

HU e-ISSN 1787-2413

Miskolc Mathematical Notes

A Publication of the University of Miskolc

VOLUME 23 (2022), NUMBER 1



MISKOLC UNIVERSITY PRESS



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CONTENTS

<i>Abdullah Açikel and Nurettin Irmak</i>	
Common terms of Tribonacci and Perrin sequences	5
<i>Doria Affane and Loubna Boulkemm</i>	
First order sweeping process with subsmooth sets	13
<i>Sajan Aggarwal, Izhar Uddin, and Juan J. Nieto</i>	
Existence and convergence of fixed point for a G-non-Lipschitzian mapping	29
<i>Akbar Ali, Kinkar Ch. Das, and Sohail Akhter</i>	
On the Extremal Graphs for Second Zagreb Index with Fixed Number of Vertices and Cyclomatic Number	41
<i>Mohammad Hossein Alizadeh and Judit Makó</i>	
On e -convexity	51
<i>Ryszard R. Andruszkiewicz and Alicja Smoktunowicz</i>	
The Yang-Baxter matrix equation for involutions	61
<i>Mohammad Arif, Mohammad Imdad, and Aftab Alam</i>	
Fixed point theorems under locally T -transitive binary relations employing Matkowski contractions	71
<i>Souad Ayadi and Ozgur Ege</i>	
Existence of solution for a Dirichlet boundary value problem involving the $p(x)$ Laplacian via a fixed point approach	85
<i>Songul Batik and Fulya Yoruk Deren</i>	
Semipositone Fractional Boundary Value Problems with n Point Fractional Integral Boundary Conditions	93
<i>Omar Bazighifan and Shyam Sundar Santra</i>	



CONTENTS

Second-order differential equations: Asymptotic behavior of the solutions	105
<i>T. Bouali, R. Guefaïfa, A. Choucha, S. Boulaaras, and M. Abdalla</i> The Existence of One Non-Trivial Weak Solution of Generalized Yamabe Equations	117
<i>S. I. Butt, T. Rasheed, Đ. Pečarić, and J. Pečarić</i> Measure Theoretic Generalizations of Jensen's Inequality by Fink's Identity	131
<i>Ivan Chajda and Helmut Länger</i> Kleene posets and pseudo-Kleene posets	155
<i>S. Tumurbat and T. Khulan and D. Dayantsolmon</i> A note on radicals of associative rings and alternative rings	175
<i>Alexandru-Darius Filip</i> Some applications of Maia's fixed point theorem in the case of operators with Volterra property with respect to a subinterval	183
<i>Özlem Ak Gümüş, Qianqian Cui, George Maria Selvam, and Abraham Vianny</i> Global stability and bifurcation analysis of a discrete time SIR epidemic model	193
<i>Arezki Kheloufi and Abderrahmane Ikassoulene</i> L^p –Regularity Results for $2m$ -th Order Parabolic Equations in Time-Varying Domains	211
<i>Nazlı Irkil and E. Pişkin</i> Local Existence and Blow up for p -Laplacian Equation with Logarithmic Nonlinearity	231



CONTENTS

Ilkay Yaslan Karaca and Sezgi Aksoy

Positive Solutions for Second Order Impulsive Differential Equations with
Integral Boundary Conditions on an Infinite Interval 253

O. Kohutych

About one characteristic initial value problem with prehistory and its in-
vestigation 271

Cahit Köme

Some combinatorial identities via k -order Fibonacci matrices 281

Sebastián Lalvay, Adrián Padilla-Segarra, and Walid Zouhair

On the existence and uniqueness of solutions for non-autonomous semi-
linear systems with non-instantaneous impulses, delay, and non-local
conditions 295

M. A. Latif, S. Hussain, and Madeeha

Weighted Simpson's type integral inequalities for harmonically-preinvex
functions 311

Xiuwen Li and Biao Zeng

General history-dependent operators with applications to differential equa-
tions 327

Xiaoting Lu, Yongjian Liu, Aimin Liu, and Chunsheng Feng

New Geometric Viewpoints to Chen Chaotic System 339

M. A. Mersin, D. Irk, and M. Zorsahin Gorgulu

Galerkin methods for the numerical solution of the Schrödinger equation
by using trigonometric B-splines 363

Celil Nebiyev and Hasan Hüseyin Ökten

Cofinitely Essential g -Supplemented Modules 381



CONTENTS

Ismail Nikoufar

Lipschitz Stability of n -Cubic Functional Equations and Applications . . . 389

Sofian Obeidat, M. A. Latif, and S. S. Dragomir

Fejér and Hermite-Hadamard type inequalities for differentiable h -Convex
and quasi convex Functions With Applications 401

Mehmet Zeki Sarıkaya, Hatice Öğülmüş, and Fatma Ertuğral

On (p, ϕ_h) -convex function and its applications 417

Benedek Schultz and Jenő Szirmai

Geodesic ball packings generated by regular prism tilings in **NIL** geometry 429

H. M. Srivastava, Bidu Bhusan Jena, and Susanta Kumar Paikray

Deferred Cesàro Statistical Convergence of Martingale Sequence and Korovkin-
Type Approximation Theorems 443

Saba Sultan, Martin Bača, Ali Ahmad, and Muhammad Imran

Minimal doubly resolving sets of antiprism graphs and Möbius ladders . . 457

E. Taşdemir, M. Göcen, and Y. Soykan

Global Dynamical Behaviours and Periodicity of a Certain Quadratic-Rational
Difference Equation with Delay 471

Guo-Cheng Wu, Chuan-Yun Gu, Lan-Lan Huang, and Dumitru Baleanu

Fractional differential equations of variable order: existence results, nu-
merical method and asymptotic stability conditions 485

Xiaohui Zhang and Zhixia Xing

Bounds for the generalized elliptic integral of the second kind 495

Zonghong Xiong and Wei Wei



Miskolc Mathematical Notes

VOLUME 23 (2022), NUMBER 1

CONTENTS

Optimal control problem governed by weakly coupled system on induction heating	505
<i>Fatih Yılmaz and Meltem Sertbaş</i>	
Singular Degenerate Normal Operators for First-order	521



COMMON TERMS OF TRIBONACCI AND PERRIN SEQUENCES

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Received 20 December, 2020

Abstract. Assume that T_n is the n^{th} term of Tribonacci sequence and R_m is the m^{th} term of Perrin sequence. In this paper, we solve the equation $T_n = R_m$ completely.

2010 Mathematics Subject Classification: 11B39; 11J86

Keywords: Tribonacci number, Perrin number, Baker method

1. INTRODUCTION

The question of finding common terms of two sequences has an extensive literature in number theory. One of the results of this topic was given by Mignotte who (see [8]) showed that if $(u_n)_{n \geq 0}$ and $(v_n)_{n \geq 0}$ are two linearly recurrence sequences then, under some weak technical assumptions, the equation $u_n = v_m$ has only finitely many solutions in positive integers m, n . In [9], authors surveyed the equation $Au_x + Bu_y + Cu_z = 0$ for fixed $A, B, C \in \mathbb{Z}$. Szalay [12] gave an algorithm to compute the intersection of the binary sequences with arbitrary initial conditions while Alekseyev [1] described how to compute the intersection of two sequences U_n and V_n that satisfy the recurrence $Z_n = PZ_{n-1} \pm Z_{n-2}$ (Z_n is U_n or V_n) with initial conditions $U_0 = 0, U_1 = 0, V_0 = 2$ and $V_1 = P$. Both of these papers depend on two homogeneous quadratic equations. Marques [6] proved the conjecture about the k -generalized Fibonacci sequence $\{F^{(k)}\}$ (defined by the initial values $0, 0, \dots, 0, 1$ (k terms) and such that each term afterwards is the sum of the k preceding terms) and showed that the non-trivial solutions of the equation $F_n^{(l)} = F_m^{(k)}$ satisfying $l > k > 1, n > l + 1, m > k + 1$ are $(m, n, l, k) = (7, 6, 3, 2)$ and $(12, 11, 7, 3)$. These papers give a motivation us to solve the equation

$$T_n = R_m,$$

where T_n is n^{th} Tribonacci number and R_m is m^{th} Perrin number. The difference of our problem is that the recurrence relations of the sequences are not the same although the orders of the sequences are same. Now, we present the recurrence relations of Tribonacci and Perrin sequences.

Tribonacci sequence $\{T_n\}$ is defined by the recurrence relation

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n$$

with $T_0 = 0, T_1 = T_2 = 1$. The characteristic equation $x^3 - x^2 - x - 1 = 0$ of Tribonacci sequence has one real root α and two conjugate complex roots β and γ . In 1982, Spickerman [11] found the Binet formula of Tribonacci numbers as

$$T_n = \frac{\alpha^{n+1}}{(\alpha - \beta)(\alpha - \gamma)} + \frac{\beta^{n+1}}{(\beta - \alpha)(\beta - \gamma)} + \frac{\gamma^{n+1}}{(\gamma - \alpha)(\gamma - \beta)}, \quad \text{for all } n \geq 0.$$

Afterwards, Dresden and Du [3] presented a Binet-style formula that can be used to produce the k -generalized Fibonacci numbers. The case $k = 3$ corresponds to Binet formula of Tribonacci sequence as follows.

$$T_n = \frac{(\alpha - 1)\alpha^{n-1}}{2 + 4(\alpha - 2)} + \frac{(\beta - 1)\beta^{n-1}}{2 + 4(\beta - 2)} + \frac{(\gamma - 1)\gamma^{n-1}}{2 + 4(\gamma - 2)}, \quad \text{for all } n \geq 0.$$

In fact, these two Binet formulas are same. The difference is the representation. Namely, while each fraction of Dresden and Du's Binet formula contains only one of the roots of the characteristic equation, each fraction in the Binet formula of Spickerman contains all of the roots of the characteristic equation. Moreover, Dresden and Du [3] proved that the terms of Tribonacci sequence can also be written as

$$T_n = c_3 \alpha^{n-1} + e_n, \quad \text{with } |e_n| < \frac{1}{\alpha^{1/2}}, \quad \text{for all } n \geq 1 \quad (1.1)$$

where $c_3 = (\alpha - 1)/(4\alpha - 6) \cong 0.61$.

Perrin sequence $\{R_m\}$ is defined by

$$R_m = R_{m-2} + R_{m-3}$$

with $R_0 = 3, R_1 = 0$ and $R_2 = 2$.

Perrin numbers were studied by several authors in the beginning of the nineteenth century (for details, see [10]). The characteristic equation $x^3 - x - 1 = 0$ of the Perrin sequence has distinct roots a, b , and c where $a \approx 1.3247 \dots$ (called a plastic constant) is a unique real root. This constant was first defined in 1924 by Gérard Cordonnier. He described applications to architecture in 1958, he gave a lecture tour that illustrated the use of the plastic constant in many buildings and monuments. The Binet formula of Perrin sequence is

$$R_m = a^m + b^m + c^m, \quad \text{for all } m \geq 0. \quad (1.2)$$

In this paper, we investigate the common terms of the Tribonacci and Perrin sequences. Our result is following.

Theorem 1. *The solutions exceed 1 of the equation $T_n = R_m$ are $T_3 = R_2 = R_4 = 2$, and $T_5 = R_7 = 7$.*

2. PRELIMINARY RESULTS

Before going further, we give several notations that we will use. Let γ be an algebraic number of degree d over $\mathbb{Q}$ with minimal primitive polynomial, where the leading coefficient a_0 is positive.

The logarithmic height of γ is given by

$$h(\gamma) := \frac{1}{d} \left(\log a_0 + \sum_{i=1}^d \log(\max \{ |\gamma^i|, 1 \}) \right). \quad (2.1)$$

Here γ^i are all conjugates of γ number.

Now, we give the upper and lower bounds for the terms of Tribonacci and Perrin numbers.

Lemma 1. *For $n \geq 2$, the inequality*

$$a^{n-1} < R_n < a^{n+1}$$

and

$$\alpha^{n-2} \leq T_n < \alpha^{n-1}$$

hold.

Proof. The proof of the first inequality is in [5] (see Lemma 2.8) and second one is in [2] (the case $k = 3$). $\square$

We use a lower bound for a linear form in logarithms à la Baker and such a bound was given by the following result of Matveev (see [7]).

Lemma 2. *Let $\gamma_1, \gamma_2, \gamma_3, \dots, \gamma_t$ be positive real algebraic numbers in a real algebraic number field $\mathbb{K}$ of degree D , further $b_1, b_2, b_3, \dots, b_t$ be non-zero integers, and assume that*

$$\Lambda := \gamma_1^{b_1} \gamma_2^{b_2} \gamma_3^{b_3} \dots \gamma_t^{b_t} - 1 \quad (2.2)$$

is non-zero. Then

$$\Lambda > \exp \left(-1.4 \cdot 30^{t+3} t^{4.5} D^2 (1 + \log D) (1 + \log B) A_1 A_2 A_3 \dots A_t \right), \quad (2.3)$$

where

$$B \geq \max \{ |b_1|, |b_2|, |b_3|, \dots, |b_t| \} \quad (2.4)$$

and

$$A_i \geq \max \{ Dh(\gamma_i), |\log \gamma_i|, 0.16 \} \quad (2.5)$$

for all $i = 1, \dots, t$.

The following lemma was given by Pethő and Dujella [4] that we will use later.

Lemma 3. *Let γ be an irrational number, M be a positive integer and p/q be a convergent of the continued fraction of γ such that $q > 6M$. Let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\varepsilon := \|\mu q\| - M \|\gamma q\|$, where $\|\cdot\|$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there is no solution of the inequality*

$$0 < |m\gamma - n + \mu| < AB^{-m}$$

in positive integers m, n with $m \leq M$ and $m \geq \log(Aq/\varepsilon)/\log B$.

3. PROOF OF THEOREM 1

Firstly, assume that $m \leq 7$. In this case, we see that $T_3 = R_2 = 2$, $T_3 = R_4 = 2$ and $T_5 = R_7 = 7$ are the solutions exceeds 1 in the equation $T_n = R_m$. From now on, we can assume that $m \geq 8$. If we put Binet formula of Perrin sequence and the equation (1.1) in the equation $T_n = R_m$, then we have

$$c_3 \alpha^{n-1} - a^m = b^m + c^m - e_n.$$

After dividing the above equation with a^m

$$\frac{c_3 \alpha^{n-1}}{a^m} - 1 = \frac{b^m + c^m - e_n}{a^m},$$

then

$$|c_3 \alpha^{n-1} a^{-m} - 1| = \left| \frac{b^m + c^m - e_n}{a^m} \right| \leq \left| \frac{b^m}{a^m} \right| + \left| \frac{c^m}{a^m} \right| + \left| \frac{e_n}{a^m} \right| < \frac{3}{a^m}$$

follows. Let

$$\Lambda := c_3 \alpha^{n-1} a^{-m} - 1. \quad (3.1)$$

To show $\Lambda \neq 0$, we contrary assume $\Lambda = 0$. It yields that $\tau = a^m \alpha^{1-n} = a^m (1/\alpha)^{n-1}$. It is obvious that a and α are algebraic integers. $1/\alpha$ is also algebraic integer since $1/\alpha$ is the root of the monic polynomial $x^3 + x^2 + x - 1$. Since product of algebraic integers is also algebraic integer, then it requires that $\tau = a^m (1/\alpha)^{n-1}$ is algebraic integer. But this is a contradiction. Because the minimal polynomial $44x^3 - 44x^2 + 12x - 1$ of c_3 is not monic. So, c_3 is not algebraic integer, but algebraic number. We arrive the claimed fact $\Lambda \neq 0$.

