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Harmonic univalent functions defined by post quantum calculus operators

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Abstract. We study a family of harmonic univalent functions in the open unit disc defined by using post quantum calculus operators. We first obtained a coefficient characterization of these functions. Using this, coefficients estimates, distortion and covering theorems were also obtained. The extreme points of the family and a radius result were also obtained. The results obtained include several known results as special cases.

1 Introduction

Let $\mathcal{A}$ be the class of functions f that are analytic in the open unit disc $\mathbb{D} := \{z : |z| < 1\}$ with the normalization $f(0) = f'(0) - 1 = 0$. A function $f \in \mathcal{A}$ can be expressed in the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in \mathbb{D}. \tag{1}$$

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The theory of (p, q)-calculus (or post quantum calculus) operators are used in various areas of science and also in the geometric function theory. Let $0 < q \leq p \leq 1$. The (p, q)-bracket or twin-basic number $[k]_{p,q}$ is defined by

$$[k]_{p,q} = \frac{p^k - q^k}{p - q} \quad (q \neq p), \quad \text{and} \quad [k]_{p,p} = kp^{k-1}.$$

Notice that $\lim_{q\to p}[k]_{p,q} = [k]_{p,p}$. For $0 < q \leq 1$, q-bracket $[k]_q$ for $k = 0, 1, 2, \cdots$ is given by

$$[k]_q = [k]_{1,q} = \frac{1 - q^k}{1 - q} \quad (q \neq 1), \quad \text{and} \quad [k]_1 = [k]_{1,1} = k.$$

The (p, q)-derivative operator $D_{p,q}$ of a function $f \in \mathcal{A}$ is given by

$$D_{p,q}f(z) = 1 + \sum_{k=2}^{\infty} [k]_{p,q} a_k z^{k-1}. \tag{2}$$

For a function $f \in \mathcal{A}$, it can be easily seen that

$$D_{p,q}f(z) = \frac{f(pz) - f(qz)}{(p - q)z}, \quad (p \neq q, z \neq 0), \tag{3}$$

$(D_{p,q}f)(0) = 1$ and $(D_{p,p}f)(z) = f'(z)$. For definitions and properties of (p, q)-calculus, one may refer to [6]. The $(1, q)$-derivative operator $D_{1,q}$ is known as the q-derivative operator and is denoted by D_q; for $z \neq 0$, it satisfies

$$(D_qf)(z) = \frac{f(z) - f(qz)}{(1 - q)z}. \tag{4}$$

For definitions and properties of q-derivative operator, one may refer to [3, 9, 10, 11, 8].

For a function h analytic in $\mathbb{D}$ and an integer $m \geq 0$, we define the (p, q)-Sălăgean differential operator $L^m_{p,q}$, using (p, q)-derivative operator, by

$$L^0_{p,q}h(z) = h(z) \quad \text{and} \quad L^m_{p,q}h(z) = zD_{p,q}(L^{m-1}_{p,q}(h(z))).$$

For analytic function $g(z) = \sum_{k=1}^{\infty} b_k z^k$, we have

$$L^m_{p,q}g(z) = \sum_{k=1}^{\infty} [k]^m_{p,q} b_k z^k. \tag{5}$$

In particular, for $h \in \mathcal{A}$ with $h(z) = z + \sum_{k=2}^{\infty} a_k z^k$, we have

$$L_{p,q}^m h(z) = z + \sum_{k=2}^{\infty} [k]_{p,q}^m a_k z^k. \tag{6}$$

Let $\mathcal{H}$ be the family of complex-valued harmonic functions $f = h + \overline{g}$ defined in $\mathbb{D}$, where h and g has the following power series expansion

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad \text{and} \quad g(z) = \sum_{k=1}^{\infty} b_k z^k. \tag{7}$$

Note that $f = h + \overline{g}$ is sense-preserving in $\mathbb{D}$ if and only if $h'(z) \neq 0$ in $\mathbb{D}$ and the second dilatation w of f satisfies the condition $|g'(z)/h'(z)| < 1$ in $\mathbb{D}$. Let $\mathcal{S}_{\mathcal{H}}$ be a subclass of functions f in $\mathcal{H}$ that are sense-preserving and univalent in $\mathbb{D}$. Clunie and Sheil-Small studied the class $\mathcal{S}_{\mathcal{H}}$ in their remarkable paper [5]. For a survey or comprehensive study of the theory of harmonic univalent functions, one may refer to the papers [1, 2, 7]. We introduce and study a new subclass of harmonic univalent functions by using (p, q)-Sălăgean harmonic differential operator $L_{p,q}^m : \mathcal{H} \to \mathcal{H}$. For the functions in the newly introduced family, a coefficient characterization is obtained (Theorem 3). Using this, coefficients estimates (Corollary 4), distortion (Theorem 6) and covering (Corollary 7) theorems were also obtained. The extreme points of the family (Theorem 5) and a radius result (Theorem 8) were also obtained. The results obtained include several known results as special cases. Our results can be extended, for example, by using fractional q-integral operator (see Ravikumar [16]).

2 Main results

We define the (p, q)-Sălăgean harmonic differential operator $L_{p,q}^m$ of a harmonic function $f = h + \overline{g} \in \mathcal{H}$ by

$$\begin{aligned} L_{p,q}^m f(z) &= L_{p,q}^m h(z) + (-1)^m \overline{L_{p,q}^m g(z)} \\ &= z + \sum_{k=2}^{\infty} [k]_{p,q}^m a_k z^k + (-1)^m \sum_{k=1}^{\infty} [k]_{p,q}^m \overline{b_k z^k}. \end{aligned} \tag{8}$$

This last expression is obtained by using (6) and (5) and is motivated by Sălăgean[17]. Recall that convolution (or the Hadamard product) of two complex-valued harmonic functions

$$f_1(z) = z + \sum_{k=2}^{\infty} a_{1k} z^k + \sum_{k=1}^{\infty} \overline{b_{1k} z^k} \quad \text{and} \quad f_2(z) = z + \sum_{k=2}^{\infty} a_{2k} z^k + \sum_{k=1}^{\infty} \overline{b_{2k} z^k}$$

is defined by

$$f_1(z) * f_2(z) = (f_1 * f_2)(z) = z + \sum_{k=2}^{\infty} a_{1k}a_{2k}z^k + \sum_{k=1}^{\infty} \overline{b_{1k}b_{2k}z^k}, \quad z \in \mathbb{D}.$$

We now introduce a family of (p, q)-Sălăgean harmonic univalent functions by using convolution and the (p, q)-Sălăgean harmonic differential operator $L^m_{p,q}$.

Definition 1 *Suppose* $i, j \in \{0, 1\}$. *Let the function* Φ_i, Ψ_j *given by*

$$\Phi_i(z) = z + \sum_{k=2}^{\infty} \lambda_k z^k + (-1)^i \sum_{k=1}^{\infty} \mu_k \overline{z}^k, \tag{9}$$

$$\Psi_j(z) = z + \sum_{k=2}^{\infty} u_k z^k + (-1)^j \sum_{k=1}^{\infty} \nu_k \overline{z}^k \tag{10}$$

be harmonic in $\mathbb{D}$ *with* $\lambda_k > u_k \geq 0$ $(k \geq 2)$ *and* $\mu_k > \nu_k \geq 0$ $(k \geq 1)$. *For* $\alpha \in [0, 1)$, $0 < q \leq p \leq 1$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $m > n$ *and* $z \in \mathbb{D}$, *let* $\mathcal{S}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$ *denote the family of harmonic functions* f *in* $\mathcal{H}$ *that satisfy the condition*

$$\operatorname{Re}\left\{ \frac{(L^m_{p,q} f * \Phi_i)(z)}{(L^n_{p,q} f * \Psi_j)(z)} \right\} > \alpha, \tag{11}$$

where $L^m_{p,q}$ *is defined by* (8).

Using (8), (9) and (10), we obtain

$$(L^m_{p,q} f * \Phi_i)(z) = z + \sum_{k=2}^{\infty} \lambda_k [k]^m_{p,q} a_k z^k + (-1)^{m+i} \sum_{k=1}^{\infty} \mu_k [k]^m_{p,q} b_k \overline{z}^k, \tag{12}$$

and

$$(L^n_{p,q} f * \Psi_j)(z) = z + \sum_{k=2}^{\infty} u_k [k]^n_{p,q} a_k z^k + (-1)^{n+j} \sum_{k=1}^{\infty} \nu_k [k]^n_{p,q} b_k \overline{z}^k. \tag{13}$$

Definition 2 *Let* $\mathcal{TS}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$ *be the family of harmonic functions* $f_m = h + \overline{g}_m \in \mathcal{TS}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$ *such that* h *and* g_m *are of the form*

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k \quad \text{and} \quad g_m(z) = (-1)^{m+i-1} \sum_{k=1}^{\infty} |b_k| z^k, \quad |b_1| < 1. \tag{14}$$

The families of $\mathcal{S}_H(m,n,\Phi_i,\Psi_j,p,q,\alpha)$ and $\mathcal{TS}_H(m,n,\Phi_i,\Psi_j,p,q,\alpha)$ include a variety of well-known subclasses of harmonic functions as well as many new ones. For example,

(1) $\mathcal{S}_H(m,n,\alpha) \equiv \mathcal{S}_H(m,n,\frac{z}{(1-z)^2}-\frac{\bar{z}}{(1-\bar{z})^2},\frac{z}{1-z}+\frac{\bar{z}}{1-\bar{z}},1,1,\alpha)$,
$\mathcal{TS}_H(m,n,\alpha) \equiv \mathcal{TS}_H(m,n,\frac{z}{(1-z)^2}-\frac{\bar{z}}{(1-\bar{z})^2},\frac{z}{1-z}+\frac{\bar{z}}{1-\bar{z}},1,1,\alpha)$, [18].

(2) $\mathcal{S}^*_H(\alpha) \equiv \mathcal{S}_H(1,0,\frac{z}{(1-z)^2}-\frac{\bar{z}}{(1-\bar{z})^2},\frac{z}{1-z}+\frac{\bar{z}}{1-\bar{z}},1,1,\alpha)$,
$\mathcal{TS}^*_H(\alpha) \equiv \mathcal{TS}_H(1,0,\frac{z}{(1-z)^2}-\frac{\bar{z}}{(1-\bar{z})^2},\frac{z}{1-z}+\frac{\bar{z}}{1-\bar{z}},1,1,\alpha)$, [12].

(3) $\mathcal{K}_H(\alpha) \equiv \mathcal{S}_H(2,1,\frac{z+z^2}{(1-z)^3}+\frac{\bar{z}+\bar{z}^2}{(1-\bar{z})^3},\frac{z}{(1-z)^2}-\frac{\bar{z}}{(1-\bar{z})^2},1,1,\alpha)$,
$\mathcal{TK}_H(\alpha) \equiv \mathcal{TS}_H(2,1,\frac{z+z^2}{(1-z)^3}+\frac{\bar{z}+\bar{z}^2}{(1-\bar{z})^3},\frac{z}{(1-z)^2}-\frac{\bar{z}}{(1-\bar{z})^2},1,1,\alpha)$, [13].

(4) $\mathcal{S}^*_{H_q}(\alpha) \equiv \mathcal{S}_H(1,0,\frac{z}{(1-z)^2}-\frac{\bar{z}}{(1-\bar{z})^2},\frac{z}{1-z}+\frac{\bar{z}}{1-\bar{z}},1,q,\alpha)$,
$\mathcal{TS}^*_{H_q}(\alpha) \equiv \mathcal{TS}_H(1,0,\frac{z}{(1-z)^2}-\frac{\bar{z}}{(1-\bar{z})^2},\frac{z}{1-z}+\frac{\bar{z}}{1-\bar{z}},1,q,\alpha)$, [4].

(5) $\mathcal{K}_{H_q}(\alpha) \equiv \mathcal{S}_H(2,1,\frac{z+z^2}{(1-z)^3}+\frac{\bar{z}+\bar{z}^2}{(1-\bar{z})^3},\frac{z}{(1-z)^2}-\frac{\bar{z}}{(1-\bar{z})^2},1,q,\alpha)$,
$\mathcal{TK}_{H_q}(\alpha) \equiv \mathcal{TS}_H(2,1,\frac{z+z^2}{(1-z)^3}+\frac{\bar{z}+\bar{z}^2}{(1-\bar{z})^3},\frac{z}{(1-z)^2}-\frac{\bar{z}}{(1-\bar{z})^2},1,q,\alpha)$.

(6) $\mathcal{S}_H(n+1,n,q,\alpha) \equiv \mathcal{S}_H(n+1,n,\frac{z}{(1-z)^2}-\frac{\bar{z}}{(1-\bar{z})^2},\frac{z}{1-z}+\frac{\bar{z}}{1-\bar{z}},1,q,\alpha)$,
$\mathcal{TS}_H(n+1,n,q,\alpha) \equiv \mathcal{TS}_H(n+1,n,\frac{z}{(1-z)^2}-\frac{\bar{z}}{(1-\bar{z})^2},\frac{z}{1-z}+\frac{\bar{z}}{1-\bar{z}},1,q,\alpha)$, [14].

(7) $\mathcal{S}_H(\Phi_i,\Psi_j,\alpha) \equiv \mathcal{S}_H(0,0,\Phi_i,\Psi_j,1,1,\alpha)$,
$\mathcal{TS}_H(\Phi_i,\Psi_j,\alpha) \equiv \mathcal{TS}_H(0,0,\Phi_i,\Psi_j,1,1,\alpha)$, [15].

We first prove coefficient conditions for the functions in $\mathcal{S}_H(m,n,\Phi_i,\Psi_j,p,q,\alpha)$ and $\mathcal{TS}_H(m,n,\Phi_i,\Psi_j,p,q,\alpha)$.

Theorem 3 *Let the function* $f = h + \overline{g}$ *be such that the functions* h *and* g *are given by* (7). *Also, let the* (p,q)*-coefficient inequality*

$$\sum_{k=2}^{\infty} \frac{\lambda_k [k]^m_{p,q} - \alpha u_k [k]^n_{p,q}}{1-\alpha} |a_k| + \sum_{k=1}^{\infty} \frac{\mu_k [k]^m_{p,q} - (-1)^{n+j-(m+i)} \alpha \nu_k [k]^n_{p,q}}{1-\alpha} |b_k| \leq 1, \tag{15}$$

be satisfied for $\alpha \in [0,1)$, $0 < q \leq p \leq 1$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $m > n$, $\lambda_k > u_k \geq 0$ $(k \geq 2)$ *and* $\mu_k > \nu_k \geq 0$ $(k \geq 1)$. *Then*

(i) *the function* $f = h + \overline{g}$ *given by* (7) *is a sense-preserving harmonic univalent functions in* $\mathbb{D}$ *and* $f \in \mathcal{S}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$ *if the inequality in* (15) *is satisfied.*

(ii) *the function* $f_m = h+\overline{g}_m$ *given by* (14) *is in the* $\mathcal{TS}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$ *if and only if the inequality in* (15) *is satisfied.*

Proof. (i). Using the techniques used in [14] and [15], it is a routine step to prove that $f = h + \overline{g}$ given by (7) is sense-preserving and locally univalent in $\mathbb{D}$. Using the fact $\text{Re}(w) > \alpha$ if and only if $|1-\alpha+w| \geq |1+\alpha-w|$, it suffices to show that

$$\left|1-\alpha+\frac{(L^m_{p,q}f*\Phi_i)(z)}{(L^n_{p,q}f*\Psi_j)(z)}\right|-\left|1+\alpha-\frac{(L^m_{p,q}f*\Phi_i)(z)}{(L^n_{p,q}f*\Psi_j)(z)}\right|\geq 0. \tag{16}$$

In view of (12) and (13), left side of (16) yields

$$\begin{aligned}
&\left|(L^m_{p,q}f*\Phi_i)(z)+(1-\alpha)(L^n_{p,q}f*\Psi_j)(z)\right| \\
&\qquad -\left|(L^m_{p,q}f*\Phi_i)(z)-(1+\alpha)(L^n_{p,q}f*\Psi_j)(z)\right| \\
&=\Bigg|(2-\alpha)z+\sum_{k=2}^{\infty}(\lambda_k[k]^m_{p,q}+(1-\alpha)u_k[k]^n_{p,q})a_kz^k \\
&\qquad +(-1)^{m+i}\sum_{k=1}^{\infty}(\mu_k[k]^m_{p,q}+(-1)^{n+j-(m+i)}(1-\alpha)\nu_k[k]^n_{p,q})b_k\overline{z}^k\Bigg| \\
&\qquad -\Bigg|-\alpha z+\sum_{k=2}^{\infty}(\lambda_k[k]^m_{p,q}-(1+\alpha)u_k[k]^n_{p,q})a_kz^k \\
&\qquad +(-1)^{m+i}\sum_{k=1}^{\infty}(\mu_k[k]^m_{p,q}-(-1)^{n+j-(m+i)}(1+\alpha)\nu_k[k]^n_{p,q})b_k\overline{z}^k\Bigg| \\
&\geq(2-2\alpha)|z|-2\sum_{k=2}^{\infty}(\lambda_k[k]^m_{p,q}-\alpha u_k[k]^n_{p,q})|a_k||z|^k \\
&\qquad -\sum_{k=1}^{\infty}(\mu_k[k]^m_{p,q}+(-1)^{n+j-(m+i)}(1-\alpha)\nu_k[k]^n_{p,q})|b|_k|z|^k \\
&\qquad -\sum_{k=1}^{\infty}(\mu_k[k]^m_{p,q}-(-1)^{n+j-(m+i)}(1+\alpha)\nu_k[k]^n_{p,q})|b|_k|z|^k \\
&\geq(1-\alpha)|z|\Bigg[1-\sum_{k=2}^{\infty}\frac{\lambda_k[k]^m_{p,q}-\alpha u_k[k]^n_{p,q}}{1-\alpha}|a_k||z|^{k-1}
\end{aligned}$$

$$-\sum_{k=1}^{\infty}\frac{\mu_k[k]_{p,q}^m-(-1)^{n+j-(m+i)}\alpha\nu_k[k]_{p,q}^n}{1-\alpha}|b_k||z|^{k-1}\Bigg]$$

$$>(1-\alpha)|z|\Bigg[1-\Bigg(\sum_{k=2}^{\infty}\frac{\lambda_k[k]_{p,q}^m-\alpha u_k[k]_{p,q}^n}{1-\alpha}|a_k|$$

$$+\sum_{k=1}^{\infty}\frac{\mu_k[k]_{p,q}^m-(-1)^{n+j-(m+i)}\alpha\nu_k[k]_{p,q}^n}{1-\alpha}|b_k|\Bigg)\Bigg].$$

This last expression is non-negative because of the condition given in (15). This completes the proof of part (i) of theorem.

(ii). Since

$$\mathcal{TS}_H(m,n,\Phi_i,\Psi_j,p,q,\alpha)\subset\mathcal{S}_H(m,n,\Phi_i,\Psi_j,p,q,\alpha),$$

the sufficient part of part (ii) follows from part (i). In order to prove the necessary part of part (ii), we assume that $f_m\in\mathcal{TS}_H(m,n,\Phi_i,\Psi_j,p,q,\alpha)$. We notice that

$$\mathrm{Re}\left\{\frac{(L_{p,q}^m f*\Phi_i)(z)}{(L_{p,q}^n f*\Psi_j)(z)}-\alpha\right\}$$

$$=\mathrm{Re}\left\{\frac{(1-\alpha)z-\sum_{k=2}^{\infty}(\lambda_k[k]_{p,q}^m-\alpha u_k[k]_{p,q}^n)a_kz^k}{z-\sum_{k=2}^{\infty}u_k[k]_{p,q}^na_kz^k+(-1)^{m+i+n+j-1}\sum_{k=1}^{\infty}\nu_k[k]_{p,q}^nb_k\bar{z}^k}\right.$$

$$\left.+\frac{(-1)^{2m+2i-1}\sum_{k=1}^{\infty}(\mu_k[k]_{p,q}^m-(-1)^{n+j-(m+i)}\alpha\nu_k[k]_{p,q}^n)b_k\bar{z}^k}{z-\sum_{k=2}^{\infty}u_k[k]_{p,q}^na_kz^k+(-1)^{m+i+n+j-1}\sum_{k=1}^{\infty}\nu_k[k]_{p,q}^nb_k\bar{z}^k}\right\}$$

$$\geq\frac{(1-\alpha)-\sum_{k=2}^{\infty}(\lambda_k[k]_{p,q}^m-\alpha u_k[k]_{p,q}^n)a_kr^{k-1}-\sum_{k=1}^{\infty}(\mu_k[k]_{p,q}^m-(-1)^{n+j-(m+i)}\alpha\nu_k[k]_{p,q}^n)b_kr^{k-1}}{1-\sum_{k=2}^{\infty}u_k[k]_{p,q}^na_kr^{k-1}-(-1)^{m+i+n+j}\sum_{k=1}^{\infty}\nu_k[k]_{p,q}^nb_kr^{k-1}}$$

$$\geq 0,$$

by (11). The above inequality must hold for all $z\in\mathbb{D}$. In particular, choosing the values of z on the positive real axis and $z\to 1^-$, we obtain the required condition (15). This completes the proof of part (ii) of theorem.

The harmonic mappings

$$\begin{aligned} f(z) = z &+ \sum_{k=2}^{\infty} \frac{1-\alpha}{\lambda_k [k]_{p,q}^m - \alpha u_k [k]_{p,q}^n} x_k z^k \\ &+ \sum_{k=1}^{\infty} \frac{1-\alpha}{\mu_k [k]_{p,q}^m - (-1)^{n+j-(m+i)} \alpha \nu_k [k]_{p,q}^n} y_k \overline{z}^k, \end{aligned} \tag{17}$$

where $\sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1$, show that the coefficient bound given by (15) is sharp. □

Theorem 3 also yields the following corollary.

Corollary 4 *For the function $f_m = h + \overline{g}_m$ given by (14), we have*

$$|a_k| \leq \frac{1-\alpha}{\lambda_k [k]_{p,q}^m - \alpha u_k [k]_{p,q}^n}, \quad k \geq 2$$

and

$$|b_k| \leq \frac{1-\alpha}{\mu_k [k]_{p,q}^m - (-1)^{n+j-(m+i)} \alpha \nu_k [k]_{p,q}^n}, \quad k \geq 1.$$

The result is sharp for each k.

Using Theorem 3 (part ii), it is seen that the class $\mathcal{TS}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$ is convex and closed with respect to the topology of locally uniform convergence so that the closed convex hulls of $\mathcal{TS}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$ equals itself. The next theorem determines the extreme points of $\mathcal{TS}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$.

Theorem 5 *Let the function $f_m = h + \overline{g}_m$ be given by (14). Then the function $f_m \in \operatorname{clco} \mathcal{TS}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$ if and only if $f_m(z) = \sum_{k=1}^{\infty} (x_k h_k(z) + y_k g_{m_k}(z))$, where*

$$h_1(z) = z, \quad h_k(z) = z - \frac{1-\alpha}{\lambda_k [k]_{p,q}^m - \alpha u_k [k]_{p,q}^n} z^k, \quad k \geq 2,$$

$$g_{m_k}(z) = z + (-1)^{m+i-1} \frac{1-\alpha}{\mu_k [k]_{p,q}^m - (-1)^{n+j-(m+i)} \alpha \nu_k [k]_{p,q}^n} \overline{z}^k, \quad k \geq 1,$$

and $\sum_{k=1}^{\infty} (x_k + y_k) = 1$ where $x_k \geq 0$ and $y_k \geq 0$. In particular, the extreme points of $\mathcal{TS}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$ are $\{h_k\}$ and $\{g_{m_k}\}$.

Proof. For a function f_m of the form $f_m(z) = \sum\limits_{k=1}^{\infty}(x_k h_k(z) + y_k g_{m_k}(z))$, where $\sum\limits_{k=1}^{\infty}(x_k + y_k) = 1$, we have

$$f_m(z) = z - \sum_{k=2}^{\infty} \frac{1-\alpha}{\lambda_k [k]_{p,q}^m - \alpha u_k [k]_{p,q}^n} x_k z^k$$
$$+ \sum_{k=1}^{\infty} (-1)^{m+i-1} \frac{1-\alpha}{\mu_k [k]_{p,q}^m - (-1)^{n+j-(m+i)} \alpha v_k [k]_{p,q}^n} y_k \bar{z}^k.$$

Then $f_m \in \operatorname{clco} \mathcal{TS}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$ because

$$\sum_{k=2}^{\infty} \frac{\lambda_k [k]_{p,q}^m - \alpha u_k [k]_{p,q}^n}{1-\alpha} \left(\frac{1-\alpha}{\lambda_k [k]_{p,q}^m - \alpha u_k [k]_{p,q}^n} x_k \right) +$$

$$\sum_{k=1}^{\infty} \frac{\mu_k [k]_{p,q}^m - (-1)^{n+j-(m+i)} \alpha v_k [k]_{p,q}^n}{1-\alpha} \left(\frac{1-\alpha}{\mu_k [k]_{p,q}^m - (-1)^{n+j-(m+i)} \alpha v_k [k]_{p,q}^n} y_k \right)$$

$$= \sum_{k=2}^{\infty} x_k + \sum_{k=1}^{\infty} y_k = 1 - x_1 \leq 1.$$

Conversely, suppose $f_m \in \operatorname{clco} \mathcal{TS}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$. Then

$$|a_k| \leq \frac{1-\alpha}{\lambda_k [k]_{p,q}^m - \alpha u_k [k]_{p,q}^n} \quad \text{and} \quad |b_k| \leq \frac{1-\alpha}{\mu_k [k]_{p,q}^m - (-1)^{n+j-(m+i)} \alpha v_k [k]_{p,q}^n}.$$

Set

$$x_k = \frac{\lambda_k [k]_{p,q}^m - \alpha u_k [k]_{p,q}^n}{1-\alpha} |a_k| \text{ and } y_k = \frac{\mu_k [k]_{p,q}^m - (-1)^{n+j-(m+i)} \alpha v_k [k]_{p,q}^n}{1-\alpha} |b_k|.$$

By Theorem 3 (ii), $\sum\limits_{k=2}^{\infty} x_k + \sum\limits_{k=1}^{\infty} y_k \leq 1$. Therefore we define $x_1 = 1 - \sum\limits_{k=2}^{\infty} x_k - \sum\limits_{k=1}^{\infty} y_k \geq 0$. Consequently, we obtain $f_m(z) = \sum\limits_{k=1}^{\infty}(x_k h_k(z) + y_k g_{m_k}(z))$ as required. □

For functions in the class $\mathcal{TS}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$, the following theorem gives distortion bounds which in turns yields the covering result for this class.

Theorem 6 *Let the function* $f_m \in \mathcal{TS}_H(m,n,\Phi_i,\Psi_j,p,q,\alpha)$, $\gamma_k = \lambda_k[k]^m_{p,q} - \alpha u_k[k]^n_{p,q}$, $k \geq 2$ *and* $\phi_k = \mu_k[k]^m_{p,q} - (-1)^{n+j-(m+i)}\alpha v_k[k]^n_{p,q}$, $k \geq 1$. *If* $\{\gamma_k\}$ *and* $\{\phi_k\}$ *are non-decreasing sequences, then we have*

$$|f_m(z)| \leq (1+|b_1|)|z| + \frac{1-\alpha}{\beta}\left(1 - \frac{\mu_1 - (-1)^{n+j-(m+i)}\alpha v_1}{\beta}|b_1|\right)|z|^2 \quad (18)$$

and

$$|f_m(z)| \geq (1-|b_1|)|z| - \frac{1-\alpha}{\beta}\left(1 - \frac{\mu_1 - (-1)^{n+j-(m+i)}\alpha v_1}{\beta}|b_1|\right)|z|^2, \quad (19)$$

for all $z \in \mathbb{D}$, *where* $b_1 = f_{\bar{z}}(0)$ *and*

$$\beta = \min\{\gamma_2, \phi_2\} = \min\{\lambda_2[2]^m_{p,q} - \alpha u_2[2]^n_{p,q}, \mu_2[2]^m_{p,q} - (-1)^{n+j-(m+i)}\alpha v_2[2]^n_{p,q}\}.$$

Proof. Let the function $f_m \in \mathcal{TS}_H(m,n,\Phi_i,\Psi_j,p,q,\alpha)$. Taking the absolute value of f_m, we obtain

$$\begin{aligned}
|f_m(z)| &\leq (1+|b_1|)|z| + \sum_{k=2}^{\infty}(|a_k|+|b_k|)|z|^k \\
&\leq (1+|b_1|)|z| + \sum_{k=2}^{\infty}(|a_k|+|b_k|)|z|^2 \\
&\leq (1+|b_1|)|z| + \frac{1-\alpha}{\beta}\sum_{n=2}^{\infty}\left(\frac{\beta}{1-\alpha}|a_k| + \frac{\beta}{1-\alpha}|b_k|\right)|z|^2 \\
&\leq (1+|b_1|)|z| + \frac{1-\alpha}{\beta}\sum_{k=2}^{\infty}\left(\frac{\lambda_k[k]^m_{p,q} - \alpha u_k[k]^n_{p,q}}{1-\alpha}|a_k|\right. \\
&\quad \left. + \frac{\mu_k[k]^m_{p,q} - (-1)^{n+j-(m+i)}\alpha v_k[k]^n_{p,q}}{1-\alpha}|b_k|\right)|z|^2 \\
&\leq (1+|b_1|)|z| + \frac{1-\alpha}{\beta}\left(1 - \frac{\mu_1 - (-1)^{n+j-(m+i)}\alpha v_1}{1-\alpha}|b_1|\right)|z|^2.
\end{aligned}$$

This proves (18). The proof of (19) is omitted as it is similar to the proof of (18). □

The following covering result follows from the inequality (19).

Corollary 7 *Under the hypothesis of Theorem 6, we have*

$$\left\{w : |w| < \frac{1}{\beta}\left(\beta - 1 + \alpha + (\mu_1 - (-1)^{n+j-(m+i)}\alpha v_1 - \beta)|b_1|\right)\right\} \subset f(\mathbb{D}).$$

Theorem 8 *If the function* $f_m \in \mathcal{TS}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$, *then the function* f_m *is convex in the disc*

$$|z| \leq \min_k \left\{ \frac{1 - b_1}{k[1 - \frac{\mu_1 - (-1)^{n+j-(m+i)} \alpha \nu_1}{1-\alpha} b_1]} \right\}^{\frac{1}{k-1}}, \quad k \geq 2.$$

Proof. Let $f_m \in \mathcal{TS}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$ and let r, $0 < r < 1$, be fixed. Then $r^{-1} f_m(rz) \in \mathcal{TS}_H(m, n, \Phi_i, p, q, \alpha)$ and we have

$$\sum_{k=2}^{\infty} k^2(|a_k| + |b_k|) = \sum_{k=2}^{\infty} k(|a_k| + |b_k|) k r^{k-1}$$
$$\leq \sum_{k=2}^{\infty} \left(\frac{\lambda_k [k]_{p,q}^m - \alpha u_k [k]_{p,q}^n}{1 - \alpha} |a_k| + \frac{\mu_k [k]_{p,q}^m - (-1)^{n+j-(m+i)} \alpha \nu_k [k]_{p,q}^n}{1 - \alpha} |b_k| \right) k r^{k-1}$$
$$\leq \sum_{k=2}^{\infty} \left(1 - \frac{\mu_1 - (-1)^{n+j-(m+i)} \alpha \nu_1}{1 - \alpha} |b_1| \right) k r^{k-1}$$
$$\leq 1 - b_1$$

provided

$$k r^{k-1} \leq \frac{1 - b_1}{1 - \frac{\mu_1 - (-1)^{n+j-(m+i)} \alpha \nu_1}{1-\alpha} b_1}$$

which is true if

$$r \leq \min_k \left\{ \frac{1 - b_1}{k[1 - \frac{\mu_1 - (-1)^{n+j-(m+i)} \alpha \nu_1}{1-\alpha} b_1]} \right\}^{\frac{1}{k-1}}, \quad k \geq 2.$$

□

Remark 9 *Our results naturally includes several results known for those subclasses of harmonic functions listed after Definition 2.*

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gr-n-ideals in graded commutative rings

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Abstract. Let G be a group with identity e and let R be a G-graded ring. In this paper, we introduce and study the concept of gr-n-ideals of R. We obtain many results concerning gr-n-ideals. Some characterizations of gr-n-ideals and their homogeneous components are given.

1 Introduction and preliminaries

Throughout this article, rings are assumed to be commutative with $1 \neq 0$. Let R be a ring, I be a proper ideal of R. By $\sqrt{I}$, we mean the radical of I which is $\{r \in R : r^n \in I$ for some positive integer $n\}$. In particular, $\sqrt{0}$ is the set of nilpotent elements in R. Recall from [11] that a proper ideal I of R is said to be an n-ideal if whenever $a, b \in R$ and $ab \in I$ with $a \notin \sqrt{0}$ implies $b \in I$. For $a \in R$, we define $Ann(a) = \{r \in R : ra = 0\}$.

The scope of this paper is devoted to the theory of graded commutative rings. One use of rings with gradings is in describing certain topics in algebraic

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geometry. Here, in particular, we are dealing with gr-n-ideals in a G-graded commutative ring.

First, we recall some basic properties of graded rings which will be used in the sequel. We refer to [6]-[8] for these basic properties and more information on graded rings.

Let G be a group with identity e. A ring R is called graded (or more precisely, G-graded) if there exists a family of subgroups $\{R_g\}$ of R such that $R = \oplus_{g\in G} R_g$ (as abelian groups) indexed by the elements $g \in G$, and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The summands R_g are called homogeneous components and elements of these summands are called homogeneous elements. If $a \in R$, then a can be written uniquely $a = \sum_{g\in G} a_g$ where a_g is the component of a in R_g. Also, we write $h(R) = \cup_{g\in G} R_g$. Let $R = \underset{g\in G}{\oplus} R_g$ be a G-graded ring. An ideal I of R is said to be a graded ideal if $I = \oplus_{g\in G}(I \cap R_g) := \oplus_{g\in G} I_g$. An ideal of a graded ring need not be graded.

If I is a graded ideal of R, then the quotient ring R/I is a G-graded ring. Indeed, $R/I = \underset{g\in G}{\oplus} (R/I)_g$ where $(R/I)_g = \{x + I : x \in R_g\}$. A G-graded ring R is called *a graded integral domain* (gr-*integral domain)* if whenever $r_g, s_h \in h(R)$ with $r_g s_h = 0$, then either $r_g = 0$ or $s_h = 0$.

The *graded radical* of a graded ideal I, denoted by $Gr(I)$, is the set of all $x = \sum_{g\in G} x_g \in R$ such that for each $g \in G$ there exists $n_g \in \mathbb{N}$ with $x_g^{n_g} \in I$. Note that, if r is a homogeneous element, then $r \in Gr(I)$ if and only if $r^n \in I$ for some $n \in \mathbb{N}$, (see [10].)

Let R be a G-graded ring. A graded ideal I of R is said to be *a graded prime (gr-prime)* if $I \neq R$; and whenever $r_g, s_h \in h(R)$ with $r_g s_h \in I$, then either $r_g \in I$ or $s_h \in I$, (see [10].)

The concepts of graded primary ideals and graded weakly primary ideals of a graded ring have been introduced in [9] and [5], respectively. Let I be a proper graded ideal of a graded ring R. Then I is called *a graded primary (*gr*-primary) (resp. graded weakly primary) ideal* if whenever $r_g, s_h \in h(R)$ and $r_g s_h \in I$ (resp. $0 \neq r_g s_h \in I$), then either $r_g \in I$ or $s_h \in Gr(I)$.

Graded 2-absorbing and graded weakly 2-absorbing ideals of a commutative graded rings have been introduced in [2]. According to that paper, I is said to be *a graded 2-absorbing (resp. graded weakly 2-absorbing) ideal of* R if whenever $r_g, s_h, t_i \in h(R)$ with $r_g s_h t_i \in I$ (resp. $0 \neq r_g s_h t_i \in I$), then $r_g s_h \in I$ or $r_g t_i \in I$ or $s_h t_i \in I$.

Then the graded 2-absorbing primary and graded weakly 2-absorbing primary ideals defined and studied in [4]. A graded ideal I is said to be *a graded 2-absorbing primary (resp. graded weakly 2-absorbing primary) ideal of* R if whenever $r_g, s_h, t_i \in h(R)$ with $r_g s_h t_i \in I$ (resp. $0 \neq r_g s_h t_i \in I$), then $r_g s_h \in I$ or $r_g t_i \in Gr(I)$ or $s_h t_i \in Gr(I)$.

Recently, R. Abu-Dawwas and M. Bataineh in [1] introduced and studied the concepts of graded r-ideals of a commutative graded rings. A proper graded ideal I of R is said to be *a graded* r*-ideal (*gr*–*r*-ideal) of* R if whenever $r_g, s_h \in h(R)$ such that $r_g s_h \in I$ and $Ann(a) = \{0\}$, then $s_h \in I$.

In this paper, we introduce the concept of graded n-ideals (gr-n-ideals) and investigate the basic properties and facts concerning gr-n-ideals.

2 Results

Definition 1 *Let* R *be a* G*-graded ring. A proper graded ideal* I *of* R *is called* a graded n-ideal of R *if whenever* $r_g, s_h \in h(R)$ *with* $r_g s_h \in I$ *and* $r_g \notin Gr(0)$, *then* $r_g \in I$. *In short, we call it a* gr*-*n*-ideal.*

Example 1 (i) *Suppose that* (R, M) *is a graded local ring with unique graded prime ideal. Then every graded ideal is a* gr*-*n*-ideal.*

(ii) *In any graded integral domain* D*, the graded zero ideal is a* gr*-*n*-ideal.*

(iii) *Any graded ring* R *need not have a* gr*-*n*-ideal. For instance, let* $G = \mathbb{Z}_2$, $R = \mathbb{Z}_6$ *be a* G*-graded ring with* $R_0 = \mathbb{Z}_6$ *and* $R_1 = \{0\}$. *Then* R *has not any* gr*-*n*-ideal.*

Lemma 1 *Let* R *be a* G*-graded ring and* I *be a graded ideal of* R. *If* I *is a* gr*-*n*-ideal of* R, *then* $I \subseteq Gr(0)$.

Proof. Assume that I is a gr-n-ideal and $I \nsubseteq Gr(0)$. Then there exists $r_g \in h(R) \cap I$ such that $r_g \notin Gr(0)$. Since $r_g 1 = r_g \in I$ and I is a gr-n-ideal, we get $1 \in I$, so $I = R$, a contradiction. Hence $I \subseteq Gr(0)$. □

Theorem 1 *Let* R *be a* G*-graded ring and* I *be a gr-prime ideal of* R. *Then* I *is a* gr*-*n*-ideal of* R *if and only if* $I = Gr(0)$.

Proof. Assume that I is a gr-prime ideal of R. It is easy to see $Gr(0) \subseteq Gr(I) = I$. If I is a gr-n-ideal of R, by Lemma 1, we have $I \subseteq Gr(0)$ and so $I = Gr(0)$. For the converse, assume that $I = Gr(0)$. Let $r_g, s_h \in h(R)$ such that $r_g s_h \in I$ and $r_g \notin Gr(0)$. Since I is a gr-prime ideal and $r_g \notin Gr(0) = I$, we get $s_h \in I$. □

Corollary 1 *Let* R *be a* G*-graded ring. Then* $Gr(0)$ *is a* gr-n*-ideal of* R *if and only if it is a* gr*-prime ideal of* R.

Proof. Assume that $Gr(0)$ is a gr-n-ideal of R. Let $r_g, s_h \in h(R)$ such that $r_g s_h \in Gr(0)$ and $r_g \notin Gr(0)$.Then $s_h \in Gr(0)$ as $Gr(0)$ is a gr-n-ideal of R. Hence $Gr(0)$ is a gr-prime ideal of R. Conversely, Assume that $Gr(0)$ is a gr-prime ideal of R, by Theorem 1, we conclude that $Gr(0)$ is a gr-n-ideal of R. □

The following theorem give us a characterization of gr-n-ideal of a graded rings.

Theorem 2 *Let* R *be a graded ring and* I *be a proper graded ideal of* R*. Then the following statements are equivalent:*

(i) I *is a* gr-n*-ideal of* R.

(ii) $I = (I :_R r_g)$ *for every* $r_g \in h(R) - Gr(0)$.

(iii) *For every graded ideals* J *and* K *of* R *such that* $JK \subseteq I$ *and* $J \cap (h(R) - Gr(0)) \neq \emptyset$ *implies* $K \subseteq I$.

Proof. (i) ⇒ (ii) Assume that I is a gr-n-ideal of R. Let $r_g \in h(R) - Gr(0)$. Clearly, $I \subseteq (I :_R r_g)$. Now, Let $s = \sum_{h \in G} s_h \in (I :_R r_g)$. This yields that $r_g s_h \in I$ for each $h \in G$. Since I is a gr-n-ideal of R and $r_g \in h(R) - Gr(0)$, we have $s_h \in I$ for each $h \in G$ and so $s \in I$. This implies that $(I :_R r_g) \subseteq I$. Therefore, $I = (I :_R r_g)$.

(ii) ⇒ (iii) Assume that $JK \subseteq I$ with $J \cap (h(R) - Gr(0)) \neq \emptyset$ for graded ideals J and K of R. Then there exists $r_g \in J \cap h(R)$ such that $r_g \notin Gr(0)$. Hence $r_g K \subseteq I$, it follows that $K \subseteq (I :_R r_g)$. By our assumption, we obtain $K \subseteq (I :_R r_g) = I$.

(iii) ⇒ (i) Let $r_g, s_h \in h(R)$ such that $r_g s_h \in I$ and $r_g \notin Gr(0)$. Let $J = r_g R$ and $K = s_h R$ be two graded ideals of R generated by r_g and s_h, respectively. Then $JK \subseteq I$. By our assumption, we obtain, $K \subseteq I$ and so $s_h \in I$. Thus I is a gr-n-ideal of R. □

Theorem 3 *Let* R *be a* G*-graded ring and* $\{I_\alpha\}_{\alpha \in \Lambda}$ *be a non empty set of* gr-n*-ideals of* R*. Then* $\cap_{i \in \Delta} I_i$ *is* gr-n*-ideal of* R.

Proof. Clearly, $\cap_{\alpha \in \Lambda} I_\alpha$ is a graded ideal of R. Let $r_g, s_h \in h(R)$ such that $r_g s_h \in \cap_{\alpha \in \Lambda} I_\alpha$ and $r_g \notin Gr(0)$. Then $r_g s_h \in I_\alpha$ for every $\alpha \in \Lambda$. Since I_α is a gr-n-ideal of R, we have $s_h \in I_\alpha$ for every $\alpha \in \Lambda$ thus $s_h \in \cap_{\alpha \in \Lambda} I_\alpha$. □

Theorem 4 *Let* R *be a* G*-graded ring and* I *be a graded ideal of* R*. If* I *is a* gr-n*-ideal of* R*, then* I *is a* gr-r*-ideal of* R*.*

Proof. Assume that I is a gr-n-ideal of R. Let $r_g, s_h \in h(R)$ such that $r_g s_h \in I$ and $ann(r_g) = 0$. Since $ann(r_g) = 0$, $r_g \notin Gr(0)$. Then $s_h \in I$ as I is a gr-n-ideal. Thus I is a gr-r-ideal of R. □

Remark 1 *It is easy to see that every graded nilpotent element is also a graded zero divisor. So graded zero divisors and graded nilpotent elements are equal in case* $< 0 >$ *is a graded primary ideal of* R*. Thus the* gr-n*-ideals and* gr-r*-ideals are equivalent in any graded commutative ring whose graded zero ideal is graded primary.*

Recall that a G-graded ring R is called a G-graded reduced ring if $r^2 = 0$ implies $r = 0$ for any $r \in h(R)$; i.e. $Gr(0) = 0$.

Theorem 5 *Let* R *be a* G*-graded ring. Then the following hold:*

(i) *Any* G*-graded reduced ring* R*, which is not graded integral domain, has no* gr-n*-ideal.*

(ii) *If* R *is a* G*-graded reduced ring, then* R *is a graded integral domain if and only if* 0 *is a* gr-n*-ideal.*

Proof. (i) Let R be a G-graded reduced ring such that R is not graded integral domain. Assume that there exists a gr-n-ideal I of R. Since R is a G-graded reduced ring, $Gr(0) = 0$. By Lemma 1, we get, $I \subseteq Gr(0) = 0$ and so $Gr(0) = 0 = I$. Since $Gr(0) = 0$ is not gr-prime ideal of R, by Corollary 1, we get $I = Gr(0)$ is not a gr-n-ideal, a contradiction.

(ii) Assume that R is a G-graded reduced ring. If R is a graded integral domain, then $Gr(0) = 0$ is a gr-prime ideal, and hence by Corollary 1, $0 = Gr(0)$ is a gr-n-ideal of R. For the converse if 0 is a gr-n-ideal of R, then by part (i) R is a graded integral domain. □

Theorem 6 *Let* R *be a* G*-graded ring,* I *be a* gr-n*-ideal of* R *and* $t_g \in h(R) - I$*. Then* $(I :_R t_g)$ *is a* gr-n*-ideal of* R*.*

Proof. By [9, Proposition 1.13], $(I :_R t_g)$ is a graded ideal. Since $t_g \notin I$, $(I :_R t_g) \neq R$. Now, let $r_h, s_\lambda \in h(R)$ such that $r_h s_\lambda \in (I :_R t_g)$ and $r_h \notin Gr((I :_R t_g))$. Then $r_h s_\lambda t_g \in I$. Since I is a gr-n-ideal of R and $r_h \notin Gr(0)$, we get $s_\lambda t_g \in I$. This yields that $s_\lambda \in (I :_R t_g)$. Therefore, $(I :_R t_g)$ is a gr-n-ideal of R. □

Theorem 7 *Let* R *be* G*-graded ring and* I *be a graded ideal of* R*. If* I *is a maximal* gr-n*-ideal of* R*, then* $I = Gr(0)$.

Proof. Assume that I is a maximal gr-n-ideal of R. Let $r_g, s_h \in h(R)$ such that $r_g s_h \in I$ and $r_g \notin I$. Since I is a gr-n-ideal and $r_g \notin I$, by Theorem 6, we have $(I :_R r_g)$ is a gr-n-ideal. Thus $s_h \in (I :_R r_g) = I$ by maximality of I. This yields that I is a gr-prime ideal of R. By Theorem 1, we get $I = Gr(0)$. □

Lemma 2 *Let* R *be a* G*-graded ring and* $\{I_i : i \in \Lambda\}$ *be a directed collection of* gr-n*-ideals of* R*. Then* $I = \cup_{i\in\Lambda} I_i$ *is a* gr-n*-ideal of* R*.*

Proof. Suppose that $r_g s_h \in I$ and $r_g \notin Gr(0)$ for some $r_g, s_h \in h(R)$. Hence $r_g s_h \in I_k$ for some $k \in \Lambda$. Since I_k is a gr-n-ideal of R, we conclude that $s_h \in I_k \subseteq \cup_{i\in\Lambda} I_i = I$. Thus I is a gr-n-ideal. □

Theorem 8 *Let* R *be a* G*-graded ring. Then the following statements are equivalent:*

(i) $Gr(0)$ *is a* gr*-prime ideal of* R*.*

(ii) *There exists a* gr-n*-ideal of* R*.*

Proof. (i) ⇒ (ii) It is clear by Corollary 1.

(ii) ⇒ (i) First we show that R has a maximal gr-n-ideal. Let D be the set of all gr-n-ideals of R. Then by our assumption, $D \neq \emptyset$. Since D is a poset by the set inclusion, take a chain $I_1 \subseteq I_2 \subseteq \cdots$ in D. We conclude that the upper bound of this chain is $I = \cup_{i=1}^{\infty} I_i$ by Lemma 2. Then D has a maximal element which is a maximal gr-n-ideal. Thus that ideal is $Gr(0)$ by Corollary 1 and Theorem 7. □

In view of Lemma 1 and Theorem 8, we have the following result.

Theorem 9 *Let* R *be a* G*-graded ring and* I *a graded ideal of* R *such that* $I \subseteq Gr(0)$.

(i) I *is a* gr-n*-ideal if and only if* I *is a* gr*-primary ideal.*

(ii) *If* I *is a* gr-n*-ideal, then* I *is a graded weakly primary (so graded weakly 2-absorbing primary) and graded 2-absorbing primary ideal.*

(iii) *If* $Gr(0)$ *is* gr*-prime, then* I *is a graded weakly 2-absorbing primary ideal if and only if* I *is a graded 2-absorbing primary ideal of* R*.*

(iv) *If* R *has at least one* gr-$\mathfrak{n}$*-ideal, then* I *is a graded weakly 2-absorbing primary ideal if and only if* I *is a graded 2-absorbing primary ideal of* R.

Proof. Straightforward. □

Theorem 10 *Let* R *be a* G*-graded ring. Then* R *is a graded integral domain if and only if* 0 *is the only* gr-$\mathfrak{n}$*-ideal of* R.

Proof. Let R be a graded integral domain. Assume that I is a nonzero gr-$\mathfrak{n}$-ideal of R. Then we have $\mathrm{I} \subseteq \mathsf{Gr}(0) = 0$ by Lemma 1, a contradiction. Hence 0 is a gr-$\mathfrak{n}$-ideal by Example 1 (ii). Conversely, if 0 is the only gr-$\mathfrak{n}$-ideal, we get $\mathsf{Gr}(0)$ is a gr-prime ideal and also a gr-$\mathfrak{n}$-ideal by Corollary 1 and Theorem 8. Hence $\mathsf{Gr}(0) = 0$ is a gr-prime ideal. Thus R is a graded integral domain. □

Theorem 11 *Let* R *be a* G*-graded ring and* J *be a graded ideal of* R *with* $\mathrm{J} \cap (\mathsf{h}(\mathsf{R}) - \mathsf{Gr}(0)) \neq \emptyset$*. Then the following statements hold:*

(i) *If* I_1 *and* I_2 *are* gr-$\mathfrak{n}$*-ideals of* R *such that* $\mathrm{I}_1\mathrm{J} = \mathrm{I}_2\mathrm{J}$*, then* $\mathrm{I}_1 = \mathrm{I}_2$.

(ii) *If* IJ *is a* gr-$\mathfrak{n}$*-ideal of* R*, then* $\mathrm{IJ} = \mathrm{I}$.

Proof.
(i) Suppose that $\mathrm{I}_1\mathrm{J} = \mathrm{I}_2\mathrm{J}$. Since $\mathrm{I}_2\mathrm{J} \subseteq \mathrm{I}_1$, $\mathrm{J} \cap (\mathsf{h}(\mathsf{R}) - \mathsf{Gr}(0)) \neq \emptyset$, and I_1 is a gr-$\mathfrak{n}$-ideal, by Theorem 2, we conclude that $\mathrm{I}_2 \subseteq \mathrm{I}_1$. Similarly, since I_2 is a gr-$\mathfrak{n}$-ideal, we have the inverse inclusion.
(ii) It is clear from (i). □

For G-graded rings R and R', a G-graded ring homomorphism $\mathsf{f} : \mathsf{R} \to \mathsf{R}'$ is a ring homomorphism such that $\mathsf{f}(\mathsf{R}_g) \subseteq \mathsf{R}'_g$ for every $g \in \mathsf{G}$.

The following result studies the behavior of gr-$\mathfrak{n}$-ideals under graded homomorphism.

Theorem 12 *Let* R_1 *and* R_2 *be two* G*-graded rings and* $\mathsf{f} : \mathsf{R}_1 \to \mathsf{R}_2$ *a graded ring homomorphism. Then the following statements hold:*

(i) *If* f *is a graded epimorphism and* I_1 *is a* gr-$\mathfrak{n}$*-ideal of* R_1 *containing* $\ker\mathsf{f}$*, then* $\mathsf{f}(\mathrm{I}_1)$ *is a* gr-$\mathfrak{n}$*-ideal of* R_2.

(ii) *If* f *is a graded monomorphism and* I_2 *is a* gr-$\mathfrak{n}$*-ideal of* R_2*, then* $\mathsf{f}^{-1}(\mathrm{I}_2)$ *is a* gr-$\mathfrak{n}$*-ideal of* R_1.

Proof. (i) Suppose that $r_g s_h \in f(I_1)$ and $r_g \notin Gr(0_{R_2})$ for some $r_g, s_h \in h(R_2)$. Since f is onto, $f(x_g) = r_g$, $f(y_h) = s_h$ for some $x_g, y_h \in h(R_1)$. Hence $f(x_g y_h) \in f(I_1)$ implies that $x_g y_h \in I_1$ as $\mathrm{Ker} f \subseteq I_1$. It is clear that $x_g \notin Gr(0_{R_1})$. Since I_1 is a gr-n-ideal of R_1, we conclude that $y_h \in I_1$; and so $s_h = f(y_h) \in f(I_1)$. Thus $f(I_1)$ is a gr-n-ideal of R_2.

(ii) Suppose that $r_g s_h \in f^{-1}(I_2)$ and $r_g \notin Gr(0_{R_1})$ for some $r_g, s_h \in h(R_1)$. Since $\ker f = \{0\}$, we have $f(r_g) \notin Gr(0_{R_2})$. Since $f(r_g s_h) = f(r_g) f(s_h) \in I_2$ and I_2 is a gr-n-ideal of R_2, we conclude that $f(s_h) \in I_2$. It means $s_h \in f^{-1}(I_2)$, we are done. □

Corollary 2 *Let I_1 and I_2 be two graded ideals of a G-graded ring R with $I_1 \subseteq I_2$. Then the following statements hold:*

(i) *If I_2 is a gr-n-ideal of R, then I_2/I_1 is a gr-n-ideal of R/I_1.*

(ii) *If I_2/I_1 is a gr-n-ideal of R/I_1 and $I_1 \subseteq Gr(0)$, then I_2 is a gr-n-ideal of R.*

(iii) *If I_2/I_1 is a gr-n-ideal of R/I_1 and I_1 is a gr-n-ideal of R, then I_2 is a gr-n-ideal of R.*

Proof. (i) Considering the natural graded epimorphism $\Pi : R \to R/I_1$, the result is clear by Theorem 12.

(ii) Suppose that $r_g s_h \in I_2$ and $r_g \notin Gr(0)$ for some $r_g, s_h \in h(R)$. Hence $(r_g + I_1)(s_h + I_1) = r_g s_h + I_1 \in I_2/I_1$ and $r_g \notin Gr(0_{R/I_1})$. It implies that $s_h + I_2 \in I_1/I_2$. Thus $s_h \in I_1$, we are done.

(iii) Let I_2/I_1 be a gr-n-ideal of R/I_1 and I_1 a gr-n-ideal of R. Assume that I_2 is not gr-n-ideal. Then $I_1 \nsubseteq Gr(0)$ by (ii). From Lemma 1, we conclude that I_1 is not a gr-n-ideal, a contradiction. Thus I_2 is a gr-n-ideal of R. □

Corollary 3 *Let R be a G-graded ring, I be a gr-n-ideal of R and S a subring of R with $S \nsubseteq I$. Then $I \cap S$ is a gr-n-ideal of S.*

Proof. Consider the injection $i : S \to R$. Then i is a graded homomorphism. Since I is a gr-n-ideal of R, $i^{-1}(I) = I \cap S$ is a gr-n-ideal of S by Theorem 12 (ii). □

Let R be a G-graded ring and $S \subseteq h(R)$ a multiplicatively closed subset of R. Then graded ring of fractions is denoted by $S^{-1}R$ which defined by $S^{-1}R = \oplus_{g \in G}(S^{-1}R)_g$ where $(S^{-1}R)_g = \{\frac{a}{s} : a \in R,\ s \in S,\ g = (\deg s)^{-1}(\deg a)\}$. A homogeneous element $r_g \in h(R)$ is said to be gr-regular if $ann(r_g) = 0$.

Observe that the set of all gr-regular elements of R is a multiplicatively closed subset of R.

The following result studies the behaviour of gr-n-ideal under localization.

Theorem 13 *Let R be a G-graded ring, $S \subseteq h(R)$ a multiplicatively closed subset of R. Then the following statements hold:*

(i) *If I is a gr-n-ideal of R, then $S^{-1}I$ is a gr-n-ideal of $S^{-1}R$.*

(ii) *Let S be the set of all gr-regular elements of R. If J is a gr-n-ideal of $S^{-1}R$, then J^c is a gr-n-ideal of R.*

Proof. (i) Suppose that $\frac{a}{s}\frac{b}{t} \in S^{-1}I$ with $\frac{a}{s} \notin Gr(0_{S^{-1}R})$ for some $\frac{a}{s}, \frac{b}{t} \in h(S^{-1}R)$. Hence there exists $u \in h(S)$ such that $uab \in I$. Clearly, we have $a \notin Gr(0)$. It implies that $ub \in I$; so $\frac{b}{t} = \frac{ub}{ut} \in S^{-1}I$. Thus $S^{-1}I$ is a gr-n-ideal of $S^{-1}R$.

(ii) Suppose that $a, b \in h(R)$ with $ab \in J^c$ and $b \notin J^c$. Then $\frac{b}{1} \notin J$. Since J is a gr-n-ideal, we have $\frac{a}{1} \in Gr(0_{S^{-1}R})$. Hence $ua^k = 0$ for some $u \in S$ and $k \geq 1$. Since u is gr-regular, $a^k = 0$; i.e. $a \in Gr(0)$. Thus J^c is a gr-n-ideal of R. □

Definition 2 *Let S be a nonempty subset of a G-graded ring R with $h(R) - Gr(0) \subseteq S \subseteq h(R)$. Then we call S gr-n-multiplicatively closed subset of R if whenever $r_g \in h(R) - Gr(0)$ and $s_h \in S$, then $r_g s_h \in S$.*

Theorem 14 *Let I be a graded ideal of a G-graded ring R. Then the following statements are equivalent:*

(i) *I is a gr-n-ideal of R.*

(ii) *$h(R) - I$ is a gr-n-multiplicatively closed subset of R.*

Proof. (i) $\Rightarrow$ (ii) Let I be a gr-n-ideal of R. Suppose that $r_g \in h(R) - Gr(0)$ and $s_h \in h(R) - I$. Since $r_g \notin Gr(0)$, $s_h \notin I$, and I is a gr-n-ideal of R, we conclude that $r_g s_h \notin I$. Therefore $r_g s_h \in h(R) - I$. Since I is a gr-n-ideal of R, we have $I \subseteq Gr(0)$ by Lemma 1. Then $h(R) - Gr(0) \subseteq h(R) - I$.

(ii) $\Rightarrow$ (i) Suppose that $r_g, s_h \in h(R)$ with $r_g s_h \in I$ and $r_g \notin Gr(0)$. If $s_h \in h(R) - I$, then from our assumption (ii), we have $r_g s_h \in h(R) - I$, a contradiction. Thus $s_h \in I$ which means that I is a gr-n-ideal of R. □

Theorem 15 *Let I be a graded ideal of a G-graded ring R and S a gr-n-multiplicatively closed subset of R with $I \cap S = \emptyset$. Then there exists a gr-n-ideal K of R such that $I \subseteq K$ and $K \cap S = \emptyset$.*

Proof. Let $D = \{J : J$ is a graded ideal of R with $I \subseteq J$ and $J \cap S = \emptyset\}$. Observe that $D \neq \emptyset$ as $I \in D$. Suppose $J_1 \subseteq J_2 \subseteq \cdots$ is a chain in D. Then $\cup_{i=1}^{\infty} J_i$ is a $\mathfrak{gr}$-$\mathfrak{n}$-ideal of R by Lemma 2. Since $I \subseteq \cup_{i=1}^{\infty} J_i$ and $(\cup_{i=1}^{\infty} J_i) \cap S = \cup_{i=1}^{\infty} (J_i \cap S) = \emptyset$, we get $\cup_{i=1}^{\infty} J_i$ is the upper bound of this chain. From Zorn's Lemma, there is a maximal element K of D. We show that this maximal element K is a $\mathfrak{gr}$-$\mathfrak{n}$-ideal of R. Suppose that $r_g s_h \in K$ and $s_h \notin K$ for some $r_g, s_h \in h(R)$. Then $K \subsetneq (K :_R r_g)$. Since K is maximal, it implies that $(K :_R r_g) \cap S \neq \emptyset$. Hence there is an element $t_\lambda \in (K :_R r_g) \cap S$. Then $r_g t_\lambda \in K$. If $r_g \in Gr(0)$, then we are done. So assume that $r_g \notin Gr(0)$. Since S is $\mathfrak{gr}$-$\mathfrak{n}$-multiplicatively closed, we conclude that $r_g t_\lambda \in S$. Thus $r_g t_\lambda \in S \cap K$, a contradiction. Therefore K is a $\mathfrak{gr}$-$\mathfrak{n}$-ideal of R. □

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All intra-regular generalized hypersubstitutions of type (2)

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Abstract. A generalized hypersubstitution of type τ maps each operation symbol of the type to a term of the type, and can be extended to a mapping defined on the set of all terms of this type. The set of all such generalized hypersubstitutions forms a monoid. An element a of a semigroup S is intra-regular if there is $b \in S$ such that $a = baab$. In this paper, we determine the set of all intra-regular elements of this monoid for type $\tau = (2)$.

1 Introduction

A solid variety is a variety in which every identity holds as a hyperidentity, that is, we substitute not only elements for the variables but also term operations for the operation symbols. The notions of hyperidentities and hypervarieties of a given type τ without nullary operations were studied by J. Aczèl [1], V. D. Belousov [2], W.D. Neumann [8] and W. Taylor [13]. The main tool used to study hyperidentities and hypervarieties is the concept of a hypersubstitution, introduced by K. Denecke et al. [5]. The concept of a generalized hypersubstitution was introduced by S. Leeratanavalee and K. Denecke [7]. The authors

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defined a binary operation on the set of all generalized hypersubstitutions and proved that this set together with the binary operation forms a monoid. In 2010, W. Puninagool and S. Leeratanavalee determined all regular elements of this monoid for type $\tau = (n)$, see [10]. The set of all completely regular elements of this monoid of type $\tau = (n)$ was determined by A. Boonmee and S. Leeratanavalee [3]. Furthermore, we found that every completely regular element is intra-regular. In the present paper, we show that the set of all completely regular elements and the set of all intra-regular elements of type $\tau = (2)$ are the same.

Let $n \geq 1$ be a natural number and let $X_n := \{x_1, x_2, \ldots, x_n\}$ be an n-element set which is called an n-*element alphabet* and let its elements be called *variables*. Let $X := \{x_1, x_2, \ldots\}$ be a countably infinite set of variables and $\{f_i \mid i \in I\}$ be a set of n_i-ary operation symbols, which is disjoint from X, indexed by the set I. To every n_i-ary operation symbol f_i we assign a natural number $n_i \geq 1$, called the *arity* of f_i. The sequence $\tau = (n_i)_{i\in I}$ is called the *type*. For $n \geq 1$, an n-*ary term* of type τ is defined in the following inductive way:

(i) Every variable $x_i \in X_n$ is an n-ary term of type τ.

(ii) If $t_1, \ldots, t_{n_i}$ are n-ary terms of type τ then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term of type τ.

The smallest set which contains $x_1, \ldots, x_n$ and is closed under any finite number of applications of (ii) is denoted by $W_\tau(X_n)$, and is called the set of all n- ary terms of type τ. The set $W_\tau(X) := \cup_{n=1}^{\infty} W_\tau(X_n)$ is called the set of all terms of type τ.

A generalized hypersubstitution of type $\tau = (n_i)_{i\in I}$ is a mapping $\sigma : \{f_i \mid i \in I\} \to W_\tau(X)$ which does not necessarily preserve the arity. Let $Hyp_G(\tau)$ be the set of all generalized hypersubstitutions of type τ. In general, the usual composition of mappings can be used as a binary operation on mappings. But in the case of $Hyp_G(\tau)$ this can not be done immediately. To define a binary operation on this set, we define inductively the concept of a generalized superposition of terms $S^m : W_\tau(X)^{m+1} \to W_\tau(X)$ by the following steps:

(i) If $t = x_j$, $1 \leq j \leq m$, then $S^m(x_j, t_1, \ldots, t_m) := t_j$.

(ii) If $t = x_j$, $m < j \in \mathbb{N}$, then $S^m(x_j, t_1, \ldots, t_m) := x_j$.

(iii) If $t = f_i(s_1, s_2, \ldots, s_{n_i})$, then
$S^m(t, t_1, \ldots, t_m) := f_i(S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, \ldots, t_m))$.

We extend any generalized hypersubstitution σ to a mapping $\widehat{\sigma} : W_\tau(X) \to W_\tau(X)$ inductively defined as follows:

(i) $\widehat{\sigma}[x] := x \in X$,

(ii) $\widehat{\sigma}[f_i(t_1, t_2, \ldots, t_{n_i})] := S^{n_i}(\sigma(f_i), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}])$, for any n_i-ary operation symbol f_i assuming that $\widehat{\sigma}[t_j]$, $1 \leq j \leq n_i$ are already defined.

Now, we define a binary operation $\circ_G$ on $Hyp_G(\tau)$ by $\sigma_1 \circ_G \sigma_2 := \widehat{\sigma}_1 \circ \sigma_2$ where $\circ$ denotes the usual composition of mappings. Let σ_{id} be the hypersubstitution which maps each n_i-ary operation symbol f_i to the term $f_i(x_1, x_2, \ldots, x_{n_i})$. Then $\underline{Hyp_G(\tau)} = (Hyp_G(\tau), \circ_G, \sigma_{id})$ is a monoid [7].

From now on, we introduce some notations which will be used throughout this paper. For a type $\tau = (n)$ with an $n-$ary operation symbol f and $t \in W_{(n)}(X)$, we denote
σ_t - the generalized hypersubstitution σ of type $\tau = (n)$ which maps f to the term t,
var(t) - the set of all variables occurring in the term t,
vb$^t(x)$ - the total number of x-variable occurring in the term t.

For a term $t \in W_{(n)}(X)$, the set sub(t) of its subterms is defined as follows ([11], [12]):

(i) if $t \in X$, then sub$(t) = \{t\}$,

(ii) if $t = f(t_1, \ldots, t_n)$, then sub$(t) = \{t\} \cup$ sub$(t_1) \cup \ldots \cup$ sub(t_n).

Example 1 Let $\tau = (2)$ and $t \in W_{(2)}(X)$ where $t = f(t_1, t_2)$ with $t_1 = f(x_3, f(x_1, x_4))$ and $t_2 = f(f(x_7, x_1), f(x_2, x_1))$. Then
var$(t) = \{x_1, x_2, x_3, x_4, x_7\}$
vb$^t(x_1) = 3$, $vb^t(x_2) = 1$, $vb^t(x_3) = 1$, $vb^t(x_4) = 1$, $vb^t(x_7) = 1$,
sub$(t_1) = \{t_1, f(x_1, x_4), x_1, x_3, x_4\}$,
sub$(t_2) = \{t_2, f(x_7, x_1), f(x_2, x_1), x_1, x_2, x_7\}$,
sub$(t) = \{t, t_1, t_2, f(x_1, x_4), f(x_7, x_1), f(x_2, x_1), x_1, x_2, x_3, x_4, x_7\}$.

2 Sequence of terms

In this section, we construct some tools used to characterize all intra-regular elements in $Hyp_G(2)$. These tools are called the *sequence* of a term and the *depth* of a term, respectively.

Definition 1 *Let* $t \in W_{(n)}(X) \setminus X$ *where* $t = f(t_1 \ldots, t_n)$ *for some* $t_1, \ldots t_n \in W_{(n)}(X)$*. For each* $s \in \mathrm{sub}(t)$*,* $s \neq t$*, a set* $\mathrm{seq}^t(s)$ *of sequences of* s *in* t *is defined by where* $\pi_{i_l} : W_{(n)}(X) \setminus X \to W_{(n)}(X)$ *by the formula* $\pi_{i_l}(f(t_1, \ldots, t_n)) = t_{i_l}$*. Maps* π_{i_l} *are defined for* $i_l = 1, 2, \ldots, n$*.*

Example 2 *Let* $t \in W_{(4)}(X)$ *where* $t = f(t_1, t_2, t_3, t_4)$ *such that* $t_1 = f(x_3, x_1, s, x_4)$*,* $t_2 = x_4$*,* $t_3 = (f(x_7, s, x_1, x_4), x_4, f(x_8, f(x_3, x_1, s, x_4), x_2, f(x_3, x_1, s, x_4)), s)$ *and* $t_4 = s$ *for some* $s \in W_{(4)}(X)$*. Then*

$$\mathrm{seq}^t(s) = \{(1,3), (3,1,2), (3,3,2,3), (3,3,4,3), (3,4), (4)\},$$
$$\mathrm{seq}^{t_3}(s) = \{(1,2), (3,2,3), (3,4,3), (4)\},$$
$$\mathrm{seq}^t(t_1) = \{(1), (3,3,2), (3,3,4)\}$$
$$\mathrm{seq}^t(x_4) = \{(1,4), (2), (3,1,3)\}.$$

Lemma 1 ([4]) *Let* $t, s \in W_{(n)}(X) \backslash X$*,* $x \in \mathrm{var}(t)$ *and* $\mathrm{var}(s) \cap X_n = \{x_{z_1}, \ldots, x_{z_k}\}$*. If* $(i_1, \ldots, i_m) \in \mathrm{seq}^t(x)$ *where* $i_1, \ldots, i_m \in \{z_1, \ldots, z_k\}$ *then* $x \in \mathrm{var}(\widehat{\sigma}_s[t]) = \mathrm{var}(\sigma_s \circ_G \sigma_t)$ *and there is* $(a_{i_1}, \ldots, a_{i_m}) \in \mathrm{seq}^{\widehat{\sigma}_s[t]}(x)$ *where* a_{i_j} *is a sequence of natural numbers* $j_1, \ldots, j_h$ *such that* $(j_1, \ldots, j_h) \in \mathrm{seq}^s(x_{i_j})$ *for all* $j \in \{1, \ldots, m\}$*.*

Let $t \in W_{(n)}(X) \setminus X$, and $t_i \in \mathrm{sub}(t)$. It can be possible that t_i occurs in the term t more than once, we denote

$t_i^{(j)}$ - subterm t_i occurring in the j^{th} order of t (from the left).

