Acta Universitatis Sapientiae

Mathematica

Volume 3, Number 1, 2011

Sapientia Hungarian University of Transylvania
Scientia Publishing House

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Acta Univ. Sapientiae, Mathematica, **3**, 1 (2011) 5–25

The discrete Fourier transform of r-even functions

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Abstract. We give a detailed study of the discrete Fourier transform (DFT) of r-even arithmetic functions, which form a subspace of the space of r-periodic arithmetic functions. We consider the DFT of sequences of r-even functions, their mean values and Dirichlet series. Our results generalize properties of the Ramanujan sum. We show that some known properties of r-even functions and of the Ramanujan sum can be obtained in a simple manner via the DFT.

1 Introduction

The discrete Fourier transform (DFT) of periodic functions is an important tool in various branches of pure and applied mathematics. For instance, in number theory, the DFT of a Dirichlet character $\chi \pmod r$ is the Gauss sum (character sum) given by

$$G(\chi, n) = \sum_{k \pmod r} \chi(k) \exp(2\pi i k n/r), \tag{1}$$

and if $\chi = \chi_0$ is the principal character (mod r), then (1) reduces to the Ramanujan sum $c_r(n)$.

2010 Mathematics Subject Classification: 11A25, 11L03, 11N37
Key words and phrases: periodic function, multiplicative function, r-even function, discrete Fourier transform, Cauchy convolution, Ramanujan sum, mean value, Dirichlet series

For the history, properties and various applications, including signal and image processing, of the DFT see for example the books of Briggs and Henson [6], Broughton and Bryan [7], Sundararajan [25], Terras [26]. For recent number theoretical papers concerning the DFT see [4, 13, 21].

It is the aim of the present paper to give a detailed study of the DFT of r-even arithmetic functions, to be defined in Section 2, which form a subspace of the space of r-periodic arithmetic functions.

Some aspects of the DFT of r-even functions were given by Haukkanen [13], Lucht [15] and were considered also by Samadi, Ahmad and Swamy [20] in the context of signal processing methods. Schramm [21] investigated the DFT of certain special r-even functions, without referring to this notion.

Our results generalize and complete those of [13, 15, 20, 21]. Note that the Ramanujan sum $c_r(n)$ is r-even and it is the DFT of χ_0, which is also r-even. Therefore, our results generalize properties of the Ramanujan sum.

The paper is organized as follows. Section 2 presents an overview of the basic notions and properties needed throughout the paper. In Section 3 we give a new simple characterization of r-even functions. Section 4 contains properties of the DFT of r-even functions, while in Sections 5 and 6 we consider sequences of r-even functions and their DFT, respectively. Mean values and Dirichlet series of the DFT of r-even functions and their sequences are investigated in Sections 7 and 8.

We also show that some known properties of r-even functions and of the Ramanujan sum can be obtained in a simple manner via the DFT.

2 Preliminaries

In this section we recall some known properties of arithmetic functions, periodic arithmetic functions, even functions, Ramanujan sums and the DFT. We also fix the notations, most of them being those used in the book by Schwarz and Spilker [22].

2.1 Arithmetic functions

Consider the $\mathbb{C}$-linear space $\mathcal{F}$ of arithmetic functions $f\colon \mathbb{N} = \{1, 2, \ldots\} \to \mathbb{C}$ with the usual linear operations. It is well known that with the Dirichlet convolution defined by

$$(f * g)(n) = \sum_{d \mid n} f(d)g(n/d) \tag{2}$$

the space $\mathcal{F}$ forms a unital commutative $\mathbb{C}$-algebra. The unity is the function ε given by $\varepsilon(1) = 1$ and $\varepsilon(n) = 0$ for $n > 1$. The group of invertible functions is $\mathcal{F}^* = \{f \in \mathcal{F} : f(1) \neq 0\}$. The Möbius function μ is defined as the inverse of the function $\mathbf{1} \in \mathcal{F}^*$ (constant 1 function). The divisor function is $\tau = \mathbf{1} * \mathbf{1}$, Euler's function is $\varphi = \mu * \mathrm{id}$ and $\sigma = \mathbf{1} * \mathrm{id}$ is the sum-of-divisors function, where $\mathrm{id}(n) = n$ $(n \in \mathbb{N})$. A function $f \in \mathcal{F}$ is called multiplicative if $f(1) = 1$ and $f(mn) = f(m)f(n)$ for any $m, n \in \mathbb{N}$ such that $\gcd(m, n) = 1$. The set $\mathcal{M}$ of multiplicative functions is a subgroup of $\mathcal{F}^*$ with respect to the Dirichlet convolution. Note that $\mathbf{1}, \mathrm{id}, \mu, \tau, \sigma, \varphi \in \mathcal{M}$. For an $f \in \mathcal{F}$ we will use the notation $f' = \mu * f$.

2.2 Periodic functions

A function $f \in \mathcal{F}$ is called r-periodic if $f(n + r) = f(n)$ for every $n \in \mathbb{N}$, where $r \in \mathbb{N}$ is a fixed number (this periodicity extends f to a function defined on $\mathbb{Z}$). The set $\mathcal{D}_r$ of r-periodic functions forms an r-dimensional subspace of $\mathcal{F}$. A function $f \in \mathcal{F}$ is called periodic if $f \in \bigcup_{r\in\mathbb{N}} \mathcal{D}_r$. The functions δ_k with $1 \leq k \leq r$ given by $\delta_k(n) = 1$ for $n \equiv k \pmod r$ and $\delta_k(n) = 0$ for $n \not\equiv k \pmod r$ form a basis of $\mathcal{D}_r$ (standard basis).

The functions e_k with $1 \leq k \leq r$ defined by $e_k(n) = \exp(2\pi i k n/r)$ (additive characters) form another basis of the space $\mathcal{D}_r$. Therefore, every r-periodic function f has a Fourier expansion of the form

$$f(n) = \sum_{k \pmod r} g(k) \exp(2\pi i k n/r) \qquad (n \in \mathbb{N}), \tag{3}$$

where the Fourier coefficients $g(k)$ are uniquely determined and are given by

$$g(n) = \frac{1}{r} \sum_{k \pmod r} f(k) \exp(-2\pi i k n/r) \qquad (n \in \mathbb{N}) \tag{4}$$

and the function g is also r-periodic.

For a function $f \in \mathcal{D}_r$ its discrete (finite) Fourier transform (DFT) is the function $\widehat{f} \in \mathcal{F}$ defined by

$$\widehat{f}(n) = \sum_{k \pmod r} f(k) \exp(-2\pi i k n/r) \qquad (n \in \mathbb{N}), \tag{5}$$

where by (5) and (4) one has $\widehat{f} = rg$.

For any $r \in \mathbb{N}$ the DFT is an automorphism of $\mathcal{D}_r$ satisfying $\widehat{\widehat{f}} = rf$. The inverse discrete Fourier transform (IDFT) is given by

$$f(n) = \frac{1}{r} \sum_{k \pmod r} \widehat{f}(k) \exp(2\pi i k n/r) \qquad (n \in \mathbb{N}). \tag{6}$$

If $f \in \mathcal{D}_r$, then

$$\sum_{n=1}^{r} |\widehat{f}(n)|^2 = r \sum_{n=1}^{r} |f(n)|^2, \tag{7}$$

which is a version of Parseval's formula.

Let $f, h \in \mathcal{D}_r$. The Cauchy convolution of f and h is given by

$$(f \otimes h)(n) = \sum_{a \pmod r} f(a) h(n-a) \qquad (n \in \mathbb{N}), \tag{8}$$

where $(\mathcal{D}_r, \otimes)$ is a unital commutative semigroup, the unity being the function ε_r given by $\varepsilon_r(n) = 1$ for $r \mid n$ and $\varepsilon_r(n) = 0$ otherwise. Also, $\widehat{f \otimes h} = \widehat{f}\,\widehat{h}$ and $\widehat{f} \otimes \widehat{h} = r\widehat{fh}$.

For the proofs of the above statements and for further properties of r-periodic functions and the DFT we refer to the books by Apostol [3, Ch. 8], Montgomery and Vaughan [17, Ch. 4], Schwarz and Spilker [22].

2.3 Even functions

A function $f \in \mathcal{F}$ is said to be an r-even function if $f(\gcd(n, r)) = f(n)$ for all $n \in \mathbb{N}$, where $r \in \mathbb{N}$ is fixed. The set $\mathcal{B}_r$ of r-even functions forms a $\tau(r)$ dimensional subspace of $\mathcal{D}_r$, where $\tau(r)$ is the number of positive divisors of r. A function $f \in \mathcal{F}$ is called even if $f \in \bigcup_{r \in \mathbb{N}} \mathcal{B}_r$. The functions g_d with $d \mid r$ given by $g_d(n) = 1$ if $\gcd(n, r) = d$ and $g_d(n) = 0$ if $\gcd(n, r) \neq d$ form a basis of $\mathcal{B}_r$. This basis can be replaced by the following one. The functions c_q with $q \mid r$ form a basis of the subspace $\mathcal{B}_r$, where c_q are the Ramanujan sums, quoted in the Introduction, defined explicitly by

$$c_q(n) = \sum_{\substack{k \pmod q \\ \gcd(k,q)=1}} \exp(2\pi i k n/q) \qquad (n, q \in \mathbb{N}). \tag{9}$$

Consequently, every r-even function f has a (Ramanujan-)Fourier expansion of the form

$$f(n) = \sum_{d \mid r} h(d) c_d(n) \qquad (n \in \mathbb{N}), \tag{10}$$

where the (Ramanujan-)Fourier coefficients $h(d)$ are uniquely determined and are given by

$$h(d) = \frac{1}{r} \sum_{e|r} f(e) c_{r/e}(r/d) \qquad (d \mid r) \tag{11}$$

and the function h is also r-even. Notation: $h(d) = \alpha_f(d)$ $(d \mid r)$. Note that $(\mathcal{B}_r, \otimes)$ is a subsemigroup of $(\mathcal{D}_r, \otimes)$ and $\alpha_{f \otimes h}(d) = r\alpha_f(d)\alpha_h(d)$ $(d \mid r)$, cf. Application 4.

Recall the following properties of Ramanujan sums $c_r(n)$. They can be represented as

$$c_r(n) = \sum_{d|\gcd(n,r)} d\mu(r/d) \qquad (n, r \in \mathbb{N}), \tag{12}$$

and as

$$c_r(n) = \frac{\mu(m)\varphi(r)}{\varphi(m)}, \quad m = r/\gcd(n,r), \qquad (n, r \in \mathbb{N}), \tag{13}$$

where (13) is Hölder's identity. It follows that $c_r(n) = \varphi(r)$ for $r \mid n$ and $c_r(n) = \mu(r)$ for $\gcd(n,r) = 1$.

Let $\eta_r(n) = r$ if $r \mid n$ and $\eta_r(n) = 0$ otherwise. For any fixed $n \in \mathbb{N}$, $c_{\bullet}(n) = \mu * \eta_{\bullet}(n)$ and $r \mapsto c_r(n)$ is a multiplicative function. On the other hand, $n \mapsto c_r(n)$ is multiplicative if and only if $\mu(r) = 1$.

As it was already mentioned, $c_r(\bullet)$ is the DFT of the principal character (mod r) to be denoted in what follows by ρ_r and given explicitly by $\rho_r(n) = 1$ if $\gcd(n,r) = 1$ and $\rho_r(n) = 0$ otherwise. Note that $\rho_r = g_1$ with the notation of above (for r fixed). Thus

$$\widehat{\rho}_r = c_r, \qquad \widehat{c}_r = r\rho_r. \tag{14}$$

The concept of r-even functions originates from Cohen [8] and was further studied by Cohen in subsequent papers [9, 10, 11]. General accounts of r-even functions and of Ramanujan sums can be found in the books by McCarthy [16], Schwarz and Spilker [22], Sivaramakrishnan [23], Montgomery and Vaughan [17, Ch. 4]. See also the papers [12, 24, 27].

3 Characterization of r-even functions

For an $r \in \mathbb{N}$ let $\mathcal{B}'_r = \{f \in \mathcal{F} : f(n) = 0 \text{ for any } n \nmid r\}$. We have

Proposition 1 *Let* $f \in \mathcal{F}$ *and* $f' = \mu * f$. *Then the following assertions are equivalent:*

i) $f \in \mathcal{B}_r$,
ii) $f(n) = \sum_{d|\gcd(n,r)} f'(d) \quad (n \in \mathbb{N})$,
iii) $f' \in \mathcal{B}'_r$.

Proof. If $f' \in \mathcal{B}'_r$, then for any $n \in \mathbb{N}$,

$$f(n) = \sum_{d|n} f'(d) = \sum_{d|n,\, d|r} f'(d) = \sum_{d|\gcd(n,r)} f'(d)$$

$$= (f' * \mathbf{1})(\gcd(n,r)) = f(\gcd(n,r)).$$

This shows that iii) $\Rightarrow$ ii) $\Rightarrow$ i).

Now we show that i) $\Rightarrow$ iii). Assume that $f \in \mathcal{B}_r$ and $f' \notin \mathcal{B}'_r$, i.e., $f'(n) \neq 0$ for some $n \in \mathbb{N}$ with $n \nmid r$. Consider the minimal $n \in \mathbb{N}$ with this property. Then all proper divisors d of n with $f'(d) \neq 0$ divide r so that

$$f(n) = \sum_{d|n} f'(d) = \sum_{d|\gcd(n,r)} f'(d) + f'(n) = f(\gcd(n,r)) + f'(n) \neq f(\gcd(n,r)),$$

which gives $f \notin \mathcal{B}_r$. $\square$

Remark 1 Let $f \in \mathcal{B}_r$. Assume that $f(n) = \sum_{d|\gcd(n,r)} g(d)$ $(n \in \mathbb{N})$ for a function $g \in \mathcal{F}$. Then $f = g\varepsilon_{\bullet}(r) * \mathbf{1}$ and $f = f' * \mathbf{1}$, by Proposition 1. Hence $g\varepsilon_{\bullet}(r) = f'$ and obtain that $g(n) = f'(n)$ for any $n \mid r$.

For $f = c_r$ (Ramanujan sum) we have by (12), Proposition 1 and Remark 1 the next identity, which can be shown also directly.

Application 1 *For any* $n, r \in \mathbb{N}$,

$$\sum_{d|n} c_r(d)\mu(n/d) = \begin{cases} n\mu(r/n), & n \mid r, \\ 0, & n \nmid r. \end{cases} \tag{15}$$

4 The DFT of r-even functions

We investigate in this section general properties of the DFT of r-even functions.

Proposition 2 *For each* $r \in \mathbb{N}$ *the DFT is an automorphism of* $\mathcal{B}_r$. *For any* $f \in \mathcal{B}_r$,

$$\widehat{f}(n) = \sum_{d|r} f(d) c_{r/d}(n) \qquad (n \in \mathbb{N}) \tag{16}$$

and the IDFT is given by

$$f(n) = \frac{1}{r} \sum_{d \mid r} \widehat{f}(d) c_{r/d}(n) \qquad (n \in \mathbb{N}). \tag{17}$$

Proof. By the definition of r-even functions and grouping the terms according to the values $d = \gcd(k, r)$,

$$\widehat{f}(n) = \sum_{d \mid r} f(d) \sum_{\substack{1 \le j \le r/d \\ \gcd(j, r/d) = 1}} \exp(-2\pi i j n/(r/d)) = \sum_{d \mid r} f(d) c_{r/d}(n)$$

giving (16) and also that $\widehat{f} \in \mathcal{B}_r$. Now applying (16) for $\widehat{f}$ (instead of f) and using that $\widehat{\widehat{f}} = rf$ we have (17). □

Proposition 2 is given by Lucht [15, Th. 4]. Formulas (16) and (17) are implicitly given by Haukkanen [13, Th. 3.2 and Eq. (9)], Samadi, Ahmad and Swamy [20, Eq. (18)] for r-even functions, and by Schramm [21] for functions $n \mapsto F(\gcd(n, r))$, where $F \in \mathcal{F}$ is arbitrary, without referring to the notion of even functions.

Remark 2 By Proposition 2, for a function $f \in \mathcal{D}_r$ one has $f \in \mathcal{B}_r$ if and only if $\widehat{f} \in \mathcal{B}_r$. This can be used to show that a given function is r-even, cf. Application 4. Furthermore, it follows that the Fourier coefficients $\alpha_f(d)$ of $f \in \mathcal{B}_r$ can be represented as

$$\alpha_f(d) = \frac{1}{r} \widehat{f}(r/d) \qquad (d \mid r). \tag{18}$$

Corollary 1 *Let* $f \in \mathcal{B}_r$. *Then*

$$\widehat{f}(n) = \sum_{d \mid r} f(d) \varphi(r/d) \qquad (r \mid n), \tag{19}$$

$$\widehat{f}(n) = \sum_{d \mid r} f(d) \mu(r/d) \qquad (\gcd(n, r) = 1). \tag{20}$$

Corollary 2 *If* f *is a real (integer) valued* r*-even function, then* $\widehat{f}$ *is also real (integer) valued.*

Proof. Use that $c_r(n) \in \mathbb{Z}$ for any $n, r \in \mathbb{N}$. □

Corollary 3 *Let* f *be an* r*-even function. Then*

$$\widehat{f}(n) = \sum_{d \mid \gcd(n,r)} d\, f'(r/d) \qquad (n \in \mathbb{N}), \tag{21}$$

and $(\widehat{f})'(n) = nf'(r/n)$ *for any* $n \mid r$ *and* $(\widehat{f})'(n) = 0$ *otherwise.*

Proof. Recall that $c_{\bullet}(n) = \mu * \eta_{\bullet}(n)$, see (12). We obtain $\widehat{f}(n) = (f * c_{\bullet}(n))(r)$ $= (f * \mu * \eta_{\bullet}(n))(r) = (f' * \eta_{\bullet}(n))(r)$, and apply Remark 1. □

Note that by (21) the DFT of any $f \in \mathcal{B}_r$ can be written in the following forms:

$$\widehat{f}(n) = (f' * \eta_{\bullet}(n))(r), \tag{22}$$

and

$$\widehat{f} = h * \mathbf{1}, \tag{23}$$

where $h(n) = nf'(r/n)$ for $n \mid r$ and $h(n) = 0$ otherwise.

Proposition 3 *Let* f *be an* r*-even function. Then*

$$\sum_{d \mid n} \widehat{f}(d) = \sum_{d \mid \gcd(n,r)} d\, f'(r/d)\tau(n/d) \qquad (n \in \mathbb{N}). \tag{24}$$

Proof. Using (23),

$$\sum_{d \mid n} \widehat{f}(d) = (\widehat{f} * \mathbf{1})(n) = (h * \mathbf{1} * \mathbf{1})(n) = (h * \tau)(n) = \sum_{d \mid n} h(d)\tau(n/d)$$

$$= \sum_{d \mid \gcd(n,r)} d\, f'(r/d)\tau(n/d).$$

□

In the special case $f = \rho_r$ we reobtain (cf. [2, Th. 1] – where σ should be replaced by τ, [16, p. 91]),

$$\sum_{d \mid n} c_r(d) = \sum_{d \mid \gcd(n,r)} d\mu(r/d)\tau(n/d) \qquad (n \in \mathbb{N}). \tag{25}$$

The DFT can be used to obtain short direct proofs of certain known properties for Ramanujan sums and special r-even functions. We give the following examples.

Application 2 By $\widehat{\rho}_r = c_r$, cf. (14), we obtain $\widehat{\widehat{\rho}}_r = r\rho_r$. Therefore, by Proposition 2,

$$\sum_{d \mid r} c_r(r/d) c_d(n) = \begin{cases} r, & \gcd(n,r) = 1, \\ 0, & \text{otherwise}, \end{cases} \tag{26}$$

see [16, p. 94].

Application 3 Let $f(n) = (-1)^n$, which is r-even for any even number r. Its DFT is

$$\widehat{f}(n) = \sum_{k=1}^{r} (-1)^k \exp(-2\pi i k n/r) = \sum_{k=1}^{r} (-\exp(-2\pi i n/r))^k, \tag{27}$$

which is r for $n = r/2 + mr$ ($m \in \mathbb{Z}$) and 0 otherwise. Using Proposition 2 we obtain for any even number r,

$$\sum_{d \mid r} (-1)^d c_{r/d}(n) = \begin{cases} r, & n \equiv r/2 \pmod{r}, \\ 0, & \text{otherwise}, \end{cases} \tag{28}$$

cf. [18, Th. IV], [16, p. 90].

Application 4 Let $f, h \in \mathcal{B}_r$. We show that their Cauchy product $f \otimes h \in \mathcal{B}_r$ and the Fourier coefficients of $f \otimes h$ are given by $\alpha_{f \otimes h}(d) = r\alpha_f(d)\alpha_h(d)$ for any $d \mid r$, cf. Section 2.3.

To obtain this use that $(\widehat{f \otimes h})(n) = \widehat{f}(n)\widehat{h}(n)$ ($n \in \mathbb{N}$), valid for functions $f, h \in \mathcal{D}_r$, cf. Section 2.2. Hence for any $n \in \mathbb{N}$,

$$(\widehat{f \otimes h})(\gcd(n,r)) = \widehat{f}(\gcd(n,r))\widehat{h}(\gcd(n,r)) = \widehat{f}(n)\widehat{h}(n) = (\widehat{f \otimes h})(n),$$

showing that $\widehat{f \otimes h}$ is r-even. It follows that $f \otimes h$ is also r-even. Furthermore, by (18), for every $d \mid r$,

$$\alpha_{f \otimes h}(d) = \frac{1}{r}(\widehat{f \otimes h})(r/d) = \frac{1}{r}\widehat{f}(r/d)\widehat{h}(r/d) = r\alpha_f(d)\alpha_h(d).$$

Application 5 Let $N_r(n,k)$ denote the number of (incongruent) solutions (mod r) of the congruence $x_1 + \ldots + x_k \equiv n \pmod{r}$ with $\gcd(x_1, r) = \ldots = \gcd(x_k, r) = 1$. Then it is immediate from the definitions that

$$N_r(\cdot, k) = \underbrace{\rho_r \otimes \cdots \otimes \rho_r}_{k}. \tag{29}$$

Therefore, $\widehat{N_r}(\bullet, k) = (\widehat{\rho_r})^k = (c_r)^k$. Now the IDFT formula (17) gives at once

$$N_r(n,k) = \frac{1}{r}\sum_{d|r}((c_r(r/d))^k c_d(n) \qquad (n \in \mathbb{N}), \tag{30}$$

formula which goes back to the work of H. Rademacher (1925) and A. Brauer (1926) and has been recovered several times. See [16, Ch. 3], [22, p. 41], [24].

Application 6 We give a new proof of the following inversion formula of Cohen [9, Th. 3]: If f and g are r-even functions and if f is defined by

$$f(n) = \sum_{d|r} g(d)c_d(n) \qquad (n \in \mathbb{N}), \tag{31}$$

then

$$g(m) = \frac{1}{r}\sum_{d|r} f(r/d)c_d(n), \quad m = r/\gcd(n,r), \qquad (n \in \mathbb{N}). \tag{32}$$

To show this consider the function $G(n) = g(r/\gcd(n,r))$ which is also r-even. By Proposition 2,

$$\widehat{G}(n) = \sum_{d|r} G(r/d)c_d(n) = \sum_{d|r} g(d)c_d(n) = f(n). \tag{33}$$

Hence

$$rg(m) = rG(n) = \widehat{\widehat{G}}(n) = \widehat{f}(n) = \sum_{d|r} f(r/d)c_d(n). \tag{34}$$

Application 7 Anderson and Apostol [1] and Apostol [2] investigated properties of r-even functions $S_{g,h}$ given by

$$S_{g,h}(n) = \sum_{d|\gcd(n,r)} g(d)h(r/d) \qquad (n \in \mathbb{N}), \tag{35}$$

where $g, h \in \mathcal{F}$ are arbitrary functions.

For $f = S_{g,h}$ we have according to (21) and Remark 1, $f'(n) = g(n)h(r/n)$ $(n \mid r)$ and obtain at once

$$\widehat{S_{g,h}}(n) = \sum_{d|\gcd(n,r)} df'(r/d) = \sum_{d|\gcd(n,r)} dg(r/d)h(d), \tag{36}$$

which is proved in [1, Th. 4] by other arguments.

Application 8 If f is any r-even function, then

$$\sum_{n=1}^{r} |\widehat{f}(n)|^2 = r \sum_{d|r} |f(d)|^2 \varphi(r/d). \tag{37}$$

This follows by the Parseval formula (7) and grouping the terms of the right hand side according to the values $\gcd(n,r)$. For $f = \rho_r$ we reobtain the familiar formula

$$\sum_{n=1}^{r} (c_r(n))^2 = r\varphi(r) \qquad (r \in \mathbb{N}). \tag{38}$$

5 Sequences of r-even functions

In this section we consider sequences of functions $(f_r)_{r\in\mathbb{N}}$ such that $f_r \in \mathcal{B}_r$ for any $r \in \mathbb{N}$. Note that the sequence $(f_r)_{r\in\mathbb{N}}$ can be viewed also as a function of two variables: $f\colon \mathbb{N}^2 \to \mathbb{C}$, $f(n,r) = f_r(n)$.

We recall here the following concept: A function $f\colon \mathbb{N}^2 \to \mathbb{C}$ of two variables is said to be multiplicative if $f(mn, rs) = f(m,r)f(n,s)$ for every $m,n,r,s \in \mathbb{N}$ such that $\gcd(mr, ns) = 1$. For example, the Ramanujan sum $c(n,r) = c_r(n)$ is multiplicative, viewed as a function of two variables.

The next result includes a generalization of this property of the Ramanujan sum.

Proposition 4 *Let* $(f_r)_{r\in\mathbb{N}}$ *be a sequence of functions. Assume that*
i) $f_r \in \mathcal{B}_r$ *(*$r \in \mathbb{N}$*),*
ii) $r \mapsto f_r(n)$ *is multiplicative (*$n \in \mathbb{N}$*).*
Then
1) the function $f\colon \mathbb{N}^2 \to \mathbb{C}$, $f(n,r) = f_r(n)$ *is multiplicative as a function of two variables,*
2) $f_r(m)f_r(n) = f_r(1)f_r(mn)$ *holds for any* $m,n,r \in \mathbb{N}$ *with* $\gcd(m,n) = 1$,
3) $n \mapsto f_r(n)$ *is multiplicative if and only if* $f_r(1) = 1$.

Proof. 1) For any $m,n,r,s \in \mathbb{N}$ such that $\gcd(mr,ns) = 1$ we have by i) and ii),

$$f_{rs}(mn) = f_r(mn)f_s(mn) = f_r(\gcd(mn,r))f_s(\gcd(mn),s)$$
$$= f_r(\gcd(m,r))f_s(\gcd(n,s)) = f_r(m)f_s(n).$$

2) By the definition of multiplicative functions of two variables $f\colon \mathbb{N}^2 \to \mathbb{C}$ it is immediate that $f(n,r) = \prod_p f(p^a, p^b)$ for $n = \prod_p p^a$, $r = \prod_p p^b$, and the given quasi-multiplicative property is a direct consequence of this equality.

3) Follows by 2). □

Part 1) of Proposition 4 is given also in [14] and for parts 2) and 3) cf. [23, Th. 80, 81].

We say that the sequence $(f_r)_{r\in\mathbb{N}}$ of functions is completely even if there exists a function $F \in \mathcal{F}$ of a single variable such that $f_r(n) = F(\gcd(n,r))$ for any $n, r \in \mathbb{N}$. This concept originates from Cohen [9] (for a function of two integer variables $f(n,r)$ satisfying $f(n,r) = F(\gcd(n,r))$ for any $n, r \in \mathbb{N}$ he used the term completely r-even function, which is ambiguous).

If the sequence $(f_r)_{r\in\mathbb{N}}$ is completely even, then $f_r \in \mathcal{B}_r$ for any $r \in \mathbb{N}$, but the converse is not true. For example, the Ramanujan sums $c_r(n)$ do not form a completely even sequence. To see this, assume the contrary and let p be any prime. Then for $n = r = p$, $F(p) = c_p(p) = p - 1$ and for $n = p, r = p^2$, $F(p) = c_{p^2}(p) = -p$, a contradiction.

If $(f_r)_{r\in\mathbb{N}}$ is completely even, then $f_r(n) = F(\gcd(n,r)) = \sum_{d\mid\gcd(n,r)} F'(d)$ $(n, r \in \mathbb{N})$ and by Remark 1 we have $f'_r(n) = F'(n)$ for any $n \mid r$, where $F' = \mu * F$.

6 The DFT of sequences of r-even functions

First we consider multiplicative properties of the DFT of sequences of r-even functions.

Proposition 5 *Let $(f_r)_{r\in\mathbb{N}}$ be a sequence of functions. Assume that*

i) $f_r \in \mathcal{B}_r$ *(*$r \in \mathbb{N}$*),*

ii) $r \mapsto f_r(n)$ *is multiplicative (*$n \in \mathbb{N}$*).*

Then

1) the function $r \mapsto \widehat{f}_r(n)$ *is multiplicative (*$n \in \mathbb{N}$*),*

2) the function $\widehat{f}\colon \mathbb{N}^2 \to \mathbb{C}$, $\widehat{f}(n,r) = \widehat{f}_r(n)$ *is multiplicative as a function of two variables,*

3) $\widehat{f}_r(m)\widehat{f}_r(n) = f'_r(r)\widehat{f}_r(mn)$ *holds for any* $m, n, r \in \mathbb{N}$ *with* $\gcd(m,n) = 1$*,*

4) $n \mapsto \widehat{f}_r(n)$ *is multiplicative if and only if* $f'_r(r) = 1$*.*

Proof. 1) Let $r, s \in \mathbb{N}$, $\gcd(r,s) = 1$. Then, for any fixed $n \in \mathbb{N}$, by Proposition 2 and using that $c_r(n)$ is multiplicative in r,

$$\widehat{f}_{rs}(n) = \sum_{d\mid rs} f_{rs}(d)c_{rs/d}(n) = \sum_{\substack{a\mid r\\ b\mid s}} f_{rs}(ab)c_{(r/a)(s/b)}(n)$$

$$= \sum_{\substack{a|r \\ b|s}} f_r(a) f_s(b) c_{r/a}(n) c_{s/b}(n) = \sum_{a|r} f_r(a) c_{r/a}(n) \sum_{b|s} f_s(b) c_{s/b}(n)$$

$$= \widehat{f}_r(n) \widehat{f}_s(n).$$

2), 3), 4) If $f_r \in \mathcal{B}_r$, then $\widehat{f}_r \in \mathcal{B}_r$ $(r \in \mathbb{N})$ and by 1) we know that the function $r \mapsto \widehat{f}_r(n)$ is multiplicative $(n \in \mathbb{N})$. Now apply Proposition 4 for the sequence $(\widehat{f}_r)_{r\in\mathbb{N}}$ and use that $\widehat{f}_r(1) = f'_r(r)$. □

Proposition 6 *Let* $(f_r)_{r\in\mathbb{N}}$ *be a sequence of functions such that* $f_r \in \mathcal{B}_r$ $(r \in \mathbb{N})$. *Then*

$$\sum_{d|r} \widehat{f}_d(n) = \sum_{d|\gcd(n,r)} d\, f_r(r/d) \qquad (n, r \in \mathbb{N}), \tag{39}$$

which is also r*-even* ($r \in \mathbb{N}$). *Furthermore,*

$$\sum_{d|n} \sum_{e|r} \widehat{f}_e(d) = \sum_{d|\gcd(n,r)} d\, f_r(r/d) \tau(n/d) \qquad (n, r \in \mathbb{N}). \tag{40}$$

Proof. Similar to the proof of Proposition 3. □

In the special case $f_r = \rho_r$ we reobtain the following known identities for the Ramanujan sum:

$$\sum_{d|r} c_d(n) = \begin{cases} r, & r \mid n, \\ 0, & r \nmid n, \end{cases} \tag{41}$$

$$\sum_{d|n} \sum_{e|r} c_e(d) = \begin{cases} r\,\tau(n/r), & r \mid n, \\ 0, & r \nmid n, \end{cases} \tag{42}$$

(41) being a familiar one and for (42) see [16, p. 91].

