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NEW EXISTENCE AND UNIQUENESS RESULT FOR FRACTIONAL BAGLEY–TORVIK DIFFERENTIAL EQUATION

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Abstract. Inspired by the paper of Fazli and Nieto in [*Open Math.* 17 (2019) 499–512], we establish new existence and uniqueness result for a type of fractional Bagley–Torvik differential equation. Reported result not only generalizes previous results but also adopts different technique. We finish this study by concluding remarks which discuss the preference of our theorem compared to previous results. An example is constructed with specific parameters that requires weaker conditions for the existence of a unique solution. Meanwhile, we construct an iterative sequence that converges to the unique solution and can not be commented via the results of Fazli and Nieto.

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1. INTRODUCTION

Guided by their popular applications, fractional differential equations have attracted the awareness of many researchers working in fields of science and engineering. The existing literature on this topic not only covers its theoretical manifestation but also describes its extensive applications in describing realistic phenomena. The apparatus of fractional calculus, in particular, are efficiently used and applied in modeling many engineering and scientific processes; we propose the popular monographs [13, 19, 23] for readers' consciousness.

Fractional differential equations have proved significant applications in many real life aspects. For this, these type of equations has attracted the attention of many researchers who produced several remarkable results in this regard. A particular emphasize of initial and boundary value problems which have been treated within fractional order settings such as the Langevin equation, the Basset equation and Bagley–Torvik equation which are popular in fluctuating environments, a incompressible viscous fluid and viscoelasticity, respectively. Fractional Langevin equations have been

systematically studied [1–5, 11, 16–18]. The latter two equations, however, have comparably gained less attention amongst researchers [6–10, 20, 25].

The Bagley–Torvik equation, which is our concern herein, of the form

$$\xi u''(\varsigma) + \zeta \mathcal{D}^{\frac{3}{2}} u(\varsigma) + \eta u(\varsigma) = \omega(\varsigma), \quad 0 < \varsigma \leq 1, \quad 1 < \alpha < 2, \quad (1.1)$$

appears in the modelling of the motion of a rigid plate immersed in a Newtonian fluid. Here the prime denotes the classical derivative, $\mathcal{D}^{\frac{3}{2}}$ is the Caputo fractional derivative of order $\frac{3}{2}$, $\omega: [0, 1] \rightarrow \mathbb{R}$ is a given function and $\xi, \zeta, \eta \in \mathbb{R}$, $\xi \neq 0$. Equation (1.1) was initially introduced in [25] and is thoroughly discussed in [23]. Particularly, the investigators have studied the analytical and numerical solutions of equation (1.1); see for instance [12, 14, 21, 22]. In [7], the equivalence between the Caputo and Riemann derivatives are discussed and pointed out that they are identical in describing the linear viscoelastic material just under two minimal restrictions.

Recently in [15], Fazli and Nieto studied fractional Bagley–Torvik equation which is in the form

$$\begin{cases} (\mathcal{D}^2 + \Lambda_1 \mathcal{D}^\alpha + \Lambda_2) u(\varsigma) = \omega(\varsigma), & 0 < \varsigma \leq 1, \quad 1 < \alpha < 2, \\ u(0) = \mu, u'(0) = \nu, \end{cases} \quad (1.2)$$

under the same assumptions, μ, ν, Λ_1 and Λ_2 are real numbers and $\mathcal{D}^2 := \frac{d^2}{d\varsigma^2}$. It is clear that equation (1.2) is a generalization of (1.1) to an arbitrary order in fractional derivative settings. The existence and approximations of solutions are proved for equation (1.2) admitting only the existence of a lower (coupled lower and upper) solution. The main results are obtained under certain assumptions and by use of an appropriate fixed point theorem in partially ordered normed linear spaces. Equation (1.2) is referred to in the text as FBTE.

In this work, we extend the main results of [15] and establish an improved existence-uniqueness theorem. Our result complements and generalizes the result of Fazli and Nieto and also proves the main theorem using different technique. We end the paper by a concluding remark that shows the preference of our theorem over the results of [15]. An example is constructed with specific parameters that requires less restrictive conditions for the existence of a unique solution. Meanwhile, we construct an iterative sequence that converges to the unique solution and can not be commented via the results of [15].

2. ESSENTIAL BACKGROUND AND RESULTS OF [15]

This section assembles some basic definitions and concepts concerning with theory of fractional calculus. For the terms and terminologies, the reader can consult one of the monographs [13, 19, 23].

The Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $u : [0, 1] \rightarrow \mathbb{R}$ is defined as

$$I^\alpha u(\varsigma) = \int_0^\varsigma \frac{(\varsigma-s)^{\alpha-1}}{\Gamma(\alpha)} u(s) ds, \quad 0 \leq \varsigma \leq 1, \quad (2.1)$$

where Γ is the Gamma function. For a function $u : [0, 1] \rightarrow \mathbb{R}$, the Caputo derivative of fractional order $\alpha \in \mathbb{R}^+$ is defined as

$$\begin{aligned} \mathcal{D}^\alpha u(\varsigma) &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{d\varsigma^n} \int_0^\varsigma (\varsigma-s)^{n-\alpha-1} \left(u(s) - \sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{k!} s^k \right) ds \\ &= \mathcal{D}^n I^{n-\alpha} \left(u(\varsigma) - \sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{k!} \varsigma^k \right), \end{aligned} \quad (2.2)$$

where $n-1 < \alpha < n$, provided that the right side is pointwise defined on $[0, 1]$. Note that if $n-1 < \alpha < n$ and $u \in AC^{n-1}[0, 1]$, then

$$\begin{aligned} \mathcal{D}^\alpha u(\varsigma) &= \frac{1}{\Gamma(n-\alpha)} \int_0^\varsigma (\varsigma-s)^{n-\alpha-1} u^{(n)}(s) ds \\ &= I^{n-\alpha} u^{(n)}(\varsigma). \end{aligned}$$

For the relationship between (2.1) and (2.2) and other properties of these notions, we refer the reader to [13, 19, 23].

Lemma 1. [15] u is a solution of FBTE if and only if it is a solution of

$$u(\varsigma) = \mu + \nu \left(\varsigma + \frac{\Lambda_1}{\Gamma(4-\alpha)} \varsigma^{3-\alpha} \right) - \Lambda_1 I^{3-\alpha} u'(\varsigma) + I^2 [\omega(\varsigma) - \Lambda_2 u(\varsigma)], \quad (2.3)$$

in the set $\mathcal{C} := C^1[0, 1]$.

The main results of [15] are proved by the help of some fixed point theorems in partially ordered metric spaces. Because the definition of coupled solutions becomes different based on the sign of the product $\Lambda_1 \cdot \Lambda_2$, the proofs are presented in two folds. We review the herein the two cases for the sake of completeness.

2.1. The case $\Lambda_1 \cdot \Lambda_2 < 0$

Definition 1. [15] An element $(u_0, v_0) \in \mathcal{C} \times \mathcal{C}$ is said to be a coupled lower and upper solution of FBTE if

$$\begin{cases} u'_0(\varsigma) \leq \nu \left(1 + \frac{\Lambda_1}{\Gamma(3-\alpha)} \varsigma^{2-\alpha} \right) - \frac{\Lambda_1}{\Gamma(2-\alpha)} \int_0^\varsigma (\varsigma-\tau)^{1-\alpha} v'_0(\tau) d\tau + \int_0^\varsigma [\omega(\tau) + |\Lambda_2| u_0(\tau)] d\tau, \\ u_0(0) \leq \mu \end{cases}$$

and

$$\begin{cases} v'_0(\varsigma) \geq \nu \left(1 + \frac{\Lambda_1}{\Gamma(3-\alpha)} \varsigma^{2-\alpha} \right) - \frac{\Lambda_1}{\Gamma(2-\alpha)} \int_0^\varsigma (\varsigma-\tau)^{1-\alpha} u'_0(\tau) d\tau + \int_0^\varsigma [\omega(\tau) + |\Lambda_2| v_0(\tau)] d\tau, \\ v_0(0) \geq \mu \end{cases}$$

for all $\varsigma \in [0, 1]$.

Theorem 1. [15] Let $(u_0, v_0) \in C \times C$ be a coupled lower and upper solution of FBTE and $k_1 := \max\{2|\Lambda_2|, \frac{2|\Lambda_1|}{\Gamma(3-\alpha)}\} < 1$.

- I. Then FBTE has a unique solution $u^* \in C$.
- II. Moreover, there exist two monotone sequence $\{u_n\}$ and $\{v_n\}$ such that both sequences converge to u^* in C .
- III. Further, the following error estimates hold,

$$\begin{aligned} \|u_n - u^*\|_C &\leq \frac{1}{2} \frac{k_1^n}{1 - k_1} (\|u_1 - u_0\|_C + \|v_1 - v_0\|_C), \\ \|v_n - u^*\|_C &\leq \frac{1}{2} \frac{k_1^n}{1 - k_1} (\|u_1 - u_0\|_C + \|v_1 - v_0\|_C), \\ \|u_n - v_n\|_C &\leq \frac{k_1^n}{1 - k_1} (\|u_1 - u_0\|_C + \|v_1 - v_0\|_C). \end{aligned} \quad (2.4)$$

2.2. The case $\Lambda_1 \cdot \Lambda_2 > 0$

Definition 2. [15] An element $u_0 \in C$ is said to be a lower solution of FBTE if

$$\begin{aligned} u'_0(\varsigma) &\leq H(\varsigma) := v \left(1 + \frac{\Lambda_1}{\Gamma(3-\alpha)} \varsigma^{2-\alpha} \right) - \frac{\Lambda_1}{\Gamma(2-\alpha)} \int_0^\varsigma (\varsigma - \tau)^{1-\alpha} u'_0(\tau) d\tau \\ &\quad + \int_0^\varsigma [\omega(\tau) - \Lambda_2 u_0(\tau)] d\tau, \\ u_0(0) &\leq \mu \end{aligned} \quad (2.5)$$

for all $\varsigma \in [0, 1]$.

Theorem 2. [15] Let $u_0 \in C$ be a lower solution of FBTE and $k_2 := |\Lambda_2| + \frac{|\Lambda_1|}{\Gamma(3-\alpha)} < 1$.

- I. Then FBTE has a unique solution $u^* \in C$.
- II. Moreover, the iterative sequence $\{u_n\}$ defined by

$$\begin{aligned} u_n(\varsigma) &= \mu + v \left(\varsigma + \frac{\Lambda_1}{\Gamma(4-\alpha)} \varsigma^{3-\alpha} \right) - \frac{\Lambda_1}{\Gamma(3-\alpha)} \int_0^\varsigma (\varsigma - \tau)^{2-\alpha} u'_{n-1}(\tau) d\tau \\ &\quad + \int_0^\varsigma (\varsigma - \tau) [\omega(\tau) - \Lambda_2 u_{n-1}(\tau)] d\tau, \end{aligned}$$

converges to u^* in C .

- III. Further, the following error estimates hold,

$$\begin{aligned} \|u_n - u^*\|_C &\leq \frac{k_2^n}{1 - k_2} \|u_1 - u_0\|_C, \\ \|u_{n+1} - u_n\|_C &\leq k_2^n \|u_1 - u_0\|_C. \end{aligned}$$

3. MAIN RESULT

This segment is dedicated to our main theorem.

Theorem 3. Let $k := |\Lambda_2| + \frac{|\Lambda_1|}{\Gamma(3-\alpha)} < 1$.

- I. Then, FBTE has a unique solution $u^* \in C$.
- II. Each sequence $\{u_n\}$ along with an initial point u_0 (not necessarily a lower solution) given by

$$u_n(\varsigma) = \mu + \nu \left(\varsigma + \frac{\Lambda_1}{\Gamma(4-\alpha)} \varsigma^{3-\alpha} \right) - \frac{\Lambda_1}{\Gamma(3-\alpha)} \int_0^\varsigma (\varsigma - \tau)^{2-\alpha} u'_{n-1}(\tau) d\tau \\ + \int_0^\varsigma (\varsigma - \tau) [\omega(\tau) - \Lambda_2 u_{n-1}(\tau)] d\tau,$$

converges to u^* in C . In addition, the following error estimates hold,

$$\|u_n - u^*\|_C \leq \frac{k^n}{1-k} \|u_1 - u_0\|_C, \\ \|u_{n+1} - u_n\|_C \leq k^n \|u_1 - u_0\|_C.$$

- III. Further, if $k_1 = \max\{2|\Lambda_2|, \frac{2|\Lambda_1|}{\Gamma(3-\alpha)}\} < 1$, then, $\exists$ two sequences $\{U_n\}$ and $\{V_n\}$ such that both sequences converge to u^* in C . Also, the following error estimates are valid,

$$\|U_n - u^*\|_C \leq \frac{1}{2} \frac{k_1^n}{1-k_1} (\|U_1 - U_0\|_C + \|V_1 - V_0\|_C), \\ \|V_n - u^*\|_C \leq \frac{1}{2} \frac{k_1^n}{1-k_1} (\|U_1 - U_0\|_C + \|V_1 - V_0\|_C), \quad (3.1)$$

$$\|U_n - V_n\|_C \leq \frac{k_1^n}{1-k_1} (\|U_1 - U_0\|_C + \|V_1 - V_0\|_C). \quad (3.2)$$

Proof. Let $U := C$ be denoted the class of continuously differentiable functions on finite interval $[0, 1]$. Then, $(U, \|\cdot\|_C)$ is a Banach space with metric $\|u\|_C = \max\{\|u\|_\infty, \|u'\|_\infty\}$, where $\|u\|_\infty = \max_{0 \leq \varsigma \leq 1} |u(\varsigma)|$.

Suppose $\Theta : U \rightarrow U$ is defined by

$$\Theta u(\varsigma) = \mu + \nu \left(\varsigma + \frac{\Lambda_1}{\Gamma(4-\alpha)} \varsigma^{3-\alpha} \right) - \frac{\Lambda_1}{\Gamma(3-\alpha)} \int_0^\varsigma (\varsigma - \tau)^{2-\alpha} u'(\tau) d\tau \\ + \int_0^\varsigma (\varsigma - \tau) [\omega(\tau) - \Lambda_2 u(\tau)] d\tau$$

for each $\varsigma \in [0, 1]$.

It is clear that if u is a solution of FBTE, then it is equivalent to a fixed point u of the operator Θ .

Now, $\forall u, v \in U$ and $\varsigma \in [0, 1]$, we have

$$|\Theta(u)(\varsigma) - \Theta(v)(\varsigma)| \leq |\Lambda_1| |I^{3-\alpha} u'(\varsigma) - I^{3-\alpha} v'(\varsigma)| + |\Lambda_2| |I^2 u(\varsigma) - I^2 v(\varsigma)|$$

$$\begin{aligned}
&\leq \frac{|\Lambda_1|}{\Gamma(4-\alpha)} \|u' - v'\|_\infty + \frac{|\Lambda_2|}{2} \|u - v\|_\infty \\
&\leq \left(\frac{|\Lambda_2|}{2} + \frac{|\Lambda_1|}{\Gamma(4-\alpha)} \right) \|u - v\|_C.
\end{aligned}$$

Then, for each $u, v \in U$, $\|\Theta(u) - \Theta(v)\|_\infty \leq \left(\frac{|\Lambda_2|}{2} + \frac{|\Lambda_1|}{\Gamma(4-\alpha)} \right) \|u - v\|_C$.

On the other hand, $\forall u, v \in U$ and $\varsigma \in [0, 1]$, we have

$$\begin{aligned}
|\Theta'(u)(\varsigma) - \Theta'(v)(\varsigma)| &\leq |\Lambda_1| |I^{2-\alpha} u'(\varsigma) - I^{2-\alpha} v'(\varsigma)| + |\Lambda_2| |I^1 u(\varsigma) - I^1 v(\varsigma)| \\
&\leq \frac{|\Lambda_1|}{\Gamma(3-\alpha)} \|u' - v'\|_\infty + |\Lambda_2| \|u - v\|_\infty \\
&\leq \left(|\Lambda_2| + \frac{|\Lambda_1|}{\Gamma(3-\alpha)} \right) \|u - v\|_C.
\end{aligned}$$

Then, for each $u, v \in U$, $\|\Theta'(u) - \Theta'(v)\|_\infty \leq \left(|\Lambda_2| + \frac{|\Lambda_1|}{\Gamma(3-\alpha)} \right) \|u - v\|_C$.

Therefore, we get for each $u, v \in U$,

$$\|\Theta(u) - \Theta(v)\|_C \leq \left(|\Lambda_2| + \frac{|\Lambda_1|}{\Gamma(3-\alpha)} \right) \|u - v\|_C = k \|u - v\|_C.$$

By Banach contraction principle, $\exists u^* \in U$ such that $u^* = \Theta u^*$. Also, Θ is a Picard operator with $\lim_{n \rightarrow \infty} \Theta^n u = u^*$, $u \in U$.

The assertion II. of the theorem is an obvious consequence of the Banach contraction mapping principle. Now, we show that the assertion III. of the theorem is correct.

Let (U_0, V_0) be an arbitrary element of $U \times U$ (not necessarily a coupled lower and upper solution). For each $n \in \mathbb{N}$, we let

$$\begin{aligned}
U_n(\varsigma) &= \mu + \nu \left(\varsigma + \frac{\Lambda_1}{\Gamma(4-\alpha)} \varsigma^{3-\alpha} \right) - \frac{\Lambda_1}{\Gamma(3-\alpha)} \int_0^\varsigma (\varsigma - \tau)^{2-\alpha} V'_{n-1}(\tau) d\tau \\
&\quad + \int_0^\varsigma (\varsigma - \tau) [\omega(\tau) - \Lambda_2 U_{n-1}(\tau)] d\tau,
\end{aligned}$$

and

$$\begin{aligned}
V_n(\varsigma) &= \mu + \nu \left(\varsigma + \frac{\Lambda_1}{\Gamma(4-\alpha)} \varsigma^{3-\alpha} \right) - \frac{\Lambda_1}{\Gamma(3-\alpha)} \int_0^\varsigma (\varsigma - \tau)^{2-\alpha} U'_{n-1}(\tau) d\tau \\
&\quad + \int_0^\varsigma (\varsigma - \tau) [\omega(\tau) - \Lambda_2 V_{n-1}(\tau)] d\tau
\end{aligned}$$

for all $\varsigma \in [0, 1]$.

It is easy to see that

$$\|U_n - u^*\|_\infty \leq \max \left\{ \frac{|\Lambda_2|}{2}, \frac{|\Lambda_1|}{\Gamma(4-\alpha)} \right\} \left(\|U_{n-1} - u^*\|_C + \|V_{n-1} - u^*\|_C \right), \quad (3.3)$$

$$\|V_n - u^*\|_\infty \leq \max \left\{ \frac{|\Lambda_2|}{2}, \frac{|\Lambda_1|}{\Gamma(4-\alpha)} \right\} \left(\|U_{n-1} - u^*\|_C + \|V_{n-1} - u^*\|_C \right), \quad (3.4)$$

$$\|U'_n - (u^*)'\|_\infty \leq \max \left\{ |\Lambda_2|, \frac{|\Lambda_1|}{\Gamma(3-\alpha)} \right\} \left(\|U_{n-1} - u^*\|_C + \|V_{n-1} - u^*\|_C \right), \quad (3.5)$$

and

$$\|V'_n - (u^*)'\|_\infty \leq \max \left\{ |\Lambda_2|, \frac{|\Lambda_1|}{\Gamma(3-\alpha)} \right\} \left(\|U_{n-1} - u^*\|_C + \|V_{n-1} - u^*\|_C \right). \quad (3.6)$$

Here, using (3.3) and (3.5), for each $n \in \mathbb{N}$, we have

$$\begin{aligned} \|U_n - u^*\|_C &\leq \max \left\{ |\Lambda_2|, \frac{|\Lambda_1|}{\Gamma(3-\alpha)} \right\} \left(\|U_{n-1} - u^*\|_C + \|V_{n-1} - u^*\|_C \right) \\ &\leq \frac{k_1}{2} \left(\|U_{n-1} - u^*\|_C + \|V_{n-1} - u^*\|_C \right). \end{aligned} \quad (3.7)$$

Furthermore, using (3.4) and (3.6), for each $n \in \mathbb{N}$, we have

$$\begin{aligned} \|V_n - u^*\|_C &\leq \max \left\{ |\Lambda_2|, \frac{|\Lambda_1|}{\Gamma(3-\alpha)} \right\} \left(\|U_{n-1} - u^*\|_C + \|V_{n-1} - u^*\|_C \right) \\ &\leq \frac{k_1}{2} \left(\|U_{n-1} - u^*\|_C + \|V_{n-1} - u^*\|_C \right). \end{aligned} \quad (3.8)$$

By induction, (3.7) and (3.8), we obtain

$$\|U_n - u^*\|_C \leq \frac{k_1^n}{2} \left(\|U_0 - u^*\|_C + \|V_0 - u^*\|_C \right), \quad n \in \mathbb{N},$$

and

$$\|V_n - u^*\|_C \leq \frac{k_1^n}{2} \left(\|U_0 - u^*\|_C + \|V_0 - u^*\|_C \right).$$

Letting $n \rightarrow \infty$ in above equations, we see that the sequences $\{U_n\}$ and $\{V_n\}$ converge to u^* .

A similar argument can also obtain the following relations:

$$\|U_{n+1} - U_n\|_C \leq \frac{k_1^n}{2} \left(\|U_1 - U_0\|_C + \|V_1 - V_0\|_C \right),$$

and

$$\|V_{n+1} - V_n\|_C \leq \frac{k_1^n}{2} \left(\|U_1 - U_0\|_C + \|V_1 - V_0\|_C \right).$$

Then for any $m > n \geq 1$,

$$\begin{aligned} \|U_m - U_n\|_C &\leq \sum_{j=0}^{m-n-1} \|U_{n+j+1} - U_{n+j}\|_C \\ &\leq \sum_{j=0}^{m-n-1} \frac{k_1^{n+j}}{2} \left(\|U_1 - U_0\|_C + \|V_1 - V_0\|_C \right) \end{aligned}$$

$$\leq \frac{1}{2} \frac{k_1^n - k_1^m}{1 - k_1} \left(\|U_1 - U_0\|_C + \|V_1 - V_0\|_C \right),$$

and

$$\begin{aligned} \|V_m - V_n\|_C &\leq \sum_{j=0}^{m-n-1} \|U_{n+j+1} - U_{n+j}\|_C \\ &\leq \sum_{j=0}^{m-n-1} \frac{k_1^{n+j}}{2} \left(\|U_1 - U_0\|_C + \|V_1 - V_0\|_C \right) \\ &\leq \frac{1}{2} \frac{k_1^n - k_1^m}{1 - k_1} \left(\|U_1 - U_0\|_C + \|V_1 - V_0\|_C \right). \end{aligned}$$

Letting $m \rightarrow \infty$, we conclude the error estimate (3.1) as following.

$$\|U_n - u^*\|_C \leq \frac{1}{2} \frac{k_1^n}{1 - k_1} \left(\|U_1 - U_0\|_C + \|V_1 - V_0\|_C \right),$$

and

$$\|V_n - u^*\|_C \leq \frac{1}{2} \frac{k_1^n}{1 - k_1} \left(\|U_1 - U_0\|_C + \|V_1 - V_0\|_C \right).$$

Finally, (3.2) follows immediately from (3.1) □

Remark 1. Consider the following equations

$$\begin{cases} u''(\varsigma) + \Lambda_1 \mathcal{D}^\alpha u(\varsigma) = h(\varsigma), & 0 < \varsigma < 1, 1 < \alpha < 2, \\ u(0) = \mu, u'(0) = \nu, \end{cases} \quad (3.9)$$

and

$$\begin{cases} v''(\varsigma) + \Lambda_1 \mathcal{D}^\alpha v(\varsigma) = h(\varsigma), & 0 < \varsigma < 1, 1 < \alpha < 2, \\ v(1) = 0, v'(0) = 0. \end{cases} \quad (3.10)$$

One may observe that if u is a solution of (3.9), then $v(\varsigma) = u(\varsigma) - u(1) + (1 - \varsigma)u'(0)$ is a solution of (3.10). Further, if v is a solution of (3.10), then $u(\varsigma) = v(\varsigma) - v(0) + \mu + \nu t$ is a solution of (3.9). Note that, $v''(\varsigma) = u''(\varsigma)$ and $\mathcal{D}^\alpha v(\varsigma) = \mathcal{D}^\alpha u(\varsigma)$.

4. A CONCLUDING REMARK AND SOME EXAMPLES

An improved existence-uniqueness theorem is established for FBTE under more relaxed assumptions. We prove our main result of this paper by using entirely different approach. The feature of Theorem 3 in this paper compared to the consequences of Theorem 3.4 and Theorem 3.6 of [15] can be viewed under:

- In Theorem 3.4 and Theorem 3.6 of [15], the authors needed the concept of coupled solutions for FBTE, while Theorem 3 does not require this notion.
- Theorem 3.4 and Theorem 3.6 of [15] are based on the product $\Lambda_1 \cdot \Lambda_2$. The proof of Theorem 3, however, is freelance of the $\Lambda_1 \cdot \Lambda_2$.

- $\ominus$ is a Picard operator, that is,

$$\lim_{n \rightarrow \infty} \ominus^n u = u^*, \quad u \in U.$$

Therefore, for each $u_0 \in U$ (not necessarily a lower solution), $\lim_{n \rightarrow \infty} u_n = u^*$, where

$$\begin{aligned} u_n(\varsigma) = & \mu + \nu \left(\varsigma + \frac{\Lambda_1}{\Gamma(4-\alpha)} \varsigma^{3-\alpha} \right) - \frac{\Lambda_1}{\Gamma(3-\alpha)} \int_0^\varsigma (\varsigma - \tau)^{2-\alpha} u'_{n-1}(\tau) d\tau \\ & + \int_0^\varsigma (\varsigma - \tau) [\omega(\tau) - \Lambda_2 u_{n-1}(\tau)] d\tau, \end{aligned}$$

for each $n \in \mathbb{N}$.

- That $\max\{2|\Lambda_2|, \frac{2|\Lambda_1|}{\Gamma(3-\alpha)}\} < 1$ is an essential condition to prove the existence and uniqueness in Theorem 3.4 of [15]. In Theorem 3, the existence and uniqueness is established under the condition $|\Lambda_2| + \frac{|\Lambda_1|}{\Gamma(3-\alpha)} < 1$ which is less restrictive.
- Remark 1 demonstrates that part (i) of Theorem 3.4 and Theorem 3.6 in [15] are a weak version of Theorem 4.1 of [24] where we put $T = 1, \mu = 0, \varphi_1 = |\Lambda_2|$ and $\varphi_2 = \varphi_3 = 0$.

Here, we provide an example that can not be discussed using Theorem 3.4 and Theorem 3.6 of [15].

Consider the following particular FBTE

$$\begin{cases} u''(\varsigma) + \frac{3}{5} \mathcal{D}^{\frac{4}{3}} u(\varsigma) - \frac{1}{4} u(\varsigma) = \frac{1}{10} \varsigma^2 - \frac{1}{4} \varsigma - \frac{8\sqrt{\varsigma}}{\sqrt{\pi}}, & 0 < \varsigma < 1, \\ u(0) = 0, \quad u'(0) = 1. \end{cases} \quad (4.1)$$

Therefore, we get $k = |\Lambda_2| + \frac{|\Lambda_1|}{\Gamma(3-\alpha)} = 0.9146393007 < 1$. Thus, by Theorem 3, equation (4.1) has a unique solution in $\mathcal{C}$.

We observe that the product $\Lambda_1 \cdot \Lambda_2 < 0$ and

$$k_1 = \max \left\{ 2|\Lambda_2|, \frac{2|\Lambda_1|}{\Gamma(3-\alpha)} \right\} = 1.354055000 > 1.$$

Thus, Theorem 3.4 of [15] does not work herein.

Consider the following particular FBTE

$$\begin{cases} u''(\varsigma) - \frac{2}{5} \mathcal{D}^{\frac{3}{2}} u(\varsigma) - \frac{1}{2} u(\varsigma) = \omega(\varsigma), & 0 < \varsigma < 1, \\ u(0) = 0, \quad u'(0) = \frac{9}{16}, \end{cases} \quad (4.2)$$

where $\omega(\varsigma) = \frac{-1}{2} \varsigma^3 - \frac{3}{4} \varsigma^2 + \frac{183}{32} \varsigma - 3 - \frac{4\sqrt{\varsigma}(-3+4\varsigma)}{5\sqrt{\pi}}$. Here $\Lambda_1 = \frac{-2}{5}, \Lambda_2 = \frac{-1}{2}, \alpha = \frac{3}{2}, \mu = 0$ and $\nu = \frac{9}{16}$. Therefore, we can obtain that $k = |\Lambda_2| + \frac{|\Lambda_1|}{\Gamma(3-\alpha)} = 0.95135 < 1$ and the exact solution is $u^*(\varsigma) = \varsigma^3 - \frac{3}{2} \varsigma^2 + \frac{9}{16} \varsigma$. Thus, by Theorem 3, equation (4.2) has a

unique solution $u^* \in C$. Moreover, the iterative sequence $\{u_n\}$ with an initial function u_0 (not necessarily a lower solution) defined by

$$u_n(\varsigma) = \mu + \nu \left(\varsigma + \frac{\Lambda_1}{\Gamma(4-\alpha)} \varsigma^{3-\alpha} \right) - \frac{\Lambda_1}{\Gamma(3-\alpha)} \int_0^\varsigma (\varsigma - \tau)^{2-\alpha} u'_{n-1}(\tau) d\tau \\ + \int_0^\varsigma (\varsigma - \tau) [\omega(\tau) - \Lambda_2 u_{n-1}(\tau)] d\tau,$$

converges to u^* in C . The graphs of $u_n, n = 0, 1, 2$ with the initial point $u_0(\varsigma) = \sin(\varsigma)$ are shown in Figure 1 whereas graphs with the initial point $u_0(\varsigma) = \cos(\varsigma)$ are shown in Figure 2. Here, it is important to say that Figure 3. shows that equation (2.5) is not true for initial function $u_0(\varsigma) = \sin(\varsigma)$. Then the mapping $u_0(\varsigma) = \sin(\varsigma)$ is not a lower solution of the problem (4.2). Hence, we can not conclude this approximations by the consequences reported in [15].

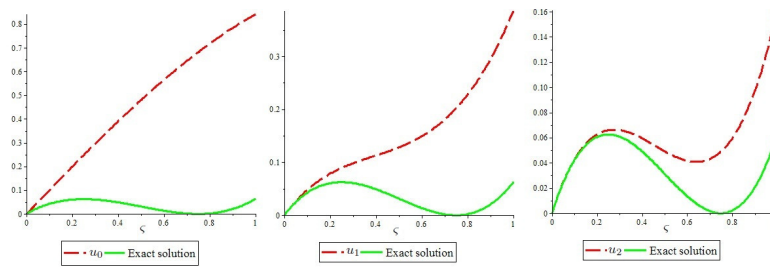


FIGURE 1. The initial function is $u_0(\varsigma) = \sin(\varsigma)$

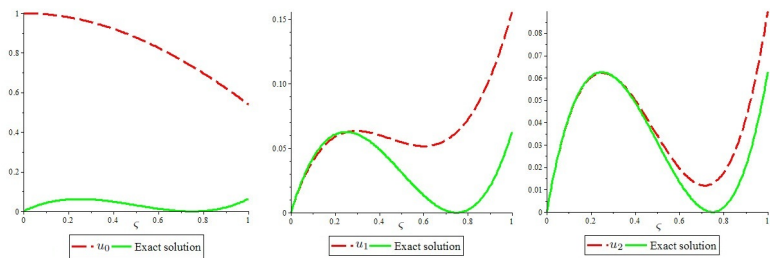


FIGURE 2. The initial function is $u_0(\varsigma) = \cos(\varsigma)$

DECLARATIONS

Availability of data and material

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

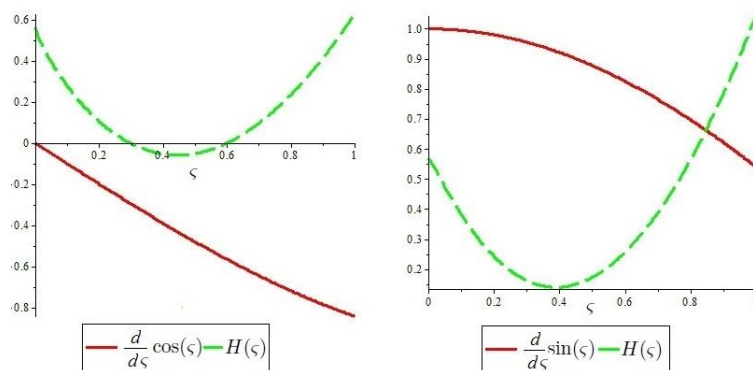


FIGURE 3. $u_0(\zeta) = \sin(\zeta)$ is not a lower solution of the problem (4.2)

Competing Interests

The authors declare no competing interests in this work.

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Authors' contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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A NOTE ON THE COEFFICIENTS OF POWER SUMS OF ARITHMETIC PROGRESSIONS

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Abstract. In this note we show a simple formula for the coefficients of the polynomial associated with the sums of powers of the terms of an arbitrary arithmetic progression. This formula consists of a double sum involving only ordinary binomial coefficients and binomial powers. Arguably, this is the simplest formula that can probably be found for the said coefficients. Furthermore, we give an explicit formula for the Bernoulli polynomials involving the Stirling numbers of the first and second kind.

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Keywords: Bernoulli numbers and polynomials, sums of powers of integers, arithmetic progression, Stirling numbers

1. INTRODUCTION

Consider the sum of k th powers of the first n terms of an arithmetic progression with initial term r and common difference m

$$S_k^{m,r}(n) = r^k + (m+r)^k + (2m+r)^k + \cdots + ((n-1)m+r)^k,$$

where k, m, r , and n are assumed to be integer variables with $m, n \geq 1$ and $k, r \geq 0$. $S_k^{m,r}(n)$ turns out to be a polynomial in n of degree $k+1$ without constant term, that is, it can be expressed in the form $S_k^{m,r}(n) = \sum_{t=1}^{k+1} c_{k,t}^{m,r} n^t$ for certain rational coefficients $c_{k,t}^{m,r}$.

Griffiths [5] obtained the following formula for $c_{k,t}^{m,r}$ (in our notation)

$$c_{k,t}^{m,r} = m^k \sum_{j=t}^{k+1} \sum_{i=j}^{k+1} \frac{1}{i} \left(\frac{r}{m}\right)^{j-t} \binom{j}{t} S_1(i, j) S_2(k, i-1), \quad (1.1)$$

for $t = 1, \dots, k+1$, where $S_1(i, j)$ are the (signed) Stirling numbers of the first kind and $S_2(k, i-1)$ are the (unsigned) Stirling numbers of the second kind. On the other

hand, the authors of [1] gave the following formula for $c_{k,t}^{m,r}$

$$c_{k,t}^{m,r} = \sum_{j=0}^k \frac{m^j W_{m,r}(k, j)}{j+1} S_1(j+1, t), \quad (1.2)$$

where $S_1(j+1, t)$ are the (signed) Stirling numbers of the first kind and $W_{m,r}(k, j)$ are the so-called r -Whitney numbers of the second kind, which are defined in [8] by

$$W_{m,r}(k, j) = \frac{1}{m^j j!} \sum_{i=0}^j (-1)^{j-i} \binom{j}{i} (mi+r)^k.$$

Moreover, from [11, Theorems 5 and 6], it can be shown that

$$c_{k,t}^{m,r} = \frac{(-1)^{k+1-t}}{(k+1)!} \sum_{j=0}^k \sum_{s=t}^{k+1} A_{m,r}(k, k-j) \binom{s}{t} (k-j)^{s-t} S_1(k+1, s), \quad (1.3)$$

where $S_1(k+1, s)$ are the (signed) Stirling numbers of the first kind, and

$$A_{m,r}(k, j) = \sum_{i=0}^j (-1)^i \binom{k+1}{i} [m(j-i) + r]^k.$$

It is to be noted that the polynomial $S_k^{m,r}(n) = \sum_{t=1}^{k+1} c_{k,t}^{m,r} n^t$ with coefficients given by (1.1), (1.2), and (1.3) can equivalently be expressed, for $k \geq 0$, as

$$S_k^{m,r}(n) = m^k \sum_{j=0}^k j! S_2(k, j) \left[\binom{n + \frac{r}{m}}{j+1} - \binom{\frac{r}{m}}{j+1} \right],$$

$$S_k^{m,r}(n) = \sum_{j=0}^k m^j j! W_{m,r}(k, j) \binom{n}{j+1},$$

and

$$S_k^{m,r}(n) = \sum_{j=0}^k A_{m,r}(k, k-j) \binom{n+j}{k+1},$$

respectively. For other explicit formulas concerning the polynomial $S_k^{m,r}(n)$ as a whole see, for example, [3, 7, 14].

2. A SIMPLER FORMULA FOR $c_{k,t}^{m,r}$

The main purpose of this note is to show that a simpler formula for $c_{k,t}^{m,r}$, as compared to the formulas in (1.1), (1.2), and (1.3), can be derived from the following alternative expression for $S_k^{m,r}(n)$ (see, e.g., [1, Equation 11] and [2, Note 3])

$$S_k^{m,r}(n) = \frac{m^k}{k+1} \sum_{t=1}^{k+1} \binom{k+1}{t} B_{k+1-t} \left(\frac{r}{m} \right) n^t, \quad k \geq 0, \quad (2.1)$$

where $B_k(x)$ are the Bernoulli polynomials with generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}, \quad (|t| < 2\pi).$$

Equation (2.1) generalizes the well-known formula for the ordinary power sum $S_k^{1,0}(n) = 0^k + 1^k + \cdots + (n-1)^k$, namely

$$S_k^{1,0}(n) = \frac{1}{k+1} \sum_{t=1}^{k+1} \binom{k+1}{t} B_{k+1-t} n^t, \quad k \geq 0,$$

where $B_k = B_k(0)$ are the Bernoulli numbers.

Indeed, by using the following specific representation for $B_k(x)$ (see, e.g. [13, Equation (38)] and [12, Equation (18)])

$$B_k(x) = \sum_{j=0}^k \sum_{i=0}^j \frac{(-1)^i}{j+1} \binom{j}{i} (i+x)^k, \quad k \geq 0, \quad (2.2)$$

and setting $x \rightarrow \frac{r}{m}$, we quickly obtain from (2.1)

$$c_{k,t}^{m,r} = \frac{m^k}{k+1} \binom{k+1}{t} \sum_{j=0}^{k+1-t} \sum_{i=0}^j \frac{(-1)^i}{j+1} \binom{j}{i} \left(i + \frac{r}{m}\right)^{k+1-t}, \quad (2.3)$$

which gives us $c_{k,t}^{m,r}$ as a double sum involving only ordinary binomial coefficients and binomial powers. Alternatively, we can write (2.3) as

$$c_{k,t}^{m,r} = \frac{m^{t-1}}{k+1} \binom{k+1}{t} \sum_{j=0}^{k+1-t} (-1)^j \frac{m^j j!}{j+1} W_{m,r}(k+1-t, j). \quad (2.4)$$

Formally, this last formula is simpler than those in (1.1), (1.2), and (1.3) because each of the numbers $S_1(k, j)$, $S_2(k, j)$, and $A_{m,r}(k, j)$, carries at least one extra summation of its own. In particular, the (signed) Stirling numbers of the first kind are given explicitly by

$$S_1(k, j) = \sum_{t=0}^{k-j} \sum_{r=0}^t \frac{(-1)^r}{t!} \binom{k+t-1}{k+t-j} \binom{2k-j}{k-t-j} \binom{t}{r} r^{k-j+t},$$

so that each of the formulas (1.1), (1.2), and (1.3) turns out to be more complex than (2.4). Indeed, as shall be argued shortly, (2.3), or its equivalent form (2.4), can be regarded as the simplest formula attainable for $c_{k,t}^{m,r}$.

Moreover, using (2.3) allows one to calculate the first few coefficients of highest degree “by hand”. These are given by

$$\begin{aligned} c_{k,k+1}^{m,r} &= \frac{m^k}{k+1}, \quad k \geq 0, \\ c_{k,k}^{m,r} &= m^{k-1} \left(r - \frac{m}{2}\right), \quad k \geq 1, \end{aligned}$$

$$c_{k,k-1}^{m,r} = \frac{1}{12} k m^{k-2} (m^2 - 6mr + 6r^2), \quad k \geq 2,$$

$$c_{k,k-2}^{m,r} = \frac{1}{12} k(k-1) m^{k-3} r (m^2 - 3mr + 2r^2), \quad k \geq 3,$$

$$c_{k,k-3}^{m,r} = \frac{1}{720} k(k-1)(k-2) m^{k-4} (m^4 - 30m^2 r^2 + 60mr^3 - 30r^4), \quad k \geq 4,$$

etc.

3. GOULD'S CONJECTURE

Setting $x = 0$ in (2.2) yields the following well-known explicit formula for the Bernoulli numbers [4, Equation 1]

$$B_k = \sum_{j=0}^k \sum_{i=0}^j \frac{(-1)^i}{j+1} \binom{j}{i} i^k, \quad k \geq 0, \quad (3.1)$$

or, equivalently,

$$B_k = \sum_{j=0}^k (-1)^j \frac{j!}{j+1} S_2(k, j), \quad k \geq 0. \quad (3.2)$$

Regarding the structure of explicit formulas for Bernoulli numbers, it is worth recalling the conjecture made by Gould at the end of his survey paper [4]:

- Gould's conjecture: “the writer has seen no formula for B_k which does not require at least two actual summations.”

This statement hints at the recognition that, in fact, there does not exist any elementary formula for B_k involving just one (finite) summation. Hence, in the spirit of Gould's conjecture, we can make the ansatz that the above formula (3.1) for B_k (or its equivalent form (3.2)), as well as the above formula (2.2) for $B_k(x)$, constitutes, in a suitably defined sense, the simplest explicit formula for the Bernoulli numbers and the Bernoulli polynomials, respectively. Consequently, with this proviso, and noting that

$$c_{k,t}^{m,r} = \frac{m^k}{k+1} \binom{k+1}{t} B_{k+1-t} \left(\frac{r}{m} \right), \quad (3.3)$$

we may conclude that the expression in (2.3), or its equivalent form (2.4), constitutes in turn the simplest formula for $c_{k,t}^{m,r}$ one is ever likely to find.

4. CONCLUDING REMARKS

We end this note with the following three remarks.

Remark 1. From (1.1) and (3.3) we can readily obtain an explicit formula for the Bernoulli polynomials involving the Stirling numbers of the first and second kind.

Indeed, for $t = 1$, (1.1) becomes

$$c_{k,1}^{m,r} = m^k \sum_{j=1}^{k+1} \sum_{i=j}^{k+1} \frac{j}{i} \left(\frac{r}{m}\right)^{j-1} S_1(i, j) S_2(k, i-1),$$

while, from (3.3), it follows that

$$c_{k,1}^{m,r} = m^k B_k \left(\frac{r}{m}\right).$$

Therefore, equating the right-hand sides of the last two equations, and replacing the rational $\frac{r}{m}$ by the arbitrary variable x , we find that

$$B_k(x) = \sum_{j=0}^k \sum_{i=j}^k \frac{j+1}{i+1} S_1(i+1, j+1) S_2(k, i) x^j, \quad k \geq 0, \quad (4.1)$$

which reduces to (3.2) when $x = 0$. Let us also observe that, by using the property $B_k(1) = (-1)^k B_k$, from (4.1) we get the identity

$$B_k = (-1)^k \sum_{j=0}^k \sum_{i=j}^k \frac{j+1}{i+1} S_1(i+1, j+1) S_2(k, i), \quad k \geq 0.$$

Moreover, from (4.1) and $B_k(x) = \sum_{j=0}^k \binom{k}{j} B_{k-j} x^j$, it turns out that

$$\binom{k}{j} B_{k-j} = (j+1) \sum_{i=j}^k \frac{1}{i+1} S_1(i+1, j+1) S_2(k, i), \quad 0 \leq j \leq k,$$

which should be compared with the alternative expression [10, Equation (15.41)]

$$\binom{k}{j} B_{k-j} = k \sum_{i=j}^k \frac{1}{i} S_1(i, j) S_2(k-1, i-1), \quad 1 \leq j \leq k.$$

Remark 2. For $t = 1$, (1.2), as well as (2.4), becomes

$$c_{k,1}^{m,r} = \sum_{j=0}^k (-1)^j \frac{m^j j!}{j+1} W_{m,r}(k, j),$$

and then

$$B_k \left(\frac{r}{m}\right) = \sum_{j=0}^k (-1)^j \frac{j!}{j+1} \frac{W_{m,r}(k, j)}{m^{k-j}}, \quad k \geq 0,$$

which was obtained in [9] as a consequence of [9, Theorem 1]. Moreover, when $m = 1$ the last formula reduces to

$$B_k(r) = \sum_{j=0}^k (-1)^j \frac{j!}{j+1} S_{2,r}(k+r, j+r),$$

thus retrieving [6, Theorem 1], where $S_{2,r}(k+r, j+r)$ denotes the r -Stirling numbers of the second kind, namely

$$S_{2,r}(k+r, j+r) = \frac{1}{j!} \sum_{i=0}^j (-1)^{j-i} \binom{j}{i} (i+r)^k.$$

Notice that, by combining the last two equations, we recover (2.2) with $x = r$.

Remark 3. Clearly, we have that $S_k^{m,r}(1) = r^k$, and then $\sum_{t=1}^{k+1} c_{k,t}^{m,r} = r^k$. Hence, it follows from (3.3) that

$$\frac{1}{k+1} \sum_{j=0}^k \binom{k+1}{j} B_j\left(\frac{r}{m}\right) = \left(\frac{r}{m}\right)^k,$$

which generalizes to the well-known relation

$$\frac{1}{k+1} \sum_{j=0}^k \binom{k+1}{j} B_j(x) = x^k.$$

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JOIN-SEMILATTICES WHOSE PRINCIPAL FILTERS ARE PSEUDOCOMPLEMENTED LATTICES

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Abstract. This paper deals with join-semilattices whose sections, i.e. principal filters, are pseudocomplemented lattices. The pseudocomplement of $a \vee b$ in the section $[b, 1]$ is denoted by $a \rightarrow b$ and can be considered as the connective implication in a certain kind of intuitionistic logic. Contrary to the case of Brouwerian semilattices, sections need not be distributive lattices. This essentially allows possible applications in non-classical logics. We present a connection of the semilattices mentioned in the beginning with the so-called non-classical implication semilattices which can be converted into I-algebras having everywhere defined operations. Moreover, we relate our structures to sectionally and relatively residuated semilattices which means that our logical structures are closely connected with substructural logics. We show that I-algebras form a congruence distributive, 3-permutable and weakly regular variety.

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Keywords: join-semilattice, pseudocomplemented lattice, intuitionistic implication, non-classical implication semilattice, sectionally residuated join-semilattice, relatively residuated join-semilattice, I-algebra, variety of I-algebras, congruence distributive, 3-permutable, weakly regular, R-algebra

1. INTRODUCTION

The concept of implication algebra was introduced by J. C. Abbott ([1]) under the name semi-boolean algebra. It is in fact an axiomatization of the connective implication in classical logic, which was formalized by means of a Boolean algebra already by G. Boole. As shown by Abbott, this algebra is in fact a join-semilattice with top element 1 where every principal filter, the so-called *section* $[x, 1]$, is a Boolean algebra. Hence, the operation $\vee$ is everywhere defined and it can be expressed by means of implication $x \rightarrow y$ as $x \vee y = (x \rightarrow y) \rightarrow y$. On the other hand, the operation $\wedge$ is only partial and $x \wedge y$ is defined if and only if the elements x, y have a common

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lower bound p ; in such a case $x \wedge y = (x \rightarrow (y \rightarrow p)) \rightarrow p$ and the complement of x in $[p, 1]$ is $x^p = x \rightarrow p$.

Later on, Abbott formalized in a similar way also the connective implication in the logic of quantum mechanics based on an orthomodular lattice, see [2]. In an analogous way, implication was axiomatized in ortholattices and in orthomodular lattices without the compatibility condition by the authors and R. Halaš in [9] and [10], see also [5] and [7] for similar cases.

Concerning implication semilattices, i.e. join-semilattices where every section is a Boolean algebra, and concerning the structure and properties of such algebras, the reader is referred to [14–16].

It is a natural question whether the connective implication in intuitionistic logic can be axiomatized in a similar way. It is known that the so-called Brouwerian semilattices can serve for this sake, see e.g. [13]. In this case we obtain a join-semilattice with top element 1 whose sections are relatively pseudocomplemented posets or lattices, see [3] for these concepts. However, relatively pseudocomplemented lattices are distributive (see e.g. [3]), this fact restricting their application in non-classical logics.

An attempt to extend the concept of relative pseudocomplementation to non-distributive lattices was made by the first author in [6]. This kind of pseudocomplementation was called *sectional pseudocomplementation* and it was extended also to posets with top element 1 by the authors and J. Paseka in [12].

Hence, we can extend our investigation of intuitionistic implication also to semilattices whose sections need not be distributive lattices. We show that in this case, these join-semilattices can be converted into the so-called sectionally residuated join-semilattices which are in one-to-one correspondence with relatively residuated join-semilattices introduced by the authors and J. Kühr for lattices in [11].

Since in our structures the lattice operation $\wedge$ is only partial, the algebras under consideration are determined by so-called *partial identities* and hence they form so-called *partial varieties*. To avoid drawbacks with partial operations, we introduce a new ternary everywhere defined operation and we show that our non-classical implication algebra can be converted in an algebra with everywhere defined operations characterized by identities only, and hence the corresponding class forms a variety. We show that this variety has nice congruence properties.

We believe that our approach brings a new insight in this topic and enables to understand the structure of the investigated objects.

2. BASIC CONCEPTS

A *pseudocomplemented lattice* (see e.g. [3] and [4]) is an algebra $(L, \vee, \wedge, *, 0)$ of type $(2, 2, 1, 0)$ such that $(L, \vee, \wedge, 0)$ is a lattice with bottom element 0 and for every $x \in L$, x^* is the greatest element $y \in L$ satisfying $x \wedge y = 0$, the so-called *pseudocomplement* of x . It is easy to see that $x \leq y$ implies $y^* \leq x^*$ and that $x \leq x^{**}$.

Let $(S, \vee, 1)$ be a join-semilattice with top element 1 whose principal filters are pseudocomplemented lattices. This means that for every $x \in S$ there exists a binary operation $\wedge_x$ and a unary operation x on $[x, 1]$ such that $([x, 1], \vee, \wedge_x, ^x)$ is a pseudocomplemented lattice with bottom element x . It is natural to assume the following *compatibility condition*:

$$x \leq y \text{ and } z, u \in [y, 1] \Rightarrow z \wedge_x u = z \wedge_y u.$$

We then also have

$$z, u \in [x, 1] \cap [y, 1] \Rightarrow z \wedge_x u = z \wedge_y u$$

since $z, u \in [x, 1] \cap [y, 1]$ implies $z, u \in [x, 1] \cap [y, 1] \cap [x \vee y, 1]$ and hence

$$z \wedge_x u = z \wedge_{x \vee y} u = z \wedge_y u.$$

In the following $z \wedge u$ denotes the infimum of z and u provided it exists. In general, the binary operation $\wedge$ is only partial. But now we can prove

$$z, u \in [x, 1] \Rightarrow z \wedge_x u = z \wedge u.$$

This can be seen as follows: Because of $z, u \in [x, 1]$ we have $z \wedge_x u \leq z, u$. If $y \in S$ and $y \leq z, u$ then $z, u \in [y, 1]$ and hence $y \leq z \wedge_y u = z \wedge_x u$. This shows $z \wedge_x u = z \wedge u$. Hence $z \wedge u$ exists if and only if z and u have a common lower bound. This motivates the following definition.

3. JOIN-SEMILATTICES WITH PSEUDOCOMPLEMENTED SECTIONS

We start with the following fundamental definition.

Definition 1. A *join-semilattice whose sections are pseudocomplemented lattices* is an ordered quintuple $(S, \vee, \wedge, (^x; x \in S), 1)$ where $(S, \vee, 1)$ is a join-semilattice with top element 1, $x \wedge y$ is defined if and only if x and y have a common lower bound and for every $x \in S$, $([x, 1], \vee, \wedge, ^x)$ is a pseudocomplemented lattice.

Here and in the followings the word “*section*” means the same as “principal filter”.

In order to avoid problems with such a number of unary operations (pseudocomplementations in sections), we introduce a new binary operation $\rightarrow$ satisfying four simple axioms. This new structure would represent an implication reduct of a certain kind of non-classical (intuitionistic) logic. Because the name *implication semilattice* was already used by W. C. Nemitz (see e.g. [14–16]) for a semilattice whose sections are Boolean algebras (which formalizes the implication reduct of classical logic), we use a different name for our structure.

Definition 2. A *non-classical implication semilattice* is an ordered quintuple $(S, \vee, \wedge, \rightarrow, 1)$ where $(S, \vee, 1)$ is a join-semilattice with top element 1, $x \wedge y$ exists if and only if x and y have a common lower bound and $\rightarrow$ is a binary operation on S satisfying (1) – (4):

$$(1) \ y \leq x \rightarrow y,$$

- (2) $(x \vee y) \wedge (x \rightarrow y) \approx y$,
- (3) $(x \vee y) \rightarrow y \approx x \rightarrow y$,
- (4) $y \leq (x \vee z) \rightarrow ((x \vee z) \wedge (y \vee z))$.

Remark 1. The conditions (1) and (3) are identities whereas (2) and (4) can be considered as identities where the operation $\wedge$ is only partial. Hence the class of non-classical implication semilattices forms a partial variety.

Observe that contrary to our case the sections of a relatively residuated join-semilattice need not be lattices and hence the meet of two elements which have a common lower bound need not exist.

For every join-semilattice $\mathbf{S} = (S, \vee, \wedge, (\rightarrow; x \in S), 1)$ whose sections are pseudocomplemented lattices put $\mathbb{I}(\mathbf{S}) := (S, \vee, \wedge, \rightarrow, 1)$ where $\rightarrow$ denotes the binary operation on S defined by $x \rightarrow y := (x \vee y)^y$ for all $x, y \in S$.

We can relate this concept with that one defined above.

Theorem 1. *Let $\mathbf{S} = (S, \vee, \wedge, (\rightarrow; x \in S), 1)$ be a join-semilattice whose sections are pseudocomplemented lattices. Then $\mathbb{I}(\mathbf{S}) = (S, \vee, \wedge, \rightarrow, 1)$ is a non-classical implication semilattice satisfying (5) – (9):*

- (5) $x \leq y \Leftrightarrow x \rightarrow y = 1$,
- (6) $x \leq y \Rightarrow y \rightarrow z \leq x \rightarrow z$,
- (7) $x \leq (x \rightarrow y) \rightarrow y$,
- (8) $((x \rightarrow y) \rightarrow y) \rightarrow y \approx x \rightarrow y$,
- (9) $1 \rightarrow x \approx x$.

Proof. Let $a, b, c \in S$.

- (1) $b \leq (a \vee b)^b = a \rightarrow b$
- (2) $(a \vee b) \wedge (a \rightarrow b) = (a \vee b) \wedge (a \vee b)^b = b$
- (3) $(a \vee b) \rightarrow b = ((a \vee b) \vee b)^b = (a \vee b)^b = a \rightarrow b$
- (4) Since $b \vee c \in [(a \vee c) \wedge (b \vee c), 1]$ and $(a \vee c) \wedge (b \vee c) = (a \vee c) \wedge (b \vee c)$ we have

$$b \vee c \leq (a \vee c)^{(a \vee c) \wedge (b \vee c)}$$

and hence

$$\begin{aligned} b \leq b \vee c &\leq (a \vee c)^{(a \vee c) \wedge (b \vee c)} = \left(a \vee c \vee ((a \vee c) \wedge (b \vee c)) \right)^{(a \vee c) \wedge (b \vee c)} \\ &= (a \vee c) \rightarrow ((a \vee c) \wedge (b \vee c)). \end{aligned}$$

- (5) If $a \leq b$ then $a \rightarrow b = (a \vee b)^b = b^b = 1$. If, conversely, $a \rightarrow b = 1$ then $a \leq a \vee b = (a \vee b) \wedge 1 = (a \vee b) \wedge (a \rightarrow b) = (a \vee b) \wedge (a \vee b)^b = b$.
- (6) If $a \leq b$ then $b \rightarrow c = (b \vee c)^c \leq (a \vee c)^c = a \rightarrow c$.
- (7) $a \leq a \vee b \leq (a \vee b)^{bb} = ((a \vee b)^b \vee b)^b = (a \rightarrow b) \rightarrow b$
- (8) By (7) and (6) we have

$$a \rightarrow b \leq ((a \rightarrow b) \rightarrow b) \rightarrow b \leq a \rightarrow b,$$

i.e.

$$((a \rightarrow b) \rightarrow b) \rightarrow b = a \rightarrow b.$$

$$(9) \quad 1 \rightarrow a = (1 \vee a)^a = 1^a = a$$

□

The following example shows that in a join-semilattice whose sections are pseudocomplemented lattices these lattices need not be modular.

Example 1. The join-semilattice depicted in Figure 1

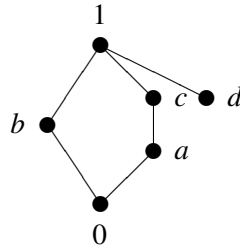


Fig. 1.

is a join-semilattice whose sections are pseudocomplemented lattices and the operation tables for $\rightarrow$ and $\wedge$ look as follows:

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	b	1	b	1	d	1
b	c	a	1	c	d	1
c	b	a	b	1	d	1
d	0	a	b	c	1	1
1	0	a	b	c	d	1

$\wedge$	0	a	b	c	d	1
0	0	0	0	0	—	0
a	0	a	0	a	—	a
b	0	0	b	0	—	b
c	0	a	0	c	—	c
d	—	—	—	—	d	d
1	0	a	b	c	d	1

Now we assign to every non-classical implication semilattice a join-semilattice whose sections are pseudocomplemented lattices. For this sake, let us introduce the following concept.

For every non-classical implication semilattice $\mathbf{I} = (I, \vee, \wedge, \rightarrow, 1)$ put

$$\mathbb{S}(\mathbf{I}) := (I, \vee, \wedge, (^x; x \in I), 1)$$

where for every $x \in I$, x denotes the unary operation on $[x, 1]$ defined by $y^x := y \rightarrow x$ for all $y \in [x, 1]$.

Theorem 2. Let $\mathbf{I} = (I, \vee, \wedge, \rightarrow, 1)$ be a non-classical implication semilattice. Then $\mathbb{S}(\mathbf{I})$ is a join-semilattice whose sections are pseudocomplemented lattices.

Proof. Let $\mathbb{S}(\mathbf{I}) = (I, \vee, \wedge, (^x; x \in I), 1)$ and $a, b, c \in I$ with $c \leq a, b$. If $a \wedge b = c$ then according to (4) we obtain

$$b \leq (a \vee c) \rightarrow ((a \vee c) \wedge (b \vee c)) = a \rightarrow (a \wedge b) = a \rightarrow c = a^c.$$

If, conversely, $b \leq a^c$ then

$$c \leq a \wedge b \leq a \wedge a^c \leq (a \vee c) \wedge (a \rightarrow c) = c$$

according to (2) and therefore $a \wedge b = c$. Hence, for each $a \in [c, 1]$ the element $a \rightarrow c$ is the pseudocomplement of a in this interval. $\square$

We justify our conversions by the following result.

Theorem 3. *The correspondence described in Theorems 1 and 2 is one-to-one.*

Proof. If $\mathbf{S} = (S, \vee, \wedge, (^x; x \in S), 1)$ is a join-semilattice whose sections are pseudo-complemented lattices and

$$\begin{aligned} \mathbb{I}(\mathbf{S}) &= (S, \vee, \wedge, \rightarrow, 1), \\ \mathbb{S}(\mathbb{I}(\mathbf{S})) &= (S, \vee, \wedge, (^x; x \in S), 1) \end{aligned}$$

then $y_x = y \rightarrow x = (y \vee x)^x = y^x$ for all $x \in S$ and $y \in [x, 1]$ and hence $\mathbb{S}(\mathbb{I}(\mathbf{S})) = \mathbf{S}$. If, conversely, $\mathbf{I} = (I, \vee, \wedge, \rightarrow, 1)$ is a non-classical implication semilattice and

$$\begin{aligned} \mathbb{S}(\mathbf{I}) &= (I, \vee, \wedge, (^x; x \in S), 1), \\ \mathbb{I}(\mathbb{S}(\mathbf{I})) &= (I, \vee, \wedge, \Rightarrow, 1) \end{aligned}$$

then $x \Rightarrow y \approx (x \vee y)^y \approx (x \vee y) \rightarrow y \approx x \rightarrow y$ according to (3) and hence $\mathbb{I}(\mathbb{S}(\mathbf{I})) = \mathbf{I}$ where $(x \vee y) \Rightarrow y \approx ((x \vee y) \vee y)^y \approx (x \vee y)^y$. $\square$

4. SECTIONALLY AND RELATIVELY RESIDUATED JOIN-SEMILATTICES

The aim of this section is to establish a relationship between the structures defined in the previous section and residuated structures which formalize certain substructural logics.

We start with definitions of our key concepts relating our implication algebras with a certain kind of residuation.

Definition 3. A *sectionally residuated join-semilattice* is an ordered quintuple $(S, \vee, (\odot_x; x \in S), \rightarrow, 1)$ such that $(S, \vee, 1)$ is a join-semilattice with top element 1, for each $x \in S$, $([x, 1], \odot_x, 1)$ is a commutative monoid, $\rightarrow$ is a binary operation on S and for all $x, y, z, u \in S$ the following conditions hold:

- (i) if $z \leq u \leq x, y$ then $x \odot_z y = x \odot_u y$ (*compatibility condition*),
- (ii) if $u \leq x, y, z$ and $x \leq y$ then $x \odot_u z \leq y \odot_u z$ (*monotonicity*),
- (iii) $(x \vee z) \odot_z (y \vee z) \leq z$ if and only if $x \vee z \leq y \rightarrow z$ (*sectional adjointness*),
- (iv) $(x \vee y) \rightarrow y \approx x \rightarrow y$.

The next concept is similar to the corresponding one from [11] which, however, was introduced for lattices and hence was equipped with an everywhere defined meet-operation.

Definition 4. A *relatively residuated join-semilattice* is a partial algebra $\mathbf{R} = (R, \vee, \odot, \rightarrow, 1)$ of type $(2, 2, 2, 0)$ where $x \odot y$ is defined if and only if x and y have a common lower bound and in this case $z \leq x \odot y$ for all $z \in R$ with $z \leq x, y$ and $\rightarrow$ is a binary operation on R such that for all $x, y, z \in R$ the following conditions hold:

- (10) $(R, \vee, 1)$ is a join-semilattice with top element 1,
- (11) $x \odot 1 \approx 1 \odot x \approx x$,
- (12) $x \odot y = y \odot x$ whenever x and y have a common lower bound,
- (13) $(x \odot y) \odot z = x \odot (y \odot z)$ whenever x, y and z have a common lower bound,
- (14) $x \leq y$ implies $x \odot z \leq y \odot z$ whenever x and z have a common lower bound (*monotonicity*),
- (15) $(x \vee z) \odot (y \vee z) \leq z$ if and only if $x \vee z \leq y \rightarrow z$ (*relative adjointness*),
- (16) $(x \vee y) \rightarrow y \approx x \rightarrow y$.

$\mathbf{R}$ is called *divisible* if it satisfies the identity

$$(x \vee y) \odot (x \rightarrow y) \approx y.$$

In the next theorem we describe several basic properties of relatively residuated join-semilattices.

Theorem 4. Let $\mathbf{S} = (S, \vee, \odot, \rightarrow, 1)$ be a relatively residuated join-semilattice and $a, b, c \in S$. Then the following conditions hold:

- (i) $a \leq b \Leftrightarrow a \rightarrow b = 1$,
- (ii) $a \odot b \leq a$,
- (iii) if $a \wedge b$ exists then $a \odot b \leq a \wedge b$,
- (iv) $a \leq b \rightarrow a$,
- (v) $a \odot (a \rightarrow b) \leq (a \vee b) \odot (a \rightarrow b) \leq b$,
- (vi) $a \leq (a \rightarrow b) \rightarrow b$,
- (vii) $a \leq b \Rightarrow b \rightarrow c \leq a \rightarrow c$,
- (viii) $((a \rightarrow b) \rightarrow b) \rightarrow b = a \rightarrow b$.

Proof.

- (i) The following conditions are equivalent:

$$\begin{aligned} a &\leq b, \\ (1 \vee b) \odot (a \vee b) &\leq b, \\ 1 \vee b &\leq a \rightarrow b, \\ a \rightarrow b &= 1. \end{aligned}$$

- (ii) By commutativity and monotonicity of $\odot$ we have $a \odot b = b \odot a \leq 1 \odot a = a$.
- (iii) This follows from (ii) and from commutativity of $\odot$.
- (iv) We have $(a \vee b) \odot a \leq a$ according to (ii). Now the following conditions are equivalent:

$$(a \vee a) \odot (b \vee a) \leq a,$$

$$\begin{aligned} a \vee a &\leq b \rightarrow a, \\ a &\leq b \rightarrow a. \end{aligned}$$

(v) We have $b \leq a \rightarrow b$ according to (iv). Now by (iv) the following conditions are equivalent:

$$\begin{aligned} (a \rightarrow b) \vee b &\leq a \rightarrow b, \\ ((a \rightarrow b) \vee b) \odot (a \vee b) &\leq b, \\ (a \vee b) \odot (a \rightarrow b) &\leq b. \end{aligned}$$

By monotonicity of $\odot$ we have $a \odot (a \rightarrow b) \leq (a \vee b) \odot (a \rightarrow b)$.

(vi) We have $(a \vee b) \odot (a \rightarrow b) \leq b$ according to (v). Now by (iv) the following conditions are equivalent:

$$\begin{aligned} (a \vee b) \odot ((a \rightarrow b) \vee b) &\leq b, \\ a \vee b &\leq (a \rightarrow b) \rightarrow b. \end{aligned}$$

(vii) Assume $a \leq b$. Then $a \vee c \leq (b \rightarrow c) \rightarrow c$ according to (iv) and (vi). Now the following conditions are equivalent:

$$\begin{aligned} a \vee c &\leq (b \rightarrow c) \rightarrow c, \\ (a \vee c) \odot ((b \rightarrow c) \vee c) &\leq c, \\ ((b \rightarrow c) \vee c) \odot (a \vee c) &\leq c, \\ (b \rightarrow c) \vee c &\leq a \rightarrow c, \\ b \rightarrow c &\leq a \rightarrow c. \end{aligned}$$

(viii) According to (vi) and (vii) we have

$$a \rightarrow b \leq ((a \rightarrow b) \rightarrow b) \rightarrow b \leq a \rightarrow b$$

proving (viii). □

In the next theorem we show how the class of relatively residuated join-semilattices can be described without using relative adjointness.

Theorem 5. *Let $\mathbf{R} = (R, \vee, \odot, \rightarrow, 1)$ be a partial algebra of type $(2, 2, 2, 0)$ where $x \odot y$ is defined if and only if x and y have a common lower bound and in this case $z \leq x \odot y$ for all $z \in R$ with $z \leq x, y$ and $\rightarrow$ is a binary operation on R such that for all $x, y, z \in R$ (10) – (14) and (16) hold. Then $\mathbf{R}$ is a relatively residuated join-semilattice if and only if it satisfies the following identities:*

- (17) $x \vee z \leq y \rightarrow ((x \vee z) \odot (y \vee z)) \vee z$,
- (18) $x \leq y \rightarrow x$,
- (19) $(x \vee y) \odot (x \rightarrow y) \leq y$.

Proof. Let $a, b, c, d \in R$. If $\mathbf{R}$ is a relatively residuated join-semilattice then (18) and (19) follow from Theorem 4, and if $d := ((a \vee c) \odot (b \vee c)) \vee c$ then $a \vee d = a \vee c$ and $b \vee d = b \vee c$ because of (ii) of Theorem 4 and (12) whence $(a \vee d) \odot (b \vee d) = (a \vee c) \odot (b \vee c) \leq d$ by definition of d which implies

$$a \vee c = a \vee d \leq b \rightarrow d = b \rightarrow ((a \vee c) \odot (b \vee c)) \vee c$$

by (15) showing (17). If, conversely, $\mathbf{R}$ satisfies (17) – (19) then, if $(a \vee c) \odot (b \vee c) \leq c$ then $a \vee c \leq b \rightarrow c$ according to (17), and if, conversely, $a \vee c \leq b \rightarrow c$ then

$$(a \vee c) \odot (b \vee c) \leq (b \rightarrow c) \odot (b \vee c) = (b \vee c) \odot (b \rightarrow c) \leq c$$

according to (14), (12) and (19) proving (15), thus showing that $\mathbf{R}$ is a relatively residuated join-semilattice. $\square$

Theorem 5 implies that the class of relatively residuated join-semilattices is determined by partial identities and hence forms a partial variety.

The following results show that the concepts just defined can be converted into each other.

Theorem 6.

- (i) Let $\mathbf{S} = (S, \vee, (\odot_x; x \in S), \rightarrow, 1)$ be a sectionally residuated join-semilattice and define a partial binary operation $\odot$ on S as follows: If x and y have a common lower bound z then $x \odot y := x \odot_z y$. Then $\odot$ is well-defined and $\mathbb{R}(\mathbf{S}) := (S, \vee, \odot, \rightarrow, 1)$ is a relatively residuated join-semilattice.
- (ii) Let $\mathbf{R} = (R, \vee, \odot, \rightarrow, 1)$ be a relatively residuated join-semilattice and put $\odot_x := \odot|_{[x, 1]}$ for all $x \in S$. Then $\mathbb{S}(\mathbf{R}) := (R, \vee, (\odot_x; x \in R), \rightarrow, 1)$ is a sectionally residuated join-semilattice.

Proof. (16) coincides with (iv) of Definition 3.

- (i) Let $a, b, c, d \in S$. If c and d are common lower bounds of a and b then also $c \vee d$ is a common lower bound of a and b and hence $a \odot_c b = a \odot_{c \vee d} b = a \odot_d b$. This shows that $\odot$ is well-defined. Moreover, we have $a \odot 1 = a \odot_a 1 = a$ and $1 \odot a = 1 \odot_a a = a$. If a and b have a common lower bound c then $a \odot b = a \odot_c b = b \odot_c a = b \odot a$. If, finally, a , b and c have a common lower bound d then

$$(a \odot b) \odot c = (a \odot_d b) \odot_d c = a \odot_d (b \odot_d c) = a \odot (b \odot c)$$

and $a \leq b$ implies $a \odot c = a \odot_d c \leq b \odot_d c = b \odot c$. Relative adjointness follows from $(a \vee c) \odot (b \vee c) = (a \vee c) \odot_c (b \vee c)$.

- (ii) Let $a, b, c, d \in R$. Then $([a, 1], \odot_a, 1)$ is a commutative monoid. Moreover, if $c \leq d \leq a, b$ then $a \odot_c b = a \odot b = a \odot_d b$. Further, if $d \leq a, b, c$ and $a \leq b$ then $a \odot_d c = a \odot c \leq b \odot c = b \odot_d c$. Finally, sectional adjointness follows from $(a \vee c) \odot_c (b \vee c) = (a \vee c) \odot (b \vee c)$. $\square$

The next result shows that the conversions described in Theorem 6 are nothing else than a translation from one language into the other.

Theorem 7. *The correspondence described in Theorem 6 is one-to-one.*

Proof. If $\mathbf{S} = (S, \vee, (\odot_x; x \in S), \rightarrow, 1)$ is a sectionally residuated join-semilattice,

$$\mathbb{R}(\mathbf{S}) = (S, \vee, \odot, \rightarrow, 1),$$

$$\mathbb{S}(\mathbb{R}(\mathbf{S})) = (S, \vee, (\circ_x; x \in S), \rightarrow, 1),$$

$a, b, c \in S$ and $c \leq a, b$ then $a \circ_c b = a \odot b = a \odot_c b$ showing $\mathbb{S}(\mathbb{R}(\mathbf{S})) = \mathbf{S}$. If, conversely $\mathbf{R} = (R, \vee, \odot, \rightarrow, 1)$ is a relatively residuated join-semilattice,

$$\mathbb{S}(\mathbf{R}) = (R, \vee, (\odot_x; x \in R), \rightarrow, 1),$$

$$\mathbb{R}(\mathbb{S}(\mathbf{R})) = (R, \vee, \circ, \rightarrow, 1),$$

$a, b, c \in R$ and $c \leq a, b$ then $a \circ b = a \odot_c b = a \odot b$ showing $\mathbb{R}(\mathbb{S}(\mathbf{R})) = \mathbf{R}$. $\square$

In the next result we show that non-classical implication semilattices form a partial subvariety of the class of divisible relatively residuated join-semilattices.

According to (iii) of Theorem 4 $x \odot y \leq x \wedge y$ whenever $x \wedge y$ exists. The question when $x \odot y = x \wedge y$ is answered in the next theorem; this equality turns out to hold if $\odot$ is idempotent (and satisfies one more simple identity).

Theorem 8. *The non-classical implication semilattices are exactly the divisible relatively residuated join-semilattices $(R, \vee, \odot, \rightarrow, 1)$ satisfying the following identities:*

- (i) $x \odot x \approx x$,
- (ii) $y \leq (x \vee z) \rightarrow ((x \vee z) \odot (y \vee z))$.

Proof. Let $(S, \vee, \wedge, \rightarrow, 1)$ be a non-classical implication semilattice. Then $x \wedge y$ is defined if and only if x and y have a common lower bound, $z \leq x \wedge y$ whenever $z \leq x, y$, $\rightarrow$ is a binary operation on S and (10) – (14) of Definition 4 hold. Now let $a, b, c \in S$. First assume $(a \vee c) \wedge (b \vee c) \leq c$. Then $(b \vee c) \wedge (a \vee c) = c$. Since

$$a \leq (b \vee c) \rightarrow ((b \vee c) \wedge (a \vee c)) = (b \vee c) \rightarrow c = b \rightarrow c$$

according to (4) and (3) and $c \leq b \rightarrow c$ according to (1), we conclude $a \vee c \leq b \rightarrow c$. If, conversely, $a \vee c \leq b \rightarrow c$ then

$$(a \vee c) \wedge (b \vee c) = (b \vee c) \wedge (a \vee c) \leq (b \vee c) \wedge (b \rightarrow c) = c$$

by (2). This shows relative adjointness. (16) coincides with (3). Divisibility of $\mathbf{S}$ follows from (2). Obviously, (i) holds. Moreover, (ii) follows from (4). Conversely, let $(R, \vee, \odot, \rightarrow, 1)$ be a divisible relatively residuated join-semilattice satisfying identities (i) and (ii). Then $(R, \vee, 1)$ is a join-semilattice with top element 1 according to (10), $x \odot y$ is defined if and only if x and y have a common lower bound and $\rightarrow$ is a binary operation on R . Let $a, b, c, d \in R$ with $c \leq a, b$. Because of (14), (11) and (12)

we have $a \odot b \leq 1 \odot b = b$ and $a \odot b = b \odot a \leq a$. If, moreover, $d \leq a, b$, then because of (i), (14) and (12) we have

$$d = d \odot d \leq a \odot d = d \odot a \leq b \odot a = a \odot b.$$

This shows $a \odot b = a \wedge b$. Finally, (1) follows from (iv) of Theorem 4, (2) from divisibility, (3) from (16) and (4) from (ii). $\square$

Remark 2. In Theorem 8 identity (i) can be replaced by condition (i')

(i') $x \wedge y$ exists if and only if x and y have a common lower bound.

This can be seen as follows: Let $(R, \vee, \odot, \rightarrow, 1)$ be a relatively residuated join-semilattice satisfying (i') and $a, b \in R$ having a common lower bound. Then $a \wedge b$ exists by (i') and $a \wedge b$ is a common lower bound of a and b whence $a \wedge b \leq a \odot b$. But $a \odot b \leq a \wedge b$ by (iii) of Theorem 4 which shows $a \odot b = a \wedge b$.

Example 2. The join-semilattice depicted in Figure 2

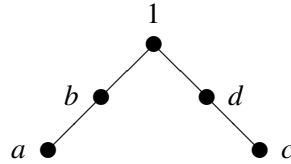


Fig. 2.

is a join-semilattice whose sections are pseudocomplemented lattices and the operation tables for $\rightarrow$ and $\wedge$ look as follows:

$\rightarrow$	a	b	c	d	1
a	1	1	c	d	1
b	a	1	c	d	1
c	a	b	1	1	1
d	a	b	c	1	1
1	a	b	c	d	1

$\wedge$	a	b	c	d	1
a	a	a	—	—	a
b	a	b	—	—	b
c	—	—	c	c	c
d	—	—	c	d	d
1	a	b	c	d	1

It is easy to check adjointness in the sense that whenever $x \wedge y$ is defined we have $x \wedge y \leq z$ if and only $x \leq y \rightarrow z$. It should be mentioned that $\rightarrow$ is a proper extension of the pseudocomplementation within sections. Namely, the pseudocomplement of b within $[a, 1]$ is $b \rightarrow a = a$. On the other hand, $b \notin [d, 1]$, but $b \rightarrow d = d$.

5. THE VARIETY OF NON-CLASSICAL IMPLICATION ALGEBRAS

Up to now, we investigated algebras where the operations $\vee$ and $\rightarrow$ were everywhere defined, but $\wedge$ or $\odot$ were only partial operations defined just in the case when both operands belong to the same section. To avoid this difficulty, we introduce a new ternary operation $r(x, y, z)$ which is everywhere defined and replaces the partial operation $\wedge$.

Definition 5. An *I-algebra* is an algebra $(A, \vee, r, \rightarrow, 1)$ of type $(2, 3, 2, 0)$ such that $(A, \vee, 1)$ is a join-semilattice with top element 1 and the following identities are satisfied:

- (1') $y \leq x \rightarrow y$,
- (2') $r(x, x \rightarrow y, y) \approx y$,
- (3') $(x \vee y) \rightarrow y \approx x \rightarrow y$,
- (4') $y \leq (x \vee z) \rightarrow r(x, y, z)$,
- (5') $r(x, y, z) \leq x \vee z$,
- (6') $r(x, y, z) \leq y \vee z$,
- (7') $r(x, x \vee y, z) \approx x \vee z$,
- (8') $r(x, y, z) \approx r(x \vee z, y \vee z, z)$,
- (9') $z \leq r(x, y, z)$,
- (10') $r(u, r(x, y, z), z) \approx r(r(u, x, z), r(u, y, z), z)$.

Considering a non-classical implication semilattice as introduced in Definition 2, we can define a ternary operation $r(x, y, z)$ as follows: $r(x, y, z) := (x \vee z) \wedge (y \vee z)$. For every non-classical implication semilattice $\mathbf{I} = (I, \vee, \wedge, \rightarrow, 1)$ put $\mathbb{A}(\mathbf{I}) := (I, \vee, r, \rightarrow, 1)$.

The next theorem shows a conversion of non-classical implication semilattices into I-algebras.

Theorem 9. Let $\mathbf{I} = (I, \vee, \wedge, \rightarrow, 1)$ be a non-classical implication semilattice. Then $\mathbb{A}(\mathbf{I})$ is an I-algebra.

Proof. Let $\mathbb{A}(\mathbf{I}) = (I, \vee, r, \rightarrow, 1)$ and $a, b, c, d \in I$. Then, clearly,

$$r(x, x, y) \approx x \vee y \approx r(y, y, x).$$

- (1') coincides with (1).
- (2') $r(a, a \rightarrow b, b) = (a \vee b) \wedge ((a \rightarrow b) \vee b) = (a \vee b) \wedge (a \rightarrow b) = b$ according to (1) and (2).
- (3') coincides with (3).
- (4') $b \leq (a \vee c) \rightarrow ((a \vee c) \wedge (b \vee c)) = (a \vee c) \rightarrow r(a, b, c)$ according to (4).
- (5') $r(a, b, c) = (a \vee c) \wedge (b \vee c) \leq a \vee c$
- (6') $r(a, b, c) = (a \vee c) \wedge (b \vee c) \leq b \vee c$
- (7') $r(a, a \vee b, c) = (a \vee c) \wedge ((a \vee b) \vee c) = a \vee c$
- (8') $r(a, b, c) = (a \vee c) \wedge (b \vee c) = ((a \vee c) \vee c) \wedge ((b \vee c) \vee c) = r(a \vee c, b \vee c, c)$
- (9') $c \leq (a \vee c) \wedge (b \vee c) = r(a, b, c)$
- (10')

$$\begin{aligned} r(d, r(a, b, c), c) &= (d \vee c) \wedge (((a \vee c) \wedge (b \vee c)) \vee c) = (d \vee c) \wedge ((a \vee c) \wedge (b \vee c)) \\ &= (d \vee c) \wedge (a \vee c) \wedge (b \vee c) = ((d \vee c) \wedge (a \vee c)) \wedge ((d \vee c) \wedge (b \vee c)) \\ &= (((d \vee c) \wedge (a \vee c)) \vee c) \wedge (((d \vee c) \wedge (b \vee c)) \vee c) \end{aligned}$$

$$= r(r(d, a, c), r(d, b, c), c)$$

□

For every I-algebra $\mathbf{A} = (A, \vee, r, \rightarrow, 1)$ put $\mathbb{J}(\mathbf{A}) := (A, \vee, \wedge, \rightarrow, 1)$ where $x \wedge y := r(x, y, z)$ for all $x, y, z \in A$ with $z \leq x, y$. The fact that $\mathbb{J}(\mathbf{A})$ is well-defined follows from the next theorem.

Theorem 10. *Let $\mathbf{A} = (A, \vee, r, \rightarrow, 1)$ be an I-algebra. Then $\mathbb{J}(\mathbf{A})$ is a well-defined non-classical implication semilattice.*

Proof. Let $\mathbb{J}(\mathbf{A}) = (A, \vee, \wedge, \rightarrow, 1)$ and $a, b, c, d \in A$. First we prove $r(a, b, c) = (a \vee c) \wedge (b \vee c)$. Because of (5') and (6') we have $r(a, b, c) \leq a \vee c, b \vee c$. Now assume $d \leq a \vee c, b \vee c$. Then

$$\begin{aligned} d &\leq d \vee c = (d \vee c) \vee c = r(d \vee c, (d \vee c) \vee c, c) = r((d \vee c) \vee c, (d \vee c) \vee c, c) \\ &= r(r(d \vee c, (d \vee c) \vee (a \vee c), c), r(d \vee c, (d \vee c) \vee (b \vee c), c), c) \\ &= r(r(d \vee c, a \vee c, c), r(d \vee c, b \vee c, c), c) = r(r(d, a, c), r(d, b, c), c) \\ &= r(d, r(a, b, c), c) \leq r(a, b, c) \vee c = r(a, b, c) \end{aligned}$$

because of (7'), (8'), (10'), (6') and (9'). This shows $r(a, b, c) = (a \vee c) \wedge (b \vee c)$. Hence $a \wedge b = \inf(a, b)$ whenever there exists a common lower bound of a and b and therefore $\wedge$ and $\mathbb{J}(\mathbf{A})$ are well-defined.

- (1) coincides with (1').
- (2) $(a \vee b) \wedge (a \rightarrow b) = r(a, a \rightarrow b, b) = b$ because of (1') and (2').
- (3) coincides with (3').
- (4) $b \leq (a \vee c) \rightarrow r(a, b, c) = (a \vee c) \rightarrow ((a \vee c) \wedge (b \vee c))$ by (4').

□

Theorem 11. *The correspondence described in Theorems 9 and 10 is one-to-one.*

Proof. If $\mathbf{I} = (I, \vee, \wedge, \rightarrow, 1)$ is a non-classical implication semilattice and

$$\begin{aligned} \mathbb{A}(\mathbf{I}) &= (I, \vee, r, \rightarrow, 1), \\ \mathbb{J}(\mathbb{A}(\mathbf{I})) &= (I, \vee, \cap, \rightarrow, 1) \end{aligned}$$

then $x \cap y = r(x, y, z) = (x \vee z) \wedge (y \vee z) = x \wedge y$ for all $x, y, z \in I$ with $z \leq x, y$ and hence $\mathbb{J}(\mathbb{A}(\mathbf{I})) = \mathbf{I}$. If, conversely, $\mathbf{A} = (A, \vee, r, \rightarrow, 1)$ is an I-algebra and

$$\begin{aligned} \mathbb{J}(\mathbf{A}) &= (A, \vee, \wedge, \rightarrow, 1), \\ \mathbb{A}(\mathbb{J}(\mathbf{A})) &= (A, \vee, R, \rightarrow, 1) \end{aligned}$$

then $R(x, y, z) \approx (x \vee z) \wedge (y \vee z) \approx r(x \vee z, y \vee z, z) \approx r(x, y, z)$ according to (8') and hence $\mathbb{A}(\mathbb{J}(\mathbf{A})) = \mathbf{A}$. □

The advantage of using a ternary operation $r(x, y, z)$ instead of the partial binary operation $\wedge$ is that now we have an algebra with everywhere defined operations which is determined by a finite set of identities. Hence these algebras form a variety $\mathcal{V}$.

Recall that an algebra $\mathbf{A}$ is called

- *3-permutable* if $\Theta \circ \Phi \circ \Theta = \Phi \circ \Theta \circ \Phi$ for all $\Theta, \Phi \in \text{Con } \mathbf{A}$,
- *congruence distributive* if the congruence lattice $\text{Con } \mathbf{A}$ of $\mathbf{A}$ is distributive.

An algebra $\mathbf{A}$ with an equationally definable constant 1 is called *weakly regular* if $\Theta, \Phi \in \text{Con } \mathbf{A}$ and $[1]\Theta = [1]\Phi$ imply $\Theta = \Phi$. A variety is called *3-permutable* or *congruence distributive* if every of its members has the respective property. A variety with an equationally definable constant 1 is called *weakly regular* if every of its members has this property. A variety is 3-permutable (cf. [8], Theorem 3.1.18) if and only if there exist ternary terms t_1, t_2 satisfying

$$\begin{aligned} t_1(x, y, y) &\approx x, \\ t_1(x, x, y) &\approx t_2(x, y, y), \\ t_2(x, x, y) &\approx y. \end{aligned}$$

A variety is congruence distributive (cf. [8], Theorem 3.2.2) if and only if there exists a positive integer n and there exist ternary terms $t_0, \dots, t_n$ (so-called *Jónsson terms*) satisfying

$$\begin{aligned} t_0(x, y, z) &\approx x, \\ t_i(x, x, y) &\approx t_{i+1}(x, x, y) \text{ for all even } i \in \{0, \dots, n-1\}, \\ t_i(x, y, y) &\approx t_{i+1}(x, y, y) \text{ for all odd } i \in \{1, \dots, n-1\}, \\ t_n(x, y, z) &\approx z, \\ t_i(x, y, x) &\approx x \text{ for all } i \in \{1, \dots, n-1\}. \end{aligned}$$

A variety with an equationally definable constant 1 is weakly regular (cf. [8], Theorem 6.4.3) if and only if there exists a positive integer n and there exist binary terms $t_1, \dots, t_n$ such that $t_1(x, y) = \dots = t_n(x, y) = 1$ is equivalent to $x = y$.

Theorem 12. *The variety $\mathcal{V}$ of I-algebras is 3-permutable, congruence distributive and weakly regular.*

Proof. Let $\mathbf{A} = (A, \vee, r, \rightarrow, 1)$ be an I-algebra and $\mathbb{J}(\mathbf{A}) = (A, \vee, \wedge, \rightarrow, 1)$. If we put

$$\begin{aligned} t_1(x, y, z) &:= r(z, y \rightarrow x, x), \\ t_2(x, y, z) &:= r(x, y \rightarrow z, z) \end{aligned}$$

then

$$\begin{aligned} t_1(x, y, z) &\approx (z \vee x) \wedge ((y \rightarrow x) \vee x) \approx (z \vee x) \wedge (y \rightarrow x) \text{ by (1),} \\ t_2(x, y, z) &\approx (x \vee z) \wedge ((y \rightarrow z) \vee z) \approx (x \vee z) \wedge (y \rightarrow z) \text{ by (1),} \end{aligned}$$

$$\begin{aligned}
t_1(x, y, y) &\approx (y \vee x) \wedge (y \rightarrow x) \approx x \text{ by (2),} \\
t_1(x, x, y) &\approx (y \vee x) \wedge (x \rightarrow x) \approx (x \vee y) \wedge 1 \approx (x \vee y) \wedge (y \rightarrow y) \approx t_2(x, y, y) \text{ by (5),} \\
t_2(x, x, y) &\approx (x \vee y) \wedge (x \rightarrow y) \approx y \text{ by (2)}
\end{aligned}$$

and hence $\mathcal{V}$ is 3-permutable.

If we put

$$\begin{aligned}
t_0(x, y, z) &:= x, \\
t_1(x, y, z) &:= r(z, y, x), \\
t_2(x, y, z) &:= r(x, y \rightarrow z, z), \\
t_3(x, y, z) &:= z
\end{aligned}$$

then

$$\begin{aligned}
t_1(x, y, z) &\approx (z \vee x) \wedge (y \vee x), \\
t_2(x, y, z) &\approx (x \vee z) \wedge ((y \rightarrow z) \vee z) \approx (x \vee z) \wedge (y \rightarrow z) \text{ by (1),} \\
t_0(x, x, y) &\approx x \approx (x \vee y) \wedge x \approx (y \vee x) \wedge (x \vee x) \approx t_1(x, x, y), \\
t_1(x, y, y) &\approx (y \vee x) \wedge (y \vee x) \approx x \vee y \approx (x \vee y) \wedge 1 \approx (x \vee y) \wedge (y \rightarrow y) \approx t_2(x, y, y) \\
&\quad \text{by (5),} \\
t_2(x, x, y) &\approx (x \vee y) \wedge (x \rightarrow y) \approx y \approx t_3(x, x, y) \text{ by (2),} \\
t_1(x, y, x) &\approx (x \vee x) \wedge (y \vee x) \approx x \wedge (x \vee y) \approx x, \\
t_2(x, y, x) &\approx (x \vee x) \wedge (y \rightarrow x) \approx x \wedge (y \rightarrow x) \approx x \text{ by (1)}
\end{aligned}$$

and hence $\mathcal{V}$ is congruence distributive.

Finally, if we put

$$\begin{aligned}
t_1(x, y) &:= x \rightarrow y, \\
t_2(x, y) &:= y \rightarrow x
\end{aligned}$$

then $t_1(x, y) = t_2(x, y) = 1$ is equivalent to $x \leq y$ and $y \leq x$ according to Theorem 10 and (5), i.e. to $x = y$ and hence $\mathcal{V}$ is weakly regular. $\square$

6. ALGEBRAS ASSIGNED TO RELATIVELY RESIDUATED JOIN-SEMILATTICES

Similarly as for non-classical implication semilattices, we are going to show that also relatively residuated join-semilattices can be converted into algebras with everywhere defined operations by using of a certain ternary term $q(x, y, z)$ which replaces the partial operation $\odot$.

Definition 6. An R-algebra is an algebra $(R, \vee, q, \rightarrow, 1)$ of type $(2, 3, 2, 0)$ such that $(R, \vee, 1)$ is a join-semilattice with top element 1 and the following conditions are satisfied for all $x, y, z, u \in R$:

$$(20) \quad z \leq q(x, y, z),$$

- (21) $q(z \vee u \vee x, z \vee u \vee y, z) \approx q(z \vee u \vee x, z \vee u \vee y, z \vee u)$,
- (22) $q(x, 1, x) \approx q(1, x, x) \approx x$,
- (23) $q(x, y, z) \approx q(y, x, z)$,
- (24) $q(q(x, y, u), z, u) \approx q(x, q(y, z, u), u)$,
- (25) $q(x, z, u) \leq q(x \vee y, z, u)$,
- (26) $x \vee z \leq y \rightarrow (q(x, y, z) \vee z)$,
- (27) $x \leq y \rightarrow x$,
- (28) $q(x, x \rightarrow y, y) \leq y$,
- (29) $q(x, y, z) \approx q(x \vee z, y \vee z, z)$,
- (30) $(x \vee y) \rightarrow y \approx x \rightarrow y$.

The following result is an immediate consequence of Definition 6.

Corollary 1. *The class $\mathcal{W}$ of $\mathbf{R}$ -algebras forms a variety.*

Considering a relatively residuated join-semilattice as introduced in Definition 4, we can define a ternary operation $q(x, y, z)$ as follows: $q(x, y, z) := (x \vee z) \odot (y \vee z)$. For every relatively residuated join-semilattice $\mathbf{R} = (R, \vee, \odot, \rightarrow, 1)$ put $\mathbb{B}(\mathbf{R}) := (R, \vee, q, \rightarrow, 1)$.

Theorem 13. *Let $\mathbf{R} = (R, \vee, \odot, \rightarrow, 1)$ be a relatively residuated join-semilattice. Then $\mathbb{B}(\mathbf{R})$ is an $\mathbf{R}$ -algebra.*

Proof. Let $\mathbb{B}(\mathbf{R}) = (R, \vee, q, \rightarrow, 1)$ and $a, b, c, d \in R$. We have

- (20) $c \leq (a \vee c) \odot (b \vee c) = q(a, b, c)$.
- (21) $q(c \vee d \vee a, c \vee d \vee b, c) = (c \vee d \vee a) \odot (c \vee d \vee b) = q(c \vee d \vee a, c \vee d \vee b, c \vee d)$.
- (22) $q(a, 1, a) = (a \vee a) \odot (1 \vee a) = a \odot 1 = a$ and $q(1, a, a) = (1 \vee a) \odot (a \vee a) = 1 \odot a = a$.
- (23) $(a, b, c) = (a \vee c) \odot (b \vee c) = (b \vee c) \odot (a \vee c) = q(b, a, c)$.
- (24) $q(q(a, b, d), c, d) = ((a \vee d) \odot (b \vee d)) \vee d \odot (c \vee d) = ((a \vee d) \odot (b \vee d)) \odot (c \vee d) = (a \vee d) \odot ((b \vee d) \odot (c \vee d)) = (a \vee d) \odot ((b \vee d) \odot (c \vee d)) \vee d = q(a, q(b, c, d), d)$.
- (25) $q(a, c, d) = (a \vee d) \odot (c \vee d) \leq ((a \vee b) \vee d) \odot (c \vee d) = q(a \vee b, c, d)$.
- (26) According to (17) we have

$$a \vee c \leq b \rightarrow ((a \vee c) \odot (b \vee c)) \vee c = b \rightarrow (q(a, b, c) \vee c).$$

(27) coincides with (18).

(28) According to (18) and (19) we have

$$q(a, a \rightarrow b, b) = (a \vee b) \odot ((a \rightarrow b) \vee b) = (a \vee b) \odot (a \rightarrow b) \leq b.$$

(29) $q(a, b, c) = (a \vee c) \odot (b \vee c) = ((a \vee c) \vee c) \odot ((b \vee c) \vee c) = q(a \vee c, b \vee c, c)$.

(30) coincides with (16).

□

For every $\mathbf{R}$ -algebra $\mathbf{R} = (R, \vee, q, \rightarrow, 1)$ put $\mathbb{Q}(\mathbf{R}) := (R, \vee, \odot, \rightarrow, 1)$ where $x \odot y := q(x, y, z)$ for all $x, y, z \in R$ with $z \leq x, y$. That $\mathbb{Q}(\mathbf{R})$ is well-defined follows from the next theorem.

Theorem 14. *Let $\mathbf{R} = (R, \vee, q, \rightarrow, 1)$ be an $\mathbf{R}$ -algebra. Then $\mathbb{Q}(\mathbf{R})$ is a well-defined relatively residuated join-semilattice.*

Proof. Let $\mathbb{Q}(\mathbf{R}) = (R, \vee, \odot, \rightarrow, 1)$ and $a, b, c, d \in R$. Then $\odot$ is well-defined since in case $c, d \leq a, b$ we have

$$\begin{aligned} q(a, b, c) &= q(a \vee c \vee d, b \vee c \vee d, c) = q(a \vee c \vee d, b \vee c \vee d, c \vee d) \\ &= q(a \vee c \vee d, b \vee c \vee d, d) = q(a, b, d) \end{aligned}$$

by (21). Then because of (20), $c \leq q(a, b, c) = a \odot b$ whenever $c \leq a, b$.

(10) follows from the definition of an $\mathbf{R}$ -algebra.

(11) $a \odot 1 = q(a, 1, a) = a$ and $1 \odot a = q(1, a, a) = a$ by (22).

(12) Because of (23) we have $a \odot b = q(a, b, c) = q(b, a, c) = b \odot a$ whenever $c \leq a, b$.

(13) Because of (24) we have $(a \odot b) \odot c = q(q(a, b, d), c, d) = q(a, q(b, c, d), d) = a \odot (b \odot c)$ whenever $d \leq a, b, c$.

(14) Because of (25) we have $a \odot c = q(a, c, d) \leq q(a \vee b, c, d) = q(b, c, d) = b \odot c$ whenever $d \leq a \leq b$ and $d \leq c$.

(16) coincides with (30).

(17) Because of (26) and (29) we have

$$\begin{aligned} a \vee c \leq b \rightarrow (q(a, b, c) \vee c) &= b \rightarrow (q(a \vee c, b \vee c, c) \vee c) \\ &= b \rightarrow ((a \vee c) \odot (b \vee c)) \vee c. \end{aligned}$$

(18) coincides with (27).

(19) Because of (27), (29) and (28) we have

$$\begin{aligned} (a \vee b) \odot (a \rightarrow b) &= (a \vee b) \odot ((a \rightarrow b) \vee b) = q(a \vee b, (a \rightarrow b) \vee b, b) \\ &= q(a, a \rightarrow b, b) \leq b. \end{aligned}$$

□

Theorem 15. *The correspondence described in Theorems 13 and 14 is one-to-one.*

Proof. If $\mathbf{R} = (R, \vee, \odot, \rightarrow, 1)$ is a relatively residuated join-semilattice and

$$\mathbb{B}(\mathbf{R}) = (R, \vee, q, \rightarrow, 1),$$

$$\mathbb{Q}(\mathbb{B}(\mathbf{R})) = (R, \vee, \otimes, \rightarrow, 1)$$

then $x \otimes y = q(x, y, z) = (x \vee z) \odot (y \vee z) = x \odot y$ for all $x, y, z \in R$ with $z \leq x, y$ and hence $\mathbb{Q}(\mathbb{B}(\mathbf{R})) = \mathbf{R}$. If, conversely, $\mathbf{R} = (R, \vee, q, \rightarrow, 1)$ is an $\mathbf{R}$ -algebra and

$$\mathbb{Q}(\mathbf{R}) = (R, \vee, \odot, \rightarrow, 1),$$

$$\mathbb{B}(\mathbb{Q}(\mathbf{R})) = (R, \vee, \mathcal{Q}, \rightarrow, 1)$$

then $\mathcal{Q}(x, y, z) \approx (x \vee z) \odot (y \vee z) \approx q(x \vee z, y \vee z, z) \approx q(x, y, z)$ according to (29) and hence $\mathbb{B}(\mathbb{Q}(\mathbf{R})) = \mathbf{R}$. $\square$

By Theorems 13, 14 and 15 the variety $\mathcal{W}$ of $\mathbf{R}$ -algebras is equivalent to the class of relatively residuated join-semilattices.

Theorem 16. *The variety $\mathcal{W}$ of $\mathbf{R}$ -algebras is weakly regular and its subvariety determined by the additional identity $q(x, x \rightarrow y, y) \approx y$ is 3-permutable.*

Proof. The proof is analogous to that of Theorem 12 and uses also (i) of Theorem 4. Since $y \leq x \rightarrow y$ according to (18), the identity $q(x, x \rightarrow y, y) \approx y$ is equivalent to the identity

$$(x \vee y) \odot (x \rightarrow y) \approx y$$

which is nothing else than divisibility of the corresponding relatively residuated join-semilattice. $\square$

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YAKUBOVICH'S THEOREM FOR IMPULSIVE DIFFERENTIAL EQUATIONS WITH PIECEWISE CONSTANT ARGUMENT OF GENERALIZED TYPE

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Abstract. The main goal of our paper is to obtain sufficient conditions for the asymptotic equivalence of a linear differential system and a quasilinear system of impulsive differential equations with piecewise constant argument of generalized type, in short IDEPCAG. A deviating argument is of the advanced and delayed type.

As an auxiliary result, the structure of the set of solutions of the quasilinear system is specified. Several examples are also given to show the feasibility of results.

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Keywords: impulsive differential equations, piecewise constant argument of generalized type, equivalence, asymptotic behavior, Gronwall's inequality

1. INTRODUCTION

The problem of asymptotic equivalence between two differential (or difference) systems is one of the most important part in the study of asymptotic property for solutions of differential (or difference) systems, and it shows an asymptotic relationship between solutions of differential (or difference) systems.

If we know that two systems are asymptotically equivalent, and if we can understand the asymptotic behavior of the solutions of one of them, then we will get information about the asymptotic behavior of the solutions of the other system. Apparently, the first results regarding the asymptotic behaviour of systems on the basis of one-to-one correspondence between sets of solutions were obtained in [13, 17, 21], see also [12, 25].

The purpose of this paper is to generalize well-known theorems of Yakubovich, valid for asymptotically equivalent of solutions of linear ODE system and quasilinear ODE system, to a linear ODE system and a perturbed system of impulsive differential equations with piecewise alternately advanced and retarded argument of generalized type.

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Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of all natural, real and complex numbers, respectively. Denote by $|\cdot|$ the Euclidean norm in $\mathbb{C}^n$, $n \in \mathbb{N}$ and $\mathbb{R}^+ = [0, \infty)$. Fix two real valued non-negative sequences $\{t_i\}$, $\{\gamma_i\}$, $i \in \mathbb{N}$ such that $t_i < t_{i+1}$, $t_i \leq \gamma_i \leq t_{i+1}$ and $t_i \rightarrow \infty$ as $i \rightarrow \infty$. Let $\gamma: \mathbb{R}^+ \rightarrow \mathbb{R}^+$, be a given general step function $\gamma|_{I_i} = \gamma_i$, $I_i = [t_i, t_{i+1})$, $I_{-1} = [0, t_0]$, $\gamma(s) = s$ for s in I_{-1} and $\mathbb{R}^+ = \bigcup_{i=-1}^{\infty} I_i$. In this case we speak of differential equations with piecewise constant arguments of generalized type, in short DEPCAG. Theory and practice of the DEPCAGs have been discussed extensively in [1, 4, 8–11, 22].

The theory of impulsive differential equations developed rapidly, in recent years. This development, in particular, is due to the fact that many phenomena and processes arise in a variety of real world applications such as in the study of theoretical physics, population dynamics, biological systems, mechanical systems, and control theory. Furthermore, moment problem approaches appear also as a natural instrument in control theory of neutral type systems. For a good account on these theories, which have seen a significant development over the past decades we refer the interested reader to the monographs [16, 26], the papers [14, 15, 18–20] and the references cited therein.

To the best of our knowledge, there are only a few papers involving the DEPCAG with impulsive effects (IDEPCAG) [3, 5–7] to investigate the problem of asymptotic equivalence between the solutions of two differential systems.

In 2008, M. U. Akhmet [1] demonstrated the asymptotic equivalence of solutions of the linear ODE system and the DEPCAG system

$$\begin{aligned}x'(t) &= Cx(t), \\z'(t) &= Cz(t) + f(t, z(t), z(\gamma(t))),\end{aligned}$$

where $x, z \in \mathbb{R}^n$, $t \in \mathbb{R}$, C is a constant $n \times n$ real valued matrix and $f \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n)$ is a real valued $n \times 1$ function, $\gamma(t) = \gamma_i$, $t \in [t_i, t_{i+1})$, $i \in \mathbb{N}$.

In 2009, M. Pinto [22] proved the asymptotic equivalence of solutions of the linear ODE system and the following DEPCAG system

$$\begin{aligned}x'(t) &= A(t)x(t), \\z'(t) &= A(t)z(t) + f(t, z(t), z(\gamma(t))),\end{aligned}$$

where $x, z \in \mathbb{C}^n$, $A = A(t)$ is a $n \times n$ complex matrix on $\mathbb{R}^+$, $f: \mathbb{R}^+ \times \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a continuous function and $\gamma(t) = \gamma_i$, $t \in [t_i, t_{i+1})$, if $i \in \mathbb{N}$.

In the present paper, we consider the following linear differential system

$$x'(t) = Ax(t) \tag{1.1}$$

and the linear perturbed IDEPCAG system

$$y'(t) = Ay(t) + f(t, y(t), y(\gamma(t))), \quad t \neq t_k, \tag{1.2a}$$

$$\Delta y|_{t=t_k} = \mathfrak{J}_k(y(t_k^-)), \quad k \in \mathbb{N}, \tag{1.2b}$$

where $x, y \in \mathbb{C}^n$, $t \in \mathbb{R}^+$, A is a constant $n \times n$ complex valued matrix and $f: \mathbb{R}^+ \times \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a continuous function and

$$\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-), \quad y(t_k^+) = \lim_{t \rightarrow t_k^+} y(t) \quad \text{and} \quad y(t_k^-) = \lim_{t \rightarrow t_k^-} y(t).$$

We assume the following hypotheses.

($\mathfrak{L}_1$) There exist two positive locally integrable functions η_1 and η_2 on $\mathbb{R}^+$ such that

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq \eta_1(t)|x_1 - x_2| + \eta_2(t)|y_1 - y_2|$$

for all $t \in \mathbb{R}^+$, $x_1, x_2, y_1, y_2 \in \mathbb{C}^n$ and $f(t, 0, 0) = 0$.

($\mathfrak{L}_2$) The impulsive operator $\mathfrak{J}_k$, $k \in \mathbb{N}$, satisfies

$$|\mathfrak{J}_k(u) - \mathfrak{J}_k(v)| \leq \mathfrak{L}_k |u - v|$$

for the positive constant $\mathfrak{L}_k$, for all $u, v \in \mathbb{C}^n$ and $\mathfrak{J}_k(0) = 0$.

($\mathfrak{D}$) Let $\rho_i^+(A) = \exp(A \cdot (\gamma(t_i) - t_i))$, $\rho_i^-(A) = \exp(A \cdot (t_{i+1} - \gamma(t_i)))$, $\rho_i(A) = \rho_i^+(A)\rho_i^-(A)$, $i \in \mathbb{N}$. Suppose

$$\rho(A) = \sup_{i \in \mathbb{N}} \rho_i(A) < \infty$$

and

$$\rho = \rho(A) \sup_{i \in \mathbb{N}} \left(\int_{t_i}^{\gamma_i} [\eta_1(s) + \eta_2(s)] ds \right) < 1.$$

To investigate the asymptotic properties of the solutions, the following definitions can be efficiently applied.

Definition 1. A function y is a solution of the IDEPCAG system (1.2a)–(1.2b) in $[\tau, \infty)$ if

- (i) $y(t)$ is continuous for $t \in [\tau, \infty)$ with the possible exception of the points $t = t_k$, $k \in \mathbb{N}$.
- (ii) $y(t)$ is right continuous and has left-hand limits at the points $t = t_k$, $k \in \mathbb{N}$.
- (iii) $y(t)$ is differentiable and satisfies (1.2a) for any $t \in [\tau, \infty)$, with the possible exception of the points $t = t_k$, $k \in \mathbb{N}$, where one-sided derivatives exist.
- (iv) $y(t_k)$ satisfies (1.2b), $k \in \mathbb{N}$.

Definition 2. ([21, Page 288]) Systems (1.1) and (1.2a)–(1.2b) will be called equivalent if there exists a homeomorphism between the sets of solutions x and y , and systems (1.1) and (1.2a)–(1.2b) will be called asymptotically equivalent if, in addition, $x(t) - y(t) \rightarrow 0$ as $t \rightarrow \infty$.

The main purpose of this paper is to obtain a theorem of asymptotic equivalence of the stable solutions of the ODE system (1.1) and the IDEPCAG system (1.2a)–(1.2b) by using the IDEPCAG's integral inequalities of Gronwall type and the method of investigation introduced by K.-S. Chiu [2] (2013). We also remark that our results extend their asymptotic formulae given by M. U. Akhmet [1] (2008). See Remark 4.

The paper is organized as follows. In the next section, we focus on some preliminary results which will be used to obtain the criteria for the existence and uniqueness of the solutions of the IDEPCAG system (1.2a)–(1.2b). Here, the IDEPCAG Gronwall-type inequality is very useful. The third section is devoted to the problem of the asymptotic equivalence of the linear ODE system (1.1) and the IDEPCAG system (1.2a)–(1.2b). The fourth section, two examples are given to illustrate the validity of our results.

2. INTEGRAL EQUATIONS AND GRONWALL INTEGRAL INEQUALITY

To study the nonlinear IDEPCAG system, we will use the approach based on the construction of an equivalent integral equation. Let us give the following proposition. We omit the proof of this assertion, since it can be proved in the same way as Proposition 2 in [2].

For the sake of convenience, we adopt the following notation. For every $t \in \mathbb{R}$, let $i = i(t) \in \mathbb{N}$ be the unique integer such that $t \in I_i = [t_i, t_{i+1})$.

Proposition 2.1. *For any $(\tau, y_0) \in \mathbb{R}^+ \times \mathbb{C}^n$ the solution $y(t) = y(t, \tau, y_0)$ of the IDEPCAG system (1.2a)–(1.2b) is defined on $[\tau, \infty)$ and given by*

$$y(t) = \begin{cases} e^{A(t-\tau)}y_0 + \int_{\tau}^t e^{A(t-s)}f(s, y(s), y(\gamma(s)))ds \\ \quad + \sum_{k=i(\tau)+1}^{i(t)} e^{A(t-t_k)}\mathfrak{J}_k(y(t_k^-)), & i(t) > i(\tau), \\ e^{A(t-\tau)}y_0 + \int_{\tau}^t e^{A(t-s)}f(s, y(s), y(\gamma_{i(\tau)}))ds, & i(t) = i(\tau). \end{cases}$$

In the next, we give the following lemma about IDEPCAG integral inequality of Gronwall type, which is one of the most important auxiliary results of the present paper. The proof of the assertion is similar to that of Lemma 3 in K.-S. Chiu [2] (2013).

Lemma 2.1 (IDEPCAG's Gronwall Inequality). *Let $v, \alpha_j: \mathbb{R}^+ \rightarrow \mathbb{R}^+$, $j = 1, 2$, be three functions such that v is continuous with possible points of discontinuity of the first kind at $t = t_k$, $k \in \mathbb{N}$ and α_j are locally integrable, for which the inequality satisfying*

$$v(t) \leq v(\tau) + \left| \int_{\tau}^t [\alpha_1(s)v(s) + \alpha_2(s)v(\gamma(s))] ds \right| + \sum_{k=i(\tau)+1}^{i(t)} \rho_k v(t_k^-),$$

where $i(t) > i(\tau)$ and ρ_k are non-negative constants. Then for $t \geq \tau$,

$$v(t) \leq v(\tau) \left\{ \prod_{k=i(\tau)+1}^{i(t)} (1 + \rho_k) \right\} \exp \left(\int_{\tau}^t \left[\alpha_1(s) + \frac{\alpha_2(s)}{1 - v} \right] ds \right) \quad (2.1)$$

and

$$v(\gamma(t)) \leq \frac{v(\tau)}{1-v} \left\{ \prod_{k=i(\tau)+1}^{i(t)} (1+\rho_k) \right\} \exp \left(\int_{\tau}^t \left[\alpha_1(s) + \frac{\alpha_2(s)}{1-v} \right] ds \right)$$

where

$$\sup_{i(\tau) \leq k} \int_{t_k}^{\gamma_k} [\alpha_1(s) + \alpha_2(s)] ds \leq v < 1.$$

Remark 1. Consider $\rho_k \equiv 0$ in the IDEPCAG's Gronwall Inequality (2.1), we can obtain the following DEPCAG's inequality of Gronwall type:

$$u(t) \leq u(\tau) \exp \left(\int_{\tau}^t \left(\alpha_1(s) + \frac{\alpha_2(s)}{1-v} \right) ds \right).$$

The following theorem provides the existence of a unique solution when the initial moment is an arbitrary positive real number τ . Proof of this affirmation is omitted, as it can be demonstrated in the same way as Theorem 3.1 in K.-S. Chiu [7] (2021).

Theorem 2.1. *Let us assume that the conditions $(\mathfrak{L}_1)$, $(\mathfrak{L}_2)$ and $(\mathfrak{D})$ are satisfied. Then, given an initial condition $(\tau, y_0) \in \mathbb{R} \times \mathbb{C}^n$, there exists a unique solution $y(\cdot) = y(\cdot, \tau, y_0)$ of the IDEPCAG system (1.2a)–(1.2b) in the sense of Definition 1 such that $y(\tau) = y_0$.*

3. ASYMPTOTIC EQUIVALENCE

In this section we establish sufficient conditions under which the linear ODE system (1.1) and the IDEPCAG system (1.2a)–(1.2b) are asymptotically equivalent. Following V. Yakubovich's theorem [21, 27] and Ráb [23, 24], as in [1, 22, 28], we do the use of analogical change of variable $y(t) = \exp(At)u(t)$. See Lemma 3.1. Along the lines we also derive a formula for asymptotic representation of solutions of the IDEPCAG system (1.2a)–(1.2b).

Consider the following nonlinear impulsive differential system with piecewise constant argument of generalized type

$$u'(t) = \hat{g}(t, u(t), u(\gamma(t))) := e^{-At} f(t, e^{At} u(t), e^{A\gamma(t)} u(\gamma(t))), \quad t \neq t_k, \quad (3.1a)$$

$$\Delta u|_{t=t_k} = \hat{\mathfrak{J}}_k(u(t_k^-)) := e^{-At_k} \mathfrak{J}_k(e^{At_k^-} u(t_k^-)), \quad k \in \mathbb{N}, \quad (3.1b)$$

where $u \in \mathbb{C}^n$, e^{At} is the fundamental matrix of solutions of the linear ODE system (1.1) and $f: \mathbb{R}^+ \times \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}^n$ satisfies the Lipschitz condition $(\mathfrak{L}_1)$ and the impulsive operator $\mathfrak{J}_k$, $k \in \mathbb{N}$, satisfies the Lipschitz condition $(\mathfrak{L}_2)$.

Assume the following hypotheses.

($\mathfrak{H}_1$) The two integrable functions $\hat{\eta}_1, \hat{\eta}_2$ are defined on $\mathbb{R}^+$, where

$$\hat{\eta}_1(t) = |e^{-At}| |e^{At} \eta_1(t)|, \quad \hat{\eta}_2(t) = |e^{-At}| |e^{A\gamma(t)} \eta_2(t)|.$$

($\mathfrak{H}_2$) There exists a summable sequence of non-negative real numbers $(\mathfrak{L}_k)_{k=1}^\infty$ such that

$$|e^{-At_k}| |e^{At_k^-}| \mathfrak{L}_k \in l^1(\mathbb{R}^+).$$

($\mathfrak{H}_3$) There exist positive constants κ_1, κ_2 such that

$$|e^{At}| \leq \kappa_1 t^{m_+-1} e^{\mu_+ t} \quad \text{and} \quad |e^{-At}| \leq \kappa_2 t^{m_- -1} e^{-\mu_- t} \quad \text{for all } t \in \mathbb{R}^+,$$

where $\mu_- = \min_{1 \leq k \leq n} \Re \lambda_k$, $\mu_+ = \max_{1 \leq k \leq n} \Re \lambda_k$ ($\Re \lambda_k$ is the real part of the eigenvalue λ_k of the matrix A) and $m_\pm$ are the maxima of the degrees of elementary divisors of the matrix A corresponding to eigenvalues λ with $\Re \lambda$ equal to $\mu_\pm$, respectively.

Let

$$\alpha(t) = \exp \left(\int_\tau^t \left[\hat{\eta}_1(s) + \frac{\hat{\eta}_2(s)}{1-\bar{v}} \right] ds \right) \left\{ \exp \left(\int_t^\infty \left[\hat{\eta}_1(s) + \frac{\hat{\eta}_2(s)}{1-\bar{v}} \right] ds \right) - 1 \right\},$$

where

$$\bar{v} = \sup_{i \in \mathbb{N}} \int_{t_i}^{\gamma_i} [\hat{\eta}_1(s) + \hat{\eta}_2(s)] ds < 1 \quad (3.2)$$

is the smallness condition for the IDEPCAG system (3.1a)–(3.1b). From the Lipschitz condition ($\mathfrak{L}_1$), the function $\hat{g}$ satisfies

$$|\hat{g}(t, u_1, v_1) - \hat{g}(t, u_2, v_2)| \leq \hat{\eta}_1(t) |u_1 - u_2| + \hat{\eta}_2(t) |v_1 - v_2| \quad (3.3)$$

where $t \in \mathbb{R}^+$, $u_1, u_2, v_1, v_2 \in \mathbb{C}^n$ and $\hat{g}(t, 0, 0) = f(t, 0, 0) = 0$.

From the Lipschitz condition ($\mathfrak{L}_2$), the function $\hat{\mathfrak{J}}_k$ satisfies

$$|\hat{\mathfrak{J}}_k(u) - \hat{\mathfrak{J}}_k(v)| \leq \hat{\mathfrak{L}}_k |u - v| \quad (3.4)$$

where $k \in \mathbb{N}$, $u, v \in \mathbb{C}^n$, $\hat{\mathfrak{L}}_k := |e^{-At_k}| |e^{At_k^-}| \mathfrak{L}_k$ and $\hat{\mathfrak{J}}_k(0) = \mathfrak{J}_k(0) = 0$.

The following lemma can be easily proved by direct substitution.

Lemma 3.1. *If $y(t)$ is a solution of the IDEPCAG system (1.2a)–(1.2b), then there is a solution $u(t)$ of the IDEPCAG system (3.1a)–(3.1b) such that*

$$y(t) = e^{At} u(t). \quad (3.5)$$

Conversely, if $u(t)$ is a solution of the IDEPCAG system (3.1a)–(3.1b), then $y(t)$ in (3.5) is a solution of the IDEPCAG system (1.2a)–(1.2b).

Theorem 3.1. *Under the conditions ($\mathfrak{H}_1$), ($\mathfrak{H}_2$), (3.2), (3.3) and (3.4), all the solutions u of the IDEPCAG system (3.1a)–(3.1b) are defined on $I_\tau = [\tau, \infty)$ and convergent to some $u_\infty \in \mathbb{C}^n$ as $t \rightarrow \infty$*

$$u(t) = u_\infty + |u(\tau)| 0 \left(\left\{ \exp \left(\int_t^\infty \left[\hat{\eta}_1(s) + \frac{\hat{\eta}_2(s)}{1-\bar{v}} \right] ds \right) - 1 \right\} + \sum_{k=i(t)+1}^\infty \hat{\mathfrak{L}}_k \right). \quad (3.6)$$

Conversely, for any $u_\infty \in \mathbb{C}^n$, there exists a unique solution u of the IDEPCAG system (3.1a)–(3.1b) defined on $I_\tau = [\tau, \infty)$ for τ big enough such that $u(t) \rightarrow u_\infty$ as $t \rightarrow \infty$ and (3.6) holds. Furthermore, the correspondence $u \rightarrow u_\infty$ is an asymptotic equivalence.

Proof. Let $u(t) = u(t, \tau, u_0)$ denote a solution of the nonlinear IDEPCAG system (3.1a)–(3.1b) satisfying $u(\tau) = u_0$. By the smallness condition (3.2) and using the same technique of Theorem 2.1, the solution

$$\begin{aligned} u(t) &= u_0 + \int_\tau^t e^{-As} f(s, e^{As} u(s), e^{A\gamma(s)} u(\gamma(s))) ds + \sum_{k=i(\tau)+1}^{i(t)} e^{-At_k} \mathfrak{J}_k(e^{At_k^-} u(t_k^-)) \\ &= u_0 + \int_\tau^t \hat{g}(s, u(s), u(\gamma(s))) ds + \sum_{k=i(\tau)+1}^{i(t)} \hat{\mathfrak{J}}_k(u(t_k^-)), \quad i(t) > i(\tau), \end{aligned}$$

exists on $I_\tau = [\tau, \infty)$ and is unique. By using (3.3) and (3.4), $\hat{g}(t, 0, 0) = 0$ and $\hat{\mathfrak{J}}_k(0) = 0$, we have

$$|u(t)| \leq |u_0| + \int_\tau^t [\hat{\eta}_1(s) |u(s)| + \hat{\eta}_2(s) |u(\gamma(s))|] ds + \sum_{k=i(\tau)+1}^{i(t)} \hat{\mathfrak{L}}_k |u(t_k^-)|, \quad i(t) > i(\tau).$$

By virtue of the IDEPCAG integral inequality of Gronwall type (Lemma 2.1), we obtain

$$|u(t)| \leq |u_0| \left\{ \prod_{\kappa=i(\tau)+1}^{i(t)} (1 + \hat{\mathfrak{L}}_\kappa) \right\} \exp \left(\int_\tau^t \left[\hat{\eta}_1(s) + \frac{\hat{\eta}_2(s)}{1 - \bar{\nu}} \right] ds \right)$$

and

$$|u(\gamma(t))| \leq \frac{1}{1 - \bar{\nu}} |u_0| \left\{ \prod_{\kappa=i(\tau)+1}^{i(t)} (1 + \hat{\mathfrak{L}}_\kappa) \right\} \exp \left(\int_\tau^t \left[\hat{\eta}_1(s) + \frac{\hat{\eta}_2(s)}{1 - \bar{\nu}} \right] ds \right)$$

where $\bar{\nu}$ is the smallness condition (3.2).

By (H₂), we have $\hat{\mathfrak{L}}_k \in l^1([\tau, \infty))$, then $\prod_{k \geq i(\tau)+1} (1 + \hat{\mathfrak{L}}_k) < \infty$. As $\hat{\eta}_1, \hat{\eta}_2 \in l^1([\tau, \infty))$ and $\prod_{k \geq i(\tau)+1} (1 + \hat{\mathfrak{L}}_k) < \infty$, we conclude that u is bounded on $\mathbb{R}^+$, i.e., $|u(t)| \leq M$, for all $t \in [\tau, \infty)$. Therefore

$$u_\infty = u_0 + \int_\tau^\infty \hat{g}(s, u(s), u(\gamma(s))) ds + \sum_{k=i(\tau)+1}^\infty \hat{\mathfrak{J}}_k(u(t_k^-)) \quad (3.7)$$

exists and

$$u(t) = u_\infty - \int_t^\infty \hat{g}(s, u(s), u(\gamma(s))) ds - \sum_{k=i(t)+1}^\infty \hat{\mathfrak{J}}_k(u(t_k^-)). \quad (3.8)$$

Moreover

$$\begin{aligned}
& \int_t^\infty |\hat{g}(s, u(s), u(\gamma(s)))| ds \\
& \leq |u_0| \int_t^\infty \left[\left(\hat{\eta}_1(s) + \frac{1}{1-\bar{v}} \hat{\eta}_2(s) \right) \exp \left(\int_\tau^s \left[\hat{\eta}_1(r) + \frac{1}{1-\bar{v}} \hat{\eta}_2(r) \right] dr \right) \right] ds \\
& = |u_0| \exp \left(\int_\tau^t \left[\hat{\eta}_1(s) + \frac{1}{1-\bar{v}} \hat{\eta}_2(s) \right] ds \right) \\
& \quad \times \left\{ \exp \left(\int_t^\infty \left[\hat{\eta}_1(s) + \frac{1}{1-\bar{v}} \hat{\eta}_2(s) \right] ds \right) - 1 \right\} \\
& = |u_0| 0 \left\{ \exp \left(\int_t^\infty \left[\hat{\eta}_1(s) + \frac{1}{1-\bar{v}} \hat{\eta}_2(s) \right] ds \right) - 1 \right\}, \quad \text{as } t \rightarrow \infty.
\end{aligned}$$

$$\begin{aligned}
& \sum_{k=i(t)+1}^\infty |\hat{\mathfrak{J}}_k(u(t_k^-))| \leq \sum_{k=i(t)+1}^\infty \hat{\mathfrak{L}}_k |u(t_k^-)| \leq M \sum_{k=i(t)+1}^\infty \hat{\mathfrak{L}}_k \\
& = |u_0| \cdot \left(\rho \sum_{k=i(t)+1}^\infty \hat{\mathfrak{L}}_k \right) = |u_0| 0 \left\{ \sum_{k=i(t)+1}^\infty \hat{\mathfrak{L}}_k \right\}, \quad \text{as } t \rightarrow \infty,
\end{aligned}$$

where $\rho = M/|u_0|$. Then the first conclusion and (3.6) follow.

For $u_0^i \in \mathbb{C}^n$, $i = 1, 2$ and $u_i(t) = u_i(t, \tau, u_0^i)$, we have

$$\begin{aligned}
|u_1(t) - u_2(t)| & \leq |u_0^1 - u_0^2| + \int_\tau^t |\hat{g}(s, u_1(s), u_1(\gamma(s))) - \hat{g}(s, u_2(s), u_2(\gamma(s)))| ds \\
& \quad + \sum_{k=i(\tau)+1}^{i(t)} |\hat{\mathfrak{J}}_k(u_1(t_k^-)) - \hat{\mathfrak{J}}_k(u_2(t_k^-))| \\
& \leq |u_0^1 - u_0^2| \\
& \quad + \int_\tau^t [\hat{\eta}_1(s) |u_1(s) - u_2(s)| + \hat{\eta}_2(s) |u_1(\gamma(s)) - u_2(\gamma(s))|] ds \\
& \quad + \sum_{k=i(\tau)+1}^{i(t)} \hat{\mathfrak{L}}_k |u_1(t_k^-) - u_2(t_k^-)|.
\end{aligned}$$

From the IDEPCAG integral inequality of Gronwall type (Lemma 2.1), for $i(t) > i(\tau)$

$$|u_1(t) - u_2(t)| \leq |u_0^1 - u_0^2| \left\{ \prod_{\kappa=i(\tau)+1}^{i(t)} (1 + \hat{\mathfrak{L}}_\kappa) \right\} \exp \left(\int_\tau^t \left[\hat{\eta}_1(s) + \frac{\hat{\eta}_2(s)}{1-\bar{v}} \right] ds \right)$$

and

$$|u_1(\gamma(t)) - u_2(\gamma(t))| \leq \frac{1}{1-\bar{v}} |u_0^1 - u_0^2| \left\{ \prod_{\kappa=i(\tau)+1}^{i(t)} (1 + \hat{\mathfrak{L}}_\kappa) \right\}$$

$$\times \exp \left(\int_{\tau}^t \left[\hat{\eta}_1(s) + \frac{\hat{\eta}_2(s)}{1 - \bar{v}} \right] ds \right).$$

Furthermore, as in (3.6), by (3.7), it is not difficult to obtain

$$\begin{aligned} \left(1 - \alpha(\tau) - M \sum_{k=i(\tau)+1}^{\infty} \hat{\mathfrak{L}}_k \right) |u_0^1 - u_0^2| &\leq |u_{\infty}^1 - u_{\infty}^2| \\ &\leq |u_0^1 - u_0^2| \left(1 + \alpha(\tau) + M \sum_{k=i(\tau)+1}^{\infty} \hat{\mathfrak{L}}_k \right), \end{aligned}$$

if $\alpha(\tau) + M \sum_{k=i(\tau)+1}^{\infty} \hat{\mathfrak{L}}_k < 1$. This establishes the equivalence of $u_0 \rightarrow u_{\infty}$. Finally, for any $u_{\infty} \in \mathbb{C}^n$, Eq. (3.8) has a unique solution u defined on $I_{\tau} = [\tau, \infty)$, if, for example, τ satisfies

$$\int_{\tau}^{\infty} \left[\hat{\eta}_1(s) + \frac{\hat{\eta}_2(s)}{1 - \bar{v}} \right] ds + M \sum_{k=i(\tau)+1}^{\infty} \hat{\mathfrak{L}}_k < 1.$$

Then the smallness condition (3.2) holds and hence the same technique of Theorem 2.1 is applicable to guarantee the uniqueness of solution u . This proves the asymptotic equivalence of the correspondence $u(t)$ and u_{∞} . $\square$

Corollary 3.1. *If conditions $(\mathfrak{H}_1)$, $(\mathfrak{H}_2)$, (3.2), (3.3) and (3.4) are valid, then every solution of the IDEPCAG system (3.1a)–(3.1b) is bounded on $I_{\tau} = [\tau, \infty)$ and for each solution u of the IDEPCAG system (3.1a)–(3.1b), there exists a constant vector u_{∞} such that $u(t) \rightarrow u_{\infty}$ as $t \rightarrow \infty$.*

Theorem 3.2. *If conditions $(\mathfrak{H}_1)$, $(\mathfrak{H}_2)$, (3.2), (3.3) and (3.4) are fulfilled, then each solution y of the IDEPCAG system (1.2a)–(1.2b) is defined on $I_{\tau} = [\tau, \infty)$. Furthermore, the linear ODE system (1.1) and the IDEPCAG system (1.2a)–(1.2b) are equivalent and we have the following asymptotic formula*

$$y(t) = e^{At} [v + \varepsilon(t)], \quad \text{as } t \rightarrow \infty, \quad (3.9)$$

where $v \in \mathbb{C}^n$ is a constant vector and the error function ε verifies

$$\begin{aligned} \varepsilon(t) = 0 \left\{ \left(\exp \left(\int_t^{\infty} \left[|e^{-As}| |e^{As}| \eta_1(s) + \frac{|e^{-As}| |e^{A \cdot \gamma(s)}| \eta_2(s)}{1 - \bar{v}} \right] ds \right) - 1 \right) \right. \\ \left. + \sum_{k=i(t)+1}^{\infty} |e^{-At_k}| |e^{At_k^-}| \mathfrak{L}_k \right\}. \end{aligned} \quad (3.10)$$

Moreover, if $\varepsilon_0(t) \rightarrow 0$ as $t \rightarrow \infty$, where

$$\varepsilon_0(t) = \int_t^{\infty} \left| e^{A \cdot (t-s)} \right| \left[|e^{As}| \eta_1(s) + |e^{A \cdot \gamma(s)}| \eta_2(s) \right] ds$$

$$+ \sum_{k=i(t)+1}^{\infty} \left| e^{A \cdot (t-t_k)} \right| \left| e^{A \cdot t_k^-} \right| \mathfrak{L}_k, \quad (3.11)$$

then the linear ODE system (1.1) and the IDEPCAG system (1.2a)–(1.2b) are asymptotically equivalent and we have the asymptotic formula

$$y(t) = x(t) + o(\varepsilon_0(t)), \quad \text{as } t \rightarrow \infty, \quad (3.12)$$

for some constant vector $v \in \mathbb{C}^n$.

Proof. By (3.1) and $y = e^{At}u$, we get

$$y(t) = e^{At} \left[u_{\infty} - \int_t^{\infty} \hat{g}(s, u(s), u(\gamma(s))) ds - \sum_{k=i(t)+1}^{\infty} \hat{\mathfrak{J}}_k(u(t_k^-)) \right], \quad (3.13)$$

where $u = u(t, \tau, u_0)$ is a solution of (3.1). Remark that in (3.13) $x(t) = e^{At}u_{\infty}$ is a solution of the linear ODE system (1.1) and $y = e^{At}u$ and so $y(\tau) = u_0$. By Theorem 3.1, the correspondence $u_0 \leftrightarrow u_{\infty}$ is a homeomorphism for some τ sufficiently large. Then, the correspondence $x \leftrightarrow y$ is also a homeomorphism and formulae (3.9) and (3.12) follow. $\square$

Remark 2. Suppose now that

$$|e^{At}| = o(\varphi_+(t)), \quad |e^{-At}| = o(\varphi_-(t)) \quad \text{as } t \rightarrow \infty \quad (3.14)$$

and

$$o(\gamma(t)) = o(t) \quad \text{as } t \rightarrow \infty. \quad (3.15)$$

Thus, by (3.14) and (3.15), the functions ε and ε_0 in (3.10) and (3.11) of Theorem 3.2 may be considered as

$$\begin{aligned} \varepsilon(t) = & 0 \left\{ \left(\exp \left[\int_t^{\infty} \varphi_-(s) \varphi_+(s) \left(\eta_1(s) + \frac{\eta_2(s)}{1-\bar{v}} \right) ds \right] - 1 \right) \right. \\ & \left. + \sum_{k=i(t)+1}^{\infty} \varphi_-(t_k) \varphi_+(t_k) \mathfrak{L}_k \right\} \end{aligned}$$

and

$$\varepsilon_0(t) = \int_t^{\infty} \varphi_-(s-t) \varphi_+(s) (\eta_1(s) + \eta_2(s)) ds + \sum_{k=i(t)+1}^{\infty} \varphi_-(t_k-t) \varphi_+(t_k) \mathfrak{L}_k.$$

This gives a practical way to apply Theorem 3.2.

Corollary 3.2. *If conditions $(\mathfrak{H}_1)$, $(\mathfrak{H}_2)$, (3.2), (3.3) and (3.4) are satisfied, then every solution y of the IDEPCAG system (1.2a)–(1.2b) possesses an asymptotic representation of the form*

$$y(t) = e^{At}[v + \varepsilon(t)], \quad \text{as } t \rightarrow \infty,$$

where $v \in \mathbb{C}^n$ is a constant vector and for a solution u of the IDEPCAG system (3.1a)–(3.1b).

$$\varepsilon(t) = - \int_t^\infty e^{-As} f(s, e^{As} u(s), e^{A\gamma(s)} u(\gamma(s))) ds - \sum_{k=i(t)+1}^\infty e^{-At_k} \mathfrak{J}_k(e^{At_k^-} u(t_k^-)).$$

Remark 3. Note that we can easily deduce the condition (3.2) for the following form

$$v = \sup_{i \in \mathbb{N}} \int_{t_i}^{\gamma_i} |e^{-As}| \left(\eta_1(s) |e^{As}| + \eta_2(s) |e^{A\gamma(s)}| \right) ds < 1. \quad (3.16)$$

By $(\mathfrak{H}_3)$, the condition (3.16) can be deduced by

$$v = \kappa_1 \kappa_2 \sup_{i \in \mathbb{N}} \int_{t_i}^{\gamma_i} s^{m_+ + m_- - 2} e^{(\mu_+ - \mu_-)s} \left(\eta_1(s) + \eta_2(s) \frac{\gamma(s)^{m_+ - 1} e^{\mu_+ \gamma(s)}}{s^{m_+ - 1} e^{\mu_+ s}} \right) ds < 1 \quad (3.17)$$

and we can find a positive number κ such that

$$\frac{\gamma(s)^{m_+ - 1} e^{\mu_+ \gamma(s)}}{s^{m_+ - 1} e^{\mu_+ s}} = \left(\frac{\gamma(s)}{s} \right)^{m_+ - 1} e^{\mu_+ (\gamma(s) - s)} \leq \sup_{i \in \mathbb{N}} \left(\frac{\gamma_i}{t_i} \right)^{m_+ - 1} e^{\mu_+ \sup_{j \in \mathbb{N}} (\gamma_j - t_j)} = \kappa.$$

Then the conditions $(\mathfrak{H}_1)$ and $(\mathfrak{H}_2)$ can be deduced by the following forms

$(\mathfrak{H}'_1)$ The two integrable functions η_1, η_2 are defined on $\mathbb{R}^+$ such that

$$t^{m_+ + m_- - 2} e^{(\mu_+ - \mu_-)t} (\eta_1(t) + \eta_2(t)) \in L_1(\mathbb{R}^+).$$

$(\mathfrak{H}'_2)$ There exists a summable sequence of non-negative real numbers $(\mathfrak{L}_k)_{k=1}^\infty$ such that

$$t_k^{m_+ + m_- - 2} e^{(\mu_+ - \mu_-)t_k} \mathfrak{L}_k \in l^1(\mathbb{R}^+).$$

Theorem 3.3. Under the conditions $(\mathfrak{H}'_1)$, $(\mathfrak{H}'_2)$, (3.2), (3.3) and (3.4), all the solutions u of the IDEPCAG system (3.1a)–(3.1b) are defined on $I_\tau = [\tau, \infty)$ and convergent to some $u_\infty \in \mathbb{C}^n$ as $t \rightarrow \infty$

$$u(t) = u_\infty + |u(\tau)| 0 \left(\left\{ \exp \left(\int_t^\infty \left[s^{m_+ + m_- - 2} e^{(\mu_+ - \mu_-)s} \left(\eta_1(s) + \frac{\kappa}{1 - v} \eta_2(s) \right) \right] ds \right) - 1 \right\} + \sum_{k=i(t)+1}^\infty t_k^{m_+ + m_- - 2} e^{(\mu_+ - \mu_-)t_k} \mathfrak{L}_k \right). \quad (3.18)$$

Conversely, for any $u_\infty \in \mathbb{C}^n$, there exists a unique solution u of the IDEPCAG system (3.1a)–(3.1b) defined on $I_\tau = [\tau, \infty)$ for τ big enough such that $u(t) \rightarrow u_\infty$ as $t \rightarrow \infty$ and (3.18) holds. Furthermore, the correspondence $u \rightarrow u_\infty$ is an asymptotic equivalence.

The proof follows from Lemma 3.1, Theorem 3.1 and Remark 3.

Theorem 3.4. *If the conditions $(\mathfrak{H}'_1)$, $(\mathfrak{H}'_2)$, (3.2), (3.3) and (3.4) are fulfilled, then each solution y of the IDEPCAG system (1.2a)–(1.2b) is defined on $I_\tau = [\tau, \infty)$. Furthermore, the linear ODE system (1.1) and the IDEPCAG system (1.2a)–(1.2b) are equivalent and we have the following asymptotic formula*

$$y(t) = e^{At}[c + \varepsilon(t)], \quad \text{as } t \rightarrow \infty,$$

where $c \in \mathbb{C}^n$ is a constant vector and the error function ε verifies

$$\begin{aligned} \varepsilon(t) = 0 \left\{ \left\langle \exp \left(\int_t^\infty \left[s^{m_+ + m_- - 2} e^{(\mu_+ - \mu_-) \cdot s} \left(\eta_1(s) + \frac{\kappa}{1 - \nu} \eta_2(s) \right) \right] ds \right) - 1 \right\rangle \right. \\ \left. + \sum_{k=i(t)+1}^\infty t_k^{m_+ + m_- - 2} e^{(\mu_+ - \mu_-) \cdot t_k} \mathfrak{L}_k \right\}. \end{aligned}$$

Moreover, if $\varepsilon_0(t) \rightarrow 0$ as $t \rightarrow \infty$, where

$$\begin{aligned} \varepsilon_0(t) = \int_t^\infty (s - t)^{m_- - 1} e^{-\mu_- \cdot (s - t)} s^{m_+ - 1} e^{\mu_+ \cdot s} (\eta_1(s) + \eta_2(s)) ds \\ + \sum_{k=i(t)+1}^\infty (t_k - t)^{m_- - 1} e^{-\mu_- \cdot (t_k - t)} t_k^{m_+ - 1} e^{\mu_+ \cdot t_k} \mathfrak{L}_k, \end{aligned} \quad (3.19)$$

then the linear ODE system (1.1) and the IDEPCAG system (1.2a)–(1.2b) are asymptotically equivalent and we have the asymptotic formula

$$y(t) = e^{At} c + o(\varepsilon_0(t)), \quad \text{as } t \rightarrow \infty, \quad (3.20)$$

for some constant vector $c \in \mathbb{C}^n$.

The proof follows from Lemma 3.1, Theorem 3.2 and Remark 3.

Remark 4. Let $\eta(s) = \max_{i=1,2} \eta_i(s)$ and assume that conditions $(\mathfrak{H}'_1)$, $(\mathfrak{H}'_2)$, (3.3), (3.4) and (3.17) are fulfilled. Moreover

$$\begin{aligned} \int_t^\infty (s - t)^{m_- - 1} e^{-\mu_- \cdot (s - t)} s^{m_+ - 1} e^{\mu_+ \cdot s} \eta(s) ds \\ + \sum_{k=i(t)+1}^\infty (t_k - t)^{m_- - 1} e^{-\mu_- \cdot (t_k - t)} t_k^{m_+ - 1} e^{\mu_+ \cdot t_k} \mathfrak{L}_k \rightarrow 0 \quad \text{as } t \rightarrow \infty, \end{aligned}$$

then the linear ODE system (1.1) and the IDEPCAG system (1.2a)–(1.2b) are asymptotically equivalent.

Let $\eta(s) \equiv \eta_i(s)$, $i = 1, 2$, and assume that conditions $(\mathfrak{H}'_1)$, (3.3) and (3.17) are fulfilled and

$$\int_t^\infty (s - t)^{m_- - 1} e^{-\mu_- \cdot (s - t)} s^{m_+ - 1} e^{\mu_+ \cdot s} \eta(s) ds \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (3.21)$$

Then the linear ODE system (1.1) and the DEPCAG system (1.2a) are asymptotically equivalent.

The previous result has been obtained in [1, Theorems 2.1 and 2.2], our condition (3.17) to guarantee the existence and uniqueness of the quasilinear DEPCAG system (1.2a) is less restrictive than (C3) mentioned in [1].

Remark 5. In [27], we can find Yakubovich's theorem on the asymptotic equivalence of the linear ODE system and quasilinear ODE system. The sufficient condition for asymptotic equivalence was

$$\int_t^\infty s^{m_++m_- - 2} e^{\mu_+ s} \eta(s) ds \rightarrow 0 \quad \text{as } t \rightarrow \infty, \quad (3.22)$$

where $\eta(s) = \max_{i=1,2} \eta_i(s)$.

We can easily to prove that the conditions $(\mathfrak{H}'_1)$ or (3.21) with $\mu_- > 0$ are less restrictive than (3.22) in Ref. [27].

Then Theorem 3.4 extends and improves some existing Yakubovich's results on asymptotic equivalence of different classes of linear and quasilinear differential systems studied in [21, 27]. Also, with this research we extend the results for the DEPCAG system in [1] and the DEPCAG system to the impulsive case (the IDEPCAG system).

4. EXAMPLES

As a direct application of Theorem 3.2 and Theorem 3.4, here we will present two particular cases.

Example 4.1. Consider the following scalar IDEPCAG

$$\begin{aligned} y'(t) &= ay + b(t)y(\gamma(t)), \quad t \neq t_k \\ \Delta y|_{t=t_k} &= \mathfrak{L}_k \cdot y(t_k^-), \quad k \in \mathbb{N}, \end{aligned} \quad (4.1)$$

with $a > 0$ is a constant, $b(t) \in L_1(\mathbb{R}^+)$, $\mathfrak{L}_k \in l^1(\mathbb{R}^+)$ and $\hat{b}(t) = 0(e^{-at})$ where

$$\hat{b}(t) = \int_t^\infty |b(s)| ds + \sum_{k=i(t)+1}^\infty \mathfrak{L}_k.$$

Then the IDEPCAG (4.1) is equivalent to the linear scalar ODE

$$x'(t) = ax(t) \quad (4.2)$$

and all solutions $y(t)$ of the IDEPCAG (4.1) have the asymptotic formula

$$y(t) = e^{at} (v + \varepsilon(t)), \quad \text{as } t \rightarrow \infty,$$

where $v \in \mathbb{R}$ and the error function ε verifies

$$\varepsilon(t) = 0 \left\{ \left\langle \exp \left(\int_t^\infty e^{a \cdot (\gamma(s) - s)} |b(s)| ds \right) - 1 \right\rangle + \sum_{k=i(t)+1}^\infty \mathfrak{L}_k \right\}.$$

Moreover, if $\varepsilon_0(t) \rightarrow 0$ as $t \rightarrow \infty$, where

$$\varepsilon_0(t) = 0 \left(\int_t^\infty \left[e^{a \cdot (t-s)} |b(s)| e^{a \cdot \gamma(s)} \right] ds + \sum_{k=i(t)+1}^\infty e^{at} \mathfrak{L}_k \right),$$

then, by Theorem 3.2, the linear scalar ODE (4.2) and the scalar IDEPCAG (4.1) are asymptotically equivalent and we have the asymptotic formula

$$y(t) = x(t) + o(\varepsilon_0(t)), \quad \text{as } t \rightarrow \infty.$$

Example 4.2. Consider the following impulsive second-order differential equations with piecewise alternately advanced and retarded argument:

$$\begin{aligned} y''(t) - 5y' + 6y &= \eta(t) \cos^2 \left(y \left(3 \left[\frac{t+1}{3} \right] \right) \right), \quad t \neq 3k-1, \\ \Delta y|_{t=t_k} &= \left(\frac{1}{4} \right)^{3k} \cdot y(3k-1^-), \\ \Delta y'|_{t=t_k} &= \left(\frac{1}{5} \right)^{3k} \cdot y'(3k-1^-), \quad k \in \mathbb{N}, \end{aligned} \quad (4.3)$$

where $[\cdot]$ is the greatest integer function, $\eta(t)$ is a continuous function defined on $\mathbb{R}^+$ and

$$|\eta(t)| < \frac{\kappa}{1+t^2} e^{-3t}, \quad \text{for } t \in \mathbb{R}^+,$$

where $\kappa \in \mathbb{R}^+$. According to the IDEPCA (4.3), we have $\gamma(t) = 3 \left[\frac{t+1}{3} \right]$, then $t_j = 3j-1$, $\gamma_j = 3j$ for all $j \in \mathbb{N}$.

Letting $z(\cdot) = (z_1(\cdot), z_2(\cdot))^T = (y(\cdot), y'(\cdot))^T$, we write the IDEPCA (4.3) in the system form

$$\begin{aligned} z'(t) &= \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} z(t) + \begin{pmatrix} 0 \\ \eta(t) \cos^2(z_1(3 \left[\frac{t+1}{3} \right])) \end{pmatrix}, \\ \Delta z|_{3k-1} &= \begin{pmatrix} \left(\frac{1}{4} \right)^{3k} & 0 \\ 0 & \left(\frac{1}{5} \right)^{3k} \end{pmatrix} z(3k-1^-), \end{aligned}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix}, \quad f \left(t, z(t), z \left(3 \left[\frac{t+1}{3} \right] \right) \right) = \begin{pmatrix} 0 \\ \eta(t) \cos^2(z_1(3 \left[\frac{t+1}{3} \right])) \end{pmatrix}.$$

We can easily verify that $\mu_- = 2$, $\mu_+ = 3$, $m_\pm = 1$ and the conditions $(\mathfrak{H}'_1)$, $(\mathfrak{H}'_2)$, (3.2), (3.3) and (3.4) are fulfilled. Moreover we can check that the condition (3.19) holds.

$$\begin{aligned} \int_t^\infty (s-t)^{m_- - 1} e^{-\mu_-(s-t)} s^{m_+ - 1} e^{\mu_+ s} (\eta(s)) ds &\leq \int_t^\infty e^{-2(s-t)} e^{3s} \frac{\kappa}{1+s^2} e^{-3s} ds \\ &\leq \frac{\kappa}{1+t^2} \xrightarrow{t \rightarrow \infty} 0, \end{aligned}$$

and

$$\sum_{k=i(t)+1}^{\infty} (t_k - t)^{m_- - 1} e^{-\mu_-(t_k - t)} t_k^{m_+ - 1} e^{\mu_+ t_k} \mathfrak{L}_k \leq \sum_{k=i(t)+1}^{\infty} e^{2t + (3k-1)} \left(\frac{1}{4}\right)^{3k} \xrightarrow{t \rightarrow \infty} 0.$$

Thus, by Theorem 3.4, the IDEPCA (4.3) is asymptotically equivalent to the following ODE equation

$$x''(t) - 5x' + 6x = 0$$

which has solutions

$$x(t) = c_1 e^{2t} + c_2 e^{3t}.$$

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GLOBAL EXISTENCE AND EXPONENTIAL DECAY OF SOLUTIONS FOR HIGHER-ORDER PARABOLIC EQUATION WITH LOGARITHMIC NONLINEARITY

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Abstract. This paper deals with the initial boundary value problem for a higher-order parabolic equation with logarithmic source term

$$u_t + (-\Delta)^m u = u^{r-2} u \ln |u|.$$

By employing the potential wells technique we show the global existence of the weak solution. Also, we obtain the exponential decay for the weak solutions.

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1. INTRODUCTION

In this article, we deal with the following higher-order parabolic equation with logarithmic source term

$$\begin{cases} u_t + Pu = u^{r-2} u \ln |u|, & x \in \Omega, \quad t > 0, \\ D^\gamma u(x, t) = 0, \quad |\gamma| \leq m-1, & x \in \partial\Omega, \quad t > 0, \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \quad (1.1)$$

where $P = (-\Delta)^m$, $m \geq 1$ a positive integer, $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) is a bound domain with smooth boundary $\partial\Omega$, $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$ is multi-index, γ_i ($i = 1, 2, \dots, n$) are positive integers, $|\gamma| = \gamma_1 + \gamma_2 + \dots + \gamma_n$, $D^\gamma = \frac{\partial^{|\gamma|}}{\partial x_1^{\gamma_1} \partial x_2^{\gamma_2} \dots \partial x_n^{\gamma_n}}$ are derivative operators, $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, and r satisfies

$$\begin{cases} 2 \leq r \leq +\infty, & n = 1, 2, \\ 2 \leq r \leq \frac{2n}{n-2}, & n \geq 3. \end{cases}$$

When $m = 1$, equation (1.1) becomes a heat equation as follows

$$u_t - \Delta u = u^{r-2} u \ln |u|,$$

where $2 \leq r$, which case was considered by many authors [1, 4, 10]. In the case of $r = 2$, Chen et al. [1] obtained under some suitable conditions for the global

existence, decay estimate and blow-up at $+\infty$ of weak solutions, via the logarithmic Sobolev inequality and potential well technique. In the case of $2 < k$, Peng and Zhou [10] studied the existence of the unique global weak solutions and blow-up in the finite time of weak solutions, via potential well technique and energy technique.

When $m = 2$, Li and Liu [7] established the equation

$$u_t + \Delta^2 u = u^{r-2} u \ln |u|,$$

where $2 < r$. They studied the existence of global solutions, by using potential well technique. In addition, they also studied result of decay and finite time blow-up for weak solutions.

Nhan and Truong [9] studied the following nonlinear pseudo-parabolic equation

$$u_t - \Delta u_t - \operatorname{div} \left(|\nabla u|^{r-2} \nabla u \right) = |u|^{r-2} u \log |u|,$$

where $2 < r$. They obtained results as regard the existence or non-existence of global solutions, by using the potential well technique and a logarithmic Sobolev inequality. Also, He et al. [5] proved the decay and the finite time blow-up for weak solutions of the equation, by using the potential well technique and concave technique.

Recently many other authors investigated higher-order hyperbolic and parabolic type equation [2, 3, 6, 11–15]. Ishige et al. [6] studied the Cauchy problem for non-linear higher-order heat equation as follows

$$u_t + (-\Delta)^m u = |u|^r.$$

They obtained existence of solutions of the Cauchy problem by introducing a new majorizing kernel. In addition, they studied the local existence of solutions under the different conditions.

Xiao and Li [13] considered the initial boundary value problem for nonlinear higher-order heat equations of

$$u_t + (-\Delta)^m u_t + (-\Delta)^m u = f(u).$$

They established the existence of a weak solution to the static problem, by using the potential well technique.

The remainder of our work is organized as follows. In Section 2, some important Lemmas are given. In Section 3, the main result is proved.

2. PRELIMINARIES

Let $\|u\|_{H^m(\Omega)} = \left(\sum_{|\gamma| \leq m} \|D^\gamma u\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}$ denote $H^m(\Omega)$ norm, let $H_0^m(\Omega)$ denote the closure in $H^m(\Omega)$ of $C_0^\infty(\Omega)$. Let $\|\cdot\|_r$ and $\|\cdot\|$ denote the usual $L^r(\Omega)$ norm and $L^2(\Omega)$ norm.

For $u \in H_0^m(\Omega) \setminus \{0\}$, we define the energy functional

$$J(u) = \frac{1}{2} \left\| P^{\frac{1}{2}} u \right\|^2 - \frac{1}{r} \int_{\Omega} |u|^r \ln |u| dx + \frac{1}{r^2} \|u\|_r^r, \quad (2.1)$$

and Nehari functional

$$I(u) = \left\| P^{\frac{1}{2}} u \right\|^2 - \int_{\Omega} |u|^r \ln |u| dx. \quad (2.2)$$

By (2.1) and (2.2), we obtain

$$J(u) = \frac{1}{r} I(u) + \left(\frac{1}{2} - \frac{1}{r} \right) \left\| P^{\frac{1}{2}} u \right\|^2 + \frac{1}{r^2} \|u\|_r^r. \quad (2.3)$$

Further, let

$$d = \inf_{u \in \mathcal{N}} J(u), \quad (2.4)$$

denote the potential depth, where $\mathcal{N}$ is the Nehari manifold, which is defined by

$$\mathcal{N} = \{u \in H_0^m(\Omega) \setminus \{0\} : I(u) = 0\}.$$

Lemma 1. *Let k be a number with $2 \leq k < +\infty$, $n \leq 2m$ and $2 \leq k \leq \frac{2n}{n-2m}$, $n > 2m$. Then there is a constant C depending*

$$\|u\|_k \leq C \left\| P^{\frac{1}{2}} u \right\|, \quad \forall u \in H_0^m(\Omega).$$

Lemma 2. *$J(t)$ is a nonincreasing function for $t \geq 0$ and*

$$J'(u) = - \int_{\Omega} u_t^2 dx \leq 0. \quad (2.5)$$

Proof. Multiplying equation (1.1) by u_t and integrating on Ω , we get

$$\int_{\Omega} u_t^2 dx + \int_{\Omega} P u u_t dx = \int_{\Omega} u^{r-1} u_t \ln |u| dx.$$

By straightforward calculation, we obtain

$$\int_{\Omega} u_t^2 dx + \frac{1}{2} \frac{d}{dt} \left\| P^{\frac{1}{2}} u \right\|^2 = \frac{1}{r} \frac{d}{dt} \int_{\Omega} |u|^r \ln |u| dx - \frac{1}{r^2} \frac{d}{dt} \|u\|_r^r,$$

which yields that

$$\frac{1}{2} \frac{d}{dt} \left\| P^{\frac{1}{2}} u \right\|^2 - \frac{1}{r} \frac{d}{dt} \int_{\Omega} |u|^r \ln |u| dx + \frac{1}{r^2} \frac{d}{dt} \|u\|_r^r = - \int_{\Omega} u_t^2 dx.$$

Thus, we get

$$\frac{d}{dt} \left(\frac{1}{2} \left\| P^{\frac{1}{2}} u \right\|^2 - \frac{1}{r} \int_{\Omega} |u|^r \ln |u| dx + \frac{1}{r^2} \|u\|_r^r \right) = - \int_{\Omega} u_t^2 dx. \quad (2.6)$$

By (2.1) and (2.6), we obtain

$$\frac{d}{dt}J(u) = - \int_{\Omega} u_t^2 dx. \quad (2.7)$$

Moreover, integrating (2.7) with respect to t on $[0, t]$, we arrive at

$$J(u(t)) + \int_0^t \|u_s(\tau)\|^2 d\tau = J(u_0). \quad (2.8)$$

□

Lemma 3. *Let $u \in H_0^m(\Omega) \setminus \{0\}$ and $j(\lambda) = J(\lambda u)$. Then we get*

- (i) $\lim_{\lambda \rightarrow 0^+} j(\lambda) = 0$ and $\lim_{\lambda \rightarrow +\infty} j(\lambda) = -\infty$,
- (ii) *there is a unique $\lambda^* > 0$ such that $j'(\lambda^*) = 0$,*
- (iii) *$j(\lambda)$ is decreasing on $(\lambda^*, +\infty)$, increasing on $(0, \lambda^*)$ and taking the maximum at λ^* ,*
- (iv) $I(\lambda u) < 0$ for $\lambda \in (\lambda^*, +\infty)$, $I(\lambda u) > 0$ for $\lambda \in (0, \lambda^*)$ and $I(\lambda^* u) = 0$.

Proof. By the definition of j , for $u \in H_0^1(\Omega) \setminus \{0\}$, we get

$$j(\lambda) = \frac{\lambda^2}{2} \|P^{\frac{1}{2}} u\|^2 - \frac{\lambda^r}{r} \int_{\Omega} |u|^r \ln |u| dx - \frac{\lambda^r}{r} \ln \lambda \|u\|_r^r + \frac{\lambda^r}{r^2} \|u\|_r^r. \quad (2.9)$$

By (2.9), we have

$$\begin{aligned} \frac{d}{d\lambda} j(\lambda) &= \lambda \|P^{\frac{1}{2}} u\|^2 - \lambda^{r-1} \int_{\Omega} |u|^r \ln |u| dx - \lambda^{r-1} \ln \lambda \|u\|_r^r - \frac{\lambda^{r-1}}{r} \|u\|_r^r + \frac{\lambda^{r-1}}{r} \|u\|_r^r \\ &= \lambda \left(\|P^{\frac{1}{2}} u\|^2 - \lambda^{r-2} \int_{\Omega} |u|^r \ln |u| dx - \lambda^{r-2} \ln \lambda \|u\|_r^r \right). \end{aligned}$$

Let $\phi(\lambda) = \lambda^{-1} j'(\lambda)$, thus we obtain

$$\phi(\lambda) = \|P^{\frac{1}{2}} u\|^2 - \lambda^{r-2} \int_{\Omega} |u|^r \ln |u| dx - \lambda^{r-2} \ln \lambda \|u\|_r^r.$$

Then

$$\phi'(\lambda) = -(r-2)\lambda^{r-3} \int_{\Omega} |u|^r \ln |u| dx - (r-2)\lambda^{r-3} \ln \lambda \|u\|_r^r - \lambda^{r-3} \|u\|_r^r,$$

which yields that there exists a $\lambda^* > 0$ such that $\phi'(\lambda) < 0$ on $(\lambda^*, +\infty)$, $\phi'(\lambda) > 0$ on $(0, \lambda^*)$ and $\phi'(\lambda) = 0$. Thus, $\phi(\lambda)$ is decreasing on $(\lambda^*, +\infty)$, increasing on $(0, \lambda^*)$. Since $\lim_{\lambda \rightarrow 0^+} \phi(\lambda) > 0$, $\lim_{\lambda \rightarrow +\infty} \phi(\lambda) = -\infty$, there exists a unique $\lambda^* > 0$ such that $\phi(\lambda^*) = 0$, i.e., $j'(\lambda^*) = 0$. Then, $j'(\lambda) = \lambda \phi(\lambda)$ is negative on $(\lambda^*, +\infty)$, positive on $(0, \lambda^*)$. Thus, $j(\lambda)$ is decreasing on $(\lambda^*, +\infty)$, increasing on $(0, \lambda^*)$ and taking the maximum at λ^* . By (2.2), we get

$$I(\lambda u) = \|P^{\frac{1}{2}}(\lambda u)\|^2 - \int_{\Omega} |\lambda u|^r \ln |\lambda u| dx$$

$$\begin{aligned}
&= \lambda^2 \left\| P^{\frac{1}{2}} u \right\|^2 - \lambda^r \int_{\Omega} |u|^r \ln |u| dx - \lambda^r \ln \lambda \|u\|_r^r \\
&= \lambda j'(\lambda).
\end{aligned}$$

So, $I(\lambda u) < 0$ for $\lambda \in (\lambda^*, +\infty)$, $I(\lambda u) > 0$ for $\lambda \in (0, \lambda^*)$ and $I(\lambda^* u) = 0$. Therefore, the proof is completed. $\square$

Lemma 4 ([8]). *Let μ be a constant and $g : R^+ \rightarrow R^+$ be a nonincreasing function such that*

$$\int_t^{+\infty} g^{1+\mu}(\tau) d\tau \leq \frac{1}{\zeta} g^\mu(0) g(t), \text{ for all } t \geq 0.$$

Hence

- (i) $g(t) \leq g(0) \left(\frac{1+\mu}{1+\zeta t} \right)^{\frac{1}{\mu}}, \forall t \geq 0$, whenever $\mu > 0$,
- (ii) $g(t) \leq g(0) e^{1-\zeta t}, \forall t \geq 0$, whenever $\mu = 0$.

3. MAIN RESULTS

As in [9], we consider the following notations:

$$\begin{aligned}
\mathcal{W}_1 &= \{u \in H_0^1(\Omega) \setminus \{0\} : J(u) < d\}, & \mathcal{W}_2 &= \{u \in H_0^1(\Omega) \setminus \{0\} : J(u) = d\}, \\
\mathcal{W}_1^+ &= \{u \in \mathcal{W}_1 : I(u) > 0\}, & \mathcal{W}_2^+ &= \{u \in \mathcal{W}_2 : I(u) > 0\}, \\
\mathcal{W}_1^- &= \{u \in \mathcal{W}_1 : I(u) < 0\}, & \mathcal{W}_2^- &= \{u \in \mathcal{W}_2 : I(u) < 0\}, \\
\mathcal{W} &= \mathcal{W}_1 \cup \mathcal{W}_2, & \mathcal{W}^+ &= \mathcal{W}_1^+ \cup \mathcal{W}_2^+, \quad \mathcal{W}^- = \mathcal{W}_1^- \cup \mathcal{W}_2^-.
\end{aligned}$$

We refer to $\mathcal{W}$ as the potential well and d as the depth of the well.

Definition 1 (Weak Solution). We say that function $u(t)$ is a weak solution of problem (1.1) on $\Omega \times [0, T]$, if $u \in L^\infty(0, T; H_0^m(\Omega))$ with $u_t \in L^2(0, T; H_0^m(\Omega))$ and implies the initial condition $u(0) = u_0 \in H_0^m(\Omega) \setminus \{0\}$, and the following equality

$$(u_t, w) + \left(P^{\frac{1}{2}} u, P^{\frac{1}{2}} w \right) = \left(|u|^{r-2} u \ln |u|, w \right),$$

for all $w \in H_0^m(\Omega)$ holds for a.e. $t \in [0, T]$, and $(\cdot, \cdot)$ means the inner product $(\cdot, \cdot)_{L^2(\Omega)}$, that is

$$(\eta, \xi) = \int_{\Omega} \eta(x) \xi(x) dx.$$

Definition 2 (Maximal Existence Time). Suppose that $u(t)$ is a weak solutions of problem (1.1). We define the following the maximal existence time $T_{\max}$

$$T_{\max} = \sup\{T > 0 : u(t) \text{ exists on } [0, T]\}.$$

Then

- (a) If $T_{\max} = \infty$, we say that $u(t)$ is global;
- (b) If $T_{\max} < \infty$, we say that $u(t)$ blows up in finite time.

Theorem 1 (Global Existence). *Let $u_0 \in \mathcal{W}^+$. Then problem (1.1) admits a global weak solution. We get $u(t) \in \mathcal{W}^+$ holds for any $t \in [0, +\infty)$, and the energy estimate*

$$J(u(t)) + \int_0^t \|u_s(s)\|^2 ds \leq J(u_0), \quad t \in [0, +\infty).$$

Also, the solution decays exponential provided $u_0 \in \mathcal{W}_1^+$.

Proof. We will investigate the following two cases:

Firstly, we address the case of the initial data $u_0 \in \mathcal{W}_1^+$.

The Faedo-Galerkin's methods is used. In the space $H_0^m(\Omega)$, we take a bases $\{w_j\}_{j=1}^\infty$ and define the finite orthogonal space

$$V_s = \text{span}\{w_1, w_2, \dots, w_s\}.$$

Let u_{0s} be an element of V_s such that

$$u_{0s} = \sum_{j=1}^s a_{sj} w_j \rightarrow u_0, \quad \text{in } H_0^m(\Omega), \quad (3.1)$$

as $s \rightarrow \infty$. We construct the following approximate solution $u_s(x, t)$ of problem (1.1)

$$u_s(x, t) = \sum_{j=1}^s a_{sj}(t) w_j(x), \quad (3.2)$$

where the coefficients a_{sj} ($1 \leq j \leq s$) imply the ODEs

$$\int_{\Omega} u_{st} w_i dx + \int_{\Omega} P u_s w_i dx = \int_{\Omega} |u_s|^{r-2} u_s \ln |u_s| w_i dx, \quad (3.3)$$

for $i \in \{1, 2, \dots, s\}$, with the initial condition

$$a_{sj}(0) = a_{sj}, \quad j \in \{1, 2, \dots, s\}. \quad (3.4)$$

We multiply both sides of (3.3) by a'_{si} , sum for $i = 1, \dots, s$ and integrating with respect to time variable on $[0, t]$, we get

$$J(u_s(t)) + \int_0^t \|u_{s\tau}(\tau)\|^2 d\tau \leq J(u_{0s}), \quad 0 \leq t \leq T_{\max}, \quad (3.5)$$

where $T_{\max}$ is the maximal existence time of solution $u_s(t)$. We shall prove that $T_{\max} = +\infty$. From (3.1), (3.5) and the continuity of J , we obtain

$$J(u_s(0)) \rightarrow J(u_{0s}), \quad \text{as } s \rightarrow \infty. \quad (3.6)$$

Thanks to $J(u_0) < d$ and the continuity of functional J , it follows from (3.6) that

$$J(u_{0s}) < d, \quad \text{for sufficiently large } m.$$

And therefore, from (3.5), we get

$$J(u_s(t)) + \int_0^t \|u_{s\tau}(\tau)\|^2 d\tau < d, \quad 0 \leq t \leq T_{\max}, \quad (3.7)$$

for sufficiently large s . Next, we will study

$$u_s(t) \in \mathcal{W}_1^+, \quad t \in [0, T_{\max}), \quad (3.8)$$

for sufficiently large s . We assume that (3.8) does not process and think that there exists a sufficiently small time t_0 such that $u_s(t_0) \notin \mathcal{W}_1^+$. Then, by continuity of $u_s(t_0) \in \partial \mathcal{W}_1^+$. So, we get

$$J(u_s(t_0)) = d, \quad (3.9)$$

and

$$I(u_s(t_0)) = 0. \quad (3.10)$$

Nevertheless, by definition of d , we see that (3.9) could not consist by (3.7) while if (3.10) holds then, we get

$$d = \inf_{u \in \mathcal{N}} J(u) \leq J(u_s(t_0)),$$

which also contradicts with (3.7). Moreover, we have (3.8), i.e., $I(u_s(t)) > 0$, and $J(u_s(t)) < d$, for all $t \in [0, T_{\max})$, for sufficiently large s . Then, from (2.3), we obtain

$$\begin{aligned} d &> J(u_s(t)) \\ &= \frac{1}{r} I(u_s(t)) + \left(\frac{1}{2} - \frac{1}{r} \right) \left\| P^{\frac{1}{2}} u_s(t) \right\|^2 + \frac{1}{r^2} \|u_s(t)\|_r^r \\ &\geq \left(\frac{1}{2} - \frac{1}{r} \right) \left\| P^{\frac{1}{2}} u_s(t) \right\|^2 + \frac{1}{r^2} \|u_s(t)\|_r^r, \end{aligned}$$

which gives

$$\|u_s(t)\|_r^r < r^2 d, \quad (3.11)$$

and

$$\left\| P^{\frac{1}{2}} u_s(t) \right\|^2 < \frac{2r}{r-2} d. \quad (3.12)$$

Since $u_s(x, t) \in \mathcal{W}_1^+$ for s large enough, it follows from (2.3) that $J(u_s) \geq 0$ for s large enough. So, by (3.7) it follows for s large enough

$$\int_0^t \|u_{s\tau}(\tau)\|^2 d\tau < d. \quad (3.13)$$

By (3.12), we know that

$$T_{\max} = +\infty.$$

It follows from (3.11) and (3.13) that there exist a function $u \in H_0^m(\Omega)$ and a subsequence of $\{u_s\}_{j=1}^\infty$ is indicated by $\{u_s\}_{j=1}^\infty$ such that

$$u_s \rightarrow u \text{ weakly* in } L^\infty(0, \infty; H_0^m(\Omega)), \quad (3.14)$$

$$u_{st} \rightarrow u_t \text{ weakly in } L^2(0, \infty; L^2(\Omega)). \quad (3.15)$$

By (3.14), (3.15) and the Aubin-Lions compactness theorem, we obtain

$$u_s \rightarrow u \text{ strongly in } C([0, +\infty]; L^2(\Omega)).$$

This yields that

$$|u_s|^{r-2} u_s \ln |u_s| \rightarrow |u|^{r-2} u \ln |u| \quad \text{a.e. } (x, t) \in \Omega \times (0, +\infty). \quad (3.16)$$

Moreover, since

$$\alpha^{r-1} \ln \alpha = -(e(r-1))^{-1} \quad \text{for } \alpha > 1,$$

and

$$\ln \alpha = 2 \ln \left(\alpha^{\frac{1}{2}} \right) \leq 2 \alpha^{\frac{1}{2}} \quad \text{for } \alpha > 0.$$

By (3.11), we have

$$\begin{aligned} \int_{\Omega} \left(|u_s(t)|^{r-1} \ln |u_s(t)| \right)^{\frac{2r}{2r-1}} dx &= \int_{\Omega_1} \left(|u_s(t)|^{r-1} \ln |u_s(t)| \right)^{\frac{2r}{2r-1}} dx \\ &\quad + \int_{\Omega_2} \left(|u_s(t)|^{r-1} \ln |u_s(t)| \right)^{\frac{2r}{2r-1}} dx \\ &\leq [e(r-1)]^{-\frac{2r}{2r-1}} |\Omega| + 2^{\frac{2r}{2r-1}} \int_{\Omega_2} |u_s(t)|^{\frac{2r(r-1+\frac{1}{2})}{2r-1}} dx \\ &= [e(r-1)]^{-\frac{2r}{2r-1}} |\Omega| + 2^{\frac{2r}{2r-1}} \int_{\Omega_2} |u_s(t)|^r dx \\ &\leq C_d := [e(r-1)]^{-\frac{2r}{2r-1}} |\Omega| + 2^{\frac{2r}{2r-1}} r^2 d, \end{aligned} \quad (3.17)$$

where

$$\Omega_1 = \{x \in \Omega : |u_s(t)| \leq 1\}, \quad \text{and} \quad \Omega_2 = \{x \in \Omega : |u_s(t)| \geq 1\}.$$

Hence, it follows from (3.16) and (3.17) that

$$|u_s|^{r-2} u_s \ln |u_s| \rightarrow |u|^{r-2} u \ln |u| \quad \text{weakly* in } L^\infty(0, +\infty; L^{\frac{2r}{2r-1}}(\Omega)).$$

Then integrating (3.3) respect to t for $0 \leq t < \infty$, we obtain

$$(u_t, w_i) + \left(P^{\frac{1}{2}} u, P^{\frac{1}{2}} w_i \right) = \left(|u|^{r-2} u \ln |u|, w_i \right).$$

On the other hand, there exists a global weak solution $u_0 \in \mathcal{W}_1^+$ of problem (1.1).

Now we address the case of the initial data $u_0 \in \mathcal{W}_2^+$.

First we can choose a sequence $\{\omega_s\}_{s=1}^\infty \subset (0, 1)$ and $\lim_{s \rightarrow \infty} \omega_s = 1$. Next, we investigate the following problem:

$$\begin{cases} u_t + Pu = |u|^{r-2} u \ln |u|, & x \in \Omega, \quad t > 0, \\ D^\gamma u(x, t) = 0, \quad |\gamma| \leq m-1, & x \in \partial\Omega, \quad t > 0, \\ u(x, 0) = u_{0s}(x), & x \in \Omega, \end{cases} \quad (3.18)$$

where $u_{0s} = \omega_s u_0$. By $I(u_0) > 0$ and Lemma 3, it is clear that there exists a $\lambda^* > 1$. Also, $J(u_{0s}) = J(\omega_s u_0) < J(u_0) = d$ and $I(u_{0s}) = I(\omega_s u_0) > 0$ hold. So, we have $u_0 \in \mathcal{W}_2^+$. Similarly to the previous situation, it is clear that problem (3.18) implies that, for all $s > 0$, there exists a global u_s which implies $u_s \in L^\infty(0, \infty; H_0^m(\Omega))$,

$u_{st} \in L^2(0, \infty; L^2(\Omega))$, $u_s(0) = u_{0s} = \omega_s u_0 \rightarrow u_0$ strongly in $H_0^m(\Omega)$, and the following equality

$$\int_{\Omega} u_{st} w dx + \int_{\Omega} P u_s w dx = \int_{\Omega} |u_s|^{r-2} u_s \ln |u_s| w dx, \quad (3.19)$$

with any $w \in H_0^m(\Omega)$ holds for a.e. $0 \leq t < \infty$. Also, we get

$$u_s(t) \in \mathcal{W}_2^+, \quad t \in [0, \infty),$$

and

$$J(u_s(t)) + \int_0^t \|u_{s\tau}(\tau)\|^2 d\tau \leq J(u_{0s}) < d.$$

On the other hand, we can deduce (3.12), (3.13) and (3.17) for each s . Also, there exist u and a subsequence still denoted by $\{u_s\}$, such that, as $s \rightarrow \infty$,

$$u_s \rightarrow u \text{ weakly* in } L^\infty(0, \infty; H_0^m(\Omega)),$$

$$u_{st} \rightarrow u_t \text{ weakly in } L^2(0, \infty; L^2(\Omega)),$$

$$|u_s|^{r-2} u_s \ln |u_s| \rightarrow |u|^{r-2} u \ln |u| \text{ weakly* in } L^\infty(0, \infty; L^{\frac{2r}{2r-1}}(\Omega)).$$

Then integrating (3.19) respect to t for $0 \leq t < \infty$, we obtain

$$(u_t, w) + \left(P^{\frac{1}{2}} u, P^{\frac{1}{2}} w \right) = \left(|u|^{r-2} u \ln |u|, w \right).$$

Therefore, there exists a global weak solution $u_0 \in \mathcal{W}_2^+$ of problem (1.1).