Definition 2 Let $t \in W_{(n)}(X) \backslash X$ where $t = f(t_1, \ldots, t_n)$ for some $t_1, \ldots, t_n \in W_{(n)}(X)$ and let $\pi_{i_l} : W_{(n)}(X) \backslash X \to W_{(n)}(X)$ by the formula $\pi_{i_l}(t) = \pi_{i_l}(f(t_1, \ldots, t_n)) = t_{i_l}$. Maps π_{i_l} are defined for $i_l = 1, 2, \ldots, n$. For each $s^{(j)} \in \mathrm{sub}(t)$ for some $j \in \mathbb{N}$, we denote the sequence of $s^{(j)}$ in t by $\mathrm{seq}^t(s^{(j)})$ and denote the depth of $s^{(j)}$ in t by $\mathrm{depth}^t(s^{(j)})$. If $s^{(j)} = \pi_{i_m} \circ \ldots \circ \pi_{i_1}(t)$ for some $m \in \mathbb{N}$, then

$$\mathrm{seq}^t(s^{(j)}) = (i_1, \ldots, i_m) \quad \text{and} \quad \mathrm{depth}^t(s^{(j)}) = m.$$

Example 3 *Let* $\tau = (3)$ *and let* $t \in W_{(3)}(X) \backslash X$ *where* $t = f(t_1, t_2, t_3)$ *such that* $t_1 = x_5$*,* $t_2 = f(x_3, f(x_4, f(x_2, x_7, x_{10}), x_5), x_5)$ *and* $t_3 = f(f(x_5, x_4, f(x_2, x_7, x_{10})), x_1, x_6)$*. Then*

$$\mathrm{seq}^t(x_5^{(1)}) = (1) \quad \text{and} \quad \mathrm{depth}^t(x_5^{(1)}) = 1;$$
$$\mathrm{seq}^t(x_5^{(2)}) = (2,2,3) \quad \text{and} \quad \mathrm{depth}^t(x_5^{(2)}) = 3;$$
$$\mathrm{seq}^t(x_5^{(3)}) = (2,3) \quad \text{and} \quad \mathrm{depth}^t(x_5^{(3)}) = 2;$$

$$\begin{array}{rcl}
seq^t(x_5^{(4)}) = (3,1,1) & \text{and} & depth^t(x_5^{(4)}) = 3; \\
seq^t(f(x_2,x_7,x_{10})^{(1)}) = (2,2,2) & \text{and} & depth^t(f(x_2,x_7,x_{10})^{(1)}) = 3; \\
seq^t(f(x_2,x_7,x_{10})^{(2)}) = (3,1,3) & \text{and} & depth^t(f(x_2,x_7,x_{10})^{(2)}) = 3; \\
seq^{t_3}(f(x_2,x_7,x_{10})^{(1)}) = (1,3) & \text{and} & depth^{t_3}(f(x_2,x_7,x_{10})^{(1)}) = 2; \\
seq^t(x_{10}^{(1)}) = (2,2,2,3) & \text{and} & depth^t(x_{10}^{(1)}) = (4); \\
seq^t(x_{10}^{(2)}) = (3,1,3,3) & \text{and} & depth^t(x_{10}^{(2)}) = 4; \\
seq^{t_3}(x_{10}^{(1)}) = (1,3,3) & \text{and} & depth^{t_3}(x_{10}^{(1)}) = 3.
\end{array}$$

Let $t, s_1, s_2, \ldots, s_k \in W_{(n)}(X) \setminus X$ and $x_i \in var(t)$. We donote

$x_i^{(j)}$ - variable x_i occurring in the j^{th} order of t (from the left);

$x_i^{(j,j_1)}$ - variable $x_i^{(j)}$ occurring in the j_1^{th} order of $\widehat{\sigma}_{s_1}[t]$ (from the left);

$x_i^{(j,j_1,j_2)}$ - variable $x_i^{(j,j_1)}$ occurring in the j_2^{th} order of $\widehat{\sigma}_{s_2}[\widehat{\sigma}_{s_1}[t]]$ (from the left).

Similarly,

$x_i^{(j,j_1,j_2,\ldots,j_k)}$ - variable $x_i^{(j,j_1,\ldots,j_{k-1})}$ occurring in the j_k^{th} order of $\widehat{\sigma}_{s_k}[\widehat{\sigma}_{s_{k-1}}[\ldots[\widehat{\sigma}_{s_2}[\widehat{\sigma}_{s_1}[t]]\ldots]$ (from the left).

Theorem 1 *Let* $t, s \in W_{(n)}(X) \setminus X$ *and* $x_i^{(j)} \in var(t)$ *for some* $i, j \in \mathbb{N}$ *and let* $seq^t(x_i^{(j)}) = i_1, \ldots, i_m$. *Then* $x_{i_1}, \ldots, x_{i_m} \in var(s) \cap X_n$ *if and only if* $x_i^{(j,j_1)} \in var(\widehat{\sigma}_s[t]) = var(\sigma_s \circ_G \sigma_t)$ *for some* $j_1 \in \mathbb{N}$ *and* $seq^{\widehat{\sigma}_s[t]}(x_i^{(j,j_1)}) = (a_{i_1}, \ldots, a_{i_m})$ *where* a_{i_l} *is a sequence of natural number* $p_1, \ldots, p_q$ *such that* $(p_1, \ldots, p_q) = seq^s(x_{i_l}^{h_l})$ *for some* $h_l \in \mathbb{N}$ *and for all* $l \in \{1, \ldots, m\}$.

Proof.($\Rightarrow$) By Lemma 1.
($\Leftarrow$) Assume that $x_i^{(j,j_1)} \in var(\widehat{\sigma}_s[t]) = var(\sigma_s \circ_G \sigma_t)$ for some $j_1 \in \mathbb{N}$ and $seq^{\widehat{\sigma}_s[t]}(x_i^{(j,j_1)}) = (a_{i_1}, \ldots, a_{i_m})$ where a_{i_l} is a sequence of natural number $p_1, \ldots, p_q$ such that $(p_1, \ldots, p_q) = seq^s(x_{i_l}^{h_l})$ for some $h_l \in \mathbb{N}$ and for all $l \in \{1, \ldots, m\}$. Then

$$\nu b^{\widehat{\sigma}_s[t]}(x_i^{(j)}) = \nu b^s(x_{i_1}) \times \nu b^s(x_{i_2}) \times \ldots \times \nu b^s(x_{i_m}).$$

Suppose that $x_{i_k} \notin var(s) \cap X_n$ for some $1 \leq k \leq m$, so $\nu b^s(x_{i_z}) = 0$, i.e. $\nu b^{\widehat{\sigma}_s[t]}(x_i^{(j)}) = 0$, which contradicts to our assumption. Hence $x_{i_1}, \ldots, x_{i_m} \in var(s) \cap X_n$. □

Example 4 Let $\tau = (3)$ and let $t = f(x_2, f(x_4, x_5, x_2), f(x_2, x_6, x_7))$ and $s = f(x_3, x_1, x_3)$. Then $seq^t(x_2^{(1)}) = (1)$, $seq^t(x_2^{(2)}) = (2,3)$, $seq^t(x_2^{(3)}) = (3,1)$

and $\mathrm{seq}^t(x_7^{(1)}) = (3,3)$. By Theorem 1 , there is $x_2^{(1,h)}, x_2^{(3,k_1)}, x_2^{(3,k_2)}, x_7^{(1,l_1)}, x_7^{(1,l_2)}, x_7^{(1,l_3)}$, $x_7^{(1,l_4)} \in \mathrm{var}(\widehat{\sigma}_s[t])$ for some $h, k_1, k_2, l_1, l_{2,3}, l_4 \in \mathbb{N}$ and

$$\mathrm{seq}^{\widehat{\sigma}_s[t]}(x_2^{(1,h)}) = (2) = \mathrm{seq}^{\widehat{\sigma}_s[t]}(x_2^{(1,2)}) \text{ where } \mathrm{seq}^s(x_1^{(1)}) = (2)$$
$$\mathrm{seq}^{\widehat{\sigma}_s[t]}(x_2^{(3,k_1)}) = (1,2) = \mathrm{seq}^{\widehat{\sigma}_s[t]}(x_2^{(3,1)}) \text{ where } \mathrm{seq}^s(x_3^{(1)}) = (1) \text{ and } \mathrm{seq}^s(x_1^{(1)}) = (2)$$
$$\mathrm{seq}^{\widehat{\sigma}_s[t]}(x_2^{(3,k_2)}) = (3,2) = \mathrm{seq}^{\widehat{\sigma}_s[t]}(x_2^{(3,3)}) \text{ where } \mathrm{seq}^s(x_3^{(2)}) = (3) \text{ and } \mathrm{seq}^s(x_1^{(1)}) = (2)$$
$$\mathrm{seq}^{\widehat{\sigma}_s[t]}(x_7^{(1,l_1)}) = (1,1) = \mathrm{seq}^{\widehat{\sigma}_s[t]}(x_7^{(1,1)}) \text{ where } \mathrm{seq}^s(x_3^{(1)}) = (1) \text{ and } \mathrm{seq}^s(x_3^{(1)}) = (1)$$
$$\mathrm{seq}^{\widehat{\sigma}_s[t]}(x_7^{(1,l_2)}) = (1,3) = \mathrm{seq}^{\widehat{\sigma}_s[t]}(x_7^{(1,2)}) \text{ where } \mathrm{seq}^s(x_3^{(1)}) = (1) \text{ and } \mathrm{seq}^s(x_3^{(2)}) = (3)$$
$$\mathrm{seq}^{\widehat{\sigma}_s[t]}(x_7^{(1,l_3)}) = (3,1) = \mathrm{seq}^{\widehat{\sigma}_s[t]}(x_7^{(1,3)}) \text{ where } \mathrm{seq}^s(x_3^{(2)}) = (3) \text{ and } \mathrm{seq}^s(x_3^{(1)}) = (1)$$
$$\mathrm{seq}^{\widehat{\sigma}_s[t]}(x_7^{(1,l_4)}) = (3,3) = \mathrm{seq}^{\widehat{\sigma}_s[t]}(x_7^{(1,4)}) \text{ where } \mathrm{seq}^s(x_3^{(2)}) = (3) \text{ and } \mathrm{seq}^s(x_3^{(2)}) = (3).$$

Since $x_2 \notin \mathrm{var}(s)$, so $x_2^{(2,i)} \notin \mathrm{var}(\widehat{\sigma}_s[t])$ for all $i \in \mathbb{N}$. Consider,

$$\begin{aligned}
\widehat{\sigma}_s[t] &= \widehat{\sigma}_s[f(x_2^{(1)}, f(x_4, x_5, x_2^{(2)}), f(x_2^{(3)}, x_6, x_7^{(1)}))] \\
&= S^3(f(x_3, x_1, x_3), \widehat{\sigma}_s[x_2^{(1)}], \widehat{\sigma}_s[f(x_4, x_5, x_2^{(2)})], \widehat{\sigma}_s[f(x_2^{(3)}, x_6, x_7^{(1)})]) \\
&= f(f(x_7^{(1,1)}, x_2^{(3,1)}, x_7^{(1,2)}), x_2^{(1,2)}, f(x_7^{(1,3)}, x_2^{(3,3)}, x_7^{(1,4)})) \\
&= f(f(x_7, x_2, x_7), x_2, f(x_7, x_2, x_7)).
\end{aligned}$$

Corollary 1 *Let* $t, s \in W_{(n)}(X)\backslash X$ *and* $x_i^{(j)} \in \mathrm{var}(t)$ *for some* $i, j \in \mathbb{N}$ *such that* $\mathrm{seq}^t(x_i^{(j)}) = (i_1, i_2, \ldots, i_m)$ *for some* $i_1, i_2, \ldots, i_m \in \{1, \ldots, n\}$ *and* $x_{i_k} \in \mathrm{var}(s)$ *for all* $1 \le k \le m$. *Then there is* $j_1 \in \mathbb{N}$ *such that*

$$\mathrm{depth}^{\widehat{\sigma}_s[t]}(x_i^{(j,j_1)}) = \mathrm{depth}^s(x_{i_1}^{(l_1)}) + \mathrm{depth}^s(x_{i_2}^{(l_2)}) + \ldots + \mathrm{depth}^s(x_{i_m}^{(l_m)})$$

for some $l_1, l_2, \ldots, l_m \in \mathbb{N}$, *and*

$$vb^{\widehat{\sigma}_s[t]}(x_i^{(j)}) = vb^s(x_{i_1}) \times vb^s(x_{i_2}) \times \ldots \times vb^s(x_{i_m}).$$

Let $vb^t(x_i) = d$.

If $x_i \in X_n$, *then* $vb^{\widehat{\sigma}_s[t]}(x_i) = \sum_{j=1}^{d} vb^{\widehat{\sigma}_s[t]}(x_i^{(j)})$.

If $x_i \in X \setminus X_n$ *where* $x_i \notin \text{var}(s)$, *then* $\nu b^{\widehat{\sigma}_s[t]}(x_i) = \sum_{j=1}^{d} \nu b^{\widehat{\sigma}_s[t]}(x_i^{(j)})$.

3 Main results

In this section, we will show that the set of all completely regular elements and the set of all intra-regular elements in $\text{Hyp}_G(2)$ are the same. First, we recall definitions of regular and completely regular elements and then we characterize all completely regular elements in $\text{Hyp}_G(2)$.

Definition 3 [6] An element a of a semigroup S is called *regular* if there exists $x \in S$ such that $axa = a$.

Definition 4 [9] An element a of a semigroup S is called *completely regular* if there exists $b \in S$ such that $a = aba$ and $ab = ba$.

Let $\sigma_t \in \text{Hyp}_G(2)$. We denote

$R_1 := \{\sigma_{x_i} | x_i \in X\}$;

$R_2 := \{\sigma_t | \text{var}(t) \cap X_2 = \emptyset\}$;

$R_3 := \{\sigma_t | t = f(t_1, t_2)$ where $t_i = x_j$ for some $i, j \in \{1, 2\}$ and $\text{var}(t) \cap X_2 = \{x_j\}\} \cup \{\sigma_{f(x_1,x_2)}, \sigma_{f(x_2,x_1)}\}$

$CR(R_3) := \{\sigma_t | t = f(t_1, t_2)$ where $t_i = x_i$ for some $i \in \{1, 2\}$ and $\text{var}(t) \cap X_2 = \{x_i\}\} \cup \{\sigma_{f(x_1,x_2)}, \sigma_f(x_2, x_1)\}$.

It was shown in [10] and [3] that $\bigcup_{i=1}^{3} R_i$ is the set of all regular elements in $\text{Hyp}_G(2)$ and $CR(\text{Hyp}_G(2)) := CR(R_3) \cup R_1 \cup R_2$ is the set of all completely regular elements in $\text{Hyp}_G(2)$, respectively.

Definition 5 [9] An element a of a semigroup S is called *intra-regular* if there is $b \in S$ such that $a = baab$.

Theorem 2 [3] *Let* S *be a semigroup and* $a \in S$. *If* a *is completely regular, then* a *is intra-regular.*

Corollary 2 [3] *Let* $\sigma_t \in CR(\text{Hyp}_G(2))$. *Then* σ_t *is intra-regular in* $\text{Hyp}_G(2)$.

Lemma 2 *Let* $t = f(t_1, x_1)$ *where* $t_1 \in W_{(2)}(X) \setminus X_2$. *Then* σ_t *is not intra-regular in* $\text{Hyp}_G(2)$.

Proof. Let $t = f(t_1, x_1)$ where $t_1 \in W_{(2)}(X) \setminus X_2$. For each $u \in X$, we get $\sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v \neq \sigma_t$ and $\sigma_v \circ_G \sigma_t^2 \circ_G \sigma_u \neq \sigma_t$ for all $v \in W_{(2)}(X)$. Let $u, v \in W_{(2)}(X) \setminus X$ where $u = f(u_1, u_2)$ and $v = f(v_1, v_2)$ for some $u_1, u_2, v_1, v_2 \in W_{(2)}(X)$, we will show that $\sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v \neq \sigma_t$. If $t_1 \in X \setminus X_2$ then $x_2 \notin \mathrm{var}(t)$. By Theorem 1, $x_1 \notin \mathrm{var}(\widehat{\sigma}_t[t]) = \mathrm{var}(\sigma_t^2)$, i.e. $\mathrm{var}(\sigma_t^2) \cap X_2 = \emptyset$. Hence $\sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v \neq \sigma_t$. If $t_1 \in W_{(2)}(X) \setminus X$,

$$\sigma_t^2(f) = \widehat{\sigma}_t[t] = S^2(f(t_1, x_1), \widehat{\sigma}_t[t_1], x_1) = f(w_1, w_2)$$

where $w_1 = S^2(t_1, \widehat{\sigma}_t[t_1], x_1)$ and $w_2 = S^2(x_1, \widehat{\sigma}_t[t_1], x_1) = \widehat{\sigma}_t[t_1]$. Let $w = f(w_1, w_2)$. Since $t_1 \notin X$, so $w_1 \notin X$ and $w_2 = \widehat{\sigma}_t[t_1] \notin X$. Consider

$$\sigma_t^2 \circ_G \sigma_v(f) = \widehat{\sigma}_w[v] = S^2(f(w_1, w_2), \widehat{\sigma}_w[v_1], \widehat{\sigma}_w[v_2]) = f(s_1, s_2)$$

where $s_i = S^2(w_i, \widehat{\sigma}_w[v_1], \widehat{\sigma}_w[v_2])$ for all $i \in \{1, 2\}$. Since $w_i \notin X$ for all $i \in \{1, 2\}$, $s_i \notin X$ for all $i \in \{1, 2\}$. Then $\widehat{\sigma}_u[s_i] \notin X$ for all $i \in \{1, 2\}$. Consider

$$\sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v(f) = S^2(f(u_1, u_2), \widehat{\sigma}_u[s_1], \widehat{\sigma}_u[s_2]) = f(r_1, r_2)$$

where $r_i = S^2(u_i, \widehat{\sigma}_u[s_1], \widehat{\sigma}_u[s_2])$ for all $i \in \{1, 2\}$. If $u_2 \in W_{(2)}(X) \setminus X$ or $u_2 \in X_2$ then $r_2 \notin X$. If $u_2 \in X \setminus X_2$ then $u_2 = r_2$. So $r_2 \neq x_1$. Therefore $\sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v \neq \sigma_t$. Hence σ_t is not intra-regular in $\mathrm{Hyp}_G(2)$. □

Lemma 3 *Let* $t = f(x_2, t_2)$ *where* $t_2 \in W_{(2)}(X) \setminus X_2$. *Then* σ_t *is not intra-regular in* $\mathrm{Hyp}_G(2)$.

Proof. The proof is similar to the proof of Lemma 2. □

Lemma 4 *Let* $t = f(x_1, t_2)$ *where* $t_2 \in W_{(2)}(X) \setminus X_2$ *and* $x_2 \in \mathrm{var}(t)$. *Then* σ_t *is not intra-regular in* $\mathrm{Hyp}_G(2)$.

Proof. Assume that $t = f(x_1, t_2)$ where $t_2 \in W_{(2)}(X) \setminus X_2$ and $x_2 \in \mathrm{var}(t)$. Let $m = \max\{\mathrm{depth}^t(x_2^{(i)}) | x_2^{(i)} \in \mathrm{var}(t) \text{ for some } i \in \mathbb{N}\}$ $(*)$, then there exists $h \in \mathbb{N}$ such that $\mathrm{seq}^t(x_2^{(h)}) = (i_1, i_2, \ldots, i_m)$ where $i_1, i_2, \ldots, i_m \in \{1, 2\}$. It means $x_2^{(h)} = \pi_{i_m} \circ \pi_{i_{m-1}} \circ \ldots \circ \pi_{i_1}(t)$ where maps $\pi_{i_1}, \ldots, \pi_{i_{m-1}}, \pi_{i_m}$ are defined on $W_{(2)}(X) \setminus X_2$ to $W_{(2)}(X)$. Since $x_2^{(h)} \in \mathrm{var}(t_2)$, $\pi_{i_1}(t) = t_2$, i.e. $i_1 = 2$. So $\mathrm{seq}^t(x_2^{(h)}) = (2, i_2, \ldots, i_m)$. By Theorem 1, there is $x_2^{(h,h_1)} \in \mathrm{var}(\widehat{\sigma}_t[t]) = \mathrm{var}(\sigma_t^2)$ for some $h_1 \in \mathbb{N}$ such that

$$\mathrm{seq}^{\sigma_t^2}(x_2^{(h,h_1)}) = (2, i_2, \ldots, i_m, a_{i_2}, \ldots, a_{i_m})$$

where $(2, i_2, \ldots, i_m) = \text{seq}^t(x_2^{(h)})$ and a_{i_z} is a sequence of natural numbers such that $(a_{i_z}) = \text{seq}^s(x_{i_z}^{(h_{i_z})})$ for some $h_{i_z} \in \mathbb{N}$ and for all $2 \leq z \leq m$. $\big[$Note: $x_2^{(h)}$ is a variable x_2 occurring in the h^{th} order of t (from the left) and $x_2^{(h,h_1)}$ is a variable $x_2^{(h)}$ occurring in the h_1^{th} order of σ_t^2 (from the left)$\big]$. Instead of a sequence $a_{i_2}, \ldots, a_{i_m}$, we write a sequence of natural numbers $w_1, \ldots, w_d$ for some $d \in \mathbb{N}$ and $w_1, \ldots, w_d \in \{1, 2\}$. Then

$$\text{seq}^{\sigma_t^2}(x_2^{(h,h_1)}) = (2, i_2, \ldots, i_m, w_1, \ldots, w_d).$$

Suppose that there exist $u, v \in W_{(2)}(X)$ such that $\sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v = \sigma_t$ $(**)$, i.e. $u = f(x_1, u_2)$ and $v = f(x_1, v_2)$ for some $u_2, v_2 \in W_2(X)$ where $x_2 \in \text{var}(u_2) \cap \text{var}(v_2)$. Choose $x_2^{(j)} \in \text{var}(v)$ for some $j \in \mathbb{N}$. Then $\text{seq}^v(x_2^{(j)}) = (2, p_1, \ldots, p_q)$ for some $p_1, \ldots, p_q \in \{1, 2\}$ and for some $q \in \mathbb{N}$. By Theorem 1, there is $x_2^{(j,j_1)} \in \text{var}(\sigma_t^2 \circ_G \sigma_v)$ for some $j_1 \in \mathbb{N}$ such that

$$\text{seq}^{\sigma_t^2 \circ_G \sigma_v}(x_2^{(j,j_1)}) = (2, i_2, \ldots, i_m, w_1, \ldots, w_d, a_{p_1}, \ldots, a_{p_q})$$

where $(2, i_2, \ldots, i_m, w_1, \ldots, w_d) = \text{seq}^{\sigma_t^2}(x_2^{(h,h_1)})$ and a_{p_z} is a sequence of natural numbers such that $(a_{p_z}) = \text{seq}^s(x_{p_z}^{(l_z)})$ for some $l_z \in \mathbb{N}$ and for all $1 \leq z \leq q$. $\big[$Note: $x_2^{(j)}$ is a variable x_2 occurring in the j^{th} order of v (from the left) and $x_2^{(j,j_1)}$ is a variable $x_2^{(j)}$ occurring in the j_1^{th} order of $\sigma_t^2 \circ_G \sigma_v$ (from the left)$\big]$. Instead of a sequence $a_{p_1}, \ldots, a_{p_q}$ we write a sequence of natural numbers $w_{d+1}, \ldots, w_k$ for some $k \in \mathbb{N}$ and $w_{d+1}, \ldots, w_k \in \{1, 2\}$. Then

$$\text{seq}^{\sigma_t^2 \circ_G \sigma_v}(x_2^{(j,j_1)}) = (2, i_2, \ldots, i_m, w_1, \ldots, w_d, w_{d+1}, \ldots, w_k).$$

By Theorem 1, we have $x_2^{(j,j_1,j_2)} \in \text{var}(\sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v)$ for some $j_2 \in \mathbb{N}$. By Corollary 1, we have

$$\begin{aligned}
\text{depth}^{\sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v}(x_2^{(j,j_1,j_2)}) &= \text{depth}^u(x_2^{(b_1)}) + \text{depth}^u(x_{i_2}^{(b_2)}) + \ldots + \text{depth}^u(x_{i_m}^{(b_m)}) \\
&\quad + \text{depth}^u(x_{w_1}^{(b_{m+1})}) + \ldots + \text{depth}^u(x_{w_d}^{(b_{m+d})}) \\
&\quad + \text{depth}^u(x_{w_{d+1}}^{(b_{m+d+1})}) + \ldots + \text{depth}^u(x_{w_k}^{(b_{m+k})}) \\
&> m
\end{aligned}$$

for some $b_1, \ldots, b_m, b_{m+1}, \ldots, b_{m+d}, b_{m+d+1}, \ldots, b_{m+k} \in \mathbb{N}$, which contradicts to $(*)$ and $(**)$. Therefore σ_t is not intra-regular in $\text{Hyp}_G(2)$. □

Lemma 5 *Let* $t = f(t_1, x_2)$ *where* $t_1 \in W_{(2)}(X) \setminus X_2$ *and* $x_1 \in \text{var}(t)$. *Then* σ_t *is not intra-regular in* $\text{Hyp}_G(2)$.

Proof. The proof is similar to the proof of Lemma 4. □

Lemma 6 *If* $t = f(t_1, t_2)$ *where* $t_1, t_2 \in W_{(2)}(X) \setminus X_2$ *and* $\mathrm{var}(t) \cap X_2 \neq \emptyset$ *then* σ_t *is not intra-regular in* $\mathrm{Hyp}_G(2)$.

Proof. Let $t = f(t_1, t_2)$ where $t_1, t_2 \in W_{(2)}(X) \setminus X_2$ and $\mathrm{var}(t) \cap X_2 \neq \emptyset$.
Case1: $\mathrm{var}(t) \cap X_2 = \{x_i\}$ for some $i \in \{1, 2\}$. Let $j \in \{1, 2\}$ where $i \neq j$.

If j is occurring in $\mathrm{seq}^t(x_i^{(h)})$ for all $x_i^{(h)} \in \mathrm{var}(t)$ then $\mathrm{var}(\sigma_t^2) \cap X_2 = \emptyset$, i.e. $\sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v \neq \sigma_t$ for all $u, v \in W_{(2)}(X)$.

If j is not occurring in $\mathrm{seq}^t(x_i^{(h)})$ for some $x_i^{(h)} \in \mathrm{var}(t)$ then $\mathrm{seq}^t(x_i^{(h)}) = (i_1, i_2, \ldots, i_m)$ where $i_1, i_2, \ldots, i_m \in \{i\}$ for some $m \in \mathbb{N}$. We can prove similar to the proof of Lemma 4, then $\sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v \neq \sigma_t$ for all $u, v \in W_{(2)}(X)$.
Case2: $\mathrm{var}(t) \cap X_2 = X_2$. We can prove similar to the proof of Lemma 4, then $\sigma_u \circ_G \sigma_t^2 \circ_G \sigma_v \neq \sigma_t$ for all $u, v \in W_{(2)}(X)$.

Therefore σ_t is not intra-regular in $\mathrm{Hyp}_G(2)$. □

Theorem 3 $CR(\mathrm{Hyp}_G(2))$ *is the set of all intra-regular elements in* $\mathrm{Hyp}_G(2)$.

Proof. By Corollary 2 and by Lemma 2 to 6. □

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Zero forcing number of degree splitting graphs and complete degree splitting graphs

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Abstract. A subset $\mathbb{Z} \subseteq V(G)$ of initially colored black vertices of a graph G is known as a zero forcing set if we can alter the color of all vertices in G as black by iteratively applying the subsequent color change condition. At each step, any black colored vertex has exactly one white neighbor, then change the color of this white vertex as black. The zero forcing number $\mathbb{Z}(G)$, is the minimum number of vertices in a zero forcing set $\mathbb{Z}$ of G (see [11]). In this paper, we compute the zero forcing number of the degree splitting graph ($\mathcal{DS}$-Graph) and the complete degree splitting graph ($\mathcal{CDS}$-Graph) of a graph. We prove that for any simple graph, $\mathbb{Z}[\mathcal{DS}(G)] \leq k + t$, where $\mathbb{Z}(G) = k$ and t is the number of newly introduced vertices in $\mathcal{DS}(G)$ to construct it.

1 Introduction

In this article, we consider only simple, finite and undirected graphs. In graph theory, the notion of zero forcing was introduced by the AIM Minimum Rank-Special Graph Work Group (see [11]). For a graph G the zero forcing number $\mathbb{Z}(G)$ can be defined as follows:

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- Color change rule: Consider a colored graph G in which every vertex is colored as either white or black. If u is a black vertex of G and exactly one neighbor v of u is white, then change the color of v to black.

- For a given a coloring of G, the *derived coloring* is the result of applying the color-change rule until no more changes are possible.

- A primarily colored black vertex set $\mathbb{Z} \subseteq V(G)$ is called a zero forcing set if all vertices's of G changes to black after limited applications of the color-change rule. The zero forcing number $\mathbb{Z}(G)$, is the minimum $|\mathbb{Z}|$ over all zero forcing sets in G (see [11]).

The zero forcing number $\mathbb{Z}(G)$ can be used to bound the minimum rank for numerous families of graphs (see [11]), also it can be use as a tool for logic circuits (see [2]).

We use the following definitions and notations from [3].

- Open neighborhood and closed neighborhood. The set of all vertices adjacent to a vertex v excluding the vertex v is called the open neighborhood of v and is denoted by $N(v)$. The set of all vertices adjacent to a vertex v including the vertex v is called the closed neighborhood of v and is denoted by $N[v]$, i.e, $N[v] = \{v \cup N(v)\}$.

- Cartesian product. The *Cartesian product* $G \square H$ of two graphs G and H is the graph with vertex set equal to the Cartesian product $V(G) \times V(H)$ and where two vertices (g_1, h_1) and (g_2, h_2) are adjacent in $G \square H$ if and only if, either g_1 is adjacent to g_2 in G or h_1 is adjacent to h_2 in H, that is, if $g_1 = g_2$ and h_1 is adjacent to h_2 or $h_1 = h_2$ and g_1 is adjacent to g_2.

- Tensor product. Let G and H be two distinct graphs. The tensor product $G \oplus H$ has vertex set $V(G \oplus H) = V(G) \times V(H)$, edge set $E(G \oplus H) = \{(u, v)(w, x) \mid uw \in E(G) \text{ and } vx \in E(H)\}$.

- Join of two graphs. Let G and H be two distinct graphs. The graph obtained by joining every vertex of G to every vertex of H is called the join of two graphs G and H and is denoted by $G \vee H$, i.e, $G \vee H$ is the graph union $G \cup H$ together with all the edges xy where $x \in v(G)$ and $y \in V(H)$.

- The circular ladder graph or the prism graphs are the graphs obtained by taking Cartesian product of a cycle graph C_n with a single edge K_2 i.e, $CL_n = C_n \square K_2$.

- When the color change rule is applied to a vertex u to change the color of v, we say u forces v and write $u \rightarrowtail v$.

The Splitting graph $\mathcal{S}(G)$ of G was introduced by E. Sampathkumar and H.B. Walikar [8] and is the graph $\mathcal{S}(G)$ obtained by taking a new vertex v' corresponding to each vertex $v \in G$ and join v' to all vertices of G adjacent to v. The graph thus obtained is the splitting graph (see [8]). It is immediate that $S(G) - E(G) = G \oplus K_2$.

In [5], Premodkumar et al. studied the concept of the zero forcing number of the splitting graph of a graph G and gave the exact values of the zero forcing number of several classes of splitting graphs.

The degree splitting graph was introduced by R. Ponraj and S. Somasundaram [4]. Let G be a graph with $V(G) = D_1 \cup D_2 \cup \ldots \cup D_t \cup B$ where each D_i is a set of vertices of the same degree with minimum two elements and $B = V(G) \setminus \cup_{i=1}^{t} D_i$. The degree splitting graph of G, denoted by $\mathcal{DS}(G)$, is obtained from G by adding vertices $d_1, d_2, ..., d_t$ and joining the vertex d_i to each vertex of D_i for $1 \leq i \leq t$.

For a graph $G = (V, E)$, let A_i denote the set of vertices in G having degree i, $0 \leq i \leq \Delta(G)$, $A_1 \cup A_2 \cup \ldots \cup A_{\Delta(G)} = V(G)$ and $A_1 \cap A_2 \cap \ldots \cap A_{\Delta(G)} = \emptyset$. The complete degree splitting graph of a graph G is the graph $\mathcal{CDS}(G)$ obtained from the graph G by adding new vertices v_i' corresponding to each set A_i in G and joining v_i' to all vertices of A_i.

Example 1 *Consider the tree T depicted in the following figure. The degree splitting graph and the complete degree splitting graph of the tree T are shown in the Figure 1.*

This paper aims to discuss the zero forcing number of the degree splitting graph $\mathcal{DS}(G)$ and the complete degree splitting graph $\mathcal{CDS}(G)$ of a graph G. For more definitions on graphs refer to [3]. For a detailed study of zero forcing refer to [11, 6, 7].

Proposition 1 *The zero forcing number can be easily determined for the following degree and complete degree splitting graphs:*

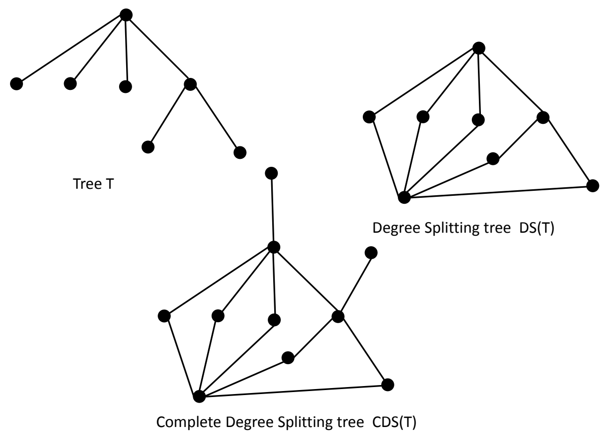

Figure 1:

- *For* P_n*, a path on* $n \geq 5$ *vertices,* $\mathbb{Z}[\mathcal{DS}(P_n)] = \mathbb{Z}[\mathcal{CDS}(P_n)] = 3$.
- *For* C_n *a cycle on* $n \geq 3$ *vertices,* $\mathbb{Z}[\mathcal{DS}(C_n)] = \mathbb{Z}[\mathcal{CDS}(C_n)] = 3$.

If G is a totally disconnected graph, then the degree splitting graph of G is the star graph. By using this fact we have the following

Proposition 2 *If* G *is a totally disconnected graph with at least two vertices, then* $\mathbb{Z}[\mathcal{DS}(G)] = \mathbb{Z}[\mathcal{CDS}(G)] = n-1$*, where* n *is the number of vertices of the graph* G.

Theorem 3 *Let* G *be any simple graph of order* $n \geq 2$ *with* $\mathbb{Z}(G) = k$ *and let* t *be the number of vertices introduced in* G *to construct* $\mathcal{DS}(G)$*. Then* $\mathbb{Z}[\mathcal{DS}(G)] \leq k + t$.

Proof. With out loss of generality assume that G is a simple graph of order $n \geq 2$ and let $\mathbb{Z}$ be an optimal zero forcing set of G with vertices $\{v_1, v_2, \dots, v_k\}$. The degree splitting graph $\mathcal{DS}(G)$ of G is obtained from G by taking new vertices $d_1, d_2, \dots, d_t$ and joining it to each D_i. Consider the degree splitting graph $\mathcal{DS}(G)$ and color the vertices $d_1, d_2, \dots, d_t$ black. Since $\mathbb{Z}$ is a zero forcing set of G and $d_1, d_2, \dots, d_t$ are black vertices, $\{v_1, v_2, \dots, v_k\} \cup \{d_1, d_2, \dots, d_t\}$ forms a zero forcing of $\mathcal{DS}(G)$. Hence the result follows. □

The above proof remains valid for the complete degree splitting graph $\mathcal{CDS}(G)$. Therefore we have the following

Theorem 4 *Let* G *be any simple graph of order* $n \geq 2$ *with* $\mathbb{Z}(G) = k$ *and let* t *be the number of vertices introduced in* G *to construct* $\mathcal{CDS}(G)$. *Then* $\mathbb{Z}[\mathcal{CDS}(G)] \leq k + t$.

Corollary 5 *Let* G *be the degree splitting graph of the cartesian product of* P_n *with* P_m, $n \leq m$. *Then* $\mathbb{Z}(\mathcal{DS}(P_n \Box P_m)) \leq n + 3$.

We recall the following result from [9] to prove the next result.

Theorem 6 [9] *Let* G_1 *and* G_2 *be two connected graphs. Then* $\mathbb{Z}(G_1 \vee G_2) = min\{|G_2| + \mathbb{Z}(G_1), |G_1| + \mathbb{Z}(G_2)\}$.

Theorem 7 *Let* G *be a regular graph of order* $n > 1$ *and let* $\mathbb{Z}(G) = k$, $k > 1$ *be a positive integer. Then* $\mathbb{Z}(\mathcal{DS}(G)) = k + 1$.

Proof. Assume that G is a regular graph. The graph $\mathcal{DS}(G)$ is obtained from G by taking a new vertex v and joining v to all other vertices in G that is, $\mathcal{DS}(G) = G \vee H$, where H is a graph with a single vertex v. Therefore, $\mathbb{Z}(H) = 1$. We have from theorem 6,

$$\mathbb{Z}(G \vee H) = \min\{|H| + \mathbb{Z}(G), |G| + \mathbb{Z}(H)\} = \min\{1 + k, n + 1\} = 1 + k.$$

□

Now we give special attention to the zero forcing number of the regular graphs considered in [11]. We recall the following results from [11].

Theorem 8 [11]

(i) *For the hypercube* Q_n, $\mathbb{Z}(Q_n) = 2^{n-1}$.

(ii) *If* G *is the prism graph* CL_n, *then* $\mathbb{Z}(G) = 4$.

(iii) *If* G *is the Petersen graph, then* $\mathbb{Z}(G) = 5$.

(iv) *If* G *is the Complete bipartite graph* $K_{m,n}$, *then* $\mathbb{Z}(G) = m + n - 2$.

The following results are the immediate consequence of the above two theorems

Corollary 9 (i) *If* G *is the Petersen graph, then* $\mathbb{Z}(\mathcal{DS}(G)) = \mathbb{Z}(G)+1 = 6$.

(ii) *If* G *is the complete bipartite graph* $K_{n,n}$, $n \geq 2$, *then* $\mathbb{Z}(\mathcal{DS}(G)) = 2n - 1$.

(iii) *If* G *is the degree splitting graph of the prism graph* CL_n, *then* $\mathbb{Z}(G) = 5$.

(iv) *If* G *is the degree splitting graph of the* n*-regular Hypercube graph* Q_n, *then* $\mathbb{Z}(G) = 2^{n-1} + 1$.

If G is a regular graph, then we have the following:

Corollary 10 *Let* G *be a regular graph and let* $\mathbb{Z}(G) = k$. *Then* $\mathbb{Z}[\mathcal{DS}(G)] = \mathbb{Z}[\mathcal{CDS}(G)] = k + 1$.

We use the following observation from [11] to prove the next proposition.

Observation 11 [11] *For any simple graph* G, $\delta(G) \leq \mathbb{Z}(G)$, *where* $\delta(G)$ *denote the minimum degree of* G.

The degree splitting graph of the cycle C_k, is known as the wheel graph W_n, where $n = k + 1$.

Proposition 12 *If* G *is the wheel graph* W_n *on* n *vertices, then* $\mathbb{Z}[\mathcal{DS}(G)] = \mathbb{Z}[\mathcal{CDS}(G)] = 4$.

Proof. Let G be the wheel graph W_n on n vertices. Then $\delta[\mathcal{DS}(G)] = 4$, and we have from Observation 11

$$4 \leq \mathbb{Z}(\mathcal{DS}(G)). \tag{1}$$

Since $\mathcal{DS}(G)$ is a graph obtained from G by taking a single vertex v and joining v to all vertices of the cycle C_k. From Proposition 1 and Theorem 3 we conclude that

$$\mathbb{Z}(\mathcal{DS}(G)) \leq \mathbb{Z}(W_n) + 1 = 4. \tag{2}$$

Hence from Equatins (1) and (2) the result follows. □

Proposition 13 *If* G *is the star graph* $K_{1,n}$ *on* $n + 1$ *vertices, where* $n \geq 2$, *then* $\mathbb{Z}[\mathcal{DS}(G)] = \mathbb{Z}[\mathcal{CDS}(G)] = n$.

Proof. The degree splitting graph of the star graph is the complete bipartite graph $K_{2,n}$, in [11], the AIM group observed that $\mathbb{Z}(K_{2,n}) = 2 + n - 2 = n$. Therefore the result follows. □

In the next Proposition we consider complete graphs of order n. In [11] the AIM group observed that for the complete graph K_n, $\mathbb{Z}(K_n) = n - 1$. Using this fact and considering that the degree splitting graph of K_n is K_{n+1}, we have the following:

Proposition 14 *For a complete graph of order* n, $\mathbb{Z}[\mathcal{DS}(K_n)] = n$.

We recall the following result from [11].

Proposition 15 [11] *For the complete graph* K_n *of order* $n \geq 2$ *and for the path* P_k *of order* $k \geq 2$, $\mathbb{Z}(K_n \square P_k) = n$.

Now we consider the degree splitting graph of the ladder graph and find its zero forcing number. The cartesian product graph $P_n \square K_2$ is known as the ladder graph.

Proposition 16 *Let* G *be the degree splitting graph of the ladder graph* $P_n \square K_2$ *with* $n \geq 4$ *vertices. Then* $\mathbb{Z}(G) = 4$.

Proof. We have from Proposition 15, $\mathbb{Z}(K_2 \square P_k) = 2$. Assume that G be the degree splitting graph of $K_2 \square P_k$. The degree splitting graph of $K_2 \square P_k$ contains two newly introduced vertices and hence $t = 2$. Therefore, from Theorem 4

$$\mathbb{Z}(G) \leq \mathbb{Z}(K_2 \square P_k) + 2 = 4. \tag{3}$$

Consider the n-ladder graph as $L_n = P_n \square K_2$. Let $v_1, v_2, \ldots, v_n$ be the vertices of the path P_n in L_n and $v'_1, v'_2, \ldots, v'_n$ be the corresponding vertices of $v_1, v_2, \ldots, v_n$ in L_n. Let $B_1 = \{v_1, v'_1, v_n, v'_n\}$ be the set of vertices of degree 2 in L_n and let $B_2 = \{v_2, v_3, \ldots, v_{n-1}, v'_2, v'_3, \ldots, v'_{n-1}\}$ be the set of vertices of degree 3 in L_n. Consider the graph $G \equiv \mathcal{DS}(L_n)$. Let $A_1 = \{B_1 \cup \{a_1\}\}$ be the set of vertices in G with $\deg(a_1) = 4$ and $A_2 = \{B_2 \cup \{a_2\}\}$ with $\deg(a_2) = 2(n-2)$.

To prove the reverse part assume that there exist a zero forcing set consisting of three vertices u, v and w. Degree of each vertex in G is at least three, therefore, to force at least one vertex it is necessary that uv and vw should form edges in G.

Case 1 Assume that the vertices u, v and w are in A_2. In A_2 each vertices have degree at least four, therefore u, v and w does not form a zero forcing set, a contradiction.

Case 2 Assume that the vertices u and v are in A_2 and the vertex w is in A_1. In this case u and v have degree at lest four and w has degree three therefore, u, v and w does not form a zero forcing set, a contradiction.

Case 3 Assume that the vertices u and v are in A_1 and the vertex w is in

A_2. $u = v_1$, $v = v_2$ and $w = v_1'$. Now v_1 forces the vertex a_1 and v_1' forces the vertex v_2' after this forcing, no more color changing is possible, a contradiction.

Case 4 Assume that the vertices u, v and w are in A_1. We have the following two sub cases.

Subcase 4.1 $u = v_1, v = v_1'$ and $w = a_1$. Now v_1 forces v_2 and v_1' forces v_2' after this forcing, no more color changing is possible, a contradiction.

Subcase 4.2 $u = v_1, v = a_1$ and $w = v_n$. In this case $\deg(u) = 3$, $\deg(v) = 4$ and $\deg(w) = 3$ and each of these vertices have two white neighbors, color changing is not possible, a contradiction.
Hence

$$4 \leq \mathbb{Z}(G). \tag{4}$$

Therefore, from (3) and (4) the result follows. □

2 Classes of graphs with $\mathbb{Z}[\mathcal{DS}(G)] < k + t$

In this section, we study simple graphs with $\mathbb{Z}[\mathcal{DS}(G)] < k+t$, where $\mathbb{Z}(G) = k$ and t be the newly introduced vertices in $\mathcal{DS}(G)$. Let G be the path P_4 and $\mathcal{DS}(P_4)$ be the degree splitting graph of P_4 as shown in Figure 2. Then the black vertices depicted in Figure 2 will act as a zero forcing set for $\mathcal{DS}(P_4)$ and hence, $\mathbb{Z}\mathcal{DS}(P_4) = 2 < 1 + 2$.

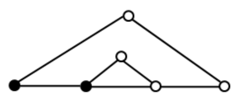

Figure 2:

Example 2 *Let* $G \equiv \mathcal{DS}(C_5 \circ K_1)$ *be the graph depicted in Figure 3. One can easily verify that the set* $\{v_7, v_4, v_8, v_9\}$ *forms a zero forcing set since there is no smaller zero forcing set exist for the graph* G*, therefore,* $\mathbb{Z}(G) = 4$*. Here* v_1 *and* v_{10} *are the newly introduced vertices in* $C_5 \circ K_1$ *to form* $\mathcal{DS}(C_5 \circ K_1)$ *, therefore* $t = 2$*. We have from [11],* $\mathbb{Z}(C_5 \circ K_1) = k = 3$*. Therefore,* $\mathbb{Z}(G) = 4 < k+t = 5$*.*

Proposition 17 *If* G *is the complete bipartite graph* $K_{m,n}$*, where* $m, n \geq 2$ *and* $m \neq n$*, then* $\mathbb{Z}(\mathcal{DS}(G)) = m + n - 1$*.*

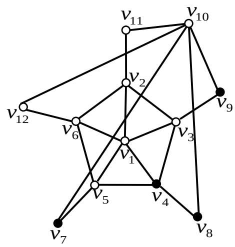

Figure 3:

Proof. Without loss of generality assume G is the complete bipartite graph $K_{m,n}$ and $H = \mathcal{DS}(G)$. Assume that we have a zero forcing set $\mathbb{Z}$ of H consisting of $m+n-2$ vertices. Then the number of white vertices in H is $m+n+2-(m+n-2) = 4$. Now we divide the vertex set of H into four sets A, B, C and D as depicted in Figure 4. Where $A = \{u\}$, $B = \{u_1, u_2, \ldots, u_m\}$, $C = \{v_1, v_2, \ldots, v_n\}$ and $D = \{v\}$. Assume that the four white vertices are distributed among the sets A, B, C and D.

Claim 1. If H has a zero forcing set, then the total number of white vertices in the set B will never exceed one. On the contrary assume that there exist two white vertices u_i and u_j in the set B. Then for all vertices $v_i, 1 \leq i \leq n$ in the set C, the open neighborhood of $N(v_i)$ contains two white neighbors in the set B also the vertex u will never force the vertices u_i and u_j. Therefore, color changing rule is not applicable in this case, a contradiction to our assumption that there exist two white vertices u_i and u_j in the set B.

Claim 2. If H has a zero forcing set, then the total number of white vertices in the set C will never exceed one. On the contrary assume that there exist two white vertices v_i and v_j in the set C. Then for all vertices $u_i, 1 \leq i \leq n$ in the set B, the open neighborhood of $N(u_i)$ contains two white neighbors in the set C also the vertex v will never force the vertices v_i and v_j. Therefore, color changing rule is not applicable in this case, a contradiction to our assumption that there exist two white vertices v_i and v_j in the set C.

Now assume that we have distributed the white vertices one each in all sets A, B, C and D. Immediately, we can see that any black vertices in the set B and the set C have two white neighbors also the vertices u and v are white, color changing rule is not applicable, a contradiction to our assumption that there exist a zero forcing set in H consisting of $m+n-2$ vertices. Therefore,

$$\mathbb{Z}(\mathcal{DS}(G)) \geq m+n-1. \tag{5}$$

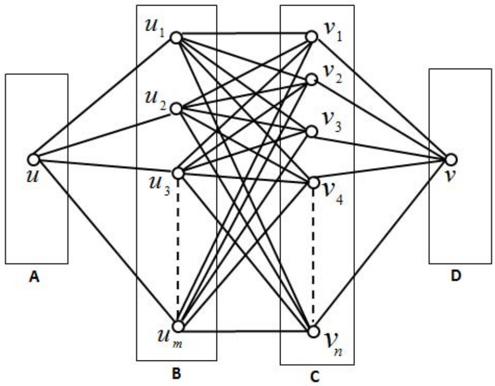

Figure 4:

To prove the reverse part consider the set $\mathbb{E} = \{u_2, u_3, \dots, u_m, v_2, v_3, \dots, v_n, v\}$ of vertices from the Figure 4. Color the vertices in the set $\mathbb{E}$ as black. Clearly the vertex $v \rightarrowtail v_1$, $v_1 \rightarrowtail u_1$, and $u_1 \rightarrowtail u$. Now the set $\mathbb{E}$ forms a zero forcing set and $|\mathbb{E}| = m-1+n-1+1 = m+n-1$. Therefore,

$$\mathbb{Z}(\mathcal{DS}(G)) \leq m+n-1. \tag{6}$$

Hence from (5) and (6) the result follows. □

The following Lemma can be found in [7].

Lemma 1 [7] *Let $G = (V, E)$ be a connected graph with a cut-vertex $v \in V(G)$. Let $C_1, \dots, C_k$ be the vertex sets for the connected components of $G - v$, and for $1 \leq i \leq k$, let $G_i = G[C_i \cup \{v\}]$. Then $\mathbb{Z}(G) \geq \sum_{i=1}^{k} \mathbb{Z}(G_i) - k + 1$.*

Definition 1 *The Pineapple graph K_m^n is obtained by coalescing any vertex of the complete graph K_m with the star $K_{1,n}$ at the vertex of degree n. The number of vertices in K_m^n is $m+n$, number of edges in K_m^n is $\frac{m^2-m+2n}{2}$. These graphs were defined and studied in [12] and [10].*

The authors in [12] and [10] studied about the spectral properties of Pineapple Graphs.

We recall the following results from [13].

Proposition 18 [13] *If* G *is the Pineapple graph* K_m^n *with* $n \geq 2, m \geq 3$*, then* $\mathbb{Z}(G) = m + n - 3$.

Proposition 19 *If* G *is the Pineapple graph* K_m^1 *with* $m \geq 3$*, then* $\mathbb{Z}(G) = m - 1$.

Proposition 20 *If* G *is the Pineapple graph* K_m^n *with* $m \geq 3$ *and* $n \geq 1$*, then* $\mathbb{Z}(\mathcal{DS}(K_m^n)) = m + n - 2$.

Proof. Case 1 Without of loss of generality assume that $n = 1$. Let $\mathcal{DS}(K_m^1)$ be the degree splitting graph of K_m^1 and let v be the newly introduced vertex in $\mathcal{DS}(K_m^1)$ to construct it. Let u' be the coalesced vertex of the complete graph K_m with the star $K_{1,n}$ in K_m^1 and let u be the corresponding vertex of u' in $\mathcal{DS}(K_m^1)$. Let w be the pendant vertex in $\mathcal{DS}(K_m^1)$ and let x be an arbitrary vertex of $\mathcal{DS}(K_m^1)$ other than u, v and w. Color all vertices except u, x and w in $\mathcal{DS}(K_m^1)$ as black. Clearly the vertex $v \rightarrowtail x$, $x \rightarrowtail u$ and $u \rightarrowtail w$ and hence

$$\mathbb{Z}(\mathcal{DS}(K_m^1)) \leq m - 1. \tag{7}$$

To prove the reverse part we use the following

$$\mathbb{Z}(K_{m+1} - e) = m - 1 \tag{A}$$
$$\mathbb{Z}(K_2) = 1. \tag{B}$$

Now Lemma 1, (A) and (B) yields,

$$\mathbb{Z}(\mathcal{DS}(K_m^1)) \geq \sum_{i=1}^{2} \mathbb{Z}(G_i) - 2 + 1 = \mathbb{Z}(K_{m+1} - e) + \mathbb{Z}(K_2) - 1 = m - 1. \tag{8}$$

Thus the result follows from (7) and (8).

Case 2 Assume that $n = 2$. Let $\mathcal{DS}(K_m^2)$ be the degree splitting graph of K_m^2. Let u' be the coalesced vertex of the complete graph K_m with the star $K_{1,n}$ in K_m^1 and let u be the corresponding vertex of u' in $\mathcal{DS}(K_m^2)$. Let w_1, w_2 and w_3 be the vertices of degree two in $\mathcal{DS}(K_m^2)$. The subgraph induced by the vertices w_1, w_2, w_3 and u forms a cycles C_4 in $\mathcal{DS}(K_m^2)$. Let x be an arbitrary vertex of $\mathcal{DS}(K_m^2)$ other than w_1, w_2, w_3 and u. Color all vertices except w_2, w_3, x and

u in $\mathcal{DS}(K_m^2)$ black. Let y be an arbitrary black colored vertex other than w_1 in $\mathcal{DS}(K_m^2)$. Clearly $y \rightarrowtail x$, $x \rightarrowtail u$, $u \rightarrowtail w_3$ and $w_3 \rightarrowtail w_2$, hence

$$\mathbb{Z}(\mathcal{DS}(K_m^2)) \leq m. \tag{9}$$

To prove the reverse inequality use the the following

$$\mathbb{Z}(K_{2,2}) = 2. \tag{C}$$

Now Lemma 1, (A) and (C) yields the following,

$$\mathbb{Z}(\mathcal{DS}(K_m^2)) \geq \sum_{i=1}^{2} \mathbb{Z}(G_i) - 2 + 1 = \mathbb{Z}(K_{m+1} - e) + \mathbb{Z}(K_{2,2}) - 1 = m - 1 + 2 - 1 = m. \tag{10}$$

Therefore, from (9) and (10) the result follows.

Case 3 Assume $n \geq 3$. Let $\mathcal{DS}(K_m^n)$ be the degree splitting graph of K_m^n. Let u' be the coalesced vertex of the complete graph K_m with the star $K_{1,n}$ in K_m^n and let u be the corresponding vertex of u' in $\mathcal{DS}(K_m^n)$. Similarly let t be the newly introduced vertex in $\mathcal{DS}(K_m^n)$ obtained by joining the pendant vertices in K_m^n. Let $w_1, w_2, \ldots, w_n$ be the vertices of degree two in $\mathcal{DS}(K_m^n)$. The subgraph induced by the vertices $\{w_1, w_2, \ldots, w_n\} \cup \{t, u\}$ forms the complete bipartite graph $K_{2,n}$ in $\mathcal{DS}(K_m^n)$.

Let x be the newly introduced vertex in $\mathcal{DS}(K_m^n)$ other than the vertex t in $\mathcal{DS}(K_m^n)$. Let y be a vertex in $\mathcal{DS}(K_m^n)$ other than $w_1, w_2, \ldots, w_n, u, x$ and t. Color all vertices except the vertices w_n, t, y and u in $\mathcal{DS}(K_m^n)$ as black. Clearly $x \rightarrowtail y$, $y \rightarrowtail u$, $u \rightarrowtail w_n$, $w_n \rightarrowtail t$ hence

$$\mathbb{Z}(\mathcal{DS}(K_m^n)) \leq m + n - 2. \tag{11}$$

To prove the reverse inequality use the the following result from [13]

$$\mathbb{Z}(K_{m,n}) = m + n - 2. \tag{D}$$

Now Lemma 1, (A) and (D) yields the following,

$$\begin{aligned} \mathbb{Z}(\mathcal{DS}(K_m^n)) &\geq \sum_{i=1}^{2} \mathbb{Z}(G_i) - 2 + 1 = \mathbb{Z}(K_{m+1} - e) + \mathbb{Z}(K_{m,n}) - 1 \\ &= (m-1) + (2 + n - 2) - 1 = m + n - 1. \end{aligned} \tag{12}$$

Therefore, from (11) and (12) the result follows. □

3 Conclusion and open problems

This paper deals with the problem of determination of the zero forcing number of graphs and their degree splitting graphs. Characterization of classes graphs for which $\mathbb{Z}[\mathcal{DS}(G)] = k + t$ is an open problem.

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Multiplicative inequalities for weighted arithmetic and harmonic operator means

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Abstract. In this paper we establish some multiplicative inequalities for weighted arithmetic and harmonic operator means under various assumption for the positive invertible operators A, B. Some applications when A, B are bounded above and below by positive constants are given as well.

1 Introduction

Throughout this paper A, B are positive invertible operators on a complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$. We use the following notations for operators

$$A\nabla_\nu B := (1-\nu)A + \nu B,$$

the *weighted operator arithmetic mean,*

$$A\sharp_\nu B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\nu A^{1/2},$$

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the *weighted operator geometric mean* and

$$A!_{\nu}B := \left((1-\nu)A^{-1} + \nu B^{-1}\right)^{-1}$$

the *weighted operator harmonic mean,* where $\nu \in [0,1]$.

When $\nu = \frac{1}{2}$, we write $A\nabla B$, $A\sharp B$ and $A!B$ for brevity, respectively.

The following fundamental inequality between the weighted arithmetic, geometric and harmonic operator means holds

$$A!_{\nu}B \le A\sharp_{\nu}B \le A\nabla_{\nu}B \tag{1}$$

for any $\nu \in [0,1]$.

For various recent inequalities between these means we recommend the recent papers [3]-[6], [8]-[12] and the references therein.

The following additive double inequality has been obtained in the recent paper [7]:

$$\nu(1-\nu)\frac{(b-a)^2}{\max\{b,a\}} \le A_{\nu}(a,b) - H_{\nu}(a,b) \le \nu(1-\nu)\frac{(b-a)^2}{\min\{b,a\}}, \tag{2}$$

for any a, $b>0$ and $\nu \in [0,1]$, where $A_{\nu}(a,b)$ and $H_{\nu}(a,b)$ are the scalar weighted arithmetic mean and harmonic mean, respectively, namely

$$A_{\nu}(a,b) := (1-\nu)a + \nu b \text{ and } H_{\nu}(a,b) := \frac{ab}{(1-\nu)b + \nu a}.$$

In particular,

$$\frac{1}{4}\frac{(b-a)^2}{\max\{b,a\}} \le A(a,b) - H(a,b) \le \frac{1}{4}\frac{(b-a)^2}{\min\{b,a\}}, \tag{3}$$

where

$$A(a,b) := \frac{a+b}{2} \text{ and } H(a,b) := \frac{2ab}{b+a}.$$

We consider the *Kantorovich's constant* defined by

$$K(h) := \frac{(h+1)^2}{4h},\ h>0. \tag{4}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \ge 1$ for any $h>0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h>0$.

Observe that for any $h > 0$

$$K(h) - 1 = \frac{(h-1)^2}{4h} = K\left(\frac{1}{h}\right) - 1.$$

Observe that

$$K\left(\frac{b}{a}\right) - 1 = \frac{(b-a)^2}{4ab} \text{ for } a,\ b > 0.$$

Since, obviously

$$ab = \min\{a, b\} \max\{a, b\} \text{ for } a,\ b > 0,$$

then we have the following version of (2):

$$4\nu(1-\nu)\min\{a,b\}\left[K\left(\frac{b}{a}\right) - 1\right] \le A_\nu(a,b) - H_\nu(a,b) \le 4\nu(1-\nu)\max\{a,b\}\left[K\left(\frac{b}{a}\right) - 1\right]. \tag{5}$$

for any $a,\ b > 0$ and $\nu \in [0,1]$.

For positive invertible operators A, B we define

$$A\nabla_\infty B := \frac{1}{2}(A+B) + \frac{1}{2}A^{1/2}\left|A^{-1/2}(B-A)A^{-1/2}\right|A^{1/2}$$

and

$$A\nabla_{-\infty} B := \frac{1}{2}(A+B) - \frac{1}{2}A^{1/2}\left|A^{-1/2}(B-A)A^{-1/2}\right|A^{1/2}.$$

If we consider the continuous functions f_∞, $f_{-\infty} : [0,\infty) \to [0,\infty)$ defined by

$$f_\infty(x) = \max\{x, 1\} = \frac{1}{2}(x+1) + \frac{1}{2}|x-1|$$

and

$$f_{-\infty}(x) = \max\{x, 1\} = \frac{1}{2}(x+1) - \frac{1}{2}|x-1|,$$

then, obviously, we have

$$A\nabla_{\pm\infty}B = A^{1/2} f_{\pm\infty}\left(A^{-1/2}BA^{-1}\right)A^{1/2}.$$

If A and B are commutative, then

$$A\nabla_{\pm\infty}B = \frac{1}{2}(A+B) \pm \frac{1}{2}|B-A| = B\nabla_{\pm\infty}A.$$

The following additive inequality between the weighted arithmetic and harmonic operator means holds [7]:

Theorem 1 *Let* A, B *be positive invertible operators and* $M > m > 0$ *such that the condition*

$$mA \leq B \leq MA \tag{6}$$

holds. Then we have

$$\begin{aligned} 4\nu(1-\nu)\, g(m,M)\, A\nabla_{-\infty}B &\leq A\nabla_\nu B - A!_\nu B \\ &\leq 4\nu(1-\nu)\, G(m,M)\, A\nabla_\infty B, \end{aligned} \tag{7}$$

where

$$g(m,M) := \begin{cases} K(M) - 1 \text{ if } M < 1, \\ 0 \text{ if } m \leq 1 \leq M, \\ K(m) - 1 \text{ if } 1 < m \end{cases}$$

and

$$G(m,M) := \begin{cases} K(m) - 1 \text{ if } M < 1, \\ \max\{K(m), K(M)\} - 1 \text{ if } m \leq 1 \leq M, \\ K(M) - 1 \text{ if } 1 < m. \end{cases}$$

In particular,

$$g(m,M)\, A\nabla_{-\infty}B \leq A\nabla B - A!B \leq G(m,M)\, A\nabla_\infty B. \tag{8}$$

Motivated by the above facts, we establish in this paper some multiplicative inequalities for weighted arithmetic and harmonic operator means under various assumption for the positive invertible operators A, B. Some applications when A, B are bounded above and below by positive constants are given as well.

2 Multiplicative inequalities

The following result is of interest in itself:

Lemma 1 *For any* a, $b > 0$ *and* $\nu \in [0,1]$ *we have*

$$\nu(1-\nu)\left(1 - \frac{\min\{a,b\}}{\max\{a,b\}}\right)^2 \leq \frac{A_\nu(a,b)}{H_\nu(a,b)} - 1 \leq \nu(1-\nu)\left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2. \tag{9}$$

In particular,

$$\frac{1}{4}\left(1 - \frac{\min\{a,b\}}{\max\{a,b\}}\right)^2 \leq \frac{A(a,b)}{H(a,b)} - 1 \leq \frac{1}{4}\left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2. \tag{10}$$

Proof. We have for any $a,\ b > 0$ and $\nu \in [0,1]$ that

$$\begin{aligned}\frac{A_\nu(a,b)}{H_\nu(a,b)} &= \frac{[(1-\nu)a+\nu b][(1-\nu)b+\nu a]}{ab}\\ &= \frac{(1-\nu)^2 ab+\nu(1-\nu)b^2+\nu(1-\nu)a^2+\nu^2 ab}{ab}\\ &= \frac{\nu(1-\nu)\left(b^2+a^2\right)+\left(1-2\nu+2\nu^2\right)ab}{ab},\end{aligned}$$

which is equivalent with

$$\frac{A_\nu(a,b)}{H_\nu(a,b)}-1=\nu(1-\nu)\frac{(b-a)^2}{ab} \tag{11}$$

for any $a,\ b > 0$ and $\nu \in [0,1]$.

Since $\min^2\{a,b\} \le ab \le \max^2\{a,b\}$ hence

$$\begin{aligned}\nu(1-\nu)\frac{(b-a)^2}{ab} &\le \nu(1-\nu)\frac{(b-a)^2}{\min^2\{a,b\}}\\ &= \nu(1-\nu)\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2\end{aligned}$$

and

$$\begin{aligned}\nu(1-\nu)\frac{(b-a)^2}{ab} &\ge \nu(1-\nu)\frac{(b-a)^2}{\max^2\{a,b\}}\\ &= \nu(1-\nu)\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2\end{aligned}$$

and by (11) we get the desired result (9). □

We observe that the inequality (9) can be written in an equivalent form as

$$\begin{aligned}&\left[\nu(1-\nu)\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2+1\right]H_\nu(a,b)\\ &\le A_\nu(a,b)\\ &\le \left[\nu(1-\nu)\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2+1\right]H_\nu(a,b)\end{aligned} \tag{12}$$

for any a, $b > 0$ and $\nu \in [0,1]$, while (10) as

$$\left[\frac{1}{4}\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2+1\right] H(a,b) \tag{13}$$
$$\leq A(a,b)$$
$$\leq \left[\frac{1}{4}\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2+1\right] H(a,b)$$

for any a, $b > 0$.

Corollary 1 *For any* a, $b \in [k,K] \subset (0,\infty)$ *and* $\nu \in [0,1]$ *we have*

$$\frac{A_\nu(a,b)}{H_\nu(a,b)} - 1 \leq \nu(1-\nu)\left(\frac{K}{k}-1\right)^2. \tag{14}$$

In particular,

$$\frac{A(a,b)}{H(a,b)} - 1 \leq \frac{1}{4}\left(\frac{K}{k}-1\right)^2. \tag{15}$$

We have the following multiplicative inequality between the weighted arithmetic and harmonic operator means:

Theorem 2 *Let* A, B *be positive invertible operators and* $M > m > 0$ *such that the condition* (6) *holds. Then we have*

$$\left[\nu(1-\nu)\left(1-\frac{\min\{M,1\}}{\max\{m,1\}}\right)^2+1\right] A!_\nu B \tag{16}$$
$$\leq A\nabla_\nu B$$
$$\leq \left[\nu(1-\nu)\left(\frac{\max\{M,1\}}{\min\{m,1\}}-1\right)^2+1\right] A!_\nu B$$

for any $\nu \in [0,1]$.

In particular,

$$\left[\frac{1}{4}\left(1-\frac{\min\{M,1\}}{\max\{m,1\}}\right)^2+1\right] A!B \tag{17}$$
$$\leq A\nabla B$$
$$\leq \left[\frac{1}{4}\left(\frac{\max\{M,1\}}{\min\{m,1\}}-1\right)^2+1\right] A!B.$$

Proof. If we write the inequality (12) for $a = 1$ and $b = x \in (0, \infty)$ then we have

$$\left[\nu(1-\nu)\left(1-\frac{\min\{1,x\}}{\max\{1,x\}}\right)^2+1\right]\left(1-\nu+\nu x^{-1}\right)^{-1} \tag{18}$$
$$\leq 1-\nu+\nu x$$
$$\leq \left[\nu(1-\nu)\left(\frac{\max\{1,x\}}{\min\{1,x\}}-1\right)^2+1\right]\left(1-\nu+\nu x^{-1}\right)^{-1}.$$

for any $\nu \in [0, 1]$.

If $x \in [m, M] \subset (0, \infty)$, then $\max\{m, 1\} \leq \max\{x, 1\} \leq \max\{M, 1\}$ and $\min\{m, 1\} \leq \min\{x, 1\} \leq \min\{M, 1\}$.

We have

$$\left(\frac{\max\{1,x\}}{\min\{1,x\}}-1\right)^2 \leq \left(\frac{\max\{M,1\}}{\min\{m,1\}}-1\right)^2$$

and

$$\left(1-\frac{\min\{M,1\}}{\max\{m,1\}}\right)^2 \leq \left(1-\frac{\min\{1,x\}}{\max\{1,x\}}\right)^2$$

for any $x \in [m, M] \subset (0, \infty)$.