From now on, we suppose that

$$\begin{aligned} \gamma_1 &= c_3, & b_1 &= 1, & D &= 3, \\ \gamma_2 &= \alpha, & b_2 &= n-1, & t &= 3, \\ \gamma_3 &= a, & b_3 &= -m. \end{aligned}$$

Since the leading coefficient a_0 is 44 of the minimal polynomial of c_3 , then

$$\begin{aligned} h(c_3) &= \frac{1}{3} \{ \log 44 + \log(\max\{c_3, 1\}) \} = \frac{1}{3} \log 44 \cong 1.26, \\ h(\alpha) &= \frac{1}{3} \{ \log 1 + \log \alpha + \log 1 + \log 1 \} = \frac{1}{3} \log \alpha \cong 0.201, \end{aligned}$$

$$h(a) = \frac{1}{3} \{ \log 1 + \log a + \log 1 + \log 1 \} = \frac{1}{3} \log a \cong 0.092.$$

only hold. We choose A_1, A_2 and A_3 as follows.

$$\begin{aligned} A_1 &\geq \max \{3h(c_3), \log c_3, 0, 16\} \cong 3.78, & A_1 &= 4, \\ A_2 &\geq \max \{3h(\alpha), \log \alpha, 0, 16\} \cong 0.604, & A_2 &= 0.61, \\ A_3 &\geq \max \{3h(a), \log a, 0, 16\} \cong 0.281, & A_3 &= 0.29. \end{aligned}$$

Now, we determine the value of B such that

$$B \geq \{1, |n-1|, |-m|\}.$$

It is obvious that $T_4 > R_4$, $T_5 > R_5$ and $T_6 > R_6$. The recursive relation of the two sequence yields that $T_n > R_n$. If $T_n = R_m$, then $n < m$ holds. So, we choose $B = m$.

If we write all the values into the inequality (2.3), then we have

$$\frac{3}{a^m} > \Lambda > \exp(-1.4 \cdot 30^6 3^{4.5} 3^2 (1 + \log 3)(1 + \log m) \cdot 3.79 \cdot 0.61 \cdot 0.29).$$

By taking logarithm it leads to

$$1.81 \cdot 10^{12}(1 + \log m) > m \log a.$$

After using the fact $1 + \log m < 2 \log m$, we can write

$$3.8 \cdot 10^{12} \log m > m \log a.$$

Finally, we obtain

$$m < 1.36 \cdot 10^{13} \log m,$$

which yields that

$$m < 4.6 \cdot 10^{14}.$$

From now on, we will reduce the upper bound on m by using the result of Pethő and Dujella.

Let $M = 4.6 \cdot 10^{14}$ and $\Gamma := \log c_3 + (n-1) \log \alpha - m \log a$. It is known that $|x| \leq |e^x - 1|$ holds for all reals x . So,

$$|\Gamma| < |e^\Gamma - 1| < \frac{3}{a^m} < \frac{1}{2}$$

follows. In this case, we have

$$|\Gamma| < \frac{6}{a^m}.$$

The fact $n < m$ gives that

$$(n-1) \log \alpha - m \log a + \log c_3 < 6a^{-(n-1)}.$$

After dividing both sides with $\log a$,

$$(n-1) \frac{\log \alpha}{\log a} - m + \frac{\log c_3}{\log a} < \frac{6}{\log a} a^{-(n-1)}$$

holds. Let

$$\gamma = \frac{\log \alpha}{\log a} \quad \mu = \frac{\log c_3}{\log a} \quad A = \frac{6}{\log a} \quad B = a,$$

and q_n be the denominator of the n^{th} convergent of the continued fraction of γ . We found

$$q_{24} = 4220424136597687 > 6M$$

and then $\varepsilon := \|\mu q_{24}\| - M \|\gamma q_{24}\| \cong 0.42266 \dots$. We note that the conditions to apply Lemma 3 are fulfilled for $A = \frac{6}{\log a}$ and $B = a$. According to Lemma 3, there is no solution m in the range

$$\left[\frac{\log(Aq_{24}/\varepsilon)}{\log a} + 1, M \right).$$

Thus $m < 142$.

Finally, we use a program written in Mathematica to find the solution of the equation $T_n = R_m$ with $m < 142$, but there is no solution satisfies $m \geq 8$. So, the proof is completed.

ACKNOWLEDGEMENT

The authors express their gratitude to the anonymous reviewer for the instructive suggestions.

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FIRST ORDER SWEEPING PROCESS WITH SUBSMOOTH SETS

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Received 28 March, 2020

Abstract. In this paper, we introduce a perturbed first order non-convex sweeping process for a class of subsmooth moving sets depending on the time and the state. In the first result we study the existence of solution and the compactness of the attainable set, the perturbation considered here is an upper semi-continuous set-valued mapping with nonempty closed convex values unnecessarily bounded. In the second result, we prove the existence of solution for the autonomous problem under assumptions that do not require the convexity of the values of the perturbation and that weaken the assumption on the upper semi-continuity. Then, we deduce a solution of the time optimality problem and we describe the attainable set.

2010 *Mathematics Subject Classification:* 34A60; 49J53

Keywords: sweeping process, subsmooth, perturbation, normal cone, almost convex, attainable set, time optimal problem

1. INTRODUCTION

The perturbed state-dependent sweeping process is an evolution differential inclusion governed by the normal cone to a mobile set depending on both time and state variables, of the following form:

$$\begin{cases} -\dot{u}(t) \in N_{C(t,u(t))}(u(t)) + G(t, u(t)), & \text{a.e } t \in [T_0, T] \\ u(t) \in C(t, u(t)), \forall t \in [T_0, T], \quad u(T_0) = u_0, \end{cases} \quad (1.1)$$

where $N_{C(t,u(t))}(u(t))$ is the normal cone to $C(t, u(t))$ at $u(t)$ and G is a set-valued mapping playing the role of a perturbation to the problem, that is an external force applied on the system. This type of problems was initiated by J. J. Moreau (see [18]) for time-dependent sets $C(t)$ and $G \equiv \{0\}$. After, many generalizations have been obtained, see for example [1, 2, 5, 7–10, 19] and the references therein. When the moving set C depends also on the state, one obtain a generalization of the classical sweeping process known as the state-dependent sweeping process. This problem has been studied in [12, 13, 16] when $C(t, u(t))$ are convex or non-convex sets.

In the study of existence of solutions for differential inclusions, the use of convexity assumptions on the perturbation is widely known, it is the property required in order to pass to a weak limit along a sequence, be it a minimizing sequence or

a sequence of successive approximations, preserving the properties that are needed. Because of its generality, this approach need not always provide the best results, since it does not take into account possible additional information for example the presence of symmetries in the problem. In [11], a generalization of convexity has been defined, that is the almost convexity of sets, the authors have shown the existence of solution to the upper semi-continuous differential inclusions $\dot{u}(t) \in G(u(t))$, $u(0) = u_0$. This almost convexity condition has been used successfully by [2–4] to study the perturbed first order Moreau’s sweeping process, the right-hand side contains a set-valued perturbation with almost convex values.

In this work, we extend the results in [2] in many direction. In the setting of a finite dimensional space, we provide the existence of solution and the compactness of the attainable sets for the problem (1.1), when the moving sets $C(t, u)$ are equi-uniform-subsmooth and the perturbation G is a set-valued mapping with nonempty closed convex values, upper semi-continuous and such that its element of minimum norm satisfies a linear growth condition.

On the other hand, under the almost convexity of the values of the perturbation and weakening the assumption of the upper semi-continuity, we provide the existence of solution for the autonomous problem:

$$\begin{cases} -\dot{u}(t) \in N_{C(u(t))}(u(t)) + G(u(t)), & \text{a.e } t \in [T_0, T] \\ u(t) \in C(u(t)), \forall t \in [T_0, T], & u(T_0) = u_0 \in C(u_0). \end{cases} \quad (1.2)$$

And, we deduce the existence of solutions to the time optimal control problem

$$\dot{u}(t) \in -N_{C(u(t))}(u(t)) + h(u(t), z(t)), \text{ for } z(t) \in Z(t), \quad (1.3)$$

where the set $G(u(t)) = h(u(t), Z(u(t)))$ is compact and almost convex, then we use the minimal time function to describe the attainable set of this problem.

The paper is organized as follows. In Section 2, we introduce notation and preliminaries needed throughout the paper. Section 3 is devoted to the study of (1.1) when the perturbation is convex. In Section 4, we prove the existence of solutions of (1.2) under the almost convexity and the upper semi-continuity of $co(G)$. Then, we present an application to an optimal time problem.

2. PRELIMINARIES

Let $\mathbb{R}^n$ be a n -dimensional Euclidean space and $I = [T, T_0]$, $T > T_0 \geq 0$ be an interval. We denote by $\mathbb{B}$ the unit closed ball of $\mathbb{R}^n$, $B(x, r)$ the open ball of center $x \in \mathbb{R}^n$ and radius $r > 0$, $C_{\mathbb{R}^n}(I)$ the Banach space of all continuous mapping from I and $L^1_{\mathbb{R}^n}(I)$ the space of all Lebesgue integrable $\mathbb{R}^n$ -valued mappings defined on I . We say that the mapping $u : I \rightarrow \mathbb{R}^n$ is absolutely continuous if there is $f \in L^1_{\mathbb{R}^n}(I)$ such that $u(t) = u(T_0) + \int_{T_0}^t f(s) ds$, for all $t \in I$.

Let S be a nonempty closed subset of $\mathbb{R}^n$, $\mathbb{I}_S$ is the characteristic function of the set S , that is, $\mathbb{I}_S(x) = 1$ if $x \in S$ and $\mathbb{I}_S(x) = 0$ otherwise. We denote by $d_S(\cdot)$ or $d(\cdot, S)$ the

usual distance function associated with S , i.e., $d_S(x) = \inf_{y \in S} \|y - x\|$ and by $Proj_S(x)$ the projection of x on S defined by $Proj_S(x) = \{y \in S : d_S(x) = \|y - x\|\}$. The convex hull of S is denoted by $co(S)$ and the closed convex hull by $\overline{co}(S)$ is characterized by

$$\overline{co}(S) = \{x \in \mathbb{R}^n, \forall x' \in \mathbb{R}^n, \langle x', x \rangle \leq \delta^*(x', S)\},$$

where $\delta^*(x', S) = \sup_{y \in S} \langle x', y \rangle$ is the support function of S at $x' \in \mathbb{R}^n$. Let $\tilde{t} \in I$, we denoted by

$$Acc_{u_0}(\tilde{t}) = \{z \in \mathbb{R}^n : z = u(\tilde{t}) \text{ such that } u(\cdot) \in S_{\tilde{t}}(u_0)\},$$

the attainable set of (1.1) at the time $\tilde{t}$, where $S_{\tilde{t}}(u_0)$ is the set of the trajectories of (1.1) on the interval $[T_0, \tilde{t}]$ and by $\mathfrak{T} : Acc_{u_0} \rightarrow I$ the minimal time mapping defined by

$$\mathfrak{T}(z) = \inf\{t \in I : z \in Acc_{u_0}(t)\}.$$

For a locally Lipschitzian function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$, the Clarke subdifferential $\partial\varphi(x)$ of φ at x is defined by (see [15])

$$\partial\varphi(x) = \{\xi \in \mathbb{R}^n : \varphi^0(x, v) \geq \langle v, \xi \rangle, \text{ for all } v \in \mathbb{R}^n\},$$

where $\varphi^0(x, v) = \limsup_{\substack{y \rightarrow x \\ t \downarrow 0}} \frac{\varphi(y + tv) - \varphi(y)}{t}$ is the generalized directional derivative of

φ at x in the direction v . The Clarke normal cone $N_S(x)$ at $x \in S$ is defined from T_S^C by polarity, that is,

$$N_S(x) = \{\xi \in \mathbb{R}^n : \langle \xi, v \rangle \leq 0, \text{ for all } v \in T_S^C(x)\},$$

where $T_S^C(x)$ is the Clarke tangent cone at $x \in S$ given by

$$T_S^C(x) = \{v \in \mathbb{R}^n : d_S^0(x, v) = 0\},$$

where $d_S^0(x, v)$ is the generalized directional derivative of d_S at x in the direction v .

A vector $\xi \in \mathbb{R}^n$ is said to be in the Fréchet subdifferential of φ at x (see [15, 17]) denoted by $\partial^F \varphi(x)$, provided that for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\langle \xi, y - x \rangle \leq \varphi(y) - \varphi(x) + \varepsilon \|y - x\|, \text{ for all } y \in B(x, \delta).$$

We always have the inclusion $\partial^F \varphi(x) \subset \partial\varphi(x)$, for all $x \in S$. The Fréchet normal cone $N_S^F(x)$ at $x \in S$ is given by $N_S^F(x) = \partial^F \psi_S(x)$, where ψ_S is the indicator function of S , that is, $\psi_S(x) = 0$ if $x \in S$ and $\psi_S(x) = +\infty$, otherwise. So we have the inclusion $N_S^F(x) \subset N_S(x)$, for all $x \in S$. On the other hand, the Fréchet normal cone is also related ([17]) to the Fréchet subdifferential, since for all $x \in S$

$$\partial^F d_S(x) = N_S^F(x) \cap \mathbb{B}. \quad (2.1)$$

Another important property is that, whenever $y \in Proj_S(x)$, one has

$$x - y \in N_S^F(y) \quad \text{then} \quad x - y \in N_S(y). \quad (2.2)$$

Now, we introduce the definition of equi-uniform subsmoothness for a family of sets. We begin with some basic definitions from subsmoothness while referring the reader to [6].

Definition 1. Let S be a closed subset of $\mathbb{R}^n$. The set S is called subsmooth at $x_0 \in S$, if for every $\varepsilon > 0$ there exists $\delta > 0$, such that for all $x_1, x_2 \in B(x_0, \delta) \cap S$ and $\xi_i \in N_S(x_i) \cap \mathbb{B}$, $i \in \{1, 2\}$, one has

$$\langle \xi_1 - \xi_2, x_1 - x_2 \rangle \geq -\varepsilon \|x_1 - x_2\|. \quad (2.3)$$

The set S is called subsmooth if it is subsmooth at every $x_0 \in S$. We say that S is uniformly-subsmooth if for every $\varepsilon > 0$ there exists $\delta > 0$, such that (2.3) holds for all $x_1, x_2 \in S$ satisfying $\|x_1 - x_2\| < \delta$ and all $\xi_i \in N_S(x_i) \cap \mathbb{B}$, $i \in \{1, 2\}$.

The following subdifferential regularity of the distance function also holds true for subsmooth sets (see [6]).

Proposition 1. Let S be a closed set of $\mathbb{R}^n$. If S is subsmooth at $x \in S$, then

$$\partial d_S(x) = \partial^F d_S(x) \quad \text{and} \quad N_S(x) = N_S^F(x). \quad (2.4)$$

Now we introduce the definition of equi-uniformly-subsmoothness.

Definition 2. Let $(S(q))_{q \in Q}$ be a family of closed sets of $\mathbb{R}^n$ with parameter $q \in Q$. This family is called equi-uniformly-subsmooth, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for each $q \in Q$, the inequality (2.3) holds for all $x_1, x_2 \in S(q)$ satisfying $\|x_1 - x_2\| < \delta$ and all $\xi_i \in N_{S(q)}(x_i) \cap \mathbb{B}$, $i \in \{1, 2\}$.

For the proof of the next proposition, we refer the reader to [16].

Proposition 2. Let $\{S(t, x) : (t, x) \in I \times \mathbb{R}^n\}$ be a family of closed and nonempty sets of $\mathbb{R}^n$, which is equi-uniformly-subsmooth and let a real $\eta \geq 0$. Assume that there exist a real constant $L \geq 0$ and a continuous function $v : I \rightarrow \mathbb{R}$ such that, for any $x, x', y, y' \in \mathbb{R}^n$ and $s, t \in I$

$$|d_{S(t, x)}(y) - d_{S(s, x')}(y')| \leq \|y - y'\| + |v(t) - v(s)| + L\|x - x'\|.$$

Then,

- (i) for all $(t, x, y) \in \text{Gph } S$, we have $\eta \partial d_{S(t, x)}(y) \subset \eta \mathbb{B}$;
- (ii) for any sequence $(t_n, x_n)_n$ in $I \times \mathbb{R}^n$ converging to (t, x) , $(y_n)_n$ converging to $y \in S(t, x)$ with $y_n \in S(t_n, x_n)$ and any $\xi \in \mathbb{R}^n$, we have

$$\limsup_{n \rightarrow +\infty} \delta^* \left(\xi, \eta \partial d_{S(t_n, x_n)}(y_n) \right) \leq \delta^* \left(\xi, \eta \partial d_{S(t, x)}(y) \right).$$

Next we give the definition of the almost convex sets.

Definition 3 ([11]). For a vector space X , a set $Q \subset X$ is called almost convex if for every $\xi \in \text{co}(Q)$ there exist λ_1 and λ_2 , $0 \leq \lambda_1 \leq 1 \leq \lambda_2$, such that $\lambda_1 \xi \in Q$ and $\lambda_2 \xi \in Q$.