Decay estimates

Thanks to $u_0 \in \mathcal{W}_1^+$, we deduce from (2.3) that

$$\begin{aligned} J(u_0) &> J(u(t)) \\ &= \frac{1}{r} I(u(t)) + \left(\frac{1}{2} - \frac{1}{r} \right) \left\| P^{\frac{1}{2}} u(t) \right\|^2 + \frac{1}{r^2} \|u(t)\|_r^r \\ &\geq \left(\frac{1}{2} - \frac{1}{r} \right) \left\| P^{\frac{1}{2}} u(t) \right\|^2 + \frac{1}{r^2} \|u(t)\|_r^r. \end{aligned} \quad (3.20)$$

From Lemma 2, (2.4) and $I(u(t)) > 0$, there exists a $\lambda^* > 1$ such that $I(\lambda^* u(t)) = 0$. We get

$$\begin{aligned} d &\leq J(\lambda^* u(t)) \\ &= (\lambda^*)^r \left((\lambda^*)^{2-r} \left(\frac{1}{2} - \frac{1}{r} \right) \left\| P^{\frac{1}{2}} u(t) \right\|^2 + \frac{1}{r^2} \|u(t)\|_r^r \right) \\ &\leq (\lambda^*)^r \left(\left(\frac{1}{2} - \frac{1}{r} \right) \left\| P^{\frac{1}{2}} u(t) \right\|^2 + \frac{1}{r^2} \|u(t)\|_r^r \right). \end{aligned} \quad (3.21)$$

Using (3.20) and (3.21), we get

$$d \leq (\lambda^*)^r J(u_0),$$

which yields that

$$\lambda^* \geq \left(\frac{d}{J(u_0)} \right)^{\frac{1}{r}}. \quad (3.22)$$

By (2.2), we get

$$0 = I(\lambda^* u(t)) = (\lambda^*)^r I(u(t)) + [(\lambda^*)^2 - (\lambda^*)^r] \left\| P^{\frac{1}{2}} u(t) \right\|^2 - (\lambda^*)^r \ln(\lambda^*) \|u(t)\|_r^r. \quad (3.23)$$

From (3.22), (3.23) and Lemma 1, we obtain

$$\begin{aligned} I(u(t)) &\geq [1 - (\lambda^*)^{2-r}] \left\| P^{\frac{1}{2}} u(t) \right\|^2 \\ &\geq \left[1 - \left(\frac{d}{J(u_0)} \right)^{\frac{2-r}{r}} \right] \left\| P^{\frac{1}{2}} u(t) \right\|^2 \\ &\geq C_1 \|u(t)\|^2, \end{aligned} \quad (3.24)$$

where C_1 is constant. Integrating the $I(u(\tau))$ with respect to τ over (t, T) , we obtain

$$\int_t^T I(u(\tau)) d\tau = - \int_t^T \int_{\Omega} u_{\tau}(\tau) u(\tau) dx d\tau \leq \frac{C_2}{2} \|u(t)\|^2. \quad (3.25)$$

where C_2 is constant. From (3.24) and (3.25), we have

$$\int_t^T C_1 \|u(t)\|^2 ds \leq \frac{C_2}{2} \|u(t)\|^2, \text{ for all } t \in [0, T]. \quad (3.26)$$

Let $T \rightarrow +\infty$ in (3.26), we can have

$$\int_t^{\infty} \|u(t)\|^2 ds \leq C_3 \|u(t)\|^2,$$

where $C_3 = \frac{C_2}{2C_1}$. By Lemma 4, we have

$$\|u(t)\|^2 \leq \|u(0)\|^2 e^{1 - \frac{t}{C_3}}, \quad t \in [0, \infty).$$

□

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APPROXIMATION OF JENSEN TYPE RECIPROCAL MAPPINGS VIA FIXED POINT TECHNIQUE

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Abstract. In this paper, we solve a new Jensen type m -dimensional multiplicative inverse functional equation and then its various stability problems in the setting of non-negative real numbers and non-Archimedean spaces via fixed point method. The functional equation dealt in this study is linked with the famous relationship between arithmetic and harmonic mean of m values. The role of harmonic mean is very significant in many other fields such as traffic flow theory, industrial engineering, communication system, etc. By inverting the arithmetic mean of reciprocal values, we attain the harmonic mean. This property could be analyzed as an inverse problem via the functional equation dealt in this investigation.

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1. INTRODUCTION & PRELIMINARIES

The query of Ulam [25] regarding homomorphisms and subsequent brilliant reply provided by Hyers [12] are the foundations for the emerging research area of the stability of various forms of functional, difference, differential and integral equations. The result presented by Hyers is known as Ulam-Hyers Stability (or UHS) and the method of proof adopted by Hyers is called as ‘Direct method’. Later, Hyers’ result is generalized in different versions by many mathematicians, viz., Aoki [1], T. Rassias (Ulam-Hyers-T. Rassias Stability or UHTRS) [20], J. Rassias (Ulam-Hyers-J. Rassias Stability or UHJRS) [19], Găvruta (Ulam-Rassias-Găvruta Stability or URGs) [11] and these results are the fundamental stabilities of a functional equation. These results inspired many mathematicians and there are a lot of novel, interesting and interesting results in the literature pertinent to different category of functional equations and their fundamental stabilities, refer [2, 15–18].

Fixed point method is another method of proving stability of functional equations. The stability problems of Jensen’s and Cauchy functional equations were dealt in [9, 10] via fixed point technique.

For the first time, the fundamental stabilities of the ensuing multiplicative inverse functional equation

$$m_i(u+v) = \frac{m_i(u)m_i(v)}{m_i(u) + m_i(v)} \quad (1.1)$$

were dealt in [22], where $m_i : \mathbb{R}^* \rightarrow \mathbb{R}$ is a multiplicative inverse function with $\mathbb{R}^* = \mathbb{R} - \{0\}$.

The stability problem of equation (1.1) triggered many authors to solve URGs of various types of multiplicative inverse functional equations such as generalized version in several variables, adjoint, difference, quadratic, cubic, quartic, quintic, sextic functional equations and their stabilities and non-stabilities (one can see [3–8, 13, 21]). Multiplicative inverse functional equations and their applications in real time are dealt in [23, 24].

In this paper, we deal with a new rational type functional equation of the form

$$r_j \left(\frac{1}{m} \sum_{k=1}^m \alpha_k \right) = \frac{m \prod_{k=1}^m r_j(\alpha_k)}{\sum_{k=1}^m \left[\prod_{i=1, i \neq k}^m r_j(\alpha_i) \right]} \quad (1.2)$$

where $m \geq 2$ is a positive integer and $r_j : \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}$ is a function with the conditions $\sum_{k=2}^{m-1} \alpha_k \neq -\alpha_1$, $\sum_{k=1}^m \left[\prod_{i=1, i \neq k}^m r_j(\alpha_i) \right] \neq 0$ and $r_j(\alpha_k) \neq 0$ for $k = 1, \dots, m$.

We solve equation (1.2) and obtain its various fundamental stabilities in the domain of non-negative real numbers and non-Archimedean fields through fixed point technique. We also demonstrate the relevance of equation (1.2) with arithmetic and harmonic means.

In the sequel, we evoke a theorem which serves as the backbone in obtaining our desired results.

Theorem 1. *Let (T, ρ) be a complete generalized metric space and let $\mu : T \rightarrow T$ be a strictly contractive mapping (that is $\rho(\mu(\alpha), \mu(\beta)) \leq L\rho(\alpha, \beta)$, for all $\alpha, \beta \in T$ and a Lipschitz constant $L < 1$). Then for each given element $\alpha \in T$, either*

$$\rho(\mu^k \alpha, \mu^{k+1} \alpha) = \infty$$

for all integers $k \geq 0$ or there exists an integer $k_0 > 0$ such that

1. $\rho(\mu^k \alpha, \mu^{k+1} \alpha) < \infty$ for all $k \geq k_0$;
2. the sequence $\{\mu^k \alpha\}$ converges to a fixed point β^* of μ ;
3. β^* is the unique fixed point of μ in the set $\Gamma = \{\beta \in T : \rho(\mu^{k_0} \alpha, \beta) < \infty\}$;
4. $d(\beta, \beta^*) \leq \frac{1}{1-L} d(\beta, \mu\beta)$ for all $\beta \in \Gamma$.

Now, we recall a few fundamental notions of non-Archimedean field and fixed point alternative theorem in non-Archimedean settings. Throughout this paper, let $\mathbb{R}$ denote the set of real numbers.

Definition 1. Let $\mathbb{P}$ be a field with a mapping (valuation) $|\cdot|$ from $\mathbb{P}$ into $[0, \infty)$. Then $\mathbb{P}$ is said to be a non-Archimedean field if the ensuing conditions hold.

- (i) $|\alpha| = 0$ if and only if $\alpha = 0$;
- (ii) $|\alpha_1 \alpha_2| = |\alpha_1| |\alpha_2|$;
- (iii) $|\alpha_1 + \alpha_2| \leq \max\{|\alpha_1|, |\alpha_2|\}$

for all $\alpha_1, \alpha_2 \in \mathbb{P}$.

It is obvious that $|1| = |-1| = 1$ and $|\alpha| \leq 1$ for all $\alpha \in \mathbb{N}$. Furthermore, we assume that $|\cdot|$ is non-trivial, that is, there exists an $\ell_0 \in \mathbb{P}$ such that $|\ell_0| \neq 0, 1$.

Suppose W is a vector space over a scalar field $\mathbb{P}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\| : W \rightarrow \mathbb{R}$ is a *non-Archimedean norm (valuation)* if it satisfies the following conditions:

- (i) $\|p\| = 0$ if and only if $p = 0$;
- (ii) $\|\lambda p\| = |\lambda| \|p\|$ ($\lambda \in \mathbb{P}, p \in W$);
- (iii) the strong triangle inequality (ultrametric); namely,

$$\|p + q\| \leq \max\{\|p\|, \|q\|\} \quad (p, q \in W).$$

Then $(W, \|\cdot\|)$ is known as a non-Archimedean space. In lieu of the following inequality

$$\|u_\ell - u_k\| \leq \max\{\|u_{j+1} - u_j\| : k \leq j \leq \ell - 1\} \quad (\ell > k),$$

a sequence $\{u_k\}$ is Cauchy if and only if $\{u_{k+1} - u_k\}$ converges to 0 in a non-Archimedean space. If every Cauchy sequence is convergent in the space, then it is called as complete non-Archimedean space.

Definition 2. Let K be a nonempty set. Suppose $\rho : K \times K \rightarrow [0, \infty]$ satisfies the following axioms:

- (i) $\rho(r, s) = 0$ if and only if $r = s$;
- (ii) $\rho(r, s) = \rho(s, r)$ (symmetry);
- (iii) $\rho(r, t) \leq \max\{\rho(r, s), \rho(s, t)\}$ (strong triangle inequality)

for all $r, s, t \in K$. Then (K, ρ) is called a generalized non-Archimedean metric space. Suppose every Cauchy sequence in K is convergent, then (K, ρ) is called complete.

Example 1. Let $\mathbb{P}$ be a non-Archimedean field and U_1 and U_2 be two non-Archimedean spaces over $\mathbb{P}$. If U_2 is complete and $\chi : U_1 \rightarrow [0, \infty)$, for every $r, s : U_1 \rightarrow U_2$, define $\rho(r, s) = \inf\{\mu > 0 : |r(\alpha) - s(\alpha)| \leq \mu\chi(\alpha), \forall \alpha \in U_1\}$. Then (U_1, ρ) is a generalized non-Archimedean metric space.

Using (Theorem 2.5, [10]), Mirmostafaei [14] introduced a different version of the alternative fixed point principle in the setting of non-Archimedean space which is presented in the following theorem.

Theorem 2. [14] (Alternative fixed point principle in non-Archimedean scheme) Suppose (H, ρ) is a non-Archimedean generalized metric space. Let a mapping $\Lambda : H \rightarrow H$ be a strictly contractive, (that is $\rho(\Lambda(u), \Lambda(v)) \leq L\rho(v, u)$, for all $u, v \in H$ and a Lipschitz constant $\rho < 1$), then either

(i) $\rho(\Lambda^p(u), \Lambda^{p+1}u) = \infty$ for all $p \geq 0$, or;
(ii) there exists some $p_0 \geq 0$ such that $\rho(\Lambda^p(u), \Lambda^{p+1}(u)) < \infty$ for all $p \geq p_0$;
the sequence $\{\Lambda^p(u)\}$ is convergent to a fixed point u^* of Λ ; u^* is the distinct invariant point of Λ in the set $Y = \{y \in X : \rho(\Lambda^{p_0}(u), y) < \infty\}$ and $\rho(y, u^*) \leq \rho(y, \Lambda(y))$ for all y in this set.

2. SOLUTION OF EQUATION (1.2)

In this section, we achieve the solution of equation (1.2). Then using the solution of equation (1.2), we define Jensen type m -dimensional multiplicative inverse mapping and functional equation.

Theorem 3. Let a rational function $r_j : \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}$ be defined by $r_j(\alpha) = \frac{1}{c_1\alpha + c_2}$, for $\alpha \in \mathbb{R}^+ \cup \{0\}$ and c_1, c_2 be any arbitrary constants with $c_2 \neq 0$ and $c_1\alpha + c_2 \neq 0$. Then r_j is a solution of equation (1.2).

Proof. In order to prove the result, first let us rewrite equation (1.2) as

$$\frac{1}{r_j\left(\frac{1}{m} \sum_{k=1}^m \alpha_k\right)} = \frac{1}{m} \left[\sum_{k=1}^m \frac{1}{r_j(\alpha_k)} \right]. \quad (2.1)$$

Assuming $r_j(0) = \frac{1}{c_2}$, letting $\alpha_2 = \alpha_3 = \dots = \alpha_m = 0$ and then plugging α_1 into α in (2.1), we obtain

$$\frac{1}{r_j\left(\frac{\alpha}{m}\right)} = \frac{1}{m} \left[\frac{1}{r_j(\alpha)} + (m-1)c_2 \right]. \quad (2.2)$$

Again, equation (2.2) can be modified as

$$r_j\left(\frac{\alpha}{m}\right) = \frac{m}{\frac{1}{r_j(\alpha)} + (m-1)c_2}. \quad (2.3)$$

By virtue of equation (2.3), the equation (2.1) can be written as

$$\frac{\prod_{k=1}^m r_j(\alpha_k)}{\sum_{k=1}^m \prod_{i=1, i \neq k}^m r_j(\alpha_i)} = \frac{1}{\frac{1}{r_j\left(\frac{1}{m} \sum_{k=1}^m \alpha_k\right)} + (m-1)c_2}. \quad (2.4)$$

Equation (2.4) can be transformed into

$$\frac{1}{r_j\left(\frac{1}{m} \sum_{k=1}^m \alpha_k\right)} = \left(\sum_{k=1}^m \frac{1}{r_j(\alpha_k)} \right) - (m-1)c_2. \quad (2.5)$$

Let $\mathcal{A}(\alpha) = \frac{1}{r_j(\alpha)} - c_2$. Then, we have

$$\mathcal{A}\left(\sum_{k=1}^m \alpha_k\right) = \frac{1}{r_j\left(\frac{1}{m} \sum_{k=1}^m \alpha_k\right)} - c_2$$

$$\begin{aligned}
 &= \left[\left(\sum_{k=1}^m \frac{1}{r_j(\alpha_k)} \right) - (m-1)c_2 \right] - c \\
 &= \sum_{k=1}^m \left(\frac{1}{r_j(\alpha)} - c_2 \right) \\
 &= \sum_{k=1}^m \mathcal{A}(\alpha_k)
 \end{aligned} \tag{2.6}$$

which indicates that $\mathcal{A}$ is a Cauchy functional equation in m variables and its solution is an additive function $\mathcal{A}(\alpha) = c_1\alpha$, where c_1 is a constant. Hence, it implies that the rational function $r_j(\alpha) = \frac{1}{c_1\alpha + c_2}$ is a solution of equation (1.2). $\square$

The upcoming definition follows from the above theorem.

Definition 3. A function r_j is called as a Jensen type m -dimensional multiplicative inverse mapping if it satisfies equation (1.2). Hence equation (1.2) is said to be Jensen type m -dimensional multiplicative inverse functional equation.

3. VARIOUS CLASSICAL STABILITIES OF EQUATION (1.2)

In order to prove our main results, let us assume that $\sum_{k=2}^{m-1} \alpha_k \neq -\alpha_1$ for all $\alpha_k \in \mathbb{R}^+ \cup \{0\}$, $k = 1, \dots, m$. We solve URGS problem of equation (1.2) in the domain of non-zero real numbers. Then, we apply URGS result of equation (1.2), to investigate the stability problems involving a positive fixed constant and sum of powers of norms.

Theorem 4. Suppose that a function $r_j : \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}$ satisfies the condition that $r_j(\alpha)$ approaches $\frac{1}{c_2}$ as $\alpha \rightarrow 0$, where c_2 is a constant and an inequality

$$\left| \frac{1}{m} r_j \left(\frac{1}{m} \sum_{k=1}^m \alpha_k \right) - \frac{1}{\sum_{k=1}^m \frac{1}{r_j(\alpha_k)}} \right| \leq \mathfrak{v}(\alpha_1, \alpha_2, \dots, \alpha_m) \tag{3.1}$$

for all $\alpha_k \in \mathbb{R}^+ \cup \{0\}$, where $\mathfrak{v} : \underbrace{\mathbb{R}^+ \cup \{0\} \times \dots \times \mathbb{R}^+ \cup \{0\}}_{m\text{-times}} \rightarrow [0, \infty)$ is a given func-

tion. Suppose that there exists $L < 1$ such that the mapping $\alpha \mapsto \Upsilon(\alpha) = \mathfrak{v}(\alpha, \underbrace{0, \dots, 0}_{(m-1)\text{-times}})$

has the property $\Upsilon\left(\frac{\alpha}{m}\right) \leq mLY(\alpha)$ for all $\alpha \in \mathbb{R}^+ \cup \{0\}$. If the function $\mathfrak{v}$ has the property

$$\lim_{n \rightarrow \infty} m^{-n} \mathfrak{v}(m^{-n}\alpha_1, \dots, m^{-n}\alpha_m) = 0, \tag{3.2}$$

for all $\alpha_k \in \mathbb{R}^+ \cup \{0\}$, then there exists a unique Jensen type m -dimensional multiplicative inverse mapping $R_j : \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}$ such that

$$|r_j(\alpha) - R_j(\alpha)| \leq \frac{1}{1-L} \Upsilon(\alpha) \tag{3.3}$$

for all $\alpha \in \mathbb{R}^+ \cup \{0\}$.

Proof. Let T be a set defined as $T = \{\phi : \mathbb{R}^+ \cup \{0\} \longrightarrow \mathbb{R}\}$. Assume ρ is a generalized metric on T which is described as:

$$\rho(\psi, \phi) = \rho_r(\psi, \phi) = \inf\{\lambda > 0 : |\psi(\alpha) - \phi(\alpha)| \leq \lambda \Upsilon(\alpha), \text{ for all } \alpha \in \mathbb{R}^+ \cup \{0\}\}. \quad (3.4)$$

The definition of ρ implies that the set T is complete. Now, let us define a mapping $\mu : T \longrightarrow T$ by

$$\mu\phi(\alpha) = \frac{1}{m}\phi\left(\frac{\alpha}{m}\right) \quad (\alpha \in \mathbb{R}^+ \cup \{0\}) \quad (3.5)$$

for all $\phi \in T$. Next, let us show that μ is a strictly contractive mapping on the set T . For given $\psi, \phi \in T$, suppose $0 \leq \lambda_{\psi\phi} \leq \infty$ is an arbitrary constant with $\rho(\psi, \phi) \leq \lambda_{\psi\phi}$. Therefore, we have

$$\begin{aligned} \rho(\psi, \phi) < \lambda_{\psi\phi} &\implies |\psi(\alpha) - \phi(\alpha)| \leq \lambda_{\psi\phi} \Upsilon(\alpha), \quad (\forall \alpha \in \mathbb{R}^+ \cup \{0\}) \\ &\implies \left| \frac{1}{m}\psi\left(\frac{\alpha}{m}\right) - \frac{1}{m}\phi\left(\frac{\alpha}{m}\right) \right| \leq \frac{1}{m}\lambda_{\psi\phi} \Upsilon\left(\frac{\alpha}{m}\right), \quad (\forall \alpha \in \mathbb{R}^+ \cup \{0\}) \\ &\implies \left| \frac{1}{m}\psi\left(\frac{\alpha}{m}\right) - \frac{1}{m}\phi\left(\frac{\alpha}{m}\right) \right| \leq L\lambda_{\psi\phi} \Upsilon(\alpha), \quad (\forall \alpha \in \mathbb{R}^+ \cup \{0\}) \\ &\implies \rho(\mu\psi, \mu\phi) \leq L\lambda_{\psi\phi}. \end{aligned}$$

The above inequality implies that $\rho(\mu\psi, \mu\phi) \leq L\rho(\psi, \phi)$ for all $\psi, \phi \in T$, which in turn indicates that μ is a strictly contractive mapping of T , with the Lipschitz constant L . Now, plugging $(\alpha_1, \dots, \alpha_m)$ by $(\alpha, \underbrace{0, \dots, 0}_{(m-1)\text{-times}})$ in (3.1), we get

$$\left| \frac{1}{m}r_j\left(\frac{\alpha}{m}\right) - \frac{1}{\frac{1}{r_j(\alpha)} + (m-1)c_2} \right| \leq \nu(\alpha, \underbrace{0, \dots, 0}_{(m-1)\text{-times}}) = \Upsilon(\alpha)$$

which produces

$$\left| \frac{1}{2}r_j\left(\frac{\alpha}{2}\right) - r_j(\alpha) \right| \leq \left| \frac{1}{m}r_j\left(\frac{\alpha}{m}\right) - \frac{1}{\frac{1}{r_j(\alpha)} + (m-1)c_2} \right| \leq \Upsilon(\alpha)$$

for all $\alpha \in \mathbb{R}^+ \cup \{0\}$. Hence (3.4) produces that $\rho(\mu r_j, r_j) \leq 1$. So, by employing the fixed point alternative Theorem 1, there exists a function $R_j : \mathbb{R}^+ \cup \{0\} \longrightarrow \mathbb{R}$ satisfying the following:

(1) R_j is a fixed point of ρ , that is,

$$R_j\left(\frac{\alpha}{m}\right) = mR_j(\alpha) \quad (3.6)$$

for all $\alpha \in \mathbb{R}^+ \cup \{0\}$. The mapping R_j is the unique fixed point of μ in the set

$$\Gamma = \{\phi \in T : \rho(R_j, r_j) < \infty\}.$$

This implies that R_j is the unique mapping satisfying (3.6) such that there exists $0 < \lambda < \infty$ satisfying

$$|R_j(\alpha) - r_j(\alpha)| \leq \lambda \Upsilon(\alpha), \quad \forall \alpha \in \mathbb{R}^+ \cup \{0\}.$$

(2) $\rho(\mu^n r_j, R_j) \rightarrow 0$ as $n \rightarrow \infty$. Thus, we have

$$\lim_{n \rightarrow \infty} m^{-n} r_j(m^{-n} \alpha) = R_j(\alpha) \quad (3.7)$$

for all $\alpha \in \mathbb{R}^+ \cup \{0\}$.

(3) $\rho(R_j, r_j) \leq \frac{1}{1-L} \rho(R_j, \mu r_j)$, which implies $\rho(R_j, r_j) \leq \frac{1}{1-L}$.

So, the inequality (3.3) holds. On the other hand, from (3.1), (3.2) and (3.7), we have

$$\begin{aligned} & \left| \frac{1}{m} R_j \left(\frac{1}{m} \sum_{k=1}^m \alpha_k \right) - \frac{1}{\sum_{k=1}^m \frac{1}{R_j(\alpha_k)}} \right| \\ &= \lim_{n \rightarrow \infty} m^{-n} \left| \frac{1}{m} r_j \left(\frac{1}{m} \sum_{k=1}^m m^{-n} \alpha_k \right) - \frac{1}{\sum_{k=1}^m \frac{1}{r_j(m^{-n} \alpha_k)}} \right| \\ &\leq \lim_{n \rightarrow \infty} m^{-n} \mathfrak{v}(m^{-n} \alpha_1, \dots, m^{-n} \alpha_m) = 0 \end{aligned}$$

for all $\alpha_k \in \mathbb{R}^+ \cup \{0\}$, which shows that R_j is a solution of the equation (1.2) and hence $R_j : \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}$ is a Jensen type m -dimensional multiplicative inverse function. $\square$

The upcoming theorem is the dual of Theorem 4. The proof is obtained by similar arguments as in Theorem 4 and hence for completeness, we present only the statement.

Theorem 5. Suppose that the mapping $r_j : \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}$ satisfies the condition $r_j(0) = \frac{1}{c_2}$ and the inequality (3.1), with a given function $\mathfrak{v} : \underbrace{\mathbb{R}^+ \cup \{0\} \times \dots \times \mathbb{R}^+ \cup \{0\}}_{m\text{-times}}$

$\rightarrow [0, \infty)$. If there exists $L < 1$ such that the mapping $\alpha \mapsto \Upsilon(\alpha) = m \mathfrak{v}(m\alpha, 0)$ has the property $\Upsilon(m\alpha) \leq \frac{1}{m} L \Upsilon(\alpha)$, for all $\alpha \in \mathbb{R}^+ \cup \{0\}$. If the function $\mathfrak{v}$ has the property

$$\lim_{n \rightarrow \infty} m^n \mathfrak{v}(m^n \alpha_1, \dots, m^n \alpha_m) = 0, \quad (3.8)$$

for all $\alpha_k \in \mathbb{R}^+ \cup \{0\}$, then there exists a unique Jensen type m -dimensional multiplicative inverse function mapping $R_j : \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}$ such that

$$|r_j(\alpha) - R_j(\alpha)| \leq \frac{1}{1-L} \Upsilon(\alpha) \quad (3.9)$$

for all $\alpha \in \mathbb{R}^+ \cup \{0\}$.

The following corollaries are the investigation of various stabilities of equation (1.2) pertinent to UHS and UHR stability. The proofs of corollaries directly follow from the above theorems.

Corollary 1. Let $r_j : \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}$ be a mapping for which there exists a constant η (independent of $\alpha_1, \dots, \alpha_m \geq 0$) such that the functional inequality

$$\left| \frac{1}{m} r_j \left(\frac{1}{m} \sum_{k=1}^m \alpha_k \right) - \frac{1}{\sum_{k=1}^m \frac{1}{r_j(\alpha_k)}} \right| \leq \frac{\eta}{2}$$

holds for all $\alpha_k \in \mathbb{R}^+ \cup \{0\}$. Then there exists a unique Jensen type m -dimensional multiplicative inverse function $R_j : \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}$ satisfying (1.2) and

$$|r_j(\alpha) - R_j(\alpha)| \leq \eta$$

for all $\alpha \in \mathbb{R}^+ \cup \{0\}$.

Proof. The proof is obtained by considering $\mathfrak{v}(\alpha_1, \dots, \alpha_m) = \frac{\eta}{2}$ and $L = \frac{1}{m}$ in Theorem 4, we attain the desired result. $\square$

Corollary 2. Let $r_j : \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}$ be a mapping and let there exist real numbers $\ell \neq -1$ and $\delta \geq 0$ such that

$$\left| \frac{1}{m} r_j \left(\frac{1}{m} \sum_{k=1}^m \alpha_k \right) - \frac{1}{\sum_{k=1}^m \frac{1}{r_j(\alpha_k)}} \right| \leq \delta \sum_{k=1}^m \alpha_k^\ell$$

for all $\alpha_k \in \mathbb{R}^+ \cup \{0\}$. Then, there exists a unique Jensen type m -dimensional multiplicative inverse function $R_j : \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}$ which satisfies (1.2) and

$$|r_j(\alpha) - R_j(\alpha)| \leq \begin{cases} \frac{m^{\ell+1}\delta}{m^{\ell+1}-1} \alpha^\ell, & \ell > -1 \\ \frac{m^\ell \delta}{1-m^{\ell+1}} \alpha^\ell, & \ell < -1 \end{cases}$$

for all $\alpha \in \mathbb{R}^+ \cup \{0\}$.

Proof. The required result is obtained by the application of Theorems 4 and 5 with $\mathfrak{v}(\alpha_1, \dots, \alpha_m) = \delta \sum_{k=1}^m \alpha_k^\ell$, for all $\alpha_k \in \mathbb{R}^+ \cup \{0\}$ and then $L = m^{-\ell-1}$ and $L = m^{\ell+1}$, respectively. $\square$

Remark 1. If upper bound is taken as $\mathfrak{v}(\alpha_1, \dots, \alpha_m) = \delta_2 \prod_{k=1}^m \alpha_k^\ell$ and then letting $\alpha_k = 0$, $k = 2, 3, \dots, m$ in Theorems 4 and 5, we observe that the upper bound becomes 0 and it is obvious that the function r_j satisfies the Jensen type m -dimensional multiplicative inverse functional equation (1.2). Similarly, if upper bound is taken as $\mathfrak{v}(\alpha_1, \dots, \alpha_m) = \delta_3 \left(\sum_{k=1}^m \alpha_k^\ell + \left(\prod_{k=1}^m \alpha_k^\ell \right) \right)$ in Theorems 4 and 5, then the result is the same as obtained Corollary 2. Hence in this study, we exclude the investigation of stabilities involving product of powers of norms and mixed product-sum of powers of norms.

4. NON-ARCHIMEDEAN STABILITIES OF EQUATION (1.2)

In this section, we solve various stability problems of equation (1.2) via fixed point scheme in non-Archimedean fields. In the upcoming results, let us presume that $\mathbb{K}$ and $\mathbb{L}$ are a non-Archimedean field and a complete non-Archimedean field, respectively. We denote $\mathbb{K}^* = \mathbb{K} - \{0\}$. For the sake of simplicity, we describe the difference operator $\Delta_{r_j} : \underbrace{\mathbb{K}^* \times \cdots \times \mathbb{K}^*}_{m\text{-times}} \longrightarrow \mathbb{L}$ by

$$\Delta_{r_j}(\alpha_1, \dots, \alpha_m) = \frac{1}{m} r_j \left(\frac{1}{m} \sum_{k=1}^m \alpha_k \right) - \frac{1}{\sum_{k=1}^m \frac{1}{r_j(\alpha_k)}}$$

for all $\alpha_k \in \mathbb{K}^*$.

Theorem 6. Suppose a function $r_j : \mathbb{K}^* \longrightarrow \mathbb{L}$ satisfies the following inequality

$$|\Delta_{r_j}(\alpha_1, \dots, \alpha_m)| \leq \varphi(\alpha_1, \dots, \alpha_m) \quad (4.1)$$

for all $\alpha_k \in \mathbb{K}^*$ with the condition that $r_j(0) = \frac{1}{c_2}$, where $\varphi : \underbrace{\mathbb{K}^* \times \cdots \times \mathbb{K}^*}_{m\text{-times}} \longrightarrow \mathbb{L}$ is

a given function. If $L \in (0, 1)$,

$$|m|^{-1} \varphi(m^{-1} \alpha_1, \dots, m^{-1} \alpha_m) \leq L \varphi(\alpha_1, \dots, \alpha_m) \quad (4.2)$$

for all $\alpha_k \in \mathbb{K}^*$, then there exists a unique Jensen type m -dimensional multiplicative inverse function $R_j : \mathbb{K}^* \longrightarrow \mathbb{L}$ satisfying (1.2) and

$$|r_j(\alpha) - R_j(\alpha)| \leq L \varphi(\alpha, \underbrace{0, \dots, 0}_{(m-1)\text{-times}}) \quad (4.3)$$

for all $\alpha \in \mathbb{K}^*$.

Proof. Let the set $\mathcal{A}$ be defined as: $\mathcal{A} = \{p | p : \mathbb{K}^* \longrightarrow \mathbb{L}\}$, and define

$$\rho(r_j, s_j) = \inf\{\mu > 0 : |r_j(\alpha) - s_j(\alpha)| \leq \mu \varphi(\alpha, \underbrace{0, \dots, 0}_{(m-1)\text{-times}}), \text{ for all } \alpha \in \mathbb{K}^*\}.$$

Example 1 produces ρ to be a complete generalized non-Archimedean complete metric on $\mathcal{A}$. Let $\Gamma : \mathcal{A} \longrightarrow \mathcal{A}$ be a mapping defined by $\Gamma(r_j)(\alpha) = m^{-1} r_j(m^{-1} \alpha)$ for all $\alpha \in \mathbb{K}^*$ and $r_j \in \mathcal{A}$. We claim that Γ is strictly contractive on $\mathcal{A}$. For given $r_j, s_j \in \mathcal{A}$, let $\mu > 0$ be a constant with $|r_j(\alpha) - s_j(\alpha)| \leq \mu \varphi(\alpha, \underbrace{0, \dots, 0}_{(m-1)\text{-times}})$, for all $\alpha \in \mathbb{K}^*$. Then

using (4.2), we obtain

$$\begin{aligned} |\Gamma(r_j)(\alpha) - \Gamma(s_j)(\alpha)| &= |m|^{-1} |r_j(m^{-1} \alpha) - s_j(m^{-1} \alpha)| \\ &\leq \mu |m|^{-1} \varphi \left(m^{-1} \alpha, \underbrace{0, \dots, 0}_{(m-1)\text{-times}} \right) \end{aligned}$$

$$\leq \mu L \varphi(\alpha, \underbrace{0, \dots, 0}_{(m-1)\text{-times}}) \quad (\alpha \in \mathbb{K}^*).$$

The above inequality implies that $\rho(\Gamma(r_j), \Gamma(s_j)) \leq L\rho(r_j, s_j)$, for $r_j, s_j \in \mathcal{A}$. Consequently, the mapping Γ is strictly contractive with Lipschitz constant L . Now, plugging $(\alpha_1, \dots, \alpha_m)$ into $(\alpha, \underbrace{0, \dots, 0}_{(m-1)\text{-times}})$ in (4.1), we have

$$\left| \frac{1}{m} r_j \left(\frac{\alpha}{m} \right) - \frac{1}{\frac{1}{r_j(\alpha)} + (m-1)c_2} \right| \leq \varphi(\alpha, \underbrace{0, \dots, 0}_{(m-1)\text{-times}}) \leq L \varphi(\alpha, \underbrace{0, \dots, 0}_{(m-1)\text{-times}}) \quad \alpha \in \mathbb{K}^*$$

which inturn implies

$$\begin{aligned} |\Gamma(r_j)(\alpha) - r_j(\alpha)| &\leq \left| \frac{1}{m} r_j \left(\frac{\alpha}{m} \right) - \frac{1}{\frac{1}{r_j(\alpha)} + (m-1)c_2} \right| \\ &\leq L \varphi(\alpha, \underbrace{0, \dots, 0}_{(m-1)\text{-times}}) \quad (\alpha \in \mathbb{K}^*). \end{aligned}$$

This indicates that $d(\Gamma(r_j), r_j) \leq L$. Due to Theorem 2 (ii), Γ has a unique fixed point $R_j : \mathbb{K}^* \rightarrow \mathbb{L}$ in the set $G = \{g \in \mathcal{A} : \rho(r_j, g) < \infty\}$ and for each $\alpha \in \mathbb{K}^*$, $R_j(\alpha) = \lim_{s \rightarrow \infty} \Gamma^s r_j(\alpha) = \lim_{s \rightarrow \infty} m^{-s} r_j(m^{-s} \alpha)$ ($\alpha \in \mathbb{K}^*$). Therefore, for all $\alpha_1, \alpha_2, \dots, \alpha_m \in \mathbb{K}^*$,

$$\begin{aligned} |\Delta_{R_j}(\alpha_1, \dots, \alpha_m)| &= \lim_{s \rightarrow \infty} |m|^{-s} |\Delta_{r_j}(m^{-s} \alpha_1, \dots, m^{-s} \alpha_m)| \\ &\leq \lim_{s \rightarrow \infty} |m|^{-s} \varphi(m^{-s} \alpha_1, \dots, m^{-s} \alpha_m) \\ &\leq \lim_{s \rightarrow \infty} L^s \varphi(\alpha_1, \dots, \alpha_m) = 0 \end{aligned}$$

which implies that the mapping R_j is a Jensen type m -dimensional multiplicative inverse function. Theorem 2 (ii) implies $\rho(r_j, R_j(\alpha)) \leq \rho(\Gamma(r_j), r_j)$, that is,

$$|r_j(\alpha) - R_j(\alpha)| \leq L \varphi(\alpha, \underbrace{0, \dots, 0}_{(m-1)\text{-times}}) \quad (\alpha \in \mathbb{K}^*).$$

Let $R'_j : \mathbb{K}^* \rightarrow \mathbb{L}$ be another Jensen type m -dimensional multiplicative inverse function which satisfies (4.3), then R'_j is a fixed point of Γ in $\mathcal{A}$. However, by Theorem 2, Γ has only one fixed point in G . This completes the uniqueness assertion of the theorem. $\square$

The succeeding theorem is dual of Theorem 6. We omit the proof as it is similar to Theorem 3.

Theorem 7. Suppose the mapping $r_j : \mathbb{K}^* \rightarrow \mathbb{L}$ satisfies the inequality (4.2) with the condition $r_j(0) = \frac{1}{c_2}$. If $L \in (0, 1)$,

$$|m| \varphi(m\alpha_1, \dots, m\alpha_m) \leq L \varphi(\alpha_1, \dots, \alpha_m)$$

for all $\alpha_k \in \mathbb{K}^*$, then there exists a unique Jensen type m -dimensional multiplicative inverse function $R_j : \mathbb{K}^* \rightarrow \mathbb{L}$ satisfying (1.2) and

$$|r_j(\alpha) - R_j(\alpha)| \leq L\varphi\left(\frac{\alpha}{m}, \underbrace{0, \dots, 0}_{(m-1)\text{-times}}\right),$$

for all $\alpha \in \mathbb{K}^*$.

The ensuing corollaries are immediate consequences of Theorems 6 and 7. In the following corollaries, we presume that $|m| < 1$ for a non-Archimedean field $\mathbb{K}^*$.

Corollary 3. Let ε (independent of $\alpha_1, \dots, \alpha_m$) be a non-negative constant. Suppose that there exists a mapping $r_j : \mathbb{K}^* \rightarrow \mathbb{L}$ such that the inequality

$$|\Delta_{r_j}(\alpha_1, \dots, \alpha_m)| \leq \frac{\varepsilon}{2},$$

for all $\alpha_k \in \mathbb{K}^*$. Then there exists a unique Jensen type m -dimensional multiplicative inverse function $R_j : \mathbb{K}^* \rightarrow \mathbb{L}$ satisfying (1.2) and $|r_j(\alpha) - R_j(\alpha)| \leq \varepsilon$, for all $\alpha \in \mathbb{K}^*$.

Proof. Considering $\varphi(\alpha_1, \dots, \alpha_m) = \varepsilon$ and then choosing $L = |m|^{-1}$ in Theorem 6, we get the desired result. $\square$

Corollary 4. Let $\lambda \neq -1$ and $\theta_1 \geq 0$ be real numbers. Suppose that there exists a mapping $r_j : \mathbb{K}^* \rightarrow \mathbb{L}$ such that $|\Delta_{r_j}(\alpha_1, \dots, \alpha_m)| \leq \theta_1 \left(\sum_{k=1}^m |\alpha_k|^\lambda\right)$, for all $\alpha_k \in \mathbb{K}^*$. Then there exists a unique Jensen type m -dimensional multiplicative inverse function $R_j : \mathbb{K}^* \rightarrow \mathbb{L}$ satisfying (1.2) and

$$|r_j(\alpha) - R_j(\alpha)| \leq \begin{cases} \frac{\theta_1}{|m|^{\lambda+1}} |\alpha|^\lambda, & \lambda > -1 \\ |m|\theta_1 |\alpha|^\lambda, & \lambda < -1 \end{cases}$$

for all $\alpha \in \mathbb{K}^*$.

Proof. The proof follows by assigning $\varphi(\alpha_1, \dots, \alpha_m) = \theta_1 \left(\sum_{k=1}^m |\alpha_k|^\lambda\right)$ in Theorems 6 and 7 and then assuming $L = |m|^{-\lambda-1}$, $\lambda > -1$ and $L = |m|^{\lambda+1}$, $\lambda < -1$, respectively. $\square$

5. PERTINENCE OF EQUATION (1.2)

We close this article, with an interpretation of equation (1.2) with arithmetic and harmonic mean of m values. Equation (1.2) can be considered as

$$r_j\left(\frac{1}{m} \sum_{k=1}^m \alpha_k\right) = \frac{m}{\sum_{k=1}^m \frac{1}{r_j(\alpha_k)}}.$$

We construe that the above equation maps the arithmetic mean of m values to harmonic mean of the images of m values. The vital applications of harmonic mean

like calculating average speed in traffic flow theory, manipulating the performance measures of objective factors in industrial engineering, analyzing the performance of transmission communication system with relays in electrical engineering render that the equation (1.2) dealt in our study is constructive in these fields.

6. CONCLUSION

In this study, we have obtained the solution of equation (1.2) as a rational function. Also, from the results obtained in this investigation, we infer that the URG stability problem and other associated stability problems of equation (1.2) have solution in the setting of non-zero real numbers and non-Archimedean fields. The equation (1.2) is interpreted with the relationship between arithmetic and harmonic mean of m values.

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SOLUTION OF COMPLEX DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS BY USING REDUCED DIFFERENTIAL TRANSFORM

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Abstract. In this article, solution of complex partial derivative equations with variable coefficients from the first and second order have been investigated. For this solution, an iteration relation was obtained using the reduced differential transform method. This method also was been applied for ordinary complex differential equations which examined in the literature. The solution which has been obtained has been seen compatible with the literature.

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1. INTRODUCTION

Some equations that don't have a general solution in real space may have a general solution in complex space. For example, although the two-dimensional Laplace equation doesn't have a general solution in real space, the equation has general solution in complex space. Two basic sources of Complex variable differential equations theory are "Theory of Pseudo Analytic Functions" and "Generalized Analytic Functions", written by L. Bers and I. N. Vekua [3, 16]. Even today, the Pompeiu integral operator, an important operator of Complex variable differential equations theory, was discovered by D. Pompeiu, an important mathematician in this theory. The Pompeiu operator has been defined as

$$T_G : C^{(\alpha)}(\overline{G}) \longrightarrow C^{(\alpha)}(\overline{G}) \quad (1.1)$$
$$f \longrightarrow T_G f = -\frac{1}{\pi} \int_G \int \frac{f(\xi)}{\xi - z} d\xi d\eta$$

This T_G operator plays a crucial role for the existence and uniqueness of equations in following form.

$$w_{\bar{z}} = F(z, w, \frac{\partial w}{\partial z}). \quad (1.2)$$

Existence and uniqueness of solutions of Equation (1.2) are investigated in [1, 2, 15].

Generalized Beltrami equation which is defined by equality

$$\frac{\partial w}{\partial \bar{z}} + q_1 \frac{\partial w}{\partial z} + q_2 \overline{\left(\frac{\partial w}{\partial \bar{z}} \right)} + aw + b\bar{w} + c = 0 \quad (1.3)$$

was reduced to singular integral equation by using Pompeiu integral operator.

In [16] linear complex differential equations have been reduced to a linear system of algebraic equations. Later approximation solution has been found by using appropriate iterative method. In [7] approximate solutions of equations of the same type have been found by using Taylor polynomial approximation. In [9], The Existence of solutions of the same type equations with analytical coefficients was investigated. In [8, 14]

$$f''(z) + A(z)f(z) = 0 \quad (1.4)$$

which $A(z)$ analytic equation has been investigated.

In this paper, it has been studied to find a special solution of the equations in the following form below that meets the given conditions.

$$A(z, \bar{z}) \cdot \frac{\partial w}{\partial z} + B(z, \bar{z}) \cdot \frac{\partial w}{\partial \bar{z}} + C(z, \bar{z}) \cdot w = F(z, \bar{z}) \quad (1.5)$$

and

$$\begin{aligned} A(z, \bar{z}) \frac{\partial^2 w}{\partial z^2} + B(z, \bar{z}) \frac{\partial^2 w}{\partial z \partial \bar{z}} + C(z, \bar{z}) \frac{\partial^2 w}{\partial \bar{z}^2} \\ + D(z, \bar{z}) \frac{\partial w}{\partial z} + E(z, \bar{z}) \frac{\partial w}{\partial \bar{z}} + F(z, \bar{z}) w = G(z, \bar{z}). \end{aligned} \quad (1.6)$$

Previously, such equations with constant coefficients were also examined by RDTM in [6]. Equations (1.5) and (1.6) were previously solved using the Adomian decomposition method in [4, 5]. In this study, RDTM is used for solution of Equations (1.5) and (1.6). With this RDTM, the solution directly is obtained without having to find the real and imaginary parts separately. RDTM was proposed firstly by Keskin in [11].

2. BASIC DEFINITIONS AND THEOREMS

Suppose $u(x, y)$ can be separated into variables as $f(x)g(y)$. Using a one-dimensional differential transformation, $u(x, y)$ can be written as follows.

$$u(x, y) = \left(\sum_{i=0}^{\infty} F_i x^i \right) \left(\sum_{j=0}^{\infty} G_j y^j \right) = \left(\sum_{k=0}^{\infty} U_k(x) \cdot y^k \right)$$

where $U_k(x)$ is called y dimensional spectrum function $u(x, y)$, and G_j, F_i are differential transform of $g(y)$ and $f(x)$, respectively.

Definition 1. If reduced differential transform of $u(x, y)$ is $U_k(x)$ than

$$U_k(x) = \frac{1}{k!} \left(\frac{\partial^k}{\partial y^k} u(x, y) \right)_{y=0}. \quad (2.1)$$

Definition 2. The differential inverse transform of $U_k(x)$ is defined as follows:

$$u(x, y) = \sum_{k=0}^{\infty} U_k(x) y^k. \quad (2.2)$$

From (2.1) and (2.2), we get

$$u(x, y) = \sum_{k=0}^{\infty} \frac{y^k}{k!} \left(\frac{\partial^k}{\partial y^k} u(x, y) \right)_{y=0} \quad (2.3)$$

Theorem 1. [11, 12] If $f(x, y) = a.g(x, y) + b.h(x, y)$ then $F_k(x) = a.G_k(x) + b.H_k(x)$.

Theorem 2. [11, 12] If $f(x, y) = x^m.y^n$, then $F_k(x) = x^m \delta(k - n)$.

Theorem 3. [11, 12] If $f(x, y) = \frac{\partial^n g(x, y)}{\partial y^n}$, then $F_k(x) = (k + 1)(k + 2) \dots (k + n) G_{k+n}(x)$.

Theorem 4. [11, 12] If $f(x, y) = \frac{\partial^n g(x, y)}{\partial x^n}$, then $F_k(x) = \frac{\partial^n G_k(x)}{\partial x^n}$.

Theorem 5. [11, 12] If $f(x, y) = g(x, y).h(x, y)$, then

$$F_k(x) = \sum_{r=0}^k G_r(x).H_{k-r}(x).$$

Now let's give the equivalent of complex derivatives in terms of real derivatives.

Definition 3. Partial derivatives of $w = w(z, \bar{z})$ function as $z = x + iy$ are defined as follows.

$$\frac{\partial w}{\partial z} = \frac{1}{2} \left(\frac{\partial w}{\partial x} - i \frac{\partial w}{\partial y} \right) \quad (2.4)$$

$$\frac{\partial w}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial w}{\partial x} + i \frac{\partial w}{\partial y} \right) \quad (2.5)$$

$$\frac{\partial^2 w}{\partial z^2} = \frac{1}{4} \left[\frac{\partial^2 w}{\partial x^2} - 2i \frac{\partial^2 w}{\partial x \partial y} - \frac{\partial^2 w}{\partial y^2} \right] \quad (2.6)$$

$$\frac{\partial^2 w}{\partial \bar{z}^2} = \frac{1}{4} \left[\frac{\partial^2 w}{\partial x^2} + 2i \frac{\partial^2 w}{\partial x \partial y} - \frac{\partial^2 w}{\partial y^2} \right] \quad (2.7)$$

$$\frac{\partial^2 w}{\partial z \partial \bar{z}} = \frac{1}{4} \left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right] \quad (2.8)$$

Theorem 6. *Let the complex function w be a holomorph in the G region. In this case, for every $z_0 \in G$ following equalities are satisfied.*

$$\begin{aligned}\frac{\partial w}{\partial z}(z_0) &= \frac{\partial w}{\partial x}(z_0) = -i \frac{\partial w}{\partial y}(z_0) \\ \frac{\partial w}{\partial z}(z_0) &= \frac{dw}{dz}(z_0), \quad \frac{\partial w}{\partial \bar{z}}(z_0) = 0\end{aligned}$$

3. SOLUTION OF COMPLEX DIFFERENTIAL EQUATIONS FROM FIRST AND SECOND ORDER WHICH IS VARIABLE COEFFICIENTS.

In this section, a separate theorem is given for each of the equations (1.5) and (1.6) giving the solution satisfying the given conditions with an iteration relation. Then, the solution is found by applying the iteration relation obtained in these theorems on the examples.

Theorem 7. *A special solution of equation*

$$A(z, \bar{z}) \frac{\partial w}{\partial z} + B(z, \bar{z}) \frac{\partial w}{\partial \bar{z}} + C(z, \bar{z}) w = F(z, \bar{z})$$

with condition

$$w(x, 0) = f(x)$$

is

$$w(z, \bar{z}) = \sum_{k=0}^{\infty} W_k(x) y^k$$

where

$$\begin{aligned}& [B^{**}(x, 0) - A^{**}(x, 0)](k+1)W_{k+1} \\ &= 2F^{**}(x, k) - \sum_{r=0}^k [A^{**}(x, r) + B^{**}(x, r)] \frac{\partial W_{k-r}}{\partial x} - 2 \sum_{r=0}^k C^{**}(x, r) W_{k-r} \\ & - \sum_{r=1}^k i [B^{**}(x, r) - A^{**}(x, r)](k+1-r) W_{k+1-r}\end{aligned}$$

and $W_0(x) = f(x)$. Here, $A^{**}(x, k)$, $B^{**}(x, k)$, $C^{**}(x, k)$, $F^{**}(x, k)$ are the differential transformations of the equalities which are obtained when are written $x + iy$ instead of z and $x - iy$ instead of $\bar{z}$ in $A(z, \bar{z})$, $B(z, \bar{z})$, $C(z, \bar{z})$, $F(z, \bar{z})$ functions.

Proof. Let's rewrite equation in which theorem.

$$A(z, \bar{z}) \frac{\partial w}{\partial z} + B(z, \bar{z}) \frac{\partial w}{\partial \bar{z}} + C(z, \bar{z}) w = F(z, \bar{z})$$

If, (2.4), (2.5) definition of complex derivatives are used, than following equality is obtained.

$$(A^*(x, y)) \frac{1}{2} \left(\frac{\partial w}{\partial x} - i \frac{\partial w}{\partial y} \right) + (B^*(x, y)) \frac{1}{2} \left(\frac{\partial w}{\partial x} + i \frac{\partial w}{\partial y} \right) + C^*(x, y)w = F^*(x, y) \quad (3.1)$$

The above equation (3.1) is equivalent to following equation.

$$\begin{aligned} [A^*(x, y) + B^*(x, y)] \frac{\partial w}{\partial x} + i[B^*(x, y) - A^*(x, y)] \frac{\partial w}{\partial y} \\ + 2C^*(x, y)w = 2F^*(x, y) \end{aligned} \quad (3.2)$$

If RDTM is applied to equation (3.2), the following equation is obtained

$$\begin{aligned} \sum_{r=0}^k [A^{**}(x, r) + B^{**}(x, r)] \frac{\partial W_{k-r}}{\partial x} \\ + \sum_{r=0}^k i[B^{**}(x, r) - A^{**}(x, r)](k+1-r)W_{k+1-r} \\ + 2 \sum_{r=0}^k C^{**}(x, r)W_{k-r} = 2F^{**}(x, k) \end{aligned} \quad (3.3)$$

If the term with the largest index is left alone in Equality (3.3), the following equality is obtained.

$$\begin{aligned} \sum_{r=0}^k i[B^{**}(x, r) - A^{**}(x, r)](k+1-r)W_{k+1-r} \\ = 2F^*(x, k) - \sum_{r=0}^k [A^{**}(x, r) + B^{**}(x, r)] \frac{\partial W_{k-r}}{\partial x} \\ - 2 \sum_{r=0}^k C^{**}(x, r)W_{k-r} \end{aligned} \quad (3.4)$$

Thus, the iteration relation required for the solution is obtained as follows from (3.4).

$$\begin{aligned} i[B^{**}(x, 0) - A^{**}(x, 0)](k+1)W_{k+1} \\ = 2F^*(x, k) - \sum_{r=0}^k [A^{**}(x, r) + B^{**}(x, r)] \frac{\partial W_{k-r}}{\partial x} - 2 \sum_{r=0}^k C^{**}(x, r)W_{k-r} \\ \sum_{r=1}^k [B^{**}(x, r) - A^{**}(x, r)](k+1-r)W_{k+1-r} \end{aligned}$$

Here $W_0(x) = f(x)$ due to condition. Thus the proof is completed. $\square$

Theorem 8. *A special solution of equation*

$$A(z, \bar{z}) \frac{\partial^2 w}{\partial z^2} + B(z, \bar{z}) \frac{\partial^2 w}{\partial z \partial \bar{z}} + C(z, \bar{z}) \frac{\partial^2 w}{\partial \bar{z}^2} + D(z, \bar{z}) \frac{\partial w}{\partial z} + E(z, \bar{z}) \frac{\partial w}{\partial \bar{z}} + F(z, \bar{z})w = G(z, \bar{z})$$

with conditions

$$w(x, 0) = f(x) \\ \frac{\partial w}{\partial y}(x, 0) = g(x).$$

is

$$w(z, \bar{z}) = \sum_{k=0}^{\infty} W_k(x) y^k$$

where

$$\begin{aligned} & [-A^{**}(x, 0) + B^{**}(x, 0) - C^{**}(x, 0)](k+1)(k+2)W_{k+2} = 4G^{**}(x, k) \\ & - 2 \sum_{r=0}^k [D^{**}(x, r) + E^{**}(x, r)] \frac{\partial W_{k-r}}{\partial x} \\ & - 2i \sum_{r=0}^k [E^{**}(x, r) - D^{**}(x, r)](k+1-r)W_{k+1-r} - 4 \sum_{r=0}^k F^{**}(x, r)W_{k-r} \\ & - \sum_{r=1}^k [-A^{**}(x, r) + B^{**}(x, r) - C^{**}(x, r)](k+1-r)(k+2-r)W_{k+2-r} \\ & - \sum_{r=0}^k [A^{**}(x, r) + B^{**}(x, r) + C^{**}(x, r)] \frac{\partial^2 W_{k-r}}{\partial x^2} \\ & - 2i \sum_{r=0}^k [C^{**}(x, r) - A^{**}(x, r)](k+1-r) \frac{\partial W_{k+1-r}}{\partial x} \end{aligned}$$

and

$$W_0(x) = f(x).$$

Here, $A^{**}(x, k), B^{**}(x, k), C^{**}(x, k), D^{**}(x, k), E^{**}(x, k), F^{**}(x, k), G^{**}(x, k)$ are the differential transformations of the equalities which are obtained when are written $x + iy$ instead of z and $x - iy$ instead of $\bar{z}$ in $A(z, \bar{z}), B(z, \bar{z}), C(z, \bar{z}), D(z, \bar{z}), E(z, \bar{z}), F(z, \bar{z}), G(z, \bar{z})$ functions.

Proof. Let's rewrite the equation in which theorem .

$$A(z, \bar{z}) \frac{\partial^2 w}{\partial z^2} + B(z, \bar{z}) \frac{\partial^2 w}{\partial z \partial \bar{z}} + C(z, \bar{z}) \frac{\partial^2 w}{\partial \bar{z}^2}$$

$$+D(z, \bar{z}) \frac{\partial w}{\partial z} + E(z, \bar{z}) \frac{\partial w}{\partial \bar{z}} + F(z, \bar{z}) w = G(z, \bar{z}).$$

If Equalities (2.4), (2.5), (2.6), (2.7), (2.8) is used above equality, than following equality is obtained.

$$\begin{aligned} A^*(x, y) \frac{1}{4} \left[\frac{\partial^2 w}{\partial x^2} - 2i \frac{\partial^2 w}{\partial x \partial y} - \frac{\partial^2 w}{\partial y^2} \right] + B^*(x, y) \frac{1}{4} \left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right] \\ + C^*(x, y) \frac{1}{4} \left[\frac{\partial^2 w}{\partial x^2} + 2i \frac{\partial^2 w}{\partial x \partial y} - \frac{\partial^2 w}{\partial y^2} \right] + D^*(x, y) \frac{1}{2} \left(\frac{\partial w}{\partial x} - i \frac{\partial w}{\partial y} \right) \\ + E^*(x, y) \frac{1}{2} \left(\frac{\partial w}{\partial x} + i \frac{\partial w}{\partial y} \right) + F^*(x, y) w = G^*(x, y). \end{aligned} \quad (3.5)$$

Where $A^*(x, y), B^*(x, y), C^*(x, y), D^*(x, y), E^*(x, y), F^*(x, y), G^*(x, y)$ expressions are obtained by writing $x + iy$ in place of z in $A(z, \bar{z}), B(z, \bar{z}), C(z, \bar{z}), D(z, \bar{z}), E(z, \bar{z}), F(z, \bar{z}), G(z, \bar{z})$ (3.5) equality is equivalent to following equality.

$$\begin{aligned} \left(\frac{A^*(x, y) + B^*(x, y) + C^*(x, y)}{4} \right) \frac{\partial^2 w}{\partial x^2} + 2i \left(\frac{C^*(x, y) - A^*(x, y)}{4} \right) \frac{\partial^2 w}{\partial x \partial y} \\ \left(\frac{-A^*(x, y) + B^*(x, y) - C^*(x, y)}{4} \right) \frac{\partial^2 w}{\partial y^2} + \left(\frac{D^*(x, y) + E^*(x, y)}{2} \right) \frac{\partial w}{\partial x} \\ + i \left(\frac{E^*(x, y) - D^*(x, y)}{2} \right) \frac{\partial w}{\partial y} + F^*(x, y) w = G^*(x, y) \end{aligned} \quad (3.6)$$

Let's apply RDTM to (3.6) equality.

$$\begin{aligned} 4G^{**}(x, k) = \sum_{r=0}^k [A^{**}(x, r) + B^{**}(x, r) + C^{**}(x, r)] \frac{\partial^2 W_{k-r}}{\partial x^2} \\ + \sum_{r=0}^k 2i [C^{**}(x, r) - A^{**}(x, r)] (k+1-r) \frac{\partial W_{k+1-r}}{\partial x} \\ + \sum_{r=0}^k [-A^{**}(x, r) + B^{**}(x, r) - C^{**}(x, r)] (k+1-r) (k+2-r) W_{k+2-r} \\ + 2 \sum_{r=0}^k [D^{**}(x, r) + E^{**}(x, r)] \frac{\partial W_{k-r}}{\partial x} \\ + 2i \sum_{r=0}^k [E^{**}(x, r) - D^{**}(x, r)] (k+1-r) W_{k+1-r} \\ + 4 \sum_{r=0}^k F^{**}(x, r) W_{k-r} \end{aligned} \quad (3.7)$$

By writing as following the expression (3.7), iteration relation in the theorem has been obtained.

$$\begin{aligned}
 & [-A^{**}(x, 0) + B^{**}(x, 0) - C^{**}(x, 0)](k+1)(k+2)W_{k+2} \\
 & = 4G^{**}(x, k) - 2 \sum_{r=0}^k [D^{**}(x, r) + E^{**}(x, r)] \frac{\partial W_{k-r}}{\partial x} \\
 & - 2i \sum_{r=0}^k [E^{**}(x, r) - D^{**}(x, r)](k+1-r)W_{k+1-r} - 4 \sum_{r=0}^k F^{**}(x, r)W_{k-r} \\
 & - \sum_{r=1}^k [-A^{**}(x, r) + B^{**}(x, r) - C^{**}(x, r)](k+1-r)(k+2-r)W_{k+2-r} \\
 & - \sum_{r=0}^k [A^{**}(x, r) + B^{**}(x, r) + C^{**}(x, r)] \frac{\partial^2 W_{k-r}}{\partial x^2} \\
 & - \sum_{r=0}^k 2i[C^{**}(x, r) - A^{**}(x, r)](k+1-r) \frac{\partial W_{k+1-r}}{\partial x} \quad (3.8)
 \end{aligned}$$

Thus the proof is completed. $\square$

Example 1. Solve the following problem

$$z \cdot w_z - \bar{z} \cdot w_{\bar{z}} = 2z^2 + 5\bar{z}$$

with the condition

$$w(x, 0) = 2x^2 - 5x.$$

Solution 1. The coefficients of the equation are $A = z, B = -z, C = 0, F = 2z^2 + 5\bar{z}$. If $x + iy, x - iy$ is written in place of $z, \bar{z}$ respectively than

$$A^* = x + iy, B^* = -x + iy, C^* = 0, F^* = 2x^2 + 5x - 2y^2 + i(4xy - 5y)$$

Reduced differential transforms of the A^*, B^*, C^*, F^* are

$$A^{**} = x\delta(k) + i\delta(k-1), B^{**} = -x\delta(k) + i\delta(k-1), C^{**} = 0,$$

$$F^{**} = (2x^2 + 5x)\delta(k) + i(4x - 5)\delta(k-1) - 2\delta(k-2).$$

Following equality has been obtained from iteration relation from theorem.

$$\begin{aligned}
 -2ix(k+1)W_{k+1} & = (4x^2 + 10x)\delta(k) + i(8x - 10)\delta(k-1) - 4\delta(k-2) \\
 & - \sum_{r=0}^k 2i\delta(r-1) \frac{\partial W_{k-r}}{\partial x} - \sum_{r=1}^k -2ix\delta(r)(k+1-r)W_{k+1-r}
 \end{aligned}$$

Because of condition in the example

$$W_0(x) = 2x^2 - 5x. \quad (3.9)$$

If $k = 0$ than

$$-2ixW_1 = 4x^2 + 10x, W_1(x) = i(2x + 5). \quad (3.10)$$

If $k = 1$ than

$$2i \frac{\partial W_0}{\partial x} - 2ixW_2 = 8ix - 10i, W_2(x) = 0. \quad (3.11)$$

If $k \geq 2$ than

$$W_k(x) = 0. \quad (3.12)$$

Solution is obtained from (3.9),(3.10),(3.11),(3.12) as following.

$$w(x, y) = \sum_{k=0}^{\infty} W_k(x) y^k = W_0(x) + W_1(x)y + W_2(x)y^2 \quad (3.13)$$

$$= 2x^2 - 5x + i(2x + 5)y. \quad (3.14)$$

Then,

$$w(z, \bar{z}) = z^2 + z\bar{z} - 5\bar{z}.$$

Example 2. Let's solve the second order variable coefficient problem below.

$$\begin{aligned} z^2 \frac{\partial^2 w}{\partial z^2} - 2\bar{z}^2 \frac{\partial^2 w}{\partial \bar{z}^2} + 5 \frac{\partial w}{\partial z} + 3z \frac{\partial w}{\partial \bar{z}} + 4w &= 10z^3 - 8z\bar{z} - 25\bar{z} \\ w(x, 0) &= x^3 - 3x^2 \\ \frac{\partial w}{\partial y}(x, 0) &= i(3x^2 - 4x) \end{aligned}$$

Solution 2. Coefficients of the equation are $A = z^2, B = 0, C = -2\bar{z}^2, D = 5, E = 3z, F = 4, G = 10z^3 - 8z\bar{z} - 25\bar{z}$.

If $x + iy$ in place of z , $x - iy$ in place of $\bar{z}$ is written than the coefficients are obtained as following.

$A^* = x^2 + 2ixy - y^2, B^* = 0, C^* = -2x^2 + 4ixy + 2y^2, D^* = 5, E^* = 3x + 3iy, F^* = 4, G^* = 10x^3 + 30x^2iy - 30xy^2 - 10iy^3 - 8x^2 - 8y^2 - 25x + 25iy$.

Let's RDTM of above equalities are

$$\begin{aligned} A^{**}(x, k) &= x^2 \delta(k) + 2ix \delta(k-1) - \delta(k-2), \\ B^{**}(x, k) &= 0, \\ C^{**}(x, k) &= -2x^2 \delta(k) + 4ix \delta(k-1) + 2\delta(k-2), \\ D^{**}(x, k) &= 5\delta(k), \\ E^{**}(x, k) &= 3x \delta(k) + 3i \delta(k-1) \end{aligned}$$

From conditions in which are given

$$W_0(x) = x^3 - 3x^2, W_1(x) = i(3x^2 - 4x) \quad (3.15)$$

From theorem

$$2x^2 W_2 = 40x^3 - 32x^2 - 100x - 18x^3 + 36x^2 - 30x^2 + 60x + 18x^3 - 24x^2$$

$$\begin{aligned}
 & -30x^2 + 40x - 16x^3 + 48x^2 + 6x^3 - 6x^2 - 36x^3 + 24x^2 \\
 W_2 = & -3x - 7
 \end{aligned} \tag{3.16}$$

$$\begin{aligned}
 6x^2W_3 = & i(120x^2 + 100) - 2i(5 + 3x)(6x - 4) - 6i(3x^2 - 6x) \\
 & - 2i(3x - 5)(-3x - 7) + 6i(3x^2 - 4x) - 16i(3x^2 - 4x) \\
 & + 12ix(-3x - 7) + 6ix^2 - 6ix(6x - 6) - 18ix^2 + 4ix(6x - 4) \\
 W_3 = & -i
 \end{aligned} \tag{3.17}$$

For $n > 3$

$$W_n = 0 \tag{3.18}$$

Thus solution is obtained by using (3.15), (3.16), (3.17), (3.18) as that.

$$\begin{aligned}
 W_0(x) + W_1(x)y + W_2(x)y^2 + W_3(x)y^3 \\
 = & x^3 - 3x^2 + i(3x^2 - 4x)y + (-3x - 7)y^2 - iy^3 \\
 = & x^3 + 3ix^2y - 3xy^2 - iy^3 - 5x^2 - 5y^2 + 2(x^2 - 2ixy - y^2)
 \end{aligned}$$

If $\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}$ is written in place of x, y respectively than solution is obtained as that.

$$w(z, \bar{z}) = z^3 - 5z\bar{z} + 2\bar{z}^2.$$

Studies in the literature are generally about ordinary complex differential equations. The two theorems and solved differential equations given in the article so far were partial differential equations. Now let's solve two ordinary complex differential equations, which have been studied by other methods in the literature, using the reduced differential transformation method.

Example 3. [7, 13] Let's solve the second order variable coefficient problem below.

$$f''(z) + zf(z) = e^z + ze^z \tag{3.19}$$

$$f(0) = f'(0) = 1 \tag{3.20}$$

Solution 3. This equation is a ordinary differential equation. The f function in the equation depends on only the variable z . This indicates that the function f is holomorph. From (2.2) it can be written that

$$f(z) = f(x + iy) = F_0(x) + F_1(x)y + F_2(x)y^2 + \dots = \left(\sum_{k=0}^{\infty} F_k(x) y^k \right)$$

Due to from $f(0) = 1$ condition

$$f(0) = F_0(0) = 1. \tag{3.21}$$

Due to from $f'(0) = 1$ condition

$$f'(0) = \frac{\partial f}{\partial x}(0, 0) = F'_0(0) = 1. \tag{3.22}$$

Let's rewrite the given equation.

$$f''(z) + zf(z) = e^z + ze^z.$$

Since f function is analytic $f''(z) = \frac{\partial^2 f}{\partial x^2} = -\frac{\partial^2 f}{\partial y^2}$. If $x + iy$, $\frac{\partial^2 f}{\partial x^2}$ is written in place of z , $f''(z)$ respectively the following equation is obtained.

$$\frac{\partial^2 f}{\partial x^2} + (x + iy)f(x + iy) = (1 + x + iy)e^x e^{iy}. \quad (3.23)$$

Let's apply the RDTM to the above equation.

$$\begin{aligned} \frac{d^2 F_k(x)}{dx^2} + xF_k(x) + i \sum_{r=0}^k \delta(r-1)F_{k-r}(x) \\ = (1+x)e^x \frac{i^k}{k!} + ie^x \sum_{r=0}^k \delta(r-1) \frac{i^{k-r}}{(k-r)!}. \end{aligned} \quad (3.24)$$

Similarly, if $x + iy$, $-\frac{\partial^2 f}{\partial y^2}$ is written in place of z , $f''(z)$ respectively, than the following equality is obtained.

$$-\frac{\partial^2 f}{\partial y^2} + (x + iy)f(x + iy) = (1 + x + iy)e^x e^{iy} \quad (3.25)$$

Let's apply the RDTM to the above (3.25) equation

$$\begin{aligned} - (k+1)(k+2)F_{k+2}(x) + xF_k(x) + i \sum_{r=0}^k \delta(r-1)F_{k-r}(x) \\ = (1+x)e^x \frac{i^k}{k!} + ie^x \sum_{r=0}^k \delta(r-1) \frac{i^{k-r}}{(k-r)!}. \end{aligned} \quad (3.26)$$

If $k = 0$ is written in equality (3.24), than following equality is obtained.

$$\frac{d^2 F_0(x)}{dx^2} + xF_0(x) = (1+x)e^x. \quad (3.27)$$

It is clear that

$$F_0(x) = e^x \quad (3.28)$$

from $F_0(0) = F'_0(0) = 1$ and (3.27). If $k = 0$ is written in the equation (3.26), following equality is obtained.

$$-2F_2(x) + xF_0(x) = (1+x)e^x \quad (3.29)$$

Therefore

$$F_2(x) = -\frac{e^x}{2}. \quad (3.30)$$

If $k = 2$ is written in the equation (3.24), following equality is obtained

$$\frac{d^2 F_2(x)}{dx^2} + xF_2(x) + iF_1(x) = -\frac{(1+x)e^x}{2} - e^x \quad (3.31)$$

From (3.30), (3.31) it is seen that

$$F_1(x) = ie^x \quad (3.32)$$

If $k = 1$ is written in the equation (3.26), following equality is obtained

$$-6F_3(x) + xF_1(x) + iF_0(x) = i(1+x)e^x + ie^x \quad (3.33)$$

Therefore

$$F_3(x) = -i\frac{e^x}{6} \quad (3.34)$$

Similarly it can be seen that

$$F_4(x) = \frac{e^x}{24}, F_5(x) = i\frac{e^x}{120}, \dots, F_n(x) = i^n \frac{e^x}{n!} \quad (3.35)$$

Thus, from (3.28), (3.30), (3.32), (3.34), (3.35) solution of the problem is that:

$$\begin{aligned} f(z) &= f(x+iy) = F_0(x) + F_1(x)y + F_2(x)y^2 + \dots \\ &= e^x + ie^x y + i^2 e^x \frac{y^2}{2} + \dots = e^x \left(1 + iy - \frac{y^2}{2} - i\frac{y^3}{6} + \dots \right) \\ &= e^x e^{iy} = e^{x+iy} = e^z \end{aligned}$$

Example 4. [13] Let's solve the second order variable coefficient problem below.

$$f''(z) + zf(z) = \frac{z^5}{12} - \frac{z^4}{6} \quad (3.36)$$

$$f(0) = 1, f'(0) = -1 \quad (3.37)$$

Solution 4. This equation is an ordinary complex differential equation like the equation in the previous example. The f function in the equation depends on only the variable z . This indicates that the function f is holomorph. From (2.2) it can be written that

$$f(z) = f(x+iy) = F_0(x) + F_1(x)y + F_2(x)y^2 + \dots = \left(\sum_{k=0}^{\infty} F_k(x) y^k \right)$$

Due to from $f(0) = 1$ condition

$$f(0) = F_0(0) = 1. \quad (3.38)$$

Due to from $f'(0) = -1$ condition

$$f'(0) = \frac{\partial f}{\partial x}(0,0) = F'_0(0) = -1. \quad (3.39)$$

Let's rewrite the given equation.

$$f''(z) + zf(z) = \frac{z^5}{12} - \frac{z^4}{6}.$$

Since f function is analytic $f''(z) = \frac{\partial^2 f}{\partial x^2} = -\frac{\partial^2 f}{\partial y^2}$. If $x + iy, \frac{\partial^2 f}{\partial x^2}$ is written in place of $z, f''(z)$ respectively the following equation is obtained.

$$\frac{\partial^2 f}{\partial x^2} + (x + iy)f(x + iy) = \frac{(x + iy)^5}{12} - \frac{(x + iy)^4}{6}. \quad (3.40)$$

Let's apply the RDTM to the above equation.

$$\begin{aligned} & \frac{d^2 F_k(x)}{dx^2} + xF_k(x) + i \sum_{r=0}^k \delta(r-1)F_{k-r}(x) \\ &= \left(\frac{x^5}{12} - \frac{x^4}{6}\right)\delta(k) + \left(\frac{5x^4}{12} - i\frac{4x^3}{6}\right)\delta(k-1) + \left(-\frac{10x^3}{12} + \frac{6x^2}{6}\right)\delta(k-2) \\ &+ \left(-\frac{10x^2}{12} + i\frac{4x}{6}\right)\delta(k-3) + \left(\frac{5x}{12} - \frac{1}{6}\right)\delta(k-4) + \frac{i}{12}\delta(k-5). \end{aligned} \quad (3.41)$$

Similarly, if $x + iy, -\frac{\partial^2 f}{\partial y^2}$ is written in place of $z, f''(z)$ respectively, than the following equality is obtained.

$$-\frac{\partial^2 f}{\partial y^2} + (x + iy)f(x + iy) = \frac{(x + iy)^5}{12} - \frac{(x + iy)^4}{6} \quad (3.42)$$

Let's apply the RDTM to the above (3.42) equation

$$\begin{aligned} & -(k+1)(k+2)F_{k+2}(x) + xF_k(x) + i \sum_{r=0}^k \delta(r-1)F_{k-r}(x) \\ &= \left(\frac{x^5}{12} - \frac{x^4}{6}\right)\delta(k) + \left(\frac{5x^4}{12} - i\frac{4x^3}{6}\right)\delta(k-1) + \left(-\frac{10x^3}{12} + \frac{6x^2}{6}\right)\delta(k-2) \\ &+ \left(-\frac{10x^2}{12} + i\frac{4x}{6}\right)\delta(k-3) + \left(\frac{5x}{12} - \frac{1}{6}\right)\delta(k-4) + \frac{i}{12}\delta(k-5). \end{aligned} \quad (3.43)$$

If $k = 0$ is written in equality (3.41), than following equality is obtained.

$$\frac{d^2 F_0(x)}{dx^2} + xF_0(x) = \left(\frac{x^5}{12} - \frac{x^4}{6}\right). \quad (3.44)$$

It is clear that $F_0(x)$ is selectable as $F_0(x) = ax^4 + bx^3 + cx^2 - x + 1$ since $F_0(0) = 1, F_0'(0) = -1$. If $F_0(x)$ is written in the equation (3.44), than a, b, c are obtained as $a = \frac{1}{12}, b = \frac{-1}{6}, c = 0$. Thus $F_0(x)$ is obtained $F_0(x) = \frac{x^4}{12} - \frac{x^3}{6} - x + 1$. If $k = 0$ is written in the equation (3.43), following equality is obtained.

$$-2F_2(x) + xF_0(x) = \frac{x^5}{12} - \frac{x^4}{6} \quad (3.45)$$

Therefore

$$F_2(x) = -\frac{x^2}{2} + \frac{x}{2}. \quad (3.46)$$

If $k = 2$ is written in the equation (3.41), following equality is obtained

$$\frac{d^2 F_2(x)}{dx^2} + xF_2(x) + iF_1(x) = -\frac{10x^3}{12} + x^2 \quad (3.47)$$

From (3.46), (3.47) it is seen that

$$F_1(x) = \frac{ix^3}{3} - \frac{ix^2}{2} - i \quad (3.48)$$

If $k = 1$ is written in the equation (3.43), following equality is obtained

$$-6F_3(x) + xF_1(x) + iF_0(x) = i \left(\frac{5x^4}{12} - \frac{4x^3}{6} \right) \quad (3.49)$$

Therefore

$$F_3(x) = i \left(\frac{1-2x}{6} \right) \quad (3.50)$$

Similarly it can be seen that

$$F_4(x) = \frac{1}{12}, F_5(x) = F_6(x) = F_7(x) = \dots = 0 \quad (3.51)$$

Thus, solution of the problem is that:

$$\begin{aligned} f(z) &= f(x+iy) = F_0(x) + F_1(x)y + F_2(x)y^2 + \dots \\ &= \frac{x^4}{12} - \frac{x^3}{6} - x + 1 + \left(\frac{ix^3}{3} - \frac{ix^2}{2} - i \right) y + \left(-\frac{x^2}{2} + \frac{x}{2} \right) y^2 + i \left(\frac{1-2x}{6} \right) y^3 + \frac{1}{12} y^4 \\ &= \frac{x^4 + 4ix^3y - 6x^2y^2 - 4ix^3y^3 + y^4}{12} - \frac{x^3 + 3ix^2y - 3xy^2 - iy^3}{6} - (x + iy) + 1 \\ &= \frac{z^4}{12} - \frac{z^3}{6} - z + 1 \end{aligned}$$

Conclusion 1. In this study, first and second order complex partial derivative equations with variable coefficients were solved by the reduced differential conversion method. The results were seen to be consistent. In addition, two ordinary derivative complex differential equations were solved by the method of reduced differential transformation. Using the method in [10], converting complex partial differential equations into ordinary differential equations and solving them can be considered as another study.

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SOME RESULTS ON I -BALLS, RADICALS, SEQUENCES AND TOPOLOGY IN BL -ALGEBRAS

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Abstract. In this paper, we obtain some new results on radicals of an ideal in BL -algebras. Further, we introduce I -balls in BL -algebras and prove that I -balls constitutes a basis for a topology on BL -algebras. We also derive some new relations of sequences in BL -algebras.

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1. INTRODUCTION

In 1958, C. C. Chang [2] devised the notion of MV -algebra in order to provide an algebraic proof of the completeness theorem of Łukasiewicz axioms for infinite valued propositional logic. In 1998, P. Hájek [7] introduced a very general many-valued logic, called Basic Logic (or BL), with the idea to formalize the many-valued semantics induced by a continuous t-norm on the unit real interval $[0, 1]$. This Basic Logic turns to be a fragment common to three important many-valued logics: $\aleph_0$ -valued Łukasiewicz logic, Gödel logic and Product logic. The Lindenbaum-Tarski algebras for Basic Logic are called BL -algebras. Apart from their logical attention, BL -algebras have important algebraic properties and they have been hard studied from an algebraic point of view. Some well-known examples of a BL -algebra are Łukasiewicz, Gödel and Product structures. These examples are defined by the unit interval $[0, 1]$ endowed with the structure induced by a continuous t-norm [7].

In 2013, C. Lele and J. B. Nganou [9], introduced the concept of ideals in BL -algebras by generalizing the notion of an ideal in MV -algebras and showed that, unlike what is in MV -algebras, in BL -algebras, the notions of filter and ideal are not dual.

A. Paad in [14], defined the notion of $rad(I)$ on a BL -algebra L , where $rad(I)$ is the radical of an ideal I of L . G. Georgescu and A. Popescu [6] defined the sequences in BL -algebras. C. Luan and Y. Yang in [10] defined the I -balls on MV -algebras. Also, the same authors, introduce a topology on MV -algebras by filters and studied some properties of the filters in MV -algebras.

Proved by Hohle [8], a BL -algebra becomes an MV -algebra if, we adjoin to the axioms the double negation law, $x = x^{**}$. Thus, a BL -algebra is in some intuitive way, a non-double negation MV -algebra. Hence the theory of MV -algebras, becomes one of the guides to the development of the theory of BL -algebras. Therefore, we define the notion of an I -ball on a BL -algebra L and derive a topology with respect to I -balls on L . We also obtain some new results of radicals and sequences in BL -algebras.

This paper is organized as follows:

In Section 2, we recall some definitions and results related to the BL -algebra and operations, which we need for the rest of the paper. In Section 3, we derive some relations on ideals, quotient ideals and $rad(I)$ on BL -algebras. We further define open I -balls and prove that open I -balls constitutes a basis for a topology. In Section 4, we obtain some new results on sequences in BL -algebras.

2. PRELIMINARIES

In this section, we recall some definitions and properties of BL -algebras which will be used throughout of the paper.

Definition 1 ([7]). An algebraic structure $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ is called a BL -algebra, if it satisfies the following conditions for all $x, y, z \in L$:

BL_1 : $(L, \wedge, \vee, 0, 1)$ is a bounded lattice relative to the order $\leq$;

BL_2 : $(L, \odot, 1)$ is a commutative monoid;

BL_3 : $x \odot y \leq z$ if and only if $x \leq y \rightarrow z$;

BL_4 : $x \wedge y = x \odot (x \rightarrow y)$;

BL_5 : $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

By L , we denote the universe of a BL -algebra $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ for any $x \in L$ and a natural number n , we define $x^* = x \rightarrow 0$, $x^n = x^{n-1} \odot x$, for $n \geq 1$, $x^0 = 1$. Let $x \in L$, if there is the least integer $n \in \mathbb{N}$ such that $x^n = 0$. We set $ord(x) = n$, if there is no such an integer, we set $ord(x) = \infty$.

A BL -algebra L is called MV -algebra, if $x^{**} = x$, for all $x \in L$. Also an element $x \in L$ is called an nilpotent element of L , if $x^n = 0$, for some $n \in \mathbb{N}$.

A BL -algebra L is linear, if for every $x, y \in L$, $x \leq y$ or $y \leq x$ [3].

An MV -algebra L is locally finite iff every element $0 \neq x \in L$ has a finite order, or equivalently, for every $0 \neq x \in L$, $nx = 1$, for some $n \in \mathbb{N}$ [2].

The following properties are well known in BL -algebras.

Proposition 1 ([7, 15]). Let L be a BL -algebra. For all $x, y, z \in L$, and $n \in \mathbb{N}$, the following statements hold:

- (1) $x \odot y \leq z$ iff $x \leq y \rightarrow z$;
- (2) $x \odot y \leq x \wedge y \leq x, y$;
- (3) $x \leq y$ implies that $x \odot z \leq y \odot z$;
- (4) $x \leq y$ iff $x \rightarrow y = 1$;

- (5) $1 \rightarrow x = x$, $x \rightarrow x = 1$, $x \leq y \rightarrow x$ and $x \rightarrow 1 = 1$;
- (6) $x \odot x^* = 0$ and $x \odot 0 = 0$;
- (7) $x \odot y = 0$ iff $x \leq y^*$;
- (8) $1^* = 0$, $0^* = 1$, $x \leq x^{**}$, $x^* = x^{***}$;
- (9) $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z = y \rightarrow (x \rightarrow z)$;
- (10) $x \leq y$ implies that $z \rightarrow x \leq z \rightarrow y$, $y \rightarrow z \leq x \rightarrow z$ and $y^* \leq x^*$;
- (11) $x \vee y = [(x \rightarrow y) \rightarrow y] \wedge [(y \rightarrow x) \rightarrow x]$;
- (12) $(x \wedge y)^{**} = x^{**} \wedge y^{**}$, $(x \vee y)^{**} = x^{**} \vee y^{**}$, $(x \odot y)^{**} = x^{**} \odot y^{**}$ and $(x \rightarrow y)^{**} = x^{**} \rightarrow y^{**}$;
- (13) $\bigvee_{i \in I} (y_i \rightarrow x) \leq (\bigwedge_{i \in I} y_i) \rightarrow x$, where L is complete and $\{y_i\}_{i \in I} \subseteq L$; $x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x)$.

Definition 2 ([4]). Let L be a BL-algebra. We define the following operations which is known for any $x, y \in L$:

- (i) $x \oplus y = (x^* \odot y^*)^*$;
- (ii) $x \ominus y = x \odot y^*$.

From [9], for every $x, y \in L$, $x \odot y = x^* \rightarrow y$.

Proposition 2 ([9]). In every BL-algebra L , the following statements hold for any $x, y, z, t \in L$:

- (i) The operation $\odot$ is associative;
- (ii) $x \leq y$ and $z \leq t$ imply $x \odot z \leq y \odot t$.

Lemma 1 ([15]). Let L be a BL-algebra. For all $x, y, z \in L$, if $x \leq y$, then the following hold:

- (i) $x \odot z \leq y \odot z$;
- (ii) $x \oplus z \leq y \oplus z$;
- (iii) $z \ominus y \leq z \ominus x$;
- (iv) $z \odot x \leq z \odot y$ and $y \odot z \leq x \odot z$.

Definition 3 ([9]). Let L be a BL-algebra and I be a nonempty subset of L , then I is an ideal of L if it satisfies the following conditions:

- (I₁) For every $x, y \in I$, $x \odot y \in I$;
- (I₂) For every $x, y \in L$, if $x \leq y$ and $y \in I$, then $x \in I$.

It is trivial to see that for any ideal I , $0 \in I$ and for every $x \in L$, $x \in I$ if and only if $x^{**} \in I$ [9]. A proper ideal I is called a maximal ideal of L , if it is not properly contained in any other ideal of L .

From [5, 15], BL-algebras are distributive lattices and a distributive lattice $\langle L, \leq, \wedge, \vee \rangle$ in which for every element $x \in L$ there is an associated element $x^* \in L$ such that for every $y \in L$, $(x \wedge x^*) \vee y = y$ and $(x \vee x^*) \wedge y = y$ is called a Boolean

algebra. The element x^* is called the lattice complement of x . The set of all complemented elements of the corresponding distributive lattice to the BL -algebra L , is a Boolean algebra and denoted by $B(L)$.

Theorem 1 ([5, 7]). *Let L be a BL -algebra. Then for $x \in L$, the following statements are equivalent:*

- (i) $x \in B(L)$;
- (ii) $x \odot x = x$ and $x^{**} = x$;
- (iii) $x \odot x = x$, $x^* \rightarrow x = x$;
- (iv) $x^* \vee x = 1$;
- (v) $(x \rightarrow y) \rightarrow x = 0$, for any $y \in L$.

Theorem 2 ([13]). *Let M be a proper ideal of a BL -algebra L . Then the following conditions are equivalent:*

- (i) M is a maximal ideal of L ;
- (ii) For all $x \notin M$, there exists $n \in \mathbb{N}$, $(x^*)^n \in M$;
- (iii) $\frac{L}{M}$ is a locally finite MV -algebra.

Definition 4 ([9]). Let I be a proper ideal of a BL -algebra L . Then the intersection of all maximal ideals of L that contain I is called the radical of I and is denoted by $rad(I)$.

From [14], $rad(I)$ is an ideal of L and $I \subseteq rad(I)$.

Theorem 3 ([14]). *Let L be a BL -algebra and I be a proper ideal of L . Then $rad(I) = \{x \in L \mid (x \rightarrow (x^*)^n)^* \in I, \text{ for all } n \in \mathbb{N}\}$.*

Definition 5 ([14]). An element a of a BL -algebra L is called unity if, for all $n \in \mathbb{N}$, $((a^n)^*)^k = 0$, for some $k \in \mathbb{N}$, i.e., $(a^n)^*$ is a nilpotent element of L .

Definition 6 ([7]). Let X and Y be two BL -algebras. A map $f : X \rightarrow Y$ is called a BL -homomorphism if, for all $x, y \in X$:

- (i) $f(x \odot y) = f(x) \odot f(y)$;
- (ii) $f(x \rightarrow y) = f(x) \rightarrow f(y)$;
- (iii) $f(0_X) = 0_Y$.

If $f : X \rightarrow Y$ is a BL -homomorphism, then the kernel of f is the set

$$\ker(f) = \{x \in X \mid f(x) = 0_Y\}.$$

From [9] the following, as immediate consequent of Definition 6, are hold, for all $x, y \in X$:

- (i) $f(x \wedge y) = f(x) \wedge f(y)$;
- (ii) $f(x \vee y) = f(x) \vee f(y)$;
- (iii) $f(x^*) = (f(x))^*$;
- (iv) $f(1_X) = 1_Y$;

- (v) If $x \leq y$, then $f(x) \leq f(y)$;
- (vi) $f(x \odot y) = f(x) \odot f(y)$.

Definition 7 ([3]). In a BL-algebra L , the distance function $d : L \times L \rightarrow L$ is defined by $d(x, y) = (x \rightarrow y) \wedge (y \rightarrow x)$, for all $x, y \in L$.

Proposition 3 ([3, 6]). Let L be a BL-algebra. Then the following statements hold:

- (i) $d(x, y) = d(y, x)$;
- (ii) $d(x, y) = 1$ if and only if $x = y$;
- (iii) $d(x, 1) = x, d(x, 0) = x^*$;
- (iv) $d(x, z) \odot d(z, y) \leq d(x, y)$;
- (v) $d(x, y) \leq d(x \odot u, y \odot u)$;
- (vi) $d(x, u) \odot d(y, v) \leq d(y \rightarrow x, v \rightarrow u)$;
- (vii) $d(x, u) \wedge d(y, v) \leq d(x \wedge y, u \wedge v)$;
- (viii) $d(x, u) \wedge d(y, v) \leq d(x \vee y, u \vee v)$;
- (ix) $d(x, y) \leq d(x^*, y^*)$.

Definition 8 ([3]). Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in a BL-algebra L . If $(x_n)_{n \in \mathbb{N}}$ is increasing, we denote $(x_n)_{n \in \mathbb{N}} \uparrow$. Similarly, if $(x_n)_{n \in \mathbb{N}}$ is decreasing, we denote $(x_n)_{n \in \mathbb{N}} \downarrow$. If $(x_n)_{n \in \mathbb{N}}$ is increasing, $\bigwedge_n x_n$ exists and $\bigwedge_n x_n = x$, we denote $(x_n)_{n \in \mathbb{N}} \uparrow x$. Similarly, if $(x_n)_{n \in \mathbb{N}}$ is decreasing, $\bigvee_n x_n$ exists and $\bigvee_n x_n = x$, we denote $(x_n)_{n \in \mathbb{N}} \downarrow x$.

Definition 9 ([3]). Let L be a BL-algebra and $(x_n)_{n \in \mathbb{N}}$ be a sequence in L . Then $(x_n)_{n \in \mathbb{N}}$ converges to $x \in L$, if there exists a sequence $(s_n)_{n \in \mathbb{N}}$ in L such that $(s_n)_{n \in \mathbb{N}} \uparrow 1$ and $d(x_n, x) \geq s_n$ for all $n \in \mathbb{N}$, which we denote by $x_n \rightarrow_s x$.

Definition 10 ([3]). A BL-algebra L is called special if, $(x \rightarrow y)^* = (y \rightarrow x)^*$, for all $x, y \in L$.

We denote a special BL-algebra L by L^* .

Proposition 4 ([11]). For a BL-algebra L , the following conditions are equivalent:

- (i) L is a special BL-algebra;
- (ii) $x^* = 0$, for any $0 \neq x \in L$.

Note that if I is an ideal of a special BL-algebra L , then $\frac{L}{I}$ is a special BL-algebra.

3. SOME RESULTS ON RADICALS AND TOPOLOGY ON BL-ALGEBRAS

In this section, we obtain some new results on radical of an ideal of a BL-algebra L . We also define the open I -balls on L and prove that these open I -balls constitute a basis for a topology on L .

Lemma 2. Let L be a BL-algebra. Then for all $x, y, z \in L$, the following hold:

- (i) $x \odot (y \odot z) = (x \odot y) \odot z$;
- (ii) $x \ominus (y \oplus z) = (x \ominus y) \ominus z$;

- (iii) $x \odot (y \odot z) = y \odot (x \odot z)$;
 (iv) $x \odot (y \odot z) \leq (x \odot y) \odot z$.

Proof. Let $x, y, z \in L$, then by Definition 2 and Proposition 1, we conclude:

- (i) $x \odot (y \odot z) = x^* \rightarrow (y \odot z) = x^* \rightarrow (y^* \rightarrow z) = (x^* \odot y^*) \rightarrow z = (x^{***} \odot y^{***}) \rightarrow z$. By Proposition 1, (12), we have, $(x^{***} \odot y^{***}) \rightarrow z = (x^* \odot y^*)^{**} \rightarrow z = (x^* \odot y^*)^* \odot z = (x \oplus y) \odot z$.
 (ii) $x \odot (y \oplus z) = x \odot (y \oplus z)^* = x \odot (y^* \odot z^*) = (x \odot y^*) \odot z^* = (x \odot y^*) \ominus z = (x \oplus y) \ominus z$.
 (iii) $x \odot (y \odot z) = x \odot (y^* \rightarrow z) = x^* \rightarrow (y^* \rightarrow z)$. By Proposition 1, (9), $x^* \rightarrow (y^* \rightarrow z) = y^* \rightarrow (x^* \rightarrow z) = y \odot (x^* \rightarrow z) = y \odot (x \odot z)$.
 (iv) By applying BL_3 , we have, $0 = z \odot 0 = z \odot (y \odot y^*) = (y \odot z) \odot y^* \leq z$ if and only if $y \odot z \leq y^* \rightarrow z = y \odot z$, therefore, $x \odot (y \odot z) \leq x \odot (y \odot z) = (x \odot y) \odot z$. $\square$

Lemma 3. Let L be a BL-algebra and $x, y \in L$. Then, the following conditions are equivalent:

- (i) $x^* \oplus y = 1$;
 (ii) $x \ominus y = 0$;
 (iii) $x \leq y^{**}$.

Proof. (i) $\Rightarrow$ (ii) By Definition 2, $1 = x^* \oplus y = (x^{**} \odot y^*)^* = (x^{**} \odot y^{***})^* = (x \odot y^*)^{***} = (x \odot y^*)^*$. This means that $x \odot y^* = 0$, i.e., $x \ominus y = 0$.
 (ii) $\Rightarrow$ (iii) Suppose $x \ominus y = 0$. Thus $x \odot y^* = 0$, i.e., $x \leq y^{**}$.
 (iii) $\Rightarrow$ (i) Since $x \leq y^{**}$, so $y^{***} \leq x^*$, i.e., $y^* \leq x^*$. This means that $y^* \odot x = 0$. By Proposition 1, $(y^* \odot x)^* = (y^* \odot x)^{***} = (y^{***} \odot x^{**})^* = (y^* \odot x^{**})^* = 1$, therefore $x^* \oplus y = 1$. $\square$

Corollary 1. Let L be a BL-algebra and $x, y, z \in L$. If $x \ominus y \leq z$, then $x \leq y \odot z$.

Proof. $x \ominus y \leq z$ iff $x \odot y^* \leq z$ iff $x \leq y^* \rightarrow z$, which is equivalent to $x \leq y \odot z$. $\square$

Theorem 4. Let I and K be two ideals of a BL-algebra L such that $K \subseteq I$. Then $\frac{I}{K}$ is a proper ideal of $\frac{L}{K}$ if and only if $rad(\frac{I}{K})$ is a proper ideal of $\frac{L}{K}$.

Proof. Let $\frac{I}{K}$ be a proper ideal of $\frac{L}{K}$ and $rad(\frac{I}{K})$ is not a proper ideal, then $rad(\frac{I}{K}) = \frac{L}{K}$, i.e., $\bigcap_{\frac{M}{K} \in Max(\frac{L}{K})} \frac{M}{K} = \frac{L}{K}$, such that $\frac{I}{K} \subseteq \frac{M}{K}$ and $Max(\frac{L}{K})$ is the set of all maximal ideals of $\frac{L}{K}$, this means that $\frac{M}{K} = \frac{I}{K}$ and $rad(\frac{I}{K}) = \frac{I}{K}$, therefore $\frac{I}{K} = \frac{L}{K}$, which is a contradiction.

Conversely, let $rad(\frac{I}{K})$ be a proper ideal of $\frac{L}{K}$. Then $\frac{y}{K} \notin rad(\frac{I}{K})$, for some $\frac{y}{K} \in \frac{L}{K}$. There exists a maximal ideal $\frac{M}{K}$ such that $\frac{I}{K} \subseteq \frac{M}{K}$ with $\frac{y}{K} \notin \frac{M}{K}$. We suppose that $\frac{I}{K}$ is not proper, then $\frac{I}{K} = \frac{L}{K}$, i.e., for all $\frac{x}{K} \in \frac{L}{K}$. Since $\frac{I}{K} \subseteq \frac{M}{K}$, so $\frac{x}{K} \in \frac{M}{K}$ for all $\frac{x}{K} \in \frac{L}{K}$. Therefore, $\frac{y}{K} \in \frac{M}{K}$, which is a contradiction. $\square$

Theorem 5. Let I, J and K be proper ideals of a BL-algebra L such that $K \subseteq I$, $K \subseteq J$. Then the following assertions hold:

- (i) If for all $x \in L$, $x^* = 1$, then $\text{rad}(I) = L$;
- (ii) If $\frac{I}{K} \subseteq \frac{J}{K}$, then $\text{rad}(\frac{I}{K}) \subseteq \text{rad}(\frac{J}{K})$;
- (iii) $\text{rad}(\frac{I}{K}) = \frac{L}{K}$ if and only if $\frac{I}{K} = \frac{L}{K}$;
- (iv) $\text{rad}(\text{rad}(\frac{I}{K})) = \text{rad}(\frac{I}{K})$.

Proof. (i) Let for all $x \in L$, $x^* = 1$, then $(x \rightarrow (x^*)^n)^* = (x \rightarrow 1)^* = 1^* = 0 \in I$. So by Theorem 3, $x \in \text{rad}(I)$. Then $L \subseteq \text{rad}(I)$. Thus $\text{rad}(I) = L$.

(ii) If $\frac{I}{K} \subseteq \frac{J}{K}$ and $\frac{x}{K} \in \text{rad}(\frac{I}{K})$, then by Theorem 3, $(\frac{x}{K} \rightarrow (\frac{x}{K})^n)^* \in \frac{I}{K} \subseteq \frac{J}{K}$, for all $n \in \mathbb{N}$, i.e., $\frac{x}{K} \in \text{rad}(\frac{J}{K})$. Hence $\text{rad}(\frac{I}{K}) \subseteq \text{rad}(\frac{J}{K})$.

(iii) Suppose $\text{rad}(\frac{I}{K}) = \frac{L}{K}$. Since $\frac{0}{K} \in \frac{L}{K}$, so $\frac{0}{K} \in \text{rad}(\frac{I}{K})$ and by Theorem 3, $\frac{0}{K} \rightarrow ((\frac{0}{K})^*)^n \in \frac{I}{K}$ for all $n \in \mathbb{N}$. We conclude $(\frac{0}{K} \rightarrow \frac{1}{K})^* \in \frac{I}{K}$, i.e., $(\frac{1}{K})^* \in \frac{I}{K}$ and $\frac{0}{K} \in \frac{I}{K}$, therefore, $\frac{I}{K} = \frac{L}{K}$.

Conversely, let $\frac{I}{K} = \frac{L}{K}$. Since $\frac{I}{K} \subseteq \frac{M}{K}$, so we have, $\text{rad}(\frac{I}{K}) = \bigcap_{\frac{M}{K} \in \text{Max}(\frac{L}{K})} \frac{M}{K} = \frac{L}{K}$.

(iv) By (ii) and the fact $\frac{I}{K} \subseteq \text{rad}(\frac{I}{K})$, we conclude, $\text{rad}(\frac{I}{K}) \subseteq \text{rad}(\text{rad}(\frac{I}{K}))$.

Conversely, let $\frac{x}{K} \in \text{rad}(\text{rad}(\frac{I}{K}))$, then $\frac{x}{K} \in \frac{M}{K}$ for any $\frac{M}{K} \in \text{Max}(\frac{L}{K})$ with $\text{rad}(\frac{I}{K}) \subseteq \frac{M}{K}$. Now, let $\frac{N}{K}$ be a arbitrary maximal ideal of $\frac{L}{K}$ such that $\frac{I}{K} \subseteq \frac{N}{K}$. Then by (ii), $\text{rad}(\frac{I}{K}) \subseteq \text{rad}(\frac{N}{K}) = \frac{N}{K}$, so $\frac{x}{K} \in \frac{N}{K}$ and $\frac{x}{K} \in \text{rad}(\frac{I}{K})$. Thus $\text{rad}(\text{rad}(\frac{I}{K})) \subseteq \text{rad}(\frac{I}{K})$ and hence $\text{rad}(\text{rad}(\frac{I}{K})) = \text{rad}(\frac{I}{K})$. $\square$

Theorem 6. Let I and J be two ideals of a BL-algebra L . If $a \in I$ and $a \leq b$, for some $b \in J$, then $\text{rad}(I) \subseteq \text{rad}(J)$.

Proof. Let $a \leq b$ for some $b \in J$. Since J is an ideal, so $a \in J$, i.e., $I \subseteq J$. Thus by Theorem 5, $\text{rad}(I) \subseteq \text{rad}(J)$. $\square$

Theorem 7. Let I, J and K be ideals of a BL-algebra L such that $K \subseteq I, J$. Then $\text{rad}(\frac{I \cap J}{K}) = \text{rad}(\frac{I}{K}) \cap \text{rad}(\frac{J}{K})$.

Proof. We know that $\frac{I}{K} \subseteq \text{rad}(\frac{I}{K})$ and $\frac{J}{K} \subseteq \text{rad}(\frac{J}{K})$. Then $\frac{I \cap J}{K} \subseteq \frac{I}{K} \subseteq \text{rad}(\frac{I}{K})$. So by Theorem 5, $\text{rad}(\frac{I \cap J}{K}) \subseteq \text{rad}(\text{rad}(\frac{I}{K})) = \text{rad}(\frac{I}{K})$. Similarly, $\text{rad}(\frac{I \cap J}{K}) \subseteq \text{rad}(\frac{J}{K})$. Thus $\text{rad}(\frac{I \cap J}{K}) \subseteq \text{rad}(\frac{I}{K}) \cap \text{rad}(\frac{J}{K})$.

Conversely, let $\frac{x}{K} \in \text{rad}(\frac{I}{K}) \cap \text{rad}(\frac{J}{K})$, then $\frac{x}{K} \in \text{rad}(\frac{I}{K})$ and $\frac{x}{K} \in \text{rad}(\frac{J}{K})$. By Theorem 3, $(\frac{x}{K} \rightarrow (\frac{x}{K})^n)^* \in \frac{I}{K}$ and $(\frac{x}{K} \rightarrow (\frac{x}{K})^n)^* \in \frac{J}{K}$ for all $n \in \mathbb{N}$. Therefore $(\frac{x}{K} \rightarrow (\frac{x}{K})^n)^* \in \frac{I}{K} \cap \frac{J}{K} = \frac{I \cap J}{K}$, for all $n \in \mathbb{N}$, i.e., $\frac{x}{K} \in \text{rad}(\frac{I \cap J}{K})$ and hence $\text{rad}(\frac{I}{K}) \cap \text{rad}(\frac{J}{K}) \subseteq \text{rad}(\frac{I \cap J}{K})$. $\square$

Theorem 8. Let I and K be two ideals of a BL-algebra L such that $K \subseteq I$. Then $\text{rad}(\frac{I}{K}) \cap B(\frac{L}{K}) \subseteq \frac{I}{K}$, where $B(\frac{L}{K})$ is the Boolean center of BL-algebra $\frac{L}{K}$.

Proof. Let $\frac{x}{K} \in \text{rad}(\frac{I}{K}) \cap B(\frac{L}{K})$, then $\frac{x}{K} \in \text{rad}(\frac{I}{K})$ and $\frac{x}{K} \in B(\frac{L}{K})$. By Theorem 1 and Theorem 3, we conclude that $(\frac{x}{K} \rightarrow (\frac{x}{K})^n)^* \in \frac{I}{K}$, for all $n \in \mathbb{N}$, $\frac{x}{K} \odot \frac{x}{K} = \frac{x}{K}$ and

$\frac{x^{**}}{K} = \frac{x}{K}$. This means that $(\frac{x}{K} \rightarrow (\frac{x^*}{K})^n)^* \in \frac{I}{K}$, for all $n \in \mathbb{N}$, i.e., $\frac{x^n}{K} = \frac{x}{K}$ and $\frac{x^{**}}{K} = \frac{x}{K}$. Therefore $(\frac{x}{K} \rightarrow \frac{x^*}{K})^* \in \frac{I}{K}$. Since $\frac{x^*}{K} \rightarrow \frac{x}{K} = \frac{x}{K}$, so $\frac{x^{**}}{K} \rightarrow \frac{x^*}{K} = \frac{x^*}{K}$. We have, $\frac{x^{**}}{K} = \frac{x}{K}$, i.e., $\frac{x}{K} \rightarrow \frac{x^*}{K} = \frac{x^*}{K}$, therefore $(\frac{x^*}{K})^* \in I$ and hence $\frac{x}{K} \in I$. $\square$

From Theorem 8, we conclude the following results:

Corollary 2. $rad\{1\} \cap B(L) = \{1\}$.

Corollary 3. $rad\{0\} \subseteq \{x \in L \mid x \text{ is a nilpotent element}\}$.

Proof. Let $x \in rad\{0\}$. Then by Theorem 3, $(x \rightarrow (x^*)^n)^* = 0$, for all $n \in \mathbb{N}$, i.e., $x \rightarrow (x^*)^n = 1$, and $x \rightarrow (x^n)^* = 1$, thus by Proposition 1, (4), $x \leq (x^n)^*$ and hence $x \odot x^n = 0$. This means that $x^{n+1} = 0$ and x is a nilpotent element. $\square$

Theorem 9. Let I be an ideal of a BL-algebra L . Then the following assertions hold:

- (i) $D(I) \subseteq rad(I)$, where $D(I) = \{x \in I \mid x^* = 1\}$;
- (ii) $rad\left(\frac{I}{D(I)}\right) = \frac{rad(I)}{D(I)}$.

Proof. (i) Let $x \in D(I)$. Then $x^* = 1$ and $(x \rightarrow (x^*)^n)^* = (x \rightarrow 1)^* = 1^* = 0 \in I$, i.e., $x \in rad(I)$.

(ii) We know that $rad\left(\frac{I}{D(I)}\right) = \bigcap \frac{N}{D(I)}$. Since N is a maximal ideal and $D(I) \subseteq I \subseteq N$, so $rad\left(\frac{I}{D(I)}\right) = \frac{\bigcap N}{D(I)} = \frac{rad(I)}{D(I)}$. $\square$

Theorem 10. Let I be an ideal of a BL-algebra L . If $D(L) \subseteq I$, then $\frac{L}{I}$ is an MV-algebra.

Proof. Suppose $D(L) \subseteq I$ and $\frac{L}{I}$ is not an MV-algebra. Then there exists $x \in L$ such that $(\frac{x}{I})^{**} \neq \frac{x}{I}$. We have $\frac{x}{I} \leq (\frac{x}{I})^{**}$ and $(\frac{x}{I})^{**} \not\leq \frac{x}{I}$. Thus $(\frac{x}{I})^{**} \rightarrow \frac{x}{I} \neq 1$, i.e., $x^{**} \rightarrow x \notin I$ and $x^{**} \rightarrow x \notin D(I)$. Therefore $(x^{**} \rightarrow x)^* \neq 1$, i.e., $x^{**} \rightarrow x \neq 0$ which is a contradiction. $\square$

Theorem 11. Let L be a linear BL-algebra and I be a proper ideal of BL-algebra L . If x is a unity element of L , then $x^* < x$.

Proof. Let x be a unity element in L and $x < x^*$, then $(x^n)^* = 0$, for all $n \in \mathbb{N}$. We have $x \leq x^*$, i.e., $x^2 = x \odot x = 0$. Therefore, $(x^2)^* = 1$, which is a contradiction with $(x^n)^* = 0$. Since L is a linear BL-algebra and $x \not\leq x^*$, thus $x^* < x$. $\square$

Lemma 4. Let X and Y be two BL-algebras and $f : X \rightarrow Y$ be a BL-homomorphism. Then $f(d(x, y)) = d(f(x), f(y))$, for all $x, y \in X$.

Proof. By Definitions 6 and 7, we have:

$$\begin{aligned} f(d(x, y)) &= f((x \rightarrow y) \wedge (y \rightarrow x)) \\ &= f(x \rightarrow y) \wedge f(y \rightarrow x) \end{aligned}$$

$$\begin{aligned}
&= (f(x) \rightarrow f(y)) \wedge (f(y) \rightarrow f(x)) \\
&= d(f(x), f(y)).
\end{aligned}$$

□

Theorem 12. Let X and Y be two BL-algebras, and $f : X \longrightarrow Y$ be a BL-homomorphism. If K and J are two ideals of Y such that $K \subseteq J$, then the following assertions hold:

- (i) $\text{rad}(f^{-1}(\frac{J}{K})) = f^{-1}(\text{rad}(\frac{J}{K}))$;
- (ii) $\text{rad}(\ker(f)) = f^{-1}(\text{rad}\{1\})$.

Proof. (i) Let $\frac{x}{K} \in \text{rad}(f^{-1}(\frac{J}{K}))$, then by Theorem 3, $(\frac{x}{K} \rightarrow ((\frac{x}{K})^*)^n)^* \in f^{-1}(\frac{J}{K})$, for all $n \in \mathbb{N}$. So $f(\frac{x}{K} \rightarrow ((\frac{x}{K})^*)^n)^* \in \frac{J}{K}$ and by Definition 6, $f((\frac{x}{K}) \rightarrow ((\frac{x}{K})^*)^n)^* = (\frac{f(x)}{K} \rightarrow (\frac{f(x^*)}{K})^n)^* \in \frac{J}{K}$. Therefore $f(\frac{x}{K}) \in \text{rad}(\frac{J}{K})$ and hence $\frac{x}{K} \in f^{-1}(\text{rad}(\frac{J}{K}))$. Conversely, it is clear by the similar way.

(ii) Let $x \in \text{rad}(\ker(f))$, then by Theorem 3, $(x \rightarrow (x^*)^n)^* \in \ker(f)$, for all $n \in \mathbb{N}$. So by Definition 6, $(f(x) \rightarrow f(x^*)^n)^* = 0$, for all $n \in \mathbb{N}$, i.e., $(f(x) \rightarrow f(x^*))^n = 1$, for all $n \in \mathbb{N}$. This means that $f(x) \in \text{rad}\{1\}$ and $x \in f^{-1}(\text{rad}\{1\})$. Conversely, it is clear by the same way. □

Definition 11. Let L be a BL-algebra and I be an ideal of L . By $U_{x_0, r}^*$, we define the open I -ball of radius $r \in I$, with center x_0 (around x_0), by

$$U_{x_0, r}^* = \{x \in L \mid (r \odot d(x, x_0))^* \in I\}.$$

Proposition 5. Let L be a BL-algebra and I be an ideal of L . Then the following assertions hold, for all $x, y \in L$ and $r, s \in I$:

- (i) $U_{1, 0}^* = \{x \in L \mid x^* \in I\}$;
- (ii) If $y \in U_{x, r}^*$ then $y^* \in U_{x^*, r}^*$;
- (iii) If $s \leq r$ then $U_{x, s}^* \subseteq U_{x, r}^*$.

Proof. (i) By Definition 11,

$$\begin{aligned}
U_{1, 0}^* &= \{x \in L \mid (0 \odot d(x, 1))^* \in I\} = \{x \in L \mid (0^* \rightarrow d(x, 1))^* \in I\} \\
&= \{x \in L \mid (1 \rightarrow x)^* \in I\} = \{x \in L \mid x^* \in I\}.
\end{aligned}$$

(ii) Suppose $y \in U_{x, r}^*$, then $(r \odot d(x, y))^* \in I$. By Proposition 3, since $d(x, y) \leq d(x^*, y^*)$, so $r \odot d(x, y) \leq r \odot d(x^*, y^*)$. Therefore $(r \odot d(x^*, y^*))^* \leq (r \odot d(x, y))^*$. I is an ideal, thus $(r \odot d(x^*, y^*))^* \in I$. Therefore $y^* \in U_{x^*, r}^*$.

(iii) Suppose $y \in U_{x, s}^*$, then $(s \odot d(x, y))^* \in I$. By assumption $s \leq r$, then $s \odot d(x, y) \leq r \odot d(x, y)$. Thus $(r \odot d(x, y))^* \leq (s \odot d(x, y))^*$. Since I is an ideal, so $(r \odot d(x, y))^* \in I$. Therefore $y \in U_{x, r}^*$. □

From [12], we recall that if X is a set, a basis for a topology on X is a collection $\mathcal{B}$ of subsets of X (called basis element) such that the following hold:

- (i) For each $x \in X$, there is at least one basis element B containing x .
- (ii) If x belongs to the intersection of two basis element B_1 and B_2 , then there is a basis element B_3 containing x such that $B_3 \subseteq B_1 \cap B_2$.

Proposition 6. *Let L be a BL-algebra and I be an ideal of L . Then the open I -balls constitute a basis for a topology on L (we call this topology, ideal topology).*

Proof. Let $x \in L$ and $r \in I$. By Propositions 1 and 3, $0 = 1^* = (r^* \rightarrow 1)^* = (r^* \rightarrow d(x, x))^* = (r \odot d(x, x))^*$. Since I is an ideal and $0 \in I$, so $(r \odot d(x, x))^* \in I$, i.e., $x \in U_{x,r}^*$. Thus there exists an element of the I -balls of topology, which contains x , for all $x \in L$. Now, let $t \in U_{x,r}^* \cap U_{y,s}^*$, then $(r \odot d(x, t))^* \in I$ and $(s \odot d(t, y))^* \in I$. This means that there exist $c^*, d^* \in I$, such that $c = r \odot d(x, t)$, $d = s \odot d(t, y)$. We put $e = c^* \vee d^*$ and claim that $U_{t,e}^* \subseteq U_{x,r}^* \cap U_{y,s}^*$. Let $z \in U_{t,e}^*$, then $(e \odot d(z, t))^* = k^* \in I$, for some $k^* \in I$. By Proposition 3, since $d(z, t) \odot d(t, x) \leq d(z, x)$, so $r \odot (d(z, t) \odot d(t, x)) \leq r \odot d(z, x)$. Also, by Lemma 2, $(r \odot d(x, t))^* \odot d(z, t) \leq r \odot (d(x, t) \odot d(z, t))$. Since $c^* \leq c^* \vee d^*$, we conclude $(c^* \vee d^*) \odot d(z, t) \leq c^* \odot d(z, t)$, hence $e \odot d(z, t) \leq r \odot d(z, t)$. Therefore $(r \odot d(z, t))^* \leq (e \odot d(z, t))^*$. Since I is an ideal, so $(r \odot d(z, t))^* \in I$ and it follows that $z \in U_{x,r}^*$. By the similar way, we conclude that, if $z \in U_{t,e}^*$, then $z \in U_{y,s}^*$. Therefore, $t \in U_{x,r}^* \cap U_{y,s}^*$. $\square$

Proposition 7. *Every ideal topology on a BL-algebra of L , makes L into a topological BL-algebra.*

Proof. By [1, 16], it is enough to show that the operations $\odot$ and $*$ are continuous. First, we consider the mapping $\odot : L \times L \rightarrow L$ by $(x, y) \mapsto x \odot y$. For $e \in I$, let U be an open I -ball of radius e around $t \odot s$ and V be an open I -ball of radius e around t . Then, $V = \{x \in L \mid (e \odot d(x, t))^* \in I\}$ and $U = \{x \in L \mid (e \odot d(x, t \odot s))^* \in I\}$. Take $x \in V$ and we assume that $e \odot d(x, t) = e_x \in I$. Let W be an open I -ball of radius e_x around s , then $V \times W$ is an open neighborhood around (t, s) . So, by Proposition 3 and Lemma 2, we have

$$\begin{aligned} (e \odot d(x \odot y, t \odot s))^* &\leq (e \odot (d(x, t) \odot d(y, s)))^* \\ &\leq ((e \odot d(x, t)) \odot d(y, s))^* \\ &= (e_x \odot d(y, s))^* \in I. \end{aligned}$$

Since I is an ideal of L , so $(e \odot d(x \odot y, t \odot s))^* \in I$ and hence $\odot(V \times W) \subseteq U$.

Now, we prove that the mapping $*$: $L \rightarrow L, x \mapsto x^*$ is continuous. Let $e \in L, t \in I$ and U be an open I -ball of radius e around t^* and V be an open I -ball of radius e around t . By Propositions 2 and 3, since $d(x, t) \leq d(x^*, t^*)$, so $e \odot d(x, t) \leq e \odot d(x^*, t^*) \in I$. Then $(e \odot d(x^*, t^*))^* \leq (e \odot d(x, t))^* \in I$. Since I is an ideal of L , so $(e \odot d(x^*, t^*))^* \in I$ and $x^* \in U$. Therefore $V^* \subseteq U$ and hence the mapping $*$ is continuous. $\square$

4. SEQUENCES IN BL-ALGEBRAS

In this section we derive some new results on sequences in BL-algebras.

Theorem 13. *Let I be an ideal of a BL-algebra L and $(\frac{x_n}{I})_{n \in \mathbb{N}}$, $(\frac{y_n}{I})_{n \in \mathbb{N}}$ be two sequences in $\frac{L}{I}$ such that $(\frac{x_n}{I})_{n \in \mathbb{N}} \uparrow 1_I$ and $(\frac{y_n}{I})_{n \in \mathbb{N}} \uparrow 1_I$, then $(\frac{x_n \odot y_n}{I})_{n \in \mathbb{N}} \uparrow 1_I$.*

Proof. Let $(\frac{x_n}{I})_{n \in \mathbb{N}} \uparrow 1_I$, $(\frac{y_n}{I})_{n \in \mathbb{N}} \uparrow 1_I$ and $\frac{t}{I} \in \frac{L}{I}$ such that for each $n \in \mathbb{N}$, $(\frac{x_n \odot y_n}{I}) \leq \frac{t}{I}$. We show that $\frac{t}{I} = 1_I$. Since $(\frac{y_n}{I})_{n \in \mathbb{N}} \uparrow 1_I$, so by Definition 8, there exists $m \in \mathbb{N}$ such that $\bigvee_n (\frac{y_n}{I}) = 1_I$, for all $n \in \mathbb{N}$ with $n \geq m$. By the assumption $\frac{x_n}{I} \odot \frac{y_n}{I} = \frac{x_n \odot y_n}{I} \leq \frac{t}{I}$, we have $\frac{y_n}{I} \leq \frac{x_n}{I} \rightarrow \frac{t}{I}$. So $\frac{y_n}{I} \leq \bigvee_{n \geq m} (\frac{x_n}{I} \rightarrow \frac{t}{I})$ and $\bigvee_{n \geq m} \frac{y_n}{I} \leq \bigvee_{n \geq m} (\frac{x_n}{I} \rightarrow \frac{t}{I})$. Therefore $\bigvee_{n \geq m} (\frac{x_n}{I} \rightarrow \frac{t}{I}) = 1_I$. By Proposition 1, (13), $\bigvee_{n \geq m} (\frac{x_n}{I} \rightarrow \frac{t}{I}) \leq (\bigwedge_{n \geq m} \frac{x_n}{I}) \rightarrow \frac{t}{I}$ for $n \geq m$ and $1_I \leq (\frac{x_m}{I} \wedge \frac{x_{m+1}}{I} \wedge \dots) \rightarrow \frac{t}{I}$. So $1_I \leq \frac{x_m}{I} \rightarrow \frac{t}{I}$ and $\frac{x_m}{I} \rightarrow \frac{t}{I} = 1_I$, i.e., $\frac{x_m}{I} \leq \frac{t}{I}$. This means that $\bigvee_{n \geq m} \frac{x_n}{I} \leq \frac{t}{I}$ and $1_I \leq \frac{t}{I}$, thus $\frac{t}{I} = 1_I$. $\square$

Proposition 8. *Let I be an ideal of a BL-algebra L and $(\frac{x_n}{I})_{n \in \mathbb{N}}$ be a sequence in $\frac{L}{I}$. If $\frac{x_n}{I} \rightarrow_s \frac{x_1}{I}$ and $\frac{x_n}{I} \rightarrow_s \frac{x_2}{I}$, then $\frac{x_1}{I} = \frac{x_2}{I}$.*

Proof. By the assumption, since $\frac{x_n}{I} \rightarrow \frac{x_1}{I}$, $\frac{x_n}{I} \rightarrow \frac{x_2}{I}$, so by Definition 9, $(\frac{s_n}{I})_{n \in \mathbb{N}} \uparrow 1_I$, $(\frac{t_n}{I})_{n \in \mathbb{N}} \uparrow 1_I$ with $d(\frac{x_n}{I}, \frac{x_1}{I}) \geq \frac{s_n}{I}$, $d(\frac{x_n}{I}, \frac{x_2}{I}) \geq \frac{t_n}{I}$. By Proposition 3, $d(\frac{x_1}{I}, \frac{x_2}{I}) \geq d(\frac{x_1}{I}, \frac{x_n}{I}) \odot d(\frac{x_n}{I}, \frac{x_2}{I})$, then $d(\frac{x_1}{I}, \frac{x_2}{I}) \geq \frac{s_n}{I} \odot \frac{t_n}{I}$. By Theorem 13, $(\frac{s_n \odot t_n}{I})_{n \in \mathbb{N}} \uparrow 1_I$, therefore $d(\frac{x_1}{I}, \frac{x_2}{I}) = 1_I$ and hence $\frac{x_1}{I} = \frac{x_2}{I}$. $\square$

Proposition 9. *Let L be a BL-algebra and $(x_n)_{n \in \mathbb{N}}$, $(y_n)_{n \in \mathbb{N}}$ be two sequences in L such that $x_n \rightarrow_s x$, $y_n \rightarrow_s y$. Then $(x_n \leftrightarrow y_n) \rightarrow_s (x \leftrightarrow y)$.*

Proof. First we show that if $x_n \rightarrow_s x$, $y_n \rightarrow_s y$, then, (i) $x_n \wedge y_n \rightarrow_s x \wedge y$, (ii) $(x_n \rightarrow y_n) \rightarrow_s (x \rightarrow y)$. Since $x_n \rightarrow_s x$ and $y_n \rightarrow_s y$, by Definition 9, there exist $(s_n)_{n \in \mathbb{N}}$ and $(t_n)_{n \in \mathbb{N}}$ such that $(s_n)_{n \in \mathbb{N}} \uparrow 1$, $(t_n)_{n \in \mathbb{N}} \uparrow 1$ and $d(x_n, x) \geq s_n$, $d(y_n, y) \geq t_n$. By Proposition 3, $d(x_n \wedge y_n, x \wedge y) \geq d(x_n, x) \wedge d(y_n, y) \geq s_n \wedge t_n$. Since $(s_n \wedge t_n)_{n \in \mathbb{N}} \uparrow 1$, so $x_n \wedge y_n \rightarrow_s x \wedge y$.

From Proposition 3, $d(x_n \rightarrow y_n, x \rightarrow y) \geq d(x_n, x) \odot d(y_n \rightarrow y) \geq s_n \odot t_n$. Since $(s_n \odot t_n)_{n \in \mathbb{N}} \uparrow 1$, so $(x_n \rightarrow y_n) \rightarrow_s (x \rightarrow y)$. By Proposition 1, (14), $d(x_n \leftrightarrow y_n, x \leftrightarrow y) = d((x_n \rightarrow y_n) \wedge (y_n \rightarrow x_n), (x \rightarrow y) \wedge (y \rightarrow x)) \geq d(x_n \rightarrow y_n, x \rightarrow y) \wedge d(y_n \rightarrow x_n, y \rightarrow x) \geq (s_n \odot t_n) \wedge (t_n \odot s_n) = s_n \odot t_n \uparrow 1$, therefore $(x_n \leftrightarrow y_n) \rightarrow_s (x \leftrightarrow y)$. $\square$

Theorem 14. *Let X and Y be two BL-algebras and $f : X \rightarrow Y$ be a BL-homomorphism. If $(x_n)_{n \in \mathbb{N}}$ is a sequence in X , then $(f(x_n))_{n \in \mathbb{N}}$ is a sequence in Y such that $f(x_n) \rightarrow_s f(x)$.*

Proof. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X . There exists a sequence $(s_n)_{n \in \mathbb{N}}$ of X such that $(s_n)_{n \in \mathbb{N}} \uparrow 1$, $d(x_n, x) \geq s_n$. Since $f(1) = 1$, so $(f(s_n)) \uparrow f(1) = 1$. We also have $d(x_n, x) \geq s_n$, then by Definition 6, $d(f(x_n), f(x)) = f(d(x_n), d(x)) \geq f(s_n)$, i.e., $f(x_n) \rightarrow_s f(x)$. $\square$

Corollary 4. Let X and Y be two BL-algebras, $f : X \rightarrow Y$ be a BL-homomorphism and $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}$ be sequences in X , in which $x_n \rightarrow_s x$, $y_n \rightarrow_s y$, then $f(x_n \leftrightarrow y_n) \rightarrow_s f(x \leftrightarrow y)$.

Proof. By Proposition 9 and Theorem 14, it is clear. $\square$

Proposition 10. Let I be a proper ideal of special BL-algebra L^* . Then $\frac{L^*}{I}$ is not an MV-algebra.

Proof. Suppose that there exists a proper ideal I of L^* such that $\frac{L^*}{I}$ is an MV-algebra. Then we have $\frac{x}{I} = \frac{x^{**}}{I}$, for all $x \in L^*$ and $x \rightarrow x^{**} \in I$. Since $x \in L^*$, so $x^* = 0$ and $x^{**} = 1$. Thus $x \rightarrow 1 \in I$ and $1 \in I$. This means that $I = L^*$, which is a contradiction. $\square$

Theorem 15. Let I be an ideal of MV-algebra L . Then $\frac{L}{I}$ is a special BL-algebra if and only if I is a maximal ideal of L .

Proof. We know that $\frac{L}{I}$ is special BL-algebra iff $\frac{x^*}{I} = (\frac{x}{I})^* = \frac{0}{I}$, for all $0 \neq x \in L$. It is equal to $x^* \rightarrow 0 \in I$, for all $0 \neq x \in L$ iff $x^{**} = x \in L$, for all $0 \neq x \in L$ which in turn equals to $I = L - \{0\}$. $\square$

Definition 12. Let L be a BL-algebra and I be an ideal of L . Then I is a special ideal if, for all $x, y \in I$, $(x \rightarrow y)^* = (y \rightarrow x)^*$.

Example 1. Let $L = \{0, a, b, 1\}$. Define " $\odot$ " and " $\rightarrow$ " as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	0	a	1	1
1	0	a	b	1

It is easy to see that L is a BL-algebra and $I = \{0, a\}$ is a special ideal of L .

Theorem 16. Let I be an ideal of BL-algebra L . Then I is a special ideal of L if and only if $D^*(I) = I$, where $D^*(I) = \{x \in I \mid x^{**} = 0\}$.

Proof. Since $0 \in D^*(I)$, it is clear that $\emptyset \neq D^*(I) \subseteq I$. Let I be a special ideal of L and $t \in I$, then for every $x, y \in I$, $(x \rightarrow y)^* = (y \rightarrow x)^*$. We put $x = t$ and $y = 0$, then, by Proposition 1, $(t \rightarrow 0)^* = (0 \rightarrow t)^*$, i.e., $t^{**} = 1^* = 0$. This means that $t \in D^*(I)$ and hence, $I \subseteq D^*(I)$.

Conversely, let $D^*(I) = I$, and $x, y \in I$, then $x^{**} = y^{**} = 0$. By Proposition 1, we have $(x \rightarrow y)^* = (x \rightarrow y)^{***} = ((x \rightarrow y)^{**})^* = (x^{**} \rightarrow y^{**})^* = (0 \rightarrow 0)^* = (y^{**} \rightarrow x^{**})^* = ((y \rightarrow x)^{**})^* = (y \rightarrow x)^{***} = (y \rightarrow x)^*$. Therefore, I is a special ideal of L . $\square$

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FURTHER IMPROVEMENTS OF GENERALIZED NUMERICAL RADIUS INEQUALITIES FOR SEMI-HILBERTIAN SPACE OPERATORS

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Abstract. Several new improvements of the A -numerical radius inequalities for operators acting on a semi-Hilbertian space, i.e., a space induced by a positive operator A , are proved. In particular, among other inequalities, we show that

$$\frac{1}{4} \|T^{\sharp_A} T + T T^{\sharp_A}\|_A \leq \frac{1}{4} \left(2\omega_A^2(T) + \gamma(T) \right) \leq \omega_A^2(T),$$

where

$$\gamma(T) = \sqrt{(\|\Re_A(T)\|_A^2 - \|\Im_A(T)\|_A^2)^2 + 4\|\Re_A(T)\Im_A(T)\|_A^2}.$$

Here $\omega_A(T)$ and $\|T\|_A$ denote respectively the A -numerical radius and the A -seminorm of an operator T . Also, $\Re_A(T) := \frac{T+T^{\sharp_A}}{2}$ and $\Im_A(T) := \frac{T-T^{\sharp_A}}{2i}$, where $T^{\sharp_A}$ is a distinguished A -adjoint operator of T . Further, some new refinements of the triangle inequality related to $\|\cdot\|_A$ are established.

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1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and associated norm $\|\cdot\|$. The C^* -algebra of all bounded linear operators acting $\mathcal{H}$ will be denoted by $\mathbb{B}(\mathcal{H})$. Let $T \in \mathbb{B}(\mathcal{H})$, the numerical radius and the usual operator norm of T are defined respectively by

$$\omega(T) = \sup_{\|x\|=1} |\langle Tx, x \rangle| \quad \text{and} \quad \|T\| = \sup_{\|x\|=1} \|Tx\|.$$

An operator $T \in \mathbb{B}(\mathcal{H})$ is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in \mathcal{H}$. In all that follows, the range of every operator $T \in \mathbb{B}(\mathcal{H})$ is denoted by $\mathcal{R}(T)$, its null space by $\mathcal{N}(T)$ and T^* is the adjoint of T . If $\mathcal{M}$ is an arbitrary linear subspace of $\mathcal{H}$, then $\overline{\mathcal{M}}$ denotes its closure in the norm topology of $\mathcal{H}$. Given a closed subspace $\mathcal{M}$ of $\mathcal{H}$, $P_{\mathcal{M}}$ stands for the orthogonal projection onto $\mathcal{M}$. For the rest of this paper, by an operator we mean a bounded linear operator and we assume

that $A \in \mathbb{B}(\mathcal{H})$ is a nonzero positive operator. It is clear that A induces a semi-inner product on $\mathcal{H}$ given by $\langle x, y \rangle_A = \langle Ax, y \rangle$ for all $x, y \in \mathcal{H}$. By $\|\cdot\|_A$ we denote the seminorm induced by $\langle \cdot, \cdot \rangle_A$, i.e. $\|x\|_A = \sqrt{\langle x, x \rangle_A}$ for every $x \in \mathcal{H}$. One can verify that $\|x\|_A = 0$ if and only if $x \in \mathcal{N}(A)$. This implies that $\|\cdot\|_A$ is a norm on $\mathcal{H}$ if and only if A is injective. Further, we observe that the semi-Hilbertian space $(\mathcal{H}, \|\cdot\|_A)$ is a complete space if and only if $\mathcal{R}(A)$ is closed in $\mathcal{H}$. For $T \in \mathbb{B}(\mathcal{H})$, an operator $S \in \mathbb{B}(\mathcal{H})$ is called an A -adjoint operator of T if the identity $\langle Tx, y \rangle_A = \langle x, Sy \rangle_A$ holds for every $x, y \in \mathcal{H}$, that is, $AS = T^*A$ (see [2]). In general, the existence of an A -adjoint operator is not guaranteed. The set of all operators that admit A -adjoints will be denoted by $\mathbb{B}_A(\mathcal{H})$. By applying Douglas' theorem [13], we get

$$\mathbb{B}_A(\mathcal{H}) = \{T \in \mathbb{B}(\mathcal{H}) ; \mathcal{R}(T^*A) \subseteq \mathcal{R}(A)\}.$$

If $T \in \mathbb{B}_A(\mathcal{H})$, then the “reduced” solution of the equation $AX = T^*A$ is a distinguished A -adjoint operator of T , which will be denoted by $T^\sharp_A$. We remark that if $T \in \mathbb{B}_A(\mathcal{H})$, then $T^\sharp_A \in \mathbb{B}_A(\mathcal{H})$, $(T^\sharp_A)^\sharp_A = P_{\overline{\mathcal{R}(A)}} T P_{\overline{\mathcal{R}(A)}}$ and $((T^\sharp_A)^\sharp_A)^\sharp_A = T^\sharp_A$. Moreover, if $S \in \mathbb{B}_A(\mathcal{H})$, then $TS \in \mathbb{B}_A(\mathcal{H})$ and $(TS)^\sharp_A = S^\sharp_A T^\sharp_A$. An operator $T \in \mathbb{B}(\mathcal{H})$ is said to be A -selfadjoint if AT is selfadjoint, that is, $AT = T^*A$. Obviously if T is A -selfadjoint, then $T \in \mathbb{B}_A(\mathcal{H})$. However, in general, the equality $T = T^\sharp_A$ may not hold. More precisely, one can verify that if $T \in \mathbb{B}_A(\mathcal{H})$, then $T = T^\sharp_A$ if and only if T is A -selfadjoint and $\mathcal{R}(T) \subseteq \overline{\mathcal{R}(A)}$. Further, we recall that an operator T is called A -positive if $AT \geq 0$ and we write $T \geq_A 0$. Clearly, A -positive operators are always A -selfadjoint. Now, if we denote by $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ the set of all operators admitting $A^{1/2}$ -adjoints, then another application of Douglas' theorem [13] gives

$$\mathbb{B}_{A^{1/2}}(\mathcal{H}) = \{T \in \mathbb{B}(\mathcal{H}) ; \exists c > 0 ; \|Tx\|_A \leq c\|x\|_A, \forall x \in \mathcal{H}\}.$$

It is clear that $\mathbb{B}_A(\mathcal{H})$ and $\mathbb{B}_{A^{1/2}}(\mathcal{H})$ are two subalgebras of $\mathbb{B}(\mathcal{H})$ which are neither closed nor dense in $\mathbb{B}(\mathcal{H})$. Further, the proper inclusions $\mathbb{B}_A(\mathcal{H}) \subseteq \mathbb{B}_{A^{1/2}}(\mathcal{H}) \subseteq \mathbb{B}(\mathcal{H})$ hold with equality if A is injective and has a closed range in $\mathcal{H}$. For more details, we refer the reader to [1, 2, 18].

Given $T \in \mathbb{B}(\mathcal{H})$. If there exists $\lambda > 0$ such that $\|Tx\|_A \leq \lambda\|x\|_A$, for all $x \in \overline{\mathcal{R}(A)}$, then it holds:

$$\|T\|_A := \sup_{\substack{x \in \overline{\mathcal{R}(A)}, \\ x \neq 0}} \frac{\|Tx\|_A}{\|x\|_A} = \sup_{\substack{x \in \overline{\mathcal{R}(A)}, \\ \|x\|_A = 1}} \|Tx\|_A < \infty.$$

Of course, if $A = I$ we reach the definition of the classical operator norm. It was shown in [14] that for $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, we have

$$\begin{aligned} \|T\|_A &= \sup \{ \|Tx\|_A ; x \in \mathcal{H}, \|x\|_A = 1 \} \\ &= \sup \{ |\langle Tx, y \rangle_A| ; x, y \in \mathcal{H}, \|x\|_A = \|y\|_A = 1 \}. \end{aligned} \quad (1.1)$$

It is useful to note that for every $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$, we have

$$\|TS\|_A \leq \|T\|_A \|S\|_A. \quad (1.2)$$

Further, clearly for $T \in \mathbb{B}_A(\mathcal{H})$, the operators $T^{\sharp_A}T$ and $TT^{\sharp_A}$ are A -positive. In addition, it was shown in [1] that

$$\|T^{\sharp_A}T\|_A = \|TT^{\sharp_A}\|_A = \|T\|_A^2 = \|T^{\sharp_A}\|_A^2. \quad (1.3)$$

Recently, the A -spectral radius of A -bounded operators is introduced by the present author in [18] as follows

$$r_A(T) := \inf_{n \geq 1} \|T^n\|_A^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \|T^n\|_A^{\frac{1}{n}}. \quad (1.4)$$

We mention that the second equality in (1.4) is also proved by the present author in [18, Theorem 1]. In addition, $r_A(\cdot)$ satisfies the commutativity property, which asserts that

$$r_A(TS) = r_A(ST), \quad (1.5)$$

for every $T, S \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$ (see [18]). In all that follows, for any arbitrary operator $T \in \mathbb{B}_A(\mathcal{H})$, we denote

$$\Re_A(T) := \frac{T + T^{\sharp_A}}{2} \quad \text{and} \quad \Im_A(T) := \frac{T - T^{\sharp_A}}{2i}.$$

For simplicity, if n is any positive integer, then we will write $\Re_A^n(T)$ and $\Im_A^n(T)$ instead of $[\Re_A(T)]^n$ and $[\Im_A(T)]^n$, respectively. Also, $\omega_A^n(T)$ means $[\omega_A(T)]^n$. For $T \in \mathbb{B}(\mathcal{H})$, the A -numerical radius of an operator T was firstly defined by Saddi in [23] by

$$\omega_A(T) := \sup \{ |\langle Tx, x \rangle_A|; x \in \mathcal{H}, \|x\|_A = 1 \}.$$

Recently, this concept received considerable attention by many authors. For more details, we refer the reader to [3, 5, 7, 9–12, 16, 18] and the references therein. It should be mentioned here that it may happen that $\|T\|_A$ and $\omega_A(T)$ are equal to $+\infty$ for some $T \in \mathbb{B}(\mathcal{H})$ (see [16, 18]). However, it was shown in [3] that the above quantities are equivalent seminorms on $\mathbb{B}_{A^{1/2}}(\mathcal{H})$. More precisely, we have

$$\frac{1}{2} \|T\|_A \leq \omega_A(T) \leq \|T\|_A. \quad (1.6)$$

Recently, several refinements of the inequalities (1.6) have been proved by many authors (e.g., see [6, 9, 16], and the references therein). In particular, it has been shown that for $T \in \mathbb{B}_A(\mathcal{H})$, we have

$$\frac{1}{2} \sqrt{\|T^{\sharp_A}T + TT^{\sharp_A}\|_A} \leq \omega_A(T) \leq \frac{\sqrt{2}}{2} \sqrt{\|T^{\sharp_A}T + TT^{\sharp_A}\|_A}, \quad (1.7)$$

(see [19]). If $A = I$, we get the well-known inequalities proved by Kittaneh in [21, Theorem 1]. One main target of the present paper is to prove some new refinements of the first inequality in (1.7). The inspiration of our investigation comes from the recent works by Moradi et al. [20, 22]. Some of the obtained results are new even in

the case that the underlying operator A is the identity operator. In particular, among other inequalities, we prove that for every $T \in \mathbb{B}_A(\mathcal{H})$ we have

$$\begin{aligned} & \frac{1}{2} \sqrt{\|T^{\sharp_A} T + T T^{\sharp_A}\|_A} \\ & \leq \frac{1}{2} \sqrt{2\omega_A^2(T) + \sqrt{(\|\Re_A(T)\|_A^2 - \|\Im_A(T)\|_A^2)^2 + 4\|\Re_A(T)\Im_A(T)\|_A^2}} \leq \omega_A(T). \end{aligned}$$

In addition, several new refinements of the triangle inequality related to $\|\cdot\|_A$ are proved. Mainly, we prove that for every $T, S \in \mathbb{B}_A(\mathcal{H})$ we have

$$\begin{aligned} \|T + S\|_A & \leq \sqrt{\frac{1}{2} \left(\|T\|_A^2 + \|S\|_A^2 + \sqrt{(\|T\|_A^2 - \|S\|_A^2)^2 + 4\|T S^{\sharp_A}\|_A^2} \right)} + 2\omega_A(S^{\sharp_A} T) \\ & \leq \|T\|_A + \|S\|_A. \end{aligned}$$

Several applications of the obtained inequalities are also given.

2. RESULTS

In this section, we present our results. In order to achieve the goals of the present section, we need the following lemma.

Lemma 1. ([4, 16]) *Let $T \in \mathbb{B}(\mathcal{H})$ be an A -selfadjoint operator. Then, the following assertions hold:*

- (i) $T^{\sharp_A}$ is A -selfadjoint and $(T^{\sharp_A})^{\sharp_A} = T^{\sharp_A}$.
- (ii) $\|T\|_A = \omega_A(T) = r_A(T)$.
- (iii) $\|T^n\|_A = \|T\|_A^n$ for any positive integer n .
- (vi) $T^{2n} \geq_A 0$ for any positive integer n .

Our first result in this section provides a refinement of the first inequality in (1.7) and reads as follows.

Theorem 1. *Let $T \in \mathbb{B}_A(\mathcal{H})$. Then*

$$\frac{1}{2} \sqrt{\|T^{\sharp_A} T + T T^{\sharp_A}\|_A} \leq \frac{\sqrt{2}}{2} \sqrt{\|\Re_A(T)\|_A^2 + \|\Im_A(T)\|_A^2} \leq \omega_A(T). \quad (2.1)$$

Proof. Observe first that T can be decomposed as $T = \Re_A(T) + i\Im_A(T)$. Further, it is not difficult to verify that $\Re_A(T)$ and $\Im_A(T)$ are A -selfadjoint operators. Thus, by applying Lemma 1 (i) we get

$$([\Re_A(T)]^{\sharp_A})^{\sharp_A} = [\Re_A(T)]^{\sharp_A} \quad \text{and} \quad ([\Im_A(T)]^{\sharp_A})^{\sharp_A} = [\Im_A(T)]^{\sharp_A}.$$

So, a short calculations shows that

$$\frac{1}{2} \left(T T^{\sharp_A} + T^{\sharp_A} T \right)^{\sharp_A} = \left([\Re_A(T)]^{\sharp_A} \right)^2 + \left([\Im_A(T)]^{\sharp_A} \right)^2. \quad (2.2)$$

Thus, one observes that

$$\begin{aligned} \frac{1}{4} \|T^{\sharp_A} T + T T^{\sharp_A}\|_A &= \frac{1}{4} \left\| \left(T T^{\sharp_A} + T^{\sharp_A} T \right)^{\sharp_A} \right\|_A \\ &= \frac{1}{2} \left\| \left([\Re_A(T)]^{\sharp_A} \right)^2 + \left([\Im_A(T)]^{\sharp_A} \right)^2 \right\|_A \\ &\leq \frac{1}{2} \left\| \left([\Re_A(T)]^{\sharp_A} \right)^2 \right\|_A + \frac{1}{2} \left\| \left([\Im_A(T)]^{\sharp_A} \right)^2 \right\|_A \\ &= \frac{1}{2} \|\Re_A^2(T)\|_A + \frac{1}{2} \|\Im_A^2(T)\|_A, \end{aligned}$$

where the last equality follows from the fact that $\|X^{\sharp_A}\|_A = \|X\|_A$ for all $X \in \mathbb{B}_A(\mathcal{H})$. So, by applying Lemma 1 (iii), we get

$$\frac{1}{4} \|T^{\sharp_A} T + T T^{\sharp_A}\|_A \leq \frac{1}{2} \|\Re_A(T)\|_A^2 + \frac{1}{2} \|\Im_A(T)\|_A^2. \quad (2.3)$$

On the other hand, let $x \in \mathcal{H}$ be such that $\|x\|_A = 1$. Then, we verify that

$$\begin{aligned} |\langle T x, x \rangle_A|^2 &= |\langle [\Re_A(T) + i\Im_A(T)] x, x \rangle_A|^2 \\ &= |\langle \Re_A(T) x, x \rangle_A + i \langle \Im_A(T) x, x \rangle_A|^2 \\ &= |\langle \Re_A(T) x, x \rangle_A|^2 + |\langle \Im_A(T) x, x \rangle_A|^2 \\ &\geq |\langle \Re_A(T) x, x \rangle_A|^2. \end{aligned}$$

So, by taking the supremum over all $x \in \mathcal{H}$ with $\|x\|_A = 1$ in the above inequality and then using Lemma 1 (ii), we obtain

$$\|\Re_A(T)\|_A^2 \leq \omega_A^2(T). \quad (2.4)$$

Similarly, one can prove that

$$\|\Im_A(T)\|_A^2 \leq \omega_A^2(T). \quad (2.5)$$

By combining (2.3) together with (2.4) and (2.5), we get the desired result. This finishes the proof of the theorem. $\square$

In order to derive a new improvement of the first inequality in (1.7), we need the following lemma.

Lemma 2. *Let $T, S \in \mathbb{B}(\mathcal{H})$ be A -selfadjoint operators. Then,*

$$\|T^2 + S^2\|_A \leq \frac{1}{2} \left(\|T^2\|_A + \|S^2\|_A + \sqrt{(\|T^2\|_A - \|S^2\|_A)^2 + 4\|TS\|_A^2} \right).$$

Proof. Since, $T, S \in \mathbb{B}(\mathcal{H})$ are A -selfadjoint operators, then by applying Lemma 1 (vi) one see that $T^2 + S^2 \geq_A 0$. So, by Lemma 1 (ii) we have

$$\|T^2 + S^2\|_A = r_A(T^2 + S^2). \quad (2.6)$$

On the other hand, it can be checked that

$$r_A(T^2 + S^2) = r_{\mathbb{A}} \left[\begin{pmatrix} T^2 + S^2 & 0 \\ 0 & 0 \end{pmatrix} \right] = r_{\mathbb{A}} \left[\begin{pmatrix} T & S \\ 0 & 0 \end{pmatrix} \begin{pmatrix} T & 0 \\ S & 0 \end{pmatrix} \right],$$

where $\mathbb{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \in \mathbb{B}(\mathcal{H} \oplus \mathcal{H})$ is the 2×2 positive diagonal operator matrix whose each diagonal entry is the positive operator A . Further, by using (1.5) we get

$$\begin{aligned} r_A(T^2 + S^2) &= r_{\mathbb{A}} \left[\begin{pmatrix} T & 0 \\ S & 0 \end{pmatrix} \begin{pmatrix} T & S \\ 0 & 0 \end{pmatrix} \right] \\ &= r_{\mathbb{A}} \left[\begin{pmatrix} T^2 & TS \\ ST & S^2 \end{pmatrix} \right]. \end{aligned} \quad (2.7)$$

Further, by [17] we have

$$r_{\mathbb{A}} \left[\begin{pmatrix} T^2 & TS \\ ST & S^2 \end{pmatrix} \right] \leq r \left[\begin{pmatrix} \|T^2\|_A & \|TS\|_A \\ \|ST\|_A & \|S^2\|_A \end{pmatrix} \right]. \quad (2.8)$$

Therefore, by applying (2.6) together with (2.7) and (2.8) we see that

$$\|T^2 + S^2\|_A \leq r \left[\begin{pmatrix} \|T^2\|_A & \|TS\|_A \\ \|ST\|_A & \|S^2\|_A \end{pmatrix} \right]. \quad (2.9)$$

Since $T \geq_A 0$ and $S \geq_A 0$, then T and S are A -selfadjoint. This implies, through Lemma 1 (i) that $T^{\sharp_A}$ and $S^{\sharp_A}$ are also A -selfadjoint. Thus, Lemma 1 (i) gives $(T^{\sharp_A})^{\sharp_A} = T^{\sharp_A}$ and $(S^{\sharp_A})^{\sharp_A} = S^{\sharp_A}$. So, we obtain

$$\|TS\|_A = \|(TS)^{\sharp_A}\|_A = \|S^{\sharp_A} T^{\sharp_A}\|_A = \|(T^{\sharp_A})^{\sharp_A} (S^{\sharp_A})^{\sharp_A}\|_A = \|T^{\sharp_A} S^{\sharp_A}\|_A = \|ST\|_A.$$

Hence, $\begin{pmatrix} \|T^2\|_A & \|TS\|_A \\ \|ST\|_A & \|S^2\|_A \end{pmatrix}$ is a symmetric matrix. This yields that

$$r \left[\begin{pmatrix} \|T^2\|_A & \|TS\|_A \\ \|ST\|_A & \|S^2\|_A \end{pmatrix} \right] = \frac{1}{2} \left(\|T^2\|_A + \|S^2\|_A + \sqrt{(\|T^2\|_A - \|S^2\|_A)^2 + 4\|TS\|_A^2} \right).$$

This finishes the proof by taking into consideration (2.9). $\square$

Now, we are in a position to prove one of our main results in this paper.

Theorem 2. *Let $T \in \mathbb{B}_A(\mathcal{H})$. Then*

$$\begin{aligned} &\frac{1}{4} \|T^{\sharp_A} T + T T^{\sharp_A}\|_A \\ &\leq \frac{1}{4} \left(2\omega_A^2(T) + \sqrt{(\|\Re_A(T)\|_A^2 - \|\Im_A(T)\|_A^2)^2 + 4\|\Re_A(T)\Im_A(T)\|_A^2} \right) \\ &\leq \omega_A^2(T). \end{aligned}$$

Proof. Notice first that T can be written as $T = [\Re_A(T)] + i[\Im_A(T)]$. By using an argument similar to that used in proof of Theorem 1, we get

$$\begin{aligned} \frac{1}{4} \|T^{\sharp_A} T + T T^{\sharp_A}\|_A &= \frac{1}{2} \left\| \left([\Re_A(T)]^{\sharp_A} \right)^2 + \left([\Im_A(T)]^{\sharp_A} \right)^2 \right\|_A \\ &= \frac{1}{2} \|\Re_A^2(T) + \Im_A^2(T)\|_A. \end{aligned}$$

Since the operators $\Re_A(T)$ and $\Im_A(T)$ are A -selfadjoint, then an application of Lemma 2 gives

$$\begin{aligned} &\frac{1}{4} \|T^{\sharp_A} T + T T^{\sharp_A}\|_A \\ &\leq \frac{1}{4} \left(\|\Re_A^2(T)\|_A + \|\Im_A^2(T)\|_A + \sqrt{(\|\Re_A^2(T)\|_A - \|\Im_A^2(T)\|_A)^2 + 4\|\Re_A(T)\Im_A(T)\|_A^2} \right) \\ &= \frac{1}{4} \left(\|\Re_A(T)\|_A^2 + \|\Im_A(T)\|_A^2 + \sqrt{(\|\Re_A(T)\|_A^2 - \|\Im_A(T)\|_A^2)^2 + 4\|\Re_A(T)\Im_A(T)\|_A^2} \right), \end{aligned}$$

where the last equality follows by applying Lemma 1 (iii). Further, by applying (2.4) and (2.5) we see that

$$\begin{aligned} &\frac{1}{4} \|T^{\sharp_A} T + T T^{\sharp_A}\|_A \\ &\leq \frac{1}{4} \left(2\omega_A^2(T) + \sqrt{(\|\Re_A(T)\|_A^2 - \|\Im_A(T)\|_A^2)^2 + 4\|\Re_A(T)\Im_A(T)\|_A^2} \right). \end{aligned}$$

This proves the first inequality in Theorem 2. Now, by using (1.2) and then making simple calculations we get

$$\begin{aligned} &\frac{1}{4} \left(2\omega_A^2(T) + \sqrt{(\|\Re_A(T)\|_A^2 - \|\Im_A(T)\|_A^2)^2 + 4\|\Re_A(T)\Im_A(T)\|_A^2} \right) \\ &\leq \frac{1}{4} \left(2\omega_A^2(T) + \sqrt{(\|\Re_A(T)\|_A^2 + \|\Im_A(T)\|_A^2)^2} \right) \\ &= \frac{1}{4} \left(2\omega_A^2(T) + \sqrt{(\|\Re_A(T)\|_A^2 + \|\Im_A(T)\|_A^2)^2} \right) \\ &= \frac{1}{4} (2\omega_A^2(T) + \|\Re_A(T)\|_A^2 + \|\Im_A(T)\|_A^2) \\ &\leq \frac{1}{4} (2\omega_A^2(T) + 2\omega_A^2(T)) = \omega_A^2(T). \end{aligned}$$

This finishes the proof of the theorem. $\square$

The following lemma plays a crucial role in proving our next result and provides a new refinement of the triangle inequality related to $\|\cdot\|_A$.

Lemma 3. Let $T \in \mathbb{B}_{A^{1/2}}(\mathcal{H})$. Then

$$\begin{aligned} \|T + S\|_A &\leq \int_0^1 \left\| \lambda T + (1 - \lambda) \frac{T + S}{2} \right\|_A d\lambda + \int_0^1 \left\| \lambda S + (1 - \lambda) \frac{T + S}{2} \right\|_A d\lambda \\ &\leq \|T\|_A + \|S\|_A. \end{aligned}$$

Proof. Let $\lambda \in [0, 1]$. Then, one observes that

$$\begin{aligned} \|T + S\|_A &= \left\| \lambda T + (1 - \lambda) \frac{T + S}{2} + \lambda S + (1 - \lambda) \frac{T + S}{2} \right\|_A \\ &\leq \left\| \lambda T + (1 - \lambda) \frac{T + S}{2} \right\|_A + \left\| \lambda S + (1 - \lambda) \frac{T + S}{2} \right\|_A. \end{aligned}$$

This proves the first inequality in Lemma 3 by taking integral over $\lambda \in [0, 1]$. The second inequality in Lemma 3 follows immediately by applying the triangle inequality related to $\|\cdot\|_A$ and then making simple calculations. $\square$

Another improvement of the first inequality in (1.7) can be stated as follows.

Theorem 3. Let $T \in \mathbb{B}_A(\mathcal{H})$. Then

$$\begin{aligned} \frac{1}{4} \|T^{\sharp_A} T + T T^{\sharp_A}\|_A &\leq \frac{1}{2} \int_0^1 \left\| \lambda \Re_A^2(T) + (1 - \lambda) \frac{\Re_A^2(T) + \Im_A^2(T)}{2} \right\|_A d\lambda \\ &\quad + \frac{1}{2} \int_0^1 \left\| \lambda \Im_A^2(T) + (1 - \lambda) \frac{\Re_A^2(T) + \Im_A^2(T)}{2} \right\|_A d\lambda \\ &\leq \omega_A^2(T). \end{aligned}$$

Proof. Since $T = \Re_A(T) + i\Im_A(T)$, then by using an argument similar to that used in proof of Theorem 1, we get

$$\begin{aligned} \frac{1}{4} \|T^{\sharp_A} T + T T^{\sharp_A}\|_A &= \frac{1}{2} \left\| \left([\Re_A(T)]^{\sharp_A} \right)^2 + \left([\Im_A(T)]^{\sharp_A} \right)^2 \right\|_A \\ &= \frac{1}{2} \| [\Re_A^2(T)]^{\sharp_A} + [\Im_A^2(T)]^{\sharp_A} \|_A \\ &= \frac{1}{2} \| \Re_A^2(T) + \Im_A^2(T) \|_A. \end{aligned}$$

So, by applying Lemma 3, we see that

$$\begin{aligned} \frac{1}{4} \|T^{\sharp_A} T + T T^{\sharp_A}\|_A &\leq \frac{1}{2} \int_0^1 \left\| \lambda \Re_A^2(T) + (1 - \lambda) \frac{\Re_A^2(T) + \Im_A^2(T)}{2} \right\|_A d\lambda \\ &\quad + \frac{1}{2} \int_0^1 \left\| \lambda \Im_A^2(T) + (1 - \lambda) \frac{\Re_A^2(T) + \Im_A^2(T)}{2} \right\|_A d\lambda \\ &\leq \frac{1}{2} (\| \Re_A^2(T) \|_A + \| \Im_A^2(T) \|_A) \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2} (\|\Re_A(T)\|_A^2 + \|\Im_A(T)\|_A^2) \quad (\text{by Lemma 1(iii)}) \\
 &\leq \omega_A^2(T),
 \end{aligned}$$

where the last inequality follows by applying (2.4) together with (2.5). Hence the proof is complete. $\square$

In the following theorem, we prove another refinement of the triangle inequality related to $\|\cdot\|_A$.

Theorem 4. Let $T, S \in \mathbb{B}_A(\mathcal{H})$. Then

$$\|T \pm S\|_A \leq \sqrt{\|T^{\sharp_A} T + S^{\sharp_A} S\|_A + \|T^{\sharp_A} S + S^{\sharp_A} T\|_A} \leq \|T\|_A + \|S\|_A. \quad (2.10)$$

Proof. Let $x \in \mathcal{H}$ be such that $\|x\|_A = 1$. Then we see that

$$\begin{aligned}
 \|Tx + Sx\|_A^2 &= \|Tx\|_A^2 + \langle Tx, Sx \rangle_A + \langle Sx, Tx \rangle_A + \|Sx\|_A^2 \\
 &\leq \left\langle \left(T^{\sharp_A} T + S^{\sharp_A} S \right) x, x \right\rangle_A + \left| \left\langle \left(T^{\sharp_A} S + S^{\sharp_A} T \right) x, x \right\rangle_A \right|.
 \end{aligned}$$

Further, by applying the Cauchy-Schwarz inequality we get

$$\|Tx + Sx\|_A^2 \leq \|T^{\sharp_A} T + S^{\sharp_A} S\|_A + \|T^{\sharp_A} S + S^{\sharp_A} T\|_A.$$

So, by taking the supremum over all $x \in \mathcal{H}$ with $\|x\|_A = 1$ in the above inequality we get

$$\|T + S\|_A^2 \leq \|T^{\sharp_A} T + S^{\sharp_A} S\|_A + \|T^{\sharp_A} S + S^{\sharp_A} T\|_A.$$

Similarly, we show that

$$\|T - S\|_A^2 \leq \|T^{\sharp_A} T + S^{\sharp_A} S\|_A + \|T^{\sharp_A} S + S^{\sharp_A} T\|_A.$$

Hence, we get the first inequality in (2.10). Moreover, by applying the triangle inequality together with (1.2) and (1.3) we see that

$$\begin{aligned}
 \|T \pm S\|_A &\leq \sqrt{\|T^{\sharp_A} T + S^{\sharp_A} S\|_A + \|T^{\sharp_A} S + S^{\sharp_A} T\|_A} \\
 &\leq \sqrt{\|T\|_A^2 + \|S\|_A^2 + 2\|T\|_A\|S\|_A} \\
 &= \sqrt{(\|T\|_A + \|S\|_A)^2} \\
 &= \|T\|_A + \|S\|_A.
 \end{aligned}$$

This immediately proves the required result. $\square$

Remark 1. We note that the inequalities obtained in Theorem 4 cover and refine the recent inequalities due to Bhunia et al. (see [8, Theorem 2.4]).

As an application of Theorem 4, we derive another improvement of the first inequality in (1.7).

Corollary 1. *Let $T \in \mathbb{B}_A(\mathcal{H})$. Then*

$$\frac{1}{4} \|T^{\sharp_A} T + T T^{\sharp_A}\|_A \leq \frac{\sqrt{2}}{2} \sqrt{\omega_A^4(T) + \|\mathfrak{S}_A^2(T) \mathfrak{R}_A^2(T)\|_A} \leq \omega_A^2(T).$$

Proof. Since $T = \mathfrak{R}_A(T) + i\mathfrak{S}_A(T)$, then by using (2.2) we observe that

$$\frac{1}{4} \|T^{\sharp_A} T + T T^{\sharp_A}\|_A = \frac{1}{2} \left\| [\mathfrak{R}_A^2(T)]^{\sharp_A} + [\mathfrak{S}_A^2(T)]^{\sharp_A} \right\|_A.$$

Moreover, in view of Lemma 1 (vi), the operators $\mathfrak{R}_A^2(T)$ and $\mathfrak{S}_A^2(T)$ are A -positive and

$$[\mathfrak{R}_A^2(T)]^{\sharp_A} = [\mathfrak{R}_A^2(T)]^{\sharp_A} \quad \text{and} \quad [\mathfrak{S}_A^2(T)]^{\sharp_A} = [\mathfrak{S}_A^2(T)]^{\sharp_A}. \quad (2.11)$$

So, by using Theorem 4 together with (2.11) we observe that

$$\begin{aligned} & \frac{1}{16} \|T^{\sharp_A} T + T T^{\sharp_A}\|_A^2 \\ & \leq \frac{1}{4} \|([\mathfrak{R}_A^2(T)]^{\sharp_A})^2 + ([\mathfrak{S}_A^2(T)]^{\sharp_A})^2\|_A \\ & \quad + \frac{1}{4} \|[\mathfrak{R}_A^2(T)]^{\sharp_A} [\mathfrak{S}_A^2(T)]^{\sharp_A} + [\mathfrak{S}_A^2(T)]^{\sharp_A} [\mathfrak{R}_A^2(T)]^{\sharp_A}\|_A \\ & \leq \frac{1}{4} \|([\mathfrak{R}_A^2(T)]^{\sharp_A})^2 + ([\mathfrak{S}_A^2(T)]^{\sharp_A})^2\|_A \\ & \quad + \frac{1}{4} \|[\mathfrak{R}_A^2(T)]^{\sharp_A} [\mathfrak{S}_A^2(T)]^{\sharp_A}\|_A + \frac{1}{4} \|[\mathfrak{S}_A^2(T)]^{\sharp_A} [\mathfrak{R}_A^2(T)]^{\sharp_A}\|_A. \end{aligned}$$

On the other hand, since $\|X\|_A = \|X^{\sharp_A}\|_A$ for all $X \in \mathbb{B}_A(\mathcal{H})$, then by (2.11) we have

$$\begin{aligned} \|[\mathfrak{R}_A^2(T)]^{\sharp_A} [\mathfrak{S}_A^2(T)]^{\sharp_A}\|_A &= \left\| [\mathfrak{S}_A^2(T)]^{\sharp_A} [\mathfrak{R}_A^2(T)]^{\sharp_A} \right\|_A \\ &= \|[\mathfrak{S}_A^2(T)]^{\sharp_A} [\mathfrak{R}_A^2(T)]^{\sharp_A}\|_A. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} & \frac{1}{16} \|T^{\sharp_A} T + T T^{\sharp_A}\|_A^2 \\ & \leq \frac{1}{4} \|([\mathfrak{R}_A^2(T)]^{\sharp_A})^2 + ([\mathfrak{S}_A^2(T)]^{\sharp_A})^2\|_A + \frac{1}{2} \|[\mathfrak{R}_A^2(T)]^{\sharp_A} [\mathfrak{S}_A^2(T)]^{\sharp_A}\|_A \\ & \leq \frac{1}{4} \|\mathfrak{R}_A^4(T) + \mathfrak{S}_A^4(T)\|_A + \frac{1}{2} \|\mathfrak{S}_A^2(T) \mathfrak{R}_A^2(T)\|_A. \end{aligned}$$

This implies that

$$\begin{aligned} \frac{1}{4} \|T^{\sharp_A} T + T T^{\sharp_A}\|_A &\leq \frac{1}{2} \sqrt{\|\mathfrak{R}_A^4(T) + \mathfrak{S}_A^4(T)\|_A + 2\|\mathfrak{S}_A^2(T) \mathfrak{R}_A^2(T)\|_A} \\ &\leq \frac{1}{2} \sqrt{\|\mathfrak{R}_A^4(T)\|_A + \|\mathfrak{S}_A^4(T)\|_A + 2\|\mathfrak{S}_A^2(T) \mathfrak{R}_A^2(T)\|_A} \end{aligned}$$

$$= \frac{1}{2} \sqrt{\|\Re_A(T)\|_A^4 + \|\Im_A(T)\|_A^4 + 2\|\Im_A^2(T)\Re_A^2(T)\|_A},$$

where the last equality follows from Lemma 1 (iii) since $\Re_A(T)$ and $\Im_A(T)$ are A -selfadjoint operators. By using (2.4) and (2.5) we obtain

$$\frac{1}{4} \|T^{\sharp_A} T + T T^{\sharp_A}\|_A \leq \frac{\sqrt{2}}{2} \sqrt{\omega_A^4(T) + \|\Im_A^2(T)\Re_A^2(T)\|_A}.$$

This shows the first inequality in Corollary 1. Finally, by using (1.2) and similar arguments as above, we see that

$$\begin{aligned} \frac{\sqrt{2}}{2} \sqrt{\omega_A^4(T) + \|\Im_A^2(T)\Re_A^2(T)\|_A} &\leq \frac{\sqrt{2}}{2} \sqrt{\omega_A^4(T) + \|\Re_A(T)\|_A^2 \|\Im_A(T)\|_A^2} \\ &\leq \frac{\sqrt{2}}{2} \sqrt{\omega_A^4(T) + \omega_A^4(T)} = \omega_A^2(T). \end{aligned}$$

Hence, the proof is complete. $\square$

In order to prove our second main result in this paper, we need the following paper.

Lemma 4. *Let $T, S \in \mathbb{B}_A(\mathcal{H})$. Then*

$$\|T \pm S\|_A \leq \sqrt{\|T T^{\sharp_A} + S S^{\sharp_A}\|_A + 2\omega_A(T S^{\sharp_A})} \leq \|T\|_A + \|S\|_A.$$

Proof. It was shown in [15, Lemma 2.14.] that for $T, S \in \mathbb{B}_A(\mathcal{H})$ we have

$$\max \left\{ \|T + S\|_A^2, \|T - S\|_A^2 \right\} - \|T T^{\sharp_A} + S S^{\sharp_A}\|_A \leq 2\omega_A(T S^{\sharp_A}).$$

This immediately proves the first inequality in Lemma 4. The second inequality in Lemma 4 can be seen easily by remarking that $\omega_A(T S^{\sharp_A}) \leq \|T S^{\sharp_A}\|_A$ and then proceeding as in the proof of Theorem 4.

This finishes the proof of the desired result. $\square$

Another improvement of the first inequality in (1.7), that involves $\Re_A(T)$ and $\Im_A(T)$ can be seen as follows.

Theorem 5. *Let $T \in \mathbb{B}_A(\mathcal{H})$. Then*

$$\frac{1}{4} \|T^{\sharp_A} T + T T^{\sharp_A}\|_A \leq \frac{\sqrt{2}}{2} \sqrt{\omega_A^4(T) + \omega_A(\Im_A^2(T)\Re_A^2(T))} \leq \omega_A^2(T).$$

Proof. Since $T = \Re_A(T) + i\Im_A(T)$. Then, by using an argument similar to that used in proof of Theorem 1, we get

$$\begin{aligned} \frac{1}{4} \|T^{\sharp_A} T + T T^{\sharp_A}\|_A &= \frac{1}{4} \left\| \left(T T^{\sharp_A} + T^{\sharp_A} T \right)^{\sharp_A} \right\|_A \\ &= \frac{1}{2} \left\| \left([\Re_A(T)]^{\sharp_A} \right)^2 + \left([\Im_A(T)]^{\sharp_A} \right)^2 \right\|_A \end{aligned}$$

$$= \frac{1}{2} \left\| [\Re_A^2(T)]^{\sharp_A} + [\Im_A^2(T)]^{\sharp_A} \right\|_A.$$

So, by applying Lemma 4 together with (2.11) and then using similar arguments as above, we get

$$\begin{aligned} & \frac{1}{4} \|T^{\sharp_A} T + T T^{\sharp_A}\|_A \\ & \leq \frac{1}{2} \sqrt{\left\| ([\Re_A^2(T)]^{\sharp_A})^2 + ([\Im_A^2(T)]^{\sharp_A})^2 \right\|_A + 2\omega_A([\Re_A^2(T)]^{\sharp_A} [\Im_A^2(T)]^{\sharp_A})} \\ & \leq \frac{1}{2} \sqrt{\|\Re_A^2(T)\|^2 + \|\Im_A^2(T)\|^2 + 2\omega_A(\Im_A^2(T) \Re_A^2(T))} \\ & \leq \frac{1}{2} \sqrt{\|\Re_A(T)\|_A^4 + \|\Im_A(T)\|_A^4 + 2\omega_A(\Im_A^2(T) \Re_A^2(T))} \\ & \leq \frac{\sqrt{2}}{2} \sqrt{\omega_A^4(T) + \omega_A(\Im_A^2(T) \Re_A^2(T))}, \end{aligned}$$

where the last inequality follows by applying (2.4) together with (2.5). Now, we will prove the second inequality in Theorem 5. By using the second inequality in (1.6), we see that

$$\begin{aligned} \frac{\sqrt{2}}{2} \sqrt{\omega_A^4(T) + \omega_A(\Im_A^2(T) \Re_A^2(T))} & \leq \frac{\sqrt{2}}{2} \sqrt{\omega_A^4(T) + \|\Im_A^2(T) \Re_A^2(T)\|_A} \\ & \leq \frac{\sqrt{2}}{2} \sqrt{\omega_A^4(T) + \|\Re_A(T)\|_A^2 \|\Im_A(T)\|_A^2} \\ & \leq \frac{\sqrt{2}}{2} \sqrt{\omega_A^4(T) + \omega_A^4(T)} = \omega_A^2(T). \end{aligned}$$

Hence, the proof is complete. $\square$

Another refinement of the triangle inequality related to $\|\cdot\|_A$ can be stated as follows.

Theorem 6. *Let $T, S \in \mathbb{B}_A(\mathcal{H})$. Then,*

$$\begin{aligned} \|T + S\|_A & \leq \sqrt{\frac{1}{2} \left(\|T\|_A^2 + \|S\|_A^2 + \sqrt{(\|T\|_A^2 - \|S\|_A^2)^2 + 4\|T S^{\sharp_A}\|_A^2} \right) + 2\omega_A(S^{\sharp_A} T)} \\ & \leq \|T\|_A + \|S\|_A. \end{aligned}$$

Proof. Let $x \in \mathcal{H}$ be such that $\|x\|_A = 1$. Then, we see that

$$\begin{aligned} \|(T + S)x\|_A^2 & = \|Tx\|_A^2 + \|Sx\|_A^2 + 2|\langle Tx, Sx \rangle_A| \\ & = \langle T^{\sharp_A} Tx, x \rangle_A + \langle S^{\sharp_A} Sx, x \rangle_A + 2|\langle S^{\sharp_A} Tx, x \rangle_A| \\ & \leq \langle (T^{\sharp_A} T + S^{\sharp_A} S)x, x \rangle_A + 2\omega_A(S^{\sharp_A} T) \end{aligned}$$

$$= \|T^{\sharp_A}T + S^{\sharp_A}S\|_A + 2\omega_A(S^{\sharp_A}T),$$

where the last equality follows from Lemma 1 (ii) since $T^{\sharp_A}T + S^{\sharp_A}S \geq_A 0$. Hence,

$$\|(T + S)x\|_A^2 \leq \|T^{\sharp_A}T + S^{\sharp_A}S\|_A + 2\omega_A(S^{\sharp_A}T).$$

So, by taking the supremum over all $x \in \mathcal{H}$ with $\|x\|_A = 1$ in the last inequality we get

$$\|T + S\|_A^2 \leq \|T^{\sharp_A}T + S^{\sharp_A}S\|_A + 2\omega_A(S^{\sharp_A}T). \quad (2.12)$$

On the other hand, let $\mathbb{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$. Since $T^{\sharp_A}T + S^{\sharp_A}S \geq_A 0$, then by applying Lemma 1 (ii) we observe that

$$\begin{aligned} \|T^{\sharp_A}T + S^{\sharp_A}S\|_A &= r_A(T^{\sharp_A}T + S^{\sharp_A}S) \\ &= r_{\mathbb{A}} \left[\begin{pmatrix} T^{\sharp_A}T + S^{\sharp_A}S & 0 \\ 0 & 0 \end{pmatrix} \right] \\ &= r_{\mathbb{A}} \left[\begin{pmatrix} T^{\sharp_A} & S^{\sharp_A} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} T & 0 \\ S & 0 \end{pmatrix} \right]. \end{aligned}$$

Moreover, an application of (1.5) gives

$$\begin{aligned} \|T^{\sharp_A}T + S^{\sharp_A}S\|_A &= r_{\mathbb{A}} \left[\begin{pmatrix} T & 0 \\ S & 0 \end{pmatrix} \begin{pmatrix} T^{\sharp_A} & S^{\sharp_A} \\ 0 & 0 \end{pmatrix} \right] \\ &= r_{\mathbb{A}} \left[\begin{pmatrix} TT^{\sharp_A} & TS^{\sharp_A} \\ ST^{\sharp_A} & SS^{\sharp_A} \end{pmatrix} \right] \\ &\leq r \left[\begin{pmatrix} \|T\|_A^2 & \|TS^{\sharp_A}\|_A \\ \|ST^{\sharp_A}\|_A & \|S\|_A^2 \end{pmatrix} \right], \end{aligned} \quad (2.13)$$

where the last inequality follows from [17] together with (1.3). In addition, since $\mathcal{R}(T^{\sharp_A}) \subseteq \overline{\mathcal{R}(A)}$ then $P_{\overline{\mathcal{R}(A)}}T^{\sharp_A} = T^{\sharp_A}$. So, by applying (1.1) we see that

$$\begin{aligned} \|TS^{\sharp_A}\|_A &= \|P_{\overline{\mathcal{R}(A)}}SP_{\overline{\mathcal{R}(A)}}T^{\sharp_A}\|_A \\ &= \|P_{\overline{\mathcal{R}(A)}}ST^{\sharp_A}\|_A \\ &= \sup \left\{ |\langle P_{\overline{\mathcal{R}(A)}}ST^{\sharp_A}x, y \rangle_A|; x, y \in \mathcal{H}, \|x\|_A = \|y\|_A = 1 \right\} = \|ST^{\sharp_A}\|_A, \end{aligned}$$

where the last equality holds since $AP_{\overline{\mathcal{R}(A)}} = A$. So, it is not difficult to verify that

$$r \left[\begin{pmatrix} \|T\|_A^2 & \|TS^{\sharp_A}\|_A \\ \|ST^{\sharp_A}\|_A & \|S\|_A^2 \end{pmatrix} \right] = \frac{1}{2} \left(\|T\|_A^2 + \|S\|_A^2 + \sqrt{(\|T\|_A^2 - \|S\|_A^2)^2 + 4\|TS^{\sharp_A}\|_A^2} \right).$$

This implies, through (2.13), that

$$\|T^{\sharp_A}T + S^{\sharp_A}S\|_A \leq \frac{1}{2} \left(\|T\|_A^2 + \|S\|_A^2 + \sqrt{(\|T\|_A^2 - \|S\|_A^2)^2 + 4\|TS^{\sharp_A}\|_A^2} \right).$$

Therefore, we prove the first inequality in Theorem 6 by combining the last inequality together with (2.12). Now, one observes that

$$\begin{aligned}
 & \frac{1}{2} \left(\|T\|_A^2 + \|S\|_A^2 + \sqrt{(\|T\|_A^2 - \|S\|_A^2)^2 + 4\|TS^{\sharp_A}\|_A^2} \right) + 2\omega_A(S^{\sharp_A}T) \\
 & \leq \frac{1}{2} \left(\|T\|_A^2 + \|S\|_A^2 + \sqrt{(\|T\|_A^2 - \|S\|_A^2)^2 + 4\|T\|_A^2\|S\|_A^2} \right) + 2\|S^{\sharp_A}T\|_A \\
 & \leq \frac{1}{2} \left(\|T\|_A^2 + \|S\|_A^2 + \sqrt{(\|T\|_A^2 + \|S\|_A^2)^2} \right) + 2\|T\|_A\|S\|_A \\
 & = \|T\|_A^2 + \|S\|_A^2 + 2\|T\|_A\|S\|_A \\
 & = (\|T\|_A + \|S\|_A)^2.
 \end{aligned}$$

This immediately proves the second inequality in Theorem 6. Therefore, the proof is finished. $\square$

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SOME APPLICATIONS OF MAIA'S FIXED POINT THEOREM IN THE CASE OF OPERATORS WITH VOLTERRA PROPERTY WITH RESPECT TO A SUBINTERVAL

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Abstract. The aim of this paper is to improve some fixed point results given by I.A. Rus in (*On a fixed point theorem of Maia*, Studia Univ. Babeş-Bolyai Math., 22(1977), 40–42), (*Basic problem for Maia's theorem*, Sem. on Fixed Point Theory, Preprint 3(1981), Babeş-Bolyai Univ. Cluj-Napoca, 112–115) and (*Data dependence of the fixed points in a set with two metrics*, Fixed Point Theory, 8(2007), no. 1, 115–123), by applying, instead of contraction principle, some variants of Maia's fixed point theorem.

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Keywords: space of continuous functions, max-norm, Bielecki norm, contraction, G -contraction, fiber contraction, fixed point, Picard operator, weakly Picard operator, operator with Volterra property, integral equation, Maia's fixed point theorem

1. INTRODUCTION AND PRELIMINARIES

By following the paper of I.A. Rus [15], we give some variants of Maia's fixed point theorem and an application of these variants, concerning the existence and uniqueness of fixed points for some integral equations. Our results improve some fixed point results given in [6], [7] and [12].

First, let us recall gradually some notions, notations and results which will be used in the sequel of this paper. We will start with the notion of L -space, the most simple structure that allows us to present the Picard and weakly Picard operators. Notice that we do not need the context of metric space to do this, but we will use this context to recall the following notion and results: the G -contraction, the Fiber Contraction Principle and the Maia's fixed point theorem.