Therefore, by (18) we have

$$\left[\nu(1-\nu)\left(1-\frac{\min\{M,1\}}{\max\{m,1\}}\right)^2+1\right]\left(1-\nu+\nu x^{-1}\right)^{-1} \tag{19}$$
$$\leq 1-\nu+\nu x$$
$$\leq \left[\nu(1-\nu)\left(\frac{\max\{M,1\}}{\min\{m,1\}}-1\right)^2+1\right]\left(1-\nu+\nu x^{-1}\right)^{-1},$$

for any $x \in [m, M]$ and for any $\nu \in [0, 1]$.

If we use the continuous functional calculus for the positive invertible operator X with $mI \leq X \leq MI$, then we have from (19) that

$$\left[\nu(1-\nu)\left(1-\frac{\min\{M,1\}}{\max\{m,1\}}\right)^2+1\right]\left((1-\nu)I+\nu X^{-1}\right)^{-1} \tag{20}$$
$$\leq (1-\nu)I+\nu X$$
$$\leq \left[\nu(1-\nu)\left(\frac{\max\{M,1\}}{\min\{m,1\}}-1\right)^2+1\right]\left((1-\nu)I+\nu X^{-1}\right)^{-1},$$

for any $\nu \in [0,1]$.

If we multiply (6) both sides by $A^{-1/2}$ we get $MI \geq A^{-1/2}BA^{-1/2} \geq mI$.

By writing the inequality (20) for $X = A^{-1/2}BA^{-1/2}$ we obtain

$$\begin{aligned} &\left[\nu(1-\nu)\left(1-\frac{\min\{M,1\}}{\max\{m,1\}}\right)^2+1\right] \\ &\times\left((1-\nu)I+\nu\left(A^{-1/2}BA^{-1/2}\right)^{-1}\right)^{-1} \\ &\leq (1-\nu)I+\nu A^{-1/2}BA^{-1/2} \\ &\leq \left[\nu(1-\nu)\left(\frac{\max\{M,1\}}{\min\{m,1\}}-1\right)^2+1\right] \\ &\times\left((1-\nu)I+\nu\left(A^{-1/2}BA^{-1/2}\right)^{-1}\right)^{-1}, \end{aligned} \tag{21}$$

for any $\nu \in [0,1]$.

If we multiply the inequality (21) both sides with $A^{1/2}$, then we get

$$\begin{aligned} &\left[\nu(1-\nu)\left(1-\frac{\min\{M,1\}}{\max\{m,1\}}\right)^2+1\right] \\ &\times A^{1/2}\left((1-\nu)I+\nu\left(A^{-1/2}BA^{-1/2}\right)^{-1}\right)^{-1}A^{1/2} \\ &\leq (1-\nu)A+\nu B \\ &\leq \left[\nu(1-\nu)\left(\frac{\max\{M,1\}}{\min\{m,1\}}-1\right)^2+1\right] \\ &\times A^{1/2}\left((1-\nu)I+\nu\left(A^{-1/2}BA^{-1/2}\right)^{-1}\right)^{-1}A^{1/2}, \end{aligned} \tag{22}$$

for any $\nu \in [0,1]$.

Since

$$\begin{aligned} &A^{1/2}\left((1-\nu)I+\nu\left(A^{-1/2}BA^{-1/2}\right)^{-1}\right)^{-1}A^{1/2} \\ &= A^{1/2}\left((1-\nu)I+\nu A^{1/2}B^{-1}A^{1/2}\right)^{-1}A^{1/2} \\ &= A^{1/2}\left(A^{1/2}\left((1-\nu)A^{-1}+\nu B^{-1}\right)A^{1/2}\right)^{-1}A^{1/2} \end{aligned}$$

$$= A^{1/2}\left(A^{-1/2}\left((1-\nu)A^{-1}+\nu B^{-1}\right)^{-1}A^{-1/2}\right)A^{1/2}$$
$$= \left((1-\nu)A^{-1}+\nu B^{-1}\right)^{-1} = A!_\nu B$$

hence by (22) we get the desired result (16). □

We also have:

Theorem 3 *Let* A, B *be positive invertible operators and* $M > m > 0$ *such that the condition (6) holds. Then we have*

$$d_\nu(m, M)\, A!_\nu B \le A\nabla_\nu B \le D_\nu(m, M)\, A!_\nu B \tag{23}$$

for any $\nu \in [0, 1]$, *where*

$$d_\nu(m, M) := 4\left[\left(\nu - \frac{1}{2}\right)^2 + \nu(1-\nu)\times\begin{cases} K(M) & \textit{if } M < 1, \\ 1 & \textit{if } m \le 1 \le M, \\ K(m) & \textit{if } 1 < m \end{cases}\right]$$

and

$$D_\nu(m, M) := 4\left[\left(\nu - \frac{1}{2}\right)^2 + \nu(1-\nu)\times\begin{cases} K(m) & \textit{if } M < 1, \\ \max\{K(m), K(M)\} & \textit{if } m \le 1 \le M, \\ K(M) & \textit{if } 1 < m. \end{cases}\right]$$

In particular, we have

$$d(m, M)\, A!B \le A\nabla B \le D(m, M)\, A!B \tag{24}$$

where

$$d(m, M) := \begin{cases} K(M) & \textit{if } M < 1, \\ 1 & \textit{if } m \le 1 \le M, \\ K(m) & \textit{if } 1 < m \end{cases}$$

and

$$D(m, M) := \begin{cases} K(m) & \textit{if } M < 1, \\ \max\{K(m), K(M)\} & \textit{if } m \le 1 \le M, \\ K(M) & \textit{if } 1 < m. \end{cases}$$

Proof. From (11) we have for any $x \in (0, \infty)$ and for any $\nu \in [0, 1]$ that

$$\frac{A_\nu(1, x)}{H_\nu(1, x)} - 1 = \nu(1-\nu)\frac{(x-1)^2}{x}. \tag{25}$$

Since $K(x) - 1 = \frac{(x-1)^2}{4x}$, $x > 0$, then by (25) we have

$$\begin{aligned}
\frac{A_\nu(1,x)}{H_\nu(1,x)} &= 1 + 4\nu(1-\nu)[K(x)-1] \\
&= 4\nu(1-\nu)K(x) + 4\left(\nu - \frac{1}{2}\right)^2 \\
&= 4\left[\nu(1-\nu)K(x) + \left(\nu - \frac{1}{2}\right)^2\right]
\end{aligned}$$

or, equivalently,

$$A_\nu(1,x) = 4\left[\nu(1-\nu)K(x) + \left(\nu - \frac{1}{2}\right)^2\right] H_\nu(1,x) \tag{26}$$

for any $x \in (0,\infty)$ and for any $\nu \in [0,1]$.

From (26) we then have for any $x \in [m,M] \subset (0,\infty)$ that

$$\begin{aligned}
&4\left[\nu(1-\nu)\min_{x\in[m,M]} K(x) + \left(\nu - \frac{1}{2}\right)^2\right] H_\nu(1,x) \\
&\le A_\nu(1,x) \le 4\left[\nu(1-\nu)\max_{x\in[m,M]} K(x) + \left(\nu - \frac{1}{2}\right)^2\right] H_\nu(1,x).
\end{aligned} \tag{27}$$

Since

$$\min_{x\in[m,M]} K(x) = \begin{cases} K(M) \text{ if } M < 1, \\ 1 \text{ if } m \le 1 \le M, \\ K(m) \text{ if } 1 < m \end{cases}$$

and

$$\max_{x\in[m,M]} K(x) = \begin{cases} K(m) \text{ if } M < 1, \\ \max\{K(m), K(M)\} \text{ if } m \le 1 \le M, \\ K(M) \text{ if } 1 < m, \end{cases}$$

then by (27) we have

$$\begin{aligned}
d_\nu(m,M)\left(1-\nu+\nu x^{-1}\right)^{-1} &\le 1-\nu+\nu x \\
&\le D_\nu(m,M)\left(1-\nu+\nu x^{-1}\right)^{-1}
\end{aligned} \tag{28}$$

for any $x \in [m,M]$ and for any $\nu \in [0,1]$.

By a similar argument to the one from Theorem 2 we deduce the desired operator inequality (23). The details are omitted. □

3 Some particular cases

Let A, B positive invertible operators and positive real numbers m, m', M, M' such that the condition $0 < mI \le A \le m'I < M'I \le B \le MI$ holds.

Put $h := \frac{M}{m}$ and $h' := \frac{M'}{m'}$, then we have

$$A < h'A = \frac{M'}{m'}A \le B \le \frac{M}{m}A = hA.$$

By (16) we get

$$\left[\nu(1-\nu)\left(\frac{h'-1}{h'}\right)^2 + 1\right] A!_\nu B \le A\nabla_\nu B \le \left[\nu(1-\nu)(h-1)^2 + 1\right] A!_\nu B \tag{29}$$

for any $\nu \in [0,1]$.

By (23) we get

$$4\left[\left(\nu - \frac{1}{2}\right)^2 + \nu(1-\nu) K(h')\right] A!_\nu B \le A\nabla_\nu B \le 4\left[\left(\nu - \frac{1}{2}\right)^2 + \nu(1-\nu) K(h)\right] A!_\nu B \tag{30}$$

for any $\nu \in [0,1]$.

If $0 < mI \le B \le m'I < M'I \le A \le MI$, then for $h := \frac{M}{m}$ and $h' := \frac{M'}{m'}$ we also have

$$\frac{1}{h}A \le B \le \frac{1}{h'}A < A.$$

Finally, by (16) we get (29) while from (23) we get (30) as well.

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On approximate solution of Drygas functional equation according to the Lipschitz criteria

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Abstract. Let G be an Abelian group with a metric d and E be a normed space. For any $f : G \to E$ we define the Drygas difference of the function f by the formula

$$\Lambda f(x, y) := 2f(x) + f(y) + f(-y) - f(x + y) - f(x - y)$$

for all $x, y \in G$. In this article, we prove that if Λf is Lipschitz, then there exists a Drygas function $D : G \to E$ such that $f - D$ is Lipschitz with the same constant. Moreover, some results concerning the approximation of the Drygas functional equation in the Lipschitz norms are presented.

1 Introduction

The stability theory of functional equations began with the well-known Ulam's Problem [21], concerning the stability of homomorphisms in metric groups:

Problem. *Let* $(G_1, *)$, $(G_2, \star)$ *be two groups and* $d : G_2 \times G_2 \to [0, \infty)$ *be a metric. Given* $\epsilon > 0$, *does there exist* $\delta > 0$ *such that if a function* $f : G_1 \to G_2$ *satisfies the inequality*

$$d(f(x * y), f(x) \star f(y)) \leq \delta$$

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for all $x, y \in G_1$*, then there is a homomorphism* $h : G_1 \to G_2$ *with*

$$d(f(x), h(x)) \leq \epsilon$$

for all $x \in G_1$*?*

Ulam's problem was partially solved by Hyers [14] in 1941 in the context of Banach spaces with $\delta = \epsilon$. Aoki [1], Z. Gajda [11] and Th.M. Rassias [17] provided a generalization of the result of Hyers for additive and linear mappings, respectively, by allowing the Cauchy difference to be unbounded. Since then many authors have studied the question of stability of various functional equations (see [9, 15] for the survey of stability results).

Let G and Y be an Abelian group and a Banach space respectively. We say that a function $f : G \to Y$ satisfies the Drygas equation if

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y), \qquad x, y \in G, \tag{1}$$

and every solution of the Drygas equation is called a Drygas function. The above equation was introduced in [4] to obtain a characterization of the quasi-inner-product spaces. The general solution of (1), obtained by Ebanks et al. in [5] (see also[18]). The stability in the Hyers–Ulam sense of the Drygas equation has been investigated, for example, in [6, 7, 10, 16, 19, 22].

In Lipschitz spaces the stability type problems for some functional equations was studied by a number of mathematicians (see, e.g., [3, 8, 20])

In the present paper, we establish the stability problem of (1) in Lipschitz spaces.

2 Preliminaries

In this section we are going to introduce some basic definitions and notations needed for further considerations.

Definition 1 [2] *Let* $\mathbb{R}$ *be the set of real numbers,* E *a vector space and* $\mathcal{S}(E)$ *a family of subsets of* E*. We say that this family is linearly invariant if*

(1) $x + \alpha V \in S(E)$ *for* $x \in E$, $\alpha \in \mathbb{R}$ *and* $V \in \mathcal{S}(E)$*,*

(2) $V + W \in \mathcal{S}(E)$ *for* $V, W \in \mathcal{S}(E)$*.*

Definition 2 *Let* G *be a set,* E *a vector space and* $\mathcal{S}(E)$ *any linearly invariant family. By* $\mathcal{B}(G, \mathcal{S}(E))$ *we denote the family*

$$\mathcal{B}\big(G, \mathcal{S}(E)\big) := \{f : G \to E; \text{Im } f \subset V \text{ for some } V \in \mathcal{S}(E)\}.$$

It is easy to verify that $\mathcal{B}(G, \mathcal{S}(E))$ *is a vector space. For any* $f : G \to E$, $a \in G$, *where* G *is a group, we put*

$$f^a(x) := f(x + a), \quad x \in G.$$

Definition 3 [13, 20] *Let* G *be a group,* E *a vector space, and let* $\mathcal{S}(E)$ *be a linearly invariant family of subsets of* E. *We say that* $\mathcal{B}(G, \mathcal{S}(E))$ *admits a left invariant mean (LIM for short) if there exists a linear operator* $M : \mathcal{B}(G, \mathcal{S}(E)) \to E$ *such that*

(i) *if* $\operatorname{Im} f \subset V$ *for some* $V \in \mathcal{S}(E)$, *then* $M[f] \in V$,

(ii) *if* $f \in \mathcal{B}(G, \mathcal{S}(E))$ *and* $a \in G$, *then* $M[f^a] = M[f]$.

Analogously we can define so-called right invariant mean. For more information about spaces which admit LIM see, e.g., [2, 12, 13].

Example 1 *Let* G *be a finite group, let* E *be a vector space, and let* $\mathcal{S}(E)$ *be any linearly invariant family of convex subsets of* E. *Let* $f \in \mathcal{B}(G, \mathcal{S}(E))$ *be arbitrary. We define*

$$M[f] := \frac{1}{|G|} \sum_{g \in G} f(g).$$

One can easily check that M *is a LIM on* $\mathcal{B}(G, \mathcal{S}(E))$, *where* $|G| =$ *cardinality of* G.

Definition 4 *Let* G *be a group,* E *a vector space and let* $\mathcal{S}(E)$ *be a linearly invariant family. We say that* $\mathbf{d} : G \times G \to \mathcal{S}(E)$ *is translation invariant if*

$$\mathbf{d}(x + a, y + a) = \mathbf{d}(a + x, a + y) = \mathbf{d}(x, y), \text{ for all } x, y, a \in G.$$

The function $f : G \to E$ *is* $\mathbf{d}$*-Lipschitz if for all* $x, y \in G$,

$$f(x) - f(y) \in \mathbf{d}(x, y).$$

Definition 5 *Let* G *be a group with a metric* d *and* E *a normed space.*

a/ *We say that* $w : \mathbb{R}_+ \to \mathbb{R}_+$ *is the module of continuity of* $f : G \to E$ *if for every* $\delta \in \mathbb{R}_+$

$$d(x, y) \leq \delta \Rightarrow \|f(x) - f(y)\| \leq w(\delta) \quad x, y \in G.$$

b/ *A function* $f : G \to E$ *called a Lipschitz function if there exists an* $L \in \mathbb{R}_+$ *such that*

$$\|f(x) - f(y)\| \le Ld(x, y), \quad x, y \in G.$$

The smallest constant L *with this property is denoted by* $lip(f)$*. By* $Lip(G, E)$ *we mean the space of all bounded Lipschitz functions with the norm*

$$\|f\|_{Lip} := \|f\|_{sup} + lip(f).$$

Moreover, by $Lip^0(G, E)$ *we denote the space of all Lipschitz functions* $f : G \to E$ *with the norm defined by the formula*

$$\|f\|_{Lip^0} := \|f(0)\| + lip(f).$$

Finally, we introduce the following remarks.

Remark 1 (i) *If E is a vector space and* $\mathcal{S}(E)$ *is a linearly invariant family, then for every* $x \in E$*, the set* $\{x\} \in \mathcal{S}(E)$*.*
(ii) *The family* $\mathcal{B}(G, \mathcal{S}(E))$ *contains all constant functions.*

Remark 2 *Let* $(G, +)$ *be a group and* E *a vector space. Assume that* $\mathcal{S}(E)$ *is a linearly invariant family such that* $\mathcal{B}(G, \mathcal{S}(E))$ *satisfies the condition LIM or RIM. If* $f : G \to E$ *is constant, then* $M[f] = \mathrm{Im}\, f$ *(i.e., if* $f(x) = c$ *for* $x \in G$*, where* $c \in E$*, then* $M[f] = c$*).*

Remark 3 *Let* G *be a group with metric* d *and let* E *be a normed space. Let* $L \in \mathbb{R}_+$*, and*

$$\mathbf{d}(x, y) := Ld(x, y)B(0, 1),$$

where $B(0, 1)$ *is the closed ball with the center at 0 and the radius 1. Then the function* $f : G \to E$ *is Lipschitz with the constant* L *if and only if it is* $\mathbf{d}$*-Lipschitz.*

3 Lipschitz approximation of Eq. (1)

In this section, we can prove one of the main results of this paper.

Theorem 1 *Let* G *be an Abelian group and* E *a vector space. Assume that* $\mathcal{S}(E)$ *is a linearly invariant family such that* $\mathcal{B}(G, \mathcal{S}(E))$ *admits LIM. Let* $f : G \to E$ *be an arbitrary function. If* $\Lambda f(\cdot, y) : G \to E$ *is* $\mathbf{d}$*-Lipschitz for every* $y \in G$*, then there exists a Drygas function* $D : G \to E$ *such that* $f - D$ *is* $\frac{1}{2}\mathbf{d}$*-Lipschitz. Moreover, if* $\mathrm{Im}(\Lambda f) \subset V$ *for some* $V \in \mathcal{S}(E)$*, then* $\mathrm{Im}(f - D) \subset \frac{1}{2}V$*.*

Proof. For every $a \in G$ we define $F_a : G \to E$ by

$$F_a(y) := \frac{1}{2}f(a+y) + \frac{1}{2}f(a-y) - \frac{1}{2}f(y) - \frac{1}{2}f(-y), \quad y \in G.$$

We will prove that F_a belongs to $\mathcal{B}(G, \mathcal{S}(E))$. In fact, we have for $y, a \in G$,

$$F_a(y) = \frac{1}{2}\Lambda f(0, y) - \frac{1}{2}\Lambda f(a, y) + f(a) - f(0).$$

So, $F_a \in \mathcal{B}(G, \mathcal{S}(E))$ for $a \in G$.

According to the assumptions, there exists a linear operator $M : \mathcal{B}(G, \mathcal{S}(E)) \to E$ such that

(i) $\mathrm{Im}(g) \subset V \Rightarrow M[g] \in V$,

(ii) if $g \in \mathcal{B}(G, \mathcal{S}(E))$ and $g_a : G \to E$ for $a \in G$ is defined by

$$g_a(x) := g(a + x), \quad x \in G,$$

then $g_a \in \mathcal{B}(G, \mathcal{S}(E))$ and $M[g_a] = M[g]$.

Consider the function $D : G \to E$ given by

$$D(x) := M[F_x], \quad \text{for } x \in G.$$

We will verify that $f - D$ is $\frac{1}{2}\mathbf{d}$-Lipschitz.

In view of our assumptions it follows that $\frac{1}{2}\Lambda f(\cdot, y)$ is $\frac{1}{2}\mathbf{d}$-Lipschitz for every $y \in G$, which means that

$$\frac{1}{2}\Lambda f(x, y) - \frac{1}{2}\Lambda f(z, y) \in \frac{1}{2}\mathbf{d}(x, z) \tag{2}$$

for all $x, z \in G$. Let $l : G \to E$ be the function

$$l(x) := f(x) - M[F_x] = f(x) - D(x), \quad x \in G,$$

and for any $x \in G$, $R_x : G \to E$ be defined by

$$R_x(y) := f(x), \quad y \in G.$$

Therefore, applying Remarks 1 and 2, one gets for all $x \in G$,

$$\begin{aligned} l(x) &= f(x) - M[F_x] = M[R_x - F_x] \\ &= M\Big[f(x) + \frac{1}{2}f(\cdot) + \frac{1}{2}f(-\cdot) - \frac{1}{2}f(x + \cdot) - \frac{1}{2}f(x - \cdot)\Big] \\ &= M\left[\frac{1}{2}\Lambda f(x, \cdot)\right]. \end{aligned} \tag{3}$$

Immediately from (2) and (3) we obtain

$$l(x) - l(z) = M\left[\frac{1}{2}\Lambda f(x,\cdot) - \frac{1}{2}\Lambda f(z,\cdot)\right], \quad x, z \in G. \tag{4}$$

For any $x, z \in G$, we define $A_{(x,z)} : G \to E$ by

$$A_{(x,z)}(y) := \frac{1}{2}\Lambda f(x,y) - \frac{1}{2}\Lambda f(z,y), \quad y \in G.$$

By (2) we have $\operatorname{Im} A_{(x,z)} \subset \frac{1}{2}\mathbf{d}(x,z)$, which together with (4) implies

$$l(x) - l(z) = M[A_{(x,z)}] \in \frac{1}{2}\mathbf{d}(x,z),$$

for all $y, z \in G$. This proves that

$$(f(x) - D(x)) - (f(z) - D(z)) \in \frac{1}{2}\mathbf{d}(x,z), \quad \text{for all } x, z \in G,$$

i.e., $f - D$ is $\frac{1}{2}\mathbf{d}$-Lipschitz.

Now we will verify that D is a Drygas function. We have the equalities

$$\begin{aligned} D(x+z) + D(x-z) &= M[F_{x+z}(y)] + M[F_{x-z}(y)] \\ &= M\Big[\frac{1}{2}f(x+z+y) + \frac{1}{2}f(x+z-y) - \frac{1}{2}f(y) - \frac{1}{2}f(-y)\Big] \\ &= M\Big[\frac{1}{2}f(x-z+y) + \frac{1}{2}f(x-z-y) - \frac{1}{2}f(y) - \frac{1}{2}f(-y)\Big] \end{aligned}$$

and

$$\begin{aligned} &2D(x) + D(z) + D(-z) = 2M[F_x(y)] + M[F_z(y)] + M[F_{-z}(y)] \\ &\quad = 2M\Big[\frac{1}{2}f(x+y) + \frac{1}{2}f(x-y) - \frac{1}{2}f(y) - \frac{1}{2}f(-y)\Big] \\ &\quad\quad + M\Big[\frac{1}{2}f(z+y) + \frac{1}{2}f(z-y) - \frac{1}{2}f(y) - \frac{1}{2}f(-y)\Big] \\ &\quad\quad + M\Big[\frac{1}{2}f(-z+y) + \frac{1}{2}f(-z-y) - \frac{1}{2}f(y) - \frac{1}{2}f(-y)\Big] \\ &\quad = M\Big[\frac{1}{2}f(x+y+z) + \frac{1}{2}f(x-y+z) - \frac{1}{2}f(y+z) - \frac{1}{2}f(-y+z)\Big] \\ &\quad\quad + M\Big[\frac{1}{2}f(x+y-z) + \frac{1}{2}f(x-y-z) - \frac{1}{2}f(y-z) - \frac{1}{2}f(-y-z)\Big] \\ &\quad\quad + M\Big[\frac{1}{2}f(z+y) + \frac{1}{2}f(z-y) - \frac{1}{2}f(y) - \frac{1}{2}f(-y)\Big] \end{aligned}$$

$$+ M\left[\frac{1}{2}f(-z+y)+\frac{1}{2}f(-z-y)-\frac{1}{2}f(y)-\frac{1}{2}f(-y)\right]$$
$$= D(x+z)+D(x-z).$$

It follows that D is a Drygas function.

To finish the proof assume that $\mathrm{Im}(\Lambda f) \subset V$ for some $V \in \mathcal{S}(E)$. Then we have $\mathrm{Im}\left(\frac{1}{2}\Lambda f\right) \subset \frac{1}{2}V$. In view of (3) we get $f(x) - D(x) = M\left[\frac{1}{2}\Lambda f(x,\cdot)\right] \in \frac{1}{2}V$ for all $x \in G$. Thus $\mathrm{Im}(f-D) \subset \frac{1}{2}V$, which completes the proof of the theorem. □

Corollary 1 *Let* G *be an Abelian group and* $(E, \|.\|)$ *a normed space. Assume that* $\mathcal{S}(E)$ *is a family of closed balls such that* $\mathcal{B}(G, \mathcal{S}(E))$ *admits LIM. Let* $f : G \to E$ *and* $g : G \to \mathbb{R}_+$ *satisfy the inequality*

$$\|\Lambda f(x,y) - \Lambda f(z,y)\| \leq g(x-z) \tag{5}$$

for all $x, y, z \in G$. *Then there exists a Drygas function* $D : G \to E$ *such that*

$$\|(f-D)(x) - (f-D)(y)\| \leq (1/2)g(x-y) \tag{6}$$

for all $x, y \in G$, *where* $(f-D)(x) \equiv f(x) - D(x)$.

Proof. We put

$$\mathbf{d}(x,y) := g(x-y)B(0,1), \quad x, y \in G$$

where $B(0,1)$ is the closed unit ball with center at zero. By (5) we obtain

$$\Lambda f(x,y) - \Lambda f(z,y) \in \mathbf{d}(x,z), \quad x, y, z \in G,$$

which means that $\Lambda f(\cdot, y)$ is a $\mathbf{d}$-Lipschitz. Therefore, from Theorem 1 there exists a Drygas function $D : G \to E$ such that $f - D$ is $(1/2)\mathbf{d}$-Lipschitz. By the definition of $\mathbf{d}$ we get the desired result. □

4 Approximation with Lipschitz norm

We shall introduce the following definition (see also [20]).

Definition 6 *A group* $(G, +, d, \widetilde{d}\,)$ *is said to be a metric pair if*

(1) $(G, +, d)$ *is an Abelian metric group,*

(2) $\widetilde{d} : (G \times G) \times (G \times G) \to \mathbb{R}_+$ *is a metric in* $G \times G$,

(3) $\widetilde{d}((a,x),(a,y)) = \widetilde{d}((x,a),(y,a)) = d(x,y)$ *for* $x,y,a \in G$.

The following lemma is needed to establish the next results.

Lemma 1 *Let* $(G,+,d,\widetilde{d}\ )$ *be a metric pair and* $(E,\|\cdot\|)$ *a normed space. Assume that* $\mathcal{S}(E)$ *is a family of closed balls such that* $\mathcal{B}(G,\mathcal{S}(E))$ *admits LIM. Let* $f : G \to E$ *be a function and* $w : \mathbb{R}_+ \to \mathbb{R}_+$ *be the module of continuity of the function* $\Lambda f : G \times G \to E$. *Then there exists a Drygas function* $D : G \to E$ *such that the function* $(1/2)w$ *is the module of continuity of* $f - D$. *Moreover, if* $\Lambda f \in \mathcal{B}(G \times G, \mathcal{S}(E))$, *then*

$$\|f - D\|_{\sup} \leq (1/2)\|\Lambda f\|_{\sup}. \tag{7}$$

Proof. Define $\mathbf{d} : G \times G \to \mathcal{S}(E)$ by the formula

$$\mathbf{d}(x,y) := \left(\inf_{t \geq d(x,y)} w(t)\right) B(0,1),$$

where $B(0,1)$ is the closed unit ball with center at zero. Since w is the module of continuity of $\Lambda f(\cdot,y)$ for $y \in G$, we have

$$\|\Lambda f(x,y) - \Lambda f(z,y)\| \leq \inf_{t \geq \widetilde{d}((x,y),(z,y))} w(t), \quad x,y,z \in G. \tag{8}$$

This implies that

$$\|\Lambda f(x,y) - \Lambda f(z,y)\| \leq \inf_{t \geq d(x,z)} w(t), \quad x,y,z \in G, \tag{9}$$

i.e.,

$$\Lambda f(x,y) - \Lambda f(z,y) \in \mathbf{d}(x,z), \quad \text{for } x,y,z \in G.$$

This shows that $Qf(\cdot,y)$ is $\mathbf{d}$-Lipschitz.

Now, in view of Theorem 1, there exists a Drygas function $D : G \to E$ such that $f - D$ is $(1/2)\mathbf{d}$-Lipschitz and consequently

$$(f(x) - D(x)) - (f(y) - D(y)) \in (1/2)\mathbf{d}(x,y), \quad x,y \in G.$$

This is equivalent to the condition

$$\|(f(x) - D(x)) - (f(y) - D(y))\| \leq \inf_{t \geq d(x,y)} (1/2)w(t), \quad x,y \in G.$$

This shows that $(1/2)w$ is the module of continuity of $f - D$.

Finally, assume that $\Lambda f \in \mathcal{B}(G \times G, \mathcal{S}(E))$. Thus the following set is well defined:

$$W := B\left(0, \|\Lambda f\|_{\sup}\right) \quad \text{with} \quad \mathrm{Im}(\Lambda f) \subset W.$$

Thus from Theorem 1, we get

$$\mathrm{Im}(f - D) \subset (1/2)W$$

which implies the inequality (7) and completes the proof. □

In the remaining part of the paper, we investigate two results about the stability of the generalized quadratic functional equation in the Lipschitz norms.

Theorem 2 *Let* $(G, +, d, \widetilde{d}\ )$ *be a metric pair and* $(E, \|\cdot\|)$ *a normed space. Assume that* $\mathcal{S}(E)$ *is a family of closed balls such that* $\mathcal{B}(G, \mathcal{S}(E))$ *admits LIM.*

(i) *Let* $f : G \to E$ *be a function satisfying the condition* $\Lambda f \in \mathrm{Lip}(G \times G, E)$. *Then there exists a Drygas function* $D : G \to E$ *such that*

$$\|f - D\|_{\mathrm{Lip}} \leq (1/2)\|\Lambda f\|_{\mathrm{Lip}}. \tag{10}$$

(ii) *Let* $f : G \to E$ *be a function satisfying the condition* $\Lambda f \in \mathrm{Lip}^0(G \times G, E)$. *Then there exists a Drygas function* $D : G \to E$ *such that*

$$\|f - D\|_{\mathrm{Lip}^0} \leq (1/2)\|\Lambda f\|_{\mathrm{Lip}^0}. \tag{11}$$

Proof. (i) Consider $w : \mathbb{R}_+ \to \mathbb{R}_+$ by the formula

$$w(x) := \mathrm{lip}(\Lambda f)x, \quad \text{for } x \in \mathbb{R}_+.$$

Since $\Lambda f \in \mathrm{Lip}(G \times G, E)$, we obtain

$$\begin{aligned} \|\Lambda f(x, y) - \Lambda f(t, z)\| &\leq \mathrm{lip}(\Lambda f)\widetilde{d}((x, y), (t, z)) \\ &= w\big(\widetilde{d}((x, y), (t, z))\big), \quad x, y, t, z \in G, \end{aligned}$$

which means that w is the module of continuity of Λf. Thus, by Lemma 1, there exists a Drygas function $D : G \to E$ such that $(1/2)w$ is the module of continuity of $f - D$. Thus we have the inequality

$$\begin{aligned} \|(f(x) - D(x)) - (f(y) - D(y))\| &\leq (1/2)w(d(x, y)) \\ &= (1/2)\mathrm{lip}(\Lambda f)d(x, y), \quad x, y \in G. \end{aligned}$$

This inequality implies that $f - D$ is a Lipschitz function and

$$\mathrm{lip}(f - D) \le \frac{1}{2}\mathrm{lip}(\Lambda f). \tag{12}$$

Taking into account that $\Lambda f \in \mathrm{Lip}(G\times G, E)$, we have also $\Lambda f \in \mathcal{B}(G\times G, \mathcal{S}(E))$. Therefore by Lemma 1 we obtain

$$\|f - D\|_{\sup} \le \frac{1}{2}\|\Lambda f\|_{\sup}, \tag{13}$$

that is, $f - D \in \mathrm{Lip}(G, E)$. Finally, from (12) and (13), we obtain the desired result.

(ii) By the same reasoning as in the proof of (i) we can prove that there exists a Drygas function $D : G \to E$ such that $f - D$ is Lipschitz and

$$\mathrm{lip}(f - D) \le \frac{1}{2}\mathrm{lip}(\Lambda f).$$

Since $D(0) = 0$, we obtain

$$\|f(0) - D(0)\| = \|f(0)\| = (1/2)\|\Lambda f(0, 0)\|.$$

Thus

$$\begin{aligned}
\|f - D\|_{\mathrm{Lip}^0} &= \|f(0) - D(0)\| + \mathrm{lip}(f - D) \\
&\le \frac{1}{2}\|\Lambda f(0, 0)\| + \frac{1}{2}\mathrm{lip}(\Lambda f) \\
&= \frac{1}{2}\|\Lambda f\|_{\mathrm{Lip}^0},
\end{aligned}$$

which completes the proof. □

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On certain subclasses of analytic functions associated with Poisson distribution series

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Abstract. In this paper, we find the necessary and sufficient conditions, inclusion relations for Poisson distribution series $\mathcal{K}(m,z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n$ to be in the subclasses $\mathcal{S}(k,\lambda)$ and $\mathcal{C}(k,\lambda)$ of analytic functions with negative coefficients. Further, we obtain necessary and sufficient conditions for the integral operator $\mathcal{G}(m,z) = \int_0^z \frac{\mathcal{F}(m,t)}{t} dt$ to be in the above classes.

1 Introduction and definitions

Let $\mathcal{A}$ denote the class of the normalized functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. Further, let $\mathcal{T}$ be a subclass of $\mathcal{A}$ consisting of functions of the form,

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \qquad z \in \mathcal{U}. \tag{2}$$

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A function f of the form (2) is in $\mathcal{S}(k,\lambda)$ if it satisfies the condition

$$\left|\frac{\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)}-1}{\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)}+1}\right|<k,\quad (0<k\le 1,\ 0\le\lambda<1, z\in\mathcal{U})$$

and $f\in\mathcal{C}(k,\lambda)$ if and only if $zf'\in\mathcal{S}(k,\lambda)$. The class $\mathcal{S}(k,\lambda)$ was introduced by Frasin et al. [3].

We note that $\mathcal{S}(k,0)=\mathcal{S}(k)$ and $\mathcal{C}(k,0)=\mathcal{C}(k)$, where the classes $\mathcal{S}(k)$ and $\mathcal{C}(k)$ were introduced and studied by Padmanabhan [9] (see also, [5], [8]).

A function $f\in\mathcal{A}$ is said to be in the class $\mathcal{R}^{\tau}(A,B)$,$\tau\in\mathbb{C}\backslash\{0\}$, $-1\le B<A\le 1$, if it satisfies the inequality

$$\left|\frac{f'(z)-1}{(A-B)\tau-B[f'(z)-1]}\right|<1,\qquad z\in\mathcal{U}.$$

This class was introduced by Dixit and Pal [2].

A variable x is said to be Poisson distributed if it takes the values $0,1,2,3,\ldots$ with probabilities e^{-m}, $m\frac{e^{-m}}{1!}$, $m^2\frac{e^{-m}}{2!}$, $m^3\frac{e^{-m}}{3!},\ldots$ respectively, where m is called the parameter. Thus

$$P(x=r)=\frac{m^r e^{-m}}{r!},\ r=0,1,2,3,\ldots.$$

Very recently, Porwal [10] (see also, [6, 7]) introduce a power series whose coefficients are probabilities of Poisson distribution

$$\mathcal{K}(m,z)=z+\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}e^{-m}z^n,\qquad z\in\mathcal{U},$$

where $m>0$. By ratio test the radius of convergence of above series is infinity. In [10], Porwal also defined the series

$$\mathcal{F}(m,z)=2z-\mathcal{K}(m,z)=z-\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}e^{-m}z^n,\qquad z\in\mathcal{U}.$$

Using the Hadamard product, Porwal and Kumar [12] introduced a new linear operator $\mathcal{I}(m,z):\mathcal{A}\to\mathcal{A}$ defined by

$$\mathcal{I}(m,z)f=\mathcal{K}(m,z)*f(z)=z+\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}e^{-m}a_n z^n,\qquad z\in\mathcal{U},$$

where $*$ denote the convolution or Hadamard product of two series.

Motivated by several earlier results on connections between various subclasses of analytic and univalent functions by using hypergeometric functions (see [1, 4, 13, 14]) and by the recent investigations of Porwal ([10, 12, 11]), in the present paper we determine the necessary and sufficient conditions for $\mathcal{F}(m,z)$ to be in our new classes $\mathcal{S}(k,\lambda)$ and $\mathcal{C}(k,\lambda)$ and connections of these subclasses with $\mathcal{R}^{\tau}(A,B)$. Finally, we give conditions for the integral operator $\mathcal{G}(m,z)=\int_0^z \frac{\mathcal{F}(m,t)}{t}dt$ to be in the classes $\mathcal{S}(k,\lambda)$ and $\mathcal{C}(k,\lambda)$.

To establish our main results, we will require the following Lemmas.

Lemma 1 [3] *A function* f *of the form* (2) *is in* $\mathcal{S}(k,\lambda)$ *if and only if it satisfies*

$$\sum_{n=2}^{\infty}[n((1-\lambda)+k(1+\lambda))-(1-\lambda)(1-k)]\,|a_n|\leq 2k \tag{3}$$

where $0<k\leq 1$ and $0\leq\lambda<1$. The result is sharp.

Lemma 2 [3] *A function* f *of the form* (2) *is in* $\mathcal{C}(k,\lambda)$ *if and only if it satisfies*

$$\sum_{n=2}^{\infty}n[n((1-\lambda)+k(1+\lambda))-(1-\lambda)(1-k)]\,|a_n|\leq 2k \tag{4}$$

where $0<k\leq 1$ *and* $0\leq\lambda<1$. *The result is sharp.*

Lemma 3 [2] *If* $f\in\mathcal{R}^{\tau}(A,B)$ *is of the form, then*

$$|a_n|\leq (A-B)\frac{|\tau|}{n}, \qquad n\in\mathbb{N}-\{1\}.$$

The result is sharp.

2 The necessary and sufficient conditions

Theorem 1 *If* $m>0$, $0<k\leq 1$ *and* $0\leq\lambda<1$, *then* $\mathcal{F}(m,z)$ *is in* $\mathcal{S}(k,\lambda)$ *if and only if*

$$((1-\lambda)+k(1+\lambda))me^{m}\leq 2k. \tag{5}$$

Proof. Since

$$\mathcal{F}(m,z)=z-\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}e^{-m}z^{n} \tag{6}$$

according to (3) of Lemma 1, we must show that

$$\sum_{n=2}^{\infty}[n((1-\lambda)+k(1+\lambda))+(1-\lambda)(k-1)]\frac{m^{n-1}}{(n-1)!}\leq 2ke^{m}. \tag{7}$$

Writing $n=(n-1)+1$, we have

$$\begin{aligned}
&\sum_{n=2}^{\infty}[n((1-\lambda)+k(1+\lambda))+(1-\lambda)(k-1)]\frac{m^{n-1}}{(n-1)!}\\
&\quad = \sum_{n=2}^{\infty}[(n-1)((1-\lambda)+k(1+\lambda))+2k]\frac{m^{n-1}}{(n-1)!}\\
&\quad = [(1-\lambda)+k(1+\lambda)]\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-2)!}+2k\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}\\
&\quad = ((1-\lambda)+k(1+\lambda))me^{m}+2k(e^{m}-1).
\end{aligned} \tag{8}$$

But this last expression is bounded above by $2ke^{m}$ if and only if (5) holds. □

Theorem 2 *If* $m>0$, $0<k\leq 1$ *and* $0\leq\lambda<1$, *then* $\mathcal{F}(m,z)$ *is in* $\mathcal{C}(k,\lambda)$ *if and only if*

$$((1-\lambda)+k(1+\lambda))m^{2}e^{m}+2(1+2k+k\lambda-\lambda)me^{m}\leq 2k. \tag{9}$$

Proof. In view of Lemma 2, it suffices to show that

$$\sum_{n=2}^{\infty}n[n((1-\lambda)+k(1+\lambda))+(1-\lambda)(k-1)]\frac{m^{n-1}}{(n-1)!}\leq 2ke^{m}.$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty}n[n((1-\lambda)+k(1+\lambda))+(1-\lambda)(k-1)]\frac{m^{n-1}}{(n-1)!}\\
&\quad =\sum_{n=2}^{\infty}n^{2}((1-\lambda)+k(1+\lambda))+n(1-\lambda)(k-1)]\frac{m^{n-1}}{(n-1)!}.
\end{aligned} \tag{10}$$

Writing $n=(n-1)+1$ and $n^{2}=(n-1)(n-2)+3(n-1)+1$, in (10) we see that

$$\sum_{n=2}^{\infty}n^{2}((1-\lambda)+k(1+\lambda))+n(1-\lambda)(k-1)]\frac{m^{n-1}}{(n-1)!}$$

$$= \sum_{n=2}^{\infty}(n-1)(n-2)((1-\lambda)+k(1+\lambda))\frac{m^{n-1}}{(n-1)!}$$
$$+\sum_{n=2}^{\infty}(n-1)[3((1-\lambda)+k(1+\lambda)+(1-\lambda)(k-1)]\frac{m^{n-1}}{(n-1)!}+\sum_{n=2}^{\infty}2k\frac{m^{n-1}}{(n-1)!}$$
$$= ((1-\lambda)+k(1+\lambda))\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-3)!}+2(1+2k+k\lambda-\lambda)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-2)!}$$
$$+2k\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}$$
$$= ((1-\lambda)+k(1+\lambda))m^2e^m+2(1+2k+k\lambda-\lambda)me^m+2k\,(e^m-1).$$

But this last expression is bounded above by $2ke^m$ if and only if (9) holds. □

By specializing the parameter $\lambda = 0$ in Theorems 1 and 2 , we have the following corollaries.

Corollary 1 *If* $m > 0$ *and* $0 < k \leq 1$, *then* $\mathcal{F}(m, z)$ *is in* $\mathcal{S}(k)$ *if and only if*

$$(1+k)me^m \leq 2k. \tag{11}$$

Corollary 2 *If* $m > 0$ *and* $0 < k \leq 1$, *then* $\mathcal{F}(m, z)$ *is in* $\mathcal{C}(k)$ *if and only if*

$$(1+k)m^2e^m + 2(1+2k)me^m \leq 2k. \tag{12}$$

3 Inclusion properties

Theorem 3 *Let* $m > 0$, $0 < k \leq 1$ *and* $0 \leq \lambda < 1$. *If* $f \in \mathcal{R}^{\tau}(A, B)$, *then* $\mathcal{I}(m, z)f$ *is in* $\mathcal{S}(k, \lambda)$ *if and only if*

$$\begin{aligned}(A-B)\,|\tau|\,\Big[&((1-\lambda)+k(1+\lambda))(1-e^{-m})\\ &+\frac{(1-\lambda)(k-1)}{m}(1-e^{-m}(1+m))\Big] \leq 2k.\end{aligned} \tag{13}$$

Proof. In view of Lemma 1, it suffices to show that

$$\sum_{n=2}^{\infty}[n((1-\lambda)+k(1+\lambda))+(1-\lambda)(k-1)]\frac{m^{n-1}}{(n-1)!}\,|a_n| \leq 2ke^m.$$

Since $f \in \mathcal{R}^{\tau}(A, B)$, then by Lemma 3, we get

$$|a_n| \leq \frac{(A-B)\,|\tau|}{n}. \tag{14}$$

Thus, we have

$$\sum_{n=2}^{\infty}[n((1-\lambda)+k(1+\lambda))+(1-\lambda)(k-1)]\frac{m^{n-1}}{(n-1)!}|a_n|$$
$$\leq (A-B)|\tau|\sum_{n=2}^{\infty}[n((1-\lambda)+k(1+\lambda))+(1-\lambda)(k-1)]\frac{m^{n-1}}{n!}$$
$$=(A-B)|\tau|\left[((1-\lambda)+k(1+\lambda))\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}+\frac{(1-\lambda)(k-1)}{m}\sum_{n=2}^{\infty}\frac{m^n}{n!}\right]$$
$$=(A-B)|\tau|\left[((1-\lambda)+k(1+\lambda))(e^m-1)+\frac{(1-\lambda)(k-1)}{m}(e^m-1-m)\right].$$

But this last expression is bounded above by $2ke^m$ if and only if (13) holds. □

Theorem 4 *Let* $m>0$, $0<k\leq 1$ *and* $0\leq\lambda<1$. *If* $f\in\mathcal{R}^{\tau}(A,B)$, *then* $\mathcal{F}(m,z)f$ *is in* $\mathcal{C}(k,\lambda)$ *if and only if*

$$(A-B)|\tau|\,[((1-\lambda)+k(1+\lambda))m+2k(1-e^{-m})]\leq 2k. \tag{15}$$

Proof. In view of Lemma 2, it suffices to show that

$$\sum_{n=2}^{\infty}n[n((1-\lambda)+k(1+\lambda))+(1-\lambda)(k-1)]\frac{m^{n-1}}{(n-1)!}|a_n|\leq 2ke^m.$$

Using (14), we have

$$\sum_{n=2}^{\infty}n[n((1-\lambda)+k(1+\lambda))+(1-\lambda)(k-1)]\frac{m^{n-1}}{(n-1)!}|a_n|$$
$$\leq\sum_{n=2}^{\infty}n[n((1-\lambda)+k(1+\lambda))+(1-\lambda)(k-1)]\frac{m^{n-1}}{(n-1)!}\frac{(A-B)|\tau|}{n}$$
$$=(A-B)|\tau|\sum_{n=2}^{\infty}[n((1-\lambda)+k(1+\lambda))+(1-\lambda)(k-1)]\frac{m^{n-1}}{(n-1)!}$$
$$=(A-B)|\tau|\sum_{n=2}^{\infty}[(n-1)((1-\lambda)+k(1+\lambda))+2k]\frac{m^{n-1}}{(n-1)!}$$
$$=(A-B)|\tau|\left[((1-\lambda)+k(1+\lambda))\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-2)!}+2k\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}\right]$$

$$= (A-B)\,|\tau|\,[((1-\lambda)+k(1+\lambda))me^{m}+2k(e^{m}-1)].$$

But this last expression is bounded above by $2ke^{m}$ if and only if (15) holds. □

By taking $\lambda = 0$ in Theorems 3 and 4, we obtain the following corollaries.

Corollary 3 *Let* $m>0$ *and* $0<k\le 1$. *If* $f\in\mathcal{R}^{\tau}(A,B)$, *then* $\mathcal{I}(m,z)f$ *is in* $\mathcal{S}(k)$ *if and only if*

$$(A-B)\,|\tau|\left[(1+k)(1-e^{-m})+\frac{(k-1)}{m}(1-e^{-m}(1+m))\right]\le 2k. \tag{16}$$

Corollary 4 *Let* $m>0$ *and* $0<k\le 1$. *If* $f\in\mathcal{R}^{\tau}(A,B)$, *then* $\mathcal{I}(m,z)f$ *is in* $\mathcal{C}(k)$ *if and only if*

$$(A-B)\,|\tau|\,[(1+k)m+2k(1-e^{-m})]\le 2k. \tag{17}$$

4 An integral operator

In this section, we obtain the necessary and sufficient conditions for the integral operator $\mathcal{G}(m,z)$ defined by

$$\mathcal{G}(m,z)=\int_0^z\frac{\mathcal{F}(m,t)}{t}dt \tag{18}$$

to be in the class $\mathcal{C}(k,\lambda)$.

Theorem 5 *If* $m>0$, $0<k\le 1$ *and* $0\le\lambda<1$, *then the integral operator* $\mathcal{G}(m,z)$ *defined by* (18) *is in* $\mathcal{C}(k,\lambda)$ *if and only if* (5) *is satisfied.*

Proof. Since

$$\mathcal{G}(m,z)=z-\sum_{n=2}^{\infty}\frac{e^{-m}m^{n-1}}{n!}z^{n}$$

then by Lemma 2, we need only to show that

$$\sum_{n=2}^{\infty}n[n((1-\lambda)+k(1+\lambda))+(1-\lambda)(k-1)]\frac{m^{n-1}}{n!}\le 2ke^{m}.$$

or, equivalently

$$\sum_{n=2}^{\infty}[n((1-\lambda)+k(1+\lambda))+(1-\lambda)(k-1)]\frac{m^{n-1}}{(n-1)!}\le 2ke^{m}.$$

From (8) it follows that

$$\sum_{n=2}^{\infty}[n((1-\lambda)+k(1+\lambda))+(1-\lambda)(k-1)]\frac{m^{n-1}}{(n-1)!}$$
$$= ((1-\lambda)+k(1+\lambda))me^{m}+2k(e^{m}-1)$$

and this last expression is bounded above by $2ke^m$ if and only if (5) holds. □

The proof of Theorem 6 (below) is much similar to that of Theorem 5 and so the details are omitted.

Theorem 6 *If $m > 0$, $0 < k \leq 1$ and $0 \leq \lambda < 1$, then the integral operator $\mathcal{G}(m, z)$ defined by (18) is in $\mathcal{S}(k, \lambda)$ if and only if*

$$((1-\lambda)+k(1+\lambda))(1-e^{-m})+\frac{(1-\lambda)(k-1)}{m}(1-e^{-m}-me^{-m})\leq 2k.$$

By taking $\lambda = 0$ in Theorems 5 and 6, we obtain the following corollaries.

Corollary 5 *If $m > 0$ and $0 < k \leq 1$, then the integral operator defined by (18) is in $\mathcal{C}(k)$ if and only if (11) is satisfied.*

Corollary 6 *If $m > 0$ and $0 < k \leq 1$,then the integral operator defined by (18) is in $\mathcal{S}(k)$ if and only if*

$$(1+k)(1-e^{-m})+\frac{(k-1)}{m}(1-e^{-m}-me^{-m})\leq 2k.$$

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Sharp bounds of Fekete-Szegő functional for quasi-subordination class

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Abstract. In the present paper, we introduce a certain subclass $\mathcal{K}_q(\lambda, \gamma, h)$ of analytic functions by means of a quasi-subordination. Sharp bounds of the Fekete-Szegő functional for functions belonging to the class $\mathcal{K}_q(\lambda, \gamma, h)$ are obtained. The results presented in the paper give improved versions for the certain subclasses involving the quasi-subordination and majorization.

1 Introduction and definitions

Let $\mathcal{A}$ denote the family of normalized functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$. If $f \in \mathcal{A}$ satisfies $f(z_1) \neq f(z_2)$ for any $z_1 \in \mathbb{U}$ and $z_2 \in \mathbb{U}$ with $z_1 \neq z_2$, then f is said to be univalent in $\mathbb{U}$ and denoted by $f \in \mathcal{S}$.

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Let g and f be two analytic functions in $\mathbb{U}$ then function g is said to be subordinate to f if there exists an analytic function w in the unit disk $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that

$$g(z) = f(w(z)) \quad (z \in \mathbb{U}).$$

We denote this subordination by $g \prec f$. In particular, if the f is univalent in $\mathbb{U}$, the above subordination is equivalent to $g(0) = f(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$. Further, function g is said to be quasi-subordinate [18] to f in the unit disk $\mathbb{U}$ if there exist the functions w (with constant coefficient zero) and ϕ which are analytic and bounded by one in the unit disk $\mathbb{U}$ such that

$$g(z) = \phi(z)f(w(z))$$

and this is equivalent to

$$\frac{g(z)}{\phi(z)} \prec f(z) \quad (z \in \mathbb{U}).$$

We denote this quasi-subordination by $g \prec_q f$. It is observed that if $\phi(z) = 1$ $(z \in \mathbb{U})$, then the quasi-subordination $\prec_q$ become the usual subordination $\prec$, and for the function $w(z) = z$ $(z \in \mathbb{U})$, the quasi-subordination $\prec_q$ become the majorization '$\ll$'. In this case

$$g(z) = \phi(z)f(w(z)) \Rightarrow g(z) \ll f(z), \quad (z \in \mathbb{U}).$$

Some typical problems in geometric function theory are to study functionals made up of combinations of the coefficients of f. In 1933, Fekete and Szegő [5] obtained a sharp bound of the functional $\lambda a_2^2 - a_3$, with real $\lambda (0 \leq \lambda \leq 1)$ for a univalent function f. Since then, the problem of finding the sharp bounds for this functional of any compact family of functions $f \in \mathcal{A}$ with any complex λ is known as the classical Fekete-Szegő problem or inequality. Lawrence Zalcman posed a conjecture in 1960 that the coefficients of $\mathcal{S}$ satisfy the sharp inequality

$$|a_n^2 - a_{2n-1}| \leq (n-1)^2, \quad n \geq 2.$$

More general versions of Zalcman conjecture have also been considered ([4, 12, 13, 14]) for the functional such as

$$\lambda a_n^2 - a_{2n-1} \text{ and } \lambda a_m a_n - a_{m+n-1}$$

for certain positive value of λ. These functionals can be seen as generalizations of the Fekete-Szegő functional $\lambda a_2^2 - a_3$. Several authors including [1]–[4], [9]–[15], [17, 20] have investigated the Fekete-Szegő and Zalcman functionals for various subclasses of univalent and multivalent functions.

Throughout this paper it is assumed that functions ϕ and h are analytic in $\mathbb{U}$. Also let

$$\phi(z) = A_0 + A_1 z + A_2 z^2 + \cdots \quad (|\phi(z)| \le 1,\ z \in \mathbb{U}) \tag{2}$$

and

$$h(z) = 1 + B_1 z + B_2 z^2 + \cdots \qquad (B_1 \in \mathbb{R}^+). \tag{3}$$

Motivated by earlier works in ([6], [7], [15], [17], [19]) on quasi-subordination, we introduce here the following subclass of analytic functions:

Definition 1 *For $0 \le \lambda \le 1$ and $\gamma \in \mathbb{C}\backslash\{0\}$, a function $f \in \mathcal{A}$ given by (1) is said to be in the class $\mathcal{K}_q(\lambda, \gamma, h)$ if the following condition are satisfied:*

$$\frac{1}{\gamma}\left(\frac{zf'(z) + z^2 f''(z)}{(1-\lambda)z + \lambda z f'(z)} - 1\right) \prec_q (h(z) - 1), \tag{4}$$

where h is given by (3) and $z \in \mathbb{U}$.

It follows that a function f is in the class $\mathcal{K}_q(\lambda, \gamma, h)$ if and only if there exists an analytic function ϕ with $|\phi(z)| \le 1$, in $\mathbb{U}$ such that

$$\frac{\frac{1}{\gamma}\left(\frac{zf'(z)+z^2f''(z)}{(1-\lambda)z+\lambda zf'(z)} - 1\right)}{\phi(z)} \prec (h(z) - 1)$$

where h is given by (3) and $z \in \mathbb{U}$.
If we set $\phi(z) \equiv 1$ $(z \in \mathbb{U})$, then the class $\mathcal{K}_q(\lambda, \gamma, h)$ is denoted by $\mathcal{K}(\lambda, \gamma, h)$ satisfying the condition that

$$1 + \frac{1}{\gamma}\left(\frac{zf'(z) + z^2 f''(z)}{(1-\lambda)z + \lambda z f'(z)} - 1\right) \prec h(z) \quad (z \in \mathbb{U}).$$

In the present paper, we find sharp bounds on the Fekete-Szegő functional for functions belonging in the class $\mathcal{K}_q(\lambda, \gamma, h)$. Several known and new consequences of these results are also pointed out. In order to derive our main results, we have to recall here the following well-known lemma:

Let Ω be class of analytic functions of the form

$$w(z) = w_1 z + w_2 z^2 + \cdots \tag{5}$$

in the unit disk $\mathbb{U}$ satisfying the condition $|w(z)| < 1$.

Lemma 1 ([8], p.10) *If $w \in \Omega$, then for any complex number ν:*

$$|w_1| \leq 1, |w_2 - \nu w_1^2| \leq 1 + (|\nu| - 1)|w_1^2| \leq \max\{1, |\nu|\}.$$

The result is sharp for the functions $w(z) = z$ or $w(z) = z^2$.

2 Main results

Theorem 1 *Let $0 \leq \lambda \leq 1$ and $\gamma \in \mathbb{C} \backslash \{0\}$. If $f \in \mathcal{A}$ of the form (1) belonging to the class $\mathcal{K}_q(\lambda, \gamma, h)$, then*

$$|a_2| \leq \frac{|\gamma| B_1}{2(2-\lambda)} \tag{6}$$

and for any $\nu \in \mathbb{C}$

$$|a_3 - \nu a_2^2| \leq \frac{|\gamma| B_1}{3(3-\lambda)} \max\left\{1, \left|\frac{B_2}{B_1} - Q B_1\right|\right\}, \tag{7}$$

where

$$Q = \gamma\left(\frac{3\nu(3-\lambda)}{4(2-\lambda)^2} - \frac{\lambda}{2-\lambda}\right). \tag{8}$$

The results are sharp.

Proof. Let $f \in \mathcal{K}_q(\lambda, \gamma, h)$. In view of Definition 1, there exist then Schwarz functions w and an analytic function ϕ such that

$$\frac{1}{\gamma}\left(\frac{zf'(z) + z^2 f''(z)}{(1-\lambda)z + \lambda z f'(z)} - 1\right) = \phi(z)(h(w(z)) - 1) \quad (z \in \mathbb{U}). \tag{9}$$

Series expansions for f and its successive derivatives from (1) gives us

$$\frac{1}{\gamma}\left(\frac{zf'(z) + z^2 f''(z)}{(1-\lambda)z + \lambda z f'(z)} - 1\right) = \frac{1}{\gamma}\left[2(2-\lambda)a_2 z + \left(3(3-\lambda)a_3 - 4\lambda(2-\lambda)a_2^2\right)z^2 + \ldots\right]. \tag{10}$$

Similarly from (2), (3) and (5), we obtain

$$h(w(z)) - 1 = B_1 w_1 z + (B_1 w_2 + B_2 w_1^2) z^2 + \cdots$$

and

$$\phi(z)\big(h(w(z)) - 1\big) = A_0 B_1 w_1 z + [A_1 B_1 w_1 + A_0(B_1 w_2 + B_2 w_1^2)] z^2 + \cdots . \quad (11)$$

Equating (10) and (11) in view of (9) and comparing the coefficients of z and z^2, we get

$$a_2 = \frac{\gamma A_0 B_1 w_1}{2(2-\lambda)} \quad (12)$$

and

$$a_3 = \frac{\gamma B_1}{3(3-\lambda)} \left[A_1 w_1 + A_0 \{ w_2 + \left(\frac{\gamma \lambda A_0 B_1}{2-\lambda} + \frac{B_2}{B_1} \right) w_1^2 \} \right]. \quad (13)$$

Thus, for any $\nu \in \mathbb{C}$, we have

$$\begin{aligned} a_3 - \nu a_2^2 &= \frac{\gamma B_1}{3(3-\lambda)} \left[A_1 w_1 + \left(w_2 + \frac{B_2}{B_1} w_1^2 \right) A_0 - \left(\frac{3(3-\lambda)\gamma}{4(2-\lambda)^2} \nu - \frac{\gamma\lambda}{2-\lambda} \right) B_1 A_0^2 w_1^2 \right] \\ &= \frac{\gamma B_1}{3(3-\lambda)} \left[A_1 w_1 + \left(w_2 + \frac{B_2}{B_1} w_1^2 \right) A_0 - Q B_1 A_0^2 w_1^2 \right], \end{aligned} \quad (14)$$

where Q is given by (8).

Since $\phi(z) = A_0 + A_1 z + A_2 z^2 + \cdots$ is analytic and bounded by one in $\mathbb{U}$, therefore we have (see [16], p 172)

$$|A_0| \leq 1 \text{ and } A_1 = (1 - A_0^2) y \qquad (y \leq 1). \quad (15)$$

From (14) and (15), we obtain

$$a_3 - \nu a_2^2 = \frac{\gamma B_1}{3(3-\lambda)} \left[y w_1 + \left(w_2 + \frac{B_2}{B_1} w_1^2 \right) A_0 - \left(B_1 Q w_1^2 + y w_1 \right) A_0^2 \right]. \quad (16)$$

If A_0=0 in (16), we at once get

$$|a_3 - \nu a_2^2| \leq \frac{|\gamma| B_1}{3(3-\lambda)}. \quad (17)$$

But if $A_0 \neq 0$, let us then suppose that

$$G(A_0) = y w_1 + \left(w_2 + \frac{B_2}{B_1} w_1^2 \right) A_0 - \left(B_1 Q w_1^2 + y w_1 \right) A_0^2$$

which is a quadratic polynomial in A_0 and hence analytic in $|A_0| \leq 1$ and maximum value of $|G(A_0)|$ is attained at $A_0 = e^{\iota\theta}$ $(0 \leq \theta < 2\pi)$, we find that

$$\begin{aligned} \max|G(A_0)| &= \max_{0\leq\theta<2\pi} |G(e^{\iota\theta})| = |G(1)| \\ &= \left| w_2 - \left(QB_1 - \frac{B_2}{B_1}\right) w_1^2 \right|. \end{aligned}$$

Therefore, it follows from (16) that

$$|a_3 - \nu a_2^2| \leq \frac{|\gamma| B_1}{3(3-\lambda)} \left| w_2 - \left(QB_1 - \frac{B_2}{B_1}\right) w_1^2 \right|, \tag{18}$$

which on using Lemma 1, shows that

$$|a_3 - \nu a_2^2| \leq \frac{|\gamma| B_1}{3(3-\lambda)} \max\left\{1, \left|\frac{B_2}{B_1} - QB_1\right|\right\},$$

and this last above inequality together with (17) establish the results. The result are sharps for the function f given by

$$1 + \frac{1}{\gamma}\left(\frac{zf'(z) + z^2 f''(z)}{(1-\lambda)z + \lambda z f'(z)} - 1\right) = h(z),$$

$$1 + \frac{1}{\gamma}\left(\frac{zf'(z) + z^2 f''(z)}{(1-\lambda)z + \lambda z f'(z)} - 1\right) = h(z^2)$$

and

$$1 + \frac{1}{\gamma}\left(\frac{zf'(z) + z^2 f''(z)}{(1-\lambda)z + \lambda z f'(z)} - 1\right) = z(h(z) - 1).$$

This completes the proof of Theorem 1. □

For $\lambda = 0$ the Theorem 1 reduces to following corollary:

Corollary 1 *If* $f \in \mathcal{A}$ *of the form* (1) *satisfies*

$$\frac{1}{\gamma}(f'(z) + zf''(z) - 1) \prec_q (h(z) - 1) \qquad (z \in \mathbb{U},\ \gamma \in \mathbb{C}\backslash\{0\}),$$

then

$$|a_2| \leq \frac{|\gamma| B_1}{4},$$

and for some $\nu \in \mathbb{C}$

$$|a_3 - \nu a_2^2| \leq \frac{|\gamma| B_1}{9} \max\left\{1, \left|\frac{B_2}{B_1} - \frac{9\nu|\gamma| B_1}{16}\right|\right\}.$$

The results are sharp.

Remark 1 *In Corollary 1, if we set* $\phi \equiv 1$*, then above result match with the result given in* [3].

Remark 2 *For* $\phi \equiv 1$*,* $\gamma = \lambda = 1$*, Theorem 1 reduces to an improved result of given in* [15].

The next theorem gives the result based on majorization.

Theorem 2 *Let* $0 \le \lambda \le 1$ *and* $\gamma \in \mathbb{C}\backslash\{0\}$*. If* $f \in \mathcal{A}$ *of the form* (1) *satisfies*

$$\frac{1}{\gamma}\left(\frac{zf'(z)+z^2f''(z)}{(1-\lambda)z+\lambda zf'(z)}-1\right) \ll (h(z)-1) \quad (z \in \mathbb{U}), \tag{19}$$

then

$$|a_2| \le \frac{|\gamma|B_1}{2(2-\lambda)}$$

and for any $\nu \in \mathbb{C}$

$$|a_3 - \nu a_2^2| \le \frac{|\gamma|B_1}{3(3-\lambda)} \max\left\{1, \left|\frac{B_2}{B_1} - QB_1\right|\right\},$$

where Q is given by (8)*. The results are sharp.*

Proof. Assume that (19) holds. From the definition of majorization, there exist an analytic function ϕ such that

$$\frac{1}{\gamma}\left(\frac{zf'(z)+z^2f''(z)}{(1-\lambda)z+\lambda zf'(z)}-1\right) = \phi(z)(h(z)-1) \quad (z \in \mathbb{U}).$$

Following similar steps as in the proof of Theorem 1, and by setting $w(z) \equiv z$, so that $w_1 = 1, w_n = 0, n \ge 2$, we obtain

$$a_2 = \frac{\gamma A_0 B_1}{2(2-\lambda)}$$

and also we obtain that

$$a_3 - \nu a_2^2 = \frac{\gamma B_1}{3(3-\lambda)}\left[A_1 + \frac{B_2}{B_1}A_0 - QB_1A_0^2\right].$$

On putting the value of A_1 from (15), we obtain

$$a_3 - \nu a_2^2 = \frac{\gamma B_1}{3(3-\lambda)}\left[y + \frac{B_2}{B_1}A_0 - (QB_1 + y)A_0^2\right]. \tag{20}$$

If A_0=0 in (20), we at once get

$$|a_3 - \nu a_2^2| \leq \frac{|\gamma|B_1}{3(3-\lambda)}, \tag{21}$$

But if $A_0 \neq 0$, let us then suppose that

$$T(A_0) = y + \frac{B_2}{B_1}A_0 - \Big(QB_1 + y\Big)A_0^2,$$

which is a quadratic polynomial in A_0, hence analytic in $|A_0| \leq 1$ and maximum value of $|T(A_0)|$ is attained at $A_0 = e^{\iota\theta}$ $(0 \leq \theta < 2\pi)$, we find that

$$\max|T(A_0)| = \max_{0\leq\theta<2\pi} |T(e^{\iota\theta})| = |T(1)|.$$

Hence, from (20), we obtain

$$|a_3 - \nu a_2^2| \leq \frac{|\gamma|B_1}{3(3-\lambda)}\left|QB_1 - \frac{B_2}{B_1}\right|.$$

Thus, the assertion of Theorem 2 follows from this last above inequality together with (21). The results are sharp for the function given by

$$1 + \frac{1}{\gamma}\left(\frac{zf'(z) + z^2f''(z)}{(1-\lambda)z + \lambda zf'(z)} - 1\right) = h(z),$$

which completes the proof of Theorem 2. □

Theorem 3 *Let* $0 \leq \lambda \leq 1$ *and* $\gamma \in \mathbb{C}\backslash\{0\}$. *If* $f \in \mathcal{A}$ *of the form* (1) *belonging to the class* $\mathcal{K}(\lambda, \gamma, h)$, *then*

$$|a_2| \leq \frac{|\gamma|B_1}{2(2-\lambda)}$$

and for any $\nu \in \mathbb{C}$

$$|a_3 - \nu a_2^2| \leq \frac{|\gamma|B_1}{3(3-\lambda)} \max\left\{1, \left|\frac{B_2}{B_1} - QB_1\right|\right\},$$

where Q is given by (8), *the results are sharp.*

Proof. The proof is similar to Theorem 1, Let $f \in \mathcal{K}(\lambda, \gamma, h)$.
If $\phi(z) = 1$, then $A_0 = 1, A_n = 0$ $(n \in \mathbb{N})$. Therefore, in view of (12) and (14) and by application of Lemma 1, we obtain the desired assertion. The results are sharp for the function f given by

$$1 + \frac{1}{\gamma}\left(\frac{zf'(z) + z^2 f''(z)}{(1-\lambda)z + \lambda z f'(z)} - 1\right) = h(z),$$

or

$$1 + \frac{1}{\gamma}\left(\frac{zf'(z) + z^2 f''(z)}{(1-\lambda)z + \lambda z f'(z)} - 1\right) = h(z^2).$$

Thus, the proof of Theorem 3 is completed. □

Now, we determine the bounds on the functional $|a_3 - \nu a_2^2|$ for real ν.

Theorem 4 *Let* $0 \le \lambda \le 1$. *If* $f \in \mathcal{A}$ *of the form* (1) *belonging to the class* $\mathcal{K}_q(\lambda, \gamma, h)$, *then for real* ν *and* γ, *we have*

$$|a_3 - \nu a_2^2| \le \begin{cases} \frac{|\gamma| B_1}{3(3-\lambda)}\left[B_1\gamma\left(\frac{\lambda}{2-\lambda} - \frac{3(3-\lambda)}{4(2-\lambda)^2}\nu\right) + \frac{B_2}{B_1}\right] & (\nu \le \sigma_1), \\ \frac{|\gamma| B_1}{3(3-\lambda)} & (\sigma_1 \le \nu \le \sigma_1 + 2\rho), \\ -\frac{|\gamma| B_1}{3(3-\lambda)}\left[B_1\gamma\left(\frac{\lambda}{2-\lambda} - \frac{3(3-\lambda)}{4(2-\lambda)^2}\nu\right) + \frac{B_2}{B_1}\right] & (\nu \ge \sigma_1 + 2\rho), \end{cases} \tag{22}$$

where

$$\sigma_1 = \frac{4\lambda(2-\lambda)}{3(3-\lambda)} - \frac{4(2-\lambda)^2}{3\gamma(3-\lambda)}\left(\frac{1}{B_1} - \frac{B_2}{B_1^2}\right) \tag{23}$$

and

$$\rho = \frac{4(2-\lambda)^2}{3\gamma(3-\lambda)B_1}. \tag{24}$$

Each of the estimates in (22) *are sharp.*

Proof. For real values of ν and γ the above bounds can be obtained from (7), respectively, under the following cases:

$$B_1 Q - \frac{B_2}{B_1} \le -1, \; -1 \le B_1 Q - \frac{B_2}{B_1} \le 1 \text{ and } B_1 Q - \frac{B_2}{B_1} \ge 1,$$

where Q is given by (8). We also note the following:

(i) When $\nu < \sigma_1$ or $\nu > \sigma_1 + 2\rho$, then the equality holds if and only if $\phi(z) \equiv 1$ and $w(z) = z$ or one of its rotations.