Trivially, any convex set is almost convex, since $Q = co(K)$. If K is a convex set not containing the origin, $Q = \partial K$ is almost convex, and if the convex set K contains the origin, one takes $Q = \{0\} \cup \partial K$.

The following lemma is a direct consequence of discrete Gronwall's inequality proved in [14] (by using the inequality $1 + b_k \leq \exp(b_k)$).

Lemma 1. *Let $A > 0$ and let (a_n) and (b_n) be two nonnegative sequences such that*

$$a_n \leq A + \sum_{k=0}^{n-1} b_k a_k, \quad \text{for all } n \in \mathbb{N}.$$

Then,

$$a_n \leq A \exp \left(\sum_{k=0}^{n-1} b_k \right).$$

3. SWEEPING PROCESS WITH CONVEX PERTURBATION

In this section, we prove the existence of solution for (1.1), where the perturbation is an upper semicontinuous set-valued mapping with nonempty closed convex values unnecessarily bounded and without any compactness assumptions.

Theorem 1. *Let $C : I \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued mapping with nonempty closed values such that:*

- ($\mathcal{H}_1$) *for all $(t, x) \in I \times \mathbb{R}^n$, the sets $C(t, x)$ are equi-uniformly-subsmooth;*
- ($\mathcal{H}_2$) *there are two constants $L_1 \geq 0$, $L_2 \in [0, 1[$ such that, for all $s, t \in I$, and any $x_1, x_2, y \in \mathbb{R}^n$*

$$|d_{C(t, x_1)}(y) - d_{C(s, x_2)}(y)| \leq L_1 |t - s| + L_2 \|x_1 - x_2\|.$$

And let $G : I \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued mapping with nonempty closed convex values, upper semi-continuous such that:

- ($\mathcal{H}_3$) *for some real $\alpha \geq 0$, $d(0, G(t, x)) \leq \alpha(1 + \|x\|)$, $\forall (t, x) \in I \times \mathbb{R}^n$.*

Then, for every $u_0 \in C(T_0, u_0)$,

- (1) *(1.1) admits an absolutely continuous solution $u : I \rightarrow \mathbb{R}^n$.*
- (2) *For $\tilde{t} \in I$ fixed the attainable set $Acc_{u_0}(\tilde{t})$ is compact on $\mathbb{R}^n$.*

Proof. (1) For each integer $n \geq 1$, we consider the following partition of I by $I_k^n = [t_k^n, t_{k+1}^n[, t_k^n = T_0 + ke_n, e_n = \frac{T - T_0}{n}, k \in \{0, 1, \dots, n-1\}$ and $I_n^n = \{t_n^n\} = \{T\}$. For all $(t, x) \in I \times \mathbb{R}^n$, we denote by $g(t, x)$ the element of minimal norm of the closed convex set $G(t, x)$, i.e., $g(t, x) = Proj_{G(t, x)}(0)$.

Step 1. We construct the sequence (x_k^n) as follows: set $x_0^n = u_0 \in C(t_0^n, x_0^n)$, as $C(t_1^n, x_0^n)$ is closed, we can choose

$$x_1^n \in Proj_{C(t_1^n, x_0^n)}(x_0^n + e_n g(t_0^n, x_0^n)),$$

according to (2.2), we get $x_1^n \in C(t_1^n, x_0^n)$, $x_0^n + e_n g(t_0^n, x_0^n) - x_1^n \in N_{C(t_1^n, x_0^n)}(x_1^n)$, and $\|g(t_0^n, x_0^n)\| \leq \alpha(1 + \|x_0^n\|)$. By $(\mathcal{H}_2)$ and the last inequalities, we obtain

$$\begin{aligned} \|x_1^n - x_0^n\| &\leq \|x_1^n - (x_0^n + e_n g(t_0^n, x_0^n))\| + \|e_n g(t_0^n, x_0^n)\| \\ &= d_{C(t_1^n, x_0^n)}(x_0^n + e_n g(t_0^n, x_0^n)) + \|e_n g(t_0^n, x_0^n)\| \\ &\leq \left| d_{C(t_1^n, x_0^n)}(x_0^n) - d_{C(t_0^n, x_0^n)}(x_0^n) \right| + 2e_n \|g(t_0^n, x_0^n)\| \\ &\leq L_1 e_n + 2\alpha e_n (1 + \|x_0^n\|). \end{aligned} \quad (3.1)$$

By induction, we define x_{k+1}^n satisfying

$$x_{k+1}^n \in C(t_{k+1}^n, x_k^n); \quad (3.2)$$

$$x_k^n + e_n g(t_k^n, x_k^n) - x_{k+1}^n \in N_{C(t_{k+1}^n, x_k^n)}(x_{k+1}^n). \quad (3.3)$$

Using $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$, we obtain

$$\begin{aligned} \|x_{k+1}^n - x_k^n\| &\leq \|x_{k+1}^n - (x_k^n + e_n g(t_k^n, x_k^n))\| + \|e_n g(t_k^n, x_k^n)\| \\ &= d_{C(t_{k+1}^n, x_k^n)}(x_k^n + e_n g(t_k^n, x_k^n)) + \|e_n g(t_k^n, x_k^n)\| \\ &\leq \left| d_{C(t_{k+1}^n, x_k^n)}(x_k^n) - d_{C(t_k^n, x_{k-1}^n)}(x_k^n) \right| + 2e_n \|g(t_k^n, x_k^n)\| \\ &\leq L_1 e_n + L_2 \|x_k^n - x_{k-1}^n\| + 2\alpha e_n (1 + \|x_k^n\|). \end{aligned}$$

Then, for $0 \leq k \leq n-1$, we have

$$\|x_{k+1}^n - x_k^n\| \leq (L_1 + 2\alpha)e_n \sum_{j=0}^k L_2^j + 2\alpha e_n \sum_{j=0}^k L_2^{k-j} \|x_j^n\|,$$

since $L_2 \in [0, 1[$, we get

$$\|x_{k+1}^n - x_k^n\| \leq \frac{L_1 + 2\alpha}{1 - L_2} e_n + 2\alpha e_n \sum_{j=0}^k L_2^{k-j} \|x_j^n\|. \quad (3.4)$$

On the other hand, we have

$$\begin{aligned} \|x_k^n - x_0^n\| &\leq \|x_k^n - x_{k-1}^n\| + \|x_{k-1}^n - x_{k-2}^n\| + \cdots + \|x_1^n - x_0^n\| \\ &\leq \frac{L_1 + 2\alpha}{1 - L_2} e_n + 2\alpha e_n \sum_{j=0}^{k-1} L_2^{k-j} \|x_j^n\| + \frac{L_1 + 2\alpha}{1 - L_2} e_n + 2\alpha e_n \sum_{j=0}^{k-2} L_2^{k-j} \|x_j^n\| \\ &\quad + \cdots + e_n (L_1 + 2\alpha) + 2\alpha e_n \|x_0^n\| \\ &\leq \frac{L_1 + 2\alpha}{1 - L_2} e_n (k-1) + 2\alpha e_n \|x_0^n\| \sum_{j=0}^{k-1} L_2^j + 2\alpha e_n \|x_1^n\| \sum_{j=0}^{k-1} L_2^j \end{aligned}$$

$$\begin{aligned}
& + 2\alpha e_n \|x_2^n\| \sum_{j=0}^{k-1} L_2^j + \cdots + 2\alpha e_n \|x_{k-1}^n\| \sum_{j=0}^{k-1} L_2^j \\
& \leq T \frac{L_1 + 2\alpha}{1 - L_2} + \frac{2\alpha T}{1 - L_2} \sum_{j=0}^{k-1} \|x_j^n\|.
\end{aligned}$$

So,

$$\|x_k^n\| \leq \|x_0^n\| + T \frac{L_1 + 2\alpha}{1 - L_2} + \frac{2\alpha T}{1 - L_2} \sum_{j=0}^{k-1} \|x_j^n\|.$$

By Lemma 1, for all $k \in \{0, 1, \dots, n-1\}$, we can write

$$\|x_k^n\| \leq \left(\|x_0^n\| + T \frac{L_1 + 2\alpha}{1 - L_2} \right) \exp \left(\frac{2\alpha T}{1 - L_2} \right) = \beta. \quad (3.5)$$

Step 2. Construction of sequence $(u_n(\cdot))_{n \geq 0}$.

For every $n \geq 1$ and for any $t \in I_k^n$ with $k \in \{0, 1, \dots, n-1\}$, we define

$$u_n(t) = \frac{t_{k+1}^n - t}{e_n} x_k^n + \frac{t - t_k^n}{e_n} x_{k+1}^n.$$

Thus, $u_n(t_k^n) = x_k^n$ and on $]t_k^n, t_{k+1}^n[$

$$\dot{u}_n(t) = \frac{x_{k+1}^n - x_k^n}{e_n}. \quad (3.6)$$

By (3.2) and (3.3) we obtain

$$u_n(t_{k+1}^n) \in C(t_{k+1}^n, u_n(t_k^n)) \quad (3.7)$$

$$\dot{u}_n(t) \in -N_{C(t_{k+1}^n, u_n(t_k^n))}(u_n(t_{k+1}^n)) + g(t_k^n, u_n(t_k^n)) \text{ a.e. } t \in I_k^n, \quad (3.8)$$

and by (3.4) and (3.5) we get

$$\frac{\|x_{k+1}^n - x_k^n\|}{e_n} \leq \frac{L_1 + 2\alpha}{1 - L_2} + 2\alpha \sum_{j=0}^k L_2^{k-j} \beta \leq \frac{1}{1 - L_2} (L_1 + 2\alpha + 2\alpha\beta). \quad (3.9)$$

The relations (3.6) and (3.9) give

$$\|\dot{u}_n(t)\| \leq \frac{1}{1 - L_2} (L_1 + 2\alpha + 2\alpha\beta) = \gamma. \quad (3.10)$$

For each $t \in I$ and $n \geq 1$, let define the function

$$\begin{aligned}
\delta_n(t) &= \begin{cases} t_k^n & \text{if } t \in I_k^n, \\ t_{n-1}^n & \text{if } t = T. \end{cases} \\
\theta_n(t) &= \begin{cases} t_{k+1}^n & \text{if } t \in I_k^n; \\ T & \text{if } t = T, \end{cases}
\end{aligned}$$

and we have

$$\lim_{n \rightarrow +\infty} |\delta_n(t) - t| = \lim_{n \rightarrow +\infty} |\theta_n(t) - t| = 0. \quad (3.11)$$

Furthermore, the definitions of $\delta_n(\cdot)$ and $\theta_n(\cdot)$ combined with (3.5), (3.7) and (3.8) yield

$$u_n(\theta_n(t)) \in C(\theta_n(t), u_n(\delta_n(t))), \quad \text{for all } t \in I, \quad (3.12)$$

$$\dot{u}_n(t) \in -N_{C(\theta_n(t), u_n(\delta_n(t)))}(u_n(\theta_n(t))) + g(\delta_n(t), u_n(\delta_n(t))) \quad \text{a.e. } t \in I, \quad (3.13)$$

$$\|g(\delta_n(t), u_n(\delta_n(t)))\| \leq \alpha(1 + \beta) = \eta \quad \text{for all } t \in I. \quad (3.14)$$

Step 3. Convergence of the sequences.

From (3.5) it follows that $(u_n(\theta_n(t)))$ is relatively compact and by (3.10) we have

$$\|u_n(\theta_n(t)) - u_n(t)\| \leq \int_t^{\theta_n(t)} \|\dot{u}_n(s)\| ds \leq \gamma(\theta_n(t) - t),$$

then

$$\lim_{n \rightarrow +\infty} \|u_n(\theta_n(t)) - u_n(t)\| = 0. \quad (3.15)$$

So, $(u_n(t))_{n \geq 1}$ is relatively compact for all $t \in I$, on the other hand $(u_n(\cdot))_{n \geq 1}$ is equi-continuous according to (3.10). Using Ascoli-Arzelà's theorem $(u_n)_{n \geq 1}$ is relatively compact in $C_{\mathbb{R}^n}(I)$, so we can extract a subsequence of $(u_n)_{n \geq 1}$ (that we do not relabel) which converges uniformly to some mapping $u \in C_{\mathbb{R}^n}(I)$ and $(\dot{u}_n)_{n \geq 1}$ converges weakly in $L^1_{\mathbb{R}^n}(I)$ to mapping $z \in L^1_{\mathbb{R}^n}(I)$ with $\|z(t)\| \leq \gamma$ a.e. $t \in I$. Fixing $t \in I$ and taking any $\xi \in \mathbb{R}^n$, the above weak convergence in $L^1_{\mathbb{R}^n}(I)$ yields

$$\lim_{n \rightarrow +\infty} \int_{T_0}^T \langle \mathbb{I}_I(s) \xi, \dot{u}_n(s) \rangle ds = \int_{T_0}^T \langle \mathbb{I}_I(s) \xi, z(s) \rangle ds$$

or equivalently

$$\lim_{n \rightarrow +\infty} \langle \xi, u_0 + \int_{T_0}^t \dot{u}_n(s) ds \rangle = \langle \xi, u_0 + \int_{T_0}^t z(s) ds \rangle.$$

Then, $\lim_{n \rightarrow +\infty} \int_{T_0}^t \dot{u}_n(s) ds = \int_{T_0}^t z(s) ds$, since $u_n(\cdot)$ is an absolutely continuous mapping, we get

$$\lim_{n \rightarrow +\infty} (u_n(t) - u_0) = \lim_{n \rightarrow +\infty} \int_{T_0}^t \dot{u}_n(s) ds = \int_{T_0}^t z(s) ds,$$

so $u(\cdot)$ is an absolutely continuous mapping, and $z = \dot{u}$.

Let put $(g(\delta_n(\cdot), u_n(\delta_n(\cdot))))_n = (h_n(\cdot))_n$, so $(h_n)_{n \geq 1}$ is a measurable because $G(\cdot, \cdot)$ is upper semi-continuous and $\|h_n(t)\| \leq \eta$, then $(h_n(\cdot)) \in L^\infty_{\mathbb{R}^n}(I)$, taking a subsequence if necessary we conclude that $(h_n(\cdot))_{n \geq 1}$ weakly converges to a mapping $h \in L^\infty_{\mathbb{R}^n}(I)$ with $\|h(t)\| \leq \eta$ a.e.

Step 4. We prove in this step that u is a solution of (1.1).

