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On the distribution of q-additive functions under some conditions III.

Imre Kátai
Eötvös Lóránd University
Department of Computer Algebra
Budapest, Hungary
email: katai@compalg.inf.elte.hu

Abstract. The existence of the limit distribution of a q-additive function over the set of integers characterized by the sum of digits is investigated.

1 Introduction

Notation

$\mathbb{N}, \mathbb{R}, \mathbb{C}$, as usual denote the set of natural, real and complex numbers, respectively. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

q-additive and q-multiplicative functions

Let $q \geq 2$ be an integer, the q-ary expansion of $n \in \mathbb{N}_0$ is defined as

$$n = \sum_{j=0}^{\infty} \varepsilon_j(n) q^j, \tag{1}$$

where the digits $\varepsilon_j(n)$ are taken from $\mathbb{A}_q = \{0, 1, \ldots, q-1\}$. It is clear that the right hand side of (1) is finite.

Let $\mathcal{A}_q$ be the set of q-additive, and $\mathcal{M}_q$ be the set of q-multiplicative functions.

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$f:\mathbb{N}_0\to\mathbb{R}$ belongs to $\mathcal{A}_q$ if $f(0)=0$ and

$$f(n)=\sum_{j=0}^{\infty} f\left(\varepsilon_j(n)\,q^j\right)\quad (n\in\mathbb{N}_0)\,. \tag{2}$$

We say that $g:\mathbb{N}_0\to\mathbb{C}$ belongs to $\mathcal{M}_q$, if $g(0)=1$,

$$g(n)=\prod_{j=0}^{\infty} g\left(\varepsilon_j(n)\,q^j\right)\quad (n\in\mathbb{N}_0)\,. \tag{3}$$

Let $\bar{\mathcal{M}}_q\subseteq\mathcal{M}_q$ be the set of those q-multiplicative functions g, for which $|g(n)|=1\quad (n\in\mathbb{N}_0)$.

Let $\beta_h(n)=\sum_{\varepsilon_j(n)=h} 1\quad (h=1,\dots,q-1)\,,\ \alpha(n)=\sum_{j=0}^{\infty}\varepsilon_j(n)$. We say that $f\in\mathcal{A}_q$ has a limit distribution, if

$$\lim_{x\to\infty}\frac{1}{x}\#\{n\le x|f(n)<y\}\quad (=G(y)) \tag{4}$$

exists for almost all y, and G is a distribution function, i.e. it is monotonic, furthermore $\lim_{y\to-\infty} G(y)=0,\ \lim_{y\to\infty} G(y)=1$.

H. Delange [1] proved that $f\in\mathcal{A}_q$ has a limit distribution if and only the series

$$\sum_{j=0}^{\infty}\sum_{a\in A_q} f\left(aq^j\right), \tag{5}$$

$$\sum_{j=0}^{\infty}\sum_{a\in A_q} f^2\left(aq^j\right) \tag{6}$$

are convergent. He proved that for some $g\in\bar{\mathcal{M}}_q$, the limit

$$\lim_{x\to\infty}\frac{1}{x}\sum_{n\le x} g(n)=M(g)$$

exists and $M(g)\ne 0$, if and only if

$$m_j:=\frac{1}{q}\sum_{c\in A_q} g\left(cq^j\right)\ne 0\quad (j=0,1,2,\dots) \tag{7}$$

and

$$\sum_{j=0}^{\infty}(1-m_j)=\sum_{j=0}^{\infty}\frac{1}{q}\left(\sum_{c\in A_q}\left(1-g\left(cq^j\right)\right)\right) \tag{8}$$

is convergent. Furthermore,

$$M(g)=\prod_{j=0}^{\infty}m_j, \tag{9}$$

if (7) holds and (8) is convergent.

Distribution of q-additive functions under the conditions that $\beta_h(n)$ are fixed.

For some fixed N, let $r_1,\dots,r_{q-1}$ be such nonnegative integers for which $r_1+\dots+r_{q-1}\le N$. Let $r_0=N-(r_1+\dots+r_{q-1})$, $\underline{r}=(r_1,r_2,\dots,r_{q-1})$.

Let

$$S_N(\underline{r})=\left\{n<q^N|\beta_h(n)=r_h,\ h=1,\dots,q-1\right\}. \tag{10}$$

Then

$$M(N|\underline{r})=\#S_N(\underline{r})=\frac{N!}{r_0!r_1!\dots r_{q-1}!}. \tag{11}$$

In [2] we proved the following

Lemma 1 *Let* $f\in\mathcal{A}_q$, $E_N=\sum_{b\in A_q}\frac{r_b}{N}\sum_{j=0}^{N-1}f\left(bq^j\right)$,

$$\Delta_N(\underline{r})=\frac{1}{M(N|\underline{r})}\sum_{n\in S_N(\underline{r})}(f(n)-E_N)^2. \tag{12}$$

Then

$$\Delta_N(\underline{r})<c\sum_{j=0}^{N-1}\sum_{b=0}^{q-1}f^2\left(bq^j\right), \tag{13}$$

c *is a constant which may depend only on* q.

We shall prove

Theorem 1 *Let* $g\in\bar{\mathcal{M}}_q$, *assume that*

$$\sum_{j=0}^{\infty}\sum_{b\in A_q}\left(1-g\left(bq^j\right)\right) \tag{14}$$

is convergent. Let $\lambda_0, \lambda_1, \dots, \lambda_{q-1}$ *be positive numbers, such that* $\lambda_0 + \cdots + \lambda_{q-1} = 1$. *Let*

$$H\left(g|\lambda_0, \dots, \lambda_{q-1}\right) := \prod_{j=0}^{\infty}\left(\sum_{b\in A_q} \lambda_j g\left(bq^j\right)\right). \tag{15}$$

If $\underline{r}^{(N)} = \left(r_1^{(N)}, \dots r_{q-1}^{(N)}\right)$ *is such a sequence for which* $\frac{r_j^{(N)}}{N} \to \lambda_j$ $(j = 1, \dots, q-1)$, *then*

$$\lim_{N\to\infty} \frac{1}{M\left(N|\underline{r}^{(N)}\right)} \sum_{\substack{n<q^N \\ n\in S_N\left(\underline{r}^{(N)}\right)}} g(n) = H\left(g|\lambda_0, \dots, \lambda_{q-1}\right). \tag{16}$$

Hence we obtain

Theorem 2 *Let* $f \in \mathcal{A}_q$, *assume that* (5),(6) *are convergent. Let* $\lambda_0, \dots, \lambda_{q-1}$ *be positive numbers such that* $\lambda_0 + \cdots + \lambda_{q-1} = 1$. *Let* $\eta_0, \eta_1 \dots$ *be independent random variables,* $P\left(\eta_l = f\left(bq^l\right)\right) = \lambda_b \quad (b \in A_q)$.

Let

$$\Theta = \sum_{l=0}^{\infty} \eta_l, \tag{17}$$

$$F_{\underline{\lambda}}(y) := P(\Theta < y), \ \underline{\lambda} = (\lambda_1, \dots, \lambda_{q-1}). \tag{18}$$

From the 3 series theorem of Kolmogorov it follows that the sum (17) *is convergent with probability 1, thus* $F_{\underline{\lambda}}(y)$ *exists.*

If $\frac{r_j^{(N)}}{N} \to \lambda_j \quad (j = 0, \dots, q-1)$, *then*

$$\lim_{N\to\infty} \frac{1}{M\left(N|\underline{r}^{(N)}\right)} \#\left\{n < q^N | n \in S_N\left(\underline{r}^{(N)}\right),\ f(n) < y\right\} = F_\lambda(y),$$

if y *is a continuity point of* F_λ.

F_λ *is continuous, if* $f\left(bq^j\right) \neq 0$ *holds for infinitely many elements of* $\left\{bq^j | j = 0, 1, 2, \dots, b \in A_q\right\}$.

In [2] we proved Theorem 1 for $\lambda_1 = \dots = \lambda_{q-1} = \frac{1}{q}$, and in the case $q = 2$ for $0 < \lambda_1 < 1$.

Furthermore, in [2] we proved the following assertion.

Theorem A *Let* $f \in \mathcal{A}_2,\ f\left(2^j\right) = \mathcal{O}(1) \quad (j \in \mathbb{N}),\ \eta_N = \frac{1}{N}\sum_{j=0}^{N-1} f\left(2^j\right),$

$$B_N^2 := \frac{1}{4}\sum_{j=0}^{N-1}\left(f\left(2^j\right) - \eta_N\right)^2.$$

Assume that $B_N \to \infty$. *Let* $\rho_N \to 0$.

Then

$$\lim_{N\to\infty} \frac{1}{\binom{N}{k}} \# \left\{ n < 2^N \left| \frac{f(n) - k\eta_N}{B_N} < y, \alpha(n) = k \right. \right\} = \Phi(y)$$

holds uniformly as $N \to \infty,\ k = k^{(N)}$ *satisfies*

$$\left| \frac{k}{N} - \frac{1}{2} \right| < \rho_N.$$

In [3] we mentioned that we are able to prove that under the conditions of Theorem A

$$\lim_{n\to\infty} \sup_{\frac{k}{N}\in[\delta,1-\delta]} \sup_{y\in\mathbb{R}} \left| \frac{1}{\binom{N}{k}} \# \left\{ n < 2^N, \alpha(n) = k, \frac{f(n) - k\eta_N}{2B_N\sqrt{(1-\eta)\eta}} < y \right\} - \Phi(y) \right|.$$

This assertion is not true, the correct assertion is

Theorem 3 *Let* $f \in \mathcal{A}_2,\ f\left(2^j\right) = \mathcal{O}(1)\ (j = 0, 1, 2, \ldots)$. *Let* $m_N = \sum_{j=0}^{N-1} f\left(2^j\right),$
$\sigma_N^2 = \sum_{j=0}^{N-1}\left(f\left(2^j\right) - \frac{m_N}{N}\right)^2$. *Let* $0 < \lambda < 1$,

$$F_{r,N}(y) = \frac{1}{\binom{N}{r}} \# \left\{ n < 2^N,\ \alpha(n) = r,\ \frac{f(n) - \frac{r}{N} m_N}{\sigma_N} < y \right\}.$$

Furthermore, let $F_\lambda(y)$ *be the distribution the characteristic function* $\varphi_\lambda(\tau) = \sum_{l=0}^{\infty} \alpha_l \frac{(i\tau)^l}{l!}$ *of which is given by the following formulas:*

$$\begin{aligned} \alpha_l &= 0, \quad \textit{if} \quad l \textit{ is odd},\ \alpha_0 = 1, \\ \alpha_{2k} &= \sum_{t=1}^{2k} \frac{\lambda^t}{t!2^t} \sum_{2m\le t} (-1)^m \binom{t}{2m} \cdot 2^{2m}(2m-1)!! \quad (k = 1, 2, \ldots). \end{aligned}$$

Since α_{2k} *is bounded as* $k \to \infty$*, therefore the series defining* $\varphi_\lambda(\tau)$ *is absolutely convergent in* $|\tau| < 1$*.*

We have

$$\lim_{\substack{\frac{r}{N}\to\lambda \\ N\to\infty}} F_{r,N}(y) = F_\lambda(y).$$

2 Proof of Theorem 1 and 2

Let us define $f\left(bq^j\right)$ as the argument of $g\left(bq^j\right)$, i.e. $g\left(bq^j\right) = e^{if\left(bq^j\right)}$. The condition (8) implies the convergence of (5) and (6). We can extend f as a q-additive function. Then $g(n) = e^{if(n)}$.

Let $g_M(n) = \prod_{j=0}^{M-1} g\left(\varepsilon_j(n) q^j\right)$. Thus $g_M\left(nq^M\right) = 1$ $(n \in \mathbb{N}_0)$. Let $f_M(n) = \sum_{j=0}^{M-1} f\left(\varepsilon_j(n) q^j\right)$; $h_M(n) = \sum_{j\geq M} f\left(\varepsilon_j(n) q^j\right)$.

Let M be fixed, and consider the integers $n < q^{N+M}$. Let $\delta > 0$ be an arbitrary (small) number. We shall estimate the number of those $n \in S_{N+M}\left(\underline{r}^{(N+M)}\right)$ for which $|g(n) - g_M(n)| \geq \delta$. If n is such an integer, then $|h_M(n)| \geq \delta_k$.

Assume that M is so large that for

$$E_M^{(N+M)} := \sum_{j=M}^{M+N-1} \sum_{b\in A_q} f\left(bq^j\right)$$

$\left|E_M^{(N+M)}\right| < \frac{\delta}{4}$. We shall apply (12), (13) for $h_M(n)$ and $E_M^{(N+M)}$. Then, in the right hand side of (13)

$$\sum_{j=M}^{N+M-1} \sum_{b\in A_q} f^2\left(bq^j\right)$$

tends to zero as $M \to \infty$. Consequently, the following assertion is true.

Let $\delta > 0, \varepsilon > 0$ be arbitrary constants. Then there exists such an M for which

$$\limsup_{N\to\infty} \frac{1}{M\left(N + M|\underline{r}^{(N+M)}\right)} \# \left\{ n \in S_{N+M}\left(\underline{r}^{(N+M)}\right) \middle| |g(n) - g_M(n)| > \delta \right\} < \varepsilon.$$

Now we estimate

$$\frac{1}{M\left(N + M|\underline{r}^{(N+M)}\right)} \sum_{n\in S_{N+M}\left(\underline{r}^{(n+M)}\right)} g_M(n).$$

Let us subdivide the integers $n \in S_{N+M}\left(\underline{r}^{(N+M)}\right)$ according to the digits $\varepsilon_0(n), \dots, \varepsilon_{M-1}(n)$. Let $n = t + m \cdot q^M$. Then $n \in S_{N+M}\left(\underline{r}^{(N+M)}\right)$, if and only if

$$m \in S_N\left(r_1^{(N+M)} - \beta_1(t), \dots, r_{q-1}^{(N+M)} - \beta_{q-1}(t)\right). \tag{19}$$

For fixed t the number of the m satisfying the condition (19) is

$$(\Psi_N(t) :=) \frac{N!}{\prod_{i=0}^{q-1}\left(r_i^{(N+M)} - \beta_i(t)\right)!},$$

where $\beta_0(t)$ is so defined that $\sum_{i=0}^{q-1} \beta_i(t) = M$.

Let $\frac{r_b^{(N+M)}}{N+M} \to \lambda_b$. Then

$$\begin{aligned}\frac{\Psi_N(t)}{S_{N+M}\left(\underline{r}^{(N+M)}\right)} &= \frac{1}{(N+1)\cdots(N+M)} \prod_{b=0}^{q-1} \frac{r_b^{(N+M)}!}{\left(r_b^{(N+M)} - \beta_b(t)\right)!} \\ &= \frac{1}{(N+1)\cdots(N+M)} \prod_{b=0}^{q-1} \prod_{l=0}^{\beta_b(t)-1} \left(r_b^{(N+M)} - l\right) \\ &= (1 + \mathcal{O}_N(1)) \prod_{b=0}^{q-1} \lambda_b^{\beta_b(t)},\end{aligned}$$

and so

$$\lim_{N\to\infty} \frac{1}{M\left(N + M|\underline{r}^{(N+M)}\right)} \sum_{n \in S_{N+M}\left(\underline{r}^{(N+M)}\right)} g_M(n) = \prod_{j=0}^{M-1} \left\{\sum_b \lambda_b g\left(bq^j\right)\right\}.$$

Finally, let us to tend $M \to \infty$. Then (16) follows. Theorem 1 is proved.

Theorem 2 is a direct consequence of Theorem 1.

3 Some lemmas

Lemma 2 (Wintner, Frechet-Shohat) *Let* $F_n(z) \quad (n = 1, 2, \dots)$ *be a sequence of distribution functions. For each non-negative integer* k *let*

$$\alpha_k = \lim_{n\to\infty} \int_{-\infty}^{\infty} z^k dF_n(z)$$

exist. Then there is a subsequence $F_{n_j}(z) \quad (n_1 < n_2 < \cdots)$ *which converges weakly to a limiting distribution* $F(z)$ *for which*

$$\alpha_k = \int_{-\infty}^{\infty} z^k dF(z) \quad (k = 0, 1, 2, \ldots).$$

Moreover, if the set of moments α_k *determine* $F(z)$ *uniquely, then as* $n \to \infty$ *the distributions* $F_n(z)$ *converge weakly to* $F(z)$.

Lemma 3 *In the notations of Lemma 2 let the series*

$$\varphi(\tau) = \sum_{l=0}^{\infty} \alpha_l \frac{(i\tau)^l}{l!}$$

converge absolutely in a disc of complex τ *values in* $|\tau| < c$, $c > 0$. *Then the* α_k *determine the distribution function* $F(u)$ *uniquely. Moreover, the characteristic function* $\varphi(t)$ *of this distribution had the above representation in the disc* $|\tau| < t$, *and can be analytically continued into the strip* $|Im(t)| < \tau$.

The proof of Lemma 2 can be found in [5] while the proof of Lemma 3 is given in [6]. (Vol. I., page 60).

4 Proof of Theorem 3

Let

$$m_N = \sum_{j=0}^{N-1} f\left(2^j\right), \tag{20}$$

$$F\left(2^j\right) = f\left(2^j\right) - \frac{m_N}{N}, \tag{21}$$

$$\sigma_N^2(f) = \sum_{j=0}^{N-1} F^2\left(2^j\right), \tag{22}$$

$$G\left(2^j\right) = \frac{F\left(2^j\right)}{\sigma_N(f)}. \tag{23}$$

Then

$$\sigma_N^2(G) = \sum_{j=0}^{N-1} G^2\left(2^j\right) = 1. \tag{24}$$

Let

$$T_k := \frac{1}{\binom{N}{r}} \sum_{\substack{n<2^N \\ \alpha(n)=r}} G^k(n). \tag{25}$$

T_k depends on N and on r, also. Let

$$\alpha_k := \frac{1}{k!} \lim_{\substack{N\to\infty \\ \frac{r}{N}\to\infty}} T_k.$$

We shall prove that α_k exists for every $k \in \mathbb{N}$, and that the function $\varphi(\tau)$ in Lemma 3 with these α_k is regular in a circle $|\tau| < c,\ c > 0$. It is enough to prove that α_k is bounded. The theorem will follow from Lemma 2, 3 immediately.

It is clear that $T_1 = 0$ and so $\alpha_1 = 0$.

We observe that

$$\sum_{l_1,\dots,l_t\in\{0,1,\dots,N-1\}} G\left(2^{l_1}\right)^{j_1} \dots G\left(2^{l_t}\right)^{j_t} \kappa\binom{l_1,\dots,l_t}{j_1,\dots,j_t} \tag{26}$$
$$= \begin{cases} \mathcal{O}(1), & \text{if} \quad \min j_l \geq 2, \\ o_N(1), & \text{if} \quad \min j_l \geq 2 \quad \text{and} \quad \max j_l \geq 3, \end{cases}$$

if $0 \leq \kappa\binom{l_1,\dots,l_t}{j_1,\dots,j_t} \leq 1$.
Since $\max_l |G(2^l)| \leq \frac{c}{\sigma_N(f)} \to 0 \quad (N\to\infty),\ \sigma_N^2(G) = 1$, this assertion is clear.

Let $D_N := \{0, 1, \dots, N-1\}$.
Let us consider sums of type

$$A_\nu := \sum_{\substack{l_1,\dots,l_t \\ j_1,\dots,j_t \\ u_1,\dots,u_\nu}} B\binom{l_1,\dots,l_t}{j_1,\dots,j_t} G\left(2^{u_1}\right)\cdots G\left(2^{u_\nu}\right) \tag{27}$$

where $l_1,\dots,l_t, u_1,\dots,u_\nu$ run over all possible distinct choices of $l_1,\dots,l_t$, $u_1,\dots,u_\nu \in D_N$, $\min\limits_{l=1,\dots,t} j_l \geq 2$

$$B\binom{l_1,\dots,l_t}{j_1,\dots,j_t} = G\left(2^{l_1}\right)^{j_1} \cdots G\left(2^{l_t}\right)^{j_t} \kappa\binom{l_1,\dots,l_t}{j_1,\dots,j_t}, \tag{28}$$

$0 \leq \kappa\binom{l_1,\dots,l_t}{j_1,\dots,j_t}$.

Assume that $\nu = 1$. Let us sum $G(2^{u_1})$ over all possible values, $u_1 \in D_N \setminus \{l_1, \dots, l_t\}$.
We have

$$A_1 = -\sum_{j=1}^{t} \sum_{\substack{l_1,\dots,l_t \\ j_1,\dots,j_t}} B\binom{l_1,\dots,l_t}{j_1,\dots,j_t} G\left(2^{l_j}\right),$$

and so $A_\nu \to 0 \quad (N \to \infty)$ follows from (26). Let now $\nu = 2$.
We obtain that

$$\sum_{\substack{u_2 \notin \{l_1,\dots,l_t\} \\ u_2 \neq u_1}} G(2^{u_2}) = -G\left(2^{l_1}\right) - \dots - G\left(2^{l_t}\right) - G(2^{u_1})$$

and so

$$A_2 = \sum_{\substack{l_1,\dots,l_t,u_1 \\ j_1,\dots,j_t}} B\binom{l_1,\dots,l_t}{j_1,\dots,j_t} G^2(2^{u_1}) + o_N(1).$$

Let $\nu > 2$. For fixed $l_1, \dots, l_t, u_1, \dots, u_{\nu-1}$ the variable u_ν run over $D_N \setminus (\{l_1, \dots, l_t\} \cup \{u_1, \dots, u_{\nu-1}\})$. Since

$$\sum_{u_\nu} G(2^{u_\nu}) = -\sum_{j=1}^{t} G\left(2^{l_j}\right) - G(2^{u_1}) - \dots - G(2^{u_{\nu-1}}),$$

we have

$$A_\nu = -\sum_{\substack{l_1,\dots,l_t \\ j_1,\dots,j_t}} B\binom{l_1,\dots,l_t}{j_1,\dots,j_t} G(2^{u_1}) \dots G(2^{u_{\nu-1}}) \left(G(2^{u_1}) + \dots + G(2^{u_{\nu-1}})\right)$$
$$+ o_N(1),$$

and so

$$A_\nu = -(\nu-1) \sum_{\substack{l_1,\dots,l_t,l_{t+1} \\ j_1,\dots,j_t,2 \\ u_1,\dots,u_{\nu-2}}} B\binom{l_1,\dots,l_t}{j_1,\dots,j_t} G^2\left(2^{l_{t+1}}\right) G(2^{u_1}) \dots G(2^{u_{\nu-2}})$$
$$+ o_N(1).$$

Thus the sum A_ν can be substituted by $(\nu - 1)$ sums of type $A_{\nu-2}$, with the error $o_N(1)$.

Let us continue the reduction. We obtain that $A_\nu = o_N(1)$, if ν is an odd number, furthermore, $A_\nu = o_N(1)$, if $\max_{j=1,\dots,t} l_j \geq 3$.

We can write

$$T_k = \frac{1}{\binom{N}{r}} \sum_{\substack{\alpha(n)=r \\ n<2^N}} G^k(n) = \frac{1}{\binom{N}{r}} \sum_{\substack{n<2^n \\ \alpha(n)=r}} \left\{ \sum_{j=0}^{N-1} G\left(\varepsilon_j(n)\cdot 2^j\right) \right\}^k \tag{29}$$

$$= \sum_{t=1}^{k} \nu(t,N) \sum_{u_1,\dots,u_k}^{*} G(2^{u_1})\dots G(2^{u_k}),$$

where $*$ indicates that the summation is over those $u_1,\dots,u_k \in D_N$, for which the number of distinct element of $u_1,\dots,u_k$ is t, and $\nu(t,N) = \frac{r}{N}\cdot\frac{r-1}{N-1}\cdots\frac{r-(t-1)}{N-(t-1)}$. Thus $\nu(t,N) = \lambda^t + o_N(1)$.

The sum $\sum_{u_1,\dots,u_k}^{*}$ can be rewritten in the form $\sum_{\substack{l_1<\dots<l_t \\ j_1,\dots,j_t}}$, where the multiplicity of the occurrence of l_h is j_h, thus $j_1+\dots+j_t = k$. It is clear that $G\left(2^{l_1}\right)^{j_1}\dots G\left(2^{l_t}\right)^{j_t}$ occurs for

$$\binom{k}{j_1}\binom{k-j_1}{j_2}\cdots\binom{k-(j_1+\dots+j_{t-1})}{j_t}$$
$$= \frac{k!}{j_1!\,(k-j_1)!}\cdot\frac{(k-j_1)!}{(k-(j_1+j_2))!\,j_2!}\cdots\frac{(k-(j_1+\dots+j_{t-1}))!}{j_t!}$$
$$= \frac{k!}{j_1!j_2!\dots j_t!}$$

distinct choices of $u_1,\dots,u_k$ as $G(2^{u_1})\dots G(2^{u_k})$. Thus

$$T_k = \sum_{t=1}^{k} \nu(t,N)\, k! \sum_{\substack{l_1<\dots<l_t \\ j_1,\dots,j_t}} \frac{G(2^{l_1})^{j_1}}{j_1!}\cdots\frac{G(2^{l_t})^{j_t}}{j_t!} \tag{30}$$
$$= k! \sum_{t=1}^{k} \frac{\nu(t,N)}{t!} \sum_{\substack{l_1<\dots<l_t \\ j_1,\dots,j_t}} \frac{G(2^{l_1})^{j_1}}{j_1!}\cdots\frac{G(2^{l_t})^{j_t}}{j_t!}.$$

In the last sum $l_1,\dots,l_t$ run over all those elements of D_N for which $l_u \neq l_v$, if $u \neq v$.

Let $E(j_1,\dots,j_t) = \sum_{\substack{l_1,\dots,l_t \\ j_1,\dots,j_t}} G\left(2^{l_1}\right)^{j_1}\dots G\left(2^{l_t}\right)^{j_t}$. As we have seen earlier, $E(j_1,\dots,j_t) \to 0$ if $\max j_u \geq 3$, or if $\#\{u|j_u = 1\} =$ odd number. Hence we obtain that $T_k \to 0$ if k is odd. Thus $\alpha_k = 0$ for odd k. Let us write now $2k$ into the place of k.

Then

$$T_{2k} = (2k)!\sum_{t=1}^{2k}\frac{\nu(t,N)}{t!}\sum_{j_1+\cdots+j_t=2k}^{*}\frac{E(j_1,\dots,j_t)}{j_1!\dots j_t!} + o_N(1)$$

where $*$ indicates that we have to sum over those $j_1,\dots,j_t$ for which $j_\nu = 1,2$. It is clear that $E(j_1,\dots,j_t)$ is symmetric in the variables, i.e. $E(j_{m_1},\dots,j_{m_t}) = E(j_1,\dots,j_t)$ if $m_1,\dots,m_t$ is a permutation of $\{1,\dots,t\}$.

Let

$$\sigma_{h,m} = E\left(\overbrace{2,\dots,2}^{h},\ \overbrace{1,\dots,1}^{m}\right).$$

If $j_1+\cdots+j_t = 2k$, then $2h+m = 2k,\ t = h+m$, thus

$$T_{2k} = (2k)!\sum_{t=1}^{2k}\frac{\nu(t,N)}{t!}\sum_{h\le t}\binom{t}{h}\frac{1}{2^h}\sigma_{h,t-h} + o_N(1). \tag{31}$$

It is clear that

$$\begin{aligned}\sigma_{h,0} &= \sum_{l_1,\dots,l_h} G\left(2^{l_1}\right)^2\dots G\left(2^{l_h}\right)^2\\ &= \left\{\sum G^2\left(2^l\right)\right\}^h = 1 + o_N(1).\end{aligned}$$

Furthermore, as we observed earlier, $\sigma_{h,m}\to 0 \quad (N\to\infty)$ if m =odd.

Let $m = 2$. We have

$$\begin{aligned}\sigma_{h,2} &= \sum_{l_1,\dots,l_h,u_1,u_2} G^2\left(2^{l_1}\right)\dots G^2\left(2^{l_h}\right)G\left(2^{u_1}\right)G\left(2^{u_2}\right)\\ &= -\sum_{l_1,\dots,l_h,u_1} G^2\left(2^{l_1}\right)\dots G^2\left(2^{l_h}\right)G^2\left(2^{u_1}\right) + o_N(1)\\ &= -\sigma_{h+1,0} + o_N(1) = -1 + o_N(1).\end{aligned}$$

Let $m = 2\nu, \nu\ge 2$.

$$\sigma_{h,2\nu} = \sum_{\substack{l_1,\dots,l_h\\ u_1,\dots,u_{2\nu}}} G^2\left(2^{l_1}\right)\dots G^2\left(2^{l_h}\right)G\left(2^{u_1}\right)\dots G\left(2^{u_{2\nu}}\right).$$

Since $G(2^{u_{2\nu}})$ should be summed over $D_N \setminus \{l_1, \ldots, l_h\} \cup \{u_1 \ldots, u_{2\nu-1}\}$, and so $\sum_{u_{2\nu}} G(2^{u_{2\nu}}) = -\sum G\left(2^{l_i}\right) - \sum_1^{2\nu-1} G(2^{u_j})$, we obtain that

$$\sigma_{h,2\nu} = -(2\nu - 1)\,\sigma_{h+1,2(\nu-1)} + o_N(1) \quad (\nu = 1, 2, \ldots).$$

Thus we have

$$\begin{aligned} &\sigma_{h,0} = 1 + o_N(1), \quad \sigma_{h,2} = -1 + o_N(1), \\ &\sigma_{h,4} = -3 \cdot \sigma_{h+1,2} = 3 + o_N(1), \\ &\sigma_{h,6} = -5 \cdot \sigma_{h+1,4} = -3 \cdot 5 + o_N(1), \end{aligned}$$

and in general

$$\sigma_{h,2\nu} = (-1)^\nu (2\nu - 1)!! + o_N(1).$$

Here $(2m-1)!! = (2m-1)(2m-3)\ldots \cdot 3 \cdot 1$.

Let us write $t - h = 2m$ in (31). Then

$$\begin{aligned} \binom{t}{h}\frac{1}{2^h}\sigma_{h,t-h} &= \binom{t}{2m}\frac{2^{2m}}{2^t}\sigma_{h,2m} \\ &= (-1)^m \binom{t}{2m}\frac{2^{2m}}{2^t}(2m-1)!! + o_N(1), \end{aligned}$$

and so

$$T_{2k} = (2k)!\sum_{t=1}^{2k}\lambda^t \cdot \frac{1}{t!2^t}\sum_{2m\le t}(-1)^m\binom{t}{2m}2^{2m}\cdot(2m-1)!! + o_N(1).$$

Let us apply Lemma 3. In the notation of Lemma 3 we have

$$\begin{aligned} \alpha_{2k} &= \lim_{N\to\infty}\frac{T_{2k}}{(2k)!} \\ &= \sum_{t=1}^{2k}\frac{\lambda^t}{t!2^t}\sum_{2m\le t}(-1)^m\binom{t}{2m}2^{2m}\cdot(2m-1)!!. \end{aligned}$$

We shall prove that α_{2k} is bounded as $2k \to \infty$. Indeed

$$\frac{(2m-1)!!}{(2m)!} = \frac{1}{2^m m!}, \quad \frac{2^{2m}}{2^t} \le 1,$$

thus

$$|\alpha_{2k}| \leq \sum_{t=1}^{2k} \frac{\lambda^t}{t!} \sum_{2m\leq t} \frac{t!\,(2m-1)\,!!}{(2m)\,!\,(t-2m)\,!}$$
$$\leq \sum_{t=1}^{2k} \lambda^t \sum_{2m\leq t} \frac{1}{(t-2m)\,!\,(2^m m!)}.$$

Here $m = 0$ can be occur, $0! = 1$.
We obtain that

$$|\alpha_{2k}| < c\lambda$$

with some c, c may depend on λ.

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References

[1] H. Delange, Sur les fonctions q-additives on q-multiplicatives, *Acta Arith.*, **21** (1972), 285–298.

[2] I. Kátai, M. V. Subbarao, Distribution of additive and q-additive functions under some conditions, *Publ. Math. Debrecen*, **64** (2004), 167–187.

[3] I. Kátai, M. V. Subbarao, Distribution of 2-additive functions under some conditions, *Annales Univ. Sci. Budapest, Sect. Comp.*, **26** (2006), 137–143.

[4] I. Kátai, M. V. Subbarao, Distribution of additive and q-additive functions under some conditions II., *Publ. Math. Debrecen*, **73** (2008), 59–88.

[5] M. Frechet, J. Shohat, A proof of the generalized central limit theorem, *Trans. Amer. Math. Soc.*, **33** (1931), 533–543

[6] P. D. T. A. Elliott, *Probabilistic number theory*, Springer Verlag, Berlin, 1979.

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Notes on functions preserving density

József Bukor
Department of Mathematics
J. Selye University
945 01 Komárno, Slovakia
email: bukor.jozsef@selyeuni.sk

Peter Csiba
Department of Mathematics
J. Selye University
945 01 Komárno, Slovakia
email: csiba.peter@selyeuni.sk

Abstract. Let $d(A)$ denote the asymptotic density of the set of positive integers. Let $\mathcal{AD}$ denote the set of all sets A having asymptotic density, and let $\mathcal{D}_\delta$ denote the set of all sets A for which the difference between its upper and lower density is less than δ. In the paper are studied fuctions $f : \mathbb{N} \to \mathbb{N}$ (not necessary a one-to-one functions) such that $A \in \mathcal{AD}$ implies $f(A) \in \mathcal{AD}$ and fuctions $f : \mathbb{N} \to \mathbb{N}$ for that $A \in \mathcal{AD}$ implies $f(A) \in \mathcal{D}_\delta$. Our results generalize a theorem in [M. B. Nathanson, R. Parikh, *Density of sets of natural numbers and Lévy group*, J. Number Theory **124** (2007), 151–158.]

