |\nabla w_n|^2 + V_1(x)w_n^2 + V_2(x)w_n^2]\mathrm{d}x \\
&\quad - t^2\int_{\mathbb{R}^2}\lambda(x)w_n^2\mathrm{d}x - \int_{\mathbb{R}^2}[F_1(x,tw_n) + F_2(x,tw_n)]\mathrm{d}x \\
&\leq t^2(1+2\delta_n) - \int_{\mathbb{R}^2}[F_1(x,tw_n) + F_2(x,tw_n)]\mathrm{d}x.
\end{aligned} \tag{3.8}$$

There are four cases to distinguish. In the sequel, we agree that all inequalities hold for large $n \in \mathbb{N}$ without mentioning.

Case i). $t \in \left[0, \sqrt{\frac{2\pi}{\alpha_0}}\right]$. Then it follows from (3.8) that

$$\begin{aligned}
\Phi(tw_n, tw_n) &\leq t^2(1+2\delta_n) - \int_{\mathbb{R}^2}[F_1(x,tw_n) + F_2(x,tw_n)]\mathrm{d}x \\
&\leq t^2\left(1 + \frac{1}{2\log n}\right) + O\left(\frac{1}{n\log n}\right) \\
&\leq \frac{2\pi}{\alpha_0} + O\left(\frac{1}{n\log n}\right).
\end{aligned} \tag{3.9}$$

Clearly, there exists $\bar{n} \in \mathbb{N}$ such that (3.5) hold.

Case ii). $t \in \left[\sqrt{\frac{2\pi}{\alpha_0}}, \sqrt{\frac{4\pi}{\alpha_0}}\right]$. Then $tw_n(x) \geq t_\epsilon$ for $x \in B_{\sqrt{2/(V_M n)}}$ and for large $n \in \mathbb{N}$, it follows (3.1) and (3.7) that

$$\begin{aligned}
\int_{\mathbb{R}^2} F_1(x,tw_n)\mathrm{d}x &\geq \int_{B_{\sqrt{2/(V_M n)}}} F_1(x,tw_n)\mathrm{d}x \\
&\geq \int_{B_{\sqrt{2/(V_M n)}}} \frac{(\kappa-\epsilon)e^{\alpha_0 t^2 w_n^2}}{t^2 w_n^2}\mathrm{d}x \\
&\geq \frac{(\kappa-\epsilon)\alpha_0}{2\log n}\int_{B_{\sqrt{2/(V_M n)}}} e^{\alpha_0 t^2 w_n^2}\mathrm{d}x \\
&= \frac{\pi(\kappa-\epsilon)\alpha_0}{V_M n^2\log n}\left[e^{(2\pi)^{-1}\alpha_0 t^2\log n} + 2n^2\log n\int_{1/2}^1 n^{(2\pi)^{-1}\alpha_0 t^2 s^2 - 2s}\mathrm{d}s\right]
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{\pi(\kappa-\epsilon)\alpha_0}{V_M n^2 \log n}\left[e^{(2\pi)^{-1}\alpha_0 t^2 \log n} + 2n^2 \log n \int_{1/2}^{1} n^{(2\pi)^{-1}\alpha_0 t^2 s - 2} \mathrm{d}s\right] \\
&= \frac{\pi(\kappa-\epsilon)\alpha_0}{V_M n^2 \log n}\left[e^{(2\pi)^{-1}\alpha_0 t^2 \log n} + \frac{4\pi}{\alpha_0 t^2}\left(n^{(2\pi)^{-1}\alpha_0 t^2} - n^{(4\pi)^{-1}\alpha_0 t^2}\right)\right] \\
&\geq \frac{2\pi(\kappa-\epsilon)\alpha_0}{V_M n^2 \log n} e^{(2\pi)^{-1}\alpha_0 t^2 \log n} - O\left(\frac{1}{n \log n}\right). \qquad (3.10)
\end{aligned}
$$

Similarly, we have

$$
\int_{\mathbb{R}^2} F_2(x, t w_n) \mathrm{d}x \geq \frac{2\pi(\kappa-\epsilon)\alpha_0}{V_M n^2 \log n} e^{(2\pi)^{-1}\alpha_0 t^2 \log n} - O\left(\frac{1}{n \log n}\right). \qquad (3.11)
$$

It follows from (3.8), (3.10) and (3.11) that

$$
\begin{aligned}
\Phi(t w_n, t w_n) &\leq t^2\left(1 + \frac{1}{2\log n}\right) - \frac{4\pi(\kappa-\epsilon)\alpha_0}{V_M n^2 \log n} e^{(2\pi)^{-1}\alpha_0 t^2 \log n} + 2O\left(\frac{1}{n \log n}\right) \\
&=: \varphi_n(t) + 2O\left(\frac{1}{n \log n}\right). \qquad (3.12)
\end{aligned}
$$

Let $t_n > 0$ such that $\varphi_n'(t_n) = 0$. Then

$$
1 + \frac{1}{2\log n} = \frac{2(\kappa-\epsilon)\alpha_0^2}{V_M n^2} e^{(2\pi)^{-1}\alpha_0 t_n^2 \log n}. \qquad (3.13)
$$

It follows that

$$
\lim_{n\to\infty} t_n^2 = \frac{4\pi}{\alpha_0}. \qquad (3.14)
$$

From (3.13) and (3.14), we have

$$
\begin{aligned}
t_n^2 &= \frac{4\pi}{\alpha_0}\left[1 + \frac{\log\left(V_M + \frac{V_M}{2\log n}\right) - \log(2(\kappa-\epsilon)\alpha_0^2)}{2\log n}\right] \\
&\leq \frac{4\pi}{\alpha_0} + \frac{2\pi}{\alpha_0 \log n} \log \frac{V_M(1+\epsilon)}{2(\kappa-\epsilon)\alpha_0^2}. \qquad (3.15)
\end{aligned}
$$

and

$$
\varphi_n(t) \leq \varphi_n(t_n) = t_n^2\left(1 + \frac{1}{2\log n}\right) - \frac{2\pi}{\alpha_0 \log n}\left(1 + \frac{1}{2\log n}\right). \qquad (3.16)
$$

From (3.14), (3.15) and (3.16), we have

$$
\begin{aligned}
\varphi_n(t) &\leq t_n^2\left(1 + \frac{1}{2\log n}\right) - \frac{2\pi}{\alpha_0 \log n}\left(1 + \frac{1}{2\log n}\right) \\
&\leq \left[\frac{4\pi}{\alpha_0} + \frac{2\pi}{\alpha_0 \log n} \log \frac{V_M(1+\epsilon)}{2(\kappa-\epsilon)\alpha_0^2}\right]\left(1 + \frac{1}{2\log n}\right) - \frac{2\pi}{\alpha_0 \log n}\left(1 + \frac{1}{2\log n}\right) \\
&\leq \left[\frac{4\pi}{\alpha_0} + \frac{2\pi}{\alpha_0 \log n} \log \frac{V_M(1+\epsilon)}{2(\kappa-\epsilon)\alpha_0^2}\right]\left(1 + \frac{1}{2\log n}\right) - \frac{2\pi}{\alpha_0 \log n}(1-\epsilon) \\
&\leq \frac{4\pi}{\alpha_0} + \frac{2\pi}{\alpha_0 \log n}\left[\log \frac{V_M(1+\epsilon)}{2(\kappa-\epsilon)\alpha_0^2} + \epsilon\right] + O\left(\frac{1}{\log^2 n}\right). \qquad (3.17)
\end{aligned}
$$

Hence, combining (3.12) with (3.17), one has

$$
\Phi(t w_n, t w_n) \leq \frac{4\pi}{\alpha_0} + \frac{2\pi}{\alpha_0 \log n}\left[\log \frac{V_M(1+\epsilon)}{2(\kappa-\epsilon)\alpha_0^2} + \epsilon\right] + O\left(\frac{1}{\log^2 n}\right). \qquad (3.18)
$$

Obviously, (3.6) and (3.18) imply that there exists $\bar{n} \in \mathbb{N}$ such that (3.5) hold.

Case iii). $t \in \left[\sqrt{\frac{4\pi}{\alpha_0}}, \sqrt{\frac{4\pi}{\alpha_0}(1+\epsilon)}\right]$. Then $tw_n(x) \geq t_\epsilon$ for $x \in B_{\sqrt{2/(V_M n)}}$ and for large $n \in \mathbb{N}$, it follows (3.7) that

$$\begin{aligned}
\int_{\mathbb{R}^2} F_1(x, tw_n)\mathrm{d}x \geq & \int_{B_{\sqrt{2/(V_M n)}}} F_1(x, tw_n)\mathrm{d}x \\
\geq & \int_{B_{\sqrt{2/(V_M n)}}} \frac{(\kappa-\epsilon)e^{\alpha_0 t^2 w_n^2}}{t^2 w_n^2}\mathrm{d}x \\
\geq & \frac{(\kappa-\epsilon)\alpha_0}{2(1+\epsilon)\log n} \int_{B_{\sqrt{2/(V_M n)}}} e^{\alpha_0 t^2 w_n^2}\mathrm{d}x \\
= & \frac{\pi(\kappa-\epsilon)\alpha_0}{(1+\epsilon)V_M n^2 \log n}\left[e^{(2\pi)^{-1}\alpha_0 t^2 \log n} + 2n^2 \log n \int_{1/2}^{1} n^{(2\pi)^{-1}\alpha_0 t^2 s^2 - 2s}\mathrm{d}s\right] \\
\geq & \frac{\pi(\kappa-\epsilon)\alpha_0}{(1+\epsilon)V_M n^2 \log n}\left[e^{(2\pi)^{-1}\alpha_0 t^2 \log n} + 2\log n \int_{1-\epsilon}^{1} n^{[(1-\epsilon)(2\pi)^{-1}\alpha_0 t^2 + 2\epsilon]s}\mathrm{d}s\right] \\
= & \frac{\pi(\kappa-\epsilon)\alpha_0}{(1+\epsilon)V_M n^2 \log n}\left[e^{(2\pi)^{-1}\alpha_0 t^2 \log n} + \frac{1}{1+\epsilon}e^{[(1-\epsilon)(2\pi)^{-1}\alpha_0 t^2 + 2\epsilon]\log n}\right] \\
& - O\left(\frac{1}{n^{2\epsilon^2}\log n}\right) \\
\geq & \frac{2\pi(\kappa-\epsilon)\alpha_0}{(1+\epsilon)^{3/2}V_M n^{2-\epsilon}\log n}e^{(2-\epsilon)(4\pi)^{-1}\alpha_0 t^2 \log n} - O\left(\frac{1}{n^{2\epsilon^2}\log n}\right). \qquad (3.19)
\end{aligned}$$

Similarly, we have

$$\int_{\mathbb{R}^2} F_2(x, tw_n)\mathrm{d}x \geq \frac{2\pi(\kappa-\epsilon)\alpha_0}{(1+\epsilon)^{3/2}V_M n^{2-\epsilon}\log n}e^{(2-\epsilon)(4\pi)^{-1}\alpha_0 t^2 \log n} - O\left(\frac{1}{n^{2\epsilon^2}\log n}\right). \qquad (3.20)$$

It follows from (3.6) , (3.19) and (3.20) that

$$\begin{aligned}
\Phi(tw_n, tw_n) \leq & t^2\left(1 + \frac{1}{2\log n}\right) - \frac{4\pi(\kappa-\epsilon)\alpha_0}{(1+\epsilon)^{3/2}V_M n^{2-\epsilon}\log n}e^{(2-\epsilon)(4\pi)^{-1}\alpha_0 t^2 \log n} + 2O\left(\frac{1}{n^{2\epsilon^2}\log n}\right) \\
=: & \psi_n(t) + 2O\left(\frac{1}{n^{2\epsilon^2}\log n}\right). \qquad (3.21)
\end{aligned}$$

Let $\hat{t}_n > 0$ such that $\psi_n'(\hat{t}_n) = 0$. Then

$$1 + \frac{1}{2\log n} = \frac{(\kappa-\epsilon)(2-\epsilon)\alpha_0^2}{(1+\epsilon)^{3/2}V_M n^{2-\epsilon}}e^{(2-\epsilon)(4\pi)^{-1}\alpha_0 \hat{t}_n^2 \log n}. \qquad (3.22)$$

It follows that

$$\lim_{n\to\infty} \hat{t}_n^2 = \frac{4\pi}{\alpha_0}. \qquad (3.23)$$

From (3.22) and (3.23), we have

$$\begin{aligned}
\hat{t}_n^2 = & \frac{4\pi}{\alpha_0}\left[1 + \frac{(1+\epsilon)^{3/2}V_M(1+\frac{1}{2\log n}) - \log((2-\epsilon)(\kappa-\epsilon)\alpha_0^2)}{(2-\epsilon)\log n}\right] \\
\leq & \frac{4\pi}{\alpha_0} + \frac{4\pi}{\alpha_0(2-\epsilon)\log n}\log\frac{V_M(1+\epsilon)^2}{(2-\epsilon)(\kappa-\epsilon)\alpha_0^2}. \qquad (3.24)
\end{aligned}$$

It follows from (3.21), (3.23) and (3.24), we have

$$\begin{aligned}\psi_n(t) \le \psi_n(\hat{t}_n) &= \hat{t}_n^2\left(1+\frac{1}{2\log n}\right) - \frac{4\pi}{(2-\epsilon)\alpha_0\log n}\left(1+\frac{1}{2\log n}\right)\\ &\le \left[\frac{4\pi}{\alpha_0} + \frac{4\pi}{\alpha_0(2-\epsilon)\log n}\log\frac{V_M(1+\epsilon)^2}{(2-\epsilon)(\kappa-\epsilon)\alpha_0^2}\right]\left(1+\frac{1}{2\log n}\right) - \frac{4\pi(1-\epsilon)}{(2-\epsilon)\alpha_0\log n}\\ &\le \frac{4\pi}{\alpha_0} + \frac{4\pi}{(2-\epsilon)\alpha_0\log n}\left[\epsilon + \log\frac{V_M(1+\epsilon)^2}{(2-\epsilon)(\kappa-\epsilon)\alpha_0^2}\right] + O\left(\frac{1}{\log^2 n}\right). \qquad (3.25)\end{aligned}$$

Hence, combining (3.21) with (3.25), one has

$$\Phi(tw_n, tw_n) \le \frac{4\pi}{\alpha_0} + \frac{4\pi}{(2-\epsilon)\alpha_0\log n}\left[\epsilon + \log\frac{V_M(1+\epsilon)^2}{(2-\epsilon)(\kappa-\epsilon)\alpha_0^2}\right] + O\left(\frac{1}{\log^2 n}\right). \qquad (3.26)$$

Clearly, (3.6) and (3.26) imply that there exists $\bar{n} \in \mathbb{N}$ such that (3.5) hold.

Case iv). $t \in \left(\sqrt{\frac{4\pi}{\alpha_0}(1+\epsilon)}, +\infty\right)$. Then $tw_n(x) \ge t_\epsilon$ for $x \in B_{\sqrt{2}/(\sqrt{V_M}n)}$ and for large $n \in \mathbb{N}$, it follows (3.1) and (3.8) that

$$\begin{aligned}\Phi(tw_n, tw_n) &\le t^2\left(1+\frac{1}{2\log n}\right) - \int_{\mathbb{R}^2}[F_1(x,tw_n)+F_2(x,tw_n)]\mathrm{d}x\\ &\le t^2\left(1+\frac{1}{2\log n}\right) - \frac{8\pi^2(\kappa-\epsilon)}{V_M n^2 t^2\log n}e^{(2\pi)^{-1}\alpha_0 t^2\log n} + 2O\left(\frac{1}{n\log n}\right)\\ &\le \frac{4\pi(1+\epsilon)}{\alpha_0}\left(1+\frac{1}{2\log n}\right) - \frac{2\alpha_0\pi(\kappa-\epsilon)}{V_M(1+\epsilon)\log n}e^{2\epsilon\log n} + 2O\left(\frac{1}{n\log n}\right)\\ &\le \frac{3\pi}{\alpha_0}. \qquad (3.27)\end{aligned}$$

which implies that there exists $\bar{n} \in \mathbb{N}$ such that (3.5) hold. In the above derivation process, we use the fact that the function

$$t^2\left(1+\frac{1}{2\log n}\right) - \frac{8\pi^2(\kappa-\epsilon)}{V_M n^2 t^2\log n}e^{(2\pi)^{-1}\alpha_0 t^2\log n} + O\left(\frac{1}{n\log n}\right) \qquad (3.28)$$

is decreasing on $t \in \left(\sqrt{\frac{4\pi}{\alpha_0}(1+\epsilon)}, +\infty\right)$, since its stagnation tend to $\sqrt{\frac{4\pi}{\alpha_0}}$ as $n \to \infty$. □

From Lemma 2.5 and 3.1, we have the following corollary immediately.

Corollary 3.2. *Assume that* (V), (F1), (F2), (F4) *and* (F5) *hold. Then*

$$b := \inf_{\mathcal{N}} \Phi < \frac{4\pi}{\alpha_0}. \qquad (3.29)$$

4 Proofs of the main results

Lemma 4.1. *The weak solution $(\tilde{u}, \tilde{v})$ is nontrival.*

Proof. By Lemmas 2.1 and 2.7, there exist a subsequence $\{u_n, v_n\} \subset E$ satisfying (2.7) and $\|(u_n, v_n)\| + \|u_n\|_2 + \|v_n\|_2 \le C$ for some constant $C_4 > 0$, it follows from (2.5) and (2.7) that

$$\int_{\mathbb{R}^2} f_1(x,u_n)u_n\mathrm{d}x \le C_5, \qquad \int_{\mathbb{R}^2} f_2(x,v_n)v_n\mathrm{d}x \le C_5. \qquad (4.1)$$

We may assume, passing to a subsequence if necessary, that $(u_n, v_n) \rightharpoonup (\tilde{u}, \tilde{v})$ in E, $(u_n, v_n) \to (\tilde{u}, \tilde{v})$ in $L^s_{loc}(\mathbb{R}^2)$ for $s \in [1, +\infty)$ and $(u_n, v_n) \to (\tilde{u}, \tilde{v})$ a.e. on $\mathbb{R}^2$.

If

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B_1(y)} (|u_n|^2 + |v_n|^2)\mathrm{d}x = 0,$$

then by Lions's concentration compactness principle, $(u_n, v_n) \to (0, 0)$ in $L^s(R^2)$ for $2 < s < \infty$. For any given $\varepsilon > 0$, we choose $M_\varepsilon > M_0 C_5/\varepsilon$, then it follows from (F3) and (4.1) that

$$\int_{|u_n|\geq M_\varepsilon} F_1(x, u_n)\mathrm{d}x \leq M_0 \int_{|u_n|\geq M_\varepsilon} |f_1(x, u_n)|\mathrm{d}x \leq \frac{M_0}{M_\varepsilon} \int_{|u_n|\geq M_\varepsilon} f_1(x, u_n)u_n\mathrm{d}x < \varepsilon. \tag{4.2}$$

$$\int_{|v_n|\geq M_\varepsilon} F_2(x, v_n)\mathrm{d}x \leq M_0 \int_{|v_n|\geq M_\varepsilon} |f_2(x, v_n)|\mathrm{d}x \leq \frac{M_0}{M_\varepsilon} \int_{|v_n|\geq M_\varepsilon} f_2(x, v_n)v_n\mathrm{d}x < \varepsilon. \tag{4.3}$$

By (F2), we can choose $N_\varepsilon \in (0, 1)$ such that

$$\int_{|u_n|\leq N_\varepsilon} F_1(x, u_n)\mathrm{d}x \leq \int_{|u_n|\leq N_\varepsilon} f_1(x, u_n)u_n\mathrm{d}x \leq \frac{\varepsilon}{C_4^2}\|u_n\|_2^2 < \varepsilon. \tag{4.4}$$

$$\int_{|v_n|\leq N_\varepsilon} F_2(x, v_n)\mathrm{d}x \leq \int_{|v_n|\leq N_\varepsilon} f_2(x, v_n)v_n\mathrm{d}x \leq \frac{\varepsilon}{C_4^2}\|v_n\|_2^2 < \varepsilon. \tag{4.5}$$

By (F1), we have

$$\int_{N_\varepsilon\leq|u_n|\leq M_\varepsilon} F_1(x, u_n)\mathrm{d}x \leq C_6\|u_n\|_3^3 = o(1), \quad \int_{N_\varepsilon\leq|v_n|\leq M_\varepsilon} F_2(x, v_n)\mathrm{d}x \leq C_6\|v_n\|_3^3 = o(1), \tag{4.6}$$

$$\int_{N_\varepsilon\leq|u_n|\leq 1} f_1(x, u_n)u_n\mathrm{d}x \leq C_7\|u_n\|_3^3 = o(1), \quad \int_{N_\varepsilon\leq|v_n|\leq 1} f_2(x, v_n)v_n\mathrm{d}x \leq C_7\|v_n\|_3^3 = o(1). \tag{4.7}$$

Due to the arbitrariness of $\varepsilon > 0$, from (4.2), (4.4), (4.6), we obtain

$$\int_{\mathbb{R}^2} F_1(x, u_n)\mathrm{d}x = o(1), \qquad \int_{\mathbb{R}^2} F_2(x, v_n)\mathrm{d}x = o(1). \tag{4.8}$$

Hence, it follows from (V), (1.7), (2.7) and (4.8) that

$$\begin{aligned}\frac{1}{2}(\|\nabla u_n\|_2^2 + \|\nabla v_n\|_2^2) &< \frac{1}{2}\|(u_n, v_n)\|^2 - \int_{\mathbb{R}^2} \lambda(x)u_n v_n\mathrm{d}x \\ &= c^* + \int_{\mathbb{R}^2} [F_1(x, u_n) + F_2(x, v_n)]\mathrm{d}x + o(1) \\ &= c^* + o(1).\end{aligned}$$

Which, together with (3.5), implies that $\limsup_{n\to\infty} \|\nabla u_n\|_2^2 + \|\nabla v_n\|_2^2 < \frac{8\pi}{\alpha_0}$. Hence, there exist $\bar{\varepsilon} > 0$ and $n_0 \in \mathbb{N}$ such that

$$\|\nabla u_n\|_2^2 + \|\nabla v_n\|_2^2 \leq \frac{8\pi}{\alpha_0}(1 - 3\bar{\varepsilon}), \qquad \forall n \geq n_0.$$

Let us choose $q \in (1, 2)$ such that

$$\frac{(1 + \bar{\varepsilon})(1 - 3\bar{\varepsilon})q}{1 - \bar{\varepsilon}} < 1. \tag{4.9}$$

By (F1), there exists $C_8 > 0$ such that

$$|f_i(x, t)|^q \leq C_8\left[e^{\alpha_0(1+\bar{\varepsilon})qt^2} - 1\right], \qquad \forall |t| \geq 1, \quad i = 1, 2. \tag{4.10}$$

It follows from (4.9), (4.10) and Lemma 1.1 ii) that

$$\begin{aligned}\int_{|u_n|\geq 1} f_i(x,u_n)^q \mathrm{d}x &\leq C_8 \int_{\mathbb{R}^2} \left[e^{\alpha_0(1+\bar{\varepsilon})qu_n^2} - 1\right]\mathrm{d}x \\ &= C_8 \int_{\mathbb{R}^2} \left[e^{\alpha_0(1+\bar{\varepsilon})q\|u_n\|^2(u_n/\|u_n\|)^2} - 1\right]\mathrm{d}x \\ &\leq C_9. \end{aligned} \tag{4.11}$$

Let $q' = q/(q-1)$. Then we have

$$\int_{|u_n|\geq 1} f_i(x,u_n)u_n \mathrm{d}x \leq \left[\int_{|u_n|\geq 1} |f_i(x,u_n)|^q \mathrm{d}x\right]^{\frac{1}{q}} \|u_n\|_{q'} = o(1). \tag{4.12}$$

Now we derive

$$\begin{aligned} c^* + o(1) &= \Phi(u_n,v_n) - \frac{1}{2}\left\langle \Phi'(u_n,v_n),(u_n,v_n)\right\rangle \\ &= \int_{\mathbb{R}^2}\left[\frac{1}{2}f_1(x,u_n)u_n - F_1(x,u_n)\right]\mathrm{d}x + \int_{\mathbb{R}^2}\left[\frac{1}{2}f_2(x,v_n)v_n - F_2(x,v_n)\right]\mathrm{d}x \\ &< \varepsilon + o(1). \end{aligned} \tag{4.13}$$

This contradiction shows that $\delta > 0$.

Going if necessary to a subsequence, we may assume that there exists $\{l_n\} \subset \mathbb{Z}^2$ such that $\int_{B_{1+\sqrt{2}}(l_n)} |(|u_n|^2 + |v_n|^2)\mathrm{d}x > \frac{\delta}{2}$. Let us define $\tilde{u}_n(x) = u_n(x+l_n)$ and $\tilde{v}_n(x) = v_n(x+l_n)$ so that

$$\int_{B_{1+\sqrt{2}}(0)} (|\tilde{u}_n|^2 + |\tilde{v}_n|^2)\mathrm{d}x \geq \frac{\delta}{2}. \tag{4.14}$$

Since $\lambda(x)$ and $f_i(x,u)$ are 1-periodic on x , we have $\|\tilde{u}_n\|_{V_1} = \|u_n\|_{V_1}$, $\|\tilde{v}_n\|_{V_2} = \|v_n\|_{V_2}$ and

$$\Phi(\tilde{u}_n,\tilde{v}_n) \to c^*, \qquad \|\Phi'(\tilde{u}_n,\tilde{v}_n)\|(1+\|(\tilde{u}_n,\tilde{v}_n)\|) \to 0. \tag{4.15}$$

Passing to a subsequence, we have $(\tilde{u}_n,\tilde{v}_n) \rightharpoonup (\tilde{u},\tilde{v})$ in E, $(\tilde{u}_n,\tilde{v}_n) \to (\tilde{u},\tilde{v})$ in $L^s_{loc}(\mathbb{R}^2), 2 \leq s \leq \infty$ and $(\tilde{u}_n,\tilde{v}_n) \to (\tilde{u},\tilde{v})$ a.e. on $\mathbb{R}^2$. Thus (4.14) implies that $(\tilde{u},\tilde{v}) \neq (0,0)$.

For any $\phi,\psi \in C_0^\infty(\mathbb{R}^2)$, let $\{e_n\}_{n=1}^\infty$ be the complete standard orthogonal basis of $C_0^\infty(\mathbb{R}^2)$, we have

$$\phi = \sum_{j=1}^\infty (\phi,e_j)e_j, \qquad \|\phi\|^2 = \sum_{j=1}^\infty |(\phi,e_j)|^2 \tag{4.16}$$

and

$$\psi = \sum_{j=1}^\infty (\psi,e_j)e_j, \qquad \|\psi\|^2 = \sum_{j=1}^\infty |(\psi,e_j)|^2. \tag{4.17}$$

Let

$$\phi_n = \sum_{j=1}^{k_n} (\phi,e_j)e_j, \qquad \tilde{\phi}_n = \sum_{j=k_n+1}^\infty (\phi,e_j)e_j \tag{4.18}$$

and

$$\psi_n = \sum_{j=1}^{k_n} (\psi,e_j)e_j, \qquad \tilde{\psi}_n = \sum_{j=k_n+1}^\infty (\psi,e_j)e_j. \tag{4.19}$$

For any given $\varepsilon > 0$, there holds

$$\int_{|\tilde{u}_n|\geq C_{10}\|\phi\|_\infty \varepsilon^{-1}} |f_1(x,\tilde{u}_n)\phi_n|\mathrm{d}x \leq \frac{\varepsilon}{C_{10}} \int_{|\tilde{u}_n|\geq C_{10}\|\phi\|_\infty \varepsilon^{-1}} f_1(x,\tilde{u}_n)\tilde{u}_n \mathrm{d}x < \varepsilon. \tag{4.20}$$

On the other hand, it follows from (F1) and (F2) that

$$
\begin{aligned}
\int_{|\tilde{u}_n|<C_{10}\|\phi\|_\infty\varepsilon^{-1}}|f_1(x,\tilde{u}_n)\tilde{\phi}_n|\mathrm{d}x &\le \int_{|\tilde{u}_n|<C_{10}\|\phi\|_\infty\varepsilon^{-1}}|u_n\tilde{\phi}_n|\mathrm{d}x + C_{11}\int_{|\tilde{u}_n|<C_{10}\|\phi\|_\infty\varepsilon^{-1}}(e^{\alpha u_n^2}-1)|\tilde{\phi}_n|\mathrm{d}x\\
&\le \left\{\|u_n\|_2 + C_{11}\left[\int_{\mathbb{R}^2}(e^{\alpha u_n^2}-1)^2\mathrm{d}x\right]^{\frac{1}{2}}\right\}\left(\int_{\mathbb{R}^2}\tilde{\phi}_n^2\mathrm{d}x\right)^{\frac{1}{2}}\\
&\le \left\{\|u_n\|_2 + C_{11}\left[\int_{\mathbb{R}^2}(e^{2\alpha u_n^2}-1)\mathrm{d}x\right]^{\frac{1}{2}}\right\}\|\tilde{\phi}_n\|_2\\
&\le \left\{\|u_n\|_2 + C_{11}\left[\int_{\mathbb{R}^2}\left(e^{2\alpha\rho_0^2\|u_n\|^2(\frac{u_n}{\rho_0\|u_n\|})^2}-1\right)\mathrm{d}x\right]^{\frac{1}{2}}\right\}\|\tilde{\phi}_n\|\\
&\le C_{12}\|\tilde{\phi}_n\| = o(1).
\end{aligned}
\tag{4.21}
$$

Similarly, we have

$$
\int_{|\tilde{v}_n|<C_{10}\|\psi\|_\infty\varepsilon^{-1}}|f_2(x,\tilde{v}_n)\tilde{\psi}_n|\mathrm{d}x = o(1). \tag{4.22}
$$

From (4.20), (4.21) and (4.22), one has

$$
\lim_{n\to\infty}\int_{\mathbb{R}^2}f_1(x,\tilde{u}_n)\tilde{\phi}_n\mathrm{d}x = 0, \quad \lim_{n\to\infty}\int_{\mathbb{R}^2}f_2(x,\tilde{v}_n)\tilde{\psi}_n\mathrm{d}x = 0. \tag{4.23}
$$

Due to the arbitrariness of $\varepsilon > 0$. Therefore, (2.4), (4.15) and (4.23) yield

$$
\begin{aligned}
\langle\Phi'(\tilde{u},\tilde{v}),(\phi,\psi)\rangle &= \int_{\mathbb{R}^2}(\nabla\tilde{u}\nabla\phi + V_1(x)\tilde{u}\phi)\,\mathrm{d}x + \int_{\mathbb{R}^2}(\nabla\tilde{v}\nabla\psi + V_2(x)\tilde{v}\psi)\,\mathrm{d}x\\
&\quad - \int_{\mathbb{R}^2}\lambda(x)(\tilde{u}\psi+\tilde{v}\phi)\mathrm{d}x - \int_{\mathbb{R}^2}(f_1(x,\tilde{u})\phi + f_2(x,\tilde{v})\psi)\,\mathrm{d}x\\
&= \lim_{n\to\infty}\Bigg[\int_{\mathbb{R}^2}(\nabla\tilde{u}_n\nabla\phi + V_1(x)\tilde{u}_n\phi)\,\mathrm{d}x + \int_{\mathbb{R}^2}(\nabla\tilde{v}_n\nabla\psi + V_2(x)\tilde{v}_n\psi)\,\mathrm{d}x\\
&\quad - \int_{\mathbb{R}^2}\lambda(x)(\tilde{u}_n\psi+\tilde{v}_n\phi)\mathrm{d}x - \int_{\mathbb{R}^2}(f_1(x,\tilde{u}_n)\phi + f_2(x,\tilde{v}_n)\psi)\,\mathrm{d}x\Bigg]\\
&= \lim_{n\to\infty}\langle\Phi'(\tilde{u}_n,\tilde{v}_n),(\phi,\psi)\rangle\\
&= \lim_{n\to\infty}\left[\langle\Phi'(\tilde{u}_n,\tilde{v}_n)(\phi_n,\psi_n)\rangle + \langle\Phi'(\tilde{u}_n,\tilde{v}_n)(\tilde{\phi}_n,\tilde{\psi}_n)\rangle\right]\\
&= \lim_{n\to\infty}\langle\Phi'(\tilde{u}_n,\tilde{v}_n)(\tilde{\phi}_n,\tilde{\psi}_n)\rangle\\
&= \lim_{n\to\infty}\left\{\int_{\mathbb{R}^2}[(\nabla\tilde{u}_n\nabla\tilde{\phi}_n + \nabla\tilde{v}_n\nabla\tilde{\psi}_n) + V_1(x)\tilde{u}_n\tilde{\phi}_n + V_2(x)\tilde{v}_n\tilde{\psi}_n]\mathrm{d}x\right\}\\
&\quad - \lim_{n\to\infty}\int_{\mathbb{R}^2}\lambda(x)(\tilde{u}_n\tilde{\psi}_n + \tilde{v}_n\tilde{\phi}_n)\mathrm{d}x - \lim_{n\to\infty}\int_{\mathbb{R}^2}[f_1(x,\tilde{u}_n)\tilde{\phi}_n + f_2(x,\tilde{v}_n)\tilde{\psi}_n]\mathrm{d}x\\
&= -\lim_{n\to\infty}\int_{\mathbb{R}^2}[f_1(x,\tilde{u}_n)\tilde{\phi}_n + f_2(x,\tilde{v}_n)\tilde{\psi}_n]\mathrm{d}x\\
&= 0.
\end{aligned}
$$

It is easy to show that $\Phi'(\tilde{u},\tilde{v}) = 0$. Since $\lambda \neq 0$, then from (1.1) we know that $(\tilde{u},\tilde{v})$ is a nontrivial solution. □

Lemma 4.2. *The weak solution $(\tilde{u},\tilde{v})$ is a ground state solution.*

Proof. Since that $(\tilde{u},\tilde{v}) \neq (0,0)$ and $\Phi'(\tilde{u},\tilde{v}) = 0$, we have $(\tilde{u},\tilde{v}) \in \mathcal{N}$. Therefore $b \leq \Phi(\tilde{u},\tilde{v})$. On the other hand, it follows from (2.20) and Fatou's lemma that

$$\begin{aligned} b \geq \bar{c} + o(1) &= \Phi(\tilde{u}_n,\tilde{v}_n) - \frac{1}{2}\left\langle \Phi'(\tilde{u}_n,\tilde{v}_n),(\tilde{u}_n,\tilde{v}_n)\right\rangle \\ &= \int_{\mathbb{R}^2}\left[\frac{1}{2}f_1(x,\tilde{u}_n)\tilde{u}_n - F_1(x,\tilde{u}_n)\right]\mathrm{d}x + \int_{\mathbb{R}^2}\left[\frac{1}{2}f_2(x,\tilde{v}_n)\tilde{v}_n - F_2(x,\tilde{v}_n)\right]\mathrm{d}x \\ &\geq \int_{\mathbb{R}^2}\left[\frac{1}{2}f_1(x,\tilde{u})\tilde{u} - F_1(x,\tilde{u})\right]\mathrm{d}x + \int_{\mathbb{R}^2}\left[\frac{1}{2}f_2(x,\tilde{v})\tilde{v} - F_2(x,\tilde{v})\right]\mathrm{d}x + o_n(1) \\ &= \Phi(\tilde{u},\tilde{v}) - \frac{1}{2}\left\langle \Phi'(\tilde{u},\tilde{v}),(\tilde{u},\tilde{v})\right\rangle + o_n(1) \\ &= \Phi(\tilde{u},\tilde{v}) + o_n(1). \end{aligned} \tag{4.24}$$

Therefore $\Phi(\tilde{u},\tilde{v}) = b$.

We have proved that $(\tilde{u},\tilde{v})$ is a ground state solution for system (1.1). In order to seek a positive ground state, we note by assumptions (F1) and (F2) that

$$F_i(x,s) \leq F_i(x,|s|), \qquad \forall (x,s) \in \mathbb{R}^2 \times \mathbb{R}, \quad i = 1,2.$$

Thus, we can deduce that $\Phi(|\tilde{u}|,|\tilde{v}|) \leq \Phi(\tilde{u},\tilde{v})$. □

5 The regularity and asymptotic behavior

In this section, we use strong maximum principle to get a unique positive ground state solution, we will introduce methods to show that a weak solution of (1.1) is in fact smooth. Moreover, we establish a priori estimate in $W^{2,p}$ for the solution of system (1.1), we show that if the functions $f_1(x,u)$ and $f_2(x,v)$ are in $L^p_{loc}(\mathbb{R}^2)$, then $(u,v) \in W^{2,p}_{loc}(\mathbb{R}^2)$ is a strong solution of (1.1), that is, there exists a constant C such that

$$\|u\|_{W^{2,p}(B_R)} \leq C(\|u\|_{L^p(B_{2R})} + \|p_1(x)\|_{L^p(B_{2R})}), \qquad \|v\|_{W^{2,p}(B_R)} \leq C(\|v\|_{L^p(B_{2R})} + \|p_2(x)\|_{L^p(B_{2R})}),$$

where $p_i(x)$ can be defined as (5.3). We will establish this for a Newtonian potential, finally, we use a bootstrap regularity lifting methods to boost the regularity of solution. The bootstrap method can be found in [6, Subsection 3.3.1], which uses a lot of Sobolev imbedding to enhance the regularity of the weak solution repeatedly, finally, Schauder's estimate will lift the solution to be a classical solution.

Lemma 5.1. *There exists a positive ground state solution* $(\bar{u},\bar{v}) \in C^{1,\beta}_{loc}(\mathbb{R}^2) \times C^{1,\beta}_{loc}(\mathbb{R}^2)$ *for some* $\beta \in (0,1)$ *with the following asymptotic behavior*

$$\|\bar{u}\|_{C^{1,\beta}(\overline{B_R})} \to 0 \quad \text{and} \quad \|\bar{v}\|_{C^{1,\beta}(\overline{B_R})} \to 0, \quad \text{as } |x| \to \infty. \tag{5.1}$$

Proof. Let $(\tilde{u},\tilde{v}) \in E$ be the ground state obtained in Lemma 4.2 It follows from Lemma 2.4 that there exists a unique $t_0 > 0$ such that $(t_0|\tilde{u}|, t_0|\tilde{v}|) \in \mathcal{N}$. Moreover, since $(\tilde{u},\tilde{v}) \in \mathcal{N}$, we point out that $\max_{t\geq 0}\Phi(t\tilde{u},t\tilde{v}) = \Phi(\tilde{u},\tilde{v})$. Thus we have that

$$\Phi(t_0|\tilde{u}|, t_0|\tilde{v}|) \leq \Phi(t_0\tilde{u}, t_0\tilde{v}) \leq \max_{t\geq 0}\Phi(t\tilde{u},t\tilde{v}) = \Phi(\tilde{u},\tilde{v}) = b.$$

Therefore, $(t_0|\tilde{u}|, t_0|\tilde{v}|) \in \mathcal{N}$ is a nonnegative ground state solution for (1.1). Next, we denote $(\bar{u},\bar{v}) = (t_0|\tilde{u}|, t_0|\tilde{v}|)$. In order to use the strong maximum principle, we note that $-\bar{u} \in$

$H_{V_1}(\mathbb{R}^2) \setminus \{0\}$ and take $(\varphi, 0)$ as a test function. Here, $\varphi \in C_0^\infty(\mathbb{R}^2), \varphi \geq 0$. Then we have

$$-\int_{\mathbb{R}^2} \nabla(-\bar{u})\nabla\varphi \mathrm{d}x - \int_{\mathbb{R}^2} V_1(x)(-\bar{u})\varphi \mathrm{d}x = \int_{\mathbb{R}^2} f(x,\bar{u})\varphi \mathrm{d}x + \int_{\mathbb{R}^2} \lambda(x)\bar{u}\varphi \mathrm{d}x \geq 0. \tag{5.2}$$

Moreover, since $V_1(x) > 0$, it follows that

$$-\int_{\mathbb{R}^2} V_1(x)\varphi \mathrm{d}x \leq 0, \qquad \forall \varphi \geq 0,\ \varphi \in C_0^1(\mathbb{R}^2).$$

Now suppose by contradiction that there exists $x_0 \in \mathbb{R}^2$ such that $\bar{u}(x_0) = 0$. Thus, since $-\bar{u} \leq 0$ in $\mathbb{R}^2$, for any $R > 0$ we have that

$$0 = \sup_{B_R(x_0)} (-\bar{u}) = \sup_{\mathbb{R}^2}(-\bar{u})$$

By the strong maximum principle we conclude that $-\bar{u} \equiv 0$ in $\mathbb{R}^2$, which is a contradiction. Therefore $\bar{u} > 0$ in $\mathbb{R}^2$. Similarly, we can prove that $\bar{v} > 0$ in $\mathbb{R}^2$. Therefore, $(\bar{u}, \bar{v})$ is positive.

In order to obtain the regularity, we use a bootstrap method. The ground state solution $(\bar{u}, \bar{v})$ is a weak solution of the restricted problem

$$\begin{cases} -\Delta\bar{u} = f_1(x,\bar{u}) + \lambda(x)\bar{v} - V_1(x)\bar{u} = p_1(x), & B_{2R}, \\ -\Delta\bar{v} = f_2(x,\bar{v}) + \lambda(x)\bar{u} - V_2(x)\bar{v} = p_2(x), & B_{2R}, \end{cases} \tag{5.3}$$

where and in the continuation $B_{2R} = B_{2R}(x) \subset \mathbb{R}^2$ denote the ball centered in a fixed point $x \in \mathbb{R}^2$. Since $V_i(x) \in C(\mathbb{R}^2)$, then $V_i(x), \lambda(x) \in L^\infty_{loc}(\mathbb{R}^2)$. For $\bar{u}, \bar{v} \in L^p(\mathbb{R}^2)$, $p \geq 2$, we have that $\lambda(x)\bar{v}, V_1(x)\bar{u} \in L^p(B_{2R})$ for all $p \geq 2$. By (F1) and (F2), for $\varepsilon > 0, p, q \geq 2, r > p$ and $\alpha > \alpha_1$, we have that

$$\begin{aligned} \int_{B_{2R}} |f_1(x,\bar{u})|^p \mathrm{d}x &\leq \int_{B_{2R}} |\varepsilon\bar{u} + C_\varepsilon(e^{\alpha\bar{u}^2} - 1)|\bar{u}|^{q-1}|^p \mathrm{d}x \\ &\leq C_{13} \int_{B_{2R}} \varepsilon^p |\bar{u}|^p \mathrm{d}x + C_{13} \int_{B_{2R}} C_\varepsilon^p (e^{\alpha\bar{u}^2} - 1)^p |\bar{u}|^{p(q-1)} \mathrm{d}x \\ &\leq C_{13}\varepsilon^p \|\bar{u}\|^p_{L^p(B_{2R})} + C_{13} \int_{B_{2R}} C_\varepsilon^p (e^{r\alpha\bar{u}^2} - 1)|\bar{u}|^{p(q-1)-1}|\bar{u}| \mathrm{d}x. \end{aligned} \tag{5.4}$$

By using Hölder's inequality, it follows from Lemma 1.1 that

$$\begin{aligned} \int_{B_{2R}} C_\varepsilon^p (e^{r\alpha\bar{u}^2} - 1)|\bar{u}|^{p(q-1)-1}|\bar{u}| \mathrm{d}x &\leq \left(\int_{B_{2R}} C_\varepsilon^{2p} (e^{r\alpha\bar{u}^2} - 1)^2 |\bar{u}|^{2(p(q-1)-1)} \mathrm{d}x \right)^{\frac{1}{2}} \|\bar{u}\|_{L^2(B_{2R})} \\ &\leq C_{14} \|\bar{u}\|_{L^2(B_{2R})}. \end{aligned} \tag{5.5}$$

Thus, we have

$$\int_{B_{2R}} |f_1(x,\bar{u})|^p \mathrm{d}x \leq C_{13} \|\bar{u}\|^p_{L^p(B_{2R})} + C_{14} \|\bar{u}\|_{L^2(B_{2R})}. \tag{5.6}$$

Since the right-hand side is finite for all $p \geq 2$, we have that $f_1(x,\bar{u}) \in L^p(B_{2R})$ for all $p \geq 2$, together with $\lambda(x)\bar{v}, V_1(x)\bar{u} \in L^p(B_{2R})$, we have that $p_1(x) \in L^p(B_{2R})$ for all $p \geq 2$. Let f_{p_1} be the Newtonian potential of $p_1(x)$. In light of L^p-regularity theory [6, Theorem 3.1.1],

$$\Delta f_{p_1} = p_1(x), \quad x \in B_{2R}, \tag{5.7}$$

and $f_{p_1} \in W^{2,p}(B_{2R})$, for all $p \geq 2$. Combining (5.3) and (5.7) we deduce that

$$\int_{B_{2R}} \nabla(\bar{u} - f_{p_1})\phi \mathrm{d}x = 0, \quad \forall \phi \in C_0^\infty(B_{2R}).$$

which implies that $\bar{u} - f_{p1}$ is a weak solution of $-\Delta z = 0$ in B_{2R}. Since $\bar{u} - f_{p_1} \in W^{1,2}(B_{2R})$. It follows from Weyl's Lemma [13, Corollary 1.2.1] that $\bar{u} - f_{p_1} \in C^\infty(B_{2R})$. Therefore, $\bar{u} \in W^{2,p}(B_{2R}), \forall p \geq 2$. Noticing that $2/p < 2$, as $p > 2$. Thus, by Sobolev imbedding we obtain that $\bar{u} \in C^{1,\beta}(B_{2R})$, for some $\beta \in (0,1)$. The same argument can be used to prove that $\bar{v} \in C^{1,\beta}(B_{2R})$. By interior L^p-estimates [6, Theorem 3.1.2], we have that

$$\|\bar{u}\|_{W^{2,p}(B_R)} \leq C_{15}(\|\bar{u}\|_{L^p(B_{2R})} + \|p_1\|_{L^p(B_{2R})}). \tag{5.8}$$

On the other hand, by the Sobolev's imbedding theorem, there exists $C_{16} > 0$ such that

$$\|\bar{u}\|_{C^{1,\beta}(\overline{B_R})} \leq C_{16}\|\bar{u}\|_{W^{2,p}(B_R)}. \tag{5.9}$$

Therefore, it follows from (5.8) and (5.9), we deduce that

$$\|\bar{u}\|_{C^{1,\beta}(\overline{B_R})} \leq C_{17}(\|\bar{u}\|_{L^p(B_{2R})} + \|\bar{u}\|_{L^2(B_{2R})}).$$

Now we show that $\lim_{|x|\to\infty} \bar{u} = 0$. Suppose on the contrary that there exists $\{x_j\} \subset \mathbb{R}^2$ with $|x_j| \to \infty$ as $j \to \infty$ and $\liminf_{j\to\infty} \bar{u}(x_j) > 0$. Letting $w_j(x) = \bar{u}(x + x_j)$, then

$$-\Delta w_j + V_1(x + x_j)w_j = f_1(x + x_j, w_j) + \lambda(x + x_j)\bar{v}(x + x_j), \quad w_j \in H^1(\mathbb{R}^2). \tag{5.10}$$

Assume that $w_j \to w$ weakly in $H^1(\mathbb{R}^2)$. Then, by elliptic estimates we have $w \neq 0$. However, for fixed $R > 0$,

$$\begin{aligned}
\int_{\mathbb{R}^2} \bar{u}^2 \mathrm{d}x &\geq \liminf_{j\to\infty} \Big(\int_{B_R(0)} \bar{u}^2 \mathrm{d}x + \int_{B_R(x_j)} \bar{u}^2 \mathrm{d}x \Big) \\
&= \int_{B_R(0)} \bar{u}^2 \mathrm{d}x + \liminf_{j\to\infty} \int_{B_R(0)} w_j^2 \mathrm{d}x \\
&= \int_{B_R(0)} \bar{u}^2 \mathrm{d}x + \int_{B_R(0)} w^2 \mathrm{d}x \\
&\to \int_{\mathbb{R}^2} \bar{u}^2 \mathrm{d}x + \int_{\mathbb{R}^2} w^2 \mathrm{d}x, \quad \text{as} \quad R \to \infty.
\end{aligned}$$

Which is a contradiction. Thus, letting $|x| \to \infty$, we get $\bar{u} \to 0$, therefore, $\|\bar{u}\|_{C^{1,\beta}(\overline{B_R})} \to 0$ as $|x| \to \infty$. Similarly, we can prove that $\|\bar{v}\|_{C^{1,\beta}(\overline{B_R})} \to 0$ as $|x| \to \infty$. □

Proof of Theorem 1.2. It follows from Lemmas 4.1, 4.2 and 5.1. □

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Stabilization via delay feedback for highly nonlinear stochastic time-varying delay systems with Markovian switching and Poisson jump

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Abstract. Little work seems to be known about stabilization results of highly nonlinear stochastic time-varying delay systems (STVDSs) with Markovian switching and Poisson jump. This paper is concerned with the stabilization problem for a class of STVDSs with Markovian switching and Poisson jump. The coefficients of such systems do not satisfy the conventional linear growth conditions, but are subject to high nonlinearity. The aim of this paper is to design a delay feedback controller to make an unstable highly nonlinear STVDSs with Markovian switching and Poisson jump H_∞-stable and asymptotically stable. Besides, an illustrative example is provided to support the theoretical results.

Keywords: stochastic systems, time-varying delay, delay feedback control, Markovian switching, Poisson jump.

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1 Introduction

Many dynamical systems are inevitably influenced by internal and external random disturbance. Such perturbation can drastically alter the deterministic dynamics and even produce new interesting dynamical behavior. Such systems are often described by stochastic differential equations (see monograph [22]) and the stability analysis of stochastic differential equations has received a great deal of attention, see [1, 12, 15, 16, 23, 32, 36, 38] and the references

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therein. In addition, the evolution process of a stochastic system is not only related to the present state, but also to the past states. In this case, stochastic delay systems (SDSs) are introduced, which have been widely applicable to genetic regulatory networks, complex dynamical networks, biological systems, control and so on ([6,10,13,28,35]). Accordingly, many results on the stability of SDSs have been obtained, see, e.g. [7–9,14,29,31].

It is well known that a Brownian motion is a continuous stochastic process, however, some systems may suffer from the jump type abrupt perturbations and the phenomenon of discontinuous random pulse excitation. In such cases, incorporating jumps into SDSs seems to be necessary, and it is therefore valuable to discuss the SDSs with Poisson jump, see, e.g., [2,11,17,26]. In the case of the SDSs with Poisson jump experiencing abrupt changes in their structure and parameters due to sudden changes of system factors, SDSs with Markovian switching and Poisson jump (SDSwMSPJs) can be applied to model them. This kind of models are more realistic, and the stability research of them has aroused great concern (see, e.g., [19,21,34,37]).

Consider an unstable STVDS with Markovian switching and Poisson jump

$$\begin{aligned} dx(t) = &f(x(t), x(t-\delta(t)), r(t), t)dt + g(x(t), x(t-\delta(t)), r(t), t)dB(t) \\ &+ h(x(t), x(t-\delta(t)), r(t), t)dN(t) \end{aligned}$$

on $t \geq 0$, where the state $x(t) \in \mathbb{R}^n$, $r(t)$ is a Markov chain, $N(t)$ is a scalar Poisson process, $\delta(\cdot) : \mathbb{R}^+ \to [0, \delta]$ be continuous function with $\delta > 0$. For details, see the system (2.1) below. To make this given unstable system become stable, it is conventional to design a feedback control $u(x(t), r(t), t)$ in the drift term, based on the current state $x(t)$, for the controlled system to become stable. Due to the fact that there exists a time lag $\tau(\tau > 0)$ between the observation of the state is made and the time when the feedback control reaches the system, it is thus more realistic to take into account the control depends on a past state $x(t-\tau)$ (see, e.g. [18,33]). Therefore, the control should be of the form $u(x(t-\tau), r(t), t)$. In this paper, we assume that $\tau \leq \delta$. Hence, the stabilization problem becomes to design a delay feedback control $u(x(t-\tau), r(t), t)$ for the controlled system

$$\begin{aligned} dx(t) = &[f(x(t), x(t-\delta(t)), r(t), t) + u(x(t-\tau), r(t), t)]dt \\ &+ g(x(t), x(t-\delta(t)), r(t), t)dB(t) \\ &+ h(x(t), x(t-\delta(t)), r(t), t)dN(t) \end{aligned}$$

to be stable. In [24], Mao et al. designed a delay feedback controller to stabilize an unstable SDSs with Markovian switching for the first time, where both the drift and diffusion coefficients of the given unstable system meet the linear growth condition. Notice that in many economic and ecological systems, the coefficients of these systems are characterized by nonlinearity, e.g. [4] and [5]. Therefore, the stabilization problems of a class of highly nonlinear stochastic systems or SDSs with Markovian switching via delay feedback control have received considerable research interests.

Recently, Lu et al. [20] used the delay feedback control to make unstable highly nonlinear stochastic systems with Markovian switching asymptotically stable. Later, Li and Mao [18] made a progress and used the delay feedback control to tackle the stabilization problem for a given unstable highly nonlinear SDSs with Markovian switching. Shen et al. [33] explored the stabilization of highly nonlinear neutral SDSs with Markovian switching by delay feedback control. Zhao and Zhu [39] designed a delay feedback control function to study the stability of highly nonlinear switched stochastic systems with time delays. Mei et al. [27] further studied the exponential stabilization problem for a class of highly nonlinear infinite delay stochastic functional differential systems with Markovian switching. It should be noted that, though the coefficients of the given unstable systems in [18,20,27,33] are highly nonlinear, little work has focused on the stabilization problem of SDSs with Markovian switching and Poisson jump simultaneously, not to mention the case where the SDSs under consideration are highly nonlinear and the delay of the SDSs is time-varying. As we know, the increment of Poisson jump has a nonzero mean, which brings significant difficulties for the stabilization of STVDSs with Markovian switching and Poisson jump. Therefore, the motivation of this paper is to overcome the identified difficulties by launching a systematic investigation.

Inspired by the analysis above, this paper investigates the stabilization problems via delay feedback control for a class of highly nonlinear STVDSs with Markovian switching and Poisson jump. Different from the existing literature, a new stabilization problem is studied for a class of highly nonlinear SDSs, where both the Markovian switching and Poisson jump are taken into consideration, which advances the results of the system considered in [24] and covers the results in [18,20,27,33,39]. Moreover, the delay of the SDSs is time-varying, which also covers the results in [18, 20, 27, 33]. The main contributions of this paper are summarized: (1) Very few results seem to be known about the stabilization problem of STVDSs with Markovian switching and Poisson jump simultaneously, not to mention the case where the coefficients of such systems are highly nonlinear. This paper investigates the stabilization of highly nonlinear STVDSs with Markovian switching and Poisson jump; (2) A delay feedback controller is designed to make an unstable highly nonlinear STVDS with Markovian switching and Poisson jump H_∞-stable and asymptotically stable.

In this paper, we first present some notations and preliminaries in Section 2. Then in Section 3, we prove that the controlled highly nonlinear STVDSs with Markovian switching and Poisson jump is H_∞-stable and asymptotically stable, respectively. Finally, an example is provided to illustrate the obtained results in Section 4.

2 Preliminaries

Throughout this paper, unless otherwise specified, we use the following notations. Let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, and $|x|$ denotes the Euclidean norm of a vector x. $\langle x, y\rangle$ or $x^T y$ represents the inner product of $\forall x, y \in \mathbb{R}^n$. For $a, b \in \mathbb{R}$, $a \vee b$ and $a \wedge b$ stand for $\max\{a, b\}$ and $\min\{a, b\}$, respectively. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. its right continuous and $\mathcal{F}_0$ contains all P-null sets). For $\tau > 0$, let $C([-\tau, 0]; \mathbb{R}^n)$ denote the family of all continuous functions $\varphi : [-\tau, 0] \to \mathbb{R}^n$ with the norm $\|\varphi\| = \sup_{-\tau\leq\theta\leq 0} |\varphi(\theta)|$. Let $C^b_{\mathcal{F}_0}(\Omega; \mathbb{R}^n)$ be the family of all $\mathcal{F}_0$-measurable bounded $C([-\tau, 0]; \mathbb{R}^n)$. For $\forall t \geq 0$ and $\delta > 0$, let $\delta(\cdot) : \mathbb{R}^+ \to [0, \delta]$ be continuous function and $\dot{\delta}(t) = d\delta(t)/dt \leq \bar{\delta} < 1$. In the case when $\delta(t) \equiv$ constant, we assert $\bar{\delta} = 0$. Let $\{r(t)\}_{t\geq 0}$ be a right-continuous Markov chain on the complete probability space taking values in a finite state space $S = \{1, 2, \ldots, N\}$ with the generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta), & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta), & \text{if } i = j, \end{cases}$$

where $\Delta > 0$, $\lim_{\Delta\to 0} o(\Delta)/\Delta = 0$, $\gamma_{ij} \geq 0$ is the transition rate from i to j for $i \neq j$, and $\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}$. It is well known that almost every sample path of $r(\cdot)$ is a right-continuous step function with finite number of simple jumps in any finite subinterval of $\mathbb{R}^+$ (see [25]).

Consider the following unstable n-dimensional stochastic time-varying delay systems with Markovian switching and Poisson jump

$$\begin{aligned} dx(t) = & f(x(t), x(t-\delta(t)), r(t), t)dt + g(x(t), x(t-\delta(t)), r(t), t)dB(t) \\ & + h(x(t), x(t-\delta(t)), r(t), t)dN(t), \qquad t \geq 0 \end{aligned} \tag{2.1}$$

with the initial value

$$x_0 = \varphi = \{x(t) : -\delta \leq t \leq 0\} \in C^b_{\mathcal{F}_0}(\Omega; \mathbb{R}^n) \quad \text{and} \quad r(0) = i_0 \in S, \tag{2.2}$$

where $f, g, h : \mathbb{R}^n \times \mathbb{R}^n \times S \times \mathbb{R}^+ \to \mathbb{R}^n$ are Borel measurable functions, $B(t)$ is a scalar Brownian motion and $N(t)$ is a scalar Poisson process with intensity $\lambda > 0$. $\tilde{N}(t) = N(t) - \lambda t$ is a compensated Poisson process satisfying the property of martingale. Moreover, $B(t)$, $N(t)$ and $r(t)$ are assumed to be mutually independent. For the purpose of the stability, we also assume that $f(0,0,i,t) = g(0,0,i,t) = h(0,0,i,t) = 0$ for $\forall (i,t) \in S \times \mathbb{R}^+$. We are required to design a delay feedback $u(x(t-\tau), r(t), t)$ in the drift term so that the controlled system which is described by

$$\begin{aligned} dx(t) = & [f(x(t), x(t-\delta(t)), r(t), t) + u(x(t-\tau), r(t), t)]dt \\ & + g(x(t), x(t-\delta(t)), r(t), t)dB(t) \\ & + h(x(t), x(t-\delta(t)), r(t), t)dN(t) \end{aligned} \tag{2.3}$$

becomes stable.

For the existence and uniqueness of the global solution, we assume that the local Lipschitz condition and the polynomial growth condition are true. For $\forall x, \bar{x}, y, \bar{y} \in \mathbb{R}^n$ and $(i,t) \in S \times \mathbb{R}^+$, we also impose the following assumptions:

Assumption 1: For any real number $h > 0$, there is a constant $L_h > 0$ such that

$$\begin{aligned} & |f(x,y,i,t) - f(\bar{x},\bar{y},i,t)| \vee |g(x,y,i,t) - g(\bar{x},\bar{y},i,t)| \\ & \qquad \vee |h(x,y,i,t) - h(\bar{x},\bar{y},i,t)| \leq L_h(|x-\bar{x}| + |y-\bar{y}|) \end{aligned} \tag{2.4}$$

with $|x| \vee |\bar{x}| \vee |y| \vee |\bar{y}| \leq h$. Moreover, there exists a positive constant β such that

$$|u(x,i,t) - u(y,i,t)| \leq \beta |x-y|. \tag{2.5}$$

For the stability purpose, we also require that $u(0,i,t) = 0$. Then we can obtain

$$|u(x,i,t)| \leq \beta |x|, \ \forall (x,i,t) \in \mathbb{R}^n \times S \times \mathbb{R}^+. \tag{2.6}$$

Assumption 2: There exist positive constants K and $q_i (i = 1,2,3)$ satisfying

$$\begin{aligned} |f(x,y,i,t)| &\leq K(1 + |x|^{q_1} + |y|^{q_1}), \\ |g(x,y,i,t)| &\leq K(1 + |x|^{q_2} + |y|^{q_2}), \\ |h(x,y,i,t)| &\leq K(1 + |x|^{q_3} + |y|^{q_3}) \end{aligned} \tag{2.7}$$

with $q_1 \geq 1$, $q_2 \geq 1$ and $q_3 \geq 1$.

Remark 2.1. If $q_i = 1 (i = 1, 2, 3)$, the condition (2.7) is the linear growth condition. In this paper, we consider that the coefficients of the stochastic time-varying delay systems (2.1) with Markovian switching and Poisson jump are highly nonlinear, so we refer to the (2.7) as the polynomial growth condition with $\max_{1 \le i \le 3}\{q_i\} > 1$.

Let $C^{2,1}(\mathbb{R}^n \times S \times \mathbb{R}^+; \mathbb{R}^+)$ be the family of all nonnegative functions $V(x, i, t)$ on $\mathbb{R}^n \times S \times \mathbb{R}^+$, which are continuously twice differentiable in x and once in t. Define an operator $LV : \mathbb{R}^n \times \mathbb{R}^n \times S \times \mathbb{R}^+ \to \mathbb{R}$ by

$$\begin{aligned} LV(x, y, i, t) = {} & V_t(x, i, t) + V_x(x, i, t) f(x, y, i, t) \\ & + \frac{1}{2}\text{trace}[g^T(x, y, i, t) V_{xx}(x, i, t) g(x, y, i, t)] \\ & + \lambda[V(x + h(x, y, i, t), i, t) - V(x, i, t)] + \sum_{j=1}^{N} \gamma_{ij} V(x, j, t), \end{aligned}$$

where

$$V_t(x, i, t) = \frac{\partial V(x, i, t)}{\partial t}, \qquad V_x(x, i, t) = \left(\frac{\partial V(x, i, t)}{\partial x_1}, \ldots, \frac{\partial V(x, i, t)}{\partial x_n}\right),$$
$$V_{xx}(x, i, t) = \left(\frac{\partial^2 V(x, i, t)}{\partial x_i \partial x_j}\right)_{n \times n}.$$

In the following, we can cite the generalized Itô formula

$$V(x(t), r(t), t) = V(x(0), r(0), 0) + \int_0^t LV(x(s), x(s - \delta(s)), r(s), s) + G_t, \tag{2.8}$$

where

$$\begin{aligned} G_t = {} & \int_0^t V_x(x(s), r(s), s) g(x(s), x(s - \delta(s)), r(s), s) dB(s) \\ & + \int_0^t [V_x(x(s) + h(x(s), x(s - \delta(s)), r(s), s), r(s), s) - V_x(x(s), r(s), s)] \times d\tilde{N}(s) \\ & + \int_0^t \int_{\mathbb{R}} [V_x(x(s), r_0 + k(r(s), u), s) - V_x(x(s), r(s), s)] \mu(ds, du) \end{aligned} \tag{2.9}$$

with $r_0 = r(0)$. The detailed representation of the functions μ and k can be found in [25]. Moreover, $\mu(ds, du)$ is a martingale measure and $\{G_t\}_{t \ge 0}$ is a local martingale.

It is well known that under the Assumption 1 that the (2.3) with the given initial condition (2.2) admits a unique maximal local solution, which may explode to infinity at a finite time. To avoid such a possible explosion, we impose another assumption:

Assumption 3: Let $H(\cdot) \in C(\mathbb{R}^n \times [-\delta, \infty); \mathbb{R}^+)$. There is a function $V \in C^{2,1}(\mathbb{R}^n \times S \times \mathbb{R}^+; \mathbb{R}^+)$, as well as $q \ge 2(q_1 \vee q_2 \vee q_3)$, and positive numbers c_1, c_2, c_3, c_4 such that $c_3 + c_4 < c_2$, $|x|^q \le V(x, i, t) \le H(x, t), \forall (x, i, t) \in \mathbb{R}^n \times S \times \mathbb{R}^+$, and $LV(x, y, i, t) + V_x(x, i, t) u(z, i, t) \le c_1 - c_2 H(x, t) + c_3(1 - \bar{\delta}) H(y, t - \delta(t)) + c_4 H(z, t - \tau), \forall (x, y, i, t) \in \mathbb{R}^n \times \mathbb{R}^n \times S \times \mathbb{R}^+, z \in \mathbb{R}^n$.

Theorem 2.2. *Let the Assumptions 1–3 hold. Under the initial value (2.2), the system (2.3) admits a unique global solution $x(t)$ on $t \ge -\delta$ and the solution $x(t)$ satisfies*

$$\sup_{-\delta \le t < \infty} \mathbb{E}|x(t)|^q < \infty. \tag{2.10}$$

Proof. (1) Existence and uniqueness. Fix any initial value (2.2). It follows from Assumption 1 that Eq. (2.3) has an unique maximal local solution $x(t)$ on $t \in [-\delta, \sigma_e]$, where σ_e is the explosion time. If we prove that the solution $x(t)$ is global, we only need to show that $\sigma_e = \infty$. Let m_0 be a sufficiently large integer such that $\|x_0\| = \|\varphi\| = \sup_{-\tau \le s \le 0} x(s) < m_0$. For each integer $m > m_0$, define the stopping time $\sigma_m = \inf\{t \in [0, \sigma_e) : |x(t)| \ge m\}$. As usual we set $\inf \emptyset = \infty$, here $\emptyset$ is an empty set. Clearly, σ_m's are increasing and $\sigma_\infty = \lim_{m\to\infty} \sigma_m \le \sigma_e$. By Itô's formula, we can get that for $\forall t > 0$,

$$\begin{aligned}
&\mathbb{E}V(x(t\wedge\sigma_m), r(t\wedge\sigma_m), t\wedge\sigma_m)\\
&\quad= V(x(0), r(0), 0) + \mathbb{E}\int_0^{t\wedge\sigma_m} [LV(x(s), x(s-\delta(s)), r(s), s)\\
&\qquad + V_x(x(s), r(s), s)u(x(s-\tau), r(s), s)]ds. \qquad (2.11)
\end{aligned}$$

Applying Assumption 3, we can obtain that

$$\begin{aligned}
&\mathbb{E}V(x(t\wedge\sigma_m), r(t\wedge\sigma_m), t\wedge\sigma_m)\\
&\quad\le V(x(0), r(0), 0) + c_1 t - c_2\int_0^{t\wedge\sigma_m} H(x(s), s)ds\\
&\qquad + c_3(1-\bar{\delta})\int_0^{t\wedge\sigma_m} H(x(s-\delta(s)), s-\delta(s))ds\\
&\qquad + c_4\int_0^{t\wedge\sigma_m} H(x(s-\tau), s-\tau)ds. \qquad (2.12)
\end{aligned}$$

Noting that

$$\begin{aligned}
\int_0^{t\wedge\sigma_m} H(x(s-\delta(s)), s-\delta(s))ds &\le \frac{1}{1-\bar{\delta}}\int_{-\delta(s)}^{t\wedge\sigma_m-\delta(s)} H(x(u), u)du \qquad (2.13)\\
&\le \frac{1}{1-\bar{\delta}}\int_{-\delta}^{0} H(x(s), s)ds + \frac{1}{1-\bar{\delta}}\int_0^{t\wedge\sigma_m} H(x(s), s)ds
\end{aligned}$$

and

$$\begin{aligned}
\int_0^{t\wedge\sigma_m} H(x(s-\tau), s-\tau)ds &= \int_{-\tau}^{t\wedge\sigma_m-\tau} H(x(u), u)du\\
&\le \int_{-\tau}^{0} H(x(s), s)ds + \int_0^{t\wedge\sigma_m} H(x(s), s)ds. \qquad (2.14)
\end{aligned}$$

Substituting (2.13) and (2.14) into (2.12) that we have

$$\begin{aligned}
\mathbb{E}V(x(t\wedge\sigma_m), r(t\wedge\sigma_m), t\wedge\sigma_m) &\le V(x(0), r(0), 0) + c_3\int_{-\delta}^{0} H(x(s), s)ds\\
&\quad - (c_2 - c_3 - c_4)\int_0^{t\wedge\sigma_m} H(x(s), s)ds\\
&\quad + c_4\int_{-\tau}^{0} H(x(s), s)ds + c_1 t. \qquad (2.15)
\end{aligned}$$

For $c_3 + c_4 < c_2$, we can further get

$$\mathbb{E}V(x(t\wedge\sigma_m), r(t\wedge\sigma_m), t\wedge\sigma_m) \le M_1 + c_1 t, \qquad (2.16)$$

where $M_1 = V(x(0), r(0), 0) + c_3\int_{-\delta}^{0} H(x(s), s)ds + c_4\int_{-\tau}^{0} H(x(s), s)ds$. Therefore,

$$\mathbb{E}[V(x(\sigma_m), r(\sigma_m), \sigma_m)I_{\{\sigma_m \le t\}}] \le M_1 + c_1 t.$$

For $|x|^q \le V(x,i,t)$, then we can obtain

$$\mathbb{E}[|x(\sigma_m)|^q I_{\{\sigma_m \le t\}}] \le M_1 + c_1 t,$$

By the definition of σ_m, we have $m^q \mathbb{P}(\sigma_m \le t) \le M_1 + c_1 t$. When $m \to \infty$, we have $\mathbb{P}(\sigma_\infty \le t) \to 0$, that is $\sigma_\infty > t$ a.s. Letting $t \to \infty$, we obtain that $\sigma_\infty = \infty$ a.s.

(2) Prove $\sup_{-\tau \le t \le \infty} \mathbb{E}|x(t)|^q < \infty$. Set $f(u) = c_2 - c_3 e^{u\delta} - c_4 e^{u\tau} - u$ for $\forall u > 0$. Obviously, $f(u)$ is continuous in u. Since $f(0) = c_2 - c_3 - c_4 > 0$, by the local sign preserving property of a continuous function, there is a sufficiently small positive number ε such that $f(\varepsilon) = c_2 - c_3 e^{\varepsilon\delta} - c_4 e^{\varepsilon\tau} - \varepsilon > 0$. For $\forall t > 0$, applying Itô's formula to $e^{\varepsilon t} V(x(t), r(t), t)$, we gain

$$\begin{aligned}
&\mathbb{E}e^{\varepsilon(t\wedge\sigma_m)} V(x(t\wedge\sigma_m), r(t\wedge\sigma_m), t\wedge\sigma_m) \\
&= V(x(0), r(0), 0) + \mathbb{E}\int_0^{t\wedge\sigma_m} \varepsilon e^{\varepsilon s} V(x(s), r(s), s) ds \\
&+ \mathbb{E}\int_0^{t\wedge\sigma_m} e^{\varepsilon s}[LV(x(s), x(s-\delta(s)), r(s), s) \\
&+ V_x(x(s), r(s), s) u(x(s-\tau), r(s), s)] ds.
\end{aligned} \tag{2.17}$$

Applying Assumption 3, we obtain

$$\begin{aligned}
&\mathbb{E}e^{\varepsilon(t\wedge\sigma_m)} V(x(t\wedge\sigma_m), r(t\wedge\sigma_m), t\wedge\sigma_m) \\
&\le V(x(0), r(0), 0) + \varepsilon\mathbb{E}\int_0^{t\wedge\sigma_m} e^{\varepsilon s} V(x(s), r(s), s) ds \\
&+ \mathbb{E}\int_0^{t\wedge\sigma_m} e^{\varepsilon s}[c_1 - c_2 H(x(s), s) + c_3(1-\bar{\delta}) H(x(s-\delta(s)), s-\delta(s)) \\
&+ c_4 H(x(s-\tau), s-\tau)] ds \\
&\le V(x(0), r(0), 0) + \varepsilon\mathbb{E}\int_0^{t\wedge\sigma_m} e^{\varepsilon s} V(x(s), r(s), s) ds + \frac{c_1}{\varepsilon}(e^{\varepsilon t} - 1) \\
&- c_2\mathbb{E}\int_0^{t\wedge\sigma_m} e^{\varepsilon s} H(x(s), s) ds + c_3(1-\bar{\delta})\mathbb{E}\int_0^{t\wedge\sigma_m} e^{\varepsilon\delta(s)} e^{\varepsilon(s-\delta(s))} \\
&\times H(x(s-\delta(s)), s-\delta(s)) ds \\
&+ c_4\mathbb{E}\int_0^{t\wedge\sigma_m} e^{\varepsilon\tau} e^{\varepsilon(s-\tau)} H(x(s-\tau), s-\tau) ds \\
&\le V(x(0), r(0), 0) + \varepsilon\mathbb{E}\int_0^{t\wedge\sigma_m} e^{\varepsilon s} V(x(s), r(s), s) ds \\
&+ \frac{c_1}{\varepsilon} e^{\varepsilon t} - c_2\mathbb{E}\int_0^{t\wedge\sigma_m} e^{\varepsilon s} H(x(s), s) ds \\
&+ c_3 e^{\varepsilon\delta}(1-\bar{\delta})\mathbb{E}\int_0^{t\wedge\sigma_m} e^{\varepsilon(s-\delta(s))} H(x(s-\delta(s)), s-\delta(s)) ds \\
&+ c_4 e^{\varepsilon\tau}\mathbb{E}\int_0^{t\wedge\sigma_m} e^{\varepsilon(s-\tau)} H(x(s-\tau), s-\tau) ds.
\end{aligned} \tag{2.18}$$

Noting that

$$\begin{aligned}
&\int_0^{t\wedge\sigma_m} e^{\varepsilon(s-\delta(s))} H(x(s-\delta(s)), s-\delta(s)) ds \\
&\le \frac{1}{1-\bar{\delta}} \int_{-\delta(s)}^{t\wedge\sigma_m - \delta(s)} e^{\varepsilon u} H(x(u), u) du \\
&\le \frac{1}{1-\bar{\delta}} \int_{-\delta}^{0} e^{\varepsilon s} H(x(s), s) ds + \frac{1}{1-\bar{\delta}} \int_0^{t\wedge\sigma_m} e^{\varepsilon s} H(x(s), s) ds
\end{aligned} \tag{2.19}$$

and

$$\int_0^{t\wedge\sigma_m} e^{\varepsilon(s-\tau)}H(x(s-\tau),s-\tau)ds = \int_{-\tau}^{t\wedge\sigma_m-\tau} e^{\varepsilon u}H(x(u),u)du \leq \int_{-\tau}^{0} e^{\varepsilon s}H(x(s),s)ds + \int_0^{t\wedge\sigma_m} e^{\varepsilon s}H(x(s),s)ds. \quad (2.20)$$

Substituting (2.19) and (2.20) into (2.18) that we have

$$\begin{aligned}
&\mathbb{E}e^{\varepsilon(t\wedge\sigma_m)}V(x(t\wedge\sigma_m),r(t\wedge\sigma_m),t\wedge\sigma_m)\\
&\leq V(x(0),r(0),0) + c_3e^{\varepsilon\delta}\int_{-\delta}^{0} e^{\varepsilon s}H(x(s),s)ds\\
&\quad + c_4e^{\varepsilon\tau}\int_{-\tau}^{0} e^{\varepsilon s}H(x(s),s)ds + \frac{c_1}{\varepsilon}e^{\varepsilon t}\\
&\quad + \varepsilon\mathbb{E}\int_0^{t\wedge\sigma_m} e^{\varepsilon s}V(x(s),r(s),s)ds\\
&\quad - (c_2 - c_3e^{\varepsilon\delta} - c_4e^{\varepsilon\tau})\mathbb{E}\int_0^{t\wedge\sigma_m} e^{\varepsilon s}H(x(s),s)ds\\
&= M_2 + \frac{c_1}{\varepsilon}e^{\varepsilon t} + \varepsilon\mathbb{E}\int_0^{t\wedge\sigma_m} e^{\varepsilon s}V(x(s),r(s),s)ds\\
&\quad - (c_2 - c_3e^{\varepsilon\delta} - c_4e^{\varepsilon\tau})\mathbb{E}\int_0^{t\wedge\sigma_m} e^{\varepsilon s}H(x(s),s)ds,
\end{aligned}$$

where $M_2 = V(x(0),r(0),0) + c_3e^{\varepsilon\delta}\int_{-\delta}^{0} e^{\varepsilon s}H(x(s),s)ds + c_4e^{\varepsilon\tau}\int_{-\tau}^{0} e^{\varepsilon s}H(x(s),s)ds$. For $|x|^q \leq V(x,i,t) \leq H(x,t)$, we further compute

$$\begin{aligned}
&\mathbb{E}e^{\varepsilon(t\wedge\sigma_m)}V(x(t\wedge\sigma_m),r(t\wedge\sigma_m),t\wedge\sigma_m)\\
&\leq M_2 + \frac{c_1}{\varepsilon}e^{\varepsilon t} - (c_2 - c_3e^{\varepsilon\delta} - c_4e^{\varepsilon\tau} - \varepsilon)\mathbb{E}\int_0^{t\wedge\sigma_m} e^{\varepsilon s}H(x(s),s)ds.
\end{aligned}$$

For $c_2 - c_3e^{\varepsilon\delta} - c_4e^{\varepsilon\tau} - \varepsilon > 0$,

$$\mathbb{E}e^{\varepsilon(t\wedge\sigma_m)}V(x(t\wedge\sigma_m),r(t\wedge\sigma_m),t\wedge\sigma_m) \leq M_2 + \frac{c_1}{\varepsilon}e^{\varepsilon t}.$$

Letting $m \to \infty$. Then

$$\mathbb{E}V(x(t),r(t),t) \leq M_2e^{-\varepsilon t} + \frac{c_1}{\varepsilon} < M_2 + \frac{c_1}{\varepsilon}.$$

Applying $|x(t)|^q \leq V(x(t),r(t),t)$ again, we can get $\mathbb{E}|x(t)|^q < M_2 + \frac{c_1}{\varepsilon}$, $\forall t > 0$. Together with $t \in [-\delta, 0]$, $\sup_{t\in[-\delta,0]}\mathbb{E}|x(t)|^q \leq \|\varphi\|^q$, therefore, $\sup_{t\in[-\delta,\infty)}\mathbb{E}|x(t)|^q < \infty$. The proof is complete. □

3 Main results

In this section, we will investigate the H_∞-stabilization and asymptotic stabilization.

To proceed, a Lyapunov functional $\bar{V}(x_t,r_t,t)$ need to be constructed on the segment $x_t := \{x(t+s) : -2\delta \leq s \leq 0\}$ and $r_t = \{r(t+s) : -2\delta \leq s \leq 0\}$ for $t \geq 0$. For x_t and r_t to be well defined for $0 \leq t < 2\delta$, we set $x(s) = \varphi(-\delta)$ for $s \in [-2\delta, -\delta)$ and $r(s) = r_0$ for $s \in [-2\delta, 0)$. Let

$$\bar{V}(x_t,r_t,t) = \bar{U}(x(t),r(t),t) + \theta\int_{-\tau}^{0}\int_{t+s}^{t} Q(v)dvds, \qquad t \geq 0, \quad (3.1)$$

where θ is a positive number to be determined later, $\bar{U} \in C^{2,1}(\mathbb{R}^n \times S \times \mathbb{R}^+; \mathbb{R}^+)$ such that

$$\lim_{|x|\to\infty} \left[\inf_{(i,t)\in S\times\mathbb{R}^+} \bar{U}(x,i,t) \right] = \infty \tag{3.2}$$

and

$$\begin{aligned} Q(t) = \tau|&f(x(t), x(t-\delta(t)), r(t), t) + u(x(t-\tau), r(t), t)|^2 \\ &+ |g(x(t), x(t-\delta(t)), r(t), t)|^2 \\ &+ 2\lambda(1+\lambda\tau)|h(x(t), x(t-\delta(t)), r(t), t)|^2. \end{aligned} \tag{3.3}$$

Set

$$\begin{aligned} &f(x,y,i,s) = f(x,y,i,0),\ u(z,i,s) = u(z,i,0), \\ &g(x,y,i,s) = g(x,y,i,0), h(x,y,i,s) = h(x,y,i,0) \end{aligned}$$

for $(x,y,i,s) \in \mathbb{R}^n \times \mathbb{R}^n \times S \times [-2\delta, 0)$. Applying Itô's formula to $\bar{U}(x(t), r(t), t)$, we obtain

$$\begin{aligned} d\bar{U}&(x(t), r(t), t) \\ &= \bar{U}_t(x(t), r(t), t)dt + \bar{U}_x(x(t), r(t), t) \\ &\quad \times [f(x(t), x(t-\delta(t)), r(t), t) + u(x(t-\tau), r(t), t)]dt \\ &\quad + \frac{1}{2}\text{trace}[g^T(x(t), x(t-\delta(t)), r(t), t) \\ &\quad \times \bar{U}_{xx}(x(t), r(t), t) g(x(t), x(t-\delta(t)), r(t), t)]dt \\ &\quad + \lambda[\bar{U}(x(t) + h(x(t), x(t-\delta(t)), r(t), t), r(t), t) - \bar{U}(x(t), r(t), t)]dt \\ &\quad + \sum_{j=1}^{N} \gamma_{r(t)j} \bar{U}(x(t), j, t)dt + \bar{U}_x(x(t), r(t), t) g(x(t), x(t-\delta(t)), r(t), t)dB(t) \\ &\quad + [\bar{U}(x(t) + h(x(t), x(t-\delta(t)), r(t), t), r(t), t) - \bar{U}(x(t), r(t), t)]d\tilde{N}(t) \\ &\quad + \int_{\mathbb{R}} [\bar{U}(x(t), r_0 + h(r(t), u), t) - \bar{U}(x(t), r(t), t)]\mu(dt, du) \\ &= [\bar{U}_x(x(t), r(t), t)(u(x(t-\tau), r(t), t) - u(x(t), r(t), t)) \\ &\quad + L\bar{U}(x(t), x(t-\delta(t)), r(t), t)]dt + dM(t),\ t \ge 0, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} L\bar{U}&(x(t), x(t-\delta(t)), r(t), t) \\ &= \bar{U}_t(x(t), r(t), t) + \bar{U}_x(x(t), r(t), t)[f(x(t), x(t-\delta(t)), r(t), t) \\ &\quad + u(x(t), r(t), t)] + \frac{1}{2}\text{trace}[g^T(x(t), x(t-\delta(t)), r(t), t) \\ &\quad \times \bar{U}_{xx}(x(t), r(t), t) g(x(t), x(t-\delta(t)), r(t), t)] \\ &\quad + \lambda[\bar{U}(x(t) + h(x(t), x(t-\delta(t)), r(t), t), r(t), t) - \bar{U}(x(t), r(t), t)] \\ &\quad + \sum_{j=1}^{N} \gamma_{r(t)j} \bar{U}(x(t), j, t) \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} M(t) = & \int_0^t \bar{U}_x(x(s), r(s), s) g(x(s), x(s-\delta(s)), r(s), s) dB(s) \\ & + \int_0^t \int_{\mathbb{R}} [\bar{U}(x(t), r_0 + h(r(t), u), t) - \bar{U}(x(t), r(t), t)] \mu(ds, du) \\ & + \int_0^t [\bar{U}(x(s) + h(x(s), x(s-\delta(s)), r(s), s), r(s), s) - \bar{U}(x(s), r(s), s)] d\tilde{N}(s). \end{aligned}$$

Here $M(t)$ is a local martingale with $M(0) = 0$.

To investigate the H_∞-stability and asymptotic stability of system (2.3), the following assumption is also given.

Assumption 4: For all $(x, y, i, t) \in \mathbb{R}^n \times \mathbb{R}^n \times S \times \mathbb{R}^+$, assume that there exist functions $U(x, i, t) \in C^{2,1}(\mathbb{R}^n \times S \times \mathbb{R}^+; \mathbb{R}^+)$, $W(x) \in C(\mathbb{R}^n; \mathbb{R}^+)$ and positive constants α and $\rho_i (i = 1, 2, \ldots, 6)$ such that

$$\begin{aligned} & L\bar{U}(x, y, i, t) + \rho_1 |\bar{U}_x(x, i, t)|^2 + \rho_2 |f(x, y, i, t)|^2 \\ & \quad + \rho_3 |g(x, y, i, t)|^2 + \rho_4 |h(x, y, i, t)|^2 \\ & \leq -\rho_5 |x|^2 + \rho_6 (1 - \bar{\delta}) |y|^2 - W(x) + \alpha (1 - \bar{\delta}) W(y), \end{aligned} \tag{3.6}$$

where $\alpha < 1$ and $\rho_6 < \rho_5$.

The following theorem shows that the controlled system (2.3) is stable in the sense of H_∞.

Theorem 3.1. *Suppose that Assumptions 1–2 and Assumption 4 hold, if positive number τ is small enough for*

$$\tau \leq \frac{1}{\beta} \sqrt{\frac{2\rho_1 \rho_2}{3}} \wedge \frac{4\rho_1 \rho_3}{3\beta^2} \wedge \frac{1}{\beta^2} \sqrt{\frac{2\rho_1 (\rho_5 - \rho_6)}{3}} \tag{3.7}$$

and $\frac{3\lambda(1+\lambda\tau)\tau\beta^2}{2\rho_1} \leq \rho_4$. Then for any given initial data (2.2), the solution of the controlled system (2.3) has the property

$$\int_0^\infty \mathbb{E}[|x(t)|^2 + W(x(t))] dt < \infty. \tag{3.8}$$

Moreover, there exist positive constants c and $\tilde{p} > 2$ such that $c|x|^{\tilde{p}} \leq W(x) (\forall (x, t) \in \mathbb{R}^n \times \mathbb{R}^+)$, then the controlled system (2.3) is H_∞-stable, namely

$$\int_0^\infty \mathbb{E}|x(t)|^p dt < \infty, \qquad p \in [2, \tilde{p}] \tag{3.9}$$

for any given initial value (2.2).

Proof. Given any initial value (2.2). Applying Itô's formula to $\bar{V}(x_{t\wedge\sigma_m}, r_{t\wedge\sigma_m}, t \wedge \sigma_m)$ defined by (3.1) yields

$$\mathbb{E}\bar{V}(x_{t\wedge\sigma_m}, r_{t\wedge\sigma_m}, t \wedge \sigma_m) = \bar{V}(x_0, r_0, 0) + \mathbb{E} \int_0^{t\wedge\sigma_m} L\bar{V}(x_s, r_s, s) ds, \qquad t \geq 0, \tag{3.10}$$

where σ_m is defined as the same as in Theorem 2.2, and

$$\begin{aligned} L\bar{V}(x_t, r_t, t) = & L\bar{U}(x(t), x(t-\delta(t)), r(t), t) \\ & + \bar{U}_x(x(t), r(t), t)[u(x(t-\tau), r(t), t) - u(x(t), r(t), t)] \\ & + \theta\tau Q(t) - \theta \int_{t-\tau}^t Q(r) dr. \end{aligned} \tag{3.11}$$

Using (2.5) that we can gain

$$\begin{aligned}
&\bar{U}_x(x(t),r(t),t)[u(x(t-\tau),r(t),t)-u(x(t),r(t),t)] \\
&\quad\leq \rho_1|U_x(x(t),r(t),t)|^2+\frac{1}{4\rho_1}|u(x(t-\tau),r(t),t)-u(x(t),r(t),t)|^2 \\
&\quad\leq \rho_1|U_x(x(t),r(t),t)|^2+\frac{\beta^2}{4\rho_1}|x(t)-x(t-\tau)|^2.
\end{aligned} \tag{3.12}$$

Then we can obtain

$$\begin{aligned}
L\bar{V}(x_t,r_t,t) \leq\ & L\bar{U}(x(t),x(t-\delta(t)),r(t),t)+\rho_1|\bar{U}_x(x(t),r(t),t)|^2 \\
&+\frac{\beta^2}{4\rho_1}|x(t)-x(t-\tau)|^2+2\theta\tau^2|f(x(t),x(t-\delta(t)),r(t),t)|^2 \\
&+2\theta\tau^2|u(x(t-\tau),r(t),t)|^2+\theta\tau|g(x(t),x(t-\delta(t)),r(t),t)|^2 \\
&+2\lambda(1+\lambda\tau)\theta\tau|h(x(t),x(t-\delta(t)),r(t),t)|^2-\theta\int_{t-\tau}^{t}Q(r)dr.
\end{aligned} \tag{3.13}$$

According to (3.7), we can further gain

$$\begin{aligned}
L\bar{V}(x_t,r_t,t) \leq\ & L\bar{U}(x(t),x(t-\delta(t)),r(t),t)+\rho_1|\bar{U}_x(x(t),r(t),t)|^2 \\
&+\frac{\beta^2}{4\rho_1}|x(t)-x(t-\tau)|^2+\rho_2|f(x(t),x(t-\delta(t)),r(t),t)|^2 \\
&+\rho_3|g(x(t),x(t-\delta(t)),r(t),t)|^2+\rho_4|h(x(t),x(t-\delta(t)),r(t),t)|^2 \\
&+2\theta\tau^2\beta^2|x(t-\tau)|^2-\theta\int_{t-\tau}^{t}Q(r)dr \\
\leq\ & -\rho_5|x(t)|^2+\rho_6(1-\bar{\delta})|x(t-\delta(t))|^2-W(x(t)) \\
&+\alpha(1-\bar{\delta})W(x(t-\delta(t)))+\frac{\beta^2}{4\rho_1}|x(t)-x(t-\tau)|^2 \\
&+2\theta\tau^2\beta^2|x(t-\tau)|^2-\theta\int_{t-\tau}^{t}Q(r)dr.
\end{aligned} \tag{3.14}$$

Substituting (3.14) into (3.10), we can obtain

$$\mathbb{E}\bar{V}(x_{t\wedge\sigma_m},r_{t\wedge\sigma_m},t\wedge\sigma_m) \leq \bar{V}(x_0,r_0,0)+v_1+v_2+v_3-v_4, \tag{3.15}$$

where

$$\begin{aligned}
v_1 &= \mathbb{E}\int_0^{t\wedge\sigma_m}[-\rho_5|x(r)|^2+\rho_6(1-\bar{\delta})|x(r-\delta(r))|^2+2\theta\tau^2\beta^2|x(r-\tau)|^2]dr, \\
v_2 &= \mathbb{E}\int_0^{t\wedge\sigma_m}[-W(x(r))+\alpha(1-\bar{\delta})W(x(r-\delta(r)))]dr, \\
v_3 &= \frac{\beta^2}{4\rho_1}\mathbb{E}\int_0^{t\wedge\sigma_m}|x(r)-x(r-\tau)|^2dr, \\
v_4 &= \theta\mathbb{E}\int_0^{t\wedge\sigma_m}\int_{r-\tau}^{r}Q(v)dvdr.
\end{aligned}$$

Noting that

$$\begin{aligned}
\int_0^{t\wedge\sigma_m} |x(s-\delta(s))|^2 ds &\le \frac{1}{1-\bar{\delta}}\int_{-\delta(0)}^{t\wedge\sigma_m-\delta(t\wedge\sigma_m)} |x(r)|^2 dr \le \frac{1}{1-\bar{\delta}}\int_{-\delta}^{t\wedge\sigma_m} |x(r)|^2 dr,\\
\int_0^{t\wedge\sigma_m} |x(s-\tau)|^2 ds &\le \int_{-\tau}^{t\wedge\sigma_m-\tau} |x(r)|^2 dr \le \int_{-\delta}^{t\wedge\sigma_m} |x(r)|^2 dr,\\
\int_0^{t\wedge\sigma_m} W(x(s-\delta(s)))ds &\le \frac{1}{1-\bar{\delta}}\int_{-\delta(0)}^{t\wedge\sigma_m-\delta(t\wedge\sigma_m)} W(x(r))dr\\
&\le \frac{1}{1-\bar{\delta}}\int_{-\delta}^{t\wedge\sigma_m} W(x(r))dr,
\end{aligned}$$

$$\begin{aligned}
v_1 &\le (\rho_6+2\theta\tau^2\beta^2)\int_{-\delta}^0 |x(r)|^2 dr - (\rho_5-\rho_6-2\theta\tau^2\beta^2)\mathbb{E}\int_0^{t\wedge\sigma_m} |x(r))|^2 dr,\\
v_2 &\le \alpha\int_{-\delta}^0 W(x(r))dr - (1-\alpha)\int_0^{t\wedge\sigma_m} W(x(r))dr.
\end{aligned} \tag{3.16}$$

Substituting (3.16) into (3.15) that we obtain

$$\begin{aligned}
\mathbb{E}\bar{V}(x_{t\wedge\sigma_m}, r_{t\wedge\sigma_m}, t\wedge\sigma_m) \le C &- (\rho_5-\rho_6-2\theta\tau^2\beta^2)\mathbb{E}\int_0^{t\wedge\sigma_m} |x(r)|^2 dr\\
&- (1-\alpha)\mathbb{E}\int_0^{t\wedge\sigma_m} W(x(r))dr + v_3 - v_4,
\end{aligned} \tag{3.17}$$

where $C = \bar{V}(x_0, r_0, 0) + (\rho_6+2\theta\tau^2\beta^2)\int_{-\tau}^0 |x(r)|^2 dr + \alpha\int_{-\tau}^0 W(x(r))dr$. Letting $m\to\infty$ and applying the classical Fatou lemma, we gain

$$\begin{aligned}
\mathbb{E}V(x_t, r_t, t) \le C &- (\rho_5-\rho_6-2\theta\tau^2\beta^2)\mathbb{E}\int_0^t |x(r)|^2 dr\\
&- (1-\alpha)\mathbb{E}\int_0^t W(x(r))dr + \bar{v}_3 - \bar{v}_4,
\end{aligned} \tag{3.18}$$

where

$$\bar{v}_3 = \frac{\beta^2}{4\rho_1}\mathbb{E}\int_0^t |x(r)-x(r-\tau)|^2 dr, \qquad \bar{v}_4 = \theta\mathbb{E}\int_0^t\int_{r-\tau}^r Q(v)dvdr.$$

For $t\in[0,\tau]$, we have

$$\begin{aligned}
\bar{v}_3 &\le \frac{\beta^2}{4\rho_1}\int_0^t \mathbb{E}|x(r)-x(r-\tau)|^2 dr\\
&\le \frac{\beta^2}{2\rho_1}\int_0^\tau (\mathbb{E}|x(r)|^2 + \mathbb{E}|x(r-\tau)|^2)dr\\
&\le \frac{\beta^2}{\rho_1}\tau\left(\sup_{-\tau\le r\le\tau}\mathbb{E}|x(r)|^2\right).
\end{aligned}$$

For $t>\tau$, we gain

$$\bar{v}_3 \le \frac{\tau\beta^2}{\rho_1}\left(\sup_{-\tau\le r\le\tau}\mathbb{E}|x(r)|^2\right) + \frac{\beta^2}{4\rho_1}\mathbb{E}\int_\tau^t |x(r)-x(r-\tau)|^2 dr.$$

It follows from (2.3) and Hölder inequality that we can obtain

$$\mathbb{E}\int_{\tau}^{t}|x(r)-x(r-\tau)|^2dr \le 3\mathbb{E}\int_{\tau}^{t}\int_{r-\tau}^{r}Q(v)dvdr \le 3\mathbb{E}\int_{0}^{t}\int_{r-\tau}^{r}Q(v)dvdr.$$

Then we can get

$$\bar{v}_3 \le \left(\frac{\beta^2\tau}{\rho_1}\sup_{-\tau\le r\le\tau}\mathbb{E}|x(r)|^2\right)+\frac{3\beta^2}{4\rho_1}\mathbb{E}\int_0^t\int_{r-\tau}^{r}Q(v)dvdr. \tag{3.19}$$

Substituting (3.19) into (3.18) we obtain

$$\begin{aligned}0\le \mathbb{E}\bar{V}(x_t,r_t,t) \le C &+ \frac{\tau\beta^2}{\rho_1}\left(\sup_{-\tau\le r\le\tau}\mathbb{E}|x(r)|^2\right)\\ &-(\rho_5-\rho_6-2\theta\tau^2\beta^2)\mathbb{E}\int_0^t|x(r)|^2dr\\ &-(1-\alpha)\mathbb{E}\int_0^t W(x(r))dr\\ &+\frac{3\beta^2}{4\rho_1}\mathbb{E}\int_0^t\int_{r-\tau}^{r}Q(v)dvdr-\theta\mathbb{E}\int_0^t\int_{r-\tau}^{r}Q(v)dvdr.\end{aligned} \tag{3.20}$$

Let $\theta=\frac{3\beta^2}{4\rho_1}$. For $\tau<\frac{1}{\beta^2}\sqrt{\frac{2(\rho_5-\rho_6)\rho_1}{3}}$, then

$$\begin{aligned}&\min\{\rho_5-\rho_6-2\theta\tau^2\beta^2,1-\alpha\}\int_0^t\mathbb{E}[|x(r)|^2+W(x(r))]dr\\ &\le(\rho_5-\rho_6-2\theta\tau^2\beta^2)\mathbb{E}\int_0^t|x(r)|^2dr+(1-\alpha)\mathbb{E}\int_0^t W(x(r))dr\\ &\le C+\frac{\tau\beta^2}{\rho_1}\left(\sup_{-\tau\le r\le\tau}\mathbb{E}|x(r)|^2\right)\end{aligned} \tag{3.21}$$

which implies the desired conclusion (3.8). Moreover, for $c|x|^{\bar{p}}\le W(x)$, applying the inequality $|v|^b\le|v|^a+|v|^c(\forall 0<a\le b\le c)$, we can get for any $p\in[2,\bar{p}]$,

$$\begin{aligned}\min\{1,c\}\int_0^\infty\mathbb{E}|x(t)|^pdt &\le \min\{1,c\}\int_0^\infty\mathbb{E}[|x(t)|^2+|x(t)|^{\bar{p}}]dt\\ &\le\int_0^\infty\mathbb{E}[|x(t)|^2+c|x(t)|^{\bar{p}}]dt\\ &\le\int_0^\infty\mathbb{E}[|x(t)|^2+W(x(t))]dt<\infty\end{aligned}$$

which implies (3.9) is true. The proof is complete. □

The next theorem illustrates that the controlled system (2.3) is asymptotically stable.

Theorem 3.2. *Let all the conditions of Theorem 3.1 hold. If $p\ge 2$ and $q\ge(p+q_1-1)\vee(p+2q_2-2)\vee pq_3$, then for the any given initial date (2.2), the solution of the system (2.3) satisfies*

$$\lim_{t\to\infty}\mathbb{E}|x(t)|^p=0. \tag{3.22}$$

Namely, the system (2.3) is asymptotically stable.

Proof. Applying Itô's formula to $|x(t)|^p$ that we obtain for any $0 \le t_1 < t_2 < \infty$,

$$\begin{aligned}
&\mathbb{E}|x(t_2)|^p - \mathbb{E}|x(t_1)|^p \\
&\quad \le \mathbb{E}\int_{t_1}^{t_2} p|x(t)|^{p-2}x^T(t)[f(x(t),x(t-\delta(t)),r(t),t) + u(x(t-\tau),r(t),t)]dt \\
&\qquad + \frac{p(p-1)}{2}\mathbb{E}\int_{t_1}^{t_2} |x(t)|^{p-2}|g(x(t),x(t-\delta(t)),r(t),t)|^2 dt \\
&\qquad + \lambda\mathbb{E}\int_{t_1}^{t_2} (|x(t)+h(x(t),x(t-\delta(t)),r(t),t)|^p - |x(t)|^p)\, dt.
\end{aligned} \tag{3.23}$$

It follows from (2.7) that we gain

$$\begin{aligned}
\mathbb{E}|x(t_2)|^p - \mathbb{E}|x(t_1)|^p \le{}& p\mathbb{E}\int_{t_1}^{t_2} |x(t)|^{p-1}[K(1+|x(t)|^{q_1} + |x(t-\delta(t))|^{q_1}) + \beta|x(t-\tau)|]dt \\
&+ \frac{3p(p-1)K^2}{2}\mathbb{E}\int_{t_1}^{t_2} |x(t)|^{p-2}(1+|x(t)|^{2q_2} + |x(t-\delta(t))|^{2q_2})dt \\
&+ \lambda 6^{p-1}K^p\mathbb{E}\int_{t_1}^{t_2} (1+|x(t)|^{pq_3} + |x(t-\delta(t))|^{pq_3})dt \\
&+ \lambda(2^{p-1}-1)\mathbb{E}\int_{t_1}^{t_2} |x(t)|^p dt.
\end{aligned} \tag{3.24}$$

By Young's inequality, we can get

$$\begin{aligned}
&\mathbb{E}\int_{t_1}^{t_2} |x(t)|^{p-1}|x(t-\delta(t))|^{q_1} dt \\
&\quad \le \frac{p-1}{p+q_1-1}\int_{t_1}^{t_2} \mathbb{E}|x(t)|^{p+q_1-1}dt + \frac{q_1}{p+q_1-1}\int_{t_1}^{t_2} \mathbb{E}|x(t-\delta(t))|^{p+q_1-1}dt.
\end{aligned} \tag{3.25}$$

Using Theorem 2.2 and $p+q_1-1 \le q$ that we get

$$\mathbb{E}\int_{t_1}^{t_2} |x(t)|^{p-1}|x(t-\delta(t))|^{q_1} dt \le \sup_{-\delta\le t<\infty} \mathbb{E}|x(t)|^{p+q_1-1}(t_2-t_1). \tag{3.26}$$

Similarly, according to $p+2q_2-2 \le q$ and $pq_3 \le q$ that we can get

$$\mathbb{E}\int_{t_1}^{t_2} |x(t)|^{p-2}|x(t-\delta(t))|^{2q_2} dt \le \sup_{-\delta\le t<\infty} \mathbb{E}|x(t)|^{p+2q_2-2}(t_2-t_1) \tag{3.27}$$

and

$$\mathbb{E}\int_{t_1}^{t_2} |x(t)|^{pq_3} dt \le \sup_{0\le t<\infty} \mathbb{E}|x(t)|^{pq_3}(t_2-t_1). \tag{3.28}$$

It follows from (3.26)–(3.28) that we can obtain

$$\mathbb{E}|x(t_2)|^p - \mathbb{E}|x(t_1)|^p \le \widetilde{C}(t_2-t_1) \tag{3.29}$$

where $\widetilde{C}$ is a constant independent t_1, t_2. That is, $\mathbb{E}|x(t)|^p$ is uniformly continuous in t on $\mathbb{R}^+$. Together with (3.9), we can assert $\lim_{t\to\infty} \mathbb{E}|x(t)|^p = 0$. The proof is thus complete. □

Remark 3.3. Different from the existing literature, this paper studies the H_∞-stability and asymptotic stability for a class of highly nonlinear SDSs, where both the Markovian switching and Poisson jump are taken into consideration, which advances the results of the system with the coefficients satisfying the linear growth condition in [24] and covers the results in [18,20,27,33,39].

4 An example

In this section, we give an example to illustrate the the obtained results. Let $B(t)$ be a scalar Brown motion and $N(t)$ be a Poisson process with intensity $\lambda = 1$. For the sake of simplicity, here we consider $\delta(t) \equiv \delta$. We thus consider the following scalar system

$$\begin{aligned} dx(t) &= f(x(t), x(t-\delta), r(t), t)dt + g(x(t), x(t-\delta), r(t), t)dB(t) \\ &\quad + h(x(t), x(t-\delta), r(t), t)dN(t), \qquad t \ge 0 \end{aligned} \tag{4.1}$$

with the initial data $x(t) = 3 + 2\cos(t)$, $t \in [-1, 0]$ and $r(0) = 1$, where $f(x, y, 1, t) = x + \frac{1}{2}y^3 - 2x^3 - 2x^7$, $f(x, y, 2, t) = x + y^3 - 2x^3 - x^7$, $g(x, y, 1, t) = g(x, y, 2, t) = \frac{1}{4}y^2$, $h(x, y, 1, t) = h(x, y, 2, t) = \frac{1}{2}x$, $r(t)$ is a Markov chain on the state space $S = \{1, 2\}$ with

$$\Gamma = (\gamma_{ij})_{2\times 2} = \begin{bmatrix} cc-2 & 2 \\ 1 & -1 \end{bmatrix}.$$

It is easy to see that $q_1 = 7$, $q_2 = 2$ and $q_3 = 1$. The sample paths of the Markov chain and the solution of the system (4.1) are shown in Fig. 4.1. From this figure, we can see that the system (4.1) is unstable. We are in position to design a control function $u : \mathbb{R} \times S \times \mathbb{R}^+ \to \mathbb{R}$ defined by $u(x, 1, t) = -4x, u(x, 2, t) = -5x$ to make the system (4.1) become stable. It is also easy to see that $\beta = 5$. The controlled system of the form

$$\begin{aligned} dx(t) &= [f(x(t), x(t-\delta), r(t), t) + u(x(t-\tau), r(t), t)]dt \\ &\quad + g(x(t), x(t-\delta), r(t), t)dB(t) \\ &\quad + h(x(t), x(t-\delta), r(t), t)dN(t), \qquad t \ge 0. \end{aligned} \tag{4.2}$$

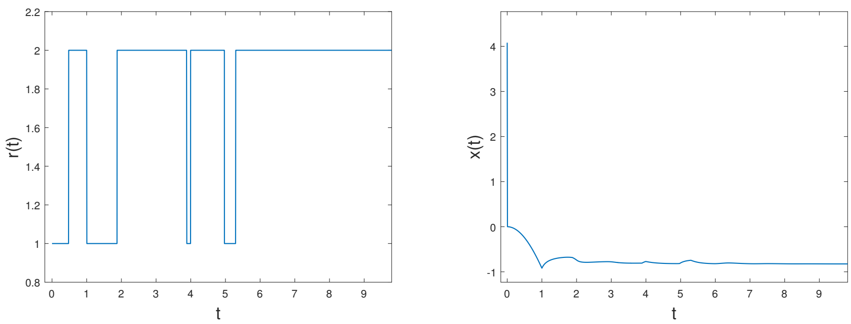

Figure 4.1: The sample paths of the Markov chain (left) and the solution of the system (4.1) (right) with $\delta = 1$.

Let $V(x, i, t) = x^{14} (i = 1, 2)$. By Young inequality, we compute

$$\begin{aligned} &LV(x, y, i, t) + V_x(x, i, t)u(z, i, t) \\ &\le \begin{cases} -28x^{20} - 18x^{16} + 356.93x^{14} + 2.74y^{16} + 4z^{14}, & i = 1, \\ -14x^{20} - 12x^{16} + 369.93x^{14} + 4.05y^{16} + 5z^{14}, & i = 2, \end{cases} \\ &\le c_1 - 12(x^{16} + x^{14}) + 4.05(y^{14} + y^{16}) + 5(z^{14} + z^{16}), \end{aligned}$$

where $c_1 = \sup_{x\in R}\{-14x^{20} + 381.93x^{14}\} < \infty$. Therefore, Assumption 3 is fulfilled with $c_2 = 12$, $c_3 = 4.05$, $c_4 = 5$, $H(x, t) = x^{14} + x^{16}$ and $q = 14$.

In the following, we define $\bar{U}(x,1,t) = x^2 + x^4 + x^8$ and $\bar{U}(x,2,t) = \frac{1}{2}(x^2 + x^4 + x^8)$. Then

$$L\bar{U}(x,y,i,t) \leq \begin{cases} -\frac{21}{4}x^{12} - \frac{195}{16}x^4 - \frac{7}{2}x^8 - \frac{403}{20}x^{10} - 16x^{14} + \frac{13}{16}y^4 + \frac{5}{4}y^6 + \frac{19}{10}y^{10}, & i = 1, \\ -\frac{19}{8}x^2 - \frac{215}{8}x^4 - \frac{47}{16}x^6 - \frac{7}{2}x^8 - \frac{267}{40}x^{10} - 4x^{14} + \frac{25}{32}y^4 + \frac{9}{8}y^6 + \frac{31}{20}y^{10}, & i = 2. \end{cases}$$

Moreover,

$$|\bar{U}_x(x,i,t)|^2 \leq \begin{cases} 12x^2 + 48x^6 + 192x^{14}, & i = 1, \\ 3x^2 + 12x^6 + 48x^{14}, & i = 2, \end{cases}$$

and

$$|f(x,y,i,t)|^2 \leq \begin{cases} 4x^2 + y^6 + 16x^6 + 16x^{14}, & i = 1, \\ 4x^2 + 4y^6 + 16x^6 + 4x^{14}, & i = 2, \end{cases}$$

and for $\forall i = 1,2$, $|g(x,y,i,t)|^2 = \frac{1}{16}y^4$, $|h(x,y,i,t)|^2 = \frac{1}{4}x^2$. Let $\rho_1 = \frac{1}{650}$, $\rho_2 = \frac{1}{80}$, $\rho_3 = \frac{1}{5}$ and $\rho_4 = \frac{1}{2}$. Then we can compute

$$\begin{aligned} &L\bar{U}(x,y,i,t) + \rho_1|\bar{U}_x(x,i,t)|^2 + \rho_2|f(x,y,i,t)|^2 + \rho_3|g(x,y,i,t)|^2 + \rho_4|h(x,y,i,t)|^2 \\ &\quad \leq -2.2|x|^2 - 2.7(x^4 + x^6 + x^8 + x^{10} + x^{14}) + 1.9(y^4 + y^6 + y^8 + y^{10} + y^{14}). \end{aligned} \tag{4.3}$$

It follows from (4.3) that we can assert that Assumption 4 is satisfied with $W(x) = 2.7(x^4 + x^6 + x^8 + x^{10} + x^{14})$, $\rho_5 = \frac{11}{5}$, $\rho_6 = 0$ and $\alpha = \frac{19}{27}$. By computing, we can set $\tau = 10^{-5}$ to satisfy (3.7) and $\frac{3\lambda(1+\lambda\tau)\tau\beta^2}{2\rho_1} \leq \rho_4$. Then according to Theorem 3.1, we therefore conclude that the solution of the controlled system (4.2) satisfies the following property

$$\int_0^\infty \mathbb{E}[x^2(t) + x^4(t) + x^6(t) + x^8(t) + x^{10}(t) + x^{14}(t)]dt < \infty.$$

Thus, we can get

$$\int_0^\infty \mathbb{E}[x^2(t) + x^4(t) + x^6(t) + x^8(t) + x^{10}(t)]dt < \infty.$$

Moreover, as $|x(t)|^p \leq x^2(t) + x^4(t) + x^6(t) + x^8(t) + x^{10}(t)$ for any $p \in [2,10]$, we obtain $\int_0^\infty \mathbb{E}|x(t)|^p dt < \infty$.

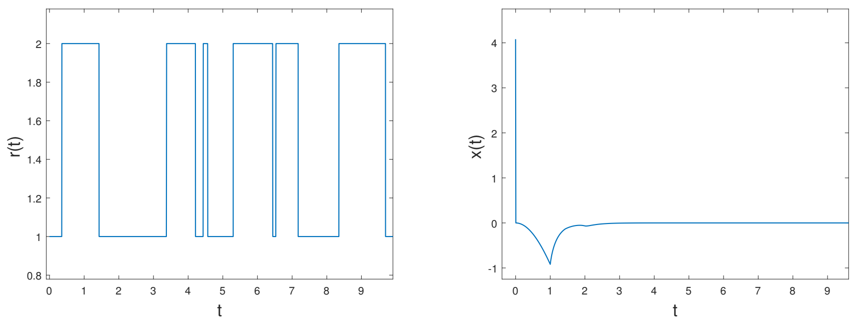

Figure 4.2: The sample paths of the Markov chain (left) and the solution of the system (4.2) (right) with $\delta = 1$ and $\tau = 10^{-5}$.

Let $p = 4$. Recalling $q_1 = 7$, $q_2 = 2$, $q_3 = 1$, then all the conditions of Theorem 3.2 are satisfied, so we can get $\lim_{t\to\infty} \mathbb{E}|x(t)|^4 = 0$.

The sample paths of the Markov chain and the solution of the controlled system (4.2) are shown in Fig. 4.2. The simulation supports the theoretical results.

5 Conclusion

Up to now, very few stabilization results seem to be known about the STVDSs with Markovian switching and Poisson jump, not to mention the case where the coefficients of such systems are highly nonlinear. This paper discussed the stabilization problem of such systems. In this paper, we designed a delay feedback controller to make an unstable highly nonlinear STVDS with Markovian switching and Poisson jump H_∞-stable and asymptotically stable, which enriches the stabilization results on such systems. Moreover, an illustrative example has been presented to verify the effectiveness of the obtained results.

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Positive solutions of a Kirchhoff–Schrödinger–Newton system with critical nonlocal term

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Abstract. This paper deals with the following Kirchhoff–Schrödinger–Newton system with critical growth

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = \phi|u|^{2^*-3}u + \lambda|u|^{p-2}u, & \text{in } \Omega, \\ -\Delta\phi = |u|^{2^*-1}, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a smooth bounded domain, $M(t) = 1 + bt^{\theta-1}$ with $t > 0$, $1 < \theta < \frac{N+2}{N-2}$, $b > 0$, $1 < p < 2$, $\lambda > 0$ is a parameter, $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent. By using the variational method and the Brézis–Lieb lemma, the existence and multiplicity of positive solutions are established.

Keywords: Kirchhoff–Schrödinger–Newton, positive solutions, critical growth.

2020 Mathematics Subject Classification: 35J20, 35J60, 35B09.

1 Introduction and main result

Consider the following Kirchhoff–Schrödinger–Newton system involving critical growth

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = \phi|u|^{2^*-3}u + \lambda|u|^{p-2}u, & \text{in } \Omega, \\ -\Delta\phi = |u|^{2^*-1}, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a smooth bounded domain, $M(t) = 1 + bt^{\theta-1}$ with $t > 0$, $1 < \theta < \frac{N+2}{N-2}$, $b > 0$, $1 < p < 2$, $\lambda > 0$ is a parameter, $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent.

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This system is derived from the Schrödinger–Poisson system

$$\begin{cases} -\Delta u + V(x)u + \eta\phi f(u) = h(x,u), & \text{in } \mathbb{R}^3, \\ -\Delta\phi = 2F(u), & \text{in } \mathbb{R}^3. \end{cases} \tag{1.2}$$

System as (1.2) has been studied extensively by many researchers because (1.2) has a strong physical meaning, which describes quantum particles interacting with the electromagnetic field generated by the motion. The Schrödinger–Poisson system (also called Schrödinger–Maxwell system) was first introduced by Benci and Fortunato in [6] as a physical model describing a charged wave interacting with its own electrostatic field in quantum mechanic. For more information on the physical aspects about (1.2), we refer the reader to [6,7].

Many recent studies of (1.2) have focused on existence of multiple solutions, ground states, positive and non-radial solutions. When $h(x,u) = |u|^{p-2}u$, Alves et al. in [4] considered the existence of ground state solutions for (1.2) with $4 < p < 6$. In [10], Cerami and Vaira proved the existence of positive solutions of (1.2) when $h(x,u) = a(x)|u|^{p-2}u$ with $4 < p < 6$ and $a(x)$ is a nonnegative function. The same result was established in [11,18,22,23] for $2 < p < 6$. In [20,25,26,28], by using variational methods, the authors proved the existence of ground state solutions of (1.2) with subcritical and critical growths. In addition, the existence of solutions for Schrödinger–Poisson system involving critical nonlocal term has been paid much attention by many authors, we can see [2,13,16,19,24,27] and so on.

In [5], Arora et al. considered a nonlocal Kirchhoff type equation with a critical Sobolev nonlinearity, using suitable variational techniques, the authors showed how to overcome the lack of compactness at critical levels. In [15], by using the variational method and the concentration compactness principle, Lei and Suo established the existence and multiplicity of nontrivial solutions. Luyen and Cuong [21] obtained the existence of multiple solutions for a given boundary value problem, using the minimax method and Rabinowitz's perturbation method. In [29], Zhou, Guo and Zhang combined the variational method and the mountain pass theorem, to get the existence of weak solutions, this time on the Heisenberg group.

Specially, Azzollini, D'Avenia and Vaira [3] studied the following Schrödinger–Newton type system with critical growth

$$\begin{cases} -\Delta u = \lambda u + |u|^{2^*-3}u\phi, & \text{in } \Omega, \\ -\Delta\phi = |u|^{2^*-1}, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a smooth bounded domain. By the variational method, they obtained the existence and nonexistence results of positive solutions when $N = 3$ and the existence of solutions in both the resonance and the non-resonance case for higher dimensions.

Lei and Gao [14] considered the Schrödinger–Newton system with sign-changing potential

$$\begin{cases} -\Delta u = f_\lambda(x)|u|^{p-2}u + |u|^3u\phi, & \text{in } \Omega, \\ -\Delta\phi = |u|^5, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain, $1 < p < 2$, $f_\lambda = \lambda f^+ + f^-$, $\lambda > 0$, $f\pm = \max\{\pm f, 0\}$. By using the variational method and analytic techniques, the authors proved the existence and multiplicity of positive solutions.

In [17], Li et al. proved the existence, nonexistence and multiplicity of positive radially symmetric solutions for the following Schrödinger–Poisson system

$$\begin{cases} -\Delta u + u + \lambda\phi|u|^3 u = \mu|u|^{p-2}u, & \text{in } \mathbb{R}^3, \\ -\Delta\phi = |u|^5, & \text{in } \mathbb{R}^3, \end{cases}$$

where $p \in (2,6)$, $\lambda \in \mathbb{R}$ and $\mu \geq 0$ are parameters.

With the help of the Lax–Milgram theorem, for every $u \in H_0^1(\Omega)$, the second equation of system (1.1) has a unique solution $\phi_u \in H_0^1(\Omega)$, we substitute ϕ_u to the first equation of system (1.1), then system (1.1) transforms into the following equation

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = \phi_u|u|^{2^*-3}u + \lambda|u|^{p-2}u, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

The variational functional associated with (1.3) is defined by

$$I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{b}{2\theta}\left(\int_\Omega |\nabla u|^2 dx\right)^\theta - \frac{1}{2(2^*-1)}\int_\Omega \phi_u|u|^{2^*-1}dx - \frac{\lambda}{p}\int_\Omega |u|^p dx.$$

We say that $u \in H_0^1(\Omega)$ is a weak solution of (1.3), for all $\psi \in H_0^1(\Omega)$, then u satisfies

$$\left[1 + b\left(\int_\Omega |\nabla u|^2 dx\right)^{\theta-1}\right]\int_\Omega \nabla u\nabla\psi dx = \int_\Omega \phi_u|u|^{2^*-3}u\psi dx + \lambda\int_\Omega |u|^{p-2}u\psi dx.$$

Our technique based on the Ekeland variational principle and the mountain pass theorem. Since system (1.1) contains a nonlocal critical growth term, which leads to the cause of the lack of compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ and the Palais–Smale condition for the corresponding energy functional could not be checked directly. Then we overcome the compactness by using the Brézis–Lieb lemma.

Now we state our main result.

Theorem 1.1. *Assume that* $1 < \theta < \frac{N+2}{N-2}$, $\frac{N}{N-2} < p < 2$ *and* $N > 4$, $b > 0$ *is small enough. Then there exists* $\Lambda_* > 0$ *such that for all* $\lambda \in (0, \Lambda_*)$, *system* (1.1) *has at least two positive solutions.*

Throughout this paper, we make use of the following notations:

- The space $H_0^1(\Omega)$ is equipped with the norm $\|u\|^2_{H_0^1(\Omega)} = \int_\Omega |\nabla u|^2 dx$, the norm in $L^p(\Omega)$ is denoted by $\|\cdot\|_p$.
- Let $D^{1,2}(\mathbb{R}^N)$ be the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|u\|^2_{D^{1,2}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\nabla u|^2 dx$.
- $C, C_1, C_2, \ldots$ denote various positive constants, which may vary from line to line.
- We denote by S_ρ (respectively, B_ρ) the sphere (respectively, the closed ball) of center zero and radius ρ, i.e. $S_\rho = \{u \in H_0^1(\Omega) : \|u\| = \rho\}$, $B_\rho = \{u \in H_0^1(\Omega) : \|u\| \leq \rho\}$.
- Let S be the best constant for Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$, namely

$$S = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega |u|^{2^*} dx\right)^{2/2^*}}.$$

2 Proof of the theorem

Firstly, we have the following important lemma in [3].

Lemma 2.1. *For all $u \in H_0^1(\Omega)$, there exists a unique solution $\phi_u \in H_0^1(\Omega)$ of*

$$\begin{cases} -\Delta\phi = |u|^{2^*-1}, & \text{in } \Omega, \\ \phi = 0, & \text{on } \partial\Omega. \end{cases}$$

Moreover,

(1) $\phi_u \geq 0$ *for* $x \in \Omega$ *and for each* $t > 0$, $\phi_{tu} = t^{2^*-1}\phi_u$.

(2)
$$\int_\Omega |\nabla\phi_u|^2 dx = \int_\Omega \phi_u |u|^{2^*-1} dx \leq S^{-2^*}\|u\|^{2(2^*-1)}.$$

(3) *If* $u_n \rightharpoonup u$ *in* $H_0^1(\Omega)$, *then*

$$\int_\Omega \phi_{u_n}|u_n|^{2^*-1}dx - \int_\Omega \phi_{u_n-u}|u_n-u|^{2^*-1}dx = \int_\Omega \phi_u|u|^{2^*-1}dx + o_n(1).$$

Lemma 2.2. *There exist constants $\delta, \rho, \Lambda_0 > 0$, for all $\lambda \in (0, \Lambda_0)$ such that the functional I_λ satisfies the following conditions:*

(i) $I_\lambda|_{u\in S_\rho} \geq \delta > 0$; $\inf_{u\in B_\rho} I_\lambda(u) < 0$.

(ii) *There exists $e \in H_0^1(\Omega)$ with $\|e\| > \rho$ such that $I_\lambda(e) < 0$.*

Proof. (i) Using the Hölder inequality and the Sobolev inequality, we get

$$\int_\Omega |u|^p dx \leq \left(\int_\Omega |u|^{2^*}dx\right)^{\frac{p}{2^*}} \left(\int_\Omega 1^{\frac{2^*}{2^*-p}}dx\right)^{\frac{2^*-p}{2^*}} \leq |\Omega|^{\frac{2^*-p}{2^*}} S^{-\frac{p}{2}}\|u\|^p. \tag{2.1}$$

Therefore, it follows from (2.1) and the Sobolev inequality that

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{b}{2\theta}\left(\int_\Omega |\nabla u|^2 dx\right)^\theta - \frac{1}{2(2^*-1)}\int_\Omega \phi_u|u|^{2^*-1}dx - \frac{\lambda}{p}\int_\Omega |u|^p dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{2(2^*-1)}S^{-2^*}\|u\|^{2(2^*-1)} - \frac{\lambda}{p}|\Omega|^{\frac{2^*-p}{2^*}}S^{-\frac{p}{2}}\|u\|^p \\ &= \|u\|^p\left(\frac{1}{2}\|u\|^{2-p} - \frac{1}{2(2^*-1)}S^{-2^*}\|u\|^{2(2^*-1)-p} - \frac{\lambda}{p}|\Omega|^{\frac{2^*-p}{2^*}}S^{-\frac{p}{2}}\right). \end{aligned}$$

Let $H(t) = \frac{1}{2}t^{2-p} - \frac{1}{2(2^*-1)}S^{-2^*}t^{2(2^*-1)-p}$ for $t > 0$, thus, there exists a constant

$$\rho = \left[\frac{(2^*-1)(2-p)S^{2^*}}{(2(2^*-1)-p)}\right]^{\frac{1}{2(2^*-2)}} > 0$$

such that $\max_{t>0} h(t) = h(\rho) > 0$. Setting $\Lambda_0 = \frac{pS^{\frac{p}{2}}}{|\Omega|^{\frac{2^*-p}{2^*}}}h(\rho)$, there exists a constant $\delta > 0$ such that $I_\lambda|_{u\in S_\rho} \geq \delta$ for each $\lambda \in (0, \Lambda_0)$. Moreover, for every $u \in H_0^1(\Omega)\backslash\{0\}$, we get

$$\lim_{t\to 0^+} \frac{I_\lambda(tu)}{t^p} = -\frac{\lambda}{p}\int_\Omega |u|^p dx < 0.$$

So we obtain $I_\lambda(tu) < 0$ for all $u \neq 0$ and tu small enough. Hence, for $\|u\|$ small enough, we have

$$m \triangleq \inf_{u \in B_\rho} I_\lambda(u) < 0.$$

(ii) Set $u \in H_0^1(\Omega)$, for all $t > 0$, we get

$$I_\lambda(tu) = \frac{t^2}{2}\|u\|^2 + \frac{bt^{2\theta}}{2\theta}\|u\|^{2\theta} - \frac{t^{2(2^*-1)}}{2(2^*-1)}\int_\Omega \phi_u |u|^{2^*-1}dx - \frac{\lambda t^p}{p}\int_\Omega |u|^p dx \to -\infty$$

as $t \to \infty$, which implies that $I_\lambda(tu) < 0$ for $t > 0$ large enough. Consequently, we can find $e \in H_0^1(\Omega)$ with $\|e\| > \rho$ such that $I_\lambda(e) < 0$. The proof is complete. □

Definition 2.3. A sequence $\{u_n\} \subset H_0^1(\Omega)$ is called $(PS)_c$ sequence of I_λ if $I_\lambda(u_n) \to c$ and $I'_\lambda(u_n) \to 0$ as $n \to \infty$. We say that I_λ satisfies $(PS)_c$ condition if every $(PS)_c$ sequence of I_λ has a convergent subsequence in $H_0^1(\Omega)$.

Lemma 2.4. *Assume that $1 < \theta < \frac{N+2}{N-2}$ and $1 < p < 2$, the functional I_λ satisfies the $(PS)_c$ condition for each $c < c_* = \frac{2}{N+2}S^{\frac{N}{2}} - D\lambda^{\frac{2}{2-p}}$, where $D = \frac{[2(2^*-1)-p]^{\frac{2}{2-p}}}{2(2^*-1)(2^*-2)^{\frac{p}{2-p}}p^{\frac{2}{2-p}}}(S^{-\frac{p}{2}}|\Omega|^{\frac{2^*-p}{2^*}})^{\frac{2}{2-p}}$.*

Proof. Let $\{u_n\} \subset H_0^1(\Omega)$ be a (PS) sequence for I_λ at the level c, that is

$$I_\lambda(u_n) \to c \quad \text{and} \quad I'_\lambda(u_n) \to 0 \quad \text{as } n \to \infty. \tag{2.2}$$

Combining with (2.1) and (2.2), we have

$$\begin{aligned} c + 1 + o(\|u_n\|) &\geq I_\lambda(u_n) - \frac{1}{2(2^*-1)}\langle I'_\lambda(u_n), u_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{2(2^*-1)}\right)\|u_n\|^2 + b\left(\frac{1}{2\theta} - \frac{1}{2(2^*-1)}\right)\|u_n\|^{2\theta} \\ &\quad - \lambda\left(\frac{1}{p} - \frac{1}{2(2^*-1)}\right)|\Omega|^{\frac{2^*-p}{2^*}}S^{-\frac{p}{2}}\|u_n\|^p \\ &\geq \left(\frac{1}{2} - \frac{1}{2(2^*-1)}\right)\|u_n\|^2 - \lambda\left(\frac{1}{p} - \frac{1}{2(2^*-1)}\right)|\Omega|^{\frac{2^*-p}{2^*}}S^{-\frac{p}{2}}\|u_n\|^p. \end{aligned}$$

Therefore $\{u_n\}$ is bounded in $H_0^1(\Omega)$ for all $1 < p < 2$. Thus, we may assume up to a subsequence, still denoted by $\{u_n\}$, that there exists $u \in H_0^1(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u, & \text{weakly in } H_0^1(\Omega), \\ u_n \to u, & \text{strongly in } L^q(\Omega)\ (1 \leq q < 2^*), \\ u_n(x) \to u(x), & \text{a.e. in } \Omega, \end{cases} \tag{2.3}$$

as $n \to \infty$. By (2.1) and the Young inequality, one has

$$\lambda\int_\Omega |u|^p dx \leq \lambda S^{-\frac{p}{2}}|\Omega|^{\frac{2^*-p}{2^*}}\|u\|^p \leq \eta\|u\|^2 + C(\eta)\lambda^{\frac{2}{2-p}}, \tag{2.4}$$

where $C(\eta) = \eta^{-\frac{p}{2-p}}(S^{-\frac{p}{2}}|\Omega|^{\frac{2^*-p}{2^*}})^{\frac{2}{2-p}}$, it follows from (2.2) and (2.4) that

$$\begin{aligned} I_\lambda(u) &= I_\lambda(u) - \frac{1}{2(2^*-1)}\langle I'_\lambda(u), u\rangle \\ &\geq \left(\frac{1}{2} - \frac{1}{2(2^*-1)}\right)\|u\|^2 - \left(\frac{1}{p} - \frac{1}{2(2^*-1)}\right)\lambda\int_\Omega |u|^p dx \\ &\geq \left(\frac{2^*-2}{2(2^*-1)} - \frac{2(2^*-1)-p}{2(2^*-1)p}\eta\right)\|u\|^2 - \frac{2(2^*-1)-p}{2(2^*-1)p}C(\eta)\lambda^{\frac{2}{2-p}}. \end{aligned}$$

Letting $\eta = \frac{p(2^*-2)}{2(2^*-1)-p}$ and $D = \frac{[2(2^*-1)-p]^{\frac{2}{2-p}}}{2(2^*-1)(2^*-2)^{\frac{p}{2-p}} p^{\frac{2}{2-p}}}(S^{-\frac{p}{2}}|\Omega|^{\frac{2^*-p}{2^*}})^{\frac{2}{2-p}}$, we have $I_\lambda(u) \geq -D\lambda^{\frac{2}{2-p}}$. Next, we prove that $u_n \to u$ strongly in $H_0^1(\Omega)$. Set $w_n = u_n - u$ and $\lim_{n\to\infty} \|w_n\| = l$, by using the Brézis–Lieb lemma [9], we have

$$\|u_n\|^2 = \|w_n\|^2 + \|u\|^2 + o(1),$$
$$\|u_n\|^{2\theta} = \left(\|w_n\|^2 + \|u\|^2 + o(1)\right)^\theta,$$
$$\int_\Omega \phi_{u_n}|u_n|^{2^*-1}dx = \int_\Omega \phi_{w_n}|w_n|^{2^*-1}dx + \int_\Omega \phi_u|u|^{2^*-1}dx + o(1).$$

From (2.2), (2.3) and Lemma 2.1, one has

$$\|w_n\|^2 + \|u\|^2 + b\left(\|w_n\|^2 + \|u\|^2 + o(1)\right)^\theta - \int_\Omega \phi_{w_n}|w_n|^{2^*-1}dx - \int_\Omega \phi_u|u|^{2^*-1}dx - \lambda\int_\Omega |u|^p dx = o(1), \tag{2.5}$$

and

$$\|u\|^2 + b\|u\|^{2\theta} - \int_\Omega \phi_u|u|^{2^*-1}dx - \lambda\int_\Omega |u|^p dx = 0. \tag{2.6}$$

It follows from (2.5) and (2.6) that

$$\|w_n\|^2 + b\left[\left(\|w_n\|^2 + \|u\|^2 + o(1)\right)^\theta - \|u\|^{2\theta}\right] - \int_\Omega \phi_{w_n}|w_n|^{2^*-1}dx = o(1). \tag{2.7}$$

Since $\|w_n\| \to l$, we have

$$\left(\|w_n\|^2 + \|u\|^2 + o(1)\right)^\theta - \|u\|^{2\theta} \to \left(l^2 + \|u\|^2 + o(1)\right)^\theta - \|u\|^{2\theta} = l_1 \geq 0, \quad \text{as } n \to \infty.$$

If follows from (2.7) that

$$\int_\Omega \phi_{w_n}|w_n|^{2^*-1}dx \to l^2 + bl_1.$$

Applying the Sobolev inequality, we get

$$\|w_n\|^{2(2^*-1)} \geq S^{2^*}\int_\Omega \phi_{w_n}|w_n|^{2^*-1}dx + o(1). \tag{2.8}$$

Thus, by (2.8), we can deduce that

$$l^{2(2^*-1)} \geq S^{2^*}(l^2 + bl_1) \geq S^{2^*}l^2 \quad \text{as } n \to \infty,$$

which implies that $l \geq S^{\frac{N}{4}}$ as $n \to \infty$. Since $I(u_n) = c + o(1)$, we obtain

$$\frac{1}{2}\|w_n\|^2 + \frac{b}{2\theta}[(\|w_n\|^2 + \|u\|^2 + o(1))^\theta - \|u\|^{2\theta}] - \frac{1}{2(2^*-1)}\int_\Omega \phi_{w_n}|w_n|^{2^*-1}dx = c - I_\lambda(u) + o(1).$$

Hence, there holds

$$c = \left(\frac{1}{2} - \frac{1}{2(2^*-1)}\right)l^2 + \left(\frac{1}{2\theta} - \frac{1}{2(2^*-1)}\right)bl_1 + I_\lambda(u)$$
$$\geq \frac{2}{N+2}S^{\frac{N}{2}} - D\lambda^{\frac{2}{2-p}} \geq c_*,$$

as $n \to \infty$. This is a contradiction. Hence, we can conclude that $u_n \to u$ in $H_0^1(\Omega)$. The proof is complete. □

Choose the extremal function

$$U_\varepsilon(x) = \frac{[N(N-2)\varepsilon^2]^{\frac{N-2}{4}}}{(\varepsilon^2+|x|^2)^{\frac{N-2}{2}}}, \qquad x \in \mathbb{R}^N,\ \varepsilon > 0.$$

It is a positive solution of the following problem

$$-\Delta U_\varepsilon = U_\varepsilon^{2^*-1} \quad \text{in } \mathbb{R}^N,$$

and satisfies

$$\int_{\mathbb{R}^N} |\nabla U_\varepsilon|^2 dx = \int_{\mathbb{R}^N} |U_\varepsilon|^{2^*} dx = S^{\frac{N}{2}}.$$

Pick a cut-off function $\varphi \in C_0^\infty(\Omega)$ such that $\varphi(x) = 1$ on $B(0, \frac{r}{2})$, $\varphi(x) = 0$ on $\mathbb{R}^N - B(0,r)$ and $0 \le \varphi(x) \le 1$ on $\mathbb{R}^N$. Set $u_\varepsilon(x) = \varphi(x)U_\varepsilon(x)$, from [8], we have

$$\begin{cases} \int_\Omega |\nabla u_\varepsilon|^2 dx = S^{\frac{N}{2}} + O(\varepsilon^{N-2}), \\ \int_\Omega |u_\varepsilon|^{2^*} dx = S^{\frac{N}{2}} + O(\varepsilon^N). \end{cases} \tag{2.9}$$

To estimate the value c observe that, multiplying the second equation of system (1.1) by $|u|$ and integrating, we get

$$\int_\Omega |u|^{2^*} dx = \int_\Omega \nabla\phi_u \nabla|u| dx \le \frac{1}{2}\|\phi_u\|^2 + \frac{1}{2}\|u\|^2. \tag{2.10}$$

Then, we define a new functional $H_\lambda : H_0^1(\Omega) \to \mathbb{R}$ by

$$\begin{aligned} H_\lambda(u) &\triangleq \frac{2^*}{2(2^*-1)}\|u\|^2 + \frac{b}{2\theta}\|u\|^{2\theta} - \frac{1}{2^*-1}\int_\Omega |u|^{2^*}dx - \frac{\lambda}{p}\int_\Omega |u|^p dx \\ &= \frac{2^*}{2^*-1}\left[\frac{1}{2}\|u\|^2 + \frac{(2^*-1)b}{2\theta 2^*}\|u\|^{2\theta} - \frac{1}{2^*}\int_\Omega |u|^{2^*}dx - \lambda\frac{2^*-1}{2^*p}\int_\Omega |u|^p dx\right] \\ &\triangleq \frac{2^*}{2^*-1}J_\lambda(u), \end{aligned}$$

where

$$J_\lambda(u) = \frac{1}{2}\|u\|^2 + \frac{(2^*-1)b}{2\theta 2^*}\|u\|^{2\theta} - \frac{1}{2^*}\int_\Omega |u|^{2^*}dx - \lambda\frac{2^*-1}{2^*p}\int_\Omega |u|^p dx.$$

By (2.10), which implies that

$$I_\lambda(u) \le H_\lambda(u) = \frac{2^*}{2^*-1}J_\lambda(u), \tag{2.11}$$

for every $u \in H_0^1(\Omega)$, and $c \le \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \sup_{t\ge 0} J_\lambda(tu)$. If we consider the following problem

$$\begin{cases} -\left[1 + \frac{(2^*-1)b}{2^*}\left(\int_\Omega |\nabla u|^2 dx\right)^{\theta-1}\right]\Delta u = |u|^{2^*-2}u + \lambda\frac{2^*-1}{2^*}|u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.12}$$

Then we find that the weak solution of problem (2.12) correspond to the critical points of the functional J_λ. Next, we compute $\sup_{t\ge 0} J_\lambda(tu_\varepsilon) = J_\lambda(t_\varepsilon u_\varepsilon)$.

Lemma 2.5. *Assume that* $1 < \theta < \frac{N+2}{N-2}$, $\frac{N}{N-2} < p < 2$ *and* $N > 4$, *then there exist* $\Lambda_3, b_0 > 0$ *such that for all* $\lambda \in (0, \Lambda_3)$ *and* $b \in (0, b_0)$, *it holds*

$$\sup_{t\geq 0} J_\lambda(tu_\varepsilon) < \frac{1}{N}S^{\frac{N}{2}} - \frac{N+2}{2N}D\lambda^{\frac{2}{2-p}}.$$

In particular,

$$\sup_{t\geq 0} I_\lambda(tu_\varepsilon) < \frac{2}{N+2}S^{\frac{N}{2}} - D\lambda^{\frac{2}{2-p}}.$$

Proof. For convenience, we consider the functional $J_b^* : H_0^1(\Omega) \to \mathbb{R}$ as follows

$$J_b^*(u) = \frac{1}{2}\|u\|^2 + \frac{(2^*-1)b}{2\theta 2^*}\|u\|^{2\theta} - \frac{1}{2^*}\int_\Omega |u|^{2^*}dx.$$

Define

$$h_b(t) = J_b^*(tu_\varepsilon) = \frac{t^2}{2}\|u_\varepsilon\|^2 + \frac{(2^*-1)bt^{2\theta}}{2\theta 2^*}\|u_\varepsilon\|^{2\theta} - \frac{t^{2^*}}{2^*}\int_\Omega |u_\varepsilon|^{2^*}dx, \quad \text{for all } t \geq 0.$$

It is clear that $\lim_{t\to 0} h_b(t) = 0$ and $\lim_{t\to\infty} h_b(t) = -\infty$. Therefore there exists $t_{b,\varepsilon} > 0$ such that $h(t_{b,\varepsilon}) = \max_{t\geq 0} h_b(t)$, that is

$$0 = h_0'(t_{0,\varepsilon}) = t_{0,\varepsilon}\left(\|u_\varepsilon\|^2 - t_{0,\varepsilon}^{2^*-2}\int_\Omega |u_\varepsilon|^{2^*}dx\right),$$

one has

$$t_{0,\varepsilon} = \left(\frac{\|u_\varepsilon\|^2}{\int_\Omega |u_\varepsilon|^{2^*}dx}\right)^{\frac{1}{2^*-2}}.$$

Hence, we deduce from (2.9) that

$$\begin{aligned}\sup_{t\geq 0} J_b^*(tu_\varepsilon) &= h_b(t_{b,\varepsilon}u_\varepsilon) \leq h_0(t_{b,\varepsilon}u_\varepsilon) \leq h_0(t_{0,\varepsilon}u_\varepsilon) \\ &= \frac{1}{N}S^{\frac{N}{2}} - \frac{N+2}{2N}D\lambda^{\frac{2}{2-p}} + \frac{N+2}{2N}D\lambda^{\frac{2}{2-p}} + O(\varepsilon^{N-2}).\end{aligned} \tag{2.13}$$

By using the definitions of J and u_ε, we have

$$J_\lambda(tu_\varepsilon) \leq \frac{t^2}{2}\|u_\varepsilon\|^2 + \frac{b(2^*-1)t^{2\theta}}{2\theta 2^*}\|u_\varepsilon\|^{2\theta},$$

for all $t \geq 0$ and $\lambda > 0$. It follows from (2.9) that there exist $T \in (0,1)$, Λ_1, $b_0 > 0$ and $\varepsilon_1 > 0$ such that

$$\sup_{0\leq t\leq T} J_\lambda(tu_\varepsilon) < \frac{1}{N}S^{\frac{N}{2}} - \frac{N+2}{2N}D\lambda^{\frac{2}{2-p}},$$

for every $0 < \lambda < \Lambda_1$, $0 < b < b_0$ and $0 < \varepsilon < \varepsilon_1$. According to the definition of u_ε, there exists $C_1 > 0$, such that we have

$$\begin{aligned}\int_\Omega |u_\varepsilon|^p dx &\geq C \int_{B_{r/2}(0)} \frac{\varepsilon^{\frac{p(N-2)}{2}}}{(\varepsilon^2+|x|^2)^{\frac{p(N-2)}{2}}} dx \\ &= C\varepsilon^{\frac{p(N-2)}{2}} \int_0^{r/2} \frac{t^{N-1}}{(\varepsilon^2+t^2)^{\frac{p(N-2)}{2}}} dt \\ &= C\varepsilon^{N-\frac{p(N-2)}{2}} \int_0^{r/2\sqrt{\varepsilon}} \frac{y^{N-1}}{(1+y^2)^{\frac{p(N-2)}{2}}} dy \\ &\geq C\varepsilon^{N-\frac{p(N-2)}{2}} \int_0^{1} \frac{y^{N-1}}{(1+y^2)^{\frac{p(N-2)}{2}}} dy \\ &\geq C_1\varepsilon^{N-\frac{p(N-2)}{2}}.\end{aligned} \tag{2.14}$$

Thus, it follows from (2.13) and (2.14) that

$$\begin{aligned}\sup_{t\geq T} J(tu_\varepsilon) &= \sup_{t\geq T}\left(J_b(tu_\varepsilon) - \lambda\frac{2^*-1}{2^*p}t^p\int_\Omega |u_\varepsilon|^p dx\right) \\ &\leq \frac{1}{N}S^{\frac{N}{2}} - \frac{N+2}{2N}D\lambda^{\frac{2}{2-p}} \\ &\quad + \frac{N+2}{2N}D\lambda^{\frac{2}{2-p}} + C_2\varepsilon^{N-2} - C_1\lambda\varepsilon^{N-\frac{p(N-2)}{2}} \\ &< \frac{1}{N}S^{\frac{N}{2}} - \frac{N+2}{2N}D\lambda^{\frac{2}{2-p}},\end{aligned} \tag{2.15}$$

where the constant $C_2 > 0$. Here we have used the fact that $\frac{N}{N-2} < p < 2$ and $\frac{(N-2)(2-2p)+2N}{(N-2)(2-p)} < \frac{2}{2-p}$, let $\varepsilon = \lambda^{\frac{2}{(N-2)(2-p)}}$, $0 < \lambda < \Lambda_2 = \min\{1, (\frac{C_1}{C_3})^{\frac{(N-2)(2-p)}{2p(N-2)-2N}}\}$, then

$$\begin{aligned}\frac{N+2}{2N}D\lambda^{\frac{2}{2-p}} + C_2\varepsilon^{N-2} - C_1\lambda\varepsilon^{N-\frac{p(N-2)}{2}} &\leq C_3\lambda^{\frac{2}{2-p}} - C_1\lambda\varepsilon^{N-\frac{p(N-2)}{2}} \\ &= C_3\lambda^{\frac{2}{2-p}} - C_1\lambda^{\frac{(N-2)(2-2p)+2N}{(N-2)(2-p)}} \\ &< 0,\end{aligned} \tag{2.16}$$

where $C_3 > 0$. Therefore, we have

$$\sup_{t\geq 0} J_\lambda(tu_\varepsilon) < \frac{1}{N}S^{\frac{N}{2}} - \frac{N+2}{2N}D\lambda^{\frac{2}{2-p}},$$

for all $0 < \lambda < \Lambda_3 = \min\{\Lambda_1, \Lambda_2, \varepsilon_1\}$ and $0 < b < b_0$. The proof is complete. □

Theorem 2.6. *Assume that* $0 < \lambda < \Lambda_0$ *(*Λ_0 *is as in Lemma 2.2). Then system* (1.1) *has a positive solution* u_λ *satisfying* $I_\lambda(u_\lambda) < 0$.

Proof. Applying Lemma 2.2, we have

$$m \triangleq \inf_{u\in\overline{B_\rho(0)}} I_\lambda(u) < 0.$$

By the Ekeland variational principle [12], there exists a minimizing sequence $\{u_n\} \subset \overline{B_\rho(0)}$ such that

$$I_\lambda(u_n) \leq \inf_{u\in\overline{B_\rho(0)}} I_\lambda(u) + \frac{1}{n}, \qquad I_\lambda(v) \geq I_\lambda(u_n) - \frac{1}{n}\|v-u_n\|, \qquad v \in \overline{B_\rho(0)}.$$

Thus, we obtain that $I_\lambda(u_n) \to m$ and $I'_\lambda(u_n) \to 0$. By Lemma 2.4, we have $u_n \to u_\lambda$ in $H_0^1(\Omega)$ with $I_\lambda(u_n) \to m < 0$, which implies that $u_\lambda \not\equiv 0$. Note that $I_\lambda(u_n) = I_\lambda(|u_n|)$, we have $u_\lambda \geq 0$. Then, by using the strong maximum principle, we obtain that u_λ is a positive solution of system (1.1) such that $I_\lambda(u_\lambda) < 0$. □

Theorem 2.7. *Assume that* $0 < \lambda < \Lambda_*(\Lambda_* = \min\{\Lambda_0, \Lambda_3\})$. *Then the system* (1.1) *has a positive solution* $u_* \in H_0^1(\Omega)$ *with* $I_\lambda(u_*) > 0$.

Proof. According to the mountain pass theorem [1] and Lemma 2.2, there exists a sequence $\{u_n\} \subset H_0^1(\Omega)$ such that

$$I_\lambda(u_n) \to c > 0 \quad \text{and} \quad I'_\lambda(u_n) \to 0 \quad \text{as } n \to \infty,$$

where

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t)),$$

and

$$\Gamma = \left\{ \gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0, \gamma(1) = e \right\}.$$

From Lemma 2.4, we know that $\{u_n\} \subset H_0^1(\Omega)$ has a convergent subsequence, still denoted by $\{u_n\}$, such that $u_n \to u_*$ in $H_0^1(\Omega)$ as $n \to \infty$,

$$I_\lambda(u_*) = \lim_{n\to\infty} I_\lambda(u_n) = c > 0,$$

which implies that $u_* \not\equiv 0$. It is similar to Theorem 2.6 that $u_* > 0$, we obtain that u_* is a positive solution of system (1.1) such that $I_\lambda(u_*) > 0$. Combining the above facts with Theorem 2.6 the proof of Theorem 1.1 is complete. □

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Ground states solutions for some non-autonomous Schrödinger–Bopp–Podolsky system

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Abstract. In this paper we study the existence of ground states solutions for non-autonomous Schrödinger–Bopp–Podolsky system

$$\begin{cases} -\Delta u + u + \lambda K(x)\phi u = b(x)|u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta \phi + a^2\Delta^2\phi = 4\pi K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $\lambda > 0, 2 < p \leq 4$ and both $K(x)$ and $b(x)$ are nonnegative functions in $\mathbb{R}^3$. Assuming that $\lim_{|x|\to+\infty}K(x) = K_\infty > 0$ and $\lim_{|x|\to+\infty}b(x) = b_\infty > 0$ and satisfying suitable assumptions, but not requiring any symmetry property on them. We show that the existence of a positive solution depends on the parameters λ and p. We also establish the existence of ground state solutions for the case $3.18 \approx \frac{1+\sqrt{73}}{3} < p \leq 4$.

Keywords: non-autonomous Schrödinger–Bopp–Podolsky system, variational methods, Pohožaev identity, Nehari manifold.

2020 Mathematics Subject Classification: 35J48, 35J50, 35Q60.

1 Introduction and main results

In this paper we are concerned with the existence of ground states for Schrödinger–Bopp–Podolsky (SBP) system

$$\begin{cases} -\Delta u + u + \lambda K(x)\phi u = b(x)|u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta \phi + a^2\Delta^2\phi = 4\pi K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $a > 0$ is the Bopp-Podolsky (BP) parameter, u represents the modulus of the wave function and ϕ the electrostatic situation. The Schrödinger–Bopp–Podolsky system has been studied in [13] for the first time in the mathematical literature. The system appears when one

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looks for stationary solutions $u(x)e^{i\omega t}$ of the Schrödinger equation coupled with the Bopp–Podolsky Lagrangian of the electromagnetic field in the purely electrostatic situation.

The Bopp–Podolsky theory is a second order for the electromagnetic field, and was proposed to deal with the so called infinity problem that appears in the classical Maxwell theory which is similar to the Mie theory [21] and its generalizations given by Born and Infeld [3–6]. In fact, by the well-known Gauss law (or Poisson's equation), the electrostatic potential ϕ for a given charge distribution whose density is ρ satisfies the equation

$$-\Delta\phi = \rho \quad \text{in } \mathbb{R}^3. \tag{1.2}$$

If $\rho = 4\pi\delta_{x_0}$, with $x_0 \in \mathbb{R}^3$, the fundamental solution of (1.2) is $\mathcal{G}(x - x_0)$, where

$$\mathcal{G}(x) = \frac{1}{|x|},$$

and the electrostatic energy is

$$\mathcal{E}_M(\mathcal{G}) = \frac{1}{2}\int_{R^3} |\nabla\mathcal{G}|^2 = +\infty.$$

Thus, equation (1.2) is replaced by

$$-\operatorname{div}\left(\frac{\nabla\phi}{\sqrt{1-|\nabla\phi|^2}}\right) = \rho \quad \text{in } \mathbb{R}^3$$

in the Bopp–Infeld theory and by

$$-\Delta\phi + a^2\Delta^2\phi = \rho \quad \text{in } \mathbb{R}^3,$$

in the Bopp–Podolsky theory. In both cases, if $\rho = 4\pi\delta_{x_0}$, their solutions can be written explicitly, and the corresponding energy is finite. In this paper, we focus on the Bopp–Podolsky theory $-\Delta + a^2\Delta^2$, the fundamental solution of the equation

$$-\Delta\phi + a^2\Delta^2\phi = 4\pi\delta_{x_0}$$

is $\mathcal{L}(x - x_0)$, where

$$\mathcal{L}(x) := \frac{1 - e^{-\frac{|x|}{a}}}{|x|},$$

which presents no singularities at x_0, since

$$\lim_{x \to x_0} \mathcal{L}(x - x_0) = \frac{1}{a}.$$

Furthermore, its energy is

$$\mathcal{E}_{\mathrm{BP}}(\mathcal{L}) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\mathcal{L}|^2 \,\mathrm{d}x + \frac{a^2}{2}\int_{\mathbb{R}^3} |\Delta\mathcal{L}|^2 \,\mathrm{d}x < \infty.$$

We refer to [13] for more details.

In recent years, there has been increasing attention to problems like (1.1) on the existence of positive solutions, ground state solutions, multiple solutions and normalized solutions, see e.g. [1, 10, 16, 18–20, 27] and the references therein. According to [25], we know that there are two parameters $K(x)$ and $b(x)$ have an effect on the nonlocal term and nonlinear term. Hence, we take advantage of the idea of [25]. And we know that a typical way to deal with (1.1) is to use Nehari manifold and variational methods. In this paper, we mainly solve the Pohožaev identity of (1.1), because the non-local terms and nonlinear terms are affected by $K(x)$ and $b(x)$. It has not been studied before.

Then we are concerned with existence of ground states for following generalized nonlinear system in $\mathbb{R}^3$

$$\begin{cases} -\Delta u + u + \lambda K(x)\phi u = b(x)|u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi K(x)u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{1.3}$$

It is known that system (1.3) can be transformed into a nonlinear Schrödinger equation with a non-local term, for example, see [2,11,24]. Then we can use the same method as in [13] to find the solution of the second equation of the system (1.3). For all $u \in H^1(\mathbb{R}^3)$, the unique $\phi_{K,u} \in \mathcal{D}$ (where $\mathcal{D}$ is a function space that will be introduced in Section 2) is given by

$$\phi_{K,u}(x) = \int_{\mathbb{R}^3} \frac{1 - e^{-\frac{|x-y|}{a}}}{|x-y|} K(y)u^2(y)dy,$$

such that $-\Delta\phi + a^2\Delta^2\phi = 4\pi K(x)u^2$ and that, substituting it into the first equation of system (1.3), gives

$$-\Delta u + u + \lambda K(x)\phi_{K,u}u = b(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^3. \tag{1.4}$$

Equation (1.4) has solutions are the critical points of functional $\mathcal{J}(u)$ defined in $H^1(\mathbb{R}^3)$ as

$$\mathcal{J}(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2\right) dx + \frac{\lambda}{4}\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2 dx - \frac{1}{p}\int_{\mathbb{R}^3} b(x)|u|^p dx. \tag{1.5}$$

Furthermore, one can see that $\mathcal{J}$ is a C^1 functional with the derivative given by

$$\langle \mathcal{J}'(u), \varphi \rangle = \int_{\mathbb{R}^3} \left(\nabla u \nabla\varphi + u\varphi + \lambda K(x)\phi_{K,u}u\varphi - b(x)|u|^{p-2}u\varphi\right) dx$$

for all $\varphi \in H^1(\mathbb{R}^3)$, where $\mathcal{J}'$ denotes the Fréchet derivative of $\mathcal{J}$. We say that a pair $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}$ is a solution of system (1.3) if and only if u is a critical point of $\mathcal{J}$. Furthermore, for system(1.3), we find that the corresponding Pohožaev identity (see section 6 for more details) is

$$\begin{aligned} 0 = &\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{3}{2}\int_{\mathbb{R}^3} u^2 dx + \frac{5\lambda}{4}\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2 dx \\ &+ \frac{\lambda}{2}\int_{\mathbb{R}^3} \langle \nabla K(x), x\rangle \phi_{K,u}u^2 dx + \frac{\lambda}{4a}\int_{\mathbb{R}^3} K(x)\psi_{K,u}u^2 dx \\ &- \frac{3}{p}\int_{\mathbb{R}^3} b(x)|u|^p dx - \frac{1}{p}\int_{\mathbb{R}^3} \langle \nabla b(x), x\rangle |u|^p dx, \end{aligned}$$

where $K(x), b(x) \in C^1(\mathbb{R}^3)$ and $\psi_{K,u} := e^{-\frac{|x|}{a}} * Ku^2 = \int_{\mathbb{R}^3} e^{-\frac{|x-y|}{a}} K(y)u^2(y)dy$. It appears that the Pohožaev identity of the non-autonomous case looks more complicated than that of the autonomous case [24].

Therefore, we introduce a new set that can be seen as the filtration of the Nehari manifold. That is

$$\mathcal{N}(c_\tau) = \{u \in \mathcal{N} : \mathcal{J}(u) < c_\tau\},$$

where $\mathcal{N} = \{u \in H^1(\mathbb{R}^3)\backslash\{0\} : \langle \mathcal{J}'(u), u\rangle = 0\}$ (we can see in [22]) is the Nehari manifold and c_τ is the energy level of the functional $\mathcal{J}$. Apparently, $\mathcal{N}(c_\tau)$ is a subset of the Nehari manifold. We will show that $\mathcal{N}(c_\tau)$ can be divided into two parts

$$\mathcal{N}^{(1)}(c_\tau) = \{u \in \mathcal{N}(c_\tau) : \|u\| < C_1\} \quad \text{and} \quad \mathcal{N}^{(2)}(c_\tau) = \{u \in \mathcal{N}(c_\tau) : \|u\| > C_2\},$$

where each local minimizer of the functional $\mathcal{J}$ is a critical point of $\mathcal{J}$ in $H^1(\mathbb{R}^3)$. The approach we take is to minimize the energy functional $\mathcal{J}$ on the $\mathcal{N}^{(1)}(c_\tau)$, where the $\mathcal{J}$ is bounded below and the minimization sequence is bounded.

This paper gives the following assumptions about $b(x)$ and $K(x)$:

(G_1) $b(x)$ is a positive continuous function on $\mathbb{R}^3$ such that

$$\lim_{|x|\to\infty} b(x) = b_\infty > 0 \quad \text{uniformly on } \mathbb{R}^3,$$

and

$$b_{\max} := \sup_{x\in\mathbb{R}^3} b(x) < \frac{b_\infty}{A(p)^{\frac{p-2}{2}}},$$

where

$$A(p) = \begin{cases} \left(\frac{4-p}{2}\right)^{\frac{1}{p-2}} & \text{if } 2 < p \leq 3, \\ \frac{1}{2} & \text{if } 3 < p \leq 4. \end{cases}$$

(G_2) $K(x) \in L^\infty(\mathbb{R}^3)\backslash\{0\}$ is a non-negative function on $\mathbb{R}^3$ such that

$$\lim_{|x|\to\infty} K(x) = K_\infty \geq 0 \quad \text{uniformly on } \mathbb{R}^3.$$

Remark 1.1. A direct calculation shows that for $2 < p \leq 4$, there holds

$$A(p) < \frac{1}{\sqrt{e}} < 1 \quad \text{and} \quad A(p)\left(\frac{2}{4-p}\right)^{\frac{2}{p-2}} > 1.$$

Let w_0 be tha unique positive solution of the following Schrödinger equation

$$-\triangle u + u = b_\infty |u|^{p-2}u \quad \text{in } \mathbb{R}^3. \tag{1.6}$$

Available from [17]

$$w_0(0) = \max_{x\in\mathbb{R}^3} w_0(x),$$

$$\|w_0\|^2 = \int_{\mathbb{R}^3} b_\infty |w_0|^p dx = \left(\frac{S_p^p}{b_\infty}\right)^{\frac{2}{p-2}}, \tag{1.7}$$

and

$$\alpha_\infty^0 := \inf_{u\in\mathcal{M}_\infty^0} \mathcal{J}_0^\infty(u) = \frac{p-2}{2p}\left(\frac{S_p^p}{b_\infty}\right)^{\frac{2}{p-2}},$$

where $\mathcal{J}_0^\infty$ is the energy functional of equation (1.6) in $H^1(\mathbb{R}^3)$ in the form

$$\mathcal{J}_0^\infty(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + u^2)dx - \frac{1}{p}\int_{\mathbb{R}^3} b_\infty |u|^p dx, \tag{1.8}$$

with

$$\mathcal{M}_\infty^0 = \{u \in H^1(\mathbb{R}^3)\backslash\{0\} \mid \langle (\mathcal{J}_0^\infty)'(u), u\rangle = 0\}.$$

Definition 1.2. (u,ϕ) is called a ground state solution of system (1.3) , if (u,ϕ) is a solution of system (1.3) which has the least energy among all nontrivial solutions of system (1.3) .

Now, we give our main results as follows.

Theorem 1.3. *Suppose that $2 < p \le 4, K(x) \equiv K_\infty > 0$ and $b(x) \equiv b_\infty > 0$. Then for each $0 < \lambda < \Lambda$, system (1.3) has a positive solution $(w, \phi_{K_\infty,w}) \in H^1(\mathbb{R}^3) \times \mathcal{D}$, and when $2 < p < 4$ it satisfies*

$$0 < \|w\| < \left(\frac{2S_p^p}{b_\infty(4-p)}\right)^{\frac{1}{p-2}},$$

and

$$\alpha_\infty^0 < \alpha_\infty^- := \mathcal{J}(w) < \frac{A(p)(p-2)}{2p}\left(\frac{2S_p^p}{b_\infty(4-p)}\right)^{\frac{2}{p-2}}.$$

In particular, when $p = 4$ we have

$$\alpha_\infty^- = \mathcal{J}(w) > \alpha_\infty^0,$$

and $(w, \phi_{K_\infty,w})$ is a ground state solution of system (1.3).

Theorem 1.4. *Suppose that $2 < p \le 4, K_\infty > 0$ and conditions (G_1)–(G_2) hold. Furthermore, we assume that*

(G_3) $\int_{\mathbb{R}^3}[b(x)-b_\infty]w^p dx \ge 0$ and $\int_{\mathbb{R}^3} K(x)\phi_{K,w}w^2 dx \le \int_{\mathbb{R}^3} K_\infty \phi_{K_\infty,w} w^2 dx$, but the equality signs can not hold at the same time, where w is the positive solution as described in Theorem 1.3.

Then for each $0 < \lambda < \Lambda$, system (1.3) has a positive solution $(v, \phi_{K,v}) \in H^1(\mathbb{R}^3) \times \mathcal{D}$, and when $2 < p < 4$ it satisfies

$$0 < \|v\| < \left(\frac{2S_p^p}{b_{\max}(4-p)}\right)^{\frac{1}{p-2}},$$

and

$$\frac{p-2}{4p}\left(\frac{S_p^p}{b_{\max}}\right)^{2/(p-2)} \le \mathcal{J}(v) < \alpha_\infty^- \quad \text{for } 2 < p < 4.$$

In particular, when $p = 4$ we have

$$\frac{1}{4}\left(\frac{S_p^p}{b_{\max}}\right)^{2/(p-2)} \le \mathcal{J}(v) < \alpha_\infty^-,$$

and $(v, \phi_{K,v})$ is a ground state solution of system (1.3).

Theorem 1.5. *Suppose that $\frac{1+\sqrt{73}}{3} < p < 4$ and conditions (G_1)–(G_2) hold. Furthermore, we assume that*

(G_4) the functions $b(x), K(x) \in C^1(\mathbb{R}^3)$ satisfy $\langle \nabla b(x), x\rangle \le 0$ and

$$\frac{3p^2-2p-24}{2(6-p)}K(x) + \frac{p(p-2)}{6-p}\langle \nabla K(x), x\rangle \ge 0.$$

If $(v, \phi_{K,v})$ is the positive solution as described in Theorem 1.4, then $(v, \phi_{K,v})$ is a ground state solution of system (1.3).

The paper is organized as follows. First, we present some notations and the lemma for the later proof in section 2. In Section 3, we give the proof Theorem 1.3. In Section 4, is devoted to proof Theorem 1.4. Section 5 is dedicated to the proof of Theorem 1.5.

2 Notations and preliminaries

We use the following notation:

- $H^1(\mathbb{R}^3)$ is the usual Sobolev space endowed with the standard scalar product and norm

$$\langle u, v\rangle = \int_{\mathbb{R}^3} (\nabla u \nabla v + uv) dx; \qquad \|u\|^2 = \int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2\right) dx.$$

- H^{-1} denotes the dual space of $H^1(\mathbb{R}^3)$.
- $L^p(\Omega), 1 \leq p \leq +\infty, \Omega \subseteq \mathbb{R}^3$, demotes a Lebesgue space, the norm in L^p is denoted by $|u|_{p,\Omega}$ when Ω is a proper subset of $\mathbb{R}^3$, by $|\cdot|_p$ when $\Omega = \mathbb{R}^3$.
- C, C', C_i are various positive constants.
- For any $\theta > 0$ and for any $\xi \in \mathbb{R}^3, B_\theta(\xi)$ denotes the ball of radius θ centered at ξ.
- $\hat{S}$ is the best constant for the embedding of $H^1(\mathbb{R}^3)$ in $L^{\frac{12}{5}}\left(\mathbb{R}^3\right)$.
- $\bar{S}$ is the best Sobolev constant for the embedding of $D^{1,2}\left(\mathbb{R}^3\right)$ in $L^6\left(\mathbb{R}^3\right)$, that is

$$\bar{S} = \inf_{u \in D^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\|u\|_{D^{1,2}}}{|u|_6}.$$

- S_p is the best Sobolev constant for the embedding of $H^1(\mathbb{R}^3)$ is continuously embedded into $L^p(\mathbb{R}^3)$ $(2 \leq p \leq 6)$, that is

$$S_p = \inf_{u \in H^1(\mathbb{R}^3)\setminus\{0\}} \frac{\|u\|}{|u|_p}$$

  where

$$D^{1,2}\left(\mathbb{R}^3\right) := \left\{u \in L^6\left(\mathbb{R}^3\right) : \nabla u \in L^2\left(\mathbb{R}^3\right)\right\}.$$

Then we let

$$\Lambda = \begin{cases} \frac{(p-2)K_\infty^{-2}\bar{S}^2\hat{S}^4}{2(4-p)} \left(\frac{b_\infty(4-p)^2}{2pS_p^p}\right)^{2/(p-2)} & \text{if } 2 < p < 4, \\ b_\infty K_\infty^{-2}\bar{S}^2\hat{S}^4 S_4^{-4} & \text{if } p = 4, \end{cases}$$

and

$$\Lambda_0 = \left[1 - A(p)\left(\frac{b_{\max}}{b_\infty}\right)^{2/(p-2)}\right]\left(\frac{b_\infty}{S_p^p}\right)^{2/(p-2)} \frac{\bar{S}^2\hat{S}^4}{K_{\max}^2}, \tag{2.1}$$

where $K_{\max} = \sup_{x\in\mathbb{R}^3} K(x)$. When $p = 12/5$, we may take $S_{12/5} = \hat{S}$. In particular, if $K(x) \equiv K_\infty$ and $b(x) \equiv b_\infty$, then equality (2.1) becomes

$$\Lambda_0 = (1 - A(p))\left(\frac{b_\infty}{S_p^p}\right)^{2/(p-2)} \frac{\bar{S}^2\hat{S}^4}{K_\infty^2}.$$

- $\mathcal{D}(\mathbb{R}^3)$ is the completion of $C_c^\infty(\mathbb{R}^3)$ with respect to the norm $\|\cdot\|_{\mathcal{D}}$ induced by the scalar product

$$\langle \eta, \zeta\rangle_{\mathcal{D}} = \int_{\mathbb{R}^3} (\nabla\eta\nabla\zeta + a^2\Delta\eta\Delta\zeta) dx.$$

Then $\mathcal{D}$ is a Hilbert space continuously embedded into $D^{1,2}(\mathbb{R}^3)$ and consequently in $L^6(\mathbb{R}^3)$. It is interesting to point out the following properties.

Lemma 2.1 ([13]). *The space $\mathcal{D}$ is continuously embedded in $L^\infty(\mathbb{R}^3)$.*

The next lemma gives a useful characterization of the space $\mathcal{D}$.

Lemma 2.2 ([13]). *The space C_c^∞ is dense in*

$$\mathcal{A} := \{\phi \in \mathcal{D}^{1,2}(\mathbb{R}^3) : \Delta\phi \in L^2(\mathbb{R}^3)\}$$

named by $\sqrt{\langle \phi, \phi \rangle_{\mathcal{D}}}$ and, therefore, $\mathcal{D} = \mathcal{A}$.

Now, by combining Lemma 3.4 in [13] with Proposition 2.1 in [27], the following lemma can be obtained.

Lemma 2.3. *For every $u \in H^1(\mathbb{R}^3)$ we have:*

(i) for every $y \in \mathbb{R}^3$, $\phi_{K,u(\cdot+y)} = \phi_{K,u}(\cdot + y)$;

(ii) $\phi_{K,u} \geq 0$ in $\mathbb{R}^3$;

(iii) $\phi_{K,u} \in \mathcal{D}$;

(iv) $\|\phi_{K,u}\|_6 \leq C\|u\|^2$;

(v) $\phi_{K,u}$ is the unique minimizer in $\mathcal{D}$ of the functional

$$E(\phi) = \frac{1}{2}\|\nabla\phi\|_2^2 + \frac{a^2}{2}\|\Delta\phi\|_2^2 - \int_{\mathbb{R}^3} \phi u^2 \, dx, \quad \phi \in \mathcal{D}.$$

Moreover,

(vi) if $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, then $\phi_{K(x),u_n} \rightharpoonup \phi_{K(x),u}$ in $\mathcal{D}$,

$$\int_{\mathbb{R}^3} K(x)\phi_{K,u_n}u_n^2 dx \to \int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2 dx,$$

and

$$\int_{\mathbb{R}^3} K(x)\phi_{K,u_n}u_n\zeta dx \to \int_{\mathbb{R}^3} K(x)\phi_{K,u}u\zeta dx, \qquad \forall \zeta \in H^1(\mathbb{R}^3).$$

Next, we define the Nehari manifold

$$\mathcal{M} := \left\{ u \in H^1(\mathbb{R}^3) \setminus \{0\} : \langle \mathcal{J}'(u), u \rangle = 0 \right\}.$$

Then, $u \in \mathcal{M}$ if and only if $\|u\|^2 + \lambda \int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2 dx - \int_{\mathbb{R}^3} b(x)|u|^p dx = 0$. It follows the Sobolev inequality that

$$\begin{aligned}\|u\|^2 &\leq \|u\|^2 + \lambda \int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2 dx \\ &= \int_{\mathbb{R}^3} b(x)|u|^p dx \\ &\leq S_p^{-p} b_{\max} \|u\|^p,\end{aligned}$$

for all $u \in \mathcal{M}$. Then we can get

$$\int_{\mathbb{R}^3} b(x)|u|^p dx \geq \|u\|^2 \geq \left(\frac{S_p^p}{b_{\max}}\right)^{\frac{2}{p-2}} \quad \text{for all } u \in \mathcal{M}. \tag{2.2}$$

The Nehari manifold $\mathcal{M}$ is closely linked to the behavior of the function of the form $h_u : t \to \mathcal{J}(tu)$ for $t > 0$. Such maps are known as fibering maps and were introduced by Drábek–Pohožaev [14], and were further discussed by Brown–Zhang [9] and Brown–Wu [7,8] etc. For $u \in H^1\left(\mathbb{R}^3\right)$, we find

$$h_u(t) = \frac{t^2}{2}\|u\|^2 + \frac{\lambda t^4}{4}\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2 dx - \frac{t^p}{p}\int_{\mathbb{R}^3} b(x)|u|^p dx,$$
$$h_u'(t) = t\|u\|^2 + \lambda t^3\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2 dx - t^{p-1}\int_{\mathbb{R}^3} b(x)|u|^p dx,$$
$$h_u''(t) = \|u\|^2 + 3\lambda t^2\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2 dx - (p-1)t^{p-2}\int_{\mathbb{R}^3} b(x)|u|^p dx.$$

As a direct consequence, we have

$$th_u'(t) = \|tu\|^2 + \lambda\int_{\mathbb{R}^3} K(x)\phi_{K,tu}(tu)^2 dx - \int_{\mathbb{R}^3} b(x)|tu|^p dx,$$

and so, for $u \in H^1\left(\mathbb{R}^3\right)\setminus\{0\}$ and $t > 0, h_u'(t) = 0$ holds if and only if $tu \in \mathcal{M}$. In particular, $h_u'(1) = 0$ holds if and only if $u \in \mathcal{M}$. It is convenient to divide $\mathcal{M}$ into three parts, corresponding to local minima, local maxima, and inflection points. Following [26], we define

$$\mathcal{M}^+ = \left\{u \in \mathcal{M} : h_u''(1) > 0\right\},$$
$$\mathcal{M}^0 = \left\{u \in \mathcal{M} : h_u''(1) = 0\right\},$$
$$\mathcal{M}^- = \left\{u \in \mathcal{M} : h_u''(1) < 0\right\}.$$

Lemma 2.4. *Suppose that u_0 is a local minimizer for $\mathcal{J}$ on $\mathcal{M}$ and $u_0 \notin \mathcal{M}^0$. Then $\mathcal{J}'(u_0) = 0$ in $H^{-1}\left(\mathbb{R}^3\right)$.*

The proof of Lemma 2.4 is essentially the same as in Brown–Zhang [9], so we omitted it here.

For each $u \in \mathcal{M}$, we find that

$$\begin{aligned} h_u''(1) &= \|u\|^2 + 3\lambda\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2 dx - (p-1)\int_{\mathbb{R}^3} b(x)|u|^p dx \\ &= -(p-2)\|u\|^2 + (4-p)\lambda\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2 dx \\ &= -2\|u\|^2 + (4-p)\int_{\mathbb{R}^3} b(x)|u|^p dx. \end{aligned} \tag{2.3}$$

For each $u \in \mathcal{M}^-$and $2 < p < 4$, using (2.2) and (2.3) gives

$$\begin{aligned} \mathcal{J}(u) &= \frac{1}{4}\|u\|^2 - \frac{(4-p)}{4p}\int_{\mathbb{R}^3} b(x)|u|^p dx \\ &> \frac{p-2}{2p}\|u\|^2 \\ &\geq \frac{p-2}{2p}\left(\frac{S_p^p}{b_{\max}}\right)^{2/(p-2)}. \end{aligned}$$

Moreover, for each $u \in \mathcal{M}^-$ and $p = 4$, by virtue of (2.2) we have

$$\mathcal{J}(u) = \frac{1}{4}\|u\|^2 \geq \frac{1}{4}\left(\frac{S_p^p}{b_{\max}}\right)^{2/(p-2)}.$$

From this, the following lemma are obtained.

Lemma 2.5. *Suppose that $2 < p \leq 4$. Then the energy functional $\mathcal{J}(u)$ is coercive and bounded below on $\mathcal{M}^-$. Furthermore, for all $u \in \mathcal{M}^-$, when $2 < p < 4$, there holds*

$$\mathcal{J}(u) > \frac{p-2}{4p}\left(\frac{S_p^p}{b_{\max}}\right)^{\frac{2}{p-2}},$$

if $p = 4$, there holds

$$\mathcal{J}(u) \geq \frac{1}{4}\left(\frac{S_p^p}{b_{\max}}\right)^{\frac{2}{p-2}}.$$

From the Lemma 2.3 and [24], the following properties can be obtained

Lemma 2.6. *For each $u \in H^1(\mathbb{R})^3$, the following two inequalities are true.*

(i) $\phi_{K,u} \geq 0$;

(ii) $\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx \leq \bar{S}^{-2}\hat{S}^{-4}K_{\max}^2\|u\|^4$.

Citing the lemma in [25], the same inequality can be obtained here, because

$$\phi_{K,u} = \int_{\mathbb{R}^3} \frac{1 - e^{-\frac{|x-y|}{a}}}{|x-y|}K(y)u^2(y)dy \leq \int_{\mathbb{R}^3} \frac{1}{|x-y|}K(y)u^2(y)dy.$$

For any $u \in \mathcal{M}$ and $2 < p < 4$ with $\mathcal{J}(u) < A(p)\frac{(p-2)}{2p}\left(\frac{2S_p^p}{b_\infty(4-p)}\right)^{\frac{2}{p-2}}$, we inference that

$$\begin{aligned}
A(p)&\frac{(p-2)}{2p}\left(\frac{2S_p^p}{b_\infty(4-p)}\right)^{\frac{2}{p-2}} \\
&> \mathcal{J}(u) \\
&= \frac{1}{2}\|u\|^2 + \frac{\lambda}{4}\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx - \frac{1}{p}\int_{\mathbb{R}^3} b(x)|u|^pdx \\
&= \frac{p-2}{2p}\|u\|^2 - \frac{\lambda(4-p)}{4p}\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx \\
&\geq \frac{p-2}{2p}\|u\|^2 - \lambda\left(\frac{4-p}{4p}\right)\bar{S}^{-2}\hat{S}^{-4}K_{\max}^2\|u\|^4. \qquad (2.4)
\end{aligned}$$

In addition, consider the quadratic equation as follows

$$\frac{1}{4}\left(1 - A(p)\left(\frac{b_{\max}}{b_\infty}\right)^{\frac{2}{p-2}}\right)\left(\frac{b_\infty(4-p)}{pS_p^p}\right)^{\frac{2}{p-2}}x^2 - x + A(p)\left(\frac{2S_p^p}{b_\infty(4-p)}\right)^{\frac{2}{p-2}} = 0.$$

It is easy to get one of the solutions expressed as

$$x_1 = \frac{2\left(1+\sqrt{1-A(p)\left(1-A(p)\left(\frac{b_{\max}}{b_\infty}\right)^{\frac{2}{p-2}}\right)\left(\frac{2}{p}\right)^{\frac{2}{p-2}}}\right)}{\left(1-A(p)\left(\frac{b_{\max}}{b_\infty}\right)\right)\left(\frac{2}{p}\right)^{\frac{2}{p-2}}}\left(\frac{2S_p^p}{b_\infty(4-p)}\right)^{\frac{2}{p-2}}$$
$$> 2\left(\frac{2S_p^p}{b_\infty(4-p)}\right)^{\frac{2}{p-2}}, \tag{2.5}$$

then we have used condition (G_1), Remark 1.1 and the fact of $\left(\frac{2}{p}\right)^{\frac{2}{p-2}} < 1$ in the last inequality.

From (2.4) and (2.5), if $2 < p < 4$ and $0 < \lambda < \frac{p-2}{2(4-p)}\left(\frac{4-p}{p}\right)^{\frac{2}{p-2}}\Lambda_0$, then there exist two positive number $D^{(1)}$ and $D^{(2)}$ satisfying

$$\sqrt{A(p)}\left(\frac{2S_p^p}{b_\infty(4-p)}\right)^{\frac{1}{p-2}} < D^{(1)} < \left(\frac{2S_p^p}{b_{\max}(4-p)}\right)^{\frac{1}{p-2}} < \sqrt{2}\left(\frac{2S_p^p}{b_\infty(4-p)}\right)^{\frac{1}{p-2}} < D^{(2)}$$

such that

$$\|u\| < D^{(1)} \quad \text{or} \quad \|u\| > D^{(2)}.$$

Obviously, it can be seen that when $p \to 4^-$, then $D^{(1)} \to \infty$.

So, we have

$$\begin{aligned}\mathcal{M}&\left[\frac{A(p)(p-2)}{2p}\left(\frac{2S_p^p}{b_\infty(4-p)}\right)^{\frac{2}{p-2}}\right]\\ &= \left\{u \in \mathcal{M} : \mathcal{J}(u) < \frac{A(p)(p-2)}{2p}\left(\frac{2S_p^p}{b_\infty(4-p)}\right)^{\frac{2}{p-2}}\right\}\\ &= \mathcal{M}^{(1)} \bigcup \mathcal{M}^{(2)},\end{aligned} \tag{2.6}$$

where

$$\mathcal{M}^{(1)} := \left\{u \in \mathcal{M}\left[\frac{A(p)(p-2)}{2p}\left(\frac{2S_p^p}{b_\infty(4-p)}\right)^{\frac{2}{p-2}}\right] : \|u\| < D^{(1)}\right\},$$

and

$$\mathcal{M}^{(2)} := \left\{u \in \mathcal{M}\left[\frac{A(p)(p-2)}{2p}\left(\frac{2S_p^p}{b_\infty(4-p)}\right)^{\frac{2}{p-2}}\right] : \|u\| > D^{(2)}\right\}.$$

Because of $2 < p < 4$ and $0 < \lambda < \frac{p-2}{2(4-p)}\left(\frac{4-p}{p}\right)^{\frac{2}{p-2}}\Lambda_0$, we have

$$\|u\| < D^{(1)} < \left(\frac{2S_p^p}{b_{\max}(4-p)}\right)^{\frac{1}{p-2}} \quad \text{for all } u \in \mathcal{M}^{(1)}, \tag{2.7}$$

and

$$\|u\| > D^{(2)} > \sqrt{2}\left(\frac{2S_p^p}{b_\infty(4-p)}\right)^{\frac{1}{p-2}} \quad \text{for all } u \in \mathcal{M}^{(2)}. \tag{2.8}$$

From the Sobolev inequality, (2.3) and (2.7)

$$h_u''(1) \leq -2\|u\|^2 + (4-p)S_p^{-p} b_{\max}\|u\|^p < 0 \quad \text{for all } u \in \mathcal{M}^{(1)}.$$

Using (2.8) we deduce that

$$\begin{aligned}\frac{1}{4}\|u\|^2 - \frac{(4-p)}{4p}\int_{\mathbb{R}^3} b(x)|u|^p dx \\ = \mathcal{J}(u) < \frac{A(p)(p-2)}{2p}\left(\frac{2S_p^p}{b_\infty(4-p)}\right)^{\frac{2}{p-2}} \\ < \frac{p-2}{2p}\left(\frac{2S_p^p}{b_\infty(4-p)}\right)^{\frac{2}{p-2}} \\ < \frac{p-2}{4p}\|u\|^2 \quad \text{for all } u \in \mathcal{M}^{(2)},\end{aligned}$$

this implies

$$2\|u\|^2 < (4-p)\int_{\mathbb{R}^3} b(x)|u|^p dx \quad \text{for all } u \in \mathcal{M}^{(2)}. \tag{2.9}$$

Combining (2.3) and (2.9) results in

$$h_u''(1) = -2\|u\|^2 + (4-p)\int_{\mathbb{R}^3} b(x)|u|^p dx > 0 \quad \text{for all } u \in \mathcal{M}^{(1)}.$$

Therefore, we get the following result.

Lemma 2.7.

(i) If $2 < p < 4$ and $0 < \lambda < \frac{p-2}{2(4-p)}\left(\frac{4-p}{p}\right)^{\frac{2}{p-2}}\Lambda_0$, then $\mathcal{M}^{(1)} \subset \mathcal{M}^-$ and $\mathcal{M}^{(2)} \subset \mathcal{M}^+$ are C^1 sub-manifolds. Furthermore, each local minimizer of the functional $\mathcal{J}$ in the sub-manifolds $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$ is a critical point of $\mathcal{J}$ in $H^1(\mathbb{R}^3)$.

(ii) If $p = 4$ and $\lambda > 0$, then $\mathcal{M}^{(1)} = \mathcal{M}^- = \mathcal{M}$ is a C^1 manifold and so the Nehari manifold $\mathcal{M}^{(1)}$ is a natural constraint for the functional $\mathcal{J}$.

There we define

$$Q_b(u) = \left(\frac{\|u\|^2}{\int_{\mathbb{R}^3} b(x)|u|^p dx}\right)^{\frac{1}{p-2}} \quad \text{for } u \in H^1(\mathbb{R}^3)\backslash\{0\}.$$

Lemma 2.8. *Suppose that $2 < p < 4$. then for each $\lambda > 0$ and $u \in H^1(\mathbb{R}^3)\backslash\{0\}$ satisfying*

$$\int_{\mathbb{R}^3} b(x)|u|^p dx > \frac{p}{4-p}\left(\frac{2\lambda(4-p)K_{\max}^2}{(p-2)\bar{S}^2\hat{S}^4}\right)^{\frac{p-2}{2}}\|u\|^p,$$

there exists a constant $\bar{q}^{(1)} > \left(\frac{p}{4-p}\right)^{\frac{1}{p-2}} Q_b(u)$ such that

$$\inf_{t\geq 0}\mathcal{J}(tu) = \inf_{\left(\frac{p}{4-p}\right)^{\frac{1}{p-2}} Q_b(u) < t < \bar{q}^{(1)}} \mathcal{J}(tu) < 0. \tag{2.10}$$

Proof. For any $u \in H^1(\mathbb{R}^3)\backslash\{0\}$ and $t > 0$, it has

$$\begin{aligned}\mathcal{J}(tu) &= \frac{t^2}{2}\|u\|^2 + \frac{\lambda t^4}{4}\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx - \frac{t^p}{p}\int_{\mathbb{R}^3} b(x)|u|^p dx \\ &= t^4\left[b(t) + \frac{\lambda}{4}\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx\right] \\ &= h_u(t),\end{aligned}$$

where $b(t) = \frac{t^{-2}}{2}\|u\|^2 - \frac{t^{p-4}}{p}\int_{\mathbb{R}^3} b(x)|u|^p dx$.

Apparently, $\mathcal{J}(tu) = 0$ if and only if $b(t) + \frac{\lambda}{4}\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx = 0$. It is not difficult to observe that $b(\hat{t}) = 0, \lim_{t\to 0^+} b(t) = \infty$ and $\lim_{t\to\infty} b(t) = 0$, where $\hat{t} = (\frac{p}{2})^{\frac{1}{p-2}}Q_b(u)$. Considering the derivative of $b(t)$, we get

$$\begin{aligned}b'(t) &= -t^{-3}\|u\|^2 + \frac{(4-p)}{p}t^{p-5}\int_{\mathbb{R}^3} b(x)|u|^p dx \\ &= t^{-3}\left[\frac{(4-p)t^{p-2}}{p}\int_{\mathbb{R}^3} b(x)|u|^p dx - \|u\|^2\right],\end{aligned}$$

it means that $b(t)$ is decreasing when $0 < t < (\frac{p}{4-p})^{\frac{1}{p-2}}Q_b(u)$ and is increasing when $t > (\frac{p}{4-p})^{\frac{1}{p-2}}Q_b(u)$, and so

$$\begin{aligned}\inf_{t>0} b(t) &= b\left[\left(\frac{p}{4-p}\right)^{\frac{1}{p-2}} Q_b(u)\right] \\ &= -\frac{p-2}{2(4-p)}\left(\frac{p\|u\|^2}{(4-p)\int_{\mathbb{R}^3} b(x)|u|^p dx}\right)^{\frac{-2}{p-2}}\|u\|^2.\end{aligned}$$

From Lemma 2.6 (ii) and the Sobolev inequality that for each $u \in H^1(\mathbb{R}^3)\backslash\{0\}$ satisfying

$$\int_{\mathbb{R}^3} b(x)|u|^p dx > \frac{p}{4-p}\left(\frac{2\lambda(4-p)K_{\max}^2}{(p-2)\bar{S}^2\hat{S}^4}\right)^{\frac{p-2}{2}}\|u\|^p,$$

we have

$$\begin{aligned}\inf_{t>0} b(t) &= -\frac{p-2}{2(4-p)}\left(\frac{p\|u\|^2}{(4-p)\int_{\mathbb{R}^3} b(x)|u|^p dx}\right)^{\frac{-2}{p-2}}\|u\|^2 \\ &< -\lambda K_{\max}^2\bar{S}^{-2}\hat{S}^{-4}\|u\|^2 \\ &< -\frac{\lambda}{4}\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx.\end{aligned}$$

Then, there exist $\bar{q}^{(1)}$ and $\bar{q}^{(2)}$ satisfying

$$0 < \bar{q}^{(2)} < \left(\frac{p}{4-p}\right)^{\frac{1}{p-2}} Q_b(u) < \bar{q}^{(1)} \tag{2.11}$$

such that

$$b(\bar{q}^{(j)}) + \frac{\lambda}{4}\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx = 0 \quad \text{for } j = 1,2.$$

That is $\mathcal{J}(\bar{q}^{(j)}u) = 0$ for $j = 1,2$. □

So, for each $\lambda > 0$ and $u \in H^1(\mathbb{R}^3)\backslash\{0\}$ satisfying

$$\int_{\mathbb{R}^3} b(x)|u|^p dx > \frac{p}{4-p}\left(\frac{2\lambda(4-p)K_{\max}^2}{(p-2)\bar{S}^2\hat{S}^4}\right)^{\frac{p-2}{2}} \|u\|^p,$$

we have

$$\begin{aligned}&\mathcal{J}\left[\left(\frac{p}{4-p}\right)^{\frac{1}{p-2}} Q_b(u)\right]\\ &\quad = \left[\left(\frac{p}{4-p}\right)^{\frac{1}{p-2}} Q_b(u)\right]\left[b\left(\left(\frac{p}{4-p}\right)^{\frac{1}{p-2}} Q_b(u)\right) + \frac{\lambda}{4}\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2 dx\right]\\ &\quad < 0,\end{aligned}$$

and so $\inf_{t\geq 0}\mathcal{J}(tu) < 0$.

