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A Tauberian theorem for the statistical generalized Nörlund-Euler summability method

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Abstract. Let (p_n) and (q_n) be any two non-negative real sequences with

$$R_n := \sum_{k=0}^{n} p_k q_{n-k} \neq 0 \ \ (n \in \mathbb{N}).$$

With E_n^1- we will denote the Euler summability method. Let (x_n) be a sequence of real or complex numbers and set

$$N_{p,q}^n E_n^1 := \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} x_\nu$$

for $n \in \mathbb{N}$. In this paper, we present necessary and sufficient conditions under which the existence of the $st-$ limit of (x_n) follows from that of $st - N_{p,q}^n E_n^1-$ limit of (x_n). These conditions are one-sided or two-sided if (x_n) is a sequence of real or complex numbers, respectively.

1 Introduction

In what follows we give the concept of the summability method known as the generalized Nörlund summability method (N, p, q) (see [1]). Given two

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non-negative sequences (p_n) and (q_n), the convolution $(p \star q)$ is defined by

$$R_n := (p \star q)_n = \sum_{k=0}^{n} p_k q_{n-k} = \sum_{k=0}^{n} p_{n-k} q_k.$$

In this paper we suppose $R_n \to \infty$ as $n \to \infty$. With E_n^1- we will denote the Euler summability method. Let (x_n) be a sequence. When $(p \star q)_n \neq 0$ for all $n \in \mathbb{N}$, the generalized Nörlund-Euler transform of the sequence (x_n) is the sequence $N_{p,q}^n E_n^1$ obtained by putting

$$N_{p,q}^n E_n^1 = \frac{1}{(p \star q)_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} x_\nu. \tag{1}$$

We say that the sequence (x_n) is generalized Nörlund-Euler summable to L determined by the sequences (p_n) and (q_n) or briefly summable $N_{p,q}^n E_n^1$ to L if

$$\lim_{n\to\infty} N_{p,q}^n E_n^1 = L. \tag{2}$$

Suppose throughout the paper we assume that the sequence $q = (q_n)$ satisfies the following conditions:

$$q_{t_n-k} \leq 2q_{n-k}, k = 0, 1, 2, 3, \cdots, n; t > 1, \tag{3}$$

$$q_{n-k} \leq 2q_{t_n-k}, k = 0, 1, 2, 3, \cdots, t_n; 0 < t < 1, \tag{4}$$

where $t_n = [t \cdot n]$.

If

$$\lim_{n\to\infty} x_n = L \tag{5}$$

implies (2), then the summation method generated by $N_{p,q}^n E_n^1$ is regular, and it is satisfied under certain conditions. However, the converse is not always true. We can show by the following example

Example 1 *Let us consider that* $x = (x_k) = (-1)^k$, *then we have*

$$\left| \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} (-1)^\nu \right| \leq \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} 1 \to 1 \text{ as } n \to \infty.$$

And as we know $x = (x_k)$, *is not convergent.*

Notice that (2) may imply (5) under a certain condition, which is called a Tauberian condition. Any theorem which states that convergence of a sequence follows from its $N^n_{p,q}E^1_n$ summability and some Tauberian condition is said to be a Tauberian theorem for the $N^n_{p,q}E^1_n$ summability method. The inclusion and Tauberian type theorems are proved in the papers [6, 7, 2, 3, 4, 5, 8], and some theorems of inclusion, Tauberian and convexity type for certain families of generalized Nörlund methods are obtained in [9].

2 Results

In this paper, we present necessary and sufficient conditions under which the existence of the limit $st - \lim_{n\to\infty} x_n = L$ follows from that of $st - \lim_{n\to\infty} N^n_{p,q}E^1_n = L$. These conditions are one-sided or two-sided if (x_n) is a sequence of real or complex numbers, respectively.

Definition 1 *A sequence* (x_n) *is statistically convergent to* L, *if for every* $\epsilon > 0$, *we have*

$$\lim_{n\to\infty} \frac{|\{k \le n : |x_k - L| \ge \epsilon\}|}{n} = 0,$$

where $|A|$, *stands for cardinality of the set.*

The theory of Tauberian is extensively studied by many authors ([1], [2], [3], [4], [7], [9]). In this section our aim is to find conditions (so-called Tauberian) under which the converse implication holds, for defined convergence. Exactly, we will prove under which conditions convergence of sequences (x_n), follows from $N^n_{p,q}E^1_n-$ convergence.

Definition 2 *A sequence* (x_n) *is weighted* $N^n_{p,q}E^1_n-$*statistically convergent to* L *if for every* $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{R_n} \left| \left\{ k \le R_n : \left| \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} x_v - L \right| \ge \epsilon \right\} \right| = 0.$$

And we say that the sequence (x_n) is statistically summable to L by the weighted summability method $N^n_{p,q}E^1_n$, if $st - \lim_n N^n_{p,q}E^1_n = L$. We denote by $N^n_{p,q}E^1_n(st)$ the set of all sequences which are statistically summable.

Theorem 1 *If sequence* $x = (x_n)$ *is* $N^n_{p,q}E^1_n$ *summable to* L, *then sequence* $x = (x_n)$ *is* $N^n_{p,q}E^1_n-$ *statistically convergent to* L. *But not conversely.*

Proof. The first part of the proof is obvious. To prove the second part we will show this example:

Example 2 *Let us consider that* $(p_k) = (1)$, $(q_{n-k}) = (2^k)$, *and define*

$$x_k = \begin{cases} \sqrt{2^k} & , \quad \textit{for} \quad k = 2^n \\ 0 & , \quad \textit{otherwise} \end{cases}$$

Under this conditions we get:

$$\frac{1}{2^n - 1}\left|\left\{k \leq 2^n - 1 : \left|\frac{1}{2^n - 1}\sum_{k=0}^{n} 1 \cdot \sum_{\nu=0}^{k}\binom{k}{\nu}x_\nu - 0\right| \geq \epsilon\right\}\right| \leq \frac{\sqrt{2^n}}{2^n - 1} \to 0.$$

Hence, it is $N^n_{p,q}E^1_n$ *statistically summable to 0. On the other hand, if we take in consideration that* $k = 2^n$, *we get*

$$\frac{1}{2^n - 1}\sum_{k=0}^{n} 1 \cdot \sum_{\nu=0}^{k}\binom{k}{\nu}x_\nu \to \infty, \quad as \quad n \to \infty.$$

From last relation follows that $x = (x_n)$ *is not* $N^n_{p,q}E^1_n$ *summable to 0.*

□

Theorem 2 *Let us suppose that sequence* (x_n)*-statistically convergent to* L, *and*

$$\sup_{\substack{0 \leq \nu \leq k \\ 0 \leq k \leq n \\ n \in \mathbb{N}}} (\nu + k + n)|x_\nu - L| < \infty.$$

Then it converges $N^n_{p,q}E^1_n$*-statistically to* L. *Converse is not true.*

Proof. From fact that (x_n) converges statistically to L, we get

$$\lim_{n \to \infty}\frac{|\{k \leq n : |x_k - L| \geq \epsilon\}|}{n} = 0.$$

Let us denote by $B_\epsilon = \{k \leq n : |x_k - L| \geq \epsilon\}$ and $\overline{B_\epsilon} = \{k \leq n : |x_k - L| < \epsilon\}$. Then

$$\left|\frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k}\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}x_\nu - L\right| = \left|\frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k}\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}(x_\nu - L)\right| \leq$$

$$\frac{1}{R_n}\sum_{\substack{k=0\\k\in B_\epsilon}}^{n} p_k q_{n-k}\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}|x_\nu - L| + \frac{1}{R_n}\sum_{\substack{k=0\\k\in \overline{B}_\epsilon}}^{n} p_k q_{n-k}\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}|x_\nu - L| \le$$

(from given conditions for sequence (x_ν), there exists a constant C such that $|x_\nu - L| \le \frac{C}{\nu+k+n}$)

$$\frac{C}{R_n}\sum_{\substack{k=0\\k\in B_\epsilon}}^{n} p_k q_{n-k}\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}\frac{1}{\nu+k+n} + \frac{\epsilon}{R_n}\sum_{\substack{k=0\\k\in \overline{B}_\epsilon}}^{n} p_k q_{n-k}\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu} \le$$

$$\frac{C}{R_n}\sum_{\substack{k=0\\k\in B_\epsilon}}^{n} p_k q_{n-k}\frac{1}{2^k}\frac{1}{k+n}\sum_{\nu=0}^{k}\binom{k}{\nu} + \frac{\epsilon}{R_n}\sum_{k=0}^{n} p_k q_{n-k}\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu} \le$$

$$\le \frac{C|B_\epsilon|}{n}\cdot\frac{\max_{0\le k\le n}\{p_k q_{n-k}\}}{R_n} + \epsilon \to 0+\epsilon, \quad \text{as} \quad n\to\infty.$$

To show that converse is not true we will use into consideration this

Example 3 *Let us consider that* $(p_n) = 1$, $(q_{n-k}) = (2^k)$ *for* $n \in \mathbb{N}$ *and we define the sequence* $x = (x_n)$, *as follows:*

$$x_k = \begin{cases} 1 & , \quad \text{for} \quad k = m^2 - m, \cdots, m^2 - 1 \\ -m & , \quad \text{for} \quad k = m^2, m = 2, \cdots \\ 0 & , \quad \text{otherwise} \end{cases}.$$

Under this conditions, after some calculations we get:

$$\left|\frac{1}{2^n-1}\sum_{k=0}^{n} 1\cdot\sum_{\nu=0}^{k}\binom{k}{\nu}x_\nu - 1\right| \le \left|\frac{1}{2^n-1}\sum_{k=0}^{n} 1\cdot\sum_{\nu=0}^{k}\binom{k}{\nu} - 1\right| = 0.$$

From last relation follows that $x = (x_n)$ *is* $N^n_{p,q}E^1_n-$ *summable to* 1. *Hence from Theorem 1,* (x_n) *is* $N^n_{p,q}E^1_n-$ *statistically convergent. On the other hand, the sequence* $(m^2; m = 2, 3\cdots,)$ *has natural density zero and it is clear that* $st-\liminf_n x_n = 0$ *and* $st-\limsup_n x_n = 1$. *Thus,* (x_k) *is not statistically convergent.*

□

Theorem 3 *If*

$$st - \liminf_n \frac{R_{t_n}}{R_n} > 1, t > 1 \tag{6}$$

where t_n, *denotes the integral parts of the* $[tn]$ *for every* $n \in \mathbb{N}$, *and let* (x_k) *be a sequence of real numbers which converges to* L, $N^n_{p,q}E^1_n-$ *statistically. Then* (x_k) *is* $st-$ *convergent to the same number* L *if and only if the following two conditions hold:*

$$\inf_{t>1} \limsup_n \frac{1}{R_n} \left| \left\{ k \leq R_n : \frac{1}{R_{t_k} - R_k} \sum_{j=k+1}^{t_k} p_j q_{t_k - j} \frac{1}{2^j} \sum_{v=0}^{j} \binom{j}{v} (x_v - x_k) \leq -\epsilon \right\} \right| = 0 \tag{7}$$

and

$$\inf_{0<t<1} \limsup_n \frac{1}{R_n} \left| \left\{ k \leq R_n : \frac{1}{R_k - R_{t_k}} \sum_{j=t_k+1}^{k} p_j q_{k-j} \frac{1}{2^j} \sum_{v=0}^{j} \binom{j}{v} (x_k - x_v) \leq -\epsilon \right\} \right| = 0. \tag{8}$$

Remark 1 *Let us suppose that* $st - \lim_k x_k = L$*;* (x_n) *is* $N^n_{p,q}E^1_n-$ *statistically convergent and relation* (6) *satisfies, then for every* $t > 1$, *is valid the following relation:*

$$st - \lim_k \frac{1}{R_{t_k} - R_k} \sum_{j=k+1}^{t_k} p_j q_{t_k - j} \frac{1}{2^j} \sum_{v=0}^{j} \binom{j}{v} (x_v - x_k) = 0 \tag{9}$$

and in case where $0 < t < 1$,

$$st - \lim_k \frac{1}{R_k - R_{t_k}} \sum_{j=t_k+1}^{k} p_j q_{k-j} \frac{1}{2^j} \sum_{v=0}^{j} \binom{j}{v} (x_k - x_v) = 0. \tag{10}$$

In the next result, we will consider the case where $x = (x_n)$ is a sequence of complex numbers.

Theorem 4 *Let us suppose that relation* (6) *is satisfied. And* (x_n) *be a sequence of complex numbers, which is* $N^n_{p,q}E^1_n-$ *statistically convergent to* L. *Then* (x_n) *is* $st-$ *convergent to the same number* L *if and only if the following two conditions hold:*

$$\inf_{t>1} \limsup_n \frac{1}{R_n} \left| \left\{ k \leq R_n : \left| \frac{1}{R_{t_k} - R_k} \sum_{j=k+1}^{t_k} p_j q_{t_k - j} \frac{1}{2^j} \sum_{v=0}^{j} \binom{j}{v} (x_v - x_k) \right| \geq \epsilon \right\} \right| = 0 \tag{11}$$

and

$$\inf_{0<t<1} \limsup_n \frac{1}{R_n} \left| \left\{ k \leq R_n : \left| \frac{1}{R_k - R_{t_k}} \sum_{j=t_k+1}^{k} p_j q_{k-j} \frac{1}{2^j} \sum_{\nu=0}^{j} \binom{j}{\nu} (x_k - x_\nu) \right| \geq \epsilon \right\} \right| = 0. \tag{12}$$

In what follows we will show some auxiliary lemmas which are needful in the sequel.

Lemma 1 *Condition given by relation (6) is equivalent to this one:*

$$st - \liminf_n \frac{R_n}{R_{t_n}} > 1, \quad 0 < t < 1. \tag{13}$$

Proof. Let us suppose that relation (6) is valid, $0 < t < 1$ and $m = t_n = [t \cdot n]$, $n \in \mathbb{N}$. Then it follows that

$$\frac{1}{t} > 1 \Rightarrow \frac{m}{t} = \frac{[t \cdot n]}{t} \leq n,$$

from above relation we obtain:

$$\frac{R_n}{R_{t_n}} \geq \frac{R_{[\frac{m}{t}]}}{R_{t_n}} \Rightarrow st - \liminf_n \frac{R_n}{R_{t_n}} \geq st - \liminf_n \frac{R_{[\frac{m}{t}]}}{R_{t_n}} > 1.$$

Conversely, let us suppose that relation (13) is valid. Let $t > 1$ be given number and let t_1 be chosen such that $1 < t_1 < t$. Set $m = t_n = [t \cdot n]$. From $0 < \frac{1}{t} < \frac{1}{t_1} < 1$, it follows that:

$$n \leq \frac{tn - 1}{t_1} < \frac{[tn]}{t_1} = \frac{m}{t_1},$$

provided $t_1 \leq t - \frac{1}{n}$, which is a case where if n is large enough. Under this conditions we have:

$$\frac{R_{t_n}}{R_n} \geq \frac{R_{t_n}}{R_{\left[\frac{m}{t_1}\right]}} \Rightarrow st_\lambda - \liminf_n \frac{R_{t_n}}{R_n} \geq st_\lambda - \liminf_n \frac{R_{t_n}}{R_{\left[\frac{m}{t_1}\right]}} > 1.$$

□

Lemma 2 *Let us suppose that relation (6) is satisfied and let $x = (x_k)$ be a sequence of complex numbers which is $N^n_{p,q}E^1_n$–statistically convergent to L. Then for every $t > 0$,*

$$st - \lim_n N^{t_n}_{p,q} E^1_n = L.$$

Proof. (I) Let us consider that $t > 1$. Then

$$\lim_{n\to\infty} \frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k}\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}x_\nu = \lim_{n\to\infty} \frac{1}{R_{t_n}}\sum_{k=0}^{t_n} p_k q_{t_n-k}\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}x_\nu, \quad (14)$$

and for every $\epsilon > 0$ we have:

$$\{k \le R_{t_n} : |N_{p,q}^{t_n}E_n^1 - L| \ge \epsilon\} \subset \{k \le R_n : |N_{p,q}^{n}E_n^1 - L| \ge \epsilon\} \cup$$

$$\left\{k \le R_n : \frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k}\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}x_\nu \ne \frac{1}{R_{t_n}}\sum_{k=0}^{t_n} p_k q_{t_n-k}\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}x_\nu\right\}.$$

Now proof of the lemma in this case follows from relation (14) and $st - \lim_n N_{p,q}^{n}E_n^1 = L$.
(II) In this case we have that $0 < t < 1$. For $t_n = [t \cdot n]$, for any natural number n, we can conclude that $N_{p,q}^{t_n}E_n^1$ does not appears more than $[1 + t^{-1}]$ times in the sequence $N_{p,q}^{n}E_n^1$. In fact if there exist integers k, l such that

$$n \le t\cdot k < t(k+1) < \cdots < t(k+l-1) < n+1 \le t(k+l),$$

then

$$n + t(l-1) \le t(k+l-1) < n+1 \Rightarrow l < 1 + \frac{1}{t}.$$

And we have this estimation

$$\begin{aligned} &\frac{1}{R_n}\left|\left\{k \le R_n : |N_{p,q}^{t_n}E_n^1 - L| \ge \epsilon\right\}\right| \\ &\le \left(1+\frac{1}{t}\right)\frac{1}{R_n}\left|\left\{k \le R_{t_n} : |N_{p,q}^{n}E_n^1 - L| \ge \epsilon\right\}\right| \\ &\le 2(1+t)\frac{1}{R_{t_n}}\left|\left\{k \le R_{t_n} : |N_{p,q}^{n}E_n^1 - L| \ge \epsilon\right\}\right|, \end{aligned}$$

provided $\frac{1}{R_n}(\frac{t+1}{t}) \le 2(t+1)\frac{1}{R_{t_n}}$, which is the case where n is large enough. From last relation it follows: $st - \lim_n N_{p,q}^{t_n}E_n^1 = L$. □

Proposition 1 *Let us suppose that relation (6) is satisfied and let $x = (x_k)$ be a sequence of complex numbers which is $N_{p,q}^{t_n}E_n^1-$statistically convergent to L. Then for every $t > 1$,*

$$st - \lim_k \frac{1}{R_{t_k} - R_k}\sum_{j=k+1}^{t_k} p_j q_{t_k-j}\frac{1}{2^j}\sum_{\nu=0}^{j}\binom{j}{\nu}x_\nu = L; \quad (15)$$

and for every $0 < t < 1$,

$$st - \lim_k \frac{1}{R_k - R_{t_k}} \sum_{j=t_k+1}^{k} p_j q_{k-j} \frac{1}{2^j} \sum_{\nu=0}^{j} \binom{j}{\nu} x_\nu = L. \tag{16}$$

Proof. (I) Let us consider the case where $t > 1$. Then we obtain

$$\begin{aligned}
&\frac{1}{R_{t_n} - R_n} \sum_{k=n+1}^{t_n} p_k q_{t_n-k} \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} (x_\nu - L) \\
&= \frac{R_{t_n}}{R_{t_n} - R_n} \frac{1}{R_{t_n}} \sum_{k=0}^{t_n} p_k q_{t_n-k} \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} (x_\nu - L) \\
&\quad - \frac{R_n}{R_{t_n} - R_n} \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} (x_\nu - L) \\
&= \frac{R_{t_n}}{R_{t_n} - R_n} \frac{1}{R_{t_n}} \sum_{k=0}^{t_n} p_k q_{t_n-k} \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} (x_\nu - L) \\
&\quad - \frac{R_{t_n}}{R_{t_n} - R_n} \frac{1}{R_{t_n}} \sum_{k=0}^{n} p_k (q_{t_n-k} + q_{n-k} - q_{n-k}) \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} (x_\nu - L) \\
&= \frac{R_{t_n}}{R_{t_n} - R_n} \frac{1}{R_{t_n}} \sum_{k=0}^{t_n} p_k q_{t_n-k} \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} (x_\nu - L) \\
&\quad - \frac{R_n}{R_{t_n} - R_n} \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} (x_\nu - L) \\
&\quad - \frac{R_n}{R_{t_n} - R_n} \frac{1}{R_n} \sum_{k=0}^{n} p_k (q_{t_n-k} - q_{n-k}) \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} (x_\nu - L).
\end{aligned} \tag{17}$$

From

$$\limsup_n \frac{R_{t_n}}{R_{t_n} - R_n} < \infty, \tag{18}$$

definition of the sequence (q_n), Lemma 2 and relation (17), we get relation (15).

(II) In this case we have that $0 < t < 1$. Then

$$\frac{1}{R_n - R_{t_n}} \sum_{k=t_n+1}^{n} p_k q_{n-k} \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} x_\nu = \frac{R_n}{R_n - R_{t_n}} \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} x_\nu -$$

$$\frac{R_{t_n}}{R_n - R_{t_n}} \frac{1}{R_{t_n}} \sum_{k=0}^{t_n} p_k q_{n-k} \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} x_v = \frac{R_{t_n}}{R_n - R_{t_n}} \frac{1}{R_{t_n}} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} x_v -$$

$$\frac{R_{t_n}}{R_n - R_{t_n}} \frac{1}{R_{t_n}} \sum_{k=0}^{t_n} p_k q_{t_n-k} \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} x_v - \frac{1}{R_n - R_{t_n}} \sum_{k=0}^{t_n} p_k (q_{n-k} - q_{t_n-k}) \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} x_v.$$

Now proof of the proposition is similar to the first part. □

3 Proofs of the theorems

Proof of Theorem 3. *Necessity.* Suppose that $\lim_{n\to\infty} x_n = L$, and (6) holds. Following Proposition 1, we have

$$st - \lim_{n\to\infty} \frac{1}{R_{\lambda_n} - R_n} \sum_{k=n+1}^{\lambda_n} p_k q_{\lambda_n - k} \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} (x_v - x_n) =$$

$$st - \lim_{n\to\infty} \frac{1}{R_{\lambda_n} - R_n} \sum_{k=n+1}^{\lambda_n} p_k q_{\lambda_n - k} \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} x_v - x_n = 0,$$

for every $\lambda > 1$. In case where $0 < \lambda < 1$, we find that

$$st - \lim_{n\to\infty} \frac{1}{R_n - R_{\lambda_n}} \sum_{k=\lambda_n+1}^{n} p_k q_{n-k} \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} (x_n - x_v) =$$

$$x_n - st - \lim_{n\to\infty} \frac{1}{R_n - R_{\lambda_n}} \sum_{k=\lambda_n+1}^{n} p_k q_{n-k} \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} x_v = 0.$$

Sufficiency. Assume that conditions (7) and (8) are satisfied. In what follows we will prove that $st - \lim_{n\to\infty} x_n = L$. Or equivalently, $st - \lim (N^n_{p,q} E^1_n - x_n) = 0$. First we will consider the case where $t > 1$. We will start from this estimation:

$$x_n - N^n_{p,q} E^1_n$$
$$= \frac{R_{t_n}}{R_{t_n} - R_n} \left[\frac{1}{R_{t_n}} \sum_{j=0}^{t_n} p_j q_{t_n - j} \frac{1}{2^j} \sum_{v=0}^{j} \binom{j}{v} x_v - \frac{1}{R_n} \sum_{j=0}^{n} p_j q_{n-j} \frac{1}{2^j} \sum_{v=0}^{j} \binom{j}{v} x_v \right]$$
$$- \frac{1}{R_{t_n} - R_n} \sum_{j=n+1}^{t_n} p_j q_{t_n - j} \frac{1}{2^j} \sum_{v=0}^{j} \binom{j}{v} (x_v - x_n).$$

For any $\epsilon > 0$, we obtain:

$$\{k \leq R_n : x_k - N^n_{p,q}E^1_n \geq \epsilon\} \subset \left\{k \leq R_n : \frac{R_{t_k}}{R_{t_k} - R_k}(N^{t_k}_{p,q}E^1_k - N^k_{p,q}E^1_k) \geq \frac{\epsilon}{2}\right\} \cup$$

$$\left\{k \leq R_n : \frac{1}{R_{t_k} - R_k}\sum_{j=k+1}^{t_k} p_j q_{t_k - j}\frac{1}{2^j}\sum_{\nu=0}^{j}\binom{j}{\nu}(x_\nu - x_k) \leq -\frac{\epsilon}{2}\right\}.$$

From relation (7), it follows that for every $\gamma > 0$, exists a $t > 1$ such that

$$\limsup_n \frac{1}{R_n}\left|\left\{k \leq R_n : \frac{1}{R_{t_k} - R_k}\sum_{j=k+1}^{t_k} p_j q_{t_k - j}\frac{1}{2^j}\sum_{\nu=0}^{j}\binom{j}{\nu}(x_\nu - x_k) \leq -\epsilon\right\}\right| \leq \gamma.$$

By Lemma 2 and relation (18) we get

$$\limsup_n \frac{1}{R_n}\left|\left\{k \leq R_n : |R_{t_k}(R_{t_k} - R_k)^{-1}(N^{t_k}_{p,q}E^1_k - N^k_{p,q}E^1_k)| \geq \frac{\epsilon}{2}\right\}\right| = 0.$$

Combining last three relations we have:

$$\limsup_n \frac{1}{R_n}\left|\left\{k \leq R_n : x_k - N^k_{p,q}E^1_k \geq \epsilon\right\}\right| \leq \gamma,$$

and γ is arbitrary, we conclude that for every $\epsilon > 0$,

$$\limsup_n \frac{1}{R_n}\left|\left\{k \leq R_n : x_k - N^k_{p,q}E^1_k \geq \epsilon\right\}\right| = 0. \tag{19}$$

Now we consider case where $0 < t < 1$. From above we get that:

$$x_n - N^n_{p,q}E^1_n$$
$$= \frac{R_{t_n}}{R_n - R_{t_n}}\left[\frac{1}{R_n}\sum_{j=0}^{n} p_j q_{n-j}\frac{1}{2^j}\sum_{\nu=0}^{j}\binom{j}{\nu}x_\nu - \frac{1}{R_{t_n}}\sum_{j=0}^{t_n} p_j q_{t_n - j}\frac{1}{2^j}\sum_{\nu=0}^{j}\binom{j}{\nu}x_\nu\right]$$
$$+ \frac{1}{R_n - R_{t_n}}\sum_{j=t_n+1}^{n} p_j q_{n-j}\frac{1}{2^j}\sum_{\nu=0}^{j}\binom{j}{\nu}(x_n - x_\nu).$$

For any $\epsilon > 0$,

$$\{k \leq R_n : x_k - N^k_{p,q}E^1_k \geq \epsilon\} \subset \left\{k \leq R_n : \frac{R_{t_k}}{R_k - R_{t_k}}(N^k_{p,q}E^1_k - N^{t_k}_{p,q}E^1_k) \geq \frac{\epsilon}{2}\right\} \cup$$

$$\left\{k \le R_n : \frac{1}{R_k - R_{t_k}} \sum_{j=t_k+1}^{k} p_j q_{k-j} \frac{1}{2^j} \sum_{\nu=0}^{j} \binom{j}{\nu} (x_k - x_\nu) \le -\frac{\epsilon}{2}\right\}.$$

For same reasons as in the case where $t > 1$, by Lemma 2, we have that for every $\epsilon > 0$,

$$\limsup_n \frac{1}{R_n} \left| \left\{ k \le R_n : x_k - N^k_{p,q} E^1_k \le -\epsilon \right\} \right| = 0. \tag{20}$$

Finally from relations (19) and (20) we get:

$$\limsup_n \frac{1}{R_n} \left| \left\{ k \le R_n : |x_k - N^k_{p,q} E^1_k| \ge \epsilon \right\} \right| = 0. \quad \square$$

Remark 2 *Let us suppose that* $st - \lim_k x_k = L$, $st - \lim_k N^k_{p,q} E^1_k = L$ *and relation* (6) *satisfies. Then for every* $t > 1$, *relation* (9) *holds, and in case where* $0 < t < 1$, *relation* (10) *is valid.*

Proof of Theorem 4. We omit it because it is similar to the Theorem 3. □

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Commutative feebly nil-clean group rings

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Abstract. An arbitrary unital ring R is called *feebly nil-clean* if any its element is of the form $q + e - f$, where q is a nilpotent and e, f are idempotents with $ef = fe$. For any commutative ring R and any abelian group G, we find a necessary and sufficient condition when the group ring $R(G)$ is feebly nil-clean only in terms of R, G and their sections. Our result refines establishments due to McGovern et al. in J. Algebra Appl. (2015) on nil-clean rings and Danchev-McGovern in J. Algebra (2015) on weakly nil-clean rings, respectively.

1 Introduction and background

Throughout the text of this short paper, all rings R are assumed to be associative and commutative, containing identity element which differs from the zero element. Our terminology and notations are mainly in agreement with [10] and [11]. For instance, $J(R)$ denotes the Jacobson radical of R, and $N(R)$ denotes the nil-radical of R. Also, let everywhere in the text G be a multiplicative abelian group, and let $R(G)$ be the group ring of G over R. As usual, G_p stands for the p-torsion component of the group G with p-socle $G[p] = \{a \in G \mid a^p = 1\}$, and we shall say that the group G is a p-group, provided $G = G_p$. Likewise, we set $G^p = \{g^p \mid g \in G\}$ to be the p-th power subgroup of the group G.

A ring R is known to be *nil-clean* if, for every $r \in R$, there are a nilpotent $q \in R$ and an idempotent $e \in R$ such that $r = q + e$. The next necessary and

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sufficient condition for a commutative group ring to be nil-clean was recently obtained in [9]. Specifically, the following holds: *The commutative group ring* $R(G)$ *is nil-clean if, and only if, the ring* R *is nil-clean and the group* G *is a* 2*-group.*

Generalizing this, a ring R is known to be *weakly nil-clean* if, for every $r \in R$, there are a nilpotent $q \in R$ and an idempotent $e \in R$ such that $r = q + e$ or $r = q - e$. Alternatively, a necessary and sufficient condition for a commutative group ring to be weakly nil-clean was recently obtained in [4]. Precisely, the following holds: *The commutative group ring* $R(G)$ *is weakly nil-clean if, and only if, either* $G = \{1\}$ *and* R *is weakly nil-clean, or* R *is nil-clean and* G *is a non-trivial* 2*-group, or* $R/N(R) \cong \mathbb{Z}_3$ *and* G *is a non-trivial* 3*-group.*

As a common generalization of these two definitions for nil-clean and weakly nil-clean rings, a ring R is known to be *feebly nil-clean* (see, for example, [1] and [2]) if, for every $r \in R$, there are a nilpotent $q \in R$ and two idempotents $e, f \in R$ such that $r = q + e - f$. So, the leitmotif of writing up this brief article is to generalize somewhat the above two claims by obtaining a criterion for an arbitrary commutative group ring to be feebly nil-clean.

For completeness of the exposition, it is worthwhile noticing that an extension of the aforementioned nil-clean rings are the so-called *UU rings* that are rings whose units are unipotents (i.e., the sum of 1 and some nilpotent). In [3] were examined commutative UU group rings. Exactly, it was proved in Corollary 2.3 there that $R(G)$ *is a commutative UU ring if, and only if,* R *is a commutative UU ring and* G *is an abelian* 2*-group.*

2 Main results

Before proving our chief statement, we need the following two key formulas from [7] and [8], respectively. In fact, appealing to [7], one writes the formula

$$J(R(G)) = J(R)(G) + \langle r(g-1) \mid g \in G_p, pr \in J(R) \rangle.$$

In that aspect, consulting with [8], one writes the formula

$$N(R(G)) = N(R)(G) + \langle r(g-1) \mid g \in G_p, pr \in N(R) \rangle.$$

Standardly, $I(R(G); G)$ designates the fundamental (augmentation) ideal of $R(G)$ with respect to G with basis consisting of all elements of the type $1 - g$, where $g \in G$. It is well known that the isomorphisms

$$R(G)/[N(R)(G) + I(R(G); G)] \cong R/N(R)$$

and

$$R(G)/I(R(G);G) \cong R$$

are valid. We will now prove something similar and helpful for us for further use.

Proposition 1 *Let R be a ring and G a group. Then the following isomorphism is fulfilled:*

$$R(G)/N(R)(G) \cong (R/N(R))(G).$$

Proof. There is the natural ring surjection $R \to R/N(R)$ which induces by the usual element-wise manipulation the ring surjective homomorphism $R(G) \to (R/N(R))(G)$. This epimorphism obviously has kernel $N(R)G$ and henceforth the well-known Homomorphism Theorem applies to get the desired assertion. □

A ring is *boolean* if each its element is an idempotent. Let us recall that a ring is said to be *tripotent* if each its element x satisfies the equation $x^3 = x$. These rings are necessarily commutative being also a subdirect product of a family of single or isomorphic copies of the fields $\mathbb{Z}_2$ and $\mathbb{Z}_3$ (see, e.g., [6]). Likewise, as $6 = 0$ here, any tripotent ring R can be decomposed as the direct product of two rings $R_1 \times R_2$, where R_1 is boolean and R_2 is tripotent of characteristic 3. It is pretty evident that reduced feebly nil-clean rings are themselves tripotent.

We begin our work with a few useful technicalities.

Lemma 1 *The next two statements are true:*

(i) *A direct factor of a feebly nil-clean ring is a feebly nil-clean ring as well.*

(ii) *The direct product of two feebly nil-clean rings is also a feebly nil-clean ring.*

Proof. Straightforward by a direct check, so that we leave it to the interested reader. □

Lemma 2 *An epimorphic image of a feebly nil-clean ring is too a feebly nil-clean ring.*

Proof. Since nilpotents and idempotents map under any homomorphism again into nilpotents and idempotents, respectively, the claim follows elementarily. □

Proposition 2 *Suppose that R is a commutative ring. Then the following three points are equivalent:*

(i) R *is feebly nil-clean.*
(ii) $J(R)$ *is nil and* $R/J(R)$ *is tripotent.*
(iii) $R/N(R)$ *is tripotent.*

Proof. It suffices to prove only the equivalence (i) $\iff$ (iii), because whenever $J(R)$ is nil we will have that $J(R) = N(R)$ as well as, in accordance with [1] or [2], R being feebly nil-clean yields that $J(R)$ is nil. To this purpose, the implication (i) $\Rightarrow$ (iii) follows at once by the usage of Lemma 2.

As for the converse implication (i) $\Leftarrow$ (iii), we may write by consulting with [6, Theorem 1] accomplished with a simple trick that every element of $R/N(R)$ is the difference of two idempotents, say $r+N(R) = (e_1+N(R))-(e_2+N(R)) = e_1 - e_2 + N(R)$, where $r \in R$ is an arbitrary element and $e_1, e_2 \in R$ are some elements. But as it is well-known as a folklore fact, we may choose these e_1 and e_2 to be idempotents. Consequently, one follows that $r = t + e_1 - e_2$ for some nilpotent t in R, as expected. □

As an immediate consequence, one yields the following.

Corollary 1 *Let* I *be a nil ideal of a ring* R*. Then* R *is feebly nil-clean if, and only if,* R/I *is feebly nil-clean.*

Proof. The "necessity" follows by virtue of Lemma 2. As for the "sufficiency", because of the inclusion $I \subseteq N(R)$, there exists an epimorphism $R/I \to R/N(R)$ with kernel $N(R)/I = N(R/I)$. Hence $R/N(R)$ is feebly nil-clean, i.e., tripotent. Furthermore, we apply Proposition 2 to conclude the claim. □

We now have all the ingredients necessary to proceed by proving the following chief assertion, which gives a necessary and sufficient condition when a commutative group ring will be feebly nil-clean.

Theorem 1 *Suppose* R *is a commutative ring and* G *is an abelian group. Then the group ring* $R(G)$ *is feebly nil-clean if, and only if, exactly one of the following three items is valid:*

(1) $G = \{1\}$ *and* R *is feebly nil-clean.*

(2) $G \neq \{1\}$ *and* $R/N(R) \cong R_1 \times R_2$*, where* R_1 *is boolean and* R_2 *is tripotent of characteristic* 3 *such that*

(a) $R_1 = \{0\}$*, or* $R_1 \neq \{0\}$ *and* G *is a* 2*-group;*

(b) $R_2 = \{0\}$*, or* $R_2 \neq \{0\}$ *and either* $G = G_3$ *or* $G = G_3 \times G[2]$*.*

Proof. "Left-to-right". The assumption that G is the trivial group leads to $R(G) \cong R$, so that we may assume without loss of generality that G is non-trivial.

The epimorphism $R(G) \to R$ implies that R is feebly nil-clean and thus Proposition 2 (iii) enables us that $R/N(R)$ is tripotent. Therefore, the main result in [6] allows us to write that $R/N(R) \cong R_1 \times R_2$, where R_1 is a boolean ring and R_2 is a tripotent ring of characteristic 3.

On the other hand, as in the proof of Proposition 1, the surjection $R \to R/N(R)$ induces a surjection $R(G) \to (R/N(R))(G)$ and so in view of Lemma 2 the group ring $(R/N(R))(G) \cong R_1(G) \times R_2(G)$ has to be feebly nil-clean, too. Since $2 = 0$ in R_1, with the aid of Lemma 1 (i) it must be that $R_1(G)$ is feebly nil-clean of characteristic 2 whence it is necessarily nil-clean, because under these circumstances the sum of two idempotents is again an idempotent. Employing now the quoted above result from [9], we derive that either R_1 is zero, or R_1 is non-trivial and $G = G_2$ is a 2-primary group.

Further, concerning the second direct factor R_2, let us assume that it is non-zero and hence a subdirect product of the field $\mathbb{Z}_3$. Since there exist two epimorphisms, namely $R_2 \to \mathbb{Z}_3$ and $G \to G/G_3$, one infers that there is an induced epimorphism $R_2(G) \to \mathbb{Z}_3(G/G_3)$ which gives with the help of Lemma 2 that the epimorphic image $\mathbb{Z}_3(G/G_3)$ is feebly nil-clean as so is $R_2(G)$ being a direct factor of $(R/N(R))(G)$. According to the listed above formula of May from [8], we obtain that $\mathbb{Z}_3(G/G_3)$ is reduced and thus it is certainly tripotent by using once again Proposition 2 (iii). Consequently, the equation $z^3 = z$ holds in the factor-group G/G_3, that is, $z^2 = 1$. We may have $G/G_3 = \{\overline{1}\}$, that is, $G = G_3$. If now $G \neq G_3$, letting g be an arbitrary element in G, one deduces that $(gG_3)^2 = G_3$, i.e., $g^2G_3 = G_3$, i.e., $g^2 \in G_3$. But the 3-component G_3 is always 2-divisible, that is, $G_3 = G_3^2$ (see, e.g., [5]). This, in turn, forces that $g = g_3a \subseteq G_3G[2]$ for some $g_3 \in G_3$ and $a \in G[2]$ assuring the direct decomposition $G = G_3 \times G[2]$, as wanted.

"**Right-to-left**". Because item (1) implies at once that $R(G) \cong R$, the claim follows immediately.

We, therefore, will be concentrated on the non-trivial case for G, which is exactly point (2). With Proposition 1 at hand, we have that $R(G)/N(R)(G)$ is isomorphic to $(R/N(R))(G) \cong R_1(G) \times R_2(G)$ with nil $N(R)(G) \subseteq N(R(G))$. Therefore, applying Corollary 1, one needs to show the feebly nil-cleanness of $(R/N(R))(G)$ only. To that aim, condition (a) along with the major result from [9] rich us that the group ring $R_1(G)$ is nil-clean and so feebly nil-clean.

On the other side, concerning condition (b), the two possibilities $G = G_3$ and $G = G_3 \times G[2]$ will imply that either $R_2(G) \cong R_2(G_3)$ or $R_2(G) \cong R_2(G_3) \times R_2(G[2])$, where the validity of the latter isomorphism is formally assumed. Moreover, as the characteristic of R_2 is 3 and the equality $x^3 = x$ holds both in R_2 and in $G[2]$, it is readily verified by utilizing only technical arguments that it will hold in the group ring $R_2(G[2])$ as well. Thus $R_2(G[2])$ is feebly nil-clean. We claim, besides, that $R_2(G_3)$ is also feebly nil-clean. Indeed, referring to the noticed above formula of May from [8], one detects by using routine argumentation that $N(R_2(G_3)) = I(R_2(G_3); G_3)$. However, as noted before, $R_2(G_3)/N(R_2(G_3)) = R_2(G_3)/I(R_2(G_3); G_3) \cong R_2$ implies the tripotent property, which invoking Proposition 2 substantiates our claim, as expected. We, finally, just need to apply once again Lemma 1 (ii) to get the desired feebly nil-cleanness of the group ring $R_2(G)$, thus concluding the initial assertion for feebly nil-cleanness of the group ring $R(G)$, as promised.

As a new and somewhat more direct and comfortable confirmation that the group ring $R_2(G)$ is feebly nil-clean in the case when G is a decomposable group as stated above, we may process like this: Since $G = G_3 \times G[2]$, it follows at once that $R_2(G) \cong (R_2(G[2]))(G_3) = R_2'(G_3)$, where we putted $R_2' := R_2(G[2])$. As we already showed above, R_2' is a ring of characteristic 3 in which the equality $x^3 = x$ holds for all its elements. Thus, in particular, it should be reduced as well. Furthermore, as we have demonstrated, $N(R_2'(G_3)) = I(R_2'(G_3); G_3)$ and, consequently, $R_2'(G_3)/N(R_2'(G_3)) = R_2'(G_3)/I(R_2'(G_3); G_3) \cong R_2'$ is tripotent (i.e., reduced feebly nil-clean), as expected. This gives the desired feebly nil-cleanness of the group ring $R_2'(G_3)$ which, in turn, substantiates the promised feebly nil-cleanness of $R_2(G)$ after all. □

We close with some more comments.

Remark 1 *Utilizing the stated above formula of Karpilovsky from* [7]*, we can deduce an equivalent necessary and sufficient condition for a commutative group ring to be feebly nil-clean in terms of* $J(R)$ *instead of* $N(R)$*.*

We end the work with a problem of interest.

Problem 1 *Find a criterion when an arbitrary (not necessarily commutative) group ring is feebly nil-clean.*

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Some inequalities for double integrals and applications for cubature formula

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Abstract. We establish two Ostrowski type inequalities for double integrals of second order partial derivable functions which are bounded. Then, we deduce some inequalities of Hermite-Hadamard type for double integrals of functions whose partial derivatives in absolute value are convex on the co-ordinates on rectangle from the plane. Finally, some applications in Numerical Analysis in connection with cubature formula are given.

1 Introduction

Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$. Then, the inequality holds:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2} \right](b-a)\left\|f'\right\|_\infty \tag{1}$$

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for all $x \in [a, b]$ [15]. The constant $\frac{1}{4}$ is the best possible. This inequality is well known in the literature as the *Ostrowski inequality*.

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$, with $a < b$. The following double inequality is well known in the literature as the Hermite-Hadamard inequality [9]:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

This inequality has attracted considerable attention and interest from mathematicians and other researchers as shown by hundreds of papers published in the last decade one can find by making a simple search in the MathSciNet database of the American Mathematical Society. For example, Bakula et al. presented some Hermite-Hadamard type inequalities for m-convex and (α, m)-convex functions in [3].

In a recent paper [2], Barnett and Dragomir proved the following Ostrowski type inequality for double integrals:

Theorem 1 *Let* $f : [a, b] \times [c, d] \to \mathbb{R}$ *be continuous on* $[a, b] \times [c, d]$, $f''_{x,y} = \frac{\partial^2 f}{\partial x \partial y}$ *exists on* $(a, b) \times (c, d)$ *and is bounded, i.e.,*

$$\left\|f''_{x,y}\right\|_\infty = \sup_{(x,y)\in(a,b)\times(c,d)} \left|\frac{\partial^2 f(x,y)}{\partial x \partial y}\right| < \infty.$$

Then, we have the inequality:

$$\begin{aligned}
&\left|\int_a^b\int_c^d f(s,t)dtds - (d-c)(b-a)f(x,y)\right. \\
&\quad \left. - \left[(b-a)\int_c^d f(x,t)dt + (d-c)\int_a^b f(s,y)ds\right]\right| \\
&\leq \left[\frac{1}{4}(b-a)^2 + (x - \frac{a+b}{2})^2\right]\left[\frac{1}{4}(d-c)^2 + (y - \frac{d+c}{2})^2\right]\left\|f''_{x,y}\right\|_\infty
\end{aligned} \tag{2}$$

for all $(x, y) \in [a, b] \times [c, d]$.

In [2], the inequality (2) is established by the use of integral identity involving Peano kernels. In [16], Pachpatte obtained a new inequality in the view of (2) by using elementary analysis. Latif et al. proved some Ostrowski type inequalities for functions that are co-ordinated convex in [12]. Sarikaya gave integral inequalities for bounded functions in [20]. Authors deduced weighted version of Ostrowski type inequalities for double integrals involving functions of two independent variables by using fairly elementary analysis in [1], [18], [19] and [24].

Let us now consider a bidimensional interval $\Delta =: [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. A mapping $f : \Delta \to \mathbb{R}$ is said to be convex on Δ if the following inequality:

$$f(tx + (1-t)z, ty + (1-t)w) \leq tf(x,y) + (1-t)f(z,w)$$

holds, for all $(x,y), (z,w) \in \Delta$ and $t \in [0,1]$. A function $f : \Delta \to \mathbb{R}$ is said to be on the co-ordinates on Δ if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ are convex where defined for all $x \in [a,b]$ and $y \in [c,d]$ (see, [5]).

A formal definition for co-ordinated convex function may be stated as follows:

Definition 1 *A function* $f : \Delta \to \mathbb{R}$ *will be called co-ordinated convex on* Δ*, for all* $t, s \in [0,1]$ *and* $(x,y), (u,v) \in \Delta$*, if the following inequality holds:*

$$\begin{aligned} &f(tx + (1-t)y, su + (1-s)v) \\ &\quad \leq tsf(x,u) + s(1-t)f(y,u) + t(1-s)f(x,v) + (1-t)(1-s)f(y,v). \end{aligned}$$

Clearly, every convex function is a co-ordinated convex. Furthermore, there exists co-ordinated convex function which is not convex, (see, [5]).
Also, in [5], Dragomir established the following similar inequality of Hadamard's type for co-ordinated convex mapping on a rectangle from the plane $\mathbb{R}^2$.

Theorem 2 *Suppose that* $f : \Delta \to \mathbb{R}$ *is co-ordinated convex on* Δ*. Then one has the inequalities:*

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] \\ &\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dy\,dx \\ &\leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx + \frac{1}{b-a}\int_a^b f(x,d)\,dx \right. \\ &\quad \left. + \frac{1}{d-c}\int_c^d f(a,y)\,dy + \frac{1}{d-c}\int_c^d f(b,y)\,dy\right] \\ &\leq \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4}. \end{aligned} \tag{3}$$

The above inequalities are sharp.

In recent years, researchers have studied some integral inequalities by using variety convex functions on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$. For example, authors gave some Hadamard's type inequalities involving Riemann-Liouville fractional integrals for convex and s-convex functions on the co-ordinates in [4] and [21]. in [6], Dragomir et al. worked an Ostrowski type inequality for two dimensional integrals in term of L_p-norms. Erden and Sarikaya deduced weighted version of Hermite-Hadamard type inequalities for functions whose partial derivatives in absolute value are convex on the co-ordinates on rectangle from the plane in [7] and [8]. In [10], [12]-[14], [22] and [23], some integral inequalities for differentiable co-ordinated convex mappings are obtained. In [21], Sarikaya et al. proved some new inequalities that give estimate of the difference between the middle and the most right terms of (3) for differentiable co-ordinated convex functions. In [7], [11] and [17], some Hermite-Hadamard type inequalities are developed for variety co-ordinated convex functions.

In this study, we firstly establish an identity for second order partial derivative functions. Then, two inequalities of Ostrowski type for double integrals is gotten by using this identity. Also, Hermite-Hadamard type inequalities for convex mappings on the co-ordinates on the rectangle from the plane are obtained. Finally, some applications of the Ostrowski type inequality developed in this work for cubature formula are given.

2 Main results

We need the following lemma so as to prove our main results.

Lemma 1 *Let* $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ *be an absolutely continuous function such that the partial derivative of order* 2 *exists for all* $(t, s) \in [a, b] \times [c, d]$. *Then, for all* $(x, y) \in [a, b] \times [c, d]$, *we have the equality*

$$\begin{aligned}&\int_a^b\int_c^d P_h(x,t)\,Q_h(y,s)\,f_{ts}(t,s)\,dsdt\\&=\int_a^b\int_c^d f(t,s)\,dsdt+m_h(x)\int_c^d\left[f(b,s)-f(a,s)\right]ds\end{aligned} \tag{4}$$

$$+ m_h(y) \int_a^b [f(t,d) - f(t,c)]\, dt - (d-c) \int_a^b f(t,y)\, dt - (b-a) \int_c^d f(x,s)\, ds$$
$$+ (b-a)(d-c) f(x,y) + m_h(x) m_h(y) [f(a,c) - f(a,d) - f(b,c) + f(b,d)]$$
$$- (d-c)\, m_h(x) [f(b,y) - f(a,y)] - (b-a)\, m_h(y) [f(x,d) - f(x,c)]$$
$$= S_h(x,y,s,t)$$

for

$$P_h(x,t) := \begin{cases} (t - a - m_h(x)), & a \le t < x \\ (t - b - m_h(x)), & x \le t \le b \end{cases}$$

$$Q_h(y,s) := \begin{cases} (s - c - m_h(y)), & c \le s < y \\ (s - d - m_h(y)), & y \le s \le d \end{cases}$$

where $m_h(x) = h\left(x - \frac{a+b}{2}\right)$ *and* $m_h(y) = h\left(y - \frac{c+d}{2}\right)$, $h \in [0,2]$.

Proof. By definitions of $P_h(x,t)$ and $Q_h(y,s)$, we have

$$\int_a^b \int_c^d P_h(x,t)\, Q_h(y,s)\, f_{ts}(t,s)\, ds dt$$
$$= \int_a^x \int_c^y [t - a - m_h(x)] [s - c - m_h(y)] f_{ts}(t,s)\, ds dt$$
$$+ \int_a^x \int_y^d [t - a - m_h(x)] [s - d - m_h(y)] f_{ts}(t,s)\, ds dt \tag{5}$$
$$+ \int_x^b \int_c^y [t - b - m_h(x)] [s - c - m_h(y)] f_{ts}(t,s)\, ds dt$$
$$+ \int_x^b \int_y^d [t - b - m_h(x)] [s - d - m_h(y)] f_{ts}(t,s)\, ds dt.$$

Now, we examine the above integrals. Applying integration by parts twice for the first integral in the right hand side of (5), we find that

$$\begin{aligned}
&\int_a^x\int_c^y [t-a-m_h(x)]\,[s-c-m_h(y)]\,f_{ts}\,(t,s)\,dsdt \\
&\quad= [x-a-m_h(x)]\,[y-c-m_h(y)]\,f\,(x,y)+[y-c-m_h(y)]\,m_h(x)f\,(a,y) \\
&\qquad -[y-c-m_h(y)]\int_a^x f\,(t,y)\,dt+m_h(y)\,[x-a-m_h(x)]\,f\,(x,c) \\
&\qquad + m_h(x)m_h(y)f\,(a,c)-m_h(y)\int_a^x f\,(t,c)\,dt-[x-a-m_h(x)]\int_c^y f\,(x,s)\,ds \\
&\qquad - m_h(x)\int_c^y f\,(a,s)\,ds+\int_a^x\int_c^y f\,(t,s)\,dsdt.
\end{aligned}$$

If we calculate the other integrals in a similar way and then we substitute the results in (5), we obtain desired equality (4) which completes the proof. □

Now, we establish a new integral inequality for double integrals and also give some results related to this theorem.

Theorem 3 *Suppose that all the assumptions of Lemma 1 hold. If* $f_{ts}=\frac{\partial^2 f}{\partial t\partial s}$ *exists on* $(a,b)\times(c,d)$ *and is bounded, i.e.,*

$$\|f_{ts}\|_\infty=\sup_{(t,s)\in(a,b)\times(c,d)}\left|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right|<\infty.$$

Then, we have the inequality:

$$\begin{aligned}
&|S_h\,(x,y,s,t)| \\
&\le\left[\left(\frac{b-a}{2}\right)^2+\left(x-\frac{a+b}{2}\right)^2+(h-2)\left(x-\frac{a+b}{2}\right)m_h(x)\right] \\
&\quad\times\left[\left(\frac{d-c}{2}\right)^2+\left(y-\frac{c+d}{2}\right)^2+(h-2)\left(y-\frac{c+d}{2}\right)m_h(y)\right]\|f_{ts}\|_\infty
\end{aligned}\tag{6}$$

for all $(x,y)\in[a,b]\times[c,d]$, *where* $m_h(x)=h\left(x-\frac{a+b}{2}\right)$ *and* $m_h(y)=h\left(y-\frac{c+d}{2}\right)$, $h\in[0,2]$.

Proof. Taking absolute value of (4) and using bounded of the mapping f_{ts}, we find that

$$\begin{aligned}
|S_h(x,y,s,t)| &\le \|f_{ts}\|_\infty \int_a^b\int_c^d |P_h(x,t)|\,|Q_h(y,s)|\,dsdt \\
&= \|f_{ts}\|_\infty \left[\int_a^x |t-a-m_h(x)|\,dt + \int_x^b |t-b-m_h(x)|\,dt\right] \\
&\quad\times \left[\int_c^y |s-c-m_h(y)|\,dt + \int_y^d |s-d-m_h(y)|\,ds\right].
\end{aligned} \tag{7}$$

We shall observe the above integrals for the cases $a \le x \le \frac{a+b}{2}$ and $\frac{a+b}{2} \le x \le b$;

For all $a \le x \le \frac{a+b}{2}$, we have

$$\int_a^x |t-a-m_h(x)|\,dt = \frac{(x-a)^2}{2} - (x-a)\,m_h(x)$$

and

$$\int_x^b |t-b-m_h(x)|\,dt = \frac{(b-x)^2}{2} + (b-x)\,m_h(x) + [m_h(x)]^2 .$$

For all $\frac{a+b}{2} \le x \le b$, we write

$$\int_a^x |t-a-m_h(x)|\,dt = \frac{(x-a)^2}{2} - (x-a)\,m_h(x) + [m_h(x)]^2$$

and

$$\int_x^b |t-b-m_h(x)|\,dt = \frac{(b-x)^2}{2} + (b-x)\,m_h(x).$$

Then, we get

$$\begin{aligned}
&\int_a^x |t-a-m_h(x)|\,dt + \int_x^b |t-b-m_h(x)|\,dt \\
&= \frac{(b-x)^2+(x-a)^2}{2} + 2\left(\frac{a+b}{2}-x\right) m_h(x) + [m_h(x)]^2 .
\end{aligned} \tag{8}$$

Similarly, we obtain

$$\begin{aligned}&\int_c^y |s-c-m_h(y)|\,dt+\int_y^d |s-d-m_h(y)|\,ds\\&=\frac{(d-y)^2+(y-c)^2}{2}+2\left(\frac{c+d}{2}-y\right)m_h(y)+[m_h(y)]^2.\end{aligned}\tag{9}$$

If we substitute the equality (8) and (9) in (7), we easily deduce required inequality (6) which completes the proof. □

Remark 1 *If we take* $x=\frac{a+b}{2}$ *and* $y=\frac{c+d}{2}$ *in Theorem 3, then we have the mid-point inequality*

$$\begin{aligned}&\Bigg|\int_a^b\int_c^d f(t,s)\,dsdt+(b-a)(d-c)\,f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\&\quad-(d-c)\int_a^b f\left(t,\frac{c+d}{2}\right)dt-(b-a)\int_c^d f\left(\frac{a+b}{2},s\right)ds\Bigg|\\&\leq\frac{1}{16}(b-a)^2(d-c)^2\|f_{ts}\|_\infty\end{aligned}$$

which was given by Barnett and Dragomir in [2].

Remark 2 *Under the same assumptions of Theorem 3 with* $h=1$ *and* $(x,y)=(a,c)$, *then the following inequality holds:*

$$\begin{aligned}&\Bigg|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt\\&\quad-\frac{1}{2}\left[\frac{1}{d-c}\int_c^d[f(b,s)+f(a,s)]\,ds+\frac{1}{b-a}\int_a^b[f(t,d)+f(t,c)]\,dt\right]\Bigg|\\&\leq\frac{(b-a)(d-c)}{16}\|f_{ts}\|_\infty.\end{aligned}$$

Similarly, if we choose $(x,y)=(a,d)$ or $(x,y)=(b,c)$ or $(x,y)=(b,d)$ for $h=1$ in Theorem 3, then we deduce inequalities which are the same as the above result.