Consider in what follows the DFT of completely even sequences, defined in Section 5. Note that formulae (16) and (17) for the DFT and IDFT, respectively of such sequences (that is, functions with values $F(\gcd(n, r))$) were given by Schramm [21]. He considered also special cases of F.

Corollary 4 *Let* $(f_r)_{r\in\mathbb{N}}$ *be a sequence of functions. Assume that*

i) $(f_r)_{r\in\mathbb{N}}$ *is completely even with* $f_r(n) = F(\gcd(n, r))$ ($n, r \in \mathbb{N}$),

ii) F *is multiplicative.*

Then

1) the function $f\colon \mathbb{N}^2 \to \mathbb{C}$, $f(n,r) = f_r(n)$ *is multiplicative in both variables, with the other variable fixed, and is multiplicative as a function of two variables,*

2) the function $r \mapsto \widehat{f}_r(n)$ *is multiplicative* ($n \in \mathbb{N}$),

3) the function $\widehat{f}\colon \mathbb{N}^2 \to \mathbb{C}$, $\widehat{f}(n,r) = \widehat{f}_r(n)$ *is multiplicative as a function of two variables.*

4) $n \mapsto \widehat{f}_r(n)$ *is multiplicative if and only if* $F'(r) = 1$.

Proof. Follows from the definitions and from Proposition 5. □

The results of Section 4 can be applied for completely even sequences.

Corollary 5 *Let* $(f_r)_{r\in\mathbb{N}}$ *be a completely even sequence with* $f_r(n) = F(\gcd(n,r))$ $(n, r \in \mathbb{N})$. *Then*

$$\widehat{f}_r(n) = \sum_{d\mid\gcd(n,r)} d\, F'(r/d) \qquad (n, r \in \mathbb{N}), \tag{43}$$

$$\sum_{d\mid r} \widehat{f}_{r/d}(d) = \sum_{e^2 k = r} e\, F(k) \qquad (r \in \mathbb{N}). \tag{44}$$

Proof. Here (43) follows at once by Corollary 3, while (44) is a simple consequence of it. □

In particular, for $f_r = \rho_r$ (44) gives

$$\sum_{d\mid r} c_{r/d}(d) = \begin{cases} \sqrt{r}, & r \text{ is a square,} \\ 0, & \text{otherwise,} \end{cases} \tag{45}$$

see [16, p. 91].

It follows from (43) that the DFT of a completely even sequence of functions is a special case of the functions $S_{g,h}$ defined by (35), investigated by Anderson and Apostol [1], Apostol [2].

The example of $c_r(n)$ shows that the DFT sequence of a completely even sequence is, in general, not completely even ($c_r(n) = \widehat{\rho}_r(n)$, where $\rho_r(n) = \varepsilon(\gcd(n,r))$).

Consider now the completely even sequence $f_r(n) = \tau(\gcd(n,r))$. Then using (43),

$$\widehat{f}_r(n) = \sum_{d\mid\gcd(n,r)} d(\mu * \tau)(r/d) = \sum_{d\mid\gcd(n,r)} d = \sigma(\gcd(n,r)) \tag{46}$$

is completely even.

Next we characterize the completely even sequences such that their DFT is also a completely even sequence.

Proposition 7 *Let* $(f_r)_{r\in\mathbb{N}}$ *be a completely even sequence of functions with* $f_r(n) = F(\gcd(n,r))$. *Then the DFT sequence* $(\widehat{f}_r)_{r\in\mathbb{N}}$ *is completely even if and only if* $F = c\,\tau$, *where* $c \in \mathbb{C}$. *In this case* $\widehat{f}_r(n) = c\,\sigma(\gcd(n,r))$.

Proof. Assume that there is a function $G \in \mathcal{F}$ such that

$$\widehat{f}_r(n) = \sum_{d|\gcd(n,r)} d\,F'(r/d) = G(\gcd(n,r)).$$

Then for any $n = r \in \mathbb{N}$, $G(r) = \widehat{f}_r(r) = \sum_{d|r} d\,F'(r/d) = (\mathrm{id}*F')(r)$, hence G has to be $G = \mathrm{id}*F'$. Now for $n = 1$ and any $r \in \mathbb{N}$, $G(1) = \widehat{f}_r(1) = F'(r)$. Denoting $G(1) = c$ we obtain that F' is the constant function c. Therefore, $F = c\,\mathbf{1} * \mathbf{1} = c\,\tau$.

Conversely, for $F = c\,\tau$ we have $F' = \mu * c\,\tau = c\,\mathbf{1}$ and $\widehat{f}_r(n) = c\,\sum_{d|\gcd(n,r)} d = c\,\sigma(\gcd(n,r))$. □

We now give a Hölder-type identity, see (13), for the DFT of completely even sequences, which is a special case of [1, Th. 2], adopted to our case. We recall that a function $F \in \mathcal{F}$ is said to be strongly multiplicative if F is multiplicative and $F(p^a) = F(p)$ for every prime p and every $a \in \mathbb{N}$.

Proposition 8 *Let* $(f_r)_{r\in\mathbb{N}}$ *be a completely even sequence with* $f_r(n) = F(\gcd(n,r))$ $(n, r \in \mathbb{N})$. *Suppose that*

i) F *is strongly multiplicative,*

ii) $F(p) \neq 1 - p$ *for any prime* p.

Then

$$\widehat{f}_r(n) = \frac{(F*\mu)(m)(F*\varphi)(r)}{(F*\varphi)(m)}, \qquad m = r/\gcd(n,r), \qquad (n, r \in \mathbb{N}). \tag{47}$$

Furthermore, for every prime power p^a $(a \in \mathbb{N})$,

$$\widehat{f}_{p^a}(n) = \begin{cases} p^{a-1}(p + F(p) - 1), & p^a \mid n, \\ p^{a-1}(F(p) - 1), & p^{a-1} \parallel n, \\ 0, & p^{a-1} \nmid n. \end{cases} \tag{48}$$

Proof. Here for any prime p, $(F*\mu)(p) = F(p) - 1$, $(F*\mu)(p^a) = 0$ for any $a \geq 2$ and $(F*\varphi)(p^a) = p^{a-1}(F(p) + p - 1)$ for any $a \geq 1$. The function F is multiplicative, thus $\widehat{f}_r(n)$ is multiplicative in r, cf. Corollary 4. Therefore, it is sufficient to verify the given identity for $r = p^a$, a prime power. Consider three cases: Case 1) $p^a \mid n$, where $\gcd(n, p^a) = p^a$; Case 2) $p^a \parallel n$, where $\gcd(n, p^a) = p^{a-1}$; Case 3) $p^a \mid n$, where $\gcd(n, p^a) = p^\delta$ with $\delta \le a - 2$. $\square$

Recall that a function $f \in \mathcal{F}$ is said to be semi-multiplicative if $f(m)f(n) = f(\gcd(m,n))f(\operatorname{lcm}[m,n])$ for any $m, n \in \mathbb{N}$. For example, $r \mapsto c_r(n)$ is semi-multiplicative for any $n \in \mathbb{N}$. As a generalization of this property we have:

Corollary 6 *Let $(f_r)_{r\in\mathbb{N}}$ be a completely even sequence with $f_r(n) = F(\gcd(n,r))$ $(n, r \in \mathbb{N})$ satisfying conditions i) and ii) of Proposition 8. Then $r \mapsto \widehat{f}_r(n)$ is semi-multiplicative for any $n \in \mathbb{N}$.*

Proof. If $g \in \mathcal{F}$ is multiplicative, then it is known that for any constant C and any $r \in \mathbb{N}$, the function $n \mapsto C\, g(r/\gcd(n,r))$ is semi-multiplicative, cf. [19], and apply (47). $\square$

7 Mean values of the DFT of r-even functions

The mean value of a function $f \in \mathcal{F}$ is $m(f) = \lim_{x\to\infty} \frac{1}{x}\sum_{n\le x} f(n)$ if this limit exists. It is known that $\sum_{n\le x} c_r(n) = \mathcal{O}(1)$ for any $r > 1$. It follows from (10) that the mean value of any r-even function f exists and is given by $m(f) = \alpha_f(1) = \frac{1}{r}\widehat{f}(r) = \frac{1}{r}(f * \varphi)(r)$, using (18), (19) (see also [27, Prop. 1]). Therefore, if f is r-even, then the mean value of $\widehat{f}$ exists and is given by $m(\widehat{f}) = \frac{1}{r}\widehat{\widehat{f}}(r) = f(r)$. This follows also by Proposition 2. More exactly, we have

Proposition 9 *Let $f \in \mathcal{B}_r$ (with $r \in \mathbb{N}$ fixed).*

i) If $x \in \mathbb{N}$ and $r \mid x$, then

$$\sum_{n=1}^{x} \widehat{f}(n) = f(r)x. \tag{49}$$

ii) For any real $x \ge 1$,

$$\sum_{n\le x} \widehat{f}(n) = f(r)x + T_f(x), \quad |T_f(x)| \le \sum_{d\mid r} d|f'(r/d)|. \tag{50}$$

iii) The mean value of the DFT function $\widehat{f}$ is $f(r)$.

Proof. For any $x \geq 1$, by Corollary 3,

$$\sum_{n\le x} \widehat{f}(n) = \sum_{\substack{n\le x\\ d\mid \gcd(n,r)}} d\, f'(r/d) = \sum_{d\mid r} d\, f'(r/d)[x/d] = \sum_{d\mid r} d\, f'(r/d)(x/d - \{x/d\})$$

$$= x\sum_{d\mid r} f'(r/d) - \sum_{d\mid r} d\, f'(r/d)\{x/d\} = xf(r) + T_f(x),$$

where $T_f(x)$ is identically zero for $x \in \mathbb{N}$, $r \mid x$. Furthermore, $T_f(x) = \mathcal{O}(1)$ for $x \to \infty$. □

Now we generalize Ramanujan's formula

$$\sum_{n=1}^{\infty} \frac{c_r(n)}{n} = -\Lambda(r) \qquad (r > 1), \tag{51}$$

where Λ is the von Mangoldt function.

Proposition 10 *Let f be an r-even function ($r \in \mathbb{N}$).*
i) Then uniformly for x and r,

$$\sum_{n\le x} \frac{\widehat{f}(n)}{n} = f(r)(\log x + C) - (f*\Lambda)(r) + \mathcal{O}\left(x^{-1}V_f(x)\right), \quad V_f(x) = \sum_{d\mid r} d\,|f'(r/d)|, \tag{52}$$

where C is Euler's constant.
ii) If $f(r) = 0$, then

$$\sum_{n=1}^{\infty} \frac{\widehat{f}(n)}{n} = -(f*\Lambda)(r). \tag{53}$$

Proof. i) By Corollary 3,

$$\sum_{n\le x} \frac{\widehat{f}(n)}{n} = \sum_{n\le x} \frac{1}{n} \sum_{d\mid (n,r)} d\, f'(r/d) = \sum_{d\mid r} f'(r/d) \sum_{j\le x/d} \frac{1}{j}$$

$$= \sum_{d\mid r} f'(r/d)\big(\log(x/d) + C + \mathcal{O}(d/x)\big)$$

$$= (\log x + C)\sum_{d\mid r} f'(r/d) - \sum_{d\mid r} f'(r/d)\log d + \mathcal{O}\left(x^{-1}\sum_{d\mid r} d|f'(r/d)|\right)$$

$$= (\log x + C)f(r) - (f*\mu*\log)(r) + \mathcal{O}\left(x^{-1}\sum_{d\mid r} d|f'(r/d)|\right).$$

ii) Part ii) follows from i) with $x \to \infty$. □

Remark 3 There is no simple general formula for $\sum_{r \le x} \widehat{f}_r(n)$, where $n \in \mathbb{N}$ is fixed and $(f_r)_{r\in\mathbb{N}}$ is a sequence of r-even functions (for example, $c_r(0) = \varphi(r)$ and $c_r(1) = \mu(r)$ have different asymptotic behaviors). For asymptotic formulae concerning special functions of type $\sum_{k=1}^{n} F(\gcd(k,n))$ see the recent papers [5, 28].

8 Dirichlet series of the DFT of sequences of r-even functions

We consider the Dirichlet series of the DFT of sequences $(f_r)_{r\in\mathbb{N}}$ such that $f_r \in \mathcal{B}_r$ for any $r \in \mathbb{N}$. By $\widehat{f}_r(n) = (\eta_{\bullet}(n) * \mu * f_r)(r)$, cf. (22), we have formally,

$$\sum_{r=1}^{\infty} \frac{\widehat{f}_r(n)}{r^s} = \sum_{r=1}^{\infty} \frac{\eta_r(n)}{r^s} \sum_{r=1}^{\infty} \frac{(f_r * \mu)(r)}{r^s} = \frac{\sigma_{s-1}(n)}{n^{s-1}} \sum_{r=1}^{\infty} \frac{1}{r^s} \sum_{k\ell=r} \mu(k) f_r(\ell) \tag{54}$$
$$= \frac{\sigma_{s-1}(n)}{n^{s-1}} \sum_{k=1}^{\infty} \frac{\mu(k)}{k^s} \sum_{\ell=1}^{\infty} \frac{f_{k\ell}(\ell)}{\ell^s},$$

where $\sigma_k(n) = \sum_{d\mid n} d^k$. This can be written in a simpler form by considering the DFT of completely even sequences of functions.

Proposition 11 *Let $(f_r)_{r\in\mathbb{N}}$ be a completely even sequence of functions with $f_r(n) = F(\gcd(n,r))$ and let a_F denote the absolute convergence abscissa of the Dirichlet series of F. Then*

$$\sum_{r=1}^{\infty} \frac{\widehat{f}_r(n)}{r^s} = \frac{\sigma_{s-1}(n)}{n^{s-1}\zeta(s)} \sum_{r=1}^{\infty} \frac{F(r)}{r^s} \tag{55}$$

for any $n \in \mathbb{N}$, absolutely convergent for $\operatorname{Re} s > \max\{1, a_F\}$,

$$\sum_{n=1}^{\infty} \frac{\widehat{f}_r(n)}{n^s} = \zeta(s)(F * \phi_{1-s})(r) \tag{56}$$

for any $r \in \mathbb{N}$, absolutely convergent for $\operatorname{Re} s > 1$, where $\phi_k(r) = \sum_{d\mid r} d^k \mu(r/d)$ is a generalized Euler function,

$$\sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \frac{\widehat{f}_r(n)}{n^s r^t} = \frac{\zeta(s)\zeta(s+t-1)}{\zeta(t)} \sum_{n=1}^{\infty} \frac{F(n)}{n^t} \tag{57}$$

absolutely convergent for $\operatorname{Re} s > 1$, $\operatorname{Re} t > \max\{1, a_F\}$.

Proof. Apply (22) and (23). □

For $F = \varepsilon$ we reobtain the known formulae for the Ramanujan sum.

Acknowledgement

The authors thank Professor Lutz G. Lucht for very helpful suggestions on the presentation of this paper.

References

[1] D. R. Anderson, T. M. Apostol, The evaluation of Ramanujan's sums and generalizations, *Duke Math. J.*, **20** (1953), 211–216.

[2] T. M. Apostol, Arithmetical properties of generalized Ramanujan sums, *Pacific J. Math.*, **41** (1972), 281–293.

[3] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, 1976.

[4] M. Beck, M. Halloran, Finite trigonometric character sums via discrete Fourier analysis, *Int. J. Number Theory*, **6** (2010), 51–67.

[5] O. Bordellès, The composition of the gcd and certain arithmetic functions, *J. Integer Sequences*, **13** (2010), Article 10.7.1, 22 pp.

[6] W. L. Briggs, V. E. Henson, *The DFT – An Owner's Manual for the Discrete Fourier Transform*, Society for Industrial and Applied Mathematics (SIAM), 1995.

[7] S. A. Broughton, K. Bryan, *Discrete Fourier Analysis and Wavelets – Applications to Signal and Image Processing*, John Wiley & Sons, 2009.

[8] E. Cohen, A class of arithmetical functions, *Proc. Nat. Acad. Sci. U.S.A.*, **41** (1955), 939–944.

[9] E. Cohen, Representations of even functions (mod r), I. Arithmetical identities, *Duke Math. J.*, **25** (1958), 401–421.

[10] E. Cohen, Representations of even functions (mod r), II. Cauchy products, *Duke Math. J.*, **26** (1959), 165–182.

[11] E. Cohen, Representations of even functions (mod r), III. Special topics, *Duke Math. J.*, **26** (1959), 491–500.

[12] P. Haukkanen, An elementary linear algebraic approach to even functions (mod r), *Nieuw Arch. Wiskd.*, (5) **2** (2001), 29–31.

[13] P. Haukkanen, Discrete Ramanujan-Fourier transform of even functions (mod r), *Indian J. Math. Math. Sci.*, **3** (2007), 75–80.

[14] P. Haukkanen, L. Tóth, An analogue of Ramanujan's sum with respect to regular integers (mod r), *Ramanujan J.*, accepted.

[15] L. G. Lucht, Discrete Fourier transform of periodic functions (Memo, unpublished), Clausthal University of Technology, Clausthal, 2000.

[16] P. J. McCarthy, *Introduction to Arithmetical Functions*, Universitext, Springer, 1986.

[17] H. L. Montgomery, R. C. Vaughan, *Multiplicative Number Theory, I. Classical Theory*, Cambridge Studies in Advanced Mathematics 97, Cambridge University Press, 2007.

[18] C. A. Nicol, H. S. Vandiver, A von Sterneck arithmetical function and restricted partitions with respect to a modulus, *Proc. Nat. Acad. Sci. U.S.A.*, **40** (1954), 825–835.

[19] D. Rearick, Semi-multiplicative functions, *Duke Math. J.*, **33** (1966), 49–53.

[20] S. Samadi, M. O. Ahmad, M. N. S. Swamy, Ramanujan sums and discrete Fourier transforms, *IEEE Signal Processing Letters*, **12** (2005), 293–296.

[21] W. Schramm, The Fourier transform of functions of the greatest common divisor, *Integers*, **8** (2008), #A50, 7 pp.

[22] W. Schwarz, J. Spilker, *Arithmetical Functions*, London Mathematical Society Lecture Note Series, 184, Cambridge University Press, 1994.

[23] R. Sivaramakrishnan, *Classical Theory of Arithmetic Functions*, Monographs and Textbooks in Pure and Applied Mathematics, Vol. 126, Marcel Dekker, 1989.

[24] J. Spilker, Eine einheitliche Methode zur Behandlung einer linearen Kongruenz mit Nebenbedingungen, *Elem. Math.*, **51** (1996), 107–116.

[25] D. Sundararajan, *The Discrete Fourier Transform – Theory, Algorithms and Applications*, World Scientific Publishing Co., 2001.

[26] A. Terras, *Fourier Analysis on Finite Groups and Applications*, London Mathematical Society Student Texts, 43, Cambridge University Press, 1999.

[27] L. Tóth, Remarks on generalized Ramanujan sums and even functions, *Acta Math. Acad. Paedagog. Nyházi. (N.S.)*, electronic, **20** (2004), 233–238.

[28] L. Tóth, A survey of gcd-sum functions, *J. Integer Sequences*, **13** (2010), Article 10.8.1, 23 pp.

Received: September 27, 2010

Acta Univ. Sapientiae, Mathematica, **3**, 1 (2011) 26–33

Large families of almost disjoint large subsets of $\mathbb{N}$

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Abstract. In the paper we study the question of possible cardinality of a family of almost disjoint subsets of positive integers each being large with respect to a given criterion. For example, it is shown that there are continuum many almost disjoint subsets of $\mathbb{N}$ where each set is large in both the sense of (R)-density and in the sense of the upper weighted density. On the other hand, when considering sets with positive lower weighted density, the result is completely different.

1 Introduction

There is a simple standard fact that any family of disjoint subsets of a given countable set, e.g. the set of all positive integers, can be at most countable. On the other hand, slightly relaxing the condition of disjointness so that any pair of sets in the considered family can overlap in a finite set, the possible cardinality of such a family is that of continuum. In this paper we will study the question of maximal possible cardinality of almost disjoint families of sets of integers, so that each set in the family is large with respect to some criterion.

Denote by $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, the sets of all positive integers, rational numbers and real numbers, respectively. Two subsets of $\mathbb{N}$ are said to be *almost disjoint* if their intersection is finite. A family of subsets of $\mathbb{N}$ is said to be an *almost*

2010 Mathematics Subject Classification: 11B05
Key words and phrases: almost disjoint family, weighted density, (R)-dense set

disjoint family if it consists of pairwise almost disjoint sets. It is well known that there are almost disjoint families having cardinality of continuum. An easy way how to construct such a family may be described as follows.

- Map the set $\mathbb{N}$ by a one to one mapping b on the set $\mathbb{Q}$.
- For each real number r choose a sequence $\{s(r)\}$ of rational numbers converging to r.

Then the family $\{\{b^{-1}(s(r))\}\}_{r\in\mathbb{R}}$ consists of $\mathfrak{c}$ almost disjoint sets. Here $\mathfrak{c} = 2^{\aleph_0}$ stands for the cardinality of continuum.

In fact, in this example the almost disjoint family is constructed on the set of rationals and all sets in the family are very small from the natural point of view the topological density, all they are nowhere dense sets.

A natural question arises: can such a large almost disjoint family consist of sets which are "large in some sense" as subsets of $\mathbb{N}$?

In this paper any family $\mathcal{F}$ of subsets of $\mathbb{N}$ satisfying the condition

$$\text{If} \quad A \in \mathcal{F} \quad \text{and} \quad A \subset B \quad \text{then} \quad B \in \mathcal{F}.$$

will be called a *family of large sets in* $\mathbb{N}$.

The purpose of this paper is to investigate the largest possible cardinality of almost disjoint families consisting of sets large in some natural sense.

2 Families of (R)-dense sets

Denote by $R(A) = \left\{\frac{a}{b}; a \in A,\ b \in A\right\}$ the *ratio set of* A and say that a set A is (R)-dense if $R(A)$ is (topologically) dense in the set $(0, \infty)$. It is manifest that the class $\mathcal{D}$ of all (R)-dense sets forms a family of large sets in $\mathbb{N}$.

Theorem 1 *There exists an almost disjoint family of* $\mathfrak{c}$ *many (R)-dense sets.*

Proof. Let $\{J_n\}_{n=1}^{\infty}$ be a family of open subintervals of the interval $(0, \infty)$ forming a base for the ordinary topology on $(0, \infty)$. First, we will construct by induction a family $\{M_n\}_{n=1}^{\infty}$ of disjoint subsets of $\mathbb{N}$ such that each M_n is (R)-dense set. As a general rule used in the each step of the construction is the following:

Each element in the choice is different from all elements previously chosen.

Step 1. Choose $p_{11} \in \mathbb{N}$ and $q_{11} \in \mathbb{N}$ such that $\dfrac{p_{11}}{q_{11}} \in J_1$.

Step 2. Choose $p_{12} \in \mathbb{N}$ and $q_{12} \in \mathbb{N}$ such that $\dfrac{p_{12}}{q_{12}} \in J_2$. Then choose successively $p_{21} \in \mathbb{N}$ and $q_{21} \in \mathbb{N}$ such that $\dfrac{p_{21}}{q_{21}} \in J_1$ and $p_{22} \in \mathbb{N}$ and $q_{22} \in \mathbb{N}$ such that $\dfrac{p_{22}}{q_{22}} \in J_2$.

Step n. Choose for $i = 1, 2, 3, \ldots n-1$ successively $p_{in} \in \mathbb{N}$ and $q_{in} \in \mathbb{N}$ such that $\dfrac{p_{in}}{q_{in}} \in J_n$. Then choose for $j = 1, 2, 3, \ldots n$ successively $p_{nj} \in \mathbb{N}$ and $q_{nj} \in \mathbb{N}$ such that $\dfrac{p_{nj}}{q_{nj}} \in J_j$.

For each $n \in \mathbb{N}$ set $M_n = \{p_{n1}, q_{n1}, p_{n2}, q_{n2}, p_{n3}, q_{n3}, \ldots\}$. Then, by construction, all sets M_n, $n = 1, 2, \ldots$ are pairwise disjoint (R)-dense sets.

Now let $\mathcal{D}$ be any fixed almost disjoint family with cardinality of continuum. Let $D = \{d_1 < d_2 < d_3 < \ldots\} \in \mathcal{D}$. Define

$$\varphi(D) = \{p_{d_1 1}, q_{d_1 1}, p_{d_2 2}, q_{d_2 2}, p_{d_3 3}, q_{d_3 3}, \ldots\}.$$

We will show that for each $D \in \mathcal{D}$ the set $\varphi(D)$ is (R)-dense. Let U be an open set of real numbers. Then there exists a positive integer m such that $J_m \subset U$ and, by Step n, $\dfrac{p_{d_m m}}{q_{d_m m}} \in J_m \subset U$. Thus $\varphi(D)$ is an (R)-dense set.

Now let $D = \{d_1 < d_2 < d_3 < \ldots\}$ and $E = \{e_1 < e_2 < e_3 < \ldots\}$ be two sets in $\mathcal{D}$ and suppose that $k \in \varphi(D) \cap \varphi(E)$. Then there are positive integers m and n such that

$$k = p_{d_m m} = p_{e_n n} \qquad (\text{or } k = q_{d_m m} = q_{e_n n}),$$

consequently, by the above construction, we have $d_m = e_n$. As the sets D and E are almost disjoint, there are only finitely many such numbers $d_m = e_n$, $d_m \in D$, $e_n \in E$. Thus we have shown that $\{\varphi(D)\}_{D \in \mathcal{D}}$ is almost disjoint, so it is a required family. $\square$

3 Families of sets with large densities

For the rest of the paper let $f\colon \mathbb{N} \to (0,\infty)$. Denote by χ_A the characteristic function of a set A. For $A \subset \mathbb{N}$ define

$$\underline{d}_f(A) = \liminf_{x\to\infty} \frac{\sum\limits_{i\le x} f(i)\chi_A(i)}{\sum\limits_{i\le x} f(i)}, \qquad \overline{d}_f(A) = \limsup_{x\to\infty} \frac{\sum\limits_{i\le x} f(i)\chi_A(i)}{\sum\limits_{i\le x} f(i)}$$

$$d_f(A) = \lim_{x\to\infty} \frac{\sum\limits_{i\le x} f(i)\chi_A(i)}{\sum\limits_{i\le x} f(i)}$$

the *weighted lower* f*-density*, *weighted upper* f*-density*, and *weighted* f*-density* (if defined), respectively.

In this paper we will consider only functions f satisfying the condition

$$\sum_{n=1}^{\infty} f(n) = \infty. \tag{D}$$

Remark 1 *The most important special cases of weighted densities are those for* $f(i) \equiv 1$*, so called asymptotic density and for* $f(i) = \frac{1}{i}$*, so called logarithmic density. Also notice that the condition* (D) *guarantees that sets differing in a finite number of elements have the same upper and lower* f*-densities.*

Let $r \in (0,1]$. Then the classes $\mathbb{L}_f(r) = \{A \subset \mathbb{N}\ ;\ \underline{d}_f(A) \ge r\}$ and $\mathbb{U}_f(r) = \{A \subset \mathbb{N}\ ;\ \overline{d}_f(A) \ge r\}$ form families of large subsets in $\mathbb{N}$.

3.1 Sets with large lower f-densities

We will denote by $[x]$ the integer part of x, i.e. the largest integer less than or equal to x.

Theorem 2 *Let* f *fulfils the condition* (D) *and let* $\mathcal{S}$ *be an almost disjoint family. Then* $\sum\limits_{A\in\mathcal{S}} \underline{d}_f(A) \le 1$ *for every* $f\colon \mathbb{N} \to (0,\infty)$.

Proof. Suppose there exists an almost disjoint subfamily $\mathcal{S}$ of $\mathbb{L}_f(r)$ such that $\sum\limits_{A\in\mathcal{S}} \underline{d}_f(A) > 1$. Then it contains a finite subfamily $\mathcal{F} = \{F_1,\ F_2,\dots,F_n\}$ with $\sum\limits_{j=1}^{n} \underline{d}_f(F_j) = s > 1$. Since $\mathcal{F}$ is finite and almost disjoint, there is an integer

k_0 such that for every pair of distinct integers $i \in [1,n]$, $j \in [1,n]$ we have $F_i \cap F_j \cap [1,k_0] = \emptyset$. Let

$$G_j = F_j \cap (k_0, \infty) \quad \text{for each } j = 1, 2, \dots n.$$

Then the sets $\{G_j\}_{j=1}^n$ are pairwise disjoint and

$$\underline{d}_f(G_j) = \underline{d}_f(F_j) \quad \text{for each } j = 1, 2, \dots n.$$

Choose a positive number ε such that $s - n\varepsilon > 1$. Then there exists a positive integer m_0 such that for every $m > m_0$ and every $j = 1, 2, \dots, n$ we have

$$\frac{\sum\limits_{i \le m} f(i)\chi_{G_j}(i)}{\sum\limits_{i \le m} f(i)} > \underline{d}(G_j) - \varepsilon$$

Denote by $G = \bigcup\limits_{j=1}^{n} G_j$ and calculate

$$1 \ge \frac{\sum\limits_{i \le m} f(i)\chi_G(i)}{\sum\limits_{i \le m} f(i)} = \sum_{j=1}^{n} \frac{\sum\limits_{i \le m} f(i)\chi_{G_j}(i)}{\sum\limits_{i \le m} f(i)} > \sum_{j=1}^{n} (\underline{d}(G_j) - \varepsilon) = s - n\varepsilon > 1,$$

a contradiction. □

The following statement is a straightforward corollary to the previous theorem.

Corollary 1 *Let $r \in (0,1]$. Then every almost disjoint subfamily of $\mathbb{L}_f(r)$ consists of at most $\left[\frac{1}{r}\right]$ sets.*

Theorem 3 *Every almost disjoint family consisting of subsets of $\mathbb{N}$ with positive lower f-densities is at most countable.*

Proof. Let $\mathcal{S}$ be an almost disjoint family of subsets of $\mathbb{N}$ and let $\underline{d}_f(S) > 0$ for every $S \in \mathcal{S}$. Then $\mathcal{S} = \bigcup\limits_{n=1}^{\infty} \left(\mathcal{S} \cap \mathbb{L}_f\left(\frac{1}{n}\right)\right)$. By Corollary 1 every set in the union on the right side is finite, so $\mathcal{S}$ is at most countable. □

Remark 2 *It is easy to find a countable disjoint family of subsets with positive lower f-densities. Thus in the class of sets with positive lower f-density the maximum cardinality of disjoint families is the same as the maximum cardinality of almost disjoint families.*

3.2 Sets with large upper f-densities

In the case of the upper f-density our considerations will substantially differ from those in the case of the lower f-density.

Theorem 4 *Let* f *satisfy the condition* (D). *Then there exists an almost disjoint family of* $\mathfrak{c}$ *many sets each of which has the upper* f*-density equal to* 1.

Proof. First notice that due to the condition (D) for every $p \in \mathbb{N}$ and for every $\varepsilon > 0$ there exists $q \in \mathbb{N}$ such that $\frac{\sum\limits_{i=p+1}^{q} f(i)}{\sum\limits_{i=1}^{q} f(i)} > 1 - \varepsilon$. Choose by induction a sequence $\{k_n\}_{n=1}^{\infty}$ of positive integers as follows

Step 1. Put $k_1 = 1$.