Then, we know that $h_u'(t) = 4t^3[b(t) + \frac{\lambda}{4}\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2 dx] + t^4 b'(t)$, which leads to $h_u'(t) < 0$ for all $t \in (\bar{q}^{(2)}, (\frac{p}{4-p})^{\frac{1}{p-2}} Q_b(u)]$ and $h_u'(\bar{q}^{(1)}) > 0$. Finally, we get the inequality (2.10).

Lemma 2.9. *For each $\lambda > 0$ and $u \in H^1(\mathbb{R}^3)\backslash\{0\}$ satisfying*

$$\int_{\mathbb{R}^3} b(x)|u|^p dx > \frac{p}{4-p}\left(\frac{2\lambda(4-p)K_{\max}^2}{(p-2)\bar{S}^2\hat{S}^4}\right)^{\frac{p-2}{2}} \|u\|^p \quad \text{if } 2 < p < 4,$$

or

$$\int_{\mathbb{R}^3} b(x)|u|^4 dx > \lambda K_{\max}^2 \bar{S}^{-2}\hat{S}^{-4}\|u\|^4 \quad \text{if } p = 4,$$

the following two statements are true.

(i) if $2 < p < 4$, then there exist two constants t^+ and t^- which satisfy

$$Q_b(u) < t^- < \sqrt{A(p)}\left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} Q_b(u) < \left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} Q_b(u) < t^+ \tag{2.12}$$

such that

$$t^{\pm}u \in \mathcal{M}^{\pm}, \qquad \mathcal{J}(t^-u) = \sup_{0\leq t\leq t^+}\mathcal{J}(tu),$$

and

$$\mathcal{J}(t^+u) = \inf_{t\geq t^-}\mathcal{J}(tu) = \inf_{t\geq 0}\mathcal{J}(tu) < 0.$$

(ii) if $p = 4$, then there is a unique constant

$$\bar{t} = \left(\frac{\|u\|^2}{\int_{\mathbb{R}^3} b(x)u^4 dx - \lambda\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2 dx}\right)^{\frac{1}{2}} > Q_b(u)$$

such that

$$\bar{t}u \in \mathcal{M}^{(1)} = \mathcal{M}^- = \mathcal{M},$$

and

$$\mathcal{J}(\bar{t}u) = \sup_{t\geq 0}\mathcal{J}(tu) = \sup_{t\geq Q_b(u)}\mathcal{J}(tu).$$

Proof. (i) Define $f(t) = t^{-2}\|u\|^2 - t^{p-4}\int_{\mathbb{R}^3} b(x)|u|^p dx$ for $t > 0$. Obviously, $tu \in \mathcal{M}$ if and only if $f(t) + \lambda \int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2 dx = 0$. A straightforward evaluation gives $f(Q_b(u)) = 0, \lim_{t\to 0^+} f(t) = \infty$ and $\lim_{t\to\infty} f(t) = 0$.

Since $2 < p < 4$ and $f'(t) = t^{-3}(-2\|u\|^2 + (4-p)t^{p-2}\int_{\mathbb{R}^3} b(x)|u|^p dx)$, we know that $f(t)$ is decreasing when $0 < t < (\frac{2}{4-p})^{\frac{1}{p-2}} Q_b(u)$ and is increasing when $t > (\frac{2}{4-p})^{\frac{1}{p-2}} Q_b(u)$. This gives

$$\inf_{t>0} f(t) = f\left[\left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} Q_b(u)\right]. \tag{2.13}$$

For each $\lambda > 0$ and $u \in H^1(\mathbb{R}^3)\backslash\{0\}$ satisfying

$$\int_{\mathbb{R}^3} b(x)|u|^p dx > \frac{p}{4-p}\left(\frac{2\lambda(4-p)K_{\max}^2}{(p-2)\bar{S}^2\hat{S}^4}\right)^{\frac{p-2}{2}} \|u\|^p \quad \text{if } 2 < p < 4,$$

from Lemma 2.6 (ii) and Sobolev's inequality we get

$$\begin{aligned} f\left(\left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} Q_b(u)\right) &= -\left(\frac{p-2}{4-p}\right)\left(\frac{2\|u\|^2}{(4-p)\int_{\mathbb{R}^3} b(x)|u|^p dx}\right)^{\frac{-2}{p-2}} \|u\|^2 \\ &< -2\left(\frac{p}{2}\right)^{\frac{2}{p-2}} \lambda K_{\max}^2 \bar{S}^{-2}\hat{S}^{-4}\|u\|^4 \\ &< -\lambda \int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2 dx, \end{aligned}$$

where we also used the fact that $(\frac{2}{p})^{\frac{2}{p-2}} > 1$. However, for each $2 < p < 4$, by Remark 1.1 we have

$$Q_b(u) < \sqrt{A(p)}\left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} Q_b(u) < \left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} Q_b(u), \tag{2.14}$$

and directly calculated

$$\frac{\left(\frac{2}{4-p}\right) A(p)^{\frac{p-2}{2}} - 1}{A(p)\left(\frac{2}{4-p}\right)^{\frac{2}{p-2}}} > \frac{p-2}{2(4-p)}\left(\frac{4-p}{p}\right)^{\frac{2}{p-2}}. \tag{2.15}$$

Then, from (2.13)–(2.15) that

$$\begin{aligned} f\left(\sqrt{A(p)}\left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} Q_b(u)\right) &= -\frac{\left(\frac{2}{4-p}\right) A(p)^{\frac{p-2}{2}} - 1}{A(p)\left(\frac{2}{4-p}\right)^{\frac{2}{p-2}}}\left(\frac{\int_{\mathbb{R}^3} b(x)|u|^p dx}{\|u\|^2}\right)^{\frac{2}{p-2}} \|u\|^2 \\ &< -\lambda K_{\max}^2 \bar{S}^{-2}\hat{S}^{-4}\|u\|^4 \\ &\le -\lambda \int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2 dx. \end{aligned}$$

Therefore, there exist two constants t^+ and $t^- > 0$ which satisfy

$$Q_b(u) < t^- < \sqrt{A(p)}\left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} Q_b(u) < \left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} Q_b(u) < t^+ \tag{2.16}$$

such that $f(t^\pm) + \lambda \int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2 dx = 0$. That is $t^\pm u \in \mathcal{M}$.

By calculating the second derivative, we find that

$$\begin{aligned} h''_{t^-u}(1) &= -2\|t^-u\|^2 + (4-p)\int_{\mathbb{R}^3} b(x)|t^-u|^p dx \\ &= (t^-)^5 f'(t^-) < 0, \end{aligned}$$

and

$$\begin{aligned} h''_{t^+u}(1) &= -2\|t^+u\|^2 + (4-p)\int_{\mathbb{R}^3} b(x)|t^+u|^p dx \\ &= (t^+)^5 f'(t^+) > 0. \end{aligned}$$

This means that $t^{\pm}u \in \mathcal{M}^{\pm}$ and $h'_u(t) = t^3(f(t) + \lambda \int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx)$. It is known that $h'_u(t) > 0$ holds for all $t \in (0, t^-) \cup (t^+, \infty)$ and $h'_u(t) < 0$ holds for all $t \in (t^-, t^+)$. It leads to $\mathcal{J}(t^-u) = \sup_{0\le t\le t^+} \mathcal{J}(tu)$ and $\mathcal{J}(t^+u) = \inf_{t\ge t^-} \mathcal{J}(tu)$, and so $\mathcal{J}(t^+u) < \mathcal{J}(t^-u)$. From Lemma 2.8 that $\mathcal{J}(t^+u) = \inf_{t\ge 0} \mathcal{J}(tu) < 0$.
(ii) Let

$$\bar{f}(t) = t^{-2}\|u\|^2 \quad \text{for} \quad t > 0. \tag{2.17}$$

Apparently, $tu \in \mathcal{M}^{(1)} = \mathcal{M}^- = \mathcal{M}$ if and only if $\bar{f}(t) - \int_{\mathbb{R}^3} b(x)u^4dx + \lambda \int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx = 0$. By (2.17), we know that $\bar{f} > 0 (t > 0)$ is decreasing, and $\lim_{t\to 0^+} \bar{f}(t) = \infty$ and $\lim_{t\to\infty} \bar{f}(t) = 0$.

For each $\lambda > 0$ and $u \in H^1(\mathbb{R}^3)\backslash\{0\}$ satisfying $\int_{\mathbb{R}^3} b(x)|u|^4dx > \lambda K^2_{\max}\bar{S}^{-2}\hat{S}^{-4}\|u\|^4$, by using Lemma 2.6 (ii) and (2.15), we obtain $\int_{\mathbb{R}^3} b(x)|u|^4dx > \lambda K^2_{\max}\bar{S}^{-2}\hat{S}^{-4}\|u\|^4 \ge \lambda \int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx$. Then we can get that equation $\bar{f}(t) - \int_{\mathbb{R}^3} b(x)u^4dx + \lambda \int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx = 0$ has a unique positive solution $\bar{t} = \left(\frac{\|u\|^2}{\int_{\mathbb{R}^3} b(x)u^4dx - \lambda \int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2dx}\right)^{\frac{1}{2}} > Q_b(u)$. This means that $\bar{t}u \in \mathcal{M}^{(1)} = \mathcal{M}^- = \mathcal{M}$. Similar to the discussion of Case (i), we get that $\mathcal{J}(\bar{t}u) = \sup_{t\ge 0} \mathcal{J}(tu) = \sup_{t\ge Q_b(u)} \mathcal{J}(tu)$. This completes the proof. □

3 Proofs of main results

3.1 Proof of Theorem 1.3

In this section, we first consider that $K(x) \equiv K_\infty > 0$ and $b(x) \equiv b_\infty > 0$. The existence of the positive ground state solutions of system (1.3) at infinity, namely,

$$\begin{cases} -\Delta u + u + \lambda K_\infty \phi u = b_\infty |u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi K_\infty u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{3.1}$$

Then we consider the following equation at infinity

$$-\Delta u + u + \lambda K_\infty \phi_{K_\infty,u} u = b_\infty |u|^{p-2}u. \tag{3.2}$$

We define the associated energy functional in $H^1(\mathbb{R}^3)$ by

$$\mathcal{J}^\infty(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2\right) dx + \frac{\lambda}{4}\int_{\mathbb{R}^3} K_\infty \phi_{K_\infty,u} u^2 dx - \frac{1}{p}\int_{\mathbb{R}^3} b_\infty |u|^p dx,$$

we know that solutions of equation (3.2) are critical points of the functional $\mathcal{J}^\infty(u)$.

Define

$$\mathcal{M}_\infty := \{u \in H^1(\mathbb{R}^3)\backslash\{0\} : \langle(\mathcal{J}^\infty)'(u), u\rangle = 0\},$$

where $(\mathcal{J}^\infty)'$ denotes the Fréchet derivative of $\mathcal{J}^\infty$. Then, $u \in \mathcal{M}_\infty$ if and only if

$$\|u\|^2 + \lambda \int_{\mathbb{R}^3} K_\infty \phi_{K_\infty,u} u^2 dx - \int_{\mathbb{R}^3} b_\infty |u|^p dx = 0.$$

Notice that $\mathcal{M}_\infty = \mathcal{M}$ with $K(x) \equiv K_\infty$ and $b(x) \equiv b_\infty$. We denote by $\mathcal{M}_\infty^{(j)} = \mathcal{M}^{(j)}$ with $K(x) \equiv K_\infty$ and $b(x) = b_\infty$ for $j = 1, 2$.

Since w_0 is the unique positive solution of equation (1.6), for $2 < p < 4$ and $0 < \lambda < \frac{p-2}{2(4-p)} \left(\frac{4-p}{p}\right)^{\frac{2}{p-2}} \Lambda_0$, from (1.7), we get

$$\begin{aligned}\int_{\mathbb{R}^3} b_\infty |w_0|^p dx &= b_\infty S_p^{-p} \|w_0\|^p \\ &> \frac{p}{4-p} \left(\frac{2\lambda(4-p)K_\infty^2}{\bar{S}^2 \hat{S}^4 (p-2)} \right)^{\frac{p-2}{2}} \|w_0\|^p.\end{aligned}$$

From Lemma 2.9 (i) there exist two constants t_∞^- and t_∞^+ satisfy

$$1 < t_\infty^- < \sqrt{A(p)} \left(\frac{2}{4-p} \right)^{\frac{1}{p-2}} < t_\infty^+,$$

such that $t_\infty^\pm w_0 \in \mathcal{M}_\infty^\pm$, where $\mathcal{M}_\infty^\pm = \mathcal{M}^\pm$ with $K(x) \equiv K_\infty$ and $b(x) \equiv b_\infty$. However, we obtain $\mathcal{J}^\infty(t_\infty^- w_0) = \sup_{0 \le t \le t_\infty^+} \mathcal{J}^\infty(t w_0)$ and $\mathcal{J}^\infty(t_\infty^+ w_0) = \inf_{t \ge t_\infty^-} \mathcal{J}^\infty(t w_0) = \inf_{t \ge 0} \mathcal{J}^\infty(t w_0) < 0$. Then we can get

$$\begin{aligned}\mathcal{J}^\infty(t_\infty^- w_0) &= \frac{1}{2} \|t_\infty^- w_0\|^2 + \frac{\lambda}{4} \int_{\mathbb{R}^3} K_\infty \phi_{K_\infty, t_\infty^- w_0} (t_\infty^- w_0)^2 dx - \frac{1}{p} \int_{\mathbb{R}^3} b_\infty |t_\infty^- w_0|^p dx \\ &= \frac{(t_\infty^-)^2}{4} \left[1 - \frac{4-p}{p} (t_\infty^-)^{p-2} \right] \|w_0\|^2 \\ &< A(p) \left(\frac{2}{4-p} \right)^{\frac{2}{p-2}} \frac{p-2}{2p} \|w_0\|^2 \\ &= A(p) \frac{p-2}{2p} \left(\frac{2 S_p^p}{b_\infty (4-p)} \right)^{\frac{2}{p-2}}. \end{aligned} \tag{3.3}$$

This indicates that $t_\infty^- w_0 \in \mathcal{M}_\infty^{(1)}$. Namely, $\mathcal{M}_\infty^{(1)}$ is nonempty.

For $p = 4$ and $0 < \lambda < b_\infty K_\infty^{-2} \bar{S}^2 \hat{S}^4 S_4^{-4}$, there holds

$$\int_{\mathbb{R}^3} b_\infty |w_0|^4 dx = b_\infty S_4^{-4} \|w_0\|^4 > \lambda K_\infty^2 \bar{S}^{-2} \hat{S}^{-4} \|w_0\|^4.$$

Then, from Lemma 2.9 (ii), there exists a unique constant

$$\bar{t}^\infty = \frac{\|w_0\|^2}{\int_{\mathbb{R}^3} b_\infty |w_0|^4 dx - \lambda \int_{\mathbb{R}^3} K_\infty \phi_{K_\infty, w_0} w_0^2 dx} > 1$$

such that $\bar{t}^\infty w_0 \in \mathcal{M}_\infty^{(1)} = \mathcal{M}_\infty^- = \mathcal{M}_\infty$ and $\mathcal{J}^\infty(\bar{t}^\infty w_0) = \sup_{t \ge 0} \mathcal{J}^\infty(t w_0) = \sup_{t > 1} \mathcal{J}^\infty(t w_0)$.

Then we define

$$\alpha_\infty^- = \inf_{u \in \mathcal{M}_\infty^{(1)}} \mathcal{J}^\infty(u) = \inf_{u \in \mathcal{M}_\infty^-} \mathcal{J}^\infty(u) \quad \text{for } 2 < p < 4,$$

$$\alpha_\infty^+ = \inf_{u\in\mathcal{M}_\infty^{(2)}} \mathcal{J}^\infty(u) = \inf_{u\in\mathcal{M}_\infty^+} \mathcal{J}^\infty(u) \quad \text{for } 2 < p < 4,$$

and

$$\alpha_\infty^- = \inf_{u\in\mathcal{M}_\infty^{(1)}} \mathcal{J}^\infty(u) = \inf_{u\in\mathcal{M}_\infty} \mathcal{J}^\infty(u) \quad \text{for } p = 4.$$

It follows from Lemma 2.5 and (3.3), we have

$$\frac{p-2}{4p}\left(\frac{S_p^p}{b_\infty}\right)^{\frac{2}{p-2}} \le \alpha_\infty^- < \frac{A(p)(p-2)}{2p}\left(\frac{2S_p^p}{b_\infty(4-p)}\right)^{\frac{2}{p-2}} \quad \text{for } 2 < p < 4, \tag{3.4}$$

and $\alpha_\infty^+ = -\infty$. For $p = 4$, it follows from Lemma 2.5 that $\alpha_\infty^- \ge \frac{1}{4}(\frac{S_p^p}{b_\infty})^{\frac{2}{p-2}}$.

Then we are ready to prove Theorem 1.3.

Let $u_n \in \mathcal{M}_\infty^{(1)}$ be a sequence, for $2 < p < 4$, we have

$$\mathcal{J}^\infty(u_n) = \alpha_\infty^- + o(1) \quad \text{and} \quad (\mathcal{J}^\infty)'(u_n) = o(1) \quad \text{in } H^{-1}(\mathbb{R}^3). \tag{3.5}$$

According to Theorem 7.2 in [25], we obtain that for

$$0 < \lambda < \frac{(p-2)\bar{S}^2\hat{S}^4}{2(4-p)K_\infty^2}\left(\frac{b_\infty(4-p)^2}{2pS_p^p}\right)^{\frac{2}{p-2}} \quad \text{if } 2 < p < 4,$$

or $\lambda > 0$ if $p = 4$, the compactness of the sequence $\{u_n\}$ holds. Then there exist a positive constant $\xi = \xi(\theta)(\theta > 0)$ and a sequence $\{y_n\} \subset \mathbb{R}^3$ such that

$$\int_{[B(y_n;\xi)]^c} (|\nabla u_n(x)|^2 + u_n^2(x))dx < \theta \quad \text{uniformly for } n \ge 1. \tag{3.6}$$

Now, we define a new sequence of functions

$$v_n := u_n(\cdot + y_n) \in H^1(\mathbb{R}^3).$$

We find that $\{v_n\} \subset \mathcal{M}_\infty^{(1)}$, and

$$\phi_{K_\infty,v_n} = \phi_{K\infty,u_n}(\cdot + y_n) \quad \text{and} \quad \mathcal{J}^\infty(v_n) = \alpha_\infty^- + o(1).$$

By inequality (3.6), there exists a positive constant $\xi = \xi(\theta)(\theta > 0)$ such that

$$\int_{[B(0;\xi)]^c} (|\nabla v_n(x)|^2 + v_n^2(x))dx < \theta \quad \text{uniformly for } n \ge 1. \tag{3.7}$$

For $\{v_n\}$ is bounded in $H^1(\mathbb{R}^3)$, we can assume that there exist a subsequence $\{v_n\}$ and $w \in H^1(\mathbb{R}^3)$ such that

$$v_n \rightharpoonup w \quad \text{in } H^1(\mathbb{R}^3), \tag{3.8}$$

$$v_n \to w \quad \text{in } L^r_{\text{loc}},\ \forall\, 2 \le r < 6, \tag{3.9}$$

$$v_n \to w \quad \text{a.e. in } \mathbb{R}^3.$$

For any $\theta > 0$ and sufficiently large $n(\geq 1)$, by Fatou's Lemma and (3.7)–(3.9), there exists a constant $\xi > 0$ such that

$$\begin{aligned}\int_{\mathbb{R}^3} |v_n - w|^p dx &\leq \int_{B(0;\xi)} |v_n - w|^p dx + \int_{[B(0;\xi)]^c} |v_n - w|^p dx \\ &\leq \theta + S_p^{-p} \left[\int_{[B(0;\xi)]^c} (|\nabla v_n|^2 + v_n^2) dx + \int_{[B(0;\xi)]^c} (|\nabla w|^2 + w^2) dx \right]^{\frac{p}{2}} \\ &\leq \theta + S_p^{-p} (2\theta)^{\frac{p}{2}},\end{aligned}$$

then we obtain

$$v_n \to w \quad \text{in } L^r(\mathbb{R}^3),\ \forall r \in (2,6). \tag{3.10}$$

We know that $\phi : L^{\frac{12}{5}}(\mathbb{R}^3) \to \mathcal{D}$ is a continuous function. It follows from (3.10) that

$$\phi_{K_\infty,v_n} \to \phi_{K_\infty,w} \quad \text{in } \mathcal{D},$$

and

$$\int_{\mathbb{R}^3} \phi_{K_\infty,v_n} v_n^2 dx \to \int_{\mathbb{R}^3} \phi_{K_\infty,w} w^2 dx. \tag{3.11}$$

Since $\{v_n\} \subset \mathcal{M}_\infty^{(1)}$, using (2.2) and (3.10) gives

$$\int_{\mathbb{R}^3} b_\infty |w|^p dx \geq \left(\frac{S_p^p}{b_\infty} \right)^{\frac{2}{p-2}} > 0.$$

This implies that $w \neq 0$ and

$$\int_{\mathbb{R}^3} b_\infty |w|^p dx - \lambda \int_{\mathbb{R}^3} K_\infty \phi_{K_\infty,w} w^2 dx \geq \|w\|^2 > 0.$$

Next, we proof that

$$v_n \to w \quad \text{in } H^1(\mathbb{R}^3).$$

For this, we assume the opposite. Then we have

$$\|w\| < \liminf_{n\to\infty} \|v_n\|. \tag{3.12}$$

An argument similar to Lemma 2.9, there exists a unique $t^- > 0$ such that

$$t^- w \in \mathcal{M}_\infty^- \quad \text{and} \quad (h_w^\infty)'(t^-) = 0. \tag{3.13}$$

For $v_n \in \mathcal{M}_\infty^{(1)}$, from (3.12) we get

$$(h_w^\infty)'(1) < 0. \tag{3.14}$$

Using (3.13), (3.14) and the contour of $h_w^\infty(t)$ results in $t^- < 1$. By (3.10)–(3.12), we know $(h_{v_n}^\infty)'(t^-) > 0$ for sufficiently large n. Obviously, there holds

$$(h_{v_n}^\infty)'(1) = 0 \tag{3.15}$$

due to $v_n \in \mathcal{M}_\infty^{(1)}$. Similar to the proof of Lemma 2.9, for $2 < p < 4$, we have

$$(h_{v_n}^\infty)'(t) = t^3 (f^\infty(t) + \lambda \int_{\mathbb{R}^3} K_\infty \phi_{K_\infty,v_n} v_n^2 dx,$$

where $f^{\infty}(t) := t^{-2}\|v_n\|^2 - t^{p-4}\int_{\mathbb{R}^3} b_\infty |v_n|^p dx$ is decreasing for $0<t<\left(\frac{2}{4-p}\right)^{\frac{1}{p-2}}\left(\frac{\|v_n\|^2}{\int_{\mathbb{R}^3} b_\infty |v_n|^p dx}\right)^{\frac{1}{p-2}}$, and $\left(\frac{2}{4-p}\right)^{\frac{1}{p-2}}\left(\frac{\|v_n\|^2}{\int_{\mathbb{R}^3} b_\infty |v_n|^p dx}\right)^{\frac{1}{p-2}} > 1$ by using (2.12) and (3.15). This implies that $(h^{\infty}_{v_n})'(t) > 0$ $(0 < t < 1)$, which indicates that $h^{\infty}_{v_n}$ is increasing on $(t^-, 1)$ for sufficiently large n. When $p = 4$, we have

$$(h^{\infty}_{v_n})'(t) = t^3\Big(\bar{f}^{\infty}(t) - \int_{\mathbb{R}^3} b_\infty |v_n|^4 dx + \lambda \int_{\mathbb{R}^3} K_\infty \phi_{K_\infty, v_n} v_n^2 dx\Big) \quad \text{for } t > 0,$$

where $\bar{f}^{\infty}(t) := t^{-2}\|v_n\|^2$ is decreasing for $t > 0$. This means that $(h^{\infty}_{v_n})'(t) > 0$ $(0 < t < 1)$, and $h^{\infty}_{v_n}$ is increasing on $(t^-, 1)$ for sufficiently large n. So, for,$2 < p \le 4$, $h^{\infty}_{v_n}(t^-) < h^{\infty}_{v_n}(1)$ holds for sufficiently large n. This means that $\mathcal{J}^{\infty}(t^- v_n) < \mathcal{J}^{\infty}(v_n)$ for sufficiently large n.

Using (3.10)–(3.12) we again obtain

$$\mathcal{J}^{\infty}(t^- w) < \liminf_{n\to\infty} \mathcal{J}^{\infty}(t^- v_n) \le \liminf_{n\to\infty} \mathcal{J}^{\infty}(v_n) = \alpha^-_\infty,$$

which is a contradiction. However, we get that $v_n \to w$ in $H^1(\mathbb{R}^3)$ and $\mathcal{J}^{\infty}(v_n) \to \mathcal{J}^{\infty}(w) = \alpha^-_\infty$ as $n \to \infty$.

In addition, we find that for $2 < p < 4$,

$$\frac{(p-2)\bar{S}^2\hat{S}^4}{2(4-p)K^2_\infty}\left(\frac{b_\infty(4-p)^2}{2pS^p_p}\right)^{\frac{2}{p-2}} < \frac{p-2}{2(4-p)}\left(\frac{4-p}{p}\right)^{\frac{2}{p-2}}\Lambda_0.$$

So, w is a minimizer for $\mathcal{J}^{\infty}$ on $\mathcal{M}^-_\infty$ for each $0 < \lambda < \Lambda$. For $2 < p < 4$, it follows from (3.2) that

$$\mathcal{J}^{\infty}(w) = \alpha^-_\infty \le \mathcal{J}^{\infty}(t^-_\infty w_0) < \frac{A(p)(p-2)}{2p}\left(\frac{2S^p_p}{b_\infty(4-p)}\right)^{\frac{2}{p-2}},$$

which indicates that $w \in \mathcal{M}^{(1)}_\infty$. Since $|w| \in \mathcal{M}^-_\infty$ and $\mathcal{J}^{\infty}(|w|) = \mathcal{J}^{\infty}(w) = \alpha^-_\infty$, we can see that w is a positive solution of equation (3.2) according to Lemma 2.4. It also implies that $(w, \phi_{K_\infty, w})$ is a positive solution of system (3.1).

Note that for $2 < p < 4$, there holds

$$(4-p)\int_{\mathbb{R}^3} b_\infty |w|^p dx < 2\|w\|^2 \quad \text{and} \quad t_{b_\infty}(w)w \in \mathcal{M}^0_\infty,$$

where

$$\left(\frac{4-p}{2}\right)^{\frac{1}{p-2}} < t_{b_\infty}(w) := \left(\frac{|w|^2}{\int_{\mathbb{R}^3} b_\infty |w|^p dx}\right)^{\frac{1}{p-2}} < 1. \tag{3.16}$$

According to Lemma 2.9, for $2 < p < 4$, we have $\mathcal{J}^{\infty}(w) = \sup_{0\le t\le t^+} \mathcal{J}^{\infty}(tw)$, where $t^+ > \left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} t_{b_\infty}(w) > 1$ by (3.16). Using this, together with (3.37), we get $\mathcal{J}^{\infty}(w) > \mathcal{J}^{\infty}(t_{b_\infty}(w)w)$. Similarly, for $p = 4$, we can also get the above inequality. So, we have

$$\begin{aligned}\alpha^-_\infty &= \mathcal{J}^{\infty}(w) > \mathcal{J}^{\infty}(t_{b_\infty}(w)w) \\ &\ge \alpha^{\infty}_0 + \frac{\lambda[t_{b_\infty}(w)]^4}{4}\int_{\mathbb{R}^3} K_\infty \phi_{K_\infty, w} w^2 dx \\ &> \alpha^0_\infty.\end{aligned}$$

Consequently, we complete the proof.

3.2 Proof of Theorem 1.4

Definition 3.1.

(1) For $\beta \in \mathbb{R}$, a sequence $\{u_n\}$ is a $(PS)_\beta$-sequence in $H^1(\mathbb{R}^3)$ for $\mathcal{J}$ if $\mathcal{J}(u_n) = \beta + o(1)$ and $\mathcal{J}(u_n)'(u_n) = o(1)$ strongly in $H^{-1}(\mathbb{R}^3)$ as $n \to \infty$.

(2) If every $(PS)_\beta$-sequence in $H^1(\mathbb{R}^3)$ for $\mathcal{J}$ contains a convergent subsequence, we can say that $\mathcal{J}$ satisfies the $(PS)_\beta$-sequence in $H^1(\mathbb{R}^3)$.

Lemma 3.2. *Let $\{u_n\}$ be a bounded $(PS)_\beta$-sequence in $H^1(\mathbb{R}^3)$ for $\mathcal{J}$. There exists a subsequence $\{u_n\}$, a number $l \in \mathbb{N}$, a sequence $\{x_n^{(k)}\} \subset \mathbb{R}^3$ for $1 \le k \le l$, a function $v_0 \in H^1(\mathbb{R}^3)$, and $0 \neq w^i \in H^1(\mathbb{R}^3)$ when $1 \le i \le l$ such that*

(i) $|x_n^k| \to \infty$ *and* $|x_n^k - x_n^h| \to \infty$, *as* $n \to \infty, 1 \le k \neq h \le l$;

(ii) $-\Delta v_0 + v_0 + \lambda K(x)\phi_{K,v_0} v_0 = b(x)|v_0|^{p-2}v_0$ *in* $\mathbb{R}^3$;

(iii) $-\Delta w^i + w^i + \lambda K_\infty \phi_{K_\infty,w^i} w^i = b(x)|w^i|^{p-2}w^i$ *in* $\mathbb{R}^3$;

(iv) $u_n = v_0 + \sum_{i=1}^{l}(\cdot - x_n^i) + o(1)$ *strongly in* $H^1(\mathbb{R}^3)$;

(v) $\mathcal{J}(u_n) = \mathcal{J}(v_0) + \sum_{i=1}^{l} \mathcal{J}^\infty(w^i) + o(1)$.

The proof is similar to the argument of [13] Lemma 4.5, so we omit it here.

Corollary 3.3. *Suppose that $\{u_n\} \subset \mathcal{M}^-$ is a $(PS)_\beta$-sequence in $H^1(\mathbb{R}^3)$ for $\mathcal{J}$ with $0 < \beta < \alpha_\infty^-$. Then there exist a subsequence $\{u_n\}$ and a nonzero u_0 in $H^1(\mathbb{R}^3)$ such that $u_n \to u_0$ strongly in $H^1(\mathbb{R}^3)$ and $\mathcal{J}(u_0) = \beta$. However, (u_0, ϕ_{u_0}) is a nonzero solution of equation (1.4).*

By Theorem 1.3, we know that equation (3.2) have a positive solution $w(x) \in \mathcal{M}_\infty^-$ (up to translation) such that for $2 < p \le 4$, there holds

$$\mathcal{J}^\infty(w) = \alpha_\infty^- \quad \text{and} \quad \frac{4-p}{2}\int_{\mathbb{R}^3} b_\infty |w|^p dx < \|w\|^2.$$

Define $Q_b(w)$ as

$$\left(\frac{(4-p)b_\infty}{2b_{\max}}\right)^{\frac{1}{p-2}} < Q_b(w) := \left(\frac{\|w\|^2}{\int_{\mathbb{R}^3} b(x)|w|^p dx}\right)^{\frac{1}{p-2}}.$$

Lemma 3.4. *Suppose that $0 < \lambda < \Lambda$. Then the following two statements are true.*

(i) If $2 < p < 4$, then there exists $t^\infty > \left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} t_{b_\infty}(w) > 1$ such that

$$\mathcal{J}^\infty(w) = \sup_{0 \le t \le t^\infty} \mathcal{J}^\infty(tw) = \alpha_\infty^-, \tag{3.17}$$

where $t_{b_\infty}(w)$ is defined as (3.16).

(ii) If $p = 4$, then it has

$$\mathcal{J}^\infty(w) = \sup_{t \ge 0} \mathcal{J}^\infty(tw) = \sup_{t \ge 1} \mathcal{J}^\infty(tw) = \alpha_\infty^-. \tag{3.18}$$

Proof. (i) Let

$$g^{\infty}(t) = t^{-2}\|w\|^2 - t^{p-4}\int_{\mathbb{R}^3} b_\infty |w|^p dx \quad \text{for } t > 0. \tag{3.19}$$

Obviously, it satisfies

$$g^{\infty}(1) + \lambda \int_{\mathbb{R}^3} K_\infty \phi_{K_\infty,w} w^2 dx = 0 \quad \text{for all } 0 < \lambda < \Lambda. \tag{3.20}$$

Then we get $g^{\infty}(t_{b_\infty}(w)) = 0$, $\lim_{t\to 0^+} g^{\infty}(t) = 0$ and $\lim_{t\to\infty} g^{\infty}(t) = 0$.

For $2 < p < 4$ and the equality $(g^{\infty})'(t) = t^{-3}(-2\|w\|^2 + (4-p)t^{p-2}\int_{\mathbb{R}^3} b_\infty |w|^p dx)$, we find that g^{∞} is decreasing when $0 < t < (\frac{2}{4-p})^{\frac{1}{p-2}} t_{b_\infty}(w)$ and is increasing when $t > (\frac{2}{4-p})^{\frac{1}{p-2}} t_{b_\infty}(w)$. This means that

$$\inf_{t>0} g^{\infty}(t) = g^{\infty}\left(\left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} t_{b_\infty}(w)\right). \tag{3.21}$$

Moreover, from (3.16) we know that

$$\left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} t_{b_\infty}(w) > 1. \tag{3.22}$$

So from (3.20)–(3.22) that

$$\inf_{t>0} g^{\infty}(t) < g^{\infty}(1) = -\lambda \int_{\mathbb{R}^3} K_\infty \phi_{K_\infty,w} w^2 dx. \tag{3.23}$$

This means that there exists $t^{\infty} > (\frac{2}{4-p})^{\frac{1}{p-2}} t_{b_\infty}(w) > 1$ such that $g^{\infty}(t^{\infty}) + \lambda \int_{\mathbb{R}^3} K_\infty \phi_{K_\infty,w} w^2 dx = 0$. Using a similar argument as the proof of Lemma (2.9) (i), we get (3.17).

(ii) Let $\bar{g}^{\infty}(t) = t^{-2}\|w\|^2$ for $t > 0$. Then we get $\bar{g}^{\infty}(1) - \int_{\mathbb{R}^3} b_\infty |w|^4 dx + \lambda \int_{\mathbb{R}^3} K_\infty \phi_{K_\infty,w} w^2 dx = 0$ for all $0 < \lambda < \Lambda$. we can observe that $\bar{g}^{\infty}(t)$ is decreasing when $t > 0$ and $\lim_{t\to 0^+} \bar{g}^{\infty}(t) = \infty$ and $\lim_{t\to\infty} \bar{g}^{\infty}(t) = 0$. Since w is the positive solution of equation (3.2), we have $\int_{\mathbb{R}^3} b_\infty |w|^4 dx - \lambda \int_{\mathbb{R}^3} K_\infty \phi_{K_\infty,w} w^2 dx = \|w\|^2 > 0$, which shows that $t = 1$ is a unique positive solution of the equation $\bar{g}^{\infty}(t) - \int_{\mathbb{R}^3} b_\infty |w|^4 dx + \lambda \int_{\mathbb{R}^3} K_\infty \phi_{K_\infty,w} w^2 dx = 0$. By the proof of Lemma 2.9 (ii) , we get (3.18). □

Lemma 3.5. *Suppose that $0 < \lambda < \Lambda$ and conditions (G_1)–(G_3) hold. Then the following two statements are true.*

(i) If $2 < p < 4$, then there exist two constants $t^{(1)}$ and $t^{(2)}$ satisfying

$$Q_b(w) < t^{(1)} < \left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} Q_b(w) < t^{(2)},$$

such that $t^{(i)}w \in \mathcal{M}^{(i)} (i = 1, 2)$, $\mathcal{J}(t^{(1)}w) = \sup_{0\le t\le t^{(2)}} \mathcal{J}(tw) < \alpha_\infty^-$, and $\mathcal{J}(t^{(2)}w) = \inf_{t\ge t^{(1)}} \mathcal{J}(tw)$.

(ii) If $p = 4$, then there exists a unique constant

$$\tilde{t} = \left(\frac{\|w\|^2}{\int_{\mathbb{R}^3} b(x)|w|^4 dx - \lambda \int_{\mathbb{R}^3} K(x)\phi_{K(x),w} w^2 dx}\right)^{\frac{1}{2}} > Q_b(w)$$

such that $\tilde{t}w \in \mathcal{M}^{(1)} = \mathcal{M}^- = \mathcal{M}$ and $\mathcal{J}(\tilde{t}w) = \sup_{t\ge 0} \mathcal{J}(tw) = \sup_{t\ge Q_b(w)} \mathcal{J}(tw) < \alpha_\infty^-$.

Proof. (i) Let $g(t) = t^{-2}\|w\|^2 - t^{p-4}\int_{\mathbb{R}^3} b(x)|w|^p dx$ for $t > 0$. Clearly, $tw \in \mathcal{M}$ if and only if

$$g(t) + \lambda \int_{\mathbb{R}^3} K(x)\phi_{K(x),w} w^2 dx = 0. \tag{3.24}$$

From (3.24) gives $g(Q_b(w)) = 0$, $\lim_{t\to 0^+} g(t) = \infty$ and $\lim_{t\to\infty} g(t) = 0$.

In view of $2 < p < 4$ and $g'(t) = t^{-3}(-2\|w\|^2 + (4-p)t^{p-2}\int_{\mathbb{R}^3} b(x)|w|^p dx)$, we see that $g(t)$ is decreasing on $0 < t < \left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} Q_b(w)$ and is increasing on $t > \left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} Q_b(w)$. Then from condition (G_3) that $Q_b(w) \le Q_{b_\infty}(w) < 1$ and $g(t) \le g^\infty(t)$, where $g^\infty(t)$ is given in (3.19). Using condition (G_3) and (3.23) again, we deduce that

$$\begin{aligned}
\inf_{t>0} g(t) &= g\left(\left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} Q_b(w)\right)\\
&\le -\frac{p-2}{4-p}\left(\frac{4-p}{2}\right)^{\frac{2}{p-2}} \|w\|^2 \left(\frac{\|w\|^2}{\int_{\mathbb{R}^3} b(x)|w|^p dx}\right)^{\frac{-2}{p-2}}\\
&= \inf_{t>0} g^\infty(t) < -\lambda \int_{\mathbb{R}^3} K_\infty \phi_{K_\infty,w} w^2 dx\\
&\le -\lambda \int_{\mathbb{R}^3} K(x)\phi_{K(x),w} w^2 dx.
\end{aligned}$$

Then, it can be concluded that there are two constants $t^{(1)}$ and $t^{(2)}$ satisfying $Q_b(w) < t^{(1)} < \left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} Q_b(w) < t^{(2)}$ such that $g(t^{(i)}) + \lambda \int_{\mathbb{R}^3} K(x)\phi_{K,w} w^2 dx = 0$ for $i = 1, 2$. That is, $t^{(i)}w \in \mathcal{M}$ $(i = 1, 2)$.

Direct calculation of the second derivative gives

$$h''_{t^{(1)}w}(1) = -2\|t^{(1)}w\|^2 + (4-p)\int_{\mathbb{R}^3} b(x)|t^{(1)}w|^p dx = (t^{(1)})^5 g'(t^{(1)}) < 0,$$

and

$$h''_{t^{(2)}w}(1) = -2\|t^{(2)}w\|^2 + (4-p)\int_{\mathbb{R}^3} b(x)|t^{(2)}w|^p dx = (t^{(2)})^5 g'(t^{(2)}) > 0.$$

Then we get $t^{(1)}w \in \mathcal{M}^-$ and $t^{(2)}w \in \mathcal{M}$.

Note that

$$t^{(1)} < \left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} Q_b(w) \le \left(\frac{2}{4-p}\right)^{\frac{1}{p-2}} t_{b_\infty}(w) < \min\left\{\left(\frac{2}{4-p}\right)^{\frac{1}{p-2}}, t^\infty\right\},$$

where t^∞ is the same as described in Lemma 3.4. For each $0 < \lambda < \Lambda$ and from Lemma 3.4 and condition (G_3), there holds

$$\begin{aligned}
\mathcal{J}(t^{(1)}w) &= \mathcal{J}^\infty(t^{(1)}w) - \frac{[t^{(1)}]^p}{p}\int_{\mathbb{R}^3}[b(x) - b_\infty]|w|^p dx\\
&\quad + \frac{\lambda [t^{(1)}]^4}{4}\left(\int_{\mathbb{R}^3} K(x)\phi_{K,w} w^2 dx - \int_{\mathbb{R}^3} K_\infty \phi_{K_\infty,w} w^2 dx\right)\\
&\le \sup_{0\le t\le t^\infty} \mathcal{J}^\infty(tw) - \frac{[t^{(1)}]^p}{p}\int_{\mathbb{R}^3}[b(x) - b_\infty]|w|^p dx\\
&\quad + \frac{\lambda [t^{(1)}]^4}{4}\left(\int_{\mathbb{R}^3} K(x)\phi_{K,w} w^2 dx - \int_{\mathbb{R}^3} K_\infty \phi_{K_\infty,w} w^2 dx\right)\\
&< \alpha_\infty^- < A(p)\frac{p-2}{2p}\left(\frac{2S_p^p}{b_\infty(4-p)}\right)^{\frac{2}{p-2}}.
\end{aligned}$$

In other words, $t^{(1)}w \in \mathcal{M}^{(1)}$ and $\mathcal{J}(t^{(1)}w) < \alpha_\infty^-$. From the equation $h'_w(t) = t^3(g(t) + \lambda \int_{\mathbb{R}^3} K(x)\phi_{K,w}w^2dx)$, we notice that $h'_w(t) > 0$ for all $t \in (0, t^{(1)}) \cup (t^{(2)}, \infty)$ and $h'_w(t) < 0$ for all $t \in (t^{(1)}, t^{(2)})$. Finally, we get $\mathcal{J}(t^{(1)}w) = \sup_{0\le t\le t^{(2)}} \mathcal{J}(tw)$ and $\mathcal{J}(t^{(2)}w) = \inf_{t\ge t^{(1)}} \mathcal{J}(tw)$. That is, $\mathcal{J}(t^{(2)}w) \le \mathcal{J}(t^{(1)}w) < \alpha_\infty^-$, and so $t^{(2)}w \in \mathcal{M}^{(2)}$.

(ii) Let $\hat{g}(t) = t^{-2}\|w\|^2$ for $t > 0$. Clearly, $tw \in \mathcal{M}^{(1)} = \mathcal{M}^- = \mathcal{M}$ if and only if $\hat{g}(t) - \int_{\mathbb{R}^3} b(x)w^4dx + \lambda \int_{\mathbb{R}^3} K(x)\phi_{K,w}w^2dx = 0$. After analysis $\hat{g}(t)$, we know that $\hat{g}(t) > 0$ is decreasing for $t > 0$, and $\lim_{t\to 0^+} \hat{g}(t) = \infty$ and $\lim_{t\to\infty} \hat{g}(t) = 0$. For $0 < \lambda < \Lambda$ and from condition (G_3) we have

$$\begin{aligned}\int_{\mathbb{R}^3} b(x)w^4dx - \lambda \int_{\mathbb{R}^3} K(x)\phi_{K,w}w^2dx &> \int_{\mathbb{R}^3} b_\infty w^4dx - \lambda \int_{\mathbb{R}^3} K_\infty\phi_{K_\infty,w}w^2dx \\ &= \|w\|^2 > 0.\end{aligned}$$

This implies that the equation $\hat{g}(t) - \int_{\mathbb{R}^3} b(x)w^4dx + \lambda \int_{\mathbb{R}^3} K(x)\phi_{K,w}w^2dx = 0$ has a unique positive solution $\hat{t} = \left(\frac{\|w\|^2}{\int_{\mathbb{R}^3} b(x)w^4dx - \lambda \int_{\mathbb{R}^3} K(x)\phi_{K,w}w^2dx}\right)^{\frac{1}{2}} > Q_b(w)$. Then we get $\hat{t}w \in \mathcal{M}^{(1)} = \mathcal{M}^- = \mathcal{M}$. Similar to the discussion of case (i), we get that $\mathcal{J}(\hat{t}w) = \sup_{t\ge 0} \mathcal{J}(tw) = \sup_{t\ge Q_b(w)} \mathcal{J}(tw) < \alpha_\infty^-$. This completes the proof. □

Learning [23,26] we get the following result.

Lemma 3.6 ([25]). *Suppose that $4 < p \le 4$ and $0 < \lambda < \Lambda$. Then for each $u \in \mathcal{M}^{(1)}$, there exist $v > 0$ and a differentiable function: $t_* : B(0; v) \subset H^1(\mathbb{R}^3) \to \mathbb{R}^+$ such that $t_*(0) = 1$ and $t_*(v)(u - v) \in \mathcal{M}^{(1)}$ for all $v \in B(0; v)$, and there holds*

$$\langle (t_*)'(0), \varphi\rangle = \frac{2\int_{\mathbb{R}^3}(\nabla u\nabla\varphi + u\varphi)dx + 4\lambda \int_{\mathbb{R}^3} K(x)\phi_{K,u}u\varphi dx - p\int_{\mathbb{R}^3} b(x)|u|^{p-2}u\varphi dx}{\|u\|^2 - (p-1)\int_{\mathbb{R}^3} b(x)|u|^p dx}$$

for all $\varphi \in H^1(\mathbb{R}^3)$.

By (2.6) and Lemma 2.7, for $2 < p < 4$, we define $\alpha^- = \inf_{u\in\mathcal{M}^{(1)}} \mathcal{J}(u) = \inf_{u\in\mathcal{M}^-} \mathcal{J}(u)$ and $\alpha^+ = \inf_{u\in\mathcal{M}^{(2)}} \mathcal{J}(u) = \inf_{u\in\mathcal{M}^+} \mathcal{J}(u)$. When $p = 4$, we define $\alpha^- = \inf_{u\in\mathcal{M}^{(1)}} \mathcal{J}(u) = \inf_{u\in\mathcal{M}} \mathcal{J}(u)$.

Proposition 3.7. *Suppose that $2 < p \le 4$ and $0 < \lambda < \Lambda$. Then there exists a sequence $\{u_n\} \subset \mathcal{M}^{(1)}$ such that*

$$\mathcal{J}(u_n) = \alpha^- + o(1) \quad \text{and} \quad \mathcal{J}'(u_n) = o(1) \quad \text{in } H^{-1}(\mathbb{R}^3). \tag{3.25}$$

Proof. According to the Ekeland variational principle [15], it follows from Lemma 2.5 that there exists a minimization sequence $\{u_n\} \subset \mathcal{M}^{(1)}$ such that $\mathcal{J}(u_n) < \alpha^- + \frac{1}{n}$ and

$$\mathcal{J}(u_n) \le \mathcal{J}(w) + \frac{1}{n}\|w - u_n\| \ \text{ for all } \ w \in \mathcal{M}^{(1)}. \tag{3.26}$$

By Lemma 3.6 with $u = u_n$, there exists a function $\bar{t}_* : B(0; \epsilon) \to \mathbb{R}$ for some $\epsilon > 0$ such that $\bar{t}_*(w)(u_n - w) \in \mathcal{M}^{(1)}$. Let $0 < \rho < \epsilon$ and $u \in H^1(\mathbb{R}^3)$ with $u \ne 0$. we set $w_\rho = \frac{\rho u}{\|u\|}$ and $z_\rho = \bar{t}_*(w_\rho)(u_n - w_\rho)$. Since $z_\rho \in \mathcal{M}^{(1)}$, from (3.26) we can get that $\mathcal{J}(z_\rho - \mathcal{J}(u_n) \ge -\frac{1}{n}\|z_\rho - u_n\|$. Generated using the median theorem

$$\langle \mathcal{J}'(u_n), z_\rho - u_n\rangle + o(\|z_\rho - u_n\|) \ge -\frac{1}{n}\|z_\rho - u_n\|,$$

and

$$\langle \mathcal{J}'(u_n), -w_\rho\rangle + (\bar{t}_*(w_\rho)-1)\langle \mathcal{J}'(u_n), u_n - w_\rho\rangle \geq -\frac{1}{n}\|z_\rho - u_n\| + o(\|z_\rho - u_n\|). \tag{3.27}$$

Observed $\bar{t}_*(w_\rho)(u_n - w) \in \mathcal{M}^{(1)}$. From (3.27) it gives

$$\begin{aligned} &-\rho\left\langle \mathcal{J}'(u_n), \frac{u}{\|u\|}\right\rangle + \frac{(\bar{t}_*(w_\rho)-1)}{\bar{t}_*(w_\rho)}\langle \mathcal{J}'(z_\rho), \bar{t}_*(w_\rho)(u_n - w)\rangle \\ &\quad + (\bar{t}_*(w_\rho)-1)\langle \mathcal{J}'(u_n) - \mathcal{J}'(z_\rho), u_n - w_\rho\rangle \\ &\geq -\frac{1}{n}\|z_\rho - u_n\| + o(\|z_\rho - u_n\|). \end{aligned}$$

Rewrite the above inequality as

$$\begin{aligned} \left\langle \mathcal{J}'(u_n), \frac{u}{\|u\|}\right\rangle \leq \frac{\|z_\rho - u_n\|}{\rho n} + \frac{o(\|z_\rho - u_n\|)}{\rho} \\ + \frac{(\bar{t}_*(w_\rho)-1)}{\rho}\langle \mathcal{J}'(u_n) - \mathcal{J}'(z_\rho), u_n - w_\rho\rangle. \end{aligned} \tag{3.28}$$

Then, there exist a constant $C>0$ independent of ρ such that $\|z_\rho - u_n\| \leq \rho + C(|\bar{t}_*(w_\rho)-1|)$ and $\lim_{\rho\to 0}\frac{|\bar{t}_*(w_\rho)-1|}{\rho} \leq \|(\bar{t}_*)'(0)\| \leq C$. Letting $\rho \to 0$ in (3.28) and using the fact that $\lim_{\rho\to 0}\|z_\rho - u_n\| = 0$, we get $\langle \mathcal{J}'(u_n), \frac{u}{\|u\|}\rangle \leq \frac{C}{n}$, this allows us to get (3.25). □

Therefore, we begin to prove the proof of Theorem 1.4.

By Proposition 3.7, for $2 < p \leq 4$, there exists a sequence $\{u_n\} \subset \mathcal{M}^{(1)}$ satisfying

$$\mathcal{J}(u_n) = \alpha^- + o(1) \quad \text{and} \quad \mathcal{J}'(u_n) = o(1) \quad \text{in } H^{-1}(\mathbb{R}^3).$$

From Corollary 3.3 and Lemma 3.4, 3.5, we know that equation (1.4) has a non-trivial solution $v \in \mathcal{M}^-$ such that $\mathcal{J}(v) = \alpha^-$. So, v is a minimizer for $\mathcal{J}$ on $\mathcal{M}^-$. In particular, for $2 < p < 4$, using $\alpha^- < \alpha_\infty^- < A(p)\frac{p-2}{2p}\left(\frac{2S_p^p}{b_\infty(4-p)}\right)^{\frac{2}{p-2}}$, we obtain $v \in \mathcal{M}^-$. Through similar discussions, we get $|v| \in \mathcal{M}^-$ and $\mathcal{J}(|v|) = \mathcal{J}(v) = \alpha^-$. According to Lemma 2.4, v is a positive solution to equation (1.4). Therefore, $(v, \phi_{K,v})$ is a positive solution to the system (1.3).

3.3 Proof of Theorem 1.5

Lemma 3.8. *Suppose that* $\frac{1+\sqrt{73}}{3} < p < 4$ *and condition* (G_4) *holds. Let* u_0 *be a nontrivial solution of equation* (1.4) *. Then* $u_0 \in \mathcal{M}^-$.

Proof. Since u_0 is a nontrivial solution of equation (1.4), there holds

$$\int_{\mathbb{R}^3}|\nabla u_0|^2\,dx + \int_{\mathbb{R}^3}u_0^2dx + \lambda\int_{\mathbb{R}^3}K(x)\phi_{K,u_0}u_0^2dx - \int_{\mathbb{R}^3}b(x)\,|u_0|^p\,dx = 0. \tag{3.29}$$

Following the argument of [12] it is not difficult to verify that equation (1.4) satisfies the following Pohožaev type identity:

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u_0|^2dx + \frac{3}{2}\int_{\mathbb{R}^3}u_0^2dx + \frac{5\lambda}{4}\int_{\mathbb{R}^3}K(x)\phi_{K,u_0}u_0^2dx \\ &\quad + \frac{\lambda}{2}\int_{\mathbb{R}^3}\langle\nabla K(x), x\rangle\phi_{K,u_0}u_0^2dx + \frac{\lambda}{4a}\int_{\mathbb{R}^3}K(x)\psi_{K,u_0}u_0^2dx \\ &= \frac{3}{p}\int_{\mathbb{R}^3}b(x)|u_0|^pdx + \frac{1}{p}\int_{\mathbb{R}^3}\langle\nabla b(x), x\rangle|u_0|^pdx. \end{aligned} \tag{3.30}$$

Combining (3.29) and (3.30) we get

$$\begin{aligned}\int_{\mathbb{R}^3}|\nabla u_0|^2dx = &\frac{3(p-2)}{6-p}\int_{\mathbb{R}^3}u_0^2dx + \frac{5p-12}{2(6-p)}\lambda\int_{\mathbb{R}^3}K(x)\phi_{K,u_0}u_0^2dx\\ &+\frac{p\lambda}{6-p}\int_{\mathbb{R}^3}\langle\nabla K(x),x\rangle\phi_{K,u_0}u_0^2dx + \frac{p\lambda}{2a(6-p)}\int_{\mathbb{R}^3}K(x)\psi_{K,u_0}u_0^2dx\\ &-\frac{2}{6-p}\int_{\mathbb{R}^3}\langle\nabla b(x),x\rangle|u_0|^pdx. \end{aligned}\tag{3.31}$$

From (2.3), (3.31) and condition (G_4) we obtain that

$$\begin{aligned}h''_{u_0}(1) =& -(p-2)\|u_0\|^2 + (4-p)\int_{\mathbb{R}^3}K(x)\phi_{K,u_0}u_0^2dx\\ =& -(p-2)\int_{\mathbb{R}^3}|\nabla u_0|^2dx - (p-2)\int_{\mathbb{R}^3}u_0^2dx + (4-p)\int_{\mathbb{R}^3}K(x)\phi_{K,u_0}u_0^2dx\\ =& -\frac{2p(p-2)}{6-p}\int_{\mathbb{R}^3}u_0^2dx - \frac{p(p-2)\lambda}{2a(6-p)}\int_{\mathbb{R}^3}K(x)\psi_{K,u_0}u_0^2dx\\ &-\lambda\int_{\mathbb{R}^3}\left(\frac{3p^2-2p-24}{2(6-p)}K(x) + \frac{p(p-2)}{6-p}\langle\nabla K(x),x\rangle\right)\phi_{K,u_0}u_0^2dx\\ &+\frac{2(p-2)}{6-p}\int_{\mathbb{R}^3}\langle\nabla b(x),x\rangle|u_0|^pdx\\ <& \ 0.\end{aligned}$$

So, we get $u_0 \in \mathcal{M}^-$. □

We now come to the proof of Theorem 1.5.

Proof. Let v be a positive solution of equation (1.4), then we get $v \in \mathcal{M}^-$ and $\mathcal{J}(v) = \inf_{u\in\mathcal{M}^-}\mathcal{J}(u) = \alpha^-$. Next by Lemma 3.8, we know that v is a ground state solution of equation (1.4). Therefore, $(v, \phi_{K,v})$ is a positive solution of system (1.3). □

4 Appendix

In this section, we give the calculation procedure of Pohožaev identity.

Let $(u,\phi) \in H^1_\phi(\mathbb{R}^3)\times\mathcal{D}$ be a nontrivial solution of (1.1). Recall that $\phi = \phi_{K,u}$. we have

$$\|\nabla u\|_2^2 + \|u\|_2^2 + \lambda\int K(x)\phi u^2 - b(x)\|u\|_p^p = 0 \tag{4.1}$$

and

$$\|\nabla\phi\|_2^2 + a^2\|\Delta\phi\|_2^2 = 4\pi\int K(x)\phi u^2, \tag{4.2}$$

that are usually called Nehari identities.

In fact, if (u,ϕ) solve (1.1), recalling the regularity proved in Appredix A.1. [13], for every $R>0$, we have

$$\int_{B_R} -\Delta u\langle x\cdot\nabla u\rangle = -\frac{1}{2}\int_{B_R}|\nabla u|^2 - \frac{1}{R}\int_{\partial B_R}|x\cdot\nabla u|^2 + \frac{R}{2}\int_{\partial B_R}|\nabla u|^2, \tag{4.3}$$

$$\int_{B_R} K(x)\phi u\langle x\cdot\nabla u\rangle = -\frac{1}{2}\int_{B_R} K(x)u^2\langle x\cdot\nabla\phi\rangle - \frac{1}{2}\int_{B_R}\phi u^2\langle x\cdot\nabla K(x)\rangle \tag{4.4}$$

$$-\frac{3}{2}\int_{B_R} K(x)\phi u^2 + \frac{R}{2}\int_{\partial B_R} K(x)\phi u^2, \tag{4.5}$$

$$\int_{B_R} u\langle x\cdot\nabla u\rangle = -\frac{3}{2}\int_{B_R} u^2 + \frac{R}{2}\int_{\partial B_R} u^2, \tag{4.6}$$

$$\int_{B_R} b(x)|u|^{p-2}u\langle x\cdot\nabla u\rangle = -\frac{1}{p}\int_{B_R}\langle x\cdot\nabla b(x)\rangle|u|^p - \frac{3}{p}\int_{B_R} b(x)|u|^p dx + \frac{R}{p}\int_{\partial B_R}|u|^p, \tag{4.7}$$

where B_R is the ball of $\mathbb{R}^3$ centered in the origin and with radius R (see also [12]), and, since

$$\Delta^2\phi\langle x\cdot\nabla\phi\rangle = \operatorname{div}\left(\nabla\Delta\phi\langle x\cdot\nabla\phi\rangle - \Delta\phi\nabla\phi - \mathbb{F} + x\frac{(\Delta\phi)^2}{2}\right) + \frac{(\Delta\phi)^2}{2},$$

where $\mathbb{F}_i = \Delta\phi\langle x\cdot\nabla(\partial_i\phi)\rangle, i = 1,2,3$, then

$$\int_{B_R}\Delta^2\phi\langle x\cdot\nabla\phi\rangle = \frac{1}{2}\int_{B_R}(\Delta\phi)^2 + \int_{\partial B_R}\left(\nabla\Delta\phi\langle x\cdot\nabla\phi\rangle - \Delta\phi\nabla\phi - \mathbb{F} + x\frac{(\Delta\phi)^2}{2}\right)\cdot v. \tag{4.8}$$

Multiplying the first equation of (1.1) by $x\cdot\nabla u$ and the second equation by $x\cdot\nabla\phi$ and integrating on B_R, by (4.3), (4.4), (4.6), (4.7), and (4.8) we obtain

$$\begin{aligned} &-\frac{1}{2}\int_{B_R}|\nabla u|^2 - \frac{1}{R}\int_{\partial B_R}|x\cdot\nabla u|^2 + \frac{R}{2}\int_{\partial B_R}|\nabla u|^2 - \frac{3}{2}\int_{B_R}u^2 + \frac{R}{2}\int_{\partial B_R}u^2 \\ &\quad -\frac{\lambda}{2}\int_{B_R}K(x)u^2\langle x\cdot\nabla\phi\rangle - \frac{\lambda}{2}\int_{B_R}\phi u^2\langle x\cdot\nabla K(x)\rangle - \frac{3\lambda}{2}\int_{B_R}K(x)\phi u^2 + \frac{\lambda R}{2}\int_{\partial B_R}K(x)\phi u^2 \\ &= -\frac{1}{p}\int_{B_R}\langle x\cdot\nabla b(x)\rangle|u|^p - \frac{3}{p}\int_{B_R}b(x)|u|^p dx + \frac{R}{p}\int_{\partial B_R}|u|^p \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} &4\pi\int_{B_R}K(x)u^2\langle x\cdot\nabla\phi\rangle \\ &\quad= -\frac{1}{2}\int_{B_R}|\nabla\phi|^2 - \frac{1}{R}\int_{\partial B_R}|x\cdot\nabla\phi|^2 + \frac{R}{2}\int_{\partial B_R}|\nabla\phi|^2 \\ &\qquad + \frac{a^2}{2}\int_{B_R}(\Delta\phi)^2 + a^2\int_{\partial B_R}\left(\nabla\Delta\phi\langle x\cdot\nabla\phi\rangle - \Delta\phi\nabla\phi - \mathbb{F} + x\frac{(\Delta\phi)^2}{2}\right)\cdot v. \end{aligned} \tag{4.10}$$

Substituting (4.10) into (4.9) we get

$$\begin{aligned} &-\frac{1}{2}\int_{B_R}|\nabla u|^2 - \frac{3}{2}\int_{B_R}u^2 + \frac{\lambda}{16\pi}\int_{B_R}|\nabla\phi|^2 - \frac{\lambda a^2}{16\pi}\int_{B_R}(\Delta\phi)^2 - \frac{3\lambda}{2}\int_{B_R}K(x)\phi u^2 \\ &\quad -\frac{\lambda}{2}\int_{B_R}\phi u^2\langle x\cdot\nabla K(x)\rangle + \frac{1}{p}\int_{B_R}\langle x\cdot\nabla b(x)\rangle|u|^p + \frac{3}{p}\int_{B_R}b(x)|u|^p dx \\ &= \frac{1}{R}\int_{\partial B_R}|x\cdot\nabla u|^2 - \frac{R}{2}\int_{\partial B_R}|\nabla u|^2 - \frac{R}{2}\int_{\partial B_R}u^2 - \frac{\lambda R}{2}\int_{\partial B_R}K(x)\phi u^2 \\ &\quad + \frac{R}{p}\int_{\partial B_R}|u|^p - \frac{\lambda}{8\pi R} - \int_{\partial B_R}|x\cdot\nabla\phi|^2 + \frac{\lambda R}{16\pi}\int_{\partial B_R}|\nabla\phi|^2 \\ &\quad + \frac{\lambda a^2}{8\pi}\int_{\partial B_R}\left(\nabla\Delta\phi\langle x\cdot\nabla\phi\rangle - \Delta\phi\nabla\phi - \mathbb{F} + x\frac{(\Delta\phi)^2}{2}\right)\cdot v. \end{aligned}$$

Using the same arguments as in [12, Proof of Theorem 1.2] we have that right-hand side tends to zero as $R \to +\infty$, since

$$\begin{aligned} \int_{\partial B_R} \nabla\Delta\phi \langle x\cdot\nabla\phi\rangle \cdot v &= R\int_{\partial B_R} \frac{\partial\Delta\phi}{\partial v}\frac{\partial\phi}{\partial v} \to 0,\\ \int_{\partial B_R} \Delta\phi\nabla\phi\cdot v &= \int_{\partial B_R} \Delta\phi\frac{\partial\phi}{\partial v} \to 0,\\ \int_{\partial B_R} \mathbb{F}\cdot v &= R\int_{\partial B_R} \frac{\partial^2\phi}{\partial v^2} \to 0,\\ \frac{1}{2}\int_{\partial B_R} (\Delta\phi)^2 x\cdot v &= \frac{R}{2}\int_{\partial B_R} (\Delta\phi)^2 \to 0. \end{aligned}$$

Finally, using formula (A.3) in [13] , the Pohožaev identity can be written as

$$\begin{aligned} 0 = &\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{3}{2}\int_{\mathbb{R}^3} u^2 dx + \frac{5\lambda}{4}\int_{\mathbb{R}^3} K(x)\phi_{K,u}u^2 dx\\ &+\frac{\lambda}{2}\int_{\mathbb{R}^3}\langle\nabla K(x), x\rangle\phi_{K,u}u^2 dx + \frac{\lambda}{4a}\int_{\mathbb{R}^3} K(x)\psi_{K,u}u^2 dx\\ &-\frac{3}{p}\int_{\mathbb{R}^3} b(x)|u|^p dx - \frac{1}{p}\int_{\mathbb{R}^3}\langle\nabla b(x), x\rangle |u|^p dx, \end{aligned}$$

where $\psi_{K,u} := e^{-\frac{|x|}{a}} * Ku^2 = \int_{\mathbb{R}^3} e^{-\frac{|x-y|}{a}} K(y)u^2(y)dy$.

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Attractivity of solutions of Riemann–Liouville fractional differential equations

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Abstract. Some new weakly singular integral inequalities are established by a new method, which generalize some results of this type in some previous papers. By these new integral inequalities, we present the attractivity of solutions for Riemann-Liouville fractional differential equations. Finally, several examples are given to illustrate our main results.

Keywords: weakly singular integral inequalities, Riemann–Liouville fractional derivative, fractional differential equations, attractivity.

2020 Mathematics Subject Classification: 26A33, 39A30, 34A08.

1 Introduction

The study of fractional differential equations has been of great interest in the past three decades. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications in various sciences. In particular, the existence, uniqueness and stability results of fractional differential equations have been studied by many papers and books. In recent years, many researchers have began to investigate the attractivity of solutions of fractional differential equations. For example, Furati and Tatar [4] investigated the asymptotic behavior for solutions of a weighted Cauchy-type nonlinear fractional problem. Kassim, Furati and Tatar [8] studied the asymptotic behavior of solutions for a class of nonlinear fractional differential equations involving two Riemann–Liouville fractional derivatives of different orders. Zhou et al. [13] studied the attractivity of solutions for fractional evolution equations with Riemann–Liouville fractional derivative. Gallegos and Duarte-Mermoud [5] studied the asymptotic behavior of solutions to Riemann–Liouville fractional systems. Tuan et al. [11] presented some results for existence of global solutions and attractivity for multidimensional fractional differential equations involving Riemann–Liouville derivative. Cong, Tuan and Trinh [2] presented some distinct asymptotic properties of solutions to Caputo fractional differential equations.

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In this paper, we first study the following weakly singular integral inequality

$$u(t) \le at^{-\alpha} + bt^{-\delta} \int_0^t (t-s)^{\beta-1} l(s) u^{\mu}(s) ds, \qquad t \in (0, +\infty), \tag{1.1}$$

where $a, b > 0$, $\alpha > 0$, $\delta \ge 0$, $0 < \beta < 1$ and $0 < \mu \le 1$. We know that weakly singular integral inequalities are well-known tools in the study of the fractional differential equations. The pioneering work of weakly singular integral inequalities was investigated by Henry [7]. In 1981, Henry [7, p. 190] studied the following weakly singular integral inequality

$$u(t) \le at^{\alpha-1} + b \int_0^t (t-s)^{\beta-1} s^{\gamma-1} u(s) ds, \qquad t \in (0, +\infty), \tag{1.2}$$

where α, β, γ are positive with $\beta + \gamma > 1$ and $\alpha + \gamma > 1$. Webb [12] also studied the following weakly singular Gronwall inequality

$$u(t) \le at^{-\alpha} + b + c \int_0^t (t-s)^{-\beta} s^{-\gamma} u(s) ds, \qquad \text{for a.e. } t \in (0, T], \tag{1.3}$$

where $0 < \alpha, \beta, \gamma < 1$ with $\alpha + \gamma < 1$ and $\beta + \gamma < 1$. Recently, Zhu [14] considered the following inequality

$$u(t) \le at^{-\alpha} + bt^{-\delta} \int_0^t (t-s)^{\beta-1} l(s) u(s) ds, \qquad t \in (0, +\infty), \tag{1.4}$$

where $\alpha > \delta \ge 0$ and $0 < \beta < 1$. Zhu [15] also considered the following weakly singular integral inequality

$$u(t) \le at^{-\alpha} + bt^{-\delta} \int_0^t (t-s)^{\beta-1} l(s) u^{\mu}(s) ds, \qquad t \in (0, +\infty), \tag{1.5}$$

where $1 > \alpha \ge \delta \ge 0$, $0 < \mu < 1$ and $0 < \beta < 1$. Some results of this type are also proved by Denton and Vatsala [3], Haraux [6], Kong and Ding [9].

Applying weakly singular integral inequality (1.1), we begin to investigate the attractivity of solutions of fractional differential equation

$$\begin{cases} D_{0^+}^{\beta} x(t) = f(t, x(t)), \\ \lim_{t \to 0^+} t^{1-\beta} x(t) = x_0, \end{cases} \tag{1.6}$$

where $\beta \in (0, 1)$ and $t \in (0, +\infty)$. As far as I know, there have been few papers to study the attractivity of fractional differential equation (1.6) by weakly singular integral inequalities. The conclusion and the method of the proof in this paper seem to be new.

The outline of this paper is as follows. In Section 2, we introduce some notations, definitions and theorems needed in our proofs. In Section 3, we obtain some new results concerning weakly singular integral inequalities. In the last Section, we give some sufficient conditions on the attractivity of solutions of fractional differential equation (1.6). Finally, some examples are given to illustrate our main results.

2 Preliminaries

In this section, we introduce some notations, definitions and theorems which will be needed later.

Let $\alpha \in (0, 1)$, we denote $C_{\alpha}(0, +\infty) = \{x(t) : x(t) \in C(0, +\infty) \text{ and } t^{\alpha} x(t) \in C[0, +\infty)\}$. $L^p_{Loc}[0, +\infty)$ $(p \ge 1)$ is the space of all real valued functions which are Lebesgue integrable over every bounded subinterval of $[0, +\infty)$.

Definition 2.1. [10, p. 33] Let $\beta \in (0,1)$, The operator $I^{\beta}_{0^+}$, defined on $L^1[0,T]$ by

$$I^{\beta}_{0^+}\varphi(t) = \frac{1}{\Gamma(\beta)}\int_0^t \frac{\varphi(s)}{(t-s)^{1-\beta}}ds, \qquad \text{a.e. } t \in [0,T]$$

is called the Riemann–Liouville fractional integral operator of order β.

Definition 2.2. [10, p. 35] Let $\beta \in (0,1)$, The operator $D^{\beta}_{0^+}$, defined by

$$D^{\beta}_{0^+}\varphi(t) = \frac{d}{dt}I^{1-\beta}_{0^+}\varphi(t) = \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t \frac{\varphi(s)}{(t-s)^{\beta}}ds, \qquad \text{a.e. } t \in [0,T],$$

where $I^{1-\beta}_{0^+}\varphi(t)$ is an absolutely continuous function, is called the Riemann–Liouville fractional differential operator of order β.

Definition 2.3. The solution $x(t) \in C_{1-\beta}(0,+\infty)$ of fractional differential equation (1.6) is said to be attractive if $\lim_{t\to+\infty} x(t) = 0$.

Using the Hölder inequality, Zhu [15] obtained the following inequality.

Lemma 2.4. *Let $0 < \beta < 1$. Suppose that $s^{1-\beta}\rho(s) \in L^p[0,1]$, where $p > \frac{1}{\beta}$. Then*

$$\left|\int_0^t (\frac{t}{t-s})^{1-\beta}\rho(s)ds\right| \le \frac{2^{\frac{1}{q}}t^{\beta-\frac{1}{p}}}{(q\beta-q+1)^{\frac{1}{q}}}\left(\int_0^t s^{p(1-\beta)}|\rho(s)|^p ds\right)^{\frac{1}{p}} \tag{2.1}$$

for $t \in [0,1]$, where $q = \frac{p}{p-1}$.

Recently, Zhu [15, Corollary 4.5] obtained the following result which is very useful for the study of the main purpose of this paper.

Theorem 2.5. *Let $0 < \beta < 1$ and $0 < \mu \le 1$. Suppose $f : (0,+\infty)\times\mathbb{R} \to \mathbb{R}$ is a continuous function, and there exist nonnegative functions $l(t)$ and $k(t)$ such that*

$$|f(t,x)| \le l(t)|x|^{\mu} + k(t)$$

for all $(t,x) \in (0,+\infty)\times\mathbb{R}$, where $t^{(1-\mu)(1-\beta)}l(t) \in C(0,+\infty)\bigcap L^p_{Loc}[0,+\infty)$ and $t^{1-\beta}k(t) \in C(0,+\infty)\bigcap L^p_{Loc}[0,+\infty)$, $p > \frac{1}{\beta}$. Then the fractional differential equation (1.6) has at least one global solution in $C_{1-\beta}(0,+\infty)$.

3 Weakly singular integral inequalities

In this section, we are now to prove some results concerning weakly singular integral inequalities, which can be used to study the attractivity of solutions for fractional differential equation (1.6). We first study the weakly singular integral inequality (1.1) for the case $\mu = 1$.

Theorem 3.1. *Let $a, b > 0$, $\alpha > 0$, $\delta \ge 0$ and $0 < \beta < 1$. Let $l(t)$ be a nonnegative, continuous function on $(0,+\infty)$ and $t^{\alpha_1}l(t) \in L^p_{Loc}[0,+\infty)$, where $\alpha_1 = \min\{1-\alpha-\beta, -\delta\}$ and $p > \frac{1}{\beta}$. Let $t^{\alpha}u(t)$ be a continuous, nonnegative function on $[0,+\infty)$ with*

$$u(t) \le at^{-\alpha} + bt^{-\delta}\int_0^t (t-s)^{\beta-1}l(s)u(s)ds, \qquad t \in (0,\infty). \tag{3.1}$$

Then

$$u(t) \le at^{-\alpha} + \frac{2^{\frac{1}{q}} bt^{2\beta-\delta-1-\frac{1}{p}}}{(q\beta-q+1)^{\frac{1}{q}}} A^{\frac{1}{p}}(t) \exp\left(\int_0^t \frac{L(s)}{p} ds\right), \qquad t \in (0,+\infty), \tag{3.2}$$

where $A(t) = \int_0^t 2^{p-1} a^p s^{p(1-\alpha-\beta)} l^p(s) ds$, $L(t) = \frac{4^{p-1} b^p t^{p(\beta-\delta)-1} l^p(t)}{(q\beta-q+1)^{\frac{p}{q}}}$ *and* $q = \frac{p}{p-1}$.

Proof. Applying Lemma 2.4, we have

$$\begin{aligned} u(t) &\le at^{-\alpha} + bt^{-\delta} \int_0^t (t-s)^{\beta-1} l(s)u(s)ds \\ &= at^{-\alpha} + bt^{\beta-\delta-1} \int_0^t (\frac{t}{t-s})^{1-\beta} l(s)u(s)ds \\ &\le at^{-\alpha} + \frac{2^{\frac{1}{q}} bt^{2\beta-\delta-1-\frac{1}{p}}}{(q\beta-q+1)^{\frac{1}{q}}} \left(\int_0^t s^{p(1-\beta)} l^p(s)u^p(s)ds\right)^{\frac{1}{p}}. \end{aligned} \tag{3.3}$$

From (3.3), we obtain

$$t^{1-\beta} l(t)u(t) \le at^{1-\alpha-\beta} l(t) + \frac{2^{\frac{1}{q}} bt^{\beta-\delta-\frac{1}{p}} l(t)}{(q\beta-q+1)^{\frac{1}{q}}} \left(\int_0^t s^{p(1-\beta)} l^p(s)u^p(s)ds\right)^{\frac{1}{p}}. \tag{3.4}$$

Since $t^{\alpha_1} l(t) \in L^p_{Loc}[0,+\infty)$, then $t^{p(1-\alpha-\beta)} l^p(t) \in L^1_{Loc}[0,+\infty)$ and $t^{p(\beta-\delta)-1} l^p(t) \in L^1_{Loc}[0,+\infty)$, where $p > \frac{1}{\beta}$. Therefore we get

$$\begin{aligned} \int_0^t s^{p(1-\beta)} l^p(s)u^p(s)ds &\le \int_0^t \left[as^{1-\alpha-\beta} l(s) + \frac{2^{\frac{1}{q}} bs^{\beta-\delta-\frac{1}{p}} l(s)}{(q\beta-q+1)^{\frac{1}{q}}} \left(\int_0^s \tau^{p(1-\beta)} l^p(\tau)u^p(\tau)d\tau\right)^{\frac{1}{p}}\right]^p ds \\ &\le \int_0^t 2^{p-1} a^p s^{p(1-\alpha-\beta)} l^p(s)ds \\ &+ \int_0^t \frac{4^{p-1} b^p s^{p(\beta-\delta)-1} l^p(s)}{(q\beta-q+1)^{\frac{p}{q}}} \int_0^s \tau^{p(1-\beta)} l^p(\tau)u^p(\tau)d\tau ds. \end{aligned} \tag{3.5}$$

Let $W(t) = \int_0^t s^{p(1-\beta)} l^p(s)u^p(s)ds$, then we get

$$W(t) \le A(t) + \int_0^t L(s)W(s)ds. \tag{3.6}$$

In (3.6), we know that $A(t)$ is a nondecreasing function on $[0,+\infty)$ and using the Gronwall integral inequality [1, Corollary 1.2], we obtain

$$W(t) \le A(t) \exp\left(\int_0^t L(s)ds\right). \tag{3.7}$$

From (3.3) and (3.7), we get

$$u(t) \le at^{-\alpha} + \frac{2^{\frac{1}{q}} bt^{2\beta-\delta-1-\frac{1}{p}}}{(q\beta-q+1)^{\frac{1}{q}}} A^{\frac{1}{p}}(t) \exp\left(\int_0^t \frac{L(s)}{p} ds\right). \tag{3.8}$$

Thus, we complete the proof. □

As a consequence of Theorem 3.1, we can immediately obtain the following result for the case $\alpha = 1-\beta$ and $\delta = 0$.

Theorem 3.2. *Let $a, b > 0$ and $0 < \beta < 1$. Let $l(t)$ be a nonnegative and continuous function on $(0, +\infty)$ with $l(t) \in L^p_{Loc}[0, +\infty)$, where $p > \frac{1}{\beta}$, and $t^{1-\beta}u(t)$ be a continuous, nonnegative function on $[0, +\infty)$ with*

$$u(t) \le at^{\beta-1} + b\int_0^t (t-s)^{\beta-1} l(s)u(s)ds, \qquad t \in (0, \infty). \tag{3.9}$$

Then

$$u(t) \le at^{\beta-1} + \frac{2^{\frac{1}{q}} bt^{2\beta-1-\frac{1}{p}}}{(q\beta-q+1)^{\frac{1}{q}}} A^{\frac{1}{p}}(t)\exp\left(\int_0^t \frac{L(s)}{p}ds\right), \qquad t \in (0, +\infty), \tag{3.10}$$

where $A(t) = \int_0^t 2^{p-1}a^p l^p(s)ds$, $L(t) = \frac{4^{p-1}b^p t^{p\beta-1} l^p(t)}{(q\beta-q+1)^{\frac{p}{q}}}$ and $q = \frac{p}{p-1}$.

Example 3.3. Suppose that $t^{\frac{1}{4}}u(t)$ is a continuous, nonnegative function on $[0, +\infty)$ and $u(t)$ satisfies the following inequality

$$u(t) \le t^{-\frac{1}{4}} + t^{-\frac{1}{3}}\int_0^t (t-s)^{-\frac{1}{3}} \frac{u(s)}{1+s}ds, \qquad t \in (0, +\infty). \tag{3.11}$$

By Theorem 3.1, let $p = 2$, then we get

$$u(t) \le t^{-\frac{1}{4}} + 6^{\frac{1}{2}} t^{-\frac{1}{2}} \left(\int_0^t \frac{2s^{\frac{1}{6}}}{(1+s)^2}ds\right)^{\frac{1}{2}} \exp\left(\int_0^t \frac{6s^{\frac{-1}{3}}}{(1+s)^2}ds\right), \qquad t \in (0, +\infty). \tag{3.12}$$

We know

$$\int_0^t \frac{s^{\frac{1}{6}}}{(1+s)^2}ds \le \int_0^{+\infty} \frac{s^{\frac{1}{6}}}{(1+s)^2}ds = \mathrm{B}(7/6, 5/6) = \frac{\pi}{3}$$

and

$$\int_0^t \frac{s^{\frac{-1}{3}}}{(1+s)^2}ds \le \int_0^{+\infty} \frac{s^{\frac{-1}{3}}}{(1+s)^2}ds = \mathrm{B}(2/3, 4/3) = \frac{2\sqrt{3}\pi}{9},$$

where $\mathrm{B}(p, q) = \int_0^1 (1-s)^{p-1}s^{q-1}ds = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ $(p, q > 0)$ is the Beta function, and $\Gamma(p) = \int_0^{+\infty} s^{p-1}\exp(-s)ds$ $(p > 0)$ is the Gamma function.

Then we obtain

$$u(t) \le t^{-\frac{1}{4}} + 2\sqrt{\pi}\exp\left(\frac{4\sqrt{3}\pi}{3}\right) t^{-\frac{1}{2}}, \qquad t \in (0, +\infty), \tag{3.13}$$

and $u(t) \to 0$ as $t \to +\infty$.

Now, we investigate the weakly singular integral inequality (1.1) when $0 < \mu < 1$.

Theorem 3.4. *Let $a, b > 0$, $\alpha > 0$, $\delta \ge 0$, $0 < \beta < 1$ and $0 < \mu < 1$. Let $l(t)$ be a nonnegative, continuous function on $(0, +\infty)$ with $t^{\alpha_2}l(t) \in L^p_{Loc}[0, +\infty)$, where $\alpha_2 = \min\{1-\alpha\mu-\beta, (\beta-\delta-1)\mu+1-\beta\}$ and $p > \frac{1}{\beta}$. Let $t^\alpha u(t)$ be a continuous, nonnegative function on $[0, +\infty)$ with*

$$u(t) \le at^{-\alpha} + bt^{-\delta}\int_0^t (t-s)^{\beta-1} l(s)u^\mu(s)ds, \qquad t \in (0, \infty). \tag{3.14}$$

Then

$$u(t) \le at^{-\alpha} + \frac{2^{\frac{1}{q}} b t^{2\beta-\delta-1-\frac{1}{p}}}{(q\beta - q + 1)^{\frac{1}{q}}} \left(A^{1-\mu}(t) + (1-\mu) \int_0^t L(s)ds \right)^{\frac{1}{p(1-\mu)}}, \qquad t \in (0, +\infty), \tag{3.15}$$

where $A(t) = \int_0^t 2^{p-1} a^{p\mu} s^{p(1-\alpha\mu-\beta)} l^p(s)ds$, $L(t) = \frac{4^{p-1} b^{p\mu} t^{(2p\beta-p\delta-p-1)\mu+p-p\beta} l^p(t)}{(q\beta-q+1)^{\frac{p\mu}{q}}}$ *and* $q = \frac{p}{p-1}$.

Proof. From the inequality (3.14), using the same procedure as in the proof of the inequality (3.3), we have

$$u(t) \le at^{-\alpha} + \frac{2^{\frac{1}{q}} b t^{2\beta-\delta-1-\frac{1}{p}}}{(q\beta - q + 1)^{\frac{1}{q}}} \left(\int_0^t s^{p(1-\beta)} l^p(s) u^{p\mu}(s) ds \right)^{\frac{1}{p}}. \tag{3.16}$$

From (3.16), we know

$$u^{\mu}(t) \le a^{\mu} t^{-\alpha\mu} + \frac{2^{\frac{1}{q}} b^{\mu} t^{(2\beta-\delta-1-\frac{1}{p})\mu}}{(q\beta - q + 1)^{\frac{\mu}{q}}} \left(\int_0^t s^{p(1-\beta)} l^p(s) u^{p\mu}(s) ds \right)^{\frac{\mu}{p}} \tag{3.17}$$

and

$$t^{1-\beta} l(t) u^{\mu}(t) \le a^{\mu} t^{-\alpha\mu+1-\beta} l(t) + \frac{2^{\frac{1}{q}} b^{\mu} t^{(2\beta-\delta-1-\frac{1}{p})\mu+1-\beta} l(t)}{(q\beta - q + 1)^{\frac{\mu}{q}}} \left(\int_0^t s^{p(1-\beta)} l^p(s) u^{p\mu}(s) ds \right)^{\frac{\mu}{p}}. \tag{3.18}$$

Since $t^{\alpha_2} l(t) \in L^p_{Loc}[0, +\infty)$, then $t^{p(1-\alpha\mu-\beta)} l^p(t) \in L^1_{Loc}[0, +\infty)$ and $t^{(2p\beta-p\delta-p-1)\mu+p-p\beta} l^p(t) \in L^1_{Loc}[0, +\infty)$, where $p > \frac{1}{\beta}$. Then we obtain

$$\begin{aligned} &\int_0^t s^{p(1-\beta)} l^p(s) u^{p\mu}(s) ds \\ &\quad \le \int_0^t 2^{p-1} a^{p\mu} s^{p(1-\alpha\mu-\beta)} l^p(s) ds \\ &\qquad + \int_0^t \frac{4^{p-1} b^{p\mu} s^{(2p\beta-p\delta-p-1)\mu+p-p\beta} l^p(s)}{(q\beta - q + 1)^{\frac{p\mu}{q}}} \left(\int_0^s \tau^{p(1-\beta)} l^p(\tau) u^{p\mu}(\tau) d\tau \right)^{\mu} ds. \end{aligned} \tag{3.19}$$

Let $W(t) = \int_0^t s^{p(1-\beta)} l^p(s) u^{p\mu}(s) ds$, then we get

$$W(t) \le A(t) + \int_0^t L(s) W^{\mu}(s) ds. \tag{3.20}$$

Using the Bihari integral inequality [1, Corollary 5.3], we obtain

$$W(t) \le \left(A^{1-\mu}(t) + (1-\mu) \int_0^t L(s) ds \right)^{\frac{1}{1-\mu}}. \tag{3.21}$$

From (3.16) and (3.21), we get

$$u(t) \le at^{-\alpha} + \frac{2^{\frac{1}{q}} b t^{2\beta-\delta-1-\frac{1}{p}}}{(q\beta - q + 1)^{\frac{1}{q}}} \left(A^{1-\mu}(t) + (1-\mu) \int_0^t L(s) ds \right)^{\frac{1}{p(1-\mu)}}. \tag{3.22}$$

Thus, we complete the proof. □

As a consequence of Theorem 3.4, we can obtain the following result when $\alpha = 1 - \beta$ and $\delta = 0$.

Theorem 3.5. *Let $a > 0$, $b > 0$, $0 < \beta < 1$ and $0 < \mu < 1$. Let $l(t)$ be a nonnegative and continuous function on $(0, +\infty)$ with $t^{(1-\mu)(1-\beta)} l(t) \in L^p_{Loc}[0, +\infty)$, where $p > \frac{1}{\beta}$, and $t^{1-\beta} u(t)$ be a continuous, nonnegative function on $[0, +\infty)$ with*

$$u(t) \le at^{\beta-1} + b \int_0^t (t-s)^{\beta-1} l(s) u^\mu(s) ds, \qquad t \in (0, \infty). \tag{3.23}$$

Then

$$u(t) \le at^{\beta-1} + \frac{2^{\frac{1}{q}} b t^{2\beta-1-\frac{1}{p}}}{(q\beta - q + 1)^{\frac{1}{q}}} \left(A^{1-\mu}(t) + (1-\mu) \int_0^t L(s) ds \right)^{\frac{1}{p(1-\mu)}}, \qquad t \in (0, +\infty), \tag{3.24}$$

where $A(t) = \int_0^t 2^{p-1} a^{p\mu} s^{p(1-\mu)(1-\beta)} l^p(s) ds$, $L(t) = \frac{4^{p-1} b^{p\mu} t^{(2p\beta - p - 1)\mu + p - p\beta} l^p(t)}{(q\beta - q + 1)^{\frac{p\mu}{q}}}$ and $q = \frac{p}{p-1}$.

Example 3.6. Suppose that $t^{\frac{1}{3}} u(t)$ is a continuous, nonnegative function on $[0, +\infty)$ and $u(t)$ satisfies the inequality

$$u(t) \le t^{-\frac{1}{3}} + t^{-\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{3}} s^{\frac{-1}{2}} u^{\frac{1}{3}}(s) ds, \qquad t \in (0, +\infty). \tag{3.25}$$

Let $p = 2$, using Theorem 3.4, we get

$$\begin{aligned} u(t) &\le t^{\frac{-1}{3}} + 6^{\frac{1}{2}} t^{\frac{-2}{3}} \left((\frac{9}{2})^{\frac{2}{3}} t^{\frac{8}{27}} + 12 \cdot 3^{\frac{1}{3}} t^{\frac{2}{9}} \right)^{\frac{3}{4}} \\ &\le t^{\frac{-1}{3}} + \sqrt{27} t^{\frac{-4}{9}} + 12\sqrt{3} t^{\frac{-1}{2}}, \qquad t \in (0, +\infty) \end{aligned} \tag{3.26}$$

and $u(t) \to 0$ as $t \to +\infty$.

In [15, Theorem 3.4], Zhu studied the weakly singular integral inequality (3.14) when $t^{(1-\mu)\alpha - \delta} l(t) \in L^p_{Loc}[0, +\infty)$, where $1 > \alpha \ge \delta \ge 0$ and $p > \max\{\frac{1}{\beta}, \frac{1}{1-\alpha+\delta}\}$. In fact, the conclusion is also correct when $1 > \alpha > 0$ and $1 > \delta \ge 0$. In the inequality (3.25), since $t^{\frac{-7}{9}} \notin L^p[0, +\infty)$ when $p > \frac{3}{2}$, then Theorem 3.4 in [15] cannot be used to solve the inequality (3.25).

Remark 3.7. In Theorem 3.4, since $0 < \mu < 1$, then $\alpha_1 < \alpha_2$. If $l(t)$ is a nonnegative and continuous function on $(0, +\infty)$ satisfying $t^{\alpha_1} l(t) \in L^p_{Loc}[0, +\infty)$, where $p > \frac{1}{\beta}$, then we can get $t^{\alpha_2} l(t) \in L^p_{Loc}[0, +\infty)$. Therefore, the hypothesis of function $l(t)$ in Theorem 3.4 is weaker than that imposed in Theorem 3.1.

Zhu [14, Theorem 3.4] obtained some results for the inequality (3.1) when $\alpha > \delta \ge 0$. Zhu [15, Theorem 3.3] studied the inequality (3.1) when $1 > \alpha \ge \delta \ge 0$. In Theorem 3.1, we study the inequality (3.1) when $\alpha > 0$ and $\delta \ge 0$. Therefore, our result generalizes some results in [14, 15].

Denton and Vatsala [3, Theorem 2.8] studied the inequality (3.1) for the special case $\alpha = 1 - \beta$ and $\delta = 0$. Henry [7, Exercise 3, p. 190] discussed the inequality (3.1) for the case $\delta = 0$ and $l(t) = t^{\gamma-1}$. Some similar results of the inequality (3.1) were proved in Haraux [6, Lemma 10, p. 112], Kong and Ding [9, Theorem 2.7], Webb [13, Theorem 3.9] and Zhu [14, Theorem 3.6]. As far as I know, there have been few papers to study the inequality (1.1), and the methods of proof in Theorem 3.1 and Theorem 3.4 seem to be new.

4 Attractivity of fractional differential equations

In this section, we present the main results of this paper. We first study the attractivity of solutions of fractional differential equation (1.6) when $|f(t,x)| \le l(t)|x|$.

Theorem 4.1. *Let $0 < \beta < 1$ and $\lambda > \beta$. Let $l(t)$ be a nonnegative function with $l(t) \in C(0,+\infty) \bigcap L^p_{Loc}[0,+\infty)$, where $p > 1$ and $\beta > \frac{1}{p} > 2\beta - 1$, and there exists a nonnegative constant K such that*

$$t^\lambda l(t) \le K \tag{4.1}$$

for all $t \in [1,+\infty)$. Suppose $f : (0,+\infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous function and

$$|f(t,x)| \le l(t)|x|$$

for all $(t,x) \in (0,+\infty) \times \mathbb{R}$. Then the solution of fractional differential equation (1.6) is attractive.

Proof. Using Theorem 2.5, we know that the fractional differential equation (1.6) has at least one global solution $x(t) \in C_{1-\beta}(0,+\infty)$ and $x(t)$ also satisfies the following Volterra integral equation

$$x(t) = x_0 t^{\beta-1} + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s,x(s))ds, \qquad t \in (0,+\infty). \tag{4.2}$$

Then we have

$$|x(t)| \le |x_0| t^{\beta-1} + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} l(s)|x(s)|ds, \qquad t \in (0,+\infty). \tag{4.3}$$

Then by Theorem 3.2, we obtain

$$|x(t)| \le |x_0| t^{\beta-1} + \frac{2^{\frac{1}{q}} t^{2\beta-1-\frac{1}{p}}}{\Gamma(\beta)(q\beta-q+1)^{\frac{1}{q}}} A^{\frac{1}{p}}(t) \exp\left(\int_0^t \frac{L(s)}{p} ds\right), \qquad t \in (0,+\infty), \tag{4.4}$$

where $A(t) = \int_0^t 2^{p-1}|x_0|^p l^p(s)ds$, $L(t) = \frac{4^{p-1} t^{p\beta-1} l^p(t)}{\Gamma^p(\beta)(q\beta-q+1)^{\frac{p}{q}}}$ and $q = \frac{p}{p-1}$.

From (4.1) and $\lambda > \beta > \frac{1}{p}$, we know

$$l^p(t) \le K^p t^{-p\lambda}, \qquad t \in [1,+\infty)$$

and $\int_1^{+\infty} K^p s^{-p\lambda} ds$ is convergent. Then we obtain that $\int_1^{+\infty} 2^{p-1}|x_0|^p l^p(s)ds$ is also convergent and there exists a nonnegative constant M_1 such that $A(t) \le M_1$ for all $t \in (0,+\infty)$. Since $\lambda > \beta$ and

$$t^{p\beta-1} l^p(t) \le K^p t^{p\beta-p\lambda-1}, \qquad t \in [1,+\infty),$$

we know that $\int_1^{+\infty} K^p s^{p\beta-p\lambda-1} ds$ is convergent. Then we obtain that $\int_1^{+\infty} s^{p\beta-1} l^p(s)ds$ and $\int_1^{+\infty} L(s)ds$ are also convergent, and there exists a nonnegative constant M_2 such that $\int_0^t \frac{L(s)}{p} ds \le M_2$ for all $t \in (0,+\infty)$.

Therefore, from (4.4) and $\beta > \frac{1}{p} > 2\beta - 1$, we get

$$|x(t)| \le |x_0| t^{\beta-1} + \frac{2^{\frac{1}{q}} t^{2\beta-1-\frac{1}{p}}}{\Gamma(\beta)(q\beta-q+1)^{\frac{1}{q}}} M_1^{\frac{1}{p}} \exp(M_2), \qquad t \in (0,+\infty), \tag{4.5}$$

and

$$\lim_{t\to+\infty} |x(t)| = 0. \tag{4.6}$$

Thus, we complete the proof. □

We now discuss the case when $|f(t,x)| \leq l(t)|x|^{\mu}$ for all $(t,x) \in (0,+\infty) \times \mathbb{R}$, where $0 < \mu < 1$.

Theorem 4.2. *Let $0 < \mu < 1$, $0 < \beta < 1$ and $\lambda > \beta$. Let $l(t)$ be a nonnegative function with $t^{(1-\mu)(1-\beta)} l(t) \in C(0,+\infty) \bigcap L^p_{Loc}[0,+\infty)$, where $p > 1$ with $\beta > \frac{1}{p} > 2\beta - 1$, and there exists a nonnegative constant K such that*

$$t^{\lambda} l(t) \leq K \tag{4.7}$$

for all $t \in [1,+\infty)$. Suppose $f : (0,+\infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous function with

$$|f(t,x)| \leq l(t)|x|^{\mu}$$

for all $(t,x) \in (0,+\infty) \times \mathbb{R}$. Then the solution of fractional differential equation (1.6) is attractive.

Proof. Using the same procedure as in the proof of Theorem 4.1, we know that the global solution $x(t) \in C_{1-\beta}(0,+\infty)$ of equation (1.6) satisfies the following Volterra integral equation

$$x(t) = x_0 t^{\beta-1} + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s,x(s))ds, \qquad t \in (0,+\infty), \tag{4.8}$$

and

$$|x(t)| \leq |x_0| t^{\beta-1} + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} l(s)|x(s)|^{\mu} ds, \qquad t \in (0,+\infty). \tag{4.9}$$

Then by Theorem 3.5, for $t \in (0,+\infty)$, we obtain

$$\begin{aligned} |x(t)| &\leq |x_0| t^{\beta-1} + \frac{2^{\frac{1}{q}} t^{2\beta-1-\frac{1}{p}}}{\Gamma(\beta)(q\beta-q+1)^{\frac{1}{q}}} \left(A^{1-\mu}(t) + (1-\mu)\int_0^t L(s)ds \right)^{\frac{1}{p(1-\mu)}} \\ &= |x_0| t^{\beta-1} + \frac{2^{\frac{1}{q}}}{\Gamma(\beta)(q\beta-q+1)^{\frac{1}{q}}} \left(\left(\frac{A(t)}{t^{p+1-2p\beta}}\right)^{1-\mu} + \frac{(1-\mu)\int_0^t L(s)ds}{t^{(p+1-2p\beta)(1-\mu)}} \right)^{\frac{1}{p(1-\mu)}}, \end{aligned} \tag{4.10}$$

where $A(t) = \int_0^t 2^{p-1} |x_0|^{p\mu} s^{p(1-\mu)(1-\beta)} l^p(s)ds$, $L(t) = \frac{4^{p-1} t^{(2p\beta-p-1)\mu+p-p\beta} l^p(t)}{\Gamma^{p\mu}(\beta)(q\beta-q+1)^{\frac{p\mu}{q}}}$ and $q = \frac{p}{p-1}$.

Since $1 > \beta > \frac{1}{p} > 2\beta - 1$ and $\lambda > \beta$, using L'Hôspital's rule, we get

$$\begin{aligned} \lim_{t\to+\infty} \frac{\int_0^t s^{p(1-\mu)(1-\beta)} l^p(s)ds}{t^{p+1-2p\beta}} &= \lim_{t\to+\infty} \frac{t^{p(1-\mu)(1-\beta)} l^p(t)}{(p+1-2p\beta)t^{p-2p\beta}} \\ &\leq \lim_{t\to+\infty} \frac{K^p t^{p(1-\mu)(1-\beta)-p\lambda}}{(p+1-2p\beta)t^{p-2p\beta}} \\ &= \lim_{t\to+\infty} \frac{K^p t^{p\mu(\beta-1)+p(\beta-\lambda)}}{(p+1-2p\beta)} \\ &= 0. \end{aligned} \tag{4.11}$$

In (4.11), if $\int_0^t s^{p(1-\mu)(1-\beta)} l^p(s)ds$ is a bounded function for $t \in [0,+\infty)$, we can also obtain this conclusion.

Since $\lambda > \beta$, using L'Hôspital's rule, we obtain

$$\begin{aligned}
\lim_{t\to+\infty} \frac{\int_0^t s^{(2p\beta-p-1)\mu+p-p\beta} l^p(s)ds}{t^{(p+1-2p\beta)(1-\mu)}} &= \lim_{t\to+\infty} \frac{t^{(2p\beta-p-1)\mu+p-p\beta} l^p(t)}{(p+1-2p\beta)(1-\mu)t^{(p+1-2p\beta)(1-\mu)-1}} \\
&\leq \lim_{t\to+\infty} \frac{K^p t^{(2p\beta-p-1)\mu+p-p(\beta+\lambda)}}{(p+1-2p\beta)(1-\mu)t^{(p+1-2p\beta)(1-\mu)-1}} \\
&= \lim_{t\to+\infty} \frac{K^p t^{p(\beta-\lambda)}}{(p+1-2p\beta)(1-\mu)} \\
&= 0.
\end{aligned} \tag{4.12}$$

In (4.12), if $\int_0^t s^{(2p\beta-p-1)\mu+p-p\beta} l^p(s)ds$ is a bounded function for $t \in [0, +\infty)$, we can also obtain this conclusion.

In (4.10), using (4.11) and (4.12), we obtain

$$\lim_{t\to+\infty} |x(t)| = 0. \tag{4.13}$$

Thus, we complete the proof. □

Example 4.3. Consider the following Riemann–Liouville fractional differential equation

$$\begin{cases} D_{0^+}^{\frac{2}{3}} x(t) = \frac{x(t)}{\sqrt{t}(1+\sqrt{t})}, \\ \lim_{t\to 0^+} t^{\frac{1}{3}} x(t) = 1. \end{cases} \tag{4.14}$$

Let $\lambda = 1$ and $\frac{3}{2} < p < 2$, using Theorem 4.1 and the inequality (4.4), we know that the solution $x(t) \in C_{\frac{1}{3}}(0, +\infty)$ of the equation (4.14) is attractive, and

$$x(t) \leq t^{\frac{-1}{3}} + Mt^{\frac{p-3}{3p}}, \tag{4.15}$$

where $M = M(p)$ is a nonnegative constant and $\lim_{p\to\frac{3}{2}^+} M(p) = +\infty$.

Example 4.4. Consider the following Riemann–Liouville fractional differential equation

$$\begin{cases} D_{0^+}^{\frac{1}{2}} x(t) = t^{\frac{-2}{3}} x^{\frac{1}{2}}(t), \\ \lim_{t\to 0^+} t^{\frac{1}{2}} x(t) = 1. \end{cases} \tag{4.16}$$

Let $\lambda = \frac{2}{3}$ and $2 < p < \frac{12}{5}$, using Theorem 4.2 and the inequality (4.10), we get that the solution $x(t) \in C_{\frac{1}{2}}(0, +\infty)$ of the equation (4.16) is attractive, and

$$x(t) \leq t^{\frac{-1}{2}} + M_1 t^{\frac{-1}{3}} + M_2 t^{\frac{-5}{12}}, \tag{4.17}$$

where $M_1 = M_1(p)$ and $M_2 = M_2(p)$ are nonnegative constants.

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Existence and uniqueness for a semilinear sixth-order ODE

To my dear professor Dan Tiba with infinite admiration

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Abstract. By using variational methods and maximum principles we discuss the existence, uniqueness and multiplicity of solutions for a semilinear sixth-order ODE. The main difference between our work and other related papers is that we treat a general case and we do not impose sign restrictions on the nonlinearity f or on its potential F.

Keywords: ODE, sixth-order, semilinear, variational method.

2020 Mathematics Subject Classification: 34A12, 34B15.

1 Introduction

In this paper, we study the existence and uniqueness of solutions of the following boundary value problem

$$\begin{cases} u^{(6)} + Au^{(4)} + Bu'' - C(x)u + f(x,u) = 0 & \text{in } \Omega \\ u = u'' = u^{(4)} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where A, B are some given constants, $C(x)$ is a given function, f is a continuous function on $[0,L] \times \mathbb{R}$ and $\Omega = (0,L)$.

The treatment of (1.1) is motivated by the study of stationary solutions (which leads to sixth-order ODEs) of the sixth-order parabolic differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^6 u}{\partial x^6} + A\frac{\partial^4 u}{\partial x^4} + B\frac{\partial^2 u}{\partial x^2} + f(x,u), \tag{1.2}$$

arising in the formation of spatial periodic patterns in bistable systems and is also a model for describing the behaviour of phase fronts in materials that are undergoing a transition between the liquid and the solid state. The case $f(u) = u - u^3$ was treated by Gardner and Jones [13] as well as by Caginalp and Fife [7].

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We also note that the deformation of the equilibrium state of an elastic circular ring segment with its two ends simply supported can be described by a boundary value of sixth-order (see [1]):

$$\begin{cases} u^{(6)} + 2u^{(4)} + u'' = f(x,u) & \text{in } \Omega = (0,1) \\ u = u'' = u^{(4)} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Boundary value problems of sixth-order also arise in sandwich beam deflection under transverse shear [2].

The existence and multiplicity of solutions to (1.1) were obtained in [20], when $f(u) = -u^3$, $A^2 < 4B$ and $C = -1$ in Ω and in [9] when $C < 0$, $f(u) = -b(x)u^3$ where b is an even continuous $2L$ periodic function. A more general existence and multiplicity result was given in [14] by using variational methods and the Brézis and Nirenberg's linking theorems in the case

$$-\frac{F(x,u)}{u^2} \to +\infty, \quad \text{uniformly with respect to } x \text{ as } |u| \to \infty, \tag{1.4}$$

where $F(x,u) = \int_0^u f(x,s)ds \le 0$.

In [15], the authors studied the existence of positive solutions of the nonlinear boundary value problem

$$\begin{cases} u^{(6)} + f(x,u,u'',u^{(4)}) = 0 & \text{in } \Omega = (0,1) \\ u = u'' = u^{(4)} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

using the Krein–Rutman Theorem and the Global Bifurcation Theory under the assumptions (again a sign restriction is assumed)

1). $f : \overline{\Omega} \times [0,\infty) \times (-\infty,0] \times [0,\infty) \to [0,\infty)$ is continuous and there exist functions a, b, c, d, m, n with $a(t)+b(t)+c(t) > 0$ and $d(t)+m(t)+n(t) > 0$ in Ω such that

$$f(t,u,p,q) = a(t)u - b(t)p + c(t)q + o(|(u,p,q)|), as\, |(u,p,q)| \to 0,$$

uniformly for $t \in \overline{\Omega}$, and

$$f(t,u,p,q) = d(t)u - m(t)p + n(t)q + o(|(u,p,q)|), as\, |(u,p,q)| \to 0,$$

uniformly for $t \in \overline{\Omega}$. Here $|(u,p,q)|^2 = u^2 + p^2 + q^2$.

2). $f > 0$ in $\overline{\Omega}$ and $[0,\infty) \times (-\infty,0] \times [0,\infty) \setminus \{(0,0,0)\}$.

3). there exists constants $a_0, b_0, c_0 \ge 0$ satisfying $a_0^2 + b_0^2 + c_0^2 > 0$ and

$$f(t,u,p,q) = a_0 u - b_0 p + c_0 q + o(|(u,p,q)|).$$

It is worth mentioning the new paper of Bonanno and Livrea [4], where the problem

$$\begin{cases} -u^{(6)} + Au^{(4)} - Bu'' + Cu = \lambda f(x,u) & \text{in } \Omega = (0,1) \\ u = u'' = u^{(4)} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

is treated.

The authors prove the existence of infinitely many solutions to problem (1.6) under different assumptions on A, B, C and by requiring an oscillation on $f(x,\cdot)$ at infinity. More precisely if

i). $F(x,t) \ge 0$ for every $(x,t) \in ([0,5/12] \cup [7/21,1]) \times \mathbb{R}$.

ii).

$$\liminf_{t\to\infty} \frac{\int_0^1 \max_{|s|<t} F(x,s)dx}{t^2} < \tau \limsup_{t\to\infty} \frac{\int_{5/12}^{7/12} F(x,t)dx}{t^2},$$

then for every

$$\lambda \in \left(\frac{2\delta^4}{\tau} \frac{1}{\limsup_{t\to\infty} \frac{\int_{5/12}^{7/12} F(x,t)dx}{t^2}}, \frac{2\delta\pi^4}{\liminf_{t\to\infty} \frac{\int_0^1 \max_{|s|<t} F(x,s)dx}{t^2}} \right)$$

the problem (1.6) admits an unbounded sequence of classical solutions. Here τ and δ are technical constants depending on A, B and C.

Using variational methods we present here some new existence results (Section 3.1). The main difference between our work and the above mentioned papers is that we treat a general case and we do not impose sign restrictions on f or F. We note that we cover nonlinearities that are not treated elsewhere, e.g., the cases $f(u) = \ln(|u|+1) + \frac{|u|}{|u|+1} + u$ and $f(x,u) = a(x)\cos(u^n + C)u^{n-1}$, where a is a bounded function, C is a constant and n is a natural number. We see that these cases are not covered in [14] since the assumption (H1) in [14], i.e. (1.4) is not satisfied. In particular, since (2.10) holds (here $A = 2, B = 1, C = 0, L = 1$), our results apply to (1.3).

We obtain our main existence results under the restriction

$$F(x,s) \le K_1|s|^r + K_2, \qquad \forall\,(x,s) \in \Omega \times \mathbb{R}, \tag{1.7}$$

where $K_1, K_2, r > 0$.

In Section 3.2 we will briefly present some uniqueness results for the corresponding non-homogeneous linear equation.

The last section is devoted to a multiplicity result. As we mentioned above, the available multiplicity results (see [20, Theorem 3] and [9, Theorem B]) are stated under the restriction $F \le 0$. Here we strengthen relation (1.7), more precisely we impose

$$-K|s|^p \le F(x,s) \le K_1|s|^r + K_2, \qquad \forall\,(x,s) \in \Omega \times \mathbb{R}, \tag{1.8}$$

where $K, K_1, K_2 > 0, 0 < r < 2, p > 2$ and obtain for sufficiently large L a multiplicity result that holds without a sign restriction on F. We also note that the multiplicity result holds if (1.4) is satisfied without the sign restriction on F.

2 Variational settings and auxiliaries

We consider the Hilbert space $H(\Omega) = \{u \in H^3(\Omega) \,|\, u = u'' = 0 \text{ on } \partial\Omega\}$, endowed with the standard inner product

$$(u,v)_{H^3(\Omega)} = \int_\Omega \left(u'''v''' + u''v'' + u'v' + uv \right) dx$$

and standard norm

$$\|u\|_{H^3(\Omega)} = (u,u)_{H^3(\Omega)}^{\frac{1}{2}}.$$

Definition 2.1. A weak solution of (1.1) is a function $u \in H(\Omega)$ such that

$$\int_{\Omega} \Big(u'''v''' - Au''v'' + Bu'v' + C(x)uv - f(x,u)v \Big) dx = 0, \qquad \forall v \in H(\Omega).$$

A classical solution of (1.1) is a function $u \in C^6(\overline{\Omega})$ that satisfies (1.1).

We note that if f is a continuous function on $[0, L] \times \mathbb{R}$, then a weak solution is a classical solution (for a proof see [20]).

The problem (1.1) has a variational structure and the weak solutions in the space $H(\Omega)$ can be found as critical points of the functional

$$J : H(\Omega) \to \mathbb{R}$$

$$J(u) = \frac{1}{2} \int_{\Omega} \Big((u''')^2 - A(u'')^2 + B(u')^2 + C(x)u^2 \Big) dx - \int_{\Omega} F(x,u) dx,$$

which is Fréchet differentiable and its Fréchet derivative is given by

$$\langle J'(u), v \rangle = \int_{\Omega} \Big(u'''v''' - Au''v'' + Bu'v' + C(x)uv - f(x,u)v \Big) dx, \qquad \forall v \in H(\Omega).$$

Throughout the paper C denotes a universal positive constant depending on the indicated quantities, unless otherwise specified.

The following results will be useful.

Lemma 2.2. *The following relations hold true for any $u \in H(\Omega)$.*

$$\int_{\Omega} u^2 dx \leq \left(\frac{L}{\pi} \right)^{2k} \int_{\Omega} (u^{(k)})^2 dx, \qquad k = 1, 2, 3. \tag{2.1}$$

$$\int_{\Omega} (u')^2 dx \leq \left(\frac{L}{\pi} \right)^{2} \int_{\Omega} (u'')^2 dx. \tag{2.2}$$

$$\int_{\Omega} (u'')^2 dx \leq \left(\frac{L}{\pi} \right)^{2} \int_{\Omega} (u''')^2 dx, \tag{2.3}$$

where L represents the length of Ω.

Lemma 2.2 is proved in [4, Proposition 2.1] in the case when $\Omega = (0, 1)$. By similar calculations we can get the result for the case $\Omega = (0, L)$.

From Lemma 2.2 it follows that the scalar product

$$(u, v)_{H(\Omega)} = \int_{\Omega} u'''v''' dx$$

induces a norm equivalent to the norm $\|u\|_{H^3(\Omega)}$ in the space $H(\Omega)$.

The next key result is more general version for bounded domains of the result presented in [20], Lemma 5 and will be used to handle the existence in the case $r > 2$ as well the multiplicity result.

Lemma 2.3. *Let $u \in H(\Omega)$. Suppose that* *a).*

$$A > 0, \qquad \frac{A^2}{C_1} < 4B, \qquad C_1 = 1 - C_m\left(\frac{L}{\pi}\right)^6 > 0, \qquad C \geq -C_m, \qquad C_m > 0. \tag{2.4}$$

Then there exists a constant k_1 such that

$$\int_\Omega \left[(u''')^2 - A(u'')^2 + B(u')^2 + C(x)u^2\right]dx \geq k_1\|u\|^2_{H^3(\Omega)}. \tag{2.5}$$

If $C \geq 0$ then $\frac{A^2}{C_1} < 4B$ may be replaced by $A^2 < 4B$.

A similar estimate holds if we assume that

$$A, B > 0, \qquad A^2 < 4C \qquad B, \geq C, \qquad \frac{A^2}{4C} \leq C - 1. \tag{2.6}$$

b).

$$A = 0, \qquad B < 0, \qquad B^2 < 2C_m, \qquad \frac{B^2}{2C_m} \leq \frac{C_m}{2} - 1, \tag{2.7}$$

where $C_m = \inf_\Omega C(x) > 0$.

Then there exists a constant $k_2 > 0$ such that

$$\int_\Omega \left[(u''')^2 + B(u')^2 + C(x)u^2\right]dx \geq k_2\|u\|^2_{H^3(\Omega)}. \tag{2.8}$$

The inequality (2.8) also holds if

$$A = 0, \qquad B < 0, \qquad C - 1 \geq \left(\frac{-2B}{3}\right)^{4/3}. \tag{2.9}$$

c).

$$C = 0, \qquad A > 0, \qquad B \geq 0, \qquad 1 - \frac{AL^2}{\pi^2} > 0. \tag{2.10}$$

Then there exists a constant $k_3 > 0$ such that

$$\int_\Omega \left[(u''')^2 - A(u'')^2 + B(u')^2\right]dx \geq k_3\|u\|^2_{H^3(\Omega)}. \tag{2.11}$$

Remark 2.4. Of course if $A \leq 0$, $B, C \geq 0$, then Lemma 2.3 is always true, i.e., there is nothing to prove.

Proof. a). We borrow some ideas from the paper of Bonheure (see [5, Lemma 5]).

It is easy to see that for any real α

$$\int_\Omega \left(u''' + \alpha u'\right)^2 dx = \int_\Omega \left((u''')^2 - 2\alpha(u'')^2 + \alpha^2(u')^2\right)dx.$$

Hence for any α the quantity

$$Q_\alpha = \int_\Omega \left((u''')^2 - 2\alpha(u'')^2 + \alpha^2(u')^2\right)dx$$

is positive.

For arbitrary $\varepsilon > 0$ we have by Lemma 2.2

$$\int_\Omega \left[(u''')^2 - A(u'')^2 + B(u')^2 + C(x)u^2\right] dx$$
$$\geq C_1 \left[\int_\Omega (u''')^2 - \frac{A}{C_1}(u'')^2 + \frac{B}{C_1}(u')^2\right] dx$$
$$= C_1 \Bigg\{ \varepsilon \int_\Omega \left[(u''')^2 + (u'')^2 + (u')^2\right] dx$$
$$+ (1-\varepsilon) \int_\Omega \left[(u''')^2 - \frac{\frac{A}{C_1}+\varepsilon}{1-\varepsilon}(u'')^2 + \frac{1}{4}\left(\frac{\frac{A}{C_1}+\varepsilon}{1-\varepsilon}\right)^2 (u')^2 \right] dx$$
$$+ \left[\frac{B}{C_1} - \varepsilon - \frac{1}{4}\frac{\left(\frac{A}{C_1}+\varepsilon\right)^2}{1-\varepsilon}\right] \int_\Omega (u')^2 dx \Bigg\}$$
$$\geq \varepsilon C_1 \int_\Omega (u''')^2 dx + (1-\varepsilon) C_1 Q_{\frac{\frac{A}{C_1}+\varepsilon}{1-\varepsilon}} + C_1 \left[\frac{B}{C_1} - \varepsilon - \frac{1}{4}\frac{\left(\frac{A}{C_1}+\varepsilon\right)^2}{1-\varepsilon}\right] \int_\Omega (u')^2 dx.$$

Choosing ε sufficiently small, using that $Q_{\frac{\frac{A}{C_1}+\varepsilon}{1-\varepsilon}} \geq 0$ and the equivalence of norms $\|\cdot\|_{H^3(\Omega)}$ and $\|\cdot\|_{H(\Omega)}$ we get the desired result.

b). The proof of inequality (2.8) under the assumption (2.7) is deduced by different means, namely by using the Fourier transform.

We first note that if one of the inequalities (2.5), (2.8) or (2.11) holds for $u \in H^3(\mathbb{R})$, then it follows that the inequalities are also true for $u \in H(\Omega)$.

Indeed, for $u \in H(\Omega)$, we have

$$\int_\Omega [\,*\,] dx = \int_{\mathbb{R}} [\,*\,] dx \geq k\|u\|^2_{H^3(\mathbb{R})} \geq k\|u\|^2_{H^3(\Omega)}. \tag{2.12}$$

Here $*$ stands for one of the expressions in the inequalities (2.5), (2.8) or (2.11) that is inside the square brackets.

We now prove the required inequalities for $u \in H^3(\mathbb{R})$.

We note that the proof of (2.5) under the conditions (2.6) is similar to the proof of (2.8) under the hypothesis (2.7) and hence is omitted.

To prove inequality (2.8) we see that for all $\xi \in \mathbb{R}$

$$-B\xi^2 \leq \frac{B^2}{2C_m}\xi^4 + \frac{C_m}{2} \leq \frac{B^2}{2C_m}\xi^6 + \frac{C_m}{2} + \frac{B^2}{2C_m} \leq \frac{B^2}{2C_m}\xi^6 + C_m - 1. \tag{2.13}$$

Hence

$$\xi^6 + B\xi^2 + C_m \geq \xi^6 - \frac{B^2}{2C_m}\xi^6 - C_m + 1 + C_m \geq \left(1 - \frac{B^2}{2C_m}\right)\left(\xi^6 + 1\right). \tag{2.14}$$

As a consequence, we get

$$\xi^6 + B\xi^2 + C_m \geq \frac{1}{3}\left(1 - \frac{B^2}{2C_m}\right)\left(1 + \xi^2 + \xi^4 + \xi^6\right), \qquad \forall \xi \in \mathbb{R}. \tag{2.15}$$

Let $\hat{u}(\xi)$ be the Fourier transform of $u(x) \in H^3(\mathbb{R})$.
By Parseval's identity and (2.15) we get

$$\int_{\mathbb{R}} \Big((u''')^2 + B(u')^2 + C(x)u^2\Big)dx \tag{2.16}$$

$$\geq \int_{\mathbb{R}} \Big((u''')^2 + B(u')^2 + C_m u^2\Big)dx = \int_{\mathbb{R}} \Big(\xi^6 + B\xi^2 + C_m\Big)\|\hat{u}(\xi)\|^2 d\xi$$

$$\geq \frac{1}{3}\left(1 - \frac{B^2}{2C_m}\right) \int_{\mathbb{R}} \Big(1 + \xi^2 + \xi^4 + \xi^6\Big)\|\hat{u}(\xi)\|^2 d\xi$$

$$= \frac{1}{3}\left(1 - \frac{B^2}{2C_m}\right) \int_{\mathbb{R}} \Big(u^2 + (u')^2 + (u'')^2 + (u''')^2\Big)dx \tag{2.17}$$

$$= \frac{1}{3}\left(1 - \frac{B^2}{2C_m}\right)\|u\|^2_{H^3(\mathbb{R})},$$

which is the desired result.

If (2.9) holds then we can achieve the proof in a similar way by showing that

$$-B\xi^2 \leq \frac{1}{2}\xi^6 + C - 1, \qquad \forall\, \xi \in \mathbb{R}. \tag{2.18}$$

To prove (2.18) we easily see that the function $\varphi(t) = \frac{1}{2}t^3 + Bt + C - 1$, $t \geq 0$ has a global minimum at $(\frac{-2B}{3})^{1/2}$.

To prove the estimate (2.11) we use inequality (2.3)

$$\int_{\Omega} \Big[(u''')^2 - A(u'')^2 + B(u')^2\Big]dx \geq \left(1 - \frac{AL^2}{\pi^2}\right) \int_{\Omega} (u''')^2 dx \geq k_3\|u\|^2_{H^3(\Omega)}. \qquad \square$$

Lemma 2.5. *Let $u \in H(\Omega)$. Then we have the estimates*

a).

$$\int_{\Omega} u^2 dx \leq \left(\frac{L}{\pi}\right)^6 \|u\|^2_{H(\Omega)}, \tag{2.19}$$

b).

$$\int_{\Omega} u^r dx \leq C(L,r)\mathcal{S}^{r-2}\|u\|^r_{H(\Omega)}, \qquad r > 2, \tag{2.20}$$

where C is a positive constant depending only on the indicated quantities and $\mathcal{S}$ is the best constant in the imbedding $H^3(\Omega) \subset C^0(\overline{\Omega})$.

Proof. a). Follows from inequality (2.1).

b). By the Sobolev imbedding and Lemma 2.2 we get

$$\int_{\Omega} u^r dx \leq \|u\|^{r-2}_{C^0(\overline{\Omega})} \int_{\Omega} u^2 dx$$

$$\leq \mathcal{S}^{r-2}\Big(\|u\|^2_{H^3(\Omega)}\Big)^{(r-2)/2} \int_{\Omega} u^2 dx \tag{2.21}$$

$$\leq \mathcal{S}^{r-2}C(L)^{(r-2)/2}\Big(\|u\|^2_{H(\Omega)}\Big)^{(r-2)/2} \int_{\Omega} u^2 dx$$

$$\leq \mathcal{S}^{r-2}C(L,r)\|u\|^r_{H(\Omega)}. \qquad \square$$

3 Main results

3.1 Existence

We split the study of existence into three cases:

Case $0 \le r < 2$

Lemma 3.1. *Suppose that F satisfies*

$$F(x,s) \le K_1|s|^r + K_2, \qquad \forall\,(x,s) \in \Omega \times \mathbb{R},$$

where $K_1, K_2, 0 \le r < 2$ *and* $A \le 0, B, C \ge 0, C \in C^0(\Omega)$. *Then the boundary value problem* (1.1) *has at least one solution.*

Proof. The result is a consequence of the Weierstrass theorem, which tells us that if the functional J is coercive and weakly lower semicontinuous on $H(\Omega)$, then J has a global minimum.

We first establish that $J(u)$ is coercive.

By Young's inequality

$$\int_\Omega F(x,u)dx \le \varepsilon \int_\Omega u^2 dx + \int_\Omega \left(C(r,\varepsilon) K_1^{\frac{2}{2-r}} + K_2 \right) dx. \tag{3.1}$$

Using Lemma 2.2 it follows that

$$\begin{aligned} J(u) &\ge \frac{1}{2}\int_\Omega \left((u''')^2 + C(x)u^2\right) dx - \varepsilon \int_\Omega u^2 dx - \int_\Omega \left(C(r,\varepsilon) K_1^{\frac{2}{2-r}} + K_2 \right) dx \\ &\ge \int_\Omega (u''')^2 \left(\frac{1}{2} - \varepsilon \left(\frac{L}{\pi}\right)^6 \right) dx - C(K_1, K_2, r, L, \varepsilon) \\ &\ge \|u\|^2_{H(\Omega)} \left(\frac{1}{2} - \varepsilon \left(\frac{L}{\pi}\right)^6 \right) - C(K_1, K_2, r, L, \varepsilon). \end{aligned}$$

If we choose now $\varepsilon > 0$ sufficiently small we get that $J(u)$ is coercive on $H(\Omega)$.

We now show that $J(u)$ is weakly lower semicontinuous on the reflexive space $H(\Omega)$. Since $A \le 0$ and $B \ge 0$ we get that

$$J_1(u) = \frac{1}{2}\int_\Omega \left((u''')^2 - A(u'')^2 + B(u')^2 \right) dx$$

is convex.

Hence $J(u)$ can be represented as the sum $J(u) = J_1(u) + J_2(u)$, where $J_1(u)$ is convex and

$$J_2(u) = \frac{1}{2}\int_\Omega \left(C(x)u^2 - 2F(x,u) \right) dx$$

is sequentially weakly continuous.

Therefore, $J(u)$ is weakly lower semicontinuous by the result in [3, Criterion 6.1.3, p. 30], and the proof follows. □

Remark 3.2. From the proof of Lemma 3.1 it can easily be seen that Lemma 3.1 still works if C takes negative values. More precisely, if $C \geq -C_m$, where $C_m > 0$ and

$$C_1 = 1 - C_m \left(\frac{L}{\pi}\right)^6 > 0. \tag{3.2}$$

The next lemma ensures that the solution we have found is nontrivial.

Lemma 3.3. *Suppose that the following condition holds:*

$$\lim_{s\to\infty} \frac{f(x,s)}{s^\alpha} = q(x) \quad \text{and} \quad \lim_{s\to\infty} F(x,s) = \infty \quad \text{uniformly in } \overline{\Omega}, \tag{3.3}$$

where $q(x) \geq 0$, $\|q\|_{L^\infty(\Omega)} > 0$, $\alpha > 1$.

Then there exists $e \in H(\Omega)$ *such that* $J(e) < 0$.

Proof. We can find a function $\varphi > 0$ in Ω such that $\varphi \in H(\Omega)$ and $\int_\Omega q(x)\varphi^{\alpha+1}(x)dx \geq \delta$, where δ is a positive constant.

A candidate for φ is

$$\varphi(x) = \sin \frac{\pi x}{L}.$$

We note that by the first part of relation (3.3) and by the fact that $f(x,s)/s^\alpha$ is continuous in $\overline{\Omega} \times (0,\infty)$ we get that there exists a strictly positive function $Q(x) \in L^1(\Omega)$ such that

$$f(x,s) \leq Q(x)s^\alpha \quad \text{in } \Omega \times [N,\infty),$$

where N is a positive constant.

Integrating with respect to s the last inequality, we obtain that $F(x,s)/s^{\alpha+1}$ is "dominated" by the L^1 function $Q(x)/(\alpha+1)$ in $\Omega \times [N,\infty)$.

Hence by the dominated convergence theorem and (3.3)

$$\begin{aligned}
\lim_{s\to\infty} \frac{J(s\varphi)}{s^{\alpha+1}} &= \frac{1}{2} \lim_{s\to\infty} \frac{s^2 \int_\Omega \left((\varphi''')^2 - A(\varphi'')^2 + B(\varphi')^2 + C(x)\varphi^2\right)dx}{s^{\alpha+1}} \\
&\quad - \lim_{s\to\infty} \int_\Omega \frac{F(x,s\varphi)}{s^{\alpha+1}}dx \\
&= -\int_\Omega \lim_{s\to\infty} \frac{F(x,s\varphi)}{s^{\alpha+1}}dx = -\int_\Omega \lim_{s\to\infty} \frac{f(x,s\varphi)\varphi}{(\alpha+1)s^\alpha}dx \\
&= -\frac{1}{\alpha+1}\int_\Omega q(x)\varphi^{\alpha+1}(x)dx < 0,
\end{aligned}$$

which is the desired result.

Hence there exists $e = s\varphi \in H(\Omega)$ such that $J(e) < 0$. □

Our first main existence result reads.

Theorem 3.4. *Suppose that* F *satisfies*

$$F(x,s) \leq K_1|s|^r + K_2, \quad \forall\,(x,s) \in \Omega \times \mathbb{R},$$

where $K_1, K_2, 0 \leq r < 2$ *and* $A \leq 0, B, C \geq 0, C \in C^0(\Omega)$. *If in addition (3.3) holds, then the boundary value problem (1.1) has at least one nontrivial solution.*

Case $r = 2$

Lemma 3.5. *Suppose that F satisfies*

$$F(x,s) \le K_1|s|^2 + K_2, \qquad \forall\,(x,s) \in \Omega \times \mathbb{R}, \tag{3.4}$$

where K_1, K_2, $A \le 0$, $B, C \ge 0, C \in C^0(\Omega)$. If in addition we assume that

$$1 - 2K_1\left(\frac{L}{\pi}\right)^6 > 0, \tag{3.5}$$

then the boundary value problem (1.1) has at least one solution.

Proof. Since relations and (3.4) and (3.5) ensure the coercivity of $J(u)$, we can imitate the proof of Lemma 3.1. □

Similarly, we get the corresponding existence result in the case $r = 2$.

Theorem 3.6. *Suppose that F satisfies*

$$F(x,s) \le K_1|s|^2 + K_2, \qquad \forall\,(x,s) \in \Omega \times \mathbb{R},$$

where K_1, K_2, $A \le 0$, $B, C \ge 0$ in Ω. If in addition we assume that

$$1 - 2K_1\left(\frac{L}{\pi}\right)^6 > 0,$$

and that (3.3) holds, then the boundary value problem (1.1) has at least one nontrivial solution.

Proof. Follows from Lemma 3.3 and Lemma 3.5. □

Case $r > 2$, $K_2 = 0$

The existence for the case $r > 2$ will be treated differently. We shall see that $J(u)$ has a mountain-pass structure and the nontrivial critical points of $J(u)$ will be found by using the Mountain-Pass theorem of Brézis and Nirenberg.

The following two lemmas show when $J(u)$ has a mountain-pass structure.

Lemma 3.7. *Let F satisfy*

$$F(x,s) \le K_1|s|^r, \qquad \forall\,(x,s) \in \Omega \times \mathbb{R}, \tag{3.6}$$

where $K_1 > 0, r > 2$.

If $A \le 0$, $B, C \ge 0$, or if one of the relations (2.4), (2.6), (2.7), (2.9) or (2.10) is satisfied, then there exist two positive constants ρ and η such that

$$J(u)_{|_{\|u\|_i = \rho}} \ge \eta, \qquad i = 1,2,3. \tag{3.7}$$

Here $\|u\|_i$ denotes one of the following norms

$$\|u\|_1^2 = \int_\Omega \Big((u''')^2 - A(u'')^2 + B(u')^2 + C(x)u^2\Big)dx,$$

when $A \le 0$, $B, C \ge 0$ or when one of the relations (2.4) or (2.6) is satisfied;

$$\|u\|_2^2 = \int_\Omega \Big((u''')^2 + B(u')^2 + C(x)u^2 \Big) dx,$$

when one of the relations (2.7) *or* (2.9) *is satisfied;*

$$\|u\|_3^2 = \int_\Omega \Big((u''')^2 - A(u'')^2 + B(u')^2 \Big) dx,$$

when the relation (2.10) *is satisfied.*

Proof. By virtue of Lemma 2.3 we see that $H(\Omega)$ endowed with one of the scalar products $(u,v)_i$, $i = 1,2,3$, becomes a Hilbert space.

We give the proof in the case when (2.4) is satisfied. The cases when relations (2.7), (2.9) or (2.10) hold can be treated similarly.

We note that (2.5) reads

$$\|u\|_1 \geq k_1 \|u\|^2_{H^3(\Omega)}. \tag{3.8}$$

$J(u)$ becomes

$$J(u) = \frac{1}{2}\|u\|_1^2 - \int_\Omega F(x,u)dx.$$

Since $r > 2$, we can choose $q > 1$ such that $r = 1 + q$.

From (3.8), (1.7) and Young's inequality it follows that

$$\begin{aligned}\int_\Omega F(x,u)dx &\leq \varepsilon K_1 \int_\Omega u^2 dx + \frac{K_1}{4\varepsilon}\int_\Omega u^{2q} dx \\ &\leq \varepsilon K_1 C(L)\|u\|^2_{H(\Omega)} + \frac{K_1}{4\varepsilon}\mathcal{S}^{2q-2}C(L)^{\frac{q}{2}}\|u\|^{2q}_{H(\Omega)} \\ &\leq \varepsilon C(L,k_1,K_1)\|u\|_1^2 + \frac{1}{4\varepsilon}C(L,\mathcal{S},k_1,K_1,q)\|u\|_1^{2q}.\end{aligned}$$

Hence

$$J(u) \geq \|u\|_1^2 \left(\frac{1}{2} - \varepsilon C(L,k_1,K_1) - \frac{1}{4\varepsilon}C(L,\mathcal{S},k_1,K_1,q)\|u\|_1^{2q-2} \right).$$

We choose now ε small such that $\frac{1}{2} - \varepsilon C(L,k_1,K_1) > 0$.

If we choose ρ sufficiently small we see that the required inequality holds. □

Lemma 3.8. *Let F satisfy*

$$F(x,s) \leq K_1|s|^r, \qquad \forall\, (x,s) \in \Omega \times \mathbb{R},$$

where $K_1 > 0, r > 2$.

Suppose that $A \leq 0$, $B, C \geq 0$, or one of the relations (2.4), (2.6), (2.7), (2.9) *or* (2.10) *is satisfied. Suppose in addition that the condition* (3.3) *is satisfied and let ρ be as in Lemma* 3.7. *Then there exists $e \in H(\Omega)$ with $||e||_i > \rho$, $i = 1,2,3$ such that $J(e) < 0$.*

Proof. The condition $A \leq 0$, $B, C \geq 0$, or one of the relations (2.4), (2.6), (2.7), (2.9) or (2.10) assures in view of Lemma 3.7 the existence of ρ, while relation (3.3) assures the existence of e with $J(e) < 0$. Since $e = s\varphi$ where s is large we get that $\|e\|_i > \rho$, $i = 1,2,3$. □

The following celebrated result is useful.

Theorem 3.9 (Mountain Pass Theorem [6]). *Let E be a real Banach space with its dual E^* and suppose that $J \in C^1(E, \mathbb{R})$ satisfies*

$$\max\{J(0), J(e)\} \leq \mu < \eta \leq \inf_{\|u\|=\rho} J(u),$$

for some constants $\mu < \eta$, $\rho > 0$ and $e \in E$ with $\|e\| > \rho$. Let $\lambda \geq \eta$ be characterized by

$$\lambda = \inf_{\gamma \in \Gamma} \max_{0 \leq \tau \leq 1} J(\gamma(\tau)),$$

where $\Gamma = \{\gamma \in C^0([0,1], E) : \gamma(0) = 0, \gamma(1) = e\}$ is the set of continuous paths joining 0 and e.
Then there exists a sequence $\{u_n\} \subset E$ such that

$$J(u_n) \to \lambda \geq \eta \quad \textit{and} \quad \|J'(u_n)\|_{E^*} \to 0, \quad \textit{as } n \to \infty.$$

We can now apply the Mountain Pass Theorem (Theorem 3.9) in $H(\Omega)$ to find a Cerami type sequence, i.e.,

$$\text{there exists } \{u_n\} \subset H(\Omega) \text{ such that } J(u_n) \to \lambda \text{ and } \|J'(u_n)\|_{H^*(\Omega)} \to 0. \tag{3.9}$$

Lemma 3.10. *Suppose that we are under the hypotheses of Lemma 3.8. Let $\alpha \in (0,2)$. If in addition there exist the constants $\beta > 0$, $\gamma > 0$, $\theta \geq 2$ such that*

$$F(x,s) - \frac{1}{\theta} f(x,s)s \leq \gamma |s|^{\alpha-1} s, \qquad \forall x \in \Omega,\ s \in \mathbb{R},\ s \neq 0, \tag{3.10}$$

then the sequence $\{u_n\}$ defined by (3.9) is bounded in $H(\Omega)$.

Proof. We give the proof in the case when (2.4) is satisfied.
By the Mountain Pass Theorem 3.9 there exists

$$\{u_n\} \subset H(\Omega) \text{ such that } J(u_n) \to \lambda \text{ and } \langle J'(u_n), u_n \rangle \to 0. \tag{3.11}$$

Hence for sufficiently large n we have,

$$\lambda + 1 \geq J(u_n) - \frac{1}{\theta} \langle J'(u_n), u_n \rangle. \tag{3.12}$$

Since

$$\langle J'(u_n), u_n \rangle = \|u_n\|_1^2 - \int_\Omega f(x, u_n) u_n dx, \tag{3.13}$$

we get that

$$J(u_n) - \frac{1}{\theta} \langle J'(u_n), u_n \rangle = \left(\frac{1}{2} - \frac{1}{\theta} \right) \|u_n\|_1^2 - \int_\Omega \left(F(x, u_n) - \frac{1}{\theta} f(x, u_n) u_n \right) dx. \tag{3.14}$$

By (3.10)

$$\begin{aligned} -\int_\Omega \left(F(x, u_n) - \frac{1}{\theta} f(x, u_n) u_n \right) dx &\geq -\gamma \int_\Omega |u_n|^{\alpha-1} u_n dx \\ &\geq -\gamma \int_\Omega |u_n|^\alpha dx. \end{aligned} \tag{3.15}$$

Now by Young's inequality we get

$$\int_\Omega |u_n|^\alpha dx \le \varepsilon \int_\Omega (u_n)^2 dx + C(\varepsilon, \alpha, L),$$

by Lemma 2.2 we get

$$\int_\Omega (u_n)^2 dx \le \left(\frac{L}{\pi}\right)^6 \int_\Omega (u_n''')^2 dx = \left(\frac{L}{\pi}\right)^6 \|u_n\|^2_{H(\Omega)}, \tag{3.16}$$

and since (2.5) reads

$$\|u_n\|_1^2 \ge k\|u_n\|^2_{H^3(\Omega)} \ge k\|u_n\|^2_{H(\Omega)}, \qquad k = \frac{1}{3}\left(1 - \frac{A^2}{4B}\right),$$

we obtain from (3.15) that

$$\begin{aligned} -\int_\Omega \left(F(x,u_n) - \frac{1}{\theta} f(x,u_n)u_n\right) dx &\ge -\varepsilon\gamma \int_\Omega (u_n)^2 dx - \gamma C(\varepsilon, \alpha, L) \\ &\ge -\varepsilon\gamma \mathcal{C}(\mathcal{L})\|u_n\|^2_{H^3(\Omega)} - C(\varepsilon, \alpha, \gamma, L) \\ &\ge -\varepsilon\frac{\gamma}{k}\mathcal{C}(\mathcal{L})\|u_n\|_1^2 - C(\varepsilon, \alpha, \gamma, L), \end{aligned} \tag{3.17}$$

where $\mathcal{C}(\mathcal{L}) = \left(\frac{L}{\pi}\right)^6$.

Combining relations (3.12), (3.14), (3.17) we have the estimate

$$\lambda + 1 \ge \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|_1^2 - \varepsilon\frac{\gamma}{k}\mathcal{C}(\mathcal{L})\|u_n\|_1^2 - C(\varepsilon, \alpha, \gamma, L). \tag{3.18}$$

We can choose $\delta > 0$ such that $\theta = 2 + 2\delta$.

If we now choose

$$\varepsilon = \frac{\delta k}{2\gamma(2+2\delta)\mathcal{C}(\mathcal{L})},$$

it follows that

$$\begin{aligned} \lambda + 1 &\ge \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|_1^2 - \frac{\delta}{2(2+2\delta)}\|u_n\|_1^2 - C(k, \alpha, \gamma, \delta, L) \\ &\ge \frac{\delta}{2(2+2\delta)}\|u_n\|_1^2 - C(k, \alpha, \gamma, \delta, L), \end{aligned} \tag{3.19}$$

which shows that $\{u_n\}$ is bounded. □

Remark 3.11. Instead of (3.10) we could have imposed the following hypotheses:

Let $\alpha \in (0,2)$ and suppose that there exist the constants $\beta > 0$, $\gamma > 0$, $\theta \ge 2$ such that

$$F(x,s) - \frac{1}{\theta} f(x,s)s \le \gamma s^\alpha, \qquad \forall x \in \Omega,\ s > 0, \tag{3.20}$$

$$F(x,s) - \frac{1}{\theta} f(x,s)s \le \beta, \qquad \forall x \in \Omega,\ s \le 0. \tag{3.21}$$

As (3.10) requires that $F(x,s) - \frac{1}{\theta} f(x,s)s$ is negative for $s < 0$ we see that (3.21) is less restrictive that (3.10).

Sketch of proof. For each fixed n we define $\Omega^+ = \{x \in \Omega \mid u_n(x) > 0\}$ and $\Omega^- = \{x \in \Omega \mid u_n(x) \leq 0\}$.

By (3.20) and (3.21)

$$\begin{aligned} -\int_\Omega \left(F(x,u_n) - \frac{1}{\theta} f(x,u_n)u_n\right)dx &\geq -\gamma \int_{\Omega^+} (u_n)^\alpha dx - \beta \int_{\Omega^-} dx \\ &= -\gamma \int_{\Omega^+} (u_n)^\alpha dx - \beta\, \mathrm{meas}(\Omega^-) \\ &\geq -\gamma \int_{\Omega^+} (u_n)^\alpha dx - \beta L. \end{aligned}$$

By Young's inequality we get

$$\int_{\Omega^+} (u_n)^\alpha dx \leq \varepsilon \int_{\Omega^+} (u_n)^2 dx + C(\varepsilon,\alpha,L) \leq \varepsilon \int_\Omega (u_n)^2 dx + C(\varepsilon,\alpha,L).$$

Hence

$$-\int_\Omega \left(F(x,u_n) - \frac{1}{\theta} f(x,u_n)u_n\right)dx \geq -\varepsilon\gamma \int_\Omega (u_n)^2 dx - C(\gamma,\varepsilon,\alpha,L,\beta),$$

which is similar to the first inequality in (3.17). Now the proof follows exactly as the proof of Lemma 3.10. □

Remark 3.12. Lemma 3.10 still holds when $\alpha = 2$, if we impose the restriction

$$\frac{1}{2} - \frac{1}{\theta} - \frac{\gamma}{k}\left(\frac{L}{\pi}\right)^6 > 0. \tag{3.22}$$

Lemma 3.13. *Under the hypotheses of Lemma 3.10, there exists a sequence $\{u_n\}$ such that $u_n \to u_0$ strongly in $H(\Omega)$.*

Proof. By Lemma 3.10 there is a bounded Cerami type sequence $\{u_n\}$. Hence we can extract a subsequence, still denoted $\{u_n\}$, such that

$$\begin{aligned} u_n &\rightharpoonup u_0 \quad \text{weakly in } H(\Omega), \\ u_n &\to u_0 \quad \text{strongly in } C^2(\overline{\Omega}). \end{aligned}$$

Let $v_n = u_n - u_0$.

Using (3.13) with u_n replaced by v_n and the fact that

$$\langle J'(v_n), v_n \rangle \to 0 \quad \text{as } n \to \infty$$

we can find a sequence $\{\alpha_n\}$, $\alpha_n > 0$, $\alpha_n \to 0$ as $n \to \infty$ such that (for sufficiently large n)

$$\alpha_n \geq \|v_n\|_1^2 - \int_\Omega f(x,v_n)v_n dx.$$

Hence

$$\alpha_n \geq \|v_n\|_1^2 - K_1 \int_\Omega |v_n|^r dx. \tag{3.23}$$

By (2.20) we have the estimate

$$\begin{aligned}\int_\Omega |v_n(x)|^r dx &= \left(\int_\Omega |v_n(x)|^r dx\right)^{\frac{r-2}{r}} \left(\int_\Omega |v_n(x)|^r dx\right)^{\frac{2}{r}} \\ &\le \|v_n(x)\|^{r-2}_{C^0(\overline{\Omega})} L^{\frac{r-2}{2}} \left(\frac{C(L,r)}{k}\mathcal{S}^{r-2}\|v_n(x)\|_1^r\right)^{\frac{2}{r}} \\ &= C(L,r,k,\mathcal{S})\|v_n(x)\|^{r-2}_{C^0(\overline{\Omega})}\|v_n(x)\|_1^2.\end{aligned} \tag{3.24}$$

Combining (3.23) and (3.24)

$$\alpha_n \ge \|v_n\|_1^2 \left(1 - C(L,r,k,K_1,\mathcal{S})\|v_n(x)\|^{r-2}_{C^0(\overline{\Omega})}\right) > 0.$$

Thus $v_n \to 0$ strongly in $H(\Omega)$. This completes the proof. □

We can now conclude the existence result in the case $r > 2$, $K_2 = 0$.

Theorem 3.14. *Let F satisfy*

$$F(x,s) \le K_1|s|^r, \qquad \forall\,(x,s) \in \Omega \times \mathbb{R},$$

where $K_1 > 0, r > 2$. *Suppose that one of the conditions of Lemma 2.3 is satisfied and that* (3.10) *holds. If the condition* (3.3) *is satisfied, then problem* (1.1) *has a nontrivial solution in* $H(\Omega)$.

We end this section by giving the following examples as an application of the results.

Example 3.15. We see that the theory presented includes the typical example

$$f(x,s) = b(x)s\,|s|^{p-2}, \qquad p > 2,$$

where b is a bounded function which is either strictly positive or strictly negative in Ω (no sign changing is allowed).

For the sake of simplicity we take p even. We can check that

$$F(x,s) = b(x)\frac{s^p}{p}$$

satisfies (1.7) and relation (3.20) becomes

$$b(x)s^p\left(\frac{1}{p} - \frac{1}{\theta}\right) \le \gamma s, \qquad x \in \Omega,\ s > 0. \tag{3.25}$$

If $b > 0$ then we can choose $2 < \theta < p$ and see that the left hand side of (3.25) becomes negative and hence (3.25) is satisfied. Due to the negativity of the left hand side of (3.25) for $s \le 0$ it is also obvious that (3.21) is satisfied.

We can argue similarly if $b < 0$ by choosing $\theta > p$.

Also since (2.10) holds with $A = 2, B = 1, C = 0, L = 1$ we get by Theorem 3.14 that the boundary value problem that describes the deformation of the equilibrium state of an elastic circular ring segment with its two ends simply supported (see [1])

$$\begin{cases} u^{(6)} + 2u^{(4)} + u'' = b(x)u\,|u|^{p-2} & \text{in}\,\Omega = (0,1) \\ u = u'' = u^{(4)} = 0 & \text{on}\,\partial\Omega, \end{cases} \tag{3.26}$$

has at least one nontrivial solution.

Example 3.16. We consider the following function

$$g(x,s) = a(x)\cos(s^n + C)s^{n-1},$$

where a is a bounded function, C is a constant and n is a natural number.

Since the potential of g is

$$G(x,s) = \frac{a(x)}{n}\left(\sin(s^n + C) - \sin(C)\right)$$

and satisfies the requirements of Lemma 3.1, we get that problem (1.1) with f replaced by g has a nontrivial solution in $H(\Omega)$ if $A \le 0, B, C \ge 0, C \in C^0(\Omega)$.

Example 3.17. If we consider

$$h(s) = \ln(|s|+1) + \frac{|s|}{|s|+1} + s$$

we see that its potential is $H(s) = s\ln(|s|+1) + s^2/2$. Due to the inequality $\ln(|s|+1) \le |s|$, we see that H satisfies the requirements of Lemma 3.3. Hence problem (1.1) with f replaced by h has a nontrivial solution in $H(\Omega)$ if $A \le 0, B, C \ge 0, C \in C^0(\Omega)$.

3.2 Uniqueness

Our first uniqueness result reads

Theorem 3.18. *Suppose that F satisfies*

$$F(x,s) \le K_1|s|^r + K_2, \qquad \forall\,(x,s) \in \Omega \times \mathbb{R},$$

where $K_1, K_2, 0 \le r < 2$ and $A \le 0, B, C \ge 0, C \in C^0(\Omega)$ and that relation (3.3) holds. If in addition

$$\frac{\partial f(x,s)}{\partial s} < 0 \quad \text{in } \Omega \times \mathbb{R} \tag{3.27}$$

holds, then problem (1.1) *has a unique nontrivial solution in $H(\Omega)$.*

Proof. By the proof of Lemma 3.1, $J(u)$ can be represented as the sum $J(u) = J_1(u) + J_2(u)$, where $J_1(u)$ is convex and

$$J_2(u) = \frac{1}{2}\int_\Omega \Big(C(x)u^2 - 2F(x,u)\Big)dx.$$

Condition (3.27) assures that the function $s \to F(x,s)$ is strictly convex and hence $J_2(u)$ is strictly convex. The last statement implies that $J(u)$ is strictly convex and the uniqueness follows. □

The next uniqueness result is a consequence of the following one dimensional generalized maximum principle (for results concerning the generalized maximum principle see [18, p. 73]) and collects several author's uniqueness results in the case when the coefficients A, B, C are nonconstant or have arbitrary sign and $f = f(x)$.

Theorem 3.19. *Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy the inequality $\mathrm{L}u \equiv u'' + \gamma(x)u \geq 0$ in Ω, where $\gamma \geq 0$ in Ω.*

Suppose that

$$\sup_{\Omega} \gamma < \frac{\pi^2}{L^2} \tag{3.28}$$

holds.

Then, there exists a function $w > 0$ in $\overline{\Omega}$, $w \in C^\infty(\overline{\Omega})$ such that u/w satisfies a generalized maximum principle in Ω, i.e., there exists a constant $k \in \mathbb{R}$ such that $u/w \equiv k$ in Ω or u/w does not attain a nonnegative maximum in Ω.

Proof. The proof follows directly from [11, Theorem 2.1] (which holds for all dimensions $n \geq 1$). $\square$

The interested reader may consult the paper [16] for a different kind of one dimensional maximum principle for sixth order operators. The authors prove (Theorem 3.1) the positivity of the solution u that satisfies a sixth order differential inequality assuming that u, u' are positive on the boundary of the domain $\Omega = (a, b)$ and (in particular) $u'''(a) \leq 0, u'''(b) \geq 0$.

Theorem 3.20. *The boundary value problem*

$$\begin{cases} u^{(6)} + A(x)u^{(4)} + B(x)u'' - C(x)u = f(x) & \text{in } \Omega \\ u = g_1,\ u'' = g_2,\ u^{(4)} = g_3 & \text{on } \partial\Omega, \end{cases} \tag{3.29}$$

has at most one solution if one of the following conditions is satisfied (here $g_i, i = 1, 2, 3$ are arbitrary constants)

1).

$$\sup_{\Omega} \frac{A(B+C)^2}{2B^2(A+1)} < \frac{\pi^2}{L^2} \quad \text{in } \Omega. \tag{3.30}$$

Here $A < -1, B > 0$ are constants and $C > 0$ in Ω is a function.

2). Suppose that the functions A, C satisfy $-A = C > 0$ in Ω and that the function B satisfies

$$B > 1 \text{ in } \quad \overline{\Omega}, \qquad \Big(1/(B-1)\Big)'' \leq 0 \quad \text{in } \Omega. \tag{3.31}$$

3). Suppose that the functions $A < 0, B, C > 0$ in Ω satisfy

$$\sup_{\Omega} \frac{-(C-A)^2}{2A(B-1)} < \frac{\pi^2}{L^2} \quad \text{in } \Omega \tag{3.32}$$

and also (3.31) holds.

4).

$$\sup_{\Omega} \frac{-2C}{A+C+1} < \frac{\pi^2}{L^2} \text{ in } \Omega, \tag{3.33}$$

where the functions $A < 0, C > 0$ satisfy $A + C + 1 < 0$ in $\overline{\Omega}$ and $\big(1/(A+C+1)\big)'' \geq 0$ in Ω.

Proof. 1). The proof uses the P-function method introduced by L. E. Payne [17]. Many results concerning the P-function method and its applications can be found in the book [19].

We give the proof when (3.30) holds.

We define $u = u_1 - u_2$, where u_1 and u_2 are solutions of (3.29). Then u satisfies (3.29), where $f = 0$ and with zero boundary data $u = u'' = u^{(4)} = 0$ on $\partial\Omega$.

According to [11], Lemma 3.1, i), the function

$$\mathrm{P} = (-Au^{(4)} + Bu)^2 + AB(A+1)(u'')^2 - B^2(A+1)u^2$$

satisfies the inequality

$$\mathrm{P}'' + \frac{A(B+C)^2}{2B^2(A+1)}\mathrm{P} \geq 0 \quad \text{in } \Omega.$$

Hence by Theorem 3.19 there exists $w > 0$ in $\overline{\Omega}$ such that P/w satisfies a generalized maximum principle in Ω, i.e., either there exists a constant $k \in \mathbb{R}$ such that

$$\frac{\mathrm{P}}{w} \equiv k \quad \text{in } \Omega, \tag{3.34}$$

or

$$\frac{\mathrm{P}}{w} \quad \text{does not attain a maximum in } \Omega. \tag{3.35}$$

If (3.34) holds then since the function P/w is smooth (3.34) holds in $\overline{\Omega}$. By the zero boundary conditions we have $\mathrm{P} = 0$ on $\partial\Omega$, i.e., $k = 0$. It follows that $\mathrm{P} = 0$ in Ω. Since P is a sum of squares multiplied by positive constants, $\mathrm{P} = 0$ in Ω implies $u \equiv 0$ in Ω. Hence $u_1 = u_2$ in Ω.

Alternatively, if (3.35) holds, then

$$\max_{\overline{\Omega}} \frac{\mathrm{P}}{w} = \max_{\partial\Omega} \frac{\mathrm{P}}{w} = 0$$

by the zero boundary conditions. It follows that

$$0 \leq \max_{\overline{\Omega}} \frac{\mathrm{P}}{w} = 0,$$

i.e., $\mathrm{P} = 0$ in Ω. Using the same arguments as above, we get $u \equiv 0$ in Ω, i.e., $u_1 = u_2$ in Ω.

2). If (3.31) holds then, by [11, Lemma 3.1, ii)], the P-function

$$\mathrm{P}_1 = (u^{(4)} + u)^2 + (B-1)(u'')^2 + (B-1)u^2$$

satisfies the classical maximum principle, which means that it attains its maximum on the boundary of Ω, i.e., $\max_{\overline{\Omega}} \mathrm{P}_1 = \max_{\partial\Omega} \mathrm{P}_1 = 0$.

3). If (3.32) holds then [11, Lemma 3.1, ii)] tells that P_1/w satisfies a generalized maximum principle and we can argue as in Case 1).

4). If (3.33) holds then we can use [11, Lemma 3.2, ii)] which shows that P_2/w satisfies a generalized maximum principle and the proof follows. Here

$$\mathrm{P}_2 = (u^{(4)} - u'')^2 + C(u'' - u)^2 - (A+C+1)(u'')^2.$$

□

3.3 Multiplicity

Finally, we present a multiplicity result for (1.1) that is based on the result presented in [20, Theorem 3], Lemma 2.3 and the next result.

Lemma 3.21. *Let A, B, C be real constants such that $C > 0$. The polynomial*

$$P(L) = L^6 - \frac{B}{C}\pi^2 L^4 + \frac{A}{C}\pi^4 L^2 - \frac{\pi^6}{C}$$

has exactly one positive zero ξ_0 if either

$$B \le 0, \qquad A > 0, \tag{3.36}$$

$$A, B > 0, \qquad B^2 \le 3AC, \tag{3.37}$$

$$A, B > 0, \qquad B^2 > 3AC, \qquad \frac{1}{C} \in (0, \gamma_-) \cup (\gamma_+, \infty), \tag{3.38}$$

where

$$\gamma_\pm = \frac{1}{27}\left[\frac{9AB}{C^2} - \frac{2B^3}{C^3} \pm \left(\frac{2B^2}{C^2} - \frac{3A}{C}\right)^{\frac{3}{2}}\right],$$

or

$$A \le 0, \qquad B \in \mathbb{R} \tag{3.39}$$

holds.

Moreover, for $L > \xi_0$ we have $P(L) > 0$.

The proof is a direct consequence of the result presented in [8, Lemma 4.3].

To prove the multiplicity result we need the following

Definition 3.22. Let X be a Banach space and $J \in C^1(X, \mathbb{R})$. We say that J satisfies a Palais–Smale condition if any sequence $\{u_n\}_n$ in X for which $J(u_n)$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$, has a convergent subsequence.

Theorem 3.23 (Clark [10])**.** *Let X be a Banach space, $J \in C^1(X, \mathbb{R})$ be even, bounded below and satisfy the Palais–Smale condition. Suppose that $J(0) = 0$ and there is a set $Y \subset X$ such that Y is homeomorphic to S^{m-1} by an odd map and $\sup_Y J < 0$. Then J possesses at least m distinct pairs of critical points.*

Our multiplicity result reads

Theorem 3.24. *Let $L > m\xi_0$, for some positive natural number m. Suppose that $F(x, 0) = 0$, $s \to F(x, s)$ is even for all $x \in \Omega$ and A, B, C are constants. If in addition one of the following relations holds*

$$\text{one of the hypotheses of Lemma 3.21, relation (1.4) and } -F(x, s) \le K|s|^p, \ p > 2, \tag{3.40}$$

$$A \le 0, \ B \ge 0, 0 \ge C \ge -C_m \text{ and relation (1.8)}, \tag{3.41}$$

$$A^2 \le 4BC_1, \ B^2 \le -3AC \text{ and relation (1.8)}, \tag{3.42}$$

where $C_1 = 1 - C_m\left(\frac{L}{\pi}\right)^6$, then problem (1.1) has m distinct nontrivial solutions.

Proof. We first note that if relation (1.4) holds, then by [14, Lemma 7], we get that $J(u)$ is bounded from below and satisfies the Palais–Smale condition for any real constants A, B, C.

If one of the relations (3.41) or (3.42) is assumed, in view of Lemma 2.3 and the structure condition (1.8) we can use a similar argument that was used in the proof of Lemma 3.1 to show that $J(u)$ is bounded from below on $H(\Omega)$ by a negative constant. Since f is continuous on $\mathbb{R}^2$ we can follow [20], proof of Theorem 3, to get that $J(u)$ satisfies the Palais–Smale condition.

We now use the same techniques as in [20, Theorem 3] and prove the case when (3.40) holds. We can treat similarly the other cases.

Consider the set $Y \subset H(\Omega)$,

$$Y = \left\{ \lambda_1 \sin\frac{\pi x}{L} + \cdots + \lambda_m \sin\frac{m\pi x}{L} : \lambda_1^2 + \cdots + \lambda_m^2 = \rho^2 \right\},$$

where ρ is a positive number to be chosen later. Y is a subset of the finite-dimensional space X_m

$$X_m = \operatorname{span}\left\{ \sin\frac{\pi x}{L}, \ldots, \sin\frac{m\pi x}{L} \right\}$$

equipped with the norm

$$\left\| \lambda_1 \sin\frac{\pi x}{L} + \cdots + \lambda_m \sin\frac{m\pi x}{L} \right\|_m^2 = \lambda_1^2 + \cdots + \lambda_m^2.$$

Since

$$J(v) = \frac{1}{2}\int_\Omega \left((v''')^2 - A(v'')^2 + B(v')^2 + Cv^2 \right) dx - \int_\Omega F(x, v) dx,$$

we get by computation and by (1.8) that for any $v \in Y$

$$J(v) \le \frac{L}{4}\|v\|_m^2 \left[\left(\frac{\pi}{L}\right)^6 - A\left(\frac{\pi}{L}\right)^4 + B\left(\frac{\pi}{L}\right)^2 + C \right] + K\int_\Omega |v|^p dx.$$

Using Hölder's inequality we have

$$\begin{aligned} |v| &= \left| \lambda_1 \sin\frac{\pi x}{L} + \cdots + \lambda_m \sin\frac{m\pi x}{L} \right| \\ &\le \left(\lambda_1^2 + \cdots + \lambda_m^2 \right)^{\frac{1}{2}} \left(\sin^2\frac{\pi x}{L} + \cdots + \sin^2\frac{m\pi x}{L} \right)^{\frac{1}{2}} \\ &\le m\|v\|_m. \end{aligned}$$

Hence we get

$$J(v) \le \frac{L}{4}\|v\|_m^2 Q(L) + C(K, m, p, L)\|v\|_m^p,$$

where

$$Q(L) = \left(\frac{\pi}{L}\right)^6 - A\left(\frac{\pi}{L}\right)^4 + B\left(\frac{\pi}{L}\right)^2 - C.$$

It is easy to check that $Q(L) < 0$ iff $P(L) > 0$.

Hence by Lemma 3.21 we see that for $L > \xi_0$ $Q(L) < 0$ and by choosing ρ sufficiently small, we get

$$J(v) \leq \|v\|_m^2 \left(\frac{L}{4} Q(L) + C(K, m, p, L) \|v\|_m^{p-2} \right) < 0,$$

for any $v \in Y$.

Now the proof follows from Clark's theorem. □

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Ground state sign-changing solutions for critical Choquard equations with steep well potential

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Abstract. In this paper, we study sign-changing solution of the Choquard type equation

$$-\Delta u + (\lambda V(x) + 1)\, u = \left(I_\alpha * |u|^{2^*_\alpha}\right)|u|^{2^*_\alpha - 2}u + \mu |u|^{p-2}u \quad \text{in } \mathbb{R}^N,$$

where $N \geq 3$, $\alpha \in ((N-4)^+, N)$, I_α is a Riesz potential, $p \in \left[2^*_\alpha, \frac{2N}{N-2}\right)$, $2^*_\alpha := \frac{N+\alpha}{N-2}$ is the upper critical exponent in terms of the Hardy–Littlewood–Sobolev inequality, $\mu > 0$, $\lambda > 0$, $V \in C(\mathbb{R}^N, \mathbb{R})$ is nonnegative and has a potential well. By combining the variational methods and sign-changing Nehari manifold, we prove the existence and some properties of ground state sign-changing solution for λ, μ large enough. Further, we verify the asymptotic behaviour of ground state sign-changing solutions as $\lambda \to +\infty$ and $\mu \to +\infty$, respectively.

Keywords: Choquard equation, upper critical exponent, steep well potential, ground state sign-changing solution, asymptotic behaviour.

2020 Mathematics Subject Classification: 35J15, 35J20, 35B33, 35D30.

1 Introduction and main results

The Choquard equation has a physical prototype, namely the Hartree type evolution equation

$$-i\partial_t \psi = \Delta\psi + \left(I_2 * |\psi|^2\right)\psi, \qquad (x,t) \in \mathbb{R}^3 \times \mathbb{R}_+, \tag{1.1}$$

where $\mathbb{R}_+ = [0, +\infty)$, $I_2(x) = \frac{1}{4\pi|x|}$, $\forall\, x \in \mathbb{R}^3 \backslash \{0\}$, and $*$ is convolution in $\mathbb{R}^3$. Eq. (1.1) was firstly proposed by Pekar to describe a resting polaron in [24]. Two decades later, Choquard [16] introduced Eq. (1.1) as a certain approximation to Hartree–Fock theory of one component plasma, and used it to characterize an electron trapped in its own hole. Afterwards, viewing the quantum state reduction as a gravitational phenomenon in quantum gravity, Penrose et al. [20] proposed Eq. (1.1) in the form of Schrödinger–Newton system to model a single particle moving in its own gravitational field.

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As we know, standing wave solution of Eq. (1.1) corresponds to solution of the Choquard equation

$$-\Delta u + u = \left(I_2 * |u|^2\right) u \quad \text{in } \mathbb{R}^3. \tag{1.2}$$

In detail, with a suitable scaling, the wave function $\psi(x,t) = e^{-it}u(x)$ is a solution of Eq. (1.1) once u is a solution of Eq. (1.2). Lieb demonstrated the seminal work on Eq. (1.2) in [16], in which he certified the existence and uniqueness (up to translations) of positive radial ground state solution by applying symmetrically decreasing rearrangement inequalities. After this, Lions [18] studied the same problem and further proved the existence of infinitely many radial solutions via the variational methods.

From mathematical perspective, scholars prefer to study the general Choquard equation

$$-\Delta u + W(x)u = \gamma \left(I_\alpha * G(u)\right) g(u) \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

where $N \geq 3$, $\gamma \in \mathbb{R}^+$, I_α is the Riesz potential of order $\alpha \in (0, N)$ defined for $x \in \mathbb{R}^N \backslash \{0\}$ by

$$I_\alpha(x) = \frac{A_\alpha}{|x|^{N-\alpha}} \quad \text{with} \quad A_\alpha = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})2^\alpha \pi^{\frac{N}{2}}},$$

Γ is the Gamma function, $*$ is convolution, $W \in C(\mathbb{R}^N, \mathbb{R})$, $g \in C(\mathbb{R}, \mathbb{R})$ and $G(u) = \int_0^u g(s)ds$.

To establish the variational framework for Choquard equations, we need the following celebrated Hardy–Littlewood–Sobolev inequality.

Proposition 1.1 ([17, Theorem 4.3]). *Let $r, s > 1$, $0 < \alpha < N$ satisfy $\frac{1}{r} + \frac{1}{s} = 1 + \frac{\alpha}{N}$. Then there exists a sharp constant $C(N, \alpha, r, s) > 0$ such that, for all $f \in L^r(\mathbb{R}^N)$ and $h \in L^s(\mathbb{R}^N)$, there holds*

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^{N-\alpha}} dxdy \right| \leq C(N, \alpha, r, s)|f|_r |h|_s. \tag{1.4}$$

In particular, if $r = s = \frac{2N}{N+\alpha}$, then the constant $C(N, \alpha, r, s)$ admits a precise expression, namely,

$$C(N, \alpha) := C(N, \alpha, r, s) = \pi^{\frac{N-\alpha}{2}} \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{N+\alpha}{2})} \left[\frac{\Gamma(\frac{N}{2})}{\Gamma(N)} \right]^{-\frac{\alpha}{N}}.$$

Thanks to (1.4), the integral $\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p dx$ is well defined in $H^1(\mathbb{R}^N)$ once $p \in [2_*^\alpha, 2_\alpha^*]$, where $2_\alpha^* := \frac{N+\alpha}{N-2}$ and $2_*^\alpha := \frac{N+\alpha}{N}$ are usually called upper and lower critical exponents with respect to the Hardy–Littlewood–Sobolev inequality, respectively. It is easy to clarify that the critical terms $\int_{\mathbb{R}^N} (I_\alpha * |u|^{2_\alpha^*})|u|^{2_\alpha^*} dx$ and $\int_{\mathbb{R}^N} (I_\alpha * |u|^{2_*^\alpha})|u|^{2_*^\alpha} dx$ are invariant under the scaling actions $\sigma^{\frac{N-2}{2}} u(\sigma \cdot)$ and $\sigma^{\frac{N}{2}} u(\sigma \cdot)$ $(\sigma > 0)$, respectively, and these two scaling actions served as group actions are noncompact on $H^1(\mathbb{R}^N)$. Consequently, from the perspective of variational methods, the critical exponents 2_*^α and 2_α^* may provoke two kinds of lack of compactness. However, fortunately, similar to the Sobolev critical case studied in [3], these two kinds of loss of compactness can be recovered to some extent by using the extremal functions of the Hardy–Littlewood–Sobolev inequality.

In [21], Moroz and Van Schaftingen studied the case of Eq. (1.3) that $W(x) \equiv 1$, $\gamma = \frac{1}{p}$ and $G(u) = |u|^p$ $(p > 1)$, they proved the existence, regularity, radially symmetry and decaying property at infinity of ground state solution when $p \in (2_*^\alpha, 2_\alpha^*)$. Meanwhile, based on the regularity of solutions, they established a Nehari–Pohožaev type identity and then showed the nonexistence of nontrivial solutions for Eq. (1.3) when $p \notin (2_*^\alpha, 2_\alpha^*)$. Afterwards, in [22], they extended the existence results in [21] to the case of Eq. (1.3) that g satisfies the so-called almost necessary conditions of Berestycki–Lions type. For the critical cases of Eq. (1.3), with the nonexistence result of [21] in hand, an increasing number of scholars devote to studying Eq. (1.3) with critical term and a noncritical perturbed term. We refer the interested readers to [4, 9, 14, 30] for upper critical case, [23, 26] for lower critical case and [15, 25, 31] for doubly critical case.

When it comes to the case $W(x) \not\equiv$ const., we focus our attention on steep well potential of the form $\lambda V(x) + b$, where $\lambda > 0$, $b \in \mathbb{R}$ and $V \in C(\mathbb{R}^N, \mathbb{R})$ satisfies the following hypotheses:

(V_1) V is bounded from below, $\Omega := \text{int}\, V^{-1}(0)$ is nonempty and $\overline{\Omega} = V^{-1}(0)$,

(V_2) there exists some constant $M > 0$ such that $\left|\{x \in \mathbb{R}^N : V(x) \leq M\}\right| < +\infty$.

This type of potential was firstly introduced by Bartch and Wang in [2] to study the existence and multiplicity of nontrivial solutions for subcritical Schrödinger equations in the case of $b > 0$. Later, Ding and Szulkin further considered the case $b = 0$ in [8]. Since $|\Omega| < +\infty$, then $-\Delta$ possesses a sequence of positive Dirichlet eigenvalues $\mu_1 < \mu_2 < \cdots < \mu_n \to +\infty$. Assuming $b < 0$ and $b \neq -\mu_i$ for any $i \in \mathbb{N}_+$, Clapp and Ding [6], together with Tang [27], studied the existence and concentration of ground state solution for critical Schrödinger equation. Recently, the pre-existing results on Schrödinger equations have been extended to the Choquard equations, see e.g. [1, 14, 15, 19] and the references therein.

As we concerned here, sign-changing solution of elliptic equation is a focusing topic due to its wide application in biology and physics etc. In [7], Clapp and Salazar investigated the Choquard equation

$$-\Delta u + W(x)u = (I_\alpha * |u|^p)\,|u|^{p-2}u \quad \text{in } \Omega,$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is an exterior domain, $p \in [2, 2_\alpha^*)$, $\alpha \in ((N-4)^+, N)$ and $W \in C(\mathbb{R}^N, \mathbb{R})$. Under symmetrical assumptions on Ω and decaying properties on W, they derived multiple sign-changing solutions. After this, many scholars considered the same topic in the whole Euclidean space, namely,

$$-\Delta u + W(x)u = (I_\alpha * |u|^p)\,|u|^{p-2}u \quad \text{in } \mathbb{R}^N. \tag{1.5}$$

In [11], Ghimenti and Van Schaftingen studied the case that $N \geq 1$, $\alpha \in ((N-4)^+, N)$, $W(x) \equiv 1$ and $p \in (2, 2_\alpha^*)$ of Eq. (1.5). There, by introducing a new minimax principle and concentration-compactness lemmas for sign-changing Palais–Smale sequences, they obtained a ground state sign-changing solution. Also, they proved that the least energy in the sign-changing Nehari manifold has no minimizers when $p \in (2_*^\alpha, \max\{2, 2_\alpha^*\})$. Further, Ghimenti, Moroz and Van Schaftingen [10] constructed a ground state sign-changing solution of Eq. (1.5) when $p = 2$ by approaching the case $p = 2$ with the cases $p \in (2, 2_\alpha^*)$. Van Schaftingen and Xia [28] assumed that $N \geq 1$, $\alpha \in ((N-4)^+, N)$, $p \in [2, 2_\alpha^*)$ and $W \in C(\mathbb{R}^N, \mathbb{R})$ satisfies the coercive condition $\lim_{|x|\to\infty} W(x) = +\infty$. By using a constrained minimization argument in sign-changing Nehari manifold, they derived a ground state sign-changing solution of Eq. (1.5) (see the similar result in [32]). Moreover, Zhong and Tang [33] studied the following Choquard equation

$$-\Delta u + (\lambda V(x) + 1)\,u = \big(I_\alpha * (K|u|^p)\big)K(x)|u|^{p-2}u + |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N,$$

where $N \geq 3$, $2^* = \frac{2N}{N-2}$, $\alpha \in ((N-4)^+, N)$, $p \in (2, 2_\alpha^*)$, $\lambda < 0$ and the functions V, K satisfy

(V_3) $V \in L^{\frac{N}{2}}(\mathbb{R}^N)\backslash\{0\}$ is nonnegative,

(V_4) there exist constants $\rho, \beta, C > 0$ such that $V(x) \geq C|x|^{-\beta}$ for all $|x| < \rho$,

(K_1) $K \in L^r(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)\backslash\{0\}$ for some $r \in \big[\frac{2N}{N+\alpha-p(N-2)}, +\infty\big)$ and K is nonnegative.

It follows from (V_3) that the first eigenvalue λ_1 of $-\Delta u + u = \lambda V(x)u$ in $H^1(\mathbb{R}^N)$ is positive. When $\lambda \in (-\lambda_1, 0)$ and $\beta \in \big(2 - \min\{\frac{N+\alpha}{2p} - \frac{N-2}{2}, \frac{N-2}{2}\}, 2\big)$, following the ideas in [5], they derived a ground state sign-changing solution by using minimization arguments in sign-changing Nehari manifold.

Motivated by the above works, in the present paper, we study the Choquard equation

$$-\Delta u + (\lambda V(x) + 1)\, u = \big(I_\alpha * |u|^{2_\alpha^*}\big)|u|^{2_\alpha^*-2}u + \mu|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \tag{1.6}$$

where $\lambda > 0$, $\mu > 0$, $N \geq 3$, $\alpha \in ((N-4)^+, N)$, $p \in [2_\alpha^*, 2^*)$, and $V \in C(\mathbb{R}^N, \mathbb{R})$ satisfies the hypotheses

(V_5) $V(x) \geq 0$ in $\mathbb{R}^N$ and there exists some $M > 0$ such that $\big|\{x \in \mathbb{R}^N : V(x) \leq M\}\big| < +\infty$,

(V_6) $\Omega := \operatorname{int} V^{-1}(0)$ is a nonempty set with smooth boundary and $\overline{\Omega} = V^{-1}(0)$.

Let $E_\lambda := \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} \lambda V(x)u^2 dx < +\infty\}$ be equipped with the inner product

$$(u,v)_\lambda := \int_{\mathbb{R}^N} \nabla u \cdot \nabla v + (\lambda V(x) + 1)\, uv dx, \qquad \forall\, u, v \in E_\lambda,$$

and the norm $\|\cdot\|_\lambda = (\cdot,\cdot)_\lambda^{\frac{1}{2}}$ for any $\lambda > 0$. Since $V \geq 0$ in $\mathbb{R}^N$, it is easy to see that $E_\lambda \hookrightarrow H^1(\mathbb{R}^N)$ and, for any $s \in [2, 2^*]$, there is some constant $\nu_s > 0$ such that, for all $\lambda > 0$,

$$|u|_s \leq \nu_s\|u\| \leq \nu_s\|u\|_\lambda, \qquad \forall\, u \in E_\lambda. \tag{1.7}$$

By (1.4) and (1.7), we deduce the energy functional $\mathcal{J}_{\lambda,\mu}$ of Eq. (1.6) belongs to $C^1(E_\lambda, \mathbb{R})$, where

$$\mathcal{J}_{\lambda,\mu}(u) = \frac{1}{2}\|u\|_\lambda^2 - \frac{1}{2 \cdot 2_\alpha^*}\int_{\mathbb{R}^N} \big(I_\alpha * |u|^{2_\alpha^*}\big)|u|^{2_\alpha^*}dx - \frac{\mu}{p}\int_{\mathbb{R}^N} |u|^p dx.$$

Now we are prepared to state our main results.

Theorem 1.2. *Assume that $N \geq 3$, $\alpha \in ((N-4)^+, N)$, $p \in [2_\alpha^*, 2^*)$ and (V_5), (V_6) hold. Then there exist $\Lambda > 0$ and $\mu_* > 0$ such that Eq. (1.6) admits a ground state sign-changing solution $u_{\lambda,\mu}$ for any $\lambda \geq \Lambda$ and $\mu \geq \mu_*$. Further, for any $\mu \geq \mu_*$ and sequence $\{\lambda_n\} \subset [\Lambda, +\infty)$ satisfying $\lambda_n \to +\infty$, the sequence $\{u_{\lambda_n,\mu}\}$ of ground state sign-changing solutions to Eq. (1.6) strongly converges to some u_μ in $H^1(\mathbb{R}^N)$ in the sense of subsequence, where u_μ is a ground state sign-changing solution of*

$$\begin{cases} -\Delta u + u = A_\alpha \displaystyle\int_\Omega \frac{|u(y)|^{2_\alpha^*}}{|x-y|^{N-\alpha}}dy|u|^{2_\alpha^*-2}u + \mu|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.8}$$

Moreover, for any $\lambda \geq \Lambda$ and sequence $\{\mu_n\} \subset [\mu_*, +\infty)$ with $\mu_n \to +\infty$, the sequence $\{u_{\lambda,\mu_n}\}$ of ground state sign-changing solutions to Eq. (1.6) strongly converges to 0 in $H^1(\mathbb{R}^N)$ up to a subsequence.

Remark 1.3. Similar to the proof of Theorem 1.1 in [14], by minimizing $\mathcal{J}_{\lambda,\mu}$ on the Nehari manifold

$$\mathcal{N}_{\lambda,\mu} = \left\{ u \in E_\lambda \setminus \{0\}, \left\langle \mathcal{J}'_{\lambda,\mu}(u), u \right\rangle = 0 \right\},$$

we can demonstrate that Eq. (1.6) has a positive ground state solution $v_{\lambda,\mu}$ for any $\lambda, \mu > 0$ large enough. It is easy to show $\mathcal{J}_{\lambda,\mu}(u_{\lambda,\mu}) > \mathcal{J}_{\lambda,\mu}(v_{\lambda,\mu})$. Indeed, if $\mathcal{J}_{\lambda,\mu}(u_{\lambda,\mu}) = \mathcal{J}_{\lambda,\mu}(v_{\lambda,\mu})$, then $|u_{\lambda,\mu}| \in \mathcal{N}_{\lambda,\mu}$ satisfies $\mathcal{J}_{\lambda,\mu}(|u_{\lambda,\mu}|) = \inf_{\mathcal{N}_{\lambda,\mu}} \mathcal{J}_{\lambda,\mu}$. Thereby, in a standard way, we may deduce $\mathcal{J}'_{\lambda,\mu}(|u_{\lambda,\mu}|) = 0$. Whereas, the strong maximum principle implies $|u_{\lambda,\mu}| > 0$ in $\mathbb{R}^N$, and the regular estimates for Choquard equations (see e.g. [21,22]) implies $u_{\lambda,\mu} \in C(\mathbb{R}^N, \mathbb{R})$, thus $u_{\lambda,\mu}$ has constant sign in $\mathbb{R}^N$, which contradicts with $u^{\pm}_{\lambda,\mu} \neq 0$. Furthermore, due to the presence of the perturbed term $\mu|u|^{p-2}u$, the methods introduced in [11,32] to verify that the least energy of sign-changing solutions is less than twice the least energy of nontrivial solutions seem invalid here, we propose an open question whether $\mathcal{J}_{\lambda,\mu}(u_{\lambda,\mu}) < 2\mathcal{J}_{\lambda,\mu}(v_{\lambda,\mu})$.

Remark 1.4. To our knowledge, there seem to be no results on (ground state) sign-changing solutions for Choquard equations with upper critical exponent, even on the bounded domain. Our present work extends and improves the existence results of sign-changing solutions verified in [7,10,11,28,33]. In [5], the authors studied the ground state sign-changing solutions for a class of critical Schrödinger equations

$$\begin{cases} -\Delta u - \lambda u = |u|^{2^*-2}u & \text{in } \mathcal{D}, \\ u = 0 & \text{on } \partial\mathcal{D}, \end{cases}$$

where $\mathcal{D} \subset \mathbb{R}^N$ $(N \geq 6)$ is a bounded domain and $\lambda \in (0, \lambda_1)$, with λ_1 denoting the first eigenvalue of $-\Delta$ on $\mathcal{D}$. They proved that any sign-changing $(PS)_c$ sequence is relatively compact once $c < c_0 + \frac{1}{N} S^{\frac{N}{2}}$, where c_0 is the least energy of nontrivial solutions. As a counterpart for the work in [5], Zhong and Tang studied a class of Choquard equations with critical Sobolev exponent in [33], where they showed the relative compactness of sign-changing $(PS)_c$ sequence with c less than the similar threshold. However, in this paper, due to the presence of the upper critical nonlocal term $(I_\alpha * |u|^{2^*_\alpha})|u|^{2^*_\alpha - 2}u$ in Eq. (1.6), the relative compactness of sign-changing $(PS)_c$ sequence with

$$c \in \left[\frac{2+\alpha}{2(N+\alpha)} S_\alpha^{\frac{N+\alpha}{2+\alpha}}, \inf_{\mathcal{N}_{\lambda,\mu}} \mathcal{J}_{\lambda,\mu} + \frac{2+\alpha}{2(N+\alpha)} S_\alpha^{\frac{N+\alpha}{2+\alpha}} \right)$$

cannot be deduced as expected, where S_α is defined by (2.12) hereinafter. Also, it seems intractable to search for sign-changing $(PS)_c$ sequence such that $c < \frac{2+\alpha}{2(N+\alpha)} S_\alpha^{(N+\alpha)/(2+\alpha)}$ for small $\mu > 0$. Naturally, we attempt to construct a sign-changing $(PS)_c$ sequence with $c < \frac{2+\alpha}{2(N+\alpha)} S_\alpha^{(N+\alpha)/(2+\alpha)}$ by assuming that $\mu > 0$ is sufficiently large. Therefrom, by applying the properties of steep well potential λV, we can standardly prove the relative compactness of this type of sign-changing $(PS)_c$ sequence and then obtain ground state sign-changing solution.

We will give the proof of Theorem 1.2 in the forthcoming section. Throughout this paper, we use the following notations:

- ♠ $L^p(\mathbb{R}^N)$ is the usual Lebesgue space with the norm $|u|_p = \left(\int_{\mathbb{R}^N} |u|^p dx \right)^{\frac{1}{p}}$ for $p \in [1, +\infty)$.
- ♠ $L^\infty(\mathbb{R}^N)$ is the space of measurable functions with the norm $|u|_\infty = \text{ess sup}_{x \in \mathbb{R}^N} |u(x)|$.

♠ $C_0^\infty(\mathbb{R}^N)$ consists of infinitely times differentiable functions with compact support in $\mathbb{R}^N$.

♠ $H^1(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : |\nabla u| \in L^2(\mathbb{R}^N)\}$ endowed with the inner product and norm

$$(u,v) = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v + uvdx \quad \text{and} \quad \|u\| = (u,u)^{\frac{1}{2}}.$$

♠ $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$ with the norm $\|u\|_\Omega = \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}}$.

♠ $D^{1,2}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|u\|_D = |\nabla u|_2$.

♠ The best Sobolev constant $S = \inf\left\{\|u\|_D^2 : u \in D^{1,2}(\mathbb{R}^N) \text{ and } |u|_{2^*} = 1\right\}$.

♠ $u^\pm(x) := \pm\max\{\pm u(x), 0\}$ and $(E^*, \|\cdot\|_*)$ is the dual space of Banach space $(E, \|\cdot\|)$.

♠ $o(1)$ is a quantity tending to 0 as $n \to \infty$ and $|\Omega|$ is the Lebesgue measure of $\Omega \subset \mathbb{R}^N$.

♠ $\mathbb{B}_r(y) = \{x \in \mathbb{R}^N : |x-y| < r\}$, $\mathbb{B}_r^c(y) = \mathbb{R}^N \backslash \mathbb{B}_r(y)$ and $\mathbb{B}_r(0) = \mathbb{B}_r$ for $r > 0$, $y \in \mathbb{R}^N$.

2 Proof of Theorem 1.2

For the limiting problem of Eq. (1.6) as $\lambda \to +\infty$, namely Eq. (1.8), its energy functional is

$$\mathcal{J}_{\infty,\mu}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + u^2 dx - \frac{A_\alpha}{2\cdot 2_\alpha^*}\int_\Omega\int_\Omega \frac{|u(x)|^{2_\alpha^*}|u(y)|^{2_\alpha^*}}{|x-y|^{N-\alpha}}dxdy - \frac{\mu}{p}\int_\Omega |u|^p dx.$$

Due to (1.4) and $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$, $\mathcal{J}_{\infty,\mu} \in C^1(H_0^1(\Omega), \mathbb{R})$. Define the sign-changing Nehari manifolds

$$\mathcal{M}_{\lambda,\mu} = \left\{u \in E_\lambda : u^\pm \neq 0, \left\langle \mathcal{J}'_{\lambda,\mu}(u), u^\pm \right\rangle = 0\right\},$$
$$\mathcal{M}_{\infty,\mu} = \left\{u \in H_0^1(\Omega) : u^\pm \neq 0, \left\langle \mathcal{J}'_{\infty,\mu}(u), u^\pm \right\rangle = 0\right\}.$$

Clearly, $\mathcal{M}_{\lambda,\mu}$ and $\mathcal{M}_{\infty,\mu}$ contain all of the sign-changing solutions of Eqs. (1.6) and (1.8), respectively. To search for ground state sign-changing solutions, we consider the following minimization problems:

$$m_{\lambda,\mu} = \inf\left\{\mathcal{J}_{\lambda,\mu}(u) : u \in \mathcal{M}_{\lambda,\mu}\right\},$$
$$m_{\infty,\mu} = \inf\left\{\mathcal{J}_{\infty,\mu}(u) : u \in \mathcal{M}_{\infty,\mu}\right\}.$$

Before completing the proof of Theorem 1.2, we establish several preliminary lemmas.