Fix any $t \in I$, by $(\mathcal{H}_2)$ and (3.12), we have

$$\begin{aligned} d(u_n(t), C(t, u(t))) &\leq \|u_n(\theta_n(t)) - u_n(t)\| + d(u_n(\theta_n(t)), C(t, u(t))) \\ &\leq \|u_n(\theta_n(t)) - u_n(t)\| + \|d(u_n(\theta_n(t)), C(t, u(t))) - d(u_n(\theta_n(t)), C(\theta_n(t), u(\delta_n(t))))\| \end{aligned}$$

$$\leq \|u_n(\theta_n(t)) - u_n(t)\| + L_1(\theta_n(t) - t) + L_2\|u_n(\delta_n(t)) - u(t)\|.$$

Using (3.11) and (3.15) and by passing to the limit in the last inequality, we get $u(t) \in C(t, u(t))$ for all $t \in I$ thanks to the closedness of $C(t, u(t))$. We have

$$\|-\dot{u}_n(t) + h_n(t)\| \leq (\gamma + \eta) = M \text{ a.e. } t \in I,$$

that is, $-\dot{u}_n(t) + h_n(t) \in M\mathbb{B}$, then

$$-\dot{u}_n(t) + h_n(t) \in N_{C(\theta_n(t), u_n(\delta_n(t)))}(u_n(\theta_n(t))) \cap M\mathbb{B},$$

we get by relation (2.1) and Proposition 1

$$-\dot{u}_n(t) + h_n(t) \in M\partial d_{C(\theta_n(t), u_n(\delta_n(t)))}(u_n(\theta_n(t))) \text{ a.e. } t \in I. \quad (3.16)$$

Note that $(-\dot{u}_n + h_n, h_n)$ weakly converges in $L^1_{\mathbb{R}^n \times \mathbb{R}^n}(I)$ to $(-\dot{u} + h, h)$ according to Mazur's lemma, there exists a sequence $(\xi_n, \zeta_n)_{n \geq 1}$ which strongly converges in $L^1_{\mathbb{R}^n \times \mathbb{R}^n}(I)$ to $(-\dot{u} + h, h)$ such that

$$\xi_n \in co\{-\dot{u}_q + h_q, q \geq n\} \quad \text{and} \quad \zeta_n \in co\{h_q, q \geq n\}. \quad (3.17)$$

Extracting a subsequence if necessary, we may assume that $(\xi_n(\cdot), \zeta_n(\cdot))_{n \geq 1}$ converges almost every to $(-\dot{u}(\cdot) + h(\cdot), h(\cdot))$. Then, there is, a Lebesgue negligible set $N \subset I$ such that for every $t \in I \setminus N$

$$-\dot{u}(t) + h(t) \in \bigcap_{n \geq 0} \overline{\{\xi_q(t), q \geq n\}} \subset \bigcap_{n \geq 0} \overline{co}\{-\dot{u}_q(t) + h_q(t), q \geq n\}, \quad (3.18)$$

and

$$h(t) \in \bigcap_{n \geq 0} \overline{\{\zeta_q(t), q \geq n\}} \subset \bigcap_{n \geq 0} \overline{co}\{h_q(t), q \geq n\}. \quad (3.19)$$

Fix any $t \in I \setminus N$ and $\mu \in \mathbb{R}^n$. Then, relations (3.16) and (3.18) give

$$\langle \mu, -\dot{u}(t) + h(t) \rangle \leq \limsup_{n \rightarrow +\infty} \delta^*\left(\mu, M\partial d_{C(\theta_n(t), u_n(\delta_n(t)))}(u_n(\theta_n(t)))\right).$$

By using Proposition 2, we obtain

$$\langle \mu, -\dot{u}(t) + h(t) \rangle \leq \delta^*\left(\mu, M\partial d_{C(t, u(t))}(u(t))\right).$$

Since $M\partial d_{C(t, u(t))}(u(t))$ is closed and convex, we conclude that

$$-\dot{u}(t) + h(t) \in M\partial d_{C(t, u(t))}(u(t)) \subset N_{C(t, u(t))}(u(t)) \text{ a.e. } t \in I \setminus N. \quad (3.20)$$

Further, by relation (3.19) we have

$$\langle \mu, h(t) \rangle \leq \limsup_{n \rightarrow +\infty} \delta^*\left(\mu, G(\delta_n(t), u_n(\delta_n(t)))\right).$$

Since G is upper semi-continuous with closed values, we obtain $h(t) \in G(t, u(t))$. Consequently, $\dot{u}(t) \in -N_{C(t, u(t))}(u(t)) + G(t, u(t))$ a.e. $t \in I$.

(2) To prove that the attainable set $Acc_{u_0}(\tilde{t})$ is compact it suffices to show that $S_{\tilde{t}}(u_0)$ is compact for $\tilde{t} \in I$. By the part (1), we get $S_{\tilde{t}}(u_0) \neq \emptyset$. Let $(u_n)_n \subset S_{\tilde{t}}(u_0)$, then, for each $n \in \mathbb{N}$, u_n is an absolutely continuous solution of (1.1) with

$$\|\dot{u}_n(t)\| \leq \gamma \text{ a.e. } t \in [T_0, \tilde{t}]. \quad (3.21)$$

and

$$\|u_n(t)\| \leq \|u_0\| + \int_{T_0}^t \|\dot{u}_n(s)\| ds \leq \|u_0\| + \gamma(t - T_0). \quad (3.22)$$

Then, $(u_n(t))_n$ is relatively compact in $\mathbb{R}^n$, in addition, it is equi-continuous according to (3.21). By Arzelà-Ascoli theorem $(u_n)_n$ is relatively compact in $C_{\mathbb{R}^n}([T_0, \tilde{t}])$, so, we can extract a subsequence of $(u_n)_n$ (that we do not relabel) which converges uniformly to some mapping u on $[T_0, \tilde{t}]$ and $(\dot{u}_n)_n$ converges in $L^1_{\mathbb{R}^n}([T_0, \tilde{t}])$ to $\dot{u}(\cdot)$ with $\|\dot{u}(t)\| \leq \gamma$ a.e. $t \in [T_0, \tilde{t}]$ and $u(t) = u_0 + \int_{T_0}^t \dot{u}(s) ds$. For the rest of the demonstration we can follow the proof of part (1) to get

$$\dot{u}(t) \in -N_{C(t, u(t))}(u(t)) + G(t, u(t)) \text{ a.e. } t \in [T_0, \tilde{t}].$$

□

4. SWEEPING PROCESS WITH ALMOST CONVEX PERTURBATION

In the following theorem we prove the existence of solution of (1.2), when the perturbation G takes almost convex values and with weaker assumption on upper semi-continuity. In the previous results, one takes the perturbation upper semi-continuous, an analysis of the proof, shows that we need only the upper semi-continuity of the $co(G)$.

Theorem 2. *Let $C : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued mapping with nonempty closed values satisfying:*

- ($\mathcal{A}_1$) *for all $x \in \mathbb{R}^n$ the sets $C(x)$ are equi-uniformly-subsmooth;*
- ($\mathcal{A}_2$) *there is a constant $L_2 \in [0, 1[$ such that, for all $x_1, x_2, y \in \mathbb{R}^n$*

$$|d(y, C(x_1)) - d(y, C(x_2))| \leq L_2 \|x_1 - x_2\|.$$

And let $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a measurable set-valued mapping with compact and almost convex values such that

- ($\mathcal{A}_3$) *the set-valued mapping $co(G(\cdot))$ is upper semi-continuous on $\mathbb{R}^n$;*
- ($\mathcal{A}_4$) *for some real $\alpha \geq 0$, $d(0, co(G(x))) \leq \alpha(1 + \|x\|)$ for all $x \in \mathbb{R}^n$.*

Then, for every $u_0 \in C(u_0)$,

- (1) *(1.2) admits an absolutely continuous solution;*
- (2) *for $\tilde{t} \in I$ fixed, the attainable set of (1.2) at $\tilde{t}$, $Acc_{u_0}(\tilde{t})$, coincides with $Acc_{u_0}^{co}(\tilde{t})$ the attainable set at $\tilde{t}$ of the convexified problem.*

Proof. (1) **Step 1.** We start by proving the existence of two integrable functions $\lambda_1(\cdot)$ and $\lambda_2(\cdot)$ defined on I , such that $0 \leq \lambda_1(t) \leq 1 \leq \lambda_2(t)$ and for $t \in I$

$$\lambda_1(t)m(u(t)) \in G(u(t)), \quad \lambda_2(t)m(u(t)) \in G(u(t)), \quad (4.1)$$

where $m(u(t)) = \text{Proj}_{\text{co}(G(u(t)))}(0)$. Since for every $t \in I$, $G(u(t))$ is almost convex, there exists two nonempty sets $\Lambda_1(t)$ and $\Lambda_2(t)$ such that

$$\Lambda_1(t) = \{\lambda_1 \in [0, 1] : \lambda_1 m(u(t)) \in G(u(t))\},$$

$$\Lambda_2(t) = \{\lambda_2 \in [1, +\infty[: \lambda_2 m(u(t)) \in G(u(t))\}.$$

Consider $\text{Gph}(\Lambda_1)$ the graph of Λ_1 defined by

$$\begin{aligned} \text{Gph}(\Lambda_1) &= \{(t, \lambda_1) \in I \times [0, 1] : \lambda_1 m(u(t)) \in G(u(t))\} \\ &= \{(t, \lambda_1) \in I \times [0, 1] : d(\lambda_1 m(u(t)), G(u(t))) = 0\} \\ &= \varphi^{-1}(\{0\}) \cap (I \times [0, 1]), \end{aligned}$$

where $\varphi : (t, \lambda_1) \rightarrow d(\lambda_1 m(u(t)), G(u(t)))$ is measurable. Then $\text{Gph}(\Lambda_1)$ is measurable. So, there exists an integrable selection $\lambda_1(\cdot)$ of $\Lambda_1(\cdot)$ satisfying $0 \leq \lambda_1(t) \leq 1$ and $\lambda_1(t)m(u(t)) \in G(u(t))$ for $t \in I$. The existence of an integrable selection $\lambda_2(\cdot)$ of $\Lambda_2(\cdot)$, satisfying $\lambda_2(t) \geq 1$ and $\lambda_2(t)m(u(t)) \in G(u(t))$, can be proved using the same reasoning as above with the fact that $G(u(t))$ is bounded.

Step 2. (a) Let $[a, b] \subset I$ be an interval and assume that there exist two integrable functions $\lambda_1(\cdot)$ and $\lambda_2(\cdot)$ define on $[a, b]$, such that $0 \leq \lambda_1(\tau) \leq 1 \leq \lambda_2(\tau)$, for all $\tau \in I$. Using the same procedure as in the proof of Theorem 4.1. in [2], we conclude the existence of two measurable subsets of $[a, b]$, having characteristic functions $\mathbb{I}_1$ and $\mathbb{I}_2$ such that $\mathbb{I}_1(\cdot) + \mathbb{I}_2(\cdot) = \mathbb{I}_{[a, b]}(\cdot)$ and an absolutely continuous function $s : [a, b] \rightarrow [a, b]$ with $s(b) - s(a) = b - a$, such that

$$\dot{s}(\tau) = \frac{1}{\lambda_1(\tau)} \mathbb{I}_1(\tau) + \frac{1}{\lambda_2(\tau)} \mathbb{I}_2(\tau).$$

(b) Using Theorem 1, there exists an absolutely continuous solution $x : I \rightarrow \mathbb{R}^n$ of the convexified problem

$$\begin{cases} \dot{u}(t) \in -N_{C(u(t))}(u(t)) + \text{co}(G(u(t))), & \text{a.e. } t \in I; \\ u(t) \in C(u(t)), \quad \forall t \in I; u(T_0) = u_0 \in C(u_0). \end{cases}$$

Consider the set $\Omega = \{\tau \in I : 0 \in \text{co}(G(x(\tau)))\}$. Since $\text{co}(G(\cdot))$ is an upper semi-continuous set-valued mapping with compact values and $x(\cdot)$ is continuous, we obtain the closedness of Ω .

If Ω is empty, in this case $\lambda_1(\tau) > 0$ on I , so, we can apply part (a) to interval I . Set $s(\tau) = T_0 + \int_{T_0}^{\tau} \dot{s}(\omega) d\omega$, s is increasing and we have $(s(T_0), s(T)) = (T_0, T)$, so s defined from I into itself. Let $\gamma : I \rightarrow I$ be its inverse, then $(\gamma(T_0), \gamma(T)) =$

(T_0, T) , $\frac{d}{d\tau} s(\gamma(\tau)) = \dot{s}(\gamma(\tau)) \dot{\gamma}(\tau) = 1$, and

$$\dot{\gamma}(\tau) = \frac{1}{\dot{s}(\gamma(\tau))} = \lambda_1(\gamma(\tau)) \mathbb{I}_1(\gamma(\tau)) + \lambda_2(\gamma(\tau)) \mathbb{I}_2(\gamma(\tau)).$$

Consider $\tilde{x} : I \rightarrow \mathbb{R}^n$ the mapping defined by $\tilde{x}(\tau) = x(\gamma(\tau))$, then we get

$$\frac{d}{d\tau} \tilde{x}(\tau) = \dot{x}(\gamma(\tau)) (\lambda_1(\gamma(\tau)) \mathbb{I}_1(\gamma(\tau)) + \lambda_2(\gamma(\tau)) \mathbb{I}_2(\gamma(\tau))),$$

using the properties of the normal cone and (4.1), we obtain

$$\begin{aligned} \frac{d}{d\tau} \tilde{x}(\tau) &\in (-N_{C(x(\gamma(\tau)))}(x(\gamma(\tau))) + m(x(\gamma(\tau)))) (\lambda_1(\gamma(\tau)) \mathbb{I}_1(\gamma(\tau)) + \lambda_2(\gamma(\tau)) \mathbb{I}_2(\gamma(\tau))) \\ &\in -N_{C(x(\gamma(\tau)))}(x(\gamma(\tau))) + G(x(\gamma(\tau))) = -N_{C(\tilde{x}(\tau))}(\tilde{x}(\tau)) + G(\tilde{x}(\tau)). \end{aligned}$$

If Ω is nonempty, let $l = \sup\{\tau, \tau \in \Omega\} \in \Omega$. The complement of Ω is open relatively to I , it consists of at most countably many overlapping open intervals $]a_i, b_i[$, with the possible exception of the form $[a_i, b_i[$ with $a_i = T_0$ and one of the form $]a_{i_f}, b_{i_f}]$ with $a_{i_f} = l$. For each i , we can apply the part (a) to the $]a_i, b_i[$, so, there exist two measurable subsets of $]a_i, b_i[$ with characteristic functions $\mathbb{I}_1^i(\cdot)$ and $\mathbb{I}_2^i(\cdot)$ such that $\mathbb{I}_1^i(\cdot) + \mathbb{I}_2^i(\cdot) = \mathbb{I}_{]a_i, b_i[}(\cdot)$. Setting $\dot{s}(\tau) = \frac{1}{\lambda_1(\tau)} \mathbb{I}_1^i(\tau) + \frac{1}{\lambda_2(\tau)} \mathbb{I}_2^i(\tau)$, we obtain $\int_{a_i}^{b_i} \dot{s}(\omega) d\omega = b_i - a_i$. For all $\tau \in [T_0, l]$, set

$$\dot{s}(\tau) = \frac{1}{\lambda_2(\tau)} \mathbb{I}_\Omega(\tau) + \sum_i \left(\frac{1}{\lambda_1(\tau)} \mathbb{I}_1^i(\tau) + \frac{1}{\lambda_2(\tau)} \mathbb{I}_2^i(\tau) \right),$$

where the sum is over all intervals of the complementary of Ω contained in $[T_0, l]$, in addition to that $\lambda_2(\tau) \geq 1$ and $\int_{T_0}^l \dot{s}(\omega) d\omega = p \leq l - T_0$. Setting $s(t) = T_0 + \int_{T_0}^t \dot{s}(\omega) d\omega$, then $s(\cdot)$ is an invertible map from $[T_0, l]$ to $[T_0, p]$. Let define $\gamma : [T_0, p] \rightarrow [T_0, l]$ to be the inverse function of $s(\cdot)$. Extend $\gamma(\cdot)$ as an absolutely continuous map $\tilde{\gamma}(\cdot)$ on $[T_0, l]$, setting $\tilde{\gamma}(\tau) = 0$ for $\tau \in [p, l]$. Let us show that the mapping $\tilde{x}(\tau) = x(\tilde{\gamma}(\tau))$ is a solution of the problem (1.2) for all $\tau \in [T_0, l]$.

On $[T_0, p]$, we have $\tilde{\gamma}(\tau) = \gamma(\tau)$, γ is invertible and

$$\dot{\gamma}(\tau) = \lambda_2(\gamma(\tau)) \mathbb{I}_\Omega(\gamma(\tau)) + \sum_i (\lambda_1(\gamma(\tau)) \mathbb{I}_1^i(\gamma(\tau)) + \lambda_2(\gamma(\tau)) \mathbb{I}_2^i(\gamma(\tau))),$$

as $\frac{d}{d\tau} \tilde{x}(\tau) = \dot{x}(\gamma(\tau)) \dot{\gamma}(\tau)$, we get

$$\begin{aligned} \frac{d}{d\tau} \tilde{x}(\tau) &\in \dot{\gamma}(\tau) (-N_{C(x(\gamma(\tau)))}(x(\gamma(\tau))) + m(x(\gamma(\tau)))) \\ &\in -N_{C(x(\gamma(\tau)))}(x(\gamma(\tau))) + G(x(\gamma(\tau))) = -N_{C(\tilde{x}(\tau))}(\tilde{x}(\tau)) + G(\tilde{x}(\tau)). \end{aligned}$$

On $]p, l]$, we have $\gamma(p) = l$ and $\tilde{\gamma}(\tau) = 0$, then we get $\tilde{\gamma}(\tau) = \tilde{\gamma}(p) = \gamma(p)$, we obtain $\tilde{x}(\tau) = x(\tilde{\gamma}(\tau)) = x(\gamma(p)) = l$, then $\tilde{x}$ is constant on $]p, l]$, and we have $\frac{d}{d\tau} \tilde{x}(\tau) = 0 \in$

$co(G(\tilde{x}(\tau)))$. Using (4.1), we conclude that for $\tau \in]p, l]$

$$\frac{d}{d\tau} \tilde{x}(\tau) = 0 \in -N_{C(\tilde{x}(\tau))}(\tilde{x}(\tau)) + G(\tilde{x}(\tau)).$$

On $[l, T]$, the set Ω is empty, and $\lambda_1(\tau) > 0$, then we can repeat the arguments of part (a). We conclude, that $\tilde{x}$ is an absolutely continuous solution of the problem (1.2).