Remark 3 *If we choose $h = 0$ in Theorem 3, then the inequality (6) reduce to (2).*

Theorem 4 *Suppose that all the assumptions of Lemma 1 hold. If $f_{ts} \in L_p(\Delta)$, $\frac{1}{p} + \frac{1}{q} = 1$ and $q > 1$, then we have the inequality*

$$|S_h(x, y, s, t)| \leq \left[\frac{[b - x + m_h(x)]^{q+1} + [x - a - m_h(x)]^{q+1}}{q + 1} \right]^{\frac{1}{q}} \left[\frac{[d - y + m_h(y)]^{q+1} + [y - c - m_h(y)]^{q+1}}{q + 1} \right]^{\frac{1}{q}} \|f_{ts}\|_p \tag{10}$$

for all $(x, y) \in [a, b] \times [c, d]$, where $m_h(x) = h\left(x - \frac{a+b}{2}\right)$ and $m_h(y) = h\left(y - \frac{c+d}{2}\right)$, $h \in [0, 2]$. Also, $\|f_{ts}\|_p$ is defined by

$$\|f_{ts}\|_p = \left(\int_a^b \int_c^d \left| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right|^p ds dt \right)^{\frac{1}{p}}.$$

Proof. Taking absolute value of (4) and using Hölder's inequality, we find that

$$\begin{aligned}
|S_h(x, y, s, t)| &\leq \left(\int_a^b \int_c^d |P_h(x, t)|^q |Q_h(y, s)|^q ds dt \right)^{\frac{1}{q}} \left(\int_a^b \int_c^d \left| \frac{\partial^2 f(t, s)}{\partial t \partial s} \right|^p ds dt \right)^{\frac{1}{p}} \\
&= \left[\int_a^x |t - a - m_h(x)|^q dt + \int_x^b |t - b - m_h(x)|^q dt \right]^{\frac{1}{q}} \\
&\quad \times \left[\int_c^y |s - c - m_h(y)| dt + \int_y^d |s - d - m_h(y)| ds \right]^{\frac{1}{q}} \|f_{ts}\|_p .
\end{aligned}$$

We need to examine the above integrals for the cases $a \leq x \leq \frac{a+b}{2}$ and $\frac{a+b}{2} \leq x \leq b$;

For the case of $a \leq x \leq \frac{a+b}{2}$, we get

$$\int_a^x |t - a - m_h(x)|^q dt = \frac{[x - a - m_h(x)]^{q+1} - [-m_h(x)]^{q+1}}{q + 1}$$

and

$$\int_x^b |t-b-m_h(x)|^q\,dt = \frac{[b-x+m_h(x)]^{q+1}+[-m_h(x)]^{q+1}}{q+1}.$$

For the case of $\frac{a+b}{2} \le x \le b$, we obtain

$$\int_a^x |t-a-m_h(x)|^q\,dt = \frac{[x-a-m_h(x)]^{q+1}+[m_h(x)]^{q+1}}{q+1}$$

and

$$\int_x^b |t-b-m_h(x)|^q\,dt = \frac{[b-x+m_h(x)]^{q+1}-[m_h(x)]^{q+1}}{q+1}.$$

Then, we can write

$$\begin{aligned}&\int_a^x |t-a-m_h(x)|^q\,dt + \int_x^b |t-b-m_h(x)|^q\,dt \\ &= \frac{[b-x+m_h(x)]^{q+1}+[x-a-m_h(x)]^{q+1}}{q+1}.\end{aligned} \tag{11}$$

Similarly, we easily deduce the identity

$$\begin{aligned}&\int_c^y |s-c-m_h(y)|^q\,dt + \int_y^d |s-d-m_h(y)|^q\,ds \\ &= \frac{[d-y+m_h(y)]^{q+1}+[y-c-m_h(y)]^{q+1}}{q+1}.\end{aligned} \tag{12}$$

Using the equality (11) and (12), we easily deduce required inequality (10). Hence, the proof is completed. □

Remark 4 *If we take $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$ in Theorem 4, then we have the mid-point inequality*

$$\left|\int_a^b\int_c^d f(t,s)\,ds\,dt + (b-a)(d-c)\,f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.$$

$$- (d-c) \int_a^b f\left(t, \frac{c+d}{2}\right) dt - (b-a) \int_c^d f\left(\frac{a+b}{2}, s\right) ds \Bigg|$$
$$\leq \frac{(b-a)^{1+\frac{1}{q}} (d-c)^{1+\frac{1}{q}}}{4(q+1)^{\frac{2}{q}}} \|f_{ts}\|_p$$

which was given by Dragomir et al. in [6].

Remark 5 *If we choose* $h = 0$ *in Theorem 4, then we have*

$$\Bigg| (b-a)(d-c) f(x,y) - (d-c) \int_a^b f(t,y)dt$$
$$- (b-a) \int_c^d f(x,s)ds + \int_a^b \int_c^d f(t,s)dsdt \Bigg|$$
$$\leq \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_p \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right]^{\frac{1}{q}}$$
$$\times \left[\frac{(y-c)^{q+1} + (d-y)^{q+1}}{q+1} \right]^{\frac{1}{q}}$$

which was proved by Dragomir et al. in [6].

Remark 6 *Under the same assumptions of Theorem 4 with* $h = 1$ *and* $(x,y) = (a,c)$, *then the following inequality holds:*

$$\Bigg| \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)\, dsdt$$
$$- \frac{1}{2} \left[\frac{1}{d-c} \int_c^d [f(b,s) + f(a,s)]\, ds + \frac{1}{b-a} \int_a^b [f(t,d) + f(t,c)]\, dt \right] \Bigg|$$
$$\leq \frac{(b-a)^{1+\frac{1}{q}} (d-c)^{1+\frac{1}{q}}}{4(q+1)^{\frac{2}{q}}} \|f_{ts}\|_\infty .$$

Similarly, if we choose $(x,y) = (a,d)$ or $(x,y) = (b,c)$ or $(x,y) = (b,d)$ for $h = 1$ in Theorem 4, then we deduce inequalities which are the same as the above result.

For convenience, we give the following notations used to simplify the details of the next theorem,

$$
\begin{aligned}
A &= (b-a)\left[\frac{(x-a)^2}{2}-(x-a)\,m_h(x)\right]+\frac{(b-x)^3-(x-a)^3}{3}\\
&\quad+\left[\left(\frac{b-a}{2}\right)^2+\left(x-\frac{a+b}{2}\right)^2\right]m_h(x)-\frac{[m_h(x)]^3}{3},\\
B &= (b-a)\left[\frac{(b-x)^2}{2}+(b-x)\,m_h(x)\right]-\frac{(b-x)^3-(x-a)^3}{3}\\
&\quad-\left[\left(\frac{b-a}{2}\right)^2+\left(x-\frac{a+b}{2}\right)^2\right]m_h(x)+\frac{[m_h(x)]^3}{3},\\
C &= (d-c)\left[\frac{(y-c)^2}{2}-(y-c)\,m_h(y)\right]+\frac{(d-y)^3-(y-c)^3}{3}\\
&\quad+\left[\left(\frac{d-c}{2}\right)^2+\left(y-\frac{c+d}{2}\right)^2\right]m_h(y)-\frac{[m_h(y)]^3}{3}
\end{aligned}
$$

and

$$
\begin{aligned}
D &= (d-c)\left[\frac{(d-y)^2}{2}+(d-y)\,m_h(y)\right]-\frac{(d-y)^3-(y-c)^3}{3}\\
&\quad-\left[\left(\frac{d-c}{2}\right)^2+\left(y-\frac{c+d}{2}\right)^2\right]m_h(y)+\frac{[m_h(y)]^3}{3}.
\end{aligned}
$$

We give some inequalities by using convexity of $|f_{ts}(t,s)|$ in the following theorem.

Theorem 5 *Suppose that all the assumptions of Lemma 1 hold. If* $|f_{ts}(t,s)|$ *is a convex function on the co-ordinates on* $[a,b]\times[c,d]$, *then the following inequalities hold:*

$$
\begin{aligned}
&|S_h(x,y,s,t)|\\
&\leq \frac{|f_{ts}(a,c)|}{(b-a)(d-c)}AC+\frac{|f_{ts}(a,d)|}{(b-a)(d-c)}A\left[D+(d-c)\,[m_h(y)]^2\right]\\
&\quad+\frac{|f_{ts}(b,c)|}{(b-a)(d-c)}\left[B+(b-a)\,[m_h(x)]^2\right]C\\
&\quad+\frac{|f_{ts}(b,d)|}{(b-a)(d-c)}\left[B+(b-a)\,[m_h(x)]^2\right]\left[D+(d-c)\,[m_h(y)]^2\right]
\end{aligned}
\tag{13}
$$

for all $a \le x \le \frac{a+b}{2}$ *and* $c \le y \le \frac{c+d}{2}$

$$\begin{aligned} &|S_h(x,y,s,t)| \\ &\quad \le \frac{|f_{ts}(a,c)|}{(b-a)(d-c)} A \left[C + (d-c)[m_h(y)]^2\right] + \frac{|f_{ts}(a,d)|}{(b-a)(d-c)} AD \\ &\qquad + \frac{|f_{ts}(b,c)|}{(b-a)(d-c)} \left[B + (b-a)[m_h(x)]^2\right]\left[C + (d-c)[m_h(y)]^2\right] \\ &\qquad + \frac{|f_{ts}(b,d)|}{(b-a)(d-c)} \left[B + (b-a)[m_h(x)]^2\right] D \end{aligned} \tag{14}$$

for all $a \le x \le \frac{a+b}{2}$ *and* $\frac{c+d}{2} \le y \le d$

$$\begin{aligned} &|S_h(x,y,s,t)| \\ &\quad \le \frac{|f_{ts}(a,c)|}{(b-a)(d-c)} \left[A + (b-a)[m_h(x)]^2\right] C \\ &\qquad + \frac{|f_{ts}(a,d)|}{(b-a)(d-c)} \left[A + (b-a)[m_h(x)]^2\right]\left[D + (d-c)[m_h(y)]^2\right] \\ &\qquad + \frac{|f_{ts}(b,c)|}{(b-a)(d-c)} BC + \frac{|f_{ts}(b,d)|}{(b-a)(d-c)} B \left[D + (d-c)[m_h(y)]^2\right] \end{aligned} \tag{15}$$

for all $\frac{a+b}{2} \le x \le b$ *and* $c \le y \le \frac{c+d}{2}$

$$\begin{aligned} &|S_h(x,y,s,t)| \\ &\quad \le \frac{|f_{ts}(a,c)|}{(b-a)(d-c)} \left[A + (b-a)[m_h(x)]^2\right]\left[C + (d-c)[m_h(y)]^2\right] \\ &\qquad + \frac{|f_{ts}(a,d)|}{(b-a)(d-c)} \left[A + (b-a)[m_h(x)]^2\right] D \\ &\qquad + \frac{|f_{ts}(b,c)|}{(b-a)(d-c)} B \left[C + (d-c)[m_h(y)]^2\right] + \frac{|f_{ts}(b,d)|}{(b-a)(d-c)} BD \end{aligned} \tag{16}$$

for all $\frac{a+b}{2} \le x \le b$ *and* $\frac{c+d}{2} \le y \le d$, *where* $m_h(x) = h\left(x - \frac{a+b}{2}\right)$ *and* $m_h(y) = h\left(y - \frac{c+d}{2}\right)$, $h \in [0,2]$.

Proof. If we take absolute value of (4), then we get

$$|S_h(x,y,s,t)| \le \int_a^b \int_c^d |P_h(x,t)|\,|Q_h(y,s)|\,|f_{ts}(t,s)|\,ds\,dt.$$

Since $|f_{ts}(t,s)|$ is a convex function on the co-ordinates on $[a,b]\times[c,d]$, we have

$$\begin{aligned}
&\left|f_{ts}\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b,\frac{d-s}{d-c}c+\frac{s-c}{d-c}d\right)\right| \\
&\leq \frac{(b-t)(d-s)}{(b-a)(d-c)}|f_{ts}(a,c)|+\frac{(b-t)(s-c)}{(b-a)(d-c)}|f_{ts}(a,d)| \\
&\quad+\frac{(t-a)(d-s)}{(b-a)(d-c)}|f_{ts}(b,c)|+\frac{(t-a)(s-c)}{(b-a)(d-c)}|f_{ts}(b,d)|.
\end{aligned} \tag{17}$$

Utilizing the inequality (17), we obtain

$$\begin{aligned}
&|S_h(x,y,s,t)| \\
&\leq \frac{|f_{ts}(a,c)|}{(b-a)(d-c)}\left[\int_a^b (b-t)|P_h(x,t)|\,dt\right]\left[\int_c^d (d-s)|Q_h(y,s)|\,ds\right] \\
&\quad+\frac{|f_{ts}(a,d)|}{(b-a)(d-c)}\left[\int_a^b (b-t)|P_h(x,t)|\,dt\right]\left[\int_c^d (s-c)|Q_h(y,s)|\,ds\right] \\
&\quad+\frac{|f_{ts}(b,c)|}{(b-a)(d-c)}\left[\int_a^b (t-a)|P_h(x,t)|\,dt\right]\left[\int_c^d (d-s)|Q_h(y,s)|\,ds\right] \\
&\quad+\frac{|f_{ts}(b,d)|}{(b-a)(d-c)}\left[\int_a^b (t-a)|P_h(x,t)|\,dt\right]\left[\int_c^d (s-c)|Q_h(y,s)|\,ds\right].
\end{aligned} \tag{18}$$

We observe that

$$\begin{aligned}
&\int_a^b (b-t)|P_h(x,t)|\,dt=(b-a)\int_a^x |t-a-m_h(x)|\,dt \\
&\quad-\int_a^x (t-a)|t-a-m_h(x)|\,dt+\int_x^b (b-t)|t-b-m_h(x)|\,dt.
\end{aligned} \tag{19}$$

Now, let us observe that

$$\begin{aligned}
\int_p^r |t-p||t-q|\,dt&=\int_p^q (t-p)(q-t)\,dt+\int_q^r (t-p)(t-q)\,dt \\
&=\frac{(q-p)^3}{3}+\frac{(r-p)^3}{3}-\frac{(q-p)(r-p)^2}{2}
\end{aligned} \tag{20}$$

for all r, p, q such that $p \leq q \leq r$.

We investigate integrals given in the equality (19) for the cases $a \leq x \leq \frac{a+b}{2}$ and $\frac{a+b}{2} \leq x \leq b$;

For all $a \leq x \leq \frac{a+b}{2}$, we have

$$\int_a^x |t - a - m_h(x)|\, dt = \frac{(x-a)^2}{2} - (x-a)\, m_h(x),$$
$$\int_a^x (t-a)\, |t - a - m_h(x)|\, dt = \frac{(x-a)^3}{3} - \frac{(x-a)^2}{2} m_h(x)$$

and using the equality (20) for second integral, we get

$$\int_x^b |b-t|\, |t - b - m_h(x)|\, dt = -\frac{[m_h(x)]^3}{3} + \frac{(b-x)^3}{3} + \frac{(b-x)^2}{2} m_h(x).$$

For all $\frac{a+b}{2} \leq x \leq b$, we have

$$\int_a^x |t - a - m_h(x)|\, dt = \frac{(x-a)^2}{2} - (x-a)\, m_h(x) + [m_h(x)]^2,$$
$$\int_x^b |b-t|\, |t - b - m_h(x)|\, dt = \frac{(b-x)^3}{3} + \frac{(b-x)^2}{2} m_h(x)$$

and using the equality (20), we obtain

$$\int_a^x |a-t|\, |t - a - m_h(x)|\, dt = \frac{[m_h(x)]^3}{3} + \frac{(x-a)^3}{3} - \frac{(x-a)^2}{2} m_h(x).$$

Then, we can write

$$\int_a^b (b-t)\, |P_h(x,t)|\, dt = A$$

for all $a \leq x \leq \frac{a+b}{2}$ and

$$\int_a^b (b-t)\, |P_h(x,t)|\, dt = A + (b-a)\, [m_h(x)]^2$$

for all $\frac{a+b}{2} < x \leq b$.

Similarly, we can easily find the other integrals given in the inequality (18) for cases $a \leq x \leq \frac{a+b}{2}$, $\frac{a+b}{2} < x \leq b$, $c \leq y \leq \frac{c+d}{2}$ and $\frac{c+d}{2} \leq y \leq d$. If we substitute the resulting inequalities for all cases in (18), we obtain desired inequalities. The proof is thus completed. □

Remark 7 *If we take* $x = \frac{a+b}{2}$ *and* $y = \frac{c+d}{2}$ *in Theorem 5, then we have the mid-point inequality*

$$\left| \int_a^b \int_c^d f(t,s)\, ds dt + (b-a)(d-c) f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right.$$
$$\left. -(d-c)\int_a^b f\left(t, \frac{c+d}{2}\right) dt - (b-a)\int_c^d f\left(\frac{a+b}{2}, s\right) ds \right|$$
$$\leq \frac{(b-a)^2 (d-c)^2}{16} \left[\frac{|f_{ts}(a,c)| + |f_{ts}(a,d)| + |f_{ts}(b,c)| + |f_{ts}(b,d)|}{4} \right]$$

which was given by Latif and Dragomir in [12].

Corollary 1 *Under the same assumptions of Theorem 5 with* $h = 0$, *we get the inequality*

$$\left| \int_a^b \int_c^d f(s,t) dt ds + (d-c)(b-a) f(x,y) \right.$$
$$\left. - \left[(b-a)\int_c^d f(x,t) dt + (d-c)\int_a^b f(s,y) ds \right] \right|$$
$$\leq \left[(b-a)\frac{(x-a)^2}{2} + \frac{(b-x)^3 - (x-a)^3}{3} \right]$$
$$\times \left\{ \frac{|f_{ts}(a,c)|}{(b-a)(d-c)} \left[(d-c)\frac{(y-c)^2}{2} + \frac{(d-y)^3 - (y-c)^3}{3} \right] \right.$$
$$\left. + \frac{|f_{ts}(a,d)|}{(b-a)(d-c)} \left[(d-c)\frac{(d-y)^2}{2} - \frac{(d-y)^3 - (y-c)^3}{3} \right] \right\}$$
$$+ \left[(b-a)\frac{(b-x)^2}{2} - \frac{(b-x)^3 - (x-a)^3}{3} \right]$$

$$\times \left\{ \frac{|f_{ts}(b,c)|}{(b-a)(d-c)} \left[(d-c)\frac{(y-c)^2}{2} + \frac{(d-y)^3-(y-c)^3}{3} \right] + \frac{|f_{ts}(b,d)|}{(b-a)(d-c)} \left[(d-c)\frac{(d-y)^2}{2} - \frac{(d-y)^3-(y-c)^3}{3} \right] \right\}$$

for all $(x,y) \in [a,b] \times [c,d]$.

Remark 8 *If we take* $(x,y) = (a,c)$ *for* $h = 1$ *in the inequality* (13), *then we have the result*

$$\left| \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)\,dsdt - \frac{1}{2}\left[\frac{1}{d-c}\int_c^d [f(b,s)+f(a,s)]\,ds + \frac{1}{b-a}\int_a^b [f(t,d)+f(t,c)]\,dt \right] \right|$$
$$\leq \frac{(b-a)(d-c)}{16} \left[\frac{|f_{ts}(a,c)|+|f_{ts}(a,d)|+|f_{ts}(b,c)|+|f_{ts}(b,d)|}{4} \right]$$

which was proved Sarikaya et al. in [21].

Similarly, if we choose $(x,y) = (a,d)$ in (14) or $(x,y) = (b,c)$ in (15) or $(x,y) = (b,d)$ in (16) for $h = 1$, then we obtain inequalities which are the same as the above result.

Theorem 6 *Suppose that all the assumptions of Lemma 1 hold. If* $|f_{ts}(t,s)|^q$ *is a convex function on the co-ordinates on* $[a,b]\times[c,d]$, $\frac{1}{p}+\frac{1}{q} = 1$ *and* $q > 1$, *then the following inequality holds:*

$$\begin{aligned} &|S_h(x,y,s,t)| \\ &\leq (b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}} \left[\frac{[b-x+m_h(x)]^{p+1}+[x-a-m_h(x)]^{p+1}}{p+1} \right]^{\frac{1}{p}} \\ &\quad\times \left[\frac{[d-y+m_h(y)]^{p+1}+[y-c-m_h(y)]^{p+1}}{p+1} \right]^{\frac{1}{p}} \\ &\quad\times \left\{ \frac{|f_{ts}(a,c)|^q+|f_{ts}(a,d)|^q+|f_{ts}(b,c)|^q+|f_{ts}(b,d)|^q}{4} \right\}^{\frac{1}{q}} \end{aligned} \tag{21}$$

for all $(x,y) \in [a,b]\times[c,d]$, *where* $m_h(x) = h\left(x - \frac{a+b}{2}\right)$ *and* $m_h(y) = h\left(y - \frac{c+d}{2}\right)$, $h \in [0,2]$.

Proof. Taking absolute value of (4) and using Hölder's inequality, we find that

$$\left|S_h(x,y,s,t)\right| \leq \left(\int_a^b\int_c^d \left|P_h(x,t)\right|^p \left|Q_h(y,s)\right|^p dsdt\right)^{\frac{1}{q}} \left(\int_a^b\int_c^d \left|f_{ts}(t,s)\right|^q dsdt\right)^{\frac{1}{q}}.$$

By similar methods in the proof of Theorem 4, we obtain

$$\left[\int_a^b\int_c^d \left|P_h(x,t)\right|^p \left|Q_h(y,s)\right|^p dsdt\right]^{\frac{1}{q}}$$
$$= \left[\frac{[b-x+m_h(x)]^{p+1}+[x-a-m_h(x)]^{p+1}}{p+1}\right]^{\frac{1}{p}}$$
$$\times \left[\frac{[d-y+m_h(y)]^{p+1}+[y-c-m_h(y)]^{p+1}}{p+1}\right]^{\frac{1}{p}}.$$

Since $\left|f_{ts}(t,s)\right|^q$ is a convex function on the co-ordinates on Δ, we have

$$\begin{aligned}
&\left|f_{ts}\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b, \frac{d-s}{d-c}c+\frac{s-c}{d-c}d\right)\right|^q \\
&\leq \frac{(b-t)(d-s)}{(b-a)(d-c)}\left|f_{ts}(a,c)\right|^q + \frac{(b-t)(s-c)}{(b-a)(d-c)}\left|f_{ts}(a,d)\right|^q \qquad (22)\\
&+ \frac{(t-a)(d-s)}{(b-a)(d-c)}\left|f_{ts}(b,c)\right|^q + \frac{(t-a)(s-c)}{(b-a)(d-c)}\left|f_{ts}(b,d)\right|^q.
\end{aligned}$$

Using the inequality (22), it follows that

$$\left(\int_a^b\int_c^d \left|f_{ts}(t,s)\right|^q dsdt\right)^{\frac{1}{q}} \leq (b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}$$
$$\times \left\{\frac{\left|f_{ts}(a,c)\right|^q+\left|f_{ts}(a,d)\right|^q+\left|f_{ts}(b,c)\right|^q+\left|f_{ts}(b,d)\right|^q}{4}\right\}^{\frac{1}{q}}.$$

The proof is thus completed. □

Remark 9 *If we take $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$ in Theorem 6, then we have the mid-point inequality*

$$\left|\int_a^b\int_c^d f(t,s)\,dsdt + (b-a)(d-c)\,f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.$$

$$-(d-c)\int_a^b f\left(t,\frac{c+d}{2}\right)dt-(b-a)\int_c^d f\left(\frac{a+b}{2},s\right)ds\Bigg|$$

$$\leq \frac{(b-a)^2(d-c)^2}{4(q+1)^{\frac{2}{q}}}\left\{\frac{|f_{ts}(a,c)|^q+|f_{ts}(a,d)|^q+|f_{ts}(b,c)|^q+|f_{ts}(b,d)|^q}{4}\right\}^{\frac{1}{q}}$$

which was deduced by Latif and Dragomir in [12].

Corollary 2 *If we choose* $h=0$ *in Theorem 6, then we have*

$$\Bigg|(b-a)(d-c)f(x,y)-(d-c)\int_a^b f(t,y)dt$$

$$-(b-a)\int_c^d f(x,s)ds+\int_a^b\int_c^d f(t,s)dsdt\Bigg|$$

$$\leq (b-a)^{\frac{1}{q}}\left[\frac{(b-x)^{p+1}+(x-a)^{p+1}}{p+1}\right]^{\frac{1}{p}}$$

$$\times (d-c)^{\frac{1}{q}}\left[\frac{(d-y)^{p+1}+(y-c)^{p+1}}{p+1}\right]^{\frac{1}{p}}$$

$$\times\left\{\frac{|f_{ts}(a,c)|^q+|f_{ts}(a,d)|^q+|f_{ts}(b,c)|^q+|f_{ts}(b,d)|^q}{4}\right\}^{\frac{1}{q}}$$

which is a Ostrowski type inequality for co-ordinated convex mappings.

Remark 10 *Under the same assumptions of Theorem 6 with* $h=1$ *and* $(x,y)=(a,c)$, *then the following inequality holds:*

$$\Bigg|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt$$

$$-\frac{1}{2}\left[\frac{1}{d-c}\int_c^d[f(b,s)+f(a,s)]\,ds+\frac{1}{b-a}\int_a^b[f(t,d)+f(t,c)]\,dt\right]\Bigg|$$

$$\leq \frac{(b-a)^2(d-c)^2}{4(q+1)^{\frac{2}{q}}}\left\{\frac{|f_{ts}(a,c)|^q+|f_{ts}(a,d)|^q+|f_{ts}(b,c)|^q+|f_{ts}(b,d)|^q}{4}\right\}^{\frac{1}{q}}$$

which was proved Sarikaya et al. in [21].

Similarly, if we choose $(x,y)=(a,d)$ or $(x,y)=(b,c)$ or $(x,y)=(b,d)$ for $h=1$ in Theorem 6, then we deduce inequalities which are the same as the above result.

3 Applications to cubature formulae

We now consider applications of the integral inequalities developed in the previous section, to obtain estimates of cubature formula which, it turns out have a markedly smaller error than that which may be obtained by the classical results.

Let $I_n : a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ and $J_m : c = y_0 < y_1 < \ldots < y_{m-1} < y_m = d$ be divisions of the intervals $[a,b]$ and $[c,d]$, $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, \ldots, n-1)$ and $\eta_j \in [y_j, y_{j+1}]$ $(j = 0, \ldots, m-1)$. Consider the sum

$$\begin{aligned} T(f, I_n, J_m, \xi, \eta) :=& \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} l_j \int_{x_i}^{x_{i+1}} f(t,\eta_j)\,dt \\ &+ \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} k_i \int_{y_j}^{y_{j+1}} f(\xi_i, s)\,ds - \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} k_i l_j f(\xi_i,\eta_j) \\ &- \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} m_h(\xi_i) \int_{y_j}^{y_{j+1}} [f(x_{i+1},s) - f(x_i,s)]\,ds \\ &- \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} m_h(\eta_j) \int_{x_i}^{x_{i+1}} [f(t,y_{j+1}) - f(t,y_j)]\,dt \\ &+ \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} l_j m_h(\xi_i) [f(x_{i+1},\eta_j) - f(x_i,\eta_j)] \\ &+ \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} k_i m_h(\eta_j) [f(\xi_i,y_{j+1}) - f(\xi_i,y_j)] \\ &- \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} m_h(\xi_i) m_h(\eta_j) [f(x_i,y_j) - f(x_i,y_{j+1}) \\ &- f(x_{i+1},y_j) + f(x_{i+1},y_{j+1})] \end{aligned} \tag{23}$$

where $k_i = x_{i+1} - x_i$, $l_j = y_{j+1} - y_j$ $(i = 0, \ldots, n-1;\ j = 0, \ldots, m-1)$,

$m_h(\xi_i) = h\left(\xi_i - \frac{x_i+x_{i+1}}{2}\right)$ and $m_h(\eta_j) = h\left(\eta_j - \frac{y_j+y_{j+1}}{2}\right)$.

Theorem 7 *Let* $f : [a,b]\times[c,d] \to \mathbb{R}$ *be an absolutely continuous function such that the partial derivative of order* 2 *exists for all* $(t,s)\in[a,b]\times[c,d]$. *If* $f_{ts} = \frac{\partial^2 f}{\partial t\partial s}$ *exists on* $(a,b)\times(c,d)$ *and is bounded, i.e.,*

$$\|f_{ts}\|_\infty = \sup_{(t,s)\in(x_i,x_{i+1})\times(y_j,y_{j+1})}\left|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right| < \infty.$$

Then we have the representation

$$\int_a^b\int_c^d f(t,s)dsdt = T(f,I_n,J_m,\xi,\eta) + R(f,I_n,J_m,\xi,\eta)$$

where $S(f,f',\xi,I_n)$ *is defined as in* (23) *and the remainder satisfies the estimations:*

$$\begin{aligned}&|R(f,I_n,J_m,\xi,\eta)|\\ &\le \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[\frac{k_i^2}{4} + \left(\xi_i - \frac{x_i+x_{i+1}}{2}\right)^2 + (h-2)\left(\xi_i - \frac{x_i+x_{i+1}}{2}\right)m_h(\xi_i)\right]\\ &\quad\times\left[\frac{l_j^2}{4} + \left(\eta_j - \frac{y_j+y_{j+1}}{2}\right)^2 + (h-2)\left(\eta_j - \frac{y_j+y_{j+1}}{2}\right)m_h(\eta_j)\right]\|f_{ts}\|_\infty\end{aligned} \tag{24}$$

for all $(\xi_i,\eta_j)\in[x_i,x_{i+1}]\times[y_j,y_{j+1}]$ *with* $(i=0,\ldots,n-1;\ j=0,\ldots,m-1)$, *where* $m_h(\xi_i) = h\left(\xi_i - \frac{x_i+x_{i+1}}{2}\right)$ *and* $m_h(\eta_j) = h\left(\eta_j - \frac{y_j+y_{j+1}}{2}\right)$ *with* $h\in[0,2]$.

Proof. Applying Theorem 3 on the interval $[x_i,x_{i+1}]\times[y_j,y_{j+1}]$, $(i=0,\ldots,n-1;\ j=0,\ldots,m-1)$, we obtain

$$\begin{aligned}&\left|\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} f(t,s)dsdt - l_j\int_{x_i}^{x_{i+1}} f(t,\eta_j)\,dt - k_i\int_{y_j}^{y_{j+1}} f(\xi_i,s)\,ds + k_il_jf(\xi_i,\eta_j)\right.\\ &\quad + m_h(\xi_i)\int_{y_j}^{y_{j+1}}[f(x_{i+1},s)-f(x_i,s)]\,ds + m_h(\eta_j)\int_{x_i}^{x_{i+1}}[f(t,y_{j+1})-f(t,y_j)]\,dt\\ &\quad - l_jm_h(\xi_i)[f(x_{i+1},\eta_j)-f(x_i,\eta_j)] - k_im_h(\eta_j)[f(\xi_i,y_{j+1})-f(\xi_i,y_j)]\end{aligned}$$

$$+m_h(\xi_i)m_h(\eta_j)\left[f(x_i,y_j)-f(x_i,y_{j+1})-f(x_{i+1},y_j)+f(x_{i+1},y_{j+1})\right]|$$
$$\leq \|f_{ts}\|_\infty\left[\frac{k_i^2}{4}+\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)^2+(h-2)\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)m_h(\xi_i)\right]$$
$$\times\left[\frac{l_i^2}{4}+\left(\eta_j-\frac{y_j+y_{j+1}}{2}\right)^2+(h-2)\left(\eta_j-\frac{y_j+y_{j+1}}{2}\right)m_h(\eta_j)\right]$$

for all $i=0,\ldots,n-1;\ j=0,\ldots,m-1$.

Summing over i from 0 to $n-1$ and over j from 0 to $m-1$ using the generalized triangle inequality we obtain the estimation (24). □

Remark 11 *If we take* $h=0$ *in Theorem 7, then we recapture the cubature formula*

$$\int_a^b\int_c^d f(t,s)\,ds\,dt = T(f,I_n,J_m,\xi,\eta)+R(f,I_n,J_m,\xi,\eta)$$

where the remainder $R(f,I_n,J_m,\xi,\eta)$ *satisfies the estimation:*

$$|R(f,I_n,J_m,\xi,\eta)|$$
$$\leq \|f_{ts}\|_\infty\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[\frac{k_i^2}{4}+\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)^2\right]\left[\frac{l_j^2}{4}+\left(\eta_j-\frac{y_j+y_{j+1}}{2}\right)^2\right] \quad (25)$$

which was given by Barnett and Dragomir in [2].

Remark 12 *If we choose* $\xi_i=\frac{x_i+x_{i+1}}{2}$ *and* $\eta_j=\frac{y_j+y_{j+1}}{2}$ *in Theorem 7, then we recapture the midpoint cubature formula*

$$\int_a^b\int_c^d f(t,s)\,ds\,dt = T_M(f,I_n,J_m)+R_M(f,I_n,J_m)$$

where the remainder $R_M(f,I_n,J_m)$ *satisfies the estimation:*

$$|R_M(f,I_n,J_m)|\leq\frac{\|f_{ts}\|_\infty}{16}\sum_{i=0}^{n-1}k_i^2\sum_{j=0}^{m-1}l_j^2.$$

Theorem 8 *Let* $f:[a,b]\times[c,d]\to\mathbb{R}$ *be an absolutely continuous function such that the partial derivative of order 2 exists for all* $(t,s)\in[a,b]\times[c,d]$. *If*

$|f_{ts}(t,s)|^q$ *is a convex function on the co-ordinates on* $[a,b]\times[c,d]$, $\frac{1}{p}+\frac{1}{q}=1$ *and* $q>1$, *then we have the representation*

$$\int_a^b\int_c^d f(t,s)dsdt = T(f,I_n,J_m,\xi,\eta)+R(f,I_n,J_m,\xi,\eta)$$

where $S(f,f',\xi,I_n)$ *is defined as in* (23) *and the remainder satisfies the estimations:*

$$\begin{aligned}&|R(f,I_n,J_m,\xi,\eta)|\\ &\leq \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}k_i^{\frac{1}{q}}l_j^{\frac{1}{q}}\left[\frac{[x_{i+1}-\xi_i+m_h(\xi_i)]^{p+1}+[\xi_i-x_i-m_h(\xi_i)]^{p+1}}{p+1}\right]^{\frac{1}{p}}\\ &\times\left[\frac{[y_{j+1}-\eta_j+m_h(\eta_j)]^{p+1}+[\eta_j-y_j-m_h(\eta_j)]^{p+1}}{p+1}\right]^{\frac{1}{p}}\\ &\times\left\{\frac{|f_{ts}(x_i,y_j)|^q+|f_{ts}(x_i,y_{j+1})|^q+|f_{ts}(x_{i+1},y_j)|^q+|f_{ts}(x_{i+1},y_{j+1})|^q}{4}\right\}^{\frac{1}{q}}\end{aligned}$$

for all $(\xi_i,\eta_j)\in[x_i,x_{i+1}]\times[y_j,y_{j+1}]$ *with* $(i=0,\ldots,n-1;\ j=0,\ldots,m-1)$, *where* $m_h(\xi_i)=h\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)$ *and* $m_h(\eta_j)=h\left(\eta_j-\frac{y_j+y_{j+1}}{2}\right)$ *with* $h\in[0,2]$.

Proof. Applying similar methods in the proof of Theorem 7 and then using the inequality (21), we obtain desired result. □

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Some results concerning the Tremblay operator and some of its applications to certain analytic functions

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Abstract. The aim of this scientific note is first to present certain information associated with the Tremblay operator in the complex plane and then to determine several results constituted by the related operator for certain analytic functions and also to point some implications of them out.

1 Introduction, definitions and motivation

As is known, in the literature, we often encounter many scientific studies in relation with fractional calculus that is that fractional integral or fractional derivative. They have important roles in both science and practice and are also used to define various new definitions or transformations. They or some of their applications naturally lead researchers to new studies or practices. In this respect, this study is one of the studies that we have indicated. Within this context and also in this present investigation, some new definitions and

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also applications of an operator, which is related to fractional calculus and known as the Tremblay operator in the literature, will be considered for certain functions with complex variable. For certain results relating to the Tremblay operator and also fractional calculus, one may check the paper in [1]-[3], [5], [7], [9]-[11], [15] and [16] in the references.

Let us now recall some definitions and information that will be relevant to our scientific research.

Firstly, let

$$\mathbb{C}, \quad \mathbb{R}, \quad \mathbb{N} \quad \text{and} \quad \mathbb{U}$$

be the set of *complex numbers,* be the set of *real numbers* and the set of *positive integers* and the *unit open* disk:

$$\Big\{z : z \in \mathbb{C} \text{ and } |z| < 1\Big\},$$

respectively. Moreover, we denote by $\mathcal{A}(n)$ the family of the functions $f(z)$ normalized by the following Taylor-Maclaurin series:

$$f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots \quad (a_{n+1} \in \mathbb{C};\ n \in \mathbb{N}), \tag{1}$$

which are *analytic and univalent* in $\mathbb{U}$.

We next recall, for a complex function $\kappa := \kappa(z)$, the definition of *fractional derivative* is denoted by the following symbol:

$$\mathcal{D}_z^{\mu}[\kappa] \equiv \mathcal{D}_z^{\mu}\big[\kappa(z)\big] \quad \big(0 \le \mu < 1\big)$$

and also defined as in (cf., e.g., [1]-[3], [9]-[11], [14] and [15]):

Let $\kappa(z)$ be an analytic function in a *simply-connected region* of the $z-$plane containing the origin. Then, the *fractional derivative* of order μ is defined by

$$\mathcal{D}_z^{\mu}[\kappa] = \frac{1}{\Gamma(1-\mu)}\frac{d}{dz}\int_0^z \frac{\kappa(\xi)}{(z-\xi)^{\mu}}\, d\xi \quad (0 \le \mu < 1), \tag{2}$$

where the multiplicity of $(z-\xi)^{-\mu}$ above is removed by requiring $\log(z-\xi)$ to be real *when* $z-\xi > 0$. All right, *here and throughout this paper,* the notation Γ denotes the well-known gamma function in the literature.

In the light of information above, for an analytic function $\kappa(z)$, the well-known derivative, which is the Srivastava-Owa derivative of order $m+\mu$, is then presented by

$$\mathcal{D}_z^{m+\mu}[\kappa] \equiv \frac{d^m}{dz^m}\Big(\mathcal{D}_z^{\mu}[\kappa]\Big) \quad \big(0 \le \mu < 1; m \in \mathbb{N} \cup \{0\}\big),$$

which readily yields

$$\mathcal{D}_z^{0+\mu}[\kappa] \equiv \mathcal{D}_z^{\mu}[\kappa] \quad \text{and} \quad \mathcal{D}_z^{1+\mu}[\kappa] \equiv \frac{d}{dz}\Big(\mathcal{D}_z^{\mu}[\kappa]\Big) \qquad \big(0 \le \mu < 1\big).$$

Recently, by means of the Srivastava-Owa derivative [14], Tremblay [5] and [16], introduced and also studied an interesting fractional derivative operator:

$$\mathcal{T}_{\tau,\mu}[\cdot] \ \ \big(0 < \tau \le 1; 0 < \mu \le 1; 0 \le \tau - \mu < 1\big),$$

which was defined in the domain of the complex plane and whose properties in several spaces were discussed systematically. For instance, one may refer to the works given by [5], [9]-[3] and [14]-[16].

For a function $f(z)$ in $\mathcal{A}(n)$, the Trambley operator:

$$\mathcal{T}_{\tau,\mu}[f] \quad \text{or} \quad \mathcal{T}_{\tau,\mu}\big[f(z)\big]$$

is defined by

$$\mathcal{T}_{\tau,\mu}[f] := \frac{\Gamma(\mu)}{\Gamma(\tau)} z^{1-\mu} \mathcal{D}_z^{\tau-\mu}\big[z^{\tau-1} f\big], \tag{3}$$

where $0 < \tau \le 1$, $0 < \mu \le 1$, $0 \le \tau - \mu < 1$ and $z \in \mathbb{U}$. As a result of a simple focus, the operator $\mathcal{D}_z^{\tau-\mu}[\cdot]$ is equivalent to the operator, which is the Srivastava-Owa operator of fractional derivative of order $\tau - \mu$ $(0 \le \tau - \mu < 1)$ represented by (2). In special, it is clear that $\mathcal{T}_{1,1}[f] = f(z)$.

For our results specified by the Tremblay operator and fractional calculus, the following assertions will be required for both stating and proving.

Lemma 1 [5] *Let the function* f *be in the class* $\mathcal{A}(n)$. *Then,*

$$\mathcal{T}_{\tau,\mu}[f] = \frac{\tau}{\mu} z + \sum_{k=n+1}^{\infty} \frac{\Gamma(k+\tau)\Gamma(\mu)}{\Gamma(k+\mu)\Gamma(\tau)} a_k z^k, \tag{4}$$

where $0 < \tau \le 1$, $0 < \mu \le 1$, $0 \le \tau - \mu < 1$ *and* $z \in \mathbb{U}$.

Lemma 2 [5] *Let the function* f *be in the class* $\mathcal{A}(n)$. *Then,*

$$\frac{d}{dz}\Big(\mathcal{T}_{\tau,\mu}[f]\Big) = \frac{\tau}{\mu} + \sum_{k=n+1}^{\infty} \frac{k\Gamma(k+\tau)\Gamma(\mu)}{\Gamma(k+\mu)\Gamma(\tau)} a_k z^{k-1}, \tag{5}$$

where $0 < \tau \le 1$, $0 < \mu \le 1$, $0 \le \tau - \mu < 1$ *and* $z \in \mathbb{U}$.

Lemma 3 [13] *Let* p *be (non-constant and) analytic function in the set* $\mathbb{U}$ *with* $p(0) = 1$. *If there exists a point* $z_0 \in \mathbb{U}$ *such that*

$$\Re e\big(p(z)\big) > 0 \quad \big(|z| < |z_0| < 1\big), \quad \Re e\big(p(z_0)\big) = 0 \quad \text{and} \quad p(z_0) \neq 0, \tag{6}$$

then

$$p(z_0) = ia \quad \text{and} \quad \left.\frac{zp'(z)}{p(z)}\right|_{z=z_0} = i\,\frac{c}{2}\left(a + \frac{1}{a}\right), \tag{7}$$

where $a \in \mathbb{R} - \{0\}$ *and* $c \in \mathbb{R}$ *with* $c \geq 1$.

2 Certain results and implications

In this section, we shall state and then prove certain results in relation with both the Tremblay operator and some of its applications, which are given by the following theorems.

Theorem 1 *If any one of the following inequalities:*

$$\Re e\left(\frac{z\frac{d^2}{dz^2}\big(\mathcal{T}_{\tau,\mu}[f]\big)}{\frac{d}{dz}\big(\mathcal{T}_{\tau,\mu}[f]\big)}\right) > -\frac{\alpha}{2(1-\alpha)} \quad \left(0 \leq \alpha \leq \frac{1}{2}\right) \tag{8}$$

and

$$\Re e\left(\frac{z\frac{d^2}{dz^2}\big(\mathcal{T}_{\tau,\mu}[f]\big)}{\frac{d}{dz}\big(\mathcal{T}_{\tau,\mu}[f]\big)}\right) > -\frac{1-\alpha}{2\alpha} \quad \left(\frac{1}{2} \leq \alpha < 1\right) \tag{9}$$

is provided, then

$$\Re e\left(\frac{d}{dz}\Big(\mathcal{T}_{\tau,\mu}[f]\Big)\right) > \alpha\frac{\tau}{\mu} \tag{10}$$

is also provided, where $0 < \tau \leq 1$, $0 < \mu \leq 1$, $0 \leq \tau - \mu < 1$, $0 \leq \alpha < 1$, $z \in \mathbb{U}$ *and* $f \in \mathcal{A}$.

Proof. In consideration of (5) in Lemma 2, define a function $p(z)$ in the implicit form:

$$\frac{d}{dz}\Big(\mathcal{T}_{\tau,\mu}[f]\Big) = \frac{\tau}{\mu}\big[\alpha + (1-\alpha)p(z)\big], \tag{11}$$

where $0 < \tau \leq 1$, $0 < \mu \leq 1$ and $0 \leq \tau - \mu < 1$, $0 \leq \alpha < 1$, $f \in \mathcal{A}$ and $z \in \mathbb{U}$. It is clear that the function $p(z)$ satisfies the condition $p(0) = 1$ in the hypothesis of Lemma 3. Then, it follows from (11) that

$$z\frac{d^2}{dz^2}\big(\mathcal{T}_{\tau,\mu}[f]\big) = (1-\alpha)\frac{\tau}{\mu}zp'(z) \quad \big(0 \leq \alpha < 1; f \in \mathcal{A}; z \in \mathbb{U}\big). \tag{12}$$

By combining (11) and (12), the following equality:

$$\frac{z\frac{d^2}{dz^2}\left(\mathcal{T}_{\tau,\mu}[f]\right)}{\frac{d}{dz}\left(\mathcal{T}_{\tau,\mu}[f]\right)} = \frac{(1-\alpha)zp'(z)}{\alpha+(1-\alpha)p(z)} \quad \left(0 \leq \alpha < 1; f \in \mathcal{A}; z \in \mathbb{U}\right) \tag{13}$$

is easily obtained, *where,* of course,

$$\frac{d}{dz}\left(\mathcal{T}_{\tau,\mu}[f]\right) \neq 0 \quad \left(\forall z \in \mathbb{U}\right).$$

Assume now that there exists a point z_0 belonging to $\mathbb{U}$, which satisfies the condition:

$$\mathfrak{Re}\big(p(z_0)\big) = 0 \quad \big(z_0 \in \mathbb{U}; p\big(z_0\big) \neq 0\big),$$

indicated by (6) of Lemma 3. Then, by applying of the assertions of Lemma 3, given in (7), which are

$$\mathfrak{Re}\big(p(z_0)\big) = ia \quad \text{and} \quad \left.\frac{zp'(z)}{p(z)}\right|_{z=z_0} = i\,\frac{c}{2}\left(a+\frac{1}{a}\right)$$

$$\left(a \neq 0\ ;\ c \in \mathbb{R}\ ;\ c \geq 1\right),$$

in the equation (13), the following equivalent equations:

$$\begin{aligned} \mathfrak{Re}\left(\left.\frac{z\frac{d^2}{dz^2}\left(\mathcal{T}_{\tau,\mu}[f]\right)}{\frac{d}{dz}\left(\mathcal{T}_{\tau,\mu}[f]\right)}\right|_{z:=z_0}\right) &= \mathfrak{Re}\left(\left.\frac{(1-\alpha)zp'(z)}{\alpha+(1-\alpha)p(z)}\right|_{z:=z_0}\right) \\ &= \mathfrak{Re}\left(\left.\frac{zp'(z)}{p(z)}\,\frac{(1-\alpha)p(z)}{\alpha+(1-\alpha)p(z)}\right|_{z:=z_0}\right) \\ &= -\frac{c}{2}\,\frac{\alpha(1-\alpha)\left(1+a^2\right)}{\alpha^2+(1-\alpha)^2a^2} \end{aligned} \tag{14}$$

are easily obtained. By taking in consideration the following inequalities:

$$\frac{1+a^2}{a^2+\left(\frac{\alpha}{1-\alpha}\right)^2} \geq 1 \ \text{ if } \ 0 \leq \alpha \leq \frac{1}{2}$$

and

$$\frac{1+a^2}{1+\left(\frac{1-\alpha}{\alpha}\right)^2a^2} \geq 1 \ \text{ if } \ \frac{1}{2} \leq \alpha \leq 1$$

for (14), the inequality:

$$- \Re e \left(\frac{z \frac{d^2}{dz^2}\left(\mathcal{T}_{\tau,\mu}[f]\right)}{\frac{d}{dz}\left(\mathcal{T}_{\tau,\mu}[f]\right)} \Bigg|_{z:=z_0} \right) = \frac{c}{2} \frac{\alpha(1-\alpha)\left(1+a^2\right)}{\alpha^2+(1-\alpha)^2 a^2} \geq \begin{cases} \frac{\alpha}{2(1-\alpha)} & \text{if } \; 0 \leq \alpha \leq \frac{1}{2} \\ \frac{1-\alpha}{2\alpha} & \text{if } \; \frac{1}{2} \leq \alpha \leq 1 \end{cases} \tag{15}$$

is also determined. But, the cases above are contradiction, respectively, with the inequalities given in (8) and (9). In this case, there is no any point $z_0 \in \mathbb{U}$ satisfying the condition $\Re e\big(p(z_0)\big) = 0$. This means that $\Re e\big(p(z)\big) > 0$ for all z in $\mathbb{U}$. Thus, (1) gives us the inequality given in (10). Therefore, this completes the proof of Theorem 1. □

For the proofs of the following theorems (Theorems 2-4 below), under those important conditions:

$$0 < \tau \leq 1 \;,\; 0 < \mu \leq 1 \;,\; 0 \leq \tau - \mu < 1, \; 0 \leq \alpha < 1 \;,\; z \in \mathbb{U} \; \text{ and } \; f \in \mathcal{A}(n) \,,$$

in view of Lemma 1 *or* Lemma 2, and by taking into account the different definitions of the function $p(z)$ that we have defined just above, namely, in the proof of Theorem 1, when one defines that function $p(z)$ again, respectively, as in the following implicit forms:

$$\frac{\mathcal{T}_{\tau,\mu}[f]}{z} = \frac{\tau}{\mu}\left[\alpha + (1-\alpha)p(z)\right],$$

$$\left[\frac{d}{dz}\left(\mathcal{T}_{\tau,\mu}[f]\right)\right]^{\chi} = \left(\frac{\tau}{\mu}\right)^{\chi}\left[\alpha + (1-\alpha)p(z)\right]$$

and

$$\left(\frac{\mathcal{T}_{\tau,\mu}[f]}{z}\right)^{\chi} = \left(\frac{\tau}{\mu}\right)^{\chi}\left[\alpha + (1-\alpha)p(z)\right]$$

and then follows the ways *or* steps used in the proof of Theorem 1, of course, by the help of Lemma 3, the following theorems (Theorems 2-4 below) can be easily proven. Their proofs are here omitted.

Theorem 2 *If any one of the following inequalities:*

$$\Re e\left(\frac{z \frac{d}{dz}\left(\mathcal{T}_{\tau,\mu}[f]\right)}{\mathcal{T}_{\tau,\mu}[f]}\right) > \frac{2-3\alpha}{2(1-\alpha)} \quad \left(0 \leq \alpha \leq \frac{1}{2}\right)$$

and

$$\Re e\left(\frac{z\frac{d}{dz}\left(\mathcal{T}_{\tau,\mu}[f]\right)}{\mathcal{T}_{\tau,\mu}[f]}\right) > \frac{3\alpha-1}{2\alpha} \quad \left(\frac{1}{2} \leq \alpha < 1\right)$$

is ensured, then

$$\Re e\left(\frac{\mathcal{T}_{\tau,\mu}[f]}{z}\right) > \alpha\frac{\tau}{\mu}$$

is also ensured, where $0 < \tau \leq 1$, $0 < \mu \leq 1$, $0 \leq \tau - \mu < 1$, $0 \leq \alpha < 1$, $z \in \mathbb{U}$ *and* $f \in \mathcal{A}$.

Theorem 3 *If any one of the following inequalities:*

$$\Re e\left(\chi \cdot \frac{z\frac{d^2}{dz^2}\left(\mathcal{T}_{\tau,\mu}[f]\right)}{\frac{d}{dz}\left(\mathcal{T}_{\tau,\mu}[f]\right)}\right) > -\frac{\alpha}{2(1-\alpha)} \quad \left(0 \leq \alpha \leq \frac{1}{2}\right)$$

and

$$\Re e\left(\chi \cdot \frac{z\frac{d^2}{dz^2}\left(\mathcal{T}_{\tau,\mu}[f]\right)}{\frac{d}{dz}\left(\mathcal{T}_{\tau,\mu}[f]\right)}\right) > -\frac{1-\alpha}{2\alpha} \quad \left(\frac{1}{2} \leq \alpha < 1\right)$$

is supplied, then

$$\Re e\left[\left(\frac{d}{dz}\left(\mathcal{T}_{\tau,\mu}[f]\right)\right)^{\chi}\right] > \alpha\left(\frac{\tau}{\mu}\right)^{\chi}$$

is also supplied, where $0 < \tau \leq 1$, $0 < \mu \leq 1$, $0 \leq \tau - \mu < 1$, $0 \leq \alpha < 1$, $\chi \in \mathbb{C} - \{0\}$, $z \in \mathbb{U}$, *and* $f \in \mathcal{A}$ *and also the value of the above complex power is taken to be its principal value.*

Theorem 4 *If any one of the following inequalities:*

$$\Re e\left(\chi \cdot \frac{z\frac{d}{dz}\left(\mathcal{T}_{\tau,\mu}[f]\right)}{\mathcal{T}_{\tau,\mu}[f]}\right) > \Re e(\chi) - \frac{\alpha}{2(1-\alpha)} \quad \left(0 \leq \alpha \leq \frac{1}{2}\right)$$

and

$$\Re e\left(\chi \cdot \frac{z\frac{d}{dz}\left(\mathcal{T}_{\tau,\mu}[f]\right)}{\mathcal{T}_{\tau,\mu}[f]}\right) > \Re e(\chi) - \frac{1-\alpha}{2\alpha} \quad \left(\frac{1}{2} \leq \alpha < 1\right)$$

is verified, then

$$\Re e\left[\left(\frac{\mathcal{T}_{\tau,\mu}[f]}{z}\right)^{\chi}\right] > \alpha\left(\frac{\tau}{\mu}\right)^{\chi}$$

is also verified, where $0 < \tau \leq 1$, $0 < \mu \leq 1$, $0 \leq \tau - \mu < 1$, $0 \leq \alpha < 1$, $\chi \in \mathbb{C} - \{0\}$, $z \in \mathbb{U}$, *and* $f \in \mathcal{A}$ *and also the value of the above complex power is taken to be its principal value.*

As certain implications of Theorems 1-4 above, when one centers on all theorems, it is easy to observe that they include several useful consequences of them, which relate to certain inequalities specified by the functions belonging to the class $\mathcal{A}(n)$. Particularly, some of them deal with the theory of (analytic and) univalent functions. For their details, see the works in [4] and [6]. As example and special consequences of them, we want to emphasize only two appertaining to the study of the relations between the analytic properties of a function $f(z)$ in the class $\mathcal{A}(n)$ and the geometric properties of the image domain $f(\mathbb{U})$. One may see their details in [4], [6] and [8]-[12]. The others possible consequences, which are well-known in the literature and are also omitted here, are presented to reveal to the attention of the researchers who are interested in the topics of this work.

By setting $\tau := \mu := 1$ in Theorem 1, for a function $f := f(z) \in \mathcal{A}(n)$, $f \equiv \mathcal{T}_{1,1}[f]$ is then received. In the circumstances, as the first consequence of the main results, which is in relation with close-to-starlikeness of a function f in the class $\mathcal{A}(n)$, it can be obtained by the following-well-known result (Proposition 1 below).

Proposition 1 *If a function* $f \in \mathcal{T}(n)$ *satisfies any one of the cases in the following inequality:*

$$\Re e\left(\frac{zf''(z)}{f'(z)}\right) > \begin{cases} -\frac{\alpha}{2(1-\alpha)} & \text{if} \quad 0 \le \alpha \le \frac{1}{2} \\ -\frac{1-\alpha}{2\alpha} & \text{if} \quad \frac{1}{2} \le \alpha \le 1 \end{cases},$$

then

$$\Re e\big(f'(z)\big) > \alpha \quad \big(0 \le \alpha < 1; z \in \mathbb{U}\big),$$

that is, that f *is a close-to-starlike function of order* α *in* $\mathbb{U}$.

By setting $\tau := \mu := \chi := 1$ *in Theorem 4, because of* $f \equiv \mathcal{T}_{1,1}[f] \in \mathcal{A}(n)$, *as the second consequence of the main results concerning close-to-convexity of a function* $f \in \mathcal{A}(n)$, *it can be also revealed by the following result (Proposition 2 below).*

Proposition 2 *If a function* $f \in \mathcal{T}(n)$ *satisfies any one of the cases in the following inequality:*

$$\Re e\left(\frac{zf'(z)}{f(z)}\right) > \begin{cases} \frac{2-3\alpha}{2(1-\alpha)} & \text{if} \quad 0 \le \alpha \le \frac{1}{2} \\ \frac{3\alpha-1}{2\alpha} & \text{if} \quad \frac{1}{2} \le \alpha \le 1 \end{cases},$$

then

$$\Re e\left(\frac{f(z)}{z}\right) > \alpha \quad \big(0 \le \alpha < 1; z \in \mathbb{U}\big),$$

that is, that f *is a close-to-convex function of order* α *in* $\mathbb{U}$.

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Study of Γ-hyperrings by fuzzy hyperideals with respect to a t-norm

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Abstract. In this paper, we study the Γ-hyperrings via T-fuzzy hyperideals. By means of the use of a triangular norm T, we define, characterize and study the T-fuzzy left and right hyperideals, T-fuzzy quasi-hyperideal and bi-hyperideal in Γ-hyperrings and some related properties are investigated. Regular Γ-hyperrings are characterized in terms of T-fuzzy quasi-hyperideal and T-fuzzy bi-hyperideal. We also introduce the T-(λ, μ)-fuzzy bi-hyperideals in Γ-hyperrings and investigate some of their properties.

1 Introduction and preliminaries

The applications of mathematics in other disciplines, for example in informatics, play a key role and they represent, in the last decades, one of the purposes

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of the study of the experts of Hyperstructures Theory all over the world. Hyperstructures, as a natural extension of classical algebraic structures, in particular hypergroups, were introduced in 1934 by a French mathematician, Marty, at the 8th Congress of Scandinavian Mathematicians [29]. Since then, a lot of papers and several books have been written on this topic. Nowadays, hyperstructures have a lot of applications to several domains of mathematics and computer science(see [10, 24, 25, 33, 41, 44]) and they are studied in many countries of the world. This theory has been subsequently developed by Corsini [11, 9, 10], Mittas [27, 28], Stratigopoulos [40] and by various authors. Basic definitions and propositions about the hyperstructures are found in [9, 10, 15, 29, 41]. Krasner [22] has studied the notion of hyperfields, hyperrings, and then some researchers, namely, Ameri [2], Dasic [12], Davvaz [14, 30, 13], Gontineac [19], Massouros [26], Pianskool et al. [34], Sen and Dasgupta [38], Vougiouklis [41, 42] and others followed him.