1 Introduction

Denote by $\mathbb{N}$ the set of all positive integers. For $A \subset \mathbb{N}$ let $A(n)$ denote the counting function of the set A. The lower asymptotic density of A is

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{A(n)}{n},$$

the upper asymptotic density of A is

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{A(n)}{n}.$$

If $\overline{d}(A) = \underline{d}(A)$, we say that A has an asymptotic density and we denote it by $d(A)$. For more details on the asymptotic density we refer to the paper [1].

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Let the group $\mathcal{L}^\sharp$ consists of all permutations of positive integers f such that $A \in \mathcal{AD}$ if and only if $f(A) \in \mathcal{AD}$, and the Lévy group $\mathcal{L}^\star$ consists of all permutations $f \in \mathcal{L}^\sharp$ such that $d(f(A)) = d(A)$ for all $A \in \mathcal{AD}$. Nathanson and Parikh [3] proved that the groups $\mathcal{L}^\sharp$ and $\mathcal{L}^\star$ coincide. Remark, more complicated result in the same direction was proved in [4], but with different assumptions on the transformation f. Connection between the Lévy group and finitely additive measures on integers extending the asymptotic density was studied in [5].

The mentioned Natanson and Parikh's result follows from the following stronger theorem.

Theorem A [2, Theorem 2] *Let $f : \mathbb{N} \to \mathbb{N}$ be a one-to-one function such that if $A \in \mathcal{AD}$, then $f(A) \in \mathcal{AD}$, that is, if the set A of positive integers has asymptotic density, then the set $f(A)$ also has asymptotic density. Let $\lambda = d(f(\mathbb{N}))$. Then*

$$d(f(A)) = \lambda d(A)$$

for all $A \in \mathcal{AD}$.

We generalize this result showing that the condition for f to be one-to-one function is not necessary and we will consider the set of functions $\mathcal{D}_\delta$ instead of $\mathcal{AD}$.

2 Results

Theorem 1 *Let $h : \mathbb{N} \to \mathbb{N}$ be a function (not necessary a one-to-one) such that if the set A of positive integers has asymptotic density, then the set $h(A)$ also has asymptotic density. Let $\lambda = d(h(\mathbb{N}))$. Then*

$$d(h(A)) = \lambda d(A)$$

for all $A \in \mathcal{AD}$.

Proof. Let the symmetric differerence of the sets X and Y be denoted by $X \ominus Y$. We construct a one-to-one function $f : \mathbb{N} \to \mathbb{N}$ such that

$$d(f(\mathbb{N}) \ominus h(\mathbb{N})) = 0.$$

Then the assertion follows immediately from the Theorem A.

First, we construct a function $g : \mathbb{N} \to \mathbb{N}$ for that $\mathbb{N} \setminus g(\mathbb{N})$ is infinite and the density of the symmetric differerence of the sets $h(\mathbb{N})$ and $g(\mathbb{N})$ is zero. It can be done easily using an infinite set

$$S = \{a_1, a_2, a_3, \dots\}$$

with the property $d(S) = 0$. Obviously, we may define the set S as the set of all squares or as the set of all primes,... Let us define

$$g(n) = \begin{cases} a_{2k}, & \text{if } h(n) = a_k \\ h(n), & \text{if } h(n) \notin S \end{cases}.$$

Let

$$B = \{a_1, a_3, a_5, \dots, a_{2k+1}, \dots\}.$$

We have $B \subset \mathbb{N} \setminus g(\mathbb{N})$ and $d(B) = 0$.

We construct the injective function f and a sequence of sets $B_1, B_2, \dots$ by induction.
Let $f(1) = g(1)$ and $B_1 = B$. For $n \geq 1$

$$\begin{array}{lll} \text{if } g(n+1) \notin g(\mathbb{N} \cap [1,n]) & \text{let} & f(n+1) = g(n+1) \text{ and} \\ & & \quad B_{n+1} = B_n \\ \text{if } g(n+1) \in g(\mathbb{N} \cap [1,n]) & \text{let} & f(n+1) = \min B_n \text{ and} \\ & & \quad B_{n+1} = B_n \setminus \{f(n+1)\} \end{array}.$$

From the above construction follows that for any $A \subset \mathbb{N}$ the set $h(A)$ has asymptotic density if and only if $f(A)$ has asymptotic density and moreover $d(f(A)) = d(h(A))$ for arbitrary $A \in \mathcal{AD}$, so the assertion follows. □

By the above proved theorem the property that $A \in \mathcal{AD}$ implies $f(A) \in \mathcal{AD}$ is strong enough to ensure that in sense of asymptotic density large irregularities in the image set $f(\mathbb{N})$ cannot occur.

The main idea of the paper [3] was to show that if for a function f the density of the set A implies the density of the set $f(A)$ then the asymptotic density of $f(A)$ depends only on $d(A)$. Equivalently, if $A, B \in \mathcal{AD}$ and $d(A) = d(B)$, then $d(f(A)) = d(f(B))$.
In what follows we consider the question: Having a function $f : \mathbb{N} \to \mathbb{N}$ such that $A \in \mathcal{AD}$ implies $f(A) \in \mathcal{D}_f$, in the case $d(A) = d(B)$ what can we say about the upper and lower densities of the image sets $f(A)$ and $f(B)$?

In our studies the following "intertwinning lemma" will be fundamental.

Lemma 1 [3] *Let* A *and* B *be sets of positive integers such that* $d(A) = d(B) = \gamma$*. Then for a sufficiently fast growing sequence* (p_i) *if*

$$C = \bigcup_{i=1}^{\infty} A \cap (p_{2i-1}, p_{2i}] \cup \bigcup_{i=1}^{\infty} B \cap (p_{2i}, p_{2i+1}]$$

then

$$d(C) = \gamma.$$

Theorem 2 *Let* $\delta > 0$ *and let* $f : \mathbb{N} \to \mathbb{N}$ *be a one-to-one function such that if* $A \in \mathcal{AD}$ *then* $f(A) \in \mathcal{D}_\delta$*. Let* A, B *are arbitrary sets of positive integers with the property* $d(A) = d(B) = \gamma$*. Then*

$$\overline{d}(B) - \underline{d}(A) \leq \delta.$$

Proof. Let $\underline{d}(A) = \alpha$ and $\overline{d}(B) = \beta$. Suppose, contrary to our claim that

$$\beta > \alpha + \delta.$$

We will construct a set C for that $d(C) = \gamma$ but the set $f(C) \notin \mathcal{D}_\delta$. We will define the sequence (p_i) by induction and using this define the set C

$$C = \bigcup_{i=1}^{\infty} A \cap (p_{2i-1}, p_{2i}] \cup \bigcup_{i=1}^{\infty} B \cap (p_{2i}, p_{2i+1}]. \tag{1}$$

Induction hypothesis:

Suppose we have constructed sequences $p_1, \dots, p_{2k+1}$, further $m_1, \dots, m_{2k}$ and $n_2, \dots, n_{2k+1}$ such that

$$\frac{|[m_{2i-1}, n_{2i}] \cap f(A)|}{n_{2i}} < \alpha + \frac{1}{i}, \tag{2}$$

$$\frac{|[m_{2i}, n_{2i+1}] \cap f(B)|}{n_{2i+1}} > \beta - \frac{1}{i} \tag{3}$$

for $i = 1, \dots, k$ and

$$f(\mathbb{N} \setminus [p_j, p_{j+1}]) \cap [m_j, n_{j+1}] = \emptyset, \tag{4}$$

for $j = 1, \dots 2k$.

Induction step: Let

$$m_{2k+1} = 1 + \max f(\mathbb{N} \cap [1, p_{2k+1}]).$$

From the fact that $\underline{d}(f(A)) = \alpha$ we get that for sufficiently large n_{2k+2} we have

$$\frac{|[m_{2k+1}, n_{2k+2}] \cap f(A)|}{n_{2k+2}} < \alpha + \frac{1}{k+1}$$

and moreover let $n_{2k+2} > (k+2).m_{2k+1}$.

Define p_{2k+2} as the least positive integer t satisfying

$$\min f([t, \infty) \cap \mathbb{N}) > n_{2k+2}.$$

From the definition of the numbers $m_{2k+1}, n_{2k+2}, p_{2k+2}$ follows that

$$f(\mathbb{N} \setminus [p_{2k+1}, p_{2k+2}]) \cap [m_{2k+1}, n_{2k+2}] = \emptyset.$$

Similarly, let

$$m_{2k+2} = 1 + \max f(\mathbb{N} \cap [1, p_{2k+2}]).$$

From $\overline{d}(f(B)) = \beta$ we have that for sufficiently large n_{2k+3} we have

$$\frac{|[m_{2k+2}, n_{2k+3}] \cap f(B)|}{n_{2k+3}} > \beta - \frac{1}{k+1}.$$

Define p_{2k+3} as the least positive integer t for that

$$\min f([t, \infty) \cap \mathbb{N}) > n_{2k+3}.$$

Analogously, from the definition of the numbers $m_{2k+2}, n_{2k+3}, p_{2k+3}$ we have

$$f(\mathbb{N} \setminus [p_{2k+2}, p_{2k+3}]) \cap [m_{2k+2}, n_{2k+3}] = \emptyset.$$

After completing induction the relations (2)-(4) hold for every $k \in \mathbb{N}$.

We estimate the upper and lower density of the constructed set C. Using (1) together with (2) and (4) we have

$$\begin{aligned} \liminf_{n\to\infty} \frac{f(C)(n)}{n} &\leq \liminf_{k\to\infty} \frac{f(C)(n_{2k})}{n_{2k}} \leq \liminf_{k\to\infty} \frac{m_{2k-1} + |[m_{2k-1}, n_{2k}] \cap f(A)|}{n_{2k}} \\ &\leq \liminf_{k\to\infty} \left(\frac{1}{k+1} + \alpha + \frac{1}{k} \right) = \alpha. \end{aligned}$$

On the other hand, by (1), (3) and (4)

$$\limsup_{n\to\infty} \frac{f(C)(n)}{n} \geq \limsup_{k\to\infty} \frac{f(C)(n_{2k+1})}{n_{2k+1}} \geq$$

$$\geq \limsup_{k\to\infty} \frac{|[m_{2k}, n_{2k+1}] \cap f(B)|}{n_{2k+1}} \geq \limsup_{k\to\infty} \left(\beta - \frac{1}{k}\right) = \beta. \qquad (5)$$

By Lemma 1 the set $C \in \mathcal{AD}$ but (5) and (5) yield to the fact that

$$\overline{d}(f(C)) - \underline{d}(f(C)) > \beta - \alpha > \delta$$

and therefore $f(C) \notin \mathcal{D}_\delta$. This contradiction completes the proof. □

Remarks. It is worth pointing out that

$$\bigcap_{n=1}^{\infty} \left\{f : \mathbb{N} \to \mathbb{N};\ \text{if } A \in \mathcal{AD} \text{ then } f(A) \in \mathcal{D}_{\frac{1}{n}}\right\} =$$
$$= \{f : \mathbb{N} \to \mathbb{N};\ \text{if } A \in \mathcal{AD} \text{ then } f(A) \in \mathcal{AD}\}.$$

In Theorem 2 the condition for the function f to be an injection is not necessary. It can be shown by the same way as in Theorem 1.

We have proved that for given $f : \mathbb{N} \to \mathbb{N}$ (if $A \in \mathcal{AD}$ then $f(A) \in \mathcal{D}_\delta$) the upper bound for $\overline{d}(f(A))$ and the lower bound for $\underline{d}(f(A))$ depends only on the asymptotic density of A. Clearly, for any dense set A and for any $\theta \in [0, 1]$ there is a set $B \subset A$ such that $d(B) = \theta.d(A)$ (see e.g. [2], Proposition 1), but using this fact we can only deduce, that these bounds are nondecreasing.

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References

[1] G. Grekos, On various definitions of density (survey), *Tatra Mt. Math. Publ.*, **31** (2005), 17–27.

[2] G. Grekos, L. Mišík, J. T. Tóth, Density sets of positive integers, *J. Number Theory*, **130** (2010), 1399–1407.

[3] M. B. Nathanson, R. Parikh, Density of sets of natural numbers and Lévy group, *J. Number Theory*, **124** (2007), 151–158.

[4] R. Giuliano Antonini, M. Paštéka, A comparison theorem for matrix limitation methods with applications, *Unif. Distrib. Theory,* **1** (2006), 87–109.

[5] M. Sleziak, M. Ziman, Lévy group and density measures, *J. Number Theory*, **128** (2008), 3005–3012.

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Malmquist-Takenaka functions on local fields

Ilona Simon
University of Pécs
Institute of Mathematics and Informatics
Pécs, Hungary
email: simoni@gamma.ttk.pte.hu

Abstract. The complex variant of the discrete Malmquist-Takenaka system plays an important role in system identification. We introduce the analogue of these functions on two dyadic local fields using the analogue of the Blaschke-functions on these fields. This results a generalization of the discrete Laguerre system. Properties of these systems, Fourier expansion and summability questions are presented.

1 Introduction

The discrete Laguerre functions and their generalizations, the Malmquist-Takenaka and Kautz systems are often used in control theory to identify the transfer function. Let us recall, that the discrete Laguerre functions $L_n^{(a)}$ ($n \in \mathbb{N}$) contain a complex parameter $a \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and can be expressed by the Blaschke functions

$$B_a(z) := \frac{z-a}{1-\bar{a}z} \quad (z \in \mathbb{C}, a \in \mathbb{D}).$$

The discrete Laguerre functions $L_n^{(a)}$ associated to B_a on $\mathbb{C}$ are defined by

$$L_k^{(a)}(z) := m_a(z)B_a^k(z), \text{ where } m_a(z) := \frac{\sqrt{1-|a|^2}}{1-\bar{a}z} \quad (z \in \mathbb{C}, k \in \mathbb{Z})$$

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for $a \in \mathbb{D}$. The boundary of $\mathbb{D}$ is denoted by $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$.

The discrete Malmquist-Takenaka functions $\Psi_n^{(p)}$ on $\mathbb{C}$ are defined by

$$\Psi_0^p(z) := \frac{\sqrt{1-|a_0|^2}}{1-\bar{a}_0 z}, \qquad \Psi_n^{(p)}(z) := \frac{\sqrt{1-|a_n|^2}}{1-\bar{a}_n z}\prod_{j=0}^{n-1} B_{a_j}(z), \quad (z \in \mathbb{C}, k \in \mathbb{Z})$$

for $(a_j \in \mathbb{D}, j \in \mathbb{N})$ and $p = (a_0, a_1, a_2, \dots)$.

The discrete Malmquist-Takenaka system is orthogonal with respect to the scalar product $\langle F, G\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{it})\overline{G}(e^{it})\ dt$. Note, that using the same parameters $a_j = a$ $(j \in \mathbb{N})$, the $\Psi_n^{(p)}$ functions give the discrete Laguerre system $(L_n^{(a)}, n \in \mathbb{N})$. For more on these systems see [1].

The analogue of the discrete Laguerre function is constructed in [4] as a composition of the corresponding characters and the Blaschke functions, inspired by the fact, that if a belongs to $\mathbb{D}$, then B_a is a bijection on $\mathbb{T}$, and B_a can be written in the form (see [1])

$$B_a(e^{is}) = e^{i\beta_a(s)}(s \in \mathbb{R}, a \in \mathbb{D}) \tag{1}$$

with some bijection $\beta_a : [-\pi, \pi] \to [-\pi, \pi]$. Obviously $L_k^{(0)}(z) = z^k$ $(k \in \mathbb{Z})$ coincides with the trigonometric system on $\mathbb{T}$. Thus the discrete Laguerre system except the factor m_a can be obtained from the trigonometric system by an argument transformation $T(z) = B_a(z)(z \in \mathbb{T})$.

We will construct the analogue of the discrete Malmquist-Takenaka functions starting from the generator system of the characters of the dyadic and 2-adic group and using an argument transformation. This is a UDMD product system, thus also a complete orthonormal system, which gives the discrete Laguerre system for identical parameters $a_n = a$ $(n \in \mathbb{N})$. Fourier expansion with respect these systems and summability questions are examined.

2 The Blaschke functions on the 2-series and on the 2-adic field

We use the basic notations, definitions and the description of the algebraic structure of the handbooks [3] and [2]. Denote by

$$\mathbb{B} := \left\{ a = (a^{(j)}, j \in \mathbb{Z}) \mid a^{(j)} \in \{0,1\} \text{ and } \lim_{j\to-\infty} a^{(j)} = 0 \right\}$$

the set of bytes, and by $\mathbb{A} := \{0,1\}$ the set of bits. The numbers $a^{(j)}$ are called the additive digits of $a \in \mathbb{B}$. Also use the notion: $\mathbb{P} := \mathbb{N}\setminus\{0\}$. The zero element of $\mathbb{B}$ is $\theta := (x^{(j)} \in \mathbb{Z})$ where $x^{(j)} = 0$ for $j \in \mathbb{Z}$, that is, $\theta = (\cdots, 0, 0, 0, \cdots)$. The order of a byte $x \in \mathbb{B}$ is defined in the following way: For $x \neq \theta$ let $\pi(x) := n$ if and only if $x^{(n)} = 1$, and $x^{(j)} = 0$ for all $j < n$, furthermore set $\pi(\theta) = +\infty$. The norm of a byte x is introduced by the following rule: $\|x\| := 2^{-\pi(x)}$ for $x \in \mathbb{B}\setminus\{\theta\}$, and $\|\theta\| := 0$.

The sets $I_n(x) := \{y \in \mathbb{B} : y^{(k)} = x^{(k)}$ for $k < n\}$ are the intervals in $\mathbb{B}$ of rank $n \in \mathbb{Z}$ and center $x \in \mathbb{B}$. Consider $\mathbb{I}_n := \{x \in \mathbb{B} : \|x\| \leqq 2^{-n}\}$ $(n \in \mathbb{Z})$. $\mathbb{I} := \mathbb{I}_0$ can be identified with the set of sequences $\mathbb{I} = \{a = (a^{(j)}, j \in \mathbb{N}) | \ a^{(j)} \in \mathbb{A}\}$ via the map $(\dots, 0, 0, a^{(0)}, a^{(1)}, \dots) \to (a^{(0)}, a^{(1)}, \dots)$.

The 2-series (or logical) sum $a \overset{\circ}{+} b$ and product $a \circ b$ of elements $a, b \in \mathbb{B}$ are defined by

$$a \overset{\circ}{+} b := \Big(a^{(n)} + b^{(n)} \ (\text{mod } 2),\ n \in \mathbb{Z}\Big)$$
$$a \circ b := (c^{(n)}, n \in \mathbb{Z}),\ \text{where } c^{(n)} := \sum_{k\in\mathbb{Z}} a^{(k)} b^{(n-k)} \ (\text{mod } 2) \quad (n \in \mathbb{Z}).$$

$(\mathbb{B}, \overset{\circ}{+}, \circ)$ is a non-Archimedian normed field, i.e. $\|a \overset{\circ}{+} b\| \leqq \max\{\|a\|, \|b\|\}$, $|a \circ b\| = \|a\|\,\|b\| \quad (a, b \in \mathbb{B})$. The multiplicative identity of $\mathbb{B}$ is the element $e := (\delta_{n0}, n \in \mathbb{N})$.

The (logical) Blaschke function with parameter $a \in \mathbb{I}_1$ is defined in [4] by:

$$B_a(x) := \frac{x \overset{\circ}{+} a}{e \overset{\circ}{+} a \circ x} \quad (x \in \mathbb{I}).$$

Set $y = B_a(x)$ with $x \in \mathbb{I}, a \in \mathbb{I}_1$. Then we have $y = x \overset{\circ}{+} a \overset{\circ}{+} y \circ a \circ x$ and consequently for the n-th digit of y we get

$$\begin{cases} y^{(n)} = 0, \text{ for } n < 0, \\ y^{(n)} = x^{(n)} + a^{(n)} + (y \circ a \circ x)^{(n)} \ (\text{mod } 2), \text{ for } n \geqq 0. \end{cases}$$

This is recursion for the bits of $y = B_a(x)$, since to compute $(y \circ a \circ x)^{(n)}$ we only need $y^{(k)}$-s with $k < n$. The bits $y^{(n)} = (B_a(x))^{(n)}$ can be written in the form

$$y^{(n)} = x^{(n)} + a^{(n)} + f_n(x^{(0)}, \cdots, x^{(n-1)}) \ (\text{mod } 2) \tag{2}$$

where the functions $f_n : \mathbb{A}^n \to \mathbb{A}$ $(n = 1, 2, \cdots)$ depend only on the bits of a. The definition of the logical Blaschke functions and details about the recursion are considered in [4].

The 2-adic (or arithmetic) sum $a \overset{\bullet}{+} b$ of elements $a = (a^{(n)}, n \in \mathbb{Z}), b = (b^{(n)}, n \in \mathbb{Z}) \in \mathbb{B}$ is defined by $a \overset{\bullet}{+} b := (s_n, n \in \mathbb{Z})$, where the bits $q_n, s_n \in \mathbb{A}$ $(n \in \mathbb{Z})$ are obtained recursively as follows: $q_n = s_n = 0$ for $n < m := \min\{\pi(a), \pi(b)\}$, and

$$a^{(n)} + b^{(n)} + q_{n-1} = 2q_n + s_n \qquad \text{for } n \geq m.$$

The 2-adic (or arithmetic) product of $a, b \in \mathbb{B}$ is $a \bullet b := (p_n, n \in \mathbb{Z})$, where the sequences $q_n \in \mathbb{N}$ and $p_n \in \mathbb{A}$ $(n \in \mathbb{Z})$ are defined recursively by $q_n = p_n = 0$ $(n < m := \pi(a) + \pi(b))$ and

$$\sum_{j=-\infty}^{\infty} a^{(j)} b^{(n-j)} + q_{n-1} = 2q_n + p_n \qquad (n \geq m).$$

Note, that $\pi(a \bullet b) = \pi(a) + \pi(b)$ and $(\mathbb{B}, \overset{\bullet}{+}, \bullet)$ is a non-Archimedian normed field.

For $x \in \mathbb{I}$ and $a \in \mathbb{I}_1$ we have that $e \overset{\bullet}{-} a \bullet x \neq \theta$, thus $e \overset{\bullet}{-} a \bullet x$ has a multiplicative inverse in $\mathbb{B}$. The (arithmetical) Blaschke function with parameter $a \in \mathbb{I}_1$ is defined in [4] by:

$$B_a(x) := (x \overset{\bullet}{-} a) \bullet (e \overset{\bullet}{-} a \bullet x)^{-1} = \frac{x \overset{\bullet}{-} a}{e \overset{\bullet}{-} a \bullet x} \qquad (x \in \mathbb{I}). \tag{3}$$

The Blaschke function $B_a : \mathbb{I} \to \mathbb{I}$ is a bijection for any $a \in \mathbb{I}_1$ on $\mathbb{I}$ and on $\mathbb{S}_0$. The maps B_a $(a \in \mathbb{I}_1)$ form a commutative group with respect to the function composition. The byte $y = B_a(x)$ can be given in a recursive form (2) like on the logical field. The definition of the arithmetical Blaschke functions and details about the recursion are considered in [4].

Consider the Haar-measure μ on the fields $(\mathbb{B}, \overset{\bullet}{+}, \bullet)$ and $(\mathbb{B}, \overset{\circ}{+}, \circ)$. More details on the algebraic structure can be found in [3].

In the following we will present UDMD systems, which are considered in [3]. Denote with $\mathcal{A}$ the σ-algebra generated by the intervals $I_n(a)$ $(a \in \mathbb{I}, n \in \mathbb{N})$. Let $\lambda(I_n(a)) := 2^{-n}$ be the measure of $I_n(a)$. Extending this measure to $\mathcal{A}$ we get a probability measure space $(\mathbb{I}, \mathcal{A}, \lambda)$. Let $\mathcal{A}_n$ be the sub-σ-algebra of

$\mathcal{A}$ generated by the intervals $I_n(a)$ $(a \in \mathbb{I})$. Let $L(\mathcal{A}_n)$ denote the set of $\mathcal{A}_n$-measurable functions on $\mathbb{I}$. The conditional expectation of an $f \in L^1(\mathbb{I})$ with respect to $\mathcal{A}_n$ is of the form

$$(\mathcal{E}_n f)(x) := \frac{1}{\lambda(I_n(x))} \int_{I_n(x)} f d\lambda.$$

A sequence of functions $(f_n, n \in \mathbb{N})$ is called a dyadic martingale if each f_n is $\mathcal{A}_n$-measurable and

$$(\mathcal{E}_n f_{n+1}) = f_n \qquad (n \in \mathbb{N}).$$

The sequence of martingale differences of f_n $(n \in \mathbb{N})$ is the sequence

$$\phi_n := f_{n+1} - f_n \qquad (n \in \mathbb{N}).$$

We notice that every dyadic martingale difference sequence has the form $\phi_n = r_n g_n$ $(n \in \mathbb{N})$ where $(g_n, n \in \mathbb{N})$ is a sequence of functions such that each g_n is $\mathcal{A}_n$-measurable and $(r_n, n \in \mathbb{N})$ denotes the Rademacher system on $\mathbb{I}$:

$$r_n(x) := (-1)^{x^{(n)}} \ (n \in \mathbb{N}).$$

The martingale difference sequence $(\phi_n, n \in \mathbb{N})$ is called a unitary dyadic martingale difference sequence or a UDMD sequence if $|\phi_n(x)| = 1$ $(n \in \mathbb{N})$. Thus $(\phi_n, n \in \mathbb{N})$ is a UDMD sequence if and only if

$$\phi_n = r_n g_n, \ g_n \in L(\mathcal{A}_n), \ |g_n| = 1 \ \ (n \in \mathbb{N}). \tag{4}$$

A system $\psi = (\psi_m, m \in \mathbb{N})$ is said to be a UDMD product system if it is a product system generated by a UDMD system, i.e., there is a UDMD system $(\phi_n, n \in \mathbb{N})$ such that for each $m \in \mathbb{N}$, with binary expansion is given by $m = \sum_{j=0}^{\infty} m^{(j)} 2^j$ $(m^{(j)} \in \mathbb{A}, j \in \mathbb{N})$, the function ψ_m satisfies

$$\psi_m = \prod_{j=0}^{\infty} \phi_j^{m^{(j)}} \qquad (m \in \mathbb{N}).$$

The author constructed orthonormal systems in this way inspired by martingales in [4, 5].

3 The discrete Malmquist-Takenaka functions on the 2-series and 2-adic field

Let us define the discrete Malmquist-Takenaka functions with parameters $p = (a_0, a_1, \ldots)$ $(a_i \in \mathbb{I}_1, i \in \mathbb{N})$ on the 2-series field $(\mathbb{I}, \overset{\circ}{+}, \circ)$ in the following way: the system $\left(\Psi_k^{(p)}, k \in \mathbb{N}\right)$ is the product system generated by

$$(\Phi_{n,a_n} := r_n \circ B_{a_n}, n \in \mathbb{N}) \tag{5}$$

That is, $\Psi_k^{(p)}(x) = \prod_{j=0}^{\infty} \left[r_j(B_{a_j}(x))\right]^{k^{(j)}}$.

Theorem 1 *For every* $a_n \in \mathbb{I}_1$ $(n \in \mathbb{N}))$ *the functions* $\Phi_{n,a_n}(x) = r_n(B_{a_n}(x))$ $(x \in \mathbb{I},\ n \in \mathbb{N})$ *form a UDMD system on* $(\mathbb{I}, \overset{\circ}{+}, \circ)$.

Proof. Using recursion form (2) of $y = B_{a_n}(x)$ we get

$$\Phi_{n,a_n}(x) = (-1)^{y^{(n)}} = (-1)^{x^{(n)}}(-1)^{(a_n)^{(n)} + f_n(x^{(0)},\cdots,x^{(n-1)})} = r_n(x)g_n(x)$$

where $g_n(x) := (-1)^{(a_n)^{(n)} + f_n(x^{(0)},\cdots,x^{(n-1)})}$ is $\mathcal{A}_n$-measurable, $g_n \in L(\mathcal{A}_n)$. Clearly, $|g_n(x)| = 1$ $(x \in \mathbb{I})$, thus $(\Phi_{n,a_n}, n \in \mathbb{N})$ is a UDMD sequence. □

Corollary 1 *The logical Malmquist-Takenaka system, that is the product system* $(\Psi_k^{(p)}, k \in \mathbb{N})$ *generated by the system* $(\Phi_{n,a_n}, n \in \mathbb{N})$ *is a UDMD product system, consequently it is a complete orthonormal system on* $(\mathbb{I}, \overset{\circ}{+}, \circ)$.

We consider $\epsilon(t) := \exp(2\pi i t)$ $(t \in \mathbb{R})$. We will use the functions $(\nu_{2^n}(x), n \in \mathbb{N})$:

$$\nu_{2^n}(x) := \epsilon\left(\frac{x^{(n)}}{2} + \frac{x^{(n-1)}}{2^2} + \cdots + \frac{x^{(0)}}{2^{n+1}}\right) \qquad (x \in \mathbb{I}, n \in \mathbb{N}), \tag{6}$$

known as a generator system of the characters of the group $(\mathbb{I}, \overset{\bullet}{+})$. Let us define the arithmetical Malmquist-Takenaka functions with parameters $p = (a_0, a_1, \ldots)$ $(a_n \in \mathbb{I}_1, n \in \mathbb{N})$ on the 2-adic field $(\mathbb{I}, \overset{\bullet}{+}, \bullet)$ in the following way: the system $\left(\Psi_k^{(p)}, k \in \mathbb{N}\right)$ is now the product system generated by

$$(\Phi_{n,a_n} := \nu_{2^n} \circ B_{a_n}, n \in \mathbb{N}) \tag{7}$$

That is, $\Psi_n^{(p)}(x) = \prod_{j=0}^{\infty} \left[\nu_{2^j}(B_{a_j}(x))\right]^{n^{(j)}}$ $.(x \in (\mathbb{I}, \overset{\bullet}{+}, \bullet))$

Theorem 2 *For every* $a_n \in \mathbb{I}_1$ $(n \in \mathbb{N})$ *the functions* $\Phi_{n,a_n}(x) = \nu_{2^n}(B_{a_n}(x))$ $(x \in \mathbb{I},\ n \in \mathbb{N})$ *form a UDMD system on* $(\mathbb{I}, \overset{\bullet}{+}, \bullet)$.

The proof is similar like on the 2-series field.

Corollary 2 *The arithmetical Malmquist-Takenaka functions, the product system* $(\Psi_k^{(p)}, k \in \mathbb{N})$ *generated by the system* $(\Phi_{n,a_n}, n \in \mathbb{N})$ *is a UDMD product system, consequently it is a complete orthonormal system on* $(\mathbb{I}, \overset{\bullet}{+}, \bullet)$.

In the following we consider the corresponding Malmquist-Takenaka-systems on both fields $(\mathbb{I}, \overset{\bullet}{+}, \bullet)$ and $(\mathbb{I}, \overset{\circ}{+}, \circ)$.

4 Summability

The Malmquist-Takenaka-Fourier coefficients of an $f \in L^q(\mathbb{I})$ $(1 \leq q \leq \infty)$ are defined by

$$\widehat{f^{(p)}}(n) := \int_{\mathbb{I}} f(x)\Psi_n^{(p)}(x)d\mu(x). \quad (n \in \mathbb{N})$$

The n-th partial sums of the Malmquist-Takenaka-Fourier series $S^{(p)}f$ is now

$$S_n^{(p)}f := \sum_{k=0}^{n-1} \widehat{f^{(p)}}(k)\Psi_k^{(p)} \ (n \in \mathbb{N}^*).$$

Furthermore, the Malmquist-Takenaka-Cesaro (or $(MT - C, 1)$) means of $S^{(p)}f$ are defined by $\sigma_0^{(p)}f := 0$ and

$$\sigma_n^{(p)}f := \frac{1}{n}\sum_{k=1}^{n} S_k^{(p)}f, \qquad (n \in \mathbb{N}^*)$$

for $p = (a_0, a_1, \dots)$ with $a_n \in \mathbb{I}_1$ $(n \in \mathbb{N})$, $f \in L^q(\mathbb{I})$.

Properties of UDMD product systems are valid for the Malmquist-Takenaka system $(\Psi_k^{(p)}, k \in \mathbb{N})$, thus applying the general theorem on convergence presented in [3], holds the following:

Theorem 3 *For any* $f \in L^q(\mathbb{I})$ $(1 \leq q < \infty)$ *we have*

$$\begin{aligned} &\lim_{n\to\infty} \|S_{2^n}^{(p)}f - f\|_q = 0, \quad \textit{and} \\ &\lim_{n\to\infty} \|\sigma_n^{(p)}f - f\|_q = 0. \end{aligned} \tag{8}$$

Moreover, (8) holds for $q = \infty$ *when* f *is continuous on* $\mathbb{I}$.

Clearly, a.e. convergence holds for $S_{2^n}^{(p)}f$ for any integrable f and for $S_m^{(p)}f$, $(m \in \mathbb{P})$ if $f \in L^q(\mathbb{I})$ and $q > 1$. This is a consequence of a general theorem in [3], pp.101-105 or [2]. This holds for $q = 1$ with identical parameters $a_n = a \in \mathbb{I}_1$ $(n \in \mathbb{N})$, that is, in the case of the discrete Laguerre system $L_n^{(a)}(x)$. See [4].

We will see in the next proposition, that the Malmquist-Takenaka system is a generalization of the discrete Laguerre system on both fields defined in [4] as follows.

The functions corresponding the trigonometric system mentioned in the Introduction, are the characters of the corresponding groups. Namely, the Walsh-Paley functions $(w_k, k \in \mathbb{N})$ defined by

$$w_k(x) = \prod_{n=0}^{\infty} r_n(x)^{k^{(n)}} = (-1)^{\sum_{j=0}^{+\infty} k^{(j)}x^{(j)}} \; (x \in \mathbb{I},\; k = \sum_{j=0}^{\infty} k^{(j)}2^j \in \mathbb{N}\; (k^{(j)} \in \mathbb{A})), \tag{9}$$

are the characters of $(\mathbb{I}, \overset{\circ}{+})$. In particular, the Walsh-Paley functions form a product system generated by the Rademacher system $(r_n, n \in \mathbb{N})$.