(ii) When $\sigma_1 < \nu < \sigma_1 + 2\rho$, then the equality holds if and only if $\phi(z) \equiv 1$ and $w(z) = z^2$ or one of its rotations.

(iii) Equality holds for $\nu = \sigma_1$ if and only if $\phi(z) \equiv 1$ and $w(z) = \frac{z(z+\epsilon)}{1+\epsilon z}$ $(0 \leq \epsilon \leq 1)$, or one of its rotations, while for $\nu = \sigma_1 + 2\rho$, the equality holds if and only if $\phi(z) \equiv 1$ and $w(z) = -\frac{z(z+\epsilon)}{1+\epsilon z}$ $(0 \leq \epsilon \leq 1)$, or one of its rotations. □

The bounds of the functional $a_3 - \nu a_2^2$ for real values of ν and γ for the middle range of the parameter ν can be improved further as follows:

Theorem 5 *Let $0 \leq \lambda \leq 1$. If $f \in \mathcal{A}$ of the form (1) belonging to the class $\mathcal{K}_q(\lambda, \gamma, h)$, then for real ν and γ, we have*

$$|a_3 - \nu a_2^2| + (\nu - \sigma_1)|a_2|^2 \leq \frac{|\gamma| B_1}{3(3-\lambda)} \quad (\sigma_1 \leq \nu \leq \sigma_1 + \rho) \tag{25}$$

and

$$|a_3 - \nu a_2^2| + (\sigma_1 + 2\rho - \nu)|a_2|^2 \leq \frac{|\gamma| B_1}{3(3-\lambda)} \quad (\sigma_1 + \rho \leq \nu \leq \sigma_1 + 2\rho), \tag{26}$$

where σ_1 and ρ are given by (23) and (24), respectively.

Proof. Let $f \in \mathcal{K}_q(\lambda, \gamma, h)$. For real ν satisfying $\sigma_1 + \rho \leq \nu \leq \sigma_1 + 2\rho$ and using (12) and (18) we get

$$|a_3 - \nu a_2^2| + (\nu - \sigma_1)|a_2|^2$$
$$\leq \frac{|\gamma| B_1}{3(3-\lambda)} \left[|w_2| - \frac{3|\gamma| B_1 (3-\lambda)}{4(2-\lambda)^2} (\nu - \sigma_1 - \rho)|w_1|^2 + \frac{3|\gamma| B_1 (3-\lambda)}{4(2-\lambda)^2} (\nu - \sigma_1)|w_1|^2 \right].$$

Therefore, by virtue of Lemma 1, we get

$$|a_3 - \nu a_2^2| + (\nu - \sigma_1)|a_2|^2 \leq \frac{|\gamma| B_1}{3(3-\lambda)} [1 - |w_1|^2 + |w_1|^2],$$

which yields the assertion (25).

If $\sigma_1 + \rho \leq \nu \leq \sigma_1 + 2\rho$, then again from (12), (18) and the application of Lemma 1, we have

$$\begin{aligned} |a_3 - \nu a_2^2| + (\sigma_1 + 2\rho - \nu)|a_2|^2 &\leq \frac{|\gamma| B_1}{3(3-\lambda)} \left[|w_2| + \frac{3|\gamma| B_1 (3-\lambda)}{4(2-\lambda)^2} (\nu - \sigma_1 - \rho)|w_1|^2 \right. \\ &\quad \left. + \frac{3|\gamma| B_1 (3-\lambda)}{4(2-\lambda)^2} (\sigma_1 + 2\rho - \nu)|w_1|^2 \right] \\ &\leq \frac{|\gamma| B_1}{3(3-\lambda)} [1 - |w_1|^2 + |w_1|^2], \end{aligned}$$

which estimates (26). □

Conflicts of interest

The authors declare that there are no conflict of interest regarding the publication of this paper.

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Fractional natural decomposition method for solving a certain class of nonlinear time-fractional wave-like equations with variable coefficients

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Abstract. In this paper, we propose a new approximate method, namely fractional natural decomposition method (FNDM) in order to solve a certain class of nonlinear time-fractional wave-like equations with variable coefficients. The fractional natural decomposition method is a combined form of the natural transform method and the Adomian decomposition method. The nonlinear term can easily be handled with the help of Adomian polynomials which is considered to be a clear advantage of this technique over the decomposition method. Some examples are given to illustrate the applicability and the easiness of this approach.

1 Introduction

Fractional differential equations, as generalizations of classical integer order differential equations, are gradually employed to model problems in fluid flow, finance, physical, hydrological, biological processes and systems [6, 7, 8, 9].

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The most frequent used methods for investigating fractional differential equations are: Adomian decomposition method (ADM) [1] variational iteration method (VIM) [12], generalized differential transform method (GDTM) [10], homotopy analysis method (HAM) [3], homotopy perturbation method (HPM) [11]. Also, there are some other classical solution techniques such as Laplace transform method, fractional Green's function method, Mellin transform method and method of orthogonal polynomials [8].

In this paper, the main objective is to solve a certain class of nonlinear time-fractional wave-like equation with variable coefficients by using a modified method called fractional natural decomposition method (FNDM) which is a combination of two powerful methods, the Natural transform and the Adomian decomposition method.

Consider the following nonlinear time-fractional wave-like equations

$$\begin{aligned} D_t^{\alpha} v = & \sum_{i,j=1}^{n} F_{1ij}(X,t,v) \frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m} F_{2ij}(v_{x_i}, v_{x_j}) \\ & + \sum_{i=1}^{n} G_{1i}(X,t,v) \frac{\partial^p}{\partial x_i^p} G_{2i}(v_{x_i}) + H(X,t,v) + S(X,t), \end{aligned} \tag{1}$$

with initial conditions

$$v(X,0) = a_0(X), \;\; v_t(X,0) = a_1(X), \tag{2}$$

where D_t^{α} is the Caputo fractional derivative operator of order α, $1 < \alpha \leq 2$.

Here $X = (x_1, x_2, ..., x_n)$, F_{1ij}, G_{1i} $i,j \in \{1,2,...,n\}$ are nonlinear functions of X, t and v, F_{2ij}, G_{2i} $i,j \in \{1,2,...,n\}$, are nonlinear functions of derivatives of v with respect to x_i and x_j $i,j \in \{1,2,...,n\}$, respectively. Also H, S are nonlinear functions and k, m, p are integers.

For $\alpha = 2$, these types of equations are of considerable significance in various fields of applied sciences, mathematical physics, nonlinear hydrodynamics, engineering physics, biophysics, human movement sciences, astrophysics and plasma physics. These equations describe the evolution of erratic motions of small particles that are immersed in fluids, fluctuations of the intensity of laser light, velocity distributions of fluid particles in turbulent flows.

2 Basic definitions

In this section, we introduce some definitions and important properties of the fractional calculus, the natural transform, and the natural transform of fractional derivatives, which are used further in this paper.

2.1 Fractional calculus

Definition 1 [8] *A real function* $f(t), t > 0$*, is considered to be in the space* C_μ*,* $\mu \in \mathbb{R}$ *if there exists a real number* $p > \mu$*, so that* $f(t) = t^p h(t)$*, where* $h(t) \in C\left([0, \infty[\right)$*, and it is said to be in the space* C_μ^n *if* $f^{(n)} \in C_\mu$*,* $n \in \mathbb{N}$*.*

Definition 2 [8] *The Riemann-Liouville fractional integral operator* I^α *of order* α *for a function* $f \in C_\mu, \mu \geq -1$ *is defined as follows*

$$I^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-\xi)^{\alpha-1} f(\xi) d\xi, & \alpha > 0, \ t > 0, \\ f(t), & \alpha = 0, \end{cases} \tag{3}$$

where $\Gamma(.)$ *is the well-known Gamma function.*

Definition 3 [8] *The fractional derivative of* $f(t)$ *in the Caputo sense is defined as follows*

$$D^\alpha f(t) = I^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int\limits_0^t (t-\xi)^{n-\alpha-1} f^{(n)}(\xi) d\xi, \ t > 0, \tag{4}$$

where $n - 1 < \alpha \leq n$*,* $n \in \mathbb{N}$*,* $f \in C_{-1}^n$*.*

For the Riemann-Liouville fractional integral and Caputo fractional derivative, we have the following relation

$$I^\alpha D^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{t^k}{k!}, \ t > 0. \tag{5}$$

Definition 4 [8] *The Mittag-Leffler function is defined as follows*

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + 1)}, \alpha \in \mathbb{C}, \mathrm{Re}(\alpha) > 0. \tag{6}$$

A further generalization of (6) *is given in the form*

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + \beta)}, \alpha, \beta \in \mathbb{C}, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0. \tag{7}$$

For $\alpha = 1$*,* $E_\alpha(z)$ *reduces to* e^z*.*

2.2 Natural transform

Definition 5 [2] *The natural transform is defined over the set of functions is defined over the set of functions*

$$A = \left\{ f(t)/\exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{\frac{|t|}{\tau_j}},\ \textit{if}\ t \in (-1)^j \times [0, \infty) \right\},$$

by the following integral

$$\mathcal{N}^+ [f(t)] = R^+(s,u) = \frac{1}{u} \int_0^{+\infty} e^{-\frac{st}{u}} f(t) dt,\ \ s,u \in (0,\infty). \tag{8}$$

Some basic properties of the natural transform are given as follows [2].

Property 1 The natural transform is a linear operator. That is, if λ and μ are non–zero constants, then

$$\mathcal{N}^+ [\lambda f(t) \pm \mu g(t)] = \lambda \mathcal{N}^+ [f(t)] \pm \mu \mathcal{N}^+ [g(t)] .$$

Property 2 If $f^{(n)}(t)$ is the n-th derivative of function $f(t)$ w.r.t. "t" then its natural transform is given by

$$\mathcal{N}^+ \left[f^{(n)}(t) \right] = R_n^+(s,u) = \frac{s^n}{u^n} R^+(s,u) - \sum_{k=0}^{n-1} \frac{s^{n-(k+1)}}{u^{n-k}} f^{(k)}(0).$$

Property 3 (Convolution property) Suppose $F^+(s,u)$ and $G^+(s,u)$ are the natural transforms of $f(t)$ and $g(t)$, respectively, both defined in the set A. Then the natural transform of their convolution is given by

$$\mathcal{N}^+ [(f * g)(t)] = u F^+(s,u) G^+(s,u),$$

where the convolution of two functions is defined by

$$(f * g)(t) = \int_0^t f(\xi) g(t-\xi) d\xi = \int_0^t f(t-\xi) g(\xi) d\xi.$$

Property 4 Some special natural transforms

$$\mathcal{N}^+ [1] \ = \ \frac{1}{s},$$

$$\mathcal{N}^{+}[t] = \frac{u}{s^{2}},$$
$$\mathcal{N}^{+}\left[\frac{t^{n-1}}{(n-1)!}\right] = \frac{u^{n-1}}{s^{n}}, n = 1, 2, ...$$

Property 5 If $\alpha > -1$, then the natural transform of t^{α} is given by

$$\mathcal{N}^{+}[t^{\alpha}] = \Gamma(\alpha+1)\frac{u^{\alpha}}{s^{\alpha+1}}.$$

2.3 Natural transform of fractional derivatives

Theorem 1 *If $R^{+}(s,u)$ is the natural transform of $f(t)$, then the natural transform of the Riemann-Liouville fractional integral for $f(t)$ of order α, is given by*

$$\mathcal{N}^{+}[I^{\alpha}f(t)] = \frac{u^{\alpha}}{s^{\alpha}}R^{+}(s,u). \tag{9}$$

Proof. The Riemann-Liouville fractional integral for the function $f(t)$, as in (3), can be expressed as the convolution

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}t^{\alpha-1} * f(t). \tag{10}$$

Applying the natural transform in the Eq. (10) and using Properties 3 and 5, we have

$$\begin{aligned}\mathcal{N}^{+}[I^{\alpha}f(t)] &= \mathcal{N}^{+}\left[\frac{1}{\Gamma(\alpha)}t^{\alpha-1} * f(t)\right] = u\frac{1}{\Gamma(\alpha)}\mathcal{N}^{+}\left[t^{\alpha-1}\right]\mathcal{N}^{+}[f(t)]\\ &= u\frac{u^{\alpha-1}}{s^{\alpha}}R^{+}(s,u) = \frac{u^{\alpha}}{s^{\alpha}}R^{+}(s,u).\end{aligned}$$

The proof is complete. □

Theorem 2 *$n \in \mathbb{N}^{*}$ and $\alpha > 0$ be such that $n-1 < \alpha \leq n$ and $R^{+}(s,u)$ be the natural transform of the function $f(t)$, then the natural transform denoted by $R^{+}_{\alpha}(s,u)$ of the Caputo fractional derivative of the function $f(t)$ of order α, is given by*

$$\mathcal{N}^{+}[D^{\alpha}f(t)] = R^{+}_{\alpha}(s,u) = \frac{s^{\alpha}}{u^{\alpha}}R^{+}(s,u) - \sum_{k=0}^{n-1}\frac{s^{\alpha-(k+1)}}{u^{\alpha-k}}\left[D^{k}f(t)\right]_{t=0}. \tag{11}$$

Proof. Let $g(t) = f^{(n)}(t)$, then by the Definition 3 of the Caputo fractional derivative, we obtain

$$\begin{aligned} D^{\alpha}f(t) &= \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\xi)^{n-\alpha-1} f^{(n)}(\xi)d\xi \\ &= \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\xi)^{n-\alpha-1} g(\xi)d\xi \\ &= I^{n-\alpha}g(t). \end{aligned} \tag{12}$$

Applying the natural transform on both sides of (12) using Eq. (9), we get

$$\mathcal{N}^+\left[D^{\alpha}f(t)\right] = \mathcal{N}^+\left[I^{n-\alpha}g(t)\right] = \frac{u^{n-\alpha}}{s^{n-\alpha}}G^+(s,u). \tag{13}$$

Also, we have from the Property 2

$$\begin{aligned} &\mathcal{N}^+\left[g(t)\right] = \mathcal{N}^+\left[f^{(n)}(t)\right], \\ &G^+(s,u) = \frac{s^n}{u^n}R^+(s,u) - \sum_{k=0}^{n-1}\frac{s^{n-(k+1)}}{u^{n-k}}\left[f^{(k)}(t)\right]_{t=0}. \end{aligned} \tag{14}$$

Hence, 13 becomes

$$\begin{aligned} \mathcal{N}^+\left[D^{\alpha}f(t)\right] &= \frac{u^{n-\alpha}}{s^{n-\alpha}}\left(\frac{s^n}{u^n}R^+(s,u) - \sum_{k=0}^{n-1}\frac{s^{n-(k+1)}}{u^{n-k}}f^{(k)}(0)\right) \\ &= \frac{s^{\alpha}}{u^{\alpha}}R^+(s,u) - \sum_{k=0}^{n-1}\frac{s^{\alpha-(k+1)}}{u^{\alpha-k}}\left[D^k f(t)\right]_{t=0} = R^+_{\alpha}(s,u), \\ &-1 < n-1 < \alpha \leq n. \end{aligned}$$

The proof is complete. □

3 FNDM of nonlinear time-fractional wave-like equations with variable coefficients

Theorem 3 *Consider the following nonlinear time-fractional wave-like equations* (1) *with the initial conditions* (2).

Then, by FNDM, the solution of Eqs. (1)-(2) *is given in the form of infinite series as follows*

$$v(X,t) = \sum_{n=0}^{\infty} v_n(X,t).$$

Proof. In order to to achieve our goal, we consider the following nonlinear time-fractional wave-like equations (1) with the initial conditions (2).

First we define

$$\begin{aligned} Nv &= \sum_{i,j=1}^{n} F_{1ij}(X,t,v)\frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m}F_{2ij}(v_{x_i}, v_{x_j}), \\ Mv &= +\sum_{i=1}^{n} G_{1i}(X,t,v)\frac{\partial^p}{\partial x_i^p}G_{2i}(v_{x_i}), \\ Kv &= H(X,t,v). \end{aligned} \tag{15}$$

Eq. (1) is written in the form

$$\begin{aligned} D_t^{\alpha} v(X,t) &= Nv(X,t) + Mv(X,t) + Kv(X,t) + S(X,t), \\ & t > 0, 1 < \alpha \leq 2. \end{aligned} \tag{16}$$

Applying the natural transform on both sides of (16) and using the Theorem 2, we get

$$\begin{aligned} \mathcal{N}^{+}[v(X,t)] = \frac{u^{\alpha}}{s^{\alpha}} \sum_{k=0}^{n-1} \frac{s^{\alpha-(k+1)}}{u^{\alpha-k}} \left[D^k v(X,t)\right]_{t=0} \\ + \frac{u^{\alpha}}{s^{\alpha}} \mathcal{N}^{+}[Nv(X,t) + Mv(X,t) + Kv(X,t) + S(X,t)]. \end{aligned} \tag{17}$$

After that, let us take the inverse natural transform on both sides of (17) we have

$$v(X,t) = L(X,t) + \mathcal{N}^{-1}\left(\frac{u^{\alpha}}{s^{\alpha}}\mathcal{N}^{+}[Nv(X,t) + Mv(X,t) + Kv(X,t)]\right), \tag{18}$$

where $L(X,t)$ is a term arising from the source term and the prescribed initial conditions.

Now, we represent the solution in an infinite series form

$$v(X,t) = \sum_{n=0}^{\infty} v_n(X,t), \tag{19}$$

and the nonlinear terms can be decomposed as

$$Nv(X,t) = \sum_{n=0}^{\infty} A_n, Mv(X,t) = \sum_{n=0}^{\infty} B_n, Kv(X,t) = \sum_{n=0}^{\infty} C_n, \tag{20}$$

where A_n, B_n and C_n are Adomian polynomials [13], of $v_0, v_1, v_2, .., v_n$, and it can be calculated by formula given below

$$A_n = B_n = C_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty} \lambda^i v_i\right)\right]_{\lambda=0}, \ n = 0, 1, 2, ... \tag{21}$$

Using Eqs. (19) and (20), we can rewrite Eq. (18) as

$$\sum_{n=0}^{\infty} v_n(X,t) = L(X,t) + \mathcal{N}^{-1}\left(\frac{u^\alpha}{s^\alpha}\mathcal{N}^+\left[\sum_{n=0}^{\infty} A_n + \sum_{n=0}^{\infty} B_n + \sum_{n=0}^{\infty} C_n\right]\right). \tag{22}$$

By comparing both sides of Eq. (22) we have the following relation

$$\begin{aligned} v_0(X,t) &= L(X,t), \\ v_1(X,t) &= \mathcal{N}^{-1}\left(\frac{u^\alpha}{s^\alpha}\mathcal{N}^+[A_0 + B_0 + C_0]\right), \\ v_2(X,t) &= \mathcal{N}^{-1}\left(\frac{u^\alpha}{s^\alpha}\mathcal{N}^+[A_1 + B_1 + C_1]\right), \\ v_3(X,t) &= \mathcal{N}^{-1}\left(\frac{u^\alpha}{s^\alpha}\mathcal{N}^+[A_2 + B_2 + C_2]\right), \\ &\dots, \end{aligned} \tag{23}$$

and so on.

In general the recursive relation is given by

$$\begin{aligned} v_0(X,t) &= L(X,t), \\ v_{n+1}(X,t) &= \mathcal{N}^{-1}\left(\frac{u^\alpha}{s^\alpha}\mathcal{N}^+[A_n + B_n + C_n]\right), \ n \geq 0. \end{aligned} \tag{24}$$

Then, the solution of Eqs. (1)-(2) is given in the form of infinite series as follows

$$v(X,t) = \sum_{n=0}^{\infty} v_n(X,t). \tag{25}$$

The proof is complete. □

Theorem 4 *Let $\mathcal{B}$ be a Banach space, Then the series solution of the Eqs.* (1)-(2) *converges to* $S \in \mathcal{B}$ *, if there exists* γ*,* $0 < \gamma < 1$ *such that*

$$\|v_n\| \leq \gamma \|v_{n-1}\|, \forall n \in \mathbb{N}.$$

Proof. Define the sequences S_n of partial sums of the series given by the recursive relation (24) as

$$S_n(X,t) = v_0(X,t) + v_2(X,t) + v_3(X,t) + ... + v_n(X,t),$$

and we need to show that $\{S_n\}$ are a Cauchy sequences in Banach space $\mathcal{B}$. For this purpose, we consider

$$\|S_{n+1} - S_n\| \leq \|v_{n+1}\| \leq \gamma \|v_n\| \leq \gamma^2 \|v_{n-1}\| \leq ... \leq \gamma^{n+1} \|v_0\|. \tag{26}$$

For every $n, m \in \mathbb{N}$, $n \geq m$, by using (26) and triangle inequality successively, we have

$$\begin{aligned}
\|S_n - S_m\| &= \|S_{m+1} - S_m + S_{m+2} - S_{m+1} + ... + S_n - S_{n-1}\| \\
&\leq \|S_{m+1} - S_m\| + \|S_{m+2} - S_{m+1}\| + ... + \|S_n - S_{n-1}\| \\
&\leq \gamma^{m+1} \|v_0\| + \gamma^{m+2} \|v_0\| + ... + \gamma^n \|v_0\| \\
&= \gamma^{m+1} \left(1 + \gamma + ... + \gamma^{n-m-1}\right) \|v_0\| \\
&\leq \gamma^{m+1} \left(\frac{1-\gamma^{n-m}}{1-\gamma}\right) \|v_0\|.
\end{aligned}$$

Since $0 < \gamma < 1$, so 1 $1 - \gamma^{n-m} \leq 1$ then

$$\|S_n - S_m\| \leq \frac{\gamma^{m+1}}{1-\gamma} \|v_0\|.$$

Since v_0 is bounded, then

$$\lim_{n,m \longrightarrow \infty} \|S_n - S_m\| = 0.$$

Therefore, the sequences $\{S_n\}$ are Cauchy sequences in the Banach space $\mathcal{B}$, so the series solution defined in (25) converges. This completes the proof. □

Remark 1 *The* $m-$*term approximate solution of Eqs.* (1)-(2) *is given by*

$$v(X,t) = \sum_{n=0}^{m-1} v_n(X,t) = v_0(X,t) + v_1(X,t) + v_2(X,t) + ...$$

4 Appliquations and numerical results

In this section, we apply the (FNDM) on three examples of nonlinear time-fractional wave-like equations with variable coefficients and then compare our approximate solutions with the exact solutions.

Example 1 *Consider the 2-dimensional nonlinear time-fractional wave-like equation with variable coefficients*

$$D_t^\alpha v = \frac{\partial^2}{\partial x \partial y}(v_{xx}v_{yy}) - \frac{\partial^2}{\partial x \partial y}(xyv_x v_y) - v, \; 1 < \alpha \leq 2, \tag{27}$$

with initial conditions

$$v(x,y,0) = e^{xy}, \; v_t(x,y,0) = e^{xy}, \tag{28}$$

where D_t^α *is the Caputo fractional derivative operator of order* α, *and* v *is a function of* $(x,y,t) \in \mathbb{R}^2 \times \mathbb{R}^+$.

By applying the steps involved in (FNDM) as presented in Section 3 to Eqs. (27)-(28), we have

$$\begin{aligned}
v_0(x,y,t) &= (1+t)e^{xy}, \\
v_1(x,y,t) &= -\left(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right)e^{xy}, \\
v_2(x,y,t) &= \left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right)e^{xy}, \\
&\cdots
\end{aligned}$$

So, the solution of Eqs. (27)-(28) can be expressed by

$$\begin{aligned}
v(x,y,t) &= \sum_{n=0}^{\infty} v_n(x,y,t) \qquad (29) \\
&= \left(1+t-\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - \dots\right)e^{xy}. \\
&= \left(E_\alpha(-t^\alpha) + tE_{\alpha,2}(-t^\alpha)\right)e^{xy},
\end{aligned}$$

where $E_\alpha(-t^\alpha)e^{xy}$ *and* $E_{\alpha,2}(-t^\alpha)$ *are the Mittag-Leffler functions, defined by Eqs. (6) and (7).*

Taking $\alpha = 2$ *in (29), the solution of Eqs. (27)-(28) has the general pattern form which is coinciding with the following exact solution in terms of infinite series*

$$v(x,y,t) = \left(1+t-\frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} - \dots\right)e^{xy}.$$

So, the exact solution of Eqs. (27)-(28) *in a closed form of elementary function will be*

$$v(x, y, t) = (\cos t + \sin t)\, e^{xy},$$

which is the same result obtained by (ADM) [4] *and (HPTM)* [5], *for the same test problem.*

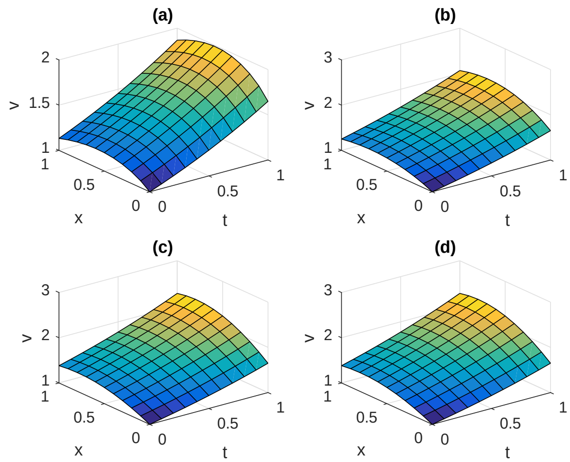

Figure 1: The surface graph of the 4−term approximate solution by (FNDM) and the exact solution for Example 1 when $y = 0.5$: (a) v when $\alpha = 1.5$, (b) v when $\alpha = 1.75$, (c) v when $\alpha = 2$, and (d) v exact.

t	$\alpha = 1.7$	$\alpha = 1.8$	$\alpha = 1.95$	$\alpha = 2$	exact solution	$\lvert v_{exact} - v_{FNDM} \rvert$
0.1	1.3953	1.3999	1.4046	1.4058	1.4058	3.2196×10^{-13}
0.3	1.5522	1.5735	1.5991	1.6061	1.6061	2.1569×10^{-9}
0.5	1.6359	1.6755	1.7272	1.7424	1.7424	1.3095×10^{-7}
0.7	1.6540	1.7088	1.7854	1.8093	1.8093	1.9680×10^{-6}
0.9	1.6137	1.6775	1.7728	1.8040	1.8040	1.4947×10^{-5}

Table 1: The numerical values of the 4−term approximate solution and the exact solution for Example 1 when $x = y = 0.5$.

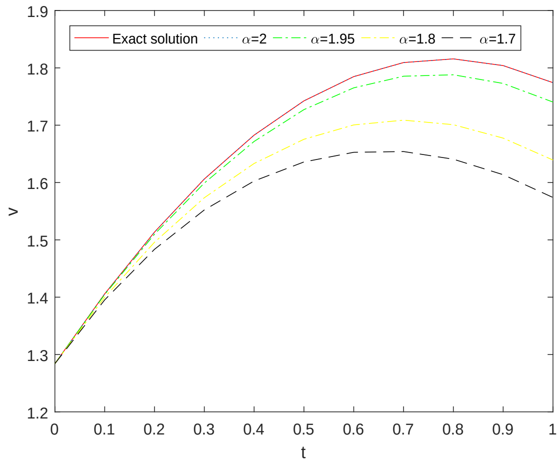

Figure 2: The behavior of the 4−term approximate solution by (FNDM) and the exact solution for Example 1 for different values of α when $x = y = 0.5$.

Example 2 *Consider the following nonlinear time-fractional wave-like equation with variable coefficients*

$$D_t^{\alpha} v = v^2 \frac{\partial^2}{\partial x^2}(v_x v_{xx} v_{xxx}) + v_x^2 \frac{\partial^2}{\partial x^2}(v_{xx}^3) - 18v^5 + v, \ 1 < \alpha \leq 2, \tag{30}$$

with initial conditions

$$v(x,0) = e^x, \ v_t(x,0) = e^x, \tag{31}$$

where D_t^{α}*is the Caputo fractional derivative operator of order* α*, and* v *is a function of* $(x,t) \in]0,1[\times \mathbb{R}^+$.

By applying the steps involved in (FNDM) as presented in Section 3 to Eqs. (30)-(31)*, we have*

$$\begin{aligned} v_0(x,t) &= (1+t)\, e^x, \\ v_1(x,t) &= \left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \right) e^x, \\ v_2(x,t) &= \left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \right) e^x, \\ &\cdots \end{aligned}$$

So, the solution of Eqs. (30)-(31) *can be expressed by*

$$v(x,t) = \left(1 + t + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \ldots\right) e^{x}$$
$$= \left(E_{\alpha}(t^{\alpha}) + tE_{\alpha,2}(t^{\alpha})\right) e^{x}, \tag{32}$$

where $E_{\alpha}(t^{\alpha})$ *and* $E_{\alpha,2}(t^{\alpha})$ *are the Mittag-Leffler functions, defined by Eqs.* (6) *and* (7).

Taking $\alpha = 2$ *in* (32), *the solution of Eqs.* (30)-(31) *has the general pattern form which is coinciding with the following exact solution in terms of infinite series*

$$v(x,t) = \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \ldots\right) e^{x}.$$

So, the exact solution of Eqs. (30)-(31) in a closed form of elementary function will be

$$v(x,t) = e^{x+t},$$

which is the same result obtained by (ADM) [4] and (HPTM) [5], for the same test problem.

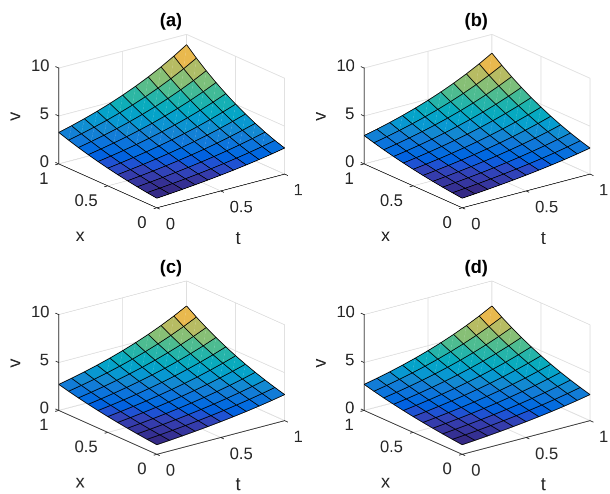

Figure 3: The surface graph of the 4−term approximate solution by (FNDM) and the exact solution for Example 2: (a) v when $\alpha = 1.5$, (b) v when $\alpha = 1.75$, (c) v when $\alpha = 2$, and (d) v exact.

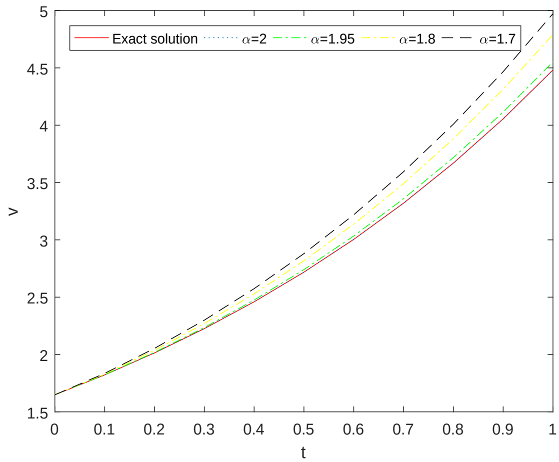

Figure 4: The behavior of the 4−term approximate solution by (FNDM) and the exact solution for Example 2 for different values of α when $x = 0.5$.

t	$\alpha = 1.7$	$\alpha = 1.8$	$\alpha = 1.95$	$\alpha = 2$	exact solution	$\|v_{exact} - v_{FNDM}\|$
0.1	1.8357	1.8298	1.8236	1.8221	1.8221	4.1350×10^{-13}
0.3	2.2994	2.2697	2.2350	2.2255	2.2255	2.7750×10^{-9}
0.5	2.8800	2.8174	2.7402	2.7183	2.7183	1.6907×10^{-7}
0.7	3.5940	3.4901	3.3585	3.3201	3.3201	2.5543×10^{-6}
0.9	4.4670	4.3129	4.1140	4.0552	4.0552	1.9535×10^{-5}

Table 2: The numerical values of the 4−term approximate solution and the exact solution for Example 2 when $x = 0.5$.

Example 3 *Consider the following one dimensional nonlinear time-fractional wave-like equation with variable coefficients*

$$D_t^{\alpha} v = x^2 \frac{\partial}{\partial x}(v_x v_{xx}) - x^2 (v_{xx})^2 - v, \ \ 1 < \alpha \leq 2, \tag{33}$$

with initial conditions

$$v(x, 0) = 0, \ \ v_t(x, 0) = x^2, \tag{34}$$

where D_t^{α} *is the Caputo fractional derivative operator of order* α, *and* v *is a function of* $(x, t) \in]0, 1[\times \mathbb{R}^+$.

By applying the steps involved in (FNDM) as presented in Section 3 to Eqs. (33)-(34)*, we have*

$$\begin{aligned}
v_0(x,t) &= tx^2,\\
v_1(x,t) &= -\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}x^2,\\
v_2(x,t) &= \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}x^2,\\
&\cdots
\end{aligned}$$

So, the solution of Eqs. (33)-(34) *can be expressed by*

$$\begin{aligned}
v(x,t) &= \sum_{n=0}^{\infty} v_n(x,t)\\
&= x^2\left(t - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - \ldots\right)\\
&= x^2\left(tE_{\alpha,2}(-t^{\alpha})\right),
\end{aligned} \tag{35}$$

where $E_{\alpha,2}(-t^{\alpha})$ *is the Mittag-Leffler function, defined by Eq.* (6).

Taking $\alpha = 2$ *in* (35)*, the solution of Eqs.* (33)-(34) *has the general pattern form which is coinciding with the following exact solution in terms of infinite series*

$$v(x,t) = x^2\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \ldots\right).$$

So, the exact solution of Eqs. (33)-(34) *in a closed form of elementary function will be*

$$v(x,t) = x^2 \sin t,$$

which is the same result obtained by (ADM) [4] *and (HPTM)* [5]*, for the same test problem.*

Remark 2 *The numerical results (See Figures* 1, 2,..., 6*) and (Tables* 1, 2 *and* 3*), affirm that when* α *approaches* 2*, our results approach the exact solutions.*

Remark 3 *In this paper, we only apply four terms to approximate the solutions, if we apply more terms of the approximate solutions, the accuracy of the approximate solutions will be greatly improved.*

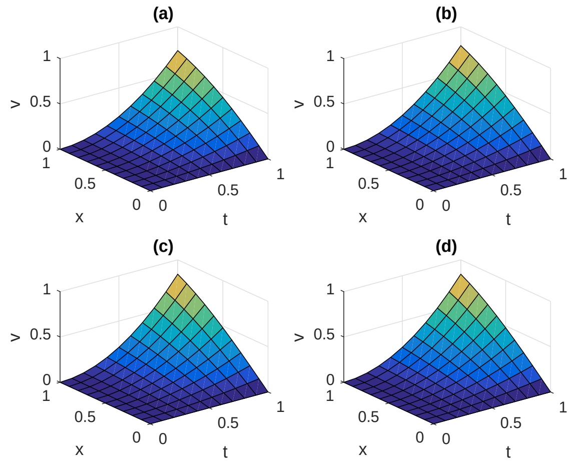

Figure 5: The surface graph of the 4−term approximate solution by (FNDM) and the exact solution for Example 3: (a) v when $\alpha = 1.5$, (b) v when $\alpha = 1.75$, (c) v when $\alpha = 2$, and (d) v exact.

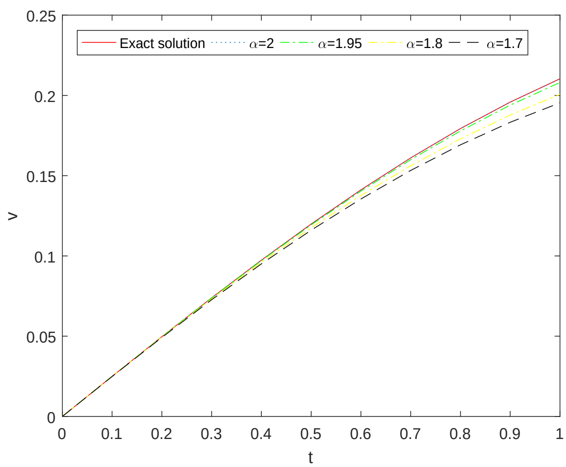

Figure 6: The behavior of the 4−term approximate solution by (FNDM) and the exact solution for Example 3 for different values of α when $x = 0.5$.

t	$\alpha = 1.7$	$\alpha = 1.8$	$\alpha = 1.95$	$\alpha = 2$	*exact solution*	$\lvert v_{exact} - v_{FNDM} \rvert$
0.1	0.02488	0.02492	0.02495	0.02496	0.02496	6.8887×10^{-16}
0.3	0.07271	0.07319	0.07374	0.07388	0.07388	1.3549×10^{-11}
0.5	0.11604	0.11752	0.11934	0.11986	0.11986	1.3425×10^{-9}
0.7	0.15325	0.15615	0.15994	0.16105	0.16105	2.7677×10^{-8}
0.9	0.18327	0.18777	0.19394	0.19583	0.19583	2.6495×10^{-7}

Table 3: The numerical values of the 4−term approximate solution and the exact solution for Example 3 when $x = 0.5$.

5 Conclution

In this paper, the (FNDM) has been successfully applied to study a certain class of nonlinear time-fractional wave-like equations with variable coefficients. The results show that the (FNDM) is an efficient and easy to use technique for finding approximate and exact solutions for this equation. The obtained approximate solutions using the suggested method is in excellent agreement with the exact solution. This confirms our belief that the efficiency of our technique gives it much wider applicability for general classes of nonlinear problems.

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Solving Riemann-Hilbert problems with meromorphic functions

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Abstract. In this paper, we introduce the use of a powerful tool from theoretical complex analysis, the Blaschke product, for the solution of Riemann-Hilbert problems. Classically, Riemann-Hilbert problems are considered for analytic functions. We give a factorization theorem for meromorphic functions over simply connected nonempty proper open subsets of the complex plane and use this theorem to solve Riemann-Hilbert problems where the given data consists of a meromorphic function.

1 Introduction

Approximation of holomorphic functions of a complex variable by a sequence of polynomials has a long history [23], some notable theorems in this regard are the Runge theorem [20], the Mergelyan theorem [19], and the Arakelyan theorem [2]. A different approach to approximation of a holomorphic function is to find and truncate an expansion or a factorization.

Since holomorphic functions are complex analytic, they admit Taylor expansion on an open disk. Furthermore, they admit Fourier expansions on the unit circle. Over the open unit disk, a holomorphic function can be written

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formally as a series of Blaschke products [9]. Moreover, entire functions can be factorized by Weierstrass factorization theorem [25]. In this paper, we give a factorization of meromorphic functions by Blaschke products over simply connected nonempty proper open subsets of the complex plane and use this theorem to solve Riemann-Hilbert problems with meromorphic functions.

One of the shortcomings of the classical solutions to Riemann-Hilbert problems is their dependence on the index of the coefficients and the Hölder continuity requirement in the application of Sokhotski-Plemelj formula. In [16], we proposed solutions to overcome these shortcomings. The current work can be considered as a sequel to [16], focusing on the complex variable case.

This paper is organized as follows. In Section 1, we define the classical Riemann-Hilbert problem, recall results on Blaschke products and state the Riemann Mapping Theorem. In Section 2, we use Blaschke products and the Riemann Mapping Theorem to give factorization theorems for meromorphic functions of bounded type over simply connected nonempty proper open subsets of the complex plane. In Section 3, we define a Riemann-Hilbert problem with meromorphic data and give a general solution by employing the results of Section 2. In Section 4, we give several results for positive definite functions on absolutely convex subsets of the complex plane. Our main results are Theorem (3), Theorem (4) , Theorem (7) and their applications which are discussed in Section 3.

1.1 Riemann-Hilbert problems with analytic functions

The Riemann-Hilbert problem was first introduced by Bernhard Riemann in connection with the Riemann's Monodromy problem which later was generalized to the Riemann-Hilbert problem by Hilbert [1, A.1.3].

Definition 1 [10, 14.1.] *Suppose that we are given a simple smooth closed contour* L *dividing the plane of the complex variable into an interior domain* D^+ *and an exterior domain* D^-*, and two functions of on the contour,* $\mathsf{G}(\mathsf{t})$ *and* $\mathsf{g}(\mathsf{t})$ *which satisfy the Hölder condition, where* $\mathsf{G}(\mathsf{t})$ *does not vanish. It is required to find two functions:* $\Phi^+(z)$*, analytic in the domain* D^+*; and* $\Phi^-(z)$*, analytic in the domain* D^-*, including* $z = \infty$*, which satisfy on the contour* L *either the linear relation*

$$\Phi^+(z) = \mathsf{G}(\mathsf{t})\Phi^-(z)$$

or

$$\Phi^+(z) = \mathsf{G}(\mathsf{t})\Phi^-(z) + \mathsf{g}(\mathsf{t})$$

The function $G(t)$ *will be called the coefficient of the Riemann problem, and the function* $g(t)$ *its free (inhomogeneous) term.*

The following theorem is of particular importance in the solution of analytic Riemann-Hilbert problems.

Theorem 1 [10, 13.2, Generalized Liouville's Theorem] *Let the function* $f(z)$ *be analytic in the entire complex plane, except at the points* $a_0 = \infty$, a_k $(k := 1, 2, \ldots, n)$, *where it has poles, and suppose that the principal parts of the expansions of the function* $f(z)$ *in the vicinities of the poles have the form:*
at the point a_0

$$G_0(z) = c_1^0 z + c_2^0 z^2 + \ldots + c_{n_0}^0 z^{n_0}$$

at the point a_k

$$G_0\left(\frac{1}{z - a_k}\right) = \frac{c_1^k}{z - a_k} + \frac{c_2^k}{(z - a_k)^2} + \ldots + \frac{c_{m_k}^k}{(z - a_k)^{m_k}}.$$

Then the function $f(z)$ *is a rational function and is representable by the relation*

$$f(z) = C + G_0(z) + \sum_{k=1}^{n} G_k(\frac{1}{z - a_k}).$$

In particular, if the only singularity of the function $f(z)$ *is a pole of order* m *at infinity, then* $f(z)$ *is a polynomial of degree* m*:*

$$f(z) = c_0 + c_1 z + \ldots + c_m z^m.$$

1.2 Blaschke products

Definition 2 [11] *A Blaschke product is a function of the form*

$$B(z) = e^{i\alpha} z^K \prod_{n \geq 1} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \bar{z}_n z}$$

in which $\alpha \in \mathbb{R}$, $K \in \mathbb{N}_0$, *and* $\{z_1, z_2, \ldots\}$ *is a sequence (finite or infinite) in* $\{0 < |z| < 1\}$ *that satisfies the Blaschke condition*

$$\sum_{n \geq 1} (1 - |z_n|) < \infty.$$

Finite Blaschke products can be considered as generalizations of polynomials in the unit disk because of their remarkable similar properties to polynomials [18, p. 249]. We only mention few of these similarities:

Proposition 1 *The following hold:*

(i) *Let* f *be analytic in* $\mathbb{C}$ *and suppose that* $\lim_{|z|\to\infty} |f(z)| = \infty$ *then* f *is a polynomial* [18, Theorem 3].

(ii) *Let* f *be analytic in* $\mathbb{D}$ *and suppose that* $\lim_{|z|\to 1} |f(z)| = 1$ *then* f *is a finite Blaschke product* [18, Theorem 13].

(iii) *Let* P *be a polynomial of degree* n *with zeros* $z_1, \dots, z_n$ *in* $\mathbb{C}$*. The critical points of* P *lie in the convex hull of the set* $\{z_1, \dots, z_n\}$ [18, Theorem 9].

(iv) *Let* B *be a finite Blaschke product of degree* n *with zeros* $z_1, \dots, z_n$ *in* $\mathbb{D}$*. Then* $B(z)$ *has exactly* $n-1$ *critical points in* $\mathbb{D}$ *and these all lie in the hyperbolic convex hull* [1] *of the set* $\{z_1, \dots, z_n\}$ [18, Theorem 19].

1.3 Riemann Mapping Theorem

We recall the Riemann Mapping Theorem.

Theorem 2 [3, 14.2] *For any simply connected domain* $R(\neq \mathbb{C})$ *and* $z_0 \in \mathbb{R}$*, there exists a unique conformal mapping* ϕ *of* R *onto* U *such that* $\phi(z_0) = 0$ *and* $\phi'(z_0) > 0$*.*

Example 1 *The map* $f(z) = \frac{z-i}{z+i}$ *is a conformal map of the unit disk to the upper half plane* $\mathbb{H}$*. In fact, all conformal maps from the upper half plane to the unit disk take the form* $e^{i\theta}\frac{z-\beta}{z-\overline{\beta}}$ *where* $\theta \in \mathbb{R}$ *and* $\beta \in \mathbb{H}$ [22, Chapter 8, Exercise 14].

For simple domains such as polygons, one can construct a Riemann map by using the Schwarz-Christoffel formula. The construction of a Riemann map for a general simply connected domain has been studied extensively and numerous algorithms are known rm [13, 6, 5, 8, 7].

[1] Recall that the Poincaré disk provides a model of the hyperbolic plane in the disk $|z| < 1$; we refer to a line in the Poincaré model as a hyperbolic line and to the associated subregions as hyperbolic half-planes. The hyperbolic convex hull of a point set is the intersection of all hyperbolic half-planes containing the point set [24].

2 Factorization of meromorphic functions

In this section, we give some theorems on factorization of meromorphic functions satisfying certain boundedness conditions in terms of (finite or infinite) Blaschke products.

Lemma 1 *Let* $f : X \subset \mathbb{C} \to \mathbb{C}$ *be a holomorphic function where* X *is a simply connected bounded open set. If* $\lim\limits_{|z|\to|a|} |f(z)| \neq 0$ *for all* $a \in \partial X$*, then* f *has finitely many zeros in* X*.*

Proof. Since f is holomorphic on X, it is continuous on X. Assume that f has infinitely many zeros. The zero set $Z = \{z_k\}$ of f is bounded; therefore, it has an accumulation point by the Bolzano-Weierstrass Theorem. The accumulation point of zeros of f does not belong to ∂X because $\lim\limits_{k\to\infty} |f(z_k)| = 0$ but $\lim\limits_{|z|\to|a|} |f(z)| \neq 0$. Therefore, the accumulation point must belong to X. By the Identity Theorem, $f \equiv 0$, on X which is a contradiction. □

Lemma 2 *Let* $f : X \subset \mathbb{C} \to \mathbb{C}$ *be holomorphic on* X *where* X *is a simply connected open set. If* f *has no zeros in* X*, then there exists a holomorphic function* h *on* X *such that* $f = e^h$*. Furthermore, if* X *is bounded,* f *is continuous on* $\overline{X}$*, and constant on* ∂X *then* f *is constant on* X*.*

Proof. The first part of the lemma is a standard result and its proof can be found in [17, XIII, Theorem 2.1]. For the second part, we note that if f has no zeros in X, then $\frac{1}{f}$ is holomorphic on X. By the maximum modulus principle, the maximum of the harmonic function $\frac{1}{|f(z)|}$ is on the boundary of X. But also the maximum of the $|f|$ is on the boundary. If $|f|$ is constant on the boundary then $|f(z)| = c$ for all $z \in X$. □

Theorem 3 *Let* $f : X \subset \mathbb{C} \to \mathbb{C}$ *be a meromorphic function where* X *is a simply connected bounded open Jordan domain. If* $\lim\limits_{|x|\to|a|} |f(x)|$ *where* $a \in \partial X$ *exists and it is not zero or infinity, then*

$$f(\phi(z)) = e^{q(z)} \prod_{i=1}^{n} \frac{z_i - z}{1 - \bar{z_i} z} \prod_{j=1}^{m} \frac{\bar{p_j} - \frac{1}{z}}{1 - \frac{p_j}{z}}$$

where $\phi : \mathbb{D} \to X$ *is a Riemann map,* $q : \mathbb{D} \to \mathbb{C}$ *is a holomorphic function,* $\{z_i\}_{i=1}^{n}$ *is the set of zeros and* $\{p_j\}_{j=1}^{m}$ *is the set of poles of* $f \circ \phi$*.*

Proof. By the Riemann mapping theorem, there exists a conformal bijective map $\phi : \mathbb{D} \to X$. By Carathéodory's theorem, there exists a homeomorphism $\tilde{\phi} : \bar{\mathbb{D}} \to \bar{X}$ that extends ϕ. Therefore, if $|z| \to |1|$ then $|\phi(z)| \to |a|$ where $a \in \partial X$ and hence $\lim\limits_{|z|\to 1} |f(\phi(z))| \neq 0, \infty$. Since $g = f \circ \phi : \mathbb{D} \to \mathbb{C}$ is meromorphic, it is the ratio of two holomorphic functions, i.e. $g = \frac{h}{k}$ where h and k are holomorphic. Since $\lim\limits_{|z|\to|1|} |g(z)| \neq 0, \infty$, we conclude $\lim\limits_{|z|\to|1|} |h(z)| \neq 0$ and $\lim\limits_{|z|\to|1|} |k(z)| \neq 0$. By Lemma (1), h and k have finitely many zeros in $\mathbb{D}$, denoted by $\{z_i\}_{i=1}^n$ and $\{p_j\}_{j=1}^m$ respectively.

The function $h_n := \frac{h}{B_h}$ where $B_h(z) = \prod_{i=1}^n \frac{z_i - z}{1-\bar{z_i}z}$, is holomorphic in $\mathbb{D}$ and has no zeros in $\mathbb{D}$. By Lemma (2), there exists a holomorphic function q_h such that $h_n = e^{q_h}$. Therefore, $h = e^{q_h} B_h$ and we can proceed similarly to prove $k = e^{q_k} B_k$. Hence, $g = e^{q_h - q_k} \frac{B_h}{B_k}$. Since $\bar{B_k}(\frac{1}{\bar{z}}) = \frac{1}{B_k(z)}$, we have $g(z) = e^{q(z)} B_h(z) \bar{B_k}(\frac{1}{\bar{z}})$ where $q(z) = q_h(z) - q_k(z)$. □

Definition 3 *A function defined on a simply connected open subset* X *of the complex plane is said to be of bounded type if it is equal to the ratio of two analytic functions bounded in* X. *The class of all such functions is called the Nevanlinna class for* X.

Lemma 3 [22, p. 156] *If* f *is holomorphic in the unit disk, bounded and not identically zero, and* $z_1, z_2, \ldots, z_n, \ldots$ *are its zeros* ($|z_k| < 1$)*, then*

$$\sum_n (1 - |z_n|) < \infty.$$

Lemma 4 [14, p. 64] *Let* $\{\alpha_n\}$ *be a sequence of non-zeros complex numbers in the open unit disc* $\mathbb{D}$. *A necessary and sufficient condition that the infinite product*

$$B(z) = \prod_{n=1}^{\infty} \left[\frac{\bar{\alpha}_n}{|\alpha_n|} \frac{(\alpha_n - z)}{(1 - \bar{\alpha}_n z)} \right]$$

should converge uniformly on compact subsets of the unit disc is that $\sum\limits_{n=1}^{\infty} (1 - |\alpha_n|) < \infty$. *When this condition is satisfied, the product defines an inner function whose zeros are exactly* $\alpha_1, \alpha_2, \ldots$

We now obtain a factorization theorem that has useful applications to the Riemann-Hilbert problem. This is discussed further in Section 3.

Theorem 4 *Let* $f : X \subset \mathbb{C} \to \mathbb{C}$ *be a meromorphic function where* X *is a simply connected open set. If* f *is of bounded type then*

$$f(\phi(z)) = z^{r-s} q(z) \prod_{i=1}^{\infty} \frac{\bar{z}_i}{|z_i|} \frac{z_i - z}{1 - \bar{z}_i z} \prod_{j=1}^{\infty} \frac{|p_j|}{p_j} \frac{\bar{p}_j - \frac{1}{z}}{1 - \frac{p_j}{z}}$$

where $\phi : \mathbb{D} \to X$ *is a Riemann map,* r, $s \in \mathbb{N}_0$, q *is a bounded holomorphic function without zeros,* $\{z_i\}$ *is the set of zeros and* $\{p_j\}$ *is the set of poles of* $f \circ \phi$.

Proof. By Riemann mapping theorem, there exists a conformal bijective map $\phi : \mathbb{D} \to X$. Since $g = f \circ \phi : \mathbb{D} \to \mathbb{C}$ is meromorphic of bounded type, it's the ratio of two bounded holomorphic functions, i.e. $g = \frac{h}{k}$ where h and k are holomorphic and bounded. The functions h and k can be factorized as $h(z) = z^r h_1(z)$ and $k(z) = z^s k_1(z)$ where $h_1(0) \neq 0$ and $k_1(0) \neq 0$. Let $\{z_i\}$ and $\{p_j\}$ be the zeros of h_1 and k_1. By Lemma (3), $\sum_i (1 - |z_i|) < \infty$ and $\sum_j (1 - |p_j|) < \infty$. By Lemma (4), the following products are convergent:

$$B_{h_1}(z) = \prod_{i=1}^{\infty} \left[\frac{\bar{z}_i}{|z_i|} \frac{(z_i - z)}{(1 - \bar{z}_i z)} \right]$$

$$B_{k_1}(z) = \prod_{j=1}^{\infty} \left[\frac{\bar{p}_j}{|p_j|} \frac{(p_j - z)}{(1 - \bar{p}_j z)} \right]$$

Hence, we can write $h_1(z) = u(z) B_{h_1}(z)$ and $k_1(z) = v(z) B_{k_1}(z)$ where $u(z) = \frac{h_1(z)}{B_{h_1}(z)}$ and $v(z) = \frac{k_1(z)}{B_{k_1}(z)}$ are bounded holomorphic functions. Therefore,

$$f(\phi(z)) = z^{r-s} \frac{u(z)}{v(z)} \frac{B_{h_1}(z)}{B_{k_1}(z)} = z^{r-s} q(z) B_{h_1}(z) \bar{B}_{k_1}(\frac{1}{\bar{z}})$$

where $q(z) = \frac{u(z)}{v(z)}$ is a bounded holomorphic function. □

3 Applications in Riemann-Hilbert problems with meromorphic functions

In engineering, a transfer function is a representation of the relation between the input and output of a linear time-invariant (LTI) system and it is a primary tool in classical control engineering. In this section, we employ Theorem (4) to find the transfer function of a differential system.

Lemma 5 [4, Theorem 5.1] *Suppose* $f : \mathbb{R} \to \mathbb{C}$ *is continuous,* $\mathrm{supp}(f) \subset [M, \infty)$*, and has exponential order* a*, i.e.* $|f(t)| \le Ke^{at}$ *for all* $t \in \mathbb{R}$*. Then the Laplace transform* $\mathfrak{L}(f)(z) := \int_{-\infty}^{\infty} e^{-zt} f(t)dt$ *is holomorphic in the half plane* $\{z \mid \Re(z) > a\}$*. The derivative is*

$$(\mathfrak{L}(f))'(z) = -\int_{-\infty}^{\infty} e^{-zt} t f(t) dt$$

and the Laplace transform satisfies the estimate

$$|\mathfrak{L}(f)(z)| \le K \frac{e^{M(a-\Re(z))}}{(\Re(z) - a)}, \Re(z) > a$$

Remark 1 *If* $\Re(z) > a + \epsilon > a$ *and* $M > 0$ *where* $\epsilon > 0$ *, then* $|\mathfrak{L}(f)(z)| \le \frac{K}{\epsilon e^{M\epsilon}}$*, i.e. the Laplace transform is bounded.*

Lemma 6 [21, Theorem 2.12] *Suppose that* $f(t), \acute{f}(t), \ldots, f^{(n-1)}(t)$ *are continuous on* $(0, \infty)$ *and of exponential order, while* $f^{(n)}(t)$ *is piecewise continuous on* $[0, \infty)$*. Then* $\mathfrak{L}(f^{(n)}(t)) = s^n \mathfrak{L}(f(t)) - s^{n-1} f(0^+) - s^{n-2} \acute{f}(0^+) - \ldots - f^{(n-1)}(0^+)$*.*

Theorem 5 *Suppose* $f_k, g_k : \mathbb{R} \to \mathbb{C}$ *are continuous, have left bounded support on the positive real line, and have positive exponential orders* a_k *and* b_k*. Furthermore, assume that* $u, y : \mathbb{R} \to \mathbb{C}$ *are* n*-times continuously differentiable, with* n*th derivative of exponential order. Then the transfer function of the following differential system with zero initial conditions, i.e.* $u^{(k)}(0) = 0$*,* $y^{(k)}(0) = 0$*,*

$$\sum_{k=0}^{n} f_k(t) * \frac{d^k u(t)}{dt^k} = \sum_{k=0}^{n} g_k(t) * \frac{d^k y(t)}{dt^k}$$

is a meromorphic function of bounded type of the form

$$T(\phi(z)) = z^{r-s} q(z) \prod_{i=1}^{\infty} \frac{\bar{z}_i}{|z_i|} \frac{z_i - z}{1 - \bar{z}_i z} \prod_{j=1}^{\infty} \frac{|p_j|}{p_j} \frac{\bar{p}_j - \frac{1}{z}}{1 - \frac{p_j}{z}}$$

where $\phi : \mathbb{D} \to X$ *is defined by* $\phi(z) := \frac{1+(z-\alpha)}{1-(z-\alpha)}$ *where* $X = \{z \in \mathbb{C} \mid \Re(z) > \alpha\}$*,* $\alpha = \min\{a_k, b_k\} + \epsilon$*,* $\epsilon > 0$ *is sufficiently small,* r*,* $s \in \mathbb{N}_0$*,* q *is a bounded holomorphic function without zeros,* $\{z_i\}$ *is the set of zeros and* $\{p_j\}$ *is the set of poles of* $T \circ \phi$*. The transfer function* $T : X \to \mathbb{C}$ *appears as the coefficient of the Riemann-Hilbert problem* $\Phi^+(z) = G(z)\Phi^-(z)$ *where*

$$G(z) = \frac{T(\phi(z))}{q(z)}$$

$$\Phi^{+}(z) = z^{r} \prod_{i=1}^{\infty} \frac{\bar{z}_i}{|z_i|} \frac{z_i - z}{1 - \bar{z}_i z}$$

and

$$\Phi^{-}(z) = z^{s} \prod_{j=1}^{\infty} \frac{\bar{p}_j}{|p_j|} \frac{p_j - z}{1 - \bar{p}_j z}$$

Proof. The transfer function is defined as the ratio of the Laplace transform of the output signal to the Laplace transform of the input signal, i.e. $T(s) := \frac{\mathfrak{L}(y)(s)}{\mathfrak{L}(u)(s)}$. If we take Laplace transform of the differential system, apply Lemma (6), and the properties of Laplace transform with respect to convolution and addition, we derive the following equation

$$\left(\sum_{k=0}^{n} s^k \mathfrak{L}(f_k)(s) \right) \mathfrak{L}(u)(s) = \left(\sum_{k=0}^{n} s^k \mathfrak{L}(g_k)(s) \right) \mathfrak{L}(y)(s)$$

Therefore, the transfer function is of the following form

$$T(s) = \frac{\mathfrak{L}(y)(s)}{\mathfrak{L}(u)(s)} = \frac{\sum_{k=0}^{n} s^k \mathfrak{L}(f_k)(s)}{\sum_{k=0}^{n} s^k \mathfrak{L}(g_k)(s)}$$

On the domain $X = \{z \in \mathbb{C} \mid \Re(z) > \alpha\}$, where $\alpha = \min\{a_k, b_k\} + \epsilon$, and $\epsilon > 0$ is sufficiently small, the transfer function

$$T(s) = \frac{\mathfrak{L}(y)(s)}{\mathfrak{L}(u)(s)} = \frac{\sum_{k=0}^{n} s^{k-n} \mathfrak{L}(f_k)(s)}{\sum_{k=0}^{n} s^{k-n} \mathfrak{L}(g_k)(s)}$$

is a meromorphic function of bounded type by Lemma (5). It suffices to apply Theorem (4) to the transfer function T. □

4 Positive definite functions of a complex variable

In this section, we give some results on positive definite functions over absolutely convex subsets of $\mathbb{C}$. It is interesting to see whether one can factorize a meromorphic positive definite function; in the sense of Definition (4), such that all factors are positive definite. Unfortunately, it is difficult to determine positive definiteness of Blaschke products; even the determination of hermitianness is difficult because it requires finding all the zeros of the equation $B(-z) - \overline{B(z)} = 0$. Nevertheless, we give a theorem (Theorem (7)) that can simplify the determination of positive definiteness for holomorphic hermitian functions.

Definition 4 *A set $X \subseteq \mathbb{C}$ is called absolutely convex if for any points x_1, x_2 in X and any numbers λ_1, λ_2 in $\mathbb{C}$ satisfying $|\lambda_1|+|\lambda_2| \leq 1$, the sum $\lambda_1 x_1+\lambda_2 x_2$ belongs to X.*

If $X \subseteq \mathbb{C}$ is absolutely convex then rX is absolutely convex for all $r \in \mathbb{C}$.

Definition 5 *A function $f : X \to \mathbb{C}$ is positive definite, where $X \subseteq \mathbb{C}$ is absolutely convex, if $\sum_{j,k=1}^{n} f(\frac{x_j-x_k}{2})\xi_j\bar{\xi}_k \geq 0$ for every choice of $x_1, \ldots, x_n$ in X and $\xi_1, \ldots, \xi_n$ in $\mathbb{C}$.*

If we set $\omega^*=[\bar{\xi}_1, \ldots, \bar{\xi}_n]$ and $A = [f(\frac{x_j-x_k}{2})]_{j,k}$ then we can rewrite the above condition as $\omega^* A\omega \geq 0$, i.e. A is positive-semidefinite. In the following proposition, we review some of the properties of positive definite functions.

Proposition 2 *If $f : X \to \mathbb{C}$ is positive definite, where $X \subseteq \mathbb{C}$ is absolutely convex, then the following hold:*

(i) $f(0) \geq 0$, $f(-\frac{z}{2}) = \bar{f}(\frac{z}{2})$, *and* $|f(\frac{z}{2})|^2 \leq f(0)^2$.

(ii) *If $f, g : X \to \mathbb{C}$ are positive definite then fg and $c_1 f+c_2 g$ where $c_1, c_2 \in \mathbb{N}$ are positive definite.*

(iii) *If $X = \mathbb{R}$, f and g are integrable and positive definite then $f*g$ is positive definite.*

(iv) *If $X = \mathbb{R}$, and f is integrable then $x^{2k+1}f(x)$ with $k \in \mathbb{N}$ is not positive definite.*

(v) *If $X = \mathbb{R}$, and f is C^n-differentiable then $\frac{d^n f(x)}{dx^n}$ is positive definite only if $n = 4k$ where $k \in \mathbb{N}$.*

(vi) *If $X = \mathbb{R}$, and f is integrable then $e^{iax}f(x)$ is positive definite.*

Proof. (i) If we set $n = 1$ in Definition (5) then $f(0)|\xi_1|^2 \geq 0$ which implies $f(0) \geq 0$. If we set $n = 2$ and consider the set of points $\{z, 0\}$ then $\alpha = f(0)^2(|\xi_1|^2 + |\xi_2|^2) + f(-\frac{z}{2})\xi_1\bar{\xi}_2 + f(\frac{z}{2})\xi_2\bar{\xi}_1 \geq 0$. Since $\alpha = \bar{\alpha}$, we have $f(-\frac{z}{2}) = \bar{f}(\frac{z}{2})$. Since the following matrix is positive-semidefinite, its determinant is nonnegative, i.e. $|f(\frac{z}{2})|^2 \leq f(0)^2$.

$$A = \begin{pmatrix} f(0) & f(\frac{z}{2}) \\ f(\frac{-z}{2}) & f(0) \end{pmatrix}$$

(ii) Since positive linear combination and product of positive definite functions correspond to positive linear combination and Hadamard product of positive-semidefinite matrices, positive-definiteness is preserved under these operations.

(iii) By convolution theorem $\widehat{f * g} = \hat{f} * \hat{g}$ which is positive by Bochner's theorem.

(iv) The Fourier transform of $x^{2k+1}f(x)$ is $(\frac{i}{2\pi})^{2k+1}\frac{d^{2k+1}\hat{f}(\xi)}{d\xi^{2k+1}}$ which is purely imaginary by Bochner's theorem.

(v) The Fourier transform of $\frac{d^n f(x)}{dx^n}$ is $(2\pi i\xi)^n\hat{f}(\xi)$ which is positive only if $n = 4k$, $k \in \mathbb{N}$.

(vi) The Fourier transform of $e^{iax}f(x)$ is $\hat{f}(\xi - \frac{a}{2\pi})$. □

Proposition 3 *If the Möbius transform* $f(z) = \frac{az+b}{cz+d}$ *is positive definite and not identically zero on* $\mathbb{C} \cup \{\infty\}$ *then* $a = 0$.

Proof. Assume $a \neq 0$ then we can set $z = \frac{2b}{a}$ in the condition $f(-\frac{z}{2}) = \bar{f}(\frac{z}{2})$ which implies $b = 0$. Therefore, the condition $|f(\frac{z}{2})|^2 \leq f(0)^2$ implies that f is identically zero which is a contradiction. □

Lemma 7 *The function* $f(z) = e^{ia}\frac{z_k - z}{1 - \bar{z}_k z}$ *where* $z_k, z \in \mathbb{D}$, *and* $a \in \mathbb{R}$ *is not positive definite.*

Proof. For the set $\{\frac{ie^{-ia}}{2}, 0\}$ in $\mathbb{D}$, the determinant of the associated matrix A is $\frac{|z_k|^4 - 1}{4 + \bar{z}_k^2 e^{-2ia}}$. The numerator of $\det(A)$ is negative for $z_k \in \mathbb{D}$, but the denominator is not negative for any value of $z_k \in \mathbb{D}$, and $a \in \mathbb{R}$. Therefore, $\det(A)$ is not nonegative for all $z_k \in \mathbb{D}$, and $a \in \mathbb{R}$. □

Theorem 6 *A conformal map from* $\mathbb{D}$ *to* $\mathbb{D}$ *is not positive definite.*

Proof. A conformal map from $\mathbb{D}$ to $\mathbb{D}$ is of the form $e^{ia_k}\frac{z_k - z}{1 - \bar{z}_k z}$ which is not positive definite by Lemma (7). □

Lemma 8 [15, Exercise 1.1.8], [12, Lemma 24] *For any nontrivial holomorphic function* $f : U \to \mathbb{C}$ *where* $U \subset \mathbb{C}^n$ *is open and connected,* $U - Z(f)$ *is connected and dense in* U *where* $Z(f)$ *denotes the zero set of* f*.*

Theorem 7 *Let* $X \subseteq \mathbb{C}$ *be an absolutely convex simply connected set and* $f : X \subseteq \mathbb{C} \to \mathbb{C}$ *be a holomorphic hermitian, i.e.* $f(-z) = \overline{f(z)}$*, function. Define the function* $W_n(f) : X^n \to M_n(\mathbb{C})$ *by* $W_n(f)(x) = [f(\frac{x_j - x_k}{2})]_{j,k}$*. If there exists a point* $x \in X^n$ *at which* $W_n(f)$ *is positive definite for all* $n \in \mathbb{N}$ *then* f *is positive definite on* X*.*

Proof. For simplicity, we denote $W_n(f)$ by g. Since $\det(g)$ is a polynomial of f and f is holomorphic, $\det(g)$ is holomorphic. By Lemma (8), $S = \mathrm{supp}(\det(g))$ is connected, open and dense in X^n. Since f is hermitian, $\mathrm{Spec}(g) = \{\lambda_x \in \mathbb{C} \mid \lambda_x$ is an eigenvalue of $g(x)$, $x \in X^n\}$ is in $\mathbb{R}$. By definition of S, g is invertible on S. Therefore, $0 \notin \mathrm{Spec}(g_{|S})$. We claim that either $\mathrm{Spec}(g_{|S}) \subseteq \mathbb{R}_+$ or $\mathrm{Spec}(g_{|S}) \subseteq \mathbb{R}_-$. Assume otherwise, then there exist $x, y \in S$ such that $\lambda_x \in \mathbb{R}_+$ and $\lambda_y \in \mathbb{R}_-$. Since S is path connected, there exists a path in S that connects x to y in S. But, there is no path that connects λ_x to λ_y because $0 \notin \mathrm{Spec}(g_{|S})$ which gives a contradiction. If there exists a point $x_0 \in X^n$ for which g is positive definite, then either $x_0 \in S$ or x_0 is a limit point of S because S is dense in X^n. In either case, $\mathrm{Spec}(g_{|S}) \subseteq \mathbb{R}_+$ and by density of S we conclude $\mathrm{Spec}(g) \subseteq \mathbb{R}_+$ for all $n \in \mathbb{N}$, i.e. f is positive definite on X. □

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Construction of (M, N)-hypermodule over (R, S)-hyperring

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Abstract. The aim of this paper is to introduce a new class of hypermodules that may be called (M, N)-hypermodules over (R, S)-hyperrings. Then, we investigate some properties of this new class of hyperstructures. Since the main tools in the theory of hyperstructures are the fundamental relations, we give some results about them with respect to the fundamental relations.