Hyperrings are essentially rings with approximately modified axioms. There are different notions of hyperrings $(R, +, \cdot)$. If the addition $+$ is a hyperoperation and the multiplication $\cdot$ is a binary operation, then the hyperring is called Krasner (additive) hyperring [22]. Rota [35] introduced a multiplicative hyperring, where $+$ is a binary operation and the multiplication $\cdot$ is a hyperoperation. De Salvo [37] studied hyperrings in which the additions and the multiplications were hyperoperations. These hyperrings were also studied by Barghi [7] and by Asokkumar and Velrajan [5, 6]. In 2007, Davvaz and Leoreanu-Fotea [15] published a book titled Hyperring Theory and Applications.

Γ-rings were introduced by Nobusawa [31] as an algebraic tool for observing the relationship between the groups of homomorphisms $\mathrm{hom}(B, C)$ and $\mathrm{hom}(C, B)$ of commutative groups B and C. The class of Γ-rings contains not only all rings but also Hestenes ternary rings. Barnes [8] weakened slightly the conditions in the definition of Γ-ring in the sense of Nobusawa. The study of Γ-hyperrings as a generalization of rings, ternary rings, Γ-rings was initiated by Ameri et. al. [4].

After the introduction of the concept of fuzzy sets by Zadeh in 1965 [45], it has found manifold applications in the field of mathematics and related areas. This provides sufficient motivations for researchers to review various concepts and results from the realm of abstract algebra to a broader framework of fuzzy setting. The study of fuzzy algebraic structures was started with the introduction of the concepts of fuzzy subgroups by Rosenfeld [36]. The study of fuzzy hyperstructures is an interesting research topic of fuzzy sets.There is a considerable amount of work on the connections between fuzzy sets and hyperstructures. Davvaz [17] introduced the concept of fuzzy hyperideals in a semihypergroup. In 2009, Davvaz [18] gave the concept of fuzzy hyperideals in ternary semihyperrings. The relationships between the fuzzy sets and algebraic hyperstructures have been considered by Corsini, Davvaz, Leoreanu, Zhan, Zahedi, Ameri, Cristea and many other researchers. For more on fuzzy hyperstructures one can see [16]. The fuzzy hyperring notion is defined and studied in [23]. The fuzzy Γ-hyperring notion is defined and studied in [4, 43].

On the other hand, in 1960, Schweizer and Sklar [39] introduced the notion of triangular norm (t-norm) and triangular conorm (t-conorm) in order to generalize the ordinary triangle inequality in a metric space to the more general probabilistic metric space. Using t-norm, Anthony and Sherwood [1] first redefined Rosenfeld's [36] notion of fuzzy groups. Since then t-norm has played an important role in fuzzy algebra. In application, t-norm T and t-conorm S are the functions that map the unit square into the unit interval. In fuzzy sets theory, triangular norm (t-norm) is extensively used to model the logical connective: conjunction (AND). There are many applications of triangular norms in several fields of mathematics and artificial intelligence [21].

In this paper, we inquire further into the properties on some kind fuzzy hyperideals and we study the Γ-hyperrings via T-fuzzy hyperideals. By means of the use of a triangular norm T, we define, characterize and study the T-fuzzy left and right hyperideals, T-fuzzy quasi-hyperideal and bi-hyperideal in Γ-hyperrings and some related properties are investigated. We compare fuzzy hyperideal to T-fuzzy hyperideals. We have shown that Γ-hyperring is regular if and only if intersection of any T-fuzzy right hyperideal with T-fuzzy left hyperideal is equal to its product. We introduce the notion of T-fuzzy quasi-hyperideal and T-fuzzy bi-hyperideal. We discuss some of its properties. We have shown that the meet of T-fuzzy right and T-fuzzy left ideal is a T-fuzzy quasi hyperideal of a Γ-hyperring. We characterize regular Γ-hyperring with T-fuzzy quasi-hyperideal and T-fuzzy bi-hyperideal. We also introduce the T-(λ, μ)-fuzzy bi-hyperideals in Γ-hyperrings and investigate some of their properties.

Recall first the basic terms and definitions from the hyperstructure theory. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set.

A map $\circ : H \times H \to \mathcal{P}^*(H)$ is called *hyperoperation* or *join operation* on the set H, where H is a non-empty set and $\mathcal{P}^*(H) = \mathcal{P}(H)\backslash\{\emptyset\}$ denotes the set of all nonempty subsets of H. A *hyperstructure* is called the pair $(H, \circ)$ where $\circ$ is a hyperoperation on the set H. A hyperstructure $(H, \circ)$ is called a *semihypergroup* if for all $x, y, z \in H$, $(x \circ y) \circ z = x \circ (y \circ z)$, which means that

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v.$$

If $x \in H$ and A, B are nonempty subsets of H, then

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, A \circ x = A \circ \{x\}, \text{ and } x \circ B = \{x\} \circ B.$$

A non-empty subset B of a semihypergroup H is called a *sub-semihypergroup* of H if $B \circ B \subseteq B$ and H is called in this case *super-semihypergroup* of B. Let $(H, \circ)$ be a semihypergroup. Then H is called a *hypergroup* if it satisfies the reproduction axiom, for all $a \in H$, $a \circ H = H \circ a = H$. An element e in a semihypergroup H is called *identity* if

$$x \circ e = e \circ x = \{x\}, \forall x \in H.$$

An element 0 in a semihypergroup H is called *zero element* if

$$x \circ 0 = 0 \circ x = \{0\}, \forall x \in H.$$

A non-empty set H with a hyperoperation $+$ is said to be a *canonical hypergroup* if the following conditions hold:

1. for every $x, y \in H, x + y = y + x$,
2. for every $x, y, z \in H, x + (y + z) = (x + y) + z$,
3. there exists $0 \in H$, (called neutral element of H) such that $0 + x = \{x\} = x + 0$ for all $x \in H$,
4. for every $x \in H$, there exists a unique element denoted by $-x \in H$ such that $0 \in x + (-x) \cap (-x) + x$,
5. for every $x, y, z \in H$, $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$.

A comprehensive review of the theory of hypergroups appears in [9]. For any subset A of a canonical hypergroup H, $-A$ denotes the set $\{-a : a \in A\}$. A non-empty subset N of a canonical hypergroup of H is called a subcanonical hypergroup of H if N is a canonical hypergroup under the same hyperoperation as that of H. Equivalently, for every $x, y \in N, x - y \subseteq N$. In particular, for any $x \in N, x - x \subseteq N$. Since $0 \in x - x$, it follows that $0 \in N$.

There are several kinds of hyperrings that can be defined on a non-empty set R. In what follows, we shall consider one of the most general types of hyperrings. The definition of a hyperring given below is equivalent to one formulated by De Salvo [37] (see Corsini [9]).

Definition 1 *A hyperring is a triple* $(R, +, \cdot)$, *where* R *is a non-empty set with a hyperaddition* $+$ *and a hypermultiplication* $\cdot$ *satisfying the following axioms:*

1. $(R, +)$ *is a canonical hypergroup,*
2. $(R, \cdot)$ *is a semihypergroup such that* $x \cdot 0 = 0 \cdot x = 0$ *for all* $x \in R$, *(i.e, 0 is a bilaterally absorbing element),*
3. *The hypermultiplication* $\cdot$ *is distributive with respect to the hyperoperation* $+$. *That is, for every* $x, y, z \in R, x \cdot (y + z) = x \cdot y + x \cdot z$, *and* $(x + y) \cdot z = x \cdot z + y \cdot z$.

Definition 2 [37, 9] *A non-empty subset* R' *of* R *is called a subhyperring of* $(R, +, \cdot)$ *if* $(R', +)$ *is a subhypergroup of* $(R, +)$ *and* $\forall x, y \in R', x \cdot y \in \mathcal{P}^*(R')$.

Example 1 *Let* $R = \{0, 1\}$ *be a set with two hyperoperations defined as follows:*

$+$	0	1
0	$\{0\}$	$\{0,1\}$
1	$\{0,1\}$	$\{0,1\}$

$\cdot$	0	1
0	$\{0\}$	$\{0\}$
1	$\{0\}$	$\{0,1\}$

Clearly, $(R, +, \cdot)$ *is a hyperring.*

Example 2 *Let* $R = \{0, a, b\}$ *be a set with two hyperoperations defined as follows.*

$+$	0	a	b
0	$\{0\}$	$\{a\}$	$\{b\}$
a	$\{a\}$	$\{a,b\}$	R
b	$\{b\}$	R	$\{a,b\}$

$\cdot$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{0\}$	R	R
b	$\{0\}$	R	R

Clearly, $(R, +, \cdot)$ *is a hyperring.*

Example 3 *Let* $(R, +, \cdot)$ *be a hyperring. Then* $(M_n(R), \oplus, \odot)$ *is a hyperring, where* $M_n(R)$ *is the set of all* $n \times n$ *matrices over* R *for some natural number* n *and the hyperoperations* $\oplus$ *and* $\odot$ *are defined as follows:*

For $x = (x_{ij}), y = (y_{ij}) \in M_n(R)$, $x \oplus y = \{z \in M_n(R) : z = (z_{ij}), z_{ij} \in x_{ij} + y_{ij}, 1 \leq i, j \leq n\}$ *and* $x \odot y = \{z \in M_n(R) : z = (z_{ij}), z_{ij} \in \sum_{k=1}^{n} x_{ik} \cdot y_{kj}, 1 \leq i, j \leq n\}$.

Definition 3 [37, 9] *Let* $(R, +, \cdot)$ *be a hyperring. A non-empty subset* A *of* R *is called a hyperideal of* R *if* $(A, +)$ *is a subhypergroup of* $(R, +)$ *and* $\forall x \in R, \forall y \in A$, *both* $x \cdot y$ *and* $y \cdot x$ *are elements of* $\mathcal{P}^*(A)$.

Let R and Γ be two non-empty sets. A map from $R \times \Gamma \times R \rightarrow P^*(R)$ will be called a Γ-hypermultiplication in R and is denoted by $(\cdot)_\Gamma$. The result of this Γ-hypermultiplication for every two elements $a, b \in R$ and every element $\gamma \in \Gamma$ is denoted by $a\gamma b$. In the following we give the definition of Γ-hyperrings in a different way.

Definition 4 (cf. [4]) *A Γ-hyperring is called a five tuple $(R, \Gamma, +, \oplus, (\cdot)_\Gamma)$ where R, Γ are nonempty sets, $+$ is the hyperaddition in R, $\oplus$ is the hyperaddition in Γ, $(\cdot)_\Gamma$ is a Γ-hypermultiplication in R, such that:*

1. *$(R, +)$ is a canonical hypergroup;*
2. *$(\Gamma, \oplus)$ is a canonical hypergroup;*
3. *$\forall (x, y, z, \alpha) \in R^3 \times \Gamma, (x + y)\alpha z = x\alpha z + y\alpha z, x\alpha(y + z) = x\alpha y + x\alpha z$;*
4. *$\forall (x, y, \alpha, \beta) \in R^2 \times \Gamma^2, x(\alpha \oplus \beta)y = x\alpha y + x\beta y$;*
5. *$\forall (x, y, z, \alpha, \beta) \in R^3 \times \Gamma^2, (x\alpha y)\beta z = x\alpha(y\beta z)$.*

Definition 5 [20] *A Γ-semihypergroup R is called an ordered pair $(R, (\cdot)_\Gamma)$ where R and Γ are nonempty sets and $(\cdot)_\Gamma$ is a Γ-hypermultiplication on R which satisfies the following property: $\forall (a, b, c, \alpha, \beta) \in R^3 \times \Gamma^2, (a\alpha b)\beta c = a\alpha(b\beta c)$.*

Definition 6 *An weakly Γ-hyperring is called any triple $(R, +, (\cdot)_\Gamma)$ where R, Γ are nonempty sets, $+$ is the hyperaddition in R, $(\cdot)_\Gamma$ is a Γ-hypermultiplication in R, such that:*

1. *$(R, +)$ is a canonical hypergroup;*
2. *$(R, (\cdot)_\Gamma)$ is a Γ-semihypergroup;*
3. *$\forall (x, y, z, \alpha) \in R^3 \times \Gamma, (x + y)\alpha z = x\alpha z + y\alpha z, x\alpha(y + z) = x\alpha y + x\alpha z$.*

Examples of Γ-hyperrings can be found in [4, 43]. It is clear tht every hyperring is a Γ-hyperring.

In what follows, unless otherwise stated, an weakly Γ-hyperring $(R, +, \Gamma)$ always denotes a Γ-hyperring.

Definition 7 [4] *A non-empty subset R' of $(R, +, \Gamma)$ is called a sub-Γ-hyperring of R if $(R', +)$ is a subhypergroup of $(R, +)$ and $\forall x, y \in R', \gamma \in \Gamma, x\gamma y \subseteq \mathcal{P}^*(R')$.*

Definition 8 [4] *Let $(R, +, \Gamma)$ be a Γ-hyperring. A non-empty subset A of R is called a right (left) hyperideal of R if $(A, +)$ is a subhypergroup of $(R, +)$ and $A\Gamma R \subseteq A(R\Gamma A \subseteq A)$. A is called a hyperideal if it is both a left and a right hyperideal of R.*

Definition 9 *A non-empty subset Q of $(R, +, \Gamma)$ is said to be a quasi-hyperideal of R if $(Q, +)$ is a subhypergroup of $(R, +)$ and $Q\Gamma R \cap R\Gamma Q \subseteq Q$. A non-empty subset B of R is said to be a bi-hyperideal of $(R, +, \Gamma)$ if $(B, +)$ is a subhypergroup of $(R, +)$ and $B\Gamma R\Gamma B \subseteq B$.*

Definition 10 [32] *An element $a \in R$ is said to be regular if $a \in a\Gamma R\Gamma a$. That is, there exist an element $b \in R$ and $\alpha, \beta \in \Gamma$ such that $a \in a\alpha b\beta a$. A Γ-hyperring R is said to be regular if every element of R is regular.*

A mapping $\mu : X \to [0, 1]$, where X is an arbitrary non-empty set and is called a *fuzzy set* in X. For $\alpha \in [0, 1]$, the set $U(\mu; \alpha) = \{x \in X | \mu(x) \geq \alpha\}$ is called *level set* of μ. A fuzzy set μ in a Γ-hyperring R is called a *fuzzy left* (resp. *right*) *hyperideal* of R if it satisfies:

- $\inf\limits_{a \in x-y} \mu(a) \geq \min\{\mu(x), \mu(y)\}$,
- $\inf\limits_{a \in x\gamma y} \mu(a) \geq \mu(y)$ (resp. $\inf\limits_{a \in x\gamma y} \mu(a) \geq \mu(x)$) for all $x, y \in R$ and $\gamma \in \Gamma$.

A fuzzy set μ in a Γ-hyperring R is called a *fuzzy hyperideal* of R if μ is both a fuzzy left and a fuzzy right hyperideal of R [4].

A triangular norm (briefly, t-norm) (cf. Schweizer and Sklar [39]) is a function $T : [0, 1] \times [0, 1] \to [0, 1]$ satisfying for every $x, y, z \in [0, 1]$ the following conditions:

(T1) $T(x, y) = T(y, x)$ (commutative),

(T2) $T(x, y) \leq T(x, z)$ if $y \leq z$ (monotone in the right factor),

(T3) $T(x, T(y, z)) = T(T(x, y), z)$ (associative),

(T4) $T(x, 1) = x$ (having 1 as a right identity).

These four axioms are independent in the sense that none of them can be deduced from the other three. Obviously, the function min defined on $[0, 1] \times [0, 1]$ is a t-norm. Other t-norms which are frequently encountered in the study of probabilistic spaces are T_m and T_p defined by $T_m(a, b) = \max(a + b - 1, 0), T_p(a, b) = ab$ for every $a, b \in [0, 1]$. Replacing 1 by 0 in condition (T4) we obtain the concept of triangular conorm (t-conorm). In general every t-norm T satisfies the following conditions:

(i) $T(x, 0) = 0; T(0, 0) = 0$ and $T(1, 1) = 1$;
(ii) $T(x, y) \leq \min(x, y), \forall x, y \in [0, 1]$.

For a t-norm T on [0,1], we denote it $E_T = \{\alpha \in [0, 1] | T(\alpha, \alpha) = \alpha\}$.

Let T_1 and T_2 be two t-norms. T_2 is said to be *dominate* T_1 and write $T_1 \ll T_2$ if for all $a, b, c, d \in [0, 1]$, $T_1(T_2(a, c), T_2(b, d)) \leq T_2(T_1(a, b), T_1(c, d))$ and T_1 is said *weaker* then T_2 or T_2 is *stronger* then T_1 and write $T_1 \leq T_2$ if for all $x, y \in [0, 1], T_1(x, y) \leq T_2(x, y)$. Since a triangular norm T is a generalization of the minimum function, Anthony and Sherwood in [1] replaced the axiom $\min\{\mu(x), \mu(y)\} \leq \mu(xy)$ occurring in the definition of a fuzzy subgroup by the inequality $T(\mu(x), \mu(y)) \leq \mu(xy)$.

Definition 11 *Let* μ, λ *be the fuzzy subsets of a set* X. *A fuzzy subset* $\mu \cap \lambda$ *is defined as* $(\mu \cap \lambda)(x) = \min(\mu(x), \lambda(x))$.

Definition 12 *Let* μ, λ *be the fuzzy subsets of a set* X. *A fuzzy subset* $\mu \wedge \lambda$ *is defined as* $(\mu \wedge \lambda)(x) = T(\mu(x), \lambda(x))$.

Definition 13 *Let* μ, λ *be the fuzzy subsets of a set* X. *The product of the fuzzy subset* μ *and* λ *is defined as* $(\mu \circ \lambda)(x) = \sup\limits_{x \in y\gamma z} T(\mu(y), \lambda(z))$, $\gamma \in \Gamma$.

2 T-fuzzy right and left hyperideals in Γ-hyperrings

In this section, we introduce the notions of T-fuzzy left hyperideal and T-fuzzy right hyperideal in Γ-hyperrings and some properties of them are studied. Also, the regular Γ-hyperrings are studied in terms of T-fuzzy left hyperideals and T-fuzzy right hyperideals.

Definition 14 *A fuzzy set* μ *in a* Γ*-hyperring* R *is called a fuzzy left (resp. right) hyperideal of* R *with respect to a* t*-norm* T *(briefly, a* T*-fuzzy left (resp. right) hyperideal of* R*) if it satisfies:*

1. $\inf\limits_{a \in x - y} \mu(a) \geq T(\mu(x), \mu(y))$,
2. $\inf\limits_{a \in x\gamma y} \mu(a) \geq \mu(y)$ (*resp.* $\inf\limits_{a \in x\gamma y} \mu(a) \geq \mu(x)$)

for all $x, y \in R$ *and* $\gamma \in \Gamma$.

Remark 1 1. *If we take* t*-norm as min-norm,* T*-fuzzy right hyperideal coincides with fuzzy right hyperideal* [3].
2. T*-fuzzy hyperideal is both* T*-fuzzy right and left hyperideal.*

Lemma 1 *Let* $(R, +, \Gamma)$ *be a* Γ*-hyperring. Every fuzzy right hyperideal of* R *is a* T*-fuzzy right hyperideal.*

Proof. Let μ be a fuzzy right hyperideal of R. Then $\inf_{a \in x-y} \mu(a) \geq \min\{\mu(x), \mu(y)\} \geq T(\mu(x), \mu(y)\}$ and $\inf_{a \in x\gamma y} \mu(a) \geq \mu(x)$ for all $x, y \in R$ and $\gamma \in \Gamma$. Hence μ is a T-fuzzy hyperideal. □

Corollary 1 *Let* $(R, +, \Gamma)$ *be a hyperring. If* A *is a right hyperideal of* R*, then* χ_A *is a* T*-fuzzy right hyperideal.*

Proof. Let A be a right hyperideal of a Γ-hyperring R. Then χ_A is a fuzzy right hyperideal. Therefore by Lemma 1, χ_A is a T-fuzzy right hyperideal. □

Note: Every T-fuzzy right (resp. left) hyperideal need not be a fuzzy hyperideal by the following examples.

Example 4 *Let* $R = \{0, a, b, c, d\}$ *be a set with two hyperoperations defined as follows.*

$+$	0	a	b	c	d
0	0	a	b	c	d
a	a	$\{0, a\}$	b	c	d
b	b	b	$\{0, a\}$	d	c
c	c	c	d	$\{0, a\}$	b
d	d	d	c	b	$\{0, a\}$

$\cdot$	0	a	b	c	d
0	0	0	0	0	0
a	0	$\{0, a\}$	$\{0, a\}$	$\{0, a\}$	$\{0, a\}$
b	0	$\{0, a\}$	$\{0, a\}$	$\{0, a\}$	$\{0, a\}$
c	0	$\{0, a\}$	$\{0, a\}$	$\{0, a\}$	$\{0, a\}$
d	0	$\{0, a\}$	$\{0, a\}$	$\{0, a\}$	$\{0, a\}$

Clearly, $(R, +, \cdot)$ *is a hyperring. Let* $\Gamma = \{0, a, b\}$ *be an hyperideal of* R*. Then* R *is a* Γ*-hyperring. Define* $\mu : R \to [0, 1]$ *on* R *as follows:*

$$\mu(0) = 0.8, \mu(a) = 0.6, \mu(b) = 0.5, \mu(c) = 0.4, \mu(d) = 0.3$$

It can be easily verified that μ *is a* T*-fuzzy right hyperideal under* T_p*. But* μ *is not a fuzzy hyperideal of* R*.*

Example 5 [4] *Let* $I_1 \subset I_2 \subset \ldots. \subset I_n \subset \ldots$ *be a strictly increasing sequence of left hyperideals of an arbitrary* Γ*-hyperring* R *and* $\{t_j\}_{j=1}^{\infty}$ *be a strictly increasing sequence in* $[0, 1]$*. Define* μ *in* R *as follows:*

$$\mu(x) = t_j \text{ if } x \in I_j \backslash I_{j-1}, \text{ where } t_{j-1} < t_j, j = 1, 2, \ldots \text{ and } \mu(x) = 0, \text{ if } x \in R \backslash \bigcup_{j=1}^{\infty} I_j$$

It can be easily verified that μ *is a* T*-fuzzy right hyperideal under* T_p*. But* μ *is not a fuzzy hyperideal of* R*, it is only a left fuzzy hyperideal of* R*.*

Theorem 1 *Let* $(R, +, \Gamma)$ *be a* Γ*-hyperring. If* μ, λ *are* T*-fuzzy right hyperideals of* R*, then* $\mu \wedge \lambda$ *is a* T*-fuzzy right hyperideal of* R*.*

Proof. Let $x, y \in R, \gamma \in \Gamma$,

$$\begin{aligned}
\inf_{a \in x-y} (\mu \wedge \lambda)(a) &\geq T(\inf_{a \in x-y} \mu(a), \inf_{a \in x-y} \lambda(a)) \\
&\geq T(T(\mu(x), \mu(y)), T(\lambda(x), \lambda(y))) \\
&= T(T(T(\mu(x), \mu(y)), \lambda(x)), \lambda(y)) \\
&\geq T(T(T(\mu(x), \lambda(x)), \mu(y), \lambda(y))) \\
&= T(T(\mu(x), \lambda(x)), T(\mu(y), \lambda(y))) \\
&= T((\mu \wedge \lambda)(x), (\mu \wedge \lambda)(y)).
\end{aligned}$$

Since $\inf_{a \in x\gamma y} \mu(a) \geq \mu(x)$ and $\inf_{a \in x\gamma y} \lambda(a) \geq \lambda(x)$, we have $T(\inf_{a \in x\gamma y} \mu(a), \inf_{a \in x\gamma y} \lambda(a)) \geq T(\mu(x), \lambda(x))$. Then $\inf_{a \in x\gamma y} (\mu \wedge \lambda)(a) \geq (\mu \wedge \lambda)(x)$. Thus $\mu \wedge \lambda$ is a T-fuzzy right hyperideal of R. □

Corollary 2 *Let* $(R, +, \Gamma)$ *be a* Γ*-hyperring. If* μ, λ *are fuzzy right hyperideals of* R*, then* $\mu \cap \lambda$ *is a fuzzy right hyperideal of* R*.*

Proof. By taking min as t-norm T in Theorem 1, we get the required result. □

Lemma 2 *Let* $(R, +, \Gamma)$ *be a hyperring.* R *is regular if and only if* $A\Gamma B = A \cap B$ *for any right hyperideal* A *and left hyperideal* B *of* R*.*

Proof. Let $(R, +, \Gamma)$ be a regular Γ-hyperring and A, B be the right and left hyperideals of R respectively. Clearly, $A\Gamma B \subseteq A \cap B$. Since R is regular, for $x \in R$ we have $x \in x\alpha a\beta x$ for some $a \in R$ and $\alpha, \beta \in \Gamma$. Now let $x \in A \cap B$. Then $x\alpha a \subseteq A$ and $x \in B$, thus $x \in x\alpha a\beta x \subseteq A \cdot B$. Hence $A\Gamma B = A \cap B$.

Conversely, let $x \in R$. Now $\langle x \rangle_r = \{w | w \in x\gamma r + nx | r \in R, n \in Z, \gamma \in \Gamma\}$ is a right hyperideal generated by x and $\langle x \rangle_l = \{w | w \in r\gamma x + nx | r \in R, n \in Z, \gamma \in \Gamma\}$ is a left hyperideal generated by x. Then $x\gamma 0 + 1 \cdot x = 0\gamma x + 1 \cdot x = x \in \langle x \rangle_r \cap \langle x \rangle_l = \langle x \rangle_r \Gamma \langle x \rangle_l$. Therefore $x \in x\Gamma R\Gamma x$ or $x \in n_1 \cdot x^3, n_1 \in Z$ where $x^3 = x\Gamma x\Gamma x$. Hence R is regular. □

Theorem 2 *Let* $(R, +, \Gamma)$ *be a* Γ*-hyperring.* R *is regular if and only if* $\lambda \circ \mu = \lambda \wedge \mu$ *for any* T*-fuzzy right hyperideal* λ *and* T*-fuzzy left hyperideal* μ *of* R*.*

Proof. Let R be a regular Γ-hyperring. Let λ, μ be the T-fuzzy right and left hyperideals of Γ-hyperring R respectively. Let $x \in R, \gamma \in \Gamma$.

$$\begin{aligned}(\lambda \circ \mu)(x) &= \sup_{x \in y\gamma z} T(\lambda(y), \mu(z)) \leq \sup_{x \in y\gamma z} T(\lambda(x), \mu(x)) \\ &= (\lambda \wedge \mu)(x).\end{aligned}$$

Thus $\lambda \circ \mu \subseteq \lambda \wedge \mu$. Since R is regular, for $x \in R$ we have $x \in x\alpha a\beta x$ for some $a \in R, \alpha, \beta \in \Gamma$.

$$\begin{aligned}(\lambda \circ \mu)(x) &= \sup_{x \in y\gamma z} T(\lambda(y), \mu(z)) \\ &\geq T(\sup_{s \in x\alpha a} \lambda(s), \mu(x)) \\ &\geq T(\lambda(x), \mu(x)) \\ &= (\lambda \wedge \mu)(x).\end{aligned}$$

Hence $\lambda \circ \mu = \lambda \wedge \mu$.

Conversely, let us assume that $\lambda \circ \mu = \lambda \wedge \mu$ for any T-fuzzy right hyperideal λ and T-fuzzy left hyperideal μ. Let A, B be the right and left hyperideals of Γ-hyperring R respectively. Then χ_A, χ_B are the T-fuzzy right and left hyperideals of Γ-hyperring R respectively. Clearly, $A\Gamma B \subseteq A \cap B$. Now $x \in A \cap B$. Then $\chi_A(x) = \chi_B(x) = 1$. Thus $(\lambda \wedge \mu)(x) = T(\lambda(x), \mu(x)) = 1$. Therefore $(\lambda \circ \mu)(x) = 1$. Then there is $a \in A, b \in B, \gamma \in \Gamma$ such that $x \in a\gamma b$. Thus $x \in A\Gamma B$. Hence $A\Gamma B = A \cap B$. Then by Lemma 2, R is regular. □

Corollary 3 *Let* $(R, +, \Gamma)$ *be a* Γ*-hyperring.* R *is regular if and only if* $\lambda \circ \mu = \lambda \cap \mu$ *for any fuzzy right hyperideal* λ *and fuzzy left hyperideal* μ *of* R.

Proof. By taking min as t-norm T in Theorem 2, we get the required result. □

Let $\{\mu_i | i \in \Lambda\}$ be a family of fuzzy sets in the Γ-hyperring $(R, +, \Gamma)$. We define the join $\bigvee_{i \in \Lambda} \mu_i$ and meet $\bigwedge_{i \in \Lambda} \mu_i$ as follows:

$$\left(\bigvee_{i \in \Lambda} \mu_i\right)(x) = \sup\{\mu_i(x) | i \in \Lambda\}, \quad \left(\bigwedge_{i \in \Lambda} \mu_i\right)(x) = \inf\{\mu_i(x) | i \in \Lambda\},$$

for all $x \in R$, where Λ is any index set.

Theorem 3 *Let* $(R, +, \Gamma)$ *be a* Γ*-hyperring. Then the family of* T*-fuzzy hyperideals in* R *is a completely distributive lattice with respect to meet* "$\wedge$" *and join* "$\vee$".

Proof. Since [0,1] is a completely distributive lattice with respect to the usual ordering in [0,1], it is sufficient to show that $\bigvee_{i\in\Lambda}\mu_i$ and $\bigwedge_{i\in\Lambda}\mu_i$ are T-fuzzy hyperideals of R for a family of T-fuzzy hyperideals $\{\mu_i|i\in\Lambda\}$. For any $x,y\in R$, we have

$$\begin{aligned}
\inf_{a\in x-y}\left(\bigvee_{i\in\Lambda}\mu_i\right)(a) &= \sup\left\{\inf_{a\in x-y}\mu_i(a)|i\in\Lambda\right\}\\
&\geq \sup\{T(\mu_i(x),\mu_i(y))|i\in\Lambda\}\\
&\geq T(\sup\{\mu_i(x)|i\in\Lambda\},\sup\{\mu_i(y)|i\in\Lambda\})\\
&= T\left(\left(\bigvee_{i\in\Lambda}\mu_i\right)(x),\left(\bigvee_{i\in\Lambda}\mu_i\right)(y)\right),\\
\inf_{a\in x-y}\left(\bigwedge_{i\in\Lambda}\mu_i\right)(a) &= \inf\left\{\inf_{a\in x-y}\mu_{ia}(a)|i\in\Lambda\right\}\\
&\geq \inf\{T(\mu_i(x),\mu_i(y))|i\in\Lambda\}\\
&\geq T(\inf\{\mu_i(x)|i\in\Lambda\},\inf\{\mu_i(y)|i\in\Lambda\})\\
&= T\left(\left(\bigwedge_{i\in\Lambda}\mu_i\right)(x),\left(\bigwedge_{i\in\Lambda}\mu_i\right)(y)\right).
\end{aligned}$$

Now let $x,y\in R,\gamma\in\Gamma$. Then

$$\begin{aligned}
\inf_{a\in x\gamma y}\left(\bigvee_{i\in\Lambda}\mu_i\right)(a) &= \sup\{\inf_{a\in x\gamma y}\mu_i(a)|i\in\Lambda\}\\
&\geq \sup\{\mu_i(y)|i\in\Lambda\}\\
&= \left(\bigvee_{i\in\Lambda}\mu_i\right)(y),\\
\inf_{a\in x\gamma y}\left(\bigwedge_{i\in\Lambda}\mu_i\right)(a) &= \inf\{\inf_{a\in x\gamma y}\mu_i(a)|i\in\Lambda\}\\
&\geq \inf\{\mu_i(y)|i\in\Lambda\}\\
&= \left(\bigwedge_{i\in\Lambda}\mu_i\right)(y).
\end{aligned}$$

Hence $\bigvee_{i\in\Lambda}\mu_i$ and $\bigwedge_{i\in\Lambda}\mu_i$ are T-fuzzy hyperideals of R. □

Definition 15 *Let* $(R, +, \Gamma)$ *be a* Γ*-hyperring and* T *be a* t*-norm. A fuzzy set* μ *in* R *is said to satisfy imaginable property if* $Im(\mu) \subseteq E_T = \{\alpha \in [0,1] \mid T(\alpha, \alpha) = \alpha\}$.

Theorem 4 *Let* $(R, +, \Gamma)$ *be a* Γ*-hyperring,* T *be a* t*-norm and* μ *be an imaginable fuzzy set in* R*. If each non-empty upper level set* $U(\mu; \alpha)$ *of* μ *is a hyperideal of* R*, then* μ *is imaginable* T*-fuzzy hyperideal of* R.

Proof. Let us suppose that each non-empty upper level set $U(\mu; \alpha)$ of μ is a hyperideal of R. Then we first show that $\inf\limits_{a \in x-y} \mu(a) \geq \min(\mu(x), \mu(y))$ for all $x, y \in R$. In fact, if not, then there exist $x_0, y_0 \in R$ such that $\inf\limits_{a \in x_0 - y_0} \mu(a) < \min(\mu(x_0), \mu(y_0))$. Taking

$$\alpha_0 = \frac{1}{2}\left(\inf_{a \in x_0 - y_0} \mu(a) + \min(\mu(x_0), \mu(y_0)) \right),$$

we get $\inf\limits_{a \in x_0 - y_0} \mu(a) < \alpha_0 < \min(\mu(x_0), \mu(y_0))$ and thus $x_0, y_0 \in U(\mu; \alpha_0)$ and $x_0 - y_0 \nsubseteq U(\mu; \alpha_0)$. This is a contradiction. Hence

$$\inf_{a \in x-y} \mu(a) \geq \min(\mu(x), \mu(y)) \geq T(\mu(x), \mu(y))$$

for all $x, y \in R$. Now if the condition (2) of Definition 14 is not true, then $\inf\limits_{b \in x_0 \gamma y_0} \mu(b) < \mu(y_0)$ for some $x_0, y_0 \in R, \gamma \in \Gamma$. Taking $s_1 = \frac{1}{2}\Big(\inf\limits_{b \in x_0 \gamma y_0} \mu(b) + \mu(y_0) \Big)$, then $0 \leq s_1 < \mu(y_0)$ and $\inf\limits_{b \in x_0 \gamma y_0} \mu(b) < s_1$. Hence $y_0 \in U(\mu; s_1)$ and $x_0 \gamma y_0 \nsubseteq U(\mu; s_1)$, a contradiction. This completes the proof. □

3 T-fuzzy quasi-hyperideals and T-fuzzy bi-hyperideals in Γ-hyperrings

In this section, we introduce the notions of fuzzy quasi(bi)-hyperideal and T-fuzzy quasi(bi)-hyperideal in Γ-hyperrings and some properties of them are studied. Also, the regular Γ-hyperrings are studied in terms of T-fuzzy quasi-hyperideals and T-fuzzy bi-hyperideals.

Definition 16 *Let* $(R, +, \Gamma)$ *be a* Γ*-hyperring. A fuzzy subset* μ *of* R *is called* T*-fuzzy quasi-hyperideal if*

1. $\inf\limits_{a \in x-y} \mu(a) \geq T(\mu(x), \mu(y))$,

2. $(\mu \circ \chi_R) \wedge (\chi_R \circ \mu) \subseteq \mu$ *for all* $x, y \in R$.

If $T = \min$, then μ is called *fuzzy quasi-hyperideal* of R.

Lemma 3 *Let* $(R, +, \cdot)$ *be a hyperring. A fuzzy subset* μ *is a* T*-fuzzy quasi-hyperideal of* R *if and only if*

(a) $\mu(x) \geq T[\sup_{x \in y\gamma z} \mu(y), \sup_{x \in y\gamma z} \mu(z)]$, $\forall x \in R, \gamma \in \Gamma$ *and*

(b) $\inf_{a \in x - y} \mu(a) \geq T(\mu(x), \mu(y))$, $\forall x, y \in R$.

Proof. Let μ be a T-fuzzy quasi-hyperideal of R. Let $x \in R$. Then

$$\begin{aligned}\mu(x) &\geq T((\mu \circ \chi_R)(x), (\chi_R \circ \mu(x))) \\ &= T\left(\sup_{x \in y\gamma z} T(\mu(y), \chi_R(z)\right), \sup_{x \in y\gamma z} T((\chi_R)(y), \mu(z)) \\ &= T\left(\sup_{x \in y\gamma z} \mu(y), \sup_{x \in y\gamma z} \mu(z)\right).\end{aligned}$$

Conversely,

$$\begin{aligned}\mu(x) &\geq T(\sup_{x \in y\gamma z} \mu(y), \sup_{x \in y\gamma z} \mu(z)) \\ &= T\left(\sup_{x \in y\gamma z} T(\mu(y), \chi_R(z)\right), \sup_{x \in y\gamma z} T((\chi_R)(y), \mu(z))) \\ &= T((\mu \circ \chi_R)(x), (\chi_R \circ \mu)(x)) \\ &= ((\mu \circ \chi_R) \wedge (\chi_R \circ \mu))(x).\end{aligned}$$

Then μ is a T-fuzzy quasi-hyperideal of R. □

Corollary 4 *Let* $(R, +, \cdot)$ *be a* Γ*-hyperring. A fuzzy subset* μ *is a fuzzy quasi-hyperideal of* R *if and only if*

1. $\mu(x) \geq \min\{\sup_{x \in y\gamma z} \mu(y), \sup_{x \in y\gamma z} \mu(z)\}$, $\forall x \in R, \gamma \in \Gamma$ *and*

2. $\inf_{a \in x - y} \mu(a) \geq \min\{\mu(x), \mu(y)\}$, $\forall x, y \in R$.

Proof. By taking min as t-norm T in Lemma 3, we get the required result. □

Lemma 4 *Let* $(R, +, \Gamma)$ *be a* Γ*-hyperring. Every fuzzy quasi-hyperideal of* R *is a* T*-fuzzy quasi-hyperideal of* R.

Proof. Let μ be a fuzzy quasi-hyperideal of R. Let $x, y \in R$. Then

$$\inf_{a \in x-y} \mu(a) \geq \min\{\mu(x), \mu(y)\} \geq T(\mu(x), \mu(y))$$

and

$$\mu(x) \geq \min\{\sup_{x \in y\gamma z} \mu(y), \sup_{x \in y\gamma z} \mu(z)\} \geq T[\sup_{x \in y\gamma z} \mu(y), \sup_{x \in y\gamma z} \mu(z)].$$

Thus μ is a T-fuzzy quasi-hyperideal of R. □

Corollary 5 *Let* $(R, +, \Gamma)$ *be a* Γ*-hyperring. If* Q *is a quasi-hyperideal in* R*, then* χ_Q *is a* T*-fuzzy quasi-hyperideal of* R.

Theorem 5 *Let* $(R, +, \Gamma)$ *be a* Γ*-hyperring. Every* T*-fuzzy right hyperideal of* R *is* T*-fuzzy quasi-hyperideal. Moreover, every* T*-fuzzy left hyperideal of* R *is* T*-fuzzy quasi-hyperideal.*

Proof. Let μ be a T-fuzzy right hyperideal of R. Let $x \in R$. Then $\inf_{a \in y\gamma z} \mu(a) \geq \mu(y)$. If $x \in y\gamma z$, then

$$\begin{aligned}\mu(x) \geq \mu(y) &\geq \min\left\{\sup_{x \in y\gamma z} \mu(y), \sup_{x \in y\gamma z} \mu(z)\right\} \\ &\geq T\left[\sup_{x \in y\gamma z} \mu(y), \sup_{x \in y\gamma z} \mu(z)\right].\end{aligned}$$

Therefore μ is a T-fuzzy quasi-hyperideal. Similarly, if λ is a T-fuzzy left hyperideal of R, then λ is a T-fuzzy quasi-hyperideal. □

Theorem 6 *Let* $(R, +, \Gamma)$ *be a* Γ*-hyperring. If* μ, λ *are* T*-fuzzy quasi-hyperideals of* R*, then* $\mu \wedge \lambda$ *is a* T*-fuzzy quasi-hyperideal of* R.

Proof. Let μ, λ be the T-fuzzy quasi-hyperideal of R. Let $x, y \in R$. From the proof of the Theorem 1, we have $\inf_{a \in x-y} (\mu \wedge \lambda)(a) \geq T((\mu \wedge \lambda)(x), (\mu \wedge \lambda)(y))$. We have

$$(\mu \wedge \lambda)(x) = T(\mu(x), \lambda(x))$$

$$\begin{aligned}
&\geq T\Big(T\Big(\sup_{x\in y\gamma z}\mu(y), \sup_{x\in y\gamma z}\mu(z)\Big), T\Big(\sup_{x\in y\gamma z}\lambda(y), \sup_{x\in y\gamma z}\lambda(z)\Big)\Big)\\
&= T\Big(T\Big(T\Big(\sup_{x\in y\gamma z}\mu(y), \sup_{x\in y\gamma z}\mu(z)\Big), \sup_{x\in y\gamma z}\lambda(y)\Big), \sup_{x\in y\gamma z}\lambda(z)\Big)\\
&= T\Big(T\Big(T\Big(\sup_{x\in y\gamma z}\mu(y), \sup_{x\in y\gamma z}\lambda(y)\Big), \sup_{x\in y\gamma z}\mu(z)\Big), \sup_{x\in y\gamma z}\lambda(z)\Big)\\
&= T\Big(T\Big(\sup_{x\in y\gamma z}\mu(y), \sup_{x\in y\gamma z}\lambda(y)\Big), T\Big(\sup_{x\in y\gamma z}\mu(z), \sup_{x\in y\gamma z}\lambda(z)\Big)\Big)\\
&\geq T(T(\mu(y),\lambda(y)), T(\mu(z),\lambda(z)))\\
&\geq T\Big(\sup_{x\in y\gamma z}T(\mu(y),\lambda(y)\Big), \sup_{x\in y\gamma z}T(\mu(z),\lambda(z)\Big)\Big)\\
&= T\Big(\sup_{x\in y\gamma z}(\mu\wedge\lambda)(y), \sup_{x\in y\gamma z}(\mu\wedge\lambda)(z)\Big).
\end{aligned}$$

Therefore $\mu \wedge \lambda$ is a T-fuzzy quasi-hyperideal of R. □

Theorem 7 *Let* $(R, +, \Gamma)$ *be a* Γ*-hyperring. If* μ, λ *are* T*-fuzzy right and* T*-fuzzy left hyperideals of* R *respectively, then* $\mu\wedge\lambda$ *is a* T*-fuzzy quasi-hyperideal of* R*.*

Proof. Let μ and λ be T-fuzzy right and T-fuzzy left hyperideal of R respectively. Then by Theorem 5, μ and λ are T-fuzzy quasi-hyperideals of R. By Theorem 6, $\mu \wedge \lambda$ is a T-fuzzy quasi-hyperideal of R. □

Definition 17 *Let* $(R, +, \Gamma)$ *be a* Γ*-hyperring. A fuzzy subset* μ *of* R *is called* T*-fuzzy bi-hyperideal if*

1. $\inf_{a\in x-y}\mu(a) \geq T(\mu(x), \mu(y))$,
2. $\inf_{a\in x\gamma_1 y\gamma_2 z}\mu(a) \geq T(\mu(x), \mu(z))$ *for all* $x, y, z \in R, \gamma_1, \gamma_2 \in \Gamma$.

If $T = \min$, then μ is called *fuzzy bi-hyperideal* of R.

Lemma 5 *Let* B *be a bi-hyperideal of a* Γ*-hyperring* R*. For any* $0 < \alpha < 1$*, there exists a fuzzy bi-hyperideal* μ *such that* $U(\mu;\alpha) = B$*.*

Proof. Let B be a bi-hyperideal of R. Define $\mu : R \to [0, 1]$ by

$$\mu(x) = \begin{cases} \alpha, & \text{if } x \in B\\ 0, & \text{if } x \notin B\end{cases}$$

where α is a fixed number in $(0,1)$. It is clear that $U(\mu;\alpha)=B$. Let $x,y\in R$. If $x,y\in B$, then $\inf_{a\in x-y}\mu(a)=\alpha=\min\{\mu(x),\mu(y)\}$. If at least one of x and y is not in B, then $\inf_{a\in x-y}\mu(a)=0=\min\{\mu(x),\mu(y)\}$. Let $x,y,z\in R$. If $x,z\in B$, then $\mu(x)=\alpha,\mu(z)=\alpha$. Also $\inf_{a\in x\gamma y\cdot z}\mu(a)=\alpha\geq\min\{\mu(x),\mu(z)\}$. If at least one of x and z is not in B, then $\inf_{a\in x\gamma y\cdot z}\mu(a)=0=\min\{\mu(x),\mu(z)\}$. Thus μ is a fuzzy bi-hyperideal of R. □

Lemma 6 *Let* B *be a non-empty subset of a* Γ*-hyperring* R. B *is a bi-hyperideal of* R *if and only if* χ_B *is a fuzzy bi-hyperideal of* R.

Proof. (i) Let $x,y\in R$.

Case 1: $x,y\in B$. Then $\chi_B(x)=\chi_B(y)=1$. Therefore $\inf_{a\in x-y}\chi_B(a)=1\geq\min\{\chi_B(x),\chi_B(y)\}$.

Case 2: $x\in B$ and $y\notin B$. Then $\chi_B(x)=1$ and $\chi_B(y)=0$. Therefore $\inf_{a\in x-y}\chi_B(a)=0\geq\min\{\chi_B(x),\chi_B(y)\}$.

Case 3: $x\notin B$ and $y\in B$. Then $\chi_B(x)=0$ and $\chi_B(y)=1$. Therefore $\inf_{a\in x-y}\chi_B(a)=0\geq\min\{\chi_B(x),\chi_B(y)\}$.

Case 4: $x\notin B$ and $y\notin B$. Then $\chi_B(x)=0$ and $\chi_B(y)=0$. Therefore $\inf_{a\in x-y}\chi_B(a)=0\geq\min\{\chi_B(x),\chi_B(y)\}$.

(ii) Let $x,y,z\in R,\gamma_1,\gamma_2\in\Gamma$.

Case 1: $x,z\in B$. Then $\chi_B(x)=\chi_B(z)=1$. Therefore $\inf_{a\in x\gamma_1y\gamma_2z}\chi_B(a)=1\geq\min\{\chi_B(x),\chi_B(z)\}$.

Case 2: $x\in B$ and $z\notin B$. Then $\chi_B(x)=1$ and $\chi_B(z)=0$. Therefore $\inf_{a\in x\gamma_1y\gamma_2z}\chi_B(a)=0\geq\min\{\chi_B(x),\chi_B(z)\}$.

Case 3: $x\notin B$ and $z\in B$. Then $\chi_B(x)=0$ and $\chi_B(z)=1$. Therefore $\inf_{a\in x\gamma_1y\gamma_2z}\chi_B(a)=0\geq\min\{\chi_B(x),\chi_B(z)\}$.

Case 4: $x\notin B$ and $z\notin B$. Then $\chi_B(x)=0$ and $\chi_B(z)=0$. Therefore $\inf_{a\in x\gamma_1y\gamma_2z}\chi_B(a)=0\geq\min\{\chi_B(x),\chi_B(z)\}$.

Thus χ_B is a fuzzy bi-hyperideal of R. Conversely, let us suppose that χ_B is a fuzzy bi-hyperideal of R. Then, by Lemma 5, χ_B is two-valued. Hence B is a bi-hyperideal of R. □

Lemma 7 *Let* $(R,+,\Gamma)$ *be a* Γ*-hyperring. A fuzzy subset* μ *is a* T*-fuzzy bi-hyperideal of* R *if and only if* $\mu\circ\chi_R\circ\mu\subseteq\mu$.

Proof. Let μ be a T-fuzzy bi-hyperideal of R. Let $x \in R, \gamma_1, \gamma_2$. If $x \notin a\gamma_1 b\gamma_2 c$ for $a, b, c \in R$, then $\mu(x) \geq (\mu \circ \chi_R \circ \mu)(x) = 0$. If $x \in a\gamma_1 b\gamma_2 c$, then

$$\begin{aligned}\mu(x) &\geq T(\mu(a), \mu(c)) = T(\mu(a), T(1, \mu(c))) \\ &= T(\mu(a), T(\chi_R(b), \mu(c))).\end{aligned}$$

Therefore,

$$\mu(x) \geq \sup_{x \in a\gamma_1 y} T(\mu(a), \sup_{y \in b\gamma_2 c} T(\chi_R(b), \mu(c))) \geq (\mu \circ \chi_R \circ \mu)(x).$$

Conversely, it is clear that $\inf\limits_{a \in x\alpha y\beta z} \mu(a) \geq T(\mu(x), \mu(z))$. □

Theorem 8 *Let* $(R, +, \Gamma)$ *be a* Γ*-hyperring. Every* T*-fuzzy quasi-hyperideal of* R *is a* T*-fuzzy bi-hyperideal of* R*.*

Proof. Let μ be a T-fuzzy quasi-hyperideal of R. Let $x, y, z \in R, \alpha, \beta$. Then $\inf\limits_{a \in x\alpha y\beta z} \mu(a) \geq T(\mu(x), \inf\limits_{s \in y\beta z} \mu(s)) \geq T(\mu(x), \mu(z))$. Therefore μ is a T-fuzzy bi-hyperideal of R. □

Theorem 9 *Let* $(R, +, \Gamma)$ *be a* Γ*-hyperring. Every fuzzy bi-hyperideal of* R *is a* T*-fuzzy bi-hyperideal of* R*.*

Proof. Let μ be a fuzzy bi-hyperideal. Let $x, y, z \in R, \alpha, \beta \in \Gamma$. Then

$$\inf_{a \in x - y} \mu(a) \geq \min\{\mu(x), \mu(y)\} \geq T(\mu(x), \mu(y))$$

and

$$\inf_{a \in x\alpha y\beta z} \mu(a) \geq \min\{\mu(x), \mu(z)\} \geq T(\mu(x), \mu(z)).$$

Thus μ is a T-fuzzy bi-hyperideal of R. □

Corollary 6 *Let* $(R, +, \Gamma)$ *be a* Γ*-hyperring. If* B *is a bi-hyperideal in* R*, then* χ_B *is a* T*-fuzzy bi-hyperideal of* R*.*

Theorem 10 *Let* $(R, +, \Gamma)$ *be a* Γ*-hyperring. If* μ *and* λ *are* T*-fuzzy bi-hyperideals of* R*, then* $\mu \wedge \lambda$ *is a* T*-fuzzy bi-hyperideal of* R*.*

Proof. Let μ and λ be T-fuzzy bi-hyperideals of R. Then

$$\begin{aligned}\inf_{a\in x\alpha y\beta z}(\mu\wedge\lambda)(a) &\geq T(T(\mu(x),\mu(z)),T(\lambda(x),\lambda(z)))\\ &= T(T(\mu(x),\lambda(x)),T(\mu(x),\lambda(z)))\\ &= T((\mu\wedge\lambda)(x),(\mu\wedge\lambda)(z)).\end{aligned}$$

Hence $\mu\wedge\lambda$ is a T-fuzzy bi-hyperideal of R. □

Corollary 7 *Let* $(R,+,\Gamma)$ *be a* Γ*-hyperring. If* μ *and* λ *are fuzzy bi-hyperideals of* R*, then* $\mu\cap\lambda$ *is a fuzzy bi-hyperideal of* R*.*

Lemma 8 *Let* $(R,+,\cdot)$ *be a* Γ*-hyperring. Then the following statements are equivalent:*

1. R *is regular.*
2. $B = B\Gamma R\Gamma B$*, for any bi-hyperideal* B *in* R*.*
3. $Q = Q\Gamma R\Gamma Q$ *for any quasi-hyperideal* Q *in* R*.*

Proof. $(1)\Rightarrow(2)$. Let us assume that R is a regular Γ-hyperring. Let B be a bi-hyperideal of R and $a\in B$. Then there exist $x\in R,\alpha,\beta\in\Gamma$ such that $a\in a\alpha x\beta a\subseteq B\Gamma R\Gamma B$. Hence, $B\subseteq B\Gamma R\Gamma B$. Since B is a bi-hyperideal, then $B\Gamma R\Gamma B\subseteq B$. Therefore $B = B\Gamma R\Gamma B$.

$(2)\Rightarrow(3)$. Let Q be a quasi-hyperideal of R, that is $Q\Gamma R\cap R\Gamma Q\subseteq Q$. Since every quasi-hyperideal is a bi-hyperideal of R, then we have $Q\Gamma R\Gamma Q = Q$.

$(3)\Rightarrow(1)$. Let us assume that (3) holds. Let I and J be any right hyperideal and left hyperideal of R, respectively. Then we have $(I\cap J)\Gamma R\cap R\Gamma(I\cap J)\subseteq I\cap J$. It is easy to see that $I\cap J$ is a quasi-hyperideal of R. By (3) and Lemma 2, we have $I\cap J\subseteq(I\cap J)\Gamma R\Gamma(I\cap J)\subseteq I\Gamma R\Gamma J\subseteq I\Gamma J\subseteq I\cap J$. Hence, $I\Gamma J = I\cap J$ and so R is regular. □

Theorem 11 *Let* $(R,+,\Gamma)$ *be a* Γ*-hyperring. Then the following statements are equivalent:*

1. R *is regular.*
2. $\lambda\wedge\lambda\subseteq\lambda\circ\chi_R\circ\lambda\subseteq\lambda$ *for any* T*-fuzzy bi-hyperideal* λ *of* R*.*
3. $\mu\wedge\mu\subseteq\mu\circ\chi_R\circ\mu\subseteq\mu$ *for any* T*-fuzzy quasi-hyperideal* μ *of* R*.*

Proof. (1) ⇒ (2). Let $(R, +, \Gamma)$ be a regular Γ-hyperring and let λ be a T-fuzzy bi-hyperideal of R. Let $x \in R$. By Lemma 7, $\lambda(x) \geq (\lambda \circ \chi_R \circ \lambda)(x)$. For $x \in R$ there is $a \in R, \alpha, \beta \in \Gamma$ such that $x \in x\alpha a\beta x$. Thus $\lambda(x) \geq \inf_{s \in x\alpha a\beta x} \lambda(s) \geq T(\lambda(x), \lambda(x)) = (\lambda \wedge \lambda)(x)$. Hence $\lambda \wedge \lambda \subseteq \lambda \circ \chi_R \circ \lambda \subseteq \lambda$.

(2) ⇒ (3). It is trivial.

(3) ⇒ (1). Let Q be any quasi-hyperideal in R. Then by Corollary 5, χ_Q is a T-fuzzy quasi-hyperideal. It is clear that $Q\Gamma R\Gamma Q \subseteq Q$. Let $x \in Q$. Then $\chi_Q(x) = 1$. Thus $(\chi_Q \circ \chi_R \circ \chi_Q)(x) \geq T(\chi_Q(x), \chi_Q(x)) = 1$. Then $(\chi_Q \circ \chi_R \circ \chi_Q)(x) = 1$. Thus $x \in Q\Gamma R\Gamma Q$. Therefore $Q \subseteq Q\Gamma R\Gamma Q$. Hence $Q = Q\Gamma R\Gamma Q$. By Lemma 8, R is regular. □

Corollary 8 *Let* $(R, +, \Gamma)$ *be a* Γ*-hyperring. Then the following statements are equivalent:*

1. R *is regular.*
2. $\lambda = \lambda \circ \chi_R \circ \lambda$ *for any fuzzy bi-hyperideal* λ *of* R.
3. $\mu = \mu \circ \chi_R \circ \mu$ *for any fuzzy quasi-hyperideal* μ *of* R.

Proof. By taking min as t-norm T in Theorem 11, we get the required result. □

Theorem 12 *Let* $(R, +, \Gamma)$ *be a regular* Γ*-hyperring. If* μ *is an imaginable* T*-fuzzy bi-hyperideal of* R*, then* μ *is a* T*-fuzzy quasi-hyperideal of* R.

Proof. Let μ be an imaginable T-fuzzy bi-hyperideal of a regular Γ-hyperring R and $a \in R$. Suppose that $(\mu \circ \chi_R)(a) \leq \mu(a)$. Then we get:

$$\begin{aligned} \mu(a) &\geq (\mu \circ \chi_R)(a) \geq \min\{(\chi_R \circ \mu)(a), (\mu \circ \chi_R)(a)\} \\ &\geq T((\chi_R \circ \mu)(a), (\mu \circ \chi_R)(a)) \\ &= (\chi_R \circ \mu \wedge \mu \circ \chi_R)(a) \end{aligned}$$

Suppose that $(\mu \circ \chi_R)(a) > \mu(a)$. Then $\sup_{a \in x\gamma y} T(\mu(x), \chi_R(y)) > \mu(a)$. Thus $T(\mu(x), \chi_R(y)) > \mu(a)$. So $\mu(x) > \mu(a)$.

Since R is a regular Γ-hyperring, then there exists $b \in R, \alpha, \beta \in \Gamma$, such that $a \in a\alpha b\beta a \subseteq x\gamma y\alpha b\beta a$. Since μ is an imaginable T-fuzzy bi-hyperideal, then $\mu(a) \geq T(\mu(x), \mu(a))$. Then $T(\mu(a), \mu(a)) \geq T(\mu(x), \mu(a))$. This is a contradiction to $\mu(x) > \mu(a)$. Thus $(\mu \circ \chi_R)(x) \leq \mu(a)$.