1.1. L -space

The notion of L -space was introduced in 1906 by M. Fréchet in [3]. It is an abstract space in which works one of the basic tools in the theory of operatorial equations, especially in the fixed point theory: the sequence of successive approximations method.

Let X be a nonempty set. Let $s(X) := \{\{x_n\}_{n \in \mathbb{N}} \mid x_n \in X, n \in \mathbb{N}\}$. Let $c(X)$ be a subset of $s(X)$ and $\text{Lim} : c(X) \rightarrow X$ be an operator. By definition the triple $(X, c(X), \text{Lim})$ is called an L -space (denoted by $(X, \rightarrow)$) if the following conditions are satisfied:

- (i) if $x_n = x$, for all $n \in \mathbb{N}$, then $\{x_n\}_{n \in \mathbb{N}} \in c(X)$ and $\text{Lim}\{x_n\}_{n \in \mathbb{N}} = x$.
- (ii) if $\{x_n\}_{n \in \mathbb{N}} \in c(X)$ and $\text{Lim}\{x_n\}_{n \in \mathbb{N}} = x$, then for all subsequences $\{x_{n_i}\}_{i \in \mathbb{N}}$ of $\{x_n\}_{n \in \mathbb{N}}$, we have that $\{x_{n_i}\}_{i \in \mathbb{N}} \in c(X)$ and $\text{Lim}\{x_{n_i}\}_{i \in \mathbb{N}} = x$.

A simple example of an L -space is the pair $(X, \xrightarrow{d})$, where X is a nonempty set and $\xrightarrow{d}$ is the convergence structure induced by a metric d on X .

In general, an L -space is any nonempty set endowed with a structure implying a notion of convergence for sequences. Other examples of L -spaces are: Hausdorff topological spaces, generalized metric spaces ($d(x, y) \in \mathbb{R}_+^m$ or $d(x, y) \in \mathbb{R}_+ \cup \{+\infty\}$), K -metric spaces ($d(x, y) \in K$, where K is a cone in an ordered Banach space), gauge spaces, 2-metric spaces, D - R -spaces, probabilistic metric spaces, syntopogenous spaces.

1.2. Picard operators and weakly Picard operators

Let $(X, \rightarrow)$ be an L -space. An operator $f : X \rightarrow X$ is called weakly Picard operator (WPO) if the sequence of successive approximations, $\{f^n(x)\}_{n \in \mathbb{N}}$, converges for all $x \in X$ and its limit (which generally depend on x) is a fixed point of f .

If an operator f is WPO and the fixed point set of f is a singleton, $F_f = \{x^*\}$, then by definition, f is called Picard operator (PO).

For a WPO , $f : X \rightarrow X$, we define the operator $f^\infty : X \rightarrow X$, by $f^\infty(x) := \lim_{n \rightarrow \infty} f^n(x)$. Notice that, $f^\infty(X) = F_f$, i.e., f^∞ is a set retraction of X on F_f .

If X is a nonempty set, then the triple $(X, \rightarrow, \leq)$ is an ordered L -space if $(X, \rightarrow)$ is an L -space and $\leq$ is a partial order relation on X which is closed with respect to the convergence structure of the L -space.

In the setting of ordered L -spaces, we have some properties concerning WPO s and PO s.

Lemma 1 (Abstract Gronwall Lemma.). *Let $(X, \rightarrow, \leq)$ be an ordered L -space and $f : X \rightarrow X$ be an increasing WPO . Then:*

- (i) $\forall x \in X, x \leq f(x) \Rightarrow x \leq f^\infty(x)$;
- (ii) $\forall x \in X, x \geq f(x) \Rightarrow x \geq f^\infty(x)$.

Lemma 2 (Abstract Comparison Lemma.). *Let $(X, \rightarrow, \leq)$ be an ordered L -space and $f, g, h : X \rightarrow X$ be such that:*

- (1) $f \leq g \leq h$;
- (2) the operators f, g, h are WPO s;
- (3) the operator g is increasing.

Then $\forall x, y, z \in X, x \leq y \leq z \Rightarrow f^\infty(x) \leq g^\infty(y) \leq h^\infty(z)$.

Regarding the theory of *WPOs* and *POs* see [9], [10], [13], [14], [17], [8], [16] and [1].

1.3. *G*-contractions

Let (X, d) be a metric space and $G \subset X \times X$ be a nonempty subset. An operator $f : X \rightarrow X$ is a *G*-contraction if there exists $l \in]0, 1[$ such that,

$$d(f(x), f(y)) \leq ld(x, y), \forall (x, y) \in G.$$

Notice that if $G = X \times X$ then the notion of *G*-contraction is identical with the notion of contraction (or *l*-contraction, if we want explicitly to mention the contraction constant l), i.e., there exists $l \in]0, 1[$ such that, $d(f(x), f(y)) \leq ld(x, y), \forall x, y \in X$.

Example 1. Let $a, b, c \in \mathbb{R}$, $a < c < b$. We consider the set of all continuous real-valued functions defined on the interval $[a, b]$, $X := C[a, b]$, endowed with the metric

$$\rho(x, y) := \left(\int_a^b |x(s) - y(s)|^2 ds \right)^{\frac{1}{2}}.$$

For $K, H \in C([a, b] \times [a, b] \times \mathbb{R}, \mathbb{R})$, let $f : C[a, b] \rightarrow C[a, b]$ be defined by

$$f(x)(t) := \int_a^c K(t, s, x(s)) ds + \int_a^t H(t, s, x(s)) ds, \quad t \in [a, b].$$

We suppose that there exists $L_H \in C([a, b] \times [a, b])$ such that

$$|H(t, s, u) - H(t, s, v)| \leq L_H(t, s)|u - v|, \quad \forall t, s \in [a, b], \quad \forall u, v \in \mathbb{R}.$$

If $G := \{(x, y) \in C[a, b] \times C[a, b] \mid x|_{[a, c]} = y|_{[a, c]}\}$ and $\int_c^b \int_c^b |L_H(t, s)|^2 dt ds < 1$, then f is a *G*-contraction with respect to ρ .

Indeed, let $(x, y) \in G$. Then, $\rho(f(x), f(y)) = \left(\int_c^b |f(x)(t) - f(y)(t)|^2 dt \right)^{\frac{1}{2}}$ and

$$\begin{aligned} |f(x)(t) - f(y)(t)| &\leq \int_c^t |H(t, s, x(s)) - H(t, s, y(s))| ds \\ &\leq \int_c^b L_H(t, s)|x(s) - y(s)| ds \\ &\stackrel{\text{Hölder's inequality}}{\leq} \left(\int_c^b |L_H(t, s)|^2 ds \right)^{\frac{1}{2}} \left(\int_c^b |x(s) - y(s)|^2 ds \right)^{\frac{1}{2}}. \end{aligned}$$

Hence,

$$\begin{aligned} \rho(f(x), f(y)) &\leq \left(\int_c^b \left(\int_c^b |L_H(t, s)|^2 ds \right) \rho(x, y)^2 dt \right)^{\frac{1}{2}} \\ &= \left(\int_c^b \int_c^b |L_H(t, s)|^2 dt ds \right)^{\frac{1}{2}} \rho(x, y), \quad \text{for all } (x, y) \in G. \end{aligned}$$

For other examples of G -contractions see [15], [11] and [16], pp. 282-284.

1.4. Fiber Contraction Principle

Another important result (see [13], [17]) concerning WPO s and PO s, is the following one.

Lemma 3 (Fiber Contraction Theorem.). *Let $(X, \rightarrow)$ be an L -space, (Y, ρ) be a metric space, $g : X \rightarrow X$, $h : X \times Y \rightarrow Y$ and $f : X \times Y \rightarrow X \times Y$, $f(x, y) := (g(x), h(x, y))$. We suppose that:*

- (1) (Y, ρ) is a complete metric space;
- (2) g is WPO ;
- (3) $h(x, \cdot) : Y \rightarrow Y$ is a contraction, $\forall x \in X$;
- (4) $h : X \times Y \rightarrow Y$ is continuous.

Then, f is WPO . Moreover, if g is a PO , then f is a PO .

Comment 1 (Generalized Fiber Contraction Theorem.). *Let $(X, \rightarrow)$ be an L -space, (X_i, d_i) , $i = \overline{1, m}$, $m \geq 1$ be metric spaces. Let, $f_i : X_0 \times \dots \times X_i \rightarrow X_i$, $i = \overline{0, m}$, be some operators. We suppose that:*

- (1) (X_i, d_i) , $i = \overline{1, m}$, are complete metric spaces;
- (2) f_0 is a WPO ;
- (3) $f_i(x_0, \dots, x_{i-1}, \cdot) : X_i \rightarrow X_i$, $i = \overline{1, m}$, are l_i -contractions;
- (4) f_i , $i = \overline{1, m}$, are continuous.

Then, the operator $f : X_0 \times \dots \times X_m \rightarrow X_0 \times \dots \times X_m$, defined by, $f(x_0, \dots, x_m) := (f_0(x_0), f_1(x_0, x_1), \dots, f_m(x_0, \dots, x_m))$ is a WPO .

If f_0 is a PO , then f is a PO .

1.5. Maia's fixed point theorem

The following result was introduced by M.G. Maia in [4].

Theorem 1. *Let X be a nonempty set, d and ρ be two metrics on X and $V : X \rightarrow X$ be an operator. We suppose that:*

- (1) *there exists $c > 0$ such that, $d(x, y) \leq c\rho(x, y)$, $\forall x, y \in X$;*
- (2) *(X, d) is a complete metric space;*
- (3) *$V : (X, d) \rightarrow (X, d)$ is continuous;*
- (4) *$V : (X, \rho) \rightarrow (X, \rho)$ is a contraction.*

Then:

- (i) $F_V = \{x^*\}$;
- (ii) $V : (X, d) \rightarrow (X, d)$ is PO .

Maia's Theorem 1 remains true if we replace the condition (1) with the following one:

- (1') *there exists $c > 0$ such that, $d(V(x), V(y)) \leq c\rho(x, y)$, $\forall x, y \in X$.*

Hence, we obtain the so called Rus' variant of Maia's fixed point theorem. More considerations can be found in [8], [6], [7], [12], [5] and [2].

2. FIXED POINT THEOREMS FOR OPERATORS WITH VOLTERRA PROPERTY WITH RESPECT TO A SUBINTERVAL

By following [15], we present first some notions and notations.

Let $a, b, c \in \mathbb{R}$, with $a < c < b$.

Let $C[a, b] := \{f : [a, b] \rightarrow \mathbb{R} \mid f \text{ is continuous on } [a, b]\}$.

On $C[a, b]$ and $C[a, c]$ we consider norms of uniform convergence such as the max-norms, $\|\cdot\|$, or the Bielecki norm, $\|\cdot\|_\tau$.

In $C[a, b] \times C[a, b]$ we consider the subset

$$G := \{(x, y) \in C[a, b] \times C[a, b] \mid x|_{[a, c]} = y|_{[a, c]}\}$$

and for each $x \in C[a, b]$ we consider the subset

$$X_x := \{y \in C[a, b] \mid y|_{[a, c]} = x\}.$$

Definition 1. An operator, $V : C[a, b] \rightarrow C[a, b]$, has the Volterra property on the subinterval, $[a, c]$, if the following implication holds,

$$x, y \in C[a, b], x|_{[a, c]} = y|_{[a, c]} \Rightarrow V(x)|_{[a, c]} = V(y)|_{[a, c]}.$$

Definition 2. An operator, $V : C[a, b] \rightarrow C[a, b]$, has the Volterra property if it has the Volterra property on each subinterval, $[a, t]$, for $a < t < b$.

For example, in the terms of the above considerations, let $K, H \in C([a, b] \times [a, b] \times \mathbb{R}, \mathbb{R})$ and $V : C[a, b] \rightarrow C[a, b]$ be defined by,

$$V(x)(t) := \int_a^c K(t, s, x(s))ds + \int_a^t H(t, s, x(s))ds, \quad t \in [a, b].$$

This operator V has the Volterra property on the subinterval $[a, c]$, but V has not the Volterra property.

Remark 1. If $V : C[a, b] \rightarrow C[a, b]$ is an operator with Volterra property on $[a, c]$, then the operator V induces an operator, V_1 , on $C[a, c]$, defined by

$$V_1(x) := V(\tilde{x})|_{[a, c]}, \text{ where } \tilde{x} \in C[a, b] \text{ with } \tilde{x}|_{[a, c]} = x.$$

Remark 2. If $V : C[a, b] \rightarrow C[a, b]$ has the Volterra property on $[a, c]$ and V is a G -contraction, then the operator

$$V|_{X_x} : X_x \rightarrow X_{V_1(x)},$$

is a contraction on X_x , for all $x \in C[a, c]$.

If $x^* \in F_{V_1}$, then $V(X_{x^*}) \subset X_{x^*}$.

The first abstract main result of our paper is the following.

Theorem 2. Let $a, b, c \in \mathbb{R}$, with $a < c < b$. On $C[a, b]$ we consider the max-norm, $\|\cdot\|$ (or the Bielecki norm, $\|\cdot\|_\tau$). On $C[a, c]$ we consider the max-norm, $\|\cdot\|$ and a metric $\rho : C[a, c] \times C[a, c] \rightarrow \mathbb{R}_+$.

Let $V : C[a, b] \rightarrow C[a, b]$ be an operator. We suppose that:

- (1) V has the Volterra property on $[a, c]$;
- (2) V_1 is a contraction with respect to $\|\cdot\|$;
- (3) $\exists M > 0$ such that $\|x - y\| \leq M\rho(x, y)$, $\forall x, y \in C[a, c]$;
- (4) V is a G -contraction with respect to ρ .

Then

- (i) $F_V = \{x^*\}$;
- (ii) V_1 is PO with respect to the uniform convergence and $x^*|_{[a, c]} = V_1^\infty(x)$, $\forall x \in C[a, c]$;
- (iii) $x^* = V^\infty(x)$, $\forall x \in X_{x^*}|_{[a, c]}$.

Proof. Let $x_0 \in C[a, c]$. By (1) and (4), V_1 is a contraction with respect to ρ on $C[a, c]$. By following the proof of Maia's Theorem 1, it can be shown that the sequence of successive approximations $\{V_1^n(x_0)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $(C[a, c], \rho)$. By (3), we get that $\{V_1^n(x_0)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $(C[a, c], \|\cdot\|)$, so it converges uniformly in the Banach space $(C[a, c], \|\cdot\|)$, to an element $x_1^* \in C[a, c]$, i.e., $V_1^\infty(x_0) = x_1^*$. By (2), $x_1^* \in F_{V_1}$ and by (4) we get $F_{V_1} = \{x_1^*\}$. Hence, V_1 is PO with respect to the uniform convergence on $(C[a, c], \|\cdot\|)$.

By (1), (4) and the Remark 2, the operator $W := V|_{X_{x_1^*}} : X_{x_1^*} \rightarrow X_{x_1^*}$, is a contraction with respect to ρ , where $X_{x_1^*} = \{y \in C[a, b] \mid y|_{[a, c]} = x_1^*\}$.

Let $y_0 \in X_{x_1^*}$. Since W is a contraction with respect to ρ , it can be shown that $\{W^n(y_0)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $(X_{x_1^*}, \rho)$, and by (3), we get that $\{W^n(y_0)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in the Banach space $(X_{x_1^*}, \|\cdot\|)$, i.e., it converges uniformly to an element $x^* \in X_{x_1^*}$. Since W is a contraction, W is also continuous, so $F_W = \{x^*\}$. Hence, $x^*|_{[a, c]} = x_1^*$.

From these we have (i), (ii) and (iii). □

From the above theorem, the following conjecture arises.

Conjecture 1. In the conditions of Theorem 2, the operator V is PO with respect to the uniform convergence on $(C[a, b], \|\cdot\|)$, i.e., $x^* = V^\infty(x)$, $\forall x \in C[a, b]$.

Also, we have the Rus' variant for Theorem 2.

Theorem 3. Let $a, b, c \in \mathbb{R}$, with $a < c < b$. On $C[a, b]$ we consider the max-norm, $\|\cdot\|$ (or the Bielecki norm, $\|\cdot\|_\tau$). On $C[a, c]$ we consider the max-norm, $\|\cdot\|$ and a metric $\rho : C[a, c] \times C[a, c] \rightarrow \mathbb{R}_+$.

Let $V : C[a, b] \rightarrow C[a, b]$ be an operator. We suppose that:

- (1) V has the Volterra property on $[a, c]$;

- (2) V_1 is a contraction with respect to $\|\cdot\|$;
- (3) $\exists M > 0$ such that $\|V_1(x) - V_1(y)\| \leq M\rho(x, y)$, $\forall x, y \in C[a, c]$;
- (4) V is a G -contraction with respect to ρ .

Then

- (i) $F_V = \{x^*\}$;
- (ii) V_1 is PO with respect to the uniform convergence and $x^*|_{[a, c]} = V_1^\infty(x)$, $\forall x \in C[a, c]$;
- (iii) $x^* = V^\infty(x)$, $\forall x \in X_{x^*|_{[a, c]}}$.

Proof. Let $x_0 \in C[a, c]$. By (4), $\{V_1^n(x_0)\}_{n \in \mathbb{N}}$ is a Cauchy sequence with respect to ρ . On the other hand, for $n, p \in \mathbb{N}$, we have:

$$\begin{aligned} \rho(V_1^{n+p}(x_0), V_1^n(x_0)) &\leq \\ &\leq \rho(V_1^{n+p}(x_0), V_1^{n+p-1}(x_0)) + \dots + \rho(V_1^{n+1}(x_0), V_1^n(x_0)) \\ &\leq l^{n+p-1}\rho(V_1(x_0), x_0) + \dots + l^n\rho(V_1(x_0), x_0) \\ &\leq \frac{l^n}{1-l}\rho(V_1(x_0), x_0) \rightarrow 0 \text{ as } n, p \rightarrow \infty. \end{aligned}$$

By (3), it follows that $\|V_1^{n+1}(x_0), V_1^{n+p+1}(x_0)\| \rightarrow 0$ as $n, p \rightarrow \infty$. Hence, $\{V_1^n(x_0)\}_{n \in \mathbb{N}}$ is a Cauchy sequence with respect to $\|\cdot\|$. We follow the proof of Theorem 2. $\square$

3. APPLICATIONS

In this section, we present an application for Theorem 2, Fiber Contraction Theorem and Abstract Gronwall Lemma.

Let $a, b, c \in \mathbb{R}$, with $a < c < b$.

For $K, H \in C([a, b] \times [a, b] \times \mathbb{R}, \mathbb{R})$, we consider the following functional integral equation,

$$x(t) = \int_a^c K(t, s, x(s))ds + \int_a^t H(t, s, x(s))ds, \quad t \in [a, b]. \quad (3.1)$$

We are looking for solutions for this equation in $C[a, b]$.

We suppose that there exists $L_K \in C([a, b] \times [a, b])$ such that

$$|K(t, s, u) - K(t, s, v)| \leq L_K(t, s)|u - v|, \quad \forall t \in [a, b], \forall s \in [a, c], \forall u, v \in \mathbb{R}$$

and there exists $L_H \in C([a, b] \times [a, b])$ such that

$$|H(t, s, u) - H(t, s, v)| \leq L_H(t, s)|u - v|, \quad \forall t, s \in [a, b], \forall u, v \in \mathbb{R}.$$

In addition, we suppose that $\int_c^b \int_c^b |L_H(t, s)|^2 dt ds < 1$.

Let $V : C[a, b] \rightarrow C[a, b]$ be the operator defined by,

$$V(x)(t) := \int_a^c K(t, s, x(s))ds + \int_a^t H(t, s, x(s))ds, \quad \forall t \in [a, b].$$

The operator V_1 , induced by V on the interval $[a, c]$ is

$$V_1(x)(t) := \int_a^c K(t, s, x(s))ds + \int_a^t H(t, s, x(s))ds, \quad \forall t \in [a, c].$$

On $C[a, b]$ we consider the max-norm, $\|x - y\| := \max_{t \in [a, b]} |x(t) - y(t)|$ and the metric

$$\rho(x, y) := \left(\int_a^b |x(t) - y(t)|^2 ds \right)^{\frac{1}{2}}.$$

It is clear that V has the Volterra property on $[a, c]$ and V_1 is continuous with respect to $\|\cdot\|$.

Let $x, y \in C[a, c]$. We have

$$\begin{aligned} |V_1(x)(t) - V_1(y)(t)| &\leq \\ &\leq \int_a^c |K(t, s, x(s)) - K(t, s, y(s))|ds + \int_a^t |H(t, s, x(s)) - H(t, s, y(s))|ds \\ &\leq \int_a^c L_K(t, s)|x(s) - y(s)|ds + \int_a^t L_H(t, s)|x(s) - y(s)|ds \\ &\stackrel{\text{Hölder's inequality}}{\leq} \left(\int_a^c |L_K(t, s)|^2 ds \right)^{\frac{1}{2}} \left(\int_a^c |x(s) - y(s)|^2 ds \right)^{\frac{1}{2}} + \\ &\quad + \left(\int_a^t |L_H(t, s)|^2 ds \right)^{\frac{1}{2}} \left(\int_a^t |x(s) - y(s)|^2 ds \right)^{\frac{1}{2}} \\ &\leq \left(\left(\int_a^c |L_K(t, s)|^2 ds \right)^{\frac{1}{2}} + \left(\int_a^t |L_H(t, s)|^2 ds \right)^{\frac{1}{2}} \right) \rho(x, y). \end{aligned}$$

By taking the $\max_{t \in [a, c]}$ in the above inequalities, there exists

$$M := \max_{t \in [a, c]} \left(\left(\int_a^c |L_K(t, s)|^2 ds \right)^{\frac{1}{2}} + \left(\int_a^t |L_H(t, s)|^2 ds \right)^{\frac{1}{2}} \right) > 0$$

such that $\|V_1(x) - V_1(y)\| \leq M\rho(x, y)$, for all $x, y \in C[a, c]$.

On the other hand, by Example 1, the operator V is a G -contraction with respect to ρ , where $G := \{(x, y) \in C[a, b] \times C[a, b] \mid x|_{[a, c]} = y|_{[a, c]}\}$.

By applying Theorem 3, it follows that the equation (3.1) has a unique solution x^* in $C[a, b]$. Moreover, for $t \in [a, c]$, $x^*(t) = \lim_{n \rightarrow \infty} x_n(t)$, for each $x_0 \in C[a, c]$, where $\{x_n\}_{n \in \mathbb{N}}$ is defined by,

$$x_{n+1}(t) = \int_a^c K(t, s, x_n(s))ds + \int_a^t H(t, s, x_n(s))ds,$$

and for $t \in [a, b]$, $x^*(t) = \lim_{n \rightarrow \infty} y_n(t)$, where $\{y_n\}_{n \in \mathbb{N}}$ is defined by

$y_0 \in C[a, b]$, with $y_0|_{[a, c]} = x^*|_{[a, c]}$, and

$$y_{n+1}(t) = \int_a^c K(t, s, x^*(s))ds + \int_a^t H(t, s, y_n(s))ds.$$

Remark 3. In the case of the operator V , considered above, Conjecture 1 is a theorem. Indeed, let $X := C[a, c]$, $Y := C[c, b]$ and $C[a, b]$ be endowed with max-norm. We take, $g : C[a, c] \rightarrow C[a, c]$, defined by $g := V_1$ and $h : C[a, c] \times C[c, b] \rightarrow C[c, b]$ be defined by

$$h(x, y)(t) := \int_a^c K(t, s, x(s))ds + \int_a^t H(t, s, x(s))ds + \int_c^t H(t, s, y(s))ds.$$

We remark that g is PO when

$$\max_{t \in [a, c]} \left(\int_a^c L_K(t, s)ds + \int_a^t L_H(t, s)ds \right) < 1 \quad (3.2)$$

and the operator $h(x, \cdot) : C[c, b] \rightarrow C[c, b]$ is a contraction, when

$$l := \max_{t \in [c, b]} \int_c^t L_H(t, s)ds < 1 \quad (3.3)$$

By the Fiber Contraction Theorem, in the conditions (3.2) and (3.3), the operator $f : C[a, c] \times C[c, b] \rightarrow C[a, c] \times C[c, b]$, defined by $f(x, y) = (g(x), h(x, y))$, for all $(x, y) \in C[a, c] \times C[c, b]$ is PO .

Now, let

$$x_0 \in C[a, c], \quad x_{n+1} = g(x_n), \quad n \in \mathbb{N},$$

and

$$y_0 \in C[c, b], \quad y_{n+1} = h(x_n, y_n), \quad n \in \mathbb{N}.$$

Then, $x_n \rightarrow x^*|_{[a, c]}$ as $n \rightarrow \infty$ and $y_n \rightarrow x^*|_{[c, b]}$ as $n \rightarrow \infty$.

We denote,

$$u_n(t) = \begin{cases} x_n(t), & t \in [a, c], \\ y_n(t), & t \in [c, b]. \end{cases}$$

Then, $u_n \in C[a, b]$, for $n \in \mathbb{N}^*$, and, $u_{n+1} = V(u_n)$ with $u_n \rightarrow x^*$ as $n \rightarrow \infty$, i.e., V is a PO .

This result is very important because we can apply for V , the Abstract Gronwall Lemma. So we have:

Theorem 4. Let us consider the equation (3.1) in the following conditions: (3.2), (3.3) and $K(t, s, \cdot), H(t, s, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ be increasing functions, for all $t, s \in [a, b]$. Let us denote by x^* the unique solution of (3.1). Then the following implications hold:

- (i) $x \in C[a, b]$, $x(t) \leq \int_a^c K(t, s, x(s))ds + \int_a^t H(t, s, x(s))ds, t \in [a, b], \Rightarrow x \leq x^*$;
- (ii) $x \in C[a, b]$, $x(t) \geq \int_a^c K(t, s, x(s))ds + \int_a^t H(t, s, x(s))ds, t \in [a, b], \Rightarrow x \geq x^*$.

Also, from the Abstract Comparison Lemma it can be established a comparison result for equation (3.1).

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EXISTENCE OF SOLUTIONS FOR THIRD ORDER MULTI POINT IMPULSIVE BOUNDARY VALUE PROBLEMS ON TIME SCALES

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Abstract. In this paper, we obtain sufficient conditions for existence of solutions of a third order m -point impulsive boundary value problem on time scales. To the best of our knowledge, there is hardly any work dealing with third order multi point dynamic impulsive BVPs. The reason may be the complex arguments caused by both impulsive perturbations and calculations on time scales. As an application, we give an example demonstrating our results.

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1. INTRODUCTION

Time scales theory was initiated by Stefan Hilger [5] to combine the analysis over continuous and discrete sets (see also [6]). Later, this theory was rapidly developed by many mathematicians and scientists working in other disciplines such as mechanics, electronics, neural networks, population models, economics, etc. [2]. In recent years impulsive dynamic equations have also attracted huge interest because the non-impulsive equations are usually insufficient for modelling evolutionary processes. In this work, we focus on multi-point dynamic impulsive boundary value problems. They are more adequate for modelling many real-world phenomena as it is more likely for a dynamical system to have multi-points of freedom. We may refer the reader to the book [1] which offers an excellent review and several examples of applications modelled by boundary value problems, and to the famous book of Bainov and Simeonov [11] for extensive knowledge about impulsive differential equations. Many good papers in literature deal with third-order m -point impulsive boundary value problems, however most of them have focused on differential BVPs on the real line $\mathbb{R}$, see for example [4, 7, 12, 13]. Regarding to dynamic BVPs on arbitrary time scales there are very few studies some of which are mentioned below.

In [9], Li and Li studied the following boundary value problem for the nonlinear third order impulsive dynamic system on time scales

$$\begin{cases} -u^{\Delta^3}(t) = f(t, u(t), u^{\Delta}(t), u^{\Delta^2}(t)), & t \in [0, T]_{\mathbb{T}} \setminus \Omega, \\ \Delta u(t_k) = I_k, \quad \Delta u^{\Delta}(t_k) = J_k, \quad \Delta u^{\Delta^2}(t_k) = L_k & k = 1, 2, \dots, m, \\ u(0) = \lambda u(\sigma(T)), \quad u^{\Delta}(0) = \lambda u^{\Delta}(\sigma(T)), \quad u^{\Delta^2}(0) = \lambda u^{\Delta^2}(\sigma(T)). \end{cases}$$

They obtained some sufficient conditions for the existence of solutions by using Schauder's fixed point theorem.

In [10], Liang and Zhang studied the third order m -point impulsive boundary value problem

$$\begin{cases} (\phi(-u''(t)))' + a(t)f(u(t)) = 0, & t \neq t_k, \quad 0 < t < 1, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), & k = 1, 2, \dots, N, \\ u(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \quad u'(1) = 0, \quad u''(0) = 0, \end{cases}$$

where $\phi: \mathbb{R} \rightarrow \mathbb{R}$ is an increasing positive homomorphism with $\phi(0) = 0$. By using the five-functionals fixed point theorem, they provided sufficient conditions for existence of three positive solutions.

Later, Karaca and Fen [8] studied the following nonlinear third order m -point impulsive boundary value problem on time scales

$$\begin{cases} (\phi(u^{\Delta\Delta}(t)))^{\Delta} + q(t)f(t, u(t), u^{\Delta}(t), u^{\Delta^2}(t)) = 0, & t \in J = [0, 1]_{\mathbb{T}}, \quad t \neq t_k, \quad 0 < t < 1, \\ \Delta u(t_k) = I_k(u(t_k)), & k = 1, 2, \dots, n, \\ \Delta u^{\Delta}(t_k) = -J_k(u(t_k), u^{\Delta}(t_k)), & k = 1, 2, \dots, n, \\ au(0) - bu^{\Delta}(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \quad cu(1) + du^{\Delta}(1) = \sum_{i=1}^{m-2} \beta_i u(\xi_i), \\ u^{\Delta^2}(0) = 0, \end{cases}$$

where $\phi: \mathbb{R} \rightarrow \mathbb{R}$ is an increasing positive homomorphism with $\phi(0) = 0$. They obtained sufficient conditions for existence of solutions by using four functionals fixed point theorem to reach existence of at least one positive solution.

Motivated by the above studies, in this work we deal with the third order multi-point boundary value problem for impulsive dynamic equations of the form

$$\begin{cases} x^{\Delta^3}(t) = f(t, x(t), x^{\Delta}(t), x^{\Delta^2}(t)), & t \in J_0, \\ x^{\Delta^2}(t_k^+) = x^{\Delta^2}(t_k) + I_k(x(t_k), x^{\Delta}(t_k), x^{\Delta^2}(t_k)), & k \in \{1, \dots, m\}, \\ x^{\Delta^2}(0) = \lambda x^{\Delta^2}(T), \\ x^{\Delta}(0) = \sum_{j=1}^n \alpha_j x^{\Delta}(\xi_j), \\ x(0) = 0, \end{cases} \quad (1.1)$$

where $z(t_k^+) = \lim_{t \rightarrow t_k^+} z(t)$, $z(t_k^-) = z(t_k)$, $k \in \{1, \dots, m\}$, and $t_k, k \in \{1, \dots, m\}$, are right dense.

We define $J_0 = [0, T]_{\mathbb{T}} \setminus \{t_k\}_{k=1}^m$ and assume the following:

- (A1): $f \in C([0, T]_{\mathbb{T}} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})$,
- (A2): $I_k \in C(\mathbb{R} \times \mathbb{R} \times \mathbb{R})$, $k \in \{1, \dots, m\}$,
- (A3): $0 \leq \xi_1 < \xi_2 < \dots < \xi_n \leq T$,
- (A4): there exist nonnegative constants α and γ_k , $k \in \{1, \dots, n\}$, so that

$$\alpha = \limsup_{|u|+|v|+|w| \rightarrow \infty} \left(\max_{t \in [0, T]_{\mathbb{T}}} \frac{f(t, u, v, w)}{|u| + |v| + |w|} \right),$$

$$\gamma_k = \limsup_{|u|+|v|+|w| \rightarrow \infty} \frac{I_k(u, v, w)}{|u| + |v| + |w|}, \quad k \in \{1, \dots, n\},$$

- (A5): $\lambda, \alpha_j \in \mathbb{R}$, $j \in \{1, \dots, n\}$, $\lambda \neq 1$, $1 - \sum_{j=1}^n \alpha_j \neq 0$.

Using an integral representation of the solutions of the BVP (1.1), sufficient conditions for the existence of at least one positive solution is obtained.

The paper is organized as follows. In the next section, we give some preliminary results. In Section 3, we formulate and prove our main result and we give an example that illustrates our main result.

2. PRELIMINARIES

In the sequel we give some basic definitions for time scales extracted from [2, 3] and discuss further the models introduced above.

A time scale $\mathbb{T}$ is any closed subset of the real numbers. Some very well known examples are $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{N}_0$ and the Cantor set.

Definition 1. For $t \in \mathbb{T}$, the forward jump operator $\sigma: \mathbb{T} \rightarrow \mathbb{T}$ is defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$, and the backward jump operator $\rho: \mathbb{T} \rightarrow \mathbb{T}$ by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$. We note that $\sigma(t) \geq t$ and $\rho(t) \leq t$ for any $t \in \mathbb{T}$.

Definition 2. t is said to be right scattered if $\sigma(t) > t$; right dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$; left scattered if $\rho(t) < t$, and left dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$. The graininess function $\mu: \mathbb{T} \rightarrow [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$ for any $t \in \mathbb{T}$.

Definition 3. Assume that $a \leq b$. Then we define the interval $[a, b]_{\mathbb{T}}$ in $\mathbb{T}$ by

$$[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}.$$

Definition 4. If $\mathbb{T}$ has a left scattered maximum m , then $\mathbb{T}^{\kappa} = \mathbb{T} - \{m\}$. Otherwise $\mathbb{T}^{\kappa} = \mathbb{T}$. In other words,

$$\mathbb{T}^{\kappa} = \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}], & \text{if } \sup \mathbb{T} < \infty \\ \mathbb{T}, & \text{if } \sup \mathbb{T} = \mathbb{T}. \end{cases}$$

Definition 5. For a function $f: \mathbb{T} \rightarrow \mathbb{R}$ we define $f^{\sigma}: \mathbb{T} \rightarrow \mathbb{R}$ by $f^{\sigma}(t) = f(\sigma(t))$.

Definition 6. If $f: \mathbb{T} \rightarrow \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, we define $f^\Delta(t)$ to be the number with the property that given any $\varepsilon > 0$ there is a neighbourhood U of t such that

$$|[f(\sigma(t)) - f(s)] - f^\Delta[\sigma(t) - s]| \leq \varepsilon |\sigma(t) - s|$$

for all $s \in U$. Here, $f^\Delta(t)$ is called the delta derivative of f at t , and f is called delta differentiable in $\mathbb{T}^\kappa$ provided that $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. We shall talk about the second derivative f^{Δ^2} provided f^Δ is differentiable on $\mathbb{T}^{\kappa^2} = (\mathbb{T}^\kappa)^\kappa$ with derivative $f^{\Delta^2} = (f^\Delta)^\Delta: \mathbb{T}^{\kappa^2} \rightarrow \mathbb{R}$. Similarly, we define higher order derivatives $f^{\Delta^n}: \mathbb{T}^{\kappa^n} \rightarrow \mathbb{R}$.

Definition 7. A function $f: \mathbb{T} \rightarrow \mathbb{R}$ is called regulated provided that its right-sided limits exist at all right-dense points in $\mathbb{T}$ and its left-sided limits exist at all left-dense points in $\mathbb{T}$.

Definition 8. A function $F: \mathbb{T} \rightarrow \mathbb{R}$ is called an antiderivative of $f: \mathbb{T} \rightarrow \mathbb{R}$ if

$$F^\Delta(t) = f(t) \text{ for all } t \in \mathbb{T}^\kappa.$$

We define the indefinite integral of f by

$$\int f(t) \Delta t = F(t) + C,$$

where C is an arbitrary constant. We define the Cauchy integral by

$$\int_r^s f(t) \Delta t = F(s) - F(r) \quad \text{for any } s, r \in \mathbb{T}.$$

Definition 9. $f: \mathbb{T} \rightarrow \mathbb{R}$ is said to be rd-continuous provided that f is continuous at each right-dense point of $\mathbb{T}$ and has a finite left-dense limit at each left-dense point of $\mathbb{T}$. The set of rd-continuous functions will be denoted by $C_{rd}(\mathbb{T})$ and the set of differentiable functions that possesses rd-continuous derivatives is denoted by $C_{rd}^1(\mathbb{T})$.

In the following theorem, we list some properties of differentiation and integration on time scales that will be utilized in the paper. Its proof can be found in [2].

Theorem 1. Let $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$ and $f, g \in C_{rd}$. Then,

- (i) $\int_a^b [f(t) + g(t)] \Delta t = \int_a^b f(t) \Delta t + \int_a^b g(t) \Delta t$;
- (ii) $\int_a^b (\alpha f)(t) \Delta t = \alpha \int_a^b f(t) \Delta t$;
- (iii) $\int_a^b f(t) \Delta t = - \int_b^a f(t) \Delta t$;
- (iv) $\int_a^b f(t) \Delta t = \int_a^c f(t) \Delta t + \int_c^b f(t) \Delta t$;
- (v) $\int_a^b f(\sigma(t)) g^\Delta(t) \Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t) g(t) \Delta t$;
- (vi) $\int_a^b f(t) g^\Delta(t) \Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t) g(\sigma(t)) \Delta t$;
- (vii) $\int_a^a f(t) \Delta t = 0$;
- (viii) if $|f(t)| \leq g(t)$ on $[a, b)$, then $\left| \int_a^b f(t) \Delta t \right| \leq \int_a^b g(t) \Delta t$;
- (ix) if $f(t) \geq 0$ for all $a \leq t < b$, then $\int_a^b f(t) \Delta t \geq 0$;
- (x) if $t \in \mathbb{T}^\kappa$, then $\int_t^{\sigma(t)} f(t) \Delta t = \mu(t) f(t)$.

3. INTEGRAL REPRESENTATION AND EXISTENCE OF SOLUTIONS

This section consists of our main results and their illustrative examples. First, an integral representation of solutions is obtained and then, by an application of the Schauder fixed point theorem, the existence of the solutions is proved.

3.1. Integral representation

To find an integral representation of the solutions for the BVP (1.1), we first introduce the following BVP for the sake of brevity. Consider

$$\begin{cases} x^{\Delta^3}(t) = h(t), & t \in J_0, \\ x^{\Delta^2}(t_k^+) = x^{\Delta^2}(t_k) + I_k(x(t_k), x^{\Delta}(t_k), x^{\Delta^2}(t_k)), & k \in \{1, \dots, m\}, \\ x^{\Delta^2}(0) = \lambda x^{\Delta^2}(T), \\ x^{\Delta}(0) = \sum_{j=1}^n \alpha_j x^{\Delta}(\xi_j), \\ x(0) = 0, \end{cases} \quad (3.1)$$

where it is assumed that

(A6): $h \in C([0, T]_{\mathbb{T}})$.

For convenience, we introduce the notation $\beta = 1 - \sum_{j=1}^n \alpha_j$.

Lemma 1. Suppose that the hypotheses (A2), (A3), (A5) and (A6) hold. Then x is a solution to the BVP (3.1) if and only if it is a solution to the integral equation

$$\begin{aligned} x(t) = & \frac{\lambda}{1-\lambda} \left(t \left(\frac{\sum_{l=1}^n \alpha_l \xi_l}{\beta} \right) + p(t) \right) \left(\int_0^T h(s) \Delta s + \sum_{j=1}^m I_j(x(t_j), x^{\Delta}(t_j), x^{\Delta^2}(t_j)) \right) \\ & + \frac{t}{\beta} \sum_{l=1}^n \alpha_l \left[\int_0^{\xi_l} (\xi_l - \sigma(s)) h(s) \Delta s + \sum_{0 < t_j < \xi_l} (\xi_l - t_j) I_j(x(t_j), x^{\Delta}(t_j), x^{\Delta^2}(t_j)) \right] \\ & + \int_0^t q(t, \sigma(s)) h(s) \Delta s + \sum_{0 < t_j < t} q(t, t_j) I_j(x(t_j), x^{\Delta}(t_j), x^{\Delta^2}(t_j)), \end{aligned} \quad (3.2)$$

where $p(t) = \int_0^t s \Delta s$ and $q(t, s) = p(t) - p(s) - (t-s)s$.

Proof. (1) Let x be a solution to the BVP (3.1). We integrate step by step both sides of the first equation of (3.1) from 0 to t to find

$$\begin{aligned} x^{\Delta^2}(t) &= x^{\Delta^2}(0) + \int_0^t h(s) \Delta s + \sum_{0 < t_j < t} (x^{\Delta^2}(t_j^+) - x^{\Delta^2}(t_j)) \\ &= x^{\Delta^2}(0) + \int_0^t h(s) \Delta s + \sum_{0 < t_j < t} I_j(x(t_j), x^{\Delta}(t_j), x^{\Delta^2}(t_j)), \quad t \in [0, T]_{\mathbb{T}}, \end{aligned} \quad (3.3)$$

which implies that

$$x^{\Delta^2}(T) = x^{\Delta^2}(0) + \int_0^T h(s)\Delta s + \sum_{j=1}^m I_j(x(t_j), x^{\Delta}(t_j), x^{\Delta^2}(t_j)).$$

Using the boundary condition

$$x^{\Delta^2}(0) = \lambda x^{\Delta^2}(T)$$

in the equation above, it follows that

$$(1 - \lambda)x^{\Delta^2}(0) = \lambda \int_0^T h(s)\Delta s + \lambda \sum_{j=1}^m I_j(x(t_j), x^{\Delta}(t_j), x^{\Delta^2}(t_j)).$$

Then, setting the last equation in (3.3), we find

$$\begin{aligned} x^{\Delta^2}(t) &= \frac{\lambda}{1 - \lambda} \left(\int_0^T h(s)\Delta s + \sum_{j=1}^m I_j(x(t_j), x^{\Delta}(t_j), x^{\Delta^2}(t_j)) \right) \\ &\quad + \int_0^t h(s)\Delta s + \sum_{0 < t_j < t} I_j(x(t_j), x^{\Delta}(t_j), x^{\Delta^2}(t_j)), \quad t \in [0, T]_{\mathbb{T}}. \end{aligned} \quad (3.4)$$

Now, integrating the last equation from 0 to t leads to

$$\begin{aligned} x^{\Delta}(t) &= x^{\Delta}(0) + t \frac{\lambda}{1 - \lambda} \left(\int_0^T h(s)\Delta s + \sum_{j=1}^m I_j(x(t_j), x^{\Delta}(t_j), x^{\Delta^2}(t_j)) \right) \\ &\quad + \int_0^t (t - \sigma(s))h(s)\Delta s \\ &\quad + \sum_{0 < t_j < t} I_j(x(t_j), x^{\Delta}(t_j), x^{\Delta^2}(t_j))(t - t_j), \quad t \in [0, T]_{\mathbb{T}} \end{aligned} \quad (3.5)$$

which implies that

$$\begin{aligned} \sum_{l=1}^n \alpha_l x^{\Delta}(\xi_l) &= x^{\Delta}(0) \sum_{l=1}^n \alpha_l \\ &\quad + \left(\sum_{l=1}^n \alpha_l \xi_l \right) \frac{\lambda}{1 - \lambda} \left(\int_0^T h(s)\Delta s + \sum_{j=1}^m I_j(x(t_j), x^{\Delta}(t_j), x^{\Delta^2}(t_j)) \right) \\ &\quad + \sum_{l=1}^n \alpha_l \left[\int_0^{\xi_l} (\xi_l - \sigma(s))h(s)\Delta s \right. \\ &\quad \left. + \sum_{0 < t_j < \xi_l} (\xi_l - t_j) I_j(x(t_j), x^{\Delta}(t_j), x^{\Delta^2}(t_j)) \right]. \end{aligned}$$

Applying the boundary condition

$$x^{\Delta}(0) = \sum_{l=1}^n \alpha_l x^{\Delta}(\xi_l)$$

one has

$$\begin{aligned} x^\Delta(0) = & \frac{\lambda}{1-\lambda} \left(\frac{\sum_{l=1}^n \alpha_l \xi_l}{\beta} \right) \left(\int_0^T h(s) \Delta s + \sum_{j=1}^m I_j(x(t_j), x^\Delta(t_j), x^{\Delta^2}(t_j)) \right) \\ & + \frac{1}{\beta} \sum_{l=1}^n \alpha_l \left[\int_0^{\xi_l} (\xi_l - \sigma(s)) h(s) \Delta s \right. \\ & \left. + \sum_{0 < t_j < \xi_l} (\xi_l - t_j) I_j(x(t_j), x^\Delta(t_j), x^{\Delta^2}(t_j)) \right]. \end{aligned}$$

Hence, using the above equation in (3.5), we obtain

$$\begin{aligned} x^\Delta(t) = & \frac{\lambda}{1-\lambda} \left(\frac{\sum_{l=1}^n \alpha_l \xi_l}{\beta} + t \right) \left(\int_0^T h(s) \Delta s + \sum_{j=1}^m I_j(x(t_j), x^\Delta(t_j), x^{\Delta^2}(t_j)) \right) \\ & + \frac{1}{\beta} \sum_{l=1}^n \alpha_l \left[\int_0^{\xi_l} (\xi_l - \sigma(s)) h(s) \Delta s \right. \\ & \left. + \sum_{0 < t_j < \xi_l} (\xi_l - t_j) I_j(x(t_j), x^\Delta(t_j), x^{\Delta^2}(t_j)) \right] \\ & + \int_0^t (t - \sigma(s)) h(s) \Delta s + \sum_{0 < t_j < t} (t - t_j) I_j(x(t_j), x^\Delta(t_j), x^{\Delta^2}(t_j)), \quad t \in [0, T]_{\mathbb{T}}. \end{aligned}$$

Finally, integration of the last equation results in

$$\begin{aligned} x(t) = & \frac{\lambda}{1-\lambda} \left(t \left(\frac{\sum_{l=1}^n \alpha_l \xi_l}{\beta} \right) + p(t) \right) \left(\int_0^T h(s) \Delta s + \sum_{j=1}^m I_j(x(t_j), x^\Delta(t_j), x^{\Delta^2}(t_j)) \right) \\ & + \frac{t}{\beta} \sum_{l=1}^n \alpha_l \left[\int_0^{\xi_l} (\xi_l - \sigma(s)) h(s) \Delta s \right. \\ & \left. + \sum_{0 < t_j < \xi_l} (\xi_l - t_j) I_j(x(t_j), x^\Delta(t_j), x^{\Delta^2}(t_j)) \right] \\ & + \int_0^t q(t, \sigma(s)) h(s) \Delta s + \sum_{0 < t_j < t} I_j(x(t_j), x^\Delta(t_j), x^{\Delta^2}(t_j)) \int_{t_j}^t (s - t_j) \Delta s, \end{aligned}$$

where $t \in [0, T]_{\mathbb{T}}$. In view of

$$q(t, t_j) = p(t) - p(t_j) - (t - t_j)t_j = \int_{t_j}^t s \Delta s - \int_{t_j}^t t_j \Delta s = \int_{t_j}^t (s - t_j) \Delta s,$$

we obtain the integral representation of $x(t)$ defined by (3.2).

(2) Let x be a solution to the integral equation (3.2). Then, from $p(0) = 0$, we have

$$x(0) = 0.$$

So, differentiation of (3.1) gives

$$\begin{aligned} x^\Delta(t) &= \frac{\lambda}{1-\lambda} \left(\frac{\sum_{l=1}^n \alpha_l \xi_l}{\beta} \right) \left(\int_0^T h(s) \Delta s + \sum_{j=1}^m I_j(x(t_j), x^\Delta(t_j), x^{\Delta^2}(t_j)) \right) \\ &\quad + \frac{1}{\beta} \sum_{l=1}^n \alpha_l \left[\int_0^{\xi_l} (\xi_l - \sigma(s)) h(s) \Delta s + \sum_{0 < t_j < \xi_l} I_j(x(t_j), x^\Delta(t_j), x^{\Delta^2}(t_j)) (\xi_l - t_j) \right] \\ &\quad + t \frac{\lambda}{1-\lambda} \left(\int_0^T h(s) \Delta s + \sum_{j=1}^m I_j(x(t_j), x^\Delta(t_j), x^{\Delta^2}(t_j)) \right) \\ &\quad + \int_0^t (t - \sigma(s)) h(s) \Delta s + \sum_{0 < t_j < t} (t - t_j) I_j(x(t_j), x^\Delta(t_j), x^{\Delta^2}(t_j)). \end{aligned} \quad (3.6)$$

Then, setting $t = 0$ and $t = \xi_l$, respectively, it is seen that

$$\begin{aligned} x^\Delta(0) &= \frac{\lambda}{1-\lambda} \left(\frac{\sum_{l=1}^n \alpha_l \xi_l}{\beta} \right) \left(\int_0^T h(s) \Delta s + \sum_{j=1}^m I_j(x(t_j), x^\Delta(t_j), x^{\Delta^2}(t_j)) \right) \\ &\quad + \frac{1}{\beta} \sum_{l=1}^n \alpha_l \left[\int_0^{\xi_l} (\xi_l - \sigma(s)) h(s) \Delta s \right. \\ &\quad \left. + \sum_{0 < t_j < \xi_l} I_j(x(t_j), x^\Delta(t_j), x^{\Delta^2}(t_j)) (\xi_l - t_j) \sum_{l=1}^n \alpha_l x^\Delta(\xi_l) \right] \\ &= \sum_{l=1}^n \alpha_l x^\Delta(\xi_l). \end{aligned}$$

Also, it is not difficult to see that differentiation of (3.6) results in (3.4). Thus, setting $t = t_k$ in equation (3.4), one has

$$\begin{aligned} x^{\Delta^2}(t_k^+) &= \frac{\lambda}{1-\lambda} \left(\int_0^T h(s) \Delta s + \sum_{j=1}^m I_j(x(t_j), x^\Delta(t_j), x^{\Delta^2}(t_j)) \right) \\ &\quad + \int_0^{t_k} h(s) \Delta s + \sum_{0 < t_j < t_k^+} I_j(x(t_j), x^\Delta(t_j), x^{\Delta^2}(t_j)), \end{aligned}$$

$k \in \{1, \dots, m\}$. Hence,

$$x^{\Delta^2}(t_k^+) - x^{\Delta^2}(t_k^-) = I_k(x(t_k), x^\Delta(t_k), x^{\Delta^2}(t_k)), \quad k \in \{1, \dots, m\}$$

holds, as well. Finally, note that

$$x^{\Delta^3}(t) = h(t), \quad t \in [0, T]_{\mathbb{T}}.$$

Therefore, x is a solution to the BVP (3.1). This completes the proof. $\square$

3.2. Existence of solutions

Lemma 2. Suppose (A1), (A2) and (A4) hold. Then there exist positive constants Q and R such that

$$\begin{aligned} |f(t, u, v, w)| &\leq Q(|u| + |v| + |w|) + R, \\ |I_k(u, v, w)| &\leq Q(|u| + |v| + |w|) + R, \quad k \in \{1, \dots, m\}, \end{aligned}$$

where $t \in [0, T]_{\mathbb{T}}$ and $u, v, w \in \mathbb{R}$.

Proof. By the first condition of (A4), it follows that there exists a positive constant N_1 such that

$$|f(t, u, v, w)| \leq \alpha(|u| + |v| + |w|), \quad t \in [0, 1], \quad |u| + |v| + |w| > N_1.$$

On the other hand, since $f \in C([0, T]_{\mathbb{T}} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})$, there exists a positive constant Q_1 such that

$$|f(t, u, v, w)| \leq Q_1, \quad t \in [0, 1], \quad |u| + |v| + |w| \leq N_1.$$

Therefore

$$|f(t, u, v, w)| \leq \alpha(|u| + |v| + |w|) + Q_1, \quad t \in [0, T]_{\mathbb{T}}, \quad u, v, w \in \mathbb{R}.$$

Similarly, from the second condition of (A4), there exist positive constants N_{2j} , $j \in \{1, \dots, m\}$, such that

$$|I_j(u, v, w)| \leq \delta_j(|u| + |v| + |w|), \quad |u| + |v| + |w| > N_{2j}, \quad j \in \{1, \dots, m\},$$

and from $I_j \in C(\mathbb{R} \times \mathbb{R} \times \mathbb{R})$, $j \in \{1, \dots, m\}$, there exist positive constants Q_{2j} , $j \in \{1, \dots, m\}$, such that

$$|I_j(u, v, w)| \leq Q_{2j}, \quad |u| + |v| + |w| \leq N_{2j}, \quad j \in \{1, \dots, m\}.$$

Consequently

$$|I_j(u, v, w)| \leq \delta_j(|u| + |v| + |w|) + Q_{2j}, \quad u, v, w \in \mathbb{R},$$

$j \in \{1, \dots, m\}$. If we define

$$\begin{aligned} Q &= \max\{\alpha, \delta_j, \quad j \in \{1, \dots, m\}\}, \\ R &= \max\{Q_1, Q_{2j}, \quad j \in \{1, \dots, m\}\}, \end{aligned}$$

we get the desired inequalities. This completes the proof. $\square$

In order to prove the existence of solutions, we will utilize the Schauder fixed point theorem. For this purpose, we first introduce the function spaces

$$PC([0, T]_{\mathbb{T}}) = \left\{ x \in C(J_0), \quad \exists \lim_{t \rightarrow t_j^-} x(t), \quad \lim_{t \rightarrow t_j^+} x(t), x(t_j^-) = x(t_j), \quad j \in \{1, \dots, m\} \right\},$$

$$PC^1([0, T]_{\mathbb{T}}) = \left\{ x \in PC([0, T]_{\mathbb{T}}) : x^\Delta \in C(J_0), \quad \exists \lim_{t \rightarrow t_j^+} x^\Delta(t), \right. \\ \left. \lim_{t \rightarrow t_j^-} x^\Delta(t), x^\Delta(t_j^-) = x^\Delta(t_j), \quad j \in \{1, \dots, m\} \right\}$$

and

$$PC^2([0, T]_{\mathbb{T}}) = \left\{ x \in PC^1([0, T]_{\mathbb{T}}) : x^{\Delta^2} \in C(J_0), \quad \exists \lim_{t \rightarrow t_j^+} x^{\Delta^2}(t), \quad \lim_{t \rightarrow t_j^-} x^{\Delta^2}(t), \right. \\ \left. x^{\Delta^2}(t_j^-) = x^{\Delta^2}(t_j), \quad j \in \{1, \dots, m\} \right\},$$

endowed with the norms

$$\|x\| = \sup_{t \in [0, T]_{\mathbb{T}}} |x(t)|,$$

$$\|x\| = \max \left\{ \sup_{t \in [0, T]_{\mathbb{T}}} |x(t)|, \quad \sup_{t \in [0, T]_{\mathbb{T}}} |x^\Delta(t)| \right\},$$

and

$$\|x\| = \max \left\{ \sup_{t \in [0, T]_{\mathbb{T}}} |x(t)|, \quad \sup_{t \in [0, T]_{\mathbb{T}}} |x^\Delta(t)|, \quad \sup_{t \in [0, T]_{\mathbb{T}}} |x^{\Delta^2}(t)| \right\},$$

respectively. Let $x \in PC^2([0, T]_{\mathbb{T}})$ and define the operator

$$Tx(t) = \frac{\lambda}{1-\lambda} \left(t \left(\frac{\sum_{l=1}^n \alpha_l \xi_l}{\beta} \right) + p(t) \right) \\ \times \left(\int_0^T f(s, x(s), x^\Delta(s), x^{\Delta^2}(s)) \Delta s + \sum_{j=1}^m I_j(x(t_j), x^\Delta(t_j), x^{\Delta^2}(t_j)) \right) \\ + \frac{t}{\beta} \sum_{l=1}^n \alpha_l \left[\int_0^{\xi_l} (\xi_l - \sigma(s)) f(s, x(s), x^\Delta(s), x^{\Delta^2}(s)) \Delta s \right. \\ \left. + \sum_{0 < t_j < \xi_l} (\xi_l - t_j) I_j(x(t_j), x^\Delta(t_j), x^{\Delta^2}(t_j)) \right] \\ + \int_0^t q(t, \sigma(s)) f(s, x(s), x^\Delta(s), x^{\Delta^2}(s)) \Delta s$$

$$+ \sum_{0 < t_j < t} q(t, t_j) I_j(x(t_j), x^\Delta(t_j), x^{\Delta^2}(t_j)), \quad t \in [0, T]_{\mathbb{T}}.$$

Take a sufficiently large $A > 0$ such that

$$\begin{aligned} A \geq & \max \left\{ \left| \frac{\lambda}{1-\lambda} \right| \left(\sigma(T) \frac{\sum_{l=1}^n |\alpha_l| \xi_l}{|\beta|} + (\sigma(T))^2 \right) (m + \sigma(T)) \right. \\ & + \frac{\sigma(T)}{|\beta|} \sum_{l=1}^n |\alpha_l| \left[\xi_l^2 + \sum_{j=1}^n (\xi_l + t_j) \right] + 4(\sigma(T))^3 + (\sigma(T))^2, \\ & \left| \frac{\lambda}{1-\lambda} \right| \frac{\sum_{l=1}^n |\alpha_l| \xi_l}{|\beta|} (m + \sigma(T)) + \frac{1}{|\beta|} \sum_{l=1}^n |\alpha_l| \left[\xi_l^2 + \sum_{j=1}^m (\xi_l + t_j) \right] \\ & \left. + \left(\left| \frac{\lambda}{1-\lambda} \right| + 1 \right) \sigma(T) (m + \sigma(T)), \left(\left| \frac{\lambda}{1-\lambda} \right| + 1 \right) (m + \sigma(T)), 1 \right\}. \end{aligned}$$

Lemma 3. Suppose (A1)-(A5) hold. Then, $T : PC^2([0, T]_{\mathbb{T}}) \rightarrow PC^2([0, T]_{\mathbb{T}})$ is a completely continuous operator.

Proof. Let $D \subset PC^2([0, T]_{\mathbb{T}})$ be a bounded set. Then, there exists a positive constant B such that

$$\|x\| \leq B, \quad x \in D.$$

Take $x \in D$ arbitrarily. Then

$$\begin{aligned} |Tx(t)| \leq & \left\{ \left| \frac{\lambda}{1-\lambda} \right| \left(\sigma(T) \frac{\sum_{l=1}^n |\alpha_l| \xi_l}{|\beta|} + (\sigma(T))^2 \right) (\sigma(T) + m) \right. \\ & + \frac{\sigma(T)}{|\beta|} \sum_{l=1}^n |\alpha_l| \left[\xi_l^2 + \sum_{j=1}^n (\xi_l + t_j) \right] + 4(\sigma(T))^3 + (\sigma(T))^2 \Big\} (3QB + R) \\ \leq & A(3QB + R), \quad t \in [0, T]_{\mathbb{T}}, \end{aligned}$$

and

$$\begin{aligned} |(Tx)^\Delta(t)| \leq & \left\{ \left| \frac{\lambda}{1-\lambda} \right| \left(\frac{\sum_{l=1}^n |\alpha_l| \xi_l}{|\beta|} + \sigma(T) \right) (\sigma(T) + m) \right. \\ & + \frac{1}{|\beta|} \sum_{l=1}^n |\alpha_l| \left[\xi_l^2 + \sum_{j=1}^m (\xi_l + t_j) \right] + \sigma(T) (\sigma(T) + m) \Big\} (3QB + R) \\ \leq & A(3QB + R), \quad t \in [0, T]_{\mathbb{T}}, \end{aligned}$$

and

$$|(Tx)^{\Delta^2}(t)| \leq \left(\left| \frac{\lambda}{1-\lambda} \right| + 1 \right) (\sigma(T) + m)(3QB + R) \leq A(3QB + R), \quad t \in [0, T]_{\mathbb{T}}.$$

Consequently

$$\|Tx\| \leq A(3QB + R). \quad (3.7)$$

Moreover,

$$|(Tx)^{\Delta^3}(t)| = |f(t, x(t), x^{\Delta}(t), x^{\Delta^2}(t))| \leq 3QB + R \leq A(3QB + R), \quad t \in [0, T]_{\mathbb{T}}.$$

Hence, by the Arzela-Ascoli theorem, the operator $T : PC^2([0, T]_{\mathbb{T}}) \rightarrow PC^2([0, T]_{\mathbb{T}})$ is completely continuous. $\square$

Theorem 2. Suppose that (A1)-(A5) hold, and the nonnegative constants A , B , Q and R satisfy

$$A(3QB + R) \leq B.$$

Then, the BVP (1.1) has at least one solution.

Proof. Let

$$S = \{x \in PC^2([0, T]_{\mathbb{T}}) : \|x\| \leq B\}.$$

By Lemma 2, $T : S \rightarrow PC^2([0, T]_{\mathbb{T}})$ is a completely continuous operator, and from (3.7), it is seen that

$$\|Tx\| \leq A(3QB + R) \leq B, \quad x \in D,$$

which means $T : S \rightarrow S$. Thus, applying Schauder fixed point theorem, we conclude that the operator T has a fixed point in S . This completes the proof. $\square$

3.3. An example

We provide the following example in order to show that our main result is applicable for impulsive dynamic boundary value problems of the form (1.1).

Example 1. Let

$$\mathbb{T} = \left[0, \frac{1}{32}\right] \cup \left[\frac{1}{16}, \frac{1}{8}\right] \cup \left[\frac{1}{4}, \frac{1}{3}\right] \cup \left[\frac{1}{2}, 1\right],$$

where $[0, \frac{1}{32}]$, $[\frac{1}{16}, \frac{1}{8}]$, $[\frac{1}{4}, \frac{1}{3}]$ and $[\frac{1}{2}, 1]$ are the real-valued intervals. Let also,

$$\begin{aligned} T &= \frac{1}{3}, \quad m = 4, \quad n = 3, \\ t_0 &= 0, \quad t_1 = \frac{1}{36}, \quad t_2 = \frac{1}{16}, \quad t_3 = \frac{1}{10}, \quad t_4 = \frac{1}{4}, \quad t_5 = T = \frac{1}{3}, \\ J &= \left[0, \frac{1}{3}\right], \quad J_0 = \left(0, \frac{1}{32}\right) \cup \left(\frac{1}{16}, \frac{1}{8}\right) \cup \left(\frac{1}{4}, \frac{1}{3}\right), \\ \xi_1 &= \frac{1}{38}, \quad \xi_2 = \frac{3}{32}, \quad \xi_3 = \frac{7}{24}. \end{aligned}$$

Consider the BVP

$$\begin{cases} x^{\Delta^3}(t) = \frac{x(t)+x^{\Delta}(t)+x^{\Delta^2}(t)}{10^{20}(1+(x(t))^2)(1+(x^{\Delta}(t))^2)(1+(x^{\Delta^2}(t))^2)}, & t \in J_0, \\ x^{\Delta^2}(t_k^+) = x^{\Delta^2}(t_k) + \frac{x(t_k)}{10^{40}(1+(x^{\Delta}(t_k))^4)(1+(x^{\Delta^2}(t_k))^8)}, & k \in \{1, \dots, 4\}, \\ x^{\Delta^2}(0) = \frac{1}{2}x^{\Delta^2}\left(\frac{1}{3}\right), \\ x^{\Delta}(0) = \frac{1}{2}x^{\Delta}(\xi_1) + \frac{1}{8}x^{\Delta}(\xi_2) + \frac{1}{16}x^{\Delta}(\xi_3), \\ x(0) = 0. \end{cases} \quad (3.8)$$

Clearly,

$$\begin{aligned} f(t, u, v, w) &= \frac{u + v + w}{10^{20}(1+u^2)(1+v^2)(1+w^2)}, \quad t \in J, \quad u, v, w \in \mathbb{R}, \\ I_k(u, v, w) &= \frac{u}{10^{40}(1+v^4)(1+w^8)}, \quad u, v, w \in \mathbb{R}, \\ \sigma\left(\frac{1}{3}\right) &= \frac{1}{2}, \quad \alpha_1 = \frac{1}{2}, \quad \alpha_2 = \frac{1}{8}, \quad \alpha_3 = \frac{1}{16}, \quad \lambda = \frac{1}{2}. \end{aligned}$$

Thus, we have

$$|f(t, u, v, w)| \leq \frac{1}{10^{20}}(|u| + |v| + |w|), \quad (3.9)$$

$$|I_k(u, v, w)| \leq \frac{1}{10^{40}}|u| \quad (3.10)$$

which implies that

$$Q = \frac{1}{10^{20}}, \quad R = 0.$$

If we take

$$A = 10^{10} \quad \text{and} \quad B = \frac{1}{2},$$

then we obtain

$$A(3QB + R) = 3AQB = 3 \cdot 10^{10} \cdot \frac{1}{10^{20}} \cdot \frac{1}{2} < \frac{1}{2} = B. \quad (3.11)$$

Therefore, by (3.9)-(3.11) it follows that all the hypotheses of Theorem 2 hold. Hence, the BVP (3.8) has at least one solution.

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DECOMPOSITION OF FUNCTIONAL EQUATIONS WITH APPLICATIONS

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Abstract. In this paper the Decomposition Theorem for functional equations is shown. As an application of this Theorem the two times continuously differentiable solution of the functional equation

$$G_1(x(x+y)) + F_1(y) = G_2(y(x+y)) + F_2(y)$$

can be given with unknown functions $G_i, F_i : \mathbb{R}_+ \rightarrow \mathbb{R}$ ($i = 1, 2$) where the Equation is fulfilled for all $x, y \in \mathbb{R}_+$ (where $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x > 0\}$).

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1. INTRODUCTION

The main purpose of this paper is to show the decomposition of Pexider functional equations.

The Pexider equations are functional equations contained more than one unknown functions. For example, consider the well-known Lobachevsky functional equation

$$f(x+y)f(x-y) = f^2(x)$$

where the unknown function $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies the Equation for all $x, y \in \mathbb{R}$ [1]. The Pexider version [17] of this Equation is

$$F(x+y)G(x-y) = H^2(x),$$

where the unknown functions $F, G, H : \mathbb{R} \rightarrow \mathbb{R}$ satisfy the Equation for all $x, y \in \mathbb{R}$.

The decomposition of functional equations is a method for Pexider equations with certain symmetric structure. Such an equation results in two other, simpler equations. The general (twice continuously differentiable) solution of the original function equation can be expressed by the general (twice continuously differentiable) solutions of the obtained equations.

In section 2 the Decomposition Theorem is given. In the rest of the paper any applications of the Decomposition Theorem can be found.

2. THE DECOMPOSITION THEOREMS

Theorem 1. *Let X be a set; $\circ$ and $*$ be binary operations on the set X ; $(Y, +)$ be a uniquely two divisible Abelian group.*

- (1) *If the functions $G_i, F_i : X \rightarrow Y$ ($i = 1, 2$) are solutions of the functional equation*

$$G_1(x \circ y) + F_1(x * y) = G_2(y \circ x) + F_2(y * x) \quad (x, y \in X), \quad (2.1)$$

and the functions $g, \gamma, f, \phi : X \rightarrow Y$ are defined by

$$\begin{aligned} g &:= \frac{1}{2}(G_1 + G_2), & f &:= \frac{1}{2}(F_1 + F_2), \\ \gamma &:= \frac{1}{2}(G_1 - G_2), & \phi &:= \frac{1}{2}(F_1 - F_2), \end{aligned} \quad (2.2)$$

then this functions satisfy the Equation

$$\begin{aligned} g(x \circ y) + f(x * y) &= g(y \circ x) + f(y * x), \\ \gamma(x \circ y) + \gamma(y \circ x) &= -\phi(x * y) - \phi(y * x) \end{aligned} \quad (x, y \in X).$$

- (2) *If the functions $g, \gamma, f, \phi : X \rightarrow Y$ are solutions of Equation (1), and the functions $G_i, F_i : X \rightarrow Y$ are defined by*

$$\begin{aligned} G_1 &:= g + \gamma, & F_1 &:= f + \phi, \\ G_2 &:= g - \gamma, & F_2 &:= f - \phi, \end{aligned} \quad (2.3)$$

then this functions satisfy Equation (2.1).

Proof. (1) Assume that the functions $G_i, F_i : X \rightarrow Y$ ($i = 1, 2$) are solutions of the functional equation (2.1). Consider the Equation (2.1) and the Equation which can be obtained from Equation (2.1) by changing x and y .

$$\begin{aligned} G_1(x \circ y) + F_1(x * y) &= G_2(y \circ x) + F_2(y * x) \\ G_2(x \circ y) + F_2(x * y) &= G_1(y \circ x) + F_1(y * x) \end{aligned} \quad (2.4)$$

for all $x, y \in X$. Add and subtract the Equations of (2.4). Thus we have

$$\begin{aligned} (G_1 + G_2)(x \circ y) + (F_1 + F_2)(y) &= (G_1 + G_2)(y \circ x) + (F_1 + F_2)(x) \\ (G_1 - G_2)(x \circ y) + (F_1 - F_2)(y) &= (G_2 - G_1)(y \circ x) + (F_2 - F_1)(x) \end{aligned} \quad (2.5)$$

for all $x, y \in X$. Define the functions $g, \gamma, f, \phi : X \rightarrow Y$ by (2.2) thus by Equation (2.5) we obtain the Equations of (1).

- (2) Assume that the functions $g, \gamma, f, \phi : X \rightarrow Y$ are solutions of the Equations of (2.3) and define the functions $G_i, F_i : X \rightarrow Y$ ($i = 1, 2$) by (1). Then

$$\begin{aligned} G_1(x \circ y) + F_1(x * y) &= (g(x \circ y) + \gamma(x \circ y)) + (f(x * y) + \phi(x * y)) \\ &= (g(x \circ y) + f(x * y)) + (\gamma(x \circ y) + \phi(x * y)) \end{aligned}$$

$$\begin{aligned}
 &= (g(y \circ x) + f(y * x)) + (-\gamma(y \circ x) - \varphi(y * x)) \\
 &= (g(y \circ x) - \gamma(y \circ x)) + (f(y * x) - \varphi(y * x)) \\
 &= G_2(y \circ x) + F_2(y * x)
 \end{aligned}$$

for all $x, y \in X$ which completes the proof. $\square$

The following Theorem is a special case of Theorem 1 with $x * y := y$ for all $x, y \in X$.

Theorem 2. *Let X be a set $\circ$ be a binary operation on the set X , $(Y, +)$ be a uniquely two divisible Abelian group. The functions $G_i, F_i : X \rightarrow Y$ ($i = 1, 2$) are solutions of the functional equation*

$$G_1(x \circ y) + F_1(y) = G_2(y \circ x) + F_2(x) \quad (x, y \in X) \quad (2.6)$$

if and only if they are of the form

$$\begin{aligned}
 G_1(x) &= g(x) + \gamma(x), & F_1(x) &= f(x) - \gamma(x \circ x), \\
 G_2(x) &= g(x) - \gamma(x), & F_2(x) &= f(x) + \gamma(x \circ x)
 \end{aligned}$$

for all $x \in X$ where the functions $f, g, \gamma : X \rightarrow Y$ are solutions of the Equations

$$\begin{aligned}
 g(x \circ y) + f(y) &= g(y \circ x) + f(x) & (x, y \in X) \\
 \gamma(x \circ x) + \gamma(y \circ y) &= \gamma(x \circ y) + \gamma(y \circ x) & (x, y \in X).
 \end{aligned}$$

Proof. This Theorem can be similarly proven as Theorem 1. $\square$

We investigate the case $X := \mathbb{R}_+ := \{x \in \mathbb{R} \mid x > 0\}$, $x \circ y := x(x + y)$ for all $x, y \in \mathbb{R}_+$, $Y(+) := \mathbb{R}(+)$, and we want to find the twice continuously differentiable solutions. By Theorem 2 instead of Equation

$$G_1(x(x + y)) + F_1(y) = G_2(y(x + y)) + F_2(x) \quad (x, y \in \mathbb{R}_+) \quad (2.7)$$

with unknown functions $G_i, F_i : \mathbb{R}_+ \rightarrow \mathbb{R}$ ($i = 1, 2$) where the Equation is fulfilled for all $x, y \in \mathbb{R}_+$ it is enough to give the twice continuously differentiable solutions of the Equations

$$g(x(x + y)) + f(y) = g(y(x + y)) + f(y) \quad (x, y \in \mathbb{R}_+) \quad (2.8)$$

and

$$\gamma(x(x + y)) + \gamma(y(x + y)) = \gamma(2x^2) + \gamma(2y^2) \quad (x, y \in \mathbb{R}_+). \quad (2.9)$$

In Equation (2.9) we will use the function $\delta : \mathbb{R}_+ \rightarrow \mathbb{R}$ defined by $\delta(x) := \gamma(2x^2)$ for all $x \in \mathbb{R}_+$.

Equation (2.9) (and with it its Pexider version, Equation 2.7) is related to equations containing means and equations containing the Gauss composition of these means (see [3], [4], [5], [7], and [8]), so these equations are very important.

The measurable solutions of Equation of (2.7) can be easily obtained by using the method of A. J  rai [15], [16], but we do not deal with finding the measurable

solutions, because we want to know the general solutions in more general settings see our Conjecture in the last section of this paper.

3. THE TWICE CONTINUOUSLY DIFFERENTIABLE SOLUTIONS OF EQUATION (2.8)

In his paper [18] S. Narumi was the first to use differential calculus to solve functional equations. Nowadays, there are many different methods for constructing differential operators that can be used to reduce a functional equation to a differential equation [10], [14], [19]. Instead of them we will use only two similar and natural differential operators.

Proposition 1. *If the functions $F_i, G_i : \mathbb{R}_+ \rightarrow \mathbb{R}$ are twice differentiable solutions of Equation (2.7) then*

$$0 = \frac{-2x+y}{x+2y}F_1'(y) + \frac{-2x-y}{x+2y}F_1''(y) + \frac{-x+2y}{x+2y}F_2'(x) + \frac{x}{y}F_2''(x) \quad (3.1)$$

for all $x, y \in \mathbb{R}_+$.

Proof. Apply the differential operator

$$D_1 := \frac{1}{2x+y} \frac{\partial}{\partial x} - \frac{1}{x} \frac{\partial}{\partial y}$$

for Equation (2.7), whence we have that

$$-\frac{1}{x}F_1'(y) = -\frac{2(x+y)^2}{x(2x+y)}G_2'(xy+y^2) + \frac{1}{2x+y}F_2'(x) \quad (x, y \in \mathbb{R}_+),$$

whence by ordering we have that

$$G_2'(xy+y^2) = \frac{2x+y}{2(x+y)^2}F_1'(y) + \frac{x}{2(x+y)^2}F_2'(x) \quad (x, y \in \mathbb{R}_+). \quad (3.2)$$

Apply the differential operator

$$D_2 := \frac{1}{y} \frac{\partial}{\partial x} - \frac{1}{x+2y} \frac{\partial}{\partial y}$$

for Equation (3.2) thus we obtain Equation (3.1). $\square$

Proposition 2. *If the functions $f, g : \mathbb{R}_+ \rightarrow \mathbb{R}$ are twice differentiable solutions of the Equation (2.8), and the constants c_1, c_2 ; and functions $a_0, a_1, \Phi : \mathbb{R}_+ \rightarrow \mathbb{R}$ are defined by*

$$\begin{aligned} c_1 &:= f'(1), \quad c_2 := f''(1), \\ a_0(x) &:= c_1 \frac{2x-1}{x(x+2)} + c_2 \frac{2x+1}{x(x+2)} \quad (x \in \mathbb{R}_+) \\ a_1(x) &:= \frac{x-2}{x(x+2)} \quad (x \in \mathbb{R}_+), \\ \Phi(x) &:= f'(x) \quad (x \in \mathbb{R}_+), \end{aligned} \quad (3.3)$$

then

$$\Phi'(x) = a_0(x) + a_1(x)\Phi(x) \quad (x \in \mathbb{R}_+) \quad (3.4)$$

Proof. Write $f(y)$ instead of $F_1(y)$ and $f(x)$ instead of $F_2(x)$ in Equation (3.1) thus we have that

$$0 = \frac{-2x+y}{x+2y}f_1'(y) + \frac{-2x-y}{x+2y}f_1''(y) + \frac{-x+2y}{x+2y}f'(x) + \frac{x}{y}f''(x) \quad (3.5)$$

for all $x, y \in \mathbb{R}_+$. Write 1 instead of y in Equation (3.5) after order the obtained Equation. Thus we have that

$$f''(x) = f'(1)\frac{2x-1}{x(x+2)} + f''(1)\frac{2x+1}{x(x+2)} + \frac{x-2}{x(x+2)}f'(x) \quad (x \in \mathbb{R}_+),$$

whence by the notations of (3.3) we obtain the differential Equation (3.4). $\square$

For to solve the above differential equation we can apply the well-known formula

$$\Phi(x) = \exp\left(\int_1^x a_1(u)du\right) \left(c_1 + \int_1^x \frac{a_0(v)}{\exp(\int_1^v a_1(u)du)}dv\right) \quad (x \in \mathbb{R}_+).$$

Proposition 3. *Preserve the notation of Proposition 2. The solution of differential equation (3.4) (with boundary values $\Phi(1) = c_1$, $\Phi'(1) = c_2$) is*

$$\Phi(x) = c_1 \frac{x^2+1}{2x} + c_2 \frac{x^2-1}{2x} \quad (x \in \mathbb{R}_+). \quad (3.6)$$

Proof. It is easy to see that

$$\begin{aligned} \exp\left(\int_1^x a_1(u)du\right) &= \frac{(x+2)^2}{9x}, \\ \int_1^x \frac{a_0(v)}{\exp(\int_1^v a_1(u)du)}dv &= c_1 \int_1^x \frac{18v-9}{(v+2)^3}dv + c_2 \int_1^x \frac{18v+9}{(v+2)^3}dv \\ &= c_1 \left(-\frac{7x^2+8x-1}{2(x+2)^2}\right) + c_2 \left(-\frac{9(x^2-1)}{2(x+2)^2}\right), \end{aligned}$$

whence we obtain Equation (3.6). $\square$

Theorem 3. *The twice continuously differentiable solution of Equation (2.8) is*

$$\begin{aligned} g(x) &= C_1x + C_2\ln(x) + C_4 \quad (x \in \mathbb{R}_+), \\ f(x) &= C_1x^2 + C_2\ln(x) + C_3 \quad (x \in \mathbb{R}_+) \end{aligned} \quad (3.7)$$

where C_i are arbitrary constants for $i = 1, 2, 3$.

Proof. Since $\Phi(x) = f'(x)$ for all $x \in \mathbb{R}_+$ thus by Equation (3.6) we obtain that

$$\begin{aligned} f(x) &= c_1 \int_1^x \frac{u^2+1}{2u}du + c_2 \int_1^x \frac{u^2-1}{2u}du + f(1) \\ &= \frac{1}{4}(c_1+c_2)x^2 + \frac{1}{2}(c_1-c_2)\ln(x) - \frac{1}{4}(c_1+c_2) + f(1) \end{aligned}$$

for all $x \in \mathbb{R}_+$ whence we obtain the second Equation of (3.7) with constants $C_1 := \frac{1}{4}(c_1 + c_2)$, $C_2 := \frac{1}{2}(c_1 - c_2)$ and $C_3 := -\frac{1}{4}(c_1 + c_2) + f(1)$.

Take the substitution $x \leftarrow \frac{x}{\sqrt{x+1}}$, $y \leftarrow \frac{1}{\sqrt{x+1}}$, that is, take $x \leftarrow \frac{u}{\sqrt{u+1}}$ and $y \leftarrow \frac{1}{\sqrt{u+1}}$ after $u \leftarrow x$ thus we obtain the Equation

$$g(x) = -f\left(\frac{1}{\sqrt{x+1}}\right) + f\left(\frac{x}{\sqrt{x+1}}\right) + g(1) \quad (x \in \mathbb{R}_+). \quad (3.8)$$

From the second Equation of (3.7) and Equation (3.8) we obtain that

$$\begin{aligned} g(x) &= -C_1 \frac{1}{x+1} - C_2 \ln\left(\frac{1}{\sqrt{x+1}}\right) \\ &\quad + C_1 \frac{x^2}{x+1} + C_2 \ln\left(\frac{x}{\sqrt{x+1}}\right) + C_3 + g(1) - C_1 \\ &= C_1 x + C_2 \ln(x) + C_3 + g(1) - C_1, \end{aligned}$$

which is the first Equation of (3.7) with constants $C_4 := C_3 + g(1) - C_1$.

The converse statement can be obtained by trivial calculation. $\square$

4. THE TWICE CONTINUOUSLY DIFFERENTIABLE SOLUTIONS OF EQUATION (2.9) AND (2.7)

Proposition 4. *If the function $\gamma: \mathbb{R}_+ \rightarrow \mathbb{R}$ is twice differentiable solutions of Equation (2.9), and the functions $b_0, b_1, \Phi: \mathbb{R}_+ \rightarrow \mathbb{R}$ and the constants d_1, d_2 are defined by*

$$\begin{aligned} \delta(x) &:= \gamma(2x^2), \\ d_1 &:= \delta'(1), \quad d_2 := \delta''(1), \\ b_0(x) &:= d_1 \frac{2x-1}{x(x+2)} + d_2 \frac{2x+1}{x(x+2)} \quad (x \in \mathbb{R}_+) \\ b_1(x) &:= \frac{x-2}{x(x+2)} \quad (x \in \mathbb{R}_+), \\ \Psi(x) &:= \delta'(x) \quad (x \in \mathbb{R}_+), \end{aligned} \quad (4.1)$$

then

$$\Psi'(x) = b_0(x) + b_1(x)\Phi(x) \quad (x \in \mathbb{R}_+) \quad (4.2)$$

Proof. Write $-\delta(y)$ instead of $F_1(y)$ and $\delta(x)$ instead of $F_2(x)$ in Equation (3.1) thus we have that

$$0 = -\frac{-2x+y}{x+2y}\delta'(y) + \frac{2x+y}{x+2y}\delta''(y) + \frac{-x+2y}{x+2y}\delta'(x) + \frac{x}{y}\delta''(x)$$

for all $x, y \in \mathbb{R}_+$. Write 1 instead of y in Equation (3.5) after order the obtained Equation. Thus we have that

$$\delta''(x) = \delta'(1) \frac{2x+1}{x(x+2)} + \delta''(1) \frac{-2x-1}{x(x+2)} + \frac{x-2}{x(x+2)} f'(x) \quad (x \in \mathbb{R}_+),$$

whence by the notations of (4.1) we obtain the differential equation (4.2). $\square$

Proposition 5. *Preserve the notation of Proposition 4. The solution of differential equation (3.4) (with boundary values $\Psi(1) = d_1$, $\Psi'(1) = d_2$) is*

$$\Psi(x) = d_1 \frac{-5x^2 + 16x + 7}{18x} + d_2 \frac{-(x-2)^2}{2x} \quad (x \in \mathbb{R}_+). \quad (4.3)$$

Proof. It is easy to see that

$$\begin{aligned} \exp\left(\int_1^x b_1(u) du\right) &= \frac{(x+2)^2}{9x}, \\ \int_1^x \frac{b_0(v)}{\exp(\int_1^v b_1(u) du)} dv &= d_1 \int_1^x \frac{-18v+9}{(v+2)^3} dv + d_2 \int_1^x \frac{-18v-9}{(v+2)^3} dv \\ &= d_1 \left(-\frac{7x^2+8x-1}{2(x+2)^2}\right) + d_2 \left(-\frac{9(x^2-1)}{2(x+2)^2}\right), \end{aligned}$$

whence we obtain Equation (4.3). $\square$

In the proof of following Theorem we shall use the difference operator Δ_λ defined by

$$\Delta_\lambda F(x) := F(\lambda x) - F(x) \quad (x \in \mathbb{R}_+),$$

where $\lambda > 0$ and $F: \mathbb{R}_+ \rightarrow \mathbb{R}$ is an arbitrary function. This difference operator was used first in [6] where was applied to give the general solution of Vajzović equation, see also [9] and have been also used in [11].

Theorem 4. *If the function $\gamma: \mathbb{R}_+ \rightarrow \mathbb{R}$ is a twice continuously differentiable solution of Equation (2.9), then it is a constant function.*

Proof. Preserve the notations of Proposition 4 and Proposition 5. Since $\Phi(x) = \delta'(x)$ for all $x \in \mathbb{R}_+$ thus by Equation (4.3) we obtain that

$$\begin{aligned} \delta(x) &= d_1 \int_1^x \frac{-5u^2 + 16u + 7}{18u} du + d_2 \int_1^x \frac{-(u-2)^2}{2u} du + \delta(1) \\ &= \frac{d_1}{36} (-5x^2 + 32x + 14 \ln(x) - 27) + \frac{d_2}{4} (-x^2 + 2 \ln(x) + 1) + \delta(1) \end{aligned}$$

for all $x \in \mathbb{R}_+$. Since $\gamma(x) = \delta(\sqrt{\frac{x}{2}})$ for all $x \in \mathbb{R}_+$ thus we have that

$$\begin{aligned} \gamma(x) &= \left(-\frac{5}{72}d_1 - \frac{1}{8}d_2\right) + \frac{4\sqrt{2}}{9}d_1\sqrt{x} \\ &\quad + \left(\frac{7}{36}d_1 + \frac{1}{4}d_2\right)(\ln(x) - \ln(2)) + \left(-\frac{3}{4}d_1 + \frac{1}{2}d_2 + \delta(1)\right) \end{aligned} \quad (4.4)$$

for all $x \in \mathbb{R}_+$. Define the function $\gamma_2 : \mathbb{R}_+ \rightarrow \mathbb{R}$ by $\gamma_2 := \Delta_2 \gamma$. Thus by Equation (4.4) we have that

$$\begin{aligned} \gamma_2(x) &= \left(-\frac{5}{72}d_1 - \frac{1}{8}d_2\right)x + \frac{4\sqrt{2}}{9}(\sqrt{2}-1)d_1\sqrt{x} \\ &\quad + \left(\frac{7}{36}d_1 + \frac{1}{4}d_2\right)\ln(2) \\ &= D_1x + D_2\sqrt{x} + D_3 \quad (x \in \mathbb{R}_+) \end{aligned} \quad (4.5)$$

with constants

$$\begin{aligned} D_1 &:= -\frac{5}{72}d_1 - \frac{1}{8}d_2, & D_2 &:= \frac{4\sqrt{2}}{9}(\sqrt{2}-1)d_1, \\ D_3 &:= \left(\frac{7}{36}d_1 + \frac{1}{4}d_2\right)\ln(2). \end{aligned} \quad (4.6)$$

□

It is easy to see that the function γ_2 satisfies the Equation (2.9) thus we can write the function γ_2 instead of the function γ in this Equation. Thus by simple calculation we obtain that

$$D_1 = D_2 \frac{(\sqrt{x} + \sqrt{y})\sqrt{x+y} - \sqrt{2}(x+y)}{(x-y)_2} \quad (x, y \in \mathbb{R}_+, x \neq y)$$

which shows that $D_1 = D_2 = 0$ whence by (4.6) we have that $d_1 = d_2 = 0$ whence by Equation (4.4) we obtain that $\gamma(x) = \delta(1) = \gamma(2)$ for all $x \in \mathbb{R}_+$ which was to be proven.

Now, we can give the twice continuously differentiable solution of Equation (2.7).

Theorem 5. *The twice continuously differentiable solution of Equation (2.7) is*

$$\begin{aligned} G_1(x) &= C_1x + C_2\ln(x) + D_1, & F_1(x) &= C_1x^2 + C_2\ln(x) + D_2, \\ G_2(x) &= C_1x + C_2\ln(x) + D_3, & F_2(x) &= C_1x^2 + C_2\ln(x) + D_4 \end{aligned}$$

for all $x \in \mathbb{R}_+$ where C_i $i = 1, 2$, and $D_j \in \mathbb{R}$ $j = 1, 2, 3, 4$ are arbitrary constants such that $D_1 + D_2 = D_3 + D_4$.

Proof. The assertion can be easily obtained by Theorem 2, Theorem 3, and Theorem 4. □

5. ADDITIONAL APPLICATIONS AND CONJECTURES

In this chapter we shall apply the notations of Theorem 2.

In paper [13] was investigated the case when $X = \mathbb{F}_+$ where $\mathbb{F} = \mathbb{F}(+, \cdot)$ is an ordered field (the operation $\cdot$ is commutative) the $\mathbb{T}_+$ is the set of positive elements

of the $\mathbb{F}$, $x \circ y := \frac{x}{y+1}$ for all $x, y \in \mathbb{F}_+$, $Y = Y(+)$ is a uniquely two-divisible Abelian group. In this settings the general solution of Equation (2.6) is

$$\begin{aligned} G_1(x) &= l_1x + l_2(x+1) + l_3(x) + d_1, \\ G_2(x) &= l_1x + l_2(x+1) - l_3(x) + d_3, \\ F_1(x) &= l_1(x(x+1)) + l_2(x) - l_3\left(\frac{x+1}{x}\right) + d_2, \\ F_2(x) &= l_1(x(x+1)) + l_2(x) + l_3\left(\frac{x+1}{x}\right) + d_4, \end{aligned}$$

for all $x \in \mathbb{F}_+$ where $l_i : \mathbb{F}_+ \rightarrow Y$ ($i = 1, 2, 3$) are arbitrary logarithmic functions, $d_i \in Y$ ($i = 1, 2, 3, 4$) are arbitrary constants such that $d_1 + d_2 = d_3 + d_4$. (Concerning the logarithmic functions see [1], and [17]). This result can not be improved.

In paper [11] was investigated the case when $X = \mathbb{F}_+$ where $\mathbb{F} = \mathbb{F}(+, \cdot)$ is an Archimedean ordered field (the operation $\cdot$ is commutative), $x \circ y := x(y+1)$ for all $x, y \in \mathbb{F}_+$, $Y = Y(+)$ is a uniquely two-divisible Abelian group. In this settings the general solution of Equation (2.6) is

$$\begin{aligned} G_1(x) &= a(x) + l_2(x) + l_3(x) + d_1, \\ G_2(x) &= a(x) + l_2(x) - l_3(x) + d_3, \\ F_1(x) &= -a(x) + l_2\left(\frac{x}{x+1}\right) - l_3(x(x+1)) + d_2, \\ F_2(x) &= -a(x) + l_2\left(\frac{x}{x+1}\right) + l_3(x(x+1)) + d_4, \end{aligned} \tag{5.1}$$

for all $x \in \mathbb{F}_+$ where $a : \mathbb{F}_+ \rightarrow Y$ is an arbitrary additive function, $l_i : \mathbb{F}_+ \rightarrow Y$ ($i = 2, 3$) are arbitrary logarithmic functions, $d_i \in Y$ ($i = 1, 2, 3, 4$) are arbitrary constants such that $d_1 + d_2 = d_3 + d_4$. (Concerning the additive and logarithmic functions see [1], and [17]). This result could perhaps be improved writing simply ordered field instead of Archimedean ordered field, that is,

Conjecture 1. *If $X = \mathbb{F}_+$ where $\mathbb{F} = \mathbb{F}(+, \cdot)$ is an ordered field, $x \circ y := x(y+1)$ for all $x, y \in \mathbb{F}_+$, $Y = Y(+)$ is a uniquely two-divisible Abelian group, then the general solution of Equation (2.6) is of the form (5.1) for all $x \in \mathbb{F}_+$ where $a : \mathbb{F}_+ \rightarrow Y$ is an arbitrary additive function, $l_i : \mathbb{F}_+ \rightarrow Y$ ($i = 2, 3$) are arbitrary logarithmic functions, $d_i \in Y$ ($i = 1, 2, 3, 4$) are arbitrary constants such that $d_1 + d_2 = d_3 + d_4$.*

The above two problems have probability background see [18], [2], [13], and [11]. Two pairs of new equations can be derived from the above two equations, the Pexider versions of the second of these equations (which contain the unknown gamma function) characterize the logarithmic functions see [12].

Finally, we can formulate our Conjecture concerning the main equation of this paper.

Conjecture 2. *If $X = \mathbb{F}_+$ where $\mathbb{F} = \mathbb{F}(+, \cdot)$ is an ordered field, $x \circ y := x(x+y)$ for all $x, y \in \mathbb{F}_+$, $Y = Y(+)$ is a uniquely two-divisible Abelian group, then the general solution of Equation (2.6) is*

$$\begin{aligned} G_1(x) &= a(x) + l(x) + d_1, & F_1(x) &= a(x^2) + l(x) + d_2, \\ G_2(x) &= a(x) + l(x) + d_3, & F_2(x) &= a(x^2) + l(x) + d_4 \end{aligned}$$

for all $x \in \mathbb{F}_+$ where $a : \mathbb{F}_+ \rightarrow Y$ is an arbitrary additive function, $l : \mathbb{F}_+ \rightarrow Y$ is an arbitrary logarithmic function, $d_i \in Y$ $i = 1, 2, 3, 4$ are arbitrary constants such that $d_1 + d_2 = d_3 + d_4$.

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SOME INEQUALITIES OF ANTI-INVARIANT RIEMANNIAN SUBMERSIONS IN COMPLEX SPACE FORMS

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Abstract. The aim of the present paper is to analyze sharp type inequalities including the scalar and Ricci curvatures of anti-invariant Riemannian submersions in complex space forms.

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1. INTRODUCTION

B.-Y. Chen revealed the intrinsic and extrinsic invariants who established an inequality including Ricci curvature and squared mean curvature of a submanifold in a real space form $R^n(c)$ in 1999 (see [4]). In 2005 by B.-Y. Chen, a generalization of this inequality was proved for arbitrary submanifolds in an arbitrary Riemannian manifold (see [5]). Subsequently, this inequality has been comprehensively examined for different ambient spaces by some authors who are achieved some results (see [3, 7, 16, 19, 22, 25]).

A C^∞ -submersion ϕ can be defined according to the following conditions: a (pseudo)-Riemannian submersion [1, 8, 12, 17, 20, 21], an almost Hermitian submersion [23], a quaternionic submersion [13], a slant submersion [11], a Clairaut Submersion [10], an anti-invariant submersion [6], conformal anti-invariant submersion [2], a semi-invariant submersion [18], etc. As far as we know, Riemannian submersions were presented by B. O'Neill [17] and A. Gray [8] in 1960s, independently. Especially, by utilizing the notion of almost Hermitian submersions, B. Watson [23] presented some differential geometric features among fibers, base manifolds, and total manifolds. Subsequently, many results occur on this topic.

The main goal of the current paper is to study sharp type inequalities including the scalar and Ricci curvatures of anti-invariant Riemannian submersions in complex space forms. The structure of the paper is as follows: After recalling some basic definitions and formulas in the second part, we investigate several inequalities including the Ricci and the scalar curvatures on $\ker \phi_*$ and $(\ker \phi_*)^\perp$ distributions of

anti-invariant Riemannian submersions in complex space forms and then, we obtain Chen-Ricci inequalities on $\ker \phi_*$ and $(\ker \phi_*)^\perp$ of anti-invariant Riemannian submersions in complex space forms.

2. PRELIMINARIES

Let (B_1, g_1) be an almost Hermitian manifold. This implies [24] that B_1 admits a tensor field J of type $(1, 1)$ on B_1 such that $\forall Z_1, Z_2 \in \chi(B_1)$, we obtain

$$J^2 = -I, \quad g_1(JZ_1, Z_2) + g_1(Z_1, JZ_2) = 0. \quad (2.1)$$

An almost Hermitian manifold B_1 is called Kaehler manifold if

$$(\nabla_{Z_1}^1)Z_2 = 0, \quad \forall Z_1, Z_2 \in \chi(B_1),$$

here ∇^1 is the Levi-Civita connection on B_1 . If $\{Z_1, JZ_1\}$ spans a plane section, the sectional curvature $F_{B_1}(Z_1) = K_{B_1}(Z_1 \wedge JZ_1)$ of $\text{span}\{Z_1, JZ_1\}$ is called a sectional curvature. The Riemannian-Christoffel curvature tensor of a Kaehler manifold [24] $B_1(v)$ of constant holomorphic sectional curvature v satisfies

$$\begin{aligned} R_{B_1}(Z_1, Z_2, Z_3, Z_4) = \frac{v}{4} \{ & g_1(Z_1, Z_4)g_1(Z_2, Z_3) - g_1(Z_1, Z_3)g_1(Z_2, Z_4) \\ & + g_1(JZ_2, Z_3)g_1(JZ_1, Z_4) - g_1(JZ_1, Z_3)g_1(JZ_2, Z_4) \\ & + 2g_1(Z_1, JZ_2)g_1(JZ_3, Z_4) \} \end{aligned} \quad (2.2)$$

for all $Z_1, Z_2, Z_3, Z_4 \in \chi(B_1)$.

Let (B_1, g_1) and (B_2, g_2) be Riemannian manifolds. A Riemannian submersion is a smooth map $\phi: B_1 \rightarrow B_2$ which is onto and satisfies the following conditions:

- (i) $\phi_{*p}: T_p B_1 \rightarrow T_{\phi(p)} B_2$ is onto for all $p \in B_1$;
- (ii) the fibres $\phi_x^{-1}, x \in B_2$, are Riemannian submanifolds of B_1 ;
- (iii) ϕ_{*p} preserves the length of the horizontal vectors.

The vectors tangent to the fibres are called vertical and those normal to the fibres are called horizontal. The tangent bundle of B_1 splits as the Whitney sum of two distributions, the vertical one $\ker \phi_*$ and the orthogonal complementary distribution $(\ker \phi_*)^\perp$ called horizontal, and we denote by h and v the horizontal and vertical projections, respectively. A horizontal vector field Z_1 on B_1 is called as basic if Z_1 is ϕ -related to a vector field Z_{1*} on B_2 [17]. A Riemannian submersion $\phi: B_1 \rightarrow B_2$ specifies two $(1, 2)$ tensor fields $\mathcal{T}$ and $\mathcal{A}$ on B_1 , by the formulae [17]:

$$\mathcal{T}(Z_1, Z_2) = \mathcal{T}_{Z_1} Z_2 = h\nabla_{vZ_1}^1 vZ_2 + v\nabla_{vZ_1}^1 hZ_2$$

and

$$\mathcal{A}(Z_1, Z_2) = \mathcal{A}_{Z_1} Z_2 = v\nabla_{hZ_1}^1 hZ_2 + h\nabla_{hZ_1}^1 vZ_2$$

for all $Z_1, Z_2 \in \chi(B_1)$.

Lemma 1 (Lemma 4 in [17]). *Let $\varphi: (B_1, g_1) \rightarrow (B_2, g_2)$ be a Riemannian submersion. Then we have:*

$$\mathcal{A}_{Z_1}Z_2 = -\mathcal{A}_{Z_2}Z_1, \quad Z_1, Z_2 \in \chi((\ker \varphi_*)^\perp); \quad (2.3)$$

$$\mathcal{T}_{F_1}F_2 = \mathcal{T}_{F_2}F_1, \quad F_1, F_2 \in \chi(\ker \varphi_*); \quad (2.4)$$

$$g_1(\mathcal{T}_{F_1}Z_2, Z_3) = -g_1(\mathcal{T}_{F_1}Z_3, Z_2), \quad F_1 \in \chi(\ker \varphi_*), \quad Z_2, Z_3 \in \chi(B_1);$$

$$g_1(\mathcal{A}_{Z_1}Z_2, Z_3) = -g_1(\mathcal{A}_{Z_1}Z_3, Z_2), \quad Z_1 \in \chi((\ker \varphi_*)^\perp), \quad Z_2, Z_3 \in \chi(B_1).$$

Let $R^{B_1}, R^{B_2}, R^{\ker \varphi_*}$ and $R^{(\ker \varphi_*)^\perp}$ stand for the Riemannian curvature tensors of Riemannian manifolds B_1, B_2 , the vertical distribution $\ker \varphi_*$ and the horizontal distribution $(\ker \varphi_*)^\perp$, respectively.

Lemma 2 (Theorem 2 in [17]). *Let $\varphi: (B_1, g_1) \rightarrow (B_2, g_2)$ be a Riemannian submersion. Then we have:*

$$\begin{aligned} R^{B_1}(F_1, F_2, F_3, F_4) &= R^{\ker \varphi_*}(F_1, F_2, F_3, F_4) + g_1(\mathcal{T}_{F_1}F_4, \mathcal{T}_{F_2}F_3) \\ &\quad - g_1(\mathcal{T}_{F_2}F_4, \mathcal{T}_{F_1}F_3), \end{aligned} \quad (2.5)$$

$$\begin{aligned} R^{B_1}(Z_1, Z_2, Z_3, Z_4) &= R^{(\ker \varphi_*)^\perp}(Z_1, Z_2, Z_3, Z_4) - 2g_1(\mathcal{A}_{Z_1}Z_2, \mathcal{A}_{Z_3}Z_4) \\ &\quad + g_1(\mathcal{A}_{Z_2}Z_3, \mathcal{A}_{Z_1}Z_4) - g_1(\mathcal{A}_{Z_1}Z_3, \mathcal{A}_{Z_2}Z_4), \end{aligned} \quad (2.6)$$

$$\begin{aligned} R^{B_1}(Z_1, F_1, Z_2, F_2) &= g_1((\nabla_{Z_1}^1 \mathcal{T})(F_1, F_2), Z_2) + g_1((\nabla_{F_1}^1 \mathcal{A})(Z_1, Z_2), F_2) \\ &\quad - g_1(\mathcal{T}_{F_1}Z_1, \mathcal{T}_{F_2}Z_2) + g_1(\mathcal{A}_{Z_2}F_2, \mathcal{A}_{Z_1}F_1) \end{aligned} \quad (2.7)$$

for all $Z_1, Z_2, Z_3, Z_4 \in \chi((\ker \varphi_*)^\perp)$ and $F_1, F_2, F_3, F_4 \in \chi(\ker \varphi_*)$.

Further, the $\mathcal{H}$ mean curvature of every fibre of φ Riemannian submersion is defined

$$\mathcal{H} = \frac{1}{s} \mathcal{N}, \quad \mathcal{N} = \sum_{p=1}^s \mathcal{T}_{E_p}E_p, \quad (2.8)$$

where $\{E_1, E_2, \dots, E_s\}$ forms an orthonormal basis for the vertical distribution $\ker \varphi_*$. Also, φ has totally geodesic fibres if $\mathcal{T} = 0$ on $\ker \varphi_*$ and $(\ker \varphi_*)^\perp$.

Definition 1 (Definition 3.1 in [6]). Let (B_1, g_1, J) be a Kaehler manifold and (B_2, g_2) be a Riemannian manifold. $\varphi: (B_1, g_1, J) \rightarrow (B_2, g_2)$ is called anti-invariant, if $\ker \varphi_*$ is anti-invariant with respect to J , i.e. $J(\ker \varphi_*) \subseteq (\ker \varphi_*)^\perp$.

From above definition, we get $J(\ker \varphi_*) \cap (\ker \varphi_*)^\perp \neq \{0\}$. We denote the complementary orthogonal distribution to $J(\ker \varphi_*)$ in $(\ker \varphi_*)^\perp$ by η . Then we obtain

$$(\ker \varphi_*)^\perp = J(\ker \varphi_*) \oplus \eta.$$

It is straightforward to show that η is an invariant distribution of $(\ker \varphi_*)^\perp$ under the endomorphism J . So, for $Z_1 \in \chi(\ker \varphi_*)^\perp$, we can state

$$JZ_1 = \alpha Z_1 + \beta Z_1, \quad (2.9)$$

here $\alpha Z_1 \in \chi(\ker \varphi_*)$ and $\beta Z_1 \in \chi(\eta)$. Using (2.1) and (2.9), we have

$$\beta^2 Z_1 = -Z_1 - J\alpha Z_1. \quad (2.10)$$

Example 1. Let B_1 be a 4-dimensional Euclidean space given by $B_1 = \{(x, y, z, w) \in \mathcal{R}^4 : z \in \mathcal{R} - \{k\frac{\pi}{2}, k\pi\}, k \in \mathbb{Z} \text{ and } x \neq 0\}$. We define the Kaehler structure (J, g_1) on B_1 given by

$$g_1 = (dx)^2 + (dy)^2 + (dz)^2 + (dw)^2 \quad \text{and} \quad J(b_1, b_2, b_3, b_4) = (-b_4, b_3, -b_2, b_1).$$

Let B_2 be $\{(x, v) \in \mathcal{R}^2 : x \neq 0\}$. We choose the Riemannian metric g_2 on B_2 in the following form

$$g_2 = e^{-2x}((dx)^2 + (dv)^2).$$

Now we define the map $\varphi: (B_1, g_1, J) \rightarrow (B_2, g_2)$ by

$$\varphi(x, y, z, w) = (e^x \cos z, e^x \sin z).$$

Then the kernel of φ_* is

$$\ker \varphi_* = \text{Span}\{F_1 = -e^x \cos z \frac{\partial}{\partial y} - e^x \sin z \frac{\partial}{\partial w}, F_2 = e^x \sin z \frac{\partial}{\partial y} - e^x \cos z \frac{\partial}{\partial w}\},$$

and the horizontal distribution is spanned by

$$(\ker \varphi_*)^\perp = \text{Span}\{Z_1 = e^x \cos z \frac{\partial}{\partial x} - e^x \sin z \frac{\partial}{\partial z}, Z_2 = e^x \sin z \frac{\partial}{\partial x} + e^x \cos z \frac{\partial}{\partial z}\}.$$

Thus, φ is a Riemannian submersion. Moreover, $JF_1 = Z_2$ and $JF_2 = Z_1$ imply that $(\ker \varphi_*)^\perp = J(\ker \varphi_*)$. Hence φ is an anti-invariant Riemannian submersion.

3. BASIC INEQUALITIES

First we give the following result. Since φ is an anti-invariant Riemannian submersion, and using (2.2) and (2.5) we have:

Lemma 3. $(B_1(\mathfrak{v}), g_1)$ and (B_2, g_2) denote a complex space form and a Riemannian manifold and let $\varphi: (B_1(\mathfrak{v}), g_1) \rightarrow (B_2, g_2)$ be an anti-invariant Riemannian submersion. Then any for $F_1, F_2, F_3, F_4 \in \chi(\ker \varphi_*)$ we obtain

$$\begin{aligned} R^{\ker \varphi_*}(F_1, F_2, F_3, F_4) &= \frac{\mathbf{v}}{4} \{g_1(F_1, F_4)g_1(F_2, F_3) - g_1(F_1, F_3)g_1(F_2, F_4)\} \\ &\quad - g_1(\mathcal{T}_{F_1} F_4, \mathcal{T}_{F_2} F_3) + g_1(\mathcal{T}_{F_2} F_4, \mathcal{T}_{F_1} F_3), \\ K^{\ker \varphi_*}(F_1, F_2) &= \frac{\mathbf{v}}{4} \{g_1^2(F_1, F_2) - \|F_1\|^2 \|F_2\|^2\} - \|\mathcal{T}_{F_1} F_2\|^2 \\ &\quad + g_1(\mathcal{T}_{F_2} F_2, \mathcal{T}_{F_1} F_1), \end{aligned} \quad (3.1)$$

here $K^{\ker \varphi_*}$ is a bi-sectional curvature of $\ker \varphi_*$.

Let $\varphi: B_1(\mathbf{v}) \rightarrow B_2$ be an anti-invariant Riemannian submersion. For every node $k \in B_1$, let $\{E_1, \dots, E_s, e_1, \dots, e_m\}$ be an orthonormal basis of $T_k B_1(\mathbf{v})$ such that $\ker \varphi_* = \text{span}\{E_1, \dots, E_s\}$, $(\ker \varphi_*)^\perp = \text{span}\{e_1, \dots, e_m\}$.

Now, if we take $F_4 = F_1$ and $F_2 = F_3 = E_i$, $i = 1, 2, \dots, s$ in (3.1), and using (2.8) then we arrive at

$$\text{Ric}^{\ker \varphi_*}(F_1) = \frac{\mathbf{v}}{4}(s-1)g_1(F_1, F_1) - sg_1(\mathcal{T}_{F_1}F_1, \mathcal{H}) + \sum_{i=1}^s g_1(\mathcal{T}_{F_1}F_1, \mathcal{T}_{F_1}E_i). \quad (3.2)$$

From here, we get:

Theorem 1. *Let $\varphi: (B_1(\mathbf{v}), g_1) \rightarrow (B_2, g_2)$ be an anti-invariant Riemannian submersion. Then we have*

$$\text{Ric}^{\ker \varphi_*}(F_1) \geq \frac{\mathbf{v}}{4}(s-1)g_1(F_1, F_1) - sg_1(\mathcal{T}_{F_1}F_1, \mathcal{H}).$$

For a unit vertical vector $F_1 \in \chi(\ker \varphi_)$, the equality status of the inequality holds if and only if every fibre is totally geodesic.*

Taking $F_1 = E_j$, $j = 1, \dots, s$ in (3.2) and using (2.4), then we obtain

$$2\rho^{\ker \varphi_*} = \frac{\mathbf{v}}{4}s(s-1) - s^2\|\mathcal{H}\|^2 + \sum_{i,j=1}^s g_1(\mathcal{T}_{E_i}E_j, \mathcal{T}_{E_i}E_j).$$

Therefore, we can state the following result.

Theorem 2. *Let $\varphi: (B_1(\mathbf{v}), g_1) \rightarrow (B_2, g_2)$ be an anti-invariant Riemannian submersion. Then we have*

$$2\rho^{\ker \varphi_*} \geq \frac{\mathbf{v}}{4}s(s-1) - s^2\|\mathcal{H}\|^2.$$

The equality status of the inequality satisfies if and only if every fibre is totally geodesic.

Since φ is an anti-invariant submersion, and using (2.2), (2.6), (2.9) we obtain:

Lemma 4. *Let $\varphi: (B_1(\mathbf{v}), g_1) \rightarrow (B_2, g_2)$ be an anti-invariant Riemannian submersion. Then for $Z_1, Z_2, Z_3, Z_4 \in \chi((\ker \varphi_*)^\perp)$ we have*

$$\begin{aligned} R^{(\ker \varphi_*)^\perp}(Z_1, Z_2, Z_3, Z_4) &= \frac{\mathbf{v}}{4}\{g_1(Z_1, Z_4)g_1(Z_2, Z_3) - g_1(Z_1, Z_3)g_1(Z_2, Z_4) \\ &\quad + g_1(\beta Z_2, Z_3)g_1(\beta Z_1, Z_4) - g_1(\beta Z_1, Z_3)g_1(\beta Z_2, Z_4) \\ &\quad + 2g_1(Z_1, \beta Z_2)g_1(\beta Z_3, Z_4)\} + 2g_1(\mathcal{A}_{Z_1}Z_2, \mathcal{A}_{Z_3}Z_4) \\ &\quad - g_1(\mathcal{A}_{Z_2}Z_3, \mathcal{A}_{Z_1}Z_4) + g_1(\mathcal{A}_{Z_1}Z_3, \mathcal{A}_{Z_2}Z_4), \\ B^{(\ker \varphi_*)^\perp}(Z_1, Z_2) &= \frac{\mathbf{v}}{4}\{g_1^2(Z_1, Z_2) - \|Z_1\|^2\|Z_2\|^2 \\ &\quad - 3g_1^2(\beta Z_1, Z_2)\} + 3\|\mathcal{A}_{Z_1}Z_2\|^2, \end{aligned} \quad (3.3)$$

here $B^{(\ker \varphi_*)^\perp}$ is a bi-sectional curvature of $(\ker \varphi_*)^\perp$.

Now, if we take $Z_4 = Z_1$ and $Z_2 = Z_3 = e_j$, $j = 1, 2, \dots, m$ in (3.3), and using (2.3), (2.10) then we get

$$\begin{aligned} Ric^{(\ker \varphi_*)^\perp}(Z_1) &= \frac{v}{4} \{ (m+2)g_1(Z_1, Z_1) + 3g_1(J\alpha Z_1, Z_1) \} \\ &\quad - 3 \sum_{j=1}^m g_1(\mathcal{A}_{Z_1} e_j, \mathcal{A}_{Z_1} e_j). \end{aligned} \quad (3.4)$$

Taking $Z_1 = e_i$, $i = 1, 2, \dots, m$ in (3.4), then we have:

$$2\rho^{(\ker \varphi_*)^\perp} = \frac{v}{4} \{ m(m+2) + 3tr(J\alpha) \} - 3 \sum_{i,j=1}^m g_1(\mathcal{A}_{e_i} e_j, \mathcal{A}_{e_i} e_j). \quad (3.5)$$

Then we write

$$2\rho^{(\ker \varphi_*)^\perp} \leq \frac{v}{4} \{ m(m+2) + 3tr(J\alpha) \}. \quad (3.6)$$

Thus, we can give:

Theorem 3. *Let $\varphi: (B_1(v), g_1) \rightarrow (B_2, g_2)$ be an anti-invariant Riemannian submersion. Then*

$$2\rho^{(\ker \varphi_*)^\perp} \leq \frac{v}{4} \{ m(m+2) + 3tr(J\alpha) \}.$$

The equality status of (3.6) satisfies if and only if $(\ker \varphi_)^\perp$ is integrable.*

4. CHEN-RICCI INEQUALITIES

Let $(B_1(v), g_1)$ be a complex space form, (B_2, g_2) a Riemannian manifold and $\varphi: B_1(v) \rightarrow B_2$ be an anti-invariant Riemannian submersion. For every node $k \in B_1$, let $\{E_1, \dots, E_s, e_1, \dots, e_m\}$ be an orthonormal basis of $T_k B_1(v)$ such that $\ker \varphi_* = \text{span}\{E_1, \dots, E_s\}$ and $(\ker \varphi_*)^\perp = \text{span}\{e_1, \dots, e_m\}$. Let's denote $\mathcal{T}_{ij}^t$ by

$$\mathcal{T}_{ij}^t = g_1(\mathcal{T}_{E_i} E_j, e_t), \quad (4.1)$$

where $1 \leq i, j \leq s$ and $1 \leq t \leq m$. Similarly, let's denote $\mathcal{A}_{ij}^\alpha$ by

$$\mathcal{A}_{ij}^\alpha = g_1(\mathcal{A}_{e_i} e_j, E_\alpha), \quad (4.2)$$

in which $1 \leq i, j \leq m$ and $1 \leq \alpha \leq s$ and we employee

$$\delta(\mathcal{N}) = \sum_{i=1}^m \sum_{k=1}^s ((\nabla_{e_i}^1 \mathcal{T})_{E_k} E_k, e_i). \quad (4.3)$$

Now, from (3.1), we get

$$2\rho^{\ker \varphi_*} = \frac{v}{4} s(s-1) - s^2 \|\mathcal{H}\|^2 + \sum_{i,j=1}^s g_1(\mathcal{T}_{E_i} E_j, \mathcal{T}_{E_i} E_j).$$

Using (2.4) and (4.1), we arrive at

$$2\rho^{\ker \varphi_*} = \frac{v}{4} s(s-1) - s^2 \|\mathcal{H}\|^2 + \sum_{t=1}^m \sum_{i,j=1}^s (\mathcal{T}_{ij}^t)^2. \quad (4.4)$$

From [9] we know that

$$\begin{aligned} \sum_{t=1}^m \sum_{i,j=1}^s (\mathcal{T}_{ij}^t)^2 &= \frac{1}{2} s^2 \|\mathcal{H}\|^2 + \frac{1}{2} \sum_{t=1}^m [\mathcal{T}_{11}^t - \mathcal{T}_{22}^t - \dots - \mathcal{T}_{ss}^t]^2 \\ &\quad + 2 \sum_{t=1}^m \sum_{j=2}^s (\mathcal{T}_{1j}^t)^2 - 2 \sum_{t=1}^m \sum_{2 \leq i < j \leq s} [\mathcal{T}_{ii}^t \mathcal{T}_{jj}^t - (\mathcal{T}_{ij}^t)^2]. \end{aligned} \quad (4.5)$$

If we put (4.5) in (4.4), we obtain

$$\begin{aligned} 2\rho^{\ker\varphi_*} &= \frac{\mathbf{v}}{4} s(s-1) - \frac{1}{2} s^2 \|\mathcal{H}\|^2 + \frac{1}{2} \sum_{t=1}^m [\mathcal{T}_{11}^t - \mathcal{T}_{22}^t - \dots - \mathcal{T}_{ss}^t]^2 \\ &\quad + 2 \sum_{t=1}^m \sum_{j=2}^s (\mathcal{T}_{1j}^t)^2 - 2 \sum_{t=1}^m \sum_{2 \leq i < j \leq s} [\mathcal{T}_{ii}^t \mathcal{T}_{jj}^t - (\mathcal{T}_{ij}^t)^2]. \end{aligned}$$

From here, we have

$$2\rho^{\ker\varphi_*} \geq \frac{\mathbf{v}}{4} s(s-1) - \frac{1}{2} s^2 \|\mathcal{H}\|^2 - 2 \sum_{t=1}^m \sum_{2 \leq i < j \leq s} [\mathcal{T}_{ii}^t \mathcal{T}_{jj}^t - (\mathcal{T}_{ij}^t)^2]. \quad (4.6)$$

On the other hand, from (2.5), taking $F_1 = F_4 = E_i, F_2 = F_3 = E_j$ and using (4.1), we have

$$\begin{aligned} 2 \sum_{2 \leq i < j \leq s} R^{B_1}(E_i, E_j, E_j, E_i) &= 2 \sum_{2 \leq i < j \leq s} R^{\ker\varphi_*}(E_i, E_j, E_j, E_i) \\ &\quad + 2 \sum_{t=1}^m \sum_{2 \leq i < j \leq s} [\mathcal{T}_{ii}^t \mathcal{T}_{jj}^t - (\mathcal{T}_{ij}^t)^2]. \end{aligned}$$

From the last equality, (4.6) can be written as

$$\begin{aligned} 2\rho^{\ker\varphi_*} &\geq \frac{\mathbf{v}}{4} s(s-1) - \frac{1}{2} s^2 \|\mathcal{H}\|^2 + 2 \sum_{2 \leq i < j \leq s} R^{\ker\varphi_*}(E_i, E_j, E_j, E_i) \\ &\quad - 2 \sum_{2 \leq i < j \leq s} R^{B_1}(E_i, E_j, E_j, E_i). \end{aligned} \quad (4.7)$$

Furthermore, we know that

$$2\rho^{\ker\varphi_*} = 2 \sum_{2 \leq i < j \leq s} R^{\ker\varphi_*}(E_i, E_j, E_j, E_i) + 2 \sum_{j=1}^s R^{\ker\varphi_*}(E_1, E_j, E_j, E_1).$$

If we put the last equality in (4.7), then we have

$$2Ric^{\ker\varphi_*}(E_1) \geq \frac{\mathbf{v}}{4} s(s-1) - \frac{1}{2} s^2 \|\mathcal{H}\|^2 - 2 \sum_{2 \leq i < j \leq s} R^{B_1}(E_i, E_j, E_j, E_i).$$

Since B_1 is a complex space form, curvature tensor R^{B_1} of B_1 provides equation (2.2), therefore we acquire

$$Ric^{\ker\varphi_*}(E_1) \geq \frac{\mathbf{v}}{4} (s-1) - \frac{1}{4} s^2 \|\mathcal{H}\|^2.$$

Thus, we can give the following result:

Theorem 4. *Let $\varphi: B_1(\mathbf{v}) \rightarrow B_2$ be an anti-invariant Riemannian submersion from a complex space form $(B_1(\mathbf{v}), g_1)$ onto a Riemannian manifold (B_2, g_2) . Then we have*

$$\text{Ric}^{\ker \varphi_*}(E_1) \geq \frac{\mathbf{v}}{4}(s-1) - \frac{1}{4}s^2 \|\mathcal{H}\|^2.$$

The equality status of the inequality satisfies if and only

$$\begin{aligned} \mathcal{T}_{11}^t &= \mathcal{T}_{22}^t + \cdots + \mathcal{T}_{ss}^t \\ \mathcal{T}_{1j}^t &= 0, j = 2, \dots, s. \end{aligned}$$

From (3.5), we have

$$2\rho^{(\ker \varphi_*)^\perp} = \frac{\mathbf{v}}{4}\{m(m+2) + 3\text{tr}(J\alpha)\} - 3 \sum_{i,j=1}^m g_1(\mathcal{A}_{e_i}e_j, \mathcal{A}_{e_i}e_j).$$

Using (2.10) and (4.2), then we have

$$2\rho^{(\ker \varphi_*)^\perp} = \frac{\mathbf{v}}{4}\{m(m+2) + 3\text{tr}(J\alpha)\} - 3 \sum_{\alpha=1}^s \sum_{i,j=1}^m (\mathcal{A}_{ij}^\alpha)^2. \quad (4.8)$$

From (2.3) then (4.8) turns into

$$2\rho^{(\ker \varphi_*)^\perp} = \frac{\mathbf{v}}{4}\{m(m+2) + 3\text{tr}(J\alpha)\} - 6 \sum_{\alpha=1}^s \sum_{j=2}^m (\mathcal{A}_{1j}^\alpha)^2 - 6 \sum_{\alpha=1}^s \sum_{2 \leq i < j \leq m} (\mathcal{A}_{ij}^\alpha)^2. \quad (4.9)$$

Moreover, from (2.6), taking $Z_1 = Z_4 = e_i, Z_2 = Z_3 = e_j$ and using (4.2) we obtain

$$\begin{aligned} 2 \sum_{2 \leq i < j \leq m} R^{B_1}(e_i, e_j, e_j, e_i) &= 2 \sum_{2 \leq i < j \leq m} R^{(\ker \varphi_*)^\perp}(e_i, e_j, e_j, e_i) \\ &\quad + 6 \sum_{\alpha=1}^s \sum_{2 \leq i < j \leq m} (\mathcal{A}_{ij}^\alpha)^2. \end{aligned} \quad (4.10)$$

If we consider (4.10) in (4.9), then we have

$$\begin{aligned} 2\rho^{(\ker \varphi_*)^\perp} &= \frac{\mathbf{v}}{4}\{m(m+2) + 3\text{tr}(J\alpha)\} - 6 \sum_{\alpha=1}^s \sum_{j=2}^m (\mathcal{A}_{1j}^\alpha)^2 \\ &\quad - 2 \sum_{2 \leq i < j \leq m} R^{B_1}(e_i, e_j, e_j, e_i) + 2 \sum_{2 \leq i < j \leq m} R^{(\ker \varphi_*)^\perp}(e_i, e_j, e_j, e_i). \end{aligned}$$

Since B_1 is a complex space form, curvature tensor R^{B_1} of B_1 satisfies (2.2), hence we get

$$2\text{Ric}^{(\ker \varphi_*)^\perp}(e_1) = \frac{\mathbf{v}}{4}(2m-2 + 6\|\beta e_1\|^2) - 6 \sum_{\alpha=1}^s \sum_{j=2}^m (\mathcal{A}_{1j}^\alpha)^2.$$

Then we can write

$$\text{Ric}^{(\ker \varphi_*)^\perp}(e_1) \leq \frac{\mathbf{v}}{4}(m-1 + 3\|\beta e_1\|^2).$$

Thus, we can give the following result:

Theorem 5. *Let $\varphi: B_1(\mathbf{v}) \rightarrow B_2$ be an anti-invariant Riemannian submersion from a complex space form $(B_1(\mathbf{v}), g_1)$ onto a Riemannian manifold (B_2, g_2) . Then we have*

$$\text{Ric}^{(\ker \varphi_*)^\perp}(e_1) \leq \frac{\mathbf{v}}{4}(m-1+3\|\beta e_1\|^2),$$

the equality status of the inequality satisfies if and only

$$\mathcal{A}_{1j} = 0, j = 2, \dots, m.$$

Next, we can state the inequality of Chen Ricci among the $\ker \varphi_*$ and $(\ker \varphi_*)^\perp$. The ρ scalar curvature of $B_1(\mathbf{v})$ is defined as

$$\begin{aligned} 2\rho &= \sum_{t=1}^m \text{Ric}(e_t, e_t) + \sum_{k=1}^s \text{Ric}(E_k, e_k), \\ 2\rho &= \sum_{j,k=1}^s R^{B_1}(E_j, E_k, E_k, E_j) + \sum_{i=1}^m \sum_{k=1}^s R^{B_1}(e_i, E_k, E_k, e_i) \\ &\quad + \sum_{i,t=1}^m R^{B_1}(e_i, e_t, e_t, e_i) + \sum_{t=1}^m \sum_{j=1}^s R^{B_1}(E_j, e_t, e_t, E_j). \end{aligned} \quad (4.11)$$

Since $B_1(\mathbf{v})$ is a complex space form, using (4.11) and (2.2), we have

$$2\rho = \frac{\mathbf{v}}{4}\{s(s-1) + m(m+2) + 2sm + 3tr(J\alpha)\}. \quad (4.12)$$

On the other hand, using the equations (2.5), (2.6) and (2.7), we obtain also the ρ scalar curvature of $B_1(\mathbf{v})$ as

$$\begin{aligned} 2\rho &= 2\rho^{\ker \varphi_*} + 2\rho^{(\ker \varphi_*)^\perp} + s^2\|\mathcal{H}\|^2 \\ &\quad + \sum_{j,k=1}^s g_1(\mathcal{T}_{E_k}E_j, \mathcal{T}_{E_k}E_j) + 3 \sum_{i,t=1}^m g_1(\mathcal{A}_{e_i}e_t, \mathcal{A}_{e_i}e_t) \\ &\quad - \sum_{i=1}^m \sum_{k=1}^s g_1((\nabla_{e_i}^1 \mathcal{T})_{E_k}E_k, e_i) + \sum_{i=1}^m \sum_{k=1}^s \{g_1(\mathcal{T}_{E_k}e_i, \mathcal{T}_{E_k}e_i) - g_1(\mathcal{A}_{e_i}E_k, \mathcal{A}_{e_i}E_k)\} \\ &\quad - \sum_{t=1}^m \sum_{j=1}^s g_1((\nabla_{e_t}^1 \mathcal{T})_{E_j}E_j, e_t) + \sum_{t=1}^m \sum_{j=1}^s \{g_1(\mathcal{T}_{E_j}e_t, \mathcal{T}_{E_j}e_t) - g_1(\mathcal{A}_{e_t}E_j, \mathcal{A}_{e_t}E_j)\}. \end{aligned}$$

Using (4.3) and (4.5), we obtain

$$\begin{aligned} 2\rho &= 2\rho^{\ker \varphi_*} + 2\rho^{(\ker \varphi_*)^\perp} + \frac{1}{2}s^2\|\mathcal{H}\|^2 - \frac{1}{2} \sum_{t=1}^m [\mathcal{T}_{11}^t - \mathcal{T}_{22}^t - \dots - \mathcal{T}_{ss}^t]^2 \\ &\quad - 2 \sum_{t=1}^m \sum_{j=2}^s (\mathcal{T}_{1j}^t)^2 + 2 \sum_{t=1}^m \sum_{2 \leq j < k \leq s} [\mathcal{T}_{jj}^t \mathcal{T}_{kk}^t - (\mathcal{T}_{jk}^t)^2] + 6 \sum_{\alpha=1}^s \sum_{t=2}^m (\mathcal{A}_{1t}^\alpha)^2 \end{aligned} \quad (4.13)$$

$$\begin{aligned}
& + 6 \sum_{\alpha=1}^s \sum_{2 \leq i < t \leq m} (\mathcal{A}_{it}^\alpha)^2 + \sum_{i=1}^m \sum_{k=1}^s \{g_1(\mathcal{T}_{E_k} e_i, \mathcal{T}_{E_k} e_i) - g_1(\mathcal{A}_{e_i} E_k, \mathcal{A}_{e_i} E_k)\} \\
& - 2\delta(\mathcal{N}) + \sum_{t=1}^m \sum_{j=1}^s \{g_1(\mathcal{T}_{E_j} e_t, \mathcal{T}_{E_j} e_t) - g_1(\mathcal{A}_{e_t} E_j, \mathcal{A}_{e_t} E_j)\}.
\end{aligned}$$

Using (4.7), (4.10) and (4.12) in the (4.13) then we have

$$\begin{aligned}
\frac{\mathbf{v}}{4} \{sm + m + s - 1 + 3\|\beta e_1\|^2\} & = Ric^{\ker \varphi_*}(E_1) + Ric^{(\ker \varphi_*)^\perp}(e_1) + \frac{1}{4}s^2\|\mathcal{H}\|^2 \\
& - \frac{1}{4} \sum_{t=1}^m [\mathcal{T}_{11}^t - \mathcal{T}_{22}^t - \cdots - \mathcal{T}_{ss}^t]^2 - \sum_{t=1}^m \sum_{j=2}^s (\mathcal{T}_{1j}^t)^2 \\
& + 3 \sum_{\alpha=1}^s \sum_{t=2}^m (\mathcal{A}_{1t}^\alpha)^2 - 2\delta(\mathcal{N}) + \|\mathcal{T}^V\|^2 - \|\mathcal{A}^H\|^2,
\end{aligned}$$

where $\|\mathcal{T}^V\|^2 = \sum_{i=1}^m \sum_{k=1}^s g_1(\mathcal{T}_{E_k} e_i, \mathcal{T}_{E_k} e_i)$, $\|\mathcal{A}^H\|^2 = \sum_{i=1}^m \sum_{k=1}^s g_1(\mathcal{A}_{e_i} E_k, \mathcal{A}_{e_i} E_k)$. Since $B_1(\mathbf{v})$ is a complex space form, from (2.2), we have following result readily:

Theorem 6. Let $\varphi: B_1(\mathbf{v}) \rightarrow B_2$ be an anti-invariant Riemannian submersion from a complex space form $(B_1(\mathbf{v}), g_1)$ onto a Riemannian manifold (B_2, g_2) . Then we have

$$\begin{aligned}
\frac{\mathbf{v}}{4} \{sm + m + s - 1 + 3\|\beta e_1\|^2\} & \leq Ric^{\ker \varphi_*}(E_1) + Ric^{(\ker \varphi_*)^\perp}(e_1) + \frac{1}{4}s^2\|\mathcal{H}\|^2 \\
& + 3 \sum_{\alpha=1}^s \sum_{t=2}^m (\mathcal{A}_{1t}^\alpha)^2 - \delta(\mathcal{N}) + \|\mathcal{T}^V\|^2 - \|\mathcal{A}^H\|^2
\end{aligned}$$

the equality status of the inequality satisfies if and only

$$\mathcal{T}_{11}^t = \mathcal{T}_{22}^t + \cdots + \mathcal{T}_{ss}^t \quad \mathcal{T}_{1j}^t = 0, \quad j = 2, \dots, s.$$

Remark 1. Recently, Chen-Ricci inequalities were stated for Riemannian maps from complex space forms in [14]. Recall that Riemannian maps generalize the well-known concepts of isometric immersions and Riemannian submersions (see, e.g., the recent work of Lee et. al., [15]). Therefore, a natural problem is to extend the results of this work in the general setting of anti-invariant Riemannian maps.

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ENERGY CONCENTRATION OF THE P-LANDAU-LIFSCHITZ FUNCTIONAL WITH RADIAL STRUCTURE

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Abstract. This paper is concerned with the asymptotic behavior of a p-Landau-Lifschitz type functional with radial structure as parameter goes to zero. We study the concentration compactness and give several global properties in the case of $p > 2$.

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1. INTRODUCTION

Let $B = \{x \in \mathbb{R}^2; x_1^2 + x_2^2 < 1\}$. Denote $\mathbb{S}^1 = \{x = (x_1 + ix_2, x_3) \in \mathbb{C} \times \mathbb{R}; x_1^2 + x_2^2 = 1, x_3 = 0\}$ and $\mathbb{S}^2 = \{x \in \mathbb{C} \times \mathbb{R}; x_1^2 + x_2^2 + x_3^2 = 1\}$. Let $g(x) = (e^{id\theta}, 0)$ where $x = (\cos \theta, \sin \theta)$ on ∂B , $d \in \mathbb{N}$. We are concerned with the minimizer of the energy functional of p-Landau-Lifschitz type

$$E_\varepsilon(u, B) = \frac{1}{p} \int_B |\nabla u|^p dx + \frac{1}{2\varepsilon^p} \int_B u^2 dx \quad (p > 2)$$

in the function class

$$W = \{u(x) = (\sin f(r)e^{id\theta}, \cos f(r)) \in W^{1,p}(B, \mathbb{S}^2); u|_{\partial B} = g\},$$

which is named the radial minimizer of $E_\varepsilon(u, B)$.

When $p = 2$, the functional $E_\varepsilon(u, B)$ was introduced in the study of some simplified model of high-energy physics, which controls the statics of planar ferromagnets and antiferromagnets (cf. [8] and [15]). In addition, it is helpful to understand the dynamics of singularities appearing in the liquid crystals (cf. [2, 7, 12, 14] and [6]). In particular, the authors of [7] discussed the asymptotic behaviour of the radial minimizer of $E_\varepsilon(u, B)$ in §5. When the penalization term $\frac{1}{2\varepsilon^2} \int_B u^2 dx$ is replaced by $\frac{1}{4\varepsilon^2} \int_B (1 - |u|^2)^2 dx$ and $\mathbb{S}^2$ is replaced by $\mathbb{C}$, the functional becomes the Ginzburg-Landau energy introduced in the theory of superconductors (cf. [3] and the references therein). Nineteen problems were proposed in [3]. Comte and Mironescu studied Problem 7 in [4, 5, 13]. Problem 7 and Theorems VII.2 and VII.3 in [3] describe the

global properties of the Ginzburg-Landau functional. For the Landau-Lifschitz functional, Theorem 4.2 in [7] shows analogous results of Theorems VII.2 and VII.3 in [3].

When $p > 2$, Lei studied the behaviour of minimizers of $E_\varepsilon(u, B)$ as $\varepsilon \rightarrow 0$ (cf. [10]). In addition, he also proved the $W_{loc}^{1,p}$ convergence of the radial minimizers, and obtained some estimates of the convergent rate of the radial minimizer (cf. [9]). For the p-Ginzburg-Landau functional, the behaviour of radial minimizers was studied in [1] and [11]. In particular, the analogous global properties are shown in [11].

In polar coordinates, for $u(x) = (\sin f(r)e^{id\theta}, \cos f(r))$, we have

$$|\nabla u| = (f_r^2 + d^2 r^{-2} \sin^2 f)^{1/2}.$$

Sometimes we denote $\sin f(r)e^{id\theta}$ by u' . If we denote

$$V = \{f \in W_{loc}^{1,p}(0, 1]; r^{1/p} f_r, r^{(1-p)/p} \sin f \in L^p(0, 1), f(r) \geq 0, f(1) = \frac{\pi}{2}\},$$

then $V = \{f(r); u(x) = (\sin f(r)e^{id\theta}, \cos f(r)) \in W\}$.

Substituting $u(x) = (\sin f(r)e^{id\theta}, \cos f(r)) \in W$ into $E_\varepsilon(u, B)$ we obtain

$$E_\varepsilon(u, B) = 2\pi E_\varepsilon(f, (0, 1)),$$

where

$$E_\varepsilon(f, (0, 1)) = \int_0^1 \left[\frac{1}{p} (f_r^2 + d^2 r^{-2} \sin^2 f)^{p/2} + \frac{1}{2\varepsilon^p} \cos^2 f \right] r dr.$$

This shows that $u = (\sin f(r)e^{id\theta}, \cos f(r)) \in W$ is the minimizer of $E_\varepsilon(u, B)$ if and only if $f(r) \in V$ is the minimizer of $E_\varepsilon(f, (0, 1))$. Applying the direct method in the calculus of variations we can see that the functional $E_\varepsilon(u, B)$ achieves its minimum on W by a function $u_\varepsilon(x) = (\sin f_\varepsilon(r)e^{id\theta}, \cos f_\varepsilon(r))$, hence $f_\varepsilon(r)$ is the minimizer of $E_\varepsilon(f, (0, 1))$.

Recall some results in [9]. Let $u_\varepsilon = (\sin f_\varepsilon(r)e^{id\theta}, \cos f_\varepsilon(r))$ be a radial minimizer of $E_\varepsilon(u, B)$ on W . Then Theorem 1.1 in [9] shows that for any $\gamma \in (0, 1)$, there exists a constant $h = h(\gamma)$ which is independent of $\varepsilon \in (0, 1)$ such that

$$Z_\varepsilon = \{x \in B; |u_{\varepsilon 3}| > \gamma\} \subset B(0, h\varepsilon). \quad (1.1)$$

This implies that all the points where $u_{\varepsilon 3}^2 = 1$ are contained in $B(0, h\varepsilon)$. Hence as $\varepsilon \rightarrow 0$, these points converge to 0. Furthermore, Proposition 3.2 and Theorem 1.3 in [9] show that for any compact subset $K \subset \bar{B} \setminus \{0\}$, there exists a positive constant C (independent of ε), such that

$$E_\varepsilon(u_\varepsilon, K) \leq C \quad (1.2)$$

$$\sup_{x \in K} |u_{\varepsilon 3}(x)| \leq C\varepsilon^{\frac{p-2}{2}}. \quad (1.3)$$

Here $K = \bar{B} \setminus B(0, \eta)$. In addition, Proposition 2.1 in [9] shows

$$E_\varepsilon(u_\varepsilon, B) \leq C\varepsilon^{2-p}. \quad (1.4)$$

In this paper, we will study the global properties of the p-Landau-Lifschitz model, which are described by the concentration properties.

Theorem 1. *Let $u_\varepsilon = (\sin f_\varepsilon(r)e^{id\theta}, \cos f_\varepsilon(r))$ be a radial minimizer of $E_\varepsilon(u, B)$ on W . Then as $\varepsilon \rightarrow 0$, there exists a subsequence ε_k such that*

$$\frac{1}{2\varepsilon_k^2} |u_{\varepsilon_k 3}|^2 \rightarrow L_1 \delta_o, \quad \text{weakly star in } C(\bar{B}), \quad (1.5)$$

$$\varepsilon_k^{p-2} |\nabla u_{\varepsilon_k}|^p \rightarrow \frac{2p}{p-2} L_1 \delta_o, \quad \text{weakly star in } C(\bar{B}). \quad (1.6)$$

Here δ_o is the Dirac mass at the origin, and the positive constant L_1 satisfies

$$\frac{\pi d^p}{p} \sup_{\gamma \in (0,1)} (1 - \gamma^2)^{p/2} h^{2-p}(\gamma) \leq L_1 \leq (1 - \frac{2}{p}) \min_W E_1(u, B) + \frac{2\pi d^p}{p^2}, \quad (1.7)$$

where $h(\gamma)$ is a positive constant in (1.1).

Theorem 2. *Let $u_\varepsilon = (\sin f_\varepsilon(r)e^{id\theta}, \cos f_\varepsilon(r))$ be a radial minimizer of $E_\varepsilon(u, B)$ on W . Then for any $\alpha \geq 2 - 4/p$, we can find a subsequence ε_k of ε , and constants $L_3 > 0$ and $L_4 \geq 0$ which are independent of ε , such that as $k \rightarrow \infty$,*

$$|u_{\varepsilon_k 3}|^\alpha |\nabla u_{\varepsilon_k}|^2 \rightarrow L_3 \delta_o, \quad \text{weakly star in } C(\bar{B}), \quad (1.8)$$

$$\varepsilon_k^{p-2} |\det(\nabla u'_{\varepsilon_k})|^{p/2} \rightarrow L_4 \delta_o, \quad \text{weakly star in } C(\bar{B}). \quad (1.9)$$

The related results in higher dimension to Theorems 2 and 1 can be found in [16] and [17].

2. PROOF OF THEOREM 1

2.1. Proofs of (1.5) and (1.6)

In view of (1.4), there exist two Radon measures ω_1 and ω_2 , such that as $\varepsilon \rightarrow 0$,

$$\varepsilon_k^{p-2} |\nabla u_{\varepsilon_k}|^p \rightarrow \omega_1, \quad \text{weakly star in } C(\bar{B}), \quad (2.1)$$

$$\frac{1}{2\varepsilon_k^2} u_{\varepsilon_k 3}^2 \rightarrow \omega_2, \quad \text{weakly star in } C(\bar{B}), \quad (2.2)$$

for some subsequence ε_k of ε . Sometimes we also denote u_{ε_k} by u_ε for convenience. Furthermore, (1.2) implies that as $\varepsilon \rightarrow 0$,

$$\begin{aligned} \varepsilon^{p-2} \int_K |\nabla u_\varepsilon|^p dx &\rightarrow 0, \\ \frac{1}{2\varepsilon^2} \int_K u_{\varepsilon 3}^2 dx &\rightarrow 0, \end{aligned}$$

where K is an arbitrary compact subset of $B \setminus \{0\}$. These results lead to $\text{supp}(\omega_i) \subset \{0\}$ for $i = 1, 2$. Then we can find constants L_1 and L_2 such that

$$\omega_1 = L_2 \delta_o, \quad \omega_2 = L_1 \delta_o. \quad (2.3)$$

Next, we shall point out the relation between L_1 and L_2 . It is not difficult to see that the radial minimizer u_ε solves the system

$$-\text{div}[|\nabla u|^{p-2} \nabla u] = u|\nabla u|^p + \frac{1}{\varepsilon^p} (uu_3^2 - u_3 e_3) \quad \text{in } B. \quad (2.4)$$

Multiplying (2.4) by $x \cdot \nabla u$ and integrating by parts, we can obtain the Pohozaev type identity

$$\begin{aligned} & - \int_{\partial B_R(0)} |x| |\nabla u|^{p-2} |\partial_\nu u|^2 ds + \int_{B_R(0)} |\nabla u|^p dx - \frac{2}{p} \int_{B_R(0)} |\nabla u|^p dx \\ & + \frac{1}{p} \int_{\partial B_R(0)} |x| |\nabla u|^p ds = - \frac{1}{2\varepsilon^p} \int_{\partial B_R(0)} |x| u_3^2 ds + \frac{1}{\varepsilon^p} \int_{B_R(0)} u_3^2 dx \end{aligned} \quad (2.5)$$

for any $R \in (0, 1]$. Hereafter, we denote f_ε by f . By (1.2) and the mean value theorem, there exists $\sigma \in (1/4, 1/2)$ such that

$$r[(f_r)^2 + d^2 r^{-2} (n-1) \sin^2 f]^{p/2} \big|_{r=\sigma} + \frac{r}{\varepsilon^p} \cos^2 f \big|_{r=\sigma} \leq C. \quad (2.6)$$

Then, we take $R = \sigma$ in (2.5) and multiply it by ε^{p-2} to obtain

$$\begin{aligned} & - \varepsilon^{p-2} \sigma^2 [(f_r)^2 + \frac{d^2}{r^2} \sin^2 f]^{p/2} \big|_{r=\sigma} + (1 - \frac{2}{p}) \varepsilon^{p-2} \int_0^\sigma [(f_r)^2 + \frac{d^2}{r^2} \sin^2 f]^{p/2} r dr \\ & + \frac{\sigma^2}{p} \varepsilon^{p-2} [(f_r)^2 + \frac{d^2}{r^2} \sin^2 f]^{p/2} \big|_{r=\sigma} = - \frac{\sigma^2}{2\varepsilon^2} \cos^2 f \big|_{r=\sigma} + \frac{1}{\varepsilon^2} \int_0^\sigma \cos^2 f r dr. \end{aligned}$$

Using (2.6), we get

$$(1 - \frac{2}{p}) \varepsilon^{p-2} \int_{B_\sigma(0)} |\nabla u_\varepsilon|^p dx - \frac{1}{\varepsilon^2} \int_{B_\sigma(0)} u_{\varepsilon 3}^2 dx \rightarrow 0 \quad (2.7)$$

as $\varepsilon \rightarrow 0$. Combining this result with (2.1)-(2.3), we obtain

$$L_2 = \frac{2p}{p-2} L_1.$$

Thus, (1.5) and (1.6) are proved.

2.2. Proof of (1.7)

Step 1. Upper bound

Similar to the proof of Proposition 2.1 in [9], it is easy to derive

$$\varepsilon^{p-2} E_\varepsilon(u_\varepsilon, B) \leq \frac{2\pi d^p}{p(p-2)} + \min_W E_1(u, B) + C\varepsilon^{p-2}. \quad (2.8)$$

Here $C > 0$ is independent of ε . On the other hand, (1.5) and (1.6) lead to

$$\lim_{\varepsilon \rightarrow 0} \left[\frac{\varepsilon^{p-2}}{p} |\nabla u_\varepsilon|^p + \frac{1}{2\varepsilon^2} u_{\varepsilon 3}^2 \right] = \frac{p}{p-2} L_1 \delta_o, \quad \text{weakly star in } C(\bar{B}). \quad (2.9)$$

This result, together with (2.8), implies the upper bound of L_1 in (1.7).

Step 2. Lower bound

From (1.1), we can deduce that, for any $\sigma > 0$, there exists $C = C(\sigma) > 0$ independent of ε , such that

$$\begin{aligned} \int_{h\varepsilon}^\sigma [(f_r)^2 + d^2 r^{-2} \sin^2 f]^{p/2} r dr &\geq d^p \int_{h\varepsilon}^\sigma r^{1-p} \sin^p f dr \\ &\geq \frac{d^p}{p-2} (1 - \gamma^2)^{p/2} h^{2-p} (\gamma) \varepsilon^{2-p} - C(\sigma). \end{aligned} \quad (2.10)$$

Applying (2.7), we obtain that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{p-2} E_\varepsilon(u_\varepsilon, B_\sigma(0)) = \pi \lim_{\varepsilon \rightarrow 0} \varepsilon^{p-2} \int_0^\sigma [(f_r)^2 + d^2 r^{-2} \sin^2 f]^{p/2} r dr.$$

Inserting (2.10) into this result, we deduce that for any $\eta \in (0, 1)$,

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{p-2} E_\varepsilon(u_\varepsilon, B_\sigma(0)) \geq \frac{\pi d^p}{p-2} (1 - \gamma^2)^{p/2} h^{2-p} (\gamma).$$

Taking the supremum and writing

$$H := \sup_{\gamma \in (0,1)} (1 - \gamma^2)^{p/2} h^{2-p} (\gamma),$$

we have

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{p-2} E_\varepsilon(u_\varepsilon, B_\sigma(0)) \geq \frac{\pi d^p}{p-2} H.$$

Combining this with (2.9), we can get $\frac{p}{p-2} L_1 \geq \frac{\pi d^p}{p-2} H$. This means $L_1 \geq \frac{\pi d^p}{p} H$, thus we obtain the lower bound of L_1 in (1.7).

3. PROOF OF THEOREM 2

3.1. Proof of (1.8)

According to Proposition 2.2 in [18], there exists a constant $C = C(h) > 0$ which is independent of ε , such that

$$\|\nabla u_\varepsilon\|_{L^\infty(B(0, h\varepsilon))} \leq C\varepsilon^{-1}. \quad (3.1)$$

Therefore,

$$\int_{B(0, h\varepsilon)} |\nabla u_\varepsilon|^2 |u_{\varepsilon 3}|^\alpha dx \leq \frac{C}{\varepsilon^2} \pi (h\varepsilon)^2 \leq C. \quad (3.2)$$

Next, using Hölder's inequality and (1.3) and (1.2), we see that as $\varepsilon \rightarrow 0$,

$$\int_{B \setminus B(0, \sigma)} |\nabla u_\varepsilon|^2 |u_{\varepsilon 3}|^\alpha dx \leq \left[\int_{B \setminus B(0, \sigma)} |\nabla u_\varepsilon|^p dx \right]^{\frac{2}{p}} \left[\int_{B \setminus B(0, \sigma)} |u_{\varepsilon 3}|^{\frac{p\alpha}{p-2}} dx \right]^{\frac{p-2}{p}} \rightarrow 0. \quad (3.3)$$

By the same derivation of (13) in [13], we also get from $\|u_{\varepsilon 3}\|_{L^2(B)} \leq C\varepsilon$ (which is deduced by (1.4)) that

$$\int_{h\varepsilon}^{\sigma} \frac{d^2}{r^2} (\sin f)^2 (\cos f)^{\alpha} r dr \leq C \quad (3.4)$$

by using Hölder's inequality. In addition, noting $\alpha \geq 2 - \frac{4}{p}$, using Hölder's inequality and (1.4), we also deduce that

$$\begin{aligned} \int_{h\varepsilon}^{\sigma} (f_r)^2 (\cos f)^{\alpha} r dr &\leq C \int_{h\varepsilon}^{\sigma} (f_r)^2 (\cos f)^{2-\frac{4}{p}} r dr \\ &\leq C \left(\int_{h\varepsilon}^{\sigma} (\cos f)^2 r dr \right)^{1-\frac{2}{p}} \left(\int_{h\varepsilon}^{\sigma} (f_r)^p r dr \right)^{\frac{2}{p}} \leq C \varepsilon^{2(1-\frac{2}{p})+\frac{2}{p}(2-p)} \leq C. \end{aligned} \quad (3.5)$$

Combining this result with (3.2)-(3.4), and noting $|\nabla u|^2 = (f_r)^2 + \frac{d^2}{r^2} (\sin f)^2$, we obtain that $|\nabla u_{\varepsilon}|^2 |u_{\varepsilon 3}|^{\alpha}$ is bounded in $L^1(B)$. Thus, there exists a Radon measure ω_3 such that

$$\lim_{\varepsilon \rightarrow 0} |\nabla u_{\varepsilon}|^2 |u_{\varepsilon 3}|^{\alpha} = \omega_3, \quad \text{weakly star in } C(\overline{B}).$$

By virtue of (3.3), $\text{supp}(\omega_3) \subset \{0\}$. Hence we can find $L_3 \geq 0$ such that $\omega_3 = L_3 \delta_0$.

We claim $L_3 > 0$. Since $f(r) \in C[0, 1]$ and $f(0) = 0$ (see Remark in p.68 of [9]), $f(h\varepsilon) \geq 1/2$ (which can be deduced by (1.1) with $\gamma = \cos(1/2)$), there must exist $r_{\varepsilon} \in (0, h\varepsilon)$ such that $f(r_{\varepsilon}) = 1/4$. Using (3.1), we can find a sufficiently small positive constant δ which is independent of ε , such that

$$\frac{1}{8} \leq f(x) \leq \frac{3}{8}, \quad r \in (r_{\varepsilon}(1-\delta), r_{\varepsilon}(1+\delta)).$$

Therefore,

$$\int_{B(0, r_{\varepsilon}(1+\delta)) \setminus B(0, r_{\varepsilon}(1-\delta))} (\cos f)^{\alpha} |\nabla u_{\varepsilon}|^2 dx \geq 2\pi d^2 \left(\sin \frac{1}{8}\right)^2 \left(\cos \frac{3}{8}\right)^{\alpha} \int_{r_{\varepsilon}(1-\delta)}^{r_{\varepsilon}(1+\delta)} \frac{dr}{r} > 0.$$

This implies $L_3 > 0$. Equation (1.8) is proved.

3.2. Proof of (1.9)

By a direct calculation, it follows

$$\det(\nabla u'_{\varepsilon}) = \frac{d}{r^2} (\sin f \cos f) (x \cdot \nabla f). \quad (3.6)$$

Using Hölder's inequality and (1.2), we get

$$\int_{B \setminus B(0, \sigma)} |\det(\nabla u'_{\varepsilon})|^{p/2} dx \leq C.$$

This means that when $\varepsilon \rightarrow 0$,

$$\varepsilon^{p-2} \int_{B \setminus B(0, \sigma)} |\det(\nabla u'_{\varepsilon})|^{p/2} dx \rightarrow 0. \quad (3.7)$$

In addition, in view of $|\det(\nabla u'_\varepsilon)| \leq \frac{1}{2} |\nabla u'_\varepsilon|^2$, we can deduce from (1.4) that

$$\varepsilon^{p-2} \int_{B(0,\sigma)} |\det(\nabla u'_\varepsilon)|^{p/2} dx \leq C \varepsilon^{p-2} \int_{B(0,\sigma)} |\nabla u'_\varepsilon|^p dx \leq C.$$

Combining this with (3.7) yields the upper bound of $\varepsilon^{p-2} |\det(\nabla u'_\varepsilon)|^{p/2}$ in $L^1(B)$. Then, we can find a Radon measure ω_4 such that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{p-2} |\det(\nabla u'_\varepsilon)|^{p/2} = \omega_4, \text{ weakly star in } C(\bar{B}).$$

In view of (3.7), $\text{supp}(\omega_4) \subset \{0\}$. There exists a constant $L_4 \geq 0$ such that $\omega_4 = L_4 \delta_0$. The proof of Theorem 2 is completed.

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GROUND STATE SOLUTIONS FOR CHOQUARD EQUATION WITH LOGARITHMIC NONLINEARITY

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Abstract. In this paper, we study the following Choquard equation with logarithmic nonlinearity

$$\begin{cases} -\Delta u + V(x)u = k(I_\alpha * |u|^p)|u|^{p-2}u + H(x)u \log |u|, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain with boundary $\partial\Omega$, $k > 0$ is real parameter, $\alpha \in (0, 3)$, I_α is the Riesz potential. Under some mild assumptions on V and H , we obtain a ground state solution by variational method and logarithmic inequality. In addition, we investigate the limit profiles of Choquard equations as $\alpha \rightarrow 0$ or $\alpha \rightarrow N$.

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Keywords: Choquard equation, logarithmic nonlinearity, variational method, ground state solution.

1. INTRODUCTION

In this paper, we consider the following Choquard equation

$$\begin{cases} -\Delta u + V(x)u = k(I_\alpha * |u|^p)|u|^{p-2}u + H(x)u \log |u|, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain with boundary $\partial\Omega$, $k > 0$ is real parameter, $p \in (1 + \alpha/3, 3 + \alpha)$, $\alpha \in (0, 3)$, I_α is the Riesz potential, V, H are potential functions.

Choquard equation was firstly introduced by Pekar [16] in 1954 for describing the quantum mechanics of a polaron at rest, an electron trapped in its hole by Choquard [12]. Li and Tang [11] considered Choquard equation with the upper critical exponent, where the nonlinearity satisfies (f_0) $f \in C(\mathbb{R}, \mathbb{R})$ is odd, (f_1) $\lim_{t \rightarrow 0} f(t)/t = \lim_{t \rightarrow +\infty} f(t)/t^{2^*-1} = 0$, where $2^* = 2N/(N-2)$, (f_2) there exists $\mu > 4$ such that

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$0 < \mu F(t) \leq f(t)t$ for all $t \neq 0$. When the nonlinear perturbation satisfies (f_0) $f(t) = o(t)$ as $t \rightarrow 0$, (f_1) $|f(t)| \leq a(|t| + |t|^{q-1})$ for some $a > 0$ and $q > 2$ with $1/q > 1/2 - 1/N$, (f_2) there exists $\mu > 4$ such that $0 < \mu F(t) \leq f(t)t$ for all $t \neq 0$, Choquard equations with lower critical exponent was studied in [22]. Multiple solutions to Kirchhoff type equations with Hardy-Littlewood-Sobolev critical nonlinearity was solved [5, 6]. Next, [10] proved the existence and concentrate behavior of ground state solutions for critical Choquard equations. Li, Gao and Liang [9] considered the existence and concentration of nontrivial nonnegative ground state solutions to Kirchhoff-type system with Hartree-type nonlinearity, where f satisfies (f_0) $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $C > 0$ such that $|f'(t)| \leq C(1 + |t|^{q-2})$ for all $t \in \mathbb{R}_+$, (f_1) $f(t)/t^{2p-1}$ is increasing on $(0, \infty)$, $\lim_{t \rightarrow \infty} f(t)/t^{2p-1} = \infty$ and $p \in (3 + \alpha)$. More results about Choquard equation, we can refer to [15, 20, 25].

Recently, logarithmic nonlinearity appears frequently in partial differential equations, which describe physical phenomena, for example continuum mechanics, phase transition phenomena, and population dynamics. On the other hand, for the parabolic equations, we can refer to [2, 3, 8, 14] and the references therein. For partial equations with logarithmic nonlinearity, we can refer to [1, 7, 13, 18, 21, 24] and the references therein. Specially, multiple solutions for the semilinear elliptic equations with logarithmic nonlinearity by Nehari manifold was studied in [18]. Liu and Xiao [13] analyzed ground state solutions for a fourth-order nonlinear elliptic problem with logarithmic nonlinearity, where $H = 1$. When $H \in C^1(\mathbb{R}^3)$, $\min_{\mathbb{R}^3} H > 0$ and $\min_{\mathbb{R}^3} (V + H) > 0$, multiple solutions to Schrödinger equation with periodic potential and logarithmic nonlinearity was solved in [21]. Furthermore, the $\max_{\mathbb{R}^3} V \in (-1, \infty)$ and $H = 1$ with ground state solutions was seen [1]. Le [24] studied a fractional p-Laplacian equation in the whole space with the sign-changing logarithmic nonlinearity by using Nehari manifold.

However, the existence of ground state solutions for Choquard equation with logarithmic nonlinearity has not been studied. In this paper, we assume that V and H satisfy:

(V): $V \in L^{3/2}(\Omega)$ and $|V^-|_{3/2} < S$, where $V^+ = \max\{V, 0\}$, $V^- = \min\{V, 0\}$, and $S = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \|u\|_{H_0^1}^2 / |u|_6^2$;

(H): $H \in C(\overline{\Omega})$, $\mu := \inf_{\Omega} H > 0$, satisfying

$$\max_{\Omega} H \leq 2\pi(1 - S^{-1}|V^-|_{3/2})/e^{-8|\Omega|^{1/2}+2}.$$

According to the condition (V), there exists $H \in C(\overline{\Omega})$ and $\min_{\Omega} (V + H) < 0$. So, our conditions are more complex than [1, 13, 21]. Notice that the logarithmic nonlinearity does not satisfy (f_0) , (f_1) and (f_2) , which means that the logarithmic nonlinearity does not satisfy the subcritical or Ambrosetti-Rabinowitz condition. Therefore, the logarithmic nonlinearity can not be replaced by the general nonlinear term [9–11, 22]. Meanwhile, there is no logarithmic Sobolev inequality concerning to the logarithmic

nonlinearity with H . In order to overcome the above difficulties, we introduce a new logarithmic inequality.

In this paper, it's worth noting that we obtain a new existence Theorem 3 on the ground state solution for the equation (1.1), which is quite different from these in the polynomial case. On the other hand, our method and conclusion are different from [18, 24]. In addition, we are interested in limit of behaviors of ground solution to (1.1) as either $\alpha \rightarrow 0$ or $\alpha \rightarrow N$.

Theorem 1. *Suppose that (V) and (H) hold. Then there exists a ground solution to (1.1).*

Remark 1. Notice that the logarithmic nonlinearity contains potential H . Therefore, the logarithmic Sobolev inequality cannot be applied directly. By condition (V), it is not difficult to find that our potential function is sign-change potential function. In addition, (1.1) contains Hatree nonlinearity. To get our conclusions, it is crucial to deal with the relationship between H , V and Hatree nonlinearity.

Theorem 2. *Let $\{u_\alpha\}$ be a family of ground solution to (1.1) for α close to 0. Then there exists u_0 such that $u_\alpha \rightarrow u_0$ and u_0 is a nontrivial solution of the equation*

$$\begin{cases} -\Delta u + V(x)u = k|u|^{2p-2}u + H(x)u \log |u|, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \quad (1.2)$$

Let $\{u_\alpha\}$ be a family of ground solution to (1.1) for α close to N . Then there exists u_N such that $u_\alpha \rightarrow u_N$ and u_N is a nontrivial solution of the equation

$$\begin{cases} -\Delta u + V(x)u = k(\int_\Omega |u|^p) |u|^{p-2}u + H(x)u \log |u|, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \quad (1.3)$$

Remark 2. Seok [19] considered the limit profiles and uniqueness of ground states to the nonlinear Choquard equations, namely, $\alpha \rightarrow 0, \alpha \rightarrow N$. When logarithmic nonlinearity does not exist, as $\alpha \rightarrow 0$, it reduces to the Euler-Lagrange equation. On the other hand, inspired by [19], we are interested to consider the case of $\alpha \rightarrow N$. In addition, by studying the limit profiles, we can better understand the properties of logarithmic nonlinearity.

Theorem 3. *When $k = 0$, the equation (1.1) has a ground state solution.*

Remark 3. Similarly, we can also consider the following fractional Schrödinger equation

$$\begin{cases} (-\Delta)^s u + V(x)u = H(x)u \log |u|, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \quad (1.4)$$

where $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain with boundary $\partial\Omega$, $s \in (0, 1)$. We assume that the potential functions H and V satisfy:

(V_1) $V \in L^{3/(2s)}(\Omega)$ and $|V^-|_{3/(2s)} < S$, where $V^+ = \max\{V, 0\}$, $V^- = \min\{V, 0\}$ and $S = \inf_{u \in H_0^s(\Omega) \setminus \{0\}} \|u\|_{H_0^s}^2 / \|u\|_{6/(3-2s)}^2$.

(H_1) $H \in C(\overline{\Omega})$, $\mu := \inf_{\Omega} H > 0$, satisfying

$$\max_{\overline{\Omega}} H \leq \pi^s (1 - S^{-1} |V^-|_{3/(2s)}) / \left(\frac{\Gamma(\frac{3}{2s})}{(\frac{3}{2})} e^{\frac{2|\Omega|^{1/2-2}}{3}} \right)^{6/s}.$$

Assume that $(V_1), (H_1)$ hold, (1.4) has a ground state solution. A complete introduction to fractional Sobolev space and fractional logarithmic Sobolev inequality, we can refer to [4, 17, 23].

2. PRELIMINARIES

In this paper, we make use of the following notation:

- $H_0^1(\Omega)$ is the usual Hilbert space with the norm

$$\|u\|_{H_0^1} = \left(\int_{\Omega} |\nabla u|^2 dx \right)^{1/2}.$$

- $L^p(\Omega)$ ($1 \leq p < \infty$) is the Lebesgue space with the norm

$$|u|_p = \left(\int_{\Omega} |u|^p dx \right)^{1/p}.$$

- $C, C_i, i = 1, 2, \dots$, denote various positive constants.
- M denotes $\max_{\overline{\Omega}} H$.
- $|\Omega|$ denotes the Lebesgue measure of Ω .
- From the Sobolev and Rellich embedding theorem, the embedding $H_0^1(\Omega) \hookrightarrow L^s(\Omega)$ is continuous for $s \in [2, 6]$ and is compact for $s \in [2, 6)$.

A weak solution to the equation (1.1) is a critical point of the following energy functional J defined on $H_0^1(\Omega)$ by

$$\begin{aligned} J(u) &= \frac{1}{2} \int_{\Omega} (|\nabla u|^2 + Vu^2) dx - \frac{k}{2p} \int_{\Omega} (I_{\alpha} * |u|^p) |u|^p dx \\ &\quad - \frac{1}{2} \int_{\Omega} Hu^2 \log |u| dx + \frac{1}{4} \int_{\Omega} Hu^2 dx, \quad u \in H_0^1(\Omega). \end{aligned}$$

It is easy to proof that J is well defined on $H_0^1(\Omega)$ and $J \in C^1(H_0^1(\Omega), \mathbb{R})$. Furthermore,

$$\begin{aligned} \langle J'(u), v \rangle &= \int_{\Omega} (\nabla u \cdot \nabla v + Vuv) dx - k \int_{\Omega} (I_{\alpha} * |u|^p) |u|^{p-2} uv dx \\ &\quad - \int_{\Omega} Huv \log |u| dx, \quad v \in H_0^1(\Omega). \end{aligned}$$

For each $u \in H_0^1(\Omega)$, by the Hölder inequality and the condition (V), we have

$$\begin{aligned} \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} V u^2 dx &\geq \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} |V^-| u^2 dx \\ &\geq (1 - S^{-1} |V^-|_{3/2}) \int_{\Omega} |\nabla u|^2 dx := \delta \int_{\Omega} |\nabla u|^2 dx, \end{aligned} \quad (2.1)$$

where $\delta = 1 - S^{-1} |V^-|_{3/2}$.

Proposition 1. (Logarithmic Sobolev inequality [8, 14]) Let u be a function in $H^1(\mathbb{R}^3) \setminus \{0\}$ and $a > 0$ be a constant. Then

$$2 \int_{\mathbb{R}^3} u^2 \log \frac{|u|}{|u|_2} dx + 3(1 + \log a) |u|_2^2 \leq \frac{a^2}{\pi} \int_{\mathbb{R}^3} |\nabla u|^2 dx.$$

For any $u \in H_0^1(\Omega) \setminus \{0\}$, we define $u(x) = 0$ for $x \in \mathbb{R}^3 \setminus \Omega$. According to the logarithmic Sobolev inequality, we have

$$2 \int_{\Omega} u^2 \log \frac{|u|}{|u|_2} dx + 3(1 + \log a) |u|_2^2 \leq \frac{a^2}{\pi} \int_{\Omega} |\nabla u|^2 dx. \quad (2.2)$$

Proposition 2. (Hardy-Littlewood-Sobolev inequality [5, 6]) Let $s, t > 1$ and $\alpha \in (0, 3)$ with $1/s + 1/r = 1 + \alpha/3$. Then there exists a sharp constant $C(\alpha, s, r) > 0$ such that for any $g \in L^s(\Omega)$ and $h \in L^r(\Omega)$

$$\int_{\Omega} \int_{\Omega} \frac{g(x)h(y)}{|x-y|^{3-\alpha}} dx dy \leq C(\alpha, s, r) |g|_s |h|_r.$$

3. PROOFS OF MAIN RESULTS

Lemma 1. There exists a $C > 0$ such that,

$$|t \log |t|| \leq C(1 + t^2), \quad t \in \mathbb{R}. \quad (3.1)$$

Proof. Define

$$f(t) = \begin{cases} \frac{|t \log |t||}{1+t^2}, & t \neq 0. \\ 0, & t = 0. \end{cases}$$

Clearly,

$$\lim_{|t| \rightarrow 0} f(t) = 0, \quad \lim_{|t| \rightarrow \infty} f(t) = 0. \quad (3.2)$$

It follows from (3.2) that (3.1) holds. $\square$

Lemma 2. For any $u \in H_0^1(\Omega) \setminus \{0\}$, then we have

$$\int_{\Omega} H u^2 \log \frac{|u|}{|u|_2} dx \leq M \left(\frac{a^2}{2\pi} \int_{\Omega} |\nabla u|^2 dx - \frac{3(1 + \log a)}{2} |u|_2^2 + 2|\Omega|^{1/2} |u|_2^2 \right).$$

Proof. For $\theta > 0$, we have

$$\log \theta < \theta. \quad (3.3)$$

For any $u \in H_0^1(\Omega) \setminus \{0\}$, then we have

$$\int_{\Omega} Hu^2 \log \frac{|u|}{|u|_2} dx = \int_{\Omega_1} Hu^2 \log \frac{|u|}{|u|_2} dx + \int_{\Omega \setminus \Omega_1} Hu^2 \log \frac{|u|}{|u|_2} dx, \quad (3.4)$$

where $\Omega_1 = \{x \in \Omega : |u(x)|/|u|_2 < 1\}$. According to the Hölder inequality and (3.3), we deduce

$$\int_{\Omega_1} Hu^2 \log \frac{|u|}{|u|_2} dx \leq M \int_{\Omega_1} u^2 \log \frac{|u|_2}{|u|} dx \leq M |\Omega|^{1/2} |u|_2^2. \quad (3.5)$$

Observing (2.2), (3.4) and (3.5) that

$$\begin{aligned} \int_{\Omega \setminus \Omega_1} Hu^2 \log \frac{|u|}{|u|_2} dx &\leq M \left(\int_{\Omega} u^2 \log \frac{|u|}{|u|_2} dx - \int_{\Omega_1} u^2 dx \log \frac{|u|}{|u|_2} \right) \\ &\leq M \left(\int_{\Omega} u^2 \log \frac{|u|}{|u|_2} dx + |\Omega|^{1/2} |u|_2^2 \right) \\ &\leq M \left(\frac{a^2}{2\pi} \int_{\Omega} |\nabla u|^2 dx - \frac{3(1+\log a)}{2} |u|_2^2 + |\Omega|^{1/2} |u|_2^2 \right). \end{aligned} \quad (3.6)$$

It follows from (3.5) and (3.6) that

$$\int_{\Omega} Hu^2 \log \frac{|u|}{|u|_2} dx \leq M \left(\frac{a^2}{2\pi} \int_{\Omega} |\nabla u|^2 dx - \frac{3(1+\log a)}{2} |u|_2^2 + 2|\Omega|^{1/2} |u|_2^2 \right).$$

□

Lemma 3. *The functional J has the mountain pass geometry:*

- (i) *there exists $\rho > 0$ such that $\inf_{\|u\|=\rho} J(u) > 0$.*
- (ii) *for any $u \in H_0^1(\Omega) \setminus \{0\}$, it holds $\lim_{t \rightarrow \infty} J(tu) = -\infty$.*

Proof. According to Hardy-Littlewood-Sobolev Inequality, we have

$$\int_{\Omega} (I_{\alpha} * |u|^p) |u|^p dx \leq C_1 |u|_{pr}^{2p}, \quad (3.7)$$

where $r = 6/(3 + \alpha)$. For any $u \in H_0^1(\Omega) \setminus \{0\}$, by (2.1), (3.3), (3.7) and Lemma 2, we get

$$\begin{aligned} J(u) &\geq \frac{1}{2} \int_{\Omega} \delta |\nabla u|^2 dx - \frac{1}{2} \int_{\Omega} Hu^2 \log \frac{|u|}{|u|_2} dx - \frac{1}{2} \int_{\Omega} Hu^2 \log |u|_2 dx \\ &\quad - C_1 k |u|_{pr}^{2p} \\ &\geq \frac{1}{4} \left(2\delta - \frac{Ma^2}{\pi} \right) \int_{\Omega} |\nabla u|^2 dx + \frac{1}{4} (3 + 3 \log a - 4|\Omega|^{1/2}) |u|_2^2 \\ &\quad - \frac{M}{2} \left(\int_{\Omega} u^2 dx \right)^{3/2} - C_2 k \|u\|^{2p}. \end{aligned} \quad (3.8)$$

According to the arbitrariness of a in logarithmic Sobolev inequality, we take $a = e^{\frac{4|\Omega|^{1/2}-3}{3}}$, we know

$$3 + 3 \log a - 4|\Omega|^{1/2} > 0.$$

According to the condition (H), we have

$$2\delta - \frac{Ma^2}{\pi} > 0.$$

When $\|u\| = \rho > 0$ is small enough, according to (3.8) and the condition (H), we complete the proof of (i). For any $u \in H_0^1(\Omega) \setminus \{0\}$ and $t > 0$, we have

$$\begin{aligned} J(tu) &= \frac{1}{2} \int_{\Omega} (|\nabla tu|^2 + V(tu)^2) dx - \frac{k}{2p} \int_{\Omega} (I_{\alpha} * |tu|^p) |tu|^p dx \\ &\quad - \frac{1}{2} \int_{\Omega} H(tu)^2 \log |tu| dx + \frac{1}{4} \int_{\Omega} H(tu)^2 dx \\ &= \frac{t^2}{2} \int_{\Omega} (|\nabla u|^2 + Vu^2) dx - \frac{t^{2p}k}{2p} \int_{\Omega} (I_{\alpha} * |u|^p) |u|^p dx \\ &\quad - \frac{t^2 \log t}{2} \int_{\Omega} Hu^2 dx - \frac{t^2}{2} \int_{\Omega} Hu^2 \log |u| dx + \frac{t^2}{4} \int_{\Omega} Hu^2 dx, \end{aligned}$$

and the conclusion (ii) follows. $\square$

Lemma 4. Suppose that $\{u_n\}$ is a sequence in $H_0^1(\Omega)$ such that $u_n \rightharpoonup u$, then we have

$$\lim_{n \rightarrow \infty} \int_{\Omega} (I_{\alpha} * |u_n|^p) |u_n|^p dx = \int_{\Omega} (I_{\alpha} * |u|^p) |u|^p dx.$$

For any $v \in H_0^1(\Omega)$, we have

$$\lim_{n \rightarrow \infty} \int_{\Omega} (I_{\alpha} * |u_n|^p) |u_n|^{p-1} u_n v dx = \int_{\Omega} (I_{\alpha} * |u|^p) |u|^{p-1} u v dx.$$

Proof. Firstly, by Fatou's lemma, then we have

$$\int_{\Omega} (I_{\alpha} * |u|^p) |u|^p dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} (I_{\alpha} * |u_n|^p) |u_n|^p dx. \quad (3.9)$$

From that $u_n \rightharpoonup u$ in $H_0^1(\Omega)$ and $6p/(3+\alpha) \in [2, 6)$, we have that $u_n \rightarrow u$ in $L^{6p/(3+\alpha)}(\Omega)$. Therefore, $|u_n|^p \rightarrow |u|^p$ in $L^{6/(3+\alpha)}(\Omega)$. Combining with Proposition 2, we obtain

$$\int_{\Omega} [I_{\alpha} * (|u_n|^p - |u|^p)] |u_n|^p dx \leq C_{\alpha,p} \| |u_n|^p - |u|^p \|_{6/(3+\alpha)} \|u_n^p\|_{6/(3+\alpha)}, \quad (3.10)$$

According to (3.10), as $n \rightarrow \infty$, we have

$$\int_{\Omega} [I_{\alpha} * (|u_n|^p - |u|^p)] |u_n|^p dx \rightarrow 0. \quad (3.11)$$

Similarly, we get

$$\int_{\Omega} (I_{\alpha} * |u|^p) |u_n|^p - |u|^p dx = o(1). \quad (3.12)$$

According to (3.9), (3.11) and (3.12), we obtain

$$\lim_{n \rightarrow \infty} \int_{\Omega} [I_{\alpha} * (|u_n|^p)] |u_n|^p dx = \int_{\Omega} [I_{\alpha} * (|u|^p)] |u|^p dx.$$

It follows from Proposition 2 that $I_{\alpha} * (|u|^{p-2}uv) \in L^{6/(3+\alpha)}(\Omega)$.

As $n \rightarrow \infty$, we get

$$\begin{aligned} & \left| \int_{\Omega} (I_{\alpha} * |u_n|^p) |u_n|^{p-1} u_n v dx - \int_{\Omega} (I_{\alpha} * |u|^p) |u|^{p-1} u v dx \right| \\ & \leq \int_{\Omega} |(I_{\alpha} * |u_n|^p) [|u_n|^{p-1} u_n v - |u|^{p-1} u v]| dx \\ & \quad + \left| \int_{\Omega} [I_{\alpha} * |u_n|^p - I_{\alpha} * |u|^p] |u|^{p-1} u v dx \right| \\ & \leq |I_{\alpha} * |u_n|^p|_{6/(3-\alpha)} \| |u_n|^{p-1} u_n v - |u|^{p-1} u v \|_{6/(3-\alpha)} \\ & \quad + \left| \int_{\Omega} [I_{\alpha} * |u_n|^p - I_{\alpha} * |u|^p] |u|^{p-1} u v dx \right| \rightarrow 0. \end{aligned} \quad (3.13)$$

The proof is complete. $\square$

Now, we prove that the functional J satisfies (PS) condition.