Step n. Suppose that positive integers $k_1 < k_2 < \cdots < k_{n-1}$ have already been chosen. Let k_n be the smallest positive integer

$$\text{such that} \qquad \frac{\sum\limits_{i=k_{n-1}+1}^{k_n} f(i)}{\sum\limits_{i=1}^{k_n} f(i)} > 1 - \frac{1}{n}. \tag{I}$$

Now for each $n \in \mathbb{N}$ put $I_n = [k_{n-1} + 1, k_n] \cap \mathbb{N}$. Let $\mathcal{D}$ be any almost disjoint family with cardinality of continuum. For every $D \in \mathcal{D}$ define $\psi(D) = \bigcup\limits_{d \in D} I_d$. To see that $\mathcal{F} = \{\psi(D)\}_{D \in \mathcal{D}}$ is an almost disjoint family notice that the intersection of each pair of sets in $\mathcal{F}$ consists of union of finitely many finite intervals in $\mathbb{N}$, consequently it is finite. Let $D = \{d_1 < d_2 < \ldots\} \in \mathcal{D}$ and calculate

$$\overline{d}_f(\psi(D)) = \limsup_{n \to \infty} \frac{\sum\limits_{i=1}^{n} f(i)\chi_{\psi(D)}(i)}{\sum\limits_{i=1}^{n} f(i)} \geq \limsup_{n \to \infty} \frac{\sum\limits_{i=1}^{k_{d_n}} f(i)\chi_{\psi(D)}(i)}{\sum\limits_{i=1}^{k_{d_n}} f(i)} \geq$$

$$\limsup_{n\to\infty} \frac{\sum\limits_{i=k_{d_n-1}+1}^{k_{d_n}} f(i)}{\sum\limits_{i=1}^{k_{d_n}} f(i)} = 1,$$

thus $\mathcal{F}$ is a required family. □

Remark 3 *Putting* $r = 1$ *in Corollary 1 for lower* f*-densities and comparing its statement to that of Theorem 4 for upper* f*-densities shows a huge difference between the lower and upper* f*-densities relative to the question in our investigation.*

Remark 4 *In [1] it is proved that every subset of* $\mathbb{N}$ *with the upper asymptotic density equal to* 1 *is necessary (R)-dense. By this result, Theorem 1 is a corollary to Theorem 4, even we can say more.*

Theorem 5 *Let a function* f *fulfil the condition* (D)*. Then there exists an almost disjoint family of* $\mathfrak{c}$ *many sets each of which is (R)-dense and at the same time it has the upper* f*-density equal to* 1*.*

Proof. In the proof of this theorem we will follow the same idea as in the previous one. The only difference is in the induction step where the condition (I) should be changed to the stronger one: Let k_n be the smallest positive integer

$$\text{greater than } n(k_{n-1}+1) \text{ such that } \quad \frac{\sum\limits_{i=k_{n-1}+1}^{k_n} f(i)}{\sum\limits_{i=1}^{k_n} f(i)} > 1 - \frac{1}{n}. \tag{II}$$

Using the previous proof, we need only to prove that each set in the family $\mathcal{F} = \{\psi(D)\}_{D\in\mathcal{D}}$ is (R)-dense. Let $D = \{d_1, d_2, \dots\} \in \mathcal{D}$ and let $1 < a < b$ be given real numbers. Choose an integer $d_l \in D$ so that

$$b < d_l \qquad \text{and} \qquad \frac{1}{d_l} < b - a. \tag{1}$$

Condition (II) guarantees $k_{d_l} > d_l(k_{d_l-1} + 1)$, consequently

$$\frac{k_{d_l}}{k_{d_l-1}+1} > d_l > b. \tag{2}$$

Clearly $k_{d_{l-1}} + 1 \geq d_l$, thus we also have

$$\frac{1}{k_{d_l - 1} + 1} \leq \frac{1}{d_l}. \tag{3}$$

As $I_{d_l} = [k_{d_l - 1} + 1, k_{d_l}] \cap \mathbb{N} \subset \psi(D)$, by (1), (2) and (3) the set

$$\left\{ \frac{k_{d_l - 1} + 1}{k_{d_l - 1} + 1} < \frac{k_{d_l - 1} + 2}{k_{d_l - 1} + 1} < \cdots < \frac{k_{d_l} - 1}{k_{d_l - 1} + 1} < \frac{k_{d_l}}{k_{d_l - 1} + 1} \right\} \subset R(\psi(D))$$

intersects (a, b), thus $\psi(D)$ is (R)-dense. □

Remark 5 *In the case when* $\sum_{n=1}^{\infty} f(n) < \infty$ *the statement corresponding to that in Theorem 4 does not hold. In this case the statement corresponding to that in Theorem 3 for lower* f*-densities takes place.*

Acknowledgement

Supported by grants MSM6198898701 and VEGA 1/0753/10

References

[1] Šalát, T. On ratio set of sets of natural numbers, *Acta Arithmetica* **XV** (1969), 273-278.

Received: October 20, 2010

Acta Univ. Sapientiae, Mathematica, **3**, 1 (2011) 34–42

On imbalances in oriented multipartite graphs

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Abstract. An oriented k-partite graph(multipartite graph) is the result of assigning a direction to each edge of a simple k-partite graph. Let $D(V_1, V_2, \cdots, V_k)$ be an oriented k-partite graph, and let $d^+_{v_{ij}}$ and $d^-_{v_{ij}}$ be respectively the outdegree and indegree of a vertex v_{ij} in V_i. Define $b_{v_{ij}}$ (or simply b_{ij} as $b_{ij} = d^+_{v_{ij}} - d^-_{v_{ij}}$ as the imbalance of the vertex v_{ij}. In this paper, we characterize the imbalances of oriented k-partite graphs and give a constructive and existence criteria for sequences of integers to be the imbalances of some oriented k-partite graph. Also, we show the existence of an oriented k-partite graph with the given imbalance set.

1 Introduction

A digraph without loops and without multi-arcs is called a simple digraph. Mubayi et al. [1] defined the imbalance of a vertex v_i in a digraph as b_{v_i} (or simply b_i) $= d^+_{v_i} - d^-_{v_i}$, where $d^+_{v_i}$ and $d^-_{v_i}$ are respectively the outdegree and indegree of v_i. The imbalance sequence of a simple digraph is formed by

2010 Mathematics Subject Classification: 05C20
Key words and phrases: digaph, imbalance, outdegree, indegree, oriented graph, oriented multipartite graph, arc

listing the vertex imbalances in non-increasing order. A sequence of integers $F = [f_1, f_2, \cdots, f_n]$ with $f_1 \geq f_2 \geq \cdots \geq f_n$ is feasible if it has sum zero and satisfies $\sum_{i=1}^{k} f_i \leq k(n-k)$, for $1 \leq k < n$.

The following result [1] provides a necessary and sufficient condition for a sequence of integers to be the imbalance sequence of a simple digraph.

Theorem 1 *A sequence is realizable as an imbalance sequence if and only if it is feasible.*

The above result is equivalent to saying that a sequence of integers $B = [b_1, b_2, \cdots, b_n]$ with $b_1 \geq b_2 \geq \cdots \geq b_n$ is an imbalance sequence of a simple digraph if and only if for $1 \leq k < n$

$$\sum_{i=1}^{k} b_i \leq k(n-k),$$

with equality when $k = n$.

On arranging the imbalance sequence in non-decreasing order, we have the following observation.

Theorem 2 *A sequence of integers* $B = [b_1, b_2, \cdots, b_n]$ *with* $b_1 \leq b_2 \leq \cdots \leq b_n$ *is an imbalance sequence of a simple digraph if and only if for* $1 \leq k < n$

$$\sum_{i=1}^{k} b_i \geq k(n-k),$$

with equality when $k = n$.

Various results for imbalances in digraphs and oriented graphs can be found in [2, 3, 4, 5].

2 Imbalance sequences in oriented multipartite graphs

An oriented multipartite (k-partite) graph is the result of assigning a direction to each edge of a simple multipartite (k-partite) graph, $k \geq 2$. Throughout this paper we denote an oriented k-partite graph by k-OG, unless otherwise stated. Let $V_i = \{v_{i1}, v_{i2}, \cdots, v_{in_i}\}$, $1 \leq i \leq k$, be k parts of k-OG $D(V_1, V_2, \cdots, V_k)$, and let $d^+_{v_{ij}}$ and $d^-_{v_{ij}}$, $1 \le j \le n_i$, be respectively the outdegree and indegree of a vertex v_{ij} in V_i. Define $b_{v_{ij}}$ (or simply b_{ij} as $b_{ij} = d^+_{v_{ij}} - d^-_{v_{ij}}$ as the imbalance of the vertex v_{ij}. The sequences $B_i = [b_{i1}, b_{i2}, \cdots, b_{in_i}]$, $1 \le i \le k$, in nondecreasing order are called the imbalance sequences of $D(V_1, V_2, \cdots, V_k)$.

The k sequences of integers $B_i = [b_{i1}, b_{i2}, \cdots, b_{in_i}]$, $1 \le i \le k$, in nondecreasing order are said to be realizable if there exists an k-OG with imbalance sequences B_i, $1 \le i \le k$. Various criterions for imbalance sequences in k-OG can be found in [2].

For any two vertices v_{ij} in V_i and v_{lm} in V_l ($i \ne l$, $1 \le i \le l \le k$, $1 \le j \le n_i$, $1 \le m \le n_l$) of k-OG $D(V_1, V_2, \cdots, V_k)$, we have one of the following possibilities.

(i). An arc directed from v_{ij} to v_{lm}, denoted by $v_{ij}(1-0)v_{lm}$.

(ii). An arc directed from v_{lm} to v_{ij}, denoted by $v_{ij}(0-1)v_{lm}$.

(iii). There is no arc from v_{ij} to v_{lm} and there is no arc from v_{lm} to v_{ij} and this is denoted by $v_{ij}(0-0)v_{lm}$.

A triple in k-OG is an induced suboriented graph of three vertices with exactly one vertex from each part. For any three vertices v_{ij}, v_{lm} and v_{pq} in k-OG D, the triples of the form $v_{ij}(1-0)v_{lm}(1-0)v_{pq}(1-0)v_{ij}$, or $v_{ij}(1-0)v_{lm}(1-0)v_{pq}(0-0)v_{ij}$ are said to be oriented intransitive, while as the triples of the form $v_{ij}(1-0)v_{lm}(1-0)v_{pq}(0-1)v_{ij}$, or $v_{ij}(1-0)v_{lm}(0-1)v_{pq}(0-0)v_{ij}$, or $v_{ij}(1-0)v_{lm}(0-0)v_{pq}(0-1)v_{ij}$, or $v_{ij}(1-0)v_{lm}(0-0)v_{pq}(0-0)v_{ij}$, or $v_{ij}(0-0)v_{lm}(0-0)v_{pq}(0-0)v_{ij}$ are said to be oriented transitive. An k-OG is said to be oriented transitive if all its triples are oriented transitive, otherwise oriented intransitive.

We have the following observation.

Theorem 3 *Let D and D'be two k-OG with the same imbalance sequences. Then D can be transformed to D' by successively transforming appropriate triples in one of the following ways. Either (a) by changing a cyclic triple $v_{ij}(1-0)v_{lm}(1-0)v_{pq}(1-0)v_{ij}$ to an oriented transitive triple $v_{ij}(0-0)v_{lm}(0-0)v_{pq}(0-0)v_{ij}$ which has the same imbalance sequences, or vice versa, or (b) by changing an oriented intransitive triple $v_{ij}(1-0)v_{lm}(1-0)v_{pq}(0-0)v_{ij}$ to an oriented transitive triple $v_{ij}(0-0)v_{lm}(0-0)v_{pq}(0-1)v_{ij}$ which has the same imbalance sequences, or vice versa.*

Proof. Let B_i be the imbalance sequences of k-OG D whose parts are V_i, $1 \le i \le k$ and $|V_i| = n_i$. Let D' be k-OG with parts V'_i, $1 \le i \le k$. To prove the result, it is sufficent to show that D' can be obtained from D by successively transforming triples in any one of the ways as given in (a), or (b).

We fix n_i, $2 \le i \le k$ and use induction on n_1. For $n_1 = 1$, the result is obvious. Assume that the result holds when there are fewer than n_1 vertices in the first part. Let $j_2, j_3, \cdots, j_k$ be such that for $l_2 > j_2, l_3 > j_3, \cdots, l_k > j_k$, $1 \le j_2 < l_2 \le n_2, 1 \le j_3 < l_3 \le n_3, \cdots, 1 \le j_k < l_k \le n_k$, the corresponding arcs have the same orientations in D and D'. For $j_2, j_3, \cdots, j_k$ and $2 \le i, p, q \le k$, $p \ne q$, we have three cases to consider.

(i). $v_{1n_1}(1-0)v_{ij_p}(1-0)v_{ij_q}$ and $v'_{1n_1}(0-0)v'_{ij_p}(0-0)v'_{ij_q}$, (ii). $v_{1n_1}(0-0)v_{ij_p}(0-1)v_{ij_q}$ and $v'_{1n_1}(1-0)v'_{ij_p}(0-0)v'_{ij_q}$ and (iii). $v_{1n_1}(1-0)v_{ij_p}(0-0)v_{ij_q}$ and $v'_{1n_1}(0-0)v'_{ij_p}(0-1)v'_{ij_q}$.

Case (i). Since v_{1n_1} and v'_{1n_1} have equal imbalances, we have $v_{1n_1}(0-1)v_{ij_q}$ and $v'_{1n_1}(0-0)v'_{ij_q}$, or $v_{1n_1}(0-0)v_{ij_q}$ and $v'_{1n_1}(1-0)v'_{ij_q}$. Thus there is a triple $v_{1n_1}(1-0)v_{ij_p}(1-0)v_{ij_q}(1-0)v_{1n_1}$, or $v_{1n_1}(1-0)v_{ij_p}(1-0)v_{ij_q}(0-0)v_{1n_1}$ in D, and corresponding to these $v'_{1n_1}(0-0)v'_{ij_p}(0-0)v'_{ij_q}(0-0)v'_{1n_1}$, or $v'_{1n_1}(0-0)v'_{ij_p}(0-0)v'_{ij_q}(0-1)v'_{1n_1}$ respectively is a triple in D'.

Case (ii). Since v_{1n_1} and v'_{1n_1} have equal imbalances, we have $v_{1n_1}(1-0)v_{ij_q}$ and $v'_{1n_1}(0-0)v'_{ij_q}$. Thus there is a triple $v_{1n_1}(0-0)v_{ij_p}(0-1)v_{ij_q}(0-1)v_{1n_1}$ in D and corresponding to this $v'_{1n_1}(1-0)v'_{ij_p}(0-0)v'_{ij_q}(0-0)v'_{1n_1}$ is a triple in D'.

Case (iii). Since v_{1n_1} and v'_{1n_1} have equal imbalances, therefore we have $v_{1n_1}(0-1)v_{ij_q}$ and $v'_{1n_1}(0-0)v'_{ij_q}$. Thus $v_{1n_1}(1-0)v_{ij_p}(0-0)v_{ij_q}(1-0)v_{1n_1}$ is a triple in D, and corresponding to this $v'_{1n_1}(0-0)v'_{ij_p}(0-1)v'_{ij_q}(0-0)v'_{1n_1}$ is a triple in D'.

Therefore from (i), (ii) and (iii) it follows that there is an k-OG that can be obtained from D by any one of the transformations (a) or (b) with the imbalances remaining unchanged. Hence the result follows by induction. $\square$

Corollary 1 *Among all* k-*OG with given imbalance sequences, those with the fewest arcs are oriented transitive.*

A transmitter is a vertex with indegree zero. In a transitive oriented k-OG with imbalance sequences $B_i = [b_{i1}, b_{i2}, \cdots, b_{in_i}]$, $1 \le i \le k$, any of the vertices with imbalances b_{in_i}, can act as a transmitter.

The next result provides a useful recursive test of checking whether the sequences of integers are the imbalance sequences of k-OG.

Theorem 4 *Let* $B_i = [b_{i1}, b_{i2}, \cdots, b_{in_i}]$, $1 \le i \le k$, *be* k *sequences of integers in non-decreasing order with* $b_{1n_1} > 0$ *and* $b_{jn_j} \le \sum_{r=1, r\ne j}^{k} n_r$, *for all* j, $2 \le i \le k$. *Let* B'_1 *be obtained from* B_1 *by deleting one entry* b_{1n_1}, *and let*

$B'_2, B'_3, \cdots, B'_k$, *be obtained from* $B_2, B_3, \cdots, B_k$ *by increasing* b_{1n_1} *smallest entries of* $B_2, B_3, \cdots, B_k$ *by one each. Then* B_i *are imbalance sequences of some* k*-OG if and only if* B'_i *are imbalance sequences.*

Proof. Suppose B'_i be the imbalance sequences of some k-OG D' with parts V'_i, $1 \le i \le k$. Then k-OG D with imbalance sequences B_i can be obtained by adding a vertex v_{1n_1} in V'_1 such that $v_{1n_1}(1-0)v_{ij}$ for those vertices v_{ij} in V'_i, $i \neq 1$ whose imbalances are increased by one in going from B_i to B'_i.

Conversely, let B_i be the imbalance sequences of k-OG D with parts V_i, $1 \le i \le k$. By Corollary 4, any of the vertices v_{in_i} in V_i with imbalances b_{in_i}, $1 \le i \le k$ can be a transmitter. Assume that the vertex v_{1n_1} in V_1 with imbalance b_{1n_1} be a transmitter. Clearly, $d^+_{v_{1n_1}} > 0$ and $d^-_{v_{1n_1}} = 0$ so that $b_{1n_1} = d^+_{v_{1n_1}} - d^-_{v_{1n_1}} > 0$. Also, $d^+_{v_{jn_j}} \le \sum_{r=1,r\neq j}^{k} n_r$ and $d^-_{v_{jn_j}} \ge 0$ for $2 \le i \le k$ so that $b_{jn_j} = d^+_{v_{jn_j}} - d^-_{v_{jn_j}} \le \sum_{r=1,r\neq j}^{k} n_r$.

Let U be the set of v_{1n_1} vertices of smallest imbalances in V_j, $2 \le i \le k$ and let $W = V_2 \cup V_3 \cup \cdots \cup V_k - U$. Now construct D such that $v_{1n_1}(1-0)u$ for all u in U. Clearly $D - \{v_{1n_1}\}$ realizes V'_i, $1 \le i \le k$. $\square$

Theorem 5 provides an algorithm for determining whether or not the sequences B_i, $1 \le i \le k$ of integers in non-decreasing order are the imbalance sequences and for constructing a corresponding k-OG.

Suppose $B_i = [b_{i1}, b_{i2}, \cdots, b_{in_i}]$, $1 \le i \le k$, be imbalance sequences of k-OG with parts $V_i = \{v_{i1}, v_{i2}, \cdots, v_{in_i}\}$, where $b_{1n_1} > 0$ and $b_{jn_j} \le \sum_{r=1,r\neq j}^{k} n_r$, $2 \le i \le k$. Deleting b_{1n_1} and increasing b_{1n_1} smallest entries of $B_2, B_3, \cdots, B_k$ by 1 each to form $B'_2, B'_3, \cdots, B'_k$. Then arcs are defined by $v_{1n_1}(1-0)v_{ij}$, for which $b'_{v_{ij}} = b_{v_{ij}} + 1$, where $i \neq 1$. If at least one of the conditions $b_{1n_1} > 0$, or $b_{jn_j} \le \sum_{r=1,r\neq j}^{k} n_r$ does not hold, then we delete b_{in_i} for that i for which the conditions get satisfied and the same argument is used for defining arcs. If this method is applied recursively, then (i) it tests whether B_i are the imbalance sequences, and if B_i are the imbalance sequences (ii) k-OG $\triangle(B_i)$ with imbalance sequences B_i is constructed.

We illustrate this reduction and resulting construction as follows.

Consider the four sequences $B_1 = [1, 3, 4]$, $B_2 = [-3, 2, 2]$, $B_3 = [-4, -3]$ and $B_4 = [-3, 1]$.

(i) $[1, 3, 4]$, $[-3, 2, 2]$, $[-4, -3]$, $[-3, 1]$

(ii) $[1, 3]$, $[-2, 2, 2]$, $[-3, -2]$, $[-2, 1]$, $v_{13}(1-0)v_{21}$, $v_{13}(1-0)v_{31}$, $v_{13}(1-0)v_{32}$, $v_{13}(1-0)v_{41}$

(iii) $[1]$, $[-1, 2, 2]$, $[-2, -1]$, $[-2, 1]$, $v_{12}(1-0)v_{21}$, $v_{12}(1-0)v_{31}$, $v_{12}(1-0)v_{32}$

(iv) $\emptyset$, $[-1, 2, 2]$, $[-2, -1]$, $[-2, 1]$, $v_{11}(1-0)v_{31}$

(v) $\emptyset$, $[-1,2]$, $[0,-1]$, $[-1,1]$, $v_{23}(1-0)v_{31}$, $v_{23}(1-0)v_{41}$ or, $\emptyset$, $[-1,2]$, $[-1,0]$, $[-1,1]$

(vi)$\emptyset$, $[-1]$, $[0,0]$, $[0,1]$, $v_{22}(1-0)v_{32}$, $v_{22}(1-0)v_{41}$

(vii)$\emptyset$, $[0]$, $[0,0]$, $[0,0]$, $v_{42}(1-0)v_{21}$.

Clearly 4-OG with parts $V_1 = \{v_{11}, v_{12}, v_{13}\}$, $V_2 = \{v_{21}, v_{22}, v_{23}\}$, $V_3 = \{v_{31}, v_{32}\}$ and $V_4 = \{v_{41}, v_{42}\}$ in which $v_{13}(1-0)v_{21}$, $v_{13}(1-0)v_{31}$, $v_{13}(1-0)v_{32}$, $v_{13}(1-0)v_{41}$, $v_{12}(1-0)v_{21}$, $v_{12}(1-0)v_{31}$, $v_{12}(1-0)v_{32}$, $v_{11}(1-0)v_{31}$, $v_{23}(1-0)v_{31}$, $v_{23}(1-0)v_{41}$, $v_{22}(1-0)v_{32}$, $v_{22}(1-0)v_{41}$, $v_{42}(1-0)v_{21}$ are arcs has imbalance sequences $[1,3,4]$, $[-3,2,2]$, $[-4,-3]$ and $[-3,-1]$.

The next result gives a combinatorial criterion for determining whether k sequences of integers are realizable as imbalances.

Theorem 5 *Let* $B_i = [b_{i1}, b_{i2}, \cdots, b_{in_i}]$, $1 \le i \le k$, *be* k *sequences of integers in non-decreasing order. Then* B_i *are the imbalance sequences of some* k*-OG if and only if*

$$\sum_{i=1}^{k}\sum_{j=1}^{m_i} b_{ij} \ge 2\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} m_i m_j - \sum_{i=1}^{k} n_i \sum_{j=1}^{k} m_j - \sum_{i=1}^{k} m_i n_i, \tag{1}$$

for all sets of k *integers* m_i, $0 \le m_i \le n_i$ *with equality when* $m_i = n_i$.

Proof. The necessity of the condition follows from the fact that the k-OG induced by m_i vertices for $1 \le i \le k$, $1 \le m_i \le n_i$ has a sum of imbalances $2\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} m_i m_j - \sum_{i=1}^{k} n_i \sum_{j=1}^{k} m_j - \sum_{i=1}^{k} m_i n_i$.

For sufficiency, assume that $B_i = [b_{i1}, b_{i2}, \cdots, b_{in_i}]$, $1 \le i \le k$ be the sequences of integers in non-decreasing order satisfying conditions (1) but are not the imbalance sequences of any k-OG. Let these sequences be chosen in such a way that n_i, $1 \le i \le k$ are the smallest possible and b_{11} is the least for the choice of n_i. We consider the following two cases.

Case (i). Suppose equality in (1) holds for some $m_j \le n_j$, $1 \le i \le k-1$, $m_k \le n_k$, so that

$$\sum_{i=1}^{k}\sum_{j=1}^{m_i} b_{ij} = 2\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} m_i m_j - \sum_{i=1}^{k} n_i \sum_{j=1}^{k} m_j - \sum_{i=1}^{k} m_i n_i.$$

By the minimality of n_i, $1 \le i \le k$ the sequences $B'_i = [b_{i1}, b_{i2}, \cdots, b_{im_i}]$ are the imbalance sequences of some k-OG $D'(V'_1, V'_2, \cdots, V'_k)$.

Define $B''_i = [b_{i(m_i+1)}, b_{i(m_i+2)}, \cdots, b_{i(n_i)}]$, $1 \le i \le k$.

Consider the sum

$$\sum_{i=1}^{k}\sum_{j=1}^{f_i} b_{i(m_i+j)} = \sum_{i=1}^{k}\sum_{j=1}^{m_i+f_i} b_{ij} - \sum_{i=1}^{k}\sum_{j=1}^{m_i} b_{ij}$$

$$\geq 2\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}(m_i+f_i)(m_j+f_j) - \sum_{i=1}^{k} n_i \sum_{j=1}^{k}(m_j+f_j)$$

$$-\sum_{i=1}^{k}(m_i+f_i)n_i - 2\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} m_i m_j + \sum_{i=1}^{k} n_i \sum_{j=1}^{k} m_j + \sum_{i=1}^{k} m_i n_i$$

$$= 2\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} m_i m_j + 2\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}(m_i f_j + f_i m_j + f_i f_j) - \sum_{i=1}^{k} n_i \sum_{j=1}^{k} f_j$$

$$-\sum_{i=1}^{k} m_i n_i - \sum_{i=1}^{k} f_i n_i - 2\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} m_i m_j + \sum_{i=1}^{k} n_i \sum_{j=1}^{k} m_j +$$

$$+\sum_{i=1}^{k} m_i n_i \geq 2\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} f_i f_j - \sum_{i=1}^{k} n_i \sum_{j=1}^{k} f_j - \sum_{i=1}^{k} f_i n_i,$$

for $1 \leq f_i \leq n_i - m_i$, with equality when $f_i = n_i - m_i$ for all i, $1 \leq i \leq k$. So by the minimality of n_i, $1 \leq i \leq k$, the sequences B''_i form the imbalance sequence of some k-OG $D''(V''_1, V''_2, \cdots, V''_k)$.

Construct a new k-OG $D(V_1, V_2, \cdots, V_k)$ as follows. Let $V_1 = V'_1 \cup V''_1, V_2 = V'_2 \cup V''_2, \cdots V_k = V'_k \cup V''_k$ with $V'_i \cap V''_i = \emptyset$ and the arc set containing those arcs which are among $V'_1, V'_2, \cdots, V'_k$ and among $V''_1, V''_2, \cdots, V''_k$. Then $D(V_1, V_2, \cdots, V_k)$ has imbalance sequences B_i, $1 \leq i \leq k$, which is a contradiction.

Case (ii). Assume that the strict inequality holds in (1) for some $m_i \neq n_i$, $1 \leq i \leq k$. Let $B'_1 = [b_{11} - 1, b_{12}, \cdots, b_{1n_1-1}, b_{1n_1}]$ and let $B'_j = [b_{j1}, b_{j2}, \cdots, b_{jn_j}]$ for all j, $2 \leq j \leq k$. Clearly the sequences B'_i, $1 \leq i \leq k$ satisfy conditions (1). Therefore, by the minimality of b_{11}, the sequences B'_i, $1 \leq i \leq k$ are the imbalance sequences of some k-OG $D'(V'_1, V'_2, \cdots, V'_k)$. Let $b_{v_{11}} = b_{11} - 1$ and $b_{v_{1n_1}} = b_{1n_1} + 1$. Since $b_{v_{1n_1}} > b_{v_{11}} + 1$, there exists a vertex v_{ij} either in V_i, $1 \leq i \leq k$, $1 \leq j \leq n_i$, such that $v_{1n_1}(0-0)v_{ij}(1-0)v_{11}$, or $v_{1n_1}(1-0)v_{ij}(0-0)v_{11}$, or $v_{1n_1}(1-0)v_{ij}(1-0)v_{11}$, or $v_{1n_1}(0-0)v_{ij}(0-0)v_{11}$ in $D'(V'_1, V'_2, \cdots, V'_k)$, and if these are changed to $v_{1n_1}(0-1)v_{ij}(0-0)v_{11}$, or $v_{1n_1}(0-0)v_{ij}(0-1)v_{11}$, or $v_{1n_1}(0-0)v_{ij}(0-0)v_{11}$, or $v_{1n_1}(0-1)v_{ij}(0-1)v_{11}$ respectively, the result is k-OG with imbalance sequences B_i, which is a contradiction. This completes the proof. □

3 Imbalance sets in oriented multipartite graphs

The set of distinct imbalances of the vertices in k-OG is called its imbalance set. Now we give the existence of k-OG with a given imbalance set.

Theorem 6 *Let* $S = \{s_1, s_2, \cdots, s_n\}$ *and* $T = \{-t_1, -t_2, \cdots, -t_n\}$, *where* $s_1, s_2, \cdots, s_n, t_1, t_2, \cdots, t_n$ *are positive integers with* $s_1 < s_2 < \cdots < s_n$ *and* $t_1 < t_2 < \cdots < t_n$. *Then there exists* k-*OG with imbalance set* $S \cup T$.

Proof. First assume that $k \geq 2$ is even. Construct k-OG $D(V_1, V_2, \cdots, V_k)$ as follows. Let

$$V_1 = V_{11} \cup V_{12} \cup \cdots \cup V_{1n},$$

$$V_2 = V_{21} \cup V_{22} \cup \cdots \cup V_{2n},$$

$$\cdots$$

$$V_k = V_{k1} \cup V_{k2} \cup \cdots \cup V_{kn},$$

with $V_{ij} \cap V_{lm} = \emptyset$, $|V_{ij}| = t_i$ for all odd i, $1 \leq i \leq k-1$, $1 \leq j \leq n$ and $|V_{ij}| = s_i$ for all even i, $2 \leq i \leq k$, $1 \leq j \leq n$. Let there be an arc from each vertex of V_{ij} to every vertex of $V_{(i+1)}j$ for all odd i, $1 \leq i \leq k-1$, $1 \leq j \leq n$ so that we obtain k-OG with imbalance of vertices as follows.

For odd i, $1 \leq i \leq k-1$ and $1 \leq j \leq n$

$$b_{v_{ij}} = |V_{(i+1)j}| - 0 = s_i,$$

for all $v_{ij} \in V_{ij}$; and for even i, $2 \leq i \leq k$ and $1 \leq j \leq n$

$$b_{v_{ij}} = 0 - |V_{(i+1)j}| = -t_i,$$

for all $v_{ij} \in V_{ij}$

Therefore imbalance set of $D(V_1, V_2, \cdots, V_k)$ is $S \cup T$.

Now assume $k \geq 3$ is odd. Construct k-OG $D(V_1, V_2, \cdots, V_k)$ as below. Let

$$\begin{aligned}
&V_1 = V_{11} \cup V'_{11} \cup V_{12} \cup V'_{12} \cup \ldots \cup V_{1n} \cup V'_{1n},\\
&V_2 = V_{21} \cup V_{22} \cup \ldots \cup V_{2n},\\
&\ldots\\
&V_{k-1} = V_{(k-1)1} \cup V_{(k-1)2} \cup \ldots \cup V_{(k-1)n},\\
&V_k = V'_{k1} \cup V'_{k2} \cup \ldots \cup V'_{kn},
\end{aligned}$$

with $V_{ij} \cap V_{lm} = \emptyset$, $V'_{ij} \cap V'_{lm} = \emptyset$, $V_{ij} \cap V'_{lm} = \emptyset$, $|V_{ij}| = t_i$ for all i, $1 \leq i \leq k-2$, $1 \leq j \leq n$, $|V_i j| = s_i$ for all even i, $2 \leq i \leq k-1$, $1 \leq j \leq n$, $|V'_{ij}| = t_i$ for all j, $1 \leq j \leq n$ and $|V'_{kj}| = s_j$ for all j, $1 \leq j \leq n$. Let there be an arc from each vertex of V_{ij} to every vertex of $V_{(i+1)j}$ for all i, $1 \leq i \leq k-2$, $1 \leq j \leq n$ and let there be an arc from each vertex of V'_{1j} to every vertex of V'_{kj} for all j, $1 \leq j \leq n$, so that we obtain k-OG with imbalance set $S \cup T$, as above. □

References

[1] D. Mubayi, T. G. Will, D. B. West, Realizing degree imbalances in directed graphs, *Discrete Math.*, **239** (2001), 147–153.