In similar way, we can prove $(\chi_R \circ \mu)(a) \leq \mu(a)$. So $\mu \circ \chi_R \wedge \chi_R \circ \mu \subseteq \mu$. Hence μ is a T-fuzzy quasi-hyperideal of R. □

4 On T-(λ,μ)-fuzzy bi-hyperideals

Definition 18 *Let* $(R,+,\Gamma)$ *be a* Γ*-hyperring and* A *be a fuzzy subset of* R*. Then* A *is called a* T-(λ,μ)*-fuzzy bi-hyperideal of* R *if for all* $x,y,z\in R$, $\alpha,\beta\in\Gamma$*, we have*

1. $\inf_{a\in x-y} A(a)\vee\lambda \geq T(T(A(x),A(y)),\mu)$
2. $\inf_{a\in x\alpha y\beta z} A(a)\vee\lambda \geq T(T(A(x),A(z)),\mu)$

Theorem 13 *Let* $(R,+,\Gamma)$ *be a* Γ*-hyperring and* A *be a fuzzy subset of* R*. Then* A *is a* T-(λ,μ)*-fuzzy bi-hyperideal of* R *iff* A_α *is a bi-hyperideal of* R *for all* $\alpha\in(\lambda,\mu)$

Proof. Let A be a T-(λ,μ)-fuzzy bi-hyperideal of R. Let $x,y\in A_\alpha$. Then $A(x)\geq\alpha, A(y)\geq\alpha$. Consider $\inf_{a\in x-y} A(a)\vee\lambda \geq T(T(A(x),A(y)),\mu) \geq T(T(\alpha,\alpha),\mu)=\alpha$ (since $\alpha>\lambda$). That is, $\inf_{a\in x-y} A(a)\vee\lambda\geq\alpha$. This implies that $\inf_{a\in x-y} A(a)\geq\alpha$, that is, $x-y\subseteq A_\alpha$. Hence A_α is a subhypergroup of R. Let $x,z\in A_\alpha$, $y\in R$ and $\gamma,\beta\in\Gamma$. This implies that $A(x)\geq\alpha, A(z)\geq\alpha$. Then $\inf_{a\in x-y} A(a)\vee\lambda\geq T(T(A(x),A(z)),\mu)\geq T(T(\alpha,\alpha),\mu)=\alpha$ (since $\alpha>\lambda$). That is, $\inf_{a\in x\gamma y\beta z} A(a)\vee\lambda\geq\alpha$. This implies that $\inf_{a\in x\gamma y\beta z} A(a)\geq\alpha$ and so $x\gamma y\beta z\subseteq A_\alpha$. Hence A_α is a bi-hyperideal of R.

Conversely, let us suppose that A_α is a bi-hyperideal of R for all $\alpha\in(\lambda,\mu)$. Suppose that $\inf_{a\in x-y} A(a)\vee\lambda < T(T(A(x),A(y)),\mu)=\alpha$. Then $\inf_{a\in x-y} A(a)\vee\lambda<\alpha$ which implies that $\inf_{a\in x-y} A(a)<\alpha$ (since $\alpha>\lambda$). This implies that $x-y\nsubseteq A_\alpha$ which is impossible since A_α is a bi-hyperideal of R. Hence $\inf_{a\in x-y} A(a)\vee\lambda\geq T(T(A(x),A(y)),\mu)$. Similarly, we can prove that $\inf_{a\in x\alpha y\beta z} A(a)\vee\lambda\geq T(T(A(x),A(z)),\mu)$. Hence A is a T-(λ,μ)-fuzzy bi-hyperideal of R. □

Definition 19 *Let* R *and* R' *be* Γ *and* Γ'*-hyperrings, respectively,* $\varphi: R\to R'$ *and* $f:\Gamma\to\Gamma'$ *be two maps. Then,* (φ,f) *is called a* (Γ,Γ')*-homomorphism if*

1. $\forall(x,y)\in R^2$, $\varphi(x+y)=\{\varphi(z)|z\in x+y\}\subseteq\varphi(x)+\varphi(y)$,
2. $\forall(x,y,\alpha)\in R^2\times\Gamma$, $\varphi(x\alpha y)=\{\varphi(z)|z\in x\alpha y\}\subseteq\varphi(x)f(\alpha)\varphi(y)$.

Let φ be a mapping from a set X to a set Y. Let μ be a fuzzy subset of X and λ be a fuzzy subset of Y. Then the inverse image $\varphi^{-1}(\lambda)$ of λ is the fuzzy subset of X defined by $\varphi^{-1}(\lambda)(x) = \lambda(\varphi(x))$ for all $x \in X$. The image $\varphi(\mu)$ of μ is the fuzzy subset of Y defined by

$$\varphi(\mu)(y) = \begin{cases} \sup\{\mu(t) | t \in \varphi^{-1}(y)\}, & \text{if } \varphi^{-1}(y) \neq \emptyset; \\ 0, & \text{otherwise.} \end{cases}$$

for all $y \in Y$.

Theorem 14 *Let* (φ, f) *be a onto* (Γ, Γ')*-homomorphism of* $(R, +, \Gamma)$ *and* $(R', +, \Gamma')$ *respectively and let* A *be a* T-(λ, μ)*-fuzzy bi-hyperideal of* R*. Then* $\varphi(A)$ *is a* T-(λ, μ)*-fuzzy bi-hyperideal of* R'.

Proof. Let $y_1, y_2 \in R'$. Then we have

$$\begin{aligned} \inf_{z \in y_1 - y_2} \{\varphi(A)(z)\} \vee \lambda &= \sup_{z \in y_1 - y_2} \left\{ \inf_{a \in \varphi^{-1}(y_1) - \varphi^{-1}(y_2)} A(a) \right\} \vee \lambda \\ &= \sup_{z \in y_1 - y_2} \left\{ \inf_{a \in \varphi^{-1}(y_1) - \varphi^{-1}(y_2)} A(a) \vee \lambda \right\} \\ &\geq \sup_{\varphi(z) \in \varphi(y_1) - \varphi(y_2)} \{A((\varphi)(z))\} \vee \lambda \\ &\geq \sup\{T(T(A((\varphi)(x_1)), A((\varphi)(x_2))), \mu) \mid \\ &\qquad x_1 \in (\varphi)^{-1}(y_1), x_2 \in (\varphi)^{-1}(y_2)\} \\ &\geq T(T(\sup\{A((\varphi)(x_1))\}, \sup\{A((\varphi)(x_2))\}), \mu) \mid \\ &\qquad x_1 \in (\varphi)^{-1}(y_1), x_2 \in (\varphi)^{-1}(y_2)\} \\ &= T(T((\varphi(A))(y_1), (\varphi(A))(y_2)), \mu) \end{aligned}$$

Therefore, $\inf_{z \in y_1 - y_2} \{\varphi(A)(z)\} \vee \lambda \geq T(T((\varphi(A))(y_1), (\varphi(A))(y_2)), \mu)$.

Similarly, we can prove that

$$\inf_{a \in y_1 \alpha y_2 \beta y_3} (\varphi(A))(a) \vee \lambda \geq T(T((\varphi(A))(y_1), (\varphi(A))(y_3)), \mu).$$

Hence $\varphi(A)$ is a T-(λ, μ)-fuzzy bi-hyperideal of R'. □

Theorem 15 *Let* (φ, f) *be a onto* (Γ, Γ')*-homomorphism of* $(R, +, \Gamma)$ *and* $(R', +, \Gamma')$ *respectively and let* B *be a* T-(λ, μ)*-fuzzy bi-hyperideal of* R'*. Then* $\varphi^{-1}(B)$ *is a* T-(λ, μ)*-fuzzy bi-hyperideal of* R.

Proof. (1) Suppose that $x, y \in R$ and $\gamma \in \Gamma$. Then we have

$$\inf_{z \in x - y} \{\varphi^{-1}(B)(z)\} \vee \lambda = \inf_{z \in x - y} \{B((\varphi)(z))\} \vee \lambda$$

$$\begin{aligned} &\geq \inf_{\varphi(z)\in\varphi(x)-\varphi(y)}\{B((\varphi)(z))\}\vee\lambda \\ &\geq T(T(B((\varphi)(x)), B((\varphi)(y))), \mu) \\ &= T(T((\varphi^{-1}(B))(x), (\varphi^{-1}(B))(y)), \mu) \end{aligned}$$

Therefore, $\inf_{z\in x-y}\{\varphi^{-1}(B)(z)\}\vee\lambda \geq T(T((\varphi^{-1}(B))(x), (\varphi^{-1}(B))(y)), \mu)$.

(2) Similarly we can prove that

$$\inf_{a\in x\alpha y\beta z}(\varphi^{-1}(B))(a)\vee\lambda \geq T(T((\varphi^{-1}(B))(x), (\varphi^{-1}(B))(z)), \mu).$$

Hence $\varphi^{-1}(B)$ is a T-(λ, μ)-fuzzy bi-hyperideal of R. □

5 Conclusions

In this paper, we have studied the Γ-hyperrings via T-fuzzy left and right hyperideals, T-fuzzy quasi-hyperideal and bi-hyperideal and some related properties were investigated. As a future work, one can extend these results applying the intuitionistic fuzzy theory and also to extend these results in other algebraic hyperstructure such as (m, n)-hyperrings etc.

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On a non flat Riemannian warped product manifold with respect to quarter-symmetric connection

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Abstract. In this paper, we study generalized quasi-Einstein warped products with respect to quarter symmetric connection for dimension $n \geq 3$ and Ricci-symmetric generalized quasi-Einstein manifold with quarter symmetric connection. We also investigate that in what conditions the generalized quasi-Einstein manifold to be nearly Einstein manifold with respect to quarter symmetric connection. Example of warped product on generalized quasi-Einstein manifold with respect to quarter symmetric connection are also discussed.

1 Introduction

A Riemannian manifold (M^n, g), $(n > 2)$ is Einstein if its Ricci tensor S of type (0,2) is of the form $S = \alpha g$, where α is smooth function, which turns

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into $S = \frac{r}{n}g$, r being the scalar curvature of the manifold. The notion of quasi Einstein manifold was introduced by M. C. Chaki and R. K. Maity [2]. A non-flat Riemannian manifold (M^n, g), $(n > 2)$ is defined to be a quasi Einstein manifold if its Ricci tensor S of type (0, 2) is not identically zero and satisfies the condition

$$S(X, Y) = \alpha g(X, Y) + \beta A_1(X)A_1(Y) \tag{1}$$

where α, β are scalars of which $\beta \neq 0$ and A_1 is a non-zero 1-form such that $g(X, U) = A_1(X)$ for all vector fields X with $g(U, U) = 1$. Such an n-dimensional quasi-Einstein manifold is denoted by $(QE)_n$.

In [5], De and Ghosh introduced generalized quasi-Einstein manifold, denoted by $G(QE)_n$, where the Ricci tensor S of type (0,2) which is not identically zero satisfies the condition

$$S(X, Y) = \alpha g(X, Y) + \beta A_1(X)A_1(Y) + \gamma B_1(X)B_1(Y), \tag{2}$$

where α, β, γ are scalars such that β, γ are nonzero and A_1, B_1 are two nonzero 1-forms such that

$$g(X, \mu) = A_1(X) \qquad \text{and} \qquad g(X, \rho) = B_1(X),$$

μ, ρ being unit vectors which are orthogonal, i.e., $g(\mu, \rho) = 0$.

Here α, β, γ are called the associated scalars, and A_1, B_1 are called the associated main and auxiliary 1-forms respectively, μ, ρ are called the main and the auxiliary generators of the manifold.

The notion of warped product generalizes that of a surface of revolution. It was introduced in [1] for studying manifolds of negative curvature. Let (B, g_B) and (F, g_F) be two Riemannian manifolds and f is a positive differentiable function on B. Consider the product manifold $B \times F$ with its projections $\pi : B \times F \to B$ and $\sigma : B \times F \to F$. The warped product $B \times_f F$ is the manifold $B \times F$ with the Riemannian structure such that $||X||^2 = ||\pi^*(X)||^2 + f^2(\pi(p))||\sigma^*(X)||^2$, for any vector field X on M. Thus we have

$$g = g_B + f^2 g_F \tag{3}$$

holds on M. The function f is called the warping function of the warped product [9].

Since $B \times_f F$ is a warped product, then we have $\nabla_X Z = \nabla_Z X = (X\ln f)Z$ for unit vector fields X, Z on B and F, respectively. Hence, we find $K(X \wedge Z) = g(\nabla_Z\nabla_X X - \nabla_X\nabla_Z X, Z) = (1/f)\{(\nabla_X X_f - X^2 f\}$. If we chose a local orthonormal frame $e_1, \ldots, e_n$ such that $e_1, \ldots, e_{n_1}$ are tangent to B and $e_{n_1+1}, \ldots, e_n$ are tangent to F, then we have

$$\frac{\Delta f}{f} = \sum_{i=1}^{n} K(e_i \wedge e_j), \tag{4}$$

for each $s = n_1 + 1, \ldots, n$ [9].

In 1924, Friedmann and Schouten introduced the notion of a semi-symmetric linear connection on a differentiable manifold [15]. In 1975, Golab introduced the definition of a quarter-symmetric linear connection on a differentiable manifold which is a generalization of semi-symmetric connection in [8]. Many authors like Q. Qu and Y. Wang [14], S. Pahan et al. [16, 17] and S. Dey et al. [18] studied on warped product manifolds with affine connections.

In this paper we study generalized quasi-Einstein warped products with respect to quarter symmetric connection. We discuss some preliminary concepts and results which are useful for proving our main results. We obtain a necessary and sufficient condition for the warped product manifold to be a generalized quasi-Einstein manifold with respect to a quarter-symmetric connection. Next we prove generalized quasi-Einstein manifold with respect to quarter symmetric connection to be nearly quasi Einstein manifold with respect to quarter symmetric connection under some certain conditions. In the last section we give an example of warped product on generalized quasi-Einstein manifold with respect to quarter symmetric connection.

2 Preliminaries

Let (M^n, g) be a Riemannian manifold with the Levi-Civita connection ∇. A linear connection $\breve{\nabla}$ on (M^n, g) is said to be a quarter-symmetric connection if its torsion tensor T with respect to the connection $\breve{\nabla}$ defined by

$$T(X, Y) = \breve{\nabla}_X Y - \breve{\nabla}_Y X - [X, Y],$$

satisfies

$$T(X, Y) = \omega(Y)\phi X - \omega(X)\phi Y,$$

where ω is a 1-form on M^n with the associated vector field P defined by $\omega(X) = g(X, P)$, for all vector field X, and ϕ is a $(1, 1)$ tensor field.

A quarter-symmetric connection $\breve{\nabla}$ is called a quarter-symmetric metric connection if $\breve{\nabla} g = 0$. $\breve{\nabla}$ is called a quarter-symmetric non-metric connection if $\breve{\nabla} g \neq 0$. The relation between a quarter-symmetric connection $\breve{\nabla}$ and the Levi-Civita connection ∇ of M^n is given by [19]

$$\breve{\nabla}_X Y = \nabla_X Y + \lambda_1 \omega(Y) X - \lambda_2 g(X, Y) P, \tag{5}$$

where $g(X, P) = \omega(X)$ and $\lambda_1 \neq 0, \lambda_2 \neq 0$ are scalar functions.

We can easily see that: when $\lambda_1 = \lambda_2 = 1$, $\breve{\nabla}$ is a semi-symmetric metric connection.

When $\lambda_1 = \lambda_2 \neq 1$, $\breve{\nabla}$ is a quarter-symmetric metric connection.

When $\lambda_1 \neq \lambda_2$, $\breve{\nabla}$ is a quarter-symmetric non-metric connection.

Further, a relation between the curvature tensors R and $\breve{R}$ of type (1,3) of the connections ∇ and $\breve{\nabla}$ respectively is given by [19],

$$\begin{aligned} \breve{R}(X, Y)Z = {} & R(X, Y)Z + \lambda_1 g(Z, \nabla_X P)Y - \lambda_2 g(Z, \nabla_Y P)X + \lambda_2[g(X, Z)\nabla_Y P \\ & - g(Y, Z)\nabla_X P] + \lambda_1\lambda_2 \omega(P)[g(X, Z)Y - g(Y, Z)X] \\ & + \lambda_2^2[g(Y, Z)\omega(X) - g(X, Z)\omega(Y)]P + \lambda_1^2 \omega(Z)[\omega(Y)X \\ & - \omega(X)Y], \end{aligned} \tag{6}$$

for vector fields X, Y, Z on M.

3 Generalized quasi-Einstein manifold with respect to quarter-symmetric connection

In this section, we consider the following propositions from Proposition 3.5., Proposition 3.6., Proposition 3.7., Proposition 3.8. of [14], which will be helpful to prove our main results. Here we consider generalized quasi-Einstein warped product manifolds and prove some results concerning these type manifolds.

Theorem 1 *Let* (M, g) *be a warped product* $I \times_f F$ *where* I *is an open interval in* $\mathbb{R}$, $\dim I = 1$ *and* $\dim F = \bar{n} - 1$, $\bar{n} \geq 3$. *Then* (M, g) *is a generalized quasi-Einstein manifold with respect to a quarter-symmetric connection if and only if* F *is a generalized quasi-Einstein manifold for* $P = \frac{\partial}{\partial t}$ *with respect to the Levi-Civita connection or the warping function* f *is a constant on* I *for* $P \in \chi(F)$, $\lambda_2 \neq (\bar{n} - 1)\lambda_1$.

Proof. Assume that $P \in \chi(B)$ and taking $f = e^{\frac{q}{2}}$ and using the Proposition 3.1. of [16], we get

$$\breve{S}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = (1 - \bar{n})\left[\frac{1}{2}q'' + \frac{1}{4}q'^2 - \frac{1}{2}\lambda_2 q' + \lambda_1\lambda_2 - \lambda_1^2\right] g_I\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right), \tag{7}$$

$$\breve{S}\left(\frac{\partial}{\partial t}, V\right) = 0, \tag{8}$$

$$\begin{aligned}\breve{S}(V,W) = S^{F}(V,W) + e^{q}\Bigg[\frac{\bar{n}-1}{4}(q')^2 + \frac{1}{2}\Big[(\bar{n}-1)\lambda_1 + (\bar{n}-2)\lambda_2\Big]q' \\ + \lambda_2^2 + \frac{1}{2}q'' + (1-\bar{n})\lambda_1\lambda_2\Bigg]g_F(V,W),\end{aligned} \tag{9}$$

for vector fields V, W on F.

Since M is generalized quasi-Einstein admitting quarter-symmetric connection, from (2) we have

$$\breve{S_M}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = \alpha g\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) + \beta A_1\left(\frac{\partial}{\partial t}\right)A_1(\frac{\partial}{\partial t}) + \gamma B_1\left(\frac{\partial}{\partial t}\right)B_1\left(\frac{\partial}{\partial t}\right), \tag{10}$$

and

$$\breve{S_M}(V,W) = \alpha g(V,W) + \beta A_1(V)A_1(W) + \gamma B_1(V)B_1(W). \tag{11}$$

Decomposing the vector fields U and $\acute{U}$ uniquely into its components U_I, U_F and $\acute{U}_I$, $\acute{U}_F$ on I and F, respectively, we can write $U = U_I + U_F$ and $\acute{U} = \acute{U}_I + \acute{U}_F$ and also $\acute{U} = \eta_2\frac{\partial}{\partial t} + U_F$, where η_1 and η_2 are functions on M. Then we can write

$$\begin{aligned}A_1\left(\frac{\partial}{\partial t}\right) &= g\left(\frac{\partial}{\partial t}, U\right) = \eta_1, \\ B_1\left(\frac{\partial}{\partial t}\right) &= g\left(\frac{\partial}{\partial t}, \acute{U}\right) = \eta_2.\end{aligned} \tag{12}$$

On the other hand, by the use of (3) and (12), the equations (10) and (11) reduce to

$$\breve{S_M}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = \alpha + \beta\eta_1^2 + \gamma\eta_2^2 \tag{13}$$

and

$$\breve{S_M}(V,W) = \alpha e^{q} g_F(V,W) + \beta A_1(V)A_1(W) + \gamma B_1(V)B_1(W). \tag{14}$$

Comparing the right hand side of the equations (7) and (13) we get

$$\alpha + \beta\eta_1^2 + \gamma\eta_2^2 = -\frac{n-1}{4}[2q^{''} + (q^{'})^2]. \tag{15}$$

Similarly, comparing the right hand sides of (9) and (14) we obtain

$$\begin{aligned}S_F(V,W) = e^{q}\Bigg[\alpha - \Bigg\{\frac{\bar{n}-1}{4}(q')^2 + \frac{1}{2}\{(\bar{n}-1)\lambda_1 + (\bar{n}-2)\lambda_2\}q' \\ + \lambda_2^2 + \frac{1}{2}q'' + (1-\bar{n})\lambda_1\lambda_2\Bigg\}\Bigg]g_F(V,W) \\ + \beta A_1(V)A_1(W) + \gamma B(V)_1 B_1(W),\end{aligned} \tag{16}$$

which gives that F is a generalized quasi-Einstein manifold with respect to the Levi-Civita connection for $P \in \chi(B)$.

Taking $P \in \chi(F)$ and by the use of Proposition 3.1. of [16], we get

$$\breve{S}\left(\frac{\partial}{\partial t}, V\right) = \frac{q'}{2}\left[(\bar{n}-1)\lambda_1 - \lambda_2\right]\omega(V) \tag{17}$$

and

$$\breve{S}\left(V, \frac{\partial}{\partial t}\right) = \frac{q'}{2}\left[\lambda_2 - (\bar{n}-1)\lambda_1\right]\omega(V), \tag{18}$$

for any vector field $V \in \chi(F)$.

Since M is a generalized quasi-Einstein manifold, we have

$$\breve{S}\left(\frac{\partial}{\partial t}, V\right) = \tilde{S}\left(V, \frac{\partial}{\partial t}\right) = \alpha g\left(V, \frac{\partial}{\partial t}\right) + \beta A_1(V)A_1\left(\frac{\partial}{\partial t}\right) + \gamma B_1(V)B_1\left(\frac{\partial}{\partial t}\right). \tag{19}$$

Now $g(V, \frac{\partial}{\partial t}) = 0$ as $\frac{\partial}{\partial t} \in \chi(B)$ and $V \in \chi(F)$.
Hence, from the last equation, we get

$$\breve{S}\left(\frac{\partial}{\partial t}, V\right) = \breve{S}\left(V, \frac{\partial}{\partial t}\right) = \beta A_1(V)A_1\left(\frac{\partial}{\partial t}\right) + +\gamma B_1(V)B_1\left(\frac{\partial}{\partial t}\right). \tag{20}$$

Therefore, we have

$$\beta A_1(V)A_1\left(\frac{\partial}{\partial t}\right) + \gamma B_1(V)B_1\left(\frac{\partial}{\partial t}\right) = \frac{q'}{2}[(\bar{n}-1)\lambda_1 - \lambda_2]\omega(V), \tag{21}$$

$$\beta A_1(V)A_1\left(\frac{\partial}{\partial t}\right) + \gamma B_1(V)B_1\left(\frac{\partial}{\partial t}\right) = \frac{q'}{2}[\lambda_2 - (\bar{n}-1)\lambda_1]\omega(V). \tag{22}$$

From the equations (21) and (22), we get

$$q' = 0,$$

when $\lambda_2 - (\bar{n}-1)\lambda_1 \neq 0$. It follows that q is a constant on I. Then f is constant on I. This completes the proof. □

Now, we consider the warped product $M = B \times_f I$ with $\dim B = \bar{n} - 1$, $\dim I = 1$, $\bar{n} \geq 3$. Under this assumption, we obtain the following theorem.

Theorem 2 *Let* (M, g) *be a warped product* $B \times_f I$*, where* $\dim I = 1$ *and* $\dim B = n - 1$, $n \geq 3$*, then*

i) *if* (M, g) *is a generalized quasi-Einstein manifold with respect to a quarter-symmetric connection,* $P \in \chi(B)$ *is parallel on* B *with respect to the Levi-Civita connection on* B *and* f *is a constant on* B*, then,*

$$\alpha = [(n-1)\lambda_1\lambda_2 - \lambda_2^2)]\omega(P).$$

ii) *If* (M, g) *is a generalized quasi-Einstein manifold with respect to a quarter-symmetric connection for* $P \in \chi(I)$*, and* $\lambda_2 \neq (n-1)\lambda_1$ *then* f *is a constant on* B.

iii) *If* f *is a constant on* B *and* B *is a generalized quasi-Einstein manifold with respect to the Levi-Civita connection for* $P \in \chi(I)$*, then* M *is a generalized quasi-Einstein manifold with respect to a quarter-symmetric connection.*

Proof. Assume that (M, g) is a generalized quasi-Einstein manifold with respect to a quarter-symmetric connection. Then we write

$$\breve{S}(X, Y) = \alpha g(X, Y) + \beta A_1(X)_1A(Y) + \gamma B_1(X)B_1(Y). \tag{23}$$

Decomposing the vector field U and V uniquely into its components U_B and U_I on B and I, respectively, we have

$$U = U_B + U_I, \quad V = V_B + V_I \tag{24}$$

Since $\dim I = 1$, we can take $U = U_B + \eta_1 \frac{\partial}{\partial t}$ and $V = V_B + \eta_2 \frac{\partial}{\partial t}$, where η_1, η_2 is a functions on M. From (23), (24) and Proposition 3.1. of [16], we have

$$\begin{aligned}\breve{S}^B(X, Y) &= \alpha g_B(X, Y) + \beta g_B(X, U_B)g_B(Y, U_B) + \gamma g_B(X, V_B)g_B(Y, V_B) \\ &\quad - \left[\frac{H^f(X, Y)}{f} + \lambda_2 \frac{Pf}{f} g(X, Y) + \lambda_1\lambda_2\omega(P)g(X, Y) \right. \\ &\quad \left. + \lambda_1 g(Y, \nabla_X P) - \lambda_1^2\omega(X)\omega(Y)\right].\end{aligned} \tag{25}$$

By contraction over X and Y, we get

$$\begin{aligned}\breve{r}^B &= \alpha(n-1) + \beta g_B(U_B, U_B) + \gamma g_B(X, V_B)g_B(Y, V_B) \\ &\quad - \left[\frac{\Delta_B f}{f} + \lambda_2(n-1)\frac{Pf}{f} + [(n-1)\lambda_1\lambda_2 - \lambda_1^2]\omega(P) + \lambda_1 \sum_{i=1}^{n-1} g(e_i, \nabla_{e_i} P)\right].\end{aligned} \tag{26}$$

Also from (23), we have

$$\breve{r}^M = \alpha n + \beta g_B(U_B, U_B) + \gamma g_B(X, V_B) g_B(Y, V_B). \tag{27}$$

Now, putting the value of (27) in (26), we get

$$\begin{aligned} \breve{r}^B = \breve{r}^M - \alpha - \frac{\Delta_B f}{f} - \lambda_2(n-1)\frac{Pf}{f} - [(n-1)\lambda_1\lambda_2 - \lambda_1^2]\omega(P) \\ - \lambda_1 \sum_{i=1}^{n-1} g(e_i, \nabla_{e_i} P)]. \end{aligned} \tag{28}$$

On the other hand, from Proposition 1., we get

$$\begin{aligned} \breve{r}^M &= \breve{r}^B + (n-1)(\lambda_1 + \lambda_2)\frac{Pf}{f} \\ &+ 2\frac{\Delta_B f}{f} + [2(n-1)\lambda_1\lambda_2 - (\lambda_1^2 + \lambda_2^2)]\omega(P) + (\lambda_1 + \lambda_2)\sum_{i=1}^{n-1} g(\nabla_{e_i} P, e_i). \end{aligned}$$

Then from the above two relations, we get

$$\begin{aligned} &\alpha + \frac{\Delta_B f}{f} + \lambda_2(n-1)\frac{Pf}{f} + \left[(n-1)\lambda_1\lambda_2 - \lambda_1^2]\omega(P) + \lambda_1 \sum_{i=1}^{n-1} g(e_i, \nabla_{e_i} P)\right] \\ &= (n-1)(\lambda_1 + \lambda_2)\frac{Pf}{f} + 2\frac{\Delta f}{f} + [2(n-1)\lambda_1\lambda_2 - (\lambda_1^2 + \lambda_2^2)]\omega(P) \\ &\quad + (\lambda_1 + \lambda_2)\sum_{i=1}^{n-1} g(\nabla_{e_i} P, e_i). \end{aligned}$$

Since $P \in \chi(B)$ is parallel and f is a constant on B, then we get

$$\alpha = [(n-1)\lambda_1\lambda_2 - \lambda_2^2)]\omega(P).$$

ii) Let $P \in \chi(I)$. By the use of Proposition 3.1. of [16], we get

$$\breve{S}(X, P) = [(n-1)\lambda_1 - \lambda_2]\omega(P)\frac{Xf}{f}, \tag{29}$$

and

$$\breve{S}(P, X) = [\lambda_2 - (n-1)\lambda_1]\omega(P)\frac{Xf}{f}. \tag{30}$$

Since M is a generalized quasi-Einstein manifold, we have

$$\breve{S}(X,P)=\breve{S}(P,X)=\alpha g(P,X)+\beta A_1(P)A_1(X)+\gamma B_1(P)B_1(X).$$

Again, we have $g(P,X)=0$ for $X\in\chi(B)$ and $P\in\chi(I)$. Hence, we have

$$Xf=0,$$

where $\lambda_2\neq(n-1)\lambda_1$. This implies that f is a constant on B.

iii) Assume that B is a generalized quasi-Einstein manifold with respect to the Levi-Civita connection. Then we have

$$S^B(X,Y)=\alpha g(X,Y)+\beta A_1(X)A_1(Y)+\gamma B_1(X)B_1(Y), \tag{31}$$

for vector fields X,Y tangent to B.

From Proposition 3.1. of [16], we get

$$\breve{S}^M(X,Y)=S^B(X,Y)+[(n-1)\lambda_1\lambda_2-\lambda_2^2]\omega(P)g(X,Y)+\frac{H^f(X,Y)}{f},$$

for any vector field $P\in\chi(I)$. Since f is a constant, $H^f(X,Y)=0$ for all $X,Y\in\chi(B)$.

The above equation reduces to

$$\breve{S}^M(X,Y)=S^B(X,Y)+[(n-1)\lambda_1\lambda_2-\lambda_2^2]\omega(P)g(X,Y). \tag{32}$$

Using the value of (31) in (32), we get

$$\breve{S}^M(X,Y)=\{\alpha+[(n-1)\lambda_1\lambda_2-\lambda_2^2]\omega(P)\}g(X,Y)+\beta A_1(X)A_1(Y)+\gamma B_1(X)B_1(Y), \tag{33}$$

which shows that M is a generalized quasi-Einstein manifold with respect to a quarter-symmetric connection. □

Next we find the relation between scalars of generalized quasi-Einstein manifold with respect to a quarter-symmetric connection.

Suppose the generator U is a parallel vector field, then $\breve{R}(X,Y)U=0$. Hence

$$\breve{S}(X,U)=0. \tag{34}$$

Let

$$U=U_B+f^2U_F,\quad V=V_B+f^2V_F. \tag{35}$$

From (2), we have

$$\breve{S}_M(X,Y) = \alpha g(X,Y) + \beta A_1(X)A_1(Y) + \gamma B_1(X)B_1(Y).$$

Putting $Y = U$ and using (35), we have

$$\begin{aligned}\breve{S}_M(X,U) &= \alpha g(X,U) + \beta A_1(X)A_1(U) + \gamma B_1(X)B_1(U)\\ &= \{\alpha + \beta(f^4+1)\}g_F(X,U_F)f^2,\end{aligned} \tag{36}$$

where $X \in \chi(F)$ and $Y \in \chi(B)$. From (9), we have

$$\begin{aligned}\breve{S_M}(X,Y) = S^F(X,Y) + e^q\Big[&\frac{n-1}{4}(q')^2 + \frac{1}{2}\{(n-1)\lambda_1 + (n-2)\lambda_2\}q'\\ &+ \lambda_2^2 + \frac{1}{2}q'' + (1-n)\lambda_1\lambda_2\Big]g_F(X,Y),\end{aligned} \tag{37}$$

for vector fields X, Y on F.
As U is parallel to F, we have from (37)

$$\begin{aligned}\breve{S_M}(X,U) &= e^q\Big[\frac{n-1}{4}(q')^2 + \frac{1}{2}\{(n-1)\lambda_1 + (n-2)\lambda_2\}q' + \lambda_2^2 + \frac{1}{2}q''\\ &\quad + (1-n)\lambda_1\lambda_2\Big]g_F(X, U_B + f^2U_F),\\ &= f^2e^q\Big[\frac{n-1}{4}(q')^2 + \frac{1}{2}\{(n-1)\lambda_1 + (n-2)\lambda_2\}q'\\ &\quad + \lambda_2^2 + \frac{1}{2}q'' + (1-n)\lambda_1\lambda_2\Big]g_F(X,U)\end{aligned} \tag{38}$$

Now comparing (36) and (38), we have

$$\begin{aligned}\{\alpha + \beta(f^4+1)\} = e^q\Big[&\frac{n-1}{4}(q')^2 + \frac{1}{2}\{(n-1)\lambda_1 + (n-2)\lambda_2\}q'\\ &+ \lambda_2^2 + \frac{1}{2}q'' + (1-n)\lambda_1\lambda_2\Big]\end{aligned} \tag{39}$$

So, we get the relation between two non-zero smooth functions a and b of the manifold M with respect to a quarter-symmetric connection. Similarly, if V is parallel to F, we have

$$\begin{aligned}\{\alpha + \gamma(f^4+1)\} = e^q\Big[&\frac{n-1}{4}(q')^2 + \frac{1}{2}\{(n-1)\lambda_1 + (n-2)\lambda_2\}q'\\ &+ \lambda_2^2 + \frac{1}{2}q'' + (1-n)\lambda_1\lambda_2\Big]\end{aligned} \tag{40}$$

So, we also get the relation between two non-zero smooth functions α and γ of the manifold M with respect to a quarter-symmetric connection. Now we have a following proposition:

Proposition 1 *Let* (M, g) *be a warped product manifold* $B \times_f I$. *If the generators* U, V *are parallel to* F *in a generalized quasi-Einstein manifold with respect to a quarter-symmetric connection, then we get the relation between three non-zero smooth functions* α, β *and* γ *of the manifold* M *with respect to a quarter-symmetric connection given by* (39) *and* (40).

4 Ricci-semisymmetric $G(QE)_n$ with respect to quarter symmetric connection

A Riemannian manifold is said to be Ricci-semisymmetric if $R \cdot S = 0$ holds. In this section we study Ricci-semisymmetric $G(QE)_n$ with respect to quarter symmetric connection and prove the following theorem:

Theorem 3 *A Ricci-semisymmetric* $G(QE)_n$ *with respect to quarter symmetric connection is nearly quasi-Einstein manifold with respect to quarter symmetric connection under the following condition holds:*

(i) $P \in \chi(F)$ *i.e., parellel vector field.*

(ii) f *is constant.*

Proof. Suppose that $\breve{R} \cdot \breve{S} = 0$. Then we get

$$\breve{S}(\breve{R}(X, Y)Z, W) + \breve{S}(Z, \breve{R}(X, Y)W) = 0, \tag{41}$$

where $X, W \in \chi(F)$, $Y, Z \in \chi(B)$.
From (2), we have

$$\begin{aligned}\breve{S}(\breve{R}(X, Y)Z, W) = {} & \alpha g(\breve{R}(X, Y)Z, W) + \beta g(\breve{R}(X, Y)Z, U)g(W, U) \\ & + \gamma g(\breve{R}(X, Y)Z, V)g(W, V).\end{aligned} \tag{42}$$

Now using (35), we have

$$\begin{aligned}\breve{S}(\breve{R}(X, Y)Z, W) = {} & \alpha g(\breve{R}(X, Y)Z, W) + \beta g(\breve{R}(X, Y)Z, U_B + f^2 U_F)g(W, U_F) \\ & + \gamma g(\breve{R}(X, Y)Z, V_B + f^2 V_F)g(W, V_F),\end{aligned} \tag{43}$$

i. e.,

$$\begin{aligned}\breve{S}(\breve{R}(X,Y)Z,W) = \alpha g(\breve{R}(X,Y)Z,W) + \beta f^4 g(W,U_F)g(Y,Z)[\lambda_2^2\Gamma(X)g(P,U_F)\\ -\lambda_1\lambda_2\Gamma(P)g(X,U_F)] + cf^4 g(W,V_F)g(Y,Z)[\lambda_2^2\Gamma(X)\\ g(P,V_F) - \lambda_1\lambda_2\Gamma(P)g(X,V_F)]\end{aligned} \tag{44}$$

Now, using the proposition 1 and proposition 3.3 in [14], we have

$$\begin{aligned}\breve{S}(Z,\breve{R}(X,Y)W) = -\alpha g(Z, -\lambda_1\lambda_2\Gamma(P)g(X,W)Y + \lambda_1^2\Gamma(X)\Gamma(W)Y\\ + \beta g(Z,U_B)g(Y,U_B)[-\lambda_1\lambda_2\Gamma(P)g(X,W)\\ + \lambda_1^2\Gamma(X)\Gamma(W)] + \gamma g(Z,V_B)g(Y,V_B)\\ [-\lambda_1\lambda_2\Gamma(P)g(X,W) + \lambda_1^2\Gamma(X)\Gamma(W)].\end{aligned} \tag{45}$$

Let e_i be an orthonormal basis of the tangent space at each point of the manifold where $1 \le i \le n$. Now putting $X = W = e_i$ in (44) and (45) and using the equation (41) and proposition 1 and proposition 3.3 in [14], we get

$$\begin{aligned}\alpha\breve{S}(Y,Z) + g(Y,Z)\beta f^4[\lambda_2^2 g(P,U_F)g(P,U_F) - \lambda_1\lambda_2\Gamma(P).1]\\ + \gamma g(Y,Z)f^4[\lambda_2^2 g(P,V_F)g(P,V_F) + \alpha[\lambda_1\lambda_2\Gamma(P)ng(Y,Z)\\ - \lambda_1^2\Gamma(P)g(Y,Z)] + \beta g(Z,U_B)g(Y,U_B)[\lambda_1^2\Gamma(P)\\ - \lambda_1\lambda_2\Gamma(P)] + \gamma g(Z,V_B)g(Y,V_B)[\lambda_1^2\Gamma(P)\\ - \lambda_1\lambda_2\Gamma(P)] = 0,\end{aligned} \tag{46}$$

i. e.,

$$\breve{S}(Y,Z) = A'g(Y,Z) + B'E(Y,Z), \tag{47}$$

where A', B' are non-zero functions and $E(Y,Z)$ is a symmetric tensor function. So, the manifold becomes nearly quasi Einstein manifold with respect to quarter symmetric connection. This completes the proof. □

5 $G(QE)_n$ with the condition $\breve{P}\cdot\breve{S} = 0$ with respect to quarter symmetric connection

The projective curvature tensor $\breve{P}$ of type (1, 3) of an n-dimensional Riemannian manifold (M^n, g), $(n > 3)$ with respect to quarter symmetric connection is defined by

$$\breve{P}(X,Y)Z = \breve{R}(X,Y)Z - \frac{1}{n-1}[\breve{S}(Y,Z)X - \breve{S}(X,Z)Y] \tag{48}$$

for any vector fields $X, Y, Z \in \chi(M)$.

In this section, we consider a generalized quasi-Einstein manifold satisfying the condition $\breve{P} \cdot \breve{S} = 0$ with respect to quarter symmetric connection and we have a following theorem.

Theorem 4 *A* $G(QE)_n$ *satisfying* $\breve{P} \cdot \breve{S} = 0$ *with respect to quarter symmetric connection is nearly quasi-Einstein manifold with respect to quarter symmetric connection under the following condition holds:*

(i) $P \in \chi(F)$ *i.e., parallel vector field.*

(ii) f *is constant,* B *is one-dimensional base and* $X, W \in \chi(F)$, $Y, Z \in \chi(B)$.

Proof. Suppose that

$$\breve{P} \cdot \breve{S} = 0. \tag{49}$$

Now using the equation (2), (44) and (35), we have

$$\begin{aligned} \breve{S}(\breve{P}(X,Y)Z, W) = \alpha g(\breve{R}(X,Y)Z, W) - \frac{\alpha}{(n-1)}[\breve{S}(Y,Z)g(X,W)] \\ + f^2\beta g(W, U_F)A(\breve{P}(X,Y)Z) \\ + \gamma f^2 g(W, V_F)B(\breve{P}(X,Y)Z), \end{aligned} \tag{50}$$

as $\breve{S}(X,Z) = 0$.
Again using (48) and proposition 1 and proposition 3.3 in [14], we have

$$\begin{aligned} \breve{S}(\breve{P}(X,Y)Z, W) \;=\; & \alpha g(Y,Z)[\lambda_1\lambda_2\Gamma(P)g(X,W) - \lambda_2^2\Gamma(X)g(P,W)] \\ & - \frac{\alpha}{(n-1)}[\breve{S}_B(Y,Z) + \{(n-1)\lambda_1\lambda_2 - \lambda_2^2\}\Gamma(P)g(Y,Z)]g(X,W) \\ & - \beta g(W,U_F)f^4[g(Y,Z)\{\lambda_1\lambda_2\Gamma(P)g(X,U_F) - \lambda_2^2\Gamma(X)g(P,U_F)\} \\ & + \frac{g(X,U_F)}{(n-1)}\{\breve{S}_B(Y,Z) + \{(n-1)\lambda_1\lambda_2 - \lambda_2^2\}\Gamma(P)g(Y,Z)\}] \\ & - \gamma g(W,V_F)f^4[g(Y,Z)\{\lambda_1\lambda_2\Gamma(P)g(X,V_F) - \lambda_2^2\Gamma(X)g(P,V_F)\} \\ & + \frac{g(X,V_F)}{(n-1)}\{\breve{S}_B(Y,Z) + \{(n-1)\lambda_1\lambda_2 - \lambda_2^2\}\Gamma(P)g(Y,Z)\}]. \end{aligned} \tag{51}$$

Similarly using the equation (2), (48), (35) and we have proposition 1 and proposition 3.3 in [14], we have

$$\begin{aligned}
\breve{S}(Z,\breve{P}(X,Y)W) \;=\; & \alpha g(Y,Z)[\lambda_1\lambda_2\Gamma(P)g(X,W)-\lambda_1^2\Gamma(W)\Gamma(X)] \\
&+\frac{\alpha}{(n-1)}g(Y,Z)[\breve{S}_F(X,W)+\{(n-1)\lambda_1\lambda_2-\lambda_2^2\}\Gamma(P)g(X,W)] \\
&+\{(1-n)\lambda_1^2+\lambda_2^2\}\Gamma(W)\Gamma(X)]+\beta g(Z,U_B)g(Y,U_B)[\lambda_1\lambda_2 g(X,W)\Gamma(P) \\
&-\lambda_1^2\Gamma(W)\Gamma(X)]+\frac{\beta}{(n-1)}g(Z,U_B)g(Y,U_B)[\breve{S}_F(X,W)+\{(n-1)\lambda_1\lambda_2 \\
&-\lambda_2^2\}\Gamma(P)g(X,W)+\{(1-n)\lambda_1^2+\lambda_2^2\}\Gamma(W)\Gamma(X)] \\
&+\gamma g(Z,V_B)g(Y,Y_B)[\lambda_1\lambda_2 g(X,W)\Gamma(P)-\lambda_1^2\Gamma(W)\Gamma(X)] \\
&+\frac{\gamma}{(n-1)}g(Z,V_B)g(Y,V_B)[[\breve{S}_F(X,W)+\{(n-1)\lambda_1\lambda_2 \\
&-\lambda_2^2\}\Gamma(P)g(X,W)+\{(1-n)\lambda_1^2+\lambda_2^2\}\Gamma(W)\Gamma(X)].
\end{aligned} \tag{52}$$

Since, $\breve{S}(X,Y)=0$, from (51) and (52), we have

$$\breve{S}(X,W)=A''g(X,W)+B''E(X,W), \tag{53}$$

where A'', B'' are non-zero functions and $E(Y,Z)$ is a symmetric tensor function.

So, the manifold becomes nearly quasi-Einstein manifold with respect to quarter symmetric connection. This completes the proof. □

6 Example of warped product on generalized quasi-Einstein manifold with respect to quarter symmetric connection

Taking a local coordinate system in M such that g, ∇, $\breve{\nabla}$, ω, ϕ, T have the local expression, respectively, g_{ij}, Γ^h_{ji}, $\breve{\Gamma}^h_{ji}$, ω_i, ϕ^h_j, T^h_{ji} then, by a direct computation, we have

$$T^h_{ji}=\omega_j\phi^h_i-\omega_i\phi^h_j.$$

In a local coordinate, the relation between a quarter-symmetric metric connection and the Levi-Civita connection is [13],

$$\breve{\Gamma}^h_{ji}=\Gamma^h_{ji}+\frac{1}{2}\omega_j(\phi_{ki}+\phi_{ik})g^{kh}-\frac{1}{2}\omega_i(\phi_{kj}+\phi_{jk})g^{kh}-\frac{1}{2}\omega_k(\phi_{ji}+\phi_{ij})g^{kh} \tag{54}$$

Now, we define a Riemannian metric g on M^4 by the formula

$$ds^2 = g_{ij}dx^i dx^j = (1+2p)\big[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2\big], \tag{55}$$

where $i, j = 1, 2, 3, 4$ and x^1, x^2, x^3, x^4 are the standard coordinates of R^4 and $p = \frac{e^{x^1}}{k^2}$ and k is a non-zero constant. Then the only non-vanishing components of the Christoffel symbols, the curvature tensor, the Ricci tensor and scalar curvature are given by

$$\Gamma^1_{22} = -\frac{p}{1+2p} = \Gamma^1_{33} = \Gamma^1_{44} = -\Gamma^1_{11} = -\Gamma^2_{12} = -\Gamma^3_{13} = -\Gamma^4_{14},$$
$$R_{1221} = R_{1331} = R_{1441} = \frac{p}{1+2p}, \quad S_{11} = \frac{3p}{(1+2p)^2},$$
$$S_{22} = S_{33} = S_{44} = \frac{p}{(1+2p)^2}, \quad r = \frac{6p}{(1+3p)^3} \neq 0.$$

Let us consider the 1-form and the associated tensor ϕ as follows:

$$\omega_1 = c_1, \omega_2 = 0, \omega_3 = 0, \omega_4 = 0,$$

where c_1 is arbitrary scalar and

$$\phi = (\phi_{ij}) \begin{pmatrix} 0 & \phi_{12} & \phi_{13} & \phi_{14} \\ -\phi_{12} & 0 & \phi_{23} & \phi_{24} \\ -\phi_{13} & -\phi_{23} & 0 & \phi_{34} \\ -\phi_{14} & -\phi_{24} & -\phi_{34} & 0, \end{pmatrix}$$

where $\phi_{ij} \neq 0$, where $i, j \in \{1, 2, 3, 4\}$, and $i \neq j$.

From (54), we have $\breve{\Gamma}^h_{ji} = \Gamma^h_{ji}$.

The non-vanishing curvature tensors and the Ricci tensors with respect to a quarter symmetric metric connection are

$$\breve{R}_{1221} = R_{1221}, \ \breve{R}_{1331} = R_{1331}, \ \breve{R}_{1441} = R_{1441} = \frac{p}{1+2p}$$

and

$$\breve{S}_{11} = S_{11} = \frac{3p}{(1+2p)^2}, \ \breve{S}_{22} = \breve{S}_{33} = \breve{S}_{44} = \frac{p}{(1+2p)^2}.$$

Let us now consider the associated scalars as follows:

$$\alpha = \frac{p}{(1+2p)^3}, \ \beta = -3, \ \gamma = 5p.$$

In terms of local coordinate system, let us consider the 1-forms A and B as follows:

$$A_i(x) = \begin{cases} \frac{\sqrt{p}}{1+2p}, & \text{if } i = 1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$B_i(x) = \begin{cases} \frac{1}{1+2p}, & \text{if } i = 1 \\ 0, & \text{otherwise} \end{cases}$$

then we have

$$\breve{S}_{11} = \alpha g_{11} + \beta A_1 A_1 + \gamma B_1 B_1,$$

$$\breve{S}_{22} = \alpha g_{22} + \beta A_2 A_2 + \gamma B_2 B_2,$$

$$\breve{S}_{33} = \alpha g_{33} + \beta A_3 A_3 + \gamma B_3 B_3,$$

$$\breve{S}_{44} = \alpha g_{44} + \beta A_4 A_4 + \gamma B_4 B_4.$$

Since all the cases other than (i)-(iv) are trivial, we can say that

$$S_{ij} = \alpha g_{ij} + \beta A_i A_j + \gamma B_i B_j, \quad \text{for} \quad i, j = 1, 2, 3, 4.$$

Example 1 *Let* (M^4, g) *be a Riemannian manifold endowed with the metric given by*

$$ds^2 = g_{ij} dx^i dx^j = (1 + 2p)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2],$$

where $i, j = 1, 2, 3, 4$ *and* x^1, x^2, x^3, x^4 *are the standard coordinates of* R^4 *and* $p = \frac{e^{x^1}}{k^2}$ *and* k *is a non-zero constant. Then* (M^4, g) *is an* $G(QE)_4$ *with respect to quarter symmetric connection and also with nonvanishing and nonconstant scalar curvature.*

So, (M^4, g) be a Riemannian manifold endowed with the metric given by

$$ds^2 = g_{ij} dx^i dx^j = (1 + 2p)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2],$$

where $(i, j = 1, 2, 3, 4)$, $p = \frac{e^{x^1}}{k^2}$, k constant is $G(QE)_4$ with respect to quarter symmetric connection.

Now, to define warped product on $G(QE)_4$, we consider the warping function $f : R^3 \longrightarrow (0, \infty)$ by $f(x^1, x^2, x^3) = \sqrt{(1 + 2p)}$ and we observe that $f > 0$ is a smooth function. The line element defined on $R^3 \times R$ which is of the form $B \times_f F$, where $B = R^3$ is the base and $F = R$ is the fibre.

Therefore the metric ds^2_M can be expressed as $ds^2_B + f^2 ds^2_F$ i.e.,

$$ds^2 = g_{ij}dx^i dx^j = (1+2p)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + [\surd(1+2p)]^2(dx^4)^2,$$

which is the example of warped product on generalized quasi-Einstein manifold with respect to quarter symmetric connection.

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Modified Hadamard product properties of certain class of analytic functions with varying arguments defined by the convolution of Sălăgean and Ruscheweyh derivative

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Abstract. In this paper we study the Hadamard product properties of certain class of analytic functions with varying arguments defined by the convolution of Sălăgean and Ruscheweyh derivative.
The obtained results are sharp and they improve known results.

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic and univalent in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Let $g \in \mathcal{A}$ where

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k. \tag{2}$$

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The Hadamard product of two functions f and g of the form (1) and (2) is defined by (see also [3, 7, 8, 9])

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z).$$

The modified Hadamard product is

$$(f \otimes g)(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k = (g \otimes f)(z).$$

Definition 1 [8]

For $f \in \mathcal{A}, n \in \mathbb{N}$, the Sălăgean differential operator $\mathcal{S}^n$ is defined by $\mathcal{S}^n : \mathcal{A} \to \mathcal{A}$,

$$\mathcal{S}^0 f(z) = f(z),$$
$$\mathcal{S}^{n+1} f(z) = z\left(\mathcal{S}^n f(z)\right)', z \in U$$

Remark 1 *If $f \in \mathcal{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then*

$$\mathcal{S}^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \ z \in U.$$

Definition 2 [6]

For $f \in \mathcal{A}, n \in \mathbb{N}$, the operator $\mathcal{R}^n$ is defined by $\mathcal{R}^n : \mathcal{A} \to \mathcal{A}$,

$$\mathcal{R}^0 f(z) = f(z),$$
$$(n+1)\,\mathcal{R}^{n+1} f(z) = z\left(\mathcal{R}^n f(z)\right)' + n\mathcal{R}^n f(z), z \in U.$$

Remark 2 *If $f \in \mathcal{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then*

$$\mathcal{R}^n f(z) = z + \sum_{k=2}^{\infty} \frac{(n+k-1)!}{n!\,(k-1)!} a_k z^k, \ z \in U.$$

Definition 3 *Let $n \in \mathbb{N}$. Denote by $\mathcal{SR}^n$ the operator given by the Hadamard product (convolution) of the Sălăgean operator $\mathcal{S}^n$ and the Ruscheweyh operator $\mathcal{R}^n$, $\mathcal{SR}^n : \mathcal{A} \to \mathcal{A}$,*

$$\mathcal{SR}^n f(z) = \mathcal{S}^n\left(\frac{z}{1-z}\right) * \mathcal{R}^n f(z), z \in U.$$

Remark 3 *If* $f \in \mathcal{A}$ *and* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, *then*

$$\mathcal{SR}^n f(z) = z + \sum_{k=2}^{\infty} \frac{k^n (n+k-1)!}{n!(k-1)!} a_k z^k, \; z \in U.$$

Definition 4 [4] *Let* f *and* g *be analytic functions in* U. *We say that the function* f *is subordinate to the function* g, *if there exists a function* w, *which is analytic in* U *and* $w(0) = 0; |w(z)| < 1; z \in U$, *such that* $f(z) = g(w(z))$; $\forall z \in U$. *We denote by* $\prec$ *the subordination relation.*

Definition 5 *For* $\lambda \geq 0; -1 \leq A < B \leq 1; 0 < B \leq 1; n \in \mathbb{N}_0$ *let* $P(n, \lambda, A, B)$ *denote the subclass of* $\mathcal{A}$ *which contain functions* $f(z)$ *of the form* (1) *such that*

$$(1-\lambda)(\mathcal{SR}^n f(z))' + \lambda(\mathcal{SR}^{n+1} f(z))' \prec \frac{1+Az}{1+Bz}. \tag{3}$$

Attiya and Aouf defined in [2] the class $\mathcal{R}(n, \lambda, A, B)$ with a condition like (3), but there instead of the operator $\mathcal{SR}^n$ they used the Ruscheweyh operator.

Definition 6 [10] *A function* $f(z)$ *of the form* (1) *is said to be in the class* $V(\theta_k)$ *if* $f \in \mathcal{A}$ *and* $\arg(a_k) = \theta_k$,$\forall k \geq 2$. *If* $\exists \delta \in \mathbb{R}$ *such that* $2\theta_k + (k-1)\delta \equiv \pi (\mathrm{mod}\ 2\pi), \forall k \geq 2$ *then* $f(z)$ *is said to be in the class* $V(\theta_k, \delta)$. *The union of* $V(\theta_k, \delta)$ *taken over all possible sequences* $\{\theta_k\}$ *and all possible real numbers* δ *is denoted by* V.

Let $VP(n, \lambda, A, B)$ denote the subclass of V consisting of functions $f(z) \in P(n, \lambda, A, B)$.

Theorem 1 [5] *Let the function* $f(z)$ *defined by* (1) *be in V. Then* $f(z) \in VP(n, \lambda, A, B)$, *if and only if*

$$T(f) = \sum_{k=2}^{\infty} (1+B) k^{n+1} C_k |a_k| \leq B - A, \tag{4}$$

where

$$C_k = [n + 1 + \lambda (k-1)(n+k+1)] \frac{(n+k-1)!}{(n+1)!(k-1)!}.$$

The extremal functions are:

$$f(z) = z + \frac{B-A}{k^{n+1} C_k (1+B)} e^{i\theta_k} z^k, (k \geq 2).$$

Main results

Theorem 2 *If* $f \in VP(n,\lambda,A_1,B), g \in VP(n,\lambda,A_2,B)$ *then*

$$f \otimes g \in VP(n,\lambda,A^*,B), \ \textit{where} \ A^* = B - \frac{(B-A_1)(B-A_2)}{2^{n+1}C_2\,(1+B)}.$$

The result is sharp.