Lemma 2.1. *For any $\lambda > 0$, $\mu > 0$ and $u \in E_\lambda$ with $u^\pm \neq 0$, there exists a unique pair $(s_{\lambda,\mu,u}, t_{\lambda,\mu,u})$ of positive numbers such that $s_{\lambda,\mu,u}^{\frac{1}{2_\alpha^*}}u^+ + t_{\lambda,\mu,u}^{\frac{1}{2_\alpha^*}}u^- \in \mathcal{M}_{\lambda,\mu}$, also,*

$$\mathcal{J}_{\lambda,\mu}(s_{\lambda,\mu,u}^{\frac{1}{2_\alpha^*}}u^+ + t_{\lambda,\mu,u}^{\frac{1}{2_\alpha^*}}u^-) = \max_{s,t\geq 0}\mathcal{J}_{\lambda,\mu}(s^{\frac{1}{2_\alpha^*}}u^+ + t^{\frac{1}{2_\alpha^*}}u^-).$$

Proof. Firstly, we certify the existence of such pair of numbers. For any $\lambda > 0$, $\mu > 0$ and

$u \in E_\lambda$ with $u^\pm \neq 0$, define the function $\mathcal{F}_{\lambda,\mu,u}(s,t)$ for any $(s,t) \in [0,+\infty)^2$ by

$$\begin{aligned}\mathcal{F}_{\lambda,\mu,u}(s,t) &= \mathcal{J}_{\lambda,\mu}(s^{\frac{1}{2_\alpha^*}}u^+ + t^{\frac{1}{2_\alpha^*}}u^-)\\ &= \frac{s^{\frac{2}{2_\alpha^*}}}{2}\|u^+\|_\lambda^2 - \frac{s^2}{2\cdot 2_\alpha^*}\int_{\mathbb{R}^N}\left(I_\alpha * |u^+|^{2_\alpha^*}\right)|u^+|^{2_\alpha^*}dx - \frac{\mu s^{\frac{p}{2_\alpha^*}}}{p}\int_{\mathbb{R}^N}|u^+|^p dx\\ &\quad + \frac{t^{\frac{2}{2_\alpha^*}}}{2}\|u^-\|_\lambda^2 - \frac{t^2}{2\cdot 2_\alpha^*}\int_{\mathbb{R}^N}\left(I_\alpha * |u^-|^{2_\alpha^*}\right)|u^-|^{2_\alpha^*}dx - \frac{\mu t^{\frac{p}{2_\alpha^*}}}{p}\int_{\mathbb{R}^N}|u^-|^p dx\\ &\quad - \frac{st}{2_\alpha^*}\int_{\mathbb{R}^N}\left(I_\alpha * |u^+|^{2_\alpha^*}\right)|u^-|^{2_\alpha^*}dx.\end{aligned}$$

It is easy to derive $\lim_{|(s,t)|\to 0}\mathcal{F}_{\lambda,\mu,u}(s,t) = 0$ and $\lim_{|(s,t)|\to+\infty}\mathcal{F}_{\lambda,\mu,u}(s,t) = -\infty$. Then there exists some point $(s_{\lambda,\mu,u}, t_{\lambda,\mu,u}) \in [0,+\infty)^2$ such that

$$\mathcal{F}_{\lambda,\mu,u}(s_{\lambda,\mu,u}, t_{\lambda,\mu,u}) = \max_{(s,t)\in[0,+\infty)^2}\mathcal{F}_{\lambda,\mu,u}(s,t).$$

Since $\mathcal{F}_{\lambda,\mu,u}(s, t_{\lambda,\mu,u})$ is increasing in s for $s > 0$ small enough, there results $s_{\lambda,\mu,u} \neq 0$. Similarly, we deduce $t_{\lambda,\mu,u} \neq 0$. Thereby, $(s_{\lambda,\mu,u}, t_{\lambda,\mu,u}) \in (0,+\infty)^2$. Then

$$\frac{\partial\mathcal{F}_{\lambda,\mu,u}}{\partial s}(s_{\lambda,\mu,u}, t_{\lambda,\mu,u}) = \frac{\partial\mathcal{F}_{\lambda,\mu,u}}{\partial t}(s_{\lambda,\mu,u}, t_{\lambda,\mu,u}) = 0.$$

Naturally, $s_{\lambda,\mu,u}^{\frac{1}{2_\alpha^*}}u^+ + t_{\lambda,\mu,u}^{\frac{1}{2_\alpha^*}}u^- \in \mathcal{M}_{\lambda,\mu}$.

Further, we claim such pair of numbers is unique. For brevity, we introduce the notation

$$B(u,v) := \frac{1}{2_\alpha^*}\int_{\mathbb{R}^N}\left(I_\alpha * |u|^{2_\alpha^*}\right)|v|^{2_\alpha^*}dx, \qquad \forall\, u,v \in E_\lambda.$$

Through direct calculation, we deduce that the Hessian matrix of $\mathcal{F}_{\lambda,\mu,u}$ at $(s,t) \in (0,+\infty)^2$ is

$$\begin{aligned}H_{\lambda,\mu,u}(s,t) &= \frac{2-2_\alpha^*}{(2_\alpha^*)^2}\begin{pmatrix} s^{\frac{2}{2_\alpha^*}-2}\|u^+\|_\lambda^2 & 0\\ 0 & t^{\frac{2}{2_\alpha^*}-2}\|u^-\|_\lambda^2\end{pmatrix}\\ &\quad - \begin{pmatrix} B(u^+,u^+) & B(u^+,u^-)\\ B(u^+,u^-) & B(u^-,u^-)\end{pmatrix} - \frac{\mu(p-2_\alpha^*)}{(2_\alpha^*)^2}\begin{pmatrix} s^{\frac{p}{2_\alpha^*}-2}|u^+|_p^p & 0\\ 0 & t^{\frac{p}{2_\alpha^*}-2}|u^-|_p^p\end{pmatrix}.\end{aligned}$$

It follows from [17, Theorem 9.8] that $B(u^+,u^-)^2 < B(u^+,u^+)B(u^-,u^-)$. Then, noting $p \geq 2_\alpha^*$, we conclude that $H_{\lambda,\mu,u}(s,t)$ is negative defined for any $(s,t) \in (0,+\infty)^2$. Thereby, it is easy to know that $\mathcal{F}_{\lambda,\mu,u}$ has at most one critical point on $(0,+\infty)^2$. Thus, $(s_{\lambda,\mu,u}, t_{\lambda,\mu,u})$ is the unique pair of positive numbers such that $s_{\lambda,\mu,u}^{\frac{1}{2_\alpha^*}}u^+ + t_{\lambda,\mu,u}^{\frac{1}{2_\alpha^*}}u^- \in \mathcal{M}_{\lambda,\mu}$, and this lemma is proved. □

As a by-product, we may derive $\mathcal{M}_{\infty,\mu} \neq \emptyset$. Indeed, since $\mathcal{J}_{\lambda,\mu} = \mathcal{J}_{\infty,\mu}$ in $H_0^1(\Omega)$, we have

Remark 2.2. For any $\mu > 0$ and $u \in H_0^1(\Omega)$ with $u^\pm \neq 0$, there exists a unique pair $(s_{\mu,u}, t_{\mu,u})$ of positive numbers such that $s_{\mu,u}^{\frac{1}{2_\alpha^*}}u^+ + t_{\mu,u}^{\frac{1}{2_\alpha^*}}u^- \in \mathcal{M}_{\infty,\mu}$ and

$$\mathcal{J}_{\infty,\mu}(s_{\mu,u}^{\frac{1}{2_\alpha^*}}u^+ + t_{\mu,u}^{\frac{1}{2_\alpha^*}}u^-) = \max_{s,t\geq 0}\mathcal{J}_{\infty,\mu}(s^{\frac{1}{2_\alpha^*}}u^+ + t^{\frac{1}{2_\alpha^*}}u^-).$$

To facilitate the subsequent discussion, we show some properties of $\mathcal{M}_{\lambda,\mu}$ in the following

Lemma 2.3. *For any $\lambda > 0$ and $\mu > 0$, if $\{u_n\} \subset \mathcal{M}_{\lambda,\mu}$ and $\lim_{n\to\infty} \mathcal{J}_{\lambda,\mu}(u_n) = m_{\lambda,\mu}$, then $m_{\lambda,\mu} > 0$ and there exist some constants $C_{\lambda,\mu,1}, C_{\lambda,\mu,2} > 0$ such that $C_{\lambda,\mu,2} \le \|u_n^\pm\|_\lambda, \|u_n\|_\lambda \le C_{\lambda,\mu,1}$ for all n.*

Proof. From $\mathcal{M}_{\lambda,\mu} \neq \emptyset$, we know $m_{\lambda,\mu} < +\infty$ for any $\lambda, \mu > 0$. Since $\{u_n\} \subset \mathcal{M}_{\lambda,\mu}$, there holds

$$m_{\lambda,\mu} + o(1) = \mathcal{J}_{\lambda,\mu}(u_n) - \frac{1}{p}\left\langle \mathcal{J}'_{\lambda,\mu}(u_n), u_n \right\rangle \ge \frac{p-2}{2p}\|u_n\|_\lambda^2. \tag{2.1}$$

Then there is constant $C_{\lambda,\mu,1} > 0$ such that $\sup_n \|u_n\|_\lambda \le C_{\lambda,\mu,1}$. Thereby, (1.4) and (1.7) imply

$$\begin{aligned}\|u_n^\pm\|_\lambda^2 &= \int_{\mathbb{R}^N} \left(I_\alpha * |u_n|^{2_\alpha^*}\right)|u_n^\pm|^{2_\alpha^*}dx + \mu\int_{\mathbb{R}^N} |u_n^\pm|^p dx \\ &\le A_\alpha C(N,\alpha)\nu_{2^*}^{2\cdot 2_\alpha^*}\|u_n\|_\lambda^{2_\alpha^*}\|u_n^\pm\|_\lambda^{2_\alpha^*} + \mu\nu_p^p\|u^\pm\|_\lambda^p \\ &\le A_\alpha C(N,\alpha)\nu_{2^*}^{2\cdot 2_\alpha^*}C_{\lambda,\mu,1}^{2_\alpha^*}\|u_n^\pm\|_\lambda^{2_\alpha^*} + \mu\nu_p^p\|u^\pm\|_\lambda^p.\end{aligned}$$

As a consequence, there exists some constant $C_{\lambda,\mu,2} > 0$ such that $\inf_n \|u_n^\pm\|_\lambda \ge C_{\lambda,\mu,2}$. Further, we deduce from (2.1) that $m_{\lambda,\mu} > 0$. Thus we complete the proof of this lemma. □

Next, following [5], we construct a sign-changing $(PS)_c$ sequence $\{u_n\}$ for $\mathcal{J}_{\lambda,\mu}$, (i.e. $u_n^\pm \neq 0$ for any n, $\mathcal{J}_{\lambda,\mu}(u_n) \to c$ and $\mathcal{J}'_{\lambda,\mu}(u_n) \to 0$ in E_λ^* as $n \to \infty$). Let P_λ be the cone of nonnegative functions in E_λ, $Q = [0,1]^2$ and $\Gamma_{\lambda,\mu}$ be the set of continuous maps $\gamma : Q \to E_\lambda$ such that, for any $(s,t) \in Q$,

(a) $\gamma(s,0) = 0$, $\gamma(0,t) \in P_\lambda$ and $\gamma(1,t) \in -P_\lambda$,

(b) $(\mathcal{J}_{\lambda,\mu} \circ \gamma)(s,1) \le 0$ and

$$\frac{\int_{\mathbb{R}^N}\left[\left(I_\alpha * |\gamma(s,1)|^{2_\alpha^*}\right)|\gamma(s,1)|^{2_\alpha^*} + \mu|\gamma(s,1)|^p\right]dx}{\|\gamma(s,1)\|_\lambda^2} \ge 2.$$

For any $u \in E_\lambda$ with $u^\pm \neq 0$, define $\gamma_{\sigma,u}(s,t) = \sigma t(1-s)u^+ + \sigma tsu^-$ for $\sigma > 0$ and $(s,t) \in Q$. It is easy to show $\gamma_{\sigma,u} \in \Gamma_{\lambda,\mu}$ for $\sigma > 0$ large enough. Therefore, $\Gamma_{\lambda,\mu} \neq \emptyset$. Define the functional

$$\mathcal{L}_{\lambda,\mu}(u,v) = \begin{cases} \dfrac{\int_{\mathbb{R}^N}\left[(I_\alpha * |u|^{2_\alpha^*})(|u|^{2_\alpha^*} + |v|^{2_\alpha^*}) + \mu|u|^p\right]dx}{\|u\|_\lambda^2}, & u \neq 0, \\ 0, & u = 0. \end{cases}$$

Clearly, $\mathcal{L}_{\lambda,\mu} > 0$ if $u \neq 0$. Moreover, $u \in \mathcal{M}_{\lambda,\mu}$ if and only if $\mathcal{L}_{\lambda,\mu}(u^+,u^-) = \mathcal{L}_{\lambda,\mu}(u^-,u^+) = 1$.

As a start point, we display a minimax characterization on $m_{\lambda,\mu}$ for any $\lambda > 0$ and $\mu > 0$.

Lemma 2.4. *For any $\lambda > 0$ and $\mu > 0$, there holds*

$$m_{\lambda,\mu} = \inf_{\gamma\in\Gamma_{\lambda,\mu}}\max_{(s,t)\in Q}\mathcal{J}_{\lambda,\mu}(\gamma(s,t)). \tag{2.2}$$

Proof. On the one hand, for every $u \in \mathcal{M}_{\lambda,\mu}$, $\gamma_u(s,t) = \sigma t(1-s)u^+ + \sigma tsu^- \in \Gamma_{\lambda,\mu}$ for some $\sigma > 0$ large enough. Then it follows from Lemma 2.1 that

$$\mathcal{J}_{\lambda,\mu}(u) = \max_{s,t\ge 0}\mathcal{J}_{\lambda,\mu}(su^+ + tu^-) \ge \max_{(s,t)\in Q}\mathcal{J}_{\lambda,\mu}(\gamma_u(s,t)) \ge \inf_{\gamma\in\Gamma_{\lambda,\mu}}\max_{(s,t)\in Q}\mathcal{J}_{\lambda,\mu}(\gamma(s,t)).$$

Thereby, due to the arbitrariness of $u \in \mathcal{M}_{\lambda,\mu}$, there results

$$m_{\lambda,\mu} \geq \inf_{\gamma\in\Gamma_{\lambda,\mu}} \max_{(s,t)\in Q} \mathcal{J}_{\lambda,\mu}(\gamma(s,t)).$$

On the other hand, for each $\gamma \in \Gamma_{\lambda,\mu}$ and $t \in [0,1]$, since $\gamma(0,t) \in P_\lambda$ and $\gamma(1,t) \in -P_\lambda$, we conclude

$$\mathcal{L}_{\lambda,\mu}(\gamma(0,t)^+,\gamma(0,t)^-) - \mathcal{L}_{\lambda,\mu}(\gamma(0,t)^-,\gamma(0,t)^+) = \mathcal{L}_{\lambda,\mu}(\gamma(0,t)^+,\gamma(0,t)^-) \geq 0, \tag{2.3}$$
$$\mathcal{L}_{\lambda,\mu}(\gamma(1,t)^+,\gamma(1,t)^-) - \mathcal{L}_{\lambda,\mu}(\gamma(1,t)^-,\gamma(1,t)^+) = -\mathcal{L}_{\lambda,\mu}(\gamma(1,t)^-,\gamma(1,t)^+) \leq 0. \tag{2.4}$$

Meanwhile, due to $\gamma(s,0) = 0$ for all $s \in [0,1]$, there holds

$$\mathcal{L}_{\lambda,\mu}(\gamma(s,0)^+,\gamma(s,0)^-) + \mathcal{L}_{\lambda,\mu}(\gamma(s,0)^-,\gamma(s,0)^+) - 2 = -2, \qquad \forall\, s \in [0,1]. \tag{2.5}$$

And, for each $\gamma \in \Gamma_{\lambda,\mu}$, by the definition of $\mathcal{L}_{\lambda,\mu}$ and the property (b) we have, for all $s \in [0,1]$,

$$\begin{aligned}&\mathcal{L}_{\lambda,\mu}(\gamma(s,1)^+,\gamma(s,1)^-) + \mathcal{L}_{\lambda,\mu}(\gamma(s,1)^-,\gamma(s,1)^+) - 2\\ &\geq \frac{\int_{\mathbb{R}^N}\left[(I_\alpha * |\gamma(s,1)|^{2_\alpha^*})|\gamma(s,1)|^{2_\alpha^*} + \mu|\gamma(s,1)|^p\right]dx}{\|\gamma(s,1)\|_\lambda^2} - 2 \geq 0.\end{aligned} \tag{2.6}$$

Moreover, it is easy to verify that, for any $(s,t) \in \partial Q$,

$$\begin{pmatrix}\mathcal{L}_{\lambda,\mu}(\gamma(s,t)^+,\gamma(s,t)^-) - \mathcal{L}_{\lambda,\mu}(\gamma(s,t)^-,\gamma(s,t)^+)\\ \mathcal{L}_{\lambda,\mu}(\gamma(s,t)^+,\gamma(s,t)^-) + \mathcal{L}_{\lambda,\mu}(\gamma(s,t)^-,\gamma(s,t)^+) - 2\end{pmatrix} \neq \begin{pmatrix}0\\0\end{pmatrix}. \tag{2.7}$$

Then, by combining (2.3)–(2.7) with the Miranda theorem (see e.g. Lemma 2.4 in [13]), we derive that there exists some $(s_\gamma,t_\gamma) \in (0,1)^2$ satisfying

$$\mathcal{L}_{\lambda,\mu}(\gamma(s_\gamma,t_\gamma)^+,\gamma(s_\gamma,t_\gamma)^-) - \mathcal{L}_{\lambda,\mu}(\gamma(s_\gamma,t_\gamma)^-,\gamma(s_\gamma,t_\gamma)^+) = 0,$$
$$\mathcal{L}_{\lambda,\mu}(\gamma(s_\gamma,t_\gamma)^+,\gamma(s_\gamma,t_\gamma)^-) + \mathcal{L}_{\lambda,\mu}(\gamma(s_\gamma,t_\gamma)^-,\gamma(s_\gamma,t_\gamma)^+) = 2.$$

In view of this fact, we easily obtain

$$\mathcal{L}_{\lambda,\mu}(\gamma(s_\gamma,t_\gamma)^+,\gamma(s_\gamma,t_\gamma)^-) = \mathcal{L}_{\lambda,\mu}(\gamma(s_\gamma,t_\gamma)^-,\gamma(s_\gamma,t_\gamma)^+) = 1,$$

which implies $\gamma(s_\gamma,t_\gamma) \in \mathcal{M}_{\lambda,\mu}$. Consequently, from the arbitrariness of $\gamma \in \Gamma_{\lambda,\mu}$, we deduce

$$\inf_{\gamma\in\Gamma_{\lambda,\mu}} \max_{(s,t)\in Q} \mathcal{J}_{\lambda,\mu}(\gamma(s,t)) \geq m_{\lambda,\mu}.$$

Now, by combining the above two sides, we know (2.2) holds. Thus this lemma is showed. □

Lemma 2.5. *For any $\lambda > 0$ and $\mu > 0$, $\mathcal{J}_{\lambda,\mu}$ possesses a sign-changing $(PS)_{m_{\lambda,\mu}}$ sequence $\{u_n\} \subset E_\lambda$.*

Proof. We will end the proof in two steps. Firstly, we construct a $(PS)_{m_{\lambda,\mu}}$ sequence for $\mathcal{J}_{\lambda,\mu}$. Take a minimizing sequence $\{w_n\} \subset \mathcal{M}_{\lambda,\mu}$ for $m_{\lambda,\mu}$ and set $\gamma_{\sigma,n}(s,t) = \sigma t(1-s)w_n^+ + \sigma tsw_n^-$. By Lemma 2.3, it is easy to choose a sufficiently large constant $\bar\sigma > 0$ such that $\{\gamma_{\bar\sigma,n}\} \subset \Gamma_{\lambda,\mu}$. Due to Lemmas 2.1 and 2.4, there holds

$$\lim_{n\to\infty} \max_{(s,t)\in Q} \mathcal{J}_{\lambda,\mu}(\gamma_{\bar\sigma,n}(s,t)) = \lim_{n\to\infty} \mathcal{J}_{\lambda,\mu}(w_n) = m_{\lambda,\mu}. \tag{2.8}$$

We assert that there exists some sequence $\{u_n\} \subset E_\lambda$ such that, as $n \to \infty$,

$$\mathcal{J}_{\lambda,\mu}(u_n) \to m_{\lambda,\mu}, \qquad \mathcal{J}'_{\lambda,\mu}(u_n) \to 0, \qquad \min_{(s,t)\in Q} \|u_n - \gamma_{\bar{\sigma},n}(s,t)\|_\lambda \to 0. \tag{2.9}$$

If not, there exists some constant $\delta_{\lambda,\mu} > 0$ such that, for n suitably large, $\gamma_{\bar{\sigma},n}(Q) \cap U_{\delta_{\lambda,\mu}} = \emptyset$, in which

$$U_{\delta_{\lambda,\mu}} \triangleq \{u \in E_\lambda : \exists\, v \in E_\lambda \text{ s.t. } \|v - u\|_\lambda \le \delta_{\lambda,\mu}, \|\nabla \mathcal{J}_{\lambda,\mu}(v)\| \le \delta_{\lambda,\mu}, |\mathcal{J}_{\lambda,\mu}(v) - m_{\lambda,\mu}| \le \delta_{\lambda,\mu}\}.$$

Then, by a variant of the classical deformation lemma due to Hofer (see [12, Lemma 1]), there exists a continuous map $\eta_{\lambda,\mu} : [0,1] \times E_\lambda \to E_\lambda$, which satisfies that, for some $\varepsilon_{\lambda,\mu} \in (0, \frac{m_{\lambda,\mu}}{2})$,

(i) $\eta_{\lambda,\mu}(0,u) = u$, $\eta_{\lambda,\mu}(\tau,-u) = -\eta_{\lambda,\mu}(\tau,u)$, $\forall\, \tau \in [0,1]$, $u \in E_\lambda$,

(ii) $\eta_{\lambda,\mu}(\tau,u) = u$, $\forall\, u \in \mathcal{J}_{\lambda,\mu}^{m_{\lambda,\mu}-\varepsilon_{\lambda,\mu}} \cup \left(E_\lambda \backslash \mathcal{J}_{\lambda,\mu}^{m_{\lambda,\mu}+\varepsilon_{\lambda,\mu}}\right)$, $\forall\, \tau \in [0,1]$,

(iii) $\eta_{\lambda,\mu}\left(1, \mathcal{J}_{\lambda,\mu}^{m_{\lambda,\mu}+\frac{\varepsilon_{\lambda,\mu}}{2}} \backslash U_{\delta_{\lambda,\mu}}\right) \subset \mathcal{J}_{\lambda,\mu}^{m_{\lambda,\mu}-\frac{\varepsilon_{\lambda,\mu}}{2}}$,

(iv) $\eta_{\lambda,\mu}\left(1, (\mathcal{J}_{\lambda,\mu}^{m_{\lambda,\mu}+\frac{\varepsilon_{\lambda,\mu}}{2}} \cap P_\lambda) \backslash U_{\delta_{\lambda,\mu}}\right) \subset \mathcal{J}_{\lambda,\mu}^{m_{\lambda,\mu}-\frac{\varepsilon_{\lambda,\mu}}{2}} \cap P_\lambda$,

where the sublevel set $\mathcal{J}_{\lambda,\mu}^d := \{u \in E_\lambda : \mathcal{J}_{\lambda,\mu}(u) \le d\}$ for $d \in \mathbb{R}$. By (2.8), we choose large n such that

$$\gamma_{\bar{\sigma},n}(Q) \subset \mathcal{J}_{\lambda,\mu}^{m_{\lambda,\mu}+\frac{\varepsilon_{\lambda,\mu}}{2}} \quad \text{and} \quad \gamma_{\bar{\sigma},n}(Q) \cap U_{\delta_{\lambda,\mu}} = \emptyset. \tag{2.10}$$

Set the continuous map $\widetilde{\gamma}_{\lambda,\mu,n}(s,t) = \eta_{\lambda,\mu}(1, \gamma_{\bar{\sigma},n}(s,t))$ for any $(s,t) \in Q$. We claim $\widetilde{\gamma}_{\lambda,\mu,n} \in \Gamma_{\lambda,\mu}$.

Indeed, from $\gamma_{\bar{\sigma},n}(s,0) = 0$ and (ii), it follows that $\widetilde{\gamma}_{\lambda,\mu,n}(s,0) = \eta_{\lambda,\mu}(1,0) = 0$ for any $s \in [0,1]$. Since $\gamma_{\bar{\sigma},n}(0,t)$, $-\gamma_{\bar{\sigma},n}(1,t) \in P_\lambda$ and (2.10) implies $\gamma_{\bar{\sigma},n}(0,t)$, $-\gamma_{\bar{\sigma},n}(1,t) \in \mathcal{J}_{\lambda,\mu}^{m_{\lambda,\mu}+\frac{\varepsilon_{\lambda,\mu}}{2}} \backslash U_{\delta_{\lambda,\mu}}$, we deduce from (i), (iv) that $\widetilde{\gamma}_{\lambda,\mu,n}(0,t) \in P_\lambda$ and $\widetilde{\gamma}_{\lambda,\mu,n}(1,t) \in -P_\lambda$ for all $t \in [0,1]$. Also, $\mathcal{J}_{\lambda,\mu}(\gamma_{\bar{\sigma},n}(s,1)) \le 0$ and (ii) imply $\widetilde{\gamma}_{\lambda,\mu,n}(s,1) = \eta_{\lambda,\mu}(1, \gamma_{\bar{\sigma},n}(s,1)) = \gamma_{\bar{\sigma},n}(s,1)$ for any $s \in [0,1]$. Then, by $\gamma_{\bar{\sigma},n} \in \Gamma_{\lambda,\mu}$, we know $\widetilde{\gamma}_{\lambda,\mu,n}$ satisfies the property (b). From the above arguments, we derive our claim $\widetilde{\gamma}_{\lambda,\mu,n} \in \Gamma_{\lambda,\mu}$.

Thereby, since (2.10) and (iii) imply $\widetilde{\gamma}_{\lambda,\mu,n}(Q) \subset \mathcal{J}_{\lambda,\mu}^{m_{\lambda,\mu}-\frac{\varepsilon_{\lambda,\mu}}{2}}$, we conclude

$$m_{\lambda,\mu} \le \max_{(s,t)\in Q} \mathcal{J}_{\lambda,\mu}(\widetilde{\gamma}_{\lambda,\mu,n}(s,t)) \le m_{\lambda,\mu} - \frac{\varepsilon_{\lambda,\mu}}{2},$$

which is a contradiction. Thus there is a sequence $\{u_n\} \subset E_\lambda$ possessing the properties in (2.9).

Secondly, we prove $u_n^\pm \ne 0$ for all large n. By (2.9), there exists a sequence $\{v_n\}$ such that

$$v_n = \alpha_n w_n^+ + \beta_n w_n^- \in \gamma_{\bar{\sigma},n}(Q) \quad \text{and} \quad \|v_n - u_n\|_\lambda \xrightarrow{n} 0. \tag{2.11}$$

Due to $\{w_n\} \subset \mathcal{M}_{\lambda,\mu}$ and $p \in (2, 2^*)$, from (1.4), Lemma 2.3 and the Young inequality we have

$$\|w_n^\pm\|_\lambda^2 \le A_\alpha C(N,\alpha)(\nu_{2^*} C_{\lambda,\mu,1})^{2_\alpha^*} |w_n^\pm|_{2^*}^{2_\alpha^*} + \frac{2^* - p}{2^* - 2} |w_n^\pm|_2^2 + \frac{\mu^{\frac{2^*-2}{p-2}}(p-2)}{2^* - 2} |w_n^\pm|_{2^*}^{2^*}.$$

Then, by (1.7), there holds

$$\frac{p-2}{(2^*-2)\nu_{2^*}^2}|w_n^\pm|_{2^*}^2 \le A_\alpha C(N,\alpha)(\nu_{2^*}C_{\lambda,\mu,1})^{2_\alpha^*}|w_n^\pm|_{2^*}^{2_\alpha^*} + \frac{\mu^{\frac{2^*-2}{p-2}}(p-2)}{2^*-2}|w_n^\pm|_{2^*}^{2^*},$$

which implies $\inf_n |w_n^\pm|_{2^*} > 0$. In view of this fact, the second limiting formula in (2.11) and (1.7), to show $u_n^\pm \neq 0$ for n large enough, it suffices to verify that $\alpha_n \nrightarrow 0$ and $\beta_n \nrightarrow 0$ up to subsequences. Suppose inversely $\alpha_n \to 0$ up to a subsequence. Then it follows from $\mathcal{J}_{\lambda,\mu} \in C(E_\lambda, \mathbb{R})$ and Lemma 2.3 that

$$m_{\lambda,\mu} = \lim_{n\to\infty}\mathcal{J}_{\lambda,\mu}(v_n) = \lim_{n\to\infty}\mathcal{J}_{\lambda,\mu}(\alpha_n w_n^+ + \beta_n w_n^-) = \lim_{n\to\infty}\mathcal{J}_{\lambda,\mu}(\beta_n w_n^-),$$

which together with $m_{\lambda,\mu} > 0$ implies $\bar{\beta} := \sup_n \beta_n < +\infty$. Further, by Lemma 2.1, the Fubini theorem, Lemma 2.3, (1.4) and (1.7), we deduce

$$\begin{aligned}
m_{\lambda,\mu} &= \lim_{n\to\infty}\mathcal{J}_{\lambda,\mu}(w_n)\\
&= \lim_{n\to\infty}\max_{s,t\ge 0}\mathcal{J}_{\lambda,\mu}(sw_n^+ + tw_n^-)\\
&\ge \lim_{n\to\infty}\max_{s\ge 0}\mathcal{J}_{\lambda,\mu}(sw_n^+ + \beta_n w_n^-)\\
&= \lim_{n\to\infty}\max_{s\ge 0}\Bigg[\frac{s^2}{2}\|w_n^+\|_\lambda^2 - \frac{s^{2\cdot 2_\alpha^*}}{2\cdot 2_\alpha^*}\int_{\mathbb{R}^N}(I_\alpha * |w_n^+|^{2_\alpha^*})|w_n^+|^{2_\alpha^*}dx - \frac{\mu s^p}{p}\int_{\mathbb{R}^N}|w_n^+|^p dx\\
&\quad + \frac{\beta_n^2}{2}\|w_n^-\|_\lambda^2 - \frac{\beta_n^{2\cdot 2_\alpha^*}}{2\cdot 2_\alpha^*}\int_{\mathbb{R}^N}(I_\alpha * |w_n^-|^{2_\alpha^*})|w_n^-|^{2_\alpha^*}dx - \frac{\mu\beta_n^p}{p}\int_{\mathbb{R}^N}|w_n^-|^p dx\\
&\quad - \frac{s^{2_\alpha^*}\beta_n^{2_\alpha^*}}{2_\alpha^*}\int_{\mathbb{R}^N}(I_\alpha * |w_n^+|^{2_\alpha^*})|w_n^-|^{2_\alpha^*}dx\Bigg]\\
&= \lim_{n\to\infty}\max_{s\ge 0}\Bigg[\frac{s^2}{2}\|w_n^+\|_\lambda^2 - \frac{s^{2\cdot 2_\alpha^*}}{2\cdot 2_\alpha^*}\int_{\mathbb{R}^N}(I_\alpha * |w_n^+|^{2_\alpha^*})|w_n^+|^{2_\alpha^*}dx\\
&\quad - \frac{s^{2_\alpha^*}\beta_n^{2_\alpha^*}}{2_\alpha^*}\int_{\mathbb{R}^N}(I_\alpha * |w_n^+|^{2_\alpha^*})|w_n^-|^{2_\alpha^*}dx - \frac{\mu s^p}{p}\int_{\mathbb{R}^N}|w_n^+|^p dx + \mathcal{J}_{\lambda,\mu}(\beta_n w_n^-)\Bigg]\\
&\ge \max_{s\ge 0}\Bigg[\frac{1}{2}C_{\lambda,\mu,2}^2 s^2 - \frac{1}{2_\alpha^*}A_\alpha C(N,\alpha)\left(\nu_{2^*}C_{\lambda,\mu,1}\right)^{2\cdot 2_\alpha^*}\bar{\beta}^{2_\alpha^*}s^{2_\alpha^*} - \frac{\mu}{p}\left(\nu_p C_{\lambda,\mu,1}\right)^p s^p\\
&\quad - \frac{1}{2\cdot 2_\alpha^*}A_\alpha C(N,\alpha)\left(\nu_{2^*}C_{\lambda,\mu,1}\right)^{2\cdot 2_\alpha}s^{2\cdot 2_\alpha}\Bigg] + \lim_{n\to\infty}\mathcal{J}_{\lambda,\mu}(\beta_n w_n^-)\\
&> m_{\lambda,\mu},
\end{aligned}$$

a contradiction. Naturally, $\{\alpha_n\}$ has no subsequence tending to 0. Similarly, we can show $\{\beta_n\}$ has no subsequence tending to 0. Thus $u_n^\pm \neq 0$ for n large enough. This lemma is proved. □

Now, we estimate the least energy $m_{\lambda,\mu}$ from above. By [9, Lemma 1.2], the best constant

$$S_\alpha := \inf\left\{\int_{\mathbb{R}^N}|\nabla u|^2 dx : u \in D^{1,2}(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N}(I_\alpha * |u|^{2_\alpha^*})|u|^{2_\alpha^*}dx = 1\right\} \tag{2.12}$$

is attained by the functions

$$U_\varepsilon(\cdot) = \frac{\left[N(N-2)\varepsilon^2\right]^{\frac{N-2}{4}}}{\left[C(N,\alpha)A_\alpha S_\alpha^{\frac{\alpha}{2}}\right]^{\frac{N-2}{4+2\alpha}}(\varepsilon^2 + |\cdot|^2)^{\frac{N-2}{2}}}, \qquad \varepsilon > 0.$$

Take $\delta > 0$ such that $\mathbb{B}_{5\delta} \subset \Omega$, and extract two cut-off functions $\varphi, \psi \in C_0^\infty(\Omega, [0,1])$ satisfying

$$\varphi(x) = \begin{cases} 1, & x \in \mathbb{B}_\delta, \\ 0, & x \in \mathbb{B}_{2\delta}^c \end{cases} \quad \text{and} \quad \psi(x) = \begin{cases} 0, & x \in \mathbb{B}_{2\delta}, \\ 1, & x \in \mathbb{B}_{4\delta} \backslash \mathbb{B}_{3\delta}, \\ 0, & x \in \mathbb{B}_{5\delta}^c. \end{cases}$$

Define $u_\varepsilon = \varphi U_\varepsilon$ and $v_\varepsilon = \psi U_\varepsilon$. As in [3,4], through direct computation, we obtain, as $\varepsilon \to 0^+$,

$$\int_\Omega |\nabla u_\varepsilon|^2 dx = S_\alpha^{\frac{N+\alpha}{2+\alpha}} + O(\varepsilon^{N-2}), \tag{2.13}$$

$$\int_\Omega |u_\varepsilon|^2 dx = \begin{cases} O(\varepsilon), & N = 3, \\ O(\varepsilon^2 |\ln \varepsilon|), & N = 4, \\ O(\varepsilon^2), & N \geq 5 \end{cases} \tag{2.14}$$

and

$$\int_\Omega \int_\Omega \frac{|u_\varepsilon(x)|^{2_\alpha^*} |u_\varepsilon(y)|^{2_\alpha^*}}{|x-y|^{N-\alpha}} dxdy = A_\alpha^{-1} S_\alpha^{\frac{N+\alpha}{2+\alpha}} + O(\varepsilon^{\frac{N+\alpha}{2}}). \tag{2.15}$$

Additionally, as $\varepsilon \to 0^+$,

$$\int_\Omega |\nabla v_\varepsilon|^2 + v_\varepsilon^2 dx = O(\varepsilon^{N-2}) \quad \text{and} \quad \int_\Omega |v_\varepsilon(x)|^p dx \geq d_p \varepsilon^{\frac{(N-2)p}{2}} \text{ for some } d_p > 0. \tag{2.16}$$

Lemma 2.6. *There exists some $\mu_* > 0$ independent of λ such that, for any $\lambda > 0$ and $\mu \geq \mu_*$,*

$$m_{\lambda,\mu} \leq m_{\infty,\mu} < m_* := \frac{2+\alpha}{2(N+\alpha)} S_\alpha^{\frac{N+\alpha}{2+\alpha}}.$$

Proof. Since $\mathcal{M}_{\infty,\mu} \subset \mathcal{M}_{\lambda,\mu}$ and $\mathcal{J}_{\lambda,\mu} = \mathcal{J}_{\infty,\mu}$ on $\mathcal{M}_{\infty,\mu}$, we easily derive $m_{\lambda,\mu} \leq m_{\infty,\mu}$. For any $\varepsilon > 0$ and $\mu > 0$, by Remark 2.2, there exist some constants $s_{\mu,\varepsilon} > 0, t_{\mu,\varepsilon} > 0$ such that $s_{\mu,\varepsilon} u_\varepsilon - t_{\mu,\varepsilon} v_\varepsilon \in \mathcal{M}_{\infty,\mu}$ and $\mathcal{J}_{\infty,\mu}(s_{\mu,\varepsilon} u_\varepsilon - t_{\mu,\varepsilon} v_\varepsilon) = \max_{s,t>0} \mathcal{J}_{\infty,\mu}(su_\varepsilon - tv_\varepsilon)$. It suffices to show $\max_{s,t>0} \mathcal{J}_{\infty,\mu}(su_\varepsilon - tv_\varepsilon) < m_*$ for $\varepsilon > 0$ small enough. Noting $\operatorname{spt} u_\varepsilon \cap \operatorname{spt} v_\varepsilon = \emptyset$, we deduce

$$\max_{s,t>0} \mathcal{J}_{\infty,\mu}(su_\varepsilon - tv_\varepsilon) \leq \max_{s>0} \mathcal{J}_{\infty,\mu}(su_\varepsilon) + \max_{t>0} \mathcal{J}_{\infty,\mu}(tv_\varepsilon). \tag{2.17}$$

It easily follows from (2.13)−(2.15) that, for $\varepsilon > 0$ sufficiently small and all $\mu > 0$, $s > 0$,

$$\mathcal{J}_{\infty,\mu}(su_\varepsilon) \leq S_\alpha^{\frac{N+\alpha}{2+\alpha}} \left(s^2 - \frac{1}{4 \cdot 2_\alpha^*} s^{2 \cdot 2_\alpha^*} \right).$$

In view of this, there exist some sufficiently small $s_1 > 0$ and sufficiently large $s_2 > 0$ independent of ε, μ such that, for $\varepsilon > 0$ small enough and all $\mu > 0$,

$$\max_{s \in (0,s_1)} \mathcal{J}_{\infty,\mu}(su_\varepsilon) < m_* \quad \text{and} \quad \max_{s \in (s_2,+\infty)} \mathcal{J}_{\infty,\mu}(su_\varepsilon) < 0.$$

Moreover, from (2.13)−(2.15) again we conclude, for $\varepsilon > 0$ sufficiently small and any $\mu > 0$,

$$\begin{aligned}\max_{s\in[s_1,s_2]} \mathcal{J}_{\infty,\mu}(su_\varepsilon) &\le \max_{s>0}\left(\frac{s^2}{2}\int_\Omega |\nabla u_\varepsilon|^2dx - \frac{s^{2\cdot 2^*_\alpha}A_\alpha}{2\cdot 2^*_\alpha}\int_\Omega\int_\Omega \frac{|u_\varepsilon(x)|^{2^*_\alpha}|u_\varepsilon(y)|^{2^*_\alpha}}{|x-y|^{N-\alpha}}dxdy\right)\\ &\quad + \frac{s_2^2}{2}\int_\Omega |u_\varepsilon|^2dx - \frac{\mu s_1^p}{p}\int_\Omega |u_\varepsilon|^p dx\\ &\le \frac{2+\alpha}{2(N+\alpha)}S_\alpha^{\frac{N+\alpha}{2+\alpha}}\left[1+O(\varepsilon^{N-2})\right]\left[1-O(\varepsilon^{\frac{N+\alpha}{2}})\right]\\ &\quad + \frac{s_2^2}{2}\int_\Omega |u_\varepsilon|^2dx - \frac{\mu s_1^p \varepsilon^{N-\frac{(N-2)p}{2}}}{p}\int_{\mathbb{B}_1} |U_1|^p dx\\ &= \frac{2+\alpha}{2(N+\alpha)}S_\alpha^{\frac{N+\alpha}{2+\alpha}} + O(\varepsilon^{N-2}) + \frac{s_2^2}{2}\int_\Omega |u_\varepsilon|^2dx - \frac{\mu s_1^p \varepsilon^{N-\frac{(N-2)p}{2}}}{p}\int_{\mathbb{B}_1} |U_1|^p dx.\end{aligned}$$

If $N \ge 4$, or $N = 3$ and $\alpha \in (1,3)$, by (2.14) and $p \ge 2^*_\alpha$ we deduce, for $\varepsilon > 0$ small enough and $\mu > 0$,

$$\eta_N(\varepsilon) := O(\varepsilon^{N-2}) + \frac{s_2^2}{2}\int_\Omega |u_\varepsilon|^2dx - \frac{\mu s_1^p \varepsilon^{N-\frac{(N-2)p}{2}}}{p}\int_{\mathbb{B}_1} |U_1|^p dx < 0.$$

If $N = 3$ and $\alpha \in (0,1]$, take $\mu = \varepsilon^{\frac{\alpha-3}{2}}$, by (2.14), there exists small $\varepsilon_1 > 0$ such that $\eta_3(\varepsilon) < 0$ for all $\varepsilon \in (0,\varepsilon_1]$. Based on the above discussion, for $\varepsilon > 0$ small enough and any $\mu \ge \varepsilon_1^{\frac{2}{\alpha-3}}$ if $N = 3$ and $\alpha \in (0,1)$, also, for $\varepsilon > 0$ small enough and any $\mu > 0$ if $N \ge 4$ or $N = 3$ and $\alpha \in (1,3)$, we conclude

$$\max_{s>0} \mathcal{J}_{\infty,\mu}(su_\varepsilon) < m_*. \tag{2.18}$$

In addition, due to (2.16), there exists some $C_1 > 0$ such that, for $\varepsilon > 0$ small enough and any $\mu > 0$,

$$\max_{t>0} \mathcal{J}_{\infty,\mu}(tv_\varepsilon) \le \max_{t>0}\left[C_1\varepsilon^{N-2}t^2 - \mu d_p\left(\varepsilon^{N-2}t^2\right)^{\frac{p}{2}}\right] \le \frac{(p-2)(2C_1)^{\frac{p}{p-2}}}{2p(\mu p d_p)^{\frac{2}{p-2}}}. \tag{2.19}$$

Now, by combining (2.17), (2.18) and (2.19), there exists some large $\mu_* \in \left[\frac{1}{\varepsilon_1}, +\infty\right)$ such that $\max_{s,t>0} \mathcal{J}_{\infty,\mu}(su_\varepsilon - tv_\varepsilon) < m_*$ for any $\mu \ge \mu_*$ and small $\varepsilon > 0$. Thus this lemma is proved. □

In the forthcoming lemma, we show that $\mathcal{J}_{\lambda,\mu}$ satisfies the local $(PS)_c$ condition for λ large.

Lemma 2.7. *There exists some $\Lambda > 0$ independent of μ such that, for any $\lambda \ge \Lambda$ and $\mu \ge \mu_*$, each $(PS)_c$ sequence $\{u_n\} \subset E_\lambda$ for $\mathcal{J}_{\lambda,\mu}$, with level $c \in (0, m_*)$, has a convergent subsequence.*

Proof. From the definition of $\{u_n\}$, there results

$$m_* + o(1) + o(\|u_n\|_\lambda) \ge \mathcal{J}_{\lambda,\mu}(u_n) - \frac{1}{p}\left\langle \mathcal{J}'_{\lambda,\mu}(u_n), u_n\right\rangle \ge \frac{p-2}{2p}\|u_n\|_\lambda^2.$$

Then there exists some $C_2 > 0$ independent of λ and μ such that $\limsup_n \|u_n\|_\lambda \le C_2$. Naturally, $\{u_n\}$ is bounded in E_λ. Hence, there exists some $u \in E_\lambda$ such that, up to subsequences,

$$\begin{cases} u_n \rightharpoonup u & \text{in } E_\lambda,\\ u_n \to u & \text{in } L^s_{loc}(\mathbb{R}^N),\ \forall\, s \in [1, 2^*), \text{ as } n \to \infty.\\ u_n(x) \to u(x) & \text{a.e. in } \mathbb{R}^N, \end{cases} \tag{2.20}$$

Set $v_n = u_n - u$. Clearly, $\limsup_n \|v_n\|_\lambda \le 2C_2$. We will show $\|v_n\|_\lambda \xrightarrow{n} 0$ up to a subsequence. Define

$$\beta = \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{\mathbb{B}_1(y)} v_n^2 dx.$$

We assert $\beta = 0$. Otherwise, $\beta > 0$. Due to (V_5), there exists some large $R > 0$ such that

$$\left|\{x \in \mathbb{B}_R^c(0) : V(x) \le M\}\right| \le \left(\frac{\beta S}{16C_2^2}\right)^{\frac{N}{2}}.$$

Then it follows from the Hölder and Sobolev inequalities that

$$\limsup_{n\to\infty} \int_{\{x\in\mathbb{B}_R^c(0):V(x)\le M\}} v_n^2 dx \le \left|\{x \in \mathbb{B}_R^c(0) : V(x) \le M\}\right|^{\frac{2}{N}} S^{-1} \limsup_{n\to\infty} \|v_n\|_\lambda^2 \le \frac{\beta}{4}. \quad (2.21)$$

Moreover, if taking $\Lambda = \frac{1}{M}\left(16C_2^2\beta^{-1} - 1\right)$ and letting $\lambda \ge \Lambda$, we have

$$\limsup_{n\to\infty} \int_{\{x\in\mathbb{B}_R^c(0):V(x)> M\}} v_n^2 dx \le \frac{1}{\lambda M + 1} \limsup_{n\to\infty} \|v_n\|_\lambda^2 \le \frac{\beta}{4}. \quad (2.22)$$

Consequently, combining (2.20)−(2.22) leads to

$$\beta \le \limsup_{n\to\infty} \int_{\mathbb{R}^N} v_n^2 dx = \limsup_{n\to\infty} \int_{\mathbb{B}_R^c(0)} v_n^2 dx \le \frac{\beta}{2},$$

which contradicts $\beta > 0$. That is, our claim $\beta = 0$ is true. Then, thanks to [29, Lemma 1.21],

$$v_n \to 0 \quad \text{in } L^s(\mathbb{R}^N), \qquad \forall\, s \in (2, 2^*). \quad (2.23)$$

By (2.20), it is easy to show $\mathcal{J}'_{\lambda,\mu}(u) = 0$. Further, with $\langle \mathcal{J}'_{\lambda,\mu}(u_n), u_n\rangle = o(1)$ in hand, we deduce from (2.20), (2.23) and the nonlocal version of the Brézis–Lieb lemma (see e.g. [4, Lemma 2.2]) that

$$o(1) = \|v_n\|_\lambda^2 - \int_{\mathbb{R}^N} \left(I_\alpha * |v_n|^{2_\alpha^*}\right)|v_n|^{2_\alpha^*} dx. \quad (2.24)$$

Set $\kappa = \limsup_{n\to\infty} \|v_n\|_\lambda$. Due to (2.24) and the definition of S_α, there results $\kappa = 0$ or $\kappa \ge S_\alpha^{\frac{N+\alpha}{2(2+\alpha)}}$. We claim $\kappa = 0$. If not, because $\mathcal{J}_{\lambda,\mu}(u) \ge 0$, it follows from (2.20), (2.24) and Lemma 2.2 in [4] that

$$c = \lim_{n\to\infty} \mathcal{J}_{\lambda,\mu}(u_n) = \mathcal{J}_{\lambda,\mu}(u) + \frac{2+\alpha}{2(N+\alpha)} \limsup_{n\to\infty} \|v_n\|_\lambda^2 \ge \frac{2+\alpha}{2(N+\alpha)} S_\alpha^{\frac{N+\alpha}{2+\alpha}},$$

which contradicts $c < m_*$. Thus $u_n \to u$ in E_λ up to a subsequence. This lemma is proved. □

Based on the above preliminary lemmas, we shall complete the proof of main results below.

Proof of Theorem 1.2. Let $\lambda \ge \Lambda$ and $\mu \ge \mu_*$. Thanks to Lemmas 2.5 and 2.6, $\mathcal{J}_{\lambda,\mu}$ has a sign-changing $(PS)_{m_{\lambda,\mu}}$ sequence $\{u_n\} \subset E_\lambda$, with $m_{\lambda,\mu} < m_*$. From Lemma 2.7, we derive that $u_n \to u_{\lambda,\mu}$ in E_λ in the sense of subsequence. Then, there result $\mathcal{J}'_{\lambda,\mu}(u_{\lambda,\mu}) = 0$ in E_λ^* and $\mathcal{J}_{\lambda,\mu}(u_{\lambda,\mu}) = m_{\lambda,\mu}$. Further, Lemma 2.3 implies $u_{\lambda,\mu}^\pm \ne 0$. That is, Eq. (1.6) has a ground state sign-changing solution $u_{\lambda,\mu}$.

Next, we show the concentration of ground state sign-changing solutions for Eq. (1.6) as $\lambda \to +\infty$. Given $\mu \ge \mu_*$ arbitrarily. For sequence $\{\lambda_n\} \subset [\Lambda, +\infty)$ with $\lambda_n \to +\infty$, let $u_{\lambda_n,\mu} \in E_{\lambda_n}$ be such that

$$u^{\pm}_{\lambda_n,\mu} \neq 0, \qquad \mathcal{J}'_{\lambda_n,\mu}(u_{\lambda_n,\mu}) = 0 \quad \text{in } E^*_{\lambda_n}, \qquad \mathcal{J}_{\lambda_n,\mu}(u_{\lambda_n,\mu}) = m_{\lambda_n,\mu}.$$

By Lemma 2.6, it is easy to obtain

$$m_* > \mathcal{J}_{\lambda_n,\mu}(u_{\lambda_n,\mu}) - \frac{1}{p}\left\langle \mathcal{J}'_{\lambda_n,\mu}(u_{\lambda_n,\mu}), u_{\lambda_n,\mu}\right\rangle > \frac{p-2}{2p}\|u_{\lambda_n,\mu}\|^2_{\lambda_n}. \tag{2.25}$$

Obviously, $\{u_{\lambda_n,\mu}\}$ is bounded in $H^1(\mathbb{R}^N)$. Then, there exists some $u_\mu \in H^1(\mathbb{R}^N)$ such that, up to subsequences,

$$\begin{cases} u_{\lambda_n,\mu} \stackrel{n}{\rightharpoonup} u_\mu & \text{in } H^1(\mathbb{R}^N), \\ u_{\lambda_n,\mu} \stackrel{n}{\to} u_\mu & \text{in } L^s_{loc}(\mathbb{R}^N),\ \forall\, s \in [1, 2^*), \\ u_{\lambda_n,\mu}(x) \stackrel{n}{\to} u_\mu(x) & \text{a.e. in } \mathbb{R}^N. \end{cases} \tag{2.26}$$

It follows from the Fatou lemma, (2.25) and (2.26) that

$$0 \le \int_{\Omega^c} V(x)u_\mu^2 dx \le \liminf_{n\to\infty}\int_{\mathbb{R}^N} V(x)u^2_{\lambda_n,\mu}dx \le \liminf_{n\to\infty}\frac{\|u_{\lambda_n,\mu}\|^2_{\lambda_n}}{\lambda_n} = 0,$$

which together with (V_6) implies $u_\mu|_{\Omega^c} = 0$. Then, $u_\mu \in H^1_0(\Omega)$, since $\partial\Omega$ is smooth. Thereby, for any $\omega \in H^1_0(\Omega)$, we derive from $\langle \mathcal{J}'_{\lambda_n,\mu}(u_{\lambda_n,\mu}), \omega\rangle = 0$ and (2.26) that $\mathcal{J}'_{\infty,\mu}(u_\mu) = 0$.

Set $v_{\mu,n} = u_{\lambda_n,\mu} - u_\mu$. For any $\varepsilon > 0$, by (V_5), there exists some large $R_\varepsilon > 0$ such that

$$\left|\left\{x \in \mathbb{B}^c_{R_\varepsilon} : V(x) \le M\right\}\right| < \left[\frac{(p-2)S\varepsilon}{4pm_*}\right]^{\frac{N}{2}}.$$

Then, due to the Hölder and Sobolev inequalities, the weakly lower semicontinuity of norm and (2.25), there holds

$$\int_{\left\{x\in\mathbb{B}^c_{R_\varepsilon}:V(x)\le M\right\}} v^2_{\mu,n}dx \le \left|\left\{x \in \mathbb{B}^c_{R_\varepsilon} : V(x) \le M\right\}\right|^{\frac{2}{N}} S^{-1}\|v_{n,\mu}\|^2_{\lambda_n} < \varepsilon.$$

From the weakly lower semicontinuity of norm and (2.25), it follows that

$$\int_{\left\{x\in\mathbb{B}^c_{R_\varepsilon}:V(x)\ge M\right\}} v^2_{\mu,n}dx \le \frac{\|v_{n,\mu}\|^2_{\lambda_n}}{\lambda_n M} \le \frac{4pm_*}{(p-2)M\lambda_n} \to 0 \quad \text{as } n \to \infty.$$

Thereby, we deduce from (2.26) that $|v_{\mu,n}|_2 \stackrel{n}{\to} 0$. Further, by (2.25), the Hölder and Sobolev inequalities, there holds

$$\begin{aligned} \limsup_{n\to\infty}\int_{\mathbb{R}^N} |v_{\mu,n}|^p dx &\le \limsup_{n\to\infty}\left(|v_{\mu,n}|_2^{\frac{2(2^*-p)}{2^*-2}}|v_{\mu,n}|_{2^*}^{\frac{2^*(p-2)}{2^*-2}}\right) \\ &\le \left[\frac{4pm_*}{(p-2)S}\right]^{\frac{2^*(p-2)}{2(2^*-2)}}\limsup_{n\to\infty}|v_{\mu,n}|_2^{\frac{2(2^*-p)}{2^*-2}} = 0. \end{aligned} \tag{2.27}$$

By (2.26), (2.27), the nonlocal type of the Brézis–Lieb Lemma 2.2 in [4] and $\mathcal{J}'_{\infty,\mu}(u_\mu) = 0$, we have

$$0 = \left\langle \mathcal{J}'_{\lambda_n,\mu}(u_{\lambda_n,\mu}), u_{\lambda_n,\mu}\right\rangle = \|v_{\mu,n}\|^2_{\lambda_n} - \int_{\mathbb{R}^N}\left(I_\alpha * |v_{\mu,n}|^{2^*_\alpha}\right)|v_{\mu,n}|^{2^*_\alpha}dx + o(1). \tag{2.28}$$

Denote $\kappa_\mu = \limsup_{n\to\infty} \|v_{\mu,n}\|_{\lambda_n}$. It follows from (2.28) and the definition of S_α that $\kappa_\mu^2 \le S_\alpha^{-2_\alpha^*}\kappa_\mu^{2\cdot 2_\alpha^*}$. Then, by (2.25), there results $\kappa_\mu = 0$ or $\kappa_\mu \ge S_\alpha^{\frac{N+\alpha}{2(2+\alpha)}}$. We assert $\kappa_\mu = 0$. If not, from Lemma 2.6, (2.25)−(2.28), the nonlocal type of the Brézis–Lieb lemma and $\mathcal{J}'_{\infty,\mu}(u_\mu) = 0$, we have

$$
\begin{aligned}
m_* &> \lim_{n\to\infty} \mathcal{J}_{\lambda_n,\mu}(u_{\lambda_n,\mu}) \\
&= \mathcal{J}_{\infty,\mu}(u_\mu) + \frac{2+\alpha}{2(N+\alpha)} \limsup_{n\to\infty} \|v_{\mu,n}\|_{\lambda_n}^2 \\
&= \mathcal{J}_{\infty,\mu}(u_\mu) - \frac{1}{p}\left\langle \mathcal{J}'_{\infty,\mu}(u_\mu), u_\mu \right\rangle + \frac{2+\alpha}{2(N+\alpha)} k_\mu^2 \\
&\ge m_*,
\end{aligned}
$$

a contradiction. Hence, $\|u_{\lambda_n,\mu} - u_\mu\|_{\lambda_n} \xrightarrow{n} 0$. Then, it is easy to show $u_{\lambda_n,\mu} \to u_\mu$ in $H^1(\mathbb{R}^N)$.

From $\langle \mathcal{J}'_{\lambda_n,\mu}(u_{\lambda_n,\mu}), u^\pm_{\lambda_n,\mu}\rangle = 0$, (1.4), the Young and Sobolev inequalities, we deduce that

$$
\begin{aligned}
S|u^\pm_{\lambda_n,\mu}|_{2^*}^2 \le \|u^\pm_{\lambda_n,\mu}\|_{\lambda_n}^2 &= \int_{\mathbb{R}^N} \left(I_\alpha * |u_{\lambda_n,\mu}|^{2_\alpha^*}\right)|u^\pm_{\lambda_n,\mu}|^{2_\alpha^*} dx + \mu |u^\pm_{\lambda_n,\mu}|_p^p \\
&\le A_\alpha C(N,\alpha)|u_{\lambda_n,\mu}|_{2^*}^{2_\alpha^*}|u^\pm_{\lambda_n,\mu}|_{2^*}^{2_\alpha^*} + \frac{2^*-p}{2^*-2}\|u^\pm_{\lambda_n,\mu}\|_{\lambda_n}^2 + \frac{p-2}{2^*-2}\mu^{\frac{2^*-2}{p-2}}|u^\pm_{\lambda_n,\mu}|_{2^*}^{2^*},
\end{aligned}
$$

which together with (2.25) implies

$$
S|u^\pm_{\lambda_n,\mu}|_{2^*}^2 \le \frac{A_\alpha C(N,\alpha)(2^*-2)}{p-2}\left[\frac{2pm_*}{S(p-2)}\right]^{\frac{2_\alpha^*}{2}} |u^\pm_{\lambda_n,\mu}|_{2^*}^{2_\alpha^*} + \mu^{\frac{2^*-2}{p-2}}|u^\pm_{\lambda_n,\mu}|_{2^*}^{2^*}.
$$

In view of this, there holds $\inf_n |u^\pm_{\lambda_n,\mu}|_{2^*} > 0$. Thereby, $\|u_{\lambda_n,\mu} - u_\mu\| \xrightarrow{n} 0$ implies $|u^\pm_\mu|_{2^*} > 0$. Naturally, $u^\pm_\mu \ne 0$ and then $u_\mu \in \mathcal{M}_{\infty,\mu}$. Thus we derive from (2.26), the Fatou lemma and Lemma 2.6 that

$$
\begin{aligned}
m_{\infty,\mu} &\le \mathcal{J}_{\infty,\mu}(u_\mu) - \frac{1}{p}\left\langle \mathcal{J}'_{\infty,\mu}(u_\mu), u_\mu \right\rangle \\
&= \frac{p-2}{2p}\int_\Omega \left(|\nabla u_\mu|^2 + u_\mu^2\right) dx + \frac{(2\cdot 2_\alpha^* - p)A_\alpha}{2p\cdot 2_\alpha^*}\int_\Omega\int_\Omega \frac{|u_\mu(x)|^{2_\alpha^*}|u_\mu(y)|^{2_\alpha^*}}{|x-y|^{N-\alpha}} dxdy \\
&\le \lim_{n\to\infty}\left[\frac{p-2}{2p}\|u_{\lambda_n,\mu}\|_{\lambda_n}^2 + \frac{2\cdot 2_\alpha^* - p}{2p\cdot 2_\alpha^*}\int_{\mathbb{R}^N}\left(I_\alpha * |u_{\lambda_n,\mu}|^{2_\alpha^*}\right)|u_{\lambda_n,\mu}|^{2_\alpha^*} dx\right] \\
&= \lim_{n\to\infty}\left[\mathcal{J}_{\lambda_n,\mu}(u_{\lambda_n,\mu}) - \frac{1}{p}\left\langle \mathcal{J}'_{\lambda_n,\mu}(u_{\lambda_n,\mu}), u_{\lambda_n,\mu}\right\rangle\right] \\
&\le m_{\infty,\mu},
\end{aligned}
$$

which leads to $\mathcal{J}_{\infty,\mu}(u_\mu) = m_{\infty,\mu}$. Therefore, u_μ is a ground state sign-changing solution for Eq. (1.8).

Further, we certify the asymptotic behavior of ground state sign-changing solutions for Eq. (1.6) as $\mu \to +\infty$. Fix $\lambda \ge \Lambda$. For any sequence $\{\mu_n\} \subset [\mu_*, +\infty)$ with $\mu_n \to +\infty$, let $\{u_{\lambda,\mu_n}\} \subset E_\lambda$ satisfy

$$
u^\pm_{\lambda,\mu_n} \ne 0, \qquad \mathcal{J}'_{\lambda,\mu_n}(u_{\lambda,\mu_n}) = 0 \quad \text{in } E_\lambda^*, \qquad \mathcal{J}_{\lambda,\mu_n}(u_{\lambda,\mu_n}) = m_{\lambda,\mu_n}.
$$

It easily follows that

$$m_{\lambda,\mu_n} = \mathcal{J}_{\lambda,\mu_n}(u_{\lambda,\mu_n}) - \frac{1}{p}\left\langle \mathcal{J}'_{\lambda,\mu_n}(u_{\lambda,\mu_n}), u_{\lambda,\mu_n}\right\rangle \geq \frac{p-2}{2p}\|u_{\lambda,\mu_n}\|^2_\lambda. \tag{2.29}$$

We assert that $\lim_{n\to\infty} m_{\lambda,\mu_n} \to 0$ in the sense of subsequence. Take $\omega \in H^1_0(\Omega)$ such that $\omega^\pm \neq 0$. Due to Remark 2.2, there exist $s_n > 0$ and $t_n > 0$ such that $s_n\omega^+ + t_n\omega^- \in \mathcal{M}_{\infty,\mu_n}$. Then we have

$$\begin{aligned} s_n^2 &\int_\Omega |\nabla\omega^+|^2 + |\omega^+|^2 dx \\ &= A_\alpha s_n^{2\cdot 2^*_\alpha}\int_\Omega\int_\Omega \frac{|\omega^+(x)|^{2^*_\alpha}|\omega^+(y)|^{2^*_\alpha}}{|x-y|^{N-\alpha}}dxdy \\ &\quad + A_\alpha(s_nt_n)^{2^*_\alpha}\int_\Omega\int_\Omega \frac{|\omega^+(x)|^{2^*_\alpha}|\omega^-(y)|^{2^*_\alpha}}{|x-y|^{N-\alpha}}dxdy + \mu_n s_n^p\int_\Omega |\omega^+|^p dx, \end{aligned} \tag{2.30}$$

$$\begin{aligned} t_n^2 &\int_\Omega |\nabla\omega^-|^2 + |\omega^-|^2 dx \\ &= A_\alpha t_n^{2\cdot 2^*_\alpha}\int_\Omega\int_\Omega \frac{|\omega^-(x)|^{2^*_\alpha}|\omega^-(y)|^{2^*_\alpha}}{|x-y|^{N-\alpha}}dxdy \\ &\quad + A_\alpha(t_ns_n)^{2^*_\alpha}\int_\Omega\int_\Omega \frac{|\omega^+(x)|^{2^*_\alpha}|\omega^-(y)|^{2^*_\alpha}}{|x-y|^{N-\alpha}}dxdy + \mu_n t_n^p\int_\Omega |\omega^-|^p dx. \end{aligned} \tag{2.31}$$

From (2.30) and (2.31), we easily deduce that both $\{s_n\}$ and $\{t_n\}$ are bounded. Thereby, $s_n \to s_0$ and $t_n \to t_0$ up to subsequences. By using (2.30) and (2.31) again, we derive $s_0 = t_0 = 0$. Consequently, Lemmas 2.3 and 2.6 imply

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} m_{\lambda,\mu_n} \leq \limsup_{n\to\infty} m_{\infty,\mu_n} \leq \limsup_{n\to\infty} \mathcal{J}_{\infty,\mu_n}(s_n\omega^+ + t_n\omega^-) \\ &\leq \limsup_{n\to\infty}\left(s_n^2\int_\Omega |\nabla\omega^+|^2 + |\omega^+|^2dx + t_n^2\int_\Omega |\nabla\omega^-|^2 + |\omega^-|^2dx\right) = 0. \end{aligned}$$

Now, from (2.29) we conclude $u_{\lambda,\mu_n} \xrightarrow{n} 0$ in E_λ. Naturally $u_{\lambda,\mu_n} \xrightarrow{n} 0$ in $H^1(\mathbb{R}^N)$ in the sense of subsequence. Thus, based on the above arguments, we complete the proof of Theorem 1.2. □

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On a viscoelastic heat equation with logarithmic nonlinearity

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Abstract. This work deals with the following viscoelastic heat equations with logarithmic nonlinearity

$$u_t - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = |u|^{p-2}u \ln |u|.$$

In this paper, we show the effects of the viscoelastic term and the logarithmic nonlinearity to the asymptotic behavior of weak solutions. Our results extend the results of Peng and Zhou [*Appl. Anal.* **100**(2021), 2804–2824] and Messaoudi [*Progr. Nonlinear Differential Equations Appl.* **64**(2005), 351–356.].

Keywords: viscoelastic heat equation, global existence, blow-up, blow-up time, exponential decay, logarithmic nonlinearity.

2020 Mathematics Subject Classification: 35K05, 35B40, 35B44, 74Dxx.

1 Introduction

In this paper, we study the following heat equations with viscoelastic term and logarithmic nonlinearity

$$\begin{cases} u_t - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = |u|^{p-2}u \ln |u|, & \text{in} \quad \Omega \times (0,\infty), \\ u = 0, & \text{on} \quad \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

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where $u_0 \in H_0^1(\Omega)$ and $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, and the parameter p satisfy

$$2 < p < \begin{cases} \infty, & \text{if} \quad n \leq 2, \\ \frac{2(n-1)}{n-2}, & \text{if} \quad n > 2. \end{cases} \tag{1.2}$$

The equation of the form

$$u_t - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = f(u), \tag{1.3}$$

is used to model many natural phenomena in physical science and engineering. For example, in the study of heat conduction in materials with memory, from the heat balance equation the temperature $u(x,t)$ will satisfy Eq. (1.3) (see [3,5,12,13] for more detail).

In the last several decades, the initial-boundary valued problem to Eq. (1.3) has been studied extensively when the source $f(u)$ is the power functions $f(u) = |u|^{p-2}u$, or power like-functions satisfying:

(1) $f \in C^1$ and $f(0) = f'(0) = 0$.

(2) (a) f is monotone and is convex for $u > 0$, and concave for $u < 0$; or (b) f is convex.

(3) $(p+1)\int_0^u f(z)dz \leq uf(u)$, and $|uf(u)| \leq \kappa \int_0^u f(z)dz$, where

$$2 < p+1 \leq \kappa < 2^* =: \begin{cases} \infty, & \text{if} \quad n \leq 2, \\ \frac{2n}{n-2}, & \text{if} \quad n \geq 3. \end{cases}$$

For example, Messaoudi [12] studied Eq. (1.3) in the case $f(u) = |u|^{p-2}u$ associated with homogeneous Dirichlet boundary condition. By the convexity method, the author showed that if the relaxation function g is non-negative and non-increasing satisfying

$$\int_0^\infty g(s)ds < \frac{2(p-2)}{2p-3},$$

then weak solution to (1.3) blows up in finite time provided initial energy is positive. In [20], Truong and Y also studied the problem of the above type with $f(u)$ in the general polynomial type and they obtained the existence, blow up and asymptotic behavior for weak solution under suitable conditions. For further results on the existence, blow-up or asymptotic behavior of solutions, we refer the reader to [5,13,16,19] in case of power or power-like sources.

With regard to the logarithmic nonlinearity, there are a few results (see [1,2,7,9,15]). In case the relaxation function g vanishes, the problem (1.1) reduces to the following:

$$\begin{cases} u_t - \Delta u = |u|^{p-2}u \ln|u|, & \text{in} \quad \Omega \times (0,\infty), \\ u = 0, & \text{on} \quad \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), & \text{in} \quad \Omega. \end{cases} \tag{1.4}$$

In case $p = 2$, self-similar solutions and their asymptotic stability for $(1.4)_1$ has been studied by Samarskii et al. [17]. With regard to weak solutions, by using the potential well method and the logarithmic Sobolev inequality in $H_0^1(\Omega)$ (see [6,11]), Chen et al. [1] prove that the weak solution blows up at infinite time and exists globally provided that the initial data start in the stable sets and unstable sets respectively. This result is so interesting because it showed the different effect of logarithmic nonlinearity compared to the power one. Inspired by this result the second and third authors [9] extended (1.4) to the evolution p-Laplacian equations and showed a different result compared to the case $p = 2$, confirming that weak solutions blow up in finite time. Afterward the PDEs with logarithmic nonlinearity have been attracted many researchers, see [2,7,15] for example. In particular, Peng and Zhou [15] have showed recently that in case $p > 2$ the solutions of (1.4) behave like the nonlinear case $f(u) = |u|^{p-2}u$. These results shows that $p = 2$ is the critical exponent for the blow-up at infinite time.

Motivated by all these works, our aim in this paper is to study the effect of the viscoelastic term $\int_0^t g(t-s)\Delta u(s)ds$ and the logarithmic nonlinearity $|u|^{p-2}u\ln|u|$ to the blow-up and global existence of weak solutions to (1.1). Firstly, the presence of logarithmic nonlinearity help us relax conditions on g compared to [12], that is,

$$\int_0^\infty g(s)ds < \frac{p(p-2)}{(p-1)^2},$$

where $\frac{p(p-2)}{(p-1)^2} > \frac{2(p-2)}{2p-3}$ since $p > 2$. Secondly, because of the presence of $\int_0^t g(t-s)\Delta u(s)ds$ we need more restriction on the range of p and for small energy levels $E(0) < d_\delta \le d$ (see (2.2) below) compared to [15].

Our result is twofold in the sense that it is not only study the blow-up in finite time but also global existence of weak solutions. In addition, we also give the lower and upper bound for blow-up time and decay estimate of global solutions. Also notice that our method differs from [12]. To obtain the main results, we employ the ideas from the potential well method due to Sattinger [18] (see also [14]). However, since the presence of the relaxation g we could not apply the stable and unstable sets as in [14]. To overcome this difficulty we construct a family of potential wells (see (2.3) and (2.4)) that is more suitable for the PDEs involving viscoelastic terms. Also notice that the asymptotic behavior of global solutions in [15] has not been studied and it can be done by using the method employed in this paper.

This paper is organized as follows. In the next section, we present some preliminaries and define the family of modified potential wells. Our main results are stated in the Section 3 and the rest of the paper is devoted to their proofs.

Notation. Throughout this paper, we denote $L^p(\Omega)$-norm by $\|\cdot\|_p$, especially $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$. And let $\langle\cdot,\cdot\rangle$ denote L^2-inner product.

2 Preliminaries and Modified potential wells

2.1 Preliminary lemmas

The following lemmas will be needed in our proof of the main results.

Lemma 2.1 ([21, Lemma 3.1.7 and Remark 3.1.4])**.** *Let* $\mathbf{B}$ *be a reflexive Banach space and* $0 < T < \infty$. *Suppose* $1 < q < \infty$, $\varphi \in L^q(0,T;\mathbf{B})$, *and the sequence* $\{\varphi_m\}_{m=1}^\infty \subset L^q(0,T;\mathbf{B})$ *satisfy (as* $m \to \infty$*)*

$$\begin{cases} \varphi_m \to \varphi & \text{weakly in} \quad L^q(0,T;\mathbf{B}), \\ \varphi_{mt} \to \varphi_t & \text{weakly in} \quad L^q(0,T;\mathbf{B}). \end{cases}$$

Then $\varphi_m(0) \to \varphi(0)$ *weakly in* $\mathbf{B}$.

Lemma 2.2 ([21, Thereom 3.1.1])**.** *Let* (1.2) *hold and* $T \in (0,\infty)$ *be fixed. Then the embedding*

$$\left\{\varphi \,\middle|\, \varphi \in L^2\left(0,T;H_0^1(\Omega)\right),\, \varphi_t \in L^2\left(0,T;L^2(\Omega)\right)\right\} \hookrightarrow L^2\left(0,T;L^p(\Omega)\right)$$

is compact.

Lemma 2.3 ([9])**.** *Let* ρ *be a positive number. Then we have the following elementary inequalities:*

$$\Psi^p \ln \Psi \le \frac{e^{-1}}{\rho}\Psi^{p+\rho},\ \forall \Psi \ge 1 \quad \text{and} \quad |\Psi^p \ln \Psi| \le (ep)^{-1},\ \forall 0 < \Psi < 1.$$

Lemma 2.4 ([8, 10])**.** *Suppose that* $\Phi(t) \in C^2[0,\infty)$ *is a positive function satisfying the following inequality*

$$\Phi(t)\Phi''(t) - (1+\gamma)\left(\Phi'(t)\right)^2 \ge 0,$$

where $\gamma > 0$ *is a constant. If* $\Phi(0) > 0, \Phi'(0) > 0$, *then* $\Phi(t) \to \infty$ *for* $t \to t_* \le t^* = \frac{\Phi(0)}{\gamma\Phi'(0)}$.

2.2 Modified potential wells

For $0 < \delta \le \ell$ with $\ell := 1 - \int_0^\infty g(s)ds$, we define potential energy functional

$$J_\delta(u) = \frac{\delta}{2}\|\nabla u\|^2 - \frac{1}{p}\int_\Omega |u|^p \ln|u|dx + \frac{1}{p^2}\|u\|_p^p,$$

and the associated Nehari functional

$$I_\delta(u) = \delta\|\nabla u\|^2 - \int_\Omega |u|^p \ln|u|dx.$$

then we have that

$$J_\delta(u) = \left(\frac{1}{2} - \frac{1}{p}\right)\delta\|\nabla u\|^2 + \frac{1}{p}I_\delta(u) + \frac{1}{p^2}\|u\|_p^p.$$

We have the following lemma.

Lemma 2.5. *Let* $u \in H_0^1(\Omega)\setminus\{0\}$. *Then we have:*

(i) $\lim_{\lambda\to 0^+} J_\delta(\lambda u) = 0$ *and* $\lim_{\lambda\to\infty} J_\delta(\lambda u) = -\infty$.

(ii) there is a unique $\lambda_1 = \lambda_1(u) > 0$ *such that* $\frac{d}{d\lambda}J_\delta(\lambda u)\Big|_{\lambda=\lambda_1} = 0$.

(iii) $J_\delta(\lambda u)$ *is strictly increasing on* $(0,\lambda_1)$ *and strictly decreasing on* (λ_1,∞), *and attains its the maximum value at* $\lambda = \lambda_1$. *In addition, one has*

$$I_\delta(\lambda u)\begin{cases} > 0, & \text{if} \quad 0 \le \lambda < \lambda_1, \\ = 0, & \text{if} \quad \lambda = \lambda_1, \\ < 0, & \text{if} \quad \lambda_1 < \lambda < \infty. \end{cases}$$

Proof. (i) From the definition of J_δ, we have for $\lambda > 0$ that

$$J_\delta(\lambda u) = \frac{\delta\lambda^2}{2}\|\nabla u\|^2 - \frac{\lambda^p}{p}\int_\Omega |u|^p \ln|u|dx - \frac{\lambda^p}{p}\ln\lambda\|u\|_p^p + \frac{\lambda^p}{p^2}\|u\|_p^p,$$

which implies $\lim_{\lambda\to 0^+} J_\delta(\lambda u) = 0$ and $\lim_{\lambda\to\infty} J_\delta(\lambda u) = -\infty$ thanks to $p > 2$.

For (ii). An easy calculation shows that

$$\frac{d}{d\lambda}J_\delta(\lambda u) = \lambda\left(\delta\|\nabla u\|^2 - \lambda^{p-2}\int_\Omega |u|^p\ln|u|dx - \lambda^{p-2}\ln\lambda\|u\|_p^p\right) := \lambda K_\delta(\lambda u),$$

where

$$K_\delta(\lambda u) = \delta\|\nabla u\|^2 - \lambda^{p-2}\int_\Omega |u|^p \ln|u|dx - \lambda^{p-2}\ln\lambda\|u\|_p^p. \tag{2.1}$$

A direct calculations yields

$$\frac{d}{d\lambda}K_\delta(\lambda u) = -\lambda^{p-3}\left((p-2)\int_\Omega |u|^p\ln|u|dx + (p-2)\ln\lambda\|u\|_p^p + \|u\|_p^p\right),$$

Hence if we choose

$$\lambda_* = \exp\left(\frac{(2-p)\int_\Omega |u|^p\ln|u|dx - \|u\|_p^p}{(p-2)\|u\|_p^p}\right),$$

then one has $\frac{d}{d\lambda}K_\delta(\lambda_* u) = 0$, $\frac{d}{d\lambda}K_\delta(\lambda u) > 0$ for $0 < \lambda < \lambda_*$ and $\frac{d}{d\lambda}K_\delta(\lambda u) < 0$ for $\lambda_* < \lambda < \infty$. On the other hand, from the definition of K, we have

$$\lim_{\lambda\to 0^+} K_\delta(\lambda u) = \delta\|\nabla u\|^2 > 0 \quad \text{and} \quad \lim_{\lambda\to\infty} K_\delta(\lambda u) = -\infty.$$

By these facts we obtain that there exists a unique $\lambda_1 > \lambda_*$ such that $K_\delta(\lambda_1 u) = 0$. Hence we obtain (ii).

The last statement (iii) follows from (i)–(ii) and the relation

$$I_\delta(\lambda u) = \lambda\frac{d}{d\lambda}J_\delta(\lambda u).$$

The proof is complete. □

Let us state here the Sobolev imbedding which can be found in [4].

Lemma 2.6. *Assume that p is a constant such that*

$$1 \le p \le \begin{cases} \frac{2n}{n-2}, & \text{if} \quad n > 2, \\ \widetilde{p}, & \text{if} \quad n = 2, \\ \infty, & \text{if} \quad n = 1, \end{cases}$$

where $\widetilde{p} \in [1,\infty)$ can be any constant. Then $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ continuously, and there exists a positive constant C_p depending on n, p and Ω such that

$$\|u\|_p \le C_p\|\nabla u\|$$

holds for all $u \in H_0^1(\Omega)$. We choose C_p be the optimal constant satisfying the above inequality, i.e.

$$C_p = \sup_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\|u\|_p}{\|\nabla u\|}.$$

Since $p < \frac{2(n-1)}{n-2} < 2^*$, let

$$\sigma^* = \begin{cases} \frac{2n}{n-2} - p, & \text{if} \quad n > 2, \\ \infty, & \text{if} \quad n = 1, 2, \end{cases}$$

then $\sigma^* > 0$ and by Lemma 2.6, we have $H_0^1(\Omega) \hookrightarrow L^{p+\sigma}(\Omega)$ continuously for any $\sigma \in [0, \sigma^*)$. Denote $C_{p+\sigma}$ by C_*, then we have the following lemma.

Lemma 2.7. *Let* (1.2) *hold and* $u \in H_0^1(\Omega) \setminus \{0\}$. *Then we have*

(i) if $I_\delta(u) < 0$, *then* $\|\nabla u\| > r_\delta(\sigma)$,

(ii) if $\|\nabla u\| \le r_\delta(\sigma)$ *then* $I_\delta(u) \ge 0$,

where $r_\delta(\sigma) = \left(\frac{e\sigma\delta}{C_*^{p+\sigma}}\right)^{\frac{1}{p+\sigma-2}}$ *for* $0 < \sigma < \sigma^*$.

Proof. For $0 < \sigma < \sigma^*$, by Lemma 2.3 and the Sobolev inequality, we have

$$\begin{aligned} \int_\Omega |u|^p \ln |u| dx &= \int_{\{\Omega:|u|\le 1\}} |u|^p \ln |u| dx + \int_{\{\Omega:|u|\ge 1\}} |u|^p \ln |u| dx \\ &\le \frac{e^{-1}}{\sigma} \|u\|_{p+\sigma}^{p+\sigma} \le \frac{e^{-1}}{\sigma} C_*^{p+\sigma} \|\nabla u\|^{p+\sigma}. \end{aligned}$$

It follows that

$$\begin{aligned} I_\delta(u) &= \delta \|\nabla u\|^2 - \int_\Omega |u|^p \ln |u| dx \\ &\ge \delta \|\nabla u\|^2 - \frac{e^{-1}}{\sigma} C_*^{p+\sigma} \|\nabla u\|^{p+\sigma} = \|\nabla u\|^2 \left(\delta - \frac{e^{-1}}{\sigma} C_*^{p+\sigma} \|\nabla u\|^{p+\sigma-2}\right). \end{aligned}$$

The conclusions then follow from the above inequality. □

Let us define the so-called Nehari manifold associated to the energy functional J_δ by

$$\mathcal{N}_\delta = \left\{ u \in H_0^1(\Omega) \setminus \{0\} : I_\delta(u) = \langle J'_\delta(u), u \rangle = 0 \right\}.$$

By Lemma 2.5 we know that $\mathcal{N}_\delta$ is not empty set. It is clear that $J_\delta(u)$ is coercive on the Nehari manifold $\mathcal{N}_\delta$, hence we can define

$$d_\delta = \inf_{u \in \mathcal{N}_\delta} J_\delta(u). \tag{2.2}$$

The standard variational method shows that d_δ is a positive finite number and therefore it is well-defined.

We end this section by giving the definitions of the modified stable and unstable sets as in [14].

$$\mathcal{W}_\delta = \left\{ u \in H_0^1(\Omega) : J_\delta(u) < d_\delta,\ I_\delta(u) > 0 \right\} \cup \{0\}, \tag{2.3}$$

$$\mathcal{U}_\delta = \left\{ u \in H_0^1(\Omega) : J_\delta(u) < d_\delta,\ I_\delta(u) < 0 \right\}. \tag{2.4}$$

3 Main results

Throughout this paper, we make the following usual assumptions on the relaxation function g:

(G) $g : \mathbb{R}^+ \to \mathbb{R}^+$ belongs to $C^1(\mathbb{R}^+)$ and satisfies the conditions

(i) $g(0) \geq 0, \quad \ell := 1 - \int_0^\infty g(s)ds > 0, \quad g'(t) \leq 0,$

(ii) $\int_0^\infty g(s)ds < \frac{p(p-2)}{(p-1)^2},$

(iii) There exists a positive differentiable function $\xi(t)$ such that

$$g'(t) \leq -\xi(t)g(t), \quad \xi'(t) \leq 0, \quad \int_0^\infty \xi(t)dt = \infty, \ \forall t > 0.$$

Let us now give the definition of weak solutions to (1.1).

Definition 3.1. Let $0 < T \leq \infty$, a function u is called a weak solution of problem (1.1) on $\Omega \times (0,T)$ if $u \in L^\infty(0,T;H_0^1(\Omega))$ with $u_t \in L^2(0,T;L^2(\Omega))$ satisfies $u(x,0) = u_0(x) \in H_0^1(\Omega)$ and the equality

$$\langle u_t, \varphi \rangle + \langle \nabla u, \nabla \varphi \rangle - \int_0^t g(t-s) \langle \nabla u(\tau), \nabla u(s) \rangle \, ds = \left\langle |u|^{p-2} u \ln |u|, \varphi \right\rangle, \tag{3.1}$$

holds for a.e. $t \in (0,T)$ and any $\varphi \in H_0^1(\Omega)$.

Let u be a weak solution of problem (1.1), we define the total energy functional as follows

$$\begin{aligned} E(t) = &\frac{1}{2}\left(1 - \int_0^t g(\tau)d\tau\right) \|\nabla u(t)\|^2 + \frac{1}{2}(g \circ \nabla u)(t) \\ &- \frac{1}{p}\int_\Omega |u(t)|^p \ln |u(t)| dx + \frac{1}{p^2}\|u(t)\|_p^p, \end{aligned} \tag{3.2}$$

where

$$(g \circ \nabla u)(t) = \int_0^t g(t-s) \|\nabla u(t) - \nabla u(s)\|^2 \, ds.$$

By the Definition 3.1, $u \in L^\infty(0,T;H_0^1(\Omega))$ and $u_t \in L^2(0,T;L^2(\Omega))$. So $E(t)$ is well-define for a.e. $t \in [0,T)$. In addition, the next lemma shows that $E(t)$ is a non-increasing functional.

Lemma 3.2. *Let (G, (i)) hold. The energy functional $E(t)$ defined in (3.2) is nonincreasing and*

$$\frac{d}{dt}E(t) = -\|u_t(t)\|^2 + \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|^2 \leq 0. \tag{3.3}$$

Proof. By substituting $\varphi = u_t$ in (3.1), we get after some simple calculations that

$$\frac{d}{dt}E(t) = \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|^2 - \|u_t(t)\|^2.$$

Then, using the assumption (G, (i)), it follows that $E(t)$ is an non-increasing functional and satisfies the energy inequality

$$E(t) + \int_0^t \|u_t(s)\|^2 \, ds \leq E(0). \tag{3.4}$$

The proof is complete. □

We are now in the position to state the main theorems of this paper.

Theorem 3.3 (Global existence)**.** *Assume that* (1.2) *and (G, (i)) hold. Let* $u_0 \in H_0^1(\Omega)$ *and*

$$E(0) = \frac{1}{2}\|\nabla u_0\|^2 - \frac{1}{p}\int_\Omega |u_0|^p \ln|u_0|dx + \frac{1}{p^2}\|u_0\|_p^p < d_\delta, \quad I_\delta(u_0) > 0.$$

Then problem (1.1) *has a global weak solution* u *such that* $u \in L^\infty(0,\infty; H_0^1(\Omega))$ *with* $u_t \in L^2(0,\infty; L^2(\Omega))$.

Theorem 3.4 (Blow-up)**.** *Assume that* (1.2) *hold and* g *satisfies (G, (i), (ii)). Assume further that* $u_0 \in H_0^1(\Omega)$ *and*

$$E(0) = \frac{1}{2}\|\nabla u_0\|^2 - \frac{1}{p}\int_\Omega |u_0|^p \ln|u_0|dx + \frac{1}{p^2}\|u_0\|_p^p < d_\kappa, \quad I_\kappa(u_0) < 0,$$

where

$$0 < \kappa = \ell - \frac{1}{p(p-2)}\int_0^\infty g(s)ds. \tag{3.5}$$

Then the weak solution $u(t)$ *to* (1.1) *blows up in finite time and the lifespan time* T *satisfies*

$$T \le \frac{8\|u_0\|^2}{(p-2)^2(d_\kappa - E(0))}.$$

Furthermore, T *is bounded below by*

$$T \ge \int_{R(0)}^\infty \frac{1}{K_1 z^{p-1+\sigma} + K_2}dz, \tag{3.6}$$

for some $0 < \sigma < \frac{2(n-1)}{n-2} - p$, *where* $R(0) = \frac{1}{2}\|\nabla u_0\|^2$ *and*

$$K_1 = \frac{1}{2}(e\sigma)^{-2} S_{2(p-1+\sigma)}^{2(p-1+\sigma)}\left(2(p-1)^2\right)^{p-1+\sigma}, \quad K_2 = \frac{1}{2}(e(p-1))^{-2}|\Omega|.$$

Here $S_{2(p-1+\sigma)}$ *is the optimal embedding constants of* $H_0^1(\Omega) \hookrightarrow L^{2(p-1+\sigma)}(\Omega)$.

Theorem 3.5 (Decay estimate)**.** *Assume that* (1.2) *holds and* g *satisfies (G,(i), (iii)). Assume further that* $u_0 \in H_0^1(\Omega)$ *with* $u_0 \in \mathcal{W}_\delta$ $(0 < \delta \le \ell)$ *and*

$$E(0) < \left(\frac{\ell}{2\delta}\right)^{\frac{p}{p-2}} d_\delta.$$

Then solution $u(t)$ *to* (1.1) *decays exponentially.*

4 Proof of Theorem 3.3

Based on the Faedo–Galerkin method, this proof consists of three steps.

Step 1. Finite-dimensional approximations. Let $\{w_j\}$ be the orthogonal complete system of eigenfunctions of $-\Delta$ in $H_0^1(\Omega)$, which is orthonormal in $L^2(\Omega)$. We find the approximate solution of the problem (1.1) in the forms

$$u_m(t) = \sum_{j=1}^m c_{mj}(t)w_j, \tag{4.1}$$

where the coefficients functions c_{mj}, $1 \le j \le m$, satisfy the system of integro-differential equations

$$\langle u_{mt}, w_j \rangle + \langle \nabla u_m, \nabla w_j \rangle - \int_0^t g(t-s) \langle \nabla u_m(s), \nabla w_j \rangle \, ds = \left\langle |u_m|^{p-2} u_m \ln |u_m|, w_j \right\rangle, \tag{4.2}$$

and

$$u_m(0) = u_{0m} = \sum_{j=1}^{m} \alpha_{mj} w_j \longrightarrow u_0 \quad \text{strongly in} \quad H_0^1(\Omega). \tag{4.3}$$

It is obvious that for each m, there exists a solution u_m of the form (4.1) which satisfies (4.2) and (4.3) almost everywhere on $t \in [0, T_m]$, for some sufficiently small $T_m > 0$. In what follows, we present a brief proof that a solution of (4.2)–(4.3) of the form (4.1) exists. It is obvious that the system (4.2)–(4.3) can be rewritten in the vectorial form

$$c_m'(t) + A_m c_m(t) = A_m \int_0^t g(t-s) c_m(s) ds + \mathcal{F}(c_m(t)),$$

with the initial condition

$$c_m(0) = \alpha_m,$$

where

$$\begin{cases} c_m(t) = (c_{m1}(t), c_{m1}(t), \ldots, c_{m1}(t))^T, \ \alpha = (\alpha_{m1}, \alpha_{m2}, \ldots, \alpha_{mm})^T, \\ A_m = \left[\langle \nabla w_i, \nabla w_j \rangle\right]_{i,j=1}^{m}, \ \mathcal{F}(c_m(t)) = (\mathcal{F}_1(c_m(t)), \mathcal{F}_2(c_m(t)), \ldots, \mathcal{F}_m(c_m(t)))^T, \\ \mathcal{F}_j(c_m(t)) = \left\langle |u_m|^{p-2} u_m \ln |u_m|, w_j \right\rangle, \quad \forall j = \overline{1,m}, \end{cases}$$

which is also equivalent to the integral equation

$$c_m(t) = \alpha_m - \int_0^t A_m c_m(s) ds + \int_0^t A_m \int_0^s g(s-\tau) c_m(\tau) d\tau ds + \int_0^t \mathcal{F}(c_m(s)) ds. \tag{4.4}$$

By the Schauder theorem, the integral equation (4.4) has a solution $c_m(t)$ in a certain closed ball of the Banach space $C([0, T_m]; \mathbb{R}^m)$ with $T_m \in (0, T]$. Therefore, there exists $u_m(t)$ of the form (4.1) which satisfies (4.2)–(4.3) on $0 \le t \le T_m$.

Step 2. A priori estimate. Multiplying (4.2) by $c_{mj}'(t)$ and summing for j from 1 to m, we get

$$\langle u_{mt}, u_{mt} \rangle + \langle \nabla u_m, \nabla u_{mt} \rangle - \int_0^t g(t-s) \langle \nabla u_m(s), \nabla u_{mt} \rangle \, ds = \left\langle |u_m|^{p-2} u_m \ln |u_m|, u_{mt} \right\rangle. \tag{4.5}$$

Integrating (4.5) with respect to time variable on $[0, t]$, we have

$$E_m(t) + \int_0^t \|u_{mt}(s)\|^2 ds = E_m(0) - \frac{1}{2} \int_0^t g(s) \|\nabla u_m(s)\|^2 ds + \frac{1}{2} \int_0^t (g' \circ \nabla u_m)(s) ds, \tag{4.6}$$

where we have for $0 < \delta \le \ell$

$$\begin{aligned} E_m(t) &= \frac{1}{2} \left(1 - \int_0^t g(s) ds\right) \|\nabla u_m(t)\|^2 + \frac{1}{2} (g \circ \nabla u_m)(t) \\ &\quad - \frac{1}{p} \int_\Omega |u_m|^p \ln |u_m| dx + \frac{1}{p^2} \|u_m(t)\|_p^p \\ &\ge J_\delta(u_m(t)) + \frac{1}{2} (g \circ \nabla u_m)(t). \end{aligned} \tag{4.7}$$

From $E(0) < d_\delta$ and (4.3), we deduce that $E_m(0) < d_\delta$ for sufficiently large m. And then, we deduce from (4.6) and (4.7) that

$$\frac{1}{2}(g \circ \nabla u_m)(t) + J_\delta(u_m(t)) + \int_0^t \|u_{mt}(s)\|^2 ds < d_\delta, \quad 0 \le t \le T_m, \tag{4.8}$$

holds for sufficiently large m. Take note of $I_\delta(u_0) > 0$, we can conclude that $u_0 \in \mathcal{W}_\delta$. It implies from (4.3) that $u_m(0) \in \mathcal{W}_\delta$ for sufficiently large m. Now, we will show that $u_m(t) \in \mathcal{W}_\delta$ for any $t \in [0, T_m]$ and sufficiently large m. In fact, if not, there exists a $t_0 \in (0, T_m]$ and a sufficient large m such that $I_\delta(u_m(t_0)) = 0$ and $u_m(t_0) \neq 0$, then we get that $u_m(t_0) \in \mathcal{N}_\delta$. So we deduce from the definition of d_δ that $J_\delta(u_m(t_0)) \ge d_\delta$, which contradicts (4.8). Thus, $u_m(t) \in \mathcal{W}_\delta$ for any $t \in [0, T_m]$ and sufficient large m, which implies $I_\delta(u_m(t)) \ge 0$ for any $t \in [0, T_m]$ and sufficient large m.

Thanks to the definition of J_δ and $I_\delta(u_m(t)) \ge 0$, we deduce from (4.8) that

$$\begin{aligned} \frac{p-2}{2p}\delta \|\nabla u_m(t)\|^2 + \frac{1}{p^2}\|u_m(t)\|_p^p + \frac{1}{2}(g \circ \nabla u_m)(t) + \int_0^t \|u_{mt}(s)\|^2 ds < d_\delta, \\ 0 \le t \le T_m, \end{aligned} \tag{4.9}$$

From (4.9) we obtain

$$\begin{cases} \|\nabla u_m(t)\|^2 < \dfrac{2p}{(p-2)\delta} d_\delta, \ \|u_m(t)\|_p^p < p^2 d_\delta, \\ \displaystyle\int_0^t \|u_{mt}(t)\|^2 < d_\delta, \ (g \circ \nabla u_m)(t) < 2d_\delta. \end{cases} \tag{4.10}$$

So $T_m = \infty$. And hence $u_m(t) \in \mathcal{W}_\delta$ for $t \in [0, \infty)$ and (4.10) holds for $t \in [0, \infty)$.

On the other hand, by (4.10), we get

$$\begin{aligned} \int_\Omega |\rho_m(x,t)|^{p'} dx &= \int_{\Omega_1} |\rho_m(x,t)|^{p'} dx + \int_{\Omega_2} |\rho_m(x,t)|^{p'} dx \\ &\le (e(p-1))^{-p'} |\Omega_1| + (e\sigma)^{-p'} \|u_m\|_{p+p'\sigma}^{p+p'\sigma} \\ &\le (e(p-1))^{-p'} |\Omega_1| + (e\sigma)^{-p'} S_{p+p'\sigma}^{p+p'\sigma} \|\nabla u_m\|^{p+p'\sigma} \\ &\le (e(p-1))^{-p'} |\Omega_1| + (e\sigma)^{-p'} S_{p+p'\sigma}^{p+p'\sigma} \left(\frac{2pd_\delta}{(p-2)\delta}\right)^{\frac{p+p'\sigma}{2}} \equiv C_\delta, \end{aligned} \tag{4.11}$$

where $p' = \frac{p}{p-1}$, $0 < \sigma < \frac{1}{p'}\left(\frac{2n}{n-2} - p\right)$, $\rho_m(x,t) = |u_m(x,t)|^{p-1} \ln |u_m(x,t)|$,

$$\Omega_1 = \{x \in \Omega : |u_m(x,t)| \le 1\}, \quad \Omega_2 = \{x \in \Omega : |u_m(x,t)| \ge 1\},$$

and S_q is the best constant of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$.

Step 3. Passage to the limit. From (4.10) and (4.11), we deduce that for each $T > 0$, there exists a function $u(t)$ and the subsequences of $\{u_m\}$, still denoted by $\{u_m\}$ such that

$$\begin{cases} u_m \to u & \text{in } L^\infty(0,T; H_0^1(\Omega)) \text{ weakly*}, \\ u_m \to u & \text{in } L^2(0,T; H_0^1(\Omega)) \text{ weakly}, \\ u_m \to u & \text{in } L^\infty(0,T; L^p(\Omega)) \text{ weakly*}, \\ u_m \to u & \text{in } L^2(0,T; L^p(\Omega)) \text{ weakly}, \\ u_{mt} \to u_t & \text{in } L^2(0,T; L^2(\Omega)) \text{ weakly*}. \end{cases} \tag{4.12}$$

By Lemma 2.1, it follows from $(4.12)_{2,5}$ that there exists the existence of a subsequence still denoted by $\{u_m\}$, such that

$$u_m \to u \quad \text{strongly in } L^2(0,T;L^p(\Omega)) \quad \text{and} \quad u_m \to u \quad \text{a.e } (x,t) \in \Omega \times (0,T),$$

which yields

$$|u_m|^{p-2}u_m \ln|u_m| \to |u|^{p-2}u\ln|u| \quad \text{a.e } (x,t) \in \Omega \times (0,T). \tag{4.13}$$

From (4.11) and (4.13) by the Aubin–Lions Lemma, we deduce that

$$|u_m|^{p-2}u_m \ln|u_m| \to |u|^{p-2}u\ln|u| \quad \text{weakly* in } L^\infty\left(0,T;L^{p'}(\Omega)\right).$$

By using Lemma 2.1, it follows from $(4.12)_{2,5}$ that

$$u_m(0) \to u(0) \quad \text{weakly in } L^2(\Omega). \tag{4.14}$$

Passing to the limit in (4.2), by (4.3), (4.12), (4.13)–(4.14), we have u satisfying equation

$$\begin{cases} \langle u_t,\varphi\rangle + \langle \nabla u,\nabla\varphi\rangle - \int_0^t g(t-s)\langle \nabla u(s),\nabla\varphi\rangle\, ds = \left\langle |u|^{p-2}u\ln|u|,\varphi\right\rangle, \\ u(0) = u_0. \end{cases}$$

The proof is complete.

5 Proof of Theorem 3.4

We begin this section by the following useful lemma which is useful later on.

Lemma 5.1. *Under the assumptions of the Theorem 3.4 and let $u(t)$ be any weak solution of the problem (1.1) on $[0,T)$ where T is the maximum existence time. Then we possess*

$$d_\kappa \le \frac{p-2}{2p}\kappa\|\nabla u(t)\|^2 + \frac{1}{p^2}\|u(t)\|_p^p, \tag{5.1}$$

where κ is defined by (3.5).

Proof. Firstly, we show that $u(t) \in \mathcal{U}_\kappa$ for all $t \in [0,T)$. Indeed, if it is false, then there exists a $t_0 > 0$ such that $I_\kappa(u(t)) < 0$ for $t \in [0,t_0)$ and $I_\kappa(u(t_0)) = 0$. By Lemma 2.7, we have $\|\nabla u(t)\| > r_\kappa(\sigma) > 0$, for $t \in [0,t_0)$ and $\|\nabla u(t_0)\| \ge r_\kappa(\sigma) > 0$, which yields $u(t_0) \in \mathcal{N}_\kappa$. So by the definition of d_κ we get $J_\kappa(u(t_0)) \ge d_\kappa$, which contradicts to $J_\kappa(u(t_0)) \le E(t_0)) \le E(0) < d_\kappa$. Hence, we obtain $u(t) \in \mathcal{U}_\kappa$ for $t \in [0,T)$.

By Lemma 2.5 we imply that there is a unique $\lambda_1 < 1$ such that $I_\kappa(\lambda_1 u(t)) = 0$. We next define $j(\lambda) = J_\kappa(\lambda u) - \frac{1}{p}I_\kappa(\lambda u)$, for $\lambda > 0$. By direct calculation, we have that

$$j(\lambda) = \frac{\kappa(p-2)}{2p}\lambda^2\|\nabla u(t)\|^2 + \frac{\lambda^p}{p^2}\|u(t)\|_p^p.$$

Since $u(t) \in \mathcal{U}_\kappa$, by Lemma 2.7 we have

$$j'(\lambda) = \frac{\kappa(p-2)}{p}\lambda\|\nabla u(t)\|^2 + \frac{\lambda^{p-1}}{p}\|u(t)\|_p^p > \kappa(p-2)\lambda r_\kappa^2(\sigma) > 0.$$

Hence, $j(\lambda)$ is strictly increasing on $(0,\infty)$ which implies $j(1) > j(\lambda_1)$, that is

$$J_\kappa(u(t)) - \frac{1}{p}I_\kappa(u(t)) > J_\kappa(\lambda_1 u(t)) - \frac{1}{p}I_\kappa(\lambda_1 u(t)) = J_\kappa(\lambda_1 u) \ge d_\kappa.$$

The proof of lemma is complete. □

We now divide the proof of the Theorem 3.4 into two following steps:

Step 1: Blow-up in finite time and upper bound estimate of the blow-up time.

By contradiction, we assume that $u(t)$ exists globally and define the function

$$\theta(t) = \int_0^t \|u(s)\|^2 ds + (T-t)\|u_0\|^2 + b(t+T_0)^2, \quad t \in [0,T], \tag{5.2}$$

where b and T_0 are positive constants to be determined later. Then we have

$$\begin{aligned}\theta'(t) &= \|u(t)\|^2 - \|u_0\|^2 + 2b(t+T_0) = \int_0^t \frac{d}{dt}\|u(s)\|^2 ds + 2b(t+T_0) \\ &= 2\int_0^t \langle u_t(s), u(s)\rangle ds + 2b(t+T_0),\end{aligned} \tag{5.3}$$

and

$$\theta''(t) = 2\int_\Omega u(t)u_t(t)dx + 2b. \tag{5.4}$$

By using (1.1), we deduce from (5.4) that

$$\theta''(t) = -2\|\nabla u(t)\|^2 + 2\int_0^t g(t-s)\langle \nabla u(s), \nabla u(t)\rangle ds + 2\int_\Omega |u(t)|^p \ln|u(t)|dx + 2b. \tag{5.5}$$

On the other hand, by the Hölder inequality and the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\frac{1}{4}(\theta'(t))^2 &\le \left(\int_0^t \|u(s)\|^2 ds + b(t+T_0)^2\right)\left(\int_0^t \|u_t(s)\|^2 ds + b\right) \\ &\le \theta(t)\left(\int_0^t \|u_t(s)\|^2 ds + b\right),\end{aligned} \tag{5.6}$$

and by the Young inequality, one has

$$\begin{aligned}&2\int_0^t g(t-s)\langle \nabla u(s), \nabla u(t)\rangle ds \\ &\quad = 2\int_0^t g(s)ds\,\|\nabla u(t)\|^2 + 2\int_0^t g(t-s)\langle \nabla u(s) - \nabla u(t), \nabla u(t)\rangle ds \\ &\quad \ge \left(2 - \frac{1}{p}\right)\int_0^t g(s)ds\,\|\nabla u(t)\|^2 - p(g\circ\nabla u)(t).\end{aligned} \tag{5.7}$$

It follows from (5.2)–(5.7) that

$$\theta''(t)\theta(t) - \frac{p+2}{4}(\theta'(t))^2 \ge \theta(t)\zeta(t), \tag{5.8}$$

where $\zeta : [0,T] \to \mathbb{R}$ is the function defined by

$$\begin{aligned}\zeta(t) = -2\|\nabla u(t)\|^2 &+ \left(2-\frac{1}{p}\right)\left(\int_0^t g(s)ds\right)\|\nabla u(t)\|^2 - p(g\circ\nabla u)(t) \\ &+ 2\int_\Omega |u(t)|^p \ln|u(t)|dx - (p+2)\int_0^t \|u_t(s)\|^2 ds - pb.\end{aligned} \tag{5.9}$$

On the other hand, from (3.2) we have that

$$\begin{aligned}\int_\Omega |u(t)|^p \ln|u(t)|dx = &-pE(t) + \frac{p}{2}\left(1 - \int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|^2 \\ &+ \frac{p}{2}(g\circ\nabla u)(t) + \frac{1}{p}\|u(t)\|_p^p.\end{aligned} \tag{5.10}$$

And hence, (5.9) and (5.10) yield

$$\zeta(t) = -2pE(t) + \left[p-2-\left(p-2+\frac{1}{p}\right)\int_0^t g(s)ds\right]\|\nabla u(t)\|^2 + \frac{2}{p}\|u(t)\|_p^p - (p+2)\int_0^t \|u_t(s)\|^2 ds - pb. \tag{5.11}$$

By virtue of the energy inequality (3.4), we deduce from (5.11) that

$$\begin{aligned}\zeta(t) \geq & -2pE(0) + \left[p-2-\left(p-2+\frac{1}{p}\right)\int_0^t g(s)ds\right]\|\nabla u(t)\|^2 \\ & + \frac{2}{p}\|u(t)\|_p^p + (p-2)\int_0^t \|u_t(s)\|^2 ds - pb \\ \geq & \, 2p\left[\frac{p-2}{2p}\left(1-\int_0^t g(s)ds - \frac{1}{p(p-2)}\int_0^t g(s)ds\right)\|\nabla u(t)\|^2 + \frac{1}{p^2}\|u(t)\|_p^p - E(0) - \frac{b}{2}\right] \\ \geq & \, 2p\left[\frac{p-2}{2p}\kappa\|\nabla u(t)\|^2 + \frac{1}{p^2}\|u(t)\|_p^p - E(0) - \frac{b}{2}\right],\end{aligned} \tag{5.12}$$

where κ is a constant given by

$$0 < \kappa = \ell - \frac{1}{p(p-2)}\int_0^\infty g(s)ds \leq \ell$$

thanks to $p > 2$ and $\ell = 1 - \int_0^\infty g(s)ds$.

By virtue of Lemma 5.1, it follows from (5.12) that

$$\zeta(t) \geq 2p\left(d_k - E(0) - \frac{b}{2}\right).$$

Since $E(0) < d_\kappa$, choosing b small enough such that

$$0 < b \leq 2\left(d_\kappa - E(0)\right), \tag{5.13}$$

we get

$$\zeta(t) > \rho > 0. \tag{5.14}$$

Combining (5.8) and (5.14), we arrive at

$$\theta''(t)\theta(t) - \frac{p+2}{4}\left(\theta'(t)\right)^2 \geq \rho\theta(t) \geq 0.$$

Applying Lemma 2.4 with $\gamma = \frac{p-2}{4}$ we have that $\theta(t) \to \infty$ for $t \to t^* < \infty$, which contradicts $T = \infty$. And hence $u(t)$ blows up at finite time T. Moreover, we have also

$$T \leq \frac{4\theta(0)}{(p-2)\theta'(0)} = \frac{4\left(T\|u_0\|^2 + bT_0^2\right)}{2(p-2)bT_0} = \frac{2\|u_0\|^2}{(p-2)bT_0}T + \frac{2T_0}{p-2}.$$

By choosing $T_0 \in \left(\frac{2\|u_0\|^2}{(p-2)b}, \infty\right)$, we get

$$T \leq \frac{2bT_0^2}{(p-2)bT_0 - 2\|u_0\|^2}.$$

Since b satisfies (5.13), by minimizing the above inequality for $T_0 > \frac{2\|u_0\|^2}{(p-2)b}$, we arrive at

$$T \leq \frac{8\|u_0\|^2}{(p-2)^2(d_\kappa - E(0))}.$$

Step 2: Lower bound estimate of the blow up time.

By Step 1 we know that $\lim_{t\to T^-}\|u(t)\|^2 = \infty$ which implies

$$\lim_{t\to T^-}\|\nabla u(t)\|^2 = \infty, \tag{5.15}$$

thanks to the continuous embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$.

Let us now define an auxiliary function

$$\begin{aligned} R(t) &= \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u(t)\|^2 + \frac{1}{2}(g\circ\nabla u)(t) \\ &= E(t) + \frac{1}{p}\int_\Omega |u(t)|^p \ln|u(t)|dx - \frac{1}{p^2}\|u(t)\|_p^p. \end{aligned}$$

Then by assumption (G, (ii)), we have

$$\frac{1}{2(p-1)^2}\|\nabla u(t)\|^2 \leq \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u(t)\|^2 + \frac{1}{2}(g\circ\nabla u)(t) = R(t).$$

which implies $\lim_{t\to T^-} R(t) = \infty$ thanks to (5.15).

Recalling the Lemma 3.2, we have

$$\begin{aligned} R'(t) &= E'(t) + \int_\Omega |u(t)|^{p-2}u(t)u_t(t)\ln|u(t)|dx \\ &\leq -\|u_t(t)\|^2 + \int_\Omega |u(t)|^{p-2}u(t)u_t(t)\ln|u(t)|dx. \end{aligned}$$

Let us divide Ω into two parts as follows:

$$\Omega_1 = \{x\in\Omega : |u(x,t)| \leq 1\} \quad \text{and} \quad \Omega_2 = \{x\in\Omega : |u(x,t)| \geq 1\}.$$

Applying Lemma 2.3, Hölder's inequality, Young's inequality, we reach

$$\begin{aligned} R'(t) &\leq -\|u_t\|^2 + \int_\Omega |u|^{p-2}uu_t\ln|u|dx \\ &= -\|u_t\|^2 + \int_{\Omega_1}|u|^{p-2}uu_t\ln|u|dx + \int_{\Omega_2}|u|^{p-2}uu_t\ln|u|dx \\ &\leq -\|u_t\|^2 + (e(p-1))^{-1}\int_{\Omega_1}|u_t|dx + (e\sigma)^{-1}\int_{\Omega_2}|u|^{p-1+\sigma}|u_t|dx \\ &\leq -\|u_t\|^2 + (e(p-1))^{-1}|\Omega_1|^{\frac{1}{2}}\|u_t\| + (e\sigma)^{-1}\|u\|_{2(p-1+\sigma)}^{p-1+\sigma}\|u_t\| \\ &\leq -\|u_t\|^2 + \frac{1}{2}(e(p-1))^{-2}|\Omega_1| + \frac{1}{2}\|u_t\|^2 + \frac{1}{2}(e\sigma)^{-2}\|u\|_{2(p-1+\sigma)}^{2(p-1+\sigma)} + \frac{1}{2}\|u_t\|^2 \\ &\leq \frac{1}{2}(e(p-1))^{-2}|\Omega| + \frac{1}{2}(e\sigma)^{-2}\|u\|_{2(p-1+\sigma)}^{2(p-1+\sigma)}. \end{aligned} \tag{5.16}$$

Here, for simplicity, we write u instead of $u(t)$.

By $2 < 2p-2 < \frac{2n}{n-2}$, there exists $\sigma > 0$ such that $2 < 2(p-1+\sigma) < \frac{2n}{n-2}$. Using the embedding $H_0^1(\Omega) \hookrightarrow L^{2(p-1+\sigma)}(\Omega)$, we deduce from (5.16) that

$$\begin{aligned} R'(t) &\le \frac{1}{2}\left(e(p-1)\right)^{-2}|\Omega| + \frac{1}{2}(e\sigma)^{-2} S_{2(p-1+\sigma)}^{2(p-1+\sigma)} \|\nabla u(t)\|^{2(p-1+\sigma)} \\ &\le K_1 R^{p-1+\sigma}(t) + K_2, \end{aligned} \tag{5.17}$$

where S_q is the optimal constant of embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$, and

$$K_1 = \frac{1}{2}(e\sigma)^{-2} S_{2(p-1+\sigma)}^{2(p-1+\sigma)} \left(2(p-1)^2\right)^{p-1+\sigma}, \quad K_2 = \frac{1}{2}\left(e(p-1)\right)^{-2}|\Omega|.$$

Integrating (5.17) from 0 to t, we get

$$\int_{R(0)}^{R(t)} \frac{1}{K_1 z^{p-1+\sigma} + K_2} dz \le t,$$

combining with the fact $\lim_{t\to T^-} R(t) = \infty$ we obtain (3.6). Thus the proof is complete.

6 Proof of Theorem 3.5

We begin with the following lemma which is helpful to the proof of Theorem 3.5.

Lemma 6.1. *Under the assumptions of the Theorem 3.3. For any $0 < \delta \le \ell$, we have that*

$$I_\delta(u(t)) \ge \left[1 - \left(\frac{d_\delta}{E(0)}\right)^{\frac{2-p}{p}}\right] \delta \|\nabla u(t)\|^2.$$

Proof. It is first noticed that $u_0 \in \mathcal{W}_\delta$ thanks to $E(0) < d_\delta$ and $I(u_0) > 0$. By using the similar method as in the proof of Lemma 5.1, we can show that $u(t) \in \mathcal{W}_\delta$ for $t \ge 0$. Taking this into account and using the Lemma 2.5 (iii), we imply that there is a constant $\lambda_1 > 1$ such that $I_\delta(\lambda_1 u(t)) = 0$.

On the other hand, from the definition of I_δ, we have

$$\begin{aligned} I_\delta(\lambda_1 u(t)) &= \delta(\lambda_1)^2 \|\nabla u(t)\|^2 - (\lambda_1)^p \int_\Omega |u(t)|^p \ln|u(t)| dx - (\lambda_1)^p \ln\lambda_1 \|u(t)\|_p^p \\ &= \left((\lambda_1)^2 - (\lambda_1)^p\right)\delta\|\nabla u(t)\|^2 + (\lambda_1)^p I_\delta(u(t)) - (\lambda_1)^p \ln\lambda_1 \|u(t)\|_p^p, \end{aligned}$$

which implies, thanks to $I_\delta(\lambda_1 u(t)) = 0$ and $\lambda_1 > 1$, that

$$I_\delta(u(t)) \ge \left[1 - (\lambda_1)^{2-p}\right]\delta\|\nabla u(t)\|^2 + \ln\lambda_1 \|u(t)\|_p^p \ge \left[1 - (\lambda_1)^{2-p}\right]\delta\|\nabla u(t)\|^2. \tag{6.1}$$

To end the proof it remains to estimate λ_1. By variational characterization of d_δ, we have

$$\begin{aligned} d_\delta \le J_\delta(\lambda_1 u(t)) &= \frac{1}{p} I_\delta(\lambda_1 u(t)) + \delta\left(\frac{1}{2} - \frac{1}{p}\right)(\lambda_1)^2\|\nabla u(t)\|^2 + \frac{(\lambda_1)^p}{p^2}\|u(t)\|_p^p \\ &\le \left[\delta\left(\frac{1}{2} - \frac{1}{p}\right)\|\nabla u(t)\|^2 + \frac{1}{p^2}\|u(t)\|_p^p\right](\lambda_1)^p. \end{aligned} \tag{6.2}$$

On the other hand, by the non-increasing property of functional energy $E(t)$, we have that

$$E(0) \geq E(t) \geq J_\delta(u(t)) = \frac{1}{p} I_\delta(u(t)) + \delta\left(\frac{1}{2} - \frac{1}{p}\right)\|\nabla u(t)\|^2 + \frac{1}{p^2}\|u(t)\|_p^p$$
$$> \delta\left(\frac{1}{2} - \frac{1}{p}\right)\|\nabla u(t)\|^2 + \frac{1}{p^2}\|u(t)\|_p^p. \tag{6.3}$$

From (6.2)–(6.3), we deduce that

$$\lambda_1 \geq \left(\frac{d_\delta}{E(0)}\right)^{1/p} > 1. \tag{6.4}$$

The proof follows from (6.1) and (6.4). □

As a consequence of this lemma, we get the following estimates.

Lemma 6.2. *Under the assumptions of the Theorem 3.3. For any $0 < \delta \leq \ell$, we possess*

$$\int_\Omega |u(t)|^p \ln|u(t)| dx \leq \delta\left(\frac{d_\delta}{E(0)}\right)^{\frac{2-p}{p}}\|\nabla u(t)\|^2 \quad \text{and} \quad \|u(t)\|_p^p \leq C(p, d_\delta)\|\nabla u(t)\|^2, \tag{6.5}$$

where $C(p, d_\delta)$ is the constant given by

$$C(p, d_\delta) = S_p^p\left[\frac{p\delta^{-1}d_\delta}{1 - \left(\frac{d_\delta}{E(0)}\right)^{\frac{2-p}{p}}}\right]^{\frac{p-2}{2}}.$$

Here S_p is the best constant in the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$.

Proof. The first estimate in (6.5) follows from the Lemma 6.1 and the identity

$$\int_\Omega |u(t)|^p \ln|u(t)| dx = \delta\|\nabla u(t)\|^2 - I_\delta(u(t))$$
$$\leq \delta\|\nabla u(t)\|^2 - \left[1 - \left(\frac{d_\delta}{E(0)}\right)^{\frac{2-p}{p}}\right]\delta\|\nabla u(t)\|^2 = \delta\left(\frac{d_\delta}{E(0)}\right)^{\frac{2-p}{p}}\|\nabla u(t)\|^2,$$

and since $2 < p < \frac{2(n-1)}{n-2}$, the second one follows from the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ and the Lemma 6.1

$$\|u(t)\|_p^p \leq S_p^p\|\nabla u(t)\|^p \leq S_p^p\left[\frac{p\delta^{-1}d_\delta}{1 - \left(\frac{d_\delta}{E(0)}\right)^{\frac{2-p}{p}}}\right]^{\frac{p-2}{2}}\|\nabla u(t)\|^2 \equiv C(p, d_\delta)\|\nabla u(t)\|^2,$$

where S_p is the best constant in the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$. □

For the proof of Theorem 3.5, we define the following auxiliary functional

$$L(t) = E(t) + \varepsilon\rho(t),$$

where ρ is given by

$$\rho(t) = \frac{1}{2}\xi(t)\|u(t)\|^2.$$

The next lemma tells us that $E(t)$ and $L(t)$ are equivalent functions.

Lemma 6.3. *For ε_1 and ε_2 small enough, we have*

$$\alpha_1 E(t) \leq L(t) \leq \alpha_2 E(t)$$

holds for two positive constants α_1 and α_2.

Proof. By virtue of Lemma 6.1 and the definition of $E(t)$, we have that

$$E(t) \geq \frac{\delta}{p}\left[1-\left(\frac{d_\delta}{E(0)}\right)^{\frac{2-p}{p}}\right]\|\nabla u(t)\|^2.$$

Taking this into account, we deduce from the definition of $\rho(t)$ that

$$|\rho(t)| \leq \frac{S_2^2}{2}\xi(t)\|\nabla u(t)\|^2 \leq \frac{pS_2^2}{2\delta}\left[1-\left(\frac{d_\delta}{E(0)}\right)^{\frac{2-p}{p}}\right]^{-1}\xi(t)E(t),$$

where S_2 is the optimal constant in the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$.

From (G, iii) we have $\xi(t) \leq \xi(0) \leq M$ for some constant $M > 0$. Combining with the above estimate to obtain

$$|L(t) - E(t)| \leq \varepsilon|\rho(t)| \leq \varepsilon C(M)E(t),$$

that is

$$(1-\varepsilon C(M))\,E(t) \leq L(t) \leq (1+\varepsilon C(M))\,E(t).$$

By choosing ε small such that $0 < \varepsilon < 1/C(M)$ we claim the lemma. □

The next lemma allow us to estimate $\rho'(t)$.

Lemma 6.4. *Let (G, (i, iii)) hold. Then we have that*

$$\rho'(t) \leq -\frac{\ell}{2}\xi(t)\|\nabla u(t)\|^2 + \xi(t)\int_\Omega |u(t)|^p \ln|u(t)|dx + \frac{1-\ell}{2\ell}\xi(t)\,(g\circ\nabla u)\,(t).$$

Proof. By using the differential equation in (1.1), we easily see that

$$\int_\Omega u_t(t)u(t)dx = -\|\nabla u(t)\|^2 + \int_\Omega |u(t)|^p \ln|u(t)|dx + \int_0^t g\,(t-s)\,\langle\nabla u(s),\nabla u(t)\rangle\,ds.$$

By using the Hölder and Young inequalities, we obtain for any $\eta > 0$

$$\begin{aligned}
&\int_0^t g\,(t-s)\,\langle\nabla u(s),\nabla u(t)\rangle\,ds\\
&\quad= \int_0^t g\,(t-s)\,\langle\nabla u(s)-\nabla u(t),\nabla u(t)\rangle\,ds + \left(\int_0^t g\,(s)\,ds\right)\|\nabla u(t)\|^2\\
&\quad\leq \frac{1}{2\eta}(g\circ\nabla u)(t) + \left(1+\frac{\eta}{2}\right)\left(\int_0^t g(s)ds\right)\|\nabla u(t)\|^2.
\end{aligned}$$

And hence, we arrive at

$$\int_\Omega u_t(t)u(t)dx \leq -\left[1-\frac{(1-\ell)(2+\eta)}{2}\right]\|\nabla u(t)\|^2 + \int_\Omega |u(t)|^p\ln|u(t)|dx + \frac{1}{2\eta}\,(g\circ\nabla u)\,(t).$$

By assumption (G,iii) and definition of $\rho(t)$, we deduce that

$$\begin{aligned}\rho'(t) &= \frac{1}{2}\xi'(t)\left\|u(t)\right\|^2 + \xi(t)\int_\Omega u_t(t)u(t)dx \\ &\leq -\left[1 - \frac{(1-\ell)(2+\eta)}{2}\right]\xi(t)\left\|\nabla u(t)\right\|^2 \\ &\quad + \xi(t)\int_\Omega |u(t)|^p \ln|u(t)|dx + \frac{1}{2\eta}\xi(t)\left(g\circ\nabla u\right)(t).\end{aligned}$$

Choosing $\eta = \frac{\ell}{1-\ell}$, we obtain

$$\rho'(t) \leq -\frac{\ell}{2}\xi(t)\left\|\nabla u(t)\right\|^2 + \xi(t)\int_\Omega |u(t)|^p \ln|u(t)|dx + \frac{1-\ell}{2\ell}\xi(t)\left(g\circ\nabla u\right)(t).$$

The proof is complete. □

We are now ready to give the proof of Theorem 3.5.

Proof of Theorem 3.5. Taking into account (3.3), we deduce from Lemma 6.4 that

$$\begin{aligned}L'(t) &= E'(t) + \varepsilon\rho'(t) \\ &\leq -\left\|u_t(t)\right\|^2 - \varepsilon\frac{\ell}{2}\xi(t)\left\|\nabla u(t)\right\|^2 + \varepsilon\xi(t)\int_\Omega |u(t)|^p \ln|u(t)|dx \\ &\quad + \frac{1}{2}\left(g'\circ\nabla u\right)(t) + \varepsilon\frac{1-\ell}{2\ell}\xi(t)\left(g\circ\nabla u\right)(t).\end{aligned} \tag{6.6}$$

By (G,iii) we have $\left(g'\circ\nabla u\right)(t) \leq -\xi(t)\left(g\circ\nabla u\right)(t)$. Using (3.2), (6.6) and Lemma 6.2, we have

$$\begin{aligned}L'(t) \leq &-\varepsilon\Lambda\xi(t)E(t) + \frac{\varepsilon\Lambda}{2}\xi(t)\left(1-\int_0^t g(s)ds\right)\left\|\nabla u(t)\right\|^2 + \frac{\varepsilon\Lambda}{2}\xi(t)\left(g\circ\nabla u\right)(t) \\ &-\frac{\varepsilon\Lambda}{p}\xi(t)\int_\Omega |u(t)|^p \ln|u(t)|dx + \frac{\varepsilon\Lambda}{p^2}\xi(t)\left\|u(t)\right\|_p^p - \left\|u_t(t)\right\|^2 \\ &-\frac{\varepsilon\ell}{2}\xi(t)\left\|\nabla u(t)\right\|^2 + \varepsilon\xi(t)\int_\Omega |u(t)|^p \ln|u(t)|dx - \frac{1}{2}\left(1-\varepsilon\frac{1-\ell}{\ell}\right)\xi(t)\left(g\circ\nabla u\right)(t) \\ \leq &-\varepsilon\Lambda\xi(t)E(t) - \varepsilon\left(\frac{\ell}{2} - \frac{\Lambda\ell}{2} - \frac{\Lambda C(p,d_\delta)}{p^2}\right)\xi(t)\left\|\nabla u(t)\right\|^2 \\ &+\varepsilon\left(1-\frac{\Lambda}{p}\right)\xi(t)\int_\Omega |u(t)|^p \ln|u(t)|dx - \frac{1}{2}\left(1-\varepsilon\frac{1-\ell}{\ell} - \varepsilon\Lambda\right)\xi(t)\left(g\circ\nabla u\right)(t),\end{aligned}$$

for any $\Lambda > 0$. By choosing $\varepsilon > 0$ and $\Lambda < p$ small enough such that

$$1-\varepsilon\frac{1-\ell}{\ell} > 0 \quad \text{and} \quad 1-\varepsilon\frac{1-\ell}{\ell} - \varepsilon\Lambda > 0$$

we obtain

$$\begin{aligned}L'(t) \leq &-\varepsilon\Lambda\xi(t)E(t) - \varepsilon\left(\frac{\ell}{2} - \frac{\Lambda\ell}{2} - \frac{\Lambda C(p,d_\delta)}{p^2}\right)\xi(t)\left\|\nabla u(t)\right\|^2 \\ &+\varepsilon\delta\left(1-\frac{\Lambda}{p}\right)\left(\frac{d_\delta}{E(0)}\right)^{\frac{2-p}{p}}\xi(t)\left\|\nabla u(t)\right\|^2 \\ \leq &-\varepsilon\Lambda\xi(t)E(t) - \varepsilon\left(\frac{\ell}{2} - \delta\left(\frac{d_\delta}{E(0)}\right)^{\frac{2-p}{p}} - \frac{\Lambda\ell}{2} - \frac{\Lambda C(p,d_\delta)}{p^2}\right)\xi(t)\left\|\nabla u(t)\right\|^2.\end{aligned}$$

Since $E(0) < \left(\frac{\ell}{2\delta}\right)^{\frac{p}{p-2}} d_\delta$, we can pick $0 < \Lambda < p$ such that

$$\frac{\ell}{2} - \delta\left(\frac{d_\delta}{E(0)}\right)^{\frac{2-p}{p}} - \frac{\Lambda\ell}{2} - \frac{\Lambda C(p, d_\delta)}{p^2} > 0.$$

Therefore, we get

$$L'(t) \le -\varepsilon\Lambda\xi(t)E(t) \le -\frac{\varepsilon\Lambda}{\alpha_2}\xi(t)L(t), \quad \forall t \ge t_0,$$

which implies

$$L(t) \le L(0)e^{-\frac{\varepsilon\Lambda}{\alpha_2}\int_0^t \xi(s)ds}, \quad \forall t \ge t_0.$$

This completes the proof. □

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Invariant measures and random attractors of stochastic delay differential equations in Hilbert space

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Abstract. This paper is devoted to a general stochastic delay differential equation with infinite-dimensional diffusions in a Hilbert space. We not only investigate the existence of invariant measures with either Wiener process or Lévy jump process, but also obtain the existence of a pullback attractor under Wiener process. In particular, we prove the existence of a non-trivial stationary solution which is exponentially stable and is generated by the composition of a random variable and the Wiener shift. At last, examples of reaction-diffusion equations with delay and noise are provided to illustrate our results.

Keywords: random dynamical system, delay, invariant measure, random attractor.

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1 Introduction

Delay differential equations arise from evolution phenomena in physical process and biological systems (see e.g. [19, 21, 25]), in which time-delay is used for mathematical modelling to describe the dynamical influence from the past. Recently, the effect of noise on such functional differential equations is increasingly a focus of investigation, in particular, in the combined influence of noise and delay in dynamical systems (see e.g. [5, 6, 13, 35, 37]). In this paper, we consider the following stochastic delay differential equation in a separable Hilbert space H:

$$\begin{cases} \mathrm{d}X(t) = [AX(t) + F(X_t)]\mathrm{d}t + G(X_t)\mathrm{d}Z(t), \quad t > 0, \\ X(t) = \varphi(t), \quad t \in [-\tau, 0], \end{cases} \tag{1.1}$$

where A: $\mathrm{Dom}(A) \subseteq H \to H$ is the infinitesimal generator of a semigroup, $X_t(s) = X(t+s)$ for $s \in [-\tau, 0]$ and $t \geq 0$. Here, $\mathrm{Dom}(A)$ denotes the domain of A and is a Banach space under the usual graph norm. Let $\mathcal{L} = L^2([-\tau, 0], H)$, and $\|\cdot\|$, $\|\cdot\|_{\mathcal{L}}$ denote the norms in H and $\mathcal{L}$, respectively. For a process $X(t) \in H$, we denote by $\{X_t : t \geq 0\}$ the segment process, which takes values in $\mathcal{L}$ for each t. $Z = \{Z(t), \mathcal{F}_t, t \geq 0\}$ could be an abstract Q-Wiener process or Lévy jump process with values in some separable Hilbert space $\mathcal{U}$, and

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$\varphi = \{\varphi(t) : t \in [-\tau, 0]\}$ is a given real-valued stochastic process, both defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where filtration $\{\mathcal{F}_t : t \geq 0\}$ is the $\mathbb{P}$-completion of the Borel σ-algebra on Ω.

For stochastic delay differential equations, there has been a rather comprehensive mathematical literature on both theories and applications. The existence of invariant measures is well studied in both finite and infinite dimensions by using Krylov–Bogoliubov theorem (see e.g. [7,17,18]). Scheutzow [35] formulated a sufficient condition ensuring the existence of an invariant probability measure with additive noise. For a similar approach and connections to stochastic partial differential equations, see Bakhtin and Mattingly [4]. For stochastic delay differential equations driven by Brownian motion, Mohammed [31] investigated the existence and uniqueness of strong or weak solutions under random functional Lipschitz conditions, Mao [30] discussed the method of steps, which provides a unique solution without a regular dependence of the coefficients on values in the past, Liptser and Shiryaev [26] considered weak solutions, Itô and Nisio [23] investigated the existence of weak solutions for equations with finite and infinite delay, Butkovsky and Scheutzow [8] established a general sufficient conditions ensuring the existence of an invariant measure for stochastic functional differential equations and exponential or subexponential convergence to the equilibrium. For stochastic delay differential equations driven by a Lévy process, Gushchin and Küchler [20] established some necessary and sufficient conditions ensuring the existence and uniqueness of stationary solutions, Reiß, Riedle, and van Gaans [32] proved that the segment process is eventually Feller, but in general not eventually strong Feller on the Skorokhod space, and also investigated the existence of an invariant measure by proving the tightness of the segments using semimartingale characteristics and the Krylov-Bogoliubov method. Existence and uniqueness of global solutions have been established under local Lipschitz and linear growth conditions (see e.g. [30,40]) or weak one-sided local Lipschitz (or monotonicity) conditions. Recently, Liu [28] considered stationary distributions of a class of second-order stochastic delay evolution equations driven by Wiener process or Lévy jump process in Hilbert space. In this paper we shall prove the existence of an invariant measure for (1.1) without boundedness conditions on the diffusion coefficient. Note that the segment process takes values in the infinite dimensional space $\mathcal{L}$, boundedness in probability does not generally imply tightness. In this case, one usually uses compactness of the orbits of the underlying deterministic equation to obtain tightness. However, such a compactness property does not hold for functional differential equation (1.1). For more details, see [7]. In this work, we will study the existence of invariant measures of (1.1) by applying the Krylov–Bogoliubov method.

A criterion for the existence of random attractors for random dynamical systems is established by Crauel and Flandoli [14], who also obtained the invariant Markov measures supported by the random attractor. Caraballo, Kloeden and Keal [10] proved the existence of random attractors of an ordinary differential equation with a random stationary delay. Kloeden and Lorenz [24] pointed out that the classical theory of pathwise random dynamical systems with a skew product (see e.g. [3]) does not apply to nonlocal dynamics such as when the dynamics of a sample path depends on other sample paths through an expectation or when the evolution of random sets depends on nonlocal properties such as the diameter of the sets. In [24], Kloeden and Lorenz showed that such nonlocal random dynamics can be characterized by a deterministic two-parameter process from the theory of nonautonomous dynamical systems acting on a state space of random variables or random sets with the mean-square topology and provided a definition of mean-square random dynamical systems and their attractors. Wu and Kloeden [39] investigated the existence of a random attractor for a mean-square random dynamical system (MS-RDS) generated by a stochastic delay differential equation with random delay for which the drift term is dominated by a nondelay component satisfying a one-sided dissipative Lipschitz condition. The exponential stability of trivial stationary solutions for stochastic partial differential equations has been extensively analyzed (see e.g. [12, 22, 29]). Caraballo, Kloeden and Schmalfuß [11] obtained the existence of a nontrivial stationary solution and a random fixed point which is exponentially stable. In this paper, we shall generalize the relevant results of Caraballo, Kloeden and Schmalfuß [11] to such a stochastic evolution equation with delay as (1.1). In particular, we shall prove the existence of a random fixed point, which generates the exponentially stable stationary solution of (1.1). Moreover, this stationary solution attracts bounded sets of initial conditions.

In this paper, we first establish a non-autonomous random dynamical system generated by equation (1.1). Then we show the existence of an invariant measure of (1.1) driven by Wiener process. In particular, we obtain a random pullback attractor consisting of a single point which is exponentially stable. Next, the existence of invariant measures of (1.1) driven by Lévy jump process is obtained by using Lévy–Itô decomposition formula. Finally, we apply our results to reaction-diffusion equations with noise and delay.

2 Preliminaries

Throughout this paper, we always assume that H is a separable Hilbert space, and there exists a Gelfand triplet $V \subset H \subset V'$ of separable Hilbert spaces, where V' denotes the dual of V and $V = \mathrm{Dom}(A^{\frac{1}{2}})$ (see page 55 of [38] for more details). The inner product in H is denoted by $\langle \cdot, \cdot \rangle$, and the duality mapping between V' and V by $\langle \cdot, \cdot \rangle_V$. We denote by $a_1 > 0$ the constant of the injection $V \subset H$, i.e.,

$$a_1 \|u\|^2 \leq \|u\|_V^2 \quad \text{for } u \in V,$$

and let $-A : V \to V'$ be a positive, linear and continuous operator for which there exists an $a_2 > 0$ such that

$$\langle -Au, u \rangle_V \geq a_2 \|u\|_V^2 \quad \text{for all } u \in V.$$

It is well known (see, for instance, [6, 9, 15]) that A is the generator of a strongly continuous semigroup $\Phi(t) = e^{tA}$ on H satisfying that

$$\| \exp\{tA\} \|_{\mathscr{L}(H)} \leq e^{-\lambda t}, \tag{2.1}$$

where $\lambda = a_1 a_2 > 0$ and $\mathscr{L}(H)$ is a space of bounded linear operators on H.

For any $\varphi \in L^2([-\tau, 0], H)$, the mild solution $X(t, \varphi)$ of (1.1) with the intimal data φ satisfies

$$\begin{cases} X(t,\varphi) = \Phi(t)\varphi(0) + \displaystyle\int_0^t \Phi(t-s) F(X_s(\varphi)) \mathrm{d}s \\ \qquad\qquad + \displaystyle\int_0^t \Phi(t-s) G(X_s(\varphi)) \mathrm{d}Z(s), \qquad t \geq 0, \\ X(t,\varphi) = \varphi(t), \qquad t \in [-\tau, 0], \end{cases} \tag{2.2}$$

where $X_t(\varphi)$ represents $X_t(\varphi)(s) = X(t+s, \varphi)$ for $s \in [-\tau, 0]$ and $t \geq 0$.

Definition 2.1. A measure μ is called an invariant measure for (1.1) if

$$\mu(f) = \mu(\mathbb{P}_t f), \qquad t \geq 0,$$

where

$$\mu(f) = \int_{\mathcal{L}} f(\phi) \mu(d\phi) \quad \text{and} \quad \mathbb{P}_t f(\phi) = \mathbb{E} f(X_t(\phi))$$

for $f \in C_b(\mathcal{L})$, where $\mathbb{P}_t$ is called the transition operator of (1.1) and $C_b(\mathcal{L})$ denotes the set of all bounded and continuous real-valued functions on $\mathcal{L}$. Let $\mu_{X_t(\phi)}$ be the distribution of $X_t(\phi)$, $t \geq 0$. If an $\mathcal{F}_0$-measurable $\phi \in L^2(\Omega, \mathcal{L})$ is such that $\mu_{X_t(\phi)} = \mu_\phi$ for all $t \geq 0$, then μ_ϕ is called a stationary distribution of (1.1) and $X(t, \phi)$ is then called a stationary solution.

It follows from the above definition that an invariant measure μ is a stationary distribution of (1.1) if and only if

$$\int_{\mathcal{L}} \|\phi\|_{\mathcal{L}}^2 \mu(\mathrm{d}\phi) < \infty,$$

when $\mathcal{F}_0$ is assumed to be rich enough to allow the existence of an $\mathcal{F}_0$-measurable random variable with distribution μ.

Definition 2.2. Denote by $\mathbb{P}(\mathcal{L})$ the set of Borel probability measures on $\mathcal{L}$ endowed with the topology of weak convergence of measures. For $\mu_1, \mu_2 \in \mathbb{P}(\mathcal{L})$ define a metric on $\mathbb{P}(\mathcal{L})$ by

$$d(\mu_1, \mu_2) = \sup_{f \in \mathcal{M}} \left| \int_{\mathcal{L}} f(\phi)\mu_1(\mathrm{d}\phi) - \int_{\mathcal{L}} f(\phi)\mu_2(\mathrm{d}\phi) \right|,$$

where

$$\mathcal{M} = \{f \in C(\mathcal{L}, \mathbb{R}) : |f(\phi) - f(\psi)| \leq \|\phi - \psi\|_{\mathcal{L}} \text{ for all } \phi, \psi \in \mathcal{L} \text{ and } |f(\cdot)| \leq 1\}.$$

It is well known that $\mathbb{P}(\mathcal{L})$ is complete under the metric $d(\cdot, \cdot)$ (see [16, Theorem 2.4.9]).

In order to show the existence of an invariant measure, we consider the segments of a solution. In contrast to the scalar solution process, the process of segment $\{X_t(\phi) : t \geq 0\}$ is a Markov process [17, 18]. It is shown that the segment process is also Feller and there exists a solution of which the segments are tight (see, for example, [17] for more details). Then we apply the Krylov–Bogoliubov method. In fact, we have the following result.

Lemma 2.3. *Suppose that for any bounded subset U of $\mathcal{L}$,*

(i) $\lim_{t\to\infty} \sup_{\phi,\psi \in U} \mathbb{E}\|X_t(\phi) - X_t(\psi)\|_{\mathcal{L}}^2 = 0$;

(ii) $\sup_{t \geq 0} \sup_{\phi \in U} \mathbb{E}\|X_t(\phi)\|_{\mathcal{L}}^2 < \infty$.

Then, for any initial condition $\phi \in \mathcal{L}$, the solution of equation (1.1) *converges to an invariant measure.*

Proof. It suffices to show that for any initial condition $\phi \in \mathcal{L}$, $\{\mathbb{P}(\phi, t, \cdot) : t \geq 0\}$ is Cauchy in the space $\mathbb{P}(\mathcal{L})$ with the metric $d(\cdot, \cdot)$ in Definition 2.2. For this purpose, we only need to show that for any initial data $\phi \in \mathcal{L}$ and $\varepsilon > 0$, there exists a time $T > 0$ such that

$$d(\mathbb{P}(\phi, t+s, \cdot), \mathbb{P}(\phi, t, \cdot)) = \sup_{f \in \mathcal{M}} |\mathbb{E}f(X_{t+s}(\phi)) - \mathbb{E}f(X_t(\phi))| \leq \varepsilon, \qquad \forall t \geq T, s > 0. \tag{2.3}$$

The proof is referred to Lemma 5.1 in [28]. Here we shall provide the details for the sake of completeness. For any $f \in \mathcal{M}$ and $t, s > 0$, note that

$$\begin{aligned}
&|\mathbb{E}f(X_{t+s}(\phi)) - \mathbb{E}f(X_t(\phi))| \\
&\quad = |\mathbb{E}[\mathbb{E}f(X_{t+s}(\phi))|\mathcal{F}_s] - \mathbb{E}f(X_t(\phi))| \\
&\quad = \left| \int_{\mathcal{L}} \mathbb{E}f(X_t(\psi))\mathbb{P}(X_s(\phi), \mathrm{d}\psi) - \mathbb{E}f(X_{t+s}(\phi)) \right| \\
&\quad \leq \int_{\mathcal{L}} |\mathbb{E}f(X_t(\psi)) - \mathbb{E}f(X_t(\phi))|\mathbb{P}(X_s(\phi), \mathrm{d}\psi) \\
&\quad \leq 2\mathbb{P}(X_s(\phi), \mathcal{L}_R^c) + \int_{\mathcal{L}_R} |\mathbb{E}f(X_t(\psi)) - \mathbb{E}f(X_t(\phi))|\mathbb{P}(X_s(\phi), \mathrm{d}\psi),
\end{aligned} \tag{2.4}$$

where $\mathcal{L}_R = \{\phi \in \mathcal{L} : \|\phi\|_{\mathcal{L}} \leq R\}$ and $\mathcal{L}_R^c = \mathcal{L} - \mathcal{L}_R$. By virtue of condition (i), there exists a time $T_2 > 0$ such that

$$\sup_{f \in \mathcal{M}} |\mathbb{E} f(X_t(\phi)) - \mathbb{E} f(X_t(\psi))| \leq \frac{\varepsilon}{2}, \qquad t \geq T_2.$$

On the other hand, condition (ii) implies that there exists a positive sufficiently large constant R such that

$$\mathbb{P}(X_s(\phi), \mathcal{L}_R^c) \leq \frac{\varepsilon}{4}, \qquad \forall s > 0.$$

Hence (2.3) holds and the transition probability $\mathbb{P}(X_s(\phi), \cdot)$ of $X_t(\phi)$ converges weakly to some $\mu \in \mathbb{P}(\mathcal{L})$. For every $f \in C_b(\mathcal{L})$ the Markovian property of $X_t(\phi), t \geq 0$ gives that

$$\mathbb{P}_{t+s} f(\phi) = \mathbb{P}_t \mathbb{P}_s f(\phi) \qquad t, s \geq 0, \phi \in \mathcal{L}.$$

Let $s \to \infty$, it follows that

$$\mu(f) = \mu(\mathbb{P}_t f), \qquad f \in C_b(\mathcal{L}).$$

That is, μ is an invariant measure for $X_t(\phi), t \geq 0$. The proof is completed. □

3 Stochastic systems driven by Wiener process

In this section we consider equation (1.1) with $Z = \{W(t) : t \geq 0\}$, which denotes a $\mathcal{U}$-valued $\{\mathcal{F}_t : t \geq 0\}$-Wiener process defined on $\{\Omega, \mathcal{F}, \mathbb{P}\}$ with covariance operator Q, i.e.,

$$\mathbb{E}\langle W(t), x\rangle\langle W(s), y\rangle = (t \wedge s)\langle Qx, y\rangle \quad \text{for all } x, y \in \mathcal{U},$$

where Q is a linear, symmetric and nonnegative bounded operator on U. In particular, we shall call $\{W(t) : t \geq 0\}$, a $\mathcal{U}$-valued Q-Wiener process with respect to $\{\mathcal{F}_t : t \geq 0\}$.

First, we shall show the solution process is tight. Let $\mathfrak{L}_2^Q(\mathcal{U}, H)$ is the space of all Hilbert–Schmidt operators from $\mathcal{U}$ to H with $\|G\|^2_{\mathfrak{L}_2^Q} := \mathrm{Tr}_H(GQG^*)$. For any $t \geq 0$ and $G(t) \in \mathfrak{L}_2^Q(\mathcal{U}, H)$, let

$$Q_t = \int_0^t \Phi(s)G(s)QG^*(s)\Phi^*(s)\mathrm{d}s,$$

where $G^*(s)$ and $\Phi^*(s)$ are the adjoint operators of $G(s)$ and $\Phi(s)$, respectively. We suppose that

$$\mathrm{Tr}(Q_t) = \int_0^t \mathrm{Tr}[\Phi(s)G(s)QG^*(s)\Phi^*(s)]\mathrm{d}s < \infty \quad \text{for any } t \geq 0. \tag{3.1}$$

Throughout this section, the operator F: $\mathcal{L} \to H$ is supposed to be Lipschitz continuous while the operator G: $\mathcal{L} \to \mathfrak{L}_2^Q(\mathcal{U}, H)$ is supposed to be Lipschitz continuous with respect to the Hilbert-Schmidt norm $\mathfrak{L}_2^Q(\mathcal{U}, H)$ of linear operators from $\mathcal{U}$ to H:

$$\begin{aligned} \|F(x) - F(y)\| + \|G(x) - G(y)\|_{\mathfrak{L}_2^Q} &\leq K\|x - y\|_{\mathcal{L}}, \\ \|F(x)\| + \|G(x)\|_{\mathfrak{L}_2^Q} &\leq K_1\|x\|_{\mathcal{L}} + K_2 \end{aligned} \tag{3.2}$$

for all $x, y \in \mathcal{L}$, where K, K_1, K_2 are nonpositive constants. Note that under hypotheses (2.1) and (3.2), (1.1) has a unique mild solution of which the segment is a Markov and Feller process (see [33, 34, 39] for more details). In the subsequent two subsections, we investigate the existence of invariant measure and random attractor as well as the exponential stability of stationary solutions.

3.1 Invariant measure

Lemma 3.1. *Assume that* $2K^2e^{2\lambda\tau}(1+\lambda^{-1}e^{-\lambda\tau}) < \lambda$*, Then all trajectories of solution processes* (2.2) *converge exponentially together in the mean-square sense. In particular, for any bounded subset* U *of* $\mathcal{L}$*,*

$$\lim_{t\to\infty} \sup_{\phi,\psi\in U} \mathbb{E}\|X_t(\phi) - X_t(\psi)\|^2_{\mathcal{L}} = 0.$$

Proof. It follows from $2(1+\varepsilon)K^2e^{2\lambda\tau}(1+\lambda^{-1}e^{-\lambda\tau}) < \lambda$ that there exists $\varepsilon > 0$ such that $2(1+\varepsilon)K^2e^{2\lambda\tau}(1+\lambda^{-1}e^{-\lambda\tau}) < \lambda$. Note that

$$(A+B+C)^2 \le (1+\varepsilon)(A+B)^2 + \left(1+\frac{1}{\varepsilon}\right)C^2 \le 2(1+\varepsilon)(A^2+B^2) + \left(1+\frac{1}{\varepsilon}\right)C^2$$

for all $A, B, C \ge 0$. Then it follows from (2.2) that for $t > \tau$,

$$\begin{aligned}
&\mathbb{E}\|X_t(\phi) - X_t(\psi)\|^2_{\mathcal{L}} \\
&\le \mathbb{E}\bigg\{ \left(1+\frac{1}{\varepsilon}\right) \int_{-\tau}^{0} \|\Phi(t+\theta)(\phi(0)-\psi(0))\|^2 d\theta \\
&\quad + 2(1+\varepsilon)\int_{-\tau}^{0} \left\| \int_0^{t+\theta} \Phi(t+\theta-s)(F(X_s(\phi)) - F(X_s(\psi)))ds \right\|^2 d\theta \\
&\quad + 2(1+\varepsilon)\int_{-\tau}^{0} \left\| \int_0^{t+\theta} \Phi(t+\theta-s)(G(X_s(\phi)) - G(X_s(\psi)))dW \right\|^2 \bigg\} d\theta \\
&=: \left(1+\frac{1}{\varepsilon}\right) I_1 + 2(1+\varepsilon)I_2 + 2(1+\varepsilon)I_3.
\end{aligned} \tag{3.3}$$

Following (2.1) we have

$$I_1 \le \frac{e^{-2\lambda t}(e^{2\lambda\tau}-1)}{2\lambda}\|\phi(0)-\psi(0)\|^2. \tag{3.4}$$

From (2.1), (3.2) and Hölder's inequality, it follows that for $t > \tau$,

$$\begin{aligned}
I_2 &\le \mathbb{E}\int_{-\tau}^{0} \bigg[\int_0^{t+\theta} \|\Phi(t+\theta-s)\|_{\mathscr{L}(H)} ds \\
&\qquad \int_0^{t+\theta} \|\Phi(t+\theta-s)\|_{\mathscr{L}(H)} \|F(X_s(\phi)) - F(X_s(\psi))\|^2 ds \bigg] d\theta \\
&\le \frac{K^2}{\lambda}\int_{-\tau}^{0}\int_0^{t+\theta} (1-e^{-\lambda(t+\theta)})e^{-\lambda(t+\theta-s)}\mathbb{E}\|X_s(\phi) - X_s(\psi)\|^2_{\mathcal{L}} ds d\theta \\
&\le \frac{K^2e^{\lambda\tau}}{\lambda}\int_0^t e^{-\lambda(t-s)}\mathbb{E}\|X_s(\phi) - X_s(\psi)\|^2_{\mathcal{L}} ds.
\end{aligned} \tag{3.5}$$

Using (2.1), (3.2) and the Burkholder–Davis–Gundy inequatlity (see, for example [27, Theorem 6.1]), we get

$$\begin{aligned}
I_3 &\le \mathbb{E}\int_{-\tau}^{0}\int_0^{t+\theta} \|\Phi(t+\theta-s)\|^2_{\mathscr{L}(H)} \|G(X_s(\phi)) - G(X_s(\psi))\|^2_{\mathfrak{L}_2^Q} ds d\theta \\
&\le K^2 \int_{-\tau}^{0}\int_0^{t+\theta} e^{-2\lambda(t+\theta-s)}\mathbb{E}\|X_s(\phi) - X_s(\psi)\|^2_{\mathcal{L}} ds d\theta \\
&\le K^2e^{2\lambda\tau}\int_0^t e^{-\lambda(t-s)}\mathbb{E}\|X_s(\phi) - X_s(\psi)\|^2_{\mathcal{L}} ds
\end{aligned} \tag{3.6}$$

for $t > \tau$. Then from (3.4), (3.5) and (3.6), we obtain

$$e^{\lambda t}\mathbb{E}\|X_t(\phi) - X_t(\psi)\|^2 \le \frac{(e^{2\lambda\tau}-1)(1+\varepsilon)}{2\lambda\varepsilon}\|\phi(0)-\psi(0)\|^2 + 2(1+\varepsilon)K^2(e^{2\lambda\tau}+\lambda^{-1}e^{\lambda\tau})\int_0^t e^{\lambda s}\mathbb{E}\|X_s(\phi)-X_s(\psi)\|_{\mathcal{L}}^2\mathrm{d}s.$$

Using Gronwall's inequality, we have

$$e^{\lambda t}\mathbb{E}\|X_t(\phi) - X_t(\psi)\|_{\mathcal{L}}^2 \le \frac{(e^{2\lambda\tau}-1)(1+\varepsilon)}{2\lambda\varepsilon}\|\phi(0)-\psi(0)\|^2 \tag{3.7}$$

The proof is completed. □

Then we will prove the segment process of solution to (1.1) is bounded with Wiener process.

Lemma 3.2. *Assume that $2K_1^2e^{2\lambda\tau}(1+\lambda^{-1}e^{-\lambda\tau}) < \lambda$. Then the solution process (2.2) is ultimately bounded in the mean-square sense, i.e., for any bounded set U of $\mathcal{L}$,*

$$\sup_{t\ge 0}\sup_{\phi\in U}\mathbb{E}\|X_t(\phi)\|_{\mathcal{L}}^2 < \infty.$$

Proof. It follows from $2K_1^2e^{2\lambda\tau}(1+\lambda^{-1}e^{-\lambda\tau}) < \lambda$ that there exists $\varepsilon > 0$ such that $2K_1^2(e^{2\lambda\tau} + \lambda^{-1}e^{\lambda\tau})(1+\varepsilon)^2 < \lambda$. Similar to (3.3), it follows from (2.2) that for all $t \ge 0$

$$\begin{aligned}\mathbb{E}\|X_t(\phi)\|_{\mathcal{L}}^2 \le \mathbb{E}\Big\{ &\left(1+\frac{1}{\varepsilon}\right)\int_{-\tau}^0 \|\Phi(t+\theta)\phi(0)\|^2\mathrm{d}\theta \\ &+2\,(1+\varepsilon)\int_{-\tau}^0\left\|\int_0^{t+\theta}\Phi(t+\theta-s)F(X_s(\phi))\mathrm{d}s\right\|^2\mathrm{d}\theta \\ &+2\,(1+\varepsilon)\int_{-\tau}^0\left\|\int_0^{t+\theta}\Phi(t+\theta-s)G(X_s(\phi))dW\right\|^2\mathrm{d}\theta\Big\} \\ =:&\left(1+\frac{1}{\varepsilon}\right)J_1 + 2\,(1+\varepsilon)\,J_2 + 2\,(1+\varepsilon)\,J_3.\end{aligned} \tag{3.8}$$

From (2.1) we have

$$J_1 \le \frac{e^{-2\lambda t}(e^{2\lambda\tau}-1)}{2\lambda}\|\phi(0)\|^2. \tag{3.9}$$

Note that

$$(A+B)^2 \le (1+\varepsilon)A^2 + \left(1+\frac{1}{\varepsilon}\right)B^2.$$

Following (2.1), (3.2) and Hölder's inequality we have

$$\begin{aligned}J_2 &\le \mathbb{E}\int_{-\tau}^0\left\|\int_0^{t+\theta}\Phi(t+\theta-s)F(X_s(\phi))\mathrm{d}s\right\|^2\mathrm{d}\theta \\ &\le \mathbb{E}\int_{-\tau}^0\left\|\int_0^{t+\theta}\Phi(t+\theta-s)[K_1\|X_s(\phi)\|_{\mathcal{L}}+K_2]\mathrm{d}s\right\|^2\mathrm{d}\theta \\ &\le \left(1+\frac{1}{\varepsilon}\right)K_2^2\int_{-\tau}^0\int_0^{t+\theta}\|\Phi(t+\theta-s)\|_{\mathscr{L}(H)}^2\,\mathrm{d}s\mathrm{d}\theta \\ &\quad + (1+\varepsilon)K_1^2\int_{-\tau}^0\int_0^{t+\theta}\|\Phi(t+\theta-s)\|_{\mathscr{L}(H)}\,\mathrm{d}s\int_0^{t+\theta}\|\Phi(t+\theta-s)\|_{\mathscr{L}(H)}\|X_s(\phi)\|_{\mathcal{L}}^2\mathrm{d}s\mathrm{d}\theta\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{(1+\varepsilon)\tau K_2^2}{2\varepsilon\lambda} + \frac{(1+\varepsilon)K_1^2}{\lambda}\int_{-\tau}^{0}\int_{0}^{t+\theta}(1-e^{-\lambda(t+\theta)})e^{-\lambda(t+\theta-s)}\mathbb{E}\|X_s(\phi)\|_{\mathcal{L}}^2\mathrm{d}s\mathrm{d}\theta \\
&\leq \frac{(1+\varepsilon)\tau K_2^2}{2\varepsilon\lambda} + \frac{(1+\varepsilon)e^{\lambda\tau}K_1^2}{\lambda}\int_{0}^{t}e^{-\lambda(t-s)}\mathbb{E}\|X_s(\phi)\|_{\mathcal{L}}^2\mathrm{d}s
\end{aligned} \tag{3.10}
$$

for $t > \tau$. It follows from (2.1), (3.2) and the Burkholder–Davis–Gundy inequality that

$$
\begin{aligned}
J_3 &\leq \int_{-\tau}^{0}\int_{0}^{t+\theta}\mathbb{E}\left\|\Phi(t+\theta-s)G(X_s(\phi))\right\|^2\mathrm{d}s\mathrm{d}\theta \\
&\leq \int_{-\tau}^{0}\int_{0}^{t+\theta}e^{-2\lambda(t+\theta-s)}\left[(1+\varepsilon)K_1^2\mathbb{E}\|X_s(\phi)\|_{\mathcal{L}}^2 + \left(1+\frac{1}{\varepsilon}\right)K_2^2\right]\mathrm{d}s\mathrm{d}\theta \\
&\leq \frac{(1+\varepsilon)\,K_2^2}{2\varepsilon\lambda}\int_{-\tau}^{0}(1-e^{-2\lambda(t+\theta)})\mathrm{d}\theta + (1+\varepsilon)K_1^2\int_{-\tau}^{0}\int_{0}^{t+\theta}e^{-2\lambda(t+\theta-s)}\mathbb{E}\|X_s(\phi)\|_{\mathcal{L}}^2\mathrm{d}s\mathrm{d}\theta \\
&\leq \frac{(1+\varepsilon)\,\tau K_2^2}{2\varepsilon\lambda} + (1+\varepsilon)K_1^2e^{2\lambda\tau}\int_{0}^{t}e^{-\lambda(t-s)}\mathbb{E}\|X_s(\phi)\|_{\mathcal{L}}^2\mathrm{d}s
\end{aligned} \tag{3.11}
$$

for $t > \tau$. Thus, (3.9), (3.10) and (3.11) together imply that for $t > \tau$,

$$
\begin{aligned}
e^{\lambda t}\mathbb{E}\|X_t(\phi)\|^2 &\leq \frac{(1+\varepsilon)\,(e^{2\lambda\tau-1})}{2\varepsilon\lambda}\|\phi(0)\|^2 + \frac{(1+\varepsilon)^2K_2^2\tau}{\varepsilon\lambda}e^{\lambda t} \\
&\quad + 2K_1^2(1+\varepsilon)^2(\lambda^{-1}e^{\lambda\tau}+e^{2\lambda\tau})\int_{0}^{t}e^{\lambda s}\mathbb{E}\|X_s(\phi)\|_{\mathcal{L}}^2\mathrm{d}s \\
&= \alpha_1 + \gamma_1e^{\lambda t} + \beta_1\int_{0}^{t}e^{\lambda s}\mathbb{E}\|X_s(\phi)\|_{\mathcal{L}}^2\mathrm{d}s,
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_1 &= \frac{(1+\varepsilon)\,(e^{2\lambda\tau-1})}{2\varepsilon\lambda}\|\phi(0)\|^2, \\
\gamma_1 &= \frac{(1+\varepsilon)^2K_2^2\tau}{\varepsilon\lambda}, \\
\beta_1 &= 2K_1^2(1+\varepsilon)^2(e^{2\lambda\tau}+\lambda^{-1}e^{\lambda\tau}).
\end{aligned}
$$

Then Gronwall's inequality gives that

$$
e^{\lambda t}\mathbb{E}\|X_t(\phi)\|_{\mathcal{L}}^2 \leq \alpha_1 + \gamma_1e^{\lambda t} + \beta_1\int_{0}^{t}\left(\gamma_1e^{\lambda s}+\alpha_1\right)e^{\beta_1(t-s)}\mathrm{d}s
$$

and hence that

$$
\begin{aligned}
\mathbb{E}\|X_t(\phi)\|_{\mathcal{L}}^2 &\leq \gamma_1 + \alpha_1e^{-\lambda t} + \beta_1e^{(\beta_1-\lambda)t}\int_{0}^{t}\left(\gamma_1e^{\lambda s}+\alpha_1\right)e^{-\beta_1 s}\mathrm{d}s \\
&\leq \gamma_1 + 2\alpha_1 + \frac{\beta_1\gamma_1}{\lambda-\beta_1}
\end{aligned}
$$

for $t > \tau$. This completes the proof. □

By Lemmas 2.3, 3.1 and 3.2, we can have the following result about the existence of invariant measures of equation (1.1) driven by Wiener process. Now we show the uniqueness of invariant measures. If $\mu, \mu' \in \mathbb{P}(\mathcal{L})$ are two different invariant measures for X_t of (1.1), for any $f \in \mathcal{M}$, by virtue of (3.7), Hölder's inequality and the invariance of $\mu(\cdot), \mu'(\cdot)$, it follows that

$$
|\mu(f)-\mu'(f)| \leq \int_{\mathcal{L}\times\mathcal{L}}|\mathbb{P}_tf(\phi)-\mathbb{P}_tf(\psi)|\mu(\mathrm{d}\phi)\mu(\mathrm{d}\psi) \leq K_3e^{\alpha t}, \qquad t \geq 0,
$$

for some constant $K_3 > 0$, where $\alpha = K^2(e^{2\lambda\tau}+\lambda^{-1}e^{\lambda\tau}) - \frac{1}{2}\lambda < 0$ under the assumption in Lemma 3.1. We obtain the uniqueness of invariant measures by letting $t \to \infty$.

Theorem 3.3. *Under the assumptions of Lemmas 3.1 and 3.2, equation (1.1) driven by Wiener process has a unique invariant measure.*

3.2 Random attractor

We consider the canonical probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where

$$\Omega = \{\omega \in C(\mathbb{R}; H) : \omega(0) = 0\},$$

and $\mathcal{F}$ is the Borel σ-algebra induced by the compact open topology of Ω (see [3]), while $\mathbb{P}$ is the corresponding Wiener measure on $(\Omega, \mathcal{F}, \mathbb{P})$. Define a shift operators by flow $\theta = \{\theta_t\}_{t\in\mathbb{R}}$ on Ω:

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \qquad t \in \mathbb{R},\ \omega \in \Omega.$$

Then, $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is a metric dynamical system, that is, $\theta : \mathbb{R}\times\Omega \to \Omega$ is $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$-measurable, θ_0 is the identity on Ω, $\theta_{s+t} = \theta_t\theta_s$ for all $s, t \in \mathbb{R}$, and $\theta_t(\mathbb{P}) = \mathbb{P}$ for all $t \in \mathbb{R}$. More precisely, $\mathbb{P}$ is ergodic with respect to θ. In addition, with respect to the filtration we have that

$$\theta_s^{-1}\bar{\mathcal{F}}_t = \bar{\mathcal{F}}_{t+s} \tag{3.12}$$

for any $t, s \in \mathbb{R}$, where $\bar{\mathcal{F}}$ is the completion of $\mathcal{F}$, see [3, Definition 2.3.4] for more details. For the sake of convenience, from now on, we will abuse the notation slightly and write the space $\overline{\Omega}$ as Ω.

The Wiener process with covariance Q is adapted to the filtration $\{\bar{\mathcal{F}}_{s+t}\}_{t\geq 0}$. First we define a mean-square random dynamical system referring to [24,39]. Let

$$\mathbb{R}^2_{\geq} \triangleq \{(t, t_0) \in \mathbb{R}^2 : t \geq t_0\},$$

and

$$\Pi \triangleq L^2((\Omega, \mathcal{F}, \mathbb{P}); \mathcal{L}), \qquad \Pi_t \triangleq L^2((\Omega, \mathcal{F}_t, \mathbb{P}); \mathcal{L})$$

for each $t \in \mathbb{R}$.

Definition 3.4 ([39, Definition 10])**.** A mean-square random dynamical system (MS-RDS) Ψ on $\mathcal{L}$ with probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ is a family of mappings

$$\Psi(t, t_0, \cdot) : \Pi_{t_0} \to \Pi_t, \qquad (t, t_0) \in \mathbb{R}^2_{\geq},$$

which satisfies

(i) initial value property: $\Psi(t, t_0, \psi) = \psi$ for every $\psi \in \Pi_{t_0}$;

(ii) two-parameter semigroup property: $\Psi(t_2, t_0, \psi) = \Psi(t_2, t_1, \Psi(t_1, t_0, \psi))$ for all $t_2 \geq t_1 \geq t_0$;

(iii) continuity property: $(t, t_0, \psi) \mapsto \Psi(t, t_0, \psi)$ is continuous in the space $\mathbb{R}^2_{\geq} \times \Pi$.

Definition 3.5 ([39, Definition 11])**.** A family $\mathcal{A} = \{A_t\}_{t\in\mathbb{R}}$ of nonempty subsets of Π with $A_t \subset \Pi_t$ is said to be Ψ-invariant if

$$\Psi(t, t_0, A_{t_0}) = A_t \quad \text{for all } (t, t_0) \in \mathbb{R}^2_{\geq},$$

and Ψ-positively invariant if

$$\Psi(t, t_0, A_{t_0}) \subset A_t \quad \text{for all } (t, t_0) \in \mathbb{R}^2_{\geq}.$$

Definition 3.6 ([39, Definition 12])**.** A Ψ-invariant family $\mathcal{A} = \{A_t\}_{t\in\mathbb{R}}$ of nonempty compact subsets of $\{\Pi_t\}_{t\in\mathbb{R}}$ is called a mean-square pullback attractor if it pullback attracts all families $\mathcal{B} = \{B_t\}_{t\in\mathbb{R}}$ of uniformly bounded subsets of $\{\Pi_t\}_{t\in\mathbb{R}}$, i.e., for any fixed $t \in \mathbb{R}$

$$\mathrm{dist}(\Psi(t,t_0,B_{t_0}),A_t) \to 0 \quad \text{as } t_0 \to -\infty.$$

Let $X(\cdot,t_0,\phi_0)$ be the solution of the following equation with initial value $\phi_0 \in \Pi_{t_0}$,

$$\begin{cases} \mathrm{d}X(t) = [AX(t) + F(X_t)]\mathrm{d}t + G(X_t)\mathrm{d}W(t), & t > t_0, \\ X(t_0+s) = \phi_0(s), \quad s \in [-\tau,0]. \end{cases} \tag{3.13}$$

For each $(t,t_0,\phi_0) \in \mathbb{R}^2_{\geq} \times \Pi_{t_0}$, define solution mapping of (3.13):

$$\Psi(t,t_0,\phi_0) = X_t(\cdot,t_0,\phi_0) = X(t+\cdot,t_0,\phi_0).$$

It is easy to see that Ψ satisfies the initial value property,

$$\Psi(t_0,t_0,\phi_0) = X_{t_0}(\cdot,t_0,\phi_0) = X(t_0+\cdot) = \phi_0$$

for all $(t_0,\phi_0) \in \mathbb{R} \times \Pi_{t_0}$. Existence and uniqueness of solution of (3.13) show that Ψ satisfies the two-parameter semigroup evolution property. Moreover, Ψ is continuous for all $(t,t_0,\phi_0) \in \mathbb{R}^2_{\geq} \times \Pi_{t_0}$ since solution $X_t(\cdot,t_0,\phi_0)$ is continuous with respect to t,ϕ_0. Thus, (3.13) generates a continuous MS-RDS $\Psi = \{\Psi(t,t_0,\cdot),(t,t_0) \in \mathbb{R}^2_{\geq}\}$ with state space $\mathcal{L}$.

It follows from Lemma 3.2 that for any bounded set U of Π there exist constants $B > 0$ and $T_U \geq 0$ such that for all $t \geq t_0 + T_U$ and $\phi_0 \in U \cap \Pi_{t_0}$,

$$\mathbb{E}\|\Psi(t,t_0,\phi_0)\|^2 < B,$$

which can be represented in the pullback sense that

$$\mathbb{E}\|\Psi(t,t_n,\phi_n)\|^2 < B$$

for all $t_n \leq t - T_B$ and $\phi_n \in U \cap \Pi_{t_n}$. Lemma 3.1 shows that any two solutions converge together in the mean-square sense uniformly for different initial conditions at the same starting time. Namely, for any $\phi_0,\psi_0 \in \Pi_{t_0}$,

$$\mathbb{E}\|\Psi(t,t_0,\phi_0) - \Psi(t,t_0,\psi_0)\|^2 \to 0 \quad \text{as } t \to \infty,$$

with the convergence being uniform for initial values in a common bounded subset as well as in the initial time t_0. Let U_B be a bounded ball about the origin of radius B in $\mathcal{L}$. Consider a sequence $t_n \to -\infty$ as $n \to \infty$ with $t_n < -T_U - \tau$ and $t_{n+1} \leq t_n - T_{B_U}$ and define a sequence $\{\chi_n\}_{n=1}^{\infty}$ in $U_B \cap \Pi_0$ by

$$\chi_n \triangleq \Psi(0,t_n,\phi_n) \tag{3.14}$$

for an arbitrary $\phi_n \in U_B \cap \Pi_{t_n}$. Namely,

$$\chi_n(s) = \Psi(s,t_n,\phi_n)$$

for all $s \in [-\tau,0]$. Then $\{\chi_n\}_{n=1}^{\infty}$ are obviously mean-square bounded by B for all ϕ_n taking values in $U_B \cap \Pi_{t_n}$.

Lemma 3.7. *$\{\chi_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence with values in $U_B \cap \Pi_0$ and there exists a unique limit $\chi_0^* \in U_B \cap \Pi_0$ such that*

$$\mathbb{E}\|\chi_n - \chi_0^*\|^2 \to 0 \quad \text{as } n \to \infty.$$

Proof. It suffices to prove that for every $\varepsilon > 0$ there exists $N_\varepsilon > 0$ such that

$$\mathbb{E}\|\chi_n - \chi_m\|^2 \leq \varepsilon \quad \text{for all } n, m \geq N_\varepsilon. \tag{3.15}$$

Let $t_m < t_n < 0$. Then we have

$$\chi_m = \Psi(0, t_m, \phi_m) = \Psi(0, t_n, \Psi(t_n, t_m, \phi_m)) = \Psi(0, t_n, \widehat{\phi_{n,m}}),$$

where $\widehat{\phi_{n,m}} := \Psi(t_n, t_m, \phi_m) \in U_B \cap \Pi_{t_n}$. Indeed,

$$\mathbb{E}\|\chi_n - \chi_m\|^2 = \mathbb{E}\|\Psi(0, t_n, \phi_n) - \Psi(0, t_m, \phi_m)\|^2 = \mathbb{E}\|\Psi(0, t_n, \phi_n) - \Psi(0, t_n, \widehat{\phi_{n,m}})\|^2.$$

Thus, it follows from Lemma 3.1 that (3.15) holds and all solutions starting in the common bounded subset U_B converge in Π_0. Since Π_0 is complete, the Cauchy sequence has a unique limit $\chi_0^* \in U_B \cap \Pi_0$. The proof is completed. □

From the above process, we can repeat with 0 in (3.14) replaced by -1 to obtain a limit $\chi_{-1}^* \in \Pi_{-1}$. It is easy to see from the construction that $\chi_0^* = \Psi(0, -1, \chi_{-1}^*)$. Follow this way, we can construct a sequence $\{\chi_{-n}^*\}_{n\in\mathbb{N}}$ and hence obtain an entire MS-RDS χ_t^* for all $t \in \mathbb{R}$. Moreover, all other MS-RDS trajectories converge to χ_t^* in the mean-square sense.

Theorem 3.8. *Under the assumptions of Lemmas 3.1 and 3.2, there exists a pullback random attractor for the random dynamical system generated by (1.1) which consists of singleton sets. Furthermore, the random attractor pullback attracts all other solution processes in mean-square sense.*

Proof. The above arguments shows the existence of random attractor consisting of singleton sets $A_t = \{\chi_t^*\}$ and attracts all other solution processes in the mean-square sense. Next we show the random attractor is unique. Suppose there is another entire trajectory $\bar{\chi}_t^* \in A_t$ for all $t \in \mathbb{R}$ and there exists a constant $\varepsilon_0 > 0$ such that

$$\mathbb{E}\|\chi_0^* - \bar{\chi}_0^*\|^2 \geq \varepsilon_0.$$

On the other hand, it follows from the convergence in Lemma 3.1 that there exists $T \geq 0$ such that

$$\mathbb{E}\|\Psi(0, -t, \chi_{-t}^*) - \Psi(0, -t, \bar{\chi}_{-t}^*)\|^2 \leq \frac{\varepsilon_0}{2}$$

for all $t \geq T$. Note that $\chi_0^* = \Psi(0, -t, \chi_{-t}^*)$ and $\bar{\chi}_0^* = \Psi(0, -t, \bar{\chi}_{-t}^*)$. Hence

$$\varepsilon_0 \leq \mathbb{E}\|\chi_0^* - \bar{\chi}_0^*\|^2 = \mathbb{E}\|\Psi(0, -t, \chi_{-t}^*) - \Psi(0, -t, \bar{\chi}_{-t}^*)\|^2 \leq \frac{\varepsilon_0}{2}$$

for $t \geq T$, which is a contradiction. This completes the proof. □

Remark 3.9. If the random attractor $\mathcal{A}(\omega), \omega \in \Omega$ consists of a single point, then $\mathcal{A}$ defines a random fixed point which attracts tempered random sets.

3.3 Exponential stability of stationary solutions

Note that zero is not a solution to the equation (1.1). In this subsection we shall prove that the non-trivial stationary solutions to equation (1.1) with Wiener process are exponentially stable. Consider the process $\theta_s W(\cdot, \omega) = W(\cdot, \theta_s \omega) = W(\cdot + s, \omega) - W(s, \omega)$, for $s \in \mathbb{R}$ which is adapted to the filtration $\{\mathcal{F}_{t+s}\}_{t\geq 0}$. The following equality holds for $\psi_0 \in \Pi_0$, $t \geq 0$, $s \in \mathbb{R}$:

$$\Psi(t, 0, \psi_0)(\theta_s \cdot) = \Psi(t, s, \psi_s)(\cdot) \quad \text{almost surely } t, s \in \mathbb{R}, \tag{3.16}$$

where $\psi_s(\cdot) := \psi_0(\theta_s \cdot)$. It follows from (3.12) that $x_0(\theta_s \cdot)$ is $\bar{\mathcal{F}}_s$-measurable. The following lemma is obvious from the proof of Lemma 3.1.

Lemma 3.10. *For $s \in \mathbb{R}$, $\epsilon \geq 0$, $\psi \in \Pi_s$,*

$$\Psi(\cdot, s+\epsilon, \Psi(\epsilon, s, \psi)) = \Psi(\cdot + \epsilon, s, \psi) \text{ almost surely.}$$

Now we can show the existence of the fixed point.

Theorem 3.11. *Under the assumptions in Lemma 3.1, there exists an exponentially attracting fixed point $X^* \in \Pi_0$ which generates an exponentially stable stationary solution for (1.1). In addition, the process $(t,\omega) \to X^*(\theta_t\omega)$ has a continuous version given by $\Psi(\cdot, 0, X^*)$.*

Proof. First we claim that $(\Psi(k, -k, \psi_0(\theta_{-k}\cdot)))_{k\in\mathbb{N}}$ is a Cauchy sequence in Π_0. It follows from Lemma 3.10 and (3.16) that

$$\begin{aligned}
&\mathbb{E}\|\Psi(k,-k,\psi_0(\theta_{-k}\cdot)) - \Psi(k-1, 1-k, \psi_0(\theta_{1-k}\cdot))\|^2 \\
&\quad = \mathbb{E}\|\Psi(k-1, 1-k, \Psi(k,-k,\psi_0(\theta_{-k}\cdot)))) - \Psi(k-1, 1-k, \psi_0(\theta_{1-k}\cdot))\|^2 \\
&\quad \leq e^{\sigma(k-1)}\mathbb{E}\|\Psi(1, -k, \psi_0(\theta_{-k}\cdot)) - \psi_0(\theta_{1-k}\cdot))\|^2 \\
&\quad = e^{\sigma(k-1)}\mathbb{E}\|\Psi(1, 0, \psi_0(\cdot)) - \psi_0(\theta_1\cdot))\|^2.
\end{aligned}$$

It follows from Lemma 3.1 that $\sigma < 0$ and then the Cauchy sequence property holds. Let $X^* \in \Pi_0$ be the limit of this sequence, i.e. in L^2-norm sense,

$$X^*(\theta_t\cdot) = \lim_{k\to\infty} \Psi(k, -k, \psi_0(\theta_{-k}\cdot))(\theta_t\omega),$$

which is equal to

$$X^*(\theta_t\cdot) = \lim_{k\to\infty} \Psi(k, t-k, \psi_0(\theta_{t-k}\cdot))(\omega).$$

For $\psi_0, \phi_0 \in \Pi_0$, we have

$$\begin{aligned}
&\mathbb{E}\|\Psi(k,-k,\psi_0(\theta_{-k}\cdot)) - \Psi(k,-k,\phi_0(\theta_{-k}\cdot))\|^2 \\
&\quad = \mathbb{E}\|\Psi(k,0,\psi_0(\cdot)) - \Psi(k,0,\phi_0(\cdot))\|^2 \leq e^{\sigma k}\mathbb{E}\|\psi_0 - \phi_0\|^2_{\mathcal{L}},
\end{aligned}$$

which tends to zero as k goes to infinity. Thus $X^* \in \Pi_0$ is exponentially stable and independent of the choice of $\psi_0 \in \Pi_0$.

Next, we show that X^* is a fixed point, i.e., for any $t \in \mathbb{R}^+$

$$\Psi(t, 0, X^*)(\cdot) = X^*(\theta_t\cdot) \quad \text{almost surely.}$$

Indeed, for any fixed t, from (3.16), Lemma 3.10 and semigroup property we have

$$\begin{aligned}
&\mathbb{E}\|\Psi(t,0,X^*) - X^*(\theta_t\cdot)\|^2 \\
&\quad = \mathbb{E}\left\|\Psi\left(t, 0, \lim_{k\to\infty}\Psi(k,-k,\psi_0(\theta_{-k}\cdot))\right) - \lim_{k\to\infty}\Psi(k, t-k, \psi_0(\theta_{t-k}\cdot))\right\|^2 \\
&\quad = \lim_{k\to\infty}\mathbb{E}\|\Psi(k, t-k, \Psi(t,-k,\psi_0(\theta_{-k}\cdot)) - \Psi(k, t-k, \psi_0(\theta_{t-k}\cdot))\|^2 \\
&\quad \leq \lim_{k\to\infty} e^{\sigma k}\mathbb{E}\|\Psi(t,-k,\psi_0) - \psi_0(\theta_t\cdot)\|^2 = 0.
\end{aligned}$$

This completes the proof. □

4 Systems driven by Lévy jump process

In this section, we will give the existence of invariant measures of (1.1) with Lévy jump process in separable Hilbert space $\mathcal{U}$. To this end, it suffices to verify the assertions in Lemma 2.3 hold.

Let $Z = \{Z(t) : t \geq 0\}$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We say that Z is a *Lévy process* if:

(i) $Z(0) = 0$ (a.s.);

(ii) Z has independent and stationary increments;

(iii) Z is stochastically continuous, i.e. for all $a > 0$ and for all $s \geq 0$

$$\lim_{t \to s} \mathbb{P}(\|Z(t) - Z(s)\|_{\mathcal{U}} > a) = 0.$$

We have the following property of Lévy measures on separable Hilbert spaces (see [36]).

Lemma 4.1. *Let $\mathcal{U}$ be a separable Hilbert space. Then a σ-finite measure ν with $\nu(\{0\}) = 0$ is on $\mathcal{U}$ if and only if*

$$\int_{\mathcal{U}} (1 \wedge \|y\|_{\mathcal{U}}^2)\nu(\mathrm{d}y) < \infty.$$

$\nu(\cdot)$ is also called a Lévy measure.

The jump process $\Delta Z = \{\Delta Z(t) : t \geq 0\}$ is defined by

$$\Delta Z(t) = Z(t) - Z(t-)$$

for each $t \geq 0$, where $Z(t-)$ is the left limit at the point t. Furthermore, ΔZ is a Poisson point process. The Poisson process of intensity $\lambda > 0$ is a Lévy process with N taking values in $\mathbb{N} \cup \{0\}$ wherein each $N(t) \sim \pi(\lambda t)$. For $t > 0$ and $\Gamma \in \mathcal{B}(\mathcal{U} - \{0\})$, define

$$N(t, \Gamma)(\omega) = \sum_{s \in (0,t]} 1_{\Gamma}(\Delta Z(s)(\omega)), \tag{4.1}$$

if $\omega \in \Omega_0$, and $N(t, \Gamma)(\omega) = 0$, if $\omega \in \Omega_0^c$, where $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ such that $t \to Z(t)(\omega)$ is càdlàg[1] for all $\omega \in \Omega_0^c$. We write $\nu(\cdot) = \mathbb{E}(N(1, \cdot))$ and call it the intensity measure associated with Z. We say that $\Gamma \in \mathcal{B}(\mathcal{U} - \{0\})$ is bounded below if $0 \notin \bar{\Gamma}$. The following results are from [2].

Lemma 4.2. **(i)** *If Z is a Lévy process, then for fixed $t > 0$, $\Delta Z(t) = 0$ (a.s.);*

(ii) *If Γ is bounded, then $N(t, \Gamma) < \infty$ (a.s.) for all $t \geq 0$;*

(iii) *If Γ is bounded, then $\{N(t, \Gamma) : t \geq 0\}$ is a Poisson process with intensity $\nu(\Gamma)$.*

Let S be a set and $\mathcal{A}$ be a ring of subsets of S. Clearly, if $\mathcal{F}$ is a σ-algebra then it is also a ring. *A random measure* M on $(S, \mathcal{A})$ is a collection of random variables $\{M(B) : B \in \mathcal{A}\}$ such that (i) $M(\varnothing) = 0$, (ii) given any disjoint A, $B \in \mathcal{A}$, $M(A \cup B) = M(A) + M(B)$. A random measure is said to be *independently scattered* if for each disjoint family $\{B_1, \ldots, B_n\}$ in $\mathcal{A}$, the random variables $M(B_1), \ldots, M(B_n)$ are independent.

[1] Let $I = [a, b]$ be an interval in $\mathbb{R}^+$. A mapping $f : I \to \mathbb{R}^d$ is said to be càdlàg if, for all $t \in [a, b]$, f has a left limit at t and f is right-continuous at t.

Let $\mathcal{S}$ be a σ-algebra of subsets of set S. Fix a non-trivial ring $\mathcal{A} \subseteq \mathcal{S}$, an independently scattered σ-finite random measure M on $(S, \mathcal{S})$ is called a *Poisson random measure* if $M(B) < \infty$ for each $B \in \mathcal{A}$ and each $M(B)$ has a Poisson distribution. It follows from (4.1) and Lemma 4.2 that $N(t,\Gamma)$ is a Poisson random measure and $\lambda(\cdot) = t\nu(\cdot)$.

Now introduce the *compensated Poisson process* $\widetilde{N} = \{\widetilde{N}(t) : t \geq 0\}$ where $\widetilde{N}(t) = N(t) - \nu t$. Note that $\mathbb{E}[\widetilde{N}(t)] = 0$ and $\mathbb{E}[\widetilde{N}(t)^2] = \nu t$ for each $t \geq 0$. Then $\widetilde{N}(t)$ is martingale, that is, for all $0 \leq s < t < \infty$, $\mathbb{E}(\widetilde{N}(t)|\mathcal{F}_s) = \widetilde{N}(s)$ a.s. For each $t \geq 0$ and Γ bounded, we define the *compensated Poisson random measure* by

$$\widetilde{N}(t,\Gamma) = N(t,\Gamma) - t\nu(\Gamma).$$

It is easy to see that $\widetilde{N}(t,\Gamma)$ is a σ-finite independently scattered martingale-valued measure.

Let $\Gamma \in \mathcal{B}(\mathcal{U} - \{0\})$ with $0 \notin \bar{\Gamma}$ and $f : \Gamma \to \mathcal{U}$ measurable. Define the following integral

$$\int_\Gamma f(z)N(t,\mathrm{d}z) = \sum_{0<s\leq t} f(\Delta Z(s))1_\Gamma(\Delta Z(s)).$$

This is a finite sum $\mathbb{P}$-a.s. since the number of summands is finite $\mathbb{P}$-a.s. For $f \in L^2_\nu \triangleq L^2(\mathcal{U} - \{0\}, \nu|_{\mathcal{U}-\{0\}}; \mathcal{U})$, the next proposition defines the integral with respect to the compensated Poisson random measure (see [36] for more details).

Proposition 4.3. *Let f be strongly square-integrable with respect to $\widetilde{N}(t,\mathrm{d}z)$ and $f \in L^2_\nu$. Then for any $\Gamma \in \mathcal{B}(\mathcal{U} - \{0\})$ with $0 \notin \bar{\Gamma}$ we have*

$$\int_\Gamma f(z)\widetilde{N}(t,\mathrm{d}z) = \sum_{0<s\leq t} f(\Delta Z(s))1_\Gamma(\Delta Z(s)) - t\int_\Gamma f(z)\nu(\mathrm{d}z).$$

Proposition 4.4 (cf. [1] and [36])**.** *Let $f \in L^2_\nu$ then for any $\Gamma \in \mathcal{B}(\mathcal{U} - \{0\})$ the integral $\int_\Gamma f(z)\widetilde{N}(t,\mathrm{d}z)$ exists and*

$$\mathbb{E}\left[\left\|\int_\Gamma f(z)\widetilde{N}(t,\mathrm{d}z)\right\|^2\right] = t\int_\Gamma \|f(z)\|^2\nu(\mathrm{d}z) < \infty.$$

The following is a very important for Lévy processes called Lévy–Itô decomposition (see e.g. [1,2,36]).

Theorem 4.5. *Let $Z = \{Z(t) : t \geq 0\}$ be a Lévy process on a separable Hilbert space $\mathcal{U}$ where the distribution of $Z(t)$ has generating triplet $[tb, tQ, t\nu]$ for each $t \geq 0$,*

$$Z(t) = bt + W_Q(t) + \int_{\|z\|_\mathcal{U}<1} z\widetilde{N}(t,\mathrm{d}z) + \int_{\|z\|_\mathcal{U}\geq 1} zN(t,\mathrm{d}z),$$

where

$$b = \mathbb{E}\left[Z(1) - \int_{\|z\|_\mathcal{U}\geq 1} zN(1,\mathrm{d}z)\right]$$

and $W_Q = \{W_Q(t) : t \geq 0\}$ is a Wiener process with covariance operator Q independent of $N(\cdot,\Gamma)$ for all $\Gamma \in \mathcal{B}(\mathcal{U} - \{0\})$ with $0 \notin \bar{\Gamma}$.

Let Z be a $\mathcal{U}$-valued Lévy process with its Lévy triplet $(0, Q, \nu)$ below. By Lemma 4.1, $\nu(\Gamma)$ is a Lévy measure with $\Gamma \in \mathcal{B}(\Gamma - \{0\})$. Note that an adapted Lévy process with zero mean is martingale, and that a Lévy process is martingale if and only if it is integrable and

$$b + \int_{\|z\|_\mathcal{U}\geq 1} z\nu(\mathrm{d}z) = 0.$$

It follows from Lévy–Itô decomposition that the Lévy process can be written as

$$Z(t) = W_Q(t) + \int_{\mathcal{U}-\{0\}} z\widetilde{N}(t, \mathrm{d}z). \tag{4.2}$$

In view of Proposition 4.4, we have

$$K_z \triangleq \int_{\mathcal{U}-\{0\}} \|z\|_{\mathcal{U}}^2 \nu(\mathrm{d}z) < \infty.$$

Throughout this section, we always assume that the operators F and G in (1.1) satisfy

$$\begin{aligned} \|F(x) - F(y)\| + \|G(x) - G(y)\|_{\mathscr{L}(\mathcal{U},H)} &\le K\|x - y\|_{\mathcal{L}}, \\ \|F(x)\| + \|G(x)\|_{\mathscr{L}(\mathcal{U},H)} &\le K_1\|x\|_{\mathcal{L}} + K_2 \end{aligned} \tag{4.3}$$

for all $x, y \in \mathcal{L}$, where K, K_1, K_2 are nonnegative constants, $\mathscr{L}(\mathcal{U}, H)$ is the space of bounded linear operators from $\mathcal{U}$ to H.

The boundedness of solution with Lévy jump process is given as follows.