[2] S. Pirzada, On imbalances in digraphs, *Kragujevac J. Math.*, **31** (2008), 143–146.

[3] S. Pirzada, T. A. Naikoo, U. Samee, A. Iványi, Imbalances in multidigraphs, *Acta Univ. Sapientiae, Math.*, **2** (2010), 137–146.

[4] S. Pirzada, T. A. Naikoo, N. A. Shah, Imbalances in oriented tripartite graphs, *Acta Math. Sinica*, **27** (2011), 927–932.

[5] U. Samee, T. A. Cahisti, On imbalances in oriented bipartite graphs, *Eurasian Math. J.*, **1** (2010), 136–141.

Received: October 20, 2010

Acta Univ. Sapientiae, Mathematica, **3**, 1 (2011) 43–59

On generalized Hurwitz–Lerch Zeta distributions occuring in statistical inference

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Abstract. The object of the present paper is to define certain new incomplete generalized Hurwitz–Lerch Zeta functions and incomplete generalized Gamma functions. Further, we introduce two new statistical distributions named as, generalized Hurwitz–Lerch Zeta Beta prime distribution and generalized Hurwitz–Lerch Zeta Gamma distribution and investigate their statistical functions, such as moments, distribution and survivor function, characteristic function, the hazard rate function and the mean residue life functions. Finally, Moment Method parameter estimators are given by means of a statistical sample of size n. The results obtained provide an elegant extension of the work reported earlier by Garg *et al.* [3] and others.

2010 Mathematics Subject Classification: 11M35, 33C05, 60E05, 60E10
Key words and phrases: Riemann zeta function, Lerch zeta function, Hurwitz–Lerch Zeta function, hazard function, mean residual life function, characteristic function, Planck distribution, generalized Beta prime distribution, moment method parameter estimation

1 Introduction and preliminaries

A generalized Hurwitz–Lerch Zeta function $\Phi(z,s,a)$ is defined [1, p. 27, Eq. 1.11.1] as the power series

$$\Phi(z,s,a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}, \tag{1}$$

where $a \in \mathbb{C}\backslash\mathbb{Z}_0^-$; $\Re\{s\} > 1$ when $|z| = 1$ and $s \in \mathbb{C}$ when $|z| < 1$ and continues meromorphically to the complex s–plane, except for the simple pole at $s = 1$, with its residue equal to 1.

The function $\Phi(z,s,a)$ has many special cases such as Riemann Zeta [1], Hurwitz–Zeta [23] and Lerch Zeta function [27, p. 280, Example 8]. Some other special cases involve the polylogarithm (or Jonqière's function) and the generalized Zeta function [27, p. 280, Example 8], [23, p. 122, Eq. 2.5] discussed for the first time by Lipschitz and Lerch.

Lin and Srivastava investigated [12, p. 727, Eq. 8] the Hurwitz–Lerch Zeta function in the following form

$$\Phi_{\mu,\nu}^{(\rho,\sigma)}(z,s,a) = \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}}{(\nu)_{\sigma n}} \frac{z^n}{(n+a)^s}, \tag{2}$$

where $\mu \in \mathbb{C}; a, \nu \in \mathbb{C}\backslash\mathbb{Z}_0^-; \rho, \sigma \in \mathbb{R}_+; \rho < \sigma$ for $s, z \in \mathbb{C}$; $\rho = \sigma$ for $z \in \mathbb{C}$; $\rho = \sigma$, $s \in \mathbb{C}$ for $|z| < 1$; $\rho = \sigma$, $\Re\{s - \mu + \nu\} > 1$ for $|z| = 1$. Here $(\theta)_{\kappa n} = \Gamma(\theta + \kappa n)/\Gamma(\theta)$ denotes the generalized Pochhammer symbol, with the convention $(\theta)_0 = 1$.

Recently, Srivastava *et al.* [24] studied a new family of the Hurwitz–Lerch Zeta function

$$\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a) = \sum_{n=0}^{\infty} \frac{(\lambda)_{\rho n}(\mu)_{\sigma n}}{(\nu)_{\kappa n}} \frac{z^n}{(n+a)^s n!}, \tag{3}$$

where $\lambda, \mu \in \mathbb{C}$; $a, \nu \in \mathbb{C}\backslash\mathbb{Z}_0^-$; $\rho, \sigma, \kappa > 0$; for $|z| < 1$ and $\Re\{s + \nu - \lambda - \mu\} > 1$ for $|z| = 1$. Function (3) is a generalization of Hurwitz–Lerch Zeta function $\Phi_{\lambda,\mu,\nu}(z,s,a) := \Phi_{\lambda,\mu,\nu}^{(1,1,1)}(z,s,a)$ which has been studied by Garg *et al.* [2]. Special attention will be given to the special case of (3) (studied earlier by Goyal and Laddha [4, p. 100, Eq. (1.5)])

$$\Phi_\mu^*(z,s,a) := \Phi_{1,\mu,1}^{(1,1,1)}(z,s,a) = \sum_{n=1}^{\infty} \frac{(\mu)_n}{(n+a)^s} \frac{z^n}{n!}. \tag{4}$$

Another case of the Hurwitz–Lerch Zeta function (3), which differs in the choice of parameters, have been considered in [24] as well. Moreover, the article [24] contains the integral representation

$$\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-at}\,{}_2\Psi_1^*\left[\begin{array}{c}(\lambda,\rho),(\mu,\sigma)\\(\nu,\kappa)\end{array}\Big|\, ze^{-t}\right]dt, \tag{5}$$

valid for all $a, s \in \mathbb{C}, \Re\{a\} > 0, \Re\{s\} > 0$, when $|z| \le 1$, $z \ne 1$; and $\Re\{s\} > 1$ for $z = 1$. Here

$${}_p\Psi_q^*\left[\begin{array}{c}(a,A)_p\\(b,B)_q\end{array}\Big|\, z\right] = \sum_{n=0}^\infty \frac{\prod_{j=1}^p (a_j)_{A_j n}}{\prod_{j=1}^q (b_j)_{B_j n}}\,\frac{z^n}{n!} \tag{6}$$

stands for the *unified variant of the Fox–Wright generalized hypergeometric function* with p upper and q lower parameters; $(a,A)_p$ denotes the parameter p–tuple $(a_1,A_1),\cdots,(a_p,A_p)$ and $a_j \in \mathbb{C}$, $b_i \in \mathbb{C}\setminus\mathbb{Z}_0^-$, $A_i, B_j > 0$ for all $j = \overline{1,p}, i = \overline{1,q}$, while the series converges for suitably bounded values of $|z|$ when

$$\Delta := 1 + \sum_{j=1}^q B_j - \sum_{j=1}^p A_j > 0\,.$$

In the case $\Delta = 0$, the converegence holds in the open disc $|z| < \beta = \prod_{j=1}^q B_j^{B_j} \cdot \prod_{j=1}^p A_j^{-A_j}$.

Remark 1 *Let us point out that the original definition of the Fox–Wright function ${}_p\Psi_q[z]$ (consult monographs [1, 11, 15]) contains Gamma functions instead of the here used generalized Pochhammer symbols. However, these two functions differ only up to constant multiplying factor, that is*

$${}_p\Psi_q\left[\begin{array}{c}(a,A)_p\\(b,B)_q\end{array}\Big|\, z\right] = \frac{\prod_{j=1}^p \Gamma(a_j)}{\prod_{j=1}^q \Gamma(b_j)}\,{}_p\Psi_q^*\left[\begin{array}{c}(a,A)_p\\(b,B)_q\end{array}\Big|\, z\right].$$

The unification's motivation is clear - for $A_1 = \cdots = A_p = B_1 = \cdots = B_q = 1$, ${}_p\Psi_q^*[z]$ one reduces exactly to the generalized hypergeometric function ${}_pF_q[z]$, see recent articles [12, 24].

Finally, we recall the integral expression for function (3), derived by Srivastava *et al.* [24]:

$$\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a) = \frac{\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\nu-\lambda)}\int_0^\infty \frac{t^{\lambda-1}}{(1+t)^\nu}\,\Phi_{\mu,\nu-\lambda}^{(\sigma,\kappa-\rho)}\Big(\frac{zt^\rho}{(1+t)^\kappa}, s, a\Big)dt, \tag{7}$$

where $\Re\{\nu\} > \Re\{\lambda\} > 0,\ \kappa \geq \rho > 0,\ \sigma > 0, s \in \mathbb{C}$.

Now, we study generalized incomplete functions and the associated statistical distributions based mainly on integral expressions (5) and (7).

2 Families of incomplete φ and ξ functions

By virtue of integral (7), we define the *lower incomplete generalized Hurwitz–Lerch Zeta function* as

$$\varphi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a|x) = \frac{\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\nu-\lambda)}\int_0^x \frac{t^{\lambda-1}}{(1+t)^\nu}\,\Phi_{\mu,\nu-\lambda}^{(\sigma,\kappa-\rho)}\Big(\frac{zt^\rho}{(1+t)^\kappa},s,a\Big)\mathrm{d}t, \quad (8)$$

and the *upper (complementary) generalized Hurwitz–Lerch Zeta function* in the form

$$\overline{\varphi}_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a|x) = \frac{\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\nu-\lambda)}\int_x^\infty \frac{t^{\lambda-1}}{(1+t)^\nu}\,\Phi_{\mu,\nu-\lambda}^{(\sigma,\kappa-\rho)}\Big(\frac{zt^\rho}{(1+t)^\kappa},s,a\Big)\mathrm{d}t. \quad (9)$$

In both cases one requires $\Re(\nu), \Re(\lambda) > 0,\ \kappa \geq \rho > 0;\ \sigma > 0,\ s \in \mathbb{C}$.

From (8) and (9) readily follows that

$$\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a) = \lim_{x\to\infty}\varphi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a|x) = \lim_{x\to 0+}\overline{\varphi}_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a|x)\,, \quad (10)$$

$$\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a) = \varphi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a|x) + \overline{\varphi}_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a|x), \qquad x \in \mathbb{R}_+. \quad (11)$$

In view of the integral expression (5), the *lower incomplete generalized Gamma function* and the *upper (complementary) incomplete generalized Gamma function* are defined respectively by

$$\xi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a,b|x) = \frac{b^s}{\Gamma(s)}\int_0^x t^{s-1}\mathrm{e}^{-at}\,{}_2\Psi_1^*\Big[\begin{matrix}(\alpha,\rho),(\mu,\sigma)\\(\nu,\kappa)\end{matrix}\Big|\,z\mathrm{e}^{-bt}\Big]\mathrm{d}t \quad (12)$$

and

$$\overline{\xi}_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa);x,\infty}(z,s,a,b|x) = \frac{b^s}{\Gamma(s)}\int_x^\infty t^{s-1}\mathrm{e}^{-at}{}_2\Psi_1^*\Big[\begin{matrix}(\alpha,\rho),(\mu,\sigma)\\(\nu,\kappa)\end{matrix}\Big|\,z\mathrm{e}^{-bt}\Big]\mathrm{d}t, \quad (13)$$

where $\Re\{a\}, \Re\{s\} > 0$, when $|z| \leq 1\,(z \neq 1)$ and $\Re\{s\} > 1$, when $z = 1$, provided that each side exists. By virtue of (12) and (13) we easily conclude the properties:

$$\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\rho)}(z,s,a) = \lim_{x\to\infty}\xi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a,b|x) = \lim_{x\to 0+}\overline{\xi}_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a,b|x)\,, \quad (14)$$

$$\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a/b) = \xi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a,b|x) + \overline{\xi}_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a,b|x), \qquad x \in \mathbb{R}_+. \quad (15)$$

3 Generalized Hurwitz–Lerch Zeta Beta prime distribution

Special functions and integral transforms are useful in the development of the theory of probability density functions (PDF). In this connection, one can refer to the books e.g. by Mathai and Saxena [14, 15] or by Johnson and Kotz [8, 9]. Hurwitz–Lerch Zeta distributions are studied by many mathematicians such as Dash, Garg, Gupta, Kalla, Saxena, Srivastava etc. (see e.g. [2, 3, 6, 7, 18, 19, 20, 21, 25]). Due to usefulness and popularity of Hurwitz–Lerch Zeta distribution in reliability theory, statistical inference etc. the authors are motivated to define a generalized Hurwitz–Lerch Zeta distribution and to investigate its important properties.

Let the random variable X be defined on some fixed standard probability space $(\Omega, \mathfrak{F}, P)$. The r.v. X such that possesses PDF

$$f(x) = \begin{cases} \dfrac{\Gamma(\nu)\, x^{\lambda-1}}{\Gamma(\lambda)\Gamma(\nu-\lambda)(1+x)^{\nu}} \; \dfrac{\Phi^{(\sigma,\kappa-\rho)}_{\mu,\nu-\lambda}\Big(\dfrac{zx^{\rho}}{(1+x)^{\kappa}}, s, a\Big)}{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s,a)} & x > 0, \\ 0 & x \le 0, \end{cases} \tag{16}$$

we call *generalized Hurwitz–Lerch Zeta Beta prime* and write $X \sim \mathrm{HLZB}'$. Here μ, λ are shape parameters, and z stands for the scale parameter which satisfy $\Re\{\nu\} > \Re\{\lambda\} > 0$, $s \in \mathbb{C}$, $\kappa \ge \rho > 0$, $\sigma > 0$.

The behaviour of the PDF $f(x)$ at $x = 0$ depends on λ in the manner that $f(0) = 0$ for $\lambda > 1$, while $\lim_{x\to 0+} f(x) = \infty$ for all $0 < \lambda < 1$.

Now, let us mention some interesting special cases of PDF (16).

(i) For $\sigma = \rho = \kappa = 1$ we get the following Hurwitz–Lerch Zeta Beta prime distribution discussed by Garg *et al.* [3]:

$$f_1(x) = \begin{cases} \dfrac{\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\nu-\lambda)\Phi_{\lambda,\mu,\nu}(z,s,a)} \; \dfrac{x^{\lambda-1}}{(1+x)^{\nu}} \, \Phi^{*}_{\mu}\Big(\dfrac{zx}{1+x}, s, a\Big) & x > 0, \\ 0 & \text{elsewhere} \end{cases}$$

where $a \notin \mathbb{Z}_0^-$, $\Re\{\nu\} > \Re\{\lambda\} > 0$, $x \in \mathbb{R}$, $s \in \mathbb{C}$ when $|z| < 1$ and $\Re\{s-\mu\} > 0$, when $|z| = 1$. Here $\Phi^{*}_{\mu}(\cdot, s, a)$ stands for the Goyal–Laddha type generalized Hurwitz–Lerch Zeta function described in (4).

(ii) If we set $\sigma = \rho = \kappa = \lambda = 1$ it gives a new probability distribution function, defined by

$$f_2(x) = \begin{cases} \dfrac{\nu-1}{(1+x)^{\nu}\,\Phi_{1,\mu,\nu}(z,s,a)}\,\Phi^*_{\mu}\Big(\dfrac{zx}{1+x},s,a\Big) & x>0,\\ 0 & x\le 0,\end{cases} \tag{17}$$

where $a \notin \mathbb{Z}_0^-$, $\Re\{\lambda\} > 0$, $x \in \mathbb{R}$, $s \in \mathbb{C}$ when $|z| < 1$ and $\Re\{s-\mu\} > 0$, when $|z| = 1$.

(iii) When $\sigma = \rho = \kappa = 1$ and $\nu = \mu$, from (16) it follows

$$f_3(x) = \begin{cases} \dfrac{\Gamma(\mu)}{\Gamma(\lambda)\Gamma(\mu-\lambda)\Phi^*_{\lambda}(z,s,a)}\,\dfrac{x^{\lambda-1}}{(1+x)^{\mu}}\,\Phi^*_{\mu}\Big(\dfrac{zx}{1+x},s,a\Big) & x>0,\\ 0 & x\le 0,\end{cases} \tag{18}$$

with $a \notin \mathbb{Z}_0^-$, $\Re\{\mu\} > \Re\{\lambda\} > 0$, $x \in \mathbb{R}$, $s \in \mathbb{C}$ when $|z| < 1$ and $\Re\{s-\mu\} > 0$, when $|z| = 1$.

(iv) For $\sigma = \rho = \kappa = 1$ and $\mu = 0$, we obtain the Beta prime distribution (or the Beta distribution of the second kind).

(v) For Fischer's F–distribution, which is a Beta prime distribution, we set $\sigma = \rho = \kappa = 1$ and replace $x = mx/n$, $\lambda = m/2$, $\nu = (m+n)/2$, where m and n are positive integers.

4 Statistical functions for the HLZB′ distribution

In this section we would introduce some classical statistical functions for the HLZB′ distributed random variable having the PDF given with (16). These characteristics are *moments* of positive, fractional order $m_r, r \in \mathbb{R}$, being the Mellin transform of order $r+1$ of the PDF; the *generating function* $G_X(t)$ which equals to the Laplace transform and the *characteristic function* (CHF) $\phi_X(t)$ which coincides with the Fourier transform of the PDF (16).

We point out that all three highly important characteristics of the probability distributions can be uniquely expressed *via* the operator of the mathematical expectation E. However, it is well–known that for any Borel function ψ there holds

$$\mathsf{E}\psi(X) = \int_{\mathbb{R}} \psi(x) f(x)\,dx. \tag{19}$$

To obtain explicitely $m_r, G_X(t), \phi_X(t)$ we also need in the sequel the unified Hurwitz–Lerch Zeta function, recently introduced by Srivastava *et al.* [24]. According to [24] we consider nonnegative integer parameters $p, q \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}$; $\lambda_j \in \mathbb{C}$, $\mu_k \in \mathbb{C} \setminus \mathbb{Z}_0^-$; $\sigma_j, \rho_k > 0$, $j = \overline{1, p}$, $k = \overline{1, q}$. Then the *Unified Hurwitz–Lerch Zeta Function* with $p + q$ upper and $p + q + 2$ lower parameters, reads as follows

$$\Phi^{(\rho;\sigma)}_{\lambda;\mu}(z, s, a) := \Phi^{(\rho_1, \cdots, \rho_p; \sigma_1, \cdots, \sigma_q)}_{\lambda_1, \cdots, \lambda_p; \mu_1, \cdots, \mu_q}(z, s, a) = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (\lambda_j)_{n\rho_j}}{\prod_{j=1}^{q} (\mu_j)_{n\sigma_j}} \frac{z^n}{(n + a)^s \, n!}, \tag{20}$$

where $s, \Re\{a\} > 0$ and the empty product is taken to be unity. The series (20) converges

1. for all $z \in \mathbb{C} \setminus \{0\}$ if $\Upsilon > -1$;
2. in the open disc $|z| < \nabla$ if $\Upsilon = -1$;
3. on the circle $|z| = \nabla$, for $\Upsilon = -1$, $\Re\{\Theta\} > 1/2$,

where

$$\nabla := \prod_{j=1}^{q} \sigma_j^{\sigma_j} \prod_{j=1}^{p} \rho_j^{-\rho_j}, \quad \Upsilon := \sum_{j=1}^{q} \sigma_j - \sum_{j=1}^{p} \rho_j + s, \quad \Theta := \sum_{j=1}^{q} \mu_j - \sum_{j=1}^{p} \lambda_j + \frac{p - q}{2}.$$

Theorem 1 *Let* $X \sim \mathrm{HLZB}'$ *be a r.v. defined on a standard probability space* $(\Omega, \mathfrak{F}, P)$ *and let* $r \in \mathbb{R}_+$. *Then the* r*th fractional order moment of* X *reads as follows*

$$m_r = \frac{(\lambda)_r \sin \pi(\nu - \lambda)}{(1 - \nu + \lambda)_r \sin \pi(\nu - \lambda - r)} \, \frac{\Phi^{(\sigma, \rho, \kappa - \rho;\, \kappa, \kappa - \rho)}_{\mu, \lambda + r, \nu - \lambda - r; \nu, \nu - \lambda}(z, s, a)}{\Phi^{(\rho, \sigma, \kappa)}_{\lambda, \mu, \nu}(z, s, a)}. \tag{21}$$

Proof. The fractional moment m_r of the r.v. $X \sim \mathrm{HLZB}'$ is given by

$$m_r = \mathsf{E}X^r = \frac{A\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\nu - \lambda)} \int_0^\infty \frac{x^{\lambda + r - 1}}{(1 + x)^\kappa} \, \Phi^{(\sigma, \kappa - \rho)}_{\mu, \nu - \lambda}\Big(\frac{z x^\rho}{(1 + x)^\kappa}, s, a\Big) \, dx \quad r \in \mathbb{R}_+,$$

where A is the related normalizing constant.

Expressing the Hurwitz–Lerch Zeta function in initial power series form, and interchanging the order of summation and integration, we find that:

$$\begin{aligned} m_r &= \frac{A\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\nu - \lambda)} \sum_{n=0}^{\infty} \frac{(\mu)_{\sigma n}}{(\nu - \lambda)_{(\kappa - \rho)n}} \frac{z^n}{(n + a)^s \, n!} \int_0^\infty \frac{x^{\lambda + r + \rho n - 1}}{(1 + x)^{\nu + \kappa n}} \, dx \\ &= \frac{A\Gamma(\lambda + r)\Gamma(\nu - \lambda - r)}{\Gamma(\lambda)\Gamma(\nu - \lambda)} \sum_{n=0}^{\infty} \frac{(\mu)_{\sigma n}(\lambda + r)_{\rho n}}{(\nu)_{\kappa n}} \cdot \frac{(\nu - \lambda - r)_{(\kappa - \rho)n}}{(\nu - \lambda)_{(\kappa - \rho)n}} \frac{z^n}{(n + a)^s \, n!}. \end{aligned}$$

By the Euler's reflection formula we get

$$\begin{aligned}
m_r &= \frac{A(\lambda)_r\,\Gamma(1-\nu+\lambda)\sin\pi(\nu-\lambda)}{\Gamma(1-\nu+\lambda+r)\sin\pi(\nu-\lambda-r)} \sum_{n=0}^\infty \frac{(\mu)_{\sigma n}(\lambda+r)_{\rho n}(\nu-\lambda-r)_{(\kappa-\rho)n}\, z^n}{(\nu)_{\kappa n}(\nu-\lambda)_{(\kappa-\rho)n}\,(n+a)^s\, n!} \\
&= \frac{A(\lambda)_r\,\sin\pi(\nu-\lambda)}{(1-\nu+\lambda)_r\,\sin\pi(\nu-\lambda-r)} \sum_{n=0}^\infty \frac{(\mu)_{\sigma n}(\lambda+r)_{\rho n}(\nu-\lambda-r)_{(\kappa-\rho)n}\, z^n}{(\nu)_{\kappa n}(\nu-\lambda)_{(\kappa-\rho)n}(n+a)^s\, n!}\,,
\end{aligned}$$

which is same as (21). □

We point out that for the integer $r \in \mathbb{N}$, the moment (21) it reduces to

$$m_r = \frac{(-1)^r(\lambda)_r}{(1-\nu+\lambda)_r}\, \frac{\Phi^{(\sigma,\rho,\kappa-\rho;\,\kappa,\kappa-\rho)}_{\mu,\lambda+r,\nu-\lambda-r;\nu,\nu-\lambda}(z,s,a)}{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s,a)}\,. \tag{22}$$

Theorem 2 *The generating function* $G_X(t)$ *and the* CHF $\phi_X(t)$, $t \in \mathbb{R}$ *for the r.v.* $X \sim \mathrm{HLZB}'$ *are represented in the form*

$$G_X(t) = \mathsf{E}\,e^{-tX} = \frac{1}{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s,a)} \sum_{r=0}^\infty \frac{(\lambda)_r}{(1+\lambda-\nu)_r}\frac{t^r}{r!}\Phi^{(\rho,\sigma,\kappa)}_{\lambda+r,\mu,\nu}(z,s,a), \tag{23}$$

$$\phi_X(t) = \mathsf{E}\,e^{itX} = \frac{1}{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s,a)} \sum_{r=0}^\infty \frac{(\lambda)_r}{(1+\lambda-\nu)_r}\frac{(-it)^r}{r!}\Phi^{(\rho,\sigma,\kappa)}_{\lambda+r,\mu,\nu}(z,s,a)\,. \tag{24}$$

Proof. Setting $\psi(X) = e^{-tX}$ in (19) respectively, then expanding the Laplace kernel into Maclaurin series, by legitimate interchange the order of summation and integration we obtain the generating function $G_X(t)$ in terms of (22). Because $\phi_X(t) = G_X(-it), t \in \mathbb{R}$, the proof is completed. □

The second set of important statistical functions concers the reliability applications of the newly introduced generalized Hurwitz–Lech Zeta Beta prime distribution. The functions associated with r.v. X are the cumulative distribution function (CDF) F, the survivor function $S = 1-F$, the hazard rate function $h = f/(1-F)$, and the mean residual life function $K(x) = \mathsf{E}(X-x|X \geq x)$. Their explicit formulæ are given in terms of lower and upper incomplete (complementary) φ–functions.

Theorem 3 *Let r.v.* $X \sim \mathrm{HLZB}'$. *Then we have:*

$$h(x) = \frac{f(x)}{S(x)} = \frac{\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\nu-\lambda)}\,\frac{x^{\lambda-1}}{(1+x)^{\nu}}\,\frac{\Phi_{\mu,\nu-\lambda}^{(\sigma,\kappa-\rho)}\Big(\frac{zx^{\rho}}{(1+x)^{\kappa}}, s, a\Big)}{\overline{\varphi}_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a|x)}, \tag{25}$$

$$\begin{aligned} K(x) &= \frac{\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\nu-\lambda)\,\overline{\varphi}_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a|x)}\sum_{n=0}^{\infty}\frac{(\mu))_{\sigma n}}{(\nu-\lambda)_{(\kappa-\rho)n}}\,\frac{z^n}{(n+a)^s\,n!} \\ &\quad\times \mathrm{B}_{(1+x)^{-1}}\big(\nu-\lambda-1+(\kappa-\rho)n, \lambda+1+\rho n\big) - x, \end{aligned} \tag{26}$$

where

$$\mathrm{B}_z(a,b) = \int_0^z t^{a-1}(1-t)^{b-1}\,dt, \qquad \min\big(\Re\{a\},\Re\{b\}\big) > 0,\ |z|<1$$

represents the incomplete Beta*–function.*

Proof. The CDF and the survivor functions of the r.v. X are

$$F(x) = \frac{\varphi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a|x)}{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a)}, \qquad S(x) = \frac{\overline{\varphi}_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a|x)}{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a)} \qquad x>0\,,$$

and vanishes elsewhere. Therefore, being $h(x) = f(x)/S(x)$, (25) is proved.

It is well–known that for the mean residual life function there holds [5]

$$K(x) = \frac{1}{S(x)}\int_x^{\infty} tf(t)dt - x\,.$$

The integral will be

$$\mathcal{J} = \int_x^{\infty} tf(t)dt = \frac{A\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\nu-\lambda)}\sum_{n=0}^{\infty}\frac{(\mu)_{\sigma n}\,(n+a)^{-s}\,z^n}{(\nu-\lambda)_{(\kappa-\rho)n}\,n!}\int_x^{\infty}\frac{t^{\lambda+\rho n}}{(1+t)^{\nu+\kappa n}}\,dt\,,$$

where the innermost t–integral reduces to the incomplete Beta function in the following way:

$$\int_x^{\infty}\frac{t^{p-1}}{(1+t)^q}\,dt = \int_0^{(1+x)^{-1}} t^{q-p-1}t^{p-1}\,dt = \mathrm{B}_{(1+x)^{-1}}\big(p, q-p\big)\,.$$

Therefore we conlude

$$\mathcal{J} = \frac{A\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\nu-\lambda)}\sum_{n=0}^{\infty}\frac{(\mu)_{\sigma n}\,(n+a)^{-s}\,z^n}{(\nu-\lambda)_{(\kappa-\rho)n}\,n!}\,\mathrm{B}_{(1+x)^{-1}}\big(\nu-\lambda-1+(\kappa-\rho)n, \lambda+1+\rho n\big)\,.$$

After some simplification it leads to the stated formula (26). □

5 Generalized Hurwitz–Lerch Zeta Gamma distribution

Gamma–type distributions, associated with certain special functions of science and engineering, are studied by several researchers, such as Stacy [26]. In this section a new probability density function is introduced, which extends both the well–known Gamma distribution [21, 28] and Planck distribution [9].

Consider the r.v. X defined on a standard probability space $(\Omega, \mathfrak{F}, P)$, defined by the PDF

$$f(x) = \begin{cases} \dfrac{b^s x^{s-1} e^{-ax}}{\Gamma(s)} \dfrac{{}_2\Psi_1^*\left[\begin{array}{c} (\lambda,\rho),(\mu,\sigma) \\ (\nu,\kappa) \end{array} \Big| ze^{-bx} \right]}{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a/b)} & x > 0, \\ 0, & x \le 0; \end{cases} \tag{27}$$

where a, b are scale parameters and s is shape parameter. Further $\Re\{a\}, \Re\{s\} > 0$ when $|z| \le 1$ $(z \ne 1)$ and $\Re\{s\} > 1$ when $z = 1$. Such distribution we call by convention *generalized Hurwitz–Lerch Zeta Gamma distribution* and write $X \sim$ HLZG. Notice that behavior of $f(x)$ near to the origin depends on s in the manner that $f(0) = 0$ for $s > 1$, and for $s = 1$ we have

$$f(0) = \frac{b \; {}_2\Psi_1^*\left[\begin{array}{c} (\lambda,\rho),(\mu,\sigma) \\ (\nu,\kappa) \end{array} \Big| z \right]}{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,1,a/b)},$$

and $\lim_{x\to 0+} f(x) = \infty$ when $0 < s < 1$.

Now, we list some important special cases of the HLZG distribution.

(a) For $\sigma = \rho = \kappa = 1$ we obtain the following PDF discussed by Garg *et al.* [3]:

$$f_1(x) = \frac{b^s x^{s-1} e^{-ax}}{\Gamma(s)} \frac{{}_2F_1\left[\begin{array}{c} \lambda,\mu \\ \nu \end{array} \Big| ze^{-bx} \right]}{\Phi_{\lambda,\mu,\nu}(z,s,a/b)}, \tag{28}$$

where $\Re\{a\}, \Re\{b\}, \Re\{s\} > 0$ and $|z| < 1$ or $|z| = 1$ with $\Re\{\nu - \lambda - \mu\} > 0$.

(b) If we set $\sigma = \rho = \kappa = 1$, $b = a$, $\lambda = 0$, then (27) reduces to the Gamma distribution [9, p. 32] and

(c) for $\sigma = \rho = \kappa = 1$, $\mu = \nu$, $\lambda = 1$ it reduces to the generalized Planck distribution defined by Nadarajah and Kotz [16], which is a generalization of the Planck distribution [9, p. 273].

6 Statistical functions for the HLZG distribution

In this section we will derive the statistical functions for the r.v. $X \sim$ HLZG distribution associated with PDF (27). For the moments m_r of fractional order $r \in \mathbb{R}_+$ we derive by definition

$$m_r = \int_0^\infty x^r f(x) dx = \frac{(s)_r}{b^r} \frac{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z, s+r, a/b)}{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z, s, a/b)}. \tag{29}$$

Next we present the Laplace and the Fourier transforms of the probability density function (27), that is the generating function $G_Y(t)$ and the related CHF $\phi_Y(t)$:

$$G_X(t) = \mathsf{E}\, e^{-tY} = \frac{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z, s, (a+t)/b)}{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z, s, a/b)}, \tag{30}$$

$$\phi_X(t) = G_Y(-it) = \mathsf{E}\, e^{itY} = \frac{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z, s, (a-it)/b)}{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z, s, a/b)}, \qquad t \in \mathbb{R}. \tag{31}$$

The second set of the statistical functions include the hazard function h and the mean residual life function K.