And the functions $(v_k, k \in \mathbb{N})$ are the characters of $(\mathbb{I}, \overset{\bullet}{+})$ defined as the product system generated by the functions $(v_{2^n}(x), n \in \mathbb{N})$ defined in (6).

The discrete Laguerre functions associated to B_a are defined in the following way:

$$L_k^{(a)}(x) := w_k(B_a(x)) \; (k \in \mathbb{N}, x \in (\mathbb{I}, \overset{\circ}{+}))$$

and

$$L_k^{(a)}(x) := v_k(B_a(x)) \; (k \in \mathbb{N}, x \in (\mathbb{I}, \overset{\bullet}{+}))$$

for any $a \in \mathbb{I}_1$.

Proposition 1 *Using identical parameters* $a_n = a \in \mathbb{I}_1$ $(n \in \mathbb{N})$ *the Malmquist-Takenaka functions* $\Psi_n^{(p)}(x)$ *give the discrete Laguerre system* $L_n^{(a)}(x)$ *on both fields* $(\mathbb{I}, \overset{\bullet}{+}, \bullet)$ *and* $(\mathbb{I}, \overset{\circ}{+}, \circ)$.

Clearly, with the special identical parameters $a_n = \theta$ $(n \in \mathbb{N})$ this method gives the characters of the corresponding field. Thus the Malmquist-Takenaka system is also a generalization of the character system of the corresponding additive group.

References

[1] J. Bokor, F. Schipp, Approximate linear H^∞ identification in Laguerre and Kautz basis, *Automatica J. IFAC*, **34** (1998), 463–468.

[2] F. Schipp, W. R. Wade, P. Simon, J. Pál, *Walsh Series, An Introduction to Dyadic Harmonic Analysis*, Adam Hilger Ltd., Bristol and New York, (1990).

[3] F. Schipp, W. R. Wade, *Transforms on normed fields*, Leaflets in Mathematics, Janus Pannonius University Pécs, 1995.

[4] I. Simon, Discrete Laguerre functions on the dyadic fields, Pure Math. Appl. (PU.M.A.), **17** (2006), 459–468.

[5] I. Simon, The characters of the Blaschke-group, Stud. Univ. Babes-Bolyai Math., **54** (2009), 149–160.

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Hadamard product of GCUD matrices

Antal Bege
Sapientia–Hungarian University of Transilvania
Department of Mathematics and Informatics
Târgu Mureş, Romania
email: abege@ms.sapientia.ro

Abstract. Let f be an arithmetical function. The matrix $[f(i,j)^{**}]_{n\times n}$ given by the value of f in greatest common unitary divisor of $(i,j)^{**}$, $f\big((i,j)^{**}\big)$ as its i, j entry is called the greatest common unitary divisor (GCUD) matrix. We consider the Hadamard product of these matrices and we calculate the Hadamard product and determinant of the Hadamard product of two GCUD matrices.

1 Introduction

The classical Smith determinant introduced by H. J. Smith [9] is

$$\det[(i,j)]_{n\times n} = \begin{vmatrix} (1,1) & (1,2) & \cdots & (1,n) \\ (2,1) & (2,2) & \cdots & (2,n) \\ \cdots & \cdots & \cdots & \cdots \\ (n,1) & (n,2) & \cdots & (n,n) \end{vmatrix} = \varphi(1)\cdot\varphi(2)\cdots\varphi(n), \quad (1)$$

where (i,j) is the greatest common divisor of i and j, and $\varphi(n)$ is Euler's totient function.
A divisor d of n is said to be a unitary divisor of n if $\left(d, \frac{n}{d}\right) = 1$ and we write $d \parallel n$. Let $(m,n)^*$ the greatest divisor of m which is unitary divisor of n (see E. Cohen [2]).

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We denote the greatest common unitary divisor of m and n as $(m,n)^{**}$. Let $\varphi^*(n)$ be the unitary analogue of $\varphi(n)$:

$$\varphi^*(n) = \sum_{\substack{k \le n \\ (k,n)^* = 1}} 1,$$

and

$$\mu^*(n) = (-1)^{\omega(n)}$$

be the unitary analogue of the Möbius function $\mu(n)$.
The GCUD matrix with respect to f is

$$[f(i,j)^{**}]_{n\times n} = \begin{bmatrix} f((1,1)^{**}) & f((1,2)^{**}) & \cdots & f((1,n)^{**}) \\ f((2,1)^{**}) & f((2,2)^{**}) & \cdots & f((2,n)^{**}) \\ \cdots & \cdots & \cdots & \cdots \\ f((n,1)^{**}) & f((n,2)^{**}) & \cdots & f((n,n)^{**}) \end{bmatrix}$$

If we consider the GCUD matrix $[f(i,j)^{**}]_{n\times n}$ where

$$f(n) = \sum_{d \| n} g(d),$$

H. Jager [5] proved that

$$\big[f\big((i,j)\big)^{**}\big]_{n\times n} = C_1 \operatorname{diag}[g(1), g(2), \dots, g(n)]C_1^T, \tag{2}$$

where $C_1 = [c_{ij}]_{n\times n}$,

$$c_{ij} = \begin{cases} 1, & \text{ha} \quad j \,\|\, i \\ 0, & \text{ha} \quad j \,\not\|\, i \end{cases}.$$

and

$$\det[f((i,j)^{**})]_{n\times n} = g(1)\cdot g(2)\cdots g(n).$$

For $g(n) = \varphi^*(n)$

$$f\big((i,j)^{**}\big) = \sum_{d \| (i,j)^{**}} \varphi^*(d) = (i,j)^{**}.$$

and the decomposition of matrix

$$[(i,j)^{**}]_{n\times n} = C_1 \operatorname{diag}[\varphi^*(1), \varphi^*(2), \dots, \varphi^*(n)]C_1^T,$$

$$\det[(i,j)^{**}]_{n\times n} = \varphi^*(1)\varphi^*(2)\cdots\varphi^*(n).$$

The unitary convolution of the arithmetical functions f and g is defined as

$$(f \odot g)(n) = \sum_{d \,\|\, n} f(d)g\left(\frac{n}{d}\right).$$

From this convolution we can write (2) in the following form:

$$\det\left[f\big((i,j)^{**}\big)\right]_{n\times n} = (f\odot\mu^*)(1)(f\odot\mu^*)(2)\cdots(f\odot\mu^*)(n). \tag{3}$$

Here we present some examples which are relevant in our study.

Example 1 *If*

$$g(n) = \beta^*(n) = \sum_{i=1}^{n}(i,n)^*$$

the unitary Pillai function then (see L. Tóth, [10, 11]). We have

$$\begin{aligned}\beta^*(n) &= \sum_{s\|n} d\varphi\left(\frac{n}{d})\right),\\ f(n) &= \sum_{d|n}\beta^*(n) = n\tau^*(n),\end{aligned}$$

where $\tau^*(n)$ *is the number of unitary divisors. The GCUD matrix and determinant in this case have the following form:*

$$[(i,j)^{**}\tau^*((i,j)^{**})]_{n\times n} = C_1 \operatorname{diag}\big(\beta^*(1),\ \beta^*(2),\ldots,\beta^*(n)\big)\ C_1^T, \tag{4}$$

$$\det[(i,j)^{**}\tau^*((i,j)^{**})]_{n\times n} = \beta^*(1)\beta^*(2)\cdots\beta^*(n). \tag{5}$$

Example 2 *If* $g(n) = \dfrac{\varphi^*(n)}{n}$, *then*

$$f(n) = \sum_{d\|n}\frac{\varphi^*(d)}{d} = \frac{\beta^*(n)}{n},$$

$$\left[\frac{\beta^*(i,j)}{(i,j)}\right]_{n\times n} = C_1\ \mathit{diag}\left(\frac{\varphi^*(1)}{1},\ \frac{\varphi^*(2)}{2},\ldots,\frac{\varphi^*(n)}{n}\right)\ C_1^T, \tag{6}$$

$$\det\left[\frac{\beta^*(i,j)}{(i,j)}\right]_{n\times n} = \frac{\varphi^*(1)\varphi^*(2)\cdots\varphi^*(n)}{n!}.$$

For other contributions, we mention the papers of P. Haukkanen, J. Wang and J. Sillanpää [3], A. Nalli, D. Tasci [6], P. Haukkanen, J. Sillanpää [4]. We introduce the concept of Hadamard product (see F. Zhang [12]).

Definition 1 *The Hadamard product* $C = A \circ B = [c_{ij}]_{n\times n}$ *of two matrices* $A = [a_{ij}]_{n\times n}$ *and* $B = [b_{ij}]_{n\times n}$ *is simply their elementwise product,*

$$c_{ij} = a_{ij}b_{ij}, \quad i,j \in \{1,2,\ldots,b\}.$$

A. Ocal [8], A. Nalli [7], A. Bege [1] establishes various results concerning classical GCD matrices and least common multiple (LCM) matrices. In examples 1 and 2 appears Hadamard products of special GCD matrices:

$$\det\left[[\tau((i,j)^{**})]_{n\times n}\circ[(i,j)^{**}]_{n\times n}\right]_{n\times n} = \beta^*(1)\beta^*(2)\cdots\beta^*(n),$$

$$\det\left[[\beta((i,j)^{**})]_{n\times n}\circ\left[\frac{1}{(i,j)^{**}}\right]_{n\times n}\right]_{n\times n} = \frac{\varphi^*(1)\varphi^*(2)\cdots\varphi^*(n)}{n!}.$$

Let f and g be two arithmetical functions. In this paper we calculate the Hadamard product and the determinant of Hadamard product of $[f((i,j)^{**})]_{n\times n}$ and $[g((i,j)^{**})]_{n\times n}$.

2 Main results

Theorem 1 *Let* h *and* g *be two arithmetical functions and* g *be multiplicative. If*

$$f(n) = \sum_{d\|n} h(d)g\left(\frac{n}{d}\right), \tag{7}$$

then

1.

$$\left[[f((i,j)^{**})]_{n\times n}\circ\left[\frac{1}{g((i,j)^{**})}\right]_{n\times n}\right]_{n\times n} = C_1 \operatorname{diag}\left(\frac{h(1)}{g(1)}, \frac{h(2)}{g(2)}, \ldots, \frac{h(n)}{g(n)}\right) C_1^T, \tag{8}$$

where $C_1 = [c_{ij}]_{n\times n}$*,*

$$c_{ij} = \begin{cases} 1, & \textit{if} \quad j \,\|\, i \\ 0, & \textit{if} \quad j \not\| \, i \end{cases},$$

2.

$$\det\left[[f((i,j)^{**})]_{n\times n}\circ\left[\frac{1}{g((i,j)^{**})}\right]_{n\times n}\right]_{n\times n} = \frac{h(1)}{g(1)}\frac{h(2)}{g(2)}\cdots\frac{h(n)}{g(n)}, \tag{9}$$

3. *There exist* $H(n)$ *and* $G(n)$ *arithmetical functions, such that*

$$\det\left[[f((i,j)^{**})]_{n\times n}\circ\left[\frac{1}{g((i,j)^{**})}\right]_{n\times n}\right]_{n\times n}=\frac{\det[H((i,j)^{**})]}{\det[G((i,j)^{**})]}.$$

Proof.
Let

$$h(n)=\big(f\odot(\mu^{*}g)\big)(n).$$

We have

$$g\odot(\mu^{*}g)=I,$$

which means that $\mu^{*}g$ is the inverse respecting to the unitary convolution. From this

$$h\odot g=f\odot\big((\mu^{*}g)\odot g\big)=f\odot I=f.$$

Because g is a multiplicative function

$$f(n)=\sum_{d\|n}h(d)g\left(\frac{n}{d}\right)=\sum_{d\|n}\frac{h(d)}{g(d)}g(d)g\left(\frac{n}{d}\right)=g(n)\sum_{d\|n}\frac{h(d)}{g(d)}.$$

By the definition of the Hadamard product we have

$$\frac{f((i,j)^{**})}{g((i,j)^{**})}.$$

Thus

$$\frac{f(n)}{g(n)}=\sum_{d\|n}\frac{h(d)}{g(d)},$$

and by using (2), we have (8).
If we calculate the determinant of both parts we have (9).
Let

$$H(n)=\sum_{d\|n}h(d)$$

and

$$G(n)=\sum_{d\|n}g(d).$$

Using (2) we have

$$\det[H((i,j)^{**})]_{n\times n}=h(1)h(2)\cdots h(n)$$

and

$$\det[G((i,j)^{**})]_{n\times n} = g(1)g(2)\cdots g(n).$$

which means that

$$\det\left[[f((i,j)^{**})]_{n\times n} \circ \left[\frac{1}{g((i,j)^{**})}\right]_{n\times n}\right]_{n\times n} = \frac{\det[H((i,j)^{**})]}{\det[G((i,j)^{**})]}.$$

□

Example 3 *If* $g(n) = n$ *then*

$$f(n) = \sum_{d\|n} h(d)\frac{n}{d}.$$

and

$$\begin{aligned}\left[\frac{f((i,j)^{**})}{(i,j)^{**}}\right]_{n\times n} &= \left[[f((i,j)^{**})]_{n\times n} \circ \left[\frac{1}{(i,j)^{**}}\right]_{n\times n}\right]_{n\times n} = \\ &= C_1 \operatorname{diag}\left(\frac{h(1)}{1}, \frac{h(2)}{2}, \dots, \frac{h(n)}{n}\right) C_1^T\end{aligned}$$

$$\begin{aligned}\det\left[\frac{f((i,j)^{**})}{(i,j)^{**}}\right]_{n\times n} &= \det\left[[f((i,j)^{**})]_{n\times n} \circ \left[\frac{1}{(i,j)^{**}}\right]_{n\times n}\right]_{n\times n} = \\ &= \frac{h(1)h(2)\cdots h(n)}{n!}.\end{aligned}$$

Example 4 *If* $g(n) = \frac{1}{n}$ *then*

$$f(n) = \sum_{d\|n} h(d)\frac{d}{n}$$

and

$$[f((i,j)^{**}(i,j)^{**}]_{n\times n} = C_1 \operatorname{diag}(h(1)1, h(2)2, \dots, h(n)n)\ C_1^T,$$

$$\begin{aligned}\det[f((i,j)^{**})(i,j)^{**}]_{n\times n} &= \det\left[[f((i,j)^{**}]_{n\times n} \circ [(i,j)^{**}]_{n\times n}\right]_{n\times n} = \\ &= h(1)\cdots h(n)n!.\end{aligned}$$

If we want to apply this theorem to a given f and g, using the unitary Möbius inversion formula we have

$$h(n) = \sum_{d \| n} \mu^*(d) g(d) f\left(\frac{n}{d}\right)$$

where $\mu^*(n)$ is the usual unitary Möbius function and we can formulate the following result.

Theorem 2 *Let* f *and* g *be two arithmetical functions and* g *be multiplicative. We have*

$$\begin{aligned}\left[\frac{f((i,j)^{**})}{g((i,j)^{**})}\right]_{n\times n} &= \left[[f((i,j)^{**})]_{n\times n} \circ \left[\frac{1}{g((i,j)^{**})}\right]_{n\times n}\right]_{n\times n} = \\ &= C_1 \operatorname{diag}\left(\frac{f(1)}{g(1)}, \dots, \frac{\sum\limits_{d\|n} \mu^*(d) g(d) f\left(\frac{n}{d}\right)}{g(n)}\right) C_1^T,\end{aligned}$$

and

$$\det\left[[f((i,j)^{**})]_{n\times n} \circ \left[\frac{1}{g((i,j)^{**})}\right]_{n\times n}\right]_{n\times n} = \frac{f(1)}{g(1)} \cdots \frac{\sum_{d\|n} \mu^*(d) g(d) f\left(\frac{n}{d}\right)}{g(n)}.$$

Example 5 *If* f *is a multiplicative arithmetical function and* $g(n) = \frac{1}{n}$

$$\det\left[f((i,j)^{**})(i,j)^{**}\right]_{n\times n} = 1 \cdots \left(n \prod_{p^\alpha \| n} \left(f(p^\alpha) - \frac{1}{p^\alpha}\right)\right).$$

In particular if $f(n) = 1$

$$\det\left[(i,j)^{**}\right]_{n\times n} = \prod_{k=1}^{n} \varphi^*(k).$$

Example 6 *For a power GCUD matrix and determinant we have*

$$[((i,j)^{**})^s]_{n\times n} = C_1 \ \mathit{diag}\big(J_s(1),\ J_s(2), \dots, J_s(n)\big)\ C_1^T,$$

$$\det[((i,j)^{**})^s]_{n\times n} = J_s(1) J_s(2) \cdots J_s(n).$$

where $J_s(n)$ *the Jordan totient function.*

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References

[1] A. Bege, Hadamard product of GCD matrices, *Acta Univ. Sapientiae, Math.*, **1** (2009), 43–49.

[2] E. Cohen, Arithmetical functions associated with the unitary divisors of an integer, *Math. Z.,* **74** (1960), 66–80.

[3] P. Haukkanen, J. Wang, J. Silanpää, On Smith's determinant, *Linear Algebra Appl.,* **258** (1997) 251–269.

[4] P. Haukkanen, J. Sillanpää, An inequality for the determinant of the GCD matrix and the GCUD matrix, *J. Math. Res. Exposition,* **20** (2000), 181–186.

[5] H. Jager, The unitary analogue of some identities for certain arithmetical functions, *Nederl. Akad. Wetensch. Proc. Ser. A,* **64** (1961), 508–515.

[6] A. Nalli, D. Tasci, On the GCUD-reciprocal LCUM matrices, *JP J. Algebra Number Theory Appl.,* **3** (2003), 245–257.

[7] A. Nalli, On the Hadamard product of Fibonacci Q^n matrix and Fibonacci Q^{-n} matrix, *Int. J. Contemp. Math. Sci.*, **1** 2006, 753–761.

[8] A. A. Öçal, On the Hadamard products of GCD and LCM matrices, *Commun. Fac. Sci. Univ. Ank., Sér. A1, Math. Stat.,* **52** (2003), 1–12.

[9] H. J. S. Smith, On the value of a certain arithmetical determinant, *Proc. London Math. Soc.,* **7** (1875/76), 208–212.

[10] L. Tóth, The unitary analogue of Pillai's arithmetic function, *Collect. Math.,* **40** (1989), 19–30.

[11] L. Tóth, The unitary analogue of Pillai's arithmetic function II., *Notes Number Theory Discrete Math.,* **2** (1996), 40–46.

[12] F. Zhang, *Matrix theory*, Springer, New York, Berlin, Heidelberg, 1999.

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Multivalent $\beta-$uniformly starlike functions involving the Hurwitz-Lerch Zeta function

Gangadharan Murugusundaramoorthy
School of Advanced Sciences
VIT University
Vellore - 632014, India
email: gmsmoorthy@yahoo.com

Abstract. Making use of convolution product, we introduce a novel subclass of $p-$valent analytic functions with negative coefficients and obtain coefficient bounds, extreme points and radius of starlikeness for functions belonging to the generalized class $TP^{k,p}_{b,\mu}(\lambda, \alpha, \beta)$. We also derive results for the modified Hadamard products of functions belonging to the class $TP^{k,p}_{b,\mu}(\lambda, \alpha, \beta)$.

1 Introduction

Denote by $\mathcal{A}_p$ the class of functions f normalized by

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}, \quad (p \in N = 1, 2, 3, ...) \tag{1}$$

which are analytic and $p-$valent in the open disc $U = \{z : z \in \mathbb{C}, \ |z| < 1\}$. Denote by $\mathcal{T}_p$ a subclass of $\mathcal{A}_p$ consisting of functions of the form

$$f(z) = z^p - \sum_{k=1}^{\infty} a_{p+k} z^{p+k}, \quad (a_{p+k} \geq 0; p \in N = 1, 2, 3, ..., z \in U). \tag{2}$$

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For functions $f \in \mathcal{A}_p$ given by(1) and $g \in \mathcal{A}_p$ given by $g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k}$, we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k} = (g * f)(z), \quad z \in U. \tag{3}$$

The following we recall a general Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ defined by (see [23])

$$\Phi(z, s, a) := \sum_{k=0}^{\infty} \frac{z^k}{(k+a)^s} \tag{4}$$

$$(a \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}; s \in \mathbb{C}, \Re(s) > 1 \text{ and } |z| = 1)$$

where, as usual, $\mathbb{Z}_0^- := \mathbb{Z} \setminus \{\mathbb{N}\}$ ($\mathbb{Z} := \{0, \pm 1, \pm 2, \pm 3, ...\}; \mathbb{N} := \{1, 2, 3, ...\}$). Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ can be found in the recent investigations by Choi and Srivastava [4], Ferreira and Lopez [5], Garg et al. [7], Lin and Srivastava [10], Lin et al. [11], and others.

For the class of analytic functions denote by $\mathcal{A}$ consisting of functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ Srivastava and Attiya [22] (see also Raducanu and Srivastava [17], and Prajapat and Goyal [14]) introduced and investigated the linear operator:

$$\mathcal{J}_{\mu,b} : \mathcal{A} \to \mathcal{A}$$

defined in terms of the Hadamard product (or convolution) by

$$\mathcal{J}_{\mu,b} f(z) = \mathcal{G}_{b,\mu} * f(z) \tag{5}$$

$(z \in U; b \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}; \mu \in \mathbb{C}; f \in \mathcal{A})$, where, for convenience,

$$G_{\mu,b}(z) := (1+b)^{\mu}[\Phi(z, \mu, b) - b^{-\mu}] \quad (z \in U). \tag{6}$$

It is easy to observe from (given earlier by [14], [17]) (1), (5) and (6)that

$$\mathcal{J}_b^{\mu} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+b}{k+b}\right)^{\mu} a_k z^k. \tag{7}$$

Motivated essentially by the above-mentioned Srivastava-Attiya operator, we define the operator

$$\mathcal{J}_{b,\mu}^{n,p} : \mathcal{A}_p \to \mathcal{A}_p$$

which is defined as

$$\mathcal{J}_{b,\mu}^{k,p}f(z) = z^p + \sum_{k=1}^{\infty} C_b^{\mu}(k,p)a_{p+k}z^{p+k} \quad (z \in \mathbb{U}; f(z) \in \mathcal{A}_p) \tag{8}$$

where

$$C_b^{\mu}(k,p) = \left| \left(\frac{p+b}{k+p+b} \right)^{\mu} \right| \tag{9}$$

and (throughout this paper unless otherwise mentioned) the parameters μ, b are constrained as

$$b \in \mathbb{C} \setminus \{\mathbb{Z}_0^-\}; \mu \in \mathbb{C} \quad \text{and} \quad p, \in \mathbb{N}.$$

1. For $\mu = 1$ and $b = \nu(\nu > -1)$ generalized Libera Bernardi integral operators [16]

$$\mathcal{J}_{\nu,1}^{k,p}f(z) := \frac{p+\nu}{z^{\nu}} \int_0^z t^{\nu-1}f(t)dt := z + \sum_{k=1}^{\infty} \left(\frac{\nu+p}{k+p+\nu} \right) a_{p+k}z^{p+k} := \mathcal{L}_{\nu}^{p}f(z). \tag{10}$$

2. For $\mu = \sigma(\sigma > 0)$ and $b = 1$ Jung-Kim-Srivastava integral operator [12]

$$\mathcal{J}_{1,\sigma}^{k,p}f(z) := z + \sum_{k=1}^{\infty} \left(\frac{1+p}{k+p+1} \right)^{\sigma} a_{p+k}z^{p+k} = \mathcal{I}_{\sigma}^{p}f(z) \tag{11}$$

closely related to some multiplier transformatiom studied by Flett[6]. Making use of the operator $\mathcal{J}_{b,\mu}^{k,p}$, and motivated by earlier works of [1, 2, 3, 8, 9, 15, 13, 20, 21, 24, 25, 26], we introduced a new subclass of analytic functions with negative coefficients and discuss some some usual properties of the geometric function theory of this generalized function class.

For $0 \leq \lambda \leq 1$, $0 \leq \alpha < 1$ and $\beta \geq 0$, we let $P_{b,\mu}^{k,p}(\lambda, \alpha, \beta)$ be the subclass of $\mathcal{A}_p$ consisting of functions of the form (1) and satisfying the inequality

$$\begin{aligned} &\text{Re} \left\{ \frac{(1-\lambda+\frac{\lambda}{p})z(\mathcal{J}_{b,\mu}^{k,p}f(z))' + \frac{\lambda}{p}z^2(\mathcal{J}_{b,\mu}^{k,p}f(z))''}{p(1-\lambda)\mathcal{J}_{b,\mu}^{k,p}f(z) + \lambda z(\mathcal{J}_{b,\mu}^{k,p}f(z))'} - \alpha \right\} \\ &> \beta \left| \frac{(1-\lambda+\frac{\lambda}{p})z(\mathcal{J}_{b,\mu}^{k,p}f(z))' + \frac{\lambda}{p}z^2(\mathcal{J}_{b,\mu}^{k,p}f(z))''}{p(1-\lambda)\mathcal{J}_{b,\mu}^{k,p}f(z) + \lambda z(\mathcal{J}_{b,\mu}^{k,p}f(z))'} - 1 \right| \end{aligned} \tag{12}$$

where $z \in \mathbb{U}$, $\mathcal{J}_{b,\mu}^{k,p} f(z)$ is given by (8). We further let $TP_{b,\mu}^{k,p}(\lambda, \alpha, \beta) = P_{b,\mu}^{k,p}(\lambda, \alpha, \beta) \cap \mathcal{T}_p$.

In particular, for $0 \leq \lambda \leq 1$, the class $TP_{b,\mu}^{k,p}(\lambda, \alpha, \beta)$ provides a transition from k−uniformly starlike functions to k−uniformly convex functions.

Example 1 *If* $\lambda = 0$, *then*

$$\begin{aligned} TP_{b,\mu}^{k,p}(0, \alpha, \beta) \equiv TS_{b,\mu}^{k,p}(\alpha, \beta) := Re \left\{ \frac{1}{p} \frac{z(\mathcal{J}_{b,\mu}^{k,p} f(z)')}{\mathcal{J}_{b,\mu}^{k,p} f(z)} - \alpha \right\} \\ > \beta \left| \frac{1}{p} \frac{z(\mathcal{J}_{b,\mu}^{k,p} f(z))'}{\mathcal{J}_{b,\mu}^{k,p} f(z)} - 1 \right|, \quad z \in \mathbb{U}. \end{aligned} \tag{13}$$

Example 2 *If* $\lambda = 1$, *then*

$$\begin{aligned} TP_{b,\mu}^{k,p}(1, \alpha, \beta) \equiv UCT_{b,\mu}^{k,p}(\alpha, \beta) := Re \left\{ \frac{1}{p}[1 + \frac{z(\mathcal{J}_{b,\mu}^{k,p} f(z))''}{(\mathcal{J}_{b,\mu}^{k,p} f(z))'}] - \alpha \right\} \\ > \beta \left| \frac{1}{p}[1 + \frac{z(\mathcal{J}_{b,\mu}^{k,p} f(z))''}{(\mathcal{J}_{b,\mu}^{k,p} f(z))'}] - 1 \right|, z \in \mathbb{U}. \end{aligned} \tag{14}$$

Example 3 *For* $\mu = 1, b = \nu(\nu > -1)$ *and* $f(z)$*is as defined in (10) is in* $\mathcal{L}_\nu^{k,p}(\lambda, \alpha, \beta)$ *if*

$$\begin{aligned} Re \left(\frac{(1 - \lambda + \frac{\lambda}{p}) z(\mathcal{L}_\nu^p f(z))' + \frac{\lambda}{p} z^2 (\mathcal{L}_\nu^p f(z))''}{p(1-\lambda)\mathcal{L}_\nu^p f(z) + \lambda z(\mathcal{L}_\nu^p f(z))'} - \alpha \right) \\ > \beta \left| \frac{(1 - \lambda + \frac{\lambda}{p}) z(\mathcal{L}_\nu^p f(z))' + \frac{\lambda}{p} z^2 (\mathcal{L}_\nu^p f(z))''}{p(1-\lambda)\mathcal{L}_\nu^p f(z) + \lambda z(\mathcal{L}_\nu^p f(z))'} - 1 \right|, z \in \mathbb{U}. \end{aligned} \tag{15}$$

Also, let $\mathcal{L}_\nu^p(\lambda, \alpha, \beta) \cap \mathcal{T}_p = \mathcal{T}\mathcal{L}_\nu^p(\lambda, \alpha, \beta)$.

Example 4 *For* $\mu = \sigma(\sigma > 0)$, $b = 1$ *and* $f(z)$ *is defined in (11), is in* $\mathcal{I}_\sigma^p(\lambda, \alpha, \beta)$ *if*

$$\begin{aligned} Re \left(\frac{(1 - \lambda + \frac{\lambda}{p}) z(\mathcal{I}_\sigma^p f(z))' + \frac{\lambda}{p} z^2 (\mathcal{I}_\sigma^p f(z))''}{p(1-\lambda)\mathcal{I}_\sigma^p f(z) + \lambda z(\mathcal{I}_\sigma^p f(z))'} - \alpha \right) \\ > \beta \left| \frac{(1 - \lambda + \frac{\lambda}{p}) z(\mathcal{I}_\sigma^p f(z))' + \frac{\lambda}{p} z^2 (\mathcal{I}_\sigma^p f(z))''}{p(1-\lambda)\mathcal{I}_\sigma^p f(z) + \lambda z(\mathcal{I}_\sigma^p f(z))'} - 1 \right|, z \in \mathbb{U}. \end{aligned} \tag{16}$$

Also, let $\mathcal{I}_\sigma^p(\lambda, \alpha, \beta) \cap \mathcal{T}_p = \mathcal{T}\mathcal{I}_\sigma^p(\lambda, \alpha, \beta)$.

The main object of this paper is to study the coefficient bounds, extreme points and radius of starlikeness for functions belong to the generalized class $TP^{k,p}_{b,\mu}(\lambda,\alpha,\beta)$ empolying the technique of Silverman[18] and also derive results for the modified Hadamard products of functions belonging to the class $TP^{k,p}_{b,\mu}(\lambda,\alpha,\beta)$using the techniques of Schild and Silverman [19]

2 Coefficient Bounds

In this section we obtain a necessary and sufficient condition for functions $f(z)$ in the classes $P^{k,p}_{b,\mu}(\lambda,\alpha,\beta)$ and $TP^{k,p}_{b,\mu}(\lambda,\alpha,\beta)$.

Theorem 1 *A function* $f(z)$ *of the form (1) is in* $P^{k,p}_{b,\mu}(\lambda,\alpha,\beta)$ *if*

$$\sum_{k=1}^{\infty}[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C^{\mu}_{b}(k,p)||a_{p+k}| \le p^2(1-\alpha), \qquad (1)$$

$0\le\lambda\le 1,\ -1\le\alpha<1,\ \beta\ge 0.$

Proof. It suffices to show that

$$\beta\left|\frac{(1-\lambda+\frac{\lambda}{p})z(\mathcal{J}^{k,p}_{b,\mu}f(z))'+\frac{\lambda}{p}z^2(\mathcal{J}^{k,p}_{b,\mu}f(z))''}{p(1-\lambda)\mathcal{J}^{k,p}_{b,\mu}f(z)+\lambda z(\mathcal{J}^{k,p}_{b,\mu}f(z))'}-1\right| - \mathrm{Re}\left\{\frac{(1-\lambda+\frac{\lambda}{p})z(\mathcal{J}^{k,p}_{b,\mu}f(z))'+\frac{\lambda}{p}z^2(\mathcal{J}^{k,p}_{b,\mu}f(z))''}{p(1-\lambda)\mathcal{J}^{k,p}_{b,\mu}f(z)+\lambda z(\mathcal{J}^{k,p}_{b,\mu}f(z))'}-1\right\}\le 1-\alpha$$

We have

$$\begin{aligned}&\beta\left|\frac{(1-\lambda+\frac{\lambda}{p})z(\mathcal{J}^{k,p}_{b,\mu}f(z))'+\frac{\lambda}{p}z^2(\mathcal{J}^{k,p}_{b,\mu}f(z))''}{p(1-\lambda)\mathcal{J}^{k,p}_{b,\mu}f(z)+\lambda z(\mathcal{J}^{k,p}_{b,\mu}f(z))'}-1\right|\\&-\mathrm{Re}\left\{\frac{(1-\lambda+\frac{\lambda}{p})z(\mathcal{J}^{k,p}_{b,\mu}f(z))'+\frac{\lambda}{p}z^2(\mathcal{J}^{k,p}_{b,\mu}f(z))''}{p(1-\lambda)\mathcal{J}^{k,p}_{b,\mu}f(z)+\lambda z(\mathcal{J}^{k,p}_{b,\mu}f(z))'}-1\right\}\\&\le(1+\beta)\left|\frac{(1-\lambda+\frac{\lambda}{p})z(\mathcal{J}^{k,p}_{b,\mu}f(z))'+\frac{\lambda}{p}z^2(\mathcal{J}^{k,p}_{b,\mu}f(z))''}{p(1-\lambda)\mathcal{J}^{k,p}_{b,\mu}f(z)+\lambda z(\mathcal{J}^{k,p}_{b,\mu}f(z))'}-1\right|\\&\le\frac{(1+\beta)\sum\limits_{k=1}^{\infty}k[\frac{p+k\lambda}{p}]|C^{\mu}_{b}(k,p)||a_{p+k}|}{p-\sum\limits_{k=1}^{\infty}[p+k\lambda]|C^{\mu}_{b}(k,p)||a_{p+k}|}.\end{aligned}$$

This last expression is bounded above by $(1-\alpha)$ if

$$\sum_{k=1}^{\infty}[p+k\lambda][k(1+\beta)+p(1-\alpha)]\ |C_b^{\mu}(k,p)|\ |a_{p+k}|\leq p^2(1-\alpha)$$

and hence the proof is complete. □

Theorem 2 *A necessary and sufficient condition for* $f(z)$ *of the form (2) to be in the class* $TP_{b,\mu}^{k,p}(\lambda,\alpha,\beta)$, $-1\leq\alpha<1, 0\leq\lambda\leq1, \beta\geq0$ *is that*

$$\sum_{k=1}^{\infty}[p+k\lambda][k(1+\beta)+p(1-\alpha)]\ |C_b^{\mu}(k,p)|a_{p+k}\leq p^2(1-\alpha), \tag{2}$$

Proof. In view of Theorem 1, we need only to prove the necessity. If $f\in P_{b,\mu}^{k,p}(\lambda,\alpha,\beta)$ and z is real then

$$\frac{1-\sum_{k=1}^{\infty}(\frac{p+k}{p})[\frac{p+k\lambda}{p}]|C_b^{\mu}(k,p)|a_{p+k}|z|^k}{1-\sum_{k=1}^{\infty}[\frac{p+k\lambda}{p}]|C_b^{\mu}(k,p)|a_{p+k}|z|^k}-\alpha\geq\beta\frac{\sum_{k=1}^{\infty}k[\frac{p+k\lambda}{p}]|C_b^{\mu}(k,p)|a_{p+k}|z|^k}{1-\sum_{k=1}^{\infty}[\frac{p+k\lambda}{p}]|C_b^{\mu}(k,p)a_{p+k}|z|^k}$$

Letting $z\to1$ along the real axis, we obtain the desired inequality

$$\sum_{k=1}^{\infty}[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C_b^{\mu}(k,p)|\ a_{p+k}\leq p^2(1-\alpha).$$

□

In view of the Examples 1 to 4 in Section 1 and Theorem 2, we have following corollaries for the classes defined in these examples.