Proof. Let $f \in VP(n,\lambda,A_1,B), g \in VP(n,\lambda,A_2,B)$ and suppose they have the form (1). Since $f \in VP(n,\lambda,A_1,B)$ we have

$$\frac{\sum\limits_{k=2}^{\infty} k^{n+1}C_k\,(1+B)\,|a_k|}{B-A_1} \leq 1 \tag{5}$$

and for $g \in VP(n,\lambda,A_2,B)$ we have

$$\frac{\sum\limits_{k=2}^{\infty} k^{n+1}C_k\,(1+B)\,|b_k|}{B-A_2} \leq 1. \tag{6}$$

We know from Theorem 1 that $f \otimes g \in VP(n,\lambda,A^*,B)$ if and only if

$$\frac{\sum\limits_{k=2}^{\infty} k^{n+1}C_k\,(1+B)\,|a_k b_k|}{B-A^*} \leq 1. \tag{7}$$

By using the Cauchy-Schwarz inequality for (5) and (6) we have

$$\frac{\sum\limits_{k=2}^{\infty} k^{n+1}C_k\,(1+B)\,\sqrt{|a_k b_k|}}{\sqrt{(B-A_1)(B-A_2)}} \leq 1.$$

We note that

$$\frac{\sum\limits_{k=2}^{\infty} k^{n+1}C_k\,(1+B)\,|a_k b_k|}{B-A^*} \leq \frac{\sum\limits_{k=2}^{\infty} k^{n+1}C_k\,(1+B)\,\sqrt{|a_k b_k|}}{\sqrt{(B-A_1)(B-A_2)}}$$

implies (7). But the above inequality holds provided that

$$\frac{|a_k b_k|}{B-A^*} \leq \frac{\sqrt{|a_k b_k|}}{\sqrt{(B-A_1)(B-A_2)}}$$

or

$$\sqrt{|a_k b_k|} \leq \frac{B - A^*}{\sqrt{(B - A_1)(B - A_2)}}. \tag{8}$$

From Theorem 1 we have:

$$|a_k| \leq \frac{B - A_1}{k^{n+1} C_k (1 + B)} \text{ and } |b_k| \leq \frac{B - A_2}{k^{n+1} C_k (1 + B)}, (k \geq 2)$$

this implies that

$$\sqrt{|a_k b_k|} \leq \frac{\sqrt{(B - A_1)(B - A_2)}}{k^{n+1} C_k (1 + B)}, (k \geq 2). \tag{9}$$

From (9) we obtain that (8) holds if

$$\frac{\sqrt{(B - A_1)(B - A_2)}}{k^{n+1} C_k (1 + B)} \leq \frac{B - A^*}{\sqrt{(B - A_1)(B - A_2)}}$$

or equivalently

$$A^* \leq B - \frac{(B - A_1)(B - A_2)}{k^{n+1} C_k (1 + B)}.$$

But $k^{n+1} C_k < (k + 1)^{n+1} C_{k+1}, (k \geq 2)$ so

$$B - \frac{(B - A_1)(B - A_2)}{k^{n+1} C_k (1 + B)} \geq B - \frac{(B - A_1)(B - A_2)}{2^{n+1} C_2 (1 + B)}, (k \geq 2),$$

consequently the above inequality holds provided that

$$A^* = B - \frac{(B - A_1)(B - A_2)}{2^{n+1} C_2 (1 + B)}.$$

The result is sharp, because if

$$f(z) = z + \frac{B - A_1}{2^{n+1} C_2 (1 + B)} e^{i\theta_1} z^2 \in VP(n, \lambda, A_1, B)$$

$$g(z) = z + \frac{B - A_2}{2^{n+1} C_2 (1 + B)} e^{i\theta_2} z^2 \in VP(n, \lambda, A_2, B)$$

$$f \otimes g \in VP(n, \lambda, A^*, B)$$

and satisfy (4) with equality. Indeed,

$$2^{n+1}C_2\left(1+B\right)\frac{(B-A_1)(B-A_2)}{2^{2n+2}C_2^2\left(1+B\right)^2}=B-A^*$$

because

$$B-A^*=\frac{(B-A_1)(B-A_2)}{2^{n+1}C_2\left(1+B\right)}.$$

□

Corollary 1 *If* $f,g\in VP(n,\lambda,A,B)$ *then* $f\otimes g\in VP(n,\lambda,A^*,B)$*, where*

$$A^*=B-\frac{(B-A)^2}{2^{n+1}C_2\left(1+B\right)}.$$

The result is sharp.

Theorem 3 *If* $f\in VP(n,\lambda,A,B_1), g\in VP(n,\lambda,A,B_2)$ *then* $f\otimes g\in VP(n,\lambda,A,B^*)$*, where*

$$B^*=A+\frac{(B_1-A)(B_2-A)\left(A+1\right)}{2^{n+1}C_2\left(1+B_1\right)\left(1+B_2\right)-\left(B_1-A\right)\left(B_2-A\right)}.$$

The result is sharp.

Proof. Let $f\in VP(n,\lambda,A,B_1), g\in VP(n,\lambda,A,B_2)$ and suppose they have the form (1). Since $f\in VP(n,\lambda,A,B_1)$ we have

$$\frac{\sum_{k=2}^{\infty}k^{n+1}C_k\left(1+B_1\right)|a_k|}{B_1-A}\leq 1 \tag{10}$$

and for $g\in VP(n,\lambda,A,B_2)$ we have

$$\frac{\sum_{k=2}^{\infty}k^{n+1}C_k\left(1+B_2\right)|b_k|}{B_2-A}\leq 1. \tag{11}$$

We know from Theorem 1 that $f\otimes g\in VP(n,\lambda,A,B^*)$ if and only if

$$\frac{\sum_{k=2}^{\infty}k^{n+1}C_k\left(1+B^*\right)|a_kb_k|}{B^*-A}\leq 1. \tag{12}$$

By using the Cauchy-Schwarz inequality for (10) and (11) we have

$$\frac{\sum_{k=2}^{\infty} k^{n+1} C_k \sqrt{|a_k b_k|}\sqrt{(1+B_1)(1+B_2)}}{\sqrt{(B_1-A)(B_2-A)}} \leq 1.$$

We note that

$$\frac{\sum_{k=2}^{\infty} k^{n+1} C_k (1+B^*)|a_k b_k|}{B^*-A} \leq \frac{\sum_{k=2}^{\infty} k^{n+1} C_k \sqrt{|a_k b_k|}\sqrt{(1+B_1)(1+B_2)}}{\sqrt{(B_1-A)(B_2-A)}}$$

implies (12). But the above inequality holds provided that

$$\frac{|a_k b_k|(1+B^*)}{B^*-A} \leq \frac{\sqrt{|a_k b_k|}\sqrt{(1+B_1)(1+B_2)}}{\sqrt{(B_1-A)(B_2-A)}}$$

or

$$\sqrt{|a_k b_k|} \leq \frac{(B^*-A)\sqrt{(1+B_1)(1+B_2)}}{(1+B^*)\sqrt{(B_1-A)(B_2-A)}}. \tag{13}$$

From Theorem 1 we have:

$$|a_k| \leq \frac{B_1-A}{k^{n+1} C_k (1+B_1)} \text{ and } |b_k| \leq \frac{B_2-A}{k^{n+1} C_k (1+B_2)}, (k \geq 2)$$

this implies that

$$\sqrt{|a_k b_k|} \leq \frac{\sqrt{(B_1-A)(B_2-A)}}{k^{n+1} C_k \sqrt{(1+B_1)(1+B_2)}}, (k \geq 2). \tag{14}$$

from (14) we obtain that (13) holds if

$$\frac{\sqrt{(B_1-A)(B_2-A)}}{k^{n+1} C_k \sqrt{(1+B_1)(1+B_2)}} \leq \frac{(B^*-A)\sqrt{(1+B_1)(1+B_2)}}{(1+B^*)\sqrt{(B_1-A)(B_2-A)}}$$

or equivalently

$$B^* \geq A + \frac{(B_1-A)(B_2-A)(A+1)}{k^{n+1} C_k (1+B_1)(1+B_2) - (B_1-A)(B_2-A)}.$$

But $k^{n+1} C_k < (k+1)^{n+1} C_{k+1}, (k \geq 2)$ so :

$$A + \frac{(B_1-A)(B_2-A)(A+1)}{k^{n+1} C_k (1+B_1)(1+B_2) - (B_1-A)(B_2-A)} \leq$$

$$\leq A + \frac{(B_1 - A)(B_2 - A)(A+1)}{2^{n+1}C_2(1+B_1)(1+B_2) - (B_1 - A)(B_2 - A)}, (k \geq 2),$$

consequently the above inequality holds provided that

$$B^* = A + \frac{(B_1 - A)(B_2 - A)(A+1)}{2^{n+1}C_2(1+B_1)(1+B_2) - (B_1 - A)(B_2 - A)}.$$

The result is sharp, because if

$$f(z) = z + \frac{B_1 - A}{2^{n+1}C_2(1+B_1)} e^{i\theta_1} z^2 \in VP(n, \lambda, A, B_1)$$

$$g(z) = z + \frac{B_2 - A}{2^{n+1}C_2(1+B_2)} e^{i\theta_2} z^2 \in VP(n, \lambda, A, B_2)$$

$$f * g \in VP(n, \lambda, A, B^*)$$

and satisfy (4) with equality. Indeed,

$$(1 + B^*)\, 2^{n+1} C_2 \frac{(B_1 - A)(B_2 - A)}{2^{2n+2}C_2^2(1+B_1)(1+B_2)} = B^* - A$$

because

$$B^* - A = \frac{(B_1 - A)(B_2 - A)(A+1)}{2^{n+1}C_2(1+B_1)(1+B_2) - (B_1 - A)(B_2 - A)}.$$

□

Corollary 2 *If* $f, g \in VP(n, \lambda, A, B)$ *then* $f \otimes g \in VP(n, \lambda, A, B^*)$, *where*

$$B^* = A + \frac{(B - A)^2 (A+1)}{2^{n+1}C_2(1+B)^2 - (B - A)^2}.$$

The result is sharp.

Theorem 4 *If* $f_j \in VP(n, \lambda, A_j, B), j = \overline{1, s}, s \in \{2, 3, 4, \ldots\}$ *then* $f_1 \otimes f_2 \otimes \ldots \otimes f_s \in VP(n, \lambda, A^{(s-1)*}, B)$, *where*

$$A^{(s-1)*} = B - \frac{\prod_{j=1}^{s}(B - A_j)}{2^{(n+1)(s-1)}C_2^{s-1}(1+B)^{s-1}}.$$

The result is sharp.

Proof. For the proof we use the mathematical induction method and suppose that $f_j, \forall j$ have the form (1).

Let $s = 2$. If $f_j \in VP(n, \lambda, A_j, B), j = \overline{1, 2}$ then $f_1 \otimes f_2 \in VP(n, \lambda, A^*, B)$ where $A^* = B - \dfrac{(B - A_1)(B - A_2)}{2^{n+1} C_2 (1 + B)}$, from Theorem 2 is true.

Assume, for $s = m$, that the formula displayed below holds.

If $f_j \in VP(n, \lambda, A_j, B), j = \overline{1, m}, m \in \{2, 3, 4, \ldots\}$ then $f_1 \otimes f_2 \otimes \ldots \otimes f_m \in VP(n, \lambda, A^{(m-1)*}, B)$, where

$$A^{(m-1)*} = B - \frac{\prod_{j=1}^{m} (B - A_j)}{k^{(n+1)(m-1)} C_k^{m-1} (1 + B)^{m-1}}.$$

Let $s = m + 1$: if $f_1 \otimes f_2 \otimes \ldots \otimes f_m \in VP(n, \lambda, A^{(m-1)*}, B), m \in \{2, 3, 4, \ldots\}$ and $f_{m+1} \in VP(n, \lambda, A_{m+1}, B)$ then we have to prove

$f_1 \otimes f_2 \otimes \ldots \otimes f_m \otimes f_{m+1} \in VP(n, \lambda, A^{m*}, B)$, where $A^{m*} = B - \dfrac{\prod_{j=1}^{m+1} (B - A_j)}{k^{(n+1)m} C_k^m (1 + B)^m}$.

For the proof we use the result of Theorem 2:

$$A^{m*} \le B - \frac{(B - A^{(m-1)*})(B - A_{m+1})}{k^{(n+1)} C_k (1 + B)}$$

$$A^{m*} \le B - \frac{\dfrac{\prod_{j=1}^{m} (B - A_j)}{k^{(m-1)(n+1)} C_k^{m-1} (1 + B)^{m-1}} (B - A_{m+1})}{k^{(n+1)} C_k (1 + B)}$$

$$A^{m*} \le B - \frac{\prod_{j=1}^{m+1} (B - A_j)}{k^{m(n+1)} C_k^m (1 + B)^m}.$$

But $k^{(n+1)} C_k < (k + 1)^{(n+1)} C_{k+1}, (k \ge 2)$ so:

$$B - \frac{\prod_{j=1}^{m+1} (B - A_j)}{k^{m(n+1)} C_k^m (1 + B)^m} \ge B - \frac{\prod_{j=1}^{m+1} (B - A_j)}{2^{m(n+1)} C_2^m (1 + B)^m}, (k \ge 2),$$

consequently the above inequality holds provided that

$$A^{m*} = B - \frac{\prod_{j=1}^{m+1} (B - A_j)}{2^{m(n+1)} C_2^m (1 + B)^m}.$$

The result is sharp, because if

$$f(z) = z + \frac{B - A^{(s-1)*}}{2^{n+1}C_2\,(1+B)} e^{i\theta_1} z^2 \in VP(n, \lambda, A^{(s-1)*}, B)$$
$$g(z) = z + \frac{B - A_s}{2^{n+1}C_2\,(1+B)} e^{i\theta_2} z^2 \in VP(n, \lambda, A_s, B)$$
$$f \otimes g \in VP(n, \lambda, A^{s*}, B)$$

and satisfy (4) with equality. Indeed,

$$2^{n+1}C_2\,(1+B)\,\frac{(B - A^{(s-1)*})(B - A_s)}{2^{2(n+1)}C_2^2\,(1+B)^2} = B - A^{s*}$$

because

$$B - A^{s*} = \frac{(B - A^{(s-1)*})(B - A_s)}{2^{n+1}C_2\,(1+B)} \Leftrightarrow B - A^{s*} = \frac{\prod_{j=1}^{s+1}(B - A_j)}{2^{s(n+1)}C_2^s\,(1+B)^s}.$$

□

Theorem 5 *If* $f_j \in VP(n, \lambda, A, B_j), j = \overline{1, s}, s \in \{2, 3, 4, \dots\}$ *then* $f_1 \otimes f_2 \otimes \dots \otimes f_s \in VP(n, \lambda, A, B^{(s-1)*})$, *where*

$$B^{(s-1)*} = A + \frac{(A+1)\prod_{j=1}^{s}(B_j - A)}{2^{(s-1)(n+1)}C_2^{s-1}\prod_{j=1}^{s}(1 + B_j) - \prod_{j=1}^{s}(B_j - A)}.$$

The result is sharp.

Proof. For the proof we use the mathematical induction method and suppose that $f_j, \forall j$ have the form (1).
Let $s = 2$. If $f_j \in VP(n, \lambda, A, B_j), j = \overline{1, 2}$ then $f_1 \otimes f_2 \in VP(n, \lambda, A, B^*)$ where

$$B^* = A + \frac{(A+1)(B_1 - A)(B_2 - A)}{2^{n+1}C_2(1 + B_1)(1 + B_2) - (B_1 - A)(B_2 - A)},$$

from Theorem 3 is true.
Assume, for $s = m$, that the formula displayed below holds.

If $f_j \in VP(n,\lambda,A,B_j), j=\overline{1,m}, m \in \{2,3,4,\dots\}$ then $f_1 \otimes f_2 \otimes \dots \otimes f_m \in VP(n,\lambda,A,B^{(m-1)*})$, where

$$B^{(m-1)*} = A + \frac{(A+1)\prod_{j=1}^{m}(B_j - A)}{2^{(m-1)(n+1)}C_2^{m-1}\prod_{j=1}^{m}(1+B_j) - \prod_{j=1}^{m}(B_j - A)}.$$

Let $s = m+1$: if $f_1 \otimes f_2 \otimes \dots \otimes f_m \in VP(n,\lambda,A,B^{(m-1)*}), m \in \{2,3,4,\dots\}$ and $f_{m+1} \in VP(n,\lambda,A,B_{m+1})$ then we have to prove $f_1 \otimes f_2 \otimes \dots \otimes f_m \otimes f_{m+1} \in VP(n,\lambda,A,B^{m*})$, where

$$B^{m*} = A + \frac{(A+1)\prod_{j=1}^{m+1}(B_j - A)}{2^{m(n+1)}C_2^{m}\prod_{j=1}^{m}(1+B_j) - \prod_{j=1}^{m}(B_j - A)}.$$

For the proof we use the result of Theorem 3:

$$B^{m*} \geq A + \frac{(A+1)(B^{(m-1)*} - A)(B_{m+1} - A)}{k^{n+1}C_k(1+B_1)(1+B_{m+1}) - (B^{(m-1)*} - A)(B_{m+1} - A)}$$

or equivalently

$$B^{m*} \geq A + \frac{(A+1)\prod_{j=1}^{m+1}(B_j - A)}{k^{m(n+1)}C_k^{m}\prod_{j=1}^{m}(1+B_j) - \prod_{j=1}^{m}(B_j - A)}.$$

But $k^{n+1}C_k < (k+1)^{n+1}C_{k+1}, (k \geq 2)$ so:

$$A + \frac{(A+1)\prod_{j=1}^{m+1}(B_j - A)}{k^{m(n+1)}C_k^{m}\prod_{j=1}^{m}(1+B_j) - \prod_{j=1}^{m}(B_j - A)} \leq$$

$$A + \frac{(A+1)\prod_{j=1}^{m+1}(B_j - A)}{2^{m(n+1)}C_2^{m}\prod_{j=1}^{m}(1+B_j) - \prod_{j=1}^{m}(B_j - A)}, \ (k \geq 2),$$

consequently the above inequality holds provided that

$$B^{m*} = A + \frac{(A+1)\prod_{j=1}^{m+1}(B_j - A)}{2^{m(n+1)}C_2^m \prod_{j=1}^{m+1}(1+B_j) - \prod_{j=1}^{m+1}(B_j - A)}.$$

The result is sharp, because if

$$f(z) = z + \frac{B^{(s-1)*} - A}{2^{n+1}C_2\left(1 + B^{(s-1)*}\right)} e^{i\theta_1} z^2 \in VP(n, \lambda, A, B^{(s-1)*})$$
$$g(z) = z + \frac{B_s - A}{2^{n+1}C_2\,(1 + B_{s+1})} e^{i\theta_2} z^2 \in VP(n, \lambda, A, B_{s+1})$$
$$f \otimes g \in VP(n, \lambda, A, B^{s*})$$

and satisfy (4) with equality. Indeed,

$$2^{n+1}C_2\,(1 + B^{s*})\,\frac{(B^{(s-1)*} - A)(B_{s+1} - A)}{2^{2(n+1)}C_2^2\,(1 + B_{s+1})^2} = B^{s*} - A$$

because

$$B^{s*} = A + \frac{(A+1)\prod_{j=1}^{s+1}(B_j - A)}{2^{s(n+1)}C_2^s \prod_{j=1}^{s+1}(1+B_j) - \prod_{j=1}^{s+1}(B_j - A)}.$$

□

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Construction of Barnette graphs whose large subgraphs are non-Hamiltonian

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Abstract. Barnette's conjecture states that every three connected cubic bipartite planar graph (CPB3C) is Hamiltonian. In this paper we show the existence of a family of CPB3C Hamiltonian graphs in which large and large subgraphs are non-Hamiltonian.

1 Introduction

Barnette's conjecture states that every three connected cubic bipartite planar graph (CPB3C) is Hamiltonian. Goodey [7] proved that if all the faces of a CPB3C graph are either quadrilaterals or hexagons, then the graph is Hamiltonian. Later Hertel [11] mentioned that if Barnette conjecture is true, then perhaps Goodey's result can be extended to show that successively large and large subgraphs of Barnette graphs are Hamiltonian. We show that there exists a family of CPB3C Hamiltonian graphs in which large and large subgraphs are non-Hamiltonian.

Tait in (1884) conjectured that every cubic polyhedral graph is Hamiltonian, this came to be known as Tait's conjecture. It was disproved by Tutte (1946), who constructed a counter example with 46 vertices. Other researchers

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later found even smaller counterexamples, however none of these counterexamples is bipartite. Tutte himself conjectured that every cubic 3-connected bipartite graph is Hamiltonian, but this was shown to be false by discovery of a counterexample, the Horton graph [1]. David W. Barnett (1969) proposed a weakened combination of Tait's and Tutte's conjecture, stating that every cubic bipartite polyhedral graph is Hamiltonian. This conjecture was first announced in [3] and later in [8]. Tutte [18] proved that all planar 4-connected graphs are Hamiltonian. Thomassen [14] extended this result by showing that every planar 4-connected graph is Hamiltonian connected, that is, for any pair of vertices, there is a Hamiltonian path with those vertices as end vertices. It must be noted that if any one of the property of Barnette graph is deleted, then it is non-Hamiltonian.

McKay et al. [15] confirmed through a combination of clever analysis and computer search that all Barnette graphs with 64 or less vertices are Hamiltonian. In an announcement [2, 5], McKay used computer search to extend this result to 84 vertices. This implies that if Barnette's conjecture is indeed false, then a minimal counterexample must contain at least 86 vertices, and is therefore considerable larger than the minimal counterexample to Tait and Tutte conjecture. This is not all we know about a possible counterexample, another interesting result is that of Fowler, who in an unpublished manuscript [6] provided a list of subgraphs that cannot appear in any minimal counterexample to Barnette's conjecture. For more definitions and notations of graph theory, we refer to [16].

Goody [7] considers proper subgraphs of the Barnette graphs and proved the following.

Theorem 1 *Every Barnette graph which has faces consisting exclusively of quadrilaterals and hexagons is Hamiltonian, and further more in all such graphs, any edge that is common to both a quadrilateral and a hexagon is a part of some Hamiltonian cycle.*

Theorem 2 *Every Barnette graph which has faces consisting of 7 quadrilaterals, 1 octagon and any number of hexagons is Hamiltonian, and any edge that is common to both a quadrilateral and an octagon is a part of some Hamiltonian cycle.*

Jensen and Toft [12] reported that Barnette conjecture is equivalent to the following.

Theorem 3 *Barnette's conjecture is true if and only if for every Barnette* *graph* G*, it is possible to partition its vertices into two subsets so that each induces an acyclic subgraph of* G*. (This theorem is not correct)*

Theorem 4 [10] *Barnette's conjecture holds if and only if any arbitrary edge in a Barnette graph is a part of some Hamiltonian cycle.*

Theorem 5 *Barnette's conjecture holds if and only if for any arbitrary face in a Barnette graph there is a Hamiltonian cycle which passes through any two arbitrary edges on that face.*

Theorem 6 [13] *Barnette's conjecture holds if and only if for any arbitrary face in a Barnette graph and for any arbitrary edges* e_1 *and* e_2 *on that face there is a Hamiltonian cycle which passes through* e_1 *and avoids* e_2*.*

Theorem 7 *Barnette's conjecture holds if and only if for any arbitrary path* P *of length 3 that lies on a face in a Barnette graph, there is a Hamiltonian cycle which passes through the middle edge in* P *and avoids both its leading and trailing edges.*

Theorem 8 [9] *Barnette's conjecture is true if and only if there is a constant* $c > 0$ *such that each Barnette graph* G *contains a path on at least* $c|V(G)|$ *vertices.*

Theorem 9 [14] *The edges of any bipartite graph* G *can be colored with* α *colors, where* α *is the minimum degree of vertices in* G*.*

More than thirty papers have been published to strengthening the Barnette's conjecture, not only this, also several proofs of this conjecture have been put forward so far but none of the proof is yet accepted universally. For references, see [4].

2 Main result

The following theorem is the main result, which shows the existence of a family of CPB3C Hamiltonian graphs in which large and large subgraphs are non-Hamiltonian. The proof is by construction.

Theorem 10 *There exist a family of CPB3C Hamiltonian graphs in which large and large subgraphs are non-Hamiltonian.*

Proof. Suppose G is a graph of such family, see Figure 1. The vertex set of graph G is colored with black and red as G is bipartite. If we remove a vertex x (say) in G, we get a non-Hamiltonian graph $G-x$, see Figure 2. Since G is cubic, let the edges incident on x be $e_1 = xu_1,\ e_2 = xu_2,\ e_3 = xu_3$. Since G is Hamiltonian, so at least two of the three edges e_1, e_2, e_3 are always included in the Hamiltonian cycle of G. If vertex x is colored black, it is always adjacent to red color vertices, as G is bipartite. Assume that the edges e_1 and e_2 are included in the Hamiltonian cycle of G. If there is a cycle starting from x to all other vertices of the graph G, then there exists a Hamiltonian path in $G-x$ beginning and ending with the vertices u_1, u_2 of the graph $G-x$. Thus we get a non-Hamiltonian graph $G-x$. We form a large graph by taking $2n$ copies of $G-x$ (n as large as possible) with vertices of different copies colored alternately, so that the vertices of two in different copies can be joined by edges. The construction is done as follows.

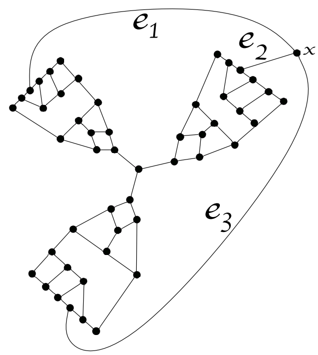

Figure 1: CPB3C Hamiltonian graph G

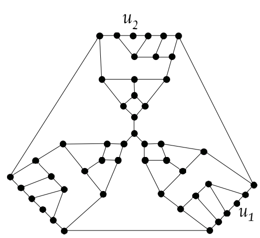

Figure 2: CPB3C non-Hamiltonian graph $G-x$

Take two copies of $G-x$ and we name them as G_1 and H_1. Let u_1^1, u_2^1, u_3^1 be the vertices of degree two in G_1 and v_1^1, v_2^1, v_3^1 be the vertices of degree two in

H_1. We assume u_1^1, u_2^1, u_3^1 are colored red and v_1^1, v_2^1, v_3^1 are colored black. (This is possible since we can choose G_1 and H_1 in such a way so that the color of vertices in G_1 is interchanged with the color of vertices in H_1). Consider the graph $S_1 = G_1 \cup H_1 \cup \{u_1^1 v_1^1, u_2^1 v_2^1, u_3^1 v_3^1\}$, the graph which consists of G_1, H_1 and the edges $u_1^1 v_1^1, u_2^1 v_2^1, u_3^1 v_3^1$. Clearly S_1 is CPB3C. Further S_1 is Hamiltonian. If any two edges from $\{u_1^1 v_1^1, u_2^1 v_2^1, u_3^1 v_3^1\}$ are removed, the resulting graph is non-Hamiltonian. See Figure 3.

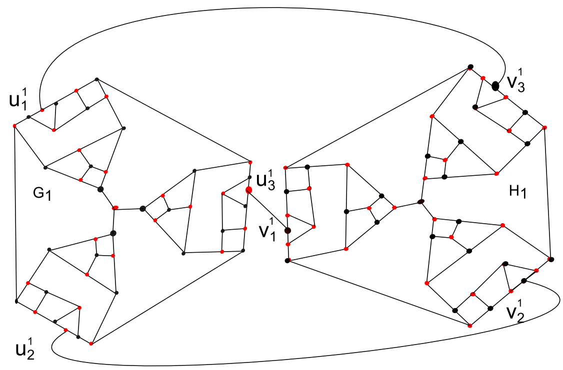

Figure 3: CPB3C Hamiltonian graph

Take four copies of $G - x$ and we name them as G_1, G_2 and H_1, H_2. Let $u_1^1, u_2^1, u_3^1 \in G_1$, $u_1^2, u_2^2, u_3^2 \in G_2$, $v_1^1, v_2^1, v_3^1 \in H_1$ and $v_1^2, v_2^2, v_3^2 \in H_2$ be vertices of degree 2. Further, we let u_1^1, u_2^1, u_3^1, v_1^2, v_2^2, v_3^2 colored red and u_1^2, u_2^2, u_3^2, v_1^1, v_2^1, v_3^1 colored black. Let $C_1 = w_1^1 w_2^1 w_3^1 w_4^1$ be a cycle of length four. Consider the graph

$$S_2 = G_1 \cup G_2 \cup H_1 \cup H_2 \cup C_1 \cup \{u_1^1 u_1^2, v_1^1 v_1^2, u_2^1 v_2^1, u_2^2 v_2^2, u_3^1 w_1^1, u_3^2 w_2^1, v_3^1 w_4^1, v_3^2 w_3^1\}.$$

By the same argument as above, S_2 is CPB3C Hamiltonian graph. Removal of three edges makes the resulting graph non-Hamiltonian. See Figure 4.

Take $2n$ copies of $G - x$ with n copies named as $G_1, G_2, \dots G_n$ and n copies named as $H_1, H_2, \dots H_n$. Let $u_1^1, u_2^1, u_3^1 \in G_1$, $u_1^2, u_2^2, u_3^2 \in G_2$ and so on $u_1^n, u_2^n, u_3^n \in G_n$ be vertices of degree 2. Similarly let $v_1^1, v_2^1, v_3^1 \in H_1$, $v_1^2, v_2^2, v_3^2 \in H_2$ and so on $v_1^n, v_2^n, v_3^n \in H_n$ be vertices of degree 2. Further consider cycles of length four as $C_1, C_2, \dots C_{n-1}$. Continuing as in Case 2, we form the graph S_n, which is clearly CPB3C Hamiltonian graph. The removal of three edges joining any two adjacent copies of $G - x$ produces a largest non-Hamiltonian subgraph of S_n.

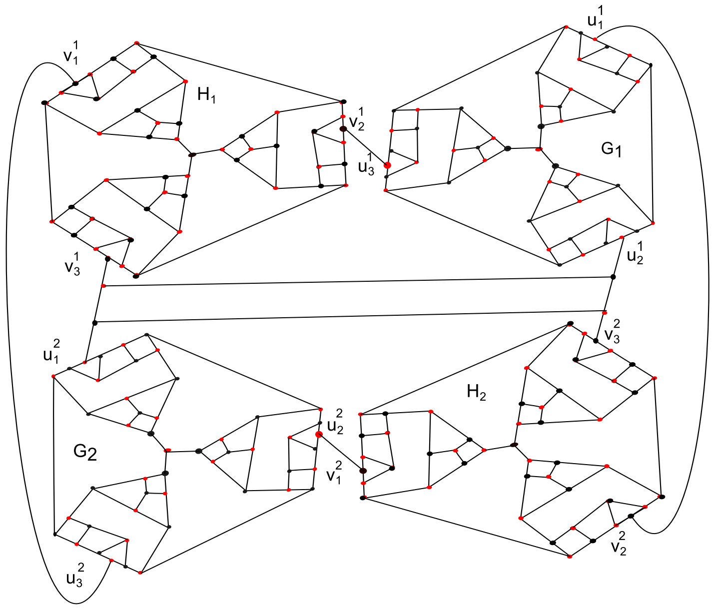

Figure 4: CPB3C Hamiltonian graph

If we remove any of the edges which connects the different components of $G - x$, we get a largest subgraph which is non-Hamiltonian. In this way, if we continuously remove three such edges again and again, we get subgraphs all of which are non-Hamiltonian. Since n is as large as possible, we get $2n$ such non-Hamiltonian graphs. Not only this, if we further remove some edges from all the smallest components of $G - x$), as shown in Figure 5, remaining large subgraphs are non-Hamiltonian. In this way, we get large and large subsets of CPB3C Hamiltonian graphs which are non- Hamiltonian. Thus we conclude that there exist a family of CPB3C Hamiltonian graphs in which large and large subsets are non-Hamiltonian.

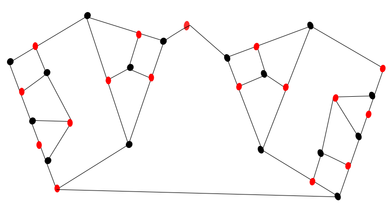

Figure 5: Resulting graph

In this way we construct a CPB3C Hamiltonian graph whose large and large subsets are non-Hamiltonian. In other words, it is possible to construct CPB3C Hamiltonian graph as large as possible, whose large and large subsets are non-Hamiltonian. □

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On bounds of the sine and cosine along straight lines on the complex plane

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Abstract. In the paper, the author discusses and computes bounds of the sine and cosine along straight lines on the complex plane.

1 Motivations

In the theory of complex functions, the sine and cosine on the complex plane $\mathbb{C}$ are denoted and defined respectively by

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2},$$

where $z = x + iy$, $x, y \in \mathbb{R}$, and $i = \sqrt{-1}$ is the imaginary unit. When $z = x \in \mathbb{R}$, these two trigonometric functions become $\sin x$ and $\cos x$ which satisfy the periodicity

$$\sin(x + 2k\pi) = \sin x, \quad \cos(x + 2k\pi) = \cos x$$

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and the boundedness

$$0 \le |\sin x| \le 1, \quad 0 \le |\cos x| \le 1 \tag{1}$$

for $k \in \mathbb{Z}$. On the other hand, when $z = iy$ for $y \in \mathbb{R}$,

$$\sin(iy) = \frac{e^{-y} - e^{y}}{2i} \to \pm i\infty \quad \text{and} \quad \cos(iy) = \frac{e^{-y} + e^{y}}{2} \to +\infty$$

as $y \to \pm\infty$. These imply that the sine and cosine are bounded on the real x-axis, but unbounded on the imaginary y-axis.

Motivated by the above boundedness, we naturally guess that the complex functions $\sin z$ and $\cos z$ for $z \in \mathbb{C}$ are

1. bounded on all straight lines parallel to the real x-axis,
2. unbounded on all straight lines whose slopes are not horizontal.

In this paper, we will verify the above guesses and compute bounds for $\sin z$ and $\cos z$ on all horizontal straight lines.

2 Unboundedness of sine and cosine on sloped and vertical lines

On the sloped straight line $y = \alpha + \beta x$ for constants $\alpha \in \mathbb{R}$ and $\beta \ne 0$ on the complex plane $\mathbb{C}$, by the triangle inequality for complex numbers, we have

$$\begin{aligned}
|\sin z| &= |\sin(x + i(\alpha + \beta x))| \\
&= \left|\frac{e^{i[x+i(\alpha+\beta x)]} - e^{-i[x+i(\alpha+\beta x)]}}{2i}\right| \\
&= \left|\frac{e^{[ix-(\alpha+\beta x)]} - e^{-[ix-(\alpha+\beta x)]}}{2i}\right| \\
&\ge \frac{1}{2}\left|\left|e^{[ix-(\alpha+\beta x)]}\right| - \left|e^{-[ix-(\alpha+\beta x)]}\right|\right| \\
&= \frac{1}{2}\left|e^{-(\alpha+\beta x)} - e^{(\alpha+\beta x)}\right| \\
&\to +\infty, \quad x \to \pm\infty
\end{aligned}$$

and

$$|\cos z| = |\cos(x + i(\alpha + \beta x))|$$

$$
\begin{aligned}
&= \left|\frac{e^{i[x+i(\alpha+\beta x)]} + e^{-i[x+i(\alpha+\beta x)]}}{2}\right| \\
&= \left|\frac{e^{[ix-(\alpha+\beta x)]} + e^{-[ix-(\alpha+\beta x)]}}{2}\right| \\
&\ge \frac{1}{2}\left|\left|e^{[ix-(\alpha+\beta x)]}\right| - \left|e^{-[ix-(\alpha+\beta x)]}\right|\right| \\
&= \frac{1}{2}\left|e^{-(\alpha+\beta x)} - e^{(\alpha+\beta x)}\right| \\
&\to +\infty, \quad x \to \pm\infty.
\end{aligned}
$$

Consequently, the functions $\sin z$ and $\cos z$ are not bounded along any sloped straight line.

On the vertical straight line $x = \gamma$ for any constant $\gamma \in \mathbb{R}$ on the complex plane, by the triangle inequality for complex numbers, we have

$$
\begin{aligned}
|\sin z| = |\sin(\gamma + iy)| &= \left|\frac{e^{i(\gamma+iy)} - e^{-i(\gamma+iy)}}{2i}\right| \\
&\ge \frac{1}{2}\left||e^{i(\gamma+iy)}| - |e^{-i(\gamma+iy)}|\right| = \frac{1}{2}|e^{-y} - e^{y}| \to +\infty
\end{aligned}
$$

and

$$
\begin{aligned}
|\cos z| = |\cos(\gamma + iy)| &= \left|\frac{e^{i(\gamma+iy)} + e^{-i(\gamma+iy)}}{2}\right| \\
&\ge \frac{1}{2}\left||e^{i(\gamma+iy)}| - |e^{-i(\gamma+iy)}|\right| = \frac{1}{2}|e^{-y} - e^{y}| \to +\infty
\end{aligned}
$$

as $y \to \pm\infty$. Consequently, the functions $\sin z$ and $\cos z$ are not bounded along any vertical straight line.

3 Bounds of the sine on horizontal straight lines

On the horizontal straight line $y = \alpha$ for any constant $\alpha \in \mathbb{R}$ on the complex plane $\mathbb{C}$, by the triangle inequality for complex numbers, we have

$$
\begin{aligned}
|\sin z| = |\sin(x + i\alpha)| &= \left|\frac{e^{i(x+i\alpha)} - e^{-i(x+i\alpha)}}{2i}\right| \\
&= \left|\frac{e^{(ix-\alpha)} - e^{-(ix-\alpha)}}{2i}\right| = \frac{1}{2}\left|\frac{e^{ix}}{e^{\alpha}} - e^{-ix}e^{\alpha}\right|
\end{aligned}
$$

$$\le \frac{1}{2}\left(\left|\frac{e^{ix}}{e^{\alpha}}\right| + \left|e^{-ix}e^{\alpha}\right|\right) = \frac{1}{2}\left(\frac{1}{e^{\alpha}} + e^{\alpha}\right)$$

and

$$\begin{aligned}|\sin z| = |\sin(x+i\alpha)| &= \left|\frac{e^{i(x+i\alpha)} - e^{-i(x+i\alpha)}}{2i}\right| \\ &= \left|\frac{e^{(ix-\alpha)} - e^{-(ix-\alpha)}}{2i}\right| = \frac{1}{2}\left|\frac{e^{ix}}{e^{\alpha}} - e^{-ix}e^{\alpha}\right| \\ &\ge \frac{1}{2}\left|\left|\frac{e^{ix}}{e^{\alpha}}\right| - \left|e^{-ix}e^{\alpha}\right|\right| = \frac{1}{2}\left|\frac{1}{e^{\alpha}} - e^{\alpha}\right|.\end{aligned}$$

Therefore, it follows that

$$\frac{1}{2}\left|\frac{1}{e^{\alpha}} - e^{\alpha}\right| \le |\sin(x+i\alpha)| \le \frac{1}{2}\left(\frac{1}{e^{\alpha}} + e^{\alpha}\right), \quad x, \alpha \in \mathbb{R}.$$

When $z = 2k\pi + i\alpha$ for $k \in \mathbb{Z}$, we have

$$\sin z = \sin(2k\pi + i\alpha) = \frac{e^{i(2k\pi+i\alpha)} - e^{-i(2k\pi+i\alpha)}}{2i} = -\frac{i}{2}\left(\frac{1}{e^{\alpha}} - e^{\alpha}\right).$$

When $z = 2k\pi + \frac{\pi}{2} + i\alpha$ for $k \in \mathbb{Z}$, we have

$$\begin{aligned}\sin z = \sin\left(2k\pi + \frac{\pi}{2} + i\alpha\right) &= \frac{e^{i(2k\pi+\pi/2+i\alpha)} - e^{-i(2k\pi+\pi/2+i\alpha)}}{2i} \\ &= \frac{e^{i(\pi/2+i\alpha)} - e^{-i(\pi/2+i\alpha)}}{2i} = \frac{e^{-\alpha} + e^{\alpha}}{2} = \frac{1}{2}\left(\frac{1}{e^{\alpha}} + e^{\alpha}\right).\end{aligned}$$

When $z = (2k+1)\pi + i\alpha$ for $k \in \mathbb{Z}$, we have

$$\begin{aligned}\sin z = \sin((2k+1)\pi + i\alpha) &= \frac{e^{i((2k+1)\pi+i\alpha)} - e^{-i((2k+1)\pi+i\alpha)}}{2i} \\ &= \frac{e^{i(\pi+i\alpha)} - e^{-i(\pi+i\alpha)}}{2i} = \frac{1}{2i}\left(e^{\alpha} - \frac{1}{e^{\alpha}}\right) = -\frac{i}{2}\left(e^{\alpha} - \frac{1}{e^{\alpha}}\right).\end{aligned}$$

When $z = (2k+1)\pi + \frac{\pi}{2} + i\alpha$ for $k \in \mathbb{Z}$, we have

$$\begin{aligned}\sin z &= \sin\left((2k+1)\pi + \frac{\pi}{2} + i\alpha\right) \\ &= \frac{e^{i((2k+1)\pi+\pi/2+i\alpha)} - e^{-i((2k+1)\pi+\pi/2+i\alpha)}}{2i} \\ &= \frac{e^{i(3\pi/2+i\alpha)} - e^{-i(3\pi/2+i\alpha)}}{2i} = -\frac{e^{-\alpha} + e^{\alpha}}{2} = -\frac{1}{2}\left(\frac{1}{e^{\alpha}} + e^{\alpha}\right).\end{aligned}$$

4 Bounds of the cosine on horizontal straight lines

On the horizontal straight line $y = \alpha$ for any constant $\alpha \in \mathbb{R}$ on the complex plane $\mathbb{C}$, by the triangle inequality for complex numbers, we have

$$\begin{aligned}
|\cos z| = |\cos(x + i\alpha)| &= \left|\frac{e^{i(x+i\alpha)} + e^{-i(x+i\alpha)}}{2}\right| \\
&= \left|\frac{e^{(ix-\alpha)} + e^{-(ix-\alpha)}}{2}\right| = \frac{1}{2}\left|\frac{e^{ix}}{e^{\alpha}} + e^{-ix}e^{\alpha}\right| \\
&\le \frac{1}{2}\left(\left|\frac{e^{ix}}{e^{\alpha}}\right| + \left|e^{-ix}e^{\alpha}\right|\right) = \frac{1}{2}\left(\frac{1}{e^{\alpha}} + e^{\alpha}\right)
\end{aligned}$$

and

$$\begin{aligned}
|\cos z| = |\cos(x + i\alpha)| &= \left|\frac{e^{i(x+i\alpha)} + e^{-i(x+i\alpha)}}{2}\right| \\
&= \left|\frac{e^{(ix-\alpha)} + e^{-(ix-\alpha)}}{2}\right| = \frac{1}{2}\left|\frac{e^{ix}}{e^{\alpha}} + e^{-ix}e^{\alpha}\right| \\
&\ge \frac{1}{2}\left|\left|\frac{e^{ix}}{e^{\alpha}}\right| - \left|e^{-ix}e^{\alpha}\right|\right| = \frac{1}{2}\left|\frac{1}{e^{\alpha}} - e^{\alpha}\right|.
\end{aligned}$$

Therefore, it follows that

$$\frac{1}{2}\left|\frac{1}{e^{\alpha}} - e^{\alpha}\right| \le |\cos(x + i\alpha)| \le \frac{1}{2}\left(\frac{1}{e^{\alpha}} + e^{\alpha}\right), \quad x, \alpha \in \mathbb{R}.$$

When $z = 2k\pi + i\alpha$ for $k \in \mathbb{Z}$, we have

$$\cos z = \cos(2k\pi + i\alpha) = \frac{e^{i(2k\pi+i\alpha)} + e^{-i(2k\pi+i\alpha)}}{2} = \frac{1}{2}\left(\frac{1}{e^{\alpha}} + e^{\alpha}\right).$$

When $z = 2k\pi + \frac{\pi}{2} + i\alpha$ for $k \in \mathbb{Z}$, we have

$$\begin{aligned}
\cos z = \cos\left(2k\pi + \frac{\pi}{2} + i\alpha\right) &= \frac{e^{i(2k\pi+\pi/2+i\alpha)} + e^{-i(2k\pi+\pi/2+i\alpha)}}{2} \\
= \frac{e^{i(\pi/2+i\alpha)} + e^{-i(\pi/2+i\alpha)}}{2} &= \frac{ie^{-\alpha} - ie^{\alpha}}{2} = \frac{i}{2}\left(\frac{1}{e^{\alpha}} - e^{\alpha}\right).
\end{aligned}$$

When $z = (2k + 1)\pi + i\alpha$ for $k \in \mathbb{Z}$, we have

$$\cos z = \cos((2k + 1)\pi + i\alpha) = \frac{e^{i((2k+1)\pi+i\alpha)} + e^{-i((2k+1)\pi+i\alpha)}}{2}$$

$$= \frac{e^{i(\pi+i\alpha)} + e^{-i(\pi+i\alpha)}}{2} = -\frac{1}{2}\left(\frac{1}{e^{\alpha}} + e^{\alpha}\right).$$

When $z = (2k+1)\pi + \frac{\pi}{2} + i\alpha$ for $k \in \mathbb{Z}$, we have

$$\begin{aligned}\cos z &= \cos\left((2k+1)\pi + \frac{\pi}{2} + i\alpha\right)\\ &= \frac{e^{i((2k+1)\pi+\pi/2+i\alpha)} + e^{-i((2k+1)\pi+\pi/2+i\alpha)}}{2}\\ &= \frac{e^{i(3\pi/2+i\alpha)} + e^{-i(3\pi/2+i\alpha)}}{2} = \frac{-ie^{-\alpha} + ie^{\alpha}}{2} = -\frac{i}{2}\left(\frac{1}{e^{\alpha}} - e^{\alpha}\right).\end{aligned}$$

5 Alternative proofs

Since $\sin\left(z + \frac{\pi}{2}\right) = \cos z$ for $z \in \mathbb{C}$, there is a similar behaviour of sine and cosine in the complex plane $\mathbb{C}$. Hence, in what follows, we just only consider sine.

It is easy to see that sine, cosine, hyperbolic sine, and hyperbolic cosine have relations

$$\sin(it) = i\sinh t, \quad \sinh(it) = i\sin t, \quad \cos(it) = \cosh t, \quad \cosh(it) = \cos t.$$

Accordingly, we have

$$\sin z = \sin(x+iy) = \sin x\cos(iy) + \cos x\sin(iy) = \sin x\cosh y + i\cos x\sinh y$$

and

$$|\sin z|^2 = \sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y.$$

On any horizontal line $y = c$, say, we have

$$\begin{aligned}|\sin z|^2 &= \sin^2 x\cosh^2 c + \cos^2 x\sinh^2 c\\ &= \sin^2 x\cosh^2 c + \cos^2 x\left(\cosh^2 c - 1\right) = \cosh^2 c - \cos^2 x\end{aligned}$$

or

$$|\sin z|^2 = 1 + \sinh^2 c - \cos^2 x = \sinh^2 c + \sin^2 x.$$

Consequently, sine is bounded on all horizontal lines.

Look at a non-horizontal line, where $z = \gamma + \alpha x + i\beta y$ for $\beta \ne 0$ (by non-horizontality). Here

$$|\sin z|^2 = \sin^2(\gamma+\alpha x)\cosh^2(\beta y) + \cos^2(\gamma+\alpha x)\sinh^2(\beta y).$$

If the line is sloped so that $\alpha \neq 0$ (by non-verticality), then both terms in the above equation are unbounded, so that sine is unbounded.

If the line is vertical, so that $\alpha = 0$, we have to be a tad careful! If γ is not a multiple of π, the term $\sin^2(\gamma + \alpha x)\cosh^2(\beta y)$ is unbounded; and if γ is a multiple of π, then the term $\cos^2(\gamma + \alpha x)\sinh^2(\beta y)$ is unbounded. In a word, sine is unbounded on all non-horizontal lines.

6 Conclusions

On the sloped straight line $y = \alpha + \beta x$ for $\alpha \in \mathbb{R}$ and $\beta \neq 0$ on the complex plane $\mathbb{C}$, the trigonometric functions $\sin z = \sin(x + i(\alpha + \beta x))$ and $\cos z = \cos(x + i(\alpha + \beta x))$ are unbounded.

On the vertical straight line $x = \gamma$ for any scalar $\gamma \in \mathbb{R}$ on the complex plane $\mathbb{C}$, the trigonometric functions $\sin z = \sin(\gamma + iy)$ and $\cos z = \cos(\gamma + iy)$ are unbounded.

On the horizontal straight line $y = \alpha$ for any constant $\alpha \in \mathbb{R}$ on the complex plane $\mathbb{C}$, the trigonometric functions $\sin z = \sin(x + i\alpha)$ and $\cos z = \cos(x + i\alpha)$ are bounded by the double inequalities

$$|\sinh \alpha| \le |\sin(x + i\alpha)| \le \cosh \alpha, \quad x, \alpha \in \mathbb{R} \tag{2}$$

and

$$|\sinh \alpha| \le |\cos(x + i\alpha)| \le \cosh \alpha, \quad x, \alpha \in \mathbb{R} \tag{3}$$

whose equalities are respectively attained at points

$$2k\pi + i\alpha, \quad 2k\pi + \frac{\pi}{2} + i\alpha, \quad (2k+1)\pi + i\alpha, \quad 2k\pi + \frac{3\pi}{2} + i\alpha$$

with concrete values

$$\begin{aligned}
\sin(2k\pi + i\alpha) &= \cos\left(2k\pi + \frac{3\pi}{2} + i\alpha\right) = i\sinh \alpha, \\
\sin\left(2k\pi + \frac{\pi}{2} + i\alpha\right) &= \cos(2k\pi + i\alpha) = \cosh \alpha, \\
\sin((2k+1)\pi + i\alpha) &= \cos\left(2k\pi + \frac{\pi}{2} + i\alpha\right) = -i\sinh \alpha, \\
\sin\left(2k\pi + \frac{3\pi}{2} + i\alpha\right) &= \cos((2k+1)\pi + i\alpha) = -\cosh \alpha
\end{aligned}$$

for $k \in \mathbb{Z}$.

Letting $\alpha \to 0$ in the double inequalities (2) and (3) recovers inequalities in (1) for $x \in \mathbb{R}$.

On the horizontal belt zones $0 \le A \le y \le B$ and $-B \le y \le -A \le 0$ on the complex plane $\mathbb{C}$, the trigonometric functions $\sin z = \sin(x + iy)$ and $\cos z = \cos(x + iy)$ are bounded by the double inequalities

$$\sinh A \le |\cos(x \pm iy)| \le \cosh B$$

and

$$\sinh A \le |\sin(x \pm iy)| \le \cosh B$$

for $x \in \mathbb{R}$.

7 An open problem

The inequalities in (1) can be refined as

$$\frac{2}{\pi}x \le \sin x \le x \quad \text{and} \quad 1 - \frac{2}{\pi}x \le \cos x \le 1 - \frac{x^2}{\pi} \tag{4}$$

for $0 \le x \le \frac{\pi}{2}$. See [1, p. 143], [3, p. 22], and [4, p. 33]. These two double inequalities in (4) are respectively called as Jordan's and Kober's inequality. These two double inequalities have been further refined, generalized, applied, and surveyed in the papers [2, 5, 6, 8, 9, 10] and closely related references therein. Motivated by these refinements, generalizations, and applications, we pose an open problem: can one refine, generalize, and apply the double inequalities (2) and (3) for $x \in \left[0, \frac{\pi}{2}\right]$ and $\alpha \ne 0$?

Finally we remark that this paper is a revised version of the preprint [7].

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Some consequences of the rank normal form of a matrix

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Abstract. If A is a rectangular matrix of rank r, then A may be written as PSQ where P and Q are invertible matrices and $S = \begin{pmatrix} I_r & O \\ O & O \end{pmatrix}$. This is the rank normal form of the matrix A.

The purpose of this paper is to exhibit some consequences of this representation form.

1 Introduction and notation

In the following we shall denote by K a commutative field and by $M(m \times n, K)$ the set of matrices with m lines and n columns with elements in K. By $O_{m,n}$ we shall denote the null matrix with m lines and n columns and by $I_{r,r}$ the unit matrix with r lines and r columns.

The purpose of this paper is to exhibit some consequences of the following representation theorem from [1], [2], [3], [4], [5], [6], [7].

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Theorem 1 (*rank normal form of a matrix*). *Let* $A \in M(m \times n, K)$ *with* $\text{rank}A = r$. *Then there are the invertible matrices* $P \in M(m \times m, K)$ *and* $Q \in M(n \times n, K)$ *and*

$$S = \begin{bmatrix} I_{r,r} & O_{r,n-r} \\ O_{m-r,r} & O_{m-r,n-r} \end{bmatrix} \in M(m \times n, K)$$

such that $A = PSQ$.

The exposition of the results is made in an unitary manner using the block matrices and contains some known inequalities as inequality of Sylvester and inequality of Frobenius.

The Theorem 9 is new.

2 Main results

In the following we give three representation theorems 2, 3 and 4 which are direct consequences of Theorem 1.

Theorem 2 *Let* $A \in M(m \times n, K)$ *with* $\text{rank}\, A = r$. *Then there are* $A_1, A_2, \ldots, A_r \in M(m \times n, K)$ *matrices of* $\text{rank}\, 1$ *such that* $A = A_1 + A_2 + \ldots + A_r$.

Proof. The proof follows from Theorem 1 if we note that the matrix

$$S = \begin{bmatrix} I_{r,r} & O_{r,n-r} \\ O_{m-r,r} & O_{m-r,n-r} \end{bmatrix},$$

may be written as the sum of r matrices of rank 1. □

Theorem 3 *Let* $A \in M(m \times n, K)$ *with* $\text{rank}\, A = r$. *Then there are two matrices with rank equal with* r, $B \in M(m \times r, K)$ *and* $C \in M(r \times n, K)$ *such that* $A = BC$.

Proof. From Theorem 1 it follows that there are the invertible matrices $P \in M(m \times m, K)$ and $Q \in M(n \times n, K)$ and

$$S = \begin{bmatrix} I_{r,r} & O_{r,n-r} \\ O_{m-r,r} & O_{m-r,n-r} \end{bmatrix} \in M(m \times n, K)$$

such that $A = PSQ$.

Note that $S = S_1 S_2$ where $S_1 = \begin{bmatrix} I_{r,r} \\ O_{m-r,r} \end{bmatrix} \in M(m \times r, K)$ and $S_2 = [I_{r,r}, O_{r,n-r}] \in M(r \times n, K)$.

If we put $B = PS_1$ and $C = S_2 Q$ we have $A = BC$ and $\operatorname{rank} B = \operatorname{rank}(PS_1) = \operatorname{rank} S_1 = r$, $\operatorname{rank} C = \operatorname{rank}(S_2 Q) = \operatorname{rank} S_2 = r$. □

Theorem 4 *Let* $A \in M(n \times n, K)$ *with* $\operatorname{rank} A = r$. *Then there exists two matrices with rank* r, $B \in M(n \times n, K)$ *and* $C \in M(n \times n, K)$ *such that* $A = BC$.

Proof. From Theorem 1 it follows that there are two invertible matrices $P, Q \in M(n \times n, K)$ and $S = \operatorname{diag}[I_{r,r}, O_{n-r,n-r}]$ such that $A = PSQ$. Let $B = PS$ and $C = SQ$.

From the equality $S^2 = S$ we have that $A = B \cdot C$. We have

$$\operatorname{rank} B = \operatorname{rank}(PS) = \operatorname{rank} S = r \text{ and } \operatorname{rank} C = \operatorname{rank}(SQ) = \operatorname{rank} S = r$$

□

Lemma 1 *Let* $A \in M(m \times n, K)$, $B \in M(p \times q, K)$, $S = \begin{bmatrix} I_{r,r} & O_{r,p-r} \\ O_{n-r,r} & O_{n-r,p-r} \end{bmatrix} \in M(n \times p, K)$, $S_1 = \begin{bmatrix} I_{r,r} \\ O_{n-r,r} \end{bmatrix} \in M(n \times r, K)$, $S_2 = [I_{r,p}, O_{r,p}] \in M(r \times p, K)$.

Then the following statements are true:

i). $\operatorname{rank}(AS) \geq \operatorname{rank} A + r - n$

ii). $\operatorname{rank}(AS_1) = \operatorname{rank}(AS_1 S_2)$

iii). $\operatorname{rank}(S_1 S_2 B) = \operatorname{rank}(S_2 B)$.

Proof.

i). Let $A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$ where $A_1 \in M(r \times r, K)$, $A_2 \in M(r \times (n-r), K)$, $A_3 \in M((m-r) \times r, K)$, $A_4 \in M((m-r) \times (n-r), K)$.

We have

$$\operatorname{rank}(AS) = \operatorname{rank}\left(\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \cdot \begin{bmatrix} I_{r,r} & O_{r,p-r} \\ O_{n-r,r} & O_{n-r,p-r} \end{bmatrix}\right)$$
$$= \operatorname{rank}\begin{bmatrix} A_1 & O_{r,p-r} \\ A_3 & O_{n-r,p-r} \end{bmatrix} = \operatorname{rank}\left(\begin{bmatrix} A_1 A_2 \\ A_3 A_4 \end{bmatrix} - \operatorname{rank}\begin{bmatrix} O_{r,r} & A_2 \\ O_{m-r,r} & A_4 \end{bmatrix}\right)$$

$$\geq \operatorname{rank}\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} - \operatorname{rank}\begin{bmatrix} A_2 \\ A_4 \end{bmatrix} \geq \operatorname{rank} A + r - n$$

ii). We have

$$\operatorname{rank}(AS_1) = \operatorname{rank}\left(\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\begin{bmatrix} I_{r,r} \\ O_{n-r,r} \end{bmatrix}\right) = \operatorname{rank}\begin{bmatrix} A_1 \\ A_3 \end{bmatrix}$$
$$= \operatorname{rank}\left(\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\begin{bmatrix} I_{r,r} & O_{r,p-r} \\ O_{n-r,r} & O_{n-r,p-r} \end{bmatrix}\right) = \operatorname{rank}(AS_1S_2)$$

iii). We can write

$$B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$$

where $B_1 \in M(r \times r, K), B_2 \in M(r \times q - r, K), B_3 \in M((p-r) \times r, K)$, $B_4 \in M((p-r) \times q - r, K)$.

We have

$$\operatorname{rank}(S_1S_2B) = \operatorname{rank}(S, B) = \operatorname{rank}\left(\begin{bmatrix} I_{r,r} & O_{r,p-r} \\ O_{n-r,r} & O_{n-r,p-r} \end{bmatrix}\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}\right)$$
$$= \operatorname{rank}\begin{bmatrix} B_1 & B_2 \\ O_{p-r,r} & O_{p-r,q-r} \end{bmatrix} = \operatorname{rank}[B_1B_2]$$
$$= \operatorname{rank}\left([I_{r,r}, O_{r,p-r}]\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}\right) = \operatorname{rank}(S_2B).$$

□

Theorems 5 and 6 may be found in the papers [1], [2], [3], [4], [5], [6], [7].