1 (M, N)-hypermodule over (R, S)-hyperring

One knows the construction of a hypergroup K having as core a fixed hypergroup H. In [10], the aforesaid construction is generalized to a large class of hypergroups obtained from a group and from a family of fixed sets, and its properties are analyzed especially in the finite case. We recall the following notions from [4, 10]. Let $(M, \oplus)$ be a hypergroup and $(N, \uplus)$ be a group with a neutral element 0_N. Also, let $\{A_n\}_{n\in N}$ be a family of non-empty subsets indexed in N such that for all $x, y \in N$, $x \neq y$, $A_x \cap A_y = \emptyset$, and $A_{0_N} = M$. We set $P = \bigcup_{n\in N} A_n$ and we define the hyperoperation $\bar{\oplus}$ in P in the following way:

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(1) for every $(x,y) \in M^2$, $x\bar{\oplus}y = x \oplus y$,

(2) for every $(x,y) \in A_{n_1} \times A_{n_2} \neq H^2$, $x\bar{\oplus}y = A_{n_1 \uplus n_2}$.

The hyperstructure $(P,\bar{\oplus})$ is a hypergroup [4, 10]. In [14], Spartalis presented a way to obtain new hyperrings, starting with other hyperrings. We recall the following notions from [8, 14]. Let $(S,\dagger,\cdot)$ be a hyperring and let $\{B_i\}_{i\in R}$ be a family of non-empty sets such that:

(1) $(R,+,\star)$ is a ring,

(2) $B_{0_R} = S$,

(3) for every $i \neq j$, $B_i \cap B_j = \emptyset$.

Let $T = \bigcup_{i\in R} B_i$ and define the following hyperoperations on T: for every $(x,y) \in B_i \times B_j$:

$$x\ddagger y = \begin{cases} x\dagger y, & \text{if } (i,j) = (0_R,0_R) \\ B_{i+j}, & \text{if } (i,j) \neq (0_R,0_R) \end{cases} \quad \text{and} \quad x\odot y = \begin{cases} x\cdot y, & \text{if } (i,j) = (0_R,0_R) \\ B_{i\star j}, & \text{if } (i,j) \neq (0_R,0_R). \end{cases}$$

The structure $(T,\ddagger,\odot)$ is a hyperring [8, 14].

Now, we introduce a way to obtain new hypermodules, starting with other hypermodules.

Definition 1 *Let* $(M,\oplus,\bullet)$ *be a hypermodule over a hyperring* $(S,\dagger,\cdot)$ *and let* $\{A_n\}_{n\in N}$ *and* $\{B_i\}_{i\in R}$ *be two families of non-empty sets such that:*

(1) $(N,\uplus,*)$ *be a module over a ring* $(R,+,\star)$,

(2) $A_{0_N} = M$ *and* $B_{0_R} = S$,

(3) *for every* $m,n \in \mathbb{N}$, $m \neq n$, $A_m \cap A_n = \emptyset$ *and for every* $i,j \in \mathbb{N}$, $i \neq j$, $B_i \cap B_j = \emptyset$.

Let $P = \bigcup_{n\in N} A_n$ *and* $T = \bigcup_{i\in R} B_i$. *We define the hyperoperation* $\bar{\oplus}$ *on* P *and the hyperoperations* $\ddagger$ *and* $\odot$ *on* T *similar to the above mentioned definitions. Also, we define a map* $\bar{\bullet} : T \times P \longrightarrow \wp^*(P)$ *as follows:*

$$t\bar{\bullet}x = \begin{cases} t \bullet x, & \text{if } (i,n) = (0_R,0_M) \\ A_{i*n}, & \text{if } (i,n) \neq (0_R,0_M), \end{cases}$$

for every $(t,x) \in B_i \times A_n$.

Theorem 1 *The structure $(P, \bar{\oplus}, \bar{\bullet})$ over the hyperring $(T, \ddagger, \odot)$ is a hypermodule.*

Proof. According to [10, 14], $(P, \bar{\oplus})$ is a hypergroup and $(T, \ddagger, \odot)$ is a hyperring. We show that for every $r, s \in T$ and $x, y \in P$:

(1) $r\bar{\bullet}(x\bar{\oplus}y) = r\bar{\bullet}x\bar{\oplus}r\bar{\bullet}x$,

(2) $(r\ddagger s)\bar{\bullet}x = r\bar{\bullet}x\ddagger s\bar{\bullet}x$,

(3) $(r \odot s)\bar{\bullet}x = r\bar{\bullet}(s\bar{\bullet}x)$.

First, we prove (1). Let $r \in T$ and $x, y \in P$. Then, we have the following cases:

(i) $r \in B_{0_R} = S$ and $x, y \in A_{0_N} = M$. Then, we have $r\bar{\bullet}(x\bar{\oplus}y) = r \bullet (x \oplus y) = r \bullet x \oplus r \bullet x = r\bar{\bullet}x\bar{\oplus}r\bar{\bullet}x$,

(ii) $r \in B_j$, where $0_R \neq j \in R$, and $x, y \in A_{0_N}$. Then, we have $r\bar{\bullet}(x\bar{\oplus}y) = r\bar{\bullet}(x \oplus y) = A_{j*0_N} = A_{0_N}$ and $r\bar{\bullet}x\bar{\oplus}r\bar{\bullet}x = A_{j*0_N}\bar{\oplus}A_{j*0_N} = A_{0_N} \oplus A_{0_N} = A_{0_N}$. So (1) is true.

(iii) $r \in B_{0_R}$ and $(x, y) \in A_a \times A_b$, where $(0_R, 0_R) \neq (a, b)$. Then, it is not difficult to see that $r\bar{\bullet}(x\bar{\oplus}y) = A_{0_N}$ and $r\bar{\bullet}x\bar{\oplus}r\bar{\bullet}x = A_{0_N}$.

(iv) $r \in B_j$, where $0_R \neq j \in R$, and $(x, y) \in A_a \times A_b$, where $(0_N, 0_N) \neq (a, b)$. Then, it is not difficult to see that $r\bar{\bullet}(x\bar{\oplus}y) = A_{j*(a\uplus b)}$ and $r\bar{\bullet}x\bar{\oplus}r\bar{\bullet}x = A_{j*a\uplus j*b}$. Since $(N, \uplus, *)$ is a module over a ring $(R, +, \star)$, then $j*(a\uplus b) = j * a \uplus j * b$ and so (1) is true.

Therefore, we show that (1). Similarly, we can prove (2) and (3). □

Example 1 *Let $N = (\mathbb{Z}_3, +)$ be a module over the ring $R = (\mathbb{Z}_3, +, \cdot)$, $M = (\mathbb{Z}_2, \oplus)$ be a hypermodule over a hyperring $S = (\mathbb{Z}_2, \oplus, \cdot)$, where $0 \oplus 0 = 0$, $0 \oplus 1 = 1 \oplus 1 = 1$ and $1 \oplus 1 = \{0, 1\}$ and set $A_0 = B_0 = \mathbb{Z}_2$, $A_1 = B_1 = \{a, b\}$ and $A_2 = B_2 = \{c\}$. Now, we have $P = T = \{0, 1, a, b, c, d, e\}$. Then, we obtain $\bar{\oplus} = \ddagger$ and $\bar{\bullet} = \odot$. Also, we have*

$$\begin{array}{lll} 0\bar{\oplus}1 = 1, & a\bar{\oplus}a = b\bar{\oplus}a = a\bar{\oplus}b = b\bar{\oplus}b = \{c\}, & c\bar{\oplus}c = \{a, b\}, \\ 0\bar{\oplus}0 = 0, & 0\bar{\oplus}a = 1\bar{\oplus}a = 0\bar{\oplus}b = 1\bar{\oplus}b = \{a, b\}, & 0\bar{\oplus}c = 1\bar{\oplus}c = \{c\}, \\ 1\bar{\oplus}1 = \{0, 1\}, & c\bar{\oplus}a = c\bar{\oplus}a = c\bar{\oplus}b = c\bar{\oplus}b = \{0, 1\}. & \end{array}$$

and

$$\begin{array}{lll} 0\bar{\bullet}1 = 0, & a\bar{\bullet}a = b\bar{\bullet}a = a\bar{\bullet}b = b\bar{\bullet}b = \{a, b\}, & c\bar{\bullet}c = \{a, b\}, \\ 0\bar{\bullet}0 = 0, & 0\bar{\bullet}a = 1\bar{\bullet}a = 0\bar{\bullet}b = 1\bar{\bullet}b = \{0, 1\}, & 0\bar{\bullet}c = 1\bar{\bullet}c = \{0, 1\}, \\ 1\bar{\bullet}1 = 1, & c\bar{\bullet}a = c\bar{\bullet}a = c\bar{\bullet}b = c\bar{\bullet}b = \{c\}, & \end{array}$$

Let $(H, +)$ be a hypergroup. We consider the fundamental relation β on H as follows: $x\beta y$ if and only if $\{x, y\} \subseteq \sum_{i=1}^{n} x_i$, for some $x_i \in H$. Let β^* be the transitive closure of β. The fundamental relation β^* is the smallest equivalence relation such that the quotient H/β^* is a group. This relation introduced by Koskas [12] and studied by others, for example see [3, 4, 5, 12, 16]. Also, we recall the definition of the fundamental relation γ on a hypergroup H as follows: $x\gamma y$ if and only if $x \in \sum_{i=1}^{n} x_i$, $y \in \sum_{i=1}^{n} x_{\sigma(i)}$, $x_i \in H$, $\sigma \in \mathbb{S}_n$. Let γ^* be the transitive closure of γ. The fundamental relation γ^* is the smallest equivalence relation such that the quotient H/γ^* is an abelian group [11], also see [6, 7].

The fundamental relation Γ on a hyperring was introduced by Vougiouklis at the fourth AHA congress (1990) [15] as follows: $x\Gamma y$ if and only if $\exists n \in \mathbb{N}$, $\exists\,(k_1, \ldots, k_n) \in \mathbb{N}^n$, and $[\exists\,(x_{i1}, \ldots, x_{ik_i}) \in R^{k_i}, (i = 1, \ldots, n)]$ such that $\{x, y\} \subseteq \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij})$. The fundamental relation Γ on a hyperring is defined as the smallest equivalence relation so that the quotient would be the (fundamental) ring. Note that the commutativity with respect to both sum and product in the fundamental ring are not assumed. In [9], Davvaz and Vougiouklis introduced a new strongly regular equivalence relation on a hyperring such that the set of quotients is an ordinary commutative ring. We recall the following definition from [9].

Definition 2 [9] *Let R be a hyperring. We define the relation α as follows: $x\alpha y$ if and only if $\exists n \in \mathbb{N}$, $\exists(k_1, \ldots, k_n) \in \mathbb{N}^n$, $\exists\sigma \in \mathbb{S}_n$ and $[\exists(x_{i1}, \ldots, x_{ik_i}) \in R^{k_i}$, $\exists\sigma_i \in \mathbb{S}_{k_i}$, $(i = 1, \ldots, n)]$ such that $x \in \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij})$ and $y \in \sum_{i=1}^{n} A_{\sigma(i)}$, where $A_i = \prod_{j=1}^{k_i} x_{i\sigma_i(j)}$.*

If α^* is the transitive closure of α, then α^* is a strongly regular relation both on $(R, +)$ and $(R, \cdot)$, and the quotient R/α^* is a commutative ring [9], also see [13].

Now, consider Definition 1 and Theorem 1. Then:

Theorem 2 *We have*

(1) $P/\beta_P^* \cong N$ *(group isomorphism).*

(2) $P/\gamma_P^* \cong N/\gamma_N^*$ *(group isomorphism) and if N is commutative then $P/\gamma_P^* \cong N$.*

(3) $T/\Gamma_T^* \cong R$ *(ring isomorphism).*

(4) $T/\alpha_T^* \cong R/\alpha_R^*$ *(ring isomorphism) and if R is commutative (with respect to the both operations) then $T/\alpha_T^* \cong R$.*

Proof. (1) We define $\phi : P/\beta^*_P \longrightarrow N$, with $\phi(\beta^*_P(a_n)) = n$, where $a_n \in A_n$ and $n \in \mathbb{N}$. Since β^*_P is a regular relation, so $(\beta^*_P(a_n))(\beta^*_P(a_m)) = (\beta^*_P(a_n a_m))$ and ϕ is a homomorphism. Let $(\beta^*_P(a_n)) = 0_N$. Then, $n = 0_N$ and so $\mathrm{Ker}\phi = (\beta^*_P(a_{0_N}))$. Hence, ϕ is one to one. Clearly, ϕ is onto.

(2) We define $\psi : P/\gamma^*_P \longrightarrow N/\gamma^*_N$, with $\psi(\gamma^*_P(a_n)) = \gamma_N(a_n)$, where $a_n \in A_n$ and $n \in \mathbb{N}$. Since γ^*_P and γ^*_N are regular relations, so $(\gamma^*_P(a_n))(\gamma^*_P(a_m))$ $= (\gamma^*_N(n)\gamma^*_N(m)) = (\gamma^*_N(nm)) = (\gamma^*_P(a_n a_m))$. Then, ϕ is a homomorphism. Let $(\gamma^*_P(a_n)) = 0_{N/\gamma^*_N} = \gamma^*_N(0_N)$. Then, $n = 0_N$ and so $\mathrm{Ker}\psi = (\gamma^*_P(a_{0_N}))$. Hence, ψ is one to one. Clearly, ψ is onto.

(3) We define $\lambda : T/\gamma^*_T \longrightarrow R$, with $\lambda(\Gamma^*_T(b_i)) = i$, where $b_i \in A_i$ and $i \in \mathbb{N}$. Since Γ^*_T is a regular relation, so $(\Gamma^*_P(a_n))(\Gamma^*_P(a_m)) = (\Gamma^*_P(a_n a_m))$ and λ is a homomorphism. Let $(\Gamma^*_P(a_i)) = 0_R$. Then, $i = 0_R$ and so $\mathrm{Ker}\lambda = (\Gamma^*_P(a_{0_R}))$. Hence λ is one to one. Clearly, λ is onto.

(4) We define $\mu : T/\alpha^*_T \longrightarrow R$, with $\mu(\alpha^*_T(b_i)) = \alpha^*_R(i)$, where $b_i \in A_i$ and $i \in \mathbb{N}$. Since α^*_T and α^*_R are regular relations, so $(\alpha^*_P(a_i))(\alpha^*_P(a_j)) = (\alpha^*_R(i))(\alpha^*_R(j)) = (\alpha^*_R(ij)) = (\alpha^*_P(a_i a_j))$. Then, μ is a homomorphism. Let $(\alpha^*_P(a_i)) = 0_{R/\alpha^*_R}$. Then, $i = 0_R$ and so $\mathrm{Ker}\mu = (\alpha^*_P(a_{0_R}))$. Thus, μ is one to one. Clearly, μ is onto. □

Now, we recall the definition of the fundamental relation ϵ on M from [16]. Let M be an R-hypermodule. Then $x\epsilon y$ if and only if $\{x, y\} \subseteq \sum_{i=1}^{n} m'_i$, where $m'_i = m_i$ or $m'_i = \sum_{j=1}^{n_i}(\prod_{k=1}^{k_{ij}} x_{ijk})m_i$, $r_{ijk} \in R$. The fundamental relation ϵ^* is defined to be the smallest equivalence relation such that the quotient M/ϵ^* is a module over the ring R/Γ^*. Also, according to [1, 2] we can consider the fundamental relation θ on hypermodules as follows: $x\theta y$ if and only if $\exists\, n \in \mathbb{N}$, $\exists(m_1, \ldots, m_n) \in M^n$, $\exists(k_1, k_2, \ldots, k_n) \in \mathbb{N}^n$, $\exists \sigma \in \mathbb{S}_n$, $\exists(x_{i1}, x_{i2}, \ldots, x_{ik}) \in R^{k_i}$, $\exists \sigma_i \in \mathbb{S}_{n_i}$, $\exists \sigma_{ij} \in \mathbb{S}_{k_{ij}}$, such that $x \in \sum_{i=1}^{n} m'_i$, $m'_i = m_i$ or $m'_i = \sum_{j=1}^{n_i}(\prod_{k=1}^{k_{ij}} x_{ijk})m_i$ and $y \in \sum_{i=1}^{n} m'_{\sigma(i)}$, where $m'_{\sigma(i)} = m_{\sigma(i)}$ if $m'_i = m_i$; $m'_{\sigma(i)} = B_{\sigma(i)} m_{\sigma(i)}$ if $m'_i = \sum_{j=1}^{n_i}(\prod_{k=1}^{k_{ij}} x_{ijk})m_i$, such that $B_i = \sum_{j=1}^{n_i} A_{i\sigma_i(j)}$ and $A_{ij} = \prod_{k=1}^{k_{ij}} x_{ij\sigma_{ij}(k)}$. Then, the (abelian group) M/θ^* is an R/α^*- module, where R/α^* is a commutative ring.

Theorem 3 (1) *The module* P/ϵ^*_P *over the ring* T/Γ^*_T *is isomorphic to the module* N *over the ring* R.

(2) *The module* P/θ^*_P *over the ring* T/α^*_T *is isomorphic to the module* N/θ^*_N *over the ring* R/α^*_R.

Proof. (1) Let $x \in P$. Then, there exists $n \in N$ such that $x \in A_n$. If $x\ \epsilon\ y$, then there exist $r_{ijk} \in T$ and $m_k \in P$ such that $\{x, y\} \subseteq \sum_{k=1}^{l} m'_k$, where $m'_k = m_k$

or $m'_k = (\sum \prod r_{ijk})m_k$. From the definition of the hyperoperations $\bar{\oplus}$, $\bar{\bullet}$, $\ddagger$ and $\odot$ it follows that $\sum_{k=1}^{l} m'_k = A_m$ for some $m \in N$. Hence, $x \in A_n \cap A_m$ and so $m = n$. Then, $y \in A_n$. Now, if $y \in \epsilon^*(x)$, then there exist $z_1, z_2, \ldots, z_s \in P$ such that $x \ \epsilon \ z_1 \ \epsilon \ z_2 \ldots z_s \ \epsilon \ y$. From $x \ \epsilon \ z_1$ and $x \in A_n$, we have $z_1 \in A_n$, so $z_2 \in A_n$ and finally we obtain $y \in A_n$. Therefore, $\epsilon^*(x) \subseteq A_n$.

Conversely, suppose that $y \in A_n$. If $n = 0$ then set $v \in A_m$ and $w \in A_{-m}$, where $m \in N - \{0\}$. Then, $\{x, y\} \subseteq A_0 = v\bar{\oplus}w$. Thus, $y \in \epsilon^*(x)$. If $n \neq 0$, then we consider $v \in A_n$ and $w \in A_0$, so $\{x, y\} \subseteq A_n = v\bar{\oplus}w$. Therefore, $y \in \epsilon^*(x)$ and consequently $A_n \subseteq \epsilon^*(x)$.

Finally, we consider the maps $\Psi : P/\epsilon^* \to N$ by $\epsilon^*(x) \to n$, where $x \in A_n$, and $\psi : T/\Gamma^* \to R$ by $\Gamma^*(r) \to i$, where $r \in B_i$. Then, Ψ is a module isomorphism and ψ is a ring isomorphism. $\square$

The following theorem from [16] gives us a connection between the fundamental relations of β^* and ϵ^*.

Theorem 4 [16]. *If for any $a \in T$ and $p \in P$, there exists $u \in P$ such that $\Gamma^*(a).\beta^*(p) \subseteq \beta^*(u)$, then $\epsilon = \beta$.*

Also, in a similar way we have:

Theorem 5 *If for any $a \in T$ and $p \in P$, there exists $u \in P$ such that $\alpha^*(a).\gamma^*(p) \subseteq \gamma^*(u)$, then $\theta = \gamma$.*

Corollary 1 *Let for any $a \in T$ and $p \in P$, there exists $u \in P$ such that $\Gamma^*(a).\beta^*(p) \subseteq \beta^*(u)$.*

(1) *The module P/β^*_P over the ring T/Γ^*_T is isomorphic to the module N over the ring R.*

(2) *The module P/γ^*_P over the ring T/α^*_T is isomorphic to the module N/θ^*_N over the ring R/α^*_R.*

By the proof of Theorem 3, we have:

Theorem 6 *For every $m_1, \ldots, m_k \in P$ and $r_{ijk} \in T$ where $k \geq 1$, one of the following cases is verified.*

(1) *There exists $t \in N$ such that $\sum_{l=1}^{k} m'_l = A_t$, where $m'_l = m_l$ or $m'_l = (\sum \prod r_{ijl})m_l$.*

(2) *There exists $B \in \wp^*(M)$ such that $\sum_{l=1}^{l} m'_l = B$, where $m'_l = m_l$ or $m'_l = (\sum \prod r_{ijl})m_l$.*

Proof. Let $m_1,\dots,m_k \in P$ and $r_{ijk} \in T$. Set $m_l' = m_l$ or $m_l' = (\sum\prod r_{ijl})m_l$. Since P is a hypermodule so $\sum_{l=1}^{k} m_l' \subseteq P$. Let $m_l \in A_{n_l}$ and $r_{ijl} \in B_{t_{ijl}}$. If $n_l \neq 0_N$ or $t_{ijl} \neq 0_R$ then by definition of the (M,N)-hypermodule over the (R,S)-hyperring, there exists $t \in N$ such that $\sum_{l=1}^{k} m_l' = A_t$. Else, for every l, i and j, we have$m_l \in A_{0_N} = M$ and $r_{ijl} \in B_{0_R} = S$. Therefore, $\sum_{l=1}^{k} m_l' \subseteq A_{0_N} = M$ and so there exists $B \in \wp^*(M)$ such that $\sum_{l=1}^{k} m_l' = B$. □

Theorem 7 (1) *For every* $x \in N$ *and* $a \in A_x$, $C_\epsilon(a) = A_i$.

(2) $w_P = M$.

Proof.

(1) By Theorem 6, it follows that for any $i \in N$, A_i is a complete part. On the other hand for any $i \in N$, there exists $(y,z) \in P^2$ such that $y\bar{\oplus}z = A_{y\uplus z} = A_i$.

(2) It obtains immediately from (1).

□

Theorem 8 *Let* $(P,\bar{\oplus},\bar{\bullet})$ *be an* (M,N)*-hypermodule over an* (R,S)*-hyperring* $(T,\ddagger,\odot)$*. Then* $\bar{\oplus}$ *is commutative if and only if* $\oplus$ *is commutative.*

Proof. It is straightforward. □

Lemma 1 *Let* $(P,\bar{\oplus},\bar{\bullet})$ *be an* (M,N)*-hypermodule over an* (R,S)*-hyperring* $(T,\ddagger,\odot)$*. Let* N *has an element* 1_N *such that for every* $r \in R$, $r * 1_N = r$*. Then,* $B_r \subseteq A_r$ *for every* $r \in R$ *if and only if for every* $t \in T$, $t \in t\bar{\bullet}u$*, for all* $u \in A_{1_N}$.

Proof. If N has an element 1_N such that $r * 1_N = r$, for every $r \in R$, then $R \subseteq N$ and so $B_0 \subseteq A_0$. Let $r \in R^*$, $t \in B_r$ and $u \in A_{1_N}$. Then, $t\bar{\bullet}u = A_{r*1_N} = A_r \supseteq B_r \ni t$.

Conversely, let $r \in R$ and $t \in B_r$. Then for every $u \in A_{1_N}$ we have $t \in t\bar{\bullet}u = A_{r*1_N} = A_r$ and so $B_r \subseteq A_r$. □

Let $(M,+,\circ)$ be a hypermodule over a hyperring $(R,+,\cdot)$ such that M has zero element 0. If $A \subseteq M$ and $B \subseteq R$ then we define the following notations:

$$(0 :_R A) = \{r \in R \mid \forall x \in A,\ r \circ x = 0\} = Ann_R(M),$$
$$(B :_M 0) = \{x \in M \mid \forall r \in B,\ r \circ x = 0\}.$$

A faithful module M is one where the action of each $r \neq 0_R$ in R on M is non-trivial (i.e., $rx \neq 0_N$ for some x in M). Equivalently, the annihilator of $M(Ann_R(M))$ is the zero hyperideal.

Lemma 2 *Let* $(M, +, \circ)$ *be a hypermodule over a hyperring* $(R, +, \cdot)$ *such that* M *has zero element* 0.

(1) *If* A *be a non-empty subset of* M, *then* $(0 :_R A)$ *is a hyperideal of* R.

(2) *If* B *be a non-empty subset of* R, *then* $(B :_M 0)$ *is a subhypermodule of* R.

Theorem 9 *Let* $(P, \bar{\oplus}, \bar{\bullet})$ *be an* (M, N)*-hypermodule over an* (R, S)*-hyperring* $(T, \ddagger, \odot)$.

(1) *Let* N *has an element* 1_N *such that* $r * 1_N = r$ *for every* $r \in R$, $t \in t\bar{\bullet}u$ *for every* $t \in T$ *and* $u \in A_{1_N}$. *Set* $E((P, \bar{\bullet})) = \{e \in P \mid \forall t \in T,\ t \in t\bar{\bullet}e\}$. *Then* $E((P, \bar{\bullet})) = \bigcup_{x \in (R:_N 0)} A_{x+1_N}$.

(2) *Let* R *has an element* 1_R *such that* $1_R * x = x$ *for every* $x \in N$, *and* $E((T, \bar{\bullet})) = \{\varepsilon \in T \mid \forall x \in P,\ \varepsilon \in \varepsilon\bar{\bullet}x\}$. *Then* $E((T, \bar{\bullet})) = \bigcup_{a \in Ann_R(N)} B_{a+1_R}$.

Proof. (1) By Lemma 1, we have $B_r \subseteq A_r$ for every $r \in R$. For every $t \in T$ there exists $r \in R$ such that $t \in B_r$. Now, let $u \in \bigcup_{x \in (R:_N 0)} A_{x+1_N}$. Then, there exists $z \in (R :_N 0)$ such that $u = A_{z+1_N}$. Thus, $t\bar{\bullet}u = B_r\bar{\bullet}A_{z+1_N} = A_{r*(z+1_N)} = A_r \supseteq B_r \ni t$. Therefore, $u \in E((P, \bar{\bullet}))$.

Conversely, suppose that $e \in E((P, \bar{\bullet}))$. Then, for every $t \in T$, $t \in t\bar{\bullet}e$. Let $t \in B_j$ and $e \in A_n$. Then, $t \in A_{j*n}$. But $t \in t\bar{\bullet}A_{1_N} = A_{j*1_N} = A_j$ so $A_j = A_{j*n}$. Therefore, $j = j*n$ for every $j \in R$. Thus, $j(n - 1_N) = 0_N$ and $n - 1_N \in (R :_N 0)$. Therefore, there exists $z \in (R :_N 0)$ such that $n = z + 1_N$.

(2) Let $t \in B_{1_R+a}$, where $a \in (0 :_R A)$. For all $x \in P$, if $x \in A_n$, then $t\bar{\bullet}x = A_{(1_R+a)*n} = A_{(1_R*n+a*n)} = A_{n+0} = A_n \ni x$. Hence, $t \in E((T, \bar{\bullet}))$. Conversely, suppose that $b \in E((T, \bar{\bullet}))$. Then, there exists $r \in R^*$, such that $b \in B_r$. Let $z \in B_{1_R}$. So, for every $n \in N$ and $x \in A_n$ we have $x \in z\bar{\bullet}x \in A_{1_R} * n = A_n$ and $x \in b\bar{\bullet}x \in A_{r*n}$. Therefore, for every $A_n \cap A_{r*n} \neq \emptyset$ and $r * n = n$ for every $n \in N$. Therefore, $(r - 1_R) * n = 0$ and $r - 1_R \in (0 :_R A)$ and there exists $a \in (0 :_R A)$ such that $r = 1_R + a$. □

Corollary 2 *Let* $(P, \bar{\oplus}, \bar{\bullet})$ *be an* (M, N)*-hypermodule over an* (R, S)*-hyperring* $(T, \ddagger, \odot)$. *If* N *has an element* 1_N *such that* $t \in t\bar{\bullet}1_N$ *for every* $t \in T$ *and* R *is a unitary ring, then* $E((P, \bar{\bullet})) = A_{1_N}$.

Corollary 3 *Let* $(P, \bar{\oplus}, \bar{\bullet})$ *be an* (M, N)*-hypermodule over an* (R, S)*-hyperring* $(T, \ddagger, \odot)$. *If* R *has an element* 1_R *such that* $1_R * x = x$ *for every* $x \in N$, *and* N *is a faithful module over the ring* R, *then* $E((T, \bar{\bullet})) = B_{1_R}$.

Lemma 3 *Let* $(M,+,\circ)$ *be a hypermodule over a commutative hyperring* $(R,+,\cdot)$ *and for every* $a\in R$ *set* $Q=a\circ M$. *Then* Q *is a subhypermodule.*

Proof. We show that $R\circ Q\subseteq Q$ and for all $q\in Q$, $Q+q=q+Q=Q$. Let $r\in R$ and $q\in Q$. Then, there exists $m\in M$ such that $q=a\circ m$. Now, we have $r\circ q=r\circ(a\circ m)=(r\cdot a)\circ m=(a\cdot r)\circ m=a\circ(r\circ m)\subseteq a\circ M=Q$. Also, $Q+q=a\circ M+a\circ m=a\circ(M+m)=a\circ M=Q$ and $q+Q=a\circ m+a\circ M=a\circ(m+M)=a\circ M=Q$. Therefore, Q is a subhypermodule of M. □

Theorem 10 *Let* $(P,\bar{\oplus},\bar{\bullet})$ *be an* (M,N)*-hypermodule over an* (R,S)*-hyperring* $(T,\ddagger,\odot)$. *Set* $P_t=t\bar{\bullet}P$. *If* S *is a commutative hyperring, then* P_t *is a subhypermodule of* P. *Also, for every* $r\in R$, $P_r=0$, *for every* $t\in(0:_P t)$, $P_t=0$.

Lemma 4 [8]. *Let* $(R,+,\cdot)$ *be a hyperring and let* $x\in R$. *Let* $I=K\cdot x$. *Then* I *is a left hyperideal of* R *if and only if for every* $y\in I$, $I\cdot y=y\cdot I=I$.

Corollary 4 *Let* $(R,+,\cdot)$ *be a commutative hyperring and let* $x\in R$. *If we set* $I=K\cdot x$ *then* I *is a hyperideal of* R *if and only if for every* $y\in I$, $I\cdot y=I$. *Moreover,* $(I,+,\circ)$ *is a hyperring.*

Theorem 11 [8] *Let* $(T,\ddagger,\odot)$ *be an* (R,S)*-hyperring and* S *be commutative. Then* $T_t=T\odot t$ *is a hyperideal of* T *and* $(T_t,\ddagger,\odot)$ *is a commutative hyperring.*

Lemma 5 *Let* $(M,+,\circ)$ *be a hypermodule over a commutative hyperring* $(R,+,\cdot)$ *and for every* $a,b\in R$ *set* $M_a=a\circ M$ *and* $R_b=R\cdot b$. *Then* M_a *is a hypermodule over a hyperring* R_b *if and only if for every* $x\in R_b$, $R_b\cdot x=R_b$.

Theorem 12 *Let* $(P,\bar{\oplus},\bar{\bullet})$ *be an* (M,N)*-hypermodule over an* (R,S)*-hyperring* $(T,\ddagger,\odot)$ *and let* $a,b\in T$. *If* S *is a commutative hyperring then* $(a\bar{\bullet}P,\bar{\oplus},\bar{\bullet})$ *is a hypermodule over a hyperring* $(T\odot b,\ddagger,\odot)$.

Proof. It obtains from Theorems 10 and 11 and Lemma 5. □

Example 2 *Let* $(M,+,\circ)$ *be a hypermodule over a commutative hyperring* $(R,+,\cdot)$ *and for every* $a\in R$ *set* $Q=a\circ M$, *and* $Q+q\neq Q$.

Lemma 6 *Let* $(P,\bar{\oplus},\bar{\bullet})$ *be an* (M,N)*-hypermodule over an* (R,S)*-hyperring* $(T,\ddagger,\odot)$. *Then* S *has a weak neutral element if and only if* P *has a weak neutral element.*

Proof. Let $e \in P$ be a weak neutral element of P. So for every $p \in P$ we have $p \in e\bar{\oplus}p \cap p\bar{\oplus}e$. Let $e \in A_n$. We show that $n = 0_N$. If $n \neq 0_N$, then $e \in e\bar{\oplus}e = A_{n+n}$ which implies that $e \in A_n \cap A_{n+n}$ and $A_n = A_{n+n}$. Thus, $n + n = n$ and $n = 0_N$. Therefore, $e \in A_{0_N} = M$.

Conversely, let $e \in M$ be a weak neutral element of M. Then, for every $p \in A_n$ when $n \neq 0_N$, we have $p\bar{\oplus}e \in A_{n+0_N} = A_n$ and so $p \in p\bar{\oplus}e$. In a similar way, we obtain $p \in e\bar{\oplus}p$. Therefore, e is a weak neutral element of P. □

Theorem 13 *Let* $(P, \bar{\oplus}, \bar{\bullet})$ *be an* (M, N)*-hypermodule over an* (R, S)*-hyperring* $(T, \ddagger, \odot)$. *If* R *is a field and* N *is a unitary* R*-module, then* P/ϵ_P^* *is a hypervector space over the field* T/Γ_T^*.

Proof. Since R is a field, T is a hyperfield. Since N is a unitary R-module, P/ϵ_P^* is a unitary T/Γ_T^*-module. Therefore, P/ϵ_P^* is a hypervector space over the field T/Γ_T^*. □

Let us denote $P_{\bar{\oplus}}$ and $P_{\bar{\bullet}}$, the sets of scalars of the (M, N)-hypermodule over the (R, S)-hyperring with respect to the hyperoperations $\bar{\oplus}$ and $\bar{\bullet}$, respectively, i.e., $P_{\bar{\oplus}} = \{u \in P \mid \text{card}(u\bar{\oplus}x) = 1, \text{ for all } x \in P\}$ and $P_{\bar{\bullet}} = \{u \in P \mid \text{card}(t\bar{\bullet}u) = 1,$ for all $t \in T\}$.

Theorem 14 *Let* $(P, \bar{\oplus}, \bar{\bullet})$ *be an* (M, N)*-hypermodule over an* (R, S)*-hyperring* $(T, \ddagger, \odot)$. *Then:*

(1) *If* $P_{\bar{\oplus}} \cap (P - M) \neq \emptyset$ *and* $P_{\bar{\oplus}} \cap (P - M) \neq \emptyset$, *then* $\bar{\oplus}$ *and* $\bar{\bullet}$ *are operations.*

(2) *If* $P_{\bar{\oplus}} \neq \emptyset$ *and* $P_{\bar{\oplus}} \cap (P - M) = \emptyset$, *then* $\text{card}A_n = 1$ *for all* $n \in N - \{0_N\}$.

Proof. (1) Let $u \in P_{\bar{\oplus}} \cap (P - M)$, i.e., $u \in A_n \neq N$. Then, for all $m \in N$, A_m is singleton, because by taking $y \in A_{m-1_N}$, we get the singleton $u\bar{\oplus}y = A_m$. Consequently, $\bar{\bullet}$ and $\bar{\oplus}$ are operations.

(2) By hypothesis, we have $P_{\bar{\oplus}} \subseteq M$. Moreover, if $u \in P_{\bar{\oplus}}$, then $u \in A_{0_N}$. For all $n \in N - \{0_N\}$, we consider $y \in A_n$. Then, we get the singleton $u\bar{\oplus}y = A_n$. □

An (M, N)-hypermodule over an (R, S)-hyperring $(T, \ddagger, \odot)$ is called a $(0, N)$-hypermodule, when M is a singleton set.

Theorem 15 *Let* $(P, \bar{\oplus}, \bar{\bullet})$ *be an* (M, N)*-hypermodule. We have*

(1) $P_{\bar{\bullet}} \neq \emptyset$, *if and only if* P *is a* $(0, N)$*-hypermodule.*

(2) *If* $P_{\bar{\bullet}} \cap A_n \neq \emptyset$, *for some* $n \in N$, *then* $A_n \subseteq P_{\bar{\bullet}}$ *and we have* $\text{card}A_k = 1$ *and* $A_k \subseteq P_{\bar{\bullet}}$, *for all* $k \in R * n$.

Proof. (1) Let $y \in P_{\bar{\bullet}}$. If $y \in M$, then for $t \in B_i \neq S$ we have $M = t\bar{\bullet}y$ is a singleton set. If $y \in P - M$, then for $s \in S = B_{0_R}$, we have $M = A_{0_N} = t\bar{\bullet}y$ is a singleton set. Hence, P is a $(0, N)$-hypermodule. Conversely, if M is a singleton set, then $P_{\bar{\bullet}} \neq \emptyset$.

(2) Let $P_{\bar{\bullet}} \cap A_n \neq \emptyset$, $n \in N$. If $n = 0_N$, then because of (1), M is a singleton set and so (2) is valid. We prove (2) for $n \in N - \{0_n\}$. Since, for all $x, y \in A_n$, $t \in T$, $t\bar{\bullet}x = t\bar{\bullet}y$, this implies that $A_n \subseteq P_{\bar{\bullet}}$. Moreover, if $x \in P_{\bar{\bullet}} \cap A_n$, then for all $r \in R$, we consider an arbitrary $t \in B_r$ and we have that $A_{r*n} = t\bar{\bullet}x$ is a singleton set. Hence, $\text{card}A_k = 1$, for all $k \in R * n$. Finally, let $A_k = \{x\}$, when $k \in R * n$. Then, for all $t \in B_r \neq S$, $t\bar{\bullet}x = A_{r*k}$ is a singleton set, because $r * k \in R * n$. Also, by (1), M is a singleton set and so $A_k \subseteq P_{\bar{\bullet}}$, when $k \in R * n$. □

Now, let $T_{\bar{\bullet}} = \{t \in T \mid \text{card}(t\bar{\bullet}u) = 1, \text{ for all } u \in P.\}$ Then, similar to Theorem 15, we have:

Theorem 16 *Let* $(P, \bar{\oplus}, \bar{\bullet})$ *be an* (M, N)*-hypermodule. Then:*

(1) $T_{\bar{\bullet}} \neq \emptyset$*, if and only if* P *is a* $(0, N)$*-hypermodule.*

(2) *If* $T_{\bar{\bullet}} \cap B_r \neq \emptyset$*, for some* $r \in R$*, then* $B_r \subseteq T_{\bar{\bullet}}$ *and for all* $k \in r * N$*, we have* $\text{card}A_k = 1$.

2 Quotient of an (M, N)-hypermodule over an (R, S)-hyperring

Proposition 1 *Let* $(P, \bar{\oplus}, \bar{\bullet})$ *be a canonical* (M, N)*-hypermodule over the Krasner* (R, S)*-hyperring* $(T, \ddagger, \odot)$ *and* $\emptyset \neq q \subseteq P$, $\emptyset \neq I \subseteq T$*. Then:*

(1) q *is a subhypermodule of* P *if and only if* $q = \bigcup_{n \in Q} A_n$*, where* Q *is a submodule of* $(N, \uplus, *)$.

(2) h *is a hyperideal of* P *if and only if* $h = \bigcup_{r \in H} B_r$*, where* H *is an ideal of* $(S, \dagger, \cdot)$.

Proof. (1) Let q be a subhypermodule of P. Then, $0 \in q$ and $r \in R^*$ which implies that $A_0 = r\bar{\bullet}0 \subseteq q$, so $M \subseteq q$. Let there exists $n \in N^*$ such that $q \cap A_n \neq \emptyset$ and $x \in q \cap A_n$. Then $-x \in g$ and $-x \in A_{-n}$ so we have $A_{-n} \in q$. Consequently, from the closure of $\bar{\oplus}$ in q, it follows $q = \bigcup_{n \in Q} A_n$, where Q is a subgroup of $(N, \uplus, *)$. Now, let $r \in R$. Then, $B_r\bar{\bullet}A_n = A_{r*n} \subseteq q$. Hence, $r * n \in Q$ and Q is a submodule of N. The converse is verified in a simple way.

(2) It obtains similar to the part (i) of Proposition 4.1 [14]. □

Proposition 2 *Let* $(P, \bar{\oplus}, \bar{\bullet})$ *be a canonical* (M, N)*-hypermodule over the Krasner* (R, S)*-hyperring* $(T, \ddagger, \odot)$*. Suppose that* G *be a submodule of* $(N, \uplus, *)$ *and* H *be an ideal of* $(R, +, \star)$*. If* $g = \bigcup_{n \in G} A_n$ *and* $h = \bigcup_{j \in H} B_j$*, then* $[P : g^*] \cong [N : G^*]$ *and* $[T : h^*] \cong [R : H^*]$*. In addition, the module* $[P : g^*]$ *over the ring* $[T : h^*]$ *is isomorphic to the module* $[M : G^*]$ *over the ring* $[R : H^*]$*.*

Proof. According to [17], $[P : g^*]$ is a hypermodule over the hyperring $[T : h^*]$ and Spartalis in [14], proved that $[T : h^*] \cong [R : H^*]$ and $\varphi : [T : h^*] \to [R : H^*]$ by $\varphi(h + t) = H + r$, is an isomorphism, where $t \in A_r$. Define the map $\phi : [P : g^*] \longrightarrow [N : G^*]$ by $g \bar{\oplus} a_i \mapsto G + i$. Then, ϕ is one to one and onto. Moreover, for every $m, n \in N$, $r, s \in R$, $x \in A_m$, $y \in A_n$, $t \in B_r$, we have $\phi((g \bar{\oplus} x) + (g \bar{\oplus} y)) = G + m + n = \phi(g \bar{\oplus} x) + \phi(g \bar{\oplus} y)$ and for any $t_r \in T$ we have $\phi((h + t) \circ (g + x)) = \phi(g + t \bar{\bullet} x) = G + rm = (H + r) \circ (G + m) = \varphi(h + t) \circ \phi(g + x)$. □

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On a class of analytic functions governed by subordination

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Abstract. The purpose of this paper is to introduce a class of functions $\mathcal{F}_\lambda$, $\lambda \in [0,1]$, consisting of analytic functions f normalized by $f(0) = f'(0) - 1 = 0$ in the open unit disk $\mathbb{U}$ which satisfies the subordination condition that

$$zf'(z)/\{(1-\lambda)f(z) + \lambda z\} \prec q(z), \quad z \in \mathbb{U},$$

where $q(z) = \sqrt{1+z^2} + z$. Some basic properties (including the radius of convexity) are obtained for this class of functions.

1 Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$ in the complex plane $\mathbb{C}$. Also, let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ comprising of functions f normalized by $f(0) = 0$, $f'(0) = 1$, and let $\mathcal{S} \subset \mathcal{A}$ denote the class of functions which are univalent in $\mathbb{U}$. We say that an analytic function f is subordinate to an analytic function g, and write $f(z) \prec g(z)$, if and only if there exists a function ω, analytic in $\mathbb{U}$ such that $\omega(0) = 0$, $|\omega(z)| < 1$ for

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$|z| < 1$ and $f(z) = g(\omega(z))$. In particular, if g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(|z| < 1) \subset g(|z| < 1). \tag{1}$$

Let a function f be analytic univalent in the unit disc $\mathbb{U} = \{z : \ |z| < 1\}$ on the complex plane $\mathbb{C}$ with the normalization $f(0) = 0$, then f maps $\mathbb{U}$ onto a starlike domain with respect to $w_0 = 0$ if and only if

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad (z \in \mathbb{U}). \tag{2}$$

It is well known that if an analytic function f satisfies (2) and $f(0) = 0$, $f'(0) \neq 0$, then f is univalent and starlike in $\mathbb{U}$.

A set E is said to be convex if and only if it is starlike with respect to each of its points, that is if and only if the linear segment joining any two points of E lies entirely in E. Let f be analytic and univalent in $\mathbb{U}_r = \{z : |z| < r \leq 1\}$. Then f maps $\mathbb{U}_r$ onto a convex domain E if and only if

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \quad (z \in \mathbb{U}_r). \tag{3}$$

If $r = 1$, then the function f is said to be convex in $\mathbb{U}$ (or briefly convex). The set of all functions $f \in \mathcal{A}$ that are starlike univalent in $\mathbb{U}$ will be denoted by $\mathcal{S}^*$ and the set of all functions $f \in \mathcal{A}$ that are convex univalent in $\mathbb{U}$ by $\mathcal{K}$.

Definition. For given $\lambda \in [0, 1]$, let $\mathcal{F}_\lambda$ denote the class of analytic functions f in the unit disc $\mathbb{U}$ normalized by $f(0) = f'(0) - 1 = 0$ and satisfying the condition that

$$\frac{zf'(z)}{(1-\lambda)f(z) + \lambda z} \prec \sqrt{1 + z^2} + z =: q(z), \quad z \in \mathbb{U}, \tag{4}$$

where the branch of the square root is chosen to be $q(0) = 1$.

We note that for $\lambda = 0$ in (4), we have the class $\mathcal{F}_0$ which connects a starlike function with the function $q(z)$ by means of a subordination and is defined by

$$\mathcal{F}_0 = \{f \in \mathcal{A} : zf'(z)/f(z) \prec \sqrt{1 + z^2} + z, \quad z \in \mathbb{U}\}. \tag{5}$$

Also, for $\lambda = 1$ in (4), we obtain a class $\mathcal{F}_1$ which depicts a subordination relationship between the function $f'(z)$ with the function $q(z)$ and this class is defined by

$$\mathcal{F}_1 = \{f \in \mathcal{A} : f'(z) \prec \sqrt{1 + z^2} + z, \quad z \in \mathbb{U}\}. \tag{6}$$

The function $w(z) = \sqrt{1+z}$ maps $\mathbb{U}$ onto a set bounded by Bernoulli lemniscate, and the class of functions $f \in \mathcal{A}$ such that $zf'(z)/f(z) \prec \sqrt{1+z}$ was considered in [14], while $zf'(z)/f(z) \prec \sqrt{1+cz}$ was considered in [1]. This way the well known class of k-starlike functions were seen to be connected with certain conic domains. For some recent results for k-starlike functions, we refer to [8, 11, 13, 15]. Certain function classes were also considered in recent papers [2, 3, 4, 5, 7, 12] which were defined by means of the subordination that $zf'(z)/f(z) \prec \widehat{q}(z)$, where $\widehat{q}(z)$ was not univalent. For a unified treatment of some special classes of univalent functions we refer to [10] (see also [16]).

2 Auxiliary results

Lemma 1 *The function*

$$h(z) = \frac{z}{\sqrt{1+z^2}} \tag{7}$$

is convex in $\mathbb{U}_r$, *where* $r = \sqrt{2}/2$..

Proof. Using (7), we have

$$1 + \frac{zh''(z)}{h'(z)} = \frac{1-2z^2}{1+z^2},$$

hence

$$\mathfrak{Re}\left\{1 + \frac{zh''(z)}{h'(z)}\right\} > 0 \quad \text{for} \quad |z| < \frac{\sqrt{2}}{2}$$

and thus $h(z)$ is convex in $\mathbb{U}_r$, where $r \le \sqrt{2}/2$. □

Corollary 1 *If* $r \le \sqrt{2}/2$ *and* $h(z) = z/\sqrt{1+z^2}$, *then we have*

$$\min_{|z|\le r} \{\mathfrak{Re}\{h(z)\}\} = \frac{-r}{\sqrt{1+r^2}}.$$

Proof. By Lemma 1, the function $h(z)$ is convex in $\mathbb{U}_r$, where $r \le \sqrt{2}/2$ and $h(\mathbb{U}_r)$ is symmetric with respect to the real axis. Since the function $h(z)$ is real for real z, therefore, $\mathfrak{Re}\{h(z)\}$ attains its extremal values at $-r$ and r, which proves the corollary. □

Lemma 2 *The function*

$$q(z) = \sqrt{1+z^2} + z$$

is convex in $\mathbb{U}_r$, *where* r *is at least* $\sqrt{2}/2$.

Proof. By elementary calculations, it can easily be shown that $q(z)$ is univalent in the unit disc. For the proof that $q(z)$ is convex, we use (3). Thus, we obtain

$$\begin{aligned} 1 + \frac{zq''(z)}{q'(z)} &= \frac{1}{1+z^2} + \frac{z}{\sqrt{1+z^2}} \\ &= \frac{1}{1+z^2} + h(z), \end{aligned}$$

where $h(z)$ is given in (7). By Corollary 1, we have

$$\begin{aligned} \min_{|z|\le\sqrt{2}/2}\left\{\Re\left\{1+\frac{zq''(z)}{q'(z)}\right\}\right\} &\ge \min_{0<x\le\sqrt{2}/2}\left\{\Re\left\{\frac{1}{1+x^2}-\frac{x}{\sqrt{1+x^2}}\right\}\right\} \\ &= \frac{2-\sqrt{3}}{3} > 0, \end{aligned} \tag{8}$$

because

$$t(x) = \frac{1}{1+x^2} - \frac{x}{\sqrt{1+x^2}}$$

decreases in $\left[0, \sqrt{(\sqrt{5}-1)/2}\right]$ from $t(0) = 1$ to $t\left(\sqrt{(\sqrt{5}-1)/2}\right) = 0$, so that $t(\sqrt{2}/2) = (2-\sqrt{3})/3$ is the smallest value of $t(x)$ for $0 < x \le \sqrt{2}/2$. Therefore, in view of (8), the function $q(z) = \sqrt{1+z^2} + z$ is convex in $\mathbb{U}_r$, where r is at least $\sqrt{2}/2$.

□

Corollary 2 *If* $r \le \sqrt{2}/2$ *and* $q(z) = \sqrt{1+z^2} + z$, *then we have*

$$\min_{|z|\le r}\{\Re\{q(z)\}\} = \sqrt{1+r^2} - r.$$

Proof. By Lemma 2, the function $q(z)$ is convex in $\mathbb{U}_r$, where $r \le \sqrt{2}/2$ and $h(\mathbb{U}_r)$ is symmetric with respect to the real axis. Therefore, $q(z)$ is real for real z, and thus, $\Re\{q(z)\}$ attains its extremal values at $-r$ and r. □

Lemma 3 *The function* $q(z) = \sqrt{1+z^2} + z$ *satisfies*

$$\Re\{q(z)\} > 0 \tag{9}$$

in $\mathbb{U}$.

Proof. Let $z = e^{it}$, $t \in [0, 2\pi)$. We assume that $\arg\{e^{2it} + 1\} \in (-\pi, \pi]$. It follows that $|e^{2it} + 1| = |2\cos t|$ and

$$\arg(e^{2it}+1) = \begin{cases} t & \text{for} \quad t \in [0, \pi/2), \\ t - \pi & \text{for} \quad t \in (\pi/2, 3\pi/2), \\ t - 2\pi & \text{for} \quad t \in (3\pi/2, 2\pi). \end{cases}$$

Therefore, we infer that

$$\begin{aligned} & e^{it} + \sqrt{e^{2it}+1} \\ = & \begin{cases} \cos t + i \sin t + \sqrt{|2\cos t|}(\cos t/2 + i \sin t/2) & \text{for} \quad t \in [0, \pi/2), \\ i & \text{for} \quad t = \pi/2, \\ \cos t + i \sin t + \sqrt{|2\cos t|}(\sin t/2 - i \cos t/2) & \text{for} \quad t \in (\pi/2, 3\pi/2), \\ -i & \text{for} \quad t = 3\pi/2, \\ \cos t + i \sin t + \sqrt{|2\cos t|}(-\cos t/2 - i \sin t/2) & \text{for} \quad t \in (3\pi/2, 2\pi). \end{cases} \end{aligned}$$

Now some simple calculations show that $\mathfrak{Re}\left\{e^{it} + \sqrt{e^{2it}+1}\right\} = 0$ if and only if $t = \pi/2$ or if $t = 3\pi/2$, which implies that $\mathfrak{Re}\{q(z)\} > 0$ in $\mathbb{U}$ (see Fig.1 below). □

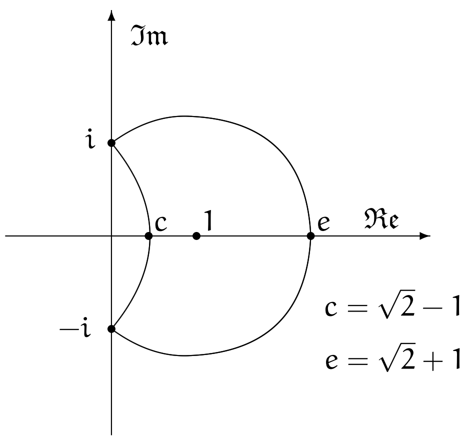

Figure 1. $q(e^{it})$.

3 Basic properties of the class $\mathcal{F}_\lambda$

Corollary 3 *Let* $n \geq 2$ *be a given positive integer. Then the function*

$$f_{n,a}(z) = z + az^n \quad (z \in \mathbb{U})$$

is in the class $\mathcal{F}_\lambda$ *if and only if*

$$|a| \le \frac{2-\sqrt{2}}{n+(1-\sqrt{2})(1-\lambda)}. \tag{10}$$

Proof. The function

$$F_{n,a}(z) := \frac{zf'_{n,a}(z)}{(1-\lambda)f_{n,a}(z)+\lambda z} = \frac{1+naz^{n-1}}{1+(1-\lambda)az^{n-1}}$$

maps $\mathbb{U}$ onto the disc $F_{n,a}(\mathbb{U})$ that is symmetric with respect to the real axis. For

$$F_{n,a}(z) \prec \sqrt{1+z^2}+z, \tag{11}$$

it is necessary that $F_{n,a}(z) \neq 0$, and so we may assume that $|na| < 1$. We have then

$$\frac{1-n|a|}{1-(1-\lambda)|a|} < \Re\{F_{n,a}(z)\} < \frac{1+n|a|}{1+(1-\lambda)|a|}.$$

It follows by applying a geometric interpretation of the subordination condition that (11) is equivalent to

$$\sqrt{2}-1 \le \frac{1-n|a|}{1-(1-\lambda)|a|} \quad \text{and} \quad \frac{1+n|a|}{1+(1-\lambda)|a|} \le \sqrt{2}+1. \tag{12}$$

Since the second inequality in (12) above is weaker, the desired inequality (10) readily follows from the first inequality of (12).

□

Theorem 1 *Let the function* f *defined by*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{U})$$

belong to the class $\mathcal{F}_\lambda$*, then*

$$|a_2| \le 1/(1+\lambda) \tag{13}$$

and

$$|a_3| \le \begin{cases} \frac{3-\lambda}{2(1+\lambda)(2+\lambda)} & \text{for} \quad \lambda \in [0, 1/3], \\ \frac{1}{2+\lambda} & \text{for} \quad \lambda \in (1/3, 1]. \end{cases} \tag{14}$$

Furthermore,

$$|a_4| \le \frac{5+9\lambda-2\lambda^2+2\left|2\lambda^2+11\lambda-1\right|}{2(1+\lambda)(2+\lambda)(3+\lambda)}. \tag{15}$$

Proof. Since the function f defined by (1) belongs to the class $\mathcal{F}_\lambda$, therefore from (4), we have

$$zf'(z) - \left\{z + (1-\lambda)\sum_{n=2}^{\infty} a_n z^n\right\}\omega(z) = \left\{z + (1-\lambda)\sum_{n=2}^{\infty} a_n z^n\right\}\sqrt{\omega^2(z)+1},$$

where ω is such that $\omega(0) = 0$ and $|\omega(z)| < 1$ for $|z| < 1$. Let us denote the function $\omega(z)$ by

$$\omega(z) = \sum_{k=1}^{\infty} c_k z^k. \tag{16}$$

Thus, (16) readily gives

$$\sqrt{\omega^2(z)+1} = 1 + \frac{1}{2}c_1^2 z^2 + c_1c_2z^3 + \left(c_1c_3 + \frac{1}{2}c_2^2 - \frac{1}{8}c_1^2\right)z^4 + \cdots .$$

Moreover,

$$\begin{aligned} &\left\{z + (1-\lambda)\sum_{n=2}^{\infty} a_n z^n\right\}\sqrt{\omega^2(z)+1} \\ &\quad = z + (1-\lambda)a_2z^2 + \left(\frac{1}{2}c_1^2 + (1-\lambda)a_3\right)z^3 \\ &\qquad + \left(c_1c_2 + \frac{1-\lambda}{2}c_1^2a_2 + (1-\lambda)a_4\right)z^4 + \cdots \end{aligned} \tag{17}$$

and

$$\begin{aligned} &zf'(z) - \left\{z + (1-\lambda)\sum_{n=2}^{\infty} a_n z^n\right\}\omega(z) \\ &\quad = z + (2a_2 - c_1)z^2 + (3a_3 - (1-\lambda)c_1a_2 - c_2)z^3 \\ &\qquad + (4a_4 - (1-\lambda)[c_1a_3 - c_2a_2] - c_3)z^4 + \cdots . \end{aligned} \tag{18}$$

Equating now the second, third and fourth coefficients in (17) and (18), we have

(i) $(1-\lambda)a_2 = 2a_2 - c_1$,

(ii) $\frac{1}{2}c_1^2 + (1-\lambda)a_3 = 3a_3 - (1-\lambda)c_1a_2 - c_2$,

(iii) $c_1c_2 + \frac{1-\lambda}{2}c_1^2a_2 + (1-\lambda)a_4 = 4a_4 - (1-\lambda)[c_1a_3 + c_2a_2] - c_3$.

From (i), we get

$$a_2 = \frac{c_1}{1+\lambda}. \tag{19}$$

It is well known that the coefficients of the bounded function $\omega(z)$ satisfies the inequality that $|c_k| \le 1$, $(k = 1, 2, 3, \dots)$, so from (19), we have the first inequality that $|a_2| \le 1/(1+\lambda)$. Now, from (ii) and (13), we obtain that

$$\begin{aligned} (2+\lambda)a_3 &= \frac{1}{2}c_1^2 + (1-\lambda)c_1 a_2 + c_2 \\ &= \frac{1}{2}c_1^2 + \frac{1-\lambda}{1+\lambda}c_1^2 + c_2 \\ &= c_2 + \frac{3-\lambda}{2(1+\lambda)}c_1^2. \end{aligned} \tag{20}$$

Also,

$$\lambda \in [0, 1/3] \;\Rightarrow\; \left|\frac{3-\lambda}{2(1+\lambda)}\right| \ge 1 \text{ and } \lambda \in (1/3, 1] \;\Rightarrow\; \left|\frac{3-\lambda}{2(1+\lambda)}\right| < 1.$$

Therefore, by using the estimate (see [9]) that if $\omega(z)$ has the form (16), then

$$|c_2 - \mu c_1^2| \le \max\{1, |\mu|\}, \quad \text{for all} \;\; \mu \in \mathbb{C},$$

we obtain (14). Also, from (i)-(iii) and (19)-(20), we find that

$$\begin{aligned} |(3+\lambda)a_4| &= \left|(1-\lambda)[c_1a_3 + c_2a_2] + c_3 + c_1c_2 + \frac{1-\lambda}{2}c_1^2a_2\right| \\ &= \left|\frac{5(1-\lambda)}{2(1+\lambda)(2+\lambda)}c_1^3 + \frac{5+2\lambda-\lambda^2}{(1+\lambda)(2+\lambda)}c_1c_2 + c_3\right| \\ &= \left|\frac{5(1-\lambda)}{2(1+\lambda)(2+\lambda)}\left(c_1^3 + 2c_1c_2 + c_3\right) + \frac{7\lambda-\lambda^2}{(1+\lambda)(2+\lambda)}c_1c_2 \right. \\ &\quad \left. + \left(1 - \frac{5(1-\lambda)}{2(1+\lambda)(2+\lambda)}\right)c_3\right| \\ &\le \frac{5(1-\lambda)}{2(1+\lambda)(2+\lambda)}\left|c_1^3 + 2c_1c_2 + c_3\right| + \frac{(7\lambda-\lambda^2)|c_1c_2|}{(1+\lambda)(2+\lambda)} \\ &\quad + \frac{\left|2\lambda^2 + 11\lambda - 1\right| |c_3|}{2(1+\lambda)(2+\lambda)}. \end{aligned} \tag{21}$$

We next use some properties of c_k involved in (16). It is known that the function $p(z)$ given by

$$\frac{1+\omega(z)}{1-\omega(z)} = 1 + p_1z + p_2z^2 + \cdots =: p(z) \tag{22}$$

defines a Caratheodory function with the property that $\mathfrak{Re}\{p(z)\} > 0$ in $\mathbb{U}$ and that $|p_k| \leq 2$ $(k = 1, 2, 3, \ldots)$. Equating of the coefficients in (22) yields that

$$p_2 = 2(c_1^2 + c_2)$$

and

$$p_3 = 2(c_1^3 + 2c_1c_2 + c_3).$$

Hence $|c_1^2 + c_2| \leq 1$ and

$$|c_1^3 + 2c_1c_2 + c_3| \leq 1. \tag{23}$$

By applying (21) and (23), we find that

$$|(3+\lambda)a_4| \leq \frac{5(1-\lambda)}{2(1+\lambda)(2+\lambda)} + \frac{7\lambda - \lambda^2}{(1+\lambda)(2+\lambda)} + \frac{\left|2\lambda^2 + 11\lambda - 1\right|}{2(1+\lambda)(2+\lambda)},$$

which gives (15). □

4 Some consequences and special cases

It may be observed from (4), (5) and (9) of Lemma 3 that

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad (z \in \mathbb{U})$$

for $f \in \mathcal{F}_0$, hence f is univalent starlike with respect to the origin, and this leads to the following result.

Corollary 4 *$\mathcal{F}_0 \subset \mathcal{S}^*$.*

In view of (5) and (6), we can deduce the coefficient estimates for functions belonging to the classes $\mathcal{F}_0$ and $\mathcal{F}_1$ from Theorem 3.1. These results are easy to obtain and we skip mentioning here their details.

Lastly, we prove the radius of convexity of a function belonging to the class $\mathcal{F}_0$.

Theorem 2 *If $f \in \mathcal{F}_0$, then f is convex in $\mathbb{U}_r$, where r is at least*

$$\sqrt{(5-\sqrt{13})/2} = 0.482\ldots.$$

Proof. Assume that $|z| < \sqrt{2}/2$. Let $f \in \mathcal{S}^*(q)$, then in view of (4), we have

$$f'(z)/f(z) = \sqrt{1+\omega^2(z)} + \omega(z),$$

where ω satisfies $\omega(0) = 0$, $|\omega(z)| < 1$ for $|z| < 1$, and by Schwarz Lemma, ω satisfies $|\omega(re^{i\varphi})| < r$. Let us recall that ([see [6], Vol. II, p. 77])

$$|\omega'(z)| \le \frac{1-|\omega(z)|^2}{1-|z|^2}. \tag{24}$$

Differentiating $zf'(z)/f(z) = \sqrt{1+\omega^2(z)} + \omega(z)$ and using (24), we obtain

$$\mathfrak{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} = \mathfrak{Re}\left\{\sqrt{1+\omega^2(z)} + \omega(z) + \frac{z\omega'(z)}{\sqrt{1+\omega^2(z)}}\right\}. \tag{25}$$

Applying now Corollary 2, we get

$$\min_{|z|<\sqrt{2}/2}\left\{\mathfrak{Re}\left\{\sqrt{1+\omega^2(z)} + \omega(z)\right\}\right\} = \sqrt{1+r^2} - r. \tag{26}$$

Hence, from (25) and (26), we have

$$\begin{aligned}
\mathfrak{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} &\ge \sqrt{1+r^2} - r - \left|\frac{z\omega'(z)}{\sqrt{1+\omega^2(z)}}\right| \\
&\ge \sqrt{1+r^2} - r - r\frac{1-|\omega^2(z)|}{1-|z^2|}\frac{1}{|\sqrt{1+\omega^2(z)}|} \\
&\ge \sqrt{1+r^2} - r - r\frac{1-|\omega^2(z)|}{1-|z^2|}\frac{1}{\sqrt{1-|\omega^2(z)|}} \\
&= \sqrt{1+r^2} - r - r\frac{\sqrt{1-|\omega^2(z)|}}{1-r^2} \\
&> \sqrt{1+r^2} - r - \frac{r}{1-r^2}.
\end{aligned}$$

Solving in $[0, \sqrt{2}/2]$ the inequality:

$$\sqrt{1+r^2} - r - \frac{r}{1-r^2} \ge 0,$$

we obtain that $3r^4 - 5r^2 + 1 \ge 0$, and so if $r \in \left[0, \sqrt{(5-\sqrt{13})/2}\right]$, then by (3) the function f is convex in $\mathbb{U}_r$. $\square$

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On some spaces of Cesàro sequences of fuzzy numbers associated with λ-convergence and Orlicz function

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Abstract. In the present paper we shall introduce some generalized difference Cesàro sequence spaces of fuzzy real numbers defined by Musielak-Orlicz function and λ-convergence. We make an effort to study some topological and algebraic properties of these sequence spaces. Furthermore, some inclusion relations between these sequence spaces are establish.

1 Introduction and preliminaries

Fuzzy set theory as compared to other mathematical theories is perhaps the most easily adaptable theory to practice. The main reason is that a fuzzy set has the property of relativity, variability and inexactness in the definition of its elements. Instead of defining an entity in calculus by assuming that its role is exactly known, we can use fuzzy sets to define the same entity by allowing possible deviations and inexactness in its role. This representation suits well the uncertainties encountered in practical life, which make fuzzy sets a valuable mathematical tool. The concepts of fuzzy sets and fuzzy set

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operations were first introduced by Zadeh [23] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming. Matloka [7] introduced bounded and convergent sequences of fuzzy numbers and studied some of their properties.

A fuzzy number is a fuzzy set on the real axis, i.e., a mapping $X : \mathbb{R}^n \to [0, 1]$ which satisfies the following four conditions:

1. X is normal, i.e., there exist an $x_0 \in \mathbb{R}^n$ such that $X(x_0) = 1$,
2. X is fuzzy convex, i.e., for $x, y \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1, X(\lambda x + (1 - \lambda)y) \geq \min[X(x), X(y)]$,
3. X is upper semi-continuous; i.e., if for each $\epsilon > 0$, $X^{-1}([0, a + \epsilon))$ for all $a \in [0, 1]$ is open in the usual topology of $\mathbb{R}^n$,
4. The closure of $\{x \in \mathbb{R}^n : X(x) > 0\}$, denoted by $[X]^0$, is compact.

Let $C(\mathbb{R}^n) = \{A \subset \mathbb{R}^n : A \text{ is compact and convex}\}$. The spaces $C(\mathbb{R}^n)$ has a linear structure induced by the operations

$$A + B = \{a + b, a \in A, b \in B\}$$

and

$$\lambda A = \{\lambda a : a \in A\}$$

for $A, B \in C(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$. The Hausdorff distance between A and B of $C(\mathbb{R}^n)$ is defined as

$$\delta_\infty(A, B) = \max\{\sup_{a\in A} \inf_{b\in B} \|a - b\|, \ \sup_{b\in B} \inf_{a\in A} \|a - b\|\},$$

where $\|.\|$ denotes the usual Euclidean norm in $\mathbb{R}^n$. It is well known that $(C(\mathbb{R}^n), \delta_\infty)$ is a complete (non separable) metric space.

For $0 < \alpha \leq 1$, the α-level set, $X^\alpha = \{x \in \mathbb{R}^n : X(x) \geq \alpha\}$ is a non empty compact convex, subset of $\mathbb{R}^n$, as is the support X^0. Let $L(\mathbb{R}^n)$ denote the set of all fuzzy numbers. The linear structure of $L(\mathbb{R}^n)$ induces addition $X+Y$ and scalar multiplication λX, $\lambda \in \mathbb{R}$, in terms of α-level sets by

$$[X + Y]^\alpha = [X]^\alpha + [Y]^\alpha$$

and

$$[\lambda X]^\alpha = \lambda[X]^\alpha.$$

Define for each $1 \leq q < \infty$

$$d_q(X,Y) = \left\{ \int_0^1 \delta_\infty(X^\alpha, Y^\alpha)^q d\alpha \right\}^{1/q}$$

and $d_\infty(X,Y) = \sup_{0<\alpha\leq 1} \delta_\infty(X^\alpha, Y^\alpha)$. Clearly $d_\infty(X,Y) = \lim_{q\to\infty} d_q(X,Y)$ with $d_q \leq d_r$ if $q \leq r$. Moreover $(L(\mathbb{R}^n), d_\infty)$ is a complete, separable and locally compact metric space. We denote by $w(f)$ the set of all sequences $X = (X_k)$ of fuzzy numbers. For more details about sequence spaces and fuzzy sequence spaces one can refer to [14, 15, 16, 17, 22].