Corollary 1 *A necessary and sufficient condition for* $f(z)$ *of the form (2) to be in the class* $TS_{b,\mu}^{k,p}(\alpha,\beta)$, $0\leq\alpha<1, \beta\geq0$ *is that*

$$\sum_{k=1}^{\infty}[k(1+\beta)+p(1-\alpha)]|C_b^{\mu}(k,p)|\ a_{p+k}\leq p(1-\alpha),$$

Corollary 2 *A necessary and sufficient condition for* $f(z)$ *of the form (2) to be in the class* $UCT_{b,\mu}^{k,p}(\alpha,\beta)$, $0\leq\alpha<1, \beta\geq0$ *is that*

$$\sum_{k=1}^{\infty}(p+k)[k(1+\beta)+p(1-\alpha)]|C_b^{\mu}(k,p)|\ a_{p+k}\leq p^2(1-\alpha),$$

Corollary 3 *A necessary and sufficient condition for* $f(z)$ *of the form (2) to be in the class* $T\mathcal{L}_{\nu}^{k,p}(\lambda, \alpha, \beta)$, $0 \le \alpha < 1, \beta \ge 0$ *is that*

$$\sum_{k=1}^{\infty} (p + k\lambda)[k(1 + \beta) + p(1 - \alpha)] \left(\frac{p + \nu}{k + p + \nu}\right) a_{p+k} \le p^2(1 - \alpha).$$

Corollary 4 *A necessary and sufficient condition for* $f(z)$ *of the form (2) to be in the class* $\mathcal{TI}_{\sigma}^{p}(\lambda, \alpha, \beta)$, $0 \le \alpha < 1, \beta \ge 0$ *is that*

$$\sum_{k=1}^{\infty} (p + k\lambda)[k(1 + \beta) + p(1 - \alpha)] \left(\frac{1 + p}{k + p + 1}\right)^{\sigma} a_{p+k} \le p^2(1 - \alpha).$$

Corollary 5 *If* $f \in TP_{b,\mu}^{k,p}(\lambda, \alpha, \beta)$, *then*

$$a_{p+k} \le \frac{p^2(1 - \alpha)}{[p + k\lambda][k(1 + \beta) + p(1 - \alpha)]|C_b^{\mu}(k, p)|}, \ k \ge 1, \tag{3}$$

where $0 \le \lambda \le 1$, $-1 \le \alpha < 1$ *and* $\beta \ge 0$. *Equality in* (3) *holds for the function*

$$f(z) = z - \frac{p^2(1 - \alpha)}{[p + k\lambda][k(1 + \beta) + p(1 - \alpha)]|C_b^{\mu}(k, p)|} z^{p+k} \qquad (p \in N). \tag{4}$$

It is of interest to note that ,when $p = 1$ and $k = n - 1$, the above results reduces to the results studied in [2, 8, 9, 20, 21] Similarly many known results can be obtained as particular cases of the following theorems, so we omit stating the particular cases for the following theorems.

3 Closure Properties

Theorem 1 *Let*

$$\begin{aligned} f_p(z) &= z^p \qquad (p \in N) \quad \text{and} \\ f_{p+k}(z) &= z^p - \frac{p^2(1 - \alpha)}{[p + k\lambda][k(1 + \beta) + p(1 - \alpha)]|C_b^{\mu}(k, p)|} z^{p+k}. \end{aligned} \tag{1}$$

Then $f \in TP_{b,\mu}^{k,p}(\lambda, \alpha, \beta)$, *if and only if it can be expressed in the form*

$$f(z) = \sum_{k=0}^{\infty} \omega_{p+k} f_{p+k}(z), \qquad \omega_{p+k} \ge 0, \quad \sum_{k=0}^{\infty} \omega_{p+k} = 1. \tag{2}$$

Proof. Let us suppose that $f(z)$ is given by (2),that is by

$$f(z) \quad = \quad z^p - \sum_{k=1}^{\infty} \frac{p^2(1-\alpha)}{[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C_b^{\mu}(k,p)|}\omega_{p+k}z^{p+k}.$$

Then, since

$$\sum_{k=1}^{\infty} \frac{[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C_b^{\mu}(k,p)|\ p^2(1-\alpha)}{p^2(1-\alpha)[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C_b^{\mu}(k,p)|}\omega_{p+k}$$
$$= \sum_{k=1}^{\infty} \omega_{p+k} = 1-\omega_p \le 1.$$

Thus $f \in TP_{b,\mu}^{k,p}(\lambda,\alpha,\beta)$. Conversely, let us have $f \in TP_{b,\mu}^{k,p}(\lambda,\alpha,\beta)$. Then by using (3), we set

$$\omega_{p+k} = \frac{[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C_b^{\mu}(k,p)|}{p^2(1-\alpha)}a_{p+k}, \quad (k \in N)$$

and $\omega_p = 1 - \sum_{k=1}^{\infty} \omega_{p+k}$,we can readily see that $f(z)$ can be expressed precisely as in (1).This evidently completes the proof of Theorem 1. □

Theorem 2 *The class* $TP_{b,\mu}^{k,p}(\lambda,\alpha,\beta)$ *is a convex set.*

Proof. Let the function

$$f_j(z) = z^p - \sum_{k=1}^{\infty} a_{p+k,j}z^{p+k}, \quad (a_{p+k,j} \ge 0, p \in N; \quad j = 1, 2...) \tag{3}$$

be in the class $TP_{b,\mu}^{k,p}(\lambda,\alpha,\beta)$. It sufficient to show that the function $h(z)$ defined by

$$h(z) = \eta f_1(z) + (1-\eta)f_2(z), \quad 0 \le \eta \le 1,$$

is in the class $TP_{b,\mu}^{k,p}(\lambda,\alpha,\beta)$. Since

$$h(z) = z^p - \sum_{k=1}^{\infty} [\eta a_{p+k,1} + (1-\eta)a_{p+k,2}]z^{p+k},$$

an easy computation with the aid of Theorem 2 gives,

$$\begin{aligned}&\sum_{k=1}^{\infty}[p+k\lambda][k(1+\beta)+p(1-\alpha)]\eta|C_b^{\mu}(k,p)|a_{p+k,1}\\&+\sum_{k=1}^{\infty}[p+k\lambda][k(1+\beta)+p(1-\alpha)](1-\eta)|C_b^{\mu}(k,p)|a_{p+k,2}\\&\leq p^2\eta(1-\alpha)+p^2(1-\eta)(1-\alpha)\\&\leq p^2(1-\alpha),\end{aligned}$$

which implies that $h\in TP_{b,\mu}^{k,p}(\lambda,\alpha,\beta)$. Hence $TP_{b,\mu}^{k,p}(\lambda,\alpha,\beta)$ is convex. □

Now we provide the radii of $p-valently$ close-to-convexity, starlikeness and convexity for the class $TP_{b,\mu}^{k,p}(\lambda,\alpha,\beta)$.

Theorem 3 *Let the function* $f(z)$ *defined by (2) be in the class* $TP_{b,\mu}^{k,p}(\lambda,\alpha,\beta)$. *Then* $f(z)$ *is p-valently close-to-convex of order* δ $(0\leq\delta<p)$ *in the disc* $|z|<r_1$, *where*

$$r_1:=\inf_{k\in\mathbb{N}}\left[\frac{(1-\delta)[k(1+\beta)+p(1-\alpha)][p+k\lambda]|C_b^{\mu}(k,p)|}{p^2(p+k)(1-\alpha)}\right]^{\frac{1}{k}}. \tag{4}$$

The result is sharp, with extremal function $f(z)$ *given by (1).*

Proof. Given $f\in\mathcal{T}_p$, and f is close-to-convex of order δ, we have

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|<p-\delta. \tag{5}$$

For the left hand side of (5) we have

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|\leq\sum_{k=1}^{\infty}(p+k)a_{p+k}|z|^k.$$

The last expression is less than $p-\delta$ if

$$\sum_{k=1}^{\infty}\frac{p+k}{p-\delta}a_{p+k}|z|^k<1.$$

Using the fact, that $f\in TP_{b,\mu}^{k,p}(\lambda,\alpha,\beta)$ if and only if

$$\sum_{k=1}^{\infty}\frac{[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C_b^{\mu}(k,p)|}{p^2(1-\alpha)}a_n\leq 1,$$

We can say (5) is true if

$$\frac{p+k}{p-\delta}|z|^k \le \frac{[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C_b^\mu(k,p)|}{p^2(1-\alpha)}a_n$$

Or, equivalently,

$$|z|^k = \left[\frac{(p-\delta)[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C_b^\mu(k,p)|}{p^2(p+k)(1-\alpha)}\right],$$

the last inequality leads us immediately to the disc $|z| < r_1$, where r_1 given by (4)and the proof of Theorem 3 is completed. □

Theorem 4 *If* $f \in TP_{b,\mu}^{k,p}(\lambda,\alpha,\beta)$, *then*

(i) f *is p-valently starlike of order* $\delta (0 \le \delta < p)$ *in the disc* $|z| < r_2$; *that is, Re* $\left\{\frac{zf'(z)}{f(z)}\right\} > \delta$, *where*

$$r_2 = \inf_{k\in N}\left[\left(\frac{p-\delta}{p+k-\delta}\right)\frac{[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C_b^\mu(k,p)|}{p^2(1-\alpha)}\right]^{\frac{1}{k}}. \quad (6)$$

(ii) f *is convex of order* δ $(0 \le \delta < p)$ *in the unit disc* $|z| < r_3$, *that is Re* $\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \delta$, *where*

$$r_3 = \inf_{k\in N}\left[\left(\frac{p-\delta}{(k+p)(p+k-\delta)}\right)\frac{[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C_b^\mu(k,p)|}{p^2(1-\alpha)}\right]^{\frac{1}{k}}. \quad (7)$$

Each of these results are sharp for the extremal function $f(z)$ *given by (1).*

Proof.(i) Given $f \in \mathcal{T}_p$, and f is starlike of order δ, we have

$$\left|\frac{zf'(z)}{f(z)} - p\right| < p-\delta. \quad (8)$$

For the left hand side of (8) we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \frac{\sum\limits_{k=1}^{\infty} k a_{p+k}\,|z|^k}{1-\sum\limits_{k=1}^{\infty} a_{p+k}\,|z|^k}.$$

The last expression is less than $p - \delta$ if

$$\sum_{k=1}^{\infty} \frac{k+p-\delta}{p-\delta} a_{p+k}\, |z|^k < 1.$$

Using the fact, that $f \in TP^{k,p}_{b,\mu}(\lambda, \alpha, \beta)$ if and only if

$$\sum_{k=1}^{\infty} \frac{[p+k\lambda][k(1+\beta)+p(1-\alpha)]}{p^2(1-\alpha)} a_{p+k} |C^{\mu}_{b}(k,p)| \le 1.$$

We can say (8) is true if

$$\frac{p+k-\delta}{p-\delta}|z|^k < \frac{[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C^{\mu}_{b}(k,p)|}{p^2(1-\alpha)}$$

Or, equivalently,

$$|z|^k = \left[\left(\frac{p-\delta}{p+k-\delta}\right) \frac{[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C^{\mu}_{b}(k,p)|}{p^2(1-\alpha)}\right]$$

which yields the starlikeness of the family.

(ii) Using the fact that f is convex if and only if zf' is starlike, we can prove (ii), on lines similar to the proof of (i). □

4 Convolution Results

Let the functions

$$f_j(z) = z^p + \sum_{k=1}^{\infty} a_{j,p+k} z^{p+k}, \quad (p \in N = 1, 2, 3, ...)(j = 1, 2) \tag{9}$$

then the modified Hadamard product of $f_1(z)$ and $f_2(z)$ is given by

$$(f_1 * f_2)(z) \;=\; z^p - \sum_{n=2}^{\infty} a_{1,p+k} a_{2,p+k}\, z^{p+k} = (f_2 * f_1)(z), (a_{1,p+k} \ge 0; a_{2,p+k} \ge 0).$$

Using the techniques of we prove the following results.

Theorem 5 *For functions* $f_j(z)(j = 1, 2)$ *defined by (9), be in the class* $TP^{k,p}_{b,\mu}(\lambda, \alpha, \beta)$*. Then* $(f_1 * f_2) \in TP^{k,p}_{b,\mu}(\lambda, \xi, \beta)$ *where*

$$\xi == 1 - \frac{p^2(1-\alpha)^2(1+\beta)}{[p+\lambda][(1+\beta)+p(1-\alpha)]^2|C^{\mu}_{b}(1,p)| - p^3(1-\alpha)^2} \tag{10}$$

where $C^{\mu}_{b}(1, p)$ *is given by (9).*

Proof. Employing the technique used earlier by Schild and Silverman[19], we need to find the largest ξ such that

$$\sum_{k=1}^{\infty} \frac{[p+k\lambda][k(1+\beta)+p(1-\xi)]|C^{\mu}_{b}(k,p)|}{p^2(1-\xi)} a_{1,p+k}a_{2,p+k} \leq 1, \quad (0 \leq \xi < 1)$$

for $f_j \in TP^{k,p}_{b,\mu}(\lambda, \alpha, \beta)(j = 1, 2)$ where ξ is defined by (10). On the other hand, under the hypothesis, it follows from (1) and the Cauchy's-Schwarz inequality that

$$\sum_{k=1}^{\infty} \frac{[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C^{\mu}_{b}(k,p)|}{p^2(1-\alpha)} \sqrt{a_{1,p+k}a_{2,p+k}} \leq 1. \tag{11}$$

Thus we need to find the largest ξ such that

$$\begin{aligned} &\sum_{k=1}^{\infty} \frac{[p+k\lambda][k(1+\beta)+p(1-\xi)]|C^{\mu}_{b}(k,p)|}{p^2(1-\xi)} a_{1,p+k}\, a_{2,p+k} \\ \leq \;& \sum_{k=1}^{\infty} \frac{[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C^{\mu}_{b}(k,p)|}{p^2(1-\alpha)} \sqrt{a_{1,p+k}a_{2,p+k}} \end{aligned}$$

or, equivalently that

$$\sqrt{a_{1,p+k}a_{2,p+k}} \quad \leq \quad \frac{(1-\xi)}{(1-\alpha)} \frac{[k(1+\beta)+p(1-\alpha)]}{[k(1+\beta)+p(1-\xi)]}, \quad (k \geq 1).$$

Hence by making use of the inequality (11), it is sufficient to prove that

$$\frac{p^2(1-\alpha)}{[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C^{\mu}_{b}(k,p)|} \leq \frac{(1-\xi)}{(1-\alpha)} \frac{[k(1+\beta)+p(1-\alpha)]}{[k(1+\beta)+p(1-\xi)]}$$

which yields

$$\xi = \Psi(k) = 1 - \frac{kp^2(1-\alpha)^2(1+\beta)}{[p+k\lambda][k(1+\beta)+p(1-\alpha)]^2|C^{\mu}_{b}(k,p)| - p^3(1-\alpha)^2} \tag{12}$$

for $k \geq 1$ is an increasing function of k and letting $k = 1$ in (12), we have

$$\xi = \Psi(1) = 1 - \frac{p^2(1-\alpha)^2(1+\beta)}{[p+\lambda][(1+\beta)+p(1-\alpha)]^2|C_b^\mu(1,p)| - p^3(1-\alpha)^2}$$

where $C_b^\mu(1,p)$ is given by (9). □

Theorem 6 *Let the function* $f(z)$ *defined by (2) be in the class* $TP_{b,\mu}^{k,p}(\lambda,\alpha,\beta)$. *Also let* $g(z) = z^p - \sum_{k=1}^{\infty} b_{p+k} z^{p+k}$ *for* $|b_{p+k}| \leq 1$. *Then* $(f*g) \in TP_{b,\mu}^{k,p}(\lambda,\alpha,\beta)$.

Proof. Since

$$\begin{aligned}
&\sum_{k=1}^{\infty}[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C_b^\mu(k,p)|\ |a_{p+k}b_{p+k}|\\
&\leq \sum_{k=1}^{\infty}[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C_b^\mu(k,p)|a_{p+k}|b_{p+k}|\\
&\leq \sum_{k=1}^{\infty}[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C_b^\mu(k,p)|a_{p+k}\\
&\leq p^2(1-\alpha)
\end{aligned}$$

it follows that $(f*g) \in TP_{b,\mu}^{k,p}(\lambda,\alpha,\beta)$, by the view of Theorem 2. □

Theorem 7 *Let the functions* $f_j(z)(j = 1,2)$ *defined by (9) be in the class* $TP_{b,\mu}^{k,p}(\lambda,\alpha,\beta)$. *Then the function* $h(z)$ *defined by*

$$h(z) = z^p - \sum_{n=2}^{\infty}(a_{1,p+k}^2 + a_{2,p+k}^2)z^{p+k}$$

is in the class $TP_{b,\mu}^{k,p}(\lambda,\xi,\beta)$, *where*

$$\xi = 1 - \frac{2p^2(1-\alpha)^2\ (1+\beta)}{[p+\lambda][(1+\beta)+p(1-\alpha)]^2|C_b^\mu(1,p)| - 2p^3(1-\alpha)^2}$$

where $C_b^\mu(1,p)$ *is given by (9).*

Proof. By virtue of Theorem 2, it is sufficient to prove that

$$\sum_{k=1}^{\infty} \frac{[p+k\lambda][k(1+\beta)+p(1-\xi)]|C_b^{\mu}(k,p)|}{p^2(1-\xi)}[a_{1,p+k}^2 + a_{2,p+k}^2] \leq 1 \tag{13}$$

where $f_j \in TP_{b,\mu}^{k,p}(\lambda,\alpha,\beta)$ we find from (9) and Theorem 2, that

$$\sum_{k=1}^{\infty} \left[\frac{[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C_b^{\mu}(k,p)|}{p^2(1-\alpha)}\right]^2 a_{j,p+k}^2$$

$$\leq \left[\sum_{k=1}^{\infty} \frac{[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C_b^{\mu}(k,p)|}{p^2(1-\alpha)} a_{j,p+k}\right]^2 \tag{14}$$

$$\leq 1, (j=1,2) \tag{15}$$

which would readily yield

$$\sum_{k=1}^{\infty} \frac{1}{2}\left[\frac{[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C_b^{\mu}(k,p)|}{p^2(1-\alpha)}\right]^2 (a_{1,p+k}^2 + a_{2,p+k}^2) \leq 1. \tag{16}$$

By comparing (14) and (16), it is easily seen that the inequality (13) will be satisfied if

$$\frac{[p+k\lambda][k(1+\beta)+p(1-\xi)]|C_b^{\mu}(k,p)|}{p^2(1-\xi)}$$

$$\leq \frac{1}{2}\left[\frac{[p+k\lambda][k(1+\beta)+p(1-\alpha)]|C_b^{\mu}(k,p)|}{p^2(1-\alpha)}\right]^2, \quad \text{for } k \geq 1.$$

That is if

$$\xi = \Psi(k) = 1 - \frac{2p^2(1-\alpha)^2\, k(1+\beta)}{[p+k\lambda][k(1+\beta)+p(1-\alpha)]^2|C_b^{\mu}(k,p)| - 2p^3(1-\alpha)^2} \tag{17}$$

Since$\Psi(k)$ is an increasing function of k $(k \geq 1)$. Taking $k = 1$ in (17), we have,

$$\xi = \Psi(1) = 1 - \frac{2p^2(1-\alpha)^2\,(1+\beta)}{[p+\lambda][(1+\beta)+p(1-\alpha)]^2|C_b^{\mu}(1,p)| - 2p^3(1-\alpha)^2}$$

which completes the proof. □

Concluding Remarks: In fact, by appropriately selecting the arbitrary sequences given in (10) and (11), suitably specializing the values of μ, α, β and p the results presented in this paper would find further applications for the class of p-valent functions stated in Examples 1 to 4 in Section 1.

References

[1] M. K. Aouf, Some families of p-valent functions with negative coefficients, *Acta Math. Univ. Comenian. (N.S.)*, **78** (2009), 121–135.

[2] R. Bharati, R. Parvatham and A. Swaminathan, On subclasses of uniformly convex functions and corresponding class of starlike functions, *Tamkang J. Math.*, **28** (1997), 17–32.

[3] N. E. Cho, On certain class of p-valent analytic functions, *Internat. J. Math. Math. Sci.*, **16** (1993), 319–328.

[4] J. Choi, H. M. Srivastava, Certain families of series associated with the Hurwitz-Lerch Zeta function, *Appl. Math. Comput.*, **170** (2005), 399–409.

[5] C. Ferreira, J. L. Lopez, Asymptotic expansions of the Hurwitz-Lerch Zeta function, *J. Math. Anal. Appl.*, **298** (2004), 210–224.

[6] T. M. Flett, The dual of an inequality of Hardy and Littlewood and some related inequalities, *J. Math. Anal. Appl.*, **38** (1972), 746–765.

[7] M. Garg, K. Jain, H. M. Srivastava, Some relationships between the generalized Apostol-Bernoulli polynomials and Hurwitz-Lerch Zeta functions, *Integral Transforms Spec. Funct.*, **17** (2006), 803–815.

[8] A. W. Goodman, On uniformly convex functions, *Ann. Polon. Math.*, **56** (1991), 87–92.

[9] A. W. Goodman, On uniformly starlike functions, *J. Math. Anal. Appl.*, **155** (1991), 364–370.

[10] S.-D. Lin, H. M. Srivastava, Some families of the Hurwitz-Lerch Zeta functions and associated fractional derivative and other integral representations, *Appl. Math. Comput.*, **154** (2004), 725–733.

[11] S.-D. Lin, H. M. Srivastava, P.-Y. Wang, Some espansion formulas for a class of generalized Hurwitz-Lerch Zeta functions, *Integral Transforms Spec. Funct.*, **17** (2006), 817–827.

[12] I. B. Jung, Y. C. Kim, H. M. Srivastava, The Hardy space of analytic functions associated with certain one-parameter families of integral operators, *J. Math. Anal. Appl.*, **176** (1993), 138–147.

[13] G. Murugusundaramoorthy, K. G. Subramanian, On a Subclass of multivalent functions with negative coefficients, *Southeast Asian Bull. Math.*, **27** (2004), 1065–1072.

[14] J. K. Prajapat, S. P. Goyal, Applications of Srivastava-Attiya operator to the classes of strongly starlike and strongly convex functions, *J. Math. Inequal.*, **3** (2009), 129–137.

[15] F. Rønning, Uniformly convex functions and a corresponding class of starlike functions, *Proc. Amer. Math. Soc.*, **118**, (1993), 189–196.

[16] G. L. Reddy, K. S. Padmanaban, On analytic functions with reference to the Bernardi integral operator, *Bull. Austral. Math. Soc.*, 25 (1982), 387–396.

[17] D. Răducanu, H. M. Srivastava, A new class of analytic functions defined by means of a convolution operator involving the Hurwitz-Lerch Zeta function, *Integral Transforms Spec. Funct.*, **18** (2007), 933–943.

[18] H. Silverman, Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.*, **51** (1975), 109–116.

[19] A. Schild, H. Silverman, Convolution of univalent functions with negative coefficients, *Ann. Univ. Mariae Curie-Sklodowska Sect. A*, **29** (1975), 99–107.

[20] K. G. Subramanian, G. Murugusundaramoorthy, P. Balasubrahmanyam, H. Silverman, Subclasses of uniformly convex and uniformly starlike functions, *Math. Japonica*, **42**, (1995), 517–522.

[21] K. G. Subramanian, T. V. Sudharsan, P. Balasubrahmanyam, H. Silverman, Classes of uniformly starlike functions, *Publ. Math. Debrecen*, **53** (1998), 309–315.

[22] H. M. Srivastava, A. A. Attiya, An integral operator associated with the Hurwitz-Lerch Zeta function and differential subordination, *Integral Transforms Spec. Funct.*, **18** (2007), 207–216.

[23] H. M. Srivastava, J. Choi, *Series associated with the Zeta and related functions*, Dordrecht, Boston, London, Kluwer Academic Publishers, 2001.

[24] H. M. Srivastava, J. Patel, G. P. Mohaptr, A certain class of p-valently analytic functions, *Math. Comput. Modelling*, **41** (2005), 321–334.

[25] H. M. Srivastava, M. K. Aouf, A certain fractional derivative operator and its applications to a new class of analytic and multivalent functions with negative coefficients. I., *J. Math. Anal. Appl.*, **171** (1992), 1–13.

[26] H. M. Srivastava and M. K. Aouf, A certain fractional derivative operator and its applications to a new class of analytic and multivalent functions with negative coefficients. II., *J. Math. Anal. Appl.*, **192** (1995), 673–688.

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On a new class of optimal eighth-order derivative-free methods

F. Soleymani
Department of Mathematics
Islamic Azad University
Zahedan Branch, Zahedan, Iran
email: fazlollah.soleymani@gmail.com

Abstract. Making use of last derivative approximation and weight function approach, we construct an eighth-order class of three-step methods, which are consistent with the optimality conjecture of Kung-Traub for constructing multi-point methods without memory. Per iteration, any method of the developed class is totally free from derivative evaluation. Such classes of schemes are more practical when the calculation of derivatives is hard. Error analysis will also be studied. Finally, numerical comparisons are made to reveal the reliability of the proposed class.

1 Introduction

The theoretical thorough study of iterative processes for simple roots goes back at least to the book of Traub [19]. Among questions and ideas which have been addressed, the problem of computing simple roots by multi-point without memory methods emerged. To illustrate further in [4], the authors have given two classes of n-step methods without memory; one including derivative calculation, also known as derivative-involved methods; and one derivative-free

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class. As an example, they gave the following family of one-parameter methods

$$\begin{cases} y_n = x_n + \beta f(x_n), \\ z_n = y_n - \beta \frac{f(x_n)f(y_n)}{f(y_n)-f(x_n)}, \\ w_n = z_n - \frac{f(x_n)f(y_n)}{f(z_n)-f(x_n)}\big[\frac{1}{f[y_n,x_n]} - \frac{1}{f[z_n,y_n]}\big], \\ x_{n+1} = w_n - \frac{f(x_n)f(y_n)f(z_n)}{f(w_n)-f(x_n)}\big[\frac{1}{f(w_n)-f(y_n)}\big\{\frac{1}{f[w_n,z_n]} - \frac{1}{f[z_n,y_n]}\big\} \\ -\frac{1}{f(z_n)-f(x_n)}\big\{\frac{1}{f[z_n,y_n]} - \frac{1}{f[y_n,x_n]}\big\}\big], \end{cases} \tag{1}$$

wherein $\beta \in \mathbb{R} - \{0\}$, by using inverse interpolation for annihilating the new-appeared first derivatives of the function in the Steffensen-Newton-Newton structure. They also conjectured that a multi-point iteration without memory can achieve the maximum order of convergence $2^{(n-1)}$, by consuming n, functional evaluations per full cycle. Therefore, (1)'s order end efficiency index are optimal.

Different methods of various orders have been introduced and improved by many authors. A complete review on the published papers in this field for the works from 2000 to 2010 have been given in the book of Iliev and Kyurkchiev [2]. In [8], the authors considered weight function approach to give some new classes of optimal Jarratt-type fourth-order methods. Authors in [13] studied a combination of last derivative approximation and weight function approach to furnish optimal eighth-order derivative-involved methods. Discussion on new multiple zero finders when the multiplicity of the roots is available have been recently introduced by Sharifi et al. in [5]. Note that Soleymani and Hosseinabadi in [9] presented a sixth-order derivative-free method including three steps. The references [10-12] also contain new derivative-free developments in this active topic of study. For more information, one may consult [15-18].

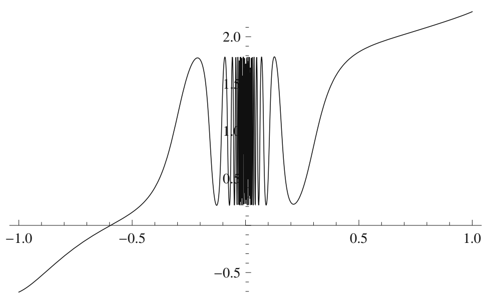

Figure 1. The graph of the function $f(x)$.

Derivative-free methods are important when we deal with complicated functions, such as $f(x) = \cos(\sin(x^2\sqrt{x})) \times \cos(x^3) \times \arctan(\sin(x^5 + x^{-1})) + x^3 + 1$, where its plot is given in Figure 1, or we try to find multiple roots of nonlinear equations (in the case that multiplicity is unknown), [1].

For this cause, this work is devoted to find an optimal three-step class of iterations without memory in which any method includes four function evaluations per cycle to obtain the eighth order of convergence and possess the optimal efficiency index 1.682. Toward this end, we make use of weight function approach alongside an approximation for the first derivative of the function for the quotients of a Steffensen-Newton-Newton structure. The efficiency of our class is then compared with those available in the literature to show better or equal results. Some methods from the suggested class are tested numerically in Section 3 to support the theoretical results given in Section 2. Section 4 includes a short conclusion of the article.

2 Main contribution

To construct a high-order class of methods for solving nonlinear scalar equations, we take into account the following three-step Steffensen-Newton-Newton structure

$$y_n = x_n - \frac{\beta f(x_n)^2}{f(w_n) - f(x_n)}, \quad z_n = y_n - \frac{f(y_n)}{f'(y_n)}, \quad x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}, \quad (2)$$

wherein $f(w_n) = f(x_n + \beta f(x_n))$; that is to say $w_n = x_n + \beta f(x_n)$, $\beta \in \mathbb{R} - \{0\}$. This structure possesses the eighth order of convergence, while it is inefficient. Because it includes 6 evaluations per step and its efficiency index therefore will be 1.4142, which is the same as Steffensen's and Newton's methods. Thus, in order to contribute and hit the assigned target, we should eliminate the existent of the derivative calculations without lowering the order, i.e. obtaining a class (family) of order eight with four evaluations of the function per full cycle only. There are many ways to do so. Among all, we first consider an approximation for the new-appeared first derivatives $f'(y_n)$ and $f'(z_n)$, and second make use of the approach of weight functions.

Let $f'(y_n) \approx (f(w_n) - f(x_n))/(\beta f(x_n))$, that is the same approximation as Steffensen used in the first step of (2). Then, an estimation of the function $f(t)$, in the open domain D, is taken into consideration as follows: $f(t) \approx w(t) = a_0 + a_1(t - y_n)$, which its first derivative is $w'(t) = a_1$. We suppose this estimation passes the points y_n and z_n. By substituting the known values $f(t)\,|_{y_n} = f(y_n), f(t)\,|_{z_n} = f(z_n)$, we could easily obtain the unknown parameters. Thus, we have $a_0 = f(y_n)$ and $a_1 = (f(y_n) - f(z_n))/(y_n - z_n) = f[y_n, z_n]$.