Theorem 5 (*Sylvester*). *Let* $A \in M(m \times n, K), B \in M(n \times p, K)$. *Then is true that*

$$\operatorname{rank}(AB) \geq \operatorname{rank} A + \operatorname{rank} B - n$$

Proof. The rank normal form of matrix B is $B = PSQ$, $\operatorname{rank} B = r$ where $P \in M(n \times n, K)$ and $Q \in M(p \times p, K)$ are invertible and

$$S = \begin{bmatrix} I_{r,r} & O_{r,p-r} \\ O_{n-r,r} & O_{n-r,p-r} \end{bmatrix} \in M(n \times p, K).$$

According with Lemma 1 i). we have

$$\operatorname{rank}(AB) = \operatorname{rank}(APSQ) = \operatorname{rank}(APS) \geq \operatorname{rank}(AP) + r - n$$
$$= \operatorname{rank} A + \operatorname{rank} B - n$$

□

Theorem 6 (*Frobenius*). *Let* $A \in M(m \times n, K), B \in M(n \times p, K), C \in M(p \times q, K)$ *then the following inequality is true*

$$\operatorname{rank}(AB) + \operatorname{rank}(BC) \leq \operatorname{rank} B + \operatorname{rank}(ABC)$$

Proof. Let $r = \operatorname{rank} B$ and the rank normal form of B is $B = PSQ$ where $P \in M(n \times n, K)$ and $Q \in M(p \times p, K)$ are invertible. We shall apply Lemma 1, ii). and iii). and Sylvester inequality and we will obtain:

$$\begin{aligned}
\operatorname{rank}(ABC) &= \operatorname{rank}(APSQC) = \operatorname{rank}(APS, S_2QC) \\
&\geq \operatorname{rank}(APS_1) + \operatorname{rank}(S_2QC) - r \\
&= \operatorname{rank}(APS_1S_2) + \operatorname{rank}(S_1S_2QC) - r \\
&= \operatorname{rank}(APSQ) + \operatorname{rank}(PSQC) - r \\
&= \operatorname{rank}(AB) + \operatorname{rank}(BC) - \operatorname{rank} B
\end{aligned}$$

□

Theorem 7 *Let* $A \in M(n \times n, K)$ *and the sequence* $(a_p)_{p\geq 1}$ *defined by* $a_p = \operatorname{rank}(A^p)$, $p \geq 1$. *Then the following statements hold:*

i). $(a_p)_{p\geq 1}$ *is decreasing*

ii). $2a_{p+1} \leq a_p + a_{p+2}$ *for each* $p \geq 1$

iii). $a_p = a_{p+1}$ *implies that* $a_p = a_{p+t}$ *for each* $t \geq 1$.

Proof. i). We have

$$a_{p+1} = \operatorname{rank}\left(A^{p+1}\right) \leq \min\left(\operatorname{rank}(A)^p ; \operatorname{rank} A\right) \leq \operatorname{rank}(A^p) = a_p$$

for each $p \geq 1$.
ii). From Theorem 5 we have for $p \geq 1$

$$\operatorname{rank}(AA^p) + \operatorname{rank}(A^pA) \leq \operatorname{rank}(A^p) + \operatorname{rank}(AA^pA)$$

this inequality is equivalent with

$$2a_{p+1} \leq a_p + a_{p+2}$$

for each $p \geq 1$.
iii). It results from i) and ii). □

Theorem 8 *Let* $A \in M(n \times n, K)$ *and* q_A *the minimal polynomial of the matrix A. Then the following inequality hold*

$$\deg(q_A) \leq 1 + \operatorname{rank} A$$

Proof. We de note

$$r = \operatorname{rank} A$$

From Theorem 2 it follows that it exists $B \in M(n \times r, K)$ and $C \in M(r \times n, K)$ such that $A = BC$.

Let $D = CB \in M(r \times r, K)$ and $f_D(t) = \det(tI_r - D)$, $t \in K$.

We have from Hamilton-Cayley-Frobenius theorem that

$$f_D(D) = O_r$$

It follows that

$$Bf_D(CB)C = O_n$$

Because $B(CB)^p C = (BC)^p BC$ for $p \geq 1$.

We obtain that

$$f_D(BC)BC = O_n$$

Let $g(t) = tf_D(t), t \in K$ and note that $\deg g = r + 1$. We have $g(A) = O_n$ and that $q_A | g$, so

$$\deg q_A \leq \deg g = r + 1 = 1 + \operatorname{rank} A$$

□

Theorem 9 *Let* $A \in M(n \times n, K)$. *Then the following statement are equivalent*

i). $A^2 = O_n$

ii). *There are* $B, C \in M(n \times n, K)$ *with the following properties*

$$A = BC \text{ and } CB = O_n$$

Proof. ii). ⇒ i). Let $A = BC$ with $CB = O_n$. We have

$$A^2 = BCBC = BO_nC = O_n$$

i). ⇒ ii). Let $A = PSQ$ the rank normal form of A. Note that $S^2 = S$. We put $B = PS$ and $C = SQ$. We have

$$O_n = A^2 = BCBC = PSSQPSSQ = PSQPSQ$$

Because P and Q are invertible, we obtain that $SQPS = O_n$. It follows that $CB = O_n$ and $A = BC$. □

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A new approach to the r-Whitney numbers by using combinatorial differential calculus

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Abstract. In the present article we introduce two new combinatorial interpretations of the r-Whitney numbers of the second kind obtained from the combinatorics of the differential operators associated to the grammar $G := \{y \to yx^m, x \to x\}$. By specializing $m = 1$ we obtain also a new combinatorial interpretation of the r-Stirling numbers of the second kind. Again, by specializing to the case $r = 0$ we introduce a new generalization of the Stirling number of the second kind and through them a binomial type family of polynomials that generalizes Touchard's polynomials. Moreover, we recover several known identities involving the r-Dowling polynomials and the r-Whitney numbers using the combinatorial differential calculus. We construct a family of posets that generalize the classical Dowling lattices. The r-Withney numbers of the first kind are obtained as the sum of the Möbius function over elements of a given rank. Finally, we prove that the r-Dowling polynomials are a Sheffer family relative to the generalized Touchard binomial family, study their umbral inverses, and introduce $[m]$-Stirling numbers of the first kind. From the relation between umbral calculus and the Riordan matrices we give several new combinatorial identities

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Key words and phrases: differential operators, r-Whitney number, r-Dowling polynomial

1 Introduction

The r-Whitney numbers of the second kind $W_{m,r}(n,k)$ were defined by Mező [24] as the connecting coefficients between some particular polynomials. For non-negative integers n, k and r with $n \geq k \geq 0$ and for any integer $m > 0$

$$(mx + r)^n = \sum_{k=0}^{n} m^k W_{m,r}(n,k) x^{\underline{k}},$$

where $x^{\underline{n}} = x(x-1)\cdots(x-n+1)$ for $n \geq 1$ and $x^{\underline{0}} = 1$.

Cheon and Jung [8] showed that the numbers $W_{m,r}(n,k)$ are related to the Dowling lattices as follows. Let $Q_n(G)$ be the Dowling lattice of rank n, where G is a finite group of order m. The coefficient of r^s of the polynomial $\sum_{k=0}^{n} W_{m,r}(n,k)$ is equal to the number of elements of $Q_n(G)$ containing $n-s$ distinct unit functions. This sequence generalizes the Whitney numbers of the second kind, $W_m(n,k) = W_{m,1}(n,k)$, that count the total number of elements of corank k in $Q_n(G)$ [11].

Mihoubi and Rahmani [27] found an interesting combinatorial interpretation for the r-Whitney numbers of the second kind by using colored set partitions. Recall that a partition of a set A is a class of disjoint subsets of A such that the union of them covers A. The subsets are called blocks. Let $r, n \geq 0$ be integers, and let $A_{n,r}$ be the set defined by $A_{n,r} := \{1, 2, \ldots, r, r+1, \ldots, n+r\}$. The elements $1, 2, \ldots, r$ will be called special elements. A block of a partition of the above set is called special if it contains special elements. Then $W_{m,r}(n,k)$ counts the number of the set partitions of $A_{n,r}$ in $k+r$ blocks, such that the elements $1, 2, \ldots, r$ are in distinct blocks (i.e., any special block contains exactly one special element). All the elements but the last one in non-special blocks are coloured with one of m colours independently and neither the elements in the special blocks nor the special blocks are coloured.

The r-Whitney numbers satisfy the following recurrence relation [24]

$$W_{m,r}(n,k) = W_{m,r}(n-1,k-1) + (km+r)W_{m,r}(n-1,k).$$

Moreover, these numbers have the exponential generating function [24]:

$$\sum_{n=k}^{\infty} W_{m,r}(n,k) \frac{z^n}{n!} = \frac{e^{rz}}{k!} \left(\frac{e^{mz}-1}{m} \right)^k. \tag{1}$$

Note that if $(m,r) = (1,0)$ we obtain the Stirling numbers of the second kind, if $(m,r) = (1,r)$ we have the r-Stirling (or noncentral Stirling) numbers

[4], and if $(m, r) = (m, 1)$ we have the Whitney numbers [1, 2]. Many properties of the r-Whitney numbers and their connections to elementary symmetric functions, matrix theory, special polynomials, combinatorial identities and generalizations can be found in [9, 10, 12, 18, 20, 21, 24, 25, 26, 28, 29, 37].

The combinatorial differential calculus was introduced by Joyal in the framework of combinatorial species (see for example [15, 3, 16]). However, by using directly exponential formal power series and the combinatorics of their coefficients (see [19]), the fundamentals of the approach can be explained without the use of the categorical framework involved in the theory of species. The combinatorial differential calculus is closely related to Chen context-free grammar method [6]. However, they are not equivalent. Differential operators that are not derivations can be combinatorially interpreted, and algebraic formulas obtained from that interpretation. Only derivations have a counterpart in the context of Chen grammars. The substitution rules in a context-free grammar can be translated into a differential operator. By the iterated application of the combinatorial version of it, the associated combinatorial objects emerge, and not infrequently, in a simple and natural way.

Using the combinatorial differential calculus we present a new combinatorial interpretation to the r-Whitney numbers of the second kind. As a special case we get a new combinatorial interpretation for the r-Stirling numbers of the second kind and introduce the $[m]$-Stirling numbers of the second kind, $S^{[m]}(n, k)$. Through them we define the $[m]$-Touchard polynomials, that generalize the classical family and are also of binomial type. Their umbral inverse give us the $[m]$-Stirling numbers of the first kind. The combinatorial interpretation of the r-Whitney numbers (and of the generalization of the Stirling numbers) of the second kind we present here are very natural. We construct a family of posets $Q_{n,r}(G)$, G a group of order m, that generalize the Dowling lattices. Our interpretation of r-Withney numbers directly count the number of elements of a given rank on those posets, and by Möbius inversion we get the r-Whitney numbers of the first kind, as the sum of their Möbius function evaluated at elements of a given rank (Theorem 2 bellow). The r-Stirling numbers of the first and second kind are interpreted in a similar way by specializing G to the trivial group $G = \{e\}$. Also a similar interpretation is given in terms of a subposet $Q_{n,0}(G)$ of $Q_{n,r}(G)$ for the $[m]$-Stirling numbers (see Remark 2).

By using classical results of umbral calculus we get that the r-Dowling polynomials are of Sheffer type relative to the $[m]$-Touchard binomial family. Finally, from the relation between umbral calculus and the Riordan matrices we give several new combinatorial identities involving the r-Whitney number of both kinds, Bernoulli polynomials and Euler polynomials.

2 New combinatorial models for r-Whitney, r-Stirling, and new $[m]$-Stirling numbers

Chen [6] introduced a combinatorial method by means of context-free grammar to study exponential structures. A context-free grammar G over an alphabet X, whose symbols are commutative indeterminates, is a set of productions or substitutions rules that replace a symbol of X by a formal function (formal power series) in the set of indeterminates X, $x \to \phi_x(X),\ x \in A$.

Here A is a subset of the alphabet X. The formal derivative D is a linear operator defined with respect to a context-free grammar G such that for a letter $x \in A$ acts by substitution by $\phi(X)$, and it is extended recursively as a derivation. For any formal functions u and v we have:

$$D(u+v) = D(u)+D(v), \quad D(uv) = D(u)v+uD(v), \quad D(f(u)) = \frac{\partial f(u)}{\partial u}D(u),$$

where $f(x)$ is a formal power series. For more applications of the context-free grammar method see for example [5, 7, 13, 17].

The grammar formal derivative D can be equivalently written as a derivation in the algebra of formal power series over a ring $\mathcal{R}$ containing $\mathbb{Q}$, $\mathcal{R}[[X]]$.

$$\mathcal{D} = \sum_{x\in A} \phi_x(X)\partial_x.$$

For A infinite, we have to make a summability assumption over the family of formal power series $\{\phi_x(X)\}_{x\in A}$. The variables in X are thought of as colors. Each operator $\phi_x(X)\partial_x$ is combinatorially interpreted as a corolla having as root a ghost vertex of color x and weighted with the coefficients of the series $\phi(X)$ according with the distribution of colors of the leaves in the corolla. A formal power series $F(X)$ are also represented by corollas in a similar way but without the ghost vertex. The combinatorial representation of the operator acting over a series $F(X)$ is obtained by dropping over the combinatorial representation of $F(X)$ corollas whose ghost vertices replace vertices of the same color on corollas representing F (see Figure 1). Details and proofs of why this combinatorial approach works can be seen in [19].

As an example let us consider the Stirling grammar ([6])

$$G := \begin{cases} y \to xy \\ x \to x. \end{cases}$$

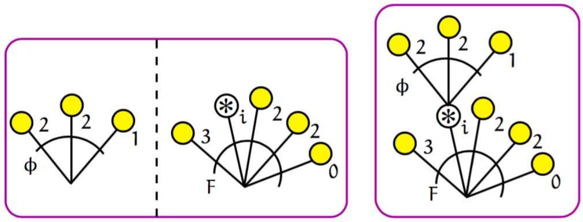

Figure 1: Corolla operator $\phi(X)\partial_i$ applied to F.

Applying the associated formal operator n times to y, it is known that

$$D^n y = \sum_{k=1}^{n} S(n,k) x^k y, \tag{2}$$

where $S(n,k)$ are the Stirling numbers of the second kind. The differential operator associated to that grammar is $\mathcal{D} = xy\partial_y + x\partial_x$. We call it the Stirling operator. The variables x and y are represented by vertices in two colors, yellow and white. The operator $\mathcal{D}$ has as combinatorial representation the sum of two corollas. The operator $xy\partial_y$ acts over a white vertex replacing it by a 'ghost' vertex $*$ and connecting it to a yellow vertex x and to white one y. The operator $x\partial_x$ places a ghost vertex in a yellow vertex and connects it to another yellow vertex (see Figure 2).

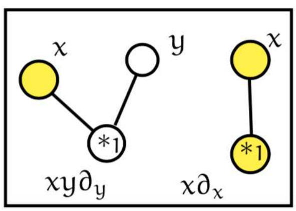

Figure 2: Combinatorial operator $\mathcal{D} = xy\partial_y + x\partial_x$.

When applying the operator more than one time, for simplicity, we replace the ghost vertices by numbers to keep track of the order in which the operator was applied. If we apply it n times to y (combinatorially represented as a singleton vertex of color white) we obtain an increasing tree with a path (spine) from the root to the white vertex on the top. Along each node of the spine there are linear branches having in the top of each of them one yellow vertex with weight y. The elements of the branches along the spine form a partition π of $[n]$. The weight of each such tree is equal to x to the number of branches (blocks of π) times y, the weight of the white vertex on top of the spine (see Figure 3). In this way we get a visual proof of Equation (2).

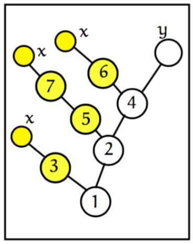

Figure 3: A tree enumerate by $\mathcal{D}^7 y = (xy\partial_y + x\partial_x)^7 y$ that corresponds to the partition $\pi = \{\{1,3\}\{2,5,7\}\{4,6\}\}$.

Let us now consider the context-free grammar G defined in [13].

$$G := \begin{cases} y \to yx^m, \\ x \to x. \end{cases}$$

Hao et al. [13] proved as a particular case that

$$D^n yx = \sum_{k=0}^{n} W_m(n,k) yx^{mk+1}.$$

This grammar G corresponds to the differential operator $\mathcal{D} = yx^m\partial_y + x\partial_x$. In Figure 4 we show the combinatorial interpretation of this operator for $m = 2$. Moreover, by the main theorem of [13] we have

$$\mathcal{D}^n yx^r = \sum_{k=0}^{n} W_{m,r}(n,k) yx^{mk+r}. \quad (3)$$

In Figures 5 and 6, we show the first and second derivative for $m = 2$ and $r = 3$, respectively.

By the iterated application of the combinatorial version of the operator $\mathcal{D}$, dropping the corollas over one white vertex and r yellow vertices (yx^r) we obtain the following combinatorial structure. A forest consisting of one

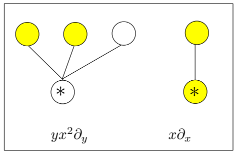

Figure 4: Combinatorial interpretation for the operator $\mathcal{D} = yx^m\partial_y + x\partial_x$, with $m = 2$.

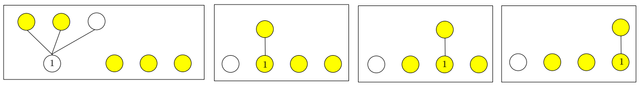

Figure 5: The first derivative.

increasing tree, grown from the initial white vertex y, followed by r linearly ordered branchless increasing trees (grown from the r yellow initial vertices, x^r). The trees of this forest are characterized as follows (see Figure 7).

1. The first tree has a spine of white vertices with an unlabeled white vertex at the top having weight y. There are m (totally ordered) branches of yellow vertices that sprout from each vertex on the spine. Each branch has an unlabeled yellow vertex at the top, with weight x.

2. Each of the r branchless increasing trees consists of a set of internal vertices (that may be empty) with an unlabeled yellow vertex at the top, having weight x.

The total weight of each of this forests is equal to yx^{mk+r}, where k is the number of vertices in the spine. Then, from Equation (3) we have that the r-Whitney numbers $W_{m,r}(n,k)$ count the number of forests as above, having k vertices in the spine, and n internal vertices in total.

Reinterpreting the forests of increasing trees we obtain the following combinatorial interpretation of $W_{m,r}(n,k)$.

Theorem 1 The r-Whitney numbers $W_{m,r}(n,k)$ count pairs of the form $(\{(B, f_B)\}_{B\in\pi}, \mathbf{V})$ where

1. The first component is a partial partition of $\{1, 2, \ldots, n\}$, having exactly k blocks, $\uplus_{B\in\pi}B = A \subseteq \{1, 2, \ldots, n\}$, plus a coloring on each block, $f_B : B \longrightarrow [m]$. The coloring on f_B assign to the minimum element of B the color 1.

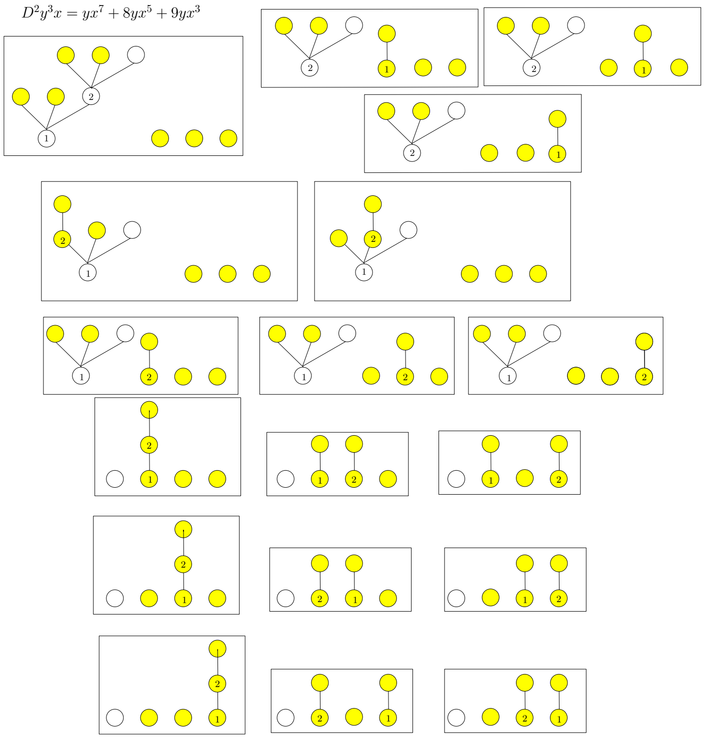

Figure 6: The second derivative.

2. The second component $\mathbf{V} = (B_1, B_2, \ldots, B_r)$ is a weak r-composition of $[n] - A$

$$\biguplus_{j=1}^{r} B_r = [n] - A.$$

Proof. We are going to prove that the first tree is in bijection with the colored partitions described in Item (1). We assign to each vertex of the spine of a given tree the set of vertices of the m branches attached to it plus the vertex itself. In this way we obtain the partial partition π with k blocks in total. Then color the vertices on the ith branch of each block with color i, $i = 1, 2, \ldots, m$, and assign color 1 to the vertex on the spine. Observe that, since the tree is increasing, the vertex on the spine has the minimum label of its block. This construction is clearly reversible (see Figure 7). The r branchless trees are naturally associated to the composition $\mathbf{V}$ by assigning the ith tree its set of internal vertices B_i.

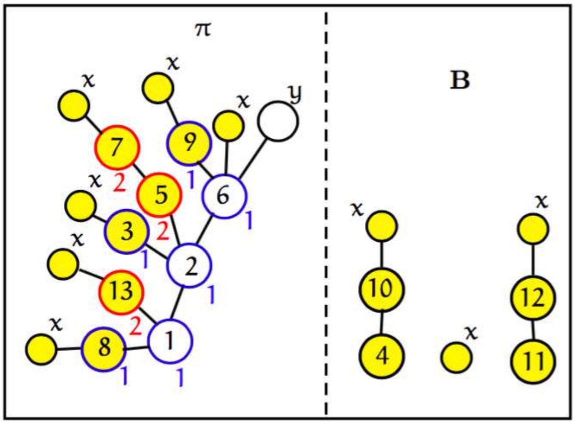

Figure 7: Whitney forest for $m = 2$ and $r = 3$.

□

We represent a colored set (B, f_B) by placing the color of each element as its exponent $(B, f_B) \equiv \{b^{f(b)} | b \in B\}$. In this way, the colored partition associated to the tree in Figure 7 is $\{\{1^1, 8^1, 13^2\}, \{2^1, 3^1, 5^2, 7^2\}, \{6^1, 9^1\}\}$. The composition is equal to $\mathbf{V} = (\{4, 10\}, \emptyset, \{11, 12\})$.

Example 1 *The r-Whitney number* $W_{2,2}(2, 2) = 1$*, the pair being* $(\{\{1^1\}, \{2^1\}\}, (\emptyset, \emptyset))$*. For* $W_{2,3}(2, 1) = 8$*, it enumerates pairs, the first component is a 2-colored partition on a subset* A *of* $\{1, 2\}$ *having one block. The second component a weak 3-composition of* $[2] - A$*. The pairs being*

1. $(\{\{1^1, 2^1\}\}, (\emptyset, \emptyset, \emptyset))$,
2. $(\{\{1^1, 2^2\}\}, (\emptyset, \emptyset, \emptyset))$,
3. $(\{\{1^1\}\}, (\{2\}, \emptyset, \emptyset))$,
4. $(\{\{1^1\}\}, (\emptyset, \{2\}, \emptyset))$,
5. $(\{\{1^1\}\}, (\emptyset, \emptyset, \{2\}))$,
6. $(\{\{2^1\}\}, (\{1\}, \emptyset, \emptyset))$,
7. $(\{\{2^1\}\}, (\emptyset, \{1\}, \emptyset))$,
8. $(\{\{2^1\}\}, (\emptyset, \emptyset, \{1\}))$.

Remark 1 *We get the following combinatorial interpretation for the generalized Stirling numbers of the second kind obtained by specializing* r *and* m*.*

1. *For* $m = 1$ *we get a combinatorial interpretation for the* r*-Stirling numbers of the second kind* $S_r(n, k)$*. They count the pairs* $(\pi, \mathbf{V})$*,* π *a partition of some subset* A *of* $[n]$*,* $\mathbf{V}$ *a weak* r*-composition of* $[n] - A$*.*
2. *For* $r = 0$ *we get a new generalization* $S^{[m]}(n, k)$ *of the Stirling numbers of the second kind. It counts the number of colored partitions, as in Theorem 1, Item (1), but over the whole set* $[n]$*.*

Definition 1 *We define the* $[m]$*-Touchard polynomials,* $T_n^{[m]}(x)$*, by*

$$T_n^{[m]}(x) := \sum_{k=1}^{n} S^{[m]}(n, k)x^k.$$

The polynomial family $\{T_n^{[m]}(x)\}_{n=0}^{\infty}$ *is of binomial type, i.e.,*

$$T_0^{[m]}(x) = 1,$$

$$T_n^{[m]}(x + y) = \sum_{k=0}^{n} \binom{n}{k} T_k^{[m]}(x)T_{n-k}^{[m]}(y).$$

We shall prove it in Section 5.

3 The r-Dowling posets

Let B_1 and B_2 be two disjoint sets, and f_i two functions, $f_i : B_i \to C$, $i = 1, 2$. Recall that the disjoint union of f_1 and f_2, $f_1 \uplus f_2 : B_1 \uplus B_2 \to C$ is defined as follows:

$$f_1 \uplus f_2(x) = \begin{cases} f_1(x), & \text{if } x \in B_1; \\ f_2(x), & \text{if } x \in B_2. \end{cases}$$

Let G be a finite group of order m. We consider the set of pairs as in Theorem 1, but where each function f_B, $B \in \pi$, colors the elements of the block B with elements of the group G (instead of $[m]$) in such a way that it assigns to the minimum element of B the identity of G. We call that kind of colorations *unital*. Consider functions $f : [n] \to G$ of the form $f = \uplus_{B \in \pi} f_B$, where each coloring $f_B : B \to G$ is unital. By simplicity, the pair (π, f) will be denoted by π^f and will be called a (unital) colored partition. Let $Q_{n,r}(G, k)$ be the set of unital colored partitions-composition pairs on $[n]$, where the colored partition has exactly k blocks. Denote by $Q_{n,r}(G)$ the union $\uplus_{k=0}^{n} Q_{n,r}(G,k)$, i.e., the set of colored partition-composition pairs $(\pi^f, \mathbf{V})$ having π an arbitrary number of blocks, $k = 0, 1, 2, \ldots, n$. Clearly, by Theorem 1, the number of unital colored partition-composition pairs in $Q_{n,r}(G,k)$ is the r-Whitney number $W_{m,r}(n,k)$, that is $W_{m,r}(n,k) = |Q_{n,r}(G,k)|$. We are going to define a partial order on $Q_{n,r}(G)$ such that its ranked Möbius function gives us the r-Whitney numbers of the first kind,

$$w_{m,r}(n,k) = \sum_{(\pi^f, \mathbf{V}),\, |\pi|=k} \mu(\widehat{0}, (\pi^f, \mathbf{V})). \tag{4}$$

This construction generalizes the classical Dowling lattice [11], recovered as a particular case of this by making $r = 1$.

For a coloration $f : A \to G$, and an element g of the group, we define a new coloration $f * g$ by

$$(f * g)(x) := f(x) * g, \; x \in A,$$

where $*$ is the operation of the group.

We introduce two kind of operations on the set $Q_{n,r}(G)$. Let $(\pi^f, \mathbf{V}) \in Q_{n,r}(G)$. We obtain another colored partition-composition pair $(\tau^h, \mathbf{V}') \in Q_{n,r}(G)$ by the following two kinds of operations.

1. The compositions remain unchanged, i.e., $\mathbf{V} = \mathbf{V}'$, and τ is obtained from π by joining some blocks of π, $B_1, B_2, \ldots, B_\ell$, $B = \uplus_{i=1}^{\ell} B_i$ (we assume that the family is listed according with the minimum element of the blocks; $\min B_i < \min B_{i+1}$). The rest of blocks remain the same. The coloring h is obtained in the block B as a *unital linear combination*

$$h_B = f_{B_1} * g_1 \uplus f_{B_2} * g_2 \uplus \cdots \uplus f_{B_\ell} * g_\ell, \tag{5}$$

   where $g_1 = e$ is the identity of G, and $g_2, g_3, \ldots, g_\ell$ are arbitrary elements of G. The coloring on the rest of the blocks of τ remain the same, $h_{B'} = f_{B'}$, for $B' \neq B$. It is clear that h_B is again unital.

2. The components of $\mathbf{V}'$ are the same as the components of $\mathbf{V}$ except one, say V_j, which is augmented by some blocks of π; while π is reduced by these blocks. Precisely:

$$\begin{aligned} V_i' &= V_i, \text{ for } i \neq j, \text{ and } V_j' = V_j \cup \bigcup_{i=1}^{\ell} B_i, \\ \pi' &= \pi - \{B_1, B_2, \ldots, B_\ell\} \end{aligned}$$

From now on we follow the convention, for concrete examples, of separating the colored partition-composition pair $(\pi^f, \mathbf{V})$ with double bars,

$$\pi^f \| \mathbf{V} := (\pi^f, \mathbf{V})$$

A colored partition π^f can be represented as a factored monomial with exponents in G

$$\pi^f \equiv \prod_{B\in\pi} \left(\prod_{b\in B_i} b^{f(b)} \right). \tag{6}$$

For example, if $G = F_5^{\times}$ is the multiplicative group of the Galois field $F_5 = \mathbb{Z}/(5)$, the colored partition $\{\{1^1, 8^3, 13^2\}, \{2^1, 3^4, 5^3, 7^2\}, \{6^1, 9^3\}\}$ is represented as the factored monomial $(1^1\, 8^3\, 13^2)(2^1\, 3^4\, 5^3\, 7^2)(6^1\, 9^3)$. A unital linear combination as in Equation (5) can be represented as the expansion of a monomial of monomials,

$$\prod_{b\in B} b^{h(b)} = \prod_{i=1}^{\ell} \left(\prod_{b\in B_i} b^{f_{B_i}(b)} \right)^{g_i} = \prod_{b\in B} b^{\uplus_{i=1}^{\ell} f_{B_i}(b) * g_i}.$$

For example, $1^1\, 8^3\, 13^2\, 2^2\, 3^3\, 5^1\, 7^4\, 6^4\, 9^2 = (1^1\, 8^3\, 13^2)^1 (2^1\, 3^4\, 5^3\, 7^2)^2 (6^1\, 9^3)^4$ represents the unital linear combination $g_1 = 1$, $g_2 = 2$, $g_3 = 1$ and $g_4 = 2$, with $B_1 = \{1\}$, $B_2 = \{2, 3, 5, 7\}$, $B_3 = \{6, 9\}$, $B_4 = \{8, 13\}$. The operations (1) and (2) are clearly closed on $Q_{n,r}(G)$. Now we can define the partial order on $Q_{n,r}(G)$.

Definition 2 *Let $(\pi^f, \mathbf{V})$ and $(\tau^h, \mathbf{V}')$ be two elements of $Q_{n,r}(G)$. We say that*

$$(\pi^f, \mathbf{V}) \leq (\tau^h, \mathbf{V}')$$

if $(\tau^h, \mathbf{V}')$ can be obtained from $(\pi^f, \mathbf{V})$ by any sequence of operations (1) or (2) above. It includes de empty sequence, that gives us $(\pi^f, \mathbf{V}) = (\tau^h, \mathbf{V}')$. In other words, $\leq$ is defined to be the transitive and reflexive clousure of the union of the binary relations defined by operations (1) and (2).

The resulting relation is antisymmetric because the operations (1) and (2) are directed. If $(\tau^h, \mathbf{V}')$ is obtained from $(\pi^f, \mathbf{V})$ by any non empty set of operation, we can not reverse those changes and go back to $(\pi^f, \mathbf{V})$ by using any sequence of these operations. This is because each of them increases the quantity $\max\{|B| : B \in \pi\} \cup \{|V_j| : j = 1, 2, \dots, r\}$, the maximum of the sizes of blocks of the partition together with those of the composition. The poset $Q_{n,r}(G)$ has a least element $\widehat{0} = 1|2|\dots|n\|(\emptyset, \emptyset, \dots, \emptyset)$. The maximal elements are of the form $\emptyset\|\mathbf{V}$, $\mathbf{V}$ being a composition of $[n]$.

Example 2 *Consider the set* $Q_{11,2}(G)$*, where* $G = F_5^{\times}$*. Its element*

$$2\,5^2|3\,4^3|6\,7^4\,8^3\,9|10\,11^2||(\{1\},\emptyset)$$

comes from

$$\pi^f||\mathbf{V} = 2\,5^2|3\,4^3|6\,7^4|\,8\,9^2|10\,11^2||(\{1\},\emptyset)$$

by an operation of type (1)*, because* $6\,7^4\,8^3\,9 = (6\,7^2)(8\,9^2)^3$*. Similarly,*

$$3\,4^3|6\,7^4|8\,9^2||(\{1,2,5,10,11\},\emptyset)$$

is obtained from $\pi^f||\mathbf{V}$ *by an operation of type* (2)*.*

It is easy to check the following proposition, that gives an equivalente way of defining the partial order on $Q_{n,r}(G)$.

Proposition 1 Let $(\pi^f,\mathbf{V})$ and $(\tau^h,\mathbf{V}')$ be two elements of $Q_{n,r}(G)$. We have that $(\pi^f,\mathbf{V}) \leq (\tau^h,\mathbf{V}')$ if and only if

1. For every $s = 1,2,\dots,r$, $V_s \subseteq V'_s$.
2. Every block B_i of $\pi = \{B_1,B_2,\dots,B_k\}$ is either contained in V'_s for some $s \in [r]$ or in some block B of τ. In the second case, we have that $B = \uplus_{B_i\subseteq B} B_i$, and for each i; $B_i \subseteq B$, there exists a $g_i \in G$ such that $h|_{B_i} = f_{B_i} * g_i$. In the case in which $\min B_i = \min B$, then $g_i = e$.

Lemma 1 *Let* $(\pi^f,\mathbf{V})$ *be an element of* $Q_{n,r}(G)$*,* π *having exactly* k *blocks. Then the set of elements* $(\tau^h,\mathbf{V}')$ *greater than or equal to* $(\pi^f,\mathbf{V})$*, such that* τ *has exactly* $j \leq k$ *blocks, is equal to the* r*-Whitney number* $W_{m,r}(k,j)$*.*

Proof. Order de blocks of π according with their minimum elements, $\pi = \{B_1,B_2,\dots,B_k\}$, $\min B_i < \min B_{i+1}$. For each block B of τ and $s = 1,2,\dots,r$, define the sets (subsets of $[k]$), $\widehat{B} := \{i|B_i \subseteq B\}$ and $\widehat{V}'_s = \{i|B_i \subseteq V'_s\}$. By part (2) of Proposition 1, we have $h_B = \uplus_{i\in\widehat{B}} f_{B_i} * g_i$. We define the quotient coloration h^q by $h^q_{\widehat{B}}(i) = g_i$, $i \in \widehat{B}$. Let i_0 be the minimum element of $\widehat{B}$. Since the elements of π are listed according with their minimum element, the minimum of B_{i_0} is also the minimum of the whole set B. Let $x = \min B_{i_0}$. Since h_B and $f_{B_{i_0}}$ are both unital, we have

$$e = h_B(x) = f_{B_{i_0}}(x) * g_{i_0} = e * g_{i_0} = g_{i_0} = h^q_{\widehat{B}}(i_0).$$

Then, $h^q_{\widehat{B}}$ is unital for every $\widehat{B} \in \widehat{\tau}$, and uniquely obtained from h and f. By part (2) of Proposition 1, $(\widehat{\tau}^{h^q},\widehat{\mathbf{V}}')$ is a colored partition-composition pair on

$[k]$. The correspondence $(\tau^h, \mathbf{V}') \mapsto (\widehat{\tau}^{h^q}, \widehat{\mathbf{V}}')$ is a bijection, and we have that $|\tau| = j = |\widehat{\tau}|$. The inverse $(\widehat{\tau}^{h^q}, \widehat{\mathbf{V}}') \mapsto (\tau^h, \mathbf{V}')$ is obtained by making

$$\begin{aligned} B &:= \biguplus_{i\in\widehat{B}} B_i \\ h_B &:= \biguplus_{i\in\widehat{B}} f_{B_i} * h^q(i) \\ V'_s &:= V_s \uplus (\biguplus_{i\in\widehat{V}'_s} B_i),\ s = 1,2,\dots,r. \end{aligned}$$

□

Example 3 *As an example of the bijection in Lemma 1, let*

$$\pi^f\|\mathbf{V} = 2\,5^2|3\,4^3|6\,7^4|\,8\,9^2|10\,11^2\|(\{1\},\emptyset)$$

and

$$\tau^h\|\mathbf{V}' = 6\,7^4\,8^3\,9\,10^2\,11^4\|(\{1,2,5\},\{3,4\}).$$

We shall verify that $\pi^f\|\mathbf{V} \leq \tau^h\|\mathbf{V}'$ *and construct* $\widehat{\tau}^{h^q}\|\widehat{\mathbf{V}}'$.

Enumerating the blocks of π *we have* $B_1 = \{2,5\}$, $B_2 = \{3,4\}$, $B_3 = \{6,7\}$, $B_4 = \{8,9\}$, $B_5 = \{10,11\}$. *Since*

$$6\,7^4\,8^3\,9\,10^2\,11^4 = (6\,7^4)^1(8\,9^2)^3(10\,11^2)^2$$

and

$$(\{1,2,5\},\{3,4\}) = (\{1\}\cup\{2,5\},\emptyset\cup\{3,4\}) = (\{1\}\cup B_1,\emptyset\cup B_2),$$

we have that $\pi^f\|\mathbf{V} \leq \tau^h\|\mathbf{V}'$, *and get* $\widehat{\tau}^{h^q}\|\widehat{\mathbf{V}}' = 3^1\,4^3\,5^2\|(\{1\},\{2\})$.

The poset $Q_{n,r}(G)$ is ranked. The rank of an element is $n-k$, k being the number of blocks of the colored partition.

Theorem 2 The sum of the Möbius function over the elements of rank $n-k$ in $Q_{n,r}(G)$ gives us the r-Withney number of the first kind $w_{m,r}(n,k)$. Denoting by μ the Möbius function of the poset $Q_{n,r}(G)$, we have

$$w_{m,r}(n,k) = \sum_{(\pi^f,\mathbf{V}),\,|\pi|=k} \mu(\widehat{0},(\pi^f,\mathbf{V})). \tag{7}$$

Proof. Define the matrix $C(n,k)$ to be the right hand side of Eq. (7). It is enough to prove that

$$\sum_{0\leq k\leq n} C(n,k)W_{m,r}(k,j) = \delta_{n,j}.$$

Let $(\tau^h, \mathbf{V}')$ be an element of $Q_{n,r}(G,j)$. By properties of the Möbius function we have that

$$\sum_{\widehat{0} \le (\pi^f, \mathbf{V}) \le (\tau^h, \mathbf{V}')} \mu(\widehat{0}, (\pi^f, \mathbf{V})) = \delta(\widehat{0}, (\tau^h, \mathbf{V}')) = \delta_{n,j}. \tag{8}$$

Summing over all the elements of $Q_{n,r}(G,j)$, interchanging sums and classifying by the size of π, we get

$$\begin{aligned}
\delta_{n,j} &= \sum_{(\tau^h,\mathbf{V}')\in Q_{n,r}(G,j)} \sum_{\widehat{0}\le(\pi^f,\mathbf{V})\le(\tau^h,\mathbf{V}')} \mu(\widehat{0}, (\pi^f, \mathbf{V})) \\
&= \sum_{k=0}^{n} \sum_{(\pi^f,\mathbf{V})\in Q_{n,r}(G,k)} \left(\sum_{\substack{(\tau^h,\mathbf{V}')\ge(\pi^f,\mathbf{V}) \\ (\tau^h,\mathbf{V}')\in Q_{n,r}(G,j)}} \mu(\widehat{0}, (\pi^f, \mathbf{V})) \right) \\
&= \sum_{k=0}^{n} \sum_{(\pi^f,\mathbf{V})\in Q_{n,r}(G,k)} \mu(\widehat{0}, (\pi^f, \mathbf{V})) \left(\sum_{\substack{(\tau^h,\mathbf{V}')\ge(\pi^f,\mathbf{V}) \\ (\tau^h,\mathbf{V}')\in Q_{n,r}(G,j)}} 1 \right) \\
&= \sum_{k=0}^{n} \sum_{(\pi^f,\mathbf{V})\in Q_{n,r}(G,k)} \mu(\widehat{0}, (\pi^f, \mathbf{V})) |\{(\tau^h, \mathbf{V}') | (\tau^h, \mathbf{V}') \ge (\pi^f, \mathbf{V}), |\tau| = j\}| \\
&= \sum_{k=0}^{n} C(n,k) W_{m,r}(k,j).
\end{aligned}$$

The last equality obtained by Lemma 1. □

Example 4 *The poset of colored partition-set pairs, $Q_{n,1}(G)$, is isomorphic to the classical Dowling lattice. See Figure 8 for the Hasse diagram of the poset $Q_{2,2}(G)$, G being the group with two elements $G = \{-1, 1\}$ (writing $\overline{2}$ instead of 2^{-1}). From the diagram we get its Möbius function and obtain $w_{2,2}(2,2) = 1$, $w_{2,2}(2,1) = -6$, and $w_{2,2}(2,0) = 8$. $Q_{n,r}(\{-1,1\})$ generalizes the signed partitions poset, $\overline{\Pi}[n]$ obtained for $r = 1$, $\overline{\Pi}[n] = Q_{n,1}(\{-1,1\})$.*

4 The r-Dowling polynomials

Cheon and Jung [8] defined the r-Dowling polynomials of degree n by

$$\mathcal{D}_{m,r}(n,u) := \sum_{k=0}^{n} W_{m,r}(n,k) u^k.$$

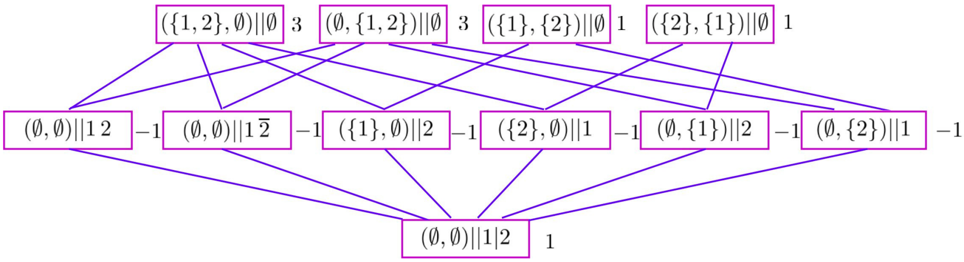

Figure 8: Poset $Q_{2,2}(G)$, $G = \{-1, 1\}$.

They found some combinatorial identities by means of Riordan arrays. Let us define the following generating function

$$\mathcal{H}_{m,r}(t) = \mathcal{H}(t; x, y) = e^{t\mathfrak{D}} y x^r = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathfrak{D}^n y x^r = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{n} W_{m,r}(n, k) y x^{km+r}.$$

It is easy to show the following

Lemma 2 *We have the identities*

$$\mathfrak{D}^n y x^r = y x^r \mathcal{D}_{m,r}(n, x^m),$$
$$\mathcal{H}_{m,r}(t) = e^{t\mathfrak{D}} y x^r = y x^r \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathcal{D}_{m,r}(n, x^m).$$

Theorem 3 *The exponential generating function for the* r*-Dowling polynomials is*

$$\sum_{n=0}^{\infty} \mathcal{D}_{m,r}(n, u) \frac{t^n}{n!} = \exp\left(rt + u \frac{e^{mt} - 1}{m} \right).$$

Proof. Since $\mathfrak{D}$ is a derivation, it is easy to show that the operator $e^{t\mathfrak{D}}$ is multiplicative. Hence

$$\begin{aligned} \mathcal{H}'_{m,r}(t) = e^{t\mathfrak{D}} \mathfrak{D} y x^r = e^{t\mathfrak{D}} (y x^{m+r} + r y x^r) = (e^{t\mathfrak{D}} y x^r) \cdot (e^{t\mathfrak{D}} x)^m + r e^{t\mathfrak{D}} y x^r \\ = \mathcal{H}_{m,r}(t)(x^m e^{mt} + r). \end{aligned}$$

From that,

$$\frac{\mathcal{H}'_{m,r}(t)}{\mathcal{H}_{m,r}(t)} = \frac{d}{dt} \ln(\mathcal{H}_{m,r}(t)) = x^m e^{mt} + r.$$

Integrating and using the initial condition $\mathcal{H}_{m,r}(0) = yx^r$, we get:

$$\mathcal{H}_{m,r}(t) = yx^r \exp\{rt + \frac{x^m}{m}(e^{mt} - 1)\}.$$

By Lemma 2, making $u = x^m$ we get the result. □

Theorem 4 *The* r*-Dowling polynomials satisfy the following relation for any integers* $r, \ell \geq 0$

$$\mathcal{D}_{m,r+\ell}(n,u) = \sum_{k=0}^{n} \binom{n}{k} \ell^{n-k} \mathcal{D}_{m,r}(k,u).$$

Proof. Let $\widehat{\mathcal{D}}_{m,r}(n,x,y) := \sum_{k=0}^{n} W_{m,r}(n,k) y x^{km+r}$. Then

$$\begin{aligned}\mathcal{H}_{m,r+\ell}(t) &= e^{t\mathfrak{D}} y x^{r+\ell} = \left(e^{t\mathfrak{D}} y x^r\right)\left(e^{t\mathfrak{D}} x^\ell\right) \\ &= x^\ell e^{\ell t} \mathcal{H}_{m,r}(t) = x^\ell \sum_{n=0}^{\infty} \frac{t^n}{n!} \left(\sum_{j=0}^{n} \binom{n}{j} \ell^{n-j} \widehat{\mathcal{D}}_{m,r}(j,x,y) \right).\end{aligned}$$

Therefore

$$\widehat{\mathcal{D}}_{m,r}(n,x,y) = x^\ell \sum_{j=0}^{n} \binom{n}{j} \ell^{n-j} \widehat{\mathcal{D}}_{m,r}(j,x,y).$$

Modifying a bit this equality we obtain the desired result. □

In particular, if $\ell = 1$ then

$$\mathcal{D}_{m,r+1}(n,u) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{D}_{m,r}(k,u). \tag{9}$$

In Theorem 5 we generalize the beautiful relation given by Spivey [34] for the Bell numbers B_n. The Spivey's formula says that:

$$B_{n+m} = \sum_{k=0}^{n} \sum_{j=0}^{m} j^{n-k} S(m,j) \binom{n}{k} B_k,$$

where $S(n,j)$ is the Stirling number of the second kind. We generalize this identity for the r-Whitney numbers by using differential operators.

Theorem 5 *The* r*-Dowling polynomials satisfy the following formula*

$$\mathcal{D}_{m,r}(n+h,u)=\sum_{k=0}^{n}\sum_{j=0}^{h}\binom{n}{k}\mathcal{D}_{m,r}(k,u)W_{m,r}(h,j)u^j j^{n-k}m^{n-k}.$$

Proof. We compute the derivative in two ways. First, in a direct way

$$\frac{d^h}{dt^h}\mathcal{H}_{m,r}(t)=yx^r\sum_{n=0}^{\infty}\mathcal{D}_{m,r}(n+h,x^m)\frac{t^n}{n!},$$

and secondly, by using the identity $\frac{d^h}{dt^h}e^{t\mathfrak{D}}=e^{t\mathfrak{D}}\mathfrak{D}^h$ and Lemma 2

$$\begin{aligned}\frac{d^h}{dt^h}\mathcal{H}_{m,r}(t)&=\frac{d^h}{dt^h}e^{t\mathfrak{D}}yx^r=e^{t\mathfrak{D}}\mathfrak{D}^h yx^r=e^{t\mathfrak{D}}yx^r\mathcal{D}_{m,r}(h,x^m)\\&=yx^r\left(\sum_{n=0}^{\infty}\mathcal{D}_{m,r}(n,x^m)\frac{t^n}{n!}\right)\left(\mathcal{D}_{m,r}(h,x^m e^{mt})\right).\end{aligned}\tag{10}$$

Making the change $u=x^m$ and from the generating functions in Equations (4) and (10), we obtain

$$\sum_{n=0}^{\infty}\mathcal{D}_{m,r}(n+h,u)\frac{t^n}{n!}=\left(\sum_{n=0}^{\infty}\mathcal{D}_{m,r}(n,u)\frac{t^n}{n!}\right)\left(\mathcal{D}_{m,r}(h,ue^{mt})\right).\tag{11}$$

Expanding $\mathcal{D}_{m,r}(h,ue^{mt})$,

$$\begin{aligned}\mathcal{D}_{m,r}(h,ue^{mt})&=\sum_{j=0}^{h}W_{m,r}(h,j)u^j e^{mjt}\\&=\sum_{j=0}^{h}W_{m,r}(h,j)u^j\sum_{k=0}^{\infty}m^k j^k\frac{t^k}{k!}=\sum_{k=0}^{\infty}\left(\sum_{j=0}^{h}W_{m,r}(h,j)m^k u^j j^k\right)\frac{t^k}{k!}.\end{aligned}$$

By plugging it into Equation (11), computing the Cauchy product and equating coefficients, we obtain the result. □

The above identity was proved by using a different approach in [37]. Moreover, this identity is a particular case of the main result of [36].

From Theorem 5 we obtain the following convolution formula.

Corollary 1 *For* $0\leq s\leq n+h$*, we have*

$$W_{m,r}(n+h,s)=\sum_{k=0}^{n}\sum_{j=0}^{h}\binom{n}{k}W_{m,r}(h,j)W_{m,r}(k,s-j)(jm)^{n-k}.$$

In particular, if $r = 1$ we obtain the Theorem 4.3 of [13]. By setting $h = 1$ in Theorem 5 and Corollary 1 we obtain the Corollary 2.

Corollary 2 *The* r*-Dowling polynomials satisfy the following recursive formula*

$$\mathcal{D}_{m,r}(n+1,u) = r\mathcal{D}_{m,r}(n,u) + u\sum_{j=0}^{n}\binom{n}{j}m^{n-j}\mathcal{D}_{m,r}(j,u).$$

Therefore, the r*-Whitney numbers of the second kind satisfy the recursive formula*

$$W_{m,r}(n+1,k) = rW_{m,r}(n,k) + \sum_{j=k-1}^{n}\binom{n}{j}m^{n-j}W_{m,r}(j,k-1).$$

The above corollary was proved in [8] by using Riordan arrays.

Theorem 6 *The* r*-Dowling polynomials satisfy the following formula*

$$\mathcal{D}_{m,r}(n,u) = \sum_{j=0}^{n}\binom{n}{j}(r-s)^{n-j}D_{m,s}(j,u).$$

Proof. From Lemma 2 we have

$$\mathcal{H}_{m,r}(t) = e^{t\mathcal{D}}yx^r = e^{t\mathcal{D}}yx^s e^{t\mathcal{D}}x^{r-s} = x^{r-s}e^{(r-s)t}\mathcal{H}_{m,s}(t). \tag{12}$$

By the Cauchy product we obtain

$$\widehat{\mathcal{D}}_{m,r}(n,x,y) = x^{r-s}\sum_{j=0}^{n}\binom{n}{j}(r-s)^{n-j}\widehat{\mathcal{D}}_{m,s}(j,x,y).$$

Therefore, we get the desired result. □

In particular if $s = 1$ we have (see Theorem 5.2 of [8])

$$\mathcal{D}_{m,r}(n,x) = \sum_{j=0}^{n}(r-1)^{n-j}\binom{n}{j}D_m(j,x),$$

where $D_m(n,u)$ are the Dowling polynomials, i.e.,

$$D_m(n,u) = \sum_{k=0}^{n}W_m(n,k)u^k.$$

Corollary 3 *The* r*-Whitney numbers satisfy the following formula*

$$W_{m,r}(n,k) = \sum_{j=0}^{n} \binom{n}{j} (r-s)^{n-j} W_{m,s}(j,k). \tag{13}$$

In particular, if $s = 1$ *we have*

$$W_{m,r}(n,k) = \sum_{j=0}^{n} \binom{n}{j} (r-1)^{n-j} W_m(j,k). \tag{14}$$

Proof. From Theorem 6

$$\begin{aligned} \sum_{k=0}^{n} W_{m,r}(n,k) y x^{mk+r} &= x^{r-s} \sum_{j=0}^{r} \binom{n}{j} (r-s)^{n-j} \sum_{k=0}^{j} W_{m,s}(j,k) y x^{km+s} \\ &= \sum_{k=0}^{n} \sum_{j=0}^{n} \binom{n}{j} (r-s)^{n-j} W_{m,s}(j,k) y x^{km+r}. \end{aligned}$$

By equating coefficients, we obtain the result. □

5 The Sheffer family of $[m]$-Touchard polynomials and r-Dowling polynomials

From the generating function of the r-Dowling polynomials (Theorem 3), making $r = 0$, we get the generating function of the $[m]$-Touchard polynomials

$$\sum_{n=0}^{\infty} T_n^{[m]}(x) \frac{t^n}{n!} = \exp\left(x \frac{e^{mt} - 1}{m} \right). \tag{15}$$

From this is immediate that they are of binomial type ([32]). From Equation (9) we have the following identity relating them with the classical Dowling polynomials

$$\mathcal{D}_{m,1}(n,x) = \mathcal{D}_m(n,x) = \sum_{k=0}^{n} \binom{n}{k} T_k^{[m]}(x).$$

and hence,

$$T_n^{[m]}(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \mathcal{D}_m(k,x).$$

Our goal is to find the umbral inverse of [m]-Touchard sequence, i.e., the polynomial family $\widehat{T}_n^{[m]}(x) = \sum_{k=1}^{n} c_{n,k}x^k$ satisfying

$$T_n^{[m]}(\widehat{\mathbf{T}}^{[m]}(x)) := \sum_{k=1}^{n} S^{[m]}(n,k)\widehat{T}_k^{[m]}(x) = x^n, \ \forall n = 0,1,2\ldots.$$
$$\widehat{T}_n^{[m]}(\mathbf{T}^{[m]}(x)) := \sum_{k=1}^{n} c_{n,k}T_k^{[m]}(x) = x^n, \ \forall n = 0,1,2\ldots.$$

Let $\mathcal{O}(t) := (e^{mt}-1)/m$, and consider its compositional inverse

$$\overline{\mathcal{O}}(t) = \ln(1+mx)^{\frac{1}{m}}.$$

Let ∂_x be the derivative operator acting on the polynomial ring $\mathbb{C}[x]$. Let $\mathcal{O}(\partial_x)$ and $\overline{\mathcal{O}}(\partial_x)$ be the shift-invariant operators acting also on the polynomial ring $\mathbb{C}[x]$,

$$\mathcal{O}(\partial_x) = \frac{e^{m\partial_x}-I}{m} = \frac{1}{m}\sum_{k=1}^{\infty} m^k\frac{\partial_x^k}{k!} = \frac{E^m - I}{m},$$
$$\overline{\mathcal{O}}(\partial_x) = \ln(1+m\partial_x)^{\frac{1}{m}} = \frac{1}{m}\sum_{k=1}^{\infty}(-1)^k m^k(k-1)!\frac{\partial_x^k}{k!} = \sum_{k=1}^{\infty}(-1)^k m^{k-1}(k-1)!\frac{\partial_x^k}{k!}.$$

Here we denote by E^a the shift operator $E^a p(x) = p(x+a)$. By the classical theory of umbral calculus, $\widehat{T}_n^{[m]}(x)$ and $T_n^{[m]}(x)$ are the sequences associated to the operators $\frac{E^m-I}{m}$, and $\ln(1+m\partial_x)^{\frac{1}{m}}$, respectively. So

$$\begin{aligned}\frac{E^m-I}{m}\widehat{T}_n^{[m]}(x) = \frac{\widehat{T}_n^{[m]}(x+m)-\widehat{T}_n^{[m]}(x)}{m} &= n\widehat{T}_{n-1}^{[m]}(x)\\ \ln(1+m\partial_x)^{\frac{1}{m}}T_n^{[m]}(x) &= nT_{n-1}^{[m]}(x).\end{aligned}$$

The derivatives of the formal power series $\mathcal{O}(x)$ and $\overline{\mathcal{O}}(x)$ are $\mathcal{O}'(x) = e^{mx}$ and $(\overline{\mathcal{O}}(x))' = \frac{1}{1+mx}$, respectively. The recurrence formula for families of binomial type ([32] Corollary 1 of Theorem 8), gives us

$$\widehat{T}_n^{[m]}(x) = xE^{-m}\widehat{T}_{n-1}^{[m]}(x) = x\widehat{T}_{n-1}^{[m]}(x-m), \tag{16}$$
$$T_n^{[m]}(x) = x(1+m\partial_x)T_{n-1}^{[m]}(x) = xT_{n-1}^{[m]}(x) + mx\partial_x T_{n-1}^{[m]}(x). \tag{17}$$

From Equation (16) we obtain the polynomial family

$$\widehat{T}_n^{[m]}(x) = x(x-m)(x-2m)\cdots(x-(n-1)m). \tag{18}$$

From Equation (16) we also have the recurrence for $[m]$-Stirling numbers of the second kind

$$S^{[m]}(n,k) = S^{[m]}(n-1,k-1) + kmS^{[m]}(n-1,k). \tag{19}$$

We define the $[m]$-Stirling numbers of the first kind as the coefficients connecting $\widehat{T}_n^{[m]}(x)$ with the powers.