Theorem 4 *Let* $X \sim$ HLZG*. Then we have:*

$$h(x) = \frac{b^s x^{s-1} e^{-ax}}{\Gamma(s)} \frac{{}_2\Psi_1^*\left[\begin{array}{c} (\lambda, \rho), (\mu, \sigma) \\ (\nu, \kappa) \end{array} \middle| z e^{-bx} \right]}{\overline{\xi}_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z, s, a/b, b|x)} \tag{32}$$

$$K(x) = \frac{1}{b\,\Gamma(s) \overline{\xi}_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z, s, a/b, b|x)} \sum_{n=0}^\infty \frac{(\lambda)_{\rho n} (\mu)_{\sigma n}}{(\nu)_{\kappa n}} \frac{\Gamma(s+1, (a+bn)x)}{(n+a/b)^{s+1}} \frac{z^n}{n!} - x\,. \tag{33}$$

Here

$$\Gamma(p, z) = \int_z^\infty t^{p-1} e^{-t}\, dt, \qquad \Re\{p\} > 0\,,$$

stands for the upper incomplete Gamma function.

Proof. From the hazard function formula a simple calculation gives:

$$\begin{aligned} K(x) &= \frac{b^s}{\Gamma(s)\overline{\xi}_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a/b,b|x)} \int_x^\infty t^s e^{-at}\, {}_2\Psi_1^*\Big[\begin{array}{c} (\lambda,\rho),\,(\mu,\sigma) \\ (\nu,\kappa) \end{array} \Big| z e^{-bt} \Big]\, dt - x \\ &= \frac{b^s}{\Gamma(s)\overline{\xi}_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a/b,b|x)} \sum_{n=0}^\infty \frac{(\lambda)_{\rho n}(\mu)_{\sigma n}}{(\nu)_{\kappa n}} \frac{z^n}{n!} \int_x^\infty t^s e^{-(a+bn)t}\, dt - x\,. \end{aligned}$$

Further simplification leads to the asserted formula (33). □

7 Statistical parameter estimation in HLZB′ and HLZG distribution models

The statistical parameter estimation becomes one of the main tools in random model identification procedures. In the study of HLZB′ and HLZG distributions the PDFs (16) and (27) are built by higher transcendental functions such as generalized Hurwitz–Lerch Zeta function $\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a)$ and Fox–Wright generalized hypergeometric function ${}_2\Psi_1^*[z]$. The power series definitions of these functions does not enable the successful implementation of the popular and efficient Maximum Likelihood (ML) parameter estimation, only the numerical system solving can reach any result for HLZB′, while ML cannot be used for HLZG distribution case, being the extrema of the likelihood function out of the parameter space.

Therefore, we consider the Moment Method estimators, such that are weakly consistent (by the Khinchin's Law of Large Numbers), also strongly consistent (by the Kolmogorov LLN) and asymptotically unbiased.

7.1 Parameter estimation in HLZB′ model

Assume that the considered statistical population possesses HLZB′ distribution, that is the r.v. $X \sim f(x)$, (16) generates n independent, identically distributed replicæ $\Xi = \big(X_j\big)_{j=\overline{1,n}}$ which forms a statistical sample of the size n. We are now interested in estimating the 9-dimensional parameter

$$\theta_9 = (a, \sigma, \kappa, \rho, \lambda, \mu, \nu, z, s)$$

or some of its coordinates by means of the sample Ξ.

First we consider the PDF (16) for small $z \to 0$. For such values we get asymptotics

$$f(x) \sim \frac{\Gamma(\nu)\, x^{\lambda-1}}{\Gamma(\lambda)\Gamma(\nu-\lambda)(1+x)^\nu} \qquad x > 0, \tag{34}$$

which is the familiar Beta distribution of the second kind (or Beta prime) $B'(\lambda, \nu)$. The moment method estimators for the remaining parameters $\lambda > 0, \nu > 2$ read:

$$\widetilde{\lambda} = \frac{\overline{X}_n\big(\overline{X^2_n} + \overline{X}_n\big)}{\overline{S}^2_n}, \quad \widetilde{\nu} = \frac{\overline{X^2_n} + \overline{X}_n}{\overline{S}^2_n}\big(\overline{X}_n + 1\big) + 1\,, \tag{35}$$

where

$$\overline{X}_n = \frac{1}{n}\sum_{j=1}^{n} X_j, \qquad \overline{S}^2_n = \frac{1}{n}\sum_{j=1}^{n}\big(X_j - \overline{X}_n\big)^2$$

expressing the sample mean and the sample variance respectively. Let us mention that for $\nu < 2$, the variance of a r.v. $X \sim B'(\lambda, \nu)$ does not exists, so for these range of parameters MM is senseless.

The case of full range parameter estimation is highly complicated. The moment method estimator can be reached by virtue of the positive integer order moments formula (22) substituting

$$\overline{X^r_n} = \frac{1}{n}\sum_{j=1}^{n} X^r_j \mapsto m_r,$$

where $\overline{X^r_n}$ is the rth sample moment. Thus, numerical solution of the system

$$\frac{(-1)^r(\lambda)_r}{(1-\nu+\lambda)_r}\,\frac{\Phi^{(\sigma,\rho,\kappa-\rho;\,\kappa,\kappa-\rho)}_{\mu,\lambda+r,\nu-\lambda-r;\nu,\nu-\lambda}(z,s,a)}{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s,a)} = \overline{X^r_n} \qquad r = \overline{1,9} \tag{36}$$

which results in the vectorial moment estimator $\widetilde{\theta}_9 = (\widetilde{a}, \widetilde{\sigma}, \widetilde{\kappa}, \widetilde{\rho}, \widetilde{\lambda}, \widetilde{\mu}, \widetilde{\nu}, \widetilde{z}, \widetilde{s})$.

7.2 Parameter estimation in HLZG distribution

To achieve Gamma distribution's PDF from the density function (27) of HLZG in a way different then $(\mathfrak{b})$ in Section 6, it is enough to consider the PDF (27) for $a = b$ and small $z \to 0$. Indeed, we have

$$\lim_{z\to 0} f(x) = \begin{cases} \dfrac{b^s x^{s-1} e^{-bx}}{\Gamma(s)} & x > 0, \\ 0, & x \le 0; \end{cases} \tag{37}$$

It is well known that the moment method estimators for parameters b, s are

$$\widetilde{b} = \frac{\overline{X}_n}{\overline{S}^2_n}, \qquad \widetilde{s} = \frac{\big(\overline{X}_n\big)^2}{\overline{S}^2_n}\,.$$

The general case includes the vectorial parameter

$$\theta_{10} = (a, b, s, \lambda, \rho, \mu, \sigma, \nu, \kappa, z)\,.$$

First we show a kind of recurrence relation for the fractional order moments between distant neighbours.

Theorem 5 *Let* $0 \le t \le r$ *be nonnegative real numbers, and* $\mathfrak{m}_r$ *denotes the fractional positive* r*th order moment of a r.v.* $X \sim$ HLZG*. Then it holds true*

$$\mathfrak{m}_r = \mathfrak{m}_{r-t} \cdot \mathfrak{m}_t. \tag{38}$$

Proof. It is not difficult to prove

$$\begin{aligned}\mathfrak{m}_r &= \frac{(s)_r}{b^r}\,\frac{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z, s+r, a/b)}{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z, s, a/b)} \\ &= \frac{\Gamma(s+r)}{b^{r-t}\,\Gamma(s+t)}\,\frac{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z, s+r, a/b)}{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z, s+t, a/b)}\,\frac{\Gamma(s+t)}{b^t\,\Gamma(s)}\,\frac{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z, s+t, a/b)}{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z, s, a/b)}\,,\end{aligned}$$

which is equivalent to the assertion of the Theorem. □

Remark 2 *Taking the integer order moments* (29)*, that is* $m_r, r \in \mathbb{N}_0$*, the recurrence relation* (38) *becomes a contiguous relation for distant neighbours:*

$$m_\ell = m_{\ell-k} \cdot m_k = \frac{(s+\ell)_{\ell-k}}{b^{\ell-k}}\,\frac{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z, s+\ell, a/b)}{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z, s+k, a/b)}\,m_k \tag{39}$$

for all $0 \le k \le \ell,\ k, \ell \in \mathbb{N}_0$*.*

Choosing a system of 10 suitable different equations like (38) in which $\mathfrak{m}_r$ is substituted with $\overline{X_n^r} \mapsto \mathfrak{m}_r$, we get

$$\frac{(s+t)_{r-t}}{b^{r-t}}\,\frac{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z, s+r, a/b)}{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z, s+t, a/b)} = \frac{\overline{X_n^r}}{\overline{X_n^t}}\,. \tag{40}$$

However, the at least complicated case of (38) occurs at the contiguous (39) with $k = 0, \ell = \overline{1, 10}$, that is, by virtue of (40) we deduce the system in unknown θ_{10}:

$$(s)_\ell\,\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z, s+\ell, a/b) = b^\ell\,\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z, s, a/b)\,\overline{X_n^\ell} \qquad \ell = \overline{1, 10}\,. \tag{41}$$

The numerical solution of system (41) with respect to unknown parameter vector θ_{10} we call moment method estimator $\widetilde{\theta}_{10}$.

References

[1] A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, *Higher Transcendental Functions*, Vol. 1, McGraw–Hill, New York, Toronto & London, 1955.

[2] M. Garg, K. Jain, S. L. Kalla, A further study of generalized Hurwitz–Lerch Zeta function, *Algebras Groups Geom.*, **25** (2008), 311–319.

[3] M. Garg, K. Jain, S. L. Kalla, On generalized Hurwitz–Lerch Zeta function, *Appl. Appl. Math.*, **4** (2009), 26–39.

[4] S. P. Goyal, R. K. Laddha, On the generalized Riemann–Zeta functions and the generalized Lambert transform, *Gaṇita Sandesh*, **11** (1997), 97–108.

[5] F. Guess, F. Proschan, Mean residual life: theory and applications, in P. R. Krishnaiah, C. R. Rao (Eds.), *Quality Control and Reliability*, Handbook of Statistics 7, North-Holland, Amsterdam, 1988.

[6] P. L. Gupta, R. C. Gupta, S. H. Ong, H. M. Srivastava, A class of Hurwitz–Lerch Zeta distribution and their applications in reliability, *App. Math. Comput.*, **196** (2008), 521–532.

[7] R. C. Gupta, S. N. U. A. Kirmam, H. M. Srivastava, Local dependence function for some families of bivariate distributions and total positivity, *App. Math. Comput.*, **216** (2010), 1267–1279.

[8] N. L. Johnson, S. Kotz, *Distribution in Statistics: Continuous Univariate Distributions*, Vol. 1, John Wiley and Sons, New York, 1970.

[9] N. L. Johnson, S. Kotz, *Distribution in Statistics: Continuous Univariate Distributions*, Vol. 2, John Wiley and Sons, New York, 1970.

[10] N. L. Johnson, S. Kotz, N. Balakrishnan, N., *Continuous Univariate Distributions*, Vol. 2 (2nd Edition), John Wiley and Sons, 1995.

[11] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applicatios of Fractioal Differential Equatios; North–Holland Mathematical Studies*, Vol. 204, Elsevier (North–Holland) Science Publishers, Amsterdam, London and New York, 2006.

[12] S. D. Lin, H. M. Srivastava, Some families of Hurwitz–Lerch Zeta functions and associated fractional derivative and other integral representations, *App. Math. Comput.*, **154** (2004), 725–733.

[13] A. M. Mathai, R. K. Saxena, A generalized Probability distribution, *Univ. National Tucuman Rev. Series (A)*, **21** (1971), 193–202.

[14] A. M. Mathai, R. K. Saxena, *Generalized Hypergeometric Fuctions with Applications in Statistics and Physical Sciences*, Springer–Verlag, New York, 1973.

[15] A. M. Mathai, R. K. Saxena, *The H–functions with Applications in Statistics and Other Disciplines*, John Wiley and Sons Inc. (Halsted Press), New York, 1978.

[16] S. Nadarajah, S. Kotz, A generalized Planck distribution, *Test*, **15** (2006), 361–374.

[17] K. Nishimoto, C. E.–E. Yen, M.–L. Lin, Some integral forms for a generalized Zeta function, *J. Fract. Calc.*, **22** (2002), 91–97.

[18] R. K. Saxena, *On a unified inverse Gaussian distribution*, Proc. 8^{th} International Conference of the Society for Special Functions and Their Applications (SSFA) Palai (India), pp. 51–65, 2007.

[19] R. K. Saxena, S. P. Dash, The distribution of the linear combination and of the ratio of product of independent random variables associated with an H–function, *Vijnana Parishad Anusandhan Patrika*, **22** (1979), 57–65.

[20] R. K. Saxena, S. L. Kalla, On a unified mixture distribution, *Appl. Math. Comput.*, **182** (2006), 325–332.

[21] R. K. Saxena, S. L. Kalla, On a generalization of Kratzel function and associated inverse Gaussian distribution, *Algebras, Groups Geom.*, **24** (2007), 303–324.

[22] R. K. Saxena, R. Saxena, An extension of Krätzel function and associated inverse Gaussian probability distributions occuring in reliability theory, *Int. J. Comput. Math. Sci.*, **3** (2009), 189–198.

[23] H. M. Srivastava, J. Choi *Series Associated with the Zeta and related functions*, Kluwer Academic Publishers, Dordrecht, Boston and London, 2001.

[24] H. M. Srivastava, R. K. Saxena, T. K. Pogány, R. Saxena, Integral and computational representations of the extended Hurwitz–Lerch Zeta function, *Integral Trasforms Spec. Funct.*, **22** (2011), 487–506.

[25] H. M. Srivastava, Ž. Tomovski, Fractional calculus with an integral operator containing a generalized Mittag–Leffler function in the kernel, *Appl. Math. Comput.*, **211** (2009), 198–210.

[26] E. W. Stacy, A generalization of the Gamma distribution, *Ann. Math. Stat.*, **33** (1962), 1187–1192.

[27] E. T. Whittaker, G. N. Watson, *A Course of Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions: With an Account of the Proncipal Transcendental Functions*, Fourth edition, Cambridge University Press, Cambridge, 1927.

[28] C.–E. Yen, M.–L. Lin, K. Nishimoto, An integral form for a generalized Zeta function, *J. Fract. Calc.*, **23** (2002), 99–102.

Received: December 2, 2010

Acta Univ. Sapientiae, Mathematica, **3**, 1 (2011) 60–76

The generalized Marcum Q−function: an orthogonal polynomial approach

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Abstract. A novel power series representation of the generalized Marcum Q-function of positive order involving generalized Laguerre polynomials is presented. The absolute convergence of the proposed power series expansion is showed, together with a convergence speed analysis by means of truncation error. A brief review of related studies and some numerical results are also provided.

1 Introduction

For ν real number let I_ν be denotes the modified Bessel function [49, p. 77] of the first kind of order ν, defined by

$$I_\nu(t) = \sum_{n\geq 0} \frac{(t/2)^{2n+\nu}}{n!\Gamma(\nu+n+1)}, \tag{1}$$

2010 Mathematics Subject Classification: 33E20
Key words and phrases: Generalized Marcum Q-function, generalized Laguerre polynomials, modified Bessel functions, incomplete gamma function, power series representation

and let $b \mapsto Q_\nu(a,b)$ be the generalized Marcum Q-function, defined by

$$Q_\nu(a,b) = \frac{1}{a^{\nu-1}} \int_b^\infty t^\nu e^{-\frac{t^2+a^2}{2}} I_{\nu-1}(at)dt, \qquad (2)$$

where $b \geq 0$ and $a, \nu > 0$. Here Γ stands for the well-known Euler gamma function. When $\nu = 1$, the function

$$b \mapsto Q_1(a,b) = \int_b^\infty t e^{-\frac{t^2+a^2}{2}} I_0(at)dt$$

is known in literature as the (first order) Marcum Q-function. The Marcum Q-function and its generalization are frequently used in the detection theories for radar systems [27] and wireless communications [12, 13], and have important applications in error performance analysis of digital communication problems dealing with partially coherent, differentially coherent, and non–coherent detections [38, 40]. Since, the precise computations of the Marcum Q-function and generalized Marcum Q-function are quite difficult, in the last few decades several authors worked on precise and stable numerical calculation algorithms for the functions. See the papers of Dillard [14], Cantrell [7], Cantrell and Ojha [8], Shnidman [34], Helstrom [17], Temme [46] and the references therein. Moreover, many tight lower and upper bounds for the Marcum Q-function and generalized Marcum Q-function were proposed as simpler alternative evaluating methods or intermediate results for further integrations. See, for example, the papers of Simon [35], Chiani [10], Simon and Alouini [37], Annamalai and Tellambura [1], Corazza and Ferrari [11], Li and Kam [22], Baricz [4], Baricz and Sun [5, 6], Kapinas et al. [19], Sun et al. [41], Li et al. [23] and the references therein. In this field, the order ν is usually the number of independent samples of the output of a square–law detector, and hence in most of the papers the authors deduce lower and upper bounds for the generalized Marcum Q-function with order ν integer. On the other hand, based on the papers [8, 27, 34] there are introduced in the Matlab 6.5 software the Marcum Q-function and positive integer order generalized Marcum Q-function[1]: `marcumq(a,b)` computes the value of the first order Marcum Q-function $Q_1(a,b)$ and `marcumq(a,b,m)` computes the value of the mth order generalized Marcum Q-function $Q_m(a,b)$, defined by (2), where m is a positive integer. However, in some important applications, the order $\nu > 0$ of the generalized Marcum Q-function is not necessarily an integer number. The

[1] See `http://www.mathworks.com/access/helpdesk/help/toolbox/signal/marcumq.html` for more details.

generalized Marcum Q-function is the complementary cumulative distribution function or reliability function of the non–central chi distribution with 2ν degrees of freedom [18, 39, 41]. Moreover, real order generalized Marcum Q-function has been used to characterize small–scale channel fading distributions with line–of–sight channel components [24, 50] or cross–channel correlations [2, 3, 19, 20, 38, 44, 45].

In this paper, we present a novel generalized Laguerre polynomial series representation of the generalized Marcum Q-function, which extends the result of the first order Marcum Q-function in Pent's paper [32] to the case of the generalized Marcum Q-function with real order $\nu > 0$. We further show the absolute convergence of the proposed power series expansion, together with a convergence speed analysis by means of truncation error. A brief review of related studies in the literature is provided, which may assist the readers to get a more complete vision of this area. Finally, some numerical results are provided as a complementary of these theoretical analysis.

2 The generalized Marcum Q-function via Laguerre polynomials

2.1 Novel series representation of the generalized Marcum Q-function

We start with the following well–known formula [43, p. 102]

$$\sum_{n\geq 0} \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)} \frac{z^n}{n!} = \Gamma(\alpha+1)e^{z}(xz)^{-\frac{\alpha}{2}} J_\alpha(2\sqrt{xz}), \tag{3}$$

where $x, z \in \mathbb{R}$ and $\alpha > -1$. Here J_α stands for the Bessel function of the first kind of order α, $L_n^{(\alpha)}$ is the generalized Laguerre polynomial of degree n and order α, defined explicitly as

$$L_n^{(\alpha)}(x) = \frac{e^x x^{-\alpha}}{n!} \left(e^{-x} x^{n+\alpha}\right)^{(n)} = \sum_{k=0}^{n} \frac{\Gamma(n+\alpha+1)}{\Gamma(k+\alpha+1)\Gamma(n-k+1)} \frac{(-x)^k}{k!}.$$

Changing in (3) z with $-z$ and taking into account $I_\nu(x) = i^{-\nu} J_\nu(ix)$ we obtain that [26]

$$\sum_{n\geq 0} \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)} \frac{(-1)^n z^n}{n!} = \Gamma(\alpha+1)e^{-z}(xz)^{-\frac{\alpha}{2}} I_\alpha(2\sqrt{xz}). \tag{4}$$

Now, if we use

$$L_n^{(\alpha)}(0) = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)},$$

and replace x with a and α with $\nu-1$, respectively, (4) can be rewritten as

$$\left(\frac{z}{a}\right)^{\frac{\nu-1}{2}} e^{-z-a} I_{\nu-1}(2\sqrt{az}) = e^{-a}\sum_{n\geq 0}(-1)^n \frac{L_n^{(\nu-1)}(a)}{\Gamma(\nu+n)} z^{n+\nu-1}, \tag{5}$$

which holds for all $a, \nu > 0$ and $z \geq 0$.

Now, consider the following formula [46, 47]

$$\begin{aligned} Q_\nu(\sqrt{2a}, \sqrt{2b}) &= e^{-a}\sum_{n\geq 0}\frac{\Gamma(\nu+n, b)}{\Gamma(\nu+n)}\frac{a^n}{n!} \\ &= \int_b^\infty \left(\frac{z}{a}\right)^{\frac{\nu-1}{2}} e^{-z-a} I_{\nu-1}(2\sqrt{az})dz, \end{aligned} \tag{6}$$

where $a, \nu > 0$ and $b \geq 0$. We note that the function $b \mapsto Q_\nu(\sqrt{a}, \sqrt{b})$, defined by

$$Q_\nu(\sqrt{a}, \sqrt{b}) = \frac{1}{2}\int_b^\infty \left(\frac{z}{a}\right)^{\frac{\nu-1}{2}} e^{-\frac{z+a}{2}} I_{\nu-1}(\sqrt{az})dz,$$

is in fact the survival function (or the complementary of the cumulative distribution function with respect to unity) of the non–central chi–square distribution with 2ν degrees of freedom and non–centrality parameter a. With other words, for all $a, \nu > 0$ and $b \geq 0$ we have

$$Q_\nu(\sqrt{a}, \sqrt{b}) = 1 - \frac{1}{2}\int_0^b \left(\frac{z}{a}\right)^{\frac{\nu-1}{2}} e^{-\frac{z+a}{2}} I_{\nu-1}(\sqrt{az})dz. \tag{7}$$

See [39] for more details. Combining (5) with (7) we obtain

$$\begin{aligned} Q_\nu(\sqrt{2a}, \sqrt{2b}) &= 1 - \int_0^b \left(\frac{z}{a}\right)^{\frac{\nu-1}{2}} e^{-z-a} I_{\nu-1}(2\sqrt{az})dz \\ &= 1 - \int_0^b e^{-a}\sum_{n\geq 0}(-1)^n \frac{L_n^{(\nu-1)}(a)}{\Gamma(\nu+n)} z^{n+\nu-1}dz \\ &\overset{(a)}{=} 1 - e^{-a}\sum_{n\geq 0}(-1)^n \frac{L_n^{(\nu-1)}(a)}{\Gamma(\nu+n)}\int_0^b z^{n+\nu-1}dz \\ &= 1 - \sum_{n\geq 0}(-1)^n e^{-a}\frac{L_n^{(\nu-1)}(a)}{\Gamma(\nu+n+1)} b^{n+\nu}, \end{aligned}$$

where in (a) the integration and summation can be interchanged, because the series on the right–hand side of (5) is uniformly convergent for $0 \leq z \leq b$. For more details see the last paragraph of Section 2.2. After some simple manipulation, we obtain a new formula of the generalized Marcum Q-function, i.e.,

$$Q_\nu(a,b) = 1 - \sum_{n\geq 0} (-1)^n e^{-\frac{a^2}{2}} \frac{L_n^{(\nu-1)}\left(\frac{a^2}{2}\right)}{\Gamma(\nu+n+1)} \left(\frac{b^2}{2}\right)^{n+\nu}, \tag{8}$$

valid for all $a, \nu > 0$ and $b \geq 0$.

In order to simplify the numerical evaluation of the series (8), we consider the expression

$$P_{\nu,n}(a,b) = \frac{b^n L_n^{(\nu-1)}(a)}{\Gamma(\nu+n+1)},$$

which satisfies the recurrence relation

$$\begin{aligned} P_{\nu,n+1}(a,b) = & \frac{(2n+\nu-a)b}{(n+1)(\nu+n+1)} P_{\nu,n}(a,b) \\ & - \frac{(n+\nu-1)b^2}{(n+1)(\nu+n)(\nu+n+1)} P_{\nu,n-1}(a,b) \end{aligned}$$

for all $a, \nu > 0$, $b \geq 0$ and $n \in \{1, 2, 3, \ldots\}$, with the initial conditions

$$P_{\nu,0}(a,b) = \frac{1}{\Gamma(\nu+1)} \quad \text{and} \quad P_{\nu,1}(a,b) = \frac{(\nu-a)b}{\Gamma(\nu+2)}.$$

Here, the recurrence relation for $P_{\nu,n}(a,b)$ were obtained from the recurrence relation [43, p. 101]

$$(n+1)L_{n+1}^{(\alpha)}(x) = (2n+\alpha+1-x)L_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x)$$

and the initial conditions from

$$L_0^{(\alpha)}(x) = 1 \quad \text{and} \quad L_1^{(\alpha)}(x) = -x+\alpha+1.$$

With the help of the expression $P_{\nu,n}(a,b)$, (8) can be easily rewritten as

$$Q_\nu(a,b) = 1 - \sum_{n\geq 0} e^{-\frac{a^2}{2}} \left(\frac{b^2}{2}\right)^\nu P_{\nu,n}\left(\frac{a^2}{2}, -\frac{b^2}{2}\right). \tag{9}$$

2.2 Convergence analysis of the new series representation

We note that for $a > 0$, $\nu \geq 1$ and $b \geq 0$ the absolute convergence of the series in (8) or (9) can be shown easily by using the following inequalities

$$
\begin{aligned}
&\left|\sum_{n\geq 0}(-1)^n e^{-\frac{a^2}{2}} \frac{L_n^{(\nu-1)}\left(\frac{a^2}{2}\right)}{\Gamma(\nu+n+1)}\left(\frac{b^2}{2}\right)^{n+\nu}\right| \\
&\leq e^{-\frac{a^2}{2}} \sum_{n\geq 0} \frac{1}{\Gamma(\nu+n+1)}\left(\frac{b^2}{2}\right)^{n+\nu}\left|L_n^{(\nu-1)}\left(\frac{a^2}{2}\right)\right| \\
&\leq e^{-\frac{a^2}{2}} \sum_{n\geq 0} \frac{1}{\Gamma(\nu+n+1)}\left(\frac{b^2}{2}\right)^{n+\nu} \frac{\Gamma(\nu+n)}{n!\Gamma(\nu)} e^{\frac{a^2}{4}} \\
&\leq e^{-\frac{a^2}{4}} \frac{1}{\Gamma(\nu)}\left(\frac{b^2}{2}\right)^{\nu-1} \sum_{n\geq 0} \frac{1}{(n+1)!}\left(\frac{b^2}{2}\right)^{n+1} \\
&= e^{-\frac{a^2}{4}} \frac{1}{\Gamma(\nu)}\left(\frac{b^2}{2}\right)^{\nu-1}\left(e^{\frac{b^2}{2}}-1\right)
\end{aligned}
$$

or

$$
\begin{aligned}
&\left|\sum_{n\geq 0}(-1)^n e^{-\frac{a^2}{2}} \frac{L_n^{(\nu-1)}\left(\frac{a^2}{2}\right)}{\Gamma(\nu+n+1)}\left(\frac{b^2}{2}\right)^{n+\nu}\right| \\
&\leq e^{-\frac{a^2}{2}} \sum_{n\geq 0} \frac{1}{\Gamma(\nu+n+1)}\left(\frac{b^2}{2}\right)^{n+\nu}\left|L_n^{(\nu-1)}\left(\frac{a^2}{2}\right)\right| \\
&\leq e^{-\frac{a^2}{2}} \sum_{n\geq 0} \frac{1}{\Gamma(\nu+n+1)}\left(\frac{b^2}{2}\right)^{n+\nu} \frac{\Gamma(\nu+n)}{n!} e^{\frac{a^2}{4}}\left(\frac{a^2}{4}\right)^{1-\nu} \\
&\leq e^{-\frac{a^2}{4}}\left(\frac{2b^2}{a^2}\right)^{\nu-1} \sum_{n\geq 0} \frac{1}{(n+1)!}\left(\frac{b^2}{2}\right)^{n+1} \\
&= e^{-\frac{a^2}{4}}\left(\frac{2b^2}{a^2}\right)^{\nu-1}\left(e^{\frac{b^2}{2}}-1\right),
\end{aligned}
$$

which contain the known inequalities of Szegő [43] for generalized Laguerre polynomials

$$|L_n^{\alpha}(x)| \leq \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+1)} e^{\frac{x}{2}}$$

and of Love [25]

$$|L_n^{\alpha}(x)| \leq \frac{\Gamma(\alpha+n+1)}{n!}\left(\frac{x}{2}\right)^{-\alpha} e^{\frac{x}{2}},$$

where in both of the inequalities $\alpha \geq 0$, $x > 0$ and $n \in \{0, 1, 2, \dots\}$.

Moreover, for $a > 0$, $0 < \nu \leq 1$ and $b \geq 0$ the absolute convergence of the series in (8) or (9) can be shown in a similar manner by using the following inequality

$$\begin{aligned}
&\left|\sum_{n\geq 0}(-1)^n e^{-\frac{a^2}{2}} \frac{L_n^{(\nu-1)}\left(\frac{a^2}{2}\right)}{\Gamma(\nu+n+1)}\left(\frac{b^2}{2}\right)^{n+\nu}\right| \\
&\leq e^{-\frac{a^2}{2}} \sum_{n\geq 0} \frac{1}{\Gamma(\nu+n+1)}\left(\frac{b^2}{2}\right)^{n+\nu}\left|L_n^{(\nu-1)}\left(\frac{a^2}{2}\right)\right| \\
&\leq e^{-\frac{a^2}{2}} \sum_{n\geq 0} \frac{1}{\Gamma(\nu+n+1)}\left(\frac{b^2}{2}\right)^{n+\nu}\left(2-\frac{\Gamma(\nu+n)}{n!\Gamma(\nu)}\right) e^{\frac{a^2}{4}} \\
&= e^{-\frac{a^2}{4}} \sum_{n\geq 0} \frac{1}{\nu+n}\left(\frac{2}{\Gamma(\nu+n)}-\frac{1}{n!\Gamma(\nu)}\right)\left(\frac{b^2}{2}\right)^{n+\nu} \\
&\leq e^{-\frac{a^2}{4}} \sum_{n\geq 0} \frac{2}{n!}\left(\frac{b^2}{2}\right)^{n+\nu} \\
&= 2e^{-\frac{a^2}{4}}\left(\frac{b^2}{2}\right)^{\nu} e^{\frac{b^2}{2}},
\end{aligned}$$

which contains the classical inequality of Szegő [43] for generalized Laguerre polynomials

$$|L_n^{\alpha}(x)| \leq \left(2-\frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+1)}\right) e^{\frac{x}{2}}$$

where $-1 < \alpha \leq 0$, $x > 0$ and $n \in \{0, 1, 2, \dots\}$. In addition here we used the fact that for all fixed $n \in \{1, 2, 3, \dots\}$ the function

$$\nu \mapsto \frac{1}{\nu+n}\left(\frac{2}{\Gamma(\nu+n)}-\frac{1}{n!\Gamma(\nu)}\right),$$

which maps 0 into $2/n!$, is decreasing on $(0, 1]$ and consequently for all $n \in \{0, 1, 2, \dots\}$ and $0 < \nu \leq 1$ we have

$$\frac{1}{\nu+n}\left(\frac{2}{\Gamma(\nu+n)}-\frac{1}{n!\Gamma(\nu)}\right) \leq \frac{2}{n!}.$$

We note that other uniform bounds for generalized Laguerre polynomials can be found in the papers of Love [25], Lewandowski and Szynal [21], Michalska and Szynal [28], Pogány and Srivastava [33]. See also the references therein.