Lemma 4.6. *Assume that $3K_1^2 e^{2\lambda\tau}\left[\lambda^{-1}e^{-\lambda\tau} + \mathrm{Tr}(Q) + K_z\right] < \lambda$. Then for any bounded set U of $\mathcal{L}$,*

$$\sup_{t\ge 0}\sup_{\phi\in U} \mathbb{E}\|X_t(\phi)\|_{\mathcal{L}}^2 < \infty.$$

Proof. Using the similar arguments as the proof of Lemma 3.2 and the Lévy–Itô decomposition (4.2), we can obtain that for all $t \ge 0$

$$\begin{aligned} \mathbb{E}\|X_t(\phi)\|_{\mathcal{L}}^2 \le{}& 3\mathbb{E}\int_{-\tau}^{0}\left\{\left\|\Phi(t+\theta)\phi(0) + \int_0^{t+\theta}\Phi(t+\theta-s)F(X_s(\phi))\mathrm{d}s\right\|^2\right. \\ &+ \left\|\int_0^{t+\theta}\Phi(t+\theta-s)G(X_s(\phi))dW_Q\right\|^2 \\ &+ \left.\left\|\int_0^{t+\theta}\int_{\mathcal{U}-\{0\}}\Phi(t+\theta-s)G(X_s(\phi))z\widetilde{N}(\mathrm{d}s,\mathrm{d}z)\right\|^2\right\}\mathrm{d}\theta \\ \le{}& 3\mathbb{E}\left\{(1+\frac{1}{\varepsilon})\int_{-\tau}^{0} e^{-2\lambda(t+\theta)}\|\phi\|_{\mathcal{L}}^2\mathrm{d}\theta\right. \\ &+ \left.(1+\varepsilon)\int_{-\tau}^{0}\left\|\int_0^{t+\theta}\Phi(t+\theta-s)F(X_s(\phi))\mathrm{d}s\right\|^2\mathrm{d}\theta\right\} \\ &+ 3\mathbb{E}\int_{-\tau}^{0}\left\|\int_0^{t+\theta}\Phi(t+\theta-s)G(X_s(\phi))dW_Q\right\|^2\mathrm{d}\theta \\ &+ 3\mathbb{E}\int_{-\tau}^{0}\left\|\int_0^{t+\theta}\int_{\mathcal{U}-\{0\}}\Phi(t+\theta-s)G(X_s(\phi))z\widetilde{N}(\mathrm{d}s,\mathrm{d}z)\right\|^2\mathrm{d}\theta \\ =:{}& \frac{3(1+\varepsilon)e^{-2\lambda t}(e^{2\lambda\tau}-1)}{2\varepsilon\lambda}\|\phi(0)\|^2 + M_1 + M_2 + M_3. \end{aligned} \tag{4.4}$$

By virtue of (3.10), we have

$$M_1 \le 3(1+\varepsilon)\left[\frac{(1+\varepsilon)\,\tau K_2^2}{2\varepsilon\lambda} + \frac{(1+\varepsilon)K_1^2 e^{\lambda\tau}}{\lambda}\int_0^t e^{-\lambda(t-s)}\mathbb{E}\|X_s(\phi)\|_{\mathcal{L}}^2\mathrm{d}s\right]. \tag{4.5}$$

Carrying out a similar argument to that of (3.11), we can easily get that

$$\begin{aligned}
M_2 &\le 3\mathrm{Tr}(Q)\int_{-\tau}^{0}\int_{0}^{t+\theta}\mathbb{E}\left\|\Phi(t+\theta-s)G(X_s(\phi))\right\|^2 \mathrm{d}s\mathrm{d}\theta \\
&\le 3\mathrm{Tr}(Q)\int_{-\tau}^{0}\int_{0}^{t+\theta} e^{-2\lambda(t+\theta-s)}\left[(1+\varepsilon)K_1^2\mathbb{E}\|X_s(\phi)\|_{\mathcal{L}}^2+\left(1+\frac{1}{\varepsilon}\right)K_2^2\right]\mathrm{d}s\mathrm{d}\theta \\
&\le \frac{3\,(1+\varepsilon)\,\tau K_2^2\mathrm{Tr}(Q)}{2\varepsilon\lambda}+3(1+\varepsilon)\mathrm{Tr}(Q)K_1^2e^{2\lambda\tau}\int_0^t e^{-\lambda(t-s)}\mathbb{E}\|X_s(\phi)\|_{\mathcal{L}}^2\mathrm{d}s.
\end{aligned} \tag{4.6}$$

Moreover,

$$\begin{aligned}
M_3 &\le 3\int_{-\tau}^{0}\int_{0}^{t+\theta}\int_{\mathcal{U}-\{0\}}\mathbb{E}\left\|\Phi(t+\theta-s)G(X_s(\phi))z\right\|^2\widetilde{N}(\mathrm{d}s,\mathrm{d}z)\mathrm{d}\theta \\
&\le 3\int_{\mathcal{U}-\{0\}}\|z\|_{\mathcal{U}}^2\nu(\mathrm{d}z)\left[\left(1+\frac{1}{\varepsilon}\right)K_2^2\int_{-\tau}^{0}\int_{0}^{t+\theta}e^{-2\lambda(t+\theta-s)}\mathrm{d}s\mathrm{d}\theta\right. \\
&\quad \left.+\,(1+\varepsilon)K_1^2\int_{-\tau}^{0}\int_{0}^{t+\theta}e^{-\lambda(t+\theta-s)}\|X_s(\phi)\|_{\mathcal{L}}^2\mathrm{d}s\mathrm{d}\theta\right] \\
&\le \frac{3\,(1+\varepsilon)\,\tau K_z K_2^2}{2\varepsilon\lambda}+3K_z(1+\varepsilon)K_1^2e^{2\lambda\tau}\int_0^t e^{-\lambda(t-s)}\mathbb{E}\|X_s(\phi)\|_{\mathcal{L}}^2\mathrm{d}s.
\end{aligned} \tag{4.7}$$

Thus (4.5), (4.6) and (4.7) together imply that

$$e^{\lambda t}\mathbb{E}\|X_t(\phi)\|_{\mathcal{L}}^2 \le \alpha_2+\gamma_2e^{\lambda t}+\beta_2\int_0^t e^{\lambda s}\mathbb{E}\|X_s(\phi)\|_{\mathcal{L}}^2\mathrm{d}s,$$

where

$$\begin{aligned}
\alpha_2 &= \frac{3\,(1+\varepsilon)\,(e^{2\lambda\tau}-1)}{2\varepsilon\lambda}\|\phi(0)\|, \\
\gamma_2 &= \frac{3\,(1+\varepsilon)\,\tau K_2^2}{2\varepsilon\lambda}(1+\varepsilon+\mathrm{Tr}(Q)+K_z), \\
\beta_2 &= 3K_1^2(1+\varepsilon)\left[(1+\varepsilon)\lambda^{-1}e^{\lambda\tau}+\mathrm{Tr}(Q)e^{2\lambda\tau}+K_ze^{2\lambda\tau}\right].
\end{aligned}$$

Then Gronwall's inequality gives that

$$e^{\lambda t}\mathbb{E}\|X_t(\phi)\|_{\mathcal{L}}^2 \le \gamma_2e^{\lambda t}+\alpha_2+\beta_2\int_0^t\left(\gamma_2e^{\lambda s}+\alpha_2\right)e^{\beta_2(t-s)}\mathrm{d}s.$$

It follows from $3K_1^2e^{2\lambda\tau}\left[\lambda^{-1}e^{-\lambda\tau}+\mathrm{Tr}(Q)+K_z\right]<\lambda$ that there exists $\varepsilon>0$ such that

$$3K_1^2e^{2\lambda\tau}(1+\varepsilon)\left[(1+\varepsilon)\lambda^{-1}e^{-\lambda\tau}+\mathrm{Tr}(Q)+K_z\right]<\lambda$$

and hence that

$$\begin{aligned}
\mathbb{E}\|X_t(\phi)\|_{\mathcal{L}}^2 &\le \gamma_2+\alpha_2e^{-\lambda t}+\beta_2e^{(\beta_2-\lambda)t}\int_0^t\left(\gamma_2e^{\lambda s}+\alpha_2\right)e^{-\beta_3 s}\mathrm{d}s \\
&\le \gamma_2+2\alpha_2+\frac{\beta_2\gamma_2}{\lambda-\beta_2}.
\end{aligned}$$

This completes the proof. □

Now we only need to show the tightness of solution (2.2) with Lévy jump process.

Lemma 4.7. *Suppose that $3K_1^2 e^{2\lambda\tau}\left[\lambda^{-1}e^{-\lambda\tau} + \mathrm{Tr}(Q) + K_z\right] < \lambda$. Then for any bounded set U of $\mathcal{L}$,*

$$\lim_{t\to\infty}\sup_{\phi,\psi\in U}\mathbb{E}\|X_t(\phi)-X_t(\psi)\|_{\mathcal{L}}^2 = 0.$$

Proof. Using the similar arguments as the proof of Lemmas 3.1 and 4.6, we have

$$\begin{aligned}
&\mathbb{E}\|X_t(\phi)-X_t(\psi)\|_{\mathcal{L}}^2\\
&\le 3\mathbb{E}\int_{-\tau}^{0}\Bigg\{\left\|\Phi(t+\theta)(\phi(0)-\psi(0))+\int_0^{t+\theta}\Phi(t+\theta-s)(F(X_s(\phi))-F(X_s(\psi)))\mathrm{d}s\right\|^2\\
&\quad+\left\|\int_0^{t+\theta}\Phi(t+\theta-s)\left[G(X_s(\phi))-G(X_s(\psi))\right]dW_Q\right\|^2\\
&\quad+\left\|\int_0^{t+\theta}\int_{\mathcal{U}-\{0\}}\Phi(t+\theta-s)\left[G(X_s(\phi))-G(X_s(\psi))\right]z\widetilde{N}(\mathrm{d}s,\mathrm{d}z)\right\|^2\Bigg\}\mathrm{d}\theta\\
&\le 3\mathbb{E}\int_{-\tau}^{0}\Bigg\{\left(1+\frac{1}{\varepsilon}\right)e^{-2\lambda(t+\theta)}\|\phi-\psi\|_{\mathcal{L}}^2 \qquad (4.8)\\
&\quad+(1+\varepsilon)\left\|\int_0^{t+\theta}\Phi(t+\theta-s)(F(X_s(\phi))-F(X_s(\psi)))\mathrm{d}s\right\|^2\\
&\quad+\left\|\int_0^{t+\theta}\Phi(t+\theta-s)\left[G(X_s(\phi))-G(X_s(\psi))\right]dW_Q\right\|^2\\
&\quad+\left\|\int_0^{t+\theta}\int_{\mathcal{U}-\{0\}}\Phi(t+\theta-s)\left[G(X_s(\phi))-G(X_s(\psi))\right]z\widetilde{N}(\mathrm{d}s,\mathrm{d}z)\right\|^2\Bigg\}\mathrm{d}\theta\\
&=:\frac{e^{-2\lambda t}3(1+\varepsilon)(e^{2\lambda\tau}-1)}{2\varepsilon\lambda}\|\phi(0)-\psi(0)\|^2+N_1+N_2+N_3.
\end{aligned}$$

Similar to (3.5) we have

$$N_1 \le \frac{3(1+\varepsilon)K^2e^{\lambda\tau}}{\lambda}\int_0^t e^{-\lambda(t-s)}\mathbb{E}\|X_s(\phi)-X_s(\psi)\|_{\mathcal{L}}^2\mathrm{d}s. \tag{4.9}$$

Burkholder–Davis–Gundy inequality implies that

$$\begin{aligned}
N_2 &\le 3\mathrm{Tr}(Q)\mathbb{E}\int_{-\tau}^{0}\int_0^{t+\theta}\|\Phi(t+\theta-s)\left[G(X_s(\phi))-G(X_s(\psi))\right]\|^2\mathrm{d}s\mathrm{d}\theta\\
&\le 3\mathrm{Tr}(Q)K^2e^{2\lambda\tau}\int_0^t e^{-\lambda(t-s)}\mathbb{E}\|X_s(\phi)-X_s(\psi)\|_{\mathcal{L}}^2\mathrm{d}s,
\end{aligned} \tag{4.10}$$

It follows from Proposition 4.4 that there exists $K_z > 0$ such that

$$\begin{aligned}
N_3 &\le 3\int_{-\tau}^{0}\int_0^{t+\theta}\int_{\mathcal{U}-\{0\}}\mathbb{E}\left\|\Phi(t+\theta-s)\left[G(X_s(\phi))-G(X_s(\psi))\right]z\right\|^2\widetilde{N}(\mathrm{d}s,\mathrm{d}z)\mathrm{d}\theta\\
&\le 3\int_{\mathcal{U}-\{0\}}\|z\|_{\mathcal{U}}^2\nu(\mathrm{d}z)\int_{-\tau}^{0}\int_0^{t+\theta}e^{-2\lambda(t+\theta-s)}K\mathbb{E}\|X_s(\phi)-X_s(\psi)\|_{\mathcal{L}}^2\mathrm{d}s\mathrm{d}\theta\\
&\le 3K_zK^2e^{2\lambda\tau}\int_0^t e^{-\lambda(t-s)}\mathbb{E}\|X_s(\phi)-X_s(\psi)\|_{\mathcal{L}}^2\mathrm{d}s.
\end{aligned} \tag{4.11}$$

Thus (4.9), (4.10) and (4.11) together imply that

$$e^{\lambda t}\mathbb{E}\|X_t(\phi)-X_t(\psi)\|_{\mathcal{L}}^2 \le \alpha_3+\beta_3\int_0^t e^{\lambda s}\mathbb{E}\|X_s(\phi)-X_s(\psi)\|_{\mathcal{L}}^2\mathrm{d}s,$$

where

$$\alpha_3 = \frac{(3(1+\varepsilon)e^{2\lambda\tau} - 1)}{2\varepsilon\lambda} \|\phi(0) - \psi(0)\|^2,$$
$$\beta_3 = 3K^2 e^{2\lambda\tau} \left[\lambda^{-1} e^{-\lambda\tau}(1+\varepsilon) + \mathrm{Tr}(Q) + K_z\right].$$

It follows from $3K^2 e^{2\lambda\tau}(\lambda^{-1}e^{-\lambda\tau} + \mathrm{Tr}(Q) + K_z) < \lambda$ that there exists $\varepsilon > 0$ such that

$$3K^2 e^{2\lambda\tau}(\lambda^{-1}e^{-\lambda\tau}(1+\varepsilon) + \mathrm{Tr}(Q) + K_z) < \lambda.$$

Then Gronwall's inequality gives that

$$e^{\lambda t}\mathbb{E}\|X_t(\phi) - X_t(\psi)\|_{\mathcal{L}}^2 \le \alpha_3 e^{\beta_3 t}. \tag{4.12}$$

This completes the proof. □

In view of Lemmas 4.6 and 4.7, it suffices to show the uniqueness of invariant measures of (1.1) driven by Lévy jump process. If $\mu, \tilde{\mu} \in \mathbb{P}(\mathcal{L})$ are two different invariant measures, then for any $f \in \mathcal{M}$, it follows from (4.12) and the invariance of $\mu, \tilde{\mu} \in \mathbb{P}(\mathcal{L})$ that

$$|\mu(f) - \tilde{\mu}(f)| = \int_{\mathcal{L}}\int_{\mathcal{L}} |\mathbb{P}_t f(\phi) - \mathbb{P}_t f(\psi)|\mu(\mathrm{d}\phi)\tilde{\mu}(\mathrm{d}\psi) \le K_4 e^{-\tilde{\alpha}t}, \qquad t \ge 0,$$

for some $K_4 > 0$, where $\tilde{\alpha} = \frac{3}{2}K^2 e^{2\lambda\tau}(\lambda^{-1}e^{-\lambda\tau} + \mathrm{Tr}(Q) + K_z) - \frac{1}{2}\lambda$. Thus, we obtain the following main result immediately.

Theorem 4.8. *Under the assumptions of Lemmas 4.6 and 4.7, equation (1.1) driven by Lévy jump process has a unique invariant measure.*

5 Application

Let $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$ be equipped with the usual Riemannian metric, and let $\mathrm{d}\xi$ denote the Lebesgue measure on $\mathbb{T}$. For any $p \ge 1$, let

$$L^p(\mathbb{T},\mathbb{R}) = \left\{x : \mathbb{T} \to \mathbb{R};\ \|x\|_p \triangleq \left[\int_{\mathbb{T}} |x(\xi)|^p \mathrm{d}\xi\right]^{1/p} < \infty\right\},$$

and

$$H = \left\{x \in L^2(\mathbb{T},\mathbb{R}) : \int_{\mathbb{T}} x(\xi)\mathrm{d}\xi = 0\right\}.$$

It is easy to see that H is a real separable Hilbert space with the inner product

$$\langle x, y\rangle = \int_{\mathbb{T}} x(\xi)y(\xi)\mathrm{d}\xi, \qquad x, y \in H,$$

and the norm $\|x\| = \sqrt{\langle x, x\rangle}$. In the following two subsections, we consider two stochastic reaction-diffusion equations on torus $\mathbb{T}$.

5.1 A Brownian motion case

Consider a stochastic reaction-diffusion equation driven by a Brownian motion $\{W(t)\}_{t\geq 0}$ on torus $\mathbb{T}$ as follows:

$$\begin{cases} du(t,\xi) = \left[\frac{\partial^2}{\partial\xi^2}u(t,\xi) + f(u(t-1,\xi))\right]dt + g(u(t-1,\xi))dW(t,\xi), & t\geq 0, \\ u(t,\xi) = \phi(t,\xi), \quad t\in[-1,0], \end{cases} \tag{5.1}$$

where $\phi \in \mathcal{C} := C([-1,0], W^{1,2}(\mathbb{T}))$, f: $\mathbb{R}\to\mathbb{R}$ and g: $\mathbb{R}\to\mathbb{R}$ are Lipschitz continuous and satisfy the linear growth, i.e., there exist positive constants K, K_1, and K_2 such that

$$\begin{aligned} f(u)-f(v)| \leq L_f|u-v|, \qquad & |g(u)-g(v)| \leq L_g|u-v|, \\ |f(u)| \leq L_1|u| + L_2, \qquad & |g(u)| \leq L_3|u| + L_4 \end{aligned} \tag{5.2}$$

for all $u,v\in\mathbb{R}$. Obviously, $A = \frac{\partial^2}{\partial\xi^2}$ is a self-adjoint operator on H with the discrete spectral. More precisely, there exist an orthogonal basis $\{\mathbf{e}_k = \exp\{ik(\cdot)\} : k\in\mathbb{Z}_*\}$ with $\mathbb{Z}_* = \mathbb{Z}\setminus\{0\}$, and a sequence of real numbers $\{\lambda_k = k^2 : k\in\mathbb{Z}_*\}$ such that $-A\mathbf{e}_k = \lambda_k\mathbf{e}_k$. Let V be the domain of the fractional operator $(-A)^{1/2}$, that is,

$$V = \left\{ \sum_{k\in\mathbb{Z}_*} \sqrt{\lambda_k}a_k\mathbf{e}_k : \{a_k\}_{k\in\mathbb{Z}_*} \subset \mathbb{R}, \sum_{k\in\mathbb{Z}_*} a_k^2 < \infty \right\}$$

with the inner product

$$\langle u,v\rangle_V = \langle(-A)^{1/2}u, (-A)^{1/2}v\rangle = \sum_{k\in\mathbb{Z}_*}\lambda_k\langle u,\mathbf{e}_k\rangle\langle v,\mathbf{e}_k\rangle,$$

and with the norm $\|u\|_V = \sqrt{\langle u,u\rangle_V} = \|(-A)^{1/2}u\|$. Clearly, V is densely and compactly embedded in H.

For every $u\in H$, there exists $\{a_k\}_{k\in\mathbb{Z}_*}\subset\mathbb{R}$ such that $u = \sum_{x\in\mathbb{Z}_*} a_k\mathbf{e}_k$. Thus, we have

$$\begin{aligned} \langle -Au, u\rangle_V &= \sum_{k\in\mathbb{Z}_*}\lambda_k\langle -Au,\mathbf{e}_k\rangle\langle u,\mathbf{e}_k\rangle \\ &= \sum_{k\in\mathbb{Z}_*}\lambda_k\langle u, -A\mathbf{e}_k\rangle\langle u,\mathbf{e}_k\rangle = \sum_{k\in\mathbb{Z}_*} a_k^2\lambda_k^2 \geq \lambda_1^2\|u\|^2. \end{aligned}$$

Thus, we obtain (2.1) with $\lambda = \lambda_1^2$.

We consider a symmetric positive linear operator Q in H such that $Q\mathbf{e}_k = q_k\mathbf{e}_k$ for $k\in\mathbb{Z}_*$, where $\{q_k\}_{k\in\mathbb{Z}_*}$ is a bounded sequence of nonnegative real numbers. Thus, $\mathrm{Tr}(Q) \triangleq \sum_{k\in\mathbb{Z}_*}\langle Q\mathbf{e}_k,\mathbf{e}_k\rangle = \sum_{k\in\mathbb{Z}_*} q_k < \infty$, and Q is also called a trace class operator. Let $\{W(t)\}_{t\geq 0}$ be a H-valued Q-Wiener process given by

$$W(t) = \sum_{x\in\mathbb{Z}_*}\sqrt{q_k}W_k(t)\mathbf{e}_k,$$

where $\{W_k(t) : t\geq 0\}_{k\in\mathbb{Z}^*}$ be a sequence of independent standard one-dimensional Brownian motions on some filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$, that is, $W_k(t)\sim\mathcal{N}(0,t)$, $\mathbb{E}W_k(t) = 0$, $\mathbb{E}[W_k(t)]^2 = t$, and $\mathbb{E}[W_k(t)W_k(s)] = \min\{t,s\}$. It is easy to see that the infinite series of $W(t)$ converges in $L^2(\Omega)$, and satisfies

$$\mathbb{E}\langle W(t), W(t)\rangle = t\mathrm{Tr}(Q), \qquad \mathbb{E}\left(\langle W(t),a\rangle\langle W(s),b\rangle\right) = (t\wedge s)\langle a,b\rangle.$$

Then we can rewrite system (5.1) into the abstract form (1.1) with $\tau = 1$, $F(u_t) = f(u(t-1,\cdot))$ and $G(u_t) = g(u(t-1,\cdot))$. Note that the segment process $u_t = u(t+s,\xi), s \in [-1,0]$ is equipped with norm in $\mathcal{C}$, i.e.,

$$\|u_t\|_{\mathcal{C}} = \max_{s\in[-1,0]} \|u(t+s,\xi)\| = \max_{s\in[-1,0]} \left\{ \int_{\mathbb{T}} |u(t+s,\xi)|^2 d\xi \right\}^{\frac{1}{2}}.$$

In what follows, we shall verify that F and G satisfy hypothesis (3.2). In fact, it follows from (5.2) and Minkowski inequality that

$$\begin{aligned}
\|F(u_t) - F(v_t)\| &= \left\{ \int_{\mathbb{T}} [f(u(t-1,\xi)) - f(v(t-1,\xi))]^2 \, \mathrm{d}\xi \right\}^{\frac{1}{2}} \\
&\le \left\{ L_f^2 \int_{\mathbb{T}} [u(t-1,\xi) - v(t-1,\xi)]^2 \, \mathrm{d}\xi \right\}^{\frac{1}{2}} \\
&\le L_f \|u_t - v_t\|_{\mathcal{C}}, \\
\|G(u_t) - G(v_t)\|_{\mathfrak{L}_2^Q} &= \left\{ \sum_{k\in\mathbb{Z}_*} \langle (G(u_t) - G(v_t)) Q (G(u_t) - G(v_t))^* e_k, e_k \rangle \right\}^{\frac{1}{2}} \\
&\le \left\{ \sum_{k\in\mathbb{Z}_*} L_g^2 \int_{\mathbb{T}} |u(t-1,\xi) - v(t-1,\xi)|^2 d\xi \, \langle Qe_k, e_k \rangle \right\}^{\frac{1}{2}} \\
&\le \left\{ \sum_{k\in\mathbb{Z}_*} L_g^2 \|u_t - v_t\|_{\mathcal{C}}^2 \, \langle Qe_k, e_k \rangle \right\}^{\frac{1}{2}} \\
&= L_g \sqrt{\mathrm{Tr}(Q)} \|u_t - v_t\|_{\mathcal{C}},
\end{aligned}$$

and

$$\begin{aligned}
\|F(u_t)\| &= \left\{ \int_{\mathbb{T}} [f(u(t-1,\xi))]^2 \, \mathrm{d}\xi \right\}^{\frac{1}{2}} \\
&\le \left\{ \int_{\mathbb{T}} (L_1 |u(t-1,\xi)| + L_2)^2 \, \mathrm{d}\xi \right\}^{\frac{1}{2}} \\
&\le \left\{ L_1 \int_{\mathbb{T}} |u(t-1,\xi)|^2 d\xi \right\}^{\frac{1}{2}} + \left(\int_{\mathbb{T}} L_2^2 d\xi \right)^{\frac{1}{2}} \le L_1 \|u_t\|_{\mathcal{C}} + L_2, \\
\|G(u_t)\|_{\mathfrak{L}_2^Q} &= \left\{ \sum_{k\in\mathbb{Z}_*} \langle G(u_t) Q G(u_t)^* e_k, e_k \rangle \right\}^{\frac{1}{2}} \\
&= \left\{ \sum_{k\in\mathbb{Z}_*} \left\langle \int_{\mathbb{T}} |g(u(t-1,\xi))|^2 d\xi Q e_k, e_k \right\rangle \right\}^{\frac{1}{2}} \\
&\le \left\{ \sum_{k\in\mathbb{Z}_*} \int_{\mathbb{T}} |L_3 u(t-1,\xi) + L_4|^2 d\xi \, \langle Qe_k, e_k \rangle \right\}^{\frac{1}{2}} \\
&\le \left[\int_{\mathbb{T}} |L_3 u(t-1,\xi) + L_4|^2 d\xi \right]^{\frac{1}{2}} \left[\sum_{k\in\mathbb{Z}_*} q_k \right]^{\frac{1}{2}} \le (L_3 \|u_t\|_{\mathcal{C}} + L_4) \sqrt{\mathrm{Tr}(Q)}.
\end{aligned}$$

Then we can set the parameter values in (3.2) as follows

$$K = L_f + L_g \left(\sum_{k\in\mathbb{Z}_*} q_k \right)^{\frac{1}{2}}, \qquad K_1 = L_1 + L_3 \left(\sum_{k\in\mathbb{Z}_*} q_k \right)^{\frac{1}{2}}, \qquad K_2 = L_2 + L_4 \left(\sum_{k\in\mathbb{Z}_*} q_k \right)^{\frac{1}{2}}.$$

Thus, from Theorems 3.3 and 3.8 we have the following result.

Corollary 5.1. *Assume that*

$$\max \left\{ L_f + L_g \left(\sum_{k\in\mathbb{Z}_*} q_k \right)^{\frac{1}{2}}, L_1 + L_3 \left(\sum_{k\in\mathbb{Z}_*} q_k \right)^{\frac{1}{2}} \right\} \le (2e^2 + 2e)^{-\frac{1}{2}}.$$

Then equation (5.1) has a unique invariant measure and a pullback attractor.

5.2 A Poisson jumps case

Let $\{N(dt, \mathrm{d}z) : t \in \mathbb{R}^+, z \in \mathbb{R}\}$ is a centered Poisson random measure with parameter $\nu(\mathrm{d}z)dt = 2m(z)\mathrm{d}z dt$, and $\widetilde{N}(dt, \mathrm{d}z) = N(dt, \mathrm{d}z) - \nu(\mathrm{d}z)dt$ be a compensated Poisson random measure, where

$$m(z) = \frac{1}{\sqrt{2\pi} z} \exp\left\{ -\frac{\ln^2 z}{2} \right\}, \qquad 0 \le z < \infty$$

is the density function of a lognormal random variable. Consider the following stochastic delay differential equations with Poisson jumps on $\mathbb{T}$:

$$\begin{cases} du(t,x) = \left[\dfrac{\partial^2}{\partial x^2} u(t,x) + f(u(t-1,x)) \right] \mathrm{d}t + \displaystyle\int_{\mathcal{U}} g(u(t-1,x)) z \widetilde{N}(\mathrm{d}t, \mathrm{d}z), \qquad t \ge 0, \\ u(t,x) = \phi(t,x), \qquad t \in [-1, 0], \end{cases} \tag{5.3}$$

where $\phi \in \mathcal{C} := C([-1,0], W^{1,2}(\mathbb{T}))$, $\mathcal{U} = \{z \in \mathbb{R} : 0 < |z| \le 1\}$, $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ satisfy (5.2).

Define $A : H \to H$ by $A = \frac{\partial^2}{\partial x^2}$. It follows from the arguments in Section 5.1 that the space H and operator A is well defined. Note that $\lambda = \lambda_1^2 = 1$. Note that A is the generator of an analytic semigroup $\Phi(t), t \ge 0$, equation (5.3) can be given by the following integral equation

$$\begin{aligned} u(t,x) = \Phi(t)\phi(t,x) &+ \int_0^t \Phi(t-s) f(u(t-1,x)) \mathrm{d}s \\ &+ \int_0^{t+} \int_{\mathcal{U}} \Phi(t-s) g(u(t-1,x)) z \widetilde{N}(\mathrm{d}z, \mathrm{d}s) \end{aligned} \tag{5.4}$$

for $t \in [0, +\infty)$ and $x \in H$. Note that $\nu(\mathcal{U}) = 1$ and

$$K_z = \int_{\mathcal{U}} z^2 \nu(\mathrm{d}z) = \int_0^1 \frac{z}{\sqrt{2\pi}} \exp\left\{ -\frac{\ln^2 z}{2} \right\} \mathrm{d}z \le e^2.$$

Then we can rewrite system (5.3) into the abstract form (1.1) with $\tau = 1$, $F(u_t) = f(u(t-1,\cdot))$ and $G(u_t) = g(u(t-1,x))$. Note that the segment process $u_t = u(t+s,\xi), s \in [-1,0]$ is equipped with norm in $\mathcal{C}$, i.e.,

$$\|u_t\|_{\mathcal{C}} = \max_{s\in[-1,0]} \|u(t+s,\xi)\| = \max_{s\in[-1,0]} \left\{ \int_{\mathbb{T}} |u(t+s,\xi)|^2 d\xi \right\}^{\frac{1}{2}}.$$

It follows from Section 5.1 that

$$\|F(u_t) - F(v_t)\| \leq L_f \|u_t - v_t\|_{\mathcal{C}}$$

and

$$\|F(u_t)\| \leq L_1 \|u_t\|_{\mathcal{C}} + L_2.$$

It is easy to check that

$$\begin{aligned} \|G(u_t) - G(v_t)\|_{\mathscr{L}(\mathbb{R},H)} &= \left\{ \int_{\mathbb{T}} [g(u(t-1,\xi)) - g(v(t-1,\xi))]^2 \, \mathrm{d}\xi \right\}^{\frac{1}{2}} \\ &\leq \left\{ L_g^2 \int_{\mathbb{T}} [u(t-1,\xi) - v(t-1,\xi)]^2 \, \mathrm{d}\xi \right\}^{\frac{1}{2}} \\ &\leq L_g \|u_t - v_t\|_{\mathcal{C}} \end{aligned}$$

and

$$\begin{aligned} \|G(u_t)\|_{\mathscr{L}(\mathbb{R},H)} &= \left\{ \int_{\mathbb{T}} [g(u(t-1,\xi))]^2 \, \mathrm{d}\xi \right\}^{\frac{1}{2}} \\ &\leq \left\{ \int_{\mathbb{T}} (L_3 |u(t-1,\xi)| + L_4)^2 \, \mathrm{d}\xi \right\}^{\frac{1}{2}} \\ &\leq \left\{ L_3 \int_{\mathbb{T}} |u(t-1,\xi)|^2 d\xi \right\}^{\frac{1}{2}} + \left(\int_{\mathbb{T}} L_4^2 \mathrm{d}\xi \right)^{\frac{1}{2}} \\ &\leq L_3 \|u_t\|_{\mathcal{C}} + L_4. \end{aligned}$$

Then we set the parameter values in (4.3) as follows

$$K = L_f + L_g, \qquad K_1 = L_1 + L_3, \qquad K_2 = L_2 + L_4.$$

Thus the result of existence of invariant measure of (5.3) follows from Theorems 4.6 and 4.7.

Corollary 5.2. *Assume that*

$$\max \left\{ L_f + L_g, L_1 + L_3 \right\} \leq (3e + 3e^4)^{-\frac{1}{2}}.$$

Then equation (5.3) has a unique invariant measure.

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Multiple nonsymmetric nodal solutions for quasilinear Schrödinger system

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Abstract. In this paper, we consider the quasilinear Schrödinger system in $\mathbb{R}^N$ ($N \geq 3$):

$$\begin{cases} -\Delta u + A(x)u - \dfrac{1}{2}\Delta(u^2)u = \dfrac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta}, \\ -\Delta v + Bv - \dfrac{1}{2}\Delta(v^2)v = \dfrac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v, \end{cases}$$

where $\alpha, \beta > 1$, $2 < \alpha + \beta < \frac{4N}{N-2}$, $B > 0$ is a constant. By using a constrained minimization on Nehari–Pohožaev set, for any given integer $s \geq 2$, we construct a non-radially symmetrical nodal solution with its $2s$ nodal domains.

Keywords: quasilinear Schrödinger system, Nehari–Pohožaev set, non-radially symmetrical nodal solutions.

2020 Mathematics Subject Classification: 35J05, 35J20, 35J60.

1 Introduction

We study the following quasilinear Schrödinger system

$$\begin{cases} -\Delta u + A(x)u - \dfrac{1}{2}\Delta(u^2)u = \dfrac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta}, \\ -\Delta v + Bv - \dfrac{1}{2}\Delta(v^2)v = \dfrac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v, \end{cases} \tag{1.1}$$

where $u(x) \to 0$, $v(x) \to 0$ as $|x| \to \infty$, $N \geq 3$, $u := u(x)$, $v := v(x)$ be real valued functions on $\mathbb{R}^N$, α, $\beta > 1$, $2 < \alpha + \beta < \frac{4N}{N-2}$, $B > 0$ is a constant. In the last two decades, much attention has been devoted to the quasilinear Schrödinger equation of the form

$$-\Delta u + V(x)u - \frac{1}{2}u\Delta(u^2) = |u|^{p-2}u, \qquad x \in \mathbb{R}^N. \tag{1.2}$$

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The equation (1.2) is related to the existence of standing waves of the following quasilinear Schrödinger equation

$$i\partial_t z = -\Delta z + V(x)z - l(|z|^2)z - \frac{1}{2}\Delta g(|z|^2)g'(|z|^2)z, \qquad x \in \mathbb{R}^N, \tag{1.3}$$

where V is a given potential, l and g are real functions. The equation (1.3) has been used as models in several areas of physics corresponding to various types of g. The superfluid film equation in plasma physics has this structure for $g(s) = s$ [9]. In the case $g(s) = (1+s)^{\frac{1}{2}}$, the equation (1.3) models the self-channeling of a high-power ultra short laser in matter [19]. The equation (1.3) also appears in fluid mechanics [9, 10], in the theory of Heidelberg ferromagnetism and magnus [11], in dissipative quantum mechanics and in condensed matter theory [14]. When considering the case $g(s) = s$, one obtains a corresponding equation of elliptic type like (1.2). For more detailed mathematical and physical interpretation of equations like (1.2), we refer to [1,3,4,12,18,21] and the references therein.

In recent years, there has been increasing interest in studying problem (1.2), see for examples, [5,6,8,15,16,24,25] and the references therein. More precisely, by the Mountain Pass Theorem and the principle of symmetric criticality, Severo [22] obtained symmetric and non-symmetric solutions for quasilinear Schrödinger equation (1.2). In [13], when $4 \le p < \frac{4N}{N-2}$, Liu, Wang and Wang established the existence results of a positive ground state solution and a sign-changing ground state solution were given by using the Nehari method for (1.2). Based on the method of perturbation and invariant sets of descending flow, Zhang and Liu [27] studied the nonautonomous case of (1.2), they obtained the existence of infinitely many sign-changing solutions for $4 < p < \frac{4N}{N-2}$. With the help of Nehari method and change of variables, Deng, Peng and Wang [7] considered

$$-\Delta u + V(x)u - u\Delta(u^2) = \lambda |u|^{p-2}u + |u|^{\frac{4N}{N-2}-2}u, \qquad x \in \mathbb{R}^N, \tag{1.4}$$

and proved that (1.4) has at least one pair of k-node solutions if either $N \ge 6$ and $4 < p < \frac{4N}{N-2}$ or $3 \le N < 6$ and $\frac{2(N+2)}{N-2} < q < \frac{4N}{N-2}$. In addition, problem (1.4) still has at least one pair of k-node solutions if $3 \le N < 6$, $4 < q \le \frac{2(N+2)}{N-2}$ and λ sufficiently large. Note that all sign-changing solutions obtained in [7, 13, 27] are only valid for $4 < p < \frac{4N}{N-2}$. When $2 < p < \frac{4N}{N-2}$, Ruiz and Siciliano [20] showed equation (1.2) has a ground states solution via Nehari–Pohožaev type constraint and concentration-compactness lemma, Wu and Wu [26] obtained the existence of radial solutions for (1.2) by using change of variables.

It is natural to pose a series of interesting questions: whether we can find an unified approach to obtain sign-changing solutions for the full subcritical range of $2 < \alpha + \beta < \frac{4N}{N-2}$? Further, whether we can extend these results to system of the quasilinear Schrödinger system? To answer these two questions, we adopt an action of finite subgroup G of $O(2)$ from Szulkin and Waliullah [23] and look for the existence of non-radially symmetrical nodal solutions for quasilinear Schrödinger system (1.1).

Before stating our main results, we make the following assumptions:

(A_1) $A \in C^1(\mathbb{R}^N, \mathbb{R}^+), 0 < A_0 \le A(x) \le A_\infty = \lim_{|x|\to\infty} A(x) < +\infty$;

(A_2) $\nabla A(x) \cdot x \in L^\infty(\mathbb{R}^N)$, $(\alpha + \beta - 2)A(x) - \nabla A(x) \cdot x \ge 0$;

(A_3) the map $s \mapsto s^{\frac{N+2}{N+\alpha+\beta}} A(s^{\frac{1}{N+\alpha+\beta}} x)$ is concave for any $x \in \mathbb{R}^N$;

(A_4) $A(x)$ is radially symmetric with respect to the first two coordinates, that is to say, if $(x_1, x_2, x_3, \dots, x_N)$, $(y_1, y_2, y_3, \dots, y_N) \in \mathbb{R}^N$ and $x_1^2 + x_2^2 = y_1^2 + y_2^2$, then

$$A(x_1, x_2, z_3, \dots, z_N) = A(y_1, y_2, z_3, \dots, z_N).$$

It is worth noting that (A_1) is used to derive the existence of a strongly convergent subsequence, while for the system, we only need one such kind of condition in our equations, which seems to be a different phenomenon due to the coupling of u and v. (A_2)–(A_3) once appeared in [20, 26] to obtain the existence of ground states solutions for the quasilinear Schrödinger equation.

Our main result reads as follows.

Theorem 1.1. *Assume that (A_1)–(A_4) hold. For any given integer $s \geq 2$, the problem (1.1) possesses a non-radially symmetrical nodal solution with its $2s$ nodal domains.*

Corollary 1.2. *If $A(x)$ is a positive constant, one can still obtain the same results as Theorem 1.1 for system (1.1).*

Remark 1.3. Since $s \in \mathbb{N}$ is arbitrary, the solution we obtained in Theorem 1.1 is actually a result of multiplicity.

Remark 1.4. As a main novelty with respect to some results in [7, 13, 27], we are able to deal with exponents $\alpha + \beta \in (2, \frac{4N}{N-2})$ and obtain the existence and multiplicity of nodal solution without any radial symmetry.

The rest of the paper is organized as follows. In Section 2, we establish some preliminary results. Theorem 1.1 is proved in Section 3.

2 Preliminaries

Throughout this paper, $\|u\|_{H^1}$ and $|u|_r$ denote the usual norms of $H^1(\mathbb{R}^N)$ and $L^r(\mathbb{R}^N)$ for $r > 1$, respectively. C and C_i $(i = 1, 2, \dots)$ denote (possibly different) positive constants and $\int_{\mathbb{R}^N} g$ denotes the integral $\int_{\mathbb{R}^N} g(z)dz$. The $\to$ and $\rightharpoonup$ denote strong convergence and weak convergence, respectively.

Let $H^1(\mathbb{R}^N)$ be the usual Sobolev space, define $X := H \times H$ with

$$H := \left\{ u \in H^1(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} u^2 |\nabla u|^2 < +\infty \right\}.$$

The term $\int_{\mathbb{R}^N} u^2 |\nabla u|^2$ is not convex and H is not even a vector space. So, the usual min-max techniques cannot be directly applied, nevertheless H is a complete metric space with distance

$$d_H(u, \omega) = \|u - \omega\|_{H^1} + |\nabla u^2 - \nabla \omega^2|_2.$$

Define

$$d_X\big((u, v), (\omega, \nu)\big) := \|u - \omega\|_{H^1} + |\nabla u^2 - \nabla \omega^2|_2 + \|v - \nu\|_{H^1} + |\nabla v^2 - \nabla \nu^2|_2.$$

Then we call $(u, v) \in X$ is a weak solution of (1.1) if for any $\varphi_1, \varphi_2 \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \left((1 + u^2) \nabla u \nabla \varphi_1 + \left(u |\nabla u|^2 + A(x) u - \frac{2\alpha |u|^{\alpha-2} u |v|^\beta}{\alpha + \beta} \right) \varphi_1 \right) = 0,$$

and

$$\int_{\mathbb{R}^N}\left((1+v^2)\nabla v\nabla\varphi_2+\left(v|\nabla v|^2+Bv-\frac{2\beta|u|^\alpha|v|^{\beta-2}v}{\alpha+\beta}\right)\varphi_2\right)=0.$$

Hence there is a one-to-one correspondence between solutions of (1.1) and critical points of the following functional $I : X \to \mathbb{R}$ defined by

$$\begin{aligned}I(u,v)=&\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2+A(x)u^2+Bv^2)\\&+\frac{1}{2}\int_{\mathbb{R}^N}(u^2|\nabla u|^2+v^2|\nabla v|^2)-\frac{2}{\alpha+\beta}\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta.\end{aligned}\tag{2.1}$$

For any $\varphi_1,\varphi_2\in C_0^\infty(\mathbb{R}^N)$, $(u,v)\in X$, and $(u,v)+(\varphi_1,\varphi_2)\in X$, we compute the Gateaux derivative

$$\begin{aligned}\langle I'(u,v),(\varphi_1,\varphi_2)\rangle=&\int_{\mathbb{R}^N}((1+u^2)\nabla u\nabla\varphi_1+(1+v^2)\nabla v\nabla\varphi_2+u|\nabla u|^2\varphi_1\\&+v|\nabla v|^2\varphi_2+A(x)u\varphi_1+Bv\varphi_2)-\frac{2\alpha}{\alpha+\beta}\int_{\mathbb{R}^N}|u|^{\alpha-2}u|v|^\beta\varphi_1\\&-\frac{2\beta}{\alpha+\beta}\int_{\mathbb{R}^N}|v|^{\beta-2}v|u|^\alpha\varphi_2.\end{aligned}$$

Then, $(u,v)\in X$ is a solution of (1.1) if and only if

$$\langle I'(u,v),(\varphi_1,\varphi_2)\rangle=0,\qquad \varphi_1,\ \varphi_2\in C_0^\infty(\mathbb{R}^N).$$

Motivated by [23], we recall that a subset U of a Banach space $\mathbb{E}$ is called invariant with respect to an action of a group G (or G-invariant) if $gU\subset U$ for all $g\in G$, and a functional $I:U\to\mathbb{R}$ is invariant (or G-invariant) if $I(gu)=I(u)$ for all $g\in G$, $u\in U$. The subspace

$$\mathbb{E}_G:=\{u\in\mathbb{E}\mid gu=u\ \text{ for all } g\in G\}$$

is called the fixed point space of this action.

Let $x=(y,z)=(y_1,y_2,z_1,\dots,z_N)\in\mathbb{R}^N$ and let $O(2)$ be the group of orthogonal transformations acting on $\mathbb{R}^2$ by $(g,y)\mapsto gy$. For any positive integer s we define G_s to be the finite subgroup of $O(2)$ generated by the two elements α and β in $O(2)$, where α is the rotation in the y-plane by the angle $\frac{2\pi}{s}$ and β is the reflection in the line $y_1=0$ if $s=2$, and in the line $y_2=\tan(\pi/s)y_1$ for other s (so in complex notation $w=y_1+iy_2$, $\alpha w=we^{\frac{2\pi i}{s}}$, $\beta w=we^{\frac{2\pi i}{s}}$).

$\forall\ g\in G_s$, $x\in\mathbb{R}^N$, $gx:=(gy,z)$. Define the action of G_s on $H^1(\mathbb{R}^N)$ by setting

$$(g(u,v))x:=(gu,gv)x=(\det(g)ug^{-1}x,\det(g)vg^{-1}x).$$

Define

$$\begin{aligned}\mathcal{V}:=&\{(u,v)\in X\mid (u,v)(gx)=(\det(g)u(x),\det(g)v(x)),\ \ g\in G_s\},\\\mathcal{M}:=&\{(u,v)\in\mathcal{V}\backslash\{(0,0)\}\mid \mathcal{G}(u,v)=0\},\end{aligned}$$

where $\mathcal{G}:X\to\mathbb{R}$ and

$$\begin{aligned}\mathcal{G}(u,v)=&\frac{N}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)+\frac{N+2}{2}\int_{\mathbb{R}^N}(A(x)u^2+Bv^2\\&+u^2|\nabla u|^2+v^2|\nabla v|^2)-\frac{2(N+\alpha+\beta)}{\alpha+\beta}\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta.\end{aligned}$$

Let

$$m := \inf_{(u,v)\in\mathcal{M}} I(u,v). \tag{2.2}$$

Then our aim is to prove that m is achieved. In the rest of this section, we will give some properties of the set $\mathcal{M}$.

For any $u \in H^1(\mathbb{R}^N)$, we define $u_t : \mathbb{R}^+ \to H^1(\mathbb{R}^N)$ by:

$$u_t(x) := tu(t^{-1}x).$$

Let $t \in \mathbb{R}^+$ and $(u,v) \in X$. We have that

$$\begin{aligned} I(u_t,v_t) = {} & \frac{t^N}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2) + \frac{t^{N+2}}{2}\int_{\mathbb{R}^N}(A(x)u^2+Bv^2 \\ & + u^2|\nabla u|^2 + v^2|\nabla v|^2) - \frac{2t^{N+\alpha+\beta}}{\alpha+\beta}\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta. \end{aligned}$$

Denote $h_{uv}(t) := I(u_t,v_t)$. Since $\alpha+\beta > 2$, we see that $h_{uv}(t) > 0$ for $t > 0$ small enough and $h_{uv}(t) \to -\infty$ as $t \to +\infty$, this implies that $h_{uv}(t)$ attains its maximum. Moreover, $h_{uv}(t) : \mathbb{R}^+ \to \mathbb{R}$ is C^1 and

$$\begin{aligned} h'_{uv}(t) = {} & \frac{N}{2}t^{N-1}\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2) + \frac{N+2}{2}t^{N+1}\int_{\mathbb{R}^N}(A(x)u^2+Bv^2 \\ & + u^2|\nabla u|^2 + v^2|\nabla v|^2) - \frac{2(N+\alpha+\beta)}{\alpha+\beta}t^{N+\alpha+\beta-1}\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta. \end{aligned}$$

Lemma 2.1. *If $(u,v) \in X$ is a weak solution of (1.1), then (u,v) satisfies the following $P(u,v) = 0$, where*

$$\begin{aligned} P(u,v) := {} & \frac{N-2}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2+u^2|\nabla u|^2+v^2|\nabla v|^2) \\ & + \frac{N}{2}\int_{\mathbb{R}^N}(A(x)u^2+Bv^2) + \frac{1}{2}\int_{\mathbb{R}^N}\nabla A(x)\cdot xu^2 \\ & - \frac{2N}{\alpha+\beta}\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta. \end{aligned} \tag{2.3}$$

Proof. The proof is standard, so we omit it here. □

The lemma below shows (2.2) is well defined.

Lemma 2.2. *For any $(u,v) \in X$ and $u,v \neq 0$, the map h_{uv} attains its maximum at exactly one point $\bar{t}$. Moreover, h_{uv} is positive and increasing for $t \in [0,\bar{t}]$ and decreasing for $t > \bar{t}$. Finally*

$$m = \inf_{(u,v)\in X}\max_{t>0} I(u_t,v_t).$$

Proof. For any $t > 0$, set $s = t^{N+\alpha+\beta}$, we obtain

$$\begin{aligned} h_{uv}(s) = {} & \frac{s^{\frac{N}{N+\alpha+\beta}}}{2}\int_{\mathbb{R}^N}|\nabla u|^2 + \frac{s^{\frac{N}{N+\alpha+\beta}}}{2}\int_{\mathbb{R}^N}|\nabla v|^2 + \frac{s^{\frac{N+2}{N+\alpha+\beta}}}{2}\int_{\mathbb{R}^N}u^2|\nabla u|^2 \\ & + \frac{s^{\frac{N+2}{N+\alpha+\beta}}}{2}\int_{\mathbb{R}^N}v^2|\nabla v|^2 + \frac{s^{\frac{N+2}{N+\alpha+\beta}}}{2}\int_{\mathbb{R}^N}A(s^{\frac{1}{N+\alpha+\beta}}x)u^2 \\ & + \frac{s^{\frac{N+2}{N+\alpha+\beta}}}{2}\int_{\mathbb{R}^N}Bv^2 - \frac{2s}{\alpha+\beta}\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta. \end{aligned}$$

This is a concave function by condition (A_3) and we already know that it attains its maximum, let $\bar{t}$ be the unique point at which this maximum is achieved. Notice that $\mathcal{G}(u_t, v_t) = th'_{uv}(t)$, then $\bar{t}$ is the unique critical point of h_{uv} and h_{uv} is positive and increasing for $0 < t < \bar{t}$ and decreasing for $t > \bar{t}$. In particular, $\bar{t} \in \mathbb{R}$ is the unique value such that $u_{\bar{t}} \in \mathcal{M}$, and $I(u_{\bar{t}}, v_{\bar{t}})$ reaches a global maximum for $t = \bar{t}$. This finishes the proof. □

Lemma 2.3. *$m > 0$.*

Proof. For every $(u, v) \in \mathcal{M}$, it follows from (A_2) that

$$\begin{aligned} I(u,v) = &\frac{\alpha+\beta}{2(N+\alpha+\beta)} \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) \\ &+ \frac{\alpha+\beta-2}{2(N+\alpha+\beta)} \int_{\mathbb{R}^N} (Bv^2 + u^2|\nabla u|^2 + v^2|\nabla v|^2) \\ &+ \frac{1}{2(N+\alpha+\beta)} \int_{\mathbb{R}^N} \big((\alpha+\beta-2)A(x) - \nabla A(x)\cdot x\big)u^2 \\ > &\ 0. \end{aligned}$$

The proof is complete. □

3 Proof of Theorem 1.1

We need the following variant of the Lions Lemma.

Lemma 3.1. *If $q \in [2, \frac{4N}{N-2})$, $\{u_n\}$ is bounded in X, $r_0 > 0$ is such that for all $r \geq r_0$*

$$\lim_{n\to\infty} \sup_{z\in\mathbb{R}} \int_{B((0,z),r)} |u_n|^q = 0, \tag{3.1}$$

then we have $u_n \to 0$ in $L^p(\mathbb{R}^N)$ for $p \in (2, \frac{4N}{N-2})$.

Proof. By using [24, Lemma 2.2], it remains to prove that for some $r > 0$,

$$\lim_{n\to\infty} \sup_{z\in\mathbb{R}^N} \int_{B(z,r)} |u_n|^q = 0.$$

Suppose that

$$\int_{B(z_n,1)} |u_n|^q \geq c > 0. \tag{3.2}$$

Observe that in the family $\{B(gz_n, 1)\}_{g\in O(2)}$, we find an increasing number of disjoint balls provided that $|(z_n^1, z_n^2)| \to \infty$. Since $\{u_n\}$ is bounded in $L^q(\mathbb{R}^N)$, $q \in [2, \frac{4N}{N-2})$, by (3.2), $|(z_n^1, z_n^2)|$ must be bounded. Then for sufficiently large $r \geq r_0$, one obtains

$$\int_{B((0,z_n^3),r)} |u_n|^q \geq \int_{B(z_n,1)} |u_n|^q \geq c > 0,$$

and we get a contradiction with (3.1). □

Lemma 3.2. *Let $u_n \rightharpoonup u, v_n \rightharpoonup v$ in X, $u_n \to u, v_n \to v$ a.e in $\mathbb{R}^N$. Then*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta - \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta = \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n - u|^\alpha |v_n - v|^\beta.$$

Proof. For $n = 1, 2, \ldots$, we have that

$$\int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta - \int_{\mathbb{R}^N} |u_n - u|^\alpha |v_n - v|^\beta$$
$$= \int_{\mathbb{R}^N} (|u_n|^\alpha - |u_n - u|^\alpha)|v_n|^\beta + \int_{\mathbb{R}^N} |u_n - u|^\alpha (|v_n|^\beta - |v_n - v|^\beta).$$

Since $u_n \rightharpoonup u, v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$, from [17, Lemma 2.5], one has

$$\int_{\mathbb{R}^N} (|u_n|^\alpha - |u_n - u|^\alpha - |u|^\alpha)^{\frac{p}{\alpha}} \to 0, \qquad n \to \infty,$$

which means that

$$|u_n|^\alpha - |u_n - u|^\alpha \to |u|^\alpha \quad \text{in } L^{\frac{p}{\alpha}}(\mathbb{R}^N).$$

Using $|v_n|^\beta \rightharpoonup |v|^\beta$ in $L^{\frac{p}{\beta}}(\mathbb{R}^N)$, it follows from $\alpha + \beta = p$ that

$$\int_{\mathbb{R}^N} (|u_n|^\alpha - |u_n - u|^\alpha)|v_n|^\beta \to \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta, \qquad n \to \infty.$$

Similarly,

$$|v_n|^\beta - |v_n - v|^\beta \to |v|^\beta \quad \text{in } L^{\frac{p}{\beta}}(\mathbb{R}^N).$$

As $|u_n - u|^\alpha \rightharpoonup 0$ in $L^{\frac{p}{\alpha}}(\mathbb{R}^N)$, we obtain that

$$\int_{\mathbb{R}^N} |u_n - u|^\alpha (|v_n|^\beta - |v_n - v|^\beta) \to 0, \qquad n \to \infty.$$

This proves the lemma. □

The following lemma is due to Poppenberg, Schmitt and Wang from [18, Lemma 2].

Lemma 3.3. *Assume that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$. Then*

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} u_n^2 |\nabla u_n|^2 \geq \liminf_{n\to\infty} \int_{\mathbb{R}^N} (u_n - u)^2 |\nabla u_n - \nabla u|^2 + \int_{\mathbb{R}^N} u^2 |\nabla u|^2. \tag{3.3}$$

Proof. The proof is analogous to that of [18, Lemma 2], so we omit it here. □

Lemma 3.4. *m is achieved at some $(u, v) \in \mathcal{M}$.*

Proof. Let $\{(u_n, v_n)\} \subset \mathcal{M}$ be a sequence such that $I(u_n, v_n) \to m$. Using $(u_n, v_n) \subset \mathcal{M}$ and (A_2), we may obtain

$$\begin{aligned}
1 + m &\geq I(u_n, v_n) \\
&= \frac{\alpha + \beta}{2(N + \alpha + \beta)} \int_{\mathbb{R}^N} (|\nabla u_n|^2 + |\nabla v_n|^2) \\
&\quad + \frac{\alpha + \beta - 2}{2(N + \alpha + \beta)} \int_{\mathbb{R}^N} (Bv_n^2 + u_n^2 |\nabla u_n|^2 + v_n^2 |\nabla v_n|^2) \\
&\quad + \frac{1}{2(N + \alpha + \beta)} \int_{\mathbb{R}^N} ((\alpha + \beta - 2)A(x) - \nabla A(x) \cdot x) u_n^2,
\end{aligned}$$

which implies that $\{u_n\}$, $\{v_n\}$, $\{u_n^2\}$ and $\{v_n^2\}$ are bounded in $H^1(\mathbb{R}^N)$, then, there exists a subsequence of (u_n, v_n), still denoted by (u_n, v_n) such that $(u_n, v_n) \rightharpoonup (u, v)$ in X. Then $\{u_n\}$ and $\{v_n\}$ are bounded in $L^{\alpha+\beta}(\mathbb{R}^N)$. The proof consists of three steps.

Step 1.

$$\int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta \not\to 0.$$

It follows from Lemma 2.3 that

$$\begin{aligned} I(u_n, v_n) = & \int_{\mathbb{R}^N} \Big(\frac{1}{2}(|\nabla u_n|^2 + |\nabla v_n|^2 + A(x)u_n^2 + Bv_n^2 \\ & + u_n^2|\nabla u_n|^2 + v_n^2|\nabla v_n|^2) - \frac{2}{\alpha+\beta}|u_n|^\alpha|v_n|^\beta \Big) \\ \to & \; m > 0, \end{aligned}$$

then

$$\int_{\mathbb{R}^N} (|\nabla u_n|^2 + |\nabla v_n|^2 + A(x)u_n^2 + Bv_n^2 + u_n^2|\nabla u_n|^2 + v_n^2|\nabla v_n|^2) \not\to 0.$$

By Lemma 2.2, for $t > 1$,

$$\begin{aligned} m & \leftarrow I(u_n, v_n) \\ & \geq I\big((u_n)_t, (v_n)_t\big) \\ & = \frac{t^N}{2}\int_{\mathbb{R}^N} (|\nabla u_n|^2 + |\nabla v_n|^2) + \frac{t^{N+2}}{2}\int_{\mathbb{R}^N} (A(tx)u_n^2 + Bv_n^2) \\ & \quad + \frac{t^{N+2}}{2}\int_{\mathbb{R}^N} (u_n^2|\nabla u_n|^2 + v_n^2|\nabla v_n|^2) - \frac{2t^{N+\alpha+\beta}}{\alpha+\beta}\int_{\mathbb{R}^N} |u_n|^\alpha|v_n|^\beta \\ & \geq \frac{t^N}{2}\int_{\mathbb{R}^N} (|\nabla u_n|^2 + |\nabla v_n|^2 + A_0u_n^2 + Bv_n^2 + u_n^2|\nabla u_n|^2 + v_n^2|\nabla v_n|^2) \\ & \quad - \frac{2t^{N+\alpha+\beta}}{\alpha+\beta}\int_{\mathbb{R}^N} |u_n|^\alpha|v_n|^\beta \\ & \geq \frac{t^N}{2}\delta - \frac{2t^{N+\alpha+\beta}}{\alpha+\beta}\int_{\mathbb{R}^N} |u_n|^\alpha|v_n|^\beta, \end{aligned}$$

where δ is a fixed constant. It suffices to choose $t > 1$ so that $\frac{t^N\delta}{2} > 2m$ to get a lower bound for

$$\int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta.$$

Therefore, we may assume (passing to a subsequence, if necessary) that

$$\int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta \to D \in (0, \infty). \tag{3.4}$$

Step 2. $u \neq 0$. By using (3.4) and Hölder's inequality, we can assume (passing to a subsequence, if necessary) that

$$\int_{\mathbb{R}^N} |u_n|^{\alpha+\beta} > \delta > 0.$$

By Lemma 3.1, there exist $\delta > 0$ and $\{z_n\} \subset \mathbb{R}$ such that

$$\limsup_{n\to+\infty} \int_{B((0,z_n),r)} |u_n|^{\alpha+\beta} > \delta > 0. \tag{3.5}$$

Define

$$y = (x_1, x_2), \qquad z = (x_3, \ldots, x_N),$$

$$w_n(x) = w_n(y,z) = u_n(y, z + z_n),$$

and

$$\sigma_n(x) = \sigma_n(y,z) = v_n(y, z + z_n),$$

then $w_n \rightharpoonup w, \sigma_n \rightharpoonup \sigma$ in X. In this case, by (A_4), we may obtain $I(u_n, v_n) = I(w_n, \sigma_n)$. By using (3.5) and $w_n \to w$ in $L^{\alpha+\beta}_{loc}(\mathbb{R}^N)$, one has

$$\begin{aligned} 0 < \delta < \limsup_{n\to+\infty} \int_{B((0,z_n),r)} |u_n|^{\alpha+\beta} \\ = \limsup_{n\to+\infty} \int_{B((0,0),r)} |w_n|^{\alpha+\beta} \\ = \int_{B((0,0),r)} |w|^{\alpha+\beta}, \end{aligned}$$

which implies $w \neq 0$, and then $u \neq 0$.

Step 3. We claim that $(u, v) \in \mathcal{M}$. Indeed, if $(u, v) \notin \mathcal{M}$, we discuss three cases:

Case 1: $\mathcal{G}(u, v) < 0$. By Lemma 2.2, there exists $t \in (0, 1)$ such that $(u_t, v_t) \in \mathcal{M}$, it follows from (A_2), $(u_n, v_n) \in \mathcal{M}$ and Fatou's Lemma that

$$\begin{aligned} m &= \liminf_{n\to+\infty} \left(I(u_n, v_n) - \frac{1}{N+\alpha+\beta}\mathcal{G}(u_n, v_n) \right) \\ &= \liminf_{n\to+\infty} \left(\frac{\alpha+\beta}{2(N+\alpha+\beta)} \int_{\mathbb{R}^N} (|\nabla u_n|^2 + |\nabla v_n|^2) \right. \\ &\quad + \frac{1}{2(N+\alpha+\beta)} \int_{\mathbb{R}^N} ((\alpha+\beta-2)A(x) - \nabla A(x)\cdot x) u_n^2 \\ &\quad \left. + \frac{\alpha+\beta-2}{2(N+\alpha+\beta)} \int_{\mathbb{R}^N} (Bv_n^2 + u_n^2|\nabla u_n|^2 + v_n^2|\nabla v_n|^2) \right) \\ &\geq \frac{\alpha+\beta}{2(N+\alpha+\beta)} \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) \\ &\quad + \frac{1}{2(N+\alpha+\beta)} \int_{\mathbb{R}^N} ((\alpha+\beta-2)A(x) - \nabla A(x)\cdot x) u^2 \\ &\quad + \frac{\alpha+\beta-2}{2(N+\alpha+\beta)} \int_{\mathbb{R}^N} (Bv^2 + u^2|\nabla u|^2 + v^2|\nabla v|^2) \\ &> \frac{\alpha+\beta}{2(N+\alpha+\beta)} t^N \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) \\ &\quad + \frac{t^{N+2}}{2(N+\alpha+\beta)} \int_{\mathbb{R}^N} ((\alpha+\beta-2)A(x) - \nabla A(x)\cdot x) u^2 \\ &\quad + \frac{\alpha+\beta-2}{2(N+\alpha+\beta)} t^{N+\alpha+\beta} \int_{\mathbb{R}^N} (Bv^2 + u^2|\nabla u|^2 + v^2|\nabla v|^2) \\ &= I(u_t, v_t) - \frac{1}{N+\alpha+\beta}\mathcal{G}(u_t, v_t) \\ &\geq m, \end{aligned}$$

which is a contradiction.

Case 2: $\mathcal{G}(u, v) > 0$. Set $\xi_n := u_n - u, \gamma_n := v_n - v$, by Lemma 3.2, the Brézis–Lieb Lemma [2], (3.3), (A_1) and (B_1), we may obtain

$$\mathcal{G}(u_n, v_n) \geq \mathcal{G}(u, v) + \mathcal{G}(\xi_n, \gamma_n) + o_n(1). \tag{3.6}$$

Then

$$\limsup_{n\to\infty} \mathcal{G}(\xi_n,\gamma_n)<0.$$

By Lemma 2.2, there exists $t_n\in(0,1)$ such that $((\xi_n)_{t_n},(\gamma_n)_{t_n})\in\mathcal{M}$. Furthermore, one has that

$$\limsup_{n\to\infty} t_n<1,$$

otherwise, along a subsequence, $t_n\to 1$ and hence

$$\mathcal{G}(\xi_n,\gamma_n)=\mathcal{G}((\xi_n)_{t_n},(\gamma_n)_{t_n})+o_n(1)=o_n(1),$$

a contradiction. It follows from $(u_n,v_n)\in\mathcal{M}$, (3.6), (A_2) that

$$\begin{aligned}
m+o_n(1) &= I(u_n,v_n)-\frac{1}{N+\alpha+\beta}\mathcal{G}(u_n,v_n)\\
&= \frac{\alpha+\beta}{2(N+\alpha+\beta)}\int_{\mathbb{R}^N}(|\nabla u_n|^2+|\nabla v_n|^2)\\
&\quad+\frac{1}{2(N+\alpha+\beta)}\int_{\mathbb{R}^N}((\alpha+\beta-2)A(x)-\nabla A(x)\cdot x)u_n^2\\
&\quad+\frac{\alpha+\beta-2}{2(N+\alpha+\beta)}\int_{\mathbb{R}^N}(Bv_n^2+u_n^2|\nabla u_n|^2+v_n^2|\nabla v_n|^2)\\
&\geq \frac{\alpha+\beta}{2(N+\alpha+\beta)}\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2+|\nabla\xi_n|^2+|\nabla\gamma_n|^2)\\
&\quad+\frac{1}{2(N+\alpha+\beta)}\int_{\mathbb{R}^N}((\alpha+\beta-2)A(x)-\nabla A(x)\cdot x)(u^2+\xi_n^2)\\
&\quad+\frac{\alpha+\beta-2}{2(N+\alpha+\beta)}\int_{\mathbb{R}^N}(Bv^2+u^2|\nabla u|^2+v^2|\nabla v|^2+\gamma_n^2+\xi_n^2|\nabla\xi_n|^2\\
&\quad+\gamma_n^2|\nabla\gamma_n|^2)\\
&> \frac{\alpha+\beta}{2(N+\alpha+\beta)}\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2+|\nabla\xi_n|^2+|\nabla\gamma_n|^2)\\
&\quad+\frac{1}{2(N+\alpha+\beta)}\int_{\mathbb{R}^N}((\alpha+\beta-2)A(x)-\nabla A(x)\cdot x)(u^2+t_n^{N+2}\xi_n^2)\\
&\quad+\frac{\alpha+\beta-2}{2(N+\alpha+\beta)}\int_{\mathbb{R}^N}(Bv^2+u^2|\nabla u|^2+v^2|\nabla v|^2+t_n^{N+2}\gamma_n^2\\
&\quad+t_n^{N+2}\xi_n^2|\nabla\xi_n|^2+t_n^{N+2}\gamma_n^2|\nabla\gamma_n|^2)\\
&= I\big((\xi_n)_{t_n},(\gamma_n)_{t_n}\big)+\frac{\alpha+\beta}{2(N+\alpha+\beta)}\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)\\
&\quad+\frac{1}{2(N+\alpha+\beta)}\int_{\mathbb{R}^N}((\alpha+\beta-2)A(x)-\nabla A(x)\cdot x)u^2\\
&\quad+\frac{\alpha+\beta-2}{2(N+\alpha+\beta)}\int_{\mathbb{R}^N}(Bv^2+u^2|\nabla u|^2+v^2|\nabla v|^2)\\
&\geq m,
\end{aligned}$$

which is also a contradiction.

Therefore, $(u,v)\in\mathcal{M}$. By using Lebesgue's dominated convergence theorem, Fatou's

Lemma, (A_2) and $(u_n, v_n) \in \mathcal{M}$, we may get

$$\begin{aligned} m &= I(u,v) - \frac{1}{N+\alpha+\beta}\mathcal{G}(u,v) \\ &= \frac{\alpha+\beta}{2(N+\alpha+\beta)} \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) \\ &\quad + \frac{1}{2(N+\alpha+\beta)} \int_{\mathbb{R}^N} \big((\alpha+\beta-2)A(x) - \nabla A(x)\cdot x\big)u^2 \\ &\quad + \frac{\alpha+\beta-2}{2(N+\alpha+\beta)} \int_{\mathbb{R}^N} (Bv^2 + u^2|\nabla u|^2 + v^2|\nabla v|^2) \\ &\le \liminf_{n\to+\infty} \left(\frac{\alpha+\beta}{2(N+\alpha+\beta)} \int_{\mathbb{R}^N} (|\nabla u_n|^2 + |\nabla v_n|^2) \right. \\ &\quad + \frac{1}{2(N+\alpha+\beta)} \int_{\mathbb{R}^N} \big((\alpha+\beta-2)A(x) - \nabla A(x)\cdot x\big)u_n^2 \\ &\quad \left. + \frac{\alpha+\beta-2}{2(N+\alpha+\beta)} \int_{\mathbb{R}^N} (Bv_n^2 + u_n^2|\nabla u_n|^2 + v_n^2|\nabla v_n|^2) \right) \\ &= \liminf_{n\to+\infty} \left(I(u_n,v_n) - \frac{1}{N+\alpha+\beta}\mathcal{G}(u_n,v_n) \right) \\ &= m, \end{aligned}$$

which implies that $(u_n, v_n) \to (u, v)$ in X and $I(u, v) = m$. □

Having a minimum of $I|_{\mathcal{M}}$, the fact that it is indeed a solution of (1.1), is based on a general idea used in [13, Lemma 2.5].

Proof of Theorem 1.1. Let $(\tilde{u}, \tilde{v}) \in \mathcal{M}$ be a minimizer of the functional $I|_{\mathcal{M}}$. We show that $I'(\tilde{u}, \tilde{v}) = 0$. By Lemma 2.2,

$$I(\tilde{u},\tilde{v}) = \inf_{(u,v)\in X} \max_{t>0} I(u_t, v_t) = m.$$

We argue by contradiction by assuming that $(\tilde{u},\ \tilde{v})$ is not a weak solution of (1.1). Then, we can chose $\phi_1,\ \phi_2 \in C_0^\infty(\mathbb{R}^N) \cap \mathcal{V}$ such that

$$\begin{aligned} \langle I'(\tilde{u},\tilde{v}), (\phi_1,\phi_2)\rangle = \int_{\mathbb{R}^N} \Big(&\nabla\tilde{u}\nabla\phi_1 + \nabla\tilde{v}\nabla\phi_2 + \nabla(\tilde{u}^2)\nabla(\tilde{u}\phi_1) + \nabla(\tilde{v}^2)\nabla(\tilde{v}\phi_2) \\ &+ A(x)\tilde{u}\phi_1 + B\tilde{v}\phi_2 - \frac{2\alpha}{\alpha+\beta}|\tilde{u}|^{\alpha-2}\tilde{u}|\tilde{v}|^\beta\phi_1 - \frac{2\beta}{\alpha+\beta}|\tilde{v}|^{\beta-2}\tilde{v}|\tilde{u}|^\alpha\phi_2 \Big) \\ < \ -1. \end{aligned}$$

Then we fix $\varepsilon > 0$ sufficiently small such that

$$\big\langle I'(\tilde{u}_t + \sigma\phi_1, \tilde{v}_t + \sigma\phi_2), (\phi_1,\phi_2)\big\rangle \le -\frac{1}{2}, \qquad \forall\ |t-1|, \|\sigma| \le \varepsilon$$

and introduce a cut-off function $0 \le \zeta \le 1$ such that $\zeta(t) = 1$ for $|t-1| \le \frac{\varepsilon}{2}$ and $\zeta(t) = 0$ for $|t-1| \ge \varepsilon$. For $t \ge 0$, we define

$$\gamma_1(t) := \begin{cases} \tilde{u}_t, & \text{if } |t-1| \ge \varepsilon, \\ \tilde{u}_t + \varepsilon\zeta(t)\phi_1, & \text{if } |t-1| < \varepsilon, \end{cases}$$

$$\gamma_2(t) := \begin{cases} \tilde{v}_t, & \text{if } |t-1| \geq \varepsilon, \\ \tilde{v}_t + \varepsilon\zeta(t)\phi_2, & \text{if } |t-1| < \varepsilon. \end{cases}$$

Note that $\gamma_1(t)$ and $\gamma_2(t)$ are continuous curve in the metric space (X, d) and, eventually choosing a smaller ε, we get that for $|t-1| < \varepsilon$,

$$d_X\big((\gamma_1(t), \gamma_2(t)), (0,0)\big) > 0.$$

Claim: $\sup_{t\geq 0} I(\gamma_1(t), \gamma_2(t)) < m$.

Indeed, if $|t-1| \geq \varepsilon$, then $I(\gamma_1(t), \gamma_2(t)) = I(\tilde{u}_t, \tilde{v}_t) < I(u,v) = m$. If $|t-1| < \varepsilon$, by using the mean value theorem to the C^1 map $[0,\varepsilon] \ni \sigma \mapsto I(\tilde{u}_t + \sigma\zeta(t)\phi_1, \tilde{v}_t + \sigma\zeta(t)\phi_2) \in \mathbb{R}$, we find, for a suitable $\bar{\sigma} \in (0, \varepsilon)$,

$$\begin{aligned} &I(\tilde{u}_t + \sigma\zeta(t)\phi_1, \tilde{v}_t + \sigma\zeta(t)\phi_2) \\ &\quad = I(\tilde{u}_t, \tilde{v}_t) + \langle I'(\tilde{u}_t + \bar{\sigma}\zeta(t)\phi_1, \tilde{v}_t + \bar{\sigma}\zeta(t)\phi_2), (\zeta(t)\phi_1, \zeta(t)\phi_2)\rangle \\ &\quad \leq I(\tilde{u}_t, \tilde{v}_t) - \frac{1}{2}\zeta(t) \\ &\quad < m. \end{aligned}$$

To conclude, we observe that $\mathcal{G}(\gamma_1(1-\varepsilon), \gamma_2(1-\varepsilon)) > 0$ and $\mathcal{G}(\gamma_1(1+\varepsilon), \gamma_2(1+\varepsilon)) < 0$. By the continuity of the map $t \mapsto \mathcal{G}(\gamma_1(t), \gamma_2(t))$ there exists $t_0 \in (1-\varepsilon, 1+\varepsilon)$ such that $\mathcal{G}(\gamma_1(t_0), \gamma_2(t_0)) = 0$. Namely,

$$(\gamma_1(t_0), \gamma_2(t_0)) = (\tilde{u}_{t_0} + \varepsilon\zeta(t_0)\phi_1, \tilde{v}_{t_0} + \varepsilon\zeta(t_0)\phi_2) \in \mathcal{M}$$

and $I(\gamma_1(t_0), \gamma_2(t_0)) < m$, this is a contradiction.

In addition, from the definition of $\mathcal{V}$ and the fact that $\det(\eta) = -1$, $(u(\eta x), v(\eta x)) = (\det(\eta)u(x), \det(\eta)v(x)) = (-u(x), -v(x))$. So (u, v) will change sign when (y_1, y_2) cross perpendicularly the half lines $y_2 = \pm y_1 \frac{\tan \pi j}{s}$ $(y_1 \geq 0)$, $j = 1, 2, \ldots, s$. Hence (u, v) is a nodal solution with at least $2s$ nodal domains. □

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Positive radial solutions for a class of quasilinear Schrödinger equations in $\mathbb{R}^3$

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Abstract. This paper is concerned with the following quasilinear Schrödinger equations of the form:

$$-\Delta u - u\Delta(u^2) + u = |u|^{p-2}u, \qquad x \in \mathbb{R}^3,$$

where $p \in (2, 12)$. By making use of the constrained minimization method on a special manifold, we prove that the existence of positive radial solutions of the above problem for any $p \in (2, 12)$.

Keywords: quasilinear Schrödinger equations, constrained minimization method.

2020 Mathematics Subject Classification: 35B09, 35J62.

1 Introduction

In this paper, we are devoted to studying the following quasilinear Schrödinger equations:

$$-\Delta u - u\Delta(u^2) + u = |u|^{p-2}u, \qquad x \in \mathbb{R}^3, \tag{1.1}$$

where $p \in (2, 12)$.

Set

$$E := \left\{ u \in H_r^1\left(\mathbb{R}^3\right) : \int_{\mathbb{R}^3} u^2 \left|\nabla u\right|^2 dx < \infty \right\},$$

where

$$H_r^1\left(\mathbb{R}^3\right) := \left\{ u \in H^1\left(\mathbb{R}^3\right) : u\left(|x|\right) = u\left(x\right) \right\}$$

with the norm

$$\|u\| = \left(\int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2 \right) dx \right)^{1/2}.$$

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A function $u \in E$ is called a weak solution of equation (1.1), if for all $\phi \in C_0^\infty\left(\mathbb{R}^3\right)$ it holds

$$\int_{\mathbb{R}^3} \nabla u \nabla \phi dx + \int_{\mathbb{R}^3} u\phi dx + 2\int_{\mathbb{R}^3} u^2 \nabla u \nabla \phi dx + 2\int_{\mathbb{R}^3} |\nabla u|^2 u\phi dx = \int_{\mathbb{R}^3} |u|^{p-2} u\phi dx.$$

Define the functional I on E by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2\right) dx + \int_{\mathbb{R}^3} u^2 |\nabla u|^2 dx - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p dx.$$

It is easy to check that I is continuous on E. Furthermore, given $u \in E$ and $\phi \in C_0^\infty\left(\mathbb{R}^3\right)$, we can compute the Gateaux derivative of I in the direction ϕ at u:

$$\left\langle I'(u), \phi \right\rangle = \int_{\mathbb{R}^3} \nabla u \nabla \phi dx + \int_{\mathbb{R}^3} u\phi dx + 2\int_{\mathbb{R}^3} \left(|\nabla u|^2 u\phi + u^2 \nabla u \nabla \phi\right) dx - \int_{\mathbb{R}^3} |u|^{p-2} u\phi dx.$$

Hence u is a weak solution of equation (1.1) if and only if this derivative is zero in every direction $\phi \in C_0^\infty\left(\mathbb{R}^3\right)$.

When $V(x) = 1$, $\alpha(s) = s$ and $f(x,z) = |z|^{p-2}z$, solutions of equation (1.1) are standing waves of the following quasilinear Schrödinger equations of the form:

$$iz_t + \Delta z - V(x)z + \Delta\alpha\left(|z|^2\right)\alpha'\left(|z|^2\right)z + f(x,z) = 0, \qquad x \in \mathbb{R}^3, \tag{1.2}$$

where $V(x)$ is a given potential, α and f are real functions. Equation (1.2) has been derived as models of several physical phenomena, such as [1,4–6]. It began with [11] for the studies on mathematics. Several methods can be used to deal with problem (1.2), such as, the existence of a positive ground state solution was studied by making use of the constrained minimization method in [8,12]; Liu et al. in [9] and Colin et al. in [3] obtained the existence results for equation (1.2) through making a change of variable and reducing the quasilinear problem (1.2) to a semilinear one; Nehari method was used to obtain the existence results of ground state solutions for equation (1.2) in [10]. Moreover, in [7], the existence results for the general form of quasilinear elliptic equations were studied by means of a perturbation method. Especially, in [13], Ruiz et al. proved the existence of positive radial solutions for the Schrödinger–Poisson equation by using the constrained minimization argument on the Nehari–Pohožaev manifold.

In the present paper, inspired by [13], our goal is to prove the existence of positive radial solutions for equation (1.1) via the constrained minimization method on the Nehari–Pohožaev manifold. Our main result reads as follows.

Theorem 1.1. *For $2 < p < 12$, problem (1.1) possesses one positive radial solution.*

2 Preliminaries and proof of main result

Lemma 2.1. *For $p \in (2, 12)$, I is unbounded from below.*

Proof. Let $u \in E$ be radial and positive, and $u_t = t^{1/2}u\left(t^{-1}x\right)$ for $t > 0$. To facilitate the estimation of $I\left(u_t\right)$, we firstly compute:

$$\int_{\mathbb{R}^3} |\nabla u_t|^2 dx = t^2 \int_{\mathbb{R}^3} |\nabla u|^2 dx, \qquad \int_{\mathbb{R}^3} u_t^2 dx = t^4 \int_{\mathbb{R}^3} u^2 dx,$$

$$\int_{\mathbb{R}^3} u_t^2 |\nabla u_t|^2 dx = t^3 \int_{\mathbb{R}^3} u^2 |\nabla u|^2 dx, \qquad \int_{\mathbb{R}^3} |u_t|^p dx = t^{\frac{p+6}{2}} \int_{\mathbb{R}^3} |u|^p dx.$$

Then one has

$$\begin{aligned} I(u_t) &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_t|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} u_t^2 dx + \int_{\mathbb{R}^3} u_t^2 |\nabla u_t|^2 dx - \frac{1}{p}\int_{\mathbb{R}^3} |u_t|^p dx \\ &= \frac{t^2}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{t^4}{2}\int_{\mathbb{R}^3} u^2 dx + t^3\int_{\mathbb{R}^3} u^2 |\nabla u|^2 dx - \frac{1}{p} t^{\frac{p+6}{2}}\int_{\mathbb{R}^3} |u|^p dx. \end{aligned}$$

Since $(p+6)/2 > 4$ for $p \in (2,12)$, we easily infer that $I(u_t) \to -\infty$ as $t \to +\infty$. □

Lemma 2.2. *Let c_1, c_2, c_3, c_4 be positive constants and $p > 2$. Then for $t > 0$, the function*

$$\eta(t) = c_1 t^2 + c_2 t^3 + c_3 t^4 - c_4 t^{\frac{p+6}{2}}$$

has a unique positive critical point which corresponds to its maximum.

Proof. The conclusion is easily obtained by elementary calculation. □

Now, in order to define the Nehari–Pohožaev manifold, we firstly need to introduce the following Pohožaev identity (see, e.g., [13, p. 1224]).

Lemma 2.3. *If $u \in E$ is a weak solution to equation (1.1), then the following Pohožaev identity holds:*

$$P(u) := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{3}{2}\int_{\mathbb{R}^3} |u|^2 dx + \int_{\mathbb{R}^3} u^2 |\nabla u|^2 dx - \frac{3}{p}\int_{\mathbb{R}^3} |u|^p dx = 0.$$

Proof. The proof is standard, so we omit it. □

As mentioned in the introduction, we will use the constrained minimization argument on a special manifold to prove the existence result of equation (1.1).

Let us justify the choice of the manifold. Assume that $u \in E$ is a critical point of I. Define, as above, $u_t(x) = t^{1/2} u(t^{-1}x)$, and consider

$$\eta(t) = I(u_t) = \frac{t^2}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{t^4}{2}\int_{\mathbb{R}^3} u^2 dx + t^3\int_{\mathbb{R}^3} u^2 |\nabla u|^2 dx - \frac{1}{p} t^{\frac{p+6}{2}}\int_{\mathbb{R}^3} |u|^p dx.$$

Obviously, $\eta(t) > 0$ for small t and $\eta(t) \to -\infty$ as $t \to +\infty$. Moreover, it follows from Lemma 2.2 that $\eta(t)$ has a unique critical point which corresponds to its maximum. But since u is a critical point of I, the maximum of $\eta(t)$ should be achieved at $t = 1$ and thus $\eta'(1) = 0$. Thus we can define the manifold $\mathcal{T}$ as

$$\mathcal{T} := \left\{ u \in E \setminus \{0\} : J(u) = 0 \right\},$$

where

$$J(u) := \eta'(1) = \int_{\mathbb{R}^3} |\nabla u|^2 dx + 2\int_{\mathbb{R}^3} u^2 dx + 3\int_{\mathbb{R}^3} u^2 |\nabla u|^2 dx - \frac{p+6}{2p}\int_{\mathbb{R}^3} |u|^p dx.$$

Clearly, $J(u) = \frac{1}{2}\langle I'(u), u\rangle + P(u)$. If u is a nontrivial solution of problem (1.1), then $u \in \mathcal{T}$. The manifold $\mathcal{T}$ can be viewed as the combination of the commonly used Nehari manifold and Pohožaev manifold. Such manifold was first introduced in [13], in which the Schrödinger–Poisson system was studied.

Lemma 2.4. *If $p \in (2,12)$, then $\mathcal{T}$ is a C^1-manifold and every critical point of $I|_{\mathcal{T}}$ is a critical point of I.*

Proof. *Step 1.* $0 \notin \partial\mathcal{T}$. By Sobolev's inequality, one has

$$J(u) \geq \|u\|^2 - C_1 \frac{p+6}{2p} \|u\|^p,$$

where C_1 is a positive constant. Choosing R small enough, then there exists $\rho > 0$ such that $J(u) > \rho$ for $\|u\| < R$, that is, $0 \notin \partial\mathcal{T}$.

Step 2. $\inf I|_{\mathcal{T}} > 0$. For any $u \in \mathcal{T}$, for convenience, we set

$$\alpha = \int_{\mathbb{R}^3} |\nabla u|^2 dx, \quad \beta = \int_{\mathbb{R}^3} u^2 dx, \quad \gamma = \int_{\mathbb{R}^3} u^2 |\nabla u|^2 dx, \quad \theta = \int_{\mathbb{R}^3} |u|^p dx, \quad s = I(u). \tag{2.1}$$

Then $\alpha, \beta, \gamma, \theta$ are positive, and we get

$$\begin{cases} I(u) = \frac{1}{2}\alpha + \frac{1}{2}\beta + \gamma - \frac{1}{p}\theta = s, \\ J(u) = \alpha + 2\beta + 3\gamma - \frac{p+6}{2p}\theta = 0. \end{cases} \tag{2.2}$$

By solving the system (2.2), we obtain

$$\gamma = \frac{2(p+6)s - (p+2)\alpha - (p-2)\beta}{2p} \tag{2.3}$$

and

$$\frac{p+2}{4}\alpha + \frac{p-2}{4}\beta + \frac{p}{2}\gamma = \frac{p+6}{2}s. \tag{2.4}$$

Since $\alpha, \beta, \gamma > 0$ and $p > 2$, we follow from (2.3) and (2.4) that

$$(p-2)(\alpha + \beta) < (p+2)\alpha + (p-2)\beta < 2(p+6)s \tag{2.5}$$

and

$$\gamma < \frac{p+6}{p}s. \tag{2.6}$$

Moreover, it follows from Step 1 that there exists $\varepsilon > 0$ such that $\alpha + \beta > \varepsilon$. Therefore, by (2.5) we get

$$I(u) = s > \frac{p-2}{2(p+6)}(\alpha + \beta) > 0, \tag{2.7}$$

which means $I|_{\mathcal{T}} > 0$.

Step 3. $\mathcal{T}$ is a C^1-manifold. It suffices to show that $J'(u) \neq 0$ for any $u \in \mathcal{T}$ by the implicit function theorem. Suppose that $J'(u) = 0$ for some $u \in \mathcal{T}$. In a weak sense, the equation $J'(u) = 0$ can be written as

$$-2\Delta u - 3u\Delta(u^2) + 4u = \frac{p+6}{2}|u|^{p-2}u. \tag{2.8}$$

Multiplying (2.8) by u and integrating, one has

$$\langle J'(u), u \rangle = 2\int_{\mathbb{R}^3} |\nabla u|^2 dx + 4\int_{\mathbb{R}^3} u^2 dx + 12\int_{\mathbb{R}^3} u^2 |\nabla u|^2 dx - \frac{p+6}{2}\int_{\mathbb{R}^3} |u|^p dx = 0. \tag{2.9}$$

The Pohožaev identity corresponding to (2.9) is

$$\int_{\mathbb{R}^3} |\nabla u|^2 dx + 6\int_{\mathbb{R}^3} u^2 dx + 3\int_{\mathbb{R}^3} u^2 |\nabla u|^2 dx - \frac{3(p+6)}{2p}\int_{\mathbb{R}^3} |u|^p dx = 0. \tag{2.10}$$

Thus, using the same notations defined in (2.1), we follow from (2.9) and (2.10) that

$$\begin{cases} I(u) = \frac{1}{2}\alpha + \frac{1}{2}\beta + \gamma - \frac{1}{p}\theta = s, \\ J(u) = \alpha + 2\beta + 3\gamma - \frac{p+6}{2p}\theta = 0, \\ 2\alpha + 4\beta + 12\gamma - \frac{p+6}{2}\theta = 0, \\ \alpha + 6\beta + 3\gamma - \frac{3(p+6)}{2p}\theta = 0. \end{cases}$$

It can be checked out that for $p \in (2, 12)$, the above system of equations admits one unique solution on θ, given by

$$\theta = \frac{-24ps}{(p-2)(p+3)}.$$

Since $s > 0$, we infer $\theta < 0$, which is impossible. So $J'(u) \neq 0$ for any $u \in \mathcal{T}$, and then we conclude that $\mathcal{T}$ is a C^1-manifold.

Step 4. $I'(u) = 0$. Assume that u is a critical point of $I|_{\mathcal{T}}$. Depending on the Lagrange multiplier argument, there exists $\mu \in \mathbb{R}$ such that $I'(u) = \mu J'(u)$. We claim that $\mu = 0$.

As above, $I'(u) = \mu J'(u)$ can be written, in a weak sense, as

$$-\Delta u - u\Delta(u^2) + u - u^{p-2}u = \mu \left[-2\Delta u - 3u\Delta(u^2) + 4u - \frac{p+6}{2}u^{p-2}u \right],$$

which means

$$-(1-2\mu)\,\Delta u - (1-3\mu)\,u\Delta(u^2) + (1-4\mu)u = \left(1 - \frac{p+6}{2}\mu\right) u^{p-2}u. \tag{2.11}$$

Combining (2.2) and (2.11), we get

$$\begin{cases} I(u) = \frac{1}{2}\alpha + \frac{1}{2}\beta + \gamma - \frac{1}{p}\theta = s, \\ J(u) = \alpha + 2\beta + 3\gamma - \frac{p+6}{2p}\theta = 0, \\ \alpha + \beta + 4\gamma - \theta = 0, \\ (1-2\mu)\alpha + (1-4\mu)\beta + (4-12\mu)\gamma - \left[1 - \frac{p+6}{2}\mu\right]\theta = 0. \end{cases} \tag{2.12}$$

The third equation corresponds to $\langle I'(u), u \rangle = 0$ for $u \in \mathcal{T}$. The fourth one follows by multiplying (2.11) by u and integrating. Now we deal with this system. Considering $\alpha, \beta, \gamma, \theta$ as unknowns and denoting by D the coefficient matrix, we can get

$$\det D = \frac{(p-2)\mu}{2}.$$

Therefore, for $p \in (2, 12)$ we infer

$$\det D = 0 \Leftrightarrow \mu = 0.$$

Now we prove that $\mu = 0$ by contradiction. If $\mu \neq 0$, then $\det D \neq 0$, which means system (2.12) has a unique solution. So we can obtain

$$\theta = -\frac{12s}{p-2}.$$

This is impossible since θ must be positive. Hence $\mu = 0$, and then $I'(u) = 0$. □

Lemma 2.5. *If $p \in (2, 12)$, then $c_{\mathcal{T}}$ is achieved, where $c_{\mathcal{T}} := \inf\{I(u) : u \in \mathcal{T}\}$.*

Proof. Let $\{u_n\} \subset \mathcal{T}$ be a minimizing sequence of $I|_{\mathcal{T}}$, namely that $I(u_n) \to c_{\mathcal{T}}$. Referring to (2.5) and (2.6), in a similar way we can deduce that

$$\|u_n\|^2 < \frac{2(p+6)}{p-2} I(u_n)$$

and

$$\int_{\mathbb{R}^3} u_n^2 |\nabla u_n|^2 dx < \frac{p+6}{p} I(u_n).$$

Then $\{u_n\}$ is bounded in E and $\{\nabla(u_n^2)\}$ is bounded in $L^2(\mathbb{R}^3)$. Moreover, by the continuous Sobolev embedding $E \hookrightarrow L^6(\mathbb{R}^3)$ and Hölder's inequality, we conclude that there exists a positive constant C such that

$$\begin{aligned}\int_{\mathbb{R}^3} |u_n^2|^2 dx &\leq \left(\int_{\mathbb{R}^3} |u_n|^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |u_n|^6 dx\right)^{\frac{1}{2}} \\ &\leq C\|u_n\|^4,\end{aligned}$$

which together with the boundedness of $\{\nabla(u_n^2)\}$ in $L^2(\mathbb{R}^3)$ means that $\{u_n^2\}$ is bounded in E. Therefore, by using the compact embedding $H_r^1(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)$ for any $s \in (2, 6)$ and interpolation inequality, we get

$$\begin{cases} u_n^2 \rightharpoonup u^2 & \text{in } E, \\ u_n \rightharpoonup u & \text{in } E, \\ u_n \to u & \text{in } L^q(\mathbb{R}^3), \text{ for } q \in (2, 12), \\ u_n \to u & \text{a.e. in } \mathbb{R}^3. \end{cases} \tag{2.13}$$

We claim that $u \in \mathcal{T}$ and $u_n \to u$ strongly in E.

Similar to (2.1), we define

$$\alpha_n = \int_{\mathbb{R}^3} |\nabla u_n|^2 dx, \qquad \beta_n = \int_{\mathbb{R}^3} u_n^2 dx, \qquad \gamma_n = \int_{\mathbb{R}^3} u_n^2 |\nabla u_n|^2 dx, \qquad \theta_n = \int_{\mathbb{R}^3} |u_n|^p dx$$

and

$$\tilde{\alpha} = \lim_{n\to\infty} \alpha_n, \qquad \tilde{\beta} = \lim_{n\to\infty} \beta_n, \qquad \tilde{\gamma} = \lim_{n\to\infty} \gamma_n, \qquad \tilde{\theta} = \lim_{n\to\infty} \theta_n.$$

In order to show $u_n \to u$ in E, we just need to prove $\|u_n\| \to \|u\|$ by the Brezis–Lieb Lemma in [2], that is, $\alpha + \beta = \tilde{\alpha} + \tilde{\beta}$. From (2.13), we infer that $\alpha \leq \tilde{\alpha}, \beta \leq \tilde{\beta}$ and $\gamma \leq \tilde{\gamma}$. Suppose by contradiction that $\alpha + \beta < \tilde{\alpha} + \tilde{\beta}$.

Noting that $\lim_{n\to\infty} I(u_n) = c_{\mathcal{T}}$ and $J(u_n) = 0$, we infer

$$\begin{cases} \frac{1}{2}\tilde{\alpha} + \frac{1}{2}\tilde{\beta} + \tilde{\gamma} - \frac{1}{p}\tilde{\theta} = c_{\mathcal{T}}, \\ \tilde{\alpha} + 2\tilde{\beta} + 3\tilde{\gamma} - \frac{p+6}{2p}\tilde{\theta} = 0. \end{cases} \tag{2.14}$$

We first show $u \neq 0$. By (2.13), we easily infer that $\theta = \tilde{\theta}$. Thanks to Step 2 in the proof of Lemma 2.4, we get $\tilde{\alpha} + \tilde{\beta} > \varepsilon > 0$, which together with (2.14) yields to $\tilde{\theta} > 0$. Thus we infer

$$\theta = \int_{\mathbb{R}^3} |u|^p dx > 0,$$

which means $u \neq 0$.

Set

$$g(t) = \frac{1}{2}t^2\alpha + \frac{1}{2}t^4\beta + t^3\gamma - \frac{1}{p}t^{\frac{p+6}{2}}\theta, \qquad \tilde{g}(t) = \frac{1}{2}t^2\tilde{\alpha} + \frac{1}{2}t^4\tilde{\beta} + t^3\tilde{\gamma} - \frac{1}{p}t^{\frac{p+6}{2}}\tilde{\theta}.$$

Depending on Lemma 2.2, we know that both g and $\tilde{g}$ have a unique critical point, corresponding to their maxima. From (2.14), we get that $\tilde{g}'(1) = 0$, namely that $\tilde{g}(1) = c_{\mathcal{T}}$. Moreover, since $\alpha + \beta < \tilde{\alpha} + \tilde{\beta}, \gamma \leq \tilde{\gamma}$ and $\theta = \tilde{\theta}$, then $g(t) < \tilde{g}(t)$ for all $t > 0$. Let $t_0 > 0$ be the maximum of g. Then $g'(t_0) = 0$ and $g(t_0) < c_{\mathcal{T}}$.

Define $v_0(x) = t_0^{1/2}u(t_0^{-1}x)$. Then one has

$$I(v_0) = \frac{1}{2}t_0^2\alpha + \frac{1}{2}t_0^4\beta + t_0^3\gamma - \frac{1}{p}t_0^{\frac{p+6}{2}}\theta = g(t_0) < c_{\mathcal{T}}$$

and

$$J(v_0) = t_0^2\alpha + 2t_0^4\beta + 3t_0^3\gamma - \frac{p+6}{2p}t_0^{\frac{p+6}{2}}\theta = g'(t_0)\,t_0 = 0.$$

Then $v_0 \in \mathcal{T}$ and $I(v_0) < c_{\mathcal{T}}$, which is a contradiction. Therefore $\alpha + \beta = \tilde{\alpha} + \tilde{\beta}$, and then $u_n \to u$ in E. □

Proof of Theorem 1.1. By Lemma 2.5, we know that $I|_{\mathcal{T}}$ attains its minimum at u and $u \neq 0$, namely that u is a nontrivial critical point of $I|_{\mathcal{T}}$. And then from Lemma 2.4, we get that u is a nontrivial solution of equation (1.1). Since the functional I and the manifold $\mathcal{T}$ are symmetric, we easily deduce that $|u|$ is also a nontrivial solution of equation (1.1). Hence we may assume that such a solution does not change sign, i.e., $u \geq 0$. Depending on the strong maximum principle, u must be strictly positive, and then u is a positive solution of equation (1.1). □

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On oscillation of solutions of scalar delay differential equation in critical case

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Abstract. In this paper we study the oscillation problem for the known scalar delay differential equation. We assume that the coefficients of this equation have an oscillatory behaviour with an amplitude of oscillation tending to zero at infinity. The asymptotic formulae for the solutions of the considered equation in the so-called critical case are constructed. We give the conditions for existence of oscillatory or nonoscillatory solutions in terms of certain numerical quantities. The obtained results are illustrated by a number of examples.

Keywords: asymptotic integration, delay differential equation, oscillation problem, center manifold, Levinson's theorem.

2020 Mathematics Subject Classification: 34K06, 34K11, 34K19, 34K25, 34C29.

1 Problem statement

In this paper, we construct the asymptotics as $t \to \infty$ for solutions of the following scalar differential equation with variable delay:

$$\dot{x} = -a(t)x(t - \tau(t)), \qquad t \geq t_0 > 0. \tag{1.1}$$

Here $a(t)$ and $\tau(t)$ are real-valued and continuous functions on $[t_0, \infty)$. Further we will impose some additional restrictions on these functions.

One of the main questions usually considered for Eq. (1.1) concerns the oscillation problem of its solutions. Choose $h > 0$ such that $0 \leq \tau(t) \leq h$ for $t \geq T \geq t_0$. By a solution of (1.1) for $t \geq T$, we mean a function $x(t)$ which is continuous on $[T - h, \infty)$, differentiable on $[T, \infty)$ and satisfies (1.1) for $t \geq T$ (by the derivative at $t = T$, we mean the right-hand side derivative). Such a solution $x(t)$ of Eq. (1.1) is said to be *oscillatory* if it has arbitrarily large zeroes. Otherwise, it is called *nonoscillatory*. Evidently, $x(t)$ is nonoscillatory if it is eventually *positive* or eventually *negative*.

The oscillation problem for Eq. (1.1) was studied by many authors. The systematic study of equation (1.1) was started by A. D. Myshkis in [23] (see also [24]). Among the works

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dealing with the oscillation problem we note the results obtained in [14, 15, 17, 21, 34], the series of papers by J. Diblík et al. [7–9, 11, 13], M. Pituk et al. [29, 31, 32], K. M. Chudinov [3–5]. In some of the mentioned papers the oscillation problem is solved by constructing the asymptotic formulae for solutions. The asymptotic properties of solutions of Eq. (1.1) are also studied in [10, 12, 18, 30]. Of course, the mentioned list of papers is not exhaustive due to the enormous amount of studies devoted to the analysis of dynamics of solutions to Eq. (1.1). More references concerning this topic can be found in the lists of cited literature in the mentioned papers. We also note paper [33] that contains the extensive review of works on this subject. Below we give two well-known criteria on oscillation of solutions to Eq. (1.1). In particular, this will allow us to refine the formulation of the problem considered in this paper.

The first of the announced results refers to the equation (1.1) with a constant delay $\tau(t) \equiv \tau$ provided that $a(t) > 0$ as $t \geq t_0$. Let us introduce the following notation. We will denote by $\ln_m t$, where $m \geq 1$, the expression, defined by the formula $\ln_m t = \ln(\ln_{m-1} t)$ and $\ln_0 t = t$. The following theorem holds [9].

Theorem 1.1.

A. *Let us assume that $a(t) \leq a_m(t)$ for $t \to \infty$ and an integer $m \geq 0$, where*

$$a_m(t) = \frac{1}{e\tau} + \frac{\tau}{8et^2} + \frac{\tau}{8e(t\ln t)^2} + \frac{\tau}{8e(t\ln t\ln_2 t)^2} + \cdots + \frac{\tau}{8e(t\ln t\ln_2 t\ldots\ln_m t)^2}.$$

Then there exists a positive solution $x = x(t)$ of (1.1). Moreover,

$$x(t) < e^{-t/\tau}\sqrt{t\ln t\ln_2 t\ldots\ln_m t}$$

as $t \to \infty$.

B. *Let us assume that*

$$a(t) > a_{m-2}(t) + \frac{\theta\tau}{8e(t\ln t\ln_2 t\ldots\ln_{m-1} t)^2}$$

if $t \to \infty$, an integer $m \geq 2$ and a constant $\theta > 1$. Then all the solutions of (1.1) oscillate.

In [13], the authors generalize certain results of Theorem 1.1 to the case of Eq. (1.1) with variable delay $\tau(t)$. One more result on the oscillation of solutions of Eq. (1.1) we would like to point out is due to Koplatadze and Chanturiya [21].

Theorem 1.2. *If $a(t) \geq 0$, $\tau(t) \geq 0$, $t - \tau(t) \to +\infty$ as $t \to +\infty$ and*

$$\liminf_{t\to+\infty} \int_{t-\tau(t)}^{t} a(s)ds > \frac{1}{e},$$

then all solutions of Eq. (1.1) oscillate. Conversely, if there exists $t_0 \geq 0$ such that

$$\int_{t-\tau(t)}^{t} a(s)ds \leq \frac{1}{e}.$$

for $t \geq t_0$ then Eq. (1.1) has a nonoscillatory solution.

The development of the ideas concerning the improvement of the results of Theorem 1.2 may be found, e.g., in [34].

The most difficult situation in the oscillation problem occurs in the so called critical case [15] when

$$\lim_{t\to+\infty} a(t) = \frac{1}{e\tau}, \qquad \lim_{t\to+\infty} \tau(t) = \tau > 0. \tag{1.2}$$

It is known that in this case equation (1.1) may have oscillatory solutions although the «limit equation»

$$\dot{x} = -\frac{1}{e\tau}x(t-\tau), \qquad \tau > 0$$

has positive solution $x(t) = e^{-t/\tau}$. To obtain any general results in this situation is a challenging task. It is necessary to take into account some additional properties of the functions $a(t)$ and $\tau(t)$, in particular, the rate of their tending to limit values in (1.2) and the character of this tending.

In our paper we consider Eq. (1.1) provided that the functions $a(t)$ and $\tau(t)$ have the following asymptotic expansions as $t \to \infty$:

$$a(t) = \frac{1}{e} + a_1(t)t^{-\rho} + a_2(t)t^{-2\rho} + \cdots + a_{k+1}(t)t^{-(k+1)\rho} + O(t^{-(k+2)\rho}), \tag{1.3}$$

$$\tau(t) = 1 + q_1(t)t^{-\rho} + q_2(t)t^{-2\rho} + \cdots + q_{k+1}(t)t^{-(k+1)\rho} + O(t^{-(k+2)\rho}), \tag{1.4}$$

where $\rho > 0$ and $k \in \mathbb{N}$ is chosen such that

$$(k+1)\rho > 1. \tag{1.5}$$

Functions $a_j(t)$, $q_j(t)$, $j = 1, \ldots, k+1$, are finite trigonometric polynomials. Since functions $a(t)$ and $\tau(t)$, in general, oscillate around the limit values, Theorem 1.1 and Theorem 1.2, as well as some other similar results, fail in this case. In this paper we construct the asymptotics as $t \to \infty$ for solutions of Eq. (1.1). The obtained asymptotic formulae will allow us to solve the oscillation problem for Eq. (1.1) in terms of certain numerical quantities that include the information about the coefficients $a_j(t)$, $q_j(t)$ of expansions (1.3) and (1.4) with account of the values of parameter ρ.

This paper is organized as follows. In Section 2 we describe the asymptotic integration method that we use throughout the paper to get the asymptotic formulae for solutions of Eq. (1.1). Asymptotic representations for solutions are constructed in Section 3. In the final section of the paper we summarize the obtained results and indicate the conditions for existence of oscillatory (nonoscillatory) solutions of Eq. (1.1). Moreover, we also give some examples in this section.

2 Description of the asymptotic integration method

In (1.1), we make the change of variable

$$x(t) = e^{-t}y(t), \tag{2.1}$$

to get

$$\dot{y} = y(t) - a(t)e^{\tau(t)}y(t-\tau(t)). \tag{2.2}$$

After some trivial manipulations with the right-hand side of Eq. (2.2) we rewrite it in the form of the functional differential equation

$$\dot{y} = B_0 y_t + G(t, y_t), \tag{2.3}$$

where $y_t(\theta) = y(t+\theta)$ $(-h \le \theta \le 0)$ denotes the element of the space $C_h \equiv C\big([-h,0],\mathbb{C}\big)$ consisting of all continuous functions defined on $[-h,0]$ and acting to $\mathbb{C}$. We choose the delay $h > 0$ such that the inequalities $0 \le \tau(t) \le h$ hold $t \ge t_0$. The norm in C_h is introduced in the standard way:

$$\|\varphi\|_{C_h} = \sup_{-h\le\theta\le 0} |\varphi(\theta)|. \tag{2.4}$$

Further, B_0 is a bounded linear functional acting from C_h to $\mathbb{C}$ that is defined by the formula

$$B_0\varphi(\theta) = \varphi(0) - \varphi(-1), \qquad \varphi(\theta) \in C_h. \tag{2.5}$$

Finally, functional $G(t,\varphi(\theta))$, acting from C_h to $\mathbb{C}$, has the form

$$G(t,\varphi(\theta)) = \varphi(-1) - a(t)e^{\tau(t)}\varphi(-\tau(t)). \tag{2.6}$$

The asymptotic integration method that we apply in this work to study the dynamics of Eq. (2.3) was suggested by the author in [26,27]. In these papers Eq. (2.3) is considered as a perturbation of the linear autonomous equation

$$\dot{y} = B_0 y_t. \tag{2.7}$$

The main assumption concerning the unperturbed Eq. (2.7) is the following. The characteristic equation should have the finite number of roots (with account of their multiplicities) with zero real parts and all other roots should have negative real parts. Linear bounded functional $G(t,\varphi(\theta))$ is, in some sense, a «small» perturbation consisting of two terms. The first term is a functional that oscillatorily tends to zero as $t \to \infty$ for each $\varphi(\theta)$. The second term is an absolutely integrable on $[t_0,\infty)$ in a certain sense functional, i.e., its values as functions of t belong to $L_1[t_0,\infty)$. Here and in what follows we write that scalar function, vector-function or matrix $F(t)$ belongs to $L_1[t_0,\infty)$, if the integral

$$\int_{t_0}^{\infty} |F(t)|dt,$$

where $|\cdot|$ is an absolute value or certain vector or matrix norm, is finite.

Proposition 2.1. *The characteristic equation*

$$\Delta(\lambda) = 0, \qquad \Delta(\lambda) = \lambda - 1 + e^{-\lambda}, \tag{2.8}$$

constructed for the unperturbed equation (2.7) *with functional* (2.5)*, has roots* $\lambda_{1,2} = 0$ *(i.e., zero root of multiplicity two) and all the other roots have negative real parts.*

Proof. It is obvious that $\Delta(0) = \Delta'(0) = 0$ and $\Delta''(0) = 1 \ne 0$. Hence, $\lambda = 0$ is a root of characteristic equation (2.8) with multiplicity two. Note that this equation does not have any other real roots λ. Since $\Delta'(\lambda) = 1 - e^{-\lambda}$, the function $\Delta(\lambda)$ decreases monotonically in the interval $(-\infty, 0)$ and increases monotonically in the interval $(0,+\infty)$. At the point $\lambda = 0$ this function has global minimum $\Delta(0) = 0$. Consequently, $\Delta(\lambda) > 0$ for all $\lambda \ne 0$.

Suppose that equation (2.8) has complex root $\lambda = \alpha + i\beta$, where $\alpha, \beta \in \mathbb{R}$ and $\beta > 0$. By equating the real and the imaginary parts in (2.8), we obtain

$$\begin{cases} \alpha - 1 + e^{-\alpha}\cos\beta = 0, \\ \beta - e^{-\alpha}\sin\beta = 0. \end{cases}$$

Due to the well-known inequality, it follows that

$$e^{\alpha} = \frac{\sin\beta}{\beta} < 1.$$

Hence, $\alpha < 0$ and all complex roots have negative real parts. □

Verification of the fact that functional $G(t, \varphi(\theta))$ is a small perturbation is not actually trivial due to the presence of the variable delay $\tau(t)$. The corresponding problems are discussed in paper [27]. It turns out that in this case the choice of the space C_h as the phase space for Eq. (2.3) is not appropriate. We should act in another manner. We remind that function $\varphi \in C_h$ is called Lipschitz continuous if there is a positive constant K (Lipschitz constant) such that

$$|\varphi(\theta_1) - \varphi(\theta_2)| \leq K\,|\theta_1 - \theta_2|, \quad -h \leq \theta_1, \theta_2 \leq 0. \tag{2.9}$$

Note that constant K in (2.9) depends on function $\varphi(\theta)$. Let us introduce the following notation.

Definition 2.2. Denote by LC_h the subspace of C_h consisting of all Lipschitz continuous functions and equipped with the norm

$$\|\varphi\|_{LC_h} = \max\left(\|\varphi\|_{C_h}, K_\varphi\right), \tag{2.10}$$

where $K_\varphi = \inf K$ and infimum is taken over all constants K for which inequality (2.9) holds. Symbol $\|\varphi\|_{C_h}$ stands for norm (2.4).

We remark that with norm (2.10) the space LC_h is a Banach space. Let $y_t(\theta)$ be the solution of Eq. (2.3) with initial value $y_T = \varphi$, where $\varphi \in C_h$ and $T \geq t_0$. Then, due to continuity property of functions $a(t)$, $\tau(t)$ and the form of functional $G(t, \varphi)$, defined by (2.6), solution $y_t(\theta)$ belongs to the space LC_h for $t \geq T + h$. Therefore, the dynamics of Eq. (2.3) is defined by the behaviour of solutions in LC_h. We can now easily check that $G(t, \varphi)$, as the functional acting from LC_h, is a small perturbation. Since, due to (1.3) and (1.4), the asymptotic formula $a(t)e^{\tau(t)} = 1 + O(t^{-\rho})$ holds as $t \to \infty$, we have

$$G(t, \varphi(\theta)) = \varphi(-1) - \varphi(-\tau(t)) + O(t^{-\rho})\varphi(-\tau(t)).$$

Thus, for each $\varphi \in LC_h$ due to (1.4) with account of (2.10) we conclude that

$$\begin{aligned} |G(t, \varphi(\theta))| &\leq |\varphi(-1) - \varphi(-\tau(t))| + O(t^{-\rho})|\varphi(-\tau(t))| \\ &\leq K_\varphi O(t^{-\rho}) + O(t^{-\rho})\|\varphi\|_{C_h} \leq O(t^{-\rho})\|\varphi\|_{LC_h} \end{aligned} \tag{2.11}$$

This proves the «smallness» of the functional $G(t, \varphi(\theta))$ as $t \to \infty$. The oscillatory decreasing character of $G(t, \varphi(\theta))$ as the function of t for each $\varphi \in LC_h$ follows from (2.6) and the corresponding properties of the functions $a_j(t)$, $q_j(t)$ in (1.3), (1.4). In what follows we will give a slightly different representation for the functional $G(t, \varphi(\theta))$. The presence of oscillatory decreasing coefficients in this representation will play an essential role for the implementation of the asymptotic integration method. We now turn to the description of this method.

The asymptotic integration method we apply in this paper is based on the existence for sufficiently large t of the positively invariant manifold in space LC_h that attracts (at the exponential rate) all the trajectories of Eq. (2.3). The dynamics of solutions of Eq. (2.3), lying in this manifold, is described by the two-dimensional linear ordinary differential system. Thus, the fundamental solutions of this system define the main parts of the asymptotic formulae for solutions of Eq. (2.3). We will now describe this method in details. First we need to decompose space C_h into direct sum of two certain subspaces.

It is known that linear autonomous equation (2.7) generates in C_h for $t \geq 0$ a strongly continuous semigroup $T(t)\colon C_h \to C_h$. The solution operator $T(t)$ of Eq. (2.7) is defined as follows: $T(t)\varphi = y_t^{\varphi}(\theta)$, where $\varphi \in C_h$ and $y_t^{\varphi}(\theta)$ is a unique solution of Eq. (2.7) with initial value $y_0^{\varphi}(\theta) = \varphi$. The infinitesimal generator A of this semigroup is defined by $A\varphi = \varphi'(\theta)$, where $\varphi \in D(A)$. The domain of A

$$D(A) = \big\{\varphi \in C_h \;\big|\; \varphi'(\theta) \in C_h, \;\; \varphi'(0) = B_0\varphi\big\}$$

is dense in C_h. Suppose that B_0 has Riesz representation

$$B_0\varphi = \int_{-h}^{0} d\eta(\theta)\varphi(\theta),$$

where $\eta(\theta)$ is the scalar function of bounded variation on $[-h, 0]$. We can associate with Eq. (2.7) the transposed equation

$$\dot{y}_* = -\int_{-h}^{0} y_*(t-\theta)d\eta(\theta), \qquad t \leq 0, \tag{2.12}$$

where $y_*(t)$ is complex scalar function. The phase space for Eq. (2.12) is $C_h' \equiv C\big([0,h],\mathbb{C}\big)$. For $\psi \in C_h'$ and $\varphi \in C_h$ we define the bilinear form

$$\big(\psi(\xi), \varphi(\theta)\big) = \psi(0)\varphi(0) - \int_{-h}^{0}\int_{0}^{\theta} \psi(\xi-\theta)d\eta(\theta)\varphi(\xi)d\xi. \tag{2.13}$$

Let

$$\Lambda = \{\lambda_1, \lambda_2\},$$

where $\lambda_1 = \lambda_2 = 0$ are the roots of characteristic equation (2.8) from Proposition 2.1. We now decompose C_h into a direct sum

$$C_h = P_\Lambda \oplus Q_\Lambda. \tag{2.14}$$

Here P_Λ is a linear span of generalized eigenfunctions of operator A corresponding to the eigenvalues from Λ and Q_Λ is certain complementary subspace of C_h such that $T(t)Q_\Lambda \subseteq Q_\Lambda$. Let $\Phi(\theta)$ be two-dimensional row-vector whose entries are the generalized eigenfunctions $\varphi_1(\theta), \varphi_2(\theta)$ of operator A corresponding to the eigenvalues from Λ. Thus, the entries of $\Phi(\theta)$ form the basis of P_Λ. Moreover, let $\Psi(\xi)$ be two-dimensional column-vector whose entries $\psi_1(\xi), \psi_2(\xi)$ form the basis of the generalized eigenspace P_Λ^T of the transposed equation (2.12) associated with Λ. We can choose vectors $\Phi(\theta)$ and $\Psi(\xi)$ such that

$$\big(\Psi(\xi), \Phi(\theta)\big) = \big\{\big(\psi_i(\xi), \varphi_j(\theta)\big)\big\}_{1\leq i,j\leq 2} = I. \tag{2.15}$$

Since $\Phi(\theta)$ is the basis of P_Λ and $AP_\Lambda \subseteq P_\Lambda$, there exists (2×2)-matrix D, whose spectrum is Λ, such that $A\Phi(\theta) = \Phi(\theta)D$. From the definition of A, we deduce that

$$\Phi(\theta) = \Phi(0)e^{D\theta}, \qquad T(t)\Phi(\theta) = \Phi(\theta)e^{Dt} = \Phi(0)e^{D(t+\theta)},$$

where $-h \leq \theta \leq 0$ and $t \geq 0$. Analogously, for column-vector $\Psi(\xi)$ we have

$$\Psi(\xi) = e^{-D\xi}\Psi(0), \tag{2.16}$$

where $0 \le \xi \le h$. Vectors $\Phi(0)$ and $\Psi(0)$ are chosen in the following way. Since the entries of row-vector $\Phi(\theta)$ are the generalized eigenfunctions of A, they should belong to $D(A)$. This implies that

$$\Phi'(0) = \Phi(0)D = B_0\Phi = \int_{-h}^{0} d\eta(\theta)\Phi(0)e^{D\theta}.$$

The same reasoning, using (2.12) and (2.16), yields

$$\Psi'(0) = -D\Psi(0) = -\int_{-h}^{0} e^{D\theta}\Psi(0)d\eta(\theta).$$

Finally, the subspaces P_Λ and Q_Λ from decomposition (2.14) may be defined as follows:

$$\begin{aligned} P_\Lambda &= \{\varphi \in C_h \mid \varphi(\theta) = \Phi(\theta)u,\ u \in \mathbb{C}^2\}, \\ Q_\Lambda &= \{\varphi \in C_h \mid (\Psi, \varphi) = 0\}. \end{aligned} \tag{2.17}$$

Here and in what follows symbol $\mathbb{C}^2$ stands for the space of two-dimensional complex column-vectors.

An easy computation yields the following formulae for vectors $\Phi(\theta)$, $\Psi(\xi)$ and matrix D for Eq. (2.7) with functional (2.5):

$$\Phi(\theta) = \begin{pmatrix} 1 & \theta \end{pmatrix}, \quad \Psi(\xi) = \begin{pmatrix} \frac{2}{3} - 2\xi \\ 2 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{2.18}$$

To calculate vectors $\Phi(\theta)$ and $\Psi(\xi)$ we also used condition (2.15). We are now in a position to define the central notion of the proposed method — the notion of critical manifold for Eq. (2.3).

Definition 2.3. Two-dimensional linear space $\mathcal{W}(t) \subset LC_h \subset C_h$ is said to be critical (or center-like) manifold of Eq. (2.3) for $t \ge t_* \ge t_0$ if the following conditions hold:

1. There exists two-dimensional row-vector $H(t, \theta)$, whose entries are continuous in $t \ge t_*$ and belong to LC_h and also subspace Q_Λ as functions of $\theta \in [-h, 0]$ for all $t \ge t_*$. Moreover, $\|H(t, \cdot)\|_{LC_h} \to 0$ as $t \to \infty$, where

$$\|H(t, \cdot)\|_{LC_h} = \|\, |H(t, \cdot)|\, \|_{LC_h}.$$

   Here $|\cdot|$ denotes some vector norm in the space of two-dimensional row-vectors.

2. The space $\mathcal{W}(t)$ for $t \ge t_*$ is defined by the formula

$$\mathcal{W}(t) = \Big\{\varphi(\theta) \in LC_h \mid \varphi(\theta) = \Phi(\theta)u + H(t, \theta)u,\ u \in \mathbb{C}^2\Big\}. \tag{2.19}$$

3. The space $\mathcal{W}(t)$ is positively invariant for trajectories of Eq. (2.3) for $t \ge t_*$, i.e., if $y_T \in \mathcal{W}(T)$, $T \ge t_*$, then $y_t \in \mathcal{W}(t)$ for $t \ge T$.

The following existence theorem holds (see [27]).

Theorem 2.4. *For sufficiently large t there exists a critical manifold $\mathcal{W}(t)$ of Eq. (2.3) in LC_h.*

Due to the positive invariance of $\mathcal{W}(t)$, the trajectories lying in this manifold for sufficiently large t are described by the formula

$$y_t(\theta) = \Phi(\theta)u(t) + H(t,\theta)u(t), \qquad t \geq T, \quad u(t) \in \mathbb{C}^2.$$

It can be shown (see, e.g., [19,20]), that the vector function $u(t)$ in the above expression satisfies the following ordinary differential system:

$$\dot{u} = \left[D + \Psi(0)G\big(t, \Phi(\theta) + H(t,\theta)\big)\right]u, \qquad t \geq T. \tag{2.20}$$

This system will be referred to as a system on critical manifold. An important property of manifold $\mathcal{W}(t)$ is that it is attractive for all trajectories of Eq. (2.3) (see [27]).

Theorem 2.5. *Suppose that $y(t)$ is a solution of Eq. (2.2), defined for $t \geq T \geq t_0$. Then there exists sufficiently large $t_* \geq T$ such that the following asymptotic formula holds for $t \geq t_*$:*

$$y_t(\theta) = \Phi(\theta)u_H(t) + H(t,\theta)u_H(t) + O\big(e^{-\beta t}\big), \qquad t \to \infty.$$

Here $u_H(t)$ ($t \geq t_*$) is a certain solution of Eq. (2.20) and $\beta > 0$ is a certain real number.

Suppose that $u^{(1)}(t), u^{(2)}(t)$ are the fundamental solutions of a system on critical manifold (2.20) and $y(t)$ is an arbitrary solution of Eq. (2.2) defined for $t \geq T$. By Theorem 2.5, this solution has the following asymptotic representation as $t \to \infty$:

$$y(t) = y_t(0) = \big(\Phi(0) + H(t,0)\big)\big(c_1u^{(1)}(t) + c_2u^{(2)}(t)\big) + O\big(e^{-\beta t}\big), \qquad t \to \infty, \tag{2.21}$$

where c_1, c_2 are arbitrary complex constants and $\beta > 0$ is a certain real number. Therefore, to solve the oscillation problem for Eq. (2.2) (evidently, for initial Eq. (1.1) as well) we need to construct the asymptotics for the fundamental solutions $u^{(1)}(t)$, $u^{(2)}(t)$ of system (2.20) that define the dynamics of all solutions of Eq. (2.2) due to (2.21). Unfortunately, having determined the type of solutions $u^{(1)}(t)$ and $u^{(2)}(t)$ (oscillatory or nonoscillatory), we cannot answer the question whether all the solutions of Eq. (1.1) are of the same type. This follows from the fact that due to (2.21) if $c_1 = c_2 = 0$ the dynamics of solutions of Eq. (1.1) is defined by the remainder term, whose form is unclear. Thus, in this paper we only give an answer concerning the existence of oscillatory or nonoscillatory solutions.

Now we need to clarify how to construct the row-vector $H(t,\theta)$ needed for system on critical manifold (2.20) and how to obtain the asymptotics for the fundamental matrix of this system. It is shown in [26, 27] that vector $H(t,\theta)$ is a solution, in certain week sense, of the following problem:

$$\Phi(\theta)\Psi(0)G\big(t, \Phi(\theta) + H(t,\theta)\big) + H(t,\theta)\left(D + \Psi(0)G\big(t, \Phi(\theta) + H(t,\theta)\big)\right) + \frac{\partial H}{\partial t}$$
$$= \begin{cases} \dfrac{\partial H}{\partial \theta}, & -h \leq \theta < 0, \\ B_0H + G\big(t, \Phi(\theta) + H(t,\theta)\big), & \theta = 0. \end{cases} \tag{2.22}$$

We can solve this problem approximately. Namely, due to the form of the functional $G(t, \varphi(\theta))$ that is defined by formula (2.6) and taking into account asymptotic representations (1.3), (1.4) we can satisfy problem (2.22) with the row-vector

$$\hat{H}(t,\theta) = H_1(t,\theta)t^{-\rho} + H_2(t,\theta)t^{-2\rho} + \cdots + H_k(t,\theta)t^{-k\rho} \tag{2.23}$$

up to the term $\hat{R}(t,\theta)$ such that $\|\hat{R}(t,\cdot)\|_{LC_h} \in L_1[t_0,\infty)$. Here $k \in \mathbb{N}$ is defined according to (1.3), (1.4) with account of (1.5) and the entries of two-dimensional row-vectors $H_j(t,\theta)$, $j = 1,\dots,k$ are trigonometric polynomials in t whose coefficients are infinitely differentiable in $\theta \in [-h,0]$. Thus, the row-vectors $H_j(t,\theta)$ has the form

$$H_j(t,\theta) = \sum_s \beta_s^{(j)}(\theta) e^{\mathrm{i}\omega_s t}, \tag{2.24}$$

where the row-vectors $\beta_s^{(j)}(\theta)$ are infinitely differentiable in $\theta \in [-h,0]$. We also note that the entries of these row-vectors belong to the subspace Q_Λ. It appears that the problem of finding the vectors $H_j(t,\theta)$ is reduced to solving certain functional boundary problems for linear ordinary differential systems. Namely, we substitute (2.23) for $H(t,\theta)$ in (2.22) and collect terms corresponding to factors $t^{-j\rho}$, $j = 1,\dots,k$. We then seek the solutions of the obtained equations in form (2.24). Substituting the latter in the mentioned equations and matching the coefficients of the corresponding exponentials, we get the functional boundary problems for linear ordinary differential systems. It is proved in[26] that each of these problems is uniquely solvable.

Row-vector $\hat{H}(t,\theta)$ is an approximation, in a certain sense, for vector $H(t,\theta)$ that describes manifold $\mathcal{W}(t)$ according to formula (2.19). To be precise the following approximation theorem holds.

Theorem 2.6. *Suppose that $\mathcal{W}(t)$ is a critical manifold of Eq. (2.3) which exists for sufficiently large t according to Theorem 2.4. Then there exists a sufficiently large t_* such that for $t \geq t_*$ row-vector $H(t,\theta)$ from (2.19) admits the following representation:*

$$H(t,\theta) = \hat{H}(t,\theta) + Z(t,\theta), \qquad t \geq t_* \geq t_0, \quad -\tau \leq \theta \leq 0. \tag{2.25}$$

Here the row-vector $\hat{H}(t,\theta)$ is defined by formula (2.23) and satisfies Eq. (2.22) up to the term $\hat{R}(t,\theta)$ such that $\|\hat{R}(t,\cdot)\|_{LC_h} \in L_1[t_0,\infty)$. Moreover, $Z(t,\theta)$ is a certain row-vector such that $\|Z(t,\cdot)\|_{LC_h} \to 0$ as $t \to \infty$ and $\|Z(t,\cdot)\|_{LC_h} \in L_1[t_*,\infty)$.

According to (1.3), (1.4), (2.6) with account of formula (2.23), describing the approximate solution of problem (2.22), it can be shown that row-vector $Z(t,\theta)$ in (2.25) has the following asymptotic estimate as $t \to \infty$:

$$\|Z(t,\cdot)\|_{LC_h} = O\Big(\frac{d}{dt}(t^{-\rho})\Big) + O\big(t^{-(k+1)\rho}\big) = O\big(t^{-(\rho+1)}\big) + O\big(t^{-(k+1)\rho}\big). \tag{2.26}$$

The asymptotic integration of system (2.20) is carried out as follows. Due to (1.3), (1.4), (2.6), (2.23) this system in the considered case has the following form:

$$\dot{u} = \Big[D + A_1(t)t^{-\rho} + A_2(t)t^{-2\rho} + \dots + A_{k+1}(t)t^{-(k+1)\rho} + R(t)\Big]u, \qquad u \in \mathbb{C}^2. \tag{2.27}$$

Here matrix D is defined in (2.18), natural number k is chosen according to (1.5) and $A_1(t),\dots$, $A_{k+1}(t)$ are (2×2)-matrices, whose entries are trigonometric polynomials, i.e., matrices having the form

$$A_j(t) = \sum_s \psi_s^{(j)} e^{\mathrm{i}\omega_s t},$$

where $\psi_s^{(j)}$ are constant complex matrices and ω_s are real numbers. Finally, $R(t)$ is a certain (2×2)-matrix that belongs to $L_1[t_*,\infty)$. It follows from (1.3), (1.4), (2.11) and (2.26) that this matrix has the following asymptotic estimate:

$$R(t) = O\big(t^{-(k+2)\rho}\big) + O\big(t^{-(2\rho+1)}\big), \qquad t \to \infty. \tag{2.28}$$

The main difficulty in the asymptotic integration of system (2.27) as $t \to \infty$ is that its coefficients have an oscillatory behaviour. Therefore, on the first step we utilize in (2.27) the averaging change of variable that makes it possible to exclude the oscillating coefficients from the main part of the system. The following theorem holds (see [25]).

Theorem 2.7. *For sufficiently large t, system (2.27) by the change of variable*

$$u = \Big[I + Y_1(t)t^{-\rho} + Y_2(t)t^{-2\rho} + \cdots + Y_{k+1}(t)t^{-(k+1)\rho}\Big]u_1 \tag{2.29}$$

can be reduced to its averaged form

$$\dot{u}_1 = \Big[D + A_1 t^{-\rho} + A_2 t^{-2\rho} + \cdots + A_{k+1}t^{-(k+1)\rho} + R_1(t)\Big]u_1 \tag{2.30}$$

with constant matrices $A_1, \dots, A_k$ and with matrix $R_1(t)$ from $L_1[t_*, \infty)$. In (2.29), I is the identity matrix and the entries of matrices $Y_1(t), \dots, Y_k(t)$ are trigonometric polynomials having zero mean value.

As a rule, to construct the asymptotics for solutions of (2.30) we need to compute only a few constant matrices. Hence, we give the explicit formulas only for matrices A_1 and A_2. We have

$$A_1 = \mathrm{M}\big[A_1(t)\big], \tag{2.31}$$
$$A_2 = \mathrm{M}\big[A_2(t) + A_1(t)Y_1(t)\big]. \tag{2.32}$$

Here symbol $\mathrm{M}\big[F(t)\big]$ denotes the mean value of the matrix $F(t)$ whose entries are trigonometric polynomials:

$$\mathrm{M}\big[F(t)\big] = \lim_{T\to\infty} \frac{1}{T}\int_0^T F(t)dt.$$

Matrix $Y_1(t)$ in (2.32) is the solution of matrix differential equation

$$\dot{Y}_1 - DY_1 + Y_1 D = A_1(t) - A_1 \tag{2.33}$$

with zero mean value. Finally, matrix $R_1(t)$ in (2.30) has the following form:

$$R_1(t) = \rho Y_1(t)t^{-(\rho+1)} + O\big(t^{-(2\rho+1)}\big) + O\big(t^{-(k+2)\rho}\big), \qquad t \to \infty. \tag{2.34}$$

Here we give the explicit formula for the first term in (2.34) since its form will be necessary for further transformation of system (2.30).

The subsequent transformations of the averaged system (2.30) aim to bring it to the form

$$\dot{u}_2 = \big[A_0 + V(t)\big]t^{-\alpha}u_2 + R_2(t)u_2, \tag{2.35}$$

where $\alpha > 0$ is a certain number, A_0 is a constant matrix, matrix $V(t)$ tends to zero matrix as $t \to \infty$ and $R_2(t) \in L_1[t_*, \infty)$. The following lemma holds (see, for instance, [1,6,16]).

Lemma 2.8 (diagonalization of variable matrices). *Suppose that all eigenvalues of the matrix A_0 are distinct. Moreover, suppose that matrix $V(t) \to 0$ as $t \to \infty$ and $V'(t) \in L_1[t_*, \infty)$. Then for sufficiently large t there exists a nonsingular matrix $C(t)$ such that*

(i) *the columns of this matrix are the eigenvectors of the matrix $A_0 + V(t)$ and $C(t) \to C_0$ as $t \to \infty$. The columns of the constant matrix C_0 are the eigenvectors of the matrix A_0;*

(ii) *the derivative* $C'(t) \in L_1[t_*, \infty)$*;*

(iii) *it brings the matrix* $A_0 + V(t)$ *to diagonal form, i.e.,*

$$C^{-1}(t)\big[A_0 + V(t)\big]C(t) = \hat{\Lambda}(t),$$

where $\hat{\Lambda}(t) = \operatorname{diag}\big(\hat{\lambda}_1(t), \hat{\lambda}_2(t)\big)$ *and* $\hat{\lambda}_1(t), \hat{\lambda}_2(t)$ *are the eigenvalues of the matrix* $A_0 + V(t)$*.*

In (2.35), we make the change of variable

$$u_2(t) = C(t)u_3(t),$$

where $C(t)$ is the matrix from Lemma 2.8. This change of variable brings system (2.35) to what is called L-diagonal form:

$$\dot{u}_3 = \big[\Lambda(t) + R_3(t)\big]u_3, \tag{2.36}$$

where $\Lambda(t) = \operatorname{diag}\big(\lambda_1(t), \lambda_2(t)\big)$, $\lambda_j(t) = \hat{\lambda}_j(t)t^{-\alpha}$ $(j = 1, 2)$ and

$$R_3(t) = -C^{-1}(t)\dot{C}(t) + C^{-1}(t)R_2(t)C(t).$$

The properties (i) and (ii) of the matrix $C(t)$ imply that matrix $R_3(t)$ belongs to $L_1[t_*, \infty)$.

To construct the asymptotics for solutions of L-diagonal system (2.36) as $t \to \infty$ the well-known Theorem of Levinson can be used. Suppose that the following dichotomy condition holds for the entries of the matrix $\Lambda(t)$: either the inequality

$$\int_{t_1}^{t_2} \operatorname{Re}\big(\lambda_i(s) - \lambda_j(s)\big)ds \le K_1, \qquad t_2 \ge t_1 \ge t_*, \tag{2.37}$$

or the inequality

$$\int_{t_1}^{t_2} \operatorname{Re}\big(\lambda_i(s) - \lambda_j(s)\big)ds \ge K_2, \qquad t_2 \ge t_1 \ge t_*, \tag{2.38}$$

is valid for each pair of indices (i, j), where K_1, K_2 are some constants. What follows is Levinson's fundamental theorem (see, e.g., [6, 16, 22]).

Theorem 2.9 (Levinson). *Let the dichotomy condition* (2.37), (2.38) *be satisfied. Then the fundamental matrix of L-diagonal system* (2.36) *has the following asymptotics as* $t \to \infty$*:*

$$U(t) = \big(I + o(1)\big)\exp\left\{\int_{t^*}^{t} \Lambda(s)ds\right\}.$$

We note that for the problem considered in this paper the dichotomy condition (2.37), (2.38) is always satisfied since quantities $\operatorname{Re}\big(\lambda_i(t) - \lambda_j(t)\big)$ do not change their signs for sufficiently large t. This follows from the fact that system (2.35) comes from the averaged system (2.30), whose coefficients in the main part do not oscillate and the utilized transformations do not change this property.

3 Construction of asymptotic formulae

In this section we obtain the asymptotic formulae for solutions of Eq. (2.2) as $t \to \infty$. The asymptotics for solutions of the initial Eq. (1.1) can be easily constructed by applying the change of variable (2.1) and, therefore, we will not write it here. First, we get another one representation for functional $G(t,\varphi(\theta))$ in (2.3) that is defined by formula (2.6). By applying Taylor's formula for $a(t)e^{\tau(t)}$ as $t \to \infty$ with account of (1.3), (1.4), we obtain

$$a(t)e^{\tau(t)} = 1 + p_1(t)t^{-\rho} + p_2(t)t^{-2\rho} + \cdots + p_{k+1}(t)t^{-(k+1)\rho} + O(t^{-(k+2)\rho}). \tag{3.1}$$

Here $p_1(t),\dots,p_{k+1}(t)$ are certain trigonometric polynomials and, in particular,

$$p_1(t) = ea_1(t) + q_1(t), \qquad p_2(t) = ea_2(t) + q_2(t) + \frac{q_1^2(t)}{2} + ea_1(t)q_1(t), \tag{3.2}$$

where $a_i(t), q_i(t)$, $i = 1,2$, are functions from asymptotic expansions (1.3), (1.4) for coefficients of the initial equation (1.1). For the sequel we need the expressions for the functions $p_1(t)$ and $q_1(t)$ in the form of the trigonometric polynomials:

$$p_1(t) = \sum_{j=-N}^{N} p_1^{(j)} e^{\mathrm{i}\omega_j t}, \qquad q_1(t) = \sum_{j=-N}^{N} q_1^{(j)} e^{\mathrm{i}\omega_j t}, \tag{3.3}$$

where $p_1^{(j)}, q_1^{(j)}$ are, in general, certain complex numbers, ω_j are real numbers and, moreover,

$$p_1^{(-j)} = \bar{p}_1^{(j)}, \quad q_1^{(-j)} = \bar{q}_1^{(j)}, \quad \omega_{-j} = -\omega_j \ (\omega_l \neq \omega_l,\ l \neq m), \quad j = 1,\dots,N. \tag{3.4}$$

here notation $\bar{a}$ stands for complex conjugate of a. Hence, we have

$$\mathrm{M}\big[p_1(t)\big] = p_1^{(0)}, \quad \mathrm{M}\big[q_1(t)\big] = q_1^{(0)}. \tag{3.5}$$

By using Taylor's formula for $\varphi(-\tau(t))$ as $t \to \infty$, and taking into account (3.1), we finally obtain the following representation for functional $G(t,\varphi(\theta))$:

$$\begin{aligned} G(t,\varphi(\theta)) = {} & \big[q_1(t)\varphi'(-1) - p_1(t)\varphi(-1)\big]t^{-\rho} \\ & + \left[p_1(t)q_1(t)\varphi'(-1) + q_2(t)\varphi'(-1) - p_2(t)\varphi(-1) - \frac{q_1^2(t)}{2}\varphi''(-1)\right]t^{-2\rho} \\ & + O\big(t^{-3\rho}\big). \end{aligned} \tag{3.6}$$

Although the functional $G(t,\varphi(\theta))$ is defined only for elements from C_h, in what follows it will be applied to infinitely differentiable functions and this makes possible to use form (3.6). We proceed now to the problem of construction of the asymptotic formulae for solutions of Eq. (2.2) as $t \to \infty$.

We write system on critical manifold (2.20) in form (2.27). To get the asymptotics for the fundamental solutions of this system we need the explicit formulae for matrices $A_1(t)$ and $A_2(t)$. We use (3.6) and also formula (2.18) to obtain

$$A_1(t) = \Psi(0)\big[q_1(t)\Phi'(-1) - p_1(t)\Phi(-1)\big] = \frac{2}{3}\begin{pmatrix} -p_1(t) & q_1(t) + p_1(t) \\ -3p_1(t) & 3\big(q_1(t) + p_1(t)\big) \end{pmatrix} \tag{3.7}$$

and

$$A_2(t) = \Psi(0)\left[p_1(t)q_1(t)\Phi'(-1) + q_2(t)\Phi'(-1) - p_2(t)\Phi(-1) - \frac{q_1^2(t)}{2}\Phi''(-1)\right]$$
$$+ \Psi(0)\left[q_1(t)\frac{\partial H_1}{\partial\theta}(t,\theta)\Big|_{\theta=-1} - p_1(t)H_1(t,-1)\right]$$
$$= \frac{2}{3}\begin{pmatrix} -p_2(t) & p_1(t)q_1(t)+q_2(t)+p_2(t) \\ -3p_2(t) & 3\big(p_1(t)q_1(t)+q_2(t)+p_2(t)\big)\end{pmatrix}$$
$$+ \frac{2}{3}\begin{pmatrix} q_1(t)\frac{\partial h_{11}}{\partial\theta}(t,\theta)\Big|_{\theta=-1} - p_1(t)h_{11}(t,-1) & q_1(t)\frac{\partial h_{12}}{\partial\theta}(t,\theta)\Big|_{\theta=-1} - p_1(t)h_{12}(t,-1) \\ 3\Big(q_1(t)\frac{\partial h_{11}}{\partial\theta}(t,\theta)\Big|_{\theta=-1} - p_1(t)h_{11}(t,-1)\Big) & 3\Big(q_1(t)\frac{\partial h_{12}}{\partial\theta}(t,\theta)\Big|_{\theta=-1} - p_1(t)h_{12}(t,-1)\Big)\end{pmatrix}. \tag{3.8}$$

Here

$$H_1(t,\theta) = \big(h_{11}(t,\theta) \quad h_{12}(t,\theta)\big) \tag{3.9}$$

is a row-vector from representation (2.23) for row-vector $\hat{H}(t,\theta)$ that is an approximation of $H(t,\theta)$ due to Theorem 2.6. Row-vector (3.9) will be defined at the end of this section.

The most simple case in constructing the asymptotic formulae for solutions of Eq. (2.2) occurs when

$$\rho > 2. \tag{3.10}$$

In this situation system on critical manifold (2.27) with account of (2.28) takes the form

$$\dot{u} = \big[D + O(t^{-\rho})\big]u, \tag{3.11}$$

where matrix D is defined by formula (2.18). Since, due to (3.10), the remainder term in (3.11) has the property that

$$O(t^{-\rho})t^{i-j} \in L_1[t_0,\infty), \quad 1 \le i,j \le 2,$$

we can use [2, Corollary 6.2, p. 213]. It follows that the fundamental solutions of system (3.11) have the following asymptotics as $t \to \infty$:

$$u^{(1)}(t) = \begin{pmatrix} 1+o(1) \\ o(t^{-1})\end{pmatrix}, \quad u^{(2)}(t) = \begin{pmatrix} t\big(1+o(1)\big) \\ 1+o(1)\end{pmatrix}. \tag{3.12}$$

We then use (2.21), with account that $H(t,0) = o(1)$, to obtain the following asymptotic representation for all solutions of Eq. (2.2) as $t \to \infty$:

$$y(t) = c_1\big(1+o(1)\big) + c_2t\big(1+o(1)\big) + O(e^{-\beta t}), \tag{3.13}$$

where c_1, c_2 are arbitrary real constants and $\beta > 0$ is a certain real number.

Thus, the main interest concerns the case

$$\rho \le 2.$$

We use Theorem 2.7 to bring system (2.27) by the change of variable (2.29) to the averaged form (2.30). In (2.30), constant matrices A_1 and A_2 are described by formulae (2.31), (2.32) and the remainder term $R_1(t)$ has form (2.34). We calculate matrix A_1 taking into account (3.7) and also expressions (3.3), (3.5). We have

$$A_1 = \mathrm{M}\big[A_1(t)\big] = \frac{2}{3}\begin{pmatrix} -p_1^{(0)} & q_1^{(0)} + p_1^{(0)} \\ -3p_1^{(0)} & 3\big(q_1^{(0)} + p_1^{(0)}\big)\end{pmatrix}. \tag{3.14}$$

The explicit form for matrix A_2 will be obtained later. The asymptotics for solutions of system (2.30) will differ depending on the mean value of the function $p_1(t)$. We now proceed to analysis of these cases.

I. $$p_1^{(0)} \neq 0$$

The eigenvalues of the matrix

$$A(t) = D + A_1 t^{-\rho} + A_2 t^{-2\rho} + \cdots + A_{k+1} t^{-(k+1)\rho}$$

in the main part of system (2.30) have the following asymptotics as $t \to \infty$:

$$\lambda_{1,2}(t) = \pm t^{-\frac{\rho}{2}} \sqrt{-2p_1^{(0)}}\big(1 + O(t^{-\rho})\big) + \Big(q_1^{(0)} + \frac{2}{3} p_1^{(0)}\Big) t^{-\rho} + O(t^{-2\rho}). \tag{3.15}$$

Here and in what follows the symbol $\sqrt{a}$, where $a \in \mathbb{R}$, stands for the quantity

$$\sqrt{a} = \begin{cases} \sqrt{a}, & a \geq 0, \\ \mathrm{i}\sqrt{-a}, & a < 0. \end{cases} \tag{3.16}$$

Since the eigenvalues (3.15) are distinct for sufficiently large t, the matrix $A(t)$ can be reduced to the diagonal form by certain non-singular matrix $C(t)$:

$$\Lambda(t) = \mathrm{diag}\big(\lambda_1(t), \lambda_2(t)\big) = C^{-1}(t) A(t) C(t). \tag{3.17}$$

Some easy calculations show that the corresponding matrix $C(t)$ has the following asymptotics as $t \to \infty$:

$$C(t) = \begin{pmatrix} 1 & 1 \\ t^{-\frac{\rho}{2}} \sqrt{-2p_1^{(0)}} + O(t^{-\rho}) & -t^{-\frac{\rho}{2}} \sqrt{-2p_1^{(0)}} + O(t^{-\rho}) \end{pmatrix}. \tag{3.18}$$

For the inverse matrix we get

$$C^{-1}(t) = \frac{1}{2\sqrt{-2p_1^{(0)}}} \begin{pmatrix} \sqrt{-2p_1^{(0)}} + O\big(t^{-\frac{\rho}{2}}\big) & t^{\frac{\rho}{2}} + O(1) \\ \sqrt{-2p_1^{(0)}} + O\big(t^{-\frac{\rho}{2}}\big) & -t^{\frac{\rho}{2}} + O(1) \end{pmatrix}, \quad t \to \infty.$$

We note that matrix $C^{-1}(t)$ is unbounded as $t \to \infty$ and has the asymptotic estimate $O\big(t^{\frac{\rho}{2}}\big)$. Keeping this fact in mind, we make in (2.30) the change of variable

$$u_1(t) = C(t) u_2(t)$$

with matrix $C(t)$ having form (3.18). Since

$$C^{-1}(t)\dot{C}(t) = \frac{\rho}{4} t^{-1} \begin{pmatrix} -1 + O\big(t^{-\frac{\rho}{2}}\big) & 1 + O\big(t^{-\frac{\rho}{2}}\big) \\ 1 + O\big(t^{-\frac{\rho}{2}}\big) & -1 + O\big(t^{-\frac{\rho}{2}}\big) \end{pmatrix}, \tag{3.19}$$

we obtain

$$\dot{u}_2 = \big[\Lambda(t) + B t^{-1} + R_2(t)\big] u_2. \tag{3.20}$$

Here the diagonal matrix $\Lambda(t)$ is defined by formula (3.17) with account of (3.15) and the constant matrix B has the following form:

$$B = \frac{\rho}{4} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{3.21}$$

Moreover, the remainder term in (3.20), due to (1.5), (2.34) and (3.19), admits the asymptotic estimate $R_2(t) = O\big(t^{-\frac{\rho}{2}-1}\big)$ as $t \to \infty$. Further, we need to study several alternatives.

Assume first that

$$\rho = 2. \tag{3.22}$$

In this situation system (3.20) takes the following form:

$$\dot{u}_2 = \left[St^{-1} + O(t^{-2})\right]u_2, \tag{3.23}$$

where

$$S = \sqrt{-2p_1^{(0)}}\,\mathrm{diag}(1,-1) + B, \tag{3.24}$$

and matrix B is defined by formula (3.21). The eigenvalues of this matrix are

$$\mu_{1,2} = \frac{1}{2} \pm \sigma, \qquad \sigma = \frac{1}{2}\sqrt{1 - 8p_1^{(0)}}. \tag{3.25}$$

We recall that the square root here means the quantity (3.16). We should consider two cases.

- $p_1^{(0)} \neq \frac{1}{8}$

  This is the case when $\mu_{1,2}$ are distinct and system (3.23) by the change of variable $u_2 = Cu_3$, where, for instance,

$$C = \begin{pmatrix} \dfrac{1}{2\sqrt{-2p_1^{(0)}} - \sqrt{1-8p_1^{(0)}}} & \dfrac{1}{2\sqrt{-2p_1^{(0)}} + \sqrt{1-8p_1^{(0)}}} \end{pmatrix},$$

can be reduced to L-diagonal form (2.36). In the corresponding L-diagonal system we have

$$\Lambda(t) = \mathrm{diag}(\mu_1, \mu_2)t^{-1}, \qquad R_3(t) = O(t^{-2}), \quad t \to \infty.$$

The asymptotics for the fundamental matrix of this system can be constructed by applying Theorem 2.9. If we return then to Eq. (2.2), we get the following asymptotics for its solutions as $t \to \infty$:

$$y(t) = c_1 t^{\frac{1}{2}} \exp\{\sigma \ln t\}\big(1 + o(1)\big) + c_2 t^{\frac{1}{2}} \exp\{-\sigma \ln t\}\big(1 + o(1)\big) + O\big(e^{-\beta t}\big),$$

where c_1, c_2 are arbitrary, in general, complex constants, $\beta > 0$ is a certain real number and quantity σ is defined by formula (3.25).

- $p_1^{(0)} = \frac{1}{8}$

  In this situation the eigenvalues of matrix (3.24) coincide:

$$\mu_{1,2} = \frac{1}{2}.$$

First, by the change of variable $u_2 = Cu_3$, where

$$C = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ -i & 2 \end{pmatrix},$$

we bring system (3.23) to the form

$$\dot{u}_3 = \left[Jt^{-1} + O(t^{-2})\right]u_3, \qquad J = \frac{1}{2}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}. \tag{3.26}$$

Next, we apply in (3.26) the transformation $u_3 = t^{\frac{1}{2}} u_4$ to obtain

$$\dot{u}_4 = \left[Dt^{-1} + O(t^{-2})\right] u_4, \tag{3.27}$$

where matrix D is defined by (2.18). Finally, in (3.27) we introduce the new time-variable $\tau = \ln t$ to get

$$u_4' = \left[D + O(e^{-\tau})\right] u_4, \tag{3.28}$$

where the dash denotes the derivative with respect to τ. The construction of the asymptotics for the fundamental matrix of system (3.28) is carried out in the same manner as for system (3.11). This results in the following asymptotic representation for solutions of Eq. (2.2) as $t \to \infty$:

$$y(t) = c_1 t^{\frac{1}{2}} \big(1 + o(1)\big) + c_2 t^{\frac{1}{2}} \ln t \big(1 + o(1)\big) + O\big(e^{-\beta t}\big),$$

where c_1, c_2 are arbitrary real constants and $\beta > 0$ is a certain real number.

Consider now the case

$$\rho < 2.$$

We can write system (3.20) in form (2.35), where, due to (3.15),

$$\alpha = \frac{\rho}{2}, \quad A_0 = \sqrt{-2p_1^{(0)}}\, \mathrm{diag}(1, -1), \quad V(t) = \left(q_1^{(0)} + \frac{2}{3} p_1^{(0)}\right) I t^{-\frac{\rho}{2}} + B t^{\frac{\rho}{2} - 1} + O(t^{-\rho}).$$

and $R_2(t) = O\big(t^{-\frac{\rho}{2} - 1}\big)$ as $t \to \infty$. Here matrix B is described by formula (3.21). The asymptotic integration of systems having form (2.35) was described at the end of the previous section. Therefore, we give only the final result concerning the asymptotic formulae for solutions of Eq. (2.2) as $t \to \infty$.

So, if

$$1 < \rho < 2,$$

we have

$$\begin{aligned} y(t) = {} & c_1 t^{\frac{\rho}{4}} \exp\left\{\frac{2}{2-\rho} \sqrt{-2p_1^{(0)}}\, t^{1-\frac{\rho}{2}}\right\} \big(1 + o(1)\big) \\ & + c_2 t^{\frac{\rho}{4}} \exp\left\{-\frac{2}{2-\rho} \sqrt{-2p_1^{(0)}}\, t^{1-\frac{\rho}{2}}\right\} \big(1 + o(1)\big) + O\big(e^{-\beta t}\big). \end{aligned}$$

If

$$\rho = 1,$$

then

$$\begin{aligned} y(t) = t^{\frac{1}{4} + q_1^{(0)} + \frac{2}{3} p_1^{(0)}} \Bigg[& c_1 \exp\left\{2\sqrt{-2p_1^{(0)} t}\right\} \big(1 + o(1)\big) \\ & + c_2 \exp\left\{-2\sqrt{-2p_1^{(0)} t}\right\} \big(1 + o(1)\big) \Bigg] + O\big(e^{-\beta t}\big). \end{aligned}$$

Finally, if

$$\rho < 1,$$

we obtain

$$\begin{aligned} y(t) = t^{\frac{\rho}{4}} \exp\left\{\frac{t^{1-\rho}}{1-\rho}\left(q_1^{(0)} + \frac{2}{3} p_1^{(0)}\right)\right\} \Bigg[& c_1 \exp\left\{\frac{2}{2-\rho} \sqrt{-2p_1^{(0)}}\, t^{1-\frac{\rho}{2}} + O\left(\int t^{-\frac{3\rho}{2}} dt\right)\right\} \big(1 + o(1)\big) \\ & + c_2 \exp\left\{-\frac{2}{2-\rho} \sqrt{-2p_1^{(0)}}\, t^{1-\frac{\rho}{2}} + O\left(\int t^{-\frac{3\rho}{2}} dt\right)\right\} \big(1 + o(1)\big) \Bigg] + O\big(e^{-\beta t}\big). \end{aligned}$$

Everywhere in these asymptotic formulae c_1, c_2 are arbitrary, in general, complex constants and $\beta > 0$ is a certain real number.

We now proceed to a case more complicated in computational sense.

II.
$$p_1^{(0)} = 0 \tag{3.29}$$

The simplest situation in this case occurs when

$$\rho > 1.$$

The averaged system (2.30) takes the form

$$\dot{u}_1 = \left[D + \hat{R}_1(t)\right] u_1, \tag{3.30}$$

where, with account of (2.34),

$$\hat{R}_1(t) = A_1 t^{-\rho} + \cdots + A_{k+1} t^{-(k+1)\rho} + O\big(t^{-(\rho+1)}\big) + O\big(t^{-(2\rho+1)}\big) + O\big(t^{-(k+2)\rho}\big).$$

We remark that, due to (3.29), matrix A_1, that is described by formula (3.14), has the following form:

$$A_1 = \frac{2}{3} q_1^{(0)} \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix}. \tag{3.31}$$

It follows that the entries $\hat{r}_{ij}(t)$ of the matrix $\hat{R}_1(t)$ have the property

$$t^{i-j} \hat{r}_{ij}(t) \in L_1[t_0, \infty), \qquad 1 \le i, j \le 2.$$

This yields that like in the case (3.10) we can use [2, Corollary 6.2, p. 213] to construct the asymptotics for the fundamental solutions of system (3.30). Hence, we obtain asymptotic formulae (3.12) for the fundamental solutions of this system. Thus, we get asymptotics (3.13) for solutions of Eq. (2.2) as $t \to \infty$.

Assume further that

$$\rho \le 1.$$

In the averaged system (2.30) we make one more averaging change of variable

$$u_1 = \left[I + Q(t) t^{-(\rho+1)}\right] u_2 \tag{3.32}$$

that allows us, due to Theorem 2.7, to exclude the summand having the asymptotic order $O(t^{-(\rho+1)})$ in the remainder term (2.34). Here matrix $Q(t)$, whose entries are trigonometric polynomials, is the solution of the matrix differential equation

$$\dot{Q} - DQ + QD = \rho Y_1(t)$$

with zero mean value. The main part of the transformed system has the same form as the main part of system (2.30) but the new remainder term has now the following asymptotic estimate as $t \to \infty$:

$$R_2(t) = O\big(t^{-(\rho+2)}\big) + O\big(t^{-(2\rho+1)}\big) + O\big(t^{-(k+2)\rho}\big). \tag{3.33}$$

Then in the obtained system we make the so-called shearing transformation

$$u_2 = \begin{pmatrix} t^{\frac{\rho}{2}} & \\ 0 & t^{-\frac{\rho}{2}} \end{pmatrix} u_3. \tag{3.34}$$

With account of formulae (2.18) and (3.31), that describe matrices D and A_1, we get the following system:

$$\dot{u}_3 = \Big[B_1 t^{-\rho} + B_2 t^{-2\rho} + \cdots + B_k t^{-k\rho} + B_0 t^{-1} + R_3(t)\Big] u_3. \tag{3.35}$$

Here $B_0, \dots, B_k$ are certain constant matrices and, in particular,

$$B_0 = \frac{\rho}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 1 \\ a_{21}^{(2)} & 2q_1^{(0)} \end{pmatrix}, \quad B_2 = \begin{pmatrix} a_{11}^{(2)} & \frac{2}{3}q_1^{(0)} \\ a_{21}^{(3)} & a_{22}^{(2)} \end{pmatrix}.$$

Symbols $a_{ij}^{(2)}$ in the above expressions denote the entries of the matrix A_2, situated in the corresponding positions, and symbol $a_{21}^{(3)}$ denotes the corresponding entry of the matrix A_3 from the averaged system (2.30). In what follows we only need the explicit formula for the entry $a_{21}^{(2)}$ of the matrix A_2. We devote the conclusive part of this section to computation of this entry. Finally, we note that the remainder term in (3.35), due to (3.33) and (3.34), has the asymptotic estimate

$$R_3(t) = O\big(t^{-2}\big) + O\big(t^{-(\rho+1)}\big) + O\big(t^{-(k+1)\rho}\big), \quad t \to \infty \tag{3.36}$$

and, therefore, belongs to $L_1[t_0, \infty)$ taking into account (1.5).

We start with the case

$$\rho = 1.$$

System (3.35) due to (3.36) gets the form

$$\dot{u}_3 = \Big[W t^{-1} + O\big(t^{-2}\big)\Big] u_3, \tag{3.37}$$

where

$$W = B_0 + B_1 = \begin{pmatrix} -\frac{1}{2} & 1 \\ a_{21}^{(2)} & \frac{1}{2} + 2q_1^{(0)} \end{pmatrix}.$$

The eigenvalues of the matrix W are

$$\nu_{1,2} = q_1^{(0)} \pm \xi, \qquad \xi = \sqrt{\Big(q_1^{(0)} + \frac{1}{2}\Big)^2 + a_{21}^{(2)}}. \tag{3.38}$$

Here the square root is defined according to (3.16). The further asymptotic analysis of system (3.37) is conducted in the same way as for the case (3.22). Thus, we give here only the final result with account of the transformation (3.34).

- $\big(q_1^{(0)} + \frac{1}{2}\big)^2 + a_{21}^{(2)} \neq 0$

  We have the following asymptotic representation for solutions of Eq. (2.2) as $t \to \infty$:

$$y(t) = t^{\frac{1}{2} + q_1^{(0)}} \Big[c_1 \exp\{\xi \ln t\}\big(1 + o(1)\big) + c_2 \exp\{-\xi \ln t\}\big(1 + o(1)\big)\Big] + O\big(e^{-\beta t}\big),$$

where c_1, c_2 are arbitrary, in general, complex constants, $\beta > 0$ is a certain real number and the quantity ξ is defined by formula (3.38).

- $\big(q_1^{(0)} + \frac{1}{2}\big)^2 + a_{21}^{(2)} = 0$

  In this case the behaviour of solutions of Eq. (2.2) as $t \to \infty$ is described by the asymptotic formula

$$y(t) = c_1 t^{\frac{1}{2} + q_1^{(0)}}\big(1 + o(1)\big) + c_2 t^{\frac{1}{2} + q_1^{(0)}} \ln t \big(1 + o(1)\big) + O\big(e^{-\beta t}\big),$$

where c_1, c_2 are arbitrary real constants and $\beta > 0$ is a certain real number.

Let

$$\rho < 1.$$

System (3.35) takes form (2.35), where

$$\alpha = \rho, \quad A_0 = B_1, \quad V(t) = B_2 t^{-\rho} + \cdots + B_k t^{(-k+1)\rho} + B_0 t^{\rho-1}, \quad R_2(t) = R_3(t). \tag{3.39}$$

The eigenvalues of the matrix $A_0 = B_1$ are

$$\nu_{1,2} = q_1^{(0)} \pm \kappa, \qquad \kappa = \sqrt{\left(q_1^{(0)}\right)^2 + a_{21}^{(2)}}, \tag{3.40}$$

where the square root means (3.16). Further in this paper we study only the case

$$\left(q_1^{(0)}\right)^2 + a_{21}^{(2)} \neq 0, \tag{3.41}$$

when these eigenvalues are distinct. Provided condition (3.41) holds, system (3.35), due to Lemma 2.8, can be reduced to L-diagonal form (2.36), where the entries of the diagonal matrix $\Lambda(t) = \operatorname{diag}\big(\lambda_1(t), \lambda_2(t)\big)$ are the eigenvalues of the matrix $\big(A_0 + V(t)\big)t^{-\rho}$. By (3.39), these eigenvalues have the following form:

$$\lambda_{1,2}(t) = q_1^{(0)} t^{-\rho} \pm \kappa t^{-\rho}\Big(1 + O\big(t^{-\rho}\big) + O\big(t^{2\rho-2}\big) + O\big(t^{-1}\big)\Big) + \frac{a_{11}^{(2)} + a_{22}^{(2)}}{2} t^{-2\rho} + O\big(t^{-3\rho}\big).$$

Here all the terms denoted by the order symbol $O(\cdot)$ are real valued. The asymptotics for the fundamental matrix of system (3.35) can be constructed according to Theorem 2.9. If we then return to Eq. (2.2), we get the following asymptotic formulae for its solutions as $t \to \infty$.

If

$$\frac{1}{2} < \rho < 1,$$

we have

$$\begin{aligned} y(t) = t^{\frac{\rho}{2}} \exp\left\{\frac{q_1^{(0)}}{1-\rho} t^{1-\rho}\right\} \bigg[& c_1 \exp\left\{\frac{\kappa}{1-\rho} t^{1-\rho}\right\} (1 + o(1)) \\ & + c_2 \exp\left\{-\frac{\kappa}{1-\rho} t^{1-\rho}\right\} (1 + o(1))\bigg] + O\big(e^{-\beta t}\big). \end{aligned} \tag{3.42}$$

If

$$\rho = \frac{1}{2},$$

then

$$\begin{aligned} y(t) = t^{\frac{1}{4} + \frac{a_{11}^{(2)} + a_{22}^{(2)}}{2}} \exp\left\{2 q_1^{(0)} \sqrt{t}\right\} \Big[& c_1 \exp\left\{2\kappa\left(\sqrt{t} + O(\ln t)\right)\right\} (1 + o(1)) \\ & + c_2 \exp\left\{-2\kappa\left(\sqrt{t} + O(\ln t)\right)\right\} (1 + o(1))\Big] + O\big(e^{-\beta t}\big). \end{aligned} \tag{3.43}$$

Finally, if

$$\rho < \frac{1}{2},$$

we obtain

$$
\begin{aligned}
y(t) = t^{\frac{\rho}{2}} \exp\left\{ \frac{q_1^{(0)}}{1-\rho} t^{1-\rho} + \frac{a_{11}^{(2)} + a_{22}^{(2)}}{2(1-2\rho)} t^{1-2\rho} + O\left(\int t^{-3\rho} dt \right) \right\} \\
\times \left[c_1 \exp\left\{ \frac{\kappa}{1-\rho} t^{1-\rho} \left(1 + O(t^{-\rho})\right) \right\} (1 + o(1)) \right. \\
\left. + c_2 \exp\left\{ -\frac{\kappa}{1-\rho} t^{1-\rho} \left(1 + O(t^{-\rho})\right) \right\} (1 + o(1)) \right] + O(e^{-\beta t}).
\end{aligned}
\tag{3.44}
$$

Everywhere in these asymptotic formulae c_1, c_2 are arbitrary, in general, complex constants, $\beta > 0$ is a certain real number and the quantity κ is defined in (3.40).

Computation of the quantity $a_{21}^{(2)}$ in case (3.29)

It follows from the asymptotic formulae (3.42)–(3.44) that the key role in the oscillation problem for Eq. (1.1) plays the quantity $a_{21}^{(2)}$. It defines, due to (3.16) and (3.40), whether the number κ is real or pure imaginary. We recall that the quantity $a_{21}^{(2)}$ is the corresponding entry of the matrix A_2. The latter is defined by formula (2.32) with account of (3.8). First, we calculate the matrix $Y_1(t)$ as the solution of the matrix differential equation (2.33) with zero mean value. Recalling the form of the matrix D (see (2.18)) and also formulae (3.7), (3.31), we conclude that the entries $y_{ij}(t)$ of the matrix $Y_1(t)$ satisfy the following linear differential system with constant coefficients:

$$
\begin{aligned}
\dot{y}_{11} &= y_{21} - \frac{2}{3} p_1(t), & \dot{y}_{12} &= y_{22} - y_{11} + \frac{2}{3}\left(q_1^{(0)}(t) + p_1(t)\right), \\
\dot{y}_{21} &= -2p_1(t), & \dot{y}_{22} &= -y_{21} + 2\left(q_1^{(0)}(t) + p_1(t)\right).
\end{aligned}
$$

Here

$$
q_1^{(0)}(t) = q_1(t) - q_1^{(0)}, \tag{3.45}
$$

function $q_1(t)$ is defined in (1.4) (see also (3.3)), and the real number $q_1^{(0)}$ is its mean value according to (3.5). After some easy calculations we obtain

$$
\begin{aligned}
y_{11}(t) &= -2 \iint p_1(t)(dt)^2 - \frac{2}{3} \int p_1(t) dt, \\
y_{12}(t) &= 4 \iiint p_1(t)(dt)^3 + \iint \left(2q_1^{(0)}(t) + \frac{8}{3} p_1(t) \right) (dt)^2 + \frac{2}{3} \int \left(q_1^{(0)}(t) + p_1(t) \right) dt, \\
y_{21}(t) &= -2 \int p_1(t) dt, \\
y_{22}(t) &= 2 \iint p_1(t)(dt)^2 + 2 \int \left(q_1^{(0)}(t) + p_1(t) \right) dt.
\end{aligned}
\tag{3.46}
$$

Symbol $\int$ denotes the antiderivative having zero mean value. Further we will use the following relations that can be proved simply by integration by parts. If $f(t)$ is a trigonometric polynomial (or T-periodic function as well) with zero mean value then the following equalities hold:

$$
\begin{aligned}
&\mathrm{M}\left[f(t) \iint f(t)(dt)^2 \right] = -\mathrm{M}\left[\left(\int f(t) dt \right)^2 \right], \quad \mathrm{M}\left[f(t) \int f(t) dt \right] = 0, \\
&\mathrm{M}\left[f(t) \iiint f(t)(dt)^3 \right] = -\mathrm{M}\left[\int f(t) dt \iint f(t)(dt)^2 \right] = 0.
\end{aligned}
\tag{3.47}
$$

We recall now (3.7), (3.31) and take into account (3.46), (3.47) to conclude that

$$\begin{aligned} &\mathrm{M}\big[A_1(t)Y_1(t)\big] \\ &\quad = \mathrm{M}\big[(A_1(t)-A_1)Y_1(t)\big] \\ &\quad = \frac{4}{9}\begin{pmatrix} -3\mathrm{M}\Big[\big(\int p_1(t)dt\big)^2\Big]+3\mathrm{M}\Big[p_1(t)\int q_1^{(0)}(t)dt\Big] & \mathrm{M}\Big[\big(\int p_1(t)dt\big)^2\Big]-\mathrm{M}\Big[p_1(t)\int q_1^{(0)}(t)dt\Big] \\ -9\mathrm{M}\Big[\big(\int p_1(t)dt\big)^2\Big]+9\mathrm{M}\Big[p_1(t)\int q_1^{(0)}(t)dt\Big] & 3\mathrm{M}\Big[\big(\int p_1(t)dt\big)^2\Big]-3\mathrm{M}\Big[p_1(t)\int q_1^{(0)}(t)dt\Big] \end{pmatrix}. \end{aligned} \tag{3.48}$$

To calculate the entries of the matrix A_2 we also need to find row-vector (3.9). This is done as follows. We substitute (2.23) in (2.22) and collect terms corresponding to factor $t^{-\rho}$. With account of (2.5) and (3.6) we obtain the following problem:

$$\begin{aligned} &\Phi(\theta)\Psi(0)\big[q_1(t)\Phi'(-1)-p_1(t)\Phi(-1)\big]+H_1(t,\theta)D+\frac{\partial H_1}{\partial t} \\ &\qquad = \begin{cases} \dfrac{\partial H_1}{\partial \theta}, & -h \le \theta < 0, \\ H_1(t,0)-H_1(t,-1)+q_1(t)\Phi'(-1)-p_1(t)\Phi(-1), & \theta = 0. \end{cases} \end{aligned}$$

We apply (2.18) to get the following partial differential system for finding the entries of row-vector (3.9):

$$\begin{aligned} \frac{\partial h_{11}}{\partial \theta} &= \frac{\partial h_{11}}{\partial t} - \left(\frac{2}{3}+2\theta\right) p_1(t), \\ \frac{\partial h_{12}}{\partial \theta} &= \frac{\partial h_{12}}{\partial t} + h_{11} + \left(\frac{2}{3}+2\theta\right) \big(q_1(t)+p_1(t)\big), \end{aligned} \tag{3.49}$$

where $-h \le \theta < 0$. At the point $\theta = 0$ the solution of this system should satisfy the condition

$$\begin{aligned} \frac{\partial h_{11}}{\partial t}(t,0) &= h_{11}(t,0) - h_{11}(t,-1) - \frac{p_1(t)}{3}, \\ \frac{\partial h_{12}}{\partial t}(t,0) &= h_{12}(t,0) - h_{11}(t,0) - h_{12}(t,-1) + \frac{1}{3}\big(q_1(t)+p_1(t)\big). \end{aligned} \tag{3.50}$$

Due to (3.3), we seek the solution of (3.49), (3.50) in the form

$$h_{11}(t,\theta) = \sum_{j=-N}^{N} g_1^{(j)}(\theta)e^{\mathrm{i}\omega_j t}, \qquad h_{12}(t,\theta) = \sum_{j=-N}^{N} g_2^{(j)}(\theta)e^{\mathrm{i}\omega_j t}, \tag{3.51}$$

where the infinitely differentiable functions $g_1^{(j)}(\theta)$ and $g_2^{(j)}(\theta)$ belong to subspace Q_Λ. Hence, by (2.17), these functions should satisfy the following additional condition:

$$\big(\Psi(\xi), g_i^{(j)}(\theta)\big) = 0, \quad i = 1,2, \quad j = -N,\dots,N. \tag{3.52}$$

Here the bilinear form $(\cdot,\cdot)$ is defined according to (2.13) and the column-vector $\Psi(\xi)$ has form (2.18).

It follows from (2.32) and (3.8) that to compute the quantity $a_{21}^{(2)}$ we need to find only the function $h_{11}(t,\theta)$. We substitute (3.3), (3.51) in (3.49), (3.50) and match the coefficients of the corresponding exponentials $e^{\mathrm{i}\omega_j t}$. Thus, we get the following boundary value problems for functions $g_1^{(j)}(\theta)$:

$$\begin{aligned} &\frac{dg_1^{(j)}}{d\theta} = \mathrm{i}\omega_j g_1^{(j)}(\theta) - \Big(\frac{2}{3}+2\theta\Big)p_1^{(j)}, \\ &(1-\mathrm{i}\omega_j)g_1^{(j)}(0) - g_1^{(j)}(-1) = \frac{p_1^{(j)}}{3}, \quad j = -N,\dots,N. \end{aligned} \tag{3.53}$$

It is easy to verify that

$$g_1^{(j)}(\theta) = \left(\frac{e^{\mathrm{i}\omega_j\theta}}{1 - \mathrm{i}\omega_j - e^{-\mathrm{i}\omega_j}} - \frac{2\mathrm{i}\theta}{\omega_j} - \frac{6 + 2\mathrm{i}\omega_j}{3\omega_j^2} \right) p_1^{(j)}, \quad j \neq 0. \tag{3.54}$$

If $j = 0$ then, by (3.4) and (3.29), we have $\omega_0 = 0$ and $p_1^{(0)} = 0$. This yields that the corresponding solution of (3.53) has the form $g_1^{(0)}(\theta) \equiv c$, where c is a certain constant. The quantity c is uniquely defined from equality (3.52). Finally, we deduce that

$$g_1^{(0)}(\theta) \equiv 0. \tag{3.55}$$

Therefore, taking into account (2.32), (3.8) and also expression (3.48), we get the following representation for the quantity $a_{21}^{(2)}$:

$$a_{21}^{(2)} = -4\,\mathrm{M}\left[\left(\int p_1(t)dt \right)^2 \right] + 4\,\mathrm{M}\left[p_1(t) \int q_1^{(0)}(t)dt \right] - 2\,\mathrm{M}\left[p_2(t) \right]$$
$$+ 2\,\mathrm{M}\left[q_1(t) \frac{\partial h_{11}}{\partial \theta}(t,\theta) \bigg|_{\theta=-1} \right] - 2\,\mathrm{M}\left[p_1(t) h_{11}(t,-1) \right]. \tag{3.56}$$

Here the function $q_1^{(0)}(t)$ is defined according to (3.45) and the function $h_{11}(t,\theta)$ has form (3.51) with account of (3.54) and (3.55). If we calculate in (3.56) all the mean values and use (3.4) we obtain the more compact form for $a_{21}^{(2)}$. Namely, we conclude that

$$a_{21}^{(2)} = 2 \sum_{\substack{j=-N \\ j \neq 0}}^{N} \frac{(\mathrm{i}\omega_j p_1^{(j)} \bar{q}_1^{(j)} - |p_1^{(j)}|^2) e^{-\mathrm{i}\omega_j}}{1 - \mathrm{i}\omega_j - e^{-\mathrm{i}\omega_j}} - 2\,\mathrm{M}\big[p_2(t)\big]. \tag{3.57}$$

Function $p_2(t)$ in this expression is defined by formula (3.2).

4 Conclusions and examples

We begin this section by analyzing the asymptotic formulae obtained in the previous section as applied to the oscillation problem of Eq. (1.1) with conditions (1.3), (1.4). The results of the analysis are given in Tables 4.1 and 4.2.

	$\rho > 2$	$\rho = 2$	$\rho < 2$
o	–	$p_1^{(0)} > \frac{1}{8}$	$p_1^{(0)} > 0$
p	+	$p_1^{(0)} \leq \frac{1}{8}$	$p_1^{(0)} < 0$

Table 4.1: Case $p_1^{(0)} \neq 0$.

In these tables the line titled «o» contains the conditions for existence of oscillatory solutions and the line titled «p» contains the conditions for existence of nonoscillatory (positive) solutions. Symbol «–» means the situation when the oscillatory solutions are not found by

	$\rho > 1$	$\rho = 1$	$\rho < 1$
o	$-$	$\left(q_1^{(0)} + \frac{1}{2}\right)^2 + a_{21}^{(2)} < 0$	$\left(q_1^{(0)}\right)^2 + a_{21}^{(2)} < 0$
p	$+$	$\left(q_1^{(0)} + \frac{1}{2}\right)^2 + a_{21}^{(2)} \geq 0$	$\left(q_1^{(0)}\right)^2 + a_{21}^{(2)} > 0$

Table 4.2: Case $p_1^{(0)} = 0$.

means of the main parts of the asymptotic formulae in the prescribed interval of the parameter ρ (highly likely oscillatory solutions don't exist at all). Symbol «+» stands for the situation when there exist nonoscillatory (positive) solutions for all values of the parameter ρ in the prescribed interval. In all the other positions of these tables the conditions for existence of oscillatory and nonoscillatory (positive) solutions of Eq. (1.1) in the prescribed intervals of the parameter ρ are collected. We also remind that the real numbers $p_1^{(0)}, q_1^{(0)}$ are defined in (3.5) and the real number $a_{21}^{(2)}$ is described by formula (3.57) with account of (3.2) and (3.3).

We now demonstrate the obtained results by a number of illustrating examples.

Example 4.1. In paper [13] the authors illustrate the obtained criteria for existence of the positive solutions by the following equation:

$$\frac{d\hat{x}}{ds} = -\hat{a}(s)\hat{x}\left(s - c - \frac{d}{s}\right), \tag{4.1}$$

where $c, d > 0$. It is claimed that if

$$\hat{a}(s) \leq \frac{1}{ec} - \frac{d}{ec^2} \cdot \frac{1}{s} + \frac{1}{e} \cdot \left(\frac{d^2}{c^3} + \frac{c}{8}\right) \cdot \frac{1}{s^2} + o\left(\frac{1}{s^2}\right) \tag{4.2}$$

or

$$\hat{a}(s) \leq \frac{1}{ec} - \frac{d}{ec^2} \cdot \frac{1}{s} + \frac{1}{e} \cdot \left(\frac{d^2}{c^3} + \frac{d}{2c}\right) \cdot \frac{1}{s^2} + o\left(\frac{1}{s^2}\right) \tag{4.3}$$

as $s \to \infty$ then Eq. (4.1) has positive solution.

We consider the case when function $\hat{a}(s)$ in Eq. (4.1) has the following asymptotic representation as $s \to \infty$:

$$\hat{a}(s) = \frac{1}{ec} + \hat{a}_1(s)s^{-1} + \hat{a}_2(s)s^{-2} + O(s^{-3}), \tag{4.4}$$

where $\hat{a}_1(s), \hat{a}_2(s)$ are real-valued trigonometric polynomials. In particular,

$$\hat{a}_1(s) = \sum_{j=-N}^{N} \hat{a}_1^{(j)} e^{i\omega_j s}, \tag{4.5}$$

and, besides,

$$\hat{a}_1^{(-j)} = \overline{\hat{a}}_1^{(j)}, \quad \omega_{-j} = -\omega_j \ (\omega_l \neq \omega_l,\ l \neq m), \quad j = 1, \ldots, N.$$

In Eq. (4.1) we make the change of independent variable $s = tc$ that transforms it to form (1.1), where

$$x(t) = \hat{x}(ct), \quad a(t) = c\hat{a}(tc), \quad \tau(t) = 1 + \frac{d}{c^2} \cdot \frac{1}{t}. \tag{4.6}$$

Due to (4.4) and (4.6), we conclude that in the considered case the coefficients in the expansions (1.3), (1.4) have the following form:

$$a_1(t) = \hat{a}_1(tc), \quad a_2(t) = \frac{\hat{a}_2(tc)}{c}, \quad q_1(t) \equiv \frac{d}{c^2}, \quad q_m(t) \equiv 0, \quad m \geq 2 \tag{4.7}$$

and $\rho = 1$. It follows from (3.5) with account of (3.2) that

$$p_1^{(0)} = ea_1^{(0)} + q_1^{(0)}, \quad q_1^{(0)} = \frac{d}{c^2},$$

where

$$a_1^{(0)} = \mathrm{M}[a_1(t)] = \hat{a}_1^{(0)}, \quad \hat{a}_1^{(0)} = \mathrm{M}[\hat{a}_1(s)].$$

We deduce from Table 4.1 that equation (4.1), (4.4) has oscillatory solutions if

$$\hat{a}_1^{(0)} > -\frac{d}{ec^2}$$

and positive solutions if

$$\hat{a}_1^{(0)} < -\frac{d}{ec^2}.$$

We also need to study the case when $p_1^{(0)} = 0$, i.e.,

$$\hat{a}_1^{(0)} = -\frac{d}{ec^2}.$$

By (3.2) and (4.7), we have

$$\mathrm{M}[p_2(t)] = ea_2^{(0)} + \frac{d^2}{2c^4} + ea_1^{(0)} \cdot \frac{d}{c^2} = ea_2^{(0)} - \frac{d^2}{2c^4},$$

where

$$a_2^{(0)} = \mathrm{M}\left[a_2(t)\right] = \frac{\hat{a}_2^{(0)}}{c}, \quad \hat{a}_2^{(0)} = \mathrm{M}\left[\hat{a}_2(s)\right]. \tag{4.8}$$

We then compute quantity (3.57) using (3.2), (4.5), (4.7) and (4.8). We obtain

$$a_{21}^{(2)} = -2e^2 \sum_{\substack{j=-N \\ j\neq 0}}^{N} \frac{\left|\hat{a}_1^{(j)}\right|^2 e^{-\mathrm{i}c\omega_j}}{1 - \mathrm{i}c\omega_j - e^{-\mathrm{i}c\omega_j}} - \frac{2e\hat{a}_2^{(0)}}{c} + \frac{d^2}{c^4}.$$

It follows from Table 4.2 that equation (4.1), (4.4) has oscillatory solutions if

$$\hat{a}_2^{(0)} > \frac{1}{e}\left(\frac{d^2}{c^3} + \frac{d}{2c} + \frac{c}{8}\right) - ec \sum_{\substack{j=-N \\ j\neq 0}}^{N} \frac{\left|\hat{a}_1^{(j)}\right|^2 e^{-\mathrm{i}c\omega_j}}{1 - \mathrm{i}c\omega_j - e^{-\mathrm{i}c\omega_j}} \tag{4.9}$$

and positive solutions if

$$\hat{a}_2^{(0)} \leq \frac{1}{e}\left(\frac{d^2}{c^3} + \frac{d}{2c} + \frac{c}{8}\right) - ec \sum_{\substack{j=-N \\ j\neq 0}}^{N} \frac{\left|\hat{a}_1^{(j)}\right|^2 e^{-\mathrm{i}c\omega_j}}{1 - \mathrm{i}c\omega_j - e^{-\mathrm{i}c\omega_j}}. \tag{4.10}$$

We now consider the special case when the following identity holds in (4.4):

$$\hat{a}_1(s) \equiv \hat{a}_1^{(0)} = -\frac{d}{ec^2}. \tag{4.11}$$

In this situation formulae (4.9), (4.10) take the simple form. It is easily seen that in this case equation (4.1), (4.4) with the coefficient $\hat{a}_1(s)$ described by (4.11) has oscillatory solutions if

$$\hat{a}_2^{(0)} > \frac{1}{e}\left(\frac{d^2}{c^3} + \frac{d}{2c} + \frac{c}{8}\right)$$

and positive solutions if

$$\hat{a}_2^{(0)} \leq \frac{1}{e}\left(\frac{d^2}{c^3} + \frac{d}{2c} + \frac{c}{8}\right).$$

This fact allows us to propose the **hypothesis** that the condition for existence of positive solutions in Eq. (4.1) is described by the inequality

$$\hat{a}(s) \leq \frac{1}{ec} - \frac{d}{ec^2}\cdot\frac{1}{s} + \frac{1}{e}\cdot\left(\frac{d^2}{c^3} + \frac{d}{2c} + \frac{c}{8}\right)\cdot\frac{1}{s^2} + o\left(\frac{1}{s^2}\right), \quad s \to \infty, \tag{4.12}$$

instead of (4.2) and (4.3).

Example 4.2. This example concerns equation (1.1), where

$$a(t) = \frac{1}{e}\left(1 + \frac{K(\sin^2 \pi t - \gamma)}{t^\rho}\right), \quad 0 < \rho \leq 2, \quad \tau(t) \equiv 1, \tag{4.13}$$

and $K > 0$, $\gamma \in \mathbb{R}$. Equation (1.1), (4.13) was considered in [15, 17]. In [17], this equation was studied provided that $\gamma = 0$. In this case it was shown that all solutions of this equation oscillate if $K > 0$ and $0 \leq \rho < 2$, and also if $K > 1$ and $\rho = 2$. If $K < \frac{1}{8}$ and $\rho = 2$, then equation (1.1), (4.13) has nonoscillatory solution. In paper [15], equation (1.1), (4.13) was studied in the case $\rho = 2$. It was shown that if $\gamma < \frac{1}{2}$ and $K > \frac{1}{4(1-2\gamma)}$ then all solutions of this equation oscillate. In particular, the authors improved the results from [17] for the case $\gamma = 0$.

We write (4.13) in form (1.3), (1.4) and obtain

$$a_1(t) = \frac{K}{e}\left(\sin^2 \pi t - \gamma\right), \quad q_1(t) \equiv 0, \quad a_m(t) = q_m(t) \equiv 0, \quad m \geq 2.$$

We deduce from (3.2) and (3.5) that

$$p_1(t) = K\left(\sin^2 \pi t - \gamma\right) = K\left(\frac{1}{2} - \gamma\right) - \frac{K}{2}\cos 2\pi t, \quad p_2(t) \equiv 0 \tag{4.14}$$

and

$$p_1^{(0)} = \mathrm{M}\left[p_1(t)\right] = K\left(\frac{1}{2} - \gamma\right), \quad q_1^{(0)} = \mathrm{M}\left[q_1(t)\right] = 0. \tag{4.15}$$

It follows from Table 4.1 that if $\rho = 2$ then equation (1.1), (4.13) has oscillatory solutions provided that inequality $4K(1-2\gamma) > 1$ holds and positive solutions if $4K(1-2\gamma) \leq 1$. Parameter γ in these inequalities may take all the values except $\gamma = \frac{1}{2}$. If $0 < \rho < 2$ then the considered equation has oscillatory solutions if $\gamma < \frac{1}{2}$ and positive solutions if $\gamma > \frac{1}{2}$. Parameter K may take all the positive values.

The more difficult case occurs when

$$\gamma = \frac{1}{2}, \tag{4.16}$$

since we have $p_1^{(0)} = 0$. We calculate coefficients in (3.3) with account of (4.14), (4.16) and conclude that

$$N = 1, \quad p_1^{(1)} = p_1^{(-1)} = -\frac{K}{4}, \quad \omega_1 = 2\pi, \quad \omega_{-1} = -2\pi. \tag{4.17}$$

We compute quantity (3.57) using (4.17) to get

$$a_{21}^{(2)} = -\frac{K^2}{8}\left(\frac{e^{2\pi i}}{1 + i2\pi - e^{2\pi i}} + \frac{e^{-2\pi i}}{1 - i2\pi - e^{-2\pi i}}\right) = 0. \tag{4.18}$$

We then deduce from Table 4.2 that equation (1.1), (4.13) under condition (4.16) has nonoscillatory solutions for all values of the parameter $K > 0$ if $1 < \rho \le 2$. If $\rho = 1$ then, by (4.15), we also conclude that this equation has nonoscillatory solutions for all values of the parameter $K > 0$.

Unfortunately, the obtained results don't allow us to analyze the oscillation problem for equation (1.1), (4.13) under condition (4.16) for the case $\rho < 1$. In this situation condition (3.41), under which the asymptotic representations were constructed in this paper, fails. Nevertheless, certain advance in the analysis of the oscillation problem for this case can still be made. Note that in relation to the studied equation system (3.35) in the case $\rho < 1$ takes the following form:

$$\dot{u}_3 = \left[B_1 t^{-\rho} + B_0 t^{-1} + O\left(t^{-2\rho}\right)\right] u_3. \tag{4.19}$$

Here matrices B_0, B_1 with account of (4.15), (4.18) are described by the formulae

$$B_0 = \frac{\rho}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We make the change of variable

$$u_3 = \begin{pmatrix} 1 & 1 \\ 0 & \rho t^{\rho-1} \end{pmatrix} u_4$$

to reduce system (4.19) to form (2.35) for new unknown variable $u_4(t)$, where

$$\alpha = 1, \quad A_0 = \begin{pmatrix} -\frac{\rho}{2} & \rho - 1 \\ 0 & 1 - \frac{\rho}{2} \end{pmatrix}, \quad V(t) \equiv 0, \quad R_2(t) = O\left(t^{1-3\rho}\right).$$

We remark that the eigenvalues of the matrix A_0 are distinct and the remainder term $R_2(t)$ belongs to $L_1[t_0, \infty)$ if $\rho > \frac{2}{3}$. Thus, we can bring the obtained system to L-diagonal form (2.36) by certain transformation with constant coefficients and then apply Levinson's Theorem to get the asymptotics for its fundamental matrix. Some easy calculations show that in this situation we obtain the asymptotic representation (3.13) for solutions of Eq. (2.2) as $t \to \infty$. Hence, equation (1.1), (4.13) under condition (4.16) has nonoscillatory solutions for all the values of the parameter $K > 0$ if $\frac{2}{3} < \rho < 1$. Evidently, to study the case $\rho \le \frac{2}{3}$ under condition (4.16) we need to compute the entries of the matrix B_2 in system (3.35).