Definition 3 *We define the* $[m]$*-Stirling numbers of the first kind* $s^{[m]}(n,k)$ *as the coefficients in the expansion of* $\widehat{T}_n^{[m]}(x) = x(x-m)(x-2m)\cdots(x-(n-1)m)$ *in terms of the power sequence, that is,*

$$x(x-m)(x-2m)\cdots(x-(n-1)m) = \sum_{k=1}^{n} s^{[m]}(n,k)x^k.$$

Remark 2 *The* $[m]$*-Stirling numbers of the first kind have the following combinatorial interpretation in terms of the Möbius function. Considering* $Q_{n,0}(G)$*, the subposet of* $Q_{n,r}$ *of elements of the form* $(\pi^f, (\emptyset, \emptyset, \ldots, \emptyset))$*,* $s^{[m]}(n,k)$ *is the sum of its Möbius function over the elements of rank* $n-k$*. The proof of this fact is completely analogous of that of Theorem 2.*

Remark 3 *From the generating function in Theorem 3,*

$$\sum_{n=0}^{\infty} \mathcal{D}_{m,r}(n,x)\frac{t^n}{n!} = e^{rx}\exp\left(x\frac{e^{mt}-1}{m}\right),$$

we have that the r*-Dowling polynomials,* $\mathcal{D}_n^{[m,r]}(x) := \mathcal{D}_{m,r}(n,x)$ *are a Sheffer family relative to the* $[m]$*-Touchard, associated to the pair of generating functions* $\left(\frac{1}{(1+mx)^{r/m}}, \ln(1+mx)^{1/m}\right)$*,*

$$\left\langle \frac{1}{(1+mx)^{r/m}}, \ln(1+mx)^{1/m} \right\rangle = \left\langle e^{rx}, \frac{e^{mx}-1}{m} \right\rangle^{-1},$$

being the inverse of $\left\langle e^{rx}, \frac{e^{mx}-1}{m}\right\rangle$ *as an exponential Riordan array, see Equation (23) bellow. For that, see [32], and [31] Theorem 2.3.4. Hence, we have the binomial identity (see [31], Theorem 2.3.9)*

$$\mathcal{D}_n^{[m,r]}(x+y) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{D}_k^{[m,r]}(x)T_{n-k}^{[m]}(y). \tag{20}$$

Its umbral inverse $\widehat{\mathcal{D}}_n^{[m,r]}(x)$ is Sheffer relative to $\widehat{T}_n^{[m]}(x)$. It is associated to the pair $\left(e^{rx}, \frac{e^{mx}-1}{m}\right)$.

From that we get next theorem

Theorem 7 *The umbral inverse of the* r*-Dowling Sheffer sequence is equal to*

$$\widehat{\mathcal{D}}_n^{[m,r]}(x) = E^{-r}\widehat{T}_n^{[m]}(x) = (x-r)(x-r-m)\cdots(x-r-(k-1)m). \qquad (21)$$

The r*-Dowling sequence satisfies the identity*

$$\mathcal{D}_n^{[m,r]}(x) = \sum_{k=0}^{n} \frac{r(r-m)(r-2m)\cdots(r-(k-1)m)}{k!}\partial_x^k T_n^{[m]}(x). \qquad (22)$$

Proof. Since $\widehat{\mathcal{D}}_n^{[m,r]}(x)$ is associated to $(e^{rx}, \frac{e^{mx}-1}{m})$, it is equal to the inverse of the operator $e^{r\partial_x} = E^r$ applied to $\widehat{T}_n^{[m]}(x)$ (see [31], Theorem 2.3.6). For the same reason we have that $\mathcal{D}_n^{[m,r]}(x) = (1+m\partial_x)^{r/m}T_n^{[m]}(x)$. Expanding the operator $(1+m\partial_x)^{r/m}$ by the binomial formula we obtain the result. □

6 Some applications from Riordan arrays

The r-Whitney numbers can be defined using exponential Riordan arrays. An infinite lower triangular matrix is called a *Riordan array* [33] if its kth column satisfies the generating function $g(z)(f(z))^k$ for $k \geq 0$, where $g(z)$ and $f(z)$ are formal power series with $g(0) \neq 0$, $f(0) = 0$ and $f'(0) \neq 0$. The matrix corresponding to the pair $f(z), g(z)$ is denoted by $(g(z), f(z))$. The product of two Riordan arrays $(g(z), f(z))$ and $(h(z), l(z))$ is defined by

$$(g(z), f(z)) * (h(z), l(z)) = (g(z)h(f(z)), l(f(z))).$$

The set of the Riordan matrices is a group under the operator "$*$" [33]. The identity element is $I = (1, z)$, and the inverse of $(g(z), f(z))$ is

$$(g(z), f(z))^{-1} = \left(1/\left(g\circ\bar{f}\right)(z), \bar{f}(z)\right), \qquad (23)$$

Sometimes, it is useful to use exponential generating functions instead of ordinary generating functions when we apply a Riordan array method. We call the resulting array an exponential Riordan array and we denote it by $\langle g(z), f(z)\rangle$. Its column k has generating function $g(z)(f(z))^k/k!, k = 0, 1, 2, \ldots$ (cf. [35]).

The r-Whitney numbers of the second kind are given by the exponential Riordan array:

$$\mathcal{W}_2 := [W_{m,r}(n,k)]_{n,k\geq 0} = \left\langle e^{rx}, \frac{e^{mx}-1}{m} \right\rangle.$$

Therefore, as we have already seen in Remark 3, the r-Dowling polynomial $\mathcal{D}_n^{[m,r]}(x)$ is a Sheffer sequence for

$$\left((1+mx)^{-r/m}, \ln(1+mx)^{1/m}\right).$$

In this section we give some explicit relations between the r-Dowling polynomials and the Bernoulli and Euler polynomials by using the connection constants $a_{n,k}$ in the expression $r_n(x) = \sum_{k=0}^{n} a_{n,k} s_k(x)$, where the polynomials $r_n(x)$ and $s_n(x)$ are Sheffer sequences. These constants can be determined by the umbral method [31, pp. 131] or equivalent by Riordan arrays [35, Theorem 6.4]. In particular, let $s_n(x)$ and $r_n(x)$ be Sheffer for $(g(t), f(t))$ and $(h(t), l(t))$, respectively. If $r_n(x) = \sum_{k=0}^{n} a_{n,k} s_k(x)$, then $a_{n,k}$ is the entry (n,k)-th of the Riordan array

$$\left(\frac{g(\bar{l}(t))}{h(\bar{l}(t))}, f(\bar{l}(t))\right). \tag{24}$$

The Bernoulli polynomials, $\mathcal{B}_n(x)$, are defined by the exponential generating function

$$\sum_{n=0}^{\infty} \mathcal{B}_n(x)\frac{t^n}{n!} = \frac{te^{xt}}{e^t-1}.$$

The Bernoulli numbers, $\mathcal{B}_n$, are define by $\mathcal{B}_n := \mathcal{B}_n(0)$. Moreover, the polynomials $\mathcal{B}_n(x)$ are Sheffer for $\left(\frac{e^t-1}{t}, t\right)$ (cf. [31, 14]). The r-Whitney numbers of the first kind are defined by exponential generating function [24]:

$$\sum_{n=k}^{\infty} w_{m,r}(n,k)\frac{z^n}{n!} = (1+mz)^{-\frac{r}{m}} \frac{\ln^k(1+mz)}{m^k k!}. \tag{25}$$

Theorem 8 *For $n \geq 0$ we have*

$$\mathcal{B}_n(x) = \sum_{k=0}^{n}\sum_{\ell=k}^{n} \binom{n}{\ell} \mathcal{B}_{n-\ell} w_{m,r}(\ell,k) \mathcal{D}_k^{[m,r]}(x). \tag{26}$$

Proof. If $\mathcal{B}_n(x) = \sum_{k=0}^{n} a_{n,k}\mathcal{D}_k^{[m,r]}(x)$, then from (24) and (25) we get

$$\begin{aligned} a_{n,k} &= \frac{1}{k!}[t^n]\left((1+mt)^{-r/m}\frac{t}{e^t-1}\cdot\left(\ln(1+mt)^{1/m}\right)^k\right) \\ &= [t^n]\left((1+mt)^{-r/m}\frac{\ln^k(1+mt)}{m^k k!}\frac{t}{e^t-1}\right) \\ &= [t^n]\left(\left(\sum_{n=k}^{\infty} w_{m,r}(n,k)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathcal{B}_n(x)\frac{t^n}{n!}\right)\right) \\ &= [t^n]\left(\sum_{n=0}^{\infty}\sum_{\ell=0}^{n}\binom{n}{\ell}\mathcal{B}_{n-\ell}w_{m,r}(\ell,k)\right)\frac{z^n}{n!}. \end{aligned}$$

Therefore, it is clear (26). □

The Euler polynomials, $\mathcal{E}_n(x)$, are defined by the exponential generating function

$$\sum_{n=0}^{\infty}\mathcal{E}_n(x)\frac{t^n}{n!} = \frac{2e^{xt}}{e^t+1}.$$

The numbers $\mathcal{E}_n$, are define by $\mathcal{E}_n := \mathcal{E}_n(0)$. Moreover, the polynomials $\mathcal{E}_n(x)$ are Sheffer for $\left(\frac{e^t+1}{2}, t\right)$.

From a similar argument as in above theorem we get the following theorem.

Theorem 9 *For* $n \geq 0$ *we have*

$$\mathcal{E}_n(x) = \sum_{k=0}^{n}\sum_{\ell=k}^{n}\binom{n}{\ell}\mathcal{E}_{n-\ell}w_{m,r}(\ell,k)\mathcal{D}_k^{[m,r]}(x).$$

In the following theorem we analyze the connecting coefficients

$$\mathcal{D}_n^{[m,r]}(x) = \sum_{k=0}^{n} a_{n,k}\mathcal{B}_k(x).$$

Theorem 10 *For* $n \geq 0$ *we have*

$$\mathcal{D}_n^{[m,r]}(x) = \frac{1}{n+1}\sum_{k=0}^{n}\sum_{\ell=0}^{n-k}\sum_{s=0}^{\ell}\binom{n+1}{\ell+1}\binom{\ell+1}{s+1}W_{m,r}(n-\ell,k)m^{\ell-s}T_{s+1}^{[m]}(1)\mathcal{B}_{\ell-s}\mathcal{B}_k(x). \quad (27)$$

Proof. From Equations (24), (1) and (15) we get

$$\begin{aligned}
a_{n,k} &= \frac{1}{k!}[t^n]\left(\frac{e^{\frac{e^m-1}{m}-1}}{\frac{e^{mt}-1}{m}}e^{rt}\left(\frac{e^{mt}-1}{m}\right)^k\right)\\
&= [t^n]\left(\frac{e^{rt}}{k!}\left(\frac{e^{mt}-1}{m}\right)^k \frac{e^{\frac{e^m-1}{m}-1}}{t}\frac{mt}{e^{mt-1}}\right)\\
&= [t^n]\left(\left(\sum_{n=k}^{\infty} W_{m,r}(n,k)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} T_n^{[m]}(1)\frac{t^n}{(n+1)!}\right)\left(\sum_{n=0}^{\infty} m^n\mathcal{B}_n\frac{t^n}{n!}\right)\right)\\
&= [t^n]\left(\left(\sum_{n=k}^{\infty} W_{m,r}(n,k)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\sum_{s=0}^{\ell}\binom{\ell}{s}m^{\ell-s}\mathcal{B}_{\ell-s}\frac{T_{s+1}^{[m]}(1)}{s+1}\frac{t^n}{n!}\right)\right)\\
&= [t^n]\left(\sum_{n=0}^{\infty}\left(\sum_{\ell=0}^{n-k}\sum_{s=0}^{\ell}\binom{n}{\ell}\binom{\ell}{s}W_{m,r}(n-\ell,k)m^{\ell-s}\mathcal{B}_{\ell-s}\frac{T_{s+1}^{[m]}(1)}{s+1}\right)\frac{t^n}{n!}\right)\\
&= [t^n]\left(\frac{1}{n+1}\sum_{n=0}^{\infty}\left(\sum_{\ell=0}^{n-k}\sum_{s=0}^{\ell}\binom{n+1}{\ell+1}\binom{\ell+1}{s+1}\right.\right.\\
&\qquad \left.\left. W_{m,r}(n-\ell,k)m^{\ell-s}\mathcal{B}_{\ell-s}T_{s+1}^{[m]}(1)\right)\frac{t^n}{n!}\right)
\end{aligned}$$

Therefore, it is clear (26). □

From a similar argument we get the following theorem.

Theorem 11 *For $n \geq 0$ we have*

$$\mathcal{D}_n^{[m,r]}(x) = \sum_{k=0}^{n}\left(\frac{1}{2}\sum_{\ell=0}^{n-k}\binom{n}{\ell}W_{m,r}(n-\ell,k)T_\ell^{[m]}(1) + \frac{1}{2}W_{m,r}(n,k)\right)\mathcal{E}_k(x).$$

6.1 Some recurrence relations

The entries in a Riordan array can be expressed as linear combination of the elements in the preceding row. That is, if $d_{n+1,k+1}$ is the $(n+1,k+1)$-th entry in a Riordan array, then there is a sequence $A=(a_n)_{n\geq 0}$ such that

$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \cdots, \quad a_0 \neq 0$$

The sequence A is called the A-sequence [30]. Additionally, any element in column 0, except the element $d_{0,0}$, can be expressed as

$$d_{n+1,0} = z_0 d_{n,0} + z_1 d_{n,1} + z_2 d_{n,2} + \cdots$$

The sequence $Z = (z_n)_{n\geq 0}$ is called the Z-sequence [22]. In general, the A and Z sequences, and the element $d_{0,0}$ completely characterize a proper Riordan array.

The above conditions can be write in terms of generating function. In particular, a triangular array $\mathcal{D} = [d_{n,k}]_{n,k\in\mathbb{N}}$ is a Riordan array if and only if

$$g(z) = \frac{g(0)}{1 - zZ(g(z))} \quad \text{and} \quad f(z) = z(A(f(z))),$$

where A and Z are the generating functions of the A-sequence and Z-sequence, respectively.

The generating function for the A-sequence of the exponential Riordan array of the r-Whitney numbers is given by

$$\frac{t}{\bar{f}(t)} = \frac{t}{\ln(1+mt)^{1/m}} = \frac{mt}{\ln(1+mt)} = \sum_{k=0}^{\infty} c_k m^k \frac{t^k}{k!},$$

where c_k are the Cauchy numbers of first kind. They are defined by $c_n = \int_0^1 x^{\underline{n}} dx$. See [23] for general information about Cauchy numbers. Taking in count that any exponential Riordan array $\langle g(x), f(x)\rangle = (d_{n,k})_{n,k\geq 0}$ satisfies the recurrence relations (see [35, Corollary 5.7])

$$d_{n+1,k+1} = \sum_{j=0}^{\infty} \frac{n+1}{k+1}\binom{k+j}{j} j!a_j d_{n,k+j},$$

$$d_{n,k} - \tilde{d}_{n-1,k} = \sum_{\ell=k}^{n} \binom{n-1}{\ell-1} f_{n-\ell+1} d_{\ell-1,k-1},$$

$$kd_{n,k} = \sum_{\ell=k}^{n} \binom{n}{\ell-1} f_{n-\ell+1} d_{\ell-1,k-1},$$

where (a_j) is the A-sequence and $(\tilde{d}_{n,k})_{n,k\geq 0} = \langle g'(x), f(x)\rangle$. Then we obtain the following corollary.

Corollary 4 *The r-Whitney numbers of the second kind satisfy the following recurrence relations*

$$W_{m,r}(n+1,k+1) = \sum_{j=0}^{\infty} \frac{n+1}{k+1}\binom{k+j}{j} c_j m^j W_{m,r}(n,k+j),$$

$$W_{m,r}(n,k) - rW_{m,r}(n-1,k) = \sum_{\ell=k}^{n} \binom{n-1}{\ell-1} m^{n-\ell} W_{m,r}(\ell-1,k-1), \ n \geq 1,$$

$$kW_{m,r}(n,k) = \sum_{\ell=k}^{n} \binom{n}{\ell-1} m^{n-\ell} W_{m,r}(\ell-1, k-1).$$

From the generating function (25) we obtain that the r-Whitney numbers of the first kind are given by the exponential Riordan array

$$W_1 := [w_{m,r}(n,k)] = \left\langle (1+mz)^{-r/m}, \ln(1+mz)^{1/m} \right\rangle.$$

In this case, the generating function for the A-sequence of the exponential Riordan array W_1 is given by

$$\frac{t}{\bar{f}(t)} = \frac{t}{\frac{e^{mt}-1}{m}} = \frac{mt}{e^{mt}-1} = \sum_{k=0}^{\infty} \mathcal{B}_k m^k \frac{t^k}{k!},$$

where $\mathcal{B}_n$ are the Bernoulli numbers.

Therefore we obtain the following corollary.

Corollary 5 *The* r*-Whitney numbers of the first kind satisfy the following recurrence relations*

$$w_{m,r}(n+1,k+1) = \sum_{j=0}^{\infty} \frac{n+1}{k+1} \binom{k+j}{j} \mathcal{B}_j m^j w_{m,r}(n,k+j),$$

$$w_{m,r}(n,k) + r\sum_{\ell=0}^{n-1} \binom{n-1}{\ell} (n-\ell-1)! w_{m,r}(\ell,k)(-m)^{n-\ell-1}$$
$$= \sum_{\ell=k}^{n} \binom{n-1}{\ell-1} (-m)^{n-\ell}(n-\ell)! w_{m,r}(\ell-1,k-1), \;\; n \geq 1,$$

$$kw_{m,r}(n,k) = \sum_{\ell=k}^{n} \binom{n}{\ell-1} (-m)^{n-\ell}(n-\ell)! w_{m,r}(\ell-1,k-1).$$

6.2 Determinantal identity

The r-Whitney numbers of both kinds satisfy the following orthogonality relation (cf. [24]):

$$\sum_{i=s}^{n} W_{m,r}(n,i) w_{m,r}(i,s) = \sum_{i=s}^{n} w_{m,r}(n,i) W_{m,r}(i,s) = \delta_{s,n},$$

where $\delta_{s,n} = 1$ if $s = n$ and 0, otherwise. From above relations we obtain the inverse relation:

$$f_n = \sum_{s=0}^{n} w_{m,r}(n,s)g_s \iff g_n = \sum_{s=0}^{n} W_{m,r}(n,s)f_s.$$

Moreover, we have the identity $W_1 = W_2^{-1}$, where W_2 is the exponential Riordan array for the r-Whitney numbers of the second kind.

From definition of the r-Dowling polynomials we obtain the equality $W_2 \cdot X = D_{m,r}$, where $X = [1, x, x^2, \ldots]^T$ and $D_{m,r} = [\mathcal{D}_0^{[m,r]}(x), \mathcal{D}_1^{[m,r]}(x), \mathcal{D}_2^{[m,r]}(x), \ldots]^T$. Then $X = W_1 D_{m,r}$ and

$$x^n = \sum_{k=0}^{n} w_{m,r}(n,k)\mathcal{D}_k^{[m,r]}(x).$$

Therefore

$$\mathcal{D}_n^{[m,r]}(x) = x^n - \sum_{k=0}^{n-1} w_{m,r}(n,k)\mathcal{D}_k^{[m,r]}(x), \quad n \geq 0. \tag{28}$$

From the above equation we obtain the following determinantal identity.

Theorem 12 *The r-Dowling polynomials polynomials satisfy*

$$\mathcal{D}_n^{[m,r]}(x) = (-1)^n \begin{vmatrix} 1 & x & \cdots & x^{n-1} & x^n \\ 1 & w_{m,r}(1,0) & \cdots & w_{m,r}(n-1,0) & w_{m,r}(n,0) \\ 0 & 1 & \cdots & w_{m,r}(n-1,1) & w_{m,r}(n,1) \\ \vdots & & \cdots & & \vdots \\ 0 & 0 & \cdots & 1 & w_{m,r}(n,n-1) \end{vmatrix}$$

Proof. This identity follows from Equation (28) and by expanding the determinant by the last column. □

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On the Elzaki transform and its applications in fractional free electron laser equation

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Abstract. This article is devoted to study Elzaki transform and its applications in Free Electron Laser equation involving Hilfer-Prabhakar fractional derivative. We derive formula of Elzaki transform for Hilfer–Prabhakar derivative and its regularized version. The solution of Free Electron Laser equation involving Hilfer-Prabhakar fractional derivative of fractional order is presented in terms of Mittag-Leffler type function. Furthermore, we find the application of the generalized Hilfer-Prabhakar derivative in linear partial differential equation and some problems of Mathematical Physics.

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1 Introduction

In past, few years fractional calculus (integration and differentiation of arbitrary order) have received considerable attention to solve the mathematics, engineering and mathematical physics problems [5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20, 21]. Hilfer-Prabhakar operator is a powerful tool to generalize the linear and non-linear fractional order differential equations. Prabhakar integral in generalized form of earlier versions and plays a key role to examine the Hilfer-Prabhakar fractional derivative problems using integral transform method.

In the literature of fractional differentiations and integrations there are several integral transforms like Laplace, Fourier, Mellin, Sumudu to name but a few. A new integral transform namely Elzaki transform [2] which is a modified form of classical Laplace and Sumudu transform and has some good features. Elzaki transform has been efficiently used to solve the integral equations and differential equations in fractional calculus. Using of Elzaki transform is also beneficial as it can solve a class of differential equation which are not solved by Sumudu transform.

The main objective of this article is to introduce the formulae for Elzaki transform and apply it to solve Hilfer-Prabhakar fractional derivative and its regularized version. Taking inspiration from these works our purpose is to find the solution of Cauchy problems of fractional order by using Elzaki transform. Furthermore, we discuss here the importance of Elzaki transform associated with Hilfer-Prabhakar fractional derivatives to solve free electron laser type integro-differential equation. The motivation of this paper is to encourage further investigation of the potential applications of this branch of mathematics.

2 Basic definition

In this section, we study some important basic definition related to fractional calculus and Elzaki transform to understand the further results, lemmas and applications.

Definition 1 *Elzaki transform [2, 3] of function* $g(t)$ *introduced by Tarig M. Elzaki is defined as*

$$p\int_0^\infty e^{-\frac{t}{p}} g\,(x,\ t)\,dt = \mathbf{E}\,[g\,(x,\ t)] = T(x,p\ ), t>0,\ \ p\in(-\tau_1,\tau_2). \quad (1)$$

Elzaki transform of first order partial derivative is defined as

$$\mathbf{E}[\frac{\partial g}{\partial t}(x,t)] = \frac{T(x, \quad p)}{p} \quad - \quad p.g\,(x, \;0)\,. \tag{2}$$

Now, we find relation between Laplace and Elzaki transform,

Let $f(t) \in A = \{f(t) \;\; : \exists\; M, \;\tau_1, \;\tau_2 > \;\; 0$ *such that* $|f(t)| < M\; e^{|t|/\tau_i}$ *if* $t \in (-i)^j \times [0, \;\infty)\}$ *and* $F(s)$ *be the Laplace transform, the Elzaki transform* $T(u)$ *is given by*

$$T\,(u) = uF\left(\frac{1}{u}\right).$$

Also, the Laplace transform and the Elzaki transform [4] *must coincide at* $u = 1$ *and we can write* $T\,(1) = F\,(1)\,.$

Definition 2 (Hilfer derivative). *Let* $\mu \in \;\; (0,1)\,,\nu \;\in\; [0,1]$, $g \;\in\; L^1\,[a,b]\,,$ $-\infty \leq a < t < b \leq \infty$, $f * K_{(1-\nu)(1-\mu)} \in AC^1\,[a,b]\,.$ *Then*

$$D_{a^+}^{\mu,\;\nu} g\,(t) = \left(I_{a^+}^{\nu(1-\;\;\mu)} \frac{d}{dt} I_{a^+}^{(1-\nu)(1-\mu)} g\right)(t) \tag{3}$$

is the Hilfer fractional derivative [7] *of order* μ.

Definition 3 *Regularized of Hilfer derivative [7] of same order* μ *is written as*

$$D_{0^+}^{\mu,\;\nu} g\,(t) - \frac{t^{-\mu} g\,(0^+)}{\Gamma(1-\mu)} =^{C} D_{0+}^{\mu} g(t).$$

Definition 4 *Prabhakar introduced the generalized Mittag-Leffler function* [8, 14] *in the following form*

$$E_{\alpha,\beta}^{\gamma}\,(z) = \sum_{k=0}^{\infty} \frac{\Gamma\,(\gamma{+}k)}{\Gamma\,(\gamma)\,\Gamma\,(\alpha k{+}\beta)} \frac{z^k}{k!}\;, \tag{4}$$

where $\alpha, \beta, \gamma \;\in\; \mathbb{C}$ and $\mathbb{R}(\alpha) > 0.$

The generalization of Hilfer derivative with a more general integral operator with kernel [5]

$$t^{\mu-1} E_{\rho,\mu}^{\gamma}\,[\omega t^{\rho}] = \; e_{\rho,\mu,\omega}^{\gamma}\,(t)\,, \;\; t \in \mathbb{R}, \rho, \mu, \omega, \gamma \in \mathbb{C}, \;\; \text{with} \;\; \mathbb{R}\,(\mu)\,, \mathbb{R}\,(\rho) > 0. \tag{5}$$

The well known Prabhakar integral is expressed in the similar way, replacing kernel by function and is defined as follows [5].

Definition 5 (Prabhakar integral) *Let* $L^1[0,b], 0 < t < b \leq \infty$. *The Prabhakar integral is written as*

$$E^{\gamma}_{\rho,\mu,\omega,0^+} f(t) = \int_0^t (t-y)^{\mu-1} E^{\gamma}_{\rho,\mu}[\omega(t-y)^{\rho}] f(y)\, dy = \left(f * e^{\gamma}_{\rho,\mu,\omega}\right)(t), \quad (6)$$

where $\rho, \mu, \omega, \gamma \in \mathbb{C}$ with $\mathbb{R}(\rho), \mathbb{R}(\mu) > 0$.

Definition 6 (Hilfer-Prabhakar derivative) *The Hilfer-Prabhakar derivative of* $g(t)$ *of order* μ *denoted by* $D^{\gamma,\mu,\nu}_{\rho,\omega,0^+} g(t)$ *is defined as*

$$D^{\gamma,\mu,\nu}_{\rho,\omega,0^+} g(t) = \left(E^{-\gamma\nu}_{\rho,\nu(1-\mu),\omega,0^+} \frac{d}{dt}\left(E^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0^+} g\right)\right)(t), \quad (7)$$

where $\mu \in (0,1)$, $\nu \in [0,1]$, *and* $\gamma, \omega \in \mathbb{R}$, $\rho > 0$, *and* $E^{0}_{\rho,0,\omega,0^+} g = g$.

Definition 7 (Regularized version of Hilfer-Prabhakar derivative) *For* $g \in AC^1[0,b]$, $0 < t < b < \infty$, $\mu \in (0,1)$, $\nu \in [0,1]$, $\omega, \gamma \in \mathbb{R}$, $\rho > 0$, *the regularized version of Hilfer-Prabhakar fractional derivative of* $f(t)$ *denoted by* ${}^{c}\mathbf{D}^{\gamma,\mu}_{\rho,\omega,0^+} f(t)$ *is defined as*

$${}^{c}\mathbf{D}^{\gamma,\mu}_{\rho,\omega,0^+} g(t) = \left(E^{-\gamma\nu}_{\rho,\nu(1-\mu),\omega,0^+} E^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0^+} \frac{d}{dt} g\right)(t), \quad (8)$$

also,

$$E^{\gamma}_{\rho,\mu,\omega,0^+} E^{\sigma}_{\rho,\nu,\omega,0^+} g(x) = E^{\gamma+\sigma}_{\rho,\mu+\nu,\omega,0^+} g(x).$$

3 Elzaki transform of fractional derivatives

In this section we introduce formula of Elzaki transform of Prabhakar integral to study Elzaki transform of Hilfer-Prabhakar fractional derivative and its regularized version.

Lemma 1 *The Elzaki transform of Prabhakar integral is given by*

$$E[e^{\gamma}_{\rho,\mu,\omega}(t)] = p^{\mu+1}[1-\omega p^{\rho}]^{-\gamma}, \quad (9)$$

for $\mu \in (0.1)$, ω, $\gamma \in \mathbb{R}$ and $\rho > 0$.

Proof. The equation (4) can be written as

$$E^{\gamma}_{\rho,\ \mu}(\omega t^{\rho}) = \sum_{k=0}^{\infty} \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)\Gamma(\rho k+\mu)} \frac{(\omega t^{\rho})^k}{k!}. \tag{10}$$

Taking Elzaki transform of Prabhakar integral using (5) and (10), we have
$\mathbf{E}\ [e^{\gamma}_{\rho,\mu,\omega}(t)] = \mathbf{E}[\sum_{k=0}^{\infty} t^{\mu-1} \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)\Gamma(\rho k+\mu)} \frac{(\omega t^{\rho})^k}{k!}]$
$= \mathbf{E}[\sum_{k=0}^{\infty} t^{\rho k+\mu-1} \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)\Gamma(\rho k+\mu)} \frac{\omega^k}{k!}]$

$$= p^{\mu+1} \sum_{k=0}^{\infty} \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)} \frac{(\omega p^{\rho})^k}{k!}$$

$$= p^{\mu+1}[1-\omega p^{\rho}]^{-\gamma}.$$

Which is the Elzaki transform formula of Prabhakar integral. We use this result to solve Free Electron Laser equation involving Prabhakar integral and Hilfer-Prabhakar fractional derivative. Even this formula is useful to solve problems associated with Hilfer-Prabhakar fractional derivative and its regularized version in engineering and mathematics. □

Lemma 2 *The Elzaki transform of Hilfer-Prabhakar derivative of fractional order (7) is given by*
$\boldsymbol{E}\left(E^{-\gamma\nu}_{\rho,\nu(1-\mu),\omega,0^+} \frac{d}{dt}\left(E^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0^+} g\right)\right)(p)$

$$= p^{-\mu}[1-\omega p^{\rho}]^{\gamma}\mathbf{E}\,[g]\,(p) - p^{\nu(1-\mu)+1}[1-\omega p^{\rho}]^{\gamma\nu}\left[E^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0^+} g\,(t)\right]_{t=0^+}. \tag{11}$$

Proof. Elzaki transform of Hilfer-Prabhakar fractional derivative using (5), (7) and convolution theorem for Elzaki transform, we have

$$\begin{aligned}
\mathbf{E}(D^{\gamma,\mu,\nu}_{\rho,\omega,0^+} g\,(t))(\,p) &= \frac{1}{p}\cdot\mathbf{E}\left[t^{\nu(1-\mu)-1}E^{-\gamma\nu}_{\rho,\nu(1-\mu)}\,(\omega t^{\rho})\right](p)\\
&\quad\cdot\mathbf{E}\left[\frac{d}{dt}\left(E^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0^+} g\right)\right](p)\\
&= p^{\nu(1-\mu)}(1-\omega p^{\rho})^{\gamma\nu}\cdot\mathbf{E}\left[\frac{d}{dt}\left(E^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0^+} g\right)\right](p)\\
&= p^{\nu(1-\mu)}(1-\omega p^{\rho})^{\gamma\nu}\left[p^{\nu\mu-\nu-\mu}(1-\omega p^{\rho})^{\gamma(1-\nu)}\mathbf{E}\,[g]\,(p)\right.
\end{aligned}$$

$$-p\left(E^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0^+} g\right)_{t=0^+}\Big] .$$

On simplification, we get the required result (11). □

Lemma 3 *The Elzaki transform of regularized version of Hilfer-Prabhakar fractional derivative* (8) *of order* μ *is expressed as*

$$\mathbf{E}[{}^{C}D^{\gamma,\mu}_{\rho,\omega,0^+} g]\,(p) = p^{-\mu}(1-\omega p^{\rho})^{\gamma}\mathbf{E}[g](p) - p^{2-\mu}[1-\omega p^{\rho}]^{\gamma} g\left(0^+\right). \tag{12}$$

Proof. Elzaki transform of regularized version of Hilfer-Prabhakar fractional derivative of order μ using (5) and (8), we have

$$\begin{aligned}\mathbf{E}[{}^{C}D^{\gamma,\mu}_{\rho,\omega,0^+} g]\,(p) &= \mathbf{E}\left(E^{-\gamma\nu}_{\rho,\nu(1-\mu),\omega,0^+} E^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0^+} \frac{d}{dt} g\right)(p) \\ &= \mathbf{E}\left(E^{-\gamma}_{\rho,(1-\mu),\omega,0^+} \frac{d}{dt} g\right)(p) \\ &= \frac{1}{p}.\mathbf{E}\left[t^{(1-\mu)-1} E^{-\gamma}_{\rho,(1-\mu)}\left(\omega t^{\rho}\right)\right](p)\cdot \mathbf{E}\left[\frac{d}{dt} g\right](p) \\ &= p^{-\mu}(1-\omega p^{\rho})^{\gamma}\mathbf{E}\,[g]\,(p) - p^{2-\mu}[1-\omega p^{\rho}]^{\gamma} g\left(0^+\right).\end{aligned}$$

□

4 Applications

In this section, we discuss generalized fractional Free Electron Laser (FEL) equation which converge to FEL integro-differential equation,

$$\begin{cases} \frac{dy}{dx} = -i\pi g \int_0^x (x-t)\, e^{-i\eta(x-t)} y\,(t)\,, g,\ \eta \in \mathbb{R},\ x \in (0,\ 1]\,, \\ y\,(0) = 1. \end{cases} \tag{13}$$

Theorem 1 *The solution of generalization the Free Electron Laser equation involving Hilfer-Prabhakar derivative* [1]

$$\begin{cases} D^{\gamma,\mu,\nu}_{\rho,\omega,0^+} y\,(x) = \gamma E^{\beta}_{\rho,\mu,\omega,0^+} y\,(x) + g\,(x)\,, x \in [0,\infty)\,, \\ [E^{\gamma(\nu-1)}_{\rho,(1-\nu)(1-\mu),\omega,0^+} y\,(x)]_{t=0} = c,\ c \geq 0, \\ \lim_{x\to\pm\infty} u\,(x,\ t) = 0, \end{cases} \tag{14}$$

with $\mu \in (0,\ 1)$, $\nu \in [0,\ 1]$, $\omega \in \mathbb{R}$, $\rho > 0$, $\gamma, \beta \geq 0$, *is given by*

$$y(p) = \sum_{n=0}^{\infty} [c.(\gamma)^n t^{\nu(1-\mu)+\mu+2\mu n-1} E^{(\beta+\gamma)n-\gamma(\nu-1)}_{\rho,\nu(1-\mu)+\mu(2n+1)} (\omega t^\rho) \\ + (\gamma)^n t^{\mu(2n+1)-1} E^{\gamma(n+1)+\beta n}_{\rho,\mu(2n+1)} (\omega t^\rho).g].$$

Proof. Taking Elzaki transform of (14) using (11), we have

$$p^{-\mu}(1-\omega p^\rho)^\gamma \mathbf{E}[y](p) - p^{\nu(1-\mu)+1}(1-\omega p^\rho)^{\gamma\nu}.c \\ = \gamma.\ p^\mu (1-\omega p^\rho)^{-\beta} \mathbf{E}[y](p) + \ \mathbf{E}[g](p)$$

$$\begin{aligned} \mathbf{E}[y](p) &= \frac{p^{\nu(1-\mu)+1}(1-\omega p^\rho)^{\gamma\nu}.c + \mathbf{E}[g](p)}{p^{-\mu}(1-\omega p^\rho)^\gamma - \gamma.\ p^\mu(1-\omega p^\rho)^{-\beta}} \\ &= \frac{p^{\nu(1-\mu)+\mu+1}(1-\omega p^\rho)^{\gamma\nu-\gamma}.c + p^\mu(1-\omega p^\rho)^{-\gamma}\mathbf{E}[g](p)}{1-\gamma.\ p^{2\mu}(1-\omega p^\rho)^{-(\beta+\gamma)}} \\ &= \sum_{n=0}^{\infty} \gamma^n c.p^{\nu(1-\mu)+\mu+2\mu n+1}(1-\omega p^\rho)^{\gamma(\nu-1)-(\beta+\gamma)n} \\ &\quad + \sum_{n=0}^{\infty} \gamma^n p^{\mu+2\mu n}(1-\omega p^\rho)^{-[\gamma+(\beta+\gamma)n]}\mathbf{E}[g], \end{aligned}$$

inverting Elzaki transform, we have

$$y(p) = \sum_{n=0}^{\infty} [c.\gamma^n t^{\nu(1-\mu)+\mu+2\mu n-1} E^{(\beta+\gamma)n-\gamma(\nu-1)}_{\rho,\nu(1-\mu)+\mu(2n+1)} (\omega t^\rho) \\ + \gamma^n t^{\mu(2\mu+1)-1} E^{\gamma(n+1)+\beta n}_{\rho,\mu(2n+1)} (\omega t^\rho) \cdot g].$$

Which is required solution. □

Theorem 2 *The solution to the Cauchy problem* [5] *of fractional order*

$$\begin{cases} D^{\gamma,\mu,\nu}_{\rho,\omega,0^+} u(x,t) = \ K\frac{\partial^2}{\partial x^2} u(x,t), \quad t > 0, x \in \mathbb{R}, \\ [E^{-\gamma(1-\nu)}_{\rho,(1-\nu)(1-\mu),\omega,0^+} u(x,t)]_{t=0} = g(x), \\ \lim_{x\to\pm\infty} u(x,\ t) = 0, \end{cases} \tag{15}$$

with $\mu \in (0,\ 1)$, $\nu \in [0,\ 1]$, $\omega \in \mathbb{R}$, K, $\rho > 0$, $\gamma \geq 0$, *is given by*

$$u(x,t) = \sum_{n=0}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(-Kk^2\right)^n e^{ikx} \hat{g}(k) dk t^{\nu(1-\mu)+\mu(n+1)-1} E_{\rho,\nu(1-\mu)+\mu(n+1)}^{\gamma[n-(\nu-1)]}(\omega t^\rho).$$

Proof. Let $\overline{u}(x,p) = \mathbf{E}(u)(x,p)$ and $\hat{u}(k,t) = F(u)(k,t)$ are the Elzaki transform and the Fourier transform respectively. Taking Elzaki transform of (15) and by using (11), we have

$$p^{-\mu}(1-\omega p^\rho)^\gamma \widehat{\overline{u}}(k,p) - p^{\nu(1-\mu)+1}(1-\omega p^\rho)^{\gamma\nu} \hat{g}(k) = -Kk^2 \widehat{\overline{u}}(k,p)$$

$$\begin{aligned} \widehat{\overline{u}}(k,p) &= \frac{p^{\nu(1-\mu)+1}(1-\omega p^\rho)^{\gamma\nu} \hat{g}(k)}{p^{-\mu}(1-\omega p^\rho)^\gamma + Kk^2} \\ &= \frac{p^{\nu(1-\mu)+1+\mu}(1-\omega p^\rho)^{\gamma(\nu-1)} \hat{g}(k)}{1 + \frac{Kk^2}{p^{-\mu}(1-\omega p^\rho)^\gamma}} \\ &= \sum_{n=0}^{\infty} \left(-Kk^2\right)^n \hat{g}(k) p^{\nu(1-\mu)+\mu(n+1)+1} (1-\omega p^\rho)^{\gamma[(\nu-1)-n]}, \end{aligned}$$

inverting Elzaki transform, we have

$$\hat{u}(k,t) = \sum_{n=0}^{\infty} \left(-Kk^2\right)^n \hat{g}(k) t^{\nu(1-\mu)+\mu(n+1)-1} E_{\rho,\nu(1-\mu)+\mu(n+1)}^{\gamma[n-(\nu-1)]}(\omega t^\rho). \tag{16}$$

inverting Fourier transform and after little simplification, finally we obtain the desired result. □

Theorem 3 *The solution of the Cauchy problem* [5] *of fractional order*

$$\begin{cases} {}^{c}D_{\rho,\omega,0^+}^{\gamma,\mu} u(x,t) = K \frac{\partial^2}{\partial x^2} u(x,t), & t > 0, x \in \mathbb{R}, \\ u(x, 0^+) = g(x), \\ \lim_{x \to \pm\infty} u(x,\ t) = 0, \end{cases} \tag{17}$$

with $\mu \in (0,\ 1)$, $\nu \in [0,\ 1]$, $\omega \in \mathbb{R}$, K, $\rho > 0$, $\gamma \geq 0$, *is given by*

$$u(x,\ t) = \sum_{n=0}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(-Kk^2\right)^n e^{ikx} \hat{g}(k) dk t^{\mu n} E_{\rho,\mu n+1}^{\gamma n}(\omega t^\rho).$$

Proof. Taking Elzaki-Fourier transform of (17) using (12), we have

$$p^{-\mu}(1-\omega p^{\rho})^{\gamma}\widehat{\widehat{u}}(k,p) - p^{2-\mu}(1-\omega p^{\rho})^{\gamma}\hat{g}\,(k) = -Kk^2\widehat{\widehat{u}}(k,p)$$

$$\begin{aligned}\widehat{\widehat{u}}(k,s) &= \frac{p^{2-\mu}(1-\omega p^{\rho})^{\gamma}\hat{g}\,(k)}{p^{-\mu}(1-\omega p^{\rho})^{\gamma}+Kk^2}\\ &= \frac{p^2\hat{g}\,(k)}{1+\frac{Kk^2}{p^{-\mu}(1-\omega p^{\rho})^{\gamma}}}\\ &= \sum_{n=0}^{\infty}\left(-Kk^2\right)^n\hat{g}(k)p^{\mu n+2}(1-\omega p^{\rho})^{-\gamma n},\end{aligned}$$

inverting Elzaki transform, we have

$$\hat{u}\,(k,t) = \sum_{n=0}^{\infty}\left(-Kk^2\right)^n\hat{g}\,(k)t^{\mu n}E^{\gamma n}_{\rho,\mu n+1}\,(\omega t^{\rho})\,, \tag{18}$$

inverting Fourier transform, we have

$$u\,(x,\ t) = \sum_{n=0}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(-Kk^2\right)^n e^{ikx}\hat{g}\,(k)dkt^{\mu n}E^{\gamma n}_{\rho,\mu n+1} \tag{19}$$

which is the claimed result. □

5 Conclusions

In this work, we studied the Elzaki transform of Prabhakar integral plays a key role to control engineering calculus problem in fractional related to Hilfer-Prabhakar fractional derivative and its regularized version. We also present some application of Hilfer-Prabhakar and its regularized version along with the integral transform mainly Elzaki transform. In this paper we consider Free Electron Laser problem. Furthermore, we discuss some application of Elzaki transform to find solution of Cauchy problem involving Hilfer-Prabhakar fractional derivative and its regularized version. We also find the point where the very popular Laplace transform and the new Elzaki transform coincide. The results show that Elzaki transform is very useful for solving fractional differential equations.

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Coefficient estimates and Fekete-Szegö inequality for a class of analytic functions satisfying subordinate condition associated with Chebyshev polynomials

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Abstract. In this paper, we define a class of analytic functions, $\mathcal{F}(\mathcal{H}, \alpha, \delta, \mu)$, satisfying the following condition

$$\left(\alpha\left[\frac{zf'(z)}{f(z)}\right]^{\delta}+(1-\alpha)\left[\frac{zf'(z)}{f(z)}\right]^{\mu}\left[1+\frac{zf''(z)}{f'(z)}\right]^{1-\mu}\right)\prec \mathcal{H}(z,t),$$

where $\alpha \in [0,1], \delta \in [1,2]$ and $\mu \in [0,1]$.

We give coefficient estimates and Fekete-Szegö inequality for this class.

1 Introduction

Let $\mathcal{A}$ denote the class of analytic functions in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$, of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, z \in \mathbb{U}. \tag{1}$$

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Definition 1 [7, p.4] *Let* f, g *analytic functions in the open unit disk. The function* f *is said to be subordinate to* g*, written* $f \prec g$ *, or* $f(z) \prec g(z)$*, if there exists a function* w*, analytic in* $\mathbb{U}$*, with* $w(0) = 0$ *and* $|w(z)| < 1$ *(i.e.* w *is a Schwarz function), such that* $f(z) = g[w(z)], z \in \mathbb{U}$.

Remark 1 [7, p.4] *If* g *is univalent, then* $f \prec g$ *if and only if* $f(0) = g(0)$ *and* $f(\mathbb{U}) \subset g(\mathbb{U})$.

Chebyshev polynomials are of four kinds, but the most common are the Chebyshev polynomials of the first kind,

$$T_n(x) = \cos n\theta, x \in [-1, 1],$$

and the second kind,

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, x \in [-1, 1],$$

where n denotes the polynomial degree and $x = \cos\theta$.
Applications of Chebyshev polynomials for analytic functions can be found in [1, 2, 3, 4].
Let

$$\mathcal{H}(z, t) = \frac{1}{1 - 2tz + z^2},$$

where $t = \cos\theta, \theta \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$.
We have

$$\begin{aligned} \mathcal{H}(z, t) &= 1 + \sum_{n=1}^{\infty} \frac{\sin(n+1)\theta}{\sin\theta} z^n \\ &= 1 + 2\cos\theta z + (3\cos^2\theta - \sin^2\theta)z^2 + \cdots \\ &= 1 + U_1(t)z + U_2(t)z^2 + \cdots, z \in \mathbb{U}, t \in \left(\frac{1}{2}, 1\right], \end{aligned} \tag{2}$$

where

$$U_{n-1} = \frac{\sin(n\cos^{-1} t)}{\sqrt{1 - t^2}}, n \in \mathbb{N},$$

are the Chebyshev polynomials of second kind.
Furthermore, we know that

$$U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t)$$

and

$$U_1(t) = 2t, U_2(t) = 4t^2 - 1, \cdots .$$

In this paper, we define a new class of analytic functions, being motivated by the following result.

Corollary 1 [5] *Let* $f \in \mathcal{A}$ *and also let* $\alpha \in [0,1], a \in [0,1], \delta \in [1,2]$ *and* $\mu \in [0,1]$. *If*

$$\Re\left(\alpha\left[\frac{zf'(z)}{f(z)}\right]^{\delta} + (1-\alpha)\left[\frac{zf'(z)}{f(z)}\right]^{\mu}\left[1+\frac{zf''(z)}{f'(z)}\right]^{1-\mu}\right) > a, z \in \mathbb{U},$$

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > a, z \in \mathbb{U},$$

so f is starlike of order a *in* $\mathbb{U}$.

Definition 2 *We say that* $f \in \mathcal{A}$ *of the form* (1) *belongs to* $\mathcal{F}(\mathcal{H}, \alpha, \delta, \mu)$ *if*

$$\left(\alpha\left[\frac{zf'(z)}{f(z)}\right]^{\delta} + (1-\alpha)\left[\frac{zf'(z)}{f(z)}\right]^{\mu}\left[1+\frac{zf''(z)}{f'(z)}\right]^{1-\mu}\right) \prec \mathcal{H}(z,t), \qquad (3)$$

the power is considered to have principal value, $\alpha \in [0,1], \delta \in [1,2]$ *and* $\mu \in [0,1]$.

Taking $\alpha = \delta = t = 1$ and $w(z) = z$, we obtain the following example.

Example 1 *The function* $f(z) = \frac{z}{1-z}e^{\frac{z}{1-z}}$ *with the series expansion* $f(z) = z + 2z^2 + \frac{7}{2}z^3 + \cdots$ *belongs to* $\mathcal{F}(\mathcal{H}, \alpha, \delta, \mu)$.

For the purpose of our results, we need the following lemma.

Lemma 1 [6] *Let the Schwarz function* w *be given by*

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots, z \in \mathbb{U}. \qquad (4)$$

Then

$$|w_1| \le 1, \; |w_2 - tw_1^2| \le 1 + (|t|-1)|w_1|^2 \le \max\{1, |t|\},$$

where $t \in \mathbb{C}$.

2 Main result

Our main result in this paper is stated as the following theorem.

Theorem 1 *Let* $f \in \mathcal{A}$ *of the form* (1) *belong to the class* $\mathcal{F}(\mathcal{H}, \alpha, \delta, \mu)$. *Then*

$$|a_2| \leq \frac{2t}{\alpha\delta + (1-\alpha)(2-\mu)}, \tag{5}$$

and, for $\lambda \in \mathbb{C}$,

$$\left|a_3 - \lambda a_2^2\right| \leq \frac{t}{\alpha\delta + (1-\alpha)(3-2\mu)} \max\left\{1, \left|2t\left(\frac{2\lambda\big(\alpha\delta + (1-\alpha)(3-2\mu)\big)}{\big(\alpha\delta + (1-\alpha)(2-\mu)\big)^2} - \frac{3 + \frac{2(1-\alpha)(1-\mu) - \alpha(\delta^2-\mu^2) - \mu^2}{\alpha\delta + (1-\alpha)(2-\mu)}}{2\big(\alpha\delta + (1-\alpha)(2-\mu)\big)}\right) - \frac{4t^2-1}{2t}\right|\right\}. \tag{6}$$

Proof. Let $f \in \mathcal{F}(\mathcal{H}, \alpha, \delta, \mu)$, then from (3) we have

$$\alpha\left[\frac{zf'(z)}{f(z)}\right]^{\delta} + (1-\alpha)\left[\frac{zf'(z)}{f(z)}\right]^{\mu}\left[1 + \frac{zf''(z)}{f'(z)}\right]^{1-\mu} = \mathcal{H}\big(w(z), t\big), z \in \mathbb{U}. \tag{7}$$

Using (2) and (4), we obtain

$$\mathcal{H}\big(w(z), t\big) = 1 + U_1(t)w_1 z + \big(U_2(t)w_1^2 + U_1(t)w_2\big)z^2 + \cdots. \tag{8}$$

Making use of (1), (7) and (8), we get

$$\big(\alpha\delta + (1-\alpha)(2-\mu)\big)a_2 = U_1(t)w_1, \tag{9}$$

and

$$\begin{aligned} &2\big(\alpha\delta + (1-\alpha)(3-2\mu)\big)a_3 + \frac{\alpha\delta(\delta-3) + (1-\alpha)(\mu^2+5\mu-8)}{2}a_2^2 \\ &\quad = U_2(t)w_1^2 + U_1(t)w_2. \end{aligned} \tag{10}$$

From (9), we obtain

$$a_2 = \frac{U_1(t)w_1}{\alpha\delta + (1-\alpha)(2-\mu)}. \tag{11}$$

Using Lemma 1 and (11), we obtain (5).
Putting (11) in (10), we have

$$2\big(\alpha\delta + (1-\alpha)(3-2\mu)\big)a_3 = U_2(t)w_1^2 + U_1(t)w_2$$

$$+\frac{\left(\alpha\delta(3-\delta)-(1-\alpha)(\mu^2+5\mu-8)\right)U_1^2(t)w_1^2}{2\left(\alpha\delta+(1-\alpha)(2-\mu)\right)^2}.$$

Therefore,

$$a_3=\frac{U_1(t)}{2\left(\alpha\delta+(1-\alpha)(3-2\mu)\right)}$$
$$\left(w_2+\left(\frac{\left(3+\frac{2(1-\alpha)(1-\mu)-\alpha(\delta^2-\mu^2)-\mu^2}{\alpha\delta+(1-\alpha)(2-\mu)}\right)U_1(t)}{2\left(\alpha\delta+(1-\alpha)(2-\mu)\right)}+\frac{U_2(t)}{U_1(t)}\right)w_1^2\right). \quad (12)$$

For $\lambda\in\mathbb{C}$, from (11) and (12) we obtain that

$$a_3-\lambda a_2^2=\frac{U_1(t)}{2\left(\alpha\delta+(1-\alpha)(3-2\mu)\right)}\left(w_2+\frac{U_2(t)}{U_1(t)}w_1^2\right.$$
$$\left.-U_1(t)w_1^2\left(\frac{2\lambda\left(\alpha\delta+(1-\alpha)(3-2\mu)\right)}{\left(\alpha\delta+(1-\alpha)(2-\mu)\right)^2}-\frac{3+\frac{2(1-\alpha)(1-\mu)-\alpha(\delta^2-\mu^2)-\mu^2}{\alpha\delta+(1-\alpha)(2-\mu)}}{2\left(\alpha\delta+(1-\alpha)(2-\mu)\right)}\right)\right). \quad (13)$$

Hence,

$$\left|a_3-\lambda a_2^2\right|=\frac{t}{\alpha\delta+(1-\alpha)(3-2\mu)}\left|w_2-\left(2t\left(\frac{2\lambda\left(\alpha\delta+(1-\alpha)(3-2\mu)\right)}{\left(\alpha\delta+(1-\alpha)(2-\mu)\right)^2}\right.\right.\right.$$
$$\left.\left.\left.-\frac{3+\frac{2(1-\alpha)(1-\mu)-\alpha(\delta^2-\mu^2)-\mu^2}{\alpha\delta+(1-\alpha)(2-\mu)}}{2\left(\alpha\delta+(1-\alpha)(2-\mu)\right)}\right)-\frac{4t^2-1}{2t}\right)w_1^2\right|. \quad (14)$$

Using Lemma 1 for inequality (14), we obtain inequality (6). □

Taking $\alpha=1-\beta,\delta=1$ and $\mu=0$ in Theorem 1, we obtain the following result:

Corollary 2 [2] *Let* $f\in\mathcal{A}$ *of the form* (1) *satisfying the condition*

$$\left((1-\beta)\frac{zf'(z)}{f(z)}+\beta\left(1+\frac{zf''(z)}{f'(z)}\right)\right)\prec\mathcal{H}(z,t),$$

where $\beta\in[0,1]$. *Then*

$$|a_2|\leq\frac{2t}{1+\beta},$$

and, for $\lambda \in \mathbb{C}$,

$$\left|a_3 - \lambda a_2^2\right| \leq \frac{t}{1+2\beta} \max\left\{1, \left|2t\left(\frac{2\lambda(1+2\beta)}{(1+\beta)^2} - \frac{1+3\beta}{(1+\beta)^2}\right) - \frac{4t^2-1}{2t}\right|\right\}.$$

Taking $\alpha = 0$ in Theorem 1, we obtain the following result:

Corollary 3 [1] *Let* $f \in \mathcal{A}$ *of the form* (1) *satisfying the condition*

$$\left(\frac{zf'(z)}{f(z)}\right)^{\mu}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\mu} \prec \mathcal{H}(z,t),$$

where $\mu \in [0,1]$. *Then*

$$|a_2| \leq \frac{2t}{2-\mu},$$

and, for $\lambda \in \mathbb{C}$,

$$\left|a_3 - \lambda a_2^2\right| \leq \frac{t}{3-2\mu} \max\left\{1, \left|2t\left(\frac{2\lambda(3-2\mu)}{(2-\mu)^2} + \frac{\mu^2+5\mu-8}{2(2-\mu)^2}\right) - \frac{4t^2-1}{2t}\right|\right\}.$$

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Congruent and non-congruent hyperball packings related to doubly truncated Coxeter orthoschemes in hyperbolic 3-space

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Abstract. In [17] we considered hyperball packings in 3-dimensional hyperbolic space. We developed a decomposition algorithm that for each saturated hyperball packing has provided a decomposition of $\mathbb{H}^3$ into truncated tetrahedra. Thus, in order to get a density upper bound for hyperball packings, it is sufficient to determine the density upper bound of hyperball packings in truncated simplices. Therefore, in this paper we examine the doubly truncated Coxeter orthoscheme tilings and the corresponding congruent and non-congruent hyperball packings. We prove that related to the mentioned Coxeter tilings the density of the densest congruent hyperball packing is ≈ 0.81335 that is – by our conjecture – the upper bound density of the relating non-congruent hyperball packings, too.

1 Introduction

In n-dimensional hyperbolic space $\mathbb{H}^n$ $(n \geq 2)$ there are 3 types of "balls (spheres)": the classical balls (spheres), the horoballs (horospheres) and the

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hyperballs (hyperspheres).

In this paper we concentrate on the hyperballs and their packings related to the complete Coxeter tilings in 3-dimensional hyperbolic space $\mathbb{H}^3$. However, first we survey the previous results related to this topic.

In the hyperbolic plane $\mathbb{H}^2$ the universal upper bound of the density of hypercycle packings is $\frac{3}{\pi}$, proved by I. Vermes in [29] and the universal lower bound of the density of hypercycle coverings is $\frac{\sqrt{12}}{\pi}$ determined also by I. Vermes in [30].

In [21] and [22] we analysed the regular prism tilings (simply truncated Coxeter orthoscheme tilings) and the corresponding optimal hyperball packings in $\mathbb{H}^n$ $(n = 3, 4)$, moreover we extended the method – developed in the former paper [22] – to 5-dimensional hyperbolic space (see [23]). In paper [24] we studied the n-dimensional hyperbolic regular prism honeycombs and the corresponding coverings by congruent hyperballs and we determined their least dense (thinnest) covering densities. Furthermore, we formulated conjectures for candidates of the least dense hyperball covering by congruent hyperballs in 3- and 5-dimensional hyperbolic spaces.