(2) For the proof of this part, we refer the reader to Theorem 3 in [2].

This completes the proof of the theorem. $\square$

5. APPLICATION TO AN OPTIMAL TIME PROBLEM

Now consider the control system (1.3) under the almost convexity assumption.

Theorem 3. *Let $C : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued mapping with nonempty closed values satisfies the hypothesis $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ in Theorem 2, $Z : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued mapping with compact valued upper semi-continuous on $\mathbb{R}^n$ such that $0 \in Z(x)$ for all $x \in \mathbb{R}^n$. And let $h : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous mapping satisfying the following assumption*

- $(\mathcal{H}_1^h)$ *there is a nonnegative constant α , such that $\|h(x, y)\| \leq \alpha(1 + \|x\|)$, for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$;*
- $(\mathcal{H}_2^h)$ *for all $x \in \mathbb{R}^n$, $h(x, 0) = 0$;*
- $(\mathcal{H}_3^h)$ *Let G be a measurable set-valued mapping with compact and almost convex values such that $G(x) = \{h(x, z)\}_{z \in Z(x)}$.*

Let u_0, u_1 be given in $\mathbb{R}^n$, and assume that for some $t \in I$, $u_1 \in Acc_{u_0}(t)$ the attainable set of the problem (1.3). Then,

- (1) *the problem of reaching u_1 from u_0 in a minimum time admits a solution;*
- (2) *for all $t \in I$*

$$Acc_{u_0}(t) = \{u \in \mathbb{R}^n : \mathfrak{T}(u) \leq t\}.$$

Proof. (1) By the assumptions on h and Z , we conclude that $co(G(\cdot))$ is an upper semicontinuous set-valued mapping values and

$$d(0, co(G(x))) \leq \alpha(1 + \|x\|), \quad \text{for all } x \in \mathbb{R}^n.$$

According to Theorem 2 the problem (1.3) admits a solution.

Let $\bar{t} = \inf\{\tau \in [T_0, t] : u_1 \in Acc_{u_0}(\tau)\}$, so by the lower bound property, there exists a decreasing sequence $(\bar{t}_n)$ in $[T_0, t]$ converging to $\bar{t}$ and for each n , $u_n(\cdot)$ is solution of the problem

$$\begin{cases} \dot{u}(t) \in -N_{C(u(t))}(u(t)) + G(u(t)) & \text{a.e. } t \in [T_0, \bar{t}_n]; \\ u(t) \in C(u(t)), \quad \forall t \in [T_0, \bar{t}_n]; & u(T_0) = u_0 \in C(u_0), \end{cases}$$

such that $u_1 = u_n(\bar{t}_n)$. We define the sequence $(y_n(\cdot))$ by $y_n(s) = u_n(s)$ for all $s \in [T_0, \bar{t}]$. So,

$$(y_n(s)) \subset Acc_{u_0}(s) = Acc_{u_0}^{co}(s).$$

By the compactness of $Acc_{u_0}^{co}(s)$, we can extract a subsequence if necessary and conclude that $(y_n(s))$ converges to $u(s) \in Acc_{u_0}^{co}(\bar{t})$ and $y(\bar{t}) = u_1 \in Acc_{u_0}^{co}(\bar{t}) = Acc_{u_0}(\bar{t})$. Consequently, u is the solution of the problem (1.3) that reaches u_1 in the minimum time, and $\bar{t}$ is the value of the minimum time.

(2) We have $u_1 \in Acc_{u_0}(t)$, this means that there exists $u(\cdot)$ solution of the problem (1.3) such that $u_1 = u(t)$. We define the mapping

$$\tilde{u}(s) = \begin{cases} u(s) & \text{for } s \in [T_0, t] \\ u_1 & \text{for } s \in [t, T]. \end{cases}$$

For $s \in [T_0, t]$, we can write

$$\dot{\tilde{u}}(s) \in -N_{C(\tilde{u}(s))}(\tilde{u}(s)) + G(\tilde{u}(s)) \quad \text{a.e.} \quad (5.1)$$

For $s \in [t, T]$, $\dot{\tilde{u}}(s) = 0$, by hypotheses $(\mathcal{H}_2^h)$ and since $0 \in N_{C(\tilde{u}(s))}(\tilde{u}(s))$ we get

$$\dot{\tilde{u}}(s) = 0 \in -N_{C(\tilde{u}(s))}(\tilde{u}(s)) + G(\tilde{u}(s)) \quad \text{a.e.} \quad (5.2)$$

By relation (5.1) and (5.2), we conclude that $\tilde{u}(\cdot)$ is a solution of (1.3) for all $t \in I$. On the other hand, for all $t < s$, $\tilde{u}(s) = u_1 \in Acc_{u_0}(s)$. Then

$$Acc_{u_0}(t) \subseteq Acc_{u_0}(s). \quad (5.3)$$

Now let $z \in \{u \in \mathbb{R}^n : \mathfrak{T}(u) \leq t\}$, according to relation (5.3), $Acc_{u_0}(\mathfrak{T}(u)) \subseteq Acc_{u_0}(t)$, then $z \in Acc_{u_0}(t)$. Hence

$$\{u \in \mathbb{R}^n : \mathfrak{T}(u) \leq t\} \subseteq Acc_{u_0}(t). \quad (5.4)$$

In addition, using the definition of the attainable set we get

$$Acc_{u_0}(t) \subseteq \{u \in \mathbb{R}^n : \mathfrak{T}(u) \leq t\}. \quad (5.5)$$

By (5.4) and (5.5) we conclude that

$$Acc_{u_0}(t) = \{u \in \mathbb{R}^n : \mathfrak{T}(u) \leq t\}.$$

□

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EXISTENCE AND CONVERGENCE OF FIXED POINT FOR A G-NON-LIPSCHITZIAN MAPPING

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Received 12 June, 2020

Abstract. The existence of fixed point in uniformly convex hyperbolic metric space endowed with graph of G -nearly asymptotically nonexpansive mapping has been obtained. Further, we prove strong and Δ -convergence of M -iteration to fixed point of G -nearly asymptotically nonexpansive mapping. We also derived some corollaries of our results in uniformly convex Banach space which are also independent new findings.

2010 *Mathematics Subject Classification:* 47H10

Keywords: hyperbolic metric space, G -nearly asymptotically nonexpansive mapping, non-Lipschitzian mapping, fixed point theorems, directed graph

1. INTRODUCTION

Fixed point theory is an active and vital branch of mathematics. In 1922, Banach gave the first fundamental fixed point theorem which is known as Banach contraction principle. Banach contraction principle is also notable for its simplicity. It requires only two conditions one on underlying space as complete metric space and other is on involved mapping as contraction mapping. Generalizations of Banach contraction principle in different directions is one of the important part of research in nonlinear analysis. The one way is to generalize Banach contraction principle is to change the underlying metric structure and second is to extend the involve mapping. The important generalizations came into picture due to Ran and Reurings [19] and Nieto and López [17]. They generalize the Banach contraction principle to partially ordered metric space. Ran and Reurings [19] applied their results to solve matrix equation while Nieto and López [17] applied to solve differential equation.

Nonexpansive mappings are those mappings whose Lipschitz constant are equal to one. In 1965, Browder [4,5], Göhde [7] and Kirk [11] independently proved existence of fixed point for nonexpansive mappings in Banach space. Further, it is well known fact that Picard iteration does not converge to fixed point of nonexpansive mappings even if it exists. In 1953, Mann [15] introduced a iteration to approximate the fixed point of nonexpansive mappings. After Mann iteration, many iteration came into

picture. Recently in 2018, Ullah and Arshad [29] introduced a new iteration namely M -iteration which is more efficient than the many existing iterations.

In 2008, Jachymski [8] proved Banach contraction principle in complete metric space endowed with graph which is natural generalization of Ran and Reurings [19] and Nieto and López [17] results.

In 2016, Alfuraidan and Shukri [2] proved Browder and Göhde fixed point theorem for G -nonexpansive mappings. Recently many authors obtained convergence results for many iterations in spaces endowed with graphs see [25, 27, 28, 30].

In 2005, Sahu [22] introduced a mapping namely nearly asymptotically nonexpansive mapping, it is a non-Lipschitzian type mapping and generalize nonexpansive mappings. In 2008 Sahu et al. [21] and in 2015 Saluja et al. [23] proved some convergence results for nearly asymptotically nonexpansive mappings. In this paper, we prove the Browder and Göhde fixed point theorem for G -nearly asymptotically nonexpansive mappings. We also prove, Δ -convergence and strong convergence of M -iteration for G -nearly asymptotically nonexpansive mappings in hyperbolic metric space.

2. PRELIMINARIES

Let us start the section by collecting some needed results.

A graph G is an ordered pair $(V(G), E(G))$ where $V(G)$ is a set called vertices and $E(G)$ is a binary relation on $V(G)$ (i.e. $E(G) \subseteq V(G) \times V(G)$) called edges of G . If the direction is imposed on each edges then we call the graph a directed graph or digraph. Here we assume that digraph have loop at every vertex (i.e. $(x, x) \in E(G)$ for each $x \in V(G)$). We also assume that G has no parallel edges. Moreover, we assume that there is a distance function d defined on the set of vertices $V(G)$ and we call it a weighted graph. G^{-1} is obtained from G by reversing the direction of edges. Thus we have

$$E(G^{-1}) = \{(y, x) : (x, y) \in E(G)\}.$$

The letter $\tilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges. It will be more convenient for us to treat $\tilde{G}$ as a directed graph for which the set of its edges is symmetric. Under this convention, we have

$$E(\tilde{G}) = E(G) \cup E(G^{-1}).$$

Further, if x and y are vertices in a graph G , then a path in G from x to y of length $N \in \mathbb{N}$ is a sequence $\{x_i\}_{i=1}^N$ of N vertices such that $x_1 = x$, $x_N = y$ and $(x_i, x_{i+1}) \in E(G)$. A graph G is connected if there is a path between any two vertices.

In this paper we will take (X, d, G) as a weighted directed graph G defined on the set X .

Definition 1. A directed graph is said to be transitive whenever $(u, v), (v, w) \in E(G)$ then this implies $(u, w) \in E(G)$.

A point $x \in X$ is said to be a fixed point of T whenever $T(x) = x$ and the set of the fixed points of T is denoted by $F(T)$.

Fixed point theory in Banach space has been developed richly, because it have linearity and convex structure. It is a great interest of researchers to develop results of a linear space (Banach Space) to a nonlinear space (metric space). But due to unavailability of convex structure in metric space it was looking impossible to develop the results of Banach space into metric space. To keep in mind this situation Reich and Shafrir [20] introduced a nonlinear space (hyperbolic metric space) by using geodesic segment and use Menger convexity [16] in this space. In 2005, Kohlenbach [12] introduced a generalization of the classical concept of hyperbolic metric space by using Takahashi [26] convex structure as follows:

Definition 2. Let (X, d) be a metric space, then (X, d, W) will be the hyperbolic metric space if the function $W : X \times X \times [0, 1] \rightarrow X$ satisfying

- (i) $d(z, W(x, y, \alpha)) \leq (1 - \alpha)d(z, x) + \alpha d(z, y)$,
- (ii) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta|d(x, y)$,
- (iii) $W(x, y, \alpha) = W(x, y, 1 - \alpha)$,
- (iv) $d(W(x, y, \alpha), W(z, w, \alpha)) \leq (1 - \alpha)d(x, z) + \alpha d(y, w)$

for all $x, y, z, w \in X$ and $\alpha, \beta \in [0, 1]$.

If only condition (i) is satisfied then it will be convex metric space in sense of Takahashi [26]. For some results of fixed point theory on convex structure see [10].

Linear example of *hyperbolic metric space* is Banach space and nonlinear examples are Hadamard manifolds, the Hilbert open unit ball equipped with the hyperbolic metric and the CAT(0) spaces.

Here we present nearly asymptotically nonexpansive mappings endowed with graphs. Originally it was introduced by Sahu [22] and is a generalization of asymptotically nonexpansive mappings.

Definition 3. A mapping $T : (X, d, G) \rightarrow (X, d, G)$ is said to be G-nearly Lipschitzian with respect to a_n if for each $n \in \mathbb{N}$, there exist a constant $k_n \geq 0$ such that

$$d(T^n x, T^n y) \leq k_n(d(x, y) + a_n),$$

where $a_n \in [0, \infty)$ with $a_n \rightarrow 0$ and for every $x, y \in X$ such that $(x, y) \in E(G)$. The infimum of constants k_n for which the last inequality hold is denoted by $\eta(T^n)$ and called the nearly Lipschitz constant. The G-nearly Lipschitz mapping T with sequence $\{(a_n, \eta(T^n))\}$ is said to be G-nearly asymptotically nonexpansive if

- (1) $\eta(T^n) \geq 1$ for all $n \in \mathbb{N}$ and
- (2) $\lim_{n \rightarrow \infty} \eta(T^n) = 1$.

If we take $a_n = 0$ for all $n \in \mathbb{N}$ then it will be G-asymptotically nonexpansive mapping and if we take $k_n = 1$, $a_n = 0$ for all $n \in \mathbb{N}$ then it will be G-nonexpansive mapping.

Lemma 1 ([3, 6]). *Let C be a closed nonempty subset of uniformly convex hyperbolic metric space (X, d) . Let $\tau : C \rightarrow [0, \infty)$ be a type function, i.e., there exist bounded sequence $\{x_n\} \in X$ such that*

$$\tau(x) = \limsup_{n \rightarrow \infty} d(x_n, x),$$

for any $x \in C$. Then τ is continuous. Since X is hyperbolic, τ is convex, i.e., the subset $\{x \in C; \tau(x) \leq r\}$ is convex for any $r \geq 0$ and there exist a unique minimum point $z \in C$ such that

$$\tau(z) = \inf\{\tau(x); x \in C\}.$$

Moreover, if $\{z_n\}$ is a minimizing sequence in C , i.e., $\lim_{n \rightarrow \infty} \tau(z_n) = \tau_0$, then $\{z_n\}$ converges strongly to z .

Due to unavailability of addition and scalar multiplication in general metric space it was impossible to study weak convergence. So to tackle this situation in 1976, Lim [14] introduced the concept of Δ -convergence in metric space. Δ -convergence is an analogue of weak convergence.

Definition 4. Let X be a complete hyperbolic metric space and $\{x_n\}$ be a bounded sequence in X . Then the type function $r(\cdot, \{x_n\}) : X \rightarrow [0, \infty)$ is defined by

$$r(x, \{x_n\}) = \limsup_{n \rightarrow \infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : \text{for } x \in X\}$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is defined as

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

Definition 5. A bounded sequence $\{x_n\}$ in X is said to Δ -converge to $x \in X$ if x is the unique asymptotic center of every subsequence $\{u_n\}$ of $\{x_n\}$. We write $x_n \rightarrow x$ ($\{x_n\}$ Δ -converges to x).

Lemma 2 ([13]). *Let (X, d) be a complete uniformly convex hyperbolic metric space and C be a nonempty, convex and closed subset X . Then, every bounded sequence $\{x_n\} \in X$ has a unique asymptotic center with respect to C .*

Proposition 1 ([1]). *Let (X, d) be complete hyperbolic metric space then (X, d, G) is said to have property $(*)$, if for each sequence $\{x_n\}$ in X Δ -converges to $x \in X$ and $(x_n, x_{n+1}) \in E(G)$, then there is a subsequence $\{x_{n_k}\}$ with $(x_{n_k}, x) \in E(G) \quad \forall n \in \mathbb{N}$. Note that if the triplet (X, d, G) has property $(*)$ and G is transitive, then we have the following property:*

*(**) For any $\{x_n\}$ in X , if x_n Δ -converges to x and $(x_n, x_{n+1}) \in E(G)$ then $(x_n, x) \in E(G)$ for $n \geq 1$.*

Lemma 3 ([9]). *Let X be a uniformly convex hyperbolic space. Let $R \in [0, \infty)$ be such that*

$$\limsup_{n \rightarrow \infty} d(x_n, a) \leq R, \limsup_{n \rightarrow \infty} d(y_n, a) \leq R \text{ and } \lim_{n \rightarrow \infty} d(a, \alpha_n x_n \oplus (1 - \alpha_n) y_n) = R$$

where $\alpha_n \in [a, b]$, with $0 < a \leq b < 1$. Then we have, $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$.