Finally, note that by using the above uniform bounds for the generalized Laguerre polynomials the uniform convergence of the series on the right-hand side of (5) can be shown easily for $0 \leq z \leq b$. This is important because in order to obtain (8) we have used tacitly that the series on the right–hand side of (5) is uniformly convergent and then we can interchange the integration with summation. For example, if we use the above Szegő's uniform bound, then for all $n \in \{0, 1, 2, \dots\}$, $a > 0$, $\nu \geq 1$ and $0 \leq z \leq b$ we have

$$\left|(-1)^n \frac{L_n^{(\nu-1)}(a)}{\Gamma(\nu+n)} z^n\right| \leq \frac{e^{\frac{a}{2}}}{\Gamma(\nu)} \frac{b^n}{n!}.$$

By the ratio test the series $e^b = \sum_{n\geq 0} b^n/n!$ is convergent and thus in view of the Weierstrass M–test the original series on the right–hand side of (5) converges uniformly for all $0 \leq z \leq b$.

2.3 Truncation error analysis

For practical evaluations of our power series expansion, we need to approximate the generalized Marcum Q-function $Q_\nu(a, b)$ by the first $n_0 \in \{1, 2, 3, \dots\}$ terms of (8), i.e.,

$$\hat{Q}_\nu(a, b) = 1 - \sum_{n=0}^{n_0} (-1)^n e^{-\frac{a^2}{2}} \frac{L_n^{(\nu-1)}\left(\frac{a^2}{2}\right)}{\Gamma(\nu+n+1)} \left(\frac{b^2}{2}\right)^{n+\nu}.$$

We note that the absolute value of the truncation error

$$\varepsilon_t = Q_\nu(a, b) - \hat{Q}_\nu(a, b) = \sum_{n\geq n_0+1} (-1)^{n+1} e^{-\frac{a^2}{2}} \frac{L_n^{(\nu-1)}\left(\frac{a^2}{2}\right)}{\Gamma(\nu+n+1)} \left(\frac{b^2}{2}\right)^{n+\nu}$$

can be upper bounded by using the upper bounds for the generalized Laguerre polynomials as in subsection 2.2. More precisely, by using the same argument as in subsection 2.2 and Sewell's inequality [29, p. 266]

$$e^x - \sum_{k=0}^{n} \frac{x^k}{k!} \leq \frac{x e^x}{n}, \quad n \in \{1, 2, 3, \dots\},\ x \geq 0,$$

we can deduce the following: if $a > 0$, $b \geq 0$ and $\nu \geq 1$, then

$$|\varepsilon_t| \leq e^{-\frac{a^2}{4}} \frac{1}{\Gamma(\nu)} \left(\frac{b^2}{2}\right)^{\nu-1} \left[e^{\frac{b^2}{2}} - \sum_{n=0}^{n_0+1} \frac{1}{n!} \left(\frac{b^2}{2}\right)^n\right] \leq \frac{e^{\frac{b^2}{2}-\frac{a^2}{4}}}{n_0+1} \frac{1}{\Gamma(\nu)} \left(\frac{b^2}{2}\right)^{\nu}$$

or

$$|\varepsilon_t| \leq e^{-\frac{a^2}{4}} \left(\frac{2b^2}{a^2}\right)^{\nu-1} \left[e^{\frac{b^2}{2}} - \sum_{n=0}^{n_0+1} \frac{1}{n!} \left(\frac{b^2}{2}\right)^n\right] \leq \frac{e^{\frac{b^2}{2}-\frac{a^2}{4}}}{n_0+1} \frac{b^2}{2} \left(\frac{2b^2}{a^2}\right)^{\nu-1}.$$

Similarly, it can be shown that if $a > 0$, $b \geq 0$ and $0 < \nu \leq 1$, then the absolute value of the truncation error is upper bounded as follows

$$|\varepsilon_t| \leq 2e^{-\frac{a^2}{4}} \left(\frac{b^2}{2}\right)^{\nu} \left[e^{\frac{b^2}{2}} - \sum_{n=0}^{n_0} \frac{1}{n!} \left(\frac{b^2}{2}\right)^n\right] \leq \frac{2e^{\frac{b^2}{2}-\frac{a^2}{4}}}{n_0} \left(\frac{b^2}{2}\right)^{\nu+1}.$$

Observe that the above upper bounds of the absolute value of the truncation error converge to zero at a speed of $1/n_0$. In practice, we can use these upper bounds to decide the number of terms, i.e. n_0, for achieving a pre–determined accuracy.

2.4 A brief review of related studies

As far as we know the formula (8), or its equivalent form (9), is new. However, if we choose $\nu = 1$ in (9), then we reobtain the main result of Pent [32]

$$Q_1(a, b) = 1 - \frac{b^2}{2} \sum_{n \geq 0} e^{-\frac{a^2}{2}} P_n\left(\frac{a^2}{2}, -\frac{b^2}{2}\right),$$

where

$$P_n(a, b) = P_{1,n}(a, b) = \frac{b^n L_n(a)}{(n+1)!},$$

which for all $a > 0$, $b \geq 0$ and $n \in \{1, 2, 3, \ldots\}$ satisfies the recurrence relation

$$P_{n+1}(a, b) = \frac{(2n+1-a)b}{(n+1)(n+2)} P_n(a, b) - \frac{nb^2}{(n+1)^2(n+2)} P_{n-1}(a, b)$$

with the initial conditions

$$P_0(a, b) = 1 \quad \text{and} \quad P_1(a, b) = \frac{(1-a)b}{2}.$$

Here $L_n = L_n^{(0)}$ is the classical Laguerre polynomial of degree n.

It should be mentioned here that another type of Laguerre expansions for the Marcum Q-function was proposed in 1977 by Gideon and Gurland [16], which involves the lower incomplete gamma function. This type of Laguerre expansions requires to use a complementary result of (4), i.e.

$$\sum_{n\geq 0} \frac{L_n^{(\alpha)}(z)}{L_n^{(\alpha)}(0)} \frac{(-1)^n x^n}{n!} = \Gamma(\alpha+1)e^{-x}(xz)^{-\frac{\alpha}{2}} I_\alpha(2\sqrt{xz}). \tag{10}$$

Now by some simple manipulation we obtain

$$\left(\frac{z}{a}\right)^{\frac{\nu-1}{2}} e^{-z-a} I_{\nu-1}(2\sqrt{az}) = z^{\nu-1}e^{-z} \sum_{n\geq 0} \frac{(-a)^n}{\Gamma(\nu+n)} L_n^{(\nu-1)}(z), \tag{11}$$

which is equivalent to Tiku's result [48], available also as equation (29.11) in the book [18]. By integrating (11) in z and by using the differentiation formula [26]

$$\frac{d}{dz}\left[z^{\alpha+1}e^{-z}L_{n-1}^{(\alpha+1)}(z)\right] = nz^{\alpha}e^{-z}L_n^{(\alpha)}(z),$$

where $n \in \{1, 2, 3, \ldots\}$, $\alpha > -1$ and $z \in \mathbb{R}$, we can obtain another generalized Laguerre polynomial series expansion of the generalized Marcum Q-function

$$Q_\nu(\sqrt{2a}, \sqrt{2b}) = 1 - \frac{1}{\Gamma(\nu)}\gamma(\nu, b) - \sum_{n\geq 1} (-1)^n e^{-b} \frac{b^\nu L_{n-1}^{(\nu)}(b)}{n\Gamma(\nu+n)} a^n,$$

which in turn implies that

$$\begin{aligned} Q_\nu(a, b) &= 1 - \frac{1}{\Gamma(\nu)}\gamma\left(\nu, \frac{b^2}{2}\right) - \sum_{n\geq 1} (-1)^n e^{-\frac{b^2}{2}} \left(\frac{b^2}{2}\right)^\nu \frac{L_{n-1}^{(\nu)}\left(\frac{b^2}{2}\right)}{n\Gamma(\nu+n)} \left(\frac{a^2}{2}\right)^n \\ &= \frac{1}{\Gamma(\nu)}\Gamma\left(\nu, \frac{b^2}{2}\right) - \sum_{n\geq 1} (-1)^n e^{-\frac{b^2}{2}} \left(\frac{b^2}{2}\right)^\nu \frac{L_{n-1}^{(\nu)}\left(\frac{b^2}{2}\right)}{n\Gamma(\nu+n)} \left(\frac{a^2}{2}\right)^n \\ &= \lim_{a\to 0} Q_\nu(a, b) - \sum_{n\geq 1} (-1)^n e^{-\frac{b^2}{2}} \left(\frac{b^2}{2}\right)^\nu \frac{L_{n-1}^{(\nu)}\left(\frac{b^2}{2}\right)}{n\Gamma(\nu+n)} \left(\frac{a^2}{2}\right)^n, \end{aligned} \tag{12}$$

where $\gamma(\cdot,\cdot)$ is the lower incomplete gamma function, defined by

$$\gamma(a, x) = \int_0^x t^{a-1}e^{-t}dt.$$

Here we used that

$$\Gamma(a,x) = \Gamma(a) - \gamma(a,x), \tag{13}$$

and

$$\lim_{a\to 0} Q_\nu(a,b) = \frac{1}{\Gamma(\nu)}\Gamma\left(\nu, \frac{b^2}{2}\right).$$

Some other Laguerre expansions for the Marcum Q-function are provided in Gideon and Gurland's paper [16], available also as equation (29.13) of [18]. Moreover, a new unified Laguerre polynomial–series–based distribution of small–scale fading envelope and power was proposed recently by Chai and Tjhung [9], which covers a wide range of small–scale fading distributions in wireless communications. Many known Laguerre polynomial–series–based probability density functions and cumulative distribution functions of small–scale fading distributions are provided, which include the multiple–waves–plus–diffuse–power fading, non–central chi and chi–square, Nakagami–m, Rician (Nakagami–n), Nakagami–q (Hoyt), Rayleigh, Weibull, Stacy, gamma, Erlang and exponential distributions as special cases. See also [42], which contains some corrections of formulas deduced in [9]. In particular, (12) is a special case of the unified cumulative distribution function given in corrected form in [42]. We note that the expression of (12) and the unified cumulative distribution function in [42] are quite different from our main result (8) or (9). This is because they are based on two different Laguerre polynomial expansions of the modified Bessel function of the first kind I_ν given in (4) and (10). Therefore, these Laguerre polynomial expansions are expanded over different variables of the generalized Marcum Q-function. Finally, we note that since Nakagami's work [30] the Laguerre polynomial series expansions of various probability density functions have been derived. We refer to the papers of Esposito and Wilson [15], Yu et al. [51], Chai and Tjhung [9] and to the references therein.

Finally, by using the infinite series representation of the modified Bessel function of the first kind (1) and the formula

$$\int_\alpha^\infty t^m e^{-\frac{t^2}{2}} dt = 2^{\frac{m-1}{2}}\Gamma\left(\frac{m+1}{2}, \frac{\alpha^2}{2}\right),$$

where $\Gamma(\cdot,\cdot)$ is the upper incomplete gamma function, defined by

$$\Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}dt,$$

we easily obtain that

$$\begin{aligned}
Q_\nu(a,b) &= \frac{1}{a^{\nu-1}}\int_b^\infty t^\nu e^{-\frac{t^2+a^2}{2}}\sum_{n\geq 0}\frac{(at)^{2n+\nu-1}}{2^{2n+\nu-1}n!\Gamma(\nu+n)}dt \\
&= e^{-\frac{a^2}{2}}\sum_{n\geq 0}\frac{a^{2n}}{2^{2n+\nu-1}n!\Gamma(\nu+n)}\int_b^\infty e^{-\frac{t^2}{2}}t^{2n+2\nu-1}dt \\
&= e^{-\frac{a^2}{2}}\sum_{n\geq 0}\frac{1}{n!}\left(\frac{a^2}{2}\right)^n\frac{\Gamma\left(\nu+n,\frac{b^2}{2}\right)}{\Gamma(\nu+n)} \\
&= 1-\sum_{n\geq 0}e^{-\frac{a^2}{2}}\left(\frac{a^2}{2}\right)^n\frac{\gamma\left(\nu+n,\frac{b^2}{2}\right)}{n!\Gamma(\nu+n)}. \qquad (14)
\end{aligned}$$

We note that (14) is usually called the canonical representation of the νth order generalized Marcum Q-function. Recently, Annamalai and Tellambura [2] (see also [3]) claimed that the series representation (14) is new, however it appears already in 1993 in the paper of Temme [46]. See also Temme's book [47] and Patnaik's [31] result from 1949, which can be found also as equation (29.2) in the book [18]. Interestingly, our novel series representation (9) for the generalized Marcum Q-function resembles to the series representation (14).

2.5 Numerical results

We now consider some numerical aspects of our generalized Laguerre polynomial expansions (8) or (9). In practice, we usually need to compute the detection probability for different values of b with fixed a to decide a proper detection threshold. Since the generalized Laguerre polynomial in (8) is determined by only a, we can save computation time by storing the values of the generalized Laguerre polynomials for computing the generalized Marcum Q-function with different values of b.

The next tables contain some values of the generalized Marcum Q-function calculated using (9) and using the Matlab `marcumq` function. For the considered choices of a and b, the numerical value of (9) is exactly the same with that of the Matlab `marcumq` function, if $\nu \in \{1, 5\}$ is integer. When $\nu = 7.7$, the Matlab `marcumq` function does not work, and the numerical value of (9) is provided in the tables. Finally, we note that more accurate intermediate terms are required for larger a and b.

$a = 0.2, b = 0.6$	$\nu = 1$	$\nu = 5$	$\nu = 7.7$
(9)	0.838249985438908	0.999998670306184	0.999999999927717
marcumq	0.838249985438908	0.999998670306184	—

$a = 1.2, b = 1.6$	$\nu = 1$	$\nu = 5$	$\nu = 7.7$
(9)	0.501536568390858	0.994346394491553	0.999944937223540
marcumq	0.501536568390858	0.994346394491553	—

$a = 2.2, b = 2.6$	$\nu = 1$	$\nu = 5$	$\nu = 7.7$
(9)	0.426794627821735	0.929671935077756	0.993735633182201
marcumq	0.426794627821735	0.929671935077756	—

Acknowledgments

The work of S. András was partially supported by the Hungarian University Federation of Cluj. The research of Á. Baricz was supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences and by the Romanian National Authority for Scientific Research CNCSIS–UEFISCSU, project number PN–II–RU–PD_388/2011. The work of Y. Sun was supported by National Basic Research Program of China (2007CB310608), National Natural Science Foundation of China (60832008) and Lab project from Tsinghua National Lab on Information Science and Technology (sub–project): Key technique for new distributed wireless communications system.

References

[1] A. Annamalai, C. Tellambura, Cauchy–Schwarz bound on the generalized Marcum Q-function with applications, *Wireless Commun. Mob. Comput.*, **1** (2001), 243–253.

[2] A. Annamalai, C. Tellambura, A simple exponential integral representation of the generalized Marcum Q-function $Q_M(a, b)$ for real–order M with applications, *Military Communications Conference MILCOM 2008*, San Diego, CA, pp. 1–7.

[3] A. Annamalai, C. Tellambura, J. Matyjas, A new twist on the generalized Marcum Q-function $Q_M(a, b)$ with fractional–order M and its applications, *Consumer Communications and Networking Conference CCNC 2009*, Las Vegas, NV, pp. 1–5.

[4] Á. Baricz, Tight bounds for the generalized Marcum Q-function, *J. Math. Anal. Appl.*, **360** (2009), 265–277.

[5] Á. Baricz, Y. Sun, New bounds for the generalized Marcum Q-function, *IEEE Trans. Inform. Theory*, **55** (2009), 3091–3100.

[6] Á. Baricz, Y. Sun, Bounds for the generalized Marcum Q-function, *Appl. Math. Comput.*, **217** (2010), 2238–2250.

[7] P. E. Cantrell, On the calculation of the generalized Q-function via Parl's method, *IEEE Trans. Inform. Theory*, **32** (1986), 817–824.

[8] P. E. Cantrell, A. K. Ojha, Comparison of generalized Q-function algorithms, *IEEE Trans. Inform. Theory*, **33** (1987), 591–596.

[9] C. C. Chai, T. T. Tjhung, Unified Laguerre polynomial–series–based distribution of small–scale fading envelopes, *IEEE Trans. Veh. Technol.*, **58** (2009), 3988–3999.

[10] M. Chiani, Integral representation and bounds for Marcum Q-function, *IEEE Electron. Lett.*, **35** (1999), 445–446.

[11] G. E. Corazza, G. Ferrari, New bounds for the Marcum Q-function, *IEEE Trans. Inform. Theory*, **48** (2002), 3003–3008.

[12] F. F. Digham, M. S. Alouini, M. K. Simon, On the energy detection of unknown signals over fading channels, in *Proc. IEEE Int. Conf. Commun.,* Anchorage, AK, May 2003, pp. 3575–3579.

[13] F. F. Digham, M. S. Alouini, M. K. Simon, On the energy detection of unknown signals over fading channels, *IEEE Trans. Commun.*, **55** (2007), 3575–3579.

[14] G. M. Dillard, Recursive computation of the generalized Q-function, *IEEE Trans. Aerosp. Electron. Syst.*, **9** (1973), 614–615.

[15] R. Esposito, L. R. Wilson, Statistical properties of two sine waves in Gaussian noise, *IEEE Trans. Inform. Theory*, **19** (1973), 176–183.

[16] R. A. Gideon, J. Gurland, Some alternative expansions for the distribution function of a noncentral chi–square variable, *SIAM J. Math. Anal.*, **8** (1977), 100–110.

[17] C. W. Helstrom, Computing the generalized Marcum Q-function, *IEEE Trans. Inform Theory*, **38** (1992), 1422–1428.

[18] N. L. Johnson, S. Kotz, N. Balakrishnan, *Continuous Univariate Distributions*, vol. 2, second ed., John Wiley & Sons, Inc., New York, 1995.

[19] V. M. Kapinas, S. K. Mihos, G. K. Karagiannidis, On the monotonicity of the generalized Marcum and Nuttall Q-functions, *IEEE Trans. Inform. Theory*, **55** (2009), 3701–3710.

[20] S. Khatalin, J. P. Fonseka, Capacity of correlated nakagami-m fading channels with diversity combining techniques, *IEEE Trans. Veh. Technol.*, **55** (2006), 142–150.

[21] Z. Lewandowski, J. Szynal, An upper bound for the Laguerre polynomials, *J. Comput. Appl. Math.*, **99** (1998), 529–533.

[22] R. Li, P. Y. Kam, Computing and bounding the generalized Marcum Q-function via a geometric approach, *Proc. IEEE Int. Symp. Inform. Theory 2006*, Seattle, USA, pp. 1090–1094.

[23] R. Li, P. Y. Kam, H. Fu, New representations and bounds for the generalized marcum Q-function via a geometric approach, and an application, *IEEE Trans. Commun.*, **58** (2010), 157–169.

[24] P. Loskot, N. C. Beaulieu, Prony and polynomial approximations for evaluation of the average probability of error over slow–fading channels, *IEEE Trans. Veh. Technol.*, **58** (2009), 1269–1280.

[25] E. R. Love, Inequalities for Laguerre functions, *J. Inequal. Appl.*, **1** (1997), 293–299.

[26] W. Magnus, F. Oberhettinger, R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Springer–Verlag, New York, 1966.

[27] J. I. Marcum, A statistical theory of target detection by pulsed radar, *IRE Trans. Inf. Theory*, **6** (1960), 59–267.

[28] M. Michalska, J. Szynal, A new bound for the Laguerre polynomials, *J. Comput. Appl. Math.*, **133** (2001), 489–493.

[29] D. S. Mitrinović, *Analytic Inequalities*, Springer–Verlag, Berlin, 1970.

[30] M. Nakagami, The $\mathfrak{m}-$distribution: A general formula of intensity distribution of rapid fading, in *Statistical Methods in Radio Wave Propagation*, Ed. W.C. Hoffman, Pergamon, New York, 1960, pp. 3–36.

[31] P. B. Patnaik, The non-central χ^2- and $\mathsf{F}-$distributions and their applications, *Biometrika*, **36** (1949), 202–232.

[32] M. Pent, Orthogonal polynomial approach for the Marcum Q-function numerical computation, *Electronic Lett.*, **4** (1968), 563–564.

[33] T. K. Pogány, H. M. Srivastava, Some improvements over Love's inequality for the Laguerre function, *Integral Transforms Spec. Funct.* **18** (2007), 351–358.

[34] D. A. Shnidman, The calculation of the probability of detection and the generalized Marcum Q-function, *IEEE Trans. Inform. Theory*, **35** (1989), 389–400.

[35] M. K. Simon, A new twist on the Marcum Q-function and its application, *IEEE Commun. Lett.*, **2** (1998), 39–41.

[36] M. K. Simon, M. S. Alouini, A unified performance analysis of digital communication with dual selective combining diversity over correlated Rayleigh and Nakagami$-\mathfrak{m}$ fading channels, *IEEE Trans. Commun.*, **47** (1999), 33–43.

[37] M. K. Simon, M. S. Alouini, Exponential–type bounds of the generalized Marcum Q-function with application to error probability analysis over fading channels, *IEEE Trans. Commun.*, **48** (2000), 359–366.

[38] M. K. Simon, M. S. Alouini, *Digital Communication over Fadding Channels: A Unified Approach to Performance Analysis*, John Wiley & Sons, New York, 2000.

[39] Y. Sun, Á. Baricz, Inequalities for the generalized Marcum Q-function, *Appl. Math. Comput.*, **203** (2008), 134–141.

[40] Y. Sun, Á. Baricz, M. Zhao, X. Xu, S. Zhou, Approximate average bit error probability for DQPSK over fading channels, *Electron. Lett.*, **45** (2009), 1177–1179.

[41] Y. Sun, Á. Baricz, S. Zhou, On the monotonicity, log-concavity and tight bounds of the generalized Marcum and Nuttall Q-functions, *IEEE Trans. Inform. Theory*, **56** (2010), 1166–1186.

[42] Y. Sun, Á. Baricz, S. Zhou, Corrections to "Unified Laguerre polynomial–series–based distribution of small–scale fading envelopes", *IEEE Trans. Veh. Technol.*, **60** (2011), 347–349.

[43] G. Szegő, *Orthogonal Polynomials*, Colloquium Publications, vol. 23, 4th ed., American Mathematical Society, Providence, RI, 1975.

[44] C. Tellambura, A. Annamalai, V. K. Bhargava, Closed–form and infinite series solutions for the MGF of a dual–diversity selection combiner output in bivariate Nakagami fading, *IEEE Trans. Commun.*, **51** (2003), 539–542.

[45] C. Tellambura, A. D. S. Jayalath, Generation of bivariate Rayleigh and Nakagami$-\mathfrak{m}$ fading envelopes, *IEEE Commun. Lett.*, **4** (2000), 170–172.

[46] N. M. Temme, Asymptotic and numerical aspects of the noncentral chi–square distribution, *Comput. Math. Appl.*, **25** (1993), 55–63.

[47] N. M. Temme, *Special Functions. An Introduction to the Classical Functions of Mathematical Physics*, John Wiley & Sons, Inc., New York, 1996.

[48] M. L. Tiku, Laguerre series forms of non–central χ^2 and F distributions, *Biometrika*, **52** (1965), 415–426.

[49] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, 1944.

[50] M. D. Yacoub, The $\kappa-\mu$ distribution and the $\eta-\mu$ distribution, *IEEE Antennas Propag. Mag.*, **49** (2007), 68–81.

[51] Z. Yu, C. C. Chai, T. T. Tjhung, Envelope probability density functions for fading model in wireless communications, *IEEE Trans. Veh. Technol.*, **56** (2007), 1907–1912.

Received: October 11, 2010

Acta Univ. Sapientiae, Mathematica, **3**, 1 (2011) 77-89

On a structured semidefinite program

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Abstract. The nonnegative biquadratics discussed comes from the Böttcher-Wenzel inequality. It is for some matrices a sum of squares of polynomials (SOS), in other cases not, depending on the nonzero pattern of the matrices at issue. Our aim was to draw a line between them. To prove the 'not a SOS' case we solve a semidefinite programming (SDP) problem. Subsequently a two-parameter version will be investigated.

1 Introduction

The Böttcher-Wenzel inequality (see [2], [7], [3], [1], [9], [5], [4], [10]) states (in its stronger form) that for real square matrices X, Y of the same order n

$$f(X,Y) \equiv 2\,||X||^2||Y||^2 - 2\,\mathrm{trace}^2(X^TY) - ||XY - YX||^2 \geq 0, \qquad (1)$$

where the norm used is the Frobenius norm. Since all our attempts to obtain a representation for f as a sum of polynomial squares (in short: SOS) failed for $n = 3$, distinguishing between the 'good' and 'bad' cases became to a natural problem.

In case of $n = 2$ we have for $X = \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix}$, $Y = \begin{pmatrix} y_1 & y_3 \\ y_2 & y_4 \end{pmatrix}$ and with variables $z_{i,j} = x_i y_j - y_i x_j,\ 1 \leq i < j \leq 4,$ that

$$f(X,Y) = 2\,z_{1,4}^2 + (z_{1,2} - z_{2,4})^2 + (z_{1,3} - z_{3,4})^2$$

is a sum of squares of quadratics.

2010 Mathematics Subject Classification: 90C22, 15A45, 65F15
Key words and phrases: sum of squares, semidefinite programming, biquadratics

Our main result is that the nonnegative form (1) is SOS for *good* matrices X, Y, whereas it isn't SOS for general *bad* matrices, where a matrix of order n will be called *good*, if nonzero elements occur only in row 1 and column n, while it is called *bad*, if, moreover, nonzero elements occur also in the main diagonal, as shown e.g. for $n = 4$:

$$good: \begin{pmatrix} * & * & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \end{pmatrix} \qquad bad: \begin{pmatrix} * & * & * & * \\ 0 & * & 0 & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}.$$

Remark 1 For convenience we re-cite the SOS representation [8] for the *good* cases in section 2. Section 3 contains the main result: the non-possibility of an SOS-representation for the *bad* cases via SDP, while in section 4 we provide the function f with two parameters and decompose the unit square into regions with different properties.

2 SOS decomposition for good matrices

Let X, Y be *good* real n-th order matrices with $m = 2n - 1$ possible nonzero elements:

$$X = \begin{pmatrix} x_1 & \dots & x_{n-1} & x_n \\ 0 & \dots & 0 & x_{n+1} \\ \vdots & \ddots & \vdots & \\ 0 & \dots & 0 & x_m \end{pmatrix}, \qquad Y = \begin{pmatrix} y_1 & \dots & y_{n-1} & y_n \\ 0 & \dots & 0 & y_{n+1} \\ \vdots & \ddots & \vdots & \\ 0 & \dots & 0 & y_m \end{pmatrix},$$

and define an $m-$th order matrix Z by help of vectors $x = (x_i)_1^m$ and $y = (y_i)_1^m$ as

$$Z = xy^T - yx^T = (z_{i,j})_{i,j=1}^m, \quad z_{i,j} = x_i y_j - y_i x_j.$$

The SOS representation for these *good* matrices is the following.

Theorem 1 *(Theorem 1, [8])*

$$||Z||^2 - \Big(\sum_{i=1}^{n} z_{i,i+n-1}\Big)^2 - \sum_{i=2}^{n-1} z_{1,i}^2 - \sum_{i=n+1}^{m-1} z_{i,m}^2 \tag{2}$$
$$= \sum_{i=1}^{n-1}\sum_{j=n+1}^{m} z_{i,j}^2 + \sum_{i=2}^{n-2}\sum_{j=i+1}^{n-1} z_{i,j}^2 + \sum_{i=n+1}^{2n-3}\sum_{j=i+1}^{2n-2} z_{i,j}^2$$
$$+ \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \big(z_{i,j} - z_{i+n-1,j+n-1}\big)^2 + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \big(z_{i,j+n-1} - z_{j,i+n-1}\big)^2.$$

Remark 2 Indeed, (1) and (2) are identical. In particular,

$$||Z||^2 = 2\,||X||^2||Y||^2 - 2\,\mathrm{trace}^2(X^TY),$$

and

$$\Big(\sum_{i=1}^{n} z_{i,i+n-1}\Big)^2 + \sum_{i=2}^{n-1} z_{1,i}^2 + \sum_{i=n+1}^{2n-1} z_{i,m}^2 = ||XY - YX||^2$$

holds, where the first is Lagrange's identity, the second is straightforward.

3 SOS decomposition impossible for bad matrices

It suffices to prove this negative result for third order matrices. Let

$$X = \begin{pmatrix} x_1 & x_2 & x_3 \\ 0 & x_6 & x_4 \\ 0 & 0 & x_5 \end{pmatrix}, \qquad Y = \begin{pmatrix} y_1 & y_2 & y_3 \\ 0 & y_6 & y_4 \\ 0 & 0 & y_5 \end{pmatrix}.$$

It turns out that the presence of x_6 and y_6 causes the impossibility of an SOS representation for (1). Since

$$XY - YX = \begin{pmatrix} 0 & z_{1,2} + z_{2,6} & z_{1,3} + z_{2,4} + z_{3,5} \\ 0 & 0 & z_{4,5} - z_{4,6} \\ 0 & 0 & 0 \end{pmatrix},$$

the nonnegative form (1) to be discussed assumes the form

$$2\sum_{1\le i<j\le 6} z_{i,j}^2 - (z_{1,3} + z_{2,4} + z_{3,5})^2 - (z_{1,2} + z_{2,6})^2 - (z_{4,5} - z_{4,6})^2 \tag{3}$$

with $z_{i,j} = x_iy_j - x_jy_i,\ 1 \le i < j \le 6.$

Theorem 2 *The biquadratic form (3), nonnegative for any real* $(x_i)_1^6$, $(y_i)_1^6$, *is not a sum of squares of any quadratics!*

We will need a lemma before proving the theorem.

Lemma 1 *An sos-representation of* f *is necessarily a sum of squares of the* $z_{i,j}$*'s. Furthermore, for the variables* $z_{i,j}$ *the basic identities*

$$z_{i,j}\, z_{k,l} + z_{i,l}\, z_{j,k} - z_{i,k}\, z_{j,l} = 0, \quad 1 \le i < j < k < l \le 6 \tag{4}$$

hold, and there are no more (quadratic) relations between them.

Proof. Note that in addition to nonnegativity: $f(X,Y) \ge 0$, we have symmetry: $f(X,Y) = f(Y,X)$, and also zero property: $f(X,X) = 0$.

By virtue of the last property, the coefficients of x_1y_2 and y_1x_2 are opposite in all terms of the representation

$$f(X,Y) = \sum_i (\alpha_i x_1 y_2 + \beta_i y_1 x_2 + \dots)^2,$$

i.e. $\beta_i = -\alpha_i$ for all i. Hence

$$f(X,Y) = \sum_i (\alpha_i z_{1,2} + \gamma_i x_1 y_3 + \delta_i y_1 x_3 + \dots)^2$$

and the procedure can be continued.

As for the relations between the $z_{i,j}$-s, assume that there holds a nontrivial quadratic identity $g(Z) = 0$ containing the term $z_{1,2}^2$. Then also $x_1^2 y_2^2$ is present, however, this latter can only occur in $z_{1,2}^2$, therefore a term $-z_{1,2}^2$ is needed to cancel it, which contradicts the non-triviality. In a similar way we see that there is no term of type $z_{1,2}\, z_{1,3}$ occurring in a non-trivial identity.

Finally, assume we have a non-trivial identity containing the term $z_{1,2}\, z_{3,4}$. (Its coefficient can be supposed to be unity.) Then $x_1y_2x_3y_4$ is a part of the (expanded) identity. In contrast to the above cases, this occurs in two additional terms: in $z_{1,4}\, z_{2,3}$ and in $z_{1,3}\, z_{2,4}$ to produce the non-trivial identity $z_{1,2}\, z_{3,4} + z_{1,4}\, z_{2,3} - z_{1,3}\, z_{2,4} = 0$.