Lemma 5. *J satisfies (PS) condition.*

Proof. Assume that $\{u_n\} \subset H_0^1(\Omega)$ is a (PS) sequence certifies $|J(u_n)| < b$ for some positive constant b and $J'(u_n) \rightarrow 0$, as $n \rightarrow \infty$. We claim $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Indeed, it follows from that for n large enough that

$$\begin{aligned} \frac{\mu}{4} \int_{\Omega} u_n^2 dx & \leq \frac{1}{4} \int_{\Omega} H u_n^2 dx = J(u_n) - \frac{1}{2} \langle J'(u_n), u_n \rangle \\ & \quad - \left(\frac{1}{2} - \frac{1}{2p} \right) k \int_{\Omega} (I_{\alpha} * |u_n|^p) |u_n|^p dx \leq b + o(1) \|u_n\|. \end{aligned} \quad (3.14)$$

It follows from (3.3) and (3.14) that

$$\begin{aligned} \int_{\Omega} H |u_n|^2 \log |u_n|_2 dx & = \log |u_n|_2 \int_{\Omega} H |u_n|^2 dx \leq M |u_n|_2 \int_{\Omega} |u_n|^2 dx \\ & \leq \frac{8M}{\mu^{3/2}} (4b + o(1) \|u_n\|)^{3/2}. \end{aligned} \quad (3.15)$$

setting $\gamma = \frac{1}{2} - \frac{1}{2p}$, according to (3.14), (3.15) and Lemma 2, we have

$$b + o(1) \|u_n\| \geq J(u_n) - \frac{1}{2p} \langle J'(u_n), u_n \rangle$$

$$\begin{aligned}
&\geq \frac{\gamma}{2} \left(2\delta - \frac{Ma^2}{\pi} \right) \int_{\Omega} |\nabla u|^2 dx + \frac{\gamma}{2} (3 + 3 \log a - 4|\Omega|^{1/2}) |u|_2^2 \\
&\quad - \gamma \int_{\Omega} H |u_n|^2 \log |u_n|_2 dx \\
&\geq \frac{\gamma}{2} \left(2\delta - \frac{Ma^2}{\pi} \right) \|u_n\|^2 - \left| \frac{\gamma}{2\mu} (3 + 3 \log a - 4|\Omega|^{1/2}) \right| (b + o(1) \|u_n\|) \\
&\quad - \frac{8M\gamma}{\mu^{3/2}} (4b + o(1) \|u_n\|)^{3/2},
\end{aligned}$$

which concludes the claim. So, going if necessary to a subsequence (still denote $\{u_n\}$), we can assume that $u_n \rightharpoonup u$ in $H_0^1(\Omega)$. It follows from (3.1) that

$$\left| \int_{\Omega} (Hu_n \log |u_n| - Hu \log |u|)(u_n - u) dx \right| \rightarrow 0. \quad (3.16)$$

By Lemma 4, we have

$$\left| \int_{\Omega} [(I_{\alpha} * |u_n|^p) |u_n|^{p-2} u_n - (I_{\alpha} * |u|^p) |u|^{p-2} u] (u_n - u) dx \right| \rightarrow 0. \quad (3.17)$$

On the other hand, we have

$$\begin{aligned}
\|u_n - u\|^2 &= \langle J'(u_n) - J'(u), u_n - u \rangle + \int_{\Omega} (u_n \log |u_n| - u \log |u|)(u_n - u) dx \\
&\quad + \int_{\Omega} [(I_{\alpha} * |u_n|^p) |u_n|^{p-2} u_n - (I_{\alpha} * |u|^p) |u|^{p-2} u] (u_n - u) dx \\
&\quad - \int_{\Omega} (Vu_n - Vu)(u_n - u) dx.
\end{aligned} \quad (3.18)$$

It is clear that

$$\langle J'(u_n) - J'(u), u_n - u \rangle \rightarrow 0. \quad (3.19)$$

and

$$\int_{\Omega} (Vu_n - Vu)(u_n - u) dx \rightarrow 0. \quad (3.20)$$

Therefore, according to (3.16)-(3.20), we get $\|u_n - u\| \rightarrow 0$. The proof is complete. $\square$

Proof of Theorem 1. According to Lemma 3 and Lemma 5, (1.1) exists a non-trivial solution. Set $K = \{u \in H_0^1(\Omega) \setminus \{0\} : J'(u) = 0\}$, we know K is nonempty. So for any $u \in K$, then

$$\begin{aligned}
J(u) - \frac{1}{2} \langle J'(u), u \rangle &= \left(\frac{1}{2} - \frac{1}{2p} \right) k \int_{\Omega} (I_{\alpha} * |u_n|^p) |u_n|^p dx + \frac{1}{4} \int_{\Omega} Hu^2 dx \\
&\geq \frac{1}{4} \int_{\Omega} Hu^2 dx \geq 0.
\end{aligned}$$

So we may define $c_\alpha = \inf\{J(u) : u \in K\}$. Let $\{u_n\} \subseteq K$ be such that $J(u_n) \rightarrow c_\alpha$, as $n \rightarrow \infty$. By Lemma 5, there exists a subsequence (still denote $\{u_n\}$) and $u_n \rightarrow u_0$ in $H_0^1(\Omega)$. By the continuation of J' , we imply that $J(u_0) = c_\alpha$ and $J'(u_0) = 0$. We show that $u_0 \neq 0$. Since $u_n \in K$, we have

$$\begin{aligned} \|u_n\|^2 &= k \int_{\Omega} (I_\alpha * |u_n|^p) |u_n|^p dx + \int_{\Omega} H u_n^2 \log |u_n| dx \\ &\leq C_2 \|u_n\|^{2p} + C_4 \|u_n\|^4. \end{aligned} \quad (3.21)$$

It follows from $u_n \in K$ and (3.21) that

$$1 \leq C_2 \|u_n\|^{2p-2} + C_4 \|u_n\|^2.$$

Therefore, we imply that $u_0 \neq 0$. It completes the proof. $\square$

Proof of Theorem 2. A weak solution to the equation (1.2) is a critical point of the following energy functional J_1 defined on $H_0^1(\Omega)$ by

$$J_1(u) = \frac{1}{2} \int_{\Omega} (|\nabla u|^2 + V u^2) dx - \frac{k}{2p} \int_{\Omega} |u|^{2p} dx - \frac{1}{2} \int_{\Omega} H u^2 \log |u| + \frac{1}{4} \int_{\Omega} H u^2 dx,$$

for all $u \in H_0^1(\Omega)$. It is easy to proof that J_1 is well defined on $H_0^1(\Omega)$ and $J \in C^1(H_0^1(\Omega), \mathbb{R})$. Furthermore,

$$\langle J_1'(u), v \rangle = \int_{\Omega} (\nabla u \cdot \nabla v + V u v) dx - k \int_{\Omega} |u|^{p-2} u v dx - \int_{\Omega} H u v \log |u| dx,$$

for all $v \in H_0^1(\Omega)$. $\square$

Lemma 6. Fix $1 < p < \frac{2N}{N-2}$. Let $\{\alpha_j\} > 0$ be a sequence converging to 0 and let $\{u_j\} \subset H_0^1(\Omega)$ be a sequence converging weakly in $H_0^1(\Omega)$ to some $u_0 \in H_0^1(\Omega)$. Then, as $j \rightarrow \infty$, the following holds:

$$\int_{\Omega} (I_{\alpha_j} * |u_j|^p) |u_j|^p dx = \int_{\Omega} |u_0|^{2p} dx.$$

For any $v \in H_0^1(\Omega)$, we have

$$\int_{\Omega} (I_{\alpha_j} * |u_j|^p) |u_j|^{p-1} u_j v dx = \int_{\Omega} |u_0|^{2p-2} u_0 v dx.$$

Proof. For the proof of the lemma, we refer to ([19], Proposition 2.7). $\square$

Lemma 7. Let $\{u_\alpha\}$ be a family of ground solution to (1.1) for α close to 0 and then there exists u_0 such that u_0 is a nontrivial solution of the equation (1.2).

Proof. Let $\{u_\alpha\}$ be a family of ground solutions to (1.1) for α close to 0. Similarly to proof of Theorem 1, define $c_{2p} = \inf\{J_1(u) : u \in K_1\}$, where $K_1 = \{u \in H_0^1(\Omega) \setminus \{0\} : J_1'(u) = 0\}$. Given $u \in H_0^1(\Omega) \setminus \{0\}$, as $\alpha \rightarrow 0$, we get

$$c_\alpha \leq \max_{t>0} J(tu) = \max_{t>0} \frac{t^2}{2} \int_{\Omega} (|\nabla u|^2 + V u^2) dx - \frac{t^{2p}k}{2p} \int_{\Omega} (I_\alpha * |u|^p) |u|^p dx$$

$$\begin{aligned}
& -\frac{t^2 \log t}{2} \int_{\Omega} H u^2 dx - \frac{t^2}{2} \int_{\Omega} H u^2 \log |u| dx + \frac{t^2}{4} \int_{\Omega} H u^2 dx \\
& \rightarrow \max_{t>0} \frac{t^2}{2} \int_{\Omega} (|\nabla u|^2 + V u^2) dx - \frac{t^{2p} k}{2p} \int_{\Omega} |u|^{2p} dx \\
& \quad - \frac{t^2 \log t}{2} \int_{\Omega} H u^2 dx - \frac{t^2}{2} \int_{\Omega} H u^2 \log |u| dx + \frac{t^2}{4} \int_{\Omega} H u^2 dx \\
& = \max_{t>0} J_1(tu).
\end{aligned}$$

Taking the infimum with respect to $u \in H_0^1(\Omega) \setminus \{0\}$, we deduce that

$$\limsup_{\alpha \rightarrow 0} c_{\alpha} \leq c_{2p}. \quad (3.22)$$

Since u_{α} is a ground solution to (1.1), by Lemma 5 and (3.22), we obtain that $\{u_{\alpha}\}$ is bounded. So, there exists $\{u_j\} \subset H_0^1(\Omega)$ be a sequence converging weakly in $H_0^1(\Omega)$ to $u_0 \in H_0^1(\Omega)$. We claim that u_0 is a weak solution of the equation (1.2). Indeed,

$$\begin{aligned}
& \lim_{j \rightarrow \infty} \int_{\Omega} (\nabla u_j \cdot \nabla v + V u_j v) dx - k \int_{\Omega} (I_{\alpha} * |u_{\alpha_j}|^p) |u_j|^{p-2} u v dx - \int_{\Omega} H u_j v \log |u_j| dx \\
& = \int_{\Omega} (\nabla u_0 \cdot \nabla v + V u_0 v) dx - k \int_{\Omega} |u_0|^{p-2} u_0 v dx - \int_{\Omega} H u_0 v \log |u_0| dx. \quad (3.23)
\end{aligned}$$

Since u_j is a ground solution of equation (1.1), then we obtain

$$\begin{aligned}
\langle J'(u_j), v \rangle &= \int_{\Omega} (\nabla u_j \cdot \nabla v + V u_j v) dx - k \int_{\Omega} (I_{\alpha_j} * |u_j|^p) |u_j|^{p-2} u v dx \\
&\quad - \int_{\Omega} H u_j v \log |u_j| dx = 0. \quad (3.24)
\end{aligned}$$

It follows from (3.23) and (3.24) that

$$\begin{aligned}
& \int_{\Omega} (\nabla u_0 \cdot \nabla v + V u_0 v) dx - k \int_{\Omega} |u_0|^{p-2} u_0 v dx - \int_{\Omega} H u_0 v \log |u_0| dx \\
& = 0 = \langle J'_1(u_0), v \rangle, \quad (3.25)
\end{aligned}$$

which concludes the claim. According to (3.1), we have

$$\lim_{j \rightarrow \infty} \int_{\Omega} H u_j^2 \log |u_j| dx = \int_{\Omega} H u_0^2 \log |u_0| dx. \quad (3.26)$$

Since u_0 is a weak solution of the equation (1.2), by the (3.26) and Lemma 6, we imply

$$\begin{aligned}
\lim_{j \rightarrow \infty} \|u_j\| &= \lim_{j \rightarrow \infty} \left(\int_{\Omega} (I_{\alpha_j} * |u_j|^p) |u_j|^p dx + \int_{\Omega} H u_j^2 \log |u_j| dx \right) \\
&= \int_{\Omega} |u_0|^{2p} dx + \int_{\Omega} H u_0^2 \log |u_0| dx = \|u_0\|. \quad (3.27)
\end{aligned}$$

According to (3.27) and $u_j \rightharpoonup u_0$, we have $u_j \rightarrow u_0$. Since u_j is a ground solution of equation (1.1), we get

$$\|u_j\|^2 = k \int_{\Omega} (I_{\alpha} * |u_j|^p) |u_j|^p dx + \int_{\Omega} H u_j^2 \log |u_j| dx \leq C_2 \|u_j\|^{2p} + C_4 \|u_j\|^4, \quad (3.28)$$

Observe from (3.28) that

$$1 \leq C_2 \|u_j\|^{2p-2} + C_4 \|u_j\|^2. \quad (3.29)$$

According to (3.29) and $u_j \rightarrow u_0$, we imply that u_0 is a nontrivial solution of the equation (1.2). $\square$

A weak solution to the equation (1.3) is a critical point of the following energy functional J_2 defined on $H_0^1(\Omega)$ by

$$\begin{aligned} J_2(u) = & \frac{1}{2} \int_{\Omega} (|\nabla u|^2 + V u^2) dx - \frac{k}{2p} \left(\int_{\Omega} |u|^p dx \right)^2 \\ & - \frac{1}{2} \int_{\Omega} H u^2 \log |u| dx + \frac{1}{4} \int_{\Omega} H u^2 dx, \end{aligned}$$

for all $u \in H_0^1(\Omega)$. It is easy to proof that J_2 is well defined on $H_0^1(\Omega)$ and $J_2 \in C^1(H_0^1(\Omega), \mathbb{R})$. Furthermore,

$$\langle J_2'(u), v \rangle = \int_{\Omega} (\nabla u \cdot \nabla v + V u v) dx - k \int_{\Omega} |u|^p \int_{\Omega} |u|^{p-2} u v dx - \int_{\Omega} H u v \log |u| dx,$$

for all $v \in H_0^1(\Omega)$.

Lemma 8. Fix $1 < p < \frac{2N}{N-2}$. Let $\{\alpha_j\} > 0$ be a sequence converging to N and let $\{u_j\} \subset H_0^1(\Omega)$ be a sequence converging weakly in $H_0^1(\Omega)$ to some $u_N \in H_0^1(\Omega)$. Then, as $j \rightarrow \infty$, the following holds:

$$\int_{\Omega} (I_{\alpha_j} * |u_j|^p) |u_j|^p dx = \left(\int_{\Omega} |u_N|^p dx \right)^2.$$

For any $v \in H_0^1(\Omega)$, we have

$$\int_{\Omega} (I_{\alpha_j} * |u_j|^p) |u_j|^{p-1} u_N v dx = \int_{\Omega} |u_N|^p \int_{\Omega} |u_N|^{p-2} u_N v dx.$$

Proof. For a proof of the lemma, we refer to ([19], Proposition 2.8). $\square$

Lemma 9. Let $\{u_a\}$ be a family of ground solution to (1.1) for α close to N and then there exists u_N such that u_N is a nontrivial solution of the equation (1.3).

Proof. Let $\{u_a\}$ be a family of ground solution to (1.1) for α close to N . Similarly to proof of Theorem 1, define $c_p = \inf\{J_2(u) : u \in K_2\}$, where $K_2 = \{u \in H_0^1(\Omega) \setminus \{0\} : J_2'(u) = 0\}$. Given $u \in H_0^1(\Omega) \setminus \{0\}$, as $\alpha \rightarrow 0$, we have

$$c_{\alpha} \leq \max_{t>0} J(tu)$$

$$\begin{aligned}
&= \max_{t>0} \frac{t^2}{2} \int_{\Omega} (|\nabla u|^2 + Vu^2) dx - \frac{t^{2p}k}{2p} \int_{\Omega} (I_{\alpha} * |u|^p) |u|^p dx \\
&\quad - \frac{t^2 \log t}{2} \int_{\Omega} Hu^2 dx - \frac{t^2}{2} \int_{\Omega} Hu^2 \log |u| dx + \frac{t^2}{4} \int_{\Omega} Hu^2 dx \\
&\rightarrow \max_{t>0} \frac{t^2}{2} \int_{\Omega} (|\nabla u|^2 + Vu^2) dx - \frac{t^{2p}k}{2p} \left(\int_{\Omega} |u|^p dx \right)^2 \\
&\quad - \frac{t^2 \log t}{2} \int_{\Omega} Hu^2 dx - \frac{t^2}{2} \int_{\Omega} Hu^2 \log |u| dx + \frac{t^2}{4} \int_{\Omega} Hu^2 dx \\
&= \max_{t>0} J_2(tu).
\end{aligned}$$

Taking the infimum with respect to $u \in H_0^1(\Omega) \setminus \{0\}$, we deduce that

$$\limsup_{\alpha \rightarrow 0} c_{\alpha} \leq c_p. \quad (3.30)$$

According to Lemma 5 and (3.30), we obtain that $\{u_{\alpha}\}$ is bounded. So, there exists $\{u_j\} \subset H_0^1(\Omega)$ be a sequence converging weakly in $H_0^1(\Omega)$ to $u_N \in H_0^1(\Omega)$. We claim that u_N is a weak solution of the equation (1.1). Indeed,

$$\begin{aligned}
&\lim_{j \rightarrow \infty} \int_{\Omega} (\nabla u_j \cdot \nabla v + Vu_j v) dx - k \int_{\Omega} (I_{\alpha} * |u_{\alpha_j}|^p) |u_j|^{p-2} u v dx \\
&\quad - \int_{\Omega} Hu_j v \log |u_j| dx \\
&= \int_{\Omega} (\nabla u_N \cdot \nabla v + Vu_N v) dx - k \int_{\Omega} |u_N|^p dx \int_{\Omega} |u_N|^{p-2} u_N v dx \\
&\quad - \int_{\Omega} Hu_N v \log |u_N| dx.
\end{aligned} \quad (3.31)$$

Since u_j is a ground solution of equation (1.1), we obtain

$$\begin{aligned}
\langle J'(u_j), v \rangle &= \int_{\Omega} (\nabla u_j \cdot \nabla v + Vu_j v) dx - k \int_{\Omega} (I_{\alpha_j} * |u_j|^p) |u_j|^{p-2} u v dx \\
&\quad - \int_{\Omega} Hu_j v \log |u_j| dx = 0.
\end{aligned} \quad (3.32)$$

It follows from (3.31) and (3.32) that

$$\begin{aligned}
&\int_{\Omega} (\nabla u_N \cdot \nabla v + Vu_N v) dx - k \int_{\Omega} |u_N|^p dx \int_{\Omega} |u_N|^{p-2} u_N v dx - \int_{\Omega} Hu_N v \log |u_N| dx \\
&= 0 = \langle J'(u_N), v \rangle,
\end{aligned}$$

which concludes the claim. Since u_0 is a weak solution of the equation (1.3), according to (3.25) and Lemma 8, we have

$$\lim_{j \rightarrow \infty} \|u_j\| = \lim_{j \rightarrow \infty} \left(\int_{\Omega} (I_{\alpha_j} * |u_j|^p) |u_j|^p dx + \int_{\Omega} Hu_j^2 \log |u_j| dx \right)$$

$$= \int_{\Omega} (|u_0|^p dx)^2 + \int_{\Omega} H u_0^2 \log |u_0| dx = \|u_0\|. \quad (3.33)$$

According to (3.33) and $u_j \rightarrow u_0$, we have $u_j \rightarrow u_0$. Since u_j is a ground solution of equation (1.1), we obtain

$$\begin{aligned} \|u_j\|^2 &= k \int_{\Omega} (I_{\alpha} * |u_j|^p) |u_j|^p dx + \int_{\Omega} H u_j^2 \log |u_j| dx \\ &\leq C_2 \|u_j\|^{2p} + C_4 \|u_j\|^4, \end{aligned} \quad (3.34)$$

By (3.34), we deduce

$$1 \leq C_2 \|u_j\|^{2p-2} + C_4 \|u_j\|^2. \quad (3.35)$$

According to (3.35) and $u_j \rightarrow u_0$, we imply that u_0 is a nontrivial solution of the equation (1.3). $\square$

According to Lemma 7 and Lemma 9, we complete the proof of Theorem 2.

Proof of Theorem 3. When $k = 0$, define $K = \{u \in H_0^1(\Omega) \setminus \{0\} : J'(u) = 0\}$. It follows from Lemma 3 and Lemma 5 that K is nonempty. So for any $u \in K$, according to (2.1), the condition (H) and Lemma 2, we have

$$\begin{aligned} 0 &= \langle J'(u), u \rangle \\ &\geq \int_{\Omega} \delta |\nabla u|^2 dx - \int_{\Omega} H u^2 \log \frac{|u|}{|u|_2} dx - \int_{\Omega} H u^2 \log |u|_2 dx + \int_{\Omega} H u^2 dx - \int_{\Omega} H u^2 dx \\ &\geq \frac{1}{4} \left(2\delta - \frac{M a^2}{\pi} \right) \int_{\Omega} |\nabla u|^2 dx + M \left(\frac{3(1 + \log a)}{2} - 2|\Omega|^{1/2} - 1 \right) |u|_2^2 \\ &\quad + (1 - \log |u|_2) \int_{\Omega} H u^2 dx. \end{aligned} \quad (3.36)$$

Take $a = e^{\frac{4|\Omega|^{1/2}-1}{3}}$, we deduce

$$1 - \log |u|_2 \leq 0. \quad (3.37)$$

It follows from (3.37) that $e^1 \leq \int_{\Omega} u^2$. Next, we claim that J is bounded from below on K . For any $u \in K$, we obtain

$$J(u) - \frac{1}{2} \langle J'(u), u \rangle = \frac{1}{4} \int_{\Omega} H u^2 dx \geq \frac{\mu e^1}{4} > 0. \quad (3.38)$$

which concludes the claim. So we may define $c_1 = \inf\{J(u) : u \in K\}$. According to (3.38), we get $c_1 > 0$. Let $\{u_n\} \subseteq K$ be such that $J(u_n) \rightarrow c_1$, as $n \rightarrow \infty$. By Lemma 5, we know that $\{u_n\}$ is bounded. Therefore, there exists a subsequence (still denote $\{u_n\}$) and $u_n \rightarrow u_0$ in $H_0^1(\Omega)$. By the continuation of J' , we imply that $J(u_0) = c_1$ and $J'(u_0) = 0$. Then, u_0 is a ground state solution of equation (1.1). This completes the proof of Theorem 3. $\square$

Proof of Remark 3. Define $K_1 = \{u \in H_0^s(\Omega) \setminus \{0\} : J'(u) = 0\}$. Similarly to proof of Lemma 3 and Lemma 5, we know that K_1 is nonempty. So for any $u \in K_1$, according to the condition (H_1) , fractional logarithmic Sobolev inequality and similarly to proof of Lemma 2, we have

$$\begin{aligned} 0 &= \langle J'(u), u \rangle \\ &\geq \delta_1 \int_{\Omega} |(-\Delta)^{s/2} u|^2 dx - \int_{\Omega} H u^2 \log \frac{|u|}{|u|_2} dx \\ &\quad - \int_{\Omega} H u^2 \log |u|_2 dx + \int_{\Omega} H u^2 dx - \int_{\Omega} H u^2 dx \\ &\geq \left(\delta_1 - \frac{a^2 M}{\pi^s} \right) \int_{\Omega} (-\Delta)^{s/2} u|^2 dx + M \left(3 + \frac{3}{s} \log a \right. \\ &\quad \left. + \log \frac{s \Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2s})} - 2|\Omega|^{\frac{1}{2}} - 1 \right) |u|_2^2 + (1 - \log |u|_2) \int_{\Omega} H u^2 dx, \end{aligned} \quad (3.39)$$

where $\delta_1 = 1 - S^{-1} |V^-|_{3/2s}$. Take $a = \left(\frac{\Gamma(\frac{3}{2s})}{(\frac{3}{2})} e^{\frac{2|\Omega|^{1/2}-2}{3}} \right)^{3/s}$, according to (3.39), we deduce

$$1 - \log |u|_2 \leq 0. \quad (3.40)$$

It follows from (3.40) that $e^1 \leq \int_{\Omega} u^2$. Next, we claim that J is bounded from below on K . For any $u \in K$, we obtain

$$J(u) - \frac{1}{2} \langle J'(u), u \rangle = \frac{1}{4} \int_{\Omega} H u^2 dx \geq \frac{\mu e^1}{4} > 0. \quad (3.41)$$

which concludes the claim. So we may define $c_1 = \inf\{J(u) : u \in K\}$. According to (3.41), we get $c_1 > 0$. Let $\{u_n\} \subseteq K$ be such that $J(u_n) \rightarrow c_1$, as $n \rightarrow \infty$. Similarly to proof of Lemma 5, we know that $\{u_n\}$ is bounded. Therefore, there exists a subsequence (still denote $\{u_n\}$) and $u_n \rightarrow u_0$ in $H_0^s(\Omega)$. By the continuation of J' , we imply that $J(u_0) = c_1$ and $J'(u_0) = 0$. Then, u_0 is a ground state solution. This completes the proof of Remark 3. $\square$

4. CONCLUSION

This paper considers the existence of ground state solutions for Choquard equation with logarithmic nonlinearity and the limit of behaviors of ground solution to (1.1) as either $\alpha \rightarrow 0$ or $\alpha \rightarrow N$ by variational method and logarithmic inequality. In addition, we obtain a new existence theorem through the property of logarithmic nonlinearity. Note that fractional Choquard has been widely studied in recent years. In the forthcoming paper, we will consider the existence of ground state solutions for fractional Choquard equation with logarithmic nonlinearity and sign-change potential function on the whole space by variational method and fractional logarithmic inequality. Lacking compactness will be the biggest difficulty.

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NEW INEQUALITIES OF WIRTINGER TYPE FOR DIFFERENT KINDS OF CONVEX FUNCTIONS

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Abstract. T.Z. Mirković [14] obtained new inequalities of Wirtinger type by using some classical inequalities and special means for convex function. So in this paper, we obtain some inequalities of Wirtinger type for s -convex function, m -convex function, (α, m) -convex function, quasi-convex function and P -function. Also several special cases are discussed, which can be deduced from our main results.

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1. INTRODUCTION

W. Wirtinger proved the following Theorem 1 regarding periodic functions. The proof of Wirtinger was published in 1916 in the book [5] by W. Blaschke.

Theorem 1. *Let f be a periodic function with period 2π and let $f' \in L^2$. Then, if $\int_0^{2\pi} f(x)dx = 0$, the following inequality holds*

$$\int_0^{2\pi} f^2(x)dx \leq \int_0^{2\pi} (f')^2(x)dx \quad (1.1)$$

with equality if and only if $f(x) = A \cos x + B \sin x$, where A and B are constants.

Inequality (1.1) is known in the literature as Wirtinger's inequality. Wirtinger's inequality compares the integral of a square of a function with that of a square of its first derivative. Last years, a large number of papers which generalize and extend Wirtinger's inequality have been appeared in the literature (see [1], [4], [13] [15], [3], [12], [17], [19]).

In 1905, E. Almansi proved the following theorem [2].

Theorem 2. Let f and f' are continuous the interval (a, b) , that $f(a) = f(b)$ and that $\int_a^b f(x)dx = 0$ then the following inequality holds

$$\int_a^b f^2(x)dx \leq \left(\frac{b-a}{2\pi}\right)^2 \int_a^b [f'(x)]^2 dx. \quad (1.2)$$

We recall some previously known definitions of different type of convexity.

Definition 1. The function $f : [a, b] \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be convex if the following inequality holds

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$. We say that f is concave if $(-f)$ is convex.

Definition 2. (see [8],[16]) Let $0 < s \leq 1$. A function $f : [0, \infty) \rightarrow \mathbb{R}$ is said to be s -Orlicz convex or s -convex in the first sense, if for every $x, y \in [0, \infty)$ and $\alpha, \beta \geq 0$ with $\alpha^s + \beta^s = 1$, we have:

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y). \quad (1.3)$$

We denote the set of all s -convex functions in the first sense by K_s^1 .

Definition 3. (see [6],[11]) Let $0 < s \leq 1$. A function $f : [0, \infty) \rightarrow \mathbb{R}$, is said to be s -Breckner convex or s -convex in the second sense, if for every $x, y \in [0, \infty)$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$, we have:

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y). \quad (1.4)$$

The set of all s -convex functions in the second sense is denoted by K_s^2 .

Definition 4. ([7]) A function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is P -function or that f belongs to the class of $P(I)$, if it is nonnegative and, for all $x, y \in I$ and $\lambda \in [0, 1]$, satisfies the following inequality;

$$f(\lambda x + (1 - \lambda)y) \leq f(x) + f(y). \quad (1.5)$$

Definition 5. (see, e.g., [9]) The function $f : [0, b] \rightarrow \mathbb{R}$ is said to be m -convex, where $m \in [0, 1]$, if for every $x, y \in [0, b]$ and $t \in [0, 1]$ we have:

$$f(tx + m(1 - t)y) \leq tf(x) + m(1 - t)f(y).$$

Denote by $K_m(b)$ the set of the m -convex functions on $[0, b]$ for which $f(0) \leq 0$.

Definition 6. (see, e.g., [9]) The function $f : [0, b] \rightarrow \mathbb{R}$ is said to be (α, m) -convex, where $(\alpha, m) \in [0, 1]^2$, if for every $x, y \in [0, b]$ and $t \in [0, 1]$ we have:

$$f(tx + m(1 - t)y) \leq t^\alpha f(x) + m(1 - t^\alpha)f(y).$$

Denote by $K_m^\alpha(b)$ the set of the (α, m) -convex functions on $[0, b]$ for which $f(0) \leq 0$.

Definition 7. [10] Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ for all $t \in [0, 1]$ and all $x, y \in I$, if the following inequality

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\}$$

holds, then f is called a quasi-convex function on I .

We recall definition of Beta function (see, e.g., [18])

Definition 8. Assume that $\Re(a) > 0$ and $\Re(b) > 0$, the Beta function is denoted by $B(a, b)$ and defined as

$$B(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1} dt.$$

An important property connecting the Gamma and Beta functions can be stated as following:

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

T.Z. Mirković [14] proved the following theorems involving inequalities of Wirtinger type for convex functions.

Theorem 3. Let f and f' are continuous on the interval (a, b) , with $f(a) = f(b)$ and $\int_a^b f(x)dx = 0$. If $(f')^2$ is convex on $[a, b]$, then the following inequality holds

$$\int_a^b [f(x)]^2 dx \leq \frac{(b-a)^3}{8\pi^2} ([f'(a)]^2 + [f'(b)]^2). \quad (1.6)$$

Theorem 4. Let f and f' are continuous on the interval (a, b) , with $f(a) = f(b)$ and $\int_a^b f(x)dx = 0$. If f' is convex on $[a, b]$, then the following inequality holds

$$\int_a^b [f(x)]^2 dx \leq \frac{(b-a)^3}{(2\pi)^2} \left(\frac{[f'(a)]^2 + [f'(a)][f'(b)] + [f'(b)]^2}{3} \right). \quad (1.7)$$

Theorem 5. Let f and f' are continuous on the interval (a, b) , with $f(a) = f(b)$ and $\int_a^b f(x)dx = 0$. If f' is positive, $(f')^{\frac{1}{\alpha}}$ and $(f')^{\frac{1}{\beta}}$ are convex on $[a, b]$, then the following inequality holds

$$\begin{aligned} \int_a^b [f(x)]^2 dx \\ \leq \alpha(b-a)^3 \frac{[f'(a)]^{\frac{1}{\alpha}} + [f'(b)]^{\frac{1}{\alpha}}}{8\pi^2} + \beta(b-a)^3 \frac{[f'(a)]^{\frac{1}{\beta}} + [f'(b)]^{\frac{1}{\beta}}}{8\pi^2} \end{aligned} \quad (1.8)$$

where $\alpha, \beta > 0$ and $\alpha + \beta = 1$.

2. MAIN RESULTS

In this section, we obtained some new inequalities of Wirtinger type for different kinds of convex functions.

Theorem 6. *Let f and f' are continuous on the interval (a, b) , with $f(a) = f(b)$ and $\int_a^b f(x)dx = 0$. If $(f')^2$ is P -function on $[a, b]$, then the following inequality holds:*

$$\int_a^b [f(x)]^2 dx \leq \frac{(b-a)^3}{(2\pi)^2} ([f'(a)]^2 + [f'(b)]^2). \quad (2.1)$$

Proof. Since $(f')^2$ is a P -function on $[a, b]$, therefore for $t \in [0, 1]$ we have

$$\begin{aligned} \frac{1}{b-a} \int_a^b [f'(x)]^2 dx &= \int_0^1 [f'(ta + (1-t)b)]^2 dt \\ &\leq \int_0^1 ([f'(a)]^2 + [f'(b)]^2) dt \\ &= [f'(a)]^2 + [f'(b)]^2. \end{aligned}$$

Multiplying the both sides of above inequality by $\frac{(b-a)^3}{(2\pi)^2}$, we get

$$\left(\frac{b-a}{2\pi}\right)^2 \int_a^b [f'(x)]^2 dx \leq \frac{(b-a)^3}{(2\pi)^2} ([f'(a)]^2 + [f'(b)]^2).$$

By using inequality (1.2), we get inequality (2.1) and the proof is completed. $\square$

Theorem 7. *Let f and f' are continuous on the interval (a, b) , $0 \leq a < b$, with $f(a) = f(b)$ and $\int_a^b f(x)dx = 0$. If $(f')^2$ is s -convex in the second sense on $[a, b]$, then the following inequality holds:*

$$\int_a^b [f(x)]^2 dx \leq \frac{(b-a)^3}{(2\pi)^2} \left(\frac{[f'(a)]^2 + [f'(b)]^2}{s+1} \right) \quad \text{where } s \in (0, 1]. \quad (2.2)$$

Proof. Since $(f')^2$ is a s -convex function on $[a, b]$, therefore for $t \in [0, 1]$, we have

$$\begin{aligned} \frac{1}{b-a} \int_a^b [f'(x)]^2 dx &= \int_0^1 [f'(ta + (1-t)b)]^2 dt \\ &\leq \int_0^1 [t^s [f'(a)]^2 + (1-t)^s [f'(b)]^2] dt \\ &= \frac{1}{s+1} [f'(a)]^2 + \frac{1}{s+1} [f'(b)]^2 \\ &= \frac{[f'(a)]^2 + [f'(b)]^2}{s+1}. \end{aligned}$$

By multiplying by $\frac{(b-a)^3}{(2\pi)^2}$, the both sides of above inequality, we can write

$$\left(\frac{b-a}{2\pi}\right)^2 \int_a^b [f'(x)]^2 dx \leq \frac{(b-a)^3}{(2\pi)^2} \frac{[f'(a)]^2 + [f'(b)]^2}{(s+1)}.$$

With the help of the inequality (1.2), we obtain (2.2) and the proof is completed. $\square$

Remark 1. In Theorem 7, if we choose $s = 1$, the inequality (2.2) reduces to the inequality (1.6).

Theorem 8. Let f and f' are continuous on the interval (a, b) , with $f(a) = f(b)$ and $\int_a^b f(x)dx = 0$. If $(f')^2$ is quasi-convex on $[a, b]$, then the following inequality holds:

$$\int_a^b [f(x)]^2 dx \leq \frac{(b-a)^3}{(2\pi)^2} \max \{ [f'(a)]^2, [f'(b)]^2 \}. \quad (2.3)$$

Proof. Since $(f')^2$ is a quasi-convex on $[a, b]$, we have

$$\begin{aligned} \frac{1}{b-a} \int_a^b [f'(x)]^2 dx &= \int_0^1 [f'(ta + (1-t)b)]^2 dt \\ &\leq \int_0^1 \max \{ [f'(a)]^2, [f'(b)]^2 \} dt \\ &= \max \{ [f'(a)]^2, [f'(b)]^2 \} \end{aligned}$$

for $t \in [0, 1]$. Multiplying the both sides of above inequality by $\frac{(b-a)^3}{(2\pi)^2}$, we get

$$\left(\frac{b-a}{2\pi}\right)^2 \int_a^b [f'(x)]^2 dx \leq \frac{(b-a)^3}{(2\pi)^2} \max \{ [f'(a)]^2, [f'(b)]^2 \}.$$

By using inequality (1.2) for the resulting inequality, the proof is completed. $\square$

Theorem 9. Let f and f' are continuous on the interval (a, mb) , $0 \leq a < mb$, with $f(a) = f(mb)$ and $\int_a^{mb} f(x)dx = 0$. If $(f')^2$ is m -convex function on $[a, b]$, then the following inequality holds

$$\int_a^{mb} [f(x)]^2 dx \leq \frac{(mb-a)^3}{(2\pi)^2} \left[\frac{[f'(a)]^2 + m[f'(b)]^2}{2} \right] \quad (2.4)$$

for $m \in [0, 1]$.

Proof. By using the m -convexity of $(f')^2$ on $[a, b]$, we have

$$\begin{aligned} \frac{1}{mb-a} \int_a^{mb} [f'(x)]^2 dx &= \int_0^1 [f'(ta + m(1-t)b)]^2 dt \\ &\leq \int_0^1 (t[f'(a)]^2 + m(1-t)[f'(b)]^2) dt \\ &= \frac{[f'(a)]^2 + m[f'(b)]^2}{2} \end{aligned}$$

for $t \in [0, 1]$. Multiplying the both sides of above inequality by $\frac{(mb-a)^3}{(2\pi)^2}$, we get

$$\left(\frac{mb-a}{2\pi}\right)^2 \int_a^{mb} [f'(x)]^2 dx \leq \frac{(mb-a)^3}{(2\pi)^2} \left[\frac{[f'(a)]^2 + m[f'(b)]^2}{2} \right]$$

By using inequality (1.2), we conclude the desired result. $\square$

Remark 2. In Theorem 9, if we set $m = 1$, the inequality (2.4) reduces to the inequality (1.6).

Theorem 10. Let f and f' are continuous on the interval (a, mb) , $0 \leq a < mb$, with $f(a) = f(mb)$ and $\int_a^{mb} f(x)dx = 0$. If $(f')^2$ is (α, m) -convex on $[a, b]$, then the following inequality holds:

$$\int_a^{mb} [f(x)]^2 dx \leq \frac{(mb-a)^3}{(2\pi)^2} \left[\frac{[f'(a)]^2 + \alpha m [f'(b)]^2}{\alpha + 1} \right] \quad (2.5)$$

where $(\alpha, m) \in [0, 1]^2$.

Proof. Since $(f')^2$ is a (α, m) -convex on $[a, b]$, we have

$$\begin{aligned} \frac{1}{mb-a} \int_a^{mb} [f'(x)]^2 dx &= \int_0^1 [f'(ta + m(1-t)b)]^2 dt \\ &\leq \int_0^1 \left(t^\alpha [f'(a)]^2 + m(1-t^\alpha) [f'(b)]^2 \right) dt \\ &= \frac{1}{\alpha+1} [f'(a)]^2 + m \frac{\alpha}{\alpha+1} [f'(b)]^2 \end{aligned}$$

for $t \in [0, 1]$ and $(\alpha, m) \in [0, 1]^2$. Multiplying the both sides of above inequality by $\frac{(mb-a)^3}{(2\pi)^2}$ and using inequality (1.2) for the resulting inequality, we get the required inequality. $\square$

Remark 3. In Theorem 10, if we take $m = 1$ and $\alpha = 1$ the inequality (2.5) reduces to the inequality (1.6).

Theorem 11. Let f and f' are continuous on the interval (a, b) , with $f(a) = f(b)$ and $\int_a^b f(x)dx = 0$. If f' is P -function on $[a, b]$, then the following inequality holds:

$$\int_a^b [f(x)]^2 dx \leq \frac{(b-a)^3}{(2\pi)^2} [f'(a) + f'(b)]^2. \quad (2.6)$$

Proof. By using the change of the variable, it is easy to see that

$$\begin{aligned} \left(\frac{b-a}{2\pi}\right)^2 \int_a^b [f'(x)]^2 dx &= \frac{(b-a)^3}{(2\pi)^2} \int_0^1 [f'(ta + (1-t)b)]^2 dt \\ &\leq \frac{(b-a)^3}{(2\pi)^2} \int_0^1 [f'(a) + f'(b)]^2 dt \end{aligned}$$

$$= \frac{(b-a)^3}{(2\pi)^2} [f'(a) + f'(b)]^2.$$

By using inequality (1.2), we get inequality (2.6) and the proof is completed. $\square$

Theorem 12. Let f and f' be continuous on the interval (a, b) , with $f(a) = f(b)$ and $\int_a^b f(x)dx = 0$. If f' is quasi-convex function on $[a, b]$, then the following inequality holds:

$$\int_a^b [f(x)]^2 dx \leq \frac{(b-a)^3}{(2\pi)^2} [\max \{f'(a), f'(b)\}]^2. \quad (2.7)$$

Proof. From the definition of quasi-convex functions and by using the change of the variable, we have

$$\begin{aligned} \left(\frac{b-a}{2\pi}\right)^2 \int_a^b [f'(x)]^2 dx &= \frac{(b-a)^3}{(2\pi)^2} \int_0^1 [f'(ta + (1-t)b)]^2 dt \\ &\leq \frac{(b-a)^3}{(2\pi)^2} \int_0^1 [\max \{f'(a), f'(b)\}]^2 dt \\ &= \frac{(b-a)^3}{(2\pi)^2} [\max \{f'(a), f'(b)\}]^2. \end{aligned}$$

By using inequality (1.2) in the above inequality, the proof is completed. $\square$

Theorem 13. Let f and f' be continuous on the interval (a, mb) , $0 \leq a < mb$, with $f(a) = f(mb)$ and $\int_a^{mb} f(x)dx = 0$. If f' is m -convex function on $[a, b]$, then the following inequality holds:

$$\int_a^{mb} [f(x)]^2 dx \leq \frac{(mb-a)^3}{(2\pi)^2} \left[\frac{[f'(a)]^2 + m[f'(a)][f'(b)] + m^2[f'(b)]^2}{3} \right] \quad (2.8)$$

for $m \in [0, 1]$.

Proof. By using the m -convexity of f' on $[a, b]$, we have

$$\begin{aligned} \left(\frac{mb-a}{2\pi}\right)^2 \int_a^{mb} [f'(x)]^2 dx &= \frac{(mb-a)^3}{(2\pi)^2} \int_0^1 [f'(ta + m(1-t)b)]^2 dt \\ &\leq \frac{(mb-a)^3}{(2\pi)^2} \int_0^1 [tf'(a) + m(1-t)f'(b)]^2 dt \\ &= \frac{(mb-a)^3}{(2\pi)^2} \int_0^1 [t^2[f'(a)]^2 \\ &\quad + (2mt - 2mt^2)f'(a)f'(b) + (m^2 - 2m^2t + m^2t^2)[f'(b)]^2] dt \\ &= \frac{(mb-a)^3}{(2\pi)^2} \left[\frac{[f'(a)]^2 + m[f'(a)][f'(b)] + m^2[f'(b)]^2}{3} \right]. \end{aligned}$$

By using inequality (1.2) we get inequality (2.8) and the proof is complete. $\square$

Remark 4. In Theorem 13, if we take $m = 1$, the inequality (2.8) reduces to the inequality (1.7)

Theorem 14. Let f and f' are continuous on the interval (a, mb) , $0 \leq a < mb$, with $f(a) = f(mb)$ and $\int_a^{mb} f(x)dx = 0$. If f' is (α, m) -convex function on $[a, b]$, then the following inequality holds:

$$\int_a^{mb} [f(x)]^2 dx \leq \frac{(mb-a)^3}{(2\pi)^2} \left[\frac{(\alpha+1)[f'(a)]^2 + 2m\alpha[f'(a)][f'(b)] + 2\alpha^2 m^2 [f'(b)]^2}{(\alpha+1)(2\alpha+1)} \right] \quad (2.9)$$

$(\alpha, m) \in [0, 1]^2$.

Proof. From the definition of f' and by using the change of the variable, we have

$$\begin{aligned} \left(\frac{mb-a}{2\pi} \right)^2 \int_a^{mb} [f'(x)]^2 dx &= \frac{(mb-a)^3}{(2\pi)^2} \int_0^1 [f'(ta + m(1-t)b)]^2 dt \\ &\leq \frac{(mb-a)^3}{(2\pi)^2} \int_0^1 [t^\alpha f'(a) + m(1-t^\alpha) f'(b)]^2 dt \\ &= \frac{(mb-a)^3}{(2\pi)^2} \int_0^1 [t^{2\alpha} [f'(a)]^2 \\ &\quad + (2mt^\alpha - 2mt^{2\alpha}) f'(a) f'(b) + (m^2 - 2m^2 t^\alpha + m^2 t^{2\alpha}) [f'(b)]^2] dt \\ &= \frac{(mb-a)^3}{(2\pi)^2} \left[\frac{(\alpha+1)[f'(a)]^2 + 2m\alpha[f'(a)][f'(b)] + 2\alpha^2 m^2 [f'(b)]^2}{(\alpha+1)(2\alpha+1)} \right]. \end{aligned}$$

By a similar argument to the proof of previous theorems, by using inequality (1.2), we get the desired result. $\square$

Remark 5. In Theorem 14, if we take $m = 1$, $\alpha = 1$ the inequality (2.9) reduces to the inequality (1.7)

Theorem 15. Let f and f' are continuous on the interval (a, b) , $0 \leq a < b$, with $f(a) = f(b)$ and $\int_a^b f(x)dx = 0$. If f' is s -convex function in the second sense on $[a, b]$, then the following inequality holds:

$$\begin{aligned} \int_a^b [f(x)]^2 dx &\quad (2.10) \\ &\leq \frac{(b-a)^3}{(2\pi)^2} \left[\frac{1}{2s+1} [f'(a)]^2 + 2B(s+1, s+1) f'(a) f'(b) + \frac{1}{2s+1} [f'(b)]^2 \right] \end{aligned}$$

for $s \in (0, 1]$.

Proof. Since f' is s -convex function in the second sense, we can write

$$\left(\frac{b-a}{2\pi} \right)^2 \int_a^b [f'(x)]^2 dx$$

$$\begin{aligned}
&= \frac{(b-a)^3}{(2\pi)^2} \int_0^1 [f'(ta + (1-t)b)]^2 dt \\
&\leq \frac{(b-a)^3}{(2\pi)^2} \int_0^1 [t^s f'(a) + (1-t)^s f'(b)]^2 dt \\
&= \frac{(b-a)^3}{(2\pi)^2} \int_0^1 [t^{2s} [f'(a)]^2 + 2t^s(1-t)^s f'(a)f'(b) + (1-t)^{2s} [f'(b)]^2] dt \\
&= \frac{(b-a)^3}{(2\pi)^2} \left[[f'(a)]^2 \int_0^1 t^{2s} dt + 2f'(a)f'(b) \int_0^1 t^s(1-t)^s dt \right. \\
&\quad \left. + [f'(b)]^2 \int_0^1 (1-t)^{2s} dt \right] \\
&= \frac{(b-a)^3}{(2\pi)^2} \left[\frac{1}{2s+1} [f'(a)]^2 + 2B(s+1, s+1) f'(a)f'(b) + \frac{1}{2s+1} [f'(b)]^2 \right].
\end{aligned}$$

By using inequality (1.2), we get inequality (2.10) and the proof is complete. $\square$

Remark 6. In Theorem 15, if we take $s = 1$, the inequality (2.10) reduces to the inequality (1.7)

Theorem 16. Let f and f' are continuous on the interval (a, b) , $0 \leq a < b$, with $f(a) = f(b)$ and $\int_a^b f(x)dx = 0$. If f' is positive, $(f')^{\frac{1}{\alpha}}$ and $(f')^{\frac{1}{\beta}}$ are s -convex in the second sense on $[a, b]$, then the following inequality holds:

$$\begin{aligned}
&\int_a^b [f(x)]^2 dx \\
&\leq \frac{(b-a)^3}{(2\pi)^2} \left\{ \frac{\alpha}{s+1} \left[(f'(a))^{\frac{1}{\alpha}} + (f'(b))^{\frac{1}{\alpha}} \right] + \frac{\beta}{s+1} \left[(f'(a))^{\frac{1}{\beta}} + (f'(b))^{\frac{1}{\beta}} \right] \right\}
\end{aligned} \tag{2.11}$$

where $s \in (0, 1]$, $\alpha, \beta > 0$ and $\alpha + \beta = 1$.

Proof. From the definition of $(f')^{\frac{1}{\alpha}}$ and $(f')^{\frac{1}{\beta}}$ on $[a, b]$, by using inequality $cd \leq \alpha c^{\frac{1}{\alpha}} + \beta d^{\frac{1}{\beta}}$, $\alpha, \beta, c, d > 0$ and $\alpha + \beta = 1$, we get

$$\begin{aligned}
&\left(\frac{b-a}{2\pi} \right)^2 \int_a^b [f'(x)]^2 dx \\
&= \frac{(b-a)^3}{(2\pi)^2} \int_0^1 f'(ta + (1-t)b) f'(ta + (1-t)b) dt \\
&\leq \frac{(b-a)^3}{(2\pi)^2} \left\{ \alpha \int_0^1 [f'(ta + (1-t)b)]^{\frac{1}{\alpha}} dt + \beta \int_0^1 [f'(ta + (1-t)b)]^{\frac{1}{\beta}} dt \right\} \\
&\leq \frac{(b-a)^3}{(2\pi)^2} \left\{ \alpha \int_0^1 [t^s (f'(a))^{\frac{1}{\alpha}} + (1-t)^s (f'(b))^{\frac{1}{\alpha}}] dt \right.
\end{aligned}$$

$$\begin{aligned}
& + \beta \int_0^1 \left[t^s (f'(a))^{\frac{1}{\beta}} + (1-t)^s (f'(b))^{\frac{1}{\beta}} \right] dt \Bigg\} \\
& = \frac{(b-a)^3}{(2\pi)^2} \left\{ \alpha \left[\frac{1}{s+1} (f'(a))^{\frac{1}{\alpha}} + \frac{1}{s+1} (f'(b))^{\frac{1}{\alpha}} \right] \right. \\
& \quad \left. + \beta \left[\frac{1}{s+1} (f'(a))^{\frac{1}{\beta}} + \frac{1}{s+1} (f'(b))^{\frac{1}{\beta}} \right] \right\} \\
& = \frac{(b-a)^3}{(2\pi)^2} \left\{ \frac{\alpha}{s+1} \left[(f'(a))^{\frac{1}{\alpha}} + (f'(b))^{\frac{1}{\alpha}} \right] + \frac{\beta}{s+1} \left[(f'(a))^{\frac{1}{\beta}} + (f'(b))^{\frac{1}{\beta}} \right] \right\}.
\end{aligned}$$

By applying (1.2), we get required inequality (2.11) and the proof is complete. $\square$

Remark 7. In Theorem 16, if we take $s = 1$, the inequality (2.11) reduces to the inequality (1.8)

Theorem 17. Let f and f' are continuous on the interval (a, b) , with $f(a) = f(b)$ and $\int_a^b f(x) dx = 0$. If f' is positive, $(f')^{\frac{1}{\alpha}}$ and $(f')^{\frac{1}{\beta}}$ are P -function on $[a, b]$, then the following inequality holds:

$$\begin{aligned}
& \int_a^b [f(x)]^2 dx \\
& \leq \frac{(b-a)^3}{(2\pi)^2} \left\{ \alpha \left[(f'(a))^{\frac{1}{\alpha}} + (f'(b))^{\frac{1}{\alpha}} \right] + \beta \left[(f'(a))^{\frac{1}{\beta}} + (f'(b))^{\frac{1}{\beta}} \right] \right\}
\end{aligned} \tag{2.12}$$

where $\alpha, \beta > 0$ and $\alpha + \beta = 1$.

Proof. By using the inequality $cd \leq \alpha c^{\frac{1}{\alpha}} + \beta d^{\frac{1}{\beta}}$, $\alpha, \beta, c, d > 0$ and $\alpha + \beta = 1$ we get

$$\begin{aligned}
& \left(\frac{b-a}{2\pi} \right)^2 \int_a^b [f'(x)]^2 dx \\
& = \frac{(b-a)^3}{(2\pi)^2} \int_0^1 f'(ta + (1-t)b) f'(ta + (1-t)b) dt \\
& \leq \frac{(b-a)^3}{(2\pi)^2} \left\{ \alpha \int_0^1 \left[f'(ta + (1-t)b) \right]^{\frac{1}{\alpha}} dt + \beta \int_0^1 \left[f'(ta + (1-t)b) \right]^{\frac{1}{\beta}} dt \right\} \\
& \leq \frac{(b-a)^3}{(2\pi)^2} \left\{ \alpha \int_0^1 \left[(f'(a))^{\frac{1}{\alpha}} + (f'(b))^{\frac{1}{\alpha}} \right] dt \right. \\
& \quad \left. + \beta \int_0^1 \left[(f'(a))^{\frac{1}{\beta}} + (f'(b))^{\frac{1}{\beta}} \right] dt \right\}
\end{aligned}$$

$$= \frac{(b-a)^3}{(2\pi)^2} \left\{ \alpha \left[(f'(a))^{\frac{1}{\alpha}} + (f'(b))^{\frac{1}{\alpha}} \right] + \beta \left[(f'(a))^{\frac{1}{\beta}} + (f'(b))^{\frac{1}{\beta}} \right] \right\}.$$

By applying (1.2), we get required inequality (2.12) and the proof is complete. $\square$

Theorem 18. Let f and f' are continuous on the interval (a, b) , with $f(a) = f(b)$ and $\int_a^b f(x)dx = 0$. If f' is positive, $(f')^{\frac{1}{\alpha}}$ and $(f')^{\frac{1}{\beta}}$ are quasi-convex function on $[a, b]$, then the following inequality holds

$$\begin{aligned} \int_a^b [f(x)]^2 dx & \\ & \leq \frac{(b-a)^3}{(2\pi)^2} \left\{ \alpha \left[\max \left\{ (f'(a))^{\frac{1}{\alpha}}, (f'(b))^{\frac{1}{\alpha}} \right\} \right] + \beta \left[\max \left\{ (f'(a))^{\frac{1}{\beta}}, (f'(b))^{\frac{1}{\beta}} \right\} \right] \right\} \end{aligned} \quad (2.13)$$

where $\alpha, \beta > 0$ and $\alpha + \beta = 1$.

Proof. By a similar way to the previous theorem, but now by using the quasi-convexity of $(f')^{\frac{1}{\alpha}}$ and $(f')^{\frac{1}{\beta}}$ on $[a, b]$, we get

$$\begin{aligned} \left(\frac{b-a}{2\pi} \right)^2 \int_a^b [f'(x)]^2 dx & \\ & = \frac{(b-a)^3}{(2\pi)^2} \int_0^1 f'(ta + (1-t)b) f'(ta + (1-t)b) dt \\ & \leq \frac{(b-a)^3}{(2\pi)^2} \left\{ \alpha \int_0^1 \left[f'(ta + (1-t)b) \right]^{\frac{1}{\alpha}} dt + \beta \int_0^1 \left[f'(ta + (1-t)b) \right]^{\frac{1}{\beta}} dt \right\} \\ & \leq \frac{(b-a)^3}{(2\pi)^2} \left\{ \alpha \int_0^1 \max \left\{ (f'(a))^{\frac{1}{\alpha}}, (f'(b))^{\frac{1}{\alpha}} \right\} dt \right. \\ & \quad \left. + \beta \int_0^1 \max \left\{ (f'(a))^{\frac{1}{\beta}}, (f'(b))^{\frac{1}{\beta}} \right\} dt \right\} \\ & = \frac{(b-a)^3}{(2\pi)^2} \left\{ \alpha \left[\max \left\{ (f'(a))^{\frac{1}{\alpha}}, (f'(b))^{\frac{1}{\alpha}} \right\} \right] \right. \\ & \quad \left. + \beta \left[\max \left\{ (f'(a))^{\frac{1}{\beta}}, (f'(b))^{\frac{1}{\beta}} \right\} \right] \right\}. \end{aligned}$$

By applying (1.2), we get required inequality (2.13) and the proof is complete. $\square$

Theorem 19. Let f and f' are continuous on the interval (a, mb) , $0 \leq a < mb$, with $f(a) = f(mb)$ and $\int_a^{mb} f(x)dx = 0$. If f' is positive, $(f')^{\frac{1}{\alpha}}$ and $(f')^{\frac{1}{\beta}}$ are m -convex function on $[a, b]$, then the following inequality holds

$$\int_a^{mb} [f(x)]^2 dx \quad (2.14)$$

$$\leq \alpha(mb-a)^3 \frac{[f'(a)]^{\frac{1}{\alpha}} + m[f'(b)]^{\frac{1}{\alpha}}}{8\pi^2} + \beta(mb-a)^3 \frac{[f'(a)]^{\frac{1}{\beta}} + m[f'(b)]^{\frac{1}{\beta}}}{8\pi^2}$$

where $m \in [0, 1]$, $\alpha, \beta > 0$ and $\alpha + \beta = 1$.

Proof. By using the inequality $cd \leq \alpha c^{\frac{1}{\alpha}} + \beta d^{\frac{1}{\beta}}$, $\alpha, \beta, c, d > 0$ and $\alpha + \beta = 1$ and m -convexity of $(f')^{\frac{1}{\alpha}}$ and $(f')^{\frac{1}{\beta}}$ on $[a, b]$, one can easily write

$$\begin{aligned} & \left(\frac{mb-a}{2\pi} \right)^2 \int_a^{mb} [f'(x)]^2 dx \\ &= \frac{(mb-a)^3}{(2\pi)^2} \int_0^1 f'(ta+m(1-t)b) f'(ta+m(1-t)b) dt \\ &\leq \frac{(mb-a)^3}{(2\pi)^2} \left\{ \alpha \int_0^1 [f'(ta+m(1-t)b)]^{\frac{1}{\alpha}} dt \right. \\ &\quad \left. + \beta \int_0^1 [f'(ta+m(1-t)b)]^{\frac{1}{\beta}} dt \right\} \\ &\leq \frac{(mb-a)^3}{(2\pi)^2} \left\{ \alpha \int_0^1 \left[t(f'(a))^{\frac{1}{\alpha}} + m(1-t)(f'(b))^{\frac{1}{\alpha}} \right] dt \right. \\ &\quad \left. + \beta \int_0^1 \left[t(f'(a))^{\frac{1}{\beta}} + m(1-t)(f'(b))^{\frac{1}{\beta}} \right] dt \right\} \\ &= \frac{(mb-a)^3}{(2\pi)^2} \left\{ \alpha \left(\frac{[f'(a)]^{\frac{1}{\alpha}} + m[f'(b)]^{\frac{1}{\alpha}}}{2} \right) \right. \\ &\quad \left. + \beta \left(\frac{[f'(a)]^{\frac{1}{\beta}} + m[f'(b)]^{\frac{1}{\beta}}}{2} \right) \right\}. \end{aligned}$$

By applying (1.2), we get required inequality (2.14) and the proof is complete. $\square$

Remark 8. In Theorem 19, if we take $m = 1$, the inequality (2.14) reduces to the inequality (1.8)

Theorem 20. Let f and f' are continuous on the interval (a, mb) , $0 \leq a < mb$, with $f(a) = f(mb)$ and $\int_a^{mb} f(x) dx = 0$. If f' is positive, $(f')^{\frac{1}{\alpha}}$ and $(f')^{\frac{1}{\beta}}$ are (α, m) -convex function on $[a, b]$, then the following inequality holds

$$\begin{aligned} & \int_a^{mb} [f(x)]^2 dx \\ &\leq \frac{(mb-a)^3}{(2\pi)^2} \left\{ \theta \left(\frac{[f'(a)]^{\frac{1}{\theta}} + m\alpha[f'(b)]^{\frac{1}{\theta}}}{\alpha+1} \right) + \beta \left(\frac{[f'(a)]^{\frac{1}{\beta}} + m\alpha[f'(b)]^{\frac{1}{\beta}}}{\alpha+1} \right) \right\} \end{aligned} \quad (2.15)$$

where $(\alpha, m) \in [0, 1]^2$, $\theta, \beta > 0$ and $\theta + \beta = 1$.

Proof. By using the same inequality and the similar computations to the proof of the Theorem 19, we have

$$\begin{aligned}
 & \left(\frac{mb-a}{2\pi} \right)^2 \int_a^{mb} [f'(x)]^2 dx \\
 &= \frac{(mb-a)^3}{(2\pi)^2} \int_0^1 f'(ta+m(1-t)b) f'(ta+m(1-t)b) dt \\
 &\leq \frac{(mb-a)^3}{(2\pi)^2} \left\{ \theta \int_0^1 [f'(ta+m(1-t)b)]^{\frac{1}{\theta}} dt \right. \\
 &\quad \left. + \beta \int_0^1 [f'(ta+m(1-t)b)]^{\frac{1}{\beta}} dt \right\} \\
 &\leq \frac{(mb-a)^3}{(2\pi)^2} \left\{ \theta \int_0^1 \left[t^\alpha (f'(a))^{\frac{1}{\theta}} + m(1-t^\alpha) (f'(b))^{\frac{1}{\theta}} \right] dt \right. \\
 &\quad \left. + \beta \int_0^1 \left[t^\alpha (f'(a))^{\frac{1}{\beta}} + m(1-t^\alpha) (f'(b))^{\frac{1}{\beta}} \right] dt \right\} \\
 &= \frac{(mb-a)^3}{(2\pi)^2} \left\{ \theta \left(\frac{[f'(a)]^{\frac{1}{\theta}} + m\alpha [f'(b)]^{\frac{1}{\theta}}}{\alpha+1} \right) \right. \\
 &\quad \left. + \beta \left(\frac{[f'(a)]^{\frac{1}{\beta}} + m\alpha [f'(b)]^{\frac{1}{\beta}}}{\alpha+1} \right) \right\}.
 \end{aligned}$$

By applying (1.2), this completes the proof. $\square$

Remark 9. In Theorem 20, if we take $m = 1$, $\alpha = 1$ the inequality (2.15) reduces to the inequality (1.8).

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A PARAMETRIC TYPE OF CAUCHY POLYNOMIALS

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Abstract. In this paper, we introduce a parametric type of Cauchy polynomials and study their characteristic and combinatorial properties. In particular, we show some determinant expressions.

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1. INTRODUCTION

By using the Taylor expansion of the two functions $e^{pt} \cos qt$ and $e^{pt} \sin qt$ in [15], a parametric type of Bernoulli polynomials is introduced and their basic properties are presented ([14]). Precisely, two kinds of bivariate Bernoulli polynomials are introduced as

$$\frac{te^{pt}}{e^t - 1} \cos qt = \sum_{n=0}^{\infty} B_n^{(c)}(p, q) \frac{t^n}{n!} \quad (1.1)$$

and

$$\frac{te^{pt}}{e^t - 1} \sin qt = \sum_{n=0}^{\infty} B_n^{(s)}(p, q) \frac{t^n}{n!}. \quad (1.2)$$

In [13], by defining two specific exponential generating functions, a kind of Euler polynomials is introduced and its basic properties are studied in detail. In [17], a kind of parametric Fubini-type polynomials is defined and some fundamental properties of these parametric-kind Fubini-type polynomials are studied. In [18], a type of generalized parametric Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials is introduced and systematically their basic properties are studied. Then, in [19], it is shown that the real and imaginary parts of a general set of complex Appell polynomials can be represented in terms of the Chebyshev polynomials of the first and second kind. In [7], some special polynomials related to Euler and Bernoulli polynomials are studied and some identities are given. Recently, in [6], identities, relations and combinatorial sums of several polynomials including trigonometric type polynomials are given. In [5], generating functions of parametric Hermite-based Milne–Thomson type polynomials are constructed.

In 1935, D. H. Lehmer [12] introduced and investigated generalized Euler numbers W_n , defined by the generating function

$$\frac{3}{e^t + e^{\omega t} + e^{\omega^2 t}} = \sum_{n=0}^{\infty} W_n \frac{t^n}{n!},$$

where $\omega = \frac{-1+\sqrt{-3}}{2}$ and $\omega^2 = \bar{\omega} = \frac{-1-\sqrt{-3}}{2}$ are the primitive cube roots of unity. Notice that $W_n = 0$ unless $n \equiv 0 \pmod{3}$. More general Lehmer's type of Euler numbers were considered in [1, 9]. Since

$$\frac{e^t + e^{\omega t} + e^{\omega^2 t}}{3} = \sum_{n=0}^{\infty} \frac{t^{3n}}{(3n)!},$$

the function on the left-hand side can be considered as 3 level generalization of the level 2 function

$$\cosh t = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!}.$$

Therefore, it may be natural to consider analogous bivariate Cauchy polynomials with level 2 of (1.1) and (1.2) with level 2. Nevertheless, it is even possible to introduce a slightly more general case. In fact, in [8], a parametric type of Bernoulli polynomials with level 3 is introduced and their characteristic and combinatorial properties are studied.

For real numbers p, q , define *bivariate Cauchy polynomials* $c_n^{(3,i)}(p, q)$ with level 3 ($i = 0, 1, 2$) as

$$\frac{t f_q^{(3,i)}(t)}{(1+t)^p \log(1+t)} = \sum_{n=0}^{\infty} c_n^{(3,i)}(p, q) \frac{t^n}{n!}, \quad (1.3)$$

where

$$f_q^{(3,i)}(t) = \sum_{n=0}^{\infty} (-1)^n \frac{(qt)^{3n+i}}{(3n+i)!}. \quad (1.4)$$

Their complementary polynomials $\hat{c}_n^{(3,i)}(p, q)$ ($i = 0, 1, 2$) are defined as

$$\frac{t \hat{f}_q^{(3,i)}(t)}{(1+t)^p \log(1+t)} = \sum_{n=0}^{\infty} \hat{c}_n^{(3,i)}(p, q) \frac{t^n}{n!}, \quad (1.5)$$

where

$$\hat{f}_q^{(3,i)}(t) = \sum_{n=0}^{\infty} \frac{(qt)^{3n+i}}{(3n+i)!}. \quad (1.6)$$

When $q = i = 0$, $c_n(x) = c_n^{(3,0)}(x, 0) = \hat{c}_n^{(3,0)}(x, 0)$ is the Cauchy polynomial, defined by

$$\frac{t}{(1+t)^x \log(1+t)} = \sum_{n=0}^{\infty} c_n(x) \frac{t^n}{n!} \quad (1.7)$$

(see [3]). When $x = 0$, $c_n = c_n(0)$ is the classical Cauchy number. In [11], convolution identities for Cauchy numbers are studied.

Since for $i = 0, 1, 2$

$$\widehat{f}_1^{(3,i)}(t) = \frac{e^t + \omega^{2i}e^{\omega t} + \omega^i e^{\omega^2 t}}{3}$$

and

$$f_1^{(3,i)}(t) = (-1)^i \frac{e^{-t} + \omega^{2i}e^{-\omega t} + \omega^i e^{-\omega^2 t}}{3},$$

we see that

$$f_q^{(3,i)}(-t) = (-1)^i \widehat{f}_q^{(3,i)}(t), \quad (1.8)$$

$$\widehat{f}_q^{(3,i)}(-t) = (-1)^i f_q^{(3,i)}(t). \quad (1.9)$$

It is trivial to see the following.

Proposition 1. For $n \geq 0$ and $i, j = 0, 1, 2$,

$$c_n^{(3,i)}(p, \omega^j q) = \omega^{ij} c_n^{(3,i)}(p, q),$$

$$\widehat{c}_n^{(3,i)}(p, \omega^j q) = \omega^{ij} \widehat{c}_n^{(3,i)}(p, q),$$

where ω is a complex root of $\omega^3 = 1$.

In the next section, we show several properties of bivariate Cauchy polynomials with level 3. In particular, Theorem 1 entails fundamental recurrence formulas. By using these formulas, we give determinant expressions of bivariate Cauchy polynomials with level 3. In special cases, we can get determinant expressions of the classical Cauchy polynomials and numbers. In a similar method, we also have determinant expressions of bivariate Cauchy polynomials with level 2.

2. BASIC PROPERTIES

In this section, we show several properties of bivariate Cauchy polynomials with level 3. We introduce the auxiliary polynomials $G_n^{(3,i)}(p, q)$ and $\widehat{G}_n^{(3,i)}(p, q)$ as

$$\frac{f_q^{(3,i)}(t)}{(1+t)^p} = \sum_{n=0}^{\infty} G_n^{(3,i)}(p, q) \frac{t^n}{n!}, \quad (2.1)$$

$$\frac{\widehat{f}_q^{(3,i)}(t)}{(1+t)^p} = \sum_{n=0}^{\infty} \widehat{G}_n^{(3,i)}(p, q) \frac{t^n}{n!}. \quad (2.2)$$

respectively. From the definition in (2.1), we have the following.

Proposition 2. For $n \geq 0$ and $i = 0, 1, 2$,

$$G_n^{(3,i)}(p, q) = \sum_{k=0}^{\lfloor \frac{n-i}{3} \rfloor} (-1)^k \binom{n}{3k+i} (-p)_{n-3k-i} q^{3k+i}, \quad (2.3)$$

$$\widehat{G}_n^{(3,i)}(p, q) = \sum_{k=0}^{\lfloor \frac{n-i}{3} \rfloor} \binom{n}{3k+i} (-p)_{n-3k-i} q^{3k+i}, \quad (2.4)$$

where $(x)_\ell = x(x-1)\cdots(x-\ell+1)$ ($\ell \geq 1$) denotes the falling factorial with $(x)_0 = 1$.

$c_n^{(3,i)}(p, q)$ (respectively, $\widehat{c}_n^{(3,i)}(p, q)$) can be written in terms of $G_n^{(3,i)}(p, q)$ (respectively, $\widehat{G}_n^{(3,i)}(p, q)$).

Proposition 3. For $n \geq 0$ and $i = 0, 1, 2$,

$$c_n^{(3,i)}(p, q) = \sum_{k=0}^n \binom{n}{k} c_{n-k} G_k^{(3,i)}(p, q), \quad (2.5)$$

$$\widehat{c}_n^{(3,i)}(p, q) = \sum_{k=0}^n \binom{n}{k} c_{n-k} \widehat{G}_k^{(3,i)}(p, q). \quad (2.6)$$

On the contrary to Proposition 3, $G_n^{(3,i)}(p, q)$ (respectively, $\widehat{G}_n^{(3,i)}(p, q)$) can be written in terms of $c_n^{(3,i)}(p, q)$ (respectively, $\widehat{c}_n^{(3,i)}(p, q)$).

Proposition 4. For $n \geq 0$ and $i = 0, 1, 2$,

$$G_n^{(3,i)}(p, q) = \sum_{k=0}^n \frac{(-1)^{n-k} n!}{(n-k+1)k!} c_k^{(3,i)}(p, q), \quad (2.7)$$

$$\widehat{G}_n^{(3,i)}(p, q) = \sum_{k=0}^n \frac{(-1)^{n-k} n!}{(n-k+1)k!} \widehat{c}_k^{(3,i)}(p, q). \quad (2.8)$$

We have a summation formula for $c_n^{(3,i)}(p, q)$ (respectively, $\widehat{c}_n^{(3,i)}(p, q)$).

Theorem 1. For $n \geq 0$ and $i = 0, 1, 2$,

$$\begin{aligned} & \sum_{k=0}^n \binom{n+1}{k} c_k^{(3,i)}(p, q) \frac{d}{dp}(p)_{n-k+1} \\ &= \begin{cases} -(-1)^{(n-i)/3} (n+1) q^n & \text{if } n \equiv i \pmod{3}; \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \quad (2.9)$$

$$\begin{aligned} & \sum_{k=0}^n \binom{n+1}{k} \widehat{c}_k^{(3,i)}(p, q) \frac{d}{dp}(p)_{n-k+1} \\ &= \begin{cases} -(n+1) q^n & \text{if } n \equiv i \pmod{3}; \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (2.10)$$

When $p = 0$ in Theorem 1, by

$$\left. \frac{d}{dp}(p)_{n-k+1} \right|_{p=0} = (-1)^{n-k}(n-k)!$$

we have simpler recurrence relations.

Corollary 1. For $n \geq 0$ and $i = 0, 1, 2$,

$$\sum_{k=0}^n \frac{(-1)^{n-k} c_k^{(3,i)}(p, q)}{(n-k+1)k!} = \begin{cases} -(-1)^{(n-i)/3} \frac{q^n}{n!} & \text{if } n \equiv i \pmod{3}; \\ 0 & \text{otherwise,} \end{cases}$$

$$\sum_{k=0}^n \frac{(-1)^{n-k} \hat{c}_k^{(3,i)}(p, q)}{(n-k+1)k!} = \begin{cases} -\frac{q^n}{n!} & \text{if } n \equiv i \pmod{3}; \\ 0 & \text{otherwise.} \end{cases}$$

Proof of Theorem 1. From the definition in (1.3) and the proof of Proposition 4, we have

$$\begin{aligned} f_q^{(3,i)}(t) &= (1+t)^p \cdot \frac{\log(1+t)}{t} \sum_{n=0}^{\infty} c_n^{(3,i)}(p, q) \frac{t^n}{n!} \\ &= \left(\sum_{m=0}^{\infty} \frac{(p)_m}{m!} t^m \right) \left(\sum_{l=0}^{\infty} \sum_{k=0}^l \frac{(-1)^{l-k+1}}{(l-k+1)k!} c_k^{(3,i)}(p, q) t^l \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \frac{(p)_{n-l}}{(n-l)!} \sum_{k=0}^l \frac{(-1)^{l-k+1} n!}{(l-k+1)k!} c_k^{(3,i)}(p, q) \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n c_k^{(3,i)}(p, q) \sum_{l=k}^n \binom{n}{l} \binom{l}{k} \frac{(-p)^{n-l}}{(l-k+1)} \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \frac{(-1)^{k-1} c_k^{(3,i)}(p, q)}{k!} \sum_{l=k}^n \frac{(-1)^l n! (p)_{n-l}}{(l-k+1)(n-l)!} \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \frac{(-1)^{k-1} c_k^{(3,i)}(p, q)}{k!} (-1)^n n! \sum_{j=0}^{n-k} \frac{(-1)^j (p)_j}{(n-k-j+1)j!} \right) \frac{t^n}{n!}. \end{aligned}$$

Since

$$\begin{aligned} \sum_{j=0}^{n-k} \frac{(-1)^j (p)_j}{(n-k-j+1)j!} &= \frac{1}{(n-k+1)!} \sum_{l=0}^{n-k} (-1)^l (l+1) \begin{bmatrix} n-k+1 \\ l+1 \end{bmatrix} p^l \\ &= \frac{1}{(n-k+1)!} \frac{d}{dp} (-1)^{n-k} (p)_{n-k+1}, \end{aligned}$$

comparing the coefficients with

$$f_q^{(3,i)}(t) = \sum_{n=0}^{\infty} (-1)^n \frac{(qt)^{3n+i}}{(3n+i)!}$$

in (1.4), we get the identity (2.9). The identity (2.10) is similarly proved. $\square$

From Theorem 1, we have the recurrence relations

$$\begin{aligned} c_n^{(3,i)}(p,q) &= -n! \sum_{k=0}^{n-1} \frac{c_k^{(3,i)}(p,q)}{k!} \frac{d}{dp} \frac{(p)_{n-k+1}}{(n-k+1)!} \\ &\quad - \begin{cases} (-1)^{(n-i)/3} q^n & \text{if } n \equiv i \pmod{3}; \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \quad (2.11)$$

$$\begin{aligned} \hat{c}_n^{(3,i)}(p,q) &= -n! \sum_{k=0}^{n-1} \frac{\hat{c}_k^{(3,i)}(p,q)}{k!} \frac{d}{dp} \frac{(p)_{n-k+1}}{(n-k+1)!} \\ &\quad - \begin{cases} q^n & \text{if } n \equiv i \pmod{3}; \\ 0 & \text{otherwise} \end{cases} \end{aligned} \quad (2.12)$$

with

$$c_0^{(3,i)}(p,q) = \hat{c}_0^{(3,i)}(p,q) = \begin{cases} 1 & \text{if } i = 0; \\ 0 & \text{if } i = 1, 2. \end{cases}$$

Using recurrence relations, we can obtain the exact values of $c_n^{(3,i)}(p,q)$ and $\hat{c}_n^{(3,i)}(p,q)$ for small n . We list some initial values in Appendix.

Theorem 2. For $n \geq 0$ and $i = 0, 1, 2$,

$$c_n^{(3,i)}(p+1,q) - c_n^{(3,i)}(p,q) = -nc_{n-1}^{(3,i)}(p+1,q), \quad (2.13)$$

$$\hat{c}_n^{(3,i)}(p+1,q) - \hat{c}_n^{(3,i)}(p,q) = -n\hat{F}_{n-1}^{(3,i)}(p+1,q). \quad (2.14)$$

Proof.

$$\begin{aligned} \sum_{n=0}^{\infty} c_n^{(3,i)}(p+1,q) \frac{t^n}{n!} - \sum_{n=0}^{\infty} c_n^{(3,i)}(p,q) \frac{t^n}{n!} &= \frac{tf_q^{(3,i)}(t)}{(1+t)^{p+1} \log(1+t)} - \frac{tf_q^{(3,i)}(t)}{(1+t)^p \log(1+t)} \\ &= -\frac{t^2 f_q^{(3,i)}(t)}{(1+t)^{p+1} \log(1+t)} \\ &= -t \sum_{n=0}^{\infty} c_n^{(3,i)}(p+1,q) \frac{t^n}{n!} \\ &= -\sum_{n=0}^{\infty} nc_{n-1}^{(3,i)}(p+1,q) \frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients on both sides, we get the identity (2.13). The identity (2.14) is similarly proved. $\square$

Theorem 3. For $n \geq 0$ and $i = 0, 1, 2$,

$$c_n^{(3,i)}(p+r, q) = \sum_{k=0}^n \binom{n}{k} c_k^{(3,i)}(p, q) (-r)_{n-k}, \quad (2.15)$$

$$\hat{c}_n^{(3,i)}(p+r, q) = \sum_{k=0}^n \binom{n}{k} \hat{c}_k^{(3,i)}(p, q) (-r)_{n-k}. \quad (2.16)$$

Proof.

$$\begin{aligned} \sum_{n=0}^{\infty} c_n^{(3,i)}(p+r, q) \frac{t^n}{n!} &= \frac{t f_q^{(3,i)}(t)}{(1+t)^{p+r} \log(1+t)} \\ &= \left(\sum_{n=0}^{\infty} c_n^{(3,i)}(p, q) \frac{t^n}{n!} \right) \left(\sum_{l=0}^{\infty} \frac{(-r)_l t^l}{l!} \right) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n}{k} c_k^{(3,i)}(p, q) (-r)_{n-k} \frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients on both sides, we get the identity (2.15). The identity (2.16) is similarly proved. $\square$

Theorem 4. For $n \geq 0$ and $i = 0, 1, 2$,

$$\sum_{k=0}^n \frac{c_k^{(3,i)}(p, q)}{k!} = -\frac{c_n^{(3,i)}(p+1, q)}{n!}, \quad (2.17)$$

$$\sum_{k=0}^n \frac{\hat{c}_k^{(3,i)}(p, q)}{k!} = -\frac{\hat{c}_n^{(3,i)}(p+1, q)}{n!}. \quad (2.18)$$

Proof. By using Theorem 3 (2.15) with $r = 1$ and $r = 0$,

$$\begin{aligned} c_{n+1}^{(3,i)}(p+1, q) - c_{n+1}^{(3,i)}(p, q) &= \sum_{k=0}^n \binom{n+1}{k} c_k^{(3,i)}(p, q) \cdot (-1)^{n-k+1} (n-k+1)! \\ &= (n+1)! \sum_{k=0}^n (-1)^{n-k+1} \frac{c_k^{(3,i)}(p, q)}{k!}. \end{aligned}$$

Together with Theorem 2 (2.13), we get the identity (2.17). The identity (2.18) is similarly proved. $\square$

Theorem 5. For $n \geq 1$,

$$\frac{\partial}{\partial p} c_n^{(3,i)}(p, q) = \sum_{k=0}^{n-1} (-1)^{n-k-1} \frac{c_k^{(3,i)}(p, q)}{(n-k)k!} \quad (i = 0, 1, 2), \quad (2.19)$$

$$\frac{\partial}{\partial q} c_n^{(3,i)}(p, q) = \begin{cases} -nc_{n-1}^{(3,2)}(p, q) & \text{if } i = 0; \\ nc_{n-1}^{(3,i-1)}(p, q) & \text{if } i = 1, 2, \end{cases} \quad (2.20)$$

$$\frac{\partial}{\partial p} \hat{c}_n^{(3,i)}(p, q) = \sum_{k=0}^{n-1} (-1)^{n-k-1} \frac{\hat{c}_k^{(3,i)}(p, q)}{(n-k)k!} \quad (i = 0, 1, 2), \quad (2.21)$$

$$\frac{\partial}{\partial q} \hat{c}_n^{(3,i)}(p, q) = \begin{cases} n\hat{c}_{n-1}^{(3,2)}(p, q) & \text{if } i = 0; \\ n\hat{c}_{n-1}^{(3,i-1)}(p, q) & \text{if } i = 1, 2. \end{cases} \quad (2.22)$$

Proof.

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\partial}{\partial p} c_n^{(3,i)}(p, q) \frac{t^n}{n!} &= -\frac{t f_q^{(3,i)}(t)}{(1+t)^p} \\ &= \left(\sum_{m=1}^{\infty} (-1)^{m-1} \frac{t^m}{m!} \right) \left(\sum_{n=0}^{\infty} c_n^{(3,i)}(p, q) \frac{t^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n-1} (-1)^{n-k-1} \frac{c_k^{(3,i)}(p, q)}{(n-k)k!} \frac{t^n}{n!}, \end{aligned}$$

yielding (2.19). Since

$$\begin{aligned} \frac{\partial}{\partial q} f_q^{(3,0)}(t) &= t \sum_{n=1}^{\infty} \frac{(-1)^n (qt)^{3n-1}}{(3n-1)!} \\ &= -t f_q^{(3,2)}(t) \end{aligned}$$

and

$$\frac{\partial}{\partial q} f_q^{(3,i)}(t) = t f_q^{(3,i-1)}(t) \quad (i = 1, 2),$$

we get (2.20). The identities (2.21) and (2.22) are similarly proved. $\square$

3. DETERMINANTS

Determinant is one of the most useful tools for analyzing numbers and polynomials. Our result is motivated by Glaisher's results in 1875 [4], where he gave determinantal expressions of several numbers, including Bernoulli, Euler and Cauchy numbers by solving the systems of equations. See [16] for a good collection of many types of determinants with their backgrounds and techniques. Our method is mainly from the recurrence relations and the inverse formula. Namely, we need the following equivalent relations (see, e.g., [10]).

Lemma 1.

$$\sum_{k=0}^n (-1)^{n-k} x_{n-k} z_k = 0 \quad \text{with} \quad x_0 = z_0 = 1$$

$$\Longleftrightarrow x_n = \begin{vmatrix} z_1 & 1 & & & \\ z_2 & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \\ z_n & \cdots & z_2 & z_1 & \end{vmatrix} \Longleftrightarrow z_n = \begin{vmatrix} x_1 & 1 & & & \\ x_2 & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \\ x_n & \cdots & x_2 & x_1 & \end{vmatrix}.$$

When $i = 0$, bivariate Cauchy polynomials $c_n^{(3,0)}(p, q)$ with level 3 and their complementary numbers $\widehat{c}_n(p) := \widehat{c}_n^{(3,0)}(p, q)$ have determinant expressions.

Theorem 6. For $n \geq 1$,

$$c_n^{(3,0)}(p, q) = n! \begin{vmatrix} r_p(2) & 1 & 0 & \cdots & 0 \\ r_p(3) & r_p(2) & 1 & & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ r_p(n) & r_p(n-1) & \cdots & r_p(2) & 1 \\ r_p^*(n+1) & r_p^*(n) & \cdots & r_p^*(3) & r_p^*(2) \end{vmatrix}$$

and

$$\widehat{c}_n^{(3,0)}(p, q) = n! \begin{vmatrix} r_p(2) & 1 & 0 & \cdots & 0 \\ r_p(3) & r_p(2) & 1 & & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ r_p(n) & r_p(n-1) & \cdots & r_p(2) & 1 \\ \widehat{r}_p(n+1) & \widehat{r}_p(n) & \cdots & \widehat{r}_p(3) & \widehat{r}_p(2) \end{vmatrix},$$

where

$$r_p(n) = \frac{d}{dp} \frac{(-1)^{n-1}(p)_n}{n!} = \frac{1}{n!} \sum_{\ell=0}^{n-1} (-1)^\ell (\ell+1) \begin{bmatrix} n \\ \ell+1 \end{bmatrix} p^\ell$$

with

$$r_p^*(n) = r_p(n) - \begin{cases} \frac{q^{n-1}}{(n-1)!} & \text{if } n \equiv 1 \pmod{3}; \\ 0 & \text{otherwise} \end{cases} \quad (n \geq 2)$$

and

$$\widehat{r}_p(n) = r_p(n) + \begin{cases} \frac{(-1)^n q^{n-1}}{(n-1)!} & \text{if } n \equiv 1 \pmod{3}; \\ 0 & \text{otherwise} \end{cases} \quad (n \geq 2).$$

When $q = i = 0$, Cauchy polynomials $c_n(p) := c_n^{(3,0)}(p, 0) = \widehat{c}_n^{(3,0)}(p, 0) = \widehat{c}_n(p)$ in (1.7) have simpler determinant expressions.

Corollary 2. For $n \geq 1$,

$$c_n(p) = \widehat{c}_n(p) = n! \begin{vmatrix} r_p(2) & 1 & 0 & \cdots & 0 \\ r_p(3) & r_p(2) & 1 & & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ r_p(n) & r_p(n-1) & \cdots & r_p(2) & 1 \\ r_p(n+1) & r_p(n) & \cdots & r_p(3) & r_p(2) \end{vmatrix}.$$

Remark 1. When $p = 0$ in Corollary 2, by

$$r_0(n) = \frac{d}{dp} \frac{(-1)^{n-1}(p)_n}{n!} \Big|_{p=0} = \frac{1}{n},$$

we have a determinant expression of Cauchy numbers c_n as follows [4, p. 50]:

$$c_n = n! \begin{vmatrix} \frac{1}{2} & 1 & 0 & \cdots & 0 \\ \frac{1}{3} & \frac{1}{2} & 1 & & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \frac{1}{n} & \frac{1}{n-1} & \cdots & \frac{1}{2} & 1 \\ \frac{1}{n+1} & \frac{1}{n} & \cdots & \frac{1}{3} & \frac{1}{2} \end{vmatrix}.$$

Proof of Theorem 6. From Theorem 1 with the recurrence relation in (2.11), we have

$$\begin{aligned} \frac{c_n^{(3,0)}(p,q)}{n!} &= \sum_{k=0}^{n-1} (-1)^{n-k+1} r_p(n-k+1) \frac{c_k^{(3,0)}(p,q)}{k!} \\ &\quad - \begin{cases} \frac{(-1)^{n/3} q^n}{n!} & \text{if } n \equiv 0 \pmod{3}; \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

By Lemma 1, this relation is equivalent to the determinant if it is expanded along the first row continually. More precisely, we see that $c_0^{(3,0)}(p,q) = 1$ and $c_1^{(p,q)} = r_p(2)$. And by induction, the first determinant in Theorem 6 is equal to

$$\begin{aligned} & r_p(2) \frac{c_{n-1}^{(3,0)}(p,q)}{(n-1)!} - \begin{vmatrix} r_p(3) & 1 & 0 & \cdots & 0 \\ r_p(4) & r_p(2) & 1 & & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ r_p(n) & r_p(n-2) & \cdots & r_p(2) & 1 \\ r_p^*(n+1) & r_p^*(n-1) & \cdots & r_p^*(3) & r_p^*(2) \end{vmatrix} \\ &= r_p(2) \frac{c_{n-1}^{(3,0)}(p,q)}{(n-1)!} - r_p(3) \frac{c_{n-2}^{(3,0)}(p,q)}{(n-2)!} + \cdots + (-1)^{n-1} r_p(n-1) \frac{c_2^{(3,0)}(p,q)}{2!} \end{aligned}$$

$$\begin{aligned}
& + (-1)^n \begin{vmatrix} r_p(n) & 1 \\ r_p^*(n+1) & r_p^*(2) \end{vmatrix} \\
& = r_p(2) \frac{c_{n-1}^{(3,0)}(p, q)}{(n-1)!} - r_p(3) \frac{c_{n-2}^{(3,0)}(p, q)}{(n-2)!} + \cdots + (-1)^{n-1} r_p(n-1) \frac{c_2^{(3,0)}(p, q)}{2!} \\
& \quad + (-1)^n r_p(n) c_1^{(3,0)}(p, q) + (-1)^{n+1} r_p^*(n+1) c_0^{(3,0)}(p, q) \\
& = \frac{c_n^{(3,0)}(p, q)}{n!}.
\end{aligned}$$

Another identity can be proved similarly. $\square$

By using Lemma 1 again, we have the inversion relation of Corollary 2.

Corollary 3. For $n \geq 1$,

$$r_p(n+1) = \begin{vmatrix} c_1(p) & 1 & 0 & \cdots & 0 \\ \frac{c_2(p)}{2!} & c_1(p) & 1 & & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \frac{c_{n-1}(p)}{(n-1)!} & \frac{c_{n-2}(p)}{(n-2)!} & \cdots & c_1(p) & 1 \\ \frac{c_n(p)}{n!} & \frac{c_{n-1}(p)}{(n-1)!} & \cdots & \frac{c_2(p)}{2!} & c_1(p) \end{vmatrix}.$$

We shall use Trudi's formula to obtain different explicit expressions and inversion relations for the numbers $c_n^{(3,i)}(p, 0)$.

Lemma 2. For $n \geq 1$, we have

$$\begin{vmatrix} a_1 & a_0 & 0 & \cdots \\ a_2 & a_1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ a_{n-1} & & \cdots & a_1 & a_0 \\ a_n & a_{n-1} & \cdots & a_2 & a_1 \end{vmatrix} = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1, \dots, t_n} (-a_0)^{n-t_1-\cdots-t_n} a_1^{t_1} a_2^{t_2} \cdots a_n^{t_n},$$

where $\binom{t_1+\cdots+t_n}{t_1, \dots, t_n} = \frac{(t_1+\cdots+t_n)!}{t_1! \cdots t_n!}$ are the multinomial coefficients.

This relation is known as Trudi's formula [16, Vol.3, p.214], [20] and the case $a_0 = 1$ of this formula is known as Brioschi's formula [2], [16, Vol.3, pp.208–209].

By Corollary 2 and Corollary 3 with Lemma 2, we get different expressions of $c_n(p)$ and $r_p(n)$.

Corollary 4. For $n \geq 1$,

$$\begin{aligned}
c_n(p) & = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1, \dots, t_n} \\
& \quad \times (-1)^{n-t_1-\cdots-t_n} (r_p(2))^{t_1} (r_p(3))^{t_2} \cdots (r_p(n+1))^{t_n},
\end{aligned}$$

$$r_p(n+1) = \sum_{t_1+2t_2+\dots+nt_n=n} \binom{t_1+\dots+t_n}{t_1, \dots, t_n} \times (-1)^{n-t_1-\dots-t_n} (c_1(p))^{t_1} \left(\frac{c_2(p)}{2!}\right)^{t_2} \dots \left(\frac{c_n(p)}{n!}\right)^{t_n}.$$

3.1. Determinant expressions for $c_n^{(3,1)}(p, q)$ and $c_n^{(3,2)}(p, q)$

When $i = 1$, bivariate Cauchy polynomials $c_n^{(3,1)}(p, q)$ with level 3 and their complementary polynomials $\hat{c}_n^{(3,1)}(p, q)$ have also determinant expressions.

Theorem 7. For $n \geq 2$,

$$c_n^{(3,1)}(p, q) = n!q \begin{vmatrix} r_p(2) & 1 & 0 & \dots & 0 \\ r_p(3) & r_p(2) & 1 & & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ r_p(n-1) & r_p(n-2) & \dots & r_p(2) & 1 \\ r_p^\dagger(n) & r_p^\dagger(n-1) & \dots & r_p^\dagger(3) & r_p^\dagger(2) \end{vmatrix}$$

and

$$\hat{c}_n^{(3,1)}(p, q) = n!q \begin{vmatrix} r_p(2) & 1 & 0 & \dots & 0 \\ r_p(3) & r_p(2) & 1 & & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ r_p(n-1) & r_p(n-2) & \dots & r_p(2) & 1 \\ \hat{r}_p^\dagger(n) & \hat{r}_p^\dagger(n-1) & \dots & \hat{r}_p^\dagger(3) & \hat{r}_p^\dagger(2) \end{vmatrix},$$

where

$$r_p^\dagger(n) = r_p(n) - \begin{cases} \frac{q^{n-1}}{n!} & \text{if } n \equiv 1 \pmod{3}; \\ 0 & \text{otherwise} \end{cases} \quad (n \geq 2)$$

and

$$\hat{r}_p^\dagger(n) = r_p(n) + \begin{cases} \frac{(-1)^n q^{n-1}}{n!} & \text{if } n \equiv 1 \pmod{3}; \\ 0 & \text{otherwise} \end{cases} \quad (n \geq 2).$$

Proof. When $i = 1$, by $c_0^{(3,1)}(p, q) = 0$ and $c_1^{(3,1)}(p, q) = q$, from Theorem 1 with the recurrence relation in (2.11), we have for $n \geq 2$

$$\begin{aligned} \frac{c_n^{(3,1)}(p, q)}{n!} &= \sum_{k=1}^{n-1} (-1)^{n-k+1} r_p(n-k+1) \frac{c_k^{(3,1)}(p, q)}{k!} \\ &\quad + \begin{cases} \frac{(-1)^{(n-1)/3} q^n}{n!} & \text{if } n \equiv 1 \pmod{3}; \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Putting $c_n^\dagger = c_n^{(3,1)}(p, q)/q$, we get for $n \geq 2$

$$\frac{c_n^\dagger}{n!} = \sum_{k=1}^{n-1} (-1)^{n-k+1} r_p(n-k+1) \frac{c_k^\dagger}{k!} - \begin{cases} \frac{(-1)^{(n-1)/3} q^{n-1}}{n!} & \text{if } n \equiv 1 \pmod{3}; \\ 0 & \text{otherwise} \end{cases}$$

with $c_1^\dagger = 1$. Then, the first identity is proved similarly to Theorem 6 because the first determinant is equal to $c_n^\dagger/n!$. The proof of the second identity is similar and omitted. $\square$

When $i = 2$, bivariate Cauchy polynomials $c_n^{(3,2)}(p, q)$ with level 3 and their complementary polynomials $\hat{c}_n^{(3,2)}(p, q)$ have also determinant expressions.

Theorem 8. For $n \geq 3$,

$$c_n^{(3,2)}(p, q) = \frac{n!q^2}{2} \begin{vmatrix} r_p(2) & 1 & 0 & \cdots & 0 \\ r_p(3) & r_p(2) & 1 & & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ r_p(n-2) & r_p(n-3) & \cdots & r_p(2) & 1 \\ r_p^*(n-1) & r_p^*(n-2) & \cdots & r_p^*(3) & r_p^*(2) \end{vmatrix}$$

and

$$\hat{c}_n^{(3,2)}(p, q) = \frac{n!q^2}{2} \begin{vmatrix} r_p(2) & 1 & 0 & \cdots & 0 \\ r_p(3) & r_p(2) & 1 & & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ r_p(n-2) & r_p(n-3) & \cdots & r_p(2) & 1 \\ \hat{r}_p^*(n-1) & \hat{r}_p^*(n-2) & \cdots & \hat{r}_p^*(3) & \hat{r}_p^*(2) \end{vmatrix},$$

where

$$r_p^*(n) = r_p(n) - \begin{cases} \frac{2q^{n-1}}{(n+1)!} & \text{if } n \equiv 1 \pmod{3}; \\ 0 & \text{otherwise} \end{cases} \quad (n \geq 2)$$

and

$$\hat{r}_p^*(n) = r_p(n) + \begin{cases} \frac{(-1)^n 2q^{n-1}}{(n+1)!} & \text{if } n \equiv 1 \pmod{3}; \\ 0 & \text{otherwise} \end{cases} \quad (n \geq 2).$$

Proof. When $i = 2$, by $c_0^{(3,1)}(p, q) = c_1^{(3,1)}(p, q) = 0$ and $c_2^{(3,1)}(p, q) = q^2$, from Theorem 1 with the recurrence relation in (2.11), we have for $n \geq 3$

$$\begin{aligned} \frac{c_n^{(3,2)}(p, q)}{n!} &= \sum_{k=2}^{n-1} (-1)^{n-k+1} r_p(n-k+1) \frac{c_k^{(3,2)}(p, q)}{k!} \\ &\quad - \begin{cases} \frac{(-1)^{(n-2)/3} q^n}{n!} & \text{if } n \equiv 2 \pmod{3}; \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Putting $c_n^* = 2c_n^{(3,2)}(p, q)/q^2$, we get for $n \geq 3$

$$\frac{c_n^*}{n!} = \sum_{k=1}^{n-1} (-1)^{n-k+1} r_p(n-k+1) \frac{c_k^*}{k!} - \begin{cases} \frac{(-1)^{(n-2)/3} 2q^{n-2}}{n!} & \text{if } n \equiv 2 \pmod{3}; \\ 0 & \text{otherwise} \end{cases}$$

with $c_2^*/2! = 1$. Then, the first identity is proved similarly to Theorem 6 because the first determinant is equal to $c_n^*/n!$. The proof of the second identity is similar and omitted. $\square$

4. BIVARIATE CAUCHY POLYNOMIALS WITH LEVEL 2

Using the functions

$$\begin{aligned} \cos qt &= f_q^{(2,0)}(t), & \sin qt &= f_q^{(2,1)}, \\ \cosh qt &= \hat{f}_q^{(2,0)}(t), & \sinh qt &= \hat{f}_q^{(2,1)}, \end{aligned}$$

we can define *bivariate Cauchy polynomials* $c_n^{(2,i)}(p, q)$ with level 2 ($i = 0, 1$) as

$$\frac{t f_q^{(2,i)}(t)}{(1+t)^p \log(1+t)} = \sum_{n=0}^{\infty} c_n^{(2,i)}(p, q) \frac{t^n}{n!}, \quad (4.1)$$

where

$$f_q^{(2,i)}(t) = \sum_{n=0}^{\infty} (-1)^n \frac{(qt)^{2n+i}}{(2n+i)!}. \quad (4.2)$$

Their complementary polynomials $\hat{c}_n^{(2,i)}(p, q)$ ($i = 0, 1$) are defined as

$$\frac{t \hat{f}_q^{(2,i)}(t)}{(1+t)^p \log(1+t)} = \sum_{n=0}^{\infty} \hat{c}_n^{(2,i)}(p, q) \frac{t^n}{n!}, \quad (4.3)$$

where

$$\hat{f}_q^{(2,i)}(t) = \sum_{n=0}^{\infty} \frac{(qt)^{2n+i}}{(2n+i)!}. \quad (4.4)$$

When $q = 0$, $c_n(x) = c_n^{(2,i)}(x, 0) = \hat{c}_n^{(2,i)}(x, 0)$.

Similarly to those with level 3, there are characteristic properties for bivariate Cauchy polynomials with level 2. We have the main recurrence relations for $c_n^{(2,i)}(p, q)$ (respectively, $\hat{c}_n^{(2,i)}(p, q)$).

Theorem 9. For $n \geq 0$ and $i = 0, 1, 2$,

$$\sum_{k=0}^n \binom{n+1}{k} c_k^{(2,i)}(p, q) \frac{d}{dp}(p)_{n-k+1} = \begin{cases} -(-1)^{(n-i)/2} (n+1) q^n & \text{if } n \equiv i \pmod{2}; \\ 0 & \text{otherwise,} \end{cases} \quad (4.5)$$

$$\sum_{k=0}^n \binom{n+1}{k} \hat{c}_k^{(2,i)}(p, q) \frac{d}{dp} (p)_{n-k+1} = \begin{cases} -(n+1)q^n & \text{if } n \equiv i \pmod{2}; \\ 0 & \text{otherwise.} \end{cases} \quad (4.6)$$

When $p = 0$ in Theorem 9, we have simpler recurrence relations.

Corollary 5. For $n \geq 0$ and $i = 0, 1, 2$,

$$\sum_{k=0}^n \frac{(-1)^{n-k} c_k^{(2,i)}(p, q)}{(n-k+1)k!} = \begin{cases} -(-1)^{(n-i)/3} \frac{q^n}{n!} & \text{if } n \equiv i \pmod{2}; \\ 0 & \text{otherwise,} \end{cases}$$

$$\sum_{k=0}^n \frac{(-1)^{n-k} \hat{c}_k^{(2,i)}(p, q)}{(n-k+1)k!} = \begin{cases} -\frac{q^n}{n!} & \text{if } n \equiv i \pmod{2}; \\ 0 & \text{otherwise.} \end{cases}$$

It is analogous to obtain determinant expressions of $c_n^{(2,i)}(p, q)$ and $\hat{c}_n^{(2,i)}(p, q)$, defined in (4.1) and (4.3), respectively. The details are similar to Theorem 6 and Theorem 7, respectively, and are omitted.

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APPENDIX

$$\begin{aligned} c_0^{(3,0)}(p, q) &= 1, \\ c_1^{(3,0)}(p, q) &= -p + \frac{1}{2}, \\ c_2^{(3,0)}(p, q) &= p^2 - \frac{1}{6}, \\ c_3^{(3,0)}(p, q) &= -p^3 - \frac{3p^2}{2} - q^3 + \frac{1}{4}, \\ c_4^{(3,0)}(p, q) &= p^4 + 4p^3 + 4p^2 + 4q^3p - 2q^3 - \frac{19}{30}, \\ c_5^{(3,0)}(p, q) &= -p^5 - \frac{15p^4}{2} - \frac{55p^3}{3} - 5(2q^3 + 3)p^2 + \frac{5q^3}{3} + \frac{9}{4}, \\ c_6^{(3,0)}(p, q) &= p^6 + 12p^5 + \frac{105p^4}{2} + 20(q^3 + 5)p^3 + 6(5q^3 + 12)p^2 + q^6 - 5q^3 - \frac{863}{84}. \\ c_0^{(3,1)}(p, q) &= 0, \end{aligned}$$

$$c_1^{(3,1)}(p, q) = q,$$

$$c_2^{(3,1)}(p, q) = -2qp + q,$$

$$c_3^{(3,1)}(p, q) = 3qp^2 - \frac{q}{2},$$

$$c_4^{(3,1)}(p, q) = -4qp^3 + 6qp^2 - q^4 + q,$$

$$c_5^{(3,1)}(p, q) = 5qp^4 + 20qp^3 + 20qp^2 + 5q^4p - \frac{5q^4}{2} - \frac{19q}{6},$$

$$c_6^{(3,1)}(p, q) = -6qp^5 - 45qp^4 - 110qp^3 - 15q(q^3 + 6)p^2 + \frac{5q^4}{2} + \frac{27q}{2}.$$

$$c_0^{(3,2)}(p, q) = 0,$$

$$c_1^{(3,2)}(p, q) = 0,$$

$$c_2^{(3,2)}(p, q) = q^2,$$

$$c_3^{(3,2)}(p, q) = -3q^2p + \frac{3q^2}{2},$$

$$c_4^{(3,2)}(p, q) = 6q^2p^2 - q^2,$$

$$c_5^{(3,2)}(p, q) = -10q^2p^3 - 15q^2p^2 - q^5 + \frac{5q^2}{2},$$

$$c_6^{(3,2)}(p, q) = 15q^2p^4 + 60q^2p^3 + 60q^2p^2 + 6q^5p - 3q^5 - \frac{19q^2}{2}.$$

$$\hat{c}_0^{(3,0)}(p, q) = 1,$$

$$\hat{c}_1^{(3,0)}(p, q) = -p + \frac{1}{2},$$

$$\hat{c}_2^{(3,0)}(p, q) = p^2 - \frac{1}{6},$$

$$\hat{c}_3^{(3,0)}(p, q) = -p^3 - \frac{3p^2}{2} + q^3 + \frac{1}{4},$$

$$\hat{c}_4^{(3,0)}(p, q) = p^4 + 4p^3 + 4p^2 - 4q^3p + 2q^3 - \frac{19}{30},$$

$$\hat{c}_5^{(3,0)}(p, q) = -p^5 - \frac{15p^4}{2} - \frac{55p^3}{3} + 5(2q^3 - 3)p^2 - \frac{5q^3}{3} + \frac{9}{4},$$

$$\hat{c}_6^{(3,0)}(p, q) = p^6 + 12p^5 + \frac{105p^4}{2} - 20(q^3 - 5)p^3 - 6(5q^3 - 12)p^2 + q^6 + 5q^3 - \frac{863}{84}.$$

$$\begin{aligned}
\hat{c}_0^{(3,1)}(p, q) &= 0, \\
\hat{c}_1^{(3,1)}(p, q) &= q, \\
\hat{c}_2^{(3,1)}(p, q) &= -2qp + q, \\
\hat{c}_3^{(3,1)}(p, q) &= 3qp^2 - \frac{q}{2}, \\
\hat{c}_4^{(3,1)}(p, q) &= -4qp^3 + 6qp^2 + q^4 + q, \\
\hat{c}_5^{(3,1)}(p, q) &= 5qp^4 + 20qp^3 + 20qp^2 - 5q^4p + \frac{5q^4}{2} - \frac{19q}{6}, \\
\hat{c}_6^{(3,1)}(p, q) &= -6qp^5 - 45qp^4 - 110qp^3 + 15q(q^3 - 6)p^2 - \frac{5q^4}{2} + \frac{27q}{2}.
\end{aligned}$$

$$\begin{aligned}
\hat{c}_0^{(3,2)}(p, q) &= 0, \\
\hat{c}_1^{(3,2)}(p, q) &= 0, \\
\hat{c}_3^{(3,2)}(p, q) &= -3q^2p + \frac{3q^2}{2}, \\
\hat{c}_4^{(3,2)}(p, q) &= 6q^2p^2 - q^2, \\
\hat{c}_5^{(3,2)}(p, q) &= -10q^2p^3 - 15q^2p^2 + q^5 + \frac{5q^2}{2}, \\
\hat{c}_6^{(3,2)}(p, q) &= 15q^2p^4 + 60q^2p^3 + 60q^2p^2 - 6q^5p + 3q^5 - \frac{19q^2}{2}.
\end{aligned}$$

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SOME IDENTITIES FOR DERANGEMENT NUMBERS

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Abstract. In this paper, we give some identities involving derangement numbers. In addition, we derive nonlinear differential equation from the generating function of r -derangement numbers and obtain some identities including generalized harmonic numbers and derangement numbers by using this differential equation. For example, for any positive integers N , n and r ,

$$\sum_{k=0}^N \sum_{i=0}^{N-k} (-1)^i \binom{r}{k} \binom{r+N-i-k}{N-i-k} \frac{d_{n-r+k}^{(r+N-i-k+1)}}{i!(n-r+k)!} = \frac{D_r(n+N)}{n!N!}.$$

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Keywords: derangement numbers, r -derangement numbers, Daehee numbers, harmonic numbers

1. INTRODUCTION

Harmonic numbers and generalized harmonic numbers have been studied since the distant past and are involved in a wide range of diverse fields such as analysis, computer science and various in a wide range of diverse fields such as analysis, computer science and various branches of number theory [1–8, 16, 17, 23–25]. The harmonic numbers are defined by

$$H_0 = 0 \quad \text{and} \quad H_n = \sum_{i=1}^n \frac{1}{i} \quad \text{for } n \geq 1.$$

For any $\alpha \in \mathbb{R}^+$, the generalized harmonic numbers $H_n(\alpha)$ [8] are defined by

$$H_0(\alpha) = 0 \quad \text{and} \quad H_n(\alpha) = \sum_{i=1}^n \frac{1}{i\alpha^i} \quad \text{for } n \geq 1.$$

For $\alpha = 1$, the usual harmonic numbers are $H_n(1) = H_n$ and the generating function of $H_n(\alpha)$ is

$$\sum_{n=1}^{\infty} H_n(\alpha) x^n = -\frac{\ln\left(1 - \frac{x}{\alpha}\right)}{1-x}.$$

For the generalized harmonic numbers $H_n(\alpha)$, Ömür et al. [16] defined the generalized hyperharmonic numbers of order r , $H_n^r(\alpha)$ as follows. For $r < 0$ or $n \leq 0$,

$H_n^r(\alpha) = 0$ and for $n \geq 1$,

$$H_n^r(\alpha) = \sum_{i=1}^n H_i^{r-1}(\alpha), \quad r \geq 1,$$

where $H_n^0(\alpha) = \frac{1}{n\alpha^n}$. For $\alpha = 1$, $H_n^r(1) = H_n^r$ are the hyperharmonic numbers of order r . The generating function of these numbers is

$$\sum_{n=1}^{\infty} H_n^r(\alpha) x^n = -\frac{\ln\left(1 - \frac{x}{\alpha}\right)}{(1-x)^r}. \quad (1.1)$$

The generalized harmonic numbers $H(n, r)$ of rank r [7, 23] are defined as for $n \geq 1$ and $r \geq 0$,

$$H(n, r) = \sum_{1 \leq n_0 + n_1 + \dots + n_r \leq n} \frac{1}{n_0 n_1 \dots n_r}.$$

It is clear that $H(n, 0) = H_n$.

$H(n, r, \alpha)$ [6] are defined as for $n \geq 1$ and $r \geq 0$,

$$H(n, r, \alpha) = \sum_{1 \leq n_0 + n_1 + \dots + n_r \leq n} \frac{1}{n_0 n_1 \dots n_r \alpha^{n_0 + n_1 + \dots + n_r}}$$

or equivalently as

$$H(n, r, \alpha) = \frac{(-1)^{r+1}}{n!} \left(\frac{d^n}{dx^n} \frac{[\ln(1 - \frac{x}{\alpha})]^{r+1}}{1-x} \right) \Big|_{x=0}.$$

For $\alpha = 1$, $H(n, r, 1) = H(n, r)$.

The generating function of these numbers is given by

$$\sum_{n=0}^{\infty} H(n, r, \alpha) x^n = \frac{(-\ln(1 - \frac{x}{\alpha}))^{r+1}}{1-x}. \quad (1.2)$$

The Cauchy numbers of order r , C_n^r , are defined by the generating functions to be

$$\sum_{n=0}^{\infty} C_n^r \frac{x^n}{n!} = \left(\frac{x}{\ln(1+x)} \right)^r. \quad (1.3)$$

For $r = 1$, $C_n^1 = C_n$ are called Cauchy numbers.

The Daehee numbers of order r , D_n^r , are defined by the generating functions to be

$$\sum_{n=0}^{\infty} D_n^r \frac{x^n}{n!} = \left(\frac{\ln(1+x)}{x} \right)^r. \quad (1.4)$$

For $r = 1$, $D_n^1 = D_n$ are called Daehee numbers.

The Stirling numbers of the first kind $S_1(n, k)$ are defined by

$$x^n = \sum_{k=0}^n S_1(n, k) x^k,$$

and the Stirling numbers of the second kind $S_2(n, k)$ are defined by

$$x^n = \sum_{k=0}^n S_2(n, k) x^k,$$

where x^n stands for the falling factorial defined by $x^0 = 1$ and $x^n = x(x-1) \cdots (x-n+1)$.

The generating function of the Stirling numbers of the first kind $S_1(n, k)$ is given by

$$\sum_{n=0}^{\infty} S_1(n, k) \frac{x^n}{n!} = \frac{(\ln(1+x))^k}{k!}, \quad k \geq 0, \quad (1.5)$$

and the generating function of the Stirling numbers of the second kind $S_2(n, k)$ is given by

$$\sum_{n=k}^{\infty} S_2(n, k) \frac{x^n}{n!} = \frac{(e^x - 1)^k}{k!}, \quad k \geq 0.$$

The generalized geometric series are given by for positive integers a and b ,

$$\sum_{n=b}^{\infty} \binom{n+a-b}{a} x^n = \frac{x^b}{(1-x)^{a+1}}. \quad (1.6)$$

The exponential generating function is

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x. \quad (1.7)$$

The generating function of the r -derangement numbers $D_r(n)$ is given by

$$\sum_{n=r}^{\infty} D_r(n) \frac{x^n}{n!} = \frac{x^r e^{-x}}{(1-x)^{r+1}}. \quad (1.8)$$

For $r \in \mathbb{N}$, the derangement numbers $d_n^{(r)}$ of order r ($n \geq 0$) [13] are defined by the generating functions to be

$$\sum_{n=0}^{\infty} d_n^{(r)} \frac{x^n}{n!} = \left(\frac{1}{1-x} \right)^r e^{-x}. \quad (1.9)$$

For $r = 1$, $d_n^{(1)} = D_0(n) = d_n$ are called derangement numbers. The derangement numbers d_n are given by the closed form formula

$$d_n = n! \sum_{i=0}^n \frac{(-1)^i}{i!}.$$

It is known that for an exponential series $f(x) = \sum_{n \geq 0} f_n \frac{x^n}{n!}$,

$$\frac{d^m}{dx^m} f(x) = \sum_{n=0}^{\infty} f_{n+m} \frac{x^n}{n!}, \quad (1.10)$$

and for an ordinary series $f(x) = \sum_{n \geq 0} f_n x^n$,

$$\frac{d^m}{dx^m} f(x) = \sum_{n=0}^{\infty} m! \binom{m+n}{n} f_{n+m} x^n. \quad (1.11)$$

There have been many studies involving derangement numbers [4–6, 14, 20, 21, 24].

Duran et al. [6] obtained sums involving generalized harmonic numbers of rank r , derangement and special numbers. For example, for any positive integers n and r ,

$$\sum_{i=0}^n (-1)^{i-r} \binom{n}{i} r! \alpha^{-i} S_1(i, r) d_{n-i} = n! \sum_{i=0}^n \frac{(-1)^{n-i}}{(n-i)!} H(i, r-1, \alpha).$$

Recently, the combinatorial properties of special numbers and polynomials have been studied using differential equations associated with the generation functions [9–12, 15, 18, 19, 22].

Kwon et al. [15] showed that for positive integer N and nonnegative integer n ,

$$D_{n+N-1} = \frac{(-1)^N (N-1)!}{n+N} \sum_{k=0}^n (-1)^k \binom{n}{k} N^k D_{n-k}^k.$$

Ömür et al. [18] gave that for any positive integers N , n , r and m ,

$$\begin{aligned} S_1(n+N, r+1) \binom{n+N+m}{m} \binom{r+m+1}{m}^{-1} \\ = \sum_{i=0}^{n+N} \binom{n+N+m}{i} C_i^m S_1(n+N+m-i, r+m+1). \end{aligned}$$

Rim et al. [22] obtained some identities involving hyperharmonic numbers and Daehee number as follows: for any positive integer N and nonnegative integer n ,

$$n! (n+N)^{\overline{N}} \sum_{k=0}^{n+N} (-1)^{n+N-k} \binom{r}{n+N-k} H_k^r = (N-1)! \sum_{k=0}^n (-1)^k \binom{n}{k} N^{m-k} D_k^{n-k}.$$

In this paper, we give some identities involving derangement numbers. In addition, we derive nonlinear differential equation from the generating function of r -derangement numbers and using this differential equation, obtain some identities including generalized harmonic numbers and derangement numbers. For example, for any positive integers N , n and r ,

$$\sum_{k=0}^N \sum_{i=0}^{N-k} (-1)^i \binom{r}{k} \binom{r+N-i-k}{N-i-k} \frac{d_{n-r+k}^{(r+N-i-k+1)}}{i!(n-r+k)!} = \frac{D_r(n+N)}{n!N!}.$$

2. SOME IDENTITIES WITH DERANGEMENT NUMBERS

In this section, we will give some identities involving Daehee, Cauchy, derangement, r -derangement and harmonic numbers.

Theorem 1. For any positive integers n, r and α , we have

$$C_n^r = \sum_{j=0}^n \sum_{k=0}^j \binom{n}{j} \binom{j}{k} C_k^r C_{j-k} D_{n-j}, \quad (2.1)$$

$$= \sum_{i=0}^n \binom{i+r}{r}^{-1} S_1(n, i) S_2(i+r, r) \quad (2.2)$$

$$= -\frac{n!}{r+1} \sum_{k=0}^n \sum_{j=0}^k \sum_{i=0}^{n-k} \frac{(-1)^{k+j} \alpha^k}{(n-k)!} \binom{i+r+1}{i}^{-1} \binom{r}{j} \quad (2.3)$$

$$\times S_1(n-k, i) S_2(i+r+1, r) H_{k-j}^r(\alpha).$$

Proof. We will give proof of (2.1). By (1.3) and (1.4), we write

$$\begin{aligned} \sum_{n=0}^{\infty} C_n^r \frac{x^n}{n!} &= \left(\frac{x}{\ln(1+x)} \right)^r = \frac{x}{\ln(1+x)} \left(\frac{x}{\ln(1+x)} \right)^r \frac{\ln(1+x)}{x} \\ &= \sum_{n=0}^{\infty} C_n^r \frac{x^n}{n!} \sum_{k=0}^{\infty} C_k \frac{x^k}{k!} \sum_{j=0}^{\infty} D_j \frac{x^j}{j!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n}{k} C_k^r C_{n-k} \frac{x^n}{n!} \sum_{j=0}^{\infty} D_j \frac{x^j}{j!} \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^n \sum_{k=0}^j \binom{n}{j} \binom{j}{k} C_k^r C_{j-k} D_{n-j} \frac{x^n}{n!}. \end{aligned}$$

Thus, comparing the coefficients on both sides, the first result is obtained. The proofs of (2.2) and (2.3) are similar to the proof of (2.1). Thus the proof is complete. $\square$

Theorem 2. For any positive integers n, r and α , we have

$$\begin{aligned} (-1)^{n+r} H(n, r-1, \alpha) &= \sum_{i=0}^n \sum_{j=0}^n (-1)^i \frac{D_{j-r}^r}{\alpha^j (j-r)!} \frac{S_1(n-j, i)}{(n-j)!} \quad (2.4) \\ &= \frac{r!}{\alpha^r} \sum_{k=0}^n \sum_{i=0}^n \sum_{l=0}^n \sum_{j=0}^l (-1)^k \frac{D_j^r}{\alpha^j j!} \frac{S_1(l-j, i)}{(l-j)!} \frac{S_1(n-l, k)}{(n-l)!} S_2(i, r). \end{aligned}$$

Proof. We will give proof of (2.4). From (1.2), (1.4) and (1.7), we get

$$\sum_{n=0}^{\infty} (-1)^n H(n, r-1, \alpha) x^n = \frac{(-\ln(1+\frac{x}{\alpha}))^r}{1+x} = \left(-\frac{x}{\alpha}\right)^r \frac{(\ln(1+\frac{x}{\alpha}))^r}{(\frac{x}{\alpha})^r} e^{-\ln(1+x)}$$

$$= \left(-\frac{x}{\alpha}\right)^r \sum_{n=0}^{\infty} D_n^r \frac{x^n}{\alpha^n n!} \sum_{i=0}^{\infty} (-1)^i \frac{(\ln(1+x))^i}{i!}.$$

By (1.5), we have

$$\begin{aligned} (-1)^r \sum_{n=0}^{\infty} (-1)^n H(n, r-1, \alpha) x^n &= \sum_{n=0}^{\infty} D_n^r \frac{x^{n+r}}{\alpha^{n+r} n!} \sum_{i=0}^{\infty} (-1)^i \sum_{n=i}^{\infty} S_1(n, i) \frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} D_{n-r}^r \frac{x^n}{\alpha^n (n-r)!} \sum_{n=0}^{\infty} \sum_{i=0}^n (-1)^i S_1(n, i) \frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^n \sum_{i=0}^j (-1)^i \frac{D_{n-j-r}^r}{\alpha^{n-j} (n-j-r)!} \frac{S_1(j, i)}{j!} x^n \\ &= \sum_{n=0}^{\infty} \sum_{i=0}^n \sum_{j=0}^n (-1)^i \frac{D_{n-j-r}^r}{\alpha^{n-j} (n-j-r)!} \frac{S_1(j, i)}{j!} x^n \\ &= \sum_{n=0}^{\infty} \sum_{i=0}^n \sum_{j=0}^n (-1)^i \frac{D_{j-r}^r}{\alpha^j (j-r)!} \frac{S_1(n-j, i)}{(n-j)!} x^n, \end{aligned}$$

as claimed. The proof of other equality is similar to proof of (2.4). $\square$

Now, for any positive integer r , we take the function $F_r(x) = \frac{e^{-x} x^r}{(1-x)^{r+1}}$. By taking the derivative of this function with respect to x , we write

$$\frac{d}{dx} F_r(x) = \frac{r+x}{1-x} F_r(x) + \frac{r}{1-x} F_{r-1}(x),$$

and

$$\begin{aligned} \frac{d^2}{dx^2} F_r(x) &= \frac{2}{(1-x)^2} \left(\binom{r+2}{2} F_r(x) + 2 \binom{r+1}{2} F_{r-1}(x) + \binom{r}{2} F_{r-2}(x) \right) \\ &\quad - \frac{2}{1-x} ((r+1) F_r(x) + r F_{r-1}(x)) + F_r(x). \end{aligned}$$

By repeating this process, we easily have

$$\frac{d^N}{dx^N} F_r(x) = N! \sum_{k=0}^N \sum_{i=0}^{N-k} \frac{(-1)^i}{i!} \binom{r}{k} \binom{r+N-i-k}{r} \frac{F_{r-k}(x)}{(1-x)^{N-i}}. \quad (2.5)$$

Theorem 3. For any positive integers N , n and r , we have

$$\begin{aligned} \frac{D_r(n+N)}{n! N!} &= \sum_{k=0}^r \sum_{i=0}^{N-k} \sum_{j=0}^n (-1)^i \binom{r}{k} \binom{r+N-i-k}{r} \binom{n-j+N-i-1}{n-j} \frac{D_{r-k}(j)}{i! j!} \\ &= \sum_{k=0}^N \sum_{i=0}^{N-k} (-1)^i \binom{r}{k} \binom{r+N-i-k}{r} \frac{d_{n-r+k}^{(r+N-i-k+1)}}{i! (n-r+k)!}. \end{aligned}$$

Proof. From (1.6), (1.8) and (2.5), we have

$$\begin{aligned} & \frac{d^N}{dx^N} \left(\frac{x^r e^{-x}}{(1-x)^{r+1}} \right) \\ &= N! \sum_{k=0}^N \sum_{i=0}^{N-k} \frac{(-1)^i}{i!} \binom{r}{k} \binom{r+N-i-k}{r} \sum_{n=0}^{\infty} D_{r-k}(n) \frac{x^n}{n!} \sum_{n=0}^{\infty} \binom{n+N-i-1}{n} x^n. \end{aligned}$$

By product of the generating functions, we write

$$\begin{aligned} & \frac{d^N}{dx^N} \left(\frac{x^r e^{-x}}{(1-x)^{r+1}} \right) \\ &= \sum_{k=0}^N \sum_{i=0}^{N-k} (-1)^i \frac{N!}{i!} \binom{r}{k} \binom{r+N-i-k}{r} \sum_{n=0}^{\infty} \sum_{j=0}^n \frac{D_{r-k}(j)}{j!} \binom{n-j+N-i-1}{n-j} x^n \\ &= N! \sum_{n=0}^{\infty} \sum_{k=0}^N \sum_{i=0}^{N-k} \sum_{j=0}^n \frac{(-1)^i}{i! j!} \binom{r}{k} \binom{r+N-i-k}{r} \binom{n-j+N-i-1}{n-j} D_{r-k}(j) x^n. \end{aligned}$$

Also from (1.10)

$$\frac{d^N}{dx^N} \left(\frac{x^r e^{-x}}{(1-x)^{r+1}} \right) = \sum_{n=0}^{\infty} \frac{d^N}{dx^N} \left(D_r(n) \frac{x^n}{n!} \right) = \sum_{n=0}^{\infty} D_r(n+N) \frac{x^n}{n!}. \quad (2.6)$$

Comparing the coefficients of x^n in the first and last series, we obtain the result. Similarly, (1.9) and (2.5) yield that

$$\begin{aligned} & \frac{d^N}{dx^N} \left(\frac{x^r e^{-x}}{(1-x)^{r+1}} \right) \\ &= \sum_{k=0}^N \sum_{i=0}^{N-k} (-1)^i \frac{N!}{i!} \binom{r}{k} \binom{r+N-i-k}{r} \frac{e^{-x}}{(1-x)^{r+N-i-k+1}} x^{r-k} \\ &= \sum_{k=0}^N \sum_{i=0}^{N-k} (-1)^i \frac{N!}{i!} \binom{r}{k} \binom{r+N-i-k}{r} \sum_{n=0}^{\infty} d_n^{(r+N-i-k+1)} \frac{x^{n+r-k}}{n!} \\ &= \sum_{k=0}^N \sum_{i=0}^{N-k} (-1)^i \frac{N!}{i!} \binom{r}{k} \binom{r+N-i-k}{r} \sum_{n=0}^{\infty} d_{n-r+k}^{(r+N-i-k+1)} \frac{x^n}{(n-r+k)!} \\ &= N! \sum_{n=0}^{\infty} \sum_{k=0}^N \binom{r}{k} \sum_{i=0}^{N-k} \frac{(-1)^i}{i!} \binom{r+N-i-k}{r} \frac{d_{n-r+k}^{(r+N-i-k+1)}}{(n-r+k)!} x^n. \end{aligned} \quad (2.7)$$

Thus, with the help of (2.6) and (2.7), comparing the coefficients of x^n on both sides, the other result is proof. $\square$

Theorem 4. For any positive integers N , n and r , we have

$$\begin{aligned}
 & \frac{1}{N!} \sum_{i=0}^n D_r(i+N) S_1(n, i) \\
 &= \sum_{k=0}^N \sum_{i=0}^{N-k} \sum_{l=0}^n \sum_{j=0}^l (-1)^i \frac{l!}{i!j!} \binom{r}{k} \binom{r+N-i-k}{r} \binom{l-j+N-i-1}{l-j} \\
 & \quad \times D_{r-k}(j) S_1(n, l) \\
 &= \sum_{k=0}^N \sum_{i=0}^{N-k} \sum_{l=0}^n \sum_{j=0}^l (-1)^i \frac{(r-k)!}{i!} \binom{n}{l} \binom{r}{k} \binom{r+N-i-k}{r} \\
 & \quad \times S_1(l, j) S_1(n-l, r-k) d_j^{(r+N-i-k+1)} \\
 &= n! \sum_{k=0}^N \sum_{i=0}^{N-k} \sum_{j=0}^n \sum_{l=0}^j \frac{(-1)^{n+i+j+r+k}}{i!} \frac{l!}{j!} \binom{r}{k} \binom{r+N-i-k}{r} \binom{l+r+N-i-k}{l} \\
 & \quad \times H(n-j, r-k-1) S_1(j, l).
 \end{aligned} \tag{2.8}$$

Proof. We will give proof of (2.8). From (1.6), (1.8) and (2.5) we have

$$\begin{aligned}
 & \frac{d^N}{dx^N} \left(\frac{(\ln(1-x))^r e^{-\ln(1-x)}}{(1-\ln(1-x))^{r+1}} \right) \\
 &= N! \sum_{k=0}^N \sum_{i=0}^{N-k} \frac{(-1)^i}{i!} \binom{r}{k} \binom{r+N-i-k}{r} \frac{F_{r-k}(\ln(1-x))}{(1-\ln(1-x))^{N-i}} \\
 &= N! \sum_{k=0}^N \sum_{i=0}^{N-k} \frac{(-1)^i}{i!} \binom{r}{k} \binom{r+N-i-k}{r} \\
 & \quad \times \sum_{n=0}^{\infty} D_{r-k}(n) \frac{(\ln(1-x))^n}{n!} \sum_{n=0}^{\infty} \binom{n+N-i-1}{n} (\ln(1-x))^n.
 \end{aligned}$$

By product of the generating functions and (1.5), we write

$$\begin{aligned}
 & \frac{d^N}{dx^N} \left(\frac{(\ln(1-x))^r e^{-\ln(1-x)}}{(1-\ln(1-x))^{r+1}} \right) \\
 &= \sum_{k=0}^N \sum_{i=0}^{N-k} (-1)^i \frac{N!}{i!} \binom{r}{k} \binom{r+N-i-k}{r} \\
 & \quad \times \sum_{l=0}^{\infty} \sum_{j=0}^l \frac{l!}{j!} \binom{l-j+N-i-1}{l-j} D_{r-k}(j) \frac{(\ln(1-x))^l}{l!} \\
 &= \sum_{k=0}^N \sum_{i=0}^{N-k} (-1)^i \frac{N!}{i!} \binom{r}{k} \binom{r+N-i-k}{r}
 \end{aligned}$$

$$\begin{aligned}
& \times \sum_{l=0}^{\infty} \sum_{j=0}^l \frac{l!}{j!} \binom{l-j+N-i-1}{l-j} D_{r-k}(j) \sum_{n=l}^{\infty} (-1)^n S_1(n, l) \frac{x^n}{n!} \\
& = \sum_{k=0}^N \sum_{i=0}^{N-k} (-1)^i \frac{N!}{i!} \binom{r}{k} \binom{r+N-i-k}{r} \\
& \quad \times \sum_{n=0}^{\infty} \sum_{l=0}^n \sum_{j=0}^l (-1)^n \frac{l!}{j!n!} \binom{l-j+N-i-1}{l-j} D_{r-k}(j) S_1(n, l) x^n \\
& = \sum_{n=0}^{\infty} \sum_{k=0}^N \sum_{i=0}^{N-k} \sum_{l=0}^n \sum_{j=0}^l (-1)^{n+i} \frac{N!l!}{i!j!n!} \binom{r}{k} \binom{r+N-i-k}{r} \\
& \quad \times \binom{l-j+N-i-1}{l-j} D_{r-k}(j) S_1(n, l) x^n. \tag{2.9}
\end{aligned}$$

By (2.9) and taking $x \rightarrow \ln(1-x)$ in (2.6), comparing the coefficients on both sides, the desired result is obtained. The proofs of other equalities are similar to the proof of (2.8). $\square$

Theorem 5. For any positive integers N, n, r and α , we have

$$\begin{aligned}
& \binom{n+N}{N}^{-1} \sum_{k=0}^N \sum_{i=0}^{N-k} \frac{(-1)^i}{i!} \binom{r}{k} \binom{r+N-i-k}{r} \frac{d_{n-r+k}^{(r+N-i-k+1)}}{(n-r+k)!} \\
& = \sum_{k=0}^{n+N} \sum_{i=0}^k (-1)^{n+N-i-r+1} \frac{C_{k-i-r+1}}{(n+N-k)!(k-i-r+1)!} H_i^{r+1} \\
& = \sum_{k=0}^{n+N} \sum_{i=0}^k (-1)^{k-i} \alpha^{r-k+i} \frac{C_{k-i}^r}{(k-i)!} \frac{d_{n+N-k}^{(r)}}{(n+N-k)!} H(i, r-1, \alpha) \tag{2.10}
\end{aligned}$$

$$= \sum_{i=0}^{n+N} \frac{(-1)^i}{i!} \binom{n+N-i}{r} \tag{2.11}$$

$$= \sum_{i=0}^{n+N} \binom{n+N-i}{r-1} \frac{d_{i-1}}{(i-1)!} \tag{2.12}$$

$$= \frac{d_{n+N-r}^{(r+1)}}{(n+N-r)!}. \tag{2.13}$$

Proof. By (1.1) and (1.3), we have

$$\begin{aligned}
\frac{e^{-x} x^r}{(1-x)^{r+1}} & = \frac{-\ln(1-x)}{(1-x)^{r+1}} \frac{-x}{\ln(1-x)} x^{r-1} e^{-x} \\
& = \sum_{n=0}^{\infty} H_n^{r+1} x^n \sum_{i=0}^{\infty} (-1)^i C_i \frac{x^{i+r-1}}{i!} \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} H_n^{r+1} x^n \sum_{i=0}^{\infty} (-1)^{i-r+1} C_{i-r+1} \frac{x^i}{(i-r+1)!} \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^n \sum_{i=0}^k (-1)^{n-i-r+1} H_i^{r+1} \frac{C_{k-i-r+1}}{(n-k)!(k-i-r+1)!} x^n,
\end{aligned}$$

and by (1.11)

$$\begin{aligned}
&\frac{d^N}{dx^N} \left(\frac{e^{-x} x^r}{(1-x)^{r+1}} \right) \\
&= \sum_{n=N}^{\infty} \sum_{k=0}^n \sum_{i=0}^k (-1)^{n-i-r+1} \binom{n}{N} N! \frac{C_{k-i-r+1} H_i^{r+1}}{(k-i-r+1)! (n-k)!} \frac{x^{n-N}}{(n-k)!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n+N} \sum_{i=0}^k (-1)^{n+N-i-r+1} \binom{n+N}{N} N! \frac{C_{k-i-r+1} H_i^{r+1}}{(k-i-r+1)! (n+N-k)!} \frac{x^n}{(n+N-k)!}.
\end{aligned}$$

From here, by (2.7), comparing the coefficients on both sides, the desired result is obtained. Similarly, from the equalities

$$\begin{aligned}
\frac{e^{-x} x^r}{(1-x)^{r+1}} &= \frac{(-\ln(1-\frac{x}{\alpha}))^r}{(1-x)} \left(\frac{(-\frac{x}{\alpha})}{\ln(1-\frac{x}{\alpha})} \right)^r \alpha^r \left(\frac{1}{1-x} \right)^r e^{-x}, \\
\frac{e^{-x} x^r}{(1-x)^{r+1}} &= \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} \sum_{n=r}^{\infty} \binom{n}{r} x^n = \sum_{n=0}^{\infty} \sum_{i=0}^n \frac{(-1)^i}{i!} \binom{n-i}{r} x^n, \\
\frac{e^{-x} x^r}{(1-x)^{r+1}} &= \sum_{n=r-1}^{\infty} \binom{n}{r-1} x^n \sum_{n=0}^{\infty} d_n \frac{x^{n+1}}{n!} = \sum_{n=0}^{\infty} \sum_{i=0}^n \binom{n-i}{r-1} d_{i-1} \frac{x^n}{(i-1)!}, \\
\frac{e^{-x} x^r}{(1-x)^{r+1}} &= \sum_{n=r}^{\infty} d_{n-r}^{(r+1)} \frac{x^n}{(n-r)!},
\end{aligned}$$

respectively, and (1.11), we have the proofs of (2.10)-(2.13). $\square$

For example, by setting $n+N \rightarrow n$ in (2.11)-(2.13), it is seen that for any positive integers n and r ,

$$\sum_{i=0}^n \binom{n-i}{r} \frac{(-1)^i}{i!} = \sum_{i=0}^n \binom{n-i}{r-1} \frac{d_{i-1}}{(i-1)!} = \frac{d_{n-r}^{(r+1)}}{(n-r)!}. \quad (2.14)$$

Corollary 1. For any positive integers n, r and α , we have

$$\begin{aligned}
&\sum_{k=0}^n \sum_{i=0}^k \sum_{j=0}^{n-k+r-1} (-1)^{j+k-i} \alpha^{r-k+i} \binom{n-k-j+r-1}{r-1} \frac{C_{k-i}^r}{(k-i)!} \frac{H(i, r-1, \alpha)}{j!} \\
&= \sum_{k=0}^n \sum_{i=0}^k (-1)^{n-i-r+1} \frac{C_{k-i-r+1}}{(k-i-r+1)!} \frac{H_i^{r+1}}{(n-k)!}.
\end{aligned}$$

Proof. By setting $n + N \rightarrow n$ in Theorem 5 and (2.14), the proof is clearly given. $\square$

Theorem 6. Let n, r and m be positive integers such that $r > m$. We have

$$\begin{aligned} \sum_{i=0}^n S_1(n, i) D_r(i) &= \sum_{j=0}^n \sum_{i=0}^j (-1)^{j-i} i! \binom{j}{i} \binom{i}{r} S_1(n, j) \\ &= \sum_{k=0}^n \sum_{j=0}^k \sum_{i=0}^j (-1)^{j-i} (i+r)! \binom{j}{i} \binom{n}{k} S_1(k, j) S_1(n-k, r) \\ &= (-1)^{r-m} m! n! \sum_{k=0}^n \sum_{j=0}^k \sum_{i=0}^{n-k} (-1)^{j-k} \frac{i!}{j!(n-k)!} \binom{i+r}{i} H(k-j, r-m-1) \\ &\quad \times S_1(j, m) S_1(n-k, i), \end{aligned}$$

and

$$\begin{aligned} \sum_{i=0}^n (-1)^n S_1(n, i) D_r(i) &= r! \sum_{i=r}^n (-1)^n \binom{i}{r} d_{i-r}^{(r+1)} S_1(n, i) \\ &= n! \sum_{k=0}^n \sum_{i=0}^k (-1)^{r+k} \frac{i!}{k!} \binom{i+r}{i} S_1(k, i) H(n-k, r-1). \end{aligned}$$

Proof. Taking $\ln(1-x)$ instead of x and setting $n + N \rightarrow n$ in Theorem 5, we can easily obtain these equalities. $\square$

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USING FUZZY-ROUGH SET EVALUATION FOR FEATURE SELECTION AND NAIVE BAYES TO CLASSIFY THE PARKINSON DISEASE

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Abstract. Feature selection is one of the issues in machine learning as well as statistical pattern recognition. This is important in many fields (such as classification) because there are many features in these areas, many of which are either unused or have little information load. Not eliminating these features does not make a problem in terms of information, but it does increase the computational burden for the intended application. Besides, it causes to store of so much useless information along with useful data. A problem for machine learning research occurs when there are many possible features with few attributes of training data. One way is to first specify the best attributes for prediction and then to classify features based on a measure of their dependence. In this study, the Fuzzy- Rough subset evaluation has been used to take features in core of similar features. Fuzzy-rough set-based feature selection (FS) has been demonstrated to be extremely advantageous at reducing dataset size but has various problems that yield it unproductive for big datasets. Fuzzy- Rough subset evaluation algorithm indicates that the techniques greatly decrease dimensionality while keeping classification accuracy. This paper considers classifying attributes by using fuzzy set similarity measures as well as the dependency degree as a relatedness measure. Here we use Artificial Neural Network, Naive Bayes as classifiers, and the performance of these techniques are compared by accuracy, precision, recall, and F-measure metrics.

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1. INTRODUCTION

These days with the increasing of computer and database technology, a large number of features can be obtained and saved in databases for various real-world implementations. Some of the features could be irrelevant or redundant for classifying learning; they could significantly lead to reduces efficiency and accuracy of classifiers and causes utmost computational confusion. Consequently, for employing a data collection it is needed to preprocess the data to extract inessential attributes. Attribute choosing or feature selection, a significant method for declining the number of unneeded attributes, is applied to figure out an ideal attribute subclass to carry out

classification on the assumption that keeping categorization accuracy. Of late years, attribute extraction has been comprehensively utilizing in data procedure, template identification, and machine learning [9], [12], [14], [24], [25].

The standard of the machine-legible input that works on it usually contributes to the success of machine learning algorithms. These items comprise whether there is indifferent, needless, untrustworthy, or noisy data. Data mining, which has started to gain great importance recently, has achieved great success in interpreting big data by processing; It makes predictions effective in solving major problems in many sectors, especially finance, health, communication, and education [15].

The use of feature selection, artificial intelligence, and machine learning applications in the field of health is carried out in many sub-fields such as medical diagnosis and disease tracking, cost estimation, imaging analysis, resource planning, and emergency management, processing of unstructured data [3, 4, 16, 18]. Artificial intelligence models, which are also used to make high-scale patient data functional, play an active role in increasing data reliability and quality [6], [23]. However, regarding the use of artificial intelligence in the field of health; The accuracy of clinical data, data management, legal and ethical processes regarding data protection limit the use of artificial intelligence in the field of health. Machine learning techniques are most commonly used in the field of health sciences for predicting, diagnosis, and determining post-illness complications, and it is aimed to provide patients with better quality healthcare services by saving time and workload [5].

Indeed, it is stated that machine learning-based classification methods play a decisive role in both decision support systems and disease diagnosis in today's medical research [13].

Recently, there has been a great deal of interest in evolving methods that able to deal with inaccuracies and uncertainties, and a significant amount of them is in the field of fuzzy and rough sets. Rough set theory's success is owing to some particulars of the theory. Merely the facts covered in the data are investigated. No further information about the inputs is needed for data analysis such as thresholds or proficient knowledge on a specific field. And it detects the minutest knowledge representation for data. While rough set theory controls only one type of handicap that exists in data, it is supplementary to other concepts for the goal, like fuzzy set theory. Fuzzy sets are deal with ambiguity, and rough sets are cover the indiscriminating [17]. Parkinson's disease (PD) appears as the decease of dopaminergic neurons in the substantiate nigra pars condensed inside the midbrain. This neuro damage caused a domain of symptoms containing coordination outcomes, bradykinesia, vocal changes, and toughness. Dysarthria is also detected in PD patients; it is specified by laxity, paralysis, and deficiency of coordination in the motor-speech system: affecting respiration, phonation, articulation, and prosody. whereas symptoms and the disease period change, PD is mostly not distinguished for many years. accordingly, there is a necessity for more accurate diagnostic implements for PD finding because,

as the disease moves forward, more symptoms become apparent that make PD more difficult to treat. Therefore, a big deal of endeavor has been made to extend methods for early detection, mostly at pre-symptomatic phases in order to slow or stop disease forward movement. The rapid advance in machine learning techniques has made it challenging to combine large-scale, high-dimensional objects. Thus, it has extended quickly in computer-aided machine learning approaches for Combined analysis. Well-known pattern analysis methods, such as Artificial Neural Network (ANN), Naive Byes (NB), have been used for early detection of PD and the prediction of PD progression [8], [19].

2. MATERIALS AND METHODS

Fuzzy sets were presented and expressed using membership functions by L.A. Zadeh in 1965 and have many convenient utilizations [27]. For the aim to implement fuzzy set similarity measures to specify concurrence between two distinct features, each feature is expressed as a fuzzy set over the patient's data sets. The dataset must be normalized to specify a degree of membership in $[0, 1]$. The patient's membership degree in the fuzzy set specifies the level of certainty for that patient's data. Suppose U is a reference set (finite and non-null collection of items) and R is a non-null set of finite features.

Definition 1. In the classical set theory an element must belong or not belong to a set. In fuzzy theory, an element can belong to a set by k degree ($0 \leq k \leq 1$). The fuzzy belonging function is shown such as $\mu_A(x) \in (0, 1)$ where A is an element collection, x is an object, and A is a fuzzy set which is given below [21].

Definition 2. A quaternary (U, R, K_S, S) gives a fuzzy information system (FIS). K_S is the set of entire fuzzy numbers and S is a knowledge operation which $F(x, r) = \mu_r(x), \forall x \in U, r \in R$ and $S: U \times R \rightarrow K_S$ and $\mu_r(x)$ is membership degree.

Definition 3. A rough set concept is another way to deal with ambiguity. Unlike the fuzzy set, the uncertainty in the rough set is determined with a boundary area, not through a partial membership function. Internal topological functions and closures as estimation can define a rough set. U is a given universe and $R \subseteq U \times U$ is an indiscernibility connection which demonstrates our information shortage about members of U . Let R is equivalence intercourse and $X \subseteq U$. Now determine the set X with about R through a primary hypothesis of rough set theory.

- R -lower approximation of $X: \underline{R}(x) = \bigcup \{R(x) : R(x) \subseteq X\}$
- R -upper approximation of $X: \overline{R}(x) = \bigcup \{R(x) : R(x) \cap X \neq \emptyset\}$
- R -boundary approximation of $X: RnR(x) = \overline{R}(x) - \underline{R}(x)$

The rough set membership function is described as:

$$\mu_{XR}: U \rightarrow (0, 1) \text{ where } \mu_{XR}(x) = \frac{|X \cap R(x)|}{|R(x)|} \text{ and } |R(x)| \text{ denotes the cardinality of } x.$$

Definition 4. A quaternary (U, R, K_S, S) gives a fuzzy information system (FIS). K_S is the set of all fuzzy numbers and S is an information function which $F(x, r) = \mu_r(x)$, $\forall x \in U, r \in R$ and $S : U \times R \rightarrow K_S$ and $\mu_r(x)$ is membership degree.

Definition 5. Let $R = (U, A)$ be an information framework, that U is a not empty collection of limited objects (the universe of discourse), and A is a nonempty limited collection of features which $a : U \rightarrow V_a$; $\forall a \in A$. V_a is the collection of values that criteria a can receive.

$$IND(P) = \{(x, y) \in U^2 \mid \forall a \in P, a(x) = a(y)\}$$

The division of U , which is produced by $IND(P)$, is demonstrated $U/IND(P)$ (or U/P for simplicity) and can be computed as:

$$U/IND(P) = \bigotimes \{U/IND(\{a\}) \mid a \in P\}$$

where $\bigotimes$ is particularly defined as follows for sets A and B :

$$A \bigotimes B = \{X \cap Y \mid X \in A, Y \in B, X \cap Y \neq \emptyset\}$$

If $(x, y) \in IND(P)$, then x and y are indistinguishable by criteria from P . The equation classes of the P indistinguishable relationship are demonstrated $[x]_P$.

Let $X \subseteq U$. X can be approximated by employing only the knowledge contained within P by setting up the P -lower and P -upper approximations of X :

$$\underline{P}X = \{x \in U \mid [x]_P \subseteq X\}$$

$$\overline{P}X = \{x \in U \mid [x]_P \cap X \neq \emptyset\}$$

$\langle \underline{P}X, \overline{P}X \rangle$ is labelled as a rough set. Let P and Q be sets of elements inducing equivalence relationship over U , then the positive, negative, and boundary regions can be described as:

$$POS_P(Q) = \bigcup_{X \in U/Q} \underline{P}X$$

$$NEG_P(Q) = U - \bigcup_{X \in U/Q} \overline{P}X$$

$$BND_P(Q) = \bigcup_{X \in U/Q} \overline{P}X - \bigcup_{X \in U/Q} \underline{P}X$$

The positive region comprises all items of U that can be categorized into classes of U/Q by employing the information in attribute P . The boundary region is the set of attributes that can possibly, but not sure, be categorized in this method. The negative

region is the set of elements that cannot be categorized into classes of U/Q . A significant subject in data mining is finding dependencies between features. intuitively, a set of features Q depends entirely on a set of attributes P , which is demonstrated by $P \Rightarrow Q$, if all data values from Q are uniquely specified by values of objects from P . If there is an operative dependency between values of Q and P , then Q depends entirely on P . $\forall P, Q \subset A$, Q depends on P in a degree $k(0 \leq k \leq 1)$, which is denoted $P \Rightarrow k Q$, if:

$$k = \gamma_P = \frac{|POS_P(Q)|}{|U|}$$

- If $k = 1$, Q depends entirely on P ;
- if $0 < k < 1$, Q depends partially (in a degree k) on P ;
- if $k = 0$, then Q does not depend on P .

By computing the change in dependency when a feature is eliminated from the set of observed conditional features, a scale of the importance of the attribute can be acquired. The vast the change in dependency, the more important the attribute is. If the importance is 0, then the attribute is dispensable. More officially, given P, Q , and an attribute $a \in P$ [2]:

$$\sigma_P(Q, a) = \gamma_P(Q) - \gamma_{P-a}(Q)$$

Definition 6. A description for fuzzy P -lower and P -upper approximations was identified as follows:

$$\begin{aligned} \mu_{PX}(F_i) &= \inf_x \max\{1 - \mu_{F_i}(x), \mu_x(x)\} \quad \forall i \\ \mu_{\bar{P}X}(F_i) &= \sup_x \min\{\mu_{F_i}(x), \mu_x(x)\} \quad \forall i \end{aligned}$$

where F_i is a fuzzy equivalency class, and X is the (fuzzy) notion to be approximated.

The $\langle \mu_{PX}, \mu_{\bar{P}X} \rangle$ is described as a fuzzy-rough set. These definitions disunite a concise from the crisp upper and lower approximations, as the memberships of single attributes to the approximations are not clearly accessible. Thereupon, the fuzzy lower and upper approximations are described as follows in [7].

$$\begin{aligned} \mu_{PX}(x) &= \sup_{F \in U/P} \min \left(\mu_F(x), \inf_{y \in U} \max\{1 - \mu_F(y), \mu_X(y)\} \right) \\ \mu_{\bar{P}X}(x) &= \sup_{F \in U/P} \min \left(\mu_F(x), \sup_{y \in U} \min\{\mu_F(y), \mu_X(y)\} \right) \end{aligned}$$

Definition 7. Fuzzy-Rough Reduction procedure: Fuzzy-rough set-based FS constructs on the idea of the fuzzy lower approximation to provide a reduction of datasets comprising real-valued features. The procedure will be the same with a crisp approach when the attributes are well-defined nominal attributes. The membership

of an attribute $x \in U$ be the property of the fuzzy positive region can be determined by:

$$\mu_{POS_P(Q)}(x) = \sup_{X \in U/Q} \mu_{\bar{P}X}(x)$$

The fuzzy- rough dependency function can be determined as come after:

$$\gamma'_P(Q) = \frac{|\mu_{POS_P(Q)}(x)|}{|U|} = \frac{\sum_{x \in U} \mu_{POS_P(Q)}(x)}{|U|}$$

If the fuzzy-rough diminution process is to be a utility, it should be able to struggle with various features by calculating the dependency between the multiple subsets of the original attribute set. In the fuzzy condition, objects could belong to various equivalence classes, and thus, the Cartesian product of $U/IND(\{a\})$ and $U/IND(\{b\})$ must be noted in assigning U/P .

$$(U/P) = \bigotimes \{U/IND(a) | a \in P\}$$

where

$$A = \{X \cap Y | X \in A, Y \in B, X \cap Y \neq \emptyset\}$$

Each set in U/P indicate an equivalence class [22].

Definition 8. A fuzzy knowledge framework is determined as (U, B, V_F, F) , which V_F is the set of whole fuzzy data and F is an knowledge assignment determined as $F : U \times B \rightarrow V_F$.

Such that $F(x, b) = \mu_b(x), \forall x \in U$ and $b \in B$, where $\mu_b(x)$ is a membership grade of the criteria x for the feature b .

Definition 9. Let FDS be a fuzzy decision system and fuzzy resemblance among two criteria for every feature could be determined as [10]:

$$SIM_a(x_i, x_j) = 1 - \frac{|\mu_a(x_i) - \mu_a(x_j)|}{|\mu_{a_{\max}} - \mu_{a_{\min}}|}$$

That, $\mu_a(x_i)$ and $\mu_a(x_j)$ are membership function of criteria x_i, x_j separately and $\mu_{a_{\max}}$ and $\mu_{a_{\min}}$ are maximum and minimum membership function for a criterion “a” respectively [20].

For a subset of attributes P ,

$$(x_i, x_j) \in SIM_P^\delta \text{ iff } SIM_a(x_i, x_j) \geq \delta$$

where δ is a resemblance entrance and tolerance classes are produced by fuzzy similarity connection as:

$$SIM_P^\delta(x_i) = \{x_j \in U | (x_i, x_j) \in SIM_P^\delta\}$$

So, lower and upper approximations of $X \subseteq U$ are expressed as:

$$\underline{P}^\delta X = \{x_i | SIM_P^\delta(x_i) \subseteq X\}$$

$$\overline{P}^{\delta}X = \{x_i | SIM_P^{\delta}(x_i) \cap X \neq \emptyset\}$$

The pair ($\underline{P}^{\delta}X$, $\overline{P}^{\delta}X$) is named tolerance fuzzy rough set. The positive region and dependency degree determined as before.

In each phase, we append one feature in the decrement collection and compute the grade of dependency, when there is not increase in the amount of dependency, the algorithm ends.

3. THE RESEARCH FINDINGS AND DISCUSSION

Fuzzy logic employs linguistic variables, determined as fuzzy sets, to approach human reasoning. The used feature selection technic gets pre-processed data set as an input and generates ranked attributes based on the got-together approach of F –Score and Accuracy on data set statement values normalized by fuzzy Gaussian membership function. we used fuzzy- rough subset evaluation to feature selection. These top- n selected data are used by the Fuzzy- discernibility and Neural- a network for classification. The process is explained with the following steps:

- a. **Normalization** Before doing anything with the dataset, it must be normalized. Occasionally, some values may be missing in the real valued dataset, there are operations such as filling or eliminating these values in preparation. And sometimes the scalars of the data are not the same, so it is necessary to bring them to the same proportions, and this is one of the works done in preparation as normalization or standardization. Sometimes there are nominal values that must be change to numeric values. Fuzzification is one of those normalization which based on fuzzy membership function. Fuzzification is the procedure of turning crisp inputs into fuzzy utilities. The membership functions supplied membership values indicate the degree of membership of a linguistic term.
The datasets used in this study comprise expression data for a set of attributes existing at <https://www.kaggle.com/c/parkinsons-detection>. This work is applied using Weka version 3.9.5.
- b. **Attribute Selection** After normalized the dataset it is time to attribute selection. In this study as said before we applied a fuzzy rough subset evaluation technique for attribute selection and the searching method is Hill Climber. In this method, the fuzzy- rough set similarity is implemented for every feature and then select the n –top attributes.
- c. **Classifying** In this study, Artificial Neural Network (ANN) and Naive Bayes (NB) are utilized as classifiers to determine the performance of attribute selection technique. Artificial Neural Network (ANN) and Naive Bayes (NB) are employed as classifiers to determine the performance of attribute selection techniques in this work. Naive Bayes is an uncomplicated learning

algorithm that makes use of Bayes principle each other by a powerful supposition which the criteria are circumstantially individualistic, dedicated the class. Meanwhile this independence supposition is frequently disturbed, Naive Bayes against all odds constantly gives passionate classification correctness. Unified by its algorithmic performance and plenty alternative favourable features, it guides to Naive Bayes becoming extensively used virtually [26].

An ANN is adaptive in disposition because it changes its anatomy and modifies its weight to keep down the error. An adaptation of weight is based on the knowledge that moves internally and externally by network within the learning period. The vantages of ANN are it needs less formal statistical training, indirectly uncover complex nonlinear connections betwixt dependent and independent variables, figure out all probable interplays among predictor variables, and the presence of various training algorithms [1].

- d. **Measurement Metrics** As mentioned in this paper, classification and accuracy, precision, recall, F -measure, and computational time metrics have been employed to analyze these techniques. The classifier performance is evaluated using training set validation. The average accuracy for the datasets is determined as the measure of correctly classified test samples. Then, the performances of every technique are evaluated with each other. The measurement metrics definitions are given below:

- d-1. **Accuracy** This evaluation parameter is used to determine how close the measurements of a value are to the true value.

$$\text{Accuracy} = \frac{TP + TN}{TP + TN + FN + FP}.$$

Here, TP is a true positive indicator that is accurately identified, TN represents a true negative that has been properly rejected, FP false positive that is misidentified, and likewise, FN represents a false negative that has been wrongly rejected.

- d-2. **Precision** This is determined by the proximity of two or more measurements to each other. Precision is also expressed as a positive predictive measure.

$$\text{Precision}(p) = \frac{TP}{TP + FP}.$$

- d-3. **Recall** is also noted as the actual positive proportion or sensibility that is retrieved to measure a division of the relevant samples.

$$\text{Recall}(r) = \frac{TP}{TP + FN}.$$

- d-4. **Recall-Precision** metric is a useful measure of success of prediction when the classes are very imbalanced. A large domain below the graph displays both great recall and big precision, that high precision shows

fallen false-positive rate, and high recall relates to a fallen false-negative rate.

- d-5. **F- Measure** This is an evaluation of test carefulness. It considers both p and r in the test to account for the measure.

$$F - Measure = \frac{2pr}{p+r}.$$

- d-6. **Computing Period** The interval indispensable to accomplish computational progress by assessing the classification implementation time [11].

Sum up Table 1 shows the Artificial Neural Network classification performance for Parkinson detection before selection attribute and after that. As can be seen from the table, a relative increase in performance is visible, and this means that the reduction of dimensions in the dataset while raising performance has caused a significant reduction in the calculated time.

Table 1. ANN Performances of Classification for Parkinson

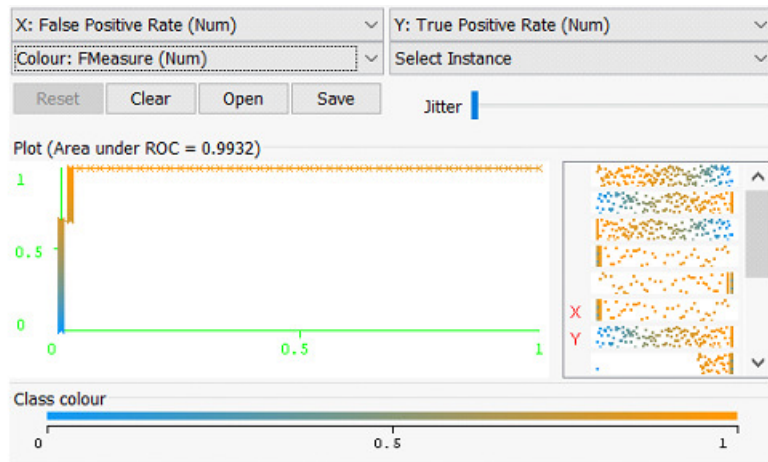
	ANN-Before Changing		ANN-After Changing	
TP Rate	1	0.979	1	0.979
FP Rate	0.021	0	0.021	0
Precision	0.993	1	0.993	1
Recall	1	0.979	1	0.979
F- Measure	0.997	0.989	0.997	0.989
ROC Area	0.993	0.993	0.994	0.994
Calculation- Time(S)	0.39	0.39	0.01	0.01
Class	Diagnosis	Not- Diagnosis	Diagnosis	Not- Diagnosis

In Table 2, you can find the Naive Bayes classification performance for Parkinson detection before selection attribute and after that. As table 2 demonstrates while there is a small change in the time of calculating, the performance of the algorithm has not been changed. So, the dimensionality reduction had not been affected by Naive Bayes' act. Hence, it can be said that Naive Bayes performed better than the Artificial Neural Network technique.

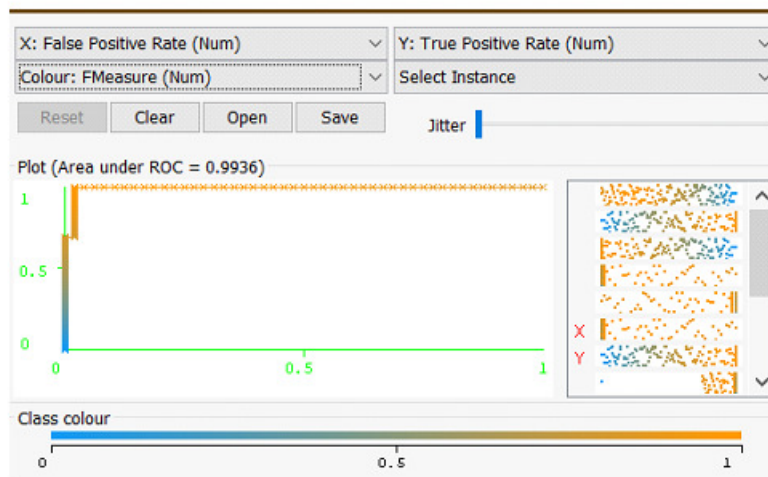
Table 2. NB Performances of Classification for Parkinson

	NB- Before Changing		NB-After Changing	
TP Rate	0.626	0.938	0.701	0.917
FP Rate	0.063	0.374	0.083	0.299
Precision	0.968	0.45	0.963	0.5
Recall	0.626	0.938	0.701	0.917
F- Measure	0.76	0.608	0.811	0.647
ROC Area	0.871	0.878	0.881	0.881
Calculation- Time(S)	0.019	0.019	0.018	0.018
Class	Diagnosis	Not- Diagnosis	Diagnosis	Not- Diagnosis

The ROC curve demonstrates the equilibration between susceptibility (or TPR) and specificity (FPR). Those curves are nearby to the top-left corner display better performance. A random classifier is awaited to have points over the diagonal (FPR = TPR). Consider that the ROC independent of the class distribution. This makes it fruitful for evaluating classifiers anticipating scarce cases such as diseases or disasters. Conversely, interpreting performance using accuracy would have preferred classifiers that regularly predict a negative result for infrequent cases.

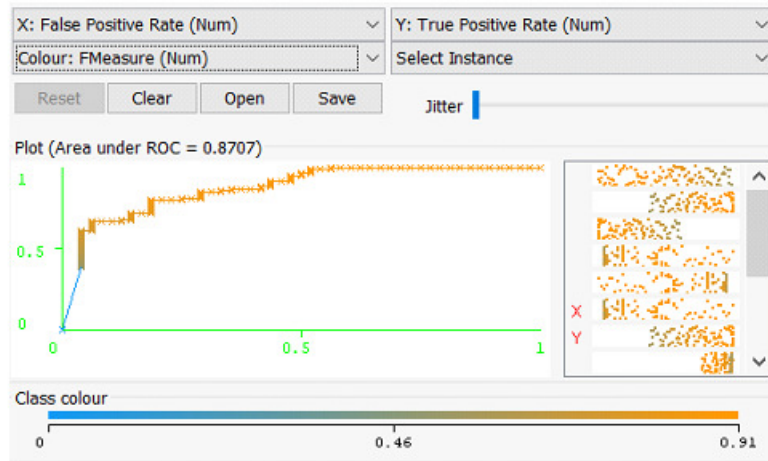


(a) ANN
Before

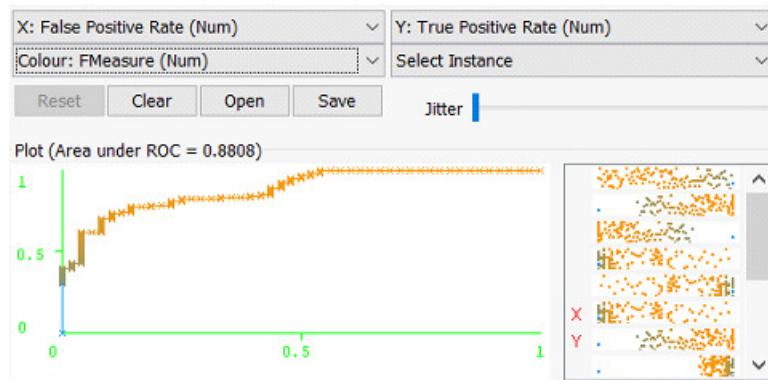


(b) ANN After

FIGURE 1. ANN ROC area before and after attribute selection



(a) NB Before



(b) NB After

FIGURE 2. The NB ROC- Area for Parkinson disease

The area under the curve (AUC), To compare distinct classifiers, it can be gainful to outline the performance of each classifier into an individual evaluation. One communal approach is to figure out the area under the ROC curve, which is called AUC. Although a classifier with a high AUC can sometimes give a worse point in a specific area than another classifier with a lower AUC, the AUC acts well as a general scale of predictive accuracy. As it can be seen Figure 1 and Figure 2 illustrate the changes after the dimension reduction. As it seems from Figures 1 and 2 while in the ANN algorithm a slight improvement in the AUC area is visible, the AUC area NB has

become better, and this means reducing the dimensions in the Naive Bayes method has had a good effect. On the other hand, the ROC curve of ANN totally has been acted better than the NB ROC curve. But here is the significant thing that should be noticed is changing the ROC and AUC before and after feature reduction. Therefore, instead of comparing Figures 1 and 2 with each other, the changes in Figures 1-a to 1-b should be considered and respectively for Figures 2-a to 2-b.

4. CONCLUSION

In this study, Fuzzy- Rough Subset Evaluation algorithm has been used for feature selection that can select a small set of datasets to prepare a highly precise classification of the instances. The proposed dataset normalized by fuzzy Gaussian membership function. The F-Score and Fuzzy-Rough Subset Evaluation are exploited on the normalized dataset to rank the objects. F-score is utilized to recognize relevant attributes and FRSE is applied to remove the redundancy among the features. In all feature selection results in Parkinson's data, the data selected may or may not be a subset of disease progression signature. So, the top n attributes are selected for classification in ANN and NB. The train set validation is used to detect the average classification accuracy. It provides 0.04% average classification accuracy for the ANN method and a 1.01% percentage for NB classifiers. It also gives the highest average AUC accuracy for NB compared to the ANN algorithms. In summary, the normalization of a dataset by using the Fuzzy Gaussian membership function can modify the classification accuracy with the suggested measure. The performance of the Naive Bayes is totally better than the Artificial Neural Network algorithm. So, the proposed method is effective and consistent for the detection of Parkinson with a small number of features.

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ROTATIONAL HYPERSURFACES WITH CONSTANT 2-MEAN CURVATURE IN $\mathbb{R}^{n+1}$

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Abstract. We study rotational hypersurfaces with constant 2-mean curvature in $\mathbb{R}^{n+1}$. We derive the ODE for the generating curves of such hypersurfaces, and find an integral expression for the inverse function of the solution.

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Keywords: higher order mean curvature, rotational hypersurface, non-linear differential equations

1. INTRODUCTION

In the field of submanifold geometry, principal curvatures tell us how the submanifold bends in each direction. Given a hypersurface, i.e. a submanifold of codimension one, the arithmetic mean of all principal curvatures is the mean curvature, while the product of principal curvatures is the Gauss-Kronecker curvature. It has been an active field of research to study various constraints on these notions of curvature. For example, the constant mean curvature (CMC) hypersurfaces in various ambient manifolds have been extensively studied. To explain our motivation, we mention a few results here. For rotational CMC surfaces in $\mathbb{R}^3$, Delaunay has proposed a beautiful classification theorem which indicates that the generating curves of these surfaces are formed geometrically by rolling a conic along a straight line without slippage [2]. In the 1980s, Wu-Yi Hsiang and Wenci Yu generalized Delaunay's theorem to rotational hypersurfaces in $\mathbb{R}^n$ [3, 4].

Other than the mean curvature and Gauss-Kronecker curvature, symmetric functions of principal curvatures are also interesting. Fixing an integer r between one and the dimension of the hypersurface, the r -th symmetric function of the principal curvatures is called (unnormalized) “ r -mean curvature”. In this paper, we consider rotational hypersurfaces in Euclidean space with constant 2-mean curvature. It can be seen as a variant of Hsiang and Yu's work mentioned above. Our main result is the following theorem:

Theorem 1. Let $M \subset \mathbb{R}^{n+1}$ be a rotational hypersurface with constant 2-mean curvature K such that its generating curve γ is a graph over the axis of rotation. Let $\gamma(s) = (\varphi(s), \psi(s))$ be a parametrization of the generating curve, where $\varphi(s)$ is the radius of the meridian $(n-1)$ -sphere, $\psi(s)$ is the height function and s is the arclength parameter. Moreover set $u(s) = \varphi(s)^{n/2}$. Then we have the following local expression for the inverse function of $u(s)$:

$$s = \int \pm \frac{du}{\sqrt{\frac{n}{n-1} \left(\frac{n(n-1)}{4} u^{2-4/n} - \frac{K}{2} u^2 + C \right)}}. \quad (1.1)$$

Here the sign of the integrand agrees with the sign of φ' , and C is a real constant.

We note that compact embedded hypersurfaces with constant r -mean curvature have been characterized in [5], while complete ones have been studied in [1, 6]. However, our examples are in general incomplete and thus are not covered by their results.

2. ROTATIONAL HYPERSURFACES AND ITS CURVATURES

We set up notations and state the formulae for principal curvatures and 2-mean curvature of a rotational hypersurface in $\mathbb{R}^{n+1}$. Throughout the paper, we take $n \geq 3$ to avoid the trivial cases. The definitions of principal curvatures and higher order mean curvatures will be provided in the Appendix. Let $x_1, x_2, \dots, x_{n+1}$ denote the standard coordinates of $\mathbb{R}^{n+1}$ and we assume that x_{n+1} is the axis of rotation. Let $f: \mathbb{R} \rightarrow (0, +\infty)$ be a smooth function.

Definition 1. A hypersurface M is called a *Rotational Hypersurface* if it is produced by rotating the *generating curve* $x_1 = f(x_{n+1})$ in the $x_1 x_{n+1}$ -plane around the x_{n+1} axis. It is characterized by the following equation

$$f(x_{n+1})^2 = \sum_{i=1}^n x_i^2.$$

Note that $f(x_{n+1})$ is the radius of the horizontal subsphere at height x_{n+1} . Throughout this paper, M will always denote a rotational hypersurface in $\mathbb{R}^{n+1}$ unless otherwise stated.

We choose an appropriate parametrization of the generating curve to facilitate the calculation. Let $\varphi(s)$ denote the radius of the $n-1$ dimensional hypersphere and $\psi(s)$ denote the corresponding height. We choose the parameter s to be the arclength parameter, that is, $\varphi'^2 + \psi'^2 = 1$. Under the above parametrization, the generating curve $x_1 = f(x_{n+1})$ is parametrized as $(x_1, x_{n+1}) = (\varphi(s), \psi(s))$. Note that $\varphi(s) \geq 0$ since it is the radius, and we require $\psi'(s) \geq 0$ so that the generating curve is a graph over the x_{n+1} axis.

We use the hypersphere coordinate $(\varphi, \theta_1, \dots, \theta_{n-1})$ to parametrize the rotational hypersurface. The position vector field of rotational hypersurface M can be written

as

$$\vec{r}(\varphi, \theta_1, \dots, \theta_{n-1}) = (\varphi \cos \theta_1 \cdots \cos \theta_{n-1}, \varphi \cos \theta_1 \cdots \cos \theta_{n-2} \sin \theta_{n-1}, \dots, \varphi \cos \theta_1 \sin \theta_2, \varphi \sin \theta_1, \psi) \quad (2.1)$$

where $\theta_1 \in [-\pi/2, \pi/2]$ and $\theta_i \in [0, 2\pi]$ for $i = 2, 3, \dots, n-1$. Note that ψ can be expressed in terms of φ since $\varphi'^2 + \psi'^2 = 1$.

Under the above parametrization, the principal curvatures and the 2-mean curvature of M are:

Theorem 2. *The principal curvatures $k_1, \dots, k_n$ of M are given below:*

- (1) $k_1 = -\frac{\varphi''}{\psi'}$;
- (2) $k_i = \frac{\psi'}{\varphi}$ for $i = 2, 3, \dots, n$.

Theorem 3. *The 2-mean curvature K of M is given below:*

$$K = \sum_{1 \leq i < j \leq n} k_i k_j = -(n-1) \frac{\varphi''}{\varphi} + \binom{n-1}{2} \frac{1-\varphi'^2}{\varphi^2}. \quad (2.2)$$

Detailed calculations can be found in the Appendix.

3. CONSTANT 2-MEAN CURVATURE

In the previous section, if we require the 2-mean curvature to be a constant K , then we obtain the following ODE:

$$(n-1)\varphi''\varphi + \binom{n-1}{2}\varphi'^2 + K\varphi^2 = \binom{n-1}{2}. \quad (3.1)$$

Set $m = \binom{n-1}{2}/(n-1) + 1 = n/2$, then

$$\frac{(\varphi^m)''}{\varphi^{m-2}} = \frac{m}{n-1} \left((n-1)\varphi''\varphi + \binom{n-1}{2}\varphi'^2 \right). \quad (3.2)$$

Thus combining (3.1) and (3.2) we have

$$\frac{n-1}{m}(\varphi^m)'' + K\varphi^m = \binom{n-1}{2}\varphi^{m-2}.$$

Set $u = \varphi^m$, then:

$$\frac{n-1}{m}u'' + Ku - \binom{n-1}{2}u^{(m-2)/m} = 0.$$

Multiply both sides by u' and then integrate:

$$\frac{n-1}{2m}u'^2 = \binom{n-1}{2} \frac{m}{2m-2} u^{2-2/m} - \frac{K}{2} u^2 + C.$$

Finally solving this equation by separation of variables gives immediately (1.1) of Theorem 1.

4. APPENDIX

In the Appendix we provide essential definitions and notations about the curvatures of hypersurfaces in $\mathbb{R}^{n+1}$ and carry out the calculation. All notations are defined in the Euclidean Space $\mathbb{R}^{n+1}$.

A hypersurface M is a codimension one submanifold of $\mathbb{R}^{n+1}$. Let U be a domain in $\mathbb{R}^n$ and

$$\vec{r}: U \rightarrow M \subset \mathbb{R}^{n+1}, \quad \vec{r} = \vec{r}(x_1, x_2, \dots, x_n)$$

be a local coordinate chart of M . We call $\vec{r}$ the *position vector field* of M in $\mathbb{R}^{n+1}$.

The *tangent vectors* of M are

$$\frac{\partial \vec{r}}{\partial x_1}, \frac{\partial \vec{r}}{\partial x_2}, \dots, \frac{\partial \vec{r}}{\partial x_n}.$$

The vector $\vec{n}$ of length one that is perpendicular to all tangent vectors of M is the *unit normal vector* of M .

Definition 2 (First Fundamental Form). Denote the first order derivatives by $\vec{r}_i = \frac{\partial \vec{r}}{\partial x_i}$. The *first fundamental form* of M is given below:

$$I = \begin{bmatrix} \vec{r}_1 \cdot \vec{r}_1 & \vec{r}_1 \cdot \vec{r}_2 & \cdots & \vec{r}_1 \cdot \vec{r}_n \\ \vec{r}_2 \cdot \vec{r}_1 & \vec{r}_2 \cdot \vec{r}_2 & \cdots & \vec{r}_2 \cdot \vec{r}_n \\ \vdots & \vdots & \ddots & \vdots \\ \vec{r}_n \cdot \vec{r}_1 & \vec{r}_n \cdot \vec{r}_2 & \cdots & \vec{r}_n \cdot \vec{r}_n \end{bmatrix}$$

Definition 3 (Second Fundamental Form). Denote the second order derivatives by $\vec{r}_{i,j} = \frac{\partial^2 \vec{r}}{\partial x_i \partial x_j}$. The *second fundamental form* of M is given below:

$$II = \begin{bmatrix} \vec{r}_{1,1} \cdot \vec{n} & \vec{r}_{1,2} \cdot \vec{n} & \cdots & \vec{r}_{1,n} \cdot \vec{n} \\ \vec{r}_{2,1} \cdot \vec{n} & \vec{r}_{2,2} \cdot \vec{n} & \cdots & \vec{r}_{2,n} \cdot \vec{n} \\ \vdots & \vdots & \ddots & \vdots \\ \vec{r}_{n,1} \cdot \vec{n} & \vec{r}_{n,2} \cdot \vec{n} & \cdots & \vec{r}_{n,n} \cdot \vec{n} \end{bmatrix}.$$

Definition 4 (Principal Curvature). Let matrix $A = -II \cdot I^{-1}$ where I^{-1} denotes the inverse matrix of I . The n eigenvalues of matrix A are the *principal curvatures* of M , denoted as $k_1, \dots, k_n$.

Definition 5 (r -mean Curvature). For an integer r with $1 \leq r \leq n$, the (unnormalized) r -mean Curvature H_r of M is the r -th symmetric function of the n principal curvatures. That is, $H_r = \sum_{1 \leq i_1 < i_2 < \dots < i_r \leq n} \prod_{j=1}^r k_{i_j}$.

We use again the hypersphere coordinate (2.1) to parametrize a rotational hypersurface M . Under this parametrization, we can compute the tangent vectors, unit normal vector, first fundamental form, second fundamental form, principal curvatures, and 2-mean curvature of M as below.