Lemma 4 ([18]). *Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be three sequences of nonnegative numbers such that*

$$y_n \geq 1 \text{ and } x_{n+1} \leq y_n x_n + z_n \text{ for all } n \in \mathbb{N}.$$

If $\sum_{n=1}^{\infty} (y_n - 1) < \infty$ and $\sum_{n=1}^{\infty} z_n < \infty$, then $\lim_{n \rightarrow \infty} x_n$ exists.

M -iteration process in hyperbolic metric space, as follows:

$$\begin{cases} x_0 \in C, \\ z_n = W(x_n, T^n x_n, \alpha_n), \\ y_n = T^n z_n, \\ x_{n+1} = T^n y_n, \forall n \in \mathbb{N}. \end{cases} \quad (2.1)$$

Through the paper we will assume that G -intervals are closed and convex. We know that a G -interval is any of the subsets

$$[a, \rightarrow) = \{x \in V(G); (a, x) \in E(G)\} \quad \text{and} \quad (\leftarrow, b] = \{x \in V(G); (x, b) \in E(G)\},$$

for every $a, b \in V(G)$.

3. EXISTENCE THEOREM

In this section we prove an existence theorem for a mapping involving graph.

Theorem 1. *Let (X, d, G) be a complete uniformly convex hyperbolic metric space endowed with directed graph G . Assume that G is convex and transitive and C be a nonempty, closed and convex subset of X which contain more than one point. If $T : C \rightarrow C$ is a continuous G -nearly asymptotically nonexpansive mapping and there exist $x_0 \in C$ such that $(x_0, T x_0) \in E(G)$. Then, T has a fixed point.*

Proof. Since $(x_0, T(x_0)) \in E(G)$, then by edge preserving of T , $(T^n x_0, T^{n+1} x_0) \in E(G)$ for all $n \in \mathbb{N}$. Since X is a complete uniformly convex hyperbolic metric space, then by property (R),

$$C_{\infty} = \bigcap_{n \geq 0} [T^n x_0, \rightarrow) \cap C = \bigcap_{n \geq 0} \{x \in C; (T^n x_0, x) \in E(G)\} \neq \emptyset.$$

Choose $x \in C_{\infty}$, then $(T^n(x_0), x) \in E(G)$. Again by using edge preserving of T , $(T^{n+1}(x_0), Tx) \in E(G)$. Thus, by transitivity of $E(G)$, $(T^n(x_0), Tx) \in E(G)$, i.e., $T(C_{\infty}) \subset C_{\infty}$. Consider the type function $\tau : C_{\infty} \rightarrow [0, \infty)$ generated by $\{T^n(x_0)\}$, that is, $\tau(x) = \limsup_{n \rightarrow \infty} d(T^n x_0, x)$. By using Lemma 1, there exist a unique minimum

point $z \in C_\infty$ that is $\tau(z) = \inf\{\tau(x) : x \in C_\infty\}$. Since $z \in C_\infty$, then we have $T^p(z) \in C_\infty$ for all $p \in \mathbb{N}$ which implies

$$\begin{aligned}\tau(T^p(z)) &= \limsup_{n \rightarrow \infty} d(T^n x_0, T^p(z)) \leq \eta(T^p) \limsup_{n \rightarrow \infty} d(T^n x_0, z) + \eta(T^p) a_p \\ &= \eta(T^p) \tau(z) + \eta(T^p) a_p.\end{aligned}$$

Since, $\tau(z)$ is minimum then, $\tau(z) \leq \tau(T^p(z)) \leq \eta(T^p) \tau(z) + \eta(T^p) a_p$ for all $p \in \mathbb{N}$. By using the condition of Definition 2, $\lim_{p \rightarrow \infty} \tau(T^p(z)) = \tau(z)$. Also, $\tau(\lim_{p \rightarrow \infty} T^p(z)) = \tau(z)$. By using continuity of T , $\tau(T \lim_{p \rightarrow \infty} T^{p-1}(z)) = \tau(z)$, that is $\tau(T(z)) = \tau(z)$. By the uniqueness of the minimum point $z = T(z)$, we get that z is a fixed point of T . $\square$

The following corollary can be directly obtained from the above result.

Corollary 1. *Let (X, d, G) be a uniformly convex Banach space endowed with directed graph G . Assume that G is convex and transitive and C be a nonempty, closed and convex subset of X which contain more than one point. If $T : C \rightarrow C$ is a continuous G -nearly asymptotically nonexpansive mapping and there exist $x_0 \in C$ such that $(x_0, T x_0) \in E(G)$. Then, T has a fixed point.*

4. Δ -CONVERGENCE AND STRONG CONVERGENCE THEOREM

In this section we are going to prove convergence results. So let's start the section with the following proposition.

Proposition 2. *Let $p \in F(T)$ be such that (x_0, p) and (p, x_0) are in $E(G)$ and $\{x_n\}$ is a sequence defined by (2.1). Then $(x_n, p), (p, x_n), (y_n, p), (p, y_n), (z_n, p), (z_n, p)$ and (x_n, x_{n+1}) are in $E(G)$.*

Proof. Given that $(x_0, p) \in E(G)$. Then, by using convexity of $E(G)$, $(z_0, p) \in E(G)$. Also, $(y_0, p) \in E(G)$. Since T is edge preserving, we have $(T y_0, p) \in E(G)$ that is $(x_1, p) \in E(G)$. Again, by edge preserving of $E(G)$, $(T x_1, p) \in E(G)$. Again by using convexity of $E(G)$ on (x_1, p) and $(T x_1, p)$, we get $(z_1, p) \in E(G)$. By using edge preserving of T , we get $(T z_1, p) \in E(G)$, that is $(y_1, p) \in E(G)$. Now suppose $(x_k, p) \in E(G)$ for fix $k \in \mathbb{N}$. Then, by edge preserving of T , $(T^k x_k, p) \in E(G)$. By convexity of $E(G)$, $(z_k, p) \in E(G)$. Again, $(T^k z_k, p) \in E(G)$, that is $(y_k, p) \in E(G)$. By edge preserving of T , $(T^k y_k, p) \in E(G)$ that is, $(x_{k+1}, p) \in E(G)$. Thus, by edge preserving of $E(G)$, $(T^{k+1} x_{k+1}, p) \in E(G)$. Again by using convexity of $E(G)$ on (x_{k+1}, p) and $(T^{k+1} x_{k+1}, p)$ we get, $(z_{k+1}, p) \in E(G)$. Again, $(T^{k+1} z_{k+1}, p) \in E(G)$ we get $(y_{k+1}, p) \in E(G)$. Hence by induction, $(x_n, p), (y_n, p), (z_n, p)$ are in $E(G)$ for all $n \in \mathbb{N}$. Using the similar argument we can show that $(p, x_n), (p, y_n), (p, z_n)$ are in $E(G)$ for all $n \in \mathbb{N}$. By transitivity of G we get $(x_n, x_{n+1}) \in E(G)$. $\square$

Theorem 2. *Let (X, d) be a complete uniformly convex hyperbolic metric space and suppose that (X, d, G) has property $(*)$. Assume that G is convex and transitive and C be a nonempty, closed and convex subset of X which contain more than*

one point. If $T : C \rightarrow C$ is a continuous G -nearly asymptotically nonexpansive mapping with sequence $\{(a_n, \eta(T^n))\}$ and $F(T) \neq \emptyset$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} (\eta(T^n) - 1) < \infty$. If sequence $\{x_n\}$ is defined by (2.1) with $(x_0, p), (p, x_0) \in E(G)$ where $0 < a \leq \alpha_n, \beta_n \leq b < 1$, $p \in F(T)$ and $x_0 \in C$, then $\{x_n\}$ Δ -converges to a fixed point x^* of T .

Proof. Let $p \in F(T)$. It follows from Proposition 1 that $(x_n, p), (y_n, p), (z_n, p)$ are in $E(G)$. Now,

$$\begin{aligned} d(x_{n+1}, p) &= d(T^n y_n, p) \\ &\leq \eta(T^n) d(y_n, p) + \eta(T^n) a_n \\ &\leq \eta(T^n) d(T^n z_n, p) + \eta(T^n) a_n \\ &\leq \eta(T^n)^2 d(z_n, p) + \eta(T^n)^2 a_n + \eta(T^n) a_n \\ &\leq \eta(T^n)^2 d(W(T^n x_n, x_n, \alpha_n), p) + \eta(T^n)^2 a_n + \eta(T^n) a_n \\ &\leq (\eta(T^n)^2 - \alpha_n \eta(T^n)^2 + \alpha_n \eta(T^n)^3) d(x_n, p) + \alpha_n \eta(T^n)^3 a_n \\ &\quad + \eta(T^n)^2 a_n + \eta(T^n) a_n \end{aligned}$$

for $n \in \mathbb{N}$. Also,

$$\begin{aligned} \sum_{n=1}^{\infty} (\eta(T^n)^2 - \alpha_n \eta(T^n)^2 + \alpha_n \eta(T^n)^3 - 1) &= \sum_{n=1}^{\infty} (\eta(T^n) + 1 + \alpha_n \eta(T^n)^2)(\eta(T^n) - 1) \\ &\leq \sup_{1 \leq n < \infty} (\eta(T^n) + 1 + \alpha_n \eta(T^n)^2) \sum_{n=1}^{\infty} ((\eta(T^n) - 1)) < \infty \end{aligned}$$

and

$$\begin{aligned} \sum_{n=1}^{\infty} (a_n)(\alpha_n \eta(T^n)^3 + \eta(T^n)^2 + \eta(T^n)) &\leq \sup_{1 \leq n < \infty} (\alpha_n \eta(T^n)^3 + \eta(T^n)^2 + \eta(T^n)) \sum_{n=1}^{\infty} a_n \\ &< \infty. \end{aligned}$$

It follows from Lemma 3 that $\lim_{n \rightarrow \infty} d(x_n, p)$ exist.

Let $\lim_{n \rightarrow \infty} d(x_n, p) = R$. Then

$$\limsup_{n \rightarrow \infty} d(T^n x_n, p) \leq \limsup_{n \rightarrow \infty} [\eta(T^n) d(x_n, p) + \eta(T^n) a_n] = \limsup_{n \rightarrow \infty} d(x_n, p) = R.$$

Now,

$$\begin{aligned} d(z_n, p) &\leq d(W(T^n x_n, x_n, \alpha_n), p) \\ &\leq (1 - \alpha_n) d(x_n, p) + \alpha_n [\eta(T^n) d(x_n, p) + \eta(T^n) a_n] \\ &= (1 - \alpha_n + \alpha_n \eta(T^n)) d(x_n, p) + \alpha_n \eta(T^n) a_n. \end{aligned}$$

Thus, $\lim_{n \rightarrow \infty} d(z_n, p) \leq R$. Again,

$$\lim_{n \rightarrow \infty} d(x_{n+1}, p) = \lim_{n \rightarrow \infty} d(T^{2n} z_n, p)$$

$$\begin{aligned}
&\leq \lim_{n \rightarrow \infty} \eta(T^n) d(T^n z_n, p) + \eta(T^n) a_n \\
&\leq \lim_{n \rightarrow \infty} \eta(T^n)^2 d(z_n, p) + \eta(T^n)^2 a_n + \eta(T^n) a_n \\
&\leq \lim_{n \rightarrow \infty} d(z_n, p) \\
R &\leq \lim_{n \rightarrow \infty} d(z_n, p).
\end{aligned}$$

Hence

$$\lim_{n \rightarrow \infty} d(z_n, p) = R.$$

By using Lemma 3,

$$\lim_{n \rightarrow \infty} d(T^n x_n, x_n) = 0.$$

From Lemma 1, $\{x_n\}$ have unique asymptotic center. Let $A(x_n) = x^*$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ such that $A(u_n) = u$. Now, claim $x^* = u$.

On contrary suppose that $x^* \neq u$. Then,

$$\begin{aligned}
\limsup_{n \rightarrow \infty} d(u_n, u) &< \limsup_{n \rightarrow \infty} d(u_n, x^*) \leq \limsup_{n \rightarrow \infty} d(x_n, x^*) \\
&< \limsup_{n \rightarrow \infty} d(x_n, u) = \limsup_{n \rightarrow \infty} d(u_n, u),
\end{aligned}$$

which is a contradiction and hence $\Delta - \lim_{n \rightarrow \infty} x_n = x^*$.

Now, we claim that $x^* \in F(T)$. Since by Property (**), $(x^*, x_n) \in E(G)$. Then,

$$\begin{aligned}
\limsup_{n \rightarrow \infty} d(T^n x^*, x_n) &\leq \limsup_{n \rightarrow \infty} d(T^n x^*, T^n x_n) + \limsup_{n \rightarrow \infty} d(T^n x_n, x_n) \\
&\leq \limsup_{n \rightarrow \infty} [\eta(T^n) d(x^*, x_n) + \eta(T^n) a_n] + \limsup_{n \rightarrow \infty} d(T^n x_n, x_n) \\
&\leq \limsup_{n \rightarrow \infty} d(x^*, x_n).
\end{aligned}$$

Since $\Delta - \lim_{n \rightarrow \infty} x_n = x^*$, $\limsup_{n \rightarrow \infty} d(x^*, x_n) < \limsup_{n \rightarrow \infty} d(T^n x^*, x_n)$. Thus, we have $T^n x^* = x^*$, which completes the proof. $\square$

The above nonlinear space result can be written as corollary in a linear space as follows:

Corollary 2. *Let (X, d) be a uniformly convex Banach space with Opial's property and suppose that (X, d, G) has property (*). Assume that G is convex and transitive and C be a nonempty, closed and convex subset of X which contain more than one point. If $T : C \rightarrow C$ is a continuous G -nearly asymptotically nonexpansive mapping with sequence $\{(a_n, \eta(T^n))\}$ and $F(T) \neq \emptyset$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} (\eta(T^n) - 1) < \infty$. If sequence $\{x_n\}$ is defined by (2.1) with (x_0, p) , $(p, x_0) \in E(G)$ where $0 < a \leq \alpha_n, \beta_n \leq b < 1$, $p \in F(T)$ and $x_0 \in C$ then, $\{x_n\}$ weakly converges to a fixed point x^* of T .*

Theorem 3. Let (X, d, G) be a complete uniformly convex hyperbolic metric space endowed with directed graph G . Assume that G is convex and transitive and C be a nonempty, closed and convex subset of X which contain more than one point. If $T : C \rightarrow C$ is a continuous G -nearly asymptotically nonexpansive mapping with sequence $\{(a_n, \eta(T^n))\}$ and $F(T) \neq \emptyset$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} (\eta(T^n) - 1) < \infty$. If sequence $\{x_n\}$ is defined by (2.1) with $(x_0, p), (p, x_0) \in E(G)$ where $0 < a \leq \alpha_n, \beta_n \leq b < 1$, $p \in F(T)$ and $x_0 \in C$, then, $\{x_n\}$ converges strongly to a fixed point x^* of T if and only if $\liminf_{n \rightarrow \infty} d(x_n, F(T)) = 0$.

Proof. It is easy to see that if $\{x_n\}$ converges to a point $x^* \in F(T)$ then

$$\liminf_{n \rightarrow \infty} d(x_n, F(T)) = 0.$$

For converse part, suppose that $\liminf_{n \rightarrow \infty} d(x_n, F(T)) = 0$. From the proof of Theorem 1, $\lim_{n \rightarrow \infty} d(x_n, x^*)$ exist. But as it is given in the hypothesis that $\liminf_{n \rightarrow \infty} d(x_n, F(T)) = 0$, therefore $\lim_{n \rightarrow \infty} d(x_n, F(T)) = 0$.