Consequently, we have $f'(z_n) \approx f[y_n, z_n]$. Therefore, we suggest the following iteration

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[x_n, w_n]}, w_n = x_n + \beta f(x_n), \beta \in \mathbb{R} - \{0\}, \\ z_n = y_n - \frac{f(y_n)}{f[x_n, w_n]} P(t), \\ x_{n+1} = z_n - \frac{f(z_n)}{f[y_n, z_n]} \big(G(\gamma) + H(t) + K(\zeta)\big), \end{cases} \tag{3}$$

where $t = f(y)/f(w)$, $\gamma = f(y)/f(x)$, and $\zeta = f(z)/f(w)$. $P(t)$, $G(\gamma)$, $H(t)$ and $K(\zeta)$ are four real-valued weight functions that should be chosen such that the order of convergence attains the value eight, this is the role of weight function approach. Taylor's series expansion around the solution for the first two steps of (3) gives us

$$\begin{aligned} e_{n+1} = &- ((c_2(1 + c_1\beta)(-1 + P(0))e_n^2)/c_1) + \\ &+ 1/c_1^2(-c_1c_3(1 + c_1\beta)(2 + c_1\beta)(-1 + P(0)) + \\ &+ c_2^2(-2 + 4P(0) + c_1\beta(-2 + 5P(0) + \\ &+ c_1\beta(-1 + 2P(0)) - P'(0)) - P'(0)))e_n^3 + O(e_n^4), \end{aligned} \tag{4}$$

where $c_j = f^{(j)}(\alpha)/j!$, $j \geq 1$, α is the solution. This shows that $P(0) = 1$, $P'(0) = 2 + \beta f[x_n, w_n]$ should be selected in order to attain at fourth-order convergence. By taking into account this, and similar expansion up to the seventh term, we obtain for (3) now that $G(0) = 1$, $G'(0) = H(0) = K(0) = H'(0) = H''(0) = G^{(3)}(0) = 0$, $K'(0) = 2 + \beta f[x_n, w_n]$ and $G''(0) = 2/(1 + \beta f[x_n, w_n])$ should be chosen in order to arrive at seventh-order convergence as follows

$$\begin{aligned} e_{n+1} = &\frac{-1}{12c_1^6}((1 + \beta c_1)c_2^4(-2c_1(1 + \beta c_1)c_3 + c_2^2(10 + 2\beta c_1(5 + \beta c_1) - \\ &- P''(0)))(3(2 + \beta c_1)(6 + 2\beta c_1(3 + \beta c_1) - \\ &- P''(0)) + H^{(3)}(0)))e_n^7 + O(e_n^8). \end{aligned} \tag{5}$$

Obviously, now to gain the optimal order eight with using only four evaluations of the function we should find $H^{(3)}(0)$ in such a way that order goes up to eight. This is summarized in Theorem 1.

Theorem 1 Let $\alpha \in D$, be a simple zero of sufficiently differentiable function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ and let $c_j = f^{(j)}(\alpha)/j!$, $j \geq 1$. If x_0 is sufficiently close to α, then, (i): the order of convergence of the solution by the three-step class of iterations without memory methods defined in (3) is eight, when $P(0) = 1$, $P'(0) = 2 + \beta f[x_n, w_n]$, $|P''(0)| < \infty$, $|P^{(3)}(0)| < \infty$, and

$$\begin{cases} G(0) = 1, G'(0) = G^{(3)}(0) = 0, G''(0) = \frac{2}{1+\beta f[x_n, w_n]}, \text{ and } |G^{(4)}(0)| < \infty, \\ H(0) = H'(0) = H''(0) = 0, \text{ and } |H^{(4)}(0)| < \infty, \\ H^{(3)}(0) = -3(2 + \beta f[x_n, w_n])(6 + 2\beta f[x_n, w_n](3 + \beta f[x_n, w_n]) - P''(0)), \\ K(0) = 0, K'(0) = 2 + \beta f[x_n, w_n], \end{cases} \tag{6}$$

and (ii): this solution reads the error equation

$$\begin{aligned} e_{n+1} = \frac{-1}{48c_1^7}&((1 + \beta c_1)c_2^2(-2c_1(1 + \beta c_1)c_3 + c_2^2(10 + 2\beta c_1(5 + \beta c_1) - P''(0))) \\ &\times(96\beta c_1(1 + \beta c_1)^2c_2c_3 - 24c_1^2(1 + \beta c_1)^2c_4 + c_2^3(-168 + 48P''(0) - 8P^{(3)}(0) \\ &+G^{(4)}(0) + c_1\beta(-4(84 + 3c_1\beta(16 + 2c_1\beta - P''(0)) - 12P''(0) + P^{(3)}(0)) \\ &+(2 + c_1\beta)(2 + c_1\beta(2 + c_1\beta))G^{(4)}(0)) + H^{(4)}(0))))e_n^8 + O(e_n^9). \end{aligned} \tag{7}$$

Proof. We expand any term of (3) around the solution α in the nth iterate by considering (6). Thus, we write

$$f(x_n) = c_1e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + c_7e_n^7 + c_8e_n^8 + O(e_n^9). \tag{8}$$

Accordingly, we attain

$$\begin{aligned} y_n = \alpha + (\beta + \frac{1}{c_1})c_2e_n^2 + \frac{(-(2 + (2 + \beta c_1)\beta c_1)c_2^2 + \beta c_1(1 + \beta c_1)(2 + \beta c_1)c_3)}{c_1^2}e_n^3 \\ + \ldots + O(e_n^9). \end{aligned} \tag{9}$$

Now we should expand $f(y_n)$ around the simple root by using (9). We obtain

$$\begin{aligned} f(y_n) = (1 + \beta c_1)c_2e_n^2 + (-\frac{(2 + \beta c_1(2 + \beta c_1))c_2^2}{c_1} + (1 + \beta c_1)(2 + \beta c_1)c_3)e_n^3 \\ +\frac{1}{c_1^2}(5 + \beta c_1(7 + \beta c_1(4 + \beta c_1)))c_2^3 - c_1c_2c_3(7 + \beta c_1(10 + \beta c_1(7 + 2\beta c_1))) \\ +c_1^2(1 + \beta c_1)(3 + \beta c_1(3 + \beta c_1))c_4)e_n^4 + \ldots + O(e_n^8). \end{aligned} \tag{10}$$

Using (10) and the second step of (3), we attain

$$y_n - \frac{f(y_n)}{f[x_n, w_n]} = \alpha + \frac{(1 + \beta c_1)(2 + \beta c_1)c_2^2}{c_1^2}e_n^3 + \ldots + O(e_n^9). \tag{11}$$

Additionally, we attain that

$$z_n = \alpha + \frac{((1+\beta c_1)c_2(-2c_1(1+\beta c_1)c_3 + c_2^2(10+2\beta c_1(5+\beta c_1) - P''(0))))}{2c_1^3} e_n^4 + \ldots + O(e_n^9). \quad (12)$$

Moreover, we obtain now

$$f(z_n) = \frac{(1+\beta c_1)c_2(-2c_1(1+\beta c_1)c_3 + c_2^2(10+2\beta c_1(5+\beta c_1) - P''(0)))}{2c_1^2} e_n^4$$
$$-\frac{1}{6c_1^3}(6c_1^2(1+\beta c_1)^2(2+\beta c_1)c_3^2 + 6c_1^2(1+\beta c_1)^2(2+\beta c_1)c_2c_4 - 3c_1(1+\beta c_1)c_2^2c_3(64$$
$$+2\beta c_1(46+\beta c_1(22+3\beta c_1)) - 3(2+\beta c_1)P''(0)) + c_2^4(6(36+c_1\beta(80+3c_1\beta(22+\beta c_1(8$$
$$+\beta c_1)))) - 3(10+3\beta c_1(5+2\beta c_1))P''(0) + (1+\beta c_1)P^{(3)}(0)))e_n^5 + \ldots + O(e_n^9). \quad (13)$$

Using (9)-(13), we have

$$z_n - \frac{f(z_n)}{f[y_n, z_n]} = \alpha$$
$$+\frac{((1+\beta c_1)^2c_2^3(-2c_1(1+\beta c_1)c_3 + c_2^2(10+2\beta c_1(5+\beta c_1) - P''(0))))e_n^6}{2c_1^5} + \ldots + O(e_n^9). \quad (14)$$

Now we also for the last step have (without considering (6)) $x_{n+1} - \alpha = -\frac{1}{2c_1^3}(c_2(1+c_1\beta)(-1+G(0)+H(0)+K(0))(-2c_1c_3(1+c_1\beta)+c_2^2(10+2c_1\beta(5+c_1\beta)-P''(0))))e_n^4 + \frac{1}{6c_1^4}(6c_1^2c_3^2(1+c_1\beta)^2(2+c_1\beta)(-1+G(0)+H(0)+K(0)) + 6c_1^2c_2c_4(1+c_1\beta)^2(2+c_1\beta)(-1+G(0)+H(0)+K(0)) - 3c_1c_2^2c_3(1+c_1\beta)(2(-32+32G(0)+32H(0)+32K(0)-G'(0)+c_1\beta(-46+46G(0)+46H(0)+46K(0)+c_1\beta((22+3c_1\beta)(-1+G(0)+H(0)+K(0))-G'(0))-2G'(0)-H'(0))-H'(0))-3(2+c_1\beta)(-1+G(0)+H(0)+K(0))P''(0))+c_2^4(6c_1^4\beta^4(3(-1+G(0)+H(0)+K(0))-G'(0))+6(-36+36G(0)+36H(0)+36K(0)-5G'(0)-5H'(0))+6c_1^3\beta^3(-24+24G(0)+24H(0)+24K(0)-7G'(0)-H'(0))+3(10-10G(0)-10H(0)-10K(0)+G'(0)+H'(0))P''(0)+3c_1^2\beta^2(4(-33+33G(0)+33H(0)+33K(0)-8G'(0)-3H'(0))+(-6(-1+G(0)+H(0)+K(0))+G'(0))P''(0))+(-1+G(0)+H(0)+K(0))P^{(3)}(0)+c_1\beta(30(16G(0)+16H(0)+16K(0)-3G'(0)-2(8+H'(0)))+3(15-15G(0)-15H(0)-15K(0)+2G'(0)+H'(0))P''(0)+(-1+G(0)+H(0)+K(0))P^{(3)}(0))))e_n^5 + \cdots + O(e_n^9)$. Therefore, by combining this, (14) and the terms of (6) in the last step of (3), we have the error equation (7). This completes the proof and shows that our multi-point class of methods arrives at optimal eighth-order convergence by using only four pieces of information and considering (6). □

Clearly, any method from our class of derivative-free methods reaches the optimal efficiency index $8^{1/4} \approx 1.682$, which is greater than that of Newton's and Steffensen's $2^{1/2} \approx 1.414$, $6^{1/4} \approx 1.565$ of the sixth-order methods given in [3, 9], $4^{1/3} \approx 1.587$ of method given in [14], and is equal to that of (1) and the classes of methods in [6, 7].

To provide the simplest case of our class of methods; by considering (6), we suggest the following method without memory including three steps

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[x_n,w_n]}, w_n = x_n + f(x_n), \\ z_n = y_n - \frac{f(y_n)}{f[x_n,w_n]}\{1 + (2 + f[x_n, w_n])\frac{f(y_n)}{f(w_n)}\}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f[y_n,z_n]}\{1 + \frac{1}{1+f[x_n,w_n]}(\frac{f(y_n)}{f(x_n)})^2 - ((2 + f[x_n, w_n])(3 \\ +f[x_n, w_n](3 + f[x_n, w_n])))(\frac{f(y_n)}{f(w_n)})^3 + (2 + f[x_n, w_n])\frac{f(z_n)}{f(w_n)}\}, \end{cases} \tag{15}$$

where its error equation satisfies

$$\begin{aligned} e_{n+1} = {} & (1/(c_1^7))(1 + c_1)^2 c_2^2((5 + c_1(5 + c_1))c_2^2 \\ & - c_1(1 + c_1)c_3)((7 + c_1(7 + c_1))c_2^3 - 4c_1(1 + c_1)c_2c_3 + \\ & + c_1^2(1 + c_1)c_4)e_n^8 + O(e_n^9). \end{aligned} \tag{16}$$

Remark 1. In order to implement and code the methods from the class (3), we should be careful that after computing $f[x_n, w_n]$ in the first step, its value will be used throughout the iteration step, which in fact does not increase the computational load of the novel optimal eighth-order derivative-free methods.

A very efficient but complicated optimal three-step method from the proposed class (3) can be

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[x_n,w_n]}, w_n = x_n + f(x_n), \\ z_n = y_n - \frac{f(y_n)}{f[x_n,w_n]}\{1 + (2 + f[x_n, w_n])\frac{f(y_n)}{f(w_n)} + (5 + f[x_n, w_n](5 \\ +f[x_n, w_n])) \left(\frac{f(y_n)}{f(w_n)}\right)^2\}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f[y_n,z_n]}\{1 + \frac{1}{1+f[x_n,w_n]}(\frac{f(y_n)}{f(x_n)})^2 - ((2 + f[x_n, w_n])(3 + f[x_n, w_n](3 \\ +f[x_n, w_n]) - 5 - f[x_n, w_n](5 + f[x_n, w_n])))(\frac{f(y_n)}{f(w_n)})^3 - (13 + f[x_n, w_n](26 \\ +f[x_n, w_n](21 + f[x_n, w_n](8 + f[x_n, w_n]))))(\frac{f(y_n)}{f(w_n)})^4 + (2 + f[x_n, w_n])\frac{f(z_n)}{f(w_n)}\}, \end{cases} \tag{17}$$

where its error equation satisfies

$$e_{n+1} = \frac{(1 + c_1)^4 c_2^2 c_3(4c_2c_3 - c_1c_4)}{c_1^5} e_n^8 + O(e_n^9). \tag{18}$$

We can easily now observe that the error equation (18) is very small. In fact, we have obtained the finest error equations for optimal three-step derivative-free methods without memory by introducing (17).

Note that if we choose very small value for the nonzero parameter β in (3), the error equations will be mostly refined and the numerical results will be better, for example we can have

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[x_n,w_n]}, w_n = x_n + 0.01f(x_n), \\ z_n = y_n - \frac{f(y_n)}{f[x_n,w_n]}\{1 + (2 + 0.01f[x_n,w_n])\frac{f(y_n)}{f(w_n)}\}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f[y_n,z_n]}\{1 + \frac{1}{1+0.01f[x_n,w_n]}(\frac{f(y_n)}{f(x_n)})^2 - ((2 + 0.01f[x_n,w_n])(3 \\ +0.01f[x_n,w_n](3 + 0.01f[x_n,w_n])))(\frac{f(y_n)}{f(w_n)})^3 + (2 + 0.01f[x_n,w_n])\frac{f(z_n)}{f(w_n)}\}. \end{cases} \quad (19)$$

Notice that if we use backward finite difference approximation in the first step of our cycle (2), by changing the weight functions suitably, we can give another class which is similar to (3), i.e. we can have

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[x_n,w_n]}, w_n = x_n - \beta f(x_n), \beta \in \mathbb{R} - \{0\}, \\ z_n = y_n - \frac{f(y_n)}{f[x_n,w_n]}P(t), \\ x_{n+1} = z_n - \frac{f(z_n)}{f[y_n,z_n]}\big(G(\gamma) + H(t) + K(\zeta)\big), \end{cases} \quad (20)$$

where $t = f(y)/f(w)$, $\gamma = f(y)/f(x)$, and $\zeta = f(z)/f(w)$. And $P(t)$, $G(\gamma)$, $H(t)$ and $K(\zeta)$ are four real-valued weight functions that should be chosen such that the order of convergence arrives at eight. This is illustrated in Theorem 2.

Theorem 2 Let $\alpha \in D$, be a simple zero of sufficiently differentiable function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ and let that $c_j = f^{(j)}(\alpha)/j!$, $j \geq 1$. If x_0 is sufficiently close to α, then, (i): the local order of convergence of the solution by the three-step class of without memory methods defined in (20) is eight, when $P(0) = 1$, $P'(0) = 2 - \beta f[x_n, w_n]$, $|P''(0)| < \infty$, $|P^{(3)}(0)| < \infty$, and

$$\begin{cases} G^{(3)}(0) = 0,\ |G(0)| < \infty,\ |G'(0)| < \infty,\ |G''(0)| < \infty,\ \text{and}\ |G^{(4)}(0)| < \infty, \\ H(0) = 1 - G(0) - K(0),\ \text{and}\ H'(0) = G'(0)(-1 + \beta f[x_n,w_n]), \\ H''(0) = -(-1 + \beta f[x_n,w_n])(2 + (-1 + \beta f[x_n,w_n])G''(0)),\ |H^{(4)}(0)| < \infty, \\ H^{(3)}(0) = 3(-2 + \beta f[x_n,w_n])(6 + 2\beta f[x_n,w_n](-3 + \beta f[x_n,w_n]) - P''(0)), \\ K'(0) = 2 - \beta f[x_n,w_n],\ |K(0)| < \infty, \end{cases} \quad (21)$$

and (ii): this solution reads the error equation

$$e_{n+1} = \frac{1}{48c_1^7}c_2^2(-1 + c_1\beta)(2c_1c_3(-1 + c_1\beta) + c_2^2(10 + 2c_1\beta(-5 + c_1\beta) - P''(0)))$$

$$\times(96c_1c_2c_3(-1+c_1\beta)^2-24c_1^2c_4(-1+c_1\beta)^2+c_2^3(-168+48P''(0)-8P^{(3)}(0)$$
$$+G^{(4)}(0)+c_1\beta(4(84-12P''(0)+3c_1\beta(-16+2c_1\beta+P''(0))+P^{(3)}(0))$$
$$+(-2+c_1\beta)(2+c_1\beta(-2+c_1\beta))G^{(4)}(0))+H^{(4)}(0)))e_n^8+O(e_n^9). \quad (22)$$

Proof. The proof of this theorem is similar to the previous one, hence it is omitted. □

An example from this new class using backward finite difference in this first step can be

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[x_n,w_n]}, w_n = x_n - f(x_n), \\ z_n = y_n - \frac{f(y_n)}{f[x_n,w_n]}\{1+(2-f[x_n,w_n])\frac{f(y_n)}{f(w_n)}\}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f[y_n,z_n]}\{1-(-1+f[x_n,w_n])(\frac{f(y_n)}{f(w_n)})^2-((-2+f[x_n,w_n])(3 \\ +f[x_n,w_n](-3+f[x_n,w_n])))(\frac{f(y_n)}{f(w_n)})^3+(2-f[x_n,w_n])\frac{f(z_n)}{f(w_n)}\}, \end{cases} \quad (23)$$

where its error equation satisfies

$$e_{n+1} = (1/(c_1^7))(-1+c_1)^2c_2^2((5+c_1(-5+c_1))c_2^2$$
$$-c_1(-1+c_1)c_3)((7+c_1(-7+c_1))c_2^3+4c_1(-1+c_1)c_2c_3 \quad (24)$$
$$-c_1^2(-1+c_1)c_4)e_n^8+O(e_n^9).$$

3 Numerical results

We check the effectiveness of the novel derivative-free method (15), (17) and (19) of our proposed class of iterative methods (3) here. Due to this, we have compared them with the optimal eighth-order family of Kung and Traub (1), where $\beta = 1$, using the examples given below. The reason that we do not include other root solvers for comparisons is that, the derivative-involved methods consists of derivative calculation, which is not mostly easy-to-calculate for hard test functions as well as the other existing derivative-free methods of lower orders do not have any dominance to the optimal 8th-order methods for sufficiently close initial guess.

$f_1(x) = \cos(\sin(x^2\sqrt{x})) \times \cos(x^3) \times \arctan(\sin(x^5+x^{-1})) + x^3 + 1$,
$\alpha_1 \approx -0.59\ldots$ $\quad x_0 = -0.65$,

$f_2(x) = \mathrm{arccot}(x^{-2}) + x^2 + x\sin(x^2) + x^3 - 6$,
$\alpha_2 \approx 1.27\ldots$ $\quad x_0 = 1.38$.

The results of comparisons are given in Tables 1 and 2 in terms of the number significant digits for each test function after the specified number of iterations, that is, e.g. $0.8e-3949$ shows that the absolute value of the given nonlinear function f_1, after four iterations is zero up to 3949 decimal places. For numerical comparisons, the stopping criterion is $|f(x_n)| < 1.E-6000$. MATLAB 7.6 has been used in all computations using VPA command. As can be seen, numerical results are in concordance with the theory developed in this paper.

In the examples, the new methods improve the corresponding classical methods. The new methods inherit the merit of the optimal fourth-order two-step methods with regards to application of divided differences, weight function and high efficiency index, which is confirmed by the results in Tables 1 and 2. According to Tables 1 and 2, under a fair comparison structure, our proposed methods from the optimal class (3) perform well.

We mention that our primary aim was to construct a general class of very efficient multi-point methods and to check the Kung-Traub conjecture for the value $n = 4$, not to show off with thousands of accurate decimal digits. The achieved accuracy of calculated approximations is certainly exceptional, maybe provocative. Nonetheless, it may initiate a new challenge for constructing more efficient methods.

Table 1. Convergence study for the test function f_1

Methods	$\lvert f_1(x_1)\rvert$	$\lvert f_1(x_2)\rvert$	$\lvert f_1(x_3)\rvert$	$\lvert f_1(x_4)\rvert$
(1)	$0.1e-6$	$0.6e-56$	$0.1e-450$	$0.1e-3608$
(15)	$0.3e-7$	$0.3e-61$	$0.3e-493$	$0.8e-3949$
(17)	$0.1e-6$	$0.2e-55$	$0.3e-446$	$0.8e-3573$
(19)	$0.2e-8$	$0.5e-71$	$0.7e-573$	$0.5e-4588$

Table 2. Convergence study for the test function f_2

Methods	$\lvert f_2(x_1)\rvert$	$\lvert f_2(x_2)\rvert$	$\lvert f_2(x_3)\rvert$	$\lvert f_2(x_4)\rvert$
(1)	$0.7e-5$	$0.1e-49$	$0.3e-407$	$0.4e-3268$
(15)	$0.3e-5$	$0.1e-51$	$0.4e-422$	$0.5e-3387$
(17)	$0.6e-5$	$0.7e-50$	$0.3e-409$	$0.3e-3284$
(19)	$0.1e-9$	$0.8e-91$	$0.1e-740$	$0.8e-5938$

Constructing with memory methods according to the main class (3) in this paper, by introducing an iteration for the nonzero parameter β can be considered for future works in this field.

4 Concluding remarks

In order to approximate the simple roots of uni-variate nonlinear equations, we have developed a class of four-point three-step methods in which no derivative evaluations per full iteration is required. Per cycle, any method of our class, such as (17), needs only four pieces of information to reach the convergence rate eight. Therefore, this class satisfies the conjecture of Kung-Traub for constructing optimal high-order multi-point without memory methods for solving nonlinear equations. Numerical examples were considered to reveal the accuracy of the methods from the class.

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References

[1] Y. Cui, L. Hou, X. Li, Sixth-order method for multiple roots, *Appl. Math. Comput. Sci.,* **2** (2010), 35–40.

[2] A. Iliev, N. Kyurkchiev, *Methods in Numerical Analysis: Selected Topics in Numerical Analysis*, LAP LAMBERT Academic Publishing, 2010.

[3] S. K. Khattri, I. K. Argyros, Sixth order derivative free family of iterative methods, *Appl. Math. Comput.,* **217** (2011), 5500–5507.

[4] H. T. Kung, J. F. Traub, Optimal order of one-point and multipoint iteration, *J. Assoc. Comput. Mach.*, **21** (1974), 643–651.

[5] M. Sharifi, D. K. R. Babajee, F. Soleymani, Finding the solution of nonlinear equations by a class of optimal methods, *Comput. Math. Appl.*, **63** (2012), 764–774.

[6] J. R. Sharma, R. Sharma, A new family of modified Ostrowski's methods with accelerated eighth order convergence, *Numer. Algorithms,* **54** (2010), 445–458.

[7] F. Soleymani, S. Karimi Vanani, Optimal Steffensen-type methods with eighth order of convergence, *Comput. Math. Appl.*, **62** (2011), 4619–4626.

[8] F. Soleymani, S. K. Khattri, S. Karimi Vanani, Two new classes of optimal Jarratt-type fourth-order methods, *Appl. Math. Lett.*, **25** (2012), 847–853.

[9] F. Soleymani, V. Hosseniabadi, New third- and sixth-order derivative-free techniques for nonlinear equations, *J. Math. Research,* **3** (2011), 107–112.

[10] F. Soleymani, On a bi-parametric class of optimal eighth-order derivative-free methods, *Int. J. Pure Appl. Math.*, **72** (2011), 27–37.

[11] F. Soleymani, An optimally convergent three-step class of derivative-free methods, *World Appl. Sci. J.*, **13** (2011), 2515–2521.

[12] F. Soleymani, V. Hosseniabadi, Robust cubically and quartically iterative techniques free from derivative, *Proyecciones*, **30** (2011), 149–161.

[13] F. Soleymani, S. Karimi Vanani, M. Khan, M. Sharifi, Some modifications of King's family with optimal eighth order of convergence, *Math. Comput. Modelling*, **55** (2012), 1373–1380.

[14] F. Soleymani, On a novel optimal quartically class of methods, *Far East J. Math. Sci. (FJMS)*, **58** (2011), 199–206.

[15] F. Soleymani, Novel computational iterative methods with optimal order for nonlinear equations, *Adv. Numer. Anal.*, (2011), Article ID 270903, 10 pages.

[16] F. Soleymani, R. Sharma, X. Li, E. Tohidi, An optimized derivative-free form of the Potra-Ptak method, *Math. Comput. Modelling*, (2011), (accepted for publication)

[17] F. Soleymani, Regarding the accuracy of optimal eighth-order methods, *Math. Comput. Modelling,* **53** (2011), 1351–1357.

[18] F. Soleymani, Concerning some sixth-order iterative methods for finding the simple roots of nonlinear equations, *Bull. Math. Anal. Appl.,* **2** (2010), 146–151.

[19] J. F. Traub, *Iterative methods for the solution of equations,* Prentice Hall, Englewood Cliffs, N.J. 1964.

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Uniqueness and weighted value sharing of differential polynomials of meromorphic functions

Abhijit Banerjee
Department of Mathematics
West Bengal State University
Barasat, North 24 Prgs.-700126
West Bengal, India.
email: abanerjee_kal@yahoo.co.in

Pulak Sahoo
Department of Mathematics
Netaji Subhas Open University
1-Woodburn Park, Kolkata-700020
West Bengal, India.
email: sahoopulak1@gmail.com

Abstract. In the paper, with the aid of weighted sharing we investigate the uniqueness problems of meromorphic functions concerning differential polynomials that share one value and prove three uniqueness results which rectify, improve and supplement some recent results of [3].

1 Introduction

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [7, 18, 21]. Let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a non-constant meromorphic function f, we denote by $T(r,f)$ the Nevanlinna characteristic of f and by $S(r,f)$ any quantity satisfying $S(r,f) = o\{T(r,f)\}(r \to \infty, r \notin E)$.

Let f and g be two non-constant meromorphic functions. We say that f and g share the value a CM (counting multiplicities), if $f - a$ and $g - a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share

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the value a IM, provided that $f - a$ and $g - a$ have the same zeros ignoring multiplicities. Throughout this paper, we need the following definition.

$$\Theta(a, f) = 1 - \limsup_{r \longrightarrow \infty} \frac{\overline{N}(r, a; f)}{T(r, f)},$$

where a is a value in the extended complex plane.

In the recent past a number of authors worked on the uniqueness problem of meromorphic functions when differential polynomials generated by them share certain values (cf. [1, 2, 4, 5, 8, 11]). In [8] following question was asked: What can be said if two nonlinear differential polynomials generated by two meromorphic functions share 1 CM ?

Since then the progress to investigate the uniqueness of meromorphic functions which are the generating functions of different types of nonlinear differential polynomials is remarkable and continuous efforts are being put in to relax the hypothesis of the results. (see [1], [4], [5], [14], [15]). In 1997, Yang and Hua [17] proved the following result.

Theorem 1 *Let* f *and* g *be two non-constant meromorphic functions,* $n(\geq 11)$ *an integer and* $a \in \mathbb{C} - \{0\}$. *If* $f^n f'$ *and* $g^n g'$ *share the value* a *CM, then either* $f = tg$ *for some* $(n+1)$*th root of unity 1 or* $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, *where* c, c_1, c_2 *are constants satisfying* $(c_1 c_2)^{n+1} c^2 = -a^2$.

In 2004 Lin-Yi [15] proved the following results.

Theorem 2 *Let* f *and* g *be two non-constant meromorphic functions satisfying* $\Theta(\infty, f) > 2/(n+1)$, $n \geq 12$ *an integer. If* $f^n(f-1)f'$ *and* $g^n(g-1)g'$ *share the value* 1 *CM, then* $f \equiv g$.

Theorem 3 *Let* f *and* g *be two non-constant meromorphic functions,* $n \geq 13$ *an integer. If* $f^n(f-1)^2 f'$ *and* $g^n(g-1)^2 g'$ *share the value* 1 *CM, then* $f \equiv g$.

Also in [4] Fang-Fang proved the following theorem.

Theorem 4 *Let* f *and* g *be two non-constant meromorphic functions and* $n(\geq 28)$ *be an integer. If* $f^n(f-1)^2 f'$ *and* $g^n(g-1)^2 g'$ *share the value* 1 *IM, then* $f \equiv g$.

Recently, in [3] Dyavanal proved the following results, which to the knowledge of the authors probably are the first approach in which in order to consider the value sharing of two differential polynomials the multiplicities of zeros and poles of f and g are taken into account.

Theorem 5 (Theorem 1.1, [3]) *Let* f *and* g *be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least* s*, where* s *is a positive integer. Let* $n \geq 2$ *be an integer satisfying* $(n+1)s \geq 12$*. If* $f^n f'$ *and* $g^n g'$ *share the value* 1 *CM, then either* $f = dg$ *for some* $(n+1)$*-th root of unity 1 or* $f(z) = c_1 e^{cz}$*,* $g(z) = c_2 e^{-cz}$*, where* c*,* c_1*,* c_2 *are constants satisfying* $(c_1 c_2)^{n+1} c^2 = -1$.

Theorem 6 (Theorem 1.2, [3]) *Let* f *and* g *be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least* s*, where* s *is a positive integer and* $\Theta(\infty, f) > 2/(n+1)$*. Let* $n \geq 4$ *be an integer satisfying* $(n+1)s \geq 12$*. If* $f^n(f-1)f'$ *and* $g^n(g-1)g'$ *share the value* 1 *CM, then* $f \equiv g$.

Theorem 7 (Theorem 1.3, [3]) *Let* f *and* g *be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least* s*, where* s *is a positive integer. Let* $n \geq 3$ *be an integer satisfying* $(n+1)s \geq 12$*. If* $f^n(f-1)^2 f'$ *and* $g^n(g-1)^2 g'$ *share the value* 1 *CM, then* $f \equiv g$.

The results are new and seems fine. However in page 7, in the proof of Theorem 1.2 [3] there is a serious lacuna when a counting function is being elaborated and then restricted in terms of Nevanlinna's characteristic function.

Actually in Page 7, line 8 onwards from bottom should be

$$\overline{N}\left(r, \frac{1}{F}\right) = \overline{N}\left[r, \frac{1}{f^{n+1}(f - \frac{n+2}{n+1})}\right] \leq \frac{1}{s(n+1)} N\left(r, \frac{1}{f^{n+1}}\right)$$
$$+\overline{N}\left(r, \frac{1}{f - \frac{n+2}{n+1}}\right) \not\leq \frac{1}{s(n+1)} N\left(r, \frac{1}{F}\right),$$

since nowhere in the paper it has been assumed that the zeros of $f - \dfrac{n+2}{n+1}$ are of multiplicities $s(n+1)$. Since the counting function just mentioned above plays a vital role in the proofs of *Theorems 1.2, 1.3* and *1.5* in [3], the validity of the three theorems namely *Theorems 1.2, 1.3* and *1.5* in [3] cease to hold.

So it would be quite natural to investigate the accurate forms of the above theorems and at the same time to combine all the theorems in [3] to a single one. In this paper we are taking up these problems. In fact, we will not only rectify the above three theorems but also improve and supplement all the theorems of [3] by relaxing the nature of sharing the values with the aid of the notion of weighted sharing of values defined as follows.

Definition 1 *Let* k *be a nonnegative integer or infinity. For* $a \in \mathbb{C} \cup \{\infty\}$ *we denote by* $E_k(a; f)$ *the set of all* a*-points of* f *where an* a*-point of multiplicity m is counted m times if* $m \leq k$ *and k+1 times if* $m > k$. *If* $E_k(a; f) = E_k(a; g)$, *we say that* f, g *share the value* a *with weight k.*

The definition implies that if f, g *share a value* a *with weight* k, *then* z_0 *is an* a*-point of* f *with multiplicity* $m(\leq k)$ *if and only if it is an* a*-point of* g *with multiplicity* $m(\leq k)$ *and* z_0 *is an a-point of* f *with multiplicity* $m(> k)$ *if and only if it is an a-point of* g *with multiplicity* $n(> k)$, *where m is not necessarily equal to n.*

We write f, g *share* (a, k) *to mean that* f, g *share the value* a *with weight k. Clearly if* f, g *share* (a, k) *then* f, g *share* (a, p) *for any integer* p, $0 \leq p < k$. *Also we note that* f, g *share a value* a *IM or CM if and only if* f, g *share* $(a, 0)$ *or* (a, ∞) *respectively.*

We now state the main results of the paper.

Theorem 8 *Let* f *and* g *be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least* s, *where* s *is a positive integer. Let* $f^n(f-1)^m f'$ *and* $g^n(g-1)^m g'$ *share* $(b, 2)$ *where* $m \geq 0$, $n > \max\{m + 1 + 2m/s, m + 1 + 9/s\}$ *are integers and* $b(\neq 0)$ *is a constant. Then each of the following holds:*
(i) If $m = 0$, *then either* $f = tg$ *for some* $(n+1)$*-th root of unity 1 or* $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, *where* c, c_1, c_2 *are constants satisfying* $(c_1 c_2)^{n+1} c^2 = -b^2$.
(ii) If $m = 1$ *and* $\Theta(\infty, f) + \Theta(\infty, g) > 4/(n+1)$ *or* $m = 2$, *then* $f \equiv g$.
(iii) If $m \geq 3$, *then*

$$f^{n+1} \sum_{i=0}^{m} {}^mC_i \frac{(-1)^i}{n+m-i+1} f^{m-i} \equiv g^{n+1} \sum_{i=0}^{m} {}^mC_i \frac{(-1)^i}{n+m-i+1} g^{m-i}.$$

Remark 1 *Putting* $s = 1$ *in the above theorem we get the rectified, improved as well as generalised form of all the theorems in [3].*

Theorem 9 *Let* f *and* g *be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least* s, *where* s *is a positive integer. Let* $f^n(f-1)^m f'$ *and* $g^n(g-1)^m g'$ *share* $(b, 1)$ *where* $m \geq 0$, $n > \max\{m + 1 + 2m/s, m + 2 + 21/2s\}$ *are integers and* $b(\neq 0)$ *is a constant. Then the conclusions (i)-(iii) of Theorem 8 hold.*

Theorem 10 *Let* f *and* g *be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least* s, *where* s *is a positive integer. Let*

$f^n(f-1)^m f'$ *and* $g^n(g-1)^m g'$ *share* $(b, 0)$ *where* $m \geq 0$, $n > \max\{m+1+2m/s, m+7+18/s\}$ *are integers and* $b(\neq 0)$ *is a constant. Then the conclusions (i)-(iii) of Theorem 8 hold.*

Though we use the standard notations and definitions of the value distribution theory available in [7], we explain the following definition and notation which is used in the paper.