Since $x_1y_2x_3y_4$ occurs only in the expansion of the three above terms, there are no more non-trivial identities containing it. $\square$

Before proving the theorem, we formulate the standard primal and dual semidefinite programs:

$$\min \{C \bullet X : \; X \ge 0, \; A_i \bullet X = b_i, \; 1 \le i \le m\} \qquad (Primal)$$

$$\max\ \{b^T y:\ S \equiv C - \sum_{i=1}^{m} y_i A_i \geq 0\} \qquad (Dual)$$

where all matrices are n-th order real symmetric, m is the number of constraints, C and $(A_i)_1^m$ are given, vector b of length m is also given, while the primal matrix X and the dual matrix S (the so-called 'slack' matrix - sometimes denoted by Z) together with the m-vector y are the output of the program, $\bullet$ denotes the standard scalar product $A \bullet B = \mathrm{trace}(AB)$ for symmetric matrices and $\geq$ stands for the Loewner ordering: $A \geq B$ iff $A - B$ is positive semidefinite, in short: psd.

Turning to our case, denote by $(A_i)_1^{15}$ the constraint matrices corresponding to the basic identities (4) mentioned in the Lemma. Since these are homogeneous equations, the b_i-s are zero. In an interesting way, both the order $n = \binom{6}{2}$ and the number of constraints $m = \binom{6}{4}$ equals 15.

Nevertheless we will need also the identity I as a constraint matrix to get a sum of squares decomposition, and – to emphasize its speciality – we associate it with index zero, i.e. we write $A_0 = I$ and get the *concrete* primal-dual pair of SDP programs:

$$\min\ \{C \bullet X:\ X \geq 0,\ \mathrm{tr}X = 1,\ A_i \bullet X = 0,\ 1 \leq i \leq 15\} \qquad (Primal)$$

$$\max\ \{y_0:\ S \equiv C - y_0 I - \sum_{i=1}^{15} y_i A_i \geq 0\} \qquad (Dual)$$

After this preparation we can prove our theorem.

Proof. To prove Theorem 2, we specify in detail the data for the SDP above and explain the results obtained. Considering the band-width of matrices C, X and S, a good order of the $z_{i,j}$'s is

$$(z_{2,5}, z_{3,4}, z_{1,2}, z_{2,6},\ z_{1,4}, z_{2,3}, z_{4,5}, z_{4,6},\ z_{1,3}, z_{3,5}, z_{2,4},\ z_{1,5}, z_{1,6}, z_{3,6}, z_{5,6}).$$

Then, denoting by z the corresponding column vector, it holds that $f(X,Y) = z^T C z$ for C appropriately defined. To this, we describe the common block-structure of the matrices C, S, X. All these matrices are block-diagonal with two 4×4 blocks and a 3×3 block, while the remaining 4×4 block is diagonal. In case of C e.g. these blocks will be denoted by C_4, C_4', C_3 and C_d. Here, C_4' is diagonally similar to C_4 through $\mathrm{diag}(1,1,1,-1)$, hence the eigenvalues of C_4' and C_4 coincide. The whole matrix is

$$C = C_4 \oplus C_4' \oplus C_3 \oplus C_d,$$

and the same direct sum representation holds for the optimal primal and dual matrices X and S. As regards C, we have

$$C_4 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \quad C_d = 2I_4.$$

Now we can explain the output of our program. The optimal value of the objective is negative: $y_0 = -\frac{1}{7}$, indicating that (1) is not a SOS, but the modified quartics

$$2\tfrac{1}{7}\, ||X||^2||Y||^2 - 2\,\mathrm{trace}^2(X^TY) - ||XY - YX||^2$$

is a sum of squares, $2\frac{1}{7}$ being the smallest number with this property. The only nonzero y-s are $y_1 = y_2 = 5/7$, they correspond to the basic relations (4) with indices $(1,2,3,4)$ and $(2,3,4,5)$.

For the optimal matrix S we have $S = S_4 \oplus S_4' \oplus S_3 \oplus S_d$ with

$$S_4 = \frac{1}{7}\begin{pmatrix} 15 & -5 & 0 & 0 \\ -5 & 15 & -5 & 0 \\ 0 & -5 & 8 & -7 \\ 0 & 0 & -7 & 8 \end{pmatrix}, \quad S_3 = \frac{1}{7}\begin{pmatrix} 8 & -7 & -2 \\ -7 & 8 & -2 \\ -2 & -2 & 8 \end{pmatrix}, \quad S_d = \frac{15}{7}I_4,$$

which yields the wanted sum of squares decomposition. The optimal primal matrix is $X = X_4 \oplus X_4' \oplus X_3 \oplus X_d$, where

$$X_4 = \frac{2}{735}\begin{pmatrix} 1 & 3 & 8 & 7 \\ 3 & 9 & 24 & 21 \\ 8 & 24 & 64 & 56 \\ 7 & 21 & 56 & 49 \end{pmatrix}, \qquad X_3 = \frac{9}{245}\begin{pmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{pmatrix},$$

and X_d is the fourth order zero matrix. Using the block-structure, the positive semidefiniteness of X and S and the complementarity condition $XS = 0$ can easily be checked (cf. the Karush-Kuhn-Tucker necessary conditions). Also, strict complementarity holds, in particular $\mathrm{rank}(X) = \mathrm{def}(S) = 3$. □

Notice that in general – unlike linear programming – rational data for a SDP problem does not necessarily result in rational solution!

4 On a parametric version

To get more insight into the problem, we insert two parameters α and β to investigate the SOS representability of the biquadratics

$$2 \sum_{1 \le i < j \le 6} z_{i,j}^2 - \alpha(z_{1,3} + z_{2,4} + z_{3,5})^2 - \beta(z_{1,2} + z_{2,6})^2 - \beta(z_{4,5} - z_{4,6})^2. \quad (5)$$

(The reason for the two β's is that these terms behave similarly.) It turns out that only the first two constraints $A_1 \bullet X = 0$ and $A_2 \bullet X = 0$ will be active with $y_1 = y_2$, and $y_i = 0,\ i \ge 3$, as in the above special case of $\alpha = \beta = 1$. This means that our problem reduces to finding the optimal y_0, y_1 for a given pair $(\alpha, \beta) \in [0,1]^2$ such that

$$(2 - y_0) f_0 - \alpha f_1 - \beta f_2 - y_1 F_1 - y_2 F_2 \ge 0 \quad (6)$$

and y_0 is maximum, where we use the abbreviations

$$f_0 = \sum_{i<j}^{6} z_{i,j}^2,\ f_1 = (z_{1,3} + z_{2,4} + z_{3,5})^2,\ f_2 = (z_{1,2} + z_{2,6})^2 + (z_{4,5} - z_{4,6})^2,$$

$$F_1 = 2(z_{1,2}\, z_{3,4} + z_{1,4}\, z_{2,3} - z_{1,3}\, z_{2,4}),\ F_2 = 2(z_{2,3}\, z_{4,5} + z_{2,5}\, z_{3,4} - z_{2,4}\, z_{3,5})$$

in connection with the notations

$$p = 2 - y_0, \qquad q = y_1 = y_2, \quad \text{and} \quad F = F_1 + F_2$$

to write (6) in the simpler form

$$p f_0 - \alpha f_1 - \beta f_2 - qF.$$

Observation. Assume that for some $(\alpha, \beta) \in [0,1]^2$ we know the optimal values of y_0, y_1, i.e. the optimal p and q. Then by multiplying through the coefficient vector (α, β, p, q) by $2/p$ we get $(\alpha', \beta', p', q')$ with

$$\alpha' = \frac{2\alpha}{p},\ \beta' = \frac{2\beta}{p},\ p' = 2,\ q' = \frac{2q}{p},$$

showing that for this new (α', β') we have $y_0' = 0$.

Example. Let us calculate the largest $\alpha = \beta$ for which (5) is SOS! (Theorem 2 tells us that this $\alpha < 1$.) For $\alpha = \beta = 1$ we know that $y_0 = -\frac{1}{7}$, thus $p = \frac{15}{7}$, and $q = y_1 = \frac{5}{7}$. The transformed variables are $\alpha' = \beta' = \frac{14}{15}$ and $q' = y_1' = \frac{2}{3}$.

Table 1: The unit square: optimal values

region	name	$p = 2 - y_0$	$q = y_1$	def(S)	rank(X)
$\beta = 0$		$2\,\alpha$	α	1	1
$\beta < 0.8\,\alpha$	R_1	$2\,\alpha$	α	1	1
$\beta = 0.8\,\alpha$		$2\,\alpha$	α	3	1
$\beta \in (0.8\alpha, 1.5\alpha)$	R_3	(7)	(8)	3	3
$\beta = 1.5\,\alpha$		$2\,\beta$	0	3	2
$\beta > 1.5\,\alpha$	R_2	$2\,\beta$	0	2	2
$\alpha = 0$		$2\,\beta$	0	2	2

Consequently the polynomial $2f_0 - \frac{14}{15}(f_1 + f_2)$ is not only nonnegative but also SOS (and the point $(\alpha, \beta) = (\frac{14}{15}, \frac{14}{15})$ lies on the border of the 'good' and 'bad' cases).

In the next theorem we summarize the results obtained for parameters (α, β) from the unit square.

Theorem 3 *Table 1 gives the optimal values for the parametrized problem (6).*

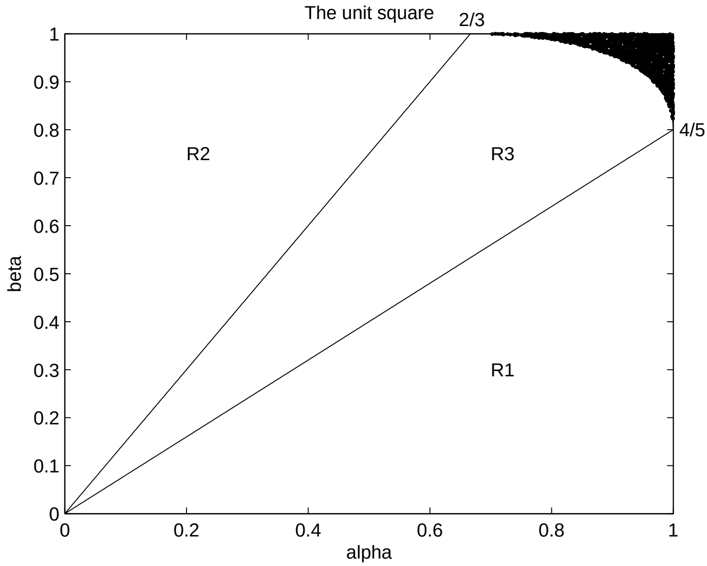
The optimal p and q for the middle sector R_3 are

$$p = -\alpha\beta\ \frac{2\alpha + 9\beta + \sqrt{(8\alpha - 9\beta)^2 + 15\beta^2}}{4\alpha^2 - 12\alpha\beta + \beta^2} \tag{7}$$

$$q = \frac{\beta(p(60\alpha^2 - 4\alpha\beta - 9\beta^2) + 15\alpha\beta(\beta - 6\alpha)}{4(4\alpha^2 - 12\alpha\beta + \beta^2)(3\beta - 2p)} \tag{8}$$

In addition to this "radial" characterization, the level lines also can be described: these are curves, along which the optimal y_0 is constant. The case $y_0 = 0$, i.e. $p = 2$ can be seen on the figure, cf. also (10). (For another instance, the dotted line starting at $\alpha = 0,\ \beta = 0.5$ and ending at $\alpha = 0.5,\ \beta = 0$ belongs to $p = 1$.) The indices of region names correspond to the defects of the optimal dual matrix S. The right upper darkened region within R_3 refers to points (α, β) for which (5), i.e. $2f_0 - \alpha f_1 - \beta f_2$ is not a sum of squares.

Proof. The block-structure of the special case $\alpha = \beta = 1$ keeps on holding,



and the blocks at issue depend just on one of the parameters α and β :

$$S_3(\alpha) = \begin{pmatrix} p-\alpha & -\alpha & q-\alpha \\ -\alpha & p-\alpha & q-\alpha \\ q-\alpha & q-\alpha & p-\alpha \end{pmatrix}, \quad S_4(\beta) = \begin{pmatrix} p & -q & 0 & 0 \\ -q & p & -q & 0 \\ 0 & -q & p-\beta & -\beta \\ 0 & 0 & -\beta & p-\beta \end{pmatrix},$$

as it easily follows by considering the polynomial (6).

Let us begin with the case $\beta = 0$. Then S_4 (and also ${S_4}'$) is psd, and our 'work matrix' is S_3. Its (1,1) entry, $p - \alpha \geq 0$, and the nonnegativity of the left upper 2-minor implies $|p-\alpha| \geq |\alpha|$. Since both sides are nonnegative, it follows that $p \geq 2\alpha$. On the other hand, the determinant of S_3 equals

$$|S_3| \equiv \det(S_3) = p\,\{p^2 - 2q^2 - \alpha(3p - 4q)\},$$

which must vanish at the optimal variables, therefore

$$\big(p - \frac{3}{2}\alpha\big)^2 = 2\big(q-\alpha\big)^2 + \frac{1}{4}\alpha^2.$$

From this we get $|p - \frac{3}{2}\alpha| \geq |\frac{1}{2}|$, and, since this holds without absolute value as well, we conclude that the optimal variables are $q = 0$ and $p = 2\alpha$ (note that maximizing y_0 is equivalent to minimizing $p = 2 - y_0$).

All this holds for R_1, i.e. for β "small" – until we arrive at $|S_4(\beta)| = 0$. Before that moment we still have $p = 2\alpha,\ q = \alpha$ and

$$S_3 = \alpha \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

giving one rank decrease. As regards S_4, for $\beta \le (4/5)\,\alpha$ we have

$$S_4 \ge \begin{pmatrix} p & -\alpha & 0 & 0 \\ -\alpha & p & -\alpha & 0 \\ 0 & -\alpha & p - \frac{4}{5}\alpha & -\frac{4}{5}\alpha \\ 0 & 0 & -\frac{4}{5}\alpha & p - \frac{4}{5}\alpha \end{pmatrix} = \frac{\alpha}{5} \begin{pmatrix} 10 & -5 & 0 & 0 \\ -5 & 10 & -5 & 0 \\ 0 & -5 & 6 & -4 \\ 0 & 0 & 4 & 6 \end{pmatrix}$$

in the sense of the semidefinite (Loewner) ordering, which easily follows by (6). Matrix S_4 becomes active, if equality is attained: $\beta = (4/5)\alpha$. Then S_4 becomes a psd matrix with rank 3, therefore S_4 and S_4' yield two further rank losses. (The situation on the border of R_1 and R_3 can be understood by thinking of the continuity of the roots depending on the parameters.)

The cases $\alpha = 0$ and $\beta > 1.5\,\alpha$ are similar, hence we give only the necessary formulas. The determinant of S_4 is

$$|S_4| = p\,\{p^2(p - 2\beta) - q^2(2p - 3\beta)\}.$$

Inequality $p \ge 2\beta$ follows similarly to the case $p \ge 2\alpha$. On the other hand, $|S_4| = 0$ can be rewritten as

$$p(p - \beta)^2 + 3q^2\beta = p(2q^2 + \beta^2),$$

whence we conclude $(p - \beta)^2 \le 2q^2 + \beta^2$, i.e. $p(p - 2\beta) \le 2q^2$, and it follows that the optimal values ($p \to \min!$) are $q = 0$ and $p = 2\beta$. The matrix

$$S_4 = \beta \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

is psd with defect one, giving (together with ${S_4}'$) rank loss=2 for the whole matrix S. As regards S_3, for $(\alpha, \beta) \in R_2$ it equals

$$S_3(\alpha) = pI_3 - \alpha\, ee^T = \alpha\big(\tfrac{p}{\alpha} I_3 - ee^T\big), \quad e = (1, 1, 1, 1)^T,$$

which is positive definite for $p/\alpha > 3$, i.e. for $\beta > (3/2)\,\alpha$, and psd (with rank=2) if equality holds:

$$S_3 = \alpha \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

In this latter case, i.e. if $\beta = (3/2)\,\alpha$, the defect of S becomes 3, since not only S_4 and S_4', but also S_3 yields a rank loss.

The middle region R_3 can be characterized by that the SDP program annihilates *both* determinants. The equations $|S_3(\alpha)| = 0$ and $|S_4(\beta)| = 0$ are equivalent to

$$p^2 - 2q^2 = \alpha(3p - 4q) \quad \text{and} \quad p(p^2 - 2q^2) = \beta(2p^2 - 3q^2), \tag{9}$$

from which we can express p and q by means of α and β to get (7) and (8), and also β by help of α and p :

$$\varphi_p(\alpha) = \tfrac{2p}{3} - \tfrac{2p^3}{3}\big(p^2 + 9\alpha p - 12\alpha^2 + 6\alpha\sqrt{2(p-\alpha)(p-2\alpha)}\,\big)^{-1}$$

In the important special case $p = 2$ we have the function $\varphi : [\frac{2}{3}, 1] \to [\frac{4}{5}, 1]$ defined by

$$\varphi(\alpha) \equiv \varphi_2(\alpha) = \tfrac{4}{3} - \tfrac{8}{3}\big(2 + 9\alpha - 6\alpha^2 + 6\alpha\sqrt{(1-\alpha)(2-\alpha)}\,\big)^{-1}. \tag{10}$$

The critical values of φ are:

$$\varphi(\tfrac{2}{3}) = 1,\ \varphi'(\tfrac{2}{3}) = 0,\ \varphi(1) = \tfrac{4}{5},\ \varphi'(1) = -\infty.$$

The set of points in the unit square, for which (5) is a SOS are delimited by the α and β axes, the horizontal line segment $(0,1)$ to $(\frac{2}{3}, 1)$, the graph of φ and the vertical line segment $(1,0)$ to $(1, \frac{4}{5})$. All other curves with different p (e.g. that with $p = 1$ dotted on the figure) are proportional to this one, since equations (9) are homogeneous. □

Remark 3 Consider the ellipse with center in $(\frac{2}{3}, \frac{4}{5})$ and vertices $(1, \frac{4}{5})$, $(\frac{2}{3}, 1)$, the right upper quarter of which is close to the graph of φ. The elementary equality $(\frac{14}{15} - \frac{2}{3})^2 + (1 - \frac{4}{5})^2 = (1 - \frac{2}{3})^2$ shows that the projection of the 'border point' $(\frac{14}{15}, \frac{14}{15})$ onto the longer axis of the ellipse is just its focus $(\frac{14}{15}, \frac{4}{5})$! As for their measures, the approximate area of the 'bad' (shadowed) region is 0.0121, while that above the ellipse amounts to $\frac{1}{15} - \frac{\pi}{60} \approx 0.0143$.

Remark 4 For the interested reader we give some 'nice' rational solutions: in addition to the known quadruples $(\alpha, \beta, p, q) = (1, 1, \frac{15}{7}, \frac{5}{7})$ and $(\frac{14}{15}, \frac{14}{15}, 2, \frac{2}{3})$ we have e.g. $(\alpha, \beta, p, q) = (\frac{34}{35}, \frac{17}{19}, 2, \frac{4}{5}), (\frac{17}{31}, \frac{68}{23}, 6, 1)$, or $(\frac{31}{16}, \frac{124}{50}, 5, \frac{5}{8})$.

Considering the derivatives also is of interest: the slope of φ_p for $p = 2\frac{1}{7}$ at $\alpha = 1$ equals $-\frac{3}{4}$, which can be used to define a new problem with the same solution! Replace to this the identity $A_0 = I$ by two matrices A_{16} and A_{17} associated with the quadratic forms f_1 and f_2, and set the corresponding coordinates of b equal to 3 and 4 (coming from the numerator and denominator of the ratio $-\frac{3}{4}$ above). The result will coincide with that of Theorem 2.

Remark 5 In [6] the authors write: "Unfortunately, the nature of a parametric SDP is far more complicated [than LP] due to regions of nonlinearity of $\phi(\gamma)$." (The function $\phi(\gamma) = C(\gamma) \bullet X(\gamma)$ is the primal objective depending on the parameter.) In light of this, present problem seems to be a refreshing exception: the nonlinearity (cf. the functions φ_p) can be handled by means of elementary functions.

References

[1] K. Audenaert, Variance bounds, with an application to norm bounds for commutators, *Linear Algebra Appl.*, **432** (2010), 1126–1143.

[2] A. Böttcher, D. Wenzel, How big can the commutator of two matrices be and how big is it typically?, *Linear Algebra Appl.*, **403** (2005), 216–228.

[3] A. Böttcher, D. Wenzel, The Frobenius norm and the commutator, *Linear Algebra Appl.*, **429** (2008), 1864–1885.

[4] Ch-M. Cheng, S-W. Vong, D. Wenzel, Commutators with maximal Frobenius norm, *Linear Algebra Appl.*, **432** (2010), 292–306.

[5] K-S. Fong, Ch-M. Cheng, I-K. Lok, Another unitarily norm attaining the minimum norm bound for commutators, *Linear Algebra Appl.*, **433**, (2010), 1793–1797.

[6] D. Goldfarb, K. Scheinberg, On parametric semidefinite programming, *Appl. Numer. Math.*, **29** (1999), 361–377.

[7] L. László, Proof of Böttcher and Wenzel's conjecture on commutator norms for 3-by-3 matrices, *Linear Algebra Appl.*, **422** (2007), 659–663.

[8] L. László, On Sum of Squares Decomposition for a Biquadratic Matrix Function, *Ann. Univ. Sci. Budapest. Sect. Comput.*, **33** (2010), 273–284.

[9] S-W. Vong, X-Q. Jin, Proof of Böttcher and Wenzel's conjecture, *Oper. Matrices*, **2** (2008), 435–442.

[10] D. Wenzel, K. M. R. Audenaert, Impressions of convexity - An illustration for commutator bounds, *Linear Algebra Appl.*, **433** (2010), 1726–1759.

Received: 22, January, 2011

Acta Univ. Sapientiae, Mathematica, **3**, 1 (2011) 90–96

Multiplying balancing numbers

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Abstract. In this paper we prove some results on multiplying balancing and cobalancing numbers and (k, l)-power numerical centers.

1 Introduction

The sequence $R = \{R_i\}_{i=0}^{\infty} = R(A, B, R_0, R_1)$ is called a second order linear recurrence sequence if the recurrence relation

$$R_i = AR_{i-1} + BR_{i-2} \ (i \geq 2)$$

holds, where $A, B \neq 0, R_0, R_1$ are fixed rational integers and $|R_0| + |R_1| > 0$. A positive integer n is called a balancing number (see [3] and [5]) if

$$1 + \cdots + (n - 1) = (n + 1) + \cdots + (n + r)$$

holds for some positive integer r. The sequence of balancing numbers is denoted by B_m $(m = 1, 2, \dots)$. As one can easily check, we have $B_1 = 6$ and $B_2 = 35$. Note that by a result of Behera and Panda [3], we have

$$B_{m+1} = 6B_m - B_{m-1} \qquad (m > 1).$$

In that paper they proved that, there are infinitely many balancing numbers.

In [7] K. Liptai searched for those balancing numbers which are Fibonacci numbers, too. Using the results of A. Baker and G. Wüstholz [2] he proved

2010 Mathematics Subject Classification: 11A41, 11A51
Key words and phrases: balancing number

that there are no Fibonacci balancing numbers. Similarly in [8] he proved that there are no Lucas balancing numbers. Using an other method L. Szalay [12] got the same result.

In [9] Liptai, Luca, Pintér and Szalay generalized the concept of balancing numbers in the following way. Let y, k, l be fixed positive integers with $y \geq 4$. A positive integer x with $x \leq y-2$ is called a (k,l)-power numerical center for y if

$$1^k + \cdots + (x-1)^k = (x+1)^l + \cdots + (y-1)^l.$$

In [9] several effective and ineffective finiteness results were proved for (k,l)-power numerical centers.

Later G.K. Panda and P.K. Ray (see [10]) slightly modified the definition of balancing number and introduced the notion of cobalancing number. A positive integer n is called a *cobalancing number* if

$$1 + 2 + \cdots + (n-1) + n = (n+1) + (n+2) + \cdots + (n+K)$$

for some $K \in \mathbb{N}$. In this case K is called the cobalancer of n.

They also proved that the cobalancing numbers fulfill the following recurrence relation

$$b_{n+1} = 6b_n - b_{n-1} + 2 \quad (n > 1),$$

where $b_0 = 1$ and $b_1 = 6$. Moreover they found that every balancer is a cobalancing number and every cobalancer is a balancing number.

In [11] G. K. Panda gave another possible generalization of balancing numbers. Let $\{a_m\}_{m=0}^{\infty}$ be a sequence of real numbers. We call an element a_n of this sequence a *sequence-balancing* number if

$$a_1 + a_2 + \cdots + a_{n-1} = a_{n+1} + a_{n+2} + \cdots + a_{n+k}$$

for some $k \in \mathbb{N}$. Similarly, one can define the notion of *sequence cobalancing numbers*. In [11] it was proved that there does not exist any sequence balancing number in the Fibonacci sequence.

As a generalization of the notion of a balancing number A. Bérczes, K. Liptai and I. Pink call a binary recurrence $R = R(A, B, R_0, R_1)$ a *balancing sequence* if

$$R_1 + R_2 + \ldots + R_{n-1} = R_{n+1} + R_{n+2} + \ldots + R_{n+k}$$

holds for some $k \geq 1$ and $n \geq 2$.

In [4] they proved that that any sequence $R = R(A, B, 0, R_1)$ with the condition $D = A^2 + 4B > 0$, $(A, B) \neq (0, 1)$ is not a balancing sequence.

T. Kovács, K. Liptai and P. Olajos in [6] extended the concept of balancing numbers to arithmetic progressions. Let $a > 0$ and $b \geq 0$ be coprime integers. If for some positive integers n and r we have

$$(a+b) + \cdots + (a(n-1)+b) = (a(n+1)+b) + \cdots + (a(n+r)+b)$$

then we say that $an+b$ is an (a,b)-balancing number. They proved several effective finiteness and explicit results about them. In the proofs they combined the Baker's method, the modular method developed by Wiles and others, the Chabauty method and the theory of elliptic curves.

In this paper we study a further generalization of balancing numbers. The idea is due to A. Behera and G. K. Panda. A positive integer n is called a *multiplying balancing number* if

$$1 \cdot 2 \cdots (n-1) = (n+1)(n+2)\cdots(n+r) \tag{1}$$

for some positive integer r. The number r is called the *balancer* corresponding to the multiplying balancing number n. The cobalancing numbers have a similar definition. A positive integer n is called a *multiplying cobalancing number* if

$$1 \cdot 2 \cdots (n-1)n = (n+1)(n+2)\cdots(n+r) \tag{2}$$

for some positive integer r. The number r is called the *cobalancer* corresponding to the multiplying cobalancing number n.

Using the concept of K. Liptai, F. Luca, . Pintér and L. Szalay ([9] we can get further generalization. Let m, k, l be fixed positive integers with $m \geq 4$. A positive integer n with $n \leq m-2$ is called a (k,l)*-power multiplying balancing number for* m if

$$1^k \cdots (n-1)^k = (n+1)^l \cdots (m-1)^l. \tag{3}$$

2 The results

Throughout the paper let p the greatest odd prime, which is less than the multiplying balancing number n, where $n \geq 4$. In the first theorem we prove that only one multiplying balancing number exists.

Theorem 1 *The only multiplying balancing number is* $n = 7$ *with the balancer* $r = 3$.

In the proof we use 4 lemmas.

Lemma 1 *There is no prime among the factors of the right side of the equation*

$$1 \cdot 2 \cdots (n-1) = (n+1)(n+2)\cdots(n+r)$$

Proof. Suppose that z is a prime among the factors of the right side. It is clear that z is not in the prime decomposition of the left side of the equation (1). Hence the prime decomposition of the right side is not the same as the left's. Thus the lemma is proved. $\square$

Let us use the function $\alpha_2\colon \mathbb{N} \to \mathbb{N}$, $\alpha_2(x) := \sum_{k=1}^{[\log_2 x]} \left[\frac{x}{2^k}\right]$, where $x \geq 2$ and $\alpha_2(x)$ shows the index of the prime 2 in $x!$.

Lemma 2 $x - \log_2 x - 2 < \alpha_2(x) < x$

Proof.

$$\begin{aligned}
\alpha_2(x) &= \left[\frac{x}{2^1}\right] + \left[\frac{x}{2^2}\right] + \left[\frac{x}{2^3}\right] + \cdots + \left[\frac{x}{2^k}\right] \leq \frac{x}{2^1} + \frac{x}{2^2} + \frac{x}{2^3} + \cdots + \frac{x}{2^k} = \\
&= x\left(\frac{1}{2^1} + \frac{1}{2^2} + \cdots + \frac{1}{2^k}\right) = x\left(1 - \frac{1}{2^k}\right) \leq x - 1 < x \\
\alpha_2(x) &> \underbrace{\left(\frac{x}{2^1} - 1\right) + \left(\frac{x}{2^2} - 1\right) + \cdots + \left(\frac{x}{2^k} - 1\right)}_{[\log_2 x]} = \\
&= x\left(1 - \frac{1}{2^k}\right) - [\log_2 x] = x - \frac{x}{2^k} - [\log_2 x] > x - \log_2 x - 2
\end{aligned}$$

$\square$

Lemma 3 *If* n *is multiplying balancing number and* r *is the balancer, furthermore* $n > 64$ *then*

$$\frac{3(n+1)}{2} < n + r.$$

Proof. From (1) it follows, that

$$(n-1)! \cdot (n)! = (n+r)!.$$

If (1) is true then

$$\alpha_2(n-1) + \alpha_2(n) = \alpha_2(n+r)$$

Using Lemma 2 we get

$$n - 2\log_2 n - 5 < r$$

We can replace $\log_2 n$ with $\frac{n}{8}$ if $n > 64$, that is

$$2n - \frac{2n}{8} - 5 < n + r$$

If $n > 64$ we get

$$\frac{3(n+1)}{2} < \frac{3(n+1)}{2} + \frac{2n}{8} - 6.5 < n + r.$$

□

Proof.[Proof of Theorem 1] Using a results of M. El Bachraoui ([1]) we get that, if $n \geq 2$ then exists a p prime satisfying the inequality

$$n < p < \frac{3(n+1)}{2}.$$

Hence the right side of (1) contains a prime if $n > 64$. But Lemma 1 says there is no prime on the right side of the equation. The conclusion is that, if $n > 64$ there is no multiplying balancing numbers. It can be checked easily if $n = 2, \ldots, 64$ then there is only one number satisfying the equation (1). We get $n = 7$, that is the theorem is proved. □

Theorem 2 *There is no multiplying cobalancing number.*

In the proof we use the following lemma.

Lemma 4 *Using our notation the following inequalities are true*

$$p < n < 2p \leq n + r < 3p.$$

Proof. Suppose that $n \geq 2p$. The interval $[p, 2p]$ always contains a prime, so there is a prime greater than p and lower than n which is impossible because of the definition of p. Hence $n < 2p$. On the left side of the equation (1) the index of p is 1, consequently on the right side the index of p is also 1 in the prime decomposition. So we can write the following inequalities $2p \leq n + r < 3p$. □

Proof. [Proof of Theorem 2] Using a result of Csebisev we get that there is a prime z between p and $2p$. Because of Lemma 4 we have to analyse three cases $z = n$, $z > n$ and $z < n$. If $z > n$ then the prime decomposition of the left and right side is not the same. Now let $z < n$.This situation contradicts the fact that p is the greatest odd prime which is less than n. The last case is $z = n$. Hence $n + r \geq 2z$ because of the prime factor z. Thus the left side of the equation (2) has as many factor as the right side has which is obviously impossible. First and last there is no cobalancing numbers. □

The following theorem deals with the (k, l)-power numerical centers.

Theorem 3 *If* $n \geq 4$ *then there is only one* (k, l)*-power numerical centers. The only solution is* $n = 7$, $m = 11$ *and* $k = l$.