Thus, for a given $\varepsilon > 0$ there exist a $K(\varepsilon) \in \mathbb{N}$ such that

$$d(x_n, F(T)) < \frac{\varepsilon}{2} \text{ whenever } n > K(\varepsilon).$$

Particularly, $\inf\{d(x_K, x^*) : x^* \in F(T)\} < \frac{\varepsilon}{2}$. So there exist $x^* \in F(T)$ such that $d(x_K, x^*) < \frac{\varepsilon}{2}$. Now, for $n, m > K(\varepsilon)$

$$d(x_n, x_m) \leq d(x_n, x^*) + d(x^*, x_m) < \varepsilon.$$

Hence, x_n is a Cauchy sequence in C . Since C is a closed subset of X , then $\lim_{n \rightarrow \infty} x_n = x^* \in C$. $\square$

Corollary 3. Let (X, d, G) be a uniformly convex Banach space endowed with directed graph G . Assume that G is convex and transitive and C be a nonempty, closed and convex subset of X which contain more than one point. If $T : C \rightarrow C$ is a continuous G -nearly asymptotically nonexpansive mapping with sequence $\{(a_n, \eta(T^n))\}$ and $F(T) \neq \emptyset$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} (\eta(T^n) - 1) < \infty$. If sequence $\{x_n\}$ is defined by (2.1) with $(x_0, p), (p, x_0) \in E(G)$ where $0 < a \leq \alpha_n, \beta_n \leq b < 1$, $p \in F(T)$ and $x_0 \in C$, then, $\{x_n\}$ converges strongly to a fixed point x^* of T if and only if $\liminf_{n \rightarrow \infty} \|x_n - F(T)\| = 0$.

5. EXAMPLE

In this section, we construct an example of a G -nearly asymptotically nonexpansive mapping which is not a nearly asymptotically nonexpansive mapping. Also, domain of our example is a hyperbolic metric space which is not in Banach space. The motivation is essentially taken from [24].

Example 1. Let $X = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 > 0\}$. Define $d : X \times X \rightarrow [0, \infty)$ by

$$d(x, y) = |x_1 - y_1| + |x_1x_2 - y_1y_2|$$

for all $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in X . Now for $\alpha \in [0, 1]$, define a function $W : X \times X \times [0, 1] \rightarrow X$ by

$$W(x, y, \alpha) = \left(\alpha x_1 + (1 - \alpha)y_1, \frac{\alpha x_1x_2 + (1 - \alpha)y_1y_2}{\alpha x_1 + (1 - \alpha)y_1} \right).$$

Then we can easily verify that (X, d, W) is a hyperbolic metric space. Now, define the graph G on X by

$$(x, y) \in E(G) \Leftrightarrow x_1 + x_2 = y_1 + y_2.$$

Define the mapping $T : [\frac{1}{2}, 1] \times [\frac{1}{2}, 1] \rightarrow [\frac{1}{2}, 1] \times [\frac{1}{2}, 1]$ by

$$T(x_1, x_2) = (1 - x_1, 1 - x_2).$$

This gives

$$T^{2n}(x_1, x_2) = (x_1, x_2)$$

and

$$T^{2n+1}(x_1, x_2) = (1 - x_1, 1 - x_2)$$

for all $n \in \mathbb{N}$. This mapping is G -nearly asymptotically nonexpansive mapping but not nearly asymptotically nonexpansive mapping because

$$d(T^{2n+1}x, T^{2n+1}y) \leq d(x, y) + |x_1 + x_2 - y_1 - y_2|.$$

6. CONCLUSION

We are concluding our paper with the following pertinent observations:

- (i) Our Theorem 1 generalize Theorem 3.3 of Alfuraiddan and Shukri [2], Theorem Alfuraiddan and Khamsi [3] and Theorem 3.1 of Dehaish and Khamsi [6].
- (ii) Our convergence results extend the domain of convergence results of Ullah and Arshad [29].
- (iii) We also furnish an example of a G -nearly asymptotically nonexpansive in hyperbolic metric space with graph.

7. ACKNOWLEDGEMENTS

The work J.J. Nieto has been partially supported by the Agencia Estatal de Investigación (AEI) of Spain, co-financed by the European Fund for Regional Development (FEDER) corresponding to the 2014–2020 multi-year financial framework, project MTM2016-75140-P, and by Xunta de Galicia under grant ED431C 2019/02. Also, we are very highly indebted to anonymous referee for providing many critical suggestions for improving the manuscript.

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ON THE EXTREMAL GRAPHS FOR SECOND ZAGREB INDEX WITH FIXED NUMBER OF VERTICES AND CYCLOMATIC NUMBER

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Received 11 July, 2017

Abstract. The cyclomatic number of a graph G (is denoted by ν) is the minimum number of edges of G whose removal makes G as acyclic. Denote by $\mathbb{G}_{n,\nu}$ the collection of all n -vertex connected graphs with cyclomatic number ν . The elements of $\mathbb{G}_{n,\nu}$ with maximum second Zagreb (M_2) index (for $\nu \leq 4$ and $\nu = \frac{k(k-3)}{2} + 1$, where $4 \leq k \leq n-2$) and with minimum M_2 index (for $\nu \leq 2$) have already been reported in the literature. The main contribution of the present article is the characterization of graphs in the collection $\mathbb{G}_{n,\nu}$ with minimum M_2 index for $\nu \geq 3$ and $n \geq 2(\nu-1)$. The obtained extremal graphs, are molecular graphs and thereby, also minimize M_2 index among all the connected molecular n -vertex graphs with cyclomatic number $\nu \geq 3$, where $n \geq 2(\nu-1)$. For $n \geq 6$, the graph having maximum M_2 value in the collection $\mathbb{G}_{n,5}$ has also been characterized and thereby a conjecture posed by Xu *et al.* [*MATCH Commun. Math. Comput. Chem.* **72** (2014) 641–654] is confirmed for $\nu = 5$.

2010 *Mathematics Subject Classification:* 05C07, 05C35

Keywords: Graph, Second Zagreb index, Cyclomatic number

1. INTRODUCTION

All the graphs considered in the present study are finite, undirected, simple and connected. Undefined notations and terminologies from (chemical) graph theory can be found in [11, 24].

Chemical compounds can be represented by graphs in which vertices correspond to the atoms while edges represent the covalent bonds between atoms [24]. In chemical graph theory, graph invariants are usually referred as *topological indices*. In 1972, Gutman and Trinajstić [9] showed that total π -electron energy of a molecule M depends on the topological index $\sum_{v \in V(G)} d_v^2$, where G is the graph corresponding to M , $V(G)$ is the vertex set of G and d_v is degree of the vertex v . Nowadays, this graph quantity is known as the *first Zagreb index* and is denoted by M_1 . The following topological index (currently known as *second Zagreb index* and is usually denoted by

M_2) was appeared in [8] within the study of molecular branching:

$$M_2(G) = \sum_{uv \in E(G)} d_u d_v,$$

where $E(G)$ is the edge set of the graph G and uv is the edge between the vertices $u, v \in V(G)$. Both of the aforementioned Zagreb indices belong to the oldest and most studied topological indices. More than five hundred papers have been devoted to these Zagreb indices, for example see the reviews [4, 7, 21] published on the occasion of their 30th anniversary, recent surveys [2, 6], recent papers [3, 12–18, 20, 22, 23] and related references mentioned therein.

A vertex $v \in V(G)$ is said to be pendent vertex if $d_v = 1$. Following Xu *et al.* [25], let K_k^{n-k} be the graph obtained by attaching $n - k$ pendent vertices to one vertex of the k -vertex complete graph K_k . For any positive integer $t < k$, let $K_k^{n-k}(t)$ be a graph obtained by adding t new edges between one pendent vertex of K_k^{n-k} and t vertices with degree $k - 1$ in it. The cyclomatic number of a graph G (is denoted by v) is the minimum number of edges of G whose removal makes G as acyclic. Denote by $\mathbb{G}_{n,v}$ the collection of all n -vertex connected graphs with cyclomatic number v . Even though the mathematical theory of Zagreb indices is well established, some important extremal graph-theoretical problems concerning these Zagreb indices are still open. One of these problems is the characterization of extremal graphs with respect to M_2 index in $\mathbb{G}_{n,v}$ (however, a similar problem concerning M_1 index has already been solved completely [25]). Indeed, partial solution of the aforementioned problem (for M_2 index) has been reported in the literature: see [5] for the characterization of graphs in $\mathbb{G}_{n,v}$ with minimum M_2 value in case of $v \leq 2$; see [5] for the characterization of graphs in $\mathbb{G}_{n,v}$ with maximum M_2 value in case of $v \leq 2$ and $v = 4$, respectively. Also, Xu *et al.* [25] characterized the members of $\mathbb{G}_{n,v}$ with maximum M_2 value for $v \leq 3$ and $v = \frac{k(k-3)}{2} + 1$, where $4 \leq k \leq n - 2$. Furthermore, the authors of [25] posed the following conjecture concerning maximum M_2 value:

Conjecture 1. Let $v = \frac{k(k-3)}{2} + t + 1$ where $1 \leq t \leq k - 1$ and $4 \leq k \leq n - 2$. The graph $K_k^{n-k}(t)$ attains maximum M_2 value among all the members of $\mathbb{G}_{n,v}$.

For $v = 4$, the Conjecture 1 was proved in [1, 10]. In this paper, we confirm the Conjecture 1 for $v = 5$ and, mainly, characterize the graphs with minimum M_2 value in the collection $\mathbb{G}_{n,v}$ for $v \geq 3$ and $n \geq 2(v - 1)$.

2. MAIN RESULTS

In order to obtain the main results, we need to establish some preliminary lemmas and thereby we recall some notations and definitions. For a vertex $u \in V(G)$, the set of all vertices adjacent with u is denoted by $N_G(u)$. The elements of $N_G(u)$ are called neighbors of u . A vertex $v \in V(G)$ is said to be branching vertex if $d_v \geq 3$. A path $v_1 v_2 \cdots v_{k+1}$ of length $k \geq 1$ in a graph G is said to be pendent path if one of the

vertices v_1, v_{k+1} is branching and the other one is pendent, and all the other vertices (if exist) $v_2, v_3, \dots, v_k$ of path have degree 2. Let G' be a graph obtained from another graph G by applying some graph transformation such that $V(G) = V(G')$. In the rest of the paper, whenever such two graphs are under discussion, by the vertex degree d_u we always mean the degree of the vertex u in G .

Lemma 1. For $v \geq 3$ and $n \geq 2(v-1)$, if G has minimum M_2 value among all the members of $\mathbb{G}_{n,v}$ then G does not contain any pendant vertex.

Proof. Suppose to the contrary that G has a pendent vertex. Let $uu_1u_2\dots u_r$ be a pendent path in G , where $d_u \geq 3$. Assume that v is a neighbor of u different from u_1 . Let G' be the graph obtained from G by removing the edge uv and adding the edge vu_r . We note that both the graphs G and G' have same cyclomatic number. If $r = 1$ then

$$\begin{aligned} M_2(G) - M_2(G') &= d_v(d_u - 2) - d_u + 2 + \sum_{x \in N_G(u): x \neq u_1, x \neq v} d_x \\ &\geq d_v - d_u + 2 + \sum_{x \in N_G(u): x \neq u_1, x \neq v} d_x > 0, \end{aligned}$$

which is a contradiction to the minimality of $M_2(G)$. If $r \geq 2$ then we have

$$M_2(G) - M_2(G') = d_v(d_u - 2) + \sum_{x \in N_G(u): x \neq u_1, x \neq v} d_x \geq d_v + \sum_{x \in N_G(u): x \neq u_1, x \neq v} d_x > 0,$$

again a contradiction. $\square$

Let n_i be the number of vertices of degree i in the graph G . The minimum and maximum vertex degree of a graph is denoted by δ and Δ respectively.

Lemma 2. For $v \geq 3$ and $n \geq 2(v-1)$, if G has minimum M_2 value among all the members of $\mathbb{G}_{n,v}$ then maximum vertex degree in G is 3.

Proof. From Lemma 1, we have $\delta \geq 2$. The assumption $v \geq 3$ and the fact that the n -vertex cycle graph C_n is the only connected graph for which minimum and maximum vertex degree is 2, implies that maximum vertex degree in G is at least 3. We have to show that $\Delta = 3$. Contrarily, suppose that $\Delta \geq 4$. The inequality $n \geq 2(v-1)$ implies that

$$\sum_{2 \leq i \leq \Delta} n_i \geq 2(m - n) = 2 \left(\sum_{2 \leq i \leq \Delta} \frac{in_i}{2} - \sum_{2 \leq i \leq \Delta} n_i \right) = 2 \left(\sum_{3 \leq i \leq \Delta} \frac{in_i}{2} - \sum_{3 \leq i \leq \Delta} n_i \right),$$

which is equivalent to

$$n_2 \geq \sum_{4 \leq i \leq \Delta} (i-3)n_i.$$

It means that G contains at least one vertex of degree 2. Suppose that the vertex $u \in V(G)$ has maximum degree.

Case 1. The vertex u has a neighbor w of degree 2.

We note that w is not adjacent to at least one neighbor, say v , of u . Let G' be the graph obtained from G by removing the edge uv and adding the edge vw . Then, we have $M_2(G) - M_2(G') > 0$, a contradiction because $\delta \geq 2$ and

$$\begin{aligned} M_2(G) - M_2(G') &= d_v(d_u - 3) - d_t - d_u + 3 + \sum_{x \in N_G(u); x \neq v, x \neq w} d_x \\ &\geq 2(d_u - 3) - d_t - d_u + 3 + \sum_{x \in N_G(u); x \neq v, x \neq w} d_x, \end{aligned}$$

where t is the neighbor of w different from u .

Case 2. All the neighbors of u have degree greater than 2.

We notice that there exists a vertex $w' \in V(G) \setminus N_G(u)$ of degree 2 which is not adjacent to at least one neighbor, say v' , of u . Let G'' be the graph obtained from G by removing the edge uv' and adding the edge $v'w'$. Due to the assumption $d_u = \Delta \geq 4$, we have

$$\begin{aligned} M_2(G) - M_2(G'') &= d_{v'}(d_u - 3) + \sum_{y \in N_G(u); y \neq v'} d_y - \sum_{z \in N_G(w')} d_z \\ &\geq 3(d_u - 3) + 3(d_u - 1) - 2d_u > 0, \end{aligned}$$

again a contradiction. This completes the proof. $\square$

Let $x_{i,j}$ be the number of edges in the graph G connecting the vertices of degrees i and j .

Lemma 3. For $v \geq 3$, let $G \in \mathbb{G}_{n,v}$ such that it contains only vertices of degree 2 and 3.

- (i) If $2(v-1) \leq n < 5(v-1)$ then at least two vertices of degree 3 are adjacent.
- (ii) If $n > 5(v-1)$ then at least two vertices of degree 2 are adjacent.
- (iii) If $n = 5(v-1)$ and if one of $x_{2,2}, x_{3,3}$ is zero then the other is also zero.

Proof. (i) From the definition of G , the following equations must holds:

$$n_2 + n_3 = n \tag{2.1}$$

$$2n_2 + 3n_3 = 2(n + v - 1) \tag{2.2}$$

$$2x_{2,2} + x_{2,3} = 2n_2 \tag{2.3}$$

$$x_{2,3} + 2x_{3,3} = 3n_3. \tag{2.4}$$

From Eqs. (2.1) and (2.2), we obtain $n_2 = n - 2(v-1)$ and $n_3 = 2(v-1)$. These two equations together with the assumption $n < 5(v-1)$ imply that

$$2n_2 < 3n_3. \tag{2.5}$$

If $x_{3,3} = 0$ then from Eqs. (2.3) and (2.4), it follows that $2n_2 \geq 3n_3$ which contradicts the Inequality (2.5). From this we conclude that at least two vertices of degree 3 are

adjacent.

- (ii) The proof is fully analogous to that of first part.
- (iii) The assumption $n = 5(v - 1)$ implies that $2n_2 = 3n_3$. Now, from Eqs. (2.3) and (2.4), the desired result follows. $\square$

Lemma 4. For $v \geq 3$ and $n > 5(v - 1)$, if G has minimum M_2 value among all the members of $\mathbb{G}_{n,v}$ then for every edge $xy \in E(G)$ at least one of the vertices x, y has degree 2.

Proof. From Lemmata 1, 2 and 3, it follows that G contains only vertices of degrees 2, 3 and contains at least two adjacent vertices of degree 2, say u and v . Suppose to the contrary that there exist two adjacent vertices $w, t \in V(G)$ of degree 3. Let $x \neq u$ be the vertex adjacent with v . The vertex x may coincides with w or t , if this is the case, then (without loss of generality) we may assume that $x = t$.