Lemma 1. Let $\vec{r}_\varphi = \frac{\partial \vec{r}}{\partial \varphi}$ and $\vec{r}_i = \frac{\partial \vec{r}}{\partial \theta_i}$. The tangent vectors of M are given below:

$$\begin{aligned}\vec{r}_\varphi &= (\cos \theta_1 \cos \theta_2 \cdots \cos \theta_{n-1}, \dots, \cos \theta_1 \sin \theta_2, \sin \theta_1, \frac{\psi'}{\varphi'}), \\ \vec{r}_1 &= (-\varphi \sin \theta_1 \cos \theta_2 \cdots \cos \theta_{n-1}, \dots, -\varphi \sin \theta_1 \sin \theta_2, \varphi \cos \theta_1, 0), \\ \vec{r}_2 &= (-\varphi \cos \theta_1 \sin \theta_2 \cos \theta_3 \cdots \cos \theta_{n-1}, \dots, -\varphi \cos \theta_1 \cos \theta_2, 0, 0), \\ &\vdots \\ \vec{r}_{n-1} &= (-\varphi \cos \theta_1 \cdots \cos \theta_{n-2} \sin \theta_{n-1}, \varphi \cos \theta_1 \cdots \cos \theta_{n-2} \cos \theta_{n-1}, 0, \dots, 0).\end{aligned}$$

Proof. Notice that $\frac{\partial \psi}{\partial \varphi} = \frac{\frac{\partial \psi}{\partial s}}{\frac{\partial \varphi}{\partial s}} = \frac{\psi'}{\varphi'}$, we can derive the tangent vectors by computing the first partial derivatives of the position vector field $\vec{r}(\varphi, \theta_1, \dots, \theta_{n-1})$ with respect to $\theta_1, \theta_2, \dots, \theta_{n-1}$ respectively. $\square$

Lemma 2. The Unit Normal Vector of M is given below:

$$\begin{aligned}\vec{n} &= \psi'(\cos \theta_1 \cos \theta_2 \cdots \cos \theta_{n-1}, \cos \theta_1 \cdots \cos \theta_{n-2} \sin \theta_{n-1}, \\ &\quad \dots, \cos \theta_1 \sin \theta_2, \sin \theta_1, -\frac{\varphi'}{\psi'}).\end{aligned}$$

Proof. We only need to show that $\vec{n} \cdot \vec{r}_\varphi = 0$ and $\vec{n} \cdot \vec{r}_i = 0$ for $i = 1, \dots, n-1$. Let $x_{a,b}$ denote the value of the b^{th} coordinate of $\vec{r}_a$. First consider the value of $\vec{n} \cdot \vec{r}_\varphi$:

$$\begin{aligned}\vec{n} \cdot \vec{r}_\varphi &= \psi' \left(\sum_{i=1}^n (x_{\varphi,i})^2 + \frac{\psi'}{\varphi'} \cdot \left(-\frac{\varphi'}{\psi'} \right) \right) \\ &= \psi' [(\cos \theta_1 \cos \theta_2 \cdots \cos \theta_{n-1})^2 + \cdots + (\cos \theta_1 \sin \theta_2)^2 + \sin^2 \theta_1 - 1] \\ &= \psi' [(\cos \theta_1 \cdots \cos \theta_{n-2})^2 + \cdots + (\cos \theta_1 \sin \theta_2)^2 + \sin^2 \theta_1 - 1] \\ &= \psi' [\sin^2 \theta_1 + \cos^2 \theta_1 - 1] \\ &= 0.\end{aligned}$$

From the above equation, we get

$$\sum_{i=1}^k (x_{\varphi,i})^2 = (\cos \theta_1 \cos \theta_2 \cdots \cos \theta_{n-k})^2.$$

Notice that:

- (1) $x_{i,j} = -\varphi x_{\psi,j} \tan \theta_i$ for $j = 1, 2, \dots, n-i$;
- (2) $x_{i,j} = \varphi x_{\psi,j} \cot \theta_i$ for $j = n-i+1$;
- (3) $x_{i,j} = 0$ for $j > n-i+1$;
- (4) The first n coordinates of $\vec{n}$ and $\vec{r}_\varphi$ are identical.

We get

$$\begin{aligned}
 \vec{n} \cdot \vec{r}_i &= \psi' \left(\sum_{j=1}^{n-i} x_{i,j} x_{\varphi,j} + x_{i,n-i+1} x_{\varphi,n-i} \right) \\
 &= \psi' \left(-\varphi \tan \theta_i \sum_{j=1}^{n-i} (x_{\psi,j})^2 + \varphi \cot \theta_i (\cos \theta_1 \cdots \cos \theta_{i-1} \sin \theta_i)^2 \right) \\
 &= \psi' \left(-\varphi \tan \theta_i (\cos \theta_1 \cdots \cos \theta_i)^2 + \varphi \cot \theta_i (\cos \theta_1 \cdots \cos \theta_{i-1} \sin \theta_i)^2 \right) = 0.
 \end{aligned}$$

Moreover, it is clear that

$$|\vec{n}| = \psi' \sqrt{1 + \left(-\frac{\varphi'}{\psi'}\right)^2} = \sqrt{\varphi'^2 + \psi'^2} = 1.$$

Therefore, $\vec{n}$ is indeed the Unit Normal Vector of M . $\square$

Lemma 3. *The First Fundamental Form of M is a diagonal matrix in the form below:*

$$I = \begin{bmatrix} |\vec{r}_\varphi|^2 & \cdots & 0 & 0 \\ \vdots & |\vec{r}_1|^2 & \cdots & 0 \\ 0 & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & |\vec{r}_{n-1}|^2 \end{bmatrix}.$$

Proof. From Lemma 2, we know that

$$\vec{r}_\varphi \cdot \vec{r}_i = \psi'^{-1} \cdot \vec{n} \cdot \vec{r}_i = 0.$$

So it remains to be shown that

$$\vec{r}_i \cdot \vec{r}_j = 0 \quad (i \neq j).$$

Assume that $i > j$. From Lemma 1 we get

$$\begin{aligned}
 \vec{r}_i \cdot \vec{r}_j &= \sum_{k=1}^{n-i} x_{i,k} x_{j,k} + x_{i,n-i} x_{j,n-i+1} \\
 &= \varphi^2 \tan \theta_i \tan \theta_j \sum_{k=1}^{n-i} (x_{\psi,i})^2 - \varphi^2 \cot \theta_i \tan \theta_j (\cos \theta_1 \cdots \cos \theta_{i-1} \sin \theta_i)^2 \\
 &= \varphi^2 \tan \theta_j [\tan \theta_i (\cos \theta_1 \cdots \cos \theta_i)^2 - \cot \theta_i (\cos \theta_1 \cdots \cos \theta_{i-1} \sin \theta_i)^2] \\
 &= 0.
 \end{aligned}$$

The above equation indicates that I is a diagonal matrix as stated in the theorem. $\square$

Lemma 4. Let $\vec{r}_{x,y}$ denotes $\frac{\partial^2 \vec{r}}{\partial x \partial y}$ where θ_i is replaced by i . The Second Fundamental Form of M is another diagonal matrix in the form below:

$$II = \begin{bmatrix} \vec{r}_{\varphi,\varphi} \cdot \vec{n} & \cdots & 0 & 0 \\ \vdots & \vec{r}_{1,1} \cdot \vec{n} & \cdots & 0 \\ 0 & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \vec{r}_{n-1,n-1} \cdot \vec{n} \end{bmatrix}.$$

Proof. From Lemma 1 and the labels in Lemma 2, we can further derive the second derivatives as below:

$$(1) \quad \vec{r}_{\varphi,\varphi} = (0, 0, \dots, 0, -\frac{\varphi''}{\varphi^3 \psi'});$$

$$(2) \quad \vec{r}_{i,i} = (-\varphi x_{\varphi,1}, -\varphi x_{\varphi,2}, \dots, -\varphi x_{\varphi,n-i+1}, 0, \dots, 0);$$

(3)

$$\vec{r}_{\varphi,i} = \vec{r}_{i,\varphi} = (-x_{\varphi,1} \tan \theta_i, -x_{\varphi,2} \tan \theta_i, \dots, -x_{\varphi,n-i} \tan \theta_i, x_{\varphi,n-i+1} \cot \theta_i, 0, \dots, 0);$$

(4)

$$\vec{r}_{i,j} = \vec{r}_{j,i} = (\varphi x_{\varphi,1} \tan \theta_i \tan \theta_j, \dots, \varphi x_{\varphi,n-i} \tan \theta_i \tan \theta_j, -\varphi \cot \theta_i \tan \theta_j, 0, \dots, 0) \quad \text{for } i > j.$$

So, we only need to prove the inner product of $\vec{n}$ and the derivatives in Items (3) and (4) is 0. From Lemma 2, we get

$$\begin{aligned} \vec{r}_{\varphi,i} \cdot \vec{n} &= \psi' [-\tan \theta_i \sum_{k=1}^{n-i} x_{\varphi,k}^2 + \cot \theta_i x_{\varphi,n-i+1}^2] \\ &= \psi' [-\tan \theta_i (\cos \theta_1 \cdots \cos \theta_i)^2 + \cot \theta_i (\cos \theta_1 \cdots \cos \theta_{i-1} \sin \theta_i)^2] \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} \vec{r}_{i,j} \cdot \vec{n} &= \psi' [\varphi \tan \theta_i \tan \theta_j \sum_{k=1}^{n-i} x_{\varphi,k}^2 + \cot \theta_i \tan \theta_j x_{\varphi,n-i+1}^2] \\ &= \psi' \varphi \tan \theta_j [\tan \theta_i (\cos \theta_1 \cdots \cos \theta_i)^2 - \cot \theta_i (\cos \theta_1 \cdots \cos \theta_{i-1} \sin \theta_i)^2] \\ &= 0. \end{aligned}$$

The above computation indicates that II is a diagonal matrix in the proposed form. $\square$

Now we can prove Theorem 2.

Proof. From Lemma 1, we can compute the entries in I as below:

$$(1) \quad |\vec{r}_{\varphi}|^2 = 1 + \frac{\psi'}{\varphi^2} = \frac{1}{\varphi^2}$$

$$(2) \quad |\vec{r}_i|^2 = \varphi^2 \prod_{a=1}^{i-1} \cos^2 \theta_a \quad (\text{Assume that } \cos \theta_0 = 1)$$

Similarly, we can compute the elements in II as below:

- (1) $\vec{r}_{\varphi, \varphi} \cdot \vec{n} = \frac{\varphi''}{\varphi^2 \psi'}$
 (2) $\vec{r}_{i,i} \cdot \vec{n} = -\varphi \psi' \prod_{a=1}^{i-1} \cos \theta_a^2$ (Assume that $\cos \theta_0 = 1$)

Then,

$$\begin{aligned}
 A &= -II \cdot I^{-1} \\
 &= \begin{bmatrix} -\frac{\varphi''}{\varphi^2 \psi'} & \cdots & 0 & 0 \\ \vdots & \varphi & \cdots & 0 \\ 0 & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varphi \prod_{a=1}^{n-1} \cos \theta_a^2 \end{bmatrix} \cdot \begin{bmatrix} \varphi'^2 & \cdots & 0 & 0 \\ \vdots & \frac{1}{\varphi^2} & \cdots & 0 \\ 0 & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\varphi^2 \prod_{a=1}^{n-1} \cos \theta_a^2} \end{bmatrix} \\
 &= \begin{bmatrix} -\frac{\varphi''}{\psi'} & \cdots & 0 & 0 \\ \vdots & \frac{\psi'}{\varphi} & \cdots & 0 \\ 0 & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\psi'}{\varphi} \end{bmatrix}
 \end{aligned}$$

Clearly, the principal curvatures are diagonal entries. $\square$

Using the formula for principal curvatures, we have derived expression (2.2) in Theorem 3 of the 2-mean curvature of M under the φ and ψ parametrization.

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A CLASS OF CONFORMABLE BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS WITH JUMPS

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Abstract. In this paper, we study conformable backward stochastic differential equations driven by a Brownian motion and a compensated random measure. We derive the conformable Itô's formula with jumps and a priori estimates and we obtain the existence and uniqueness of solutions under some assumptions in the framework of the conformable derivative. In addition we get a predictable representation of the solution. Comparison theorems for the operator g under different conditions are given. We also establish the inverse comparison theorem for the operator g under a Lipschitz condition.

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Keywords: backward stochastic differential equations, conformable derivative, existence, predictable representation, comparison theorem, inverse comparison theorem

1. INTRODUCTION

In stock transaction, the holder always expects that the option pricing problem can be calculated and analyzed more accurately in order to avoid the risk and to achieve the expected return and backward stochastic differential equations is a good tool to deal with this kind of problem. The linear backward stochastic differential equation was proposed by Bismut [4] in 1978. In 1990, Pardoux and Peng [19] obtained an existence and uniqueness theorem for non-linear backward stochastic differential equations under a Lipschitz condition; for applications in mathematical finance, stochastic control, partial differential equations, stochastic differential games and economics we refer the reader to [5, 9, 11].

Mandelbrot [14] proposed in 1963 that the distribution of returns in the financial market has a typical thick-tailed characteristic which does not obey the general normal distribution, and has the characteristics of self similarity and long-term correlation. This has led to a large number of pricing models driven by Brownian motion

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that do not conform to the real market (see [7, 10, 17, 18]). The fractional backward stochastic differential equation can give a reasonable explanation to this kind of problem. Fractional Brownian motion is neither a semi martingale nor a Markov process [15], so it can describe phenomena that cannot be described by semi martingales and Markov processes, and it plays an important role in finance, hydrology, meteorology, transportation, etc. It has self similarity and long-term memory [6], so it is practical to describe asset price changes by fractional backward stochastic differential equations. However, fractional derivatives have unique non-local characteristics (essentially different from the local properties of classical derivatives), so there are difficulties in the theoretical analysis and application [13]. Khalil et al. [12] defined a derivative called the conformable fractional derivative and when $\rho = 1$ its definition is consistent with the classical derivative. The conformable derivative has not only some properties of fractional derivative, but also some properties of integer order derivatives [2, 12]. In this paper we consider conformable backward stochastic differential equations.

This article is mainly divided into six parts. First, we give some important lemmas that are applicable to conformable backward stochastic differential equation. We give the Itô's formula with jumps, state a priori estimates, and prove the existence of a unique solution. Then, the predictable representations of different types of solutions are studied. Next, we establish comparison theorems for the operator g under different constraints, and give a class of inverse comparison theorems. Finally, some examples are given to illustrate the theoretical results.

2. PRELIMINARIES

Let $B(\cdot)$ be a standard Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, P)$ with the filtration $\{\mathcal{F}_t\}_{t \geq 0}$, $\mathfrak{B}(\mathbb{R})$ denotes the Borel sets of $\mathbb{R}$ and $\mathbb{E}$ denotes the expected value. Next we give some definitions of spaces.

$L_Q^2(\mathbb{R})$ is a space of measurable function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ satisfying $\int_{\mathbb{R}} |\varphi(s)|^2 Q(ds) < \infty$, where Q is a σ -finite measure; $\mathbb{L}^2(\mathbb{R})$ is a space of random variable $\zeta : \Omega \rightarrow \mathbb{R}$ satisfying $\mathbb{E}[|\zeta|^2] < \infty$; $\mathbb{H}^2(\mathbb{R})$ is a space of predictable process $Y : \Omega \times [a, T] \rightarrow \mathbb{R}$ satisfying $\mathbb{E}[\int_a^T |Y(t)|^2 dt] < \infty$ with $\|Y\|_{\mathbb{H}^2}^2 = \mathbb{E}[\int_a^T e^{\sigma t} |Y(t)|^2 dt]$, σ is a constant; $\mathbb{H}_N^2(\mathbb{R})$ is a space of predictable process $Z : \Omega \times [a, T] \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying

$$\mathbb{E}[\int_a^T \int_{\mathbb{R}} |Z(t, s)|^2 Q(t, ds) \eta(t) dt] < \infty$$

with $\|Z\|_{\mathbb{H}_N^2}^2 = \mathbb{E}[\int_a^T \int_{\mathbb{R}} e^{\sigma t} |Z(t, s)|^2 Q(t, ds) \eta(t) dt]$; $\mathbb{S}^2(\mathbb{R})$ is a space of adapted, càdlàg process $X : \Omega \times [a, T] \rightarrow \mathbb{R}$ satisfying $\mathbb{E}[\sup_{t \in [a, T]} |X(t)|^2] < \infty$ with $\|X\|_{\mathbb{S}^2}^2 = \mathbb{E}[\sup_{t \in [a, T]} e^{\sigma t} |X(t)|^2]$.

Definition 1 ([2, Definition 2.1]). The conformable derivative with low index ρ of a function $f : [a, \infty) \rightarrow \mathbb{R}$ is defined as

$$D_\rho^\alpha f(x) = \lim_{\varepsilon \rightarrow 0} \frac{f(x + \varepsilon(x-a)^{1-\rho}) - f(x)}{\varepsilon}, \quad x > a, \quad 0 < \rho \leq 1,$$

while $D_\rho^\alpha f(a) = \lim_{x \rightarrow a^+} D_\rho^\alpha f(x)$.

Remark 1 ([1, Theorem 2]). Fix $0 < \rho \leq 1$ and $x > a$. A function $f : [a, \infty) \rightarrow \mathbb{R}$ has a conformable derivative $D_\rho^\alpha f(x)$ if and only if it is differentiable at x and $D_\rho^\alpha f(x) = (x-a)^{1-\rho} f'(x)$ holds.

Definition 2 ([2, Definition 3.1]). The conformable integral with low index ρ of a function $f : [a, \infty) \rightarrow \mathbb{R}$ is defined as

$$I_\rho^\alpha f(x) = \int_a^x f(t) d \frac{(t-a)^\rho}{\rho} = \int_a^x f(t) (t-a)^{\rho-1} dt, \quad x > a, \quad 0 < \rho \leq 1.$$

Definition 3 ([8, p.15]). Let ϑ be compensator of the random measure N . For any $(t, s) \in [a, T] \times \mathbb{R}$, the compensated random measure $\tilde{N}$ is defined as

$$\tilde{N}(\omega, dt, ds) = N(\omega, dt, ds) - \vartheta(\omega, dt, ds).$$

Lemma 1 ([8, p.17]). If $\eta : \Omega \times [a, T] \rightarrow [a, \infty)$ is a predictable process, Q is a kernel from $(\Omega \times [a, T], \mathfrak{P})$ into $(\mathbb{R}, \mathfrak{B}(\mathbb{R}))$ satisfying $\int_a^T \int_{\mathbb{R}} z^2 Q(t, ds) \eta(t) dt < \infty$, then

$$\vartheta(dt, ds) = Q(t, ds) \eta(t) dt.$$

Lemma 2 ([16, Theorem 7.3]). Let $f \in \mathbb{L}^2(\mathbb{R})$. For any $a \leq t \leq T$, define $x(t) = \int_a^t f(s) dB(s)$ and $A(t) = \int_a^t |f(s)|^2 ds$. Then for any $p > 0$, there exist universal positive constants c_p, C_p (depending only on p), such that

$$c_p \mathbb{E}[|A(t)|^{\frac{p}{2}}] \leq \mathbb{E}[\sup_{a \leq t \leq T} |x(s)|^p] \leq C_p \mathbb{E}[|A(t)|^{\frac{p}{2}}].$$

Lemma 3 ([8, Theorem 2.3.1]). (i) Let $V : \Omega \times [a, T] \rightarrow \mathbb{R}$ be a predictable process satisfying $\int_a^T |V(s)|^2 ds < \infty$. Then $\int_a^t V(s) dB(s)$ is a continuous local martingale with the quadratic variation process

$$\left[\int_a^\cdot V(\tau) dB(\tau), \int_a^\cdot V(\tau) dB(\tau) \right](t) = \int_a^t |V(\tau)|^2 d\tau, \quad a \leq t \leq T.$$

(ii) Let $V : \Omega \times [a, T] \times \mathbb{R} \rightarrow \mathbb{R}$ be a predictable process satisfying

$$\int_a^T \int_{\mathbb{R}} |V(t, s)|^2 Q(t, ds) \eta(t) dt < \infty.$$

Then $\int_a^t \int_{\mathbb{R}} V(\tau, s) \tilde{N}(d\tau, ds)$ is a càdlàg local martingale with the quadratic variation process

$$\left[\int_a^\cdot \int_{\mathbb{R}} V(\tau, s) \tilde{N}(d\tau, ds), \int_a^\cdot \int_{\mathbb{R}} V(\tau, s) \tilde{N}(d\tau, ds) \right](t)$$

$$= \int_a^t \int_{\mathbb{R}} |V(\tau, s)|^2 N(d\tau, ds), \quad a \leq t \leq T.$$

Lemma 4 ([8, Theorem 2.3.2]). *Let $V : \Omega \times [a, T] \times \mathbb{R} \rightarrow \mathbb{R}$ be a predictable process satisfying*

$$\int_a^T \int_{\mathbb{R}} |V(t, ds)| Q(t, ds) \eta(t) dt < \infty.$$

Then

$$\left(\int_a^T \int_{\mathbb{R}} V(t, ds) \tilde{N}(dt, ds), a \leq t \leq T \right)$$

is a càdlàg local martingale and

$$\left(\int_a^T \int_{\mathbb{R}} V(t, ds) N(dt, ds), a \leq t \leq T \right)$$

is a càdlàg process.

Lemma 5 ([8, p.23]). *If V is a non-negative predictable process satisfying*

$$\mathbb{E} \left[\int_a^T \int_{\mathbb{R}} V(t, s) Q(t, ds) \eta(t) dt \right] < \infty,$$

then

$$\mathbb{E} \left[\int_a^T \int_{\mathbb{R}} V(t, s) N(dt, ds) \right] = \mathbb{E} \left[\int_a^T \int_{\mathbb{R}} V(t, s) Q(t, ds) \eta(t) dt \right].$$

Lemma 6 ([8, p.42]). *Let $\gamma > 0$ and $x_1, x_2 \in \mathbb{R}$. Then*

$$2|x_1 x_2| \leq \frac{1}{\gamma} |x_1|^2 + \gamma |x_2|^2. \quad (2.1)$$

Lemma 7 ([8, Theorem 2.5.1]). *Let B be a $(\mathbb{P}, \mathcal{F})$ -Brownian motion, N be a $(\mathbb{P}, \mathcal{F})$ -random measure with compensator $\vartheta(d\tau, ds) = Q(\tau, ds) \eta(\tau) d\tau$. Assume an equivalent probability measure $\mathbb{Q} \sim \mathbb{P}$. Then*

$$B^{\mathbb{Q}}(t) = B(t) - \int_a^t \phi(\tau) d\tau, \quad a \leq t \leq T,$$

$$\tilde{N}^{\mathbb{Q}}(t, A) = N(t, A) - \int_a^t \int_{\mathbb{R}} (1 + \kappa(\tau, s)) Q(\tau, ds) \eta(\tau) d\tau, \quad a \leq t \leq T, \quad A \in \mathfrak{B}(\mathbb{R}),$$

are $(\mathbb{Q}, \mathcal{F})$ -Brownian motion and a $(\mathbb{Q}, \mathcal{F})$ -random measure.

Lemma 8 ([8, Proposition 2.5.1]). *Let M be the stochastic exponential defined by*

$$\frac{dM(t)}{M(t)} = \phi(t) dB(t) + \int_{\mathbb{R}} \kappa(t, s) \tilde{N}(dt, ds), \quad M(a) = 1,$$

where $\phi(\cdot)$ and $\kappa(\cdot, s)$ are predictable processes satisfying

$$|\phi(t)| \leq K, \quad \int_{\mathbb{R}} |\kappa(t, s)|^2 Q(t, ds) \eta(t) \leq K, \quad a \leq t \leq T, \quad \kappa(t, s) > -1, \quad a \leq t \leq T, \quad s \in \mathbb{R}.$$

The process M is a square integrable, positive martingale.

3. EXISTENCE AND UNIQUENESS RESULTS

In this part, we consider the following conformable backward stochastic differential equation with jumps

$$\begin{cases} D_{\rho}^a X(t) = -g(t, X(t), Y(t), Z(t, \cdot)) + Y(t) \frac{dB(t)}{dt} \\ \quad + \frac{\int_{\mathbb{R}} Z(t, s) \tilde{N}(dt, ds)}{dt}, \quad 0 < \rho \leq 1, \quad t \in [a, T], \\ X(T) = \zeta \in \mathbb{L}^2(\mathbb{R}), \end{cases} \quad (3.1)$$

where D_{ρ}^a is the conformable derivative, X is an adapted process, Y and Z are given control processes, $g : \Omega \times [a, T] \times \mathbb{R} \times \mathbb{R} \times L_Q^2(\mathbb{R}) \rightarrow \mathbb{R}$ is predictable, $B(\cdot)$ is a given Brownian motion and $\tilde{N}$ is a compensated random measure.

From now on, in order to simplify the notation, we think of ω as an implicit function in function $g(\omega, t, X(t), Y(t), Z(t, s))$, so as not to cause confusion.

Using Definition 2, (3.1) has an equivalent form

$$\begin{cases} dX(t) = -(t-a)^{\rho-1} g(t, X(t), Y(t), Z(t, \cdot)) dt + (t-a)^{\rho-1} Y(t) dB(t) \\ \quad + (t-a)^{\rho-1} \int_{\mathbb{R}} Z(t, s) \tilde{N}(dt, ds), \\ X(T) = \zeta. \end{cases} \quad (3.2)$$

Definition 4. A triple $(X, Y, Z) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}_N^2(\mathbb{R})$ is called the solution of (3.1) if (X, Y, Z) satisfies

$$\begin{aligned} X(t) = \zeta + \int_t^T (\tau-a)^{\rho-1} g(\tau, X(\tau), Y(\tau), Z(\tau, \cdot)) d\tau - \int_t^T (\tau-a)^{\rho-1} Y(\tau) dB(\tau) \\ - \int_t^T (\tau-a)^{\rho-1} \int_{\mathbb{R}} Z(\tau, s) \tilde{N}(d\tau, ds), \quad 0 < \rho \leq 1, \quad a \leq t \leq T. \end{aligned} \quad (3.3)$$

Next we give some important assumptions:

(S1) The terminal value $\zeta \in \mathbb{L}^2(\mathbb{R})$.

(S2) For all $(x, y, z), (x', y', z') \in \mathbb{R} \times \mathbb{R} \times L_Q^2(\mathbb{R})$, g is Lipschitz continuous and

$$|g(t, x, y, z) - g(t, x', y', z')|^2 \leq K \left(|x - x'|^2 + |y - y'|^2 + \int_{\mathbb{R}} |z(s) - z'(s)|^2 Q(t, ds) \eta(t) \right),$$

a.s., a.e. $t \in [a, T]$, $K \in \mathbb{R}$.

(S3) $\mathbb{E}[\int_a^T |g(t, 0, 0, 0)|^2 dt] < \infty$.

Lemma 9. Suppose that $(X, Y, Z), (X', Y', Z') \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}_N^2(\mathbb{R})$, and $(\zeta, g), (\zeta', g')$ satisfy (S1), (S2) and (S3). For any $a \leq t \leq T$, if

$$e^{\sigma t} |X(t) - X'(t)|^2 + \sigma \int_t^T e^{\sigma \tau} |X(\tau) - X'(\tau)|^2 d\tau$$

$$\begin{aligned}
& + \int_t^T e^{\sigma\tau} (\tau - a)^{2(\rho-1)} |Y(\tau) - Y'(\tau)|^2 d\tau \\
& + \int_t^T \int_{\mathbb{R}} e^{\sigma\tau} (\tau - a)^{2(\rho-1)} |Z(\tau, s) - Z'(\tau, s)|^2 Q(\tau, ds) \eta(\tau) d\tau \\
& \leq e^{\sigma T} |\zeta - \zeta'|^2 + 2 \int_t^T e^{\sigma\tau} (\tau - a)^{\rho-1} |X(\tau) - X'(\tau)| \\
& \quad \times |g(\tau, X(\tau), Y(\tau), Z(\tau)) - g'(\tau, X'(\tau), Y'(\tau), Z'(\tau))| d\tau,
\end{aligned}$$

then

$$\begin{aligned}
& \mathbb{E}[e^{\sigma t} |X(t) - X'(t)|^2] + \mathbb{E}\left[\int_t^T e^{\sigma\tau} (\tau - a)^{2(\rho-1)} |Y(\tau) - Y'(\tau)|^2 d\tau\right] \\
& + \mathbb{E}\left[\int_t^T \int_{\mathbb{R}} e^{\sigma\tau} (\tau - a)^{2(\rho-1)} |Z(\tau, s) - Z'(\tau, s)|^2 Q(\tau, ds) \eta(\tau) d\tau\right] \\
& \leq \mathbb{E}[e^{\sigma T} |\zeta - \zeta'|^2] + \frac{1}{\sigma} \mathbb{E}\left[\int_t^T e^{\sigma\tau} (\tau - a)^{2(\rho-1)} \right. \\
& \quad \left. \times |g(\tau, X(\tau), Y(\tau), Z(\tau)) - g'(\tau, X'(\tau), Y'(\tau), Z'(\tau))|^2 d\tau\right].
\end{aligned}$$

Proof. From (2.1), one can see that

$$\begin{aligned}
& 2 \int_t^T e^{\sigma\tau} (\tau - a)^{\rho-1} |X(\tau) - X'(\tau)| \\
& \quad \times |g(\tau, X(\tau), Y(\tau), Z(\tau)) - g'(\tau, X'(\tau), Y'(\tau), Z'(\tau))| d\tau \\
& \leq \frac{1}{\sigma} \int_t^T e^{\sigma\tau} |X(\tau) - X'(\tau)|^2 d\tau + \sigma \int_t^T e^{\sigma\tau} (\tau - a)^{2(\rho-1)} \\
& \quad \times |g(\tau, X(\tau), Y(\tau), Z(\tau)) - g'(\tau, X'(\tau), Y'(\tau), Z'(\tau))|^2 d\tau, \quad a \leq t \leq T.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
& e^{\sigma t} |X(t) - X'(t)|^2 + \sigma \int_t^T e^{\sigma\tau} |X(\tau) - X'(\tau)|^2 d\tau \\
& + \int_t^T e^{\sigma\tau} (\tau - a)^{2(\rho-1)} |Y(\tau) - Y'(\tau)|^2 d\tau \\
& + \int_t^T \int_{\mathbb{R}} e^{\sigma\tau} (\tau - a)^{2(\rho-1)} |Z(\tau, s) - Z'(\tau, s)|^2 Q(\tau, ds) \eta(\tau) d\tau \\
& \leq e^{\sigma T} |\zeta - \zeta'|^2 + \frac{1}{\sigma} \int_t^T e^{\sigma\tau} |X(\tau) - X'(\tau)|^2 d\tau + \sigma \int_t^T e^{\sigma\tau} (\tau - a)^{2(\rho-1)} \\
& \quad \times |g(\tau, X(\tau), Y(\tau), Z(\tau)) - g'(\tau, X'(\tau), Y'(\tau), Z'(\tau))|^2 d\tau, \quad a \leq t \leq T.
\end{aligned}$$

Using Lemma 3 and Theorem I.51 in [20], we note the integrals in the above inequality are uniformly integrable martingales. Thus

$$\begin{aligned}
& \mathbb{E}[e^{\sigma t}|X(t) - X'(t)|^2] + \sigma \mathbb{E} \left[\int_t^T e^{\sigma \tau} |X(\tau) - X'(\tau)|^2 d\tau \right] \\
& + \mathbb{E} \left[\int_t^T e^{\sigma \tau} (\tau - a)^{2(\rho-1)} |Y(\tau) - Y'(\tau)|^2 d\tau \right] \\
& + \mathbb{E} \left[\int_t^T \int_{\mathbb{R}} e^{\sigma \tau} (\tau - a)^{2(\rho-1)} |Z(\tau, s) - Z'(\tau, s)|^2 Q(\tau, ds) \eta(\tau) d\tau \right] \\
& \leq \mathbb{E}[e^{\sigma T} |\zeta - \zeta'|^2] + \sigma \mathbb{E} \left[\int_t^T e^{\sigma \tau} |X(\tau) - X'(\tau)|^2 d\tau \right] \\
& + \frac{1}{\sigma} \mathbb{E} \left[\int_t^T e^{\sigma \tau} (\tau - a)^{2(\rho-1)} |g(\tau, X(\tau), Y(\tau), Z(\tau)) \right. \\
& \quad \left. - g'(\tau, X'(\tau), Y'(\tau), Z'(\tau))|^2 d\tau \right], \quad a \leq t \leq T.
\end{aligned}$$

Let $\gamma = \sigma$, and we have

$$\begin{aligned}
& \mathbb{E}[e^{\sigma t}|X(t) - X'(t)|^2] + \mathbb{E} \left[\int_t^T e^{\sigma \tau} (\tau - a)^{2(\rho-1)} |Y(\tau) - Y'(\tau)|^2 d\tau \right] \\
& + \mathbb{E} \left[\int_t^T \int_{\mathbb{R}} e^{\sigma \tau} (\tau - a)^{2(\rho-1)} |Z(\tau, s) - Z'(\tau, s)|^2 Q(\tau, ds) \eta(\tau) d\tau \right] \\
& \leq \mathbb{E}[e^{\sigma T} |\zeta - \zeta'|^2] + \frac{1}{\sigma} \mathbb{E} \left[\int_t^T e^{\sigma \tau} (\tau - a)^{2(\rho-1)} |g(\tau, X(\tau), Y(\tau), Z(\tau)) \right. \\
& \quad \left. - g'(\tau, X'(\tau), Y'(\tau), Z'(\tau))|^2 d\tau \right], \quad a \leq t \leq T,
\end{aligned}$$

as required. The proof is complete. $\square$

Lemma 10. Assume that (S2) is satisfied and let $0 < \rho \leq 1$. Then

$$\int_a^T e^{\sigma \tau} |Y(\tau) - Y'(\tau)|^2 d\tau \leq (T - a)^{2(1-\rho)} \int_a^T e^{\sigma \tau} (\tau - a)^{2(\rho-1)} |Y(\tau) - Y'(\tau)|^2 d\tau, \quad (3.4)$$

and

$$\begin{aligned}
& \int_a^T \int_{\mathbb{R}} e^{\sigma \tau} |Z(\tau, s) - Z'(\tau, s)|^2 Q(\tau, ds) \eta(\tau) d\tau \\
& \leq (T - a)^{2(1-\rho)} \int_a^T \int_{\mathbb{R}} e^{\sigma \tau} (\tau - a)^{2(\rho-1)} |Z(\tau, s) - Z'(\tau, s)|^2 Q(\tau, ds) \eta(\tau) d\tau. \quad (3.5)
\end{aligned}$$

Proof. We have

$$\int_a^T e^{\sigma \tau} (\tau - a)^{2(\rho-1)} |Y(\tau) - Y'(\tau)|^2 d\tau \geq \int_a^T e^{\sigma \tau} (T - a)^{2(\rho-1)} |Y(\tau) - Y'(\tau)|^2 d\tau$$

$$= (T-a)^{2(\rho-1)} \int_a^T e^{\sigma\tau} |Y(\tau) - Y'(\tau)|^2 d\tau.$$

Thus

$$\int_a^T e^{\sigma\tau} |Y(\tau) - Y'(\tau)|^2 d\tau \leq (T-a)^{2(1-\rho)} \int_a^T e^{\sigma\tau} (\tau-a)^{2(\rho-1)} |Y(\tau) - Y'(\tau)|^2 d\tau,$$

where the function $(\cdot - a)^{2(\rho-1)}$ monotonicity has been used. The inequality in (3.5) can be obtained in a similar manner to that in (3.4). $\square$

Lemma 11. *For any $a \leq t \leq T$, we have*

$$\mathbb{E} \left[\int_t^T e^{\sigma\tau} |X(\tau) - X'(\tau)|^2 d\tau \right] \leq T \mathbb{E} \left[\sup_{t \leq \tau \leq T} e^{\sigma\tau} |X(\tau) - X'(\tau)|^2 \right]. \quad (3.6)$$

Proof. From Lemma 2 and the Hölder inequality, we have

$$\begin{aligned} \mathbb{E} \left[\int_t^T e^{\sigma\tau} |X(\tau) - X'(\tau)|^2 d\tau \right] &\leq \sqrt{T-t} \mathbb{E} \left[\left(\int_t^T e^{2\sigma\tau} |X(\tau) - X'(\tau)|^4 d\tau \right)^{\frac{1}{2}} \right] \\ &\leq (T-t) \mathbb{E} \left[\sup_{t \leq \tau \leq T} e^{\sigma\tau} |X(\tau) - X'(\tau)|^2 \right] \\ &\leq T \mathbb{E} \left[\sup_{t \leq \tau \leq T} e^{\sigma\tau} |X(\tau) - X'(\tau)|^2 \right], \quad a \leq t \leq T. \end{aligned}$$

The proof is complete. $\square$

We introduce the conformable Itô's formula with jumps here.

Theorem 1. *Suppose $U(\cdot) = U(X(\cdot), \cdot) \in C^{2,1}(\mathbb{R} \times \mathbb{R}_+, \mathbb{R})$. Then, for any $a \leq t \leq T$, we have*

$$\begin{aligned} D_\rho^a U(t) d \frac{(t-a)^\rho}{\rho} &= \left(\frac{\partial u}{\partial t} - (t-a)^{\rho-1} g(t, X(t), Y(t), Z(t, \cdot)) \frac{\partial u}{\partial x} \right. \\ &\quad \left. + \frac{1}{2} (t-a)^{2(\rho-1)} Y^2(t) \frac{\partial^2 u}{\partial x^2} \right) dt + (t-a)^{\rho-1} Y(t) \frac{\partial u}{\partial x} dB(t) \\ &\quad + (t-a)^{\rho-1} \frac{\partial u}{\partial x} \int_{\mathbb{R}} Z(t, s) \tilde{N}(dt, ds) \\ &\quad + \frac{1}{2} (t-a)^{2(\rho-1)} \frac{\partial^2 u}{\partial x^2} \int_{\mathbb{R}} Z^2(t, s) N(dt, ds), \quad 0 < \rho \leq 1. \end{aligned} \quad (3.7)$$

Proof. For any $a \leq t \leq T$, by Remark 1 and Lemma 3, we have

$$\begin{aligned} D_\rho^a U(t) d \frac{(t-a)^\rho}{\rho} &= dU(t) \\ &= \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial x} dX(t) + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} (dX(t))^2 \\ &= \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial x} dX(t) + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} (t-a)^{2(\rho-1)} Y^2(t) dt \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2}(t-a)^{2(\rho-1)} \frac{\partial^2 u}{\partial x^2} \int_{\mathbb{R}} Z^2(t,s) N(dt, ds) \\
& = \frac{\partial u}{\partial t} dt - (t-a)^{\rho-1} \frac{\partial u}{\partial x} g(t, X(t), Y(t), Z(t, \cdot)) dt \\
& \quad + (t-a)^{\rho-1} \frac{\partial u}{\partial x} Y(t) dB(t) + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} (t-a)^{2(\rho-1)} Y^2(t) dt \\
& \quad + (t-a)^{\rho-1} \frac{\partial u}{\partial x} \int_{\mathbb{R}} Z(t,s) \tilde{N}(dt, ds) \\
& \quad + \frac{1}{2}(t-a)^{2(\rho-1)} \frac{\partial^2 u}{\partial x^2} \int_{\mathbb{R}} Z^2(t,s) N(dt, ds), \quad 0 < \rho \leq 1.
\end{aligned}$$

The proof is complete. $\square$

Theorem 2. Suppose (S1), (S2) and (S3) hold and let $(X, Y, Z), (X', Y', Z') \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}_N^2(\mathbb{R})$ be solutions to (3.1). Then

$$\begin{aligned}
& \|Y - Y'\|_{\mathbb{H}^2}^2 + \|Z - Z'\|_{\mathbb{H}_N^2}^2 \\
& \leq (T-a)^{2(1-\rho)} \left(\mathbb{E}[e^{\sigma T} |\zeta - \zeta'|^2] + \frac{1}{\sigma} \mathbb{E} \left[\int_t^T e^{\sigma \tau} (\tau-a)^{2(\rho-1)} \right. \right. \\
& \quad \left. \left. \times |g(\tau, X(\tau), Y(\tau), Z(\tau)) - g'(\tau, X'(\tau), Y'(\tau), Z'(\tau))|^2 d\tau \right] \right), \quad (3.8)
\end{aligned}$$

and

$$\begin{aligned}
& \|Y - Y'\|_{\mathbb{H}^2}^2 + \|Z - Z'\|_{\mathbb{H}_N^2}^2 \\
& \leq \hat{K} \left(\mathbb{E}[e^{\sigma T} |\zeta - \zeta'|^2] + \mathbb{E} \left[\int_t^T e^{\sigma \tau} (\tau-a)^{2(\rho-1)} \right. \right. \\
& \quad \left. \left. \times |g(\tau, X(\tau), Y(\tau), Z(\tau)) - g'(\tau, X(\tau), Y(\tau), Z(\tau))|^2 d\tau \right] \right), \quad (3.9)
\end{aligned}$$

and

$$\begin{aligned}
\|X - X'\|_{\mathbb{S}^2}^2 & \leq \left(1 + \sqrt{\sigma} (T-a)^{4(1-\rho)} \tilde{K} \right) \mathbb{E}[e^{\sigma T} |\zeta - \zeta'|^2] \\
& \quad + \sqrt{\sigma} \left(1 + \frac{(T-a)^{4(1-\rho)} \tilde{K}}{\sigma} \right) \mathbb{E} \left[\int_t^T e^{\sigma \tau} (\tau-a)^{2(\rho-1)} \right. \\
& \quad \left. \times |g(\tau, X(\tau), Y(\tau), Z(\tau)) - g'(\tau, X'(\tau), Y'(\tau), Z'(\tau))|^2 d\tau \right], \quad (3.10)
\end{aligned}$$

$$\begin{aligned}
\|X - X'\|_{\mathbb{S}^2}^2 & \leq \hat{K} \left(\mathbb{E}[e^{\sigma T} |\zeta - \zeta'|^2] + \mathbb{E} \left[\int_t^T e^{\sigma \tau} (\tau-a)^{2(\rho-1)} \right. \right. \\
& \quad \left. \left. \times |g(\tau, X(\tau), Y(\tau), Z(\tau)) - g'(\tau, X(\tau), Y(\tau), Z(\tau))|^2 d\tau \right] \right), \quad (3.11)
\end{aligned}$$

where $\tilde{K}$ depends on K_1, K_2 , and $\hat{K}$ depends on K, T, σ, a , and $a \leq t \leq T$.

Proof. Case 1. Applying Theorem 1 to $e^{\sigma t}|X(t) - X'(t)|^2$, then

$$\begin{aligned}
& e^{\sigma t}|X(t) - X'(t)|^2 + \sigma \int_t^T e^{\sigma \tau}|X(\tau) - X'(\tau)|^2 d\tau \\
& + \int_t^T e^{\sigma \tau}(\tau - a)^{2(\rho-1)}|Y(\tau) - Y'(\tau)|^2 d\tau \\
& + \int_t^T \int_{\mathbb{R}} e^{\sigma \tau}(\tau - a)^{2(\rho-1)}|Z(\tau, s) - Z'(\tau, s)|^2 Q(\tau, ds) \eta(\tau) d\tau \\
& = e^{\sigma T}|\zeta - \zeta'|^2 + 2 \int_t^T e^{\sigma \tau}(\tau - a)^{\rho-1}|X(\tau) - X'(\tau)| \\
& \quad \times |g(\tau, X(\tau), Y(\tau), Z(\tau)) - g'(\tau, X'(\tau), Y'(\tau), Z'(\tau))| d\tau \\
& \quad - 2 \int_t^T e^{\sigma \tau}(\tau - a)^{\rho-1}|X(\tau) - X'(\tau)||Y(\tau) - Y'(\tau)| dB(\tau) \\
& \quad - 2 \int_t^T e^{\sigma \tau}(\tau - a)^{\rho-1}|X(\tau) - X'(\tau)| \int_{\mathbb{R}} |Z(\tau, s) - Z'(\tau, s)| \tilde{N}(d\tau, ds) \\
& \quad - \int_t^T e^{\sigma \tau}(\tau - a)^{2(\rho-1)} \int_{\mathbb{R}} |Z(\tau, s) - Z'(\tau, s)|^2 \tilde{N}(d\tau, ds), \quad a \leq t \leq T. \quad (3.12)
\end{aligned}$$

Thus

$$\begin{aligned}
& e^{\sigma t}|X(t) - X'(t)|^2 + \sigma \int_t^T e^{\sigma \tau}|X(\tau) - X'(\tau)|^2 d\tau \\
& + \int_t^T e^{\sigma \tau}(\tau - a)^{2(\rho-1)}|Y(\tau) - Y'(\tau)|^2 d\tau \\
& + \int_t^T \int_{\mathbb{R}} e^{\sigma \tau}(\tau - a)^{2(\rho-1)}|Z(\tau, s) - Z'(\tau, s)|^2 Q(\tau, ds) \eta(\tau) d\tau \\
& \leq e^{\sigma T}|\zeta - \zeta'|^2 + 2 \int_t^T e^{\sigma \tau}(\tau - a)^{\rho-1}|X(\tau) - X'(\tau)| \\
& \quad \times |g(\tau, X(\tau), Y(\tau), Z(\tau)) - g'(\tau, X'(\tau), Y'(\tau), Z'(\tau))| d\tau, \quad a \leq t \leq T.
\end{aligned}$$

From Lemma 9, we have

$$\begin{aligned}
& \mathbb{E} \left[e^{\sigma t}|X(t) - X'(t)|^2 \right] + \mathbb{E} \left[\int_t^T e^{\sigma \tau}(\tau - a)^{2(\rho-1)}|Y(\tau) - Y'(\tau)|^2 d\tau \right] \\
& + \mathbb{E} \left[\int_t^T \int_{\mathbb{R}} e^{\sigma \tau}(\tau - a)^{2(\rho-1)}|Z(\tau, s) - Z'(\tau, s)|^2 Q(\tau, ds) \eta(\tau) d\tau \right] \\
& \leq \mathbb{E}[e^{\sigma T}|\zeta - \zeta'|^2] + \frac{1}{\sigma} \mathbb{E} \left[\int_t^T e^{\sigma \tau}(\tau - a)^{2(\rho-1)}|g(\tau, X(\tau), Y(\tau), Z(\tau)) \right. \\
& \quad \left. - g'(\tau, X'(\tau), Y'(\tau), Z'(\tau))|^2 d\tau \right], \quad a \leq t \leq T.
\end{aligned}$$

Next from Lemma 10, estimate (3.8) holds.

Case 2. By the Lipschitz condition of g' and (2.1), we get

$$\begin{aligned} & 2(t-a)^{\rho-1}|X(t)-X'(t)| \times |g(t, X(t), Y(t), Z(t)) - g'(t, X'(t), Y'(t), Z'(t))| \\ & \leq 2\gamma|X(t)-X'(t)|^2 + \frac{1}{\gamma}(t-a)^{2(\rho-1)} \\ & \quad \times |g(t, X(t), Y(t), Z(t)) - g'(t, X(t), Y(t), Z(t))|^2 \\ & \quad + \frac{K}{\gamma}(t-a)^{2(\rho-1)} \left(|X(t)-X'(t)|^2 + |Y(t)-Y'(t)|^2 \right. \\ & \quad \left. + \int_{\mathbb{R}} |Z(t, s) - Z'(t, s)|^2 Q(t, ds) \eta(t) \right), \quad a \leq t \leq T. \end{aligned}$$

Similar to the proof of the estimate (3.8), but here we take $\gamma = \frac{\sigma + \sqrt{\sigma^2 - 8K}}{4}$, $\sigma \geq 2\sqrt{2K}$ instead of $\gamma = \sigma$ in Lemma 9, then the proof of estimate (3.9) is complete.

Furthermore, we can also establish the following estimates in the same way:

$$\begin{aligned} \mathbb{E}[e^{\sigma t}|X(t)-X'(t)|^2] & \leq \mathbb{E}[e^{\sigma T}|\zeta - \zeta'|^2] + \frac{1}{\sigma} \mathbb{E} \left[\int_t^T e^{\sigma \tau} (\tau - a)^{2(\rho-1)} \right. \\ & \quad \left. \times |g(\tau, X(\tau), Y(\tau), Z(\tau)) - g'(\tau, X'(\tau), Y'(\tau), Z'(\tau))|^2 d\tau \right], \quad a \leq t \leq T, \end{aligned} \quad (3.13)$$

and

$$\begin{aligned} \mathbb{E} \left[e^{\sigma t} |X(t) - X'(t)|^2 \right] & \leq \hat{K} \left(\mathbb{E}[e^{\sigma T}|\zeta - \zeta'|^2] + \mathbb{E} \left[\int_t^T e^{\sigma \tau} (\tau - a)^{2(\rho-1)} \right. \right. \\ & \quad \left. \left. \times |g(\tau, X(\tau), Y(\tau), Z(\tau)) - g'(\tau, X(\tau), Y(\tau), Z(\tau))|^2 d\tau \right] \right), \quad a \leq t \leq T. \end{aligned} \quad (3.14)$$

Case 3. By (3.12) and Definition 3 we have

$$\begin{aligned} & e^{\sigma t}|X(t)-X'(t)|^2 + \sigma \int_t^T e^{\sigma \tau} |X(\tau)-X'(\tau)|^2 d\tau \\ & \quad + \int_t^T e^{\sigma \tau} (\tau - a)^{2(\rho-1)} |Y(\tau) - Y'(\tau)|^2 d\tau \\ & \quad + \int_t^T e^{\sigma \tau} (\tau - a)^{2(\rho-1)} \int_{\mathbb{R}} |Z(\tau, s) - Z'(\tau, s)|^2 N(d\tau, ds) \\ & = e^{\sigma T} |\zeta - \zeta'|^2 + 2 \int_t^T e^{\sigma \tau} (\tau - a)^{\rho-1} |X(\tau) - X'(\tau)| \\ & \quad \times |(g(\tau, X(\tau), Y(\tau), Z(\tau)) - g'(\tau, X'(\tau), Y'(\tau), Z'(\tau)))| d\tau \\ & \quad - 2 \int_t^T e^{\sigma \tau} (\tau - a)^{\rho-1} |X(\tau) - X'(\tau)| |Y(\tau) - Y'(\tau)| dB(\tau) \\ & \quad - 2 \int_t^T e^{\sigma \tau} (\tau - a)^{\rho-1} |X(\tau) - X'(\tau)| \int_{\mathbb{R}} |Z(\tau, s) - Z'(\tau, s)| \tilde{N}(d\tau, ds), \quad a \leq t \leq T. \end{aligned}$$

Note

$$\begin{aligned}
& \sup_{t \in [a, T]} e^{\sigma t} |X(t) - X'(t)|^2 \\
& \leq e^{\sigma T} |\zeta - \zeta'|^2 + 2 \int_t^T e^{\sigma \tau} (\tau - a)^{\rho-1} |X(\tau) - X'(\tau)| \\
& \quad \times |g(\tau, X(\tau), Y(\tau), Z(\tau)) - g'(\tau, X'(\tau), Y'(\tau), Z'(\tau))| d\tau \\
& \quad + 4 \sup_{t \in [a, T]} \left| \int_t^T e^{\sigma \tau} (\tau - a)^{\rho-1} |X(\tau) - X'(\tau)| |Y(\tau) - Y'(\tau)| dB(\tau) \right| \\
& \quad + 4 \sup_{t \in [a, T]} \left| \int_t^T e^{\sigma \tau} (\tau - a)^{\rho-1} |X(\tau) - X'(\tau)| \int_{\mathbb{R}} |Z(\tau, s) - Z'(\tau, s)| \tilde{N}(d\tau, ds) \right|.
\end{aligned}$$

According to Lemma 2, Lemma 5 and (2.1), we get

$$\begin{aligned}
& \mathbb{E} \left[\sup_{t \in [a, T]} e^{\sigma t} |X(t) - X'(t)|^2 \right] \\
& \leq \mathbb{E} [e^{\sigma T} |\zeta - \zeta'|^2] + 2 \mathbb{E} \left[\int_t^T e^{\sigma \tau} (\tau - a)^{\rho-1} |X(\tau) - X'(\tau)| \right. \\
& \quad \times |g(\tau, X(\tau), Y(\tau), Z(\tau)) - g'(\tau, X'(\tau), Y'(\tau), Z'(\tau))| d\tau \Big] \\
& \quad + \frac{K_1 + K_2}{\gamma} \mathbb{E} \left[\sup_{t \in [a, T]} e^{\sigma t} |X(t) - X'(t)|^2 \right] \\
& \quad + K_1 \sigma \mathbb{E} \left[\int_t^T e^{\sigma \tau} (\tau - a)^{2(\rho-1)} |Y(\tau) - Y'(\tau)|^2 d\tau \right] \\
& \quad + K_2 \sigma \mathbb{E} \left[\int_t^T \int_{\mathbb{R}} e^{\sigma \tau} (\tau - a)^{2(\rho-1)} |Z(\tau, s) - Z'(\tau, s)|^2 Q(\tau, ds) \eta(\tau) d\tau \right],
\end{aligned}$$

where $a \leq t \leq T$, $K_1, K_2 \in \mathbb{R}$ and $0 < \rho \leq 1$.

From Lemma 9, we have

$$\begin{aligned}
\|X - X'\|_{\mathbb{S}^2}^2 & \leq \mathbb{E} [e^{\sigma T} |\zeta - \zeta'|^2] + \frac{1}{\gamma} \mathbb{E} \left[\int_t^T e^{\sigma \tau} |X(\tau) - X'(\tau)|^2 d\tau \right] \\
& \quad + \gamma \mathbb{E} \left[\int_t^T e^{\sigma \tau} (\tau - a)^{2(\rho-1)} |g(\tau, X(\tau), Y(\tau), Z(\tau)) \right. \\
& \quad \left. - g'(\tau, X'(\tau), Y'(\tau), Z'(\tau))|^2 d\tau \right] \\
& \quad + \frac{K_1 + K_2}{\gamma} \mathbb{E} \left[\sup_{t \in [a, T]} e^{\sigma t} |X(t) - X'(t)|^2 \right] \\
& \quad + K_1 \gamma \mathbb{E} \left[\int_t^T e^{\sigma \tau} (\tau - a)^{2(\rho-1)} |Y(\tau) - Y'(\tau)|^2 d\tau \right]
\end{aligned}$$

$$+ K_2 \gamma \mathbb{E} \left[\int_t^T \int_{\mathbb{R}} e^{\sigma \tau} (\tau - a)^{2(\rho-1)} |Z(\tau, s) - Z'(\tau, s)|^2 Q(\tau, ds) \eta(\tau) d\tau \right].$$

Next by (3.8) and (3.6), one has

$$\begin{aligned} \frac{\gamma - K_1 - K_2 - T}{\gamma} \|X - X'\|^2 &\leq \left(1 + \gamma(T - a)^{4(1-\rho)} \tilde{K} \right) \mathbb{E}[e^{\sigma T} |\zeta - \zeta'|^2] \\ &\quad + \gamma \left(1 + \frac{(T - a)^{4(1-\rho)} \tilde{K}}{\sigma} \right) \mathbb{E} \left[\int_t^T e^{\sigma \tau} \times (\tau - a)^{2(\rho-1)} \right. \\ &\quad \left. \times |g(\tau, X(\tau), Y(\tau), Z(\tau)) - g'(\tau, X'(\tau), Y'(\tau), Z'(\tau))|^2 d\tau \right], \end{aligned}$$

where $\tilde{K}$ depends on K_1 and K_2 , $a \leq t \leq T$. Let $\gamma > K_1 + K_2 + T$, and then

$$\begin{aligned} \|X - X'\|_{\mathbb{S}^2}^2 &\leq \left(1 + \gamma(T - a)^{4(1-\rho)} \tilde{K} \right) \mathbb{E}[e^{\sigma T} |\zeta - \zeta'|^2] \\ &\quad + \gamma \left(1 + \frac{(T - a)^{4(1-\rho)} \tilde{K}}{\sigma} \right) \mathbb{E} \left[\int_t^T e^{\sigma \tau} \times (\tau - a)^{2(\rho-1)} \right. \\ &\quad \left. \times |g(\tau, X(\tau), Y(\tau), Z(\tau)) - g'(\tau, X'(\tau), Y'(\tau), Z'(\tau))|^2 d\tau \right]. \end{aligned}$$

Choose $\gamma = \sqrt{\sigma}$, and we have

$$\begin{aligned} \|X - X'\|_{\mathbb{S}^2}^2 &\leq \left(1 + \sqrt{\sigma}(T - a)^{4(1-\rho)} \tilde{K} \right) \mathbb{E}[e^{\sigma T} |\zeta - \zeta'|^2] \\ &\quad + \sqrt{\sigma} \left(1 + \frac{(T - a)^{4(1-\rho)} \tilde{K}}{\sigma} \right) \mathbb{E} \left[\int_t^T e^{\sigma \tau} (\tau - a)^{2(\rho-1)} \right. \\ &\quad \left. \times |g(\tau, X(\tau), Y(\tau), Z(\tau)) - g'(\tau, X'(\tau), Y'(\tau), Z'(\tau))|^2 d\tau \right], \end{aligned}$$

which proves estimate (3.10).

Case 4. By the Lipschitz property of g' and some classical inequalities, then similar to the proof process of estimate (3.10), we can directly get estimate (3.11). $\square$

Remark 2. When $\rho = 1$, the above estimates are the same as in [8, Lemma 3.1.1].

We now prove the existence of a unique solution to the conformable backward stochastic differential equation (3.1). The essence of this proof is to transform (3.1) into an equivalent integral equation (3.3), and then prove the existence and uniqueness of the solution of (3.3). Its proof idea is similar to the proof idea of the existence and uniqueness of the solution to ordinary differential equations.

The following theorem is the core in this paper.

Theorem 3. Assume that (S1), (S2) and (S3) are satisfied. The conformable backward stochastic differential equation (3.1) has a unique solution $(X, Y, Z) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}_N^2(\mathbb{R})$, provided $\frac{1}{2} < \rho \leq 1$.

Proof. We first prove the existence of the solution. Suppose $X^0(T) = \zeta$, $X^0(t) = Y^0(t) = Z^0(t, s) = 0$, $(t, s) \in [a, T] \times \mathbb{R}$ and $(X^n, Y^n, Z^n) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}_N^2(\mathbb{R})$. Let $a \leq t \leq T$, and then let

$$\begin{aligned} X^{n+1}(t) &= \zeta + \int_t^T (\tau - a)^{\rho-1} g(\tau, X^n(\tau), Y^n(\tau), Z^n(\tau, \cdot)) d\tau \\ &\quad - \int_t^T (\tau - a)^{\rho-1} Y^{n+1}(\tau) dB(\tau) \\ &\quad - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z^{n+1}(\tau, s) \tilde{N}(d\tau, ds). \end{aligned} \quad (3.15)$$

For any $m, n \in \mathbb{R}$, $|m + n|^2 \leq 2(m^2 + n^2)$. From (S2), we get

$$\begin{aligned} &|g(t, X^n(t), Y^n(t), Z^n(t))|^2 \\ &\leq |g(t, X^n(t), Y^n(t), Z^n(t)) + g(t, 0, 0, 0)|^2 \\ &\leq 2|g(t, X^n(t), Y^n(t), Z^n(t))|^2 + 2|g(t, 0, 0, 0)|^2 \\ &\leq 2|g(t, 0, 0, 0)|^2 + 2K \left(|X^n(t)|^2 + |Y^n(t)|^2 + \int_{\mathbb{R}} |Z^n(t)|^2 Q(t, ds) \eta(t) \right), \end{aligned}$$

where $a \leq t \leq T$, $K \in \mathbb{R}$ and g is independent of $(X^{n+1}, Y^{n+1}, Z^{n+1})$.

According to (3.6), we have

$$\begin{aligned} \mathbb{E} \left[\int_a^T |g(t, X^n(t), Y^n(t), Z^n(t))|^2 dt \right] &\leq 2\mathbb{E} \left[\int_a^T |g(t, 0, 0, 0)|^2 dt \right] + 2K(T \|X^n\|_{\mathbb{S}^2}^2 \\ &\quad + \|Y^n\|_{\mathbb{H}^2}^2 + \|Z^n\|_{\mathbb{H}_N^2}^2) < \infty, \quad a \leq t \leq T. \end{aligned}$$

For any sequence of stopping times $(t_n)_{n \geq 1}$, one has

$$\begin{aligned} \mathbb{E}[[X^{n+1}, X^{n+1}](t_n)] &= \zeta^2 + \mathbb{E} \left[\left(\int_{t_n}^T (\tau - a)^{\rho-1} g(\tau, X^n(\tau), Y^n(\tau), Z^n(\tau)) d\tau \right)^2 \right] \\ &\quad + \mathbb{E} \left[\int_{t_n}^T (\tau - a)^{2(\rho-1)} |Y^{n+1}(\tau)|^2 d\tau \right] \\ &\quad + \mathbb{E} \left[\int_{t_n}^T \int_{\mathbb{R}} (\tau - a)^{2(\rho-1)} |Z^{n+1}(\tau, s)|^2 Q(\tau, ds) \eta(\tau) d\tau \right]. \end{aligned}$$

By the Cauchy-Schwarz inequality and the properties of integrals, we get

$$\begin{aligned} &\mathbb{E} \left[\left(\int_{t_n}^T (\tau - a)^{\rho-1} g(\tau, X^n(\tau), Y^n(\tau), Z^n(\tau)) d\tau \right)^2 \right] \\ &\leq \mathbb{E} \left[\int_{t_n}^T (\tau - a)^{2(\rho-1)} d\tau \int_{t_n}^T |g(\tau, X^n(\tau), Y^n(\tau), Z^n(\tau))|^2 d\tau \right] \\ &\leq \frac{1}{2\rho-1} (T-a)^{2\rho-1} \mathbb{E} \left[\int_{t_n}^T |g(\tau, X^n(\tau), Y^n(\tau), Z^n(\tau))|^2 d\tau \right], \end{aligned}$$

$$\mathbb{E} \left[\int_{t_n}^T (\tau - a)^{2(\rho-1)} |Y^{n+1}(\tau)|^2 d\tau \right] \leq \frac{1}{2\rho-1} (T-a)^{2\rho-1} \mathbb{E} \left[\int_{t_n}^T |Y^{n+1}(\tau)|^2 d\tau \right],$$

and

$$\begin{aligned} & \mathbb{E} \left[\int_{t_n}^T \int_{\mathbb{R}} (\tau - a)^{2(\rho-1)} |Z^{n+1}(\tau, s)|^2 Q(\tau, ds) \eta(\tau) d\tau \right] \\ & \leq \frac{1}{2\rho-1} (T-a)^{2\rho-1} \mathbb{E} \left[\int_{t_n}^T \int_{\mathbb{R}} |Z^{n+1}(\tau, s)|^2 Q(\tau, ds) \eta(\tau) d\tau \right], \end{aligned}$$

where $\frac{1}{2} < \rho \leq 1$. Thus for any $a \leq t \leq T$, we have $\mathbb{E}[[X^{n+1}, X^{n+1}](t)] < \infty$ and $X^{n+1}(t)$ is a uniformly integrable martingale. Taking the conditional expected values on (3.15), we get

$$\mathbb{E}[X^{n+1}(t)] = \mathbb{E} \left[\zeta + \int_t^T (\tau - a)^{\rho-1} g(\tau, X^n(\tau), Y^n(\tau), Z^n(\tau, \cdot)) d\tau \middle| \mathcal{F}_t \right].$$

Also the triple $(X^{n+1}, Y^{n+1}, Z^{n+1})$ must satisfy the equation

$$\begin{aligned} & \zeta + \int_a^T (\tau - a)^{\rho-1} g(\tau, X^n(\tau), Y^n(\tau), Z^n(\tau, \cdot)) d\tau \\ & = \mathbb{E} \left[\zeta + \int_a^T (\tau - a)^{\rho-1} g(\tau, X^n(\tau), Y^n(\tau), Z^n(\tau, \cdot)) d\tau \middle| \mathcal{F}_t \right] \\ & \quad + \int_a^T (\tau - a)^{\rho-1} Y^{n+1}(\tau) dB(\tau) + \int_a^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z^{n+1}(\tau, s) \tilde{N}(d\tau, ds), \end{aligned}$$

then Y^{n+1} and Z^{n+1} can be determined by ζ and

$$\mathbb{E} \left[\zeta + \int_a^T (\tau - a)^{\rho-1} g(\tau, X^n(\tau), Y^n(\tau), Z^n(\tau, \cdot)) d\tau \middle| \mathcal{F}_t \right].$$

By Fatou's Lemma and Doob's inequality we can conclude that there exists a triple $(X^{n+1}, Y^{n+1}, Z^{n+1}) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}_N^2(\mathbb{R})$.

For any $a \leq t \leq T$, from Lemma 9, Lemma 10, (3.10) and (3.15) we have

$$\begin{aligned} & \|X^{n+1} - X^n\|_{\mathbb{S}^2}^2 + \|Y^{n+1} - Y^n\|_{\mathbb{H}^2}^2 + \|Z^{n+1} - Z^n\|_{\mathbb{H}_N^2}^2 \\ & \leq K_3 \left[\int_t^T e^{\sigma\tau} (\tau - a)^{2(\rho-1)} |g(\tau, X^n(\tau), Y^n(\tau), Z^n(\tau)) \right. \\ & \quad \left. - g(\tau, X^{n-1}(\tau), Y^{n-1}(\tau), Z^{n-1}(\tau))|^2 d\tau \right] \\ & \leq 2KK_3(T-a)^{2(1-\rho)} (T\|X^n - X^{n-1}\|_{\mathbb{S}^2}^2 \\ & \quad + \|Y^n - Y^{n-1}\|_{\mathbb{H}^2}^2 + \|Z^n - Z^{n-1}\|_{\mathbb{H}_N^2}^2) \\ & \leq 2KK_3(T+1)(T-a)^{2(1-\rho)} (\|X^n - X^{n-1}\|_{\mathbb{S}^2}^2 \\ & \quad + \|Y^n - Y^{n-1}\|_{\mathbb{H}^2}^2 + \|Z^n - Z^{n-1}\|_{\mathbb{H}_N^2}^2), \end{aligned} \tag{3.16}$$

where $K_3 = \sqrt{\sigma} + \frac{(T-a)^{4(1-\rho)}\tilde{K}}{\sqrt{\sigma}} + \frac{(T-a)^{2(1-\rho)}}{\sigma}$. Thus there exists a limit $(X, Y, Z) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}_N^2(\mathbb{R})$ of the converging sequence $(X^{n+1}, Y^{n+1}, Z^{n+1})_{n \in \mathbb{N}}$, and (X, Y, Z) satisfies (3.3).

Next, we prove the uniqueness of the solution. Suppose (X, Y, Z) and (X', Y', Z') are solutions of (3.1). For any $a \leq t \leq T$, we have

$$\begin{aligned} X(t) - X'(t) &= \int_t^T (\tau - a)^{\rho-1} [g(\tau, X(\tau), Y(\tau), Z(\tau, \cdot)) \\ &\quad - g(\tau, X'(\tau), Y'(\tau), Z'(\tau, \cdot))] d\tau \\ &\quad - \int_t^T (\tau - a)^{\rho-1} (Y(\tau) - Y'(\tau)) dB(\tau) \\ &\quad - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} (Z(\tau, s) - Z'(\tau, s)) \tilde{N}(d\tau, ds). \end{aligned}$$

Then

$$\begin{aligned} &\mathbb{E} \left[\sup_{a \leq t \leq T} e^{\sigma t} |X(t) - X'(t)|^2 \right] \\ &= \mathbb{E} \left[\sup_{a \leq t \leq T} e^{\sigma t} \left| \int_t^T (\tau - a)^{\rho-1} [g(\tau, X(\tau), Y(\tau), Z(\tau, \cdot)) \right. \right. \\ &\quad \left. \left. - g(\tau, X'(\tau), Y'(\tau), Z'(\tau, \cdot))] d\tau \right|^2 \right] \\ &\quad + \mathbb{E} \left[\sup_{a \leq t \leq T} e^{\sigma t} \int_t^T (\tau - a)^{2(\rho-1)} |Y(\tau) - Y'(\tau)|^2 d\tau \right] \\ &\quad + \mathbb{E} \left[\sup_{a \leq t \leq T} e^{\sigma t} \int_t^T \int_{\mathbb{R}} (\tau - a)^{2(\rho-1)} |Z(\tau, s) - Z'(\tau, s)|^2 Q(\tau, ds) \eta(\tau) d\tau \right]. \end{aligned}$$

From Lemma 10, (3.9) and (3.11), we can get

$$\|X - X'\|_{\mathbb{S}^2}^2 + \|Y - Y'\|_{\mathbb{H}^2}^2 + \|Z - Z'\|_{\mathbb{H}_N^2}^2 = 0,$$

which proves the uniqueness of the solution (X, Y, Z) . The proof is complete. $\square$

4. PREDICTABLE REPRESENTATION OF SOLUTION

We introduce two types of predictable representations of conformable backward stochastic differential equations. Assume that the terminal condition ζ satisfies (S1), and a generator $g : [a, T] \rightarrow \mathbb{R}$ is a predictable process.

Proposition 1. Let $a \leq t \leq T$, $\frac{1}{2} < \rho \leq 1$, and we consider the backward stochastic differential equation with zero generator

$$\begin{cases} dX(t) = (t-a)^{\rho-1}Y(t)dB(t) + (t-a)^{\rho-1} \int_{\mathbb{R}} Z(t,s)\tilde{N}(dt,ds), \\ X(T) = \zeta. \end{cases}$$

Then there exists a unique solution (X, Y, Z) given by

$$X(t) = \zeta - \int_t^T (\tau-a)^{\rho-1}Y(\tau)dB(\tau) - \int_t^T (\tau-a)^{\rho-1} \int_{\mathbb{R}} Z(\tau,s)\tilde{N}(d\tau,ds). \quad (4.1)$$

Moreover, we have

$$X(t) = \mathbb{E}[\zeta | \mathcal{F}_t],$$

and

$$\zeta = \mathbb{E}[\zeta] + \int_t^T (\tau-a)^{\rho-1}Y(\tau)dB(\tau) + \int_t^T (\tau-a)^{\rho-1} \int_{\mathbb{R}} Z(\tau,s)\tilde{N}(d\tau,ds).$$

Proof. Any solution (X, Y, Z) of the backward stochastic differential equation (4.1) satisfies

$$\zeta = X(a) + \int_a^T (\tau-a)^{\rho-1}Y(\tau)dB(\tau) + \int_a^T (\tau-a)^{\rho-1} \int_{\mathbb{R}} Z(\tau,s)\tilde{N}(d\tau,ds).$$

From Lemmas 3 and Lemma 5, for any sequence of stopping times $(t_n)_{n \geq 1}$, one has

$$\begin{aligned} \mathbb{E}[[X, X](t_n)] &= \zeta^2 + \mathbb{E} \left[\int_{t_n}^T (\tau-a)^{2(\rho-1)} |Y(\tau)|^2 d\tau \right] \\ &\quad + \mathbb{E} \left[\int_{t_n}^T \int_{\mathbb{R}} (\tau-a)^{2(\rho-1)} |Z(\tau,s)|^2 Q(\tau,ds) \eta(\tau) d\tau \right] \\ &\leq \zeta^2 + \frac{1}{2\rho-1} (T-a)^{2\rho-1} \mathbb{E} \left[\int_{t_n}^T |Y(\tau)|^2 d\tau \right] \\ &\quad + \frac{1}{2\rho-1} (T-a)^{2\rho-1} \mathbb{E} \left[\int_{t_n}^T \int_{\mathbb{R}} |Z(\tau,s)|^2 Q(\tau,ds) \eta(\tau) d\tau \right], \end{aligned} \quad (4.2)$$

where $\frac{1}{2} < \rho \leq 1$. The stochastic integrals in (4.2) are uniformly integrable martingales, then X is a square integral martingale and $\mathbb{E}[[X, X](a)] < \infty$. Furthermore, we derive $Y \in \mathbb{H}^2(\mathbb{R})$ and $Z \in \mathbb{H}_N^2(\mathbb{R})$. By Doob's inequality and Theorem I.9 in [20], we can conclude that $X \in \mathbb{S}^2(\mathbb{R})$. Taking the conditional expected value on both sides of (4.1), we get $X(t) = \mathbb{E}[\zeta | \mathcal{F}_t]$, $a \leq t \leq T$. Next taking the martingale $X(t) = \mathbb{E}[\zeta | \mathcal{F}_t]$ on both sides of (4.1), we have

$$\begin{aligned} \zeta &= \mathbb{E}[\zeta] + \int_t^T (\tau-a)^{\rho-1}Y(\tau)dB(\tau) \\ &\quad + \int_t^T (\tau-a)^{\rho-1} \int_{\mathbb{R}} Z(\tau,s)\tilde{N}(d\tau,ds), \quad a \leq t \leq T. \end{aligned}$$

In addition, the processes (Y, Z) can be uniquely expressed by ζ and $\mathbb{E}[\zeta]$. $\square$

Proposition 2. *Let $a \leq t \leq T$, $\frac{1}{2} < \rho \leq 1$, and we consider the backward stochastic differential equation with generator independent of (X, Y, Z)*

$$\begin{cases} dX(t) = -(t-a)^{\rho-1}g(t)dt + (t-a)^{\rho-1}Y(t)dB(t) \\ \quad + (t-a)^{\rho-1} \int_{\mathbb{R}} Z(t,s)\tilde{N}(dt,ds), \\ X(T) = \zeta, \end{cases}$$

where g satisfies $\mathbb{E}[\int_a^T |g(\tau)|^2 d\tau] < \infty$. Then there exists a unique solution $(X, Y, Z) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}_N^2(\mathbb{R})$ satisfying

$$\begin{aligned} X(t) = \zeta + \int_t^T (\tau-a)^{\rho-1}g(\tau)d\tau - \int_t^T (\tau-a)^{\rho-1}Y(\tau)dB(\tau) \\ - \int_t^T (\tau-a)^{\rho-1} \int_{\mathbb{R}} Z(\tau,s)\tilde{N}(d\tau,ds). \end{aligned} \quad (4.3)$$

Moreover we have

$$X(t) = \mathbb{E}\left[\zeta + \int_t^T (\tau-a)^{\rho-1}g(\tau)d\tau \middle| \mathcal{F}_t\right],$$

and

$$\begin{aligned} \zeta + \int_a^T (\tau-a)^{\rho-1}g(\tau)d\tau = \mathbb{E}\left[\zeta + \int_a^T (\tau-a)^{\rho-1}g(\tau)d\tau\right] + \int_a^T (\tau-a)^{\rho-1}Y(\tau)dB(\tau) \\ + \int_a^T (\tau-a)^{\rho-1} \int_{\mathbb{R}} Z(\tau,s)\tilde{N}(d\tau,ds). \end{aligned}$$

Proof. For any sequence of stopping times $(t_n)_{n \geq 1}$, from Lemma 3 and Lemma 5, one has

$$\begin{aligned} \mathbb{E}[[X, X](t_n)] = \zeta^2 + \mathbb{E}\left[\left(\int_{t_n}^T (\tau-a)^{\rho-1}g(\tau)d\tau\right)^2\right] \\ + \mathbb{E}\left[\int_{t_n}^T (\tau-a)^{2(\rho-1)}|Y(\tau)|^2 d\tau\right] \\ + \mathbb{E}\left[\int_{t_n}^T \int_{\mathbb{R}} (\tau-a)^{2(\rho-1)}|Z(\tau,s)|^2 Q(\tau,ds)\eta(\tau)d\tau\right]. \end{aligned} \quad (4.4)$$

Next, from the Schwarz inequality we have

$$\begin{aligned} \mathbb{E}[[X, X](t_n)] \leq \zeta^2 + \frac{1}{2\rho-1}(T-a)^{2\rho-1}\mathbb{E}\left[\int_{t_n}^T |g(\tau)|^2 d\tau\right] \\ + \frac{1}{2\rho-1}(T-a)^{2\rho-1}\mathbb{E}\left[\int_{t_n}^T e^{\sigma\tau}|Y(\tau)|^2 d\tau\right] \end{aligned}$$

$$+ \frac{1}{2\rho - 1} (T - a)^{2\rho - 1} \mathbb{E} \left[\int_{t_n}^T \int_{\mathbb{R}} e^{\sigma\tau} |Z(\tau, s)|^2 Q(\tau, ds) \eta(\tau) d\tau \right], \quad (4.5)$$

where $\frac{1}{2} < \rho \leq 1$.

Notice that $\int_{t_n}^T e^{\sigma\tau} |Y(\tau)|^2 d\tau$ and $\int_{t_n}^T \int_{\mathbb{R}} e^{\sigma\tau} |Z(\tau, s)|^2 Q(\tau, ds) \eta(\tau) d\tau$ are uniformly integrable martingales. Since $\mathbb{E}[\int_a^T |g(\tau)|^2 d\tau] < \infty$, we can get $\mathbb{E}[[X, X](a)] < \infty$. Furthermore, we also get $Y \in \mathbb{H}^2(\mathbb{R})$ and $Z \in \mathbb{H}_N^2(\mathbb{R})$. By Doob's inequality and Theorem I.9 in [20], we see that $X \in \mathbb{S}^2(\mathbb{R})$. Since $t \rightarrow \int_a^T (\tau - a)^{\rho-1} g(\tau) d\tau$ is a.s. continuous, see Sect. 4.3 in [3], then taking the conditional expected value in (5.4), we get $X(t) = \mathbb{E}[\zeta + \int_t^T (\tau - a)^{\rho-1} g(\tau) d\tau | \mathcal{F}_t]$, $a \leq t \leq T$. Next taking the martingale $X(t) = \mathbb{E}[\zeta + \int_t^T (\tau - a)^{\rho-1} g(\tau) d\tau | \mathcal{F}_t]$ on both sides of (5.4), we have

$$\begin{aligned} \zeta + \int_a^T (\tau - a)^{\rho-1} g(\tau) d\tau &= \mathbb{E} \left[\zeta + \int_a^T (\tau - a)^{\rho-1} g(\tau) d\tau \right] + \int_a^T (\tau - a)^{\rho-1} Y(\tau) dB(\tau) \\ &\quad + \int_a^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z(\tau, s) \tilde{N}(d\tau, ds). \end{aligned}$$

This finishes the proof. $\square$

5. COMPARISON THEOREM AND CONVERSE COMPARISON THEOREM

For any $(\omega, t, X(t), Y(t), Z(t)) \in \Omega \times [a, T] \times \mathbb{R} \times \mathbb{R} \times L_Q^2(\mathbb{R})$, let

$$\begin{aligned} g(\omega, t, X(t), Y(t), Z(t, s)) &= h(\omega, t, X(t), Y(t), \int_{\mathbb{R}} Z(t, s) \delta(t, s) Q(t, ds) \eta(t)) \\ &= h(\omega, t, X(t), Y(t), U(t, s)), \end{aligned}$$

where the process $\delta : \Omega \times [a, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is predictable and non-negative.

Hence equation (3.2) can be written in the following form

$$\begin{cases} dX(t) = -(t - a)^{\rho-1} h(t, X(t), Y(t), \int_{\mathbb{R}} Z(t, s) \delta(t, s) Q(t, ds) \eta(t)) dt \\ \quad + (t - a)^{\rho-1} Y(t) dB(t) + (t - a)^{\rho-1} \int_{\mathbb{R}} Z(t, s) \tilde{N}(dt, ds), \\ X(T) = \zeta. \end{cases} \quad (5.1)$$

Definition 5. A triple $(X(t), Y(t), Z(t)) \in \mathbb{R} \times \mathbb{R} \times L_Q^2(\mathbb{R})$ is called the solution of equation (5.1) if $(X(t), Y(t), Z(t))$ satisfies

$$\begin{aligned} X(t) &= \zeta + \int_t^T (\tau - a)^{\rho-1} h\left(\tau, X(\tau), Y(\tau), \int_{\mathbb{R}} Z(\tau, s) \delta(\tau, s) Q(\tau, ds) \eta(\tau)\right) d\tau \\ &\quad - \int_t^T (\tau - a)^{\rho-1} Y(\tau) dB(\tau) - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z(\tau, s) \tilde{N}(d\tau, ds), \end{aligned}$$

where $a \leq t \leq T$.

From now on, in order to simplify the notation, we think of ω as an implicit function in the function $h(\omega, t, X(t), Y(t), \int_{\mathbb{R}} Z(t, s) \delta(t, s) Q(t, ds) \eta(t))$, so as not to cause confusion.

Next we introduce the following hypotheses, denoted by H_{cs} .

- (i) the generator $h : \Omega \times [a, T] \times \mathbb{R} \times \mathbb{R} \times L_Q^2(\mathbb{R}) \rightarrow \mathbb{R}$ is predictable, continuous and satisfies the Lipschitz condition

$$|h(\omega, t, x, y, u) - h(\omega, t, x', y', u')| \leq K(|x - x'| + |y - y'| + |u - u'|),$$

a.s., a.e. $(\omega, t) \in \Omega \times [a, T]$, for any $(x, y, u), (x', y', u') \in \mathbb{R} \times \mathbb{R} \times L_Q^2(\mathbb{R})$,

- (ii) for any $(t, x, y) \in \Omega \times \mathbb{R} \times \mathbb{R}$, the mapping $u \mapsto h(t, x, y, u)$ is non-decreasing,

- (iii) the mapping $t \mapsto \int_{\mathbb{R}} |\delta(t, s)|^2 Q(t, ds) \eta(t)$ is bounded,

- (iv) $\mathbb{E}[\int_a^T |h(t, 0, 0, 0)|^2 dt] < \infty$.

Theorem 4. Consider the conformable backward stochastic differential equation

$$\begin{aligned} X(t) = & \zeta + \int_t^T (\tau - a)^{\rho-1} h(\tau, X(\tau), Y(\tau), \int_{\mathbb{R}} Z(\tau, s) \delta(\tau, s) Q(\tau, ds) \eta(\tau)) d\tau \\ & - \int_t^T (\tau - a)^{\rho-1} Y(\tau) dB(\tau) - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z(\tau, s) \tilde{N}(d\tau, ds), \end{aligned}$$

where $a \leq t \leq T$. Assume that the terminal values $\zeta, \zeta' \in \mathbb{L}^2(\mathbb{R})$ and the generators h, h' satisfy the hypotheses H_{cs} . Let $(X, Y, Z), (X', Y', Z') \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}_N^2(\mathbb{R})$ be the unique solutions with (ζ, h) and (ζ', h') . For any $(t, x, y, z) \in [a, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, if $\zeta \geq \zeta'$ and $h(t, x, y, u) \geq h'(t, x, y, u)$, then $X(t) \geq X'(t)$, where $u = \int_{\mathbb{R}} z(t, s) \delta(t, s) Q(t, ds) \eta(t)$. In addition, for any $a \leq t_0 \leq T$, if $X(t_0) = X'(t_0)$, then $X(t) = X'(t), t_0 \leq t \leq T$.

Proof. The existence and uniqueness of solutions $(X, Y, Z), (X', Y', Z')$ follows from Proposition 2. Define $\tilde{X}(t) = X(t) - X'(t)$, $\tilde{Y}(t) = Y(t) - Y'(t)$, $\tilde{Z}(t) = Z(t) - Z'(t)$ and $U(t, s) = \int_{\mathbb{R}} Z(t, s) \delta(t, s) Q(t, ds) \eta(t)$. The generator h' is Lipschitz continuous in x, y and z , and we can define the following processes

$$\alpha(t) = \begin{cases} \frac{h(t, X(t), Y(t), U(t, s)) - h(t, X'(t), Y(t), U(t, s))}{\tilde{X}(t)} & \text{if } X(t) \neq X'(t), \\ 0 & \text{otherwise,} \end{cases}$$

$$\beta(t) = \begin{cases} \frac{h(t, X'(t), Y(t), U(t, s)) - h(t, X'(t), Y'(t), U(t, s))}{\tilde{Y}(t)} & \text{if } Y(t) \neq Y'(t), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\psi(t) = \begin{cases} \frac{h(t, X'(t), Y'(t), U(t, s)) - h(t, X'(t), Y'(t), U'(t, s))}{\int_{\mathbb{R}} \tilde{Z}(t, s) \delta(t, s) Q(t, ds) \eta(t)} & \text{if } Z(t) \neq Z'(t), \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\begin{aligned}
d\tilde{X}(t) &= -(t-a)^{\rho-1} \left[h(t, X(t), Y(t), \int_{\mathbb{R}} Z(t, s) \delta(t, s) Q(t, ds) \eta(t)) \right. \\
&\quad \left. - h'(t, X'(t), Y'(t), \int_{\mathbb{R}} Z'(t, s) \delta(t, s) Q(t, ds) \eta(t)) \right] dt \\
&\quad + (t-a)^{\rho-1} \tilde{Y}(t) dB(t) + (t-a)^{\rho-1} \int_{\mathbb{R}} \tilde{Z}(t, s) \tilde{N}(dt, ds) \\
&= -(t-a)^{\rho-1} \left[\tilde{h}(t, X'(t), Y'(t), \int_{\mathbb{R}} Z'(t, s) \delta(t, s) Q(t, ds) \eta(t)) + \alpha(t) \tilde{X}(t) \right. \\
&\quad \left. + \beta(t) \tilde{Y}(t) + \psi(t) \int_{\mathbb{R}} \tilde{Z}(t, s) \delta(t, s) Q(t, ds) \eta(t) \right] dt + (t-a)^{\rho-1} \tilde{Y}(t) dB(t) \\
&\quad + (t-a)^{\rho-1} \int_{\mathbb{R}} \tilde{Z}(t, s) \tilde{N}(dt, ds),
\end{aligned}$$

where

$$\begin{aligned}
&\tilde{h}(t, X'(t), Y'(t), \int_{\mathbb{R}} Z'(t, s) \delta(t, s) Q(t, ds) \eta(t)) \\
&= h(t, X'(t), Y'(t), \int_{\mathbb{R}} Z'(t, s) \delta(t, s) Q(t, ds) \eta(t)) \\
&\quad - h'(t, X'(t), Y'(t), \int_{\mathbb{R}} Z'(t, s) \delta(t, s) Q(t, ds) \eta(t)).
\end{aligned}$$

Applying Lemma 1 to $\tilde{X}(t) \exp(\int_a^t (v-a)^{\rho-1} \alpha(v) dv)$, and we have

$$\begin{aligned}
&d\tilde{X}(t) \exp \left(\int_a^t (v-a)^{\rho-1} \alpha(v) dv \right) \\
&= (t-a)^{\rho-1} \alpha(t) \tilde{X}(t) \exp \left(\int_a^t (v-a)^{\rho-1} \alpha(v) dv \right) dt \\
&\quad + \exp \left(\int_a^t (v-a)^{\rho-1} \alpha(v) dv \right) d\tilde{X}(t) \\
&= -(t-a)^{\rho-1} \exp \left(\int_a^t (v-a)^{\rho-1} \alpha(v) dv \right) \\
&\quad \times \tilde{h} \left(t, X'(t), Y'(t), \int_{\mathbb{R}} Z'(t, s) \delta(t, s) Q(t, ds) \eta(t) \right) dt \\
&\quad - (t-a)^{\rho-1} \beta(t) \tilde{Y}(t) \exp \left(\int_a^t (v-a)^{\rho-1} \alpha(v) dv \right) dt - (t-a)^{\rho-1} \psi(t) \\
&\quad \times \exp \left(\int_a^t (v-a)^{\rho-1} \alpha(v) dv \right) \int_{\mathbb{R}} \tilde{Z}(t, s) \delta(t, s) Q(t, ds) \eta(t) dt + (t-a)^{\rho-1} \\
&\quad \times \exp \left(\int_a^t (v-a)^{\rho-1} \alpha(v) dv \right) \tilde{Y}(t) dB(t) + (t-a)^{\rho-1}
\end{aligned}$$

$$\times \exp \left(\int_a^t (v-a)^{\rho-1} \alpha(v) dv \right) \int_{\mathbb{R}} \tilde{Z}(t,s) \tilde{N}(dt, ds)$$

By integrating both sides of the equation from t to T , one has

$$\begin{aligned} & \tilde{X}(t) \exp \left(\int_a^t (v-a)^{\rho-1} \alpha(v) dv \right) \\ &= \tilde{\xi} \exp \left(\int_a^T (v-a)^{\rho-1} \alpha(v) dv \right) + \int_t^T (\tau-a)^{\rho-1} \\ & \quad \times \exp \left(\int_a^\tau (v-a)^{\rho-1} \alpha(v) dv \right) \\ & \quad \times \tilde{h} \left(\tau, X'(\tau), Y'(\tau), \int_{\mathbb{R}} Z'(\tau,s) \delta(\tau,s) Q(\tau, ds) \eta(\tau) \right) d\tau \\ & \quad + \int_t^T (\tau-a)^{\rho-1} \beta(\tau) \exp \left(\int_a^\tau (v-a)^{\rho-1} \alpha(v) dv \right) \tilde{Y}(\tau) d\tau \\ & \quad + \int_t^T (\tau-a)^{\rho-1} \psi(\tau) \exp \left(\int_a^\tau (v-a)^{\rho-1} \alpha(v) dv \right) \\ & \quad \times \int_{\mathbb{R}} \tilde{Z}(\tau,s) \delta(\tau,s) Q(\tau, ds) \eta(\tau) d\tau \\ & \quad - \int_t^T (\tau-a)^{\rho-1} \exp \left(\int_a^\tau (v-a)^{\rho-1} \alpha(v) dv \right) \tilde{Y}(\tau) dB(\tau) \\ & \quad - \int_t^T (\tau-a)^{\rho-1} \exp \left(\int_a^\tau (v-a)^{\rho-1} \alpha(v) dv \right) \int_{\mathbb{R}} \tilde{Z}(\tau,s) \tilde{N}(d\tau, ds), \quad a \leq t \leq T. \end{aligned}$$

From Lemma 7, we can introduce an equivalent probability measure

$$\begin{aligned} \frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} &= M(t), \quad M(a) = 1 \\ \frac{dM(t)}{M(t)} &= \beta(t) dB(t) + \int_{\mathbb{R}} \psi(t) \delta(t,s) \tilde{N}(dt, ds), \quad a \leq t \leq T. \end{aligned}$$

We know that $|\beta(t)| \leq K$, $\int_{\mathbb{R}} |\psi(t) \delta(t,s)|^2 Q(t, ds) \eta(t) \leq K$ and $\psi(t) \delta(t,s) > 1$. According to Proposition 8, the process M is a square integrable martingale under the equivalent probability measure $\mathbb{Q}$. Then changing the probability measure, we have

$$\begin{aligned} & \tilde{X}(t) \exp \left(\int_a^t (v-a)^{\rho-1} \alpha(v) dv \right) \\ &= \tilde{\xi} \exp \left(\int_a^T (v-a)^{\rho-1} \alpha(v) dv \right) + \int_t^T (\tau-a)^{\rho-1} \exp \left(\int_a^\tau (v-a)^{\rho-1} \alpha(v) dv \right) \\ & \quad \times \tilde{h} \left(\tau, X'(\tau), Y'(\tau), \int_{\mathbb{R}} Z'(\tau,s) \delta(\tau,s) Q(\tau, ds) \eta(\tau) \right) d\tau \end{aligned}$$

$$\begin{aligned}
& - \int_t^T (\tau - a)^{p-1} \exp \left(\int_a^\tau (v - a)^{p-1} \alpha(v) dv \right) \tilde{Y}(\tau) dB^\mathbb{Q}(\tau) \\
& - \int_t^T (\tau - a)^{p-1} \exp \left(\int_a^\tau (v - a)^{p-1} \alpha(v) dv \right) \int_{\mathbb{R}} \tilde{Z}(\tau, s) \tilde{N}^\mathbb{Q}(d\tau, ds), \quad (5.2)
\end{aligned}$$

where $a \leq t \leq T$. By Lemma 2, Schwarz inequality, the definition of expectation and the boundedness of the functions, we have

$$\begin{aligned}
& \mathbb{E}^\mathbb{Q} \left[\sup \int_t^T (\tau - a)^{p-1} \exp \left(\int_a^\tau (v - a)^{p-1} \alpha(v) dv \right) \tilde{Y}(\tau) dB^\mathbb{Q}(\tau) \right] \\
& \leq K \mathbb{E}^\mathbb{Q} \left[\sqrt{\int_t^T \left| (\tau - a)^{p-1} \exp \left(\int_a^\tau (v - a)^{p-1} \alpha(v) dv \right) \tilde{Y}(\tau) \right|^2 d\tau} \right] \\
& \leq K \mathbb{E} \left[\left| \frac{d\mathbb{Q}}{d\mathbb{P}} \right|^2 \right] \mathbb{E} \left[\int_t^T \left| (\tau - a)^{p-1} \exp \left(\int_a^\tau (v - a)^{p-1} \alpha(v) dv \right) \tilde{Y}(\tau) \right|^2 d\tau \right] \\
& \leq K \sqrt{\frac{(T-a)^{4p-3}}{4p-3}} \mathbb{E} \left[\left| \frac{d\mathbb{Q}}{d\mathbb{P}} \right|^2 \right] \mathbb{E} \left[\sqrt{\int_t^T \left| \exp \left(\int_a^\tau (v - a)^{p-1} \alpha(v) dv \right) \tilde{Y}(\tau) \right|^4 d\tau} \right],
\end{aligned}$$

and

$$\begin{aligned}
& \mathbb{E}^\mathbb{Q} \left[\sup \int_t^T (\tau - a)^{p-1} \exp \left(\int_a^\tau (v - a)^{p-1} \alpha(v) dv \right) \int_{\mathbb{R}} \tilde{Z}(\tau, s) \tilde{N}^\mathbb{Q}(d\tau, ds) \right] \\
& \leq K \mathbb{E}^\mathbb{Q} \left[\left(\int_t^T \int_{\mathbb{R}} \left| (\tau - a)^{p-1} \right. \right. \right. \\
& \quad \times \exp \left(\int_a^\tau (v - a)^{p-1} \alpha(v) dv \right) \tilde{Z}(\tau, s) \left. \right|^2 Q(\tau, ds) \eta(\tau) d\tau \left. \right] \\
& \leq K \mathbb{E} \left[\left| \frac{d\mathbb{Q}}{d\mathbb{P}} \right|^2 \right] \mathbb{E} \left[\int_t^T \int_{\mathbb{R}} \left| (\tau - a)^{p-1} \right. \right. \\
& \quad \times \exp \left(\int_a^\tau (v - a)^{p-1} \alpha(v) dv \right) \tilde{Z}(\tau, s) \left. \right|^2 Q(\tau, ds) \eta(\tau) d\tau \left. \right] \\
& \leq K \sqrt{\frac{(T-a)^{4p-3}}{4p-3}} \mathbb{E} \left[\left| \frac{d\mathbb{Q}}{d\mathbb{P}} \right|^2 \right] \mathbb{E} \left[\left(\int_t^T \int_{\mathbb{R}} \right. \right. \\
& \quad \left. \left| \exp \left(\int_a^\tau (v - a)^{p-1} \alpha(v) dv \right) \tilde{Z}(\tau, s) \right|^4 Q(\tau, ds) \eta(\tau) d\tau \right)^{\frac{1}{2}} \right].
\end{aligned}$$

Taking the conditional expectation on both sides of equation (5.2), we get

$$\tilde{X}(t) \exp \left(\int_a^t (v - a)^{p-1} \alpha(v) dv \right)$$

$$\begin{aligned}
&= \mathbb{E}^{\mathbb{Q}} \left[\tilde{\zeta} \exp \left(\int_a^T (\mathfrak{v} - a)^{\mathfrak{p}-1} \alpha(\mathfrak{v}) d\mathfrak{v} \right) \right. \\
&\quad + \int_t^T (\tau - a)^{\mathfrak{p}-1} \exp \left(\int_a^\tau (\mathfrak{v} - a)^{\mathfrak{p}-1} \alpha(\mathfrak{v}) d\mathfrak{v} \right) \\
&\quad \times \tilde{h} \left(\tau, X'(\tau), Y'(\tau), \int_{\mathbb{R}} Z'(\tau, s) \delta(\tau, s) Q(\tau, ds) \eta(\tau) \right) d\tau \Big| \mathcal{F}_t \Big]. \quad (5.3)
\end{aligned}$$

Hence for any $a \leq t \leq T$, we have $X(t) \geq X'(t)$.

For the last part note we add a hypothesis $X(t_0) = X'(t_0)$, that is, $\tilde{X}(t_0) = 0$, where $a \leq t_0 \leq T$. Notice that $\tilde{\zeta} \geq 0$ and $\tilde{h}(t, X'(t), Y'(t), \int_{\mathbb{R}} Z'(t, s) \delta(t, s) Q(t, ds) \eta(t)) \geq 0$, then we can conclude that $\tilde{\zeta} = 0$ and $\tilde{h}(t, X'(t), Y'(t), \int_{\mathbb{R}} Z'(t, s) \delta(t, s) Q(t, ds) \eta(t)) = 0$, where $t_0 \leq t \leq T$. By (5.2), one has

$$\begin{aligned}
&\tilde{X}(t) \exp \left(\int_a^t (\mathfrak{v} - a)^{\mathfrak{p}-1} \alpha(\mathfrak{v}) d\mathfrak{v} \right) \\
&= - \int_t^T (\tau - a)^{\mathfrak{p}-1} \exp \left(\int_a^\tau (\mathfrak{v} - a)^{\mathfrak{p}-1} \alpha(\mathfrak{v}) d\mathfrak{v} \right) \tilde{Y}(\tau) dB^{\mathbb{Q}}(\tau) \\
&\quad - \int_t^T (\tau - a)^{\mathfrak{p}-1} \exp \left(\int_a^\tau (\mathfrak{v} - a)^{\mathfrak{p}-1} \alpha(\mathfrak{v}) d\mathfrak{v} \right) \\
&\quad \times \int_{\mathbb{R}} \tilde{Z}(\tau, s) \tilde{N}^{\mathbb{Q}}(d\tau, ds), \quad t_0 \leq t \leq T. \quad (5.4)
\end{aligned}$$

Taking the conditional expectation in (5.4), the proof of the second conclusion is complete. $\square$

On the basis of Theorem 4, we propose an inverse principle of the comparison principle.

Theorem 5. Consider the conformable backward stochastic differential equation

$$\begin{aligned}
X(t) &= \zeta + \int_t^T (\tau - a)^{\mathfrak{p}-1} h(\tau, X(\tau), Y(\tau), \int_{\mathbb{R}} Z(\tau, s) \delta(\tau, s) Q(\tau, ds) \eta(\tau)) d\tau \\
&\quad - \int_t^T (\tau - a)^{\mathfrak{p}-1} Y(\tau) dB(\tau) - \int_t^T (\tau - a)^{\mathfrak{p}-1} \int_{\mathbb{R}} Z(\tau, s) \tilde{N}(d\tau, ds),
\end{aligned}$$

where $a \leq t \leq T$. Assume that the generators h, h' satisfy the hypotheses H_{cs} . Let $(X, Y, Z), (X', Y', Z') \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}_N^2(\mathbb{R})$ be the unique solutions with h and h' . For any $(t, x, y, u) \in [a, T] \times \mathbb{R} \times \mathbb{R} \times L_Q^2(\mathbb{R})$, if $\zeta = \zeta' \in \mathbb{L}^2(\mathbb{R})$ and $X(t) \geq X'(t)$, then $h(t, x, y, u) \geq h'(t, x, y, u)$, where $u(t) = \int_{\mathbb{R}} z(t, s) \delta(t, s) Q(t, ds) \eta(t)$.

Proof. According to the proof of Theorem 4, we can still get formula (5.3),

$$\tilde{X}(t) \exp \left(\int_a^t (\mathfrak{v} - a)^{\mathfrak{p}-1} \alpha(\mathfrak{v}) d\mathfrak{v} \right)$$

$$\begin{aligned}
&= \mathbb{E}^{\mathbb{Q}} \left[\tilde{\zeta} \exp \left(\int_a^T (\mathfrak{v} - a)^{\rho-1} \alpha(\mathfrak{v}) d\mathfrak{v} \right) + \int_t^T (\tau - a)^{\rho-1} \right. \\
&\quad \times \exp \left(\int_a^{\tau} (\mathfrak{v} - a)^{\rho-1} \alpha(\mathfrak{v}) d\mathfrak{v} \right) \\
&\quad \times \tilde{h} \left(\tau, X'(\tau), Y'(\tau), \int_{\mathbb{R}} Z'(\tau, s) \delta(\tau, s) Q(\tau, ds) \eta(\tau) \right) d\tau \Big| \mathcal{F}_t \Big].
\end{aligned}$$

Since $\zeta = \zeta'$, the above formula can be rewritten as

$$\begin{aligned}
&\tilde{X}(t) \exp \left(\int_a^t (\mathfrak{v} - a)^{\rho-1} \alpha(\mathfrak{v}) d\mathfrak{v} \right) \\
&= \mathbb{E}^{\mathbb{Q}} \left[\int_t^T (\tau - a)^{\rho-1} \exp \left(\int_a^{\tau} (\mathfrak{v} - a)^{\rho-1} \alpha(\mathfrak{v}) d\mathfrak{v} \right) \right. \\
&\quad \times \tilde{h} \left(\tau, X'(\tau), Y'(\tau), \int_{\mathbb{R}} Z'(\tau, s) \delta(\tau, s) Q(\tau, ds) \eta(\tau) \right) d\tau \Big| \mathcal{F}_t \Big].
\end{aligned}$$

Hence for any $X(t) \geq X'(t)$, we have $h(t, x, y, u) \geq h'(t, x, y, u)$. The proof is complete. $\square$

Theorem 6. Consider the following conformable backward stochastic differential equation

$$\begin{aligned}
X(t) &= \zeta + \int_t^T (\tau - a)^{\rho-1} g(\tau, X(\tau), Y(\tau), Z(\tau, \cdot)) d\tau \\
&\quad - \int_t^T (\tau - a)^{\rho-1} Y(\tau) dB(\tau) - \int_t^T (\tau - a)^{\rho-1} \\
&\quad \times \int_{\mathbb{R}} Z(\tau, s) \tilde{N}(d\tau, ds), \quad 0 < \rho \leq 1, \quad a \leq t \leq T.
\end{aligned}$$

Assume that

- (i) the terminal value $\zeta \in \mathbb{L}^2(\mathbb{R})$,
- (ii) the generator $g : \Omega \times [a, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is predictable and Lipschitz in x and y

$$|g(t, x, y, z) - g(t, x', y', z)| \leq K(|x - x'| + |y - y'|),$$

In addition, for any $(x, y, z), (x, y, z') \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, there exists constants $-1 < C_1 \leq 0$ and $C_2 \geq 0$ such that

$$g(t, x, y, z) - g(t, x, y, z') \leq \int_{\mathbb{R}} (z(t, s) - z'(t, s)) \delta^{x, y, z, z'}(t, s) Q(t, ds) \eta(t),$$

where $\delta^{x, y, z, z'}(t, s) : \Omega \times [a, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is predictable and satisfies

$$C_1(1 \wedge |s|) \leq \delta^{x, y, z, z'}(t, s) \leq C_2(1 \wedge |s|),$$

- (iii) $\mathbb{E}[\int_a^T |g(t, 0, 0, 0)|^2 dt] < \infty$.

Let $(X, Y, Z) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}_N^2(\mathbb{R})$ and $(X', Y', Z') \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}_N^2(\mathbb{R})$ be the unique solutions to the above equation with (ζ, g) and (ζ', g') . For any $(t, x, y, z) \in [a, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, if $\zeta \geq \zeta'$ and $g(t, x, y, z) \geq g'(t, x, y, z)$, then $X(t) \geq X'(t)$. In addition, for any $a \leq t_0 \leq T$, if $X(t_0) = X'(t_0)$, then $X(t) = X'(t)$, $t_0 \leq t \leq T$.

Proof. Notice that $\delta^{x,y,z,z'}(t, s)$ depends on x, y, z, z' and when it is greater than $C_1(1 \wedge |s|)$, it can also be negative. According to hypothesis (ii), we can conclude that the generator g is also Lipschitz in z such that

$$|g(t, x, y, z) - g(t, x, y, z')| \leq C_3 \int_{\mathbb{R}} |z(t, s) - z'(t, s)| (1 \wedge |s|) Q(t, ds) \eta(t),$$

where $C_3 \geq \max(-C_1, C_2)$. In other words, there exists a predictable process $\bar{\delta}^{x,y,z,z'} : \Omega \times [a, T] \times \mathbb{R} \rightarrow \mathbb{R}$ such that

$$g(t, x, y, z) - g(t, x, y, z') \geq \int_{\mathbb{R}} (z(t, s) - z'(t, s)) \bar{\delta}^{x,y,z,z'}(t, s) Q(t, ds) \eta(t),$$

where $-C_3(1 \wedge |s|) \leq \bar{\delta}^{x,y,z,z'}(t, s) \leq C_3(1 \wedge |s|)$. Hence we can get

$$\begin{aligned} |g(t, x, y, z) - g(t, x', y', z')| &\leq K(|x - x'| + |y - y'| + \int_{\mathbb{R}} |z(t, s) - z'(t, s)| \\ &\quad \times \max(|\bar{\delta}^{x,y,z,z'}(t, s)|, |\bar{\delta}^{x',y',z',z'}(t, s)|) Q(t, ds) \eta(t)). \end{aligned}$$

The existence and uniqueness of the solutions (X, Y, Z) and (X', Y', Z') follow from Theorem 3.

Let

$$\tilde{X}(t) = X(t) - X'(t), \quad \tilde{Y}(t) = Y(t) - Y'(t), \quad \tilde{Z}(t) = Z(t) - Z'(t), \quad \text{and}$$

$$U(t, s) = \int_{\mathbb{R}} Z(t, s) \delta(t, s) Q(t, ds) \eta(t).$$

Since

$$\begin{aligned} g(\omega, t, X(t), Y(t), Z(t, s)) &= h(\omega, t, X(t), Y(t), \int_{\mathbb{R}} Z(t, s) \delta(t, s) Q(t, ds) \eta(t)) \\ &= h(\omega, t, X(t), Y(t), U(t, s)), \end{aligned}$$

we also have

$$\alpha(t) = \begin{cases} \frac{g(t, X(t), Y(t), Z(t, s)) - g(t, X'(t), Y(t), Z(t, s))}{\tilde{X}(t)} & \text{if } X(t) \neq X'(t), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\beta(t) = \begin{cases} \frac{g(t, X'(t), Y(t), Z(t, s)) - g(t, X'(t), Y'(t), Z(t, s))}{\tilde{Y}(t)} & \text{if } Y(t) \neq Y'(t), \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$d\tilde{X}(t) = -(t - a)^{\rho-1} [\alpha(t)\tilde{X}(t) + \beta(t)\tilde{Y}(t) + g(t, X', Y', Z)]$$

$$-g(t, X', Y', Z') + \tilde{g}(t, X', Y', Z')]dt + (t-a)^{\rho-1} \tilde{Y}(t)dB(t) \\ + (t-a)^{\rho-1} \int_{\mathbb{R}} \tilde{Z}(t, s) \tilde{N}(t, s).$$

Applying Lemma 1 to $\tilde{X}(t) \exp(\int_a^t (v-a)^{\rho-1} \alpha(v)dv)$, we have

$$d\tilde{X}(t) \exp\left(\int_a^t (v-a)^{\rho-1} \alpha(v)dv\right) \\ = -(t-a)^{\rho-1} \beta(t) \tilde{Y}(t) \exp\left(\int_a^t (v-a)^{\rho-1} \alpha(v)dv\right) dt \\ - (t-a)^{\rho-1} \exp\left(\int_a^t (v-a)^{\rho-1} \alpha(v)dv\right) [g(t, X', Y', Z) - g(t, X', Y', Z')]dt \\ - (t-a)^{\rho-1} \exp\left(\int_a^t (v-a)^{\rho-1} \alpha(v)dv\right) \tilde{g}\left(t, X'(t), Y'(t), Z'(t, s)\right) dt \\ + (t-a)^{\rho-1} \exp\left(\int_a^t (v-a)^{\rho-1} \alpha(v)dv\right) \tilde{Y}(t)dB(t) \\ + (t-a)^{\rho-1} \exp\left(\int_a^t (v-a)^{\rho-1} \alpha(v)dv\right) \int_{\mathbb{R}} \tilde{Z}(t, s) \tilde{N}(dt, ds)$$

By integrating both sides of the equation from t to T , one has

$$\tilde{X}(t) \exp\left(\int_a^t (v-a)^{\rho-1} \alpha(v)dv\right) \\ = \tilde{\xi} \exp\left(\int_a^T (v-a)^{\rho-1} \alpha(v)dv\right) \\ + \int_t^T (\tau-a)^{\rho-1} \beta(\tau) \exp\left(\int_a^\tau (v-a)^{\rho-1} \alpha(v)dv\right) \tilde{Y}(\tau) d\tau \\ + \int_t^T (\tau-a)^{\rho-1} \exp\left(\int_a^\tau (v-a)^{\rho-1} \alpha(v)dv\right) \\ \times [g(\tau, X'(\tau), Y'(\tau), Z(\tau)) - g(\tau, X'(\tau), Y'(\tau), Z'(\tau))] d\tau \\ + \int_t^T (\tau-a)^{\rho-1} \exp\left(\int_a^\tau (v-a)^{\rho-1} \alpha(v)dv\right) \tilde{g}\left(\tau, X'(\tau), Y'(\tau), Z'(\tau, s)\right) d\tau \\ - \int_t^T (\tau-a)^{\rho-1} \exp\left(\int_a^\tau (v-a)^{\rho-1} \alpha(v)dv\right) \tilde{Y}(\tau) dB(\tau) \\ - \int_t^T (\tau-a)^{\rho-1} \exp\left(\int_a^\tau (v-a)^{\rho-1} \alpha(v)dv\right) \int_{\mathbb{R}} \tilde{Z}(\tau, s) \tilde{N}(d\tau, ds), \quad a \leq t \leq T.$$

Notice that $g(t, x, y, z) - g(t, x, y, z') \geq \int_{\mathbb{R}} \tilde{z}(t, s) \bar{\delta}^{x, y, z, z'}(t, s) Q(t, ds) \eta(t)$, then

$$\tilde{X}(t) \exp\left(\int_a^t (v-a)^{\rho-1} \alpha(v)dv\right)$$

$$\begin{aligned}
&\geq \tilde{\zeta} \exp \left(\int_a^T (\mathfrak{v} - a)^{\rho-1} \alpha(\mathfrak{v}) d\mathfrak{v} \right) + \int_t^T (\tau - a)^{\rho-1} \beta(\tau) \\
&\quad \times \exp \left(\int_a^\tau (\mathfrak{v} - a)^{\rho-1} \alpha(\mathfrak{v}) d\mathfrak{v} \right) \tilde{Y}(\tau) d\tau + \int_t^T (\tau - a)^{\rho-1} \\
&\quad \times \exp \left(\int_a^\tau (\mathfrak{v} - a)^{\rho-1} \alpha(\mathfrak{v}) d\mathfrak{v} \right) \int_{\mathbb{R}} \tilde{Z}(\tau, s) \bar{\delta}^{x, y, z, z'}(\tau, s) Q(\tau, ds) \eta(\tau) d\tau \\
&\quad + \int_t^T (\tau - a)^{\rho-1} \exp \left(\int_a^\tau (\mathfrak{v} - a)^{\rho-1} \alpha(\mathfrak{v}) d\mathfrak{v} \right) \tilde{g} \left(\tau, X'(\tau), Y'(\tau), Z'(\tau, s) \right) d\tau \\
&\quad - \int_t^T (\tau - a)^{\rho-1} \exp \left(\int_a^\tau (\mathfrak{v} - a)^{\rho-1} \alpha(\mathfrak{v}) d\mathfrak{v} \right) \tilde{Y}(\tau) dB(\tau) \\
&\quad - \int_t^T (\tau - a)^{\rho-1} \exp \left(\int_a^\tau (\mathfrak{v} - a)^{\rho-1} \alpha(\mathfrak{v}) d\mathfrak{v} \right) \int_{\mathbb{R}} \tilde{Z}(\tau, s) \tilde{N}(d\tau, ds).
\end{aligned}$$

Define

$$\begin{aligned}
\left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_{\mathcal{F}_t} &= M(t), \quad M(a) = 1 \\
\frac{dM(t)}{M(t)} &= \beta(t) dB(t) + \int_{\mathbb{R}} \bar{\delta}^{x, y, z, z'}(t, s) \tilde{N}(dt, ds), \quad a \leq t \leq T.
\end{aligned}$$

According to Lemma 7, we get

$$\begin{aligned}
&\tilde{X}(t) \exp \left(\int_a^t (\mathfrak{v} - a)^{\rho-1} \alpha(\mathfrak{v}) d\mathfrak{v} \right) \\
&\geq \tilde{\zeta} \exp \left(\int_a^T (\mathfrak{v} - a)^{\rho-1} \alpha(\mathfrak{v}) d\mathfrak{v} \right) \\
&\quad + \int_t^T (\tau - a)^{\rho-1} \exp \left(\int_a^\tau (\mathfrak{v} - a)^{\rho-1} \alpha(\mathfrak{v}) d\mathfrak{v} \right) \tilde{g} \left(\tau, X'(\tau), Y'(\tau), Z'(\tau, s) \right) d\tau \\
&\quad - \int_t^T (\tau - a)^{\rho-1} \exp \left(\int_a^\tau (\mathfrak{v} - a)^{\rho-1} \alpha(\mathfrak{v}) d\mathfrak{v} \right) \tilde{Y}(\tau) dB^{\mathbb{Q}}(\tau) \\
&\quad - \int_t^T (\tau - a)^{\rho-1} \exp \left(\int_a^\tau (\mathfrak{v} - a)^{\rho-1} \alpha(\mathfrak{v}) d\mathfrak{v} \right) \int_{\mathbb{R}} \tilde{Z}(\tau, s) \tilde{N}^{\mathbb{Q}}(d\tau, ds).
\end{aligned}$$

Taking the conditional expectation on both sides, we have

$$\begin{aligned}
&\tilde{X}(t) \exp \left(\int_a^t (\mathfrak{v} - a)^{\rho-1} \alpha(\mathfrak{v}) d\mathfrak{v} \right) \\
&\geq \mathbb{E}^{\mathbb{Q}} \left[\tilde{\zeta} \exp \left(\int_a^T (\mathfrak{v} - a)^{\rho-1} \alpha(\mathfrak{v}) d\mathfrak{v} \right) + \int_t^T (\tau - a)^{\rho-1} \right. \\
&\quad \left. \times \exp \left(\int_a^\tau (\mathfrak{v} - a)^{\rho-1} \alpha(\mathfrak{v}) d\mathfrak{v} \right) \tilde{g} \left(\tau, X'(\tau), Y'(\tau), Z'(\tau, s) \right) \middle| \mathcal{F}_t \right].
\end{aligned}$$

Since $\tilde{\xi} \geq 0$ and $\tilde{g}(t, X', Y', Z') \geq 0$, we can conclude that $\tilde{X}(t) \geq 0$. The proof of the first assertion is complete.

According to $X(t_0) = X'(t_0)$, $a \leq t_0 \leq T$, we have $\tilde{\xi} = 0$ and $\tilde{g}(t, X', Y', Z') = 0$. Then

$$\begin{aligned} & \tilde{X}(t) \exp \left(\int_a^t (\mathfrak{v} - a)^{\rho-1} \alpha(\mathfrak{v}) d\mathfrak{v} \right) \\ & \geq - \int_t^T (\tau - a)^{\rho-1} \exp \left(\int_a^\tau (\mathfrak{v} - a)^{\rho-1} \alpha(\mathfrak{v}) d\mathfrak{v} \right) \tilde{Y}(\tau) dB^\mathbb{Q}(\tau) \\ & \quad - \int_t^T (\tau - a)^{\rho-1} \exp \left(\int_a^\tau (\mathfrak{v} - a)^{\rho-1} \alpha(\mathfrak{v}) d\mathfrak{v} \right) \int_{\mathbb{R}} \tilde{Z}(\tau, s) \tilde{N}^\mathbb{Q}(d\tau, ds). \end{aligned}$$

Taking the conditional expectation on both sides of the above inequality, we can get $\tilde{X}(t) \geq 0$.

In the same way, since

$$g(t, x, y, z) - g(t, x, y, z') \leq \int_{\mathbb{R}} (z(t, s) - z'(t, s)) \delta^{x, y, z, z'}(t, s) Q(t, ds) \eta(t),$$

one has

$$\begin{aligned} & \tilde{X}(t) \exp \left(\int_a^t (\mathfrak{v} - a)^{\rho-1} \alpha(\mathfrak{v}) d\mathfrak{v} \right) \\ & \leq \tilde{\xi} \exp \left(\int_a^T (\mathfrak{v} - a)^{\rho-1} \alpha(\mathfrak{v}) d\mathfrak{v} \right) + \int_t^T (\tau - a)^{\rho-1} \beta(\tau) \\ & \quad \times \exp \left(\int_a^\tau (\mathfrak{v} - a)^{\rho-1} \alpha(\mathfrak{v}) d\mathfrak{v} \right) \tilde{Y}(\tau) d\tau + \int_t^T (\tau - a)^{\rho-1} \\ & \quad \times \exp \left(\int_a^\tau (\mathfrak{v} - a)^{\rho-1} \alpha(\mathfrak{v}) d\mathfrak{v} \right) \int_{\mathbb{R}} \tilde{Z}(\tau, s) \delta^{x, y, z, z'}(\tau, s) Q(\tau, ds) \eta(\tau) d\tau \\ & \quad + \int_t^T (\tau - a)^{\rho-1} \exp \left(\int_a^\tau (\mathfrak{v} - a)^{\rho-1} \alpha(\mathfrak{v}) d\mathfrak{v} \right) \tilde{g} \left(\tau, X'(\tau), Y'(\tau), Z'(\tau, s) \right) d\tau \\ & \quad - \int_t^T (\tau - a)^{\rho-1} \exp \left(\int_a^\tau (\mathfrak{v} - a)^{\rho-1} \alpha(\mathfrak{v}) d\mathfrak{v} \right) \tilde{Y}(\tau) dB(\tau) \\ & \quad - \int_t^T (\tau - a)^{\rho-1} \exp \left(\int_a^\tau (\mathfrak{v} - a)^{\rho-1} \alpha(\mathfrak{v}) d\mathfrak{v} \right) \int_{\mathbb{R}} \tilde{Z}(\tau, s) \tilde{N}(d\tau, ds). \end{aligned}$$

Define

$$\begin{aligned} & \left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_{\mathcal{F}_t} = M(t), \quad M(a) = 1 \\ & \frac{dM(t)}{M(t)} = \beta(t) dB(t) + \int_{\mathbb{R}} \delta^{x, y, z, z'}(t, s) \tilde{N}(dt, ds), \quad a \leq t \leq T. \end{aligned}$$

According to Lemma 7, $\tilde{\zeta} = 0$ and $\tilde{g}(t, X', Y', Z') = 0$, we get

$$\begin{aligned} & \tilde{X}(t) \exp \left(\int_a^t (v-a)^{\rho-1} \alpha(v) dv \right) \\ & \leq - \int_t^T (\tau-a)^{\rho-1} \exp \left(\int_a^\tau (v-a)^{\rho-1} \alpha(v) dv \right) \tilde{Y}(\tau) dB^\mathbb{Q}(\tau) \\ & \quad - \int_t^T (\tau-a)^{\rho-1} \exp \left(\int_a^\tau (v-a)^{\rho-1} \alpha(v) dv \right) \int_{\mathbb{R}} \tilde{Z}(\tau, s) \tilde{N}^\mathbb{Q}(d\tau, ds). \end{aligned}$$

Taking the conditional expectation on both sides, we can conclude that $\tilde{X}(t) \leq 0$. Combined with the inequality $\tilde{X}(t) \geq 0$, we get $\tilde{X}(t) = 0$, $t_0 \leq t \leq T$. The proof is complete. $\square$

Notice that the comparison theorem does not hold when the generator g only satisfies conditions (S1), S(2) and (S3). In the next section, we will give a counter example.

Remark 3. When $\rho = 1$, the above comparison theorems are the same as in [21, Theorem 2.5] and [21, Theorem 2.6].

6. EXAMPLES

Example 1. Consider the following linear conformable backward stochastic differential equation

$$\begin{aligned} X(t) = \zeta + \int_t^T (\tau-a)^{\rho-1} b(\tau) X(\tau) d\tau + \int_t^T (\tau-a)^{\rho-1} \\ \times c(\tau) Y(\tau) d\tau - \int_t^T (\tau-a)^{\rho-1} Y(\tau) dB(\tau), \quad a < t \leq T. \end{aligned}$$

Let $\zeta \in \mathbb{L}^2(\mathbb{R})$, and let $b(\cdot)$ and $c(\cdot)$ be $\mathcal{F}$ -predictable, bounded processes. From Theorem 3, there exists a unique solution $(X, Y) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R})$.

Define

$$\begin{aligned} \hat{X}(t) &= X(t) \exp \left(\int_a^t (v-a)^{\rho-1} b(v) dv \right), \quad a < t \leq T \\ \hat{Y}(t) &= Y(t) \exp \left(\int_a^t (v-a)^{\rho-1} b(v) dv \right), \quad a < t \leq T. \end{aligned}$$

Then we apply the Itô's formula to $\hat{X}(t)$, and one has

$$\begin{aligned} d\hat{X}(t) &= (t-a)^{\rho-1} b(t) \exp \left(\int_a^t (v-a)^{\rho-1} b(v) dv \right) dt \\ & \quad + \exp \left(\int_a^t (v-a)^{\rho-1} b(v) dv \right) dX(t) \end{aligned}$$

$$\begin{aligned}
&= (t-a)^{\rho-1}b(t)\exp\left(\int_a^t(v-a)^{\rho-1}b(v)dv\right)dt \\
&\quad - (t-a)^{\rho-1}b(t)\exp\left(\int_a^t(v-a)^{\rho-1}b(v)dv\right)dt \\
&\quad - (t-a)^{\rho-1}c(t)\exp\left(\int_a^t(v-a)^{\rho-1}b(v)dv\right)Y(t)dt \\
&\quad + (t-a)^{\rho-1}\exp\left(\int_a^t(v-a)^{\rho-1}b(v)dv\right)Y(t)dB(t) \\
&= -(t-a)^{\rho-1}c(t)\exp\left(\int_a^t(v-a)^{\rho-1}b(v)dv\right)Y(t)dt \\
&\quad + (t-a)^{\rho-1}\exp\left(\int_a^t(v-a)^{\rho-1}b(v)dv\right)Y(t)dB(t).
\end{aligned}$$

By integrating both sides of the equation from t to T , we get

$$\hat{X}(t) = \hat{X}(T) + \int_t^T (\tau-a)^{\rho-1}c(\tau)\hat{Y}(\tau)d\tau - \int_t^T (\tau-a)^{\rho-1}\hat{Y}(\tau)dB(\tau).$$

Let

$$\begin{aligned}
&\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = M(t), \quad M(a) = 1, \\
&\frac{dM(t)}{M(t)} = c(t)dB(t).
\end{aligned}$$

According to Lemma 7, we have

$$\hat{X}(t) = \hat{X}(T) - \int_t^T (\tau-a)^{\rho-1}\hat{Y}(\tau)dB^{\mathbb{Q}}(\tau).$$

Since the process $\hat{Y}$ is a.s. square integrable, we can conclude that $\hat{X}$ is a $\mathbb{Q}$ -local martingale, and by Lemma 2 and some important inequality we have $\hat{X}(t) = \mathbb{E}^{\mathbb{Q}}[\hat{X}(T)|\mathcal{F}_t]$. In addition, we can get

$$\begin{aligned}
\zeta \exp\left(\int_a^T(v-a)^{\rho-1}b(v)dv\right) &= \mathbb{E}^{\mathbb{Q}}\left[\zeta \exp\left(\int_a^T(v-a)^{\rho-1}b(v)dv\right)\Big|_{\mathcal{F}_t}\right] \\
&\quad - \int_t^T (\tau-a)^{\rho-1}\hat{Y}(\tau)dB^{\mathbb{Q}}(\tau).
\end{aligned}$$

The above results satisfy Proposition 1.

Example 2. Let $a \leq t \leq T$ and $0 < \rho \leq 1$, and we consider the following conformable backward stochastic differential equation

$$\begin{cases} D_{\rho}^a X(t) = -rX(t) + \beta(t-a)^{1-\rho}X(t)\frac{dB(t)}{dt} + \frac{\int_{\mathbb{R}} X(t)\tilde{N}(dt,ds)}{dt}, \\ X(T) = \zeta, \end{cases} \quad (6.1)$$

where $r, \beta \in \mathbb{R}$. From Remark 1, equation (6.1) has the equivalent form

$$\begin{cases} dX(t) = -r(t-a)^{\rho-1}X(t)dt + \beta X(t)dB(t) \\ \quad + (t-a)^{\rho-1} \int_{\mathbb{R}} X(t)\tilde{N}(dt, ds), \\ X(T) = \zeta. \end{cases}$$

Suppose $U(t, X(t)) = \ln X(t)$. From Theorem 1, we have

$$\begin{aligned} d\ln X(t) &= \frac{1}{X(t)}dX(t) - \frac{1}{2X^2(t)}(dX(t))^2 \\ &= -r(t-a)^{\rho-1}dt + \beta dB(t) + \frac{1}{X(t)}(t-a)^{\rho-1} \int_{\mathbb{R}} X(t)\tilde{N}(dt, ds) \\ &\quad - \frac{1}{2}\beta^2 dt - \frac{1}{2X^2(t)}(t-a)^{2(\rho-1)} \int_{\mathbb{R}} X^2(t)N(dt, ds) \\ &= -\left[r(t-a)^{\rho-1} + \frac{1}{2}\beta^2\right]dt + \beta dB(t) \\ &\quad + \frac{1}{X(t)}(t-a)^{\rho-1} \int_{\mathbb{R}} X(t)\tilde{N}(dt, ds) \\ &\quad - \frac{1}{2X^2(t)}(t-a)^{2(\rho-1)} \int_{\mathbb{R}} X^2(t)N(dt, ds). \end{aligned}$$

Then

$$\begin{aligned} \ln X(t) &= \ln \zeta + \int_t^T \left[r(t-a)^{\rho-1} + \frac{1}{2}\beta^2 \right] dt \\ &\quad - \int_t^T \beta dB(t) + \int_t^T \frac{1}{X(t)}(t-a)^{\rho-1} \int_{\mathbb{R}} X(t)\tilde{N}(dt, ds) \\ &\quad - \int_t^T \frac{1}{2X^2(t)}(t-a)^{2(\rho-1)} \int_{\mathbb{R}} X^2(t)N(dt, ds), \end{aligned}$$

and the solution of (6.1) is expressed as

$$\begin{aligned} X(t) &= \zeta \exp \left(\frac{r}{\rho} [(T-a)^\rho - (t-a)^\rho] + \frac{1}{2}\beta^2(T-t) - \beta(B(T) - B(t)) \right. \\ &\quad - \int_t^T \beta dB(t) + \int_t^T \frac{1}{X(t)}(t-a)^{\rho-1} \int_{\mathbb{R}} X(t)\tilde{N}(dt, ds) \\ &\quad \left. - \int_t^T \frac{1}{2X^2(t)}(t-a)^{2(\rho-1)} \int_{\mathbb{R}} X^2(t)N(dt, ds) \right). \end{aligned}$$

Taking the conditional expected value on both sides of the above equation, one has

$$X(t) = \mathbb{E}[\zeta \exp \{ \frac{r}{\rho} [(T-a)^\rho - (t-a)^\rho] + \frac{1}{2}\beta^2(T-t) \} | \mathcal{F}_t],$$

which satisfies Proposition 2.

Example 3. Consider the conformable backward stochastic differential equations

$$\begin{aligned} X_1(t) = & \zeta_1 + \int_t^T (\tau - a)^{\rho-1} [g_1(\tau, X_1(\tau), Y_1(\tau), Z_1(\tau, \cdot)) + v_1(\tau)] d\tau \\ & - \int_t^T (\tau - a)^{\rho-1} Y_1(\tau) dB(\tau) - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z_1(\tau, s) \tilde{N}(d\tau, ds), \end{aligned} \quad (6.2)$$

and

$$\begin{aligned} X_2(t) = & \zeta_2 + \int_t^T (\tau - a)^{\rho-1} [g_2(\tau, X_2(\tau), Y_2(\tau), Z_2(\tau, \cdot)) + v_2(\tau)] d\tau \\ & - \int_t^T (\tau - a)^{\rho-1} Y_2(\tau) dB(\tau) - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z_2(\tau, s) \tilde{N}(d\tau, ds), \end{aligned} \quad (6.3)$$

where $0 < \rho \leq 1$, $a \leq t \leq T$ and $v_1(\cdot), v_2(\cdot) \in \mathbb{S}^2(\mathbb{R})$. Assume that the generator g_1 and g_2 satisfy condition H_{cs} . At this time, for any $a \leq t \leq T$, let $\zeta_1 = 0$, $\zeta_2 = \frac{(t-a)^\rho}{\rho}$ and $g_1(t, X_1(t), Y_1(t), Z_1(t)) + v_1(t) = 0$, $g_2(t, X_2(t), Y_2(t), Z_2(t)) + v_2(t) = 1$, then $(X_1, Y_1, Z_1) = (-K(B(T) - B(t)), K(t-a)^{1-\rho}, 0)$ is the unique solution of equation (6.2) and $(X_2, Y_2, Z_2) = (\frac{(T-a)^\rho}{\rho} - K(B(T) - B(t)), K(t-a)^{1-\rho}, 0)$ is the unique solution of equation (6.3). Obviously, $\zeta_2 \geq \zeta_1$ and $g_2 + v_2 \geq g_1 + v_1$, and we know that $X_2(t) \geq X_1(t)$, which satisfies Theorem 4.

In the financial market, such as the Merton model, $v(\cdot)$ represents the investor's consumption rate, $X(t)$ represents his property at time t , $Y(t)$ represents the portfolio strategy at time t under the condition of small fluctuation, and $Z(t, s)$ represents the portfolio strategy at time t affected by large fluctuation. At this time, the comparison principle has the following explanation, that is, if an investor wants to achieve a higher fiscal revenue at a future moment, then the investor must either invest more money now, or reduce consumption before time T .

Example 4. Consider the following price model of a stock

$$\begin{aligned} X_i(t) = & \zeta_i + m \int_t^T (\tau - a)^{\rho-1} g_i(\tau) d\tau - n \int_t^T (\tau - a)^{\rho-1} Y_i(\tau) dB(\tau) \\ & - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z_i(\tau) \tilde{N}(d\tau, ds), \quad a \leq t \leq T, \end{aligned} \quad (6.4)$$

where $i = 1, 2$ and $m, n \in \mathbb{R}^+$. Let generator g_i satisfy the condition H_{cs} . Choose $Y_1(t) = T(t-a)^{1-\rho}$, $Y_2(t) = 0$ and $Z_1(t) = Z_2(t) = 0$. For any $\zeta_1 = \zeta_2 = \zeta$, if $X_1(t) = t$ and $X_2(t) = t - 1$, we can get

$$t = \zeta + m \int_t^T (\tau - a)^{\rho-1} g_1(\tau) d\tau - nT(B(T) - B(t)), \quad a \leq t \leq T,$$

and

$$t - 1 = \zeta + m \int_t^T (\tau - a)^{\rho-1} g_1(\tau) d\tau, \quad a \leq t \leq T.$$

Then $g_1(t) = \frac{-(t-a)^{1-\rho}}{m} [1 - nTB'(t)]$, $g_2(t) = \frac{-(t-a)^{1-\rho}}{m}$ and $\zeta = T - 1$. Clearly, $g_1(t) \geq g_2(t)$ which satisfies Theorem 5.

Example 5. Consider the following conformable backward stochastic differential equation

$$\begin{aligned} X(t) = & \zeta + \int_t^T (\tau - a)^{\rho-1} g(\tau, X(\tau), Y(\tau), Z(\tau)) d\tau \\ & - \int_t^T (\tau - a)^{\rho-1} Y(\tau) dB(\tau) - \int_t^T (\tau - a)^{\rho-1} \int_{\mathbb{R}} Z(\tau, s) \tilde{N}(d\tau, ds), \end{aligned} \quad (6.5)$$

where the intensity of the Poisson process is 1. Let $g(t, X, Y, Z) = -2Y(t)$ and g only satisfies the conditions (S1), (S2) and (S3). Then

$$\int_a^t \int_{\mathbb{R}} N(d\tau, ds), \quad a \leq t \leq T,$$

is a standard Poisson process. If we choose $\zeta = 0$ and $\zeta' = \int_a^T \int_{\mathbb{R}} N(d\tau, ds)$, then $(X(t), Y(t), Z(t)) = (0, 0, 0)$ and $(X'(t), Y'(t), Z'(t)) = (\int_a^t \int_{\mathbb{R}} N(d\tau, ds) - (T - t), 0, 1)$ are the unique solutions of the equation (6.5) with (ζ, g) and (ζ', g) . The existence and uniqueness of the solutions $(X(t), Y(t), Z(t))$ and $(X'(t), Y'(t), Z'(t))$ follow from Theorem 3. It is clear that $\zeta \leq \zeta'$, but $X(t) \geq X'(t)$.

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EXISTENCE AND NONEXISTENCE OF POSITIVE SOLUTIONS FOR THE SECOND-ORDER m -POINT EIGENVALUE IMPULSIVE BOUNDARY VALUE PROBLEM

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Abstract. This paper is devoted to the study of the existence and nonexistence of positive solutions for a second-order m -point eigenvalue impulsive boundary value problem. We use the fixed point theorems on the cones in order to achieve our results.

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1. INTRODUCTION

Impulsive differential equations have recently become important in applied mathematical models for real processes that arise from phenomena studied in physics, ecology, biological systems, biotechnology and industrial robotic. Moreover, impulsive differential equations are richer in applications compared with the corresponding theory of ordinary differential equations. Many of the mathematical problems encountered in the study of impulsive differential equations cannot be treated with the usual techniques within the standard framework of ordinary differential equations. For the introduction of the basic theory of impulsive equations, see [3, 5, 6, 15, 21], and the papers [2, 4, 8, 11–14, 24–27].

In the literature, there are many studies on second-order impulsive boundary value problems, for these, we refer to reader to [1, 7, 10, 16–19, 22, 29, 31, 32]. In recent years, several authors have been working on the existence of positive solutions for second order m -point impulsive boundary value problems [9, 20, 28, 30].

Liu and Zhao [22] studied the following impulsive boundary value problem:

$$\begin{cases} -u''(t) + u(t) + p(t)u'(t) = \lambda f(t, u) + \mu g(t, u), & t \in [0, T], \\ \Delta u'(t_i) = I_i(u(t_i)), & i = 1, 2, \dots, n, \\ u(0) = u(T) = 0 \end{cases}$$

where $\lambda > 0$, $\mu \geq 0$, $f, g \in C([0, T] \times \mathbb{R}, \mathbb{R})$, $p \in L^\infty([0, T])$ satisfies $0 = t_0 < \frac{T}{4} \leq t_1 < t_2 < \dots < t_n \leq \frac{3T}{4} < t_{n+1} = T$, $I_i: \mathbb{R} \rightarrow \mathbb{R}$, $i = 1, 2, \dots, n$ are continuous. They obtained some new existence of solutions by using variational methods combining with a three critical points theorem.

Motivated by the aforementioned results, in this study, we consider the following second-order m -point eigenvalue impulsive boundary value problem:

$$\begin{cases} \vartheta''(t) + q_1(t)\vartheta'(t) + q_2(t)\vartheta(t) + \kappa h(t, \vartheta(t)) = 0, & \kappa > 0 \\ \Delta\vartheta|_{t=t_j} = \kappa I_j(\vartheta(t_j)), \\ \Delta\vartheta'|_{t=t_j} = -\kappa J_j(\vartheta(t_j)), \\ a\vartheta(0) - b\vartheta'(0) = \sum_{i=1}^{m-2} \alpha_i \vartheta(\xi_i), \\ c\vartheta(1) + d\vartheta'(1) = \sum_{i=1}^{m-2} \beta_i \vartheta(\xi_i) \end{cases} \quad (1.1)$$

where $J = [0, 1]$, $t \neq t_j$, $j = 1, 2, \dots, n$ with $0 < t_1 < t_2 < \dots < t_n < 1$. $\Delta\vartheta|_{t=t_j}$ and $\Delta\vartheta'|_{t=t_j}$ denote the jump of $\vartheta(t)$ and $\vartheta'(t)$ at $t = t_j$, i.e.,

$$\Delta\vartheta|_{t=t_j} = \vartheta(t_j^+) - \vartheta(t_j^-), \quad \Delta\vartheta'|_{t=t_j} = \vartheta'(t_j^+) - \vartheta'(t_j^-),$$

where $\vartheta(t_j^+)$, $\vartheta'(t_j^+)$ and $\vartheta(t_j^-)$, $\vartheta'(t_j^-)$ represent the right-hand limit and left-hand limit of $\vartheta(t)$ and $\vartheta'(t)$ at $t = t_j$, $j = 1, 2, \dots, n$, respectively.

Throughout this paper we assume that following conditions hold:

- (C1) $a, b, c, d \in (0, \infty)$, $\alpha_i, \beta_i \in [0, \infty)$, $\xi_i \in (0, 1)$ for $i \in \{1, \dots, m-2\}$, κ is a positive parameter,
- (C2) $h \in C(J \times \mathbb{R}^+, \mathbb{R}^+)$, $q_1 \in C(J, \mathbb{R}^+)$, $q_2 \in C(J, (-\infty, 0))$ such that for $t \in J$: $q_1(t) + q_2(t) \geq 0$,
- (C3) $I_j \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $J_j \in C(\mathbb{R}^+, \mathbb{R}^+)$ are bounded functions for $j = 1, 2, \dots, n$.

By using Guo-Krasnosel'skii's fixed point theorem [15], some criteria of existence and nonexistence of positive solutions for the second-order m -point eigenvalue impulsive boundary value problem (1.1) are established in terms of different values of κ . The problem (1.1) is more general, includes two-point, multi-point and impulsive problems as special cases. Thus, we generalize and improve some of the known results in the literature to some degree, and therefore our results may contribute to the literature in this field.

The main structure of this paper is as follows. In section 2, we provide some definitions and preliminary lemmas which are key tools for our main results. In section 3, we give and prove our main results.

2. PRELIMINARIES

In this section, we first introduce some background definitions in Banach spaces, and then present auxiliary lemmas which will be used later.

Let $J' = J \setminus \{t_1, t_2, \dots, t_n\}$. $C(J, \mathbb{R}^+)$ denote the Banach space of all continuous mapping $\vartheta: J \rightarrow \mathbb{R}^+$ with the norm $\|\vartheta\| = \sup_{t \in J} |\vartheta(t)|$,

$$PC(J, \mathbb{R}^+) = \left\{ \vartheta: J \rightarrow \mathbb{R}^+ \mid \vartheta \in C(J'), \vartheta(t_j^+) \text{ and } \vartheta(t_j^-) \text{ exist} \right. \\ \left. \text{and } \vartheta(t_j^-) = \vartheta(t_j), j = 1, 2, \dots, n \right\}$$

is also a Banach space with norm $\|\vartheta\|_{PC} = \sup_{t \in J} |\vartheta(t)|$;

$$PC^1(J, \mathbb{R}^+) = \left\{ \vartheta \in PC(J, \mathbb{R}^+) \mid \vartheta' \in PC(J'), \vartheta'(t_j^+) \text{ and } \vartheta'(t_j^-) \text{ exist} \right. \\ \left. \text{and } \vartheta'(t_j^-) = \vartheta'(t_j), j = 1, 2, \dots, n \right\}$$

is a real Banach space with norm $\|\vartheta\|_{PC^1} = \max\{\|\vartheta\|_{PC}, \|\vartheta'\|_{PC}\}$ where

$$\|\vartheta\|_{PC} = \sup_{t \in J} |\vartheta(t)|, \quad \|\vartheta'\|_{PC} = \sup_{t \in J} |\vartheta'(t)|.$$

A function $\vartheta \in PC^1(J, \mathbb{R}^+) \cap C^2(J', \mathbb{R})$ is called a solution of problem (1.1) if it satisfies (1.1).

Lemma 1 (Lemma 2.1 in [23]). *Let (C1–C2) hold. Let φ and ψ be the unique solution of boundary value problem:*

$$\begin{cases} \varphi''(t) + q_1(t)\varphi'(t) + q_2(t)\varphi(t) = 0, & t \in J, \\ \varphi(0) = b, \quad \varphi'(0) = a \end{cases}$$

and

$$\begin{cases} \psi''(t) + q_1(t)\psi'(t) + q_2(t)\psi(t) = 0, & t \in J, \\ \psi(1) = d, \quad \psi'(1) = -c \end{cases}$$

respectively. Then

- (i) φ is strictly increasing on $[0, 1]$, and $\varphi > 0$ on $(0, 1]$;
- (ii) ψ is strictly decreasing on $[0, 1]$, and $\psi > 0$ on $[0, 1)$.

Set $\rho := q(t) \begin{vmatrix} \psi(t) & \varphi(t) \\ \psi'(t) & \varphi'(t) \end{vmatrix} > 0$ with $q(t) = \exp\left(\int_0^t q_1(s)ds\right)$, $\forall t \in J$ and

$$\Delta = \begin{vmatrix} -\sum_{i=1}^{m-2} \alpha_i \varphi(\xi_i) & \rho - \sum_{i=1}^{m-2} \alpha_i \psi(\xi_i) \\ \frac{\rho}{q(1)} - \sum_{i=1}^{m-2} \beta_i \varphi(\xi_i) & -\sum_{i=1}^{m-2} \beta_i \psi(\xi_i) \end{vmatrix}.$$

Lemma 2. *Let (C1–C3) hold. Assume*

(C4) $\Delta \neq 0$.

If $\vartheta \in PC^1(J, \mathbb{R}^+) \cap C^2(J')$ is a solution of the equation

$$\begin{aligned} \vartheta(t) = & \kappa \int_0^1 G(t, s) q(s) h(s, \vartheta(s)) ds + \kappa \sum_{j=1}^n G_s(t, s)|_{s=t_j} q(t_j) I_j(\vartheta(t_j)) \\ & + \kappa \sum_{j=1}^n G(t, t_j) q(t_j) J_j(\vartheta(t_j)) + \kappa A(h) \varphi(t) + \kappa B(h) \psi(t) \end{aligned} \quad (2.1)$$

where

$$G(t, s) = \frac{1}{\rho} \begin{cases} \varphi(s) \psi(t), & s \leq t, \\ \varphi(t) \psi(s), & t \leq s, \end{cases} \quad (2.2)$$

$$\begin{aligned} K_i = & \int_0^1 G(\xi_i, s) q(s) h(s, \vartheta(s)) ds + \sum_{j=1}^n G_s(t, s)|_{s=\xi_i} q(t_j) I_j(\vartheta(t_j)) \\ & + \sum_{j=1}^n G(\xi_i, t_j) q(t_j) J_j(\vartheta(t_j)), \end{aligned} \quad (2.3)$$

$$A(h) = \frac{1}{\Delta} \begin{vmatrix} \sum_{i=1}^{m-2} \alpha_i K_i & \rho - \sum_{i=1}^{m-2} \alpha_i \psi(\xi_i) \\ \sum_{i=1}^{m-2} \beta_i K_i & - \sum_{i=1}^{m-2} \beta_i \psi(\xi_i) \end{vmatrix} \quad (2.4)$$

and

$$B(h) = \frac{1}{\Delta} \begin{vmatrix} - \sum_{i=1}^{m-2} \alpha_i \varphi(\xi_i) & \sum_{i=1}^{m-2} \alpha_i K_i \\ \frac{\rho}{q(1)} - \sum_{i=1}^{m-2} \beta_i \varphi(\xi_i) & \sum_{i=1}^{m-2} \beta_i K_i \end{vmatrix}, \quad (2.5)$$

then ϑ is a solution of the impulsive boundary value problem (1.1).

Proof. Let ϑ satisfies the integral equation (2.1), then we have

$$\begin{aligned} \vartheta(t) = & \frac{\kappa}{\rho} \int_0^t \varphi(s) \psi(t) q(s) h(s, \vartheta(s)) ds + \frac{\kappa}{\rho} \int_t^1 \varphi(t) \psi(s) q(s) h(s, \vartheta(s)) ds \\ & + \frac{\kappa}{\rho} \sum_{t_j < t} \varphi'(t_j) \psi(t) q(t_j) I_j(\vartheta(t_j)) + \frac{\kappa}{\rho} \sum_{t < t_j} \varphi(t) \psi'(t_j) q(t_j) I_j(\vartheta(t_j)) \\ & + \frac{\kappa}{\rho} \sum_{t_j < t} \varphi(t_j) \psi(t) q(t_j) J_j(\vartheta(t_j)) + \frac{\kappa}{\rho} \sum_{t < t_j} \varphi(t) \psi(t_j) q(t_j) J_j(\vartheta(t_j)) \\ & + \kappa A(h) \varphi(t) + \kappa B(h) \psi(t). \end{aligned}$$

Also, $\vartheta'(t)$ and $\vartheta''(t)$ can be easily written. Thus by Liouville's formula, we have

$$\begin{aligned}\vartheta''(t) + q_1(t)\vartheta'(t) + q_2(t)\vartheta(t) &= \frac{\kappa}{\rho} [\varphi(t)\psi'(t) - \varphi'(t)\psi(t)] q(t)h(t, \vartheta(t)) \\ &= \frac{\kappa}{\rho} [-W(\varphi(t), \psi(t))] q(t)h(t, \vartheta(t)) \\ &= \frac{\kappa}{\rho} \left[-\rho e^{-\int_0^t q_1(s)ds} \right] q(t)h(t, \vartheta(t)) \\ &= -\kappa h(t, \vartheta(t)),\end{aligned}$$

$$\begin{aligned}\Delta \vartheta|_{t=t_j} &= \vartheta(t_j^+) - \vartheta(t_j^-) = \frac{\kappa}{\rho} [\varphi'(t_j)\psi(t_j) - \varphi(t_j)\psi'(t_j)] q(t_j)I_j(\vartheta(t_j)) \\ &= \frac{\kappa}{\rho} q(t_j) \begin{vmatrix} \psi(t_j) & \varphi(t_j) \\ \psi'(t_j) & \varphi'(t_j) \end{vmatrix} I_j(\vartheta(t_j)) \\ &= \frac{\kappa}{\rho} \begin{vmatrix} \psi(0) & \varphi(0) \\ \psi'(0) & \varphi'(0) \end{vmatrix} e^{-\int_0^{t_j} q_1(s)ds} q(t_j)I_j(\vartheta(t_j)) \\ &= \kappa I_j(\vartheta(t_j)),\end{aligned}$$

and

$$\begin{aligned}\Delta \vartheta'|_{t=t_j} &= \vartheta'(t_j^+) - \vartheta'(t_j^-) = \frac{\kappa}{\rho} [\varphi(t_j)\psi'(t_j) - \varphi'(t_j)\psi(t_j)] q(t_j)J_j(\vartheta(t_j)) \\ &= -\frac{\kappa}{\rho} \begin{vmatrix} \psi(t_j) & \varphi(t_j) \\ \psi'(t_j) & \varphi'(t_j) \end{vmatrix} q(t_j)J_j(\vartheta(t_j)) \\ &= -\frac{\kappa}{\rho} \begin{vmatrix} \psi(0) & \varphi(0) \\ \psi'(0) & \varphi'(0) \end{vmatrix} e^{-\int_0^{t_j} q_1(s)ds} q(t_j)J_j(\vartheta(t_j)) \\ &= -\kappa J_j(\vartheta(t_j)).\end{aligned}$$

Since

$$\begin{aligned}\vartheta(0) &= \frac{\kappa}{\rho} \int_0^1 \varphi(0)\psi(s)q(s)h(s, \vartheta(s))ds + \frac{\kappa}{\rho} \sum_{j=1}^n \varphi(0)\psi'(t_j)q(t_j)I_j(\vartheta(t_j)) \\ &\quad + \frac{\kappa}{\rho} \sum_{j=1}^n \varphi(0)\psi(t_j)q(t_j)J_j(\vartheta(t_j)) + \kappa A(h)\varphi(0) + \kappa B(h)\psi(0), \\ \vartheta'(0) &= \frac{\kappa}{\rho} \int_0^1 \varphi'(t)\psi(s)q(s)h(s, \vartheta(s))ds + \frac{\kappa}{\rho} \sum_{j=1}^n \varphi'(0)\psi'(t_j)q(t_j)I_j(\vartheta(t_j))\end{aligned}$$

$$+ \frac{\kappa}{\rho} \sum_{j=1}^n \varphi'(0) \psi(t_j) q(t_j) J_j(\vartheta(t_j)) + \kappa A(h) \varphi'(0) + \kappa B(h) \psi'(0),$$

we have that

$$a\vartheta(0) - b\vartheta'(0) = \kappa \rho B(h) = \sum_{i=1}^{m-2} \alpha_i (\kappa A(h) \varphi(\xi_i) + \kappa B(h) \psi(\xi_i) + \kappa K_i). \quad (2.6)$$

Since

$$\begin{aligned} \vartheta(1) &= \frac{\kappa}{\rho} \int_0^1 \varphi(s) \psi(1) q(s) h(s, \vartheta(s)) ds + \frac{\kappa}{\rho} \sum_{j=1}^n \varphi'(t_j) \psi(1) q(t_j) I_j(\vartheta(t_j)) \\ &\quad + \frac{\kappa}{\rho} \sum_{j=1}^n \varphi(t_j) \psi(1) q(t_j) J_j(\vartheta(t_j)) + \kappa A(h) \varphi(1) + \kappa B(h) \psi(1), \\ \vartheta'(1) &= \frac{\kappa}{\rho} \int_0^1 \varphi(s) \psi'(1) q(s) h(s, \vartheta(s)) ds + \frac{\kappa}{\rho} \sum_{j=1}^n \varphi'(t_j) \psi'(1) q(t_j) I_j(\vartheta(t_j)) \\ &\quad + \frac{\kappa}{\rho} \sum_{j=1}^n \varphi(t_j) \psi'(1) q(t_j) J_j(\vartheta(t_j)) + \kappa A(h) \varphi'(1) + \kappa B(h) \psi'(1), \end{aligned}$$

we have that

$$c\vartheta(1) + d\vartheta'(1) = \kappa \frac{\rho}{q(1)} A(h) = \sum_{i=1}^{m-2} \beta_i (\kappa A(h) \varphi(\xi_i) + \kappa B(h) \psi(\xi_i) + \kappa K_i). \quad (2.7)$$

From (2.6) and (2.7), we get that

$$\begin{cases} A(h) \left[-\sum_{i=1}^{m-2} \alpha_i \varphi(\xi_i) \right] + B(h) \left[\rho - \sum_{i=1}^{m-2} \alpha_i \psi(\xi_i) \right] = \sum_{i=1}^{m-2} \alpha_i K_i \\ A(h) \left[\frac{\rho}{q(1)} - \sum_{i=1}^{m-2} \beta_i \varphi(\xi_i) \right] + B(h) \left[-\sum_{i=1}^{m-2} \beta_i \psi(\xi_i) \right] = \sum_{i=1}^{m-2} \beta_i K_i \end{cases}$$

which implies that $A(h)$ and $B(h)$ satisfy (2.4) and (2.5), respectively. $\square$

Lemma 3. Let (C1)-(C4) hold. Assume

$$(C5) \quad \Delta < 0, \rho - \sum_{i=1}^{m-2} \alpha_i \psi(\xi_i) > 0, \frac{\rho}{q(1)} - \sum_{i=1}^{m-2} \beta_i \varphi(\xi_i) > 0 \text{ and}$$

$$\psi(t_j) J_j(\vartheta(t_j)) + \psi'(t_j) I_j(\vartheta(t_j)) > 0 \quad \text{for } j = 1, 2, \dots, n.$$

Then for $\vartheta \in PC^1(J, \mathbb{R}^+) \cap C^2(J')$, the solution ϑ of the problem (1.1) satisfies $\vartheta(t) \geq 0$ for $t \in J$.

Proof. It is an immediate subsequence of the facts that $G(t, s) \geq 0$ on $J \times J$ and $A(h) \geq 0, B(h), I_j, J_j \geq 0$. $\square$

Lemma 4. Suppose that (C1)-(C5) hold, then for any $t, s \in J$, we have

$$G(t, s) \leq \delta G(s, s) \quad (2.8)$$

and

$$G_s(t, s) \leq \delta G_s(t, s)|_{t=s} \quad (2.9)$$

where

$$\delta = \frac{\max\{\varphi(1), \psi(0)\}}{\min\{\varphi(0), \psi(1)\}} \quad (2.10)$$

and

$$G_s(t, s) = \frac{1}{\rho} \begin{cases} \varphi'(s)\psi(t), & s \leq t, \\ \varphi(t)\psi'(s), & t < s, \end{cases} \quad (2.11)$$

for $t, s \in J$.

Proof. By the monotonicity of φ, ψ and the definition of $G(t, s)$, we have

$$\frac{G(t, s)}{G(s, s)} = \frac{\varphi(t)}{\varphi(s)} \leq \frac{\varphi(1)}{\varphi(0)} \leq \frac{\max\{\varphi(1), \psi(0)\}}{\min\{\varphi(0), \psi(1)\}} = \delta \quad \text{for } t \leq s$$

and

$$\frac{G(t, s)}{G(s, s)} = \frac{\psi(t)}{\psi(s)} \leq \frac{\psi(0)}{\psi(1)} \leq \frac{\max\{\varphi(1), \psi(0)\}}{\min\{\varphi(0), \psi(1)\}} = \delta \quad \text{for } s \leq t.$$

Similarly, by the monotonicity of φ, ψ and the definition of $G_s(t, s)$, we have $G_s(t, s) \leq \delta G_s(t, s)|_{t=s}$ for $t, s \in J$. $\square$

Lemma 5. Suppose that (C1)-(C5) hold. Let $\sigma \in (0, \frac{1}{2})$ and $t \in [\sigma, 1 - \sigma]$. Then we have

$$G(t, s) \geq \omega G(s, s) \quad (2.12)$$

where

$$\omega = \frac{\min\{\varphi(\sigma), \psi(1 - \sigma)\}}{\max\{\varphi(1), \psi(0)\}}. \quad (2.13)$$

Proof. By the monotonicity of φ, ψ and the definition of $G(t, s)$, for $t \in [\sigma, 1 - \sigma]$, we have

$$\frac{G(t, s)}{G(s, s)} = \frac{\varphi(t)}{\varphi(s)} \geq \frac{\varphi(\sigma)}{\varphi(1)} \geq \frac{\min\{\varphi(\sigma), \psi(1 - \sigma)\}}{\max\{\varphi(1), \psi(0)\}} = \omega \quad \text{for } t \leq s$$

and

$$\frac{G(t, s)}{G(s, s)} = \frac{\psi(t)}{\psi(s)} \geq \frac{\psi(1 - \sigma)}{\psi(0)} \geq \frac{\min\{\varphi(\sigma), \psi(1 - \sigma)\}}{\max\{\varphi(1), \psi(0)\}} = \omega \quad \text{for } s \leq t.$$

$\square$

Lemma 6. *Let (C1)-(C5) hold. Assume*

$$(C6) \quad \sum_{i=1}^{m-2} (a\beta_i - c\alpha_i) \geq 0.$$

Then the solution $\vartheta \in PC^1(J, \mathbb{R}^+) \cap C^2(J', \mathbb{R})$ of the problem (1.1) satisfies $\vartheta'(t) \geq 0$ for $t \in J$.

Proof. Assume that the inequality $\vartheta'(t) < 0$ holds. Since $\vartheta(t)$ is nonincreasing on J , one can verify that $\vartheta(1) < \vartheta(0)$.

From the boundary conditions of the problem (1.1), we have

$$\frac{1}{c} \sum_{i=1}^{m-2} \beta_i \vartheta(\xi_i) - \frac{d}{c} \vartheta'(1) < \sum_{i=1}^{m-2} \alpha_i \vartheta(\xi_i) + \frac{b}{a} \vartheta'(0).$$

This yields that

$$\frac{1}{c} \sum_{i=1}^{m-2} \beta_i \vartheta(\xi_i) - \frac{1}{a} \sum_{i=1}^{m-2} \alpha_i \vartheta(\xi_i) < \frac{d}{c} \vartheta'(1) + \frac{b}{a} \vartheta'(0) < 0.$$

The last inequality yields

$$a \sum_{i=1}^{m-2} \beta_i \vartheta(\xi_i) - c \sum_{i=1}^{m-2} \alpha_i \vartheta(\xi_i) < 0.$$

Therefore, we obtain that

$$\sum_{i=1}^{m-2} (a\beta_i - c\alpha_i) \vartheta(\xi_i) < 0.$$

According to Lemma 3, we have $\vartheta(\xi_i) \geq 0$. So, $\sum_{i=1}^{m-2} (a\beta_i - c\alpha_i) < 0$. However, this contradicts to condition (C6). Consequently, $\vartheta'(t) \geq 0$ for $t \in J$. $\square$

Let $K = \{\vartheta \in PC(J, \mathbb{R}^+) \mid \vartheta(t) \text{ is nonnegative, nondecreasing and concave on } J\}$. So, K is a cone of $PC(J, \mathbb{R}^+)$.

Lemma 7. *Let (C1)-(C6) hold and $\vartheta(t) \in K$, $\sigma \in (0, \frac{1}{2})$. Then*

$$\min_{t \in [\sigma, 1-\sigma]} \vartheta(t) \geq \sigma \|\vartheta\|_{PC}$$

where $\|\vartheta\|_{PC} = \sup_{t \in J} |\vartheta(t)|$.

Proof. Since $\vartheta \in K$, we know that $\vartheta(t)$ is concave on J . Thus, $\min_{t \in [\sigma, 1-\sigma]} \vartheta(t) = \vartheta(\sigma)$ and $\|\vartheta\|_{PC} = \sup_{t \in J} |\vartheta(t)| = \vartheta(1)$. Since the graph of ϑ is concave down on J , we obtain

$$\frac{\vartheta(1) - \vartheta(0)}{1 - 0} \leq \frac{\vartheta(\sigma) - \vartheta(0)}{\sigma - 0},$$

i.e. $\vartheta(\sigma) \geq \sigma\vartheta(1) + (1 - \sigma)\vartheta(0)$. So $\vartheta(\sigma) \geq \sigma\vartheta(1)$. The proof is complete. $\square$

For any $r > 0$, let $K_r = \{\vartheta \in K \mid \|\vartheta\|_{PC} < r\}$, $\partial K_r = \{\vartheta \in K \mid \|\vartheta\|_{PC} = r\}$, and $\bar{K}_r = \{\vartheta \in K \mid \|\vartheta\|_{PC} \leq r\}$. Defining operator $T_\kappa: K \rightarrow PC(J, \mathbb{R}^+)$ as follows:

$$\begin{aligned} T_\kappa\vartheta(t) = & \kappa \int_0^1 G(t, s)q(s)h(s, \vartheta(s))ds + \kappa \sum_{j=1}^n G_s(t, s)|_{s=t_j} q(t_j)I_j(\vartheta(t_j)) \\ & + \kappa \sum_{j=1}^n G(t, t_j)q(t_j)J_j(\vartheta(t_j)) + \kappa A(h)\varphi(t) + \kappa B(h)\psi(t). \end{aligned} \quad (2.14)$$

It is well known that $\vartheta \in K$ is a positive fixed point of operator T_κ if and only if ϑ is a positive solution of problem (1.1).

Lemma 8. *Let (C1)-(C6) hold. If conditions*

$$(C7) \quad 0 < \int_0^1 \delta G(s, s)q(s)ds < \infty \text{ and}$$

$$(C8) \quad 0 < \int_0^1 q(s)ds < \infty$$

hold, then $T_\kappa: K \rightarrow K$ is completely continuous.

Proof. For any $\vartheta \in K$, from Lemmas 2, 3, 6 and the definition of T_κ , we have $(T_\kappa\vartheta)(t) \geq 0$, $(T_\kappa\vartheta)'(t) \geq 0$, and $(T_\kappa\vartheta)'(t)$ is concave on J . Therefore $T_\kappa(K) \subset K$. Next, with a similar method in Lemma 2.4 in [19], it can be shown that the $T_\kappa: K \rightarrow K$ is completely continuous. So this part is skipped and thus the Lemma is proved. $\square$

3. MAIN RESULTS

In this section, we will apply the following Guo-Krasnosel'skii's fixed point theorem [15] to state main results.

Lemma 9 (Theorem 2.3.1 in [15]). *Let X be a Banach space, and P be cone in X . Assume that Ω_1 and Ω_2 are two bounded open subsets of X with $0 \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2$. Let $A: P \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow P$ be a completely continuous operator, satisfying either*

- (i) $\|Ax\| \leq \|x\|$, $x \in P \cap \partial\Omega_1$, $\|Ax\| \geq \|x\|$, $x \in P \cap \partial\Omega_2$, or
- (ii) $\|Ax\| \geq \|x\|$, $x \in P \cap \partial\Omega_1$, $\|Ax\| \leq \|x\|$, $x \in P \cap \partial\Omega_2$.

Then A has at least one fixed point in $P \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

For suitability and simplicity in the following discussion, we indicate:

$$\begin{aligned} h_0 &= \liminf_{y \rightarrow 0} \min_{t \in [\sigma, 1-\sigma]} \frac{h(t, y)}{y}, & h_\infty &= \liminf_{y \rightarrow \infty} \min_{t \in [\sigma, 1-\sigma]} \frac{h(t, y)}{y}, \\ h^0 &= \limsup_{y \rightarrow 0} \max_{t \in J} \frac{h(t, y)}{y}, & I_j^0 &= \limsup_{y \rightarrow 0} \frac{I_j(y)}{y}, & J_j^0 &= \limsup_{y \rightarrow 0} \frac{J_j(y)}{y} \end{aligned}$$

$$h^\infty = \limsup_{y \rightarrow \infty} \max_{t \in J} \frac{h(t, y)}{y}, \quad I_j^\infty = \limsup_{y \rightarrow \infty} \frac{I_j(y)}{y}, \quad J_j^\infty = \limsup_{y \rightarrow \infty} \frac{J_j(y)}{y},$$

$$\begin{aligned} \hat{A} = \frac{1}{\Delta} & \left[\left(- \sum_{i=1}^n \beta_i \psi(\xi_i) \right) \cdot \right. \\ & \left(\sum_{i=1}^n \alpha_i \left(\int_0^1 G(\xi_i, s) q(s) ds + \sum_{j=1}^n G_s(t, s) \Big|_{t=\xi_i, s=t_j} q(t_j) + \sum_{j=1}^n G(\xi_i, t_j) q(t_j) \right) \right) \\ & - \left(\rho - \sum_{i=1}^n \alpha_i \psi(\xi_i) \right) \cdot \\ & \left. \left(\sum_{i=1}^n \beta_i \left(\int_0^1 G(\xi_i, s) q(s) ds + \sum_{j=1}^n G_s(t, s) \Big|_{t=\xi_i, s=t_j} q(t_j) + \sum_{j=1}^n G(\xi_i, t_j) q(t_j) \right) \right) \right], \end{aligned}$$

$$\begin{aligned} \hat{B} = \frac{1}{\Delta} & \left[\left(- \sum_{i=1}^n \alpha_i \varphi(\xi_i) \right) \cdot \right. \\ & \left(\sum_{i=1}^n \beta_i \left(\int_0^1 G(\xi_i, s) q(s) ds + \sum_{j=1}^n G_s(t, s) \Big|_{t=\xi_i, s=t_j} q(t_j) + \sum_{j=1}^n G(\xi_i, t_j) q(t_j) \right) \right) \\ & - \left(\frac{\rho}{q(1)} - \sum_{i=1}^n \beta_i \varphi(\xi_i) \right) \cdot \\ & \left. \left(\sum_{i=1}^n \alpha_i \left(\int_0^1 G(\xi_i, s) q(s) ds + \sum_{j=1}^n G_s(t, s) \Big|_{t=\xi_i, s=t_j} q(t_j) + \sum_{j=1}^n G(\xi_i, t_j) q(t_j) \right) \right) \right], \end{aligned}$$

$$\begin{aligned} \dot{A} = \frac{1}{\Delta} & \left[\left(- \sum_{i=1}^n \beta_i \psi(\xi_i) \right) \cdot \left(\sum_{i=1}^n \alpha_i \cdot \right. \right. \\ & \left(\int_0^1 G(\xi_i, s) q(s) h^0 ds + \sum_{j=1}^n G_s(t, s) \Big|_{t=\xi_i, s=t_j} q(t_j) I_j^0 + \sum_{j=1}^n G(\xi_i, t_j) q(t_j) J_j^0 \right) \\ & - \left(\rho - \sum_{i=1}^n \alpha_i \psi(\xi_i) \right) \cdot \left(\sum_{i=1}^n \beta_i \cdot \right. \\ & \left. \left. \left(\int_0^1 G(\xi_i, s) q(s) h^0 ds + \sum_{j=1}^n G_s(t, s) \Big|_{t=\xi_i, s=t_j} q(t_j) I_j^0 + \sum_{j=1}^n G(\xi_i, t_j) q(t_j) J_j^0 \right) \right) \right], \end{aligned}$$

$$\dot{B} = \frac{1}{\Delta} \left[\left(- \sum_{i=1}^n \alpha_i \varphi(\xi_i) \right) \cdot \left(\sum_{i=1}^n \beta_i \cdot \right. \right.$$

$$\left(\int_0^1 G(\xi_i, s) q(s) h^0 ds + \sum_{j=1}^n G_s(t, s) \Big|_{\substack{t=\xi_i \\ s=t_j}} q(t_j) I_j^0 + \sum_{j=1}^n G(\xi_i, t_j) q(t_j) J_j^0 \right) \\ - \left(\frac{\rho}{q(1)} - \sum_{j=1}^n \beta_i \varphi(\xi_i) \right) \cdot \left(\sum_{j=1}^n \alpha_i \cdot \right. \\ \left. \left(\int_0^1 G(\xi_i, s) q(s) h^0 ds + \sum_{j=1}^n G_s(t, s) \Big|_{\substack{t=\xi_i \\ s=t_j}} q(t_j) I_j^0 + \sum_{j=1}^n G(\xi_i, t_j) q(t_j) J_j^0 \right) \right) \Bigg],$$

$$\ddot{A} = \frac{1}{\Delta} \left[\left(- \sum_{j=1}^n \beta_i \psi(\xi_i) \right) \cdot \left(\sum_{j=1}^n \alpha_i \left(\int_0^1 G(\xi_i, s) q(s) (h^0 + \zeta) ds \right. \right. \right. \\ \left. \left. + \sum_{j=1}^n G_s(t, s) \Big|_{\substack{t=\xi_i \\ s=t_j}} q(t_j) (I_j^0 + \zeta) + \sum_{j=1}^n G(\xi_i, t_j) q(t_j) (J_j^0 + \zeta) \right) \right) \\ \left. - \left(\rho - \sum_{j=1}^n \alpha_i \psi(\xi_i) \right) \cdot \left(\sum_{j=1}^n \beta_i \left(\int_0^1 G(\xi_i, s) q(s) (h^0 + \zeta) ds \right. \right. \right. \right. \\ \left. \left. + \sum_{j=1}^n G_s(t, s) \Big|_{\substack{t=\xi_i \\ s=t_j}} q(t_j) (I_j^0 + \zeta) + \sum_{j=1}^n G(\xi_i, t_j) q(t_j) (J_j^0 + \zeta) \right) \right) \right],$$

$$\ddot{B} = \frac{1}{\Delta} \left[\left(- \sum_{j=1}^n \alpha_i \varphi(\xi_i) \right) \cdot \left(\sum_{j=1}^n \beta_i \left(\int_0^1 G(\xi_i, s) q(s) (h^0 + \zeta) ds \right. \right. \right. \\ \left. \left. + \sum_{j=1}^n G_s(t, s) \Big|_{\substack{t=\xi_i \\ s=t_j}} q(t_j) (I_j^0 + \zeta) + \sum_{j=1}^n G(\xi_i, t_j) q(t_j) (J_j^0 + \zeta) \right) \right) \\ \left. - \left(\frac{\rho}{q(1)} - \sum_{j=1}^n \beta_i \varphi(\xi_i) \right) \cdot \left(\sum_{j=1}^n \alpha_i \left(\int_0^1 G(\xi_i, s) q(s) (h^0 + \zeta) ds \right. \right. \right. \right. \\ \left. \left. + \sum_{j=1}^n G_s(t, s) \Big|_{\substack{t=\xi_i \\ s=t_j}} q(t_j) (I_j^0 + \zeta) + \sum_{j=1}^n G(\xi_i, t_j) q(t_j) (J_j^0 + \zeta) \right) \right) \right],$$

$$A' = \frac{1}{\Delta} \left[\left(- \sum_{j=1}^n \beta_i \psi(\xi_i) \right) \cdot \left(\sum_{j=1}^n \alpha_i \cdot \right. \right. \\ \left. \left(\int_0^1 G(\xi_i, s) q(s) h^\infty ds + \sum_{j=1}^n G_s(t, s) \Big|_{\substack{t=\xi_i \\ s=t_j}} q(t_j) I_j^\infty + \sum_{j=1}^n G(\xi_i, t_j) q(t_j) J_j^\infty \right) \right) \\ \left. - \left(\rho - \sum_{j=1}^n \alpha_i \psi(\xi_i) \right) \cdot \left(\sum_{j=1}^n \beta_i \cdot \right. \right.$$

$$\begin{aligned}
& \left(\int_0^1 G(\xi_i, s) q(s) h^\infty ds + \sum_{j=1}^n G_s(t, s) \Big|_{\substack{t=\xi_i \\ s=t_j}} q(t_j) I_j^\infty + \sum_{j=1}^n G(\xi_i, t_j) q(t_j) J_j^\infty \right) \Bigg] , \\
B' = & \frac{1}{\Delta} \left[\left(- \sum_{j=1}^n \alpha_i \varphi(\xi_i) \right) \cdot \left(\sum_{j=1}^n \beta_i \cdot \right. \right. \\
& \left(\int_0^1 G(\xi_i, s) q(s) h^\infty ds + \sum_{j=1}^n G_s(t, s) \Big|_{\substack{t=\xi_i \\ s=t_j}} q(t_j) I_j^\infty + \sum_{j=1}^n G(\xi_i, t_j) q(t_j) J_j^\infty \right) \\
& \left. - \left(\frac{\rho}{q(1)} - \sum_{j=1}^n \beta_i \varphi(\xi_i) \right) \cdot \left(\sum_{j=1}^n \alpha_i \cdot \right. \right. \\
& \left. \left(\int_0^1 G(\xi_i, s) q(s) h^\infty ds + \sum_{j=1}^n G_s(t, s) \Big|_{\substack{t=\xi_i \\ s=t_j}} q(t_j) I_j^\infty + \sum_{j=1}^n G(\xi_i, t_j) q(t_j) J_j^\infty \right) \right) \Bigg] , \\
A'' = & \frac{1}{\Delta} \left[\left(- \sum_{j=1}^n \beta_i \psi(\xi_i) \right) \cdot \left(\sum_{j=1}^n \alpha_i \left(\int_0^1 G(\xi_i, s) q(s) (h^\infty + \zeta) ds \right. \right. \right. \\
& \left. \left. + \sum_{j=1}^n G_s(t, s) \Big|_{\substack{t=\xi_i \\ s=t_j}} q(t_j) (I_j^\infty + \zeta) + \sum_{j=1}^n G(\xi_i, t_j) q(t_j) (J_j^\infty + \zeta) \right) \right) \\
& - \left(\rho - \sum_{j=1}^n \alpha_i \psi(\xi_i) \right) \cdot \left(\sum_{j=1}^n \beta_i \left(\int_0^1 G(\xi_i, s) q(s) (h^\infty + \zeta) ds \right. \right. \\
& \left. \left. + \sum_{j=1}^n G_s(t, s) \Big|_{\substack{t=\xi_i \\ s=t_j}} q(t_j) (I_j^\infty + \zeta) + \sum_{j=1}^n G(\xi_i, t_j) q(t_j) (J_j^\infty + \zeta) \right) \right) \Bigg]
\end{aligned}$$

and

$$\begin{aligned}
B'' = & \frac{1}{\Delta} \left[\left(- \sum_{j=1}^n \alpha_i \varphi(\xi_i) \right) \cdot \left(\sum_{j=1}^n \beta_i \left(\int_0^1 G(\xi_i, s) q(s) (h^\infty + \zeta) ds \right. \right. \right. \\
& \left. \left. + \sum_{j=1}^n G_s(t, s) \Big|_{\substack{t=\xi_i \\ s=t_j}} q(t_j) (I_j^\infty + \zeta) + \sum_{j=1}^n G(\xi_i, t_j) q(t_j) (J_j^\infty + \zeta) \right) \right) \\
& - \left(\frac{\rho}{q(1)} - \sum_{j=1}^n \beta_i \varphi(\xi_i) \right) \cdot \left(\sum_{j=1}^n \alpha_i \left(\int_0^1 G(\xi_i, s) q(s) (h^\infty + \zeta) ds \right. \right. \\
& \left. \left. + \sum_{j=1}^n G_s(t, s) \Big|_{\substack{t=\xi_i \\ s=t_j}} q(t_j) (I_j^\infty + \zeta) + \sum_{j=1}^n G(\xi_i, t_j) q(t_j) (J_j^\infty + \zeta) \right) \right) \Bigg] .
\end{aligned}$$

Theorem 1. Assume that (C1)-(C8) hold,

$$\omega\sigma \int_{\sigma}^{1-\sigma} G(s,s)q(s)ds h_0 > \int_0^1 \delta G(s,s)q(s)ds h^\infty + \sum_{j=1}^n \delta G_s(t,s)|_{t=t_j, s=t_j} q(t_j) I_j^\infty$$

$$+ \sum_{j=1}^n \delta G(t_j, t_j) q(t_j) J_j^\infty + (A' + B') \delta \min\{\varphi(0), \psi(1)\},$$

$$h^\infty + \sum_{j=1}^n I_j^\infty + \sum_{j=1}^n J_j^\infty < \infty \text{ and } \omega\sigma \int_{\sigma}^{1-\sigma} G(s,s)q(s)ds h_0 > 0, \text{ then problem (1.1) has}$$

$$\text{at least one positive solution for}$$

$$\left[\omega\sigma \int_{\sigma}^{1-\sigma} G(s,s)q(s)ds h_0 \right]^{-1} < \kappa < \left[\int_0^1 \delta G(s,s)q(s)ds h^\infty \right. \quad (3.1)$$

$$\left. + \sum_{j=1}^n \delta G_s(t,s)|_{t=t_j, s=t_j} q(t_j) I_j^\infty + \sum_{j=1}^n \delta G(t_j, t_j) q(t_j) J_j^\infty + (A' + B') \delta \min\{\varphi(0), \psi(1)\} \right]^{-1}$$

Proof. By (3.1), there exists $\zeta > 0$ such that

$$\left[\omega\sigma \int_{\sigma}^{1-\sigma} G(s,s)q(s)ds (h_0 - \zeta) \right]^{-1} \leq \kappa$$

$$\leq \left[\int_0^1 \delta G(s,s)q(s)ds (h^\infty + \zeta) + \sum_{j=1}^n \delta G_s(t,s)|_{t=t_j, s=t_j} q(t_j) (I_j^\infty + \zeta) \right. \quad (3.2)$$

$$\left. + \sum_{j=1}^n \delta G(t_j, t_j) q(t_j) (J_j^\infty + \zeta) + (A'' + B'') \delta \min\{\varphi(0), \psi(1)\} \right]^{-1}$$

Let ζ be fixed. Since $\omega\sigma \int_{\sigma}^{1-\sigma} G(s,s)q(s)ds h_0 > 0$, there exists $\tau_1 > 0$ such that $h(t, y) > (h_0 - \zeta)y$, $0 \leq y \leq \tau_1$, $t \in [\sigma, 1 - \sigma]$. Then for $\vartheta \in \partial K_{\tau_1}$, $t \in [\sigma, 1 - \sigma]$, we have

$$T_\kappa \vartheta(t) \geq \kappa \int_0^1 G(t,s)q(s)h(s, \vartheta(s))ds > \kappa \int_{\sigma}^{1-\sigma} \omega G(s,s)q(s)(h_0 - \zeta)\vartheta(s)ds$$

$$\geq \kappa \omega \sigma (h_0 - \zeta) \int_{\sigma}^{1-\sigma} G(s,s)q(s)ds \|\vartheta\|_{PC} \geq \|\vartheta\|_{PC}.$$

Hence, $\|T_\kappa \vartheta\|_{PC} \geq \|\vartheta\|_{PC}$, $\vartheta \in \partial K_{\tau_1}$. Since $h^\infty + \sum_{j=1}^n I_j^\infty + \sum_{j=1}^n J_j^\infty < \infty$, there exists $\bar{\tau}_2 > 0$ such that

$$h(t, y) < (h^\infty + \zeta)y, I_j(y) < (I_j^\infty + \zeta)y, J_j(y) < (J_j^\infty + \zeta)y, y \geq \bar{\tau}_2, t \in J.$$

Set

$$N = \max \left\{ \max_{\substack{t \in J \\ y \in [0, \bar{\tau}_2]}} h(t, y), \max_{y \in [0, \bar{\tau}_2]} I_1(y), \dots, \max_{y \in [0, \bar{\tau}_2]} I_n(y), \max_{y \in [0, \bar{\tau}_2]} J_1(y), \dots, \max_{y \in [0, \bar{\tau}_2]} J_n(y) \right\},$$

$$\tau_2 > \max \left\{ \tau_1, \bar{\tau}_2, \frac{N}{h^\infty + \zeta}, \frac{N}{I_1^\infty + \zeta}, \dots, \frac{N}{I_n^\infty + \zeta}, \frac{N}{J_1^\infty + \zeta}, \dots, \frac{N}{J_n^\infty + \zeta} \right\},$$

then

$$h(t, y) < (h^\infty + \zeta)\tau_2, I_j(y) < (I_j^\infty + \zeta)\tau_2, J_j(y) < (J_j^\infty + \zeta)\tau_2, 0 \leq y \leq \tau_2, t \in J.$$

Thus for $\vartheta \in \partial K_{\tau_2}$, we have

$$\begin{aligned} \|T_\kappa \vartheta\|_{PC} &\leq \kappa \int_0^1 \delta G(s, s) q(s) h(s, \vartheta(s)) ds + \kappa \sum_{j=1}^n \delta G_s(t, s)|_{t=t_j, s=t_j} q(t_j) I_j(\vartheta(t_j)) \\ &\quad + \kappa \sum_{j=1}^n \delta G(t_j, t_j) q(t_j) J_j(\vartheta(t_j)) + \kappa [A(h) + B(h)] \delta \min\{\varphi(0), \psi(1)\} \\ &\leq \kappa \left[\int_0^1 \delta G(s, s) q(s) (h^\infty + \zeta) ds + \delta \sum_{j=1}^n G_s(t, s)|_{t=t_j, s=t_j} q(t_j) (I_j^\infty + \zeta) \right. \\ &\quad \left. + \delta \sum_{j=1}^n G(t_j, t_j) q(t_j) (J_j^\infty + \zeta) + (A'' + B'') \delta \min\{\varphi(0), \psi(1)\} \right] \|\vartheta\|_{PC} \\ &\leq \|\vartheta\|_{PC}. \end{aligned}$$

By Lemma 9, the operator T_κ has at least one fixed point ϑ^* with $\tau_1 \leq \|\vartheta^*\| \leq \tau_2$, which is a positive solution of problem (1.1). $\square$

Theorem 2. Assume that (C1)-(C8) hold,

$$\begin{aligned} \omega \sigma \int_\sigma^{1-\sigma} G(s, s) q(s) ds h_\infty &> \int_0^1 \delta G(s, s) q(s) ds h^0 + \sum_{j=1}^n \delta G_s(t, s)|_{t=t_j, s=t_j} q(t_j) I_j^0 \\ &\quad + \sum_{j=1}^n \delta G(t_j, t_j) q(t_j) J_j^0 + (\dot{A} + \dot{B}) \delta \min\{\varphi(0), \psi(1)\}, \end{aligned}$$

$h^0 + \sum_{j=1}^n I_j^0 + \sum_{j=1}^n J_j^0 < \infty$ and $\omega \sigma \int_\sigma^{1-\sigma} G(s, s) q(s) ds h_\infty > 0$, then the problem (1.1) has at least one positive solution for

$$\begin{aligned} &\left[\omega \sigma \int_\sigma^{1-\sigma} G(s, s) q(s) ds h_\infty \right]^{-1} < \kappa \\ &< \left[\int_0^1 \delta G(s, s) q(s) ds h^0 + \sum_{j=1}^n \delta G_s(t, s)|_{t=t_j, s=t_j} q(t_j) I_j^0 \right. \\ &\quad \left. + \sum_{j=1}^n \delta G(t_j, t_j) q(t_j) J_j^0 + (\dot{A} + \dot{B}) \delta \min\{\varphi(0), \psi(1)\} \right]^{-1}. \end{aligned} \quad (3.3)$$

Proof. By (3.3), there exists $\zeta > 0$ such that

$$\begin{aligned} & \left[\omega \sigma \int_{\sigma}^{1-\sigma} G(s, s) q(s) ds (h_{\infty} - \zeta) \right]^{-1} \leq \kappa \\ & \leq \left[\int_0^1 \delta G(s, s) q(s) ds (h^0 + \zeta) + \sum_{j=1}^n \delta G_s(t, s) \Big|_{t=t_j, s=t_j} q(t_j) (I_j^0 + \zeta) \right. \\ & \quad \left. + \sum_{j=1}^n \delta G(t_j, t_j) q(t_j) (J_j^0 + \zeta) + (\ddot{A} + \ddot{B}) \delta \min\{\varphi(0), \psi(1)\} \right]^{-1} \end{aligned} \quad (3.4)$$

Let ζ be fixed. Since $h^0 + \sum_{j=1}^n I_j^0 + \sum_{j=1}^n J_j^0 < \infty$, there exists $\tau_1 > 0$ such that

$$h(t, y) < (h^0 + \zeta)y, \quad I_j(y) < (I_j^0 + \zeta)y, \quad J_j(y) < (J_j^0 + \zeta)y, \quad 0 \leq y \leq \tau_1, t \in J. \quad (3.5)$$

Then for $\vartheta \in \partial K_{\tau_1}$, by using (3.4) and (3.5), we have $\|T_{\kappa}\vartheta\|_{PC} \leq \|\vartheta\|_{PC}$. Since $\omega \sigma \int_{\sigma}^{1-\sigma} G(s, s) q(s) ds h_{\infty} > 0$, there exists $\bar{\tau}_2 > 0$ such that $h(t, y) > (h_{\infty} - \zeta)y$, $y \geq \bar{\tau}_2$, $t \in [\sigma, 1 - \sigma]$. Let $\tau_2 \geq \max\{2\tau_1, \frac{\bar{\tau}_2}{\sigma}\}$, then for $\vartheta \in \partial K_{\tau_2}$, $t \in [\sigma, 1 - \sigma]$, we can obtain $\|T_{\kappa}\vartheta\|_{PC} \geq \|\vartheta\|_{PC}$, $\vartheta \in \partial K_{\tau_2}$. It follows from Lemma 9 that T_{κ} has a fixed point ϑ^* with $\tau_1 \leq \|\vartheta^*\| \leq \tau_2$, which is a positive solution of problem (1.1). $\square$

Theorem 3. Let (C1)-(C8) hold. If $h_0 = \infty$ or $h_{\infty} = \infty$, then there exists $\kappa_0 > 0$ such that problem (1.1) has a positive solution for $0 < \kappa < \kappa_0$.