Example 4.3. Our last example deals with equation (1.1), where

$$a(t) = \frac{1}{e} + \frac{a \sin \omega t}{t^\rho}, \quad \tau(t) = 1 + \frac{b \sin \omega t}{t^\rho}, \quad \rho > 0 \tag{4.20}$$

and $a, b \in \mathbb{R}$, $\omega > 0$. Therefore,

$$a_1(t) = a \sin \omega t, \quad q_1(t) = b \sin \omega t, \quad a_m(t) = q_m(t) \equiv 0, \quad m \ge 2.$$

It follows from (3.2) and (3.5) that

$$p_1(t) = (ea+b)\sin \omega t, \quad p_2(t) = \left(\frac{b^2}{2} + eab\right)\sin^2 \omega t \tag{4.21}$$

and, moreover,

$$p_1^{(0)} = \mathrm{M}\left[p_1(t)\right] = 0, \quad q_1^{(0)} = \mathrm{M}\left[q_1(t)\right] = 0, \quad p_2^{(0)} = \mathrm{M}\left[p_2(t)\right] = \frac{b^2}{4} + \frac{eab}{2}. \tag{4.22}$$

By calculating coefficients in (3.3) with account of (4.21), we get

$$N = 1, \quad p_1^{(1)} = -p_1^{(-1)} = \frac{ea+b}{2\mathrm{i}}, \quad q_1^{(1)} = -q_1^{(-1)} = \frac{b}{2\mathrm{i}}, \quad \omega_1 = -\omega_{-1} = \omega.$$

We then compute quantity (3.57) and conclude that

$$\begin{aligned} a_{21}^{(2)} = {} & 2\left(\frac{\mathrm{i}\omega e^{-\mathrm{i}\omega}}{1-\mathrm{i}\omega - e^{-\mathrm{i}\omega}} - \frac{\mathrm{i}\omega e^{\mathrm{i}\omega}}{1+\mathrm{i}\omega - e^{\mathrm{i}\omega}}\right)\frac{(ea+b)b}{4} \\ & - 2\left(\frac{e^{-\mathrm{i}\omega}}{1-\mathrm{i}\omega - e^{-\mathrm{i}\omega}} + \frac{e^{\mathrm{i}\omega}}{1+\mathrm{i}\omega - e^{\mathrm{i}\omega}}\right)\frac{(ea+b)^2}{4} - \left(\frac{b^2}{2} + eab\right). \end{aligned}$$

We can write this expression in the real form. In particular, we used the mathematical package Wolfram Mathematica to obtain the following real-valued expression:

$$a_{21}^{(2)} = -\frac{(2eab+b^2)\omega^2 - 2e^2a^2 + 2\left(e^2a^2 + (eab+b^2)\omega^2\right)\cos\omega - 2\omega\left(b^2 - e^2a^2 + eab\right)\sin\omega}{2\left(\omega^2 - 2\omega\sin\omega - 2\cos\omega + 2\right)}. \tag{4.23}$$

If we consider the quantity $a_{21}^{(2)}$ as the function of ω we can write the following limit relations (again we used *Wolfram Mathematica*):

$$a_{21}^{(2)} = -\frac{2(ea+b)^2}{\omega^2} + \frac{1}{18}\left(7e^2a^2 + 20eab + 22b^2\right) + O\left(\omega^2\right), \quad \omega \to 0, \tag{4.24}$$

$$a_{21}^{(2)} = -\frac{1}{2}\left(2eab + b^2\right) - \left(eab + b^2\right)\cos\omega + O\left(\omega^{-1}\right), \quad \omega \to \infty. \tag{4.25}$$

In particular, we conclude from (4.25) that $a_{21}^{(2)}$ as the function of ω is asymptotically 2π-periodic as $\omega \to \infty$. In Fig. 4.1 we give the graph of quantity $a_{21}^{(2)}$ as the function $f(\omega) = a_{21}^{(2)}(\omega)$ for the values of parameters $a = b = 1$.

To obtain the conditions for existence of oscillatory or nonoscillatory solutions of (1.1), (4.20) we can use Table 4.2 with account of (4.22) and (4.23). In particular, if $\rho \leq 1$ then it follows from (4.24) that for all sufficiently small ω equation (1.1), (4.20) has oscillatory solutions for all the values of parameters $a, b \in \mathbb{R}$ not simultaneously equal to zero.

It is highly likely that the obtained results are still valid in the case when $a_j(t)$, $q_j(t)$, $j = 1, \ldots, k+1$ in (1.3), (1.4) are sufficiently smooth ω-periodic functions. In this situation the periodic coefficients are described in terms of the infinite Fourier series having form (3.3) with $N = +\infty$. Of course, the problem of convergence of the corresponding series (2.24) and its partial derivatives arises in these case. This question is not discussed here.

In conclusion we note that the oscillation problem in critical case can be also studied for the difference analog of equation (1.1):

$$\Delta y(n) = -g(n)y(n-k), \qquad k \in \mathbb{N},$$

where $g(n) > 0$ for all $n \in \mathbb{N}$. The corresponding results are discussed in paper [28].

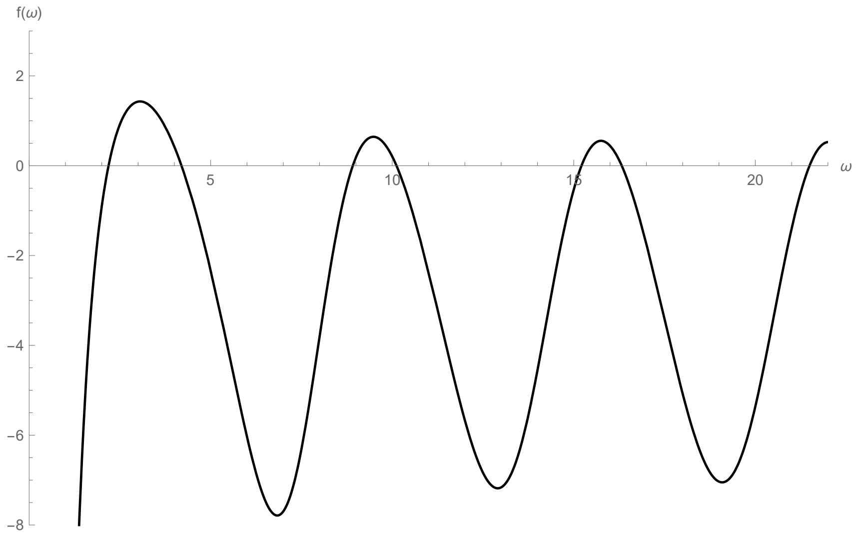

Figure 4.1: The graph of the quantity $a_{21}^{(2)}$, defined by (4.23), as the function $f(\omega) = a_{21}^{(2)}(\omega)$ for the values of parameters $a = b = 1$.

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Lyapunov functionals and practical stability for stochastic differential delay equations with general decay rate

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Abstract. This paper stands for the almost sure practical stability of nonlinear stochastic differential delay equations (SDDEs) with a general decay rate. We establish some sufficient conditions based upon the construction of appropriate Lyapunov functionals. Furthermore, we provide some numerical examples to validate the effectiveness of the abstract results of this paper.

Keywords: Keywords: stochastic delay systems, almost sure practical stability, decay function, Lyapunov functional, Itô formula.

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1 Introduction

Several applied problems are modeled by non-delay systems. Non-delay systems are governed by the assumption that the future evolution of the system is determined by the present state. Moreover, it is independent of the past states. In reality, such an assumption is the only a first approximation to the real system. A more realistic model assumes that the evolution of the future states depends not only on the current state but also on their past history. Delay differential equations (DDEs) (also called hereditary systems, systems with aftereffect, functional differential equations, retarded differential equations, differential difference equations) provide an appropriate model for physical processes whose time evolution depends on their history.

The stochastic delay differential equations (SDDEs) have been extensively used in many branches of physics, biology, as well as in dynamical structures in engineering, mechanics, automatic regulation, economy finance, ecology, sociology, medicine, etc. The stability of SDDEs has become a very prevalent theme of recent research in Mathematics and its applications. An

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important direction in the study of equations with delays is the analysis of stability. The corresponding study of the stability properties of solutions has received much attention during the last decades. The reader is referred to [15,16,20–22,24], for more details.

As it is well known, in the case without any hereditary features, Lyapunov's technique is available to obtain sufficient conditions for the stability of the solutions of stochastic differential equations. These sufficient conditions are obtained using the construction of some Lyapunov functions of functionals, being the latter a method which provides better conditions than using Lyapunov functions. Moreover, the construction of Lyapunov functionals is more complicated as Krasovskii [19] pointed out.

In this us, the construction of different Lyapunov functionals for one SDDEs allows to establish several stability conditions for the solution of this equation. There exist numerous works that tackle the construction of Lyapunov functionals for a wide range of equations containing some hereditary properties, see [10,17,23].

Several fundamental variants to Lyapunov's original concepts of practical stability were introduced in [1–6,9,11,12]. When the origin is not necessarily an equilibrium point, we can study the asymptotic stability of solutions of the SDDEs in a small neighborhood of the origin. In the investigation of the asymptotic behavior of solutions to stochastic differential systems, one can find that a solution is asymptotically stable but may not necessarily exponentially stable. Further, in the nonlinear and/or nonautonomous situations, it may happen that the stability cannot always be exponential but can be sub or super-exponential, see [7,8]. For this reason, the main aim of this paper is to discuss the almost sure practical stability with a general decay rate of stochastic delay differential equations.

The general method of Lyapunov functionals construction, which was proposed by V. Kolmanovskii and L. Shaikhet [17,18,23], is used here for stochastic differential equations with delay. This approach has already been successfully used for functional differential equations, for difference equations with discrete time, for difference equations with continuous time. Our interest in this paper is to investigate the practical stability with a general decay rate of stochastic differential equations with constant and time-varying delay by using the general method of Lyapunov functionals construction.

In [11], Caraballo et al. investigated the practical convergence to zero with a general decay rate of stochastic delay evolution equation by using Lyapunov functions. To the best of our knowledge, no work has been published about the practical stability of SDDEs in the literature by using Lyapunov functionals, which is our research topic in our paper. The novelty of our work is to investigate the practical convergence to a small ball centered at the origin with a general decay rate in terms of the existence and construction of Lyapunov functionals. Furthermore, we construct Lyapunov functionals for stochastic differential equations with constant and time-varying delay to obtain sufficient conditions ensuring the practical convergence to a small ball centered at the origin with a general decay rate. The contents of this paper are as follows: in Section 2, we introduce the necessary notations and preliminaries. In Section 3, we establish several sufficient criteria for almost sure practical stability of the stochastic delay systems with a general decay rate utilizing Lyapunov's functional. In Section 4, we aim to analyze the almost sure practical stability with a general decay rate of stochastic differential equations with constant and time-varying delay by constructing suitable Lyapunov functionals. Moreover, we exhibit some examples to illustrate the theoretical findings. Eventually, some conclusions are included in the last section.

Notations

Throughout this paper, unless otherwise specified, we use the following notations.

Let $\{\Omega, \mathcal{F}, \mathrm{P}\}$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $B(t) = (B_1(t), \ldots, B_m(t))$ be an m-dimensional Brownian motion defined on the probability space. Let $\mathbb{R}_+ = [0, +\infty)$ and $\tau > 0$. We denote by $C([-\tau, 0], \mathbb{R}^n)$ the family of all continuous functions from $[-\tau, 0]$ to $\mathbb{R}^n$ with the norm $\|\varphi\| = \sup_{-\tau\leq\theta\leq 0} |\varphi(\theta)|$. Let $p > 0$, and denote by $L^p_{\mathcal{F}_t}([-\tau, 0], \mathbb{R}^n)$ the family of all $\mathcal{F}_t$-measurable $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$-valued random variables ξ, such that $\mathbb{E}(\|\xi\|^p) < \infty$. If $x(t)$ is a continuous $\mathbb{R}^n$-valued stochastic process on $t \in [-\tau, +\infty)$, for each $t \geq 0$ we define x_t by $x_t(\theta) = x(t + \theta) : -\tau \leq \theta \leq 0$ for $t \geq 0$, which is a $C([-\tau, 0], \mathbb{R}^n)$valued process.

Let us consider the following n-dimensional stochastic differential delay equation (SDDE):

$$dx(t) = F(t, x_t)dt + G(t, x_t)dB(t), \quad t \geq 0, \tag{1.1}$$

where $F : [0, +\infty) \times C([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}^n, \ G : [0, +\infty) \times C([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}^{n\times m}$.

We assume that there exits $t \in \mathbb{R}_+$, such that $F(t, 0) \neq 0$ or $G(t, 0) \neq 0$, i.e., the stochastic differential delay equation (1.1) does not have the trivial solution $x \equiv 0$.

In order to solve equation (1.1), we require to know the initial data, then we assume that they are given by

$$x_0 = \xi, \quad \text{i.e.,} \quad x_0(\theta) = \xi(\theta) = x(\theta), \quad \forall\theta \in [-\tau, 0], \tag{1.2}$$

where ξ is a $C([-\tau, 0], \mathbb{R}^n)$ – valued random variable such that $\mathbb{E}(\|\xi\|^2) < \infty$.

For the well-posedness of system (1.1), we impose the following assumptions.

Assumptions:

1. A local Lipschitz condition:

   For every real number $T > 0$ and integer $i \geq 1$, there exists a positive constant $K_{T,i}$, such that for all $t \in [0, T]$ and all $\varphi, \ \bar{\varphi} \in C([-\tau, 0], \mathbb{R}^n)$ with $\|\varphi\| \vee \|\bar{\varphi}\| \leq i$,

   $$\|F(t, \varphi) - F(t, \bar{\varphi})\|^2 \vee \|G(t, \varphi) - G(t, \bar{\varphi})\|^2 \leq K_{T,i} \ (\|\varphi - \bar{\varphi}\|^2).$$

2. A linear growth condition:

   For every real number $T > 0$, there exists a positive constant K_T, such that for all $t \in [0, T]$ and all $\varphi \in C([-\tau, 0], \mathbb{R}^n)$,

   $$\|F(t, \varphi)\|^2 \vee \|G(t, \varphi)\|^2 \leq K_T \ (1 + \|\varphi\|^2).$$

Then, under assumptions (1) and (2), the stochastic differential delay equation (1.1) with the given initial data (1.2) has a unique global solution $x(\cdot) = x(\cdot, 0, \xi) \in \mathcal{M}^2([-\tau, +\infty), \mathbb{R}^n)$, (see Mao [21], for more details). Moreover, $x(\cdot)$ satisfies the following integral equation:

$$\begin{cases} x(t) = \xi(0) + \int_0^t F(s, x_s)ds + \int_0^t G(s, x_s)dB(s), \quad \text{a.s.,} \quad \text{and} \\ x(t) = \xi(t), \quad t \in [-\tau, 0]. \end{cases}$$

To calculate the stochastic differential of the process $\eta(t) = v(t, x(t))$, where $x(t)$ is a solution of the SDDE (1.1), and the function $v : [0, +\infty) \times \mathbb{R}^n \to \mathbb{R}_+$ has continuous partial derivatives

$$v_t(t,x) = \frac{\partial v}{\partial t}(t,x); \quad v_x(t,x) = \left(\frac{\partial v}{\partial x_1}(t,x), \ldots, \frac{\partial v}{\partial x_n}(t,x)\right); \quad v_{xx}(t,x) = \left(\frac{\partial^2 v}{\partial x_i \partial x_j}(t,x)\right)_{n\times n}.$$

The following Itô's formula [14] is used:

$$d\eta(t) = \mathcal{L}v(t,x(t))dt + v_x(t,x(t))G(t,x_t)dB(t).$$

The operator $\mathcal{L}v$ is called the generator of (1.1) and is defined in the following way:

$$\mathcal{L}v(t,x(t)) = v_t(t,x(t)) + v_x(t,x(t))F(t,x_t) + \frac{1}{2}\,\mathrm{trace}\left(G^T(t,x_t)v_{xx}(t,x(t))G(t,x_t)\right).$$

The generator $\mathcal{L}$ can be applied also for some functionals $V(t,\varphi)\colon [0,+\infty)\times C([-\tau,0],\mathbb{R}^n)\to \mathbb{R}_+$. Suppose that a functional $V(t,\varphi)$ can be represented in the form $V(t,\varphi(0),\varphi(\theta))$, $\theta < 0$, and for $\varphi = x_t$, put

$$\begin{aligned} V_\varphi(t,x) = V(t,\varphi) = V(t,x_t) = V(t,x,x(t+\theta)), \quad \theta < 0,\\ x = \varphi(0) = x(t). \end{aligned} \tag{1.3}$$

Denote by D the set of the functionals for which the function $V_\varphi(t,x)$ defined by (1) has a continuous derivative with respect to t and two continuous derivatives with respect to x (see [23]). For functionals from D, the generator $\mathcal{L}$ of (1.1) has the following form:

$$\mathcal{L}V(t,x_t) = V_{\varphi t}(t,x(t)) + V_{\varphi x}(t,x(t))F(t,x_t) + \frac{1}{2}\,\mathrm{trace}\left(G^T(t,x_t)V_{\varphi xx}(t,x(t))G(t,x_t)\right).$$

From the Itô formula it follows that for a functional V from D,

$$dV(t,x_t) = \mathcal{L}V(t,x_t)dt + V_{\varphi x}(t,x(t))G(t,x_t)dB(t).$$

The following lemma is known as the exponential martingale inequality, and will be useful in our analysis.

Lemma 1.1 (See [21]). *Let $g = (g_1, \ldots, g_m) \in L^2(\mathbb{R}_+, \mathbb{R}^m)$, and let τ, μ, η be any positive numbers. Then,*

$$\mathbb{P}\left(\sup_{0\le t\le \tau}\left[\int_0^t g(s)dB_s - \frac{\mu}{2}\int_0^t \|g(s)\|^2 ds\right] > \eta\right) \le \exp(-\mu\eta).$$

2 Practical stability of stochastic delay equations

First, we define the practical uniform exponential stability of a stochastic delay equation.

Definition 2.1.

i) The ball $B_r := \{x \in \mathbb{R}^n : \|x\| \le r\}$, $r > 0$ is said to be almost surely globally uniformly exponentially stable, if for any initial data $\xi \in C([-\tau,0],\mathbb{R}^n)$, such that $0 < \|x(t,0,\xi)\| - r$, for all $t \ge 0$,

$$\lim_{t\to+\infty} \sup \frac{1}{t}\ln(\|x(t,0,\xi)\| - r) < 0, \quad \text{a.s.}$$

ii) The system (1.1) is said to be almost surely practically uniformly exponentially stable, if there exists $r > 0$, such that B_r is almost surely uniformly exponentially stable.

Now, we state the definition of practical convergence to the ball B_r with a general decay function $\lambda(t)$.

Definition 2.2. Let $\lambda(t)$ be a positive function defined for sufficiently large $t > 0$, such that $\lambda(t) \to +\infty$ as $t \to +\infty$. A solution $x(t)$ to system (1.1) is said to decay to the ball B_r almost surely practically with decay function $\lambda(t)$ and order at least $\gamma > 0$, if its generalized Lyapunov exponent is less than or equal to $-\gamma$ with probability one, i.e.,

$$\lim_{t\to+\infty} \sup \frac{\ln(\|x(t,0,\xi)\| - r)}{\ln \lambda(t)} \le -\gamma, \quad \text{a.s.}$$

If in addition, 0 is a solution to system (1.1), the zero solution is said to be almost surely practically asymptotically stable with decay function $\lambda(t)$ and order at least γ, if every solution to system (1.1) decays to the ball B_r almost surely practically with decay function $\lambda(t)$ and order at least γ, for all $r > 0$ sufficiently small.

Remark 2.3. Clearly, replacing in the above definition, the decay function $\lambda(t)$ by $O(e^t)$ leads to the almost sure practical exponential stability.

Remark 2.4. Here we should mention that in [11] we establish sufficient conditions for practical decay to zero by using Lyapunov functions but now we will use Lyapunov functionals and decay to ball B_r.

Now, we aim to prove the practical stability of stochastic differential delay equations with general decay rate in terms of Lyapunov functionals.

Theorem 2.5. *Let $V : \mathbb{R}_+ \times C([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}_+$ be a functional from D. Assume that $\ln \lambda(t)$ is uniformly continuous on $t \ge 0$, and there exists a constant $\sigma \ge 0$, such that*

$$\lim_{t\to+\infty} \frac{\ln \ln t}{\ln \lambda(t)} \le \sigma.$$

Let $x(\cdot) = x(\cdot, 0, \xi)$ be a solution to system (1.1) and assume that there exist constants $q \in \mathbb{N}^\star$, $m \ge 0$, $\beta_1 \in \mathbb{R}$, $\beta_2 \ge 0$, a non-increasing function $\phi(t) > 0$ and a continuous non-negative function $\psi(t)$, such that, for all $t \ge 0$, the following assumptions hold:

$(\mathcal{H}_1)$
$$\lambda^m(t)\|x(t)\|^q \le V(t, x_t).$$

$(\mathcal{H}_2)$
$$\int_0^t \mathcal{L}V(s, x_s)ds + \int_0^t \phi(s)\|V_x(s, x_s)G(s, x_s)\|^2 ds \le \int_0^t \psi(s)\lambda^m(s)\|x(s)\|^q ds + \rho(t),$$

where $\rho(t)$ is a continuous non-negative function.

$(\mathcal{H}_3)$
$$\lim_{t\to+\infty} \sup \frac{\int_0^t \psi(s)ds}{\ln \lambda(t)} \le \beta_1,$$
$$\lim_{t\to+\infty} \inf \frac{\ln \phi(t)}{\ln \lambda(t)} \ge -\beta_2,$$
$$\lim_{t\to+\infty} \frac{\rho(t)}{\lambda^m(t)} = \upsilon > 0.$$

$(\mathcal{H}_4)$

$$\|x(t,0,\xi)\| > \left(\frac{\rho(t)}{\lambda^m(t)}\right)^{\frac{1}{q}}, \quad \textit{for all } t \geq 0.$$

Then,

$$\limsup_{t\to+\infty} \frac{\ln\left(\|x(t,0,\xi)\| - \left(\frac{\rho(t)}{\lambda^m(t)}\right)^{\frac{1}{q}}\right)}{\ln\lambda(t)} \leq -\left[m - (\beta_1 + (\beta_2+\sigma)\vee m)\right], \quad a.s.$$

Proof. Observe that we have

$$\begin{aligned}\lambda^m(t)\|x(t)\|^q - \rho(t) &= \lambda^m(t)\left(\|x(t)\|^q - \frac{\rho(t)}{\lambda^m(t)}\right)\\ &= \lambda^m(t)\left(\|x(t)\|^q - \left(\left(\frac{\rho(t)}{\lambda^m(t)}\right)^{\frac{1}{q}}\right)^q\right).\end{aligned}$$

Using the inequality

$$a^q - b^q = (a-b)(a^{q-1} + a^{q-2}b + a^{q-3}b^2 + \cdots + a^0 b^{q-1}),$$

we conclude

$$\begin{aligned}\lambda^m(t)\|x(t)\|^q - \rho(t) &= \lambda^m(t)\left(\|x(t)\|^q - \left(\left(\frac{\rho(t)}{\lambda^m(t)}\right)^{\frac{1}{q}}\right)^q\right)\\ &= \lambda^m(t)\left(\|x(t)\| - \left(\frac{\rho(t)}{\lambda^m(t)}\right)^{\frac{1}{q}}\right)\\ &\quad\times\left(\|x(t)\|^{q-1} + \|x(t)\|^{q-2}\left(\frac{\rho(t)}{\lambda^m(t)}\right)^{\frac{1}{q}} + \cdots + \left(\frac{\rho(t)}{\lambda^m(t)}\right)^{\frac{q-1}{q}}\right)\\ &= \lambda^m(t)\left(\|x(t)\| - \left(\frac{\rho(t)}{\lambda^m(t)}\right)^{\frac{1}{q}}\right)\sum_{k=1}^{q}\|x(t)\|^{q-k}\left(\frac{\rho(t)}{\lambda^m(t)}\right)^{\frac{k-1}{q}}.\end{aligned}$$

From condition $(\mathcal{H}_3)$, we have $\lim_{t\to+\infty}\frac{\rho(t)}{\lambda^m(t)} = v > 0$. That is, for $0 < v_0 < v$, there exits $\bar{T} \geq 0$, such that $\frac{\rho(t)}{\lambda^m(t)} \geq v_0$ for all $t \geq \bar{T}$. Then, as we are assuming that $\|x(t)\| > \left(\frac{\rho(t)}{\lambda^m(t)}\right)^{\frac{1}{q}}$, for all $t \geq 0$, it holds

$$\begin{aligned}\sum_{k=1}^{q}\|x(t)\|^{q-k}\left[\frac{\rho(t)}{\lambda^m(t)}\right]^{\frac{k-1}{q}} &= \|x(t)\|^{q-1} + \|x(t)\|^{q-2}\left[\frac{\rho(t)}{\lambda^m(t)}\right]^{\frac{1}{q}} + \cdots + \left[\frac{\rho(t)}{\lambda^m(t)}\right]^{\frac{q-1}{q}}\\ &\geq v' = q\,(v_0)^{(q-1)/q}, \quad \forall t \geq \bar{T} \geq 0.\end{aligned}$$

Therefore,

$$\lambda^m(t)\|x(t)\|^q - \rho(t) \geq \lambda^m(t)\left[\|x(t)\| - \left[\frac{\rho(t)}{\lambda^m(t)}\right]^{\frac{1}{q}}\right] v', \quad \text{for all } t \geq \bar{T} \geq 0.$$

Hence, we see that

$$V(t,x_t) \geq \lambda^m(t)\|x(t)\|^q \geq \lambda^m(t)\|x(t)\|^q - \rho(t)$$
$$\geq \lambda^m(t)\left[\|x(t)\| - \left[\frac{\rho(t)}{\lambda^m(t)}\right]^{\frac{1}{q}}\right] v'.$$

That is,

$$v'\lambda^m(t)\left[\|x(t)\| - \left[\frac{\rho(t)}{\lambda^m(t)}\right]^{\frac{1}{q}}\right] \leq V(t,x_t),$$

and,

$$\ln v' + \ln\left[\lambda^m(t)\left[\|x(t)\| - \left[\frac{\rho(t)}{\lambda^m(t)}\right]^{\frac{1}{q}}\right]\right] \leq \ln\left[V(t,x_t)\right].$$

Consequently, it follows that

$$\ln v' + m\ln\lambda(t) + \ln\left[\|x(t)\| - \left[\frac{\rho(t)}{\lambda^m(t)}\right]^{\frac{1}{q}}\right] \leq \ln\left[V(t,x_t)\right], \quad \forall t \geq \bar{T} \geq 0.$$

Applying the Itô formula, we obtain

$$V(t,x_t) = V(0,x_0) + \int_0^t \mathcal{L}V(s,x_s)ds + \int_0^t V_x(s,x_s)G(s,x_s)dB(s). \tag{2.1}$$

Based upon the uniform continuity of $\ln\lambda(t)$, we can ensure that for each $\varepsilon > 0$ there exists two positive integers $N = N(\varepsilon)$ and $k_1(\varepsilon)$, such that if $\frac{k-1}{2^N} \leq t \leq \frac{k}{2^N}$, $k \geq k_1(\varepsilon)$, it follows that

$$\left|\ln\lambda\left(\frac{k}{2^N}\right) - \ln\lambda(t)\right| \leq \varepsilon.$$

On the other side, owing to the exponential martingale inequality from Lemma 1.1, we have

$$\mathbb{P}\left\{\omega : \sup_{0\leq t\leq\tau}\left[M(t) - \frac{\mu}{2}\int_0^t \|V_x(s,x_s)G(s,x_s)\|^2 ds\right] > \eta\right\} \leq e^{-\mu\eta},$$

for any positive constants μ, η and τ, where

$$M(t) = \int_0^t V_x(s,x_s)G(s,x_s)dB(s).$$

In particular, for the preceding $\varepsilon > 0$, we set

$$\mu = 2\phi\left(\frac{k-1}{2^N}\right), \quad \eta = \phi\left(\frac{k-1}{2^N}\right)^{-1}\ln\frac{k-1}{2^N}, \quad \tau = \frac{k}{2^N}, \quad k = 2,3,\ldots$$

Then, we apply the well-known Borel–Cantelli lemma to obtain that, for almost all $\omega \in \Omega$, there exists an integer $k_0 = k(\varepsilon,\omega) > 0$, such that

$$M(t) \leq \phi\left(\frac{k-1}{2^N}\right)^{-1}\ln\frac{k-1}{2^N} + \phi\left(\frac{k-1}{2^N}\right)\int_0^t \|V_x(s,x_s))G(s,x_s)\|^2 ds$$
$$\leq \phi\left(\frac{k-1}{2^N}\right)^{-1}\ln\frac{k-1}{2^N} + \int_0^t \phi(s)\|V_x(s,x_s)G(s,x_s)\|^2 ds,$$

for $0 \leq t \leq \frac{k}{2^N}$, $k \geq k_0(\varepsilon,\omega)$.

Substituting the last inequality into Eq. (2.1), we obtain

$$V(t,x_t) \leq V(0,x_0) + \phi\left(\frac{k-1}{2^N}\right)^{-1} \ln\frac{k-1}{2^N} + \int_0^t \mathcal{L}V(s,x_s)ds + \int_0^t \phi(s)\|V_s(s,x_s)G(s,x_s)\|^2 ds,$$

for $0 \leq t \leq \frac{k}{2^N}$, $k \leq k_0(\varepsilon,\omega)$.

Using conditions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$, it follows that

$$\begin{aligned} V(t,x_t) &\leq V(0,x_0) + \phi\left(\frac{k-1}{2^N}\right)^{-1} \ln\frac{k-1}{2^N} + \rho(t) + \int_0^t \psi(s)\lambda^m(s)\|x(s)\|^q ds \\ &\leq V(0,x_0) + \phi\left(\frac{k-1}{2^N}\right)^{-1} \ln\frac{k-1}{2^N} + \rho(t) + \int_0^t \psi(s)V(s,x_s)ds, \end{aligned}$$

for $0 \leq t \leq \frac{k}{2^N}$, $k \geq k_0(\varepsilon,\omega)$.

Applying now the Gronwall lemma [13],

$$V(t,x_t) \leq \left(V(0,x_0) + \phi\left(\frac{k-1}{2^N}\right)^{-1} \ln\frac{k-1}{2^N} + \rho(t)\right) \exp\left(\int_0^t \psi(s)ds\right). \tag{2.2}$$

Based upon condition $(\mathcal{H}_3)$ we have that, for any $\varepsilon > 0$, $\lim_{t\to+\infty}\sup \frac{\int_0^t \psi(s)ds}{\ln\lambda(t)} < \beta_1 + \varepsilon$, and $\lim_{t\to+\infty}\inf \frac{\ln\phi(t)}{\ln\lambda(t)} > -\beta_2 - \varepsilon$. Thanks also to the uniform continuity of $\ln\lambda(t)$, there exists a positive integer $k_1(\varepsilon)$, such that whenever $t \geq k_1(\varepsilon)$,

$$\begin{aligned} \int_0^t \psi(s)ds &\leq (\beta_1+\varepsilon)\ln\lambda(t), \\ \phi\left(\frac{k-1}{2^N}\right)^{-1} &\leq \phi(t) \leq \lambda(t)^{\beta_2+\varepsilon}, \end{aligned}$$

for $\frac{k-1}{2^N} \leq t \leq \frac{k}{2^N}$, $k \geq k_1(\varepsilon)$.

Furthermore, we have

$$\ln\frac{k-1}{2^N} \leq \ln t \leq \ln\frac{k}{2^N}, \quad \text{for} \quad \frac{k-1}{2^N} \leq t \leq \frac{k}{2^N}.$$

Based on inequality (2.2), and the standing assumptions, we obtain for almost all $\omega \in \Omega$,

$$\ln V(t,x_t) \leq \ln\left(V(0,x_0) + \lambda(t)^{\beta_2+\sigma+2\varepsilon} + \rho(t)\right) + (\beta_1+\varepsilon)\ln\lambda(t),$$

for $\frac{k-1}{2^N} \leq t \leq \frac{k}{2^N}$, $k \geq k_1(\varepsilon)$.

Hence, we deduce that

$$\lim_{t\to+\infty}\sup\frac{\ln V(t,x_t)}{\ln\lambda(t)} \leq (\beta_2+\sigma+2\varepsilon)\vee m + \beta_1 + \varepsilon, \quad \text{a.s.}$$

Recall that, for $t \geq \bar{T} \geq 0$ and $q \in \mathbb{N}^\star$, we have

$$\ln\left(\|x(t)\| - \left(\frac{\rho(t)}{\lambda^m(t)}\right)^{\frac{1}{q}}\right) \leq \ln V(t,x_t) - m\ln\lambda(t) - \ln v'.$$

Taking into account that $\varepsilon > 0$ is arbitrary, we derive that,

$$\limsup_{t\to+\infty} \frac{\ln\left(\|x(t)\| - \left(\frac{\rho(t)}{\lambda^m(t)}\right)^{\frac{1}{q}}\right)}{\ln\lambda(t)} \leq -\left[m - (\beta_1 + (\beta_2+\sigma)\vee m)\right], \quad \text{a.s.},$$

as required. □

In the next corollary, we will deduce the practical convergence to the ball B_r with a general decay rate of stochastic differential delay equations.

Corollary 2.6. *Let $V : \mathbb{R}_+ \times C([-\tau,0],\mathbb{R}^n) \to \mathbb{R}_+$ be a functional from D. Assume that $\ln\lambda(t)$ is uniformly continuous on $t \geq 0$, and there exists a constant $\sigma \geq 0$, such that*

$$\lim_{t\to+\infty} \frac{\ln\ln t}{\ln\lambda(t)} \leq \sigma.$$

Let $x(\cdot) = x(\cdot,0,\xi)$ be a solution to system (1.1), and assume that there exist constants $q \in \mathbb{N}^\star$, $m \geq 0$, $\beta_1 \in \mathbb{R}$, $\beta_2 \geq 0$, a non–increasing function $\phi(t) > 0$ and a continuous non–negative function $\psi(t)$, such that for all $t \geq 0$, assumptions $(\mathcal{H}_1)$–$(\mathcal{H}_4)$ are satisfied. Then, if in addition there exists $\widetilde{v} > v > 0$, such that $\|x(t,0,\xi)\| > \widetilde{v}$ for all $t \geq 0$, it follows

$$\limsup_{t\to+\infty} \frac{\ln\left(\|x(t,0,\xi)\| - (\widetilde{v})^{\frac{1}{q}}\right)}{\ln\lambda(t)} \leq -\gamma, \quad ;a.s.,$$

where $\gamma = m - (\beta_1 + (\beta_2+\sigma)\vee m)$.

In particular, if $m > \beta_1 + (\beta_2+\sigma)\vee m$, then the solution to system (1.1) decays to the ball B_r, with $r = (\widetilde{v})^{\frac{1}{q}}$ almost surely practically with decay function $\lambda(t)$ and order at least γ.

Remark 2.7. Observe that the condition $m > \beta_1 + (\beta_2+\sigma)\vee m$ (or equivalently $\gamma > 0$) in the corollary holds in the following cases:

- If $\beta_2 + \sigma \leq m$, then the condition becomes $m > \beta_1 + m$. Therefore, this requires $\beta_1 < 0$.
- If $\beta_2 + \sigma > m$, then the condition turns to $m > \beta_1 + \beta_2 + \sigma$ which again requires $\beta_1 < 0$. As a conclusion, the condition ensuring that γ is positive requires that $\beta_1 < 0$, and this implies that when $\beta_2 + \sigma \leq m$, then $\gamma > 0$, and when $\beta_2 + \sigma > m$, then β_1 must be smaller than $m - \beta_2 - \sigma$.

Proof. From Theorem 2.5, it follows that

$$\limsup_{t\to+\infty} \frac{\ln\left(\|x(t)\| - \left(\frac{\rho(t)}{\lambda^m(t)}\right)^{\frac{1}{q}}\right)}{\ln\lambda(t)} \leq -\gamma, \quad \text{a.s.}$$

Since, we have $\lim_{t\to+\infty} \frac{\rho(t)}{\lambda^m(t)} = v < \widetilde{v}$, then there exists $\bar{T} \geq 0$, such that $\frac{\rho(t)}{\lambda^m(t)} \leq \widetilde{v}$, for all $t \geq \bar{T} \geq 0$. Hence, we obtain

$$\limsup_{t\to+\infty} \frac{\ln\left(\|x(t)\| - (\widetilde{v})^{\frac{1}{q}}\right)}{\ln\lambda(t)} \leq \limsup_{t\to+\infty} \frac{\ln\left(\|x(t)\| - \left(\frac{\rho(t)}{\lambda^m(t)}\right)^{\frac{1}{q}}\right)}{\ln\lambda(t)} \leq -\gamma, \quad textrma.s.$$

Hence, if $m > \beta_1 + (\beta_2+\sigma)\vee m$, then the solution to system (1.1) decays to the ball B_r, with $r = (\widetilde{v})^{\frac{1}{q}}$ almost surely practically with decay function $\lambda(t)$ and order at least γ. □

Example 2.8. Consider the following one dimensional stochastic differential delay equation with constant time delay.

$$\begin{cases} dx(t) = \left[-\frac{b+1}{2(1+t)} x(t) + \frac{1}{1+t} x(t-\tau) \right] dt + (1+t)^{-\frac{1}{2}} dB(t), \quad t \geq 0, \\ x(t) = \xi(t), \quad t \in [-\tau, 0], \end{cases} \tag{2.3}$$

where $b \in \mathbb{R}_+$, $B(t)$ is a one-dimensional Brownian motion and τ is a positive constant.

Define for $\Phi \in C([-\tau, 0], \mathbb{R})$:

$$F(t, \Phi) = -\frac{b+1}{2(1+t)} \Phi(0) + \frac{1}{1+t} \Phi(-\tau), \quad G(t, \Phi) = (1+t)^{-\frac{1}{2}}, \quad t \geq 0.$$

Now, we proceed to investigate the practical stability with a general decay rate of system (2.3) by using a Lyapunov functional.

Consider the following functional,

$$V(t, x_t) := (1+t)|x(t)|^2 + \int_{t-\tau}^{t} |x(u)|^2 du.$$

Then, it is easy to check that for arbitrary $\alpha > 1$, $\phi(t) = \frac{b}{4(1+t)^\alpha}$, we have

$$\begin{aligned} &\int_0^t \mathcal{L}V(s, x_s) ds + \int_0^t \frac{b}{4(1+s)^\alpha} |V_x(s, x_s) G(s, x_s)|^2 ds \\ &\leq \int_0^t ds - \int_0^t b|x(s)|^2 ds + \int_0^t 2|x(s)||x(s-\tau)| ds \\ &\quad + \int_0^t |x(s)|^2 ds - \int_0^t |x(s-\tau)|^2 ds + \int_0^t \frac{b}{(1+s)^{\alpha-2}} |x(s)|^2 ds \\ &\leq \int_0^t ds + \int_0^t (1-b)|x(s)|^2 ds + \int_0^t |x(s-\tau)|^2 ds \\ &\quad + \int_0^t |x(s)|^2 ds - \int_0^t |x(s-\tau)|^2 ds + \int_0^t \frac{b}{(1+s)^{\alpha-2}} |x(s)|^2 ds. \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} &\int_0^t \mathcal{L}V(s, x_s) ds + \int_0^t \frac{1}{b(1+s)^\alpha} |V_x(s, x_s) G(s, x_s)|^2 ds \\ &\leq t + \int_0^t \left[\frac{2-b}{1+s} + \frac{b}{(1+s)^{\alpha-1}} \right] (1+s)|x(s)|^2 ds. \end{aligned}$$

Hence, we see that

$$\psi(t) = \frac{b}{(1+t)^{\alpha-1}} + \frac{2-b}{1+t}, \quad \rho(t) = t.$$

Taking $\lambda(t) = (1+t)$, then by some easy computations, we can check that,

$$\sigma = 0, \quad \beta_1 = 2-b, \quad \beta_2 = \alpha, \quad v = 1, \quad m = 1.$$

Finally, Corollary 2.6 allows us to conclude that

$$\lim_{t \to +\infty} \sup \frac{\ln(|x(t)| - 1)}{\ln(1+t)} \leq -\gamma, \quad \text{a.s.}$$

Hence, we deduce that the solution to system (2.3) decays to the ball B_r almost surely practically with decay function $\lambda(t) = 1+t$, $r = 1$, and order at least $\gamma = b - 1 - \alpha$, whenever $b > 1 + \alpha$.

3 Method of Lyapunov functionals construction in practical stability of stochastic delay differential equations

Notice that Corollary 2.6 implies that the almost sure practical stability with a general decay rate of SDDE (1.1) can be reduced to the construction of appropriate Lyapunov functionals.

A formal procedure to construct Lyapunov functionals is described below, (see Krasovskii [19], and V. Kolmanovskii and L. Shaikhet [17,18,23], for more details).

3.1 The formal procedure of constructing Lyapunov functionals

The formal procedure for constructing Lyapunov functionals consists of four steps.

Step 1 : Let us represent (1.1) in the following form:

$$dz(t,x_t) = (F_1(t,x(t)) + F_2(t,x_t))\,dt + (G_1(t,x(t)) + G_2(t,x_t))\,dB(t), \tag{3.1}$$

where $z(t,x_t)$ is some functional of x_t, the functions $F_1(t,x(t))$ and $G_1(t,x(t))$, depend on t and $x(t)$ only and do not depend on the previous values $x(t+\theta)$, $\theta < 0$, of the solution, and there exists $t \in \mathbb{R}_+$, such that $F_1(t,\cdot) \neq 0$ or $G_1(t,\cdot) \neq 0$.

Step 2 : Consider the auxiliary differential equation without memory

$$dy(t) = F_1(t,y(t))dt + G_1(t,y(t))dB(t). \tag{3.2}$$

Assume that the system (3.2) is almost sure practical stable with a general decay rate and there exists a Lyapunov function $v(t,y(t))$, which satisfies the conditions of Corollary 2.6.

Step 3 : A Lyapunov functional $V(t,x_t)$ for Eq.(1.1) is constructed in the form $V = V_1 + V_2$, where $V_1(t,x_t) = v(t,z(t,x_t))$. Here the argument y of the function $v(t,y)$ is replaced on the functional $z(t,x_t)$ from the left–hand part of Eq.(3.1).

Step 4 : Usually, the functional $V_1(t,x_t)$ almost satisfies the conditions of Corollary 2.6. To fully satisfy these conditions, it is necessary to calculate $\mathcal{L}V_1(t,x_t)$ and estimate it. Then, we choose the additional functional $V_2(t,x_t)$ in a standard way.

Remark 3.1. The representation (3.1) is not unique. This fact allows, using different representations of the type of (3.1) or different ways to estimate $\mathcal{L}V_1(t,x_t)$, to construct different Lyapunov functionals and, as a result to obtain different sufficient conditions for the practical stability with a general decay rate.

3.2 Construction of Lyapunov functionals for stochastic differential equations with constant delay

Consider the following stochastic differential equation with constant delay:

$$\begin{aligned} dx(t) &= (f(t,x(t)) + F(t,x(t),x(t-h)))\,dt + G(t,x(t),x(t-\tau))dB(t), \\ x(s) &= \xi(s), \quad s \in [-\widetilde{h},0], \end{aligned} \tag{3.3}$$

where,

$$\widetilde{h} = \max[h,\tau], \quad \text{and} \quad f : \mathbb{R}_+ \times \mathbb{R}^n \to \times\mathbb{R}^n,$$

$$F : [0,+\infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \quad G : [0,+\infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}.$$

$B(t)$ is an m-dimensional Brownian motion defined on the probability space $\{\Omega, \mathcal{F}, \mathrm{P}\}$.

Observe that Eq. (3.3) is a particular case of Eq. (1.1).

We will apply the method described above to construct Lyapunov functionals for Eq. (3.3), and, as a consequence, to obtain sufficient conditions ensuring the almost sure practical stability with decay function $\lambda(t)$, where $\lambda(\cdot) \in C^1(\mathbb{R}_+)$.

Theorem 3.2. *Assume that* $\ln \lambda(t)$ *is uniformly continuous on* $t \geq 0$*, and there exists a constant* $\sigma \geq 0$*, such that*

$$\lim_{t\to+\infty} \frac{\ln \ln t}{\ln \lambda(t)} \leq \sigma.$$

Let $\psi_1(t)$ *be a continuous non–negative function, and* $\rho(t)$ *a non-negative continuous differentiable function, such that for all* $t \geq 0 \geq 0$ *the following assumptions hold:*

$(\mathcal{A}_1)$

$$\begin{aligned} 2\langle x, f(t,x)\rangle &\leq (\psi_1(t) - K)\|x\|^2 + \frac{\rho'(t)}{\lambda^m(t)}, \quad K > 0, \\ \|\widetilde{F}(t,\Phi)\| &\leq \alpha_1 \|\Phi(-h)\|, \\ \|\widetilde{G}(t,\Phi)\| &\leq \alpha_2 \|\Phi(-\tau)\|, \\ \|\Phi(0)\widetilde{G}(t,\Phi)\| &\leq \alpha_3 \|\Phi(-\tau)\|, \end{aligned} \tag{3.4}$$

where $\widetilde{F}(t,\Phi) = F(t,\Phi(0),\Phi(-h)), \quad \widetilde{G}(t,\Phi) = G(t,\Phi(0),\Phi(-\tau)).$

$(\mathcal{A}_2)$

$$\lim_{t\to+\infty} \sup \frac{\int_0^t \psi_1(s)ds}{\ln \lambda(t)} \leq \alpha, \quad \alpha \in \mathbb{R}.$$

$$\lim_{t\to+\infty} \sup \frac{t}{\ln \lambda(t)} = C \geq 0, \quad \lim_{t\to+\infty} \frac{\rho(t)}{\lambda^m(t)} = v > 0.$$

$(\mathcal{A}_3)$

$$\|x(t,0,\xi)\| > \left(\frac{\rho(t)}{\lambda^m(t)}\right)^{\frac{1}{2}}, \quad \textit{for all } t \geq 0.$$

Then, if in addition there exists $\widetilde{v} \geq v > 0$*, such that* $\|x(t,0,\xi)\| > \widetilde{v}$ *for all* $t \geq 0$*, it follows*

$$\lim_{t\to+\infty} \sup \frac{\ln\left(\|x(t,0,\xi)\| - (\widetilde{v})^{\frac{1}{2}}\right)}{\ln \lambda(t)} \leq -\gamma, \quad a.s.,$$

where $\gamma = KC - (m + \alpha + \sigma) - (2\alpha_1 + \widetilde{\alpha})C$*,* $\widetilde{\alpha} = \alpha_1^2 + \alpha_2^2$*.*

In particular, if $KC > m + \alpha + \sigma + (2\alpha_1 + \widetilde{\alpha})C$*, then the solution to system* (3.3) *decays to the ball* B_r*, with* $r = (\widetilde{v})^{\frac{1}{2}}$ *almost surely practically, with decay function* $\lambda(t)$ *and order at least* γ*.*

Proof. Based upon the procedure of Lyapunov functionals construction, we consider the auxiliary equation without memory of the type (3.2) as

$$\dot{y}(t) = f(t, y(t)). \tag{3.5}$$

Our target now is to prove that the solution to system (3.5) decays to the ball B_r, with $r = (\widetilde{v})^{\frac{1}{2}}$ almost surely with decay function $\lambda(t)$. To this end, we consider the function $v(t,y) = \lambda^m(t)\|y\|^2$, $m \geq 0$ as a Lyapunov function for Eq. (3.5). Then, we have to prove that $v(t,y)$ satisfies all conditions of Corollary 2.6.

Using (3.4), it follows that

$$\begin{aligned}
&\int_0^t v_s(s,y(s))ds + \int_0^t v_x(s,y(s))f(s,y(s))ds \\
&\quad \leq \int_0^t m\lambda'(s)\lambda^{m-1}(s)\|y(s)\|^2 ds + \int_0^t 2\lambda^m(s)\langle y(s), f(s,y(s))\rangle ds \\
&\quad \leq \int_0^t m\lambda'(s)\lambda^{m-1}(s)\|y(s)\|^2 ds + \int_0^t \left(\lambda^m(s)\left[\psi_1(s)-K\right]\|y(s)\|^2 + \rho'(s)\right) ds \\
&\quad \leq \int_0^t \left[m\frac{\lambda'(s)}{\lambda(s)} + \psi_1(s) - K\right]\lambda^m(s)\|y(s)\|^2 ds + \rho(t) - \rho(0).
\end{aligned}$$

That is,

$$\begin{aligned}
&\int_0^t v_s(s,y(s))ds + \int_0^t v_x(s,y(s))f(s,y(s))ds \\
&\quad \leq \int_0^t \left[m\frac{\lambda'(s)}{\lambda(s)} + \psi_1(s) - K\right]\lambda^m(s)\|y(s)\|^2 ds + \rho(t).
\end{aligned}$$

Thus, setting

$$\psi(t) = m\frac{\lambda'(t)}{\lambda(t)} + \psi_1(t) - K.$$

Then, using assumption $(\mathcal{A}_2)$, one obtains

$$\lim_{t\to+\infty}\sup \frac{\int_0^t \psi(s)ds}{\ln\lambda(t)} \leq m + \alpha - KC.$$

Consequently, Corollary 2.6 allows us to conclude that,

$$\lim_{t\to+\infty}\sup \frac{\ln\left(\|y(t)\| - (\widetilde{v})^{\frac{1}{2}}\right)}{\ln\lambda(t)} \leq -\gamma, \quad \text{a.s.,}$$

where $\gamma = KC - (\alpha + \sigma \vee m)$. Hence, if $KC > \alpha + \sigma \vee m$, then the solution to system (3.4) decays to the ball B_r, with $r = (\widetilde{v})^{\frac{1}{2}}$ almost surely practically with decay function $\lambda(t)$ and order at least γ.

Based on the procedure, now we construct a Lyapunov functional V for Eq. (3.3) in the form $V = V_1 + V_2$, where $V_1(t,x_t) = \lambda^m(t)\|x(t)\|^2$.

Following Corollary 2.6, we consider the function $\phi(t) = \frac{1}{4\lambda^m(t)}$, $t \geq 0$, then, it follows that

$$\begin{aligned}
&\int_0^t \mathcal{L}V_1(s,x_s)ds + \int_0^t \phi(s)\|V_{1x}(s,x_s)\widetilde{G}(s,x(s),x(s-\tau))\|^2 ds \\
&\quad = \int_0^t m\lambda'(s)\lambda^{m-1}(s)\|x(s)\|^2 ds + \int_0^t 2\lambda^m(s)\langle f(s,x(s)), x(s)\rangle ds \\
&\qquad + \int_0^t 2\lambda^m(s)\langle \widetilde{F}(s,x(s),x(s-h)), x(s)\rangle ds + \int_0^t \lambda^m(s)\|\widetilde{G}(s,x(s),x(s-\tau))\|^2 ds \\
&\qquad + \int_0^t \lambda^m(s)\|x(s)\widetilde{G}(s,x(s),x(s-\tau))\|^2 ds.
\end{aligned}$$

Taking into account assumptions (3.4), we obtain

$$\begin{aligned}
&\int_0^t \mathcal{L}V_1(s,x_s)ds + \int_0^t \frac{1}{4\lambda^m(s)}\|V_{1x}(s,x_s)\widetilde{G}(s,x(s),x(s-\tau))\|^2 ds \\
&\quad \le \int_0^t \lambda^m(s)\left[m\frac{\lambda'(s)}{\lambda(s)} + \psi_1(s) - K\right]\|x(s)\|^2 ds + \rho(t) \\
&\quad\quad + \int_0^t 2\alpha_1\lambda^m(s)\|x(s)\|\|x(s-h)\|ds + \int_0^t \alpha_2^2\lambda^m(s)\|x(s-\tau)\|^2 ds \\
&\quad\quad + \int_0^t \alpha_3^2\lambda^m(s)\|x(s-\tau)\|^2 ds \\
&\quad \le \int_0^t \lambda^m(s)\left(\left[m\frac{\lambda'(s)}{\lambda(s)} + \psi_1(s) - K\right] + \alpha_1\right)\|x(s)\|^2 ds \\
&\quad\quad + \int_0^t \alpha_1\lambda^m(s)\|x(s-h)\|^2 ds + \int_0^t \widetilde{\alpha}\lambda^m(s)\|x(s-\tau)\|^2 ds + \rho(t),
\end{aligned}$$

where, $\widetilde{\alpha} = \alpha_2^2 + \alpha_3^2$.

Set now

$$V_2(t,x_t) = \alpha_1\int_{t-h}^t \lambda^m(u+h)\|x(u)\|^2 du + \widetilde{\alpha}\int_{t-\tau}^t \lambda^m(u+\tau)\|x(u)\|^2 du.$$

Then,

$$\begin{aligned}
\int_0^t \mathcal{L}V_2(s,x_s)ds &= \alpha_1\int_0^t \lambda^m(s+h)\|x(s)\|^2 ds - \alpha_1\int_0^t \lambda^m(s)\|x(s-h)\|^2 ds \\
&\quad + \widetilde{\alpha}\int_0^t \lambda^m(s+\tau)\|x(s)\|^2 ds - \widetilde{\alpha}\int_0^t \lambda^m(s)\|x(s-\tau)\|^2 ds \\
&\simeq \alpha_1\int_0^t \lambda^m(s)\|x(s)\|^2 ds - \alpha_1\int_0^t \lambda^m(s)\|x(s-h)\|^2 ds \\
&\quad + \widetilde{\alpha}\int_0^t \lambda^m(s)\|x(s)\|^2 ds - \widetilde{\alpha}\int_0^t \lambda^m(s)\|x(s-\tau)\|^2 ds.
\end{aligned}$$

That is, for $V = V_1 + V_2$, we obtain

$$\begin{aligned}
&\int_0^t \mathcal{L}V(s,x_s)ds + \int_0^t \frac{1}{4\lambda^m(s)}\|V_x(s,x_s)\widetilde{G}(s,x(s),x(s-\tau))\|^2 ds \\
&\quad \le \int_0^t \lambda^m(s)\left[m\frac{\lambda'(s)}{\lambda(s)} + \psi_1(s) + 2\alpha_1 + \widetilde{\alpha} - K\right]\|x(s)\|^2 ds + \rho(t).
\end{aligned}$$

That is, we have

$$\psi(t) = m\frac{\lambda'(t)}{\lambda(t)} + \psi_1(t) + 2\alpha_1 + \widetilde{\alpha} - K, \quad \phi(t) = \frac{1}{4\lambda^m(t)}.$$

Therefore, we obtain

$$\begin{aligned}
\limsup_{t\to+\infty} \frac{\int_0^t \psi(s)ds}{\ln\lambda(t)} &\le m + \alpha + (2\alpha_1 + \widetilde{\alpha} - K)C, \\
\liminf_{t\to+\infty} \frac{\ln\phi(t)}{\ln\lambda(t)} &\ge -m.
\end{aligned}$$

Finally, Corollary 2.6 allows us to conclude that,

$$\lim_{t\to+\infty} \frac{\ln\left(\|x(t,0,\xi)\| - (\widetilde{v})^{\frac{1}{2}}\right)}{\ln \lambda(t)} \le -\gamma,$$

where $\gamma = KC - (m+\alpha+\sigma) - (2\alpha_1 + \widetilde{\alpha})C$. Thus, if $KC > (m+\alpha+\sigma) + (2\alpha_1+\widetilde{\alpha})C$, the solution to system (3.3) decays to the ball B_r, with $r = (\widetilde{v})^{\frac{1}{2}}$ almost surely practically with decay function $\lambda(t)$. □

Now, we provide an illustrative example that implements the previous result.

Example 3.3. Consider the following one dimensional stochastic differential delay equation with constant time delay.

$$\begin{cases} dx(t) = \left[\left(a + e^{-\frac{3}{2}t} - 4K\right)x(t) + \dfrac{1}{2(1+|x(t)|)} + \cos(t)x(t-h)\right]dt \\ \qquad +g(x(t))\dfrac{x(t-h)}{1+|x(t)|}dB(t), \quad t \ge 0, \\ x(t) = \xi(t), \quad t \in [-h, 0], \end{cases} \tag{3.6}$$

where $a, K \in \mathbb{R}_+$, $g(\cdot) : \mathbb{R} \to \mathbb{R}$ is a bounded Lipschitz continuous function, such that $g(0) \neq 0$, and $|g(x)| \le L$, $x \in \mathbb{R}$, $L > 0$, $B(t)$ is a one-dimensional Brownian motion and h is a positive constant.

We can set this problem in our formulation by taking,

$$\begin{aligned} f(t,x) &= \frac{1}{2}\left(a + e^{-\frac{3}{2}t} - 4K\right)x + \frac{1}{2(1+|x(t)|)}, \\ \widetilde{F}(t,\Phi) &= \cos(t)\Phi(-h), \\ \widetilde{G}(t,\Phi) &= g(\Phi(0))\frac{\Phi(-h)}{1+|\Phi(0)|}, \end{aligned}$$

$x \in \mathbb{R}$, $\Phi \in C([-h,0],\mathbb{R})$.

We will consider the decay function $\lambda(t) = e^t$ and $m = 1$. Indeed, we can apply Theorem 3.2 in a straightforward way since,

$$\begin{aligned} 2\langle x, f(t,x)\rangle &\le \left(a + e^{-\frac{3}{2}t} - 4K\right)|x|^2 + \frac{e^t}{e^t}, \\ |\widetilde{F}(t,\Phi)| &\le |\Phi(-h)|, \\ |\widetilde{G}(t,\Phi)| &\le L|\Phi(-h)|, \\ |\Phi(0)\widetilde{G}(t,\Phi)| &\le L|\Phi(-h)|. \end{aligned}$$

Therefore, we can set

$$\rho(t) = e^t, \quad \psi_1(t) = (a + e^{-\frac{3}{2}t}).$$

Then, we can choose constants in Theorem 3.2 as follows:

$$\sigma = 0, \quad C = 1, \quad \alpha = a, \quad \alpha_1 = 1, \quad \alpha_2 = \alpha_3 = L, \quad v = 1.$$

Eventually, we deduce that

$$\lim_{t\to+\infty} \sup \frac{\ln(|x(t)| - 1)}{t} \le -\gamma, \quad \text{a.s.},$$

where $\gamma = 4K - (3 + a + 2L^2)$. Hence, if $4K > 3 + a + 2L^2$, we deduce that the solution to system (3.6) is almost surely practically exponentially stable with decay function $\lambda(t) = e^t$ and order at least γ.

3.3 Construction of Lyapunov functionals for stochastic differential equations with time-varying delay

Consider the following stochastic differential equation with time-varying delay:

$$\begin{aligned} &dx(t) = [f(t, x(t)) + F(t, x(t), x(t-h(t)))]\,dt + G(t, x(t), x(t-\tau(t)))dB(t),\\ &h(t) \in [0, h_0], \quad \tau(t) \in [0, \tau_0], \quad h = \max[h_0, \tau_0],\\ &x(s) = \xi(s), \quad s \in [-h, 0], \end{aligned} \tag{3.7}$$

where,

$$f : \mathbb{R}_+ \times \mathbb{R}^n \to \times\mathbb{R}^n, \quad F : [0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \quad G : [0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}.$$

$B(t)$ is an m–dimensional Brownian motion defined on the probability $\{\Omega, \mathcal{F}, \mathrm{P}\}$.
Observe that Eq.(3.7) is a particular case of Eq. (1.1).

Now, we aim to apply the procedure of constructing Lyapunov functionals for Eq. (3.7), in order to obtain sufficient conditions ensuring the almost sure practical uniform exponential stability, with decay function $\lambda(t) = e^t$. The construction of Lyapunov functionals for general decay functions will be analyzed elsewhere.

Theorem 3.4. *Let $\psi_1(t)$ be a continuous non–negative function, $\psi_2(t) > 0$ a non-increasing function and $\rho(t)$ a continuous non-negative differentiable function, such that for all $t \geq 0 \geq 0$ the following assumptions hold:*

$(\mathcal{A}_1')$

$$\begin{aligned} 2\langle x, f(t,x)\rangle &\leq (\psi_1(t) - K)\|x\|^2 + \frac{\rho'(t)}{e^{mt}}, \quad K > 0,\\ \|\widetilde{F}(t,\Phi)\| &\leq \psi_2(t)\|\Phi(-h(t))\|,\\ \|\widetilde{G}(t,\Phi)\| &\leq \alpha_2\|\Phi(-\tau(t))\|,\\ \|\Phi(0)\widetilde{G}(t,\Phi)\| &\leq \alpha_3\|\Phi(-\tau(t))\|, \end{aligned} \tag{3.8}$$

where $\widetilde{F}(t,\Phi) = F(t, \Phi(0), \Phi(-h(t)))$, $\widetilde{G}(t,\Phi) = G(t, \Phi(0), \Phi(-\tau(t)))$, and

$$\begin{aligned} h(t) &\in [0, h_0], \quad \dot{h}(t) \leq h_1 \leq 1,\\ \tau(t) &\in [0, \tau_0], \quad \dot{\tau}(t) \leq \tau_1 \leq 1. \end{aligned} \tag{3.9}$$

$(\mathcal{A}_2')$

$$\limsup_{t\to+\infty} \frac{\int_0^t \psi_1(s)ds}{t} \leq \alpha, \quad \alpha > 0,$$

$$\limsup_{t\to+\infty} \frac{\int_0^t \psi_2(s)ds}{t} \leq a, \quad a > 0,$$

$$\lim_{t\to+\infty} \frac{\rho(t)}{e^{mt}} = v > 0.$$

$(\mathcal{A}_3')$

$$\|x(t, 0, \xi)\| > \left(\frac{\rho(t)}{e^{mt}}\right)^{\frac{1}{2}}, \quad \textit{for all } t \geq 0.$$

Then, if in addition there exists $\widetilde{v} \geq v > 0$, such that $\|x(t,0,\xi)\| > \widetilde{v}$ for all $t \geq 0$, it follows

$$\lim_{t\to+\infty} \frac{\ln\left(\|x(t,0,\xi)\| - (\widetilde{v})^{\frac{1}{2}}\right)}{\ln\lambda(t)} \leq -\gamma, \quad a.s.,$$

where $\gamma = K - (m+\alpha) - \left(1 + \frac{e^{mh_0}}{1-h_1}\right)a - \widetilde{\alpha}\frac{e^{m\tau_0}}{1-\tau_1}$, $\widetilde{\alpha} = \alpha_1^2 + \alpha_2^2$.

In particular, if $K > m + \alpha + \left(1 + \frac{e^{mh_0}}{1-h_1}\right)a + \widetilde{\alpha}\frac{e^{m\tau_0}}{1-\tau_1}$, *the solution to system (3.7) decays to the ball B_r, with $r = (\widetilde{v})^{\frac{1}{2}}$ almost surely uniformly practically exponentially stable, i.e., with decay function $\lambda(t) = e^t$, and order at least γ.*

Proof. Proceeding as in the proof of Theorem 3.2, we consider the auxiliary equation without memory of the type (3.2) as

$$\dot{y}(t) = f(t, y(t)). \tag{3.10}$$

We have to prove that the solution to system (3.10) decays to the ball B_r, with $r = (\widetilde{v})^{\frac{1}{2}}$ almost surely with decay function $\lambda(t)$. To this end, we consider the function $v(t,y) = e^{mt}\|y\|^2$, $m \geq 0$ as a Lyapunov function for Eq. (3.10).

Then, we have to prove that $v(t,y)$ satisfies all conditions of Corollary 2.6.

On account of (3.8), it follows that

$$\int_0^t v_s(s,y(s))ds + \int_0^t v_x(s,y(s))f(s,y(s))ds \leq \int_0^t [m + \psi_1(s) - K]\, e^{ms}\|y(s)\|^2 ds + \rho(t).$$

Thus, setting

$$\psi(t) = m + \psi_1(t) - K.$$

and using Corollary 2.6, it follows that

$$\lim_{t\to+\infty} \sup \frac{\ln\left(\|y(t)\| - (\widetilde{v})^{\frac{1}{2}}\right)}{t} \leq -\gamma, \quad \text{a.s.,}$$

where $\gamma = K - (\alpha + \sigma \vee m)$. Hence, if $K > \alpha + \sigma \vee m$, then the solution to system (3.10) decays to the ball B_r, with $r = (\widetilde{v})^{\frac{1}{2}}$ almost surely practically uniformly exponentially stable with order at least γ. Based on the procedure, now we construct a Lyapunov functional V for Eq. (3.7) in the form $V = V_1 + V_2$, where $V_1(t, x_t) = e^{mt}\|x(t)\|^2$.

Following Corollary 2.6, we consider the function $\phi(t) = \frac{1}{4e^{mt}}$, $t \geq 0$, then it follows that

$$\begin{aligned}
&\int_0^t \mathcal{L}V_1(s,x_s)ds + \int_0^t \phi(s)\|V_{1x}(s,x_s)\widetilde{G}(s,x_s)\|^2 ds \\
&\quad = \int_0^t e^{ms}\|x(s)\|^2 ds + \int_0^t 2e^{ms}\langle f(s,x(s)), x(s)\rangle ds \\
&\qquad + \int_0^t 2e^{ms}\langle \widetilde{F}(s,x_s), x(s)\rangle ds + \int_0^t e^{ms}\|\widetilde{G}(s,x_s)\|^2 ds + \int_0^t e^{ms}\|x(s)\widetilde{G}(s,x_s)\|^2 ds.
\end{aligned}$$

Taking into account assumption $(\mathcal{A}_1')$, it follows that

$$\begin{aligned}
&\int_0^t \mathcal{L}V_1(s,x_s)ds + \int_0^t \frac{1}{4e^{ms}}\|V_{1x}(s,x_s)\widetilde{G}(s,x_s)\|^2 ds \\
&\quad \le \int_0^t e^{ms}\left[m+\psi_1(s)-K\right]\|x(s)\|^2 ds + \rho(t) \\
&\qquad + \int_0^t 2\psi_2(s)e^{ms}\|x(s)\|\|x(s-h(s))\|ds + \int_0^t \alpha_2^2 e^{ms}\|x(s-\tau(s))\|^2 ds \\
&\qquad + \int_0^t \alpha_3^2 e^{ms}\|x(s-\tau(s))\|^2 ds \\
&\quad \le \int_0^t e^{ms}\left(m+\psi_1(s)-K+\psi_2(s)\right)\|x(s)\|^2 ds \\
&\qquad + \int_0^t \psi_2(s)e^{ms}\|x(s-h(s))\|^2 ds + \int_0^t \widetilde{\alpha} e^{ms}\|x(s-\tau(s))\|^2 ds + \rho(t),
\end{aligned}$$

where $\widetilde{\alpha} = \alpha_2^2 + \alpha_3^2$.

Set now

$$V_2(t,x_t) = \frac{1}{1-h_1}\int_{t-h(t)}^t e^{m(u+h_0)}\psi_2(u)\|x(u)\|^2 du + \frac{\widetilde{\alpha}}{1-\tau_1}\int_{t-\tau(t)}^t e^{m(u+\tau_0)}\|x(u)\|^2 du.$$

Then,

$$\begin{aligned}
&\int_0^t \mathcal{L}V_2(s,x_s)ds \\
&\quad = \frac{1}{1-h_1}\int_0^t e^{m(s+h_0)}\psi_2(s)\|x(s)\|^2 ds \\
&\qquad - \frac{1}{1-h_1}\int_0^t (1-\dot{h}(s))e^{m(s-h(s)+h_0)})\psi_2(s-h(s))\|x(s-h(s))\|^2 ds \\
&\qquad + \frac{\widetilde{\alpha}}{1-\tau_1}\int_0^t e^{m(s+\tau_0)}\|x(s)\|^2 ds - \frac{\widetilde{\alpha}}{1-\tau_1}\int_0^t (1-\dot{\tau}(s))e^{m(s-\tau(s)+\tau_0)}\|x(s-\tau(s))\|^2 ds \\
&\quad \le \frac{1}{1-h_1}\int_0^t e^{m(s+h_0)}\psi_2(s)\|x(s)\|^2 ds \\
&\qquad - \frac{1}{1-h_1}\int_0^t (1-h_1)e^{ms}e^{m(h_0-h(s))}\psi_2(s-h(s))\|x(s-h(s))\|^2 ds \\
&\qquad + \frac{\widetilde{\alpha}}{1-\tau_1}\int_0^t e^{m(s+\tau_0)}\|x(s)\|^2 ds - \frac{\widetilde{\alpha}}{1-\tau_1}\int_0^t (1-\tau_1)e^{ms}e^{m(\tau_0-\tau(s))}\|x(s-\tau(s))\|^2 ds \\
&\quad \le \frac{1}{1-h_1}\int_0^t e^{m(s+h_0)}\psi_2(s)\|x(s)\|^2 ds - \int_0^t e^{ms}\psi_2(s)\|x(s-h(s))\|^2 ds \\
&\qquad + \frac{\widetilde{\alpha}}{1-\tau_1}\int_0^t e^{m(s+\tau_0)}\|x(s)\|^2 ds - \widetilde{\alpha}\int_0^t e^{ms}\|x(s-\tau(s))\|^2 ds.
\end{aligned}$$

That is, for $V = V_1 + V_2$, we obtain

$$\begin{aligned}
&\int_0^t \mathcal{L}V(s,x_s)ds + \int_0^t \frac{1}{4e^{ms}}\|V_x(s,x_s)\widetilde{G}(s,x(s),x(s-\tau))\|^2 ds \\
&\quad \le \int_0^t e^{ms}\left(m+\psi_1(s)-K+\left(1+\frac{e^{mh_0}}{1-h_1}\right)\psi_2(s)+\widetilde{\alpha}\frac{e^{m\tau_0}}{1-\tau_1}\right)\|x(s)\|^2 ds + \rho(t).
\end{aligned}$$

That is, we have

$$\psi(t) = m + \psi_1(t) - K + \left(1 + \frac{e^{mh_0}}{1-h_1}\right)\psi_2(t) + \widetilde{\alpha}\frac{e^{m\tau_0}}{1-\tau_1},$$
$$\phi(t) = \frac{1}{4e^{mt}}.$$

Therefore, we obtain

$$\lim_{t\to+\infty}\sup \frac{\int_0^t \psi(s)ds}{t} \le m + \alpha - K + \left(1 + \frac{e^{mh_0}}{1-h_1}\right)a + \widetilde{\alpha}\frac{e^{m\tau_0}}{1-\tau_1},$$
$$\lim_{t\to+\infty}\inf \frac{\ln\phi(t)}{t} = -m.$$

Using Corollary 2.6, we infer that

$$\lim_{t\to+\infty} \frac{\ln\left(\|x(t,0,\xi)\| - (\widetilde{v})^{\frac{1}{2}}\right)}{\ln\lambda(t)} \le -\gamma, \quad \text{a.s.},$$

where $\gamma = K - (m+\alpha) - \left(1 + \frac{e^{mh_0}}{1-h_1}\right)a - \widetilde{\alpha}\dfrac{e^{m\tau_0}}{1-\tau_1}$.

Then, if $K > m + \alpha + \left(1 + \frac{e^{mh_0}}{1-h_1}\right)a + \widetilde{\alpha}\frac{e^{m\tau_0}}{1-\tau_1}$, the solution to system (3.7) decays to the ball B_r, with $r = (\widetilde{v})^{\frac{1}{2}}$ almost surely uniformly practically exponentially stable with decay function $\lambda(t) = e^t$, and order at least γ. □

We analyze now an example to show how the previous theorem can be implemented.

Example 3.5. Consider the following one dimensional stochastic differential delay equation with constant time delay.

$$\begin{cases} dx(t) = \left[\frac{1}{2}\left(b + |\cos(t)| - K\right)x(t) + \frac{1}{2}\frac{e^{-t}}{1+|x(t)|} + \frac{1}{t+1}x(t-h(t))\right]dt \\ \qquad\qquad + \frac{x(t-\tau(t))}{1+|x(t)|}dB(t), \quad t \ge 0, \\ x(t) = \xi(t), \quad t \in [-h, 0], \end{cases} \tag{3.11}$$

with the conditions,

$$h(t) \in [0, h_0], \quad \dot{h}(t) \le h_1 \le 1,$$
$$\tau(t) \in [0, \tau_0], \quad \dot{\tau}(t) \le \tau_1 \le 1,$$

where $b,\ K \in \mathbb{R}_+,\ x \in \mathbb{R},\ B(t)$ is a one-dimensional Brownian motion, and $h = \max[h_0, \tau_0]$.

We can set this problem in our formulation by taking,

$$f(t,x) = \frac{1}{2}\left(a + |\cos(t)| - K\right)x + \frac{1}{2}\frac{e^{-t}}{1+|x|},$$
$$\widetilde{F}(t,\Phi) = \frac{1}{t+1}\Phi(-h(t)),$$
$$\widetilde{G}(t,\Phi) = \frac{\Phi(-\tau(t))}{1+|\Phi(0)|},$$

$x \in \mathbb{R}, \quad \Phi \in C([-h,0],\mathbb{R})$.

For $m=2$, we can check that

$$\begin{aligned} 2\langle x, f(t,x)\rangle &\leq (b-K)|x|^2 + \frac{e^t}{e^{2t}}, \\ |\widetilde{F}(t,\Phi)| &\leq \frac{1}{t+1}|\Phi(-h(t))|, \\ |\widetilde{G}(t,\Phi)| &\leq |\Phi(-\tau(t))|, \\ |\Phi(0)\widetilde{G}(t,\Phi)| &\leq |\Phi(-\tau(t))|. \end{aligned}$$

Hence, we see that

$$\psi_1(t) = (b + |\cos(t)|), \quad \psi_2(t) = \frac{1}{t+1}, \quad \rho(t) = e^t.$$

Then, we can choose constants in Theorem 3.4 as follows:

$$\alpha = b, \quad a = 0, \quad \alpha_2 = \alpha_3 = 1, \quad v = 1.$$

Finally, Theorem 3.4 allows us to conclude that,

$$\lim_{t\to+\infty} \sup \frac{\ln(|x(t)|-1)}{t} \leq -\gamma, \quad \text{a.s.},$$

where $\gamma = K - (2-b) - \frac{e^{2\tau_0}}{1-\tau_1}$. Hence, if $K > 2 + b + 2\frac{e^{2\tau_0}}{1-\tau_1}$, we deduce that the solution to system (3.11) is almost surely practically exponentially stable, i.e., with decay function $\lambda(t) = e^t$, and order at least γ.

4 Conclusion

We investigated herein the practical convergence to a small ball centered at the origin with a general decay rate of stochastic differential delay equations. We then establish sufficient conditions ensuring practical stability with a general decay rate of SDDEs by using Lyapunov functionals. Furthermore, we construct suitable Lyapunov functionals for stochastic differential equations with constant and time–varying delay to obtain sufficient conditions ensuring the practical stability with a general decay rate. Finally, based on the established stability criteria, some examples are given to check the correctness of the derived results.

Data availability statement

Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

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Homoclinic orbits for periodic second order Hamiltonian systems with superlinear terms

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Abstract. We obtain the existence of nontrivial homoclinic orbits for nonautonomous second order Hamiltonian systems by using critical point theory under some different superlinear conditions from those previously used in Hamiltonian systems. In particular, an example is given to illustrate our result.

Keywords: second order Hamiltonian systems, homoclinic orbits, superlinear.

2020 Mathematics Subject Classification: 37J45, 37K05, 58E05.

1 Introduction and main result

We consider the following nonautonomous second order Hamiltonian system

$$u''(t) - A(t)u(t) + \nabla H(t, u(t)) = 0, \quad t \in \mathbb{R}, \tag{1.1}$$

where $A(t) \in C(\mathbb{R}, \mathbb{R}^{N\times N})$ is T-periodic $N \times N$ symmetric matrix, and is positive definite uniformly for $t \in [0, T]$; $H(t, u) \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ is T-periodic in t for each $u \in \mathbb{R}^N$ and $\nabla H(t, u)$ denotes its gradient with respect to the u variable. We say that a solution $u(t)$ of (1.1) is homoclinic (with 0) if $u(t) \in C^2(\mathbb{R}, \mathbb{R}^N)$ such that $u(t) \to 0$ and $u'(t) \to 0$ as $|t| \to \infty$. If $u(t) \not\equiv 0$, then $u(t)$ is called a nontrivial homoclinic solution.

In the past decades, many authors have studied the existence and multiplicity of periodic or homoclinic solutions of (1.1). In this paper, we are interested in the case where the nonlinearity ∇H is superlinear as $|u| \to \infty$. Therefore, here we only state some related results. There are some authors [1–4,7,9–11,13–16] who have obtained homoclinic orbits for (1.1) with ∇H being superlinear as $|u| \to \infty$ by critical point theory under the following A–R condition due to Ambrosetti and Rabinowitz (e.g., [2]): there exists a constant $\mu > 2$ such that

$$0 < \mu H(t, u) \le (\nabla H(t, u), u), \quad u \in \mathbb{R}^N \backslash \{0\}, \tag{1.2}$$

where $(\cdot, \cdot)$ denotes the inner product in $\mathbb{R}^N$, and the corresponding norm is denoted by $|\cdot|$. Roughly speaking the role of (1.2) is to insure that all Palais–Smale sequences for the

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corresponding function of (1.1) at the Mountain-Pass level are bounded. For related papers, we refer the readers to see [5,6] and so on.

Let $G(t,u) := \frac{1}{2}(\nabla H(t,u),u) - H(t,u)$. We weaken the condition (1.2) and obtain the following result.

Theorem 1.1. *Assume that the following conditions hold.*

(H_1) $H(t,u) \geq 0$, $\forall (t,u) \in \mathbb{R} \times \mathbb{R}^N$.

(H_2) $|\nabla H(t,u)| \leq c(1+|u|^{p-1})$ *for some* $p > 2$ *and* $c > 0$, $\forall t \in \mathbb{R}$.

(H_3) $|\nabla H(t,u)| = o(|u|)$ *as* $|u| \to 0$ *uniformly in* $t \in \mathbb{R}$.

(H_4) $\frac{H(t,u)}{|u|^2} \to +\infty$ *as* $|u| \to +\infty$ *uniformly in* $t \in \mathbb{R}$.

(H_5) *If* $|u| \leq |v|$, *then* $G(t,u) \leq DG(t,v)$ *for some* $D \geq 1$, $\forall t \in \mathbb{R}$.

Then there is at least one nontrivial homoclinic orbit of (1.1).

Remark 1.2. Note that (H_5) implies $G(t,u) \geq 0$ for all $(t,u) \in \mathbb{R} \times \mathbb{R}^N$. In fact, the condition (H_5) was used firstly to study Schrödinger equations [12], but as far as we know, the condition was not used by other authors to study the seconder order Hamiltonian system (1.1).

Example 1.3. Let

$$H(t,u) = \frac{1}{2}|u|^2 \ln(1+|u|) - \left(\frac{1}{2}|u|^2 - |u| + \ln(1+|u|)\right).$$

A simple calculation shows that H satisfies (H_1)–(H_5) but does not satisfy the superquadratic condition (1.2).

To prove our main result, we need the following theorem developed by Jeanjean [12].

Theorem A ([12]). *Let E be a Banach space equipped with the norm $\|\cdot\|$. Let $J \subset \mathbb{R}^+$ be an interval, and $I_\lambda \in C^1(E,\mathbb{R})$ $(\lambda \in J)$ is defined by*

$$I_\lambda(u) := A(u) - \lambda B(u).$$

If the following conditions hold:

(1) $B(u) \geq 0$ *for all* $u \in E$;

(2) either $A(u) \to +\infty$ *or* $B(u) \to +\infty$ *as* $\|u\| \to +\infty$;

(3) there are two points v_1 *and* v_2 *in* E *such that setting*

$$\Gamma = \{\gamma \in C([0,1],E),\ \gamma(0) = v_1,\ \gamma(1) = v_2\}$$

it holds for all $\lambda \in J$ *that*

$$c_\lambda := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t)) > \max\{I_\lambda(v_1), I_\lambda(v_2)\},$$

then, for almost every $\lambda \in J$, *there is a sequence* $\{u_j\} \subset E$ *such that*

$$\{u_j\} \text{ is bounded in } E, \quad I_\lambda(u_j) \to c_\lambda, \quad I'_\lambda(u_j) \to 0 \text{ in the dual } E^{-1} \text{ of } E.$$

Theorem A means that for a wide class of functionals, having a Mountain-Pass geometry, almost every functionals in this class has a bounded Palais–Smale sequence at the Mountain-Pass level.

The rest of our paper is organized as follows. In Section 2, we give the variational framework of (1.1) and some preliminary lemmas, and then we give the detailed proof of our result.

2 Variational frameworks and the proof of Theorem 1.1

Throughout this paper we denote by $\|\cdot\|_{L^q}$ the usual $L^q(\mathbb{R}, \mathbb{R}^N)$ norm and C for generic constants.

Let $E := H^1(\mathbb{R}, \mathbb{R}^N)$ under the usual norm

$$\|u\|_E^2 = \int_{-\infty}^{+\infty} (|u|^2 + |u'|^2)dt.$$

Thus E is a Hilbert space and it is not difficult to show that $E \subset C^0(\mathbb{R}, \mathbb{R}^N)$, the space of continuous functions u on $\mathbb{R}$ such that $u(t) \to 0$ as $|t| \to \infty$ (see, e.g., [15]). We will seek solutions of (1.1) as critical points of the functional I associated with (1.1) and given by

$$I(u) := \frac{1}{2}\int_{-\infty}^{+\infty} \left(|u'|^2 + (A(t)u, u)\right) dt - \int_{-\infty}^{+\infty} H(t, u)dt.$$

Let

$$\|u\|^2 := \int_{-\infty}^{+\infty} ((A(t)u, u) + |u'|^2)dt,$$

then $\|\cdot\|$ can and will be taken as an equivalent norm on E. Hence I can be written as

$$I(u) := \frac{1}{2}\|u\|^2 - \int_{-\infty}^{+\infty} H(t, u)dt.$$

The assumptions on H imply that $I \in C^1(E, \mathbb{R})$. Moreover, critical points of I are classical solutions of (1.1) satisfying $u'(t) \to 0$ as $|t| \to \infty$. Thus u is a homoclinic solution of (1.1).

In what follows, we always assume that (H_1)–(H_5) hold. Let us show that I has a Mountain-Pass geometry. That is a consequence of the two following results:

Lemma 2.1. *$I(u) = \frac{1}{2}\|u\|^2 + o(\|u\|^2)$ as $u \to 0$.*

Proof. By (H_2) and (H_3), we know for any $\varepsilon > 0$ there exists a $C_\varepsilon > 0$ such that

$$|\nabla H(t, u)| \le \varepsilon|u| + C_\varepsilon|u|^{p-1}, \quad \forall (t, u) \in \mathbb{R} \times \mathbb{R}^N. \tag{2.1}$$

Note that Remark 1.2 implies that $\frac{1}{2}(\nabla H(t, u), u) \ge H(t, u)$, which together with (2.1) implies that

$$|H(t, u)| \le \frac{\varepsilon}{2}|u|^2 + \frac{C_\varepsilon}{2}|u|^p, \quad \forall (t, u) \in \mathbb{R} \times \mathbb{R}^N. \tag{2.2}$$

Thus (2.2) follows from the Sobolev embedding theorem that $\int_{-\infty}^{+\infty} |H(t, u)|dt \le \frac{\varepsilon}{2}\|u\|^2 + C\|u\|^p$, that is, $\int_{-\infty}^{+\infty} H(t, u)dt = o(\|u\|^2)$. The proof is finished. □

Lemma 2.2. *There exists a function $u^0 \in E$ with $u^0 \neq 0$ satisfying $I(u^0) \le 0$.*

Proof. For every $v \in E$ with $v \neq 0$, $|sv| \to +\infty$ as $s \to \infty$. It follows from (H_4) that

$$\lim_{s\to\infty} \frac{H(t, sv)}{s^2} = \lim_{s\to\infty} \frac{H(t, sv)}{s^2|v|^2}|v|^2 = +\infty \quad \text{uniformly in } t \in \mathbb{R}.$$

Thus by (H_1) and Fatou's lemma, we have

$$\lim_{s\to\infty} \int_{-\infty}^{+\infty} \frac{H(t, sv)}{s^2} dt = +\infty.$$

It follows from the definition of I that

$$\lim_{s\to\infty}\frac{I(sv)}{s^2}=\frac{1}{2}\|v\|^2-\lim_{s\to\infty}\int_{-\infty}^{+\infty}\frac{H(t,sv)}{s^2}dt\to-\infty.$$

Thus we can choose $u^0:=sv$ with $|s|$ big enough such that $u^0\in E$ with $u^0\neq 0$ satisfying $I(u^0)\leq 0$. □

We define on E the family of functionals

$$I_\lambda(u)=A(u)-\lambda B(u):=\frac{1}{2}\|u\|^2-\lambda\int_{-\infty}^{+\infty}H(t,u)dt,\quad \lambda\in[1,2].$$

Lemma 2.3. *For almost every $\lambda\in[1,2]$, there exists a sequence $\{u_j\}\subset E$ satisfying*

$$\{u_j\}\ \textit{is bounded in}\ E,\quad 0<\lim_{j\to\infty}I_\lambda(u_j)=c_\lambda,\quad I'_\lambda(u_j)\to 0.$$

Proof. We will use Theorem A to prove this lemma. Obviously, conditions (1) and (2) in Theorem A hold. Next we prove the condition (3) holds. Let

$$\Gamma:=\{\gamma\in C([0,1],E):\ \gamma(0)=0\ \text{and}\ \gamma(1)=u^0\},\quad u^0\ \text{is obtained in Lemma 2.2},$$

$$c_\lambda:=\inf_{\gamma\in\Gamma}\max_{s\in[0,1]}I_\lambda(\gamma(s)),\quad \forall\lambda\in[1,2].$$

Lemma 2.1 implies that $I_\lambda(\gamma(s))>0$ ($\forall\lambda\in[1,2]$) for any small enough $|\gamma(s)|$ (i.e., $\gamma(s)\to 0$), and $I_\lambda(0)=I(0)=0$ ($\forall\lambda\in[1,2]$) by Lemma 2.1, besides, (H_1) and Lemma 2.2 imply that $I_\lambda(u^0)\leq 0$, $\forall\lambda\in[1,2]$. Therefore,

$$c_\lambda:=\inf_{\gamma\in\Gamma}\max_{s\in[0,1]}I_\lambda(\gamma(s))>0=\max\{I_\lambda(0),I_\lambda(u^0)\}.$$

That is the condition (3) of Theorem A holds. An application of Theorem A implies that for almost every $\lambda\in[1,2]$ there exists a sequence $\{u_j\}\subset E$ satisfying

$$\{u_j\}\ \text{is bounded in}\ E,\quad I_\lambda(u_j)\to c_\lambda,\quad I'_\lambda(u_j)\to 0.$$

Obviously, the definition of c_λ and $I_\lambda(u_j)\to c_\lambda$ imply that $0<\lim_{j\to\infty}I_\lambda(u_j)=c_\lambda$. □

Lemma 2.4. *Let $\lambda\in[1,2]$ is fixed. If $\{u_j\}\subset E$ satisfying*

$$\{u_j\}\ \textit{is bounded in}\ E,\quad 0<\lim_{j\to\infty}I_\lambda(u_j)=c_\lambda,\quad I'_\lambda(u_j)\to 0,$$

then up to a subsequence, $u_j\rightharpoonup u_\lambda\neq 0$ with $I'_\lambda(u_\lambda)=0$ and $I_\lambda(u_\lambda)\leq c_\lambda$.

Proof. If $\{u_j\}\subset E$ satisfying

$$\{u_j\}\ \text{is bounded in}\ E,\quad 0<\lim_{j\to\infty}I_\lambda(u_j)=c_\lambda,\quad I'_\lambda(u_j)\to 0,$$

then

$$\lim_{j\to\infty}\int_{-\infty}^{+\infty}G(t,u_j)dt=\lim_{j\to\infty}\left(I_\lambda(u_j)-\frac{1}{2}I'_\lambda(u_j)u_j\right)=\lim_{j\to\infty}I_\lambda(u_j)>0, \tag{2.3}$$

where $G(t,u):=\frac{1}{2}(\nabla H(t,u),u)-H(t,u)$ is defined in Section 1. To continue the proof, we need the following remark:

Remark 2.5. If $\{w_j\} \subset E$ is bounded and vanishing, then $\lim_{j\to\infty} \int_{-\infty}^{+\infty} G(t, w_j)dt = 0$.

Now, we give the proof of Remark 2.5. If $\{w_j\}$ vanishes, then Lion's concentration compactness principle implies $w_j \to 0$ in $L^q(\mathbb{R}, \mathbb{R}^N)$ for all $q \in (2, \infty)$, which together with (2.1), (2.2), $\|w_j\| < \infty$ and the Sobolev embedding theorem implies that

$$\int_{-\infty}^{+\infty} (\nabla H(t, w_j), w_j) \le \int_{-\infty}^{+\infty} |\nabla H(t, w_j)||w_j| \le \varepsilon \|w_j\|_{L^2}^2 + C_\varepsilon \|w_j\|_{L^p}^p \le \varepsilon C\|w_j\|^2 + C_\varepsilon \|w_j\|_{L^p}^p \to 0$$

and

$$\int_{-\infty}^{+\infty} H(t, w_j)dt \le \int_{-\infty}^{+\infty} |H(t, w_j)|dt \le \frac{\varepsilon}{2}\|w_j\|_{L^2}^2 + \frac{C_\varepsilon}{2}\|w_j\|_{L^p}^p \le \frac{\varepsilon}{2}C\|w_j\|^2 + \frac{C_\varepsilon}{2}\|w_j\|_{L^p}^p \to 0.$$

It follows from the definition of $G(t, w)$ that $\lim_{j\to\infty} \int_{-\infty}^{+\infty} G(t, w_j)dt = 0$.

By Remark 2.5, (2.3) and the boundedness of $\{u_j\}$ in E, we know $\{u_j\}$ does not vanish, i.e., there exist $r, \delta > 0$ and a sequence $\{s_j\} \subset \mathbb{R}$ such that

$$\lim_{j\to\infty} \int_{B_r(s_j)} u_j^2 dt \ge \delta,$$

where $B_r(s_j) := [s_j - r, s_j + r]$. Note that $\{u_j\}$ is bounded implies that $u_j \rightharpoonup u_\lambda$ in E and $u_j \to u_\lambda$ in $L^2_{loc}(\mathbb{R}, \mathbb{R}^N)$ (see [8]) after passing to a subsequence, which together with

$$\lim_{j\to\infty} \int_{B_r(s_j)} u_j^2 dt \ge \delta$$

implies $u_\lambda \ne 0$. By the fact I'_λ is weakly sequentially continuous [17] and $I'_\lambda(u_j) \to 0$, we have $I'_\lambda(u_\lambda)v = \lim_{j\to\infty} I'_\lambda(u_j)v = 0$ for all $v \in E$. Therefore, $I'_\lambda(u_\lambda) = 0$.

Next, we still need to prove $I_\lambda(u_\lambda) \le c_\lambda$. Since (H_5) implies $G(t, u) \ge 0$ for all $(t, u) \in \mathbb{R} \times \mathbb{R}^N$, it follows from Fatou's lemma, $I_\lambda(u_j) \to c_\lambda$, $I'_\lambda(u_j) \to 0$ and $I'_\lambda(u_\lambda) = 0$ that

$$\begin{aligned} c_\lambda = \lim_{j\to\infty} \left(I_\lambda(u_j) - \frac{1}{2} I'_\lambda(u_j)u_j \right) &= \lim_{j\to\infty} \lambda \int_{-\infty}^{+\infty} G(t, u_j)dt \\ &\ge \lambda \int_{-\infty}^{+\infty} G(t, u_\lambda)dt \\ &= I_\lambda(u_\lambda) - \frac{1}{2} I'_\lambda(u_\lambda)u_\lambda = I_\lambda(u_\lambda). \end{aligned}$$

The proof is finished. □

By Lemmas 2.3 and 2.4, we deduce the existence of a sequence $\{(\lambda_j, u_j)\} \subset [1, 2] \times E$ such that

- $\lambda_j \to 1$ and $\{\lambda_j\}$ is decreasing.
- $u_j \ne 0$, $I_{\lambda_j}(u_j) \le c_{\lambda_j}$ and $I'_{\lambda_j}(u_j) = 0$. (2.4)

Lemma 2.6. *The sequence $\{u_j\}$ obtained in (2.4) is bounded.*

Proof. Arguing by contradiction, suppose $\|u_j\| \to \infty$. Let $v_j := \frac{u_j}{\|u_j\|}$, then $\|v_j\| = 1$ and thus $v_j \rightharpoonup v$ and $v_j \to v$ a.e. $t \in \mathbb{R}$, up to a subsequence. So either $\{v_j\}$ vanishes or it does not vanish. Next, we shall prove that the two cases are all impossible.

Part 1. The non-vanishing of $\{v_j\}$ is impossible. By contradiction, if $\{v_j\}$ is non-vanishing, that is, there exist $r,\delta > 0$ and a sequence $\{s_j\} \subset \mathbb{R}$ such that

$$\lim_{j\to\infty} \int_{B_r(s_j)} v_j^2 dt \geq \delta. \tag{2.5}$$

Thus it follows from $v_j \to v$ in $L^2_{loc}(\mathbb{R};\mathbb{R}^N)$ that $v \neq 0$.

Since $I'_{\lambda_j}(u_j) = 0$ implies $\|u_j\|^2 = \lambda_j \int_{-\infty}^{+\infty} (\nabla H(t,u_j),u_j)dt$, thus it follows from Remark 1.2 that

$$1 = \lambda_j \int_{-\infty}^{+\infty} \frac{(\nabla H(t,u_j),u_j)}{\|u_j\|^2} dt \geq 2\int_{-\infty}^{+\infty} \frac{H(t,u_j)}{\|u_j\|^2} dt = 2\int_{-\infty}^{+\infty} \frac{H(t,u_j)}{|u_j|^2}|v_j|^2 dt. \tag{2.6}$$

On the other hand, the facts $v_j \to v$ a.e. $t \in \mathbb{R}$, $v \neq 0$ and $\|u_j\| \to \infty$ imply that $|u_j| = |v_j| \cdot \|u_j\| \to +\infty$, which together with (H_4) implies

$$\frac{H(t,u_j)}{|u_j|^2}|v_j|^2 \to +\infty \quad \text{a.e. } t \in \mathbb{R}.$$

It follows from Fatou's lemma that

$$\int_{-\infty}^{+\infty} \frac{H(t,u_j)}{|u_j|^2}|v_j|^2 dt \to +\infty \quad \text{as } j \to \infty,$$

which contradicts with (2.6).

Part 2. The vanishing of $\{v_j\}$ is impossible. If $\{v_j\}$ is vanishing. We define a sequence $\{z_j\} \subset E$ by $z_j = t_j u_j$ with $0 \leq t_j \leq 1$ satisfying

$$I_{\lambda_j}(z_j) := \max_{0\leq t\leq 1} I_{\lambda_j}(tu_j). \tag{2.7}$$

(Here, if for a $j \in N$, t_j defined by (2.7) is not unique we choose the smaller possible value). We claim that

$$\lim_{j\to\infty} I_{\lambda_j}(z_j) = +\infty. \tag{2.8}$$

Seeking a contradiction we assume for all $t_j \in [0,1]$ there exists a positive constant M such that

$$\liminf_{j\to\infty} I_{\lambda_j}(z_j) \leq M. \tag{2.9}$$

Let $\{k_j\}$ be defined by $k_j := \frac{\sqrt{4M}}{\|u_j\|}u_j$. With the relationships of $\{v_j\}$ and $\{k_j\}$, we know $\{k_j\}$ is also bounded and vanishing. Hence Remark 2.5 in Lemma 2.4 implies that $\int_{-\infty}^{+\infty} H(t,k_j)dt \to 0$. Thus for j sufficiently large,

$$I_{\lambda_j}(k_j) = 2M - \lambda_j \int_{-\infty}^{+\infty} H(t,k_j)dt \geq \frac{3}{2}M. \tag{2.10}$$

If we let $t_j := \frac{\sqrt{4M}}{\|u_j\|}$ for j sufficiently large, then $t_j \in [0,1]$. Thus (2.10) contradicts with (2.9). Therefore, (2.8) holds. Note that $I'_{\lambda_j}(z_j)z_j = 0$ for all $j \in N$ by (2.7), thus

$$I_{\lambda_j}(z_j) = I_{\lambda_j}(z_j) - \frac{1}{2}I'_{\lambda_j}(z_j)z_j = \lambda_j \int_{-\infty}^{+\infty} G(t,z_j)dt,$$

which together with (2.8) implies that

$$\int_{-\infty}^{+\infty} G(t, z_j)dt \to +\infty. \tag{2.11}$$

Note that conditions $I_{\lambda_j}(u_j) \leq c_{\lambda_j}$ and $I'_{\lambda_j}(u_j) = 0$ in (2.4) imply that

$$\frac{1}{2}\|u_j\|^2 - \lambda_j \int_{-\infty}^{+\infty} H(t, u_j)dt \leq c_{\lambda_j}, \quad \|u_j\|^2 - \lambda_j \int_{-\infty}^{+\infty} (\nabla H(t, u_j), u_j)dt = 0.$$

It follows from the definition of G that $\int_{-\infty}^{+\infty} G(t, u_j)dt \leq \frac{c_{\lambda_j}}{\lambda_j}$. Clearly, $\frac{c_{\lambda_j}}{\lambda_j}$ is increasing and bounded by $c = c_1$, thus we have

$$\int_{-\infty}^{+\infty} G(t, u_j)dt \leq c, \quad \forall j \in N.$$

It follows from (H_5) that $\int_{-\infty}^{+\infty} G(t, z_j)dt \leq D \int_{-\infty}^{+\infty} G(t, u_j)dt \leq C$, which contradicts with (2.11).

Therefore, the proof is finished by Part 1 and Part 2. □

Proof of Theorem 1.1. Since Lemma 2.6 implies that $\{u_j\}$ is bounded in E, we can assume $u_j \rightharpoonup u$ in E and $u_j \to u$ a.e. $t \in \mathbb{R}$, up to a subsequence. Obviously,

$$I(u_j) = I_{\lambda_j}(u_j) + (\lambda_j - 1) \int_{-\infty}^{+\infty} H(t, u_j)dt. \tag{2.12}$$

We distinguish two cases: either $\limsup_{j\to\infty} I_{\lambda_j}(u_j) > 0$ or $\limsup_{j\to\infty} I_{\lambda_j}(u_j) \leq 0$.

Case 1. If $\limsup_{j\to\infty} I_{\lambda_j}(u_j) > 0$, then (2.12) implies that $\limsup_{j\to\infty} I(u_j) > 0$, besides, the facts $\lambda_j \to 1$ and $I'_{\lambda_j}(u_j) = 0$ (see (2.4)) imply that $I'(u_j) \to 0$, by the similar proof of Lemma 2.4, we can get $u_j \rightharpoonup u \neq 0$ with $I'(u) = 0$, up to a subsequence.

Case 2. If $\limsup_{j\to\infty} I_{\lambda_j}(u_j) \leq 0$, we use the sequence $\{z_j\}$ defined in (2.7). Since $\{u_j\}$ is bounded, $\{z_j\}$ is also bounded. Note that $I'_{\lambda_j}(z_j)z_j = 0$ for all $j \in N$ by (2.7), thus

$$\lambda_j \int_{-\infty}^{+\infty} G(t, z_j)dt = I_{\lambda_j}(z_j) - \frac{1}{2} I'_{\lambda_j}(z_j)z_j = I_{\lambda_j}(z_j). \tag{2.13}$$

Similarly to Lemma 2.1, we have

$$I'_{\lambda_j}(u_j)u_j = \|u_j\|^2 + o(\|u_j\|^2) \quad \text{as } u_j \to 0,$$

uniformly in $j \in N$. Note that $I'_{\lambda_j}(u_j) = 0$, thus there is $\theta > 0$ such that $\|u_j\| \geq \theta$, $\forall j \in N$. Similarly to Lemma 2.1, we also get

$$I_{\lambda_j}(tu_j) = \frac{1}{2}t^2\|u_j\|^2 + o(t^2\|u_j\|^2) \quad \text{as } t \to 0, \quad t \in [0, 1],$$

uniformly in $j \in N$, thus $I_{\lambda_j}(tu_j) > 0$ for small enough t. It follows from $\limsup_{j\to\infty} I_{\lambda_j}(u_j) \leq 0$ that the maximum $I_{\lambda_j}(z_j) := \max_{0\leq t\leq 1} I_{\lambda_j}(tu_j)$ (see (2.7)) can not be obtained at $t = 1$, and there holds $\liminf_{j\to\infty} I_{\lambda_j}(z_j) > 0$. It follows from (2.13) and $\lambda_j \to 1$ that

$$\liminf_{j\to\infty} \int_{-\infty}^{+\infty} G(t, z_j)dt = \liminf_{j\to\infty} I_{\lambda_j}(z_j) > 0,$$

it follows from the fact $\{z_j\}$ is bounded and the Remark 2.5 in Lemma 2.4 that $\{z_j\}$ does not vanish. Therefore, $\{u_j\}$ does not vanish. Moreover, (2.4) implies that

$$I'(u_j)\varphi = I'_{\lambda_j}(u_j)\varphi + (\lambda_j - 1) \int_{-\infty}^{+\infty} (\nabla H(t, u_j), \varphi)\, dt \to 0 \quad \text{as } j \to \infty, \quad \forall \varphi \in E.$$

Therefore, similar to the proof of Lemma 2.4, we can easily get $u \neq 0$ and $I'(u) = 0$. □

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Uniqueness and Liouville type results for radial solutions of some classes of k-Hessian equations

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Abstract. We establish a uniqueness theorem and a Liouville type result for positive radial solutions of some classes of nonlinear autonomous equation with the k-Hessian operator. We also give some interesting qualitative properties of solutions. We provide an approach, based upon a Pohozaev-type identity, that unifies all our results.

Keywords: uniqueness, Liouville-type theorem, radial solutions, k-Hessian operators.

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1 Introduction

Let $\Omega = \mathbb{R}^n$ or $\Omega = B$ a finite ball about the origin. For $1 \le k \le n$ and $u \in C^2(\Omega)$, let $S_k(D^2u)$ denote the k-Hessian operator of u which is defined by

$$S_k(D^2u) = \sigma_k\left(\lambda[D^2u]\right) = \sum_{1\le i_1<\cdots<i_k\le n} \lambda_{i_1}\cdots\lambda_{i_k},$$

where $\lambda[D^2u] = (\lambda_1, \dots, \lambda_n)$ denotes the eigenvalues of the Hessian matrix D^2u of u and σ_k is the k^{th} elementary symmetric polynomial in n variables. This family of partial differential operators includes the Laplace and Monge–Ampère operator, respectively, when $k = 1$ and $k = n$. The study of general k-Hessian operators have many applications in geometry, optimization theory and other related fields. It began with the work of Krylov [15] and Caffarelli, Nirenberg, and Spruck [3]; and was continued by Jacobson [13], Trudinger and Wang [26,27], Tso [28] and Wang [29], among others. In this paper, we consider the Dirichlet boundary value problem

$$\begin{aligned} S_k(D^2(-u)) &= f(u) \quad \text{on } \Omega \\ u &> 0 \quad \text{on } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

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For $\Omega = \mathbb{R}^n$, the Dirichlet boundary condition is understood to mean that

$$\lim_{|x|\to\infty} u(x) = 0.$$

When $k = 1$, problem (1.1) is reduced to

$$\begin{aligned} -\Delta u &= f(u) && \text{on } \Omega \\ u &> 0 && \text{on } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{1.2}$$

The interest in radial solutions is sparked by the well known results of Gidas, Ni, and Nirenberg [11, 12]. The authors showed that any solution of problem (1.2) is necessarily radially symmetric. The uniqueness of ground state solution (radial solution in $\Omega = \mathbb{R}^n$) plays an important role in physics. This importance was mentioned, for instance, by Troy [25] and the references therein for the logarithmic Schrödinger equation $-\Delta u = u \ln u$. Over the past half century, the question of uniqueness of radial solution of problem (1.2) has been explored under a variety of conditions on the non-linearity $f(u)$. For relevant references, see [1,4,6–9,14,16–19,21–25,31]. In general, two types of datum $f(u)$ are considered: $f(u) > 0$ on the whole of the interval $(0,\infty)$, or $f(u) < 0$ on $(0,\gamma)$ and $f(u) > 0$ on (γ,∞) for some $\gamma > 0$. The fundamental examples correspond to $f(u) = u^p$ and $f(u) = u^p - u$, $p > 1$. A natural question to ask is whether uniqueness of radial solutions of problem (1.1) continues to hold for general k-Hessian operators. This question seems to have received almost no attention in the literature and little is known when $2 \leq k \leq n$. Clément, Figueiredo and Mitidieri [5] studied problem (1.1) in $\Omega = B$ and $f(u) = \lambda e^u$, $\lambda > 0$, which is known as Liouville–Gelfand problem in the literature. The authors proved the existence of $\lambda^* > 0$ such that the Liouville–Gelfand problem has exactly two radial solutions if $0 < \lambda < \lambda^*$, a unique radial solution if $\lambda = \lambda^*$, and no radial solutions if $\lambda > \lambda^*$. The question of uniqueness has been explored also by Wei [30] and Zhang [32] over the last few years. Wei apply the argument of Erbe and Tang [8, 9] to prove the uniqueness of radial solutions to problem (1.1) when $1 \leq k < n/2$ and $f(u)$ satisfies some convexity conditions. The argument is based upon a Pohozaev-type identity and the monotone separation techniques. Zhang proved existence and uniqueness of radial solution of problem (1.1) where $f(u)$ is replaced by $\lambda f(u)$ a positive continuous function which satisfies some growth conditions at ∞ and 0, and λ is a large parameter. We also note that a characterization of semi-stable radial solutions of some class of autonomous k-Hessian equation in the unit ball have been studied recently in [20]. We mention here that all the aforementioned authors investigated problem (1.1) in $\Omega = B$ and a non-linearity $f(u)$ which is always assumed to be positive on the whole of $(0,\infty)$.

In the present work, we are concerned with radial solutions of problem (1.1) in $\Omega = B$ or $\Omega = \mathbb{R}^n$ and datum $f(u)$ of the form:

(a) $f_1(u) = u^p - u^k$, $k < p$.

(b) $f_2(u) = u^k - u^p$, $0 < p < k$.

(c) $f_3(u) = u^k(\ln u + \beta)$, $\beta \in \mathbb{R}$.

Here $k \in \{1,\dots,n\}$ is the index of the Hessian operator and p is a parameter. As far as we know this is the first work dealing with non-linearity $f(u)$ which change sign on $(0,\infty)$. Our main results are the following:

Theorem 1.1 (Liouville-type results). *Let $n \geq 1$ and $k \in \{1, \ldots, n\}$.*

1. *If $f(u) = u^p - u^k$, $p > k$ and*

$$p(n-2k) \geq k(n+2), \qquad n > 2k,$$

then problem (1.1) has no radial solutions in $\Omega = B$.

2. *If $f(u) = u^k - u^p$, $0 < p < k$ then problem (1.1) has no radial solutions in $\Omega = \mathbb{R}^n$.*

We mention that similar nonexistence result of radial solutions of problem (1.1) in $\Omega = B$ was established in [5, 28] when $f(u) = u^p$, $p > 1$. In our second main result, we give some interesting qualitative properties of radial solutions of problem (1.1).

Theorem 1.2. *Let $n \geq 1$ and $k \in \{1, \ldots, n\}$. Let $f(u) = f_i(u)$, $i = 1, 2$ or 3.*

1. *Suppose that problem (1.1) admits a radial solution u_j in $\Omega = B_j$ a finite ball of radius b_j, $j = 1, 2$. If $u_1(0) < u_2(0)$ then $b_2 < b_1$, and $u_1(r)$ and $u_2(r)$ intersect exactly once in $(0, b_2)$.*

2. *If problem (1.1) admits a radial solution u in $\Omega = B$ for some ball B then there is no radial solution v in $\Omega = \mathbb{R}^n$ such that $u(0) < v(0)$.*

As an immediate consequence, we have the following uniqueness result in balls.

Corollary 1.3. *Problem (1.1) has at most one radial solution in $\Omega = B$.*

Theorem 1.4 (Uniqueness results in $\Omega = \mathbb{R}^n$). *Let $n \geq 1$ and $k \in \{1, \ldots, n\}$.*

1. *If $f(u) = u^p - u^k$, $k < p$ and $p(n-2k) < k(n+2)$ then problem (1.1) has at most one radial solution in $\Omega = \mathbb{R}^n$. Furthermore, if such a radial solution u exists then*

$$u(x) \leq C \exp\left[-\frac{k+1}{2k}\left(\frac{(p-k)}{(p+1)C_{n-1}^{k-1}}\right)^{\frac{1}{k+1}} |x|^{\frac{2k}{k+1}}\right],$$

where C is a positive constant.

2. *If $f(u) = u^k(\ln u + \beta)$ then the function*

$$u(x) = \exp\left[-\left(\frac{1}{2\, C_{n-1}^{k-1}}\right)^{\frac{1}{k}} \frac{|x|^2}{2} + \frac{n}{2k} - \beta\right]$$

is the unique radial solution of problem (1.1) in $\Omega = \mathbb{R}^n$.

This theorem is an extension of the uniqueness results established in [16, 25] when $k = 1$. We note that the kind of bound of solutions given in the first statement when $k = 1$ has been proved in the celebrated work of Berestycki and Lions [2].

The following result is an immediate consequence of Theorems 1.2 and 1.4.

Corollary 1.5. *For $f(u) = u^k(\ln u + \beta)$, problem (1.1) has no radial solution u such that*

$$u(0) < \exp\left(\frac{n}{2k} - \beta\right)$$

neither in finite balls nor in the whole of $\mathbb{R}^n$.

We provide an approach, based upon a Pohozaev-type identity due to Tso [28], that unifies all our uniqueness and Liouville type results. Our method can be used without any restriction on $k \in \{1, \dots, n\}$, including the Laplace operator when $k = 1$ and also the Monge–Ampère operator when $k = n$.

We finally note that the question of uniqueness for problems of type (1.1) has been also explored with non-local operators. In this context, Frank, Lenzmann, and Silvestre [10] showed the uniqueness of ground state solutions for the non-linear equation

$$(-\Delta)^s u + u - u^p = 0 \quad \text{in } \mathbb{R}^n,$$

where $(-\Delta)^s$ denotes the fractional Laplacian with $s \in (0,1)$ and $p > 1$ a real number. A little is known on the uniqueness of positive radial solutions for the fractional Laplacian.

2 Properties of radial solutions

For radial function $v(x) = v(r)$ with $r = |x|$, we have

$$\begin{aligned} S_k(D^2(-v))(x) &= r^{1-n} \frac{C_{n-1}^{k-1}}{k} \left(r^{n-k}(-v')^k \right)' \\ &= C_{n-1}^{k-1} \left(\frac{-v'}{r} \right)^{k-1} \left(-v'' - \frac{n-k}{k} \frac{v'}{r} \right), \end{aligned}$$

where $C_{n-1}^{k-1} = \frac{(n-1)!}{(k-1)!(n-k)!}$ and $1 \le k \le n$. Therefore, when referring to a radial solution v of problem (1.1) in Ω, we mean a C^2 function $v(|x|) = v(r)$ satisfies

$$\begin{aligned} \frac{C_{n-1}^{k-1}}{k} \left(r^{n-k}(-v')^k \right)' &= r^{n-1} f(v) \quad \text{on } (0,b) \\ v &> 0 \quad \text{on } (0,b) \\ \lim_{r \to b} v(r) &= 0, \end{aligned} \tag{2.1}$$

where b denotes the radius of Ω ($0 < b \le \infty$) and $f = f_i$, $i = 1, 2, 3$, the function defined by

(a) $f_1(v) = v^p - v^k, \ \ k < p.$

(b) $f_2(v) = v^k - v^p, \ \ 0 < p < k.$

(c) $f_3(v) = v^k \ln v + \beta v^k, \ \ \beta \in \mathbb{R}.$

Denote by γ_f the unique zero of f in $(0, \infty)$. We note that $f < 0$ on $(0, \gamma_f)$, $f > 0$ on (γ_f, ∞) and

$$\gamma_{f_1} = \gamma_{f_2} = 1, \quad \gamma_{f_3} = e^{-\beta}.$$

In the following lemma, we shall focus attention on some basic properties of solutions of problem (2.1).

Lemma 2.1. *If v is a solution of (2.1), then $v'(0) = 0$, $v(0) > \gamma_f$ and $v'(r) < 0$ for $0 < r < b$.*

Proof. The results are well known when $k = 1$, see for instance [24]. So we can assume that $k \ge 2$. We first write equation (2.1) in the following equivalent form

$$C_{n-1}^{k-1} \left(\frac{-v'}{r} \right)^{k-1} \left(-v'' - \frac{n-k}{k} \frac{v'}{r} \right) = f(v). \tag{2.2}$$

If $v'(0) \neq 0$ then by letting r tend to 0 we obtain $\infty = f(v(0))$, a contradiction. Hence $v'(0) = 0$. Suppose that there exists $r_0 \in (0, b)$ such that $v'(r_0) = 0$ and $v' \neq 0$ on $(0, r_0)$. Then integrate the equation in (2.1) from 0 to r_0 to get

$$0 = \frac{C_{n-1}^{k-1}}{k}\left[r^{n-k}(-v')^k\right]_0^{r_0} = \int_0^{r_0} r^{n-1} f(v)\, dr. \tag{2.3}$$

On the other hand, since $v'(r_0) = 0$, it follows from (2.2) that $f(v(r_0)) = 0$, yielding $v(r_0) = \gamma_f$ the unique zero of f other than 0. Since v is strictly monotone on the interval $(0, r_0)$, this implies that $f(v)$ does not change sign on $(0, r_0)$, which contradicts (2.3). Therefore $v' \neq 0$ on $(0, b)$, and hence $v' < 0$ as desired. Suppose that $v(0) \leq \gamma_f$. Since v decreases on $(0, b)$ and $f < 0$ on $(0, \gamma_f)$, this implies that $f(v) < 0$ on $(0, b)$ and thus

$$\int_0^r t^{n-1} f(v) dt < 0.$$

But this is impossible since

$$\int_0^r t^{n-1} f(v)\, dt = \frac{C_{n-1}^{k-1}}{k} r^{n-k}(-v')^k > 0.$$

Hence $v(0) > \gamma_f$. This completes the proof. ☐

Let

$$F(v) = \int_0^v f(t) dt.$$

One easily checks that

$$F_1(v) = \frac{v^{p+1}}{p+1} - \frac{v^{k+1}}{k+1}, \qquad F_2(v) = \frac{v^{k+1}}{k+1} - \frac{v^{p+1}}{p+1}, \qquad F_3(v) = \frac{v^{k+1}}{k+1}\left(\ln v + \beta - \frac{1}{k+1}\right).$$

We denote by γ_F the unique zero of F in $(0, \infty)$. It can be easily calculated

$$\gamma_{F_1} = \left(\frac{p+1}{k+1}\right)^{\frac{1}{p-k}}, \qquad \gamma_{F_2} = \left(\frac{k+1}{p+1}\right)^{\frac{1}{k-p}}, \qquad \gamma_{F_3} = e^{\frac{1}{k+1} - \beta}.$$

We note that

$$\gamma_{F_i} > \gamma_{f_i}, \qquad i = 1, 2, 3.$$

For a given solution v of problem (2.1), we define

$$E(r, v) := C_{n-1}^{k-1} r^{1-k}(-v')^{k+1} + (k+1)F(v), \qquad 0 \leq r < b. \tag{2.4}$$

Lemma 2.2. *If v is a solution of problem (2.1), then $v(0) > \gamma_F$ and*

$$E(r, v) > 0, \qquad 0 \leq r < b. \tag{2.5}$$

Proof. Let v be a solution of (2.1). By using (2.2), a straightforward computation gives

$$\frac{d}{dr}E(r, v) = -(n + k(n-2))\frac{C_{n-1}^{k-1}}{k}\frac{(-v')^{k+1}}{r^k}.$$

Since $v' < 0$ on $(0, b)$, this implies that $E(r, v)$ decreases on $(0, b)$, yielding

$$E(r, v) > \lim_{r \to b} E(r, v) = C_{n-1}^{k-1} \lim_{r \to b} r^{1-k}(-v')^{k+1} \geq 0.$$

For $r = 0$, we obtain $0 < E(0, v) = (k+1)F(v(0))$ from which we conclude that $v(0) > \gamma_F$. ☐

For $i \in \{1,2,3\}$, let

$$G_i(t) = (n-2k)tf_i(t) - n(k+1)F_i(t). \tag{2.6}$$

A straightforward computation shows that

$$G_1(t) = C(n,k,p)t^{p+1} + 2kt^{k+1},$$
$$G_2(t) = -2kt^{k+1} - C(n,k,p)t^{p+1}$$

and

$$G_3(t) = C(n,k,p)t^{k+1}\ln t + \left(\frac{n}{k+1} - 2\beta k\right)t^{k+1},$$

where

$$C(n,k,p) = n - 2k - \frac{n(k+1)}{p+1} \tag{2.7}$$

with the obvious convention $p = k$ when $f(t) = t^k \ln t + \beta t^k$.

Remark 2.3.

1. We note that $C(n,k,p) < 0$ when $f = f_2$ or $f = f_3$, that is, when $0 < p \le k$. However, the mapping $p \to C(n,k,p)$ can change sign when $p > k$.
2. If $C(n,k,p) \ge 0$ then G_1 is positive on $(0,\infty)$.
3. If $C(n,k,p) < 0$ then G_1 is positive on $(0,\gamma_{G_1})$ and negative on (γ_{G_1},∞), where
$$\gamma_{G_1} = \left(\frac{2k}{-C(n,k,p)}\right)^{\frac{1}{p-k}}.$$
4. For $i = 2,3$, G_i is positive on $(0,\gamma_{G_i})$ and negative on (γ_{G_i},∞), where
$$\gamma_{G_2} = \left(\frac{-C(n,k,p)}{2k}\right)^{\frac{1}{k-p}}, \qquad \gamma_{G_3} = \exp\left(-\beta + \frac{n}{2k(k+1)}\right).$$
5. It is worth noting that
$$\gamma_{F_1} < \gamma_{G_1}, \qquad \gamma_{G_2} < \gamma_{F_2}$$
and
$$\gamma_{F_3} < \gamma_{G_3} \quad \text{if } n-2k > 0, \qquad \gamma_{G_3} < \gamma_{F_3} \quad \text{if } n-2k < 0.$$
6. A straightforward computation shows
$$\frac{G(t)}{t^{k+1}} - \frac{G(s)}{s^{k+1}} = C(n,k,p)\left(\frac{f(t)}{t^k} - \frac{f(s)}{s^k}\right), \qquad t,s > 0. \tag{2.8}$$
This identity will be crucial in the proof of uniqueness results.

For a solution v of problem (2.1) in $(0,b)$, let

$$P(r,v) = r^n\left[C_{n-1}^{k-1}r^{1-k}(-v')^{k+1} + (k+1)F(v) - C_{n-1}^{k-1}\frac{n-2k}{k}v\left(\frac{-v'}{r}\right)^k\right].$$

We note that the radial form of the Pohozaev identity for k-Hessian equation established in [28] states that

$$P(r,v) = -\int_0^r t^{n-1}G(v)dt. \tag{2.9}$$

Lemma 2.4. *If v is a solution of problem (2.1) in $(0,b)$, $0<b<\infty$, then $v(0)>\gamma_G$ and*

$$P(r,v)>0, \qquad r\in(0,b). \tag{2.10}$$

Proof. Let v be a solution of problem (2.1) in $(0,b)$. By (2.9),

$$\frac{d}{dr}P(r,v)=-r^{n-1}G(v).$$

By Remarks 2.3, G is positive on $(0,\gamma_G)$ and negative on (γ_G,∞). Suppose that $v(0)\le\gamma_G$. Then $G(v)>0$ on $(0,b)$ which implies that $P(r,v)$ decreases on $(0,b)$. Thus,

$$0=P(0,v)>\lim_{r\to b}P(r,v)=C_{n-1}^{k-1}b^{n+1-k}(-v'(b))^{k+1}\ge 0,$$

a contradiction. Hence $v(0)>\gamma_G$. Since v is decreasing on $(0,b)$, the condition $v(0)>\gamma_G$ implies that $G(v)$ is positive-negative on $(0,b)$, yielding $P(r,v)$ increases-decreases on $(0,b)$. Since $P(0,v)=0$ and $P(b,v)\ge 0$, we immediately deduce that $P(r,v)>0$ for every $r\in(0,b)$. This completes the proof. □

We now provide some interesting estimates of radial solutions of problem (1.1) in $\Omega=\mathbb{R}^n$.

Lemma 2.5.

(a) Let $f(v)=v^p-v^k$, $k<p$. If v is a solution of problem (2.1) in $(0,\infty)$, then there exist two constants $C_1,C_2>0$ such that

$$v(r)\le C_1\exp\left[-\frac{k+1}{2k}\left(\frac{(p-k)}{(p+1)C_{n-1}^{k-1}}\right)^{\frac{1}{k+1}}r^{\frac{2k}{k+1}}\right] \tag{2.11}$$

and

$$-rv'(r)\le C_2\exp\left[-\frac{k+1}{2k}\left(\frac{(p-k)}{(p+1)C_{n-1}^{k-1}}\right)^{\frac{1}{k+1}}\left(\frac{r}{2}\right)^{\frac{2k}{k+1}}\right]. \tag{2.12}$$

(b) Let $f(v)=v^k\ln v+\beta v^k$, $\beta\in\mathbb{R}$. If v is a solution of problem (2.1) in $(0,\infty)$, then there exist two constants $C_1,C_2>0$ such that

$$v(r)\le C_1\exp\left[-\left(\frac{1}{(k+1)C_{n-1}^{k-1}}\right)^{\frac{1}{k+1}}r^{\frac{2k}{k+1}}\right] \tag{2.13}$$

and

$$-rv'(r)\le C_2\exp\left[-\left(\frac{1}{(k+1)C_{n-1}^{k-1}}\right)^{\frac{1}{k+1}}\left(\frac{r}{2}\right)^{\frac{2k}{k+1}}\right]. \tag{2.14}$$

Proof. We shall use the same lines of reasoning in the proof of both cases. It is for this reason that we omit the proof of the second statement. Let $f(v)=v^p-v^k$, $k<p$. Let v be a solution of problem (2.1) in $(0,\infty)$. We denote by b_1 the unique positive constant such that $v(b_1)=\gamma_f(=1)$. By (2.5), we have $E(r,v)>0$. This means that

$$C_{n-1}^{k-1}r^{1-k}(-v')^{k+1}>-(k+1)F(v)=v^{k+1}\left(1-\frac{k+1}{p+1}v^{p-k}\right).$$

Since v decreases, we then obtain, for every $r \geq b_1$,

$$(-v')^{k+1} \geq \frac{(p-k)}{(p+1)C_{n-1}^{k-1}} r^{k-1} v^{k+1},$$

or equivalently,

$$-\frac{v'}{v} \geq \left(\frac{(p-k)}{(p+1)C_{n-1}^{k-1}}\right)^{\frac{1}{k+1}} r^{\frac{k-1}{k+1}}.$$

Integrating from b_1 to r gives

$$-\ln v \geq \frac{k+1}{2k}\left(\frac{(p-k)}{(p+1)C_{n-1}^{k-1}}\right)^{\frac{1}{k+1}} \left(r^{\frac{2k}{k+1}} - b_1^{\frac{2k}{k+1}}\right).$$

Thus

$$v(r) \leq C \exp\left[-\frac{k+1}{2k}\left(\frac{(p-k)}{(p+1)C_{n-1}^{k-1}}\right)^{\frac{1}{k+1}} r^{\frac{2k}{k+1}}\right], \qquad r \geq b_1,$$

where

$$C = \exp\left[\frac{k+1}{2k}\left(\frac{(p-k)}{(p+1)C_{n-1}^{k-1}}\right)^{\frac{1}{k+1}} b_1^{\frac{2k}{k+1}}\right].$$

Since v is continuous on the whole of $\mathbb{R}_+$, this yields the existence of a constant $C_1 > 0$ such that (2.11) holds for all $r \geq 0$. The estimate (2.12) follows from the fact that v is convex on (b_1, ∞) together with (2.11). Indeed, v is convex on (b_1, ∞) since

$$v'' = -\frac{n-k}{k}\frac{v'}{r} - \frac{1}{C_{n-1}^{k-1}}\left(\frac{r}{-v'}\right)^{k-1} f(v) > 0.$$

Thus, for every $b_1 < t < r$, we have

$$\frac{v(r) - v(t)}{r-t} \leq v'(r).$$

For $t = \frac{r}{2}$, we get

$$2v(r) - 2v(\frac{r}{2}) \leq rv'(r).$$

Multiplying by

$$\exp\left[\frac{k+1}{2k}\left(\frac{(p-k)}{(p+1)C_{n-1}^{k-1}}\right)^{\frac{1}{k+1}} \left(\frac{r}{2}\right)^{\frac{2k}{k+1}}\right]$$

and then letting r tend to ∞, we conclude using (2.11) that

$$-\infty < \lim_{r\to\infty} rv'(r) \exp\left[\frac{k+1}{2k}\left(\frac{(p-k)}{(p+1)C_{n-1}^{k-1}}\right)^{\frac{1}{k+1}} \left(\frac{r}{2}\right)^{\frac{2k}{k+1}}\right].$$

This completes the proof of the first statement. □

We mention here that this kind of bound when $k = 1$ and $f(u) = u^p - u$ has been proved in the celebrated work [2].

Lemma 2.6. *Assume that* $f(v) = f_i(v)$, $i \in \{1,3\}$. *If* v *is a solution of* (2.1) *in* $(0,\infty)$ *then* $v(0) > \gamma_G$,

$$\lim_{r\to\infty} P(r,v) = 0 \tag{2.15}$$

and

$$P(r,v) > 0.$$

Proof. The property (2.15) follows from the estimates (2.11) and (2.12) when $f(v) = f_1(v)$, and the estimates (2.13) and (2.14) when $f(v) = f_3(v)$. The rest of the proof is similar to that of Lemma 2.4. □

3 Proof of Theorem 1.1

1. Assume that $f(v) = v^p - v^k$, $k < p$ and $p(n-2k) \geq k(n+2)$. Suppose that problem (1.1) has a radial solution v in $\Omega = B$ a finite ball of radius b. By (2.9), we have

$$\frac{d}{dr}P(r,v) = -r^{n-1}G_1(v).$$

$G_1(v) > 0$ by Remarks 2.3 since $C(n,k,p) \geq 0$ by hypothesis. Thus $P(r,v)$ is decreasing on $(0,b)$, and hence

$$0 = P(0,v) > P(b,v) = C_{n-1}^{k-1} b^{n+1-k}(-v'(b))^{k+1} \geq 0,$$

a contradiction. Therefore, problem (1.1) has no radial solutions in $\Omega = B$.

2. Assume that $f(v) = v^k - v^p$, $0 < p < k$. Striving for a contradiction, suppose that problem (1.1) admits a radial solution v in $\mathbb{R}^n$. By (2.5), we have $E(r,v) > 0$. Thus,

$$C_{n-1}^{k-1} r^{1-k}(-v')^{k+1} \geq -(k+1)F(v) = v^{p+1}\left(\frac{k+1}{p+1} - v^{k-p}\right).$$

Let $b_1 \in (0,\infty)$ so that $v(b_1) = 1$. Since v is decreasing, it follows that, for every $r \geq b_1$,

$$(-v')^{k+1} \geq \frac{(k-p)}{(p+1)C_{n-1}^{k-1}} r^{k-1} v^{p+1},$$

or equivalently,

$$-v'v^{-\frac{p+1}{k+1}} \geq \left(\frac{(k-p)}{(p+1)C_{n-1}^{k-1}}\right)^{\frac{1}{k+1}} r^{\frac{k-1}{k+1}}.$$

Integrate from b_1 to r gives

$$1 - v^{\frac{k-p}{k+1}} \geq \frac{k-p}{2k}\left(\frac{(k-p)}{(p+1)C_{n-1}^{k-1}}\right)^{\frac{1}{k+1}}\left(r^{\frac{2k}{k+1}} - b_1^{\frac{2k}{k+1}}\right).$$

Now, let r tend to ∞ we obtain $1 \geq \infty$, a contradiction. This completes the proof. □

4 Proof of Theorem 1.2

Let $k \in \{1, \cdots, n\}$ be the index of the Hessian operator. Let $f(v)$ be a function defined on $[0, \infty[$ which takes one of the following forms:

(a) $f_1(v) = v^p - v^k, \;\; k < p$ and $p(n-2k) < k(n+2)$.

(b) $f_2(v) = v^k - v^p, \;\; 0 < p < k$.

(c) $f_3(v) = v^k \ln v + \beta v^k, \;\; \beta \in \mathbb{R}$.

Assume that v and w are two solutions of problem (2.1) in $(0, b)$ and $(0, c)$ respectively. We shall prove that if $v(0) < w(0)$ and $b < \infty$ then $c < \infty$ and

$$c < b.$$

Furthermore, v and w intersect exactly once in $(0, c)$. The proof will be divided into a sequence of lemmas. Assume that $v(0) < w(0)$ and $b < \infty$. Arguing by contradiction, suppose that

$$c \geq b.$$

For $0 \leq r \leq b$, let

$$Y(r) = vw' - v'w.$$

Lemma 4.1. *Let v and w be two positive solutions of the equation*

$$\frac{C_{n-1}^{k-1}}{k}\left(r^{n-k}(-u')^k\right)' = r^{n-1} f(u).$$

If $v(0) < w(0)$ then $Y(r) < 0$ as long as $v(r) < w(r)$.

Proof. By writing the above equation in the following equivalent form

$$C_{n-1}^{k-1}\left(\frac{-v'}{r}\right)^{k-1}\left(-v'' - \frac{n-k}{k}\frac{v'}{r}\right) = f(v), \tag{4.1}$$

we see that

$$\lim_{r \to 0} \frac{-v'}{r} = \left(\frac{k}{n\, C_{n-1}^{k-1}} f(v(0))\right)^{\frac{1}{k}}.$$

Thus

$$\lim_{r \to 0} \frac{Y(r)}{r} = \left(\frac{k}{n\, C_{n-1}^{k-1}}\right)^{\frac{1}{k}} \left[w\left(f(v)\right)^{\frac{1}{k}} - v\left(f(w)\right)^{\frac{1}{k}}\right](0).$$

Together with the fact that $f(t)/t^k$ increases on $(0, \infty)$ and $v(0) < w(0)$, this implies that $Y < 0$ on some neighbourhood $(0, \varepsilon)$. Arguing by contradiction, suppose that there exists $a \in (0, b)$ such that $Y(a) = 0$, $Y < 0$ on $(0, a)$ and $v(a) < w(a)$. It is obvious that $Y'(a) \geq 0$. On the other hand, using the fact that v and w satisfy equation (4.1), we easily obtain

$$Y'(r) = \frac{r^{k-1}}{C_{n-1}^{k-1}} vw \left(\frac{f(v)}{v(-v')^{k-1}} - \frac{f(w)}{w(-w')^{k-1}}\right) - \frac{n-k}{kr} Y(r).$$

Since $Y(a) = 0$, this implies that

$$Y'(a) = \frac{a^{k-1}}{C_{n-1}^{k-1}} v(a) w(a) \left(\frac{v}{-v'}\right)^{k-1}(a) \left(\frac{f(v)}{v^k} - \frac{f(w)}{w^k}\right)(a) < 0,$$

in contradiction with $Y'(a) \geq 0$. This completes the proof. □

It is clear that Y must vanish on $(0,b]$ since $Y(0)=0$, $Y<0$ near 0 and $Y(b)\geq 0$. Let τ denote the first zero of Y in $(0,b]$. Therefore,

$$Y(\tau)=0 \quad \text{and} \quad Y(r)<0, \qquad 0<r<\tau.$$

Define

$$Z(r)=\frac{C_{n-1}^{k-1}}{k}r^{n-k}\left[(-w')^k-\left(-v'\frac{w}{v}\right)^k\right], \qquad 0\leq r\leq b. \tag{4.2}$$

Lemma 4.2.

(a) For every $r\in(0,b)$, we have

$$Z'(r)+C_{n-1}^{k-1}r^{n-k}Y\frac{w^{k-1}(-v')^k}{v^{k+1}}=r^{n-1}\left[f(w)-\left(\frac{w}{v}\right)^k f(v)\right]. \tag{4.3}$$

(b) For every $r\in(0,\tau)$, we have

$$-w'Z+C_{n-1}^{k-1}r^{n-k}Y\frac{(-v'w)^k}{v^{k+1}}>0. \tag{4.4}$$

Proof. (a) A straightforward computation using equation (2.1) gives

$$\begin{aligned} Z'(r)&=r^{n-1}\left[f(w)-\left(\frac{w}{v}\right)^k f(v)\right]-C_{n-1}^{k-1}r^{n-k}\left(\frac{w}{v}\right)'\left(\frac{w}{v}\right)^{k-1}(-v')^k\\ &=r^{n-1}\left[f(w)-\left(\frac{w}{v}\right)^k f(v)\right]-C_{n-1}^{k-1}r^{n-k}Y\frac{w^{k-1}(-v')^k}{v^{k+1}}. \end{aligned}$$

(b) For $r\in(0,\tau)$ we have $-vw'>-v'w>0$. Thus,

$$\begin{aligned} Z(r)&=\frac{C_{n-1}^{k-1}}{k}\frac{r^{n-k}}{v^k}\left[(-vw')^k-(-wv')^k\right]\\ &\geq -C_{n-1}^{k-1}\frac{r^{n-k}}{v^k}(-wv')^{k-1}Y. \end{aligned}$$

The last inequality follows immediately from the fact that $t^k-s^k\geq k(t-s)s^{k-1}$ for $t>s>0$. Hence,

$$\begin{aligned} -w'Z+C_{n-1}^{k-1}r^{n-k}Y\frac{(-wv')^k}{v^{k+1}}&\geq C_{n-1}^{k-1}r^{n-k}Y\frac{(-wv')^{k-1}}{v^{k+1}}(w'v-wv')\\ &=C_{n-1}^{k-1}r^{n-k}\frac{(-wv')^{k-1}}{v^{k+1}}Y^2>0. \end{aligned}$$ □

For $0\leq r\leq b$, we define

$$\Phi(r)=P(r,w)-\left(\frac{w}{v}\right)^{k+1}P(r,v).$$

We now show by two methods that $\Phi(\tau)>0$ and $\Phi(\tau)\leq 0$ which is the required contradiction.

Lemma 4.3. $\Phi(\tau)>0$.

Proof. By differentiation, we have

$$\Phi'(r) = r^{n-1}\left[\left(\frac{w}{v}\right)^{k+1} G(v) - G(w)\right] - (k+1)\left(\frac{w}{v}\right)'\left(\frac{w}{v}\right)^{k} P(r,v).$$

Using (2.8) and (4.3), we obtain

$$\begin{aligned}\Phi'(r) &= C(n,k,p)r^{n-1}w\left[\left(\frac{w}{v}\right)^{k} f(v) - f(w)\right] - (k+1)\left(\frac{w}{v}\right)'\left(\frac{w}{v}\right)^{k} P(r,v)\\ &= -C(n,k,p)\left[wZ'(r) + C_{n-1}^{k-1} r^{n-k} Y \frac{(-v'w)^k}{v^{k+1}}\right] - (k+1)Y\frac{w^k}{v^{k+2}}P(r,v).\end{aligned}$$

Seeing that $Z(0) = Z(\tau) = 0$, an integration by parts yields

$$\int_0^\tau wZ'dr = [wZ]_0^\tau - \int_0^\tau w'Zdr = -\int_0^\tau w'Zdr.$$

Thus

$$\Phi(\tau) = -C(n,k,p)\int_0^\tau \left[-w'Z + C_{n-1}^{k-1} r^{n-k} Y \frac{(-v'w)^k}{v^{k+1}}\right] dr - (k+1)\int_0^\tau \frac{Yw^k}{v^{k+2}}P(r,v)dr.$$

Now, $\Phi(\tau) > 0$ follows from (2.10) and (4.4) together with the fact that $Y < 0$ on $(0,\tau)$ and $C(n,k,p) < 0$. □

On the other hand, we have

Lemma 4.4. $\Phi(\tau) \le 0$.

Proof. For $r \in (0,b)$, we have

$$\begin{aligned}\Phi(r) = C_{n-1}^{k-1} r^{n+1-k}\left[(-w')^{k+1} - \left(-\frac{wv'}{v}\right)^{k+1}\right] - (k+1)r^n w^{k+1}\left[\frac{F(v)}{v^{k+1}} - \frac{F(w)}{w^{k+1}}\right]\\ - (n-2k)\frac{C_{n-1}^{k-1}}{k} r^{n-k} w\left[(-w')^k - \left(-\frac{wv'}{v}\right)^k\right].\end{aligned}$$

Thus

- If $\tau < b$ then
$$\Phi(\tau) = -(k+1)\tau^n w(\tau)^{k+1}\left[\frac{F(v)}{v^{k+1}} - \frac{F(w)}{w^{k+1}}\right](\tau).$$
By Lemma 4.1, $Y(\tau) = 0$ implies $v(\tau) \ge w(\tau)$. Since $F(t)/t^{k+1}$ increases on $(0,\infty)$, it follows that
$$\Phi(\tau) \le 0.$$

- If $\tau = b$, then $Y(b) = 0$ implies that $w(b) = 0$. If, in addition, $v'(b) \neq 0$ then
$$\Phi(\tau) = \lim_{r\to b}\Phi(r) = P(b,w) - \left(\frac{w'(b)}{v'(b)}\right)^{k+1} P(b,v) = 0.$$
If $v'(b) = 0$ then we must have $w'(b) = 0$. Otherwise, since the ratio v/w increases on $(0,b)$, we obtain
$$0 < \frac{v(0)}{w(0)} < \lim_{r\to b}\frac{v}{w}(r) = \frac{v'(b)}{w'(b)} = 0.$$

a contradiction. Thus $P(b,w) = P(b,v) = 0$, and hence

$$|\Phi(\tau)| = \lim_{r\to b} |\Phi(r)| \leq \lim_{r\to b} \left[P(r,w) + \left(\frac{w(0)}{v(0)}\right)^{k+1} P(r,v) \right] = 0.$$

Hence $\Phi(\tau) \leq 0$ as desired. □

We have shown that if $v(0) < w(0)$ and $b < \infty$ then $c < b$, this holds for any solutions v and w of problem (2.1) in $(0,b)$ and $(0,c)$ respectively. This implies that the set $\mathcal{A} := \{r \in (0,c);\ v(r) = w(r)\}$ is nonempty. Arguing by contradiction, suppose that the set $\mathcal{A}$ contains two points $r_1 < r_2$. Thus

$$\frac{v}{w}(r_1) = 1 = \frac{v}{w}(r_2).$$

This yields the existence of $\tau \in (r_1, r_2)$ such that

$$\left(\frac{v}{w}\right)'(\tau) = 0,$$

or equivalently,

$$\left(v'w - vw'\right)(\tau) = 0.$$

Using Lemma 4.1, we can assume that $v'w - vw' < 0$ on the interval $(0,\tau)$. The contradiction follows now again from Lemmas 4.3 and 4.4 above. Hence v and w intersect exactly once in $(0,c)$. This completes the proof of Theorem 1.2. □

5 Proof of Theorem 1.4

Let $k \in \{1, \cdots, n\}$ be the index of the Hessian operator. Let $f(v)$ be of the form:

(a) $f_1(v) = v^p - v^k,\ \ k < p$ and $p(n-2k) < k(n+2)$.

(b) $f_3(v) = v^k \ln v + \beta v^k,\ \ \beta \in \mathbb{R}$.

We shall first prove that problem (2.1) has at most one solution in $(0,\infty)$. The proof proceeds along the same lines as the proof of Theorem 1.2. We shall be brief here and just outline the proof. Let the notation be as in the preceding paragraph. Arguing by contradiction, suppose that problem (2.1) has two solutions v and w in $(0,\infty)$. We can assume that $v(0) < w(0)$. For $r \geq 0$, let

$$Y(r) = w'v - v'w$$

and

$$\Phi(r) = P(r,w) - \left(\frac{w}{v}\right)^{k+1} P(r,v).$$

By Lemma 4.1, we have $Y(r) < 0$ as long as $v(r) < w(r)$.

- Assume that there exists $\tau \in (0,\infty)$ such that

$$Y < 0 \quad \text{on } (0,\tau) \quad \text{and} \quad Y(\tau) = 0.$$

In this case the required contradiction follows from Lemmas 4.3 and 4.4 above.

- Assume that $Y < 0$ on the whole of $(0, \infty)$. Since the ratio w/v decreases on $(0, \infty)$, it follows that
$$|\Phi(r)| \leq |P(r,w)| + \left(\frac{w(0)}{v(0)}\right)^{k+1} |P(r,v)|.$$
This implies together with (2.15) that
$$\lim_{r\to\infty} |\Phi(r)| = 0.$$
On the other hand, following the same steps as in the proof of Lemma 4.3, we conclude that
$$\lim_{r\to\infty} \Phi(r) > 0$$
provided we have
$$\lim_{r\to\infty} w(r)Z(r) = 0, \tag{5.1}$$
where Z is given by (4.2). So it remains to show (5.1). We see that w/v is decreasing on $(0, \infty)$ since $Y < 0$, and so
$$\frac{w}{v} \leq \frac{w(0)}{v(0)}.$$
Thus
$$|Z(r)| \leq \frac{C_{n-1}^{k-1}}{k} r^{n-k} \left[(-w')^k + \left(\frac{w(0)}{v(0)}\right)^k (-v')^k\right]$$
from which we obtain (5.1) using (2.11) and (2.12) when $f(v) = f_1(v)$, and the estimates (2.13) and (2.14) when $f(v) = f_3(v)$. This completes the proof of the uniqueness result.

The upper bound of solutions stated in the first statement of the theorem is given by (2.11). So it remains to show that the function
$$u(r) := \exp\left[-\left(\frac{1}{2\,C_{n-1}^{k-1}}\right)^{\frac{1}{k}} \frac{r^2}{2} + \frac{n}{2k} - \beta\right]$$
is a solution of equation (2.1) in $(0, \infty)$ when $f(u) = u^k(\ln u + \beta)$. Let
$$v(r) = e^{-a\frac{r^2}{2} - b}.$$
Then $v'(r) = -arv(r)$, yielding
$$\begin{aligned}
\frac{C_{n-1}^{k-1}}{k}\left(r^{n-k}(-v')^k\right)' &= \frac{C_{n-1}^{k-1}}{k} a^k \left(r^n v^k\right)' \\
&= \frac{C_{n-1}^{k-1}}{k} a^k n r^{n-1} v^k + C_{n-1}^{k-1} a^k r^n v' v^{k-1} \\
&= 2a^k C_{n-1}^{k-1} v^k \left(\frac{n}{2k} - a\frac{r^2}{2}\right) r^{n-1}.
\end{aligned}$$
Now by taking $2a^k C_{n-1}^{k-1} = 1$ and $b = \beta - \frac{n}{2k}$, we obtain
$$\frac{C_{n-1}^{k-1}}{k}\left(r^{n-k}(-v')^k\right)' = r^{n-1} v^k (\ln v + \beta).$$
This completes the proof of the theorem. □

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Spatial wave solutions for generalized atmospheric Ekman equations

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Abstract. In this paper, we use Prandtl mixing-length theory and semiempirical theory to extend the classical problem of the wind in the steady atmospheric Ekman layer with constant eddy viscosity. New generalized atmospheric Ekman equations are established and qualitative properties of the corresponding ODEs are studied. Spatial wave solutions results for the nonlinear and implicit equations with different nonlinearities are presented.

Keywords: generalized atmospheric Ekman equations, nonlinear and implicit equations, spatial wave solutions.

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1 Introduction

Lamina sublayer, surface layer and Ekman layer are three important parts for the atmospheric boundary layer [24,26]. In particular, the Ekman layer covers ninety percent of the atmospheric boundary layer, which is driven by a three-way balance among frictional effects, pressure gradient and the influence of the Coriolis force in non-equatorial regions [13,24,33]. However, this balance breaks down in equatorial regions, where the Coriolis effect due to the Earths rotation vanishes, the Coriolis force changes sign across the Equator, so the nonlinear effects have to be accounted for [4–8,11,23,25].

Ekman was the first to formula and analyse a mathematical model which describes the behavior of wind-generated steady surface currents [13], the theory is the basis for our understanding of wind-driven currents, and is also relevant for the air flow in the atmospheric boundary layer.

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We consider a rotating framework with the origin at a point on the Earth's surface, with the x axis chosen horizontally due east, the y axis horizontally due north, and the z axis upward, it is known that the standard Ekman equations are given by

$$\begin{cases} f(v-v_g) = -\frac{\partial}{\partial z}(k\frac{\partial u}{\partial z}), \\ f(u-u_g) = \frac{\partial}{\partial z}(k\frac{\partial v}{\partial z}), \end{cases} \tag{1.1}$$

where $u = u(t,x,y,z)$, $v = v(t,x,y,z)$ are the components of the wind in the x and y directions respectively, P is the atmospheric pressure, ρ is the reference density, $f = 2\Omega \sin\phi$ is the Coriolis parameter at the fixed latitude ϕ, u_g and v_g are the corresponding geostrophic wind components, k denotes the eddy viscosity [22].

Ekman derived the flow from this model and obtained three characteristics, two of which have been shown to hold in the general case of depth-dependent eddy viscosity. However, regarding of the value of the deflection angle of the surface flow from the wind direction, some data in non-equatorial regions predicted significant differences [9,17,34]. It is natural to attribute this difference to the assumption of constant vorticity. Some results have been made on the explicit formula of the solution to (1.1) with the hight-dependent eddy viscosity and the classic boundary conditions $u = v = 0$ at $z = 0$ and $u \to u_g$, $v \to v_g$ for $z \to \infty$ for the atmospheric Ekman equations [9,10,16,19,20,32]. With respect to wind-driven surface current, one can refer to [1–3,12,30,31] for the depth-dependent eddy viscosity and the corresponding boundary conditions.

Noting that (1.1) is formulated by omitting the turbulent fluxes, which has obvious limitations. Recently, Guan et al. [18] introduced a new nonhomogeneous model containing turbulent flux terms, which improved the classical model proposed in [24]. Further, in this paper, we propose the following generalized model

$$\begin{cases} f(v-v_g) = -\frac{\partial}{\partial z}(k\frac{\partial u}{\partial z}) + 2l^2 u\frac{\partial u}{\partial x}, \\ f(u-u_g) = \frac{\partial}{\partial z}(k\frac{\partial v}{\partial z}) + 2l^2 v\frac{\partial v}{\partial y}, \end{cases} \tag{1.2}$$

where l is a constant number. We emphasize that (1.2) is a generalization of the standard Ekman equations since the turbulent flux term is considered. Comparing with the previous extension model in [18], (1.2) has a totally different and specific turbulent flux terms. In [18], the turbulent flux is assumed to be a function of height, but here we use semi-empirical method and assume turbulent flux to be a function of u, v and their partial derivatives, which are more reasonable than the turbulent fluxes only depending on the high z in [18] and also makes the current model more complex.

Note that explicit solution and dynamical properties of atmospheric Ekman flows with boundary conditions have been presented extensively. There are still very few contributions on the modified Ekman equation. In particular, periodic solutions and Hyers–Ulam stability are reported in a modified model in [18] by using the theory of ordinary differential equations and hyperbolic matrix theory. In this paper, we consider spatial wave solutions of (1.2), which satisfy certain ODEs, and we study qualitative properties of this corresponding ODEs. This is a novelty of this paper.

The rest of the paper is organized as follows. New generalized atmospheric Ekman equations are derived in Section 2. Section 3 deals with spatial wave solutions of (1.2). We study qualitative properties of the corresponding ODEs determining these solutions. Involving also other terms not just linear ones into (2.5), we continue our analysis in Section 4 with more general ODEs. Finally, (2.4) is investigated in Section 5. The obtained spatial wave ODEs are nonlinear and implicit, so their study is difficult. There are still many open challenging problems for further research. These aspects are presented in Section 6.

2 Model description

In the local Cartesian coordinate system, the earth's surface is approximately regarded as a plane, and the curvature term can be omitted, so the Ekman layer is governed by the following equations, see [24,26]

$$\begin{cases} \frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + 2\Omega\sin\phi v - 2\Omega\cos\phi w + F_{rx}, \\ \frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial y} - 2\Omega\sin\phi u + F_{ry}, \\ \frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial z} - g + 2\Omega\cos\phi u + F_{rz}, \end{cases} \tag{2.1}$$

where

$$\begin{cases} F_{rx} = v[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}], \\ F_{ry} = v[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}], \\ F_{rz} = v[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}], \end{cases}$$

and $v = \frac{\mu}{\rho}$ is the kinematic viscosity coefficient [24], $u = u(t,x,y,z)$, $v = v(t,x,y,z)$ and $w = w(t,x,y,z)$ are the components of the wind in the x,y and z directions respectively. Besides, $\overrightarrow{U} = (u,v,w)$ satisfies the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \overrightarrow{U}) = 0. \tag{2.2}$$

For a wide range of air movements, $w \ll u,v$ [33], so we assume $w = 0$, kinematic viscosity coefficient is negligible in the Ekman layer, so $F_{rx} = 0,\ F_{ry} = 0$, then (2.1) reduces to

$$\begin{cases} \frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + 2\Omega\sin\phi v, \\ \frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial y} - 2\Omega\sin\phi u. \end{cases} \tag{2.3}$$

Note that the Boussinesq approximation is an important simplifications in (2.2) and (2.3) for application in the boundary layer, in this approximation, density ρ in (2.2) and (2.3) are replaced by a constant mean value (everywhere except in the buoyancy term in the vertical momentum equation, see [24]). Clearly, (2.2) becomes to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

We assume that the variable consists of the mean value and the turbulence value, for example, $u = \overline{u} + u'$, the corresponding mean values are indicated by overbars and the fluctuating component by primes.

Under the Boussinesq approximation, the mean velocity fields satisfy the following continuity equations [24]

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z} = 0,$$

we separate each dependent variable into mean and fluctuating parts, and substitute into the chain rule of the differentiation, then we obtain

$$\overline{\frac{Du}{Dt}} = \frac{\partial \overline{u}}{\partial t} + \frac{\partial}{\partial x}(\overline{u'u'}) + \frac{\partial}{\partial y}(\overline{u'v'}) + \frac{\partial}{\partial z}(\overline{u'w'}),$$

where
$$\frac{\overline{D}}{Dt} = \frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x} + \overline{v}\frac{\partial}{\partial y} + \overline{w}\frac{\partial}{\partial z}$$
is the rate of change following the mean motion.

Using the above relationships and (2.3), the mean equations thus have the following form:
$$\begin{cases} \frac{\overline{D}\overline{u}}{Dt} = -\frac{1}{\rho}\frac{\partial \overline{P}}{\partial x} + f\overline{v} - [\frac{\partial \overline{u'u'}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y} + \frac{\partial \overline{u'w'}}{\partial z}], \\ \frac{\overline{D}\overline{v}}{Dt} = -\frac{1}{\rho}\frac{\partial \overline{P}}{\partial y} - f\overline{u} - [\frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{v'v'}}{\partial y} + \frac{\partial \overline{v'w'}}{\partial z}]. \end{cases}$$
We omit the inertial acceleration terms because they are much smaller that the Cariolis force and pressure gradient force terms for midlatitude synoptic-scale motions [24], using the geostrophic balance, we obtain
$$\begin{cases} f(\overline{v} - \overline{v_g}) - [\frac{\partial \overline{u'u'}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y} + \frac{\partial \overline{u'w'}}{\partial z}] = 0, \\ -f(\overline{u} - \overline{u_g}) - [\frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{v'v'}}{\partial y} + \frac{\partial \overline{v'w'}}{\partial z}] = 0. \end{cases}$$
By the Flux-Gradient theory [24], we get
$$\begin{cases} \overline{u'w'} = -k\frac{\partial u}{\partial z}, \\ \overline{v'w'} = -k\frac{\partial v}{\partial z}, \end{cases}$$
where k is the eddy viscosity coefficient, then we obtain
$$\begin{cases} f(\overline{v} - \overline{v_g}) = -\frac{\partial}{\partial z}(k\frac{\partial \overline{u}}{\partial z}) + \frac{\partial \overline{u'u'}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y}, \\ f(\overline{u} - \overline{u_g}) = \frac{\partial}{\partial z}(k\frac{\partial \overline{v}}{\partial z}) + \frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{v'v'}}{\partial y}, \end{cases}$$
usually we omit the terms $\frac{\partial \overline{u'u'}}{\partial x}$, $\frac{\partial \overline{u'v'}}{\partial y}$, $\frac{\partial \overline{u'v'}}{\partial x}$ and $\frac{\partial \overline{v'v'}}{\partial y}$ because they are small in comparison to the terms $\frac{\partial \overline{u'w'}}{\partial z}$, $\frac{\partial \overline{v'w'}}{\partial z}$, but here we retain $\frac{\partial \overline{u'u'}}{\partial x}$ and $\frac{\partial \overline{v'v'}}{\partial y}$ and obtain
$$\begin{cases} f(\overline{v} - \overline{v_g}) = -\frac{\partial}{\partial z}(k\frac{\partial \overline{u}}{\partial z}) + \frac{\partial \overline{u'u'}}{\partial x}, \\ f(\overline{u} - \overline{u_g}) = \frac{\partial}{\partial z}(k\frac{\partial \overline{v}}{\partial z}) + \frac{\partial \overline{v'v'}}{\partial y}. \end{cases}$$
By the Prandtl mixing-length theory [24], we have $u' = -l'\frac{\partial \overline{u}}{\partial z}$, $v' = -l'\frac{\partial \overline{v}}{\partial z}$, so
$$\begin{cases} f(\overline{v} - \overline{v_g}) = -\frac{\partial}{\partial z}(k\frac{\partial \overline{u}}{\partial z}) + l^2\frac{\partial}{\partial x}(\frac{\partial \overline{u}}{\partial z})^2, \\ f(\overline{u} - \overline{u_g}) = \frac{\partial}{\partial z}(k\frac{\partial \overline{v}}{\partial z}) + l^2\frac{\partial}{\partial y}(\frac{\partial \overline{v}}{\partial z})^2. \end{cases} \tag{2.4}$$
where $l = \overline{l'}$ is the mean mixing-length.

Now replacing $\overline{u}$, $\overline{v}$, $\overline{u_g}$ and $\overline{v_g}$ by u, v, u_g and v_g, respectively, and we assume that
$$\frac{\partial u}{\partial z} \approx u, \quad \frac{\partial v}{\partial z} \approx v, \tag{2.5}$$
by semiempirical theory, one can obtain
$$\begin{cases} f(v - v_g) = -\frac{\partial}{\partial z}(k\frac{\partial u}{\partial z}) + 2l^2u\frac{\partial u}{\partial x}, \\ f(u - u_g) = \frac{\partial}{\partial z}(k\frac{\partial v}{\partial z}) + 2l^2v\frac{\partial v}{\partial y}. \end{cases}$$

3 Spatial wave solutions for (1.2)

Assuming that k is a nonzero constant, (1.2) becomes

$$\begin{cases} f(v-v_g) = -k\frac{\partial^2 u}{\partial z^2} + 2l^2 u\frac{\partial u}{\partial x}, \\ f(u-u_g) = k\frac{\partial^2 v}{\partial z^2} + 2l^2 v\frac{\partial v}{\partial y}. \end{cases} \tag{3.1}$$

We are looking for spatial wave solutions of (3.1) as follows

$$\begin{aligned} u(x,y,z) &= U(\alpha x + \beta y + z), \\ v(x,y,z) &= V(\alpha x + \beta y + z), \end{aligned} \tag{3.2}$$

where α and β are parameters. Then we get

$$\begin{aligned} f(V-v_g) &= -kU'' + 2\alpha l^2 UU', \\ f(U-u_g) &= kV'' + 2\beta l^2 VV'. \end{aligned} \tag{3.3}$$

For $\alpha = 0$ and $\beta = 0$, we get the standard Ekman equations. Taking

$$X = \begin{bmatrix} U \\ V \\ U' \\ V' \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix},$$

(3.3) becomes

$$X' = F(X) = \begin{bmatrix} x_3 \\ x_4 \\ \frac{f}{k}(v_g - x_2) + \frac{2\alpha l^2}{k}x_1x_3 \\ \frac{f}{k}(x_1 - u_g) - \frac{2\beta l^2}{k}x_2x_4 \end{bmatrix}. \tag{3.4}$$

Note (3.4) has a unique equilibrium

$$X_0 = \begin{bmatrix} u_g \\ v_g \\ 0 \\ 0 \end{bmatrix}$$

and its Jacobian matrix is

$$DF(X_0) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{f}{k} & \frac{2\alpha l^2 u_g}{k} & 0 \\ \frac{f}{k} & 0 & 0 & -\frac{2\beta l^2 v_g}{k} \end{bmatrix} \tag{3.5}$$

with the characteristic polynomial

$$\chi(\lambda) = \lambda^4 + \lambda^3\frac{2\beta l^2 v_g - 2\alpha l^2 u_g}{k} - \lambda^2\frac{4\alpha\beta l^4 u_g v_g}{k^2} + \frac{f^2}{k^2}. \tag{3.6}$$

Lemma 3.1. *χ defined in (3.6) has no pure imaginary roots.*

Proof. Suppose $\lambda = \imath\omega$, $\omega \in \mathbb{R}$ is a root of χ, then we get

$$0 = \chi(\imath\omega) = \omega^4 - \imath\omega^3 \frac{2\beta l^2 v_g - 2\alpha l^2 u_g}{k} + \omega^2 \frac{4\alpha\beta l^4 u_g v_g}{k^2} + \frac{f^2}{k^2}.$$

So

$$\omega^4 + \omega^2 \frac{4\alpha\beta l^4 u_g v_g}{k^2} + \frac{f^2}{k^2} = 0,$$
$$\omega^3 \frac{2\beta l^2 v_g - 2\alpha l^2 u_g}{k} = 0.$$

Clearly $\omega \neq 0$, then $\beta v_g - \alpha u_g = 0$, so

$$\omega^4 + \omega^2 \frac{4\alpha^2\beta^2 l^4 u_g v_g}{k^2} + \frac{f^2}{k^2} = 0,$$

which is not possible. The proof is finished. □

Consequently, $DF(X_0)$ is hyperbolic. When $\alpha = \beta = 0$, we get

$$DF(X_0) = A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{f}{k} & 0 & 0 \\ \frac{f}{k} & 0 & 0 & 0 \end{bmatrix}$$

with (3.6) of the form

$$\lambda^4 + \frac{f^2}{k^2} = 0$$

and possessing four eigenvalues

$$\sqrt{\frac{f}{2k}} + \sqrt{\frac{f}{2k}}\imath, \quad \sqrt{\frac{f}{2k}} - \sqrt{\frac{f}{2k}}\imath, \quad -\sqrt{\frac{f}{2k}} + \sqrt{\frac{f}{2k}}\imath, \quad -\sqrt{\frac{f}{2k}} - \sqrt{\frac{f}{2k}}\imath.$$

Thus there are two eigenvalues of A on both sides of the imaginary axis. By Lemma 3.1 this property remains for any $DF(X_0)$ with arbitrary α and β. Consequently, X_0 has a 2-dimensional stable manifold $W^s_{X_0}$. So we have a 4-parameterized family of functions

$$X(\alpha, \beta, s_1, s_2; t)$$

such that

$$X(t) = X(\alpha, \beta, s_1, s_2; t)$$

is a solution of (3.4) with $X(0) \in W^s_{X_0}$. Then $X(t) \to X_0$ exponentially fast as $t \to \infty$. Summarizing, we arrive at the following result.

Theorem 3.2. *Functions*

$$u_{\alpha,\beta,s_1,s_2}(x, y, z) = U(\alpha, \beta, s_1, s_2; \alpha x + \beta y + z),$$
$$v_{\alpha,\beta,s_1,s_2}(x, y, z) = V(\alpha, \beta, s_1, s_2; \alpha x + \beta y + z) \tag{3.7}$$

give a 4-parameterized family of solutions for (3.1) with

$$u_{\alpha,\beta,s_1,s_2}(x, y, z) \to u_g,$$
$$v_{\alpha,\beta,s_1,s_2}(x, y, z) \to v_g,$$

as $x + y + z \to \infty$, $x \geq 0$, $y \geq 0$, $z \geq 0$, $\alpha > 0$ and $\beta > 0$. In general, the above asymptotic properties hold for $\alpha x + \beta y + z \to \infty$.

For $l = 0$, $u_g = v_g = 1, \frac{f}{2k} = 1$, we have an implicit solution [20]

$$\begin{aligned} u_{10,0,1,-1}(x,y,z) &= e^{-(10x+z)}\sin(10x+z) - e^{-(10x+z)}\cos(10x+z) + 1, \\ v_{10,0,1,-1}(x,y,z) &= -e^{-(10x+z)}\sin(10x+z) - e^{-(10x+z)}\cos(10x+z) + 1, \end{aligned} \tag{3.8}$$

visualizing their spatial wave forms on Figure 3.1.

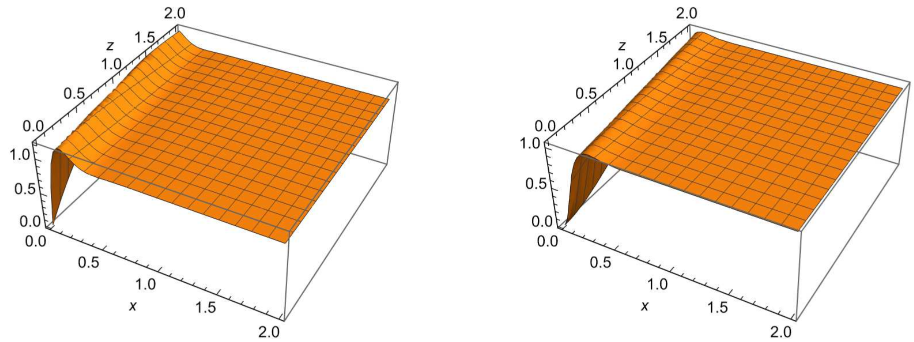

Figure 3.1: Solutions of (3.8): left $u_{10,0,1,-1}(x,y,z)$, right $v_{10,0,1,-1}(x,y,z)$

We need the next observation.

Lemma 3.3. *If $\alpha > 0$ and $\beta > 0$, then*

$$L(X) = \frac{k}{2}(x_4^2 - x_3^2) - fx_1x_2 + fv_gx_1 + fu_gx_2$$

is a Lyapunov function of (3.4) *on the set*

$$\Pi = \{x_1 \geq 0, x_2 \geq 0\} \subset \mathbb{R}^4.$$

Proof. For any solution $X(t) \in \Pi$ of (3.4), we compute

$$\begin{aligned} L(X(t))' &= k(x_4(t)x_4'(t) - x_3(t)x_3'(t)) - fx_1'(t)x_2(t) - fx_1(t)x_2'(t) + fv_gx_1'(t) + fu_gx_2'(t) \\ &= x_4(t)(f(x_1(t) - u_g) - 2\beta l^2 x_2(t)x_4(t)) - x_3(t)(f(v_g - x_2(t)) \\ &\quad +2\alpha l^2 x_1(t)x_3(t)) - fx_3(t)x_2(t) - fx_1(t)x_4(t) \\ &\quad +fv_gx_3(t) + fu_gx_4(t) = -2\beta l^2 x_2(t)x_4(t)^2 - 2\alpha l^2 x_1(t)x_3(t)^2 \leq 0. \end{aligned}$$

The proof is finished. □

Now we present a uniqueness result for nonnegative solutions in Theorem 3.2.

Theorem 3.4. *If $\alpha > 0$ and $\beta > 0$, then any bounded solution $X(t) \in \Pi$, $\forall t \geq 0$ of* (3.4) *tends to X_0 as $t \to \infty$, i.e., $X(t) \in W^s_{X_0}$, $\forall t \geq 0$.*

Proof. Set

$$\dot{L}(X) = -2\beta l^2 x_2 x_4^2 - 2\alpha l^2 x_1 x_3^2.$$

The ω-limit of set of $X(0)$ is denoted by $\omega(X(0))$. The largest invariant subset of the set

$$\{X \in \Pi \mid \dot{L}(X) = 0\}$$

is denoted by M. A simple analysis shows that $M = \{X_0\}$. Next, by Lemma 3.3 and [21, Theorem 9.22], we know $\omega(X(0)) = \{X_0\}$. The proof is finished. □

Next, using (3.2), we consider that a solution depends on z. Now we study opposite, that is we take

$$u(x,y,z) = U(\alpha x + \beta y),\\ v(x,y,z) = V(\alpha x + \beta y). \tag{3.9}$$

Then (3.3) is transformed to

$$f(V - v_g) = 2\alpha l^2 UU',\\ f(U - u_g) = 2\beta l^2 VV', \tag{3.10}$$

which is an implicit ODE (see [15,29]). Now (3.10) gives

$$\begin{aligned} 0 &= \alpha UU'(U - u_g) - \beta VV'(V - v_g) \\ &= \frac{d}{dt}\left[\alpha\left(\frac{U^3}{3} - \frac{U^2}{2}u_g\right) - \beta\left(\frac{V^3}{3} - \frac{V^2}{2}v_g\right)\right], \end{aligned}$$

thus implicit solutions are given by

$$H(U,V) = \alpha\left(\frac{U^3}{3} - \frac{U^2}{2}u_g\right) - \beta\left(\frac{V^3}{3} - \frac{V^2}{2}v_g\right) = c \in \mathbb{R}. \tag{3.11}$$

Theorem 3.5. *There is a family of periodic spatial solutions* (3.9) *of* (3.1) *given by the equation* (3.11) *under the following condition*

$$\alpha\beta u_g v_g < 0. \tag{3.12}$$

Proof. The gradient of $H(U,V)$ is

$$\nabla H(U,V) = \begin{bmatrix} \alpha U(U - u_g) \\ -\beta V(V - v_g) \end{bmatrix},$$

so

$$\begin{bmatrix} u_g \\ v_g \end{bmatrix}, \tag{3.13}$$

is a critical point of $H(U,V)$ with the Hessian

$$\operatorname{Hess} H(u_g, v_g) = \begin{bmatrix} \alpha u_g & 0 \\ 0 & -\beta v_g \end{bmatrix}.$$

Clearly, if (3.12) holds then (3.13) is a strong local extreme of $H(U,V)$ and it is a center for (3.10). If $\alpha\beta u_g v_g > 0$, then (3.13) is a non-degenerate saddle point of $H(U,V)$ and it is hyperbolic. Consequently, (3.11) are periodic for suitable $c \approx \frac{\beta v_g^3 - \alpha u_g^3}{6}$ under (3.12). The proof is finished. □

Implicit ODE (3.10) has the same phase portrait as the following ODE

$$a' = \beta b(b - v_g),\\ b' = \alpha a(a - u_g) \tag{3.14}$$

when $a \neq 0$ and $b \neq 0$. (3.14) has 4 equilibria $(0,0)$, $(u_g,0)$, $(0,v_g)$ and (u_g,v_g) which are either centers or hyperbolic. Thus, implicit ODE (3.10) has impasse solutions, so solutions terminating in singularities $U = 0$ or $V = 0$ in finite time [15,29], which are impasse spatial solutions of (3.1). This is demonstrated on Figure 3.2.

We end this section with the following notes.

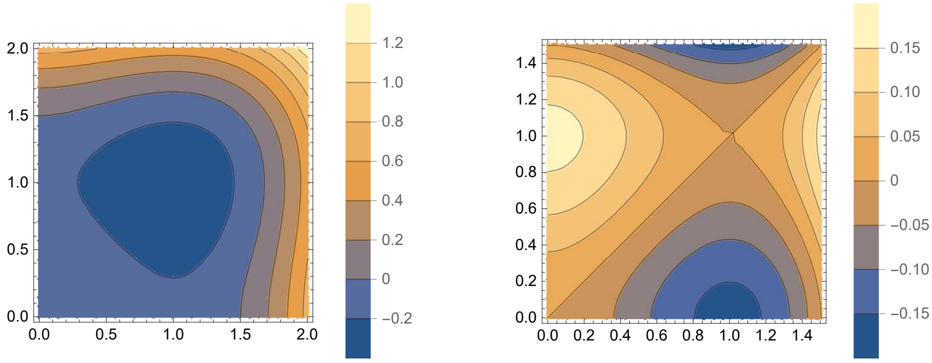

Figure 3.2: Periodic and impasse solutions of (3.10): left $\alpha = -\beta = u_g = v_g = 1$, right $\alpha = \beta = u_g = v_g = 1$

1. We can reduce parameters in (3.3) by taking

$$\begin{aligned} U(t) &= u_g + \frac{\sqrt{fk}}{2l^2} U_1 \left(\sqrt{\frac{f}{k}} t \right), \\ V(t) &= v_g + \frac{\sqrt{fk}}{2l^2} V_1 \left(\sqrt{\frac{f}{k}} t \right) \end{aligned} \tag{3.15}$$

to get

$$\begin{aligned} V_1 &= -U_1'' + \alpha U_1 U_1', \\ U_1 &= V_1'' + \beta V_1 V_1'. \end{aligned} \tag{3.16}$$

We do not consider (3.16) until instead of (3.3) to keep the role of other parameters in the above results.

2. Let (3.16) have a T-periodic solution. Then integrating (3.16) we have

$$\begin{aligned} \int_0^T V_1(t)dt &= \int_0^T (-U_1''(t) + \alpha U_1(t) U_1'(t))dt = \left[-U'(t) + \alpha \frac{U_1(t)^2}{2} \right]_{t=0}^{t=T} = 0, \\ \int_0^T U_1(t)dt &= \int_0^T (V_1''(t) + \beta V_1(t) V_1'(t))dt = \left[-V'(t) + \beta \frac{V_1(t)^2}{2} \right]_{t=0}^{t=T} = 0. \end{aligned}$$

So we can use Wirtinger inequality [27, p. 9] to derive

$$\begin{aligned} \|U_1''\|_2 &\leq \|V_1\|_2 + |\alpha| \|U_1 U_1'\|_2 \leq \frac{T^2}{4\pi^2} \|V_1''\|_2 + \frac{T}{2\pi} |\alpha| \|U_1\|_\infty \|U_1''\|_2, \\ \|V_1''\|_2 &\leq \|U_1\|_2 + |\beta| \|V_1 V_1'\|_2 \leq \frac{T^2}{4\pi^2} \|U_1''\|_2 + \frac{T}{2\pi} |\beta| \|V_1\|_\infty \|V_1''\|_2, \end{aligned} \tag{3.17}$$

where

$$\|U\|_2 = \sqrt{\int_0^T U(t)^2 dt}, \quad \|U\|_\infty = \max_{t \in [0,T]} |U(t)|.$$

Adding the two equations of (3.17), we arrive at

$$\|U_1''\|_2 + \|V_1''\|_2 \leq \left(\frac{T^2}{4\pi^2} + \frac{T}{2\pi} \max\{|\alpha| \|U_1\|_\infty, |\beta| \|V_1\|_\infty\} \right) \|U_1''\|_2 + \|V_1''\|_2. \tag{3.18}$$

So if

$$\|U_1''\|_2 + \|V_1''\|_2 \neq 0,$$

then (3.18) implies

$$1 \leq \frac{T^2}{4\pi^2} + \frac{T}{2\pi}\max\{|\alpha|\|U_1\|_\infty, |\beta|\|V_1\|_\infty\},$$

which leads to

$$\left(\sqrt{4 + (\max\{|\alpha|\|U_1\|_\infty, |\beta|\|V_1\|_\infty\})^2} - \max\{|\alpha|\|U_1\|_\infty, |\beta|\|V_1\|_\infty\}\right)\pi \leq T. \tag{3.19}$$

Using (3.15) and (3.19), we obtain

Theorem 3.6. *A period T of any nonconstant T-periodic solution of (3.3) with*

$$\max_{t\in[0,T]} |U(t) - u_g| \leq M, \quad \max_{t\in[0,T]} |V(t) - v_g| \leq N$$

satisfying

$$\pi\sqrt{\frac{k}{f}}\left(\sqrt{4 + \frac{4l^2}{fk}(\max\{|\alpha|M, |\beta|N\})^2} - \frac{2l^2}{\sqrt{fk}}\max\{|\alpha|M, |\beta|N\}\right) \leq T.$$

Results similar to Theorem 3.6 are presented in [14].

3. We are focusing in this paper on the case for fixed $f \neq 0$. This leads to a hyperbolic-like dynamics. On the other hand, if $f = 0$, then (3.4) has a form

$$\begin{aligned} x_1' &= x_3, \\ x_2' &= x_4, \\ x_3' &= \frac{2\alpha l^2}{k}x_1x_3, \\ x_4' &= -\frac{2\beta l^2}{k}x_2x_4. \end{aligned} \tag{3.20}$$

Clearly

$$\Sigma = \{x_3 = x_4 = 0\}$$

is a fixed point set of (3.20) with Jacobian matrices

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{2\alpha l^2 x_1}{k} & 0 \\ 0 & 0 & 0 & -\frac{2\beta l^2 x_2}{k} \end{bmatrix} \tag{3.21}$$

possessing eigenvalues

$$0, \quad 0, \quad \frac{2\alpha l^2 x_1}{k}, \quad -2\beta l^2 x_2$$

and the corresponding eigenvectors

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ \frac{2\alpha l^2 x_1}{k} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ -\frac{2\beta l^2 x_2}{k} \end{bmatrix}$$

for $x_1 \neq 0$ and $x_2 \neq 0$. (3.20) is decoupling to

$$\begin{aligned} x_1'' &= \frac{2\alpha l^2}{k} x_1 x_1', \\ x_2'' &= -\frac{2\beta l^2}{k} x_2 x_2'. \end{aligned} \tag{3.22}$$

Integrating (3.22), we derive

$$\begin{aligned} x_1' &= \frac{\alpha l^2}{k} x_1^2 + c_1, \\ x_2' &= -\frac{\beta l^2}{k} x_2^2 + c_2 \end{aligned} \tag{3.23}$$

(3.23) is solvable and leading to these cases [28]:

i) $c_1 = 0$:

$$\begin{aligned} x_1(t) &= \frac{k x_1(0)}{k - \alpha l^2 x_1(0) t}, \\ x_3(t) &= \frac{\alpha k l^2 x_1(0)^2}{(k - \alpha l^2 x_1(0) t)^2} \end{aligned}$$

is a blow-up solution.

ii) $\frac{\alpha l^2 c_1}{k} < 0$:

$$\begin{aligned} x_1(t) &= \frac{x_1(0)\sqrt{-\frac{\alpha l^2 c_1}{k}} + c_1 \tanh\left(\sqrt{-\frac{\alpha l^2 c_1}{k}}t\right)}{\sqrt{-\frac{\alpha l^2 x c_1}{k}} - \frac{\alpha l^2 x_1(0)}{k} \tanh\left(\sqrt{-\frac{\alpha l^2 c_1}{k}}t\right)}, \\ x_3(t) &= \frac{-\frac{\alpha l^2 c_1}{k}\left(c_1 + \frac{\alpha l^2 x_1(0)^2}{k}\right)}{\left(\sqrt{-\frac{\alpha l^2 c_1}{k}} \cosh\left(\sqrt{-\frac{\alpha l^2 c_1}{k}}t\right) - \frac{\alpha l^2 x_1(0)}{k} \sinh\left(\sqrt{-\frac{\alpha l^2 c_1}{k}}t\right)\right)^2} \end{aligned}$$

is an asymptotic solution for $|x_1(0)| < \sqrt{-\frac{\alpha l^2}{k c_1}}$ connecting two points on Σ:

$$\begin{aligned} \lim_{t \to -\infty} x_1(t) = \pm\sqrt{-\frac{c_1 k}{\alpha l^2}}, \quad \lim_{t \to \infty} x_1(t) = \mp\sqrt{-\frac{c_1 k}{\alpha l^2}}, \\ \lim_{t \to -\infty} x_3(t) = 0, \quad \lim_{t \to \infty} x_3(t) = 0 \end{aligned}$$

is a blow-up solution for $|x_1(0)| \geq \sqrt{-\frac{\alpha l^2}{k c_1}}$.

iii) $\frac{\alpha l^2 c_1}{k} > 0$:

$$\begin{aligned} x_1(t) &= \frac{x_1(0)\sqrt{\frac{\alpha l^2 c_1}{k}} + c_1 \tan\left(\sqrt{\frac{\alpha l^2 c_1}{k}}t\right)}{\sqrt{\frac{\alpha l^2 x c_1}{k}} - \frac{\alpha l^2 x_1(0)}{k} \tan\left(\sqrt{\frac{\alpha l^2 c_1}{k}}t\right)}, \\ x_3(t) &= \frac{\frac{\alpha l^2 c_1}{k}\left(c_1 + \frac{\alpha l^2 x_1(0)^2}{k}\right)}{\left(\sqrt{\frac{\alpha l^2 c_1}{k}} \cos\left(\sqrt{\frac{\alpha l^2 c_1}{k}}r\right) - \frac{\alpha l^2 x_1(0)}{k} \sin\left(\sqrt{\frac{\alpha l^2 c_1}{k}}t\right)\right)^2} \end{aligned}$$

is a blow-up solution.

iv) Similar formulas hold for $x_2(t)$ and $x_4(t)$ by exchanging (α, c_1) with $(-\beta, c_2)$.

v) Note

$$c_1 = x_3(0) - \frac{\alpha l^2}{k} x_1(0)^2, \quad c_2 = x_4(0) + \frac{\beta l^2}{k} x_2(0)^2.$$

Clearly blow up solutions persist in (3.4) for $f \neq 0$ small. It will be our next study the asymptotic solutions ii).

Finally, we note that (3.4) for l small has a hyperbolic structure on bounded sets due to the Hartman–Grobman theorem. On the other hand, when l large, say $l = \epsilon^{-1/2} > 0$ then (3.3) becomes

$$\begin{aligned} \epsilon f(V - v_g) &= -\epsilon k U'' + 2\alpha U U', \\ \epsilon f(U - u_g) &= \epsilon k V'' + 2\beta V V'. \end{aligned}$$

Scaling

$$U(t) = U_1(t/\epsilon), \quad V(t) = V_1(t/\epsilon),$$

we get

$$\begin{aligned} \epsilon^2 f(V_1 - v_g) &= -k U_1'' + 2\alpha U_1 U_1', \\ \epsilon^2 f(U_1 - u_g) &= k V_1'' + 2\beta V_1 V_1'. \end{aligned} \tag{3.24}$$

(3.24) has a form of (3.22) for $\epsilon = 0$, so we can apply above results and remarks. We see that (3.4) has different dynamics for l small and large.

4 General nonlinearities

Assuming that (2.5) involves also other terms not just linear ones, we suppose that

$$\left(\frac{\partial u}{\partial z}\right)^2 \approx p(u), \quad \left(\frac{\partial v}{\partial z}\right)^2 \approx q(v)$$

for $p, q \in C^2(\mathbb{R}, \mathbb{R})$. Then instead of (1.2), we obtain

$$\begin{cases} f(v - v_g) = -\frac{\partial}{\partial z}(k \frac{\partial u}{\partial z}) + l^2 p'(u) \frac{\partial u}{\partial x}, \\ f(u - u_g) = \frac{\partial}{\partial z}(k \frac{\partial v}{\partial z}) + l^2 q'(v) \frac{\partial v}{\partial y}. \end{cases} \tag{4.1}$$

Then (3.4) becomes

$$X' = F(X) = \begin{bmatrix} x_3 \\ x_4 \\ \frac{f}{k}(v_g - x_2) + \frac{\alpha l^2}{k} p'(x_1) x_3 \\ \frac{f}{k}(x_1 - u_g) - \frac{\beta l^2}{k} q'(x_2) x_4 \end{bmatrix}. \tag{4.2}$$

(4.2) still has a unique equilibrium X_0 with a Jacobian matrix

$$DF(X_0) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{f}{k} & \frac{\alpha l^2 p''(u_g)}{k} & 0 \\ \frac{f}{k} & 0 & 0 & -\frac{\beta l^2 q''(v_g)}{k} \end{bmatrix}.$$

We see again that X_0 is hyperbolic with 2-dimensional stable and unstable manifolds. Note (4.2) has a form

$$X' = B(X)(X - X_0) \tag{4.3}$$

for

$$B(X) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{f}{k} & \frac{\alpha l^2 p'(x_1)}{k} & 0 \\ \frac{f}{k} & 0 & 0 & -\frac{\beta l^2 q'(x_2)}{k} \end{bmatrix}.$$

For any X, $B(X)$ is hyperbolic with 2-dimensional stable and unstable manifolds.

This motivates us to show the following results. Let W_s and W_u be stable and unstable subspaces of A defined in (3.5). Let $P_s : \mathbb{R}^4 \to W_s$ and $P_u : \mathbb{R}^4 \to W_u$ be projections with $P_s + P_u = I$. Then from [20] we have

$$e^{At}P_s = \frac{e^{-\tilde{k}t}}{2} \begin{bmatrix} \cos\tilde{k}t & -\sin\tilde{k}t & \frac{-\cos\tilde{k}t+\sin\tilde{k}t}{2\tilde{k}} & \frac{\cos\tilde{k}t+\sin\tilde{k}t}{2\tilde{k}} \\ \sin\tilde{k}t & \cos[\tilde{k}t] & -\frac{\cos\tilde{k}t+\sin\tilde{k}t}{2\tilde{k}} & \frac{-\cos\tilde{k}t+\sin\tilde{k}t}{2\tilde{k}} \\ -\tilde{k}(\cos\tilde{k}t+\sin\tilde{k}t) & \tilde{k}(-\cos\tilde{k}t+\sin\tilde{k}t) & \cos\tilde{k}t & -\sin\tilde{k}t \\ \tilde{k}(\cos\tilde{k}t-\sin\tilde{k}t) & -\tilde{k}(\cos\tilde{k}t+\sin\tilde{k}t) & \sin\tilde{k}t & \cos\tilde{k}t \end{bmatrix}$$

and

$$e^{At}P_u = \frac{e^{\tilde{k}t}}{2} \begin{bmatrix} \cos\tilde{k}t & \sin\tilde{k}t & \frac{\cos\tilde{k}t+\sin\tilde{k}t}{2\tilde{k}} & \frac{-\cos\tilde{k}t+\sin\tilde{k}t}{2\tilde{k}} \\ -\sin\tilde{k}t & \cos\tilde{k}t & \frac{\cos\tilde{k}t-\sin\tilde{k}t}{2\tilde{k}} & \frac{\cos\tilde{k}t+\sin\tilde{k}t}{2\tilde{k}} \\ \tilde{k}(\cos\tilde{k}t-\sin\tilde{k}t) & \tilde{k}(\cos\tilde{k}t+\sin\tilde{k}t) & \cos\tilde{k}t & \sin\tilde{k}t \\ -\tilde{k}(\cos\tilde{k}t+\sin\tilde{k}t) & \tilde{k}(\cos\tilde{k}t-\sin\tilde{k}t) & -\sin\tilde{k}t & \cos\tilde{k}t \end{bmatrix}$$

for

$$\tilde{k} = \sqrt{\frac{f}{2k}}.$$

By considering a norm

$$\|X\| = \max_{i=1,2,3,4} |x_i|$$

on $\mathbb{R}^4$, we compute

$$\begin{aligned} \|e^{At}P_{s,u}\| &\leq Ke^{\tilde{k}t}, \quad K = \frac{1}{\sqrt{2}} + \max\left\{\frac{1}{2\tilde{k}}, \tilde{k}\right\}, \\ \|B(X) - A\| &= \frac{l^2}{k}\max\{|\alpha p'(x_1)|, |\beta q'(x_2)|\}. \end{aligned} \tag{4.4}$$

We are ready to prove the next theorem.

Theorem 4.1. *Let $M > 0$ and set*

$$S_{X_0}(M) = \{X \in \mathbb{R}^4 \mid |x_1| \leq M, |x_2| \leq M\}.$$

Suppose

$$\kappa = \frac{l^2}{k} \max_{X \in S_{X_0}(M)} \max\{|\alpha p'(x_1)|, |\beta q'(x_2)|\} < \frac{\tilde{k}}{2K}, \tag{4.5}$$

where K is given in (4.4). Then (4.2) has $X(t) = X_0$ as the only bounded solution on $\mathbb{R}$ with $X(t) \in S_{X_0}(M)$.

Proof. Rewriting (4.3) as

$$(X - X_0)' = A(X - X_0) + (B(X) - A)(X - X_0),$$

its bounded solution $X(t) \in S_{X_0}(M)$ on $\mathbb{R}$ is given by

$$X(t) - X_0 = \int_{-\infty}^{t} e^{A(t-s)} P_s (B(X(s)) - A)(X(s) - X_0) ds \\ - \int_{t}^{\infty} e^{A(t-s)} P_u (B(X(s)) - A)(X(s) - X_0) ds,$$

which by (4.4) implies

$$\|X(t) - X_0\| \leq K \int_{-\infty}^{t} e^{\tilde{k}(t-s)} \|B(X(s)) - A\| \|X(s) - X_0\| ds \\ + \int_{t}^{\infty} e^{\tilde{k}(t-s)} \|B(X(s)) - A\| \|X(s) - X_0\| ds \leq \frac{2K\kappa}{\tilde{k}} \sup_{t\in\mathbb{R}} \|X(t) - X_0\|.$$

This gives

$$\sup_{t\in\mathbb{R}} \|X(t) - X_0\| \leq \frac{2K\kappa}{\tilde{k}} \sup_{t\in\mathbb{R}} \|X(t) - X_0\|,$$

which by (4.5) implies $\sup_{t\in\mathbb{R}} \|X(t) - X_0\| = 0$, i.e., $X(t) = X_0$. The proof is finished. □

Theorem 4.1 leads to the following extension of Theorem 3.4.

Corollary 4.2. *If (4.5) holds then a bounded solution $X(t) \in S_{X_0}(M)$, $t \geq 0$ of (4.2) satisfies*

$$\lim_{t\to\infty} X(t) = X_0.$$

Proof. If $X(t) \in S_{X_0}(M)$, $t \geq 0$ is a bounded solution of (4.2), then its ω-limit set $\omega(X(0)) \subset S_{X_0}(M)$ is compact and invariant. Thus for any $\tilde{X}_0 \in \omega(X(0))$, the solution $\tilde{X}(t)$, $\tilde{X}(0) = \tilde{X}_0$, $t \in \mathbb{R}$ of (4.2) is bounded and it satisfies $X(t) \in S_{X_0}(M)$, since $\tilde{X}(t) \in \omega(X(0)) \subset S_{X_0}(M)$, $t \in \mathbb{R}$. Theorem 4.1 gives $\tilde{X}(t) = X_0$, so $\tilde{X}_0 = X_0$ and thus $\omega(X(0)) = \{X_0\}$. The proof is finished. □

Corollary 4.3. *If*

$$\Theta = \max \left\{ \sup_{x_1\in\mathbb{R}} |p'(x_1)|, \sup_{x_2\in\mathbb{R}} |q'(x_2)| \right\} < \infty,$$

then for any

$$\max\{|\alpha|, |\beta|\} < \frac{\tilde{k}k}{2Kl^2\Theta}, \tag{4.6}$$

all bounded solutions $X(t)$, $t \geq 0$ of (4.2) satisfies

$$\lim_{t\to\infty} X(t) = X_0.$$

Proof. Since condition (4.6) implies (4.5), the proof is finished by Corollary 4.2. □

We continue with utilizing a hyperbolic structure of $B(X)$ by considering a slowly variable system

$$X' = \begin{bmatrix} x_3 \\ x_4 \\ \frac{f}{k}(v_g - x_2) + \frac{\alpha l^2}{k} p'(\epsilon x_1) x_3 \\ \frac{f}{k}(x_1 - u_g) - \frac{\beta l^2}{k} q'(\epsilon x_2) x_4 \end{bmatrix} \tag{4.7}$$

for a small parameter $\epsilon \in \mathbb{R}$. Then (4.7) has a form

$$X' = B(\epsilon X)(X - X_0).$$

We have the following conclusion.