Mursaleen and Noman (see [9, 10]) introduced the notion of λ-convergent and λ-bounded sequences as follows:
Let w be the set of all complex sequences $x = (x_k)$. Let $\lambda = (\lambda_k)_{k=1}^\infty$ be strictly increasing sequence of positive real numbers tending to infinity as

$$0 < \lambda_0 < \lambda_1 < \ldots. \text{ and } \lambda_k \to \infty \text{ as } k \to \infty$$

and said that a sequence $x = (x_k) \in w$ is λ-convergent to the number L, called the λ-limit of x if $\Lambda_m(x) \to L$ as $m \to \infty$, where

$$\Lambda_m(x) = \frac{1}{\lambda_m} \sum_{k=1}^m (\lambda_k - \lambda_{k-1}) x_k.$$

The sequence $x = (x_k) \in w$ is λ-bounded if $\sup_m |\Lambda_m(x)| < \infty$. It is well known [11] that if $\lim_m x_m = a$ in the ordinary sense of convergence, then

$$\lim_m \frac{1}{\lambda_m} \left(\sum_{k=1}^m (\lambda_k - \lambda_{k-1}) |x_k - a| \right) = 0.$$

This implies that

$$\lim_m |\Lambda_m(x) - a| = \lim_m |\frac{1}{\lambda_m} \sum_{k=1}^m (\lambda_k - \lambda_{k-1})(x_k - a)| = 0$$

which yields that $\lim_m \Lambda_m(x) = a$ and hence $x = (x_k) \in w$ is λ-convergent to a.

Definition 1 *A fuzzy real number* X *is a fuzzy set on* R*, i.e. a mapping* $X : R \to I(= [0,1])$ *associating each real number* t *with its grade of membership* $X(t)$.

Definition 2 *A fuzzy real number* X *is called convex if* $X(t) \geq X(s) \wedge X(r) = \min(X(s), X(r))$*, where* $s < t < r$*.*

Definition 3 *If there exists* $t_0 \in R$ *such that* $X(t_0) = 1$*, then the fuzzy real number* X *is called normal.*

Definition 4 *A fuzzy real number* X *is said to be upper semi continuous if for each* $\epsilon > 0$*,* $X^{-1}([0, a+\epsilon))$*, for all* $a \in I$*, is open in the usual topology of* R*.*

The class of all upper semi-continuous, normal, convex fuzzy real numbers is denoted by $R(I)$.

Definition 5 *For* $X \in R(I)$*, the* α*-level set* X^α*, for* $0 < \alpha \leq 1$ *is defined by* $X^\alpha = \{t \in R : X(t) \geq \alpha\}$*. The* 0*-level, i.e.* X^0 *is the closure of strong* 0*-cut, i.e.* $X^0 = cl\{t \in R : X(t) > 0\}$*.*

Definition 6 *The absolute value of* $X \in R(I)$*, i.e.* $|X|$ *is defined by*

$$|X|(t) = \begin{cases} \max\{X(t), X(-t)\}, & \text{for } t \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Definition 7 *For* $r \in R$*,* $\bar{r} \in R(I)$ *is defined as*

$$\bar{r}(t) = \begin{cases} 1, & \text{if } t = r \\ 0, & \text{if } t \neq r. \end{cases}$$

Definition 8 *The additive identity and multiplicative identity of* $R(I)$ *are denoted by* $\bar{0}$ *and* $\bar{1}$ *respectively. The zero sequence of fuzzy real numbers is denoted by* $\bar{\theta}$*.*

Definition 9 *Let* D *be the set of all closed bounded intervals* $X = [X^L, X^R]$*. Define* $d : D \times D \longrightarrow R$ *by* $d(X, Y) = \max\{|X^L - Y^L|, |X^R - Y^R|\}$*. Then clearly* (D, d) *is a complete metric space.*
Define $\bar{d} : R(I) \times R(I)$ *by* $\bar{d}(X, Y) = \sup_{0<\alpha\leq 1} d(X^\alpha, Y^\alpha)$*, for* $X, Y \in R(I)$*. Then it is well known that* $(R(I), \bar{d})$ *is a complete metric space.*

Definition 10 *A sequence* $X = (X_k)$ *of fuzzy numbers is said to be convergent to a fuzzy number* X_0*, if for every* $\epsilon > 0$ *there exists a positive integer* k_0 *such that* $\bar{d}(X_k, X_0) < \epsilon$*, for all* $k \geq k_0$*.*

Definition 11 *A sequence* $X = (X_k)$ *of fuzzy numbers is said to be bounded if the set* $\{X_k : k \in \mathbb{N}\}$ *of fuzzy numbers is bounded.*

Definition 12 *A sequence space* E *is said to be solid(or normal) if* $(Y_n) \in E$ *whenever* $(X_n) \in E$ *and* $|Y_n| \leq |X_n|$ *for all* $n \in \mathbb{N}$.

Definition 13 *Let* $X = (X_n)$ *be a sequence, then* $S(X)$ *denotes the set of all permutations of the elements of* (X_n) *i.e.* $S(X) = \{(X_{\pi(n)}) : \pi$ *is a permutation of* $N\}$. *A sequence space* E *is said to be symmetric if* $S(X) \subset E$ *for all* $X \in E$.

Definition 14 *A sequence space* E *is said to be convergence-free if* $(Y_n) \in E$ *whenever* $(X_n) \in E$ *and* $X_n = \bar{0}$ *implies* $Y_n = \bar{0}$.

Definition 15 *A sequence space* E *is said to be monotone if* E *contains the canonical pre-images of all its step spaces.*

Lemma 1 [3] *A sequence space* E *is normal implies* E *is monotone.*

The notion of difference sequence spaces was introduced by Kizmaz [4], who studied the difference sequence spaces $\ell_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [1] by introducing the spaces $\ell_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Another type of generalization of the difference sequence spaces is due to Tripathy and Esi [19] who studied the spaces $\ell_\infty(\Delta^n_m)$, $c(\Delta^n_m)$ and $c_0(\Delta^n_m)$. Let m, n be non-negative integers, then we have sequence spaces

$$Z(\Delta^n_m) = \{x = (x_k) \in w : (\Delta^n_m x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta^n_m x = (\Delta^n_m x_k) = (\Delta^{n-1}_m x_k - \Delta^{n-1}_m x_{k+1})$ and $\Delta^0_m x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta^n_m x_k = \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} x_{k+m\nu}. \tag{1}$$

Taking $m = 1$, we get the spaces $\ell_\infty(\Delta^n), c(\Delta^n)$ and $c_0(\Delta^n)$ studied by Et and Çolak [1]. Taking $m = n = 1$, we get the spaces $\ell_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$ introduced and studied by Kizmaz [4].

Definition 16 *Ng and Lee* [12] *defined the Cesàro sequence spaces* X_p *of non-absolute type as follows:*

$$x = (x_k) \in X_p \text{ if and only if } \sigma(x) \in \ell_p, 1 \leq p < \infty,$$

where $\sigma(x) = \left(\frac{1}{n} \sum_{k=1}^{n} x_k \right)_{n=1}^{\infty}$.

Orhan [13] defined the Cesàro difference sequence spaces $X_p(\Lambda)$, for $1 \le p < \infty$ and studied their different properties and proved some inclusion results. He also obtained the duals of these sequence spaces.

Musaleen et al. [8] defined the second difference Cesàro sequence spaces $X_p(\Lambda^2)$, for $1 \le p < \infty$ and studied their different topological properties and proved some inclusion results. They also calculated their duals sequence spaces.

Later on, Tripathy et al. [20] further introduced new types of difference Cesàro sequence spaces as $C_\infty(\Delta_m^n), O_\infty(\Delta_m^n), C_p(\Delta_m^n), O_p(\Delta_m^n)$ and $\ell_\infty(\Delta_m^n)$, for $1 \le p < \infty$.

For $m = 1$, the spaces $C_p(\Delta^n)$ and $C_\infty(\Delta_m^n)$ are studied by Et [2].

An Orlicz function $M : [0, \infty) \to [0, \infty)$ is a continuous, non-decreasing and convex such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$. An Orlicz function M is said to satisfy Δ_2-condition for all values of x, if there exists a constant $K > 0$, $M(Lx) \le KLM(x)$, for all $x > 0$ and for $L > 1$. If convexity of the Orlicz function is replaced by subadditivity i.e. $M(x+y) \le M(x)+M(y)$, then this function is called as modulus function [18].

Lindenstrauss and Tzafriri [5] used the idea of Orlicz function to define the following sequence space,

$$\ell_M = \left\{ x = (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}$$

is known as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||x|| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\}.$$

Also it was shown in [5] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \ge 1)$. A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is said to be Musielak-Orlicz function (see [6]).

Let $m, n \ge 0$ be fixed integers, $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ be a bounded sequence of positive real numbers. In this paper we define the following generalized difference Cesàro sequence spaces of fuzzy real numbers:

$$C^F(\mathcal{M}, \Lambda, \Delta_m^n, p) =$$
$$\left\{ X = (X_k) \in w(F) : \sum_{i=1}^{\infty} \left(\frac{1}{i} \sum_{k=1}^{i} \left(M_k \left(\frac{\bar{d}(\Lambda_k \Delta_m^n X_k, \bar{0})}{\rho} \right) \right) \right)^{p_k} < \infty, \text{ for some } \rho > 0 \right\},$$

$C^F_\infty(\mathcal{M}, \Lambda, \Delta^n_m, p) =$

$$\left\{X=(X_k) \in w(F) : \sup_i \frac{1}{i}\left(\sum_{k=1}^{i} M_k\left(\frac{\overline{d}(\Lambda_k \Delta^n_m X_k, \overline{0})}{\rho}\right)\right)^{p_k} < \infty, \text{ for some } \rho > 0\right\},$$

$\ell^F(\mathcal{M}, \Lambda, \Delta^n_m, p) =$

$$\left\{X = (X_k) \in w(F) : \sum_{k=1}^{\infty}\left(M_k\left(\frac{\overline{d}(\Lambda_k \Delta^n_m X_k, \overline{0})}{\rho}\right)\right)^{p_k} < \infty, \text{ for some } \rho > 0\right\},$$

$O^F(\mathcal{M}, \Lambda, \Delta^n_m, p) =$

$$\left\{X=(X_k) \in w(F) : \sum_{i=1}^{\infty} \frac{1}{i}\left(\sum_{k=1}^{i}\left(M_k\left(\frac{\overline{d}(\Lambda_k \Delta^n_m X_k, \overline{0})}{\rho}\right)\right)\right)^{p_k} < \infty, \text{for some } \rho > 0\right\},$$

$O^F_\infty(\mathcal{M}, \Lambda, \Delta^n_m, p) =$

$$\left\{X = (X_k) \in w(F) : \sup_i \frac{1}{i}\sum_{k=1}^{i} M_k\left(\frac{\overline{d}(\Lambda_k \Delta^n_m X_k, \overline{0})}{\rho}\right)^{p_k} < \infty, \text{for some } \rho > 0\right\}.$$

Lemma 2 [21] *Let* $1 \leq p < \infty$. *Then,*
(i) *The space* $C^F_p(M)$ *is a complete metric space with the metric,*

$$\eta_1(X, Y) = \inf\left\{\rho > 0 : \left(\sum_{i=1}^{\infty}\frac{1}{i}\sum_{k=1}^{i}\left(M\left(\frac{\overline{d}(X_k, Y_k)}{\rho}\right)\right)^{p}\right)^{\frac{1}{p}} \leq 1\right\}.$$

(ii) *The space* $C^F_\infty(M)$ *is a complete metric space with the metric,*

$$\eta_2(X, Y) = \inf\left\{\rho > 0 : \sup_i \frac{1}{i}\sum_{k=1}^{i}\left(M\left(\frac{\overline{d}(X_k, Y_k)}{\rho}\right) \leq 1\right\}.$$

(iii) *The space* $\ell^F_p(M)$ *is a complete metric space with the metric,*

$$\eta_3(X, Y) = \inf\left\{\rho > 0 : \left(\sum_{k=1}^{\infty}\left(M\left(\frac{\overline{d}(X_k, Y_k)}{\rho}\right)\right)^{p}\right)^{\frac{1}{p}} \leq 1\right\}.$$

(iv) *The space* $O^F_p(M)$ *is a complete metric space with the metric,*

$$\eta_4(X, Y) = \inf\left\{\rho > 0 : \left(\sum_{i=1}^{\infty}\frac{1}{i}\sum_{k=1}^{i}\left(M\left(\frac{\overline{d}(X_k, Y_k)}{\rho}\right)\right)^{p}\right)^{\frac{1}{p}} \leq 1\right\}.$$

(v) *The space* $O^F_\infty(M)$ *is a complete metric space with the metric,*

$$\eta_5(X, Y) = \inf\left\{\rho > 0 : \sup_i \frac{1}{i}\sum_{k=1}^{i}\left(M\left(\frac{\bar{d}(X_k, Y_k)}{\rho}\right) \leq 1\right\}.$$

The following inequality will be used throughout the paper. Let $p = (p_k)$ be a sequence of positive real numbers with $0 < p_k \leq \sup_k p_k = H$ and let $K = \max\left\{1, 2^{H-1}\right\}$. Then, for the factorable sequences (a_k) and (b_k) in the complex plane, we have

$$|a_k + b_k|^{p_k} \leq K(|a_k|^{p_k} + |b_k|^{p_k}). \tag{2}$$

Also $|a_k|^{p_k} \leq \max\left\{1, |a|^H\right\}$ for all $a \in \mathbb{C}$.

The main aim of this paper is to study some topological properties and prove some inclusion relations between above defined sequence spaces.

2 Main results

Theorem 1 *Let* $\mathcal{M} = (M_k)$ *be a Musielak-Orlicz function and* $p = (p_k)$ *be a bounded sequence of positive real numbers. Then the classes of sequences* $C^F(\mathcal{M}, \Lambda, \Delta^n_m, p)$, $C^F_\infty(\mathcal{M}, \Lambda, \Delta^n_m, p)$, $\ell^F(\mathcal{M}, \Lambda, \Delta^n_m, p)$, $O^F(\mathcal{M}, \Lambda, \Delta^n_m, p)$ *and* $O^F_\infty(\mathcal{M}, \Lambda, \Delta^n_m, p)$ *are linear spaces over the field* $\mathbb{R}$ *of real numbers.*

Proof. We shall prove the result for the space $C^F(\mathcal{M}, \Lambda, \Delta^n_m, p)$ and for other spaces, it will follow on applying similar arguments. Suppose $X = (X_k)$, $Y = (Y_k) \in C^F(\mathcal{M}, \Lambda, \Delta^n_m, p)$ and $\alpha, \beta \in \mathbb{R}$. Then there exit positive real numbers ρ_1, ρ_2 such that

$$\sum_{i=1}^{\infty}\left(\frac{1}{i}\sum_{k=1}^{i}\left(M_k\left(\frac{\bar{d}(\Lambda_k\Delta^n_m X_k, \bar{0})}{\rho_1}\right)\right)\right)^{p_k} < \infty, \text{ for some } \rho_1 > 0$$

and

$$\sum_{i=1}^{\infty}\left(\frac{1}{i}\sum_{k=1}^{i}\left(M_k\left(\frac{\bar{d}(\Lambda_k\Delta^n_m Y_k, \bar{0})}{\rho_2}\right)\right)\right)^{p_k} < \infty, \text{ for some } \rho_2 > 0.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\mathcal{M} = (M_k)$ is a non-decreasing and convex so by using inequality (2), we have

$$\sum_{i=1}^{\infty}\left(\frac{1}{i}\sum_{k=1}^{i}\left(M_k\left(\frac{\overline{d}(\alpha\Lambda_k\Delta_m^n X_k+\beta\Lambda_k\Delta_m^n Y_k,\overline{0})}{\rho_3}\right)\right)\right)^{p_k}$$

$$\begin{aligned}
&= \sum_{i=1}^{\infty}\left(\frac{1}{i}\sum_{k=1}^{i}\left(M_k\left(\frac{\overline{d}(\alpha\Lambda_k\Delta_m^n X_k,\overline{0})}{\rho_3}+\frac{\overline{d}(\beta\Lambda_k\Delta_m^n Y_k,\overline{0})}{\rho_3}\right)\right)\right)^{p_k}\\
&\leq \sum_{i=1}^{\infty}\left(\frac{1}{i}\sum_{k=1}^{i}\frac{1}{2^{p_k}}\left(M_k\left(\frac{\overline{d}(\alpha\Lambda_k\Delta_m^n X_k,\overline{0})}{\rho_1}\right)+M_k\left(\frac{\overline{d}(\beta\Lambda_k\Delta_m^n Y_k,\overline{0})}{\rho_2}\right)\right)\right)^{p_k}\\
&\leq K\sum_{i=1}^{\infty}\left(\frac{1}{i}\sum_{k=1}^{i}\left(M_k\left(\frac{\overline{d}(\Lambda_k\Delta_m^n X_k,\overline{0})}{\rho_1}\right)\right)\right)^{p_k}\\
&+ K\sum_{i=1}^{\infty}\left(\frac{1}{i}\sum_{k=1}^{i}\left(M_k\left(\frac{\overline{d}(\Lambda_k\Delta_m^n Y_k,\overline{0})}{\rho_2}\right)\right)\right)^{p_k}\\
&< \infty.
\end{aligned}$$

Thus, $\alpha X+\beta Y\in C^F(\mathcal{M},\Lambda,\Delta_m^n,p)$. This proves that $C^F(\mathcal{M},\Lambda,\Delta_m^n,p)$ is a linear space. □

Proposition 1 *The classes of sequences* $C^F(\mathcal{M},\Lambda,\Delta_m^n,p)$, $C_\infty^F(\mathcal{M},\Lambda,\Delta_m^n,p)$, $\ell^F(\mathcal{M},\Lambda,\Delta_m^n,p)$, $O^F(\mathcal{M},\Lambda,\Delta_m^n,p)$ *and* $O_\infty^F(\mathcal{M},\Lambda,\Delta_m^n,p)$ *are metric spaces with respect to the metric,*

$$f(X,Y)=\sum_{k=1}^{mn}\overline{d}(X_k,\overline{0})+\eta(\Lambda_k\Delta_m^n X_k,\Lambda_k\Delta_m^n Y_k),$$

where $Z=C^F,C_\infty^F,O^F,O_\infty^F,\ell^F$.

Proof. The proof of the proposition is direct consequence of the Proposition 3.1 [21]. □

Theorem 2 *Let* $Z(\mathcal{M})$ *be a complete metric space with respect to the metric* η, *the space* $Z(\mathcal{M},\Lambda,\Delta_m^n,p)$ *is a complete metric space with respect to the metric,*

$$f(X,Y)=\sum_{k=1}^{mn}\overline{d}(X_k,\overline{0})+\eta(\Lambda_k\Delta_m^n X_k,\Lambda_k\Delta_m^n Y_k),$$

where $Z=C^F,C_\infty^F,O^F,O_\infty^F,\ell^F$.

Proof. Let $(X^{(u)})$ be a Cauchy sequence in $Z(\mathcal{M}, \Lambda, \Delta_m^n, p)$ such that $(X^{(u)}) = (X_n^{(u)})_{n=1}^{\infty}$. Then for $\epsilon > 0$, there exists a positive integer $n_0 = n_0(\epsilon)$ such that

$$f(X^{(u)}, X^{(v)}) < \epsilon \text{ for all } u, v \geq n_0.$$

By the definition of f, we get

$$\sum_{r=1}^{mn} \bar{d}(X_r^{(u)}, X_r^{(v)}) + \eta(\Lambda_k \Delta_m^n X_k^{(u)}, \Lambda_k \Delta_m^n X_k^{(v)}) < \epsilon, \text{ for all } u, v \geq n_0 \qquad (3)$$

$$\implies \sum_{r=1}^{mn} \bar{d}(X_r^{(u)}, X_r^{(v)}) < \epsilon \quad \forall\, u, v \geq n_0$$

$$\implies \bar{d}(X_r^{(u)}, X_r^{(v)}) < \epsilon \quad \forall\, u, v \geq n_0, \; r = 1, 2, 3, ..., mn.$$

Hence, $(X_r^{(u)})$ is a Cauchy sequence in $R(I)$, so it is convergent in $R(I)$ by the completeness property of $R(I)$, for $r = 1, 2, 3, ..., mn$.
Let

$$\lim_{u \to \infty} X_r^{(u)} = X_r, \text{ for } r = 1, 2, 3, ..., mn. \qquad (4)$$

Next, we have

$$\eta(\Lambda_k \Delta_m^n X_k^{(u)}, \Lambda_k \Delta_m^n X_k^{(v)}) < \epsilon \text{ for all } u, v \geq n_0$$

which implies that $(\Lambda_k \Delta_m^n X_k^{(u)})$ is a Cauchy sequence in $Z(\mathcal{M})$, Since $\mathcal{M} = (M_k)$ is continuous function and so it is convergent in $Z(\mathcal{M})$ by the completeness property of $Z(\mathcal{M})$.
Let $\lim_u \Lambda_k \Delta_m^n X_k^{(u)} = Y_k$ (say), in $Z(\mathcal{M})$, for each $k \in N$. We have to prove $\lim_u X^{(u)} = X$ and $X \in Z(\mathcal{M}, \Lambda, \Delta_m^n, p)$.
For $k = 1$, we have from equation (1) and (4),

$$\lim_u X_{mn+1}^{(u)} + X_{mn+1}, \text{ for } m \geq 1, n \geq 1.$$

Proceeding in this way of induction, we get

$$\lim_u X_k^{(u)} + X_k, \text{ for each } \; k \in N.$$

Also, $\lim_u \Lambda_k \Delta_m^n X_k^{(u)} = \Lambda_k \Delta_m^n X_k$ for each $k \in N$. Now, taking $v \to \infty$ and fixing u, it follows from (3),

$$\sum_{r=1}^{mn} \bar{d}(X_r^{(u)}, X_r) + \eta(\Lambda_k \Delta_m^n X_k^{(u)}, \Lambda_k \Delta_m^n X_k) < \epsilon, \text{ for all } u, v \geq n_0.$$

$$\Longrightarrow f(X^{(u)}, X) < \epsilon, \text{ for all } u \geq n_0.$$

Therefore, we have $\lim_u X^{(u)} = X$.

Now, we show that $X \in Z(\mathcal{M}, \Lambda, \Delta^n_m, p)$. Since

$$f(\Lambda_k \Delta^n_m X_k, \bar{0}) \leq f(\Lambda_k \Delta^n_m X^{(i)}_k,\ \Lambda_k \Delta^n_m X_k) + f(\Lambda_k \Delta^n_m X^{(i)}_k, \bar{0}) < \infty.$$

$\Longrightarrow X \in Z(\mathcal{M}, \Lambda, \Delta^n_m, p)$. Hence, $Z(\mathcal{M}, \Lambda, \Delta^n_m, p)$ is a complete metric space. □

Proposition 2 *Let* $1 \leq p = \sup_k p_k < \infty$. *Then,*

(i) *The space* $C^F(\mathcal{M}, \Lambda, \Delta^n_m, p)$ *is a complete metric space with the metric,*

$f_1(X, Y) =$

$$\sum_{r=1}^{mn} \bar{d}(X_r, Y_r) + \inf\left\{\rho > 0 : \left(\sum_{i=1}^{\infty} \frac{1}{i} \sum_{k=1}^{i} \left(M_k\left(\frac{\bar{d}(\Lambda_k \Delta^n_m X_k, \Lambda_k \Delta^n_m Y_k)}{\rho}\right)\right)^p\right)^{\frac{1}{p}} \leq 1\right\}.$$

(ii) *The space* $C^F_\infty(\mathcal{M}, \Lambda, \Delta^n_m, p)$ *is a complete metric space with the metric,*

$f_2(X, Y) =$

$$\sum_{r=1}^{mn} \bar{d}(X_r, Y_r) + \inf\left\{\rho > 0 : \sup_i \frac{1}{i} \sum_{k=1}^{i} \left(M_k\left(\frac{\bar{d}(\Lambda_k \Delta^n_m X_k, \Lambda_k \Delta^n_m Y_k)}{\rho}\right)\right)^{p_k} \leq 1\right\}.$$

(iii) *The space* $\ell^F(\mathcal{M}, \Lambda, \Delta^n_m, p)$ *is a complete metric space with the metric,*

$f_3(X, Y) =$

$$\sum_{r=1}^{mn} \bar{d}(X_r, Y_r) + \inf\left\{\rho > 0 : \left(\sum_{k=1}^{\infty} \left(M_k\left(\frac{\bar{d}(\Lambda_k \Delta^n_m X_k, \Lambda_k \Delta^n_m Y_k)}{\rho}\right)\right)^p\right)^{\frac{1}{p}} \leq 1\right\}.$$

(iv) *The space* $O^F(\mathcal{M}, \Lambda, \Delta^n_m, p)$ *is a complete metric space with the metric,*

$f_4(X, Y) =$

$$\sum_{r=1}^{mn} \bar{d}(X_r, Y_r) + \inf\left\{\rho > 0 : \left(\sum_{i=1}^{\infty} \frac{1}{i} \sum_{k=1}^{i} \left(M_k\left(\frac{\bar{d}(\Lambda_k \Delta^n_m X_k, \Lambda_k \Delta^n_m Y_k)}{\rho}\right)\right)^p\right)^{\frac{1}{p}} \leq 1\right\}.$$

(v) *The space* $O^F_\infty(\mathcal{M},\Lambda,\Delta^n_m,p)$ *is a complete metric space with the metric,*

$f_5(X,Y) =$

$$\sum_{r=1}^{mn}\overline{d}(X_r,Y_r)+\inf\left\{\rho>0:\sup_i\frac{1}{i}\sum_{k=1}^{i}\left(M_k\left(\frac{\overline{d}(\Lambda_k\Delta^n_mX_k,\Lambda_k\Delta^n_mY_k)}{\rho}\right)\right)^{p_k}\le 1\right\}.$$

Proof. The proof directly comes from ([21], Proposition 3.2). □

Theorem 3 (a) $\ell^F(\mathcal{M},\Lambda,\Delta^n_m,p)\subset O^F(\mathcal{M},\Lambda,\Delta^n_m,p)\subset C^F_\infty(\mathcal{M},\Lambda,\Delta^n_m,p)$ *and the inclusions are strict.*
(b) $Z(\mathcal{M},\Lambda,\Delta^{n-1}_m,p)\subset Z(\mathcal{M},\Lambda,\Delta^n_m,p)$ *(in general* $Z(\mathcal{M},\Lambda,\Delta^i_m,p)\subset Z(\mathcal{M},\Lambda,\Delta^n_m,p)$ *for* $i=1,2,3...,n-1$*), for* $Z=C^F,C^F_\infty,O^F,O^F_\infty,\ell^F$.
(c) $O^F_\infty(\mathcal{M},\Lambda,\Delta^n_m,p)\subset C^F_\infty(\mathcal{M},\Lambda,\Delta^n_m,p)$ *and the inclusion is strict.*

Proof. We shall prove the result for the space $Z=C_\infty$ only and others can be proved in the similar way. Let $(X_k)\in C^F_\infty(\mathcal{M},\Lambda,\Delta^{n-1}_m,p)$. Then, we have

$$\sup_i\frac{1}{i}\left(\sum_{k=1}^{i}M_k\left(\frac{\overline{d}(\Lambda_k\Delta^{n-1}_mX_k,\overline{0})}{\rho}\right)\right)^{p_k}<\infty,\ \text{for some } \rho>0.$$

Now, we have

$$\begin{aligned}
&\sup_i\frac{1}{i}\left(\sum_{k=1}^{i}M_k\left(\frac{\overline{d}(\Lambda_k\Delta^n_mX_k,\overline{0})}{2\rho}\right)\right)^{p_k}\\
&= \sup_i\frac{1}{i}\left(\sum_{k=1}^{i}M_k\left(\frac{\overline{d}(\Lambda_k\Delta^{n-1}_mX_k-\Lambda_k\Delta^{n-1}_mX_{k+1},\overline{0})}{2\rho}\right)\right)^{p_k}\\
&\le \sup_i\frac{1}{2}\left(\frac{1}{i}\left(\sum_{k=1}^{i}M_k\left(\frac{\overline{d}(\Lambda_k\Delta^{n-1}_mX_k,\overline{0})}{2\rho}\right)\right)\right)^{p_k}\\
&+ \sup_i\frac{1}{2}\left(\frac{1}{i}\left(\sum_{k=1}^{i}M_k\left(\frac{\overline{d}(\Lambda_k\Delta^{n-1}_mX_{k+1},\overline{0})}{2\rho}\right)\right)\right)^{p_k}\\
&< \infty.
\end{aligned}$$

Proceeding in this way, we have $Z(\mathcal{M},\Lambda,\Delta^i_m,p)\subset Z(\mathcal{M},\Lambda,\Delta^n_m,p)$, for $0\le i<n$, for $Z=C^F,C^F_\infty,O^F,O^F_\infty,\ell^F$. □

Theorem 4 (a) *If* $1 \leq p < q < \infty$, *then*
(i) $C^F(\mathcal{M}, \Lambda, \Delta^n_m, p) \subset C^F(\mathcal{M}, \Lambda, \Delta^n_m, q)$;
(ii) $\ell^F(\mathcal{M}, \Lambda, \Delta^n_m, p) \subset \ell^F(\mathcal{M}, \Lambda, \Delta^n_m, q)$;
(b) $C^F(\mathcal{M}, \Lambda, p) \subset C^F(\mathcal{M}, \Lambda, \Delta^n_m, p)$ *for all* $m \geq 1$ *and* $n \geq 1$.

Proof. (i) We shall prove the result for the space $C^F(\mathcal{M}, \Lambda, \Delta^n_m, p)$ and others can be proved in the similar way. Let $X \in C^F(\mathcal{M}, \Lambda, \Delta^n_m, p)$. Then there exists $\rho > 0$ such that

$$\sum_{i=1}^{\infty} \left(\frac{1}{i} \sum_{k=1}^{i} \left(M_k \left(\frac{\bar{d}(\Lambda_k \Delta^n_m X_k, \bar{0})}{\rho} \right) \right) \right)^{p_k} < \infty.$$

This implies that

$$\frac{1}{i} \sum_{k=1}^{i} \left(M_k \left(\frac{\bar{d}(\Lambda_k \Delta^n_m X_k, \bar{0})}{\rho} \right) \right)^{p_k} < 1$$

for sufficiently large values of i. Since (M_k) is non-decreasing, we get

$$\sum_{i=1}^{\infty} \left(\frac{1}{i} \sum_{k=1}^{i} \left(M_k \left(\frac{\bar{d}(\Lambda_k \Delta^n_m X_k, \bar{0})}{\rho} \right) \right) \right)^{q_k}$$
$$\leq \sum_{i=1}^{\infty} \left(\frac{1}{i} \sum_{k=1}^{i} \left(M_k \left(\frac{\bar{d}(\Lambda_k \Delta^n_m X_k, \bar{0})}{\rho} \right) \right) \right)^{p_k} < \infty.$$

Thus, $X \in C^F(\mathcal{M}, \Lambda, \Delta^n_m, q)$. This completes the proof. □

Theorem 5 *Let* $\mathcal{M} = (M_k)$, $\mathcal{M}' = (M'_k)$ *and* $\mathcal{M}'' = (M''_k)$ *be Musielak-Orlicz functions satisfying* Δ_2-*condition. Then for* $Z = C^F, C^F_\infty, O^F, O^F_\infty, \ell^F$, *we have*
(i) $Z(\mathcal{M}', \Lambda, \Delta^n_m, p) \subseteq Z(\mathcal{M} \circ \mathcal{M}', \Lambda, \Delta^n_m, p)$.
(ii) $Z(\mathcal{M}', \Lambda, \Delta^n_m, p) \cap Z(\mathcal{M}'', \Lambda, \Delta^n_m, p) \subseteq Z(\mathcal{M}' + \mathcal{M}'', \Lambda, \Delta^n_m, p)$.

Proof. Let $(X_k) \in Z(\mathcal{M}', \Lambda, \Delta^n_m, p)$. For $\epsilon > 0$, there exists $\eta > 0$ such that $\epsilon = \mathcal{M}(\eta)$. Then,

$$M'_k \left(\frac{\bar{d}(\Lambda_k \Delta^n_m X_k, L)}{\rho} \right)^{p_k} < \eta, \text{ for some } \rho > 0, \ L \in R(I).$$

Let $Y_k = M'_k\left(\frac{\bar{d}(\Lambda_k\Delta^n_m X_k, L)}{\rho}\right)^{p_k}$, for some $\rho > 0$, $L \in R(I)$. Since $\mathcal{M} = (M_k)$ is continuous and non-decreasing, we get

$$M_k(Y_k) = M_k\left(M'_k\left(\frac{\bar{d}(\Lambda_k\Delta^n_m X_k, L)}{\rho}\right)^{p_k} < M_k(\eta) = \epsilon, \text{ for some } \rho > 0.\right.$$

$\Longrightarrow (X_k) \in Z(\mathcal{M}\circ\mathcal{M}', \Lambda, \Delta^n_m, p)$.
(ii) Let $(X_k) \in Z(\mathcal{M}', \Lambda, \Delta^n_m, p) \cap Z(\mathcal{M}'', \Lambda, \Delta^n_m, p)$. Then,

$$M'_k\left(\frac{\bar{d}(\Lambda_k\Delta^n_m X_k, L)}{\rho}\right)^{p_k} < \epsilon, \text{ for some } \rho > 0, \ L \in R(I)$$

and

$$M''_k\left(\frac{\bar{d}(\Lambda_k\Delta^n_m X_k, L)}{\rho}\right)^{p_k} < \epsilon, \text{ for some } \rho > 0, \ L \in R(I).$$

The rest of the proof follows from the equality

$$\begin{aligned}(M'_k + M''_k)&\left(\frac{\bar{d}(\Lambda_k\Delta^n_m X_k, L)}{\rho}\right)^{p_k}\\ &= M'_k\left(\frac{\bar{d}(\Lambda_k\Delta^n_m X_k, L)}{\rho}\right)^{p_k} + M''_k\left(\frac{\bar{d}(\Lambda_k\Delta^n_m X_k, L)}{\rho}\right)^{p_k}\\ &< \epsilon + \epsilon = 2\epsilon, \text{ for some } \rho > 0\end{aligned}$$

which implies that $(X_k) \in Z(\mathcal{M}'+\mathcal{M}'', \Lambda, \Delta^n_m, p)$. This completes the proof. □

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Multiplication semimodules

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Abstract. Let S be a semiring. An S-semimodule M is called a multiplication semimodule if for each subsemimodule N of M there exists an ideal I of S such that $N = IM$. In this paper we investigate some properties of multiplication semimodules and generalize some results on multiplication modules to semimodules. We show that every multiplicatively cancellative multiplication semimodule is finitely generated and projective. Moreover, we characterize finitely generated cancellative multiplication S-semimodules when S is a yoked semiring such that every maximal ideal of S is subtractive.

1 Introduction

In this paper, we study multiplication semimodules and extend some results of [7] and [17] to semimodules over semirings. A semiring is a nonempty set S together with two binary operations addition $(+)$ and multiplication $(\cdot)$ such that $(S, +)$ is a commutative monoid with identity element 0; $(S, .)$ is a monoid with identity element $1 \neq 0$; $0a = 0 = a0$ for all $a \in S$; $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for every $a, b, c \in S$. We say that S is a commutative semiring if the monoid $(S, .)$ is commutative. In this paper we assume that all semirings are commutative. A nonempty subset I of a semiring S is called an ideal of S if $a + b \in I$ and $sa \in I$ for all $a, b \in I$ and $s \in S$. A semiring

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S is called yoked if for all $a, b \in S$, there exists an element t of S such that $a + t = b$ or $b + t = a$. An ideal I of a semiring S is subtractive if $a + b \in I$ and $b \in I$ imply that $a \in I$ for all $a, b \in S$. A semiring S is local if it has a unique maximal ideal. A semiring is entire if $ab = 0$ implies that $a = 0$ or $b = 0$. An element s of a semiring S is a unit if there exists an element s' of S such that $ss' = 1$. A semiring S is called a semidomain if for any nonzero element a of S, $ab = ac$ implies that $b = c$. An element a of a semiring S is called multiplicatively idempotent if $a^2 = a$. The semiring S is multiplicatively idempotent if every element of S is multiplicatively idempotent.

Let $(M, +)$ be an additive abelian monoid with additive identity 0_M. Then M is called an S-semimodule if there exists a scalar multiplication $S \times M \longrightarrow M$ denoted by $(s, m) \mapsto sm$, such that $(ss')m = s(s'm)$; $s(m + m') = sm + sm'$; $(s + s')m = sm + s'm$; $1m = m$ and $s0_M = 0_M = 0m$ for all $s, s' \in S$ and all $m, m' \in M$. A subsemimodule N of a semimodule M is a nonempty subset of M such that $m + n \in N$ and $sn \in N$ for all $m, n \in N$ and $s \in S$. If N and L are subsemimodules of M, we set $(N : L) = \{s \in S \mid sL \subseteq N\}$. It is clear that $(N : L)$ is an ideal of S.

Let R be a ring. An R-module M is a multiplication module if for each submodule N of M there exists an ideal I of R such that $N = IM$ [2]. Multiplication semimodules are defined similarly. These semimodules have been studied by several authors(e.g. [5], [6], [18], [20]). It is known that invertible ideals of a ring R are multiplication R-modules. Invertible ideals of semirings has been studied in [8]. In this paper, in order to study the relations between invertible ideals of semirings and multiplication semimodules, we generalize some properties of multiplication modules to multiplication semimodules (cf. Theorems 2 and 12). In Section 2, we show that if M is a multiplication S-semimodule and P is a maximal ideal of S such that $M \neq PM$, then M_P is cyclic. In Section 3, we study multiplicatively cancellative(abbreviated as MC) multiplication semimodules. We show that MC multiplication semimodules are finitely generated and projective. In Section 4, we characterize finitely generated cancellative multiplication semimodules over yoked semirings with subtractive maximal ideals.

2 Multiplication semimodule

In this section we give some results of multiplication semimodules which are related to the corresponding results in multiplication modules.

Definition 1 [6] *Let S be a semiring and M an S-semimodule. Then M is called a multiplication semimodule if for each subsemimodule N of M there exists an ideal I of S such that $N = IM$. In this case it is easy to prove that $N = (N : M)M$. For example, every cyclic S-semimodule is a multiplication S-semimodule* [20, Example 2].

Example 1 *Let S be a multiplicatively idempotent semiring. Then every ideal of S is a multiplication S-semimodule. Let J be an ideal of S and $I \subseteq J$. If $x \in I$, then $x = x^2 \in IJ$. Therefore $I = IJ$ and hence J is a multiplication S-semimodule.*

Let M and N be S-semimodules and $f : M \to N$ an S-homomorphism. If M' is a subsemimodule of M and I is an ideal of S, then $f(IM') = If(M')$. Now suppose that f is surjective and N' is a subsemimodule of N. Put $M' = \{m \in M \mid f(m) \in N'\}$. Then M' is a subsemimodule of M and $f(M') = N'$. It is well-known that every homomorphic image of a multiplication module is a multiplication module (cf. [7] and [19, Note 1.4]). A similar result holds for multiplication semimodules.

Theorem 1 *Let S be a semiring, M and N S-semimodules and $f : M \to N$ a surjective S-homomorphism. If M is a multiplication S-semimodule, then N is a multiplication S-semimodule.*

Proof. Let N' be a subsemimodule of N. Then there exists a subsemimodule M' of M such that $f(M') = N'$. Since M is a multiplication S-semimodule, there exists an ideal I of S such that $M' = IM$. Then $N' = f(M') = f(IM) = If(M) = IN$. Therefore N is a multiplication S-semimodule. □

Fractional and invertible ideals of semirings have been studied in [8]. We recall here some definitions and properties.

An element s of a semiring S is multiplicatively-cancellable (abbreviated as MC), if $sb = sc$ implies $b = c$ for all $b, c \in S$. We denote the set of all MC elements of S by $MC(S)$. The total quotient semiring of S, denoted by $Q(S)$, is defined as the localization of S at $MC(S)$. Then $Q(S)$ is an S-semimodule and S can be regarded as a subsemimodule of $Q(S)$. For the concept of the localization in semiring theory, we refer to [10] and [11]. A subset I of $Q(S)$ is called a fractional ideal of S if I is a subsemimodule of $Q(S)$ and there exists an MC element $d \in S$ such that $dI \subseteq S$. Note that every ideal of S is a fractional ideal. The product of two fractional ideals is defined by $IJ = \{a_1b_1 + \ldots + a_nb_n \mid a_i \in I, b_i \in J\}$. A fractional ideal I of a semiring S is called invertible if there exists a fractional ideal J of S such that $IJ = S$.

Now we restate the following property of invertible ideals from [8, Theorem 1.3] (see also [13, Proposition 6.3]).

Theorem 2 *Let* S *be a semiring. An ideal* I *of* S *is invertible iff it is a multiplication* S*-semimodule which contains an* MC *element of* S*.*

Let M be an S-semimodule and P a maximal ideal of S. Then similar to [7], we define $T_P(M) = \{m \in M \mid \text{ there exist } s \in S \text{ and } q \in P \text{ such that } s+q = 1 \text{ and } sm = 0\}$. Clearly $T_p(M)$ is a subsemimodule of M. We say that M is P-cyclic if there exist $m \in M$, $t \in S$ and $q \in P$ such that $t + q = 1$ and $tM \subseteq Sm$.

The following two theorems can be thought of as a generalization of [7, Theorem 1.2] (see also [5, Proposition 3]).

Theorem 3 *Let* M *be an* S*-semimodule. If for every maximal ideal* P *of* S *either* $T_P(M) = M$ *or* M *is* P*-cyclic, then* M *is a multiplication semimodule.*

Proof. Let N be a subsemimodule of M and $I = (N : M)$. Then $IM \subseteq N$. Let $x \in N$ and $J = \{s \in S \mid sx \in IM\}$. Clearly J is an ideal of S. If $J \neq S$, then by [9, Proposition 6.59] there exists a maximal ideal P of S such that $J \subseteq P$. If $M = T_P(M)$, then there exist $s \in S$ and $q \in P$ such that $s + q = 1$ and $sx = 0 \in IM$. Hence $s \in J \subseteq P$ which is a contradiction. So the second case will happen. Therefore there exist $m \in M$, $t \in S$ and $q \in P$ such that $t + q = 1$ and $tM \subseteq Sm$. Thus tN is a subsemimodule of Sm and $tN = Km$ where $K = \{s \in S \mid sm \in tN\}$. Moreover, $tKM = KtM \subseteq Km \subseteq N$. Therefore $tK \subseteq I$. Thus $t^2x \in t^2N = tKm \subseteq IM$. Hence $t^2 \in J \subseteq P$ which is a contradiction. Therefore $J = S$ and $x \in IM$. □

Theorem 4 *Suppose that* M *is an* S*-semimodule. If* M *is a multiplication semimodule, then for every maximal ideal* P *of* S *either* $M = \{m \in M \mid m = qm \text{ for some } q \in P\}$ *or* M *is* P*-cyclic.*

Proof. Let P be a maximal ideal of S and $M = PM$. If $m \in M$, then there exists an ideal I of S such that $Sm = IM$. Hence $Sm = IPM = PIM = Pm$. Therefore $m = qm$ for some $q \in P$. Now let $M \neq PM$. Thus there exists $x \in M$ such that $x \notin PM$. Then there exists ideal I of S such that $Sx = IM$. If $I \subseteq P$, then $x \in IM \subseteq PM$ which is a contradiction. Thus $I \nsubseteq P$ and since P is a maximal ideal of S, $P + I = S$. Thus there exist $t \in I$ and $q \in P$ such that $q + t = 1$. Moreover, $tM \subseteq IM = Sx$. Therefore M is P-cyclic. □

We recall the following result from [10].

Theorem 5 *A commutative semiring* S *is local iff for all* $r, s \in S$*,* $r + s = 1$ *implies* r *or* s *is a unit.*

By using Theorem 4, we obtain the following corollary.

Corollary 1 *Suppose that* $(S, \mathfrak{m})$ *is a local semiring. Let* M *be a multiplication* S*-semimodule such that* $M \neq \mathfrak{m}M$*. Then* M *is a cyclic semimodule.*

Proof. Since $M \neq \mathfrak{m}M$, M is $\mathfrak{m}$-cyclic. Thus there exist $n \in M$, $t \in S$ and $q \in \mathfrak{m}$ such that $t + q = 1$ and $tM \subseteq Sn$. Since S is a local semiring, t is unit. Hence $M = Sn$. □

Remark 1 *Let* S *be a semiring and* T *a non-empty multiplicatively closed subset of* S*, and let* M *be an* S*-semimodule. Define a relation* $\sim$ *on* $M \times T$ *as follows:* $(m, t) \sim (m', t') \Longleftrightarrow \exists s \in T$ *such that* $stm' = st'm$*. The relation* $\sim$ *on* $M \times T$ *is an equivalence relation. Denote the set* $M \times T/\sim$ *by* $T^{-1}M$ *and the equivalence class of each pair* $(m, s) \in M \times T$ *by* m/s*. We can define addition on* $T^{-1}M$ *by* $m/t + m'/t' = (t'm + tm')/tt'$*. Then* $(T^{-1}M, +)$ *is an abelian monoid. Let* $s/t \in T^{-1}S$ *and* $m/u \in T^{-1}M$*. We can define the product of* s/t *and* m/u *by* $(s/t)(m/u) = sm/tu$*. Then it is easy to check that* $T^{-1}M$ *is an* $T^{-1}S$*-semimodule [3]. Let* P *be a prime ideal in* S *and* $T = S \backslash P$*. Then* $T^{-1}M$ *is denoted by* M_P*.*

We can obtain the following results as in [15].

1. *Suppose that* I *is an ideal of a semiring* S *and* M *is an* S*-semimodule. Then* $T^{-1}(IM) = T^{-1}IT^{-1}M$*.*

2. *Let* N*,* N' *be subsemimodules of an* S*-semimodule* M*. If* $N_{\mathfrak{m}} = N'_{\mathfrak{m}}$ *for every maximal ideal* $\mathfrak{m}$*, then* $N = N'$*.*

Theorem 6 *Let* S *be a semiring and* M *a multiplication* S*-semimodule. If* P *is a maximal ideal of* S *such that* $M \neq PM$*, then* M_P *is cyclic.*

Proof. By (1), M_P is a multiplication S_P-semimodule. Since $M \neq PM$, $M_P \neq P_P M_P$ by (2). Moreover, by [10, Theorem 4.5], S_P is a local semiring. Thus by Corollary 1, M_P is cyclic. □

3 MC multiplication semimodules

In this section, we study MC multiplication semimodules and give some properties of these semimodules.

In [4] an S-semimodule M is called cancellative if for any $s, s' \in S$ and $0 \neq m \in M$, $sm = s'm$ implies $s = s'$. We will call these semimodules multiplicatively cancellative(abbreviated as MC). For example every ideal of a semidomain S is an MC S-semimodule.

Note that if M is an MC S-semimodule, then M is a faithful semimodule. Let $tM = \{0\}$ for some $t \in S$. If $0 \neq m \in M$, then $tm = 0m = 0$. Thus $t = 0$. Therefore M is faithful. But the converse is not true. For example, if S is an entire multiplicatively idempotent semiring, then every ideal of S is a faithful S-semimodule but it is not an MC semimodule.

Moreover, for an R-module M over a domain R, M is an MC semimodule iff it is torsionfree. Also we know that if R is a domain and M a faithful multiplication R-module, then M will be a torsionfree R-module and so M is an MC semimodule.

An element m of an S-semimodule M is cancellable if $m + m_1 = m + m_2$ implies that $m_1 = m_2$. The semimodule M is cancellative iff every element of M is cancellable [9, P. 172].

Lemma 1 *Let S be a yoked entire semiring and M a cancellative faithful multiplication S-semimodule. Then M is an MC semimodule.*

Proof. Let $0 \neq m \in M$ and $s, s' \in S$ such that $sm = s'm$. Since S is a yoked semiring, there exists $t \in S$ such that $s + t = s'$ or $s' + t = s$. Suppose that $s + t = s'$. Then $sm + tm = s'm$. Since M is a cancellative S-semimodule, $tm = 0$. Moreover, there exists an ideal I of S such that $Sm = IM$ since M is a multiplication S-semimodule. Then $tIM = tSm = \{0\}$ and hence $tI = \{0\}$ since M is faithful. But S is an entire semiring, so $t = 0$. Therefore $s = s'$. Now suppose that $s' + t = s$. A similar argument shows that $s = s'$. Therefore M is an MC semimodule. □

We now give the following definition similar to [12, P. 127].

Definition 2 *Let S be a semidomain. An S-semimodule M is said to be torsionfree if for any $0 \neq a \in S$, multiplication by a on M is injective, i.e., if $ax = ay$ for some $x, y \in M$, then $x = y$.*

Theorem 7 *Let S be a yoked semidomain and M a cancellative torsionfree S-semimodule. Then M is an MC semimodule.*

Proof. Let $0 \neq m \in M$ and $s, s' \in S$ such that $sm = s'm$. Since S is a yoked semiring, there exists $t \in S$ such that $s + t = s'$ or $s' + t = s$. Suppose that $s + t = s'$. Then $sm + tm = s'm$. Since M is a cancellative S-semimodule,

$tm = 0$. Since M is a torsionfree S-semimodule, $m = 0$ which is a contradiction. Thus $t = 0$ and hence $s = s'$. Now suppose that $s' + t = s$. A similar argument shows that $s = s'$. Therefore M is an MC semimodule. □

Now, similar to [7, Lemma 2.10] we give the following theorem (see also [6, Theorem 3.2]).

Theorem 8 *Let P be a prime ideal of S and M an MC multiplication semimodule. Let $a \in S$ and $x \in M$ such that $ax \in PM$. Then $a \in P$ or $x \in PM$.*

Proof. Let $a \notin P$ and put $K = \{s \in S \mid sx \in PM\}$. If $K \neq S$, there exists a maximal ideal Q of S such that $K \subseteq Q$. Let $M = QM$ and $m \in M$. Then similar to the proof of Theorem 4, there exists $q \in Q$ such that $m = qm$ which is a contradiction, since M is an MC semimodule. Therefore $M \neq QM$. Thus by Theorem 4, we can conclude that M is Q-cyclic. Therefore there exist $m \in M$, $t \in S$ and $q \in Q$ such that $t + q = 1$ and $tM \subseteq Sm$. Thus $tx = sm$ for some $s \in S$. Moreover, $tPM \subseteq Pm$. Hence $tax \in tPM \subseteq Pm$. Therefore $tax = p_1m$ for some $p_1 \in P$ and hence $asm = p_1m$. Since M is an MC semimodule, $as = p_1 \in P$ and since P is a prime ideal, $s \in P$. Then $tx = sm \in PM$ and hence $t \in K \subseteq Q$ which is a contradiction. Thus $K = S$. Therefore $x \in PM$. □

Lemma 2 (cf. [1]) *Suppose that S is a semiring. Let M be an S-semimodule and $\theta(M) = \sum_{m \in M}(Sm : M)$. If M is a multiplication S-semimodule, then $M = \theta(M)M$.*

Proof. Suppose that $m \in M$. Then $Sm = (Sm : M)M$. Thus $m \in (Sm : M)$ $M \subseteq \theta(M)M$. Therefore $M = \theta(M)M$. □

Theorem 9 (cf. [7, Theorem 3.1]) *Let S be a semiring and M an MC multiplication S-semimodule. Then the following statements hold:*

1. *If I and J are ideals of S such that $IM \subseteq JM$ then $I \subseteq J$.*
2. *For each subsemimodule N of M there exists a unique ideal I of S such that $N = IM$.*
3. *$M \neq IM$ for any proper ideal I of S.*
4. *$M \neq PM$ for any maximal ideal P of S.*
5. *M is finitely generated.*

Proof. (1) Let $IM \subseteq JM$ and $a \in I$. Set $K = \{s \in S \mid sa \in J\}$. If $K \neq S$, there exists a maximal ideal P of S such that $K \subseteq P$. By Theorem 4, M is P-cyclic since M is an MC semimodule. Thus there exist $m \in M$, $t \in S$ and $q \in P$ such that $t + q = 1$ and $tM \subseteq Sm$. Then $tam \in tIM \subseteq tJM = JtM \subseteq Jm$. Hence there exists $b \in J$ such that $tam = bm$. Since M is an MC semimodule, $ta = b \in J$. Thus $t \in K \subseteq P$ which is a contradiction. Therefore $K = S$ and hence $I \subseteq J$.

(2) Follows by (1)

(3) Follows by (2)

(4) Follows by (3)

(5) By Lemma 2, $M = \theta(M)M$, where $\theta(M) = \sum_{m\in M}(Sm : M)$. Then by 3, $\theta(M) = S$. Thus there exist a positive integer n and elements $m_i \in M$, $r_i \in (Sm_i : M)$ such that $1 = r_1 + \ldots + r_n$. If $m \in M$, then $m = r_1 m + \ldots + r_n m$. Therefore $M = Sm_1 + \ldots + Sm_n$. □

By Lemma 1, we have the following result.

Corollary 2 *Let S be a yoked entire semiring and M a cancellative faithful multiplication S-semimodule. Then the following statements hold:*

1. *If I and J are ideals of S such that $IM \subseteq JM$ then $I \subseteq J$.*
2. *For each subsemimodule N of M there exists a unique ideal I of S such that $N = IM$.*
3. *$M \neq IM$ for any proper ideal I of S.*
4. *$M \neq PM$ for any maximal ideal P of S.*
5. *M is finitely generated.*

The concept of cancellation modules was introduced in [14]. Similarly we call an S-semimodule M a cancellation semimodule if whenever $IM = JM$ for ideals I and J of S, then $I = J$.

Using the Theorem 9, we obtain the following corollary.

Corollary 3 *Let M be an MC multiplication semimodule. Then M is a cancellation semimodule.*

In [7, Lemma 4.1] it is shown that faithful multiplication modules are torsionfree. Similarly, we have the following result.

Theorem 10 *Suppose that S is a semidomain and M is an MC multiplication S-semimodule. Then M is a torsionfree S-semimodule.*

Proof. Suppose that there exist $0 \neq t \in S$ and $m, m' \in M$ such that $tm = tm'$. Then $Sm = IM$ and $Sm' = JM$ for some ideals I, J of S. Thus $tIM = tJM$ since $tm = tm'$. By Corollary 3, M is a cancellation semimodule, thus $tI = tJ$. Let $x \in I$. Then $tx = tx'$ for some $x' \in J$. Since S is a semidomain, $x = x'$. Therefore $I \subseteq J$. Similarly $J \subseteq I$. Hence $I = J$ and $Sm = Sm'$. Then there exists $s_1 \in S$ such that $m = s_1m'$. Thus $tm' = tm = ts_1m'$. Since M is an MC semimodule, $t = s_1t$. Since S is a semidomain, $s_1 = 1$. Therefore $m = m'$ and hence M is torsionfree. □

If M is a finitely generated faithful multiplication module, then M is a projective module [17, Theorem 11]. Similarly, we have the following theorem:

Theorem 11 *Let M be an MC multiplication semimodule. Then M is a projective S-semimodule.*

Proof. By Theorem 9, $\theta(M) = \sum_i^n (Sm_i : M) = S$. Thus for each $1 \leq i \leq n$, there exist $r_i \in (Sm_i : M)$ and $s_i \in S$ such that $1 = s_1r_1^2 + \ldots + s_nr_n^2$. Define a map $\phi_i : M \to S$ by $\phi_i : m \mapsto s_ir_ia$ where a is an element of S such that $r_im = am_i$. Suppose that $am_i = bm_i$ for some $b \in S$. Since M is an MC semimodule, $a = b$ and therefore ϕ_i is a well defined S-homomorphism. Let $m \in M$. Then $m = 1m = s_1r_1^2m + \ldots + s_nr_n^2m = \phi_1(m)m_1 + \ldots + \phi_n(m)m_n$. By [16, Theorem 3.4.12], M is a projective S-semimodule. □

By Lemma 1, we obtain the following result.

Corollary 4 *Let S be a yoked entire semiring and M a cancellative faithful multiplication S-semimodule. Then M is a projective S-semimodule.*

Theorem 12 [7, Lemma 3.6] *Let S be a semidomain and let M be an MC multiplication S-semimodule. Then there exists an invertible ideal I of S such that $M \cong I$.*

Proof. Suppose that $0 \neq m \in M$. Then there exists an ideal J of S such that $Sm = JM$. Let $0 \neq a \in J$. We can define an S-homomorphism $\phi : M \to Sm$ by $\phi : x \mapsto ax$. Let $x, x' \in M$ such that $ax = ax'$. By Theorem 10, M is torsionfree and hence $x = x'$. Therefore ϕ is injective and so $M \cong f(M)$. Now define an S-homomorphism $\phi' : S \to Sm$ by $\phi'(s) = sm$. Let $s, s' \in S$ such that $sm = s'm$. Since M is an MC semimodule, $s = s'$. Therefore ϕ' is injective. It is clear that ϕ' is surjective. Therefore $S \cong Sm$. Hence M is isomorphic to an ideal I of S. Thus I is a multiplication ideal and hence an invertible ideal of S. □

4 Cancellative multiplication semimodule

In this section, we investigate cancellative multiplication semimodules over some special semirings and restate some previous results. From now on, let S be a yoked semiring such that every maximal ideal of S is subtractive and let M be a cancellative S-semimodule.

Theorem 13 (See Theorems 4 and 3) *The S-semimodule M is a multiplication S-semimodule iff for every maximal ideal P of S either M is P-cyclic or $M = \{m \in M \mid m = qm \text{ for some } q \in P\}$.*

Proof. ($\Rightarrow$) Follows by Theorem 4.

($\Leftarrow$) Let N be a subsemimodule of M and $I = (N : M)$. Then $IM \subseteq N$. Let $x \in N$ and put $K = \{s \in S \mid sx \in IM\}$. If $K \neq S$, there exists a maximal ideal P of S such that $K \subseteq P$. If $M = \{m \in M \mid m = qm \text{ for some } q \in P\}$, then there exists $q \in P$ such that $x = qx$. Since S is a yoked semiring, there exists $t \in S$ such that $t + 1 = q$ or $q + t = 1$. Suppose that $q + t = 1$. Then $qx + tx = x$ and hence $tx = 0$. Therefore $t \in K \subseteq P$ which is a contradiction. Now suppose that $t + 1 = q$. Then $tx + x = qx$ and hence $tx = 0$. Therefore $t \in K \subseteq P$. But P is a subtractive ideal of S, so $1 \in P$ which is a contradiction. Therefore M is P-cyclic. Thus there exist $m \in M$, $t \in S$ and $q \in P$ such that $t + q = 1$ and $tM \subseteq Sm$. Therefore tN is a subsemimodule of Sm. Hence $tN = Jm$ where J is the ideal $\{s \in S \mid sm \in tN\}$ of S. Then $tJM = JtM \subseteq Jm \subseteq N$ and hence $tJ \subseteq I$. Thus $t^2x \in t^2N = tJm \subseteq IM$. Therefore $t^2 \in K \subseteq P$ which is a contradiction. □

Lemma 3 *If P is a maximal ideal of S, then $N = \{m \in M \mid m = qm \text{ for some } q \in P\}$ is a subsemimodule of M.*

Proof. *Let $m_1, m_2 \in N$. Then there exist $q_1, q_2 \in P$ such that $m_1 = q_1m_1$ and $m_2 = q_2m_2$. Since S is a yoked semiring, there exits an element r such that $q_1 + q_2 + r = q_1q_2$ or $q_1q_2 + r = q_1 + q_2$. Since P is a subtractive ideal, $r \in P$.*

Assume that $q_1q_2 + r = q_1 + q_2$. Then $q_1q_2(m_1 + m_2) + r(m_1 + m_2) = (q_1 + q_2)(m_1 + m_2)$. Thus $q_1q_2m_1 + q_1q_2m_2 + r(m_1 + m_2) = q_1m_1 + q_2m_1 + q_1m_2 + q_2m_2$. Hence $q_2m_1 + q_1m_2 + r(m_1 + m_2) = q_1m_1 + q_2m_1 + q_1m_2 + q_2m_2$. Since M is a cancellative S-semimodule, $r(m_1 + m_2) = q_1m_1 + q_2m_2$. Thus $r(m_1 + m_2) = m_1 + m_2$. Therefore $m_1 + m_2 \in N$.

Now assume that $q_1 + q_2 + r = q_1q_2$. Then $(q_1 + q_2 + r)(m_1 + m_2) = q_1q_2(m_1 + m_2)$. Hence $q_1m_1 + q_1m_2 + q_2m_1 + q_2m_2 + r(m_1 + m_2) = q_1q_2m_1 + q_1q_2m_2$. Thus $q_1m_1 + q_1m_2 + q_2m_1 + q_2m_2 + r(m_1 + m_2) = q_2m_1 + q_1m_2$. Since M

is a cancellative S*-semimodule,* $q_1m_1 + q_2m_2 + r(m_1 + m_2) = 0$ *and hence* $m_1 + m_2 + r(m_1 + m_2) = (1 + r)(m_1 + m_2) = 0$*. Since* P *is a subtractive ideal,* $(1 + r) \notin P$*. Therefore* $(1 + r) + P = S$ *since* P *is a maximal ideal of* S*. Thus there exist* $t \in P$ *and* $s \in S$ *such that* $s(1 + r) + t = 1$*. Hence* $s(1+r)(m_1+m_2)+t(m_1+m_2) = m_1+m_2$*. Therefore* $t(m_1+m_2) = m_1+m_2$ *and so* $m_1 + m_2 \in N$*.*

Let $s \in S$ *and* $m \in N$*. Then there exists* $q \in P$ *such that* $m = qm$*. Thus* $sm = sqm$*. Since* $sq \in P$*,* $sm \in N$*. Therefore* N *is a subsemimodule of* M*.* □

Similar to [7, Corollary 1.3], we have the following theorem.

Theorem 14 *Let* $M = \sum_{\lambda\in\Lambda} Sm_\lambda$*. Then* M *is a multiplication semimodule if and only if there exist ideals* $I_\lambda(\lambda \in \Lambda)$ *of* S *such that* $Sm_\lambda = I_\lambda M$ *for all* $\lambda \in \Lambda$*.*

Proof. ($\Rightarrow$) Obvious.

($\Leftarrow$) Assume that there exist ideals $I_\lambda(\lambda \in \Lambda)$ of S such that $Sm_\lambda = I_\lambda M(\lambda \in \Lambda)$. Let P be a maximal ideal of S and $I_\mu \nsubseteq P$ for some $\mu \in \Lambda$. Then there exists $t \in I_\mu$ such that $t \notin P$. Thus $P + (t) = S$ and hence there exist $q \in P$ and $s \in S$ such that $1 = q + st$. Then $tsM \subseteq I_\mu M = Sm_\mu$. Therefore M is P-cyclic. Now suppose that $I_\lambda \subseteq P$ for all $\lambda \in \Lambda$. Then $Sm_\lambda \subseteq PM(\lambda \in \Lambda)$. This implies that $M = PM$. But for any $\lambda \in \Lambda$, $Sm_\lambda = I_\lambda M = I_\lambda PM = Pm_\lambda$. Therefore $m_\lambda \in \{m \in M \mid m = qm \text{ for some } q \in P\}$. Since by Lemma 3, $\{m \in M \mid m = qm \text{ for some } q \in P\}$ is an S-semimodule, we conclude that $M = \{m \in M \mid m = qm \text{ for some } q \in P\}$. By Theorem 13, M is a multiplication semimodule. □

It follows from Theorem 14 that if S is a yoked semiring such that every maximal ideal of S is subtractive, then any additively cancellative ideal I generated by idempotents is a multiplication ideal.

The following is a generalization of [7, Theorem 3.1]

Theorem 15 *Let* M *be a faithful multiplication* S*-semimodule. Then the following statements are equivalent:*

1. *M is finitely generated.*
2. $M \neq PM$ *for any maximal ideal* P *of* S*.*
3. *If* I *and* J *are ideals of* S *such that* $IM \subseteq JM$ *then* $I \subseteq J$*.*
4. *For each subsemimodule* N *of* M *there exists a unique ideal* I *of* S *such that* $N = IM$*.*

5. $M \neq IM$ *for any proper ideal* I *of* S.

Proof. (1) $\to$ (2) Let P be a maximal ideal of S such that $M = PM$ and $M = Sm_1 + \ldots + Sm_n$. Since M is a multiplication S-semimodule, for each $1 \leq i \leq n$, there exists $K_i \subseteq S$ such that $Sm_i = K_iM = K_iPM = PK_iM = Pm_i$. Therefore $m_i = p_im_i$ for some $p_i \in P$. Since S is a yoked semiring, there exists $t_i \in S$ such that $t_i + p_i = 1$ or $1 + t_i = p_i$. Suppose that $t_i + p_i = 1$. Then $t_im_i + p_im_i = m_i$. Since M is a cancellative S-semimodule, $t_im_i = 0$. Now suppose that $1 + t_i = p_i$. Then $m_i + t_im_i = p_im_i$. Since M is a cancellative S-semimodule, $t_im_i = 0$. Put $t = t_1 \ldots t_n$. Then for all i, $tm_i = 0$. Thus $tM = \{0\}$. Since M is a faithful S-semimodule, $t = 0 \in P$. Since P is a prime ideal, $t_i \in P$ for some $1 \leq i \leq n$. If $t_i + p_i = 1$, then $1 \in P$ which is a contradiction. If $1 + t_i = p_i$, then, since P is a subtractive ideal of S, $1 \in P$ which is a contradiction. Therefore $M \neq PM$.

(2) $\to$ (3) Let I and J be ideals of S such that $IM \subseteq JM$. Let $a \in I$ and put $K = \{r \in S \mid ra \in J\}$. If $K \neq S$, then there exists a maximal ideal P of S such that $K \subseteq P$. By 2, $M \neq PM$. Thus M is P-cyclic and hence there exist $m \in M$, $t \in S$ and $q \in P$ such that $t + q = 1$ and $tM \subseteq Sm$. Then $tam \in tJM = JtM \subseteq Jm$. Thus there exists $b \in J$ such that $tam = bm$. Since S is a yoked semiring, there exists $c \in S$ such that $ta + c = b$ or $b + c = ta$. Suppose that $ta + c = b$. Then $t^2a + tc = tb$ and $tam + cm = bm$. Since M is cancellative, $cm = 0$. But $tcM \subseteq c(Sm) = \{0\}$. Since M is a faithful semimodule, $tc = 0$. Hence $t^2a = tb \in J$. Therefore $t^2 \in K \subseteq P$ which is a contradiction. Thus $S = K$ and $a \in J$. Now suppose that $b + c = ta$. Then $tb + tc = t^2a$ and $bm + cm = tam$. Since M is cancellative, $cm = 0$. A similar argument shows that $a \in J$.

(3) $\to$ (4) $\to$ (5) Obvious.

(5) $\to$ (1) By Lemma 2, $M = \theta(M)M$, where $\theta(M) = \sum_{m \in M}(Sm : M)$. Then by 5, $\theta(M) = S$. Thus there exist elements $m_i \in M$, $r_i \in (Sm_i : M)$ such that $1 = r_1 + \ldots + r_n$. Now let $m \in M$. Then $m = r_1m + \ldots + r_nm$. Hence M is finitely generated. □

Theorem 8 can be restated as follows:

Theorem 16 (cf. [5, Proposition 3]) *Suppose that* P *is a prime ideal and let* M *be a faithful multiplication* S*-semimodule. Let* $a \in S$ *and* $x \in M$ *such that* $ax \in PM$*. Then* $a \in P$ *or* $x \in PM$.