Proof. Choose a number $\tau_1 > 0$ and

$$\begin{aligned} \kappa_0 = \frac{\tau_1}{\bar{N}} & \left(\int_0^1 \delta G(s, s) q(s) ds + \sum_{j=1}^n \delta G_s(t, s) \Big|_{t=t_j, s=t_j} q(t_j) + \sum_{j=1}^n \delta G(t_j, t_j) q(t_j) \right. \\ & \quad \left. + (\hat{A} + \hat{B}) \delta \min\{\varphi(0), \psi(1)\} \right)^{-1}, \end{aligned}$$

where $\bar{N} = \max \left\{ \max_{\substack{t \in J \\ y \in [0, \tau_1]}} h(t, y), \max_{y \in [0, \tau_1]} I_j(y), \max_{y \in [0, \tau_1]} J_j(y) \right\}$. For $\vartheta \in \partial K_{\tau_1}$, $0 < \kappa < \kappa_0$,

$$\begin{aligned} \|T_{\kappa}\vartheta\|_{PC} & \leq \kappa \left[\int_0^1 \delta G(s, s) q(s) ds \max_{\substack{t \in J \\ y \in [0, \tau_1]}} h(t, y) + \sum_{j=1}^n \delta G_s(t, s) \Big|_{t=t_j, s=t_j} q(t_j) \max_{y \in [0, \tau_1]} I_j(y) \right. \\ & \quad \left. + \sum_{j=1}^n \delta G(t_j, t_j) q(t_j) \max_{y \in [0, \tau_1]} J_j(y) + \bar{N}(\hat{A} + \hat{B}) \delta \min\{\varphi(0), \psi(1)\} \right] \end{aligned}$$

$$\begin{aligned}
&< \kappa_0 \bar{N} \left[\int_0^1 \delta G(s, s) q(s) ds + \sum_{j=1}^n \delta G_s(t, s) \Big|_{s=t_j} q(t_j) + \sum_{j=1}^n \delta G(t_j, t_j) q(t_j) \right. \\
&\quad \left. + (\hat{A} + \hat{B}) \delta \min\{\varphi(0), \psi(1)\} \right] \\
&= \tau_1 = \|\vartheta\|_{PC}.
\end{aligned} \tag{3.6}$$

If $h_0 = \infty$, then there exists $\tau_2 \in (0, \tau_1)$ such that $h(t, y) > \eta y$ for all $y \in [0, \tau_2]$, $t \in [\sigma, 1 - \sigma]$ where $\eta > 0$ satisfying

$$\kappa \omega \sigma \eta \int_{\sigma}^{1-\sigma} G(s, s) q(s) ds > 1. \tag{3.7}$$

Then for $\vartheta \in \partial K_{\tau_2}$,

$$\begin{aligned}
T_{\kappa} \vartheta(t) &\geq \kappa \int_{\sigma}^{1-\sigma} \omega G(s, s) q(s) h(s, \vartheta(s)) ds > \kappa \int_{\sigma}^{1-\sigma} \omega G(s, s) q(s) \eta \vartheta(s) ds \\
&\geq \kappa \omega \sigma \eta \int_{\sigma}^{1-\sigma} G(s, s) q(s) ds \|\vartheta\|_{PC} > \|\vartheta\|_{PC}.
\end{aligned}$$

Hence, $\|T_{\kappa} \vartheta\|_{PC} \geq \|\vartheta\|_{PC}$, $\vartheta \in \partial K_{\tau_2}$.

If $h_{\infty} = \infty$, then there exists $\bar{\tau}_3 > 0$ such that $h(t, y) > \gamma y$ for $y \geq \bar{\tau}_3$, $t \in [\sigma, 1 - \sigma]$ where $\gamma > 0$ satisfying

$$\kappa \omega \sigma \gamma \int_{\sigma}^{1-\sigma} G(s, s) q(s) ds > 1. \tag{3.8}$$

Let $\tau_3 > \max\{\tau_1, \frac{\bar{\tau}_3}{\sigma}\}$, then for $\vartheta \in \partial K_{\tau_3}$,

$$\begin{aligned}
T_{\kappa} \vartheta(t) &\geq \kappa \int_{\sigma}^{1-\sigma} \omega G(s, s) q(s) h(s, \vartheta(s)) ds > \kappa \int_{\sigma}^{1-\sigma} \omega G(s, s) q(s) \gamma \vartheta(s) ds \\
&\geq \kappa \omega \sigma \gamma \int_{\sigma}^{1-\sigma} G(s, s) q(s) ds \|\vartheta\|_{PC} > \|\vartheta\|_{PC}.
\end{aligned}$$

Hence, $\|T_{\kappa} \vartheta\|_{PC} \geq \|\vartheta\|_{PC}$, $\vartheta \in \partial K_{\tau_3}$.

Then, Lemma 9 yields that T_{κ} has a fixed point in $\bar{K}_{\tau_1} \setminus K_{\tau_2}$ or $\bar{K}_{\tau_3} \setminus K_{\tau_1}$ according to whether $h_0 = \infty$ or $h_{\infty} = \infty$, respectively. As a result, problem (1.1) has a positive solution for $0 < \kappa < \kappa_0$. $\square$

Theorem 4. Let (C1)-(C8) hold. If $h^0 + \sum_{j=1}^n I_j^0 + \sum_{j=1}^n J_j^0 = 0$ or $h^{\infty} + \sum_{j=1}^n I_j^{\infty} + \sum_{j=1}^n J_j^{\infty} = 0$ such that problem (1.1) has a positive solution for $\kappa > \kappa_0$.

Proof. The proof is done similarly to the theorem 3. $\square$

Corollary 1. Let (C1)-(C8) hold. If $h_0 = h_{\infty} = \infty$, then there exists $\kappa_0 > 0$ such that problem (1.1) has at least two positive solutions for $0 < \kappa < \kappa_0$.

Corollary 2. Let (C1)-(C8) hold. If $h^0 + \sum_{j=1}^n I_j^0 + \sum_{j=1}^n J_j^0 = h^\infty + \sum_{j=1}^n I_j^\infty + \sum_{j=1}^n J_j^\infty = 0$, then there exists $\kappa_0 > 0$ such that problem (1.1) has at least two positive solutions for $\kappa > \kappa_0$.

Theorem 5. Let (C1)-(C8) hold. If $h_0 > 0$ and $h_\infty > 0$, then there exists $\kappa_0 > 0$ such that problem (1.1) has no positive solution for $\kappa > \kappa_0$.

Proof. Since $h_0 > 0$ and $h_\infty > 0$, then there exist positive numbers $\eta_1, \eta_2, \tau_1, \tau_2$ such that $\tau_1 < \sigma\tau_2$ and $h(t, y) \geq \eta_1 y$, $y \in [0, \tau_1]$, $t \in [\sigma, 1 - \sigma]$, $h(t, y) \geq \eta_2 y$, $y \in [\tau_1, +\infty)$, $t \in [\sigma, 1 - \sigma]$. Let the positive number η_3 be defined by

$$\eta_3 = \min \left\{ \eta_1, \eta_2, \min_{\substack{t \in [\sigma, 1 - \sigma] \\ y \in [\tau_1, \tau_2]}} \frac{h(t, y)}{y} \right\}.$$

Then, $h(t, y) \geq \eta_3 y$, $y \in [0, \tau_2]$, $t \in [\sigma, 1 - \sigma]$ and $h(t, y) \geq \eta_3 y$, $y \in [\tau_1, +\infty)$, $t \in [\sigma, 1 - \sigma]$.

Assume $\vartheta(t)$ is a positive solution of problem (1.1). We will show that this leads to a contradiction for

$$\kappa > \kappa_0 = \frac{1}{\omega \sigma \eta_3 \int_{\sigma}^{1-\sigma} G(s, s) q(s) ds}.$$

Since $T_\kappa \vartheta(t) = \vartheta(t)$ for $t \in J$, then for $\kappa > \kappa_0$ and $\|\vartheta\|_{PC} \in [0, \tau_2]$,

$$\begin{aligned} \|\vartheta\|_{PC} &= \|T_\kappa \vartheta\|_{PC} \geq T_\kappa \vartheta \left(\frac{1}{2} \right) \geq \kappa \int_{\sigma}^{1-\sigma} G \left(\frac{1}{2}, s \right) q(s) h(s, \vartheta(s)) ds \\ &> \kappa_0 \eta_3 \int_{\sigma}^{1-\sigma} \omega G(s, s) q(s) \vartheta(s) ds \\ &\geq \kappa_0 \omega \sigma \eta_3 \int_{\sigma}^{1-\sigma} G(s, s) q(s) ds \|\vartheta\|_{PC} \geq \|\vartheta\|_{PC}, \end{aligned}$$

which is a contradiction. For $\kappa > \kappa_0$ and $\|\vartheta\|_{PC} \in [\frac{\tau_1}{\sigma}, +\infty)$,

$$\begin{aligned} \|\vartheta\|_{PC} &= \|T_\kappa \vartheta\|_{PC} \geq T_\kappa \vartheta \left(\frac{1}{2} \right) \geq \kappa \int_{\sigma}^{1-\sigma} G \left(\frac{1}{2}, s \right) q(s) h(s, \vartheta(s)) ds \\ &\geq \kappa_0 \omega \sigma \eta_3 \int_{\sigma}^{1-\sigma} G(s, s) q(s) ds \|\vartheta\|_{PC} \geq \|\vartheta\|_{PC}, \end{aligned}$$

which is a contradiction. Hence, problem (1.1) has no positive solution. $\square$

Theorem 6. Let (C1)-(C8) hold. If $h^0 + \sum_{j=1}^n I_j^0 + \sum_{j=1}^n J_j^0 < \infty$ and $h^\infty + \sum_{j=1}^n I_j^\infty + \sum_{j=1}^n J_j^\infty < \infty$, then there exists $\kappa_0 > 0$ such that problem (1.1) has no positive solution for $0 < \kappa < \kappa_0$.

Proof. The proof can be done similarly to the theorem 5. □

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GAME-THEORETIC p -LAPLACE OPERATOR INVOLVING THE GRADIENT

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Abstract. The game-theoretic p -Laplacian operator is a version of classical variational p -Laplacian which is in connection with stochastic games called Tug-of-War with noise. The existence of positive singular and Hölder continuous solutions of the game-theoretic p -Laplace operator involving the gradient in a small C^2 perturbation of the unit ball in $\mathbb{R}^n$ are proved. Finally, a more case problem is introduced.

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1. INTRODUCTION

Game theoretic stochastic or deterministic methods have recently emerged as a novel approach to study and to approximate various non-linear Partial Differential Equations (PDEs). The normalized p -Laplacian or game-theoretic p -Laplacian operator is a version of classical variational p -Laplacian which was introduced recently in connection with stochastic games called Tug-of-War with noise and it can be used as a unified framework for interpolation problems in signal processing on graphs, such as image processing and machine learning (see [9]).

Generally, tug-of-war, athletic contest between two teams at opposite ends of a rope, each team trying to drag the other across a center line. In some forms of the game a tape or handkerchief is tied around the center of the rope, and two others are tied six feet (1.8 meters) on either side. Three corresponding lines are marked on the ground. The game ends when one team pulls the other so that the tape on the losers' side crosses the ground mark on the winners' side. The contest is decided by the best two out of three pulls. A rural pastime in England and Scotland, the tug-of-war was an Olympic event from 1900 to 1920, with five men to a side. It has often been an outdoor contest at Scottish Highland Games and at other large social gatherings in the 20th century.

Mathematically, tug-of-war games related to the 1-Laplacian or to the p -Laplacian (first introduced by Peres, Schramm, Shefel, and Wilson in [22, 23]) have attracted a

lot of attention and are used in many works to study the existence or the regularity of solutions for many PDEs (see [35] and references therein). Many of these games generally are formulated as well-known statistical functionals such as mean, min, or max operators. They are interpreted as a discrete approximation of the underlying PDE, and solving the latter leads to taking a suitable limit of the solution of the discrete game.

Here, we briefly review the notion of tug-of-war game and continuous PDEs problem form [8] (one can see more details in [22, 23]).

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded domain, $h : \Omega \rightarrow \mathbb{R}$ the running payoff function, and $g : \partial\Omega \rightarrow \mathbb{R}$ the payoff function. Fix a number $\varepsilon > 0$. The dynamics of the game are as follows: a token is placed at an initial position $x_0 \in \Omega$. At the k th stage of the game, Player I and Player II select points x_k^I and x_k^{II} , respectively, each belonging to a specified set $B_\varepsilon(x_{k-1}) \subset \Omega$ (where $B_\varepsilon(x_{k-1})$ is the ε -ball centered in x_{k-1}). The token is then moved to x_k , where x_k can be either x_k^I or x_k^{II} with equal probability. In other words, a fair coin is tossed to determine where the token is placed (i.e., which player won this stage).

After the k th stage of the game, if $x_k \in \Omega$ then the game continue to stage $k + 1$. Otherwise, if $x_k \in \partial\Omega$, the game ends and Player II pays Player I the amount $g(x_k) + \varepsilon^2 \sum_{j=0}^{k-1} h(x_j)$. Player I attempts to maximize the payoff while Player II attempts to minimize it. If both player are using optimal strategy, according to the Dynamic Programming Principle, the value functions for Player I and Player II for standard ε -turn tug-of-war satisfy the relation:

$$\begin{cases} u^\varepsilon(x) = \frac{1}{2} \left[\sup_{y \in B_\varepsilon(x)} u^\varepsilon(y) + \inf_{y \in B_\varepsilon(x)} u^\varepsilon(y) \right] + \varepsilon^2 h(x) & x \in \Omega, \\ u^\varepsilon(x) = g(x) & x \in \partial\Omega, \end{cases}$$

The authors of [23] have shown that if the running payoff function h is of constant sign, the value function u^ε converges to the unique viscosity solution of the normalized ∞ -Poisson equation:

$$\begin{cases} \Delta_\infty^N u(x) = -h(x) & x \in \Omega, \\ u(x) = g(x) & x \in \partial\Omega, \end{cases}$$

where $\Delta_\infty^N := \frac{1}{|\nabla u|^2} \Delta_\infty$ is the normalized ∞ -Laplacian and

$$\Delta_\infty = |\nabla u|^2 \sum_{ij} u_{x_i} u_{x_i x_j} u_{x_j}$$

In another version of tug-of-war game with noise, the game is modified as follows: at point $x_k \in \Omega$, player I and player II play ε -step tug-of-war game with probability $\beta \in [0, 1]$ and a random point in ball of radius ε centered at x_k is chosen with probability $1 - \beta$. The value functions of the game satisfy the dynamic programming

principle:

$$u^\varepsilon(x) = \frac{\beta}{2} \left[\sup_{y \in B_\varepsilon(x)} u^\varepsilon(y) + \inf_{y \in B_\varepsilon(x)} u^\varepsilon(y) \right] + \frac{1-\beta}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} u^\varepsilon(y) dy + \varepsilon^2 h(x)$$

with the boundary condition $u^\varepsilon(x) = g(x)$ for $x \in \partial\Omega$. A detailed proof for existence and uniqueness of these types of function was shown in [17].

Choosing the probability $\beta = \frac{p-2}{p+n}$, this dynamic programming principle gives a connection to viscosity solutions of the following p -Laplace equation [14, 16, 35]:

$$\begin{cases} \Delta_p^N u(x) = -h(x) & x \in \partial\Omega, \\ u(x) = g(x) & x \in \partial\Omega, \end{cases}$$

for $p \geq 2$ with $\Delta_p^N u := \frac{1}{p} |\nabla u|^{2-p} \operatorname{div}(|\nabla u|^{p-2} \nabla u)$.

Recently, in [9], the authors proposed an adaptation and generalization of the normalized p -Laplacian operator on weighted graphs, they studied the uniqueness and existence of the solution where the Dirichlet problem associated to this operator is considered. Also there is a high interest in adapting classical signal processing tools on graphs and networks such as wavelets or PDEs (see [8] and references there in). The demand for such methods is motivated by existing and potential future applications, such as in machine learning and mathematical image processing. Indeed, any kind of data can be represented by a graph in an abstract form in which the vertices are associated to the data and the edges correspond to relationships within data.

In this paper, we are interested in obtaining positive singular and Hölder continuous solutions of

$$\begin{cases} -\Delta_p^N u(y) = C |\nabla u(y)|^q & y \in \Omega, \\ u = 0 & y \in \partial\Omega, \end{cases} \quad (1.1)$$

where Ω is a small C^2 perturbation of the unit ball in $\mathbb{R}^n$ and $C > 0$ is a constant.

The equation (1.1) can be rewrite as

$$0 = |\nabla u|^2 \Delta u + \frac{(p-2)}{2} \nabla u \cdot \nabla |\nabla u|^2 + pC |\nabla u|^{q+2} \quad y \in \Omega, \quad (1.2)$$

with $u = 0$ on $y \in \partial\Omega$. We can write this in terms of the components as

$$0 = \left(|\nabla u|^2 \Delta u + (p-2) \sum_{i,j=1}^n u_{y_i} u_{y_j} u_{y_i y_j} \right) + pC |\nabla u|^{q+2}, \quad y \in \Omega. \quad (1.3)$$

Notice that we can re-write the equation as $-\Delta_p u - a(x) \cdot \nabla u = 0$ in Ω with $u = 0$ on $\partial\Omega$ where $a(x) = p |\nabla u|^{q+p-4} \nabla u$ and hence if u sufficiently smooth we see that $a(x)$ should be sufficiently smooth so as to apply the maximum principle; hence the only solution should be $u = 0$. From this informal argument we expect the only way to obtain a positive solution is for the solution to be somewhat singular. The following example gives an explicit solution on the puncture of the unit ball. Our approach will be to perturb an explicit solution on the ball.

Example 1. Let B_1 denote the unit ball centered at the origin in $\mathbb{R}^n$.

- (1) Let $1 < p < n$, $1 + \frac{(p-1)}{n-1} < q < 2$ and define $w(r) := r^{-\sigma} - 1$ where $\sigma := \frac{2-q}{q-1}$ and

$$C := \frac{(n-1)(q-1) - (p-1)}{p(q-1)\sigma^{q-1}}. \quad (1.4)$$

Then u is a singular weak solution of (1.1) with $\Omega = B_1$. Note the above restriction forces $\sigma > 0$ and the further restriction forces $C > 0$.

- (2) Let $q > \max\{2, 1 + \frac{(p-1)}{n-1}\}$ and define $w(r) := 1 - r^\sigma$ where $\sigma := \frac{q-2}{q-1}$ and

$$C := \frac{(n-1)(q-1) - (p-1)}{p(q-1)\sigma^{q-1}}. \quad (1.5)$$

Then w is a positive Hölder continuous weak solution of (1.1) with $\Omega = B_1$. Note the restriction forces $\sigma > 0$ and $C > 0$.

PDE's problems are important for their applications in other sciences. The standard mathematical techniques are not adequate to study these problems and they need new techniques. This may be the central development of mathematical ideas in active areas of pure mathematics which have had a decisive interaction with PDE's (see [1, 10–13, 15, 18, 20, 21, 24–34, 36–38] for more relevant problems). It is very remarkable to write that in the classical theory of the p -Laplace equation (as well as Laplace equation) several main parts of mathematics such as Calculus of Variations, Partial Differential Equations, Potential Theory, Function Theory are joined. The problem $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is studied for different range of p . It is worth to mention that (1.1) is a non variational equation and hence there are various standard tools which are not available anymore. Here, by an idea motivated from [2–7, 19] we study the problem.

Due to do this, we do a change of variables to reduce the problem to one on the unit ball; this is taken from [7]. Fix $\psi : \overline{B_1} \rightarrow \mathbb{R}^n$ be a smooth map and for $\varepsilon > 0$ define

$$\Omega_\varepsilon := \{x + \varepsilon\psi(x) : x \in B_1\}.$$

This domain will be the small perturbation of the unit ball we work on. There is some small $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ one has that Ω_ε is diffeomorphic to the unit ball B_1 . Let $y = x + \varepsilon\psi(x)$ for $x \in B_1$ and note there is some $\tilde{\psi}$ smooth such that $x = y + \varepsilon\tilde{\psi}(\varepsilon, y)$ for $y \in \Omega_\varepsilon$. Given $u(y)$ defined on $y \in \Omega_\varepsilon$ or $v(x)$ defined on $x \in B_1$ we define the other via $u(y) = v(x)$. So to find a positive singular solution $u(y)$ of (1.1) it is sufficient to find a positive singular solution $v(x)$ of some, to be determined equation, on the unit ball. To compute the equation for $v(x)$ we will use the chain rule, but we mention that the computation becomes quite involved. We know that

$$u_{y_i} = \sum_{k=1}^n v_{x_k} \left(\delta_{ki} + \frac{\partial \tilde{\psi}^k}{\partial y_i} \right) = v_{x_i} + \varepsilon \sum_{k=1}^n v_{x_k} \frac{\partial \tilde{\psi}^k}{\partial y_i}.$$

Also a computation shows (similar to [2, 3])

$$\sum_{i,j=1}^n u_{y_i} u_{y_j} u_{y_i y_j} = \frac{\nabla v \cdot \nabla(|\nabla v|^2)}{2} + g_0(\varepsilon) \sum_{i,j,k=1}^n \{v_{x_i x_j} v_{x_i} v_{x_j} + v_{x_i} v_{x_j} v_{x_k}\},$$

where $|g_0(\varepsilon)| \leq C\varepsilon$ for all $|\varepsilon|$ small. We now make some comments on this simplification. Our approach will be to look for solutions of the form $v(x) = w(x) + \phi(x)$ where $w(x) = w(r)$ is the above explicit singular radial solution. We will end up writing out fixed point argument but all these terms that were simplified will not affect the linearized operator; but will only show up in the nonlinear terms. So the exact nature of the terms is not overly important, and in fact if one checks all the dropped terms, they see they are all of the exact for of the two terms we left. Additionally we have dropped the smooth coefficients, but this won't affect anything either.

By [7] we can write $\Delta_y u(y) = \Delta_x v(x) + E_\varepsilon(v)$ where $E_\varepsilon(v)$ is defined by (1.6). So the equation for v on the unit ball now becomes (after taking into account the prior mentioned simplification)

$$\begin{aligned} 0 &= |\nabla v + \varepsilon A_0 \nabla v|^2 (\Delta v + E_\varepsilon(v)) + \frac{p-2}{2} \nabla v \cdot \nabla(|\nabla v|^2) \\ &\quad + g_0(\varepsilon) \sum \{v_{x_i x_j} v_{x_i} v_{x_j} + v_{x_i} v_{x_j} v_{x_k}\} + pC |\nabla v + \varepsilon A_0 \nabla v|^{q+2} \\ &= (\Delta v) |\nabla v|^2 + \frac{p-2}{2} \nabla v \cdot \nabla(|\nabla v|^2) + pC |\nabla v + \varepsilon A_0 \nabla v|^{q+2} + H_\varepsilon(v) \end{aligned}$$

where

$$\begin{aligned} H_\varepsilon(v) &:= (\Delta v) 2\varepsilon (A_0 \nabla v) \cdot \nabla v + \varepsilon^2 (\Delta v) |A_0 \nabla v|^2 + E_\varepsilon(v) |\nabla v|^2 \\ &\quad + E_\varepsilon(v) (2\varepsilon A_0 \nabla v) \cdot \nabla v + E_\varepsilon(v) \varepsilon^2 |A_0 \nabla v|^2 \\ &\quad + g_0(\varepsilon) \sum \{v_{x_i x_j} v_{x_i} v_{x_j} + v_{x_i} v_{x_j} v_{x_k}\} \\ E_\varepsilon(v) &:= 2\varepsilon \sum_{i,k} v_{x_i x_k} \partial_{y_i} \tilde{\psi}_k + \varepsilon \sum_{i,k} v_{x_k} \partial_{y_i y_i} \tilde{\psi}_k + \varepsilon^2 \sum_{i,j,k} v_{x_j x_k} \partial_{y_i} \tilde{\psi}_j \tilde{\psi}_k. \end{aligned} \quad (1.6)$$

We now hope for small enough ε we can find a solution of the form $v = w + \phi$. If we rewrite the equation putting all the linear in ϕ terms on the left we arrive at

$$\begin{cases} -L(\phi) = \sum_{k=1}^7 F_k(\phi) + I_\varepsilon(\phi) + H_\varepsilon(w + \phi) & B_1, \\ \phi = 0 & \partial B_1, \end{cases} \quad (1.7)$$

where

$$\begin{aligned} F_1(\phi) &= \Delta w |\nabla \phi|^2, \\ F_2(\phi) &= (\Delta \phi) (2 \nabla w \cdot \nabla \phi), \\ F_3(\phi) &= (\Delta \phi) |\nabla \phi|^2, \\ F_4(\phi) &= \frac{p-2}{2} \nabla w \cdot \nabla(|\nabla \phi|^2), \\ F_5(\phi) &= (p-2) \nabla \phi \cdot \nabla(\nabla w \cdot \nabla \phi), \end{aligned}$$

$$\begin{aligned}
F_6(\phi) &= \frac{(p-2)}{2} \nabla \phi \cdot \nabla |\nabla \phi|^2 \\
I_\varepsilon(\phi) &= pC |\nabla w + \nabla \phi + \varepsilon A_0(\nabla w + \nabla \phi)|^{q+2} - pC |\nabla w + \nabla \phi|^{q+2}, \\
F_7(\phi) &= pC \{ |\nabla w + \nabla \phi|^{q+2} - |\nabla w|^{q+2} - (q+2) |\nabla w|^q \nabla w \cdot \nabla \phi \}.
\end{aligned}$$

The linear operator L is given by

$$\begin{aligned}
L(\phi) &:= |\nabla w|^2 (\Delta \phi) + (\Delta w) (2 \nabla w \cdot \nabla \phi) + (p-2) \nabla w \cdot \nabla (\nabla w \cdot \nabla \phi) \\
&\quad + \frac{(p-2)}{2} \nabla \phi \cdot \nabla |\nabla w|^2 + pC(q+2) |\nabla w|^q \nabla w \cdot \nabla \phi.
\end{aligned}$$

Of crucial importance will be the linear operator L and what functions spaces we work in. Before we consider these issues we want to normalize L by dividing by $|\nabla w|^2$. So instead of considering (1.7) we will consider

$$\begin{cases} -\tilde{L}(\phi) &:= \frac{-L(\phi)}{|\nabla w|^2} = \sum_{k=1}^7 \frac{F_k(\phi)}{|\nabla w|^2} + \frac{I_\varepsilon(\phi)}{|\nabla w|^2} + \frac{H_\varepsilon(w+\phi)}{|\nabla w|^2} & B_1, \\ \phi &= 0 & \partial B_1. \end{cases} \quad (1.8)$$

To obtain a solution of this we will apply the Contraction Mapping Principle to the nonlinear mapping $J_\varepsilon(\phi) = \psi$ (for $\phi \in X$ where X is yet to be determined and of course this mapping is not well defined yet)

$$\begin{cases} -\tilde{L}(\psi) &= \sum_{k=1}^7 \frac{F_k(\psi)}{|\nabla w|^2} + \frac{I_\varepsilon(\psi)}{|\nabla w|^2} + \frac{H_\varepsilon(w+\psi)}{|\nabla w|^2} & B_1, \\ \psi &= 0 & \partial B_1. \end{cases} \quad (1.9)$$

The exact form of $\tilde{L}$ will be crucial for us. A computation shows that we can write

$$\tilde{L}(\phi) = \Delta \phi + \gamma \phi_{rr} + \frac{\alpha \phi_r}{r},$$

where $\gamma := p-2$ and

$$\alpha := 2(n-1) - 2(p-1)(\sigma+1) - pC(q+2)\sigma^{q-1} \quad (1.10)$$

where C is given by (1.4).

2. LINEAR THEORY

We study the linear theory for the problem in two different cases (1) The singular case and (2) The Hölder continuous case as follows:

For the singular case, we first define the function spaces. For $0 < s \leq \frac{1}{2}$ define $A_s := \{x \in \mathbb{R}^n : s < |x| < 2s\}$ and for $\sigma \in \mathbb{R}$ and $n < t < \infty$ define the spaces $\dot{Y} = Y_{t,\sigma}$ and $X = X_{t,\sigma}$ with norms given by

$$\begin{aligned}
\|f\|_Y^t &:= \sup_{0 < s \leq \frac{1}{2}} s^{(2+\sigma)t-n} \int_{A_s} |f(x)|^t dx \\
\|\phi\|_X^t &:= \sup_{0 < s \leq \frac{1}{2}} s^{\sigma t-n} \left\{ \int_{A_s} |\phi|^t dx + s^t \int_{A_s} |\nabla \phi|^t dx + s^{2t} \int_{A_s} |D^2 \phi|^t dx \right\}
\end{aligned}$$

where for the space X we impose the boundary condition $\phi = 0$ on ∂B_1 . We now define the closed subspaces of X and Y respectively X_1, Y_1 where we remove the first mode. So to define this properly we need to introduce the spherical harmonics.

Consider the Laplace-Beltrami operator $\Delta_{S^{n-1}} = \Delta_\theta$ on S^{n-1} and the eigenpairs

$$-\Delta_\theta \psi_k(\theta) = \lambda_k \psi_k(\theta), \quad \theta \in S^{n-1},$$

and note that $\lambda_0 = 0, \psi_0 = 1$ (multiplicity 1); $\lambda_1 = n - 1$ with multiplicity n and $\lambda_2 = 2n$. Given $\phi \in X, f \in Y$ we write

$$\phi(x) = \sum_{k=0}^{\infty} a_k(r) \psi_k(\theta), \quad f(x) = \sum_{k=0}^{\infty} b_k(r) \psi_k(\theta),$$

and so we define

$$X_1 := \left\{ \phi \in X : \phi(x) = \sum_{k=1}^{\infty} a_k(r) \psi_k(\theta) \right\}, \quad \text{note there is no } k = 0 \text{ mode}$$

and analogous for Y . Note we are abusing notation by not showing the correct multiplicity for modes which have multiplicity greater than one; but this isn't an issue for the procedures we perform. For $\gamma, \alpha \in \mathbb{R}$ we define the operator

$$L(\phi)(x) := L_{\gamma, \alpha}(\phi)(x) := \Delta \phi(x) + \gamma \phi_{rr}(x) + \frac{\alpha}{r} \phi_r(x). \quad (2.1)$$

Note we can write the operator as

$$L_{\gamma, \alpha}(\phi)(x) = \Delta \phi + \gamma \sum_{i,j=1}^n \frac{x_i x_j}{|x|^2} \phi_{x_i x_j} + \frac{\alpha}{|x|^2} x \cdot \nabla \phi(x).$$

In this section we will prove various results regarding this operator $L = L_{\gamma, \alpha}$. For explicit values of γ, α this operator $L_{\gamma, \alpha}$ will be exactly the operator $\tilde{L}$ from the previous section. In this section the values of γ, α, σ will satisfy a few constraints but are otherwise arbitrary. These constraints are: Take $1 < p < n$, $1 + \frac{(p-1)}{n-1} < q < 2$ and set

$$\begin{aligned} \gamma &:= p - 2, \quad \sigma := \frac{2-q}{q-1}, \\ \alpha &:= 2(n-1) - 2(p-1)(\sigma+1) - pC(q+2)\sigma^{q-1}, \end{aligned} \quad (2.2)$$

Note that by the above α , one can get $n - 2 - \gamma + \alpha = \frac{p-1}{q-1} - (n-1)(q-1)$ and a computation shows that $n - 2 - \gamma + \alpha$ changes sign in the interval $1 + \frac{(p-1)}{n-1} < q < 2$. Now, similar to [3] one can prove the following theorem.

Theorem 1. Suppose $-1 < \gamma < n - 2$ and $0 < \sigma < -\beta_1^-$. Then there is some $C > 0$ such that for all $f \in Y_1$ there is some $\phi \in X_1$ such that $L_{\gamma, \alpha}(\phi) = f$ in $B_1 \setminus \{0\}$ and $\|\phi\|_X \leq C\|f\|_Y$.

For the following lemma we use the exact values of the parameters.

Lemma 1. (Onto estimate for $k = 0$ mode) Suppose the parameters satisfy (2.2) and set $\beta := \frac{n-1+\alpha}{1+\gamma}$ (which implies $\beta - \sigma - 1 < 0$). There is some $C_0 > 0$ such that for all b_0 there is some a_0 which satisfies

$$(1 + \gamma)a_k''(r) + \frac{(n-1+\alpha)a_k'(r)}{r} - \frac{\lambda_k a_k(r)}{r^2} = b_k(r) \quad 0 < r < 1 \quad (2.3)$$

with $a_k(1) = 0$, for $k = 0$ and $\|a_0\|_X \leq C_0 \|b_0\|_Y$.

Corollary 1. Suppose the parameters satisfy (2.2). Then there is some $C > 0$ such that for all $f \in Y$ there is some $\phi \in X$ which satisfies $L_{\gamma,\alpha}(\phi) = f$ in $B_1 \setminus \{0\}$ with $\phi = 0$ on ∂B_1 . Moreover, one has $\|\phi\|_X \leq C \|f\|_Y$.

For the Hölder continuous case, we examine the needed linear theory to linearize around the radial Hölder continuous solution from Example 1 Case 2 where $w(r) = 1 - r^\sigma$ where $\sigma := \frac{q-2}{q-1}$. If one takes the same approach as in the singular case, they see we need to examine the operator $L = L_{\gamma,\alpha}$ (defined by (2.1)), where $\gamma := p - 2$, $\sigma := \frac{q-2}{q-1}$ and $\alpha := 2(n-1) + 2(p-1)(\sigma-1) - pC(q+2)\sigma^{q-1}$, where C defined in (1.5).

The spaces we work in are the same as before (again we have $n < t < \infty$) except now note the change of sign in front of σ ; define the spaces $Y = Y_{t,\sigma}$ and $X = X_{t,\sigma}$ with norms given by

$$\begin{aligned} \|f\|_Y^t &:= \sup_{0 < s \leq \frac{1}{2}} s^{(2-\sigma)t-n} \int_{A_s} |f(x)|^t dx \\ \|\phi\|_X^t &:= \sup_{0 < s \leq \frac{1}{2}} s^{-\sigma t-n} \left\{ \int_{A_s} |\phi|^t dx + s^t \int_{A_s} |\nabla \phi|^t dx + s^{2t} \int_{A_s} |D^2 \phi|^t dx \right\} \end{aligned}$$

where for the space X we impose the boundary condition $\phi = 0$ on ∂B_1 .

By a similar argument in [3], we have the following theorem.

Theorem 2. Let $n \geq 2$, $p > 1$ and $q > \max \left\{ 2, 1 + \frac{p-1}{n-1} \right\}$ and σ, γ, α be as above. Then there is some $C > 0$ such that for all $f \in Y$ there is some $\phi \in X$ which satisfies

$$L(\phi) = L_{\gamma,\alpha}(\phi) = f \quad \text{in } B_1 \setminus \{0\}, \quad \phi = 0 \quad \text{on } \partial B_1, \quad (2.4)$$

and $\|\phi\|_X \leq C \|f\|_Y$.

3. FIXED POINT THEORY

In this section we study the fixed point argument to show that in the singular and Hölder continuous cases J_ε has a fixed point. Due to do, we show that $J_\varepsilon(B_R) \subset B_R$ and $J_\varepsilon(B_R)$ is a contraction on B_r , where B_r is the closed ball of radius r centered at the origin in X .

For the singular case, we know that $\sigma = \frac{2-q}{q-1}$, $1 < q \leq 2$ and the norms given by

$$\|f\|_Y^t := \sup_{0 < s \leq \frac{1}{2}} s^{(2+\sigma)t-n} \int_{A_s} |f(x)|^t dx$$

and

$$\|\phi\|_X^t := \sup_{0 < s \leq \frac{1}{2}} s^{\sigma t - n} \left\{ \int_{A_s} |\phi|^t dx + s^t \int_{A_s} |\nabla \phi|^t dx + s^{2t} \int_{A_s} |D^2 \phi|^t dx \right\}.$$

Also a computation shows that

$$\frac{\Delta w}{|\nabla w|^2} = \sigma(\sigma + 2 - n)r^\sigma.$$

Recall we have defined $J_\varepsilon(\phi) = \psi$, where ψ satisfies (1.9). In order to obtain a solution ϕ of (1.7) we will show that J_ε is a contraction on B_r where B_r is the closed ball of radius r centered at the origin in X . First of all note that J_ε is into X .

Theorem 3. Assume $\phi \in B_R \subset X$. Then there exists C such that

$$\begin{aligned} (1) \quad & \left\| \frac{F_1(\phi)}{|\nabla w|^2} \right\|_Y^t \leq C \|\phi\|_X^{2t}. & (2) \quad & \left\| \frac{F_2(\phi)}{|\nabla w|^2} \right\|_Y^t \leq C \|\phi\|_X^{2t}. \\ (3) \quad & \left\| \frac{F_3(\phi)}{|\nabla w|^2} \right\|_Y^t \leq C \|\phi\|_X^{3t}. & (4) \quad & \left\| \frac{F_4(\phi)}{|\nabla w|^2} \right\|_Y^t \leq C \|\phi\|_X^{2t}. \\ (5) \quad & \left\| \frac{F_5(\phi)}{|\nabla w|^2} \right\|_Y^t \leq C \|\phi\|_X^{2t}. & (6) \quad & \left\| \frac{F_6(\phi)}{|\nabla w|^2} \right\|_Y^t \leq C \|\phi\|_X^{3t}. \\ (7) \quad & \left\| \frac{F_7(\phi)}{|\nabla w|^2} \right\|_Y^t \leq C \left(\|\phi\|_X^{2t} + \|\phi\|_X^{(q-p+4)t} \right). \\ (8) \quad & \left\| \frac{I_\varepsilon(\phi)}{|\nabla w|^2} \right\|_Y^t \leq C \varepsilon^t \left(1 + \|\phi\|_X^{(q-p+4)t} \right). \\ (9) \quad & \left\| \frac{H_\varepsilon(w + \phi)}{|\nabla w|^2} \right\|_Y^t \leq C \varepsilon^t \text{ for all } \phi \in B_R \text{ with } R \leq 1. \end{aligned}$$

Proof. The proof is straightforward. $\square$

Combining the above results we see that for $0 < R < 1$ chosen sufficiently small and then $\varepsilon > 0$ chosen sufficiently small we have $J_\varepsilon(B_R) \subset B_R$.

Contraction: We want to show that for small enough $\varepsilon > 0$ that J_ε is a contraction on $B_R \subset X$ for suitably (small) R . Let $J_\varepsilon(\phi) = \psi$ and $J_\varepsilon(\phi_0) = \psi_0$ with $\phi, \phi_0 \in B_r$. Note that

$$\tilde{L}(\psi) - \tilde{L}(\psi_0) = \sum_{k=1}^7 \frac{F_k(\phi) - F_k(\phi_0)}{|\nabla w|^2} + \frac{I_\varepsilon(\phi) - I_\varepsilon(\phi_0)}{|\nabla w|^2} + \frac{H_\varepsilon(w + \phi) - H_\varepsilon(w + \phi_0)}{|\nabla w|^2} \quad (3.1)$$

Theorem 4. $J_\varepsilon : B_R \rightarrow B_R$ is a contraction, where ε and R are small enough.

Proof. We have to show that for sufficiently small ε and R , $J_\varepsilon : B_R \rightarrow B_R$ is a contraction. In other words we need to show there exists a $k_{R,\varepsilon} < 1$ such that

$$\|J_t(\phi) - J_t(\phi_0)\|_Y \leq k_{R,\varepsilon} \|\phi - \phi_0\|_X.$$

We need to prove there exist $k_{R,\varepsilon}$ such that

$$\begin{aligned} \left\| \frac{F_k(\phi) - F_k(\phi_0)}{|\nabla w|^2} \right\|_Y &\leq k_{R,\varepsilon} \|\phi - \phi_0\|_X \quad \text{for } k = 1, 2, \dots, 7 \\ \left\| \frac{I_\varepsilon(\phi) - I_\varepsilon(\phi_0)}{|\nabla w|^2} \right\|_Y &\leq k_{R,\varepsilon} \|\phi - \phi_0\|_X \quad \text{and} \\ \left\| \frac{H_\varepsilon(w+\phi) - H_\varepsilon(w+\phi_0)}{|\nabla w|^2} \right\|_Y &\leq k_{R,\varepsilon} \|\phi - \phi_0\|_X. \end{aligned} \quad (3.2)$$

A computation shows that each of the above inequalities are hold. $\square$

Remark 1. Notice that $v > 0$. Firstly, $v \geq 0$ and since $v(x) = w(x) + \phi(x)$, the pointwise estimates on ϕ and it's gradient for $R > 0$ small enough implies $v \neq 0$ in B_1 .

By the same argument in the singular continuous case, for the Hölder continuous case one can show that for $0 < R < 1$ chosen sufficiently small and then $\varepsilon > 0$ chosen sufficiently small we have $J_\varepsilon(B_R) \subset B_R$, and for small enough $\varepsilon > 0$, J_ε is a contraction on $B_R \subset X$ for suitably (small) R . Thus the fixed point argument is working for the Hölder continuous case.

Thus, by combining the fixed point argument and the linear theory, we can state that the main result as follows:

Theorem 5. Assume $n \geq 2$.

- (1) Suppose p, q, n, σ, C are as in Example 1 part 1. Then for sufficiently small C^2 perturbations of the unit ball, say Ω_ε , there exists a positive singular weak solution u of (1.1) (with $\Omega = \Omega_\varepsilon$) which blows up at exactly one point x_ε (near the origin) and behaves like $u(x) \approx |x - x_\varepsilon|^{-\sigma}$ near x_ε . The proof gives the exact behaviour near x_ε .
- (2) Suppose p, q, n, σ, C are as in Example 1 part 2. Then for sufficiently small C^2 perturbations of the unit ball, say Ω_ε , there exists a positive weak solution u of (1.1) (with $\Omega = \Omega_\varepsilon$) with $u \in C^\infty(\Omega_\varepsilon \setminus \{x_\varepsilon\})$ and with $u \in C^{0,\sigma}(\overline{\Omega_\varepsilon})$. In addition u is not in $C^{0,\sigma+\delta}(\overline{\Omega_\varepsilon})$ for any $\delta > 0$.

4. MORE GENERAL CASE

By a similar argument in the above sections, one can study the existence of positive singular solutions of

$$\begin{cases} -\operatorname{div}(|x|^\alpha |\nabla u|^\beta \nabla u) + \varepsilon h(x) \cdot \nabla u = |x|^\gamma |\nabla u|^q + \varepsilon_1 u^\eta + \varepsilon_2 g(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.1)$$

where Ω is a small C^2 perturbation of the unit ball in $\mathbb{R}^n$, $\eta > 0$ is a positive constant, and $\varepsilon, \varepsilon_1, \varepsilon_2$ are small enough. Assume there exist C_0 such that $|h(x)| \leq C_0 |x|^{\sigma+1}$ and $|g_i(x)| \leq C_0 |x|^{\sigma+2}$, $i = 1, 2$. Under suitable conditions on α, β, γ and q , one can prove

that if Ω is a sufficiently small C^2 perturbation of the unit ball there exists a singular positive weak solution u of (4.1). For other ranges of α, β, γ and q , one may prove the existence of Hölder continuous positive solution (with optimal regularity) on a C^2 perturbation of the unit ball.

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AN INTEGRAL EQUATION IN PARTIALLY ORDERED SETS

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Abstract. Here, some fixed point theorems in partially ordered spaces are proved. As an application, the existence of a solution of an integral equation is obtained.

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1. INTRODUCTION

Fixed point problems are interesting problems in Nonlinear Analysis and Engineering Sciences. There is a broad set of applications of fixed point problems, especially in solving differential equations and integral equations. Thus fixed point theory is an interdisciplinary science and many researchers are studying them in different spaces (see [1–6, 9–11, 13–18]).

The existence of fixed points in partially ordered sets was first considered in [12] to solve an matrix equation. This study was continued in [7, 8] by assuming the existence of only a lower solution instead of the usual approach where both lower and upper solutions are assumed to exist. These fixed point theorems were applied to obtain certain uniqueness and existence results for ordinary differential equations in [7, 8].

Definition 1. A partial ordering on a nonempty set X is a relation R on X with the following properties:

- (i) xRx for all $x \in X$,
- (ii) if xRy and yRx then $x = y$,
- (iii) if xRy and yRz then xRz .

Definition 2. Let $(X, \leq)$ be a partially ordered set and $f : X \rightarrow X$. f is called monotone nondecreasing if $x \leq y \Rightarrow f(x) \leq f(y)$, where $x, y \in X$.

Ran et al. [12, Theorem 2.1] proved the following theorem.

Theorem 1. Let X be a partially ordered set such that every pair $x, y \in X$ has a lower and an upper bound. Furthermore, let d be a metric on X such that (X, d) is

a complete metric space. If $f : X \rightarrow X$ is a continuous, monotone (i.e., either order preserving or order-reversing) operator such that

$$\exists 0 < c < 1 \text{ } d(f(x), f(y)) \leq cd(x, y), \forall x \geq y,$$

$$\exists x_0 \in X \text{ } x_0 \leq f(x_0) \text{ or } x_0 \geq f(x_0),$$

then f has a unique fixed point $\bar{x}$. Moreover, for every $x \in X$, $\lim_{n \rightarrow \infty} f^n(x) = \bar{x}$.

Later on, Petruşel et al. [10, Theorem 1.2] by dropping the hypothesis that each pair of points has an upper and a lower bound, proved the following theorem.

Theorem 2. Let X be a partially ordered set and let d be a metric on X such that the metric space (X, d) is complete and the metric and the ordered structure are compatible. Let $f : X \rightarrow X$ be a continuous and monotone (i.e. either order-preserving or order-reversing) operator. Suppose that the following two assertions hold:

- 1) there exists $a \in (0, 1)$ such that $d(f(x), f(y)) \leq ad(x, y)$ for each $x, y \in X$ with $x \geq y$,
- 2) there exists $x_0 \in X$ such that $x_0 \leq f(x_0)$ or $x_0 \geq f(x_0)$.

Then f has at least a fixed point $x^* \in X$ and for each $x \in X$ and for each $x \in X$ with $x \geq x_0$ (or $x \leq x_0$) the sequence $(f^n(x))_{n \in \mathbb{N}}$ of successive approximations of f starting from x converges to $x^* \in X$.

Also Nieto et al. [7, 8, Theorem 2.1] proved the following theorem.

Theorem 3. Let $(X, \leq)$ be a partially ordered set. Assume there exists a metric d in X such that (X, d) is a complete metric space. Suppose $f : X \rightarrow X$ is a continuous and nondecreasing mapping such that there exists a $k \in [0, 1)$ with

$$d(f(x), f(y)) \leq kd(x, y),$$

for all $x \geq y$. If there exists $x_0 \in X$ with $x_0 \leq f(x_0)$, then f has a fixed point.

In this paper, we extend the fixed point theorem [7, 8, Theorem 2.1] in partially ordered sets. Finally, an integral equation is solved and an example is stated. The main result of the paper is as follows:

Theorem 4. Let $(X, \leq)$ be a partially ordered set. Assume there exists a metric d in X such that (X, d) is a complete metric space. Suppose $f : X \rightarrow X$ is a continuous and nondecreasing mapping satisfying

$$d(f(x), f(y)) \leq \alpha(d(x, y))d(x, y),$$

where $\alpha : [0, \infty) \rightarrow [0, 1)$ and $\limsup_{s \rightarrow t+} \alpha(s) < 1$ for all $t \in [0, \infty)$. If there exists $x_0 \in X$ with $x_0 \leq f(x_0)$, then f has a fixed point.

This theorem can be considered as a new version generalization of Theorem 3. Indeed, the contractive constant is replaced by a function.

2. FIXED POINT THEOREMS IN PARTIALLY ORDERED SET

In this section, we present the proof of the main result and an alternative version of the main result. Then we study the existence of a solution of an integral equation. By induction of proof of Theorem 3 or [7, 8, Theorem 2.1], we state the proof of Theorem 4.

Proof of Theorem 4. . If $x_0 = f(x_0)$, then the proof is finished. If $x_0 \neq f(x_0)$ then $x_0 \leq f(x_0)$ and since f is nondecreasing we have

$$x_0 \leq f(x_0) \leq f^2(x_0) \leq f^3(x_0) \leq \cdots \leq f^n(x_0) \leq f^{n+1}(x_0) \leq \cdots.$$

Now

$$d(f^n(x_0), f^{n+1}(x_0)) \leq \alpha(d(f^{n-1}(x_0), f^n(x_0)))d(f^{n-1}(x_0), f^n(x_0)).$$

If $f^n(x_0) = f^{n+1}(x_0)$ then $f^n(x_0)$ is a fixed point of f . Otherwise, since $\alpha(t) < 1$, we have

$$d(f^n(x_0), f^{n+1}(x_0)) < d(f^{n-1}(x_0), f^n(x_0)).$$

Thus $\{d(f^n(x_0), f^{n+1}(x_0))\}$ is a nonnegative decreasing sequence. Thus it converges to a , where $a \geq 0$, that is

$$d(f^n(x_0), f^{n+1}(x_0)) \rightarrow a.$$

Since $\limsup_{s \rightarrow t+} \alpha(s) < 1$, then $\alpha(a) < 1$. Therefore, for all $\varepsilon > 0$ there exists $r \in [0, 1)$ such that $\alpha(s) < r$ for all $s \in [a, a + \varepsilon)$. Now, choose $v \in \mathbb{N}$ such that

$$a \leq d(f^n(x_0), f^{n+1}(x_0)) \leq a + \varepsilon$$

for all $n \geq v$. Since

$$\begin{aligned} d(f^{n+1}(x_0), f^{n+2}(x_0)) &\leq \alpha(d(f^n(x_0), f^{n+1}(x_0)))d(f^n(x_0), f^{n+1}(x_0)) \\ &\leq rd(f^n(x_0), f^{n+1}(x_0)), \end{aligned}$$

thus by induction

$$d(f^{n+v}(x_0), f^{n+v+1}(x_0)) \leq r^n d(f^v(x_0), f^{v+1}(x_0)). \quad (2.1)$$

In order to show that $\{f^n(x_0)\}$ is a Cauchy sequence, consider

$$\begin{aligned} \sum_{n=1}^{n=\infty} d(f^n(x_0), f^{n+1}(x_0)) &\leq \sum_{n=1}^{n=v} d(f^n(x_0), f^{n+1}(x_0)) \\ &\quad + \sum_{n=v}^{n=\infty} d(f^n(x_0), f^{n+1}(x_0)). \end{aligned}$$

Using (2.1), the second part of the right hand side of the above inequality is estimated as follows:

$$\sum_{n=v}^{n=\infty} d(f^n(x_0), f^{n+1}(x_0)) \leq \sum_{n=1}^{n=\infty} r^n d(f^v(x_0), f^{v+1}(x_0))$$

$$= d(f^v(x_0), f^{v+1}(x_0)) \sum_{n=1}^{n=\infty} r^n \leq \infty.$$

This shows that $\sum_{n=1}^{n=\infty} d(f^n(x_0), f^n(x_0))$ is convergent which implies

$$d(f^n(x_0), f^{n+1}(x_0)) \rightarrow 0, \quad (2.2)$$

Notice that the convergence of $\sum_{n=1}^{n=\infty} d(f^n(x_0), f^n(x_0))$ and (2.2) shows that $\{f^n(x_0)\}$ is a Cauchy sequence. Since X is complete, there exists a q in X such that $f^n(x_0) \rightarrow q$ as $n \rightarrow \infty$. We prove that $q \in X$ is a fixed point of f . Let $\varepsilon > 0$. Using the continuity of f at the point q , given $\varepsilon/2 > 0$, there exists $\delta > 0$ such that $d(z, q) < \delta$ implies that $d(f(z), f(q)) < \varepsilon/2$. Now, by the convergence of $\{f^n(x_0)\}$ to q , assume $\eta = \min\{\varepsilon/2, \delta\}$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$d(f^n(x_0), q) < \eta.$$

Then

$$\begin{aligned} d(f(q), q) &\leq d(f(q), f^n(x_0)) + d(f^n(x_0), q) \\ &\leq \varepsilon/2 + \eta = \varepsilon. \end{aligned}$$

This proves that $d(f(q), q) = 0$, and q is a fixed point of f . $\square$

Next theorem is an alternative version of Theorem 4.

Theorem 5. *Let $(X, \leq)$ be a partially ordered set. Assume there exists a metric d in X such that (X, d) is a complete metric space. Suppose $f : X \rightarrow X$ is a continuous and nondecreasing mapping satisfying*

$$d(f(x), f(y)) \leq \varphi(d(x, y)),$$

where $\varphi : [0, \infty) \rightarrow [0, \infty)$ is an increasing and upper semicontinuous function such that $\varphi(t) < t$ for all $t > 0$. If there exists $x_0 \in X$ with $x_0 \leq f(x_0)$, then f has a fixed point.

Proof. If $x_0 = f(x_0)$ then the proof is finished. If $x_0 \neq f(x_0)$ then $x_0 \leq f(x_0)$ and since f is nondecreasing we have

$$x_0 \leq f(x_0) \leq f^2(x_0) \leq f^3(x_0) \leq \cdots \leq f^n(x_0) \leq f^{n+1}(x_0) \leq \cdots.$$

Using $\varphi(t) < t$,

$$\begin{aligned} d(f^n(x_0), f^{n+1}(x_0)) &\leq \varphi(d(f^{n-1}(x_0), f^n(x_0))) \\ &< d(f^{n-1}(x_0), f^n(x_0)). \end{aligned}$$

Thus $\{d(f^n(x_0), f^{n+1}(x_0))\}$ is a nonnegative decreasing sequence and then it converges to a as $n \rightarrow \infty$. We show $a = 0$. If not, since φ is upper semicontinuous, we have

$$a = \lim d(f^n(x_0), f^{n+1}(x_0))$$

$$\leq \lim \varphi(d(f^{n-1}(x_0), f^n(x_0))) \leq \varphi(a).$$

This means $a \leq \varphi(a)$, which is a contradiction. Now, we prove $\{f^n(x_0)\}$ is a Cauchy sequence. Suppose not, there exists $\varepsilon > 0$ such that for all $r \in \mathbb{N}$, there exists $m_r > n_r \geq r$ such that

$$d(f^{m_r}(x_0), f^{n_r}(x_0)) \geq \varepsilon. \quad (2.3)$$

We can suppose that for all r , m_r is the smallest number greater than n_r such that (2.3) holds. Therefore for such k we have

$$\begin{aligned} \varepsilon &\leq d(f^{m_r}(x_0), f^{n_r}(x_0)) \\ &\leq d(f^{m_r}(x_0), f^{m_{r-1}}(x_0)) + d(f^{m_{r-1}}(x_0), f^{n_r}(x_0)) \\ &\leq d(f^{m_r}(x_0), f^{m_{r-1}}(x_0)) + \varepsilon. \end{aligned}$$

Hence

$$\lim_{r \rightarrow \infty} d(f^{m_r}(x_0), f^{n_r}(x_0)) = \varepsilon^+,$$

thus we have

$$\begin{aligned} d(f^{m_r}(x_0), f^{n_r}(x_0)) &\leq d(f^{m_r}(x_0), f^{m_{r+1}}(x_0)) + d(f^{m_{r+1}}(x_0), f^{n_{r+1}}(x_0)) \\ &\quad + d(f^{n_{r+1}}(x_0), f^{n_r}(x_0)) \\ &\leq d(f^{m_{r+1}}(x_0), f^{m_r}(x_0)) \\ &\quad + d(f^{n_{r+1}}(x_0), f^{n_r}(x_0)) + \varphi(d(f^{m_r}(x_0), f^{n_r}(x_0))). \end{aligned}$$

It follows that $\varepsilon \leq \varphi(\varepsilon)$, and this a contradiction. Hence $\{f^n(x_0)\}$ is a Cauchy sequence. Since X is complete, there exists a q in X such that $f^n(x_0) \rightarrow q$ as $n \rightarrow \infty$. The rest of the proof is like as the previous theorem. $\square$

Remark 1. Consider the integral equation

$$y(x) = \int_0^x k(x, s)g(s, y(s))ds, \quad (2.4)$$

where I is a closed interval in $\mathbb{R}$, $k : I \times R \rightarrow R^+$ such that $\sup_{x \in I} \int_0^x k(x, s)ds \leq 1$ and $g : I \times R \rightarrow R$ is increasing with respect to the second component and $|g(x, y) - g(x, z)| \leq \varphi(d(y, z))$ (for all $x \in I$ and $y \leq z$) where $\varphi : R^+ \rightarrow R^+$ is an increasing and upper semicontinuous function such that $\varphi(x) < x$ for all $x > 0$. Assume there exists y_0 such that $y_0(x) \leq \int_0^x k(x, s)g(s, y_0(s))ds$. Then by Theorem 5 the integral equation (2.4) has a solution in $C(I, R)$.

By the main result we are interested in solving the ordinary differential equation

$$y'(x) = 1 + \log(x+1) \sin((\pi y)/x). \quad (2.5)$$

Set $I = [0, 1]$ and $X = C(I, R)$. We define $d(y, z) = \sup\{|y(x) - z(x)| : x \in I\}$ for every $y, z \in X$ and an order relation in X as

$$y \leq z \text{ if and only if } y(x) \leq z(x) \text{ for all } x \in I$$

where $y, z \in X$. It is obvious that $(X, \leq)$ is a partially ordered set and (X, d) is a complete metric space. Define $T : X \rightarrow X$ by

$$Ty(x) := \int_0^x g(s, y(s)) ds, \quad (2.6)$$

where $g : I \rightarrow \mathbb{R}$ is

$$g(x, y) := \begin{cases} 1 + \log(x+1) \sin((\pi y)/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Since the mapping g is increasing with respect to the second component, T is non-decreasing. Notice that $|g(x, y) - g(x, z)| \leq \log |y - z|$ for every $x \in I$ and $y \leq z$. Also $\log$ is an increasing function, $\log(x) < x$ for $x > 0$ and an upper semicontinuous function. In addition, for $y \leq z$

$$\begin{aligned} d(T(y), T(z)) &= \sup_{x \in I} |[Ty](x) - [Tz](x)| \\ &\leq \sup_{x \in I} \int_0^x |g(s, y(s)) - g(s, z(s))| ds \\ &\leq \sup_{x \in I} \int_0^x \log(|y(s) - z(s)|) ds \\ &\leq \log(\sup |y - z|) \\ &= \log d(y, z). \end{aligned}$$

If there exists y_0 such that

$$y_0(x) \leq \int_0^x g(s, y_0(s)) ds.$$

According to the Theorem 5, T has a fixed point and thus the integral equation (2.6) has a solution in $C(I, \mathbb{R})$. Now consider the following integral equation

$$y(x) = \int_0^x 1 + \log(\xi + 1) \sin((\pi y)/\xi) d\xi.$$

Notice that since this integral equation has a solution in $[0, 1]$ then equivalently (2.5) has a solution. According to numerical method this solution is $y = x$. Notice that this solution satisfies in (2.5).

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SOME APPLICATIONS OF FIRST-ORDER DIFFERENTIAL SUBORDINATIONS FOR HOLOMORPHIC FUNCTIONS IN COMPLEX NORMED SPACES

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Abstract. In Geometric Function Theory of Complex Analysis, there have been many interesting and fruitful usages of a wide variety of differential subordinations for holomorphic functions in the unit disk $\mathbb{U}$:

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Here, in this article, we derive some properties of the first-order differential subordinations for holomorphic functions which are defined in the unit ball $\mathbb{B}$:

$$\mathbb{B} = \{z : z \in \mathbb{C}^n \text{ and } \|z\| < 1\}$$

by using a certain class of admissible functions. We also make use of the theory of biholomorphic functions in our investigation here.

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Keywords: holomorphic functions, differential subordinations, admissible functions, normed spaces, biholomorphic functions, unit disk, unit ball

1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

We first introduce some of the notation that will be used in this paper. The symbol v' represents the transpose of the vector v . We denote by $\mathbb{C}^n$ the space of n complex variables $z = (z_1, z_2, \dots, z_n)'$ with an arbitrary norm $\|\cdot\|$. The origin $(0, 0, \dots, 0)$ is denoted by 0 .

Let $\mathbb{B}$ given by

$$\mathbb{B} = \{z : z \in \mathbb{C}^n \text{ and } \|z\| < 1\}$$

be the unit ball in $\mathbb{C}^n$. and Also let $\mathcal{H}(\mathbb{B})$ denote the class of holomorphic functions $f : \mathbb{B} \longrightarrow \mathbb{C}^n$ which have the following form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1.1)$$

We now review the principle of subordination between holomorphic functions in the normed space $(\mathbb{C}^n, \|\cdot\|)$. Let the functions f and g be members of the class $\mathcal{H}(\mathbb{B})$.

Then the function f is said to be subordinate to g , if there exists a function $\varphi \in \mathcal{H}(\mathbb{B})$ with

$$\varphi(0) = 0 \quad \text{and} \quad \|\varphi(z)\| < 1 \quad (z \in \mathbb{B}),$$

such that

$$f(z) = g(\varphi(z)).$$

This subordination is denoted by

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \mathbb{B}).$$

Further, if the function g is *biholomorphic* in $\mathbb{B}$, then

$$f \prec g \quad (z \in \mathbb{B}) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{B}) \subseteq g(\mathbb{B}).$$

Formally, a biholomorphic function is a function ϕ defined on an open subset U of the n -dimensional complex space $\mathbb{C}^n$ with values in $\mathbb{C}^n$, which is holomorphic and one-to-one, such that its image is an open set V in $\mathbb{C}^n$ and the inverse $\phi^{-1} : V \rightarrow U$ is also holomorphic (see, for details, [6]).

We next define the classes of starlike and convex functions in the unit ball $\mathbb{B}$ as follows.

Definition 1. A function $f \in \mathcal{H}(\mathbb{B})$ with $f(z) \neq 0$ is said to be starlike in $\mathbb{B}$, if it satisfies the following condition:

$$\Re \left(\frac{zf'(z)}{f(z)} \right) > 0 \quad (z \in \mathbb{B}).$$

Furthermore, a function $f \in \mathcal{H}(\mathbb{B})$ with $f'(z) \neq 0$ is said to be convex in $\mathbb{B}$, if it satisfies the following condition:

$$\Re \left(\frac{zf''(z)}{f'(z)} + 1 \right) > 0 \quad (z \in \mathbb{B}).$$

Our next definitions (Definitions 2, 3 and 4 below) make use of biholomorphic functions.

Definition 2 ([6]). Let $\psi : \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{B} \rightarrow \mathbb{C}$. Also let h be a biholomorphic function in $\mathbb{B}$. If p is holomorphic in $\mathbb{B}$ and satisfies the first-order differential subordination given by

$$\psi(p(z), zp'(z); z) \prec h(z) \quad (z \in \mathbb{B}), \quad (1.2)$$

then p is called a solution of the differential subordination (1.2). The function q is called a dominant of the solutions of the differential subordination or, more simply, a dominant if $p(z) \prec q(z)$ for all solution p of (1.2). A dominant $\check{q}$ that satisfies $\check{q}(z) \prec q(z)$ for all dominants q of (1.2) is said to be the best dominant.

Definition 3 ([6]). Let $k \geq 1$ and $\xi \in \partial\mathbb{B}$. Also let q be a biholomorphic function in $\mathbb{B}_\rho$ with $\rho > 1$. Then the subset $Q_k(q, \xi)$ of $\mathbb{C}^n \times \mathbb{C}^n$ is defined by

$$Q_k(q, \xi) = \left\{ (u, v) : u = q(\xi) \quad \text{and} \quad k \left\| (q'(\xi))^{-1} \right\|^{-1} \leq \|v\| \leq k \|q'(\xi)\| \right\}.$$

And the subset $Q(q)$ of $\mathbb{C}^n \times \mathbb{C}^n$ is defined by

$$Q(q) = \bigcup \{ Q_k(q, \xi) : k \geq 1 \quad (\xi \in \partial\mathbb{B}) \}.$$

Definition 4 ([6]). Let $\Omega \subset \mathbb{C}^n$ and q be a biholomorphic function in $\mathbb{B}_\rho$ with $\rho > 1$. The class of admissible functions $\Psi[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{B} \rightarrow \mathbb{C}$ that satisfy the conditions $\psi(q(0), 0; 0) \in \Omega$ and $\psi(u, v; z) \notin \Omega$ when $(u, v) \in Q(q)$ and $z \in \mathbb{B}$.

The following lemma will be used in proving our main results.

Lemma 1 ([6]). Let $\psi \in \Psi[\Omega, q]$ and suppose that $p \in \mathcal{H}(\mathbb{B})$ with $p(0) = q(0)$. If

$$\psi(p(z), zp'(z); z) \in \Omega,$$

then $p \prec q$.

The existing literature in Geometric Function Theory of Complex Analysis contains a considerably large number of interesting investigations dealing with differential subordination problems for analytic functions in the unit disk $\mathbb{U}$ (see, for example, [1–5, 7–18]). In particular, in the recently-published survey-cum-expository review article by Srivastava [7], the so-called (p, q) -calculus was exposed to be a rather trivial and inconsequential variation of the classical q -calculus, the additional parameter p being redundant or superfluous (see, for details, [7, p. 340]). In the present work, we determine certain appropriate classes of admissible functions and investigate some first-order differential subordination properties of holomorphic functions defined in unit ball $\mathbb{B}$. We also make use of the theory of biholomorphic functions in our investigation here.

2. A SET OF MAIN RESULTS

In this section, we first give the following definition.

Definition 5. Let $\Omega \subset \mathbb{C}^n$ and q be a biholomorphic function in $\mathbb{B}_\rho$ with $\rho > 1$. The class of admissible functions $\Phi_{\mathbb{B}}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{B} \rightarrow \mathbb{C}$ that satisfy the conditions $\phi(1, 0; 0) \in \Omega$ and $\phi(s, t; z) \notin \Omega$ when $(s, t) \in Q(q)$ and $z \in \mathbb{B}$.

Theorem 1. Let $\phi \in \Phi_{\mathbb{B}}[\Omega, q]$. If $f \in \mathcal{H}(\mathbb{B})$ satisfies the following inclusion relation:

$$\left\{ \phi \left(\frac{zf'(z)}{f(z)}, \frac{zf''(z)}{f'(z)} + 1; z \right) \quad (z \in \mathbb{B}) \right\} \subset \Omega, \quad (2.1)$$

then

$$\frac{zf'(z)}{f(z)} \prec q(z).$$

Proof. For a given $f \in \mathcal{H}(\mathbb{B})$, we define the function p by

$$p(z) = \frac{zf'(z)}{f(z)}. \quad (2.2)$$

It is obvious that the function p is holomorphic in $\mathbb{B}$ and $p(0) = 1$. By simple calculations using (2.2), we see that

$$p(z) + \frac{zp'(z)}{p(z)} = \frac{zf''(z)}{f'(z)} + 1. \quad (2.3)$$

We now define the transformation from $\mathbb{C}^n \times \mathbb{C}^n$ to $\mathbb{C}$ by

$$s(u, v) = u \quad \text{and} \quad t(u, v) = u + \frac{v}{u}.$$

We also set

$$\Psi(u, v; z) = \Phi(s, t; z) = \Phi\left(u, u + \frac{v}{u}; z\right). \quad (2.4)$$

If we use the equations (2.2) and (2.3), it follows from (2.4) that

$$\Psi(p(z), zp'(z); z) = \Phi\left(\frac{zf'(z)}{f(z)}, \frac{zf''(z)}{f'(z)} + 1; z\right). \quad (2.5)$$

In view of (2.1) and (2.5), we thus find that

$$\Psi(p(z), zp'(z); z) \in \Omega.$$

Hence, from (2.4), we observe that the admissibility condition for $\Phi \in \Phi_{\mathbb{B}}[\Omega, q]$ in Definition 5 is equivalent to the admissibility condition for Ψ as given in Definition 4. Thus, clearly, $\Psi \in \Psi[\Omega, q]$ and, by applying the Lemma in Section 1, we obtain $p(z) \prec q(z)$ or, equivalently,

$$\frac{zf'(z)}{f(z)} \prec q(z).$$

This completes the proof of Theorem 1. $\square$

We next consider the special situation when there is a biholomorphic mapping h from $\mathbb{B}$ onto Ω . In this case, $\Omega = h(\mathbb{B})$ and the class $\Phi_{\mathbb{B}}[h(\mathbb{B}), q]$ is written as $\Phi_{\mathbb{B}}[h, q]$. The following result is an immediate consequence of Theorem 1.

Theorem 2. Let $\Phi \in \Phi_{\mathbb{B}}[h, q]$ and $f \in \mathcal{H}(\mathbb{B})$. If

$$\Phi\left(\frac{zf'(z)}{f(z)}, \frac{zf''(z)}{f'(z)} + 1; z\right)$$

is holomorphic in $\mathbb{B}$ and

$$\Phi\left(\frac{zf'(z)}{f(z)}, \frac{zf''(z)}{f'(z)} + 1; z\right) \prec h(z), \quad (2.6)$$

then

$$\frac{zf'(z)}{f(z)} \prec q(z).$$

The next result is an extension of Theorem 1 to the case when the behaviour of q on the boundary of $\mathbb{B}$ is not known.

Theorem 3. *Let the function q be biholomorphic on $\mathbb{B}$. Also let $\phi \in \Phi_{\mathbb{B}}[\Omega, q_{\eta}]$ for some $\eta \in (0, 1)$, where $q_{\eta}(z) = q(\eta z)$. Also let $f \in \mathcal{H}(\mathbb{B})$. If*

$$\phi\left(\frac{zf'(z)}{f(z)}, \frac{zf''(z)}{f'(z)} + 1; z\right)$$

is holomorphic in $\mathbb{B}$ and

$$\phi\left(\frac{zf'(z)}{f(z)}, \frac{zf''(z)}{f'(z)} + 1; z\right) \in \Omega,$$

then

$$\frac{zf'(z)}{f(z)} \prec q(z).$$

Proof. Since q is biholomorphic on $\mathbb{B}$, the function q_{η} is biholomorphic on $\mathbb{B}_{\rho}$ for some $\rho > 1$. Thus, by applying Theorem 1, we obtain

$$\frac{zf'(z)}{f(z)} \prec q_{\eta}(z).$$

The result asserted by Theorem 3 is now deduced from the fact that $q_{\eta}(z) \prec q(z)$. $\square$

The next result gives the best dominant of the differential subordination (2.6).

Theorem 4. *Let h be a biholomorphic function on $\mathbb{B}_{\rho}$ with $\rho > 1$. Also let $\phi : \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{B} \rightarrow \mathbb{C}$. Suppose that the following differential equation:*

$$\phi\left(q(z), q(z) + \frac{zq'(z)}{q(z)}; z\right) = h(z) \quad (2.7)$$

has a solution q with $q(0) = 1$ and satisfies one of the following conditions:

- (1) $\phi \in \Phi_{\mathbb{B}}[h, q]$;
- (2) $\phi \in \Phi_{\mathbb{B}}[h, q_{\eta}]$ for some $\eta \in (0, 1)$.

If $f \in \mathcal{H}(\mathbb{B})$ and

$$\phi\left(\frac{zf'(z)}{f(z)}, \frac{zf''(z)}{f'(z)} + 1; z\right)$$

is holomorphic in $\mathbb{B}$, then the differential subordination (2.6) implies that

$$\frac{zf'(z)}{f(z)} \prec q(z)$$

and q is the best dominant.

Proof. By applying Theorem 1 and Theorem 3, we deduce that q is a dominant of the differential subordination (2.6). Since q satisfies (2.7), it is also a solution of (2.6) and hence q is the best dominant among all dominants of (2.6). $\square$

In the particular case, we introduce a class of admissible functions, which deals with differential subordinations with dominants of the form $q : \mathbb{B} \rightarrow \mathbb{C}^n$ given by $q(z) = Mz$, where M is a positive constant. In this case, q is a biholomorphic function on $\mathbb{B}$ and $q(\mathbb{B})$ is a ball in $\mathbb{C}^n$. The class $\Phi_{\mathbb{B}}[\Omega, Mz]$, which is denoted simply by $\Phi_{\mathbb{B}}[M]$, can be expressed in the following form.

Definition 6. Let $\Omega \subset \mathbb{C}^n$ and let M be a positive constant. The class $\Phi_{\mathbb{B}}[M]$ of admissible functions consists of those functions $\phi : \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{B} \rightarrow \mathbb{C}^n$ that satisfy the conditions $\phi(0, 0; 0) \in \Omega$ and $\phi(s, t; z) \notin \Omega$ when $(s, t) \in Q(M)$ and $z \in \mathbb{B}$, where

$$Q(M) \equiv Q(M, z) = \{(s, t) \in \mathbb{C}^n \times \mathbb{C}^n : \|s\| = M \text{ and } \|t\| \geq M\}.$$

Theorem 5. Let $\phi \in \Phi_{\mathbb{B}}[M]$. If $f \in \mathcal{H}(\mathbb{B})$ satisfies the following condition:

$$\phi(zf'(z), z^2f''(z); z) \in \Omega, \quad (2.8)$$

then

$$zf'(z) \prec Mz \quad (M > 0).$$

Proof. For given $f \in \mathcal{H}(\mathbb{B})$, we define the function p by

$$p(z) = zf'(z). \quad (2.9)$$

It is obvious that the function p is holomorphic in $\mathbb{B}$ and $p(0) = 1$. After some computations and by using (2.9), we conclude that

$$zp'(z) - p(z) = z^2f''(z). \quad (2.10)$$

We now define the transformation from $\mathbb{C}^n \times \mathbb{C}^n$ to $\mathbb{C}^n$ by

$$s(u, v) = u \quad \text{and} \quad t(u, v) = v - u.$$

We assume that

$$\psi(u, v; z) = \phi(s, t; z) = \phi(u, v - u; z). \quad (2.11)$$

In view of (2.9) and (2.10), we find from (2.11) that

$$\psi(p(z), zp'(z); z) = \phi(zf'(z), z^2f''(z); z). \quad (2.12)$$

Hence (2.8) becomes $\psi(p(z), zp'(z); z) \in \Omega$.

We next find from (2.11) that the admissibility condition for $\phi \in \Phi_{\mathbb{B}}[M]$ in Definition 6 is equivalent to the admissibility condition for ψ as given in Definition 4 and, by applying the Lemma in Section 1, we obtain $p(z) \prec Mz$ or, equivalently,

$$zf'(z) \prec Mz \quad (M > 0).$$

$\square$

3. CONCLUDING REMARKS AND OBSERVATIONS

In Geometric Function Theory of Complex Analysis, there are many interesting and fruitful usages of a wide variety of differential subordinations for holomorphic functions in the unit disk $\mathbb{U}$:

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Here, in this article, we have derived several properties of the first-order differential subordinations for holomorphic functions which are defined in the unit ball $\mathbb{B}$:

$$\mathbb{B} = \{z : z \in \mathbb{C}^n \text{ and } \|z\| < 1\}$$

by using a certain class of admissible functions. In our investigation, we have made use of biholomorphic functions as well. We have also indicated how one can deduce a number of special cases and consequences for our main results (Theorem 1 and Theorem 5).

We conclude our investigation by remarking that, in order to motivate further researches on the subject-matter of this article, we have chosen to draw the attention of the interested readers toward a considerably large number of related recent publications on the subjects which we have discussed here.

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LOCALLY GENERATED HERMITIAN C^1 -SPLINES ON TRIANGULAR MESHES

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Abstract. We classify all possible local linear procedures on triangular meshes resulting in polynomial C^1 -spline functions with affinely uniform shape conditions for the basic functions at the edges, and fitting the 9 value and gradient data at the vertices of the mesh triangles. There is a unique procedure among them with shape functions and basic polynomials of degree 5 and all other admissible procedures are its perturbations with higher degree.

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Keywords: Polynomial C^1 -spline, triangular mesh, gradient data.

1. INTRODUCTION

As for heuristical background: it is a well-known feature of 2-dimensional spline curves, that given a sequence $x_0 < x_1 < \dots < x_r$ of points on the real line $\mathbb{R}$ along with a sequence $[y_k, y'_k] : 0 \leq k \leq r$, there is a unique C^1 -function $f : [x_0, x_r] \rightarrow \mathbb{R}$ consisting of polynomials of 3-rd degree on the intervals $I_k := [x_{k-1}, x_k]$ ($k = 1, \dots, r$) such that $f(x_k) = y_k$, $f'(x_k) = y'_k$ ($k = 0, \dots, r$). Namely each restriction $f|_{I_k}$ is necessarily a Hermite interpolation polynomial fitting to two value and derivative data at the end points. With respect to the natural intrinsic coordinatization of the interval I_k , we can write $f((1-\lambda)x_{k-1} + \lambda x_k) = \Phi_1(\lambda)y_k + \Psi_1(\lambda)y'_k + \Phi_1(1-\lambda)y'_{k-1} - \Psi_1(1-\lambda)y'_{k-1}$ for $0 \leq \lambda \leq 1$ with the polynomials $\Phi_1(t) := t^2(3-2t)$ and $\Psi_1(t) = t^2(1-t)$. Thus, piecewise, f consists of linear combinations of affines copies of two standard polynomials of 3-rd degree. Hence even higher dimensional interpolating surfaces of Catmull-Rom type fitting gradient data on rectangular meshes [5, 6] can be constructed with tensor products of affine copies of Φ_1 and Ψ_1 . However, motivated by scan data of 2D surfaces in 3D, it is commonly agreed that a system of adjacent triangles should be the natural domain for the analogous problem with 2-variable functions instead of a rectangular mesh. In contrast with the case of 1-variable functions, there exists a large variety of ad hoc C^1 -spline procedures based on various spatial and differential scan data at the vertices of a triangular mesh which can be obtained locally by global polynomial extensions with Hermite type interpolation [4].

Roughly speaking, in this paper we are going to give the complete list for the algebraic form of analogous constructions in 2D: intervals will be replaced with triangles, a family $[[z_{\mathbf{p}}, z_{\mathbf{p}}^{(x)}, z_{\mathbf{p}}^{(y)}] : \mathbf{p} \in \{\text{vertices of the given triangles}\}]$ will play the role of the differential data $[[y_k, y'_k] : 0 \leq k \leq r]$ and we have two polynomials Φ, Ψ in one variable such that the restriction of the constructed spline function to any edge of any given triangle should be a linear combination of affine copies of Φ and Ψ . As for natural intrinsic parametrization of triangles, we consider normalized barycentric coordinates. For later use, we establish the following terminology:

By a *triangular mesh* we mean a finite family of closed non-degenerate triangles on the plane $\mathbb{R}^2$ with pairwise non-intersecting interiors and admitting only common vertices or edges. As usually, we regard $\mathbb{R}^2$ as the set of all real tuples $[\xi, \eta]$ considered also as 1×2 matrices (row vectors). We shall use the standard notations $x^{[1]} = x : [\xi, \eta] \mapsto \xi$, $x^{[2]} = y : [\xi, \eta] \mapsto \eta$ and $\langle \mathbf{u} | \mathbf{v} \rangle := \sum_{j=1}^2 x^{[j]}(\mathbf{u}) x^{[j]}(\mathbf{v})$ for the Cartesian coordinates and scalar product, respectively. We write $\|\mathbf{u}\| = \langle \mathbf{u} | \mathbf{u} \rangle^{1/2}$ for the norm of $\mathbf{u} \in \mathbb{R}^2$ and $\text{Co}(\mathbf{S})$ for the convex hull of $\mathbf{S} \subset \mathbb{R}^2$ resp. $\det(\mathbf{u}, \mathbf{v}) = x^{[1]}(\mathbf{u})x^{[2]}(\mathbf{v}) - x^{[1]}(\mathbf{v})x^{[2]}(\mathbf{u})$ for 2×2 -determinants. Given a triangular mesh $\mathcal{T} = \{\mathbf{T}_1, \dots, \mathbf{T}_M\}$, in the sequel $\text{Vert}(\mathbf{T}_k)$ and $\text{Edge}(\mathbf{T}_k)$ will denote the sets of vertices resp. closed edges of the mesh members, $\text{Dom}(\mathcal{T}) := \bigcup_{k=1}^M \mathbf{T}_k$, $\text{Edge}(\mathcal{T}) := \bigcup_{k=1}^M \partial \mathbf{T}_k$, $\text{Vert}(\mathcal{T}) := \bigcup_{k=1}^M \text{Vert}(\mathbf{T}_k)$ will stand for the domain covered by $\mathcal{T}$, the line figure covered by all edges and the collection of all vertices, respectively. Recall that, given a *gradient data*

$$F = \left\{ (\mathbf{p}, f_{\mathbf{p}}, [f'_{x,\mathbf{p}}, f'_{y,\mathbf{p}}]) : \mathbf{p} \in \text{Vert}(\mathcal{T}) \right\} \subset \text{Vert}(\mathcal{T}) \times \mathbb{R} \times \mathbb{R}^2$$

on the set of the vertices in $\mathcal{T}$, a function $f : \mathbf{D} \rightarrow \mathbb{R}$ is a $\mathcal{C}^1$ -extension of F on $\mathbf{D} := \text{Dom}(\mathcal{T})$ if f has a continuous gradient $\mathbf{p} \mapsto \nabla f(\mathbf{p}) = [\frac{\partial}{\partial x} f(\mathbf{p}), \frac{\partial}{\partial y} f(\mathbf{p})]$ on $\text{Interior}(\mathbf{D})$ which admits a continuous extension to $\mathbf{D}$ as well (denoted also by ∇f) such that

$$f(\mathbf{p}) = f_{\mathbf{p}}, \quad \nabla f(\mathbf{p}) = [f'_{x,\mathbf{p}}, f'_{y,\mathbf{p}}] \quad (\mathbf{p} \in \text{Vert}(\mathcal{T})). \quad (1.1)$$

A $\mathcal{C}^1$ -extension $f : \mathbf{D} \rightarrow \mathbb{R}$ of F is said to be a $\mathcal{C}^1$ -spline interpolation of F with respect to the mesh $\mathcal{T}$ if the restrictions $f|_{\mathbf{T}_k}$ are polynomials of the coordinates x, y .

Postulates A,B below exhibit two main features of most known tensor product spline procedures which can naturally be generalized even to procedures

$$\mathfrak{S} : (\mathcal{T}, F) \mapsto f_{\mathcal{T}, F} \quad (\mathcal{T} \text{ triang. mesh, } F \text{ grad. data on } \text{Vert}(\mathcal{T})) \quad (1.2)$$

furnishing $\mathcal{C}^1$ -spline interpolation functions from gradient data at the vertices over triangular meshes.

Postulate A. Linearity and being locally generated: *There are polynomial functions*

$$\phi_{\mathbf{T}}^{\mathbf{p}}, \psi_{\mathbf{T}}^{1,\mathbf{p}}, \psi_{\mathbf{T}}^{2,\mathbf{p}} : \mathbf{T} \rightarrow \mathbb{R} \quad (\mathbf{T} \in \{\text{non-deg. triangles}\}, \mathbf{p} \in \text{Vert}(\mathbf{T}))$$

depending only on the couple of the triangle $\mathbf{T}$ with a distinguished vertex such that the restriction of $\mathfrak{S}$ to any mesh triangle $\mathbf{T} \in \mathcal{T}$ has the form

$$f_{\mathcal{T},F}|_{\mathbf{T}} = \sum_{\mathbf{p} \in \text{Vert}(\mathbf{T})} \left[f_{\mathbf{p}} \Phi_{\mathbf{T}}^{\mathbf{p}} + f'_{x,\mathbf{p}} \Psi_{\mathbf{T}}^{1,\mathbf{p}} + f'_{y,\mathbf{p}} \Psi_{\mathbf{T}}^{2,\mathbf{p}} \right]. \quad (1.3)$$

If Postulate A holds and $\text{Vert}(\mathbf{T}) = \{\mathbf{a}, \mathbf{b}, \mathbf{p}\}$, in terms of the canonical frame vectors

$$\mathbf{e}^{[0]} := \mathbf{0} = [0, 0], \quad \mathbf{e}^{[1]} := [1, 0], \quad \mathbf{e}^{[2]} := [0, 1]$$

we necessarily have

$$\begin{aligned} \Phi_{\mathbf{T}}^{\mathbf{p}}(\mathbf{p}) &= 1, \quad \nabla \Phi_{\mathbf{T}}^{\mathbf{p}}(\mathbf{p}) = \mathbf{0}, \quad \Psi_{\mathbf{T}}^{j,\mathbf{p}}(\mathbf{p}) = 0, \quad \nabla \Psi_{\mathbf{T}}^{j,\mathbf{p}}(\mathbf{p}) = \mathbf{e}^{[j]}; \\ \Phi_{\mathbf{T}}^{\mathbf{p}}(\mathbf{x}) &= \Psi_{\mathbf{T}}^{j,\mathbf{p}}(\mathbf{x}) = 0, \quad \nabla \Phi_{\mathbf{T}}^{\mathbf{p}}(\mathbf{x}) = \nabla \Psi_{\mathbf{T}}^{j,\mathbf{p}}(\mathbf{x}) = \mathbf{0} \quad (\mathbf{x} \in \text{Co}\{\mathbf{a}, \mathbf{b}\}). \end{aligned} \quad (1.4)$$

The first statement in (1.4) is immediate from (1.3). The second one is a consequence of the fact that given any point $\tilde{\mathbf{p}}$ forming an adjacent triangle $\tilde{\mathbf{T}} := \text{Co}\{\mathbf{a}, \mathbf{b}, \tilde{\mathbf{p}}\}$, considering the mesh $\mathcal{T} := \{\mathbf{T}, \tilde{\mathbf{T}}\}$ with gradient data $F(\mathbf{q}) = (0, \mathbf{0})$ where $\mathbf{q} = \mathbf{a}, \mathbf{b}, \tilde{\mathbf{p}}$, we must have $f_{\mathcal{T},F} \equiv 0$ on $\tilde{\mathbf{T}}$ and hence also $\nabla f_{\mathcal{T},F} \equiv 0$ on the common edge $\text{Co}\{\mathbf{a}, \mathbf{b}\}$ of the triangles $\mathbf{T}, \tilde{\mathbf{T}}$.

Locally generated linear spline procedures have the computational advantage that the resulting functions can be calculated on any mesh triangle regardless to what happens at vertices outside. A practical disadvantage is that in most cases only function values are available (mostly from scanned data) and convenient gradient values must be guessed or found by optimizing procedures.

Postulate B. Uniform shape on edges: (1.3) holds and there are polynomial functions $\Phi, \Psi : [0, 1] \rightarrow \mathbb{R}$ such that

$$\Phi(0) = \Psi(0) = \Phi'(0) = \Psi'(0) = \Psi(1) = 0, \quad \Phi(1) = \Psi'(1) = 1 \quad (1.5)$$

and the graphs of the basic functions $\Phi_{\mathbf{T}}^{\mathbf{p}}$ on the edges of the triangle $\mathbf{T}$ are affine images of the graph of Φ , those of $\Psi_{\mathbf{T}}^{j,\mathbf{p}}$ ($j = 1, 2$) are affine images of the graph of Ψ .

That is, under Postulate B, for the generic points $\mathbf{y}_t := t\mathbf{p} + (1-t)\mathbf{a}$ on the edge $\text{Co}\{\mathbf{a}, \mathbf{p}\}$, resp. $\mathbf{z}_t := t\mathbf{p} + (1-t)\mathbf{b}$ on $\text{Co}\{\mathbf{b}, \mathbf{p}\}$ we have

$$\begin{aligned} \Phi_{\text{Co}\{\mathbf{a}, \mathbf{p}\}}^{\mathbf{p}}(\mathbf{y}_t) &= \text{const}_{\mathbf{a}, \mathbf{p}} \Phi(t), \quad \Psi_{\text{Co}\{\mathbf{a}, \mathbf{p}\}}^{j,\mathbf{p}}(\mathbf{y}_t) = \text{const}_{\mathbf{a}, \mathbf{p}}^{(j)} \Psi(t), \\ \Phi_{\text{Co}\{\mathbf{a}, \mathbf{p}\}}^{\mathbf{p}}(\mathbf{z}_t) &= \text{const}_{\mathbf{b}, \mathbf{p}} \Phi(t), \quad \Psi_{\text{Co}\{\mathbf{a}, \mathbf{p}\}}^{j,\mathbf{p}}(\mathbf{z}_t) = \text{const}_{\mathbf{b}, \mathbf{p}}^{(j)} \Psi(t) \end{aligned} \quad (1.6)$$

while for the points $\mathbf{x}_t := (1-t)\mathbf{a} + t\mathbf{b}$ on the edge $\text{Co}\{\mathbf{a}, \mathbf{b}\}$ we simply have

$$\begin{aligned} \Phi_{\text{Co}\{\mathbf{a}, \mathbf{b}\}}^{\mathbf{p}}(\mathbf{x}_t) &= \Psi_{\text{Co}\{\mathbf{a}, \mathbf{b}\}}^{j,\mathbf{p}}(\mathbf{x}_t) = 0, \\ \nabla \Phi_{\text{Co}\{\mathbf{a}, \mathbf{b}\}}^{\mathbf{p}}(\mathbf{x}_t) &= \nabla \Psi_{\text{Co}\{\mathbf{a}, \mathbf{b}\}}^{j,\mathbf{p}}(\mathbf{x}_t) = \mathbf{0}. \end{aligned} \quad (1.7)$$

In the sequel we call Φ, Ψ the *shape functions* of the spline procedure (1.2) satisfying Postulate B. Notice that the requirements $\Phi(0) = \Phi'(0) = \Psi(0) = \Psi'(0)$ follow automatically from the order condition (1.4) on the edge $\text{Co}\{\mathbf{a}, \mathbf{b}\}$.

Our aim in this paper is a parametric classification of all procedures satisfying Postulates A, B which produce C^1 -smooth functions.

It is remarkable that there is a unique one among them with lowest degree (degree 5) which turns out to be homothetically invariant. From the view point of applications, the results provide the complete list of hermitian C^1 -splines with shape uniformity over edges from which one can choose the best fit one with respect to various aspects. It is worth to relate the latter fact to the local linear polynomial spline interpolation procedures on the basis of the Zlámal-Ženišek (ZZ) equations [8, 9]. These methods rely upon the fact that, given a triangular mesh with gradient and Hessian data at the vertices and normal derivative values at edge middle points, there is a unique fitting spline with 5th degree polynomials. The 21 polynomial coefficients over any mesh triangle can be obtained as the unique solution of a system of 21 straightforward linear equations. Explicit formulas along with estimates concerning the approximation of smooth functions with this method were published recently [7]. Recently there are computer algebras (as MAPLE 14 or Wolfram Mathematica 12) being powerful enough for a completely symbolic solution of the ZZ-equations and it is easy to give examples where the ZZ-approximation produces only a C^1 -smooth function¹. It seems that our first order approach with the shape conditions of Postulate B provides a geometrically motivated alternative to several problems discussed in [7]. As remarked also in [1], first order approaches with a few (actually 9 in [1]) free parameters may have practical advantages versus higher order methods due to the fact that data sampling can rarely support Hessian data or even adequate guesses for them. The usual way to overcome such problems is minimizing curvature functionals with given function values at mesh vertices but varying the data of first and higher derivatives. This can be carried out with relatively low computational costs for our splines in Theorem 1, namely about at least 5 times less computational effort than with ZZ-based procedures.

2. MAIN RESULTS

Recall that given a non-degenerate triangle $\mathbf{T} \subset \mathbb{R}^2$ with $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\} = \text{Vert}(\mathbf{T})$, the *normalized barycentric coordinates* of a point $\mathbf{x}$ are the terms of the necessarily unique triple $[\lambda_{\mathbf{T}}^{\mathbf{a}}(\mathbf{x}), \lambda_{\mathbf{T}}^{\mathbf{b}}(\mathbf{x}), \lambda_{\mathbf{T}}^{\mathbf{c}}(\mathbf{x})] \in \mathbb{R}^3$ such that

$$\mathbf{x} = \lambda_{\mathbf{T}}^{\mathbf{a}}(\mathbf{x})\mathbf{a} + \lambda_{\mathbf{T}}^{\mathbf{b}}(\mathbf{x})\mathbf{b} + \lambda_{\mathbf{T}}^{\mathbf{c}}(\mathbf{x})\mathbf{c}, \quad \lambda_{\mathbf{T}}^{\mathbf{a}}(\mathbf{x}) + \lambda_{\mathbf{T}}^{\mathbf{b}}(\mathbf{x}) + \lambda_{\mathbf{T}}^{\mathbf{c}}(\mathbf{x}) = 1.$$

¹Example: $f(x, y) = \begin{cases} 0 & \text{for } x \leq 0, \\ x^2 y^2 (1 - x - y) & \text{for } x \geq 0 \end{cases}$ on the mesh $\{\mathbf{T}_k : k = 1, 2\}$, $\mathbf{T}_k = \text{Co}\{\mathbf{0}, (-1)^k \mathbf{e}^{[1]}, \mathbf{e}^{[2]}\}$ with vanishing ZZ data except for the edge middle point $\frac{1}{2}\mathbf{e}^{[1]} + \frac{1}{2}\mathbf{e}^{[2]}$ of $\mathbf{T}_2$.

We reserve the symbols λ_T^p as standard notation. It is well-known from elementary analytic plain geometry [2] that

$$\lambda_T^p(\mathbf{x}) = \text{area}(\text{Co}\{\mathbf{a}, \mathbf{b}, \mathbf{x}\}) / \text{area}(\mathbf{T}) \quad (\mathbf{x} \in \mathbf{T})$$

thus normalized barycentric coordinates can easily be calculated by means of determinants or inner products with a $(\pi/2)$ -rotation:

$$\lambda_T^p(\mathbf{x}) = \frac{\det(\mathbf{x}-\mathbf{a}, \mathbf{x}-\mathbf{b})}{\det(\mathbf{p}-\mathbf{a}, \mathbf{p}-\mathbf{b})} = \frac{\langle (\mathbf{b}-\mathbf{a})\mathbf{R} | \mathbf{x}-\mathbf{a} \rangle}{\langle (\mathbf{b}-\mathbf{a})\mathbf{R} | \mathbf{p}-\mathbf{a} \rangle} \quad \text{where } \mathbf{R} := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \quad (2.1)$$

For later use we also introduce the abbreviating notations

$$x_p^{[j]} := x^{[j]} - x^{[j]}(\mathbf{p}), \quad \xi_{p,a}^v := \frac{\langle \mathbf{v}-\mathbf{a} | \mathbf{p}-\mathbf{a} \rangle}{\|\mathbf{p}-\mathbf{a}\|^2}, \quad \bar{\xi}_{p,a}^v := \frac{\langle \mathbf{v}-\mathbf{a} | (\mathbf{p}-\mathbf{a})\mathbf{R} \rangle}{\|\mathbf{p}-\mathbf{a}\|^2}. \quad (2.2)$$

As for geometric interpretation, $\xi_{p,a}^v$ resp. $\bar{\xi}_{p,a}^v$ are the affine coordinates of the point $\mathbf{v}$ with respect to the orthogonal frame $[\mathbf{a}, \mathbf{p}, \mathbf{a} + (\mathbf{p}-\mathbf{a})\mathbf{R}]$ with origin $\mathbf{a}$ so that $\mathbf{v} = \mathbf{a} + \xi_{p,a}^v(\mathbf{p}-\mathbf{a}) + \bar{\xi}_{p,a}^v(\mathbf{p}-\mathbf{a})\mathbf{R}$.

Theorem 1. *There is a unique local linear polynomial C^1 -spline procedure acting on triangular meshes with the property of uniform shape on vertices² and having shape functions with minimal computational complexity. Its shape functions are*

$$^*\Phi(t) = t^3(10 - 15t + 6t^2), \quad ^*\Psi(t) = t^3(t-1)(4-3t).$$

The corresponding shape functions (for a non-degenerate triangle $\mathbf{T} = \text{Co}\{\mathbf{a}, \mathbf{b}, \mathbf{p}\}$ with distinguished vertex $\mathbf{p}$) have the form

$$\begin{aligned} ^*\phi_T^p &= ^*\Phi(\lambda_T^p) + 30[\lambda_T^p]^2 \lambda_T^a \lambda_T^b \left[\xi_{p,a}^b \lambda_T^a + \xi_{p,b}^a \lambda_T^b \right], \\ ^*\psi_T^{j,p} &= \frac{^*\Psi(\lambda_T^p)}{\lambda_T^p - 1} x_p^{[j]} + 12[\lambda_T^p]^2 \lambda_T^a \lambda_T^b \left[\xi_{p,a}^b x_p^{[j]}(\mathbf{a}) \lambda_T^a + \xi_{p,b}^a x_p^{[j]}(\mathbf{b}) \lambda_T^b \right]. \end{aligned}$$

Theorem 2. *A spline procedure on triangular meshes and satisfying Postulates A,B produces C^1 -smooth splines if and only if its shape functions are of the form*

$$\Phi(t) = ^*\Phi(t) + t^3(1-t)^3\Phi_1(t), \quad \Psi(t) = ^*\Psi(t) + t^3(1-t)^3\Psi_1(t) \quad (2.3)$$

and the shape functions (for a non-degenerate triangle $\mathbf{T} = \text{Co}\{\mathbf{a}, \mathbf{b}, \mathbf{p}\}$ with distinguished vertex $\mathbf{p}$) can be written in terms of the modified shape function

$$\Theta(t) := \Psi(t)/(t-1) \quad (2.3')$$

and the rotation matrix $\mathbf{R}$ in (2.1) as

$$\begin{aligned} \phi_T^p &= \Phi(\lambda_T^p) + [\lambda_T^p]^2 \lambda_T^a \lambda_T^b P_{a,b}^p(\lambda_T^b, \lambda_T^a), \\ \psi_T^{j,p} &= \Theta(\lambda_T^p) x_p^{[j]} + [\lambda_T^p]^2 \lambda_T^a \lambda_T^b Q_{a,b}^{j,p}(\lambda_T^b, \lambda_T^a) \end{aligned} \quad (2.4)$$

²That is satisfying Postulates A,B with $f_{T,F} \in C^1(\text{Dom}(\mathcal{T}))$.

$$\begin{aligned}
P_{\mathbf{a},\mathbf{b}}^{\mathbf{p}}(s,t) &= s \left\{ \xi_{\mathbf{p},\mathbf{b}}^{\mathbf{a}} \frac{\Phi'(1-s)}{(1-s)^2 s^2} + \bar{\xi}_{\mathbf{p},\mathbf{b}}^{\mathbf{a}} k_{\mathbf{b}}^{0,\mathbf{p}}(s) \right\} + \\
&\quad + t \left\{ \xi_{\mathbf{p},\mathbf{a}}^{\mathbf{b}} \frac{\Phi'(1-t)}{(1-t)^2 t^2} + \bar{\xi}_{\mathbf{p},\mathbf{a}}^{\mathbf{b}} k_{\mathbf{a}}^{0,\mathbf{p}}(t) \right\} + st R_{\mathbf{a},\mathbf{b}}^{0,\mathbf{p}}(s,t), \\
Q_{\mathbf{a},\mathbf{b}}^{j,\mathbf{p}}(s,t) &= s \left\{ \xi_{\mathbf{p},\mathbf{b}}^{\mathbf{a}} x_{\mathbf{p}}^{[j]}(\mathbf{b}) \frac{\Theta'(1-s)}{s(1-s)^2} + \bar{\xi}_{\mathbf{p},\mathbf{b}}^{\mathbf{a}} k_{\mathbf{b}}^{j,\mathbf{p}}(s) \right\} + \\
&\quad + t \left\{ \xi_{\mathbf{p},\mathbf{a}}^{\mathbf{b}} x_{\mathbf{p}}^{[j]}(\mathbf{a}) \frac{\Theta'(1-t)}{t(1-t)^2} + \bar{\xi}_{\mathbf{p},\mathbf{a}}^{\mathbf{b}} k_{\mathbf{a}}^{j,\mathbf{p}}(t) \right\} + st R_{\mathbf{a},\mathbf{b}}^{j,\mathbf{p}}(s,t)
\end{aligned} \tag{2.5}$$

with the following free options in (2.4) resp. (2.5):

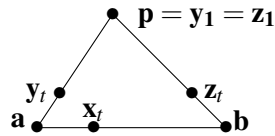
- (i) $\Phi_1, \Psi_1 : [0, 1] \rightarrow \mathbb{R}$ are arbitrary polynomial functions,
- (ii) $(\mathbf{p}, \mathbf{q}) \mapsto k_{\mathbf{q}}^{i,\mathbf{p}}$ ($i = 0, 1, 2$) are arbitrary maps assigning polynomial functions $\mathbb{R} \rightarrow \mathbb{R}$ to pairs of distinct points,
- (iii) $(\mathbf{p}, \mathbf{q}, \mathbf{r}) \mapsto R_{\mathbf{q},\mathbf{r}}^{i,\mathbf{p}}$ ($i = 0, 1, 2$) are arbitrary maps assigning polynomials $\mathbb{R}^2 \rightarrow \mathbb{R}$ to triples of non-collinear points with the symmetry $R_{\mathbf{q},\mathbf{r}}^{i,\mathbf{p}}(s,t) \equiv R_{\mathbf{r},\mathbf{q}}^{i,\mathbf{p}}(t,s)$.

Remark 1. (i) Theorem 1 is simply a corollary of Theorem 2 by setting the options (i)–(iv) to 0. We emphasize it for its potential practical and educational use.

(ii) The rational expressions in (2.4–2.5) are polynomials: $\Phi'(1-t)/[t^2(1-t)^2] = 30 - 3(1-2t)\Phi_1(1-t) + t(1-t)\Phi_1'(1-t)$, respectively $\Psi(t)/(t-1) = t^3[(4-3t) - (1-t)^2\Psi_1(t)]$, $\Theta'(1-t)/[t(1-t)^2] = 12 + (2-5t)\Psi_1(1-t) - t(1-t)\Psi_1'(1-t)$.

(iii) $\lambda_{\mathbf{T}}^{\mathbf{p}}, \lambda_{\mathbf{T}}^{\mathbf{a}}, \lambda_{\mathbf{T}}^{\mathbf{b}}$ are the affine functions determined by the properties $\text{Line}\{\mathbf{a}, \mathbf{b}\} = (\lambda_{\mathbf{T}}^{\mathbf{p}}=0)$, $\text{Line}\{\mathbf{b}, \mathbf{p}\} = (\lambda_{\mathbf{T}}^{\mathbf{a}}=0)$, $\text{Line}\{\mathbf{a}, \mathbf{p}\} = (\lambda_{\mathbf{T}}^{\mathbf{b}}=0)$, $\lambda_{\mathbf{T}}^{\mathbf{p}}(\mathbf{p}) = \lambda_{\mathbf{T}}^{\mathbf{a}}(\mathbf{a}) = \lambda_{\mathbf{T}}^{\mathbf{b}}(\mathbf{b}) = 1$. For the parametrized edge points in (1.6) we have

$$\begin{aligned}
\lambda_{\mathbf{T}}^{\mathbf{p}}(\mathbf{x}_t) &= \lambda_{\mathbf{T}}^{\mathbf{a}}(\mathbf{z}_t) = \lambda_{\mathbf{T}}^{\mathbf{b}}(\mathbf{y}_t) \equiv 0, \\
\lambda_{\mathbf{T}}^{\mathbf{p}}(\mathbf{y}_t) &= \lambda_{\mathbf{T}}^{\mathbf{p}}(\mathbf{z}_t) = \lambda_{\mathbf{T}}^{\mathbf{b}}(\mathbf{x}_t) \equiv t, \\
\lambda_{\mathbf{T}}^{\mathbf{a}}(\mathbf{x}_t) &= \lambda_{\mathbf{T}}^{\mathbf{a}}(\mathbf{y}_t) = \lambda_{\mathbf{T}}^{\mathbf{b}}(\mathbf{z}_t) \equiv 1-t.
\end{aligned}$$



On the other hand $x^{[j]}(\mathbf{y}_t) = (1-t)x^{[j]}(\mathbf{a}-\mathbf{p}) = (1-t)x_{\mathbf{p}}^{[j]}(\mathbf{a})$ resp. $x^{[j]}(\mathbf{z}_t) = (1-t)x^{[j]}(\mathbf{b}-\mathbf{p}) = (1-t)x_{\mathbf{p}}^{[j]}(\mathbf{b})$. Hence, with the formulas (2.4), the shape conditions (1.6) hold automatically with $\text{const}_{\mathbf{a},\mathbf{p}} = \text{const}_{\mathbf{b},\mathbf{p}} = 1$ and $\text{const}_{\mathbf{a},\mathbf{p}}^{(j)} = x^{[j]}(\mathbf{p}-\mathbf{a})$ resp. $\text{const}_{\mathbf{b},\mathbf{p}}^{(j)} = x^{[j]}(\mathbf{p}-\mathbf{b})$, furthermore also (1.7) is fulfilled.

(iv) One can check with symbolic computer algebra that all the spline procedures described in Theorem 2 produce C^1 -functions. It suffices to establish only that, given

any two adjacent non-degenerate triangles $\mathbf{T} := \text{Co}\{\mathbf{p}, \mathbf{a}, \mathbf{b}\}$ resp. $\tilde{\mathbf{T}} := \text{Co}\{\mathbf{p}, \mathbf{a}, \tilde{\mathbf{p}}\}$ with common edge $\text{Co}\{\mathbf{p}, \mathbf{a}\}$ and distinguished point $\mathbf{p}$, the gradient vectors of the shape functions $\varphi_{\mathbf{T}}^{\mathbf{p}}, \psi_{\mathbf{T}}^{j,\mathbf{p}}$ coincide with those of $\varphi_{\tilde{\mathbf{T}}}^{\mathbf{p}}, \psi_{\tilde{\mathbf{T}}}^{j,\mathbf{p}}$ at the points $\mathbf{y}_t = (1-t)\mathbf{a} + t\mathbf{b}$. Indeed, hence it follows that the unit spline functions $f_{\mathcal{T}, F_{\mathbf{p}}^i}(\mathbf{p} \in \text{Vert}(\mathcal{T}), i = 0, 1, 2)$ corresponding to the gradient data $F_{\mathbf{p}}^0 := \{[\mathbf{p}, 1, \mathbf{0}], [\mathbf{q}, 0, \mathbf{0}] : \mathbf{q} \in \text{Vert}(\mathcal{T}) \setminus \{\mathbf{p}\}\}$ resp. $F_{\mathbf{p}}^j := \{[\mathbf{p}, 0, \mathbf{e}^{[j]}], [\mathbf{q}, 0, \mathbf{0}] : \mathbf{q} \in \text{Vert}(\mathcal{T}) \setminus \{\mathbf{p}\}\}$ ($j = 1, 2$) are continuously differentiable.

3. PROOF OF THEOREM 2

Henceforth we consider an arbitrarily fixed procedure $\mathfrak{S} : (\mathcal{T}, F) \mapsto f_{\mathcal{T}, F}$ which satisfies Postulates A, B and produces continuous but not necessarily continuously differentiable functions. We reserve the notations $\varphi_{\mathbf{T}}^{\mathbf{p}}, \psi_{\mathbf{T}}^{j,\mathbf{p}}$ resp. $\Phi, \Psi, \Theta(t) := \Psi(t)/(t-1)$ for the basic functions resp. shape functions as established in Section 1. In accordance with (1.5) we can write

$$\Phi(t) = t^2(3-2t) + t^2(1-t)^2\Phi_0(t), \quad \Psi(t) = t^2(t-1) + t^2(1-t)^2\Psi_0(t) \quad (3.1)$$

and $\Theta(t) = t^2 + t^2(t-1)\Psi_0(t)$ with suitable polynomials Φ_0, Ψ_0 .

Next we are going to express the constraints (1.4), (1.6-1.7) on the basic functions in terms of Φ, Ψ and the barycentric coordinates. To this aim, we recall the following folklore fact from elementary analytic plane geometry relating the root curves with a product decomposition of multivariate polynomials which is an easy consequence of Bézout's Theorem [3].

Remark 2. (i) If $\mathbf{L}_0, \mathbf{L}_1, \dots, \mathbf{L}_m$ are distinct straight lines such that $\mathbf{L}_k = (\ell_k = 0)$ with the affine functions (i.e. polynomials of first degree) $\ell_k : \mathbb{R}^2 \rightarrow \mathbb{R}$ ($k = 1, \dots, m$) then a polynomial $\mathbb{R}^2 \rightarrow \mathbb{R}$ is divisible with $\prod_{k=0}^m \ell_k^{v_k}$ if and only if, for any index k , it vanishes in order v_k at the points of $\mathbf{L}_k$. In particular, given a non-degenerate triangle $\mathbf{T} := \text{Co}\{\mathbf{a}, \mathbf{b}, \mathbf{p}\}$, a polynomial $Q : \mathbb{R}^2 \rightarrow \mathbb{R}$ of two variables has the form

$$Q = [\lambda_{\mathbf{T}}^{\mathbf{p}}]^{v_0} [\lambda_{\mathbf{T}}^{\mathbf{a}}]^{v_1} [\lambda_{\mathbf{T}}^{\mathbf{b}}]^{v_2}$$

for some polynomial $q : \mathbb{R}^2 \rightarrow \mathbb{R}$ if and only if it vanishes in order v_0 at the points of $\text{Line}\{\mathbf{a}, \mathbf{b}\}$, order v_1 at $\text{Line}\{\mathbf{p}, \mathbf{b}\}$ and order v_2 at $\text{Line}\{\mathbf{p}, \mathbf{a}\}$, respectively.³

(ii) If $Q : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a polynomial of two variables, we can write

$$\begin{aligned} Q(x, y) &= Q(0, 0) + xq_1(x) + yq_2(y) + xyq_3(x, y) \quad \text{where} \\ q_1(x) &:= [Q(x, 0) - Q(0, 0)]/x, \quad q_2(y) := [Q(0, y) - Q(0, 0)]/y, \\ q_3(x, y) &:= [Q(x, y) - [Q(0, 0) + xq_1(x) + yq_2(y)]]/(xy) \end{aligned}$$

are well-defined polynomials in one resp. two variables.

³ Q vanishes in order v at the point $[x_0, y_0]$ if $\frac{\partial^{k+m}}{\partial x^k \partial y^m} Q(x_0, y_0) = 0$ whenever $k+m < v$.

We shall call the $\mathbb{R}^2$ -polynomial $Q_0(x, y) := Q(0, 0) + xq_1(x) + yq_2(y)$ of first degree the *principal part* of Q .

Lemma 1. *The basic functions $\varphi_{\mathbf{T}}^{\mathbf{p}}, \psi_{\mathbf{T}}^{j, \mathbf{p}}$ for $\mathbf{T} = \text{Co}\{\mathbf{a}, \mathbf{b}, \mathbf{p}\}$ have the form*

$$\begin{aligned}\varphi_{\mathbf{T}}^{\mathbf{p}} &= \Phi(\lambda_{\mathbf{T}}^{\mathbf{p}}) + [\lambda_{\mathbf{T}}^{\mathbf{p}}]^2 \lambda_{\mathbf{T}}^{\mathbf{a}} \lambda_{\mathbf{T}}^{\mathbf{b}} \text{Pol}(\lambda_{\mathbf{T}}^{\mathbf{b}}, \lambda_{\mathbf{T}}^{\mathbf{a}}), \\ \psi_{\mathbf{T}}^{j, \mathbf{p}} &= \Theta(\lambda_{\mathbf{T}}^{\mathbf{p}}) x_{\mathbf{p}}^{[j]} + [\lambda_{\mathbf{T}}^{\mathbf{p}}]^2 \lambda_{\mathbf{T}}^{\mathbf{a}} \lambda_{\mathbf{T}}^{\mathbf{b}} \text{Pol}(\lambda_{\mathbf{T}}^{\mathbf{b}}, \lambda_{\mathbf{T}}^{\mathbf{a}})\end{aligned}$$

in terms of the barycentric coordinates (2.1), the shape functions (3.1), the modified shape function (2.3') and with suitable polynomials of two variables.

Proof. Fix any triangle $\mathbf{T} := \text{Co}\{\mathbf{a}, \mathbf{b}, \mathbf{p}\}$. As mentioned, necessarily (3.1) holds and Θ is a polynomial. Consider the functions

$$f := \Phi(\lambda_{\mathbf{T}}^{\mathbf{p}}), \quad g^{(j)} := \Theta(\lambda_{\mathbf{T}}^{\mathbf{p}}) \cdot x_{\mathbf{p}}^{[j]}.$$

Along the edge $\text{Co}\{\mathbf{a}, \mathbf{p}\}$, at the points $\mathbf{y}_t := (1-t)\mathbf{a} + t\mathbf{p}$ we have $\lambda_{\mathbf{T}}^{\mathbf{p}}(\mathbf{y}_t) = t$, $\lambda_{\mathbf{T}}^{\mathbf{b}}(\mathbf{y}_t) = 0$, $\lambda_{\mathbf{T}}^{\mathbf{a}}(\mathbf{y}_t) = [1 - \lambda_{\mathbf{T}}^{\mathbf{a}}(\mathbf{y}_t) - \lambda_{\mathbf{T}}^{\mathbf{b}}(\mathbf{y}_t)] = 1 - t$. Observe that the functions $f, g^{(j)}$ suit the shape uniformity conditions because

$$\begin{aligned}f(\mathbf{y}_t) &= \Phi(t), \quad g^{(j)}(\mathbf{y}_t) = \Theta(t) \langle \mathbf{e}^{[j]} | \mathbf{y}_t - \mathbf{p} \rangle = \Theta(t)(1-t) \langle \mathbf{e}^{[j]} | \mathbf{a} - \mathbf{p} \rangle = \\ &= \langle \mathbf{e}^{[j]} | \mathbf{p} - \mathbf{a} \rangle \Psi(t)\end{aligned}$$

and since $f, g^{(j)}$ are polynomial multiples of $[\lambda_{\mathbf{T}}^{\mathbf{p}}]^2$. Also, since $\mathbf{y}_1 = \mathbf{p}$, $f(\mathbf{p}) = \Phi(\lambda_{\mathbf{T}}^{\mathbf{p}}(\mathbf{y}_1)) = \Phi(1) = 1$ and

$$\begin{aligned}\nabla f(\mathbf{p}) &= \Phi'(\lambda_{\mathbf{T}}^{\mathbf{p}}(\mathbf{y}_1)) \nabla \lambda_{\mathbf{T}}^{\mathbf{p}}(\mathbf{y}_1) = 0 \cdot \nabla \lambda_{\mathbf{T}}^{\mathbf{p}}(\mathbf{y}_1) = \mathbf{0}, \\ \nabla g^{(j)}(\mathbf{p}) &= \nabla_{\mathbf{x}=\mathbf{y}_1} \left[\Theta(\lambda_{\mathbf{T}}^{\mathbf{p}}(\mathbf{x})) x_{\mathbf{p}}^{[j]}(\mathbf{x}) \right] = \\ &= x_{\mathbf{p}}^{[j]}(\mathbf{p}) \nabla_{\mathbf{x}=\mathbf{y}_1} \Theta(\lambda_{\mathbf{T}}^{\mathbf{p}}(\mathbf{x})) + \Theta(\lambda_{\mathbf{T}}^{\mathbf{p}}(\mathbf{p})) \nabla_{\mathbf{x}=\mathbf{y}_1} x_{\mathbf{p}}^{[j]}(\mathbf{p}) = \\ &= 0 \cdot \Theta'(1) \nabla \lambda_{\mathbf{T}}^{\mathbf{p}}(\mathbf{y}_1) + \Theta(1) \mathbf{e}^{[j]} = \mathbf{e}^{[j]}.\end{aligned}$$

Therefore the difference functions $\varphi_{\mathbf{T}}^{\mathbf{p}} - f$ and $\psi_{\mathbf{T}}^{j, \mathbf{p}} - g^{(j)}$ vanish on the edge $\text{Co}\{\mathbf{a}, \mathbf{p}\}$ of the triangle $\mathbf{T}$. Similar arguments with the points $\mathbf{z}_t := (1-t)\mathbf{b} + t\mathbf{p}$ show that $\varphi_{\mathbf{T}}^{\mathbf{p}} - g$ and $\psi_{\mathbf{T}}^{j, \mathbf{p}} - g^{(j)}$ vanish on $\text{Co}\{\mathbf{b}, \mathbf{p}\}$. By (1.7) their gradients also vanish on the edge $\text{Co}\{\mathbf{a}, \mathbf{b}\} = (\lambda_{\mathbf{T}}^{\mathbf{p}} = 0)$. Hence (cf. Remark 2) they are polynomial multiples of $[\lambda_{\mathbf{T}}^{\mathbf{p}}]^2 \lambda_{\mathbf{T}}^{\mathbf{a}} \lambda_{\mathbf{T}}^{\mathbf{b}}$, say $\varphi_{\mathbf{T}}^{\mathbf{p}} = f + [\lambda_{\mathbf{T}}^{\mathbf{p}}]^2 \lambda_{\mathbf{T}}^{\mathbf{a}} \lambda_{\mathbf{T}}^{\mathbf{b}} \Pi_{\mathbf{p}, \mathbf{T}}^{(0)}$ and $\psi_{\mathbf{T}}^{j, \mathbf{p}} = g^{(j)} + [\lambda_{\mathbf{T}}^{\mathbf{p}}]^2 \lambda_{\mathbf{T}}^{\mathbf{a}} \lambda_{\mathbf{T}}^{\mathbf{b}} \Pi_{\mathbf{p}, \mathbf{T}}^{(j)}$, respectively. Since $\lambda_{\mathbf{T}}^{\mathbf{a}}, \lambda_{\mathbf{T}}^{\mathbf{b}}$ are linearly independent affine functionals, the mapping $\Lambda_{\mathbf{a}, \mathbf{b}}^{\mathbf{p}} : \mathbf{x} \mapsto [\lambda_{\mathbf{T}}^{\mathbf{a}}(\mathbf{x}), \lambda_{\mathbf{T}}^{\mathbf{b}}(\mathbf{x})]$ is an affine coordinatization on the plain $\mathbb{R}^2$. Thus we can express each term $\varphi_{\mathbf{T}}^{\mathbf{p}}, \psi_{\mathbf{T}}^{j, \mathbf{p}}$ as a polynomial of the coordinates $\Lambda_{\mathbf{a}, \mathbf{b}}^{\mathbf{p}}$ which completes the proof.

Notation 1. For later convenience, without danger of confusion, we introduce the unifying context-free notations

$$\lambda_{\mathbf{a},\mathbf{b}}^{\mathbf{p}} := \lambda_{\text{Co}\{\mathbf{a},\mathbf{b},\mathbf{p}\}}^{\mathbf{p}}, \quad f_{\mathbf{a},\mathbf{b}}^{0,\mathbf{p}} := \Phi_{\text{Co}\{\mathbf{a},\mathbf{b},\mathbf{p}\}}^{\mathbf{p}}, \quad f_{\mathbf{a},\mathbf{b}}^{j,\mathbf{p}} := \Psi_{\text{Co}\{\mathbf{a},\mathbf{b},\mathbf{p}\}}^{j,\mathbf{p}} \quad (j=1,2).$$

Furthermore, in view of Lemma 1, we shall write

$$f_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}} = \Phi^{[i]}(\lambda_{\mathbf{a},\mathbf{b}}^{\mathbf{p}})x_{\mathbf{p}}^{[i]} + [\lambda_{\mathbf{a},\mathbf{b}}^{\mathbf{p}}]^2 \lambda_{\mathbf{a},\mathbf{p}}^{\mathbf{b}} \lambda_{\mathbf{b},\mathbf{p}}^{\mathbf{a}} P_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}}(\lambda_{\mathbf{a},\mathbf{p}}^{\mathbf{b}}, \lambda_{\mathbf{b},\mathbf{p}}^{\mathbf{a}}) \quad (i=0,1,2) \quad (3.2)$$

where

$$\Phi^{[0]} := \Phi, \quad \Phi^{[1]} := \Phi^{[2]} := \Theta, \quad x_{\mathbf{p}}^{[0]} : \mathbf{x} \mapsto 1 \quad \text{with} \quad \mathbf{e}^{[0]} := \nabla x_{\mathbf{p}}^{[0]} = \mathbf{0}$$

and the terms $P_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}}$ ($i=0,1,2$) are polynomials with coefficients depending on the ordered tuple $(i, \mathbf{p}, \mathbf{a}, \mathbf{b})$. Notice that necessarily

$$P_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}}(s, t) \equiv P_{\mathbf{b},\mathbf{a}}^{i,\mathbf{p}}(t, s) \quad (3.3)$$

due to the trivial index symmetries $\lambda_{\mathbf{u},\mathbf{v}}^{\mathbf{w}} \equiv \lambda_{\mathbf{v},\mathbf{u}}^{\mathbf{w}}$ and $f_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}} \equiv f_{\mathbf{b},\mathbf{a}}^{i,\mathbf{p}}$.

Lemma 2. *We have $f_{T,F} \in C^1(\text{Dom}(T))$ for every triangular mesh with arbitrary gradient data if and only if*

$$\mathbf{b} \mapsto \nabla f_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}}(\mathbf{y}) \equiv \text{const}_{\mathbf{p},\mathbf{a},\mathbf{y}} \quad \text{for fixed } \mathbf{p} \neq \mathbf{a} \text{ and } \mathbf{y} \in \text{Co}\{\mathbf{p}, \mathbf{a}\}. \quad (3.4)$$

Proof. Given any non-degenerate triangle $\mathbf{T} = \text{Co}\{\mathbf{a}, \mathbf{b}, \mathbf{p}\}$, by construction, for the points $\mathbf{x}_t := (1-t)\mathbf{a} + t\mathbf{b}$, $\mathbf{y}_t := (1-t)\mathbf{a} + t\mathbf{p}$ and $\mathbf{z}_t := (1-t)\mathbf{b} + t\mathbf{p}$ on the edges of the triangle $\mathbf{T}$ we have $f_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}}(\mathbf{x}_t) = 0$ independently of $\mathbf{p}$, $f_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}}(\mathbf{y}_t) = \Phi^{[i]}(t)$ independently of $\mathbf{b}$ and $f_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}}(\mathbf{z}_t) = \Phi^{[i]}(t)$ independently of $\mathbf{a}$. Thus the shape conditions are automatic from (3.2). Moreover, given any triangle $\tilde{\mathbf{T}}$ with a common edge but disjoint interior to $\mathbf{T}$, the function pairs $\Phi_{\mathbf{T}}^{\mathbf{p}}, \Phi_{\tilde{\mathbf{T}}}^{\mathbf{p}}$ resp. $\Psi_{\mathbf{T}}^{j,\mathbf{p}}, \Psi_{\tilde{\mathbf{T}}}^{j,\mathbf{p}}$ touch continuously. The analogous necessary and sufficient condition for a C^1 -smooth touching is that the gradient pairs $\nabla \Phi_{\mathbf{T}}^{\mathbf{p}}, \nabla \Phi_{\tilde{\mathbf{T}}}^{\mathbf{p}}$ resp. $\nabla \Psi_{\mathbf{T}}^{j,\mathbf{p}}, \nabla \Psi_{\tilde{\mathbf{T}}}^{j,\mathbf{p}}$ coincide on the common edge:

$$\begin{aligned} \text{(i)} \quad & \nabla f_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}}(\mathbf{x}) = \nabla f_{\mathbf{a},\mathbf{b}}^{i,\tilde{\mathbf{p}}}(\mathbf{x}) && \text{if } \mathbf{x} \in \text{Co}\{\mathbf{a}, \mathbf{b}\} = \mathbf{T} \cap \text{Co}\{\mathbf{a}, \mathbf{b}, \tilde{\mathbf{p}}\}, \\ \text{(ii)} \quad & \nabla f_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}}(\mathbf{y}) = \nabla f_{\mathbf{a},\mathbf{p}}^{i,\tilde{\mathbf{b}}}(\mathbf{y}) && \text{if } \mathbf{y} \in \text{Co}\{\mathbf{a}, \mathbf{p}\} = \mathbf{T} \cap \text{Co}\{\mathbf{a}, \tilde{\mathbf{b}}, \mathbf{p}\}, \\ \text{(iii)} \quad & \nabla f_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}}(\mathbf{z}) = \nabla f_{\mathbf{b},\mathbf{p}}^{i,\tilde{\mathbf{a}}}(\mathbf{z}) && \text{if } \mathbf{z} \in \text{Co}\{\mathbf{b}, \mathbf{p}\} = \mathbf{T} \cap \text{Co}\{\tilde{\mathbf{a}}, \mathbf{b}, \mathbf{p}\}. \end{aligned} \quad (3.5)$$

Observe that (3.5(i)) holds automatically with the trivial value $\mathbf{0}$. Furthermore conditions (3.5(ii)) and (3.5(iii)) are analogous (by changing the roles of $\mathbf{a}$ and $\mathbf{b}$ resp. $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$). Finally we observe that, in (3.5(ii)), for fixed $\mathbf{a}, \mathbf{p}$ and $\mathbf{y} \in \text{Co}\{\mathbf{a}, \mathbf{p}\}$ we can choose the points $\mathbf{b}$ and $\tilde{\mathbf{b}}$ on different half plain components of $\mathbb{R}^2 \setminus \text{Line}\{\mathbf{a}, \mathbf{p}\}$ arbitrarily. This implies that all the vectors $\nabla f_{\mathbf{a},\mathbf{p}}^{i,\mathbf{b}}(\mathbf{y}), \nabla f_{\mathbf{a},\mathbf{p}}^{i,\tilde{\mathbf{b}}}(\mathbf{y})$ with $\mathbf{b}, \tilde{\mathbf{b}} \in \mathbb{R}^2 \setminus \text{Line}\{\mathbf{a}, \mathbf{p}\}$ must be the same. Due to the construction (1.3), the fact that all the pairs $\Phi_{\mathbf{T}}^{\mathbf{p}}, \Phi_{\tilde{\mathbf{T}}}^{\mathbf{p}}$

resp. $\psi_{\mathbf{T}}^{j,\mathbf{p}}, \psi_{\tilde{\mathbf{T}}}^{j,\mathbf{p}}$ of basic functions touch C^1 -smoothly in case of adjacent triangles $\mathbf{T}, \tilde{\mathbf{T}}$, ensures that the splines $f_{T,F}$ are all C^1 -smooth as well.

Notation 2. Given any ordered triple $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ of non-collinear points, we shall write $\mathbf{g}_{\mathbf{u},\mathbf{v}}^{\mathbf{w}} := \nabla \lambda_{\mathbf{u},\mathbf{v}}^{\mathbf{w}}$ for the constant gradient vectors of the barycentric coordinate functions. Notice that, by (2.1),

$$\mathbf{g}_{\mathbf{u},\mathbf{v}}^{\mathbf{w}} := \frac{(\mathbf{u} - \mathbf{v})\mathbf{R}}{\langle (\mathbf{u} - \mathbf{v})\mathbf{R} | \mathbf{w} - \mathbf{u} \rangle} = \frac{\sigma_{\mathbf{u},\mathbf{v}}^{\mathbf{w}}(\mathbf{u} - \mathbf{v})\mathbf{R}}{\text{area}(\text{Co}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\})}. \quad (3.6)$$

where $\sigma_{\mathbf{u},\mathbf{v}}^{\mathbf{w}} = \pm 1$ according as $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ are oriented anticlockwise or clockwise. In particular, if $\mathbf{T} = \text{Co}\{\mathbf{a}, \mathbf{b}, \mathbf{p}\}$ is a non-degenerate triangle, we have

$$\begin{aligned} \mathbf{g}_{\mathbf{a},\mathbf{b}}^{\mathbf{p}} + \mathbf{g}_{\mathbf{b},\mathbf{p}}^{\mathbf{a}} + \mathbf{g}_{\mathbf{a},\mathbf{p}}^{\mathbf{b}} &= \nabla [\lambda_{\mathbf{a},\mathbf{b}}^{\mathbf{p}} + \lambda_{\mathbf{b},\mathbf{p}}^{\mathbf{a}} + \lambda_{\mathbf{a},\mathbf{p}}^{\mathbf{b}}] = \nabla 1 = \mathbf{0}, \\ \mathbf{g}_{\mathbf{a},\mathbf{b}}^{\mathbf{p}} \perp \mathbf{b} - \mathbf{a}, \quad \mathbf{g}_{\mathbf{a},\mathbf{p}}^{\mathbf{b}} \perp \mathbf{a} - \mathbf{p}, \quad \mathbf{g}_{\mathbf{b},\mathbf{p}}^{\mathbf{a}} \perp \mathbf{b} - \mathbf{p}. \end{aligned}$$

Lemma 3. *If $\mathbf{T} = \text{Co}\{\mathbf{a}, \mathbf{b}, \mathbf{p}\}$ is a non-degenerate triangle, at the points $\mathbf{y}_t := (1-t)\mathbf{a} + t\mathbf{p}$ of the edge $\text{Co}\{\mathbf{a}, \mathbf{p}\}$ we have*

$$\begin{aligned} \nabla f_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}}(\mathbf{y}_t) &= x_{\mathbf{a},\mathbf{b}}^{[i]}((1-t)(\mathbf{a} - \mathbf{p}))[\Phi^{[i]}]'(t) \mathbf{g}_{\mathbf{a},\mathbf{b}}^{\mathbf{p}} + \\ &\quad + \Phi^{[i]}(t)\mathbf{e}^{[i]} + t^2(1-t)P_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}}(0, 1-t) \mathbf{g}_{\mathbf{a},\mathbf{p}}^{\mathbf{b}}. \end{aligned} \quad (3.7)$$

Proof. With the abbreviations

$$\ell_0 := \lambda_{\mathbf{a},\mathbf{b}}^{\mathbf{p}}, \quad \ell_1 := \lambda_{\mathbf{a},\mathbf{p}}^{\mathbf{b}}, \quad \ell_2 := \lambda_{\mathbf{b},\mathbf{p}}^{\mathbf{a}}, \quad P^{[i]} := P_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}}, \quad G^{[i]} := \ell_0^2 \ell_2 P^{[i]}(\ell_1, \ell_2)$$

we can write

$$\begin{aligned} \nabla f_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}} &= \nabla \left[x_{\mathbf{p}}^{[i]} \Phi^{[i]}(\ell_0) + \ell_1 G^{[i]} \right] = \\ &= x_{\mathbf{p}}^{[i]} \nabla [\Phi^{[i]}(\ell_0)] + \Phi^{[i]}(\ell_0) \nabla x_{\mathbf{p}}^{[i]} + \ell_1 \nabla G^{[i]} + G^{[i]} \nabla \ell_1 = \\ &= x_{\mathbf{p}}^{[i]} \Phi^{[i]'}(\ell_0) \nabla \ell_0 + \Phi^{[i]}(\ell_0) \mathbf{e}^{[i]} + \ell_1 \nabla G^{[i]} + G^{[i]} \nabla \ell_1. \end{aligned}$$

We complete the proof with the observations that

$$\begin{aligned} \ell_0(\mathbf{y}_t) &= t, \quad \ell_1(\mathbf{y}_t) = 0, \quad \ell_2(\mathbf{y}_t) = 1-t, \\ x_{\mathbf{p}}^{[i]}(\mathbf{y}_t) &= x_{\mathbf{a},\mathbf{b}}^{[i]}((1-t)(\mathbf{a} - \mathbf{p})), \quad \nabla x_{\mathbf{p}}^{[i]} \equiv \mathbf{e}^{[i]}. \end{aligned}$$

Remark 3. To prove Theorem 2, we need a precise description for the coefficients of the polynomials $P_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}}$ in terms of the variables $\mathbf{a}, \mathbf{b}, \mathbf{p}$ such that (3.4) should hold.

According to Lemma 2, the procedure $\mathfrak{S} : (\mathcal{T}, F) \mapsto f_{T,F}$ produces C^1 -splines for every admissible data if and only if, for any $t \in [0, 1]$ and for any fixed pair $\mathbf{a}, \mathbf{p}$ of distinct points, the gradient expressions (3.7) are independent of the variable $\mathbf{b}$

ranging in $\mathbb{R}^2 \setminus \text{Line}\{\mathbf{a}, \mathbf{p}\}$. This latter condition can be formulated in terms of the $\mathbf{b}$ -independent affine coordinates (2.2) as follows. By (3.6) we have

$$\begin{aligned} \mathbf{g}_{\mathbf{a}, \mathbf{p}}^{\mathbf{b}} &= \frac{(\mathbf{a} - \mathbf{p})\mathbf{R}}{\langle (\mathbf{a} - \mathbf{p})\mathbf{R} | \mathbf{b} - \mathbf{a} \rangle} = \|\mathbf{p} - \mathbf{a}\|^{-2} (1/\bar{\xi}_{\mathbf{p}, \mathbf{a}}^{\mathbf{b}}) (\mathbf{p} - \mathbf{a})\mathbf{R}, \\ \mathbf{g}_{\mathbf{a}, \mathbf{b}}^{\mathbf{p}} &= \frac{(\mathbf{a} - \mathbf{b})\mathbf{R}}{\langle (\mathbf{a} - \mathbf{b})\mathbf{R} | \mathbf{p} - \mathbf{a} \rangle} = \frac{\xi_{\mathbf{p}, \mathbf{a}}^{(\mathbf{a} - \mathbf{b})\mathbf{R}} (\mathbf{p} - \mathbf{a}) + \bar{\xi}_{\mathbf{p}, \mathbf{a}}^{(\mathbf{a} - \mathbf{b})\mathbf{R}} (\mathbf{p} - \mathbf{a})\mathbf{R}}{\|\mathbf{p} - \mathbf{a}\|^2 \bar{\xi}_{\mathbf{p}, \mathbf{a}}^{\mathbf{b}}} = \\ &= \|\mathbf{p} - \mathbf{a}\|^{-2} \left[(\mathbf{p} - \mathbf{a}) + (\xi_{\mathbf{p}, \mathbf{a}}^{\mathbf{b}} / \bar{\xi}_{\mathbf{p}, \mathbf{a}}^{\mathbf{b}}) (\mathbf{p} - \mathbf{a})\mathbf{R} \right]. \end{aligned}$$

Thus we can rewrite (3.7) in the form

$$\begin{aligned} \nabla f_{\mathbf{a}, \mathbf{b}}^{i, \mathbf{p}}(\mathbf{y}_t) &= [\mathbf{b}\text{-independent terms}] + \\ &+ \frac{x^{[i]}((1-t)(\mathbf{a} - \mathbf{p})) [\Phi^{[i]}]'(t) \xi_{\mathbf{p}, \mathbf{a}}^{\mathbf{b}} + t^2 (1-t) P_{\mathbf{a}, \mathbf{b}}^{i, \mathbf{p}}(0, 1-t)}{\|\mathbf{p} - \mathbf{a}\|^2 \bar{\xi}_{\mathbf{p}, \mathbf{a}}^{\mathbf{b}}} (\mathbf{p} - \mathbf{a})\mathbf{R}. \end{aligned} \quad (3.8)$$

Hence we conclude immediately the following.

Lemma 4. *We have (3.4) if and only if for every pair $\mathbf{p}, \mathbf{a}$ of distinct points there exist polynomials $K_{\mathbf{a}}^{i, \mathbf{p}}$ ($i=0, 1, 2$) of one variable such that*

$$K_{\mathbf{a}}^{i, \mathbf{p}}(t) = x^{[i]}((1-t)(\mathbf{a} - \mathbf{p})) [\Phi^{[i]}]'(t) \frac{\xi_{\mathbf{p}, \mathbf{a}}^{\mathbf{b}}}{\bar{\xi}_{\mathbf{p}, \mathbf{a}}^{\mathbf{b}}} + \frac{t^2 (1-t)}{\bar{\xi}_{\mathbf{p}, \mathbf{a}}^{\mathbf{b}}} P_{\mathbf{a}, \mathbf{b}}^{i, \mathbf{p}}(0, 1-t) \quad (3.9)$$

independently of the choice of $\mathbf{b}$ outside $\text{Line}\{\mathbf{a}, \mathbf{p}\}$.

We can regard (3.9) as a partial algebraic condition on the polynomials $P_{\mathbf{a}, \mathbf{b}}^{i, \mathbf{p}}$ of two variables, namely the identity $0 < t < 1$ below:

$$P_{\mathbf{a}, \mathbf{b}}^{i, \mathbf{p}}(0, 1-t) = \bar{\xi}_{\mathbf{p}, \mathbf{a}}^{\mathbf{b}} \frac{K_{\mathbf{a}}^{i, \mathbf{p}}(t)}{t^2 (1-t)} - \xi_{\mathbf{p}, \mathbf{a}}^{\mathbf{b}} \frac{x^{[i]}((1-t)(\mathbf{a} - \mathbf{p})) [\Phi^{[i]}]'(t)}{t^2 (1-t)}. \quad (3.10)$$

Since, for fixed $\mathbf{a}, \mathbf{p}$, the coordinates $(\xi_{\mathbf{p}, \mathbf{a}}^{\mathbf{b}}, \bar{\xi}_{\mathbf{p}, \mathbf{a}}^{\mathbf{b}})$ may assume arbitrary values (r, s) with $s \neq 0$, from (3.10) we obtain the polynomial divisibility relations $t^2(1-t) \mid K_{\mathbf{a}}^{i, \mathbf{p}}(t)$ and $t^2(1-t) \mid x^{[i]}((1-t)(\mathbf{a} - \mathbf{p})) [\Phi^{[i]}]'(t)$, respectively. Since $x^{[0]}((1-t)(\mathbf{a} - \mathbf{p})) \equiv 1$ and $x^{[0]}((1-t)(\mathbf{a} - \mathbf{p})) \equiv (1-t)x^{[j]}(\mathbf{a} - \mathbf{p})$ for $j = 1, 2$, with the aid of (3.22') we can state (3.10) in the form

$$P_{\mathbf{a}, \mathbf{b}}^{i, \mathbf{p}}(0, 1-t) = \frac{\langle \mathbf{b} - \mathbf{a} | \mathbf{p} - \mathbf{a} \rangle}{\|\mathbf{p} - \mathbf{a}\|^2} x_{\mathbf{p}}^{[i]}(\mathbf{a}) \chi^{[i]}(t) + \frac{\langle \mathbf{b} - \mathbf{a} | (\mathbf{p} - \mathbf{a})\mathbf{R} \rangle}{\|\mathbf{p} - \mathbf{a}\|^2} \kappa_{\mathbf{a}}^{i, \mathbf{p}}(t) \quad (3.11)$$

with suitable polynomials $\chi^{[i]}$ and $\kappa_{\mathbf{a}}^{i,\mathbf{p}}$ ($i = 0, 1, 2$; $\mathbf{a} \neq \mathbf{p} \in \mathbb{R}^2$) of one variable. Actually

$$\begin{aligned}\kappa_{\mathbf{a}}^{i,\mathbf{p}}(t) &= \frac{K_{\mathbf{a}}^{i,\mathbf{p}}(t)}{t^2(1-t)^2}, \quad \chi^{[0]}(t) = \frac{[\Phi^{[0]}]'(t)}{t^2(1-t)} = \frac{\Phi'(t)}{t^2(1-t)}, \\ x_{\mathbf{p}}^{[j]}(\mathbf{a})\chi^{[j]}(t) &= \frac{x^{[j]}((1-t)(\mathbf{a}-\mathbf{p}))[\Phi^{[j]}]'(t)}{t^2(1-t)} = x_{\mathbf{p}}^{[j]}(\mathbf{a}) \frac{[\Psi(t)/(t-1)]'}{t^2}\end{aligned}$$

for $j=1, 2$ on the basis of (3.10). In terms of the Kronecker- δ , we can write even

$$\chi^{[i]}(t) = t^{-2}(1-t)^{-\delta_{i,0}}[\Phi^{[i]}]'(t) \quad (i = 0, 1, 2).$$

Clearly, the polynomials $K_{\mathbf{c}}^{i,\mathbf{p}}$ cannot be chosen arbitrarily. There is a unique obstacle: we obtained Lemma 3 and hence (3.9) by an inspection of $\nabla f_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}}$ on one of the edges of a triangle $\mathbf{T} = \text{Co}\{\mathbf{a}, \mathbf{b}, \mathbf{p}\}$ at the distinguished point $\mathbf{p}$ (namely $\text{Co}\{\mathbf{a}, \mathbf{p}\}$ with the parametrization $\mathbf{y}_t := (1-t)\mathbf{a} + t\mathbf{b}$) while also the analogous conclusion should also be taken simultaneously in to account with the second edge (namely $\text{Co}\{\mathbf{b}, \mathbf{p}\}$ issued from $\mathbf{p}$). Applying a change $\mathbf{a} \leftrightarrow \mathbf{b}$ and taking into account the symmetry (3.3), we see that also

$$P_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}}(1-t, 0) = \frac{\langle \mathbf{a}-\mathbf{b} | \mathbf{p}-\mathbf{b} \rangle}{\|\mathbf{p}-\mathbf{b}\|^2} x_{\mathbf{p}}^{[i]}(\mathbf{b}) \chi^{[i]}(t) + \frac{\langle \mathbf{a}-\mathbf{b} | (\mathbf{p}-\mathbf{b})\mathbf{R} \rangle}{\|\mathbf{p}-\mathbf{b}\|^2} \kappa_{\mathbf{b}}^{i,\mathbf{p}}(t). \quad (3.12)$$

We obtain the complete description for the families of polynomials $K_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}}$ being admissible by Lemma 4 by the next observation.

Lemma 5. *For any pair $\mathbf{p} \neq \mathbf{c} \in \mathbb{R}^2$, in (3.11 – 3.12) we have $\chi^{[i]}(1) = \kappa_{\mathbf{c}}^{i,\mathbf{p}}(1) = 0$.*

Proof. Fix $i, \mathbf{p} \in \mathbb{R}^2$ and $\rho > 0$ arbitrarily. Consider (3.11) for pairs $\mathbf{a}, \mathbf{b}$ with $\|\mathbf{a}-\mathbf{p}\| = \|\mathbf{b}-\mathbf{p}\| = \rho$ written in the form

$$\mathbf{a} := \mathbf{c}_{\sigma}, \quad \mathbf{b} := \mathbf{c}_{\tau} \quad \text{where} \quad \mathbf{c}_{\tau} := \mathbf{p} + \rho \mathbf{u}_{\tau}, \quad \mathbf{u}_{\tau} := \cos \tau \mathbf{e}^{[1]} + \sin \tau \mathbf{e}^{[2]}.$$

Due to (3.3), with the abbreviations $\alpha := \chi^{[i]}(1)$ and $\beta(\tau) := \kappa_{\mathbf{c}_{\tau}}^{i,\mathbf{p}}(1)$ we get

$$\begin{aligned}0 &= P_{\mathbf{c}_{\sigma}, \mathbf{c}_{\tau}}^{i,\mathbf{p}}(0, 0) - P_{\mathbf{c}_{\tau}, \mathbf{c}_{\sigma}}^{i,\mathbf{p}}(0, 0) = \\ &= \left[(\langle \mathbf{u}_{\tau} | \mathbf{u}_{\sigma} \rangle - 1) x^{[i]}(\mathbf{c}_{\sigma}) \alpha + \langle \mathbf{u}_{\tau} | \mathbf{u}_{\sigma} \mathbf{R} \rangle \beta(\sigma) \right] - \\ &\quad - \left[(\langle \mathbf{u}_{\sigma} | \mathbf{u}_{\tau} \rangle - 1) x^{[i]}(\mathbf{c}_{\tau}) \alpha + \langle \mathbf{u}_{\sigma} | \mathbf{u}_{\tau} \mathbf{R} \rangle \beta(\tau) \right] = \\ &= (\langle \mathbf{u}_{\sigma} | \mathbf{u}_{\tau} \rangle - 1) [x^{[i]}(\mathbf{c}_{\sigma}) - x^{[i]}(\mathbf{c}_{\tau})] \alpha + \langle \mathbf{u}_{\tau} | \mathbf{u}_{\sigma} \mathbf{R} \rangle \beta(\sigma) - \langle \mathbf{u}_{\sigma} | \mathbf{u}_{\tau} \mathbf{R} \rangle \beta(\tau) = \\ &= [\cos(\sigma - \tau) - 1] [x^{[i]}(\mathbf{c}_{\sigma}) - x^{[i]}(\mathbf{c}_{\tau})] \alpha + \sin(\tau - \sigma) [\beta(\sigma) + \beta(\tau)].\end{aligned}$$

Since $x^{[0]} \equiv 1$, in any case we have $x_{\mathbf{p}}^{[i]}(\mathbf{c}_{\sigma}) - x_{\mathbf{p}}^{[i]}(\mathbf{c}_{\tau}) = \rho[x^{[i]}(\mathbf{u}_{\sigma}) - x^{[i]}(\mathbf{u}_{\tau})]$. It follows

$$\begin{aligned} \beta(\sigma) + \beta(\tau) &= \alpha \rho \frac{\cos(\tau - \sigma) - 1}{\sin(\tau - \sigma)} [x^{[i]}(\mathbf{u}_{\sigma}) - x^{[i]}(\mathbf{u}_{\tau})], \\ |\beta(\sigma) + \beta(\tau)| &\leq \rho |\alpha| \frac{1 - \cos(\tau - \sigma)}{\sin(|\tau - \sigma|)} \|\mathbf{u}_{\sigma} - \mathbf{u}_{\tau}\| \leq \\ &\leq 2\rho |\alpha| [1 - \cos(\tau - \sigma)]. \end{aligned} \quad (3.13)$$

Suppose indirectly $\beta(\tau) \neq 0$ for some $\tau \in \mathbb{R}$. Let $\varepsilon := |\beta(\tau)|$ and choose $\delta > 0$ such that $2\rho |\alpha| (1 - \cos \theta) < \varepsilon/4$ whenever $|\theta| \leq \varepsilon$. Observe that then we have $|\beta(\tau) + \beta(\tau \pm \delta/2)| < \varepsilon/4$ that is $\beta(\tau \pm \delta/2) \in [-\varepsilon/4, \varepsilon/4] - \beta(\tau)$. Therefore $\beta(\tau + \delta/2) + \beta(\tau - \delta/2) \in [-\varepsilon/2, \varepsilon/2] - 2\beta(\tau) \subset [-\varepsilon/2, \varepsilon/2] + \{-2\varepsilon, 2\varepsilon\} = [-5\varepsilon/2, -3\varepsilon/2] \cup [3\varepsilon/2, 5\varepsilon/2]$ i.e. $|\beta(\tau + \delta/2) + \beta(\tau - \delta/2)| \in [3\varepsilon/2, 5\varepsilon/2]$. However, we also have $|\beta(\tau + \delta/2) + \beta(\tau - \delta/2)| < \varepsilon/4$ which leads to the contradiction $|\beta(\tau + \delta/2) + \beta(\tau - \delta/2)| \in [3\varepsilon/2, 5\varepsilon/2] \cap [0, \varepsilon/4] = \emptyset$. By the arbitrariness of the radius ρ , the angle τ and the origin $\mathbf{p}$, we conclude that $\kappa_{\mathbf{c}}^{i,\mathbf{p}}(1) = 0$ in any case.

For $i = 1, 2$ we get $\alpha = 0$ i.e. $\chi^{[i]}(1) = 0$ immediately by plugging $\beta(\tau) = \beta(\sigma) = 0$ with $\sigma := \tau + \pi/4$ in the first equation of (3.13). (Remark: $x^{[0]}(\mathbf{u}_{\sigma}) - x^{[0]}(\mathbf{u}_{\tau}) = 1 - 1 = 0$, thus the argument does not work for $i = 0$). In the case $i = 0$ we conclude $\alpha = 0$ as follows. Consider the difference of equations (3.11–3.12) for $t = 1$ with $\mathbf{a} := \mathbf{p} + \mathbf{e}^{[1]}$ and $\mathbf{b} := \mathbf{p} + \mathbf{e}^{[1]} + \mathbf{e}^{[2]}$. Since $\kappa_{\mathbf{c}}^{i,\mathbf{p}} = 0$ ($\mathbf{c} = \mathbf{a}, \mathbf{b}$) is established already, we get simply $0 = -(1/2)\chi^{[0]}(1)$ which completes the proof.

Corollary 1. *The relations (3.4) hold if and only if we have (3.9) with the symmetry (3.3) where the polynomials $K_{\mathbf{c}}^{i,\mathbf{p}}(t)$ respectively $x^{[i]}((1-t)(\mathbf{a}-\mathbf{p}))[\Phi^{[i]}]'(t)$ are all divisible by $t^2(1-t)^2$.*

Proof. The relation $\kappa_{\mathbf{c}}^{i,\mathbf{p}}(1) = 0$ implies that there is a polynomial $\tilde{\kappa}_{\mathbf{c}}^{i,\mathbf{p}}$ such that $\kappa_{\mathbf{c}}^{i,\mathbf{p}}(t) = (1-t)\tilde{\kappa}_{\mathbf{c}}^{i,\mathbf{p}}(t)$ and $K_{\mathbf{c}}^{i,\mathbf{p}}(t) = t^2(1-t)\kappa_{\mathbf{c}}^{i,\mathbf{p}} = t^2(1-t)^2\tilde{\kappa}_{\mathbf{c}}^{i,\mathbf{p}}(t)$ with some polynomial. Similarly, from $\chi^{[i]}(1) = 0$ we conclude that $\chi^{[i]}(t) = (1-t)\tilde{\chi}^{[i]}(t)$ and $(1-t)x^{[i]}(\mathbf{a})[\Phi^{[i]}]'(t) = ((1-t)t^2\tilde{\chi}^{[i]}(t) = t^2(1-t)^2\tilde{\chi}^{[i]}(t)$ with some polynomial $\tilde{\chi}^{[i]}$.

Corollary 2. *We can write $K_{\mathbf{c}}^{i,\mathbf{p}}(t) = t^2(1-t)^2k_{\mathbf{c}}^{i,\mathbf{p}}(t)$ ($\mathbf{p} \neq \mathbf{c} \in \mathbb{R}^2$) and the admissible shape functions Φ, Ψ have the form*

$$\begin{aligned} \text{(i)} \quad \Phi(t) &= t^3(10 - 15t + 6t^2) + t^3(1-t)^3\Phi_1(t), \\ \text{(ii)} \quad \Psi(t) &= t^3(t-1)(4-3t) + t^3(1-t)^3\Psi_1(t) \end{aligned} \quad (3.14)$$

with suitable polynomials $k_{\mathbf{c}}^{i,\mathbf{p}}, \Phi_2, \Psi_2$.

Proof. The stated form of $K_{\mathbf{c}}^{i,\mathbf{p}}$ is clear from (1). By definition $\Phi^{[0]}(t) = \Phi(t)$ and $x^{[0]}((1-t)(\mathbf{a}-\mathbf{p})) \equiv 1$. Furthermore, for $j = 1, 2$ we have $\Phi^{[j]}(t) = \Psi(t)/(t-1)$ and $x^{[j]}((1-t)(\mathbf{a}-\mathbf{p})) = (1-t)x_{\mathbf{p}}^{[j]}(\mathbf{a})$. Thus, taking (3.1) into account, the relation

that $t^2(1-t)^2$ is a divisor of the polynomial $x^{[0]}((1-t)(\mathbf{a}-\mathbf{p}))[\Phi^{[0]}]'(t) = \Phi'(t) = 6t(1-t) + 2t(1-t)(1-2t)\Phi_0(t) + t^2(1-t)^2\Phi_0'(t)$ means simply that $t(1-t)$ is a divisor of $6 + 2(1-2t)\Phi_0(t)$. Thus we have $6 + 2(1-2t)\Phi_0(t)|_{t=0,1} = 0$ implying that $\Phi_0(0) = -3$, $\Phi_0(1) = 3$. Therefore $\Phi_0(t) = -3 + 6t + t(1-t)\Phi_1(t)$ with some polynomial Φ_1 and the generic form of Φ is (3.14(i)). Also according to (3.1), in the cases $j = 1, 2$ we can write $\Psi(t) = -t^2(1-t) + t^2(1-t)^2\Psi_0(t)$ with some polynomial Ψ_0 . Thus the relation that $t^2(1-t)^2$ is a divisor of the polynomial $x^{[j]}((1-t)(\mathbf{a}-\mathbf{p}))[\Phi^{[0]}]'(t) \equiv (1-t)x_{\mathbf{p}}^{[j]}(\mathbf{a})[\Psi(t)/(1-t)]'$ means that $t^2(1-t)|[\Psi(t)/(1-t)]' \equiv -2t + t(2-3t)\Psi_0(t) + t^2(1-t)^2\Psi_0'(t)$ is equivalent to saying $t(1-t)|-2 + (2-3t)\Psi_0(t)$ i.e. $-2 + (2-3t)\Psi_0(t)|_{t=0,1} = 0$ implying $\Psi_0(0) = 1$ and $\Psi_0(1) = -2$. Therefore $\Psi_0(t) = 1 - 3t + t^2(1-t)\Psi_1(t)$ with some polynomial Ψ_1 and the generic form of Ψ is (3.14(i)).

Finish of the proof of Theorem 2. In view of (3.11 - 3.12) and Remark 2 we can write

$$\begin{aligned} P_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}}(s,t) &= P_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}}(0,0) + s[(P_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}}(s,0) - P_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}}(0,0))/s] \\ &\quad + t[(P_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}}(0,t) - P_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}}(0,0))/t] + st\text{Pol}(s,t) \\ &= s[(P_{\mathbf{b},\mathbf{a}}^{i,\mathbf{p}}(0,s)/s] + t[(P_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}}(0,t)/t] + st\text{Pol}(s,t) \\ &= s\left[\bar{\xi}_{\mathbf{p},\mathbf{b}}^{\mathbf{a}} \frac{K_{\mathbf{b}}^{i,\mathbf{p}}(1-s)}{s^2(1-s)^2} - \xi_{\mathbf{p},\mathbf{b}}^{\mathbf{a}} \frac{x^{[i]}(\mathbf{b})[\Phi^{[i]}]'(1-s)}{s(1-s)^2}\right] \\ &\quad + t\left[\bar{\xi}_{\mathbf{p},\mathbf{a}}^{\mathbf{b}} \frac{K_{\mathbf{a}}^{i,\mathbf{p}}(1-t)}{t^2(1-t)^2} - \xi_{\mathbf{p},\mathbf{a}}^{\mathbf{b}} \frac{x^{[i]}(\mathbf{a})[\Phi^{[i]}]'(1-t)}{t(1-t)^2}\right] + stR_{\mathbf{a},\mathbf{b}}^{\mathbf{p}}(s,t) \end{aligned}$$

with suitable polynomials $K_{\mathbf{c}}^{i,\mathbf{p}}, \Phi^{[i]}, R_{\mathbf{a},\mathbf{b}}^{\mathbf{p}}$ of one- resp. two variables such that $t^2(1-t)^2|K_{\mathbf{c}}^{i,\mathbf{p}}(t)$ and $t^2(1-t)|x_{\mathbf{p}}^{[i]}(\mathbf{a})[\Phi^{[i]}]'(t)$. It is straightforward to check that the functions $f_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}}$ are polynomials in these cases and $P_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}}(s,t) = P_{\mathbf{b},\mathbf{a}}^{i,\mathbf{p}}(t,s)$ if and only if $R_{\mathbf{a},\mathbf{b}}^{\mathbf{p}}(s,t) = R_{\mathbf{b},\mathbf{a}}^{\mathbf{p}}(t,s)$. It remains to show that the expressions

$$\begin{aligned} \nabla f_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}}(\mathbf{y}_t) \quad \text{with} \quad \mathbf{y}_t &:= (1-t)\mathbf{a} + t\mathbf{p}, \\ f_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}} &= \Phi^{[i]}(\lambda_{\mathbf{a},\mathbf{b}}^{\mathbf{p}}) + [\lambda_{\mathbf{a},\mathbf{b}}^{\mathbf{p}}]^2 \lambda_{\mathbf{a},\mathbf{p}}^{\mathbf{b}} \lambda_{\mathbf{b},\mathbf{p}}^{\mathbf{a}} P_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}}(\lambda_{\mathbf{a},\mathbf{b}}^{\mathbf{b}}, \lambda_{\mathbf{b},\mathbf{p}}^{\mathbf{a}}) \end{aligned}$$

are independent of the term $\mathbf{b}$ whenever

$$\begin{aligned} K_{\mathbf{c}}^{i,\mathbf{p}}(t) &= t^2(1-t)^2 k_{\mathbf{c}}^{i,\mathbf{p}}(t), \\ \Phi^{[0]}(t) &= \Phi(t), \quad \Phi^{[1]}(t) = \Phi^{[2]}(t) \equiv [\Psi(t)/(t-1)]' \end{aligned}$$

with arbitrary polynomials $k_{\mathbf{c}}^{i,\mathbf{p}}$ and the polynomials Φ, Ψ have the form (3.14) with arbitrarily fixed polynomials Φ_1, Ψ_1 of one variable.

Repeating the calculations of Lemma 3, we see that (3.8) holds independently of the choice of $k_c^{i,p}, \Phi_1, \Psi_1, R_{a,b}^{i,p}$. Notice that we have constructed the polynomials $P_{a,b}^{i,p}(0, 1-t) = P_{b,a}^{i,p}(1-t, 0)$ in terms of $K_a^{i,p}$ in a manner such that (3.9) should be fulfilled. Thus the expression

$$[\bar{\xi}_{p,a}^b]^{-1} \left[x^{[i]} ((1-t)(a-p)) [\Phi^{[i]}]'(t) \xi_{p,a}^b + t^2 (1-t) P_{a,b}^{i,p}(0, 1-t) \right] (= K_a^{i,p}(t))$$

is independent of b automatically. This completes the proof in view of Lemma 4.

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EVENTUAL PERIODICITY OF A MAX-TYPE SYSTEM OF DIFFERENCE EQUATIONS OF HIGHER ORDER WITH FOUR VARIABLES

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Abstract. The aim of this paper is to investigate eventual periodicity of the following max-type system of difference equations of higher order with four variables

$$\begin{cases} u_n = \max \left\{ A, \frac{s_{n-k}}{v_{n-1}} \right\}, \\ v_n = \max \left\{ B, \frac{t_{n-k}}{u_{n-1}} \right\}, \\ s_n = \max \left\{ C, \frac{u_{n-k}}{t_{n-1}} \right\}, \\ t_n = \max \left\{ D, \frac{v_{n-k}}{s_{n-1}} \right\}, \end{cases} \quad n \in \{0, 1, 2, \dots\},$$

where k is a positive integer, $A, B, C, D \in (0, +\infty)$ with $A \leq B$ and $C \leq D$, and the initial values $u_{-i}, v_{-i}, s_{-i}, t_{-i} \in (0, +\infty)$ for $i \in \{1, 2, \dots, k\}$. We show that:

- (1) If $AC < 1$ or $A = B = C = D = 1$, then there exists a solution $\{(u_n, v_n, s_n, t_n)\}_{n=-k}^{+\infty}$ of this system which is not eventually periodic.
- (2) If $BD = AC = 1$ with $A \neq C$ or $BD > AC = 1$ or $AC > 1$, then every solution of this system is eventually periodic.

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1. INTRODUCTION

The aim of this paper is to investigate eventual periodicity of the following max-type system of difference equations of higher order with four variables

$$\begin{cases} u_n = \max \left\{ A, \frac{s_{n-k}}{v_{n-1}} \right\}, \\ v_n = \max \left\{ B, \frac{t_{n-k}}{u_{n-1}} \right\}, \\ s_n = \max \left\{ C, \frac{u_{n-k}}{t_{n-1}} \right\}, \\ t_n = \max \left\{ D, \frac{v_{n-k}}{s_{n-1}} \right\}, \end{cases} \quad n \in \mathbb{N}_0 \equiv \{0, 1, 2, \dots\}, \quad (1.1)$$

where $k \in \mathbb{N} \equiv \{1, 2, \dots\}$, $A, B, C, D \in \mathbb{R}^+ \equiv (0, +\infty)$ satisfying

$$A \leq B \quad \text{and} \quad C \leq D, \quad (1.2)$$

and the initial values $u_{-i}, v_{-i}, s_{-i}, t_{-i} \in (0, +\infty)$ ($i \in \mathbb{Z}(1, k)$), where $\mathbb{Z}(a, b) \equiv \{a, a+1, \dots, b\}$ for any integer $a < b$.

If $x_n = u_n = v_n$ and $y_n = s_n = t_n$ and $A = B$ and $C = D$, then (1.1) reduces to the following max-type system

$$\begin{cases} x_n = \max \left\{ A, \frac{y_{n-k}}{x_{n-1}} \right\}, \\ y_n = \max \left\{ C, \frac{x_{n-k}}{y_{n-1}} \right\}, \end{cases} \quad n \in \mathbb{N}_0. \quad (1.3)$$

In [24], Su et al. investigated the periodicity of (1.3) and showed that if $AC > 1$ or $AC = 1$ and $A \neq C$, then every positive solution of (1.3) is eventually periodic.

Recently, the study of the properties of the max-type difference equations and systems, such as global behavior, eventual periodicity and boundedness, has aroused a great deal of interest (see [1–5, 7–17, 19–21, 23, 26, 28, 30]). For example, Fotiadis and Papaschinopoulos [6] investigated the following max-type system of difference equations

$$\begin{cases} x_n = \max \left\{ A, \frac{y_{n-1}}{x_{n-2}} \right\}, \\ y_n = \max \left\{ B, \frac{x_{n-1}}{y_{n-2}} \right\}, \end{cases} \quad n \in \mathbb{N}_0, \quad (1.4)$$

and showed that every positive solution of (1.4) is eventually periodic.

In [25], Su et al., inspired by above results of (1.4), investigated the periodicity of positive solutions of the following max-type systems of difference equations

$$\begin{cases} x_n = \max \left\{ A_n, \frac{y_{n-1}}{x_{n-2}} \right\}, \\ y_n = \max \left\{ B_n, \frac{x_{n-1}}{y_{n-2}} \right\}, \end{cases} \quad n \in \mathbb{N}_0, \quad (1.5)$$

where $A_n, B_n \in \mathbb{R}^+$ are periodic with period 2 and showed that every positive solution of (1.5) is eventually periodic.

In 2015, Yazlik et al. [31] investigated the periodicity of positive solutions of the following system

$$\begin{cases} x_n = \max \left\{ \frac{1}{x_{n-1}}, \min \left\{ 1, \frac{A}{y_{n-1}} \right\} \right\}, \\ y_n = \max \left\{ \frac{1}{y_{n-1}}, \min \left\{ 1, \frac{A}{x_{n-1}} \right\} \right\}, \end{cases} \quad n \in \mathbb{N}_0 \quad (1.6)$$

and the general solution of (1.6) is obtained in an elegant manner.

The above results of (1.6) motivated Sun and Xi [27] in 2016 to investigate the following more general system

$$\begin{cases} x_n = \max \left\{ \frac{1}{x_{n-m}}, \min \left\{ 1, \frac{A}{y_{n-r}} \right\} \right\}, \\ y_n = \max \left\{ \frac{1}{y_{n-m}}, \min \left\{ 1, \frac{B}{x_{n-t}} \right\} \right\}, \end{cases} \quad n \in \mathbb{N}_0, \quad (1.7)$$

where $A, B \in \mathbb{R}^+$, $m, r, t \in \mathbb{N}$ and the initial values $x_{-i}, y_{-i} \in \mathbb{R}^+$ ($i \in \mathbb{Z}(1, d)$) with $d = \max\{m, r, t\}$ and it is shown that every positive solution of (1.7) is eventually periodic with period $2m$.

In 2013, Stević [18] investigated the boundedness character and global attractivity of the following symmetric system

$$\begin{cases} x_n = \max \left\{ B, \frac{y_{n-1}^p}{x_{n-2}^p} \right\}, \\ y_n = \max \left\{ B, \frac{x_{n-1}^p}{y_{n-2}^p} \right\}, \end{cases} \quad n \in \mathbb{N}_0, \quad (1.8)$$

where $B, p \in \mathbb{R}^+$ and the initial values $x_{-i}, y_{-i} \in \mathbb{R}^+$ ($i \in \mathbb{Z}(1, 2)$).

Also above results of (1.8) motivated Stević [22] to continue studying the behavior of the following system

$$\begin{cases} x_n = \max \left\{ B, \frac{y_{n-1}^p}{z_{n-2}^p} \right\}, \\ y_n = \max \left\{ B, \frac{z_{n-1}^p}{x_{n-2}^p} \right\}, \\ z_n = \max \left\{ B, \frac{x_{n-1}^p}{y_{n-2}^p} \right\}. \end{cases} \quad n \in \mathbb{N}_0, \quad (1.9)$$

where $B, p \in \mathbb{R}^+$ and the initial values $x_{-i}, y_{-i}, z_{-i} \in \mathbb{R}^+$ ($i \in \mathbb{Z}(1, 2)$), and showed that system (1.9) is permanent when $p \in (0, 4)$.

In this paper, we investigate eventual periodicity of (1.1) and obtain the following theorem.

Theorem 1.

- (1) If $AC < 1$, then there exists a solution $(u_n, v_n, s_n, t_n)_{n=-k}^{+\infty}$ of (1.1) such that $u_n = A$ and $s_n = C$ for any $n \geq -k$ and $\lim_{n \rightarrow \infty} v_n = \lim_{n \rightarrow \infty} t_n = \infty$
- (2) If $A = B = C = D = 1$, then there exists a solution $\{(u_n, v_n, s_n, t_n)\}_{n=-k}^{\infty}$ of (1.1) such that $u_n = v_n = s_n = t_n$ and $1 < u_{n+1} < u_n$ for any $n \geq -k$ and $\lim_{n \rightarrow \infty} u_n = 1$.
- (3) If $BD = AC = 1$ and $A \neq C$, then every solution of (1.1) is eventually periodic with period $2k$.
- (4) If $BD > AC = 1$ and $\{(u_n, v_n, s_n, t_n)\}_{n=-k}^{+\infty}$ is a solution of (1.1), then u_n and s_n are eventually periodic with period 2 and v_n and t_n are eventually periodic with period $2k$.
- (5) If $AC > 1$, then every solution of (1.1) is eventually periodic with period 1.

2. PROOF OF THEOREM 1

In this section, we study eventual periodicity of positive solutions of system (1.1). Let $u_n = Ax_n, v_n = By_n, s_n = Cp_n, t_n = Dq_n$ for any $n \geq -k$. Then (1.1) reduces to the

following system

$$\begin{cases} x_n = \max \left\{ 1, \frac{Cp_{n-k}}{AB y_{n-1}} \right\}, \\ y_n = \max \left\{ 1, \frac{Dq_{n-k}}{AB x_{n-1}} \right\}, \\ p_n = \max \left\{ 1, \frac{Ax_{n-k}}{CD q_{n-1}} \right\}, \\ q_n = \max \left\{ 1, \frac{By_{n-k}}{CD p_{n-1}} \right\}, \end{cases} \quad n \in \mathbb{N}_0, \quad (2.1)$$

where the initial values $x_{-i}, y_{-i}, p_{-i}, q_{-i} \in \mathbb{R}^+$ ($i \in \mathbb{Z}(1, k)$). Let $\{(x_n, y_n, p_n, q_n)\}_{n=-k}^{+\infty}$ be a positive solution of (2.1). To show Theorem 1, we need the following lemmas and propositions.

Lemma 1.