Theorem 4.4. *If $p'(0) = q'(0) = 0$, then for any $M > 0$ there is an $\epsilon_M > 0$ such that any bounded solution $X(t) \in S_{X_0}(M)$, $t \geq 0$ of (4.7) with $|\epsilon| < \epsilon_M$ satisfies*

$$\lim_{t\to\infty} X(t) = X_0.$$

Proof. Now (4.5) means

$$\kappa = \frac{l^2}{k} \max_{X\in S_{X_0}(M)} \max\{|\alpha p'(\epsilon x_1)|, |\beta q'(\epsilon x_2)|\} < \frac{\tilde{k}}{2K},$$

which clearly holds for any ϵ small due $p'(0) = q'(0) = 0$. The proof is finished. □

Results of this section lead to Theorem 3.2.

5 Spatial wave solutions for (2.4)

Motivated by the above method and results, we consider (2.4) for constant k

$$\begin{cases} f(v - v_g) = -k\frac{\partial^2 u}{\partial z^2} + 2l^2 \frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z \partial x}, \\ f(u - u_g) = k\frac{\partial^2 v}{\partial z^2} + 2l^2 v\frac{\partial v}{\partial z}\frac{\partial^2}{\partial z \partial y}. \end{cases} \tag{5.1}$$

We are looking again for spatial wave solutions (3.2) of (5.1) to get

$$\begin{aligned} f(V - v_g) &= -kU'' + 2\alpha l^2 U'U'', \\ f(U - u_g) &= kV'' + 2\beta l^2 V'V''. \end{aligned} \tag{5.2}$$

We observe that (5.2) is more sophisticated than (3.3). Shifting

$$U \longleftrightarrow U - u_g, \quad V \longleftrightarrow V - v_g,$$

we study

$$\begin{aligned} fV &= -kU'' + 2\alpha l^2 U'U'', \\ fU &= kV'' + 2\beta l^2 V'V''. \end{aligned} \tag{5.3}$$

Integrating both equations of (5.3), we obtain

$$\begin{aligned} f\int V(t)dt &= -kU'(t) + \alpha l^2 U'^2(t), \\ f\int U(t)dt &= kV'(t) + \beta l^2 V'^2(t). \end{aligned} \tag{5.4}$$

By introducing

$$W_1 = \int U(t)dt, \quad W_2 = \int V(t)dt,$$

we get

$$\begin{aligned} fW_2 &= -kW_1'' + \alpha l^2 W_1''^2, \\ fW_1 &= kW_2'' + \beta l^2 W_2''^2. \end{aligned} \tag{5.5}$$

When $W_1(t) = 0$ then $U(t) = 0$ and (5.3) implies $V(t) = 0$, so $W_2(t) = 0$. Consequently $W_1(t) = 0 \Longrightarrow W_2(t) = 0$. Similarly $W_2(t) = 0 \Longrightarrow W_1(t) = 0$. Thus (5.5) gives

$$\begin{aligned} W_1'' &= \frac{k - \sqrt{k^2 + 4\alpha f l^2 W_2}}{2\alpha l^2}, \\ W_2'' &= \frac{-k + \sqrt{k^2 + 4\beta f l^2 W_1}}{2\beta l^2}. \end{aligned} \tag{5.6}$$

Next, we take in (5.6)

$$Y_1 = k^2 + 4\beta f l^2 W_1, \quad Y_2 = k^2 + 4\alpha f l^2 W_2$$

to get

$$\begin{aligned} Y_1'' &= \frac{2\beta f}{\alpha}(k - \sqrt{Y_2}), \\ Y_2'' &= \frac{2\alpha f}{\beta}(-k + \sqrt{Y_1}). \end{aligned} \tag{5.7}$$

Next, we set

$$Y_i(t) = k^2 Z_i\left(\sqrt{\frac{2f}{k}}t\right), \quad i = 1,2$$

in (5.7) to obtain

$$\begin{aligned} Z_1'' &= \mu^{-1}(1 - \sqrt{Z_2}), \\ Z_2'' &= \mu(-1 + \sqrt{Z_1}). \end{aligned} \tag{5.8}$$

for

$$\mu = \frac{\alpha}{\beta}.$$

Taking

$$X = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_1' \\ Z_2' \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix},$$

(5.8) becomes

$$X' = G(X) = \begin{bmatrix} x_3 \\ x_4 \\ \mu^{-1}(1 - \sqrt{x_2}) \\ \mu(-1 + \sqrt{x_1}) \end{bmatrix}. \tag{5.9}$$

Note (5.9) has a unique equilibrium

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

and its Jacobian matrix is

$$DG(X_1) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{1}{2\mu} & 0 & 0 \\ \frac{\mu}{2} & 0 & 0 & 0 \end{bmatrix}$$

with eigenvalues

$$\frac{-1-\imath}{2}, \quad \frac{-1+\imath}{2}, \quad \frac{1-\imath}{2}, \quad \frac{1+\imath}{2}$$

and the corresponding complex eigenvectors

$$\begin{bmatrix} -\frac{1+\imath}{\mu} \\ -1+\imath \\ \frac{\imath}{\mu} \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -\frac{1-\imath}{\mu} \\ -1-\imath \\ -\frac{\imath}{\mu} \\ 1 \end{bmatrix}, \quad \begin{bmatrix} \frac{1-\imath}{\mu} \\ 1+\imath \\ -\frac{\imath}{\mu} \\ 1 \end{bmatrix}, \quad \begin{bmatrix} \frac{1+\imath}{\mu} \\ 1-\imath \\ \frac{\imath}{\mu} \\ 1 \end{bmatrix}.$$

Consequently, X_1 is a hyperbolic equilibrium. Thus we have the following result similar to the statement of Theorem 3.2.

Theorem 5.1. *There is a 4-parametrized family of spacial waves solutions of* (5.2) *asymptotic to the equilibrium.*

Furthermore, (5.9) has a first integral

$$I(x_1, x_2, x_3, x_4) = x_3 x_4 - \mu^{-1}\left[x_2 - \frac{2}{3}x_2^{3/2}\right] - \mu\left[-x_1 + \frac{2}{3}x_1^{3/2}\right].$$

Its reduction on the level

$$I(x_1, x_2, x_3, x_4) = C \tag{5.10}$$

is given by

$$\begin{aligned} x_1' &= x_3, \\ x_3 x_2' &= \mu^{-1}\left[x_2 - \frac{2}{3}x_2^{3/2}\right] + \mu\left[-x_1 + \frac{2}{3}x_1^{3/2}\right] + C, \\ x_3' &= \mu^{-1}(1 - \sqrt{x_2}). \end{aligned} \tag{5.11}$$

(5.11) is an implicit ODE [15, 29] and its analysis seems to be difficult in general. Some numerical simulations should help. On the other hand, taking

$$y_i(t) = x_i(\mu t), \quad i = 1, 2, 3, \tag{5.12}$$

we get

$$\begin{aligned} y_1' &= \mu y_3, \\ y_3 y_2' &= y_2 - \frac{2}{3}y_2^{3/2} + C\mu + \mu^2\left[-y_1 + \frac{2}{3}y_1^{3/2}\right], \\ y_3' &= 1 - \sqrt{y_2}. \end{aligned} \tag{5.13}$$

(5.13) is reducing for $\mu = 0$ to

$$\begin{aligned} y_1' &= 0 \\ y_3 y_2' &= y_2 - \frac{2}{3}y_2^{3/2} \\ y_3' &= 1 - \sqrt{y_2}. \end{aligned} \tag{5.14}$$

The first equation of (5.14) gives $y_1(t) = y_1(0)$, and the second and third ones imply

$$\frac{dy_3}{y_3} = \frac{1 - \sqrt{y_2}}{y_2 - \frac{2}{3}y_2^{3/2}} dy_2. \tag{5.15}$$

Integrating (5.15), we have

$$\ln y_3 = \ln(y_2(2\sqrt{y_2} - 3)) + \tilde{C}$$

for a constant $\tilde{C}$, which implies

$$y_3 = C_0(3 - 2\sqrt{y_2})y_2 \tag{5.16}$$

for a constant C_0. Note y_1, y_2 and y_3 are depending on t, so differentiating (5.16) with respect to t, we get

$$y_3' = 3C_0(1 - \sqrt{y_2})y_2',$$

which together with the third equation of (5.14) give

$$3C_0y_2' = 1,$$

which possesses a solution

$$y_2(t) = \frac{t}{3C_0} + y_2(0)$$

and (5.16) leads to

$$y_3(t) = C_0\left(3 - 2\sqrt{\frac{t}{3C_0} + y_2(0)}\right)\left(\frac{t}{3C_0} + y_2(0)\right).$$

Clearly

$$y_3(0) = C_0\left(3 - 2\sqrt{y_2(0)}\right)y_2(0).$$

Consequently, (5.13) has a solution

$$\begin{aligned}
y_1(t) &= y_1(0) + O(\mu),\\
y_2(t) &= \frac{t}{3C_0} + y_2(0) + O(\mu),\\
y_3(t) &= C_0\left(3 - 2\sqrt{\frac{t}{3C_0} + y_2(0)}\right)\left(\frac{t}{3C_0} + y_2(0)\right) + O(\mu),\\
C_0 &= \frac{y_3(0)}{(3 - 2\sqrt{y_2(0)})y_2(0)}.
\end{aligned} \tag{5.17}$$

Summarizing, (5.17), (5.12) and (5.10) give a first order approximate solution of (5.9) with respect to μ small. Higher orders can be computed similarly. But since the right hand side of (5.13) is not analytic, it is better instead of (5.13) to take

$$u_1^2 = y_1, \quad u_2^2 = y_2, \quad u_3 = y_3$$

and consider

$$\begin{aligned}
2u_1u_1' &= \mu u_3,\\
2u_3u_2u_2' &= u_2^2 - \frac{2}{3}u_2^3 + C\mu + \mu^2\left[-u_1^2 + \frac{2}{3}u_1^3\right],\\
u_3' &= 1 - u_2.
\end{aligned} \tag{5.18}$$

Then we expand

$$u_i(t) = \sum_{k=0}^{r} \mu^k u_{ik}(t), \quad i = 1,2,3, \quad u_{ik}(0) = 0, \quad k \geq 1 \tag{5.19}$$

and plugging (5.19) into (5.18), we derive other terms. By (5.17), we have

$$\begin{aligned}
u_{10}(t) &= u_1(0),\\
u_{20}(t) &= \sqrt{\frac{t}{3C_0} + u_2(0)^2},\\
u_{30}(t) &= C_0\left(3 - 2\sqrt{\frac{t}{3C_0} + u_2(0)^2}\right)\left(\frac{t}{3C_0} + u_2(0)^2\right),\\
C_0 &= \frac{u_3(0)}{(3 - 2u_2(0))u_2(0)^2}.
\end{aligned}$$

Note that (5.18) is not solvable at the surface $u_1u_2u_3 = 0$, so it is implicit in the terminology of [15,29]. But it is orbitally equivalent for $u_1u_2u_3 \neq 0$ to a standard ODE

$$\begin{aligned} \hat{u}_1' &= \mu\hat{u}_2\hat{u}_3^2, \\ \hat{u}_2' &= \hat{u}_1\hat{u}_2^2 - \frac{2}{3}\hat{u}_1\hat{u}_2^3 + C\mu + \mu^2\hat{u}_1\left[-\hat{u}_1^2 + \frac{2}{3}\hat{u}_1^3\right], \\ \hat{u}_3' &= \hat{u}_1\hat{u}_2\hat{u}_3 - \hat{u}_1\hat{u}_2^2\hat{u}_3. \end{aligned} \tag{5.20}$$

Hence expansion (5.19) really works for (5.18).

6 Conclusion

We use Prandtl mixing-length theory and semiempirical theory to extend the classical problem of the wind in the steady atmospheric Ekman layer with constant eddy viscosity. This establishes new generalized atmospheric Ekman equations. Then paper deals with the existence of spatial wave solutions for these generalized atmospheric Ekman equations. Such kind of solutions are determined by certain 4-dimensional autonomous ODEs with quadratic nonlinearities. We apply methods of dynamical systems for investigating qualitative properties of these ODEs. The existence of families of asymptotic and periodic spatial wave solutions is proved. Exact and approximative solutions of the corresponding ODEs are also derived. Two figures are presented for visualization of certain these solutions. The derived spatial wave ODEs are nonlinear and could be implicit, so their study is difficult in general. Consequently, there are still many open challenging problems for further research such as existence or nonexistence of quasiperiodic, homoclinic or even chaotic solutions.

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Asymptotic behavior of multiple solutions for quasilinear Schrödinger equations

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Abstract. This paper establishes the multiplicity of solutions for a class of quasilinear Schrödinger elliptic equations:

$$-\Delta u + V(x)u - \frac{\gamma}{2}\Delta(u^2)u = f(x,u), \qquad x \in \mathbb{R}^3,$$

where $V(x) : \mathbb{R}^3 \to \mathbb{R}$ is a given potential and $\gamma > 0$. Furthermore, by the variational argument and L^∞-estimates, we are able to obtain the precise asymptotic behavior of these solutions as $\gamma \to 0^+$.

Keywords: quasilinear Schrödinger equations, variational methods, L^∞-estimate, asymptotic behavior.

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1 Introduction

This paper deals with multiplicity and asymptotic behavior of solitary wave solutions for quasilinear Schrödinger equations of the form

$$i\partial_t z = -\Delta z + W(x)z - l(x,|z|^2)z - \frac{\gamma}{2}[\Delta\rho(|z|^2)]\rho'(|z|^2)z, \tag{1.1}$$

where $z : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}$, $W : \mathbb{R}^3 \to \mathbb{R}$ is a given potential, γ is a real constant and l, ρ are real functions. Quasilinear equations of the form (1.1) have been established in the past in several areas of physics with different types of ρ. For example, the case $\rho(t) = t$ was used in [18] for the superfluid film equation in plasma physics; the case $\rho(t) = (1+t)^{1/2}$ was considered for the self-channeling of a high-power ultrashort laser in matter, see [11] and [12]. These types of equations also appear in fluid mechanics [19], in the theory of Heidelberg ferromagnetism and magnus [20], in dissipative quantum mechanics [17] and in condensed matter theory [27].

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We now consider the case of the superfluid film equation in plasma physics, namely $\rho(t) = t$. If we look for standing waves, that is, solutions of the form $z(t,x) := \exp(-iEt)u(x)$ with $E > 0$, we are lead to investigate the following elliptic equation

$$-\Delta u + V(x)u - \frac{\gamma}{2}\Delta(u^2)u = f(x,u), \qquad x \in \mathbb{R}^3, \tag{1.2}$$

with $V(x) = W(x) - E$ and $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ given by $f(x,t) := l(x,|t|^2)t$ is a new nonlinear term. Later on, we shall pose precisely the hypotheses on V and f.

Taking $\gamma = 0$, the equation (1.2) is a semilinear case, scholars have obtained a large number of existence and multiplicity results based on variational methods, see e.g. [10, 14, 21, 22]. When $\gamma > 0$, the first existence of positive solutions is proved by Poppenberg, Schmitt and Wang in [28] with a constrained minimization argument. While a general existence result for (1.1) is due to Liu et al. in [25] through using of a change of variable to reformulate the quasilinear problem (1.2) to a semilinear one in an Orlicz space framework. Colin and Jeanjean in [13] used the same method of changing variables, but the classical Sobolev space $H^1(\mathbb{R}^N)$ was chosen. We refer the readers to [5, 26, 31, 33, 34] for more results. Recently, in [23], by using perturbation methods, Liu et al. proved the existence of nodal solutions for the general quasilinear problem in bounded domains.

In the above references mentioned, the γ in the quasilinear problem (1.2) was assumed to be a fixed constant. While, the constant γ represents several physical effect and is assumed to be small in some situation. This indicates the importance of the study of the asymptotic behavior of ground states as $\gamma \to 0^+$. But, asymptotic behavior of solutions for quasilinear Schrödinger equations is much less studied. In [1], Adachi et al. considered the problem for $N = 3$, $\lambda > 0$, $\gamma > 0$ and $f(x,s) = |s|^{p-2}s$ $(4 < p < 6)$:

$$-\Delta u + \lambda u - \frac{\gamma}{2}\Delta(u^2)u = |u|^{p-2}u, \qquad x \in \mathbb{R}^3. \tag{1.3}$$

They showed the ground states u_γ of (1.3) satisfies $u_\gamma \to u_0$ in $H^2(\mathbb{R}^3) \cap C^2(\mathbb{R}^3)$ as $\gamma \to 0^+$, where u_0 is a unique ground state of

$$-\Delta u + \lambda u = |u|^{p-2}u, \qquad x \in \mathbb{R}^3.$$

Then, in [34], Wang and Shen proved the asymptotic behavior of positive solutions for (1.3) when $p \in (2,4)$, which complemented the result given by Adachi et al. in [1]. By applying the blow-up analysis and the variational methods, in [2–4] Adachi et al. obtained the precise asymptotic behavior of ground states when $N \geq 3$ and the nonlinear term has H^1-critical growth or H^1-supercritical growth.

However, the work in the literature always assumed that $V(x) \equiv \lambda > 0$ and studied the asymptotic behavior of one ground state solution for (1.4). We are interested in the problem that whether or not we can find the multiplicity of solutions for (1.4) with some suitable potential conditions. Furthermore, as $\gamma \to 0^+$, whether these solutions have any asymptotic behavior. Specifically, the main purpose of the present paper is to solve the following three problems:

(Q_1) We have the multiplicity of solutions for (1.4) in unbounded domains, which complements the results given by Liu et al. in [23].

(Q_2) We obtain the asymptotic properties of solutions for (1.4) under some suitable potential conditions. Our result, in the sense that we do not need the restrictive conditions $V(x) \equiv \lambda > 0$, improves the one obtained in [1].

(Q_3) All the papers mentioned above only studied the asymptotic behavior of a positive ground state solution for (1.4). In this paper, we explore the asymptotic behavior of multiple solutions for quasilinear Schrödinger equations. More precisely, we can obtain the asymptotic behavior of sign-changing solution for (1.4).

For this purpose, we consider the multiplicity and asymptotic behavior of solutions for the following one-parameter family of elliptic equations with general nonlinearities:

$$-\Delta u + V(x)u - \frac{\gamma}{2}\Delta(u^2)u = f(x,u), \qquad x \in \mathbb{R}^3, \tag{1.4}$$

where $\gamma > 0$ and $V(x) \in C(\mathbb{R}^3, \mathbb{R})$ satisfying:

(V_0) : $V(x) \geq V_0 > 0$ for all $x \in \mathbb{R}^3$;

(V_1) : For any $M, r > 0$, there is a ball $B_r(y)$ centered at y with radius r such that

$$\mu(\{x \in B_r(y) : V(x) \leq M\}) \to 0, \quad \text{as } |y| \to \infty.$$

Remark 1.1. The condition (V_1) was firstly introduced by Bartsch, Pankov and Wang [8] to guarantee the compactness of embeddings of the work space. The limit of condition (V_1) can be replaced by one of the following simpler conditions:

(V_2) : $V(x) \in C(\mathbb{R}^3)$, $\mu(\{x \in \mathbb{R}^3 : V(x) \leq M\}) < \infty$ for any $M > 0$ (see [9]);

(V_3) : $V(x) \in C(\mathbb{R}^3)$, $V(x)$ is coercive, i.e., $\lim_{|x|\to\infty} V(x) = \infty$.

For the continuous nonlinearity f, we suppose that it satisfies the following conditions:

(f_1) : there exist a constant C and $p \in (4,6)$ such that

$$|f(x,t)| \leq C(1 + |t|^{p-1}), \quad \text{for all } x \in \mathbb{R}^3, t \in \mathbb{R};$$

(f_2) : $\lim_{t\to 0} \frac{f(x,t)}{t} = 0$ uniformly with respect to $x \in \mathbb{R}^3$;

(f_3) : there exists $\theta > 4$ such that

$$0 < \theta F(x,t) \leq t f(x,t), \quad \text{for all } x \in \mathbb{R}^3, t \neq 0,$$

where $F(x,t) = \int_0^t f(x,s)ds$.

Note that (1.4) is the Euler–Lagrange equation associated to the natural energy functional:

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} (1 + \gamma u^2)|\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)u^2 dx - \int_{\mathbb{R}^3} F(x,u)dx,$$

which is not well defined in $H^1(\mathbb{R}^3)$. Due to this fact, the usual variational methods can not be applied directly. This difficulty makes problem like (1.4) interesting and challenging. Inspired by the work of Shen [29], we first establish the existence of signed solutions for a modified quasilinear Schrödinger equation

$$-\mathrm{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = f(x,u), \qquad x \in \mathbb{R}^3, \tag{1.5}$$

where $g(t) = \sqrt{1 + \gamma t^2}$.

In what follows, instead of using the dual method, we search the existence of sign-changing solutions for the problem (1.4) via the perturbation method and invariant sets of descending flow.

For asymptotic behavior of solutions for the problem (1.4), arguments we apply are rather standard. Using a bootstrap argument, we obtain the uniform boundedness of L^∞-norm of u_γ. Then we apply the uniform estimates for the energies to show the strong convergence in $H^1_V(\mathbb{R}^3)$ ($H^1_V(\mathbb{R}^3)$ will be defined in Section 2), this is a key problem to the study.

Next, we give our main results.

Theorem 1.2. *Assume that (V_0), (V_1), and (f_1)–(f_3) hold. Then, for fixed $\gamma \in (0,1]$, the problem (1.4) has at least three solutions: a positive solution $u_{\gamma,1}$, a negative solution $u_{\gamma,2}$ and a sign-changing solution $u_{\gamma,3}$.*

Theorem 1.3. *For fixed $\gamma \in (0,1]$, $u_{\gamma,i}$ ($i = 1,2,3$) are solutions of the problem (1.4). As $\gamma \to 0^+$, then passing to a subsequence, there exist $u_i \in H^1_V(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ ($i = 1,2,3$) such that $u_{\gamma,i} \to u_i$ strongly in $H^1_V(\mathbb{R}^3)$, where u_1 is a positive solution of problem*

$$-\Delta u + V(x)u = f(x,u), \qquad x \in \mathbb{R}^3. \tag{1.6}$$

u_2 is a negative solution of the problem (1.6) and u_3 is a sign-changing solution of the problem (1.6).

Remark 1.4. In order to prove the existence of a sign-changing solution, we need a restriction $p > 4$ because of the degeneracy of the quasilinear term. Moreover we require that p is H^1-subcritical to prove the L^∞-norm of the solutions of (1.5) are uniformly bounded. Since $4 < \frac{2N}{N-2}$ if and only if $N < 4$. Hence we only show the asymptotic behavior of multiple solutions for the quasilinear Schrödinger for $N = 3$.

This paper is organized as follows. In Section 2, we describe the variational framework associated with the problem (1.4). We give the proofs of existence of signed and sign-changing solutions in Sections 3–4, respectively. Section 5 is devoted to the study of asymptotic behavior of solutions.

In what follows, C and C_i ($i = 1,2,\ldots$) denote positive generic constants. In this paper, the norms of $L^s(\mathbb{R}^N)(s \geq 1)$ is denoted by $|\cdot|_s$.

2 The modified problem

Let

$$H^1_V(\mathbb{R}^3) = \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2\right) dx < +\infty \right\}$$

with the inner product

$$\langle u, v\rangle_{H^1_V(\mathbb{R}^3)} = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + V(x)uv)\, dx$$

and the norm

$$\|u\|^2_{H^1_V} = \langle u, u\rangle_{H^1_V(\mathbb{R}^3)}.$$

From [9], we know that under the assumptions (V_0) and (V_1), the embedding $H^1_V(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)$ is compact for each $s \in [2,6)$.

Note that (1.4) is the Euler–Lagrange equation associated to the natural energy functional:

$$I_\gamma(u) = \frac{1}{2}\int_{\mathbb{R}^3} (1+\gamma u^2)|\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)u^2 dx - \int_{\mathbb{R}^3} F(x,u)dx,$$

which is not well defined in $H^1(\mathbb{R}^3)$ or $H^1_V(\mathbb{R}^3)$. Inspired by [13, 29, 30], we consider the following quasilinear Schrödinger equation:

$$-\text{div}(g_\gamma^2(u)\nabla u) + g_\gamma(u)g_\gamma'(u)|\nabla u|^2 + V(x)u = f(x,u), \qquad x \in \mathbb{R}^3. \tag{2.1}$$

Here we choose $g_\gamma(t) : \mathbb{R} \to \mathbb{R}$ given by

$$g_\gamma(t) = \sqrt{1+\gamma t^2}.$$

It follows that $g_\gamma(t) \in C^1(\mathbb{R}, [1,\infty))$, increases in $[0,+\infty)$ and decreases in $(-\infty, 0]$.

Next, we set

$$G_\gamma(t) = \int_0^t g_\gamma(s)ds.$$

It is well known that $G_\gamma(t)$ is an odd function and inverse function $G_\gamma^{-1}(t)$ exists. Moreover, we summarize some properties of $G_\gamma^{-1}(t)$ as follows.

Lemma 2.1 ([30]).

(1) $\lim_{t\to 0} \frac{G_\gamma^{-1}(t)}{t} = 1$;

(2) $\lim_{t\to +\infty} \frac{G_\gamma^{-1}(t)}{t} = 0$;

(3) $\lim_{t\to +\infty} \frac{|G_\gamma^{-1}(t)|^2}{t} = \frac{2}{\sqrt{\gamma}}$;

(4) for all $t, s \in \mathbb{R}$, *then*

$$G_\gamma(s) \le g_\gamma(s)s, \qquad |G_\gamma^{-1}(t)| \le |t|;$$

(5) $0 \le \frac{s}{g_\gamma(s)} g_\gamma'(s) \le 1$, *for all* $s \in \mathbb{R}$;

(6) there exists a positive constant C independent of γ *such that*

$$|G_\gamma^{-1}(t)| \ge \begin{cases} C|t| & \text{if } |t| \le 1, \\ C|t|^{1/2} & \text{if } |t| \ge 1; \end{cases}$$

(7) there exists $\theta > 4$ *such that*

$$0 < \frac{\theta}{2} F(x,t) g_\gamma(t) \le G_\gamma(t) f(x,t), \quad \text{for all } x \in \mathbb{R}^3, t \neq 0.$$

In what follows, taking the change variable

$$v = G_\gamma(u) = \int_0^u g_\gamma(s)ds,$$

we observe that the functional $I_\gamma(u)$ can be written of the following way

$$J_\gamma(v) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|G_\gamma^{-1}(v)|^2 dx - \int_{\mathbb{R}^3} F(x, G_\gamma^{-1}(v))dx.$$

From Lemma 2.1 and conditions (V_0), (V_1) and (f_1)–(f_3), we obtain the functional $J_\gamma(v)$ is well-defined in $H^1_V(\mathbb{R}^3)$, $J_\gamma \in C^1(H^1_V(\mathbb{R}^3), \mathbb{R})$ and

$$J_\gamma'(v)\varphi = \int_{\mathbb{R}^3} \nabla v \nabla \varphi dx + \int_{\mathbb{R}^3} V(x) \frac{G_\gamma^{-1}(v)}{g_\gamma(G_\gamma^{-1}(v))} \varphi dx - \int_{\mathbb{R}^3} \frac{f(x, G_\gamma^{-1}(v))}{g_\gamma(G_\gamma^{-1}(v))} \varphi dx,$$

for all $\varphi \in H_V^1(\mathbb{R}^3)$.

Moreover, the critical points of the functional J_γ correspond to the weak solutions of the following equation

$$-\Delta v + V(x)\frac{G_\gamma^{-1}(v)}{g_\gamma(G_\gamma^{-1}(v))} = \frac{f(x, G_\gamma^{-1}(v))}{g_\gamma(G_\gamma^{-1}(v))}, \qquad x \in \mathbb{R}^3. \tag{2.2}$$

It is clear that if v is a critical point of J_γ, $u = G_\gamma^{-1}(v)$ is a critical point of I_γ, i.e. $u = G_\gamma^{-1}(v)$ is a solution of (1.4).

3 The existence of signed solutions

In this section we fix $1 \geq \gamma > 0$. Let $u_+ = \max\{u, 0\}$ and $u_- = \min\{u, 0\}$. Set

$$I_\gamma^\pm(u) = \frac{1}{2}\int_{\mathbb{R}^3}(1+\gamma u^2)|\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)u^2 dx - \int_{\mathbb{R}^3} F(x, u_\pm)dx$$

and

$$J_\gamma^\pm(v) := I_\gamma^\pm(G_\gamma^{-1}(v)) = \frac{1}{2}\int_{\mathbb{R}^3}|\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|G_\gamma^{-1}(v)|^2 dx - \int_{\mathbb{R}^3} F(x, (G_\gamma^{-1}(v))_\pm)dx.$$

Lemma 3.1. *Assume that (f_1)–(f_3), (V_0) and (V_1) hold. Then there exist $\rho > 0$ and $e \in H_V^1(\mathbb{R}^3)$ such that*

$$J_\gamma^+(v) > 0, \quad \text{for } \|v\|_{H_V^1} = \rho,$$

and $J_\gamma^+(e) < 0$.

Proof. By conditions (f_1), (f_2) and $|G_\gamma^{-1}(s)| \leq |s|$, for $\delta > 0$ small enough, there exists $C_\delta > 0$ such that

$$|F(x, G_\gamma^{-1}(v)_+)| \leq \delta V(x)v^2 + C_\delta |v|^p, \quad \text{for all } x \in \mathbb{R}^3,$$

since we have

$$\lim_{|t|\to 0}\frac{G_\gamma^{-1}(t)}{t} = 1,$$

and

$$\lim_{|t|\to \infty}\frac{G_\gamma^{-1}(t)}{t} = 0.$$

Then, setting $H_\gamma(x,t) := -\frac{1}{2}V(x)|G_\gamma^{-1}(t)|^2 + F(x, (G_\gamma^{-1}(t))_+)$, it follows that

$$\lim_{t\to 0}\frac{H_\gamma(x,t)}{t^2} = -\frac{1}{2}V(x) < 0, \qquad \lim_{t\to +\infty}\frac{H_\gamma(x,t)}{t^6} = 0, \quad \text{for all } x \in \mathbb{R}^3$$

and we have

$$\begin{aligned}
J_\gamma^+(v) &= \frac{1}{2}\int_{\mathbb{R}^3}|\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|G_\gamma^{-1}(v)|^2 dx - \int_{\mathbb{R}^3} F(x, (G_\gamma^{-1}(v))_+)dx \\
&\geq \frac{1}{2}\int_{\mathbb{R}^3}|\nabla v|^2 dx - \int_{\mathbb{R}^3} H_\gamma(x, v)dx \\
&\geq \frac{1}{2}\int_{\mathbb{R}^3}|\nabla v|^2 dx + (\frac{1}{2} - \delta)\int_{\mathbb{R}^3} V(x)|v|^2 dx - C_\delta \int_{\mathbb{R}^3}|v|^6 dx \\
&\geq C\|v\|_{H_V^1}^2 - C\|v\|_{H_V^1}^6,
\end{aligned}$$

where we need sufficiently small $\delta > 0$ and the Sobolev inequality. Thus, it implies $J_\gamma^+(v)$ has local minimum at $v = 0$.

On the other hand, the condition (f_3) implies that

$$F(x,t) \geq Ct^\theta - C, \quad \text{for all } t > 0, x \in \mathbb{R}^3.$$

For $w \in C_0^\infty(\mathbb{R}^3)$ with $\operatorname{supp}(w) = \overline{B_1}$ and $w(x) \geq 0$,

$$\begin{aligned} J_\gamma^+(tw) &= \frac{t^2}{2}\int_{\mathbb{R}^3} |\nabla w|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|G_\gamma^{-1}(tw)|^2 dx - \int_{\mathbb{R}^3} F(x,(G_\gamma^{-1}(tw))_+)dx \\ &\leq \frac{t^2}{2}\int_{\mathbb{R}^3} |\nabla w|^2 dx + \frac{t^2}{2}\int_{\mathbb{R}^3} V(x)|w|^2 dx - Ct^{\frac{\theta}{2}}\int_{\mathbb{R}^3} |w|^{\frac{\theta}{2}} dx - C. \end{aligned}$$

Since $\theta > 4$, it follows that $J_\gamma^+(tw) \to -\infty$ as $t \to \infty$. □

As a consequence of Lemma 3.1 and the Ambrosetti–Rabinowitz Mountain Pass Theorem, for the constant

$$d_\gamma = \inf_{\eta \in \Gamma} \sup_{t \in [0,1]} J_\gamma^+(\eta(t)),$$

where

$$\Gamma = \{\eta : \ \eta \in C([0,1], H_V^1(\mathbb{R}^3)), \eta(0) = 0, J_\gamma^+(\eta(1)) < 0\},$$

there exists a Palais–Smale sequence $\{v_n\}$ at level d_γ, that is $J_\gamma^+(v_n) \to d_\gamma$ and $(J_\gamma^+)'(v_n) \to 0$, as $n \to \infty$.

Lemma 3.2. *Assume that (f_1)–(f_3), (V_0) and (V_1) hold. Then the Palais–Smale sequence of J_γ^+ is bounded.*

Proof. Let $\{v_n\} \subset H_V^1(\mathbb{R}^3)$ be a Palais–Smale sequence. Then

$$\begin{aligned} J_\gamma^+(v_n) &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|G_\gamma^{-1}(v_n)|^2 dx - \int_{\mathbb{R}^3} F(x,(G_\gamma^{-1}(v_n))_+)dx \\ &= d_\gamma + o_n(1) \end{aligned} \tag{3.1}$$

and for any $\varphi \in H_V^1(\mathbb{R}^3)$, $\langle (J_\gamma^+)'(v_n), \varphi\rangle = o_n(1)\|\varphi\|_{H_V^1}$, that is

$$\int_{\mathbb{R}^3} \left(\nabla v_n \nabla \varphi + V(x)\frac{G_\gamma^{-1}(v_n)}{g_\gamma(G_\gamma^{-1}(v_n))}\varphi\right) dx - \int_{\mathbb{R}^3} \frac{f(x,(G_\gamma^{-1}(v_n))_+)}{g(G_\gamma^{-1}(v_n))}\varphi dx = o_n(1)\|\varphi\|_{H_V^1}. \tag{3.2}$$

Fixing $\varphi = v_n$, we deduce that

$$\begin{aligned} o_n(1)\|v_n\|_{H_V^1} &= \langle (J_\gamma^+)'(v_n), v_n\rangle \\ &= \int_{\mathbb{R}^3} \left(|\nabla v_n|^2 + V(x)\frac{G_\gamma^{-1}(v_n)}{g_\gamma(G_\gamma^{-1}(v_n))}v_n\right) dx \\ &\quad - \int_{\mathbb{R}^3} \frac{f(x,(G_\gamma^{-1}(v_n))_+)}{g_\gamma(G_\gamma^{-1}(v_n))}v_n dx. \end{aligned} \tag{3.3}$$

Therefore, by (3.1)–(3.3) and Lemma 2.1-(7), we have

$$\begin{aligned}\frac{\theta}{2}d_\gamma + o_n(1) + o_n(1)\|v_n\|_{H^1_V} &= \frac{\theta}{2}J^+_\gamma(v_n) - \langle (J^+_\gamma)'(v_n), v_n\rangle \\ &\geq \frac{\theta-4}{4}\int_{\mathbb{R}^3}|\nabla v_n|^2dx \\ &\quad + \int_{\mathbb{R}^3} V(x)G^{-1}_\gamma(v_n)\left(\frac{\theta G^{-1}_\gamma(v_n)}{4} - \frac{1}{g_\gamma(G^{-1}_\gamma(v_n))}v_n\right)dx \\ &\quad - \int_{\mathbb{R}^3}\left(\frac{\theta}{2}F(x,(G^{-1}_\gamma(v_n))^+) - \frac{f(x,(G^{-1}_\gamma(v_n))^+)}{g_\gamma(G^{-1}_\gamma(v_n))}v_n\right)dx \\ &\geq \frac{\theta-4}{4}\left(\int_{\mathbb{R}^3}|\nabla v_n|^2dx + \int_{\mathbb{R}^3}V(x)(G^{-1}_\gamma(v_n))^2dx\right).\end{aligned}$$

Next, we will prove that there exists a constant $C>0$ such that

$$\int_{\mathbb{R}^3}\left(|\nabla v_n|^2 + V(x)(G^{-1}_\gamma(v_n))^2\right)dx \geq C\|v_n\|^2_{H^1_V}.$$

Otherwise, there exists a sequence $\{v_{n_k}\} \subset H^1_V(\mathbb{R}^3)$ such that

$$A_k^2 := \int_{\mathbb{R}^3}\left(|\nabla v_{n_k}|^2 + V(x)(G^{-1}_\gamma(v_{n_k}))^2\right)dx < \frac{1}{k}\|v_{n_k}\|^2_{H^1_V}. \tag{3.4}$$

Hence, by (3.4), $\frac{A_k^2}{\|v_{n_k}\|^2_{H^1_V}} \to 0$. Consequently, in Lemma 2.4 of [30], we get a contradiction. This shows that $\|v_n\|_{H^1_V} < +\infty$. □

Lemma 3.3. *Assume that (f_1)–(f_3), (V_0) and (V_1) hold. Then J^+_γ has a positive critical point.*

Proof. First, we show that the sequence $\{v_n\}$ possesses a convergent subsequence in $H^1_V(\mathbb{R}^3)$. Indeed, by the boundedness of $\{v_n\}$ and the compactness of embedding $H^1_V(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)$ $(2 \leq s < 6)$, up to subsequence, one has $v_n \rightharpoonup v$ weakly in $H^1_V(\mathbb{R}^3)$, $v_n \to v$ strongly in $L^s(\mathbb{R}^3)$ for all $s \in [2,6)$ and $v_n(x) \to v(x)$ a.e. on $\mathbb{R}^3$.

By conditions (f_1), (f_2), Lemma 2.1-(4) and $g_\gamma(s) \geq 1$, one has

$$\begin{aligned}&\left|\int_{\mathbb{R}^3}\left(\frac{f(x,(G^{-1}_\gamma(v_n))_+)}{g_\gamma(G^{-1}_\gamma(v_n))} - \frac{f(x,(G^{-1}_\gamma(v))_+)}{g_\gamma(G^{-1}_\gamma(v))}\right)(v_n - v)dx\right| \\ &\leq C\int_{\mathbb{R}^3}\left(|G^{-1}_\gamma(v_n)| + |G^{-1}_\gamma(v_n)|^{p-1} + |G^{-1}_\gamma(v)| + |G^{-1}_\gamma(v)|^{p-1}\right)|v_n - v|dx \\ &\leq C\int_{\mathbb{R}^3}\left(|v_n| + |v_n|^{p-1} + |v| + |v|^{p-1}\right)|v_n - v|dx \\ &\leq C\left((|v_n|_2 + |v|_2)|v_n - v|_2 + (|v_n|_p^{p-1} + |v|_p^{p-1})|v_n - v|_p\right).\end{aligned} \tag{3.5}$$

On the other hand, as in Lemma 2.5 of [30], we know that

$$\begin{aligned}&\int_{\mathbb{R}^3}\left(|\nabla(v_n - v)|^2 + V(x)\left(\frac{G^{-1}_\gamma(v_n)}{g_\gamma(G^{-1}_\gamma(v_n))} - \frac{G^{-1}_\gamma(v)}{g_\gamma(G^{-1}_\gamma(v))}\right)(v_n - v)\right)dx \\ &\quad \geq C\|v_n - v\|^2_{H^1_V}.\end{aligned} \tag{3.6}$$

By virtue of (3.5) and (3.6), we have

$$
\begin{aligned}
o(1) &= \langle (J_\gamma^+)'(v_n) - (J_\gamma^+)'(v), v_n - v\rangle \\
&= \int_{\mathbb{R}^3} \left(|\nabla(v_n - v)|^2 + V(x) \left(\frac{G_\gamma^{-1}(v_n)}{g_\gamma(G_\gamma^{-1}(v_n))} - \frac{G_\gamma^{-1}(v)}{g_\gamma(G_\gamma^{-1}(v))} \right) (v_n - v) \right) dx \\
&\quad - \int_{\mathbb{R}^3} \left(\frac{f(x,(G_\gamma^{-1}(v_n))_+)}{g_\gamma(G_\gamma^{-1}(v_n))} - \frac{f(x,(G_\gamma^{-1}(v))_+)}{g_\gamma(G_\gamma^{-1}(v))} \right) (v_n - v) dx \\
&\geq C\|v_n - v\|^2_{H_V^1} + o(1).
\end{aligned}
$$

This implies $v_n \to v$ strongly in $H_V^1(\mathbb{R}^3)$. By standard regular arguments, the weak limit v of $\{v_n\}$ is a critical point of J_γ^+. Furthermore, from $v_n \to v$ strongly in $H_V^1(\mathbb{R}^3)$ and v can be shown to be positive critical point of J_γ by applying the maximum principle in [16]. Hence, $u = G_\gamma^{-1}(v)$ is a positive weak solution of (1.4). By the similar argument, we know that the equation (1.4) also has a negative weak solution. □

The next two results establish the uniform boundedness of H_V^1-norm of v_γ. This important estimate will be used in Section 5.

Lemma 3.4. *Assume that (f_1)–(f_3), (V_0) and (V_1) hold. Let v_γ be a critical point of J_γ^+ with $J_\gamma^+(v_\gamma) = d_\gamma$. Then there exists $C > 0$ (independent of γ) such that*

$$\|v_\gamma\|^2_{H_V^1} \leq C d_\gamma. \tag{3.7}$$

Proof. Let v_γ be a critical point of J_γ^+. Similar with Lemma 3.2, we get the following estimates

$$
\begin{aligned}
\frac{\theta}{2} d_\gamma &= \frac{\theta}{2} J_\gamma^+(v_\gamma) - \langle (J_\gamma^+)'(v_\gamma), v_\gamma\rangle \\
&\geq \frac{\theta - 4}{4} \int_{\mathbb{R}^3} |\nabla v_\gamma|^2 dx \\
&\quad + \int_{\mathbb{R}^3} V(x) G_\gamma^{-1}(v_\gamma) \left(\frac{\theta G_\gamma^{-1}(v_\gamma)}{4} - \frac{1}{g_\gamma(G_\gamma^{-1}(v_\gamma))} v_\gamma \right) dx \\
&\quad - \int_{\mathbb{R}^3} \left(\frac{\theta}{2} F(x,(G_\gamma^{-1}(v_\gamma))_+) - \frac{f(x,(G_\gamma^{-1}(v_\gamma))_+)}{g_\gamma(G_\gamma^{-1}(v_\gamma))} v_\gamma \right) dx \\
&\geq \frac{\theta - 4}{4} \left(\int_{\mathbb{R}^3} |\nabla v_\gamma|^2 dx + \int_{\mathbb{R}^3} V(x) (G_\gamma^{-1}(v_\gamma))^2 dx \right) \\
&\geq C\|v_\gamma\|^2_{H_V^1},
\end{aligned}
$$

which implies $\|v_\gamma\|^2_{H_V^1} \leq C d_\gamma$. □

Lemma 3.5. *Assume $\gamma \in [0,1]$. Then there exist positive constants m_1, m_2 (independent on γ), such that*

$$m_1 \leq J_\gamma^+(v_\gamma) \leq m_2,$$

where v_γ is a positive critical point of J_γ^+.

Proof. For $\rho > 0$, let

$$\Sigma_\rho = \left\{ v \in H_V^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} \left(|\nabla v|^2 + V(x) v^2 \right) dx \leq \rho^2 \right\}.$$

Similar with Lemma 3.1, we have

$$\begin{aligned}
J_\gamma^+(v) &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|G_\gamma^{-1}(v)|^2 dx - \int_{\mathbb{R}^3} F(x,(G_\gamma^{-1}(v))_+)dx \\
&\geq \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2 dx - \int_{\mathbb{R}^3} H_\gamma(x,v)dx \\
&\geq \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2 dx + \left(\frac{1}{2}-\delta\right)\int_{\mathbb{R}^3} V(x)|v|^2 dx - C_\delta \int_{\mathbb{R}^3} |v|^6 dx \\
&\geq C\|v\|_{H_V^1}^2 - C\|v\|_{H_V^1}^6,
\end{aligned}$$

where we need sufficiently small $\delta > 0$ and the Sobolev inequality. Thus, if $v \in \partial\Sigma_\rho$, take ρ small enough, it implies that $J_\gamma^+(v) \geq C\rho^2 := m_1$, where m_1 does not depend on γ.

Note that

$$J_\gamma^+(v_\gamma) = \inf_{\eta\in\Gamma}\sup_{t\in[0,1]} J_\gamma^+(\eta(t)),$$

where

$$\Gamma = \{\eta:\ \eta \in C([0,1], H_V^1(\mathbb{R}^3)),\ \eta(0) = 0,\ J_\gamma^+(\eta(1)) < 0\}.$$

Since any path $\eta(t) \in \Gamma$ always passes though $\partial\Sigma_\rho$, then

$$J_\gamma^+(v_\gamma) = \inf_{\eta\in\Gamma}\sup_{t\in[0,1]} J_\gamma^+(\eta(t)) \geq \inf_{v\in\partial\Sigma_\rho} J_\gamma^+(v) \geq m_1.$$

Take $\varphi \in C_0^\infty(\mathbb{R}^3)$, $\varphi \geq 0$, and define a path $h : [0,1] \to H_V^1(\mathbb{R}^3)$ by $h(t) = tT\varphi$, where the constant $T > 0$. For T large enough, we have

$$J_\gamma^+(h(1)) \leq J_1^+(h(1)) < 0, \qquad \int_{\mathbb{R}^3} |\nabla h(1)|^2 + V(x)(G_\gamma^{-1}(h(1)))^2 dx > \rho^2.$$

Due to $h(t) \in \Gamma$, then we get

$$J_\gamma^+(v_\gamma) \leq \sup_{t\in[0,1]} J_\gamma^+(h(t)) \leq \sup_{t\in[0,1]} J_1^+(h(t)) := m_2,$$

where m_2 does not depend on γ. □

4 The existence of sign-changing solutions

The goal of this section is to consider the existence of sign-changing solutions. To do this, we define the work space E as follows

$$E = W^{1,4}(\mathbb{R}^3) \cap H_V^1(\mathbb{R}^3),$$

where

$$H_V^1(\mathbb{R}^3) := \left\{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2 dx < +\infty\right\},$$

which endowed with the norm

$$\|u\|_{H_V^1} = \left(\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2\right) dx\right)^{1/2}$$

and $W^{1,4}(\mathbb{R}^3)$ endowed with the norm

$$\|u\|_W = \left(\int_{\mathbb{R}^3} (|\nabla u|^4 + u^4) dx\right)^{1/4}.$$

The norm of E is denoted by

$$\|u\|_E = \|u\|_W + \|u\|_{H_V^1}.$$

Remark 4.1. It is noteworthy that the embedding from $H_V^1(\mathbb{R}^3)$ into $L^2(\mathbb{R}^3)$ is compact (see [9]). Applying the interpolation inequality, we obtain that the embedding from E into $L^s(\mathbb{R}^3)$ for $2 \leq s < 12$ is compact.

In what follows, we formally formulate (1.4) in variational structure as follows

$$I_\gamma(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2 + \gamma u^2|\nabla u|^2\right) dx - \int_{\mathbb{R}^3} F(x,u)dx. \tag{4.1}$$

If $u \in H_V^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ is a weak solution of (1.4), that is, for all $\varphi \in C_0^\infty(\mathbb{R}^3)$ the following equation holds

$$\int_{\mathbb{R}^3} (\nabla u \nabla \varphi + V(x)u\varphi)\, dx + \gamma \int_{\mathbb{R}^3} u^2 \nabla u \nabla \varphi dx + \gamma \int_{\mathbb{R}^3} |\nabla u|^2 u\varphi dx - \int_{\mathbb{R}^3} f(x,u)\varphi dx = 0. \tag{4.2}$$

Notice that I_γ is an ill-behaved functional in $H_V^1(\mathbb{R}^3)$. To avoid this difficulty, in the sequel, for each $\mu, \gamma > 0$ fixed, let us consider the perturbation functional $I_{\mu,\gamma} : E \to \mathbb{R}$ associated with (1.4) given by

$$I_{\mu,\gamma}(u) = \frac{\mu}{4}\int_{\mathbb{R}^3} \left(|\nabla u|^4 + u^4\right) dx + I_\gamma(u). \tag{4.3}$$

By deducing as in [15] (see also [23]), it is normal to verify that $I_{\mu,\gamma} \in C^1(E, \mathbb{R})$ and for each $\varphi \in E$, we get

$$\begin{aligned}\langle I'_{\mu,\gamma}(u), \varphi\rangle = \mu \int_{\mathbb{R}^3} (|\nabla u|^2 \nabla u \nabla \varphi + u^3\varphi)\, dx + \int_{\mathbb{R}^3} (\nabla u \nabla \varphi + V(x)u\varphi)\, dx \\ + \gamma \int_{\mathbb{R}^3} u^2 \nabla u \nabla \varphi dx + \gamma \int_{\mathbb{R}^3} |\nabla u|^2 u \varphi dx - \int_{\mathbb{R}^3} f(x,u)\varphi dx.\end{aligned} \tag{4.4}$$

In the following, we prove a compactness condition for $I_{\mu,\gamma}$.

Lemma 4.2. *For $\mu, \gamma > 0$ fixed, then $I_{\mu,\gamma}$ satisfies the (PS) conditions.*

Proof. Let $\{u_n\} \subset E$ be a (PS) sequence for $I_{\mu,\gamma}$, that is $\{u_n\}$ satisfies:

$$|I_{\mu,\gamma}(u_n)| \leq c \quad \text{and} \quad I'_{\mu,\gamma}(u_n) \to 0 \quad \text{as } n \to \infty.$$

Consider

$$\begin{aligned}&I_{\mu,\gamma}(u_n) - \frac{1}{\theta}\langle I'_{\mu,\gamma}(u_n), u_n\rangle \\ &= \left(\frac{\mu}{4} - \frac{\mu}{\theta}\right)\|u_n\|_W^4 + \left(\frac{1}{2} - \frac{1}{\theta}\right)\int_{\mathbb{R}^3} (|\nabla u_n|^2 + V(x)u_n^2)\, dx \\ &\quad + \left(\frac{1}{2} - \frac{2}{\theta}\right)\gamma \int_{\mathbb{R}^3} |\nabla u_n|^2 u_n^2 dx + \int_{\mathbb{R}^3} \left(\frac{1}{\theta}u_n f(x,u_n) - F(x,u_n)\right) dx \\ &\geq \left(\frac{\mu}{4} - \frac{\mu}{\theta}\right)\|u_n\|_W^4 + \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|_{H_V^1}^2,\end{aligned}$$

which deduces that $\{u_n\}$ is bounded in E.

By a standard argument, we can prove that every bounded (PS) sequence $\{u_n\} \subset E$ of $I_{\mu,\gamma}$ possesses a convergent subsequence, cf. [15]. This completes the proof. □

In the following, we would like to construct a descending flow guaranteeing existence of desired invariant sets for the functional $I_{\mu,\gamma}$. For this purpose, we introduce an auxiliary operator $\mathcal{A} : E \to E$, $u \mapsto \mathcal{A}u := v$ satisfies

$$\langle J'_{\mu,\gamma}(v), \omega \rangle = C_0 \int_{\mathbb{R}^3} u^3 \omega dx + \int_{\mathbb{R}^3} f(x,u)\omega dx, \quad \text{for all } \omega \in E, \tag{4.5}$$

where

$$J_{\mu,\gamma}(v) = \frac{\mu}{4} \int_{\mathbb{R}^3} \left(|\nabla v|^4 + v^4 \right) dx + \frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla v|^2 + V(x)v^2 + \gamma v^2 |\nabla v|^2 \right) dx + \frac{C_0}{4} \int_{\mathbb{R}^3} v^4 dx,$$

and $C_0 > 0$ large enough. It is normal to verify that $J_{\mu,\gamma} \in C^1(E, \mathbb{R})$ and for all $\omega \in E$ we have

$$\begin{aligned} \langle J'_{\mu,\gamma}(v), \omega \rangle = \mu &\int_{\mathbb{R}^3} \left(|\nabla v|^2 \nabla v \nabla \omega + v^3 \omega \right) dx + \int_{\mathbb{R}^3} \left(\nabla v \nabla \omega + V(x) v \omega \right) dx \\ &+ \gamma \int_{\mathbb{R}^3} \left(|\nabla v|^2 v \omega + v^2 \nabla v \nabla \omega \right) dx + C_0 \int_{\mathbb{R}^3} v^3 \omega dx. \end{aligned}$$

Clearly, we notice that the following two statements are equivalent:

u is a fixed point of $\mathcal{A}$ and u is a critical point of $I_{\mu,\gamma}$.

Lemma 4.3. *For fixed $\mu \in (0,1]$ and $\gamma > 0$, the operator $u \mapsto v = \mathcal{A}u$ is well defined and continuous. Moreover, there exist constants $c_1, c_2, c_3 > 0$ such that*

(1) $\|I'_{\mu,\gamma}(u)\|_{E^*} \le c_1(\|u\|_W^2 + \|\mathcal{A}u\|_W^2)\|u - \mathcal{A}u\|_W + c_2\|u - \mathcal{A}u\|_{H_V^1}$;

(2) $\langle I'_{\mu,\gamma}(u), u - \mathcal{A}u \rangle \ge c_3(\|u - \mathcal{A}u\|_W^4 + \|u - \mathcal{A}u\|_{H_V^1}^2)$;

(3) for all $u \in I_{\mu,\gamma}^{-1}([a,b])$, if $\|I'_{\mu,\gamma}(u)\|_{E^*} \ge \alpha > 0$, then there exists $\delta > 0$ such that $\|u - \mathcal{A}u\|_E \ge \delta$.

Proof. To prove the operator $u \mapsto v = \mathcal{A}u$ is well defined and continuous, we consider

$$\begin{aligned} \Phi_{\mu,\gamma}(v) = \frac{\mu}{4} &\int_{\mathbb{R}^3} \left(|\nabla v|^4 + v^4 \right) dx + \frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla v|^2 + V(x)v^2 + \gamma v^2 |\nabla v|^2 \right) dx \\ &+ \frac{C_0}{4} \int_{\mathbb{R}^3} v^4 dx - \frac{C_0}{4} \int_{\mathbb{R}^3} u^3 v dx - \int_{\mathbb{R}^3} f(x,u) v dx, \quad \text{for all } v \in E. \end{aligned}$$

Obviously, $\Phi_{\mu,\gamma} \in C^1(E, \mathbb{R})$. And one can see that $\Phi_{\mu,\gamma}$ is weakly lower semicontinuous.

From conditions (f_1), (f_2) and the Sobolev embeddings theorem, for any $\delta > 0$, there exists C_δ, such that

$$\int_{\mathbb{R}^3} \left(\frac{C_0}{4} u^3 + f(x,u) \right) v dx \le \frac{C_0}{4} |u|_6^3 |v|_2 + \delta |u|_2 |v|_2 + C_\delta |u|_p^{p-1} |v|_p \le C\|v\|_E.$$

This deduces

$$\Phi_{\mu,\gamma}(v) \ge C(\|v\|_W^4 + \|v\|_{H_V^1}^2) - C\|v\|_E \to +\infty, \quad \text{as } \|v\|_E \to +\infty.$$

Therefore, the functional $\Phi_{\mu,\gamma}$ is coercive. We can see that the functional $\Phi_{\mu,\gamma}$ is bounded from below and maps bounded sets into bounded sets. In the following, we shall prove that the functional $\Phi_{\mu,\gamma}$ is also strictly convex. In fact, since

$$\begin{aligned}
&\langle \Phi'_{\mu,\gamma}(v) - \Phi'_{\mu,\gamma}(\omega), v-\omega\rangle \\
&\quad = 3\mu \int_0^1 \int_{\mathbb{R}^3} |\nabla\theta_t|^2 |\nabla(v-\omega)|^2 dxdt + 3\mu \int_0^1 \int_{\mathbb{R}^3} \theta_t^2 (v-\omega)^2 dxdt \\
&\qquad + \int_{\mathbb{R}^3} (|\nabla(v-\omega)|^2 + V(x)(v-\omega)^2) dx + 4\gamma \int_0^1 \int_{\mathbb{R}^3} \nabla\theta_t \nabla(v-\omega)\theta_t(v-\omega) dxdt \\
&\qquad + \gamma \int_0^1 \int_{\mathbb{R}^3} |\nabla\theta_t|^2 (v-\omega)^2 dxdt + \gamma \int_0^1 \int_{\mathbb{R}^3} \theta_t^2 |\nabla(v-\omega)|^2 dxdt \\
&\qquad + 3C_0 \int_0^1 \int_{\mathbb{R}^3} \theta_t^2 (v-\omega)^2 dxdt,
\end{aligned}$$

where $\theta_t = tv + (1-t)\omega$ ($t \in (0,1)$). By Young's inequality, for any $\delta > 0$, there exists $C_\delta > 0$, such that

$$\begin{aligned}
&\left| 4\gamma \int_0^1 \int_{\mathbb{R}^3} \nabla\theta_t \nabla(v-\omega)\theta_t(v-\omega) dxdt \right| \\
&\qquad \le \delta \int_0^1 \int_{\mathbb{R}^3} |\nabla\theta_t|^2 |\nabla(v-\omega)|^2 dxdt + C_\delta \int_0^1 \int_{\mathbb{R}^3} \theta_t^2 (v-\omega)^2 dxdt.
\end{aligned}$$

Taking $\delta = \frac{3\mu}{2}$ and choosing $C_0 > \frac{C_{\frac{3\mu}{2}}}{3}$, if $v \neq \omega$, we get

$$\begin{aligned}
&\langle \Phi'_{\mu,\gamma}(v) - \Phi'_{\mu,\gamma}(\omega), v-\omega\rangle \\
&\quad \ge \frac{\mu}{2} \int_{\mathbb{R}^3} \left((|\nabla v|^2 \nabla v - |\nabla\omega|^2 \nabla\omega)\nabla(v-\omega) + (v^3-\omega^3)(v-\omega) \right) dx \\
&\qquad + \int_{\mathbb{R}^3} \left(|\nabla(v-\omega)|^2 + V(x)(v-\omega)^2 \right) dx \\
&\quad \ge C(\|v-\omega\|_W^4 + \|v-\omega\|_{H_V^1}^2) \\
&\quad > 0.
\end{aligned} \tag{4.6}$$

From the above analysis, we obtain that the functional $\Phi_{\mu,\gamma}$ is coercive, bounded below, weakly lower semicontinuous and strictly convex. Thus, the functional $\Phi_{\mu,\gamma}$ admits a unique minimizer $v = \mathcal{A}(u)$. Moreover, the operator $\mathcal{A}$ maps bounded sets into bounded sets.

Next, we will verify the continuity of the operator $\mathcal{A}$ on E. To prove this, let

$$K(u) = \frac{C_0}{4} \int_{\mathbb{R}^3} u^4 dx + \int_{\mathbb{R}^3} F(x,u) dx.$$

If $\{u_n\} \subset E$ satisfying $u_n \to u$ strongly in E, setting $v = \mathcal{A}(u)$ and $v_n = \mathcal{A}(u_n)$, then we can obtain

$$\langle J'_{\mu,\gamma}(v_n) - J'_{\mu,\gamma}(v), \omega\rangle = \langle K'(u_n) - K'(u), \omega\rangle, \quad \text{for all } \omega \in E. \tag{4.7}$$

Furthermore, by the similar estimates of (4.6), for C_0 large enough, we get

$$\begin{aligned}
&\langle J'_{\mu,\gamma}(v_n) - J'_{\mu,\gamma}(v), v_n - v\rangle \\
&\quad \ge \frac{\mu}{2} \int_{\mathbb{R}^3} \left((|\nabla v_n|^2 \nabla v_n - |\nabla v|^2 \nabla v)\nabla(v_n - v) + (v_n^3 - v^3)(v_n - v) \right) dx \\
&\qquad + \int_{\mathbb{R}^3} \left(|\nabla(v_n - v)|^2 + V(x)(v_n - v)^2 \right) dx \\
&\quad \ge C(\|v_n - v\|_W^4 + \|v_n - v\|_{H_V^1}^2).
\end{aligned} \tag{4.8}$$

Then, combining (4.7) with (4.8), we have

$$C(\|v_n - v\|_W^4 + \|v_n - v\|_{H_V^1}^2) \leq \langle J'_{\mu,\gamma}(v_n) - J'_{\mu,\gamma}(v), v_n - v\rangle$$
$$= \langle K'(u_n) - K'(u), v_n - v\rangle$$
$$\leq \|K'(u_n) - K'(u)\|_{E^*}\|v_n - v\|_E.$$

Since $K \in C^1(E, \mathbb{R})$ and $u_n \to u$ strongly in E, we get that $v_n \to v$ strongly in E and the operator $\mathcal{A}$ is continuous.

Next, we shall verify (1) and (2) as follows. By (4.5), we get

$$\langle I'_{\mu,\gamma}(u), \varphi\rangle = \langle J'_{\mu,\gamma}(u) - J'_{\mu,\gamma}(v), \varphi\rangle, \quad \text{for } \varphi \in E. \tag{4.9}$$

Furthermore, we have the following estimates

$$\begin{aligned}
&\langle J'_{\mu,\gamma}(u) - J'_{\mu,\gamma}(v), \varphi\rangle \\
&\quad = 3\mu \int_0^1 \int_{\mathbb{R}^3} |\nabla \omega_t|^2 \nabla(u - v)\nabla\varphi dxdt + 3\mu \int_0^1 \int_{\mathbb{R}^3} \omega_t^2 (u - v)\varphi dxdt \\
&\qquad + \int_{\mathbb{R}^3} (\nabla(u - v)\nabla\varphi + V(x)(u - v)\varphi)dx + 2\gamma \int_0^1 \int_{\mathbb{R}^3} \nabla\omega_t \nabla(u - v)\omega_t\varphi dxdt \\
&\qquad + \gamma \int_0^1 \int_{\mathbb{R}^3} |\nabla\omega_t|^2 (u - v)\varphi dxdt + 2\gamma \int_0^1 \int_{\mathbb{R}^3} \omega_t (u - v)\nabla\omega_t \nabla\varphi dxdt \\
&\qquad + \gamma \int_0^1 \int_{\mathbb{R}^3} \omega_t^2 \nabla(u - v)\nabla\varphi dxdt + 3C_0 \int_0^1 \int_{\mathbb{R}^3} \omega_t^2 (u - v)\varphi dxdt,
\end{aligned} \tag{4.10}$$

where $\omega_t = tu + (1 - t)v$. By $|\omega_t| \leq |u| + |v|$, $|\nabla\omega_t| \leq |\nabla u| + |\nabla v|$, the Hölder inequality and (4.9), we can get

$$|\langle I'_\lambda(u), \varphi\rangle| \leq c_1(\|u\|_W^2 + \|v\|_W^2)\|u - v\|_W \|\varphi\|_E + c_2\|u - v\|_{H_V^1}\|\varphi\|_E.$$

In fact, there hold

$$\begin{aligned}
&3\mu \int_0^1 \int_{\mathbb{R}^3} |\nabla\omega_t|^2 \nabla(u - v)\nabla\varphi dxdt + 3\mu \int_0^1 \int_{\mathbb{R}^3} \omega_t^2 (u - v)\varphi dxdt \\
&\quad \leq C(|\nabla u|_4^2 + |\nabla v|_4^2)|\nabla(u - v)|_4 |\nabla\varphi|_4 + C(|u|_4^2 + |v|_4^2)|u - v|_4 |\varphi|_4 \\
&\quad \leq C(\|u\|_W^2 + \|v\|_W^2)\|u - v\|_W \|\varphi\|_E
\end{aligned}$$

and

$$\int_{\mathbb{R}^3} (\nabla(u - v)\nabla\varphi + V(x)(u - v)\varphi)dx \leq C\|u - v\|_{H_V^1}\|\varphi\|_E.$$

Using similar methods, we can also estimate other terms in (4.10). Hence

$$\|I'_{\mu,\gamma}(u)\|_{E^*} \leq c_1(\|u\|_W^2 + \|v\|_W^2)\|u - v\|_W + c_2\|u - v\|_{H_V^1}.$$

For (2), by the similar estimates of (4.6), set $\varphi = u - v$, we have

$$\begin{aligned}
\langle I'_{\mu,\gamma}(u), u - v\rangle &= \langle J'_{\mu,\gamma}(u) - J'_{\mu,\gamma}(v), u - v\rangle \\
&\geq \frac{\mu}{2} \int_{\mathbb{R}^3} \left((|\nabla u|^2 \nabla u - |\nabla v|^2 \nabla v)\nabla(u - v) + (u^3 - v^3)(u - v)\right) dx \\
&\quad + \int_{\mathbb{R}^3} \left(|\nabla(u - v)|^2 + V(x)(u - v)^2\right) dx \\
&\geq c_3(\|u - v\|_W^4 + \|u - v\|_{H_V^1}^2).
\end{aligned}$$

In order to prove (3), we consider

$$
\begin{aligned}
&I_{\mu,\gamma}(u) - \frac{1}{\theta}\langle I'_{\mu,\gamma}(u), u\rangle \\
&= \left(\frac{\mu}{4} - \frac{\mu}{\theta}\right)\|u\|_W^4 + \left(\frac{1}{2} - \frac{1}{\theta}\right)\int_{\mathbb{R}^3}\left(|\nabla u|^2 + V(x)u^2\right)dx \\
&\quad + \left(\frac{\gamma}{2} - \frac{2\gamma}{\theta}\right)\int_{\mathbb{R}^3}|\nabla u|^2u^2dx + \int_{\mathbb{R}^3}\left(\frac{1}{\theta}uf(x,u) - F(x,u)\right)dx \\
&\geq \left(\frac{\mu}{4} - \frac{\mu}{\theta}\right)\|u\|_W^4 + \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u\|^2_{H^1_V}.
\end{aligned}
$$

Hence, for any $\delta > 0$, there exists C_δ, such that

$$
\begin{aligned}
\|u\|_W^4 + \|u\|^2_{H^1_V} &\leq C(|I_{\mu,\gamma}(u)| + \|I'_{\mu,\gamma}(u)\|_{E^*}\|u\|_E) \\
&= C(|I_{\mu,\gamma}(u)| + \|I'_{\mu,\gamma}(u)\|_{E^*}(\|u\|_W + \|u\|_{H^1_V})) \\
&\leq C(|I_{\mu,\gamma}(u)| + C_\delta\|I'_{\mu,\gamma}(u)\|_{E^*}^{4/3} + \delta\|u\|_W^4 + C_\delta\|I'_{\mu,\gamma}(u)\|^2_{E^*} + \delta\|u\|^2_{H^1_V}).
\end{aligned}
$$

Taking $\delta > 0$ small enough, by direct calculation, we obtain the following estimates

$$\|u\|_W^2 \leq C(1 + |I_{\mu,\gamma}(u)|^{1/2} + \|I'_{\mu,\gamma}(u)\|_{E^*}) \tag{4.11}$$

Combining (4.11) and Lemma 4.3-(1), we can obtain

$$
\begin{aligned}
\|I'_{\mu,\gamma}(u)\|_{E^*} &\leq c_1(\|u\|_W^2 + \|v\|_W^2)\|u - v\|_W + c_2\|u - v\|_{H^1_V} \\
&\leq C(1 + \|u\|_W^2 + \|u - v\|_E^2)\|u - v\|_E \\
&\leq \widetilde{C}(1 + |I_{\mu,\gamma}(u)|^{1/2} + \|I'_{\mu,\gamma}(u)\|_{E^*} + \|u - v\|_E^2)\|u - v\|_E.
\end{aligned}
$$

For $u \in I^{-1}_{\mu,\gamma}([a,b])$ and $\|I'_{\mu,\gamma}(u)\|_{E^*} \geq \alpha > 0$, without loss of generality, let $\|u - v\|_E \leq \frac{1}{2\widetilde{C}}$, we obtain

$$\|I'_{\mu,\gamma}(u)\|_{E^*} \leq \widetilde{C}\left(1 + b^{1/2} + \frac{1}{(2\widetilde{C})^2}\right)\|u - v\|_E + \frac{1}{2}\|I'_{\mu,\gamma}(u)\|_{E^*},$$

and

$$\|u - v\|_E \geq C\|I'_\lambda(u)\|_{E^*} \geq C\alpha.$$

□

Consider a positive cone P in E defined by $P := \{u \in E : \ u \geq 0 \ a.e. \text{ on } x \in \mathbb{R}^3\}$. For an arbitrary $\varepsilon > 0$, let

$$P_\varepsilon^\pm = \left\{u \in E : \ V_0\int_{\mathbb{R}^3}u_\mp^2dx + S\left(\int_{\mathbb{R}^3}|u_\mp|^6dx\right)^{\frac{1}{3}} < \varepsilon\right\},$$

where $S = \inf_{u \in D^{1,2}(\mathbb{R}^3)\setminus\{0\}}\frac{\int_{\mathbb{R}^3}|\nabla u|^2dx}{(\int_{\mathbb{R}^3}|u|^6dx)^{1/3}}$, $u_+ = \max\{u, 0\}$, $u_- = \min\{u, 0\}$.

Lemma 4.4. *There exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, then*

$$\mathcal{A}(\partial P_\varepsilon^+) \subset P_\varepsilon^+ \quad \text{and} \quad \mathcal{A}(\partial P_\varepsilon^-) \subset P_\varepsilon^-.$$

Proof. Since the proofs of the two conclusions are similar, we just give the proof of $\mathcal{A}(\partial P_\varepsilon^+) \subset P_\varepsilon^+$.

Let $u \in E$, $v = \mathcal{A}(u)$, v satisfying (4.5). Taking $\omega = v_-$, we have

$$\begin{aligned}\mu \int_{\mathbb{R}^3} \left(|\nabla v_-|^4 + v_-^4\right) dx &+ \int_{\mathbb{R}^3} \left(|\nabla v_-|^2 + V(x) v_-^2\right) dx \\ &+ 2\gamma \int_{\mathbb{R}^3} |\nabla v_-|^2 v_-^2 \, dx + C_0 \int_{\mathbb{R}^3} v_-^4 \, dx \\ &= C_0 \int_{\mathbb{R}^3} u^3 v_- \, dx + \int_{\mathbb{R}^3} f(x,u) v_- \, dx.\end{aligned} \tag{4.12}$$

Next, we will give the estimates of both sides of above equality. On one hand, we have

$$\begin{aligned}\mu \int_{\mathbb{R}^3} \left(|\nabla v_-|^4 + v_-^4\right) dx &+ \int_{\mathbb{R}^3} \left(|\nabla v_-|^2 + V(x) v_-^2\right) dx \\ &+ 2\gamma \int_{\mathbb{R}^3} |\nabla v_-|^2 v_-^2 \, dx + C_0 \int_{\mathbb{R}^3} v_-^4 \, dx \\ &\geq V_0 \int_{\mathbb{R}^3} v_-^2 \, dx + S \left(\int_{\mathbb{R}^3} |v_-|^6 dx \right)^{1/3}.\end{aligned} \tag{4.13}$$

On the other hand, by Young inequality, we obtain

$$\begin{aligned} C_0 \int_{\mathbb{R}^3} & u^3 v_- \, dx + \int_{\mathbb{R}^3} f(u) v_- \, dx \\ &\leq \delta \int_{\mathbb{R}^3} u_- v_- \, dx + C_\delta \int_{\mathbb{R}^3} u_-^5 v_- \, dx \\ &\leq \frac{1}{2}\delta \int_{\mathbb{R}^3} (u_-^2 + v_-^2) dx + \frac{S}{2} \left(\int_{\mathbb{R}^3} |v_-|^6 dx \right)^{1/3} \\ &\quad + C_\delta \left(\int_{\mathbb{R}^3} |u_-|^6 dx \right)^{5/3}, \quad \text{for any } \delta > 0.\end{aligned} \tag{4.14}$$

Fix $\delta = V_0$ and choose ε_0 such that $C_\delta (\frac{\varepsilon_0}{S})^4 \leq \frac{S}{2}$. For $0 < \varepsilon < \varepsilon_0$ and $u \in P_\varepsilon^+$, we have

$$C_\delta \left(\int_{\mathbb{R}^3} |u_-|^6 dx \right)^{4/3} \leq C_\delta \left(\frac{\varepsilon}{S} \right)^4 \leq \frac{S}{2}. \tag{4.15}$$

By (4.13)–(4.15), we get

$$V_0 \int_{\mathbb{R}^3} v_-^2 \, dx + S \left(\int_{\mathbb{R}^3} |v_-|^6 dx \right)^{\frac{1}{3}} \leq V_0 \int_{\mathbb{R}^3} u_-^2 \, dx + S \left(\int_{\mathbb{R}^3} |u_-|^6 dx \right)^{\frac{1}{3}}.$$

Therefore, for $u \in \partial P_\varepsilon^+$, $u \neq 0$, we have

$$V_0 \int_{\mathbb{R}^3} v_-^2 \, dx + S \left(\int_{\mathbb{R}^3} |v_-|^6 dx \right)^{\frac{1}{3}} < \varepsilon,$$

which implies $v \in P_\varepsilon^+$. This completes the proof. □

From the above analysis, we know that $\mathcal{A}$ is merely continuous. But $\mathcal{A}$ itself is not applicable to construct a descending flow for $I_{\mu,\gamma}$, and we have to construct a locally Lipschitz continuous operator $\mathcal{B}$ which inherits the main properties of $\mathcal{A}$.

Lemma 4.5. *Let $E_0 = E \setminus K$, $K = \{u \in E : I'_{\mu,\gamma}(u) = 0\}$. There exist a locally Lipschitz continuous operator $\mathcal{B} : E_0 \to E$ such that*

(1) $\frac{1}{2}\|u - \mathcal{B}(u)\|_E \le \|u - \mathcal{A}(u)\|_E \le 2\|u - \mathcal{B}(u)\|_E$ *for all* $u \in E_0$*;*

(2) $\langle I'_{\mu,\gamma}(u), u - \mathcal{B}(u)\rangle \ge c_3^*(\|u - \mathcal{B}u\|_W^4 + \|u - \mathcal{B}u\|^2_{H^1_V})$ *for all* $u \in E_0$*;*

(3) $\|I'_{\mu,\gamma}(u)\|_{E^*} \le c_1^*(\|u\|_W^2 + \|\mathcal{B}u\|_W^2)\|u - \mathcal{B}u\|_W + c_2^*\|u - \mathcal{B}u\|_{H^1_V}$ *for all* $u \in E_0$*;*

(4) $\mathcal{B}(\partial P_\varepsilon^+) \subset P_\varepsilon^+$, $\mathcal{B}(\partial P_\varepsilon^-) \subset P_\varepsilon^-$ *for* $\varepsilon \in (0, \varepsilon_0)$*,*

where c_1^*, c_2^*, c_3^* *are different constants.*

Proof. The proof is similar to the proofs in [6] and [7]. We omit the details. □

From the above discussions, it is worth pointing that P_ε^+ and P_ε^- are invariant sets of descending flow τ, where $\varepsilon \in (0, \varepsilon_0)$ and τ satisfies the following initial value problem

$$\begin{cases} \frac{d}{dt}\tau(t,u) = -(id - \mathcal{B})\tau(t,u), \\ \tau(0,u) = u. \end{cases}$$

By applying invariant sets of descending flow, we can find one sign-changing critical point of the functional $I_{\mu,\gamma}$. For this purpose, we adapt some abstract results in [24].

Let $I \in C^1(E, \mathbb{R})$, $P, Q \subset E$ be open sets, $M = P \cap Q$, $\Sigma = \partial P \cap \partial Q$ and $W = P \cup Q$. For $c \in \mathbb{R}$, let $K_c = \{u \in E : I(u) = c, I'(u) = 0\}$ and $I^c = \{u \in E : I(u) \le c\}$.

Definition 4.6. $\{P, Q\}$ is called an admissible family of invariant sets with respect to I at level c, provided that the following deformation property holds: if $K_c \setminus W = \emptyset$, then, there exists $\varepsilon_1 > 0$ such that for $\varepsilon \in (0, \varepsilon_1)$, there exists $\eta \in C(E, E)$ satisfying

(1) $\eta(\overline{P}) \subset \overline{P}$, $\eta(\overline{Q}) \subset \overline{Q}$;

(2) $\eta|_{I^{c-2\varepsilon}} = \mathrm{id}$;

(3) $\eta(I^{c+\varepsilon} \setminus W) \subset I^{c-\varepsilon}$.

Theorem 4.7 ([24])**.** *Assume that $\{P, Q\}$ is an admissible family of invariant sets with respect to I at any level $c \ge c_* := \inf_{u\in\Sigma} I(u)$ and there exists a map $\varphi_0 : \chi \to E$ satisfying*

(1) $\varphi_0(\partial_1\chi) \subset P$ *and* $\varphi_0(\partial_2\chi) \subset Q$*;*

(2) $\varphi_0(\partial_0\chi) \cap M = \emptyset$*;*

(3) $\sup_{u\in\varphi_0(\partial_0\chi)} I(u) < c_*$*,*

where $\chi = \{(t_1, t_2) \in \mathbb{R}^2 : t_1, t_2 \ge 0, t_1 + t_2 \le 1\}$, $\partial_1\chi = \{0\} \times [0, 1]$, $\partial_2\chi = [0, 1] \times \{0\}$ and $\partial_0\chi = \{(t_1, t_2) \in \mathbb{R}^2 : t_1, t_2 \ge 0, t_1 + t_2 = 1\}$. Define

$$c = \inf_{\varphi\in\Gamma} \sup_{u\in\varphi(\chi)\setminus W} I(u),$$

where $\Gamma := \{\varphi \in C(\chi, E) : \varphi(\partial_1\chi) \subset P,\ \varphi(\partial_2\chi) \subset Q,\ \varphi|_{\partial_0\chi} = \varphi_0|_{\partial_0\chi}\}$. Then $c \ge c_*$, and $K_c \setminus W \ne \emptyset$.

To apply Theorem 4.7 to obtain one sign-changing critical point of $I_{\mu,\gamma}$, we take $P = P_\varepsilon^+$, $Q = P_\varepsilon^-$, $I = I_{\mu,\gamma}$. Then we need to prove the following crucial lemma.

Lemma 4.8. *If $K_c \setminus W = \emptyset$, then there exists $\varepsilon_2 > 0$ such that, for $0 < \varepsilon < \varepsilon' < \varepsilon_2$, there exists a continuous map $\sigma : [0,1] \times E \to E$ satisfying*

(1) $\sigma(0,u) = u$ for $u \in E$;

(2) $\sigma(t,u) = u$ for $t \in [0,1]$, $u \notin I_{\mu,\gamma}^{-1}[c-\varepsilon', c+\varepsilon']$;

(3) $\sigma(1, I_{\mu,\gamma}^{c+\varepsilon} \setminus W) \subset I_{\mu,\gamma}^{c-\varepsilon}$;

(4) $\sigma(t, \overline{P_\varepsilon^+}) \subset \overline{P_\varepsilon^+}$ and $\sigma(t, \overline{P_\varepsilon^-}) \subset \overline{P_\varepsilon^-}$ for $t \in [0,1]$.

Proof. The proof is similar to many existing literature (see [25, 32]). For the readers' convenience, here we give the details.

Let $N_\delta(K_c) := \{u \in E : d(E, K_c) < \delta\}$. If $K_c \setminus W = \emptyset$, then $K_c \subset W$. Thus for $\delta > 0$ small enough, we get

$$N_\delta(K_c) \subset W.$$

By Lemma 4.2, we know that $I_{\mu,\gamma}$ satisfies the (PS)-condition. Hence K_c is compact and exist $\varepsilon_2, \alpha > 0$ such that

$$\|I'_{\mu,\gamma}(u)\|_{E^*} \geq \alpha, \quad \text{for all } u \in I_{\mu,\gamma}^{-1}([c-\varepsilon_2, c+\varepsilon_2]) \setminus N_{\delta/2}(K_c).$$

Using Lemma 4.3-(3) and Lemma 4.5-(1),(2), we can find $\beta > 0$ such that

$$\langle I'_{\mu,\gamma}(u), \frac{u - Bu}{\|u - Bu\|_E} \rangle \geq \beta, \quad \text{for all } u \in I_{\mu,\gamma}^{-1}([c-\varepsilon_2, c+\varepsilon_2]) \setminus N_{\delta/2}(K_c).$$

Assume

$$\varepsilon_2 < \min\{\frac{\beta\delta}{4}, \varepsilon_0\},$$

where ε_0 is defined in Lemma 4.4. Defining two Lipschitz continuous functionals $g, q : E \to [0,1]$, satisfying

$$g(u) = \begin{cases} 0, & \text{if } u \in N_{\delta/4}(K_c), \\ 1, & \text{if } u \notin N_{\delta/2}(K_c) \end{cases}$$

and

$$q(u) = \begin{cases} 0, & \text{if } u \notin I_{\mu,\gamma}^{-1}([c-\varepsilon', c+\varepsilon']), \\ 1, & \text{if } u \in I_{\mu,\gamma}^{-1}([c-\varepsilon, c+\varepsilon]). \end{cases}$$

Consider the following initial value problem

$$\begin{cases} \frac{d\tau(t,u)}{dt} = -\Phi(\tau(t,u)), \\ \tau(0,u) = u, \end{cases} \tag{4.16}$$

where $\Phi(u) = g(u)q(u)\frac{u-Bu}{\|u-Bu\|_E}$. Using the existence and uniqueness theory of ODE, we obtain that the problem (4.16) has a unique solution $\tau(\cdot, u) \in C(\mathbb{R}^+, E)$. Let $\sigma(t,u) = \tau(\frac{2\varepsilon}{\beta}t, u)$, then we verify (1)–(3). In fact, (1) and (2) are obvious. It suffices to verify (3). To do this, we consider the following two cases.

Case 1. There exists $t_0 \in [0, \frac{2\varepsilon}{\beta}]$ such that $I_{\mu,\gamma}(\tau(t_0,u)) < c - \varepsilon$. Using Lemma 4.5-(2), we obtain that $I_{\mu,\gamma}(\tau(t,u)$ is decreasing for $t \geq 0$. Therefore, $I_{\mu,\gamma}(\sigma(1,u)) \leq c - \varepsilon$.

Case 2. For $u \in I_{\mu,\gamma}^{c+\varepsilon} \setminus W$ and $t \in [0, \frac{2\varepsilon}{\beta}]$, then $I_{\mu,\gamma}(\tau(t,u)) > c - \varepsilon$. In this case, we claim that $\tau(t,u) \in N_{\delta/2}(K_c)$ for any $t \in [0, \frac{2\varepsilon}{\beta}]$. Indeed, if for some $t_0 \in [0, \frac{2\varepsilon}{\beta}]$ such that $\tau(t_0,u) \in N_{\delta/2}(K_c)$, then

$$\frac{\delta}{2} \leq \|\tau(t_0,u) - u\|_E \leq \int_0^{t_0} \|\tau'(s,u)\|_E ds \leq t_0 < \frac{\delta}{2},$$

which is a contradiction. Thus, $g(\tau(t,u))q(\tau(t,u)) \equiv 1$ for all $t \in [0, \frac{2\varepsilon}{\beta}]$. Hence,

$$\begin{aligned}
I_{\mu,\gamma}(\sigma(1,u)) &= I_{\mu,\gamma}(\tau(\frac{2\varepsilon}{\beta},u)) \\
&= I_{\mu,\gamma}(u) - \int_0^{\frac{2\varepsilon}{\beta}} \langle I'_{\mu,\gamma}(\tau(s,u)), \Phi(\tau(s,u))\rangle ds \\
&\leq c + \varepsilon - 2\varepsilon \\
&= c - \varepsilon.
\end{aligned}$$

The proof is completed. □

Next, we will construct φ_0 satisfying the hypotheses in Theorem 4.7. Choose $u_1, u_2 \in C_0^\infty(\mathbb{R}^3)$ which satisfy $\text{supp}(u_1) \cap \text{supp}(u_2) = \emptyset$ and $u_1 \leq 0, u_2 \geq 0$. Let $\varphi_0(t,s) := R(tu_1 + su_2)$ for $(t,s) \in \chi$, where $\chi = \{(t_1,t_2) \in \mathbb{R}^2 : t_1, t_2 \geq 0, t_1 + t_2 \leq 1\}$ and R is a positive constant to be determined later. Obviously, for $t, s \in [0,1]$, $\varphi_0(0,s) = Rsu_2 \in P_\varepsilon^+$ and $\varphi_0(t,0) = Rtu_1 \in P_\varepsilon^-$.

Lemma 4.9. *Assume that* (V_0), (V_1) *and* (f_1)–(f_3) *hold. Then the functional* $I_{\mu,\gamma}$ *has a sign-changing critical point.*

Proof. It is sufficient to check assumptions (2)–(3) in applying Theorem 4.7.

Notice that $\rho = \min\{|tu_1 + (1-t)u_2|_2 : 0 \leq t \leq 1\} > 0$. Then,

$$|u|_2 \geq \rho R \quad \text{for } u \in \varphi_0(\partial_0\chi).$$

Furthermore, for $u \in M = P_\varepsilon^+ \cap P_\varepsilon^-$, we have that

$$|u|_2^2 \leq \frac{2}{V_0}\varepsilon.$$

Hence, $\varphi_0(\partial_0\chi) \cap M = \emptyset$ for R large enough.

To verify (3), for any $u \in \Sigma$, from the conditions (f_1) and (f_2) and the definition of Σ, for all $\delta > 0$, there exists $C_\delta > 0$, such that

$$I_{\mu,\gamma}(u) \geq -\int_{\mathbb{R}^3} F(x,u)dx \geq -\delta \int_{\mathbb{R}^3} u^2 dx - C_\delta \int_{\mathbb{R}^3} u^6 dx \geq -C(\varepsilon + \varepsilon^3),$$

which implies that

$$c_* \geq -C(\varepsilon + \varepsilon^3). \tag{4.17}$$

On the other hand, by the condition (f_3), we have $F(x,t) \geq C|t|^\theta$ for all $x \in \mathbb{R}^3$. For any $u \in \varphi_0(\partial_0\chi)$, then

$$\begin{aligned}
I_{\mu,\gamma}(u) &= \frac{\mu}{4}\|u\|_W^4 + \frac{1}{2}\|u\|_{H_V^1}^2 + \frac{\gamma}{2}\int_{\mathbb{R}^3} u^2|\nabla u|^2 dx - \int_{\text{supp}(u_1)\cap\text{supp}(u_2)} F(x,u)dx \\
&\leq \frac{\mu}{4}\|u\|_W^4 + \frac{1}{2}\|u\|_{H_V^1}^2 + \frac{\gamma}{2}\int_{\mathbb{R}^3} u^2|\nabla u|^2 dx - C|u|_\theta^\theta \\
&\leq C\|u\|_E^4 - C|u|_\theta^\theta,
\end{aligned} \tag{4.18}$$

which together with (4.17) implies that for R large enough and ε small enough, we obtain

$$\sup_{u\in\varphi_0(\partial_0\chi)} I_{\mu,\gamma}(u) < c_*.$$

Hence, by Theorem 4.7, $I_{\mu,\gamma}$ has at least one critical point u in $E\setminus(P_\varepsilon^+\cup P_\varepsilon^-)$. $\square$

The next result establishes an important estimate associated with critical values.

Lemma 4.10. *Assume $0<\mu<1$ and $0<\gamma<1$. Then there exists a positive constant m_3 (independent on μ and γ), such that*

$$I_{\mu,\gamma}(u_{\mu,\gamma}) \le m_3,$$

where $u_{\mu,\gamma}$ is a sign-changing critical point of $I_{\mu,\gamma}$.

Proof. For fixed $0<\mu<1$ and $0<\gamma<1$, take a path $\varphi_{1,1}(s,t):[0,1]\times[0,1]\to E\setminus\{0\}$, $\varphi_{1,1}(t,s):=T(tu_1+su_2)$, where the constant $T>R$ (R is defined in the proof of Lemma 4.9). A simple computation ensures that $\varphi_{1,1}(0,s)\in P_\varepsilon^+$, $\varphi_{1,1}(t,0)\in P_\varepsilon^-$ and $\varphi_{1,1}(\partial_0\chi)\cap M=\emptyset$. By the similar estimates of (4.18), taking T sufficiently large, we obtain

$$I_{1,1}(\varphi_{1,1}(t,s)) \le -C_1 \quad \text{for all } (t,s)\in\partial_0\chi, \tag{4.19}$$

where $C_1>0$ is large enough.

On the other hand, for ε small enough, we have

$$\inf_{u\in\Sigma} I_{\mu,\gamma}(u) > -\sup_{u\in\Sigma}\int_{\mathbb{R}^3} F(x,u)dx \ge -C_2, \tag{4.20}$$

here choose C_1 large enough, such that $0<C_2<C_1$. Then estimates (4.19) and (4.20) ensure that

$$\max_{(t,s)\in\partial_0\chi} I_{\mu,\gamma}(\varphi_{1,1}(t,s)) \le \max_{(t,s)\in\partial_0\chi} I_{1,1}(\varphi_{1,1}(t,s)) \le -C_2 < \inf_{u\in\Sigma} I_{\mu,\gamma}(u).$$

This implies

$$\varphi_{1,1}(s,t)\in\Gamma,$$

where $\Gamma:=\{\varphi\in C(\chi,E):\varphi(\partial_1\chi)\subset P_\varepsilon^+,\ \varphi(\partial_2\chi)\subset P_\varepsilon^-,\ \varphi|_{\partial_0\chi}=\varphi_0|_{\partial_0\chi}\}$, and so

$$I_{\mu,\gamma}(u_{\mu,\gamma}) = \inf_{\varphi\in\Gamma}\sup_{u\in\varphi(\chi)\setminus W} I_{\mu,\gamma}(u) \le \sup_{u\in\varphi_{1,1}(\chi)} I_{\mu,\gamma}(u) \le \max_{(t,s)\in[0,1]\times[0,1]} I_{1,1}(\varphi_{1,1}(t,s)) := m_3,$$

where m_3 is independent on γ and μ. $\square$

Finally, the existence of a sign-changing critical point to the original functional I_γ is based on the following convergence result for the perturbation functional $I_{\mu,\gamma}$.

Proposition 4.11 ([23]). *Let $\mu_i\to 0$ and $\{u_i\}\subset E$ be a sequence of critical points of $I_{\mu_i,\gamma}$ satisfying $I'_{\mu_i,\gamma}(u_i)=0$ and $I_{\mu_i,\gamma}(u_i)\le C$ for some C independent of i. Then as $i\to\infty$, up to a subsequence $u_i\to u_\gamma$ in $H_V^1(\mathbb{R}^3)$, $u_i\nabla u_i\to u_\gamma\nabla u_\gamma$ in $L^2(\mathbb{R}^3)$, $\mu_i\int_{\mathbb{R}^3}\left(|\nabla u_i|^4+u_i^4\right)dx\to 0$, $I_{\mu_i,\gamma}(u_i)\to I_\gamma(u_\gamma)$ and u_γ is a critical point of I_γ.*

Lemma 4.12. *Assume $0<\gamma<1$. Then there exist a positive constant m_3 and a sign-changing critical point u_γ of I_γ, such that*

$$I_\gamma(u_\gamma)\le m_3,$$

where m_3 is independent on γ.

Proof. From Lemma 4.9 and Lemma 4.10, it permits to apply the Proposition 4.11. Therefore, there exists a critical point u_γ of I_γ such that $u_\gamma \in H_V^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. In the following, we will show that u_γ is a sign-changing critical point of I_γ. To this end, we need estimate $u_{\gamma+} \neq 0$ as follows. Consider $\langle I'_{\mu_i,\gamma}(u_i), u_{i+}\rangle = 0$, it follows from Sobolev inequality and the conditions (f_1), (f_2) that

$$\begin{aligned}
V_0 \int_{\mathbb{R}^3} |u_{i+}|^2 dx + S\left(\int_{\mathbb{R}^3} |u_{i+}|^6 dx\right)^{\frac{1}{3}} &\\
\leq V_0 \int_{\mathbb{R}^3} |u_{i+}|^2 dx + \int_{\mathbb{R}^3} |\nabla u_{i+}|^2 dx &\\
\leq \int_{\mathbb{R}^3} f(x, u_{i+}) u_{i+} dx &\\
\leq \delta \int_{\mathbb{R}^3} |u_{i+}|^2 dx + C_\delta \int_{\mathbb{R}^3} |u_{i+}|^6 dx, &
\end{aligned}$$

where $\delta > 0$ small enough. This implies $|u_{i+}|_6 \geq C > 0$. Recall that $u_{i+} \to u_{\gamma+}$ strongly in $L^6(\mathbb{R}^3)$. Therefore, we see that $u_{\gamma+} \neq 0$. By the same argument we can prove that $u_{\gamma-} \neq 0$. Hence we obtain u_γ is a sign-changing critical point of I_γ.

Moreover, by Lemma 4.10, we obtain

$$I_{\mu,\gamma}(u_{\mu,\gamma}) \leq m_3,$$

where m_3 is independent on γ and μ.

Having this in mind, taken $\mu \to 0$, from the Proposition 4.11 we have

$$I_\gamma(u_\gamma) \leq m_3,$$

where u_γ is sign-changing critical point of I_γ. □

Before concluding this section, we would like to complete the proof of Theorem 1.2.

Proof of Theorem 1.2. From Lemma 3.3 and Lemma 4.12, the problem (1.4) has at least three solutions: a positive solution $u_{\gamma,1}$, a negative solution $u_{\gamma,2}$ and a sign-changing solution $u_{\gamma,3}$. □

5 Asymptotic behavior of solutions

In this section, our goal is to study the asymptotic behavior of $u_\gamma = G^{-1}(v_\gamma)$. Having this in mind, we are going to show the L^∞ estimates of the critical points of J_γ.

Lemma 5.1. *If $v_\gamma \in H_V^1(\mathbb{R}^3)$ is a weak solution of problem (2.2), then $v_\gamma \in L^\infty(\mathbb{R}^3)$. Moreover, there exists a constant $C > 0$ independents of γ such that $|v_\gamma|_\infty \leq C\|v_\gamma\|_{H_V^1}^{\frac{4}{6-p}}$.*

Proof. The result can be proved similarly to [5,14] but we give a proof for the convenience of the readers. In what follows, for simplicity, we denote v_γ by v. Let $v \in H_V^1(\mathbb{R}^3)$ be a weak solution of $-\Delta v + V(x)\frac{G_\gamma^{-1}(v)}{g_\gamma(G_\gamma^{-1}(v))} = \frac{f(x,G_\gamma^{-1}(v))}{g_\gamma(G_\gamma^{-1}(v))}$, i.e.

$$\int_{\mathbb{R}^3} \nabla v \nabla \varphi dx + \int_{\mathbb{R}^3} V(x)\frac{G_\gamma^{-1}(v)}{g_\gamma(G_\gamma^{-1}(v))}\varphi dx = \int_{\mathbb{R}^3} \frac{f(x, G_\gamma^{-1}(v))}{g_\gamma(G_\gamma^{-1}(v))}\varphi dx, \quad \text{for all } \varphi \in H_V^1(\mathbb{R}^3). \tag{5.1}$$

Set $T > 0$, and denote

$$v_T = \begin{cases} -T, & \text{if } v \leq -T, \\ v, & \text{if } -T < v < T, \\ T, & \text{if } v \geq T. \end{cases}$$

Choosing $\varphi = |v_T|^{2(\eta-1)}v$ in (5.1), where $\eta > 1$ to be determined later, we get

$$\begin{aligned} &\int_{\mathbb{R}^3} |\nabla v|^2 \cdot |v_T|^{2(\eta-1)}dx + 2(\eta-1)\int_{\{x:\ |v(x)|<T\}} |v|^{2(\eta-1)}|\nabla v|^2 dx \\ &\quad + \int_{\mathbb{R}^3} V(x)\frac{G_\gamma^{-1}(v)}{g_\gamma(G_\gamma^{-1}(v))}|v_T|^{2(\eta-1)}v dx \\ &= \int_{\mathbb{R}^3} \frac{f(x, G_\gamma^{-1}(v))}{g_\gamma(G_\gamma^{-1}(v))}|v_T|^{2(\eta-1)}v dx. \end{aligned}$$

Combining the fact that the second term in the left side of the above equation is nonnegative and Lemma 2.1-(4), we obtain

$$\begin{aligned} &\int_{\mathbb{R}^3} |\nabla v|^2|v_T|^{2(\eta-1)}dx + \int_{\mathbb{R}^3} V(x)\frac{G_\gamma^{-1}(v)}{g_\gamma(G_\gamma^{-1}(v))}|v_T|^{2(\eta-1)}v dx \\ &\leq \int_{\mathbb{R}^3} \frac{f(x, G_\gamma^{-1}(v))}{g_\gamma(G_\gamma^{-1}(v))}|v_T|^{2(\eta-1)}v dx \\ &\leq \delta\int_{\mathbb{R}^3} \frac{G_\gamma^{-1}(v)}{g_\gamma(G_\gamma^{-1}(v))}|v_T|^{2(\eta-1)}v dx + C_\delta\int_{\mathbb{R}^3} \frac{|G_\gamma^{-1}(v)|^{p-1}}{g_\gamma(G_\gamma^{-1}(v))}|v_T|^{2(\eta-1)}v dx \\ &\leq \delta\int_{\mathbb{R}^3} \frac{G_\gamma^{-1}(v)}{g_\gamma(G_\gamma^{-1}(v))}|v_T|^{2(\eta-1)}v dx + C_\delta\int_{\mathbb{R}^3} |v|^p|v_T|^{2(\eta-1)}dx. \end{aligned} \tag{5.2}$$

Taking δ small enough in (5.2), we have

$$\int_{\mathbb{R}^3} |\nabla v|^2|v_T|^{2(\eta-1)}dx \leq C\int_{\mathbb{R}^3} |v|^p|v_T|^{2(\eta-1)}dx. \tag{5.3}$$

On the other hand, using the Sobolev inequality, we have

$$\begin{aligned} \left(\int_{\mathbb{R}^3} (|v||v_T|^{\eta-1})^6 dx\right)^{\frac{1}{3}} &\leq C\int_{\mathbb{R}^3} |\nabla(v v_T^{\eta-1})|^2 dx \\ &\leq C\int_{\mathbb{R}^3} |\nabla v|^2|v_T|^{2(\eta-1)}dx + C(\eta-1)^2\int_{\mathbb{R}^3} |\nabla v|^2|v_T|^{2(\eta-1)}dx \\ &\leq C\eta^2\int_{\mathbb{R}^3} |\nabla v|^2|v_T|^{2(\eta-1)}dx, \end{aligned}$$

where we used that $(a+b)^2 \leq 2(a^2+b^2)$ and $\eta^2 \geq (\eta-1)^2 + 1$.

By (5.3), the Hölder inequality and the Sobolev embedding theorem,

$$\begin{aligned} \left(\int_{\mathbb{R}^3} (|v||v_T|^{\eta-1})^6 dx\right)^{\frac{1}{3}} &\leq C\eta^2\int_{\mathbb{R}^3} |v|^{p-2}v^2|v_T|^{2(\eta-1)}dx \\ &\leq C\eta^2\left(\int_{\mathbb{R}^3} |v|^6 dx\right)^{\frac{p-2}{6}}\left(\int_{\mathbb{R}^3} (|v||v_T|^{\eta-1})^{\frac{12}{8-p}}dx\right)^{\frac{8-p}{6}} \\ &\leq C\eta^2\|v\|_{H_V^1}^{p-2}\left(\int_{\mathbb{R}^3} |v|^{\frac{12\eta}{8-p}}dx\right)^{\frac{8-p}{6}}, \end{aligned}$$

where we used the fact that $|v_T| \leq |v|$. In what follows, taking $\zeta = \frac{12}{8-p}$, we get

$$\left(\int_{\mathbb{R}^3} (|v||v_T|^{\eta-1})^6 dx\right)^{\frac{1}{3}} \leq C\eta^2 \|v\|_{H_V^1}^{p-2} |v|_{\eta\zeta}^{2\eta}.$$

From Fatou's lemma, it follows that

$$|v|_{6\eta} \leq (C\eta^2 \|v\|_{H_V^1}^{p-2})^{\frac{1}{2\eta}} |v|_{\eta\zeta}. \tag{5.4}$$

Let us define $\eta_{n+1}\zeta = 6\eta_n$ where $n = 0, 1, 2, \ldots$ and $\eta_0 = \frac{8-p}{2}$. By (5.4) we have

$$|v|_{6\eta_1} \leq (C\eta_1^2 \|v\|_{H_V^1}^{p-2})^{\frac{1}{2\eta_1}} |v|_{6\eta_0} \leq (C\|v\|_{H_V^1}^{p-2})^{\frac{1}{2\eta_1}+\frac{1}{2\eta_0}} \eta_0^{\frac{1}{\eta_0}} \eta_1^{\frac{1}{\eta_1}} |v|_6.$$

By Moser's iteration method we have

$$|v|_{6\eta_n} \leq (C\|v\|_{H_V^1}^{p-2})^{\frac{1}{2\eta_0}\sum_{i=0}^n (\frac{\zeta}{6})^i} (\eta_0)^{\frac{1}{\eta_0}\sum_{i=0}^n (\frac{\zeta}{6})^i} (\frac{6}{\zeta})^{\frac{1}{\eta_0}\sum_{i=0}^n i(\frac{\zeta}{6})^i} |v|_6.$$

Thus, we have

$$|v|_\infty \leq C\|v\|_{H_V^1}^{\frac{4}{6-p}}.$$

□

Now we are ready to prove H_V^1-strong convergence of the weak solution of problem (1.4).

Lemma 5.2. *Assume u_γ is a solution of (1.4), then $u_\gamma \to u_0$ strongly in $H_V^1(\mathbb{R}^3)$ as $\gamma \to 0^+$, where u_0 is a solution of (1.6).*

Proof. If u_γ is a signed solution of (1.4), Lemma 3.4 and Lemma 3.5 guarantee that

$$\|v_\gamma\|_{H_V^1} < C,$$

for some $C > 0$. This together with the fact that

$$\|u_\gamma\|_{H_V^1} = \|G^{-1}(v_\gamma)\|_{H_V^1} \leq C\|v_\gamma\|_{H_V^1},$$

gives $\{u_\gamma\}$ is uniformly bounded in $H_V^1(\mathbb{R}^3)$, that is

$$\|u_\gamma\|_{H_V^1} < C,$$

where C is independent on γ.

Similarly, if u_γ is a sign-changing solution of (1.4), from Lemma 3.4 and Lemma 4.12, it follows that $\{u_\gamma\}$ is uniformly bounded in $H_V^1(\mathbb{R}^3)$ as well.

Thus, if u_γ is a solution of (1.4), then there exists $u_0 \in H_V^1(\mathbb{R}^3)$ such that, as $\gamma \to 0^+$ passing to a subsequence

$$\begin{aligned} &u_\gamma \rightharpoonup u_0 \quad \text{weakly in } H_V^1(\mathbb{R}^3), \\ &u_\gamma \to u_0 \quad \text{strongly in } L^p(\mathbb{R}^3)\ (p \in [2,6)), \\ &u_\gamma \to u_0 \quad a.e. \text{ on } \mathcal{K} := \text{supp}\ \varphi,\ \varphi \in C_0^\infty(\mathbb{R}^3). \end{aligned}$$

Moreover, there exists a function $\phi \in L^p(\mathbb{R}^3)$ such that $|u_\gamma| \leq \phi$ *a.e.* on $\mathcal{K}$ for all γ.

Since $u_\gamma \rightharpoonup u_0$ weakly in $H_V^1(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3} (\nabla u_\gamma \nabla \varphi + V(x) u_\gamma \varphi)\, dx \to \int_{\mathbb{R}^3} (\nabla u_0 \nabla \varphi + V(x) u_0 \varphi)\, dx \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^3). \tag{5.5}$$

By conditions (f_1) and (f_2), the Lebesgue dominated theorem and the fact that $u_\gamma \to u_0$ strongly in $L^p(\mathbb{R}^3)$, we get

$$\int_{\mathbb{R}^3} f(x,u_\gamma)\varphi dx \to \int_{\mathbb{R}^3} f(x,u_0)\varphi dx \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^3). \tag{5.6}$$

In what follows, define the following functional:

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2\right) dx - \int_{\mathbb{R}^3} F(x,u)dx.$$

Next we are going to show that $\langle I'(u_0), \varphi\rangle = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^3)$. Indeed, u_γ is a critical point of I_γ, i.e. for all $\varphi \in C_0^\infty(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3} (\nabla u_\gamma \nabla\varphi + V(x)u_\gamma\varphi)\, dx + \gamma\int_{\mathbb{R}^3}\left(|\nabla u_\gamma|^2 u_\gamma\varphi + \nabla u_\gamma\nabla\varphi u_\gamma^2\right) dx - \int_{\mathbb{R}^3} f(x,u_\gamma)\varphi dx = 0. \tag{5.7}$$

On the other hand, by Lemma 5.1,

$$|u_\gamma|_\infty \le C|v_\gamma|_\infty \le C\|v_\gamma\|_{H_V^1}^{\frac{4}{6-p}} \le C$$

and so, from $\|u_\gamma\|_{H_V^1} \le C$,

$$\begin{aligned}
&\gamma\int_{\mathbb{R}^3}\left(|\nabla u_\gamma|^2 u_\gamma\varphi + \nabla u_\gamma\nabla\varphi u_\gamma^2\right) dx \\
&\quad\le C\gamma|\varphi|_\infty\int_{\mathbb{R}^3}|\nabla u_\gamma|^2 dx + C\gamma\int_{\mathbb{R}^3}|\nabla u_\gamma||\nabla\varphi| dx \\
&\quad\le C\gamma\left(|\varphi|_\infty|\nabla u_\gamma|_2^2 + |\nabla\varphi|_2|\nabla u_\gamma|_2\right) \to 0, \quad \text{as } \gamma\to 0^+.
\end{aligned} \tag{5.8}$$

In view of (5.5)–(5.8), for all $\varphi \in C_0^\infty(\mathbb{R}^3)$, we obtain

$$\int_{\mathbb{R}^3}\left(\nabla u_0 + V(x)u_0 - f(x,u_0)\right)\varphi dx = 0, \tag{5.9}$$

which yields that u_0 is a weak solution of problem (1.6).

Next we will show that the test function φ in (5.7) can be taken as arbitrary functions $\psi \in H_V^1(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$. First, without loss of generality, for $\psi \ge 0$, choose a sequence $\{\varphi_n\} \subset C_0^\infty(\mathbb{R}^3)$ such that $\varphi_n \ge 0$, $\varphi_n \to \psi$ strongly in $H_V^1(\mathbb{R}^3)$, $\varphi_n \to \psi$ a.e. $x \in \mathbb{R}^3$ and $|\varphi_n|_\infty \le |\psi|_\infty + 1$. Take φ_n as the test function in (5.7), letting $n\to\infty$ we know that (5.7) holds for $\varphi = \psi$. Hence we can take $\varphi = u_\gamma$ in (5.7), then

$$\int_{\mathbb{R}^3}\left(|\nabla u_\gamma|^2 + V(x)u_\gamma^2\right) dx + 2\gamma\int_{\mathbb{R}^3}|\nabla u_\gamma|^2 u_\gamma^2 dx - \int_{\mathbb{R}^3} f(x,u_\gamma)u_\gamma dx = 0. \tag{5.10}$$

Since u_0 is a weak solution of (1.6), taking $\varphi = u_0$ in (5.9), we have

$$\int_{\mathbb{R}^3}\left(|\nabla u_0|^2 + V(x)u_0^2\right) dx - \int_{\mathbb{R}^3} f(x,u_0)u_0 dx = 0. \tag{5.11}$$

Similar with (5.6), we obtain

$$\int_{\mathbb{R}^3} f(x,u_\gamma)u_\gamma dx \to \int_{\mathbb{R}^3} f(x,u_0)u_0 dx, \quad \text{as } \gamma\to 0^+. \tag{5.12}$$

By (5.10)–(5.12) and the lower semicontinuity of $\|u_\gamma\|_{H^1_V}$, we get

$$\gamma \int_{\mathbb{R}^3} |\nabla u_\gamma|^2 u_\gamma^2 dx \to 0, \quad \text{as } \gamma \to 0^+$$

and

$$\int_{\mathbb{R}^3} (|\nabla u_\gamma|^2 + V(x)u_\gamma^2)\, dx \to \int_{\mathbb{R}^3} (|\nabla u_0|^2 + V(x)u_0^2)\, dx, \quad \text{as } \gamma \to 0^+.$$

This combined with the fact that $u_\gamma \rightharpoonup u_0$ weakly in $H^1_V(\mathbb{R}^3)$ gives

$$u_\gamma \to u_0 \quad \text{strongly in } H^1_V(\mathbb{R}^3) \quad \text{as } \gamma \to 0^+. \qquad \square$$

Proof of Theorem 1.3. From Lemma 3.3, we know that for all $\gamma \in (0,1]$, there exists a positive critical point $u_{\gamma,1}$. Then, by Lemma 5.2, we obtain $u_{\gamma,1} \to u_1$ strongly in $H^1_V(\mathbb{R}^3)$ as $\gamma \to 0^+$, where u_1 is critical point of I. Note that at this stage, we do not know whether $u_1 \neq 0$. To this end, by Lemma 3.5, we know that

$$0 < m_1 \leq I^+_\gamma(u_{\gamma,1})$$

and so, by $u_{\gamma,1} \to u_1$ strongly in $H^1_V(\mathbb{R}^3)$ as $\gamma \to 0^+$,

$$I^+_\gamma(u_1) \geq m_1 > 0.$$

Consequently, $u_1 \neq 0$, then u_1 can be shown to be positive critical point of I^+_γ by applying the maximum principle in [16], that is, u_1 is a positive solution of (1.6). Similarly, we can show u_2 is a negative solution of problem (1.6).

On the other hand, by Lemma 4.12, for all $\gamma \in (0,1]$, there exists a positive constant m_3 such that I_γ has a sign-changing solution $u_{\gamma,3}$ with $I_\gamma(u_{\gamma,3}) \leq m_3$. By Lemma 5.2, as $\gamma_i \to 0^+$, there exists a sequence of sign-changing critical points $\{u_{\gamma_i,3}\}$ of I_{γ_i}, converges to a critical point $u_3 \in H^1_V(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ of I. Next, we will show u_3 is a sign-changing critical point of I. Taking $\varphi = (u_{\gamma,3})_+ := u^+_{\gamma,3}$ in the equation $\langle I'_\gamma(u_{\gamma,3}), \varphi\rangle = 0$, by the conditions $(f_1), (f_2)$ and Poincare inequalities and Sobolev inequalities we have

$$\begin{aligned} C\int_{\mathbb{R}^3} (u^+_{\gamma,3})^2 dx + C\left(\int_{\mathbb{R}^3} (u^+_{\gamma,3})^6 dx\right)^{1/3} &\leq \int_{\mathbb{R}^3} \left(|\nabla u^+_{\gamma,3}|^2 + V(x)(u^+_{\gamma,3})^2\right) dx \\ &\leq \int_{\mathbb{R}^3} f(x, u^+_{\gamma,3}) u^+_{\gamma,3} dx \\ &\leq \delta \int_{\mathbb{R}^3} (u^+_{\gamma,3})^2 dx + C_\delta \int_{\mathbb{R}^3} (u^+_{\gamma,3})^6 dx. \end{aligned}$$

This implies that there exists $C > 0$ such that $\int_{\mathbb{R}^3} (u^+_{\gamma,3})^6 dx \geq C$ for $\gamma \in (0,1]$. Now by Lemma 5.2, we have $u_{\gamma,3} \to u_3$ strongly in $H^1_V(\mathbb{R}^3)$ as $\gamma \to 0^+$. This combined with the Sobolev embedding gives

$$\int_{\mathbb{R}^3} (u_{3+})^6 dx = \lim_{\gamma \to 0^+} \int_{\mathbb{R}^3} (u^+_{\gamma,3})^6 dx \geq C > 0.$$

Thereby, we can infer that $u_{3+} \neq 0$. By the same argument we can show $u_{3-} \neq 0$. This completes the proof. $\square$

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On a generalized cyclic-type system of difference equations with maximum

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Abstract. In this paper we investigate the behaviour of the solutions of the following k-dimensional cyclic system of difference equations with maximum:

$$x_i(n+1) = \max\left\{A_i, \frac{x_i^p(n)}{x_{i+1}^q(n-1)}\right\}, \qquad i = 1, 2, \dots, k-1,$$
$$x_k(n+1) = \max\left\{A_k, \frac{x_k^p(n)}{x_1^q(n-1)}\right\}$$

where $n = 0, 1, \dots$, $A_i > 1$, for $i = 1, 2, \dots, k$, whereas the exponents p, q and the initial values $x_i(-1), x_i(0)$, $i = 1, 2, \dots, k$ are positive real numbers.

Keywords: difference equations with maximum, cyclic system, equilibrium, eventually equal to equilibrium.

2020 Mathematics Subject Classification: 39A10.

1 Introduction

Undoubtedly, there is a growing interest in the study of difference equations and systems of difference equations. Among others, the study of difference equations and systems of difference equations with maximum, have attracted some attention in the last few decades (see, for instance, [1, 5–9, 11–17, 20, 22, 24, 26, 28, 29, 35–51, 54–58] and the related references therein). For some differential equations with maximum see, for example, [18, 19].

At the beginning were usually studied the difference equations and systems containing several arguments of the form $A_k(n)/x(n-k)$ where $k = 0, 1, \dots,$ and $A_k(n)$ is a given sequence of real numbers (see, for example, [5, 7, 9, 15–17, 26, 28, 29, 56–58]), whereas equations and systems containing several arguments of the form $x^p(n-k)$, where p is a real number, have been usually studied recently (see, for example, [1, 6, 12–14, 35–49, 51, 52, 54, 55]).

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The motivation for the study of such difference equations and systems of difference equations stems from the study of the equations of the form

$$x(n) = a + \frac{x^p(n-k)}{x^q(n-l)}, \qquad n = 1, 2, \ldots,$$

where the parameters a, p, q, and the initial values $x(j)$, $j = -\max\{k, l\}, \ldots, 0$, are real or nonnegative numbers and k and l are positive integers, and their generalizations (see, for example, [2–4, 21, 23, 25, 27, 30–36] and the references cited therein).

In [10] was initiated studying cyclic systems of difference equations. The study was continued, for instance, in [11, 24, 46, 49, 52–55].

In [55] was studied the behaviour of the solutions of the following cyclic system of difference equations with maximum:

$$x_i(n+1) = \max\left\{A_i, \frac{x_i(n)}{x_{i+1}(n-1)}\right\}, \qquad i = 1, 2, \ldots, k,$$

where $n = 0, 1, \ldots,$ the coefficients A_i, $i = 1, 2, \ldots, k$ are positive constants, and the initial values $x_i(-1), x_i(0)$, $i = 1, 2, \ldots, k$ are real positive numbers. Moreover, for $k = 2$ under some conditions it were found solutions which converge to periodic six solutions.

In this paper we continue the investigation of cyclic systems of difference equations by studying the behaviour of the solutions of the following generalized cyclic system of difference equations with maximum:

$$x_i(n+1) = \max\left\{A_i, \frac{x_i^p(n)}{x_{i+1}^q(n-1)}\right\}, \qquad i = 1, 2, \ldots, k, \tag{1.1}$$

where $n = 0, 1, \ldots,$ for the coefficients A_i we assume that $A_i > 1$, $i = 1, 2, \ldots, k$, the exponents p, q and the initial values $x_i(-1), x_i(0)$, $i = 1, 2, \ldots, k$ are positive real numbers, and since the system is cyclic we have $A_{\lambda k+i} = A_i$, $x_{\lambda k+i}(n) = x_i(n)$, λ positive integer, $i = 1, 2, \ldots, k$. To do this we use some methods and ideas in the literature mentioned above. Finally, using the results obtained for the general system (1.1), we derive some further results for system (1.1) for $k = 2$.

2 Main results

Lemma 2.1. *Consider the system of algebraic equations*

$$x_i = \max\left\{A_i, \frac{x_i^p}{x_{i+1}^q}\right\}, \qquad i = 1, 2, \ldots, k, \tag{2.1}$$

where

$$A_{\lambda k+i} = A_i, \qquad x_{\lambda k+i} = x_i, \qquad i = 1, 2, \ldots k, \qquad \lambda \text{ is a positive integer}, \tag{2.2}$$

and

$$A_i > 1, \qquad i = 1, 2, \ldots k, \tag{2.3}$$

then

(i) if

$$0 < p \le 1, \qquad q > 0, \tag{2.4}$$

then system (2.1) *has a unique solution, which is*

$$(A_1, A_2, \dots, A_k).$$

(ii) If

$$p > 1, \qquad 0 < q < p - 1, \tag{2.5}$$

then system (2.1) *has no solutions.*

(iii) Suppose that

$$p > 1, \qquad q > p - 1. \tag{2.6}$$

If there exist m positive integers

$$r_1, r_2, \dots, r_m \in \{1, 2, \dots, k\}, \qquad r_1 < r_2 < \cdots < r_m, \qquad m \in \{1, 2, \dots, k\}, \tag{2.7}$$

such that

$$A_i < A_{r_j}^{\left(\frac{q}{p-1}\right)^{k+r_j-i}}, \quad \text{for any } i \in \{r_j, r_j + 1, \dots, k\}, \text{ and for any } j \in \{1, 2, \dots, m\}, \tag{2.8}$$

and

$$A_i < A_{r_j}^{\left(\frac{q}{p-1}\right)^{r_j-i}}, \quad \text{for any } i \in \{1, 2, \dots, r_j - 1\}, \text{ and for any } j \in \{1, 2, \dots, m\}, \tag{2.9}$$

and for any $r \in \{1, 2, \dots, k\}$, $r \neq r_j$, $j \in \{1, 2, \dots, m\}$, *there exists an integer* $i \in \{1, 2, \dots, k\}$, *such that*

$$A_i > A_r^{\left(\frac{q}{p-1}\right)^{k+r-i}}, \quad \text{for } i > r, \tag{2.10}$$

or

$$A_i > A_r^{\left(\frac{q}{p-1}\right)^{r-i}}, \quad \text{for } i < r, \tag{2.11}$$

holds, then system (2.1) *has* $2^m - 1$ *solutions.*

(iv) If

$$q = p - 1 > 0, \tag{2.12}$$

then all solutions of (2.1) *are the following*

$$(x_1, x_2, \dots, x_k) = (a, a, \dots, a), \quad \text{for any } a \ge A_w = \max\{A_1, A_2, \dots, A_k\}. \tag{2.13}$$

Proof. From (2.1) and (2.3), we get

$$x_i > 1, \quad \text{for any } i \in \{1, 2, \dots, k\}. \tag{2.14}$$

(i) Suppose that (2.4) holds, then, from (2.14), we have

$$\frac{x_i^p}{x_{i+1}^q} < x_i^p \le x_i, \quad \text{for any } i \in \{1, 2, \dots, k\}. \tag{2.15}$$

Using (2.1) and (2.15), we have

$$x_i = A_i, \quad \text{for any } i \in \{1, 2, \ldots, k\}.$$

(ii) Now, suppose that (2.5) holds. We prove that system (2.1) has no solution.

On the contrary, we assume that there exists a solution of system (2.1). From (2.1), we have

$$x_i \geq \frac{x_i^p}{x_{i+1}^q}, \quad \text{for any } i \in \{1, 2, \ldots, k\}, \tag{2.16}$$

and so from (2.5), (2.14), and (2.16), we get

$$x_{i+1} \geq x_i^{\frac{p-1}{q}} > x_i, \quad \text{for any } i \in \{1, 2, \ldots, k\},$$

and obviously,

$$x_{k+1} > x_k > x_{k-1} > \cdots > x_1. \tag{2.17}$$

From (2.2) and (2.17), we get $x_1 > x_1$. So, system (2.1) has no solution.

(iii) Now, suppose that (2.6) holds.

From (2.3) and (2.6) it is obvious that (2.8) and (2.9) hold for $r_j = w$, where

$$A_w = \max\{A_1, A_2, \ldots, A_k\}.$$

So, $m \geq 1$.

First, we prove that, for every solution of (2.1), there exists a $b \in \{1, 2, \ldots, k\}$ such that

$$x_b = A_b. \tag{2.18}$$

On the contrary, suppose that

$$x_i = \frac{x_i^p}{x_{i+1}^q} = x_{i+1}^{\frac{q}{p-1}}, \quad \text{for any } i \in \{1, 2, \ldots, k\}. \tag{2.19}$$

From (2.2) and (2.19), we get

$$x_1 = x_{k+1}^{\left(\frac{q}{p-1}\right)^k} = x_1^{\left(\frac{q}{p-1}\right)^k},$$

and since k is a positive integer and (2.14) holds, we get $q = p - 1$ which contradicts with (2.6). So (2.18) is true.

To continue, we prove that

$$x_i \leq x_{i+1}^{\frac{q}{p-1}}, \quad \text{for any } i \in \{1, 2, \ldots, k\}. \tag{2.20}$$

From (2.1), we get (2.16) and so from (2.6), relation (2.20) is obvious.

In addition, from (2.1),

$$A_i \leq x_i, \quad \text{for any } i \in \{1, 2, \ldots, k\}. \tag{2.21}$$

In what follows, we prove that if there exist $i,\ r \in \{1, 2, \ldots, k\}$, such that either (2.10) or (2.11) holds, then

$$x_r = \frac{x_r^p}{x_{r+1}^q}. \tag{2.22}$$

On the contrary, suppose that

$$x_r = A_r. \tag{2.23}$$

If (2.10) holds, then, from (2.6), (2.20), and (2.21), we have

$$A_i \le x_i \le x_{i+1}^{\frac{q}{p-1}} \le \cdots \le x_k^{(\frac{q}{p-1})^{k-i}} \le x_1^{(\frac{q}{p-1})^{k-i+1}} \le \cdots \le x_r^{(\frac{q}{p-1})^{k-i+r}} = A_r^{(\frac{q}{p-1})^{k+r-i}},$$

which contradicts with (2.10). So, necessarily, if (2.10) holds, then relation (2.22) is true.

Now, suppose that (2.11) holds, then, from (2.6), (2.20), and (2.21), we have

$$A_i \le x_{i+1}^{\frac{q}{p-1}} \le x_{i+2}^{(\frac{q}{p-1})^2} \le \cdots \le x_r^{(\frac{q}{p-1})^{r-i}} = A_r^{(\frac{q}{p-1})^{r-i}},$$

which contradicts with (2.11). So, necessarily, if (2.11) holds, then relation (2.22) is true.

Finally, suppose that there exist exactly m positive integers such that (2.7), (2.8) and (2.9) hold. For any $j \in \{1,2,\ldots,m\}$, we prove that both equations

$$x_{r_j} = A_{r_j}. \tag{2.24}$$

and

$$x_{r_j} = \frac{x_{r_j}^p}{x_{r_j+1}^q}, \tag{2.25}$$

are possible.

Since for any $i \in \{1,2,\ldots,k\}$, $i \ne r_j$, $j \in \{1,2,\ldots,m\}$, relation either (2.10) or (2.11) holds, from (2.22) we get

$$x_i = x_{i+1}^{\frac{q}{p-1}}, \quad \text{for any } i \in \{1,2,\ldots,k\},\ i \ne r_j,\ j \in \{1,2,\ldots,m\}. \tag{2.26}$$

From (2.26),

$$\begin{aligned}
&x_{r_m-1} = x_{r_m}^{\frac{q}{p-1}}, x_{r_m-2} = x_{r_m}^{(\frac{q}{p-1})^2}, \ldots, x_{r_{m-1}+1} = x_{r_m}^{(\frac{q}{p-1})^{r_m-r_{m-1}-1}},\\
&x_{r_{m-1}-1} = x_{r_{m-1}}^{\frac{q}{p-1}}, x_{r_{m-1}-2} = x_{r_{m-1}}^{(\frac{q}{p-1})^2}, \ldots, x_{r_{m-2}+1} = x_{r_{m-1}}^{(\frac{q}{p-1})^{r_{m-1}-r_{m-2}-1}},\\
&\vdots\\
&x_{r_2-1} = x_{r_2}^{\frac{q}{p-1}}, x_{r_2-2} = x_{r_2}^{(\frac{q}{p-1})^2}, \ldots, x_{r_1+1} = x_{r_2}^{(\frac{q}{p-1})^{r_2-r_1-1}},\\
&x_{r_1-1} = x_{r_1}^{\frac{q}{p-1}}, \ldots, x_1 = x_{r_1}^{(\frac{q}{p-1})^{r_1-1}}, x_k = x_{r_1}^{(\frac{q}{p-1})^{r_1}}, \ldots, x_{r_m+1} = x_{r_1}^{(\frac{q}{p-1})^{k-(r_m-r_1)-1}},
\end{aligned} \tag{2.27}$$

and so from (2.1) and (2.27) for $l = 1,2,\ldots,m-1$ we get,

$$x_{r_l} = \max\left\{A_{r_l}, \frac{x_{r_l}^p}{x_{r_l+1}^q}\right\} = \max\left\{A_{r_l}, \frac{x_{r_l}^p}{\left(x_{r_{l+1}}^{(\frac{q}{p-1})^{r_{l+1}-r_l-1}}\right)^q}\right\}.$$

Now, we prove that x_{r_l} can be equal either to A_{r_l} or to $\frac{x_{r_l}^p}{\left(x_{r_{l+1}}^{(\frac{q}{p-1})^{r_{l+1}-r_l-1}}\right)^q}$.

If $x_{r_l} = A_{r_l}$ then, from (2.1), (2.6), (2.7) we get

$$\frac{x_{r_l}^p}{\left(x_{r_{l+1}}^{(\frac{q}{p-1})^{r_{l+1}-r_l-1}}\right)^q} \le \frac{A_{r_l}^p}{\left(A_{r_{l+1}}^{(\frac{q}{p-1})^{r_{l+1}-r_l-1}}\right)^q}. \tag{2.28}$$

Using (2.6), (2.7) and (2.9) for $i = r_l$ and $j = l + 1$ we have

$$A_{r_l} < A_{r_{l+1}}^{(\frac{q}{p-1})^{r_{l+1}-r_l}}$$

and from (2.5)

$$A_{r_l}^{p-1} < A_{r_{l+1}}^{(p-1)(\frac{q}{p-1})^{r_{l+1}-r_l}} = A_{r_{l+1}}^{q\frac{(p-1)}{q}(\frac{q}{p-1})^{r_{l+1}-r_l}} = \left(A_{r_{l+1}}^{(\frac{q}{p-1})^{r_{l+1}-r_l-1}}\right)^q.$$

Then,

$$\frac{A_{r_l}^p}{\left(A_{r_{l+1}}^{(\frac{q}{p-1})^{r_{l+1}-r_l-1}}\right)^q} < A_{r_l}. \tag{2.29}$$

Therefore, from (2.28) and (2.29) we

$$\frac{x_{r_l}^p}{\left(x_{r_{l+1}}^{(\frac{q}{p-1})^{r_{l+1}-r_l-1}}\right)^q} < A_{r_l}.$$

If $x_{r_l} = \frac{x_{r_l}^p}{\left(x_{r_{l+1}}^{(\frac{q}{p-1})^{r_{l+1}-r_l-1}}\right)^q}$ then, from (2.1), (2.6), (2.7) and (2.9) for $i = r_l$ and $j = l+1$, we get

$$x_{r_l} = x_{r_{l+1}}^{(\frac{q}{p-1})^{r_{l+1}-r_l}} \geq A_{r_{l+1}}^{(\frac{q}{p-1})^{r_{l+1}-r_l}} > A_{r_l},$$

and so, for any $j \in \{1, 2, \dots, m-1\}$, both equations (2.24) and (2.25) are possible.

From (2.1) and the last equality of (2.27) we get

$$x_{r_m} = \max\left\{A_{r_m}, \frac{x_{r_m}^p}{x_{r_m+1}^q}\right\} = \max\left\{A_{r_m}, \frac{x_{r_m}^p}{\left(x_{r_1}^{(\frac{q}{p-1})^{k-r_m+r_1-1}}\right)^q}\right\}.$$

Finally, we prove that x_{r_m} can be equal either to A_{r_m} or to $\frac{x_{r_m}^p}{\left(x_{r_1}^{(\frac{q}{p-1})^{k-r_m+r_1-1}}\right)^q}$.

If $x_{r_m} = A_{r_m}$, then, from (2.1), (2.6), (2.7), we get

$$\frac{x_{r_m}^p}{\left(x_{r_1}^{(\frac{q}{p-1})^{k-r_m+r_l-1}}\right)^q} \leq \frac{A_{r_m}^p}{\left(A_{r_1}^{(\frac{q}{p-1})^{k-r_m+r_1-1}}\right)^q}. \tag{2.30}$$

Using (2.6), (2.7) and (2.8) for $i = r_m$ and $j = 1$, we have

$$A_{r_m} < A_{r_1}^{(\frac{q}{p-1})^{k-r_m+r_1}}$$

and so, arguing as to prove (2.29)

$$\frac{A_{r_m}^p}{\left(A_{r_1}^{(\frac{q}{p-1})^{k-r_m+r_1-1}}\right)^q} < A_{r_m}. \tag{2.31}$$

Therefore, from (2.30) and (2.31), we take

$$\frac{x_{r_m}^p}{\left(x_{r_1}^{\left(\frac{q}{p-1}\right)^{k-r_m+r_1-1}}\right)^q} < A_{r_m}.$$

If $x_{r_m} = \frac{x_{r_m}^p}{\left(x_{r_1}^{\left(\frac{q}{p-1}\right)^{k-r_m+r_1-1}}\right)^q}$ then, from (2.1), (2.6), (2.7) and (2.8) for $i = r_m$ and $j = 1$, we get

$$x_{r_m} = x_{r_1}^{\left(\frac{q}{p-1}\right)^{k-r_m+r_1}} \geq A_{r_1}^{\left(\frac{q}{p-1}\right)^{k-r_m+r_1}} > A_{r_m}$$

and so for any $j \in \{1, 2, \ldots, m\}$ both equations (2.24) and (2.25) are possible.

From (2.7), (2.8), (2.9), (2.24), (2.25), and (2.26), and since, for every solution of (2.1) there exists at least one r such that (2.18) holds, we have that system (2.1) has $2^m - 1$ solutions.

(iv) Finally, suppose that (2.12) holds. From (2.1) and (2.12), we get

$$x_i \geq \frac{x_i^p}{x_{i+1}^{p-1}}, \quad \text{for any } i \in \{1, 2, \ldots, k\},$$

and so

$$x_{i+1} \geq x_i, \quad \text{for any } i \in \{1, 2, \ldots, k\}. \tag{2.32}$$

From (2.2) and (2.32), we have

$$x_{k+1} = x_1 \geq x_k \geq x_{k-1} \geq \cdots \geq x_2 \geq x_1,$$

which means that

$$x_1 = x_2 = \cdots = x_k.$$

Then, from (2.1) and (2.12), if we set $x_i = a$, $i = 1, 2, \ldots, k$, we get

$$a = \max\{A_i, a\}, \qquad i = 1, 2, \ldots, k.$$

Therefore, if $a \geq A_w$, we get that all the solutions of (2.1), if (2.12), holds are given by (2.13). This completes the proof of the Lemma 2.1. □

In the following proposition we give a result concerning the global behavior of the solutions of (1.1). Since the proof is similar to the proof of Proposition 2.2 of [55], we omit it.

Proposition 2.2. *Consider the system of difference equations* (1.1). *If* (2.4) *holds, then every solution of* (1.1) *is eventually equal to the unique equilibrium* $(A_1, A_2, \ldots, A_k)$.

In the following lemma we prove some results concerning the solutions of (1.1), which can be used in order to study the behavior of these solutions.

Lemma 2.3. *Consider the system of difference equations* (1.1) *where*

$$p > 1 \quad \text{and} \quad q > 0. \tag{2.33}$$

For a solution of (1.1), *suppose that there exist a* $j \in \{1, 2, \ldots, k\}$, *a positive integer* $S_j \geq 2$, *and a constant* $a > 0$, *such that*

$$x_j(n) = a, \quad \text{for any } n \geq S_j, \tag{2.34}$$

then

(i) If

$$x_{j-1}(S_j+1) > a^{\frac{q}{p-1}}, \tag{2.35}$$

then the solution of (1.1) *is unbounded.*

(ii) If

$$x_{j-1}(S_j+1) < a^{\frac{q}{p-1}}, \tag{2.36}$$

then there exists an integer $S_{j-1} \geq S_j+1$*, such that*

$$x_{j-1}(n) = A_{j-1}, \quad \text{for any } n \geq S_{j-1}. \tag{2.37}$$

(iii) If

$$x_{j-1}(S_j+1) = a^{\frac{q}{p-1}}, \tag{2.38}$$

then

$$x_{j-1}(n) = a^{\frac{q}{p-1}}, \quad \text{for any } n \geq S_j+1. \tag{2.39}$$

Proof. (i) From (1.1) and (2.34), we get

$$x_{j-1}(S_j+2) \geq \frac{x_{j-1}^{p}(S_j+1)}{x_j^{q}(S_j)} = \frac{x_{j-1}^{p}(S_j+1)}{a^q},$$

$$x_{j-1}(S_j+3) \geq \frac{x_{j-1}^{p}(S_j+2)}{x_j^{q}(S_j+1)} \geq \frac{x_{j-1}^{p^2}(S_j+1)}{a^{q(1+p)}},$$

and working inductively we have

$$x_{j-1}(S_j+m) \geq \frac{x_{j-1}^{p^{m-1}}(S_j+1)}{a^{q(1+p+p^2+\cdots+p^{m-2})}} = \frac{x_{j-1}^{p^{m-1}}(S_j+1)}{a^{q\frac{p^{m-1}-1}{p-1}}} = a^{\frac{q}{p-1}}\left(\frac{x_{j-1}(S_j+1)}{a^{\frac{q}{p-1}}}\right)^{p^{m-1}}, \quad m \geq 2. \tag{2.40}$$

From (2.33), (2.35), and (2.40), we get

$$\lim_{n\to\infty} x_{j-1}(n) = \infty,$$

and so, the solution of (1.1) is unbounded.

(ii) Now, suppose that (2.36) holds.

First, we prove that there exists a positive integer $S_{j-1} \geq S_j+1$, such that

$$x_{j-1}(S_{j-1}) = A_{j-1}. \tag{2.41}$$

If

$$x_{j-1}(S_j+1) = A_{j-1},$$

then (2.41) holds for $S_{j-1} = S_j+1$.

Now, suppose that

$$x_{j-1}(n) > A_{j-1}, \quad \text{for any } n \geq S_j+1, \tag{2.42}$$

then, from (1.1) and (2.34), and working as to prove (2.40), we have

$$x_{j-1}(S_j+m) = a^{\frac{q}{p-1}}\left(\frac{x_{j-1}(S_j+1)}{a^{\frac{q}{p-1}}}\right)^{p^{m-1}}, \qquad m \geq 2. \tag{2.43}$$

From (2.33), (2.36) and (2.43), we have that there exists a positive integer $n_0 \geq S_j + 2$, such that

$$x_{j-1}(n) < A_{j-1}, \quad \text{for any } n \geq n_0,$$

which contradicts with (2.42). So, in any case, there exists a positive integer $S_{j-1} \geq S_j + 1$, such that (2.41) holds.

Now, we prove that (2.37) holds for any $n \geq S_{j-1}$.

From (1.1) and (2.36), we get

$$A_{j-1} < a^{\frac{q}{p-1}}. \tag{2.44}$$

From (2.34), (2.41) and (2.44), we have

$$\frac{x_{j-1}^p(S_{j-1})}{x_j^q(S_{j-1}-1)} = \frac{A_{j-1}^p}{a^q} < \frac{A_{j-1}^p}{A_{j-1}^{p-1}} = A_{j-1},$$

and so, from (1.1), we have

$$x_{j-1}(S_{j-1}+1) = A_{j-1},$$

and working inductively we get (2.37).

(iii) Finally, suppose that (2.38) holds.

From (1.1) and (2.38), we get

$$A_{j-1} \leq a^{\frac{q}{p-1}}. \tag{2.45}$$

Using (2.34), (2.38) and (2.45), we get

$$\frac{x_{j-1}^p(S_j+1)}{x_j^q(S_j)} = \frac{a^{\frac{pq}{p-1}}}{a^q} = a^{\frac{q}{p-1}} \geq A_{j-1},$$

and so, from (1.1), we get

$$x_{j-1}(S_j+2) = a^{\frac{q}{p-1}},$$

and working inductively (2.39) is true.

So, the proof of Lemma 2.3 is completed. □

In the following propositions, we give furthermore results for system (1.1), where $k = 2$ and relation (2.6) or (2.12) holds. Our aim is to present how the results of Lemma 2.3 can be used, in order to find out how a solution of (1.1) behaves.

In what follows, without loss of generality, we assume that $A_2 = \max\{A_1, A_2\}$. If, in addition, (2.6) holds, and since $A_2 > 1$, we have that

$$A_1 < A_2^{\frac{q}{p-1}}. \tag{2.46}$$

Proposition 2.4. *Consider the system of difference equations*

$$\begin{aligned} x_1(n+1) &= \max\left\{A_1, \frac{x_1^p(n)}{x_2^q(n-1)}\right\}, \\ x_2(n+1) &= \max\left\{A_2, \frac{x_2^p(n)}{x_1^q(n-1)}\right\}, \end{aligned} \tag{2.47}$$

where $n = 0, 1, \ldots$, A_1, $A_2 > 1$, and the initial values $x_i(-1), x_i(0)$, $i = 1, 2$, are positive real numbers. Suppose that (2.6) holds.

The following statements are true:

I. *Suppose that*

$$A_2 > A_1^{\frac{q}{p-1}}. \tag{2.48}$$

Then system (2.47) has a unique equilibrium which is

$$(A_2^{\frac{q}{p-1}}, A_2). \tag{2.49}$$

Furthermore, we have:

(a) *There exist solutions $(x_1(n), x_2(n))$ of (2.47), for which, there exists an integer $r \geq 2$, such that*

$$x_1(r) < A_2^{\frac{q}{p-1}}. \tag{2.50}$$

These solutions are unbounded.

(b) *There exist solutions $(x_1(n), x_2(n))$ of (2.47), such that*

$$x_1(n) \geq A_2^{\frac{q}{p-1}}, \quad \text{for any } n \geq 2, \tag{2.51}$$

and

$$x_1(z) = A_2^{\frac{q}{p-1}}, \quad \text{for an integer } z \geq 2. \tag{2.52}$$

These solutions are eventually equal to the unique equilibrium (2.49).

(c) *There exist solutions $(x_1(n), x_2(n))$ of (2.47), such that*

$$x_1(n) > A_2^{\frac{q}{p-1}}, \quad \text{for any } n \geq 2, \tag{2.53}$$

and

$$x_2(d) = A_2, \quad \text{for an integer } d \geq 2. \tag{2.54}$$

These solutions are unbounded.

II. *Suppose that*

$$A_2 < A_1^{\frac{q}{p-1}}. \tag{2.55}$$

Then system (2.47) has three equilibria, the one given by (2.49), and the following two,

$$(A_1, A_1^{\frac{q}{p-1}}), \tag{2.56}$$

and

$$(A_1, A_2). \tag{2.57}$$

Furthermore, we have:

(a) *There exist solutions $(x_1(n), x_2(n))$ of (2.47), for which, there exists an integer $r \geq 2$, such that (2.50) holds. These solutions are unbounded or eventually equal to the equilibrium (2.56) or eventually equal to the equilibrium (2.57).*

(b) *There exist solutions $(x_1(n), x_2(n))$ of (2.47), such that (2.51) and (2.52) hold. These solutions are eventually equal to the equilibrium (2.49).*

(c) *There exist solutions $(x_1(n), x_2(n))$ of (2.47), such that (2.53) and (2.54) hold. These solutions are unbounded.*

Proof. **(I.)** From (2.46), (2.48) and (iii) of Lemma 2.1, we have that system (2.47) has a unique equilibrium given by (2.49).

I(a). First, we prove that there exist solutions $(x_1(n), x_2(n))$ of (2.47), for which there exists an integer $r \geq 2$, such that (2.50) holds.
Indeed, if, for instance,

$$x_1(-1) > 0,\ x_1(0) > 0 \quad \text{and} \quad x_2(-1) \geq \frac{x_1^{\frac{p}{q}}(0)}{A_1^{\frac{1}{q}}}, \qquad x_2(0) > \frac{A_1^{\frac{p}{q}}}{A_2^{\frac{1}{p-1}}},$$

then, it is easy to prove that

$$x_1(2) < A_2^{\frac{q}{p-1}},$$

and so (2.50) is true for $r = 2$.

Now, we prove that, if for a solution of (2.47), relation (2.50) is satisfied, then the solution is unbounded.

At the beginning, we prove that there exists a positive integer $s \geq r$, such that

$$x_1(s) = A_1. \tag{2.58}$$

On the contrary, suppose that

$$x_1(n) > A_1, \quad \text{for any } n \geq r, \tag{2.59}$$

then, from (2.47), we have

$$x_1(r+1) = \frac{x_1^p(r)}{x_2^q(r-1)} \leq \frac{x_1^p(r)}{A_2^q},$$
$$x_1(r+2) = \frac{x_1^p(r+1)}{x_2^q(r)} \leq \frac{x_1^{p^2}(r)}{A_2^{q(1+p)}},$$

and working inductively and as in (2.40), we get

$$x_1(r+m) \leq A_2^{\frac{q}{p-1}} \left(\frac{x_1(r)}{A_2^{\frac{q}{p-1}}} \right)^{p^m}, \qquad m \geq 1. \tag{2.60}$$

From (2.6), (2.50) and (2.60), we have that there exists a positive integer $n_0 \geq r$, such that

$$x_1(n) < A_1, \quad \text{for any } n \geq n_0,$$

which contradicts with (2.59). So, if (2.50) holds, then there exists a positive integer $s \geq r$, such that (2.58) holds.

Now, we prove that

$$x_1(n) = A_1, \quad \text{for any } n \geq s. \tag{2.61}$$

From (2.46), (2.47) and (2.58), we get

$$\frac{x_1^p(s)}{x_2^q(s-1)} \leq \frac{A_1^p}{A_2^q} \leq \frac{A_1^p}{A_1^{p-1}} = A_1. \tag{2.62}$$

From (2.47) and (2.62), obviously,

$$x_1(s+1) = A_1,$$

and working inductively we get (2.61).

From (2.47) and (2.48), we have

$$x_2(s+1) \geq A_2 > A_1^{\frac{q}{p-1}},$$

and so, from (2.61) and (i) of Lemma 2.3 for $a = A_1$, we have that the solution is unbounded.

I(b). We show that there exist solutions $(x_1(n), x_2(n))$ of (2.47) and an integer $z \geq 2$, such that (2.51) and (2.52) hold.

Indeed, if, for instance,

$$x_1(0) > A_2^{\frac{p-1}{q}}, \qquad x_1(-1) > A_2^{\frac{p-1}{q}}, \qquad x_2(0) = A_2, \qquad x_2(-1) = \frac{x_1^{\frac{p}{q}}(0)}{A_2^{\frac{1}{p-1}}},$$

it is easy to prove that

$$x_1(n) \geq A_2^{\frac{q}{p-1}}, \qquad n \geq -1 \quad \text{and} \quad x_1(2) = A_2^{\frac{q}{p-1}}.$$

Now, we prove that, if for a solution of (2.47), relations (2.51) and (2.52) hold, then the solution is eventually equal to the unique equilibrium (2.49).

From (2.47) and (2.52), we have

$$\frac{x_1^p(z)}{x_2^q(z-1)} \leq \frac{\left(A_2^{\frac{q}{p-1}}\right)^p}{A_2^q} = A_2^{\frac{q}{p-1}},$$

and so, from (2.46) and (2.47), we get

$$x_1(z+1) \leq A_2^{\frac{q}{p-1}},$$

and from (2.51) we have

$$x_1(z+1) = A_2^{\frac{q}{p-1}}.$$

Working inductively, we get

$$x_1(n) = A_2^{\frac{q}{p-1}} > A_1, \quad \text{for any } n \geq z. \tag{2.63}$$

From (2.47) and (2.63), we get

$$A_2^{\frac{q}{p-1}} = \max\left\{A_1, \frac{A_2^{\frac{pq}{p-1}}}{x_2^q(n)}\right\}, \qquad n \geq z-1,$$

and so, from (2.46), we have

$$x_2(n) = A_2, \quad \text{for any } n \geq z-1. \tag{2.64}$$

From (2.63) and (2.64), we have that the solution is eventually equal to the unique equilibrium (2.49).

I(c). We show that there exist solutions $(x_1(n), x_2(n))$ of (2.47) and an integer $d \geq 3$, such that (2.53) and (2.54) hold.

Indeed, if, for instance,

$$x_1(-1) > A_2^{\frac{p-1}{q}}, \qquad x_1(0) > A_2^{\frac{q}{p-1}} \quad \text{and} \quad x_2(-1) \leq A_2, \qquad x_2(0) \leq A_2,$$

it is easy to prove that

$$x_1(n) > A_2^{\frac{q}{p-1}}, \quad \text{for any } n \geq 2 \quad \text{and} \quad x_2(3) = A_2.$$

Now, we prove that, if for a solution of (2.47), relations (2.53) and (2.54) hold, then the solution is unbounded.

From (2.53) and (2.54), we have

$$\frac{x_2^p(d)}{x_1^q(d-1)} < \frac{A_2^p}{\left(A_2^{\frac{q}{p-1}}\right)^q} < A_2, \tag{2.65}$$

and so, from (2.47),

$$x_2(d+1) = A_2, \tag{2.66}$$

and working inductively, obviously,

$$x_2(n) = A_2, \quad \text{for any } n \geq d. \tag{2.67}$$

Since (2.53) hold, then from (2.67) and (i) of Lemma 2.3 for $a = A_2$, we have that the solution is unbounded.

II. From (2.46), (2.55) and (iii) of Lemma 2.1 we have that system (2.47) has three equilibria, which are given by (2.49), (2.56) and (2.57).

II(a). For a solution $(x_1(n), x_2(n))$ of (2.47) suppose that there exists an integer $r \geq 2$, such that (2.50) holds. Then, arguing as in **I(a)**, we get that there exists a positive integer $s \geq r$, such that (2.61) holds.

If

$$x_2(s+1) > A_1^{\frac{q}{p-1}}, \tag{2.68}$$

then from (2.61), (2.68) and (i) of Lemma 2.3 for $a = A_1$, we have that the solution is unbounded.

If

$$x_2(s+1) < A_1^{\frac{q}{p-1}}, \tag{2.69}$$

then from (2.61), (2.69) and (ii) of Lemma 2.3 for $a = A_1$, we have that there exists an integer $s_2 \geq s+1$, such that

$$x_2(n) = A_2, \quad \text{for any } n \geq s_2. \tag{2.70}$$

From (2.61) and (2.70), we have that the solution is eventually equal to the equilibrium (2.57).

If

$$x_2(s+1) = A_1^{\frac{q}{p-1}}, \tag{2.71}$$

then from (2.61), (2.71) and (iii) of Lemma 2.3 for $a = A_1$, we have that

$$x_2(n) = A_1^{\frac{q}{p-1}}, \quad \text{for any } n \geq s+1. \tag{2.72}$$

From (2.61) and (2.72) we have that the solution is eventually equal to the equilibrium (2.56).

Now, we show that there exist solutions $(x_1(n), x_2(n))$ of (2.47) and integers r, s, $r \geq 2$, $s \geq r$, such that (2.50) and (2.68) hold.

Indeed, if, for instance,

$$x_1(0) > 0, \qquad x_2(0) > A_1^{\frac{p-1}{q}} \quad \text{and} \quad x_1(-1) < \frac{x_2^{\frac{p}{q}}(0)}{A_1^{\frac{1}{p(p-1)}} x_1^{\frac{1}{p}}(0)}, \qquad x_2(-1) \geq \frac{x_1^{\frac{p}{q}}(0)}{A_1^{\frac{1}{q}}}, \tag{2.73}$$

it is easy to prove that

$$x_1(2) = A_1 < A_2^{\frac{q}{p-1}} \quad \text{and} \quad x_2(3) > A_1^{\frac{q}{p-1}},$$

and so these solutions are unbounded.

In addition, we show that there exist solutions $(x_1(n), x_2(n))$ of (2.47) and integers r, s, $r \geq 2$, $s \geq r$, such that (2.50) and (2.69) hold.

Indeed, if, for instance,

$$x_1(0) > \frac{A_2^{\frac{p}{q}}}{A_1^{\frac{1}{p-1}}}, \qquad x_2(0) > A_1^{\frac{p-1}{q}}, \qquad x_2(-1) \geq \frac{x_1^{\frac{p}{q}}(0)}{A_1^{\frac{1}{q}}}, \qquad x_1(-1) > \frac{x_2^{\frac{p}{q}}(0)}{A_1^{\frac{1}{p(p-1)}} x_1^{\frac{1}{p}}(0)},$$

it is easy to prove that

$$x_1(2) = A_1 < A_2^{\frac{q}{p-1}} \quad \text{and} \quad x_2(3) < A_1^{\frac{q}{p-1}},$$

and so these solutions are eventually equal to the equilibrium (2.57).

Finally, we show that there exist solutions $(x_1(n), x_2(n))$ of (2.47) and integers r, s, $r \geq 2$, $s \geq r$, such that (2.50) and (2.71) hold.

Indeed, if, for instance,

$$x_1(0) > \frac{A_2^{\frac{p}{q}}}{A_1^{\frac{1}{p-1}}}, \qquad x_2(0) > A_1^{\frac{p-1}{q}}, \qquad x_2(-1) \geq \frac{x_1^{\frac{p}{q}}(0)}{A_1^{\frac{1}{q}}}, \qquad x_1(-1) = \frac{x_2^{\frac{p}{q}}(0)}{A_1^{\frac{1}{p(p-1)}} x_1^{\frac{1}{p}}(0)}, \tag{2.74}$$

it is easy to prove that

$$x_1(2) = A_1 < A_2^{\frac{q}{p-1}} \quad \text{and} \quad x_2(3) = A_1^{\frac{q}{p-1}},$$

and so these solutions are eventually equal to the equilibrium (2.56).

II(b). The proof is the same as in **I**(b).

II(c). The proof is the same as in **I**(c). □

Proposition 2.5. *Consider the system of difference equations*

$$\begin{aligned} x_1(n+1) &= \max\left\{A_1, \frac{x_1^p(n)}{x_2^{p-1}(n-1)}\right\}, \\ x_2(n+1) &= \max\left\{A_2, \frac{x_2^p(n)}{x_1^{p-1}(n-1)}\right\}, \end{aligned} \tag{2.75}$$

where $n = 0, 1, \ldots,$ $A_1, A_2 > 1$, *and the initial values* $x_i(-1), x_i(0)$, $i = 1, 2$, *are positive real numbers.*

The following statements are true.

(a) *There exist solutions* $(x_1(n), x_2(n))$ *of* (2.75)*, for which, there exists an integer* $r \geq 2$*, such that*

$$x_1(r) < A_2. \tag{2.76}$$

These solutions are unbounded.

(b) *There exist solutions* $(x_1(n), x_2(n))$ *of* (2.75)*, such that*

$$x_1(n) \geq A_2, \quad \text{for any } n \geq 2, \tag{2.77}$$

and

$$x_1(z) = A_2, \quad \text{for an integer } z \geq 2. \tag{2.78}$$

These solutions are unbounded or eventually equal to the equilibrium (A_2, A_2)*.*

(c) *There exist solutions* $(x_1(n), x_2(n))$ *of* (2.75)*, such that*

$$x_1(n) > A_2, \quad \text{for any } n \geq 2, \tag{2.79}$$

and

$$x_2(d) = A_2, \quad \text{for an integer } d \geq 2. \tag{2.80}$$

These solutions are unbounded.

(d) *The solution* $(x_1(n), x_2(n)) = (a, a)$*,* $n \geq -1$*,* $a > A_2$*, is the only solution of* (2.75)*, which is eventually equal to the equilibrium* (a, a)*.*

Proof. **(a)** Since Lemma 2.3 holds for $q = p - 1 > 0$ and, from (2.75) and (2.76), we get that $A_1 < A_2$, the proof of **(a)** is exactly the same with the proof of **I(a)** of Proposition 2.4, and we omit it.

(b) If $A_1 < A_2$, then arguing as in the proof of **I(b)** of Proposition 2.4, we can prove that, there exist solutions $(x_1(n), x_2(n))$ of (2.75), such that relations (2.77) and (2.78) hold, and these solutions are eventually equal to the equilibrium (A_2, A_2).

If $A_1 = A_2$, then for a solution $(x_1(n), x_2(n))$ of (2.75), such that (2.77) and (2.78) hold, we have

$$x_1(z) = A_1, \quad \text{for an integer } z \geq 2, \tag{2.81}$$

and so, arguing as to prove (2.61), we get

$$x_1(n) = A_1, \quad \text{for any } n \geq z. \tag{2.82}$$

If

$$x_2(z+1) = A_2 = A_1, \tag{2.83}$$

then, from (2.82), (2.83), and (iii) of Lemma 2.3 for $a = A_1$ and $q = p - 1 > 0$, we have that

$$x_2(n) = A_2 = A_1, \quad \text{for any } n \geq z + 1. \tag{2.84}$$

From (2.82) and (2.84), we have that the solution is eventually equal to the equilibrium (A_2, A_2).

If

$$x_2(z+1) > A_2 = A_1, \tag{2.85}$$

then, from (2.82), (2.85), and (i) of Lemma 2.3 for $a = A_1$ and $q = p - 1 > 0$, we have that the solution is unbounded.

Now, we show that there exist solutions $(x_1(n), x_2(n))$ of (2.75), such that (2.81) and (2.85) hold for an integer z, $z \geq 2$. Indeed, if, for instance, relations (2.74) hold for $q = p - 1 > 0$, then it is easy to prove that

$$x_1(2) = A_1 = A_2 \quad \text{and} \quad x_2(3) = A_1 = A_2,$$

and so these solutions are eventually equal to the equilibrium (A_2, A_2).

In addition, we show that there exist solutions $(x_1(n), x_2(n))$ of (2.75), such that (2.81) and (2.83) hold for an integer z, $z \geq 2$,.

Indeed, if, for instance, relations (2.73) hold for $q = p - 1 > 0$, then it is easy to prove that

$$x_1(2) = A_1 = A_2 \quad \text{and} \quad x_2(3) > A_1 = A_2,$$

and so these solutions are unbounded.

(c) Relation (2.65), for $q = p - 1 > 0$, becomes

$$\frac{x_2^p(d)}{x_1^{p-1}(d-1)} < \frac{A_2^p}{A_2^{p-1}} = A_2,$$

and so, we have that, (2.66) also holds, and since Lemma 2.3 holds for $q = p - 1 > 0$, the proof of **(**c**)** is exactly the same with the proof of **I**(c) of Proposition 2.4, and we omit it.

(d) Suppose that $(x_1(n), x_2(n))$ is a solution of (2.75) eventually equal to the equilibrium (a, a), $a > A_2$. Then, there exists a positive integer n_0, such that

$$x_1(n) = a, \qquad x_2(n) = a, \quad \text{for any } n \geq n_0. \tag{2.86}$$

Since $a > A_2$, from (2.75) and (2.86), we have

$$x_1(n_0+1) = \frac{x_1^p(n_0)}{x_2^{p-1}(n_0-1)}, \qquad x_2(n_0+1) = \frac{x_2^p(n_0)}{x_1^{p-1}(n_0-1)},$$

and so, $x_2(n_0 - 1) = a$ and $x_1(n_0 - 1) = a$. Working inductively, we get

$$x_1(n) = a, \qquad x_2(n) = a, \quad \text{for any } -1 \leq n \leq n_0 - 1. \tag{2.87}$$

From (2.86) and (2.87), we have that

$$x_1(n) = a, \qquad x_2(n) = a, \quad \text{for any } n \geq -1. \qquad \square$$

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A nontrivial solution for a nonautonomous Choquard equation with general nonlinearity

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Abstract. With the help of the monotonicity trick, a nonautonomous Choquard equations with general nonlinearity is studied and a nontrivial solution is obtained.

Keywords: Choquard equations, monotonicity trick, Pohožaev identity.

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1 Introduction and main result

In the paper, we explore nontrivial solutions for the following nonlocal problem

$$-\Delta u + V(x)u = \left(\frac{1}{|x|} * u^2\right)u + g(u) \quad \text{in } \mathbb{R}^3, \tag{1.1}$$

where $\frac{1}{|x|} * u^2 = \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|}dy$, the nonlinearity g satisfies general subcritical growth conditions

(g_1) $g \in C(\mathbb{R}, \mathbb{R})$ is odd;

(g_2) $-\infty < \lim\limits_{s\to 0^+} \frac{g(s)}{s} = -m < 0$;

(g_3) $\lim_{s\to+\infty} \frac{g(s)}{s^5} = 0$; and the potential function V verifies

(V_1) $V \in C(\mathbb{R}^3, (-m, 0])$ and $\lim_{|x|\to\infty} V(x) = 0$;

(V_2) $(\nabla V, x) \in L^{\frac{3}{2}}(\mathbb{R}^3)$ and

$$|(\nabla V, x)|_{\frac{3}{2}} := \left(\int_{\mathbb{R}^3} |(\nabla V, x)|^{\frac{3}{2}} dx\right)^{\frac{2}{3}} < 2S := 2 \inf_{0\neq u\in D^{1,2}(\mathbb{R}^3)} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^3} u^6 dx\right)^{\frac{1}{3}}}.$$

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When $V \equiv 0$ and $g(s) = -s$, Eq. (1.1) is simplified to the classical Choquard equation

$$-\Delta u + u = \left(\frac{1}{|x|} * u^2\right)u \quad \text{in } \mathbb{R}^3. \tag{1.2}$$

Eq. (1.2) appeared at least as early as in 1954, in a work by S. I. Pekar describing the quantum mechanics of a polaron at rest [11]. In 1976, P. Choquard used Eq. (1.2) to describe an electron trapped in its own hole in a certain approximation to Hartree–Fock theory of one component plasma [4]. For more details in the physics aspects, please refer to [7]. Therefore, many scholars have carried out in-depth research on Choquard equations and related problems. For recent results, we refer the readers to [6, 8, 9, 12, 14] and references therein. See also [10] for a broad survey of Choquard equations.

It is important to point out that Liu et al. in [6] considered the following special case of Eq. (1.1)

$$-\Delta u = \left(\frac{1}{|x|} * u^2\right)u + g(u) \quad \text{in } \mathbb{R}^3. \tag{1.3}$$

Under the assumptions (g_1)-(g_3), they investigated ground states of Eq. (1.3) by using the Pohožaev manifold method. In the present paper, we study Eq. (1.1) which can be regarded as the perturbation equation of Eq. (1.3). By using the monotonicity trick we obtain the following main result.

Theorem 1.1. *Suppose that (V_1)–(V_2) and (g_1)–(g_3) hold. Then Eq. (1.1) possesses a nontrivial solution.*

Set $K(x) = V(x) + m$ and $f(s) = g(s) + ms$. Then Eq. (1.1) equals to the following equation

$$-\Delta u + K(x)u = \left(\frac{1}{|x|} * u^2\right)u + f(u) \quad \text{in } \mathbb{R}^3, \tag{1.4}$$

where f satisfies

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$ is odd;

(f_2) $\lim_{s\to 0^+} \frac{f(s)}{s} = 0$;

(f_3) $\lim_{s\to +\infty} \frac{f(s)}{s^5} = 0$;

and K verifies

(K_1) $K \in C(\mathbb{R}^3, (0, m])$ and $\lim_{|x|\to\infty} K(x) = m$;

(K_2) $(\nabla K, x) \in L^{\frac{3}{2}}(\mathbb{R}^3)$ and $|(\nabla K, x)|_{\frac{3}{2}} < 2S$.

Then we convert to consider the following

Theorem 1.2. *Suppose that (K_1)–(K_2) and (f_1)–(f_3) hold. Then Eq. (1.4) has a nontrivial solution.*

Remark 1.3. If $K(x) \equiv m$, then Theorem 1.2 was proved in [6]. Thus we assume that $K(x) \not\equiv m$.

For the rest of this paper, we make the following marks. $H := H^1(\mathbb{R}^3)$ is the usual Sobolev space endowed with the standard norm $\|\cdot\|$. $L^s(\mathbb{R}^3)$, $2 \le s \le 6$, denotes the usual Lebesgue space with the norm $|\cdot|_s$. $C, C_1, C_2, \dots$ denote different positive constants whose exact value is inessential. For any $u \in H$, we define $u_t(\cdot) := u(t^{-1}\cdot)$ for $t > 0$.

It is widely known that the solutions of Eq. (1.4) correspond to the critical points of the functional defined by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + K(x)u^2)dx - \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u^2(x)u^2(y)}{|x-y|}dxdy - \int_{\mathbb{R}^3}F(u)dx, \qquad u \in H,$$

where $F(s) = \int_0^s f(t)dt$. Using the Hardy–Littlewood–Sobolev inequality [5], one has

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u^2(x)u^2(y)}{|x-y|}dxdy \le C|u|^4_{\frac{12}{5}} \le C_1\|u\|^4.$$

Combining with (f_1)–(f_3) and (K_1) we know that I is well defined and I is of C^1. But it is hard to obtain a bounded (PS) sequence for the functional I under the assumptions (f_1)–(f_3). In addition, another difficulty we face is the lack of space compactness.

From (K_1) we know that there exits $a > 0$ such that $a\|u\|^2 \le \int_{\mathbb{R}^3}(|\nabla u|^2 + K(x)u^2)dx$.

2 Preliminaries

In order to prove Theorem 1.2, we cannot directly apply the mountain pass theorem [1]. Instead we use an indirect approach which dated to Struwe [13] and was developed by Jeanjean in [2]. Exactly, we apply the following

Proposition 2.1. *Let X be a Banach space equipped with a norm $\|\cdot\|_X$ and let $J \subset \mathbb{R}^+$ be an interval. We consider a family $\{\Phi_\mu\}_{\mu\in J}$ of C^1-functionals on X of the form*

$$\Phi_\mu(u) = A(u) - \mu B(u), \qquad \forall \mu \in J,$$

where $B(u) \ge 0$ for all $u \in X$ and such that either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\|_X \to +\infty$. We assume that there are two points v_1, v_2 in X such that

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = v_1, \gamma(1) = v_2\},$$

there hold, $\forall \mu \in J$,

$$c_\mu = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\Phi_\mu(\gamma(t)) > \max\{\Phi_\mu(v_1), \Phi_\mu(v_2)\}.$$

Then for almost every $\mu \in J$, there is a sequence $\{u_n\} \subset X$ such that

(i) $\{u_n\}$ is bounded in X,

(ii) $\Phi_\mu(u_n) \to c_\mu$ and

(iii) $\Phi'_\mu(u_n) \to 0$ in the dual X^* of X.

Moreover, the map $\mu \to c_\mu$ is non-increasing and continuous from the left.

Define $f^{\pm}(s) = \max\{\pm f(s), 0\}$, $F_1(s) = \int_0^s f^+(t)dt$ and $F_2(s) = \int_0^s f^-(t)dt$, from (f_1)–(f_3) one has

$$\lim_{s\to 0} \frac{f^{\pm}(s)}{s} = 0 \quad \text{and} \quad \lim_{s\to\infty} \frac{f^{\pm}(s)}{s^5} = 0. \tag{2.1}$$

Set

$$X = H, \quad \|\cdot\|_X = \|\cdot\|, \quad \Phi_\mu = I_\mu, \; J = [2^{-1}, 1],$$
$$B(u) = \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|}dxdy + \int_{\mathbb{R}^3} F_1(u)dx$$

and

$$A(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + K(x)u^2)dx + \int_{\mathbb{R}^3} F_2(u)dx.$$

Then $A(u) \to +\infty$ as $\|u\| \to +\infty$ and

$$\begin{aligned} I_\mu(u) &= A(u) - \mu B(u) \\ &= \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + K(x)u^2)dx + \int_{\mathbb{R}^3} F_2(u)dx \\ &\quad - \frac{\mu}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|}dxdy - \mu\int_{\mathbb{R}^3} F_1(u)dx. \end{aligned}$$

Specially, $I_1(u) = I(u)$. The following limit equations

$$-\Delta u + mu = \mu\left(\frac{1}{|x|} * u^2\right)u + l_\mu(u) \quad \text{in } \mathbb{R}^3, \tag{2.2}$$

will play an important role, where $l_\mu(s) = \mu f^+(s) - f^-(s)$. The energy functional of Eq. (2.2) is defined by

$$I_\mu^\infty(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + mu^2)dx - \frac{\mu}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|}dxdy - \int_{\mathbb{R}^3} L_\mu(u)dx,$$

where $L_\mu(s) = \int_0^s l_\mu(t)dt$. Set

$$c_\mu^\infty = \inf\{I_\mu^\infty(u) : 0 \neq u \in H, (I_\mu^\infty)'(u) = 0\}.$$

Let ω be a positive ground state solution of Eq. (2.2) with $\mu = 1$. By the proof of [6, Lemma 3.3], one has $c_1^\infty = \max_{t>0} I_1^\infty(\omega(t^{-1}x))$.

3 The proof of Theorem 1.2

The following lemma is to verify the assumptions of Proposition 2.1.

Lemma 3.1. *Suppose that (K_1) and (f_1)–(f_3) hold. Then there exist $v_1, v_2 \in H$ such that for any $\mu \in J$, $c_\mu > \max\{I_\mu(v_1), I_\mu(v_2)\}$.*

Proof. From (f_1)–(f_3) it follows that

$$F(s) \leq \frac{a}{4}|s|^2 + C|s|^6 \quad \text{for all } s \in \mathbb{R}.$$

Combining with the Hardy–Littlewood–Sobolev and Sobolev inequality, for any $u \in H$ and $\mu \in J$ one has

$$\begin{aligned} I_\mu(u) &\geq I(u) \\ &\geq \frac{a}{4}\|u\|^2 - \frac{C_1}{4}\|u\|^4 - C\|u\|^6 \end{aligned}$$

which implies that there exist $\alpha, \rho > 0$ such that $I_\mu(u) \geq \alpha$ for all $\mu \in J$ and $\|u\| = \rho$. Let $\omega \in H$ be a positive ground state solution of Eq. (2.2) with $\mu = 1$. For any $\mu \in J$, one has

$$\begin{aligned} I_\mu(\omega_t) &\leq I^\infty_{\frac{1}{2}}(\omega_t) \\ &= \frac{t}{2}\int_{\mathbb{R}^3} |\nabla\omega|^2 dx + \frac{mt^3}{2}\int_{\mathbb{R}^3} \omega^2 dx - \frac{t^5}{8}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\omega^2(x)\omega^2(y)}{|x-y|} dxdy - \frac{t^3}{2}\int_{\mathbb{R}^3} L_{\frac{1}{2}}(\omega) dx. \end{aligned}$$

Combining with

$$\|\omega_t\|^2 = t\int_{\mathbb{R}^3} |\nabla\omega|^2 dx + t^3 \int_{\mathbb{R}^3} |\omega|^2 dx,$$

there exists $t_0 > 0$ such that $\|\omega_{t_0}\| > \rho$ and $I_\mu(\omega_{t_0}) < 0$. Set $v_1 = 0$ and $v_2 = \omega_{t_0}$. Thus for any $\gamma \in \Gamma$, $\max_{t\in[0,1]} I_\mu(\gamma(t)) \geq \alpha > 0$. So $c_\mu \geq \alpha > \max\{I_\mu(v_1), I_\mu(v_2)\}$. □

Lemma 3.2. *Suppose that (K_1) and (f_1)–(f_3) hold. Then there exists $\delta \in [\frac{1}{2}, 1)$ such that for any $\mu \in [\delta, 1]$, $c_\mu < c^\infty_\mu$.*

Proof. According to the proof of Lemma 3.1, for any $\mu \in J$, there exists $t_\mu \in (0, t_0)$ such that $I_\mu(\omega(t_\mu^{-1}x)) = \max_{t\in(0,1]} I_\mu(\omega((t_0 t)^{-1}x)) \geq c_\mu$. Set $\theta = \inf_{\mu\in J} t_\mu$. We claim $\theta > 0$. Otherwise, there exists $\mu_n \in J$ such that $t_{\mu_n} \to 0$ and then

$$c_1 \leq c_{\mu_n} \leq I_{\mu_n}(\omega(t_{\mu_n}^{-1}x)) \to 0.$$

It is a contradiction. Note that $K(x) \leq m$ and $K(x) \not\equiv m$. Define

$$\delta = \max\left\{\frac{1}{2}, 1 - \frac{\theta^3 \min\limits_{s\in[\theta,t_0]} \int_{\mathbb{R}^3}[m - K(sx)]\omega^2 dx}{2t_0^3 \int_{\mathbb{R}^3} F_1(\omega)dx + \frac{t_0^5}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\omega^2(x)\omega^2(y)}{|x-y|}dxdy}\right\}.$$

Then for any $\mu \in [\delta, 1]$, we get

$$\begin{aligned} c^\infty_\mu &\geq c^\infty_1 \\ &\geq I^\infty_1(\omega(t_\mu^{-1}x)) \\ &= I_\mu(\omega(t_\mu^{-1}x)) + \frac{t_\mu^3}{2}\int_{\mathbb{R}^3}[m - K(t_\mu x)]\omega^2 dx - (1-\mu)t_\mu^3 \int_{\mathbb{R}^3} F_1(\omega)dx \\ &\quad - \left(\frac{1}{4} - \frac{\mu}{4}\right) t_\mu^5 \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\omega^2(x)\omega^2(y)}{|x-y|}dxdy \\ &> c_\mu + \frac{\theta^3}{2}\min_{s\in[\theta,t_0]}\int_{\mathbb{R}^3}[m - K(sx)]\omega^2 dx - (1-\mu)t_0^3\int_{\mathbb{R}^3} F_1(\omega)dx \\ &\quad - \left(\frac{1}{4} - \frac{\mu}{4}\right) t_0^5 \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\omega^2(x)\omega^2(y)}{|x-y|}dxdy \\ &\geq c_\mu. \end{aligned}$$

□

Lemma 3.3. *Fix $\mu \in [\delta, 1]$. Suppose that (K_1) and (f_1)–(f_3) hold and that $\{u_n\} \subset H$ is a bounded $(\mathrm{PS})_{c_\mu}$ sequence for I_μ. Then there exists $u \in H$, $k \in \mathbb{N}$, $v_i \in H \backslash \{0\}$, $y_{n,i} \in \mathbb{R}^3$ for $1 \le i \le k$ such that up to a subsequence,*

(i) $|y_{n,i}| \to \infty$, $|y_{n,i} - y_{n,j}| \to \infty$, $i \neq j$, *for* $1 \le i, j \le k$,

(ii) $(I_\mu)'(u) = 0$ *and* $(I_\mu^\infty)'(v_i) = 0$ *for* $1 \le i \le k$,

(iii) $u_n - u - \sum_{i=1}^k v_i(\cdot - y_{n,i}) \to 0$ *in* H,

(iv) $c_\mu = I_\mu(u) + \sum_{i=1}^k I_\mu^\infty(v_i) + o(1)$,

where we agree that in the case $k = 0$ the above holds without v_i, $y_{n,i}$.

Proof. The proof is in the spirit of [3]. Obviously, there exists $u \in H$ such that up to a subsequence $u_n \rightharpoonup u$ in H, $u_n \to u$ in $L^p_{\mathrm{loc}}(\mathbb{R}^3)$ with $2 \le p < 6$ and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^3$. For any $\varphi \in C_0^\infty(\mathbb{R}^3)$, one has

$$0 = \langle I_\mu'(u_n), \varphi \rangle + o(1) = \langle I_\mu'(u), \varphi \rangle.$$

Set $u_{n,1} = u_n - u$. If $u_n \to 0$ in H, we are done. So we can assume that $\{u_{n,1}\}$ does not converge strongly to 0 in H. Thus up to a subsequence $u_{n,1} \rightharpoonup 0$ in H, $u_{n,1} \to 0$ in $L^p_{\mathrm{loc}}(\mathbb{R}^3)$ and $u_{n,1}(x) \to 0$ a.e. in $\mathbb{R}^3$. Then we have

$$\|u_{n,1}\|^2 = \|u_n\|^2 - \|u\|^2 + o(1),$$

$$\int_{\mathbb{R}^3} K(x) u_{n,1}^2 dx = \int_{\mathbb{R}^3} K(x) u_n^2 dx - \int_{\mathbb{R}^3} K(x) u^2 dx + o(1),$$

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_{n,1}^2(x) u_{n,1}^2(y)}{|x-y|} dxdy = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_n^2(x) u_n^2(y)}{|x-y|} dxdy - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x) u^2(y)}{|x-y|} dxdy + o(1)$$

and

$$\int_{\mathbb{R}^3} L_\mu(u_{n,1}) dx = \int_{\mathbb{R}^3} L_\mu(u_n) dx - \int_{\mathbb{R}^3} L_\mu(u) dx + o(1).$$

Therefore,

$$I_\mu(u_{n,1}) = I_\mu(u_n) - I_\mu(u) + o(1).$$

Define

$$\beta_1 = \limsup_{n \to \infty} \sup_{z \in \mathbb{R}^3} \int_{B_1(z)} u_{n,1}^2 dx.$$

If $\beta_1 = 0$, one sees $u_{n,1} \to 0$ in $L^p(\mathbb{R}^3)$ with $2 < p < 6$ from Lion's lemma [15]. Then

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_n^2(x) u_n^2(y)}{|x-y|} dxdy \to \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x) u^2(y)}{|x-y|} dxdy$$

and

$$\int_{\mathbb{R}^3} l_\mu(u_n) u_n dx \to \int_{\mathbb{R}^3} l_\mu(u) u dx.$$

Therefore,

$$0 = \langle I_\mu'(u_n), u_n \rangle + o(1) \ge \langle I_\mu'(u), u \rangle = 0,$$

which infers $u_n \to u$ in H. It is a contradiction. If $\beta_1 > 0$, we may assume the existence of $y_{n,1} \in \mathbb{R}^3$ such that $\int_{B_1(y_{n,1})} u_{n,1}^2 dx > \frac{\beta_1}{2}$. Set $w_{n,1} = u_n(\cdot + y_{n,1})$, there exists $v_1 \in H$ such that up to a subsequence $w_{n,1} \rightharpoonup v_1$ in H, $w_{n,1} \to v_1$ in $L^p_{\text{loc}}(\mathbb{R}^3)$ with $2 \le p < 6$ and $w_{n,1}(x) \to v_1(x)$ a.e. in $\mathbb{R}^3$. From

$$\int_{B_1(0)} v_1^2 dx = \lim_{n\to\infty} \int_{B_1(0)} w_{n,1}^2 dx = \lim_{n\to\infty} \int_{B_1(y_{n,1})} u_n^2 dx = \lim_{n\to\infty} \int_{B_1(y_{n,1})} (u_{n,1}^2 + u^2) dx \ge \frac{\beta_1}{2},$$

we know $v_1 \neq 0$. Since $u_{n,1} \rightharpoonup 0$ in H, $\{y_{n,1}\}$ is unbounded in $\mathbb{R}^3$ and, up to a subsequence, we can assume that $|y_{n,1}| \to \infty$. Thus

$$\begin{aligned} 0 &= \langle I'_\mu(u_n), \varphi(\cdot - y_{n,1})\rangle + o(1) \\ &= \langle (I^\infty_\mu)'(v_1), \varphi\rangle. \end{aligned}$$

Set $u_{n,2} = u_n - u - v_1(\cdot - y_{n,1})$. If $u_{n,2} \to 0$ in H, we are done. So we can assume that $\{u_{n,2}\}$ does not converge strongly to 0 in H. Thus up to a subsequence $u_{n,2} \rightharpoonup 0$ in H, $u_{n,2} \to 0$ in $L^p_{\text{loc}}(\mathbb{R}^3)$ and $u_{n,2}(x) \to 0$ a.e. in $\mathbb{R}^3$. Thus we have

$$\|u_{n,2}\|^2 = \|u_n\|^2 - \|u\|^2 - \|v_1\|^2 + o(1),$$

$$\int_{\mathbb{R}^3} K(x) u_{n,2}^2 dx = \int_{\mathbb{R}^3} K(x) u_n^2 dx - \int_{\mathbb{R}^3} K(x) u^2 dx - \int_{\mathbb{R}^3} m v_1^2 dx + o(1),$$

$$\begin{aligned} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u_{n,2}^2(x) u_{n,2}^2(y)}{|x-y|} dxdy = &\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u_n^2(x) u_n^2(y)}{|x-y|} dxdy - \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x) u^2(y)}{|x-y|} dxdy \\ &- \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{v_1^2(x) v_1^2(y)}{|x-y|} dxdy + o(1) \end{aligned}$$

and

$$\int_{\mathbb{R}^3} L_\mu(u_{n,2}) dx = \int_{\mathbb{R}^3} L_\mu(u_n) dx - \int_{\mathbb{R}^3} L_\mu(u) dx - \int_{\mathbb{R}^3} L_\mu(v_1) dx + o(1).$$

Therefore,

$$I_\mu(u_{n,2}) = I_\mu(u_n) - I_\mu(u) - I^\infty_\mu(v_1) + o(1).$$

Define

$$\beta_2 = \limsup_{n\to\infty} \sup_{z\in\mathbb{R}^3} \int_{B_1(z)} u_{n,2}^2 dx.$$

We replaced $u_{n,1}$ by $u_{n,2}$ and repeat the above arguments. If $\beta_2 = 0$, then $u_{n,2} \to 0$. It is a contradiction. If $\beta_2 > 0$, we may assume the existence of $y_{n,2} \in \mathbb{R}^3$ such that $\int_{B_1(y_{n,2})} u_{n,2}^2 dx > \frac{\beta_2}{2}$. Set $w_{n,2} = u_n(\cdot + y_{n,2})$, there exists $v_2 \in H$ such that up to a subsequence $w_{n,2} \rightharpoonup v_2$ in H, $w_{n,2} \to v_2$ in $L^p_{\text{loc}}(\mathbb{R}^3)$ with $2 \le p < 6$ and $w_{n,2}(x) \to v_2(x)$ a.e. in $\mathbb{R}^3$. From

$$\int_{B_1(0)} v_2^2 dx = \lim_{n\to\infty} \int_{B_1(0)} w_{n,2}^2 dx = \lim_{n\to\infty} \int_{B_1(y_{n,2})} u_n^2 dx = \lim_{n\to\infty} \int_{B_1(y_{n,2})} u_{n,2}^2 dx \ge \frac{\beta_2}{2},$$

we know $v_2 \neq 0$. Since $u_{n,2} \rightharpoonup 0$ in H, $\{y_{n,2}\}$ is unbounded in $\mathbb{R}^3$ and, up to a subsequence, we can assume that $|y_{n,2}| \to \infty$ and $|y_{n,2} - y_{n,1}| \to \infty$. Similarly, $(I^\infty_\mu)'(v_2) = 0$. Set

$$u_{n,3} = u_n - u - v_1(\cdot - y_{n,1}) - v_2(\cdot - y_{n,2}).$$

Again we repeat the above arguments, then there exists $k \in \mathbb{N}$, $v_i \in H\backslash\{0\}$, $y_{n,i} \in \mathbb{R}^3$ for $1 \le i \le k$ such that up to a subsequence, $|y_{n,i}| \to \infty$, $|y_{n,i} - y_{n,j}| \to \infty$, $i \ne j$, for $1 \le i,j \le k$, $(I_\mu^\infty)'(v_i) = 0$ for $1 \le i \le k$,

$$u_{n,k+1} = u_n - u - \sum_{i=1}^{k} v_i(\cdot - y_{n,i})$$

and

$$c_\mu = I_\mu(u_{n,k+1}) + I_\mu(u) + \sum_{i=1}^{k} I_\mu^\infty(v_i) + o(1).$$

Note that there exists $\alpha > 0$ such that $\|v\| \ge \alpha$ for any $v \in \{v \in H : v \ne 0 \text{ and } (I_\mu^\infty)'(v) = 0\}$. The iterations must stop after steps because $\{u_n\}$ is bounded in H. □

For almost every $\mu \in [\delta, 1]$, by Proposition 2.1 there is a sequence $\{u_n\} \subset H$ such that

$$\begin{array}{ll} (i) & \{u_n\} \text{ is bounded in } H, \\ (ii) & I_\mu(u_n) \to c_\mu, \\ (iii) & I'_\mu(u_n) \to 0 \text{ in the dual } H^* \text{ of } H. \end{array} \tag{3.1}$$

Moreover, the map $\mu \to c_\mu$ is non-increasing and continuous from the left.

Lemma 3.4. *Fix $\mu \in [\delta, 1]$. Suppose that (K_1) and (f_1)–(f_3) hold and that $\{u_n\} \subset H$ satisfies (3.1). Then there exists $u \in H$ such that $I_\mu(u) = c_\mu$ and $I'_\mu(u) = 0$.*

Proof. We assume $k \ge 1$ in Lemma 3.3. Then

$$\|u_n - u - \sum_{i=1}^{k} v_i(\cdot - y_{n,i})\| \to 0$$

and

$$c_\mu = I_\mu(u) + \sum_{i=1}^{k} I_\mu^\infty(v_i) + o(1),$$

where $I'_\mu(u) = 0$ and $(I_\mu^\infty)'(v_i) = 0$ for $1 \le i \le k$. Because $I_\mu(u) \ge 0$ and $I_\mu^\infty(v_i) \ge c_\mu^\infty$ for $1 \le i \le k$, we have $c_\mu \ge c_\mu^\infty$. It is a contradiction. Thus $k = 0$ and $\|u_n - u\| \to 0$. Therefore, $I_\mu(u) = c_\mu$ and $I'_\mu(u) = 0$. □

Lemma 3.5. *Suppose that (K_1)–(K_2) and (f_1)–(f_3) hold. Then there exists $u \in H$ such that $I(u) = c_1$ and $I'(u) = 0$.*

Proof. Choosing $\mu_n \in [\delta, 1]$ and $\mu_n \nearrow 1$, then Lemma 3.4 implies that there exists a sequence $\{u_{\mu_n} := u_n\} \subset H$ such that

$$\begin{aligned} c_{\mu_n} &= I_{\mu_n}(u_n) \\ &= \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u_n|^2 + K(x)u_n^2)dx - \frac{\mu_n}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u_n^2(x)u_n^2(y)}{|x-y|}dxdy - \int_{\mathbb{R}^3} L_{\mu_n}(u_n)dx \end{aligned} \tag{3.2}$$

and the following Pohožaev identity

$$\begin{aligned} 0 = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{3}{2}\int_{\mathbb{R}^3} K(x)u_n^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} (\nabla K(x), x)u_n^2 dx \\ - \frac{5\mu_n}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u_n^2(x)u_n^2(y)}{|x-y|}dxdy - 3\int_{\mathbb{R}^3} L_{\mu_n}(u_n)dx. \end{aligned} \tag{3.3}$$

(3.2) × 3 − (3.3) implies

$$3c_{\mu_n} = \int_{\mathbb{R}^3} |\nabla u_n|^2 dx - \frac{1}{2}\int_{\mathbb{R}^3} (\nabla K(x), x) u_n^2 dx + \frac{\mu_n}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u_n^2(x)u_n^2(y)}{|x-y|}dxdy$$
$$\geq \left(1 - \frac{|(\nabla K, x)|_{\frac{3}{2}}}{2S}\right)\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u_n^2(x)u_n^2(y)}{|x-y|}dxdy$$

which implies that

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 dx, \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u_n^2(x)u_n^2(y)}{|x-y|}dxdy \leq C, \qquad \forall\, n \in \mathbb{N}^*.$$

Combining with (2.1) and the Sobolev inequality, we get

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \int_{\mathbb{R}^3} K(x) u_n^2 dx = \mu_n \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u_n^2(x)u_n^2(y)}{|x-y|}dxdy + \int_{\mathbb{R}^3} l_{\mu_n}(u_n) u_n dx$$
$$\leq C + \frac{\min_{x\in\mathbb{R}^3} K(x)}{2}\int_{\mathbb{R}^3} u_n^2 dx + C\int_{\mathbb{R}^3} u_n^6 dx$$

which implies

$$\int_{\mathbb{R}^3} u_n^2 dx \leq C \qquad \forall\, n \in \mathbb{N}^*.$$

Then $\{u_n\}$ is bounded in H. Recall that $\mu_n \nearrow 1$,

$$I(u_n) = c_{\mu_n} + \frac{\mu_n - 1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u_n^2(x)u_n^2(y)}{|x-y|}dxdy + (\mu_n - 1)\int_{\mathbb{R}^3} F_1(u_n) dx$$

and

$$\|I'(u_n)\|_* = \sup_{\|\varphi\|_E = 1}\left|(\mu_n - 1)\int_{\mathbb{R}^3} \frac{u_n(x)\varphi(x)u_n^2(y)}{|x-y|}dxdy + (\mu_n - 1)\int_{\mathbb{R}^3} f^+(u_n)\varphi dx\right|,$$

where $\|\cdot\|_*$ denotes the norm in H^*. So $I(u_n) \to c_1$ and $\|I'(u_n)\|_* \to 0$. According to Lemma 3.3, we get that there exists $u \in H$ such that $I(u) = c_1$ and $I'(u) = 0$.

Hence, we complete the proof. □

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Infinitely many homoclinic solutions for a class of damped vibration problems

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Abstract. In this paper, we consider the multiplicity of homoclinic solutions for the following damped vibration problems

$$\ddot{x}(t) + B\dot{x}(t) - A(t)x(t) + H_x(t, x(t)) = 0,$$

where $A(t) \in (\mathbb{R}, \mathbb{R}^N)$ is a symmetric matrix for all $t \in \mathbb{R}$, $B = [b_{ij}]$ is an antisymmetric $N \times N$ constant matrix, and $H(t, x) \in C^1(\mathbb{R} \times B_\delta, \mathbb{R})$ is only locally defined near the origin in x for some $\delta > 0$. With the nonlinearity $H(t, x)$ being partially sub-quadratic at zero, we obtain infinitely many homoclinic solutions near the origin by using a Clark's theorem.

Keywords: homoclinic solutions, Clark's theorem, critical points, Palais–Smale condition.