Proof. Let $a \notin P$ and $K = \{s \in S \mid sx \in PM\}$. Assume that $K \neq S$. Then there exists a maximal ideal Q of S such that $K \subseteq Q$. A similar argument to that of Theorem 13 shows that $M \neq QM$. Thus by Theorem 4, M is Q-cyclic. Therefore there exist $m \in M$, $t \in S$ and $q \in Q$ such that $t + q = 1$ and

$tM \subseteq Sm$. Thus $tx = sm$ for some $s \in S$. Since $tPM \subseteq Pm$, $tax \in tPM \subseteq Pm$. Hence $tax = p_1m$ for some $p_1 \in P$. Then $asm = p_1m$. Since S is a yoked semiring, there exists $c \in S$ such that $as + c = p_1$ or $c + p_1 = as$. Suppose that $as + c = p_1$. Then $asm + cm = p_1m$. Since M is cancellative, $cm = 0$. Then $tcM \subseteq c(Sm) = \{0\}$. Since M is a faithful semimodule, $tc = 0$. Hence $ast = p_1t \in P$ and so $s \in P$ since P is a prime ideal. Then $tx = sm \in PM$ and hence $t \in K \subseteq Q$ which is a contradiction. Thus $K = S$. Therefore $x \in PM$. Now suppose that $c + p_1 = as$. A similar argument shows that $x \in PM$. □

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The sparing number of certain graph powers

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Abstract. Let $\mathbb{N}_0$ be the set of all non-negative integers and $\mathcal{P}(\mathbb{N}_0)$ be its power set. Then, an integer additive set-indexer (IASI) of a given graph G is an injective function $f : V(G) \to \mathcal{P}(\mathbb{N}_0)$ such that the induced function $f^+ : E(G) \to \mathcal{P}(\mathbb{N}_0)$ defined by $f^+(uv) = f(u) + f(v)$ is also injective. An IASI f is said to be a weak IASI if $|f^+(uv)| = max(|f(u)|, |f(v)|)$ for all $u, v \in V(G)$. A graph which admits a weak IASI may be called a weak IASI graph. The set-indexing number of an element of a graph G, a vertex or an edge, is the cardinality of its set-labels. The sparing number of a graph G is the minimum number of edges with singleton set-labels, required for a graph G to admit a weak IASI. In this paper, we study the admissibility of weak IASI by certain graph powers and their corresponding sparing numbers.

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1 Introduction

For all terms and definitions, not defined specifically in this paper, we refer to [3, 7, 18]. For different graph classes, we further refer to [2, 4, 19]. Unless mentioned otherwise, all graphs considered here are simple, finite and have no isolated vertices.

The sumset of two non-empty sets A and B is denoted by $A + B$ and is defined by $A + B = \{a + b : a \in A, b \in B\}$ (see [8]). Using the concept of sumsets of two sets we have the following notion.

Let $\mathbb{N}_0$ denote the set of all non-negative integers. An *integer additive set-indexer* (IASI, in short) of a graph G is defined in [5] as an injective function $f : V(G) \to \mathcal{P}(\mathbb{N}_0)$ such that the induced function $f^+ : E(G) \to \mathcal{P}(\mathbb{N}_0)$ defined by $f^+(uv) = f(u) + f(v)$ is also injective (see [5, 9]).

The cardinality of the labeling set of an element (vertex or edge) of a graph G is called the *set-indexing number* of that element (see [9, 6]).

Lemma 1 [6] *Let A and B be two non-empty finite sets of non-negative integers. Then,* $\max(|A|, |B|) \leq |A + B| \leq |A|\,|B|$. *Therefore, for an integer additive set-indexer f of a graph G, we have* $\max(|f(u)|, |f(v)|) \leq |f^+(uv)| = |f(u) + f(v)| \leq |f(u)||f(v)|$, *where* $u, v \in V(G)$.

Definition 1 [6] An IASI f is said to be a *weak IASI* if $|f^+(uv)| = |f(u) + f(v)| = \max(|f(u)|, |f(v)|)$ for all $uv \in E(G)$. A graph which admits a weak IASI may be called a *weak IASI graph*. A weak IASI f is said to be *weakly k-uniform IASI* if $|f^+(uv)| = k$, for all $u, v \in V(G)$ and for some positive integer k.

Lemma 2 [6] *An IASI f define on a graph G is a weak IASI of G if and only if, with respect to f, at least one end vertex of every edge of G has the set-indexing number 1.*

Definition 2 [10] An element (a vertex or an edge) of graph which has the set-indexing number 1 is called a *mono-indexed element* of that graph. The *sparing number* of a graph G is defined to be the minimum number of mono-indexed edges required for G to admit a weak IASI and is denoted by $\varphi(G)$.

The following are some major results on the spring number of certain graph classes, which are relevant in our present study.

Theorem 1 [10] *An odd cycle C_n contains odd number of mono-indexed edges and an even cycle contains an even number of mono-indexed edges.*

Theorem 2 [10] *The sparing number of an odd cycle* C_n *is* 1 *and that of an even cycle is* 0.

Theorem 3 [10] *The sparing number of a bipartite graph is* 0.

Theorem 4 [10] *The sparing number of a complete graph* K_n *is* $\frac{1}{2}(n-1)(n-2)$.

Now, let us recall the definition of graph powers.

Definition 3 [3] The r-th power of a simple graph G is the graph G^r whose vertex set is V, two distinct vertices being adjacent in G^r if and only if their distance in G is at most r. The graph G^2 is referred to as the *square* of G, the graph G^3 as the *cube* of G.

The following is an important theorem on graph powers.

Theorem 5 [17] *If* d *is the diameter of a graph* G*, then* G^d *is a complete graph.*

Some studies on the sparing numbers of certain graph classes and graph structures have been done in [12, 13, 14]. As a continuation of these studies, in this paper, we determine the sparing number of the powers certain graph classes. The statements of the main results of this paper can also be seen in the review paper [15]. For the concepts of graph powers which admit certain types of IASIs, see [16] also.

2 Sparing number of square of some graphs

In this section, we estimate the sparing number of the square of certain graph classes. It is to be noted that the weak IASI f which gives the minimum number of mono-indexed edges in a given graph G will not induce a weak IASI for its square graph, since some of the vertices having non-singleton set-labels will also be at a distance 2 in G. Hence, interchanging the set-labels or relabeling certain vertices may be required to obtain a weak IASI for the square graph of a given graph.

First consider a path graph P_n on n vertices. The following theorem provides the sparing number of the square of a path P_n.

Proposition 1 *The sparing number of the square of a path* P_n *is given by*

$$\varphi(P_n^2) = \begin{cases} \frac{1}{3}(2n-3) & \text{if } n \equiv 0 \pmod 3 \\ \frac{1}{3}(2n-2) & \text{if } n \equiv 1 \pmod 3 \\ \frac{1}{3}(2n-1) & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

Proof. Let $P_m : v_1v_2v_3\ldots v_n$, where $m = n-1$. In P_m^2, $d(v_1) = d(v_n) = 2$ and $d(v_2) = d(v_{n-1}) = 3$ and $d(v_r) = 4$, where $3 \le r \le n-2$. Hence, $|E(P_m^2)| = \frac{1}{2}\sum_{v\in V} d(v) = \frac{1}{2}[2\times 2 + 2\times 3 + 4(n-4)] = (2n-3)$. Also, for $1 \le i \le n-2$, the vertices $v_i, v_{i+1},$ and v_{i+2} form a triangle in P_m^2. Then, by Theorem 5, each of these triangles must have a mono-indexed edge. That is, among any three consecutive vertices $v_i, v_{i+1},$ and v_{i+2} of P_m, two vertices must be mono-indexed. We require an IASI which makes the maximum possible number of vertices that are not mono-indexed. Hence, label v_1 and v_2 by singleton sets and v_3 by a non-singleton set. Since v_4 and v_5 are adjacent to v_3, they can be labeled only by distinct singleton sets that are not used before for labeling. Now, v_6 can be labeled by a non-singleton set that has not already been used. Proceeding like this the vertices which has the form $v_{3k}, 3k \le n$ can be labeled by distinct non-singleton sets and all other vertices by singleton sets. Now, we have to consider the following cases.

Case-1: If $n \equiv 0 \pmod 3$, then $n = 3k$. Therefore, v_n can also be labeled by a non-singleton set. Then the number of vertices that are not mono-indexed is $\frac{n}{3}$. Therefore, the number of edges that are not mono-indexed is $4(\frac{n}{3} - 1) + 2 = \frac{1}{3}(4n-6)$. Therefore, the total number of mono-indexed edges is $(2n-3) - \frac{1}{3}(4n-6) = \frac{1}{3}(2n-3)$.

Case-2: If $n \equiv 1 \pmod 3$, then $n - 1 = 3k$. Then, v_{n-1} can be labeled by a non-singleton set and v_n can be labeled by a singleton set. Then the number of vertices that are not mono-indexed is $\frac{n-1}{3}$. Therefore, the number of edges that are not mono-indexed is $4(\frac{(n-1)}{3} - 1) + 3 = \frac{1}{3}(4n-7)$. Therefore, the total number of mono-indexed edges is $(2n-3) - \frac{1}{3}(4n-7) = \frac{1}{3}(2n-2)$.

Case-3: If $n \equiv 2 \pmod 3$, then $n - 2 = 3k$. Then, v_{n-2} can be labeled by a non-singleton set and v_n and v_{n-1} can be labeled by distinct singleton sets. Then the number of vertices that are not mono-indexed is $\frac{n-2}{3}$. Therefore, the number of edges that are not mono-indexed is $4(\frac{(n-2)}{3} = \frac{1}{3}(4n-8)$. Therefore, the total number of mono-indexed edges is $(2n-3) - \frac{1}{3}(4n-8) = \frac{1}{3}(2n-1)$. □

Figure 1 illustrates squares of even and odd paths which admit weak IASIs. Mono-indexed edges of the graphs are represented by dotted lines.

Next, we shall discuss the sparing number of the square of cycles. We have $C_3^2 = C_3 = K_3$, $C_4^2 = K_4$ and $C_5^2 = K_5$ and hence by Theorem 4, their sparing numbers are $1, 3$ and 6 respectively. The following theorem determines the sparing number of the square of a given cycle on n vertices, for $n \ge 5$.

Theorem 6 *Let C_n be a cycle on n vertices. Then, the sparing number of the square of C_n is given by*

$$\varphi(C_n^2) = \begin{cases} \frac{2}{3}n & \textit{if } n \equiv 0 \pmod 3 \\ \frac{2}{3}(n+2) & \textit{if } n \equiv 1 \pmod 3 \\ \frac{2}{3}(n+4) & \textit{if } n \equiv 2 \pmod 3. \end{cases}$$

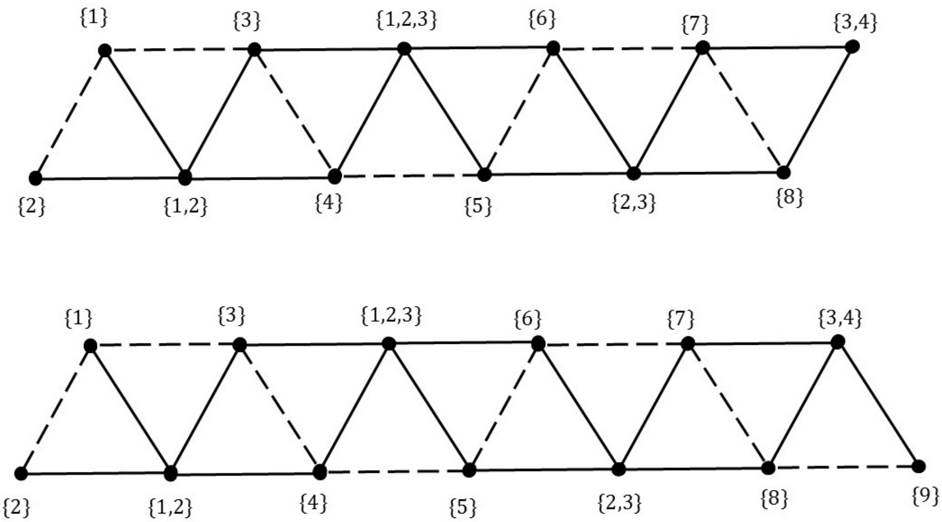

Figure 1: Squares of even and odd paths which admit weak IASI

Proof. Let $C_n : v_1v_2v_3\dots v_nv_1$ be the given cycle on n vertices. The square of C_n is a 4-regular graph. Also, $V(C_n^2) = V(C_n)$. Therefore, by the first theorem on graph theory, we have $\sum_{v\in V} d(v) = 2|E|$. That is, $2|E| = 4n \Rightarrow |E| = 2n$, $n \geq 5$.

First, label the vertex v_1 in C_n^2 by a non-singleton set. Therefore, four vertices $v_2, v_3, v_n,$ andv_{n-1} must be labeled by distinct singleton sets. Next, we can label the vertex v_4 by a non-singleton set, that is not already used for labeling. The vertices v_2 and v_3 have already been mono-indexed and the vertices v_5 and v_6 that are adjacent to v_4 in C_n^2 must be labeled by distinct singleton sets that are not used before for labeling. Proceeding like this, we can label all the vertices of the form v_{3k+1}, where k is a positive integer such that $3k+1 \leq n-2$ (since the last vertex that remains unlabeled is v_{n-2}).
Here, we need to consider the following cases.
Case-1: If $n \equiv 0 \pmod 3$, then $n-2 = 3k+1$ for some positive integer k. Then, v_{n-2} can be labeled by a non-singleton set. Therefore, the number of

vertices that are labeled by non-singleton sets is $\frac{n}{3}$. Since C_n^2 is 4-regular, we have the number of edges that are not mono-indexed in C_n^2 is $\frac{4n}{3}$. Hence, the number of mono-indexed edges is $2n - \frac{4n}{3} = \frac{2n}{3}$.
Case-2: If $n \equiv 1 \pmod 3$, then $n - 2 \neq 3k + 1$ for some positive integer k. Then, v_{n-2} can not be labeled by a non-singleton set. Here $n - 3 = 3k + 1$ for some positive integer k. Therefore, the number of vertices that are labeled by non-singleton sets is $\frac{n-1}{3}$ and the number of edges that are not mono-indexed in C_n^2 is $\frac{4(n-1)}{3}$. Hence, the number of mono-indexed edges is $2n - \frac{4(n-1)}{3} = \frac{2(n+2)}{3}$.
Case-3: If $n \equiv 2 \pmod 3$, then neither $n - 2$ nor $n - 3$ is equal to $3k + 1$ for some positive integer k. Here $n - 4 = 3k + 1$ for some positive integer k. Therefore, the number of vertices that are labeled by non-singleton sets is $\frac{n-2}{3}$ and the number of edges that are not mono-indexed in C_n^2 is $\frac{4(n-2)}{3}$. Hence, the number of mono-indexed edges is $2n - \frac{4(n-2)}{3} = \frac{2(n+4)}{3}$. □

Figure 2 illustrates the admissibility of weak IASIs by the squares of cycles. The graphs given in the figure are examples to the weak IASIs of an even cycle and an odd cycle respectively.

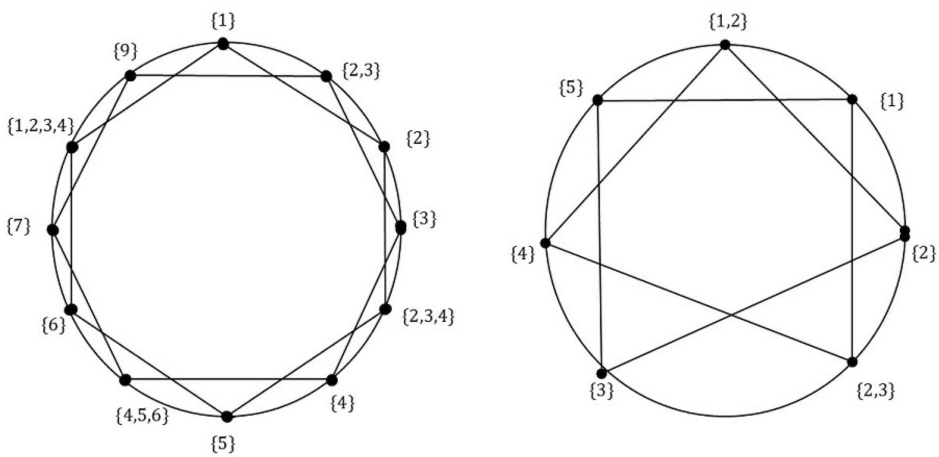

Figure 2: Weak IASIs of C_{12}^2 and C_7^2.

A question that arouses much interest in this context is about the sparing number of the powers of bipartite graphs. Invoking Theorem 5, we first verify the existence of weak IASIs for the complete bipartite graphs.

Theorem 7 *The sparing number of the square of a complete bipartite graph* $K_{m,n}$ *is* $\frac{1}{2}(m+n-1)(m+n-1)$.

Proof. The diameter of a graph $K_{m,n}$ is 2. Hence by Theorem 5, $K_{m,n}^2 = K_{m+n}$. Hence, every pair of vertices, that are not mono-indexed, are at a distance 2. The set-labels of all these vertices, except one, must be replaced by distinct singleton sets. Therefore, by Theorem 4, $\varphi(K_{m,n}^2) = \frac{1}{2}(m+n-1)(m+n-2)$. □

A *balanced bipartite graph* is the bipartite graph which has equal number of vertices in each of its bipartitions.

Corollary 1 *If* G *is a balanced complete bipartite graph on* $2n$ *vertices, then* $\varphi(G) = (n-1)(2n-1)$

Proof. Let $G = K_{n,n}$. Then by Theorem 7, $\varphi(G) = \frac{1}{2}(2n-1)(2n-2) = (n-1)(2n-1)$. □

Let G be a bipartite graph. The vertices which are at a distance 2 are either simultaneously mono-indexed or simultaneously labeled by non-singleton sets. Therefore, in G^2, among any pair of vertices which are are not mono-indexed and are at a distance 2 between them, one vertex should be relabeled by a singleton set. Hence, the sparing number of the square of a bipartite graph G depends on the adjacency pattern of its vertices. Hence, the problem of finding the sparing number of bipartite graphs does not offer much scope in this context.

Now we proceed to study the admissibility of weak IASI by the squares of certain other graph classes. First, we discuss about the sparing number of *wheel graphs*. A wheel graph can be defined as follows.

Definition 4 [4] A *wheel graph* is a graph defined by $W_{n+1} = C_n + K_1$. The following theorem discusses the sparing number of the square of a wheel graph W_{n+1}.

The sparing number of the square of a wheel graph W_{n+1} is determined in the following result.

Proposition 2 *The sparing number of the square of a wheel graph on* $n+1$ *vertices is* $\frac{1}{2}n(n-1)$.

Proof. The diameter of a wheel graph W_{n+1}, for any positive integer $n \geq 3$, is 2. Hence, by Theorem 5, the square of a wheel graph W_{n+1} is a complete graph on $n+1$ vertices. Therefore, by Theorem 4, the sparing number of the square graph W_{n+1}^2 is $\frac{1}{2}n(n-1)$. □

Next, we determine the sparing number of another graph class known as *helm graphs* which is defined as follows.

Definition 5 *A* helm graph*, denoted by* H_n*, is the graph obtained by adjoining a pendant edge to each vertex of the outer cycle* C_n *of a wheel graph* W_{n+1}*. It has* $2n+1$ *vertices and* $3n$ *edges.*

The following result determines the sparing number of a helm graph.

Theorem 8 *The sparing number of the square of a helm graph* H_n *is* $\frac{1}{2}n$ $(n+1)$.

Proof. Let v be the central vertex, $V = \{v_1, v_2, v_3, \dots, v_n\}$ be the vertex set of the outer cycle of the corresponding wheel graph and $W = \{w_1, w_2, w_3, \dots, w_n\}$ be the set of pendant vertices in H_n.

The vertex v is adjacent to all the vertices in V and is at distance 2 from all the vertices in W. Therefore, the degree of v in H_n^2 is $2n$. In H_n, for $1 \le i \le n$, each v_i is adjacent to two vertices v_{i-1} and v_{i+1} in V and is adjacent to w_i in W and to the vertex v and is at a distance 2 from all the remaining vertices in V and from the vertices w_{i-1} and w_{i+2} in W. Therefore, the degree of each $v_i \in V$ in H_n^2 is $n+3$. Now, in H_n, each vertex w_i is adjacent to the vertex v_i in V and is at a distance 2 from two vertices v_{i-1} and v_{i+2} in V and to the central vertex v. Hence, the degree of each $w_i \in W$ in H_n^2 is 4. Therefore, the number of edges in H_n, $|E| = \frac{1}{2}\sum_{u \in V(H_n)} d(u) = \frac{1}{2}[2n + n(n+3) + 4n] = \frac{1}{2}n(n+9)$.

It is to be noted that W is an independent set in H_n^2 and we can label all vertices in W by distinct non-singleton sets. It can be seen that there are more edges in H_n^2 that are not mono-indexed if we label all the vertices of W by non-singleton sets than labeling possible number of vertices of $V \cup \{v\}$ by non-singleton sets. Therefore, the number of edges of H_n^2 which are not mono-indexed is $4n$. Therefore, the number of mono-indexed edges in H_n^2 is $\frac{1}{2}n(n+9) - 4n = \frac{1}{2}n(n+1)$. □

Figure 3 illustrates the existence of a weak IASI for the square of a helm graph.

An interesting question in this context is about the sparing number of some graph classes containing complete graphs as subgraphs. An important graph class of this kind is a complete n-*sun* which is defined as follows.

Definition 6 [2] An n-*sun* or a *trampoline*, denoted by S_n, is a chordal graph on $2n$ vertices, where $n \ge 3$, whose vertex set can be partitioned into two sets $U = \{u_1, u_2, c_3, \dots, u_n\}$ and $W = \{w_1, w_2, w_3, \dots, w_n\}$ such that W is an independent set of G and w_j is adjacent to u_i if and only if $j = i$ or $j = i + 1$ (mod$n$). *A complete sun* is a sun G where the induced subgraph $\langle U \rangle$ is complete.

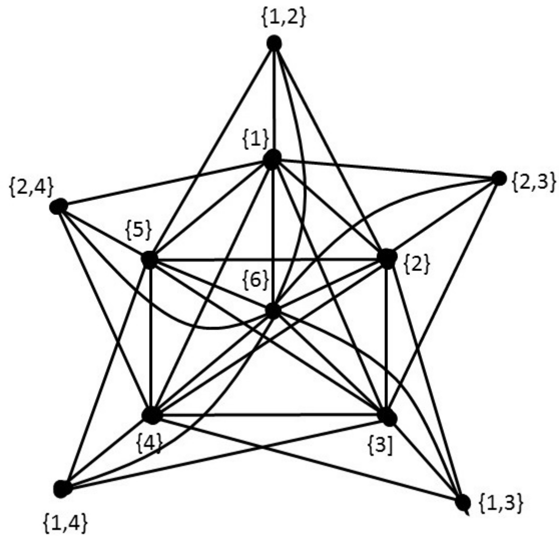

Figure 3: Square of a helm graph with a weak IASI defined on it.

The following theorem determines the sparing number of the square of complete sun graphs.

Theorem 9 *Let* G *be the complete sun graph on* $2n$ *vertices. Then sparing number of* G^2 *is*

$$\varphi(G^2) = \begin{cases} n^2+1 & \text{if } n \text{ is odd} \\ \frac{n}{2}(2n-1) & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let G be a sun graph on $2n$ vertices, whose vertex set can be partitioned into two sets $U = \{u_1, u_2, c_3, \ldots, u_n\}$ and $W = \{w_1, w_2, w_3, \ldots, w_n\}$ such that w_j is adjacent to u_i if and only if $j = i$ or $j = i+1 \pmod n$, where W is an independent set and the induced subgraph $\langle U \rangle$ is complete.

In G, the degree of each u_i is $n+1$ and the degree of each w_j is 2. It can be seen that each vertex w_j is adjacent to two vertices in U and is at a distance 2 from all other vertices in U. Hence, in G^2, each vertex w_j is adjacent to all vertices in U and to two vertices w_{j-1} and w_{j+1} (in the sense that $w_0 = w_n$ and $w_{n+1} = w_1$). That is, in G^2, the degree of each vertex w_j in W is $n+2$ and the degree of each vertex u_i in U is $2n-1$. Therefore, $|E(G^2)| = \frac{1}{2}\sum_{v \in V} d(v) = \frac{1}{2}[n(n+2) + n(2n-1)] = \frac{1}{2}n(3n+1)$.

If we label any vertex u_i by a non-singleton set, then no other vertex in G^2 can be labeled by non-singleton sets, as each u_i is adjacent to all other vertices in G^2. Therefore, we label possible number of vertices in W by non-singleton sets. Since w_j is adjacent to w_{j+1}, only alternate vertices in W can be labeled by non-singleton sets.

Case 1: If n odd, then $\frac{1}{2}(n-1)$ vertices W can be labeled by distinct non-singleton sets. Therefore, the number of edges that are not mono-indexed in G^2 is $\frac{1}{2}(n-1)(n+2)$. Hence, the number of mono-indexed edges in G^2 is $\frac{1}{2}n(3n+1) - \frac{1}{2}(n-1)(n+2) = n^2+1$.

Case 2: If n even, then $\frac{n}{2}$ vertices W can be labeled by distinct non-singleton sets. Therefore, the number of edges that are not mono-indexed in G^2 is $\frac{1}{2}n(n+2)$. Hence, the number of mono-indexed edges in G^2 is $\frac{1}{2}n(3n+1)-\frac{1}{2}n(n+2) = \frac{1}{2}n(2n-1)$. □

Theorem 9 is illustrated in Figure 4. The first and second graphs in 9 are example to the weak IASIs of the square of the complete n-sun graphs where n is odd and even respectively.

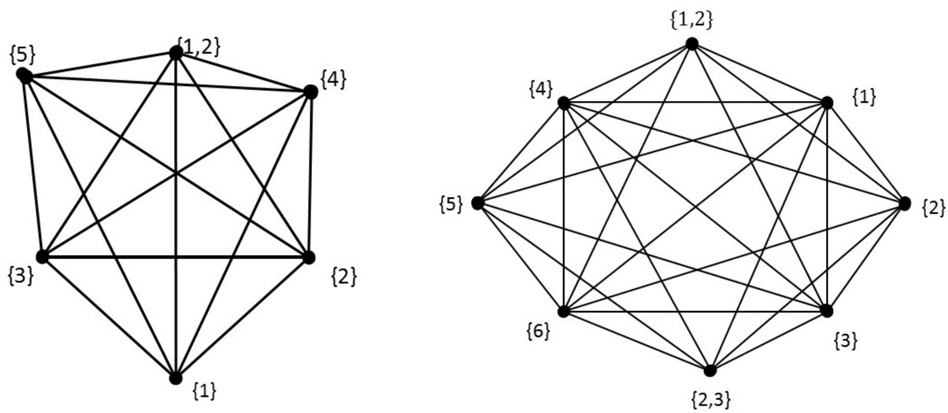

Figure 4: Weak IASIs of the square of a complete 3-sun and a complete 4-sun.

Another important graph that contains a complete graph as one of its subgraph is a split graph, which is defined as follows.

Definition 7 [2] A *split graph* is a graph in which the vertices can be partitioned into a clique K_r and an independent set S. A split graph is said to be a *complete split graph* if every vertex of the independent set S is adjacent to every vertex of the the clique K_r and is denoted by $K_S(r,s)$, where r and s are the orders of K_r and S respectively.

The following theorem establishes the sparing number of the square of a complete split graph.

Theorem 10 *Let* $G = K_S(r, s)$ *be a complete split graph without isolated vertices. Then, the sparing number of* G^2 *is* $\frac{1}{2}[(r+s-1)(r+s-2)]$, *where* $r = |V(K_r)|$ *and* $s = |S|$.

Proof. Since G has no isolated vertices, every vertex of v_i S is adjacent to at least one vertex u_j of K_r. Then, v_i is at a distance 2 from all other vertices of K_r. Hence, in G^2 each vertex v_i in S is adjacent to all the vertices of K_r. Also, in G, two vertices of S is at a distance 2 from all other vertices of S. Therefore, every pair of vertices in S are also adjacent in G^2. That is, G^2 is a complete graph on $r+s$ vertices. Hence, by Theorem 4, $\varphi(G^2) = \frac{1}{2}[(r+s-1)(r+s-2)$. □

So far we have discussed about the sparing number of square of certain graph classes. In this context, a study about the sparing number of the higher powers of these graph classes is noteworthy. In the following section, we discuss about the sparing number of arbitrary powers of certain graph classes.

3 Sparing number of arbitrary graph powers

For the descriptions of graph powers, please see [16] also. For any positive integer n, we know that the diameter of a complete graph K_n is 1. Hence, any power of K_n, denoted by K_n^r is K_n itself. Hence, we have the following result.

Proposition 3 *For a positive integer* r, $\varphi(K_n^r) = \frac{1}{2}(n-1)(n-2)$.

Proof. We have $K_n^r = K_n$. Hence, $\varphi(K_n^r) = \varphi(K_n)$. Therefore, by Theorem 4, $\varphi(K_n^r) = \frac{1}{2}(n-1)(n-2)$. □

The following results discuss about the sparing numbers of the arbitrary powers of the graph classes which are discussed in Section 2.

Proposition 4 *For a positive integer* $r > 1$, *the sparing number of the* r*-th power of a complete bipartite graph* $K_{m,n}$ *is* $\frac{1}{2}(m+n-1)(m+n-2)$.

Proof. Since $K_{m,n}^2 = K_{m+n}$, we have $K_{m,n}^r = K_{m+n}$ for any positive integer $r \geq 2$. Therefore, $\varphi(K_{m,n}^r) = \varphi(K_{m+n}) = \frac{1}{2}(m+n-1)(m+n-2)$. □

Proposition 5 *Let* G *be a split graph, without isolated vertices, that contains a clique* K_r *and an independent set* S *with* $|S| = s$*. Then, for* $r \geq 3$*, the sparing number of* G^r *is* $\frac{1}{2}(r+s-1)(r+s-2)$*.*

Proof. Since S has no isolated vertices in G, every pair vertices of S are at a distance at most 3 among themselves. Hence, G^3 is a complete graph. Therefore, For any $r \geq 3$, G^r is a complete graph. Hence by Theorem 4, the sparing number of G^r is $\frac{1}{2}(r+s-1)(r+s-2)$. □

Theorem 11 *For a positive integer* $r > 2$*, the sparing number of* H_n^r *is*

$$\varphi(H_n^r) = \begin{cases} \lfloor \frac{n}{2} \rfloor (n+3) & \text{if } r = 3 \\ n(2n-1) & \text{if } r \geq 4. \end{cases}$$

Proof. Let u be the central vertex, $V = \{v_1 v_2 v_3, \ldots, v_n\}$ be the set of vertices of the cycle C_n and $W = \{w_1, w_2, w_3, \ldots,$
$w_n\}$ be the set of pendant vertices in H_n. In H_n, the central vertex u is adjacent to each vertex v_i of V and each v_i is adjacent to a vertex w_i in W.

Since each vertex w_i in W is at a distance at most 3 from u as well as from all vertices of V, for $1 \leq i \leq n$, and from two vertices w_{i-1} and w_{i+1} of W, the subgraph of H_n^3 induced by $V \cup \{u, w_{i-1}, w_i, w_{i+1}\}$ is a complete graph. Hence only one vertex of this set can have a non-singleton set-label. We get minimum number of mono-indexed edges if we label possible number of vertices in W by non-singleton sets. Since w_i is adjacent to w_{i-1} and w_{i+1}, only alternate vertices in W can be labeled by non-singleton sets. Therefore, $\lfloor \frac{n}{2} \rfloor$ vertices in W can be labeled by non-singleton sets. Therefore, since each w_i is of degree $n+3$, total number of edges in H_n^3, that are not mono-indexed, is $\lfloor \frac{n}{2} \rfloor (n+3)$.

The distance between any two points of a helm graph is at most 4. Hence, G^4 is a complete graph. Therefore, For any $r \geq 4$, G^r is a complete graph. Hence by Theorem 4, the sparing number of G^r is $n(2n-1)$. □

We have not determined the sparing number of arbitrary powers of paths and cycles yet. The following results discusses the sparing number of the r-th power of a path on n vertices.

The diameter of a path P_m on $n = m+1$ vertices is $m = n-1$. Therefore, by Theorem 5, $P_m^m = P_{n-1}^{n-1}$ is a complete graph. Hence, we need to study about the r-th powers of P_{n-1} if $r < n-1$.

Theorem 12 *Let* P_{n-1} *be a path graph on* n *vertices. Then, its spring number is* $\frac{r-1}{2(r+1)}[r(2n-1-r)+2i]$*.*

Proof. Let $P_m : v_1v_2v_3\dots v_n$, where $m = n-1$. In P_m^2, $d(v_1) = d(v_n) = r, d(v_2) = d(v_{n-1}) = r+1, \dots, d(v_r) = d(v_{n-r+1} = r+r-1 = 2r-1$ and $d(v_j) = 2r,\ r+1 \le j \le n-r$. Hence, $\sum\limits_{v\in V(P_n)} d(v) = 2[r+(r+1)+(r+2)+ \dots + 2r-1)] + (n-2r)2r = r(2n-1-r)$. Therefore, $|E(P_m^r)| = \frac{r}{2}(2n-1-r)$.

It can be seen that among any $r+1$ consecutive vertices $v_i, v_{i+1}, \dots v_{i+r}$ of P_m, r vertices must be mono-indexed. Hence, label $v_1, v_2, \dots, v_k$ by singleton sets and v_{r+1} by a non-singleton set. Since $v_{r+2}, v_{r+3}\dots, v_{2r+1}$ are adjacent to v_{r+1}, they can be labeled only by distinct singleton sets that are not used before for labeling. Now, v_{2r+2} can be labeled by a non-singleton set that has not already been used. Proceeding like this the vertices which has the form $v_{(r+1)k}, (r+1)k \le n$ can be labeled by distinct non-singleton sets and all other vertices by singleton sets.

If $n \equiv\ i \pmod{(k+1)}$, then v_{n-i} can also be labeled by a non-singleton set. Then the number of vertices that are not mono-indexed is $\frac{n-i}{r+1}$. Therefore, the number of edges that are not mono-indexed is $2r[\frac{(n-i)}{r+1} - 1] + (r+i) = \frac{1}{r+1}[r(2n-1-r)-(r-1)i]$. Therefore, the total number of mono-indexed edges is $\frac{r}{2}(2n-1-r) - \frac{1}{r+1}[r(2n-1-r)-(r-1)i] = \frac{r-1}{2(r+1)}[r(2n-1-r)+2i]$. □

Figure 5 depicts the cube of a path with a weak IASI defined on it.

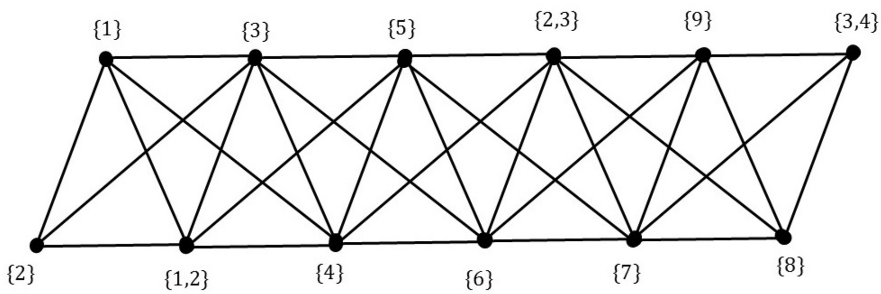

Figure 5: Cubes of a path which admits a weak IASI

The diameter of a cycle C_n is $\lfloor \frac{n}{2} \rfloor$. Therefore, by Theorem 5, $C_n^{\lfloor \frac{n}{2} \rfloor}$ (and higher powers) is a complete graph. Hence, we need to study about the r-th power of C_n if $r < \lfloor \frac{n}{2} \rfloor$. The following theorem discusses about the sparing number of an arbitrary power of a cycle.

Theorem 13 *Let C_n be a cycle on n vertices and let r be a positive integer less than $\lfloor \frac{n}{2} \rfloor$. Then the sparing number of the the r-th power of C_n is given by* $\varphi(C_n^r) = \frac{r}{r+1}((r-1)n + 2i)$ *if* $n \equiv i \pmod{(r+1)}$.

Proof. Let $C_n : v_1v_2v_3 \dots v_nv_1$ be the given cycle on n vertices. The graph C_n^r is a $2r$-regular graph. Therefore, we have $|E(C_n^r)| = \frac{1}{2}\sum_{v \in V} d(v) = rn$.

First, label the vertex v_1 in C_n^r by a non-singleton set. Therefore, $2r$ vertices $v_2, v_3, \dots v_{r+1}, v_n, v_{n-1} \dots v_{n-r+1}$ can be labeled only by distinct singleton sets. Next, we can label the vertex v_{r+2} by a non-singleton set, that is not already used for labeling. Since the vertices $v_2, v_3, \dots v_{r+1}$ have already been mono-indexed, r vertices $v_{r+3}, v_{r+4}, \dots v_{2r+2}$ that are adjacent to v_{r+2} in C_n^r must be labeled by distinct singleton sets. Proceeding like this, we can label all the vertices of the form $v_{(r+1)k+1}$, where k is a positive integer less than $\lfloor n \rfloor$, such that $(r+1)k+1 \leq n-r$ (since the last vertex that remains unlabeled is v_{n-r}).

If $n \equiv i \pmod{(k+1)}$, then $n-i = (r+1)k+1$ for some positive integer k. Then, $v_{n-(r-i)}$ can be labeled by a non-singleton set. Therefore, the number of vertices that are labeled by non-singleton set is $\frac{n-i}{r+1}$. Since C_n^r is $2r$-regular, the number of edges that are not mono-indexed in C_n^r is $2r\frac{n-i}{r+1}$. Hence, the number of mono-indexed edges is $rn - 2r\frac{n-i}{r+1} = \frac{r}{r+1}((r-1)n + 2i)$. □

Figure 6 illustrates the admissibility of weak IASIs by the squares of even and odd cycles.

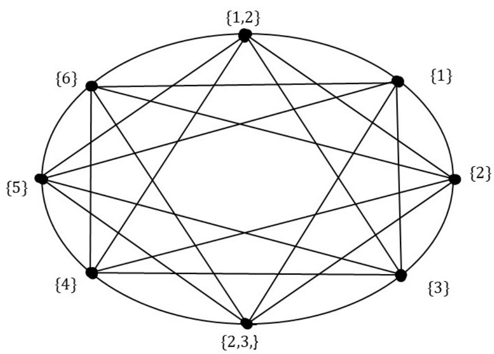

Figure 6: Cube of a cycle with a weak IASI defined on it.

4 Conclusion

In this paper, we have established some results on the admissibility of weak IASIs by certain graphs and graph powers. The admissibility of weak IASI by various graph classes, graph operations and graph products and finding the corresponding sparing numbers are still open.

In this paper, we have not addressed the following problems, which are still open. The adjacency and incidence patterns of elements of the graph concerned will matter in determining its admissibility of weak IASI and the sparing number.

Problem 1 Find the sparing number of the r-th power of trees and in particular, binary trees for applicable values of r.

Problem 2 Find the sparing number of the r-th power of bipartite graph and in general, graphs that don't have a complete bipartite graphs as their subgraphs, for applicable values of r.

Problem 3 Find the sparing number of the r-th power of an n-sun graph that is not complete, for applicable values of r.

Problem 4 Find the sparing number of the square of a split graph that is not complete.

Some other standard graph structures related to paths and cycles are lobster graph, ladder graphs, grid graphs and prism graphs. Hence, the following problems are also worth studying.

Problem 5 Find the sparing number of arbitrary powers of a lobster graph.

Problem 6 Find the sparing number of arbitrary powers of a ladder graphs L_n.

Problem 7 Find the sparing number of arbitrary powers of grid graphs (or lattice graphs) $L_{m,n}$.

Problem 8 Find the sparing number of arbitrary powers of prism graphs and anti-prism graphs.

Problem 9 Find the sparing number of arbitrary powers of armed crowns and dragon graphs.

More properties and characteristics of different IASIs, both uniform and non-uniform, are yet to be investigated. The problems of establishing the necessary and sufficient conditions for various graphs and graph classes to have certain IASIs are also open.

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Radii problems for normalized q−Bessel and Wright functions

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Abstract. In this investigation, our main objective is to ascertain the radii of k-uniform convexity of order α and the radii of strong starlikeness of the some normalized q-Bessel and Wright functions. In making this investigation we deal with the normalized Wright functions for three different kinds of normalization and six different normalized forms of q-Bessel functions. The key tools in the proof of our main results are the Mittag-Leffler expansion for Wright and q-Bessel functions and properties of real zeros of these functions and their derivatives. We also have shown that the obtained radii are the smallest positive roots of some functional equations.

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1 Introduction

Special functions have an indispensable role in many branches of mathematics and applied mathematics. Thus, it is important to examine their properties in many aspects. In the recent years, there has been a vivid interest on some special functions from the point of view of geometric function theory. For more details we refer to the papers [1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18] and references therein. However, the origins of these studies can be traced to Brown [20], to Kreyszig and Todd [22], and to Wilf [24]. These studies initiated investigation on the univalence of Bessel functions and determining the radius of starlikeness for different kinds of normalization. In other words, their results have a very important place on account of the fact that they have paved the way for obtaining other geometric properties of Bessel function such as univalence, starlikeness, convexity and so forth. Recently, in 2014, Baricz *et al.* [11], by considering a much simpler approach, succeeded to determine the radius of starlikeness of the normalized Bessel functions. In the same year, Baricz and Szász [15] obtained the radius of convexity of the normalized Bessel functions. We see in their proofs that some properties of the zeros of Bessel functions and the Mittag-Leffler expansions for Bessel function of the first kind play a crucial role in determining the radii of starlikeness and convexity of Bessel functions of the first kind. It is worth to mention that some geometric properties of other special functions involving Bessel function of first kind were investigated extensively by several authors. For instance, in 2017, Deniz and Szász [21] studied on determining the radius of uniform convexity of the normalized Bessel functions. And also, very recently, Bohara and Ravichandran in [19] determined, by using the method of Baricz *et al.* [11, 15, 16, 21], the radius of strong starlikeness and $k-$uniform convexity of order α of the normalized Bessel functions.

Inspired by the above mentioned results and considering the approach of Baricz *et al.* in this paper, we investigate the radius of strong starlikeness and $k-$uniform convexity of order α of the normalized Wright and $q-$Bessel functions.

This paper is organized as follows: The rest of this section contains some basic definitions needed for the proof of our main results. Section 2 is divided into three subsections: The first subsection is devoted to the radii of $k-$uniform convexity of order α of normalized Wright functions. The second subsection contains the study of the radii of $k-$uniform convexity of order α of normalized $q-$Bessel functions. The third subsection is dedicated to the radius of strong starlikeness of normalized Wright and $q-$Bessel functions.

Before starting to present our main results we would like to call attention to some basic concepts, which are used by us for building our main results. For $r > 0$ we denote by $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$ the open disk with radius r centered at the origin. Let $f : \mathbb{D}_r \to \mathbb{C}$ be the function defined by

$$f(z) = z + \sum_{n \geq 2} a_n z^n, \tag{1}$$

here r is less or equal than the radius of convergence of the above power series. Let $\mathcal{A}$ be the class of analytic functions of the form (1), that is, normalized by the conditions $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of univalent functions.

In this paper, for $k \geq 0$ and $0 \leq \alpha < 1$ we study on more general class $\mathcal{UCV}(k, \alpha)$ of $k-$uniformly convex functions of order α. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{UCV}(k, \alpha)$ if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right| + \alpha \quad (z \in \mathbb{D}).$$

The real number

$$r^{uc}_{k,\alpha}(f) = \sup\left\{r > 0 \,\middle|\, \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right| + \alpha \text{ for all } z \in \mathbb{D}_r\right\}$$

is called the radius of $k-$uniform convexity of order α of the function f.

Finally, let us take a look at the next lemma which is very useful in building our main results. It is worth to mention that the following lemma was proven by Deniz and Szász [21].

Lemma 1 (see [21]) *If* $a > b > r \geq |z|$, *and* $\lambda \in [0, 1]$, *then*

$$\left|\frac{z}{b - z} - \lambda\frac{z}{a - z}\right| \leq \frac{r}{b - r} - \lambda\frac{r}{a - r}. \tag{2}$$

The followings can be obtained as a natural consequence of this inequality:

$$\operatorname{Re}\left(\frac{z}{b - z} - \lambda\frac{z}{a - z}\right) \leq \frac{r}{b - r} - \lambda\frac{r}{a - r} \tag{3}$$

and

$$\operatorname{Re}\left(\frac{z}{b - z}\right) \leq \left|\frac{z}{b - z}\right| \leq \frac{r}{b - r}. \tag{4}$$

We are now in a position to present our main results.

2 Main results

2.1 The radii of k-uniform convexity of order α of normalized Wright functions

In this subsection, we will focus on the function

$$\phi(\rho,\beta,z)=\sum_{n\geq 0}\frac{z^n}{n!\Gamma(n\rho+\beta)} \qquad (\rho>-1 \quad z,\beta\in\mathbb{C})$$

named after the British mathematician E.M. Wright. It is well known that this function was introduced by him for the first time in the case $\rho>0$ in connection with his investigations on the asymptotic theory of partitions [26].

From [17, Lem. 1] we know that under the conditions $\rho>0$ and $\beta>0$, the function $z\mapsto\lambda_{\rho,\beta}(z)=\phi(\rho,\beta,-z^2)$ has infinitely many zeros which are all real. Thus, in light of the Hadamard factorization theorem, the infinite product representation of the function $\lambda_{\rho,\beta}(z)$ can be written as

$$\Gamma(\beta)\lambda_{\rho,\beta}(z)=\prod_{n\geq 1}\left(1-\frac{z^2}{\lambda^2_{\rho,\beta,n}}\right)$$

where $\lambda_{\rho,\beta,n}$ is the nth positive zero of the function $\lambda_{\rho,\beta}(z)$ (or the positive real zeros of the function $\Psi_{\rho,\beta}$). Moreover, let $\zeta'_{\rho,\beta,n}$ denote the nth positive zero of $\Psi'_{\rho,\beta}$, where $\Psi_{\rho,\beta}(z)=z^{\beta}\lambda_{\rho,\beta}(z)$, then the zeros satisfy the chain of inequalities

$$\zeta'_{\rho,\beta,1}<\zeta_{\rho,\beta,1}=\lambda_{\rho,\beta,1}<\zeta'_{\rho,\beta,2}<\zeta_{\rho,\beta,2}=\lambda_{\rho,\beta,2}<\ldots.$$

One can easily see that the function $z\mapsto\phi(\rho,\beta,-z^2)$ do not belong to $\mathcal{A}$, and thus first we perform some natural normalizations. We define three functions originating from $\phi(\rho,\beta,.)$:

$$f_{\rho,\beta}(z)=\left(z^{\beta}\Gamma(\beta)\phi(\rho,\beta,-z^2)\right)^{\frac{1}{\beta}},$$
$$g_{\rho,\beta}(z)=z\Gamma(\beta)\phi(\rho,\beta,-z^2),$$
$$h_{\rho,\beta}(z)=z\Gamma(\beta)\phi(\rho,\beta,-z).$$

Clearly, these functions are contained in the class $\mathcal{A}$.

Now, we would like to present our results regarding the $k-$uniform convexity of order α of the functions $f_{\rho,\beta}$, $g_{\rho,\beta}$ and $h_{\rho,\beta}$.

Theorem 1 *Let* $\beta, \rho > 0$*,* $\alpha \in [0, 1)$ *and* $k \geq 0$*. Then, the following statements are valid:*

- **a.** *The radius of* k*-uniform convexity of order* α *of the function* $f_{\rho,\beta}$ *is the real number* $r^{uc}_{k,\alpha}(f_{\rho,\beta})$ *which is the smallest positive root of the equation*

$$(1 + k)r\frac{\Psi''_{\rho,\beta}(r)}{\Psi'_{\rho,\beta}} + (\frac{1}{\beta} - 1)(1 + k)r\frac{\Psi'_{\rho,\beta}(r)}{\Psi_{\rho,\beta}(r)} + 1 - \alpha = 0$$

in the interval $(0, \zeta'_{\rho,\beta,1})$*, where* $\Psi_{\rho,\beta}(z) = z^{\beta}\lambda_{\rho,\beta}(z)$ *and* $\zeta'_{\rho,\beta,1}$ *stands for the smallest positive zero of the function* $\Psi'_{\rho,\beta}(z)$*.*

- **b.** *The radius of* k*-uniform convexity of order* α *of the function* $g_{\rho,\beta}$ *is the real number* $r^{uc}_{k,\alpha}(g_{\rho,\beta})$ *which is the smallest positive root of the equation*

$$(1 + k)r\frac{g''_{\rho,\beta}(r)}{g'_{\rho,\beta}(r)} + 1 - \alpha = 0$$

in the interval $(0, \vartheta_{\rho,\beta,1})$*, where* $\vartheta_{\rho,\beta,1}$ *stands for the smallest positive zero of the function* $g'_{\rho,\beta}(z)$.

- **c.** *The radius of* k*-uniform convexity of order* α *of the function* $h_{\rho,\beta}$ *is the real number* $r^{uc}_{k,\alpha}(h_{\rho,\beta})$ *which is the smallest positive root of the equation*

$$(1 + k)r\frac{h''_{\rho,\beta}(r)}{h'_{\rho,\beta}(r)} + 1 - \alpha = 0$$

in the interval $(0, \tau_{\rho,\beta,1})$*, where* $\tau_{\rho,\beta,1}$ *stands for the smallest positive zero of the function* $h'_{\rho,\beta}(z)$

Proof.

- **a.** We note that

$$1 + \frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)} = 1 + \frac{z\Psi''_{\rho,\beta}(z)}{\Psi'_{\rho,\beta}(z)} + \left(\frac{1}{\beta} - 1\right)\frac{z\Psi'_{\rho,\beta}(z)}{\Psi_{\rho,\beta}(z)}.$$

Using the following infinite product representations of $\Psi_{\rho,\beta}$ and $\Psi'_{\rho,\beta}$ [17, Theorem 5] given by

$$\Gamma(\beta)\Psi_{\rho,\beta}(z) = z^{\beta}\prod_{n\geq 1}\left(1 - \frac{z^2}{\zeta^2_{\rho,\beta,n}}\right),\ \Gamma(\beta)\Psi'_{\rho,\beta}(z) = z^{\beta-1}\prod_{n\geq 1}\left(1 - \frac{z^2}{\zeta'^2_{\rho,\beta,n}}\right),$$

where $\zeta_{\rho,\beta,n}$ and $\zeta'_{\rho,\beta,n}$ denote the nth positive roots of $\Psi_{\rho,\beta}$ and $\Psi'_{\rho,\beta}$, respectively, we have

$$\frac{z\Psi'_{\rho,\beta}(z)}{\Psi_{\rho,\beta}(z)} = \beta - \sum_{n\geq 1} \frac{2z^2}{\zeta^2_{\rho,\beta,n} - z^2}, \qquad \frac{z\Psi''_{\rho,\beta}(z)}{\Psi'_{\rho,\beta}(z)} = \beta - 1 - \sum_{n\geq 1} \frac{2z^2}{\zeta'^2_{\rho,\beta,n} - z^2}.$$

Thus we arrive at

$$1 + \frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)} = 1 - \left(\frac{1}{\beta} - 1\right) \sum_{n\geq 1} \frac{2z^2}{\zeta^2_{\rho,\beta,n} - z^2} - \sum_{n\geq 1} \frac{2z^2}{\zeta'^2_{\rho,\beta,n} - z^2}.$$

In order to prove the theorem we consider two cases $\beta \in (0, 1]$ and $\beta > 1$ separately.

Case 1 $\beta \in (0, 1]$.

Then $\lambda = \frac{1}{\beta} - 1 > 0$. By making use of inequality (4) stated in Lemma 1 we conclude that the following inequality

$$\frac{|z|^2}{\zeta^2_{\rho,\beta,n} - |z|^2} \geq \mathrm{Re}\left(\frac{z^2}{\zeta^2_{\rho,\beta,n} - z^2}\right)$$

holds true for every $\rho > 0$, $\beta > 0$, $n \in \mathbb{N}$ and $|z| < \zeta_{\rho,\beta,n}$. With the help of (4), we get

$$\begin{aligned} \mathrm{Re}\left(1 + \frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)}\right) &\geq 1 - \left(\frac{1}{\beta} - 1\right) \sum_{n\geq 1} \frac{2r^2}{\zeta^2_{\rho,\beta,n} - r^2} - \sum_{n\geq 1} \frac{2r^2}{\zeta'^2_{\rho,\beta,n} - r^2} \\ &= 1 + \frac{rf''_{\rho,\beta}(r)}{f'_{\rho,\beta}(r)}, \end{aligned} \tag{5}$$

where $|z| = r$ and $z \in \mathbb{D}_{\zeta'_{\rho,\beta,1}}$.

Moreover, by using triangle inequality $|z_1 + z_2| \leq |z_1| + |z_2|$ together with the fact that $\frac{1}{\beta} - 1 > 0$, we get

$$\begin{aligned}\left|\frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)}\right| &= \left|\sum_{n\geq1}\frac{2z^2}{\zeta'^2_{\rho,\beta,n}-z^2}+\left(\frac{1}{\beta}-1\right)\sum_{n\geq1}\frac{2z^2}{\zeta^2_{\rho,\beta,n}-z^2}\right| \\ &\leq \sum_{n\geq1}\left|\left(\frac{2z^2}{\zeta'^2_{\rho,\beta,n}-z^2}+\left(\frac{1}{\beta}-1\right)\frac{2z^2}{\zeta^2_{\rho,\beta,n}-z^2}\right)\right| \\ &\leq \sum_{n\geq1}\left(\frac{2r^2}{\zeta'^2_{\rho,\beta,n}-r^2}+\left(\frac{1}{\beta}-1\right)\frac{2r^2}{\zeta^2_{\rho,\beta,n}-r^2}\right) \\ &= -\frac{rf''_{\rho,\beta}(r)}{f'_{\rho,\beta}(r)}.\end{aligned} \tag{6}$$

From (5) and (6), we obtain

$$\mathrm{Re}\left(1+\frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)}\right)-k\left|\frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)}\right|-\alpha\geq 1+(1+k)r\frac{f''_{\rho,\beta}(r)}{f'_{\rho,\beta}(r)}-\alpha, \qquad |z|\leq r<\zeta'_{\rho,\beta,1}. \tag{7}$$

Case 2 $\beta>1$. Then, we show that the same inequality is valid in this case also. In this case, taking into consideration the inequality (3) stated in 1 we get

$$\begin{aligned}\mathrm{Re}\left(1+\frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)}\right) &\geq 1-\left(\frac{1}{\beta}-1\right)\sum_{n\geq1}\frac{2r^2}{\zeta^2_{\rho,\beta,n}-r^2}-\sum_{n\geq1}\frac{2r^2}{\zeta'^2_{\rho,\beta,n}-r^2} \\ &= 1+\frac{rf''_{\rho,\beta}(r)}{f'_{\rho,\beta}(r)}.\end{aligned} \tag{8}$$

Also, with the aid of (2) stated in the same lemma, we have

$$\begin{aligned}\left|\frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)}\right| &= \left|\sum_{n\geq1}\frac{2z^2}{\zeta'^2_{\rho,\beta,n}-z^2}-\left(1-\frac{1}{\beta}\right)\sum_{n\geq1}\frac{2z^2}{\zeta^2_{\rho,\beta,n}-z^2}\right| \\ &\leq \sum_{n\geq1}\left|\left(\frac{2z^2}{\zeta'^2_{\rho,\beta,n}-z^2}-\left(1-\frac{1}{\beta}\right)\frac{2z^2}{\zeta^2_{\rho,\beta,n}-z^2}\right)\right| \\ &\leq \sum_{n\geq1}\left(\frac{2r^2}{\zeta'^2_{\rho,\beta,n}-r^2}-\left(1-\frac{1}{\beta}\right)\frac{2r^2}{\zeta^2_{\rho,\beta,n}-r^2}\right) \\ &= -\frac{rf''_{\rho,\beta}(r)}{f'_{\rho,\beta}(r)}.\end{aligned} \tag{9}$$

From (8) and (9), we deduce

$$\operatorname{Re}\left(1+\frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)}\right)-k\left|\frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)}\right|-\alpha\geq 1+(1+k)r\frac{f''_{\rho,\beta}(r)}{f'_{\rho,\beta}(r)}-\alpha, \quad |z|\leq r<\zeta'_{\rho,\beta,1}. \tag{10}$$

Due to the minimum principle for harmonic functions, equality holds if and only if $z=r$. Now, the above deduced inequalities imply for $r\in(0,\zeta'_{\rho,\beta,1})$

$$\inf_{z\in\mathbb{D}_r}\left\{\operatorname{Re}\left(1+\frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)}\right)-k\left|\frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)}\right|-\alpha\right\}=1-\alpha+(1+k)r\frac{f''_{\rho,\beta}(r)}{f'_{\rho,\beta}(r)}.$$

On the other hand, the function $u_{\rho,\beta}:(0,\zeta'_{\rho,\beta,1})\to\mathbb{R}$ is defined by

$$\begin{aligned}u_{\rho,\beta}(r)&=1-\alpha+(1+k)r\frac{f''_{\rho,\beta}(r)}{f'_{\rho,\beta}(r)}\\&=1-\alpha+(1+k)\left(\sum_{n\geq1}\frac{2r^2}{\zeta'^2_{\rho,\beta,n}-r^2}-\left(1-\frac{1}{\beta}\right)\sum_{n\geq1}\frac{2r^2}{\zeta^2_{\rho,\beta,n}-r^2}\right).\end{aligned}$$

Then,

$$\begin{aligned}u'_{\rho,\beta}(r)=&-4(1+k)\left(\frac{1}{\beta}-1\right)\sum_{n\geq1}\frac{\zeta^2_{\rho,\beta,n}r}{(\zeta^2_{\rho,\beta,n}-r^2)^2}\\&-4(k+1)\sum_{n\geq1}\frac{\zeta'^2_{\rho,\beta,n}r}{(\zeta'^2_{\rho,\beta,n}-r^2)^2}<0\end{aligned}$$

for all $\beta\in(0,1]$ and $z\in\mathbb{D}_{\zeta'_{\rho,\beta,1}}$. Moreover, we consider that if $\beta>1$, then $0<1-1/\beta<1$ and taking into consideration the inequality $\zeta^2_{\rho,\beta,n}(\zeta'^2_{\rho,\beta,n}-r^2)^2<\zeta'^2_{\rho,\beta,n}(\zeta^2_{\rho,\beta,n}-r^2)^2$ for $r<\zeta'_{\rho,\beta,1}$, we get

$$\begin{aligned}u'_{\rho,\beta}(r)&=-4(1+k)\left(\frac{1}{\beta}-1\right)\sum_{n\geq1}\frac{\zeta^2_{\rho,\beta,n}r}{(\zeta^2_{\rho,\beta,n}-r^2)^2}-4(k+1)\sum_{n\geq1}\frac{\zeta'^2_{\rho,\beta,n}r}{(\zeta'^2_{\rho,\beta,n}-r^2)^2}\\&<4(1+k)\left(\sum_{n\geq1}\frac{\zeta^2_{\rho,\beta,n}r}{(\zeta^2_{\rho,\beta,n}-r^2)^2}-\sum_{n\geq1}\frac{\zeta'^2_{\rho,\beta,n}r}{(\zeta'^2_{\rho,\beta,n}-r^2)^2}\right)<0.\end{aligned}$$

Consequently, $u_{\rho,\beta}$ is strictly decreasing function of r for all $\beta > 0$. Also,

$$\lim_{r\searrow 0} u_{\rho,\beta}(r) = 1-\alpha \quad \text{and} \quad \lim_{r\nearrow \zeta'_{\rho,\beta,1}} u_{\rho,\beta}(r) = -\infty.$$

This means that

$$\operatorname{Re}\left(1+\frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)}\right) - k\left|\frac{zf''_{\rho,\beta}(z)}{f'_{\rho,\beta}(z)}\right| - \alpha > 0$$

for all $z \in \mathbb{D}_{r^{uc}_{k,\alpha}(f_{\rho,\beta})}$ where $r^{uc}_{k,\alpha}(f_{\rho,\beta})$ is the unique root of the equation

$$1-\alpha+(1+k)r\frac{f''_{\rho,\beta}(r)}{f'_{\rho,\beta}(r)} = 0$$

or

$$(1+k)r\frac{\Psi''_{\rho,\beta}(r)}{\Psi'_{\rho,\beta}} + (\frac{1}{\beta}-1)(1+k)r\frac{\Psi'_{\rho,\beta}(r)}{\Psi_{\rho,\beta}(r)} + 1 - \alpha = 0$$

in $(0, \zeta'_{\rho,\beta,1})$.

b. Let $\vartheta_{\rho,\beta,n}$ be the nth positive zero of the function $g'_{\rho,\beta}(z)$. In view of the Hadamard theorem we get the Weierstrassian canonical representation (see [17])

$$g'_{\rho,\beta}(z) = \prod_{n\geq 1}\left(1-\frac{z^2}{\vartheta^2_{\rho,\beta,n}}\right).$$

Logarithmic derivation of both sides yields

$$1+\frac{zg''_{\rho,\beta}(z)}{g'_{\rho,\beta}(z)} = 1-\sum_{n\geq 1}\frac{2z^2}{\vartheta^2_{\rho,\beta,n}-z^2}.$$

Application of the inequality (4) implies that

$$\operatorname{Re}\left(1+\frac{zg''_{\rho,\beta}(z)}{g'_{\rho,\beta}(z)}\right) \geq 1-\sum_{n\geq 1}\frac{2r^2}{\vartheta^2_{\rho,\beta,n}-r^2}, \tag{11}$$

where $|z| = r$. Moreover,

$$\begin{aligned}\left|\frac{zg''_{\rho,\beta}(z)}{g'_{\rho,\beta}(z)}\right| &= \left|\sum_{n\geq 1}\frac{2z^2}{\vartheta^2_{\rho,\beta,n}-z^2}\right| \leq \sum_{n\geq 1}\left|\frac{2z^2}{\vartheta^2_{\rho,\beta,n}-z^2}\right| \leq \sum_{n\geq 1}\frac{2r^2}{\vartheta^2_{\rho,\beta,n}-r^2} \\ &= -\frac{rg''_{\rho,\beta}(r)}{g'_{\rho,\beta}(r)}, \quad |z| \leq r < \vartheta_{\rho,\beta,1}.\end{aligned} \tag{12}$$

Taking into considering the inequalities (11) and (12) we arrive at

$$\operatorname{Re}\left(1+\frac{zg''_{\rho,\beta}(z)}{g'_{\rho,\beta}(z)}\right)-k\left|\frac{zg''_{\rho,\beta}(z)}{g'_{\rho,\beta}(z)}\right|-\alpha\geq 1-\alpha+(1+k)r\frac{g''_{\rho,\beta}(r)}{g'_{\rho,\beta}(r)} \quad |z|<r<\vartheta_{\rho,\beta,1}.$$

In light of the minimum principle for harmonic functions, equality holds if and only if $z=r$. Thus, for $r\in(0,\vartheta_{\rho,\beta,1})$ we get

$$\inf_{|z|<r}\left\{\operatorname{Re}\left(1+\frac{zg''_{\rho,\beta}(z)}{g'_{\rho,\beta}(z)}\right)-k\left|\frac{zg''_{\rho,\beta}(z)}{g'_{\rho,\beta}(z)}\right|-\alpha\right\}=1-\alpha+(1+k)r\frac{g''_{\rho,\beta}(r)}{g'_{\rho,\beta}(r)}.$$

The function $w_{\rho,\beta}:(0,\vartheta_{\rho,\beta,1})\to\mathbb{R}$, defined by

$$w_{\rho,\beta}(r)=1-\alpha+(1+k)r\frac{g''_{\rho,\beta}(r)}{g'_{\rho,\beta}(r)},$$

is strictly decreasing and

$$\lim_{r\searrow 0}w_{\rho,\beta}(r)=1-\alpha>0,\qquad \lim_{r\nearrow\vartheta_{\rho,\beta,1}}w_{\rho,\beta}(r)=-\infty.$$

Consequently,

$$\operatorname{Re}\left(1+\frac{zg''_{\rho,\beta}(z)}{g'_{\rho,\beta}(z)}\right)-k\left|\frac{zg''_{\rho,\beta}(z)}{g'_{\rho,\beta}(z)}\right|-\alpha>0$$

for all $\mathbb{D}_{r^{uc}_{k,\alpha}(g_{\rho,\beta})}$ where $r^{uc}_{k,\alpha}(g_{\rho,\beta})$ is the unique root of the equation

$$1-\alpha+(1+k)r\frac{g''_{\rho,\beta}(r)}{g'_{\rho,\beta}(r)}=0$$

in $(0,\vartheta_{\rho,\beta,1})$.

c. Let $\tau_{\rho,\beta,n}$ denote the nth positive zero of the function $h'_{\rho,\beta}$. By using again the fact that the zeros of the Wright function $\lambda_{\rho,\beta}$ are all real and in view of the Hadamard theorem we obtain

$$h'_{\rho,\beta}(z)=\prod_{n\geq 1}\left(1-\frac{z}{\tau_{\rho,\beta,n}}\right),$$

which implies that

$$1+\frac{zh''_{\rho,\beta}(z)}{h'_{\rho,\beta}(z)}=1-\sum_{n\geq 1}\frac{z}{\tau_{\rho,\beta,n}-z}.$$

By using again the inequaliy (4) we get

$$\mathrm{Re}\left(1+\frac{zh''_{\rho,\beta}(z)}{h'_{\rho,\beta}(z)}\right)\geq 1-\sum_{n\geq 1}\frac{r}{\tau_{\rho,\beta,n}-r}=1+r\frac{h''_{\rho,\beta}(r)}{h'_{\rho,\beta}(r)}. \tag{13}$$

Also,

$$\left|\frac{zh''_{\rho,\beta}(z)}{h'_{\rho,\beta}(z)}\right|=\left|-\sum_{n\geq 1}\frac{z}{\tau_{\rho,\beta,n}-z}\right|\leq\sum_{n\geq 1}\frac{r}{\tau_{\rho,\beta,n}-r}=-r\frac{h''_{\rho,\beta}(r)}{h'_{\rho,\beta}(r)}. \tag{14}$$

Considering the inequalities (13) and (14) we have

$$\mathrm{Re}\left(1+\frac{zg''_{\rho,\beta}(z)}{g'_{\rho,\beta}(z)}\right)-k\left|\frac{zg''_{\rho,\beta}(z)}{g'_{\rho,\beta}(z)}\right|-\alpha\geq 1-\alpha+(1+k)r\frac{g''_{\rho,\beta}(r)}{g'_{\rho,\beta}(r)}.$$

In view of the minimum principle for harmonic functions, equality holds if and only if $z=r$. Thus, for $r\in(0,\tau_{\rho,\beta,1})$ we have

$$\inf_{|z|<r}\left\{\mathrm{Re}\left(1+\frac{zh''_{\rho,\beta}(z)}{h'_{\rho,\beta}(z)}\right)-k\left|\frac{zh''_{\rho,\beta}(z)}{h'_{\rho,\beta}(z)}\right|-\alpha\right\}=1-\alpha+(1+k)r\frac{h''_{\rho,\beta}(r)}{h'_{\rho,\beta}(r)}.$$

Now define the function $\varphi_{\rho,\beta}:(0,\vartheta_{\rho,\beta,1})\to\mathbb{R}$,as

$$\varphi_{\rho,\beta}(r)=1-\alpha+(1+k)r\frac{h''_{\rho,\beta}(r)}{h'_{\rho,\beta}(r)}$$

is strictly decreasing and

$$\lim_{r\searrow 0}\varphi_{\rho,\beta}(r)=1-\alpha>0,\qquad\lim_{r\nearrow\vartheta_{\rho,\beta,1}}\varphi_{\rho,\beta}(r)=-\infty.$$

Consequently,

$$\mathrm{Re}\left(1+\frac{zh''_{\rho,\beta}(z)}{h'_{\rho,\beta}(z)}\right)-k\left|\frac{zh''_{\rho,\beta}(z)}{h'_{\rho,\beta}(z)}\right|-\alpha>0$$

for all $\mathbb{D}_{r^{uc}_{k,\alpha}(h_{\rho,\beta})}$ where $r^{uc}_{k,\alpha}(h_{\rho,\beta})$ is the unique root of equation

$$1-\alpha+(1+k)r\frac{h''_{\rho,\beta}(r)}{h'_{\rho,\beta}(r)}=0$$

in $(0,\tau_{\rho,\beta,1})$. This completes the proof. □

Remark 1 *It is clear that by choosing* $k=0$ *in the above theorem we obtain the earlier results given in* [17, Thm. 5, p. 107]. *Moreover, for* $k=1$ *and* $\alpha=0$ *in the above theorem we get the results given in* [5, Thm. 2.2].