Definition 2 [13] *Let* p *be a positive integer or infinity. We denote by* $N_p(r, a; f)$ *the counting function of* a*-points of* f*, where an* a*-point of multiplicity* m *is counted* m *times if* $m \leq p$ *and* p *times if* $m > p$*. Then*

$$N_p(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2) + ... + \overline{N}(r, a; f \mid\geq p).$$

2 Lemmas

In this section we present some lemmas which will be needed to prove the theorem.

Lemma 1 [16] *Let* f *be a non-constant meromorphic function and* $P(f) = a_0 + a_1 f + a_2 f^2 + ... + a_n f^n$*, where* a_0*,* a_1*,* a_2*, ... ,* $a_n(\neq 0)$ *are constants. Then*

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

Lemma 2 [19] *Let* f *be a non-constant meromorphic function. Then*

$$N\left(r, 0; f^{(k)}\right) \leq k\overline{N}(r, \infty; f) + N(r, 0; f) + S(r, f).$$

Lemma 3 [22] *Let* f *be a non-constant meromorphic function and* p*,* k *be a positive integers. Then*

$$N_p\left(r, 0; f^{(k)}\right) \leq k\overline{N}(r, \infty; f) + N_{p+k}(r, 0; f) + S(r, f).$$

Lemma 4 [9] *Let* f *and* g *be two non-constant meromorphic functions sharing* $(1, 2)$*. Then one of the following cases holds:*
(i) $T(r) \leq N_2(r, 0; f) + N_2(r, 0; g) + N_2(r, \infty; f) + N_2(r, \infty; g) + S(r)$,
(ii) $f \equiv g$,
(iii) $fg \equiv 1$.

Lemma 5 [1] *Let* f *and* g *be two non-constant meromorphic functions sharing* $(1, m)$ *and*

$$\frac{f''}{f'} - \frac{2f'}{f-1} \not\equiv \frac{g''}{g'} - \frac{2g'}{g-1}.$$

Now the following hold:
(i) if $m = 1$ *then* $T(r, f) \leq N_2(r, 0; f) + N_2(r, 0; g) + N_2(r, \infty; f) + N_2(r, \infty; g) + \frac{1}{2}\overline{N}(r, 0; f) + \frac{1}{2}\overline{N}(r, \infty; f) + S(r, f) + S(r, g)$;
(ii) if $m = 0$ *then* $T(r, f) \leq N_2(r, 0; f) + N_2(r, 0; g) + N_2(r, \infty; f) + N_2(r, \infty; g) + 2\overline{N}(r, 0; f) + \overline{N}(r, 0; g) + 2\overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + S(r, f) + S(r, g)$.

Lemma 6 [20] *Let* f *and* g *be two non-constant meromorphic functions. If*

$$\frac{f''}{f'} - \frac{2f'}{f-1} \equiv \frac{g''}{g'} - \frac{2g'}{g-1}$$

and

$$\limsup_{r\to\infty, r\notin E} \frac{\overline{N}(r, 0; f) + \overline{N}(r, 0; g) + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g)}{T(r)} < 1$$

then either $f \equiv g$ *or* $fg \equiv 1$, *where* $T(r)$ *is the maximum of* $T(r, f)$ *and* $T(r, g)$.

Lemma 7 *Let* f *and* g *be two non-constant meromorphic functions whose zeros and poles are of multiplicities at least* s, *where* s *is a positive integer. Let* n *and* m *are positive integers such that* $n > m + 1 + 2m/s$. *Then*

$$f^n(f-1)^m f' g^n (g-1)^m g' \not\equiv b^2,$$

where b *is a nonzero constant.*

Proof. We suppose that

$$f^n(f-1)^m f' g^n (g-1)^m g' \equiv b^2. \tag{1}$$

Let z_0 be a zero of f with multiplicity $p_0(\geq s)$. Then z_0 is a pole of g with multiplicity $q_0(\geq s)$, say. From (1) we obtain

$$np_0 + p_0 - 1 = (n + m + 1)q_0 + 1$$

and so

$$(n+1)(p_0 - q_0) = mq_0 + 2. \tag{2}$$

From (2) we get $q_0 \geq \frac{n-1}{m}$ and so again from (2) we obtain

$$p_0 \geq q_0 + 1 \geq \frac{n+m-1}{m}.$$

Let z_1 be a zero of $f-1$ with multiplicity p_1. Then z_1 is a pole of g with multiplicity $q_1 (\geq s)$, say. So from (1) we get

$$(m+1)p_1 - 1 \quad = \quad (n+m+1)q_1 + 1$$

which gives

$$p_1 \geq \frac{(n+m+1)s+2}{m+1}.$$

Since a pole of f is either a zero of $g^n(g-1)^m$ or a zero of g', we have

$$\begin{aligned}
\overline{N}(r,\infty;f) &\leq \overline{N}(r,0;g) + \overline{N}(r,1;g) + \overline{N}_0(r,0;g') \\
&\leq \frac{m}{n+m-1}N(r,0;g) + \frac{m+1}{(n+m+1)s+2}N(r,1;g) \\
&\quad + \overline{N}_0(r,0;g') \\
&\leq \left(\frac{m}{n+m-1} + \frac{m+1}{(n+m+1)s+2}\right)T(r,g) + \overline{N}_0(r,0;g'),
\end{aligned}$$

where $\overline{N}_0(r,0;g')$ denotes the reduced counting function of those zeros of g' which are not the zeros of $g(g-1)$.

Then by the second fundamental theorem of Nevanlinna we get

$$\begin{aligned}
T(r,f) &\leq \overline{N}(r,\infty;f) + \overline{N}(r,0;f) + \overline{N}(r,1;f) - \overline{N}_0(r,0;f') + S(r,f) \\
&\leq \left(\frac{m}{n+m-1} + \frac{m+1}{(n+m+1)s+2}\right)\{T(r,f)+T(r,g)\} \\
&\quad + \overline{N}_0(r,0;g') - \overline{N}_0(r,0;f') + S(r,f). \qquad (3)
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
T(r,g) &\leq \left(\frac{m}{n+m-1} + \frac{m+1}{(n+m+1)s+2}\right)\{T(r,f)+T(r,g)\} \\
&\quad + \overline{N}_0(r,0;f') - \overline{N}_0(r,0;g') + S(r,g). \qquad (4)
\end{aligned}$$

Adding (3) and (4) we obtain

$$\left(1 - \frac{2m}{n+m-1} - \frac{2(m+1)}{(n+m+1)s+2}\right)\{T(r,f)+T(r,g)\} \leq S(r,f) + S(r,g),$$

which leads to a contradiction as $n > m + 1 + 2m/s$. This proves the lemma. $\square$

Lemma 8 *Let* f *and* g *be two non-constant entire functions and* n *be a positive integer. If* $f^n f' g^n g' = b^2$*, where* b *is a nonzero constant, then* $f(z) = c_1 e^{cz}$*,* $g(z) = c_2 e^{-cz}$*, where* c*,* c_1*,* c_2 *are constants satisfying* $(c_1 c_2)^{n+1} c^2 = -b^2$*.*

Proof. We omit the proof since it can be proved in the line of the proof of Theorem 3 in [17]. □

Lemma 9 *Let* f *and* g *be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least* s*, where* s *is a positive integer and*

$$F = f^{n+1}\left(\sum_{i=0}^{m} {}^{m}C_i \frac{(-1)^i}{n+m-i+1} f^{m-i}\right);$$

$$G = g^{n+1}\left(\sum_{i=0}^{m} {}^{m}C_i \frac{(-1)^i}{n+m-i+1} g^{m-i}\right).$$

Further let $F_0 = \frac{F'}{b}$ *and* $G_0 = \frac{G'}{b}$*, where* $b(\neq 0)$ *is a constant. Then* $S(r, F_0)$ *and* $S(r, G_0)$ *are replaceable by* $S(r, f)$ *and* $S(r, g)$ *respectively.*

Proof. By Lemma 1 we get

$$\begin{aligned} T(r, F_0) &\leq T(r, F') + S(r, f) \\ &\leq 2T(r, F) + S(r, f) \\ &= 2(n+m+1)T(r, f) + S(r, f) \end{aligned}$$

and similarly

$$T(r, G_0) \leq 2(n+m+1)T(r, g) + S(r, g).$$

This proves the lemma. □

Lemma 10 *Let* F*,* G*,* F_0 *and* G_0 *be defined as in Lemma 9. We define* $F = f^{n+1}F_1$ *and* $G = g^{n+1}G_1$ *where*
$F_1 = \sum_{i=0}^{m} {}^{m}C_i \frac{(-1)^i}{n+m-i+1} f^{m-i}$ *and* $G_1 = \sum_{i=0}^{m} {}^{m}C_i \frac{(-1)^i}{n+m-i+1} g^{m-i}$*.*
Then
(i) $T(r, F) \leq T(r, F_0) + N(r, 0; f) + N(r, 0; F_1) - mN(r, 1; f) - N(r, 0; f') + S(r, f)$*,*
(ii) $T(r, G) \leq T(r, G_0) + N(r, 0; g) + N(r, 0; G_1) - mN(r, 1; g) - N(r, 0; g') + S(r, g)$*.*

Proof. We prove (i) only as the proof of (ii) is similar. By Nevanlinna's first fundamental theorem and Lemma 1 we get

$$\begin{aligned} T(r,F) &= T\left(r,\frac{1}{F}\right)+O(1) \\ &= N(r,0;F)+m\left(r,\frac{1}{F}\right)+O(1) \\ &\leq N(r,0;F)+m\left(r,\frac{F_0}{F}\right)+m(r,0;F_0)+O(1) \\ &= N(r,0;F)+T(r,F_0)-N(r,0;F_0)+S(r,F) \\ &= T(r,F_0)+N(r,0;f)+N(r,0;F_1)-mN(r,1;f) \\ &\quad -N(r,0;f')+S(r,f). \end{aligned}$$

This proves the lemma. □

Lemma 11 *Let* F *and* G *be defined as in Lemma 9, where* $m(\geq 0)$ *and* $n(\geq m+3/s)$ *are positive integers. Then* $F'\equiv G'$ *implies* $F\equiv G$.

Proof. Let $F'\equiv G'$. Then $F\equiv G+C$, where C is a constant. If possible, we suppose that $C\neq 0$. Then by the second fundamental theorem of Nevanlinna we get

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,C;F)+S(r,F) \\ &\leq \overline{N}(r,0;f)+\overline{N}(r,0;F_1)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g) \\ &\quad +\overline{N}(r,0;G_1)+S(r,f) \\ &\leq \frac{1}{s}N(r,0;f)+\overline{N}(r,0;F_1)+\frac{1}{s}N(r,\infty;f)+\frac{1}{s}N(r,0;g) \\ &\quad +\overline{N}(r,0;G_1)+S(r,f) \\ &\leq (m+2/s)T(r,f)+(m+1/s)T(r,g)+S(r,f), \end{aligned}$$

where F_1 and G_1 are defined as in Lemma 9. So by Lemma 1 we have

$$(n+1-2/s)T(r,f)\leq (m+1/s)T(r,g)+S(r,f). \quad (5)$$

Similarly, it can be shown that

$$(n+1-2/s)T(r,g)\leq (m+1/s)T(r,f)+S(r,g). \quad (6)$$

Adding (5) and (6) we obtain

$$(n-m+1-3/s)\{T(r,f)+T(r,g)\}\leq S(r,f)+S(r,g),$$

which is a contradiction. Therefore $C=0$ and the lemma follows. □

Lemma 12 *Let* f *and* g *be two non-constant meromorphic functions such that*

$$\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n+1},$$

where $n(\geq 2)$ *is an integer. Then*

$$f^{n+1}(af + b) \equiv g^{n+1}(ag + b)$$

implies $f \equiv g$*, where* a*,* b *are two nonzero constants.*

Proof. We omit the proof since it can be carried out in the line of Lemma 6 [12]. □

Lemma 13 [6] *Let*

$$Q(w) = (n-1)^2(w^n - 1)(w^{n-2} - 1) - n(n-2)(w^{n-1} - 1)^2,$$

then

$$Q(w) = (w-1)^4(w - \nu_1)(w - \nu_2)......(w - \nu_{2n-6}),$$

where $\nu_j \in C \setminus \{0, 1\}$ $(j = 1, 2, ..., 2n-6)$*, which are distinct respectively.*

3 Proof of the Theorem

Proof of Theorem 8. Let F, G, F_0 and G_0 be defined as in Lemma 9. Since F_0 and G_0 share $(1, 2)$, one of the possibilities of Lemma 4 holds. We suppose that

$$\begin{aligned} T_0(r) \leq\ & N_2(r, 0; F_0) + N_2(r, 0; G_0) + N_2(r, \infty; F_0) + N_2(r, \infty; G_0) \\ & + S(r, F_0) + S(r, G_0), \end{aligned} \tag{7}$$

where $T_0(r) = max\{T(r, F_0), T(r, G_0)\}$. Now by Lemma 2, Lemma 9 and Lemma 10 we get from (7)

$$\begin{aligned}
T(r,F) &\le T(r,F_0) + N(r,0;f) + N(r,0;F_1) - mN(r,1;f) \\
&\quad -N(r,0;f') + S(r,f) \\
&\le N_2(r,0;F_0) + N_2(r,0;G_0) + N_2(r,\infty;F_0) + N_2(r,\infty;G_0) \\
&\quad +N(r,0;f) + N(r,0;F_1) - mN(r,1;f) - N(r,0;f') \\
&\quad +S(r,f) + S(r,g) \\
&\le 2\overline{N}(r,0;f) + mN(r,1;f) + N(r,0;f') + 2\overline{N}(r,\infty;f) \\
&\quad +2\overline{N}(r,0;g) + mN(r,1;g) + N(r,0;g') + 2\overline{N}(r,\infty;g) \\
&\quad +N(r,0;f) + N(r,0;F_1) - mN(r,1;f) - N(r,0;f') \\
&\quad +S(r,f) + S(r,g) \\
&= \frac{2}{s}N(r,0;f) + \frac{2}{s}N(r,\infty;f) + N(r,0;f) + N(r,0;F_1) \\
&\quad +\frac{2}{s}N(r,0;g) + mN(r,1;g) + N(r,0;g') + \frac{2}{s}N(r,\infty;g) \\
&\quad +S(r,f) + S(r,g) \\
&\le \{m+1+4/s\}T(r,f) + \{m+1+5/s\}T(r,g) \\
&\quad +S(r,f) + S(r,g) \\
&\le \{2m+2+9/s\}T(r) + S(r),
\end{aligned}$$

where $T(r)$ is defined as in Lemma 6. So by Lemma 1 we obtain

$$(n+m+1)T(r,f) \le \{2m+2+9/s\}T(r) + S(r). \tag{8}$$

Similarly we get

$$(n+m+1)T(r,g) \le \{2m+2+9/s\}T(r) + S(r). \tag{9}$$

From (8) and (9) we see that

$$[n-m-1-9/s]T(r) \le S(r),$$

which is a contradiction. Hence (7) does not hold. So by Lemma 4 either $F_0G_0 \equiv 1$ or $F_0 \equiv G_0$. Suppose that $F_0G_0 \equiv 1$. Then

$$f^n(f-1)^m f' g^n(g-1)^m g' \equiv b^2. \tag{10}$$

If $m \ge 1$, by Lemma 7 we arrive at a contradiction. If $m = 0$, by (10) we get

$$f^n f' g^n g' \equiv b^2. \tag{11}$$

Let z_0 be a zero of f with multiplicity $p(\geq s)$. Then z_0 is a pole of g with multiplicity $q(\geq s)$, say. From (11) we obtain

$$np + p - 1 = nq + q + 1$$

and so $(n+1)(p-q) = 2$, which is impossible as $n \geq 2$ and p, q are positive integers. Therefore, we conclude that f and g are entire functions. Hence using Lemma 8, we get $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c, c_1, c_2 are constants satisfying $(c_1 c_2)^{n+1} c^2 = -b^2$.

Now we assume that $F_0 \equiv G_0$. And so by Lemma 11 we get $F \equiv G$, that is

$$f^{n+1}\left(\sum_{i=0}^{m} {}^{m}C_i \frac{(-1)^i}{n+m-i+1} f^{m-i}\right) \equiv g^{n+1}\left(\sum_{i=0}^{m} {}^{m}C_i \frac{(-1)^i}{n+m-i+1} g^{m-i}\right). \tag{12}$$

We now consider following three cases.

Case 1 *Let* $m = 0$. *Then from (12) we obtain* $f^{n+1} = g^{n+1}$, *which gives* $f = tg$ *for some* $(n+1)$ *-th root of unity 1.*

Case 2 *Let* $m = 1$. *From (12) we obtain*

$$f^{n+1}\left(\frac{1}{n+2} f - \frac{1}{n+1}\right) = g^{n+1}\left(\frac{1}{n+2} g - \frac{1}{n+1}\right),$$

which together with

$$\Theta(\infty, f) + \Theta(\infty, g) > 4/(n+1)$$

and Lemma 12 gives $f \equiv g$.

Case 3 *Let* $m = 2$. *Suppose that* $h = \frac{f}{g}$. *By (12) we get*

$$(n+2)(n+1)g^2(h^{n+3} - 1) - 2(n+3)(n+1)g(h^{n+2} - 1) + (n+2)(n+3)(h^{n+1} - 1) = 0. \tag{13}$$

By (13) and by Lemma 13, we can conclude that

$$\{(n+1)(n+2)(h^{n+3} - 1)g - (n+3)(n+1)(h^{n+2} - 1)\}^2 = -(n+3)(n+1)Q(h),$$

where $Q(h) = (h-1)^4(h-\nu_1)(h-\nu_2)......(h-\nu_{2n})$, *where* $\nu_j \in C \setminus \{0,1\}$ $(j = 1, 2, ..., 2n)$, *which are pairwise distinct.*

If h *is not a constant, this implies that every zero of* $h - \nu_j$ $(j = 1, 2, ..., 2n)$, *has a multiplicity of at least 2. By the second fundamental theorem of Nevanlinna we obtain that* $n \leq 2$, *which is again a contradiction. Therefore,* h *is a constant. We have from (13) that* $h^{n+1} - 1 = 0$ *and* $h^{n+2} - 1 = 0$, *which imply* $h = 1$, *and hence* $f \equiv g$.

This completes the proof of theorem 8. □

Proof of Theorem 9. We put

$$H = \left(\frac{F_0''}{F_0'} - \frac{2F_0'}{F_0 - 1}\right) - \left(\frac{G_0''}{G_0'} - \frac{2G_0'}{G_0 - 1}\right).$$

We suppose that $H \not\equiv 0$. Since F_0 and G_0 share $(1, 1)$, by Lemma 2, Lemma 5(i), Lemma 9 and Lemma 10 we get

$$\begin{aligned} T(r,F) &\leq T(r,F_0) + N(r,0;f) + N(r,0;F_1) - mN(r,1;f) \\ &\quad -N(r,0;f') + S(r,f) \\ &\leq N_2(r,0;F_0) + N_2(r,0;G_0) + N_2(r,\infty;F_0) + N_2(r,\infty;G_0) \\ &\quad +\frac{1}{2}\overline{N}(r,0;F_0) + \frac{1}{2}\overline{N}(r,\infty;F_0) + N(r,0;f) + N(r,0;F_1) \\ &\quad -mN(r,1;f) - N(r,0;f') + S(r,f) + S(r,g) \\ &\leq 2\overline{N}(r,0;f) + mN(r,1;f) + N(r,0;f') + 2\overline{N}(r,\infty;f) \\ &\quad +2\overline{N}(r,0;g) + mN(r,1;g) + N(r,0;g') + 2\overline{N}(r,\infty;g) \\ &\quad +\frac{1}{2}\overline{N}(r,0;f) + \frac{1}{2}\overline{N}(r,1;f) + \frac{1}{2}\overline{N}(r,0;f') + \frac{1}{2}\overline{N}(r,\infty;f) \\ &\quad +N(r,0;f) + N(r,0;F_1) - mN(r,1;f) - N(r,0;f') \\ &\quad +S(r,f) + S(r,g) \\ &\leq (m + 2 + 11/2s)T(r,f) + (m + 1 + 5/s)T(r,g) \\ &\quad +S(r,f) + S(r,g) \\ &\leq (2m + 3 + 21/2s)T(r) + S(r). \end{aligned}$$

So by Lemma 1 we get

$$(n + m + 1)T(r,f) \leq (2m + 3 + 21/2s)T(r) + S(r).$$

Similarly we get

$$(n + m + 1)T(r,g) \leq (2m + 3 + 21/2s)T(r) + S(r).$$

Combining the above two inequalities we obtain

$$(n-m-2-21/2s)T(r) \leq S(r),$$

which is a contradiction. Hence $H \equiv 0$.
Now by Lemma 1 we get

$$\begin{aligned}(n+m)T(r,f) &= T\left(r, f^{n}(f-1)^{m}\right) + S(r,f)\\ &\leq T(r,F') + T(r,f') + S(r,f)\\ &\leq T(r,F_0) + 2T(r,f) + S(r,f)\end{aligned}$$

and so

$$T(r,F_0) \geq (n+m-2)T(r,f) + S(r,f).$$

Similarly we get

$$T(r,G_0) \geq (n+m-2)T(r,g) + S(r,g).$$

Also from Lemma 2 we have

$$\begin{aligned}&\overline{N}(r,0;F_0) + \overline{N}(r,\infty;F_0) + \overline{N}(r,0;G_0) + \overline{N}(r,\infty;G_0)\\ &\quad\leq \overline{N}(r,0;f) + \overline{N}(r,1;f) + \overline{N}(r,0;f') + \overline{N}(r,\infty;f) + \overline{N}(r,0;g)\\ &\qquad + \overline{N}(r,1;g) + \overline{N}(r,0;g') + \overline{N}(r,\infty;g) + S(r,f) + S(r,g)\\ &\quad\leq (2+3/s)T(r,f) + (2+3/s)T(r,g) + S(r,f) + S(r,g)\\ &\quad\leq \frac{4+6/s}{n+m-2}T_0(r) + S(r),\end{aligned}$$

where $S_0(r) = o\{T_0(r)\}$ as $r \to \infty$ possibly outside a set of finite linear measure and $\epsilon(>0)$ is sufficiently small.
In view of the hypothesis we get from above

$$\limsup_{r\to\infty, r\notin E} \frac{\overline{N}(r,0;F_0) + \overline{N}(r,\infty;F_0) + \overline{N}(r,0;G_0) + \overline{N}(r,\infty;G_0)}{T_0(r)} < 1.$$

So by Lemma 6 we obtain either $F_0G_0 \equiv 1$ or $F_0 \equiv G_0$. Now by using Lemma 7, Lemma 8, Lemma 11 and proceeding in the same way as the proof of Theorem 8, we easily obtain the results of Theorem 9. □

Proof of Theorem 10. Using Lemma 5(ii) the theorem can be proved as the proof of Theorem 9. □

References

[1] A. Banerjee, Meromorphic functions sharing one value, *Int. J. Math. Math. Sci.*, **22** (2005), 3587–3598.

[2] A. Banerjee, On uniqueness for non-linear differential polynomials sharing the same 1-point, *Ann. Polon. Math.*, **89** (2006), 259–272.

[3] R. S. Dyavanal, Uniqueness and value sharing of differential polynomials of meromorphic functions, *J. Math. Anal. Appl.*, **372** (2010), 252–261.

[4] C. Y. Fang, M. L. Fang, Uniqueness of meromorphic functions and differential polynomials, *Comput. Math. Appl.*, **44** (2002), 607–617.

[5] M. L. Fang, W. Hong, A unicity theorem for entire functions concerning differential polynomials, *Indian J. Pure Appl. Math.*, **32** (2001), 1343–1348.

[6] G. Frank, M. Reinders, A unique range set for meromorphic function with 11 elements, *Complex Variables*, **37** (1998), 185–193.

[7] W. K. Hayman, *Meromorphic Functions*, The Clarendon Press, Oxford, 1964.

[8] I. Lahiri, Uniqueness of meromorphic functions when two linear differential polynomials share the same 1-points, *Ann. Polon. Math.*, **71** (1999), 113–128.

[9] I. Lahiri, Weighted value sharing and uniqueness of meromorphic functions, *Complex Var. Theory Appl.*, **46** (2001), 241–253.

[10] I. Lahiri, Weighted sharing and uniqueness of meromorphic functions, *Nagoya Math. J.*, **161** (2001), 193–206.

[11] I. Lahiri, Linear differential polynomials sharing the same 1-points with weight two, *Ann. Polon. Math.*, **79** (2002), 157–170.

[12] I. Lahiri, On a question of Hong Xun Yi, *Arch. Math. (Brno)*, **38** (2002), 119–128.

[13] I. Lahiri, Weighted sharing of three values, *Z. Anal. Anwendungen*, **23** (2004), 237–252.

[14] I. Lahiri, P. Sahoo, Uniqueness of meromorphic functions when two non-linear differential polynomials share a small function, *Arch. Math. (Brno)*, **44** (2008), 201–210.

[15] W. C. Lin, H. X. Yi, Uniqueness theorems for meromorphic functions, *Indian J. Pure Appl. Math.*, **35** (2004), 121–132.

[16] C. C. Yang, On deficiencies of differential polynomials II, *Math. Z.*, **125** (1972), 107–112.

[17] C.C. Yang, X. Hua, Uniqueness and value sharing of meromorphic functions, *Ann. Acad. Sci. Fenn. Math.*, **22** (1997), 395–406.

[18] L. Yang, *Value distribution theory*, Springer-Verlag / Science Press, Berlin / Beijing, 1993.

[19] H. X. Yi, Uniqueness of meromorphic functions and a question of C. C. Yang, *Complex Var. Theory Appl.*, **14** (1990), 169–176.

[20] H. X. Yi, Meromorphic functions that share one or two values, *Complex Var. Theory Appl.*, **28** (1995), 1–11.

[21] H.X. Yi, C.C. Yang, *Uniqueness theory of meromorphic functions*, Science Press, Beijing, 1995.

[22] Q.C. Zhang, Meromorphic function that shares one small function with its derivative, *J. Inequal. Pure Appl. Math.*, **6** (2005), Art. 116.

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Asymptotic behaviour of solutions of third order nonlinear differential equations

A. T. Ademola
Department of Mathematics
University of Ibadan
Ibadan, Nigeria
email: ademola672000@yahoo.com

P. O. Arawomo
Department of Mathematics
University of Ibadan
Ibadan, Nigeria
email: womopeter@gmail.com

Abstract. In this paper, Lyapunov direct method was employed. We present criteria for all solutions $x(t)$ its first and second derivatives of the third order nonlinear non autonomous differential equations to converge to zero as $t \to \infty$. Sufficient conditions are also established for the boundedness and uniform ultimate boundedness of solutions of the equations considered. Our results revise, improve and generalize existing results in the literature.

1 Introduction

Nonlinear differential equations of higher order have been extensively studied with high degree of generality. In particular, boundedness, uniform boundedness, ultimate boundedness, uniform ultimate boundedness and asymptotic behaviour of solutions have in the past and also recently been discussed by remarkable authors, see for instance Reissig *et al.* [18], Rouche *et al.* [19], Yoshizawa [26] and [27] where the general results were discussed. Authors that have worked on the qualitative behaviour of solutions of third order nonlinear differential equations include Ademola *et al.* [1, 2, 3, 4, 5, 6], Chukwu [7], Ezeilo [8, 9, 10, 11, 12], Hara [13], Mehri and Shadman [14], Omeike [15, 16], Qian [17], Swick [20, 21, 22], Tejumola [23] and Tunç [24, 25]. Complete and

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incomplete Lyapunov functions were constructed and used by these authors to establish their results. The nonlinear differential equations considered are the types where the restoring nonlinear terms do not depend explicitly on the independent real variable t, except in [1, 2, 4, 13] and [14] where the restoring nonlinear terms depend or multiplied by functions of t.

Till now, according to our observation from the relevant literature, the problem of boundedness (where the bounding constant depends on the solutions in question), uniform ultimate boundedness and asymptotic behaviour of solutions of the nonlinear non autonomous third order differential equation considered, have so far remained open. In this paper therefore, using Lyapunov direct method, a complete Lyapunov function was constructed and used to obtain criteria for boundedness, uniform ultimate boundedness and asymptotic behaviour of solutions of the third order nonlinear differential equation

$$x''' + \psi(t)f(x,x',x'')x'' + \phi(t)g(x,x') + \varphi(t)h(x,x',x'') = p(t,x,x',x'') \quad (1)$$

or its equivalent system

$$x' = y, y' = z, z' = p(t,x,y,z) - \psi(t)f(x,y,z)z - \phi(t)g(x,y) + \varphi(t)h(x,y,z) \quad (2)$$

in which $p \in C(\mathbb{R}^+ \times \mathbb{R}^3, \mathbb{R})$; $f, h \in C(\mathbb{R}^3, \mathbb{R})$; $g \in C(\mathbb{R}^2, \mathbb{R})$; $\phi, \varphi, \psi \in C(\mathbb{R}^+, \mathbb{R})$; $\mathbb{R} = (-\infty, \infty)$; $\mathbb{R}^+ = [0, \infty)$; the functions $\phi, \varphi, \psi, f, g, h$ and p depend only on the arguments displaced explicitly. The derivatives $\frac{\partial}{\partial x}f(x,y,z) = f_x(x,y,z)$, $\frac{\partial}{\partial y}f(x,y,z) = f_y(x,y,z)$, $\frac{\partial}{\partial z}f(x,y,z) = f_z(x,y,z)$, $\frac{\partial}{\partial x}g(x,y) = g_x(x,y)$,
$\frac{\partial}{\partial x}h(x,y,z) = h_x(x,y,z)$, $\frac{\partial}{\partial y}h(x,y,z) = h_y(x,y,z)$, $\frac{\partial}{\partial z}h(x,y,z) = h_z(x,y,z)$, $\frac{d}{dt}\psi(t) = \psi'(t)$, $\frac{d}{dt}\phi(t) = \phi'(t)$ and $\frac{d}{dt}\varphi(t) = \varphi'(t)$ exist and are continuous for all x, y, z and t. As usual, condition for uniqueness will be assumed and x', x'', x'''as elsewhere, stand for differentiation with respect to the independent variable t. Motivation for this studies comes from the works of Hara [13], Omeike [15, 16],Tunç [24, 25] and the recent work of Ademola and Arawomo [4] where conditions for stability and uniform ultimate boundedness of solutions of (1) were proved. Our results revise and improve the results in [4] and extend the results in [13, 14, 15, 16, 24] and [25].

2 Preliminaries

Consider the system of the form

$$X'(t) = F(t, X(t)) \quad (3)$$

where $F \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $\mathbb{R}^n$ is the $n-$dimensional Euclidean space.

Definition 1 *A solution* $X(t; t_0, X_0)$ *of* (3) *is bounded, if there exists a* $\beta > 0$ *such that* $\|X(t; t_0, X_0)\| < \beta$ *for all* $t \geq t_0$ *where* β *may depend on each solution.*

Definition 2 *The solutions* $X(t; t_0, X_0)$ *of* (3) *are uniformly bounded, if for any* $\alpha > 0$ *and* $t_0 \in \mathbb{R}^+$, *there exists a* $\beta(\alpha) > 0$ *such that if* $\|X_0\| < \alpha$ $\|X(t; t_0, X_0)\| < \beta$ *for all* $t \geq t_0$.

Definition 3 *The solutions of* (3) *are uniformly ultimately bounded for bound* B *if there exists a* $B > 0$ *and if corresponding to any* $\alpha > 0$ *and* $t_0 \in \mathbb{R}^+$, *there exists a* $T(\alpha) > 0$ *such that if* $\|X_0\| < \alpha$ *implies that* $\|X(t; t_0, X_0)\| < B$ *for all* $t \geq t_0 + T(\alpha)$.

Definition 4 (i) *A function* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, *continuous, strictly increasing with* $\phi(0) = 0$, *is said to be a function of class* $\mathbb{K}$ *for such function, we shall write* $\phi \in \mathbb{K}$.
(ii) *If in addition to (i)* $\phi(r) \to +\infty$ *as* $r \to \infty$, ϕ *is said to be a function of class* $\mathbb{K}^*$ *and we write* $\phi \in \mathbb{K}^*$.

The following lemmas are very important in the proofs of our results.

Lemma 1 [27] Suppose that there exists a Lyapunov function $V(t, X)$ defined on $\mathbb{R}^+, \|X\| \geq \rho$ were $\rho > 0$ may be large which satisfies the following conditions:

(i) $a(\|X\|) \leq V(t, X) \leq b(\|X\|)$, $a \in \mathbb{K}^*$ and $b \in \mathbb{K}$;

(ii) $V'_{(3)}(t, X) \leq 0$, for all $(t, X) \in \mathbb{R}^+ \times \mathbb{R}^n$.

Then the solutions of (3) are uniformly bounded.

Lemma 2 [27] If in addition to assumption (i) of Lemma 1, $V'_{(3)}(t, X) \leq -c(\|X\|)$, $c \in \mathbb{K}$ for all $(t, X) \in \mathbb{R}^+ \times \mathbb{R}^n$. Then the solutions of (3) are uniformly ultimately bounded.