(1) For any $n \in \mathbb{N}_0$,

$$x_n \geq 1, \quad y_n \geq 1, \quad p_n \geq 1, \quad q_n \geq 1. \quad (2.2)$$

(2) If $AC \geq 1$, then for any $r \in \mathbb{N}$ and $n \geq 2rk$,

$$\begin{cases} x_n = \max \left\{ 1, \frac{C}{AB y_{n-1}}, \left(\frac{1}{BD} \right)^r \frac{x_{n-2rk}}{y_{n-1} q_{n-k-1} y_{n-2k-1} q_{n-3k-1} \cdots y_{n-2(r-1)k-1} q_{n-(2r-1)k-1}} \right\}, \\ y_n = \max \left\{ 1, \frac{D}{AB x_{n-1}}, \left(\frac{1}{AC} \right)^r \frac{y_{n-2rk}}{x_{n-1} p_{n-k-1} x_{n-2k-1} p_{n-3k-1} \cdots x_{n-2(r-1)k-1} p_{n-(2r-1)k-1}} \right\}, \\ p_n = \max \left\{ 1, \frac{A}{CD q_{n-1}}, \left(\frac{1}{BD} \right)^r \frac{p_{n-2rk}}{q_{n-1} y_{n-k-1} q_{n-2k-1} y_{n-3k-1} \cdots q_{n-2(r-1)k-1} y_{n-(2r-1)k-1}} \right\}, \\ q_n = \max \left\{ 1, \frac{B}{CD p_{n-1}}, \left(\frac{1}{AC} \right)^r \frac{q_{n-2rk}}{p_{n-1} x_{n-k-1} p_{n-2k-1} x_{n-3k-1} \cdots p_{n-2(r-1)k-1} x_{n-(2r-1)k-1}} \right\}. \end{cases} \quad (2.3)$$

Proof.

(1) It follows from (2.1).

(2) Note $BD \geq AC \geq 1$. From this, (2.1) and (2.2) it follows that for any $r \in \mathbb{N}$ and $n \geq 2rk$,

$$\begin{aligned} x_n &= \max \left\{ 1, \frac{Cp_{n-k}}{AB y_{n-1}} \right\} \\ &= \max \left\{ 1, \frac{C}{AB y_{n-1}} \max \left\{ 1, \frac{Ax_{n-2k}}{CD q_{n-k-1}} \right\} \right\} \\ &= \max \left\{ 1, \frac{C}{AB y_{n-1}}, \frac{x_{n-2k}}{BD y_{n-1} q_{n-k-1}} \right\} \\ &= \max \left\{ 1, \frac{C}{AB y_{n-1}}, \frac{1}{BD y_{n-1} q_{n-k-1}} \max \left\{ 1, \frac{C}{AB y_{n-2k-1}}, \frac{x_{n-4k}}{BD y_{n-2k-1} q_{n-3k-1}} \right\} \right\} \\ &= \max \left\{ 1, \frac{C}{AB y_{n-1}}, \left(\frac{1}{BD} \right)^2 \frac{x_{n-4k}}{y_{n-1} q_{n-k-1} y_{n-2k-1} q_{n-3k-1}} \right\} \\ &\dots \\ &= \max \left\{ 1, \frac{C}{AB y_{n-1}}, \left(\frac{1}{BD} \right)^r \frac{x_{n-2rk}}{y_{n-1} q_{n-k-1} y_{n-2k-1} q_{n-3k-1} \cdots y_{n-(2r-2)k-1} q_{n-(2r-1)k-1}} \right\}. \end{aligned}$$

In a similar way, also we can obtain the other three formulas. The proof is complete. $\square$

Proposition 1. *If $AC < 1$, then there exists a solution $\{(x_n, y_n, p_n, q_n)\}_{n=-k}^{+\infty}$ of (2.1) such that $x_n = p_n = 1$ for any $n \geq -k$ and $\lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} q_n = \infty$.*

Proof. Let $x_{-i} = p_{-i} = 1$ and $y_{-i} = q_{-i} = \max\{\frac{AB}{D}, \frac{A}{CD}, \frac{C}{AB}, \frac{CD}{B}\} + 1$ for any $i \in \mathbb{Z}(1, k)$. Then by a simple calculation it follows from (1.2) and (2.1) that

$$\begin{aligned} & \begin{cases} x_0 = \max\left\{1, \frac{Cp_{-k}}{AB y_{-1}}\right\} = 1, \\ y_0 = \max\left\{1, \frac{Dq_{-k}}{AB x_{-1}}\right\} = \frac{Dq_{-k}}{AB} > 1, \\ p_0 = \max\left\{1, \frac{Ax_{-k}}{CD q_{-1}}\right\} = 1, \\ q_0 = \max\left\{1, \frac{By_{-k}}{CD p_{-1}}\right\} = \frac{By_{-k}}{CD} > 1. \end{cases} \\ & \dots\dots\dots \\ & \begin{cases} x_{k-1} = \max\left\{1, \frac{Cp_{-1}}{AB y_{k-2}}\right\} = 1, \\ y_{k-1} = \max\left\{1, \frac{Dq_{-1}}{AB x_{k-2}}\right\} = \frac{Dq_{-1}}{AB} > 1, \\ p_{k-1} = \max\left\{1, \frac{Ax_{-1}}{CD q_{k-2}}\right\} = 1, \\ q_{k-1} = \max\left\{1, \frac{By_{-1}}{CD p_{k-2}}\right\} = \frac{By_{-1}}{CD} > 1. \end{cases} \\ & \begin{cases} x_k = \max\left\{1, \frac{Cp_0}{AB y_{k-1}}\right\} = 1, \\ y_k = \max\left\{1, \frac{Dq_0}{AB x_{k-1}}\right\} = \frac{y_{-k}}{AC} > 1, \\ p_k = \max\left\{1, \frac{Ax_0}{CD q_{k-1}}\right\} = 1, \\ q_k = \max\left\{1, \frac{By_0}{CD p_{k-1}}\right\} = \frac{q_{-k}}{AC} > 1. \end{cases} \\ & \begin{cases} x_{k+1} = \max\left\{1, \frac{Cp_1}{AB y_k}\right\} = 1, \\ y_{k+1} = \max\left\{1, \frac{Dq_1}{AB x_k}\right\} = \frac{By_{1-k}}{CD} > 1, \\ p_{k+1} = \max\left\{1, \frac{Ax_1}{CD q_k}\right\} = 1, \\ q_{k+1} = \max\left\{1, \frac{By_1}{CD p_k}\right\} = \frac{q_{1-k}}{AC} > 1. \end{cases} \\ & \dots\dots\dots \end{aligned}$$

$$\begin{cases} x_{2k-1} = \max \left\{ 1, \frac{Cp_{k-1}}{AB y_{2k-2}} \right\} = 1, \\ y_{2k-1} = \max \left\{ 1, \frac{Dq_{k-1}}{AB x_{2k-2}} \right\} = \frac{y_{-1}}{AC} > 1, \\ p_{2k-1} = \max \left\{ 1, \frac{Ax_{k-1}}{CD q_{2k-2}} \right\} = 1, \\ q_{2k-1} = \max \left\{ 1, \frac{By_{k-1}}{CD p_{2k-2}} \right\} = \frac{q_{-1}}{AC} > 1. \end{cases}$$

By mathematical induction, we can obtain that for any $\lambda \in \mathbb{N}_0$ and any $r \in \mathbb{Z}(0, k-1)$,

$$\begin{cases} x_{2\lambda k+r} = \max \left\{ 1, \frac{Cp_{2\lambda k-r}}{AB y_{2\lambda k+r-1}} \right\} = 1, \\ y_{2\lambda k+r} = \max \left\{ 1, \frac{Dq_{2\lambda k-r}}{AB x_{2\lambda k+r-1}} \right\} = \frac{D}{AB} \left(\frac{1}{AC} \right)^\lambda q_{r-k} > 1, \\ p_{2\lambda k+r} = \max \left\{ 1, \frac{Ax_{2\lambda k-r}}{CD q_{2\lambda k+r-1}} \right\} = 1, \\ q_{2\lambda k+r} = \max \left\{ 1, \frac{By_{2\lambda k-r}}{CD p_{2\lambda k+r-1}} \right\} = \frac{B}{CD} \left(\frac{1}{AC} \right)^\lambda y_{r-k} > 1 \end{cases}$$

and

$$\begin{cases} x_{(2\lambda+1)k+r} = \max \left\{ 1, \frac{Cp_{2\lambda k+r}}{AB y_{(2\lambda+1)k+r-1}} \right\} = 1, \\ y_{(2\lambda+1)k+r} = \max \left\{ 1, \frac{Dq_{2\lambda k+r}}{AB x_{(2\lambda+1)k+r-1}} \right\} = \left(\frac{1}{AC} \right)^{\lambda+1} y_{r-k} > 1, \\ p_{(2\lambda+1)k+r} = \max \left\{ 1, \frac{Ax_{2\lambda k+r}}{CD q_{(2\lambda+1)k+r-1}} \right\} = 1, \\ q_{(2\lambda+1)k+r} = \max \left\{ 1, \frac{By_{2\lambda k+r}}{CD p_{(2\lambda+1)k+r-1}} \right\} = \left(\frac{1}{AC} \right)^{\lambda+1} q_{r-k} > 1. \end{cases}$$

From the above we have $x_n = p_n = 1$ for any $n \geq -k$ and $\lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} q_n = \infty$. The proof is complete. $\square$

In Example 3.1 of [29], we showed that the following equation

$$x_n = \frac{x_{n-k}}{x_{n-1}} \quad (2.4)$$

has a positive solution z_n ($n \geq -k$) with $1 < z_{n+1} < z_n$ for any $n \geq -k$ and $\lim_{n \rightarrow \infty} z_n = 1$. From Example 3.1 of [29], we obtain the following proposition.

Proposition 2. *If $A = B = C = D = 1$ and z_n ($n \geq -k$) is a positive solution of (2.4) with $1 < z_{n+1} < z_n$ for any $n \geq -k$ and $\lim_{n \rightarrow \infty} z_n = 1$, then there exists a solution $\{(x_n, y_n, p_n, q_n)\}_{n=-k}^\infty$ of (2.1) such that $x_n = y_n = p_n = q_n = z_n$ and $1 < x_{n+1} < x_n$ for any $n \geq -k$ and $\lim_{n \rightarrow \infty} x_n = 1$.*

Now we assume $A = B > AC = 1 > C = D$. From Lemma 1, we see that for any $i \in \mathbb{Z}(0, 2k-1)$ and $n \in \mathbb{N}$,

$$\begin{cases} x_{2nk+i} = \max \left\{ 1, \frac{x_{2(n-1)k+i}}{y_{2nk+i-1} q_{2nk+i-k-1}} \right\}, \\ y_{2nk+i} = \max \left\{ 1, \frac{y_{2(n-1)k+i}}{x_{2nk+i-1} p_{2nk+i-k-1}} \right\}, \\ p_{2nk+i} = \max \left\{ 1, \frac{A}{C^2 q_{2nk+i-1}}, \frac{p_{2(n-1)k+i}}{q_{2nk+i-1} y_{2nk+i-k-1}} \right\}, \\ q_{2nk+i} = \max \left\{ 1, \frac{A}{C^2 p_{2nk+i-1}}, \frac{q_{2(n-1)k+i}}{p_{2nk+i-1} x_{2nk+i-k-1}} \right\} \end{cases} \quad (2.5)$$

since $\frac{C}{A^2 y_{2nk+i-1}} < 1$ and $\frac{C}{A^2 x_{2nk+i-1}} < 1$. From (2.5) and (2.2) it follows that for any $i \in \mathbb{Z}(0, 2k-1)$ and $n \in \mathbb{N}$,

$$\begin{cases} 1 \leq x_{2nk+i} \leq \max \left\{ 1, x_{2(n-1)k+i} \right\} \leq x_{2(n-1)k+i}, \\ 1 \leq y_{2nk+i} \leq \max \left\{ 1, y_{2(n-1)k+i} \right\} \leq y_{2(n-1)k+i}. \end{cases}$$

Write

$$\begin{cases} \lim_{n \rightarrow \infty} x_{2nk+i} = A_i \geq 1, \\ \lim_{n \rightarrow \infty} y_{2nk+i} = B_i \geq 1. \end{cases}$$

Lemma 2. Let $A = B > AC = 1 > C = D$.

- (1) If $A_i > 1$ (resp. $B_i > 1$) for some $i \in \mathbb{Z}(0, 2k-1)$, then $x_{2nk+i+2r}$ and $p_{2nk-k+i+2r}$ (resp. $y_{2nk+i+2r}$ and $q_{2nk-k+i+2r}$) are constant sequences eventually for any $r \in \mathbb{N}$, and $q_{2nk-k+i+2r+1} = y_{2nk+i+2r+1} = 1$ (resp. $p_{2nk-k+i+2r+1} = x_{2nk+i+2r+1} = 1$) eventually for any $r \in \mathbb{N}_0$.
- (2) If $A_i = 1$ (resp. $B_i = 1$) for some $i \in \mathbb{Z}(0, 2k-1)$, then $x_{2nk+i+2r} = 1$ (resp. $y_{2nk+i+2r} = 1$) eventually for any $r \in \mathbb{N}_0$.

Proof.

- (1) If $A_i > 1$ for some $i \in \mathbb{Z}(0, 2k-1)$, then by (2.5) one has

$$x_{2nk+i} = \frac{x_{2(n-1)k+i}}{y_{2nk+i-1} q_{2nk+i-k-1}} \quad (2.6)$$

eventually and

$$y_{2nk+i+1} = \max \left\{ 1, \frac{y_{2(n-1)k+i+1}}{x_{2nk+i} p_{2nk+i-k}} \right\} = 1 \quad (2.7)$$

eventually since $p_{2nk+i-k} \geq 1$ and

$$\frac{y_{2(n-1)k+i+1}}{x_{2nk+i} p_{2nk+i-k}} \leq \frac{y_{2(n-1)k+i+1}}{x_{2nk+i}} \longrightarrow \frac{B_{i+1}}{A_i} < B_{i+1} = \lim_{n \rightarrow \infty} y_{2nk+i+1}.$$

It follows from (2.6) that

$$\lim_{n \rightarrow \infty} y_{2nk+i-1} = \lim_{n \rightarrow \infty} q_{2nk+i-k-1} = 1. \quad (2.8)$$

On the other hand, by (2.1) we see

$$x_{2nk+i} = \frac{C p_{2nk+i-k}}{A^2 y_{2nk+i-1}} \quad (2.9)$$

eventually. Furthermore by (1.2) and (2.1) and (2.8) we have

$$y_{2nk+i-1} = \max \left\{ 1, \frac{C q_{2nk+i-k-1}}{A^2 x_{2nk+i-1}} \right\} = 1 \quad (2.10)$$

eventually since $\frac{C q_{2nk+i-k-1}}{A^2 x_{2nk+i-1}} \longrightarrow \frac{C}{A^2 A_{i-1}} < 1$, which with (2.9) implies

$$x_{2nk+i} = \frac{C p_{2nk+i-k}}{A^2}. \quad (2.11)$$

It follows from (2.1) and (2.7) and (2.11) that

$$q_{2nk+i-k+1} = \max \left\{ 1, \frac{Ay_{2nk+i-2k+1}}{C^2 p_{2nk+i-k}} \right\} = 1 \quad (2.12)$$

eventually since

$$\lim_{n \rightarrow \infty} \frac{Ay_{2nk+i-2k+1}}{C^2 p_{2nk+i-k}} = \frac{1}{A_i} < 1.$$

Thus by (2.2) and (2.5) and (2.7) and (2.12) we have

$$x_{2nk+i+2} = \max \left\{ 1, \frac{x_{2(n-1)k+i+2}}{y_{2nk+i+1} q_{2nk+i-k+1}} \right\} = x_{2(n-1)k+i+2}$$

eventually.

We claim that $x_{2nk+i+2} > 1$ eventually. Indeed, if $x_{2nk+i+2} = 1$ eventually, then by (2.1) and (2.2) and (2.7) and (2.11) one has

$$q_{2nk+k+i+1} = \max \left\{ 1, \frac{Ay_{2nk+i+1}}{C^2 p_{2nk+k+i}} \right\} = \max \left\{ 1, \frac{1}{x_{2(n+1)k+i}} \right\} = 1$$

eventually and

$$p_{2nk+k+i+2} = \max \left\{ 1, \frac{A}{C^2 q_{2nk+k+i+1}} \right\} = \frac{A}{C^2}$$

eventually and

$$q_{2nk+k+i+3} = \max \left\{ 1, \frac{Ay_{2nk+i+3}}{C^2 p_{2nk+k+i+2}} \right\} = y_{2nk+i+3}$$

eventually and

$$y_{2nk+2k+i+3} = \max \left\{ 1, \frac{Aq_{2nk+k+i+3}}{C^2 x_{2nk+2k+i+2}} \right\} = \max \left\{ 1, \frac{Ay_{2nk+i+3}}{C^2} \right\} = \frac{Ay_{2nk+i+3}}{C^2}$$

eventually since $\frac{Ay_{2nk+i+3}}{C^2} \geq \frac{AB_{i+3}}{C^2} > 1$, which leads to a contradiction that $B_{i+3} = \frac{AB_{i+3}}{C^2} > B_{i+3}$.

By $x_{2nk+i+2} > 1$ eventually, in a similar way as the above also we have

$$y_{2nk+2k+i+3} = q_{2nk-k+i+3} = 1, \quad x_{2nk+i+4} = x_{2nk-2k+i+4}, \quad p_{2nk-k+i+2} = \frac{A^2}{C} x_{2nk+i+2}$$

eventually.

Continuing in a similar way, we can obtain that $x_{2nk+i+2r}$ and $p_{2nk+i+2r}$ are constant sequences eventually for any $r \in \mathbb{N}$, and $q_{2nk-k+i+2r+1} = y_{2nk+i+2r+1} = 1$ eventually for any $r \in \mathbb{N}$. The other case is treated similarly, so we omit the detail.

(2) Indeed, if $A_i = 1$ and $x_{2nk+i} > 1$ for some $i \in \mathbb{Z}(0, 2k-1)$ and any $k \in \mathbb{N}$, then by (2.1) we have

$$x_{2nk+i} = \max \left\{ 1, \frac{Cp_{2nk-k+i}}{A^2 y_{2nk+i-1}} \right\} = \frac{Cp_{2nk-k+i}}{A^2 y_{2nk+i-1}} > 1$$

eventually and by (2.5) we have

$$y_{2nk+i+1} = \max \left\{ 1, \frac{Cy_{2nk-2k+i+1}}{A^2 x_{2nk+i} p_{2nk+i-k}} \right\} = 1 \quad (2.13)$$

eventually since $x_{2nk+i} p_{2nk+i-k} = x_{2nk+i}^2 y_{2nk+i-1} \frac{A^2}{C} \geq \frac{A^2}{C}$ and

$$\frac{Cy_{2nk-2k+i+1}}{A^2 x_{2nk+i} p_{2nk+i-k}} \leq \frac{C^2 y_{2nk-2k+i+1}}{A^4} \longrightarrow \frac{C^2 B_{i+1}}{A^4} < B_{i+1} = \lim_{n \rightarrow \infty} y_{2nk+i+1}.$$

By (2.1) and (2.13) we have

$$q_{2nk+k+i+1} = \max \left\{ 1, \frac{Ay_{2nk+i+1}}{C^2 p_{2nk+i+k}} \right\} = \max \left\{ 1, \frac{AC}{C^2 A^2 x_{2nk+2k+i} y_{2nk+2k+i-1}} \right\} = 1$$

eventually and

$$p_{2nk+k+i+2} = \max \left\{ 1, \frac{Ax_{2nk+i+2}}{C^2 q_{2nk+i+k+1}} \right\} = \frac{Ax_{2nk+i+2}}{C^2} \quad (2.14)$$

eventually. Thus it follows from (2.13) and (2.14) that

$$x_{2nk+2k+i+2} = \max \left\{ 1, \frac{C p_{2nk+k+i+2}}{A^2 y_{2nk+i+2k+1}} \right\} = x_{2nk+2k+i+2}$$

eventually.

Using arguments similar to ones developed in (1), also we can show that $x_{2(n+1)k+i+2r}$ is constant sequence eventually for any $r \in \mathbb{N}$. Thus one has

$$x_{2(n+1)k+i+2k} = x_{2(n+2)k+i} = 1$$

eventually, which leads to a contradiction. The other case is treated similarly, so we omit the detail. The proof is complete. $\square$

Proposition 3. If $BD = AC = 1$ and $A \neq C$, then $\{(x_n, y_n, p_n, q_n)\}_{n=-k}^{+\infty}$ is eventually periodic with period $2k$.

Proof. Without loss of generality we assume $A > C$. There are the following three cases to consider.

Case 1. $A_{2i} > 1$ for some $i \in \mathbb{Z}(0, k-1)$ and $A_{2j+1} > 1$ for some $j \in \mathbb{Z}(0, k-1)$ or $B_{2i} > 1$ for some $i \in \mathbb{Z}(0, k-1)$ and $B_{2j+1} > 1$ for some $j \in \mathbb{Z}(0, k-1)$ or $A_{2i} > 1$ for some $i \in \mathbb{Z}(0, k-1)$ and $B_{2j} > 1$ for some $j \in \mathbb{Z}(0, k-1)$ or $A_{2i+1} > 1$ for some $i \in \mathbb{Z}(0, k-1)$ and $B_{2j+1} > 1$ for some $j \in \mathbb{Z}(0, k-1)$. By Lemma 2 we see easily that $\{(x_n, y_n, p_n, q_n)\}_{n=-k}^{+\infty}$ is eventually periodic with period $2k$.

Case 2. $A_{2i} > 1$ for some $i \in \mathbb{Z}(0, k-1)$ and $A_{2j+1} = B_{2j} = B_{2j+1} = 1$ for any $j \in \mathbb{Z}(0, k-1)$ or $B_{2i} > 1$ for some $i \in \mathbb{Z}(0, k-1)$ and $A_{2j+1} = A_{2j} = B_{2j+1} = 1$ for any $j \in \mathbb{Z}(0, k-1)$ or $A_{2i+1} > 1$ for some $i \in \mathbb{Z}(0, k-1)$ and $A_{2j} = B_{2j} = B_{2j+1} = 1$ for any $j \in \mathbb{Z}(0, k-1)$ or $B_{2i+1} > 1$ for some $i \in \mathbb{Z}(0, k-1)$ and $A_{2j} = B_{2j} = A_{2j+1} = 1$ for any $j \in \mathbb{Z}(0, k-1)$.

Without loss of generality we assume $A_{2i} > 1$ for some $i \in \mathbb{Z}(0, k-1)$ and $A_{2j+1} = B_{2j} = B_{2j+1} = 1$ for any $j \in \mathbb{Z}(0, k-1)$. Then by Lemma 2 we see that x_{2nk+2r}

and $p_{2nk-k+2r}$ are eventually constant sequences for any $r \in \mathbb{N}$, and $q_{2nk-k+2r+1} = x_{2nk+2r+1} = y_{2nk+r} = 1$ eventually for any $r \in \mathbb{N}_0$. Thus it follows from (2.1) that

$$\begin{cases} p_{2nk+k+2r+1} = \max \left\{ 1, \frac{Ax_{2nk+2r+1}}{C^2 q_{2nk+k+2r}} \right\} = \max \left\{ 1, \frac{A}{C^2 q_{2nk+k+2r}} \right\}, \\ q_{2nk+k+2r+2} = \max \left\{ 1, \frac{Ay_{2nk+2r+2}}{C^2 p_{2nk+k+2r+1}} \right\} = \max \left\{ 1, \frac{A}{C^2 p_{2nk+k+2r+1}} \right\} \end{cases} \quad (2.15)$$

eventually, from which it follows that

$$\begin{cases} p_{2nk+k+2r+1} q_{2nk+k+2r} \geq \frac{A}{C^2}, \\ q_{2nk+k+2r+2} p_{2nk+k+2r+1} \geq \frac{A}{C^2} \end{cases} \quad (2.16)$$

eventually. By (2.15) and (2.16) one has

$$\begin{cases} 1 \leq p_{2nk+k+2r+1} = \max \left\{ 1, \frac{Ap_{2nk+k+2r-1}}{C^2 q_{2nk+k+2r} p_{2nk+k+2r-1}} \right\} \\ \leq \max \left\{ 1, p_{2nk+k+2r-1} \right\} = p_{2nk+k+2r-1}, \\ 1 \leq q_{2nk+k+2r+2} = \max \left\{ 1, \frac{Aq_{2nk+k+2r}}{C^2 p_{2nk+k+2r+1} q_{2nk+k+2r}} \right\} \\ \leq \max \left\{ 1, q_{2nk+k+2r} \right\} = q_{2nk+k+2r} \end{cases} \quad (2.17)$$

eventually. On the other hand, it follows from (2.15) and (2.17) that

$$\begin{cases} p_{2nk+k+2r+1} = \max \left\{ 1, \frac{A}{C^2 q_{2nk+k+2r}} \right\} \geq \max \left\{ 1, \frac{A}{C^2 q_{2nk+k+2r-2}} \right\} = p_{2nk+k+2r-1}, \\ q_{2nk+k+2r+2} = \max \left\{ 1, \frac{A}{C^2 p_{2nk+k+2r+1}} \right\} \geq \max \left\{ 1, \frac{A}{C^2 p_{2nk+k+2r-1}} \right\} = q_{2nk+k+2r} \end{cases} \quad (2.18)$$

eventually. $p_{2nk-k+2r}$ is eventually constant sequence for any $r \in \mathbb{N}$ and $q_{2nk-k+2r+1} = 1$ eventually for any $r \in \mathbb{N}_0$ and (2.17) and (2.18) imply that p_n and q_n are eventually periodic with period $2k$.

Case 3. $A_{2i} = A_{2i+1} = B_{2i} = B_{2i+1} = 1$ for any $i \in \mathbb{Z}(0, k-1)$. Then by Lemma 2 we see that $x_n = y_n = 1$ eventually. Thus it follows from (2.1) that

$$\begin{cases} p_n = \max \left\{ 1, \frac{A}{C^2 q_{n-1}} \right\}, \\ q_n = \max \left\{ 1, \frac{A}{C^2 p_{n-1}} \right\} \end{cases} \quad (2.19)$$

and $p_n q_{n-1} \geq \frac{A}{C^2}$ and $q_n p_{n-1} \geq \frac{A}{C^2}$ eventually, from which we have

$$\begin{cases} 1 \leq p_n = \max \left\{ 1, \frac{Ap_{n-2}}{C^2 q_{n-1} p_{n-2}} \right\} \leq \max \left\{ 1, p_{n-2} \right\} = p_{n-2}, \\ 1 \leq q_n = \max \left\{ 1, \frac{Aq_{n-2}}{C^2 p_{n-1} q_{n-2}} \right\} \leq \max \left\{ 1, q_{n-2} \right\} = q_{n-2} \end{cases} \quad (2.20)$$

eventually. On the other hand, it follows from (2.19) and (2.20) that

$$\begin{cases} p_n = \max \left\{ 1, \frac{A}{C^2 q_{n-1}} \right\} \geq \max \left\{ 1, \frac{A}{C^2 q_{n-3}} \right\} = p_{n-2}, \\ q_n = \max \left\{ 1, \frac{A}{C^2 p_{n-1}} \right\} \geq \max \left\{ 1, \frac{A}{C^2 p_{n-3}} \right\} = q_{n-2} \end{cases} \quad (2.21)$$

eventually. By (2.20) and (2.21) we see that p_n and q_n are eventually periodic with period 2. The proof is complete. $\square$

Proposition 4. *If $BD > AC = 1$ and $\{(x_n, y_n, p_n, q_n)\}_{n=-k}^{+\infty}$ is a solution of (2.1), then x_n and p_n are eventually periodic with period 2 and y_n and q_n are eventually periodic with period $2k$.*

Proof. If $BD > AC = 1$, then from Lemma (2.3) we see that there exists $N \in \mathbb{N}$ such that for any $n \geq N + 2$, we have

$$\begin{cases} x_n = \max \left\{ 1, \frac{C}{AB y_{n-1}} \right\}, \\ y_n = \max \left\{ 1, \frac{D}{AB x_{n-1}}, \frac{y_{n-2rk}}{x_{n-1} p_{n-k-1} x_{n-2k-1} p_{n-3k-1} \cdots x_{n-2(r-1)k-1} p_{n-(2r-1)k-1}} \right\}, \\ p_n = \max \left\{ 1, \frac{A}{CD q_{n-1}} \right\}, \\ q_n = \max \left\{ 1, \frac{B}{CD p_{n-1}}, \frac{q_{n-2rk}}{p_{n-1} x_{n-k-1} p_{n-2k-1} p_{n-3k-1} \cdots p_{n-2(r-1)k-1} x_{n-(2r-1)k-1}} \right\}. \end{cases} \quad (2.22)$$

Now we show that p_n and x_n are eventually periodic with period 2.

If $p_M = 1$ for some $M \geq N + 2$, then from (2.22) it follows that

$$q_{M+1} \geq \frac{B}{CD}.$$

and

$$1 \leq p_{M+2} = \max \left\{ 1, \frac{A}{CD q_{M+1}} \right\} \leq \max \left\{ 1, \frac{A}{B} \right\} = 1.$$

By mathematical induction, we can obtain $s_{M+2r} = 1$ for any $r \geq 0$,

If $p_{M+2r} = \frac{A}{CD q_{M+2r-1}} > 1$ for some $M \geq N + 2$ and any $r \geq 0$, then from (2.1) and (2.22) and Lemma 1 we see

$$\begin{aligned} \max \left\{ \frac{B}{CD}, q_1, q_2, \dots, q_k \right\} &\geq q_{M+2r+1} = \max \left\{ 1, \frac{B y_{M+1+2r-k}}{CD p_{M+2r}} \right\} \\ &= \max \left\{ 1, \frac{B y_{M+1+2r-k}}{A} q_{M+2r-1} \right\} = \frac{B}{A} y_{M+1+2r-k} q_{M+2r-1} \\ &\geq \frac{B}{A} q_{M+2r-1}, \end{aligned}$$

which implies $A = B$ (since $B > A$ leads to $\lim_{r \rightarrow \infty} q_{M+2r+1} = \infty$) and

$$q_{M+2r+1} \geq q_{M+2r-1}. \quad (2.23)$$

By (2.22) we see that for any $i \in \mathbb{Z}(1, k)$,

$$\begin{aligned} q_{M+2rk+2i+1} &= \max \left\{ 1, \frac{B}{CD p_{M+2rk+2i}} \right\}, \\ &\quad \frac{q_{M+2i+1}}{p_{M+2rk+2i} x_{M+(2r-1)k+2i} p_{M+(2r-2)k+2i} p_{M+(2r-3)k+2i} \cdots p_{M+2k+2i} x_{M+k+2i}} \Big\} \\ &= \max \left\{ q_{M+2rk+2i-1}, \right. \end{aligned}$$

$$\frac{q_{M+2i+1}}{p_{M+2rk+2i}x_{M+(2r-1)k+2i}p_{M+(2r-2)k+2i}p_{M+(2r-3)k+2i}\cdots p_{M+2k+2i}x_{M+k+2i}}\}$$

By (2.23) and (2.2) one has that $\mu(i, r) = q_{M+2rk+2i-1}$ is increasing and

$$\lambda(i, r) = \frac{q_{M+2i+1}}{p_{M+2rk+2i}x_{M+(2r-1)k+2i}p_{M+(2r-2)k+2i}p_{M+(2r-3)k+2i}\cdots p_{M+2k+2i}x_{M+k+2i}}$$

is decreasing for any $i \in \mathbb{Z}(1, k)$.

If for some $i \in \mathbb{Z}(1, k)$, $q_{M+2rk+2i+1} = \lambda(i, r) > \mu(i, r)$ for any $r \in \mathbb{N}$, then by (2.23) we see that q_{M+2r+1} is an eventually constant sequence.

If for any $i \in \mathbb{Z}(1, k)$, $q_{M+2rk+2i+1} = \mu(i, r) = q_{M+2rk+2i-1} \geq \lambda(i, r)$ eventually, then we have $q_{M+2rk+2i+1} = q_{M+2rk+2i-1}$ eventually for any $i \in \mathbb{Z}(1, k)$.

From the above we see that p_n is eventually periodic with period 2. In a similar way we can show that x_n is eventually periodic with period 2. Let $L > N + 1$ such that $p_n = p_{n+2}$ and $x_n = x_{n+2}$ for any $n \geq L$. By (2.3) we see that for any $i \in \mathbb{Z}(1, k)$,

$$q_{L+2rk+2i+1} = \max \left\{ 1, \frac{B}{CDp_{L+2rk+2i}}, \frac{q_{L+2i+1}}{p_{L+2rk+2i}x_{L+(2r-1)k+2i}p_{L+(2r-2)k+2i}p_{L+(2r-3)k+2i}\cdots p_{L+2k+2i}x_{L+k+2i}} \right\}.$$

If $p_{L+2k}x_{L+k} = 1$, then for any $i \in \mathbb{Z}(1, k)$,

$$q_{L+2rk+2i+1} = \max \left\{ 1, \frac{B}{CDp_{L+2rk+2i}}, q_{L+2i+1} \right\}.$$

If $p_{L+2k}x_{L+k} > 1$, then

$$\lim_{r \rightarrow \infty} \frac{q_{L+2i+1}}{p_{L+2rk+2i}x_{L+(2r-1)k+2i}p_{L+(2r-2)k+2i}p_{L+(2r-3)k+2i}\cdots p_{L+2k+2i}x_{L+k+2i}} = 0.$$

Thus $q_{L+2rk+2i+1} = \max \left\{ 1, \frac{B}{CDp_{L+2rk+2i}} \right\}$ eventually.

From the above we see that q_n is eventually periodic with period $2k$. In a similar way we can show that y_n is eventually periodic with period $2k$. The proof is complete. $\square$

Proposition 5. *If $AC > 1$, then $\{(x_n, y_n, p_n, q_n)\}_{n=-k}^{+\infty}$ is eventually periodic with period 1. Furthermore, the following statements hold:*

- (1) $x_n = 1$ eventually and $y_n = \max \left\{ 1, \frac{D}{AB} \right\}$ eventually or $y_n = 1$ eventually and $x_n = \max \left\{ 1, \frac{C}{AB} \right\}$ eventually.
- (2) $p_n = 1$ eventually and $q_n = \max \left\{ 1, \frac{B}{CD} \right\}$ eventually or $q_n = 1$ eventually and $p_n = \max \left\{ 1, \frac{A}{CD} \right\}$ eventually.

Proof. If $AC > 1$, then from (2.3) it follows that there exists $N \in \mathbb{N}$ such that for any $n \geq N + 2$, we have

$$\begin{cases} x_n = \max \left\{ 1, \frac{C}{AB y_{n-1}} \right\}, \\ y_n = \max \left\{ 1, \frac{D}{AB x_{n-1}} \right\}, \\ p_n = \max \left\{ 1, \frac{A}{CD q_{n-1}} \right\}, \\ q_n = \max \left\{ 1, \frac{B}{CD p_{n-1}} \right\}. \end{cases} \quad (2.24)$$

We claim that $x_n = 1$ for any $n \geq N + 2$ or $p_n = 1$ for any $n \geq N + 2$. Indeed, if $x_n = \frac{C}{AB y_{n-1}} > 1$ for some $n \geq N + 2$ and $p_m = \frac{A}{CD q_{m-1}} > 1$ for some $m \geq N + 2$, then $\frac{1}{BD} = \frac{C}{AB} \frac{A}{CD} > 1$ since $y_{n-1} \geq 1$ and $q_{m-1} \geq 1$, which leads to a contradiction. In a similar way, also we can obtain that $y_n = 1$ for any $n \geq N + 2$ or $q_n = 1$ for any $n \geq N + 2$.

If $x_n = 1$ eventually, then $y_n = \max \left\{ 1, \frac{D}{AB} \right\}$ eventually. If $y_n = 1$ eventually, then $x_n = \max \left\{ 1, \frac{C}{AB} \right\}$ eventually. If $p_n = 1$ eventually, then $q_n = \max \left\{ 1, \frac{B}{CD} \right\}$ eventually. If $q_n = 1$ eventually, then $p_n = \max \left\{ 1, \frac{A}{CD} \right\}$ eventually. The proof is complete. $\square$

From Proposition 1, Proposition 2, Proposition 3, Proposition 4 and Proposition 5, we get Theorem 1 immediately.

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GRAPH WITH RESPECT TO SUPERFLUOUS ELEMENTS IN A LATTICE

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Abstract. We consider superfluous elements in a bounded lattice with 0 and 1, and introduce various types of graphs associated with these elements. The notions such as superfluous element graph ($S(L)$), join intersection graph ($JI(L)$) in a lattice, and in a distributive lattice, superfluous intersection graph ($SI(L)$) are defined. Dual atoms play an important role to find connections between the lattice-theoretic properties and those of corresponding graph-theoretic properties. Consequently, we derive some important equivalent conditions of graphs involving the cardinality of dual atoms in a lattice. We provide necessary illustrations and investigate properties such as diameter, girth, and cut vertex of these graphs.

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1. INTRODUCTION

The study of graphs associated with algebraic structures is important to understand their structural aspects. Among possible graphs obtained from various algebraic structures, zero divisor graphs, annihilator graphs, and intersection graphs are the significant ones [5, 6, 12]. Amjadi [2], defined an essential ideal graph with respect to a commutative ring. The notion of the essential submodule and its dualizing concept namely, the superfluous submodule were studied by the authors (Anderson [3], Fluery [13]). Intersection graphs of rings and module over rings were studied by Chakrabarty [12], and in commutative rings, graphs associated with proper non-small ideals were studied by Atani [4]. However, some authors [15, 16] have studied the properties of graphs with respect to lattices obtained from standard substructures of modules over associative rings. Grzeszczuk and Puczyłowski [15] have introduced essential elements and superfluous elements in a lattice. Nimborkar and Vidya [19] have investigated the properties of the essential element graph of a lattice. Alizade and Toksoy [1] have obtained interesting properties and characterization for cofinitely weak supplement lattices. The dualizing submodules notions such as superfluous submodules and hollow submodules are well known in the case of a module

over rings. Anderson [3], Fleury [13], and Bhavanari [7] have extensively studied the spanning dimension in modules associated with the notions of superfluous and hollow submodules. The authors Bhavanari and Kuncham [8] studied isomorphism theorems for a directed hypercube, and in Bhavanari et. al [10], graphs with respect to ideal symmetry and related properties were studied in the case of a generalized rings. Indeed, it is interesting to note that several module analogous developments in lattices were found in Calugareanu [11].

The purpose of this paper is to define various types of graphs associated with superfluous elements in a bounded lattice L with 0 and 1. We introduce the notions of superfluous element graph ($S(L)$), weak supplement element graph ($WSp(L)$), join intersection graph ($JI(L)$) in a lattice, and if a lattice is distributive, we introduce superfluous intersection graph ($SI(L)$). Dual atoms play an important role to find some important connections between the lattice-theoretic properties and those of corresponding graph-theoretic properties. We prove that a non-superfluous element in L is a dual atom if and only if $S(L)$ is complete. We derive important equivalent conditions of graphs involving the cardinality of dual atoms in a lattice. We obtain an equivalent condition for $SI(L)$ to be disconnected if L has exactly two dual atoms. Apart from several properties of $SI(L)$, we show that for any natural number r , $SI(L)$ can not be complete r -partite. Further, we have established some equivalent conditions that yield interrelations between the lattice and graph theoretical properties. We also investigate properties such as diameter, girth, and cut vertex of these graphs.

2. PRELIMINARIES

A lattice is called distributive if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, for all $a, b, c \in L$, and modular if for $a, b, c \in L$ with $a \leq c$, $a \vee (b \wedge c) = (a \vee b) \wedge c$. For any $x, y \in L$ and $x \leq y$, let $[x, y] = \{a \in L \mid x \leq a \leq y\}$ be the interval between x and y . An element a of a lattice L is proper if $a \neq 1$. If L is a bounded lattice, then $a \in L$ is an atom (resp. dual atom), if there does not exist $b \in L$ such that $0 < b < a$ (resp. $a < b < 1$). The set of all dual atoms in L is denoted by $D(L)$.

We consider a simple finite graph G , whose vertex set is $V(G)$ and the edge set is $E(G)$. We denote ab to represent the edge between $a, b \in V(G)$. We denote by $\deg(v)$ the number of vertices associated with v . If a vertex is adjacent to all other vertices in G , then we refer to it as a universal vertex. If there is a path between every pair of vertices of G , then G is connected; otherwise, G is called disconnected.

A graph whose vertices set is empty, is called a null graph and a graph whose edge set is empty is called an empty graph. The length of the shortest path between two vertices a, b in G , is denoted by $d(a, b)$, and $d(a, b) = \infty$, if such a path doesn't exist between a and b . Evidently, $d(a, a) = 0$. The diameter of a graph G , denoted by $\text{diam}(G)$, is equal to $\sup\{d(a, b) : a, b \in V(G)\}$. The girth of a graph G , denoted by $gr(G)$, is the length of the shortest cycle in G , provided G contains a cycle; otherwise $gr(G) = \infty$. A vertex x of a connected graph G is a cut vertex of G if

$G - \{x\}$ is disconnected. For $r \in \mathbb{N}$, the r -partite graph is a graph whose vertex set is $V(G) = P_1 \cup P_2 \cdots \cup P_r$, $\bigcap_{i=1}^r P_i = \emptyset$, and for any $xy \in E(G)$, if $x \in P_i$, $y \in P_j$ for $i \neq j$.

A complete r -partite graph is one in which each vertex is adjacent to every vertex that is not in the same subset. The complete bipartite (that is, 2-partite) graph with part size m and n is denoted by $K_{m,n}$, and when $m = 1$, we call it a star graph. G is said to be k -regular ($k \in \mathbb{N}$), if every vertex is of degree k . The eccentricity (denoted by, $e(v)$) of a vertex v in a connected graph G is $\max d(u, v)$ for all $u \in V(G)$. A vertex with minimum eccentricity is called a center of G .

We consider a bounded lattice $(L, \wedge, \vee, 0, 1)$, where $0, 1$ are the smallest and the greatest element respectively. For standard notions and terminologies in lattice theory, we refer to Grätzer [14], and for concepts in graph theory, we refer to Bhavanari and Kuncham [9].

3. SUPERFLUOUS ELEMENT GRAPH: $S(L)$

Definition 1 ([15]). An element a of a lattice L is said to be superfluous, denoted by $a \leq_s L$ if for every $1 \neq b \in L$, $a \vee b \neq 1$.

Lemma 1. If $a \leq_s L$ and $0 \leq b \leq a$, then $b \leq_s L$.

Proof. Let $c \in L$ be with $b \vee c = 1$. Now $1 = b \vee c \leq a \vee c$, we get $a \vee c = 1$. Since $a \leq_s L$, we get $c = 1$. Therefore, $b \leq_s L$. $\square$

Definition 2 ([11]). (1) An element $x \in L$ is a supplement of $y \in L$, if x is minimal with respect to the property $x \vee y = 1$.

(2) The join of all atoms of L is called socle of L , denoted by $\text{soc}(L)$. For an element, a in L , $\text{soc}(a)$ is the socle of the sublattice $[0, a]$.

Definition 3 ([11]). An element $x \in L$ is a weak supplement of $y \in L$ if $x \vee y = 1$ and $x \wedge y \leq_s L$.

Definition 4 ([18]). A proper element $x \in L$ is a soc-weak-supplement of $y \in L$ if $x \vee y = 1$ and $x \wedge y \leq \text{soc}(1)$.

Cigdem Bicer et.al [17] studied complete modular lattices and investigated the properties of generalized supplemented lattices. Nebiyev [16] has studied the properties of weak supplement elements.

Now we define the notion of superfluous element graph of a lattice as follows.

Definition 5. A superfluous element graph of L (denoted by, $S(L)$) is a graph with $V(S(L)) = \{a \in L \mid 0 \neq a \neq 1\}$ as its vertex set, and $E(S(L)) = \{ab \mid a \wedge b \leq_s L\}$ as its edge set.

Example 1. Consider $(D_{30}, \leq)$, the set of all positive divisors of 30, with $a \leq b \Leftrightarrow a$ divides b , as given in Figure 1. Then $L = (D_{30}, \wedge, \vee)$ is a lattice where meet and join are the greatest common divisor and least common multiple respectively. Here,

the least element is denoted by 0 and the greatest element is denoted by 1. The corresponding $S(L)$ is given in Figure 1.

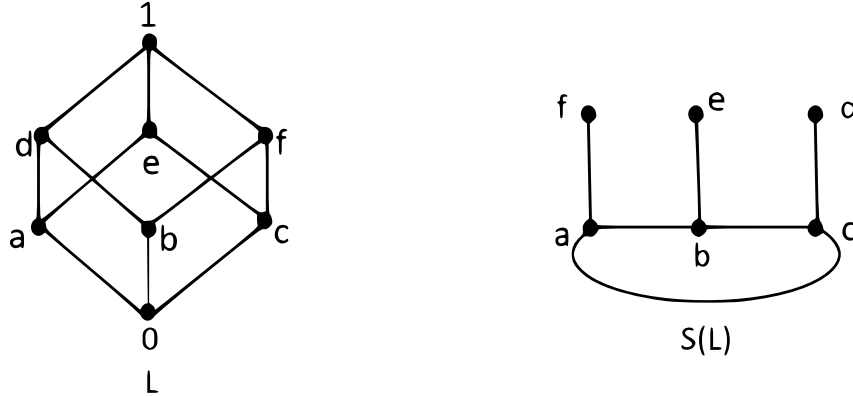


FIGURE 1.

Lemma 2. If $0 \neq a \in L$ is superfluous, then a is universal in $S(L)$.

Proof. Suppose $0 \neq a \in L$, and $a \leq_s L$. To show a is a universal vertex, we have to show that for any $b \in V(S(L))$, $ab \in E(S(L))$. Let $b \in V(S(L))$. Then, $b \neq 1$, and since $a \leq_s L$, we have $a \vee b \neq 1$. Since $a \wedge b \leq a \vee b \neq 1$, we get $a \wedge b \in V(S(L))$. Now, for any $c \in L$, if $(a \wedge b) \vee c = 1$, then $1 = (a \wedge b) \vee c \leq a \vee c$. Since $a \leq_s L$, we have $c = 1$. Therefore, $(a \wedge b) \leq_s L$, for all $b \in V(S(L))$, shows that a is a universal vertex. $\square$

Remark 1. The lattices L_1 and L_2 given in Figure 2 are not isomorphic but their superfluous element graph is the same, as shown in Figure 2.

Definition 6. The dual annihilator of an element x in L , denoted by the set, $\text{ann}_d(x) = \{y \in L \mid x \vee y = 1\}$.

Proposition 1. For a proper element x of L , $x \wedge (\bigwedge \text{ann}_d(x)) \leq_s L$.

Proof. Suppose $[x \wedge (\bigwedge \text{ann}_d(x))] \vee y = 1$, for some $y \in L$. Clearly,

$$1 = [x \wedge (\bigwedge \text{ann}_d(x))] \vee y \leq x \vee y,$$

implies that $x \vee y = 1$. Hence, $y \in \text{ann}_d(x)$. Also, since $1 = [x \wedge (\bigwedge \text{ann}_d(x))] \vee y \leq y \vee [\bigwedge \text{ann}_d(x)]$, it follows that $\bigwedge \text{ann}_d(x) \leq y$. This shows that $y \vee (\bigwedge \text{ann}_d(x)) = y$. Hence, $1 \leq y \vee (\bigwedge \text{ann}_d(x)) = y$, implies $y = 1$. $\square$

Now we define the notions of weak supplement element graph and socle-weak supplement graph of a lattice as follows.

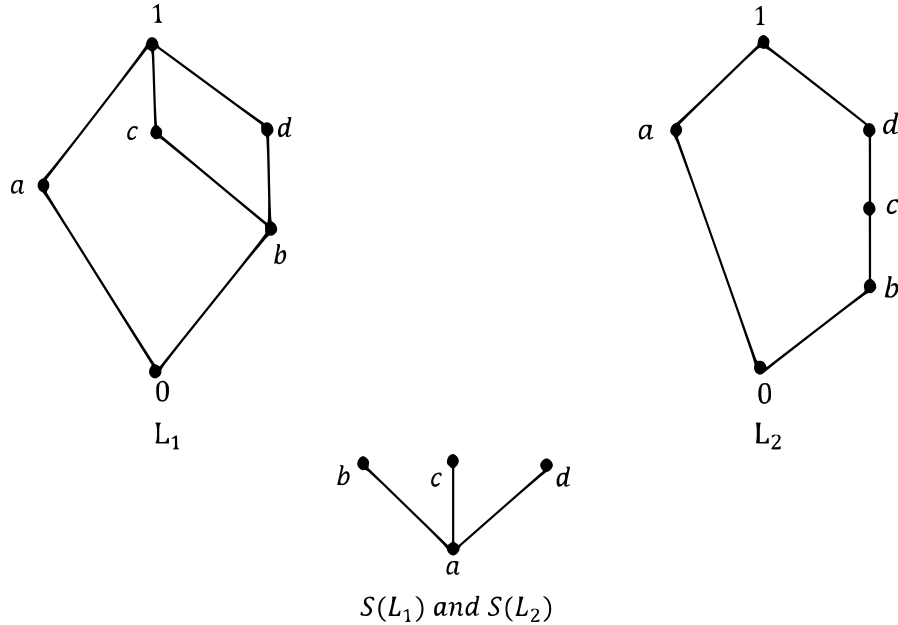


FIGURE 2.

Definition 7.

- (1) A weak supplement element graph of L (referred as, $WS_p(L)$), with vertex set $V(WS_p(L)) = \{a \in L \mid 0 \neq a \neq 1\}$, and edge set

$$E(WS_p(L)) = \{ab \mid a \vee b = 1, a \wedge b \leq_s L\}.$$

- (2) A socle-weak supplement element graph of L (referred as, $SWSP(L)$), with vertex set $V(SWSP(L)) = \{a \in L \mid 0 \neq a \neq 1\}$, and edge set

$$E(SWSP(L)) = \{ab \mid a \vee b = 1, a \wedge b \leq \text{soc}(1)\}.$$

Example 2. Consider the lattice L given in Figure 3. Then graph $WS_p(L)$ corresponding to L is given in Figure 3.

Proposition 2. Let $0 \neq a, b \neq 1$ be in L such that a is a weak supplement of b , for all b . Then a is universal in $WS_p(L)$.

Proof. Suppose $0 \neq a \neq 1 \in L$ and a is a weak supplement of b , for all $0 \neq b \neq 1 \in L$. To show a is a universal vertex, we have to show that for any $b \in V(WS_p(L))$, $ab \in E(WS_p(L))$. Let $b \in V(WS_p(L))$. Then clearly, $0 \neq b \neq 1$, and hence $a \vee b = 1$ and $a \wedge b \leq_s L$. Therefore $ab \in E(WS_p(L))$, for every $b \in V(S(L))$, proves that a is universal. $\square$

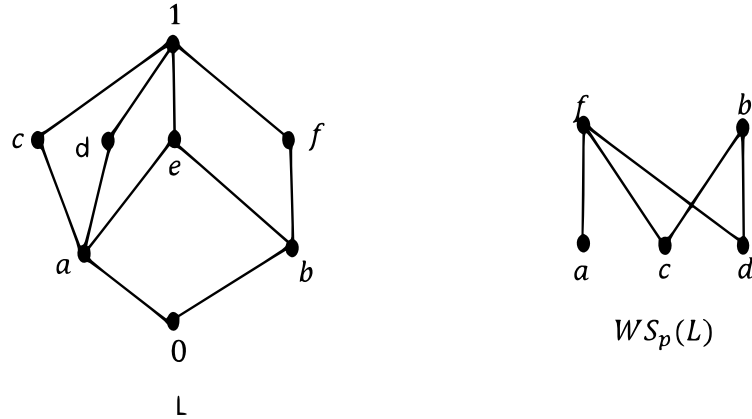


FIGURE 3.

Remark 2. If an element is both an atom as well as a dual atom in L , then it is universal in $WS_p(L)$. In the lattice L_2 given in Figure 2, a is a universal vertex in $WS_p(L_2)$.

Lemma 3. $S(L)$ is a graph with $E(S(L)) = \emptyset$ if and only if L contains a unique non-zero proper element.

Proof. Clearly, if L contains a unique non-zero proper element, then $E(S(L)) = \emptyset$. Conversely, assume that $E(S(L)) = \emptyset$. Let $a, b \in L$. If L has atoms $c \neq d$ such that $0 < c \leq a$ and $0 < d \leq b$, then $c \wedge d = 0 \leq_s L$, and so $cd \in E(S(L))$, a contradiction. Thus, $c = d$. Let $1 \neq x \neq c$. If $c \not\leq x$, then since c is an atom, x is indifferent from c , showing that $x \wedge c = 0$, and so $xc \in E(S(L))$, a contradiction. Therefore, for every $1 \neq x \in L$, $c \leq x$. Hence, $x \vee c = x \neq 1$, for all $1 \neq x \in L$, implies that $c \leq_s L$. Now $c \leq a$ implies $ac \in E(S(L))$, a contradiction. Therefore, a is the only non-zero element that is proper in L . $\square$

Remark 3. The following example shows that $S(L)$ of a lattice L can be disconnected. Consider the lattice L , given in Figure 3. Then the corresponding $S(L)$ is a disconnected graph shown in Figure 4.

Theorem 1. Any one of the following conditions implies $S(L)$ is complete.

- (1) $a \leq_s L$, for all $a \in L$, $0 \neq a \neq 1$.
- (2) $a \vee b = 1 \Leftrightarrow$ either $a = 1$ or $b = 1$ in L .
- (3) L has a non-zero superfluous element in which $S(L)$ is k -regular.

Proof. (1) Let $0 \neq a \in L$ be a superfluous element of L . Then, by Lemma 2, a is a universal vertex in $S(L)$. Hence, a is adjacent to all other vertices in $S(L)$, and this is true for all $0 \neq a \in L$, which shows that $S(L)$ is complete.

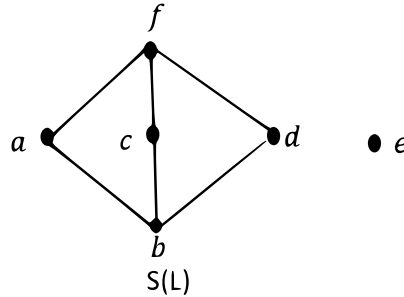


FIGURE 4.

(2) In a contrary way, suppose that $S(L)$ is not complete. Then there exist $a, b \in L$ with $ab \notin E(S(L))$. Then $a \wedge b \not\leq_s L$, implies there exists $1 \neq c \in L$ such that $(a \wedge b) \vee c = 1$. Now by hypothesis, and since $c \neq 1$, we have $a \wedge b = 1$, a contradiction, as a and b are proper. Therefore, $S(L)$ is complete.

(3) Let $S(L)$ be k -regular. Let $0 \neq x \leq_s L$, then by Lemma 2, x is universal in $S(L)$. That is, $xy \in E(S(L))$, for all $y \in V(S(L))$. Therefore, $S(L)$ contains exactly $k + 1$ vertices, which implies that $S(L)$ is a complete graph. $\square$

Remark 4. The existence of a non-zero superfluous element in Theorem 1(3) is necessary. However, the condition is not sufficient.

Consider the graph given in Figure 5. Here the non-zero element are $\{a, b\}$ which are non-superfluous in L . But $ab \in E(S(L))$, since $a \wedge b \leq_s L$. Hence, $S(L)$ is complete.

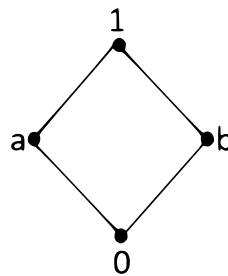


FIGURE 5.

Theorem 2. $S(L)$ is complete if and only if every $1 \neq a \not\leq_s L$, is a dual atom.

Proof. Let every $1 \neq a \not\leq_s L$ be a dual atom. Let $0 \neq x, y \neq 1$ be two distinct elements of L . If one of x or y is superfluous, then $x \vee y \neq 1$. Since $x \wedge y \leq x \vee y \neq 1$, we get $x \wedge y \neq 1$. That is, $x \wedge y$ is not a dual atom. Hence, by hypothesis, $x \wedge y \leq_s L$. Therefore, $xy \in E(S(L))$. Suppose neither x nor y is a superfluous element. Then

by assumption, $x, y \in D(L)$. Now $(x \wedge y) < x < 1$, shows that $x \wedge y$ is not a dual atom. Therefore, $x \wedge y \leq_s L$ and $xy \in E(S(L))$, which proves $S(L)$ is complete. Conversely, let $1 \neq x \not\leq_s L$ and $x < y < 1$. Since $S(L)$ is complete, $xy \in E(S(L))$. Therefore, $x \wedge y \leq_s L$. Now, since $x = x \wedge y$, it follows that $x \leq_s L$, a contradiction. Hence, x is a dual atom. $\square$

Theorem 3. *If $S(L)$ has exactly one universal vertex, then L contains a unique non-zero superfluous element, which is an atom.*

Proof. Let $x \in V(S(L))$ be unique universal. If a and b are two non-zero superfluous elements in L , then by Lemma 2, a and b are universal, a contradiction. Therefore, L contains at most one non-zero superfluous element, say a . Then, $x \wedge a \leq_s L$ and so by Lemma 2, $x \wedge a$ is also universal in $S(L)$. Thus, $x \wedge a = x$. Hence by Lemma 1, we get $x \leq_s L$. Now, by assumption, we get $x = a$. If $0 < y \leq x$, and since $x \leq_s L$, by Lemma 1, $y \leq_s L$. Since x is the only universal vertex, we get $x = y$, and it follows that x is an atom. $\square$

4. JOIN INTERSECTION GRAPH: $JL(L)$

We define the notion of a join intersection graph of a lattice as follows.

Definition 8. The join intersection graph of L , (denoted by $JL(L)$), is the graph with the vertex set $V(JL(L)) = \{a \mid a \in L\}$ and the edge set

$$E(JL(L)) = \{ab \mid a \neq b, a \vee b \neq 1\}.$$

Example 3. Consider the Lattice L given in Figure 6. Then graph $JL(L)$ corresponding to L is given in Figure 6.

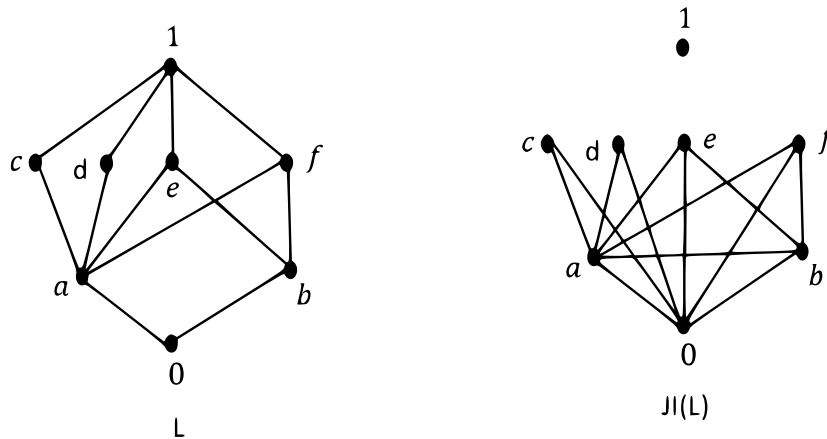


FIGURE 6.

Theorem 4. *The following conditions are equivalent in a join intersection graph of L .*

- (1) $a \leq_s L$;
- (2) If $|V(JI(L))| = n$, then $\deg(a) = n - 2$;
- (3) a is a center in $JI(L) \setminus \{1\}$.

Proof. (1) $\Rightarrow$ (2): Suppose $a \leq_s L$ and let $|V(JI(L))| = n$. Consider $V(JI(L)) = \{x_1, x_2, \dots, x_n\}$, where $x_1 = 0, x_n = 1$ and $x_2 = a$. Since $a \leq_s L$, $a \vee x_i \neq 1$, for all $1 \neq x_i \in V(JI(L))$, shows that $ax_i \in E(JI(L))$. That is, a is adjacent to all in $V(JI(L))$ except 1 and itself. Hence, $\deg(a) = n - 2$.

(2) $\Rightarrow$ (3): Let $|V(JI(L))| = n$ and $\deg(a) = n - 2$. Clearly by definition, $a1 \notin E(JI(L))$ and $aa \notin E(JI(L))$. Since a is adjacent to all other vertices, $d(a, x) = 1$, for all $x \in V(JI(L))$, $x \notin \{0, 1\}$. Therefore, a is the center of $JI(L) \setminus \{1\}$.

(3) $\Rightarrow$ (1): Let a be the center of $JI(L) \setminus \{1\}$. To show $a \leq_s L$, let $1 \neq b \in L$. Then $b \in V(JI(L))$, and $d(a, b)$ is minimum. Hence, $ab \in E(JI(L))$, implies $a \neq b$ and $a \vee b \neq 1$. Since b is arbitrary, we have $a \leq_s L$. $\square$

Theorem 5. *The following conditions are equivalent.*

- (1) $a \not\leq_s L$, for all $0 \neq a \in L$;
- (2) $\bigwedge_{d \in D(L)} d = 0$;
- (3) 0 is the only center in $JI(L) \setminus \{1\}$.

Proof. (1) $\Rightarrow$ (2): Let $a \not\leq_s L$, for all $0 \neq a \in L$. On a contrary, assume that $\bigwedge_{i=1}^n a_i = d \neq 0$, where $a_i \in D(L)$, $1 \leq i \leq n$. Then $d \leq a_i$, for all i , implies $d \vee a_i = a_i \neq 1$, for all i , shows that, $d \leq_s L$, a contradiction. Therefore, $\bigwedge_{i=1}^n a_i = 0$.

(2) $\Rightarrow$ (3): Suppose $\bigwedge_{i=1}^n a_i = 0$, where $D(L) = \{a_i \mid 1 \leq i \leq n\}$. On the contrary, assume that $a \neq 0$ is a center in $JI(L) \setminus \{1\}$. Then by Theorem 4, $a \leq_s L$, and hence $a \vee a_i \neq 1$, for all i . That is, $aa_i \in E(JI(L))$. Now since a_i 's are dual atoms for all i , we have $a \vee a_i \leq a_i$, for all i . Then $a \vee a_i = a_i$, for all i . That is, $a \leq a_i$, for all i , implies that $a \leq \bigwedge_{i=1}^n a_i = 0$. Hence, $a = 0$, a contradiction.

(3) $\Rightarrow$ (1): Suppose 0 is the only center in $JI(L) \setminus \{1\}$. Let $0 \neq b \in L$. Then $b \in V(JI(L))$. Since b is not a center in $JI(L) \setminus \{1\}$, $bx \notin E(JI(L))$, for some $x \in JI(L)$, $0 \neq x \neq 1$. That is, there exists $1 \neq x \in V(JI(L))$ such that $b \vee x = 1$. Hence, $b \not\leq_s L$. Since b is arbitrary, we get $b \leq_s L$, for all $0 \neq b \in L$. $\square$

5. SUPERFLUOUS INTERSECTION GRAPH: $SI(L)$

In this section, let L denote a distributive lattice.

Definition 9. An element a of L is said to be non-superfluous, denoted by $a \not\leq_s L$, if there exists a proper element b in L such that $a \vee b = 1$.

Lemma 4. Let a and b be two proper elements of L . If $c \in D(L)$, then $a \wedge b \leq c \Rightarrow a \leq c$ or $b \leq c$.

Proof. Suppose c is a dual atom and a, b are two proper elements of L such that $a \wedge b \leq c$. Clearly, $c \not\leq a$, $c \not\leq b$. If $a \not\leq c$ and $b \not\leq c$, then there exists $d \in L$ such that $d = a \wedge b \leq c$. This implies that there exists a diamond sublattice in L , a contradiction, as L is distributive. Therefore, $a \leq c$ or $b \leq c$. $\square$

Lemma 5. Let $D(L) = \{a_i\}_{i \in I}$ and $T \subset I$. Then, $\bigwedge_{i \in T} a_i \not\leq_s L$.

Proof. Assume the contrary, suppose $\bigwedge_{i \in T} a_i \leq_s L$. Then $\bigwedge_{i \in T} a_i \vee x \neq 1$, for all $x \neq 1 \in L$. In particular, $\bigwedge_{i \in T} a_i \vee a_j \neq 1$, for every $a_j \in D(L)$. Since a_j is a dual atom, we have $\bigwedge_{i \in T} a_i \leq a_j$, for each $j \in I \setminus T$. Then by Lemma 4 we have $a_i \leq a_j$, for some $i \in T$, a contradiction as a_i is a dual atom. $\square$

We define the notion of a superfluous intersection graph of a lattice as follows.

Definition 10. The superfluous intersection graph of L (referred as $SI(L)$) is a graph with $V(SI(L)) = \{a \in L \mid 1 \neq a \not\leq_s L\}$ as its vertex set and $E(SI(L)) = \{ab \mid a \wedge b \not\leq_s L\}$ as its edge set.

Example 4. Consider the Lattice L , the 3-cube given in Figure 1. Then graph $SI(L)$ corresponding to L is given in Figure 7.

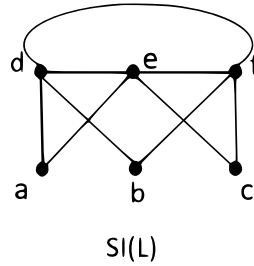


FIGURE 7.

Proposition 3. $SI(L)$ is a null graph if and only if L has a unique dual atom.

Proof. Let $SI(L)$ be a null graph. Then, $V(SI(L)) = \emptyset$. That is, there are no proper non-superfluous elements of L . Let L have two dual atoms, say a and b . Then, $0 < a < 1$ and $0 < b < 1$, implies that $a \vee b = 1$, with $b \neq 1$. Hence, $a \not\leq_s L$, a contradiction. Therefore, L has a unique dual atom. Conversely, suppose that L has

a unique dual atom, say d . To show that $SI(L)$ is a null graph. If there exists an element $1 \neq a \in L$ such that $a \not\leq_s L$, then for every $1 \neq b \in L$, $a \vee b = 1$. This shows that $a \vee d = 1$, a contradiction to the hypothesis. Therefore, $V(SI(L)) = \emptyset$. $\square$

Theorem 6. *$SI(L)$ is an empty graph if and only if $D(L) = \{a_1, a_2\}$, where a_1, a_2 are non-superfluous elements.*

Proof. Suppose $SI(L)$ is an empty graph. If $|D(L)| = 1$, then by Proposition 3, $SI(L)$ is a null graph, a contradiction. Let $D(L) = \{a_i\}_{i \in I}$, $|I| \geq 3$. Then by Lemma 5, $a_i \wedge a_j \not\leq_s L$ whenever $i \neq j$. Hence $a_i a_j \in E(SI(L))$, a contradiction, since $SI(L)$ is an empty graph. Therefore, $|D(L)| = 2$. Let $D(L) = \{a_1, a_2\}$, with $a_1 \neq a_2$. It remains to show a_1 and a_2 are non-superfluous. Since $a_1 < 1$ and $a_2 < 1$, and $a_1 \vee a_2 = 1$, with $a_2 \neq 1$, it follows that $a_1 \not\leq_s L$. Similarly, $a_2 \not\leq_s L$. Conversely, let $D(L) = \{a_1, a_2\}$, where $a_1 \not\leq_s L$ and $a_2 \not\leq_s L$. In order to show that $SI(L)$ is an empty graph, we show that a_1 and a_2 are the only non-superfluous elements of L . Assume the contrary, suppose that $x \in L$ such that $x \not\leq_s L$. Then $x < a_1$ or $x < a_2$. Let $x < a_1$. Since $a_1 \wedge a_2$ is maximal such that $a_1 \wedge a_2 < a_1$ and a_2 , we get $x \leq a_1 \wedge a_2 < a_2$. Then $x < a_2$. Therefore, $x \leq_s L$, a contradiction to our assumption. Hence, a_1 and a_2 are the only non-superfluous elements of L . This shows that $V(SI(L)) = \{a_1, a_2\}$. Further, since $a_1 \wedge a_2 \leq_s L$, a_1 and a_2 are not adjacent in $SI(L)$. Therefore, $E(SI(L)) = \emptyset$, $SI(L)$ is an empty graph. $\square$

Theorem 7. *The the following conditions are equivalent.*

- (1) $SI(L)$ is a disconnected graph.
- (2) $|D(L)| = 2$.
- (3) $SI(L) = SI_1(L) \cup SI_2(L)$, where $SI_1(L)$ and $SI_2(L)$ are two disjoint complete subgraphs of $SI(L)$.

Proof. (1) $\Rightarrow$ (2): Suppose that $SI(L)$ is disconnected. Let $SI_1(L)$ and $SI_2(L)$ be two components of $SI(L)$, and a_1, a_2 be two elements of L such that $a_1 \in SI_1(L)$ and $a_2 \in SI_2(L)$. Let x_1 and x_2 be two dual atoms of L such that $a_1 \leq x_1$ and $a_2 \leq x_2$. If $x_1 = x_2$, then $a_1 \leq x_1 < 1$ and $a_2 \leq x_1 < 1$, implies $a_1 \vee a_2 \leq x_1 \neq 1$. Hence $a_1 \leq_s L$ and $a_2 \leq_s L$, shows that $V(SI(L)) = \emptyset$, a contradiction. Therefore, $x_1 \neq x_2$. If $x_1 \wedge x_2 \not\leq_s L$, then $x_1 x_2 \in E(SI(L))$, a contradiction. Hence $x_1 \wedge x_2 \leq_s L$. Therefore, $|D(L)| \geq 2$. Let $D(L) = \{a_i\}_{i \in I}$, $|I| \geq 3$. Then by Lemma 5, $a_i \wedge a_j \not\leq_s L$ whenever $i \neq j \in I$. This shows that $SI(L)$ is connected, a contradiction. Hence $|D(L)| = 2$.

(2) $\Rightarrow$ (3): Let $|D(L)| = 2$, say x_1, x_2 . Let $SI_1(L) = \{a_j \in L \mid a_j \leq x_i \text{ and } a_j \not\leq_s L\}$, for $i = 1, 2$. Let a_1, a_2 be two elements of $SI_1(L)$. If a_1 and a_2 are not adjacent, then $a_1 \wedge a_2 \leq_s L$, which implies $a_1 \wedge a_2 \leq x_1 \wedge x_2 \leq x_2$. Now by Lemma 4, $a_1 \leq x_2$ or $a_2 \leq x_2$. This implies $a_1 \leq_s L$ or $a_2 \leq_s L$, a contradiction. Therefore, $SI_1(L)$ is a complete subgraph of $SI(L)$. In a similar way, we can prove that $SI_2(L)$ is a complete subgraph of $SI(L)$. Next, we show there is no path between $SI_1(L)$ and $SI_2(L)$. On the contrary, suppose a_1 and a_2 are adjacent for some elements $a_1 \in S(L_1)$ and $a_2 \in S(L_2)$. Since $a_1 \wedge a_2 \leq x_1 \wedge x_2$, we have $a_1 \wedge a_2 \leq_s L$, a contradiction. Therefore, none of the

vertices of $SI_1(L)$ and $SI_2(L)$ is adjacent. Hence, $SI(L) = SI_1(L) \cup SI_2(L)$, where $SI_1(L)$ and $SI_2(L)$ are complete subgraphs of $SI(L)$.

(3) $\Rightarrow$ (1): It is clear. $\square$

Remark 5. The following example shows that the graph $SI(L)$ can be totally disconnected, if L is non-distributive having $|D(L)| \geq 3$.

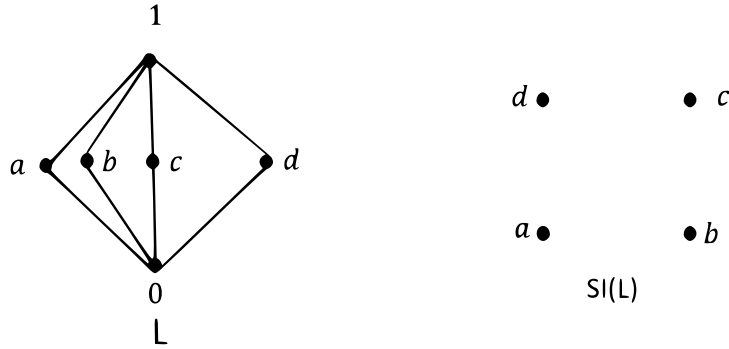


FIGURE 8.

Consider the lattice L given in Figure 8. Here $|D(L)| = 4$, but $SI(L)$ is a disconnected graph.

Theorem 8. *If $SI(L)$ is a connected graph, then $\text{diam}(SI(L)) \leq 2$.*

Proof. Let $a, b \in V(SI(L))$. If $ab \in E(SI(L))$, then we are done. Let a and b be two non-adjacent vertices in $SI(L)$. Then, $a \wedge b \leq_s L$. Let $a \leq x_1$ and $b \leq x_2$, for some dual atoms x_1, x_2 of L . If $a \wedge x_2 \not\leq_s L$, then $ax_2 \in E(SI(L))$, and since $b \wedge x_2 = b \not\leq_s L$, we have $bx_2 \in E(SI(L))$. Therefore, $a - x_2 - b$ is a path in $SI(L)$. Hence, $d(a, b) = 2$. Similarly, if $b \wedge x_1 \not\leq_s L$, then $bx_1 \in E(SI(L))$, and since $a \wedge x_1 = a \not\leq_s L$, we have $ax_1 \in E(SI(L))$. Therefore, $b - x_1 - a$ is a path in $SI(L)$. Hence $d(a, b) = 2$. Suppose that $a \wedge x_2 \leq_s L$ and $b \wedge x_1 \leq_s L$. Since $SI(L)$ is connected, by Theorem 7, $|D(L)| \geq 3$. Let x_3 be a dual atom in L . Since $a \wedge b \leq_s L$, $(a \wedge b) \vee x_3 \neq 1$. Then $a \wedge b \leq x_3$, which implies $a \leq x_3$ or $b \leq x_3$. Without loss of generality, we assume that $a \leq x_3$. Now we show that $b \wedge x_3 \not\leq_s L$. If $b \wedge x_3 \leq_s L$, then $(b \wedge x_3) \vee d \neq 1, \forall d \in D(L)$. That is, $b \wedge x_3 \leq d$. Then, by Lemma 4, $b \leq d$ or $x_3 \leq d$. Since x_3 is a dual atom, $x_3 \not\leq d$. Therefore, $b \leq d$, for all $d \in D(L)$. Hence, $b \leq_s L$, a contradiction, shows that $b \wedge x_3 \not\leq_s L$. Thus, $a - x_3 - b$ is a path in $SI(L)$, and so $d(a, b) = 2$. Hence, $\text{diam}(SI(L)) \leq 2$. $\square$

Theorem 9. *If $SI(L)$ contains a cycle, then $gr(SI(L)) = 3$.*

Proof. Case (i): Suppose $|D(L)| = 2$. Then, by Theorem 7, $SI(L)$ is a union of two disjoint complete subgraphs. Since $SI(L)$ contains a cycle, at least one component should contain a cycle of minimum length 3. Therefore, $gr(SI(L)) = 3$.

Case (ii): If $|D(L)| \geq 3$, say a_1, a_2, a_3 , then by Lemma 5, $a_1 - a_2 - a_3 - a_1$ is a cycle in $SI(L)$. So $gr(SI(L)) = 3$. $\square$

Theorem 10. *If $SI(L)$ is connected, then $SI(L)$ has no cut vertex.*

Proof. Let a be a cut vertex of $SI(L)$. Then, $SI(L) \setminus \{a\}$ is not connected. That is; there exists x and y in $SI(L)$ such that a lies on every path from x to y . By Theorem 8, $diam(SI(L)) \leq 2$, and so the shortest path from y to x is of length 2. So $x - a - y$ is a path between x and y . Thus, $x \wedge a \not\leq_s L$ and $a \wedge y \not\leq_s L$. Also since $xy \notin E(SI(L))$, $x \wedge y \leq_s L$. First to show that a is a dual atom of L . If $a \notin D(L)$, then there exists $b \in L$ such that $a < b$. Since $a \not\leq_s L$, we have $b \not\leq_s L$. Since $x \wedge a \leq x \wedge b$ and $x \wedge a \not\leq_s L$, we have $x \wedge b \not\leq_s L$. In a similar way, $y \wedge b \not\leq_s L$. So $x - b - y$ is a path in $SI(L) \setminus \{a\}$, a contradiction. Therefore, a is a dual atom of L . We claim that there exists a dual atom $z \neq a$ of L such that $x \not\leq z$. If $x \leq z$, for each $a \neq z \in D(L)$, then $x \leq \left(\bigwedge_{z \neq a} z \right)$ implies that $x \wedge a \leq \bigwedge_{z \in D(L)} z$. Hence, $x \wedge a \leq_s L$, a contradiction. In a similar way, there exists a dual atom $w \neq a \in L$ such that $y \not\leq w$. Now to show for each $d \in D(L)$, $x \leq d$, or $y \leq d$. Since $x \wedge y \leq_s L$, we have $(x \wedge y) \vee d \neq 1$, for every $d \in D(L)$. Then $(x \wedge y) \leq d$, for every $d \in D(L)$. Thus by Lemma 4, $x \leq d$ or $y \leq d$, for every $d \in D(L)$. Since $SI(L)$ is connected, by Theorem 7, $|D(L)| \geq 3$. Now, let $a \neq t, s \in D(L)$ such that $x \not\leq t$ and $y \not\leq s$. Then $x \leq s$ and $y \leq t$. Hence $x - s - t - y$ is a path in $SI(L) \setminus \{a\}$, a contradiction. Therefore, $SI(L)$ has no cut vertex. $\square$

Theorem 11. *$SI(L)$ can not be complete q -partite for any $q \in \mathbb{N}$.*

Proof. Suppose $SI(L)$ is a complete q -partite graph with q parts $P_1, P_2, \dots, P_q$.

By Lemma 5, a and b are adjacent for each $a, b \in D(L)$. Therefore, P_i contains at most one dual atom of L . Hence, $|D(L)| \leq q$, by the pigeonhole principle. Next to show $|D(L)| = q$. Suppose, in a contrary, $|D(L)| = \{a_1, a_2, \dots, a_t\}$, $t < q$. Let $a_i \in P_i$, for $1 \leq i \leq t$. That is, P_{t+1} has no dual atom. Since $|D(L)|$ is finite, by Lemma 5, $\bigwedge_{j \neq i} a_j \not\leq_s L$. Now if $x \in L$ such that $x \neq 1$ and $\bigwedge_{j \neq i} a_j \vee x = 1$, then

we have $\left(\bigwedge_{j \neq i} a_i \wedge a_j \right) \vee x < \bigwedge_{j \neq i} a_i \vee x = 1$. Therefore, $\left(\bigwedge_{j \neq i} a_i \wedge a_j \right) \vee x \neq 1$, and so $\bigwedge_{j \neq i} a_i \wedge a_j \leq_s L$. Thus $\bigwedge_{j \neq i} a_i$ and a_j are not adjacent in $SI(L)$. Since $a_i \in P_i$, we have $\bigwedge_{j \neq i} a_i \in P_i$. Let x be a vertex in P_{t+1} and $x \leq a_k$, for some $a_k \in D(L)$. Then $xa_k \in E(SI(L))$. Since $SI(L)$ is a complete q -partite graph and $a_k \in P_k$, so x is adjacent to all elements of P_k . Then x is adjacent to $\bigwedge_{j \neq k} a_j$, a contradiction, as

$x \wedge \left(\bigwedge_{j \neq k} a_j \right) \leq a_k \wedge \left(\bigwedge_{j \neq k} a_j \right) \leq_s L$. Hence $|D(L)| = q$. Now, consider $\bigwedge_{i=3}^q a_j = d$. By Lemma 5, $d \not\leq_s L$. Since $d \wedge a_1 = \bigwedge_{i \neq 2} a_i \not\leq_s L$, d is adjacent to a_1 . Similarly, d

is adjacent to a_2 . So $d \notin P_1, P_2$. Since $d \wedge a_i = d \not\leq_s L$, for each $3 \leq i \leq q$, we have $da_i \in E(SI(L))$. So $d \notin P_i$, for every $1 \leq i \leq q$, leads to a contradiction. $\square$

Theorem 12. *If L has finitely many dual atoms, then $SI(L)$ has no universal vertex.*

Proof. Let $D(L) = \{a_1, a_2, \dots, a_t\}$. Suppose, on the contrary, there exists $x \in V(SI(L))$ such that x is a universal vertex. Let $x \leq a_i$. By Lemma 5, $d = \bigwedge_{j \neq i} a_j \not\leq_s L$, whereas $x \wedge d \leq a_i \wedge \left(\bigwedge_{j \neq i} a_j \right) = a_i \wedge d \not\leq_s L$, a contradiction to x is universal. Thus, no vertex in $SI(L)$ is a universal vertex. $\square$

Corollary 1. *$SI(L)$ can not be a complete graph.*

Proof. Follows from Theorem 12. $\square$

Theorem 13. *The following statements hold for a lattice L .*

- (1) *$SI(L)$ contains a vertex of degree 1 if and only if $|D(L)| = 2$ and $SI(L) = SI_1(L) \cup SI_2(L)$, where $SI_1(L)$ and $SI_2(L)$ are two disjoint complete subgraphs of $SI(L)$ and $|V(SI_i(L))| = 2$, for some $i = 1, 2$;*
- (2) *$SI(L)$ cannot be a star graph.*

Proof. (1) Let $a \in V(SI(L))$ and $\deg(a) = 1$. If $|D(L)| = 1$, by Proposition 3, $SI(L)$ is a null graph, a contradiction. Suppose $|D(L)| \geq 3$. By Lemma 5, for each $x_i \in D(L)$, x_i is adjacent to all other x_j (dual atoms) of L , so $\deg(x_i) \geq 2$. Hence, a is not a dual atom. Without loss of generality, assume that $a \leq x_1$. Then, $ax_1 \in E(SI(L))$. As $\deg(a) = 1$, we have x_1 as the only vertex adjacent to a in $SI(L)$, and in this case, there is no dual atom $x_i \neq x_1$ of L such that $a \leq x_i$. In particular, $a \wedge x_2 \leq_s L$. Then, $(a \wedge x_2) \vee x_i \neq 1$, for every $x_i \in D(L)$, implies that $a \wedge x_2 \leq x_i$. Now by Lemma 4, $a \leq x_i$ or $x_2 \leq x_i$. Since x_2 is a dual atom, $x_2 \not\leq x_i$ for every $i \neq 2$. Therefore, $a \leq x_i$, for all $x_i \in D(L)$, a contradiction. Thus, $|D(L)| = 2$. By Theorem 7, $SI(L) = SI_1(L) \cup SI_2(L)$, where $SI_1(L)$ and $SI_2(L)$ are complete subgraphs of $SI(L)$. Let $a \in SI_i(L)$, for some $i \in \{1, 2\}$. Since $SI_i(L)$ is complete as a subgraph of $SI(L)$ and $\deg(a) = 1$, we get $|V(SI_i(L))| = 2$.

The converse is straightforward.

(2) Suppose $SI(L)$ is a star graph. Then $SI(L)$ has an end vertex. So $|D(L)| = 2$, by (1). Now by Theorem 7, $SI(L)$ is disconnected, a contradiction. Therefore, $SI(L)$ cannot be a star graph. $\square$

Definition 11. For every non-negative integer t , the graph G is called t -regular if the degree of each vertex of G is equal to t .

Theorem 14. *The following holds for a lattice L .*

- (1) *If a and b are two vertices of $SI(L)$ such that $a \leq b$ in L , then $\deg(a) \leq \deg(b)$.*
- (2) *If $SI(L)$ is a t -regular graph, then $|D(L)| = 2$ and $|V(SI(L))| = 2(t + 1)$.*

Proof. (1) Let $a, b \in V(SI(L))$ be such that $a \leq b$ in L . Let $ca \in E(SI(L))$. Since $a \wedge c \not\leq_s L$, we have $a \wedge c \leq b \wedge c \not\leq_s L$. Thus, $bc \in E(SI(L))$. Therefore, any vertex adjacent to a is also adjacent to b in $SI(L)$. Hence, $\deg(a) \leq \deg(b)$.

(2) Let $SI(L)$ be t -regular. Clearly, for each $a_i \in D(L)$, $\deg(a_i) = t$. By Lemma 5, a_i is adjacent to all other dual atoms of L , hence $|D(L)|$ is finite. If $|D(L)| = 1$, then by Proposition 3, $SI(L)$ is a null graph, a contradiction. Now suppose $|D(L)| \geq 3$. Then by (1) $\deg(a_1 \wedge a_2) \leq \deg(a_1) = t$.

Case (i): If $\deg(a_1 \wedge a_2) < t$, then we get a contradiction to the t -regularity.

Case (ii): We show that $\deg(a_1 \wedge a_2) \neq t$. If $d = \bigwedge_{j \neq 2} a_j$, then by Lemma 5, $d \not\leq_s L$. Since $(d \wedge a_1) = d \not\leq_s L$, we have $d \wedge a_1 \not\leq_s L$. Therefore, $a_1 d \in E(SI(L))$. But $d \wedge (a_1 \wedge a_2) \leq_s L$, as $d \wedge (a_1 \wedge a_2) = \bigwedge_i a_i$, and $\bigwedge_i a_i \leq a_j$, for all $a_j \in D(L)$. Therefore, $d \wedge (a_1 \wedge a_2) = \bigwedge_i a_i \leq_s L$, implies that $d(a_1 \wedge a_2) \notin E(SI(L))$. Thus, $\deg(a_1 \wedge a_2) \neq t$, a contradiction to the t -regularity. So $|D(L)| = 2$. Hence, by Theorem 7, $SI(L) = SI_1(L) \cup SI_2(L)$, where $SI_1(L)$ and $SI_2(L)$ is the union of two disjoint complete subgraphs. Let $D(L) = \{a_1, a_2\}$ such that $a_1 \in SI_1(L)$ and $a_2 \in SI_2(L)$. Since $\deg(a_1) = t$, so $|V(SI_1(L))| = t + 1$. Similarly, $|V(SI_2(L))| = t + 1$. Hence, $|V(SI(L))| = 2(t + 1)$. $\square$

6. CONCLUSION

In this paper, some properties of superfluous elements in a lattice are considered and investigated corresponding graph-theoretical properties. We have obtained important equivalent conditions of these graphs. These concepts can be extended to study module theoretical analogues spanning dimensional aspects in a lattice, in terms of superfluous and supplement elements.

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SOME NOTES ON MODULES IN WHICH ALL SUBMODULES HAVE A UNIQUE CLOSURE

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Abstract. A module M is called a UC-module if whenever every submodule of M has a unique closure. In this paper, we establish new characterizations of several well-studied classes of rings in terms of UC-modules, and show that UC is not a Morita invariant property. In addition, we study the behaviour of UC-modules under excellent extensions of rings.

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1. INTRODUCTION

Throughout this note all rings are associative with unity and R denotes such a ring. Modules are unital and M_R shall indicate that M is a right R -module. Unless stated otherwise, all R -modules are understood to be right R -modules. Let M be an R -module. A submodule C of M is a *complement* of submodule A in M if C is maximal such that $C \cap A = 0$. A submodule C of M is *closed* in M provided C has no proper essential extension in M . For a submodule C of M , C is a closed submodule if and only if C is a complement submodule [7, 1.10]. The intersection of any two closed submodules of a module may not be closed [13, Example 1.6]. It is well known that, for any submodule A of M , there exists a closed submodule C of M such that A is essential in C , and C is called a *closure* of A (in M). Smith [21] defines a module M to be a *UC-module* if every submodule has a unique closure, or equivalently, the intersection of any two closed submodules of M is also closed. Smith in his study [21] provides 20 different characterizations of UC-modules, and later UC-modules were studied by many authors [2, 6, 8, 11, 15, 16]. We should note that UC-modules are called *dimension modules* in [5]. It is well known that every direct summand of a module M is a closed submodule of M . The converse is not true, generally (for example, the $\mathbb{Z}$ -module $\mathbb{Z}_p \oplus \mathbb{Z}_{p^3}$). A module M is called an *extending* (or *CS*) *module* if every closed submodule of M is a direct summand of M [9]. Wilson [22] says that a module M has the *summand intersection property* (in short, *SIP-module*) if

the intersection of any two direct summands is again a direct summand. UC-modules and SIP-modules coincide when the module is extending [2, Lemma 17].

In this paper, we first prove in Proposition 1 that if M is a UC-module, then for every decomposition $M = A \oplus B$ and every R -homomorphism $f : A \rightarrow B$, $\text{Ker}(f)$ is a complement submodule of M . This proposition is key to our work in this paper and is used to characterize many well known classes of rings in terms of UC-modules. For example, we show in Theorem 1 that a ring R is semisimple if and only if every R -module is a UC-module; or equivalently, every injective R -module is a UC-module; or equivalently, every UC-module is injective. We prove in Theorem 2 that a ring R is a right V-ring (that is, every simple right R -module is injective) if and only if every finitely cogenerated R -module is a UC-module; or equivalently, every finitely copresented R -module is a UC-module. In Corollary 2, we prove that a ring R is an SSI-ring if and only if R is a right Noetherian ring and every finitely cogenerated (or finitely copresented) right R -module is a UC-module. Later, it is shown in Proposition 2 that if the class of UC- R -modules is closed under finite direct sums, then R is a right V-ring. It is proved in Theorem 3 that a ring R is semisimple if and only if the following are satisfied: (1) R is a right Noetherian ring, and (2) the class of UC- R -modules is closed under arbitrary direct sums. We give a new characterization of SI-rings in Theorem 4 that a ring R is a right SI-ring if and only if $Z(R_R) = 0$ and every singular right R -module is a UC-module. Analogous to an idea of Enochs [10], we introduce the notion of UC-cover, and we prove that R is a semisimple ring if and only if every right R -module has a UC-cover (Theorem 5). At the end of this section, we show in Example 2 that UC is not a Morita invariant property.

Section 4 is devoted to the behaviour of UC-modules under excellent extensions of rings. We prove in Theorem 8 that if M is a right S -module, then M_R is a UC-module if and only if M_S is a UC-module, and prove in Theorem 9 that if M is a right R -module, then $(M \otimes_R S)_S$ is a UC-module if and only if M_R is a UC-module. Let S be a right excellent extension of R . Then R is right finitely Σ -UC if and only if S is also (Theorem 10).

For a submodule A of M , the notation $A \leq M$, $A \leq^{ess} M$, $A \leq^\oplus M$ and $A \leq^c M$ mean that A is a submodule, an essential submodule, a direct summand and a complement submodule of M , respectively. For a module M , we use $Z(M)$ and $E(M)$ to denote the singular submodule and the injective hull, respectively. $\text{CFM}_\Lambda(R)$ denotes the column finite $\text{card}(\Lambda) \times \text{card}(\Lambda)$ matrix ring over R , where $\text{card}(\Lambda)$ is the cardinality of Λ . For a module M , $M^{(I)}$ is the direct sum of copies of M indexed by a set I . For definitions and notations which are not given, please see [3].

2. UC-MODULES

First, we begin by proving a useful proposition:

Proposition 1. *If M is a UC-module, then for every decomposition $M = A \oplus B$ and every R -homomorphism $g : A \rightarrow B$, $\text{Ker}(g) \leq^c M$.*

Proof. Assume M is a UC-module. Let $M = A \oplus B$ and $g : A \rightarrow B$ be an R -homomorphism. Let $C = \{u + g(u) \mid u \in A\}$. We want to show that $M = C \oplus B$. Let $x \in M$, then $x = u + v$ where $u \in A$ and $v \in B$. Now, $x = u + g(u) - g(u) + v$. But $u + g(u) \in C$ and $-g(u) + v \in B$. So, $M = C + B$. Let us choose $x \in C \cap B$. We can write $x = u + g(u)$ where $u \in A$ and hence $u = x - g(u) \in A \cap B = 0$. Therefore $g(u) = 0$ which gives $x = 0$. So, $M = C \oplus B$. Since M is a UC-module, an intersection of closed submodules is closed, thus $C \cap A$ is a closed submodule of M . It is a straightforward matter to show that $C \cap A = \text{Ker}(g)$. Thus, $\text{Ker}(g) \leq^c M$. $\square$

Corollary 1. *Let M be an R -module. If $E(M) \oplus E(E(M)/M)$ is a UC-module, then M is injective.*

Proof. It is *mutatis mutandis* the same as the proof (3) $\Rightarrow$ (1) of [14, Theorem 4.12]. $\square$

3. CHARACTERIZATIONS OF RINGS IN TERMS OF UC-MODULES

Let R be a ring. R is semisimple if and only if every R -module is semisimple if and only if every R -module is injective [23, 20.3] if and only if every injective R -module has the SIP [22, Proposition 3]. Recall from [4] that R is said to be an *SSI-ring* if every semisimple R -module is injective; or equivalently, R is a right Noetherian V-ring [4, Proposition 1]. First, we provide some characterizations of semisimple rings:

Theorem 1. *The following conditions are equivalent for a ring R :*

- (1) R is semisimple;
- (2) Every R -module is a UC-module;
- (3) Every injective R -module is a UC-module;
- (4) Every UC-module is injective.

Proof. (1) $\Rightarrow$ (2) Since R is semisimple, every R -module is semisimple. Then, every R module is a UC-module (see also [5, Corollary 3]).

(2) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (1) We want to show that every injective R -module has the SIP. Since every injective R -module is a UC-module, then every injective R -module has the SIP. So, R is semisimple by [22, Proposition 3].

(1) $\Rightarrow$ (4) Since R is semisimple then every R -module is injective.

(4) $\Rightarrow$ (1) Suppose that every UC-module is injective. Then, every semisimple module is injective. Thus, R is an SSI-ring, and hence R is a right Noetherian V-ring by [4, Proposition 1]. Since R is right Noetherian, $E(R) = \bigoplus_{\lambda \in \Lambda} E_\lambda$, where E_λ is an indecomposable injective for each $\lambda \in \Lambda$. Let $0 \neq e \in E_\lambda$. It follows that $eR \leq E_\lambda$ is a uniform submodule of E_λ . Thus, eR is a UC-module, and hence eR is injective by the hypothesis. Hence $eR \leq^\oplus E_\lambda$. Since E_λ is indecomposable, $E_\lambda = eR$. Then E_λ is a simple R -module for each $\lambda \in \Lambda$. Therefore, $E(R)$ is a semisimple R -module, and hence R is a semisimple ring. $\square$

Remark 1. The proof of (4) $\Rightarrow$ (1) can also be proved in a different way: Assume (4) holds. Then R is an SSI-ring. Since R is a product of simple rings, it is right nonsingular and thus a UC-module, whence injective. Since simple self-injective rings are von Neumann regular (that is, every principal ideal is a direct summand of R_R), the Noetherian condition implies semisimplicity.

An R -module M is finitely cogenerated if and only if $\text{Soc}(M)$ is finitely generated and essential in M [23, 21.3]. An R -module X is called *finitely copresented* if (i) X is finitely cogenerated and (ii) in every exact sequence $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ in $\text{Mod-}R$ with Y finitely cogenerated, Z is also finitely cogenerated [23, p.248]. Recall from [23] that a ring R is called a *right V-ring (co-semisimple)* if every simple right R -module is injective. A ring R is a right V-ring if and only if every finitely cogenerated R -module is semisimple if and only if every finitely cogenerated R -module is injective [23, 23.1] if and only if every finitely copresented R -module is semisimple if and only if every finitely copresented R -module is injective [23, 31.7].

Theorem 2. *The following conditions are equivalent for a ring R :*

- (1) R is a right V-ring;
- (2) Every finitely cogenerated R -module is a UC-module;
- (3) Every finitely copresented R -module is a UC-module.

Proof. (1) $\Rightarrow$ (2) This is immediate, since it has already been noted that if R is a right V-ring, then every finitely cogenerated R -module is semisimple and thus UC.
 (2) $\Rightarrow$ (3) It is clear since every finitely copresented module is finitely cogenerated.
 (3) $\Rightarrow$ (1) Let M be a finitely copresented R -module. We will show that M is injective. By [23, 30.1], $E(M)$ and $E(M)/M$ are finitely cogenerated. Since $E(M)/M$ is finitely cogenerated, $E(E(M)/M)$ is finitely cogenerated. Since any finitely cogenerated injective module is finitely copresented, by condition (3) and [23, 21.4], $E(M) \oplus E(E(M)/M)$ is a UC-module. By Corollary 1, M is injective. By [23, 31.7], R is a right V-ring. $\square$

Corollary 2. *A ring R is an SSI-ring if and only if R is a right Noetherian ring and every finitely cogenerated (or finitely copresented) right R -module is a UC-module.*

Proof. Note that R is an SSI-ring if and only if R is a right Noetherian, right V-ring [4, Proposition 1]. The equivalence holds true by Theorem 2. $\square$

Proposition 2. *If the class of UC- R -modules is closed under finite direct sums, then R is a right V-ring.*

Proof. Let M be a finitely cogenerated R -module. Then M is a finite direct sum of uniform R -modules. Since uniform modules are UC-modules, M is a UC-module by the hypothesis. Hence, by Theorem 2, R is a right V-ring. $\square$

Theorem 3. *A ring R is semisimple if and only if the following are satisfied:*

- (1) R is a right Noetherian ring,

(2) *The class of UC- R -modules is closed under arbitrary direct sums.*

Proof. ($\Rightarrow$;) (1) Clear.

(2) If R is semisimple, then every R -module is a UC-module by Theorem 1(2).

($\Leftarrow$;) Let M be an injective R -module. Since R is a right Noetherian ring, M is a direct sum of uniform modules. Since uniform modules are UC-modules, M is a UC-module. Hence, by Theorem 1(3), R is semisimple. $\square$

Any module isomorphic to the factor M/N of an essential extension $N \leq M$ is called a *singular module*. Now, we recall some facts about singular modules:

Fact 1: If $Z(R_R) = 0$, the class of all singular right R -modules is closed under essential extensions [13, Proposition 1.23(c)].

Fact 2: The class of all singular right R -modules is closed under factor modules, and direct sums [13, Proposition 1.22(b)].

Fact 3: Let M be an R -module. M/N is singular whenever $N \leq^{ess} M$. Thus, $E(M)/M$ is always singular. The converse of this assertion is not true in general, please see [13, p. 32].

A ring R is called a *right SI-ring* if every singular right R -module is injective [12]. In the next theorem we give a new characterization of SI-rings:

Theorem 4. *The following are equivalent for a ring R :*

- (1) *R is a right SI-ring;*
- (2) *$Z(R_R) = 0$ and every singular right R -module is a UC-module.*

Proof. (1) $\Rightarrow$ (2) If R is a right SI-ring, then every singular right R -module is semisimple by [20, Lemma 3.1]. On the other hand, it is clear that R is right nonsingular. Hence, (2) holds.

(2) $\Rightarrow$ (1) Let M be a singular right R -module. We want to show that M is injective. By Fact 1, $E(M)$ is singular. Moreover, by Facts 1 and 3, $E(E(M)/M)$ is a singular R -module. It follows that $E(M) \oplus E(E(M)/M)$ is a singular module from Fact 2. Then, by the hypothesis, $E(M) \oplus E(E(M)/M)$ is a UC-module. Hence, by Corollary 1, M is injective. So, R is a right SI-ring. $\square$

It is well-known that R is right Noetherian if and only if every direct sum of injective modules is injective.

Lemma 1. *If the direct sum of any family of injective envelopes of simple right R -modules is a UC-module, then R is a right Noetherian ring.*

Proof. Let $\{S_i\}_{i \in I}$ be a family of simple modules. Set $M = \bigoplus_{i \in I} E(S_i)$. We will show that M is injective. By the hypothesis, $E(M) \oplus E(E(M)/M)$ is a UC-module. By Corollary 1, M is injective. So, R is right Noetherian. $\square$

The next example shows that converse of Lemma 1 is not true generally.

Example 1. Consider $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$ as a $\mathbb{Z}$ -module. It is well-known that the ring of integer $\mathbb{Z}$ is a right Noetherian ring, and M is an injective $\mathbb{Z}$ -module. It is proved in [14, Example 2.4(1)] that M does not have the *SIP*. Then, M is not a UC-module by [2, Lemma 17].

Enochs [10] introduced the injective cover notion which is the dual to the injective envelope, and showed that a ring R is a right Noetherian ring if and only if every right R -module has an injective cover. Now, we introduce the UC-cover notion.

Definition 1. An R -homomorphism $g : E \rightarrow M$ is called a *UC-cover* of a right R -module M if E is a UC-module such that any diagram

$$\begin{array}{ccc} & E & \\ \exists \alpha \nearrow & \downarrow g & \\ E' & \xrightarrow{\theta} & M \end{array}$$

with E' a UC-module can be completed; and the diagram

$$\begin{array}{ccc} & E & \\ \exists \alpha \nearrow & \downarrow g & \\ E & \xrightarrow{g} & M \end{array}$$

can be completed only by an automorphism α .

Now, we prove in Theorem 5 that a ring R is semisimple if and only if every right R -module has a UC-cover.

Theorem 5. *The following are equivalent for a ring R :*

- (1) R is semisimple;
- (2) Every right R -module has a UC-cover.

Proof. (1) $\Rightarrow$ (2) Obvious.

(2) $\Rightarrow$ (1) First, we want to prove R is right Noetherian. Let $\{S_i\}_{i \in I}$ be a family of simple right R -module and let $M = \bigoplus_{i \in I} E(S_i)$. Call $g : E \rightarrow M$ a UC-cover of M . Consider the following diagram:

$$\begin{array}{ccc}
 & & E \\
 & \nearrow \exists \alpha_i & \downarrow g \\
 E(S_i) & \xrightarrow{\pi_i} & M
 \end{array}$$

where π_i is the canonical injection for $i \in I$. Note that all modules $E(S_i)$ are uniform and injective modules. It follows that all modules $E(S_i)$ are UC-modules. By the definition of UC-cover, there exists a homomorphism $\alpha_i : E(S_i) \rightarrow E$ such that $g\alpha_i = \pi_i$ for $i \in I$. Define $\alpha : M \rightarrow E$ by $\alpha(\sum_{i=1}^n x_i) = \sum_{i=1}^n \alpha_i(x_i)$ for $x_i \in (E(S_i))$ and $i \in I$. It can easily be checked that α is well-defined and we have

$$g\alpha(\sum_{i=1}^n x_i) = \sum_{i=1}^n g\alpha_i(x_i) = \sum_{i=1}^n \pi_i(x_i) = \sum_{i=1}^n x_i.$$

Thus, $g\alpha = 1_M$, and $\alpha : M \rightarrow E$ is a split monomorphism. Then $M \cong D \leq^\oplus E$. Since a direct summand of a UC-module is again a UC-module, M is a UC-module. By Lemma 1, R is a right Noetherian ring. With similar argument, we can prove that an arbitrary direct sum $M = \bigoplus_{i \in I} M_i$ of right UC-modules is again a UC-module. Hence, by Theorem 3, R is semisimple. $\square$

Theorem 6. Let $R = \prod_{\alpha \in \Lambda} R_\alpha$ be a product of rings. Then R is right UC if and only if each R_α is right UC.

Proof. Let π_α be the α th projection map and i_α the α th inclusion map canonically. ($\Rightarrow$.) Let $0 \neq C_\alpha$ be a closed right ideal of R_α for each R_α . We show that $C = i_\alpha(C_\alpha)$ is a closed right ideal of R . If C is not a closed right ideal of R , there is a right ideal B of R such that C is properly contained in B and $C \leq^{ess} B$. If $\beta \neq \alpha$, then $C \leq^{ess} B$ implies that $\pi_\beta(B) = 0$. Thus $B = i_\alpha \pi_\alpha(B)$. Since C is properly contained in B and $C \leq^{ess} B$, C_α is properly contained in $\pi_\alpha(B)$ and $C_\alpha \leq^{ess} \pi_\alpha(B)$. This is impossible because C_α is a closed right ideal of R_α . Thus C is a nonzero closed right ideal of R . The rest is straightforward.

($\Leftarrow$.) Let $0 \neq C$ be a closed right ideal of R . Set $C_\alpha = \pi_\alpha(C)$, $\alpha \in \Lambda$. It can easily be checked that $C = \prod_{\alpha \in \Lambda} C_\alpha$. Since C is a closed right ideal of R , clearly, C_α is a closed right ideal of R_α for each $\alpha \in \Lambda$. Since $C \neq 0$, there exists $\alpha \in \Lambda$ such that C_α is a nonzero closed right ideal of R_α . The rest is straightforward. $\square$

In this section, we give some results about UC-rings. Recall that a ring R is said to be a *right UC-ring* if the module R_R is a UC-module. Let R be a ring, e an idempotent in R such that $R = ReR$, and S the subring eRe . It is clear that if M is a right R -module, then Me is a right S -module.

Theorem 7. Using the above notation, the module $(Me)_S$ is a UC-module if and only if the module M_R is a UC-module.

Proof. Immediate by [1, Lemma 5(i)]. $\square$

Corollary 3. *Using the above notation, the ring R is a UC-ring if and only if the module $(Re)_{eRe}$ is a UC-module.*

Proof. This follows immediately from Theorem 7. $\square$

In this note, R^n denotes the set of all $n \times 1$ column matrices over R . Let R be a ring, n a positive integer, $M_n(R)$ the ring of $n \times n$ matrices over R , and e_{11} the matrix in $M_n(R)$ with $(1, 1)$ entry 1 and all other entries 0. It is well known that e_{11} is idempotent, $R \cong e_{11}M_n(R)e_{11}$ and $M_n(R) = M_n(R)e_{11}M_n(R)$. Thus, Corollary 3 gives the next two results without further proof.

Corollary 4. *The ring $M_n(R)$ is a UC-ring if and only if the free R -module R^n is a UC-module.*

Corollary 5. *Let R be a ring, and let Λ be an infinite set. Then $\mathbb{CFM}_\Lambda(R)$ is right UC if and only if $R_R^{(\Lambda)}$ is UC.*

In the following example we see that UC is not a Morita invariant property.

Example 2. Consider the ring $\mathbb{Z}_4$. Although $\mathbb{Z}_4$ is UC, the ring $R = \begin{bmatrix} \mathbb{Z}_4 & \mathbb{Z}_4 \\ \mathbb{Z}_4 & \mathbb{Z}_4 \end{bmatrix}$ of 2×2 matrices over $\mathbb{Z}_4$ is not UC. If R were UC, then by Corollary 4, the right $\mathbb{Z}_4$ -module $\mathbb{Z}_4^2$ would be UC. One can now argue that this is not so, since if A and B are the submodules of $\mathbb{Z}_4^2$ generated by $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, respectively, then A and B are both direct summands of $\mathbb{Z}_4^2$ (and thus closed submodules), yet their intersection, which is the submodule generated by $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$, is not closed.

4. UC-MODULES AND EXCELLENT EXTENSIONS

Recall from [17] that let R be a subring of a ring S such that they have the same identity. The ring S is called a right excellent extension of R if the following two conditions are satisfied:

- (1) ${}_S R$ and R_S are free modules with a basis $\{1 = a_1, a_2, \dots, a_n\}$ such that $a_i R = R a_i$ for $i = 1, \dots, n$.
- (2) For any submodule A_S of a module M_S , if A_R is a direct summand of M_R , then A_S is a direct summand of M_S .

Lemma 2 ([18, Proposition 1.6]). *Let A_S be a submodule of an S -module M . Then A_S is closed in M_S if and only if A_R is closed in M_R .*

Lemma 3 ([19, Lemma 2.4]). *Let A_R be a submodule of M_R . Then A_R is a closed submodule of M_R if and only if $(A \otimes_R S)_S$ is a closed submodule of $(M \otimes_R S)_S$.*

Theorem 8. *Let M be a right S -module. Then M_R is a UC-module if and only if M_S is a UC-module.*

Proof. Let M_R be a UC-module and A_S and B_S are closed submodules of M_S . By Lemma 2, A_R and B_R are closed submodules of M_R . Since M_R is UC, $A_R \cap B_R = C_R$ is a closed submodule of M_R . By Lemma 2, $A_S \cap B_S = C_S$ is a closed submodule of M_S . So, M_S is a UC-module. The converse can be proved similarly. $\square$

Theorem 9. *Let M be a right R -module. Then $(M \otimes_R S)_S$ is a UC-module if and only if M_R is a UC-module.*

Proof. Immediate by the definition of UC-modules and Lemma 3. $\square$

An R -module M is called *finitely Σ -UC* if every finite direct sum of copies of M is UC. The ring R is called *right finitely Σ -UC* if R_R is *finitely Σ -UC*.

Theorem 10. *Let S be a right excellent extension of R . Then R is right finitely Σ -UC if and only if S is also.*

Proof. Suppose R_R is finitely Σ -UC. Then for any $k > 0$, $(S^k)_R \cong (R^k)_R$ is UC. Thus, $(S^k)_S$ is UC by Theorem 8.

For the converse, suppose S_S is finitely Σ -UC. Then for any $k > 0$, $(R^k \otimes_R S)_S \cong (S^k)_S$ is UC. By Theorem 9, $(R^k)_R$ is UC. $\square$

It can easily be checked that the theorem still holds (with the same proof) if “finitely Σ -UC” is replaced by “countably Σ -UC” or “ Σ -UC”.

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SUMMATIONS AND TRANSFORMATIONS FOR VERY WELL-POISED HYPERGEOMETRIC FUNCTIONS ${}_{2q+5}F_{2q+4}(1)$ AND ${}_{2q+7}F_{2q+6}(1)$ WITH ARBITRARY INTEGRAL PARAMETER DIFFERENCES

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Abstract. The present paper aims to derive summation and transformation formulae for the generalized very well-poised hypergeometric functions ${}_{2q+5}F_{2q+4}(1)$ and ${}_{2q+7}F_{2q+6}(1)$ having arbitrary integral parameter differences. These results are derived with the help of Bailey's transform and the extension of Saalschütz summation theorem for the series ${}_{r+3}F_{r+2}(1)$, where r pairs of parameters differ by positive integers. The particularizations of these generalized identities give classical summation theorems due to Dougall, transformation formula due to Whipple and, other related results. Furthermore, the application of ${}_{2q+5}F_{2q+4}(1)$ summation to the limiting case, when $q \rightarrow 1$, of one of the Andrews' q -identities gives a Srivastava-Daoust type multiple hypergeometric series.

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1. INTRODUCTION

An extension of terminating ${}_2F_1(1)$ to r pairs of parameters differ by positive integers due to Minton [17], is given by:

$${}_{r+2}F_{r+1} \left[\begin{matrix} -n, a, & (b_r + m_r); \\ a + 1, & (b_r); \end{matrix} \middle| 1 \right] = \frac{n!}{(a+1)_n} \frac{(b_1 - a)_{m_1} \cdots (b_r - a)_{m_r}}{(b_1)_{m_1} \cdots (b_r)_{m_r}}, \quad (1.1)$$

where (m_r) be a non-empty sequence of r positive integers $m_1, \dots, m_r$ and $n \in \mathbb{Z}$ such that $n \geq m_1 + \cdots + m_r$. The ${}_{r+2}F_{r+1}(1)$ hypergeometric series (1.1) often appears as a solution to the problems in theoretical physics (For example, $r = 2$ case of (1.1) implies Racah coefficients, see [17]), and hence such type of series and their variants are of great importance and worthwhile to study. A reduction formula for generalized hypergeometric functions with certain parameter pairs differ by positive integral values [8, p. 270, Equation (1)] was proved by Karlsson and (in two markedly different

simpler ways) by Srivastava [22]. Furthermore, [8] generalized this result to non-terminating ${}_{r+2}F_{r+1}(1)$ under the less restrictive condition that $-n$ may be replaced by the complex number c such that $R(-c) > m_1 + \cdots + m_r - 1$, which guarantees the convergence of ${}_{r+2}F_{r+1}(1)$. Later on, several hypergeometric summation and transformation formulae with integer parameter differences have been investigated, see [9, 11–16, 18–20, 28] and, the papers cited therein. Notably, Miller and Srivastava [16] developed the Karlsson-Minton summation theorems for the generalized hypergeometric series with integer parameter differences. Srivastava et al. [27] were deduced several Srivastava-Daoust double hypergeometric functions with the help of general double series identities, derived using series iteration techniques in conjunction with Kummer and Dixon.

Very recently, Srivastava et al. [26] derived many new summation theorems; where one pair of numerator and denominator parameters differ by negative integers; for the truncated, terminating and non-terminating Gauss hypergeometric series, Kummer's first, second and third summation theorems and, their generalized versions by utilizing a general series identity involving a bounded sequence of complex numbers and its hypergeometric versions asserted and proved in [26, Section 2, pp. 468–469]. In the sequel to [26], Srivastava et al. [25] derived some corollaries [25, Corollaries 3.2–3.4] using the general series identity [26, p. 468] and, utilized them to investigate certain summation theorems with one pair of numerator and denominator parameters differ by negative integers, for the truncated, terminating and non-terminating Clausen's hypergeometric series ${}_3F_2(1)$. The applications of summation theorems, investigated by [25, 26], in deriving the closed-form evaluations of the Mellin transforms of the product of the exponential function e^{t^u} and Kummer's truncated and non-terminating confluent hypergeometric series and, the product of the exponential function e^{t^u} and Goursat's non-terminating hypergeometric function ${}_2F_2(1)$, respectively, are also shown, see [25, 26].

Recently, Srivastava et al. [28, Theorems 3.1–3.5] investigated extensions and transformations for the classical summation theorems on very well-poised hypergeometric functions, as listed by Slater [21], namely, Dougall's terminating ${}_5F_4(1)$ summation theorem [21, p. 244, Equation (III.13)], Whipple's ${}_7F_6(1)$ transformation [21, p. 61, Equation (2.4.1.1)], Dougall's ${}_7F_6(1)$ summation [21, p. 244, Equation (III.14)], Bailey's ${}_9F_8(1)$ transformation [21, p. 71, Equation (2.4.4.1)], and Dougall's non-terminating ${}_5F_4(1)$ summation [21, p. 244, Equation (III.12)], respectively, with two pairs of numerator and denominator parameters having unit difference by utilizing Bailey's transform [21, pp. 58–74], and extended Saalschütz summation theorem given by Rakha and Rathie [19] that contains a pair of parameters differ by unity. Furthermore, they produced a longer list of summation and transformation formulae as particular cases of main theorems and shown the application of generalized Dougall's non-terminating ${}_5F_4(1)$ summation theorem [28, Theorem 3.5] in deriving numerous Ramanujan type series involving π and other constants.

Motivated from [28], in this paper, the unique families of summation and transformation formulae for the generalized very well-poised hypergeometric functions, namely, Dougall's terminating ${}_{2q+5}F_{2q+4}(1)$ summation (§3, Theorem 2) and Whipple's ${}_{2q+7}F_{2q+6}(1)$ transformation (§4, Theorem 3), involving the arbitrary number of pairs of parameters with arbitrary differences, are investigated. It is accomplished by utilizing Bailey's transform [21, pp. 58-74] and extended Saalschütz summation theorem for the series ${}_{r+3}F_{r+2}(1)$ due to Kim et al. [9]. Theorem 2, proved in this paper by induction, provides an elegant method to obtain the summation s of high-order hypergeometric series when $q \in \mathbb{Z}^+$ and its particular cases are discussed in §7. The study of such type of summations is important, as whenever a generalized hypergeometric function reduces to the quotient of the products of the gamma function, the results are very important from the application point of view in various fields of science and engineering.

The generalized Dougall's terminating ${}_{2p+7}F_{2p+6}(1)$ summation (§5, Theorem 4) and Dougall's non-terminating ${}_{2p+5}F_{2p+4}(1)$ summation (§6, Theorem 5) follow from Theorem 3, which is the most generalized version of Whipple's transformation in the form of ${}_{2p+7}F_{2p+6}(1)$ into ${}_{m+4}F_{m+3}(1)$; $m = \sum_{i=1}^p m_i$. Recently, Maier [14] has obtained the generalization of Whipple's transformation in the form of ${}_{2k+7}F_{2k+6}(1)$ into ${}_4F_3(1)$ and ${}_{2k+7}F_{2k+6}(1)$ into ${}_5F_4(1)$ using a different method.

The limiting case, when $q \rightarrow 1$, of a forty five year old q -identity due to Andrews [1, Theorem 4] (see also [10, Equation (3.1)]) that relates a terminating very well-poised basic hypergeometric series to a terminating multiple basic hypergeometric series is given by:

$$\begin{aligned}
 & {}_{2s+5}F_{2s+4} \left[\begin{matrix} a, 1 + \frac{a}{2}, b_1, c_1, \dots, b_{s+1}, c_{s+1}, -N; \\ \frac{a}{2}, 1 + a - b_1, 1 + a - c_1, \dots, 1 + a - b_{s+1}, 1 + a - c_{s+1}, 1 + a + N; \end{matrix} \right] \\
 &= \frac{(1+a)_N (1+a-b_{s+1}-c_{s+1})_N}{(1+a-b_{s+1})_N (1+a-c_{s+1})_N} \sum_{k_1, \dots, k_s=0}^{\infty} \frac{(-N)_{k_1+\dots+k_s}}{(b_{s+1}+c_{s+1}-a-N)_{k_1+\dots+k_s}} \\
 & \quad \cdot \prod_{j=1}^s \frac{(1+a-b_j-c_j)_{k_j} (b_{j+1})_{k_1+\dots+k_j} (c_{j+1})_{k_1+\dots+k_j}}{k_j! (1+a-b_j)_{k_1+\dots+k_j} (1+a-c_j)_{k_1+\dots+k_j}}, \quad (1.2)
 \end{aligned}$$

where N and s are positive integers and other involved parameters are complex numbers with the restriction that none of the denominator parameters are negative integers. In §8, the generalized Dougall's ${}_{2q+5}F_{2q+4}(1)$ summation formula (Theorem 2) is applied to the left side of identity (1.2) to produce the sum of a terminating Srivastava-Daoust type multiple hypergeometric series.

2. PRELIMINARIES

The generalized Gauss hypergeometric function [23, 24] for any integer $r \in \mathbb{N}_0$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ where $\mathbb{N}$ being the set of natural numbers, and parameter vector

$$(\mathbf{a}; \mathbf{b}) = (a_1, \dots, a_{r+1}; b_1, \dots, b_r) \in \mathbb{C}^{r+1} \times \mathbb{C}^r$$

in which $b_j \notin \mathbb{Z}_0^-$ ($j = 1, \dots, r$); $\mathbb{Z}_0^- = \mathbb{Z}^- \cup \{0\}$ and $\mathbb{C}$ denotes the set of complex numbers, is defined as follows:

$$\begin{aligned} {}_{r+1}F_r \left[\begin{matrix} a_1, a_2, \dots, a_{r+1}; \\ b_1, b_2, \dots, b_r; \end{matrix} \middle| z \right] &= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{r+1} (a_j)_n}{\prod_{j=1}^r (b_j)_n} \frac{z^n}{n!} \\ &= {}_{r+1}F_r(a_1, \dots, a_{r+1}; b_1, \dots, b_r; z), \end{aligned} \quad (2.1)$$

where $(a)_n$ ($n \in \mathbb{N}_0$) is Pochhammer's symbol (also known as the *shifted factorial*) defined by

$$(a)_0 := 1 \quad \text{and} \quad (a)_n := a(a+1) \cdots (a+n-1) \quad (n \in \mathbb{N}).$$

The compact notation $(a_1, \dots, a_r)_n = (a_1)_n \cdots (a_r)_n$ will be used throughout the presentation. The series in (2.1) is absolutely convergent when $|z| < 1$, and when $|z| = 1$ if the parametric excess

$$\omega = \sum_{j=1}^r b_j - \sum_{j=1}^{r+1} a_j$$

has positive real part. Furthermore, at the point $z = -1$, the series in (2.1) converges conditionally if $\Re(\omega) > -1$ ([23, p. 20]).

The generalized hypergeometric series ${}_{r+1}F_r(1)$ has several summation and transformation formulae that have great importance in the theory of special functions (For example, Equation (1.1)). Many of the ${}_{r+1}F_r(1)$ type series like Saalschütz summation theorem are enriched with balancing property. A generalized hypergeometric function is called ω -balanced if the sum of denominator parameters exceed by ω to the sum of numerator parameters. The classical summation theorems like ${}_5F_4(1)$, ${}_7F_6(1)$, ${}_9F_8(1)$, etc. (see [21]), are not only balanced but also possess characteristic of well-poisedness. The well-poisedness of a generalized hypergeometric function lies in the fact that $1 + a_1 = a_2 + b_1 = a_3 + b_2 = \cdots = a_{r+1} + b_r$. A generalized hypergeometric series is called very well-poised if it is well-poised and, $a_2 = 1 + \frac{a_1}{2}$ and $b_1 = \frac{a_1}{2}$.

The following classical very well-poised transformation due to Whipple, the extension of Saalschütz summation theorem given by [9] and, well-known Bailey's

transform will be used in our investigation:

$$\begin{aligned} & {}_7F_6 \left[\begin{matrix} a, 1 + \frac{a}{2}, b, c, d, e, -N; \\ \frac{a}{2}, 1 + a - b, 1 + a - c, 1 + a - d, 1 + a - e, 1 + a + N; \end{matrix} \right] \\ &= \frac{(1+a)_N (1+a-d-e)_N}{(1+a-e)_N (1+a-d)_N} {}_4F_3 \left[\begin{matrix} 1 + a - b - c, d, e, -N; \\ 1 + a - b, 1 + a - c, d + e - a - N; \end{matrix} \right]. \end{aligned} \quad (2.2)$$

The Bailey-type proof of (2.2), using the Saalschütz summation theorem [21, p. 243, Equation (III.2)] and a ${}_5F_4(1)$ summation [21, p. 244, Equation (III.13)], is outlined in [21, p. 61]. The extension of Saalschütz summation theorem [9] when r pairs of parameters differ by (m_r) , a non-empty sequence of positive integers, is as follows:

$$\begin{aligned} & {}_{r+3}F_{r+2} \left[\begin{matrix} a, b, -n, & (f_r + m_r); \\ c, 1 + a + b - c + m - n, & (f_r); \end{matrix} \right] \\ &= \frac{(c-a-m)_n (c-b-m)_n ((\eta_m + 1))_n}{(c)_n (c-a-b-m)_n ((\eta_m))_n}, \end{aligned} \quad (2.3)$$

where $n \in \mathbb{Z}^+$, $f_i \neq a$ or b ($1 \leq i \leq r$) and, the lower parameters (η_m) on the right side are non-vanishing zeros of the associated parametric polynomial $Q_m(t)$ of degree m ($= \sum_{i=1}^r m_i$) given by

$$Q_m(t) = \sum_{k=0}^m B_k(a)_k (b)_k (t)_k G_{m,k}(t) \quad (2.4)$$

with

$$B_k = (-1)^k A_k (c-a-m+k)_{m-k} (c-b-m+k)_{m-k} \quad (2.5)$$

and

$$G_{m,k}(t) = {}_3F_2 \left[\begin{matrix} -(m-k), t+k, c-a-b-m; \\ c-a-m+k, c-b-m+k; \end{matrix} \right]. \quad (2.6)$$

The function $G_{m,k}(t)$ is a polynomial in t of the degree $m-k$ where $0 \leq k \leq m$. The coefficients A_k are defined by

$$A_k = \sum_{j=k}^m S_j^{(k)} \sigma_{m-j}, \quad A_0 = (f_1)_{m_1} \cdots (f_r)_{m_r}, \quad A_m = 1, \quad (2.7)$$

where $S_j^{(k)}$ are the Stirling number of second kind defined as

$$S_j^{(k)} = \sum_{l=0}^k \frac{(-1)^{k-l}}{k!} \binom{k}{l} l^j.$$

Since $S_j^{(k)}$ represents the number of ways to partition j objects into k non-empty equivalence classes, it is evident that $S_0^{(0)} = S_j^{(j)} = S_j^{(1)} = 1$, and $S_j^{(0)} = S_0^{(j)} = 0$. For further details on the Stirling number of second kind, please refer to Graham et al. [5, Chapter 6]. The coefficients σ_j ($0 \leq j \leq m$) can be obtained from the following generating relation:

$$(f_1 + x)_{m_1} \cdots (f_r + x)_{m_r} = \sum_{j=0}^m \sigma_{m-j} x^j. \quad (2.8)$$

The case of m being zero implies that (m_r) is empty.

W. N. Bailey [2, 3] developed a series transform known as Bailey's transform and gave a mechanism to derive ordinary and q -hypergeometric identities and Rogers-Ramanujan type identities. Its extensions and other types of Bailey's transform and their applications in deriving many new summation and transformation formulae can be found in [6, 7]. It states that if

$$\beta_n = \sum_{r=0}^n \alpha_r u_{n-r} v_{n+r} \quad (2.9)$$

and

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{r+n}, \quad (2.10)$$

then, subject to convergence conditions,

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n. \quad (2.11)$$

3. GENERALIZATION OF DOUGALL'S SUMMATION

Theorem 1. For any $m \in \mathbb{Z}_0^+$, it is asserted that

$$\begin{aligned} & {}_7F_6 \left[\begin{matrix} a, 1 + \frac{a}{2}, b, a - f + 1, c, f + m, -N; \\ \frac{a}{2}, 1 + a - b, f, 1 + a - c, 1 + a - f - m, 1 + a + N; \end{matrix} \right] \\ &= \frac{(1+a)_N (1+a-b-c-m)_N ((g'_m+1))_N}{(1+a-b)_N (1+a-c)_N ((g'_m))_N}, \end{aligned} \quad (3.1)$$

where the lower parameters (g'_m) on the right side are non-vanishing zeros of the associated parametric polynomial $Q_m(t)$ of degree m ($= \sum_{i=1}^1 m_i$) defined as:

$$Q_m(t) = \sum_{k=0}^m B_k (f-b)_k (c)_k (t)_k G_{m,k}(t) \quad (3.2)$$

with

$$B_k = (-1)^k A_k (1+a-f-m+k)_{m-k} (1+a-b-c-m+k)_{m-k} \quad (3.3)$$

and

$$G_{m,k}(t) = {}_3F_2 \left[\begin{matrix} -(m-k), t+k, 1+a-f-c-m; \\ 1+a-f-m+k, 1+a-b-c-m+k; \end{matrix} \middle| 1 \right]. \quad (3.4)$$

The coefficients A_k are defined by

$$A_k = \sum_{j=k}^m S_j^{(k)} \sigma_{m-j}, \quad A_0 = (f)_m, \quad A_m = 1, \quad (3.5)$$

The coefficients σ_j ($0 \leq j \leq m$) can be obtained from the following generating relation:

$$(f+x)_m = \sum_{j=0}^m \sigma_{m-j} x^j. \quad (3.6)$$

Remark 1. For $m = 1$, the Theorem 1 gives a theorem due to [28, Theorem 3.1]. Further, by letting $f \rightarrow \infty$ or $f = b$ or $m = 0$ in Theorem 1, a terminating classical ${}_5F_4(1)$ summation theorem [21, p. 244, Eq. (III.13)] follows.

Remark 2. For $m = 2$, the equation (3.5) gives the following parameters as:

$$A_0 = f(f+1), \quad A_1 = 2(f+1), \quad A_2 = 1.$$

The associated parametric polynomial $Q_2(t)$ (see (3.2)) (with zeros g'_1 and g'_2) is given by

$$\begin{aligned} Q_2(t) &= A_0 (\lambda)_2 (\lambda')_2 \left(1 - \frac{2\sigma t}{\lambda \lambda'} + \frac{(\sigma)_2 (t)_2}{(\lambda)_2 (\lambda')_2} \right) - A_1 (f-b)ct(\lambda+1)(\lambda'+1) \\ &\quad \cdot \left(1 - \frac{\sigma(1+t)}{(\lambda+1)(\lambda'+1)} \right) + A_2 (f-b)_2 (c)_2 (t)_2 \\ &= A_0 (\lambda)_2 (\lambda')_2 \left(1 - \frac{2Y_1 t}{\lambda \lambda'} + \frac{Y_2 (t)_2}{(\lambda)_2 (\lambda')_2} \right) \end{aligned} \quad (3.7)$$

where

$$Y_1 = \sigma + \frac{(f-b)c}{f}, \quad Y_2 = (\sigma)_2 + \frac{(f-b)c}{f} \left(2\sigma + \frac{(f-b+1)(c+1)}{f+1} \right)$$

and

$$\lambda = (a-f-1), \quad \lambda \neq 0, \quad \lambda' = (a-b-c-1), \quad \lambda' \neq 0, \quad \sigma = (a-f-c-1).$$

Then, the Theorem 1 gives a ${}_7F_6(1)$ summation formula, as written below:

$${}_7F_6 \left[\begin{matrix} a, 1 + \frac{a}{2}, b, a-f+1, c, f+2, -N; \\ \frac{a}{2}, 1+a-b, f, 1+a-c, a-f-1, 1+a+N; \end{matrix} \middle| 1 \right]$$

$$= \frac{(1+a)_N (a-b-c-1)_N (g'_1+1)_N (g'_2+1)_N}{(1+a-b)_N (1+a-c)_N (g'_1)_N (g'_2)_N}. \quad (3.8)$$

Similarly, the following ${}_7F_6(1)$ summation formula for $m = 3$ holds true, where the lower parameters g'_1, g'_2 , and g'_3 on the right side are non-vanishing zeros of the associated parametric polynomial $Q_3(t)$ evaluated by manipulating the equations (3.2)-(3.6) accordingly.

$$\begin{aligned} & {}_7F_6 \left[\begin{matrix} a, 1 + \frac{a}{2}, b, a-f+1, c, f+3, -N; \\ \frac{a}{2}, 1+a-b, f, 1+a-c, a-f-2, 1+a+N; \end{matrix} \right] \\ &= \frac{(1+a)_N (a-b-c-2)_N (g'_1+1)_N (g'_2+1)_N (g'_3+1)_N}{(1+a-b)_N (1+a-c)_N (g'_1)_N (g'_2)_N (g'_3)_N}. \end{aligned} \quad (3.9)$$

Proof of Theorem 1. Selecting $e = f + m, c = a - f + 1$ and, $d = c$ in the Whipple's transformation (2.2) and summing the ${}_4F_3(1)$ on the right side by using the extended Saalschütz theorem (2.3) ($r = 1$ case), followed by some simplifications, would lead to the Theorem 1. $\square$

Theorem 2. For $q, m \in \mathbb{Z}_0^+$, it is asserted that

$$\begin{aligned} & {}_{2q+5}F_{2q+4} \left[\begin{matrix} a, 1 + \frac{a}{2}, b, c, a-f_1+1, \dots, a-f_q+1, \\ \frac{a}{2}, 1+a-b, 1+a-c, 1+a-f_1-m_1, \dots, 1+a-f_q-m_q, \\ f_1+m_1, \dots, f_q+m_q, -N; \\ f_1, \dots, f_q, 1+a+N; \end{matrix} \right] \\ &= \frac{(1+a)_N (1+a-b-c-m)_N ((g_m^q+1))_N}{(1+a-b)_N (1+a-c)_N ((g_m^q))_N}, \end{aligned}$$

where the lower parameters (g_m^q) on the right side are non-vanishing zeros of the associated parametric polynomial $Q_m(t)$ of degree m ($= \sum_{i=1}^q m_i$) defined as:

$$Q_m(t) = \sum_{k=0}^m B_k (f_q - b - M)_k (c)_k (t)_k G_{m,k}(t); \quad M = \sum_{i=1}^{q-1} m_i$$

with

$$B_k = (-1)^k A_k (1+a-f_q-m_q+k)_{m-k} (1+a-b-c-m+k)_{m-k}$$

and

$$G_{m,k}(t) = {}_3F_2 \left[\begin{matrix} -(m-k), t+k, 1+a-f_q-c-m_q; \\ 1+a-f_q-m_q+k, 1+a-b-c-m+k; \end{matrix} \right]_1.$$

The coefficients A_k are defined by

$$A_k = \sum_{j=k}^m S_j^{(k)} \sigma_{m-j}, \quad A_0 = \left((g_M^{q-1}) \right)_1 (f_q)_{m_q}, \quad A_m = 1,$$

The coefficients $\sigma_j (0 \leq j \leq m)$ are generated by

$$\left((g_M^{q-1} + x) \right)_1 (f_q + x)_{m_q} = \sum_{j=0}^m \sigma_{m-j} x^j,$$

where $\left(g_M^{q-1} \right)$ themselves are non-vanishing zeros of associated parametric polynomial $Q_M(t)$ of degree $M \left(= \sum_{i=1}^{q-1} m_i \right)$. Also,

$$\begin{aligned} \left((g_M^{q-1}) \right)_1 &= (g_1^{q-1})(g_2^{q-1}) \cdots (g_M^{q-1}) \\ \left((g_M^{q-1} + x) \right)_1 &= (g_1^{q-1} + x)(g_2^{q-1} + x) \cdots (g_M^{q-1} + x) \end{aligned}$$

The above-mentioned products vanishes when $M = 0$.

Remark 3. The classical Dougall's terminating ${}_5F_4(1)$ summation theorem [21, p. 244, Equation (III.13)] follows from Theorem 2 when $m = 0$.

Proof of Theorem 2. In view of Theorem 1, it can be observed that the Theorem 2 holds true for $q = 1$. Next, assume that it holds true for an arbitrary fixed positive integer p , i.e.

$$\begin{aligned} & {}_{2p+5}F_{2p+4} \left[\begin{matrix} a, 1 + \frac{a}{2}, b, c, a - f_1 + 1, \dots, a - f_p + 1, \\ \frac{a}{2}, 1 + a - b, 1 + a - c, 1 + a - f_1 - m_1, \dots, 1 + a - f_p - m_p, \\ f_1 + m_1, \dots, f_p + m_p, -N; \\ f_1, \dots, f_p, 1 + a + N; \end{matrix} \right]_1 \\ &= \frac{(1+a)_N (1+a-b-c-m)_N ((g_m^p + 1))_N}{(1+a-b)_N (1+a-c)_N ((g_m^p))_N}, \end{aligned} \quad (3.10)$$

where the lower parameters (g_m^p) on the right side are non-vanishing zeros of the associated parametric polynomial $Q_m(t)$ of degree $m \left(= \sum_{i=1}^p m_i \right)$, which is similar

to one obtained in Theorem 2. Finally, it remains to verify that the Theorem 2 is also true for $q = p + 1$ (see (3.11)).

$$\begin{aligned}
 & {}_{2(p+1)+5}F_{2(p+1)+4} \left[\begin{matrix} a, 1 + \frac{a}{2}, b, c, a - f_1 + 1, \dots, a - f_{p+1} + 1, \\ \frac{a}{2}, 1 + a - b, 1 + a - c, 1 + a - f_1 - m_1, \dots, \\ f_1 + m_1, \dots, f_{p+1} + m_{p+1}, -N; \\ 1 + a - f_{p+1} - m_{p+1}, f_1, \dots, f_{p+1}, 1 + a + N; \end{matrix} \right] \\
 &= \frac{(1+a)_N (1+a-b-c-m)_N \left((g_m^{p+1} + 1) \right)_N}{(1+a-b)_N (1+a-c)_N \left((g_m^{p+1}) \right)_N}, \quad (3.11)
 \end{aligned}$$

where the lower parameters (g_m^{p+1}) on the right side are non-vanishing zeros of the associated parametric polynomial $Q_m(t)$ of degree $m \left(= \sum_{i=1}^{p+1} m_i \right)$ and, can be obtained by manipulating the equations (2.4)-(2.8) according to the substitution $a = f_{p+1} - b - M; M = \sum_{i=1}^p m_i$, $b = c$ and $c = 1 + a - b$. The modified equation (2.8) has the parameters (g_M^p) , which themselves are non-vanishing zeros of a parametric polynomial $Q_M(t)$ of degree $M \left(= \sum_{i=1}^p m_i \right)$ associated with equation (3.10). Selecting

$$\begin{aligned}
 \alpha_r &= \frac{\left(a, 1 + \frac{a}{2}, b, a - f_1 + 1, \dots, a - f_p + 1, f_1 + m_1, \dots, f_p + m_p, d \right)_r (-1)^r}{\left(\frac{a}{2}, 1 + a - b, f_1, \dots, f_p, 1 + a - d, 1 + a - f_1 - m_1, \dots, 1 + a - f_p - m_p \right)_r r!}, \\
 u_r &= \frac{1}{r!}, \quad v_r = \frac{1}{(1+a)_r} \quad \text{and} \quad \delta_r = \frac{(c, e, -N)_r}{(c + e - a - N)_r}
 \end{aligned}$$

in (2.9) and (2.10) and making use of the summation theorem given in equation (3.10) and Saalschütz summation theorem [21, p. 243, Equation (III.2)] to obtain the following values of β_n and γ_n :

$$\beta_n = \frac{(1+a-b-d-m)_n \left((g_m^p + 1) \right)_n}{(1+a-d, 1+a-b)_n \left((g_m^p) \right)_n n!},$$

where the parameters (g_m^p) are as defined with (3.10) and

$$\gamma_n = \frac{(1+a-c, 1+a-e)_N}{(1+a, 1+a-c-e)_N} \frac{(c, e, -N)_n (-1)^n}{(1+a-c, 1+a-e, 1+a+N)_n}.$$

Putting these values of β_n and γ_n in (2.11), a transformation of ${}_{2p+7}F_{2p+6}(1)$ follows, which is being differed in the next section (see Theorem 3). Next, selecting $e =$

$f_{p+1} + m_{p+1}$ and $d = 1 + a - f_{p+1}$ in Theorem 3 and finally, applying (2.3) followed by some simplifications, lead to the required summation . $\square$

4. GENERALIZATION OF WHIPPLE'S TRANSFORMATION OF ${}_7F_6(1)$ TO ${}_4F_3(1)$

Theorem 3. For $p, m \in \mathbb{Z}_0^+$, it is asserted that

$$\begin{aligned} & \frac{(1+a-c)_N (1+a-e)_N}{(1+a)_N (1+a-c-e)_N} {}_{2p+7}F_{2p+6} \left[\begin{matrix} a, 1 + \frac{a}{2}, b, a-f_1+1, \dots, a-f_p+1, d \\ \frac{a}{2}, 1+a-b, 1+a-d, 1+a-f_1-m_1, \dots, \\ f_1+m_1, \dots, f_p+m_p, c, e, -N; \\ 1+a-f_p-m_p, f_1, \dots, f_p, 1+a-c, 1+a-e, 1+a+N; \end{matrix} \right] \\ &= {}_{m+4}F_{m+3} \left[\begin{matrix} c, e, 1+a-b-d-m, -N, (g_m^p+1); \\ 1+a-b, 1+a-d, c+e-a-N, (g_m^p); \end{matrix} \right] \quad (4.1) \end{aligned}$$

Remark 4. The transformation in Theorem 3 is a generalization of Whipple's ${}_7F_6$ transformation (Equation (2.2)), which follows upon letting $m = 0$ in Theorem 3. The case $p = m = 1$ of Theorem 3 is previously observed by [28, Theorem 3.2], and subsequently by Mishev [18, Proposition 3.1 and 3.6] using a different method given in [4, Chapter 4]. Clearly, for the arbitrary values of p and m in Theorem 3, it is always possible to generate other high-order transformations.

5. GENERALIZATION OF DOUGALL'S TERMINATING ${}_7F_6(1)$ SUMMATION

Theorem 4. For $p, m \in \mathbb{Z}_0^+$, it is asserted that

$$\begin{aligned} & {}_{2p+7}F_{2p+6} \left[\begin{matrix} a, 1 + \frac{a}{2}, b, a-f_1+1, \dots, a-f_p+1, 1+2a-b-c-d+N-m, \\ \frac{a}{2}, 1+a-b, f_1, \dots, f_p, b+c+d-a-N+m, 1+a-f_1-m_1, \dots, \\ f_1+m_1, \dots, f_p+m_p, c, d, -N; \\ 1+a-f_p-m_p, 1+a-c, 1+a-d, 1+a+N; \end{matrix} \right] = \frac{(1+a)_N}{(1+a-b)_N} \\ & \frac{(1+a-c-d)_N (1+a-b-c-m)_N (1+a-b-d-m)_N ((\eta_m+1))_N}{(1+a-c)_N (1+a-d)_N (1+a-b-c-d-m)_N ((\eta_m))_N}, \quad (5.1) \end{aligned}$$

where the lower parameters (η_m) on the right side are non-vanishing zeros of associated parametric polynomial $Q_m(t)$ of degree m ($= \sum_{i=1}^p m_i$). The polynomial $Q_m(t)$ can be obtained easily by manipulating the equations (2.4)-(2.8) accordingly.

Remark 5. The case $m = 0$ of Theorem 4 is the classical Dougall's theorem [21, p. 244, Equation (III.14)]. Furthermore, by setting $p = m = 1$ in Theorem 4, leads to a result due to [28, Theorem 3.3].

Proof of Theorem 4. Choosing $d = 1 + 2a - b - c - e + N - m$ in Theorem 3 and applying (2.3) (for the case $m_1 = \dots = m_r = 1$) and then replacing $e \rightarrow d$, gives the Theorem 4. $\square$

6. GENERALIZATION OF DOUGALL'S NON-TERMINATING ${}_5F_4(1)$ SUMMATION

Theorem 5. For $p, m \in \mathbb{Z}_0^+$, it is asserted that

$$\begin{aligned}
 & {}_{2p+5}F_{2p+4} \left[\begin{matrix} a, 1 + \frac{a}{2}, b, a - f_1 + 1, \dots, a - f_p + 1, \\ \frac{a}{2}, 1 + a - b, f_1, \dots, f_p, 1 + a - f_1 - m_1, \dots, \\ f_1 + m_1, \dots, f_p + m_p, c, d; \\ 1 + a - f_p - m_p, 1 + a - c, 1 + a - d; \end{matrix} \right] \\
 &= \frac{\Gamma(1 + a - b)\Gamma(1 + a - c)\Gamma(1 + a - d)\Gamma(1 + a - b - c - d - m)\Gamma((\eta_m))}{\Gamma(1 + a)\Gamma(1 + a - c - d)\Gamma(1 + a - b - c - m)\Gamma(1 + a - b - d - m)\Gamma((\eta_m + 1))}, \\
 & \text{provided } \operatorname{Re}(a - b - c - d + (1 - m)) < 0, \text{ where the lower parameters } (\eta_m) \text{ on the} \\
 & \text{right side are non-vanishing zeros of associated parametric polynomial } Q_m(t) \text{ of de-} \\
 & \text{gree } m (= \sum_{i=1}^p m_i). \text{ The polynomial } Q_m(t) \text{ can be obtained easily by manipulating} \\
 & \text{the conditions obtained with Theorem 4.}
 \end{aligned}$$

Remark 6. The case $m = 0$ of Theorem 5 is the classical Dougall's non-terminating theorem [21, p. 244, Equation (III.12)]. Furthermore, by selecting $p = m = 1$, a result due to [28, Theorem 3.5] follows.

Proof of Theorem 5. The Theorem 5 follows from Theorem 4 by taking the limit when $N \rightarrow \infty$. $\square$

7. PARTICULAR CASES OF THEOREM 2

Theorem 6. It is asserted that

$$\begin{aligned}
 & {}_9F_8 \left[\begin{matrix} a, 1 + \frac{a}{2}, b, c, a - f_1 + 1, a - f_2 + 1, f_1 + m_1, f_2 + m_2, -N; \\ \frac{a}{2}, 1 + a - b, 1 + a - c, 1 + a - f_1 - m_1, 1 + a - f_2 - m_2, f_1, f_2, 1 + a + N; \end{matrix} \right] \\
 &= \frac{(1 + a)_N (1 + a - b - c - m)_N ((g_m'' + 1))_N}{(1 + a - b)_N (1 + a - c)_N ((g_m''))_N},
 \end{aligned}$$

where the lower parameters (g_m'') on the right side are non-vanishing zeros of the associated parametric polynomial $Q_m(t)$ of degree $m (= \sum_{i=1}^2 m_i)$ and, can be obtained by manipulating the equations (2.4)-(2.8) according to the substitution $a = f_2 - b - m_1, b = c$ and $c = 1 + a - b$.

Proof of Theorem 6. The Theorem 6 follows from Theorem 2 when $q = 2$. $\square$

Theorem 7. It is asserted that

$${}_9F_8 \left[\begin{matrix} a, 1 + \frac{a}{2}, b, c, a - f_1 + 1, a - f_2 + 1, f_1 + 1, f_2 + 1, -N; \\ \frac{a}{2}, 1 + a - b, 1 + a - c, a - f_1, a - f_2, f_1, f_2, 1 + a + N; \end{matrix} \right] = \frac{(1+a)_N (a-b-c-1)_N ((g_2'')_N)}{(1+a-b)_N (1+a-c)_N ((g_2'')_N)}$$

Proof of Theorem 7. The Theorem 7 follows from Theorem 6 when $m_1 = m_2 = 1$ and the associated parametric polynomial $Q_2(t)$ (with zeros g_1'' and g_2'') is as given below:

$$\begin{aligned} Q_2(t) &= A_0 (\lambda)_2 (\lambda')_2 \left(1 - \frac{2\sigma t}{\lambda \lambda'} + \frac{(\sigma)_2 (t)_2}{(\lambda)_2 (\lambda')_2} \right) - A_1 (f_2 - b - 1) \\ &\cdot ct(\lambda + 1)(\lambda' + 1) \left(1 - \frac{\sigma(1+t)}{(\lambda + 1)(\lambda' + 1)} \right) + A_2 (f_2 - b - 1)_2 (c)_2 (t)_2 \\ &= A_0 (\lambda)_2 (\lambda')_2 \left(1 - \frac{\Upsilon_1 t}{\lambda \lambda'} + \frac{\Upsilon_2 (t)_2}{(\lambda)_2 (\lambda')_2} \right), \end{aligned} \quad (7.1)$$

where

$$\Upsilon_1 = 2\sigma + X (f_2 - b - 1) c, \quad (7.2)$$

$$\Upsilon_2 = (\sigma)_2 + (f_2 - b - 1) c \left(X\sigma + \frac{(f_2 - b)(c + 1)}{g_1' f_2} \right) \quad (7.3)$$

with $X = \frac{(g_1' + f_2 + 1)}{g_1' f_2}$ and

$$\lambda = (a - f_2), \lambda \neq 0, \quad \lambda' = (a - b - c - 1), \lambda' \neq 0, \quad \sigma = (a - f_2 - c).$$

The parameter g_1' is itself a non-vanishing zero of the associated parametric polynomial $Q_1(t)$ given by

$$g_1' = \frac{f_1(f_1 - a)(b + c - a)}{f_1(a - f_1) - bc}$$

and can be obtained by setting $f = f_1, m = 1$ in the equations (3.2)-(3.6) given with Theorem 1. $\square$

Theorem 8. *It is asserted that*

$${}_9F_8 \left[\begin{matrix} a, 1 + \frac{a}{2}, b, c, a - f_1 + 1, a - f_2 + 1, f_1 + 2, f_2 + 1, -N; \\ \frac{a}{2}, 1 + a - b, 1 + a - c, a - f_1 - 1, a - f_2, f_1, f_2, 1 + a + N; \end{matrix} \right] 1$$

$$= \frac{(1+a)_N (a-b-c-2)_N ((g_3''+1))_N}{(1+a-b)_N (1+a-c)_N ((g_3''))_N}$$

Proof of Theorem 8. The Theorem 8 follows from Theorem 6 when $m_1 = 2$ and $m_2 = 1$ and the associated parametric polynomial $Q_3(t)$ (with zeros g_1'' , g_2'' and g_3'') is as given below:

$$Q_3(t) = A_0(\lambda)_3 (\lambda')_3 \left(1 - \frac{\Upsilon_1 t}{\lambda \lambda'} + \frac{(t)_2 \Upsilon_2}{(\lambda)_2 (\lambda')_2} - \frac{(t)_3 \Upsilon_3}{(\lambda)_3 (\lambda')_3} \right) \quad (7.4)$$

where

$$\Upsilon_1 = 3\sigma + X(f_2 - b - 2)c, \quad (7.5)$$

$$\Upsilon_2 = 3(\sigma)_2 + (f_2 - b - 2)c(2X\sigma + Y(f_2 - b - 1)(c + 1)), \quad (7.6)$$

$$\Upsilon_3 = (\sigma)_3 + (f_2 - b - 2)c \left(X(\sigma)_2 + Y(f_2 - b - 1)(c + 1)\sigma + \frac{(f_2 - b - 1)_2 (c + 1)_2}{g_1' g_2' f_2} \right) \quad (7.7)$$

with

$$X = \frac{g_1' g_2' + (f_2 + 1)(g_1' + g_2' + 1)}{g_1' g_2' f_2} \quad Y = \frac{(f_2 + g_1' + g_2' + 3)}{g_1' g_2' f_2}$$

and $\lambda = (a - f_2)$, $\lambda \neq 0$, $\lambda' = (a - b - c - 2)$, $\lambda' \neq 0$, $\sigma = (a - f_2 - c)$. $\square$

Theorem 9. *It is asserted that*

$${}_{11}F_{10} \left[\begin{matrix} a, 1 + \frac{a}{2}, b, c, a - f_1 + 1, a - f_2 + 1, a - f_3 + 1, \\ \frac{a}{2}, 1 + a - b, 1 + a - c, 1 + a - f_1 - m_1, 1 + a - f_2 - m_2, 1 + a - f_3 - m_3, \\ f_1 + m_1, f_2 + m_2, f_3 + m_3, -N; \\ f_1, f_2, f_3, 1 + a + N; \end{matrix} \right] 1$$

$$= \frac{(1+a)_N (1+a-b-c-m)_N ((g_m''' + 1))_N}{(1+a-b)_N (1+a-c)_N ((g_m'''))_N},$$

where the lower parameters (g_m''') on the right side are non-vanishing zeros of the associated parametric polynomial $Q_m(t)$ of degree $m (= \sum_{i=1}^3 m_i)$ and, can be obtained by manipulating the equations (2.4)-(2.8) according to the substitution $a = f_3 - b - M; M = \sum_{i=1}^2 m_i, b = c$ and $c = 1 + a - b$. The modified equation (2.8) has the parameters (g_M'') , which themselves are non-vanishing zeros of associated parametric polynomial $Q_M(t)$ of degree $M (= \sum_{i=1}^2 m_i)$, see Theorem 6.

Proof of Theorem 9. The Theorem 9 follows from Theorem 2 when $q = 3$. $\square$

Remark 7. Clearly, for the arbitrary values of q in Theorem 2, it is always possible to generate other high-order summations like ${}_{13}F_{12}(1)$, ${}_{15}F_{14}(1)$, etc.

8. A SUMMATION FOR MULTIPLE SRIVASTAVA-DAOUST TYPE SERIES

Theorem 10. It is asserted that

$$\sum_{k_1, \dots, k_s=0}^{\infty} \frac{(b, c, -N)_{k_1+\dots+k_s} (-m_s)_{k_s}}{k_s! (f_s, 1+a-f_s-m_s, b+c-a-N)_{k_1+\dots+k_s}} \Upsilon$$

$$= \frac{(1+a-b-c-m)_N ((g_m^s+1))_N}{(1+a-b-c)_N ((g_m^s))_N},$$

where

$$\Upsilon = \prod_{j=1}^{s-1} \frac{(-m_j)_{k_j} (f_{j+1} + m_{j+1}, 1+a-f_{j+1})_{k_1+\dots+k_j}}{k_j! (f_j)_{k_1+\dots+k_j} (1+a-f_j-m_j)_{k_1+\dots+k_j}}$$

and the lower parameters (g_m^s) on the right side are non-vanishing zeros of the associated parametric polynomial $Q_m(t)$ of degree $m (= \sum_{j=1}^s m_j)$, which is similar to one obtained in Theorem 2.

Proof of Theorem 10. Choosing

$$b_{s+1} = b, \quad c_{s+1} = c, \quad (b_1, \dots, b_s) = (a-f_1+1, \dots, a-f_s+1)$$

$$(c_1, \dots, c_s) = (f_1+m_1, \dots, f_s+m_s)$$

in equation (1.2) and applying Theorem 2, followed by some simplifications, would lead to the Theorem 10, which is a sum of very general class of terminating Srivastava-Daoust type multiple hypergeometric series. $\square$

Remark 8. The Dougall's summation theorem for ${}_{2p+7}F_{2p+6}(1)$ (Theorem 4) can be applied to equation (1.2) to obtain a class of another type of Srivastava-Daoust type multiple hypergeometric series.

9. CONCLUSION

In this paper, some classes of summation and transformation theorems are developed, which provide the generalizations of Dougall's terminating summation theorem for ${}_5F_4(1)$ and ${}_7F_6(1)$, Dougall's non-terminating summation theorem for ${}_5F_4(1)$ and Whipple's transformation formula for ${}_7F_6(1)$. Thus, this work has extended existing literature by adding several additional hypergeometric summations and transformations to the Slater's list [21, p. 243-245, Appendix III]. Furthermore, the generalization of Dougall's terminating summation theorem for ${}_5F_4(1)$ is applied to obtain a summation theorem for a class of Srivastava-Daoust type multiple hypergeometric series by making use of the limiting case, when $q \rightarrow 1$, of Andrews' identities.

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NUMERICAL SOLUTION OF THE CONFORMABLE FRACTIONAL DIFFUSION EQUATION

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Abstract. In this paper, a numerical approach for solving space-time fractional diffusion equation with variable coefficients is proposed. The fractional derivatives are described in the conformable sense. The numerical approach is based on shifted Chebyshev polynomials of the second kind. The space-time fractional diffusion equation with variable coefficients is reduced to a system of ordinary differential equations by using the properties of Chebyshev polynomials. The finite difference method is applied to solve this system of equations. Numerical results are provided to verify the accuracy and efficiency of the proposed approach.

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Keywords: Space-time fractional diffusion equation, Shifted Chebyshev polynomials of the second kind, Conformable fractional derivative, Finite difference method

1. INTRODUCTION

Fractional differential equations have been the focus of many studies due to their applications in various fields of science and engineering (see, for example, [2, 10, 15–17]). The main physical purpose of investigating fractional diffusion equations is to describe phenomena of anomalous diffusion in transport processes. Fractional diffusion equations have been also used in modeling turbulent flow [4], groundwater contaminant transport [3] and chaotic dynamics of classical conservative systems [25].

Recently, some different numerical methods have been proposed for solving the fractional diffusion equation. The space fractional diffusion equation which is described in Caputo sense with variable coefficients has been investigated in [1, 7, 8, 18, 22]. In [22], space fractional diffusion equation has been solved by using shifted Chebyshev polynomials of the second kind with finite difference method, respectively. In [1, 8], it has been solved by using shifted Chebyshev polynomials of first kind with finite difference method. In [18], a numerical scheme based on the shifted Legendre tau method to solve the space fractional diffusion equation has been studied. In [7], a numerical approximation for the space fractional diffusion equation via splines has been suggested.

The space fractional diffusion equations which are described in Riemann-Liouville sense have been studied in [5, 12, 13]. In [12], the authors have been used the finite difference method based on a modified Grünwald approximation. In [5], compact finite difference scheme has been used to the space fractional diffusion equation with constant coefficient. In [13], the finite difference approximation obtained from the Grünwald-Letnikov formulation has been applied to the space fractional diffusion equation with variable coefficients. In [21], the Crank-Nicolson finite difference method has been applied to the time fractional diffusion equations with constant coefficients. Generally, fractional diffusion equations have been studied for derivatives in Caputo sense or Riemann-Liouville sense in the literature.

Nowadays, solutions of conformable partial differential equations have been investigated. Conformable time fractional and space-time fractional partial differential equations have been studied by using extended reduced conformable differential transform method and fractional differential transform method in [20]-[23], respectively. Residual power series method has been applied to high-order linear conformable partial differential equations in [6]. Numerical solution of the conformable space-time fractional wave equation has been obtained by using Shifted Chebyshev polynomials and finite difference method in [24]. Furthermore, there are also studies on traveling wave solutions of the conformable partial differential equations (see, for example, [14, 19]). But, the methods used to find the traveling wave solutions can not be applied to the conformable partial differential equations with variable coefficients.

In this paper, we consider conformable space-time fractional diffusion equation with variable coefficient. Fundamental goal of this work is to obtain an analytical approximate solution in terms of the shifted Chebyshev polynomials of the second kind of the space-time fractional diffusion equation with the initial and boundary conditions. Consider the one-dimensional space-time fractional diffusion equation of the form:

$$\begin{aligned} T_t^\mu u(x, t) &= f(x, t) T_x^\alpha u(x, t) + g(x, t), \quad 0 < x < L, \quad 0 < t \leq T, \\ 0 < \mu \leq 1, \quad 1 < \alpha \leq 2, \end{aligned} \quad (1.1)$$

with initial condition

$$u(x, 0) = f_1(x), \quad 0 < x < L, \quad (1.2)$$

and the boundary conditions:

$$u(0, t) = h_1(t), \quad 0 < t \leq T, \quad (1.3)$$

$$u(1, t) = h_2(t), \quad 0 < t \leq T \quad (1.4)$$

where x is a space variable, t is a time variable; the parameters μ and α refer to the order of conformable fractional derivative with respect to time variable and space variable, respectively. The functions $f(x, t)$, $g(x, t)$, $f_1(x)$, $h_1(t)$ and $h_2(t)$ are given

functions. In case of $\mu = 1$, $\alpha = 2$, Eq. (1.1) is the classical diffusion equation:

$$\frac{\partial u(x, t)}{\partial t} = f(x, t) \frac{\partial^2 u(x, t)}{\partial t^2} + g(x, t).$$

The organization of this paper is as follows: In Section 2, definition and properties of the conformable fractional derivative are presented. In the Section 3, properties of Chebyshev polynomials of the second kind are given. In Section 4, the conformable fractional derivative is written by using shifted Chebyshev polynomials of the second kind. In Section 5, numerical scheme is given to obtain an analytical approximate solution in terms of the shifted Chebyshev polynomials of the second kind of the problem (1.1)-(1.4). In Section 6, numerical results are given to clarify the method. Conclusion is presented in Section 7. Note that numerical results have been computed by using the Matlab programming.

2. DESCRIPTION OF CONFORMAL FRACTIONAL DERIVATIVE AND ITS PROPERTIES

For a function $f : (0, \infty) \rightarrow \mathbb{R}$, the conformal fractional derivative of f of order $0 < \alpha < 1$ in variable t is defined as (see, for example, [9])

$$T_t^\alpha f(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}.$$

Some important properties of the the conformal fractional derivative are as follows:

Theorem 1. (see, for example, [9]) Let $\alpha \in (0, 1]$ and f, g be α differentiable at a point $t > 0$. Then

$$T_t^\alpha (af + bg)(t) = aT_t^\alpha f(t) + bT_t^\alpha g(t), \quad \forall a, b \in \mathbb{R}, \quad (2.1)$$

$$T_t^\alpha (fg)(t) = f(t)T_t^\alpha g(t) + g(t)T_t^\alpha f(t), \quad (2.2)$$

$$T_t^\alpha (t^p) = pt^{p-\alpha}, \quad \forall p \in \mathbb{R}. \quad (2.3)$$

If, in addition, f is differentiable, then

$$T_t^\alpha (f(t)) = t^{1-\alpha} f'(t). \quad (2.4)$$

The fractional derivative starting from b of a function $f : [b, \infty) \rightarrow \mathbb{R}$, of order α , where $f^{(n)}(t)$ exists, is defined by

$$T_t^\alpha (f(t)) = T_t^\beta (f^{(n)}(t)), \quad n < \alpha \leq n + 1, \quad \beta = \alpha - n. \quad (2.5)$$

3. SOME PROPERTIES OF CHEBYSHEV POLYNOMIALS OF THE SECOND KIND

The Chebyshev polynomials $U_n(x)$ of the second kind are orthogonal polynomials of degree n in x defined on the $[-1, 1]$ (see, for example, [11])

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta},$$

where $x = \cos\theta$ and $\theta \in [0, \pi]$. Also, these polynomials $U_n(x)$ are orthogonal on $[-1, 1]$ with respect to the inner product

$$\langle U_n(x), U_m(x) \rangle = \int_{-1}^1 \sqrt{1-x^2} U_n(x) U_m(x) dx = \begin{cases} 0, & n \neq m, \\ \frac{\pi}{2}, & n = m. \end{cases},$$

where $\sqrt{1-x^2}$ is weight function corresponding to $U_n(x)$. The polynomials $U_n(x)$ may be generated by using the recurrence relations

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), \quad n = 2, 3, \dots,$$

with $U_0(x) = 1$, $U_1(x) = 2x$.

Shifted Chebyshev polynomials of the second kind $U_n^*(x)$ of degree n in x on $[0, 1]$ is given by $U_n^*(x) = U_n(2x-1)$. These polynomials are orthogonal on the support interval $[0, 1]$ as the following inner product:

$$\langle U_n^*(x), U_m^*(x) \rangle = \int_0^1 \sqrt{x-x^2} U_n^*(x) U_m^*(x) dx = \begin{cases} 0, & n \neq m, \\ \frac{\pi}{8}, & n = m. \end{cases},$$

where $\sqrt{x-x^2}$ is weight function corresponding to $U_n^*(x)$. The polynomials $U_n^*(x)$ may be generated by using the recurrence relations

$$U_n^*(x) = 2(2x-1)U_{n-1}^*(x) - U_{n-2}^*(x), \quad n = 2, 3, \dots,$$

with $U_0^*(x) = 1$, $U_1^*(x) = 4x-2$.

The analytical form of the shifted Chebyshev polynomials of the second kind $U_n^*(x)$ of degree n is given by

$$U_n^*(x) = \sum_{k=0}^n (-1)^k 2^{2n-2k} \frac{\Gamma(2n-k+2)x^{n-k}}{2n-2k+2}. \quad (3.1)$$

The function $y(x)$ which belongs to the space of square integrable in $[0, 1]$ may be expressed in terms of shifted Chebyshev polynomials of the second kind as

$$y(x) = \sum_{i=0}^{\infty} b_i U_i^*(x),$$

where the coefficients b_i are given by

$$b_i = \frac{8}{\pi} \int_0^1 y(x) \sqrt{x-x^2} U_i^*(x) dx. \quad (3.2)$$

Consider only the first $(N+1)$ – terms of shifted Chebyshev polynomials of the second kind, so we can write

$$y_N(x) = \sum_{i=0}^N b_i U_i^*(x). \quad (3.3)$$

4. EVALUATION OF THE CONFORMABLE FRACTIONAL DERIVATIVE USING SHIFTED CHEBYSHEV POLYNOMIALS OF THE SECOND KIND

In this section, we derive an approximate formula of the conformable fractional derivative of $y_N(x)$. By using the linearity of the conformable fractional differentiation given in Eq. (2.1) and by using definition of approximated function $y_N(x)$ as in Eq. (3.3) we have:

$$T_x^\alpha y_N(x) = \sum_{i=0}^N b_i T_x^\alpha U_i^*(x), \quad \alpha > 0. \quad (4.1)$$

Moreover, from the properties of linearity of the conformable derivative in addition to Eqs. (2.3) and (2.5) we get:

$$T_x^\alpha U_i^*(x) = 0, \quad i = 0, 1, \dots, n, \quad n < \alpha \leq n+1. \quad (4.2)$$

Also we have for $n < \alpha \leq n+1$

$$T_x^\alpha U_i^*(x) = \sum_{k=0}^{i-(n+1)} (-1)^k 2^{2i-2k} \frac{\Gamma(2i-k+2)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+2)\Gamma(i-k-n)} x^{i-k-\alpha}. \quad (4.3)$$

By combinations Eqs. (4.1), (4.2) and (4.3) we obtain for $n < \alpha \leq n+1$

$$T_x^\alpha y_N(x) = \sum_{i=n+1}^N \sum_{k=0}^{i-(n+1)} b_i (-1)^k 2^{2i-2k} \frac{\Gamma(2i-k+2)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+2)\Gamma(i-k-n)} x^{i-k-\alpha},$$

which can be rewritten as the form:

$$T_x^\alpha y_N(x) = \sum_{i=n+1}^N \sum_{k=0}^{i-(n+1)} b_i A_{i,k}^\alpha x^{i-k-\alpha}, \quad n < \alpha \leq n+1, \quad (4.4)$$

where

$$A_{i,k}^\alpha = (-1)^k 2^{2i-2k} \frac{\Gamma(2i-k+2)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+2)\Gamma(i-k-n)}. \quad (4.5)$$

Test example: Consider $y(x) = x^2$ with $N = 3$ and $\alpha = 1.5$. Using Eqs. (1) and (2.5) we obtain:

$$T_x^{1.5} x^2 = T_x^{0.5} 2x = 2x^{0.5}.$$

Then, using the proposed method we obtain:

$$T_x^{1.5}x^2 = \sum_{i=2}^3 \sum_{k=0}^{i-2} b_i A_{i,k}^{1.5} x^{i-k-1.5}, \quad (4.6)$$

where

$$A_{2,0}^{1.5} = 32, \quad A_{3,0}^{1.5} = 384, \quad A_{3,1}^{1.5} = -\frac{192}{\Gamma(2)}.$$

Substituting the constants $b_2 = \frac{1}{16}$ and $b_3 = 0$ from Eq. (3.2) into Eq. (4.6) we get:

$$T_x^{1.5}x^2 = 2x^{0.5}.$$

5. NUMERICAL SCHEME

In this section, we apply Chebyshev collocation method to problem (1.1)-(1.4). Assume that the solution of the problem (1.1)-(1.4) can be written as

$$u_N(x, t) = \sum_{i=0}^N a_i(t) U_i^*(x). \quad (5.1)$$

From (1.1), (1), (4.4) and (4.5) we obtain

$$\begin{aligned} \sum_{i=0}^N t^{1-\mu} \frac{da_i(t)}{dt} U_i^*(x) &= f(x, t) \sum_{i=0}^N a_i(t) T_x^\alpha U_i^*(x) + g(x, t). \\ \sum_{i=0}^N t^{1-\mu} \frac{da_i(t)}{dt} U_i^*(x) &= f(x, t) \sum_{i=n+1}^N \sum_{k=0}^{i-(n+1)} a_i(t) A_{i,k}^\alpha x^{i-k-\alpha} + g(x, t). \end{aligned} \quad (5.2)$$

Now, we collocate Eq. (5.2) at $N - n$ points x_p as follows:

$$\sum_{i=0}^N t^{1-\mu} \frac{da_i(t)}{dt} U_i^*(x_p) = f(x_p, t) \sum_{i=n+1}^N \sum_{k=0}^{i-(n+1)} a_i(t) A_{i,k}^\alpha x_p^{i-k-\alpha} + g(x_p, t). \quad (5.3)$$

We use the roots of shifted Chebyshev polynomials of the second kinds $U_{N-n}^*(x)$ to suitable the collocation points.

The boundary conditions (1.3) and (1.4) can be written as

$$\sum_{i=0}^N (-1)^i (i+1) a_i(t) = h_1(t), \quad \sum_{i=0}^N (i+1) a_i(t) = h_2(t). \quad (5.4)$$

Since $1 < \alpha \leq 2$ in the problem (1.1), we can take $n = 1$. Let us take $0 < t_k \leq T$, $\Delta t = \frac{T}{M}$, $t_k = k\Delta t$, $k = 0, 1, \dots, M$, $a_i(t_k) = a_i^k$, $g(x_p, t_k) = g_p^k$, $h_1(t_k) = h_1^k$ and $h_2(t_k) = h_2^k$. Applying the finite difference method to the system (5.3) and considering (5.4) we have the following system

$$(A - B)V^j = CV^{j-1} + D, \quad j = 1, \dots, M, \quad (5.5)$$

where

$$\begin{aligned}
 D &= \left(\Delta t g_1^j, \Delta t g_2^j, \dots, \Delta t g_{N-1}^j, h_1^j, h_2^j \right)_{(N+1) \times 1}^*, \quad V^j = \left(a_0^j, a_1^j, \dots, a_N^j \right)_{(N+1) \times 1}^*, \\
 A &= \begin{pmatrix} t_j^{1-\mu} U_0^*(x_1) & t_j^{1-\mu} U_1^*(x_1) & \dots & t_j^{1-\mu} U_N^*(x_1) \\ \vdots & \vdots & \dots & \vdots \\ t_j^{1-\mu} U_0^*(x_{N-1}) & t_j^{1-\mu} U_1^*(x_{N-1}) & \dots & t_j^{1-\mu} U_N^*(x_{N-1}) \\ 1 & -2 & \dots & (-1)^N (N+1) \\ 1 & 2 & \dots & (N+1) \end{pmatrix}_{(N+1) \times (N+1)}, \\
 B &= \begin{pmatrix} 0 & 0 & F_2^j(x_1) & \dots & F_N^j(x_1) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & F_2^j(x_{N-1}) & \dots & F_N^j(x_{N-1}) \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}_{(N+1) \times (N+1)}, \\
 F_i^j(x_p) &= \Delta t f(x_p, t_j) \sum_{k=0}^{i-2} A_{i,k}^\alpha x_p^{i-k-\alpha}, \quad i = 2, \dots, N, p = 1, \dots, N-1, \\
 C &= t_j^{1-\mu} \begin{pmatrix} U_0^*(x_1) & U_1^*(x_1) & \dots & U_N^*(x_1) \\ \vdots & \vdots & \dots & \vdots \\ U_0^*(x_{N-1}) & U_1^*(x_{N-1}) & \dots & U_N^*(x_{N-1}) \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{pmatrix}_{(N+1) \times (N+1)}
 \end{aligned}$$

and $*$ denotes transposition of the matrix. To compute V^j , $j = 1, 2, \dots, M$, in the system (5.5), we need to obtain initial case V^0 . Substituting Eqs. (3.2) and (5.1) into initial condition (1.2) we can compute the constants $a_i(t_0) = a_i^0$ in the initial case at $j = 1$.

6. APPLICATIONS

Example 1. Let us consider the following space-time fractional diffusion equation with initial and boundary conditions

$$T_t^{0.4} u(x, t) = \frac{x^{1.7} t^{0.6}}{2} T_x^{1.7} u(x, t) + 2x^3 t^{0.6} \exp(t), \quad 0 < x < 1, \quad t > 0, \quad (6.1)$$

$$u(x, 0) = x^2 - x^3, \quad 0 < x < 1, \quad (6.2)$$

$$u(0, t) = 0, \quad u(1, t) = 0, \quad 0 < t \leq T. \quad (6.3)$$

Note that the exact solution to this problem is $u_e(x, t) = (x^2 - x^3) \exp(t)$. We apply the suggested method for $N = 3$ and approximate to the solution $u(x, t)$ as follows:

$$u_3(x, t) = \sum_{i=0}^3 a_i(t) U_i^*(x). \quad (6.4)$$

By using the obtained matrix equation (5.5) for $N = 3$ with the initial data $V^0 = (3/32, 1/32, -1/32, -1/64)^*$, we obtain the unknown coefficients $a_i(t), i = 0, 1, 2, 3$, for $t = 0.25; 0.5; 0.75; 1; 5$ which are given in Table 1. In Table 2, the absolute errors between the exact solution $u_e(x, t)$ and the approximate solution $u_3(x, t)$, at $t = 0.25; 0.5; 0.75; 1; 5$. It is clear from Table 2 that the obtained numerical solution by using our approach is in very good agreement with the exact solution even for small value $N = 3$. Furthermore, the computation is completed in a very short time since a few terms of the series solution (i.e. $N = 3$) are used. Fig.1 shows 3D plot of the obtained solution (6.4) for $0 < x < 1$ and $0 < t < 1$.

TABLE 1. The obtained coefficients $a_i (i = 0, 1, 2, 3)$ for different values t in Example 1.

t	a_0	a_1	a_2	a_3
0.25	0.120377383756128	0.040125794456509	-0.040125794585376	-0.020062897228255
0.5	0.154567620926466	0.051522539911611	-0.051522540308822	-0.025761269955805
0.75	0.198468754317411	0.066156250704570	-0.066156251439137	-0.033078125352285
1	0.254838925339978	0.084946307306756	-0.084946308446659	-0.042473153653378
5	13.913734969534085	4.637911222166822	-4.637911656511361	-2.318955611083411

TABLE 2. The absolute errors between the exact and approximate solutions in Example 1.

x	$t = 0.25$	$t = 0.5$	$t = 0.75$	$t = 1$	$t = 5$
0	10^{-17}	8.10^{-17}	0	2.10^{-17}	3.10^{-15}
0.1	2.10^{-9}	6.10^{-10}	10^{-9}	10^{-9}	6.10^{-7}
0.2	5.10^{-9}	10^{-9}	2.10^{-9}	3.10^{-9}	10^{-6}
0.3	8.10^{-9}	10^{-9}	2.10^{-9}	4.10^{-9}	10^{-6}
0.4	10^{-9}	2.10^{-9}	3.10^{-9}	4.10^{-9}	10^{-6}
0.5	10^{-9}	2.10^{-9}	3.10^{-9}	5.10^{-9}	10^{-6}
0.6	10^{-9}	2.10^{-9}	3.10^{-9}	5.10^{-9}	10^{-6}
0.7	10^{-9}	2.10^{-9}	3.10^{-9}	4.10^{-9}	10^{-6}
0.8	10^{-9}	10^{-9}	2.10^{-9}	3.10^{-9}	10^{-6}
0.9	6.10^{-10}	10^{-9}	10^{-9}	2.10^{-9}	6.10^{-7}
1	10^{-17}	8.10^{-17}	2.10^{-17}	0	0

Example 2. Consider the following space-time fractional diffusion problem

$$T_t^{0.3} u(x, t) = \frac{x^{0.8}}{2} T_x^{1.8} u(x, t) - (x^2 + 1) \sin\left(\frac{t^{0.3}}{0.3}\right) - x \cos\left(\frac{t^{0.3}}{0.3}\right) \quad (6.5)$$

$$0 < x < 1, \quad t > 0,$$

$$u(x, 0) = x^2 + 1, \quad 0 < x < 1, \quad (6.6)$$

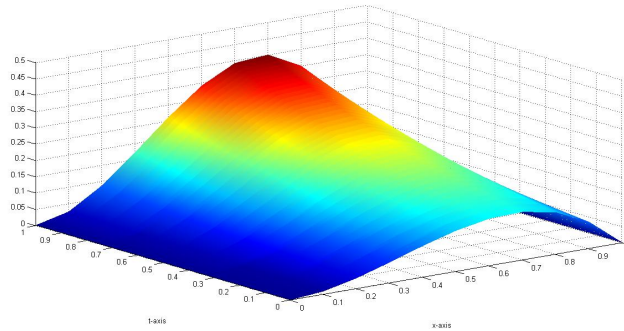


FIGURE 1. The behavior of the approximate solution of the problem (6.1)-(6.3).

$$u(0,t) = \cos\left(\frac{t^{0.3}}{0.3}\right), \quad u(1,t) = 2 \cos\left(\frac{t^{0.3}}{0.3}\right), \quad 0 < t \leq T. \quad (6.7)$$

The exact solution is $u_e(x,t) = (x^2 + 1) \cos(\frac{t^{0.3}}{0.3})$. By using the obtained matrix equation (5.5) for $N = 2$ with the initial data $V^0 = (21/16, 1/4, 1/16)^*$, we obtain the unknown coefficients $a_i(t), i = 0, 1, 2$. In Table 3, the absolute errors between the exact solution $u_e(x,t)$ and the approximate solution $u_2(x,t)$, at $t = 0.25; 0.5; 0.75; 1; 5$. From the results of Table 3, it is obvious that the presented method gives high accuracy even for the small value N . Fig. 2 shows 3D plot of the solution of (6.5)-(6.7) for $0 < x < 1$ and $0 < t < 15$.