2.2 The radii of k-uniform convexity of order α of normalized $q-$Bessel functions

In this subsection, we shall concentrate on Jackson's second and third (or Hahn-Exton) $q-$Bessel functions which are defined by

$$J_\nu^{(2)}(z;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \sum_{n\geq 0} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n+\nu}}{(q;q)_n (q^{\nu+1};q)_n} q^{n(n+\nu)}$$

and

$$J_\nu^{(3)}(z;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \sum_{n\geq 0} \frac{(-1)^n z^{2n+\nu}}{(q;q)_n (q^{\nu+1};q)_n} q^{\frac{1}{2}n(n+1)},$$

where $z \in \mathbb{C}$, $\nu > -1$, $q \in (0,1)$ and

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=1}^{n} \left(1 - aq^{k-1}\right), \quad (a,q)_\infty = \prod_{k\geq 1} \left(1 - aq^{k-1}\right).$$

These functions are $q-$analogue of the classical Bessel function of the first kind [23]

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k\geq 0} \frac{(-1)^k}{k!\Gamma(\nu+k+1)} \left(\frac{z}{2}\right)^{2k},$$

since

$$\lim_{q\nearrow 1} J_\nu^{(2)}\left((1-z)q;q\right) = J_\nu(z), \quad \lim_{q\nearrow 1} J_\nu^{(3)}\left(\frac{1-q}{2}z;q\right) = J_\nu(z).$$

Obviously, the functions $J_\nu^{(2)}(.;q)$ and $J_\nu^{(3)}(.;q)$ do not belong to $\mathcal{A}$, and thus first we perform some natural normalization. We consider the following six normalized functions, as given by [10], originating from $J_\nu^{(2)}(.;q)$ and $J_\nu^{(3)}(.;q)$: For $\nu > -1$,

$$f_\nu^{(2)}(z;q) = \left(2^\nu c_\nu(q) J_\nu^{(2)}(z;q)\right)^{\frac{1}{\nu}}, \qquad f_\nu^{(3)}(z;q) = \left(c_\nu(q) J_\nu^{(3)}(z;q)\right)^{\frac{1}{\nu}}, (\nu \neq 0)$$
$$g_\nu^{(2)}(z;q) = 2^\nu c_\nu(q) z^{1-\nu} J_\nu^{(2)}(z;q), \qquad g_\nu^{(3)}(z;q) = c_\nu(q) z^{1-\nu} J_\nu^{(3)}(z;q),$$
$$h_\nu^{(2)}(z;q) = 2^\nu c_\nu(q) z^{1-\frac{\nu}{2}} J_\nu^{(2)}(\sqrt{z};q), \quad h_\nu^{(3)}(z;q) = c_\nu(q) z^{1-\frac{\nu}{2}} J_\nu^{(3)}(\sqrt{z};q),$$

where $c_\nu(q) = (q;q)_\infty/(q^{\nu+1};q)_\infty$. It is clear that each of the above functions belong to the class $\mathcal{A}$.

In view of [10, Lem. 1, p.972], we know that the infinite product representations of the functions $z \mapsto j_\nu^{(2)}(z;q)$ and $z \mapsto j_\nu^{(3)}(z;q)$ are of the form

$$J_\nu^{(2)}(z;q) = \frac{z^\nu}{2^\nu c_\nu(q)} \prod_{n\geq 1}\left(1 - \frac{z^2}{j_{\nu,n}^2(q)}\right), \quad J_\nu^{(3)}(z;q) = \frac{z^\nu}{c_\nu(q)} \prod_{n\geq 1}\left(1 - \frac{z^2}{l_{\nu,n}^2(q)}\right)$$

where $j_{\nu,n}(q)$ and $l_{\nu,n}(q)$ denote the nth positive zeros of the functions $j_\nu^{(2)}(z;q)$ and $j_\nu^{(3)}(z;q)$, respectively.

Also, from [10, Lem. 8] we observe that the functions $z \mapsto g_\nu^{(2)}(z;q), z \mapsto h_\nu^{(2)}(z;q), z \mapsto g_\nu^{(3)}(z;q)$ and $z \mapsto h_\nu^{(3)}(z;q)$ are of the form

$$\frac{dg_\nu^{(2)}(z;q)}{dz} = \prod_{n\geq 1}\left(1 - \frac{z^2}{\alpha_{\nu,n}^2(q)}\right), \quad \frac{dg_\nu^{(3)}(z;q)}{dz} = \prod_{n\geq 1}\left(1 - \frac{z^2}{\gamma_{\nu,n}^2(q)}\right) \qquad (15)$$

$$\frac{dh_\nu^{(2)}(z;q)}{dz} = \prod_{n\geq 1}\left(1 - \frac{z}{\beta_{\nu,n}^2(q)}\right), \quad \frac{dh_\nu^{(3)}(z;q)}{dz} = \prod_{n\geq 1}\left(1 - \frac{z}{\delta_{\nu,n}^2(q)}\right) \qquad (16)$$

where $\alpha_{\nu,n}(q)$ and $\beta_{\nu,n}(q)$ represent the nth positive zeros of $z \mapsto z.dJ_\nu^{(2)}(z;q)/dz + (1-\nu)J_\nu^{(2)}(z;q)$ and $z \mapsto z.dJ_\nu^{(2)}(z;q)/dz + (2-\nu)J_\nu^{(2)}(z;q)$, while $\gamma_{\nu,n}(q)$ and $\delta_{\nu,n}(q)$ are the nth positive zeros of $z \mapsto z.dJ_\nu^{(3)}(z;q)/dz + (1-\nu)J_\nu^{(3)}(z;q)$ and $z \mapsto z.dJ_\nu^{(3)}(z;q)/dz + (2-\nu)J_\nu^{(3)}(z;q)$.

Now, we are ready to present our results related with the radius of $k-$uniform convexity of order α of the normalized $q-$Bessel functions:

Theorem 2 *Let $\nu > -1$, $s \in \{2,3\}$ and $q \in (0,1)$. Then, the following assertions holds true*

a. *Suppose that $\nu > 0$. Then, the radius of $k-$uniform convexity of order α of the function $z \mapsto f_\nu^{(s)}(z;q)$ is the real number $r_{k,\alpha}^{uc}(f_\nu^{(s)})$ which is the smallest positive root of the equation*

$$1 - \alpha + (1+k)r\frac{(f_\nu^{(s)}(r;q))''}{(f_\nu^{(s)}(r;q))'} = 0$$

in $(0, j'_{\nu,1}(q))$.

b. *The radius of k-uniform convexity of order α of the function $z \mapsto g_\nu^{(s)}(z;q)$ is the real number $r_{k,\alpha}^{uc}(g_\nu^{(s)})$ which is the smallest positive root of the equation*

$$\begin{aligned}&\left((1-\nu)(1+\alpha-(1+k)\nu)\right)J_\nu^{(s)}(r;q)\\&\quad+\left(1-\alpha+2(1+k)(1-\nu)\right)r\left(J_\nu^{(s)}(r;q)\right)'\\&\quad+(1+k)r^2\left(J_\nu^{(s)}(r;q)\right)''=0\end{aligned}$$

in $(0,\alpha_{\nu,1}(q))$.

c. *The radius of* k*-uniform convexity of order* α *of the function* $z\mapsto h_\nu^{(s)}(z;q)$ *is the real number* $r^{uc}_{k,\alpha}(h_\nu^{(s)})$ *which is the smallest positive root of the equation*

$$\begin{aligned}&\left((\nu-2)\left(\nu(1+k)-2(1-\alpha)\right)\right)J_\nu^{(s)}\\&\quad+\left((3-2\nu)(1+k)+2(1-\alpha)\right)\sqrt{r}\left(J_\nu^{(s)}\right)'\\&\quad+(1+k)r\left(J_\nu^{(s)}\right)''=0\end{aligned}$$

in $(0,\beta^2_{\nu,1}(q))$, *where* $J_\nu^{(s)}=J_\nu^{(s)}(\sqrt{r};q)$.

Proof. Since the proofs for the cases $s=2$ and $s=3$ are almost the same we are going to present the proof only for the case $s=2$.

a. In [10, p. 979] it was proven that the following equality is valid

$$1+z\frac{\left(f_\nu^{(2)}(z;q)\right)''}{\left(f_\nu^{(2)}(z;q)\right)'}=1-\left(\frac{1}{\nu}-1\right)\sum_{n\geq1}\frac{2z^2}{j^2_{\nu,n}(q)-z^2}-\sum_{n\geq1}\frac{2z^2}{j'^2_{\nu,n}(q)-z^2},$$

where $j_{\nu,n}(q)$ and $j'_{\nu,n}(q)$ are the nth positive roots of the functions $z\mapsto J_\nu^{(2)}(z;q)$ and $z\mapsto dJ_\nu^{(2)}(z;q)/dz$, respectively.

Now, suppose that $\nu\in(0,1]$. Taking into account the inequality (4), for $z\in\mathbb{D}_{j'_1(q)}$ we obtain the inequality

$$\mathrm{Re}\left(1+z\frac{\left(f_\nu^{(2)}(z;q)\right)''}{\left(f_\nu^{(2)}(z;q)\right)'}\right)\geq 1-\left(\frac{1}{\nu}-1\right)\sum_{n\geq1}\frac{2r^2}{j_{\nu,n}^2(q)-r^2}$$
$$-\sum_{n\geq1}\frac{2r^2}{j'^2_{\nu,n}(q)-r^2} \tag{17}$$
$$=1+r\frac{\left(f_\nu^{(2)}(r;q)\right)''}{\left(f_\nu^{(2)}(r;q)\right)'},$$

where $|z|=r$. Moreover, by using triangle inequality along with the fact that $\frac{1}{\nu}-1>0$, we get

$$\left|z\frac{\left(f_\nu^{(2)}(z;q)\right)''}{\left(f_\nu^{(2)}(z;q)\right)'}\right|\leq -r\frac{\left(f_\nu^{(2)}(r;q)\right)''}{\left(f_\nu^{(2)}(r;q)\right)'}. \tag{18}$$

On the other hand, observe that if we use the inequality (3), then we obtain that the above inequalities is also valid for $\nu>1$. Here we used tacitly that the zeros $j_{\nu,n}(q)$ and $j'_{\nu,n}(q)$ interlace according to [10, Lem. 9., p. 975]. The above inequalities imply for $r\in(0,j'_{\nu,1}(q))$

$$\inf_{|z|<r}\left[\mathrm{Re}\left(1+z\frac{\left(f_\nu^{(2)}(z;q)\right)''}{\left(f_\nu^{(2)}(z;q)\right)'}\right)-k\left|z\frac{\left(f_\nu^{(2)}(z;q)\right)''}{\left(f_\nu^{(2)}(z;q)\right)'}\right|-\alpha\right]=1-\alpha+(1+k)r\frac{\left(f_\nu^{(2)}(r;q)\right)''}{\left(f_\nu^{(2)}(r;q)\right)'}.$$

The function $u_\nu:(0,j'_{\nu,1}(q))\mapsto\mathbb{R}$ defined by

$$u_\nu(r)=1-\alpha+(1+k)r\frac{\left(f_\nu^{(2)}(r;q)\right)''}{\left(f_\nu^{(2)}(r;q)\right)'}$$
$$=1-\alpha-(1+k)\sum_{n\geq1}\left(\frac{2r^2}{j'^2_{\nu,n}(q)-r^2}-\left(1-\frac{1}{\nu}\right)\frac{2r^2}{j^2_{\nu,n}(q)-r^2}\right)$$

is strictly decreasing since

$$u'_\nu(r)=-(1+k)\sum_{n\geq1}\left(\frac{4rj'^2_{\nu,n}(q)}{\left(j'^2_{\nu,n}(q)-r^2\right)^2}-\left(1-\frac{1}{\nu}\right)\frac{4rj^2_{\nu,n}(q)}{\left(j^2_{\nu,n}(q)-r^2\right)^2}\right)<0$$

for $r \in (0, j'_{\nu,1}(q))$. Also, it can be observed that

$$\lim_{r \searrow 0} u_\nu(r) = 1 - \alpha \text{ and } \lim_{r \nearrow j'_{\nu,1}(q)} u_\nu(r) = -\infty.$$

Consequently, it is obvious that the equation

$$1 - \alpha + (1+k) r \frac{\left(f_\nu^{(2)}(r;q)\right)''}{\left(f_\nu^{(2)}(r;q)\right)'} = 0$$

has a unique root $r^{uc}_{k,\alpha}\left(f_\nu^{(2)}(z;q)\right)$ in $\mathbb{D}_{(0,j'_{\nu,1}(q))}$, where $r^{uc}_{k,\alpha}\left(f_\nu^{(2)}(z;q)\right)$ is the radius of $k-$uniform convexity of order α of the function $z \mapsto f_\nu^{(2)}(z;q)$.

Taking into account Equ. (15) and (16), the rest of proof is obvious and follows by considering a similar way of concluding process as in the previous theorem. This is why we omit the rest of proof here. □

Remark 2 *It is obvious that by taking* $k = 1$ *and* $\alpha = 0$ *in the above theorem we obtain the results given in* [5, Thm. 2.1].

2.3 Radius of strong starlikeness of normalized Wright and q−Bessel functions

In this subsection, our aim is to present the radius of strong starlikeness of normalized Wright and $q-$Bessel functions. It is well known from [19] that a function $f \in \mathcal{A}$ is said to be strong starlike of order γ, $0 < \gamma \le 1$, if

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi\gamma}{2}, \quad z \in \mathbb{D}$$

and the real number

$$r_\gamma(f) = \sup\left\{r > 0 \colon \left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi\gamma}{2}, \quad \forall z \in \mathbb{D}_r\right\}$$

is called the radius of strong starlikeness of f.

The following lemma have an important place for finding our main results:

Lemma 2 [19] *If a is any point in* $|\arg w| \le \frac{\pi\gamma}{2}$ *and if*

$$R_a \le \mathrm{Re}[a] \sin\frac{\pi\gamma}{2} - \mathrm{Im}[a]\cos\frac{\pi\gamma}{2}, \quad \mathrm{Im}[a] \ge 0,$$

the disk $|w - a| \le R_a$ *is contained in the sector* $|\arg w| \le \frac{\pi\gamma}{2}$, $0 < \gamma \le 1$. *In particular when* $\mathrm{Im}[a] = 0$, *the condition becomes* $R_a \le a \sin\frac{\pi\gamma}{2}$.

We are now in a position to present our main results related with the radii of strong starlikeness of normalized Wright and $q-$Bessel functions. Upcoming theorem is related with normalized Wright functions.

Theorem 3 *Let $\rho > 0$ and $\beta > 0$. The following assertions are true:*

a. *The radius of strong starlikeness of $f_{\rho,\beta}$ is the smallest positive root of the equation*

$$\frac{2}{\beta}\sum_{n\geq 1}\frac{r^2\left(\lambda^2_{\rho,\beta,n}+r^2\sin\frac{\pi\gamma}{2}\right)}{\lambda^4_{\rho,\beta,n}-r^4}-\sin\frac{\pi\gamma}{2}=0$$

in $(0,\lambda_{\rho,\beta,1})$.

b. *The radius of strong starlikeness of $g_{\rho,\beta}$ is the smallest positive root of the equation*

$$2\sum_{n\geq 1}\frac{r^2\left(\lambda^2_{\rho,\beta,n}+r^2\sin\frac{\pi\gamma}{2}\right)}{\lambda^4_{\rho,\beta,n}-r^4}-\sin\frac{\pi\gamma}{2}=0$$

in $(0,\lambda_{\rho,\beta,1})$.

c. *The radius of strong starlikeness of $h_{\rho,\beta}$ is the smallest positive root of the equation*

$$\sum_{n\geq 1}\frac{r\left(\lambda^2_{\rho,\beta,n}+r\sin\frac{\pi\gamma}{2}\right)}{\lambda^4_{\rho,\beta,n}-r^2}-\sin\frac{\pi\gamma}{2}=0$$

in $(0,\lambda^2_{\rho,\beta,1})$.

Proof. For $|z|\leq r<1$, $|z_k|=R>r$, we have from [19]

$$\left|\frac{z}{z-z_k}+\frac{r^2}{R^2-r^2}\right|\leq\frac{Rr}{R^2-r^2}. \tag{19}$$

Since the series $\sum_{n\geq 1}\frac{2r^2}{\lambda^2_{\rho,\beta,n}-r^2}$ and $\sum_{n\geq 1}\frac{r}{\lambda^2_{\rho,\beta,n}-r}$ are convergent, we arrive at

$$\left|\frac{zf'_{\rho,\beta}(z)}{f_{\rho,\beta}(z)}-\left(1-\frac{2}{\beta}\sum_{n\geq 1}\frac{r^4}{\lambda^4_{\rho,\beta,n}-r^4}\right)\right|\leq\frac{2}{\beta}\sum_{n\geq 1}\frac{\lambda^2_{\rho,\beta,n}r^2}{\lambda^4_{\rho,\beta,n}-r^4} \tag{20}$$

$$\left|\frac{zg'_{\rho,\beta}(z)}{g_{\rho,\beta}(z)}-\left(1-\sum_{n\geq1}\frac{2r^4}{\lambda^4_{\rho,\beta,n}-r^4}\right)\right|\leq 2\sum_{n\geq1}\frac{\lambda^2_{\rho,\beta,n}r^2}{\lambda^4_{\rho,\beta,n}-r^4} \tag{21}$$

$$\left|\frac{zh'_{\rho,\beta}(z)}{h_{\rho,\beta}(z)}-\left(1-\sum_{n\geq1}\frac{r^2}{\lambda^4_{\rho,\beta,n}-r^2}\right)\right|\leq \sum_{n\geq1}\frac{\lambda^2_{\rho,\beta,n}r}{\lambda^4_{\rho,\beta,n}-r^2} \tag{22}$$

for $z\in\mathbb{D}_{\lambda_{\rho,\beta,1}}$ where $|z|=r$ and $\lambda_{\rho,\beta,n}$ stands for the nth positive zero of the function $\lambda_{\rho,\beta}$. Thanks to Lemma 2, it is obvious that the disk given in (20) is contained in the sector $|\arg w|\leq\frac{\pi\gamma}{2}$, if

$$\frac{2}{\beta}\sum_{n\geq1}\frac{\lambda^2_{\rho,\beta,n}r^2}{\lambda^4_{\rho,\beta,n}-r^4}\leq\left(1-\frac{2}{\beta}\sum_{n\geq1}\frac{r^4}{\lambda^4_{\rho,\beta,n}-r^4}\right)\sin\frac{\pi\gamma}{2}$$

is satisfied. This inequality reduces to $\psi(r)\leq0$ where

$$\psi(r)=\frac{2}{\beta}\sum_{n\geq1}\frac{r^2\left(\lambda^2_{\rho,\beta,n}+r^2\sin\pi\gamma/2\right)}{\lambda^4_{\rho,\beta,n}-r^4}-\sin\frac{\pi\gamma}{2}.$$

We note that

$$\psi'(r)=\frac{2}{\beta}\sum_{n\geq1}\frac{2r\lambda^6_{\rho,\beta,n}+2r5\lambda^2_{\rho,\beta,n}+4r^3\lambda^4_{\rho,\beta,n}\sin\pi\gamma/2}{(\lambda^4_{\rho,\beta,n}-r^4)^2}\geq0.$$

Moreover, $\lim_{r\searrow0}\psi(r)<0$ and $\lim_{r\nearrow\lambda_{\rho,\beta,1}}\psi(r)=\infty$. Thus $\psi(r)=0$ has a unique root say $\mathcal{R}_{f_{\rho,\beta}}$ in $(0,\lambda_{\rho,\beta,1})$. Hence the function $f_{\rho,\beta}$ is strongly starlike in $|z|<\mathcal{R}_{f_{\rho,\beta}}$.

The disk given in (21) is contained in the sector $|\arg w|\leq\frac{\pi\gamma}{2}$, if

$$\phi(r)=2\sum_{n\geq1}\frac{r^2\left(\lambda^2_{\rho,\beta,n}+r^2\sin\pi\gamma/2\right)}{\lambda^4_{\rho,\beta,n}-r^4}-\sin\frac{\pi\gamma}{2}\leq0.$$

Also, the proof of part (b) is completed by considering the limits $\lim_{r\searrow0}\phi(r)<0$ and $\lim_{r\nearrow\lambda_{\rho,\beta,1}}\phi(r)=\infty$.

The proof of part (c) is obvious and follows by considering the same concluding process as in the proof of part (b).

□

Since it can be obtained desired results by repeating the same calculations in the previous theorem we present the following theorem without proof.

Theorem 4 *Let $\nu > -1$, $s \in \{2, 3\}$ and $q \in (0, 1)$. Moreover, let $\eta_{\nu,n}(q)$ be the nth positive root of the function $z \mapsto J_\nu^{(s)}(z; q)$. Then the following assertions are true:*

a. *The radius of strong starlikeness of the function $f_\nu^{(s)}(z; q)$ is the smallest positive root of the equation*

$$\frac{2}{\nu}\sum_{n\geq 1}\frac{r^2\left(\eta_{\nu,n}^2(q)+r^2\sin\frac{\pi\gamma}{2}\right)}{\eta_{\nu,n}^4(q)-r^4}-\sin\frac{\pi\gamma}{2}=0$$

in $(0, \eta_{\nu,1}(q))$, where $\eta_{\nu,1}(q)$ is the smallest positive zero of the function $J_\nu^{(s)}(z; q)$.

b. *The radius of strong starlikeness of $g_\nu^{(s)}(z; q)$ is the smallest positive root of the equation*

$$2\sum_{n\geq 1}\frac{r^2\left(\eta_{\nu,n}^2(q)+r^2\sin\frac{\pi\gamma}{2}\right)}{\eta_{\nu,n}^4(q)-r^4}-\sin\frac{\pi\gamma}{2}=0$$

in $(0, \eta_{\nu,1}(q))$.

c. *The radius of strong starlikeness of $h_\nu^{(s)}(z; q)$ is the smallest positive root of the equation*

$$\sum_{n\geq 1}\frac{r\left(\eta_{\eta,n}^2(q)+r\sin\frac{\pi\gamma}{2}\right)}{\eta_{\nu,n}^4(q)-r^2}-\sin\frac{\pi\gamma}{2}=0$$

in $(0, \eta_{\nu,1}^2(q))$.

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I-Rad-$\oplus$-supplemented modules

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Abstract. Let M be an R-module and I be an ideal of R. We say that M is I-Rad-$\oplus$-*supplemented*, provided for every submodule N of M, there exists a direct summand K of M such that $M = N + K$, $N \cap K \subseteq IK$ and $N \cap K \subseteq \mathrm{Rad}(K)$. The aim of this paper is to show new properties of I-Rad-$\oplus$-supplemented modules. Especially, we show that any finite direct sum of I-Rad-$\oplus$-supplemented modules is I-Rad-$\oplus$-supplemented. We also prove that an R-module M is I-Rad-$\oplus$-supplemented if and only if K and $\frac{M}{K}$ are I-Rad-$\oplus$-supplemented for a fully invariant direct summand K of M. Finally, we determine the structure of I-Rad-$\oplus$-supplemented modules over a discrete valuation ring.

1 Introduction

Throughout the whole text, all rings are to be associative, unit and all modules are left unitary. Let R be such a ring and M be an R-module. The notation $K \subseteq M$ ($K \subset M$) means that K is a (proper) submodule of M. A module M is called *extending* if every submodule is essential in a direct summand of M [4]. Here a submodule $K \leq M$ is said to be *essential* in M, denoted as $K \trianglelefteq M$, if $K \cap N \neq 0$ for every non-zero submodule $N \leq M$. Dually, a submodule S of M is called *small (in M)*, denoted as $S \ll M$, if $M \neq S + L$ for every proper submodule L of M [17]. If all non-zero submodules of M are essential in M,

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then M is called *uniform* [4, 1.5]. The Jacobson radical of M will be denoted by Rad(M). It is known that Rad(M) is the sum of all small submodules of M.

A non-zero module M is said to be *hollow* if every proper submodule of M is small in M, and it is said to be *local* if it is hollow and is finitely generated. A module M is local if and only if it is finitely generated and Rad(M) is the maximal submodule of M (see [4, 2.12 §2.15]). A ring R is said to be *local* if J is the maximal ideal of R, where J is the Jacobson radical of R.

An R-module M is called *supplemented* if every submodule of M has a supplement in M. Here a submodule $K \subseteq M$ is said to be a *supplement* of N in M if K is minimal with respect to $N+K = M$, or equivalently, if $N+K = M$ and $N \cap K \ll K$ [17]. A supplement submodule X of M is then defined when X is a supplement of some submodule of M. Every direct summand of a module M is a supplement submodule of M, and supplemented modules are a generalization of semisimple modules. In addition, every factor module of a supplemented module is again supplemented.

A module M is called *lifting* (or D_1-*module*) if, for every submodule N of M, there exists a direct summand K of M such that $K \leq N$ and $\frac{N}{K} \ll \frac{M}{K}$. Mohamed and Müller have generalized the concept of lifting modules to ⊕-supplemented modules. M is called *⊕-supplemented* if every submodule N of M has a supplement that is a direct summand of M [12]. Clearly every ⊕-supplemented module is supplemented, but a supplemented module need not be ⊕-supplemented in general (see [12, Lemma A.4 (2)]). It is shown in [12, Proposition A.7 and Proposition A.8] that if R is a Dedekind domain, every supplemented R-module is ⊕-supplemented. Hollow modules are ⊕-supplemented.

Weakening the notion of "supplement", one calls a submodule K of M a Rad-*supplement* of N in M if $M = N + K$ and $N \cap K \subseteq \text{Rad}(K)$ ([4, pp.100]).

Recall from [6] that a module M is called *Rad-⊕-supplemented*(or *generalized ⊕-supplemented* in [5]) if for every $N \subseteq M$, there exists a direct summand K of M such that $M = N + K$ and $N \cap K \subseteq \text{Rad}(K)$. In [15], various properties of Rad-⊕-supplemented modules are given. In addition, a ring R is semiperfect if and only if every finitely generated free R-module is generalized ⊕-supplemented (see [5]).

In this paper, we define I-Rad-⊕-supplemented modules which is specialized of Rad-⊕-supplemented modules. We obtain various properties of this modules adapting by [14]. We show that every finite direct sum of I-Rad-⊕-supplemented modules is a I-Rad-⊕-supplemented module. We prove that the class of I-Rad-⊕-supplemented modules is closed under extension in some constriction. Finally, we characterize I-Rad-⊕-supplemented modules over a discrete valuation ring.

2 Some results of I-Rad-$\oplus$-supplemented modules

A module M is called *semilocal* if $\frac{M}{\mathrm{Rad}(M)}$ is semisimple, and a ring R is called *semilocal* if ${}_R R$ (or R_R) is semilocal. Lomp proved in [11, Theorem 3.5] that a ring R is semilocal if and only if every left R-module is semilocal. Using this fact we obtain the following:

Lemma 1 *Let M be a module over a semilocal ring R. Then M is* Rad-$\oplus$-*supplemented if and only if for every submodule $N \subseteq M$, there exists a direct summand K of M such that $M = N + K$, $N \cap K \subseteq JK$.*

Proof. Clear by [1, Corollary 15.18]. □

By using the above lemma, we have a specialized notion which is strong of Rad-$\oplus$-supplemented modules. Now we define this notion.

Definition 1 *Let M be an R-module and I be an ideal of R. We say that M is a* I-Rad-$\oplus$-supplemented module*, provided for every submodule N of M, there exists a direct summand K of M such that $M = N + K$, $N \cap K \subseteq IK$ and $N \cap K \subseteq \mathrm{Rad}(K)$.*

Lemma 2 *Let M be an R-module and I be an ideal of R such that $IM = 0$. Then, M is* I-Rad-$\oplus$-*supplemented if and only if M is semisimple.*

Proof. ($\Rightarrow$) Let N be a submodule of M. By the hypothesis, there exists a direct summand K of M such that $M = N+K$, $N\cap K \subseteq IK$ and $N\cap K \subseteq \mathrm{Rad}(K)$. Since $IK \subseteq IM = 0$, we obtain that $M = N \oplus K$. Hence M is semisimple.

($\Leftarrow$) Let N be a submodule of M. Then there exists a submodule N' of M such that $M = N \oplus N'$. So $M = N + N'$, $N \cap N' = 0 \subseteq IN'$ and $N \cap N' = 0 \subseteq \mathrm{Rad}(N')$. Therefore M is a I-Rad-$\oplus$-supplemented module. □

Lemma 3 [14, Lemma 3.4] *Let M be an R-module and I be an ideal of R. If K is a direct summand of M, then we have $IK = K \cap IM$.*

Proposition 1 *Let M be an arbitrary R-module and I be an ideal of R such that $\mathrm{Rad}(M) \subseteq IM$. Then M is* I-Rad-$\oplus$-*supplemented if and only if M is* Rad-$\oplus$-*supplemented.*

Proof. ($\Rightarrow$) It is clear.

($\Leftarrow$) Suppose that M is I-Rad-$\oplus$-supplemented. Let N be a submodule of M. Then there exists a direct summand K of M such that $M = N + K$ and

$N \cap K \subseteq \mathrm{Rad}(K)$. Note that $IK = K \cap IM$ by Lemma 3. Since $\mathrm{Rad}(M) \subseteq IM$, we have $N \cap K \subseteq \mathrm{Rad}(K) \subseteq K \cap \mathrm{Rad}(M) \subseteq K \cap IM = IK$. Therefore M is I-Rad-⊕-supplemented. This completes the proof. □

Recall from [17] that a ring R is called *a left good ring* if $\mathrm{Rad}(M) = JM$ for every R-module M. A semilocal ring is an example of a left good ring.

Corollary 1 *Let M be an R-module. Suppose further that either*

(1) *R is a left good ring, or*

(2) *M is a projective module.*

If an ideal I of R contains the Jacobson radical J of R, then M is Rad-⊕-supplemented if and only if M is I-Rad-⊕-supplemented.

Proof. Note that $\mathrm{Rad}(M) = JM$ by [1, Proposition 17.10]. The result follows from Proposition 1. □

It is clear that every I-Rad-⊕-supplemented module is Rad-⊕-supplemented module, but the following example shows that the converse is not be always true. Firstly, we need the following crucial proposition.

Proposition 2 *Let M be an indecomposable R-module with $\mathrm{Rad}(M) \ll M$ and I be an ideal of R. Then the following statements are equivalent.*

(1) *M is I-Rad-⊕-supplemented;*

(2) *M is local with $IM = M$ or $IM = \mathrm{Rad}(M)$.*

Proof. (1)⟹(2) Let N be a proper submodule of M. By hypothesis, there exists a direct summand K of M such that $M = N + K$, $N \cap K \subseteq IK$ and $N \cap K \subseteq \mathrm{Rad}(K)$. Since M is indecomposable, we have $K = M$. Hence, $N \subseteq IM$ and $N \subseteq \mathrm{Rad}(M)$. Since $\mathrm{Rad}(M) \ll M$, we have $N \ll M$. Thus, M is a local module. Moreover, note that if $IM \neq M$, then IM contains all other proper submodules of M. Hence M is a local module and $IM = \mathrm{Rad}(M)$.

(2)⟹(1) Let N be a proper submodule of M. Then $M = N + M$ and $N \cap M = N \subseteq \mathrm{Rad}(M) \subseteq IM$. So M is I-Rad-⊕-supplemented. □

Example 1 (See [14, Example 3.8]) *Let p and q be two different prime integers. Consider the local $\mathbb{Z}$-module $M = \frac{\mathbb{Z}}{\mathbb{Z}p^3}$. We have $\mathrm{Rad}(M) = \frac{\mathbb{Z}p}{\mathbb{Z}p^3} \ll M$. Let $I_1 = \mathbb{Z}p$, $I_2 = \mathbb{Z}q$ and $I_3 = \mathbb{Z}p^2$. Then $I_1M = \mathrm{Rad}(M)$, $I_2M = M$ and $I_3M = \frac{\mathbb{Z}p^2}{\mathbb{Z}p^3}$. By Proposition 2, M is I_i-Rad-⊕-supplemented for each $i = 1, 2$ but not I_3-Rad-⊕-supplemented. On the other hand, it is clear that M is Rad-⊕-supplemented.*

Proposition 3 *Let* I *be an ideal of* R *and* M *be an* R*-module. If* M *is an* I-Rad-$\oplus$*-supplemented* R*-module, then* $\frac{M}{IM}$ *is semisimple.*

Proof. Let N be a submodule of M such that $IM \subseteq N$. By assumption, there exists a direct summand K of M such that $M = N + K$, $N \cap K \subseteq IK$ and $N \cap K \subseteq \mathrm{Rad}(K)$. Then $\frac{N}{IM} + \frac{K+IM}{IM} = \frac{M}{IM}$. Clearly, we have $N \cap (K + IM) = IM + N \cap K = IM$ and so $\frac{N}{IM} \cap \frac{K+IM}{IM} = \frac{IM}{IM}$. Therefore $\frac{M}{IM} = \frac{N}{IM} \oplus \frac{K+IM}{IM}$. It means that $\frac{M}{IM}$ is semisimple. □

Corollary 2 *Let* M *be a* Rad-$\oplus$*-supplemented* R*-module such that* $IM = M$*, where* I *is an ideal of* R*. Then* M *is* I-Rad-$\oplus$*-supplemented.*

Corollary 3 *Let* m *be a maximal ideal of a commutative ring* R *and* M *be an* R*-module. Assume that* I *is an ideal of* R *such that* $IM = mM$*. If* M *is a* Rad-$\oplus$*-supplemented* R*-module, then* M *is* I-Rad-$\oplus$*-supplemented.*

Proof. Note that $\mathrm{Rad}(M) \subseteq mM$ by [7, Lemma 3]. The result follows from Proposition 1. □

Recall from [17] that an R-module M is called *divisible* in case $rM = M$ for each non-zero element $r \in R$, where R is a commutative domain.

Proposition 4 *Let* M *be a divisible module over a commutative domain* R*. If* M *is* Rad-$\oplus$*-supplemented, then* M *is* I-Rad-$\oplus$*-supplemented for every non-zero ideal* I *of* R*.*

Proof. This follows from Corollary 2. □

Corollary 4 *Let* R *be a Dedekind domain and* M *be an injective* R*-module. Then,* M *is* I-Rad-$\oplus$*-supplemented for every non-zero ideal* I *of* R*.*

Proof. Since every injective module over a Dedekind domain is divisible, the proof follows from Proposition 4. □

Theorem 1 *Let* I *be an ideal of* R*. Then any finite direct sum of* I-Rad-$\oplus$*-supplemented* R*-modules is* I-Rad-$\oplus$*-supplemented.*

Proof. Let n be any positive integer and M_i $(1 \leq i \leq n)$ be any finite collection of I-Rad-$\oplus$-supplemented R-modules. Let $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$. Suppose that $n = 2$, that is, $M = M_1 \oplus M_2$. Let K be any submodule of M. Then $M = M_1 + M_2 + K$ and so $M_1 + M_2 + K$ has a Rad-supplement 0 in M. Since M_1

is I-Rad-⊕-supplemented, $M_1 \cap (M_2+K)$ has a Rad-supplement X in M_1 such that X is a direct summand of M_1 and $X \cap (M_2+K) = M_1 \cap (M_2+K) \cap X \subseteq IX$. By [5, Lemma 3.2], X is a Rad-supplement of $M_2 + K$ in M. Since M_2 is I-Rad-⊕-supplemented, $M_2 \cap (K+X)$ has a Rad-supplement Y in M_2 such that Y is a direct summand of M_2 and $Y \cap (K+X) = M_2 \cap (K+X) \cap Y \subseteq IY$. Again applying [5, Lemma 3.2], we obtain that $X+Y$ is a Rad-supplement of K in M. Since X is a direct summand of M_1 and Y is a direct summand of M_2, it follows that $X \oplus Y$ is a direct summand of M. Note that

$$\begin{aligned} K \cap (X+Y) &\subseteq X \cap (Y+K) + Y \cap (K+X) \\ &\subseteq X \cap (M_2+K) + Y \cap (K+X) \\ &\subseteq IX \oplus IY = I(X \oplus Y) \end{aligned}$$

So $M_1 \oplus M_2$ is I-Rad-⊕-supplemented. The proof is completed by induction on n. □

Recall from [17] that a submodule U of an R-module M is called *fully invariant* if $f(U)$ is contained in U for every R-endomorphism f of M. Let M be an R-module and τ be a preradical for the category of R-modules. Then $\tau(M)$ is fully invariant submodule of M. A module M is called *duo* if every submodule of M is fully invariant [13].

Proposition 5 *Let I be an ideal of R and $M = \oplus_{\lambda\in\Lambda} M_\lambda$ be a duo module where M is a direct sum of submodules M_λ $(\lambda \in \Lambda)$. Assume that M_λ is* I-Rad-⊕*-supplemented for every $\lambda \in \Lambda$. Then M is* I-Rad-⊕*-supplemented.*

Proof. By hypothesis, for every $\lambda \in \Lambda$, there exists a direct summand K_λ of M_λ such that $M_\lambda = (N \cap M_\lambda) + K_\lambda$, $N \cap K_\lambda \subseteq IK_\lambda$ and $N \cap K_\lambda \subseteq \mathrm{Rad}(K_\lambda)$. Put $K = \oplus_{\lambda\in\Lambda} K_\lambda$. Clearly K is a direct summand of M and $M = N + K$. Also, we have $N \cap K = \oplus_{\lambda\in\Lambda}(N \cap K_\lambda) \subseteq IK$ and $N \cap K \subseteq \mathrm{Rad}(K)$. This completes the proof. □

Now, we give an example showing that the I-Rad-⊕-supplemented property doesn't always transfer from a module to each of its factor modules.

Example 2 (see [2, Example 4.1]) *Let F be a field. Consider the local ring $R = \frac{F[x^2,x^3]}{(x^4)}$ and let m be the maximal ideal of R. Let n be an integer with $n \geq 2$ and $M = R^{(n)}$. By Proposition 2 and Theorem 1, M is* m-Rad-⊕-*supplemented. Note that R is an artinian local ring which is not a principal ideal ring. So, there exists a submodule K of M such that the factor module $\frac{M}{K}$ isn't* Rad-⊕-*supplemented. Therefore $\frac{M}{K}$ isn't* m-Rad-⊕-*supplemented.*

Recall from [17, 6.4] that a module M is called *distributive* if $(A+B)\cap C=(A\cap C)+(B\cap C)$ for all submodules A,B,C of M (or equivalently, $(A\cap B)+C=(A+C)\cap(B+C)$ for all submodules A,B,C of M).

Now, we show that a factor module of an I-Rad-$\oplus$-supplemented module is I-Rad-$\oplus$-supplemented under some conditions.

Proposition 6 *Let* I *be an ideal of* R *and* M *be an* I-Rad-$\oplus$-*supplemented module.*

(1) *Let* $X\subseteq M$ *be a submodule such that for every direct summand* K *of* M, $\frac{X+K}{X}$ *is a direct summand of* $\frac{M}{X}$. *Then* $\frac{M}{X}$ *is* I-Rad-$\oplus$-*supplemented;*

(2) *Let* $X\subseteq M$ *be a submodule such that for every decomposition* $M=M_1\oplus M_2$, *we have* $X=(X\cap M_1)\oplus(X\cap M_2)$. *Then* $\frac{M}{X}$ *is* I-Rad-$\oplus$-*supplemented;*

(3) *If* X *is a fully invariant submodule of* M, *then* $\frac{M}{X}$ *is* I-Rad-$\oplus$-*supplemented;*

(4) *If* M *is a distributive module, then* $\frac{M}{X}$ *is* I-Rad-$\oplus$-*supplemented for every submodule* X *of* M.

Proof. (1) Let N be a submodule of M such that $X\subseteq N$. Since M is I-Rad-$\oplus$-supplemented, there exists a direct summand K of M such that $M=N+K$, $N\cap K\subseteq IK$ and $N\cap K\subseteq \mathrm{Rad}(K)$. Therefore $\frac{M}{X}=\frac{N}{X}+\frac{X+K}{X}$ and $\frac{N}{X}\cap\frac{K+X}{X}=\frac{X+(N\cap K)}{X}\subseteq\frac{X+IK}{X}\subseteq I(\frac{X+K}{X})$. Consider the natural epimorphism $\pi: K\longrightarrow\frac{X+K}{X}$. Since $N\cap K\subseteq\mathrm{Rad}(K)$, we have $\pi(N\cap K)=\frac{X+(N\cap K)}{X}\subseteq\mathrm{Rad}(\frac{X+K}{X})$. Note that by assumption, $\frac{X+K}{X}$ is a direct summand of $\frac{M}{X}$. It follows that $\frac{M}{X}$ is I-Rad-$\oplus$-supplemented.

(2), (3) and (4) are consequences of (1). □

Proposition 7 *Let* M *be an* R-*module,* I *be an ideal of* R *and* K *be a fully invariant direct summand of* M. *Then the following statements are equivalent:*

(1) M *is* I-Rad-$\oplus$-*supplemented;*

(2) K *and* $\frac{M}{K}$ *are* I-Rad-$\oplus$-*supplemented.*

Proof. (1) $\Rightarrow$ (2) Let L be a submodule of K. By hypothesis, there exist submodules A and B of M such that $M=A\oplus B$, $M=A+L$, $A\cap L\subseteq IA$ and $A\cap L\subseteq\mathrm{Rad}(A)$. Clearly, we have $K=(A\cap K)+L$. Since K is fully invariant in M, we have $K=(A\cap K)\oplus(B\cap K)$. Hence $A\cap K$ is a direct summand of K. By Lemma 3, $I(A \cap K) = (A \cap K) \cap IM$. It follows that $(A \cap K) \cap L = A \cap L \subseteq (A \cap K) \cap IM = I(A \cap K)$. Since $A \cap K$ is a direct summand of K and K is a direct summand of M, $A \cap K$ is a direct summand of M such that $A \cap L \subseteq A \cap K$. Since $A \cap L \subseteq \mathrm{Rad}(M)$, we have $A \cap L \subseteq \mathrm{Rad}(A \cap K)$. Therefore, K is I-Rad-⊕-supplemented. Moreover, $\frac{M}{K}$ is I-Rad-⊕-supplemented by Proposition 6 (3).

(2) ⇒ (1) It follows from Theorem 1. □

Let I be an ideal of R. We call an R-module M is called *completely* I-Rad-⊕-*supplemented* if every direct summand of M is I-Rad-⊕-supplemented. Clearly, semisimple modules are completely I-Rad-⊕-supplemented. Also, every I-Rad-⊕-supplemented hollow module is completely I-Rad-⊕-supplemented.

Proposition 8 *Let $M = M_1 \oplus M_2$ be a direct sum of local submodules M_1 and M_2. Then the following statements are equivalent:*

(1) *M_1 and M_2 are* I-Rad-⊕-*supplemented modules;*

(2) *M is a completely* I-Rad-⊕-*supplemented module.*

Proof. (1) ⇒ (2) Let L be a non-zero direct summand of M. If $L = M$, then L is I-Rad-⊕-supplemented by Theorem 1. Assume that $L \neq M$. Let K be a submodule of M such that $M = L \oplus K$. Then L is a local module by [4, 5.4 (1)]. Let us prove that L is I-Rad-⊕-supplemented. To see this, it suffices to show that $IL = L$ or $IL = \mathrm{Rad}(L)$ by Proposition 2. Since M is I-Rad-⊕-supplemented, $\frac{M}{IM} \cong \frac{L}{IL} \oplus \frac{K}{IK}$ is semisimple by Proposition 3. Then $\frac{L}{IL}$ is semisimple and so $Rad(L) \subseteq IL$. Since L is local, we get that $L = IL$ or $Rad(L) = IL$.

(2) ⇒ (1) Obvious. □

Now, we determine the structure of all I-Rad-⊕-supplemented modules over a discrete valuation ring.

Theorem 2 *Assume that R is a discrete valuation ring with maximal ideal $\mathfrak{m}$. Let I be an ideal of R and M be an R-module.*

(1) *If $I = \mathfrak{m}$ or $I = R$, then the following statements are equivalent.*

  (i) *M is* I-Rad-⊕-*supplemented;*

  (ii) *M is* Rad-⊕-*supplemented;*

  (iii) *$M \cong R^{\mathfrak{a}} \oplus D \oplus B$, where $\mathfrak{a} \in \mathbb{N}$, B is a bounded R-module and D is an injective R-module.*

(2) *If* $I \notin \{\mathfrak{m}, R\}$*, then the following are equivalent:*

(i) M *is* I-Rad-$\oplus$-*supplemented;*

(ii) $M \cong D \oplus B$ *for some injective* R-*module* D *and some semisimple* R-*module* B.

Proof. It is well known that, for any module M over a discrete valuation ring, we have $\mathrm{Rad}(M) = JM = \mathfrak{m}M$.
(1) (i) $\Leftrightarrow$ (ii) Since local rings are a good ring, by Corollary 1 and assumption, the proof follows.

(ii) $\Leftrightarrow$ (iii) Clear by [15, Corollary 3.3].
(2) (i) $\Rightarrow$ (ii) Suppose that M is I-Rad-$\oplus$-supplemented. Applying [15, Corollary 3.3], $M \cong R^a \oplus D \oplus B$ for some bounded R-module B, some natural numbers a and an injective R-module D. Since D is a fully invariant submodule of M, it follows from Proposition 7 that $N = R^a \oplus B$ is I-Rad-$\oplus$-supplemented. Using Lemma 3 and Proposition 3, we obtain that $\frac{N}{IN}$ is semisimple. Since $I \notin \{\mathfrak{m}, R\}$, we get that $a = 0$. Now we will prove that B is semisimple. Since $\frac{B}{IB}$ is semisimple and $I < \mathfrak{m}$, we can write $\mathrm{Rad}(B) = JB = IB$. Note that B is bounded. Then, there exists an ideal H of R such that $HB = 0$. Therefore, $\mathrm{Rad}(B) = JB = HB = 0$ and so B is semisimple by Lemma 2. This completes the proof.

(ii) $\Rightarrow$ (i) By Corollary 4, D is I-Rad-$\oplus$-supplemented. Since B is semisimple, B is I-Rad-$\oplus$-supplemented. Applying Theorem 1, we obtain that M is I-Rad-$\oplus$-supplemented. □

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Initial coefficient bounds for certain class of meromorphic bi-univalent functions

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Abstract. In this paper, we introduce and investigate an interesting subclass of meromorphic bi-univalent functions defined on $\Delta = \{z \in \mathbb{C} : 1 < |z| < \infty\}$. For functions belonging to this class, estimates on the initial coefficients are obtained. The results presented in this paper generalize and improve some recent works.

1 Introduction

Let Σ be the family of meromorphic functions f of the form

$$f(z) = z + b_0 + \sum_{n=1}^{\infty} b_n \frac{1}{z^n}, \tag{1}$$

that are univalent in $\Delta = \{z \in \mathbb{C} : 1 < |z| < \infty\}$. Since $f \in \Sigma$ is univalent, it has an inverse f^{-1} that satisfy

$$f^{-1}(f(z)) = z \qquad (z \in \Delta)$$

and

$$f(f^{-1}(w)) = w \qquad (M < |w| < \infty,\ M > 0)\,.$$

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Furthermore, the inverse function f^{-1} has a series expansion of the form

$$f^{-1}(w) = w + \sum_{n=0}^{\infty} B_n \frac{1}{w^n}, \tag{2}$$

where $M < |w| < \infty$. A simple calculation shows that the function f^{-1}, is given by

$$f^{-1}(w) = w - b_0 - \frac{b_1}{w} - \frac{b_2 + b_0 b_1}{w^2} - \frac{b_3 + 2b_0 b_1 + b_0^2 b_1 + b_1^2}{w^3} + \dots . \tag{3}$$

A function $f \in \Sigma$ is said to be meromorphic bi-univalent if $f^{-1} \in \Sigma$. The family of all meromorphic bi-univalent functions is denoted by $\Sigma_{\mathfrak{B}}$.

Estimates on the coefficient of meromorphic univalent functions were widely investigated in the literature; for example, Schiffer [8] obtained the estimate $|b_2| \le 2/3$ for meromorphic univalent functions $f \in \Sigma$ with $b_0 = 0$ and Duren [2] proved that $|b_n| \le 2/(n+1)$ for $f \in \Sigma$ with $b_k = 0$, $1 \le k \le n/2$.

For the coefficients of inverses of meromorphic univalent functions, Springer [10] proved that

$$|B_3| \le 1 \quad \text{and} \quad |B_3 + \frac{1}{2}B_1^2| \le \frac{1}{2}$$

and conjectured that

$$|B_{2n-1}| \le \frac{(2n-2)!}{n!(n-1)!} \qquad (n = 1, 2, \cdots).$$

In 1977, Kubota [6] proved that the Springer conjecture is true for $n = 3, 4, 5$ and subsequently Schober [9] obtained a sharp bounds for the coefficients B_{2n-1}, $1 \le n \le 7$.

A function f in the class $\Sigma_{\mathfrak{B}}$ is said to be memorphic bi-univalent starlike of order β where $0 \le \beta < 1$, if it satisfies the flowing inequalities

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta \quad \text{and} \quad \operatorname{Re}\left(\frac{wg'(w)}{g(w)}\right) > \beta \quad (z, w \in \Delta),$$

where g is the inverse of f given by (3). We denote by $\Sigma_{\mathfrak{B}}^*(\beta)$ the class of all meromorphic bi-univalent starlike functions of order β. Similarly, a function f in the class $\Sigma_{\mathfrak{B}}$ is said to be meromorphic bi-univalent strongly starlike of order α where $0 < \alpha \le 1$, if it satisfies the following conditions

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \quad \text{and} \quad \left|\arg\left(\frac{wg'(w)}{g(w)}\right) \arg\right| < \frac{\alpha\pi}{2} \quad (z, w \in \Delta),$$

where g is the inverse of f given by (3). We denote by $\widetilde{\Sigma}^*_{\mathfrak{B}}(\alpha)$ the class of all meromorphic bi-univalent strongly starlike functions of order α. The classes $\Sigma^*_{\mathfrak{B}}(\beta)$ and $\widetilde{\Sigma}^*_{\mathfrak{B}}(\alpha)$ were introduced and studied by Halim et al. [3].

Several researchers introduced and investigated some subclasses of meromorphically bi-univalent functions. (see, for detailes [3], [4], [5], [6], [9] and [13]).

Recently, Srivastava at al. [11] introduced the following subclasses of the meromorphic bi-univalent function and obtained non sharp estimates on the initial coefficients $|b_0|$ and $|b_1|$ as follow.

Definition 1 [11, Definition 2] *A function* $f(z) \in \Sigma_{\mathfrak{B}}$ *given by (1) is said to be in the class* $\Sigma_{B,\lambda^*}(\alpha)$*, if the following conditions are satisfied:*

$$\left|\arg\left(\frac{z[f'(z)]^\lambda}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \leq 1,\ \lambda \geq 1,\ z \in \Delta)$$

and

$$\left|\arg\left(\frac{w[g'(w)]^\lambda}{g(w)}\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1,\ \lambda \geq 1,\ w \in \Delta),$$

where the function g *is the inverse of* f *given by* (3)*.*

Theorem 1 [11, Theorem 2.1] *Let* $f(z) \in \Sigma_{\mathfrak{B}}$ *given by* (1) *be in the class* $\Sigma_{B,\lambda^*}(\alpha)$*. Then*

$$|b_0| \leq 2\alpha, \qquad |b_1| \leq \frac{2\sqrt{5}\alpha^2}{1+\lambda}.$$

Definition 2 [11, Definition 3] *A function* $f(z) \in \Sigma_{\mathfrak{B}}$ *given by* (1) *is said to be in the class* $\Sigma_{B^*}(\lambda, \beta)$*, if the following conditions are satisfied:*

$$\operatorname{Re}\left(\frac{z[f'(z)]^\lambda}{f(z)}\right) > \beta \quad (0 \leq \beta < 1,\ \lambda \geq 1,\ z \in \Delta)$$

and

$$\operatorname{Re}\left(\frac{w[g'(w)]^\lambda}{g(w)}\right) > \beta \quad (0 \leq \beta < 1,\ \lambda \geq 1,\ w \in \Delta),$$

where the function g *is the inverse of* f *given by* (3)*.*

Theorem 2 [11, Theorem 3.1] *Let* $f(z)$ *given by* (1) *be in the class* $\Sigma_{B^*}(\lambda, \beta)$*. Then*

$$|b_0| \leq 2(1-\beta), \qquad |b_1| \leq \frac{2(1-\beta)\sqrt{4\beta^2 - 8\beta + 5}}{1+\lambda}.$$

The following subclass of the meromorphic bi-univalent functions was investigated by Hai-Gen Xiao and Qing-Hua Xu [12].

Definition 3 [12, Definition 3] *A function* $f(z) \in \Sigma_{\mathfrak{B}}$ *given by* (1) *is said to be in the class* $\Sigma^*_{\vartheta}(\mu, \alpha)$*, if the following conditions are satisfied:*

$$\left|\arg\left\{(1-\mu)\frac{zf'(z)}{f(z)} + \mu\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\}\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1,\ \mu \in \mathbb{R},\ z \in \Delta)$$

and

$$\left|\arg\left\{(1-\mu)\frac{wg'(w)}{g(w)} + \mu\left(1 + \frac{wg''(w)}{g'(w)}\right)\right\}\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1,\ \mu \in \mathbb{R},\ w \in \Delta),$$

where the function g *is the inverse of* f *given by* (3)*.*

Theorem 3 [12, Theorem 1] *Let* $f(z)$ *given by* (1) *be in the class* $\Sigma^*_{\vartheta}(\mu, \alpha)$*,* $\mu \in \mathbb{R} - \left\{\frac{1}{2}, 1\right\}$*. Then*

$$|b_0| \leq \frac{2\alpha}{|1-\mu|}, \qquad |b_1| \leq \frac{\sqrt{\mu^2 - 2\mu + 5}}{|1-\mu||2\mu - 1|}\alpha^2.$$

The object of the present paper is to introduce a new subclass of the function class $\Sigma_{\mathfrak{B}}$ and obtain estimates on the initial coefficients for functions in this new subclass which improve Theorem 1, Theorem 2 and Theorem 3. Our results generalize and improve those in related works of several earlier authors.

2 Coefficient bounds for the function class $M^{h,p}_{\Sigma_{\mathfrak{B}}}(\lambda, \mu)$

In this section, we introduce and investigate the general subclass $M^{h,p}_{\Sigma_{\mathfrak{B}}}(\lambda, \mu)$.

Definition 4 *Let the functions* $h, p : \Delta \to \mathbb{C}$ *be analytic functions and*

$$h(z) = 1 + \frac{h_1}{z} + \frac{h_2}{z^2} + \frac{h_3}{z^3} + \cdots, \quad p(z) = 1 + \frac{p_1}{z} + \frac{p_2}{z^2} + \frac{p_3}{z^3} + \cdots,$$

such that

$$\min\{\mathrm{Re}(h(z)),\ \mathrm{Re}\,(p(z))\} > 0, \quad z \in \Delta.$$

A function $f \in \Sigma_{\mathfrak{B}}$ *given by* (1) *is said to be in the class* $M^{h,p}_{\Sigma_{\mathfrak{B}}}(\lambda, \mu)$ $(\lambda \geq 1,$ $\mu \in \mathbb{R})$*, if the following conditions are satisfied:*

$$(1-\mu)\frac{z(f'(z))^{\lambda}}{f(z)} + \mu\left(1 + \frac{zf''(z)}{f'(z)}\right)^{\lambda} \in h(\Delta) \quad (\lambda \geq 1,\ \mu \in \mathbb{R},\ z \in \Delta) \tag{4}$$

and

$$(1-\mu)\frac{w(g'(w))^{\lambda}}{g(w)} + \mu\left(1 + \frac{wg''(w)}{g'(w)}\right)^{\lambda} \in p(\Delta) \quad (\lambda \geq 1,\ \mu \in \mathbb{R},\ w \in \Delta), \tag{5}$$

where the function g *is the inverse of* f *given by* (3)*.*

Remark 1 *There are many selections of the functions* $h(z)$ *and* $p(z)$ *which would provide interesting subclasses of the meromorphic function class* Σ*. For example, if we let*

$$h(z) = p(z) = \left(\frac{1+\frac{1}{z}}{1-\frac{1}{z}}\right)^{\alpha} = 1 + \frac{2\alpha}{z} + \frac{2\alpha^2}{z^2} + \cdots \quad (0 < \alpha \leq 1,\ z \in \Delta),$$

it is easy to verify that the functions $h(z)$ *and* $p(z)$ *satisfy the hypotheses of Definition 4.*

If $f \in M^{h,p}_{\Sigma_{\mathfrak{B}}}(\lambda, \mu)$*, then*

$$\left|\arg\left\{(1-\mu)\frac{z(f'(z))^{\lambda}}{f(z)} + \mu\left(1 + \frac{zf''(z)}{f'(z)}\right)^{\lambda}\right\}\right| < \frac{\alpha\pi}{2}$$
$$(0 < \alpha \leq 1,\ \lambda \geq 1,\ \mu \in \mathbb{R},\ z \in \Delta)$$

and

$$\left|\arg\left\{(1-\mu)\frac{w(g'(w))^{\lambda}}{g(w)} + \mu\left(1 + \frac{wg''(w)}{g'(w)}\right)^{\lambda}\right\}\right| < \frac{\alpha\pi}{2}$$
$$(0 < \alpha \leq 1,\ \lambda \geq 1,\ \mu \in \mathbb{R},\ w \in \Delta).$$

In this case, the function f *is said to be in the class* $M_{\Sigma_{\mathfrak{B}}}(\lambda, \mu, \alpha)$ *and in special case* $\lambda = 1$*, it reduces to Definition 3. We note that, by putting* $\mu = 0$*,* *the class* $M_{\Sigma_{\mathfrak{B}}}(\lambda,\mu,\alpha)$ *reduces to Definition 1, the class* $\Sigma_{B,\lambda^*}(\alpha)$ *introduced and studied by Srivastava et al.* [11].

If we let

$$h(z)=p(z)=\frac{1+\frac{1-2\beta}{z}}{1-\frac{1}{z}}$$
$$=1+\frac{2(1-\beta)}{z}+\frac{2(1-\beta)}{z^2}+\frac{2(1-\beta)}{z^3}+\dots \quad (0\leq\beta<1,\ z\in\Delta),$$

it is easy to verify that the functions $h(z)$ *and* $p(z)$ *satisfy the hypotheses of Definition 4.*

If $f\in M^{h,p}_{\Sigma_{\mathfrak{B}}}(\lambda,\mu)$, *then*

$$\mathrm{Re}\left\{(1-\mu)\frac{z(f'(z))^\lambda}{f(z)}+\mu\left(1+\frac{zf''(z)}{f'(z)}\right)^\lambda\right\}>\beta$$
$$(0\leq\beta<1,\ \lambda\geq1,\ \mu\in\mathbb{R},\ z\in\Delta)$$

and

$$\mathrm{Re}\left\{(1-\mu)\frac{w(g'(w))^\lambda}{g(w)}+\mu\left(1+\frac{wg''(w)}{g'(w)}\right)^\lambda\right\}>\beta$$
$$(0\leq\beta<1,\ \lambda\geq1,\ \mu\in\mathbb{R},\ w\in\Delta).$$

Therefore for $h(z)=p(z)=\frac{1+\frac{1-2\beta}{z}}{1-\frac{1}{z}}$ *and* $\mu=0$, *the class* $M^{h,p}_{\Sigma_{\mathfrak{B}}}(\lambda,\mu)$ *reduces to Definition 2.*

Now, we derive the estimates of the coefficients $|b_0|$ and $|b_1|$ for class $M^{h,p}_{\Sigma_{\mathfrak{B}}}(\lambda,\mu)$.

Theorem 4 *Let* $f(z)\in\Sigma_{\mathfrak{B}}$ *given by* (1) *be in the class* $M^{h,p}_{\Sigma_{\mathfrak{B}}}(\lambda,\mu)$ $(\lambda\geq1,\ \mu\in\mathbb{R}-\{1\},\ (3\lambda\mu+\mu-\lambda)\neq1)$. *Then*

$$|b_0|\leq\min\left\{\sqrt{\frac{|h_1|^2+|p_1|^2}{2(1-\mu)^2}},\sqrt{\frac{|h_2|+|p_2|}{2|1-\mu|}}\right\} \tag{6}$$

and

$$|b_1|\leq\min\left\{\frac{|h_2|+|p_2|}{2|3\lambda\mu+\mu-\lambda-1|},\frac{1}{|3\lambda\mu+\mu-\lambda-1|}\sqrt{\frac{|h_2|^2+|p_2|^2}{2}+\frac{(|h_1|^2+|p_1|^2)^2}{4(1-\mu)^2}}\right\}. \tag{7}$$

Proof. First of all, we write the argument inequalities in (4) and (5) in their equivalent forms as follows:

$$(1-\mu)\frac{z(f'(z))^{\lambda}}{f(z)} + \mu\left(1+\frac{zf''(z)}{f'(z)}\right)^{\lambda} = h(z) \quad (z \in \Delta) \tag{8}$$

and

$$(1-\mu)\frac{w(g'(w))^{\lambda}}{g(w)} + \mu\left(1+\frac{wg''(w)}{g'(w)}\right)^{\lambda} = p(w) \quad (w \in \Delta), \tag{9}$$

respectively, where functions $h(z)$ and $p(w)$ satisfy the conditions of Definition 4.
Furtheremore, the functions $h(z)$ and $p(w)$ have the forms:

$$h(z) = 1 + \frac{h_1}{z} + \frac{h_2}{z^2} + \frac{h_3}{z^3} + \cdots$$

and

$$p(w) = 1 + \frac{p_1}{w} + \frac{p_2}{w^2} + \frac{p_3}{w^2} + \cdots,$$

respectively. Now, upon equating the coefficients of

$$\begin{aligned} &(1-\mu)\frac{z(f'(z))^{\lambda}}{f(z)} + \mu\left(1+\frac{zf''(z)}{f'(z)}\right)^{\lambda} \\ &= 1 - \frac{(1-\mu)b_0}{z} + \frac{(1-\mu)b_0^2 + (3\lambda\mu+\mu-\lambda-1)b_1}{z^2} + \ldots \end{aligned} \tag{10}$$

with those of $h(z)$ and coefficients of

$$\begin{aligned} &(1-\mu)\frac{w(g'(w))^{\lambda}}{g(w)} + \mu\left(1+\frac{wg''(w)}{g'(w)}\right)^{\lambda} \\ &= 1 + \frac{(1-\mu)b_0}{w} + \frac{(1-\mu)b_0^2 - (3\lambda\mu+\mu-\lambda-1)b_1}{w^2} + \ldots \end{aligned} \tag{11}$$

with those of $p(w)$, we get

$$-(1-\mu)b_0 = h_1, \tag{12}$$

$$(1-\mu)b_0^2 + (3\lambda\mu+\mu-\lambda-1)b_1 = h_2, \tag{13}$$

$$(1-\mu)b_0 = p_1 \tag{14}$$

and

$$(1-\mu)b_0^2-(3\lambda\mu+\mu-\lambda-1)b_1=p_2 \tag{15}$$

From (12) and (14), we get

$$h_1=-p_1 \quad (b_0=-\frac{h_1}{1-\mu})$$

and

$$2(1-\mu)^2b_0^2=h_1^2+p_1^2. \tag{16}$$

Adding (13) and (15), we get

$$2(1-\mu)b_0^2=h_2+p_2. \tag{17}$$

Therefore, we find from the equations (16) and (17) that

$$|b_0|^2\leq\frac{|h_1|^2+|p_1|^2}{2(1-\mu)^2},$$

and

$$|b_0|^2\leq\frac{|h_2|+|p_2|}{2|1-\mu|}$$

respectively. So we get the desired estimate on the coefficient $|b_0|$ as asserted in (6).

Next, in order to find the bound on the coefficient $|b_1|$, we subtract (15) from (13). We thus get

$$2(3\lambda\mu+\mu-\lambda-1)b_1=h_2-p_2. \tag{18}$$

By squaring and adding (13) and (15), using (16) in the computation leads to

$$b_1^2=\frac{1}{2(3\lambda\mu+\mu-\lambda-1)^2}\left(h_2^2+p_2^2-\frac{(h_1^2+p_1^2)^2}{2(1-\mu)^2}\right). \tag{19}$$

Therefore, we find from the equations (18) and (19) that

$$|b_1|\leq\frac{|h_2|+|p_2|}{2|3\lambda\mu+\mu-\lambda-1|}$$

and

$$|b_1|\leq\frac{1}{|3\lambda\mu+\mu-\lambda-1|}\sqrt{\frac{|h_2|^2+|p_2|^2}{2}+\frac{(|h_1|^2+|p_1|^2)^2}{4(1-\mu)^2}}.$$

This evidently completes the proof of Theorem 4. □

3 Corollaries and consequences

By setting

$$h(z) = p(z) = \frac{1 + \frac{1-2\beta}{z}}{1 - \frac{1}{z}} = 1 + \frac{2(1-\beta)}{z} + \frac{2(1-\beta)}{z^2} + \dots \quad (0 \le \beta < 1,\ z \in \Delta)$$

and $\mu = 0$ in Theorem 4, we conclude the following result.

Corollary 1 *Let the function* $f(z)$ *given by* (1) *be in the class* $\Sigma_{B^*}(\lambda, \beta)$, $(0 \le \beta < 1, \lambda \ge 1)$. *Then*

$$|b_0| \le \begin{cases} \sqrt{2(1-\beta)}; & \beta \le \frac{1}{2} \\ 2(1-\beta); & \beta > \frac{1}{2} \end{cases}$$

and

$$|b_1| \le \min\left\{ \frac{2(1-\beta)}{1+\lambda}, \frac{2(1-\beta)\sqrt{4\beta^2 - 8\beta + 5}}{1+\lambda} \right\} = \frac{2(1-\beta)}{1+\lambda}.$$

Remark 2 *The bounds on* $|b_0|$ *and* $|b_1|$ *given in Corollary 1 are better than those given in Theorem 2.*

By setting $\lambda = 1$ in Corollary 1, we conclude the following result.

Corollary 2 *Let the function* $f(z)$ *given by* (1) *be in the class* $\Sigma^*_{\mathfrak{B}}(\beta)$ $(0 \le \beta < 1)$. *Then*

$$|b_0| \le \begin{cases} \sqrt{2(1-\beta)}; & \beta \le \frac{1}{2} \\ 2(1-\beta); & \beta > \frac{1}{2} \end{cases}$$

and

$$|b_1| \le \min\{1-\beta, (1-\beta)\sqrt{1 + 4(1-\beta)^2}\} = 1 - \beta.$$

Remark 3 *The bounds on* $|b_0|$ *and* $|b_1|$ *given in Corollary 2 are better than those given by Halim et al.* [3, Theorem 1].

By setting

$$h(z) = p(z) = \left(\frac{1 + \frac{1}{z}}{1 - \frac{1}{z}} \right)^{\alpha} \quad (0 < \alpha \le 1, \quad z \in \Delta),$$

in Theorem 4, we conclude the following result.

Corollary 3 *Let the function* $f(z)$ *given by* (1) *be in the class* $M_{\Sigma_{\mathfrak{B}}}(\lambda, \mu, \alpha)$ $(0 < \alpha \leq 1,\ \lambda \geq 1,\ \mu \in \mathbb{R} - \{1\},\ (3\lambda\mu + \mu - \lambda) \neq 1)$. *Then*

$$|b_0| \leq \begin{cases} \alpha\sqrt{\frac{2}{|1-\mu|}}; & |1-\mu| \leq 2 \\ \frac{2\alpha}{|1-\mu|}; & |1-\mu| > 2 \end{cases}$$

and

$$|b_1| \leq \min\left\{\frac{2\alpha^2}{|3\lambda\mu + \mu - \lambda - 1|}, \frac{2\alpha^2}{|3\lambda\mu + \mu - \lambda - 1|}\sqrt{1 + \frac{4}{(1-\mu)^2}}\right\}$$
$$= \frac{2\alpha^2}{|3\lambda\mu + \mu - \lambda - 1|}.$$

By setting $\mu = 0$ in Corollary 3, we conclude the following result.

Corollary 4 *Let the function* $f(z)$ *given by* (1) *be in the class* $\Sigma_{B,\lambda^*}(\alpha)$ $(0 < \alpha \leq 1,\ \lambda \geq 1)$. *Then*

$$|b_0| \leq \sqrt{2}\alpha$$

and

$$|b_1| \leq \frac{2\alpha^2}{\lambda + 1}.$$

Remark 4 *The bounds on* $|b_0|$ *and* $|b_1|$ *given in Corollary 4 are better than those given in Theorem 2.*

By setting $\lambda = 1$ in Corollary 3, we conclude the following result.

Corollary 5 *Let the function* $f(z)$ *given by* (1) *be in the class* $\Sigma_{\vartheta}^*(\mu, \alpha)$ $(0 < \alpha \leq 1,\ \mu \in \mathbb{R} - \{\frac{1}{2}, 1\})$. *Then*

$$|b_0| \leq \begin{cases} \alpha\sqrt{\frac{2}{|1-\mu|}}; & |1-\mu| \leq 2 \\ \frac{2\alpha}{|1-\mu|}; & |1-\mu| > 2 \end{cases}$$

and

$$|b_1| \leq \min\left\{\frac{\alpha^2}{|2\mu - 1|}, \frac{\sqrt{\mu^2 - 2\mu + 5}}{|1-\mu||2\mu - 1|}\alpha^2\right\} = \frac{\alpha^2}{|2\mu - 1|}.$$

Remark 5 *The bounds on* $|b_0|$ *and* $|b_1|$ *given in Corollary 5 are better than those given in Theorem 3.*

By setting $\mu = 0$ in Corollary 5, we conclude the following result.

Corollary 6 *Let the function* $f(z)$ *given by* (1) *be in the class* $\widetilde{\Sigma}^*_{\mathfrak{B}}(\alpha)$ $(0 < \alpha \leq 1)$. *Then*

$$|b_0| \leq \sqrt{2}\alpha \qquad \text{and} \qquad |b_1| \leq \min\left\{\alpha^2, \sqrt{5}\alpha^2\right\} = \alpha^2.$$

Remark 6 *The bounds on* $|b_0|$ *and* $|b_1|$ *given in Corollary 6 are better than those given by Halim et al.* [3, Theorem 2].

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