In [16] we discussed congruent and non-congruent hyperball packings of the truncated regular tetrahedron tilings. These are derived from the Coxeter simplex tilings $\{p, 3, 3\}$ $(7 \leq p \in \mathbb{N})$ and $\{5, 3, 3, 3, 3\}$ in 3- and 5-dimensional hyperbolic spaces, respectively. We determined the densest hyperball packing arrangement and its density with congruent hyperballs in $\mathbb{H}^5$ and determined the smallest density upper bounds of non-congruent hyperball packings generated by the above tilings in $\mathbb{H}^n$, $(n = 3, 5)$.

In [15] we dealed with the packings derived by horo- and hyperballs (briefly hyp-hor) packings in n-dimensional hyperbolic spaces $\mathbb{H}^n$ $(n = 2, 3)$ which form a new class of the classical packing problems. We constructed in the 2- and 3-dimensional hyperbolic spaces hyp-hor packings that are generated by complete Coxeter tilings of degree 1 and we determined their densest packing configurations and their densities. Using also numerical methods we proved that in the hyperbolic plane $(n = 2)$ the density of the above hyp-hor packings arbitrarily approximate the universal upper bound of the hypercycle or horocycle packing density $\frac{3}{\pi}$ and in $\mathbb{H}^3$ the optimal configuration belongs to the Coxeter tiling $\{7, 3, 6\}$ with density ≈ 0.83267. Furthermore, we analyzed the hyp-hor packings in truncated orthoschemes $\{p, 3, 6\}$ $(6 < p < 7,\ p \in \mathbb{R})$ whose density function attains its maximum for a parameter $p \in [6.05, 6.06]$, and the densities are larger that ≈ 0.85397 if p lies in this interval.

In [14] we proved that if the truncated tetrahedron is regular, then the density of the densest packing is ≈ 0.86338. This is larger than the Böröczky-Florian density upper bound but our locally optimal hyperball packing configuration cannot be extended to the whole space $\mathbb{H}^3$. However, we described a hyperball packing construction, by the regular truncated tetrahedron tiling under the extended Coxeter group $\{3, 3, 7\}$ with maximal density ≈ 0.82251.

Recently, (to the best of author's knowledge) the candidates for the densest hyperball (hypersphere) packings in the 3-, 4- and 5-dimensional hyperbolic space $\mathbb{H}^n$ are derived from regular prism tilings described in [21], [22] and [23].

In [17] we considered hyperball packings in 3-dimensional hyperbolic space. We developed a decomposition algorithm that provides a decomposition of $\mathbb{H}^3$ into truncated tetrahedra for each saturated hyperball packing. Therefore, in order to get a density upper bound for hyperball packings, it is sufficient to determine the density upper bound of hyperball packings in truncated simplices.

In [26] we studied hyperball packings related to truncated regular octahedron and cube tilings derived from the Coxeter simplex tilings $\{p, 3, 4\}$ $(7 \leq p \in \mathbb{N})$ and $\{p, 4, 3\}$ $(5 \leq p \in \mathbb{N})$ in 3-dimensional hyperbolic space $\mathbb{H}^3$. We determined the densest hyperball packing arrangement and its density with congruent and non-congruent hyperballs related to the above tilings. Moreover, we proved that the locally densest congruent or non-congruent hyperball configuration belongs to the regular truncated cube with density ≈ 0.86145. This is larger than the Böröczky-Florian density upper bound for balls and horoballs. We described a non-congruent hyperball packing construction derived from the regular cube tiling under the extended Coxeter group $\{4, 3, 7\}$ with maximal density ≈ 0.84931.

In [27] we proved, that the density upper bound of the saturated congruent hyperball (hypersphere) packings related to the corresponding truncated tetrahedron cells is realized in a regular truncated tetrahedra with density ≈ 0.86338. Furthermore, we proved that the density of locally optimal congruent hyperball arrangement in regular truncated tetrahedron is not monotonically increasing function of the height (radius) of corresponding optimal hyperball, contrary to the ball (sphere) and horoball (horosphere) packings.

In the present paper we study congruent and non-congruent hyperball packings generated by doubly truncated Coxeter orthoscheme tilings in the 3*-dimensional hyperbolic space. We prove that the densest congruent hyperball packing belongs to the Coxeter orthoscheme tiling of parameter* $\{7, 3, 7\}$ *with density* ≈ 0.81335 *(see Theorems 2-3). This density is equal – by our conjecture – with the upper bound density of the corresponding non-congruent hyperball arrangements (see Theorem 4 and Conjecture 1).*

2 Orthoschemes, hyperspheres and their volumes

An orthoscheme $\mathcal{O}$ in $\mathbb{H}^n$ $(n \geq 2)$ *"in classical sense"* is a simplex bounded by $n+1$ hyperplanes $H_0, \dots, H_n$ such that $H_i \perp H_j$, for $j \neq i-1, i, i+1$. Or, equivalently, the $n+1$ vertices of $\mathcal{O}$ can be labelled by $A_0, A_1, \dots, A_n$ in such a way that $\text{span}(A_0, \dots, A_i) \perp \text{span}(A_i, \dots, A_n)$ for $0 < i < n-1$.

Geometrically, complete orthoschemes of degree $m = 0, 1, 2$ can be described as follows:

1. For $m = 0$, they coincide with the class of classical orthoschemes introduced by Schläfli. The initial and final vertices, A_0 and A_n of the orthogonal edge-path $A_i A_{i+1}$, $i = 0, \dots, n-1$, are called principal vertices of the orthoscheme.
2. A complete orthoscheme of degree $m = 1$ can be constructed from an orthoscheme with one outer principal vertex, one of A_0 or A_n, which is simply truncated by its polar plane (see Fig. 1-2).
3. A complete orthoscheme of degree $m = 2$ can be constructed from an orthoscheme with two outer principal vertices, A_0 and A_n, which is doubly truncated by their polar planes $\mathrm{pol}(A_0)$ and $\mathrm{pol}(A_n)$ (see Fig. 1-2).

For the *complete Coxeter orthoschemes* $\mathcal{O} \subset \mathbb{H}^n$ we adopt the usual conventions and sometimes even use them in the Coxeter case: If two nodes are related by the weight $\cos\frac{\pi}{p}$ then they are joined by a $(p-2)$-fold line for $p = 3,\ 4$ and by a single line marked by p for $p \geq 5$. In the hyperbolic case if two bounding hyperplanes of O are parallel, then the corresponding nodes are joined by a line marked ∞. If they are divergent then their nodes are joined by a dotted line.

In the following we concentrate only on dimension 3 and on hyperbolic Coxeter-Schläfli symbol of the complete orthoscheme tiling $\mathcal{P}$ generated by reflections in the planes of a complete orthoscheme $\mathcal{O}$. To every scheme there is a corresponding symmetric 4×4 matrix (b^{ij}) where $b^{ii} = 1$ and, for $i \neq j \in \{0, 1, 2, 3\}$, b^{ij} equals to $-\cos\alpha_{ij}$ with all dihedral angles α_{ij} between the faces H_i,H_j of $\mathcal{O}$.

For example, (b^{ij}) in formula (1) is the so called Coxeter-Schläfli matrix with parameters $(u; v; w)$, i.e. $\alpha_{01} = \frac{\pi}{u}$, $\alpha_{12} = \frac{\pi}{v}$, $\alpha_{23} = \frac{\pi}{w}$. Now only $3 \leq u, v, w$ come into account (see [7]).

$$(b^{ij}) = \langle \mathbf{b}^i, \mathbf{b}^j \rangle := \begin{pmatrix} 1 & -\cos\frac{\pi}{u} & 0 & 0 \\ -\cos\frac{\pi}{u} & 1 & -\cos\frac{\pi}{v} & 0 \\ 0 & -\cos\frac{\pi}{v} & 1 & -\cos\frac{\pi}{w} \\ 0 & 0 & -\cos\frac{\pi}{w} & 1 \end{pmatrix}. \tag{1}$$

This 3-dimensional complete (truncated or frustum) orthoscheme $\mathcal{O} = \mathcal{O}(u,v,w)$ and its reflection group $\mathbf{G}_{uvw}$ will be described in Fig. 2, and by the symmetric Coxeter-Schläfli matrix (b^{ij}) in formula (1), furthermore by its inverse matrix (a_{ij}) in formula (2).

$$(a_{ij}) = (b^{ij})^{-1} = \langle \mathbf{a}_i, \mathbf{a}_j \rangle :$$
$$= \frac{1}{B}\begin{pmatrix} \sin^2\frac{\pi}{w} - \cos^2\frac{\pi}{v} & \cos\frac{\pi}{u}\sin^2\frac{\pi}{w} & \cos\frac{\pi}{u}\cos\frac{\pi}{v} & \cos\frac{\pi}{u}\cos\frac{\pi}{v}\cos\frac{\pi}{w} \\ \cos\frac{\pi}{u}\sin^2\frac{\pi}{w} & \sin^2\frac{\pi}{w} & \cos\frac{\pi}{v} & \cos\frac{\pi}{w}\cos\frac{\pi}{v} \\ \cos\frac{\pi}{u}\cos\frac{\pi}{v} & \cos\frac{\pi}{v} & \sin^2\frac{\pi}{u} & \cos\frac{\pi}{w}\sin^2\frac{\pi}{u} \\ \cos\frac{\pi}{u}\cos\frac{\pi}{v}\cos\frac{\pi}{w} & \cos\frac{\pi}{w}\cos\frac{\pi}{v} & \cos\frac{\pi}{w}\sin^2\frac{\pi}{u} & \sin^2\frac{\pi}{u} - \cos^2\frac{\pi}{v} \end{pmatrix}, \tag{2}$$

where

$$B = \det(b^{ij}) = \sin^2\frac{\pi}{u}\sin^2\frac{\pi}{w} - \cos^2\frac{\pi}{v} < 0, \quad \text{i.e. } \sin\frac{\pi}{u}\sin\frac{\pi}{w} - \cos\frac{\pi}{v} < 0.$$

In the following we use the above orthoscheme whose volume is derived by the next Theorem of R. Kellerhals ([8], by the ideas of N. I. Lobachevsky):

Theorem 1 (R. Kellerhals) *The volume of a three-dimensional hyperbolic complete orthoscheme $\mathcal{O} = \mathcal{O}(u,v,w) \subset \mathbb{H}^3$ can be expressed with the essential angles $\alpha_{01} = \frac{\pi}{u}$, $\alpha_{12} = \frac{\pi}{v}$, $\alpha_{23} = \frac{\pi}{w}$, $(0 \le \alpha_{ij} \le \frac{\pi}{2})$ (Fig. 1.) in the following form:*

$$\mathrm{Vol}(\mathcal{O}) = \frac{1}{4}\{\mathcal{L}(\alpha_{01}+\theta) - \mathcal{L}(\alpha_{01}-\theta) + \mathcal{L}(\frac{\pi}{2}+\alpha_{12}-\theta) +$$
$$+ \mathcal{L}(\frac{\pi}{2}-\alpha_{12}-\theta) + \mathcal{L}(\alpha_{23}+\theta) - \mathcal{L}(\alpha_{23}-\theta) + 2\mathcal{L}(\frac{\pi}{2}-\theta)\},$$

where $\theta \in [0, \frac{\pi}{2})$ is defined by:

$$\tan\theta = \frac{\sqrt{\cos^2\alpha_{12} - \sin^2\alpha_{01}\sin^2\alpha_{23}}}{\cos\alpha_{01}\cos\alpha_{23}},$$

and where $\mathcal{L}(x) := -\int\limits_0^x \log|2\sin t|\,dt$ denotes the Lobachevsky function (in J. Milnor's interpretation).

The hypersphere (or equidistant surface) is a quadratic surface at a constant distance from a plane (base plane) in both halfspaces. The infinite body of the hypersphere, containing the base plane, is called *hyperball*.

The *half hyperball* (i.e., the part of the hyperball lying on one side of its base plane) with distance h to a base plane β is denoted by $\mathcal{H}^h_+$. The volume of the intersection of $\mathcal{H}^h_+(\mathcal{A})$ and the right prism with base a 2-polygon $\mathcal{A} \subset \beta$, can be determined by the classical formula of J. Bolyai [2]

$$\mathrm{Vol}(\mathcal{H}^h_+(\mathcal{A})) = \frac{1}{4}\mathrm{Area}(\mathcal{A}) \left[k \sinh \frac{2h}{k} + 2h\right]. \tag{3}$$

The constant $k = \sqrt{\frac{-1}{K}}$ is the natural length unit in $\mathbb{H}^3$, where K denotes the constant negative sectional curvature. In the following we may assume that $k = 1$.

3 Essential points in a doubly truncated orthoscheme

Let $A_0(\mathbf{a}_0)$, $A_1(\mathbf{a}_1)$, $A_2(\mathbf{a}_2)$, $A_3(\mathbf{a}_3)$ be the vertices of the above complete orthoscheme $\mathcal{O}(u, v, w)$ (see Fig. 1,2). In the considered cases the principal vertices A_0 and A_3 are outer points $(a_{ii} > 0)$, $(i \in \{0, 3\})$.

We distinguish the following main configurations:

1. *A_3 is outer point $\frac{\pi}{u} + \frac{\pi}{v} < \frac{\pi}{2}$, then $a_3(\mathbf{a}_3) = JEQ$ is its polar plane and A_0 is also outer $\frac{\pi}{v} + \frac{\pi}{w} < \frac{\pi}{2}$, then $a_0(\mathbf{a}_0) = CLH$ is its polar plane.*

1.i $u = w$, $F_{03}F_{12}$ is half turn axis, $\mathbf{h}$ is the half turn changing $0 \leftrightarrow 3$, $1 \leftrightarrow 2$. Here a "half orthoscheme" $JQEB_{13}F_{12}B_{02}F_{03}A_2$ will be the fundamental domain of $\mathbf{G}_{u=w,v}$.

In our calculations we will use the following important lemmas, (see [11] and Fig. 1-2):

Lemma 1 *Let A_0 be an outer principal vertex of the orthoscheme W_{uvw} and let $a_0(\mathbf{a}_0) = CLH$ be its polar plane where $C = a_0 \cap A_0A_1$, $L = a_0 \cap A_0A_2$, $H = a_0 \cap A_0A_3$ whose vectors are the following:*

$$C(\mathbf{c}) = a_0 \cap A_0A_1;\ \mathbf{c} = \mathbf{a}_1 - \frac{a_{01}}{a_{00}}\mathbf{a}_0, \text{ with}$$

$$\langle \mathbf{c}, \mathbf{c} \rangle = \frac{(a_{11}a_{00} - a_{01}^2)}{a_{00}} = \langle \mathbf{c}, \mathbf{a}_1 \rangle = \frac{\sin^2 \frac{\pi}{w}}{\sin^2 \frac{\pi}{w} - \cos^2 \frac{\pi}{v}} = \frac{a_{11}}{a_{00}}.$$

$$L(\mathbf{l}) = a_0 \cap A_0A_2;\ \mathbf{l} = \mathbf{a}_2 - \frac{a_{02}}{a_{00}}\mathbf{a}_0, \text{ with}$$

$$\langle \mathbf{l}, \mathbf{l} \rangle = \frac{(a_{22}a_{00} - a_{02}^2)}{a_{00}} = \langle \mathbf{l}, \mathbf{a}_2 \rangle = \frac{1}{\sin^2 \frac{\pi}{w} - \cos^2 \frac{\pi}{v}} = \frac{1}{Ba_{00}}.$$

$$H(\mathbf{h}) = a_0 \cap A_0A_3; \ \mathbf{h} = \mathbf{a}_3 - \frac{a_{03}}{a_{00}}\mathbf{a}_0, \text{ with} \tag{4}$$

$$\langle \mathbf{h}, \mathbf{h} \rangle = \frac{(a_{33}a_{00} - a_{03}^2)}{a_{00}} = \langle \mathbf{h}, \mathbf{a}_3 \rangle = \frac{\sin^2 \frac{\pi}{v}}{\sin^2 \frac{\pi}{w} - \cos^2 \frac{\pi}{v}} = \frac{\sin^2 \frac{\pi}{v}}{Ba_{00}}.$$

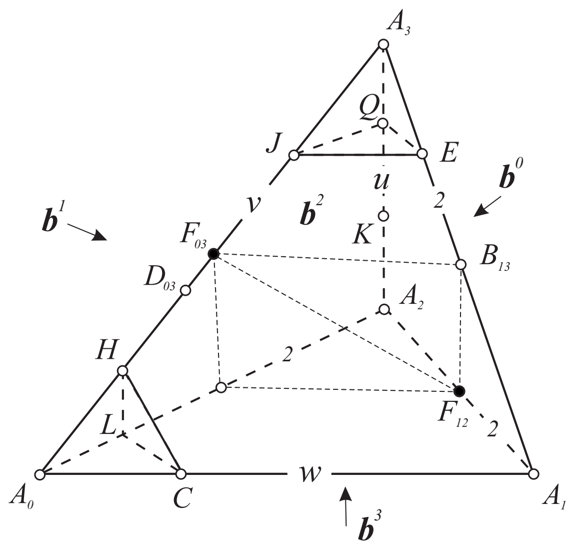


Figure 1: Double truncated complete orthoscheme with essential points

Lemma 2 *Let* A_3 *be an outer principal vertex of the orthoscheme* $\mathcal{O}(u, v, w)$ *and let* $a_3(\mathbf{a}_3) = JEQ$ *be its polar plane where* $J = a_3 \cap A_3A_0$, $E = a_3 \cap A_3A_1$, $Q = a_3 \cap A_3A_2$ *whose vectors are the following:*

$$J(\mathbf{j}) = a_3 \cap A_3A_0; \ \mathbf{j} = \mathbf{a}_0 - \frac{a_{03}}{a_{33}}\mathbf{a}_3, \text{ with}$$

$$\langle \mathbf{j}, \mathbf{j} \rangle = \frac{(a_{00}a_{33} - a_{03}^2)}{a_{33}} = \langle \mathbf{j}, \mathbf{a}_0 \rangle = \frac{\sin^2 \frac{\pi}{v}}{\sin^2 \frac{\pi}{u} - \cos^2 \frac{\pi}{v}} = \frac{\sin^2 \frac{\pi}{v}}{Ba_{33}}.$$

$$E(\mathbf{e}) = a_3 \cap A_3A_1; \ \mathbf{e} = \mathbf{a}_1 - \frac{a_{13}}{a_{33}}\mathbf{a}_3, \text{ with}$$

$$\langle \mathbf{e},\mathbf{e}\rangle = \frac{(a_{11}a_{33}-a_{13}^2)}{a_{33}} = \langle \mathbf{e},\mathbf{a}_1\rangle = \frac{1}{\sin^2\frac{\pi}{u}-\cos^2\frac{\pi}{v}} = \frac{1}{Ba_{33}}.$$

$$Q(\mathbf{h}) = a_3 \cap A_3A_2;\ \mathbf{q} = \mathbf{a}_2 - \frac{a_{23}}{a_{33}}\mathbf{a}_3,\ \text{with} \tag{5}$$

$$\langle \mathbf{q},\mathbf{q}\rangle = \frac{(a_{22}a_{33}-a_{23}^2)}{a_{33}} = \langle \mathbf{q},\mathbf{a}_2\rangle = \frac{\sin^2\frac{\pi}{u}}{\sin^2\frac{\pi}{u}-\cos^2\frac{\pi}{v}} = \frac{a_{22}}{a_{33}}.$$

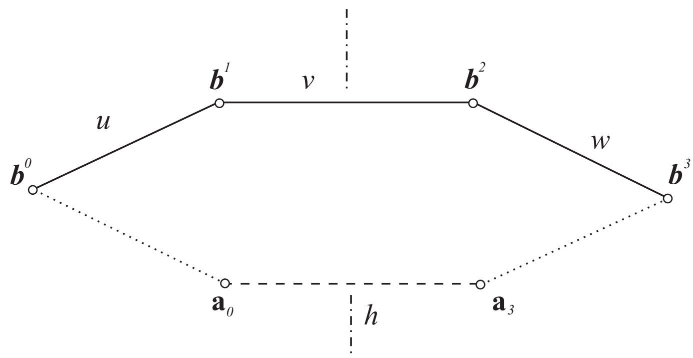


Figure 2:

Especially if $u = w$, the midpoints F_{03} of JH and F_{12} of A_1A_2 can play important roles, since $F_{03}F_{12}$ will be the axis of half turn

$$\mathbf{h}:\ 0 \leftrightarrow 3, 1 \leftrightarrow 2,\ \text{i.e. } A_0 \leftrightarrow A_3,\ \mathbf{b}^0 \leftrightarrow \mathbf{b}^3,\ A_1 \leftrightarrow A_2,\ \mathbf{b}^1 \leftrightarrow \mathbf{b}^2.$$

(Here $a_{00} = a_{33}$ and $a_{11} = a_{22}$ hold, of course.)

Lemma 3 *The midpoints* $F_{03}(\mathbf{f}_{03})$ *of* JH *and* $F_{12}(\mathbf{f}_{12})$ *of* A_1A_2 *(see Fig. 1) can be determined by the following vectors:*

$$\mathbf{f}_{03} = \mathbf{a}_0 + \mathbf{a}_3,\ \langle \mathbf{f}_{03},\mathbf{f}_{03}\rangle = 2(a_{00}+a_{03}) < 0,$$
$$\mathbf{f}_{12} = \mathbf{a}_1 + \mathbf{a}_2,\ \langle \mathbf{f}_{12},\mathbf{f}_{12}\rangle = 2(a_{11}+a_{12}) < 0.$$

4 On hyperball packings in a doubly truncated orthoscheme

Similarly to the former cases (see [21], [22], [24], [14], [16], [17]) it is interesting to study and to construct locally optimal *congruent and non-congruent* hyperball packings relating to suitable truncated polyhedron tilings in 3- and higher dimensions as well. This study fits into our program to look for the upper bound density of the congruent and non-congruent hyperball packings in $\mathbb{H}^n$.

4.1 Congruent hyperball packings

We consider a doubly truncated orthoscheme tiling $\mathcal{T}(\mathcal{O}(u,v,w))$ with Schläfli symbol $\{u,v,w\}$, $(\frac{1}{u}+\frac{1}{v}<\frac{1}{2},\ \frac{1}{v}+\frac{1}{w}<\frac{1}{2},\ 3 \le u,v,w \in \mathbb{N})$ whose fundamental domain is doubly truncated orthoschem (e.g. $CHLA_1A_2EJQ$ in Fig. 1).

Let a truncated orthoscheme $\mathcal{O}(u,v,w) \subset \mathbb{H}^3$ be a tile from the above tiling. This truncated orthoscheme can be derived also by truncation from a orthoschem $A_0A_1A_2A_3$ with outer essential vertices A_0 and A_3. The truncating planes $a_0(\boldsymbol{a}_0) = CLH$ and $a_3(\boldsymbol{a}_3) = JEQ$ are the polar planes of outer vertices A_0 and A_3, that *can be the ultraparallel base planes of hyperballs* $\mathcal{H}_i^s$ with height s $(i = 0,3)$. The distance between the two base planes is $2h^{03}(u,v,w) = d(a_0(\boldsymbol{a}_0), a_3(\boldsymbol{a}_3)) = d(H,J)$ (d is the hyperbolic distance function).

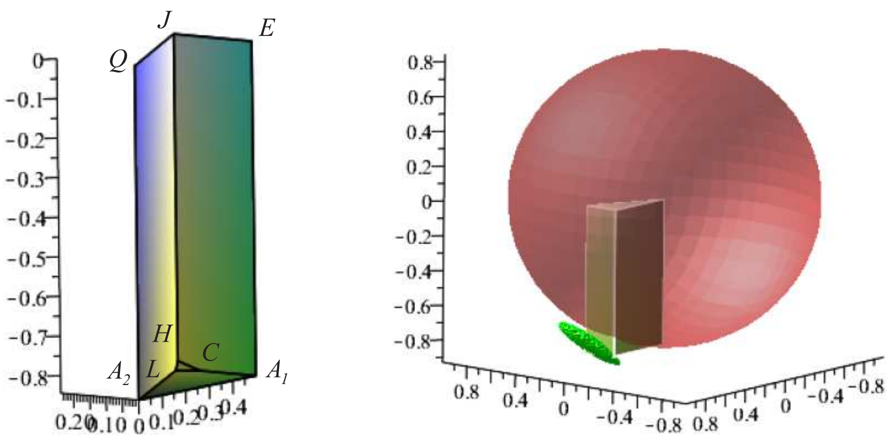


Figure 3: The densest congruent hyperball packing arrangement related to parameters $\{7,3,7\}$ with density ≈ 0.81335

In this subsection we consider congruent hyperball packings therefore we have to distinguish 2 main cases.

1. Both polar planes are assigned hyperspheres that are congruent with each other therefore the height of a hyperball is at most $h^{03}(u,v,w)$ (see Fig. 1). It is clear, that the heights $h^0 = h^3$ of optimal hyperballs $\mathcal{H}_i^{h^i}$

$(i = 0, 3)$ is

$$h = h^0(u, v, w) = h^3(u, v, w) = \\ = \min\{h^{03}(u, v, w) = d(H, J)/2, d(Q, A_2), d(C, A_1)\}, \quad (6)$$

where u, v, w are suitable given integer parameters. In this case the volume sum of the hyperball pieces lying in the orthoscheme is

$$\mathrm{Vol}(\mathcal{H}^h(\mathcal{A}_0) \cap \mathcal{O}(u, v, w)) + \mathrm{Vol}(\mathcal{H}^h(\mathcal{A}_3) \cap \mathcal{O}(u, v, w))$$

(see (3)) where $\mathcal{A}_0$ is the area of the triangle CLH and $\mathcal{A}_3$ is the area of the triangle JEQ.

2. In these cases we consider only one hyperball type:

   (a) with base plane $a_0(\mathbf{a}_0) = CLH$. The height of the optimal hyperball $\mathcal{H}_0^{h^0}$ is

   $$h^0(u, v, w) = \min\{2h^{03}(u, v, w) = d(J, H), d(C, A_1)\}, \quad (7)$$

   where u, v, w are suitable given integer parameters.

   (b) with base plane $a_3(\mathbf{a}_3) = JEQ$. The height of the optimal hyperball $\mathcal{H}_3^{h^3}$ is

   $$h^3(u, v, w) = \min\{2h^{03}(u, v, w) = d(J, H), d(Q, A_2)\}, \quad (8)$$

   where u, v, w are suitable given integer parameters.

Definition 1 *The locally density functions $\delta^i(\mathcal{O}(u, v, w))$ of the congruent hyperball packings related to $\mathcal{O}(u, v, w)$ and the above cases $(i \in \{1, 2\})$ are defined by next formulas:*

1.

$$\delta^1(\mathcal{O}(u, v, w)) := \frac{\mathrm{Vol}(\mathcal{H}^s(\mathcal{A}_0) \cap \mathcal{O}(u, v, w)) + \mathrm{Vol}(\mathcal{H}_3^s(\mathcal{A}_3) \cap \mathcal{O}(u, v, w))}{\mathrm{Vol}(\mathcal{O}(u, v, w))},$$

where $0 < s \leq h$, $\mathcal{A}_0$ is the area of the triangle CLH and $\mathcal{A}_3$ is the area of the triangle JEQ (see (6)).

2.

$$\delta_j^2(\mathcal{O}(u, v, w)) := \frac{\mathrm{Vol}(\mathcal{H}^s(\mathcal{A}_j) \cap \mathcal{O}(u, v, w))}{\mathrm{Vol}(\mathcal{O}(u, v, w))},$$

where $0 < s \leq h^j$, $j \in \{0, 3\}$ (see (7), (8)) and

$$\delta^2(\mathcal{O}(u, v, w)) := \max_{j=0,3}\{\delta_j^2(\mathcal{O}(u, v, w))\}.$$

The distance s of two proper points $X(\mathbf{x})$ and $Y(\mathbf{y})$ is calculated by the formula

$$\cosh s = \frac{-\langle \mathbf{x}, \mathbf{y} \rangle}{\sqrt{\langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle}}. \tag{9}$$

If the parameters u, v, w are given then the lengths of the line segments A_1C, A_2Q and JH can be determined by the machinery of the projective geometry using the Lemmas 1-3 and formula (9):

Lemma 4

$$d(A_1, C) = \operatorname{arcosh}\frac{1}{\sqrt{a_{00}}}, \ d(A_2, Q) = \operatorname{arcosh}\frac{1}{\sqrt{a_{33}}}, \ d(J, H) = \operatorname{arcosh}\frac{-a_{03}}{\sqrt{a_{00}a_{33}}}.$$

where (a_{ij}) $(i, j = 0, 1, 2, 3)$ *is the inverse of the corresponding Coxeter-Schläfli matrix* (*see* (2)).

In our cases the essential dihedral angles of orthoschemes $\mathcal{O}(u, v, w)$ are the following: $\alpha_{01} = \frac{\pi}{u}$, $\alpha_{12} = \frac{\pi}{v}$, $\alpha_{23} = \frac{\pi}{w}$ (see Fig. 1), therefore, the volume $\mathrm{Vol}(\mathcal{O}(u, v, w))$ of the orthoscheme $\mathcal{O}(u, v, w)$ can be determined by Theorem 1. Moreover, the maximal height h, h^0 or h^3 of congruent optimal hyperballs and the corresponding volumes of the hyperball pieces can be computed for any suitable fixed integer parameters u, v, w (see (3)). Therefore, the density $\delta^1(\mathcal{O}(u, v, w))$ or $\delta^2_j(\mathcal{O}(u, v, w))$ $(j \in \{1, 2\})$ (see Definition 1) depends only on the suitable integer parameters u, v, w of the doubly truncated orthoscheme $\mathcal{O}(u, v, w)$.

4.1.1 Numerical data of the optimal, congruent hyperball arrangements

First, we illustrate our computation method for given parameters and then we summarize the numerical data of optimal congruent hyperball arrangements for several parameters in Tables 1 and 2.

Results for parameters $u = 7$, $v = 3$, $w = 7$:

$$d(A_1, C) = \operatorname{arcosh}\frac{1}{\sqrt{a_{00}}} = d(A_2, Q) = \operatorname{arcosh}\frac{1}{\sqrt{a_{33}}} \approx 1.23469,$$
$$d(J, H)/2 = \frac{1}{2}\operatorname{arcosh}\frac{-a_{03}}{\sqrt{a_{00}a_{33}}} \approx 1.28517$$

Therefore, the optimal heights in all cases are equal: $h = h^0 = h^3 = d(A_1, C) = d(A_2, Q) \approx 1.23469$ (see (6), (7), (8)).

$$\mathrm{Vol}(\mathcal{O}(7,3,7)) \approx 0.38325, \quad \mathrm{Vol}(\mathcal{H}^h(\mathcal{A}_0)) = \mathrm{Vol}(\mathcal{H}^h(\mathcal{A}_3)) = \\ = \mathrm{Vol}(\mathcal{H}^{h^0}(\mathcal{A}_0)) = \mathrm{Vol}(\mathcal{H}^{h^3}(\mathcal{A}_3)) \approx 0.15586,$$

1. *Two congruent hyperballs:*

$$\delta^1(\mathcal{O}(7,3,7)) = \frac{2 \cdot \mathrm{Vol}(\mathcal{H}^h(\mathcal{A}_0) \cap \mathcal{O}(7,3,7))}{\mathrm{Vol}(\mathcal{O}(7,3,7))} \approx 0.81335,$$

2. *One hyperball:*

$$\delta^2_j(\mathcal{O}(7,3,7)) = \frac{\mathrm{Vol}(\mathcal{H}^h(\mathcal{A}_j) \cap \mathcal{O}(7,3,7))}{\mathrm{Vol}(\mathcal{O}(7,3,7))} \approx 0.40668, (j \in \{0,3\})$$

Remark 1 *If* $u = w$ *and* $h \geq h^0 = h^3$ *then* $\delta^2(\mathcal{O}(u,v,w)) = \frac{1}{2}\delta^1(\mathcal{O}(u,v,w))$ (*see the above example with parameters* $u = 7, v = 3, w = 7$).

In the following Table we summarize the data of the hyperball packings for some parameters $u, v, w \in \mathbb{N}$, where $\mathcal{A}_i$ $(i \in \{0,3\})$ is the area of the trigonal face (triangle CHL or EJQ) of the truncated tetrahedron related to the vertex A_i, cf. Fig. 1 (see (6), (7), (8) and Definition 1). We note here, that the role of the parameters u and w is symmetrical therefore we can assume, that $u \leq w$.

The volume $\mathrm{Vol}(\mathcal{O}(u,v,w))$ can be calculated by Theorem 1 The maximal volume sum $\sum_{i=0,3} \mathrm{Vol}(\mathcal{H}^h(\mathcal{A}_i))$ of the hyperball pieces lying in $\mathcal{O}(u,v,w)$ can be computed by the formulas (3), (6), (7), (8) and by the above described computation method for each given possible parameters u, v, w. Therefore, the maximal density of the congruent hyperball packings related to every doubly truncated orthoscheme tiling with “two hyperball types” – $\delta^1(\mathcal{O}(u,v,w))$ (see Definition 1) – can be computed for each possible parameters.

After careful analysis of the function, finally, we obtain the following

Theorem 2 *The density function* $\delta^1(\mathcal{O}(u,v,w))$, $\left(\frac{1}{u} + \frac{1}{v} < \frac{1}{2}, \ \frac{1}{v} + \frac{1}{w} < \frac{1}{2}. \ 3 \leq u, v, w \in \mathbb{N}\right)$ *attains its maximum at parameters* $\{u, v, w\} = \{7, 3, 7\}$ *with density* $\delta^1(\mathcal{O}(7,3,7)) \approx 0.81335$ (*see Table 1*).

Table 1, two congruent hyperballs				
$\{u,v,w\}$	h	$Vol(\mathcal{O}(u,v,w))$	$\sum_{i=0,3} Vol(\mathcal{H}^h(\mathcal{A}_i))$	$\delta^1(\mathcal{O}(u,v,w))$
$\{7,3,7\}$	1.23469	0.38325	0.31172	0.81335
$\{7,3,8\}$	0.93100	0.41326	0.25726	0.62251
$\{7,3,9\}$	0.76734	0.43171	0.23355	0.54099
⋮	⋮	⋮	⋮	⋮
$\{7,3,50\}$	0.11380	0.49016	0.06121	0.12488
⋮	⋮	⋮	⋮	⋮
$\{8,3,8\}$	0.94946	0.44383	0.33794	0.76143
$\{8,3,9\}$	0.78366	0.46266	0.29474	0.63704
$\{8,3,10\}$	0.67409	0.47536	0.26747	0.56266
⋮	⋮	⋮	⋮	⋮
$\{8,3,50\}$	0.11668	0.52248	0.06935	0.13274
⋮	⋮	⋮	⋮	⋮
$\{5,4,5\}$	0.88055	0.46190	0.36007	0.77955
$\{5,4,6\}$	0.73969	0.50747	0.37287	0.73476
$\{5,4,7\}$	0.59326	0.53230	0.32974	0.61947
⋮	⋮	⋮	⋮	⋮
$\{5,4,50\}$	0.07206	0.59291	0.06350	0.10710
⋮	⋮	⋮	⋮	⋮
$\{4,5,4\}$	0.80846	0.43062	0.31702	0.73620
$\{4,5,5\}$	0.69129	0.49789	0.38284	0.76893
$\{4,5,6\}$	0.53064	0.52971	0.33597	0.63426
⋮	⋮	⋮	⋮	⋮
$\{4,5,50\}$	0.05502	0.59318	0.05710	0.096256

Similarly to the above case the volume of the doubly truncated orthoscheme can be computed by Theorem 1 but here we use only one hyperball type. Now, the volume – $\max_{i=0,3}\{Vol(\mathcal{H}^{h^i}(\mathcal{A}_i)\}$ – of the optimal hyperball piece lying in $\mathcal{O}(u,v,w)$ can be computed by the formulas (3), (6), (7), (8) and by the described computation method above for each given possible parameters u,v,w. Therefore, the maximal density of the congruent hyperball packings related to a doubly truncated orthoscheme tiling with "one hyperball type" – $\delta^2(\mathcal{O}(u,v,w))$ (see Definition 1) – can be computed for each possible parameters.

After careful analysis of the function, finally, we obtain the following (see Fig. 3)

Theorem 3 *The density function* $\delta^2(\mathcal{O}(u,v,w))$, *(*$\frac{1}{u}+\frac{1}{v}<\frac{1}{2}$, $\frac{1}{v}+\frac{1}{w}<\frac{1}{2}$. $3 \le u,v,w \in \mathbb{N}$*) attains its maximum at parameters* $\{u,v,w\}=\{4,6,5\}$ *with density* $\delta^2(\mathcal{O}(4,6,5)) \approx 0.63548$ *(see Fig. 3 and Table 2).*

Table 2, one hyperball					
$\{u,v,w\}$	h^0	h^3	$Vol(\mathcal{O}(u,v,w))$	$\max_{i=0,3}\{Vol(\mathcal{H}^{h^i}(\mathcal{A}_i)\}$	$\delta^2(\mathcal{O}(u,v,w))$
$\{7,3,8\}$	0.93100	1.25596	0.41326	0.16371	0.39614
$\{7,3,9\}$	0.76734	1.27042	0.43171	0.16543	0.38320
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\{7,3,50\}$	0.11380	1.32226	0.49016	0.18040	0.36805
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\{8,3,9\}$	0.78366	0.96206	0.46266	0.17265	0.37316
$\{8,3,10\}$	0.67409	0.97104	0.47536	0.17531	0.36879
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\{8,3,50\}$	0.11668	1.00753	0.52248	0.18650	0.35695
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\{5,4,5\}$	1.02221	1.02221	0.46190	0.22942	0.49668
$\{5,4,6\}$	0.73969	1.07541	0.50747	0.25088	0.49437
$\{5,4,7\}$	0.59326	1.10694	0.53230	0.26448	0.49686
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\{5,4,50\}$	0.07206	1.19054	0.59291	0.30407	0.51284
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\{4,5,4\}$	1.06128	1.06128	0.43062	0.24500	0.56895
$\{4,5,5\}$	0.69129	1.16974	0.49789	0.29371	0.58990
$\{4,5,6\}$	0.53064	1.22646	0.52971	0.32284	0.60946
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\{4,5,50\}$	0.05502	1.19344	0.59318	0.30555	0.51510
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\{4,6,4\}$	0.88137	0.88137	0.50192	0.30049	0.59868
$\{4,6,5\}$	0.61415	0.97970	0.55992	0.35582	0.63548
$\{4,6,6\}$	0.48121	1.01251	0.58850	0.32284	0.58711
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\{4,6,50\}$	0.05138	0.88231	0.64697	0.30100	0.46522

4.2 Non-congruent hyperball packings

In this subsection we consider non-congruent hyperball packings related to the doubly truncated Coxeter orthoschemes which can be derived, similarly to the above section by truncation from an orthoschem $A_0A_1A_2A_3$ with outer essential vertices A_0 and A_3. The truncating planes are the polar planes of outer vertices A_0 and A_3, which *can be the ultraparallel base planes of hyperballs* $\mathcal{H}_i^s$ $(i=0,3)$ with height s. The distance between the two base planes is $2h^{03}(u,v,w) = d(H,J)$.

Lemma 5 *If* $u \leq w$ *then* $d(C, A_1) \leq d(Q, A_2)$ *where* u, v, w *are suitable given integer parameters.*

Proof. From Lemma 4 we obtain that $d(A_1, C) = \text{arcosh}\frac{1}{\sqrt{a_{00}}}$ and $d(A_2, Q) = \text{arcosh}\frac{1}{\sqrt{a_{33}}}$ where (a_{ij}) $(i, j = 0, 1, 2, 3)$ is the inverse of the corresponding Coxeter-Schläfli matrix (see (2)). From the matrix (a_{ij}) follows that

$$\frac{1}{\sqrt{a_{00}}} = 1/\sqrt{-\frac{\cos^2\frac{\pi}{w}\cos^2\frac{\pi}{u} - \cos^2\frac{\pi}{w} + \sin^2\frac{\pi}{v} - \cos^2\frac{\pi}{u}}{\cos^2\frac{\pi}{w} - \sin^2\frac{\pi}{v}}},$$
$$\frac{1}{\sqrt{a_{33}}} = 1/\sqrt{-\frac{\cos^2\frac{\pi}{w}\cos^2\frac{\pi}{u} - \cos^2\frac{\pi}{w} + \sin^2\frac{\pi}{v} - \cos^2\frac{\pi}{u}}{\cos^2\frac{\pi}{u} - \sin^2\frac{\pi}{v}}}.$$

Finally, we obtain the statement of the lemma directly from the above formulas. □

We may assume that $u \leq w$ because of the symmetrical role of parameters u and w. Therefore, the inequality $d(C, A_1) \leq d(Q, A_2)$ inequality holds.

We have to distinguish 2 different main cases all of which we set up from the optimal congruent hyperball packing configuration described in the former subsection.

1. $h = d(H, J)/2 \leq d(C, A_1)$ (and so $h \leq d(Q, A_2)$).

   Both polar planes are assigned hyperspheres. It is clear, that in the "starting configuration" (congruent case) the heights of optimal hyperballs $\mathcal{H}^{h^i}(\mathcal{A}_i)$ $(i = 0, 3)$ are equal: $h = h^0 = h^3$.

   (a) We consider the hyperball (hypersphere) $\mathcal{H}^{h^0}(\mathcal{A}_0)$ and blow up it keeping the hyperballs $\mathcal{H}^{h^3}(\mathcal{A}_3)$ tangent to it upto this hypersphere touch the plane $A_1A_2A_3$ or the plane EJQ (see Fig. 1). During this expansion the height of hyperball $\mathcal{H}^{h^0}$ will be $h^0 = h + x$ where $x \in [0, \min\{d(C, A_1) - h, h\}]$, furthermore the height of hyperball $\mathcal{H}^{h^3}(\mathcal{A}_3)$ will be $h^3 = h - x$. ($x = 0$ means that the hyperballs are congruent.)

   (b) We consider the hyperball $\mathcal{H}^{h^3}(\mathcal{A}_3)$ and blow up it keeping the hyperball $\mathcal{H}^{h^0}(\mathcal{A}_0)$ tangent to it upto this hypersphere touch the plane $A_0A_1A_2$ or the plane CLH (see Fig. 1). During this expansion the height of hyperball $\mathcal{H}^{h^3}(\mathcal{A}_3)$ will be $h^3 = h + x$ where $x \in [0, \min\{d(Q, A_2) - h, h\}]$, furthermore the height of hyperball $\mathcal{H}^{h^0}(\mathcal{A}_0)$ will be $h^0 = h - x$.

2. $h = d(C, A_1)$, $(d(C, A_1) \leq d(Q, A_2)$ because $u \leq w)$.

   In this case we distinguish two subcases:

   (a) We blow up the hyperball $\mathcal{H}^{h^3}(\mathcal{A}_3)$ until it touch the plane $A_0A_1A_2$ or the hyperball $\mathcal{H}^{h^0}(\mathcal{A}_0)$ (see Fig. 1). During this expansion the height of hyperball $\mathcal{H}^{h^0}(\mathcal{A}_0)$ is $h^0 = h$ constant and the height of hyperball $\mathcal{H}^{h^3}(\mathcal{A}_3)$ is $h^3 = h + x$ where $x \in [0, \min\{d(Q, A_2) - d(C, A_1), d(H, J) - 2h\}]$. (If $x = 0$ then the hyperballs are congruent.)

   (b) If in the above situation the hyperball $\mathcal{H}^{h^3}(\mathcal{A}_3)$ first touches the hyperball $\mathcal{H}^{h^0}(\mathcal{A}_0)$ $(h^3 = d(H, J) - h \leq d(Q, A_2))$ then we can continue the blowing of hyperball $\mathcal{H}^{h^3}(\mathcal{A}_3)$ until this hypersphere touch the plane $A_0A_1A_2$ or the plane CLH (see Fig. 1). During this expansion the height of hyperball $\mathcal{H}^{h^3}(\mathcal{A}_3)$ will be $h^3 = d(H, J) - h + x$ where $x \in [0, \min\{d(Q, A_2) - d(H, J) + h, h\}]$, furthermore the height of hyperball $\mathcal{H}^{h^0}(\mathcal{A}_0)$ will be $h^0 = h - x$.

We extend this arrangements to images of the hyperballs $\mathcal{H}^{h^i}(\mathcal{A}_i)$ $(i \in \{0, 3\})$ by the considered Coxeter group and obtain non-congruent hyperball packing $\mathcal{B}(x)$. Its density is defined by the following

Definition 2 *The locally density functions* $\delta^j_x(\mathcal{O}(u, v, w))$ *of the non-congruent hyperball packings related to* $\mathcal{O}(u, v, w)$ *in the above cases* $(j \in \{1, 2\})$ *are defined by the next formulas:*

$$\delta^j_x(\mathcal{O}(u, v, w)) := \frac{\mathrm{Vol}(\mathcal{H}^{h^0}(\mathcal{A}_0) \cap \mathcal{O}(u, v, w)) + \mathrm{Vol}(\mathcal{H}^{h^3}(\mathcal{A}_3) \cap \mathcal{O}(u, v, w))}{\mathrm{Vol}(\mathcal{O}(u, v, w))},$$

where $0 \leq h^0, h^3$ *are suitable real parameters related to the above main non-congruent cases (depend on* x *parameter) and* $\mathcal{A}_0$ *is the area of the triangle* CLH *and* $\mathcal{A}_3$ *is the area of the triangle* JEQ *(see (6)).*

The main problem is to find *the maximum of density function* $\delta^j_x(\mathcal{O}(u, v, w))$ *for suitable integer parameters* u, v, w *where* $x \in \mathbb{R}$, *and* $x \in [0, \min\{d(Q, A_2) - d(C, A_1), d(H, J) - 2h\}]$ *or* $x \in [0, \min\{d(Q, A_2) - d(H, J) + h, h\}]$ (see the above two main cases).

4.2.1 Numerical data of non-congruent hyperball packing arrangements

First, we illustrate our computation method for given parameters and then we summarize the numerical data of optimal non-congruent hyperball arrangements for several given parameters in Table 3.

Results for parameters $u = 5$, $v = 4$, $w = 5$:

$$d(A_1, C) = \operatorname{arcosh}\frac{1}{\sqrt{a_{00}}} = d(A_2, Q) = \operatorname{arcosh}\frac{1}{\sqrt{a_{33}}} \approx 1.02221,$$
$$d(J, H)/2 = \frac{1}{2}\operatorname{arcosh}\frac{-a_{03}}{\sqrt{a_{00}a_{33}}} \approx 0.88055$$

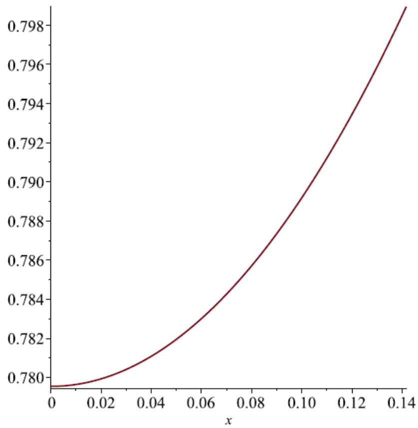


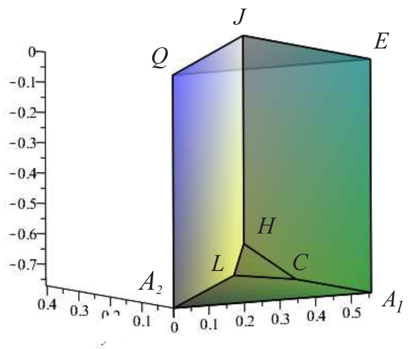
Figure 4: Density function $\delta^1_x(\mathcal{O}(5,4,5))$ where $x \in [0, d(C, A_1) - h \approx 0.14166]$

Therefore, the heights in the starting position ($x = 0$): $h = h^0 = h^3 = d(J, H)/2 \approx 0.88055$ (see (6), (7), (8)) and $\mathrm{Vol}(\mathcal{O}(5,4,5)) \approx 0.38325$.

We consider the hyperball $\mathcal{H}^{h^0}(\mathcal{A}_0)$ and blow up it keeping the hyperball $\mathcal{H}^{h^3}(\mathcal{A}_3)$ tangent to it until this hypersphere touch the plane $A_1A_2A_3$ (see Fig. 1). During this expansion the height of hyperball $\mathcal{H}^{h^0}$ will be $h^0 = h + x$ where, furthermore the height of hyperball $\mathcal{H}^{h^3}(\mathcal{A}_3)$ is $h^3 = h - x$. By the Definition 2 of the density function $\delta^1_x(\mathcal{O}(5,4,5))$ that it is a strictly increasing function in the intervall $[0, d(C, A_1) - h \approx 0.14166]$ (see Fig. 4). Thus, the optimal arrangement belongs to the parameter $x^{opt} = d(C, A_1) - h \approx 0.14166$:

$$\delta^1_{x^{opt}}(\mathcal{O}(5,4,5)) =$$
$$= \frac{\mathrm{Vol}(\mathcal{H}^{h^0}(\mathcal{A}_0) \cap \mathcal{O}(5,4,5)) + \mathrm{Vol}(\mathcal{H}^{h^3}(\mathcal{A}_3) \cap \mathcal{O}(5,4,5))}{\mathrm{Vol}(\mathcal{O}(5,4,5))} \approx 0.79895,$$

where $x^{opt} = d(C, A_1) - h \approx 0.14166$ and $\mathcal{A}_0 = \mathcal{A}_3 = \pi/2 - \pi/5 - \pi/4$. (see (6)).



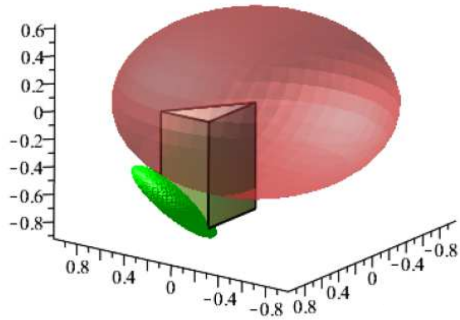


Figure 5: Locally optimal non-congruent hyperball packing configuration related to parameters $\{5, 4, 5\}$

Table 3, allowing non-congruent hyperballs				
$\{u, v, w\}$	h^0	h^3	$\sum_{i=0,3} Vol(\mathcal{H}^{h^i}(\mathcal{A}_i))$	$\delta^1(\mathcal{O}(u, v, w))$
$\{7, 3, 7\}$	1.23469	1.23469	0.31172	0.81335
$\{7, 3, 8\}$	0.93100	1.25596	0.32520	0.78690
$\{7, 3, 9\}$	0.76734	1.27042	0.32892	0.76189
⋮	⋮	⋮	⋮	⋮
$\{7, 3, 50\}$	0.11380	1.32226	0.23307	0.47549
⋮	⋮	⋮	⋮	⋮
$\{8, 3, 8\}$	0.94946	0.94946	0.33794	0.76143
$\{8, 3, 9\}$	0.78366	0.96206	0.34107	0.73718
$\{8, 3, 10\}$	0.67409	0.97104	0.33990	0.71504
⋮	⋮	⋮	⋮	⋮
$\{8, 3, 50\}$	0.11668	1.00753	0.24051	0.46032
⋮	⋮	⋮	⋮	⋮
$\{5, 4, 5\}$	0.73890	1.02221	0.36903	0.79895
$\{5, 4, 6\}$	0.73969	0.83611	0.39956	0.78736
$\{5, 4, 7\}$	0.59326	0.90486	0.41263	0.77517
⋮	⋮	⋮	⋮	⋮
$\{5, 4, 50\}$	0.07206	1.19054	0.35623	0.60082
⋮	⋮	⋮	⋮	⋮
$\{4, 5, 4\}$	0.55565	1.06128	0.34184	0.79382
$\{4, 5, 5\}$	0.69129	0.69129	0.38284	0.76893
$\{4, 5, 6\}$	0.53064	0.77568	0.39374	0.74331
⋮	⋮	⋮	⋮	⋮
$\{4, 5, 50\}$	0.05502	1.13842	0.32720	0.55161
$\{5, 5, 5\}$	0.35764	0.77537	0.41589	0.72618

Similarly to the above cases the volume of a doubly truncated orthoscheme can be computed by Theorem 1 and here we allow non-congruent hyperballs. The volume sum $\sum_{i=0,3} \mathrm{Vol}(\mathcal{H}^{h^i}(\mathcal{A}_i))$ of hyperball pieces lying in $\mathcal{O}(u,v,w)$ can be computed by the formulas (3), (6), (7), (8) and by the above described computation method for each given possible parameters u, v, w. However, the computations can be contained some subcases, so the determination of the densest hyperball configuration for given parameters u, v, w more complicated than in the congruent cases, as the above example shows.

The maximal density of the congruent hyperball packing related to the doubly truncated orthoscheme tilings with "non-congruent hyperballs" – $\delta^i_x(\mathcal{O}(u,v,w))$ (see Definition (2), $i \in \{1,2\}$) – can be computed for each possible parameters.

Finally, we can formulate only the next Theorem and Conjecture (see Fig. 5):

Theorem 4 *The density functions* $\delta^i_x(\mathcal{O}(u,v,w))$, $(\frac{1}{u}+\frac{1}{v}<\frac{1}{2},\ \frac{1}{v}+\frac{1}{w}<\frac{1}{2}.$ $3 \le u,v,w \in \mathbb{N}$ *and* $x \in [0, \min\{d(Q,A_2)-d(C,A_1), d(H,J)-2h\}]$ *or* $x \in [0, \min\{d(Q,A_2)-d(H,J)+h, h\}]$) *attain their maximum at* $\{u,v,w\}=\{7,3,7\}$ *with* $x=0$ (*congruent case*) *among the investigated parameters with density* $\delta^1_x(\mathcal{O}(7,3,7)) \approx 0.81335$ (*see Table 3*).

Conjecture 1 *The density functions* $\delta^i_x(\mathcal{O}(u,v,w))$, $(\frac{1}{u}+\frac{1}{v}<\frac{1}{2},\ \frac{1}{v}+\frac{1}{w}<\frac{1}{2}.$ $3 \le u,v,w \in \mathbb{N}$ *and* $x \in [0, \min\{d(Q,A_2)-d(C,A_1), d(H,J)-2h\}]$ *or* $x \in [0, \min\{d(Q,A_2)-d(H,J)+h, h\}]$) *attain their maximum at* $\{u,v,w\}=\{7,3,7\}$ *with* $x=0$ *(congruent case) and with density* $\delta^1(\mathcal{O}(7,3,7)) \approx 0.81335$ (*see Table 3*).

4.2.2 On non-extendable congruent hyperball packings to parameters $\{p,3,p\}$ $(6<p<7,\ p\in\mathbb{R})$

We can construct infinitely many congruent and non congruent hyperball configuration whose densities are locally larger then the Böröczky-Florian density upper bound (≈ 0.85328). Now, we describe only one, congruent locally dense hyperball arrangement related to parameters $\{p,3,p\}$ $(6<p<7,\ p\in\mathbb{R})$.

The computation method described in the former sections is suitable to determine the densities of congruent hyperball packings for real parameters $(6<p<7,\ p\in\mathbb{R})$ as well. To each such p parameter belongs a doubly truncated orthoscheme and therefore we can determine similarly to the above cases the corresponding maximal density of its optimal congruent hyperball packing. But these packings can not be extended to the 3-dimensional space.

Analysing these *non-extendable packings* for these parameters p we obtain the following (see Fig. 6).

Theorem 5 *The function* $\delta^1(\mathcal{O}(u=p, v=3, w=p))$, $\big(\frac{1}{u}+\frac{1}{v}<\frac{1}{2},\ \frac{1}{v}+\frac{1}{w}<\frac{1}{2}.\ 3 \le u,v,w \in \mathbb{N}$, *see Definition 1*$\big)$ *is attained its maximum for the parameter* $p_{opt} \approx 6.05061$ *and the density is* ≈ 0.85461. *That means that these locally optimal hyperball configurations provide larger density than the Böröczky-Florian density upper bound* (≈ 0.85328) *for ball and horoball packings* [4].

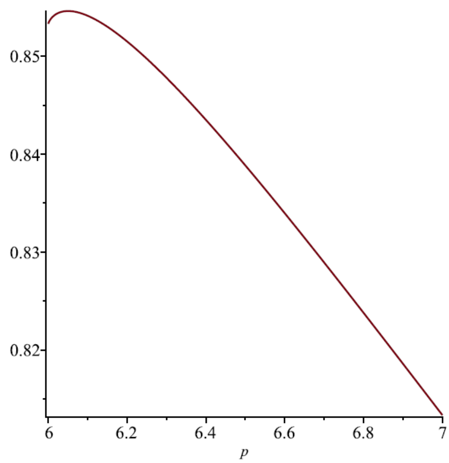


Figure 6: Locally optimal non-extendable congruent hyperball packing configuration related to parameters $\{p, 3, p\}$ $(6 < p < 7,\ p \in \mathbb{R})$

In hyperbolic spaces $\mathbb{H}^n$ $(n \ge 3)$ the problems of the densest horoball and hyperball packings have not been settled yet, in general (see e.g. [9], [19], [20]). Moreover, the optimal sphere packing problem can be extended to the other homogeneous Thurston geometries, e.g. **Nil**, **Sol**, $\widetilde{\mathbf{SL_2R}}$. For these non-Euclidean geometries only very few results are known (e.g. [25] and the references given there).

By the above investigation we can say that the revisited Kepler problem keep several interesting open questions. Detailed studies are the objective of ongoing research. Applications of the above projective method seem to be interesting in (non-Euclidean) crystallography as well.

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