Let Q be an open set in $\mathbb{R}^n$ and $Q^* \subset Q$. Consider a system of differential equation

$$X'(t) = F(t, X(t)) + G(t, X(t)) \tag{4}$$

where F, G are defined and continuous on $\mathbb{R}^+ \times Q$.

Definition 5 *A scalar function $W(X)$ defined for $X \in Q$ is said to be positive definite with respect to a set S, if $W(X) = 0$ for $X \in S$ and if corresponding to each $\epsilon > 0$ and each compact set Q^* in Q there exists a positive number $\delta(\epsilon, Q^*)$ such that*

$$W(X) \geq \delta(\epsilon, Q^*)$$

for $X \in Q^* - N(\epsilon, S)$. $N(\epsilon, S)$ is the ϵ neighborhood of S.

Let Ω be a closed set in Q, we have the following lemma

Lemma 3 *Suppose that there exist a nonnegative Lyapunov function $V(t, X)$ defined on $\mathbb{R}^+ \times Q$ such that*

$$V'_{(4)}(t, X) \leq -W(X)$$

where $W(X)$ is positive definite with respect to a closed set Ω in the space $\mathbb{R}^n$. Moreover suppose that $F(t, X)$ of system (4) is bounded for all t when X belongs to an arbitrary compact set in Q and that $F(t, X)$ satisfies conditions:

(i) *$F(t, X)$ tends to a function $H(X)$ for $X \in \Omega$ as $t \to \infty$ and on any compact set in Ω this convergence is uniform;*

(ii) *Corresponding to each $\epsilon > 0$ and each $Y \in \Omega$ there exists a $\delta(\epsilon, Y) > 0$ and a $T(\epsilon, Y) > 0$ such that if $\|X - Y\| < \delta(\epsilon, Y)$ and $t \geq T(\epsilon, Y)$, we have*

$$\|F(t, X) - F(t, Y)\| < \epsilon.$$

Then every bounded solution of (4) approaches the largest semi-invariant set of the system

$$X' = H(X), \ X \in \Omega \tag{5}$$

as $t \to \infty$. In particular, if all solutions of (4) are bounded, every solution of (4) approaches the largest semi-invariant set of (5) contained in Ω as $t \to \infty$.

3 Statement of Results

We have the following results

Theorem 1 *Further to the basic assumptions on the functions f, g, h, ϕ, φ and ψ appearing in (2), suppose that $a, a_1, b, b_1, c, \delta_0, \epsilon, \phi_0, \phi_1, \varphi_0, \varphi_1, \psi_0$ and ψ_1, are positive constants such that for all $t \geq 0$:*

(i) $a \le f(x,y,z) \le a_1$ *for all* x, y, z;

(ii) $b \le g(x,y)/y \le b_1$ *for all* $x, y \ne 0$;

(iii) $\psi_0 \le \psi(t) \le \psi_1,\ \phi_0 \le \phi(t) \le \phi_1,\ \varphi_0 \le \varphi(t) \le \varphi_1$;

(iv) $h(0,0,0) = 0,\ \delta_0 \le h(x,y,z)/x$ *for all* $x \ne 0, y$ *and* z;

(v) $\sup_{t \ge 0}[|\psi'(t)| + |\phi'(t)| + |\varphi'(t)|] < \epsilon$;

(vi) $g_x(x,y) \le 0,\ yf_x(x,y,z) \le 0,\ h_x(x,0,0) \le c$ *for all* x, y *and* $ab > c$;

(vii) $h_y(x,y,0) \ge 0,\ h_z(x,0,z) \ge 0,\ yf_z(x,y,z) \ge 0$ *for all* x, y, z;

(viii) $\int_0^\infty |p(t,x,y,z)|dt < \infty$.

Then the solution $(x(t), y(t), z(t))$ *of* (2) *is uniformly ultimately bounded.*

Theorem 2 *In addition to the assumptions of Theorem 1,* $g(0,0) = 0$, *then every solution* $(x(t), y(t), z(t))$ (2) *is uniformly bounded and satisfies*

$$\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} y(t) = 0, \quad \lim_{t\to\infty} z(t) = 0 \tag{6}$$

Theorem 3 *Suppose that* $a, b, c, \delta_0, \epsilon, \phi_0, \varphi_0, \varphi_1$ *and* ψ_0 *are positive constants such that for all* $t \ge 0$:

(i) *assumptions* (iv)-(viii) *of Theorem 1 hold;*

(ii) $a \le f(x,y,z)$ *for all* x, y, z;

(iii) $b \le g(x,y)/y$ *for all* x *and* $y \ne 0$;

(iv) $\phi_0 \le \phi(t),\ \varphi_0 \le \varphi(t) \le \varphi_1,\ \psi_0 \le \psi(t)$.

Then any solution $(x(t), y(t), z(t))$ *of* (2) *with initial conditions*

$$x(0) = x_0, \quad y(0) = y_0, \quad z(0) = z_0, \tag{7}$$

satisfies

$$|x(t)| \le D, \quad |y(t)| \le D, \quad |z(t)| \le D, \tag{8}$$

for all $t \ge 0$, *where the constant* $D > 0$ *depends on* $a, b, c, \delta_0, \epsilon, \phi_0, \varphi_0, \varphi_1, \psi_0$ *as well as on* t_0, x_0, y_0, z_0 *and on the function* p *appearing in* (2).

If the function $p(t, x, y, z) \equiv p(t) \neq 0$, (2) reduces to

$$x' = y, y' = z, z' = p(t) - \psi(t)f(x, y, z)z - \phi(t)g(x, y) + \varphi(t)h(x, y, z) \quad (9)$$

where $p \in C(\mathbb{R}^+, \mathbb{R})$, with the following results:

Corollary 1 *If hypotheses* (i)-(vii) *of Theorem 1 hold true, and in addition* $\int_0^\infty |p(t)|dt < \infty$, *then the solution* $(x(t), y(t), z(t))$ *of* (9) *is uniformly ultimately bounded.*

Corollary 2 *If in addition to assumptions of Corollary 1,* $g(0, 0) = 0$, *then every solution* $(x(t), y(t), z(t))$ *of* (9) *is uniformly bounded and satisfies* (6).

Corollary 3 *Suppose that* $a, b, c, \delta_0, \epsilon, \phi_0, \varphi_0, \varphi_1$ *and* ψ_0 *are positive constants such that for all* $t \geq 0$:

(i) *assumptions* (iv)-(vii) *of Theorem 1 hold;*

(ii) *assumptions* (ii)-(iv) *of Theorem 3 hold;*

(iii) $\int_0^\infty |p(t)|dt < \infty$.

Then every solution $(x(t), y(t), z(t))$ *of* (9) *with initial conditions* (7) *satisfies* (8) *for all* $t \geq 0$ *where* $D > 0$ *is a constant depending on* $a, b, c, \delta_0, \epsilon, \phi_0, \varphi_0, \varphi_1$, ψ_0 *as well as on* t_0, x_0, y_0, z_0 *and on the function* p *appearing in* (9).

Remark 1 (i) *The results in [5],[10]-[13] and [21] are special cases of Theorem 1. Also, if* $\phi(t) = \varphi(t) = \psi(t) \equiv 1$, *system* (2) *specializes to that discussed by Ademola and Arawomo [3] (the generalization of the results of Omeike [15] and Tunç [24]). Moreover, in [4] Ademola and Arawomo studied stability and uniform ultimate boundedness of solutions of* (2). *Theorem 1 revises Theorem 6 in [4]. In particular, the main tool used in this investigation weaken the hypothesis on the function* p *compared with the result in [4].*

(ii) *If* $f(x, y, z) \equiv p(t)$, $g(x, y) \equiv g(y)$, $h(x, y, z) \equiv h(x)$ *and* $p(t, x, y, z) \equiv 0$ *system* (2) *specializes to that discussed by Swick [22]. His result in Theorem 1 is a special case of Theorem 2. Moreover, if* $f(x, y, z) \equiv a$ $a > 0$ *is a constant or* $p(t)$, $g(x, y) \equiv yg(x)$ *or* $g(y)$, $p(t, x, y, z) \equiv e(t)$ *and* $\varphi(t) = \psi(t) \equiv 1$ *system* (2) *reduces to that discussed by Swick [20]. Moreover, when* $p(t, x, y, z) \equiv 0$ *in* (2) *conditions under which all solutions* $x(t)$, *its first and second derivatives converge to zero as* $t \to \infty$

had been discussed by Ademola and Arawomo [4]. Furthermore, whenever $f(x,y,z) \equiv \psi(x,y)$ *or* $\psi(x,y,z)$, $h(x,y,z) \equiv 0$ *and* $p(t,x,y,z) \equiv p(t)$ *system* (2) *specializes to that studied by Omeike [16], Qian [17] and Tunç [24]. Hence, Theorem 2 revises, improves and generalizes the results in [4, 16, 17, 20] and [24].*

(iii) *The results of Ademola* et al. *[5], Mehri and Shadman [14] and Swick [22] Theorem 5 are all special cases of Theorem 3.*

The proofs of our results depend on the function $V = V(t, x(t), y(t), z(t))$ defined as

$$V = e^{-P_*(t)}U \tag{10a}$$

where

$$P_*(t) = \int_0^t |p(\mu, x, y, z)| d\mu \tag{10b}$$

and the function $U \equiv U(t, x(t), y(t), z(t))$

$$\begin{aligned} 2U &= 2(\alpha + a\psi(t))\varphi(t)\int_0^x h(\xi, 0, 0)d\xi + 4\varphi(t)yh(x, 0, 0) \\ &+ 4\phi(t)\int_0^y g(x,\tau)d\tau + 2(\alpha + a\psi(t))\psi(t)\int_0^y \tau f(x,\tau,0)d\tau \\ &+ 2z^2 + \beta y^2 + b\beta\phi(t)x^2 + 2a\beta\psi(t)xy + 2\beta xz + 2(\alpha + a\psi(t))yz \end{aligned} \tag{10c}$$

where α and β are positive fixed constants satisfying

$$\frac{\varphi_1 c}{\phi_0 b} < \alpha < \psi_0 a \tag{10d}$$

and

$$0 < \beta < \min\left\{b\phi_0, (ab\psi_0\phi_0 - c\varphi_1)\eta_1^{-1}, \frac{1}{2}(a\psi_0 - \alpha)\eta_2^{-1}\right\} \tag{10e}$$

where

$$\eta_1 := 1 + a\psi_1 + \delta_0^{-1}\varphi_0^{-1}\phi_0^2\left(\frac{g(x,y)}{y} - b\right)^2 \text{ and } \eta_2 := 1 + \delta_0^{-1}\varphi_0^{-1}\psi_0^2[f(x,y,z) - a]^2.$$

Remark 2 *If* $t = 0$ *in* (10b), (10a) *coincides with* (10c) *and the main tool used in [4].*

Next, we shall show that (10) and its time derivative along a solution of (2) satisfy some fundamental inequalities as presented in the following lemma.

Lemma 4 *If all the hypotheses of Theorem 1 hold true, then for the function* V *defined in* (10) *there exist positive constants* $D_1 > 0,\ D_2 > 0$ *such that*

$$D_1(x^2(t) + y^2(t) + z^2(t)) \le V(t, x, y, z) \le D_2(x^2(t) + y^2(t) + z^2(t)) \quad (11a)$$

and

$$V(t, x(t), y(t), z(t)) \to +\infty \text{ as } x^2(t) + y^2(t) + z^2(t) \to \infty. \quad (11b)$$

Furthermore, there exists a finite constant $D_3 > 0$ *such that along a solution of* (2)

$$V' \equiv \frac{d}{dt} V(t, x(t), y(t), z(t)) \le -D_3(x^2(t) + y^2(t) + z^2(t)). \quad (11c)$$

Proof. Since $h(0, 0, 0) = 0$, (10c) can be rearranged in the form

$$\begin{aligned} 2U = &\frac{2\varphi(t)}{b\phi(t)} \int_0^x [(\alpha + a\psi(t))b\phi(t) - 2\varphi(t)h_\xi(\xi, 0, 0)]h(\xi, 0, 0)d\xi \\ &+ 4\phi(t)\int_0^y \left(\frac{g(x, \tau)}{\tau} - b\right)\tau d\tau + 2b^{-1}\phi^{-1}(t)[\varphi(t)h(x, 0, 0) + b\phi(t)y]^2 \\ &+ 2\int_0^y [(\alpha + a\psi(t))\psi(t)f(x, \tau, 0) - (\alpha^2 + a^2\psi^2(t))]\tau d\tau \\ &+ (\alpha y + z)^2 + (\beta x + a\psi(t)y + z)^2 + \beta[b\phi(t) - \beta]x^2 + \beta y^2. \end{aligned}$$

In view of the hypotheses of Theorem 1 this equation becomes

$$\begin{aligned} U \ge &\frac{1}{2}\Big\{[(\alpha + a\psi_0)b\phi_0 - 2\varphi_1 c]b^{-1}\phi_0^{-1}\varphi_0\delta_0 + \beta(b\phi_0 - \beta)\Big\}x^2 \\ &+ \frac{1}{2}\Big[\alpha(a\psi_0 - \alpha) + \beta\Big]y^2 + b^{-1}\phi_0^{-1}[\delta_0\varphi_0 x + b\phi_0 y]^2 \\ &+ \frac{1}{2}(\alpha y + z)^2 + \frac{1}{2}(\beta x + a\psi_0 y + z)^2. \end{aligned} \quad (12)$$

From (10d) and (10e) $\alpha b\phi_0 > c\varphi_1$, $ab\phi_0\psi_0 > c\varphi_1$, $a\psi_0 > \alpha$ and $b\phi_0 > \beta$ respectively, so that the quadratic in the right hand side of the inequality (12) is positive definite, hence there exists a positive constant $\lambda_0 = \lambda_0(a, b, c, \alpha, \beta, \delta_0, \phi_0, \varphi_0, \varphi_1, \psi_0)$ such that

$$U \ge \lambda_0(x^2 + y^2 + z^2) \quad (13a)$$

for all $t \geq 0, x, y$ and z. From hypothesis (viii) of Theorem 1 and (10b) there exists a constant $P_0 > 0$ such that

$$0 \leq P_*(t) \leq P_0 \tag{13b}$$

for all $t \geq 0$.Now, using (13) in (10a) we obtain

$$V \geq \delta_1(x^2 + y^2 + z^2) \tag{14a}$$

for all $t \geq 0, x, y$ and z, where $\delta_1 := \lambda_0 \exp[-P_0] > 0$. This establishes the lower inequality in (11a). From (14a), estimate (11b) follows immediately i.e

$$V(t, x, y, z) \to +\infty \text{ as } x^2 + y^2 + z^2 \to \infty. \tag{14b}$$

Furthermore, $h(0, 0, 0) = 0$ implies that $h(x, 0, 0) \leq cx$ for all $x \neq 0$, using this estimate, the hypotheses of Theorem 1 and the inequalities $2|xy| \leq x^2 + y^2$, $2|xz| \leq x^2 + z^2$ and $2|yz| \leq y^2 + z^2$, (10c) yields

$$U \leq \delta_2(x^2 + y^2 + z^2) \tag{15}$$

for all $t \geq 0, x, y$ and z, where $\delta_2 := \frac{1}{2}\max\{\lambda_1, \lambda_2, \lambda_3\} > 0$, $\lambda_1 = (2 + \alpha + a\psi_1)c\varphi_1 + (1 + a\psi_1 + b\phi_1)\beta$, $\lambda_2 = (\alpha + a\psi_1)(1 + a_1\psi_1) + (1 + a\psi_1)\beta + 2(b_1\phi_1 + c\varphi_1)$ and $\lambda_3 = 2 + \alpha + \beta + a\psi_1$. Using estimates (13b) and (15) in (10a), we obtain

$$V \leq \delta_2(x^2 + y^2 + z^2) \tag{16}$$

for all $t \geq 0, x, y$ and z. Thus by (16), the upper inequality in (11a) is established.

Moreover, the derivative of V along a solution $(x(t), y(t), z(t))$ of (2), with respect to t is given by

$$V'_{(2)} = -e^{-P_*(t)}\left[U|p(t, x, y, z)| - U'_{(2)}\right], \tag{17}$$

where $P_*(t)$ and U are the functions defined in (10b) and (10c) respectively and the derivative of the function U with respect to t, along a solution of (2) is after simplifying

$$\begin{aligned} U'_{(2)} &= \sum_{i=1}^{3} U_i - U_4x^2 - U_5y^2 - U_6z^2 - U_7 \\ &- \beta\phi(t)\left[\frac{g(x, y)}{y} - b\right]xy - \beta\psi(t)[f(x, y, z) - a]xz \\ &+ [\beta x + [\alpha + a\psi(t)]y + 2z]p(t, x, y, z), \end{aligned} \tag{18}$$

where:

$$U_1 := \left[2\int_0^y g(x,\tau)d\tau + \frac{1}{2}b\beta x^2\right]\phi'(t) + \left[[\alpha + a\psi(t)]\int_0^x h(\xi,0,0)d\xi\right.$$
$$\left. + 2yh(x,0,0) + ayz\right]\varphi'(t) + \left[a\varphi(t)\int_0^x h(\xi,0,0)d\xi + a\beta xy\right.$$
$$\left. + [\alpha + 2a\psi(t)]\int_0^y \tau f(x,\tau,0)d\tau\right]\psi'(t);$$
$$U_2 := a\beta\psi(t)y^2 + 2\beta yz;$$
$$U_3 := 2\phi(t)y\int_0^y g_x(x,\tau)d\tau + [\alpha + a\psi(t)]\psi(t)y\int_0^y \tau f(x,\tau,0)d\tau;$$
$$U_4 := \beta\varphi(t)\frac{h(x,y,z)}{x}, \quad (x \neq 0);$$
$$U_5 := [\alpha + a\psi(t)]\phi(t)\frac{g(x,y)}{y} - 2\varphi(t)h_x(x,0,0), \quad (y \neq 0);$$
$$U_6 := 2\psi(t)f(x,y,z) - [\alpha + a\psi(t)]$$

and

$$U_7 := \varphi(t)[[\alpha + a\psi(t)]y + 2z][h(x,y,z) - h(x,0,0)]$$
$$+[\alpha + a\psi(t)]\psi(t)yz[f(x,y,z) - f(x,y,0)].$$

In view of the hypotheses of Theorem 1, we have the following estimates for U_i $(i = 1,2,\cdots,6)$:

$$U_1 \leq \epsilon\lambda_4(x^2 + y^2 + z^2)$$

for all $t \geq 0, x, y$ and z, where $\lambda_4 := \max\{\lambda_{41}, \lambda_{42}, \lambda_{43}\} > 0$, $\lambda_{41} := \max\{\frac{1}{2}b\beta, b_1, 1\}$, $\lambda_{42} := \frac{1}{2}\max\{(\alpha + a\psi_1 + 2)c, a + 2c, a\}$ and $\lambda_{43} := \frac{1}{2}\max\{a(\beta + c\varphi_1), a\beta + (\alpha + 2a\psi_1)a_1, 1\}$;

$$U_2 \leq \beta[(1 + a\psi_1)y^2 + z^2]$$

for all $t \geq 0, x$ and y;

$$U_3 \leq 0$$

for all $t \geq 0, x$ and y;

$$U_4 \geq \beta\delta_0\varphi_0$$

for all $t \geq 0, x \neq 0, y$ and z;

$$U_5 \geq (\alpha + a\psi_0)b\phi_0 - 2c\varphi_1$$

for all $t \geq 0, x$ and y;

$$U_6 \geq a\psi_0 - \alpha$$

for all $t \geq 0, x, y$ and z. Finally by the mean value theorem and the hypotheses of Theorem 1, we have

$$U_7 = [\alpha + a\psi(t)]\psi(t)yz^2 f_z(x, y, \theta_1 z) + [\alpha + a\psi(t)]\varphi(t)y^2 h_y(x, \theta_2 y, 0)$$

$$+2\varphi(t)z^2 h_z(x, 0, \theta_3 z) \geq 0$$

for all $t \geq 0, x, y \neq 0 \neq z$ where $0 \leq \theta_i \leq 1$ $(i = 1, 2, 3)$, but $U_7 = 0$ for $y = 0 = z$. Using estimate U_i $(i = 1, 2, \cdots, 7)$ in (18), we obtain

$$\begin{aligned} U'_{(2)} \leq &-\frac{1}{2}\beta\delta_0\varphi_0 x^2 - [(\alpha + a\psi_0)b\phi_0 - 2c\varphi_1 - \beta(1 + a\psi_1)]y^2 \\ &- (a\psi_0 - \alpha - \beta)z^2 - \frac{1}{4}\beta\delta_0\varphi_0\left[x + 2\phi_0\varphi_0^{-1}\delta_0^{-1}\left(\frac{g(x,y)}{y} - b\right)y\right]^2 \\ &+ \beta\phi_0^2\delta_0^{-1}\varphi_0^{-1}\left(\frac{g(x,y)}{y} - b\right)^2 y^2 + \beta\psi_0^2\delta_0^{-1}\varphi_0^{-1}\Big(f(x,y,z) - a\Big)^2 z^2 \\ &- \frac{1}{4}\beta\delta_0\varphi_0\Big[x + 2\psi_0\varphi_0^{-1}\delta_0^{-1}(f(x,y,z) - a)z\Big]^2 + \epsilon\lambda_4(x^2 + y^2 + z^2) \\ &+ \lambda_5(|x| + |y| + |z|)|p(t,x,y,z)|, \end{aligned} \tag{19}$$

where $\lambda_5 = \max\{\beta, \alpha + a\psi_1, 2\}$. Since, $\beta, \delta_0, \varphi_0$ are positive constants, $[x+2\phi_0\varphi_0^{-1}\delta_0^{-1}\left(\frac{g(x,y)}{y} - b\right)y]^2 \geq 0$ and $[x+2\psi_0\varphi_0^{-1}\delta_0^{-1}(f(x,y,z)-a)z]^2 \geq 0$ for all $t \geq 0, x, y$ and z, estimate (19) reduces to

$$\begin{aligned} U'_{(2)} \leq &-\frac{1}{2}\beta\delta_0\varphi_0 x^2 - (\alpha b\phi_0 - c\varphi_1)y^2 - \frac{1}{2}(a\psi_0 - \alpha)z^2 \\ &- \left\{ab\phi_0\psi_0 - c\varphi_1 - \beta\left[1 + a\psi_1 + \phi_0^2\delta_0^{-1}\varphi_0^{-1}\left(\frac{g(x,y)}{y} - b\right)^2\right]\right\}y^2 \\ &- \left\{\frac{1}{2}(a\psi_0 - \alpha) - \beta\left[1 + \psi_0^2\delta_0^{-1}\varphi_0^{-1}\Big(f(x,y,z) - a\Big)^2\right]\right\}z^2 \\ &+ \epsilon\lambda_4(x^2 + y^2 + z^2) + \lambda_5(|x| + |y| + |z|)|p(t,x,y,z)|. \end{aligned}$$

Applying estimates (10d), (10e) and choosing $\epsilon < \lambda_4^{-1}\lambda_6$ where $\lambda_6 := \min\{\frac{1}{2}\beta\delta_0\varphi_0, \alpha b\phi_0 - c\varphi_1, \frac{1}{2}(a\psi_0 - \alpha)\}$, we obtain

$$U'_{(2)} \leq -\lambda_7(x^2 + y^2 + z^2) + \lambda_5(|x| + |y| + |z|)|p(t,x,y,z)|, \tag{20}$$

for all $t \geq 0, x, y$ and z, where $\lambda_7 := \lambda_6 - \epsilon\lambda_4 > 0$. Now, using estimates (13a) and (17), we find

$$V'_{(2)} \leq -e^{-P_*(t)}\Big\{[\lambda_0(x^2+y^2+z^2) - \lambda_5(|x|+|y|+|z|)]|p(t,x,y,z)| + \lambda_7(x^2+y^2+z^2)\Big\} \quad (21)$$

for all $t \geq 0, x, y$ and z. Using condition (viii) of Theorem 1, noting the fact that $(|x|+|y|+|z|)^2 \leq 3(x^2+y^2+z^2)$, and choosing $(x^2+y^2+z^2)^{1/2} \geq 3^{1/2}\lambda_0^{-1}\lambda_5$, estimate (21) becomes

$$V'_{(2)} \leq -\delta_3(x^2+y^2+z^2), \quad (22)$$

for all $t \geq 0, x, y$ and z where $\delta_3 = \lambda_7 \exp[-P_*(\infty)]$. (22) establishes estimate (11c) of the lemma. This completes the proof of the lemma. □

Proof of Theorem 1. Let $(x(t), y(t), z(t))$ be any solution of (2), in view of estimates (11) the hypotheses of Lemma 2 hold true. Thus, by Lemma 2, the solution $(x(t), y(t), z(t))$ of (2) is uniformly ultimately bounded. □

Proof of Theorem 2. The proof of this theorem depends on the function V defined in (10). First, by Lemma 4, and the hypotheses of Lemma 1 are satisfied so that the solution $(x(t), y(t), z(t))$ of (2) is uniformly bounded.
Furthermore, the continuity and boundedness of the functions f, g, h, ϕ, φ and ψ imply the boundedness of the function $F(t, X)$ for all t when X belongs to any compact set in $\mathbb{R}^3$.
Next, from estimate (22), let $W(X) := \delta_3(x^2+y^2+z^2)$, clearly $W(X) \geq 0$, for all $X \in \mathbb{R}^3$. Consider the set

$$\Omega := \{X = (x, y, z) \in \mathbb{R}^3 | W(X) = 0\}. \quad (23)$$

The continuity of the function $W(X)$ implies that the set Ω is closed and $W(X)$ is positive definite with respect to Ω and

$$V'_{(2)}(t, X) \leq -W(X)$$

for all $(t, X) \in \mathbb{R}^+ \times \mathbb{R}^3$. System (2) can be rewritten in the form

$$X' = F(t, X) + G(t, X)$$

where $X = (x, y, z)^T$, $F(t, X) = (y, z, -\psi(t)f(x, y, z)z - \phi(t)g(x, y) - \varphi(t)h(x, y, z))^T$ and $G(t, X) = (0, 0, p(t, x, y, z))^T$. Moreover, from the hypotheses of the theorem we have $F(t, X)$ tends to a function $F(X)$, say, for all $X \in \Omega$ as $t \to \infty$, and by (23) $W(X) = 0$ on Ω implies that $x = y = z = 0$. By system (2) and the fact that $h(0,0,0) = 0 = g(0,0)$, the largest semi invariant set of $X' = F(X)$ $X \in \Omega$ as $t \to \infty$ is the origin. Thus the hypotheses of Lemma 3 are satisfied and (6) follows. This completes the proof of the theorem. □

Proof of Theorem 3. Let $(x(t), y(t), z(t))$ be any solution of (2). Under the hypotheses of Theorem 3, estimates (14a) and (21) hold. To prove (8), since $\lambda_0(x^2 + y^2 + z^2)|p(t,x,y,z)| \geq 0$, $\lambda_7(x^2 + y^2 + z^2) \geq 0$ for all $t \geq 0, x, y, z$, the fact that $|x| \leq 1 + x^2$, $|y| \leq 1 + y^2$ and $|z| \leq 1 + z^2$, estimate (21) becomes

$$V'_{(2)} \leq \lambda_5 e^{-P_*(t)}(3 + x^2 + y^2 + z^2)]|p(t,x,y,z)|$$

for all $t \geq 0, x, y$ and z. Now, from estimates (14a) and (13b) this inequality yields

$$V'_{(2)} - \delta_1^{-1}\lambda_5|p(t,x,y,z)|V \leq 3\lambda_5|p(t,x,y,z)|.$$

Solving this first order differential inequality using integrating factor $\exp[-\delta_1^{-1}\lambda_5 P_*(t)]$ and estimate (13b), we have

$$V(t,x,y,z) \leq \lambda_8 \tag{24}$$

for all $t \geq 0, x, y$ and z, where $\lambda_8 := [V(t_0,x_0,y_0,z_0) + 3\lambda_5 P_0]\exp[\delta_1^{-1}\lambda_5 P_0] > 0$ is a constant. From estimates (14a) and (24), estimate (8) follows for all $t \geq 0$, with $D \equiv \delta_1^{-1}\lambda_8$. This completes the proof of the theorem. □

References

[1] A. T. Ademola, P. O. Arawomo, Boundedness and stability of solutions of some nonlinear differential equations of the third order, *The Pacific J. of Science and Technology,* **10** (2009), 187–193.

[2] A. T. Ademola, P. O. Arawomo, On the stability and ultimate boundedness of solutions for certain third order differential equations, *J. Math. Stat.*, **4** (2008), 201–207.

[3] A. T. Ademola, P. O. Arawomo, Stability and ultimate boundedness of solutions to certain third order differential equations, *Appl. Math. E-Notes*, **10** (2010), 61–69.

[4] A. T. Ademola, P. O. Arawomo, Stability and uniform ultimate boundedness of solutions of a third order differential equation. *International Journal of Applied Mathematics*, **23** (2010), 11–22.

[5] A. T. Ademola, M. O. Ogundiran, P. O. Arawomo, O. A. Adesina, Boundedness results for a certain third order nonlinear differential equations, *Appl. Math. Comput.*, **216** (2010), 3044–3049.

[6] A. T. Ademola, M. O. Ogundiran, P. O. Arawomo, O. A. Adesina, Stability results for the solutions of a certain third order nonlinear differential equation, *Mathematical Sciences Research Journal MSRJ.*, **12** (2008), 124–134.

[7] E. N. Chukwu, On boundedness of solutions of third order differential equations, *Ann. Mat. Pura. Appl.,* **104** (1975), 123–149.

[8] J. O. C. Ezeilo, A boundedness theorem for a certain third order differential equation, *Proc. London Math. Soc.*, **13** (1963), 99–124.

[9] J. O. C. Ezeilo, A generalization of a theorem of Resisg for a certain third order differential equation, *Ann Mat. Pura Appl.*, **87** (1970), 349–356.

[10] J. O. C. Ezeilo, An elementary proof of a boundedness theorem for a certain third order differential equation, *J. London Math. Soc.*, **38** (1963), 11–16.

[11] J. O. C. Ezeilo, H. O. Tejumola, Boundedness theorems for certain third order differential equations, *Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.*, **55** (1973), 194–201.

[12] J. O. C. Ezeilo, Further results for the solutions of a third order differential equation, *Proc. Camb. Phil. Soc.,* **59** (1963), 111–116.

[13] T. Hara, On the uniform ultimate boundedness of solutions of certain third order differential equations, *J. Math. Anal. Appl.*, **80** (1981), 533–544.

[14] B. Mehri, D. Shadman, Boundedness of solutions of certain third order differential equation, *Math. Inequal. Appl.*, **2**, (1999) 545–549.

[15] M. O. Omeike, New result in the ultimate boundedness of solutions of a third order nonlinear ordinary differential equation, *J. Inequal. Pure and Appl. Math.*, **9** (2008), Art. 15, 8 pp.

[16] M. O. Omeike, New results on the asymptotic behaviour of a third order nonlinear differential equations, *Differential Equations and Applications*, **2** (2010), 39–51.

[17] C. Qian, Asymptotic behaviour of a third order nonlinear differential equation, *J. Math. Anal. Appl.*, **284** (2003), 191–205.

[18] R. Reissig, G. Sansone, R. Conti, *Nonlinear differential equations of higher order*, Noordhoff International Publishing Leyeden, 1974.

[19] N. Rouche, N. Habets, M. Laloy, *Stability theory by Liapunov's direct method*, Applied Mathematical Sciences 22, Springer-Verlag New York. Heidelberg. Berlin, 1977.

[20] K. E. Swick, Asymptotic behaviour of the solutions of certain third order differential equations, *SIAM J. Appl.*, **19** (1970), 96–102.

[21] K. E. Swick, Boundedness and stability for a nonlinear third order differential equation, *Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.*, **56** (1974), 859–865.

[22] K. E. Swick, On the boundedness and the stability of solutions for some non-autonomous differential equations of the third order, *J. London Math. Soc.*, **44** (1969), 347–359.

[23] H. O. Tejumola, A note on the boundedness of solutions of some nonlinear differential equations of the third order, *Ghana J. of Science*, **11** (1970), 117–118.

[24] C. Tunç, Boundedness of solutions of a third order nonlinear differential equation, *J. Inequal. Pure and Appl. Math.,* **6** (2005), 1–6.

[25] C. Tunç, On the asymptotic behaviour of solutions of certain third order nonlinear differential equations, *J. Applied Math. and Stochastic Anal.*, **1** (2005), 29–35.

[26] T. Yoshizawa, *Stability theory and existence of periodic solutions and almost periodic solutions*, Springer-Verlag, New York, Heidelberg, Berlin, 1975.

[27] T. Yoshizawa, *Stability theory by Liapunov's second method*, The Mathematical Society of Japan, 1966.

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Contact warped product semi-slant submanifolds of $(LCS)_n$-manifolds

Shyamal Kumar Hui
Nikhil Banga Sikshan Mahavidyalaya
Bishnupur, Bankura – 722 122
West Bengal, India
email: shyamal_hui@yahoo.co.in

Mehmet Atceken
Gaziosmanpasa University
Faculty of Arts and Sciences,
Department of Mathematics
Tokat – 60250, Turkey
email: matceken@gop.edu.tr

Abstract. The present paper deals with a study of warped product submanifolds of $(LCS)_n$-manifolds and warped product semi-slant submanifolds of $(LCS)_n$-manifolds. It is shown that there exists no proper warped product submanifolds of $(LCS)_n$-manifolds. However we obtain some results for the existence or non-existence of warped product semi-slant submanifolds of $(LCS)_n$-manifolds.

1 Introduction

The notion of warped product manifolds were introduced by Bishop and O'Neill [3] and later it was studied by many mathematicians and physicists. These manifolds are generalization of Riemannian product manifolds. The existence or non-existence of warped product manifolds plays some important role in differential geometry as well as physics.