2020 Mathematics Subject Classification: 34C25, 35A15, 35B38.

1 Introduction

The homoclinic orbit is an important kind of trajectory in dynamical systems recognized by Poincaré at the end of the 19th century. Their presence often means the occurrence of chaos or the bifurcation behavior of periodic orbits, see [4, 7, 10, 12, 14] and references therein. In recent decades, the existence and multiplicity of homoclinic orbits has been studied in depth via variational methods. In this paper, we consider the existence of infinitely many homoclinic solutions for the following damped vibration problems

$$\ddot{x}(t) + B\dot{x}(t) - A(t)x(t) + H_x(t, x(t)) = 0, \tag{1.1}$$

where $x(t) \in C^2(\mathbb{R}, \mathbb{R}^N)$, $A(t) = [a_{ij}(t)]$ is a symmetric and positive $N \times N$ matrix-valued function with $a_{ij} \in L^\infty(\mathbb{R}, \mathbb{R})(\forall i, j = 1, 2, \dots, N)$, $B = [b_{ij}]$ is an antisymmetric $N \times N$ constant matrix, $H(t, x) \in C^1(\mathbb{R} \times B_\delta, \mathbb{R})$ with $B_\delta = \{x \in \mathbb{R}^N \mid |x| \le \delta\}$ for some $\delta > 0$, $H_x(t, x)$ denote its derivative with respect to the x variable.

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When $B = 0$, the system (1.1) is the classical second-order Hamiltonian systems which has been extensively studied in the past, see [1,5,6,8,11,13,15,16] and references therein. When $B \neq 0$, many authors have studied the existence and multiplicity of homoclinic solutions for (1.1) under various growth conditions, see [2,3,17–19] and references therein. In [17], Wu and Zhang obtained the existence and multiplicity of homoclinic solutions by using a symmetric mountain pass theorem and a generalized mountain pass theorem under the local (AR) superquadratic growth condition. In [2], by using a variant fountain theorem, Chen obtained infinitely many nontrivial homoclinic orbits for non-periodic damped vibration systems when $H(t,x)$ satisfies the subquadratic condition at infinity. In [19], Zhang and Yuan studied the existence of the homoclinic solutions via the genus properties in critical point theory when $H(t,x)$ is of subquadratic growth as $|x| \to +\infty$. In [3], Chen and Tang obtained infinitely many homoclinic solutions for (1.1) by using a fountain theorem when $H(t,x)$ satisfies a new subquadratic condition. In [18], Zhu obtained the existence of nontrivial homoclinic solutions using the mountain pass theorem when $H(t,x)$ satisfies asymptotically quadratic condition.

In this paper, we study the existence of homoclinic solutions for (1.1) when the nonlinearity $H(t,x)$ is only defined near the origin with respect to x and $H(t,x)$ is partially subquadratic at zero. To the best of our knowledge, the existence of homoclinic solutions for damped vibration systems in this case has not been considered before. Our work is motivated by [9], where the authors improved and extended Clark's theorem and applied it to the problems on solutions of elliptic equations and periodic solutions of Hamiltonian systems. Here by using the Clark's theorem in [9], we prove that (1.1) has infinitely many homoclinic solutions near the origin. Furthermore, we make the following assumptions:

(H_1) $H(t,x) \in C^1(\mathbb{R} \times B_\delta, \mathbb{R})$ is even in x, $H(t,x) = H(t,-x)$ for all $t \in \mathbb{R}$ and $x \in B_\delta$, and $H(t,0) = 0$ for all $t \in \mathbb{R}$;

(H_2) There exists constants $\alpha > 0$, such that $(A(t)x, x) \geq \alpha |x|^2$ and $\|B\| < 2\sqrt{\alpha}$ for all $(t,x) \in (\mathbb{R}, \mathbb{R}^N)$;

(H_3) There exist $t_0 \in \mathbb{R}$ and $r > 0$ such that uniformly in $t \in [t_0 - r, t_0 + r]$,

$$\lim_{|x| \to 0} \frac{H(t,x)}{|x|^2} = +\infty;$$

(H_4) For all $(t,x) \in \mathbb{R} \times B_\delta$,

$$|H_x(t,x)| \leq b(t),$$

where $b(t) : \mathbb{R} \to \mathbb{R}$ is a function such that $b \in L^\xi(\mathbb{R})$ for some $1 \leq \xi \leq 2$.

Now, we state the main result as follows.

Theorem 1.1. *Assume that (H_1)–(H_4) hold, then (1.1) has infinitely many homoclinic solution x_k with $\|x_k\|_{L^\infty} \to 0$ as $k \to \infty$.*

Remark 1.2. Now we give some comparisons between our result and other results on the system (1.1). Firstly, in the previous works [2,3,17–19], the authors needed to make assumptions about the behavior of the nonlinearity $H(t,x)$ as $|x| \to +\infty$. They assumed that $H(t,x)$ satisfies the subquadratic condition, superquadratic condition or asymptotically quadratic condition at infinity. Compared with these works, we do not need the behavior of the nonlinearity $H(t,x)$ for $|x|$ large. Secondly, our subquadratic conditions near zero are also weaker than the related papers [2,3]. In [2,3], the authors assumed that $H(t,x)$ satisfies $\lim_{|x|\to 0} \frac{H(t,x)}{|x|^2} = +\infty$ for all $t \in \mathbb{R}$. By contrast, we only assume that $\lim_{|x|\to 0} \frac{H(t,x)}{|x|^2} = +\infty$ in a interval $t \in [t_0 - r, t_0 + r]$. Thirdly, in the literature [2,3,17–19], the authors did not give the information for the obtained homoclinic solutions. However, we can prove that the homoclinic solutions found here converge to the null solution in L^∞ norm.

Example 1.3. Let $H(t,x) = \eta(t)|x|^\mu$, where $1 < \mu < 2$, $\eta(t) \in C^\infty(\mathbb{R},\mathbb{R})$ satisfies that $\eta(t) = 1$, $\forall |t| \le 1$, and $\eta(t) = 0$, $\forall |t| \ge 2$. It is not difficult to see that $H(t,x)$ satisfies all conditions of Theorem 1.1. It is worth noting that $H(t,x)$ does not satisfies $\lim_{|x|\to 0} \frac{H(t,x)}{|x|^2} = +\infty$ for all $t \in \mathbb{R}$.

The remainder of this paper is organized as follows. In Section 2, we give the variational framework for (1.1). In Section 3, we prove our main result in detail.

2 Preliminaries

In this section, we establish the variational framework for (1.1) and give a preliminary result.

Let $E = H^1(\mathbb{R},\mathbb{R}^N)$ be a Hilbert space where the function is from $\mathbb{R}$ to $\mathbb{R}^N$ with the inner product

$$\langle x,y\rangle_0 = \int_{\mathbb{R}} \Big((x(t),y(t)) + (\dot{x}(t),\dot{y}(t)) \Big) dt, \quad \forall x,y \in E_0, \tag{2.1}$$

where $(\cdot,\cdot)$ means the standard inner product in $\mathbb{R}^N$. The corresponding norm is

$$\|x\|_0 = \left(\int_{\mathbb{R}} (|x(t)|^2 + |\dot{x}(t)|^2) dt \right)^{\frac{1}{2}}, \quad \forall x \in E_0. \tag{2.2}$$

For simplicity, we define a new norm on E. Let

$$\|x\| = \left(\int_{\mathbb{R}} [|\dot{x}|^2 + (A(t)x(t),x(t)) - (B\dot{x}(t),x(t))] dt \right)^{\frac{1}{2}}, \quad \forall x \in E. \tag{2.3}$$

And the corresponding inner product is denoted by $\langle\cdot,\cdot\rangle$. Now we show that the norms $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent. Since $\|B\| < 2\sqrt{\alpha}$ from (H_2), then $\frac{\|B\|^2}{2\alpha} < 2$. Hence we can choose a constant ε_0 such that

$$\frac{\|B\|^2}{2\alpha} < \varepsilon_0 < 2. \tag{2.4}$$

Set

$$C_0 = \min\left\{1 - \frac{\varepsilon_0}{2}, \alpha - \frac{\|B\|^2}{2\varepsilon_0}\right\}. \tag{2.5}$$

By (2.4), we see that $C_0 > 0$. Then by (H_2) and mean inequality, we have

$$\begin{aligned} \|x\|^2 &= \int_{\mathbb{R}} [|\dot{x}|^2 + (A(t)x(t),x(t)) - (B\dot{x}(t),x(t))] dt \\ &\ge \int_{\mathbb{R}} [(1 - \frac{\varepsilon_0}{2})|\dot{x}(t)|^2 + (\alpha - \frac{\|B\|^2}{2\varepsilon_0})|x|^2] dt \\ &\ge C_0\|x\|_0^2. \end{aligned} \tag{2.6}$$

On the other hand,

$$\begin{aligned}\|x\|^2 &= \int_{\mathbb{R}} [|\dot{x}|^2 + (A(t)x(t), x(t)) - (B\dot{x}(t), x(t))]dt \\ &\leq \int_{\mathbb{R}} [|\dot{x}(t)|^2 + \|A(t)\|_{L^\infty(\mathbb{R})}|x|^2 + \|B\|(|\dot{x}(t)|^2 + |x|^2)]dt \\ &\leq C_1\|x\|_0^2,\end{aligned} \tag{2.7}$$

where $C_1 = (1 + \|A(t)\|_{L^\infty(\mathbb{R})} + \|B\|)$ is a constant. Therefore, the norms $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent.

To obtain the homoclinic solution of (1.1), we consider the following systems

$$\ddot{x}(t) + B\dot{x}(t) - A(t)x(t) + \hat{H}_x(t, x(t)) = 0, \tag{2.8}$$

where $\hat{H} \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ satisfies that $\hat{H}$ is even in u, $\hat{H}(t,x) = H(t,x)$ for $t \in \mathbb{R}$ and $|x| < \frac{\delta}{2}$, and $\hat{H}(t,x) = 0$ for $t \in \mathbb{R}$ and $|x| > \delta$.

Define the functional Φ on E by

$$\begin{aligned}\Phi(x) &= \frac{1}{2}\int_{\mathbb{R}} [|\dot{x}|^2 + (A(t)x(t), x(t)) - (B\dot{x}(t), x(t))]dt - \int_{\mathbb{R}} \hat{H}(t, x(t))dt \\ &= \frac{1}{2}\|x\|^2 - \int_{\mathbb{R}} \hat{H}(t, x(t))dt.\end{aligned} \tag{2.9}$$

By (H_1), $\Phi \in C^1(E, \mathbb{R})$ and the critical points of Φ correspond to the homoclinic solutions of (2.8) (see [17]). We can get that

$$\begin{aligned}\langle \Phi'(x), y\rangle = &\int_{\mathbb{R}} [(\dot{x}(t), \dot{y}(t)) + (A(t)x(t), y(t)) - (B\dot{x}(t), y(t))]dt \\ &- \int_{\mathbb{R}} (\hat{H}_x(t, x(t)), y(t))dt.\end{aligned} \tag{2.10}$$

Now we introduce a Clark's theorem established by Liu and Wang [9]. Clark's theorem is a classical theorem in the critical point theory and has a large number of applications in differential equations. In [9], Liu and Wang improved and extended Clark's theorem, and applied it to elliptic equations and Hamiltonian systems.

Let X be a Banach space, $\Phi \in C^1(X, \mathbb{R})$. We say that Φ satisfies (PS) condition if any sequence $\{x_j\}$ such that $\Phi(x_j)$ is bounded and $\Phi'(x_j) \to 0$ as $j \to \infty$ contains a convergent subsequence.

Theorem 2.1 ([9]). *Assume Φ satisfies the (PS) condition, is even and bounded from below, and $\Phi(0) = 0$. If for any $k \in \mathbb{N}$, there exists a k-dimensional subspace X^k of X and $\rho_k > 0$ such that $\sup_{X^k \cap S_{\rho_k}} \Phi < 0$, where $S_\rho = \{x \in X \mid \|x\| = \rho\}$, then at least one of the following conclusions holds.*

(1) There exists a sequence of critical points $\{x_k\}$ satisfying $\Phi(x_k) < 0$ for all n and $\|x_k\| \to 0$ as $k \to \infty$.

(2) There exists $r > 0$ such that for any $0 < a < r$ there exists a critical point x such that $\|x\| = a$ and $\Phi(x) = 0$.

Remark 2.2. Clearly, under the assumptions of Theorem 2.1 there exist infinitely many critical points x_k of Φ that satisfies $\Phi(x_k) \leq 0$, $\Phi(x_k) \to 0$ and $\|x_k\| \to 0$ as $k \to \infty$.

3 Proof of the main result

In this section, we use Theorem 2.1 to prove the main result of this paper.

Proof. Step 1. We prove that Φ is bounded from below. Let $\|\cdot\|_{L^p(\mathbb{R})}$ denote the norm of $L^p(\mathbb{R}, \mathbb{R}^N)(1 \le p \le \infty)$. By (H_4), we have that

$$|\hat{H}(t,x)| \le b(t)|x|, \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N, \tag{3.1}$$

where $b \in L^{\xi}(\mathbb{R})$ is from (H_4). If $\xi = 1$, we have

$$\int_{\mathbb{R}} \hat{H}(t,x(t))dt \le \int_{\mathbb{R}} b(t)|x|dt \le \|x\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} b(t)dt \le C_1' \|x\| \|b(t)\|_{L^1(\mathbb{R})}, \tag{3.2}$$

where the Sobolev inequality $\|x\|_{L^\infty(\mathbb{R})} \le C_1'\|x\|$ has been used. If $1 < \xi \le 2$, by the Hölder inequality and the Sobolev inequality, we have

$$\int_{\mathbb{R}} \hat{H}(t,x(t))dt \le \left(\int_{\mathbb{R}} (b(t))^{\xi} \right)^{\frac{1}{\xi}} \left(\int_{\mathbb{R}} |x|^{\frac{\xi}{\xi-1}} dt \right)^{\frac{\xi-1}{\xi}} \le C_\xi' \|x\| \|b(t)\|_{L^\xi(\mathbb{R})}. \tag{3.3}$$

Then, by (3.2), (3.3) we can see that

$$\int_{\mathbb{R}} \hat{H}(t,x(t))dt \le C_\xi' \|x\| \|b(t)\|_{L^\xi(\mathbb{R})}. \tag{3.4}$$

Therefore by (2.3) and (3.4), we have

$$\begin{aligned} \Phi(x) &= \frac{1}{2} \int_{\mathbb{R}} [|\dot{x}|^2 + (A(t)x(t), x(t)) + (Bx(t), \dot{x}(t))]dt \\ &\quad - \int_{\mathbb{R}} \hat{H}(t,x(t))dt \\ &\ge \frac{1}{2}\|x\|^2 - C_\xi' \|x\| \|b(t)\|_{L^\xi(\mathbb{R})}. \end{aligned} \tag{3.5}$$

Consequently, Φ is bounded from below.

Step 2. We prove that $\Phi(x)$ satisfies the (PS) condition. Let $\{x_n\}$ be a (PS) sequence, that is $\Phi(x_n)$ is bounded and $\Phi'(x_n) \to 0$ as $n \to \infty$. By (3.5), we see that $\{x_n\}$ is bounded in E. Hence, there exists a subsequence of $\{x_n\}$ (for simplicity still denoted by $\{x_n\}$) and some $x_0 \in E$ such that $x_n \rightharpoonup x_0$ in E, and $x_n \to x_0$ strongly in $C_{loc}(\mathbb{R}^1)$ as $n \to \infty$. Then $\Phi'(x_0) = 0$.

Notice that

$$\begin{aligned} \|x_n - x_0\|^2 &= \langle (\Phi'(x_n) - \Phi'(x_0)), (x_n - x_0) \rangle \\ &\quad + \int_{\mathbb{R}} \big((\hat{H}_x(t,x_n) - \hat{H}_x(t,x_0)), (x_n - x_0) \big) dt \end{aligned} \tag{3.6}$$

Since $x_n \rightharpoonup x_0$ in E and $\Phi'(x_n) \to 0$ as $n \to \infty$, we have

$$\langle (\Phi'(x_n) - \Phi'(x_0)), (x_n - x_0) \rangle \to 0 \quad \text{as } n \to \infty. \tag{3.7}$$

By (3.1), the Hölder inequality and the Sobolev inequality, for every $R > 0$ we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}} ((\hat{H}_x(t,x_n) - \hat{H}_x(t,x_0)), (x_n - x_0))\, dt\right| \\
&\quad \le \int_{\mathbb{R}} |\hat{H}_x(t,x_n) - \hat{H}_x(t,x_0)||x_n - x_0|dt \\
&\quad \le \int_{\mathbb{R}\setminus[-R,R]} |\hat{H}_x(t,x_n) - \hat{H}_x(t,x_0)|(|x_n| + |x_0|)dt \\
&\qquad + \int_{-R}^{R} |\hat{H}_x(t,x_n) - \hat{H}_x(t,x_0)||x_n - x_0|dt \\
&\quad \le 2\int_{\mathbb{R}\setminus[-R,R]} b(t)(|x_n| + |x_0|)dt + \int_{-R}^{R} |\hat{H}_x(t,x_n) - \hat{H}_x(t,x_0)||x_n - x_0|dt \\
&\quad \le 2\|b(t)\|_{L^{\xi}(\mathbb{R}\setminus[-R,R])}(\|x_n\| + \|x_0\|) + \int_{-R}^{R} |\hat{H}_x(t,x_n) - \hat{H}_x(t,x_0)||x_n - x_0|dt.
\end{aligned} \tag{3.8}$$

For any $\varepsilon > 0$, since $b(t) \in L^{\xi}(\mathbb{R})$ and $\{x_n\}$ is bounded in E, there exists $R_0 > 0$ large enough such that

$$(\|x_n\| + \|x_0\|)\|b(t)\|_{L^{\xi}(\mathbb{R}\setminus[-R_0,R_0])} < \frac{\varepsilon}{4}, \qquad \forall n \in \mathbb{Z}^+. \tag{3.9}$$

On the other hand, since $x_n \to x_0$ strongly in $C([-R_0, R_0])$, there must exist $n_0 \in \mathbb{Z}^+$ such that for $n \ge n_0$

$$\int_{-R_0}^{R_0} |\hat{H}_x(t,x_n) - \hat{H}_x(t,x_0)||x_n - x_0|dt < \frac{\varepsilon}{2}. \tag{3.10}$$

Then by (3.8),(3.9) and (3.10), for $n \ge n_0$ we have

$$\left|\int_{\mathbb{R}} (\hat{H}_x(t,x_n) - \hat{H}_x(t,x_0), x_n - x_0)dt\right| < \varepsilon,$$

which implies that

$$\left|\int_{\mathbb{R}} (\hat{H}_x(t,x_n) - \hat{H}_x(t,x_0), x_n - x_0)dt\right| \to 0 \quad \text{as } n \to \infty. \tag{3.11}$$

Hence, by (3.6), (3.7) and (3.11), we have $x_n \to x_0$ in E as $n \to \infty$. Therefore, $\Phi(x)$ satisfies the (PS) condition.

Step 3. We show that for every $k \in \mathbb{N}$, there exists a k-dimensional subspace X^k of X and $\rho_k > 0$ such that $\sup_{X^k \cap S_{\rho_k}} \Phi < 0$. Let X^k be a k-dimensional subspace of $C_0^\infty([t_0 - r, t_0 + r])$. Since X^k is a finite dimensional space and the norms in finite dimensional space are all equivalent, there exists a positive constant $C_k > 0$ such that

$$\|x\|^2 \le C_k\|x\|_{L^2}^2, \quad \forall x \in X^k. \tag{3.12}$$

By (H_3) and the definition of $\hat{H}(t,x)$, there exists a constant $0 < \delta_k < \frac{\delta}{2}$ such that for $t \in [t_0 - r, t_0 + r]$ and $x \in B_{\delta_k}$, we have

$$\hat{H}(t,x) \ge C_k|x|^2. \tag{3.13}$$

Recall the Sobolev inequality $\|x\|_{L^\infty(\mathbb{R})} \le C_1'\|x\|$, we take $\rho_k = \frac{\delta_k}{C_1'}$. Then for any $x \in S_{\rho_k}$, we have $\|x\|_{L^\infty} < \delta_k$. Thus by (2.9), (3.12) and (3.13), for any $x \in X^k \cap S_{\rho_k}$ we have

$$\begin{aligned}\Phi(x) &\le \frac{1}{2}\|x\|^2 - C_k\|x\|^2_{L^2(\mathbb{R})}\\ &< -\frac{1}{2}\|x\|^2\\ &= -\frac{1}{2}\rho_k^2 < 0,\end{aligned}$$

which implies that $\sup_{X^k \cap S_{\rho_k}} \Phi < 0$. Now by Theorem 2.1, we obtain infinitely many solutions $\{x_k\}$ for (2.1) such that $\|x_k\| \to 0$ as $k \to \infty$. By Sobolev's inequality, we can get that $\|x_k\|_{L^\infty(\mathbb{R})} \to 0$ as $k \to \infty$. Then there exists $k_0 \in \mathbb{N}$ such that $\|x_k\|_{L^\infty(\mathbb{R})} < \frac{\delta}{2}$, $\forall k \ge k_0$. Hence by the definition of $\hat{H}(t,x)$, for $k \ge k_0$, $\{x_k\}$ are also solutions of (1.1). □

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Existence and blow-up of global solutions for a class of fractional Lane–Emden heat flow system

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Abstract. In this paper, we consider a class of Lane–Emden heat flow system with the fractional Laplacian

$$\begin{cases} u_t + (-\Delta)^{\frac{\alpha}{2}} u = N_1(v) + f_1(x), & (x,t) \in Q, \\ v_t + (-\Delta)^{\frac{\alpha}{2}} v = N_2(u) + f_2(x), & (x,t) \in Q, \\ u(x,0) = a(x), v(x,0) = b(x), & x \in \mathbb{R}^N, \end{cases}$$

where $0 < \alpha \leq 2,\ N \geq 3,\ Q := \mathbb{R}^N \times (0, +\infty),\ f_i(x) \in L^1_{loc}(\mathbb{R}^N)\ (i = 1, 2)$ are nonnegative functions. We study the relationship between the existence, blow-up of the global solutions for the above system and the indexes p, q in the nonlinear terms $N_1(v), N_2(u)$. Here, we first establish the existence and uniqueness of the global solutions in the supercritical case by using Duhamel's integral equivalent system and the contraction mapping principle, and we further obtain some relevant properties of the global solutions. Next, in the critical case, we prove the blow-up of nonnegative solutions for the system by utilizing some heat kernel estimates and combining with proof by contradiction. Finally, by means of the test function method, we investigate the blow-up of negative solutions for the Cauchy problem of a more general higher-order nonlinear evolution system with the fractional Laplacian in the subcritical case.

Keywords: fractional Laplacian, Lane–Emden heat flow system, critical exponent, the contraction mapping principle, the test function method.

2020 Mathematics Subject Classification: 35B08, 35B44, 35B33, 35R11.

1 Introduction

The classical Lane–Emden equation

$$-\Delta u = u^p, \qquad x \in \mathbb{R}^N,\ N > 2,\ p > 1,$$

has been extensively studied, going back to the pioneering work of astronomers and astrophysicists Lane [32] and Emden [15]. It is one of the basic equations in the theory of stellar

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structure and originally used to compute the pressure, density and temperature on the surface of the Sun. It has been discussed by many scholars, see [13, 18, 35, 43, 47] and the references therein. The existence and nonexistence of the global solutions of the equation was once an significant research topic for scholars. For instance, Gidas and Spruck [22] proved that the equation has no positive classical solution in a bounded domain when $1 < p < \frac{N+2}{N-2}$, while the existence of the solution was solved by Caffarelli et al. in [6]. Thereafter, Chen and Li [8] found the form of the positive solution for $p = \frac{N+2}{N-2}$ $(N \geq 3)$ in the whole space and obtained that only trivial solutions exist for $p < \frac{N+2}{N-2}$ by using the method of moving planes. In addition, as for $p > \frac{N+2}{N-2}$, Zou [49] proved that the equation has a unique positive radial symmetric solution with polynomial decay at infinity. Meanwhile, scholars also discussed the existence and nonexistence of solutions to nonlinear elliptic equation and system with a more general nonlinearity. In [4], Bernard studied the semilinear elliptic equation $-\Delta u = u^p + f(x)$ in the whole space. He obtained the blow-up of the global solutions for $1 < p \leq \frac{N}{N-2}$, while if $p > \frac{N}{N-2}$ and $f \in C^{0,\gamma}(\mathbb{R}^N)$ with $0 < \gamma \leq 1$, he showed that the equation has a bounded positive solution. Obviously, the Lane–Emden type system is the natural counterpart of the Lane–Emden equation

$$\begin{cases} -\Delta u = v|v|^{p-1}, & x \in \mathbb{R}^N, \\ -\Delta v = u|u|^{q-1} + f(x), & x \in \mathbb{R}^N, \end{cases}$$

where $N \geq 3$, $p, q > 1$. When $f = 0$, Mitidieri [37] proved that there has no nontrivial radial positive solutions of class $C^2(\mathbb{R}^N)$ by contradiction if $1 < p \leq q$ and $\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N}$, while if $1 < p \leq q$ and $\frac{1}{p+1} + \frac{1}{q+1} \leq \frac{N-2}{N}$, the existence of positive (radial, bounded) classical solution for the system is fully solved by Serrin and Zou in [42]. As for more general cases, when $f \in L^{\frac{N(pq-1)}{2q(p+1)}}$, Ferreira et al. [19] showed the existence of the global solutions in the supercritical case $N > \max\{\frac{2q(p+1)}{pq-1}, \frac{2p(q+1)}{pq-1}\}$ by means of the fixed point theorem, here the range for (p, q) covers the critical and supercritical cases with respect to the hyperbola $\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}$. In case $N \leq \max\{\frac{2q(p+1)}{pq-1}, \frac{2p(q+1)}{pq-1}\}$, the nonexistence results has been pointed out by Mitidieri in [38]. For more researches on elliptic equations, please refer to [2, 11, 34, 40].

The parabolic equation corresponding to the classical Lane–Emden equation, namely the semilinear reaction-diffusion equation

$$u_t - \Delta u = u^q, \qquad x \in \mathbb{R}^N,\ t > 0,$$

has been studied by many scholars, since the pioneering work [20] of Fujita in 1966, where it was shown that the Cauchy problem of the equation has two cases of the solution: if $q > q_c = 1 + \frac{2}{N}$, there exist both global and blow-up solutions, corresponding to small and large initial values, respectively; while if $q < q_c = 1 + \frac{2}{N}$, then the problem does not admit nonnegative global solution. The case of $q = q_c = 1 + \frac{2}{N}$ was decided by Hayakawa [28] for $N = 1, 2$ and Kobayashi et al. [31] for all $N \geq 1$ that the problem does not admit nontrivial nonnegative global solution. Thus, it can be seen that the range of index q plays an important role in the researches of existence and blow-up of the solutions. And q_c is called Fujita critical exponent. Since then, there have been a number of extensions to the research of critical exponent in several directions. For instance, Pascucci [39] considered a semilinear Cauchy problem on nilpotent Lie groups and obtained the sharp Fujita critical exponent, which generalized the results in [20, 28, 31].

As for the semilinear parabolic system, Escobedo and Herrero [16] discussed the Cauchy problem of semilinear reaction-diffusion system in the whole space

$$\begin{cases} u_t - \Delta u = v^p, & (x,t) \in \mathbb{R}^N \times (0,\infty), \\ v_t - \Delta v = u^q, & (x,t) \in \mathbb{R}^N \times (0,\infty), \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in \mathbb{R}^N,\ N \geq 1. \end{cases}$$

They showed that the system has two significant curves, namely, the global existence curve $pq = 1$ and the Fujita curve $pq = 1 + \frac{2}{N} \max\{p+1, q+1\}$. If $0 < pq \leq 1$ or $pq > 1 + \frac{2}{N} \max\{p+1, q+1\}$ with suitably small initial values, then every solution is global by employing the integral equivalent system and Gronwall-type inequalities, respectively; while if $pq > 1 + \frac{2}{N} \max\{p+1, q+1\}$ with large initial values, the system possesses no nontrivial global solution. Meanwhile, the nonexistence of nontrivial global solution is proved based on some heat kernel estimates for $1 < pq \leq 1 + \frac{2}{N} \max\{p+1, q+1\}$. For some problems with boundary conditions or nonlinear terms different from the above, many scholars have also studied the existence and nonexistence of global solutions. For example, Deng and Fila [14] and Bai et al. [5] discussed the Fujita critical exponent of parabolic problems in the upper half space and bounded domain respectively. For more researches on parabolic system, see for example [17,29,30,45].

In mathematical physics, nonlinear evolution equations with the fractional Laplacian are extensively used to describe anomalous diffusion, see [25,26,33] and the references therein. Therefore, it is of theoretical value and practical significance to study the existence of solutions of equations with the fractional Laplacian. Amor and Kenzizi [3] studied the Cauchy problem of the fractional perturbed heat equation on a bounded domain and obtained the necessary conditions for the existence of nonnegative global solution. In [23], Greco et al. concerned the Cauchy problem of the fractional heat equation $u_t + (-\Delta)^s u = 0$ in the whole space. It was showed that the problem has a global solution if the initial value subject to a certain growth condition. In addition, many scholars have also considered the fractional nonhomogeneous parabolic equation

$$u_t + (-\Delta)^\alpha u = f(t,u).$$

When $f(t,u) = h(t)u^p$, Guedda et al. [24] and Tan et al. [44] concerned the Cauchy problem of the equation by means of the integral equivalent equation and the contraction mapping principle, respectively. Their conclusions implied that the Fujita critical exponent is $1 + \frac{2\alpha(1+\sigma)}{N}$. Here, $p > 1$ and the function $h(t) \in C([0,\infty))$ satisfied $c_0 t^\sigma \leq h(t) \leq c_1 t^\sigma$ with $c_0, c_1 > 0$, $\sigma > -1$ for t large enough. Besides, the nonexistence of nontrivial nonnegative solutions and the asymptotic symmetry of the solution were obtained in [10] and [9] under suitable assumptions on $f(t,u)$ via narrow region principles and the method of moving planes, respectively. For more works about the fractional parabolic equation, see [1,21,36,41] and the references therein.

Inspired by the above literature, we study the Cauchy problem of the Lane–Emden heat flow system with the fractional Laplacian

$$\begin{cases} u_t + (-\Delta)^{\frac{\alpha}{2}} u = N_1(v) + f_1(x), & (x,t) \in Q, \\ v_t + (-\Delta)^{\frac{\alpha}{2}} v = N_2(u) + f_2(x), & (x,t) \in Q, \\ u(x,0) = a(x), v(x,0) = b(x), & x \in \mathbb{R}^N. \end{cases} \tag{1.1}$$

This problem is used to describe the heat transfer of two mixed combustibles, where u and v represent the temperature of anomalous diffusion at which the two substances interact respectively. We are primarily concerned with the case $0 < \alpha \leq 2$, $Q := \mathbb{R}^N \times (0, +\infty)$, $N \geq 3$.

$f_i(x) \in L^1_{loc}(\mathbb{R}^N)$ for $i = 1,2$ are nonnegative functions. The nonnegative coupling terms $N_1(v)$, $N_2(u)$ satisfying

$$C_1|v-\widetilde{v}|(|v|^{p-1}+|\widetilde{v}|^{p-1}) \le |N_1(v)-N_1(\widetilde{v})| \le C_2|v-\widetilde{v}|(|v|^{p-1}+|\widetilde{v}|^{p-1}), \tag{1.2a}$$
$$\widetilde{C}_1|u-\widetilde{u}|(|u|^{q-1}+|\widetilde{u}|^{q-1}) \le |N_2(u)-N_2(\widetilde{u})| \le \widetilde{C}_2|u-\widetilde{u}|(|u|^{q-1}+|\widetilde{u}|^{q-1}), \tag{1.2b}$$

where $p > 1$, $q > 1$, and $N_1(v) = N_2(u) = 0$ if $u = v = 0$. We shall assume henceforth that both $a(x) \in L^{\omega_1}(\mathbb{R}^N)$, $b(x) \in L^{\omega_2}(\mathbb{R}^N)$ are continuous, bounded, nonnegative and not-constant zero functions with $\omega_1, \omega_2 > 1$. Here u is a curve in $L^{\omega_3}(\mathbb{R}^N)$, $u\colon [0,\infty) \to L^{\omega_3}(\mathbb{R}^N)$ while v is a curve in $L^{\omega_4}(\mathbb{R}^N)$, $v\colon [0,\infty) \to L^{\omega_4}(\mathbb{R}^N)$ for $\omega_3, \omega_4 > 1$, which assumes only non-negative values. We show the existence of a unique global solution for (1.1) in the supercritical case and the problem does not admit nonnegative global solutions in the critical case. As for the subcritical case, we consider the blow-up of the global solution for the following Cauchy problem of the higher-order nonlinear evolution system

$$\begin{cases} \frac{\partial^k u}{\partial t^k} + (-\Delta)^{\frac{\alpha}{2}} u = N_1(v) + f_1(x), & (x,t) \in Q, \\ \frac{\partial^k v}{\partial t^k} + (-\Delta)^{\frac{\beta}{2}} v = N_2(u) + f_2(x), & (x,t) \in Q, \\ \frac{\partial^{k-1} u}{\partial t^{k-1}}(x,0) \ge 0, \frac{\partial^{k-1} v}{\partial t^{k-1}}(x,0) \ge 0, & x \in \mathbb{R}^N, \end{cases} \tag{1.3}$$

where $k > 1$, $0 < \alpha, \beta \le 2$.

For simplicity, throughout the paper, we denote by C a generic positive constant which may vary in value from line to line and even within the same line, but is independent of the terms which will take part in any limit process.

The following Duhamel's integral equivalent system [44] will be used to prove the existence of a global solution for (1.1) in the supercritical case and the blow-up result in the critical case for (1.1).

$$\begin{aligned} u(x,t) = &\int_{\mathbb{R}^N} \Gamma(x-y,t)a(y)\mathrm{d}y + \int_0^t \int_{\mathbb{R}^N} \Gamma(x-y,t-s)N_1(v)(y,s)\mathrm{d}y\mathrm{d}s \\ &+ \int_0^t \int_{\mathbb{R}^N} \Gamma(x-y,t-s)f_1(y)\mathrm{d}y\mathrm{d}s, \end{aligned} \tag{1.4}$$

$$\begin{aligned} v(x,t) = &\int_{\mathbb{R}^N} \Gamma(x-y,t)b(y)\mathrm{d}y + \int_0^t \int_{\mathbb{R}^N} \Gamma(x-y,t-s)N_2(u)(y,s)\mathrm{d}y\mathrm{d}s \\ &+ \int_0^t \int_{\mathbb{R}^N} \Gamma(x-y,t-s)f_2(y)\mathrm{d}y\mathrm{d}s, \end{aligned} \tag{1.5}$$

where $\Gamma(x,t)$ is the fundamental solution to $u_t + (-\Delta)^{\frac{\alpha}{2}} u = 0$. It is well known that $\Gamma(x,t)$ is given by

$$\int_{\mathbb{R}^N} \Gamma(x,t)e^{-iz\cdot x}\mathrm{d}x = e^{-t|z|^\alpha}, \qquad 0 < \alpha \le 2,$$

From [48], we have

$$\Gamma(x,t) = \int_0^{+\infty} f_{t,\frac{\alpha}{2}}(s)T(x,s)\mathrm{d}s, \qquad 0 < \alpha \le 2,$$

and $\Gamma(x,t) = T(x,t)$ if $\alpha = 2$, where

$$f_{t,\frac{\alpha}{2}}(s) = \frac{1}{2i\pi}\int_{\tau-i\infty}^{\tau+i\infty} e^{zs-tz^{\frac{\alpha}{2}}}\mathrm{d}z \ge 0, \qquad T(x,s) = \left(\frac{1}{4\pi s}\right)^{\frac{N}{2}} e^{-\frac{|x|^2}{4s}}, \qquad \tau > 0, s > 0.$$

To facilitate writing, we set

$$u_0(x,t) = \int_{\mathbb{R}^N} \Gamma(x-y,t)a(y)\mathrm{d}y, \tag{1.6}$$

and

$$v_0(x,t) = \int_{\mathbb{R}^N} \Gamma(x-y,t)b(y)\mathrm{d}y. \tag{1.7}$$

Define

$$F(f_1) = \int_0^t \int_{\mathbb{R}^N} \Gamma(x-y,t-s)f_1(y)\mathrm{d}y\mathrm{d}s, \tag{1.8}$$

and

$$F(f_2) = \int_0^t \int_{\mathbb{R}^N} \Gamma(x-y,t-s)f_2(y)\mathrm{d}y\mathrm{d}s. \tag{1.9}$$

In this framework we can write the integral system (1.4)–(1.5) in the abstract form

$$(u,v) = (u_0,v_0) + B(u,v) + (F(f_1), F(f_2)), \tag{1.10}$$

where

$$B(u,v) = (B_1(v), B_2(u)), \tag{1.11}$$

and $B_1(v) = F(N_1(v))$, $B_2(u) = F(N_2(u))$.

If $N > \max\{\frac{\alpha(p+1)}{pq-1}, \frac{\alpha(q+1)}{pq-1}\}$, we denote

$$P_{sc} := \frac{N(pq-1)}{\alpha(p+1)}, \qquad Q_{sc} := \frac{N(pq-1)}{\alpha(q+1)},$$

and

$$P'_{sc} := \frac{N(pq-1)}{\alpha(p+1)+pq-1}, \qquad Q'_{sc} := \frac{N(pq-1)}{\alpha(q+1)+pq-1}.$$

Below we assume the basic assumptions on the range of p_1 and q_1:

$$\frac{p+1}{pq-1} - \frac{p+1}{pq+p} < \frac{N}{\alpha p_1} < \frac{p+1}{pq-1}, \qquad p_1 \geq p, \tag{1.12}$$

$$\frac{q+1}{pq-1} - \frac{q+1}{pq+q} < \frac{N}{\alpha q_1} < \frac{q+1}{pq-1}, \qquad q_1 \geq q, \tag{1.13}$$

and

$$\frac{N}{\alpha q_1} < \frac{Nq}{\alpha p_1} < 1 + \frac{N}{\alpha q_1}, \qquad \frac{N}{\alpha p_1} < \frac{Np}{\alpha q_1} < 1 + \frac{N}{\alpha p_1},$$

that is

$$\frac{1}{q_1} < \frac{q}{p_1} < \frac{1}{q_1} + \frac{\alpha}{N}, \qquad \frac{1}{p_1} < \frac{p}{q_1} < \frac{1}{p_1} + \frac{\alpha}{N}. \tag{1.14}$$

The range and some basic assumptions of the indexes p'_1 and q'_1 are

$$\frac{1}{\alpha} + \frac{p+1}{pq-1} - \frac{p+1}{pq+p} < \frac{N}{\alpha p'_1} < \frac{1}{\alpha} + \frac{p+1}{pq-1}, \qquad p'_1 \geq p_1, \tag{1.15}$$

$$\frac{1}{\alpha} + \frac{q+1}{pq-1} - \frac{q+1}{pq+q} < \frac{N}{\alpha q'_1} < \frac{1}{\alpha} + \frac{q+1}{pq-1}, \qquad q'_1 \geq q_1. \tag{1.16}$$

Therefore

$$\frac{1}{q'_1} < \frac{q}{p_1}, \qquad \frac{1}{p'_1} < \frac{p}{q_1}. \tag{1.17}$$

The above assumptions are used in the following statements. Our main results read

Theorem 1.1. *Suppose that* $N > \max\{\frac{\alpha(p+1)}{pq-1}, \frac{\alpha(q+1)}{pq-1}\}$. *Let* $C_0(\mathbb{R}^N)$ *denote the space of all continuous functions decaying to zero at infinity, and let* $a(x) \in L^{P_{sc}}(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$, $b(x) \in L^{Q_{sc}}(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$, $f_1(x) \in L^{\frac{Q_{sc}}{p}}(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$, $f_2(x) \in L^{\frac{P_{sc}}{q}}(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$.

(1) There exists $\delta > 0$ *such that if* $\|a(x)\|_{P_{sc}}, \|b(x)\|_{Q_{sc}}, \|f_1(x)\|_{\frac{Q_{sc}}{p}}, \|f_2(x)\|_{\frac{P_{sc}}{q}} \le \frac{\delta}{3K_1}$, *then the integral system* (1.10) *has a unique solution* (u, v) *satisfying* $u \in L^{p_1}(\mathbb{R}^N)$, $v \in L^{q_1}(\mathbb{R}^N)$ *and*

$$\|u\|_{p_1}, \|v\|_{q_1} \le 2\delta,$$

where the constant K_1 *is as in Lemma* 2.3.

(2) If $N > 1 + \max\left\{\frac{\alpha(p+1)}{pq-1}, \frac{\alpha(q+1)}{pq-1}\right\}$, *then* (u, v) *is a solution in the sense of distributions and satisfies*

$$\nabla u \in L^{p_1'}(\mathbb{R}^N), \qquad \nabla v \in L^{q_1'}(\mathbb{R}^N).$$

(3) Furthermore, if $a(x) \in L^{p_1}(\mathbb{R}^N)$, $b(x) \in L^{q_1}(\mathbb{R}^N)$, *then* $u, v \in C([0,\infty), C_0(\mathbb{R}^N))$.

Theorem 1.2. *Suppose that* $N = \max\{\frac{\alpha(p+1)}{pq-1}, \frac{\alpha(q+1)}{pq-1}\}$. *Then the problem* (1.1) *has no nonnegative global solution* $u, v \in C^1(Q) \cap L^\infty(Q)$ *such that* $(-\Delta)^{\frac{\alpha}{2}} u(x,t)$, $(-\Delta)^{\frac{\alpha}{2}} v(x,t) \in L^\infty(Q)$.

Theorem 1.3. *Suppose that* $N < \max\{\frac{q(\alpha+p\beta)}{pq-1} - \sigma, \frac{p(\alpha+q\beta)}{pq-1} - \sigma\}$ *with* $\sigma > \max\{\frac{\alpha}{k}, \frac{\beta}{k}\}$ *and* $f_i(x) \not\equiv 0$ *for* $i = 1, 2$. *Then* (1.3) *has no nonnegative global weak solution (see Definition* 2.1 *below).*

Remark 1.4. It is worth noting that, compared with the semilinear reaction-diffusion system of the classical Laplacian in [16], the influence of the fractional operator and the nonlinear terms for (1.1) we consider on the estimates are more complicated. Hence, when we prove Theorem 1.2, we argue by contradiction, the integral related to the initial value is estimated skillfully, which reduces a large number of calculations generated by using the method in [16], and the method here is more convenient.

Remark 1.5. From Theorem 1.3, if $\alpha = \beta$ and $k = 1$, then hypothetical condition will correspondingly change to $N < \max\{\frac{\alpha(p+1)}{pq-1}, \frac{\alpha(q+1)}{pq-1}\}$, which is consistent with the indexes in Theorems 1.1 and 1.2. So we can get that the critical curve for (1.1) is $N = \max\{\frac{\alpha(p+1)}{pq-1}, \frac{\alpha(q+1)}{pq-1}\}$. Next, we give some comments about the critical curve(exponent) for (1.1).

(1) If $\alpha = 2$, then $N = \max\{\frac{\alpha(p+1)}{pq-1}, \frac{\alpha(q+1)}{pq-1}\}$ becomes $pq = 1 + \frac{2}{N}\max\{p+1, q+1\}$, which is the critical curve for semilinear reaction-diffusion system in [16].

(2) If $u = v$ and $p = q$, then $N = \max\{\frac{\alpha(p+1)}{pq-1}, \frac{\alpha(q+1)}{pq-1}\}$ becomes $p = 1 + \frac{\alpha}{N}$, which is the critical exponent for the corresponding single parabolic equation $u_t + (-\Delta)^{\frac{\alpha}{2}} = u^p$ in [24,44].

We conclude this introduction by describing the plan of the paper. Section 2 recalls some lemmas and some properties of the fundamental solution $\Gamma(x,t)$ which we shall use in the sequel. In Section 3, we use the contraction mapping principle to prove the existence of a unique global solution for (1.1) in the supercritical case, and further obtain some relevant properties of the global solution. The blow-up of the global solutions in the critical case is discussed via Duhamel's integral equivalent equations and combined with proof by contradiction, which is gathered in Section 4. As for the blow-up result for a more general higher-order system (1.3) in the subcritical case, we utilize the test function method to obtain and make up the content of Section 5. Section 6 is an appendix, in which we prove some lemmas given in Section 2 in detail.

2 Preliminaries

In this section, we mainly introduce some lemmas, as well as some properties and estimates related to the kernel function $\Gamma(x,t)$, which will be utilized in the following proofs. For general k, we first give the definition of weak solutions for (1.3).

Definition 2.1. Let $u,v \in L^1_{loc}\left(\mathbb{R}^N \times [0,\infty)\right)$ with $N_1(v), N_2(u) \in L^1_{loc}\left(\mathbb{R}^N \times [0,\infty)\right)$, and let the locally integrable traces $\frac{\partial^i u}{\partial t^i}(x,0), \frac{\partial^i v}{\partial t^i}(x,0), i = 1,2,\ldots,k-1$ on the hyperplane $t=0$ are well defined. The function (u,v) is called a global weak solution for (1.3) in Q if for any nonnegative test function $\varphi(x,t) \in C_c^\infty\left(\mathbb{R}^N \times [0,\infty)\right)$, the following integral equalities hold:

$$\begin{aligned}\iint_Q u\left[(-1)^k\frac{\partial^k\varphi}{\partial t^k} + (-\Delta)^{\frac{\alpha}{2}}\varphi\right]\mathrm{d}x\mathrm{d}t = & \iint_Q \left(N_1(v)+f_1(x)\right)\varphi(x,t)\mathrm{d}x\mathrm{d}t \\ & + \sum_{i=1}^{k-1}(-1)^i\int_{\mathbb{R}^N}\frac{\partial^{k-1-i}u}{\partial t^{k-1-i}}(x,0)\frac{\partial^i\varphi}{\partial t^i}(x,0)\mathrm{d}x \\ & + \int_{\mathbb{R}^N}\frac{\partial^{k-1}u}{\partial t^{k-1}}(x,0)\varphi(x,0)\mathrm{d}x,\end{aligned} \tag{2.1}$$

$$\begin{aligned}\iint_Q v\left[(-1)^k\frac{\partial^k\varphi}{\partial t^k} + (-\Delta)^{\frac{\beta}{2}}\varphi\right]\mathrm{d}x\mathrm{d}t = & \iint_Q \left(N_2(u)+f_2(x)\right)\varphi(x,t)\mathrm{d}x\mathrm{d}t \\ & + \sum_{i=1}^{k-1}(-1)^i\int_{\mathbb{R}^N}\frac{\partial^{k-1-i}v}{\partial t^{k-1-i}}(x,0)\frac{\partial^i\varphi}{\partial t^i}(x,0)\mathrm{d}x \\ & + \int_{\mathbb{R}^N}\frac{\partial^{k-1}v}{\partial t^{k-1}}(x,0)\varphi(x,0)\mathrm{d}x.\end{aligned} \tag{2.2}$$

According to [24,27,46], we collect the following propositions:

Proposition 2.2.

(1) $\Gamma(x,ts) = t^{-\frac{N}{\alpha}}\Gamma(t^{-\frac{1}{\alpha}}x,s)$.
(2) $\Gamma(x,t) \geq \left(\frac{s}{t}\right)^{-\frac{N}{\alpha}}\Gamma(x,s)$ for all $t \geq s$.
(3) For all $x \in \mathbb{R}^N$ and $\alpha > 0$, $\Gamma(x,t)$ satisfies the following pointwise estimates

$$|\Gamma(x,1)| \leq C\left(1+|x|\right)^{-N-\alpha}, \qquad \left|(-\Delta)^{\frac{\alpha}{2}}\Gamma(x,1)\right| \leq C'\left(1+|x|\right)^{-N-\alpha}.$$

(4) $\|\Gamma(\cdot,t)\|_1 = 1$ for all $t>0$, and $\Gamma(x,t)$ satisfies:

$$\Gamma(x,t) \in L^p(\mathbb{R}^N), \qquad (-\Delta)^{\frac{\alpha}{2}}\Gamma(x,t) \in L^p(\mathbb{R}^N).$$

for all $t>0$ and $1 \leq p \leq \infty$.
(5) For all $x \in \mathbb{R}^N$ and $t,s>0$, the following Chapman–Kolmogorov equation holds:

$$\int_{\mathbb{R}^N}\Gamma(x-z,s)\Gamma(z,t)\mathrm{d}z = \Gamma(x,t+s).$$

(6) If $\Gamma(0,t) \leq 1$ and $\tau \geq 2$, then $\Gamma\left(\frac{1}{\tau}(x-y),t\right) \geq \Gamma(x,t)\Gamma(y,t)$.

Lemma 2.3. *Let $1 \le m \le n \le \infty$. Then for $t > 0$, $e^{-t(-\Delta)^{\frac{\alpha}{2}}} : L^m\left(\mathbb{R}^N\right) \to L^n\left(\mathbb{R}^N\right)$ is a bounded map. Furthermore, for any $T > 0$ and $h(x,t) \in L^m(\mathbb{R}^N)$, there are positive constants K_1 and K_2 depending only on m, n and l, such that*

$$\|\Gamma(x,t) * h(x,t)\|_n \le K_1 t^{-\frac{N}{\alpha}\left(1-\frac{1}{r}\right)} \|h(x,t)\|_m, \tag{2.3}$$

$$\left\|(-\Delta)^{\frac{l}{2}}\Gamma(x,t) * h(x,t)\right\|_n \le K_2 t^{-\frac{l}{\alpha}-\frac{N}{\alpha}\left(1-\frac{1}{r}\right)} \|h(x,t)\|_m, \tag{2.4}$$

for all $t \in (0,T]$ and any $l > 0$, where $1 + \frac{1}{n} = \frac{1}{m} + \frac{1}{r}$. In particular, if $l = 1$, then

$$\|\nabla\Gamma(x,t) * h(x,t)\|_n \le K_2 t^{-\frac{1}{\alpha}-\frac{N}{\alpha}\left(1-\frac{1}{r}\right)} \|h(x,t)\|_m, \qquad \forall t \in (0,T].$$

Here and hereafter, " $*$" stands for the convolution in the space variable.

See Appendix for detailed proof of Lemma 2.3.

Lemma 2.4 (See [7]). *Let $a \wedge b := \min\{a,b\}$ for $a, b \in \mathbb{R}$. Then there exist positive constants $C_{\alpha,N}$ and $C'_{\alpha,N}$, depending only on N and α, such that*

$$C'_{\alpha,N}\left(t^{-\frac{N}{\alpha}} \wedge \frac{t}{|x|^{N+\alpha}}\right) \le \Gamma(x,t) \le C_{\alpha,N}\left(t^{-\frac{N}{\alpha}} \wedge \frac{t}{|x|^{N+\alpha}}\right)$$

for all $(x,t) \in \mathbb{R}^N \times (0,+\infty)$ and $0 < \alpha < 2$.

Lemma 2.5. *Let (u,v) is a nonnegative solution to (1.1), then there exist positive constants t_0, C and τ such that*

$$u(x,t_0) \ge C\Gamma(x,\tau), \qquad v(x,t_0) \ge C\Gamma^q(x,\tau), \tag{2.5}$$

for $q > 1$ and all $x \in \mathbb{R}^N$.

Similar estimates can be found in [24, Lemma 3.2]. To make the paper self-contained, we give the proof of Lemma 2.5 in Appendix.

3 Existence of the global solution for (1.1) in the supercritical case

In this section, we utilize (1.10) and the contraction mapping principle to prove the existence of a global solution for (1.1) in the supercritical case. To achieve this, we first derive a key lemma, which provides estimates for the integrals in (1.10).

Define

$$E := L^\infty\left((0,\infty), L^{p_1}\left(\mathbb{R}^N\right)\right) \times L^\infty\left((0,\infty), L^{q_1}\left(\mathbb{R}^N\right)\right).$$

For each $\delta > 0$ fixed we consider the space D defined by

$$D := \left\{(u,v) \in E \mid \sup t^{b_1}\|u\|_{p_1} < 2\delta,\ \sup t^{b_2}\|v\|_{q_1} < 2\delta\right\},$$

where constants b_1, b_2 are given by formulas (3.9)–(3.10).

On the space D, we show the following lemma:

Lemma 3.1. *Let p_1, q_1 be as in (1.12)–(1.14) and p'_1, q'_1 be as in (1.15)–(1.16), $(u,v) \in D$. For all $v_1, v_2 \in L^{q_1}\left(\mathbb{R}^N\right)$ and $u_1, u_2 \in L^{p_1}\left(\mathbb{R}^N\right)$, there are positive constants $M_1, M_2, M'_1, M'_2 > 0$ such that*

(1)

$$\|B_1(v_1)-B_1(v_2)\|_{p_1} \leq M_1\int_0^t (t-s)^{-\frac{N}{\alpha}\left(\frac{p}{q_1}-\frac{1}{p_1}\right)}\|v_1-v_2\|_{q_1}\left(\|v_1\|_{q_1}^{p-1}+\|v_2\|_{q_1}^{p-1}\right)\mathrm{d}s, \tag{3.1}$$

$$\|B_2(u_1)-B_2(u_2)\|_{q_1} \leq M_2\int_0^t (t-s)^{-\frac{N}{\alpha}\left(\frac{q}{p_1}-\frac{1}{q_1}\right)}\|u_1-u_2\|_{p_1}\left(\|u_1\|_{p_1}^{q-1}+\|u_2\|_{p_1}^{q-1}\right)\mathrm{d}s. \tag{3.2}$$

(2)

$$\|\nabla\left[B_1(v_1)-B_1(v_2)\right]\|_{p_1'} \leq M_1'\int_0^t (t-s)^{-\frac{1}{\alpha}-\frac{N}{\alpha}\left(\frac{p}{q_1}-\frac{1}{p_1'}\right)}\|v_1-v_2\|_{q_1}\left(\|v_1\|_{q_1}^{p-1}+\|v_2\|_{q_1}^{p-1}\right)\mathrm{d}s, \tag{3.3}$$

$$\|\nabla\left[B_2(u_1)-B_2(u_2)\right]\|_{q_1'} \leq M_2'\int_0^t (t-s)^{-\frac{1}{\alpha}-\frac{N}{\alpha}\left(\frac{q}{p_1}-\frac{1}{q_1'}\right)}\|u_1-u_2\|_{p_1}\left(\|u_1\|_{p_1}^{q-1}+\|u_2\|_{p_1}^{q-1}\right)\mathrm{d}s. \tag{3.4}$$

Proof. We will only prove the estimates in (3.1) and (3.3) because the ones in (3.2) and (3.4) can be obtained analogously.

(1) According to the definition of $B_1(v)$, together with (1.2a), one can calculate

$$\begin{aligned}
&\|B_1(v_1)-B_1(v_2)\|_{p_1}\\
&=\left\|\int_0^t\int_{\mathbb{R}^N}\Gamma(x-y,t-s)\left[N_1(v_1)-N_1(v_2)\right]\mathrm{d}y\mathrm{d}s\right\|_{p_1}\\
&\leq C_2\int_0^t\left\|\int_{\mathbb{R}^N}\Gamma(x-y,t-s)\,|v_1-v_2|\left(|v_1|^{p-1}+|v_2|^{p-1}\right)\mathrm{d}y\right\|_{p_1}\mathrm{d}s\\
&\leq K_1C_2\int_0^t (t-s)^{-\frac{N}{\alpha}\left(\frac{p}{q_1}-\frac{1}{p_1}\right)}\left\||v_1-v_2|\left(|v_1|^{p-1}+|v_2|^{p-1}\right)\right\|_{\frac{q_1}{p}}\mathrm{d}s,
\end{aligned} \tag{3.5}$$

here we have used Lemma 2.3 in the second inequality. By employing Hölder's inequality we have

$$\begin{aligned}
&\left\||v_1-v_2|\left(|v_1|^{p-1}+|v_2|^{p-1}\right)\right\|_{\frac{q_1}{p}}\\
&\leq\left(\left(\int_{\mathbb{R}^N}|v_1-v_2|^{q_1}\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^N}\left(|v_1|^{p-1}+|v_2|^{p-1}\right)^{\frac{q_1}{p-1}}\mathrm{d}x\right)^{\frac{p-1}{p}}\right)^{\frac{p}{q_1}}\\
&=\|v_1-v_2\|_{q_1}\left(\int_{\mathbb{R}^N}\left(|v_1|^{p-1}+|v_2|^{p-1}\right)^{\frac{q_1}{p-1}}\mathrm{d}x\right)^{\frac{p-1}{q_1}}\\
&\leq C\,\|v_1-v_2\|_{q_1}\left(\|v_1\|_{q_1}+\|v_2\|_{q_1}\right)^{p-1}\\
&\leq C\,\|v_1-v_2\|_{q_1}\left(\|v_1\|_{q_1}^{p-1}+\|v_2\|_{q_1}^{p-1}\right).
\end{aligned} \tag{3.6}$$

Substitute (3.6) into (3.5), it yields (3.1), where $M_1=K_1C_2C$.

(2) Using Lemma 2.3, similar to (3.5)–(3.6), we can get

$$\begin{aligned}
&\|\nabla [B_1(v_1) - B_1(v_2)]\|_{p_1'} \\
&\le C_2 \int_0^t \left\| \int_{\mathbb{R}^N} \nabla\Gamma(x-y, t-s)\, |v_1 - v_2| \left(|v_1|^{p-1} + |v_2|^{p-1}\right) \mathrm{d}y \right\|_{p_1'} \mathrm{d}s \\
&\le K_2 C_2 \int_0^t (t-s)^{-\frac{1}{\alpha} - \frac{N}{\alpha}\left(\frac{p}{q_1} - \frac{1}{p_1'}\right)} \left\| |v_1 - v_2| \left(|v_1|^{p-1} + |v_2|^{p-1}\right) \right\|_{\frac{q_1}{p}} \mathrm{d}s \\
&\le M_1' \int_0^t (t-s)^{-\frac{1}{\alpha} - \frac{N}{\alpha}\left(\frac{p}{q_1} - \frac{1}{p_1'}\right)} \|v_1 - v_2\|_{q_1} \left(\|v_1\|_{q_1}^{p-1} + \|v_2\|_{q_1}^{p-1}\right) \mathrm{d}s. \qquad (3.7)
\end{aligned}$$

□

Using Lemma 3.1, we now give the proof of Theorem 1.1.

Proof of Theorem 1.1. (1) Due to $N > \max\{\frac{\alpha(p+1)}{pq-1}, \frac{\alpha(q+1)}{pq-1}\}$, combining (1.12) and (1.13) we can obtain

$$p_1 > \frac{N(pq-1)}{\alpha(p+1)} = P_{sc} > 1, \qquad q_1 > \frac{N(pq-1)}{\alpha(q+1)} = Q_{sc} > 1. \qquad (3.8)$$

Let

$$b_1 = \frac{N}{\alpha}\left(\frac{1}{P_{sc}} - \frac{1}{p_1}\right) = \frac{N}{\alpha P_{sc}} - \frac{N}{\alpha p_1} = \frac{p+1}{pq-1} - \frac{N}{\alpha p_1}, \qquad (3.9)$$

$$b_2 = \frac{N}{\alpha}\left(\frac{1}{Q_{sc}} - \frac{1}{q_1}\right) = \frac{N}{\alpha Q_{sc}} - \frac{N}{\alpha q_1} = \frac{q+1}{pq-1} - \frac{N}{\alpha q_1}. \qquad (3.10)$$

Then using (1.12)–(1.14) and (3.8)–(3.10), we conclude that

$$b_1 > 0, \qquad b_2 > 0,$$

$$b_2 p - b_1 = 1 - \frac{N}{\alpha}\left(\frac{p}{q_1} - \frac{1}{p_1}\right), \qquad b_1 q - b_2 = 1 - \frac{N}{\alpha}\left(\frac{q}{p_1} - \frac{1}{q_1}\right). \qquad (3.11)$$

As $\|a(x)\|_{P_{sc}} \le \frac{\delta}{3K_1}$, $\|b(x)\|_{Q_{sc}} \le \frac{\delta}{3K_1}$, here K_1 is determined by Lemma 2.3. Applying Lemma 2.3, we obtain for any $t > 0$

$$\sup t^{b_1} \|u_0(x,t)\|_{p_1} = \sup t^{b_1} \|\Gamma(x,t) * a(x)\|_{p_1} \le K_1 \|a(x)\|_{P_{sc}} \le \frac{\delta}{3} < \infty, \qquad (3.12)$$

$$\sup t^{b_2} \|v_0(x,t)\|_{q_1} = \sup t^{b_2} \|\Gamma(x,t) * b(x)\|_{q_1} \le K_1 \|b(x)\|_{Q_{sc}} \le \frac{\delta}{3} < \infty. \qquad (3.13)$$

Since $\|f_1(x)\|_{\frac{Q_{sc}}{p}} \le \frac{\delta}{3K_{1,}}$, $\|f_2(x)\|_{\frac{P_{sc}}{q}} \le \frac{\delta}{3K_1}$, combining (1.14) and (3.9), applying Lemma 2.3 with $n = p_1$ and $m = \frac{Q_{sc}}{p}$, we have

$$\begin{aligned}
\sup t^{b_1} \|F(f_1)\|_{p_1} &\le \sup t^{b_1} \int_0^t \|\Gamma(x, t-s) * f_1(x)\|_{p_1} \mathrm{d}s \\
&\le K_1 \sup t^{b_1} \int_0^t (t-s)^{-\frac{N}{\alpha}\left(\frac{p}{Q_{sc}} - \frac{1}{p_1}\right)} \|f_1(x)\|_{\frac{Q_{sc}}{p}} \mathrm{d}s \\
&\le K_1 \|f_1(x)\|_{\frac{Q_{sc}}{p}},
\end{aligned}$$

namely

$$\sup t^{b_1} \|F(f_1)\|_{p_1} \le K_1 \|f_1(x)\|_{\frac{Q_{sc}}{p}} \le \frac{\delta}{3}. \qquad (3.14)$$

One obtains in a similar way

$$\sup t^{b_2}\|F(f_2)\|_{q_1} \leq K_1 \|f_2(x)\|_{\frac{p_{sc}}{q}} \leq \frac{\delta}{3}. \tag{3.15}$$

Here u is a curve in $L^{p_1}\left(\mathbb{R}^N\right)$, $u\colon [0,\infty) \to L^{p_1}\left(\mathbb{R}^N\right)$, and v is also a curve in $L^{q_1}\left(\mathbb{R}^N\right)$, $v\colon [0,\infty) \to L^{q_1}\left(\mathbb{R}^N\right)$. For the above space E is endowed with the usual norm

$$\|(u,v)\|_E = \sup t^{b_1}\|u\|_{p_1} + \sup t^{b_2}\|v\|_{q_1},$$

we can define a map $\Phi\colon E \to E$ by

$$\Phi(u,v) = (u_0,v_0) + B(u,v) + \left(F\left(f_1\right), F\left(f_2\right)\right).$$

For each $\delta > 0$ fixed we consider the ball

$$B_\delta = \left\{\mathrm{u} \in E \mid \|\mathrm{u}\|_E < 2\delta\right\},$$

endowed with the metric

$$d_B(\mathrm{u},\mathrm{v}) = \|\mathrm{u}-\mathrm{v}\|_E, \qquad \forall \mathrm{u},\mathrm{v} \in B_\delta.$$

Therefore, the metric space (B_δ, d_B) is complete. We will next prove that the operator $\Phi|_{B_\delta}$ is a strict contraction for some $\delta > 0$.

In fact, for any $(u_1,v_1),(u_2,v_2) \in B_\delta$, using (3.1) with $v_2 = 0$ we get

$$\begin{aligned}\sup t^{b_1}\left\|B_1(v_1)\right\|_{p_1} &\leq M_1 \sup t^{b_1} \int_0^t (t-s)^{-\frac{N}{\alpha}\left(\frac{p}{q_1}-\frac{1}{p_1}\right)} \|v_1\|_{q_1}^p \mathrm{d}s \\ &\leq M_1(2\delta)^p \sup t^{b_1} \int_0^t (t-s)^{-\frac{N}{\alpha}\left(\frac{p}{q_1}-\frac{1}{p_1}\right)} s^{-b_2 p} \mathrm{d}s. \end{aligned} \tag{3.16}$$

Using (3.11), one obtains

$$\int_0^t (t-s)^{-\frac{N}{\alpha}\left(\frac{p}{q_1}-\frac{1}{p_1}\right)} s^{-b_2 p} \mathrm{d}s \leq C t^{-b_1}. \tag{3.17}$$

Substituting (3.17) into (3.16) we get

$$\sup t^{b_1}\left\|B_1(v_1)\right\|_{p_1} \leq M_3(2\delta)^p. \tag{3.18}$$

Similarly, we can arrive at

$$\sup t^{b_2}\left\|B_2(u_1)\right\|_{q_1} \leq M_4(2\delta)^q. \tag{3.19}$$

Estimates (3.12)–(3.15), (3.18)–(3.19) and Minkowski's inequality yield

$$\begin{aligned}\|\Phi(u_1,v_1)\|_E \leq& \sup t^{b_1}\|u_0\|_{p_1} + \sup t^{b_1}\|B_1(v_1)\|_{p_1} + \sup t^{b_1}\|F(f_1)\|_{p_1} \\ &+ \sup t^{b_2}\|v_0\|_{q_1} + \sup t^{b_2}\|B_2(u_1)\|_{q_1} + \sup t^{b_2}\|F(f_2)\|_{q_1} \\ \leq& \left(\frac{4}{3} + M_3 2^p \delta^{p-1} + M_4 2^q \delta^{q-1}\right)\delta. \end{aligned} \tag{3.20}$$

Consequently, $\|\Phi(u_1,v_1)\|_E < 2\delta$ if $\frac{4}{3} + M_3 2^p \delta^{p-1} + M_4 2^q \delta^{q-1} < 2$. This shows that $\Phi\left(B_\delta\right) \subset B_\delta$.

For all $(u_1,v_1),(u_2,v_2)\in B_\delta$, we then have $\|(u_1,v_1)-(u_2,v_2)\|_E\le 4\delta$. Combining (3.1) and (3.17) we get

$$\begin{aligned}\sup t^{b_1}\|B_1(v_1)-B_1(v_2)\|_{p_1}\le{}& M_1 2^p\delta^{p-1}t^{b_1}\|(u_1,v_1)-(u_2,v_2)\|_E\\&\cdot\int_0^t(t-s)^{-\frac{N}{\alpha}\left(\frac{p}{q_1}-\frac{1}{p_1}\right)}s^{-b_2p}ds\\\le{}& M_3 2^p\delta^{p-1}\|(u_1,v_1)-(u_2,v_2)\|_E.\end{aligned}\tag{3.21}$$

We can proceed this process similarly as in (3.21) to derive that

$$\sup t^{b_2}\|B_2(u_1)-B_2(u_2)\|_{q_1}\le M_4 2^q\delta^{q-1}\|(u_1,v_1)-(u_2,v_2)\|_E.\tag{3.22}$$

By (3.21) and (3.22), it follows that

$$\begin{aligned}\|\Phi(u_1,v_1)-\Phi(u_2,v_2)\|_E&=\|B(u_1,v_1)-B(u_2,v_2)\|_E\\&=\sup t^{b_1}\|B_1(v_1)-B_1(v_2)\|_{p_1}+\sup t^{b_2}\|B_2(u_1)-B_2(u_2)\|_{q_1}\\&\le\left(M_3 2^p\delta^{p-1}+M_4 2^q\delta^{q-1}\right)\|(u_1,v_2)-(u_1,v_2)\|_E.\end{aligned}\tag{3.23}$$

Combining (3.20) and (3.23) we obtain that the map $\Phi|_{B_\delta}$ is a strict contraction. So it has a fixed point in B_δ, which is the unique solution (u,v) for (1.10) satisfying $\|(u,v)\|_E<2\delta$.

(2) If $N>1+\max\{\frac{\alpha(p+1)}{pq-1},\frac{\alpha(q+1)}{pq-1}\}$, using (1.15)–(1.16), we have

$$p_1'>\frac{N(pq-1)}{\alpha(p+1)+pq-1}=P_{sc}'>1,\qquad q_1'>\frac{N(pq-1)}{\alpha(q+1)+pq-1}=Q_{sc}'>1.\tag{3.24}$$

Let

$$d_1=\frac{N}{\alpha}\left(\frac{1}{P_{sc}'}-\frac{1}{p_1'}\right)=\frac{1}{\alpha}+\frac{p+1}{pq-1}-\frac{N}{\alpha p_1'},\tag{3.25}$$
$$d_2=\frac{N}{\alpha}\left(\frac{1}{Q_{sc}'}-\frac{1}{q_1'}\right)=\frac{1}{\alpha}+\frac{q+1}{pq-1}-\frac{N}{\alpha q_1'}.\tag{3.26}$$

Combining (1.15)–(1.17) and (3.24)–(3.26), we conclude that

$$d_1>0,\qquad d_2>0,$$

and

$$b_2p-d_1=1-\frac{1}{\alpha}-\frac{N}{\alpha}\left(\frac{p}{q_1}-\frac{1}{p_1'}\right),\qquad b_1q-d_2=1-\frac{1}{\alpha}-\frac{N}{\alpha}\left(\frac{q}{p_1}-\frac{1}{q_1'}\right).\tag{3.27}$$

We consider the space $E_1:=L^\infty\big((0,\infty),L^{p_1'}(\mathbb{R}^N)\big)\times L^\infty\big((0,\infty),L^{q_1'}(\mathbb{R}^N)\big)$ endowed with the usual norm

$$\|(u,v)\|_{E_1}=\sup t^{d_1}\|u\|_{p_1'}+\sup t^{d_2}\|v\|_{q_1'}.$$

It is easy to see that $E_1\subset E$. Applying Lemma 2.3, similar to (1), we have

$$\sup t^{d_1}\|\nabla u_0(x,t)\|_{p_1'}=\sup t^{d_1}\|\nabla\Gamma(x,t)*a(x)\|_{p_1'}\le K_2\|a(x)\|_{P_{sc}}\le\frac{\delta K_2}{3K_1}<\infty,\tag{3.28}$$

$$\sup t^{d_2}\|\nabla v_0(x,t)\|_{q_1'}=\sup t^{d_2}\|\nabla\Gamma(x,t)*b(x)\|_{q_1'}\le K_2\|b(x)\|_{Q_{sc}}\le\frac{\delta K_2}{3K_1}<\infty,\tag{3.29}$$

for any $t > 0$. In view of the definitions of F and B, combining (1.17), (3.8), (3.3) with $v_2 = 0$, (3.25) and (3.27), we can calculate

$$\begin{aligned}\sup t^{d_1}\|\nabla F(f_1)\|_{p_1'} &\le \sup t^{d_1}\int_0^t \|\nabla\Gamma(x,t-s)*f_1(x)\|_{p_1'}\,\mathrm{d}s\\ &\le K_2 \sup t^{d_1}\int_0^t (t-s)^{-\frac{1}{\alpha}-\frac{N}{\alpha}\left(\frac{p}{Qsc}-\frac{1}{p_1'}\right)}\|f_1(x)\|_{\frac{Qsc}{p}}\,\mathrm{d}s\\ &\le \frac{\delta K_2}{3K_1},\end{aligned}\tag{3.30}$$

and

$$\begin{aligned}\sup t^{d_1}\|\nabla B_1(v)\|_{p_1'} &\le M_1' \sup t^{d_1}\int_0^t (t-s)^{-\frac{1}{\alpha}-\frac{N}{\alpha}\left(\frac{p}{q_1}-\frac{1}{p_1'}\right)}\|v\|_{q_1}^p\,\mathrm{d}s\\ &\le M_1'(2\delta)^p \sup t^{d_1}\int_0^t (t-s)^{-\frac{1}{\alpha}-\frac{N}{\alpha}\left(\frac{p}{q_1}-\frac{1}{p_1'}\right)} s^{-b_2 p}\mathrm{d}s\\ &\le M_3'(2\delta)^p,\end{aligned}\tag{3.31}$$

analogously,

$$\sup t^{d_2}\|\nabla F(f_2)\|_{q_1'} \le \frac{\delta K_2}{3K_1},\tag{3.32}$$

and

$$\sup t^{d_2}\|\nabla B_2(u)\|_{q_1'} \le M_4'(2\delta)^q.\tag{3.33}$$

It follows that

$$\begin{aligned}\|\nabla\Phi(u,v)\|_{E_1} \le& \sup t^{d_1}\|\nabla u_0\|_{p_1'} + \sup t^{d_1}\|\nabla B_1(v)\|_{p_1'} + \sup t^{d_1}\|\nabla F(f_1)\|_{p_1'}\\ &+ \sup t^{d_2}\|\nabla v_0\|_{q_1'} + \sup t^{d_2}\|\nabla B_2(u)\|_{q_1'} + \sup t^{d_2}\|\nabla F(f_2)\|_{q_1'}\\ \le& \left(\frac{4K_2}{3K_1} + M_3' 2^p\delta^{p-1} + M_4' 2^q \delta^{q-1}\right)\delta,\end{aligned}\tag{3.34}$$

so $\nabla\Phi(u,v) \in E_1$. In view of the fact that (u,v) is the unique fixed point of Φ on E, thus $\nabla(u,v) \in E_1$, and likewise $\nabla u \in L^{p_1'}(\mathbb{R}^N)$, $\nabla v \in L^{q_1'}(\mathbb{R}^N)$.

Let $\phi(x,t) \in C_c^\infty(\mathbb{R}^N)$ be a nonnegative test function. Multiplying the integral equation (1.4) by $(-\frac{\partial}{\partial t} + (-\Delta)^{\frac{\alpha}{2}})\varphi(x,t)$, and then integrating on Q, we obtain

$$\begin{aligned}&\iint_Q u\left(-\frac{\partial}{\partial t} + (-\Delta)^{\frac{\alpha}{2}}\right)\varphi(x,t)\mathrm{d}x\mathrm{d}t\\ &= \iint_Q [u_0(x,t) + F(N_1(v)+f_1)]\left(-\frac{\partial}{\partial t} + (-\Delta)^{\frac{\alpha}{2}}\right)\varphi(x,t)\mathrm{d}x\mathrm{d}t\\ &= \iint_Q u_0(x,t)\left(-\frac{\partial}{\partial t} + (-\Delta)^{\frac{\alpha}{2}}\right)\varphi(x,t)\mathrm{d}x\mathrm{d}t\\ &\quad + \iint_Q F(N_1(v)+f_1)\left(-\frac{\partial}{\partial t} + (-\Delta)^{\frac{\alpha}{2}}\right)\varphi(x,t)\mathrm{d}x\mathrm{d}t\\ &=: A_1 + A_2.\end{aligned}\tag{3.35}$$

One can invert the order of integration and utilize the self-adjointness of $(-\Delta)^{\frac{\alpha}{2}}$ to obtain that

$$A_1 = -\int_{\mathbb{R}^N}\int_0^\infty u_0(x,t)\varphi_t(x,t)\mathrm{d}t\mathrm{d}x + \iint_Q (-\Delta)^{\frac{\alpha}{2}} u_0(x,t)\varphi(x,t)\mathrm{d}x\mathrm{d}t$$
$$= \int_{\mathbb{R}^N} u_0(x,0)\varphi(x,0)\mathrm{d}x + \iint_Q \left(\frac{\partial}{\partial t} + (-\Delta)^{\frac{\alpha}{2}}\right) u_0(x,t)\cdot\varphi(x,t)\mathrm{d}x\mathrm{d}t. \tag{3.36}$$

Furthermore, A_2 is estimated as follows

$$A_2 = -\int_{\mathbb{R}^N}\int_0^\infty F\left(N_1(v)+f_1\right)\varphi_t(x,t)\mathrm{d}t\mathrm{d}x + \iint_Q (-\Delta)^{\frac{\alpha}{2}} F\left(N_1(v)+f_1\right)\varphi(x,t)\mathrm{d}x\mathrm{d}t$$
$$= \iint_Q \left(\frac{\partial}{\partial t} + (-\Delta)^{\frac{\alpha}{2}}\right) F\left(N_1(v)+f_1\right)\cdot\varphi(x,t)\mathrm{d}x\mathrm{d}t, \tag{3.37}$$

where we have used the fact that $\varphi \in C_c^\infty(Q)$ and $F\left(N_1(v)+f_1\right)(x,0) = 0$ in the last equality. Plug (3.36) and (3.37) into (3.35), one obtains

$$\begin{aligned}
&\iint_Q u\left(-\frac{\partial}{\partial t} + (-\Delta)^{\frac{\alpha}{2}}\right)\varphi(x,t)\mathrm{d}x\mathrm{d}t \\
&= \iint_Q \left(\frac{\partial}{\partial t} + (-\Delta)^{\frac{\alpha}{2}}\right)\left(u_0(x,t) + F\left(N_1(v)+f_1\right)\right)\cdot\varphi(x,t)\mathrm{d}x\mathrm{d}t \\
&\quad + \int_{\mathbb{R}^N} u_0(x,0)\varphi(x,0)\mathrm{d}x \\
&= \iint_Q \left(\frac{\partial}{\partial t} + (-\Delta)^{\frac{\alpha}{2}}\right) u(x,t)\cdot\varphi(x,t)\mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^N} u_0(x,0)\varphi(x,0)\mathrm{d}x \\
&= \iint_Q \left(N_1(v) + f_1(x)\right)\varphi(x,t)\mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^N} u(x,0)\varphi(x,0)\mathrm{d}x,
\end{aligned} \tag{3.38}$$

here we have used (1.4) with $t = 0$ in the last equality. Obviously, (3.38) is (2.1) when $k = 1$.

In the same vein, (2.2) with $k = 1$ can be deduced from the integral equation (1.5), which can be derived through a similar process as in the proof for (3.38). As a result, (u,v) satisfies (1.1) in the sense of distributions.

(3) For $0 < T < \infty$, if $a(x) \in L^{p_1}\left(\mathbb{R}^N\right)\cap C_0\left(\mathbb{R}^N\right)$, $b(x) \in L^{q_1}\left(\mathbb{R}^N\right)\cap C_0\left(\mathbb{R}^N\right)$, then repeating the fixed point argument, it is easily conclude that

$$u \in C\left([0,T], L^{p_1}\left(\mathbb{R}^N\right)\cap C_0\left(\mathbb{R}^N\right)\right), \qquad v \in C\left([0,T], L^{q_1}\left(\mathbb{R}^N\right)\cap C_0\left(\mathbb{R}^N\right)\right).$$

Next, we show that $u, v \in C\left([T,\infty), C_0\left(\mathbb{R}^N\right)\right)$ by a bootstrap argument.

Indeed, for $t \geq T$, we write

$$\begin{aligned}
u(x,t) - u_0(x,t) &= \int_0^T\int_{\mathbb{R}^N} \Gamma(x-y,t-s)\left[N_1(v)+f_1(y)\right]\mathrm{d}y\mathrm{d}s \\
&\quad + \int_T^t\int_{\mathbb{R}^N} \Gamma(x-y,t-s)\left[N_1(v)+f_1(y)\right]\mathrm{d}y\mathrm{d}s \\
&:= I_1(x,t) + I_2(x,t),
\end{aligned} \tag{3.39}$$

$$\begin{aligned}
v(x,t) - v_0(x,t) &= \int_0^T\int_{\mathbb{R}^N} \Gamma(x-y,t-s)\left[N_2(u)+f_2(y)\right]\mathrm{d}y\mathrm{d}s \\
&\quad + \int_T^t\int_{\mathbb{R}^N} \Gamma(x-y,t-s)\left[N_2(u)+f_2(y)\right]\mathrm{d}y\mathrm{d}s \\
&:= J_1(x,t) + J_2(x,t).
\end{aligned} \tag{3.40}$$

Since $u, v \in C\left([0,T], C_0\left(\mathbb{R}^N\right)\right)$, $f_1(x), f_2(x) \in C_0\left(\mathbb{R}^N\right)$, it follows that

$$I_1(x,t), J_1(x,t) \in C\left([T,\infty), C_0\left(\mathbb{R}^N\right)\right).$$

Also, if $t \geq T$, then $t^{-b_1} \leq T^{-b_1} < \infty$, $t^{-b_2} \leq T^{-b_2} < \infty$. On the metric space (B_δ, d_B), using (3.14) and (3.18), we can get

$$\begin{aligned} \sup t^{b_1} \|I_1(x,t)\|_{p_1} &\leq \sup t^{b_1} \|B_1(v)\|_{p_1} + \sup t^{b_1} \|F(f_1)\|_{p_1} \\ &\leq M_3 (2\delta)^p + \frac{\delta}{3}, \end{aligned} \tag{3.41}$$

it implies that

$$\|I_1(x,t)\|_{p_1} \leq t^{-b_1}\left(M_3 (2\delta)^p + \frac{\delta}{3}\right) \leq T^{-b_1}\left(M_3 (2\delta)^p + \frac{\delta}{3}\right).$$

Similarly, we obtain

$$\|J_1(x,t)\|_{q_1} \leq T^{-b_2}\left(M_4 (2\delta)^q + \frac{\delta}{3}\right). \tag{3.42}$$

As a result,

$$I_1(x,t) \in C\left([T,\infty), L^{p_1}\left(\mathbb{R}^N\right)\right), \qquad J_1(x,t) \in C\left([T,\infty), L^{q_1}\left(\mathbb{R}^N\right)\right).$$

Hence,

$$\begin{aligned} I_1(x,t) &\in C\left([T,\infty), L^{p_1}\left(\mathbb{R}^N\right) \cap C_0\left(\mathbb{R}^N\right)\right), \\ J_1(x,t) &\in C\left([T,\infty), L^{q_1}\left(\mathbb{R}^N\right) \cap C_0\left(\mathbb{R}^N\right)\right). \end{aligned}$$

Next, from (1.14) we can get

$$0 < \frac{N}{\alpha}\left(\frac{p}{q_1} - \frac{1}{p_1}\right) < 1, \qquad 0 < \frac{N}{\alpha}\left(\frac{q}{p_1} - \frac{1}{q_1}\right) < 1.$$

Thus, there exists $p_2 > p_1$, $q_2 > q_1$ such that

$$0 < \frac{N}{\alpha}\left(\frac{p}{q_1} - \frac{1}{p_2}\right) < 1, \qquad 0 < \frac{N}{\alpha}\left(\frac{q}{p_1} - \frac{1}{q_2}\right) < 1.$$

Due to $u \in L^\infty\left((0,\infty), L^{p_1}\left(\mathbb{R}^N\right)\right)$, $v \in L^\infty\left((0,\infty), L^{q_1}\left(\mathbb{R}^N\right)\right)$, then

$$u^q \in L^\infty\left((0,\infty), L^{\frac{p_1}{q}}\left(\mathbb{R}^N\right)\right), \qquad v^p \in L^\infty\left((0,\infty), L^{\frac{q_1}{p}}\left(\mathbb{R}^N\right)\right).$$

For $t \geq T$, using (3.8)–(3.10), we can get

$$b_1 + 1 - \frac{N}{\alpha}\left(\frac{p}{q_1} - \frac{1}{p_2}\right) < b_2 p, \qquad b_1 + 1 - \frac{N}{\alpha}\left(\frac{p}{Q_{sc}} - \frac{1}{p_2}\right) < 0. \tag{3.43}$$

Taking into account that the fixed point is in B_δ, employing Lemma 2.3, Hölder's inequality and (3.43), we have

$$\begin{aligned}\sup t^{b_1}\|I_2(x,t)\|_{p_2} &\le C\sup t^{b_1}\int_T^t \|\Gamma(x,t-s)*v^p\|_{p_2}\,\mathrm{d}s\\ &\quad+\sup t^{b_1}\int_T^t\|\Gamma(x,t-s)*f_1(x)\|_{p_2}\,\mathrm{d}s\\ &\le C\sup t^{b_1}\int_0^t (t-s)^{-\frac{N}{\alpha}\left(\frac{p}{q_1}-\frac{1}{p_2}\right)}\|v\|_{q_1}^p\mathrm{d}s\\ &\quad+K_1\sup t^{b_1}\int_T^t (t-s)^{-\frac{N}{\alpha}\left(\frac{p}{Q_{sc}}-\frac{1}{p_2}\right)}\|f_1(x)\|_{\frac{Q_{sc}}{p}}\,\mathrm{d}s\\ &\le C\sup t^{b_1}(2\delta)^p\int_0^t (t-s)^{-\frac{N}{\alpha}\left(\frac{p}{q_1}-\frac{1}{p_2}\right)}s^{-b_2p}\mathrm{d}s\\ &\quad+K_1\sup t^{b_1+1-\frac{N}{\alpha}\left(\frac{p}{Q_{sc}}-\frac{1}{p_2}\right)}\|f_1(x)\|_{\frac{Q_{sc}}{p}}\\ &\le C\,(2\delta)^p+\frac{\delta}{3},\end{aligned}\tag{3.44}$$

therefore,

$$\|I_2(x,t)\|_{p_2}\le T^{-b_1}\left((2\delta)^p+\frac{\delta}{3}\right),$$

that is $I_2(x,t)\in C\left([T,\infty),L^{p_2}\left(\mathbb{R}^N\right)\right)$. Similarly, we can get $J_2(x,t)\in C\left([T,\infty),L^{q_2}\left(\mathbb{R}^N\right)\right)$.

Combining (3.9) and $p_2>p_1$, it yields $b_1-\frac{N}{\alpha}\left(\frac{1}{P_{sc}}-\frac{1}{p_2}\right)<0$. Similar to (3.12), (3.14) and (3.16), we have

$$\sup t^{b_1}\|u_0(x,t)\|_{p_2}\le K_1\sup t^{b_1-\frac{N}{\alpha}\left(\frac{1}{P_{sc}}-\frac{1}{p_2}\right)}\|a(x)\|_{P_{sc}}\le\frac{\delta}{3}T^{b_1-\frac{N}{\alpha}\left(\frac{1}{P_{sc}}-\frac{1}{p_2}\right)},\tag{3.45}$$

$$\sup t^{b_1}\|F(f_1)\|_{p_2}\le K_1\sup t^{b_1+1-\frac{N}{\alpha}\left(\frac{p}{Q_{sc}}-\frac{1}{p_2}\right)}\|f_1(x)\|_{\frac{Q_{sc}}{p}}\le\frac{\delta}{3}T^{b_1+1-\frac{N}{\alpha}\left(\frac{p}{Q_{sc}}-\frac{1}{p_2}\right)},\tag{3.46}$$

and

$$\sup t^{b_1}\|B_1(v)\|_{p_2}\le M_3(2\delta)^p\sup t^{1+b_1-\frac{N}{\alpha}\left(\frac{p}{q_1}-\frac{1}{p_2}\right)-b_2p}\le M_3(2\delta)^pT^{1+b_1-\frac{N}{\alpha}\left(\frac{p}{q_1}-\frac{1}{p_2}\right)-b_2p}.\tag{3.47}$$

Using (3.44)–(3.47), we can calculate

$$\begin{aligned}\sup t^{b_1}\|u(x,t)\|_{p_2}&\le\sup t^{b_1}\|u_0(x,t)\|_{p_2}+\sup t^{b_1}\|I_1(x,t)\|_{p_2}+\sup t^{b_1}\|I_2(x,t)\|_{p_2}\\ &\le\frac{\delta}{3}T^{b_1-\frac{N}{\alpha}\left(\frac{1}{P_{sc}}-\frac{1}{p_2}\right)}+M_3\,(2\delta)^p\,T^{1+b_1-\frac{N}{\alpha}\left(\frac{p}{q_1}-\frac{1}{p_2}\right)-b_2p}\\ &\quad+\frac{\delta}{3}T^{b_1+1-\frac{N}{\alpha}\left(\frac{p}{Q_{sc}}-\frac{1}{p_2}\right)}+C\,(2\delta)^p+\frac{\delta}{3}\\ &\le C,\end{aligned}$$

consequently,

$$\|u(x,t)\|_{p_2}\le Ct^{-b_1}\le CT^{-b_1}.$$

Obviously, $u(x,t)\in C\left([T,\infty),L^{p_2}\left(\mathbb{R}^N\right)\right)$. Analogously, we can obtain that $v(x,t)\in C\left([T,\infty),L^{q_2}\left(\mathbb{R}^N\right)\right)$.

Iterating this procedure a finite number of times, we deduce that

$$u(x,t),v(x,t)\in C\left([T,\infty),C_0\left(\mathbb{R}^N\right)\right).$$

This completes the proof. □

4 Blow-up of nonnegative solutions for (1.1) in the critical case

Throughout this section, we shall assume $1 < p \leq q$ for definiteness. The following estimate of the solution for (1.1) is the key step in proving the blow-up theorem for (1.1) in the critical case.

Lemma 4.1. *Assume $u, v \in C^1(Q) \cap L^\infty(Q)$, and let (u, v) is a nonnegative global solution for (1.1) and satisfies $(-\Delta)^{\frac{\alpha}{2}} u(x,t), (-\Delta)^{\frac{\alpha}{2}} v(x,t) \in L^\infty(Q)$, $u_0(x,t)$, $v_0(x,t)$ be as in (1.6)–(1.7), then there exists a constant C, depending on only p and q, such that*

$$t^{\frac{p+1}{pq-1}} u_0(x,t) \leq C, \qquad t^{\frac{q+1}{pq-1}} v_0(x,t) \leq C. \tag{4.1}$$

Proof. We will only show the first estimate in (4.1) because the proof of the second one is similar. Arguing as in Lemma 2.5 one has

$$v(x,t) \geq Ct \left| \int_{\mathbb{R}^N} \Gamma(x-y,t) a(y) \mathrm{d}y \right|^q. \tag{4.2}$$

We now substitute (4.2) into (1.4), drop the first and third terms on the right there, and use (1.2a), Jensen's inequality and Tonelli's theorem to obtain

$$\begin{aligned} u(x,t) &\geq C_1 \int_0^t \int_{\mathbb{R}^N} \Gamma(x-y,t-s) v^p(y,s) \mathrm{d}y\mathrm{d}s \\ &\geq C \int_0^t \int_{\mathbb{R}^N} \Gamma(x-y,t-s) \left(s \left| \int_{\mathbb{R}^N} \Gamma(y-z,s) a(z) \mathrm{d}z \right|^q \right)^p \mathrm{d}y\mathrm{d}s \\ &\geq C \int_0^t s^p \left(\int_{\mathbb{R}^N} \Gamma(x-y,t-s) \int_{\mathbb{R}^N} \Gamma(y-z,s) a(z) \mathrm{d}z\mathrm{d}y \right)^{pq} \mathrm{d}s \\ &= \frac{C}{p+1} t^{p+1} \left(u_0(x,t)\right)^{pq}. \end{aligned} \tag{4.3}$$

We next substitute (4.3) into (1.5). Ignoring again the first and third terms, we have

$$\begin{aligned} v(x,t) &\geq \frac{C}{(p+1)^q} \int_0^t s^{(p+1)q} \int_{\mathbb{R}^N} \Gamma(x-y,t-s) \left(\int_{\mathbb{R}^N} \Gamma(y-z,s) a(z) \mathrm{d}z \right)^{pq^2} \mathrm{d}y\mathrm{d}s \\ &\geq \frac{C}{(p+1)^q} \cdot \frac{t^{(p+1)q+1}}{(p+1)q+1} \left(u_0(x,t)\right)^{pq^2}. \end{aligned} \tag{4.4}$$

Plugging (4.4) into (1.4) we obtain in turn

$$u(x,t) \geq \frac{C}{(p+1)^{pq}} \cdot \frac{1}{((p+1)q+1)^p} \cdot \frac{t^{(p+1)(pq+1)}}{(p+1)(pq+1)} \left(u_0(x,t)\right)^{p^2q^2}. \tag{4.5}$$

Iterating the previous procedure, it follows that for any integer k

$$u(x,t) \geq A_k B_k t^{(p+1)\left(1+pq+(pq)^2+\cdots+(pq)^{k-1}\right)} \left(u_0(x,t)\right)^{(pq)^k}, \tag{4.6}$$

where

$$A_k = \frac{C}{(p+1)^{(pq)^{k-1}}} \left(\frac{1}{(p+1)(1+pq)}\right)^{(pq)^{k-2}} \left(\frac{1}{(p+1)\left(1+pq+p^2q^2\right)}\right)^{(pq)^{k-3}} \cdots \left(\frac{1}{(p+1)\left(1+pq+p^2q^2+\cdots+(pq)^{k-1}\right)}\right), \tag{4.7}$$

$$B_k = \left(\frac{1}{(p+1)q+1}\right)^{\frac{(pq)^{k-1}}{q}} \left(\frac{1}{(p+1)(1+pq)q+1}\right)^{\frac{(pq)^{k-2}}{q}} \cdot \left(\frac{1}{(p+1)\left(1+pq+p^2q^2\right)q+1}\right)^{\frac{(pq)^{k-3}}{q}} \cdots \left(\frac{1}{(p+1)\left(1+pq+p^2q^2+\cdots+(pq)^{k-2}\right)q+1}\right)^{p}, \tag{4.8}$$

here constant C in A_k is changed one by one according to the different k.

We next note that for any positive integers k and l, the following equalities hold

$$\sum_{i=0}^{k-1}(pq)^i = \frac{(pq)^k-1}{pq-1},$$

$$\sum_{i=0}^{l-1}(l-i)(pq)^i = pq\frac{(pq)^l-1}{(pq-1)^2} - \frac{l}{pq-1},$$

$$\sum_{i=1}^{l-1}(l-i)(pq)^i = pq\frac{(pq)^l-1}{(pq-1)^2} - \frac{pql}{pq-1}.$$

Now set $\lambda = pq$, then (4.7) can be written as

$$A_k = C\left(\frac{1}{p+1}\right)^{\frac{\lambda^k-1}{\lambda-1}} \prod_{i=1}^{k-1}\left(\frac{\lambda-1}{\lambda^{i+1}-1}\right)^{\lambda^{k-i-1}}. \tag{4.9}$$

We note that $(p+1)\left(1+\lambda+\lambda^2+\cdots+\lambda^i\right) > 1$ for any integer $i > 1$, then

$$B_k \geq \left(\frac{1}{(p+1)(q+1)}\right)^{\frac{\lambda^{k-1}}{q}} \left(\frac{1}{(p+1)(q+1)(1+\lambda)}\right)^{\frac{\lambda^{k-2}}{q}} \cdot \left(\frac{1}{(p+1)(q+1)\left(1+\lambda+\lambda^2\right)}\right)^{\frac{\lambda^{k-3}}{q}} \cdots \left(\frac{1}{(p+1)(q+1)\left(1+\lambda+\lambda^2+\cdots+\lambda^{k-2}\right)}\right)^{\frac{\lambda}{q}},$$

and therefore

$$B_k \geq \left(\frac{1}{(p+1)(q+1)}\right)^{\frac{\lambda^k-1}{q(\lambda-1)}} \cdot \prod_{i=1}^{k-1}\left(\frac{\lambda-1}{\lambda^{i+1}-1}\right)^{\frac{\lambda^{k-i-1}}{q}}. \tag{4.10}$$

Substitution of (4.9) and (4.10) into (4.6) yields

$$u(x,t) \geq Ct^{\frac{(p+1)\left(\lambda^k-1\right)}{\lambda-1}} \left(u_0(x,t)\right)^{\lambda^k} \left(\frac{1}{p+1}\right)^{\frac{\lambda^k-1}{\lambda-1}\left(1+\frac{1}{q}\right)} \times \left(\frac{1}{q+1}\right)^{\frac{\lambda^k-1}{q(\lambda-1)}} \left(\prod_{i=1}^{k-1}\left(\frac{\lambda-1}{\lambda^{i+1}-1}\right)^{\lambda^{k-i-1}}\right)^{1+\frac{1}{q}},$$

hence

$$t^{\frac{(p+1)\left(\lambda^k-1\right)}{(\lambda-1)\lambda^k}} u_0(x,t) \le C\,(p+1)^{\frac{\lambda^k-1}{(\lambda-1)\lambda^k}\left(1+\frac{1}{q}\right)}\,(q+1)^{\frac{\lambda^k-1}{q(\lambda-1)\lambda^k}}\,\|u(x,t)\|_\infty^{\lambda^{-k}}$$
$$\times\left(\prod_{i=1}^{k-1}\left(\frac{\lambda-1}{\lambda^{i+1}-1}\right)^{\lambda^{k-i-1}}\right)^{-(1+\frac{1}{q})\lambda^{-k}}. \tag{4.11}$$

Since $\|u(x,t)\|_\infty < +\infty$ for any $t \in [0,\infty)$, letting $k \to \infty$ in (4.11) and recalling that $\lambda = pq$, we finally arrive at

$$t^{\frac{p+1}{pq-1}} u_0(x,t) \le C < +\infty$$

for some constant C that only depends on p and q. □

In the critical case, applying Lemma 4.1, the blow-up theorem (Theorem 1.2) of the non-negative solutions for (1.1) is proved as follows.

Proof of Theorem 1.2. Let $1 < p \le q$, $N = \max\{\frac{\alpha(p+1)}{pq-1}, \frac{\alpha(q+1)}{pq-1}\} = \frac{\alpha(q+1)}{pq-1}$. We suppose by contradiction that there exists a nonnegative global solution $u, v \in C^1(Q) \cap L^\infty(Q)$ for (1.1) such that $(-\Delta)^{\frac{\alpha}{2}} u(x,t)$, $(-\Delta)^{\frac{\alpha}{2}} v(x,t) \in L^\infty(Q)$. Using Lemma 4.1, there exists a constant C which depends only on p and q such that

$$t^{\frac{q+1}{pq-1}} v_0(x,t) \le C,$$

that is

$$t^{\frac{N}{\alpha}} v_0(0,t) \le C. \tag{4.12}$$

By employing Fatou's lemma and Lemma 2.4, we can derive

$$\begin{aligned}\lim_{t\to\infty} t^{\frac{N}{\alpha}} v_0(0,t) &\ge \int_{\mathbb{R}^N} \lim_{t\to\infty} t^{\frac{N}{\alpha}} \Gamma(-y,t) b(y) \mathrm{d}y \\ &\ge C'_{\alpha,N} \int_{\mathbb{R}^N} b(y) \mathrm{d}y.\end{aligned} \tag{4.13}$$

Estimates (4.12) and (4.13) yield

$$\|b(x)\|_1 \le C, \tag{4.14}$$

where $C > 0$ depends only on α and N. Regarding $v(\cdot,t)$ as initial value, by (4.14) we get

$$\|v(\cdot,t)\|_1 \le C, \qquad \forall t \ge 0. \tag{4.15}$$

Let $t_0 > 0$ be as in Lemma 2.5. For $t > 1$, we set $\tilde{u}(\cdot,t) = u(\cdot,t+t_0)$, $\tilde{v}(\cdot,t) = v(\cdot,t+t_0)$. Obviously, $(\tilde{u},\tilde{v})$ is also a solution for (1.1). Applying (1.4) and Lemma 2.5, it follows that

$$\begin{aligned}\tilde{u}(x,t) &\ge \int_{\mathbb{R}^N} \Gamma(x-y,t)\tilde{u}(y,0)\mathrm{d}y \\ &\ge C\int_{\mathbb{R}^N} \Gamma(x-y,t)\Gamma(y,\tau)\mathrm{d}y \\ &= C\Gamma(x,t+\tau),\end{aligned} \tag{4.16}$$

here we have used Proposition 2.2 (5) in the last equality. By means of (1.5), (1.2b), Tonelli's theorem and Proposition 2.2 (4), we have

$$\begin{aligned}\|\tilde{v}(x,t)\|_1 &\geq \widetilde{C}_1 \int_{\mathbb{R}^N} \left(\int_0^t \int_{\mathbb{R}^N} \Gamma(x-y,t-s)\tilde{u}^q(y,s)\mathrm{d}y\mathrm{d}s \right) \mathrm{d}x \\ &= \widetilde{C}_1 \int_0^t \int_{\mathbb{R}^N} \tilde{u}^q(y,s)\mathrm{d}y\mathrm{d}s. \end{aligned} \tag{4.17}$$

Substituting (4.16) into (4.17), combining with Proposition 2.2 (2), and observing that $N = \frac{\alpha(q+1)}{pq-1}$, we can estimate

$$\begin{aligned}\|\tilde{v}(x,t)\|_1 &\geq C \int_1^t \int_{\mathbb{R}^N} \Gamma^q(y,s+\tau)\mathrm{d}y\mathrm{d}s \\ &\geq C \int_1^t (s+\tau)^{\frac{N}{\alpha}q}\,\mathrm{d}s \int_{\mathbb{R}^N} \Gamma^q(y,1)\mathrm{d}y \\ &= C\left[(t+\tau)^{\frac{q(q+1)}{pq-1}+1} - (1+\tau)^{\frac{q(q+1)}{pq-1}+1} \right]. \end{aligned} \tag{4.18}$$

In the proceeding estimate, we consider the integral $\int_{\mathbb{R}^N} \Gamma^q(y,1)\mathrm{d}y$ as a constant.

In addition, estimate (4.15) also holds for the function $\tilde{v}$, which conflicts with (4.18) as t large enough. □

5 Blow-up of nonnegative solutions for (1.3) in the subcritical case

Next, we prove the nonexistence of nonnegative solutions for (1.3) in the subcritical case.

Proof of Theorem 1.3. Assume (1.3) admits a nonnegative global solution (u,v), we argue by contradiction. Take $\varphi(x,t) = \phi^{s_1}(\frac{|x|}{R})\phi^{ks_2}(\frac{t}{R^\sigma})$ as the test function in (2.1) and (2.2), where $s_1, s_2 \geq \max\{\frac{p}{p-1}, \frac{q}{q-1}\}$, $\sigma \geq \max\{\frac{\alpha}{k}, \frac{\beta}{k}\}$, $\phi(\rho) \in C_c^\infty(\mathbb{R})$ is the "standard cut-off function" with the following properties:

$$0 \leq \phi(\rho) \leq 1, \quad \text{and} \quad \phi(\rho) = \begin{cases} 1, & \text{if } \rho \leq 1, \\ 0, & \text{if } \rho \geq 2. \end{cases}$$

Substituting $\varphi(x,t)$ into (2.1), thanks to $\frac{\partial^i \varphi}{\partial t^i}(x,0) \equiv 0, i = 1,2,\ldots,k-1$, it follows that

$$\begin{aligned} &\iint_Q (N_1(v) + f_1(x))\,\varphi(x,t)\mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^N} \frac{\partial^{k-1}u}{\partial t^{k-1}}(x,0)\varphi(x,0)\mathrm{d}x \\ &= \iint_Q u \left[(-1)^k \frac{\partial^k \varphi}{\partial t^k} + (-\Delta)^{\frac{\alpha}{2}}\varphi \right] \mathrm{d}x\mathrm{d}t \\ &\leq \iint_Q u \left[(-1)^k \phi^{s_1}(\frac{|x|}{R}) \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\phi^{ks_2}\left(\frac{t}{R^\sigma}\right)\right) \right. \\ &\quad \left. + s_1\phi^{ks_2}\left(\frac{t}{R^\sigma}\right) \phi^{s_1-1}\left(\frac{|x|}{R}\right) (-\Delta)^{\frac{\alpha}{2}}\phi\left(\frac{|x|}{R}\right) \right] \mathrm{d}x\mathrm{d}t, \end{aligned} \tag{5.1}$$

here we have used Ju's inequality in the last inequality. Since $u(x,t) \geq 0$, $N_1(v) \geq 0$ and (1.2a), we obtain

$$\iint_Q N_1(v)\varphi(x,t)\mathrm{d}x\mathrm{d}t \geq \iint_Q (C_1|v|^p + N_1(0))\,\varphi(x,t)\mathrm{d}x\mathrm{d}t \geq C_1 \iint_Q |v|^p\varphi(x,t)\mathrm{d}x\mathrm{d}t.$$

By $\frac{\partial^{k-1}u}{\partial t^{k-1}}(x,0) \geq 0$, $f_1(x) \geq 0$ and $f_1(x) \not\equiv 0$, applying Hölder's inequality, we get that

$$\begin{aligned}
&\iint_Q v^p \varphi(x,t)\mathrm{d}x\mathrm{d}t \\
&\quad < C\left(\iint_Q u^q \varphi(x,t)\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{q}} \cdot \left(\iint_Q \left|(-1)^k \phi^{s_1}\left(\frac{|x|}{R}\right)\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\phi^{ks_2}\left(\frac{t}{R^\sigma}\right)\right)\varphi^{\frac{1-q'}{q'}}\right|^{q'}\right. \\
&\qquad \left.+\left|s_1\phi^{ks_2}\left(\frac{t}{R^\sigma}\right)\phi^{s_1-1}\left(\frac{|x|}{R}\right)(-\Delta)^{\frac{\alpha}{2}}\phi\left(\frac{|x|}{R}\right)\varphi^{\frac{1-q'}{q'}}\right|^{q'}\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{q'}}, \qquad (5.2)
\end{aligned}$$

here we have used C_p inequality in the last inequality. Set

$$\begin{aligned}
B_1 &:= C\iint_Q \left|\phi^{s_1}\left(\frac{|x|}{R}\right)\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\phi^{ks_2}\left(\frac{t}{R^\sigma}\right)\right)\varphi^{\frac{1-q'}{q'}}\right|^{q'}\mathrm{d}x\mathrm{d}t, \\
B_2 &:= C\iint_Q \left|s_1\phi^{ks_2}\left(\frac{t}{R^\sigma}\right)\phi^{s_1-1}\left(\frac{|x|}{R}\right)(-\Delta)^{\frac{\alpha}{2}}\phi\left(\frac{|x|}{R}\right)\varphi^{\frac{1-q'}{q'}}\right|^{q'}\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Then we have

$$\iint_Q v^p\varphi(x,t)\mathrm{d}x\mathrm{d}t < \left(\iint_Q u^q\varphi(x,t)\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{q}} \cdot (B_1+B_2)^{\frac{1}{q'}}. \qquad (5.3)$$

Now according to the expression of $\varphi(x,t)$, we get

$$\begin{aligned}
B_1 &= C\iint_Q \phi^{s_1}\left(\frac{|x|}{R}\right)\left|\frac{1}{R^{k\sigma}}\left(\phi^{ks_2}\right)^{(k)}\left(\frac{t}{R^\sigma}\right)\right|^{q'}\phi^{ks_2(1-q')}\left(\frac{t}{R^\sigma}\right)\mathrm{d}x\mathrm{d}t \\
&= C\int_{R^\sigma}^{2R^\sigma}\left|\frac{1}{R^{k\sigma}}\left(\phi^{ks_2}\right)^{(k)}\left(\frac{t}{R^\sigma}\right)\right|^{q'}\phi^{ks_2(1-q')}\left(\frac{t}{R^\sigma}\right)\mathrm{d}t\int_{\{x\in\mathbb{R}^N:|x|<2R\}}\phi^{s_1}\left(\frac{|x|}{R}\right)\mathrm{d}x. \qquad (5.4)
\end{aligned}$$

Since $\phi(\rho) \in [0,1]$, and if $s_2 \geq \max\{\frac{p}{p-1}, \frac{q}{q-1}\}$, namely $s_2 \geq q'$, for any $1 \leq j \leq k$, we obtain

$$\left|\left(\varphi^{ks_2}\right)^{(j)}(\rho)\right|^{q'} \leq \left|\sum_{i=1}^{j} C\frac{(ks_2)!}{(ks_2-i)!}\phi^{ks_2-i}(\rho)\right|^{q'} \leq C\phi^{ks_2(q'-1)}(\rho). \qquad (5.5)$$

Substituting (5.5) into (5.4), it yields

$$B_1 \leq C\int_{R^\sigma}^{2R^\sigma}\frac{1}{R^{k\sigma q'}}\mathrm{d}t\int_{\{x\in\mathbb{R}^N:|x|<2R\}}\phi^{s_1}\left(\frac{|x|}{R}\right)\mathrm{d}x \leq CR^{N+\sigma-k\sigma q'}. \qquad (5.6)$$

Furthermore, B_2 is estimated as follows

$$\begin{aligned}
B_2 &= Cs_1^{q'}\int_0^{2R^\sigma}\phi^{ks_2}\left(\frac{t}{R^\sigma}\right)\mathrm{d}t\int_{\{x\in\mathbb{R}^N:|x|<2R\}}\phi^{s_1-q'}\left(\frac{|x|}{R}\right)\left|(-\Delta)^{\frac{\alpha}{2}}\phi\left(\frac{|x|}{R}\right)\right|^{q'}\mathrm{d}x \\
&\leq 2Cs_1^{q'}R^\sigma\int_{\{x\in\mathbb{R}^N:|x|<2R\}}\phi^{s_1-q'}\left(\frac{|x|}{R}\right)\left|(-\Delta)^{\frac{\alpha}{2}}\phi\left(\frac{|x|}{R}\right)\right|^{q'}\mathrm{d}x. \qquad (5.7)
\end{aligned}$$

Let $y = \frac{x}{R}$. Using the definition of fractional Laplacian [12], we obtain

$$(-\Delta)^{\frac{\alpha}{2}}\phi\left(\frac{x}{R}\right) = R^{-\alpha}(-\Delta)^{\frac{\alpha}{2}}\phi(|y|).$$

Plugging the above equality into (5.7), it follows that

$$\begin{aligned} B_2 &\leq CR^{\sigma} \int_{\{x\in\mathbb{R}^N:|x|<2R\}} \left|(-\Delta)^{\frac{\alpha}{2}}\phi\left(\frac{|x|}{R}\right)\right|^{q'} \mathrm{d}x \\ &\leq CR^{\sigma} \int_{\{y\in\mathbb{R}^N:|y|<2\}} R^{N-\alpha q'} \left|(-\Delta)^{\frac{\alpha}{2}}\phi\left(|y|\right)\right|^{q'} \mathrm{d}y \\ &\leq CR^{N+\sigma-\alpha q'}. \end{aligned} \tag{5.8}$$

Here we consider the integral $\int_{\{y\in\mathbb{R}^N:|y|<2\}} \left|(-\Delta)^{\frac{\alpha}{2}}\phi\left(|y|\right)\right|^{q'} \mathrm{d}y$ as a constant.

When $\sigma \geq \max\left\{\frac{\alpha}{k}, \frac{\beta}{k}\right\}$, the powers in (5.6) and (5.8) satisfy the following inequality:

$$N+\sigma-k\sigma q' \leq N+\sigma-\alpha q'.$$

Thus, we eventually arrive at

$$B_1+B_2 \leq CR^{N+\sigma-\alpha q'} \tag{5.9}$$

for sufficiently large R.

Analogously, we next substitute $\varphi(x,t)$ into (2.2), use Hölder's inequality and the definition of the global weak solution to obtain

$$\iint_Q u^q\varphi(x,t)\mathrm{d}x\mathrm{d}t < \left(\iint_Q v^p\varphi(x,t)\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p}} \cdot (B_3+B_4)^{\frac{1}{p'}}, \tag{5.10}$$

where

$$B_3 = C\iint_Q \left|\phi^{s_1}\left(\frac{|x|}{R}\right)\frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\phi^{ks_2}\left(\frac{t}{R^{\sigma}}\right)\right)\varphi^{\frac{1-p'}{p'}}\right|^{p'}\mathrm{d}x\mathrm{d}t,$$

and

$$B_4 = C\iint_Q \left|s_1\phi^{ks_2}\left(\frac{t}{R^{\sigma}}\right)\phi^{s_1-1}\left(\frac{|x|}{R}\right)(-\Delta)^{\frac{\beta}{2}}\phi\left(\frac{|x|}{R}\right)\varphi^{\frac{1-p'}{p'}}\right|^{p'}\mathrm{d}x\mathrm{d}t.$$

Similar to (5.6) and (5.8), we can derive that

$$B_3 \leq CR^{N+\sigma-k\sigma p'}, \qquad B_4 \leq CR^{N+\sigma-\beta p'},$$

since $\sigma \geq \max\{\frac{\alpha}{k}, \frac{\beta}{k}\}$, we derive

$$B_3+B_4 \leq CR^{N+\sigma-\beta p'} \tag{5.11}$$

for sufficiently large R.

Plugging (5.3) into (5.10) we obtain

$$\left(\iint_Q u^q\varphi(x,t)\mathrm{d}x\mathrm{d}t\right)^{1-\frac{1}{pq}} < (B_1+B_2)^{\frac{1}{pq'}} \cdot (B_3+B_4)^{\frac{1}{p'}} \leq CR^{\frac{N+\sigma}{pq'}-\frac{\alpha}{p}+\frac{N+\sigma}{p'}-\beta} \tag{5.12}$$

for sufficiently large R. Similarly, substituting (5.10) into (5.3) gives

$$\left(\iint_Q v^p\varphi(x,t)\mathrm{d}x\mathrm{d}t\right)^{1-\frac{1}{pq}} < CR^{\frac{N+\sigma}{qp'}-\frac{\alpha}{q}+\frac{N+\sigma}{q'}-\beta} \tag{5.13}$$

for sufficiently large R. Letting $R \to \infty$ in (5.12) and (5.13). Using the hypothesis of Theorem 1.3, we can see that $\frac{N+\sigma}{pq'} - \frac{\alpha}{p} + \frac{N+\sigma}{p'} - \beta < 0$ or $\frac{N+\sigma}{qp'} - \frac{\alpha}{q} + \frac{N+\sigma}{q'} - \beta < 0$, we eventually have

$$\iint_Q u^q \mathrm{d}x\mathrm{d}t = \lim_{R\to+\infty} \iint_Q u^q \varphi(x,t)\mathrm{d}x\mathrm{d}t \le 0,$$

or

$$\iint_Q v^p \mathrm{d}x\mathrm{d}t = \lim_{R\to+\infty} \iint_Q v^p \varphi(x,t)\mathrm{d}x\mathrm{d}t \le 0.$$

Therefore $u(x,t) \equiv 0$ or $v(x,t) \equiv 0$ in Q. This is a contradiction with the assumption that $f_i(x) \not\equiv 0$ for $i = 1,2$, which ends the proof. □

A Appendix

Below we give the complete proof of Lemma 2.3.

***Proof of Lemma* 2.3.** (1) We first prove (2.3). When $1 + \frac{1}{n} = \frac{1}{m} + \frac{1}{r}$, using Proposition 2.2 (4), we obtain $\Gamma(x,t) \in L^r(\mathbb{R}^N)$. Since $h(x,t) \in L^m(\mathbb{R}^N)$, by applying generalized Young's inequality, we get

$$\|\Gamma(x,t) * h(x,t)\|_n \le \|\Gamma(x,t)\|_r \|h(x,t)\|_m = \left(\int_{\mathbb{R}^N} \Gamma^r(x,t)\mathrm{d}x\right)^{\frac{1}{r}} \cdot \|h(x,t)\|_m .$$

By Proposition 2.2 (1) with $s = 1$, namely $\Gamma(x,t) = t^{-\frac{N}{\alpha}}\Gamma\big(t^{-\frac{1}{\alpha}}x, 1\big)$, we have

$$\|\Gamma(x,t) * h(x,t)\|_n \le t^{-\frac{N}{\alpha}} \left(\int_{\mathbb{R}^N} \Gamma^r(t^{-\frac{1}{\alpha}}x, 1)\mathrm{d}x\right)^{\frac{1}{r}} \cdot \|h(x,t)\|_m . \tag{A.1}$$

Utilizing Proposition 2.2 (3), we estimate

$$\begin{aligned}\int_{\mathbb{R}^N} \Gamma^r(t^{-\frac{1}{\alpha}}x, 1)\mathrm{d}x &\le C \int_{\mathbb{R}^N} \frac{1}{\left(1 + t^{-\frac{1}{\alpha}}|x|\right)^{(N+\alpha)r}}\mathrm{d}x \\ &\le C \int_{\mathbb{R}^N} \frac{t^{\frac{N}{\alpha}}}{\left(1 + t^{-\frac{1}{\alpha}}|x|\right)^{(N+\alpha)r}}\mathrm{d}\left(t^{-\frac{1}{\alpha}}x\right).\end{aligned} \tag{A.2}$$

Denote $y = t^{-\frac{1}{\alpha}}x$, then (A.2) can be reduced to

$$\int_{\mathbb{R}^N} \Gamma^r(t^{-\frac{1}{\alpha}}x, 1)\mathrm{d}x \le Ct^{\frac{N}{\alpha}} \int_{\mathbb{R}^N} \frac{1}{(1 + |y|)^{(N+\alpha)r}}\mathrm{d}y. \tag{A.3}$$

Due to $1 + \frac{1}{n} = \frac{1}{m} + \frac{1}{r}$ and $1 \le m \le n \le \infty$, we have $r \ge 1$. Otherwise, it conflicts with $m \le n$. Therefore, $(N+\alpha)r > N$. Consequently, the integral on the right hand side of inequality (A.3) is integrable. We now substitute (A.3) into (A.1) to obtain

$$\|\Gamma(x,t) * h(x,t)\|_n \le K_1 t^{-\frac{N}{\alpha}(1-\frac{1}{r})} \|h(x,t)\|_m , \tag{A.4}$$

where $K_1 = \left(C \int_{\mathbb{R}^N} \frac{1}{(1+|y|)^{(N+\alpha)r}}\mathrm{d}y\right)^{\frac{1}{r}}$.

(2) The proof of (2.4) is as follows. Similarly, by using generalized Young's inequality and Proposition 2.2 (1), we conclude

$$\begin{aligned}\left\|(-\Delta)^{\frac{l}{2}}\Gamma(x,t)*h(x,t)\right\|_n &\le \left\|(-\Delta)^{\frac{l}{2}}\Gamma(x,t)\right\|_r \|h(x,t)\|_m \\ &= \left(\int_{\mathbb{R}^N}\left((-\Delta)^{\frac{l}{2}}\Gamma(x,t)\right)^r \mathrm{d}x\right)^{\frac{1}{r}}\cdot\|h(x,t)\|_m \\ &= t^{-\frac{N}{\alpha}}\left(\int_{\mathbb{R}^N}\left((-\Delta)^{\frac{l}{2}}\Gamma(t^{-\frac{1}{\alpha}}x,1)\right)^r \mathrm{d}x\right)^{\frac{1}{r}}\cdot\|h(x,t)\|_m.\end{aligned}$$

Let $y=t^{-\frac{1}{\alpha}}x$. Using the definition of fractional Laplacian, we get that

$$\begin{aligned}\int_{\mathbb{R}^N}\left((-\Delta)^{\frac{l}{2}}\Gamma(t^{-\frac{1}{\alpha}}x,1)\right)^r \mathrm{d}x &= t^{-\frac{lr}{\alpha}+\frac{N}{\alpha}}\int_{\mathbb{R}^N}\left((-\Delta)^{\frac{l}{2}}\Gamma(y,1)\right)^r \mathrm{d}y \\ &\le C't^{-\frac{lr}{\alpha}+\frac{N}{\alpha}}\int_{\mathbb{R}^N}\frac{1}{(1+|y|)^{(N+l)r}}\mathrm{d}y,\end{aligned} \tag{A.5}$$

here we have used Proposition 2.2 (3) in the last inequality. Since $r\ge 1$, then $(N+l)r>N$. Therefore, the integral on the right hand side of inequality (A.5) is integrable. It follows that

$$\left\|(-\Delta)^{\frac{l}{2}}\Gamma(x,t)*h(x,t)\right\|_n \le K_2 t^{-\frac{l}{\alpha}-\frac{N}{\alpha}\left(1-\frac{1}{r}\right)}\|h(x,t)\|_m, \tag{A.6}$$

where $K_2=\left(C'\int_{\mathbb{R}^N}\frac{1}{(1+|y|)^{(N+l)r}}\mathrm{d}y\right)^{\frac{1}{r}}$.

In particular, substituting $l=1$ into (A.6), we have

$$\|\nabla\Gamma(x,t)*h(x,t)\|_n \le K_2 t^{-\frac{1}{\alpha}-\frac{N}{\alpha}\left(1-\frac{1}{r}\right)}\|h(x,t)\|_m, \qquad \forall t\in(0,T].$$

This completes the proof of Lemma 2.3. □

Based on the method in the proof of Lemma 3.2 in [24], we now show the complete proof of Lemma 2.5 as follows.

***Proof of Lemma* 2.5**. Let $t_0>0$ be such that $\Gamma(0,t_0)\le 1$. We obtain

$$\Gamma(x-y,t_0)=\Gamma\left(\frac{1}{2}(2x-2y),t_0\right).$$

Proposition 2.2 (1) and (6) yield

$$\Gamma\left(\frac{1}{2}(2x-2y),t_0\right)\ge\Gamma(2x,t_0)\,\Gamma(2y,t_0)=2^{-N}\Gamma\left(x,\frac{t_0}{2^\alpha}\right)\Gamma(2y,t_0),$$

that is

$$\Gamma(x-y,t_0)\ge 2^{-N}\Gamma\left(x,\frac{t_0}{2^\alpha}\right)\Gamma(2y,t_0). \tag{A.7}$$

Plugging (A.7) into (1.4), and dropping the second and third terms on the right side, we obtain

$$\begin{aligned}u(x,t_0)&\ge\int_{\mathbb{R}^N}\Gamma(x-y,t_0)a(y)\mathrm{d}y \\ &\ge\int_{\mathbb{R}^N}2^{-N}\Gamma\left(x,\frac{t_0}{2^\alpha}\right)\Gamma(2y,t_0)a(y)\mathrm{d}y \\ &= C\Gamma(x,\tau),\end{aligned}$$

where $C = \int_{\mathbb{R}^N} 2^{-N}\Gamma(2y, t_0)a(y)\mathrm{d}y$, $\tau = \frac{t_0}{2^\alpha} > 0$.

In order to get the corresponding result of $v(x,t)$, for $q > 1$, by employing Jensen's inequality and Tonelli's theorem we get

$$\begin{aligned} v(x,t) &\geq \int_0^t \int_{\mathbb{R}^N} \Gamma(x-y, t-s) N_2(u)(y,s)\mathrm{d}y\mathrm{d}s \\ &\geq \widetilde{C}_1 \int_0^t \int_{\mathbb{R}^N} \Gamma(x-y,t-s) \left| \int_{\mathbb{R}^N} \Gamma(y-z,s)a(z)\mathrm{d}z \right|^q \mathrm{d}y\mathrm{d}s \\ &\geq C \int_0^t \left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Gamma(x-y,t-s)\Gamma(y-z,s)a(z)\mathrm{d}z\mathrm{d}y \right|^q \mathrm{d}s \\ &= C \int_0^t \left| \int_{\mathbb{R}^N} \Gamma(x-z,t)a(z)\mathrm{d}z \right|^q \mathrm{d}s, \end{aligned}$$

hence

$$v(x,t) \geq Ct \left| \int_{\mathbb{R}^N} \Gamma(x-y,t)a(y)\mathrm{d}y \right|^q. \tag{A.8}$$

For the above t_0, we then have

$$\begin{aligned} v(x,t_0) &\geq Ct_0 \left| \int_{\mathbb{R}^N} \Gamma(x-y,t_0)a(y)\mathrm{d}y \right|^q \\ &\geq Ct_0 \left| \int_{\mathbb{R}^N} 2^{-N}\Gamma\left(x, \frac{t_0}{2^\alpha}\right)\Gamma(2y,t_0)a(y)\mathrm{d}y \right|^q \\ &= C\Gamma^q(x,\tau), \end{aligned}$$

where $\tau = \frac{t_0}{2^\alpha} > 0$. We consider the integral $\left|\int_{\mathbb{R}^N} 2^{-N}\Gamma(2y,t_0)a(y)\mathrm{d}y\right|^q$ as a constant. □

Remark A.1. In Section 4, we assumed that $1 < p \leq q$. If $p \geq q > 1$, then the conclusion of Lemma 2.5 needs to be changed as follows:

$$v(x,t_0) \geq C\Gamma(x,\tau), \qquad u(x,t_0) \geq C\Gamma^p(x,\tau).$$

Its proof is similar to the proof of Lemma 2.5. Therefore, in the case of $p \geq q > 1$, through the homologous proof of Theorem 1.2, we can still get the conclusion of Theorem 1.2.

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Semilinear heat equation with singular terms

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Abstract. The main goal of this paper is to analyze the existence and nonexistence as well as the regularity of positive solutions for the following initial parabolic problem

$$\begin{cases} \partial_t u - \Delta u = \mu \dfrac{u}{|x|^2} + \dfrac{f}{u^\sigma} & \text{in } \Omega_T := \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded open, $\sigma \geq 0$ and $\mu > 0$ are real constants and $f \in L^m(\Omega_T)$, $m \geq 1$, and u_0 are nonnegative functions. The study we lead shows that the existence of solutions depends on σ and the summability of the datum f as well as on the interplay between μ and the best constant in the Hardy inequality. Regularity results of solutions, when they exist, are also provided. Furthermore, we prove uniqueness of finite energy solutions.

Keywords: heat equation, existence and regularity, Hardy potential, singular terms.

2020 Mathematics Subject Classification: 35K20, 35K91, 35K67, 35B65.

1 Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 3$, containing the origin. Set $\Omega_T := \Omega \times (0,T)$ where $T > 0$ is a real constant. In this paper we investigate the existence and regularity as well as the uniqueness of solutions to the following initial parabolic problem

$$\begin{cases} \partial_t u - \Delta u = \mu \dfrac{u}{|x|^2} + \dfrac{f}{u^\sigma} & \text{in } \Omega_T, \\ u = 0 & \text{on } \Gamma := \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\sigma \geq 0$ and $\mu \geq 0$. The source terms f and u_0 satisfy

$$f \geq 0, \quad f \in L^m(\Omega_T), \quad m \geq 1 \tag{1.2}$$

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and $u_0 \in L^\infty(\Omega)$ such that

$$\forall w \subset\subset \Omega \quad \exists d_w > 0 : u_0 \geq d_w \quad \text{in } w. \tag{1.3}$$

It is clear that problem (1.1) is strongly related to the following classical Hardy inequality which asserts that

$$\Lambda_{N,2} \int_\Omega \frac{|u|^2}{|x|^2} dx \leq \int_\Omega |\nabla u|^2 dx, \tag{1.4}$$

for all $u \in C_0^\infty(\Omega)$, where $\Lambda_{N,2} = (\frac{N-2}{2})^2$ is optimal and not achieved (see for instance [20,50] and [11] when $\Omega = \mathbb{R}^N$). The presence of a term with negative exponent generally induces a difficulty in defining the notion of solution for the problem (1.1).

In the literature, singular problems like (1.1) are considered and intensively studied in various situations depending on σ or μ. If $\sigma = 0$ and $\mu > 0$, the problem (1.1) is reduced to the following heat equation involving the Hardy potential

$$\begin{cases} \partial_t u - \Delta u = \mu \dfrac{u}{|x|^2} + f & \text{in } \Omega_T, \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.5}$$

and is studied first by Baras and Goldstein in their pioneering work [15]. When the data $0 \leq f \in L^1(\Omega_T)$ and u_0 is a positive L^1-function or a positive Radon measure on Ω are not both identically zero (otherwise the result is false since $u \equiv 0$ is a solution), Baras and Goldstein [15] have proved that if $0 \leq \mu \leq \Lambda_{N,2}$ then there exists a positive global solution for the problem (1.5), while if $\mu > \Lambda_{N,2}$ there is no solution.

Problem (1.5) with $-\text{div}a(x,t,\nabla u)$ instead of $-\Delta$ was studied in [45], where the author proved that all the solutions have the same asymptotic behaviour, that is they all tend to the solution of the original problem which satisfies a zero initial condition. In [46] the authors studied the influence of the presence of the Hardy potential and the summability of the datum f on the regularity of the solutions of problem (1.5) with the nonlinear operator $-\text{div}a(x,t,u,\nabla u)$ in the principal part.

The singular Hardy potential appears in the context of combustion theory (see [50] and references therein) and quantum mechanics (see [15] and [50] and references therein). There is a wide literature about problems involving the Hardy potential where the existence and regularity of solutions as well as nonexistence of solutions are analyzed, for instance, we refer to [2–7,10,17,18,32,36–38,54].

Problems involving singularities (like (1.1) with $\mu = 0$) describe naturally several physical phenomena. Stationary cases include the semilinear equation $-\Delta u = f(x)u^{-\sigma}$, $x \in \Omega \subset \mathbb{R}^N$, that can be obtained as a generalization to the higher dimension from a one dimensional ODE ($N = 1$) by some transformations of boundary layer equations for the class of non-Newtonian fluids called pseudoplastic (see [29,39]). As far as we know, semilinear equations with singularities arise in various contexts of chemical heterogeneous catalysts [9], non-Newtonian fluids as well as heat conduction in electrically conducting materials (the term u^σ describes the resistivity of the material), see for instance, [31,39]. In view of this physical interpretation various generalizations of this later equation considered in the framework of partial differential equations ($N \geq 2$) has been the subject of study in many papers. For the mathematical analysis account, the seminal papers [23,49] constitute the starting point of a wide literature about singular semilinear elliptic equations. Far from being complete we quote the list [8,17,19,21,26,27,33,34,40,42,43,52,53,56].

It is worth recalling that due to the meaning of the unknowns (concentrations, populations,...), only the positive solutions are relevant in most cases.

As far as the parabolic setting is concerned for problems as in (1.1) with $\mu = 0$, the literature is not rich enough. For problems like (1.1) with p-Laplacian operator, existence results of nonnegative solutions are obtained in [25] for data with higher summability while in [41] the authors proved the existence of nonnegative distributional solutions for non regular data (L^1 and measure) and the uniqueness is proved for energy solutions. Other related problems with singular terms can be found in [12–14].

In the case where $\sigma \neq 0$ and $\mu = 0$, problem (1.1) with a quite more general diffusion operator including the Laplacian one was studied in [24]. The authors considered nonnegative data having suitable Lebesgue-type summabilities and assumed the strict positivity on the initial data inside the parabolic cylinder. They have shown, via Harnack's inequality, that this strict positivity is inherited by the constructed solution to the problem, thus giving a meaning to the notion of solution considered. Some regularity results are obtained according to the regularity of f and the values of σ.

Our main goal in this paper is to study the problem (1.1) in the presence of the two singular terms, that is $\mu > 0$ and $\sigma \geq 0$ extending to the evolution case some results obtained for the elliptic problem (with the Δ_p operator instead of Laplacian one) studied in [1]. Abdellaoui and Attar [1] investigated the interplay between the summability of f and σ providing the largest class of the datum f for which the problem admits a solution in the sense of distributions. Uniqueness and regularity results on the distributional solutions are also established. In the same spirit, the parabolic case with $\mu = 0$ was investigated in [24]. Our work fits in the context of recent work on equations involving the Hardy potential in the case of nonexistence of solutions. We start by studying first the case $\mu < \Lambda_{N,2} := \frac{(N-2)^2}{4}$ distinguishing two cases where $\sigma \geq 1$ and $f \in L^1(\Omega_T)$ and the case where $\sigma < 1$ with $f \in L^{m_1}(\Omega_T)$, $m_1 = \frac{2N}{2N+(\sigma-1)(N-2)}$. Then we investigate the case $\mu = \Lambda_{N,2}$ and $\sigma = 1$ with data $f \in L^1(\Omega_T)$. In both cases we prove the existence of a weak solution obtained as limit of approximations that belongs to a suitable Sobolev space. The approach we use consists in approximating the singular equation with a regular problem, where the standard techniques (e.g., fixed point argument) can be applied and then passing to the limit to obtain the weak solution of the original problem. The regularity of weak solutions is analyzed according to the Lebesgue summability of f and σ. Furthermore, we prove the uniqueness of finite energy solutions when the source term f has a compact support by extending the formulation of weak solutions to a large class of test functions. Finally, in the case where $\mu > \Lambda_{N,2}$ we prove a nonexistence result.

The paper is presented as follows. Section 2 contains all the main results (existence, regularity and uniqueness) and also graphic presentations allowing to better locate the obtained results. In Section 3 we first prove an existence result for approximate regular problems of the problem (1.1) and then we give the proof of all the main results Theorem 2.2, Theorem 2.4, Theorem 2.5, Theorem 2.6, Theorem 2.8 and Theorem 2.10. At the end, some results that are necessary for the accomplishment of the work are given in an appendix to make the paper quite self contained.

2 Main results

We begin by stating the definition of weak solution and finite energy solution of the problem (1.1) and then we state and comment the main results.

Definition 2.1.

1) By a weak solution of the problem (1.1) we mean a function $u \in L^1(0,T;W^{1,1}_{loc}(\Omega))$ satisfying

$$\forall \Omega' \subset\subset \Omega \quad \exists C_{\Omega'} > 0 : u \geq C_{\Omega'} \quad \text{in } \Omega'$$

and

$$-\int_\Omega u_0(x)\phi(x,0)dx - \int_{\Omega_T} u\partial_t\phi dxdt + \int_{\Omega_T} \nabla u \cdot \nabla\phi dxdt = \int_{\Omega_T} \left(\mu\frac{u}{|x|^2} + \frac{f}{u^\sigma}\right)\phi dxdt, \tag{2.1}$$

for every $\phi \in C_0^\infty(\Omega \times [0,T))$.

2) We call a finite energy solution of the problem (1.1) a weak solution u that satisfies $u \in L^2(0,T;H_0^1(\Omega))$ with $\partial_t u \in L^2(0,T;H^{-1}(\Omega)) + L^1(0,T;L^1_{loc}(\Omega))$.

In Definition 2.1 above all the integrals make sense. Generated by the singular terms, the only difficulty is raised in the right-hand side. Indeed, by Hardy's inequality the integral $\int_{\Omega_T}\frac{u\phi}{|x|^2}dxdt$ is finite while we make use of a comparison result with a solution of a problem in [24, Proposition 2.2], where the hypothesis (1.3) is used, for the integral $\int_{\Omega_T}|\frac{f\phi}{u^\sigma}|dxdt$ to be finite. Thus one has $\frac{f}{u^\sigma} \in L^1(0,T;L^1_{loc}(\Omega))$.

Throughout this paper, we will use the two real auxiliary truncation functions T_k and G_k defined for $k > 0$ respectively as $T_k(s) = \max(-k, \min(s,k))$ and $G_k(s) = (|s|-k)^+\text{sign}(s)$. We also define

$$m_1 := \frac{2N}{2N-(1-\sigma)(N-2)}.$$

Observe that $m_1 \geq 1$ if and only if $\sigma \leq 1$. We will prove the existence of solution for the problem (1.1) under the assumption that the datum f satisfies

$$\begin{cases} f \in L^{m_1}(\Omega_T) & \text{if } 0 \leq \sigma \leq 1, \\ f \in L^1(\Omega_T) & \text{if } \sigma \geq 1. \end{cases} \tag{2.2}$$

2.1 The case $\mu < \Lambda_{N,2}$: existence of weak solutions

The first existence result is the following.

Theorem 2.2. *Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 3$, containing the origin. Assume that u_0 and f are nonnegative functions satisfying (1.3) and (2.2) respectively. If $\mu < \Lambda_{N,2}$ then the problem (1.1) has a positive weak solution u such that*

1. *if $0 \leq \sigma \leq 1$ then u is a finite energy solution,*
2. *if $\sigma > 1$ then $u \in L^2(0,T;H^1_{loc}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ with $G_k(u) \in L^2(0,T;H_0^1(\Omega))$. Moreover, if $\frac{4\sigma}{(\sigma+1)^2} - \frac{\mu}{\Lambda_{N,2}} > 0$ then we have $u^{\frac{\sigma+1}{2}} \in L^2(0,T;H_0^1(\Omega))$,*
3. *if $\sigma > 1$ and $\text{supp}(f) \subset\subset \Omega$ then u is a finite energy solution.*

Remark 2.3. Let us notice that in absence of the Hardy potential (i.e. $\mu = 0$), the result corresponding to the case $\sigma \leq 1$ is already obtained in [24, Theorem 1.3 (i)], when $p = 2$ and the source term f belongs to $L^{m_2}(\Omega_T)$, $m_2 := \frac{2(N+2)}{2(N+2)-N(1-\sigma)}$. Note that since $m_1 < m_2$, the result we prove here is a refinement of that in [24, Theorem 1.3 (i)]. While in the case $\sigma > 1$

we obtain the same result to that in [24, Theorem 1.3 (ii)]. Note that if $\sigma = 1$ the above results coincide.

Observe that $1 \leq m_1 \leq \frac{2N}{N+2}$ for any $0 \leq \sigma \leq 1$. We point out that in the case where $\sigma = 0$, which yields $m_1 = \frac{2N}{N+2}$, we find the result already established in [46, Theorem 1.2] for data $f \in L^r(0,T;L^q(\Omega))$ with $r = q \geq \frac{2N}{N+2}$. It is worth recalling here that $\frac{2N}{N+2}$ is the Hölder conjugate exponent of the Sobolev exponent $\frac{2N}{N-2}$ and by duality argument, data belonging to the Lebesgue space of exponent $\frac{2N}{N+2}$ are in force in the dual space $L^2(0,T;H^{-1}(\Omega))$.

2.2 The case $\mu = \Lambda_{N,2}$: existence of infinite energy solutions

In the following result we deal with the case where $\mu = \Lambda_{N,2}$. The weak solutions found do not generally belong to the energy space.

Theorem 2.4. *Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 3$, containing the origin. Suppose that (1.3) is fulfilled and assume that $\sigma = 1$ and $f \in L^1(\Omega_T)$. If $\mu = \Lambda_{N,2}$ then the problem (1.1) has a weak solution u such that $u \in L^q(0,T;W_0^{1,q}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$, for every $q < 2$.*

2.3 The case $\mu > \Lambda_{N,2}$: nonexistence of weak solutions

If we assume $\mu > \Lambda_{N,2}$ then the problem (1.1) has no weak solution. This is stated in the following theorem.

Theorem 2.5. *Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 3$, containing the origin. Assume that (1.3) and (2.2) hold. If $\mu > \Lambda_{N,2}$ then the problem (1.1) has no positive weak solution.*

The following Figure 2.1 summarizes the different existence results according to the interactions between the singularities.

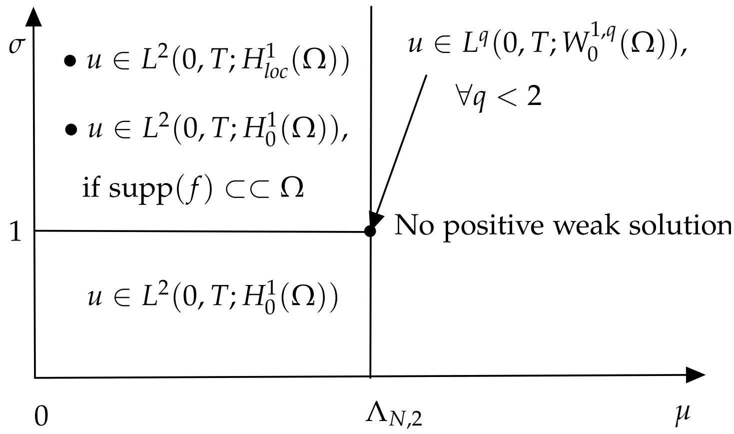

Figure 2.1: Existence and nonexistence results

2.4 Regularity of weak solutions

In the following theorem we give some regularity results for the weak solution u of the problem (1.1) obtained in Theorem 2.2.

Theorem 2.6. *Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 3$, containing the origin. Assume that (1.2) and (1.3) hold and suppose that $\sigma \geq 0$ and $\mu < \Lambda_{N,2}$. Then* *(i) if $\sigma \geq 1$ and $m \geq 1$ one has*

(a) if $m > \frac{N}{2} + 1$ then $u \in L^\infty(\Omega_T)$,

(b) if $1 \leq m < \frac{N}{2} + 1$, then $u^{\frac{\gamma+1}{2}} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$ where $\gamma = \frac{Nm(1+\sigma)-N+2m-2}{N-2m+2}$ provided that $\frac{4\gamma}{(\gamma+1)^2} - \frac{\mu}{\Lambda_{N,2}} > 0$.

(ii) If $0 \leq \sigma \leq 1$ one has

(c) if $m > \frac{N}{2} + 1$ then $u \in L^\infty(\Omega_T)$,

(d) if $m_1 \leq m < \frac{N}{2} + 1$ then $u^{\frac{\gamma+1}{2}} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$ where $\gamma = \frac{Nm(1+\sigma)-N+2m-2}{N-2m+2}$ provided that $\frac{4\gamma}{(\gamma+1)^2} - \frac{\mu}{\Lambda_{N,2}} > 0$.

Remark 2.7.

1. Observe that since $0 < \sigma \leq 1$ and $N \geq 3$ one has $1 \leq m_1 := \frac{2N}{2N-(1-\sigma)(N-2)} < \frac{N}{2} + 1$.

2. If $\sigma \geq 1$ and $1 \leq m < \frac{N}{2} + 1$ then $\gamma \geq m\sigma \geq 1$.

3. If $0 \leq \sigma \leq 1$ and $m_1 \leq m < \frac{N}{2} + 1$ then $\gamma \geq m\sigma \geq 0$.

4. Notice that $0 \leq \frac{4\gamma}{(\gamma+1)^2} \leq 1$ and since $\mu < \Lambda_{N,2}$ the assumption $\frac{4\gamma}{(\gamma+1)^2} - \frac{\mu}{\Lambda_{N,2}} > 0$ is necessary in order to get the results stated in Theorem 2.6.

In the case where $0 < \sigma \leq 1$, the regularity results obtained in the previous Theorem 2.6 concerns the weak solutions corresponding to data $f \in L^m(\Omega_T)$, with $m \geq m_1$. When we decrease the summability of the data, that is $f \in L^m(\Omega_T)$ with $1 < m < m_1$, we obtain solutions lying in a bigger space than the energy one. Actually, we have the following result.

Theorem 2.8. *Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 3$, containing the origin. Assume that (1.3) holds and $f \in L^m(\Omega_T)$, with $1 < m < m_1$ and suppose that $0 \leq \sigma \leq 1$ and $\mu < \Lambda_{N,2}$. Then if $\frac{mN(1+\sigma)}{2N-4(m-1)} > \frac{\Lambda_{N,2}}{\mu}\left(1 - \sqrt{1 - \frac{\mu}{\Lambda_{N,2}}}\right)$, the problem (1.1) has a weak solution u such that $u \in L^q(0, T; W_0^{1,q}(\Omega)) \cap L^\gamma(\Omega_T)$ with $q = \frac{m(N+2)(1+\sigma)}{N+2-m(1-\sigma)}$ and $\gamma = \frac{m(1+\sigma)(N+2)}{N-2m+2}$.*

Remark 2.9. We point out that for the particular case $\sigma = 0$ we obtain that the solution u belongs to $L^q(0, T; W_0^{1,q}(\Omega)) \cap L^\gamma(\Omega_T)$ with $q = \frac{m(N+2)}{N+2-m}$ and $\gamma = \frac{m(N+2)}{N-2m+2}$. These are exactly the same exponents as those obtained in nonsingular case in [16, Theorem 1.9] when $f \in L^{m_3}(\Omega_T)$, $m_3 := \frac{2(N+2)}{2(N+2)-N}$. Observe that since for $\sigma = 0$ we have $m_1 = \frac{2N}{N+2} < m_3$, the result we prove is a refinement of the one in [16, Theorem 1.9]. This is not surprising since the effect of Hardy's potential vanishes for $\mu < \Lambda_{N,2}$ as it is shown in the proof of Theorem 2.8. Remark that we cannot the consider case where $\sigma = 0$ and $m = 1$, since the test functions we use in order to obtain the regularity stated in Theorem 2.8 cannot be chosen.

The following Figure 2.2 summarizes the previous regularity results considering the interplay between the singularity and the summability of the source term f.

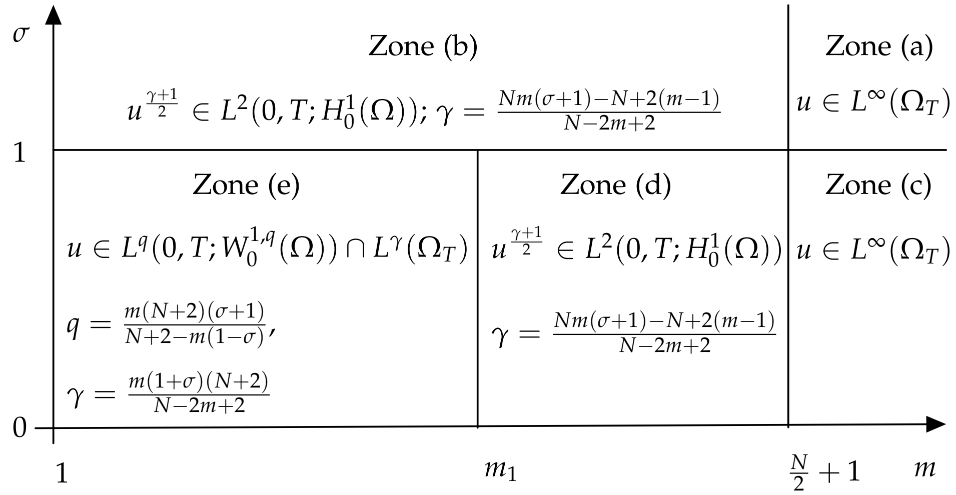

Figure 2.2: Regularity results for $\mu < \Lambda_{N,2}$. Zone (e) corresponds to the result in Theorem 2.8

2.5 Uniqueness of finite energy solutions

As far as the uniqueness is concerned, we give the following result for the finite energy solutions in the case of data with compact support.

Theorem 2.10. *Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 3$, containing the origin. Suppose that* (1.3) *is fulfilled, $\mu < \Lambda_{N,2}$ and $\sigma \geq 0$. If $f \in L^m(\Omega_T)$, with $m \geq 1$ and* $\operatorname{supp}(f) \subset\subset \Omega_T$ *then the energy solution $u \in L^2(0,T;H_0^1(\Omega))$ of the problem* (1.1) *is unique.*

3 Proofs of the results

3.1 Approximate problems

Let us consider the following sequence of approximate initial-boundary value problems

$$\begin{cases} \partial_t u_n - \Delta u_n = \mu \dfrac{T_n(u_n)}{|x|^2 + \frac{1}{n}} + \dfrac{f_n}{(|u_n| + \frac{1}{n})^\sigma} & \text{in } \Omega \times (0,T), \\ u_n(x,t) = 0 & \text{in } \partial\Omega \times (0,T), \\ u_n(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{3.1}$$

where $f_n = T_n(f) = \min(f,n)$. The case $\sigma = 0$ leads to the variational framework since $m_1 = \frac{2N}{N+2}$ is the Hölder conjugate exponent of the Sobolev exponent $2^* := \frac{2N}{N-2}$ and then by the Sobolev embedding and a duality argument we obtain $f \in L^{m_1}(\Omega_T) \hookrightarrow L^2(0,T;H^{-1}(\Omega))$ and the existence of u_n can be found in [30, Theorem 3] on page 356. If $0 < \sigma \leq 1$, the proof of the existence of a solution u_n to the approximate problem (3.1), which is based on the Schauder's fixed point theorem, is now classical. For the convenience of the reader we give it here.

Lemma 3.1. *Assume that $0 < \sigma \leq 1$ and $\mu \leq \Lambda_{N,2}$. For each integer $n \in \mathbb{N}$ the approximate problem* (3.1) *has a solution $u_n \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(\Omega_T)$ such that $\partial_t u_n \in L^2(0,T;H^{-1}(\Omega))$ satisfying* *for every* $\phi \in L^2(0,T;H_0^1(\Omega))$

$$\int_0^T \int_\Omega \partial_t u_n \phi dxdt + \int_0^T \int_\Omega \nabla u_n \nabla \phi dxdt = \mu \int_0^T \int_\Omega \frac{T_n(u_n)\phi}{|x|^2+\frac{1}{n}} dxdt + \int_0^T \int_\Omega \frac{f_n \phi}{(|u_n|+\frac{1}{n})^\sigma} dxdt \tag{3.2}$$

Moreover, u_n *is such that for every* $\Omega' \subset\subset \Omega$ *there exists* $C_{\Omega'} > 0$ *(not depending on* n*), such that* $u_n \geq C_{\Omega'}$ *in* $\Omega' \times [0,T]$.

Proof. Let $v \in L^2(\Omega_T)$ and let $n \in \mathbb{N}$ be fixed. We consider $w \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \cap \mathcal{C}([0,T];L^2(\Omega)) \cap L^\infty(\Omega_T)$ with $\partial_t w \in L^2(0,T;H^{-1}(\Omega))$ the unique weak solution (depending on v and n) of the following problem

$$\begin{cases} \partial_t w - \Delta w = \mu \frac{T_n(w)}{|x|^2+\frac{1}{n}} + \frac{f_n}{\left(|v|+\frac{1}{n}\right)^\sigma} & \text{in } \Omega_T \\ w(x,t) = 0 & \text{in } \partial\Omega \times (0,T) \\ w(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{3.3}$$

which satisfies for every $\phi \in L^2(0,T;H_0^1(\Omega))$

$$\int_0^T \int_\Omega \partial_t u_n \phi dxdt + \int_0^T \int_\Omega \nabla u_n \cdot \nabla \phi dxdt = \mu \int_0^T \int_\Omega \frac{T_n(u_n)\phi}{|x|^2+\frac{1}{n}} dxdt + \int_0^T \int_\Omega \frac{f_n \phi}{(|v|+\frac{1}{n})^\sigma} dxdt$$

The existence of w can be found in [30, Theorem 3] on page 356 (see also [35]). Let us consider the map S defined by $S(v) = w$. Taking w as test function in (3.3) we get

$$\|\nabla w\|^2_{L^2(\Omega_T)} \leq \mu \int_{\Omega_T} \frac{T_n(w)w}{|x|^2+\frac{1}{n}} dxdt + \int_{\Omega_T} \frac{f_n w}{\left(|v|+\frac{1}{n}\right)^\sigma} dxdt + \|u_0\|^2_{L^2(\Omega)}.$$

Thus, by the Hölder inequality we arrive at

$$\|\nabla w\|^2_{L^2(\Omega_T)} \leq |\Omega_T|^{\frac{1}{2}} \left(\mu n^2 + n^{\sigma+1}\right) \left(\int_{\Omega_T} w^2 dxdt\right)^{\frac{1}{2}} + \|u_0\|^2_{L^2(\Omega)},$$

so that by the Poincaré inequality one has

$$\|w\|^2_{L^2(\Omega_T)} \leq C_1 \|w\|_{L^2(\Omega_T)} + C_2,$$

where $C_1 = C_P^2 |\Omega_T|^{\frac{1}{2}} (\mu n^2 + n^{\sigma+1})$, $C_2 = C_P^2 \|u_0\|^2_{L^2(\Omega)}$ and C_p is the constant in the Poincaré inequality. Therefore by the Young inequality we obtain

$$\|w\|_{L^2(\Omega_T)} \leq C := \sqrt{C_1^2 + 2C_2}. \tag{3.4}$$

Defining the ball $B := \left\{v \in L^2(\Omega_T) : \|v\|_{L^2(\Omega_T)} \leq C\right\}$ of $L^2(\Omega_T)$ we have proved that the map $S : B \to B$ is well defined. In order to apply Schauder's fixed point theorem over S to guarantee the existence of a solution for (3.1) in the sense of (3.2), we need to check that the map S is continuous and compact on B.

Let us first prove the continuity of S. In order to do this, let $\{v_k\}_k \subset B$ be a sequence such that

$$\lim_{k\to+\infty} \|v_k - v\|_{L^2(\Omega_T)} = 0.$$

Denote by $w_k := S(v_k)$ and $w := S(v)$. Then w_k is the solution of the problem

$$\begin{cases} \partial_t w_k - \Delta w_k = \mu \frac{T_n(w_k)}{|x|^2+\frac{1}{n}} + \frac{f_n}{\left(|v_k|+\frac{1}{n}\right)^{\sigma}} & \text{in } \Omega_T \\ w_k = 0 & \text{on } \partial\Omega \times (0,T), \\ w_k(\cdot,0) = u_0(\cdot) & \text{in } \Omega. \end{cases} \tag{3.5}$$

We shall prove that

$$\lim_{k\to+\infty} \|w_k - w\|_{L^2((\Omega_T)} = 0.$$

Observe that up to a subsequence, we can assume that $v_k \to v$ a.e. in Ω_T. So that one has $\frac{f_n}{\left(|v_k|+\frac{1}{n}\right)^{\sigma}}$ converges to $\frac{f_n}{\left(|v|+\frac{1}{n}\right)^{\sigma}}$ a.e. in Ω_T. Furthermore, since

$$\frac{|f_n|}{\left(|v_k|+\frac{1}{n}\right)^{\sigma}} \le n^{\sigma+1},$$

by the dominated convergence theorem we have

$$\frac{f_n}{\left(|v_k|+\frac{1}{n}\right)^{\sigma}} \to \frac{f_n}{\left(|v|+\frac{1}{n}\right)^{\sigma}} \quad \text{in } L^2(\Omega_T). \tag{3.6}$$

Thus, testing by $w_k - w$ in the difference equations solved by w_k and w and using the fact that $w_k(x,0) = w(x,0) = u_0$ and the Hölder inequality, we obtain

$$\frac{1}{2}\int_\Omega ((w_k(x,T) - w(x,T)))^2 dx + \int_{\Omega_T} |\nabla(w_k - w)|^2 dxdt - \mu \int_{\Omega_T} \frac{(w_k - w)^2}{|x|^2 + \frac{1}{n}} dxdt$$
$$\le \left(\int_{\Omega_T} \left| \frac{f_n}{\left(|v_k|+\frac{1}{n}\right)^{\sigma}} - \frac{f_n}{\left(|v|+\frac{1}{n}\right)^{\sigma}} \right|^2 dxdt \right)^{\frac{1}{2}} \|w_k - w\|_{L^2(\Omega_T)}.$$

If $\mu < \Lambda_{N,2}$ then by the Poincaré inequality we obtain

$$\left(1 - \frac{\mu}{\Lambda_{N,2}}\right) \|w_k - w\|_{L^2(\Omega_T)} \le C_p^2 \left(\int_{\Omega_T} \left| \frac{f_n}{\left(|v_k|+\frac{1}{n}\right)^{\sigma}} - \frac{f_n}{\left(|v|+\frac{1}{n}\right)^{\sigma}} \right|^2 dxdt \right)^{\frac{1}{2}},$$

where C_p is the Poincaré constant. While if $\mu = \Lambda_{N,2}$ then by [50, Theorem 2.1] there exists a constant $C(\Omega) > 0$ such that

$$C(\Omega)\|w_k - w\|_{L^2(\Omega_T)} \le \left(\int_{\Omega_T} \left| \frac{f_n}{\left(|v_k|+\frac{1}{n}\right)^{\sigma}} - \frac{f_n}{\left(|v|+\frac{1}{n}\right)^{\sigma}} \right|^2 dxdt \right)^{\frac{1}{2}}.$$

Having in mind (3.6) we conclude that the sequence $\{w_k\}_k$ converges to w in $L^2(\Omega_T)$ and so S is continuous.

We turn now to prove that S is compact on B. Let $\{v_k\}_{k\in\mathbb{N}}$ be a bounded sequence in B. We shall prove that there exists a subsequence of $w_k := S(v_k)$ that converges in norm in $L^2(\Omega_T)$.

Taking $w_k = S(v_k)$ as a test function in (3.5) solved by w_k and using the Hölder inequality we obtain

$$\|w_k\|^2_{L^2(0,T;H^1_0(\Omega))} \le |\Omega_T|^{\frac12}\left(\mu n^2 + n^{\sigma+1}\right)\left(\int_{\Omega_T} w_k^2 dxdt\right)^{\frac12} + \|u_0\|^2_{L^2(\Omega)}.$$

Thus, from the Poincaré and Young inequalities it follows

$$\|w_k\|_{L^2(0,T;H^1_0(\Omega))} \le C, \tag{3.7}$$

where C is a positive constant not depending on k. Hence, by (3.7) the sequence $\{w_k\}_k$ is uniformly bounded in $L^2(0,T;H^1_0(\Omega))$. Now, testing by an arbitrary $\phi \in L^2(0,T;H^1_0(\Omega))$ in (3.5) we obtain

$$\int_{\Omega_T} \partial_t w_k \phi dxdt + \int_{\Omega_T} \nabla w_k \cdot \nabla\phi dxdt \le (\mu n^2 + n^{\sigma+1})\int_{\Omega_T} \phi dxdt.$$

Then,

$$\int_{\Omega_T} \partial_t w_k \phi dxdt \le \int_{\Omega_T} |\nabla w_k \cdot \nabla\phi| dxdt + (\mu n^2 + n^{\sigma+1})\int_{\Omega_T} \phi dxdt.$$

By Hölder's inequality we have

$$\int_{\Omega_T} \partial_t w_k \phi dxdt \le \left(\left(\int_{\Omega_T} |\nabla w_k|^2 dxdt\right)^{\frac12} + C(n,\Omega,T)\right)\left(\int_{\Omega_T} |\phi|^2 dxdt\right)^{\frac12},$$

so that since the sequence $\{w_k\}_k$ is uniformly bounded in $L^2(0,T;H^1_0(\Omega))$ then so is $\{\partial_t w_k\}_k$ in $L^1(0,T;H^{-1}(\Omega))$. Therefore, by [47, Corollary 4] there exists a subsequence of $\{w_k\}_{k\in\mathbb{N}}$ which converges in norm in $L^2(\Omega_T)$. So $S : B \to B$ is compact. Given these conditions on S, Schauder's fixed point theorem provides the existence of a function $u_n \in B$ such that $u_n = S(u_n)$ that is u_n solves (3.1) in the sense of (3.2). In particular we have $u_n \in L^2(0,T;H^1_0(\Omega)) \cap L^\infty(\Omega_T)$. The last assertion follows from Lemma A.5 (in Appendix). □

We also observe that from Lemma A.6 (in Appendix) the sequence $\{u_n\}_n$ is increasing.

3.2 Proof of Theorem 2.2

The main argument is to get a priori estimates on $\{u_n\}_n$ and then to pass to the limit as $n \to +\infty$. We divide the proof in four cases, the case where $\sigma = 1$, the case $\sigma < 1$, the case $\sigma > 1$ and the case $\sigma > 1$ with $\text{supp}(f) \subset\subset \Omega_T$.

Case 1 : $\sigma = 1$.

Taking $u_n\chi(0,\tau)(t)$ as test function in (3.2), with $0 \le \tau \le T$, we get

$$\frac12\int_\Omega (u_n(x,\tau))^2 dx + \int_0^\tau\int_\Omega |\nabla u_n|^2 dxdt \le \mu\int_0^\tau\int_\Omega \frac{u_n^2}{|x|^2+\frac1n} dxdt \int_{\Omega_T} f dxdt + \|u_0\|^2_{L^2(\Omega)}.$$

Then, by using (1.4) we obtain

$$\frac12\int_\Omega (u_n(x,\tau))^2 dx + \left(1 - \frac{\mu}{\Lambda_{N,2}}\right)\int_0^\tau\int_\Omega |\nabla u_n|^2 dxdt \le \|f\|_{L^1(\Omega_T)} + \|u_0\|^2_{L^2(\Omega)}.$$

Passing to the supremum in $\tau \in [0,T]$, we obtain

$$\frac{1}{2}\sup_{0\le\tau\le T}\int_\Omega (u_n(x,\tau))^2dx + \left(1-\frac{\mu}{\Lambda_{N,2}}\right)\int_{\Omega_T}|\nabla u_n|^2dxdt \le \|f\|_{L^1(\Omega_T)} + \|u_0\|^2_{L^2(\Omega)}.$$

This shows that the sequence $\{u_n\}_n$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))$. Then, there exist a subsequence of $\{u_n\}_n$ still indexed by n and a function $u\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))$ such that $u_n \rightharpoonup u$ weakly in $L^2(0,T;H_0^1(\Omega))$. Moreover, the boundedness of $\{\partial_t u_n\}_n$ in the dual space $L^2(0,T;H^{-1}(\Omega))$ implies that the sequence $\{u_n\}_n$ is relatively compact in $L^1(\Omega_T)$ (see [47, Corollary 4]) and hence for a subsequence, indexed again by n, we have $u_n \to u$ a.e. in Ω_T.

Let $\phi \in C_0^\infty(\Omega\times[0,T))$. Using ϕ as test function in (3.2) we obtain

$$\begin{aligned}&-\int_\Omega u_0(x)\phi(x,0)dx - \int_{\Omega_T} u_n\partial_t\phi dtdx + \int_{\Omega_T}\nabla u_n\cdot\nabla\phi dxdt\\ &\quad = \mu\int_{\Omega_T}\frac{T_n(u_n)\phi}{|x|^2+\frac{1}{n}}dxdt + \int_{\Omega_T}\frac{f_n\phi}{|u_n|+\frac{1}{n}}dxdt.\end{aligned} \tag{3.8}$$

Notice that since $u_n \rightharpoonup u$ weakly in $L^2(0,T;H_0^1(\Omega))$, we immediately have

$$\lim_{n\to+\infty}\int_{\Omega_T}\nabla u_n\cdot\nabla\phi dxdt = \int_{\Omega_T}\nabla u\cdot\nabla\phi dxdt$$

and

$$\lim_{n\to+\infty}\int_{\Omega_T}u_n\partial_t\phi dtdx = \int_{\Omega_T}u\partial_t\phi dtdx.$$

As regards the first integral in the right-hand side of (3.8), we know that the sequence $\{u_n\}$ is increasing to its limit u so we have

$$\left|\frac{T_n(u_n)\phi}{|x|^2+\frac{1}{n}}\right| \le \frac{|u\phi|}{|x|^2}.$$

Applying Hölder's and Hardy's inequalities we obtain

$$\int_{\Omega_T}\frac{|u\phi|}{|x|^2}dxdt \le \|\phi\|_\infty(\Lambda_{N,2})^{-\frac{1}{2}}\left(\int_{\Omega_T}|\nabla u|^2dxdt\right)^{\frac{1}{2}}\left(\int_{\Omega_T}\frac{dxdt}{|x|^2}\right)^{\frac{1}{2}}.$$

As $N\ge 3$ and Ω bounded, a straightforward calculation yields the existence of a positive constant C_1 such that

$$\int_\Omega\frac{dx}{|x|^2}\le C_1. \tag{3.9}$$

Therefore, the function $\frac{|u\phi|}{|x|^2}$ lies in $L^1(\Omega_T)$ and since $\frac{T_n(u_n)\phi}{|x|^2+\frac{1}{n}} \to \frac{u\phi}{|x|^2}$ a.e. in Ω_T the Lebesgue dominated convergence theorem gives

$$\lim_{n\to+\infty}\int_{\Omega_T}\frac{T_n(u_n)\phi}{|x|^2}dxdt = \int_{\Omega_T}\frac{u\phi}{|x|^2}dxdt.$$

On the other hand, the support $\mathrm{supp}(\phi)$ of the function ϕ is a compact subset of Ω_T and so by Lemma A.5 (in Appendix) there exists a constant $C_{\mathrm{supp}(\phi)}>0$ such that $u_n \ge C_{\mathrm{supp}(\phi)}$ in $\mathrm{supp}(\phi)$. Then,

$$\left|\frac{f_n\phi}{u_n+\frac{1}{n}}\right| \le \frac{\|\phi\|_\infty}{C_{\mathrm{supp}(\phi)}}|f| \in L^1(\Omega_T).$$

So that by the Lebesgue dominated convergence theorem we can get

$$\lim_{n\to+\infty}\int_{\Omega_T}\frac{f_n\phi}{u_n+\frac{1}{n}}dxdt=\int_{\Omega_T}\frac{f\phi}{u}dxdt.$$

Now passing to the limit as n tends to ∞ in (3.8) we obtain

$$-\int_\Omega u_0\phi(x,0)dx-\int_{\Omega_T}u\partial_t\phi dtdx+\int_{\Omega_T}\nabla u\cdot\nabla\phi dxdt$$
$$=\mu\int_{\Omega_T}\frac{u\phi}{|x|^2}dxdt+\int_{\Omega_T}\frac{f\phi}{u}dxdt$$

for all $\phi\in C_0^\infty(\Omega_T)$, namely u is a finite energy solution to (1.1).

Case 2 : $\sigma<1$.

The function $u_n\chi_{(0,\tau)}\in L^2(0,T;H_0^1(\Omega))$, $\tau\in(0,T)$, is an admissible test function in (3.2). Taking it so and using Hölder's inequality and (1.4) we arrive at

$$\frac{1}{2}\int_\Omega|u_n(x,\tau)|^2dx+\left(1-\frac{\mu}{\Lambda_{N,2}}\right)\int_0^\tau\int_\Omega|\nabla u_n|^2dxdt$$
$$\le\|f\|_{L^{m_1}(\Omega_T)}\left(\int_0^\tau\int_\Omega|u_n|^{(1-\sigma)m_1'}dxdt\right)^{\frac{1}{m_1'}}+\frac{1}{2}\|u_0\|_{L^2(\Omega)},$$

where $m_1:=\frac{2N}{2N-(1-\sigma)(N-2)}$ and $m_1':=\frac{m_1}{m_1-1}$. Setting $2^*:=\frac{2N}{N-2}$ one has

$$(1-\sigma)m_1'=2^*.$$

Then, we have

$$\frac{1}{2}\int_\Omega|u_n(x,\tau)|^2dx+\left(1-\frac{\mu}{\Lambda_{N,2}}\right)\int_0^\tau\int_\Omega|\nabla u_n|^2dxdt$$
$$\le\|f\|_{L^{m_1}(\Omega_T)}\left(\int_0^\tau\int_\Omega|u_n|^{2^*}dxdt\right)^{\frac{1-\sigma}{2^*}}+\frac{1}{2}\|u_0\|_{L^2(\Omega)}.$$

By Sobolev's inequality there exists a positive constant C such that

$$\begin{aligned}&\frac{1}{2}\int_\Omega|u_n(x,\tau)|^2dx+\left(1-\frac{\mu}{\Lambda_{N,2}}\right)\|u_n\|^2_{L^2(0,\tau;H_0^1(\Omega))}\\&\le C\|f\|_{L^{m_1}(\Omega_T)}\|u_n\|^{1-\sigma}_{L^2(0,\tau;H_0^1(\Omega))}+\frac{1}{2}\|u_0\|_{L^2(\Omega)}.\end{aligned}\tag{3.10}$$

For every real numbers a, $b\ge0$ and for every Let $\epsilon>0$ be arbitrary. For every positive real numbers a and b, the Young inequality yields

$$ab\le\epsilon a^p+C_\epsilon b^q,\tag{3.11}$$

where $p>1$, $q=\frac{p}{p-1}$ and $C_\epsilon=\frac{p-1}{p(p\epsilon)^{\frac{1}{p-1}}}$. Since $\frac{2^*}{m_1'}=1-\sigma<2$ we apply (3.11) with $p=\frac{2m_1'}{2^*}$ in the first term on the right hand side of (3.10) obtaining

$$\frac{1}{2}\int_\Omega|u_n(x,\tau)|^2dx+\left(1-\frac{\mu}{\Lambda_{N,2}}-\epsilon\right)\|u_n\|^2_{L^2(0,\tau;H_0^1(\Omega))}$$
$$\le C_\epsilon(C\|f\|_{L^{m_1}(\Omega_T)})^{\frac{2m_1'}{2m_1'-2^*}}+\frac{1}{2}\|u_0\|_{L^2(\Omega)}.$$

Choosing ϵ such that $1 - \frac{\mu}{\Lambda_{N,2}} - \epsilon > 0$ and passing to the supremum in $\tau \in [0, T]$ we obtain

$$\frac{1}{2} \sup_{0 \le \tau \le T} \int_\Omega (u_n(x,\tau))^2 dx + \left(1 - \frac{\mu}{\Lambda_{N,2}} - \epsilon\right) \int_{\Omega_T} |\nabla u_n|^2 dxdt \le C_3,$$

with $C_3 = C_\epsilon (C\|f\|_{L^{m_1}(\Omega_T)})^{\frac{2m_1'}{2m_1'-2^*}} + \frac{1}{2}\|u_0\|_{L^2(\Omega)}$. Therefore, the sequence $\{u_n\}_n$ is uniformly bounded in $L^2(0,T;H_0^1(\Omega))$ and $L^\infty(0,T;L^2(\Omega))$. Thus there exist a subsequence of $\{u_n\}_n$, still labelled by n, and a function $u \in L^2(0,T;H_0^1(\Omega))$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } L^2(0,T;H_0^1(\Omega)).$$

Now we shall prove that u is a weak solution of (1.1). For this, let us insert as a test function in (3.2) an arbitrary function $\phi \in C_0^\infty(\Omega \times [0,T))$.

$$\begin{aligned} &-\int_\Omega u_0(x)\phi(x,0)dx - \int_{\Omega_T} u_n \partial_t \phi dtdx + \int_{\Omega_T} \nabla u \cdot \nabla \phi dxdt \\ &\quad = \mu \int_{\Omega_T} \frac{T_n(u_n)\phi}{|x|^2 + \frac{1}{n}} dxdt + \int_{\Omega_T} \frac{f_n \phi}{(u_n + \frac{1}{n})^\sigma} dxdt. \end{aligned}$$

As in the first case, we can pass to the limit in the above equality to conclude that u is a finite energy solution of (1.1).

Case 3 : $\sigma > 1$.

In order to prove that $\{u_n\}_n$ is uniformly bounded in $L^2(0,T;H^1_{loc}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$, we will prove that the sequence $G_k(u_n)$ is uniformly bounded in $L^2(0,T;H_0^1(\Omega))$ $\cap L^\infty(0,T;L^2(\Omega))$ and $T_k(u_n)$ is uniformly bounded in $L^2(0,T;H^1_{loc}(\Omega)) \cap L^\infty(0,T;L^{\sigma+1}(\Omega))$. Let us first prove that $G_k(u_n)$ is uniformly bounded in $L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$. Inserting $G_k(u_n)\chi_{(0,\tau)}$, with $0 \le \tau \le T$, as a test function in (3.2) we obtain

$$\begin{aligned} &\int_0^\tau \int_\Omega \partial_t u_n G_k(u_n) dxdt + \int_{\Omega_\tau} |\nabla G_k(u_n)|^2 dxdt \\ &\quad = \mu \int_{\Omega_\tau} \frac{T_n(u_n)G_k(u_n)}{|x|^2 + \frac{1}{n}} dxdt + \int_{\Omega_\tau} \frac{f_n G_k(u_n)}{(u_n + \frac{1}{n})^\sigma} dxdt \\ &\quad \le \mu \int_{\Omega_\tau} \frac{u_n G_k(u_n)}{|x|^2} dxdt + \int_{\Omega_T} \frac{f_n G_k(u_n)}{(u_n + \frac{1}{n})^\sigma} dxdt. \end{aligned} \tag{3.12}$$

Observe that the function $G_k(u_n)$ is different from zero only on the set $B_{n,k} := \{(x,t) \in \Omega_\tau : u_n(x,t) > k\}$, and so we have

$$\begin{aligned} \int_0^\tau \int_\Omega \partial_t u_n G_k(u_n) dxdt &= \frac{1}{2} \int_{B_{n,k}} \partial_t (u_n - k)^2 dxdt = \frac{1}{2} \int_{\Omega_\tau} \partial_t (G_k(u_n(x,\tau)))^2 dxdt \\ &= \frac{1}{2} \int_\Omega (G_k(u_n(x,\tau)))^2 dx - \frac{1}{2} \int_\Omega (G_k(u_n(x,0)))^2 dx. \end{aligned}$$

Since $\int_\Omega (G_k(u_n(x,0)))^2 dx \le \int_\Omega (u_0(x))^2 dx$ and $u_n + \frac{1}{n} \ge k$ on $B_{n,k}$ inequality (3.12) becomes

$$\frac{1}{2} \int_\Omega (G_k(u_n(x,\tau)))^2 dx + \int_{\Omega_\tau} |\nabla G_k(u_n)|^2 dxdt \le \mu \int_{\Omega_\tau} \frac{u_n G_k(u_n)}{|x|^2} dxdt + C_4,$$

with $C_4 = \|u_0\|^2_{L^2(\Omega)} + \frac{1}{k^{\sigma-1}}\|f\|_{L^1(\Omega_T)}$. Moreover, since $u_n G_k(u_n) = (G_k(u_n))^2 + kG_k(u_n)$ on the set $B_{n,k}$ we get

$$\frac{1}{2}\int_\Omega (G_k(u_n(x,\tau)))^2 dx + \int_{\Omega_\tau} |\nabla G_k(u_n)|^2 dxdt - \mu \int_{\Omega_\tau} \frac{(G_k(u_n))^2}{|x|^2} dxdt \\ \leq \mu k \int_{\Omega_\tau} \frac{G_k(u_n)}{|x|^2} dxdt + C_4.$$

Taking into account that $\mu < \Lambda_{N,2}$ by (1.4) we have

$$\frac{1}{2}\int_\Omega (G_k(u_n(x,\tau)))^2 dx + \left(1 - \frac{\mu}{\Lambda_{N,2}}\right) \int_{\Omega_\tau} |\nabla G_k(u_n)|^2 dxdt \\ \leq \mu k \int_{\Omega_\tau} \frac{G_k(u_n(x,t))}{|x|^2} dxdt + C_4. \tag{3.13}$$

We shall now estimate the term $\mu k \int_{\Omega_\tau} \frac{G_k(u_n(x,t))}{|x|^2} dxdt$. Let us fix α such that $1 < \alpha < 2$ and set $\beta = \frac{\alpha}{\alpha-1}$. By Young's inequality we can write

$$\mu k \int_{\Omega_\tau} \frac{G_k(u_n)}{|x|^2} dxdt \leq \frac{\mu}{\alpha} \int_{\Omega_\tau} \frac{(G_k(u_n))^\alpha}{|x|^2} dxdt + \frac{\mu}{\beta} \int_{\Omega_\tau} \frac{k^\beta}{|x|^2} dxdt.$$

Having in mind (3.9) we get

$$\mu k \int_{\Omega_\tau} \frac{G_k(u_n)}{|x|^2} dxdt \leq \mu \int_{\Omega_\tau} \frac{(G_k(u_n))^\alpha}{|x|^2} dxdt + C_5,$$

where $C_5 = \frac{C_1 \mu k^\beta}{\beta}$. Then the Hölder inequality yields

$$\mu k \int_{\Omega_\tau} \frac{G_k(u_n)}{|x|^2} dxdt \leq \mu \left(\int_{\Omega_\tau} \frac{(G_k(u_n))^2}{|x|^2} dxdt\right)^{\frac{\alpha}{2}} \left(\int_{\Omega_\tau} \frac{dxdt}{|x|^2}\right)^{\frac{2-\alpha}{2}} + C_5 \\ \leq C_6 \left(\int_{\Omega_\tau} \frac{(G_k(u_n))^2}{|x|^2} dxdt\right)^{\frac{\alpha}{2}} + C_5,$$

where $C_6 = \mu C_1^{\frac{2-\alpha}{2}}$ and by (1.4) we obtain

$$\mu k \int_{\Omega_\tau} \frac{G_k(u_n(x,t))}{|x|^2} dxdt \leq C_7 \left(\int_{\Omega_\tau} |\nabla G_k(u_n)|^2 dxdt\right)^{\frac{\alpha}{2}} + C_5,$$

where $C_7 = \frac{C_6}{\Lambda_{N,2}}$. For arbitrary $\epsilon > 0$, applying the Young inequality (3.11) with $a = \int_{\Omega_\tau} |\nabla G_k(u_n)|^2 dxdt$, $b = C_7$ and $p = \frac{2}{\alpha}$, we get

$$\mu k \int_{\Omega_\tau} \frac{G_k(u_n(x,t))}{|x|^2} dxdt \leq \epsilon \int_{\Omega_\tau} |\nabla G_k(u_n)|^2 dxdt + C_8, \tag{3.14}$$

where $C_8 = C_5 + C_\epsilon C_7^{\frac{2-\alpha}{2}}$. Choosing ϵ such that $1 - \frac{\mu}{\Lambda_{N,2}} - \epsilon > 0$ and gathering (3.13) and (3.14), we deduce that

$$\frac{1}{2}\int_\Omega (G_k(u_n(x,\tau)))^2 dx + \left(1 - \frac{\mu}{\Lambda_{N,2}} - \epsilon\right) \int_{\Omega_\tau} |\nabla G_k(u_n)|^2 dxdt \leq C_9, \tag{3.15}$$

where $C_9 = C_8 + C_4$. Passing to the supremum in $\tau \in [0, T]$, we conclude that the sequence $\{G_k(u_n)\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^2(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$.

We now turn to prove that the sequence $\{T_k(u_n)\}_n$ is uniformly bounded in $L^2(0, T; H^1_{loc}(\Omega)) \cap L^\infty(0, T; L^{\sigma+1}(\Omega))$. Using $T_k^\sigma(u_n)\chi_{(0,\tau)}$, $0 \le \tau \le T$, as a test function in (3.2) we obtain

$$\begin{aligned}\frac{1}{\sigma+1}\int_\Omega (T_k(u_n(x,\tau)))^{\sigma+1}dx + \int_{\Omega_\tau}(T_k(u_n))^{\sigma-1}|\nabla T_k(u_n)|^2 dxdt \\ \le k^{\sigma-1}\mu\int_{\Omega_T}\frac{u_n^2}{|x|^2}dxdt + \int_{\Omega_T} f dxdt + \frac{1}{\sigma+1}\|u_0\|_{L^{\sigma+1}(\Omega)},\end{aligned} \tag{3.16}$$

where we have dropped $\sigma > 1$ in the second integral on the left-hand side and written $T_k^\sigma(u_n) = T_k^{\sigma-1}(u_n)T_k(u_n)$ in the first integral on the right-hand side of the inequality. As $u_n = T_k(u_n) + G_k(u_n)$, the first term on the right-hand side of the above inequality can be estimated as

$$\begin{aligned}\int_{\Omega_T}\frac{u_n^2}{|x|^2}dxdt &= \int_{\Omega_T}\frac{(T_k(u_n))^2}{|x|^2}dxdt + \int_{\Omega_T}\frac{(G_k(u_n))^2}{|x|^2}dxdt + 2\int_{\Omega_T}\frac{T_k(u_n)G_k(u_n)}{|x|^2}dxdt \\ &\le k^2\int_{\Omega_T}\frac{dxdt}{|x|^2} + \int_{\Omega_T}\frac{(G_k(u_n))^2}{|x|^2}dxdt + 2k\int_{\Omega_T}\frac{G_k(u_n)}{|x|^2}dxdt.\end{aligned}$$

So that by (1.4), (3.9), (3.14) and (3.15) there exists a real constant $C_{10} > 0$ such that

$$\int_{\Omega_T}\frac{u_n^2}{|x|^2}dxdt \le C_{10}.$$

Then, it follows that the inequality (3.16) reads as

$$\tfrac{1}{\sigma+1}\int_\Omega (T_k(u_n(x,\tau)))^{\sigma+1}dx + \int_{\Omega_\tau}(T_k(u_n))^{\sigma-1}|\nabla T_k(u_n)|^2 dxdt \le C_{11}, \tag{3.17}$$

with $C_{11} = k^{\sigma-1}\mu C_{10} + \|f\|_{L^1(\Omega_T)} + \frac{1}{\sigma+1}\|u_0\|_{L^{\sigma+1}(\Omega)}$. On the other hand, let $\Omega' \subset\subset \Omega$. By Lemma A.5 (in Appendix) there exists $C_{\Omega'} > 0$ such that

$$T_k(u_n(x,t)) \ge C_0 := \min\{k, C_{\Omega'}\}, \tag{3.18}$$

for all $(x, t) \in \Omega' \times [0, T]$. Thus, by (3.17) and (3.18) we get

$$\frac{1}{\sigma+1}\int_\Omega (T_k(u_n(x,\tau)))^{\sigma+1}dx + C_0^{\sigma-1}\int_0^\tau\int_{\Omega'}|\nabla T_k(u_n)|^2 dxdt \le C_{11}.$$

Passing to the supremum in $\tau \in [0, T]$, we get that the sequence $\{T_k(u_n)\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^2(0, T; H^1_{loc}(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$. Therefore, we conclude that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^2(0, T; H^1_{loc}(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$. As a consequence, there exist a subsequence of $\{u_n\}_{n\in\mathbb{N}}$, relabelled again by n, and a function $u \in L^2(0, T; H^1_{loc}(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ such that $u_n \rightharpoonup u$ weakly in $L^2(0, T; H^1_{loc}(\Omega))$.

On the other hand, let us assume that $\frac{4\sigma}{(\sigma+1)^2} - \frac{\mu}{\Lambda_{N,2}} > 0$. Taking $u_n^\sigma\chi_{(0,\tau)}(t)$, $0 \le \tau \le T$, as a test function in (3.2) and using the Hardy inequality (1.4) we arrive at

$$\begin{aligned}\frac{1}{\sigma+1}\int_\Omega (u_n(x,\tau))^{\sigma+1}dx + \left(\frac{4\sigma}{(\sigma+1)^2} - \frac{\mu}{\Lambda_{N,2}}\right)\int_0^\tau\int_\Omega |\nabla u_n^{\frac{\sigma+1}{2}}|^2 dxdt \\ \le \int_{\Omega_T} f dxdt + \frac{1}{\sigma+1}\int_\Omega (u_0(x))^{\sigma+1}dx.\end{aligned}$$

This shows that $u_n^{\frac{\sigma+1}{2}}$ is uniformly bounded in $L^2(0,T;H_0^1(\Omega))$ and so by the Poincaré inequality the sequence u_n is uniformly bounded in $L^{\sigma+1}(\Omega_T)$ and hence for a subsequence, labelled again by n, we have $u_n \to u$ a.e. in Ω_T.

Testing by an arbitrary function $\phi \in C_0^\infty(\Omega \times [0,T))$ in (3.2) we obtain

$$\begin{aligned}
&-\int_\Omega u_0(x)\phi(x,0)dx - \int_{\Omega_T} u_n\partial_t\phi dtdx + \int_{\Omega_T} \nabla u_n \cdot \nabla\phi dxdt \\
&= \mu\int_{\Omega_T} \frac{T_n(u_n)\phi}{|x|^2+\frac{1}{n}}dxdt + \int_{\Omega_T}\frac{f_n\phi}{|u_n|+\frac{1}{n}}dxdt.
\end{aligned} \tag{3.19}$$

We shall now pass of the limit in each term of (3.19). Notice that since $u_n \rightharpoonup u$ weakly in $L^2(0,T;H^1_{loc}(\Omega))$ we have

$$\lim_{n\to+\infty}\int_{\Omega_T}\nabla u_n\cdot\nabla\phi dxdt = \int_{\Omega_T}\nabla u\cdot\nabla\phi dxdt$$

and

$$\lim_{n\to+\infty}\int_{\Omega_T} u_n\partial_t\phi dtdx = \int_{\Omega_T} u\partial_t\phi dtdx.$$

For the first integral in the right-hand side of (3.19), we know that the sequence $\{u_n\}$ is increasing to its limit u so we obtain

$$\left|\frac{T_n(u_n)\phi}{|x|^2+\frac{1}{n}}\right| \le \frac{|u\phi|}{|x|^2}.$$

By Hölder's and Hardy's inequalities we get

$$\begin{aligned}
\int_{\Omega_T}\frac{|u\phi|}{|x|^2}dxdt &\le \|\phi\|_\infty\int_{\text{supp}(\phi)}\frac{|u|}{|x|^2}dxdt = \|\phi\|_\infty\int_{\text{supp}(\phi)}\frac{|u|}{|x|}\times\frac{1}{|x|}dxdt \\
&\le \|\phi\|_\infty\left(\int_{\text{supp}(\phi)}\frac{|u|^2}{|x|^2}dxdt\right)^{\frac12}\left(\int_{\Omega_T}\frac{dxdt}{|x|^2}\right)^{\frac12} \\
&\le \|\phi\|_\infty(\Lambda_{N,2})^{-\frac12}\left(\int_{\text{supp}(\phi)}|\nabla u|^2dxdt\right)^{\frac12}\left(\int_{\Omega_T}\frac{dxdt}{|x|^2}\right)^{\frac12}.
\end{aligned}$$

Since $u \in L^2(0,T;H^1_{loc}(\Omega))$, a calculation as in (3.9) allows us conclude that the function $\frac{|u\phi|}{|x|^2}$ lies in $L^1(\Omega_T)$. Moreover, $\frac{T_n(u_n)\phi}{|x|^2+\frac{1}{n}} \to \frac{u\phi}{|x|^2}$ a.e. in Ω_T, so that by the Lebesgue dominated convergence theorem one has

$$\lim_{n\to+\infty}\int_{\Omega_T}\frac{T_n(u_n)\phi}{|x|^2}dxdt = \int_{\Omega_T}\frac{u\phi}{|x|^2}dxdt.$$

As regards the last term in (3.19), by Lemma A.5 (in Appendix) there exists a constant $C_{\text{supp}(\phi)} > 0$ such that $u_n \ge C_{\text{supp}(\phi)}$ in supp(ϕ). Then,

$$\int_{\Omega_T}\left|\frac{f_n\phi}{u_n+\frac{1}{n}}\right|dxdt \le \frac{\|\phi\|_\infty}{C_{\text{supp}(\phi)}}\int_{\Omega_T}|f|dxdt < +\infty.$$

So that by the Lebesgue dominated convergence theorem we get

$$\lim_{n\to+\infty}\int_{\Omega_T}\frac{f_n\phi}{u_n+\frac{1}{n}}dxdt = \int_{\Omega_T}\frac{f\phi}{u}dxdt.$$

Finally passing to the limit as n tends to ∞ in (3.19) we obtain

$$-\int_{\Omega} u_0\phi(x,0)dx - \int_{\Omega_T} u\partial_t\phi dtdx + \int_{\Omega_T} \nabla u\cdot\nabla\phi dxdt = \mu\int_{\Omega_T}\frac{u\phi}{|x|^2}dxdt + \int_{\Omega_T}\frac{f\phi}{u}dxdt$$

for all $\phi \in C_0^\infty(\Omega_T)$. Furthermore, by Lemma A.5 there exists a constant $C_{\Omega'} > 0$ such that $u \geq C_{\Omega'}$ in $\Omega' \times [0,T]$ which shows that u is a weak solution of (1.1).

Now assume that $\sigma > 1$ is such that $\frac{4\sigma}{(\sigma+1)^2} - \frac{\mu}{\Lambda_{N,2}} > 0$. For $0 \leq \tau \leq T$ let us use $u_n^\sigma \chi_{(0,\tau)}$ as a test function in (3.2). By the Hardy inequality (1.4) we arrive at

$$\frac{1}{\sigma+1}\int_{\Omega}(u_n(x,\tau))^{\sigma+1}dx + \left(\frac{4\sigma}{(\sigma+1)^2} - \frac{\mu}{\Lambda_{N,2}}\right)\int_0^\tau\int_{\Omega}|\nabla u_n^{\frac{\sigma+1}{2}}|^2dxdt \leq C,$$

where $C = \|f\|_{L^1(\Omega_T)} + \frac{1}{\sigma+1}\|u_0\|_{L^{\sigma+1}(\Omega)}$. Therefore, we deduce that $u^{\frac{\sigma+1}{2}} \in L^2(0,T;H_0^1(\Omega))$.