TABLE 3. The absolute errors between the exact and approximate solutions in Example 2.

x	$t = 0.25$	$t = 0.5$	$t = 0.75$	$t = 1$	$t = 5$
0	0	0	0	0	0
0.1	10^{-6}	5.10^{-7}	2.10^{-7}	2.10^{-16}	4.10^{-9}
0.2	2.10^{-6}	10^{-6}	5.10^{-7}	10^{-7}	7.10^{-9}
0.3	3.10^{-6}	10^{-6}	6.10^{-7}	3.10^{-7}	9.10^{-9}
0.4	4.10^{-6}	10^{-6}	7.10^{-7}	4.10^{-7}	10^{-9}
0.5	4.10^{-6}	10^{-6}	8.10^{-7}	4.10^{-7}	10^{-8}
0.6	4.10^{-6}	10^{-6}	7.10^{-7}	4.10^{-7}	10^{-8}
0.7	3.10^{-6}	10^{-6}	6.10^{-7}	4.10^{-7}	9.10^{-9}
0.8	2.10^{-6}	10^{-6}	5.10^{-7}	4.10^{-7}	7.10^{-9}
0.9	10^{-6}	5.10^{-7}	2.10^{-7}	3.10^{-7}	4.10^{-9}
1	0	0	0	2.10^{-16}	0

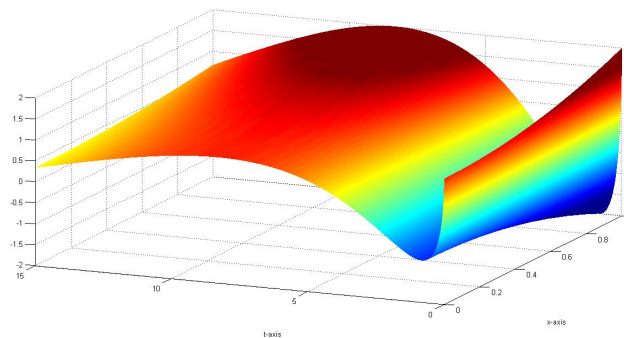


FIGURE 2. The behavior of the approximate solution of the problem (6.5)-(6.7).

7. CONCLUSION

In this paper, using the shifted Chebyshev polynomials of the second kind and its properties together with the Chebyshev collocation method, the space-time fractional diffusion equation with variable coefficients is reduced to a system of ordinary differential equations which is solved by the finite difference method. The fractional derivatives are considered in the conformable sense. The numerical results obtained by the proposed technique are compared with the exact solution to illustrate validity and applicability of the proposed technique. From the numerical results, it is obvious that the proposed method gives high accuracy and efficiency even for small values N . Furthermore, accuracy of the method can be increased by adding new terms to the series (5.1).

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INFINITELY MANY SOLUTIONS FOR A $p(x)$ -KIRCHHOFF TYPE EQUATION WITH STEKLOV BOUNDARY VALUE

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Abstract. In the present article deal with the existence and multiplicity of solutions to a class of $p(x)$ -Kirchhoff type problem with Steklov boundary-value. By variational approach and theory of the variable exponent Sobolev spaces, under appropriate assumptions on f , we obtain existence of infinitely solutions and at least one nontrivial weak solution.

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1. INTRODUCTION

The purpose of this article is to study the following nonlinear Steklov boundary value problem

$$\begin{cases} M\left(\int_{\Omega} \frac{|\nabla u|^{p(x)}}{p(x)} dx\right) \Delta_{p(x)} u = |u|^{p(x)-2} u & \text{in } \Omega, \\ |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \nu} = \lambda f(x, u) & \text{on } \partial\Omega, \end{cases} \quad (\mathbf{E})$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a smooth bounded domain, λ is a positive parameter $p \in C(\overline{\Omega})$, $\Delta_{p(x)} u = \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u)$ is a $p(x)$ -Laplacian operator, $M: (0, \infty) \rightarrow (0, \infty)$ is a continuous Kirchhoff function and $f: \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function, $\frac{\partial u}{\partial \nu}$ is the outer unit normal derivative on $\partial\Omega$.

Problem (E) is related to the stationary version of a model, the so-called Kirchhoff equation, introduced by Kirchhoff [17]. In 1883, Kirchhoff established a model given by the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0,$$

where L, E, ρ, P_0, h are constants. The above equation is an extension of the classical D’Alembert’s wave equation, by considering the effects of the changes in the length

of the strings during the vibrations. Moreover, equation (E) can be used for modelling several physical and biological systems, where u describes a process which depends on the average of itself, such as the population density [10].

Lions [19] has proposed an abstract framework for the Kirchhoff-type equations. After this work of Lions, various equations of Kirchhoff-type have attracted much attention [6, 9]. The study of Kirchhoff-type equations has initially been extended to the case involving the p -Laplacian operator, and then the equations containing the $p(x)$ -Laplacian operator [11, 16].

The study of variation problems with variable exponent has extremely been attract in recently. Because such problems are used to model dynamical phenomena arising from the study of electrorheological fluids [20], elastic mechanics [25], like image processing and stationary thermo-rheological viscous flows of non-Newtonian fluids [2, 8, 21] and in the mathematical description of the processes filtration of an idea barotropic gas through a porous medium [5].

The Steklov problems involving $p(x)$ -Laplacian have been worked by some of the authors [15, 22, 24]. Especially, the authors have studied the problems of type (E) when $M(t) = 1$. For example, in [7], the author investigated the existence and multiplicity of solutions for Steklov problem with non-standard growth condition without using the Ambrosetti-Rabinowitz type condition. In [3], the author proved the existencel of solutions by using Ekeland variational principle together with min-max method. In [4], the authors obtained the existence and multiplicity of solutions for the nonlinear Steklov boundary value problem, using Mountain Pass, Fountain and Ricceri three critical points theorems. Moreover, in [1], they showed the existence and multiplicity of solutions using variational methods under suitable assumptions on the nonlinearity,.

Inspired by the papers above mentioned , we studied the Steklov problem involving the $p(x)$ -Kirchhoff type operator. The present article is composed of two sections. In Section 2, we present some necessary preliminary knowledge of variable exponent Lebesgue-Sobolev spaces. In Section 3, using the variational method, we give the main results and their proofs.

2. PRELIMINARIES

To discuss problem (E), we define some definitions and basic properties of variable exponent Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ which will be used later. For more details, see [18].

Set

$$C_+(\overline{\Omega}) = \{p : p \in C(\overline{\Omega}), p(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $p(x) \in C_+(\overline{\Omega})$, we write

$$1 < p^- := \max_{x \in \overline{\Omega}} p(x) \leq p(x) \leq p^+ := \min_{x \in \overline{\Omega}} p(x) < \infty.$$

Define the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \left\{ u \mid u: \Omega \rightarrow \mathbb{R} \text{ is a measurable and } \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\},$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} := |u|_{p(x)} = \inf \left\{ \eta > 0 : \int_{\Omega} \left| \frac{u(x)}{\eta} \right|^{p(x)} dx \leq 1 \right\}.$$

Also, we can define $C_+(\partial\Omega)$ and p^-, p^+ for any $p(x) \in C(\partial\Omega)$, and denote

$$L^{p(x)}(\partial\Omega) = \left\{ u \mid u: \partial\Omega \rightarrow \mathbb{R} \text{ is a measurable and } \int_{\partial\Omega} |u(x)|^{p(x)} d\sigma < \infty \right\},$$

with the norm

$$|u|_{L^{p(x)}(\partial\Omega)} = |u|_{p(x)} := \inf \left\{ \delta > 0 : \int_{\partial\Omega} \left| \frac{u(x)}{\delta} \right|^{p(x)} d\sigma \leq 1 \right\},$$

where $d\sigma$ is the measure on the boundary.

Proposition 1 ([18, Theorem 2.1]). *For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ we have the following Hölder-type inequality*

$$\left| \int_{\Omega} uv dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{(p^-)'} \right) |u|_{p(x)} |v|_{p'(x)},$$

where $L^{p'(x)}(\Omega)$ denotes the conjugate space of $L^{p(x)}(\Omega)$ and $\frac{1}{p'(x)} + \frac{1}{p(x)} = 1$.

The modular of the $L^{p(x)}(\Omega)$ space, which is the mapping $\Phi_{p(x)}: L^{p(x)}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\Phi_{p(x)}(u) = \int_{\Omega} |u(x)|^{p(x)} dx \quad \text{for all } u \in L^{p(x)}(\Omega),$$

and it satisfies the following proposition.

Proposition 2 ([14, Theorem 1.3]). *For all $u, u_n \in L^{p(x)}(\Omega)$, $n = 1, 2, \dots$ we have*

- (i) $|u|_{p(x)} > 1 (=1, >1) \Leftrightarrow \Phi_{p(x)}(u) > 1 (=1, >1),$
- (ii) $\min \left(|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+} \right) \leq \Phi_{p(x)}(u) \leq \max \left(|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+} \right),$
- (iii) $|u_n - u|_{p(x)} \rightarrow 0 (\rightarrow \infty) \Leftrightarrow \Phi_{p(x)}(u_n - u) \rightarrow 0 (\rightarrow \infty).$

Proposition 3 ([12, Proposition 2.4]). *Let $\phi(u) = \int_{\partial\Omega} |u(x)|^{p(x)} d\sigma$ for all $u, u_n \in L^{p(x)}(\partial\Omega)$, $n = 1, 2, \dots$ we have*

- (i) $|u|_{L^{p(x)}(\partial\Omega)} > 1 \Rightarrow |u|_{L^{p(x)}(\partial\Omega)}^{p^-} \leq \phi(u) \leq |u|_{L^{p(x)}(\partial\Omega)}^{p^+},$
- (ii) $|u|_{L^{p(x)}(\partial\Omega)} < 1 \Rightarrow |u|_{L^{p(x)}(\partial\Omega)}^{p^+} \leq \phi(u) \leq |u|_{L^{p(x)}(\partial\Omega)}^{p^-},$
- (iii) $|u_n - u|_{p(x)} \rightarrow 0 (\rightarrow \infty) \Leftrightarrow \phi(u_n - u) \rightarrow 0 (\rightarrow \infty).$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is denined by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

and equipped with the norm

$$\|u\|_{1,p(x)} = \inf \left\{ \kappa > 0 : \int_{\Omega} \left(\left| \frac{\nabla u(x)}{\kappa} \right|^{p(x)} + \left| \frac{u(x)}{\kappa} \right|^{p(x)} \right) dx \leq 1 \right\},$$

or

$$\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}, \quad \forall u \in W^{1,p(x)}(\Omega).$$

The space $W_0^{1,p(x)}(\Omega)$ is denoted by the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ with respect to the norm $\|u\|_{1,p(x)}$. We can define an equivalent norm

$$\|u\| = |\nabla u|_{p(x)} \quad \text{for all } u \in W_0^{1,p(x)}(\Omega).$$

Proposition 4 ([18, Theorem 3.1]; [14, Theorem 2.3]; [12, Theorem 2.1]; [13, Lemma 3.1]).

- (i) If $1 < p^- \leq p^+ < \infty$, then the spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.
- (ii) If $q(x) \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$ then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous, where

$$p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } N > p(x), \\ \infty & \text{if } N \leq p(x). \end{cases}$$

- (iii) If $q(x) \in C_+(\partial\Omega)$ and $q(x) < p^\partial(x)$ for all $x \in \partial\Omega$ then the trace embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega)$ is compact and continuous, where

$$p^\partial(x) := \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & \text{if } N > p(x) \\ \infty & \text{if } N \leq p(x). \end{cases}$$

- (iv) (**Poincaré inequality**). There is a positive constant $C > 0$ such that

$$|u|_{p(x)} \leq C \|u\|, \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

Proposition 5 ([13, Lemma 2.1]). Let $p(x)$ and $q(x)$ be measurable functions such that $1 \leq p(x)q(x) \leq \infty$ and $p(x) \in L^\infty(\Omega)$ for a.e. $x \in \Omega$. Let $u \in L^{q(x)}(\Omega)$, $u \neq 0$. Then

$$\min \left(|u|_{p(x)q(x)}^{p^-}, |u|_{p(x)q(x)}^{p^+} \right) \leq \left| |u|^{p(x)} \right|_{q(x)} \leq \max \left(|u|_{p(x)q(x)}^{p^-}, |u|_{p(x)q(x)}^{p^+} \right).$$

In particular, if $p(x) = p$ is constant, then $\|u\|_{q(x)}^p = |u|_{pq(x)}^p$. Throughout this paper, we assume that f and M satisfy the following assumptions:

(**M**₀) $M: (0, \infty) \rightarrow (0, \infty)$ is a continuous function such that

$$m_1 s^{\alpha-1} \leq M(s) \leq m_2 s^{\alpha-1}, \quad \forall s > 0$$

where m_1, m_2 and α are real numbers such that $0 < m_1 \leq m_2$ and $\alpha > 1$.

(**f**₁) $f: \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies Carathéodory condition and

$$|f(x, t)| \leq c_1 (1 + |t|^{q(x)-1}), \quad \forall (x, t) \in \partial\Omega \times \mathbb{R},$$

where $c_1 > 0$ is positive constant and $q(x) \in C_+(\partial\Omega)$ such that $p^+ < q^- < q(x) < p^\partial(x)$ for all $x \in \partial\Omega$.

(**f**₂) $f(x, t) = o(|t|^{\alpha p^+ - 1})$ as $t \rightarrow 0$, uniformly for $x \in \partial\Omega$ and $\alpha p^+ < q^-$.

(**f**₃) $f(x, -t) = -f(x, t)$ for all $(x, t) \in \partial\Omega \times \mathbb{R}$.

(**AR**) Ambrosetti-Rabinowitz's Condition holds, i.e. there exists $M > 0$ and $\theta > \frac{m_2 \alpha (p^+)^{\alpha}}{m_1 (p^-)^{\alpha-1}}$ such that

$$0 < \theta F(x, t) \leq f(x, t)t, \quad |t| \geq M \quad \text{for all } x \in \partial\Omega.$$

Moreover, we will use X instead of the variable exponent Sobolev space $W_0^{1, p(x)}(\Omega)$.

3. MAIN RESULTS AND PROOFS

We present the main results of the paper:

Theorem 1. Assume that (**M**₀), (**AR**), (**f**₁), (**f**₂) and $p^+ < \alpha p^-$ hold. Then there exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$, problem (**E**) has at least one nontrivial weak solution.

Theorem 2. Assume that (**M**₀), (**AR**), (**f**₁), (**f**₂), (**f**₃) and $p^+ < \alpha p^-$ hold. Then there exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$, I_λ has a sequence of critical points $\{u_n\}$ such that $I_\lambda(u_n) \rightarrow \infty$ as $n \rightarrow \infty$ and problem (**E**) has infinite many pairs of weak solutions.

Proposition 6 ([4, Theorem 2.9]). Let $f: \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function satisfying (**f**₁). For each $u \in X$ set $\Psi(u) = \int_{\partial\Omega} F(x, u) d\sigma$. Then $\Psi(u) \in C^1(X, \mathbb{R})$ and

$$\langle \Psi'(u), v \rangle = \int_{\partial\Omega} f(x, u) v d\sigma \quad \text{for all } v \in X.$$

Moreover, the operator $\Psi: X \rightarrow X^*$ is compact.

Proposition 7 ([14, Proposition 3.1]). If one denotes

$$\Psi(u) = \int_{\Omega} \frac{|\nabla u|^{p(x)}}{p(x)} dx$$

then $\psi \in C^1(X, \mathbb{R})$ and the derivative operator of ψ , denoted by ψ' , is

$$\langle \psi'(u), v \rangle = \int_{\Omega} |\nabla u|^{p(x)-2} u v dx \quad \forall u, v \in X$$

and one has

- (i) $\psi': X \rightarrow X^*$ is a continuous, bounded, strictly monotone operator and homeomorphism,
- (ii) ψ' is a mapping of (S_+) type, that is if $u_n \rightharpoonup u$ in X and $\limsup_{n \rightarrow \infty} \langle \psi'(u_n), u_n - u \rangle \leq 0$ implies, then $u_n \rightarrow u$ in X , where $X = W_0^{1,p(x)}(\Omega)$.

We say that $u \in X$ is a weak solution of (E) if

$$M \left(\int_{\Omega} \frac{|\nabla u|^{p(x)}}{p(x)} dx \right) \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla v dx + \int_{\Omega} |u|^{p(x)-2} u v dx = \lambda \int_{\partial\Omega} f(x, u) v d\sigma$$

where $v \in X$. We associate to the problem (E) the energy functional, defined as $I_{\lambda}: X \rightarrow \mathbb{R}$,

$$I_{\lambda}(u) = \widehat{M} \left(\int_{\Omega} \frac{|\nabla u|^{p(x)}}{p(x)} dx \right) + \int_{\Omega} \frac{|u|^{p(x)}}{p(x)} dx - \lambda \int_{\partial\Omega} F(x, u) d\sigma,$$

where $\widehat{M}(t) = \int_0^t M(s) ds$ and $F(x, t) = \int_0^t f(x, s) ds$. Moreover, from (f_1) and (M_0) , Proposition 6 and Proposition 7, it is easy to see that the functional $I_{\lambda} \in C^1(X, \mathbb{R})$ and we can infer that critical points of functional I_{λ} are the weak solutions for problem (E). Then, we have

$$\begin{aligned} \langle I'_{\lambda}(u), v \rangle &= M \left(\int_{\Omega} \frac{|\nabla u|^{p(x)}}{p(x)} dx \right) \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla v dx \\ &\quad + \int_{\Omega} |u|^{p(x)-2} u v dx - \lambda \int_{\partial\Omega} f(x, u) v d\sigma \end{aligned}$$

for any $u, v \in X$.

Definition 1. Let X be a Banach spaces and $I_{\lambda} \in C^1(X, \mathbb{R})$. We say that I_{λ} satisfies Palais-Smale condition (PS) if any sequence $\{u_n\}$ in X such that $\{I_{\lambda}(u_n)\}$ is bounded and $I'_{\lambda}(u_n) \rightarrow 0$ as $n \rightarrow \infty$ has a convergent subsequence.

Lemma 1. If (M_0) , (f_1) , (AR) and $p^+ < \alpha p^-$ hold, then for any $\lambda \in (0, \infty)$ the functional I_{λ} satisfies (PS) condition.

Proof. Let us assume that there exists a sequence $\{u_n\}$ in X such that

$$|I_{\lambda}(u_n)| \leq c \quad \text{and} \quad I'_{\lambda}(u_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.1)$$

Initially we prove that $\{u_n\}$ is bounded in X . Considering $\|u_n\| > 1$ for n large enough and using $(\mathbf{M}_0)$, $(\mathbf{AR})$, (3.1) we obtain

$$\begin{aligned} c+1 &\geq I_\lambda(u_n) - \frac{1}{\theta} \langle I'_\lambda(u_n), u_n \rangle \\ &\geq \left(\frac{m_1}{\alpha(p^+)^\alpha} - \frac{m_2}{\theta(p^-)^{\alpha-1}} \right) \left(\int_\Omega |\nabla u_n|^{p(x)} dx \right)^\alpha + \frac{1}{p^+} \int_\Omega |u_n|^{p(x)} dx \\ &\quad - \frac{1}{\theta} \int_\Omega |u_n|^{p(x)} dx - \lambda \left(\int_{\partial\Omega} \frac{1}{\theta} f(x, u_n) u_n - F(x, u_n) d\sigma \right) \\ &\geq \left(\frac{m_1}{\alpha(p^+)^\alpha} - \frac{m_2}{\theta(p^-)^{\alpha-1}} \right) \|u_n\|^{\alpha p^-} - \frac{c_1}{\theta} \|u_n\|^{p^+}. \end{aligned}$$

When we divide the last inequality by $\|u_n\|^{\alpha p^-}$,

$$\frac{c+1}{\|u_n\|^{\alpha p^-}} \geq \left(\frac{m_1}{\alpha(p^+)^\alpha} - \frac{m_2}{\theta(p^-)^{\alpha-1}} \right) - \frac{c_1}{\theta} \frac{\|u_n\|^{p^+}}{\|u_n\|^{\alpha p^-}}$$

and pass to the limit as $n \rightarrow \infty$, we have

$$0 \geq \frac{m_1}{\alpha(p^+)^\alpha} - \frac{m_2}{\theta(p^-)^{\alpha-1}}$$

or

$$\theta \leq \frac{m_2 \alpha (p^+)^\alpha}{m_1 (p^-)^{\alpha-1}}.$$

Since $\theta > \frac{m_2 \alpha (p^+)^\alpha}{m_1 (p^-)^{\alpha-1}}$ in the condition $(\mathbf{AR})$, we obtain a contradiction. Thus $\{u_n\}$ is bounded in X ; from Proposition 4, there exists u in X such that, up to a subsequence, $\{u_n\}$ converges weakly to u in X . Later, we will show that $u_n \rightarrow u$ (converges strongly) in X . By relation (3.1), we have that $\langle I'_\lambda(u_n), u_n - u \rangle \rightarrow 0$. Therefore, we write

$$\begin{aligned} \langle I'_\lambda(u_n), u_n - u \rangle &= M \left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)} dx \right) \int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u) dx \\ &\quad + \int_\Omega |u_n|^{p(x)-2} (u_n - u) dx \\ &\quad - \lambda \int_{\partial\Omega} f(x, u_n) (u_n - u) d\sigma \rightarrow 0 \end{aligned} \tag{3.2}$$

By Proposition 1, Proposition 4, Proposition 4 and $(\mathbf{f}_1)$ we obtain

$$\begin{aligned} \left| \int_{\partial\Omega} f(x, u_n) (u_n - u) d\sigma \right| &\leq \left| \int_{\partial\Omega} \left(c_1 + c_1 |u_n|^{q(x)-1} \right) (u_n - u) d\sigma \right| \\ &\leq c_1 \int_{\partial\Omega} |(u_n - u)| d\sigma + c_2 \left| |u_n|^{q(x)-1} \right|_{L^{q'(x)}} \|u_n - u\|_{L^{q(x)}(\partial\Omega)}, \end{aligned}$$

where $c_2 > 0$ is a constant. If we consider the compact embedding $X \hookrightarrow L^{q(x)}(\partial\Omega)$, that is, $\|u_n - u\|_{L^{q(x)}(\partial\Omega)} \rightarrow 0$ as $n \rightarrow \infty$, we get

$$\int_{\partial\Omega} f(x, u_n)(u_n - u) \, d\sigma \rightarrow 0. \quad (3.3)$$

Similarly, by Proposition 1, Proposition 4, Proposition 5, we can write

$$\int_{\Omega} |u_n|^{p(x)-2} u_n (u_n - u) \, dx \rightarrow 0. \quad (3.4)$$

On the other hand, we use (3.1), (3.3) and (3.4) in the above inequality (3.2) we obtain

$$M \left(\int_{\Omega} \frac{|\nabla u_n|^{p(x)}}{p(x)} dx \right) \int_{\Omega} |\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u) \rightarrow 0.$$

Moreover, from $(\mathbf{M}_0)$, we conclude that

$$\int_{\Omega} |\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u) \rightarrow 0.$$

Eventually from Proposition 7 we obtain that $u_n \rightarrow u$ (strongly) in X . Therefore I_λ satisfies the **(PS)** condition. $\square$

Lemma 2. Assume that $(\mathbf{M}_0)$, $(\mathbf{AR})$, $(\mathbf{f}_1)$ and $(\mathbf{f}_2)$ hold. Then, the following statements hold:

- (i) There exist positive real numbers μ , τ and λ^* for any $\lambda \in (0, \lambda^*)$ such that

$$I_\lambda(u) \geq \tau > 0, \quad \forall u \in X \quad \text{with } \|u\| = \mu,$$

- (ii) There exists $u_1 \in X$ such that $\|u_1\| > \mu$ and $I_\lambda(u_1) < 0$.

Proof.

- (i) Let us assume that $\|u\| < 1$. Since we have the continuous embeddings $X \hookrightarrow L^{p^-}(\Omega)$, $X \hookrightarrow L^{\alpha p^+}(\partial\Omega)$ and $X \hookrightarrow L^{q^-}(\partial\Omega)$ from Proposition 4, there exist positive constants c_3 , c_4 and c_5 for all $u \in X$ such that

$$\|u\|_{L^{\alpha p^+}(\partial\Omega)} \leq c_3 \|u\|, \quad \|u\|_{L^{q^-}(\partial\Omega)} \leq c_4 \|u\|$$

and

$$\|u\|_{L^{p^-}(\Omega)} \leq c_5 \|u\|. \quad (3.5)$$

In addition, using $(\mathbf{f}_1)$ and $(\mathbf{f}_2)$, we write

$$|F(x, t)| \leq \varepsilon |t|^{\alpha p^+} + c_\varepsilon |t|^{q(x)}, \quad \forall (x, t) \in \partial\Omega \times \mathbb{R}. \quad (3.6)$$

By taking into account (3.5) and (3.6), we get

$$I_\lambda(u) \geq \frac{m_1}{\alpha(p^+)^\alpha} \|u\|^{\alpha p^+} + \frac{c_5^{p^-}}{p^+} \|u\|^{p^+} - \lambda \varepsilon \int_{\partial\Omega} |u|^{\alpha p^+} \, d\sigma - \lambda \int_{\partial\Omega} |u|^{q(x)} \, d\sigma$$

$$\geq \left(\frac{m_1}{\alpha(p^+)^\alpha} + \frac{c_5^{p^-}}{p^+} \right) \|u\|^{\alpha p^+} - \lambda \varepsilon c_3^{\alpha p^+} \|u\|^{\alpha p^+} - \lambda c_\varepsilon c_4^{q^-} \|u\|^{q^-}.$$

By the above inequality, if we choose $\varepsilon > 0$ small enough such that $0 < 2\lambda \varepsilon c_3^{\alpha p^+} < \left(\frac{m_1}{\alpha(p^+)^\alpha} + \frac{c_5^{p^-}}{p^+} \right)$, we obtain

$$I_\lambda(u) \geq \left(\left(\frac{m_1}{2\alpha(p^+)^\alpha} + \frac{c_5^{p^-}}{2p^+} \right) \|u\|^{\alpha p^+ - q^-} - \lambda c_6 \right) \|u\|^{q^-}.$$

On the other hand, we remark that $\alpha p^+ < q$

$$\lambda^* = \frac{1}{4c_6} \left(\frac{m_1}{\alpha(p^+)^\alpha} + \frac{c_5^{p^-}}{p^+} \right) \mu^{\alpha p^+ - q^-}.$$

Then, there exist $\tau > 0$ and $\mu > 0$ for any $\lambda \in (0, \lambda^*)$ such that

$$I_\lambda(u) \geq \tau > 0, \quad \forall u \in X \quad \text{with } \|u\| = \mu \in (0, 1).$$

This completes the proof.

(ii) By (AR), one easily deduces $c_7 > 0$ such that

$$F(x, t) \geq c_7 |t|^\theta, \quad \forall (x, t) \in \partial\Omega \times \mathbb{R}. \quad (3.7)$$

Using Proposition 2, Proposition 4 and (3.7) for any $\omega \in X \setminus \{0\}$ and $t > 1$ large enough, we have

$$\begin{aligned} I_\lambda(t\omega) &= \widehat{M} \left(\int_\Omega \frac{|\nabla t\omega|^{p(x)}}{p(x)} dx \right) + \int_\Omega \frac{|t\omega|^{p(x)}}{p(x)} dx - \lambda \int_{\partial\Omega} F(x, t\omega) d\sigma \\ &\leq \frac{m_2}{\alpha(p^-)^\alpha} t^{\alpha p^+} \left(\int_\Omega |\nabla \omega|^{p(x)} dx \right)^\alpha + \frac{t^{p^+}}{p^-} \int_\Omega |\omega|^{p(x)} dx \\ &\quad - t^\theta \lambda c_7 \int_{\partial\Omega} |\omega|^\theta d\sigma. \end{aligned}$$

Since $\theta > \alpha p^+$, we obtain $\lim_{t \rightarrow \infty} I_\lambda(t\omega) = -\infty$. Then, we can take $u_1 = t\omega$ such that $\|u_1\| > \mu$ and $I_\lambda(u_1) < 0$. The proof of Lemma 2 is complete. $\square$

Proof of Theorem 1. From Lemma 1, Lemma 2 and $I_\lambda(0) = 0$, I_λ satisfies the Mountain Pass theorem [23]. Therefore, I_λ has at least one nontrivial weak.

We will use the following “Fountain theorem” to prove Theorem 2.

Since X is a separable and reflexive Banach space, then there exist $e_j \subset X$ and $e_j^* \subset X^*$ such that

$$X = \overline{\text{span}\{e_j : j = 1, 2, \dots\}}, \quad X^* = \overline{\text{span}\{e_j^* : j = 1, 2, \dots\}},$$

and

$$\langle e_i, e_j^* \rangle = \begin{cases} 1, & \text{if } i \neq j \\ 0, & \text{if } i = j \end{cases}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality product between X and X^* . For convenience, we have

$$X_j = \text{span} \{e_j\}, \quad Y_k = \bigoplus_{j=1}^k X_j, \quad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}.$$

□

Lemma 3 ([7, Theorem 2.6]). Assume that $I_\lambda \in C^1(X, \mathbb{R})$ is an even functional satisfying the condition (PS). Moreover, If there exist $\rho_k > \gamma_k > 0$, for each $k = 1, 2, \dots$ such that

$$(A_1) \quad \inf_{u \in Z_k, \|u\| = \gamma_k} I_\lambda(u) \rightarrow \infty \text{ as } k \rightarrow \infty,$$

$$(A_2) \quad \max_{u \in Y_k, \|u\| = \rho_k} I_\lambda(u) \leq 0,$$

then I_λ has an unbounded sequence of critical values.

Lemma 4 ([7, Lemma 2.7]). If $q(x) \in C_+(\partial\Omega)$ and $q(x) < p^\partial(x)$ for any $x \in \partial\Omega$, denote

$$\beta_k = \sup \left\{ |u|_{L^{q(x)}(\partial\Omega)} : \|u\| = 1, u \in Z_k \right\},$$

then $\lim_{k \rightarrow \infty} \beta_k = 0$.

Proof of Theorem 2. It is enough to prove that I_λ has an unbounded sequence of critical points. Since I_λ satisfies (PS) condition from Lemma 1 and I_λ is an even functional from the assumptions (f₃), we only need to show whether it satisfies the conditions (A₁) and (A₂) in Lemma 3.

(A₁) For any $u \in Z_k$ with $\|u\| > 1$, by (M₀), (f₁) and $\alpha p^- > p^+$, we write

$$\begin{aligned} I_\lambda(u) &\geq \frac{m_1}{\alpha(p^+)^{\alpha}} \|u\|^{\alpha p^-} + \frac{1}{p^+} \|u\|^{p^-} - \lambda c_1 \int_{\partial\Omega} (1 + |t|^{q(x)-1}) d\sigma \\ &\geq \left(\frac{m_1}{\alpha(p^+)^{\alpha}} + \frac{1}{p^+} \right) \|u\|^{p^-} - \lambda c_1 \max \left\{ |u|_{L^{q(x)}(\partial\Omega)}^{q^+}, |u|_{L^{q(x)}(\partial\Omega)}^{q^-} \right\} - c_8. \end{aligned}$$

It follows that

$$\begin{aligned} I_\lambda(u) &\geq \begin{cases} \left(\frac{m_1}{\alpha(p^+)^{\alpha}} + \frac{1}{q^-} \right) \|u\|^{p^-} - \lambda c_9 - c_8 & \text{if } |u|_{L^{q(x)}(\partial\Omega)} \leq 1 \\ \left(\frac{m_1}{\alpha(p^+)^{\alpha}} + \frac{1}{q^-} \right) \|u\|^{p^-} - \lambda c_9 \beta_k^{q^+} \|u\|^{q^+} - c_8 & \text{if } |u|_{L^{q(x)}(\partial\Omega)} > 1 \end{cases} \\ &\geq \left(\frac{m_1}{\alpha(p^+)^{\alpha}} + \frac{1}{q^-} \right) \|u\|^{p^-} - \lambda c_9 \beta_k^{q^+} \|u\|^{q^+} - c_{10}. \end{aligned}$$

For $\|u\| = \gamma_k = \left(\lambda c_9 q^+ \beta_k^{q^+} \right)^{\frac{1}{p^- - q^+}}$, we obtain

$$I_\lambda(u) \geq \left(\frac{m_1}{\alpha(p^+)^{\alpha}} + \frac{1}{q^-} - \frac{1}{q^+} \right) \gamma_k^{p^-} - c_{10}$$

$$\geq \frac{m_1}{\alpha(p^+)^\alpha} \gamma_k^{p^-} - c_{10},$$

where c_8 , c_9 and c_{10} are positive constants. Since $\beta_k \rightarrow 0$ and $p^- < q^+$ we obtain $\gamma_k \rightarrow \infty$ as $k \rightarrow \infty$. Consequently,

$$I_\lambda(u) \rightarrow \infty \quad \text{as } \|u\| \rightarrow \infty \quad \text{for } u \in Z_k.$$

The statement of **(A₁)** is satisfied.

(A₂) From **(AR)**, we have

$$F(x, t) \geq c_{11} |t|^\theta - c_{12},$$

where c_{11} , c_{12} are positive constants. Let $u \in Y_k$ with $\|u\| = \rho_k > 1$. We write

$$\begin{aligned} I_\lambda(u) &\leq \frac{m_2}{\alpha(p^-)^\alpha} \|u\|^{\alpha p^+} + \frac{1}{p^-} \|u\|^{p^+} - \lambda \int_{\partial\Omega} F(x, u) d\sigma \\ &\leq \frac{m_2}{\alpha(p^-)^\alpha} \|u\|^{\alpha p^+} + \frac{1}{p^-} \|u\|^{p^+} - \lambda c_{11} \int_{\partial\Omega} |u|^\theta d\sigma + c_{12}. \end{aligned}$$

Since the space Y_k has finite dimension, all norms are equivalents. Hence, we obtain

$$I_\lambda(u) \leq \frac{m_2}{\alpha(p^-)^\alpha} \|u\|^{\alpha p^+} - \lambda c_{11}^\theta \|u\|^\theta + c_{12}.$$

Finally, we have

$$I_\lambda(u) \rightarrow -\infty \text{ as } n \rightarrow +\infty \quad \text{for any } u \in Y_k$$

because $\theta > \alpha p^+$. We show that there exists $\rho_k > \gamma_k > 0$ such that

$$\max_{u \in Y_k, \|u\| = \rho_k} I_\lambda(u) \leq 0 \quad \text{for each } k = 1, 2, \dots$$

Therefore, the statement of **(A₂)** is satisfied. We complete the proof of Theorem 2. □

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BLOW-UP AND DECAY OF SOLUTIONS FOR A DELAYED TIMOSHENKO EQUATION WITH VARIABLE-EXPONENTS

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Abstract. This work deals with a Timoshenko equation with delay term and variable exponents. Firstly, we obtain the blow up of solutions for negative initial energy in a finite time. Later, we establish the decay results by using an integral inequality due to Komornik. These, improve and extend the previous studies in the literature.

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Keywords: blow up, decay, delay term, Timoshenko equation, variable exponents

1. INTRODUCTION

This paper is interested to study the following problem:

$$\begin{cases} u_{tt} + \Delta^2 u - M\left(\|\nabla u\|^2\right) \Delta u - \Delta u_t \\ \quad + \mu_1 u_t(x, t) |u_t|^{m(x)-2}(x, t) \\ \quad + \mu_2 u_t(x, t - \tau) |u_t|^{m(x)-2}(x, t - \tau) \\ \quad = bu |u|^{p(x)-2} \\ u(x, t) = \frac{\partial u(x, t)}{\partial \nu} = 0 \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) \\ u_t(x, t - \tau) = f_0(x, t - \tau) \end{cases} \quad \begin{array}{l} \text{in } \Omega \times R^+, \\ \\ \\ \text{on } x \in \partial\Omega, t \in [0, \infty), \\ \text{in } \Omega, \\ \text{in } \Omega \times (0, \tau), \end{array} \quad (1.1)$$

which is contained a Timoshenko equation over $\Omega \times R^+$ with the Dirichlet-Neumann conditions on $\partial\Omega$ and the initial conditions on Ω and finally a initial condition related to the presence of the delay time $\tau > 0$, given in $\Omega \times (0, \tau)$ where Ω is a bounded domain in R^n with sufficiently smooth boundary. μ_1 is a positive constant, μ_2 is a real number, $b \geq 0$ is a constant and ν is the unit outward normal vector on $\partial\Omega$. $M(s)$ is a positive C^1 -function given as $M(s) = 1 + s^\gamma$ for $s \geq 0$, $\gamma > 0$. The exponents $m(\cdot)$

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and $p(\cdot)$ are given continuous functions on $\overline{\Omega}$ and satisfy

$$\begin{cases} 2 \leq m^- \leq m(x) \leq m^+ \leq m^* \\ 2 \leq p^- \leq p(x) \leq p^+ \leq p^*, \end{cases} \quad (1.2)$$

where

$$\begin{aligned} m^- &= \operatorname{ess\,inf}_{x \in \Omega} m(x), & m^+ &= \operatorname{ess\,inf}_{x \in \Omega} m(x) \\ p^- &= \operatorname{ess\,inf}_{x \in \Omega} p(x), & p^+ &= \operatorname{ess\,inf}_{x \in \Omega} p(x), \end{aligned}$$

and

$$\begin{cases} 2 < p^*, m^* < \infty & \text{if } n \leq 4, \\ 2 < p^*, m^* < \frac{2n}{n-4} & \text{if } n > 4. \end{cases}$$

The Timoshenko equation is among the famous wave equation's model which describe extensible beam theory. It has been introduced in 1921 by Timoshenko [31]. For detailed information on derivation the equation, see [9]. The problems with variable exponents arises in many branches in sciences such as nonlinear elasticity theory, electrorheological fluids and image processing [6, 8, 29]. Time delay appears in many practical problems such as thermal, biological, chemical, physical and economic phenomena [15].

Datko et al. [7], indicated that a small delay in a boundary control is a source of instability. In [20], Nicaise and Pignotti studied the equation as follows

$$u_{tt} - \Delta u + a_0 u_t(x, t) + a u_t(x, t - \tau) = 0,$$

where a_0, a are positive real parameters. They obtained that, under the condition $0 \leq a \leq a_0$, the system is exponentially stable. In the case $a \geq a_0$, they obtained a sequence of delays that shows the solution is instable. In [32], Xu et al. obtained the same result similar to the [20] for the one space dimension by adopting the spectral analysis approach. In [19], Nicaise et al. studied the wave equation in one space dimension in the case of time-varying delay. In that work, they showed that an exponential stability result under the condition

$$a \leq \sqrt{1-d} a_0,$$

where d is a constant such that

$$\tau'(t) \leq d < 1, \forall t > 0.$$

In [11], Feng studied the following equation

$$u_{tt} + \Delta^2 u - M(\|\nabla u\|^2) \Delta u - \int_0^t g(t-s) \Delta u(s) ds + \mu_1 u_t + \mu_2 u_t(t-\tau) = 0.$$

He obtained well-posedness of solutions with $|\mu_2| \leq \mu_1$, and proved decay results under the assumption $|\mu_2| < \mu_1$.

Park [21], looked into the following equation

$$u_{tt} + \Delta^2 u - M\left(\|\nabla u\|^2\right) \Delta u + \sigma(t) \int_0^t g(t-s) \Delta u(s) ds + a_0 u_t + a_1 u_t(t - \tau(t)) = 0.$$

He established decay results under the assumption $|a_1| < \sqrt{1 - da_0}$.

Antontsev et al. [2] concerned with the following equation with variable exponents

$$u_{tt} + \Delta^2 u - \Delta u_t + |u_t|^{p(x)-2} u_t = |u|^{q(x)-2} u. \quad (1.3)$$

They proved the local weak solutions and obtained the blow up results for the (1.3) under suitable conditions.

Antontsev et al. [5] examined the following equation with variable exponents

$$u_{tt} + \Delta^2 u - M\left(\|\nabla u\|^2\right) \Delta u + |u_t|^{p(x)-2} u_t = |u|^{q(x)-2} u. \quad (1.4)$$

They established the local existence and proved the nonexistence of solutions with negative initial energy for the equation (1.4).

When $M(s) \equiv 1$, and in the absence of $(+\Delta^2 u)$ term and without strong damping term $(-\Delta u_t)$, the equation (1.1) becomes the following equation

$$u_{tt} - \Delta u + \mu_1 u_t(x, t) |u_t|^{m(x)-2}(x, t) + \mu_2 u_t(x, t - \tau) |u_t|^{m(x)-2}(x, t - \tau) = bu |u|^{p(x)-2}. \quad (1.5)$$

Messaoudi and Kafini [14] established the decay estimates and proved the global nonexistence of the equation (1.5).

In recent years, some other authors investigated related studies (see [1, 2, 12, 13, 16, 22–28, 30, 33–38]).

In the present paper, we consider the blow up and the decay results for the Timoshenko equation (1.1) with delay term and variable exponents. Our aim in this work is to study the Timoshenko equation with the strong damping term $(-\Delta u_t)$, delay term $(\mu_2 u_t(x, t - \tau))$ and variable exponents.

The plan of this paper is as follows: In Section 2, the definitions of the variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$ and Sobolev spaces $W^{1,p(\cdot)}(\Omega)$, as well as some of their properties, are stated. In Section 3, we prove the blow up of solutions for negative initial energy. In Section 4, we establish the decay results by using an integral inequality due to Komornik.

2. PRELIMINARIES

Let us start by presents our functional spaces and some related results taken from [3, 4, 8, 10, 18].

Let $p: \Omega \rightarrow [1, \infty)$ be a measurable function. The variable exponent Lebesgue space with a variable exponent $p(\cdot)$ defined as

$$L^{p(\cdot)}(\Omega) = \left\{ u: \Omega \rightarrow \mathbb{R} \text{ measurable in } \Omega : \int_{\Omega} |u|^{p(\cdot)} dx < \infty \right\},$$

and inner with a Luxemburg-type norm

$$\|u\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{u}{\lambda} \right|^{p(x)} dx \leq 1 \right\},$$

is a Banach space (see [8]).

We define the variable-exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ as follows

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : \nabla u \text{ exists and } |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}.$$

Variable exponent Sobolev space with respect to the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)},$$

is a Banach space. The space $W_0^{1,p(\cdot)}(\Omega)$ is defined to be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. The dual space of $W_0^{1,p(\cdot)}(\Omega)$ is $W_0^{-1,p'(\cdot)}(\Omega)$, defined in the same way as in the classical Sobolev spaces, where

$$\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1.$$

Assume that

$$|p(x) - p(y)| \leq -\frac{A}{\log|x-y|} \quad \text{and} \quad |m(x) - m(y)| \leq -\frac{B}{\log|x-y|} \quad (2.1)$$

for all $x, y \in \Omega$, $A, B > 0$ and $0 < \delta < 1$ with $|x - y| < \delta$. (log-Hölder condition).

If $p \geq 1$ is a measurable function on Ω , then

$$\min \left\{ \|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+} \right\} \leq \rho_{p(\cdot)}(u) \leq \max \left\{ \|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+} \right\}$$

for a.e. $x \in \Omega$ and for any $u \in L^{p(\cdot)}(\Omega)$.

Let $p, q, s \geq 1$ be measurable functions defined on Ω such that

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)} \quad \text{for a.e. } y \in \Omega.$$

If $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{q(\cdot)}(\Omega)$, then $fg \in L^{s(\cdot)}(\Omega)$ and

$$\|fg\|_{s(\cdot)} \leq 2 \|f\|_{p(\cdot)} \|g\|_{q(\cdot)}.$$

(Hölder's inequality).

Lemma 1 (pp. 506 in [4], Poincaré's inequality). *Suppose that $p(\cdot)$ satisfies (2.1) and let Ω be a bounded domain of $\mathbb{R}^n$. Then,*

$$\|u\|_{p(\cdot)} \leq c \|\nabla u\|_{p(\cdot)} \quad \text{for all } u \in W_0^{1,p(\cdot)}(\Omega),$$

where $c = c(p^-, p^+, |\Omega|) > 0$.

Lemma 2 ([8]). Let $m(\cdot) \in C(\overline{\Omega})$ and $p: \Omega \rightarrow [1, \infty)$ be a measurable function, such that

$$\operatorname{ess\,inf}_{x \in \overline{\Omega}} (m^*(x) - p(x)) > 0.$$

Then, the Sobolev embedding $W_0^{1,m(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$ is continuous and compact, where

$$\begin{cases} \frac{nm^-}{n-m^-} & \text{if } m^- < n, \\ \text{any number in } [1, \infty) & \text{if } m^- \geq n. \end{cases}$$

If in addition $m(\cdot)$ satisfies log-Hölder condition, then

$$\begin{cases} \frac{nm(x)}{n-m(x)} & \text{if } m(x) < n, \\ \text{any number in } [1, \infty) & \text{if } m(x) \geq n. \end{cases}$$

Remark 1. Let c be various positive constants which may be different from line to line. Then, we use the embedding

$$H_0^2(\Omega) \hookrightarrow H_0^1(\Omega) \hookrightarrow L^p(\Omega)$$

which satisfies

$$\|u\|_p \leq c \|\nabla u\| \leq c \|\Delta u\|,$$

where $2 \leq p < \infty$ ($n = 1, 2$), $2 \leq p \leq \frac{2n}{n-2}$ ($n \geq 3$). Moreover,

$$\|u\|_p \leq c \|\Delta u\|, \quad p = \begin{cases} \infty & \text{if } n < 4, \\ \text{any number in } [1, \infty) & \text{if } n = 4, \\ \frac{2n}{n-4} & \text{if } n > 4. \end{cases}$$

3. BLOW UP RESULTS

In this part, we give the blow up result of solutions under two conditions, the first one if the initial energy is negative and the second if the weight of the external force $b > 0$. Firstly, as in [20], we introduce the new function

$$z(x, \rho, t) = u_t(x, t - \tau\rho), \quad x \in \Omega, \quad \rho \in (0, 1), \quad t > 0 \quad (3.1)$$

which gives

$$\tau z_t(x, \rho, t) + z_\rho(x, \rho, t) = 0, \quad x \in \Omega, \quad \rho \in (0, 1), \quad t > 0.$$

Then, the problem (1.1) takes the form

$$\begin{cases}
u_{tt} + \Delta^2 u - M\left(\|\nabla u\|^2\right) \Delta u \\
- \Delta u_t + \mu_1 u_t(x, t) |u_t(x, t)|^{m(x)-2} \\
+ \mu_2 z(x, 1, t) |z(x, 1, t)|^{m(x)-2} \\
= bu |u|^{p(x)-2} & \text{in } \Omega \times (0, \infty), \\
\tau z_t(x, \rho, t) + z_\rho(x, \rho, t) = 0 & \text{in } \Omega \times (0, 1) \times (0, \infty), \\
z(x, \rho, 0) = f_0(x, -\rho\tau) & \text{in } \Omega \times (0, 1), \\
u(x, t) = \frac{\partial u(x, t)}{\partial \nu} = 0 & \text{on } x \in \partial\Omega, t \in [0, \infty), \\
u(x, 0) = u_0(x), u_t(x, 0) = u_1(x) & \text{in } \Omega.
\end{cases} \quad (3.2)$$

Similar to the work of Kafini and Messaoudi [14], we can write the following definition:

Definition 1. Fix $T > 0$. We call (u, z) a strong solution of (3.2) if

$$\begin{aligned}
u &\in W^{2,\infty}([0, T]; L^2(\Omega)) \cap W^{1,\infty}([0, T]; H_0^2(\Omega)) \\
&\cap L^\infty([0, T]; H^2(\Omega) \cap H_0^2(\Omega)), \\
u_t &\in L^{m(\cdot)}(\Omega \times (0, T)), \\
z &\in W^{1,\infty}([0, 1] \times [0, T]; L^2(\Omega)) \cap L^\infty([0, 1]; L^{m(\cdot)}(\Omega) \cap [0, T])
\end{aligned}$$

and (u, z) satisfies the initial data and (3.2) in the following sense:

$$\begin{aligned}
&\int_{\Omega} u_{tt}(\cdot, t) v dx + \int_{\Omega} \Delta^2 u(\cdot, t) v dx - \int_{\Omega} M\left(\|\nabla u(\cdot, t)\|^2\right) \Delta u(\cdot, t) v dx \\
&- \int_{\Omega} \Delta u_t(\cdot, t) v dx + \mu_1 \int_{\Omega} |u_t(\cdot, t)|^{m(\cdot)-2} u_t(\cdot, t) v dx \\
&+ \mu_2 \int_{\Omega} |z(\cdot, 1, t)|^{m(\cdot)-2} z(\cdot, 1, t) v dx \\
&= b \int_{\Omega} |u(\cdot, t)|^{p(\cdot)-2} u(\cdot, t) v dx
\end{aligned}$$

and

$$\tau \int_{\Omega} z_t(\cdot, \rho, t) w dx + \int_{\Omega} z_\rho(\cdot, \rho, t) w dx = 0,$$

for a.e. $t \in [0, T]$ and for $(v, w) \in H_0^2(\Omega) \cap L^2(\Omega)$.

The energy functional related to (3.2) is given by

$$\begin{aligned}
E(t) &= \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|\Delta u\|^2 + \frac{1}{2} \|\nabla u\|^2 + \frac{1}{2(\gamma+1)} \|\nabla u\|^{2(\gamma+1)} \\
&+ \int_0^1 \int_{\Omega} \frac{\xi(x) |z(x, \rho, t)|^{m(x)}}{m(x)} dx d\rho - b \int_{\Omega} \frac{|u|^{p(x)}}{p(x)} dx,
\end{aligned}$$

for $t \geq 0$, where ξ is a continuous function satisfying

$$\tau|\mu_2|(m(x) - 1) < \xi(x) < \tau(\mu_1 m(x) - |\mu_2|), x \in \overline{\Omega}. \quad (3.3)$$

The following lemma shows that the related energy of the problem is nonincreasing under the condition $\mu_1 > |\mu_2|$.

Lemma 3. *Let (u, z) be a solution of (3.2), such that*

$$E'(t) \leq -C_0 \int_{\Omega} \left(|u_t|^{m(x)} + |z(x, 1, t)|^{m(x)} \right) dx \leq 0,$$

for some $C_0 > 0$.

Proof. Multiplying the first equation in (3.2) by u_t , integrating over Ω , then multiplying the second equation in (3.2) by $\frac{1}{\tau}\xi(x)|z|^{m(x)-2}z$ and integrating over $\Omega \times (0, 1)$, then summing, we obtain

$$\begin{aligned} & \frac{d}{dt} \left[\frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|\Delta u\|^2 + \frac{1}{2} \|\nabla u\|^2 + \frac{1}{2(\gamma+1)} \|\nabla u\|^{2(\gamma+1)} \right. \\ & \quad \left. + \int_0^1 \int_{\Omega} \frac{\xi(x) |z(x, \rho, t)|^{m(x)}}{m(x)} dx d\rho - b \int_{\Omega} \frac{|u|^{p(x)}}{p(x)} dx \right] \\ & = -\mu_1 \int_{\Omega} |u_t|^{m(x)} dx - \int_{\Omega} |\nabla u_t|^2 dx \\ & \quad - \frac{1}{\tau} \int_{\Omega} \int_0^1 \xi(x) |z(x, \rho, t)|^{m(x)-2} z z_{\rho}(x, \rho, t) d\rho dx \\ & \quad - \mu_2 \int_{\Omega} u_t z(x, 1, t) |z(x, 1, t)|^{m(x)-2} dx. \end{aligned} \quad (3.4)$$

Now, we estimate the last two terms of the right hand side of (3.4) as follows,

$$\begin{aligned} & -\frac{1}{\tau} \int_{\Omega} \int_0^1 \xi(x) |z(x, \rho, t)|^{m(x)-2} z z_{\rho}(x, \rho, t) d\rho dx \\ & = -\frac{1}{\tau} \int_{\Omega} \int_0^1 \frac{\partial}{\partial \rho} \left(\frac{\xi(x) |z(x, \rho, t)|^{m(x)}}{m(x)} \right) d\rho dx \\ & = \frac{1}{\tau} \int_{\Omega} \frac{\xi(x)}{m(x)} \left(|z(x, 0, t)|^{m(x)} - |z(x, 1, t)|^{m(x)} \right) dx \\ & = \int_{\Omega} \frac{\xi(x)}{\tau m(x)} |u_t|^{m(x)} dx - \int_{\Omega} \frac{\xi(x)}{\tau m(x)} |z(x, 1, t)|^{m(x)}. \end{aligned}$$

By using Young's inequality, $q = \frac{m(x)}{m(x)-1}$ and $q' = m(x)$ for the last term, we get

$$|u_t| |z(x, 1, t)|^{m(x)-1} \leq \frac{1}{m(x)} |u_t|^{m(x)} + \frac{m(x)-1}{m(x)} |z(x, 1, t)|^{m(x)}.$$

Consequently, we conclude that

$$\begin{aligned} & -\mu_2 \int_{\Omega} u_t z |z(x, 1, t)|^{m(x)-2} dx \\ & \leq |\mu_2| \left(\int_{\Omega} \frac{1}{m(x)} |u_t(t)|^{m(x)} dx + \int_{\Omega} \frac{m(x)-1}{m(x)} |z(x, 1, t)|^{m(x)} dx \right). \end{aligned}$$

Thus,

$$\begin{aligned} \frac{dE(t)}{dt} & \leq - \int_{\Omega} \left(\mu_1 - \left(\frac{\xi(x)}{\tau m(x)} + \frac{|\mu_2|}{m(x)} \right) \right) |u_t(t)|^{m(x)} dx \\ & \quad - \int_{\Omega} \left(\frac{\xi(x)}{\tau m(x)} - \frac{|\mu_2|(m(x)-1)}{m(x)} \right) |z(x, 1, t)|^{m(x)} dx. \end{aligned}$$

As a result, for all $x \in \overline{\Omega}$, the relation (3.3) yields

$$f_1(x) = \mu_1 - \left(\frac{\xi(x)}{\tau m(x)} + \frac{|\mu_2|}{m(x)} \right) > 0, \quad f_2(x) = \frac{\xi(x)}{\tau m(x)} - \frac{|\mu_2|(m(x)-1)}{m(x)} > 0.$$

Since $m(x)$, and hence $\xi(x)$, is bounded, we infer that $f_1(x)$ and $f_2(x)$ are bounded. Hence, if we define

$$C_0(x) = \min \{f_1(x), f_2(x)\} > 0 \text{ for any } x \in \overline{\Omega},$$

and take $C_0 = \inf_{\overline{\Omega}} C_0(x)$, then $C_0(x) \geq C_0 > 0$. Therefore,

$$E'(t) \leq -C_0 \left[\int_{\Omega} |u_t(t)|^{m(x)} dx + \int_{\Omega} |z(x, 1, t)|^{m(x)} dx \right] \leq 0.$$

□

In order to obtain the blow up result, we suppose in addition to (1.2) that $E(0) < 0$.

We set

$$H(t) = -E(t),$$

therefore,

$$H'(t) = -E'(t) \geq 0,$$

$$0 < H(0) \leq H(t) \leq b \int_{\Omega} \frac{|u|^{p(x)}}{p(x)} dx \leq \frac{b}{p^-} \rho(u),$$

where

$$\rho(u) = \rho_{p(\cdot)}(u) = \int_{\Omega} |u|^{p(x)} dx.$$

Lemma 4 (Lemma 3.2, Lemma 3.6 and Lemma 3.7 in [14]). Assume that the exponents $m(\cdot)$ and $p(\cdot)$ satisfy

$$2 \leq m^- \leq m(x) \leq m^+ < p^- \leq p(x) \leq p^+ \leq 2 + \frac{4}{n-4} \text{ if } n > 4.$$

Then, depending on Ω only, there exists a positive $C > 1$, such that

$$\rho^{s/p^-}(u) \leq C \left(\|\Delta u\|^2 + \rho(u) \right).$$

Then, for any $u \in H_0^2(\Omega)$ and $2 \leq s \leq p^-$, we have the following inequalities:

(i)

$$\|u\|_{p^-}^s \leq C \left(\|\Delta u\|^2 + \|u\|_{p^-}^{p^-} \right),$$

(ii)

$$\rho^{s/p^-}(u) \leq C \left(|H(t)| + \|u_t\|^2 + \rho(u) + \int_0^1 \int_{\Omega} \frac{\xi(x) |z(x, \rho, t)|^{m(x)}}{m(x)} dx d\rho \right),$$

(iii)

$$\|u\|_{p^-}^s \leq C \left(|H(t)| + \|u_t\|^2 + \|u\|_{p^-}^{p^-} + \int_0^1 \int_{\Omega} \frac{\xi(x) |z(x, \rho, t)|^{m(x)}}{m(x)} dx d\rho \right), \quad (3.5)$$

(iv)

$$\rho(u) \geq C \|u\|_{p^-}^{p^-}, \quad (3.6)$$

(v)

$$\int_{\Omega} |u|^{m(x)} dx \leq C \left(\rho^{m^-/p^-}(u) + \rho^{m^+/p^-}(u) \right). \quad (3.7)$$

To obtain the main result we have the theorem as follows:

Theorem 1. Let the condition (2.1) holds and Lemma 4 be provided. Suppose further $E(0) < 0$, and the exponents $m(\cdot)$ and $p(\cdot)$ satisfy

$$2 \leq m^- \leq m(x) \leq m^+ < p^- \leq p(x) \leq p^+ \leq 2 + \frac{4}{n-4} \quad \text{if } n > 4.$$

Then, the solution of (3.2) blows up in finite time.

Proof. We define

$$L(t) = H^{1-\alpha}(t) + \varepsilon \int_{\Omega} uu_t dx + \frac{\varepsilon}{2} \|\nabla u\|^2, \quad (3.8)$$

for small ε to be chosen later and

$$0 \leq \alpha \leq \min \left\{ \frac{p^- - 2}{2p^-}, \frac{p^- - m^+}{p^- (m^+ - 1)} \right\}. \quad (3.9)$$

A direct differentiation of (3.8) using the first equation in (3.2) gives

$$\begin{aligned} L'(t) = & (1 - \alpha) H^{-\alpha}(t) H'(t) + \varepsilon \int_{\Omega} u_t^2 dx - \varepsilon \int_{\Omega} |\Delta u|^2 dx - \varepsilon \int_{\Omega} |\nabla u|^2 dx \\ & - \varepsilon \int_{\Omega} |\nabla u|^{2(\gamma+1)} dx + \varepsilon b \int_{\Omega} |u|^{p(x)} dx - \varepsilon \mu_1 \int_{\Omega} uu_t(x, t) |u_t(x, t)|^{m(x)-2} dx \end{aligned}$$

$$- \varepsilon \mu_2 \int_{\Omega} u z(x, 1, t) |z(x, 1, t)|^{m(x)-2} dx.$$

Recalling the definition of $H(t)$ and for $0 < a < 1$, we get

$$\begin{aligned} L'(t) &\geq C_0(1-\alpha)H^{-\alpha}(t) \left[\int_{\Omega} |u_t(t)|^{m(x)} dx + \int_{\Omega} |z(x, 1, t)|^{m(x)} dx \right] \\ &\quad + \varepsilon \left((1-a)p^- H(t) + \frac{(1-a)p^-}{2} \|u_t\|^2 + \frac{(1-a)p^-}{2} \|\Delta u\|^2 \right) \\ &\quad + \varepsilon \left(\frac{(1-a)p^-}{2} \|\nabla u\|^2 + \frac{(1-a)p^-}{2(\gamma+1)} \|\nabla u\|^{2(\gamma+1)} \right) \\ &\quad + \varepsilon(1-a)p^- \int_0^1 \int_{\Omega} \frac{\xi(x) |z(x, \rho, t)|^{m(x)}}{m(x)} dx d\rho \\ &\quad + \varepsilon \int_{\Omega} [u_t^2 - |\Delta u|^2 - |\nabla u|^2 - |\nabla u|^{2(\gamma+1)}] dx \\ &\quad + \varepsilon ab \int_{\Omega} |u|^{p(x)} dx - \varepsilon \mu_1 \int_{\Omega} u u_t(x, t) |u_t(x, t)|^{m(x)-2} dx \\ &\quad - \varepsilon \mu_2 \int_{\Omega} u z(x, 1, t) |z(x, 1, t)|^{m(x)-2} dx. \end{aligned}$$

Therefore,

$$\begin{aligned} L'(t) &\geq C_0(1-\alpha)H^{-\alpha}(t) \left[\int_{\Omega} |u_t(t)|^{m(x)} dx + \int_{\Omega} |z(x, 1, t)|^{m(x)} dx \right] \\ &\quad + \varepsilon(1-a)p^- H(t) + \varepsilon \frac{(1-a)p^- + 2}{2} \|u_t\|^2 \\ &\quad + \varepsilon \frac{(1-a)p^- - 2}{2} \|\Delta u\|^2 + \varepsilon \frac{(1-a)p^- - 2}{2} \|\nabla u\|^2 \\ &\quad + \varepsilon \frac{(1-a)p^- - 2(\gamma+1)}{2(\gamma+1)} \|\nabla u\|^{2(\gamma+1)} \\ &\quad + \varepsilon(1-a)p^- \int_0^1 \int_{\Omega} \frac{\xi(x) |z(x, \rho, t)|^{m(x)}}{m(x)} dx d\rho + \varepsilon ab \rho(u) \\ &\quad - \varepsilon \mu_1 \int_{\Omega} u u_t(x, t) |u_t(x, t)|^{m(x)-2} dx \\ &\quad - \varepsilon \mu_2 \int_{\Omega} u z(x, 1, t) |z(x, 1, t)|^{m(x)-2} dx. \end{aligned}$$

Utilizing Young's inequality, we obtain

$$\int_{\Omega} |u_t|^{m(x)-1} |u| dx \leq \frac{1}{m^-} \int_{\Omega} \delta^{m(x)} |u|^{m(x)} dx + \frac{m^+ - 1}{m^+} \int_{\Omega} \delta^{-\frac{m(x)}{m(x)-1}} |u_t|^{m(x)} dx \quad (3.10)$$

and

$$\begin{aligned} \int_{\Omega} |z(x, 1, t)|^{m(x)-1} |u| dx &\leq \frac{1}{m^+} \int_{\Omega} \delta^{m(x)} |u|^{m(x)} dx \\ &\quad + \frac{m^+ - 1}{m^+} \int_{\Omega} \delta^{-\frac{m(x)}{m(x)-1}} |z(x, 1, t)|^{m(x)} dx. \end{aligned} \quad (3.11)$$

The estimates (3.10) and (3.11) remain valid if δ is time-dependent. Hence, taking δ such that

$$\delta^{-\frac{m(x)}{m(x)-1}} = kH^{-\alpha}(t),$$

for large $k \geq 1$ to be specified later, we obtain

$$\int_{\Omega} \delta^{-\frac{m(x)}{m(x)-1}} |u_t|^{m(x)} dx = kH^{-\alpha}(t) \int_{\Omega} |u_t|^{m(x)} dx, \quad (3.12)$$

$$\int_{\Omega} \delta^{-\frac{m(x)}{m(x)-1}} |z(x, 1, t)|^{m(x)} dx = kH^{-\alpha}(t) \int_{\Omega} |z(x, 1, t)|^{m(x)} dx \quad (3.13)$$

and

$$\int_{\Omega} \delta^{m(x)} |u|^{m(x)} dx = \int_{\Omega} k^{1-m(x)} H^{\alpha(m(x)-1)}(t) |u|^{m(x)} dx \quad (3.14)$$

$$\leq \int_{\Omega} k^{1-m^-} H^{\alpha(m^+-1)}(t) |u|^{m(x)} dx. \quad (3.15)$$

By using (3.6) and (3.7), we obtain

$$\begin{aligned} H^{\alpha(m^+-1)}(t) \int_{\Omega} |u|^{m(x)} dx &\leq \\ &C \left[(\rho(u))^{m^-/p^- + \alpha(m^+-1)} + (\rho(u))^{m^+/p^- + \alpha(m^+-1)} \right]. \end{aligned} \quad (3.16)$$

From (3.9), we infer that

$$s = m^- + \alpha p^- (m^+ - 1) \leq p^- \text{ and } s = m^+ + \alpha p^- (m^+ - 1) \leq p^-.$$

Thus, Lemma 4 yields

$$H^{\alpha(m^+-1)}(t) \int_{\Omega} |u|^{m(x)} dx \leq C \left(\|\Delta u\|^2 + \rho(u) \right). \quad (3.17)$$

By combining (3.10)-(3.17), we conclude that

$$\begin{aligned} L'(t) &\geq (1 - \alpha) H^{-\alpha}(t) \left[C_0 - \varepsilon \left(\frac{m^+ - 1}{m^+} \right) ck \right] \int_{\Omega} |u_t(t)|^{m(x)} dx \\ &\quad + (1 - \alpha) H^{-\alpha}(t) \left[C_0 - \varepsilon \left(\frac{m^+ - 1}{m^+} \right) ck \right] \int_{\Omega} |z(x, 1, t)|^{m(x)} dx \\ &\quad + \varepsilon \left(\frac{(p^- - 2) - \alpha p^-}{2} - \frac{C}{m^- k^{m^- - 1}} \right) \|\Delta u\|^2 + \varepsilon \frac{(1 - \alpha) p^- - 2}{2} \|\nabla u\|^2 \\ &\quad + \varepsilon (1 - \alpha) p^- H(t) + \varepsilon \frac{(1 - \alpha) p^- + 2}{2} \|u_t\|^2 \end{aligned} \quad (3.18)$$

$$\begin{aligned}
& + \varepsilon \frac{(1-a)p^- - 2(\gamma+1)}{2(\gamma+1)} \|\nabla u\|^{2(\gamma+1)} + \varepsilon \left(ab - \frac{C}{m^- k^{m^- - 1}} \right) \rho(u) \\
& + \varepsilon (1-a)p^- \int_0^1 \int_{\Omega} \frac{\xi(x) |z(x, \rho, t)|^{m(x)}}{m(x)} dx d\rho.
\end{aligned}$$

By choosing a small enough, such that

$$\frac{(1-a)p^- - 2}{2} > 0 \quad \text{and} \quad \frac{(1-a)p^- - 2(\gamma+1)}{2(\gamma+1)} > 0,$$

and k so large that

$$\frac{(p^- - 2) - ap^-}{2} - \frac{C}{m^- k^{m^- - 1}} > 0 \quad \text{and} \quad ab - \frac{C}{m^- k^{m^- - 1}} > 0.$$

Once k and a are fixed, we choose ε small enough that

$$C_0 - \varepsilon \left(\frac{m^+ - 1}{m^+} \right) ck > 0, \quad C_0 - \varepsilon \left(\frac{m^+ - 1}{m^+} \right) ck > 0$$

and

$$L(0) = H^{1-\alpha}(0) + \varepsilon \int_{\Omega} u_0(x) u_1(x) dx + \frac{\varepsilon}{2} \|\nabla u_0\|^2 > 0.$$

Therefore, (3.18) becomes

$$\begin{aligned}
L'(t) & \geq \varepsilon \eta \left[H(t) + \|u_t\|^2 + \|\Delta u\|^2 + \|\nabla u\|^2 + \|\nabla u\|^{2(\gamma+1)} \right. \\
& \quad \left. + \rho(u) + \int_0^1 \int_{\Omega} \frac{\xi(x) |z(x, \rho, t)|^{m(x)}}{m(x)} dx d\rho \right], \tag{3.19}
\end{aligned}$$

for a constant $\eta > 0$. As a result,

$$L(t) \geq L(0) > 0 \quad \forall t \geq 0.$$

Next, for some constants $\sigma, \Gamma > 0$, we show $L'(t) \geq \Gamma L^\sigma(t)$. For this reason, we estimate

$$\left| \int_{\Omega} uu_t(x, t) dx \right| \leq \|u\|_2 \|u_t\|_2 \leq C \|u\|_{p^-} \|u_t\|_2,$$

which implies

$$\left| \int_{\Omega} uu_t(x, t) dx \right|^{1/(1-\alpha)} \leq C \|u\|_{p^-}^{1/(1-\alpha)} \|u_t\|_2^{1/(1-\alpha)}$$

and utilizing Young's inequality yields

$$\left| \int_{\Omega} uu_t(x, t) dx \right|^{1/(1-\alpha)} \leq C \left[\|u\|_{p^-}^{mu/(1-\alpha)} + \|u_t\|_2^{\Theta/(1-\alpha)} \right],$$

where $1/\mu + 1/\Theta = 1$. From (3.9), the choice of $\Theta = 2(1 - \alpha)$ will make $\mu/(1 - \alpha) = 2/(1 - 2\alpha) \leq p^-$. Hence,

$$\left| \int_{\Omega} uu_t(x, t) dx \right|^{1/(1-\alpha)} \leq C \left[\|u\|_{p^-}^s + \|u_t\|^2 \right],$$

where $s = \mu/(1 - \alpha)$. By (3.5), we obtain

$$\begin{aligned} \left| \int_{\Omega} uu_t(x, t) dx \right|^{1/(1-\alpha)} &\leq C \left[|H(t)| + \|u_t\|^2 + \rho(u) \right. \\ &\quad \left. + \int_0^1 \int_{\Omega} \frac{\xi(x) |z(x, \rho, t)|^{m(x)}}{m(x)} dx d\rho \right]. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} L^{1/(1-\alpha)}(t) &= \left[H^{(1-\alpha)}(t) + \varepsilon \int_{\Omega} uu_t dx + \frac{\varepsilon}{2} \|\nabla u\|^2 \right]^{1/(1-\alpha)} \\ &\leq 2^{\alpha/(1-\alpha)} \left[H(t) + \varepsilon^{1/(1-\alpha)} \left| \int_{\Omega} uu_t dx \right|^{1/(1-\alpha)} \right] \\ &\leq C \left[|H(t)| + \|u_t\|^2 + \rho(u) + \int_0^1 \int_{\Omega} \frac{\xi(x) |z(x, \rho, t)|^{m(x)}}{m(x)} dx d\rho \right]. \end{aligned}$$

Thus, for some $\Psi > 0$, from (3.19) we arrive at $L'(t) \geq \Psi L^{1/(1-\alpha)}(t)$. A simple integration over $(0, t)$ yields

$$L^{\alpha/(1-\alpha)}(t) \geq \frac{1}{L^{-\alpha/(1-\alpha)}(0) - \Psi \alpha t / (1 - \alpha)}$$

which implies that the solution blows up in a finite time T^* , with

$$T^* \leq \frac{1 - \alpha}{\Psi \alpha [L(0)]^{\alpha/(1-\alpha)}}.$$

As a result, the proof is completed. $\square$

4. DECAY RESULTS

In this part, we obtain the decay results for the problem (4.1) without source term (i.e. $b = 0$). Similar to the beginning of the blow up section, we introduce a same

function z defined in (3.1), hence the problem (1.1) becomes equivalent to:

$$\begin{cases} u_{tt} + \Delta^2 u - M(\|\nabla u\|^2) \Delta u - \Delta u_t \\ + \mu_1 u_t(x, t) |u_t(x, t)|^{m(x)-2} & \text{in } \Omega \times (0, \infty), \\ + \mu_2 z(x, 1, t) |z(x, 1, t)|^{m(x)-2} = 0, \\ \tau z_t(x, \rho, t) + z_\rho(x, \rho, t) = 0 & \text{in } \Omega \times (0, 1) \times (0, \infty), \\ z(x, \rho, 0) = f_0(x, -\rho\tau) & \text{in } \Omega \times (0, 1), \\ u(x, t) = \frac{\partial u(x, t)}{\partial \nu} = 0 & \text{on } x \in \partial\Omega, t \in [0, \infty), \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x) & \text{in } \Omega. \end{cases} \quad (4.1)$$

The energy functional associated to (4.1) is given by

$$\begin{aligned} E(t) &= \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|\Delta u\|^2 + \frac{1}{2} \|\nabla u\|^2 + \frac{1}{2(\gamma+1)} \|\nabla u\|^{2(\gamma+1)} \\ &\quad + \int_0^1 \int_\Omega \frac{\xi(x) |z(x, \rho, t)|^{m(x)}}{m(x)} dx d\rho, \end{aligned} \quad (4.2)$$

where ξ is the continuous function introduced in (3.3) and $t \geq 0$.

Similar to Lemma 3, we easily establish, for $\mu_1 > |\mu_2|$ and for some $C_0 > 0$, that

$$E'(t) \leq -C_0 \int_\Omega \left(|u_t|^{m(x)} + |z(x, 1, t)|^{m(x)} \right) dx \leq 0.$$

Lemma 5 (Komornik, pp. 103 and pp. 124 in [17]). *Let $E: R^+ \rightarrow R^+$ be a nonincreasing function, such that*

$$\int_s^\infty E^{1+\sigma}(t) dt \leq \frac{1}{\Omega} E^\sigma(0) E(s) = cE(s) \quad \forall s > 0,$$

where $\sigma, \omega > 0$. Then, we have

$$\begin{cases} E(t) \leq cE(0) / (1+t)^{1/\sigma} & \text{if } \sigma > 0, \\ E(t) \leq cE(0) e^{-\omega t} & \text{if } \sigma = 0, \end{cases}$$

for all $t \geq 0$.

We need the following technical lemma, before we state the main theorem:

Lemma 6 (Lemma 4.2 in [14]). *The functional*

$$F(t) = \tau \int_0^1 \int_\Omega e^{-\rho\tau} \xi(x) |z(x, \rho, t)|^{m(x)} dx d\rho,$$

satisfies, along the solution of (4.1),

$$F'(t) \leq \int_\Omega \xi(x) |u_t|^{m(x)} dx - \tau e^{-\tau} \int_0^1 \int_\Omega \xi(x) |z(x, \rho, t)|^{m(x)} dx d\rho.$$

Theorem 2. Suppose that the condition (2.1) is satisfied and the exponents $m(\cdot)$ and $p(\cdot)$ satisfy

$$2 \leq m^- \leq m(x) \leq m^+ < p^- \leq p(x) \leq p^+ \leq 2 + \frac{4}{n-4} \quad \text{if } n > 4.$$

Then, there exist two constants $c, \alpha > 0$ independent of t , such that, any global solution of (4.1) satisfies,

$$\begin{cases} E(t) \leq ce^{-\alpha t} & \text{if } m(x) = 2, \\ E(t) \leq cE(0)/(1+t)^{2/(m^+-2)} & \text{if } m^+ > 2. \end{cases}$$

Proof. Multiplying the first equation of (4.1) by $uE^q(t)$, for $q > 0$ to be specified later, and integrating over $\Omega \times (s, T)$, $s < T$, to get

$$\begin{aligned} \int_s^T E^q(t) \int_{\Omega} \left(uu_{tt} + u\Delta^2 u - u\Delta u - \|\nabla u\|^{2\gamma} u\Delta u - u\Delta u_t \right. \\ \left. + \mu_1 uu_t |u_t|^{m(x)-2} + \mu_2 uz(x, 1, t) |z(x, 1, t)|^{m(x)-2} \right) dx dt = 0, \end{aligned}$$

which gives

$$\begin{aligned} \int_s^T E^q(t) \int_{\Omega} \left(\frac{d}{dt} (uu_t) - u_t^2 + |\Delta u|^2 + |\nabla u|^2 \right. \\ \left. + \|\nabla u\|^{2\gamma} |\nabla u|^2 + \nabla u \nabla u_t + \mu_1 uu_t(x, t) |u_t(x, t)|^{m(x)-2} \right. \\ \left. + \mu_2 uz(x, 1, t) |z(x, 1, t)|^{m(x)-2} \right) dx dt = 0. \end{aligned} \quad (4.3)$$

Recalling the definition of $E(t)$ given in (4.2), adding and subtracting some terms and using the relation

$$\frac{d}{dt} \left(E^q(t) \int_{\Omega} uu_t dx \right) = qE^{q-1}(t) E'(t) \int_{\Omega} uu_t dx + E^q(t) \frac{d}{dt} \int_{\Omega} uu_t dx,$$

the equation (4.3) becomes,

$$\begin{aligned} 2 \int_s^T E^{q+1}(t) dt = - \int_s^T \frac{d}{dt} \left(E^q(t) \int_{\Omega} uu_t dx \right) dt + q \int_s^T E^{q-1}(t) E'(t) \int_{\Omega} uu_t dx dt \\ - \frac{\gamma}{\gamma+1} \int_s^T E^q \int_{\Omega} \|\nabla u\|^{2\gamma} |\nabla u|^2 dx dt + 2 \int_s^T E^q(t) \int_{\Omega} u_t^2 dx dt \\ - \frac{1}{2} \int_s^T \frac{d}{dt} \left(E^q(t) \int_{\Omega} |\nabla u|^2 dx \right) dt \\ + \frac{q}{2} \int_s^T E^{q-1}(t) E'(t) \int_{\Omega} |\nabla u|^2 dx dt \\ - \mu_1 \int_s^T E^q(t) \int_{\Omega} uu_t |u_t|^{m(x)-2} dx dt \end{aligned} \quad (4.4)$$

$$\begin{aligned}
& -\mu_2 \int_s^T E^q(t) \int_{\Omega} u z(x, 1, t) |z(x, 1, t)|^{m(x)-2} dx dt \\
& + 2 \int_s^T E^q(t) \int_0^1 \int_{\Omega} \frac{\xi(x) |z(x, \rho, t)|^{m(x)}}{m(x)} dx d\rho dt.
\end{aligned}$$

Now, we estimate the parts on the right side of (4.4), respectively.

The first term is estimated as follows:

$$\begin{aligned}
& \left| -\int_s^T \frac{d}{dt} \left(E^q(t) \int_{\Omega} u u_t dx \right) dt \right| \\
& = \left| E^q(s) \int_{\Omega} u u_t(x, s) dx - E^q(T) \int_{\Omega} u u_t(x, T) dx \right| \\
& \leq \frac{1}{2} E^q(s) \left[\int_{\Omega} u^2(x, s) dx + \int_{\Omega} u_t^2(x, s) dx \right] \\
& \quad + \frac{1}{2} E^q(T) \left[\int_{\Omega} u^2(x, T) dx + \int_{\Omega} u_t^2(x, T) dx \right] \\
& \leq \frac{1}{2} E^q(s) [C_p \|\Delta u(s)\|_2^2 + 2E(s)] \\
& \quad + \frac{1}{2} E^q(T) [C_p \|\Delta u(T)\|_2^2 + 2E(T)] \\
& \leq E^q(s) [C_p E(s) + E(s)] + E^q(T) [C_p E(T) + E(T)],
\end{aligned}$$

where C_p is the Poincaré constant. Recalling that $E(t)$ is decreasing, we infer that

$$\left| -\int_s^T \frac{d}{dt} \left(E^q(t) \int_{\Omega} u u_t dx \right) dt \right| \leq c E^{q+1}(s) \leq c E^q(0) E(s) \leq c E(s). \quad (4.5)$$

In a similar way, we handle the term

$$\begin{aligned}
\left| q \int_s^T E^{q-1}(t) E'(t) \int_{\Omega} u u_t dx dt \right| & \leq -q \int_s^T E^{q-1}(t) E'(t) [C_p E(T) + E(T)] dt \\
& \leq -c \int_s^T E^q(t) E'(t) dt \leq c E^{q+1}(s) \leq c E(s).
\end{aligned}$$

We estimate the next term as follows,

$$\begin{aligned}
& \left| -\frac{\gamma}{\gamma+1} \int_s^T E^q \int_{\Omega} \|\nabla u\|^{2\gamma} |\nabla u^2| dx dt \right| = \left| -2\gamma \int_s^T E^q \left(\frac{\|\nabla u\|^{2\gamma}}{2(\gamma+1)} \int_{\Omega} |\nabla u^2| dx \right) dt \right| \\
& = \left| -2\gamma \int_s^T E^q \left(\frac{\|\nabla u\|^{2(\gamma+1)}}{2(\gamma+1)} \right) dt \right| \leq \left| -2\gamma \int_s^T E^q(E(t)) dt \right| \\
& \leq C^* \int_s^T E^{q+1}(t) dt \leq C^* E(s)
\end{aligned} \quad (4.6)$$

where C^* is a generic constant.

To treat the other term, we set

$$\Omega_+ = \{x \in \Omega, |u_t(x, t)| \geq 1\} \quad \text{and} \quad \Omega_- = \{x \in \Omega, |u_t(x, t)| < 1\},$$

and by using the Hölder's and Young's inequalities, to obtain

$$\begin{aligned} & \left| \int_s^T E^q(t) \int_{\Omega} u_t^2 dx dt \right| \\ &= \left| \int_s^T E^q(t) \left[\int_{\Omega_+} u_t^2 dx + \int_{\Omega_-} u_t^2 dx \right] dt \right| \\ &\leq c \int_s^T E^q(t) \left[\left(\int_{\Omega_+} |u_t|^{m^-} dx \right)^{2/m^-} + \left(\int_{\Omega_-} |u_t|^{m^+} dx \right)^{2/m^+} \right] dt \\ &\leq c \int_s^T E^q(t) \left[\left(\int_{\Omega} |u_t|^{m(x)} dx \right)^{2/m^-} + \left(\int_{\Omega} |u_t|^{m(x)} dx \right)^{2/m^+} \right] dt \\ &\leq c \int_s^T E^q(t) \left[(-E'(t))^{2/m^-} + (-E'(t))^{2/m^+} \right] dt \\ &\leq c\epsilon \int_s^T [E(t)]^{qm^-/(m^--2)} dt + c(\epsilon) \int_s^T (-E'(t)) dt \\ &\quad + c\epsilon \int_s^T E(t)^{q+1} dt + c(\epsilon) \int_s^T (-E'(t))^{2(q+1)/m^+} dt. \end{aligned}$$

For $m^- > 2$ and the choice of $q = m^+/2 - 1$ will make $\frac{qm^-}{m^--2} = q + 1 + \frac{m^+-m^-}{m^--2}$. Therefore,

$$\begin{aligned} \left| \int_s^T E^q(t) \int_{\Omega} u_t^2 dx dt \right| &\leq c\epsilon \int_s^T E(t)^{q+1} dt + c\epsilon [E(0)]^{\frac{m^+-m^-}{m^--2}} \int_s^T [E(t)]^{q+1} dt \\ &\quad + c(\epsilon) E(s) \leq c\epsilon \int_s^T E(t)^{q+1} dt + c(\epsilon) E(s). \end{aligned} \quad (4.7)$$

For the case $m^- = 2$ and the choice of $q = m^+/2 - 1$ will give a similar result.

The other term is estimated as follows:

$$\begin{aligned} & \left| -\frac{1}{2} \int_s^T \frac{d}{dt} \left(E^q(t) \int_{\Omega} |\nabla u|^2 dx \right) dt \right| \leq \\ & \quad \frac{1}{2} E^q(s) \int_{\Omega} |\Delta u(s)|^2 dx + \frac{1}{2} E^q(s) \int_{\Omega} |\Delta u(T)|^2 dx \leq \\ & \quad c E^{q+2/m^+}(s) \leq c \left(E^{q-1+2/m^+}(0) \right) E(s) \leq \lambda E(s) \end{aligned} \quad (4.8)$$

where c and λ are positive constants.

Similarly,

$$\int_s^T E^{q-1}(t) E'(t) \int_{\Omega} |\nabla u|^2 dx dt \leq$$

$$cE^{q+2/m^+}(s) \leq c \left(E^{q-1+2/m^+}(0) \right) E(s) \leq \lambda_1 E(s) \quad (4.9)$$

where c and λ_1 are positive constants.

For the other term, utilizing Young's inequality, we conclude that

$$\begin{aligned} & \left| -\mu_1 \int_s^T E^q(t) \int_{\Omega} u |u_t|^{m(x)-1} dx dt \right| \leq \\ & \varepsilon \int_s^T E^q(t) \int_{\Omega} |u(t)|^{m(x)} dx dt + c \int_s^T E^q(t) \int_{\Omega} c_{\varepsilon}(x) |u_t(t)|^{m(x)} dx dt \leq \\ & \varepsilon \int_s^T E^q(t) \left[\int_{\Omega_+} |u(t)|^{m^-} dx + \int_{\Omega_-} |u(t)|^{m^+} dx \right] dt \\ & + c \int_s^T E^q(t) \int_{\Omega} c_{\varepsilon}(x) |u_t(t)|^{m(x)} dx dt, \end{aligned}$$

where we have used Young's inequality with

$$p(x) = \frac{m(x)}{m(x)-1}, p'(x) = m(x),$$

thus,

$$c_{\varepsilon}(x) = (m(x)-1)m(x)^{m(x)/(1-m(x))} \varepsilon^{1/(1-m(x))}.$$

Hence, by using the embeddings $H_0^2(\Omega) \hookrightarrow L^{m^-}(\Omega)$ and $H_0^2(\Omega) \hookrightarrow L^{m^+}(\Omega)$, we conclude that

$$\begin{aligned} & \left| -\mu_1 \int_s^T E^q(t) \int_{\Omega} u |u_t|^{m(x)-1} dx dt \right| \\ & \leq \varepsilon \int_s^T E^q(t) \left[c \|\Delta u(s)\|_2^{m^-} + c \|\Delta u(s)\|_2^{m^+} \right] dt \\ & \quad + c \int_s^T E^q(t) \int_{\Omega} c_{\varepsilon}(x) |u_t(t)|^{m(x)} dx dt \\ & \leq \varepsilon \int_s^T E^q(t) \left[cE^{(m^- - 2)/2}(0) E(t) + cE^{(m^+ - 2)/2}(0) E(t) \right] dt \\ & \quad + c \int_s^T E^q(t) \int_{\Omega} c_{\varepsilon}(x) |u_t(t)|^{m(x)} dx dt \\ & \leq c\varepsilon \int_s^T E^{q+1}(t) dt + \int_s^T E^q(t) \int_{\Omega} c_{\varepsilon}(x) |u_t(t)|^{m(x)} dx dt. \end{aligned} \quad (4.10)$$

The next term of (4.4) can be estimated in a similar attitude

$$\begin{aligned} & \left| -\mu_2 \int_s^T E^q(t) \int_{\Omega} u |z(x, 1, t)|^{m(x)-1} dx dt \right| \\ & \leq \varepsilon \int_s^T E^q(t) \left[c \|\Delta u(s)\|_2^{m^-} + c \|\Delta u(s)\|_2^{m^+} \right] dt \end{aligned}$$

$$\begin{aligned}
 & + c \int_s^T E^q(t) \int_{\Omega} c_{\varepsilon}(x) |z(x, 1, t)|^{m(x)} dx dt \\
 & \leq c\varepsilon \int_s^T E^{q+1}(t) dt + \int_s^T E^q(t) \int_{\Omega} c_{\varepsilon}(x) |z(x, 1, t)|^{m(x)} dx dt.
 \end{aligned} \tag{4.11}$$

As $\xi(x)$ is bounded, using Lemma 6 and (4.2), we obtain

$$\begin{aligned}
 & 2 \int_s^T E^q(t) \int_0^1 \int_{\Omega} \frac{\xi(x) |z(x, \rho, t)|^{m(x)}}{m(x)} dx d\rho dt \\
 & \leq \frac{2\tau e^{-\tau}}{m^-} E^q(s) E(s) + \frac{2c}{m^-} E^{q+1}(T) \\
 & \leq \frac{2\tau e^{-\tau}}{m^-} E^q(0) E(s) + \frac{2c}{m^-} E^q(T) E(s) \leq cE(s)
 \end{aligned} \tag{4.12}$$

for some $c > 0$.

By combining (4.4)-(4.12), we conclude that

$$\begin{aligned}
 \int_s^T E^{q+1}(t) dt & \leq \varepsilon \int_s^T E^{q+1}(t) dt + cE(s) \\
 & \quad + c \int_s^T E^q(t) \int_{\Omega} c_{\varepsilon}(x) |z(x, 1, t)|^{m(x)} dx dt.
 \end{aligned}$$

At this point, the choice of ε small enough, gives

$$\int_s^T E^{q+1}(t) dt \leq cE(s) + c \int_s^T E^q(t) \int_{\Omega} c_{\varepsilon}(x) |z(x, 1, t)|^{m(x)} dx dt.$$

Once ε is fixed, $c_{\varepsilon}(x)$ becomes bounded (i.e. $c_{\varepsilon}(x) \leq M$), as $m(x)$ is bounded. Therefore, we infer that

$$\begin{aligned}
 \int_s^T E^{q+1}(t) dt & \leq cE(s) + cM \int_s^T E^q(t) \int_{\Omega} |z(x, 1, t)|^{m(x)} dx dt \\
 & \leq cE(s) - C_0 M \int_s^T E^q(t) E'(t) dt \\
 & \leq cE(s) + \frac{C_0 M}{q+1} [E^{q+1}(s) - E^{q+1}(T)] \leq cE(s).
 \end{aligned}$$

As $T \rightarrow \infty$, we obtain

$$\int_s^{\infty} E^{q+1}(t) dt \leq cE(s).$$

Thus, Komornik's Lemma is satisfied with $\sigma = q = m^+/2 - 1$, which implies the desired result. $\square$

5. CONCLUSIONS

In recent years, many papers have been published about decay or blow up of solutions for different type of wave equations (Kirchhoff, Petrovsky, Bessel, ... etc.) with different state of delay time (constant delay, time-varying delay, ... etc.). However, to the best of our knowledge, there were no blow up and decay results for the Timoshenko equation with delay term and variable exponents. Firstly, we have been proved the blow up of solutions. Later, we have been obtained the decay results by using an integral inequality due to Komornik.

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ON THE DIOPHANTINE EQUATIONS $z^2 = f(x)^2 \pm f(y)^2$ INVOLVING LAURENT POLYNOMIALS II.

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Abstract. By the theory of Pell's equation, we give conditions for $f(x) = b + \frac{c}{x}$ with $b, c \in \mathbb{Z} \setminus \{0\}$ such that the Diophantine equations $z^2 = f(x)^2 \pm f(y)^2$ have infinitely many solutions $x, y \in \mathbb{Z}$ and $z \in \mathbb{Q}$, which gives a positive answer to Question 3.2 of Zhang and Shamsi Zargar [16]. By the theory of elliptic curve, we study the non-trivial rational solutions of the above Diophantine equations for Laurent polynomials $f(x) = \frac{\prod_{i=0}^n (x+k'_i)}{x}$, $\frac{\prod_{i=0}^n (x-k'_i)(x+k'_i)}{x}$, $n \geq 1$, $k \in \mathbb{Z} \setminus \{0, \pm 1\}$, and give a positive answer to Question 3.1 of Zhang and Shamsi Zargar [16].

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1. INTRODUCTION

Let $f(x) \in \mathbb{Q}[x]$ be a polynomial without multiple roots and $\deg f \geq 2$. Many authors considered the non-trivial integer and rational (parametric) solutions of the Diophantine equations

$$z^2 = f(x)^2 + f(y)^2 \quad (1.1)$$

and

$$z^2 = f(x)^2 - f(y)^2 \quad (1.2)$$

for different polynomials $f(x)$. Let us recall that a triple (x, y, z) is a non-trivial solution of Eq. (1.1) (respectively, Eq. (1.2)) if $f(x)f(y) \neq 0$ (respectively, $f(x)^2 \neq f(y)^2$, $f(y) \neq 0$).

In 1962, W. Sierpiński [5] obtained infinitely many non-trivial positive integer solutions of Eq. (1.1) for $f(x) = \frac{x(x+1)}{2}$. In 2010, M. Ulas and A. Togbé [9] studied the non-trivial rational solutions of Eqs. (1.1) and (1.2) for $f(x)$ being quadratic and cubic

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polynomials. At the same year, B. He, A. Togbé and M. Ulas [4] further investigated the non-trivial integer solutions of Eqs. (1.1) and (1.2) for some special polynomials $f(x)$. In 2018, Y. Zhang and A. Shamsi Zargar [15] proved that Eq. (1.1) has infinitely many non-trivial rational solutions for the quartic polynomials $f(x) = x(x-1)(x+1)(x + \frac{1-k^2}{2k})$, $k \in \mathbb{Z} \setminus \{0, \pm 1\}$ and Eq. (1.2) has infinitely many rational solutions for the quartic polynomials $f(x) = x(x-1)(x+1)(x - \frac{2k}{k^2+1})$, $k \in \mathbb{Z} \setminus \{0, \pm 1\}$, which gave a positive answer to **Question 4.3** of [9] for quartic polynomials. In 2019, A. E. A. Youmbai and D. Behloul [11] extended the results of Zhang-Shamsi Zargar [15] to the polynomials $x(\prod_{t=0}^n (x-k^t)(x+k^t))$ of degree $2n+3$ and gave a positive answer to **Question 4.3** of [9] for the polynomials $x(\prod_{t=0}^n (x+k^t))$ of degree $n+2$.

As a generalization of Eqs. (1.1) and (1.2), in 2017, Sz. Tengely and M. Ulas [8] investigated the existence of the non-trivial integer solutions of the Diophantine equations $z^2 = f(x)^2 \pm g(y)^2$ and proved similar results for some special higher degree polynomials as well.

A Laurent polynomial with coefficients in a field $\mathbb{F}$ is an expression of the form

$$f(x) = \sum_k a_k x^k, \quad a_k \in \mathbb{F},$$

where x is a formal variable, the summation index k is an integer (not necessarily positive) and only finitely many coefficients a_k are non-zero.

In 2018, Y. Zhang [13] studied the non-trivial rational parametric solutions of the Diophantine equation

$$f(x)f(y) = f(z)^n \quad (1.3)$$

for $n = 1, 2$, involving the Laurent polynomials $f(x) = ax + b + c/x$. In 2019, Y. Zhang and A. Shamsi Zargar [14] further investigated the non-trivial rational (parametric) solutions of Eq. (1.3), where $f(x) = x^k + ax^{k-1} + b/x$, $k \geq 2$, $x^2 + a/x + b/x^2$ for $n = 1$, and $f(x) = x^2 + ax + b + a^3/(27x)$, $x^2 + ax + b + a^3/(16x) + a^4/(256x^2)$ for $n = 2$.

In 1783, L. Euler [2, p. 167] studied the rational solutions of Eq. (1.1) for $f(x) = x + \frac{1}{x}$. In 2020, Y. Zhang and A. Shamsi Zargar [16] continued the study of Euler and considered the non-trivial rational (parametric) solutions of Eqs. (1.1) and (1.2) for some simple Laurent polynomials, such as

$$f(x) = x + b + \frac{c}{x}, \quad \frac{(x+1)(x+b)(x+c)}{x}$$

with $b, c \in \mathbb{Z} \setminus \{0\}$.

By the theory of Pell's equation and the same method of Zhang [12, Theorem 1.1], we give a positive answer to **Question 3.2** of Zhang and Shamsi Zargar [16] in Theorem 1.

Theorem 1. 1) For

$$f(x) = b + \frac{c}{x}$$

with $b, c \in \mathbb{Z} \setminus \{0\}$, if there exists an integer y_0 such that $(2b^2y_0^2 + 2bcy_0 + c^2)x^2 + 2bcy_0^2x + c^2y_0^2 = v^2$ has a non-zero integer solution (x_0, v_0) and $2b^2y_0^2 + 2bcy_0 + c^2 > 0$ is not a perfect square, then Eq. (1.1) has infinitely many non-trivial solutions (x, y_0, z) with $x, y_0 \in \mathbb{Z}$ and $z \in \mathbb{Q}$.

2) For

$$f(x) = b + \frac{c}{x}$$

with $b, c \in \mathbb{Z} \setminus \{0\}$, if there is an integer x_0 such that $c(2bx_0 + c)y^2 - 2bcx_0^2y - c^2x_0^2 = v^2$ has a non-zero integer solution (y_0, v_0) and $c(2bx_0 + c) > 0$ is not a perfect square, then Eq. (1.2) has infinitely many non-trivial solutions (x_0, y, z) with $x_0, y \in \mathbb{Z}$ and $z \in \mathbb{Q}$.

Following the methods of Youmbai and Behloul [11, Theorems 1.1-1.4] and Zhang and Shamsi Zargar [15, Theorems 1.1-1.2], we give a positive answer to **Question 3.1** of Zhang and Shamsi Zargar [16] in Theorems 2 and 3.

Theorem 2. For

$$f(x) = \frac{\prod_{t=0}^n (x + k^t)}{x}, \quad n \geq 1$$

with $k \in \mathbb{Z} \setminus \{0, \pm 1\}$, Eqs. (1.1) and (1.2) have a rational parametric solution.

Theorem 3. When

$$f(x) = \frac{\prod_{t=0}^n (x - k^t)(x + k^t)}{x}, \quad n \geq 1,$$

for all but finitely many $k \in \mathbb{Z} \setminus \{0, \pm 1\}$, Eqs. (1.1) and (1.2) have infinitely many non-trivial rational solutions.

By the map $x \mapsto 1/x$, we have

Proposition 1. For

$$f(x) = \frac{\prod_{t=0}^n (k^t x + 1)}{x^n}, \quad n \geq 1$$

with $k \in \mathbb{Z} \setminus \{0, \pm 1\}$, Eqs. (1.1) and (1.2) have a rational parametric solution.

When

$$f(x) = \frac{\prod_{t=0}^n (1 - k^t x)(1 + k^t x)}{x^{2n+1}}, \quad n \geq 1,$$

for all but finitely many $k \in \mathbb{Z} \setminus \{0, \pm 1\}$, Eqs. (1.1) and (1.2) have infinitely many non-trivial rational solutions.

2. PRELIMINARIES

To prove Theorem 1, we need the following lemma about the integer solutions of Pell's equation.

Lemma 1 (Corollary in [3]). *Let m_1, m_2, D be positive integers, D is not a square, and $a^2 - Db^2 = M$, then there are infinitely many integer solutions (u, v) of the Pell's equation $u^2 - Dv^2 = M$ with*

$$u \equiv a \pmod{m_1} \quad \text{and} \quad v \equiv b \pmod{m_2}.$$

To simplify the proof of Theorem 3, we state the following useful Lemma.

Lemma 2. *The quartic curve $C: V^2 = aU^4 + bU^3 + cU^2 + dU + e^2$ is birationally equivalent to an elliptic curve with Weierstrass equation*

$$E: Y^2 = X^3 - 27(12ae^2 - 3bd + c^2)X + 27(2c^3 - 72ace^2 + 27b^2e^2 + 27ad^2 - 9bcd),$$

by the map $\phi: C \ni (U, V) \mapsto (X, Y) \in E$

$$X = \frac{3(cU^2 + 3dU + 6e^2 + 6eV)}{U^2},$$

$$Y = \frac{27(beU^3 + 2ceU^2 + 3deU + dUV + 4e^3 + 4e^2V)}{U^3},$$

and its inverse map ϕ^{-1} is

$$U = -\frac{3(24ce^2 + 4e^2X - 9d^2)}{54be^2 - 9cd + 3dX - 2eY},$$

$$V = -\frac{N(V)}{(54be^2 - 9cd + 3dX - 2eY)^2},$$

where

$$N(V) = -8e^3X^3 - 9e(8ce^2 - 3d^2)X^2 + 4e^3Y^2 + (-216be^4 + 108cde^2 - 27d^3)Y \\ + 27e(108b^2e^4 - 108bcde^2 + 32c^3e^2 + 27bd^3 - 9c^2d^2).$$

Proof. This is a modified version in [10, p. 37, Theorem 2.17]. □

3. PROOFS OF THE THEOREMS

Proof of Theorem 1. 1) For

$$f(x) = b + \frac{c}{x}$$

with $b, c \in \mathbb{Z} \setminus \{0\}$, Eq. (1.1) equals

$$z^2 = \frac{(2b^2y^2 + 2bcy + c^2)x^2 + 2bcy^2x + c^2y^2}{x^2y^2}.$$

To get integral values of x and y , let us consider the integer solutions (x, v) of the quadratic equation

$$(2b^2y^2 + 2bcy + c^2)x^2 + 2bcy^2x + c^2y^2 = v^2.$$

Put $U = (2b^2y^2 + 2bcy + c^2)x + bcy^2$, $V = v$, then we get the Pell's equation

$$U^2 - (2b^2y^2 + 2bcy + c^2)V^2 = -c^2y^2(by + c)^2.$$

Take $y = y_0$, then $U = (2b^2y_0^2 + 2bcy_0 + c^2)x + bcy_0^2$ and

$$U^2 - (2b^2y_0^2 + 2bcy_0 + c^2)V^2 = -c^2y_0^2(by_0 + c)^2. \quad (3.1)$$

Note that if $2b^2y_0^2 + 2bcy_0 + c^2 > 0$ is not a perfect square, then the Pell's equation

$$U^2 - (2b^2y_0^2 + 2bcy_0 + c^2)V^2 = 1$$

has infinitely many integer solutions. If there exists an integer y_0 such that $(2b^2y_0^2 + 2bcy_0 + c^2)x^2 + 2bcy_0^2x + c^2y_0^2 = v^2$ has a non-zero integer solution (x_0, v_0) , then

$$(U_0, V_0) = ((2b^2y_0^2 + 2bcy_0 + c^2)x_0 + bcy_0^2, v_0)$$

is an integer solution of Eq. (3.1). So there are infinitely many integer solutions of Eq. (3.1).

Note that $U_0 = (2b^2y_0^2 + 2bcy_0 + c^2)x_0 + bcy_0^2$ satisfies

$$U_0 \equiv bcy_0^2 \pmod{2b^2y_0^2 + 2bcy_0 + c^2}.$$

By Lemma 1, Eq. (3.1) has infinitely many integer solutions U satisfying the above condition. Therefore, there are infinitely many

$$x = \frac{U - bcy_0^2}{2b^2y_0^2 + 2bcy_0 + c^2} \in \mathbb{Z}.$$

Thus, for

$$f(x) = b + \frac{c}{x}$$

with $b, c \in \mathbb{Z} \setminus \{0\}$, Eq. (1.1) has infinitely many non-trivial solutions (x, y_0, z) with $x, y_0 \in \mathbb{Z}$ and $z \in \mathbb{Q}$.

2) For

$$f(x) = b + \frac{c}{x}$$

with $b, c \in \mathbb{Z} \setminus \{0\}$, Eq. (1.2) becomes

$$z^2 = \frac{c(2bx + c)y^2 - 2bcx^2y - c^2x^2}{x^2y^2}.$$

To get integral values of x and y , let us consider the integer solutions (y, v) of the quadratic equation

$$c(2bx + c)y^2 - 2bcx^2y - c^2x^2 = v^2.$$

Take $U = c(2bx + c)y - bcx^2$, $V = v$, then we obtain the Pell's equation

$$U^2 - c(2bx + c)V^2 = -c^2x^2(bx + c)^2.$$

By the same method as 1), we can give the proof of 2). □

Example 1. When $b = c = 1$, $f(x) = 1 + \frac{1}{x}$, Eq. (1.1) reduces to

$$z^2 = \frac{(2y^2 + 2y + 1)x^2 + 2xy^2 + y^2}{x^2y^2}.$$

Consider $(2y^2 + 2y + 1)x^2 + 2xy^2 + y^2 = v^2$. If $y_0 = 1$, we have

$$5x^2 + 2x + 1 = v^2.$$

It is easy to check that $(x_0, v_0) = (2, 5)$ is a solution of the above equation, and $2y_0^2 + 2y_0 + 1 = 5 > 0$ is not a perfect square. By the theory of Pell's equation, $5x^2 + 2x + 1 = v^2$ has infinitely many integer solutions (x, v) . Then Eq. (1.1) has infinitely many non-trivial solutions $(x, 1, z)$ with $x \in \mathbb{Z}$ and $z \in \mathbb{Q}$.

Remark 1. In fact, we can use the transformation $x = T$, $y = cT$ to study Eqs. (1.1) and (1.2). But we also need some conditions about b, c to get infinitely many non-trivial solutions. Here, we give two simple cases to display this method.

1) When $b = 1$, $c \in \mathbb{Z} \setminus \{0, 1\}$, $f(x) = 1 + \frac{c}{x}$. Let

$$x = T, \quad y = cT.$$

Then Eq. (1.1) equals

$$z^2 = \frac{2T^2 + (2c + 2)T + c^2 + 1}{T^2}.$$

To get integral values of x and y , let us consider the integer solutions (T, S) of the quadratic equation

$$2T^2 + (2c + 2)T + c^2 + 1 = S^2.$$

Let $U = 2T + c + 1$, $V = S$, then we get the Pell's equation

$$U^2 - 2V^2 = -(c - 1)^2.$$

Note that $(U, V) = (c - 1, c - 1)$ is an integer solution of the above Pell's equation, and $(U, V) = (3, 2)$ is an integer solution of the Pell's equation $U^2 - 2V^2 = 1$. Thus, an infinity of integer solutions of $U^2 - 2V^2 = -(c - 1)^2$ are given by

$$U_m + V_m\sqrt{2} = (3 + 2\sqrt{2})^m(c - 1 + (c - 1)\sqrt{2}), \quad m \geq 0.$$

Then

$$\begin{cases} U_m = 6U_{m-1} - U_{m-2}, & U_0 = c-1, \quad U_1 = 7(c-1); \\ V_m = 6V_{m-1} - V_{m-2}, & V_0 = c-1, \quad V_1 = 5(c-1). \end{cases}$$

It is easy to prove that

$$U_m \equiv c+1 \pmod{2}, \quad V_m = S_m \in \mathbb{Z}.$$

Then we have

$$T_m = \frac{U_m - (c+1)}{2} \in \mathbb{Z}, \quad m \geq 0.$$

So

$$x_m = T_m \in \mathbb{Z}, \quad y_m = cT_m \in \mathbb{Z}, \quad z_m = \frac{S_m}{T_m} \in \mathbb{Q}, \quad m \geq 0.$$

2) For

$$f(x) = 1 + \frac{2c}{x}$$

with $c \in \mathbb{Z} \setminus \{0, 1\}$, let

$$x = T, \quad y = cT.$$

Then Eq. (1.2) reduces to

$$z^2 = \frac{4(c-1)(c+T+1)}{T^2}.$$

To get integral values of x and y , let us consider the integer solutions (T, S) of the following equation

$$(c-1)(c+T+1) = S^2.$$

Solve it for T , then we have

$$T = \frac{S^2}{c-1} - c - 1.$$

Put $S = (c-1)u$, then

$$T = (c-1)u^2 - c - 1.$$

So

$$x = (c-1)u^2 - c - 1 \in \mathbb{Z}, \quad y = cx \in \mathbb{Z}, \quad z = \frac{2u(c-1)}{(c-1)u^2 - c - 1} \in \mathbb{Q},$$

where u is an integer parameter.

Proof of Theorem 2. 1) For

$$f(x) = \frac{\prod_{t=0}^n (x + k^t)}{x}, \quad n \geq 1$$

with $k \in \mathbb{Z} \setminus \{0, \pm 1\}$, let

$$x = T, \quad y = kT.$$

Then Eq. (1.1) equals

$$z^2 = \left(\frac{\prod_{t=0}^{n-1} (x + k^t)}{T} \right)^2 \left((k^{2n} + 1)T^2 + 2k^n(k^{n-1} + 1)T + k^{2n-2}(k^2 + 1) \right).$$

Consider the conic section

$$C_1 : S^2 = (k^{2n} + 1)T^2 + 2k^n(k^{n-1} + 1)T + k^{2n-2}(k^2 + 1).$$

Take $U = T + \frac{1}{k}$, $V = kS$, then we obtain

$$C_{1,k} : V^2 = k^2(k^{2n} + 1)U^2 + 2k(k^{n+1} - 1)U + (k^{n+1} - 1)^2,$$

which can be parametrized by

$$U = \frac{2(1 - k^{n+1})(t - k)}{t^2 - k^2(k^{2n} + 1)},$$

$$V = \frac{(1 - k^{n+1})(t^2 - 2kt + k^2(k^{2n} + 1))}{t^2 - k^2(k^{2n} + 1)},$$

where t is a rational parameter. Then Eq. (1.1) has a rational parametric solution

$$(x, y, z) = \left(U - \frac{1}{k}, kU - 1, \frac{\prod_{t=0}^{n-1} (U - \frac{1}{k} + k^t)}{kU - 1} V \right),$$

where U, V are given in above.

2) For

$$f(x) = \frac{\prod_{t=0}^n (x + k^t)}{x}, \quad n \geq 1$$

with $k \in \mathbb{Z} \setminus \{0, \pm 1\}$, let

$$x = T, \quad y = kT.$$

Then Eq. (1.2) becomes

$$z^2 = \left(\frac{\prod_{t=0}^{n-1} (x + k^t)}{T} \right)^2 \left((-k^{2n} + 1)T^2 + 2k^n(-k^{n-1} + 1)T + k^{2n-2}(k^2 - 1) \right).$$

Consider

$$C_2 : S^2 = (1 - k^{2n})T^2 + 2k^n(1 - k^{n-1})T + k^{2n-2}(k^2 - 1).$$

Put $U = T + \frac{1}{k}$, $V = kS$, then we have

$$C_{2,k} : V^2 = k^2(1 - k^{2n})U^2 + 2k(k^{n+1} - 1)U + (k^{n+1} - 1)^2.$$

The remainder of the proof is similar as 1), we omit it. $\square$

Proof of Theorem 3. 1) For

$$f(x) = \frac{\prod_{t=0}^n (x - k^t)(x + k^t)}{x}, \quad n \geq 1$$

with $k \in \mathbb{Z} \setminus \{0, \pm 1\}$, let

$$x = T, \quad y = kT.$$

Then Eq. (1.1) reduces to

$$z^2 = \left(\frac{\prod_{t=0}^{n-1} (x^2 - k^{2t})}{T} \right)^2 ((k^{4n+2} + 1)T^4 - 2k^{2n}(k^{2n} + 1)T^2 + k^{4n-2}(k^2 + 1)).$$

Consider the quartic curve

$$C_3 : S^2 = (k^{4n+2} + 1)T^4 - 2k^{2n}(k^{2n} + 1)T^2 + k^{4n-2}(k^2 + 1).$$

Take $U = T - \frac{1}{k}$, $V = k^2S$, then we get

$$C_3(n, k) : V^2 = k^4(k^{4n+2} + 1)U^4 + 4k^3(k^{4n+2} + 1)U^3 + 2k^2(2k^{4n+2} - k^{2n+2} + 3)U^2 - 4k(k^{2n+2} - 1)U + (k^{2n+2} - 1)^2.$$

The discriminant of $C_3(n, k)$ is

$$256k^{12n+18}(k^2 + 1)(k^{4n+2} + 1)(k^{2n+2} - 1)^4.$$

So $C_3(n, k)$ is smooth, when $k \neq 0, \pm 1$. By Lemma 2, the corresponding elliptic curve $\mathcal{E}_3(n, k)$ of $C_3(n, k)$ is

$$\begin{aligned} \mathcal{E}_3(n, k) : Y^2 = X^3 - 108k^{4n+6}(3k^{4n+4} + 4k^{4n+2} + 2k^{2n+2} + 4k^2 + 3)X \\ + 432k^{6n+10}(k^{2n} + 1)(9k^{4n+4} + 8k^{4n+2} - 2k^{2n+2} + 8k^2 + 9). \end{aligned}$$

In order to prove Theorem 3, it needs to show that the curve $C_3(n, k)$ has infinitely many rational points, equivalently, $\mathcal{E}_3(n, k)$ has a rational point of infinite order. It is easy to see that the rational points $(U, V) = (0, \pm(k^{2n+2} - 1))$ on $C_3(n, k)$ lead to rational points of order 2 on $\mathcal{E}_3(n, k)$. Since $f(k^n) = 0$, Eq. (1.1) has a trivial solution $(k^n, k^{n+1}, f(k^{n+1}))$. From this observation, we find another rational point on $C_3(n, k)$:

$$P = \left(k^n - \frac{1}{k}, k^{2n+1}(k^{2n+2} - 1) \right).$$

By the map $\varphi_1 : C_3(n, k) \ni (U, V) \mapsto (X, Y) \in \mathcal{E}_3(n, k)$, we can obtain the point

$$Q = \varphi_1(P) = (6k^{2n+3}(3k^{2n+2} + 2k^{2n+1} + 6k^{n+1} + 2k + 3), 108k^{3n+5}k^{n+1}(k^{n+1} + 1)^2)$$

on $\mathcal{E}_3(n, k)$. By the group law, we have

$$[2]Q = \left(\frac{3k^{2n+2}(3k^{4n+4} + 4k^{4n+2} + 8k^{3n+2} + 14k^{2n+2} + 8k^{n+2} + 4k^2 + 3)}{(k^n + 1)^2}, \right. \\ \left. - \frac{27k^{3n+3}(k^{2n+2} + 1)(k^{4n+4} - 4k^{3n+2} - 6k^{2n+2} - 4k^{n+2} + 1)}{(k^n + 1)^3} \right).$$

Let $\mathcal{E}_3(n, 2)$ be the specialization of $\mathcal{E}_3(n, k)$ at $k = 2$, and the specialization of $[2]Q$ at $k = 2$ is

$$[2]Q_2 = \left(\frac{3 \times 2^{2n+2}(2^{4n+6} + 2^{3n+5} + 7 \times 2^{2n+3} + 2^{n+5} + 19)}{(2^n + 1)^2}, \right. \\ \left. - \frac{27 \times 2^{3n+3}(2^{2n+2} + 1)(2^{4n+4} - 2^{3n+5} - 3 \times 2^{2n+3} - 2^{n+4} + 1)}{(2^n + 1)^3} \right).$$

A quick calculation reveals that the remainder of the division of the numerator of the X -coordinate of the point $[2]Q_2$ by its denominator equals

$$|r| = 18(2^{n+8} + 2^{n+2} + 210),$$

which is non-zero. So the X -coordinate of $[2]Q_2$ is not a polynomial. For $1 \leq n \leq 12$, one can check that

$$\frac{|r|}{(2^n + 1)^2}$$

is not an integer except for $n = 1, 2$, and that it is nonzero and is less than 1 in modulus for $n > 12$. Then for integers $n \neq 1, 2$, the point $[2]Q_2$ has non-integral X -coordinate and hence, by Nagell-Lutz Theorem (see [7, p.56]), is of infinite order. Thus, $\mathcal{E}_3(n, k)$ has a positive rank in the field $\mathbb{Q}(k)$. By the Specialization Theorem of Silverman (see [6, p.457, Theorem 20.3]), when $n \neq 1, 2$, for all but finitely many $k \in \mathbb{Z} \setminus \{0, \pm 1\}$, $\mathcal{E}_3(n, k)$ has a positive rank and infinitely many rational points.

When $n = 1, 2$, we have

$$\mathcal{E}_3(1, 2) : Y^2 = X^3 - 118886400X + 399900672000,$$

$$\mathcal{E}_3(2, 2) : Y^2 = X^3 - 29251141632X + 1385178693894144.$$

Using the package of Magma [1], the ranks of the above two elliptic curves are 2, hence, they have infinitely many rational points. Therefore, when $n \geq 1$, for all but finitely many $k \in \mathbb{Z} \setminus \{0, \pm 1\}$, $\mathcal{E}_3(n, k)$ has infinitely many rational points, i.e., the curve $\mathcal{C}_3(n, k)$ has infinitely many rational points.

Thus, when

$$f(x) = \frac{\prod_{t=0}^n (x - k^t)(x + k^t)}{x}, \quad n \geq 1,$$

for all but finitely many $k \in \mathbb{Z} \setminus \{0, \pm 1\}$, Eq. (1.1) has infinitely many non-trivial rational solutions.

2) For

$$f(x) = \frac{\prod_{t=0}^n (x - k^t)(x + k^t)}{x}, \quad n \geq 1$$

with $k \in \mathbb{Z} \setminus \{0, \pm 1\}$, let

$$x = T, \quad y = kT.$$

Then Eq. (1.2) equals

$$z^2 = \left(\frac{\prod_{t=0}^{n-1} (x^2 - k^{2t})}{T} \right)^2 ((-k^{4n+2} + 1)T^4 + 2k^{2n}(k^{2n} - 1)T^2 + k^{4n-2}(k^2 - 1)).$$

Consider the quartic curve

$$C_4 : S^2 = (-k^{4n+2} + 1)T^4 + 2k^{2n}(k^{2n} - 1)T^2 + k^{4n-2}(k^2 - 1).$$

Put $U = T - \frac{1}{k}$, $V = k^2S$, then we have

$$C_4(n, k) : V^2 = k^4(1 - k^{4n+2})U^4 + 4k^3(1 - k^{4n+2})U^3 - 2k^2(2k^{4n+2} + k^{2n+2} - 3)U^2 - 4k(k^{2n+2} - 1)U + (k^{2n+2} - 1)^2.$$

The discriminant of $C_4(n, k)$ is

$$-256k^{12n+18}(k^2 - 1)(k^{4n+2} - 1)(k^{2n+2} - 1)^4.$$

So $C_4(n, k)$ is smooth, when $k \neq 0, \pm 1$. By Lemma 2, the corresponding elliptic curve $\mathcal{E}_4(n, k)$ of $C_4(n, k)$ is

$$\begin{aligned} \mathcal{E}_4(n, k) : Y^2 = & X^3 + 108k^{4n+6}(3k^{4n+4} - 4k^{4n+2} + 2k^{2n+2} - 4k^2 + 3)X \\ & + 432k^{6n+10}(k^{2n} - 1)(9k^{4n+4} - 8k^{4n+2} - 2k^{2n+2} - 8k^2 + 9). \end{aligned}$$

By the method of Fermat [2, p. 639], from the point $P_0 = (0, k^{2n+2} - 1)$, we can get another point

$$P' = \left(-\frac{4(k^{4n+4} - 1)}{k(k^{4n+4} + 2k^{2n+2} + 4k^{4n+2} - 3)}, \frac{(k^{2n+2} - 1)N(k)}{(k^{4n+4} + 2k^{2n+2} + 4k^{4n+2} - 3)^2} \right)$$

on $\mathcal{E}_4(n, k)$, where

$$\begin{aligned} N(k) = & k^{8n+8} - 24k^{8n+6} + 16k^{8n+4} - 4k^{6n+6} - 16k^{6n+4} + 6k^{4n+4} - 24k^{4n+2} \\ & - 4k^{2n+2} + 1. \end{aligned}$$

By the map $\varphi_2 : C_4(n, k) \ni (U, V) \mapsto (X, Y) \in \mathcal{E}_4(n, k)$, we get the rational point

$$\begin{aligned} Q' = & \varphi_2(P') \\ = & \left(3(3k^{8n+8} - 40k^{8n+6} + 48k^{8n+4} + 4k^{6n+6} + 16k^{6n+4} + 50k^{4n+4} - 40k^{4n+2} \right. \\ & \left. + 4k^{2n+2} + 3)/(4(k^{2n+2} + 1)^2), \right. \\ & \left. - 27(k^{4n+4} - 4k^{4n+2} - 2k^{2n+2} + 1)(k^{8n+8} + 16k^{8n+6} - 16k^{8n+4} + 4k^{6n+6} \right. \end{aligned}$$

$$-10k^{4n+4} + 16k^{4n+2} + 4k^{2n+2} + 1)/(8(k^{2n+2} + 1)^3))$$

lying on $\mathcal{E}_4(n, k)$.

Using Nagell-Lutz Theorem and the Specialization Theorem of Silverman, we obtain the result in a similar way like in 1). $\square$

Example 2. When $n = 1$, $k \in \mathbb{Z} \setminus \{0, \pm 1\}$,

$$f(x) = \frac{(x-1)(x+1)(x-k)(x+k)}{x},$$

we have

$$C_3(1, k) : V^2 = k^4(k^6 + 1)U^4 + 4k^3(k^6 + 1)U^3 + 2k^2(2k^6 - k^4 + 3)U^2 - 4k(k^4 - 1)U + (k^4 - 1)^2$$

and

$$\mathcal{E}_3(1, k) : Y^2 = X^3 - 108k^{10}(3k^4 - 2k^2 + 3)(k^2 + 1)^2X + 432k^{16}(k^2 + 1)^3(9k^4 - 10k^2 + 9).$$

From Theorem 3, $\mathcal{E}_3(1, k)$ has rational points:

$$\begin{aligned} Q(1) &= (6k^5(k^2 + 1)(3k^2 + 2k + 3), 108k^8(k + 1)(k^2 + 1)^2), \\ [2]Q(1) &= \left(\frac{3k^4(3k^8 + 4k^6 + 8k^5 + 14k^4 + 8k^3 + 4k^2 + 3)}{(k + 1)^2}, \right. \\ &\quad \left. - \frac{27k^6(k^8 - 4k^5 - 6k^4 - 4k^3 + 1)(k^4 + 1)}{(k + 1)^3} \right). \end{aligned}$$

By the group law, we get

$$[4]Q(1) = (X([4]Q(1)), Y([4]Q(1))),$$

where

$$\begin{aligned} X([4]Q(1)) &= 3k^4(3k^{32} + 40k^{30} + 128k^{29} + 248k^{28} + 384k^{27} + 1016k^{26} + 2176k^{25} \\ &\quad + 2964k^{24} + 1664k^{23} - 2136k^{22} - 7040k^{21} - 8504k^{20} - 2688k^{19} \\ &\quad + 10296k^{18} + 23808k^{17} + 29778k^{16} + 23808k^{15} + 10296k^{14} - 2688k^{13} \\ &\quad - 8504k^{12} - 7040k^{11} - 2136k^{10} + 1664k^9 + 2964k^8 + 2176k^7 + 1016k^6 \\ &\quad + 384k^5 + 248k^4 + 128k^3 + 40k^2 + 3)/(4(k^8 - 4k^5 - 6k^4 - 4k^3 + 1)^2 \\ &\quad \times (k^4 + 1)^2(k + 1)^2) \end{aligned}$$

and

$$\begin{aligned} Y([4]Q(1)) &= -(27k^6(k^{16} + 4k^{14} + 16k^{13} + 28k^{12} + 16k^{11} - 4k^{10} - 32k^9 - 42k^8 \\ &\quad - 32k^7 - 4k^6 + 16k^5 + 28k^4 + 16k^3 + 4k^2 + 1)(k^{32} - 16k^{30} - 80k^{29} \end{aligned}$$

$$\begin{aligned}
& -136k^{28} - 144k^{27} - 80k^{26} - 32k^{25} - 68k^{24} - 544k^{23} - 1712k^{22} \\
& - 3312k^{21} - 3896k^{20} - 2096k^{19} + 2320k^{18} + 7232k^{17} + 9478k^{16} \\
& + 7232k^{15} + 2320k^{14} - 2096k^{13} - 3896k^{12} - 3312k^{11} - 1712k^{10} \\
& - 544k^9 - 68k^8 - 32k^7 - 80k^6 - 144k^5 - 136k^4 - 80k^3 - 16k^2 + 1)) \\
& / (8(k+1)^3(k^8 - 4k^5 - 6k^4 - 4k^3 + 1)^3(k^4 + 1)^3).
\end{aligned}$$

Using the method of Zhang and Shamsi Zargar [15, Theorem 1.1], we can show that, for $k \in \mathbb{Z} \setminus \{0, \pm 1\}$, $[4]Q(1)$ is a rational point of infinite order. Then $\mathcal{E}_3(1, k)$ and $\mathcal{C}_3(1, k)$ have infinitely many rational points. Thus, Eq. (1.1) has infinitely many non-trivial rational solutions.

From the rational point $[2]Q(1)$ on $\mathcal{E}_3(1, k)$, we can get the rational point

$$\begin{aligned}
(U, V) = & (2(k-1)(k^2+1)(k^8+4k^6+8k^5+6k^4+1)(k+1)^2/(kD(k)), \\
& k^2(k^4-1)(k^{24}+24k^{22}+64k^{21}+82k^{20}+24k^{19}-92k^{18}-192k^{17} \\
& -81k^{16}+288k^{15}+704k^{14}+768k^{13}+492k^{12}+144k^{11}-72k^{10}-192k^9 \\
& -177k^8-96k^7+8k^6+64k^5+66k^4+24k^3+4k^2+1)/(D(k)^2))
\end{aligned}$$

on $\mathcal{C}_3(1, k)$, where

$$D(k) = k^{13} - 4k^{11} - 16k^{10} - 15k^9 - 6k^8 + 4k^7 + 8k^6 + 11k^5 + 4k^4 + 3k + 2.$$

Therefore, for

$$f(x) = \frac{(x-1)(x+1)(x-k)(x+k)}{x}, \quad k \in \mathbb{Z} \setminus \{0, \pm 1\},$$

Eq. (1.1) has a rational solution

$$(x, y, z) = \left(U + \frac{1}{k}, \quad kU + 1, \quad \frac{(kU + 1 - k)(kU + 1 + k)V}{k^3(kU + 1)} \right),$$

where U, V are given in above.

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MULTIPLE SOLUTIONS FOR A FRACTIONAL P&Q-LAPLACIAN SYSTEM INVOLVING HARDY-SOBOLEV EXPONENTS

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Abstract. In this paper, we prove the existence of infinitely many solutions for a fractional p&q-Laplacian system involving Hardy-Sobolev exponents and obtain new conclusion under different conditions. The methods used here are based on variational methods and Ljusternik-Schnirelmann theory.

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1. INTRODUCTION

In this paper, we study the following fractional p&q-Laplacian system involving Hardy-Sobolev exponents:

$$\begin{cases} (-\Delta)_p^s u + (-\Delta)_q^s u = \lambda |u|^{r-2} u + \frac{2\alpha}{\alpha+\beta} \frac{|u|^{\alpha-2} |v|^\beta}{|x|^\theta} & \text{in } \Omega, \\ (-\Delta)_p^s v + (-\Delta)_q^s v = \mu |v|^{r-2} v + \frac{2\beta}{\alpha+\beta} \frac{|u|^\alpha |v|^{\beta-2}}{|x|^\theta} & \text{in } \Omega, \\ u = v = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (1.1)$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain containing the origin, $p \in (1, \infty)$, $s \in (0, 1)$, $1 < r < q < p$, $0 \leq \theta < sp < N$, $\alpha + \beta = p_{s,\theta}^*$ and $\lambda, \mu > 0$ are two parameters, $p_{s,\theta}^* = \frac{(N-\theta)p}{N-ps}$ is the fractional Hardy-Sobolev exponent, the fractional p -Laplacian operator $(-\Delta)_p^s$ is the nonlocal operator defined on smooth functions by

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon \searrow 0^+} \int_{B_\varepsilon^c(x)} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))}{|x - y|^{N+sp}} dy, \quad x \in \mathbb{R}^N.$$

This definition is consistent, up to normalization depending on N and s . We would like to point out that, in the last decades, problems involving fractional p -Laplacian

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are widely studied. We refer the readers to [2, 5, 6, 10, 13, 18–24] and references therein.

Then system (1.1) reduces to the critical fractional p -Laplacian equation with Hardy-Sobolev exponents

$$\begin{cases} (-\Delta)_p^s u = \lambda |u|^{r-2} u + \frac{|u|^{p_{s,\theta}^*-2}}{|x|^\theta} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.2)$$

Ning, Wang and Zhang proved the existence, multiplicity and bifurcation results for the above problem in [15]. When $\theta = 0$, in [11], Khiddi proved that problem (1.2) has infinitely many solutions with negative energy by using Ljusternik-Schnirelmann theory. When $r = p$, Ghoussoub and Yuan obtained multiple solutions in [8]. In [16], Perera and Zou proved that this problem has a nontrivial solution where $s = 1$, $r = p$, $\lambda > \lambda_1$ is not an eigenvalue and λ_1 is the first eigenvalue of the eigenvalue problem.

However, only a few articles have studied the p & q -Laplacian system, the results for the case can be seen in [4, 9, 12, 14], for example,

$$\begin{cases} (-\Delta)_p^s u + (-\Delta)_q^s u = \lambda |u|^{r-2} u + \frac{2\alpha}{\alpha+\beta} |u|^{\alpha-2} u |v|^\beta & \text{in } \Omega, \\ u = v = 0 & \text{on } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (1.3)$$

Obviously, system (1.3) is equivalent to $\theta = 0$ of system (1.1), Chen and Gui proved the existence of infinitely many solutions of problem (1.3) in [4]. Moreover, in [3], Chen and Deng proved that problem (1.3) has at least two positive solutions when $p = q$.

Yet even fewer authors study systems involving Hardy-potential and critical nonlinearities. Motivated by the above works, this paper discusses the fractional p & q -Laplacian system with Hardy-Sobolev exponents, by some techniques to establish new estimates to overcome difficulties. We prove the existence of infinitely many solutions by using Ljusternik-Schnirelmann theory. This result extend some results in the literature for the fractional p & q -Laplacian problem.

Before stating our main result, we introduce some notations. Let

$$[u]_{s,p} = \left(\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{\frac{1}{p}}$$

be the Galiardo seminorm of a measurable function $u: \mathbb{R}^N \rightarrow \mathbb{R}$, and let

$$W^{s,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N): [u]_{s,p} < \infty\}$$

be the fractional Sobolev space endowed with the norm

$$\|u\|_{s,p} = (|u|_p^p + [u]_{s,p}^p)^{\frac{1}{p}},$$

where $|\cdot|_p$ is the norm in $L^p(\mathbb{R}^N)$. We define

$$X_p^s(\Omega) = \{u \in W^{s,p}(\mathbb{R}^N): u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}$$

equivalently renormed by setting $\|\cdot\|_{s,p} = [\cdot]_{s,p}$, which is a uniformly convex Banach space. We work in the closed linear subspaces $W_p = X_p^s(\Omega) \times X_p^s(\Omega)$ and $W_q = X_q^s(\Omega) \times X_q^s(\Omega)$, which are reflexive Banach spaces endowed with the norms

$$\|(u, v)\|_p = (\|u\|_{s,p}^p + \|v\|_{s,p}^p)^{\frac{1}{p}} \text{ and } \|(u, v)\|_q = (\|u\|_{s,q}^q + \|v\|_{s,q}^q)^{\frac{1}{q}}. \quad (1.4)$$

Set $E = W_p \cap W_q$ endowed the norm $\|(u, v)\| = \|(u, v)\|_p + \|(u, v)\|_q$. Defining

$$\mathcal{A}_{p,s}(u, \phi) = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y)) (\phi(x) - \phi(y))}{|x - y|^{N+sp}} dx dy. \quad (1.5)$$

We say that $(u, v) \in E$ is a weak solutions of problem (1.1), if $\forall (\phi, \psi) \in E$, the following holds

$$\begin{aligned} \mathcal{A}_{p,s}(u, \phi) + \mathcal{A}_{q,s}(u, \phi) + \mathcal{A}_{p,s}(v, \psi) + \mathcal{A}_{q,s}(v, \psi) &= \lambda \int_{\Omega} |u|^{r-2} u \phi dx \\ &+ \mu \int_{\Omega} |v|^{r-2} v \psi dx + \frac{2\alpha}{p_{s,\theta}^*} \int_{\Omega} \frac{|u|^{\alpha-2} u |v|^{\beta} \phi}{|x|^{\theta}} dx + \frac{2\beta}{p_{s,\theta}^*} \int_{\Omega} \frac{|u|^{\alpha} |v|^{\beta-2} v \psi}{|x|^{\theta}} dx. \end{aligned}$$

The corresponding energy functional of system (1.1) is defined by

$$\begin{aligned} J_{\lambda,\mu}(u, v) &= \frac{1}{p} \|(u, v)\|_p^p + \frac{1}{q} \|(u, v)\|_q^q \\ &- \frac{1}{r} \left(\lambda \int_{\Omega} |u|^r dx + \mu \int_{\Omega} |v|^r dx \right) - \frac{2}{p_{s,\theta}^*} \int_{\Omega} \frac{|u|^{\alpha} |v|^{\beta}}{|x|^{\theta}} dx, \end{aligned} \quad (1.6)$$

for $(u, v) \in E$, it is easy to know that $J_{\lambda,\mu}$ is even, $J_{\lambda,\mu} \in C^1(E, \mathbb{R})$ and

$$\begin{aligned} \langle J'_{\lambda,\mu}(u, v), (u, v) \rangle &= \|(u, v)\|_p^p + \|(u, v)\|_q^q \\ &- \left(\lambda \int_{\Omega} |u|^r dx + \mu \int_{\Omega} |v|^r dx \right) - 2 \int_{\Omega} \frac{|u|^{\alpha} |v|^{\beta}}{|x|^{\theta}} dx. \end{aligned} \quad (1.7)$$

Our result can be stated as follows.

Theorem 1. *There exists $\Lambda_* > 0$ such that for each $0 < \lambda^{\frac{p}{p-r}} + \mu^{\frac{p}{p-r}} < \Lambda_*$, the system (1.1) has infinitely many solutions with negative energy.*

Remark 1. The system in [4] is a special case of system (1.1). In [12], when $\theta = 0$, $s = 1$, Li and Yang proved that problem (1.3) has at least $\text{cat}(\Omega) + 1$ distinct positive solutions by Lusternik-Schnirelmann category under condition (A):

$$(A) : N > p, \quad 1 < r < q < \frac{N(p-1)}{N-1} < p < p^* = \frac{Np}{N-p}.$$

The paper is organized as follows. In Section 2, we show that the $(PS)_c$ condition holds for the related energy functional in certain critical levels. In Section 3, we prove Theorem 1.

2. THE $(PS)_c$ CONDITION FOR $J_{\lambda,\mu}$

In this section, we show that the Palais-Smale condition, $(PS)_c$ holds for the related energy functional in certain critical levels.

Definition 1. Let $c \in \mathbb{R}$, E is a Banach space and $J_{\lambda,\mu} \in C^1(E, \mathbb{R})$. We say that $\{(u_k, v_k)\}$ is a $(PS)_c$ sequence for $J_{\lambda,\mu}$ in E if $J_{\lambda,\mu}(u_k, v_k) = c + o(1)$ and $J'_{\lambda,\mu}(u_k, v_k) = o(1)$ strongly in E^* (the dual space of the Sobolev space E) as $k \rightarrow \infty$. We say that $J_{\lambda,\mu}$ satisfies the $(PS)_c$ condition if any $(PS)_c$ sequence $\{(u_k, v_k)\}$ for $J_{\lambda,\mu}$ in E has a convergent subsequence.

Let

$$S_{s,\theta} = \inf_{u \in X_p^s(\Omega) \setminus \{0\}} \frac{\|u\|_{s,p}^p}{\left(\int_{\Omega} \frac{|u|^{p_{s,\theta}^*}}{|x|^{\theta}}\right)^{\frac{p}{p_{s,\theta}^*}}},$$

which is positive by the fractional Hardy-Sobolev constant of $X_p^s \hookrightarrow L^{p_{s,\theta}^*}(\mathbb{R}^N)$ and independent of Ω . In order to simplify calculation, set

$$V_{\theta}(\Omega) = \int_{\Omega} |x|^{\frac{r\theta}{p_{s,\theta}^* - r}} dx.$$

We need the following Lemmas.

Lemma 1. *If $\{(u_k, v_k)\} \subset E$ is a $(PS)_c$ sequence for $J_{\lambda,\mu}$, then $\{(u_k, v_k)\}$ is bounded in E .*

Proof. Let $\{(u_k, v_k)\} \subset E$ is a $(PS)_c$ sequence for $J_{\lambda,\mu}$ satisfying

$$J_{\lambda,\mu}(u_k, v_k) = c + o(1) \text{ and } J'_{\lambda,\mu}(u_k, v_k) = o(1) \text{ in } E^*.$$

From (1.6) and (1.7), we obtain

$$\begin{aligned} J_{\lambda,\mu}(u_k, v_k) - \frac{1}{p_{s,\theta}^*} \langle J'_{\lambda,\mu}(u_k, v_k), (u_k, v_k) \rangle &= \left(\frac{1}{p} - \frac{1}{p_{s,\theta}^*}\right) \|(u_k, v_k)\|_p^p \\ &\quad + \left(\frac{1}{q} - \frac{1}{p_{s,\theta}^*}\right) \|(u_k, v_k)\|_q^q \\ &\quad - \left(\frac{1}{r} - \frac{1}{p_{s,\theta}^*}\right) \int_{\Omega} (\lambda |u_k|^r + \mu |v_k|^r) dx. \end{aligned} \quad (2.1)$$

From the definition of $S_{s,\theta}$, Hölder and Sobolev inequalities, we conclude that

$$\begin{aligned} \int_{\Omega} (\lambda |u_k|^r + \mu |v_k|^r) dx &\leq \lambda V_{\theta}(\Omega)^{\frac{p_{s,\theta}^* - r}{p_{s,\theta}^*}} \left(\int_{\Omega} \frac{|u_k|^{p_{s,\theta}^*}}{|x|^{\theta}} dx \right)^{\frac{r}{p_{s,\theta}^*}} \\ &\quad + \mu V_{\theta}(\Omega)^{\frac{p_{s,\theta}^* - r}{p_{s,\theta}^*}} \left(\int_{\Omega} \frac{|v_k|^{p_{s,\theta}^*}}{|x|^{\theta}} dx \right)^{\frac{r}{p_{s,\theta}^*}} \end{aligned}$$

$$\begin{aligned}
 &\leq V_{\theta}(\Omega)^{\frac{p_{s,\theta}^* - r}{p_{s,\theta}^*}} S_{s,\theta}^{-\frac{r}{p}} (\lambda \|u_k\|_p^r + \mu \|v_k\|_p^r) \\
 &\leq V_{\theta}(\Omega)^{\frac{p_{s,\theta}^* - r}{p_{s,\theta}^*}} S_{s,\theta}^{-\frac{r}{p}} (\lambda^{\frac{p}{p-r}} + \mu^{\frac{p}{p-r}})^{\frac{p-r}{p}} \|(u_k, v_k)\|_p^r \\
 &\leq V_{\theta}(\Omega)^{\frac{p_{s,\theta}^* - r}{p_{s,\theta}^*}} S_{s,\theta}^{-\frac{r}{p}} (\lambda^{\frac{p}{p-r}} + \mu^{\frac{p}{p-r}})^{\frac{p-r}{p}} \|(u_k, v_k)\|^r. \quad (2.2)
 \end{aligned}$$

From (2.1) and (2.2), we deduce that

$$C(1 + \|(u_k, v_k)\| + \|(u_k, v_k)\|^r) \geq \left(\frac{1}{p} - \frac{1}{p_{s,\theta}^*}\right) \|(u_k, v_k)\|_p^p + \left(\frac{1}{q} - \frac{1}{p_{s,\theta}^*}\right) \|(u_k, v_k)\|_q^q. \quad (2.3)$$

We suppose the contrary, we may assume that $\|(u_k, v_k)\| \rightarrow \infty$ as $k \rightarrow \infty$, we need to consider the following three cases:

Case 1: If $\|(u_k, v_k)\|_p \rightarrow \infty$ and $\|(u_k, v_k)\|_q \rightarrow \infty$. For k large enough, we obtain $\|(u_k, v_k)\|_p \geq 1$ and $\|(u_k, v_k)\|_p^p \geq \|(u_k, v_k)\|_q^q$. Indeed, using the inequality $(a+b)^q \leq C_q(a^q + b^q)$ and (2.3), we deduce that

$$\begin{aligned}
 C(1 + \|(u_k, v_k)\| + \|(u_k, v_k)\|^r) &\geq \min\left\{\frac{1}{p} - \frac{1}{p_{s,\theta}^*}, \frac{1}{q} - \frac{1}{p_{s,\theta}^*}\right\} (\|(u_k, v_k)\|_p^q + \|(u_k, v_k)\|_q^q) \\
 &\geq \left(\frac{1}{p} - \frac{1}{p_{s,\theta}^*}\right) C_q^{-1} (\|(u_k, v_k)\|_p + \|(u_k, v_k)\|_q)^q \\
 &= \left(\frac{1}{p} - \frac{1}{p_{s,\theta}^*}\right) C_q^{-1} \|(u_k, v_k)\|^q,
 \end{aligned}$$

both sides are divided by $\|(u_k, v_k)\|^r$, we have

$$C\left(\frac{1}{\|(u_k, v_k)\|^r} + \frac{1}{\|(u_k, v_k)\|^{r-1}} + 1\right) \geq \left(\frac{1}{p} - \frac{1}{p_{s,\theta}^*}\right) C_q^{-1} \|(u_k, v_k)\|^{q-r},$$

we get $C \geq \infty$, as $k \rightarrow \infty$, this is impossible.

Case 2: If $\|(u_k, v_k)\|_p$ is bounded and $\|(u_k, v_k)\|_q \rightarrow \infty$. By (2.1) and (2.2), we obtain

$$C(1 + \|(u_k, v_k)\|_p + \|(u_k, v_k)\|_q + \|(u_k, v_k)\|^r) \geq \left(\frac{1}{q} - \frac{1}{p_{s,\theta}^*}\right) \|(u_k, v_k)\|_q^q,$$

both sides are divided by $\|(u_k, v_k)\|_q^q$, we conclude that

$$C\left(\frac{1}{\|(u_k, v_k)\|_q^q} + \frac{\|(u_k, v_k)\|_p}{\|(u_k, v_k)\|_q^q} + \frac{1}{\|(u_k, v_k)\|_q^{q-1}} + \frac{\|(u_k, v_k)\|_p^r}{\|(u_k, v_k)\|_q^q}\right) \geq \frac{1}{q} - \frac{1}{p_{s,\theta}^*},$$

we get $0 \geq \frac{1}{q} - \frac{1}{p_{s,\theta}^*} > 0$, as $k \rightarrow \infty$, this is a contradiction.

Case 3: If $\|(u_k, v_k)\|_p \rightarrow \infty$ and $\|(u_k, v_k)\|_q$ is bounded. Similar to Case 2.

From Cases 1-3, we can conclude that $\{(u_k, v_k)\}$ is bounded in E . $\square$

Lemma 2. *If $\{(u_k, v_k)\}$ is a $(PS)_c$ sequence for $J_{\lambda, \mu}$ with $(u_k, v_k) \rightharpoonup (u, v)$ in E , then $J'_{\lambda, \mu}(u, v) = 0$, and there exists a positive constant C_0 such that*

$$J_{\lambda, \mu}(u, v) \geq -C_0(\lambda^{\frac{p}{p-r}} + \mu^{\frac{p}{p-r}}).$$

Proof. If $\{(u_k, v_k)\}$ is a $(PS)_c$ sequence for $J_{\lambda, \mu}$ with $(u_k, v_k) \rightharpoonup (u, v)$ in E , then

$$J'_{\lambda, \mu}(u_k, v_k) = o(1) \text{ strongly in } E^* \text{ as } k \rightarrow \infty.$$

Let $(\phi, \psi) \in E$, we have

$$\begin{aligned} \langle J'_{\lambda, \mu}(u_k, v_k) - J'_{\lambda, \mu}(u, v), (\phi, \psi) \rangle &= \mathcal{A}_{p,s}(u_k, \phi) - \mathcal{A}_{p,s}(u, \phi) + \mathcal{A}_{q,s}(u_k, \phi) - \mathcal{A}_{q,s}(u, \phi) \\ &\quad + \mathcal{A}_{p,s}(v_k, \psi) - \mathcal{A}_{p,s}(v, \psi) + \mathcal{A}_{q,s}(v_k, \psi) - \mathcal{A}_{q,s}(v, \psi) \\ &\quad - \lambda \int_{\Omega} (|u_k|^{r-2} u_k - |u|^{r-2} u) \phi dx \\ &\quad - \mu \int_{\Omega} (|v_k|^{r-2} v_k - |v|^{r-2} v) \psi dx \\ &\quad - \frac{2\alpha}{p_{s,\theta}^*} \int_{\Omega} \left(\frac{|u_k|^{\alpha-2} u_k |v_k|^{\beta}}{|x|^{\theta}} - \frac{|u|^{\alpha-2} u |v|^{\beta}}{|x|^{\theta}} \right) \phi dx \\ &\quad - \frac{2\beta}{p_{s,\theta}^*} \int_{\Omega} \left(\frac{|u_k|^{\alpha} |v_k|^{\beta-2} v_k}{|x|^{\theta}} - \frac{|u|^{\alpha} |v|^{\beta-2} v}{|x|^{\theta}} \right) \psi dx, \end{aligned}$$

where $\mathcal{A}_{p,s}$ is defined in (1.5). Since $\{(u_k, v_k)\}$ is bounded in E , up to subsequence, this implies the following:

$$\begin{aligned} u_k &\rightharpoonup u \text{ in } X_p^s(\Omega), \\ u_k &\rightarrow u \text{ a.e. in } \Omega, \\ u_k &\rightarrow u \text{ in } L^r(\Omega), \quad 1 \leq r < p_{s,\theta}^* \end{aligned} \tag{2.4}$$

and

$$\frac{|u_k(x) - u_k(y)|^{p-2} (u_k(x) - u_k(y))}{|x - y|^{\frac{N+sp}{p'}}} \text{ is bounded in } L^{p'}(\mathbb{R}^N),$$

therefore,

$$\frac{|u_k(x) - u_k(y)|^{p-2} (u_k(x) - u_k(y))}{|x - y|^{\frac{N+sp}{p'}}} \rightharpoonup \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))}{|x - y|^{\frac{N+sp}{p'}}} \text{ in } L^{p'}(\mathbb{R}^N),$$

where $\frac{1}{p} + \frac{1}{p'} = 1$, and $\frac{\phi(x) - \phi(y)}{|x - y|^{\frac{N+sp}{p'}}} \in L^p(\mathbb{R}^N)$, so

$$\mathcal{A}_{p,s}(u_k, \phi) \rightarrow \mathcal{A}_{p,s}(u, \phi), \text{ as } k \rightarrow \infty. \tag{2.5}$$

Similarly, we obtain

$$\mathcal{A}_{q,s}(u_k, \phi) \rightarrow \mathcal{A}_{q,s}(u, \phi), \text{ as } k \rightarrow \infty. \tag{2.6}$$

Similar to (2.4), we have

$$\begin{aligned} v_k &\rightharpoonup v \text{ in } X_p^s(\Omega), \\ v_k &\rightarrow v \text{ a.e. in } \Omega, \\ v_k &\rightarrow v \text{ in } L^r(\Omega), \quad 1 \leq r < p_{s,\theta}^* \end{aligned} \quad (2.7)$$

and

$$\frac{|v_k(x) - v_k(y)|^{p-2}(v_k(x) - v_k(y))}{|x - y|^{\frac{N+sp}{p'}}} \text{ is bounded in } L^{p'}(\mathbb{R}^N),$$

it follows that

$$\frac{|v_k(x) - v_k(y)|^{p-2}(v_k(x) - v_k(y))}{|x - y|^{\frac{N+sp}{p'}}} \rightharpoonup \frac{|v(x) - v(y)|^{p-2}(v(x) - v(y))}{|x - y|^{\frac{N+sp}{p'}}} \text{ in } L^{p'}(\mathbb{R}^N)$$

and

$$\frac{\Psi(x) - \Psi(y)}{|x - y|^{\frac{N+sp}{p}}} \in L^p(\mathbb{R}^N),$$

so

$$\mathcal{A}_{p,s}(v_k, \Psi) \rightarrow \mathcal{A}_{p,s}(v, \Psi), \text{ as } k \rightarrow \infty. \quad (2.8)$$

Similarly, we obtain

$$\mathcal{A}_{q,s}(v_k, \Psi) \rightarrow \mathcal{A}_{q,s}(v, \Psi), \text{ as } k \rightarrow \infty. \quad (2.9)$$

Moreover, by (2.4) and (2.7), we conclude that

$$\begin{aligned} |u_k|^{r-2}u_k &\rightharpoonup |u|^{r-2}u, \quad |v_k|^{r-2}v_k \rightharpoonup |v|^{r-2}v \text{ in } L^{r'}(\Omega), \\ \frac{|u_k|^{\alpha-2}u_k|v_k|^\beta}{|x|^\theta} &\rightharpoonup \frac{|u|^{\alpha-2}u|v|^\beta}{|x|^\theta}, \quad \frac{|u_k|^\alpha|v_k|^{\beta-2}v_k}{|x|^\theta} \rightharpoonup \frac{|u|^\alpha|v|^{\beta-2}v}{|x|^\theta} \text{ in } L^{\frac{p_{s,\theta}^*}{p_{s,\theta}^*-1}}(\Omega), \end{aligned}$$

thus,

$$\begin{aligned} \int_{\Omega} (|u_k|^{r-2}u_k|u|^{r-2}u)\phi dx &\rightarrow 0, \\ \int_{\Omega} (|v_k|^{r-2}v_k|v|^{r-2}v)\psi dx &\rightarrow 0, \text{ as } k \rightarrow \infty, \end{aligned} \quad (2.10)$$

and

$$\begin{aligned} \int_{\Omega} \left(\frac{|u_k|^{\alpha-2}u_k|v_k|^\beta}{|x|^\theta} - \frac{|u|^{\alpha-2}u|v|^\beta}{|x|^\theta} \right) \phi dx &\rightarrow 0, \\ \int_{\Omega} \left(\frac{|u_k|^\alpha|v_k|^{\beta-2}v_k}{|x|^\theta} - \frac{|u|^\alpha|v|^{\beta-2}v}{|x|^\theta} \right) \psi dx &\rightarrow 0, \text{ as } k \rightarrow \infty. \end{aligned} \quad (2.11)$$

From (2.5)-(2.6) and (2.8)-(2.11), we deduce that

$$\langle J'_{\lambda,\mu}(u_k, v_k) - J'_{\lambda,\mu}(u, v), (\phi, \Psi) \rangle \rightarrow 0 \text{ for all } (\phi, \Psi) \in E,$$

furthermore, we obtain $\langle J'_{\lambda,\mu}(u, v), (u, v) \rangle = 0$, it implies that

$$2 \int_{\Omega} \frac{|u|^{\alpha}|v|^{\beta}}{|x|^{\theta}} dx = \|(u, v)\|_p^p + \|(u, v)\|_q^q - \int_{\Omega} (\lambda|u|^r + \mu|v|^r) dx,$$

therefore,

$$\begin{aligned} J_{\lambda,\mu}(u, v) &= \left(\frac{1}{p} - \frac{1}{p_{s,\theta}^*}\right) \|(u, v)\|_p^p + \left(\frac{1}{q} - \frac{1}{p_{s,\theta}^*}\right) \|(u, v)\|_q^q \\ &\quad - \left(\frac{1}{r} - \frac{1}{p_{s,\theta}^*}\right) \int_{\Omega} (\lambda|u|^r + \mu|v|^r) dx \\ &\geq \frac{ps - \theta}{(N - \theta)p} \|(u, v)\|_p^p - \left(\frac{1}{r} - \frac{1}{p_{s,\theta}^*}\right) \int_{\Omega} (\lambda|u|^r + \mu|v|^r) dx. \end{aligned} \quad (2.12)$$

By Sobolev embedding, Hölder and Young inequalities, we deduce that

$$\begin{aligned} \lambda \int_{\Omega} |u|^r dx &\leq V_{\theta}(\Omega)^{\frac{p_{s,\theta}^* - r}{p_{s,\theta}^*}} S_{s,\theta}^{-\frac{r}{p}} \lambda \|u\|_p^r \\ &\leq \frac{ps - \theta}{(N - \theta)p} \left(\frac{1}{r} - \frac{1}{p_{s,\theta}^*}\right)^{-1} \|u\|_p^p \\ &\quad + \frac{p - r}{p} \left[\frac{p}{r} \frac{ps - \theta}{(N - \theta)p} \left(\frac{1}{r} - \frac{1}{p_{s,\theta}^*}\right)^{-1} \right]^{-\frac{r}{p-r}} \cdot V_{\theta}(\Omega)^{\frac{p(p_{s,\theta}^* - r)}{p_{s,\theta}^*(p-r)}} S_{s,\theta}^{-\frac{r}{p-r}} \lambda^{\frac{p}{p-r}}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \mu \int_{\Omega} |v|^r dx &\leq \frac{ps - \theta}{(N - \theta)p} \left(\frac{1}{r} - \frac{1}{p_{s,\theta}^*}\right)^{-1} \|v\|_p^p \\ &\quad + \frac{p - r}{p} \left[\frac{p}{r} \frac{ps - \theta}{(N - \theta)p} \left(\frac{1}{r} - \frac{1}{p_{s,\theta}^*}\right)^{-1} \right]^{-\frac{r}{p-r}} \cdot V_{\theta}(\Omega)^{\frac{p(p_{s,\theta}^* - r)}{p_{s,\theta}^*(p-r)}} S_{s,\theta}^{-\frac{r}{p-r}} \mu^{\frac{p}{p-r}}, \end{aligned}$$

so,

$$\begin{aligned} \int_{\Omega} (\lambda|u|^r + \mu|v|^r) dx &\leq \frac{ps - \theta}{(N - \theta)p} \left(\frac{1}{r} - \frac{1}{p_{s,\theta}^*}\right)^{-1} \|(u, v)\|_p^p \\ &\quad + \frac{p - r}{p} \left[\frac{p}{r} \frac{ps - \theta}{(N - \theta)p} \left(\frac{1}{r} - \frac{1}{p_{s,\theta}^*}\right)^{-1} \right]^{-\frac{r}{p-r}} \\ &\quad \cdot V_{\theta}(\Omega)^{\frac{p(p_{s,\theta}^* - r)}{p_{s,\theta}^*(p-r)}} S_{s,\theta}^{-\frac{r}{p-r}} (\lambda^{\frac{p}{p-r}} + \mu^{\frac{p}{p-r}}) \end{aligned}$$

$$= \frac{ps - \theta}{(N - \theta)p} \left(\frac{1}{r} - \frac{1}{p_{s,\theta}^*} \right)^{-1} \|(u, v)\|_p^p + \tilde{C}(\lambda^{\frac{p}{p-r}} + \mu^{\frac{p}{p-r}}),$$

where $\tilde{C} = \frac{p-r}{p} \left[\frac{p}{r} \frac{ps-\theta}{(N-\theta)p} \left(\frac{1}{r} - \frac{1}{p_{s,\theta}^*} \right)^{-1} \right]^{-\frac{r}{p-r}} V_\theta(\Omega)^{\frac{p(p_{s,\theta}^*-r)}{p_{s,\theta}^*(p-r)}} S_{s,\theta}^{-\frac{r}{p-r}}$. By (2.12), we conclude that

$$J_{\lambda,\mu}(u, v) \geq - \left(\frac{1}{r} - \frac{1}{p_{s,\theta}^*} \right) \tilde{C}(\lambda^{\frac{p}{p-r}} + \mu^{\frac{p}{p-r}}),$$

it follows that, let $C_0 = \left(\frac{1}{r} - \frac{1}{p_{s,\theta}^*} \right) \tilde{C}$, we can get

$$J_{\lambda,\mu}(u, v) \geq -C_0(\lambda^{\frac{p}{p-r}} + \mu^{\frac{p}{p-r}}).$$

□

Lemma 3. $J_{\lambda,\mu}$ satisfies the $(PS)_c$ condition with

$$c < \frac{2(ps - \theta)}{(N - \theta)p} \left(\frac{S_{\alpha,\beta}}{2} \right)^{\frac{N-\theta}{ps-\theta}} - C_0(\lambda^{\frac{p}{p-r}} + \mu^{\frac{p}{p-r}}),$$

where $S_{\alpha,\beta} = \inf_{(u,v) \in E \setminus \{(0,0)\}} \frac{\|(u,v)\|_p^p}{\left(\int_\Omega \frac{|u|^\alpha |v|^\beta}{|x|^\theta} dx \right)^{\frac{p}{p_{s,\theta}^*}}}$.

Proof. Let $\{(u_k, v_k)\}$ be a $(PS)_c$ sequence for $J_{\lambda,\mu}$ in E , these hold that

$$\begin{aligned} c + o(1) &= \frac{1}{p} \|(u_k, v_k)\|_p^p + \frac{1}{q} \|(u_k, v_k)\|_q^q \\ &\quad - \frac{1}{r} \int_\Omega (\lambda |u_k|^r + |v_k|^r) dx - \frac{2}{p_{s,\theta}^*} \int_\Omega \frac{|u_k|^\alpha |v_k|^\beta}{|x|^\theta} dx \end{aligned}$$

and

$$\begin{aligned} o(1) \|(u_k, v_k)\| &= \|(u_k, v_k)\|_p^p + \|(u_k, v_k)\|_q^q \\ &\quad - \int_\Omega (\lambda |u_k|^r + |v_k|^r) dx - 2 \int_\Omega \frac{|u_k|^\alpha |v_k|^\beta}{|x|^\theta} dx. \end{aligned} \quad (2.13)$$

By Lemma 1, $\{(u_k, v_k)\}$ is bounded in E , then up to a subsequence, we have $(u_k, v_k) \rightharpoonup (u, v)$ in E and we deduce from Lemma 2 that $J'_{\lambda,\mu}(u, v) = 0$, we will show that $(u_k, v_k) \rightarrow (u, v)$ in E . Since $u_k \rightarrow u$, $v_k \rightarrow v$ strongly in $L^r(\Omega)$, so

$$\int_\Omega (\lambda |u_k|^r + |v_k|^r) dx \rightarrow \int_\Omega (\lambda |u|^r + \mu |v|^r) dx, \text{ as } k \rightarrow \infty.$$

Applying Brezis-Lieb Lemma [25, Lemma 1.32], it follows that

$$\begin{aligned} \|(u_k, v_k)\|_p^p &= \|(u_k - u, v_k - v)\|_p^p + \|(u, v)\|_p^p + o(1), \\ \|(u_k, v_k)\|_q^q &= \|(u_k - u, v_k - v)\|_q^q + \|(u, v)\|_q^q + o(1) \end{aligned} \quad (2.14)$$

and

$$\int_{\Omega} \frac{|u_k|^\alpha |v_k|^\beta}{|x|^\theta} dx = \int_{\Omega} \frac{|u_k - u|^\alpha |v_k - v|^\beta}{|x|^\theta} dx + \int_{\Omega} \frac{|u|^\alpha |v|^\beta}{|x|^\theta} dx + o(1). \quad (2.15)$$

From (2.14)-(2.15), we conclude that

$$\begin{aligned} c - J_{\lambda, \mu}(u, v) + o(1) &= \frac{1}{p} \|(u_k - u, v_k - v)\|_p^p + \frac{1}{q} \|(u_k - u, v_k - v)\|_q^q \\ &\quad - \frac{2}{p_{s, \theta}^*} \int_{\Omega} \frac{|u_k - u|^\alpha |v_k - v|^\beta}{|x|^\theta} dx. \end{aligned} \quad (2.16)$$

By (2.13), we obtain

$$\|(u_k - u, v_k - v)\|_p^p + \|(u_k - u, v_k - v)\|_q^q = 2 \int_{\Omega} \frac{|u_k - u|^\alpha |v_k - v|^\beta}{|x|^\theta} dx + o(1). \quad (2.17)$$

Therefore, we may assume that

$$\begin{aligned} \|(u_k - u, v_k - v)\|_p^p &\rightarrow d, \\ \|(u_k - u, v_k - v)\|_q^q &\rightarrow l, \\ 2 \int_{\Omega} \frac{|u_k - u|^\alpha |v_k - v|^\beta}{|x|^\theta} dx &\rightarrow m, \text{ as } k \rightarrow \infty. \end{aligned}$$

Moreover, $d, l \geq 0$ and $d + l = m$. If $d = 0$, then $(u_k, v_k) \rightarrow (u, v)$, the proof is done.

Suppose that $d > 0$, by the definition of $S_{\alpha, \beta}$, we have $d \leq m \leq 2S_{\alpha, \beta}^{\frac{p_{s, \theta}^*}{p}} d^{\frac{p_{s, \theta}^*}{p}}$, or,

$$d \geq 2 \left(\frac{S_{\alpha, \beta}}{2} \right)^{\frac{N-\theta}{ps-\theta}}.$$

From (2.16), we deduce that

$$\begin{aligned} c &= \frac{d}{p} + \frac{l}{q} - \frac{m}{p_{s, \theta}^*} + J_{\lambda, \mu}(u, v) \\ &= \left(\frac{1}{p} - \frac{1}{p_{s, \theta}^*} \right) d + \left(\frac{1}{q} - \frac{1}{p_{s, \theta}^*} \right) l + J_{\lambda, \mu}(u, v) \\ &\geq \frac{2(ps - \theta)}{(N - \theta)p} \left(\frac{S_{\alpha, \beta}}{2} \right)^{\frac{N-\theta}{ps-\theta}} - C_0(\lambda^{\frac{p}{p-r}} + \mu^{\frac{p}{p-r}}), \end{aligned}$$

this is a contradiction with the definition of c . □

3. THE PROOF OF OUR MAIN THEOREM

Given the function $J_{\lambda, \mu}$ defined by (1.6) and (2.2), we obtain that

$$J_{\lambda, \mu}(u, v) \geq \frac{1}{p} \|(u, v)\|_p^p - \frac{2}{p_{s, \theta}^*} \int_{\Omega} \frac{|u|^\alpha |v|^\beta}{|x|^\theta} dx - \frac{1}{r} \int_{\Omega} (\lambda |u|^r dx + \mu |v|^r dx)$$

$$\begin{aligned}
 &\geq \frac{1}{p} \|(u, v)\|_p^p - \frac{2}{p_{s,\theta}^*} S_{\alpha,\beta}^{-\frac{p_{s,\theta}^*}{p}} \|(u, v)\|_p^{p_{s,\theta}^*} \\
 &\quad - \frac{1}{r} C_{\theta,\Omega} (\lambda^{\frac{p}{p-r}} + \mu^{\frac{p}{p-r}})^{\frac{p-r}{p}} \|(u, v)\|_p^r \\
 &= c_1 \|(u, v)\|_p^p - c_2 \|(u, v)\|_p^{p_{s,\theta}^*} - c_3 (\lambda^{\frac{p}{p-r}} + \mu^{\frac{p}{p-r}})^{\frac{p-r}{p}} \|(u, v)\|_p^r,
 \end{aligned}$$

where $c_1 = \frac{1}{p}$, $c_2 = \frac{2}{p_{s,\theta}^*} S_{\alpha,\beta}^{-\frac{p_{s,\theta}^*}{p}}$, $c_3 = \frac{1}{r} C_{\theta,\Omega}$ with $C_{\theta,\Omega} = V_\theta(\Omega)^{\frac{p_{s,\theta}^*}{p}} S_{s,\theta}^{-\frac{r}{p}}$. If we define for $t \geq 0$, $h(t) := c_1 t^p - c_2 t^{p_{s,\theta}^*} - c_3 (\lambda^{\frac{p}{p-r}} + \mu^{\frac{p}{p-r}})^{\frac{p-r}{p}} t^r$, then

$$J_{\lambda,\mu}(u, v) \geq h(\|(u, v)\|_p). \quad (3.1)$$

It is easy to see that there exists Λ_* such that for any $0 < \lambda^{\frac{p}{p-r}} + \mu^{\frac{p}{p-r}} < \Lambda_*$, we have:

- (1) h has exactly two distinct positive zero points denoted by R_0 and R_1 ;
- (2) h attains nonnegative maximum at R and it verifies $R_0 < R < R_1$;
- (3) $\frac{2(ps-\theta)}{(N-\theta)p} \left(\frac{S_{\alpha,\beta}}{2} \right)^{\frac{N-\theta}{ps-\theta}} - C_0 (\lambda^{\frac{p}{p-r}} + \mu^{\frac{p}{p-r}}) \geq 0$, where C_0 is given in Lemma 2.

From the structure of $h(t)$, we see that there are constants $0 < R_0 < R_1$, such that $h(R_0) = h(R_1) = 0$ and

$$\begin{aligned}
 h(t) &\leq 0 & \text{if } t &\leq R_0, \\
 h(t) &> 0 & \text{if } R_0 < t < R_1, \\
 h(t) &< 0 & \text{if } t > R_1.
 \end{aligned}$$

We now introduce the following truncation of the functional $J_{\lambda,\mu}$. Taking the nonincreasing function $\tau: \mathbb{R}^+ \rightarrow [0, 1]$ and $C^\infty(\mathbb{R}^+)$ such that $\tau(t) = 1$ if $t \leq R_0$; $\tau(t) = 0$ if $t \geq R_1$. Let $\varphi(u, v) = \tau(\|(u, v)\|_p)$, consider the truncated functional

$$\begin{aligned}
 I_{\lambda,\mu}(u, v) &= \frac{1}{p} \|(u, v)\|_p^p + \frac{1}{q} \|(u, v)\|_q^q - \frac{1}{r} \int_\Omega (\lambda |u|^r dx + \mu |v|^r dx) \\
 &\quad - \frac{2}{p_{s,\theta}^*} \varphi(u, v) \int_\Omega \frac{|u|^\alpha |v|^\beta}{|x|^\theta} dx.
 \end{aligned} \quad (3.2)$$

Similarly to (3.1), we have

$$I_{\lambda,\mu}(u, v) \geq \tilde{h}(\|(u, v)\|_p), \quad (3.3)$$

where $\tilde{h}(t) := c_1 t^p - c_2 t^{p_{s,\theta}^*} \tau(t) - c_3 (\lambda^{\frac{p}{p-r}} + \mu^{\frac{p}{p-r}})^{\frac{p-r}{p}} t^r$.

By the definition of $h(t)$ and $0 \leq \tau(t) \leq 1$ for $t \geq 0$, we obtain $\tilde{h}(t) \geq h(t)$ if $t \geq 0$. It follows from $\tau(t) = 1$ if $0 \leq t \leq R_0$ that $\tilde{h}(t) = h(t)$ if $0 \leq t \leq R_0$. From $h(R_0) = h(R_1) = 0$ and $h(t) > 0$ if $R_0 < t < R_1$, we deduce that $\tilde{h}(t) \geq 0$ if $R_0 < t \leq R_1$.

Moreover, $\tilde{h}(t) = t^r(c_1 t^{p-r} - c_3(\lambda^{\frac{p}{p-r}} + \mu^{\frac{p}{p-r}})^{\frac{p-r}{p}})$ is strictly increasing if $t > R_1$, then $\tilde{h}(t) > 0$ if $t > R_1$. Consequently

$$\tilde{h}(t) \geq 0 \text{ for } t \geq R_0. \quad (3.4)$$

Lemma 4. *We have the following results:*

- (1) $I_{\lambda,\mu} \in C^1(E, \mathbb{R})$.
- (2) If $I_{\lambda,\mu}(u, v) \leq 0$, then $\|(u, v)\|_p \leq R_0$. Moreover, $I_{\lambda,\mu}(\tilde{u}, \tilde{v}) = J_{\lambda,\mu}(\tilde{u}, \tilde{v})$ for all $(\tilde{u}, \tilde{v})$ in a small enough neighborhood of (u, v) .
- (3) There exists $\Lambda_* > 0$ such that if $0 < \lambda^{\frac{p}{p-r}} + \mu^{\frac{p}{p-r}} < \Lambda_*$, then $I_{\lambda,\mu}$ satisfies a local $(PS)_c$ condition for $c < 0$.

Proof. Since $\varphi \in C^\infty$ and $\varphi(u, v) = 1$ for (u, v) near $(0, 0)$, $I_{\lambda,\mu} \in C^1(E, \mathbb{R})$ and assertion (1) holds.

By taking $I_{\lambda,\mu}(u, v) \leq 0$, we can deduce from (3.3) that

$$\tilde{h}(\|(u, v)\|_p) \leq 0$$

and by (3.4), we have

$$\|(u, v)\|_p \leq R_0$$

implying (2).

For the proof of (3), let $\{(u_k, v_k)\} \subset E$ be a $(PS)_c$ sequence of $I_{\lambda,\mu}$ with $c < 0$. Then we may assume that $I_{\lambda,\mu}(u_k, v_k) < 0$, $I'_{\lambda,\mu}(u_k, v_k) \rightarrow 0$. By (2), there exists $\Lambda_* > 0$ such that $0 < \lambda^{\frac{p}{p-r}} + \mu^{\frac{p}{p-r}} < \Lambda_*$, $\|(u_k, v_k)\|_p \leq R_0$, so $I_{\lambda,\mu}(u_k, v_k) = J_{\lambda,\mu}(u_k, v_k)$ and $I'_{\lambda,\mu}(u_k, v_k) = J'_{\lambda,\mu}(u_k, v_k)$. By Lemma 3, $J_{\lambda,\mu}$ satisfies the $(PS)_c$ condition for $c < 0$, thus $I_{\lambda,\mu}$ satisfies the $(PS)_c$ condition for $c < 0$, this completes the proof. $\square$

It is possible to prove the existence of level sets of $I_{\lambda,\mu}$ with arbitrarily large genus. Now, we use the idea in [7] to construct negative critical value of $I_{\lambda,\mu}$ via genus, more precisely:

Lemma 5. $\forall k \in \mathbb{N}, \exists \varepsilon = \varepsilon(k) > 0$ such that

$$\gamma(\{(u, v) \in E : I_{\lambda,\mu} \leq -\varepsilon(k)\}) \geq k.$$

Proof. Let $k \in \mathbb{N}$. We consider E_k be a k -dimensional subspaces of E . Let $(u, v) \in E_k$ with norm $\|(u, v)\|_p = 1$. For $0 < \rho < R_0$, we have

$$\begin{aligned} J_{\lambda,\mu}(\rho u, \rho v) &= I_{\lambda,\mu}(\rho u, \rho v) \\ &= \frac{1}{p} \rho^p + \frac{\rho^q}{q} \|(u, v)\|_q^q - \frac{\rho^r}{r} \int_{\Omega} (\lambda |u|^r + \mu |v|^r) dx \\ &\quad - \frac{2\rho^{p_{s,\theta}^*}}{p_{s,\theta}^*} \Phi(u, v) \int_{\Omega} \frac{|u|^\alpha |v|^\beta}{|x|^\theta} dx. \end{aligned}$$

Since E_k is a space of finite dimension, all the norms in E_k are equivalent. If we define

$$\alpha_k := \sup\{\|(u, v)\|_q^q : (u, v) \in E_k, \|(u, v)\|_p = 1\} < \infty$$

and

$$\beta_k := \inf\left\{\int_{\Omega} |u|^r dx : (u, v) \in E_k, \|(u, v)\|_p = 1\right\} > 0.$$

Then we have

$$I_{\lambda, \mu}(\rho u, \rho v) \leq \frac{1}{p} \rho^p + \frac{\rho^q}{q} \alpha_k - \beta_k \min\{\lambda, \mu\} \frac{\rho^r}{r}.$$

Then, there exist $\varepsilon(k) > 0$ and $0 < \rho < R_0$ such that $I_{\lambda, \mu}(\rho u, \rho v) \leq -\varepsilon(k)$ for $(u, v) \in E_k$ with $\|(u, v)\|_p = 1$. Let $S_\rho = \{(u, v) \in E : \|(u, v)\|_p = \rho\}$, so

$$S_\rho \cap E_k \subset \{(u, v) \in E : I_{\lambda, \mu}(u, v) \leq -\varepsilon(k)\},$$

therefore, by the property of genus in [17] and the fact $\gamma(S_\rho \cap E_k) = k$, it implies that

$$\gamma(\{(u, v) \in E : I_{\lambda, \mu}(u, v) \leq -\varepsilon(k)\}) \geq \gamma(S_\rho \cap E_k) = k.$$

□

Now, we prove our main result.

Proof of Theorem 1. For $k \in \mathbb{N}$, set

$$\Gamma_k = \{A \subset E \setminus \{(0, 0)\} : A \text{ is closed}, A = -A, \gamma(A) \geq k\},$$

where $\gamma(A)$ is the genus of A . Let us set

$$c_k = \inf_{A \in \Gamma_k} \sup_{(u, v) \in A} I_{\lambda, \mu}(u, v),$$

and

$$K_c = \{(u, v) \in E : I_{\lambda, \mu}(u, v) = c, I'_{\lambda, \mu}(u, v) = 0\}.$$

Suppose $0 < \lambda^{\frac{p}{p-r}} + \mu^{\frac{p}{p-r}} < \Lambda_*$, where Λ_* is the constant given by Lemma 4. In fact, if we denote $I_{\lambda, \mu}^{-\varepsilon} = \{(u, v) \in E : I_{\lambda, \mu}(u, v) \leq -\varepsilon\}$, by Lemma 5, there exists $\varepsilon(k) > 0$ such that $\gamma(I_{\lambda, \mu}^{-\varepsilon(k)}) \geq k$ for $k \in \mathbb{N}$. Because $I_{\lambda, \mu}$ is continuous and even, $I_{\lambda, \mu}^{-\varepsilon(k)} \in \Gamma_k$, then $c_k \leq -\varepsilon(k) < 0$ for all $k \in \mathbb{N}$. But $I_{\lambda, \mu}$ is bounded from below, hence $c_k > -\infty$ for all $k \in \mathbb{N}$.

Let us assume that $c = c_k = c_{k+1} = \dots = c_{k+l}$. Note that $c < 0$, therefore, $I_{\lambda, \mu}$ satisfies the $(PS)_c$ condition, and it is easy to see that K_c is a compact set.

If $\gamma(K_c) \leq l$, then there is a closed and symmetric set U with $K_c \subset U$ and $\gamma(U) \leq l$ by the continuity property of genus [17]. By [1, Lemma 1.3], there is an odd homeomorphism $\eta : E \rightarrow E$ such that $\eta(I_{\lambda, \mu}^{c+\delta} - U) \subset I_{\lambda, \mu}^{c-\delta}$ for some $\delta > 0$. By definition,

$$c = c_k = c_{k+l} = \inf_{A \in \Gamma_{k+l}} \sup_{(u, v) \in A} I_{\lambda, \mu}(u, v),$$

there exists $A \in \Gamma_{k+l}$ such that $\sup_{(u,v) \in A} I_{\lambda,\mu}(u,v) < c + \delta$, i.e. $A \subset I_{\lambda,\mu}^{c+\delta}$ and $\eta(A - U) \subset \eta(I_{\lambda,\mu}^{c+\delta} - U) \subset I_{\lambda,\mu}^{c-\delta}$, that is

$$\sup_{(u,v) \in \eta(A-U)} I_{\lambda,\mu}(u,v) \leq c - \delta. \quad (3.5)$$

But $\gamma(\overline{A-U}) \geq \gamma(A) - \gamma(U) \geq k$, and $\gamma(\eta(\overline{A-U})) = \gamma(\overline{A-U}) \geq k$, then $\eta(\overline{A-U}) \in \Gamma_k$. This is a contradiction. In fact, $\eta(\overline{A-U}) \in \Gamma_k$ implies that

$$\sup_{(u,v) \in \eta(\overline{A-U})} I_{\lambda,\mu}(u,v) \geq c_k = c,$$

which contradicts to (3.5). So we have proved that $\gamma(K_c) \geq l + 1$.

We are now ready to show that $I_{\lambda,\mu}$ has infinitely many critical points. Note c_k is nondecreasing and strictly negative. We distinguish two cases:

Case 1: Suppose that there are $1 < k_1 < \cdots < k_i < \cdots$, satisfying

$$c_{k_1} < \cdots < c_{k_i} < \cdots,$$

then $\gamma(K_c) \geq 1$, and we see that $\{c_{k_i}\}$ is a sequence of distinct critical values of $I_{\lambda,\mu}$.

Case 2: We assume in this case that, for some positive integer k_0 , there is $l \geq 1$ such that $c = c_{k_0} = c_{k_0+1} = \cdots = c_{k_0+l}$, then $\gamma(K_{c_{k_0}}) \geq l + 1$, which shows that $K_{c_{k_0}}$ contains infinitely many distinct elements.

Since $J_{\lambda,\mu}(u,v) = I_{\lambda,\mu}(u,v)$ if $I_{\lambda,\mu}(u,v) < 0$, we see that there are infinitely many critical points of $J_{\lambda,\mu}(u,v)$, that is to say, there are negative energy solutions to problem (1.1). This completes the proof of Theorem 1. $\square$

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