The notion of slant submanifolds in a complex manifold was introduced and studied by Chen [7], which is a natural generalization of both invariant and anti-invariant submanifolds. Chen [7] also found examples of slant submanifolds of complex Euclidean space C^2 and C^4. Then Lotta [9] has defined and

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studied of slant immersions of a Riemannian manifold into an almost contact metric manifold and proved some properties of such immersions. Also Cabrerizo et. al ([5], [6]) studied slant immersions in Sasakian and K-contact manifolds respectively. Again Gupta et. al [8] studied slant submanifolds of a Kenmotsu manifolds and obtained a necessary and sufficient condition for a 3-dimensional submanifold of a 5-dimensional Kenmotsu manifold to be minimal proper slant submanifold.

In 1994 Papaghuic [13] introduced the notion of semi-slant submanifolds of almost Hermitian manifolds. Then Cabrerizo et. al [4] defined and investigated semi-slant submanifolds of Sasakian manifolds. In this connection, it may be mentioned that Sahin [14] studied warped product semi-slant submanifolds of Kaehler manifolds. Also in [1], Atceken studied warped product semi-slant submanifolds in locally Riemannian product manifolds. Again Atceken [2] studied warped product semi-slant submanifolds in Kenmotsu manifolds and he has shown the non-existence cases of the warped product semi-slant submanifolds in a Kenmotsu manifold [2].

Recently Shaikh [15] introduced the notion of Lorentzian concircular structure manifolds (briefly, $(LCS)_n$-manifolds), with an example, which generalizes the notion of LP-Sasakian manifolds introduced by Matsumoto [10] and also by Mihai and Rosca [11]. Then Shaikh and Baishya ([17], [18]) investigated the applications of $(LCS)_n$-manifolds to the general theory of relativity and cosmology. The $(LCS)_n$-manifolds is also studied by Sreenivasa et. al [21], Shaikh [16], Shaikh and Binh [19], Shaikh and Hui [20] and others.

The object of the paper is to study warped product semi-slant submanifolds of $(LCS)_n$-manifolds. The paper is organized as follows. Section 2 is concerned with some preliminaries. Section 3 deals with a study of warped product submanifolds of $(LCS)_n$-manifolds. It is shown that there do not exist proper warped product submanifolds $N = N_1 \times_f N_2$ of a $(LCS)_n$-manifold M, where N_1 and N_2 are submanifolds of M. In section 4, we investigate warped product semi-slant submanifolds of $(LCS)_n$-manifolds and obtain many interesting results.

2 Preliminaries

An n-dimensional Lorentzian manifold M is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric g, that is, M admits a smooth symmetric tensor field g of type (0,2) such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to \mathbb{R}$ is a non-degenerate inner product of signature

$(-,+,\cdots,+)$, where T_pM denotes the tangent vector space of M at p and $\mathbb{R}$ is the real number space. A non-zero vector $v \in T_pM$ is said to be timelike (resp., non-spacelike, null, spacelike) if it satisfies $g_p(v,v) < 0$ (resp, ≤ 0, $=0$, >0) [12].

Definition 1 [15] *In a Lorentzian manifold* (M,g) *a vector field* P *defined by*

$$g(X,P) = A(X),$$

for any $X \in \Gamma(TM)$*, is said to be a concircular vector field if*

$$(\bar{\nabla}_X A)(Y) = \alpha\{g(X,Y) + \omega(X)A(Y)\}$$

where α *is a non-zero scalar and* ω *is a closed 1-form and* $\bar{\nabla}$ *denotes the operator of covariant differentiation with respect to the Lorentzian metric* g.

Let M be an n-dimensional Lorentzian manifold admitting a unit timelike concircular vector field ξ, called the characteristic vector field of the manifold. Then we have

$$g(\xi,\xi) = -1. \tag{1}$$

Since ξ is a unit concircular vector field, it follows that there exists a non-zero 1-form η such that for

$$g(X,\xi) = \eta(X), \tag{2}$$

the equation of the following form holds

$$(\bar{\nabla}_X \eta)(Y) = \alpha\{g(X,Y) + \eta(X)\eta(Y)\} \quad (\alpha \neq 0) \tag{3}$$

for all vector fields X, Y, where $\bar{\nabla}$ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function satisfies

$$\bar{\nabla}_X \alpha = (X\alpha) = d\alpha(X) = \rho\eta(X), \tag{4}$$

ρ being a certain scalar function given by $\rho = -(\xi\alpha)$. Let us take

$$\phi X = \frac{1}{\alpha}\bar{\nabla}_X \xi, \tag{5}$$

then from (3) and (5) we have

$$\phi X = X + \eta(X)\xi, \tag{6}$$

from which it follows that ϕ is a symmetric (1,1) tensor and called the structure tensor of the manifold. Thus the Lorentzian manifold M together with the unit timelike concircular vector field ξ, its associated 1-form η and an (1,1) tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly, $(LCS)_n$-manifold) [15]. Especially, if we take $\alpha = 1$, then we can obtain the LP-Sasakian structure of Matsumoto [10]. In a $(LCS)_n$-manifold ($n > 2$), the following relations hold [15]:

$$\eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{7}$$

$$\phi^2 X = X + \eta(X)\xi, \tag{8}$$

$$S(X, \xi) = (n-1)(\alpha^2 - \rho)\eta(X), \tag{9}$$

$$R(X, Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \tag{10}$$

$$R(\xi, Y)Z = (\alpha^2 - \rho)[g(Y, Z)\xi - \eta(Z)Y], \tag{11}$$

$$(\bar{\nabla}_X\phi)(Y) = \alpha\{g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}, \tag{12}$$

$$(X\rho) = d\rho(X) = \beta\eta(X), \tag{13}$$

$$R(X, Y)Z = \phi R(X, Y)Z + (\alpha^2 - \rho)\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}\xi, \tag{14}$$

for all X, Y, $Z \in \Gamma(TM)$ and $\beta = -(\xi\rho)$ is a scalar function, where R is the curvature tensor and S is the Ricci tensor of the manifold.

Let N be a submanifold of a $(LCS)_n$-manifold M with induced metric g. Also let ∇ and $\nabla^\perp$ are the induced connections on the tangent bundle TN and the normal bundle $T^\perp N$ of N respectively. Then the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{15}$$

and

$$\bar{\nabla}_X V = -A_V X + \nabla^\perp_X V \tag{16}$$

for all $X, Y \in \Gamma(TN)$ and $V \in \Gamma(T^\perp N)$, where h and A_V are second fundamental form and the shape operator (corresponding to the normal vector field V) respectively for the immersion of N into M. The second fundamental form h and the shape operator A_V are related by [22]

$$g(h(X, Y), V) = g(A_V X, Y) \tag{17}$$

for any $X, Y \in \Gamma(TN)$ and $V \in \Gamma(T^\perp N)$.

For any $X \in \Gamma(TN)$, we may write

$$\phi X = EX + FX, \tag{18}$$

where EX is the tangential component and FX is the normal component of ϕX.

Also for any $V \in \Gamma(T^{\perp}N)$, we have

$$\phi V = BV + CV, \tag{19}$$

where BV and CV are the tangential and normal components of ϕV respectively. From (18) and (19) we can derive the tensor fields E, F, B and C are also symmetric. The covariant derivatives of the tensor fields of E and F are defined as

$$(\nabla_X E)(Y) = \nabla_X EY - E(\nabla_X Y), \tag{20}$$

$$(\bar{\nabla}_X F)(Y) = \nabla^{\perp}_X FY - F(\nabla_X Y) \tag{21}$$

for all $X, Y \in \Gamma(TN)$. The canonical structures E and F on a submanifold N are said to be parallel if $\nabla E = 0$ and $\bar{\nabla} F = 0$ respectively.

Throughout the paper, we consider ξ to be tangent to N. The submanifold N is said to be invariant if F is identically zero, i.e., $\phi X \in \Gamma(TN)$ for any $X \in \Gamma(TN)$. Also N is said to anti-invariant if E is identically zero, that is $\phi X \in \Gamma(T^{\perp}N)$ for any $X \in \Gamma(TN)$.

Furthermore for submanifolds tangent to the structure vector field ξ, there is another class of submanifolds which is called slant submanifold. For each non-zero vector X tangent to N at x, the angle $\theta(x)$, $0 \le \theta(x) \le \frac{\pi}{2}$ between ϕX and EX is called the slant angle or wirtinger angle. If the slant angle is constant, then the submanifold is also called the slant submanifold. Invariant and anti-invariant submanifolds are particular slant submanifolds with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$ respectively. A slant submanifold is said to be proper slant if the slant angle θ lies strictly between 0 and $\frac{\pi}{2}$, i.e., $0 < \theta < \frac{\pi}{2}$ [5].

Lemma 1 [5] *Let N be a submanifold of a $(LCS)_n$-manifold M such that ξ is tangent to N. Then N is slant submanifold if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$E^2 = \lambda(I + \eta \otimes \xi). \tag{22}$$

Furthermore, if θ is the slant angle of N, then $\lambda = \cos^2\theta$.
Also from (22) we have

$$g(EX, EY) = \cos^2\theta[g(X, Y) + \eta(X)\eta(Y)], \tag{23}$$

$$g(FX, FY) = \sin^2\theta[g(X, Y) + \eta(X)\eta(Y)] \tag{24}$$

for any X, Y tangent to N.

The study of semi-slant submanifolds of almost Hermitian manifolds was introduced by Papaghuic [13], which was extended to almost contact manifold by Cabrerizo et. al [4]. The submanifold N is called semi-slant submanifold of M if there exist an orthogonal direct decomposition of TN as

$$TN = D_1 \oplus D_2 \oplus \{\xi\},$$

where D_1 is an invariant distribution, i.e., $\phi(D_1) = D_1$ and D_2 is slant with slant angle $\theta \neq 0$. The orthogonal complement of FD_2 in the normal bundle $T^\perp N$ is an invariant subbundle of $T^\perp N$ and is denoted by μ. Thus we have

$$T^\perp N = FD_2 \oplus \mu.$$

Similarly N is called anti-slant subbundle of M if D_1 is an anti-invariant distribution of N, i.e., $\phi D_1 \subset T^\perp N$ and D_2 is slant with slant angle $\theta \neq 0$.

3 Warped product submanifolds of $(LCS)_n$-manifolds

The notion of warped product manifolds were introduced by Bishop and O'Neill [3].

Definition 2 *Let* (N_1, g_1) *and* (N_2, g_2) *be two Riemannian manifolds and* f *be a positive definite smooth function on* N_1*. The warped product of* N_1 *and* N_2 *is the Riemannian manifold* $N_1 \times_f N_2 = (N_1 \times N_2, g)$*, where*

$$g = g_1 + f^2 g_2. \tag{25}$$

A warped product manifold $N_1 \times_f N_2$ is said to be trivial if the warping function f is constant.
More explicitely, if the vector fields X and Y are tangent to $N_1 \times_f N_2$ at (x, y) then

$$g(X, Y) = g_1(\pi_1 * X, \pi_1 * Y) + f^2(x) g_2(\pi_2 * X, \pi_2 * Y),$$

where π_i $(i = 1, 2)$ are the canonical projections of $N_1 \times N_2$ onto N_1 and N_2 respectively and * stands for the derivative map.

Let $N = N_1 \times_f N_2$ be warped product manifold, which means that N_1 and N_2 are totally geodesic and totally umbilical submanifolds of N respectively. For warped product manifolds, we have [3]

Proposition 1 *Let* $N = N_1 \times_f N_2$ *be a warped product manifold. Then*

(I) $\nabla_X Y \in TN_1$ *is the lift of* $\nabla_X Y$ *on* N_1

(II) $\nabla_U X = \nabla_X U = (X \ln f) U$

(III) $\nabla_U V = \nabla'_U V - g(U, V) \nabla \ln f$

for any $X,\ Y \in \Gamma(TN_1)$ *and* $U,\ V \in \Gamma(TN_2)$*, where* ∇ *and* ∇' *denote the Levi-Civita connections on* N_1 *and* N_2 *respectively.*

We now prove the following:

Theorem 1 *There exist no proper warped product submanifolds in the form* $N = N_T \times_f N_\perp$ *of a* $(LCS)_n$*-manifold* M *such that* ξ *is tangent to* N_T*, where* N_T *and* $N_\perp$ *are invariant and anti-invariant submanifolds of* M*, respectively.*

Proof. We suppose that $N = N_T \times_f N_\perp$ is a warped product submanifold of $(LCS)_n$-manifold M. For any $X \in \Gamma(TN_T)$ and $U, V \in \Gamma(TN_\perp)$, from Proposition 1 we have

$$\nabla_U X = \nabla_X U = (X \ln f)U. \tag{26}$$

On the other hand, by using (12) and (26) we have

$$\begin{aligned}
(X \ln f)g(U,V) = &\ g(\nabla_U X, V) = g(\bar{\nabla}_U X, V) = g(\phi\nabla_U X, \phi V)\\
= &\ g(\bar{\nabla}_U \phi X - (\bar{\nabla}_U \phi)X, \phi V) = g(h(U,\phi X), \phi V) - \alpha\eta(X)g(U,\phi V)\\
= &\ g(h(U,\phi X), \phi V) = g(\bar{\nabla}_{\phi X} U, \phi V) = g(\phi\bar{\nabla}_{\phi X} U, V)\\
= &\ g(\bar{\nabla}_{\phi X}\phi U - (\bar{\nabla}_{\phi X}\phi)U, V) = g(\bar{\nabla}_{\phi X}\phi U, V)\\
= &\ -g(A_{\phi U}\phi X, V) = -g(h(\phi X, V), \phi U) = -g(\bar{\nabla}_V \phi X, \phi U)\\
= &\ -g(\bar{\nabla}_V X, U) = -g(\nabla_V X, U) = -(X \ln f)g(U,V).
\end{aligned}$$

It follows that $X(\ln f) = 0$. So f is constant on N_T. Hence we get our desired assertion.

4 Warped product semi-slant submanifolds of $(LCS)_n$-manifolds

Let us suppose that $N = N_1 \times_f N_2$ be a warped product semi-slant submanifold of a $(LCS)_n$-manifold M. Such submanifolds are always tangent to the structure vector field ξ. If the manifolds N_θ and N_T (respectively $N_\perp$) are slant and invariant (respectively anti-invariant) submanifolds of a $(LCS)_n$-manifold M, then their warped product semi-slant submanifolds may be given by one of the following forms:
(i) $N_T \times_f N_\theta$ (ii) $N_\perp \times_f N_\theta$ (iii) $N_\theta \times_f N_T$ (iv) $N_\theta \times_f N_\perp$.

However, the existence or non-existence of a structure on a manifold is very important. Because the every structure of a manifold may not be admit. In this paper, we have researched cases that there exist no warped product semi-slant submanifolds in a $(LCS)_n$-manifold. Therefore we now study each of the above four cases and begin the following Theorem:

Theorem 2 *There exist no proper warped product semi-slant submanifold in the form* $N = N_T \times_f N_\theta$ *of a* $(LCS)_n$*-manifold* M *such that* ξ *is tangent to* N_T*, where* N_T *and* N_θ *are invariant and slant submanifolds of* M*, respectively.*

Proof. Let us assume that $N = N_T \times_f N_\theta$ is a proper warped product semi-slant submanifolds of a $(LCS)_n$-manifold M such that ξ is tangent to N_T. Then for any $X, \xi \in \Gamma(TN_T)$ and $U \in \Gamma(TN_\theta)$, from (5) and (15) we have

$$\bar{\nabla}_U \xi = \nabla_U \xi + h(U, \xi) = \alpha \phi U. \tag{27}$$

From the tangent and normal components of (27), respectively, we obtain

$$\xi(\ln f)U = \alpha EU \quad and \quad h(U, \xi) = \alpha FU. \tag{28}$$

On the other hand, by using (7) and (12), we have

$$\begin{aligned} (\bar{\nabla}_U \phi)\xi &= -\phi \bar{\nabla}_U \xi \\ \alpha U &= \phi(\xi(\ln f)U) + \phi h(U, \xi), \end{aligned}$$

that is,

$$B(U, \xi) + \xi(\ln f)EU = \alpha U \quad and \quad \xi(\ln f)FU + Ch(U, \xi) = 0. \tag{29}$$

Since $\Gamma(\mu)$ and $\Gamma(F(TN_\theta))$ are orthogonal subspaces, we can derive $\xi(\ln f)FU = 0$. So we conclude $\xi(\ln f) = 0$ or $FU = 0$. Here we have to show that FU for the proof. For this we assume that $FU \neq 0$.

Making use of (12), (15), (16) and (18), we obtain

$$\begin{aligned} (\bar{\nabla}_X \phi)U &= \bar{\nabla}_X \phi U - \phi \bar{\nabla}_X U \\ h(X, EU) - A_{FU}X + \nabla^\perp_X FU &= X(\ln f)FU + Bh(X, U) + Ch(X, U). \end{aligned} \tag{30}$$

Taking into account that the tangent components of (30) and making the necessary abbreviations, we get

$$A_{FU}X = -Bh(X, U). \tag{31}$$

With similar thoughts, we have

$$\begin{aligned} (\bar{\nabla}_U \phi)X &= \bar{\nabla}_U \phi X - \phi \bar{\nabla}_U X \\ \alpha\eta(X)U &= EX(\ln f)U + h(U, EX) - X(\ln f)EU - X(\ln f)FU \\ &- Bh(X, U) - Ch(X, U). \end{aligned} \tag{32}$$

From the normal components of (32), we arrive at

$$X(\ln f)FU = h(U, EX) - Ch(U, X). \tag{33}$$

Thus by using (31) and (33), we conclude

$$\begin{aligned} X(\ln f)g(FU, FU) &= g(h(U, EX), FU) = g(A_{FU}EX, U) = -g(Bh(EX, U), U) \\ &= -g(\phi h(EX, U), U) = -g(h(U, EX), FU) \\ &= -X(\ln f)g(FU, FU). \end{aligned}$$

This tell us that $X(\ln f) = 0$, that is, f is a constant function N_T because FU is a non-null vector field and N_θ is a proper slant submanifold.

Theorem 3 *There exist no proper warped product semi-slant submanifolds in the form $N = N_\perp \times_f N_\theta$ of a $(LCS)_n$-manifold M such that ξ is tangent to N, where $N_\perp$ and N_θ are anti-invariant and proper slant submanifolds of M respectively.*

Proof. Let $N = N_\perp \times_f N_\theta$ be a proper warped product semi-slant submanifold of a $(LCS)_n$-manifold M such that ξ is tangent to N. If ξ is tangent to $\Gamma(TN_\theta)$, then for any $X \in \Gamma(TN_\theta)$ and $U \in \Gamma(TN_\perp)$, from (5) and (15), we have

$$\bar{\nabla}_U \xi = \nabla_U \xi + h(U, \xi) = \alpha\phi U, \tag{34}$$

which is equivalent to $U(\ln f)\xi = 0$ because $\xi \neq 0$. So f is a constant function on $N_\perp$.

On the other hand, if $\xi \in \Gamma(TN_\perp)$, from (5) and (15), we reach

$$\begin{aligned} \bar{\nabla}_X \xi &= \nabla_X \xi + h(X, \xi) \\ \alpha\phi X &= \xi(\ln f)X + h(X, \xi), \end{aligned}$$

that is,

$$\alpha EX = \xi(\ln f)X \quad \text{and} \quad \alpha FX = h(X, \xi). \tag{35}$$

Furthermore, since $\phi\xi = 0$, by direct calculations, we obtain

$$\begin{aligned} (\bar{\nabla}_X \phi)\xi &= -\phi(\bar{\nabla}_X \xi) \\ \alpha X &= \xi(\ln f)EX + \xi(\ln f)FX + Bh(X, \xi) + Ch(X, \xi). \end{aligned}$$

It follows that

$$\alpha X = \xi(\ln f)EX + Bh(X, \xi) \quad \text{and} \quad \xi(\ln f)FX = -Ch(X, \xi). \tag{36}$$

By virtue of (36), we conclude

$$\xi(\ln f)g(FX,FX)=\sin^2\theta\xi(\ln f)g(X,X)=-g(Ch(X,\xi),FX)=0,$$

which follows $\xi(\ln f)=0$ or $\sin^2\theta g(X,X)=0$. Here if $\xi(\ln f)\neq 0$ and $\sin^2\theta g(X,X)=0$, the proof is obvious. Otherwise, making use of (36), we conclude that

$$\alpha g(X,X)=g(Bh(X,\xi),X)=0.$$

Consequently, we can easily to see that $\alpha=0$. This is a contradiction because the ambient space M is a $(LCS)_n$-manifold. Thus the proof is complete.

Theorem 4 *There exist no proper warped product semi-slant submanifolds in the form* $N_\theta\times_f N_T$ *in* $(LCS)_n$*-manifold* M *such that* ξ *tangent to* N_T*, where* N_θ *and* N_T *are proper slant and invariant submanifolds of* M*.*

Proof. Let $N=N_\theta\times_f N_T$ be warped product semi-slant submanifolds in a $(LCS)_n$-manifold M such that ξ is tangent to N_T. Then for any $\xi,X\in\Gamma(TN_T)$ and $U\in\Gamma(TN_\theta)$, taking account of relations (12), (15), (16), (18) and (19) and Proposition 1, we have

$$\begin{aligned}(\bar\nabla_U\phi)X &= \bar\nabla_U\phi X-\phi\bar\nabla_U X\\ \alpha\eta(X)U &= h(U,EX)-Bh(U,X)-Ch(U,X),\end{aligned}$$

which implies that

$$\alpha\eta(X)U=-Bh(U,X)\ \ \text{and}\ \ h(U,EX)=Ch(U,X). \tag{37}$$

In the same way, we have

$$\begin{aligned}(\bar\nabla_X\phi)U &= \bar\nabla_X\phi U-\phi\bar\nabla_X U\\ -A_{FU}X+\nabla^\perp_X FU+h(X,EU) &= Bh(X,U)+Ch(X,U),\end{aligned}$$

from here

$$Bh(X,U)=-A_{FU}X+EU(\ln f)X-U(\ln f)EX \tag{38}$$

and

$$\nabla^\perp_X FU=Ch(X,U)-h(X,EU). \tag{39}$$

Taking inner product both of sides of (37) with $V \in \Gamma(TN_\theta)$ and also using (38), we arrive at

$$\begin{aligned} \alpha\eta(X)g(U,V) &= -g(Bh(U,X),V) = -g(\phi h(U,X),V) = -g(h(U,X),\phi V) \\ &= -g(h(U,X),FV) = -g(A_{FV}X,U) = g(Bh(X,V),U) \\ &= -\alpha\eta(X)g(U,V). \end{aligned}$$

Here for $X = \xi$, we obtain $\alpha g(U,V) = 0$. Because the ambient space M is a $(LCS)_n$-manifold and N_θ is a proper slant submanifold, this also tells us the accuracy of the statement of the theorem.

Theorem 5 *There exist no proper warped product semi-slant submanifolds in the form $N = N_\theta \times_f N_\perp$ in a $(LCS)_n$-manifold M such that ξ tangent to N_θ, where N_θ and $N_\perp$ are proper slant and anti-invariant submanifolds of M, respectively.*

Proof. Let us assume that $N = N_\theta \times_f N_\perp$ be a proper warped product semi-slant submanifold in the $(LCS)_n$-manifold M such that ξ is tangent to N_θ. Then for $X \in \Gamma(TN_\theta)$ and $U \in \Gamma(TN_\perp)$, we have

$$\begin{aligned} (\bar{\nabla}_X\phi)U &= \bar{\nabla}_X\phi U - \phi\bar{\nabla}_X U \\ -A_{FU}X + \nabla^\perp_X FU &= \phi\nabla_X U + \phi h(X,U), \end{aligned}$$

which follows that

$$A_{FU}X = -Bh(X,U) \quad \text{and} \quad (\nabla_X F)U = Ch(X,U). \tag{40}$$

In the same way, we have

$$(\bar{\nabla}_U\phi)X = \bar{\nabla}_U\phi X - \phi\bar{\nabla}_U X,$$

which also follow that

$$\alpha\eta(X)U = EX(\ln f)U - A_{FX}U - Bh(X,U), \tag{41}$$

$$\nabla^\perp_U FX = X(\ln f)FU + Ch(X,U) - h(U,EX). \tag{42}$$

From (41), we can derive

$$g(h(U,X),FX) = g(h(U,X),FU) = 0. \tag{43}$$

Taking $X = \xi$ in (42), we have $\xi(\ln f)FU = -Ch(X,\xi)$, that is, $\xi(\ln f)FU = 0$. Let $X = \xi$ be in (41), then we get

$$\alpha U = Bh(U,\xi). \tag{44}$$

Taking the inner product of the both sides of (44) by $U \in \Gamma(TN_\perp)$, and using (43) we conclude

$$\alpha g(U, U) = g(Bh(U, \xi), U) = g(h(U, \xi), FU) = 0, \tag{45}$$

which implies that $\alpha = 0$. This is impossible because the ambient space is a $(LCS)_n$-manifold. Hence the proof is complete.

References

[1] M. Atceken, Warped product semi-slant submanifolds in locally Riemannian product manifolds, *Bull. Austral Math. Soc.*, **77** (2008), 177–186.

[2] M. Atceken, Warped product semi-slant submanifolds in Kenmotsu manifolds, *Turk. J. Math.*, **34** (2010), 425–432.

[3] R. L. Bishop and B. O'Neill, Manifolds of negative curvature, *Trans. Amer. Math. Soc.*, **145** (1969), 1–49.

[4] J. L. Cabrerizo, A. Carriazo, L. M. Fernandez, M. Fernandez, Semi-slant submanifolds of a Sasakian manifold, *Geom. Dedicata*, **78** (1999), 183–199.

[5] J. L. Cabrerizo, A. Carriazo, L. M. Fernandez, M. Fernandez, Slant submanifolds in Sasakian manifolds, *Glasgow Math. J.*, **42** (2000), 125–138.

[6] J. L. Cabrerizo, A. Carriazo, L. M. Fernandez, M. Fernandez, Structure on a slant submanifold of a contact manifold, *Indian J. Pure and Appl. Math.*, **31** (2000), 857–864.

[7] B. Y. Chen, *Geometry of slant submanifolds*, Katholieke Universietit Leuven, 1990.

[8] R. S. Gupta, S. M. K. Haider and M. H. Shadid, Slant submanifolds of a Kenmotsu manifold, *Radovi Matematicki*, **12** (2004), 205–214.

[9] A. Lotta, Slant submanifolds in contact geometry, *Bull. Math. Soc. Sci. Math. R. S. Roumanie*, **39** (1996), 183–198.

[10] K. Matsumoto, On Lorentzian almost paracontact manifolds, *Bull. of Yamagata Univ. Nat. Sci.*, **12** (1989), 151–156.

[11] I. Mihai, R. Rosca, *On Lorentzian para-Sasakian manifolds*, Classical Analysis, World Scientific Publ., Singapore, (1992), 155–169.

[12] B. O'Neill, *Semi Riemannian Geometry with applications to relativity*, Academic Press, New York, 1983.

[13] N. Papaghiuc, Semi-slant submanifolds of a Kaehlerian manifold, *An. Sti. Al. I. Cuza, Iasi*, **40** (1994), 55–61.

[14] B. Sahin, Non-existence of warped product semi-slant submanifolds of Kaehler manifold, *Geom. Dedicata*, **117** (2006), 195–202.

[15] A. A. Shaikh, On Lorentzian almost paracontact manifolds with a structure of the concircular type, *Kyungpook Math. J.*, **43** (2003), 305–314.

[16] A. A. Shaikh, Some results on $(LCS)_n$-manifolds, *J. Korean Math. Soc.*, **46** (2009), 449–461.

[17] A. A. Shaikh, K. K. Baishya, On concircular structure spacetimes, *J. Math. Stat.*, **1** (2005), 129–132.

[18] A. A. Shaikh, K. K. Baishya, On concircular structure spacetimes II., *American J. Appl. Sci.*, **3** (2006), 1790–1794.

[19] A. A. Shaikh, T. Q. Binh, On weakly symmetric $(LCS)_n$-manifolds, *J. Adv. Math. Studies*, **2** (2009), 75–90.

[20] A. A. Shaikh, S. K. Hui, On generalized ϕ-recurrent $(LCS)_n$-manifolds, *AIP Conference Proceedings*, **1309** (2010), 419–429.

[21] G. T. Sreenivasa, Venkatesha, C. S. Bagewadi, Some results on $(LCS)_{2n+1}$-manifolds, *Bull. Math. Analysis and Appl.*, **1(3)** (2009), 64–70.

[22] K. Yano, M. Kon, *Structures on manifolds*, World Scientific Publishing Co., Singapore, 1984.

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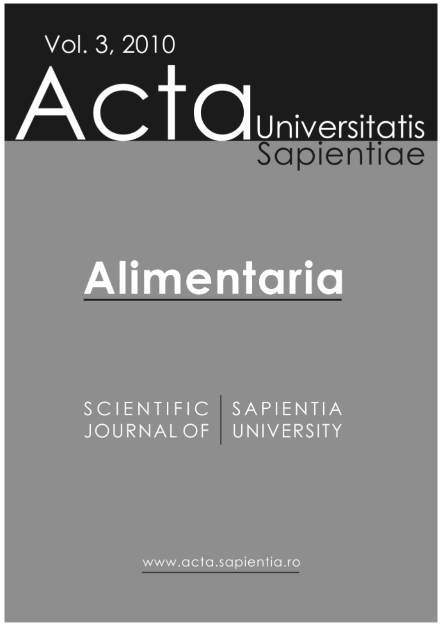

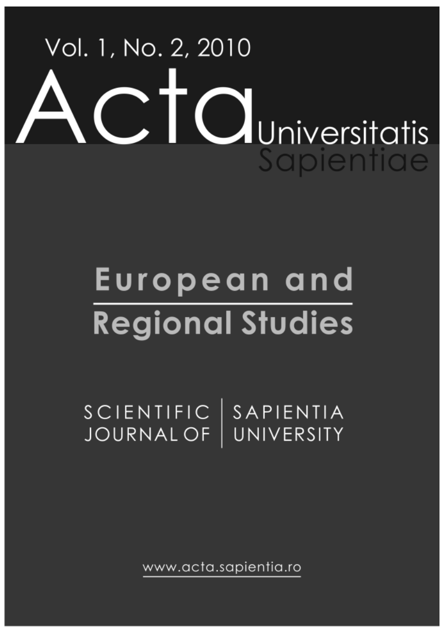

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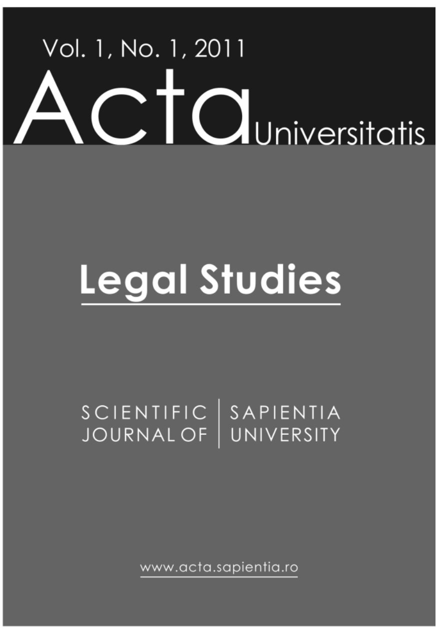

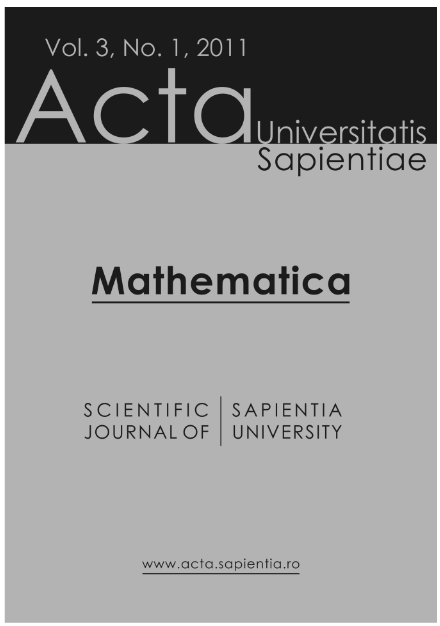

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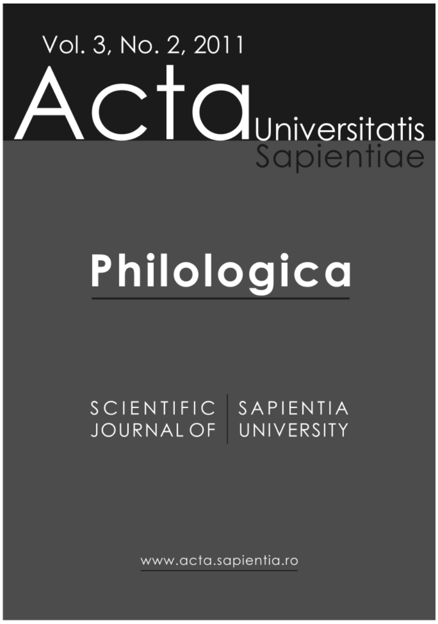

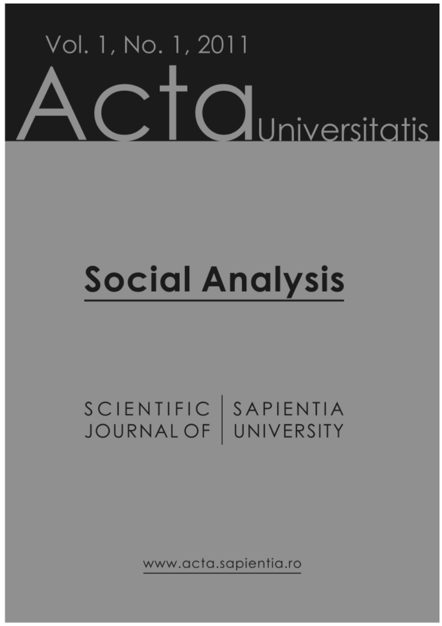

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