Case 4 :

Suppose that $\sigma > 1$ and $\operatorname{supp}(f) \subset\subset \Omega_T$. Taking $u_n\chi(0,\tau)$, $0 \leq \tau \leq T$, as a test function in (3.2) and using (1.4) we get

$$\frac{1}{2}\int_{\Omega}(u_n(x,\tau))^2dx + \left(1 - \frac{\mu}{\Lambda_{N,2}}\right)\int_0^\tau\int_{\Omega}|\nabla u_n|^2dxdt \leq \int_{\Omega_T}\frac{f}{u_n^{\sigma-1}}dxdt + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}.$$

Applying Lemma A.5 (in Appendix) there exists $C > 0$ such that $u_n \geq C$ in $\operatorname{supp}(f)$. Whence, passing to the supremum in $\tau \in [0,T]$ we obtain

$$\begin{aligned}\frac{1}{2}\sup_{0\leq\tau\leq T}\int_{\Omega}(u_n(x,\tau))^2dx + \left(1 - \frac{\mu}{\Lambda_{N,2}}\right)\int_{\Omega}|\nabla u_n|^2dxdt\\ \leq \frac{1}{C^{\sigma-1}}\int_{\operatorname{supp}(f)} fdxdt + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}.\end{aligned}$$

Thus, the sequence $\{u_n\}_n$ is bounded in $L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$. Therefore, there exist a function $u \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ and a subsequence of $\{u_n\}_n$, still indexed by n, such that $u_n \rightharpoonup u$ in $L^2(0,T;H_0^1(\Omega))$ and then u is a finite energy solution of the problem (1.1). □

3.3 Proof of Theorem 2.4

Let $0 \leq \tau \leq T$. Taking $u_n\chi_{(0,\tau)}(t)$ as a test function in (3.2), we get

$$\frac{1}{2}\int_{\Omega}(u_n(x,\tau))^2dx + \int_0^\tau\int_{\Omega}|\nabla u_n|^2dxdt - \Lambda_{N,2}\int_0^\tau\int_{\Omega}\frac{u_n^2}{|x|^2}dxdt \leq \|f\|_{L^1(\Omega_T)} + \frac{1}{2}\|u_0\|_{L^2(\Omega)}.$$

Passing to the supremum in $\tau \in [0,T]$ and using Theorem A.1 (in Appendix) we conclude that the sequence $\{u_n\}_n$ is uniformly bounded in $L^q(0,T;W_0^{1,q}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$, for all $q < 2$. As a consequence, there exist a subsequence of $\{u_n\}_n$, still indexed by n, and a function $u \in L^q(0,T;W_0^{1,q}(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ such that $u_n \rightharpoonup u$ weakly in $L^q(0,T;W_0^{1,q}(\Omega))$. Arguing in a similar way as in the case 1, we conclude that u is a weak solution of the problem (1.1). □

3.4 Proof of Theorem 2.5

Suppose that $\mu > \Lambda_{N,2}$. Arguing by contradiction, assume that (1.1) admits a positive weak solution u. Thus u is also a weak solution to the problem

$$\begin{cases} \partial_t u - \Delta u - \Lambda_{N,2}\frac{u}{|x|^2} = (\mu - \Lambda_{N,2})\frac{u}{|x|^2} + \frac{f}{u^\sigma} & \text{in } \Omega_T, \\ u = 0 & \text{in } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \partial\Omega \times (0,T). \end{cases}$$

By virtue of Lemma A.3 (in Appendix) we have

$$\left((\mu - \Lambda_{N,2})\frac{u}{|x|^2} + \frac{f}{u^\sigma} \right) |x|^{-\alpha_1} \in L^1(B_{r_1}(0) \times (t_1,t_2)),$$

for any small enough parabolic cylinder $B_{r_1}(0) \times (t_1,t_2) \subset\subset \Omega_T$ where α_1 is defined in (A.1). As in our equation $\lambda = \Lambda_{N,2}$ we have $\alpha_1 = \frac{N-2}{2}$. Since $u > 0$ and $f \geq 0$ we have in particular

$$(\mu - \Lambda_{N,2})\frac{u}{|x|^2}|x|^{-\frac{N-2}{2}} \in L^1(B_{r_1}(0) \times (t_1,t_2)). \tag{3.20}$$

On the other hand, since

$$\partial_t u - \Delta u - \Lambda_{N,2}\frac{u}{|x|^2} = (\mu - \Lambda_{N,2})\frac{u}{|x|^2} + \frac{f}{u^\sigma} \geq 0$$

by Lemma A.2 (in Appendix) there exists a constant $C > 0$ such that

$$u \geq C|x|^{-\frac{N-2}{2}}. \tag{3.21}$$

Gathering (3.20) and (3.21) we obtain

$$|x|^{-N} \in L^1(B_{r_1}(0) \times (t_1,t_2))$$

which is a contradiction. Therefore, if $\mu > \Lambda_{N,2}$ the problem (1.1) has no positive weak solution. □

3.5 Proof of Theorem 2.6

The proofs of (i) and (ii) are similar. We only give the proof of (i).

• **Proof of (a)** – We shall establish an a priori L^∞-estimate for the solution u_n of (3.2). To do so, we use standard ideas that can be found in several nonsingular cases as for instance in [22,28,48,51,55,57]. Despite being classic, we give the proof for the convenience of the reader. Let $k \geq k_0 := \max(1, \|u_0\|_\infty)$. We choose $G_k(u_n)\chi_{(0,\tau)}$, $0 \leq \tau \leq T$, as a test function in (3.2), we get

$$\int_0^\tau \int_\Omega \partial_t u_n G_k(u_n)dxdt + \int_{A_{k,n}} |\nabla G_k(u_n)|^2 dxdt$$
$$\leq \mu \int_{A_{k,n}} \frac{u_n G_k(u_n)}{|x|^2}dxdt + \int_{A_{k,n}} \frac{f G_k(u_n)}{(u_n + \frac{1}{n})^\sigma}dxdt,$$

where we have set $A_{k,n} = \{(x,t) \in \Omega_\tau : u_n(x,t) > k\}$. Observe that since $G_k(u_n)$ is different from zero only on the set $A_{k,n}$ and according to the choice of k, one has

$$\int_0^\tau \int_\Omega \partial_t u_n G_k(u_n)dxdt = \frac{1}{2}\int_\Omega G_k(u_n(x,\tau))^2 dx.$$

Note that the Hölder inequality implies

$$\int_{A_{k,n}} \frac{u_n G_k(u_n)}{|x|^2} dxdt \le \left(\int_{A_{k,n}} \frac{u_n^2}{|x|^2} dxdt \right)^{\frac{1}{2}} \left(\int_{A_{k,n}} \frac{G_k(u_n)^2}{|x|^2} dxdt \right)^{\frac{1}{2}}.$$

Taking into account that on the subset $A_{k,n}$ one has $\nabla G_k(u_n) = \nabla u_n$ a.e. in Ω, so that Hardy's inequality yields

$$\int_{A_{k,n}} \frac{u_n G_k(u_n)}{|x|^2} dxdt \le \frac{1}{\Lambda_{N,2}} \int_{A_{k,n}} |\nabla G_k(u_n)|^2 dxdt.$$

Since $u_n + \frac{1}{n} > k_0$ on the subset $A_{k,n}$ we get

$$\frac{1}{2} \int_{\Omega} G_k(u_n(x,\tau))^2 dx + \int_{A_{k,n}} |\nabla G_k(u_n)|^2 dxdt \\ \le \frac{\mu}{\Lambda_{N,2}} \int_{A_{k,n}} |\nabla G_k(u_n)|^2 dxdt + \frac{1}{k_0^\sigma} \int_{A_{k,n}} f G_k(u_n) dxdt.$$

Then passing to the supremum in $\tau \in (0,T)$ we obtain

$$\frac{1}{2} \|G_k(u_n)\|^2_{L^\infty(0,T;L^2(\Omega))} + \left(1 - \frac{\mu}{\Lambda_{N,2}}\right) \|G_k(u_n)\|^2_{L^2(0,T;H_0^1(\Omega))} \le \frac{1}{k_0^\sigma} \int_{\Omega_T} f G_k(u_n) dxdt. \quad (3.22)$$

On the other hand, since $G_k(u_n) \in L^\infty(\Omega_T) \cap L^2(0,T;H_0^1(\Omega))$ then $G_k(u_n) \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$. Therefore, by [28, Proposition 3.1] there exists a positive constant c such that

$$\int_{\Omega_T} G_k(u_n)^{\frac{2N+4}{N}} dxdt \le c^{\frac{2N+4}{N}} \left(\int_{\Omega_T} |\nabla G_k(u_n)|^2 dxdt \right) \left(\|G_k(u_n)\|^2_{L^\infty(0,T;L^2(\Omega))} \right)^{\frac{2}{N}}.$$

Setting $\Gamma_{N,2} := 1 - \frac{\mu}{\Lambda_{N,2}}$ and $C_1 := \frac{c^{\frac{2N+4}{N}} 2^{\frac{2}{N}}}{\Gamma_{N,2} k_0^{\sigma(1+\frac{2}{N})}}$, we obtain using (3.22)

$$\int_{\Omega_T} G_k(u_n)^{\frac{2N+4}{N}} dxdt \le C_1 \left(\int_{\Omega_T} f G_k(u_n) dxdt \right)^{1+\frac{2}{N}}.$$

Observe that both integrals are on the subset $A_{k,n}$. Using Hölder's inequality in the right-hand side term with exponents $\frac{2N+4}{N}$ and $\frac{2N+4}{N+4}$, we get

$$\int_{A_{k,n}} G_k(u_n)^{\frac{2N+4}{N}} dxdt \le C_1 \left(\int_{A_{k,n}} f^{\frac{2N+4}{N+4}} dxdt \right)^{\frac{N+4}{2N}} \left(\int_{A_{k,n}} G_k(u_n)^{\frac{2N+4}{N}} dxdt \right)^{\frac{1}{2}},$$

from which it follows

$$\int_{A_{k,n}} G_k(u_n)^{\frac{2N+4}{N}} dxdt \le C_1^2 \left(\int_{A_{k,n}} f^{\frac{2N+4}{N+4}} dxdt \right)^{\frac{N+4}{N}}.$$

Since $f \in L^m(\Omega_T)$ with $m > \frac{N}{2} + 1 > \frac{2N+4}{N+4}$, we use again Hölder's inequality obtaining

$$\int_{A_{k,n}} G_k(u_n)^{\frac{2N+4}{N}} dxdt \le C_1^2 \|f\|^{\frac{2N+4}{N}}_{L^m(\Omega_T)} |A_{k,n}|^{\frac{N+4}{N} - \frac{2N+4}{mN}}.$$

Now let $h > k$. It's easy to see that $A_{h,n} \subset A_{k,n}$ and $G_k(u_n) \ge h - k$ on $A_{h,n}$, so that one has

$$|A_{h,n}|(h-k)^{\frac{2N+4}{N}} \le C_1^2 \|f\|^{\frac{2N+4}{N}}_{L^m(\Omega_T)} |A_{k,n}|^{\frac{N+4}{N} - \frac{2N+4}{mN}}.$$

Setting $\psi(k) = |A_{k,n}|$, we get

$$\psi(h) \leq \frac{C_2}{(h-k)^{\alpha}} \psi(k)^{\beta},$$

where $C_2 = C_1^2 \|f\|_{L^m(\Omega_T)}^{\frac{2N+4}{N}}$, $\alpha = \frac{2N+4}{N}$ and $\beta = \frac{N+4}{N} - \frac{2N+4}{mN}$. Since $m > \frac{N}{2} + 1$ we have $\beta > 1$ and then we can apply the first item of [48, Lemma 4.1] to conclude that there exists a constant C_∞, such that $\psi(C_\infty) = 0$, that is

$$\|u_n\|_\infty \leq C_\infty.$$

□

• **Proof of (b)** – Using $u_n^{\gamma}\chi_{(0,\tau)}$, $0 < \tau < T$, as a test function in (3.2) and applying the Hölder's inequality and (1.4) we arrive at

$$\begin{aligned} &\frac{1}{\gamma+1}\int_\Omega (u_n(x,\tau))^{\gamma+1}dx + \left(\gamma\left(\frac{2}{\gamma+1}\right)^2 - \frac{\mu}{\Lambda_{N,2}}\right)\int_{\Omega_\tau} |\nabla u_n^{\frac{\gamma+1}{2}}|^2 dxdt \\ &\leq \|f\|_{L^m(\Omega_T)} \left(\int_{\Omega_T} u_n^{(\gamma-\sigma)m'} dxdt\right)^{\frac{1}{m'}} + \|u_0\|_{L^{\gamma+1}(\Omega)}^{\gamma+1}. \end{aligned} \tag{3.23}$$

Note that $1 \leq \sigma \leq \gamma = \frac{Nm(\sigma+1)-N+2m-2}{N-2m+2}$. Since we have supposed that $\gamma\left(\frac{2}{\gamma+1}\right)^2 - \frac{\mu}{\Lambda_{N,2}} > 0$, we discuss the two cases $\sigma = \gamma$ and $\sigma < \gamma$. Thus, if $\sigma = \gamma$ we immediately have

$$\begin{aligned} &\frac{1}{\gamma+1}\int_\Omega (u_n(x,\tau))^{\gamma+1}dx + \left(\gamma\left(\frac{2}{\gamma+1}\right)^2 - \frac{\mu}{\Lambda_{N,2}}\right)\int_{\Omega_\tau} |\nabla u_n^{\frac{\gamma+1}{2}}|^2 dxdt \\ &\leq |\Omega_T|^{\frac{1}{m'}} \|f\|_{L^m(\Omega_T)} + \|u_0\|_{L^{\gamma+1}(\Omega)}^{\gamma+1}. \end{aligned}$$

While If $\sigma < \gamma$, we compute $(\gamma-\sigma)m' = (\gamma+1)\frac{N+2}{N} < (\gamma+1)\frac{N}{N-2}$. Therefore, by (3.23) there exists a positive constant C such that

$$\begin{aligned} &\frac{1}{\gamma+1}\int_\Omega (u_n(x,\tau))^{\gamma+1}dx + \left(\gamma\left(\frac{2}{\gamma+1}\right)^2 - \frac{\mu}{\Lambda_{N,2}}\right)\int_{\Omega_\tau} |\nabla u_n^{\frac{\gamma+1}{2}}|^2 dxdt \\ &\leq C\|f\|_{L^m(\Omega_T)} \left(\int_{\Omega_T} (u_n^{\frac{\gamma+1}{2}})^{2^*} dxdt\right)^{\frac{2(\gamma-\sigma)}{2^*(\gamma+1)}} + \|u_0\|_{L^{\gamma+1}(\Omega)}^{\gamma+1}. \end{aligned}$$

Using the Sobolev inequality in the first term on the right hand side of the above inequality, we conclude that there exists a positive constant C_1 such that

$$\begin{aligned} &\frac{1}{\gamma+1}\int_\Omega (u_n(x,\tau))^{\gamma+1}dx + \left(\gamma\left(\frac{2}{\gamma+1}\right)^2 - \frac{\mu}{\Lambda_{N,2}}\right)\int_{\Omega_\tau} |\nabla u_n^{\frac{\gamma+1}{2}}|^2 dxdt \\ &\leq C_1\|f\|_{L^m(\Omega_T)} \left(\int_{\Omega_\tau} |\nabla u_n^{\frac{\gamma+1}{2}}|^2 dxdt\right)^{\frac{\gamma-\sigma}{\gamma+1}} + \|u_0\|_{L^{\gamma+1}(\Omega)}^{\gamma+1}. \end{aligned}$$

Applying (3.11) with $a = \left(\int_{\Omega_\tau} |\nabla u_n^{\frac{\gamma+1}{2}}|^2 dxdt\right)^{\frac{\gamma-\sigma}{\gamma+1}}$, $b = C_1\|f\|_{L^m(\Omega_T)}$, $p = \frac{\gamma+1}{\gamma-\sigma}$ and $q = \frac{\gamma+1}{\sigma+1}$ to obtain

$$\begin{aligned} &\frac{1}{\gamma+1}\|u_n^{\frac{\gamma+1}{2}}\|_{L^\infty(0,T;L^2(\Omega))}^2 + \left(\gamma\left(\frac{2}{\gamma+1}\right)^2 - \frac{\mu}{\Lambda_{N,2}} - \epsilon\right)\int_{\Omega_\tau} |\nabla u_n^{\frac{\gamma+1}{2}}|^2 dxdt \\ &\leq C_\epsilon (C\|f\|_{L^m})^{\frac{\gamma+1}{\sigma+1}} + \|u_0\|_{L^{\gamma+1}(\Omega)}^{\gamma+1}. \end{aligned}$$

Finally we choose ϵ such that $\gamma\left(\frac{2}{\gamma+1}\right)^2 - \frac{\mu}{\Lambda_{N,2}} - \epsilon > 0$. Consequently, in both cases the sequence $\{u_n^{\frac{\gamma+1}{2}}\}_n$ is uniformly bounded in $L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$. Whence, there exist a subsequence of $\{u_n^{\frac{\gamma+1}{2}}\}_n$, still indexed by n, and a function $v \in L^2(0,T;H_0^1(\Omega))$ such that $u_n^{\frac{\gamma+1}{2}} \rightharpoonup v$ weakly in $L^2(0,T;H_0^1(\Omega))$. Now according to the proof of the second item of Theorem 2.2, we know that $u_n \rightharpoonup u$ weakly in $L^2(0,T;H^1_{loc}(\Omega))$ so that identifying almost everywhere the limits one has $v = u^{\frac{\gamma+1}{2}} \in L^2(0,T;H_0^1(\Omega))$. □

3.6 Proof of Theorem 2.8

The ideas we use are standard and we follow the lines of [24, Theorem 4.1, (i)-(b)]. Let us choose $u_n^{2\delta-1}\chi_{(0,\tau)}$, $0 < \tau < T$, as a test function in (3.2) where δ is a positive real constant satisfying $\frac{\Lambda_{N,2}}{\mu}\left(1-\sqrt{1-\frac{\mu}{\Lambda_{N,2}}}\right) < \delta < 1$. This choice made possible by the fact that $\mu < \Lambda_{N,2}$ implies $\frac{1}{2} < \delta$ and $\frac{2\delta-1}{\delta^2} - \frac{\mu}{\Lambda_{N,2}} > 0$ that will be chosen after few lines. We get

$$\frac{1}{2\delta}\int_\Omega (u_n(x,\tau))^{2\delta}dx + \frac{(2\delta-1)}{\delta^2}\int_{\Omega_\tau}|\nabla u_n^\delta|^2 dxdt \leq \mu\int_{\Omega_\tau}\frac{u_n^{2\delta}}{|x|^2}dxdt + \int_{\Omega_\tau} f u_n^{(2\delta-1-\sigma)}dxdt + \frac{1}{2\delta}\|u_0^\delta\|^2_{L^2(\Omega)}.$$

Passing to the supremum in $\tau \in (0,T)$ and applying Hardy's inequality (1.4) and then Hölder's inequality, we obtain

$$\begin{aligned}\frac{1}{2\delta}\|u_n^\delta\|^2_{L^\infty(0,T;L^2(\Omega))} &+ \left(\frac{2\delta-1}{\delta^2} - \frac{\mu}{\Lambda_{N,2}}\right)\int_{\Omega_T}|\nabla u_n^\delta|^2 dxdt \\ &\leq \int_{\Omega_T} f u_n^{2\delta-1-\sigma}dxdt + \frac{1}{2\delta}\|u_0^\delta\|^2_{L^2(\Omega)} \\ &\leq \|f\|_{L^m(\Omega_T)}\left(\int_{\Omega_T} u_n^{(2\delta-1-\sigma)m'}dxdt\right)^{\frac{1}{m'}} + \frac{1}{2\delta}\|u_0^\delta\|^2_{L^2(\Omega)}.\end{aligned} \tag{3.24}$$

Since $u_n \in L^\infty(\Omega_T) \cap L^2(0,T;H_0^1(\Omega))$ then $u_n \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$. Thus, by [28, Proposition 3.1] there exists a positive constant c such that

$$\int_{\Omega_T}(u_n^\delta)^{\frac{2N+4}{N}}dxdt \leq c^{\frac{2N+4}{N}}\left(\int_{\Omega_T}|\nabla u_n^\delta|^2 dxdt\right)\left(\|u_n^\delta\|^2_{L^\infty(0,T;L^2(\Omega))}\right)^{\frac{2}{N}}.$$

Then, using (3.24) we obtain

$$\begin{aligned}\int_{\Omega_T}(u_n^\delta)^{\frac{2N+4}{N}}dxdt &\leq \frac{(2\delta)^{\frac{2}{N}}c^{\frac{2N+4}{N}}}{\Lambda_\delta}\left(\|f\|_{L^m(\Omega_T)}\left(\int_{\Omega_T}u_n^{(2\delta-1-\sigma)m'}dxdt\right)^{\frac{1}{m'}} + \frac{1}{2\delta}\|u_0^\delta\|^2_{L^2(\Omega)}\right)^{1+\frac{2}{N}} \\ &\leq \frac{(4\delta)^{\frac{2}{N}}c^{\frac{2N+4}{N}}}{\Lambda_\delta}\left(\|f\|^{1+\frac{2}{N}}_{L^m(\Omega_T)}\left(\int_{\Omega_T}u_n^{(2\delta-1-\sigma)m'}dxdt\right)^{\frac{N+2}{Nm'}} + \frac{1}{(2\delta)^{1+\frac{2}{N}}}\|u_0^\delta\|^{\frac{2N+4}{N}}_{L^2(\Omega)}\right),\end{aligned}$$

where $\Lambda_\delta = \frac{2\delta-1}{\delta^2} - \frac{\mu}{\Lambda_{N,2}}$. Now we choose δ to be such that $\delta\frac{2N+4}{N} = (2\delta-1-\sigma)m'$, that is $\delta = \frac{mN(1+\sigma)}{2N-4(m-1)}$. Observe that since $1 < m < m_1 < \frac{N}{2}+1$ one has $N-2(m-1) > 0$ and $\delta > \frac{1+\sigma}{2} \geq \frac{1}{2}$. We point out that $\frac{\mu}{\Lambda_{N,2}} > 0$ implies $\frac{\Lambda_{N,2}}{\mu}\left(1-\sqrt{1-\frac{\mu}{\Lambda_{N,2}}}\right) > \frac{1}{2}$ and the choice

$\delta > \frac{\Lambda_{N,2}}{\mu}(1 - \sqrt{1 - \frac{\mu}{\Lambda_{N,2}}})$ ensures that $\Lambda_\delta > 0$. To check the upper bound on δ, we notice that $\delta < 1$ is equivalent to $m < \frac{2N+4}{N(1+\sigma)+4}$. Such an inequality is always satisfied since for $\sigma \le 1$ we have $m < m_1 \le \frac{2N+4}{N(1+\sigma)+4}$. Therefore, with this choice of δ we obtain

$$\|u_n\|^{(2\delta-1-\sigma)m'}_{L^{(2\delta-1-\sigma)m'}(\Omega_T)} \le \frac{(4\delta)^{\frac{2}{N}} c^{\frac{2N+4}{N}}}{\Lambda_\delta} \|f_n\|^{\frac{2}{N}+1}_{L^m(\Omega_T)} \|u_n\|^{\frac{(N+2)(2\delta-1-\sigma)}{N}}_{L^{(2\delta-1-\sigma)m'}(\Omega_T)} + \frac{(4\delta)^{\frac{2}{N}} c^{\frac{2N+4}{N}}}{\Lambda_\delta} \frac{1}{(2\delta)^{1+\frac{2}{N}}} \|u_0^\delta\|^{\frac{2N+4}{N}}_{L^2(\Omega)}.$$

Since $m < \frac{N}{2} + 1$ we have

$$(2\delta - 1 - \sigma)m' > \frac{(N+2)(2\delta - 1 - \sigma)}{N}$$

and so by virtue of Young's inequality the sequence $\{u_n\}_n$ is uniformly bounded in $L^\gamma(\Omega_T)$ with

$$\gamma = (2\delta - 1 - \sigma)m' = \frac{m(N+2)(1+\sigma)}{N - 2m + 2} > 1.$$

Now we shall obtain an estimation on ∇u_n. Notice that from (3.24) we get

$$\Lambda_\delta \delta^2 \int_{\Omega_T} \frac{|\nabla u_n|^2}{u_n^{2(1-\delta)}} dxdt \le \|f_n\|^{\frac{2}{N}+1}_{L^m(\Omega_T)} \|u_n\|^{(2\delta-1-\sigma)}_{L^\gamma(\Omega_T)} + \frac{1}{2\delta}\|u_0^\delta\|^2_{L^2(\Omega)}$$

and since $\{u_n\}_n$ is uniformly bounded in $L^\gamma(\Omega_T)$, we deduce the existence of a positive constant C, not depending on n, such that

$$\int_{\Omega_T} \frac{|\nabla u_n|^2}{u_n^{2(1-\delta)}} dxdt \le C.$$

Let now $q \ge 1$ be such that $q < 2$. An application of Hölder's inequality with exponents $\frac{2}{q}$ and $\frac{2}{2-q}$ yields

$$\begin{aligned}
\int_{\Omega_T} |\nabla u_n|^q dxdt &= \int_{\Omega_T} \frac{|\nabla u_n|^q}{u_n^{q(1-\delta)}} u_n^{q(1-\delta)} dxdt \\
&\le \left(\int_{\Omega_T} \frac{|\nabla u_n|^2}{u_n^{2(1-\delta)}} dxdt \right)^{\frac{q}{2}} \left(\int_{\Omega_T} u_n^{\frac{(1-\delta)2q}{2-q}} dxdt \right)^{\frac{2-q}{2}} \\
&\le C^{\frac{q}{2}} \left(\int_{\Omega_T} u_n^{\frac{(1-\delta)2q}{2-q}} dxdt \right)^{\frac{2-q}{2}}.
\end{aligned}$$

Now we impose the condition $\gamma = \frac{(1-\delta)2q}{2-q}$ that gives $q = \frac{m(N+2)(\sigma+1)}{N+2-m(1-\sigma)}$. Observe that $q \ge m(\sigma+1) > 1$ and since $\sigma \le 1$ we have $m < m_1 \le \frac{2N+4}{N(1+\sigma)+4}$ which implies $q < 2$. Thus, the sequence $\{u_n\}_n$ is uniformly bounded in $L^q(0,T;W_0^{1,q}(\Omega)) \cap L^\gamma(\Omega_T)$. Therefore, there exist a subsequence of $\{u_n\}_n$, still indexed by n, and a function $u \in L^q(0,T;W_0^{1,q}(\Omega)) \cap L^\gamma(\Omega_T)$ such that $u_n \rightharpoonup u$ weakly in $L^q(0,T;W_0^{1,q}(\Omega)) \cap L^\gamma(\Omega_T)$ and $u_n \to u$ a.e. in Ω_T. Using $\phi \in C_0^\infty(\Omega \times [0,T))$ as test function in (3.2) we obtain

$$\begin{aligned}
&- \int_\Omega u_0(x)\phi(x,0)dx - \int_{\Omega_T} u_n \partial_t \phi dtdx + \int_{\Omega_T} \nabla u_n \cdot \nabla \phi dxdt \\
&\quad = \mu \int_{\Omega_T} \frac{T_n(u_n)\phi}{|x|^2 + \frac{1}{n}} dxdt + \int_{\Omega_T} \frac{f_n \phi}{|u_n| + \frac{1}{n}} dxdt.
\end{aligned} \tag{3.25}$$

Notice that since $u_n \rightharpoonup u$ weakly in $L^q(0,T;W_0^{1,q}(\Omega))$, we immediately have

$$\lim_{n\to+\infty}\int_{\Omega_T}\nabla u_n\cdot\nabla\phi dxdt=\int_{\Omega_T}\nabla u\cdot\nabla\phi dxdt$$

and

$$\lim_{n\to+\infty}\int_{\Omega_T}u_n\partial_t\phi dtdx=\int_{\Omega_T}u\partial_t\phi dtdx.$$

As regards the first integral in the right-hand side of (3.25), we know that the sequence $\{u_n\}$ is increasing to its limit u so we have

$$\Big|\frac{T_n(u_n)\phi}{|x|^2+\frac{1}{n}}\Big|\leq\frac{|u\phi|}{|x|^2}.$$

Applying Hölder's and Hardy's inequalities with exponents 2δ and $\frac{2\delta}{2\delta-1}$ we obtain

$$\begin{aligned}\int_{\Omega_T}\frac{|u\phi|}{|x|^2}dxdt&\leq\|\phi\|_\infty\int_{\Omega_T}\frac{|u|}{|x|^{\frac{1}{\delta}}}\times\frac{1}{|x|^{\frac{2\delta-1}{\delta}}}dxdt\\&\leq\|\phi\|_\infty\left(\int_{\Omega_T}\frac{|u|^{2\delta}}{|x|^2}dxdt\right)^{\frac{1}{2\delta}}\left(\int_{\Omega_T}\frac{dxdt}{|x|^2}\right)^{\frac{2\delta-1}{2\delta}}\\&\leq\|\phi\|_\infty(\Lambda_{N,2})^{\frac{-1}{2\delta}}\left(\int_{\Omega_T}|\nabla u^\delta|^2dxdt\right)^{\frac{1}{2\delta}}\left(\int_{\Omega_T}\frac{dxdt}{|x|^2}\right)^{\frac{2\delta-1}{2\delta}}.\end{aligned}$$

From (3.9) and (3.24) we deduce that the sequence $\{u_n^\delta\}$ is uniformly bounded in $L^2(0,T;H_0^1(\Omega))$ and thus there exist a subsequence of $\{u_n^\delta\}$, still indexed by n, and a function $v\in L^2(0,T;H_0^1(\Omega))$ such that $u_n^\delta\rightharpoonup v$ weakly in $L^2(0,T;H_0^1(\Omega))$ and $u_n^\delta\to v$ a.e. in Ω_T. But we also have $u_n^\delta\rightharpoonup v$ weakly in $L^q(0,T;W_0^{1,q}(\Omega))$ and hence follows $v=u^\delta\in L^2(0,T;H_0^1(\Omega))$. Which shows that the function $\frac{|u\phi|}{|x|^2}$ lies in $L^1(\Omega_T)$. Furthermore, since $\frac{T_n(u_n)\phi}{|x|^2+\frac{1}{n}}\to\frac{u\phi}{|x|^2}$ a.e. in Ω_T, the Lebesgue dominated convergence theorem gives

$$\lim_{n\to+\infty}\int_{\Omega_T}\frac{T_n(u_n)\phi}{|x|^2}dxdt=\int_{\Omega_T}\frac{u\phi}{|x|^2}dxdt.$$

On the other hand, the support $\mathrm{supp}(\phi)$ of the function ϕ is a compact subset of Ω_T and so by Lemma A.5 (in Appendix) there exists a constant $C_{\mathrm{supp}(\phi)}>0$ such that $u_n\geq C_{\mathrm{supp}(\phi)}$ in $\mathrm{supp}(\phi)$. Then,

$$\left|\frac{f_n\phi}{u_n+\frac{1}{n}}\right|\leq\frac{\|\phi\|_\infty}{C_{\mathrm{supp}(\phi)}}|f|\in L^1(\Omega_T).$$

So that by the Lebesgue dominated convergence theorem we get

$$\lim_{n\to+\infty}\int_{\Omega_T}\frac{f_n\phi}{u_n+\frac{1}{n}}dxdt=\int_{\Omega_T}\frac{f\phi}{u}dxdt.$$

We point out that we also have $u\geq C_{\mathrm{supp}(\phi)}$ in $\mathrm{supp}(\phi)$. Now passing to the limit as n tends to ∞ in (3.25) we obtain

$$-\int_\Omega u_0\phi(x,0)dx-\int_{\Omega_T}u\partial_t\phi dtdx+\int_{\Omega_T}\nabla u\cdot\nabla\phi dxdt=\mu\int_{\Omega_T}\frac{u\phi}{|x|^2}dxdt+\int_{\Omega_T}\frac{f\phi}{u}dxdt$$

for all $\phi\in C_0^\infty(\Omega\times[0,T))$. Namely u is a finite energy solution of the problem (1.1). □

3.7 Proof of Theorem 2.10

Let $u, v \in L^2(0,T;H_0^1(\Omega))$ be two energy solutions of the problem (1.1) corresponding to the same data u_0 satisfying (1.3) and $f \in L^m(\Omega_T)$, $m \geq 1$. Since the datum f is compactly supported in Ω_T, then $\partial_t u \in L^2(0,T;H^{-1}(\Omega)) + L^1(\Omega_T)$. Let $k > 0$ and $r > k$. The function $T_k((u-v)_+) \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(\Omega_T)$ is an admissible test function in the formulation of solution (A.8) in Lemma A.7 (in Appendix). Taking it so in the difference of formulations (A.8) solved by u and v, we obtain

$$\begin{aligned}\int_{\Omega_T} \partial_t (u-v)_+ T_k((u-v)_+)dxdt &+ \int_{\Omega_T} |\nabla T_k((u-v)_+)|^2 dxdt \\ &\leq \int_{\{(u-v)_+\leq k\}} \frac{(T_k((u-v)_+))^2}{|x|^2} dxdt + k\mu \int_{\{(u-v)_+>k\}} \frac{(u-v)_+}{|x|^2} dxdt \\ &\quad + \int_{\Omega_T} f\left(\frac{1}{u^\sigma} - \frac{1}{v^\sigma}\right) T_k((u-v)_+)dxdt\end{aligned}$$

Setting $\Theta_k(s) = \int_0^s T_k(\nu)d\nu$ and dropping the negative term, we get

$$\begin{aligned}\int_{\Omega} \Theta_k((u-v)_+(x,T))dx &+ \int_{\Omega_T} |\nabla T_k((u-v)_+)|^2 dxdt \\ &\leq \int_{\{(u-v)_+\leq k\}} \frac{(T_k((u-v)_+))^2}{|x|^2} dxdt + k\mu \int_{\{(u-v)_+>k\}} \frac{(u-v)_+}{|x|^2} dxdt \\ &\quad + \int_{\Omega} \Theta_k((u-v)_+(x,0))dx.\end{aligned}$$

Using $\int_\Omega \Theta_k((u-v)_+(x,T))dx \geq 0$, the fact that $u(x,0) = v(x,0) = u_0(x)$, Hardy's inequality (1.4) and Hölder's inequality, we arrive at

$$\begin{aligned}\int_{\Omega_T} |\nabla T_k((u-v)_+)|^2 dxdt &\leq \frac{\mu}{\Lambda_{N,2}} \int_{\Omega_T} |\nabla T_k((u-v)_+)|^2 dxdt \\ &\quad + k\mu \left(\int_{\{(u-v)_+>k\}} \frac{((u-v)_+)^2}{|x|^2} dxdt\right)^{\frac{1}{2}} \left(\int_{\{(u-v)_+>k\}} \frac{dxdt}{|x|^2}\right)^{\frac{1}{2}}.\end{aligned}$$

Having in mind (3.9) and using again (1.4) we reach that

$$\begin{aligned}\int_{\Omega_T} |\nabla T_k((u-v)_+)|^2 dxdt &\leq \frac{\mu}{\Lambda_{N,2}} \int_{\Omega_T} |\nabla T_k((u-v)_+)|^2 dxdt \\ &\quad + \frac{k\mu T^{\frac{1}{2}} C_1^{\frac{1}{2}}}{\Lambda_{N,2}^{\frac{1}{2}}} \left(\int_{\{(u-v)_+>k\}} |\nabla (u-v)_+|^2 dxdt\right)^{\frac{1}{2}}.\end{aligned} \tag{3.26}$$

On the other hand, taking $T_r(G_k((u-v)_+)) \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(\Omega_T)$ as a test function in the problems solved by u and v and subtracting the two equations we obtain

$$\begin{aligned}\int_{\Omega_T} \partial_t (u-v)_+ T_r(G_k((u-v)_+))dxdt &+ \int_{\{k<(u-v)_+<k+r\}} |\nabla (u-v)_+|^2 dxdt \\ &\leq \mu \int_{\{(u-v)_+>k\}} \frac{(u-v)_+^2}{|x|^2} dxdt + \int_{\Omega_T} f\left(\frac{1}{u^\sigma} - \frac{1}{v^\sigma}\right) T_r(G_k((u-v)_+))dxdt.\end{aligned}$$

Setting $\Theta_{k,r}(s) = \int_0^s T_r(G_k(v))dv$ and dropping the negative term, the above inequality becomes

$$\int_\Omega \Theta_{k,r}((u-v)_+(x,T))dx + \int_{\{k<(u-v)_+<k+r\}} |\nabla(u-v)_+|^2 dxdt \\ \leq \mu \int_{\{(u-v)_+>k\}} \frac{(u-v)_+^2}{|x|^2} dxdt + \int_\Omega \Theta_{k,r}((u-v)_+(x,0))dx.$$

Note that $\int_\Omega \Theta_{k,r}((u-v)_+(x,T))dx \geq 0$ and $\int_\Omega \Theta_{k,r}((u-v)_+(x,0))dx = 0$. Whence, by (1.4) we obtain

$$\int_{\{k<(u-v)_+<k+r\}} |\nabla(u-v)_+|^2 dxdt \leq \frac{\mu}{\Lambda_{N,2}} \int_{\{(u-v)_+>k\}} |\nabla(u-v)_+|^2 dxdt.$$

Then, passing to the limit as r tends to $+\infty$ we get

$$\int_{\{k<(u-v)_+\}} |\nabla(u-v)_+|^2 dxdt \leq \frac{\mu}{\Lambda_{N,2}} \int_{\{k<(u-v)_+\}} |\nabla(u-v)_+|^2 dxdt. \tag{3.27}$$

Therefore, gathering (3.26) and (3.27) we obtain

$$\int_{\Omega_T} |\nabla(u-v)_+|^2 dxdt \leq \frac{\mu}{\Lambda_{N,2}} \int_{\Omega_T} |\nabla(u-v)_+|^2 dxdt \\ + \frac{k\mu C_1}{\Lambda_{N,2}} \left(\int_{\{(u-v)_+>k\}} |\nabla(u-v)_+|^2 dxdt \right)^{\frac{1}{2}}.$$

Passing now to the limit as k tends to 0 we obtain

$$\int_{\Omega_T} |\nabla(u-v)_+|^2 dxdt \leq \frac{\mu}{\Lambda_{N,2}} \int_{\Omega_T} |\nabla(u-v)_+|^2 dxdt,$$

which, recalling that $u - v \in C([0,T]; L^1(\Omega))$ (see [44, Theorem 1.1]) implies $(u-v)_+(\cdot,\tau) = 0$ for any $\tau \in [0,T]$ and for almost every $x \in \Omega$. Since u and v play symmetrical roles we conclude that $u = v$ a.e. in Ω_T. □

A Appendix

We give here some important lemmas that are necessary for the accomplishment of the proofs of the previous results.

Theorem A.1 ([50, Theorem 2.2])**.** *Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 3$. Then for every $1 \leq q < 2$ there exists a positive constant $C = C(\Omega, q)$ such that for all $u \in H_0^1(\Omega)$ we have*

$$C \left(\int_\Omega |\nabla u|^q dx \right)^{\frac{2}{q}} \leq \int_\Omega |\nabla u|^2 dx - \Lambda_{N,2} \int_\Omega \frac{u^2}{|x|^2} dx.$$

Let

$$\alpha_1 := \frac{N-2}{2} - \sqrt{\left(\frac{N-2}{2}\right)^2 - \lambda} \tag{A.1}$$

be the smallest root of $\alpha^2 - (N-2)\alpha + \lambda = 0$. It is well known that this root yields the radial solution $|x|^{-\alpha_1}$ to the homogeneous equation

$$-\Delta v - \lambda \frac{v}{|x|^2} = 0.$$

The following lemma provides a local comparison result with this radial solution.

Lemma A.2 ([5, Lemma 2.2])**.** *Assume that u is a non-negative function defined in Ω such that $u \not\equiv 0$, $u \in L^1_{loc}(\Omega_T)$. If u satisfies*

$$\partial_t u - \Delta u - \lambda \frac{u}{|x|^2} \geq 0, \quad \text{in } \mathcal{D}'(\Omega_T)$$

with $\Omega_T := \Omega \times (0,T)$, $\lambda \leq \Lambda_{N,2}$ and $B_r(0) \subset\subset \Omega$, then there exists a constant $C = C(N,r,t_1,t_2)$ such that for each cylinder $B_{r_1}(0) \times (t_1,t_2) \subset \Omega \times (0,T)$, $0 < r_1 < r$,

$$u \geq C|x|^{-\alpha_1} \quad \text{in } B_{r_1}(0) \times (t_1,t_2),$$

where α_1 is the constant defined in (A.1).

Lemma A.3. *Let $0 < \lambda \leq \Lambda_{N,2}$ and $g \in L^1(0,T;L^1_{loc}(\Omega))$, $g \geq 0$. If u is a weak solution of the problem*

$$\begin{cases} \partial_t u - \Delta u = \lambda \dfrac{u}{|x|^2} + g & \text{in } \Omega_T := \Omega \times (0,T), \\ u = 0 & \text{in } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{A.2}$$

where $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$, then g satisfies

$$\int_{t_1}^{t_2} \int_{B_{r_1}(0)} |x|^{-\alpha_1} g dxdt < +\infty,$$

for any ball $B_{r_1}(0) \subset\subset \Omega$, where α_1 is defined in (A.1).

Proof. We use similar arguments as in [5, Remark 2.4]. Let $B_r(0) \subset\subset \Omega$ and $\phi \in L^2(0,T;H^1_0(\Omega)) \cap L^\infty(\Omega_T)$ be a weak solution of the problem

$$\begin{cases} \partial_t \phi - \Delta \phi - \lambda \frac{\phi}{|x|^2} = 1 & \text{in } \Omega_T, \\ \phi = 0 & \text{in } \partial\Omega \times (0,T), \\ \phi(x,0) = 1 & \text{in } \Omega. \end{cases} \tag{A.3}$$

Multiplying (A.2) by $T_n(\phi)$ and integrating over $B_r(0) \times (0,T)$ we obtain

$$\begin{aligned} &\int_0^T \int_{B_r(0)} \partial_t u T_n(\phi) dxdt - \int_0^T \int_{B_r(0)} \Delta u T_n(\phi) dxdt - \lambda \int_0^T \int_{B_r(0)} \frac{u}{|x|^2} T_n(\phi) dxdt \\ &\quad = \int_0^T \int_{B_r(0)} g T_n(\phi) dxdt. \end{aligned}$$

Since u is a weak solution of (A.2) the above integrals make sense for each integer n. By the classical by-parts integration formula, one has

$$\begin{aligned} &\int_{B_r(0)} u(x,T) T_n(\phi(x,T)) dx - \int_{B_r(0)} u(x,0) dx - \int_0^T \int_{B_r(0)} u \partial_t (T_n(\phi)) dxdt \\ &\qquad - \int_0^T \int_{B_r(0)} u \Delta (T_n(\phi)) dxdt - \lambda \int_0^T \int_{B_r(0)} \frac{u}{|x|^2} T_n(\phi) dxdt \\ &\quad = \int_0^T \int_{B_r(0)} g T_n(\phi) dxdt. \end{aligned} \tag{A.4}$$

Since $T_n(\phi) \to \phi$ in $L^1(\Omega_T)$ and a.e. in Ω_T and $\phi \in L^\infty(\Omega_T)$, we can apply the Lebesgue dominated convergence theorem in the (A.4) to get

$$\int_{B_r(0)} u(x,T)\phi(x,T)dx - \int_{B_r(0)} u_0(x)dx - \int_0^T \int_{B_r(0)} u\partial_t\phi dxdt$$
$$- \int_0^T \int_{B_r(0)} u\Delta\phi dxdt - \lambda \int_0^T \int_{B_r(0)} \frac{u}{|x|^2}\phi dxdt = \int_0^T \int_{B_r(0)} g\phi dxdt.$$

As ϕ is a solution of (A.3), we get

$$\int_{B_r(0)} u(x,T)\phi(x,T)dx - \int_{B_r(0)} u_0 dx - 2\int_0^T \int_{B_r(0)} u\partial_t\phi dxdt + \int_0^T \int_{B_r(0)} udxdt$$
$$= \int_0^T \int_{B_r(0)} g\phi dxdt.$$

Applying again the by-parts integration formula we obtain

$$-\int_{B_r(0)} u(x,T)\phi(x,T)dx + \int_{B_r(0)} u_0(x)dx + 2\int_0^T \int_{B_r(0)} \partial_t u\phi dxdt + \int_0^T \int_{B_r(0)} udxdt$$
$$= \int_0^T \int_{B_r(0)} g\phi dxdt.$$

By Lemma A.2, for every cylinder $B_{r_1}(0) \times (t_1, t_2) \subset B_r(0) \times (0,T)$, $0 < r_1 < r$ there exists a constant $C > 0$ such that

$$\int_{t_1}^{t_2} \int_{B_{r_1}(0)} |x|^{-\alpha_1} g dxdt \le \int_{B_r(0)} u(x,T)\phi(x,T)dx + \int_{B_r(0)} u_0 dx$$
$$+ 2\int_0^T \int_{B_r(0)} |\partial_t u\phi| dxdt + \int_0^T \int_{B_r(0)} udxdt.$$

Since $u \in L^1(0,T; L^1_{loc}(\Omega))$, $u_0 \in L^\infty(\Omega)$, $\phi \in L^\infty(\Omega_T)$ and $\partial_t u \in L^2(0,T; H^{-1}_{loc}(\Omega)) + L^1(0,T; L^1_{loc}(\Omega))$ conclude that

$$\int_{t_1}^{t_2} \int_{B_{r_1}(0)} |x|^{-\alpha_1} g dxdt < +\infty.$$ □

We will now compare the solution u_n of (3.1) with the solution w_n of the problem

$$\begin{cases} \partial_t w_n - \Delta w_n = \dfrac{f_n}{(w_n + \frac{1}{n})^\sigma} & \text{in } \Omega \times (0,T), \\ w_n(x,t) = 0 & \text{in } \partial\Omega \times (0,T), \\ w_n(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{A.5}$$

where $f = \min(f,n)$ and u_0 satisfies (1.3). Recall that (A.5) has a weak solution w_n (see [24, Lemma 2.1]).

Lemma A.4. *Let u_n be a solution of* (3.1) *and w_n be a solution of* (A.5)*. Then, $w_n \le u_n$ a.e. in Ω_T.*

Proof. Consider the problems solved by w_n and u_n, subtracting the two equations, we get

$$\begin{aligned} \partial_t(w_n - u_n) - \Delta(w_n - u_n) &= -\mu\frac{T_n(u_n)}{|x|^2 + \frac{1}{n}} + f_n\left(\frac{1}{(w_n + \frac{1}{n})^\sigma} - \frac{1}{(u_n + \frac{1}{n})^\sigma}\right) \\ &\le f_n\left(\frac{1}{(w_n + \frac{1}{n})^\sigma} - \frac{1}{(u_n + \frac{1}{n})^\sigma}\right). \end{aligned} \tag{A.6}$$

Using $(w_n - u_n)_+\chi_{(0,\tau)}$, $0 \le \tau \le T$, as test function in (A.6) it follows that

$$\begin{aligned}\frac{1}{2}\int_\Omega (w_n-u_n)_+^2(x,\tau)dx &+ \int_{\Omega_\tau}|\nabla(w_n-u_n)_+|^2 dxdt \\ &\le \int_{\Omega_\tau} f_n\left(\frac{(u_n+\frac{1}{n})^\sigma-(w_n+\frac{1}{n})^\sigma}{(u_n+\frac{1}{n})^\sigma(w_n+\frac{1}{n})^\sigma}\right)(w_n-u_n)_+dxdt \\ &\le 0,\end{aligned}$$

where we have used $w_n(x,0) = u_n(x,0) = u_0(x)$. Hence we conclude that

$$\int_{\Omega_T}|\nabla(w_n-u_n)_+|^2(x,\tau)dx = 0.$$

Recalling that $w_n - u_n \in \mathcal{C}([0,T]; L^1(\Omega))$ (see [44, Theorem 1.1]) implies $(w_n - u_n)_+(\cdot,\tau) = 0$ for every $0 \le \tau \le T$ and for almost every $x \in \Omega$. Thus, $w_n \le u_n$ a.e. in Ω_T. □

Lemma A.5. *Let u_n be the solution of (3.1) given by Lemma 3.1. Then for every $\Omega' \subset\subset \Omega$ there exists $C_{\Omega'} > 0$ (not depending on n), such that $u_n \ge C_{\Omega'}$ in $\Omega' \times [0,T]$.*

Proof. The proof follows by combining [24, Proposition 2.2] and Lemma A.4. □

Lemma A.6. *Assume that $\mu \le \Lambda_{N,2}$ and let u_n be a solution of (3.1). The sequence $\{u_n\}_{n\in\mathbb{N}}$ is nonnegative and increasing with respect to $n \in \mathbb{N}$.*

Proof. Writing (3.2) with u_n and u_{n+1} and then subtracting the two corresponding equations, we obtain

$$\begin{aligned}\int_0^T\int_\Omega \partial_t(u_n-u_{n+1})\phi dxdt &+ \int_0^T\int_\Omega \nabla(u_n-u_{n+1})\nabla\phi dxdt \\ &\le \mu\int_0^T\int_\Omega \frac{T_{n+1}(u_n)-T_{n+1}(u_{n+1})}{|x|^2+\frac{1}{n+1}}\phi dxdt \\ &\quad + \int_0^T\int_\Omega f_{n+1}\left(\frac{1}{(u_n+\frac{1}{n+1})^\sigma}-\frac{1}{(u_{n+1}+\frac{1}{n+1})^\sigma}\right)\phi dxdt\end{aligned} \tag{A.7}$$

for every $\phi \in L^2(0,T;H_0^1(\Omega))$. Inserting $(u_n - u_{n+1})_+ \in L^2(0,T;H_0^1(\Omega))$ as a test function in (A.7) and using the fact that T_{n+1} is a 1-Lipschitzian function, we get

$$\begin{aligned}\frac{1}{2}\int_{\Omega_T}\partial_t(u_n-u_{n+1})_+^2 dxdt &+ \int_{\Omega_T}|\nabla(u_n-u_{n+1})_+|^2 dxdt \\ \le \int_{\Omega_T} f_{n+1}(u_n-u_{n+1})_+&\left(\frac{1}{(u_n+\frac{1}{n+1})^\sigma}-\frac{1}{(u_{n+1}+\frac{1}{n+1})^\sigma}\right)dxdt \\ +\mu\int_{\Omega_T}\frac{(u_n-u_{n+1})_+^2}{|x|^2}dxdt.&\end{aligned}$$

Dropping the non-negative parabolic term and using the fact that

$$(u_n-u_{n+1})_+\left(\frac{1}{(u_n+\frac{1}{n+1})^\sigma}-\frac{1}{(u_{n+1}+\frac{1}{n+1})^\sigma}\right) \le 0,$$

we obtain

$$\int_{\Omega_T}|\nabla(u_n-u_{n+1})_+|^2 dxdt \le \mu\int_{\Omega_T}\frac{(u_n-u_{n+1})_+^2}{|x|^2}dxdt.$$

Thus, if $\mu < \Lambda_{N,2}$ the Hardy inequality (1.4) yields

$$\int_{\Omega_T} |\nabla(u_n - u_{n+1})_+|^2 dxdt = 0,$$

while if $\mu = \Lambda_{N,2}$ we can apply Theorem A.1 obtaining

$$\int_{\Omega_T} |\nabla(u_n - u_{n+1})_+|^q dxdt = 0,$$

for all $q < 2$. Therefore, in both cases we get $(u_n - u_{n+1})_+ = 0$ a.e. in Ω_T, that is $u_n \le u_{n+1}$ a.e. in Ω_T. In addition, as $u_n \ge u_0$ we infer that u_n is nonnegative. □

Lemma A.7. *Let $u \in L^2(0,T;H_0^1(\Omega))$ be a finite energy solution of (1.1) with a datum $f \in L^1(\Omega_T)$ such that $\mathrm{supp}(f) \subset\subset \Omega_T$. Then u satisfies $\frac{u\phi}{|x|^2} \in L^1(\Omega_T)$, $\frac{f\phi}{u^\sigma} \in L^1(\Omega_T)$ and*

$$\int_{\Omega_T} \partial_t u \phi dxdt + \int_{\Omega_T} \nabla u \cdot \nabla \phi dxdt = \int_{\Omega_T} \left(\mu \frac{u}{|x|^2} + \frac{f}{u^\sigma}\right) \phi dxdt, \tag{A.8}$$

for every $\phi \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(\Omega_T)$.

Proof. Let $\phi \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(\Omega_T)$ be a nonnegative function. A direct application of Hardy's inequality yields $\mu\frac{u\phi}{|x|^2} \in L^1(\Omega_T)$, while since f is compactly supported in Ω_T, by Lemma A.5 there exists a constant $C_{\mathrm{supp}(f)} > 0$ such that $u \ge C_{\mathrm{supp}(f)}$ in $\mathrm{supp}(f)$ so that one has

$$\int_{\Omega_T} \frac{|f\phi|}{u^\sigma} dxdt \le C^\sigma_{\mathrm{supp}(f)} \|\phi\|_\infty \|f\|_{L^1(\Omega_T)} < \infty.$$

We argue as in [41, Lemma 4.2] considering a sequence of function $\phi_n \in C_0^\infty(\Omega_T)$, with $\phi_n \ge 0$ and $\phi_n \to \phi$ in $L^2(0,T;H_0^1(\Omega))$, with $\|\phi_n\|_\infty \le \|\phi\|_\infty$. Inserting ϕ_n as a test function in (2.1) and integrating by parts, we obtain

$$\int_{\Omega_T} \partial_t u \phi_n dxdt + \int_{\Omega_T} \nabla u \cdot \nabla \phi_n dxdt = \int_{\Omega_T} \left(\mu \frac{u}{|x|^2} + \frac{f}{u^\sigma}\right) \phi_n dxdt. \tag{A.9}$$

Since $\phi_n \to \phi$ in $L^2(\Omega_T)$ then, for a subsequence still indexed by n, we may assume that $\phi_n \to \phi$ a.e. in Ω_T. As f is compactly supported in Ω_T we have

$$\left(\mu \frac{u}{|x|^2} + \frac{f}{u^\sigma}\right) \phi_n \le \|\phi\|_\infty \left(\mu \frac{u}{|x|^2} + \frac{f}{u^\sigma}\right) \in L^1(\Omega_T).$$

Thus, by the Lebesgue dominated convergence theorem we obtain

$$\lim_{n\to\infty} \int_{\Omega_T} \left(\mu \frac{u}{|x|^2} + \frac{f}{u^\sigma}\right) \phi_n dxdt = \int_{\Omega_T} \left(\mu \frac{u}{|x|^2} + \frac{f}{u^\sigma}\right) \phi dxdt.$$

Since $\partial_t u \in L^2(0,T;H^{-1}(\Omega)) + L^1(\Omega_T)$ we use the convergence $\phi_n \to \phi$ in $L^2(0,T;H_0^1(\Omega))$ and again the Lebesgue dominated convergence theorem in (A.9) obtaining

$$\int_{\Omega_T} \partial_t u \phi dxdt + \int_{\Omega_T} \nabla u \cdot \nabla \phi dxdt = \int_{\Omega_T} \left(\mu \frac{u}{|x|^2} + \frac{f}{u^\sigma}\right) \phi dxdt.$$ □

Conflict of interest

The authors declare no potential conflict of interests.

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Random invariant manifolds and foliations for slow-fast PDEs with strong multiplicative noise

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Abstract. This article is devoted to the dynamical behaviors of a class of slow-fast PDEs perturbed by strong multiplicative noise. We will accomplish the existence of random invariant manifolds and foliations, and show exponential tracking property of them. Moreover, the asymptotic approximation for both objects will be presented.

Keywords: slow-fast systems; random invariant manifolds; random invariant foliations.
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1 Introduction

Various kinds of mathematical models arising from physics, engineering and biology not only involve random effects such as uncertain parameters, stochastic perturbation, but also relate to multiple disparate time or spatial scales [2, 16, 21]. Many important physical models, such as Burger's equation, Ginzburg–Laudau equation, Swift–Hohenberg equation are highly referred in this field. In order to investigate a variety of equations in the context of random influences, by combining probability theory, functional analysis and the theory of partial differential equations, mathematicians gradually developed and perfected a systematic framework of stochastic partial differential equations (SPDEs) in recent decades [13, 30]. In terms of SPDEs evolving on multi-scales, there are many methods used to analyze the dynamical behaviours of SPDEs, such as averaging method [9, 10], amplitude equations [4, 5] and the theory of invariant manifolds [17, 29].

Among these methods, the theory of invariant manifolds is considered as a practicable tool, which can provide a geometric structure of complex systems [1, 35, 37]. For deterministic systems, the pioneering results were obtained by Hadamard [22], Lyapunov [23] and Perron [25]. Duan et al. [14, 15] extended this theory to random dynamic systems and show the existence of random invariant manifolds for SPDEs with simple multiplicative noise. Equations with more general multiplicative noise were studied by Caraballo et al. [7] and Mohammed

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et al. [24]. Also other dynamical properties of SPDEs have been already addressed in the literature, just to list a few but far from being complete: random invariant foliations [26,32,34], asymptotic dynamical behaviors [19,33,34,36], geometric shape [6,11,18,20], etc.

Applying the property that the random invariant manifolds contribute to the reduction of SPDEs, mathematicians can eliminate the fast variable of slow-fast systems to reduce the original system to a lower dimensional system. At earlier stage of the research, Schmalfuß and Schneider [31] studied a class of slow-fast systems with noise in the finite dimensional case by Hadamard method, and obtained that inertial manifolds tend to slow manifolds if the scaling parameter ε tends to 0. Fu et al. [17] applied Lyapunov–Perron method to a class of stochastic evolution equations with slow and fast components, and proved that slow manifolds asymptotically approximate to critical manifolds. Qiao et al. [28,29] obtained a reduced system of a class of SDEs under slow-fast Gaussian noisy fluctuations on the random invariant manifolds, and showed the delicate error between the filter of the original system and that of the reduced system. The slow invariant foliation, another interesting object in this field, was originally studied by Chen et al. [12]. They constructed random invariant foliations for a class of slow-fast stochastic evolutionary systems, and presented the approximation of slow foliations. Recently, slow-fast systems with non-Gaussian noise have gained substantial attention from researchers. For details, please see [27,38,39], etc.

In this paper, we investigate a class of slow-fast PDEs driven by strong multiplicative noise:

$$dX^\varepsilon = \left[\frac{A}{\varepsilon}X^\varepsilon + \frac{f(X^\varepsilon, Y^\varepsilon)}{\varepsilon}\right]dt + \frac{X^\varepsilon}{\sqrt{\varepsilon}} \circ dW, \quad \text{in } H_1, \tag{1.1}$$

$$dY^\varepsilon = [BY^\varepsilon + g(X^\varepsilon, Y^\varepsilon)]\,dt + \frac{Y^\varepsilon}{\sqrt{\varepsilon}} \circ dW, \quad \text{in } H_2, \tag{1.2}$$

where H_1 and H_2 are separable Hilbert spaces, ε is small parameter ($0 < \varepsilon \ll 1$), $W(t)$ is a two-sided Wiener process taking value in $\mathbb{R}$, $\circ$ means Stratonovich stochastic differential, and A, B, f, g will be introduced later. Briefly, the main goal of this paper is to construct the random invariant manifolds and foliations for (1.1)–(1.2) and to derive corresponding approximations for both. Compared with [17,28,29,31], the system we study is forced by multiplicative noise rather than additive noise. To the best of our knowledge, this is the first research to consider the slow manifolds and slow foliations for slow-fast SPDEs with multiplicative noise.

This paper is organized as follows. In next section, we present some assumptions and recall some basic concepts in random dynamical systems. In Section 3, the existence of random invariant manifolds of (1.1)–(1.2) is established. Moreover, we show the orbit starting from random invariant manifold can exponentially approach to the other orbits in forward time, and prove that invariant manifolds can converge to slow ones as ε tends to 0. Section 4 is aimed at the theory of random invariant foliations including existence, exponential tracking property in backward time, and asymptotic foliations.

2 Preliminaries

The section is devoted to presenting some conditions that we need later, and reviewing some background materials in random dynamic systems.

2.1 Notations and assumptions

Let H_1 and H_2 be separable Hilbert spaces in (1.1) and (1.2). Denote their norms by $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Set $H := H_1 \times H_2$ with norm $\|\cdot\| = \|\cdot\|_1 + \|\cdot\|_2$. A, B, f, g in (1.1)–(1.2) satisfy the following conditions.

Assumption 1. Suppose that linear operator A generates a C_0-semigroup $\{e^{At}\}_{t\geq 0}$ on H_1 fulfilling

$$\|e^{At}x\|_1 \leq e^{-\gamma_1 t}\|x\|_1, \quad \text{for} \quad x \in H_1, \ t \geq 0,$$

and linear operator B generates a C_0-group $\{e^{Bt}\}_{t\in\mathbb{R}}$ on H_2 fulfilling

$$\|e^{Bt}y\|_2 \leq e^{\gamma_2 t}\|y\|_2, \quad \text{for} \quad y \in H_2, \ t \leq 0,$$

where $\gamma_1 > 0, \gamma_2 \geq 0$.

Assumption 2. Suppose that nonlinear terms

$$f : H_1 \times H_2 \to H_1,$$
$$g : H_1 \times H_2 \to H_2,$$

satisfy $f(0,0) = 0$ and $g(0,0) = 0$, and there exists a constant $K > 0$ such that

$$\|f(x_1, y_1) - f(x_2, y_2)\|_1 \leq K(\|x_1 - x_2\|_1 + \|y_1 - y_2\|_2),$$
$$\|g(x_1, y_1) - g(x_2, y_2)\|_2 \leq K(\|x_1 - x_2\|_1 + \|y_1 - y_2\|_2),$$

for all $x_1, x_2 \in H_1$ and $y_1, y_2 \in H_2$.

Assumption 3. $f(x, y)$ and $g(x, y)$ are C^1 functions, and all the first order partial derivatives of them are uniformly bounded.

Assumption 4. The Lipschitz constant K and decay rate γ_1 of A satisfy

$$K < \gamma_1. \tag{2.1}$$

Assumption 5. The Lipschitz constant K, decay rate γ_1 of A and decay rate γ_2 of B satisfy

$$K < \frac{\gamma_1\gamma_2}{2\gamma_1 + \gamma_2}. \tag{2.2}$$

Remark 2.1. (1) We remark that our main theorems hold when H_1 and H_2 are real or complex separable Hilbert spaces. For simplicity, we ignore it.

(2) Assumption 3 and Assumption 4 will be imposed in Section 3 and Section 4, respectively. We would like to point out that condition (2.2) is sufficient for condition (2.1), which implies that the condition used for the study of the random invariant manifolds is weaker than that used for the study of the random invariant foliations.

(3) Moreover, we remark that there are other conditions, which can also play the same role as condition (2.2). For the details, please see Remark 4.3 and Remark 4.4 in [12].

2.2 Random dynamical systems

Referring to the literature [1,12,14,15,26], we introduce some concepts of random dynamical systems.

Definition 2.2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and a flow θ of mappings $\{\theta_t\}_{t\in\mathbb{R}}$ be defined by $\theta : \mathbb{R} \times \Omega \to \Omega$ such that

$$\begin{aligned}
&\theta_0 = \mathrm{id}_\Omega, \\
&\theta_{t_1} \circ \theta_{t_2} = \theta_{t_1+t_2}, \quad \forall t_1, t_2 \in \mathbb{R}, \\
&\text{the flow is } (\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F})\text{-measurable}, \\
&\theta_t \mathbb{P} = \mathbb{P}, \quad \forall t \in \mathbb{R}.
\end{aligned}$$

Then $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ is called a metric dynamical system.

Definition 2.3. A random dynamical system on the topological space X over a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ is a mapping

$$\varphi : \mathbb{R} \times \Omega \times X \to X, \qquad (t, \omega, x) \to \varphi(t, \omega, x),$$

such that

$$\begin{aligned}
&\varphi \text{ is } (\mathcal{B}(\mathbb{R}) \otimes \mathcal{F} \otimes \mathcal{B}(X), \mathcal{B}(X))\text{-measurable}, \\
&\varphi(0, \omega) = \mathrm{id}_X, \quad \forall \omega \in \Omega, \\
&\varphi(t+s, \omega, \cdot) = \varphi(t, \theta_s \omega, \varphi(s, \omega, \cdot)), \quad \forall s, t \in \mathbb{R}, \ \omega \in \Omega, \\
&\varphi(t, \omega, x) \text{ is continuous with respect to } t, \text{ for fixed } \omega \in \Omega, x \in X.
\end{aligned}$$

In what follows, we consider $\varphi(t, \omega, \cdot)$ as a random dynamical system on a complete separable metric space (H, d_H) over a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$.

Definition 2.4. A family of nonempty closed sets $M = \{M(\omega)\}$ contained in (H, d_H) is called a random set if

$$\omega \to \inf_{y\in M(\omega)} d_H(x, y)$$

is a random variable for $x \in H$.

Definition 2.5. A random set $M = \{M(\omega)\}$ is called a positively invariant set contained in (H, d_H) if

$$\varphi(t, \omega, M(\omega)) \subset M(\theta_t \omega), \quad t \geq 0, \quad \omega \in \Omega.$$

Furthermore, if, for every $\omega \in \Omega$, we can represent M by a graph of a Lipschitz mapping

$$\phi(\omega, \cdot) : H_2 \to H_1,$$

i.e.,

$$M(\omega) = \{(\phi(\omega, y), y) | y \in H_2\},$$

then $M(\omega)$ is called a Lipschitz continuous invariant manifold.

Definition 2.6. (i) Fixing $x \in H$, we call $\mathcal{W}_{\alpha s}(x, \omega)$ is an α-stable fiber passing through x with $\alpha \in \mathbb{R}^-$, if $\|\varphi(t, \omega, x) - \varphi(t, \omega, \bar{x})\|_H = \mathcal{O}(e^{\alpha t}), \forall \omega \in \Omega$ as $t \to +\infty$ for all $\bar{x} \in \mathcal{W}_{\alpha s}(x, \omega)$.

(ii) Fixing $x \in H$, we call $\mathcal{W}_{\beta u}(x,\omega)$ is a β-unstable fiber passing through x with $\beta \in \mathbb{R}^+$, if $\|\varphi(t,\omega,x)-\varphi(t,\omega,\bar{x})\|_H = \mathcal{O}(e^{\beta t}), \forall \omega \in \Omega$ as $t \to -\infty$ for all $\bar{x} \in \mathcal{W}_{\beta u}(x,\omega)$.

(iii) $\mathcal{W}_{\alpha s}(\omega) := \cup_{x\in H}\mathcal{W}_{\alpha s}(x,\omega)$ is called stable foliation.

(iv) $\mathcal{W}_{\beta u}(\omega) := \cup_{x\in H}\mathcal{W}_{\beta u}(x,\omega)$ is called unstable foliation.

(v) A foliation $\mathcal{W}_{\beta u}(\omega)$ is invariant with respect to random dynamical system φ if each fiber of it satisfies that

$$\varphi(t,\omega,\mathcal{W}_{\beta u}(x,\omega)) \subset \mathcal{W}_{\beta u}(\varphi(t,\omega,x),\theta_t\omega).$$

2.3 Transformation from SPDEs to RPDEs

The motivation of this subsection is to transform SPDEs (1.1)–(1.2) into random partial differential equations (RPDEs), and show the relationship between them. For our applications, we introduce the metric dynamical system induced by Wiener process. Let $W(t)$ be a two-sided Wiener process with trajectories in the space $C_0(\mathbb{R},\mathbb{R})$ which is the collection of continuous functions $\omega : \mathbb{R} \to \mathbb{R}$ with $\omega(0) = 0$. Set $\bar{\Omega} := C_0(\mathbb{R},\mathbb{R})$. This set is equipped with a compact-open topology (please see the Appendix in [1]). Let $\bar{\mathcal{F}}$ be its Borel σ-field and $\bar{\mathbb{P}}$ be the Wiener measure. Set

$$\theta_t\omega(\cdot) := \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega, \quad t \in \mathbb{R}.$$

Note that $\bar{\mathbb{P}}$ is ergodic with respect to θ_t. Then $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}, \{\theta_t\}_{t\in\mathbb{R}})$ is a metric dynamical system.

In order ot obtain RPDEs, we need the following preparation. Consider the linear stochastic differential equation:

$$\mathrm{d}z^\varepsilon = -\frac{z^\varepsilon}{\varepsilon}\mathrm{d}t + \frac{1}{\sqrt{\varepsilon}}\mathrm{d}W. \tag{2.3}$$

The solution of (2.3) is called an Ornstein–Uhlenbeck process. Following Lemma 2.1 in [14], we present the properties of $z^\varepsilon(t)$ as follows.

Lemma 2.7.

(1) There exists a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant set $\Omega \in \mathcal{B}(C_0(\mathbb{R},\mathbb{R}))$ of full measure with sublinear growth:

$$\lim_{t\to\pm\infty}\frac{|\omega(t)|}{|t|} = 0, \omega \in \Omega.$$

(2) For $\omega \in \Omega$ the random variable

$$z^\varepsilon(\omega) = -\varepsilon^{-\frac{3}{2}}\int_{-\infty}^0 e^{\frac{\tau}{\varepsilon}}\omega(\tau)\mathrm{d}\tau$$

exists and generates a unique stationary solution of (2.3) given by

$$\Omega \times \mathbb{R} \ni (\omega,t) \to z^\varepsilon(\theta_t\omega) = -\varepsilon^{-\frac{3}{2}}\int_{-\infty}^0 e^{\frac{\tau}{\varepsilon}}\theta_t\omega(\tau)\mathrm{d}\tau = -\varepsilon^{-\frac{3}{2}}\int_{-\infty}^0 e^{\frac{\tau}{\varepsilon}}\omega(\tau+t)\mathrm{d}\tau + \varepsilon^{-\frac{1}{2}}\omega(t).$$

The mapping $t \to z^\varepsilon(\theta_t\omega)$ is continuous.

(3) In particular, we have

$$\lim_{t\to\pm\infty}\frac{|z^\varepsilon(\theta_t\omega)|}{|t|} = 0 \ \textit{for}\ \omega \in \Omega.$$

(4) In addition,

$$\lim_{t\to\pm\infty}\frac{1}{t}\int_0^t z^\varepsilon(\theta_\tau\omega)\mathrm{d}\tau = 0 \ \textit{for}\ \omega\in\Omega.$$

In the followings of this paper, we consider (1.1)–(1.2) on the new metric dynamical system $(\Omega,\mathcal{F},\mathbb{P},\{\theta_t\}_{t\in\mathbb{R}})$, where Ω is given in Lemma 2.7, $\mathcal{F} := \{\Omega\cap\mathcal{A},\mathcal{A}\in\mathcal{B}(C_0(\mathbb{R},\mathbb{R}))\}$, and $\mathbb{P}$ is the restriction of the Wiener measure $\bar{\mathbb{P}}$ to $\mathcal{F}$. We proceed to show the solution of (1.1)–(1.2) can generate a random dynamical system over the metric dynamical system $(\Omega,\mathcal{F},\mathbb{P},\{\theta_t\}_{t\in\mathbb{R}})$. Letting $\bar{X}^\varepsilon = e^{-z^\varepsilon(\theta_t\omega)}X^\varepsilon$, $\bar{Y}^\varepsilon = e^{-z^\varepsilon(\theta_t\omega)}Y^\varepsilon$, we obtain RPDEs:

$$\mathrm{d}\bar{X}^\varepsilon = \left[\frac{A}{\varepsilon}\bar{X}^\varepsilon + \frac{z^\varepsilon(\theta_t\omega)\bar{X}^\varepsilon}{\varepsilon} + \frac{F(\bar{X}^\varepsilon,\bar{Y}^\varepsilon,\theta_t\omega)}{\varepsilon}\right]\mathrm{d}t, \tag{2.4}$$

$$\mathrm{d}\bar{Y}^\varepsilon = [B\bar{Y}^\varepsilon + \frac{z^\varepsilon(\theta_t\omega)\bar{Y}^\varepsilon}{\varepsilon} + G(\bar{X}^\varepsilon,\bar{Y}^\varepsilon,\theta_t\omega)]\mathrm{d}t, \tag{2.5}$$

where

$$F(\bar{X}^\varepsilon,\bar{Y}^\varepsilon,\theta_t\omega) = e^{-z^\varepsilon(\theta_t\omega)}f(e^{z^\varepsilon(\theta_t\omega)}\bar{X}^\varepsilon, e^{z^\varepsilon(\theta_t\omega)}\bar{Y}^\varepsilon),$$
$$G(\bar{X}^\varepsilon,\bar{Y}^\varepsilon,\theta_t\omega) = e^{-z^\varepsilon(\theta_t\omega)}g(e^{z^\varepsilon(\theta_t\omega)}\bar{X}^\varepsilon, e^{z^\varepsilon(\theta_t\omega)}\bar{Y}^\varepsilon).$$

Since F and G are also Lipschitz functions with the same Lipschitz constant K for $\omega\in\Omega$, there exists a unique solution $Z^\varepsilon(t) = (X^\varepsilon(t), Y^\varepsilon(t))$ of (2.4)–(2.5) for $\omega\in\Omega$. Hence, the mapping

$$(t,\omega,Z^\varepsilon(0)) \to Z^\varepsilon(t,\omega,Z^\varepsilon(0))$$

is $(\mathbb{R}\otimes\mathcal{F}\otimes\mathcal{B}(H),\mathcal{B}(H))$-measurable and generates a random dynamical system. We introduce the transform

$$T(\omega,x) = xe^{-z^\varepsilon(\omega)} \tag{2.6}$$

and its inverse transform

$$T^{-1}(\omega,x) = xe^{z^\varepsilon(\omega)} \tag{2.7}$$

for $x\in H$ and $\omega\in\Omega$.

Lemma 2.8. *Suppose that $u^\varepsilon(t,\omega,x)$ is the random dynamical system generated by (2.4)–(2.5). Then*

$$(t,\omega,x) \to T^{-1}(\theta_t\omega, u^\varepsilon(t,\omega,T(\omega,x))) =: \hat{u}^\varepsilon(t,\omega,x)$$

is a random dynamical system. For any $x\in H$ this process $(t,\omega)\to\hat{u}^\varepsilon(t,\omega,x)$ is a solution to (1.1)–(1.2).

Proof. Note that $T(\omega,\cdot)$ is a homeomorphism for any $\omega\in\Omega$, $T(\cdot,x)$, $T^{-1}(\cdot,x)$ are measurable for any $x\in H$, and $u^\varepsilon(t,\omega,x)$ is a random dynamical system. Hence, $\hat{u}^\varepsilon(t,\omega,x)$ is a random dynamical system. For $x\in H$, applying Itô's formula to $T(\theta_t\omega,\hat{u}^\varepsilon(t,\omega,T^{-1}(\omega,x)))$, we can obtain a solution of (2.4)–(2.5). Because $T(\theta_t\omega,x)$ and $u(t,\omega,x)$ are well defined for any $\omega\in\Omega$, and T^{-1} is the inverse of T, the converse is also true, which implies $\hat{u}^\varepsilon(t,\omega,x)$ is a solution of (1.1)–(1.2). □

Based on the above lemma, we can investigate (1.1)–(1.2) via (2.4)–(2.5). Then we are concerned with the random partial differential equations (RPDEs) (2.4)–(2.5) in the remainder of this paper.

3 Random invariant manifolds and slow manifolds

In this section, we use Lyapunov–Perron's method to prove the existence of random invariant manifolds for (2.4)–(2.5), and state that any orbit can be exponentially attracted by random invariant manifolds. Moreover, we show slow manifolds can approach to random invariant manifolds as the parameter ε tends to 0.

3.1 Random invariant manifolds

Let us give some notations. For $\alpha \in \mathbb{R}$, a real-valued stochastic process $p(t,\omega)$ and $i = 1,2$, define Banach Space

$$C^{i,-}_{\alpha,p} := \Big\{\phi : (-\infty,0] \to H_i | \phi \text{ is continuous and } \sup_{t\in(-\infty,0]} e^{-\alpha t - \int_0^t p(s,\omega)\mathrm{d}s}\|\phi(t)\|_i < \infty\Big\}$$

with the norm $\|\phi\|_{C^{i,-}_{\alpha,p}} = \sup_{t\in(-\infty,0]} e^{-\alpha t - \int_0^t p(s,\omega)\mathrm{d}s}\|\phi(t)\|_i$, and

$$C^{i,+}_{\alpha,p} := \Big\{\phi : [0,+\infty) \to H_i | \phi \text{ is continuous and } \sup_{t\in[0,+\infty)} e^{-\alpha t - \int_0^t p(s,\omega)\mathrm{d}s}\|\phi(t)\|_i < \infty\Big\}$$

with the norm $\|\phi\|_{C^{i,+}_{\alpha,p}} = \sup_{t\in[0,+\infty)} e^{-\alpha t - \int_0^t p(s,\omega)\mathrm{d}s}\|\phi(t)\|_i$. Furthermore, define product Banach space $C^{\pm}_{\alpha,p} := C^{1,\pm}_{\alpha,p} \times C^{2,\pm}_{\alpha,p}$ with the norm $\|z\|_{C^{\pm}_{\alpha,p}} = \|x\|_{C^{1,\pm}_{\alpha,p}} + \|y\|_{C^{2,\pm}_{\alpha,p}}$, $z = (x,y) \in C^{\pm}_{\alpha,p}$.

Let μ be a positive number satisfying $\gamma_1 - \mu > K$. Let $\bar{Z}^\varepsilon(t,\omega,\bar{Z}_0)$ be the solution of (2.4)-(2.5) with the initial value $\bar{Z}_0 \in H$. Set $M^\varepsilon(\omega) = \big\{\bar{Z}_0 \in H | \bar{Z}^\varepsilon(\cdot,\omega,\bar{Z}_0) \in C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}\big\}$. More precisely, $M^\varepsilon(\omega)$ is the set containing all initial data such that corresponding solutions belong to $C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}$.

Following the idea from [15, 17], we will show $M^\varepsilon(\omega)$ can be represented by a Lipschitz function with Lyapunov–Perron's method.

Lemma 3.1. *Suppose that Assumptions 1, 2, 4 hold.* $\bar{Z}_0 = (\bar{X}_0, \bar{Y}_0) \in M^\varepsilon(\omega)$ *if and only if there exists a function* $\bar{Z}^\varepsilon(\cdot) = (\bar{X}^\varepsilon(\cdot), \bar{Y}^\varepsilon(\cdot)) \in C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}$ *with* $\bar{Z}^\varepsilon(0) = \bar{Z}_0$ *and satisfies*

$$\bar{X}^\varepsilon(t) = \frac{1}{\varepsilon}\int_{-\infty}^t e^{\frac{A(t-s)+\int_s^t z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}} F(\bar{X}^\varepsilon, \bar{Y}^\varepsilon, \theta_s\omega)\mathrm{d}s, \tag{3.1}$$

$$\bar{Y}^\varepsilon(t) = e^{Bt + \frac{\int_0^t z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}}\bar{Y}_0 + \int_0^t e^{B(t-s)+\frac{\int_s^t z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}} G(\bar{X}^\varepsilon, \bar{Y}^\varepsilon, \theta_s\omega)\mathrm{d}s. \tag{3.2}$$

Proof. The proof can be completed by that of Theorem 3.1 in [15], so it is omitted here. □

Theorem 3.2 (Existence of invariant manifold)**.** *Under Assumptions 1, 2, 4, for sufficiently small* $\varepsilon > 0$*, there exists a Lipschitz invariant manifold* $M^\varepsilon(\omega)$ *for* (2.4)–(2.5) *represented as a graph*

$$M^\varepsilon(\omega) = \{(H^\varepsilon(\omega, Y_0), Y_0) | Y_0 \in H_2\}, \tag{3.3}$$

where $H^\varepsilon(\cdot,\cdot) : \Omega \times H_2 \to H_1$ *is a Lipschitz continuous mapping with Lipschitz constant* $\mathrm{Lip}\, H^\varepsilon(\omega,\cdot)$ *satisfying that*

$$\mathrm{Lip}\, H^\varepsilon(\omega,\cdot) \le \frac{K}{(\gamma_1-\mu)\Big[1 - K\Big(\frac{1}{\gamma_1-\mu} + \frac{\varepsilon}{\mu+\varepsilon\gamma_2}\Big)\Big]}, \quad \omega \in \Omega. \tag{3.4}$$

Moreover, $H^\varepsilon(\omega, 0) = 0$.

Proof. The proof consists of four steps.

Step1. Claim that for $\varepsilon > 0$ sufficiently small, (3.1)–(3.2) will have a unique solution $\bar{Z}^\varepsilon(\cdot,\omega,\bar{Z}_0) = (\bar{X}^\varepsilon(\cdot,\omega,\bar{Z}_0), \bar{Y}^\varepsilon(\cdot,\omega,\bar{Z}_0)) \in C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}$. We will use Banach's Fixed Point Theorem to achieve the claim.

Define two operators $J_1^\varepsilon : C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}} \to C^{1,-}_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}$ and $J_2^\varepsilon : C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}} \to C^{2,-}_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}$ satisfying

$$J_1^\varepsilon(\bar{Z}^\varepsilon(t)) = \frac{1}{\varepsilon}\int_{-\infty}^t e^{\frac{A(t-s)+\int_s^t z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}} F(\bar{X}^\varepsilon,\bar{Y}^\varepsilon,\theta_s\omega)\mathrm{d}s, \tag{3.5}$$

$$J_2^\varepsilon(\bar{Z}^\varepsilon(t)) = e^{Bt+\frac{\int_0^t z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}}\bar{Y}_0 + \int_0^t e^{B(t-s)+\frac{\int_s^t z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}} G(\bar{X}^\varepsilon,\bar{Y}^\varepsilon,\theta_s\omega)\mathrm{d}s, \tag{3.6}$$

for $t \le 0$. Define $J^\varepsilon : C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}} \to C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}$ by means of $J^\varepsilon(\bar{Z}^\varepsilon(\cdot)) = (J_1^\varepsilon(\bar{Z}^\varepsilon(\cdot)), J_2^\varepsilon(\bar{Z}^\varepsilon(\cdot)))$. Firstly, let us show that J^ε maps $C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}$ into itself. Since F, G keep the Lipschitz condition, we have

$$\begin{aligned}
\|J_1^\varepsilon(\bar{Z}^\varepsilon(\cdot))\|_{C^{1,-}_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}} &= \frac{1}{\varepsilon}\sup_{t\in(-\infty,0]}\left\{\int_{-\infty}^t e^{\frac{A(t-s)+\mu t+\int_s^0 z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}} F(\bar{X}^\varepsilon,\bar{Y}^\varepsilon,\theta_s\omega)\mathrm{d}s\right\} \\
&\le \frac{K}{\varepsilon}\sup_{t\in(-\infty,0]}\left\{\int_{-\infty}^t e^{\frac{(-\gamma_1+\mu)(t-s)}{\varepsilon}}\mathrm{d}s\right\}\|\bar{Z}^\varepsilon\|_{C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}} \\
&= \frac{K}{\gamma_1-\mu}\|\bar{Z}^\varepsilon\|_{C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}}
\end{aligned}$$

and

$$\begin{aligned}
\|J_2^\varepsilon(\bar{Z}^\varepsilon(\cdot))\|_{C^{2,-}_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}} &= \sup_{t\in(-\infty,0]}\left\{e^{Bt+\frac{\mu}{\varepsilon}t}\bar{Y}_0 + \int_0^t e^{B(t-s)+\frac{\mu t+\int_s^0 z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}} G(\bar{X}^\varepsilon,\bar{Y}^\varepsilon,\theta_s\omega)\mathrm{d}s\right\} \\
&\le K\sup_{t\in(-\infty,0]}\left\{\int_t^0 e^{(\gamma_2+\frac{\mu}{\varepsilon})(t-s)}\mathrm{d}s\right\}\|\bar{Z}^\varepsilon\|_{C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}} + \|\bar{Y}_0\|_2 \\
&= \frac{\varepsilon K}{\mu+\varepsilon\gamma_2}\|\bar{Z}^\varepsilon\|_{C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}} + \|\bar{Y}_0\|_2.
\end{aligned}$$

Thus, $\|J^\varepsilon(\bar{Z}^\varepsilon(\cdot))\|_{C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}} \le \rho(K,\mu,\gamma_1,\gamma_2,\varepsilon) + \|\bar{Y}_0\|_2$, where $\rho(K,\mu,\gamma_1,\gamma_2,\varepsilon) = \frac{K}{\gamma_1-\mu} + \frac{\varepsilon K}{\mu+\varepsilon\gamma_2}$.

Next, we verify that J^ε is a contractive mapping. Let $Z^\varepsilon(\cdot)$ and $\bar{Z}^\varepsilon(\cdot) \in C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}$. Then

$$\begin{aligned}
&\|J_1^\varepsilon(Z^\varepsilon(\cdot)) - J_1^\varepsilon(\bar{Z}^\varepsilon(\cdot))\|_{C^{1,-}_{-\frac{\mu}{\varepsilon},\frac{z}{\varepsilon}}} \\
&\le \frac{K}{\varepsilon}\sup_{t\in(-\infty,0]}\left\{\int_{-\infty}^t e^{\frac{(-\gamma_1+\mu)(t-s)}{\varepsilon}}(\|X^\varepsilon-\bar{X}^\varepsilon\|_{C^{1,-}_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}} + \|Y^\varepsilon-\bar{Y}^\varepsilon\|_{C^{2,-}_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}})\mathrm{d}s\right\} \\
&\le \frac{K}{\varepsilon}\sup_{t\in(-\infty,0]}\left\{\int_{-\infty}^t e^{\frac{(-\gamma_1+\mu)(t-s)}{\varepsilon}}\mathrm{d}s\right\}\|Z^\varepsilon-\bar{Z}^\varepsilon\|_{C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}} \\
&= \frac{K}{\gamma_1-\mu}\|Z^\varepsilon-\bar{Z}^\varepsilon\|_{C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}}
\end{aligned}$$

and

$$\begin{aligned}
&\|J_2^\varepsilon(Z^\varepsilon(\cdot)) - J_2^\varepsilon(\bar{Z}^\varepsilon(\cdot))\|_{C^{2,-}_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}} \\
&\le K \sup_{t\in(-\infty,0]} \Big\{ \int_0^t e^{(-\gamma_2+\frac{\mu}{\varepsilon})(t-s)} (\|X^\varepsilon - \bar{X}^\varepsilon\|_{C^{1,-}_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}} + \|Y^\varepsilon - \bar{Y}^\varepsilon\|_{C^{2,-}_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}}) ds \Big\} \\
&\le K \sup_{t\in(-\infty,0]} \Big\{ \int_t^0 e^{(\gamma_2+\frac{\mu}{\varepsilon})(t-s)} ds \Big\} \|Z^\varepsilon - \bar{Z}^\varepsilon\|_{C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}} \\
&= \frac{\varepsilon K}{\mu + \varepsilon\gamma_2} \|Z^\varepsilon - \bar{Z}^\varepsilon\|_{C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}}.
\end{aligned}$$

Therefore, $\|J^\varepsilon(Z^\varepsilon(\cdot)) - J^\varepsilon(\bar{Z}^\varepsilon(\cdot))\|_{C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}} \le \rho(K,\mu,\gamma_1,\gamma_2,\varepsilon)\|Z^\varepsilon - \bar{Z}^\varepsilon\|_{C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}}$. Since $\gamma_1 - \mu > K$, we have $\rho < 1$ if $\varepsilon \in (0,\varepsilon_0)$ with $\varepsilon_0 = \frac{(\gamma_1-\mu-K)\mu}{K(\gamma_1-\mu)-(\gamma_1-\mu-K)\gamma_2}$.

Consequently, we use Banach's Fixed Point Theorem to obtain the existence of the unique solution $\bar{Z}^\varepsilon(\cdot) \in C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}$ for (2.4)–(2.5), and the standard a-priori estimate:

$$\|\bar{Z}^\varepsilon(\cdot,\omega,Z_0) - \bar{Z}^\varepsilon(\cdot,\omega,\bar{Z}_0)\|_{C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}} \le \frac{1}{1-\rho(K,\mu,\gamma_1,\gamma_2,\varepsilon)}\|Y_0 - \bar{Y}_0\|_2, \tag{3.7}$$

for all $\omega \in \Omega$, $Y_0, \bar{Y}_0 \in H_2$.

Step 2. Construct the random invariant manifold $M^\varepsilon(\omega)$.

Define

$$H^\varepsilon(\omega, Y_0) := \frac{1}{\varepsilon}\int_{-\infty}^0 e^{\frac{-As - \int_0^s z^\varepsilon(\theta_r\omega)dr}{\varepsilon}} F(\bar{X}^\varepsilon(s,\omega,Y_0), \bar{Y}^\varepsilon(s,\omega,Y_0), \theta_s\omega) ds.$$

Then, the Lipschitz constant of $H^\varepsilon(\omega, Y_0)$ is given by

$$\begin{aligned}
\|H^\varepsilon(\omega,Y_0) - H^\varepsilon(\omega,\bar{Y}_0)\|_1 &\le \frac{K}{\gamma_1-\mu}\|\bar{Z}^\varepsilon(\cdot,\omega,Z_0) - \bar{Z}^\varepsilon(\cdot,\omega,\bar{Z}_0)\|_{C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}} \\
&\le \frac{K}{(\gamma_1-\mu)\Big[1 - K\Big(\frac{1}{\gamma_1-\mu} + \frac{\varepsilon}{\mu+\varepsilon\gamma_2}\Big)\Big]}\|Y_0 - \bar{Y}_0\|_2
\end{aligned}$$

for all $\omega \in \Omega$, $Y_0, \bar{Y}_0 \in H_2$. Lemma 3.1 yields that $M^\varepsilon(\omega) = \{(H^\varepsilon(\omega,Y_0), Y_0) | Y_0 \in H_2\}$.

Step 3. We need to prove $M^\varepsilon(\omega)$ is a random set. To this end, we show that

$$\omega \to \inf_{z'\in H} \|(x,y) - (H^\varepsilon(\omega,\mathcal{P}z'),\mathcal{P}z')\| \tag{3.8}$$

is measurable, where $\mathcal{P}$ is the projection from H to H_2. Let H_c be a countable dense set of the separable space H. The continuity of $H^\varepsilon(\omega,\cdot)$ yields that the right-hand side of (3.8) is equivalent to

$$\omega \to \inf_{z'\in H_c} \|(x,y) - (H^\varepsilon(\omega,\mathcal{P}z'),\mathcal{P}z')\|. \tag{3.9}$$

Since $\omega \to H^\varepsilon(\omega,\mathcal{P}z')$ is measurable, we obtain that measurability of any expression under the infimum of (3.9). Then the fact that M^ε is a random set follows from Theorem III.9 in [8].

Step 4. We are going to show that M^ε is positively invariant which means that for every $\bar{Z}_0 = (\bar{X}_0, \bar{Y}_0) \in M^\varepsilon(\omega)$, $\bar{Z}(s,\omega,\bar{Z}_0) \in M^\varepsilon(\theta_s\omega)$ for all $s \ge 0$. For every fixed $s \ge 0$, we claim that $\bar{Z}(t+s,\omega,\bar{Z}_0)$ is the solution of

$$d\bar{X}^\varepsilon = \left[\frac{A}{\varepsilon}\bar{X}^\varepsilon + \frac{z^\varepsilon(\theta_t(\theta_s\omega))\bar{X}^\varepsilon}{\varepsilon} + \frac{F(\bar{X},\bar{Y},\theta_t(\theta_s\omega))}{\varepsilon}\right]dt,$$
$$d\bar{Y}^\varepsilon = [B\bar{Y}^\varepsilon + \frac{z^\varepsilon(\theta_t(\theta_s\omega))\bar{Y}^\varepsilon}{\varepsilon} + G(\bar{X}^\varepsilon,\bar{Y}^\varepsilon,\theta_t(\theta_s\omega))]dt$$

with initial value $\bar{Z}(0) = \bar{Z}(s,\omega,\bar{Z}_0)$. Then, $\bar{Z}(t+s,\omega,\bar{Z}_0) = \bar{Z}(t,\theta_s\omega,\bar{Z}(s,\omega,\bar{Z}_0))$. Since $\bar{Z}(\cdot,\omega,\bar{Z}_0) \in C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}$, we get $\bar{Z}(\cdot,\theta_s\omega,\bar{Z}(s,\omega,\bar{Z}_0)) \in C^-_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}$. Thus $\bar{Z}(s,\omega,\bar{Z}_0) \in M^\varepsilon(\theta_s\omega)$. The proof is completed. □

Next, we want to show that the Lipschitz invariant manifolds for (2.4)–(2.5) given in (3.3) has exponential tracking property

Theorem 3.3 (Exponential tracking property in forward time)**.** *Under Assumptions 1, 2, 4, for sufficiently small $\varepsilon > 0$, there exists a positive constant C and a random process $D(t,\omega)$ such that for any $\check{Z}_0 = (\check{X}_0,\check{Y}_0) \in H$, there exists $\bar{Z}_0 = (\bar{X}_0,\bar{Y}_0) \in M^\varepsilon(\omega)$ satisfying that*

$$\|\bar{Z}^\varepsilon(t,\omega,\check{Z}_0) - \bar{Z}^\varepsilon(t,\omega,\bar{Z}_0)\| \le D(t,\omega)e^{-Ct}\|\check{Z}_0 - \bar{Z}_0\|, \quad t \ge 0.$$

Proof. Suppose that $\check{Z}^\varepsilon(t) = (\check{X}^\varepsilon(t),\check{Y}^\varepsilon(t))$ and $\bar{Z}^\varepsilon(t) = (\bar{X}^\varepsilon(t),\bar{Y}^\varepsilon(t))$ are two solutions of (2.4)–(2.5) with initial data $\check{Z}^\varepsilon(0) = \check{Z}_0$ and $\bar{Z}^\varepsilon(0) = \bar{Z}_0$, respectively. Then $\widetilde{Z}^\varepsilon(t) = \check{Z}^\varepsilon(t) - \bar{Z}^\varepsilon(t) = (\widetilde{X}^\varepsilon(t),\widetilde{Y}^\varepsilon(t))$ satisfies that the following system:

$$d\widetilde{X}^\varepsilon = \left[\frac{A}{\varepsilon}\widetilde{X}^\varepsilon + \frac{z^\varepsilon(\theta_t\omega)\widetilde{X}^\varepsilon}{\varepsilon} + \frac{\widetilde{F}(\widetilde{X}^\varepsilon,\widetilde{Y}^\varepsilon,\theta_t\omega)}{\varepsilon}\right]dt, \tag{3.10}$$

$$d\widetilde{Y}^\varepsilon = [B\widetilde{Y}^\varepsilon + \frac{z^\varepsilon(\theta_t\omega)\widetilde{Y}^\varepsilon}{\varepsilon} + \widetilde{G}(\widetilde{X}^\varepsilon,\widetilde{Y}^\varepsilon,\theta_t\omega)]dt, \tag{3.11}$$

where

$$\widetilde{F}(\widetilde{X}^\varepsilon,\widetilde{Y}^\varepsilon,\theta_t\omega) = F(\widetilde{X}^\varepsilon+\bar{X}^\varepsilon,\widetilde{Y}^\varepsilon+\bar{Y}^\varepsilon,\theta_t\omega) - F(\bar{X}^\varepsilon,\bar{Y}^\varepsilon,\theta_t\omega), \tag{3.12}$$
$$\widetilde{G}(\widetilde{X}^\varepsilon,\widetilde{Y}^\varepsilon,\theta_t\omega) = G(\widetilde{X}^\varepsilon+\bar{X}^\varepsilon,\widetilde{Y}^\varepsilon+\bar{Y}^\varepsilon,\theta_t\omega) - G(\bar{X}^\varepsilon,\bar{Y}^\varepsilon,\theta_t\omega). \tag{3.13}$$

Acoording to the variation of constants formula, we state that $\widetilde{Z}^\varepsilon(\cdot) = (\widetilde{X}^\varepsilon(\cdot),\widetilde{Y}^\varepsilon(\cdot))$ with initial value $\widetilde{Z}^\varepsilon(0) = \check{Z}_0 - \bar{Z}_0 = (\widetilde{X}^\varepsilon(0),\widetilde{Y}^\varepsilon(0))$ is the solution in $C^+_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}$ of (3.10)–(3.11) if and only if

$$\widetilde{X}^\varepsilon(t) = e^{\frac{At+\int_0^t z^\varepsilon(\theta_r\omega)dr}{\varepsilon}}\widetilde{X}^\varepsilon(0) + \frac{1}{\varepsilon}\int_0^t e^{\frac{A(t-s)+\int_s^t z^\varepsilon(\theta_r\omega)dr}{\varepsilon}}\widetilde{F}(\widetilde{X}^\varepsilon,\widetilde{Y}^\varepsilon,\theta_s\omega)ds, \tag{3.14}$$
$$\widetilde{Y}^\varepsilon(t) = \int_{+\infty}^t e^{B(t-s)+\frac{\int_s^t z^\varepsilon(\theta_r\omega)dr}{\varepsilon}}\widetilde{G}(\widetilde{X}^\varepsilon,\widetilde{Y}^\varepsilon,\theta_s\omega)ds. \tag{3.15}$$

Now, let us show that there exists a unique solution $\widetilde{Z}^\varepsilon(\cdot) = (\widetilde{X}^\varepsilon(\cdot),\widetilde{Y}^\varepsilon(\cdot))$ in $C^+_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}$ with initial value $(\widetilde{X}^\varepsilon(0),\widetilde{Y}^\varepsilon(0))$ such that

$$(\bar{X}_0,\bar{Y}_0) = (\widetilde{X}^\varepsilon(0),\widetilde{Y}^\varepsilon(0)) + (\check{X}_0,\check{Y}_0) \in M^\varepsilon(\omega). \tag{3.16}$$

It follows from Theorem 3.2 that

$$\begin{aligned}\widetilde{X}^\varepsilon(0) &= -\check{X}_0 + H^\varepsilon(\omega,\widetilde{Y}^\varepsilon(0)+\check{Y}_0)\\ &= -\check{X}_0 + \frac{1}{\varepsilon}\int_{-\infty}^0 e^{\frac{-As-\int_0^s z^\varepsilon(\theta_r\omega)dr}{\varepsilon}}F(\bar{X}^\varepsilon(s,\widetilde{Y}^\varepsilon(0)+\check{Y}_0),\bar{Y}^\varepsilon(s,\widetilde{Y}^\varepsilon(0)+\check{Y}_0),\theta_s\omega)ds.\end{aligned}$$

Let $\widetilde{Z}^\varepsilon(\cdot) = (\widetilde{X}^\varepsilon(\cdot), \widetilde{Y}^\varepsilon(\cdot)) \in C^+_{-\frac{\mu}{\varepsilon}, \frac{z^\varepsilon}{\varepsilon}}$. For $t \geq 0$, define two operators $C^+_{-\frac{\mu}{\varepsilon}, \frac{z^\varepsilon}{\varepsilon}} \to C^{1,+}_{-\frac{\mu}{\varepsilon}, \frac{z^\varepsilon}{\varepsilon}}$ and $C^+_{-\frac{\mu}{\varepsilon}, \frac{z^\varepsilon}{\varepsilon}} \to C^{2,+}_{-\frac{\mu}{\varepsilon}, \frac{z^\varepsilon}{\varepsilon}}$ by means of

$$
\begin{aligned}
\mathcal{J}_1^\varepsilon(\widetilde{Z}^\varepsilon(t)) &= e^{\frac{At + \int_0^t z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}} \widetilde{X}^\varepsilon(0) + \frac{1}{\varepsilon} \int_0^t e^{\frac{A(t-s) + \int_s^t z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}} F(\widetilde{X}^\varepsilon, \widetilde{Y}^\varepsilon, \theta_s\omega)\mathrm{d}s, \\
\mathcal{J}_2^\varepsilon(\widetilde{Z}^\varepsilon(t)) &= \int_{+\infty}^t e^{B(t-s) + \frac{\int_s^t z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}} G(\widetilde{X}^\varepsilon, \widetilde{Y}^\varepsilon, \theta_s\omega)\mathrm{d}s,
\end{aligned}
$$

where $\widetilde{X}^\varepsilon(0)$ is from (3.16). Furthermore, define $\mathcal{J}^\varepsilon : C^+_{-\frac{\mu}{\varepsilon}, \frac{z^\varepsilon}{\varepsilon}} \to C^+_{-\frac{\mu}{\varepsilon}, \frac{z^\varepsilon}{\varepsilon}}$ as

$$
\mathcal{J}^\varepsilon(\widetilde{Z}^\varepsilon(\cdot)) = (\mathcal{J}_1^\varepsilon(\widetilde{Z}^\varepsilon(\cdot)), \mathcal{J}_2^\varepsilon(\widetilde{Z}^\varepsilon(\cdot))).
$$

As the proof of Theorem 3.2, we apply Banach's Fixed Point Theorem to (3.14)–(3.15). Obviously, $\mathcal{J}^\varepsilon$ is self-map. It remains to show that $\mathcal{J}^\varepsilon$ is contractive. Note

$$
\begin{aligned}
&\|e^{\frac{A\cdot + \int_0^\cdot z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}} (\widetilde{X}_1^\varepsilon(0) - \widetilde{X}_2^\varepsilon(0))\|_{C^{1,+}_{-\frac{\mu}{\varepsilon}, \frac{z^\varepsilon}{\varepsilon}}} \\
&\quad \leq e^{\frac{(-\gamma_1+\mu)t}{\varepsilon}} \operatorname{Lip} H^\varepsilon \|\widetilde{Y}_1^\varepsilon(0) - \widetilde{Y}_2^\varepsilon(0))\|_2 \\
&\quad \leq e^{\frac{(-\gamma_1+\mu)t}{\varepsilon}} \operatorname{Lip} H^\varepsilon \| \int_{+\infty}^0 e^{-Bs + \frac{\int_s^t z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}} \big(\widetilde{G}(\widetilde{X}_1^\varepsilon, \widetilde{Y}_1^\varepsilon, \theta_s\omega) - \widetilde{G}(\widetilde{X}_2^\varepsilon, \widetilde{Y}_2^\varepsilon, \theta_s\omega)\big)\mathrm{d}s\|_2 \\
&\quad \leq e^{\frac{(-\gamma_1+\mu)t}{\varepsilon}} \operatorname{Lip} H^\varepsilon \cdot K \|\widetilde{Z}_1^\varepsilon(\cdot) - \widetilde{Z}_2^\varepsilon(\cdot)\|_{C^+_{-\frac{\mu}{\varepsilon}, \frac{z^\varepsilon}{\varepsilon}}} \int_0^{+\infty} e^{(-\gamma_2 - \frac{\mu}{\varepsilon})s}\mathrm{d}s.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\|\mathcal{J}_1^\varepsilon(\widetilde{Z}_1^\varepsilon - \widetilde{Z}_2^\varepsilon)\|_{C^{1,+}_{\frac{\mu}{\varepsilon}, \frac{z^\varepsilon}{\varepsilon}}} &\leq \operatorname{Lip} H^\varepsilon \cdot K \|\widetilde{Z}_1^\varepsilon(\cdot) - \widetilde{Z}_2^\varepsilon(\cdot)\|_{C^+_{-\frac{\mu}{\varepsilon}, \frac{z^\varepsilon}{\varepsilon}}} \sup_{t\geq 0} \left\{ e^{\frac{(-\gamma_1+\mu)t}{\varepsilon}} \int_0^{+\infty} e^{(-\gamma_2 - \frac{\mu}{\varepsilon})s}\mathrm{d}s \right\} \\
&\quad + \frac{K}{\varepsilon} \|\widetilde{Z}_1^\varepsilon(\cdot) - \widetilde{Z}_2^\varepsilon(\cdot)\|_{C^+_{-\frac{\mu}{\varepsilon}, \frac{z^\varepsilon}{\varepsilon}}} \sup_{t\geq 0} \left\{ \int_0^t e^{\frac{(-\gamma_1+\mu)(t-s)}{\varepsilon}} \mathrm{d}s \right\} \\
&\leq \Big(\frac{\operatorname{Lip} H^\varepsilon \cdot \varepsilon K}{\mu + \varepsilon\gamma_2} + \frac{K}{\gamma_1 - \mu} \Big) \|\widetilde{Z}_1^\varepsilon(\cdot) - \widetilde{Z}_2^\varepsilon(\cdot)\|_{C^+_{-\frac{\mu}{\varepsilon}, \frac{z^\varepsilon}{\varepsilon}}},
\end{aligned}
$$

and

$$
\begin{aligned}
\|\mathcal{J}_2^\varepsilon(\widetilde{Z}_1^\varepsilon - \widetilde{Z}_2^\varepsilon)\|_{C^{2,+}_{\frac{\mu}{\varepsilon}, \frac{z}{\varepsilon}}} &\leq K \|\widetilde{Z}_1^\varepsilon(\cdot) - \widetilde{Z}_2^\varepsilon(\cdot)\|_{C^+_{-\frac{\mu}{\varepsilon}, \frac{z^\varepsilon}{\varepsilon}}} \sup_{t\geq 0} \left\{ \int_t^{+\infty} e^{(-\gamma_2 - \frac{\mu}{\varepsilon})(t-s)}\mathrm{d}s \right\} \\
&\leq \frac{\varepsilon K}{\mu + \varepsilon\gamma_2} \|\widetilde{Z}_1^\varepsilon(\cdot) - \widetilde{Z}_2^\varepsilon(\cdot)\|_{C^+_{-\frac{\mu}{\varepsilon}, \frac{z^\varepsilon}{\varepsilon}}}.
\end{aligned}
$$

By (3.4), we further obtain

$$
\|\mathcal{J}^\varepsilon(\widetilde{Z}_1^\varepsilon(\cdot)) - \mathcal{J}^\varepsilon(\widetilde{Z}_2^\varepsilon(\cdot))\|_{C^+_{-\frac{\mu}{\varepsilon}, \frac{z^\varepsilon}{\varepsilon}}} \leq \beta(K, \gamma_1, \gamma_2, \mu) \|\widetilde{Z}_1^\varepsilon(\cdot) - \widetilde{Z}_2^\varepsilon(\cdot)\|_{C^+_{-\frac{\mu}{\varepsilon}, \frac{z^\varepsilon}{\varepsilon}}},
$$

where

$$
\beta(K, \gamma_1, \gamma_2, \mu) = \frac{\varepsilon K^2}{(\gamma_1 - \mu)(\mu + \varepsilon\gamma_2)[1 - K(\frac{1}{\gamma_1 - \mu} + \frac{\varepsilon}{\mu + \varepsilon\gamma_2})]} + K(\frac{1}{\gamma_1 - \mu} + \frac{\varepsilon}{\mu + \varepsilon\gamma_2}).
$$

Note $\beta(K,\gamma_1,\gamma_2,\mu)<1$ if ε is sufficiently small. Then there exists a unique solution $\widetilde{Z}^\varepsilon(\cdot)$ for (3.14)–(3.15) in $C^+_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}$.

Furthermore,

$$\|\widetilde{Z}^\varepsilon(\cdot)\|_{C^+_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}} \le \|\widetilde{Z}^\varepsilon(0)\| + K\Big(\frac{1}{\gamma_1-\mu}+\frac{\varepsilon}{\mu+\varepsilon\gamma_2}\Big)\|\widetilde{Z}^\varepsilon(\cdot)\|_{C^+_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}},$$

which implies that

$$\|\widetilde{Z}^\varepsilon(\cdot)\|_{C^+_{-\frac{\mu}{\varepsilon},\frac{z^\varepsilon}{\varepsilon}}} \le \frac{1}{1-K\left(\frac{1}{\gamma_1-\mu}+\frac{\varepsilon}{\mu+\varepsilon\gamma_2}\right)}\|\widetilde{Z}^\varepsilon(0)\|.$$

Thus, we obtain

$$\|\bar{Z}^\varepsilon(t,\omega,\check{Z}_0)-\bar{Z}^\varepsilon(t,\omega,\bar{Z}_0)\| \le \frac{e^{\frac{\int_0^t z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}}}{1-K\left(\frac{1}{\gamma_1-\mu}+\frac{\varepsilon}{\mu+\varepsilon\gamma_2}\right)}e^{-\frac{\mu}{\varepsilon}t}\|\check{Z}_0-\bar{Z}_0\|,\quad t\ge 0.$$

The proof is completed. □

3.2 Slow manifolds

In this subsection, we are going to present the approximation of $M^\varepsilon(\omega)$ in slow time-scale $T=\frac{t}{\varepsilon}$. Scaling $t=\varepsilon T$ in (2.4)–(2.5), we have

$$\mathrm{d}\bar{X}^\varepsilon = [A\bar{X}^\varepsilon + z^\varepsilon(\theta_{\varepsilon T}\omega)\bar{X}^\varepsilon + \bar{F}(\bar{X}^\varepsilon,\bar{Y}^\varepsilon,\theta_{\varepsilon T}\omega)]\mathrm{d}T, \tag{3.17}$$
$$\mathrm{d}\bar{Y}^\varepsilon = [\varepsilon B\bar{Y}^\varepsilon + z^\varepsilon(\theta_{\varepsilon T}\omega)\bar{Y}^\varepsilon + \varepsilon\bar{G}(\bar{X}^\varepsilon,\bar{Y}^\varepsilon,\theta_{\varepsilon T}\omega)]\mathrm{d}T. \tag{3.18}$$

Let $\eta(\theta_T\omega)$ be the stationary solution of

$$\mathrm{d}\eta = -\eta\mathrm{d}T + \mathrm{d}\widetilde{W}(T), \tag{3.19}$$

where $\widetilde{W}(T)$ and $\varepsilon^{-\frac{1}{2}}W(\varepsilon T)$ are identical in distribution. Replacing $z^\varepsilon(\theta_{\varepsilon T}\omega)$ by $\eta(\theta_T\omega)$ in (3.17)–(3.18), we have

$$\mathrm{d}\widehat{X}^\varepsilon = [A\widehat{X}^\varepsilon + \eta(\theta_T\omega)\widehat{X}^\varepsilon + \widehat{F}(\widehat{X}^\varepsilon,\widehat{Y}^\varepsilon,\theta_T\omega)]\mathrm{d}T, \tag{3.20}$$
$$\mathrm{d}\widehat{Y}^\varepsilon = [\varepsilon B\widehat{Y}^\varepsilon + \eta(\theta_T\omega)\widehat{Y}^\varepsilon + \varepsilon\widehat{G}(\widehat{X}^\varepsilon,\widehat{Y}^\varepsilon,\theta_T\omega)]\mathrm{d}T, \tag{3.21}$$

where

$$\widehat{F}(\widehat{X}^\varepsilon,\widehat{Y}^\varepsilon,\theta_T\omega) = e^{-\eta(\theta_T\omega)}f(e^{\eta(\theta_T\omega)}\widehat{X}^\varepsilon, e^{\eta(\theta_T\omega)}\widehat{Y}^\varepsilon),$$
$$\widehat{G}(\widehat{X}^\varepsilon,\widehat{Y}^\varepsilon,\theta_T\omega) = e^{-\eta(\theta_T\omega)}g(e^{\eta(\theta_T\omega)}\widehat{X}^\varepsilon, e^{\eta(\theta_T\omega)}\widehat{Y}^\varepsilon).$$

Since $z^\varepsilon(\theta_{\varepsilon T}\omega)$ is the same as $\eta(\theta_T\omega)$ in distribution (please see Lemma 3.2 in [31]), the distribution of the solution (3.17)–(3.18) coincides with that of (3.20)–(3.21). Using the similar procedure as the proof of Theorem 3.2, we obtain that (3.20)–(3.21) has a random invariant manifold $\widehat{M}^\varepsilon(\omega)$ represented as

$$\widehat{M}^\varepsilon(\omega) = \{(\widehat{H}^\varepsilon(\omega,\widehat{Y}_0),\widehat{Y}_0)|\widehat{Y}_0\in H_2\}$$

with

$$\widehat{H}^\varepsilon(\omega,\widehat{Y}_0)=\int_{-\infty}^0 e^{-As-\int_0^s\eta(\theta_r\omega)\mathrm{d}r}\widehat{F}(\widehat{X}^\varepsilon,\widehat{Y}^\varepsilon,\theta_s\omega)\mathrm{d}s,$$

where

$$\begin{aligned}\widehat{X}^\varepsilon(T)&=\int_{-\infty}^T e^{A(T-s)+\int_s^T\eta(\theta_r\omega)\mathrm{d}r}\widehat{F}(\widehat{X}^\varepsilon,\widehat{Y}^\varepsilon,\theta_s\omega)\mathrm{d}s,\\ \widehat{Y}^\varepsilon(T)&=e^{\varepsilon BT+\int_0^T\eta(\theta_r\omega)\mathrm{d}r}\widehat{Y}_0+\varepsilon\int_0^T e^{\varepsilon B(T-s)+\int_s^T\eta(\theta_r\omega)\mathrm{d}r}\widehat{G}(\widehat{X}^\varepsilon,\widehat{Y}^\varepsilon,\theta_s\omega)\mathrm{d}s.\end{aligned}$$

Then, for fixed $\widehat{Y}_0\in H_2$, $H^\varepsilon(\omega,\widehat{Y}_0)=\widehat{H}^\varepsilon(\omega,\widehat{Y}_0)$ in distribution. In fact,

$$\begin{aligned}H^\varepsilon(\omega,\widehat{Y}_0)&=\frac{1}{\varepsilon}\int_{-\infty}^0 e^{\frac{-As-\int_0^s z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}}F(\bar{X}(s,\omega,\widehat{Y}_0),\bar{Y}(s,\omega,\widehat{Y}_0),\theta_s\omega)\mathrm{d}s\\ &=\int_{-\infty}^0 e^{-As-\int_0^s z^\varepsilon(\theta_{\varepsilon r}\omega)\mathrm{d}r}F(\bar{X}(\varepsilon s,\omega,\widehat{Y}_0),\bar{Y}(\varepsilon s,\omega,\widehat{Y}_0),\theta_{\varepsilon s}\omega)\mathrm{d}s\\ &\overset{d}{=}\int_{-\infty}^0 e^{-As-\int_0^s \eta^\varepsilon(\theta_r\omega)\mathrm{d}r}\widehat{F}(\widehat{X}(s,\omega,\widehat{Y}_0),\widehat{Y}(s,\omega,\widehat{Y}_0),\theta_s\omega)\mathrm{d}s\\ &=\widehat{H}^\varepsilon(\omega,\widehat{Y}_0),\end{aligned}\tag{3.22}$$

where $\overset{d}{=}$ denotes the equivalence in distribution.

We proceed to want to explore the approximation form of the invariant manifold $\widehat{M}^\varepsilon(\omega)$ as $\varepsilon\to 0$. To achieve it, we observe (3.20)–(3.21) when $\varepsilon=0$.

Consider

$$\mathrm{d}\widehat{X}^0=[A\widehat{X}^0+\eta(\theta_T\omega)\widehat{X}^0+\widehat{F}(\widehat{X}^0,\widehat{Y}^0,\theta_T\omega)]\mathrm{d}T,\tag{3.23}$$

$$\mathrm{d}\widehat{Y}^0=\eta(\theta_T\omega)\widehat{Y}^0\mathrm{d}T.\tag{3.24}$$

We comment that (3.23)–(3.24) is the critical system of (3.20)–(3.21). It is clear that there exists a random invariant manifold $\widehat{M}^0(\omega)$ for (3.23)–(3.24) which is

$$\widehat{M}^0(\omega)=\{(\widehat{H}^0(\omega,\widehat{Y}_0),\widehat{Y}_0)|\widehat{Y}_0\in H_2\},\tag{3.25}$$

where

$$\widehat{H}^0(\omega,\widehat{Y}_0)=\int_{-\infty}^0 e^{-As-\int_0^s\eta(\theta_r\omega)\mathrm{d}r}\widehat{F}(\widehat{X}^0,e^{\int_0^s\eta(\theta_r\omega)\mathrm{d}r}\widehat{Y}_0,\theta_s\omega)\mathrm{d}s.\tag{3.26}$$

Furthermore, using the idea coming from Theorem 5.1 in [17], we state that $\widehat{H}^\varepsilon(\omega,\widehat{Y}_0)=\widehat{H}^0(\omega,\widehat{Y}_0)+\mathcal{O}(\varepsilon)$, for $\omega\in\Omega$.

Theorem 3.4. *Under Assumptions 1, 2, 4, for sufficiently small $\varepsilon>0$, we have*

$$\|\widehat{H}^\varepsilon(\omega,\widehat{Y}_0)-\widehat{H}^0(\omega,\widehat{Y}_0)\|_1=\mathcal{O}(\varepsilon),$$

for all $\widehat{Y}_0\in\mathcal{D}(B)$, $\omega\in\Omega$.*

Proof. For $T\le 0$ and $\widehat{Y}_0\in\mathcal{D}(B)$,

$$\begin{aligned}\|e^{\varepsilon BT+\int_0^T\eta(\theta_r\omega)\mathrm{d}r}\widehat{Y}_0-e^{\int_0^T\eta(\theta_r\omega)\mathrm{d}r}\widehat{Y}_0\|_2&\le e^{\int_0^T\eta(\theta_r\omega)\mathrm{d}r}\|\int_{\varepsilon T}^0 e^{Bs}B\widehat{Y}_0\mathrm{d}s\|_2\\ &\le e^{\int_0^T\eta(\theta_r\omega)\mathrm{d}r}\|B\widehat{Y}_0\|_2\frac{1-e^{\gamma_2\varepsilon T}}{\gamma_2}.\end{aligned}$$

*$\mathcal{D}(B)$ means the domain of operator B

Then,

$$\begin{aligned}
&\|\widehat{Y}^{\varepsilon}(T)-\widehat{Y}^{0}(T)\|_2\\
&\quad\le \|e^{\varepsilon BT+\int_0^T\eta(\theta_r\omega)\mathrm{d}r}\widehat{Y}_0-e^{\int_0^T\eta(\theta_r\omega)\mathrm{d}r}\widehat{Y}_0\|_2\\
&\qquad+\|\varepsilon\int_0^T e^{\varepsilon B(T-s)+\int_s^T\eta(\theta_r\omega)\mathrm{d}r}\widehat{G}(\widehat{X}^{\varepsilon},\widehat{Y}^{\varepsilon},\theta_s\omega)\mathrm{d}s\|_2\\
&\quad\le e^{\int_0^T\eta(\theta_r\omega)\mathrm{d}r}\|B\widehat{Y}_0\|_2\frac{1-e^{\gamma_2\varepsilon T}}{\gamma_2}\\
&\qquad+\varepsilon K\int_T^0 e^{\varepsilon\gamma_2(T-s)-\mu s+\int_0^T\eta(\theta_r\omega)\mathrm{d}r}\big(\|\widehat{X}^{\varepsilon}\|_{C^{1,-}_{-\mu,\eta}}+\|\widehat{Y}^{\varepsilon}\|_{C^{2,-}_{-\mu,\eta}}\big)\mathrm{d}s\\
&\quad\le e^{\int_0^T\eta(\theta_r\omega)\mathrm{d}r}\|B\widehat{Y}_0\|_2\frac{1-e^{\gamma_2\varepsilon T}}{\gamma_2}+\frac{\varepsilon K\|\widehat{Y}_0\|_2}{1-\rho}\int_T^0 e^{\varepsilon\gamma_2(T-s)-\mu s+\int_0^T\eta(\theta_r\omega)\mathrm{d}r}\mathrm{d}s,\\
&\quad\le e^{\int_0^T\eta(\theta_r\omega)\mathrm{d}r}\|B\widehat{Y}_0\|_2\frac{1-e^{\gamma_2\varepsilon T}}{\gamma_2}+e^{\int_0^T\eta(\theta_r\omega)\mathrm{d}r}\frac{\varepsilon K\|\widehat{Y}_0\|_2}{1-\rho}\frac{e^{-\mu T}-e^{\varepsilon\gamma_2 T}}{\mu+\varepsilon\gamma_2},
\end{aligned}\tag{3.27}$$

where we use the estimation (3.7) in the third inequality. By (3.27), we obtain

$$\begin{aligned}
&\|\widehat{X}^{\varepsilon}(\cdot)-\widehat{X}^{0}(\cdot)\|_{C^{1,-}_{-\mu,\eta}}\\
&\quad\le K\|\widehat{X}^{\varepsilon}(\cdot)-\widehat{X}^{0}(\cdot)\|_{C^{1,-}_{-\mu,\eta}}\sup_{T\le 0}\Big\{\int_{-\infty}^T e^{-(\gamma_1-\mu)(T-s)}\mathrm{d}s\Big\}\\
&\qquad+K\sup_{T\le 0}\Big\{\int_{-\infty}^T e^{-\gamma_1(T-s)+\mu T+\int_s^0\eta(\theta_r\omega)\mathrm{d}r}\|\widehat{Y}^{\varepsilon}-\widehat{Y}^{0}\|_2\mathrm{d}s\Big\}\\
&\quad\le \frac{K}{\gamma_1-\mu}\|\widehat{X}^{\varepsilon}(\cdot)-\widehat{X}^{0}(\cdot)\|_{C^{1,-}_{-\mu,\eta}}+\mathcal{R},
\end{aligned}\tag{3.28}$$

where

$$\begin{aligned}
\mathcal{R}&=\sup_{T\le 0}\Bigg\{Ke^{\mu T}\Big(\frac{\|B\widehat{Y}_0\|_2}{\gamma_1\gamma_2}-\frac{\|B\widehat{Y}_0\|_2e^{\gamma_2\varepsilon T}}{(\gamma_1+\gamma_2\varepsilon)\gamma_2}-\frac{\varepsilon K\|\widehat{Y}_0\|_2e^{\gamma_2\varepsilon T}}{(1-\rho)(\mu+\varepsilon\gamma_2)(\gamma_1+\gamma_2\varepsilon)}\Big)\\
&\qquad+\frac{\varepsilon K^2\|\widehat{Y}_0\|_2}{(1-\rho)(\mu+\varepsilon\gamma_2)(\gamma_1-\mu)}\Bigg\}\\
&:=\sup_{T\le 0}\Sigma(T).
\end{aligned}$$

Note that there exists $T_{sup}<0$ such that $\frac{\mathrm{d}\Sigma(T)}{\mathrm{d}T}|_{T=T_{sup}}=0$, which implies that $\mathcal{R}=\Sigma(T_{sup})=\mathcal{O}(\varepsilon)$. Then we have

$$\|\widehat{X}^{\varepsilon}(\cdot)-\widehat{X}^{0}(\cdot)\|_{C^{1,-}_{-\mu,\eta}}\le\frac{(\gamma_1-\mu)\mathcal{R}}{\gamma_1-\mu-K}.$$

Therefore,

$$
\begin{aligned}
&\|\widehat{H}^{\varepsilon}(\omega,\widehat{Y}_0)-\widehat{H}^{0}(\omega,\widehat{Y}_0)\|_1\\
&\leq K\int_{-\infty}^{0} e^{(\gamma_1-\mu)s}\|\widehat{X}^{\varepsilon}-\widehat{X}^{0}\|_{C^{1,-}_{-\mu,\eta}}\mathrm{d}s+K\int_{-\infty}^{0} e^{\gamma_1 s-\int_0^s\eta(\theta_r\omega)\mathrm{d}r}\|\check{Y}^{\varepsilon}-\check{Y}^{0}\|_2\mathrm{d}s\\
&\leq \frac{K\mathcal{R}}{\gamma_1-\mu-K}+\frac{\varepsilon K^2\|\widehat{Y}_0\|_2}{(1-\rho)(\mu+\varepsilon\gamma_2)(\gamma_1-\mu)}+\frac{K\|B\widehat{Y}_0\|_2}{\gamma_1\gamma_2}\\
&\quad-\frac{K\|B\widehat{Y}_0\|_2}{(\gamma_1+\gamma_2\varepsilon)\gamma_2}-\frac{\varepsilon K^2\|\widehat{Y}_0\|_2}{(1-\rho)(\mu+\varepsilon\gamma_2)(\gamma_1+\gamma_2\varepsilon)}\\
&=\mathcal{O}(\varepsilon).
\end{aligned}
$$

□

We now show the better approximation of slow manifolds. According to Assumption 3, we know $\widehat{F}(x,y)$ has the partial derivatives. Let

$$
\begin{aligned}
\widehat{X}^{\varepsilon}(T)&=\widehat{X}^{0}(T)+\varepsilon\widehat{X}^{1}(T)+\varepsilon^2\widehat{X}^{2}(T)+\cdots,\\
\widehat{Y}^{\varepsilon}(T)&=\widehat{Y}^{0}(T)+\varepsilon\widehat{Y}^{1}(T)+\varepsilon^2\widehat{Y}^{2}(T)+\cdots.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
\widehat{F}(\widehat{X}^{\varepsilon},\widehat{Y}^{\varepsilon},\theta_T\omega)&=\widehat{F}(\widehat{X}^{0},\widehat{Y}^{0},\theta_T\omega)+\varepsilon\widehat{F}_x(\widehat{X}^{0},\widehat{Y}^{0},\theta_T\omega)\widehat{X}^{1}\\
&\quad+\varepsilon\widehat{F}_y(\widehat{X}^{0},\widehat{Y}^{0},\theta_T\omega)\widehat{Y}^{1}+\mathcal{O}(\varepsilon^2),\\
\varepsilon\widehat{G}(\widehat{X}^{\varepsilon},\widehat{Y}^{\varepsilon},\theta_T\omega)&=\varepsilon\widehat{G}(\widehat{X}^{0},\widehat{Y}^{0},\theta_T\omega)+\mathcal{O}(\varepsilon^2),
\end{aligned}
$$

where $\widehat{F}_x$ and $\widehat{F}_y$ are the partial derivatives of $\widehat{F}(x,y)$ with respect to x and y, respectively. Equating the same degree of ε, we have

$$
\mathrm{d}\widehat{X}^{0}=[A\widehat{X}^{0}+\eta(\theta_T\omega)\widehat{X}^{0}+\widehat{F}(\widehat{X}^{0},\widehat{Y}^{0},\theta_T\omega)]\mathrm{d}T, \tag{3.29}
$$

$$
\mathrm{d}\widehat{Y}^{0}=\eta(\theta_T\omega)\widehat{Y}^{0}\mathrm{d}T, \tag{3.30}
$$

and

$$
\mathrm{d}\widehat{X}^{1}=[A\widehat{X}^{1}+\eta(\theta_T\omega)\widehat{X}^{1}+\widehat{F}_x(\widehat{X}^{0},\widehat{Y}^{0},\theta_T\omega)\widehat{X}^{1}+\widehat{F}_y(\widehat{X}^{0},\widehat{Y}^{0},\theta_T\omega)\widehat{Y}^{1}]\mathrm{d}T, \tag{3.31}
$$

$$
\mathrm{d}\widehat{Y}^{1}=[B\widehat{Y}^{0}+\eta(\theta_T\omega)\widehat{Y}^{1}+\widehat{G}(\widehat{X}^{0},\widehat{Y}^{0},\theta_T\omega)]\mathrm{d}T. \tag{3.32}
$$

We note (3.29)–(3.30) is the same as (3.23)–(3.24). Hence, (3.29)–(3.30) has a random invariant manifold $\widehat{M}^0(\omega)$ given in (3.25).

Let us consider

$$
\widehat{X}^{1}(T)=\int_{-\infty}^{T} e^{A(T-s)+\int_s^T\eta(\theta_r\omega)\mathrm{d}r}[\widehat{F}_x(\widehat{X}^{0},\widehat{Y}^{0},\theta_s\omega)\widehat{X}^{1}+\widehat{F}_y(\widehat{X}^{0},\widehat{Y}^{0},\theta_s\omega)\widehat{Y}^{1}]\mathrm{d}s, \tag{3.33}
$$

$$
\widehat{Y}^{1}(T)=e^{\int_0^T\eta(\theta_r\omega)\mathrm{d}r}\widehat{Y}_1+\int_0^T e^{\int_s^T\eta(\theta_r\omega)\mathrm{d}r}[B\widehat{Y}^{0}+\widehat{G}(\widehat{X}^{0},\widehat{Y}^{0},\theta_s\omega)]\mathrm{d}s, \tag{3.34}
$$

where

$$
\begin{aligned}
\widehat{X}^{0}(T)&=e^{AT+\int_0^T\eta(\theta_r\omega)\mathrm{d}r}\widehat{H}^{0}(Y_0,\omega)+\int_0^T e^{A(T-s)+\int_s^T\eta(\theta_r\omega)\mathrm{d}r}\widehat{F}(\widehat{X}^{0},\widehat{Y}^{0},\theta_s\omega)\mathrm{d}T,\\
\widehat{Y}^{0}(T)&=e^{\int_0^T\eta(\theta_r\omega)\mathrm{d}r}\widehat{Y}_0,
\end{aligned}
$$

with $\widehat{H}^0(Y_0, \omega)$ given in (3.26).

We state there exists a unique solution for (3.33)–(3.34) in $C^{+}_{-\frac{\mu}{\varepsilon}, \frac{z^\varepsilon}{\varepsilon}}$ without proof. Then, it is easy to obtain that (3.31)–(3.32) has the random invariant manifold represented as

$$M^1(\omega) = \{(\check{H}^1(\omega, \widehat{Y}_1), \widehat{Y}_1) | \widehat{Y}_1 \in H_2\},$$

where

$$\check{H}^1(\omega, \widehat{Y}_1) = \int_{-\infty}^{0} e^{-As - \int_0^s \eta(\theta_r \omega) \mathrm{d}r} [\widehat{F}_x(\widehat{X}^0, \widehat{Y}^0, \theta_s \omega) \widehat{X}^1 + \widehat{F}_y(\widehat{X}^0, \widehat{Y}^0, \theta_s \omega) \widehat{Y}^1] \mathrm{d}s.$$

Set

$$\begin{aligned} \widehat{H}^1(\omega, \widehat{Y}_0) &= \check{H}^1(\omega, 0) \\ &= \int_{-\infty}^{0} e^{-As - \int_0^s \eta(\theta_r \omega) \mathrm{d}r} \Bigg\{ \widehat{F}_x(\widehat{X}^0, \widehat{Y}^0, \theta_s \omega) \widehat{X}^1 + \widehat{F}_y(\widehat{X}^0, \widehat{Y}^0, \theta_s \omega) \\ &\quad \times \Big[\int_0^s e^{\int_\tau^s \eta(\theta_r \omega) \mathrm{d}r} \Big(B\widehat{Y}_0 + \widehat{G}(\widehat{X}^0, \widehat{Y}^0, \theta_\tau \omega) \Big) \mathrm{d}\tau \Big] \Bigg\} \mathrm{d}s. \end{aligned} \tag{3.35}$$

Then, we can formally show the first order approximation of $H^\varepsilon(\widehat{Y}_0, \omega)$ as follow:

$$\begin{aligned} &H^\varepsilon(\omega, \widehat{Y}_0) \\ &\quad \overset{d}{=} \widehat{H}^\varepsilon(\omega, \widehat{Y}_0) \\ &\quad = \int_{-\infty}^{0} e^{-As - \int_0^s \eta(\theta_r \omega) \mathrm{d}r} \widehat{F}(\widehat{X}(s, \omega, Y_0), \widehat{Y}(s, \omega, Y_0), \theta_s \omega) \mathrm{d}s \\ &\quad = \int_{-\infty}^{0} e^{-As - \int_0^s \eta(\theta_r \omega) \mathrm{d}r} \widehat{F}(\widehat{X}^0, \widehat{Y}^0, \theta_s \omega) \mathrm{d}s \\ &\qquad + \int_{-\infty}^{0} e^{-As - \int_0^s \eta(\theta_r \omega) \mathrm{d}r} [\widehat{F}_x(\widehat{X}^0, \widehat{Y}^0, \theta_s \omega) \widehat{X}^1 + \widehat{F}_y(\widehat{X}^0, \widehat{Y}^0, \theta_s \omega) \widehat{Y}^1] \mathrm{d}s + \mathcal{O}(\varepsilon^2) \\ &\quad = \widehat{H}^0(\omega, \widehat{Y}_0) + \varepsilon \widehat{H}^1(\omega, \widehat{Y}_0) + \mathcal{O}(\varepsilon^2). \end{aligned} \tag{3.36}$$

Note that $H^\varepsilon(\omega, \widehat{Y}_0)$ coincides with $\widehat{H}^\varepsilon(\omega, \widehat{Y}_0)$ in distribution. We have, in fact, proved the following theorem.

Theorem 3.5 (First order approximation of slow manifold)**.** *Under Assumptions 1-4, for sufficiently small $\varepsilon > 0$, we obtain the approximation of the random invariant manifold for (2.4)–(2.5) as*

$$\begin{aligned} M^\varepsilon(\omega) &= \{(H^\varepsilon(\omega, \widehat{Y}_0), \widehat{Y}_0) | \widehat{Y}_0 \in \mathcal{D}(B)\} \\ &\overset{d}{=} \{(\widehat{H}^\varepsilon(\omega, \widehat{Y}_0), \widehat{Y}_0) | \widehat{Y}_0 \in \mathcal{D}(B)\} \\ &= \{(\widehat{H}^0(\omega, \widehat{Y}_0) + \varepsilon \widehat{H}^1(\omega, \widehat{Y}_0) + \mathcal{O}(\varepsilon^2), \widehat{Y}_0) | \widehat{Y}_0 \in \mathcal{D}(B)\}, \end{aligned}$$

where the second equality holds in distribution, that means for fixed $\widehat{Y}_0 \in \mathcal{D}(B)$, $H^\varepsilon(\omega, \widehat{Y}_0)$ and $\widehat{H}^\varepsilon(\omega, \widehat{Y}_0)$ are identical in distribution, while the third equality holds for all $\omega \in \Omega$, $\widehat{H}^0(\omega, \widehat{Y}_0)$ is the critical manifold as (3.26), and $\widehat{H}^1(\omega, \widehat{Y}_0)$ is the first order manifold as (3.35).

We now go back to investigate the approximation of the random invariant manifold for SPDEs (1.1)–(1.2). Recall the transforms T and T^{-1} defined in (2.6) and (2.7). Let $\widetilde{M}^{\varepsilon}(\omega) := T^{-1}(\omega, M^{\varepsilon}(\omega))$, $\widetilde{Z}^{\varepsilon}(t,\omega,\cdot)$ be the solution of SPDEs (1.1)–(1.2) and $\bar{Z}^{\varepsilon}(t,\omega,\cdot)$ be the solution of RPDEs (2.4)–(2.5). By Lemma 2.8, we have

$$
\begin{aligned}
\widetilde{Z}^{\varepsilon}(t,\omega,\widetilde{M}^{\varepsilon}(\omega)) &= T^{-1}(\theta_t\omega, \bar{Z}^{\varepsilon}(t,\omega,T(\omega,\widetilde{M}^{\varepsilon}(\omega))))\\
&= T^{-1}(\theta_t\omega, \bar{Z}^{\varepsilon}(t,\omega,M^{\varepsilon}(\omega)))\\
&\subset T^{-1}(\theta_t\omega, M^{\varepsilon}(\theta_t\omega))\\
&= \widetilde{M}^{\varepsilon}(\theta_t\omega),
\end{aligned}
\tag{3.37}
$$

which implies that $\widetilde{M}^{\varepsilon}(\omega)$ is an invariant set. Moreover, we notice that

$$
\begin{aligned}
\widetilde{M}^{\varepsilon}(\omega) &= T^{-1}(\omega, M^{\varepsilon}(\omega))\\
&= \{(e^{z^{\varepsilon}(\omega)}H^{\varepsilon}(\omega,\widehat{Y}_0), e^{z^{\varepsilon}(\omega)}\widehat{Y}_0) | \widehat{Y}_0 \in H_2\}\\
&= \{(e^{z^{\varepsilon}(\omega)}H^{\varepsilon}(\omega,e^{-z^{\varepsilon}(\omega)}\widehat{Y}_0), \widehat{Y}_0) | \widehat{Y}_0 \in H_2\}\\
&= \{(\widetilde{H}^{\varepsilon}(\omega,\widehat{Y}_0), \widehat{Y}_0) | \widehat{Y}_0 \in H_2\},
\end{aligned}
\tag{3.38}
$$

where $\widetilde{H}^{\varepsilon}(\omega,\widehat{Y}_0) = e^{z^{\varepsilon}(\omega)}H^{\varepsilon}(\omega,e^{-z^{\varepsilon}(\omega)}\widehat{Y}_0)$. Then, $\widetilde{M}^{\varepsilon}(\omega)$ can be represented by a graph of a Lipschitz function $\widetilde{H}^{\varepsilon}(\omega,\cdot)$. Therefore, $\widetilde{M}^{\varepsilon}(\omega)$ is the random invariant manifold for (1.1)–(1.2). With the help of Theorem 3.5, we show the approximation of the random invariant manifold for SPDEs (1.1)–(1.2) as follows.

Theorem 3.6. *Under Assumptions 1-4, for sufficiently small $\varepsilon > 0$, we obtain the approximation of the random invariant manifold for (1.1)–(1.2) as*

$$
\begin{aligned}
\widetilde{M}^{\varepsilon}(\omega) &= \{(\widetilde{H}^{\varepsilon}(\omega,\widehat{Y}_0), \widehat{Y}_0) | \widehat{Y}_0 \in \mathcal{D}(B)\}\\
&= \{(e^{z^{\varepsilon}(\omega)}H^{\varepsilon}(\omega,e^{-z^{\varepsilon}(\omega)}\widehat{Y}_0), \widehat{Y}_0) | \widehat{Y}_0 \in \mathcal{D}(B)\}\\
&\overset{d}{=} \{(e^{\eta(\omega)}\widehat{H}^{\varepsilon}(\omega,e^{-\eta(\omega)}\widehat{Y}_0), \widehat{Y}_0) | \widehat{Y}_0 \in \mathcal{D}(B)\}\\
&= \{(e^{\eta(\omega)}\widehat{H}^{0}(\omega,e^{-\eta(\omega)}\widehat{Y}_0) + \varepsilon e^{\eta(\omega)}\widehat{H}^{1}(\omega,e^{-\eta(\omega)}\widehat{Y}_0) + \mathcal{O}(\varepsilon^2), \widehat{Y}_0) | \widehat{Y}_0 \in \mathcal{D}(B)\},
\end{aligned}
$$

where the third equality holds in distribution while the fourth equality holds for all $\omega \in \Omega$, $\eta(\theta_T\omega)$ is the stationary solution of (3.19), $e^{\eta(\omega)}\widehat{H}^{0}(\omega,e^{-\eta(\omega)}\widehat{Y}_0)$ is the critical manifold, and $e^{\eta(\omega)}\widehat{H}^{1}(\omega,e^{-\eta(\omega)}\widehat{Y}_0)$ is the first order manifold.

4 Random invariant foliations and slow foliations

In the section, we are going to show there also exist random invariant foliations for RPDEs (2.4)–(2.5), and any two orbits start from the same fiber can approach to each other as exponential rate in backward time. Then, we prove that random invariant foliations converge to slow foliations as the parameter ε tends to 0.

4.1 Random invariant foliations

For $(\bar{X}_0,\bar{Y}_0),(\check{X}_0,\check{Y}_0) \in H$, let $\widetilde{Y}^{\varepsilon}(0) = \check{Y}_0 - \bar{Y}_0$. Set $\widetilde{Z}^{\varepsilon}(t,\omega) = \bar{Z}^{\varepsilon}(t,\omega,(\check{X}_0,\check{Y}_0)) - \bar{Z}^{\varepsilon}(t,\omega,(\bar{X}_0,\bar{Y}_0)) =(\widetilde{X}^{\varepsilon}(t,\omega,(\bar{X}_0,\bar{Y}_0);\widetilde{Y}^{\varepsilon}(0)), \widetilde{Y}^{\varepsilon}(t,\omega,(\bar{X}_0,\bar{Y}_0);\widetilde{Y}^{\varepsilon}(0)))$. Introduce a set:

$$
\mathcal{W}^{\varepsilon}_{\frac{\gamma_2}{2},\frac{z^{\varepsilon}}{\varepsilon}}((\bar{X}_0,\bar{Y}_0),\omega) := \{(\check{X}_0,\check{Y}_0) \in H | \bar{Z}^{\varepsilon}(t,\omega,(\check{X}_0,\check{Y}_0)) - \bar{Z}^{\varepsilon}(t,\omega,(\bar{X}_0,\bar{Y}_0)) \in C^{-}_{\frac{\gamma_2}{2},\frac{z^{\varepsilon}}{\varepsilon}}\}. \tag{4.1}
$$

In the followings, we will prove $\mathcal{W}^{\varepsilon}_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}\big((\bar{X}_0,\bar{Y}_0),\omega\big)$ is a fiber of the random invariant foliations for (2.4)–(2.5).

Lemma 4.1. *Under Assumptions 1, 2, 5, for sufficiently small $\varepsilon > 0$, we have the following results:*

(1) $(\check{X}_0,\check{Y}_0) \in \mathcal{W}^{\varepsilon}_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}\big((\bar{X}_0,\bar{Y}_0),\omega\big)$ *if and only if* $\widetilde{Z}^\varepsilon(\cdot) \in C^-_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}$ *satisfies*

$$\widetilde{X}^\varepsilon(t) = \frac{1}{\varepsilon}\int_{-\infty}^{t} e^{\frac{A(t-s)+\int_s^t z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}}\widetilde{F}(\widetilde{X}^\varepsilon,\widetilde{Y}^\varepsilon,\theta_s\omega)\mathrm{d}s, \tag{4.2}$$

$$\widetilde{Y}^\varepsilon(t) = e^{Bt+\frac{\int_0^t z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}}\widetilde{Y}^\varepsilon(0) + \int_0^t e^{B(t-s)+\frac{\int_s^t z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}}\widetilde{G}(\widetilde{X}^\varepsilon,\widetilde{Y}^\varepsilon,\theta_s\omega)\mathrm{d}s, \tag{4.3}$$

where nonlinear functions $\widetilde{F}$ and $\widetilde{G}$ are defined in (3.12)–(3.13).

(2) There exists a unique solution $\widetilde{Z}^\varepsilon(\cdot) \in C^-_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}$ for (4.2)–(4.3).

(3) Let $\widetilde{Y}^\varepsilon_1(0)$ and $\widetilde{Y}^\varepsilon_2(0) \in H_2$. Then

$$\begin{aligned}&\|\widetilde{Z}^\varepsilon(\cdot,\omega,(\bar{X}_0,\bar{Y}_0);\widetilde{Y}^\varepsilon_1(0)) - \widetilde{Z}^\varepsilon(\cdot,\omega,(\bar{X}_0,\bar{Y}_0);\widetilde{Y}^\varepsilon_2(0))\|_{C^-_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}}\\ &\le \frac{1}{1-\widetilde{\rho}(K,\gamma_1,\gamma_2,\varepsilon)}\|\widetilde{Y}^\varepsilon_1(0)-\widetilde{Y}^\varepsilon_2(0)\|_2,\end{aligned}$$

where

$$\widetilde{\rho}(K,\gamma_1,\gamma_2,\varepsilon) = \frac{2K}{\gamma_2} + \frac{2K}{2\gamma_1+\varepsilon\gamma_2}. \tag{4.4}$$

Proof. The proof of (i) follows from the variation of constants formula. With the help of Banach's Fixed Point Theorem, we can prove (2). Using the same techniques as in the proof of Theorem 3.3, we can obtain (3). □

For $\zeta \in H_2$, we define

$$\begin{aligned}l^\varepsilon(\zeta,(\bar{X}_0,\bar{Y}_0),\omega) := \bar{X}_0 + \frac{1}{\varepsilon}\int_{-\infty}^{0} e^{\frac{-As-\int_0^s z^\varepsilon(\theta^\varepsilon_r\omega)\mathrm{d}r}{\varepsilon}}\widetilde{F}\Big(&\widetilde{X}^\varepsilon(s,\omega,(\bar{X}_0,\bar{Y}_0);\zeta-\bar{Y}_0),\\ &\widetilde{Y}^\varepsilon(s,\omega,(\bar{X}_0,\bar{Y}_0);\zeta-\bar{Y}_0),\theta_s\omega\Big)\mathrm{d}s.\end{aligned} \tag{4.5}$$

By (4.5), we can show the existence of random invariant foliations for (2.4)–(2.5) as follows.

Theorem 4.2 (Existence of random invariant foliations)**.** *Under Assumptions 1, 2, 5, for sufficiently small $\varepsilon > 0$, we have the following results:*

(1) (2.4)–(2.5) has a random invariant foliation, whose each fiber is represented as

$$\mathcal{W}^{\varepsilon}_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}\big((\bar{X}_0,\bar{Y}_0),\omega\big) = \big\{\big(l^\varepsilon(\zeta,(\bar{X}_0,\bar{Y}_0),\omega),\zeta\big)\big|\zeta \in H_2\big\}, \tag{4.6}$$

where $l^\varepsilon(\zeta,(\bar{X}_0,\bar{Y}_0),\omega)$ is defined in (4.5).

(2) l^ε is a Lipschitz mapping with respect to ζ, whose Lipschitz constant $\mathrm{Lip}\, l^\varepsilon(\cdot,\omega)$ satisfies that

$$\mathrm{Lip}\, l^\varepsilon(\cdot,\omega) \le \frac{2K}{(\varepsilon\gamma_2+2\gamma_1)[1-\widetilde{\rho}(K,\gamma_1,\gamma_2,\varepsilon)]},$$

with $\widetilde{\rho}(K,\gamma_1,\gamma_2,\varepsilon)$ given in (4.4).

Proof. (1) According to (4.2)–(4.3), we obtain

$$\begin{aligned}\check{X}_0 - \bar{X}_0 &= \frac{1}{\varepsilon}\int_{-\infty}^{0} e^{\frac{-As-\int_0^s z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}}\widetilde{F}(\widetilde{X},\widetilde{Y},\theta_s\omega)\mathrm{d}s\\ &= \frac{1}{\varepsilon}\int_{-\infty}^{0} e^{\frac{-As-\int_0^s z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}}\widetilde{F}(\widetilde{X}(s,\omega,(\bar{X}_0,\bar{Y}_0);(\check{Y}_0-\bar{Y}_0)),\\ &\qquad \widetilde{Y}(s,\omega,(\bar{X}_0,\bar{Y}_0);(\check{Y}_0-\bar{Y}_0)),\theta_s\omega)\mathrm{d}s.\end{aligned}$$

Then, replacing $\check{Y}_0$ with ζ, we have (4.6) by (4.1), (4.5) and Lemma 4.1.

We proceed to verify that each fiber is invariant. Let $(\check{X}_0,\check{Y}_0) \in \mathcal{W}^\varepsilon_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}((\bar{X}_0,\bar{Y}_0),\omega)$. Since $\bar{Z}^\varepsilon(\cdot,\omega,(\check{X}_0,\check{Y}_0)) - \bar{Z}^\varepsilon(\cdot,\omega,(\bar{X}_0,\bar{Y}_0)) \in C^-_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}$, we have $\bar{Z}^\varepsilon(\cdot+\tau,\omega,(\check{X}_0,\check{Y}_0)) - \bar{Z}^\varepsilon(\cdot+\tau,\omega,(\bar{X}_0,\bar{Y}_0)) \in C^-_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}$. Then, the cocycle property leads to

$$\begin{aligned}\bar{Z}^\varepsilon(\cdot+\tau,\omega,(\check{X}_0,\check{Y}_0)) &= \bar{Z}^\varepsilon(\cdot,\theta_\tau\omega,\bar{Z}^\varepsilon(\tau,\omega,(\check{X}_0,\check{Y}_0))),\\ \bar{Z}^\varepsilon(\cdot+\tau,\omega,(\bar{X}_0,\bar{Y}_0)) &= \bar{Z}^\varepsilon(\cdot,\theta_\tau\omega,\bar{Z}^\varepsilon(\tau,\omega,(\bar{X}_0,\bar{Y}_0))),\end{aligned}$$

which can yield that $\bar{Z}^\varepsilon(\tau,\omega,(\check{X}_0,\check{Y}_0)) \in \mathcal{W}^\varepsilon_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}(\bar{Z}^\varepsilon(\tau,\omega,(\bar{X}_0,\bar{Y}_0)),\theta_\tau\omega)$.

(2) Let ζ and $\bar{\zeta} \in H_2$. Then

$$\begin{aligned}&\|l^\varepsilon(\zeta,(\bar{X}_0,\bar{Y}_0)) - l^\varepsilon(\bar{\zeta},(\bar{X}_0,\bar{Y}_0))\|_1\\ &\quad\le \|\widetilde{X}^\varepsilon(\cdot,\omega,(\bar{X}_0,\bar{Y}_0);\zeta-\bar{X}_0) - \widetilde{X}^\varepsilon(\cdot,\omega,(\bar{X}_0,\bar{Y}_0);\bar{\zeta}-\bar{X}_0)\|_{C^{1,-}_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}}\\ &\quad\le \frac{2K}{\varepsilon\gamma_2+2\gamma_1}\|\widetilde{Z}^\varepsilon(\cdot,\omega,(\bar{X}_0,\bar{Y}_0);\zeta-\bar{X}_0) - \widetilde{Z}^\varepsilon(\cdot,\omega,(\bar{X}_0,\bar{Y}_0);\bar{\zeta}-\bar{X}_0)\|_{C^-_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}}\\ &\quad\le \frac{2K}{(\varepsilon\gamma_2+2\gamma_1)[1-\widetilde{\rho}(K,\gamma_1,\gamma_2,\varepsilon)]}\|\zeta-\bar{\zeta}\|_2,\end{aligned}$$

where the second inequality is from direct calculation and the last one is from Lemma 4.1. □

Theorem 4.3 (Exponential tracking property in backward time). *Under Assumptions 1, 2, 5, for sufficiently small $\varepsilon > 0$, any two points $(\check{X}^1_0,\check{Y}^1_0)$ and $(\check{X}^2_0,\check{Y}^2_0)$ in a same fiber $\mathcal{W}^\varepsilon_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}((\bar{X}_0,\bar{Y}_0),\omega)$, we have*

$$\|\bar{Z}^\varepsilon(t,\omega,(\check{X}^1_0,\check{Y}^1_0)) - \bar{Z}^\varepsilon(t,\omega,(\check{X}^2_0,\check{Y}^2_0))\| \le \frac{e^{\frac{\int_0^t z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}+\frac{\gamma_2 t}{2}}}{1-\widetilde{\rho}(K,\gamma_1,\gamma_2,\varepsilon)}\|\check{Y}^1_0-\check{Y}^2_0\|_2 \tag{4.7}$$

with $t \le 0$.

Proof. Let

$$\begin{aligned}\widetilde{Z}^\varepsilon_1(t) &= \bar{Z}^\varepsilon(t,\omega,(\check{X}^1_0,\check{Y}^1_0)) - \bar{Z}^\varepsilon(t,\omega,(\bar{X}_0,\bar{Y}_0)),\\ \widetilde{Z}^\varepsilon_2(t) &= \bar{Z}^\varepsilon(t,\omega,(\check{X}^2_0,\check{Y}^2_0)) - \bar{Z}^\varepsilon(t,\omega,(\bar{X}_0,\bar{Y}_0)).\end{aligned}$$

Applying Lemma 4.1, we know that

$$\begin{aligned}
&\|\bar{Z}^\varepsilon(\cdot,\omega,(\check{X}_0^1,\check{Y}_0^1)) - \bar{Z}^\varepsilon(\cdot,\omega,(\check{X}_0^2,\check{Y}_0^2))\|_{C^-_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}}\\
&= \|\widetilde{Z}_1^\varepsilon(\cdot) - \widetilde{Z}_2^\varepsilon(\cdot)\|_{C^-_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}}\\
&\le \frac{1}{\varepsilon}\|\int_{-\infty}^{\cdot} e^{\frac{A(\cdot-s)+\int_s^\cdot z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}}[\widetilde{F}(\widetilde{X}_1^\varepsilon,\widetilde{Y}_1^\varepsilon,\theta_s\omega) - \widetilde{F}(\widetilde{X}_2^\varepsilon,\widetilde{Y}_2^\varepsilon,\theta_s\omega)]\mathrm{d}s\|_{C^{1,-}_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}}\\
&\quad + \|e^{B\cdot+\frac{\int_0^\cdot z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}}(\check{Y}_0^1 - \check{Y}_0^2)\|_{C^{2,-}_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}}\\
&\quad + \|\int_0^{\cdot} e^{B(\cdot-s)+\frac{\int_s^\cdot z^\varepsilon(\theta_r\omega)\mathrm{d}r}{\varepsilon}}[\widetilde{G}(\widetilde{X}_1^\varepsilon,\widetilde{Y}_1^\varepsilon,\theta_s\omega) - \widetilde{G}(\widetilde{X}_2^\varepsilon,\widetilde{Y}_2^\varepsilon,\theta_s\omega)]\mathrm{d}s\|_{C^{2,-}_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}}\\
&\le \|\check{Y}_0^1 - \check{Y}_0^2\|_2 + \widetilde{\rho}(K,\gamma_1,\gamma_2,\varepsilon)\|\widetilde{Z}_1^\varepsilon(\cdot) - \widetilde{Z}_2^\varepsilon(\cdot)\|_{C^-_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}}.
\end{aligned}$$

For sufficiently small $\varepsilon > 0$, we have

$$\|\widetilde{Z}_1^\varepsilon(\cdot) - \widetilde{Z}_2^\varepsilon(\cdot)\|_{C^-_{\frac{\gamma_2}{2},\frac{z^\varepsilon}{\varepsilon}}} \le \frac{1}{1-\widetilde{\rho}(K,\gamma_1,\gamma_2,\varepsilon)}\|\check{Y}_0^1 - \check{Y}_0^2\|_2,$$

which implies (4.7). □

4.2 Slow foliations

The motivation of this subsection is to investigate the approximation of the random invariant foliations for RPDEs (2.4)–(2.5) in slow time-scale $T = \frac{t}{\varepsilon}$. As the arguments in Subsection 3.2, we will study the approximation of the random invariant foliations for RPDEs (2.4)–(2.5) via (3.20)–(3.21).

Let $\widehat{Z}_1^\varepsilon(T,\omega,(\check{X}_0,\check{Y}_0))$ and $\widehat{Z}_2^\varepsilon(T,\omega,(\widehat{X}_0,\widehat{Y}_0))$ be the solutions of (3.20)–(3.21) with initial data $(\check{X}_0,\check{Y}_0)$ and $(\widehat{X}_0,\widehat{Y}_0)$, respectively. Set $\widetilde{Z}^\varepsilon(T,\omega) = \widehat{Z}_1^\varepsilon(T,\omega,(\check{X}_0,\check{Y}_0)) - \widehat{Z}_2^\varepsilon(T,\omega,(\widehat{X}_0,\widehat{Y}_0))$, $\widetilde{Y}^\varepsilon(0) = \check{Y}_0 - \widehat{Y}_0$. According to the variation of constants formula, we state $\widetilde{Z}^\varepsilon(\cdot,\omega) = (\widetilde{X}^\varepsilon(\cdot,\omega,(\widehat{X}_0,\widehat{Y}_0);\widetilde{Y}^\varepsilon(0)),\widetilde{Y}^\varepsilon(\cdot,\omega,(\widehat{X}_0,\widehat{Y}_0);\widetilde{Y}^\varepsilon(0))) \in C^-_{\frac{\gamma_2}{2},\eta}$ if and only if $\widetilde{Z}^\varepsilon(T,\omega)$ satisfies

$$\widetilde{X}^\varepsilon(T) = \int_{-\infty}^{T} e^{A(T-s)+\int_s^T \eta(\theta_r\omega)\mathrm{d}r}\mathring{F}(\widetilde{X}^\varepsilon,\widetilde{Y}^\varepsilon,\theta_s\omega)\mathrm{d}s, \tag{4.8}$$

$$\widetilde{Y}^\varepsilon(T) = e^{\varepsilon BT+\int_0^T \eta(\theta_r\omega)\mathrm{d}r}\widetilde{Y}^\varepsilon(0) + \varepsilon\int_0^T e^{\varepsilon B(T-s)+\int_s^T \eta(\theta_r\omega)\mathrm{d}r}\mathring{G}(\widetilde{X}^\varepsilon,\widetilde{Y}^\varepsilon,\theta_s\omega)\mathrm{d}s, \tag{4.9}$$

where

$$\begin{aligned}
\mathring{F}(\widetilde{X}^\varepsilon,\widetilde{Y}^\varepsilon,\theta_s\omega) &= \widehat{F}(\widetilde{X}^\varepsilon + \widehat{X}_2^\varepsilon,\widetilde{Y}^\varepsilon + \widehat{Y}_2^\varepsilon,\theta_s\omega) - \widehat{F}(\widehat{X}_2^\varepsilon,\widehat{Y}_2^\varepsilon,\theta_s\omega),\\
\mathring{G}(\widetilde{X}^\varepsilon,\widetilde{Y}^\varepsilon,\theta_s\omega) &= \widehat{G}(\widetilde{X}^\varepsilon + \widehat{X}_2^\varepsilon,\widetilde{Y}^\varepsilon + \widehat{Y}_2^\varepsilon,\theta_s\omega) - \widehat{G}(\widehat{X}_2^\varepsilon,\widehat{Y}_2^\varepsilon,\theta_s\omega).
\end{aligned}$$

By Banach's Fixed Point Theorem, we can prove there exists a unique solution $\widetilde{Z}^\varepsilon(\cdot) \in C^-_{\frac{\gamma_2}{2},\eta}$ of (4.8)–(4.9). Then, by the similar arguments as in Theorem 4.2, we can prove that there exists a random invariant foliation for (3.20)–(3.21), each fiber of which is represented as

$$\begin{aligned}
\mathcal{W}^\varepsilon_{\frac{\gamma_2}{2},\eta}((\widehat{X}_0,\widehat{Y}_0),\omega) &= \{(\check{X}_0,\check{Y}_0) \in H | \widehat{Z}_1^\varepsilon(T,\omega,(\check{X}_0,\check{Y}_0)) - \widehat{Z}_2^\varepsilon(T,\omega,(\widehat{X}_0,\widehat{Y}_0)) \in C^-_{\frac{\gamma_2}{2},\eta}\}\\
&= \{(\widehat{l}^\varepsilon(\zeta,(\widehat{X}_0,\widehat{Y}_0),\omega),\zeta) | \zeta \in H_2\},
\end{aligned} \tag{4.10}$$

where

$$\widehat{l}^{\varepsilon}(\zeta,(\widehat{X}_0,\widehat{Y}_0),\omega):=\widehat{X}_0+\int_{-\infty}^{0}e^{-As-\int_0^s\eta(\theta_r\omega)\mathrm{d}r}\mathring{F}\Big(\widetilde{X}^{\varepsilon}(s,\omega,(\widehat{X}_0,\widehat{Y}_0);\zeta-\widehat{Y}_0),$$
$$\widetilde{Y}^{\varepsilon}(s,\omega,(\widehat{X}_0,\widehat{Y}_0);\zeta-\widehat{Y}_0),\theta_s\omega\Big)\mathrm{d}s.$$

Using the similar discussion as (3.22), we derive

$$\widehat{l}^{\varepsilon}(\zeta,(\widehat{X}_0,\widehat{Y}_0),\omega)\overset{d}{=}l^{\varepsilon}(\zeta,(\widehat{X}_0,\widehat{Y}_0),\omega). \tag{4.11}$$

Based on (4.11), we turn to the study of the approximation of $\widehat{l}^{\varepsilon}(\zeta,(\widehat{X}_0,\widehat{Y}_0),\omega)$. Taking into account the critical system (3.23)–(3.24). Let $\widehat{Z}_1^0(T,\omega,(\check{X}_0,\check{Y}_0))$ and $\widehat{Z}_2^0(T,\omega,(\widehat{X}_0,\widehat{Y}_0))$ be the solutions of (3.23)–(3.24) with initial data $(\check{X}_0,\check{Y}_0)$ and $(\widehat{X}_0,\widehat{Y}_0)$, respectively. Set $\widetilde{Z}^0(T,\omega)=\widehat{Z}_1^0(T,\omega,(\check{X}_0,\check{Y}_0))-\widehat{Z}_2^0(T,\omega,(\widehat{X}_0,\widehat{Y}_0))$, $\widetilde{Y}^0(0)=\check{Y}_0-\widehat{Y}_0$.

Clearly, $\widetilde{Z}^0(\cdot,\omega)=(\widetilde{X}^0(\cdot,\omega,(\widehat{X}_0,\widehat{Y}_0);\widetilde{Y}^0(0)),\widetilde{Y}^0(\cdot,\omega,(\widehat{X}_0,\widehat{Y}_0);\widetilde{Y}^0(0)))\in C^-_{\frac{\gamma_2}{2},\eta}$ if and only if $\widetilde{Z}^0(T,\omega)$ satisfies

$$\widetilde{X}^0(T)=\int_{-\infty}^{T}e^{A(T-s)+\int_s^T\eta(\theta_r\omega)\mathrm{d}r}\mathring{F}(\widetilde{X}^0,\widetilde{Y}^0,\theta_s\omega)\mathrm{d}s, \tag{4.12}$$
$$\widetilde{Y}^0(T)=e^{\int_0^T\eta(\theta_r\omega)\mathrm{d}r}\widetilde{Y}^0(0), \tag{4.13}$$

where

$$\mathring{F}(\widetilde{X}^0,\widetilde{Y}^0,\theta_T\omega)=\widehat{F}(\widetilde{X}^0+\widehat{X}_2^0,\widetilde{Y}^0+\widehat{Y}_2^0,\theta_T\omega)-\widehat{F}(\widehat{X}_2^0,\widehat{Y}_2^0,\theta_T\omega).$$

Furthermore, we claim that (3.23)–(3.24) has a random invariant foliation, whose each fiber is represented as

$$\mathcal{W}^0_{\frac{\gamma_2}{2},\eta}((\widehat{X}_0,\widehat{Y}_0),\omega)=\{(\widehat{l}^0(\zeta,(\widehat{X}_0,\widehat{Y}_0),\omega),\zeta)|\zeta\in H_2\}, \tag{4.14}$$

where

$$\widehat{l}^0(\zeta,(\widehat{X}_0,\widehat{Y}_0),\omega):=\widehat{X}_0+\int_{-\infty}^{0}e^{-As-\int_0^s\eta(\theta_r\omega)\mathrm{d}r}\mathring{F}\Big(\widetilde{X}^0(s,\omega,(\widehat{X}_0,\widehat{Y}_0);\zeta-\widehat{Y}_0),$$
$$\widetilde{Y}^0(s,\omega,(\widehat{X}_0,\widehat{Y}_0);\zeta-\widehat{Y}_0),\theta_s\omega\Big)\mathrm{d}s. \tag{4.15}$$

Inspired by the technique from Theorem 5.2 in [12], we are going to prove that the random invariant foliation for (3.20)–(3.21) converges to that for (3.23)–(3.24) as $\varepsilon\to 0$.

Theorem 4.4. *Under Assumptions 1, 2, 5, for sufficiently small $\varepsilon>0$, we have*

$$\widehat{l}^{\varepsilon}(\zeta,(\widehat{X}_0,\widehat{Y}_0),\omega)=\widehat{l}^0(\zeta,(\widehat{X}_0,\widehat{Y}_0),\omega)+\mathcal{O}(\varepsilon),$$

for all $\hat{X}_0\in H_1$, $\widehat{Y}_0,\zeta\in\mathcal{D}(B)$, $\omega\in\Omega$.

Proof. Due to the representations of $\widehat{l}^{\varepsilon}$ and $\widehat{l}^0$, we obtain

$$\|\widehat{l}^{\varepsilon}(\zeta,(\widehat{X}_0,\widehat{Y}_0),\omega)-\widehat{l}^0(\zeta,(\widehat{X}_0,\widehat{Y}_0),\omega)\|_1$$
$$=\|\widetilde{X}^{\varepsilon}(T,\omega,(\widehat{X}_0,\widehat{Y}_0);(\zeta-\widehat{Y}_0))-\widetilde{X}^0(T,\omega,(\widehat{X}_0,\widehat{Y}_0);(\zeta-\widehat{Y}_0))\|_1|_{T=0}.$$

Hence, if we can estimate the error between $\widetilde{X}^{\varepsilon}(T)$ and $\widetilde{X}^{0}(T)$, the proof will be done. For $T \leq 0$, we have

$$\begin{aligned}
&\|\widetilde{X}^{\varepsilon}(T,\omega,(\widehat{X}_0,\widehat{Y}_0);(\zeta-\widehat{Y}_0)) - \widetilde{X}^{0}(T,\omega,(\widehat{X}_0,\widehat{Y}_0);(\zeta-\widehat{Y}_0))\|_1 \\
&\quad = \int_{-\infty}^{T} e^{A(T-s)+\int_s^T \eta(\theta_r\omega)\mathrm{d}r}[\mathring{F}(\widetilde{X}^{\varepsilon}(s),\widetilde{Y}^{\varepsilon}(s),\theta_s\omega) - \mathring{F}(\widetilde{X}^{0}(s),\widetilde{Y}^{0}(s),\theta_s\omega)]\mathrm{d}s \\
&\quad = \int_{-\infty}^{T} e^{A(T-s)+\int_s^T \eta(\theta_r\omega)\mathrm{d}r}[\widehat{F}(\widetilde{X}^{\varepsilon}(s)+\widehat{X}_2^{\varepsilon}(s),\widetilde{Y}^{\varepsilon}(s)+\widehat{Y}_2^{\varepsilon}(s),\theta_s\omega) \\
&\qquad - \widehat{F}(\widetilde{X}^{0}(s)+\widehat{X}_2^{0}(s),\widetilde{Y}^{0}(s)+\widehat{Y}_2^{0}(s),\theta_s\omega) + \widehat{F}(\widehat{X}_2^{0}(s),\widehat{Y}_2^{0}(s),\theta_s\omega) \\
&\qquad - \widehat{F}(\widehat{X}_2^{\varepsilon}(s),\widehat{Y}_2^{\varepsilon}(s),\theta_s\omega)]\mathrm{d}s \\
&\quad \leq K\int_{-\infty}^{T} e^{A(T-s)+\int_s^T \eta(\theta_r\omega)\mathrm{d}r}[\|\widetilde{X}^{\varepsilon}(s)-\widetilde{X}^{0}(s)\|_1 + \|\widetilde{Y}^{\varepsilon}(s)-\widetilde{Y}^{0}(s)\|_2 \\
&\qquad + 2\|\widehat{X}_2^{\varepsilon}(s)-\widehat{X}_2^{0}(s)\|_1 + 2\|\widehat{Y}_2^{\varepsilon}(s)-\widehat{Y}_2^{0}(s)\|_2]\mathrm{d}s.
\end{aligned}$$

In order to show the bounds of $\|\widetilde{X}^{\varepsilon}(T)-\widetilde{X}^{0}(T)\|_1$, we need the estimates of $\|\widetilde{Y}^{\varepsilon}(T)-\widetilde{Y}^{0}(T)\|_2$, $\|\widehat{X}_2^{\varepsilon}(T)-\widehat{X}_2^{0}(T)\|_1$ and $\|\widehat{Y}_2^{\varepsilon}(T)-\widehat{Y}_2^{0}(T)\|_2$, respectively. Choose $\bar{\mu}$ satisfying $\bar{\mu} \in (0, \frac{2\gamma_1^2}{\gamma_2+2\gamma_1}]$, which implies $K < \gamma_1 - \bar{\mu}$. Similar to the deduction in (3.27), we derive

$$\|\widehat{Y}_2^{\varepsilon}(T)-\widehat{Y}_2^{0}(T)\|_2 \leq e^{\int_0^T \eta(\theta_r\omega)\mathrm{d}r}\|B\widehat{Y}_0\|_2\frac{1-e^{\gamma_2\varepsilon T}}{\gamma_2} + e^{\int_0^T \eta(\theta_r\omega)\mathrm{d}r}\frac{\varepsilon K\|\widehat{Y}_0\|_2}{1-\widehat{\rho}}\frac{e^{-\bar{\mu}T}-e^{\varepsilon\gamma_2 T}}{\bar{\mu}+\varepsilon\gamma_2},$$

and

$$\|\widetilde{Y}^{\varepsilon}(T)-\widetilde{Y}^{0}(T)\|_2 \leq e^{\int_0^T \eta(\theta_r\omega)\mathrm{d}r}\|B(\zeta-\widehat{Y}_0)\|_2\frac{1-e^{\gamma_2\varepsilon T}}{\gamma_2} + e^{\int_0^T \eta(\theta_r\omega)\mathrm{d}r}\frac{\varepsilon K\|\zeta-\widehat{Y}_0\|_2}{1-\widehat{\rho}}\frac{e^{-\bar{\mu}T}-e^{\varepsilon\gamma_2 T}}{\bar{\mu}+\varepsilon\gamma_2},$$

where

$$\widehat{\rho}(K,\bar{\mu},\gamma_1,\gamma_2,\varepsilon) = \frac{K}{\gamma_1-\bar{\mu}} + \frac{\varepsilon K}{\bar{\mu}+\varepsilon\gamma_2}.$$

Analogous argument as (3.28) yields

$$\|\widehat{X}_2^{\varepsilon}(\cdot)-\widehat{X}_2^{0}(\cdot)\|_{C^{1,-}_{-\bar{\mu},\eta}} \leq \frac{(\gamma_1-\bar{\mu})\widehat{\mathcal{R}}}{\gamma_1-\bar{\mu}-K},$$

where

$$\begin{aligned}
\widehat{\mathcal{R}} &= \sup_{T\leq 0}\Bigg\{Ke^{\bar{\mu}T}\Big(\frac{\|B\widehat{Y}_0\|_2}{\gamma_1\gamma_2} - \frac{\|B\widehat{Y}_0\|_2 e^{\gamma_2\varepsilon T}}{(\gamma_1+\gamma_2\varepsilon)\gamma_2} - \frac{\varepsilon K\|\widehat{Y}_0\|_2 e^{\gamma_2\varepsilon T}}{(1-\widehat{\rho})(\bar{\mu}+\varepsilon\gamma_2)(\gamma_1+\gamma_2\varepsilon)}\Big) \\
&\qquad + \frac{\varepsilon K^2\|\widehat{Y}_0\|_2}{(1-\widehat{\rho})(\bar{\mu}+\varepsilon\gamma_2)(\gamma_1-\bar{\mu})}\Bigg\} \\
&= \mathcal{O}(\varepsilon).
\end{aligned}$$

Furthermore, combining the above estimates, we obtain that

$$\begin{aligned}
\|\widetilde{X}^{\varepsilon}(\cdot)-\widetilde{X}^{0}(\cdot)\|_{C^{1,-}_{-\bar{\mu},\eta}} &\leq K\int_{-\infty}^{T} e^{(-\gamma_1+\bar{\mu})(T-s)}(2\|\widehat{X}_2^{\varepsilon}(\cdot)-\widehat{X}_2^{0}(\cdot)\|_{C^{1,-}_{-\bar{\mu},\eta}} + \|\widetilde{X}^{\varepsilon}(\cdot)-\widetilde{X}^{0}(\cdot)\|_{C^{1,-}_{-\bar{\mu},\eta}})\mathrm{d}s \\
&\quad + K\int_{-\infty}^{T} e^{-\gamma_1(T-s)+\bar{\mu}T+\int_s^0 \eta(\theta_r\omega)\mathrm{d}r}(2\|\widehat{Y}_2^{\varepsilon}-\widehat{Y}_2^{0}\|_2 + \|\widetilde{Y}^{\varepsilon}-\widetilde{Y}^{0}\|_2)\mathrm{d}s \\
&\leq \frac{K}{\gamma_1-\bar{\mu}}\|\widetilde{X}^{\varepsilon}(\cdot)-\widetilde{X}^{0}(\cdot)\|_{C^{1,-}_{-\bar{\mu},\eta}} + \frac{2K\widehat{\mathcal{R}}}{\gamma_1-\bar{\mu}-K} + 2\widehat{\mathcal{R}} + \widetilde{\mathcal{R}},
\end{aligned}$$

where

$$\begin{aligned}\widetilde{\mathcal{R}} = \sup_{T\le 0}\Bigg\{ & Ke^{\bar{\mu}T}\Big(\frac{\|B(\zeta-\widehat{Y}_0)\|_2}{\gamma_1\gamma_2} - \frac{\|B(\zeta-\widehat{Y}_0)\|_2 e^{\gamma_2\varepsilon T}}{(\gamma_1+\gamma_2\varepsilon)\gamma_2} \\ & - \frac{\varepsilon K\|\zeta-\widehat{Y}_0\|_2 e^{\gamma_2\varepsilon T}}{(1-\widehat{\rho})(\bar{\mu}+\varepsilon\gamma_2)(\gamma_1+\gamma_2\varepsilon)}\Big) + \frac{\varepsilon K^2\|\zeta-\widehat{Y}_0\|_2}{(1-\widehat{\rho})(\bar{\mu}+\varepsilon\gamma_2)(\gamma_1-\bar{\mu})}\Bigg\} \\ = \mathcal{O}(\varepsilon). &\end{aligned}$$

Therefore, we deduce

$$\begin{aligned}\|\widetilde{X}^{\varepsilon}(\cdot)-\widetilde{X}^{0}(\cdot)\|_{C^{1,-}_{-\bar{\mu},\eta}} &\le \frac{\frac{2K\widehat{\mathcal{R}}}{\gamma_1-\bar{\mu}-K}+2\widehat{\mathcal{R}}+\widetilde{\mathcal{R}}}{1-\frac{K}{\gamma_1-\bar{\mu}}} \\ &= \mathcal{O}(\varepsilon),\end{aligned}$$

which implies

$$\|\widetilde{X}^{\varepsilon}(T)-\widetilde{X}^{0}(T)\|_1 \le \frac{\frac{2K\widehat{\mathcal{R}}}{\gamma_1-\bar{\mu}-K}+2\widehat{\mathcal{R}}+\widetilde{\mathcal{R}}}{1-\frac{K}{\gamma_1-\mu}} e^{-\bar{\mu}T+\int_0^T \eta(\theta_r\omega)\mathrm{d}r}.$$

Our proof is completed. □

In the following, we study the first order approximation of the random invariant foliation for (3.20)–(3.21). Let $\widehat{Z}^1(T,\omega,(\check{X}^1_0,\check{Y}^1_0))$ and $\widehat{Z}^1(T,\omega,(\widehat{X}^1_0,\widehat{Y}^1_0))$ be the solutions of (3.31)–(3.32) with initial data $(\check{X}^1_0,\check{Y}^1_0)$ and $(\widehat{X}^1_0,\widehat{Y}^1_0)$, respectively. Set $\widetilde{Z}^1(T,\omega)=\widehat{Z}^1_1(T,\omega,(\check{X}^1_0,\check{Y}^1_0))-\widehat{Z}^1_2(T,\omega,(\widehat{X}^1_0,\widehat{Y}^1_0))$.

According to Assumption 3, we know $\widehat{F}(x,y)$ has the partial derivatives. Expanding $\widetilde{Z}^{\varepsilon}(T,\omega)=(\widetilde{X}^{\varepsilon}(T,\omega),\widetilde{Y}^{\varepsilon}(T,\omega))$ with respect to ε, we have $\widetilde{Z}^{\varepsilon}(T,\omega)=\widetilde{Z}^{0}(T,\omega)+\varepsilon\widetilde{Z}^{1}(T,\omega)+\mathcal{O}(\varepsilon^2)$, where $\widetilde{Z}^{0}(T,\omega)$ satisfies

$$\begin{aligned}\mathrm{d}\widetilde{X}^0 &= [A\widetilde{X}^0+\eta(\theta_T\omega)\widetilde{X}^0+\widehat{F}(\widetilde{X}^0+\widehat{X}^0_2,\widetilde{Y}^0+\widehat{Y}^0_2,\theta_T\omega)-\widehat{F}(\widehat{X}^0_2,\widehat{Y}^0_2,\theta_T\omega)]\mathrm{d}T, \\ \mathrm{d}\widetilde{Y}^0 &= \eta(\theta_T\omega)\widetilde{Y}^0\mathrm{d}T,\end{aligned}$$

and $\widetilde{Z}^1(T,\omega)$ satisfies

$$\begin{aligned}\mathrm{d}\widetilde{X}^1 = {} & [A\widetilde{X}^1+\eta(\theta_T\omega)\widetilde{X}^1+\widehat{F}_x(\widetilde{X}^0+\widehat{X}^0_2,\widetilde{Y}^0+\widehat{Y}^0_2,\theta_T\omega)(\widetilde{X}^1+\widehat{X}^1_2) \\ & +\widehat{F}_y(\widetilde{X}^0+\widehat{X}^0_2,\widetilde{Y}^0+\widehat{Y}^0_2,\theta_T\omega)(\widetilde{Y}^1+\widehat{Y}^1_2)-\widehat{F}_x(\widehat{X}^0_2,\widehat{Y}^0_2,\theta_T\omega)\widehat{X}^1_2 \\ & -\widehat{F}_y(\widehat{X}^0_2,\widehat{Y}^0_2,\theta_T\omega)\widehat{Y}^1_2]\mathrm{d}T, \\ \mathrm{d}\widetilde{Y}^1 = {} & [B\widetilde{Y}^0+\eta(\theta_T\omega)\widetilde{Y}^1+\widehat{G}(\widetilde{X}^0+\widehat{X}^0_2,\widetilde{Y}^0+\widehat{Y}^0_2,\theta_T\omega)-\widehat{G}(\widehat{X}^0_2,\widehat{Y}^0_2,\theta_T\omega)]\mathrm{d}T.\end{aligned}$$

The critical foliation for (3.23)–(3.24) has been presented in (4.14). We proceed to investigate the first order of the random invariant foliations for (3.23)–(3.24). Consider

$$\begin{aligned}\widetilde{X}^1(T) = {} & \int_{-\infty}^{T} e^{A(T-s)+\int_s^T \eta(\theta_r\omega)\mathrm{d}r}[\widehat{F}_x(\widetilde{X}^0+\widehat{X}^0_2,\widetilde{Y}^0+\widehat{Y}^0_2,\theta_s\omega)(\widetilde{X}^1+\widehat{X}^1_2) \\ & +\widehat{F}_y(\widetilde{X}^0+\widehat{X}^0_2,\widetilde{Y}^0+\widehat{Y}^0_2,\theta_s\omega)(\widetilde{Y}^1+\widehat{Y}^1_2)-\widehat{F}_x(\widehat{X}^0_2,\widehat{Y}^0_2,\theta_s\omega)\widehat{X}^1_2 \\ & -\widehat{F}_y(\widehat{X}^0_2,\widehat{Y}^0_2,\theta_s\omega)\widehat{Y}^1_2]\mathrm{d}s, \qquad (4.16) \\ \widetilde{Y}^1(T) = {} & \int_0^T e^{\int_s^T \eta(\theta_r\omega)\mathrm{d}r}[B\widetilde{Y}^0+\widehat{G}(\widetilde{X}^0+\widehat{X}^0_2,\widetilde{Y}^0+\widehat{Y}^0_2,\theta_s\omega) \\ & -\widehat{G}(\widehat{X}^0_2,\widehat{Y}^0_2,\theta_s\omega)]\mathrm{d}s, \qquad (4.17)\end{aligned}$$

where $\widetilde{X}^0(T)$ and $\widetilde{Y}^0(T)$ are from (4.12)–(4.13). By Banach's Fixed Point Theorem, we know that there exists a unique solution $(\widetilde{X}^1(\cdot), \widetilde{Y}^1(\cdot))$ of (4.16)–(4.17) in $C^-_{\frac{\lambda_2}{2},\eta}$. Similar to the arguments as in Theorem 4.2, we can obtain the first order approximation of the random invariant foliation for (3.20)–(3.21). Fixing $\widehat{X}_0 \in H_1$, $\widehat{Y}_0 \in \mathcal{D}(B)$, define

$$\begin{aligned}\widehat{l}^1(\check{Y}_0, (\widehat{X}_0, \widehat{Y}_0), \omega) := \int_{-\infty}^{0} & e^{A(T-s)+\int_s^T \eta(\theta_r\omega)\mathrm{d}r}[\widehat{F}_x(\widetilde{X}^0 + \widehat{X}_2^0, \widetilde{Y}^0 + \widehat{Y}_2^0, \theta_s\omega)(\widetilde{X}^1 + \widehat{X}_2^1) \\ & + \widehat{F}_y(\widetilde{X}^0 + \widehat{X}_2^0, \widetilde{Y}^0 + \widehat{Y}_2^0, \theta_s\omega)(\widetilde{Y}^1 + \widehat{Y}_2^1) - \widehat{F}_x(\widehat{X}_2^0, \widehat{Y}_2^0, \theta_s\omega)\widehat{X}_2^1 \\ & - \widehat{F}_y(\widehat{X}_2^0, \widehat{Y}_2^0, \theta_s\omega)\widehat{Y}_2^1]\mathrm{d}s. \end{aligned} \tag{4.18}$$

Similar to (3.36), we formally obtain

$$\widehat{l}^\varepsilon(\zeta, (\widehat{X}_0, \widehat{Y}_0), \omega) = \widehat{l}^0(\zeta, (\widehat{X}_0, \widehat{Y}_0), \omega) + \widehat{l}^1(\zeta, (\widehat{X}_0, \widehat{Y}_0), \omega) + \mathcal{O}(\varepsilon^2).$$

Ultimately, we have the following theorem.

Theorem 4.5 (First order approximation of slow foliation)**.** *Under Assumptions 1, 2, 3, 5, for sufficiently small $\varepsilon > 0$, we obtain the approximation fo the random invariant foliation for as*

$$\begin{aligned}\mathcal{W}^\varepsilon_{\frac{\gamma_2}{2}, \frac{z^\varepsilon}{\varepsilon}}((\widehat{X}_0, \widehat{Y}_0), \omega) &= \{(l^\varepsilon(\zeta, (\widehat{X}_0, \widehat{Y}_0), \omega), \zeta) | \zeta \in \mathcal{D}(B)\} \\ &\stackrel{d}{=} \{(\widehat{l}^\varepsilon(\zeta, (\widehat{X}_0, \widehat{Y}_0), \omega), \zeta) | \zeta \in \mathcal{D}(B)\} \\ &= \{(\widehat{l}^0(\zeta, (\widehat{X}_0, \widehat{Y}_0), \omega) + \varepsilon\widehat{l}^1(\zeta, (\widehat{X}_0, \widehat{Y}_0), \omega) + \mathcal{O}(\varepsilon^2), \zeta) | \zeta \in \mathcal{D}(B)\},\end{aligned}$$

where $\widehat{X}_0 \in H_1$, $\widehat{Y}_0 \in \mathcal{D}(B)$, the second equality holds in distribution while the third equality holds for all $\omega \in \Omega$, $\widehat{l}^0(\zeta, (\widehat{X}_0, \widehat{Y}_0), \omega)$ is the critical foliation as (4.15), and $\widehat{l}^1(\zeta, (\widehat{X}_0, \widehat{Y}_0), \omega)$ is the first order foliation as (4.18).

We are going to study the the approximation fo the random invariant foliation for SPDEs (1.1)–(1.2) in the followings. Recall the transforms T and T^{-1} defined in (2.6) and (2.7). Let $(\widehat{X}_0, \widehat{Y}_0) \in H$. By Lemma 2.8 and the similar arguments as in (3.19)–(3.37), we can obtain a random invariant foliation for SPDEs (1.1)–(1.2), each fiber of which is represented as

$$\begin{aligned}\widetilde{\mathcal{W}}^\varepsilon_{\frac{\gamma_2}{2}, \frac{z^\varepsilon}{\varepsilon}}((\widehat{X}_0, \widehat{Y}_0), \omega) &= T^{-1}\mathcal{W}^\varepsilon_{\frac{\gamma_2}{2}, \frac{z^\varepsilon}{\varepsilon}}((T\widehat{X}_0, T\widehat{Y}_0), \omega) \\ &= \{(e^{z^\varepsilon(\omega)} l^\varepsilon(\zeta, (e^{-z^\varepsilon(\omega)}\widehat{X}_0, e^{-z^\varepsilon(\omega)}\widehat{Y}_0), \omega), e^{z^\varepsilon(\omega)}\zeta) | \zeta \in H_2\} \\ &= \{(\widetilde{l}^\varepsilon(\zeta, (\widehat{X}_0, \widehat{Y}_0), \omega), \zeta) | \zeta \in H_2\},\end{aligned}$$

where l^ε is defined in (4.5), and $\widetilde{l}^\varepsilon(\zeta, (\widehat{X}_0, \widehat{Y}_0), \omega) = e^{z^\varepsilon(\omega)} l^\varepsilon(e^{-z^\varepsilon(\omega)}\zeta, (e^{-z^\varepsilon(\omega)}\widehat{X}_0, e^{-z^\varepsilon(\omega)}\widehat{Y}_0), \omega)$. By Theorem 4.5, we can obtain the first order approximation of the random invariant foliation for SPDEs (1.1)–(1.2) as follows.

Theorem 4.6. *Under Assumptions 1, 2, 3, 5, for sufficiently small $\varepsilon > 0$, we obtain the approximation of the random invariant foliation for* (1.1)–(1.2) *as*

$$\begin{aligned}&\widetilde{\mathcal{W}}^\varepsilon_{\frac{\gamma_2}{2}, \frac{z^\varepsilon}{\varepsilon}}((\widehat{X}_0, \widehat{Y}_0), \omega) \\ &= \{(\widetilde{l}^\varepsilon(\zeta, (\widehat{X}_0, \widehat{Y}_0), \omega), \zeta) | \zeta \in \mathcal{D}(B)\} \\ &= \{(e^{z^\varepsilon(\omega)} l^\varepsilon(\zeta, (e^{-z^\varepsilon(\omega)}\widehat{X}_0, e^{-z^\varepsilon(\omega)}\widehat{Y}_0), \omega), e^{z^\varepsilon(\omega)}\zeta) | \zeta \in \mathcal{D}(B)\} \\ &\stackrel{d}{=} \{(e^{\eta(\omega)}\widehat{l}^\varepsilon(\zeta, (e^{-\eta(\omega)}\widehat{X}_0, e^{-\eta(\omega)}\widehat{Y}_0), \omega), e^{\eta(\omega)}\zeta) | \zeta \in \mathcal{D}(B)\} \\ &= \{(e^{\eta(\omega)}\widehat{l}^0(\zeta, (e^{-\eta(\omega)}\widehat{X}_0, e^{-\eta(\omega)}\widehat{Y}_0), \omega) + \varepsilon e^{\eta(\omega)}\widehat{l}^1(\zeta, (e^{-\eta(\omega)}\widehat{X}_0, e^{-\eta(\omega)}\widehat{Y}_0), \omega) \\ &\quad + \mathcal{O}(\varepsilon^2), e^{\eta(\omega)}\zeta) | \zeta \in \mathcal{D}(B)\},\end{aligned}$$

where $\widehat{X}_0 \in H_1$, $\widehat{Y}_0 \in \mathcal{D}(B)$, *the third equality holds in distribution while the fourth equality holds for all* $\omega \in \Omega$, $e^{\eta(\omega)}\widehat{l}^0(\zeta,(e^{-\eta(\omega)}\widehat{X}_0, e^{-\eta(\omega)}\widehat{Y}_0),\omega)$ *is the critical foliation, and* $e^{\eta(\omega)}\widehat{l}^1(\zeta,(e^{-\eta(\omega)}\widehat{X}_0,$ $e^{-\eta(\omega)}\widehat{Y}_0),\omega)$ *is the first order foliation.*

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