Proof. First we prove that if n is a (k, l)-power numerical center for m then $k = l$. Using Lemma 4 the index of p in the equation (3) is k on the left side and l on the right side in the prime decomposition. The index of p have to be equal on the left and right side. So $k = l$.

So we get that n satisfies (1) if and only if n satisfies (3). So if $n \geq 4$ there is only one (k, l)-power numerical center. It is $n = 7$, $m = 11$ and $l = k$. □

Remark 1 *If* $p = 2$ *and* $n = 3$ *we get the equation*

$$1^k \cdot 2^k = 4^l.$$

In this case $n = 3$ (k, l)*-power numerical center for* $m - 5$ *and there are infinitely many* (k, l) *pairs with* $k = 2l$.

Acknowledgement

The author is grateful to Kálmán Liptai for his useful and helpful remarks and suggestions.

References

[1] M. El Bachraoui, Primes in the Interval $[2n, 3n]$, *Int. J. Contemp. Math. Sci.*, **1** (2006), 617–621.

[2] A. Baker, G. Wüstholz, Logarithmic forms and group varieties, *J. Reine Angew. Math.*, **442** (1993), 19–62.

[3] A. Behera, G. K. Panda, On the square roots of triangular numbers, Fibonacci Quart., **37** (1999), 98–105.

[4] A. Bérczes, K. Liptai, I. Pink, On Generalized Balancing Sequences, *Fibonacci Quart.*, **48** (2010), 121–128.

[5] R. P. Finkelstein, 'The House problem', *American Math. Monthly*, **72** (1965), 1082–1088.

[6] T. Kovács, K. Liptai, P. Olajos, On (a, b)-balancing numbers, *Publicationes Mathematicae Debrecen* (accepted).

[7] K. Liptai, Fibonacci balancing numbers, *Fibonacci Quarterly*, **42** (2004), 330–340.

[8] K. Liptai, Lucas balancing numbers, *Acta Math. Univ. Ostrav.*, **14**, (2006), 43–47.

[9] K. Liptai, F. Luca, Á. Pintér, L. Szalay, Generalized balancing numbers, *Indag. Math., New Ser.*, **20**, (2009), 87–100.

[10] G. K. Panda, P. K. Ray, Cobalancing numbers and cobalancers, *Int. J. Math. Math. Sci.*, No. 8, (2005) 1189–1200.

[11] G. K. Panda, Sequence balancing and cobalancing numbers, *Fibonacci Quarterly*, **45** (2007), 265–271.

[12] L. Szalay, On the resolution of simultaneous Pell equations, *Annales Mathematicae et Informaticae*, **34** (2007), 77–87.

Received: October 3, 2010

Acta Univ. Sapientiae, Mathematica, **3**, 1 (2011) 97–109

Some sequence spaces in 2-normed spaces defined by Musielak-Orlicz function

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Abstract. In the present paper we introduce some sequence spaces combining lacunary sequence, invariant means in 2-normed spaces defined by Musielak-Orlicz function $\mathcal{M} = (M_k)$. We study some topological properties and also prove some inclusion results between these spaces.

1 Introduction and preliminaries

The concept of 2-normed space was initially introduced by Gahler [2] as an interesting linear generalization of a normed linear space which was subsequently studied by many others see ([3], [9]). Recently a lot of activities have started to study sumability, sequence spaces and related topics in these linear spaces see ([4], [10]).

Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $\|.,.\| : X \times X \to \mathbb{R}$ which satisfies

(i) $\|x, y\| = 0$ if and only if x and y are linearly dependent

(ii) $\|x, y\| = \|y, x\|$

(iii) $\|\alpha x, y\| = |\alpha|\|x, y\|$, $\alpha \in \mathbb{R}$

(iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$ for all $x, y, z \in X$.

2010 Mathematics Subject Classification: 40A05, 40A30
Key words and phrases: Paranorm space, Difference sequence space, Orlicz function, Musielak-Orlicz function, Lacunary sequence, Invariant mean

The pair $(X, \|.,.\|)$ is then called a 2-normed space see [3]. For example, we may take $X = \mathbb{R}^2$ equipped with the 2-norm defined as $\|x, y\| =$ the area of the parallelogram spanned by the vectors x and y which may be given explicitly by the formula

$$\|x_1, x_2\|_E = \text{abs}\left(\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} \right).$$

Then, clearly $(X, \|.,.\|)$ is a 2-normed space. Recall that $(X, \|.,.\|)$ is a 2-Banach space if every cauchy sequence in X is convergent to some x in X.

Let σ be the mapping of the set of positive integers into itself. A continuous linear functional φ on l_∞,is said to be an invariant mean or σ-mean if and only if

(i) $\varphi(x) \geq 0$ when the sequence $x = (x_k)$ has $x_k \geq 0$ for all k,

(ii) $\varphi(e) = 1$, where $e = (1, 1, 1, \dots)$ and

(iii) $\varphi(x_{\sigma(k)}) = \varphi(x)$ for all $x \in l_\infty$.

If $x = (x_n)$, write $Tx = Tx_n = (x_{\sigma(n)})$. It can be shown in [11] that

$$V_\sigma = \left\{ x \in l_\infty \mid \lim_k t_{kn}(x) = l, \ \text{ uniformly } \ in \ n, \ l = \sigma - \lim x \right\},$$

where

$$t_{kn}(x) = \frac{x_n + x_{\sigma^1 n} + \dots + x_{\sigma^k n}}{k+1}.$$

In the case σ is the translation mapping $n \to n+1$, σ-mean is often called a Banach limit and V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences see [6].

By a lacunary sequence $\theta = (k_r)$ where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$. We write $h_r = k_r - k_{r-1}$. The ratio $\dfrac{k_r}{k_{r-1}}$ will be denoted by q_r. The space of lacunary strongly convergent sequence was defined in [1].

Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

(i) $p(x) \geq 0$, for all $x \in X$

(ii) $p(-x) = p(x)$, for all $x \in X$

(iii) $p(x + y) \leq p(x) + p(y)$, for all $x, y \in X$

(iv) if (σ_n) is a sequence of scalars with $\sigma_n \to \sigma$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\sigma_n x_n - \sigma x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [12], Theorem 10.4.2, P-183).

An orlicz function $M : [0, \infty) \to [0, \infty)$ is a continuous, non-decreasing and convex function such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.

Lindenstrauss and Tzafriri [5] used the idea of Orlicz function to define the following sequence space. Let w be the space of all real or complex sequences $x = (x_k)$, then

$$l_M = \left\{ x \in w \mid \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \right\}$$

which is called a Orlicz sequence space. Also l_M is a Banach space with the norm

$$\|x\| = \inf\left\{ \rho > 0 \mid \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\}.$$

Also, it was shown in [5] that every Orlicz sequence space l_M contains a subspace isomorphic to $l_p (p \ge 1)$. The Δ_2- condition is equivalent to $M(Lx) \le LM(x)$, for all L with $0 < L < 1$. An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t)dt$$

where η is known as the kernel of M, is right differentiable for $t \ge 0, \eta(0) = 0, \eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.

A sequence $\mathcal{M} = (M_k)$ of Orlicz function is called a Musielak-Orlicz function see ([7], [8]). A sequence $\mathcal{N} = (N_k)$ is called a complementary function of a Musielak-Orlicz function $\mathcal{M}$

$$N_k(v) = \sup\left\{ |v|u - M_k \mid u \ge 0 \right\},\ k = 1, 2, \ldots$$

For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows

$$\begin{aligned} t_{\mathcal{M}} &= \left\{ x \in w \mid I_{\mathcal{M}}(cx) < \infty, \text{ for some } c > 0 \right\}, \\ h_{\mathcal{M}} &= \left\{ x \in w \mid I_{\mathcal{M}}(cx) < \infty, \text{ for all } c > 0 \right\}, \end{aligned}$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$\|x\| = \inf \left\{ k > 0 \mid I_{\mathcal{M}}\left(\frac{x}{k}\right) \leq 1 \right\}$$

or equipped with the Orlicz norm

$$\|x\|^0 = \inf \left\{ \frac{1}{k}\left(1 + I_{\mathcal{M}}(kx)\right) \mid k > 0 \right\}.$$

Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $(X, \|.,.\|)$ be a 2-normed space and $p = (P_k)$ be any sequence of strictly positive real numbers. By $S(2-X)$ we denote the space of all sequences defined over $(X, \|.,.\|)$. We now define the following sequence spaces:

$$w_\sigma^o[\mathcal{M}, p, \|.,.\|]_\theta = \left\{ x \in S(2-X) \mid \lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} \left[M_k\left(\left\| \frac{t_{kn}(x)}{\rho}, z \right\| \right) \right]^{p_k} = 0, \right.$$
$$\left. \rho > 0, \text{ uniformly in } n \right\},$$
$$w_\sigma[\mathcal{M}, p, \|.,.\|]_\theta = \left\{ x \in S(2-X) \mid \lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} \left[M_k\left(\left\| \frac{t_{kn}(x-l)}{\rho}, z \right\| \right) \right]^{p_k} = 0, \right.$$
$$\left. \rho > 0, \text{ uniformly in } n \right\}, \text{ and}$$
$$w_\sigma^\infty[\mathcal{M}, p, \|.,.\|]_\theta = \left\{ x \in S(2-X) \mid \sup_{r,n} \frac{1}{h_r} \sum_{k \in I_r} \left[M_k\left(\left\| \frac{t_{kn}(x)}{\rho}, z \right\| \right) \right]^{p_k} < \infty, \right.$$
$$\left. \text{for some } \rho > 0 \right\}.$$

When $\mathcal{M}(x) = x$ for all k, the spaces $w_\sigma^o[\mathcal{M}, p, \|.,.\|]_\theta$, $w_\sigma[\mathcal{M}, p, \|.,.\|]_\theta$ and $w_\sigma^\infty[M_k, p, \|.,.\|]_\theta$ reduces to the spaces $w_\sigma^o[p, \|.,.\|]_\theta$, $w_\sigma[, p, \|.,.\|]_\theta$ and $w_\sigma^\infty[p, \|.,.\|]_\theta$ respectively.

If $p_k = 1$ for all k, the spaces $w_\sigma^o[\mathcal{M}, p, \|.,.\|]_\theta$, $w_\sigma[\mathcal{M}, p, \|.,.\|]_\theta$ and $w_\sigma^\infty[\mathcal{M}, p, \|.,.\|]_\theta$ reduces to $w_\sigma^o[\mathcal{M}, \|.,.\|]_\theta$, $w_\sigma[\mathcal{M}, \|.,.\|]_\theta$ and $w_\sigma^\infty[\mathcal{M}, \|.,.\|]_\theta$ respectively.

The following inequality will be used throughout the paper. If $0 \leq p_k \leq \sup p_k = H$, $K = \max(1, 2^{H-1})$ then

$$|a_k + b_k|^{p_k} \leq K\{|a_k|^{p_k} + |b_k|^{p_k}\} \tag{1}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

In the present paper we study some topological properties of the above sequence spaces.

2 Main results

Theorem 1 *Let $\mathcal{M} = (M_k)$ be Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers, then the classes of sequences $w_\sigma^o[\mathcal{M}, p, \|.,.\|]_\theta$, $w_\sigma[\mathcal{M}, p, \|.,.\|]_\theta$ and $w_\sigma^\infty[\mathcal{M}, p, \|.,.\|]_\theta$ are linear spaces over the field of complex numbers.*

Proof. Let $x, y \in w_\sigma^o[\mathcal{M}, p, \|.,.\|]_\theta$ and $\alpha, \beta \in \mathbb{C}$. In order to prove the result we need to find some ρ_3 such that

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \left[M_k\left(\left\|\frac{t_{kn}(\alpha x + \beta y)}{\rho_3}, z\right\|\right)\right]^{p_k} = 0, \text{ uniformly in } n.$$

Since $x, y \in w_\sigma^o[\mathcal{M}, p, \|.,.\|]_\theta$, there exist positive ρ_1, ρ_2 such that

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \left[M_k\left(\left\|\frac{t_{kn}(x)}{\rho_1}, z\right\|\right)\right]^{p_k} = 0, \text{ uniformly in } n$$

and

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \left[M_k\left(\left\|\frac{t_{kn}(y)}{\rho_2}, z\right\|\right)\right]^{p_k} = 0, \text{ uniformly in } n.$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since (M_k) is non-decreasing and convex

$$\begin{aligned}
\frac{1}{h_r} \sum_{k\in I_r} \left[M_k\left(\left\|\frac{t_{kn}(\alpha x + \beta y)}{\rho_3}, z\right\|\right)\right]^{p_k} &\leq \frac{1}{h_r} \sum_{k\in I_r} \left[M_k\left(\left\|\frac{t_{kn}(\alpha x)}{\rho_3}, z\right\| + \left\|\frac{t_{kn}(\beta y)}{\rho_3}, z\right\|\right)\right] \\
&\leq \frac{1}{h_r} \sum_{k\in I_r} \left[M_k\left(\left\|\frac{t_{kn}(x)}{\rho_1}, z\right\| + \left\|\frac{t_{kn}(y)}{\rho_2}, z\right\|\right)\right] \\
&\leq K\frac{1}{h_r} \sum_{k\in I_r} \left[M_k\left(\left\|\frac{t_{kn}(x)}{\rho_1}, z\right\|\right)\right] + \\
&\quad + K\frac{1}{h_r} \sum_{k\in I_r} \left[M_k\left(\left\|\frac{t_{kn}(y)}{\rho_2}, z\right\|\right)\right] \\
&\to 0 \text{ as } r \to \infty, \text{ uniformly in } n.
\end{aligned}$$

So that $\alpha x+\beta y\in w_\sigma^o\big[\mathcal{M},p,\|.,.\|\big]_\theta$. This completes the proof. Similarly, we can prove that $w_\sigma\big[\mathcal{M},p,\|.,.\|\big]_\theta$ and $w_\sigma^\infty\big[\mathcal{M},p,\|.,.\|\big]_\theta$ are linear spaces. □

Theorem 2 *Let* $\mathcal{M}=(M_k)$ *be Musielak-Orlicz function,* $p=(p_k)$ *be a bounded sequence of positive real numbers. Then* $w_\sigma^o\big[\mathcal{M},p,\|.,.\|\big]_\theta$ *is a topological linear spaces paranormed by*

$$g(x)=\inf\left\{\rho^{\frac{p_r}{H}}:\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(x)}{\rho},z\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}}\le 1, r=1,2,\dots,\ \ n=1,2,\dots\right\},$$

where $H=\max(1,\sup_k p_k<\infty)$.

Proof. Clearly $g(x)\ge 0$ for $x=(x_k)\in w_\sigma^o\big[\mathcal{M},p,\|.,.\|\big]_\theta$. Since $M_k(0)=0$, we get $g(0)=0$.

Conversely, suppose that $g(x)=0$, then

$$\inf\left\{\rho^{\frac{p_r}{H}}:\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(x)}{\rho},z\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}}\le 1, r\ge 1,\ n\ge 1\right\}=0.$$

This implies that for a given $\epsilon>0$, there exists some $\rho_\epsilon(0<\rho_\epsilon<\epsilon)$ such that

$$\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(x)}{\rho_\epsilon},z\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}}\le 1.$$

Thus

$$\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(x)}{\epsilon},z\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}}\le\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(x)}{\rho_\epsilon},z\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}}\le 1,$$

for each r and n. Suppose that $x_k\neq 0$ for each $k\in N$. This implies that $t_{kn}(x)\neq 0$, for each $k,n\in N$. Let $\epsilon\to 0$, then $\left\|\frac{t_{kn}(x)}{\epsilon},z\right\|\to\infty$. It follows that

$$\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(x)}{\epsilon},z\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}}\to\infty$$

which is a contradiction.

Therefore, $t_{kn}(x) = 0$ for each k and thus $x_k = 0$ for each $k \in N$. Let $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(x)}{\rho_1}, z\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1$$

and

$$\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(y)}{\rho_2}, z\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1$$

for each r . Let $\rho = \rho_1 + \rho_2$. Then, we have

$$\begin{aligned}
&\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(x+y)}{\rho}, z\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} \\
&\quad \leq \left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(x)+t_{kn}(y)}{\rho_1+\rho_2}, z\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} \\
&\quad \leq \left(\frac{1}{h_r}\sum_{k\in I_r}\left[\frac{\rho_1}{\rho_1+\rho_2}M_k\left(\left\|\frac{t_{kn}(x)}{\rho_1}, z\right\|\right) + \frac{\rho_2}{\rho_1+\rho_2}M_k\left(\left\|\frac{t_{kn}(y)}{\rho_2}, z\right\|\right)\right]^{p^k}\right)^{\frac{1}{H}} \\
&\quad \leq \left(\frac{\rho_1}{\rho_1+\rho_2}\right)\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(x)}{\rho_1}, z\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} \\
&\quad + \left(\frac{\rho_2}{\rho_1+\rho_2}\right)\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(y)}{\rho_2}, z\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1.
\end{aligned}$$

(by Minkowski's inequality)

Since $\rho's$ are non-negative, so we have

$$g(x+y)=$$

$$=\inf\left\{\rho^{\frac{p_r}{H}} \mid \left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(x)+t_{kn}(y)}{\rho},z\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1, r\geq 1, n\geq 1\right\}$$

$$\leq \inf\left\{\rho_1^{\frac{p_r}{H}} \mid \left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(x)}{\rho_1},z\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1, r\geq 1, n\geq 1\right\}+$$

$$+\inf\left\{\rho_2^{\frac{p_r}{H}} \mid \left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(x)}{\rho_2},z\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1, r\geq 1, n\geq 1\right\}.$$

Therefore,

$$g(x+y)\leq g(x)+g(y).$$

Finally, we prove that the scalar multiplication is continuous. Let λ be any complex number. By definition,

$$g(\lambda x)=\inf\left\{\rho^{\frac{p_r}{H}} \mid \left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(\lambda x)}{\rho},z\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1, r\geq 1, n\geq 1\right\}.$$

Then

$$g(\lambda x)= \inf\left\{(|\lambda|t)^{\frac{p_r}{H}} \mid \left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(x)}{t},z\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1, r\geq 1, n\geq 1\right\}.$$

where $t=\dfrac{\rho}{|\lambda|}$. Since $|\lambda|^{p_r}\leq \max(1,|\lambda|^{\sup p_r})$, we have

$$\begin{aligned} g(\lambda x) \leq\ & \max(1,|\lambda|^{\sup p_r}) \\ & \inf\left\{t^{\frac{p_r}{H}} \mid \left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(x)}{t},z\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1, r\geq 1, n\geq 1\right\}. \end{aligned}$$

So, the fact that scalar multiplication is continuous follows from the above inequality.

This completes the proof of the theorem. □

Theorem 3 *Let* $\mathcal{M} = (M_k)$ *be Musielak-Orlicz function. If*

$$\sup_k \left[M_k(t)\right] p_k < \infty \text{ for all } t > 0,$$

then

$$w_\sigma \left[\mathcal{M}, p, \|.,.\|\right]_\theta \subset w_\sigma^\infty \left[\mathcal{M}, p, \|.,.\|\right]_\theta .$$

Proof. Let $x \in w_\sigma\left[\mathcal{M}, p, \|.,.\|\right]_\theta$. By using inequality (1), we have

$$\frac{1}{h_r} \sum_{k \in I_r} \left[M_k\left(\left\|\frac{t_{kn}(x)}{\rho}, z\right\|\right)\right]^{p_k} \leq \frac{K}{h_r} \sum_{k \in I_r} \left[M_k\left(\left\|\frac{t_{kn}(x-l)}{\rho}, z\right\|\right)\right]^{p_k} + \frac{K}{h_r} \sum_{k \in I_r} \left[M_k\left(\left\|\frac{l}{\rho}, z\right\|\right)\right]^{p_k}.$$

Since $\sup_k \left[M_k(t)\right]^{p_k} < \infty$, we can take that $\sup_k \left[M_k(t)\right]^{p_k} = T$. Hence we get $x \in w_\sigma^\infty\left[\mathcal{M}, p, \|.,.\|\right]_\theta$. □

Theorem 4 *Let* $\mathcal{M} = (M_k)$ *be Musielak-Orlicz function which satisfies* Δ_2-*condition for all* k, *then*

$$w_\sigma \left[p, \|.,.\|\right]_\theta \subset w_\sigma \left[\mathcal{M}, p, \|.,.\|\right]_\theta .$$

Proof. Let $x \in w_\sigma \left[p, \|.,.\|\right]_\theta$. Then we have

$$\mathcal{T}_r = \frac{1}{h_r} \sum_{k \in I_r} \|t_{kn}(x-l), z\|^{p_k} \to \infty \text{ as } r \to \infty \text{ uniformly in } n, \text{ for some } l.$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_k(t) < \epsilon$ for $0 \leq t \leq \delta$ for all k. So that

$$\frac{1}{h_r} \sum_{k \in I_r} \left[M_k\left(\left\|\frac{t_{kn}(x-l)}{\rho}, z\right\|\right)\right]^{p_k} = \frac{1}{h_r} \sum^{1}_{\substack{k \in I_r \\ \|t_{kn}(x-l)z\| \leq \delta}} \left[M_k\left(\left\|\frac{t_{kn}(x-l)}{\rho}, z\right\|\right)\right]^{p_k} + \frac{1}{h_r} \sum^{2}_{\substack{k \in I_r \\ \|t_{kn}(x-l)z\| \leq \delta}} \left[M_k\left(\left\|\frac{t_{kn}(x-l)}{\rho}, z\right\|\right)\right]^{p_k}.$$

For the first summation in the right hand side of the above equation, we have $\sum^{1} \leq \epsilon^H$ by using continuity of M_k for all k. For the second summation, we write

$$\|t_{kn}(x-l), z\| \leq 1 + \|\frac{t_{kn}(x-l)}{\delta}, z\|.$$

Since M_k is non-decreasing and convex for all k, it follows that

$$M_k(\|t_{kn}(x-l),z\|) < M_k\left(1+\left\|\frac{t_{kn}(x-l)}{\delta},z\right\|\right)$$
$$\leq \frac{1}{2}M_k(2)+\frac{1}{2}M_k\left((2)\left\|\frac{t_{kn}(x-l)}{\delta},z\right\|\right).$$

Since M_k satisfies Δ_2-condition for all k, we can write

$$M_k\Big(\|t_{kn}(x-l),z\|\Big) \leq \frac{1}{2}L\left\|\frac{t_{kn}(x-l)}{\delta},z\right\|M_k(2)+\frac{1}{2}L\left\|\frac{t_{kn}(x-l)}{\delta},z\right\|M_k(2)$$
$$= L\left\|\frac{t_{kn}(x-l)}{\delta},z\right\|M_k(2).$$

So we write

$$\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(x-l)}{\rho},z\right\|\right)\right]^{p_k} \leq \epsilon^H + [\max(1, LM_k(2))\delta]^H \mathcal{T}_r.$$

Letting $r\to\infty$, it follows that $x \in w_\sigma\big[\mathcal{M},p,\|.,.\|\big]_\theta$.
This completes the proof. □

Theorem 5 *Let $\mathcal{M}=(M_k)$ be Musielak-Orlicz function. Then the following statements are equivalent:*

(i) $w_\sigma^\infty\big[p,\|.,.\|\big]_\theta \subset w_\sigma^o\big[\mathcal{M},p,\|.,.\|\big]_\theta$,

(ii) $w_\sigma^o\big[p,\|.,.\|\big]_\theta \subset w_\sigma^o\big[\mathcal{M},p,\|.,.\|\big]_\theta$,

(iii) $\sup_r \frac{1}{h_r}\sum_{k\in I_r}\big[M_k(t)\big]^{p_k} < \infty$ *for all* $t>0$.

Proof. (i) $\Longrightarrow$ (ii) We have only to show that $w_\sigma^o\big[p,\|.,.\|\big]_\theta \subset w_\sigma^\infty\big[p,\|.,.\|\big]_\theta$. Let $x\in w_\sigma^o\big[p,\|.,.\|\big]_\theta$. Then there exists $r\geq r_o$, for $\epsilon>0$, such that

$$\frac{1}{h_r}\sum_{k\in I_r}\left\|\frac{t_{kn}(x)}{\rho},z\right\|^{p_k} < \epsilon.$$

Hence there exists $H>0$ such that

$$\sup_{r,n}\frac{1}{h_r}\sum_{k\in I_r}\left\|\frac{t_{kn}(x)}{\rho},z\right\|^{p_k} < H$$

for all n and r. So we get $x \in w_\sigma^\infty\left[p, \|.,.\|\right]_\theta$.

(ii) $\Longrightarrow$ (iii) Suppose that (iii) does not hold. Then for some $t > 0$

$$\sup_r \frac{1}{h_r} \sum_{k \in I_r} [M_k(t)]^{p_k} = \infty$$

and therefore we can find a subinterval $I_{r(m)}$ of the set of interval I_r such that

$$\frac{1}{h_{r(m)}} \sum_{k \in I_{r(m)}} \left[M_k(\frac{1}{m})\right]^{p_k} > m, \quad m = 1, 2, \tag{2}$$

Let us define $x = (x_k)$ as follows, $x_k = \frac{1}{m}$ if $k \in I_{r(m)}$ and $x_k = 0$ if $k \notin I_{r(m)}$. Then $x \in w_\sigma^o[p, \|.,.\|]_\theta$ but by eqn. (2), $x \notin w_\sigma^\infty\left[\mathcal{M}, p, \|.,.\|\right]_\theta$. which contradicts (ii). Hence (iii) must hold. (iii) $\Longrightarrow$ (i) Suppose (i) not holds, then for $x \in w_\sigma^\infty\left[p, \|.,.\|\right]_\theta$, we have

$$\sup_{r,n} \frac{1}{h_r} \sum_{k \in I_r} \left[M_k\left(\left\|\frac{t_{kn}(x)}{\rho}, z\right\|\right)\right]^{p_k} = \tag{3}$$

Let $t = \left\|\frac{t_{kn}(x)}{\rho}, z\right\|$ for each k and fixed n, so that eqn. (3) becomes

$$\sup_r \frac{1}{h_r} \sum_{k \in I_r} [M_k(t)]^{p_k} = \infty$$

which contradicts (iii). Hence (i) must hold. □

Theorem 6 *Let $\mathcal{M} = (M_k)$ be Musielak-Orlicz function. Then the following statements are equivalent:*

(i) $w_\sigma^o[\mathcal{M}, p, \|.,.\|]_\theta \subset w_\sigma^o[p, \|.,.\|]_\theta$,

(ii) $w_\sigma^o[\mathcal{M}, p, \|.,.\|]_\theta \subset w_\sigma^\infty\left[p, \|.,.\|\right]_\theta$,

(iii) $\inf_r \sum_{k \in I_r} \left[M_k(t)\right]^{p_k} > 0$ *for all* $t > 0$.

Proof. (i) $\Longrightarrow$ (ii) : It is easy to prove.

(ii) $\Longrightarrow$ (iii) Suppose that (iii) does not hold. Then

$$\inf_r \frac{1}{h_r} \sum_{k \in I_r} [M_k(t)]^{p_k} = 0 \ \text{ for some } t > 0,$$

and we can find a subinterval $I_{r(m)}$ of the set of interval I_r such that

$$\frac{1}{h_r}\sum_{k\in I_{r(m)}}[M_k(m)]^{p_k} < \frac{1}{m}, \quad m = 1, 2, \ldots \tag{4}$$

Let us define $x_k = m$ if $k \in I_{r(m)}$ and $x_k = 0$ if $k \notin I_{r(m)}$. Thus by eqn.(4), $x \in w_\sigma^o\big[\mathcal{M}, p, \|.,.\|\big]_\theta$ but $x \notin w_\sigma^\infty\big[p, \|.,.\|\big]_\theta$ which contradicts (ii). Hence (iii) must hold.

(iii) $\Longrightarrow$ (i) It is obvious. □

Theorem 7 *Let* $\mathcal{M} = (M_k)$ *be Musielak-Orlicz function. Then* $w_\sigma^\infty\big[\mathcal{M}, p, \|.,.\|\big]_\theta \subset w_\sigma^o\big[p, \|.,.\|\big]_\theta$ *if and only if*

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\big[M_k(t)\big]^{p_k} = \infty \tag{5}$$

Proof. Let $w_\sigma^\infty\big[\mathcal{M}, p, \|.,.\|\big]_\theta \subset w_\sigma^o\big[p, \|.,.\|\big]_\theta$. Suppose that eqn. (5) does not hold. Therefore there is a subinterval $I_{r(m)}$ of the set of interval I_r and a number $t_o > 0$, where $t_o = \left|\frac{t_{kn}(x)}{\rho}, z\right|$ for all k and n, such that

$$\frac{1}{h_{r(m)}}\sum_{k\in I_{r(m)}}[M_k(t_o)]^{p_k} \leq M < \infty, \quad m = 1, 2, \ldots \tag{6}$$

Let us define $x_k = t_o$ if $k \in I_{r(m)}$ and $x_k = 0$ if $k \notin I_{r(m)}$. Then, by eqn. (6), $x \in w_\sigma^\infty\,[M_k, p, \|.,.\|]_\theta$. But $x \notin w_\sigma^o\,[p, \|.,.\|]_\theta$. Hence eqn. (5) must hold.

Conversely, suppose that eqn. (5) hold and that $x \in w_\sigma^\infty\,[M_k, p, \|.,.\|]_\theta$. Then for each r and n

$$\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\left\|\frac{t_{kn}(x)}{\rho}, z\right\|\right)\right]^{p_k} \leq M < \infty. \tag{7}$$

Now suppose that $x \notin w_\sigma^o\,[p, \|.,.\|]_\theta$. Then for some number $\epsilon > 0$ and for a subinterval I_{ri} of the set of interval I_r, there is k_o such that $\|t_{kn}(x), z\|^{p_k} > \epsilon$ for $k \geq k_o$. From the properties of sequence of Orlicz functions, we obtain

$$\left[M_k\left(\frac{\epsilon}{\rho}\right)\right]^{p_k} \leq \left[M_k\left(\left\|\frac{t_{kn}(x)}{\rho}, z\right\|\right)\right]^{p_k}$$

which contradicts eqn.(6), by using eqn. (7). This completes the proof. □

References

[1] A. R. Freedman, J. J. Sember and R. Ramphel, Cesaro-type summability spaces, *Proc. London Math. Soc.*, **37** (1978), 508–520.

[2] S. Gahler, 2- metrische Raume und ihre topologishe Struktur, *Math. Nachr.*, **26** (1963), 115–148.

[3] H. Gunawan, H. Mashadi, On finite dimensional 2-normed spaces, *Soochow J. Math.*, **27** (2001), 321–329.

[4] M. Gurdal, S. Pehlivan, Statistical convergence in 2-normed spaces, *Southeast Asian Bull. Math.*, **33** (2009), 257–264.

[5] J. Lindenstrauss, L. Tzafriri, On Orlicz seequence spaces, *Israel J. Math.*, **10** (1971), 379–390.

[6] G. G. Lorentz, A contribytion to the theory of divergent series, *Act. Math.*, **80** (1948), 167–190.

[7] L. Maligranda, *Orlicz spaces and interpolation*, Seminars in Mathematics 5, Polish Academy of Science, 1989.

[8] J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Mathematics, 1034, Springer Verlag, 1983.

[9] W. Raymond, Y. Freese, J. Cho, *Geometry of linear 2-normed spaces*, N. Y. Nova Science Publishers, Huntington, 2001.

[10] A. Sahiner, M. Gurdal, S. Saltan, H. Gunawan, Ideal Convergence in 2-normed spaces, *Taiwanese J. Math.*, **11** (2007), 1477–1484.

[11] P. Schaefer, Infinite martices and invariant means, *Proc. Amer. Math. Soc.*, **36** (1972), 104–110.

[12] A. Wilansky, *Summability through Functional Analysis*, North- Holland Math. Stnd., 1984.

Received: October 20, 2010

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