Case 1. The vertices u and v do not have common neighbor.

Let G' be the graph obtained from G by removing the edges uv, vx, wt and adding the edges ux, wv, tv . Whether $x = t$ or $x \neq t$, in both the cases, we have $M_2(G) - M_2(G') = 1$, a contradiction to the minimality of $M_2(G)$.

Case 2. The vertices u and v have common neighbor.

It is evident that $d_x = 3$. If $x \neq t$ then the graph G'' obtained from G by removing the edges vx, wt and adding the edges vt, wx , has same M_2 value as G has. The vertices u and v do not have common neighbor in the graph G'' and hence by Case 1, we arrives at a contradiction. If $x = t$ then we consider a neighbor w_1 of w different from t . If G''' is the graph obtained from G by removing the edges vt, w_1w and adding the edges wv, w_1t , then $M_2(G) = M_2(G''')$. Again, the vertices u and v do not have common neighbor in the graph G''' and hence by Case 1, we arrives at a contradiction. $\square$

Lemma 5. For $v \geq 3$ and $n = 5(v - 1)$, if G has minimum M_2 value among all the members of $\mathbb{G}_{n,v}$ then for every edge $xy \in E(G)$ one of the vertices x, y has degree 2 and the other has degree 3.

Proof. From Lemmas 1 and 2, it follows that G contains only vertices of degree 2 and 3. We claim that $x_{2,2} = x_{3,3} = 0$. Contrarily, suppose that at least one of $x_{2,2}, x_{3,3}$ is non-zero. If one of $x_{2,2}, x_{3,3}$ is zero then (due to Lemma 3) the other must also be zero, a contradiction. If both of $x_{2,2}, x_{3,3}$ are non-zero. Then, from the proof of Lemma 4, we conclude that there exists a graph $G' \in \mathbb{G}_{n,v}$ such that $M_2(G) > M_2(G')$, which is again a contradiction. $\square$

Lemma 6. For $v \geq 3$ and $2(v - 1) \leq n < 5(v - 1)$, if G has minimum M_2 value among all the members of $\mathbb{G}_{n,v}$ then G does not contain any edge connecting the vertices of degree 2.

Proof. Contrarily, suppose that G contains at least two adjacent vertices of degree 2, say u and v . From Lemmas 1, 2 and 3, it follows that G has at least two adjacent vertices of degree 3. From the proof of Lemma 4, we conclude that there exists a graph $G' \in \mathbb{G}_{n,v}$ such that $M_2(G) > M_2(G')$, which is a contradiction. $\square$

Theorem 1. *If $v \geq 3$ then among all the members of $\mathbb{G}_{n,v}$,*

- (i) *the cubic graphs uniquely attains minimum M_2 value (which is $9(v + n - 1)$) for $n = 2(v - 1)$;*
- (ii) *the graphs containing only vertices of degree 2 and 3 such that no two vertices of degree 2 are adjacent, uniquely attain minimum M_2 value (which is $3n + 21(v - 1)$) for $2(v - 1) < n < 5(v - 1)$;*
- (iii) *the graphs containing only vertices of degree 2 and 3 such that for every pair of adjacent vertices of G one vertex has degree 2 and the other vertex has degree 3, uniquely attain minimum M_2 value (which is $6(v + n - 1)$) for $n = 5(v - 1)$;*
- (iv) *the graphs containing only vertices of degree 2 and 3 such that no two vertices of degree 3 are adjacent, uniquely attain minimum M_2 value (which is $4n + 34(v - 1)$) for $n > 5(v - 1)$.*

Proof. Let $G \in \mathbb{G}_{n,v}$ be the graph with minimum M_2 value among all the members of $\mathbb{G}_{n,v}$. From Lemmas 1 and 2, it is concluded that G contains only vertices of degree 2 and 3.

- (i) If $n = 2(v - 1)$ then Eqs. (2.1) and (2.2) yield $n_2 = 0$ and hence G must be a cubic graph.
- (ii) If $2(v - 1) < n < 5(v - 1)$ then bearing in mind the Lemma 6, we find the values of $x_{2,3}$, $x_{3,3}$ from Eqs. (2.1) - (2.4): $x_{2,3} = 2(n - 2(v - 1))$, $x_{3,3} = 5(v - 1) - n$. Hence $M_2(G) = 3n + 21(v - 1)$.
- (iii) A direct consequence of Lemma 5.
- (iv) If $n > 5(v - 1)$ then bearing in mind the Lemma 4 we find the values of $x_{2,2}$, $x_{2,3}$ from Eqs. (2.1) - (2.4): $x_{2,2} = n - 5(v - 1)$, $x_{2,3} = 6(v - 1)$. Hence $M_2(G) = 4n + 34(v - 1)$. $\square$

Remark 1. We note that all the extremal graphs characterized in Theorem 1 are molecular graphs. Hence, these extremal graphs also minimize M_2 among all the (connected) molecular n -vertex graphs with cyclomatic number $v \geq 3$, where $n \geq 2(v - 1)$.

Remark 2. If a graph $G \in \mathbb{G}_{n,v}$ has size m then it holds $v = m - n + 1$ and thereby the extremal values mentioned in Theorem 1 can be rewritten in terms of order and size of the graph.

Miličević *et al.* [19] reformulated the first Zagreb index in terms of edge-degrees instead of vertex-degrees. This topological index is denoted by EM_1 and can be

defined as:

$$EM_1(G) = \sum_{uv \in E(G)} (d_u + d_v - 2)^2.$$

Remark 3. We remark that the graphs which minimize M_2 index among all the members of the collection $\mathbb{G}_{n,v}$ (where $v \geq 3$), also minimize EM_1 index in the aforementioned collection. We omit details because the proofs of the results corresponding to Lemmas 1, 2, Lemma 4 - Lemma 6 and Theorem 1 concerning EM_1 index are fully analogous to that of Lemmas 1, 2, Lemma 4 - Lemma 6 and Theorem 1.

Since all the extremal graphs in Theorem 1 have minimum degree 2, we derive a sharp lower bound on M_2 for the n -vertex graph with minimum degree at least 2 and cyclomatic number v .

Theorem 2. *If G is an n -vertex graph with minimum degree at least 2 and cyclomatic number v then $M_2(G) \geq 29(v - 1) - n$ with equality if and only if G is cubic graph.*

Proof. Bearing in mind the equations

$$\begin{aligned} M_2(G) &= \sum_{2 \leq i \leq j \leq \Delta} ij \cdot x_{i,j}, \\ v + n - 1 &= \sum_{2 \leq i \leq j \leq \Delta} x_{i,j}, \\ n &= \sum_{2 \leq i \leq j \leq \Delta} \left(\frac{1}{i} + \frac{1}{j} \right) x_{i,j}, \end{aligned}$$

we have

$$M_2(G) - 29(v + n - 1) + 30n = \sum_{2 \leq i \leq j \leq \Delta} \left(ij - 29 + \frac{30}{i} + \frac{30}{j} \right) x_{i,j}.$$

To obtain the desired result, we have to show that $f(i, j) = ij - 29 + \frac{30}{i} + \frac{30}{j} \geq 0$ for $2 \leq i \leq j \leq \Delta$ such that $f(i, j) = 0$ if and only if $i = j = 3$. Direct calculations yield $f(2, j) > 0$ for $j = 2, 3$. For all $j \geq 4$, it holds that $f(2, j) \geq f(2, 4) > 0$. Also, we note that $f(3, 3) = 0$ and $f(3, j) \geq f(3, 4) > 0$ for all $j \geq 4$. Finally, if $j \geq i \geq 4$ then extension of the function f to the interval $[4, \infty)$ is increasing in both variables and hence $f(i, j) \geq f(4, 4) > 0$. This completes the proof. $\square$

Now, we turn our attention to the prove the Conjecture 1 for $v = 5$. For this, we need the following useful lemma:

Lemma 7. [25] *If the graph G has maximum M_2 value among all the members of $\mathbb{G}_{n,v}$ then the maximum vertex degree in G is $n - 1$.*

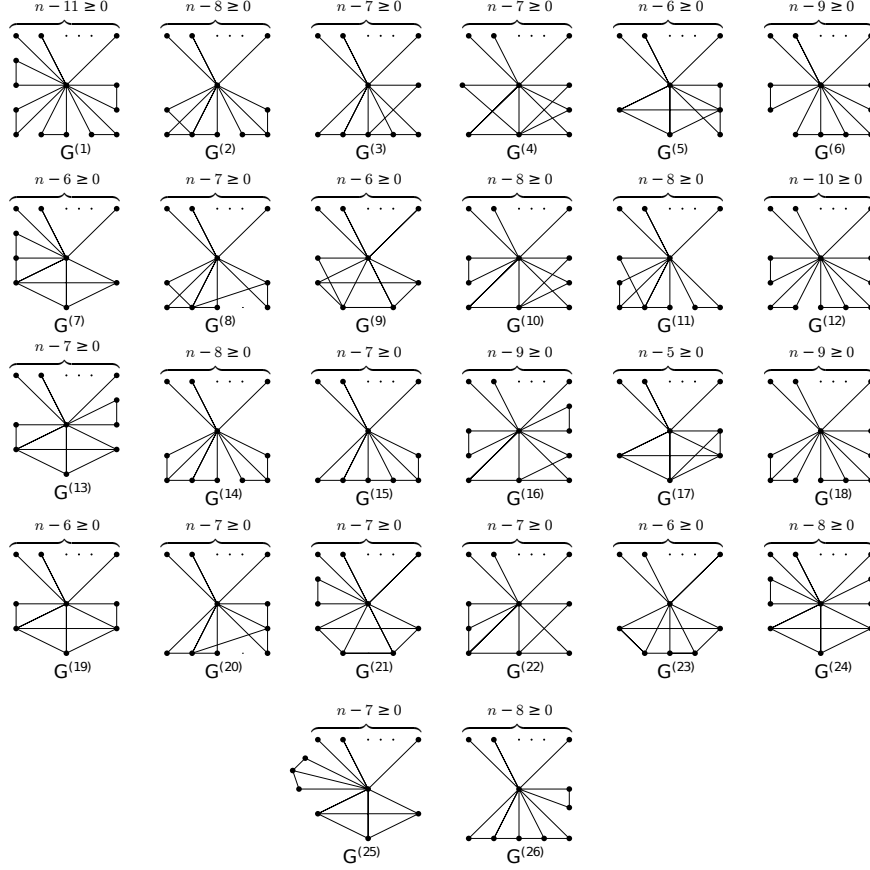


FIGURE 1. All the non-isomorphic graphs in $\mathbb{G}_{n,5}$ with maximum degree $n-1$.

Theorem 3. For $n \geq 6$, the graph $G^{(17)}$ (which is isomorphic to $K_4^{n-4}(2)$ and is depicted in Figure 1) uniquely maximizes M_2 among all the members of $\mathbb{G}_{n,5}$, where $M_2(G^{(17)}) = n^2 + 8n + 55$.

Proof. Firstly, we enumerate all the non-isomorphic members of $\mathbb{G}_{n,5}$ having maximum vertex degree $n-1$, which can be obtained from the n -vertex star graph S_n by adding five (possible) edges. All these non-isomorphic graphs are shown in Figure 1. From Lemma 7, it follows that the member of $\mathbb{G}_{n,5}$ which maximizes M_2 must be one of the graphs $G^{(1)}, G^{(2)}, \dots, G^{(26)}$. Let $a_1 = 11, a_2 = 27, a_3 = a_{10} = 35, a_4 = 51, a_5 = 50, a_6 = 20, a_7 = a_8 = 42, a_9 = 41, a_{11} = 29, a_{12} = 15, a_{13} = a_{23} = 36, a_{14} = 24, a_{15} = a_{25} = 30, a_{16} = 23, a_{17} = 55, a_{18} = 19, a_{19} = 47, a_{20} = 39, a_{21} = 31, a_{22} = 34, a_{24} = 26, a_{26} = 25$. Routine calculations yield $M_2(G^{(i)}) = n^2 + 8n + a_i$, where $i = 1, 2, 3, \dots, 26$. Simple comparison gives the desired result. $\square$

3. CONCLUDING REMARKS

We have characterized the graphs having minimum M_2 value among all the n -vertex connected graphs (and molecular graphs) with cyclomatic number $v \geq 3$, where $n \geq 2(v - 1)$. We have also characterized the graphs having maximum M_2 value among all the n -vertex connected graphs with cyclomatic number $v = 5$ and thereby confirmed the Conjecture 1 for $v = 5$. We believe that this conjecture is true. However, at the present moment, we do not have its proof.

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ON e -CONVEXITY

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Received 5 December, 2019

Abstract. In this paper, we examine a generalized convexity type inequality, called e -convexity.

2010 Mathematics Subject Classification: 39B22, 39B12.

Keywords: approximate convexity, strengthening of convexity, Hermite–Hadamard type inequalities

1. INTRODUCTION

Throughout this paper denote by $\mathbb{R}, \mathbb{R}_+, \mathbb{Z}$, and $\mathbb{N}$ the sets of real numbers, non-negative real numbers, integers, and positive integers, respectively, and denote I by a nonempty subinterval of $\mathbb{R}$.

The stability theory of convexity started with the paper [2] of Hyers and Ulam who defined the ε -convex functions: If D is a convex subset of a real linear space X and ε is a nonnegative number, then a function $f : D \rightarrow \mathbb{R}$ is called ε -convex, if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon$$

for all $x, y \in D, t \in [0, 1]$. The basic result obtained by Hyers and Ulam states that if the underlying space X is finite dimension then f can be written as $f = g + h$, where g is a convex function and h is a bounded function whose supremum norm is not larger than $k_n \varepsilon$, where the positive constant k_n depends only on the dimension of the underlying space X .

In [7], Páles introduced a more general notion than ε -convexity. Let ε, δ be non-negative constants. A function $f : D \rightarrow \mathbb{R}$ is called (ε, δ) -convex, if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \delta + \varepsilon t(1-t)\|x - y\|$$

for every $x, y \in D$ and $t \in [0, 1]$. The main results of the paper [7] obtain a complete characterization of (ε, δ) -convexity, if $D \subset \mathbb{R}$ is an open real interval by showing that these functions are of the form $f = g + h + l$, where g is convex, h is bounded with $|h| \leq \delta/2$ and l is Lipschitzian with Lipschitz modulus $\text{Lip}(l) \leq \varepsilon$.

In [1], Alizadeh and Roohi introduced a general convexity notion, the so-called σ -convexity, namely let $\sigma : D \rightarrow \mathbb{R}$ be a nonnegative function. We say that a function

$f : D \rightarrow \mathbb{R}$ is σ -convex, if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + t(1-t)\min(\sigma(x), \sigma(y))\|x - y\|$$

for all $x, y \in D$ and $t \in [0, 1]$.

In this paper the relations between the σ -monotonicity and σ -convexity were investigated. Moreover, some results on the sum and difference of two σ -monotone operator was considered. In this paper, we would like to generalize the notion of σ -convexity and we would like to consider the basic properties of this generalized convexity. Namely, we will characterize e -convexity in the real case, give a kind of strengthening of e -convexity, give Bernstein–Doetsch type result, search relations between Hermite–Hadamard type inequalities and e -convexity.

2. MAIN RESULTS

Let X be a linear space and D be a nonempty convex subset of X , moreover let $e : D \times D \rightarrow [0, \infty[$ be a nonnegative, symmetric error function. We say that $f : D \rightarrow \mathbb{R}$ is e -convex, if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + t(1-t)e(x, y) \quad (t \in [0, 1], x, y \in D). \quad (2.1)$$

If the above inequality stands for a $t \in]0, 1[$, we say that the function f is (t, e) -convex.

Remark 1. The e -convexity reduces to

- 1) convexity if $e(x, y) = 0$, for all $x, y \in D$;
- 2) ε -convexity if $e(x, y) = \varepsilon\|x - y\|$ for all $x, y \in D$ and for a fixed $\varepsilon \geq 0$;
- 3) paraconvexity if $e(x, y) = C\|x - y\|^2$ for all $x, y \in D$ and for a fixed $C \in \mathbb{R}$;
- 4) $\alpha(\cdot)$ -paraconvexity if $e(x, y) = C\alpha(\|x - y\|)$, for all $x, y \in D$, where $C > 0$ and α is a nondecreasing function mapping the interval $[0, +\infty[$ into the interval $[0, \infty[$. (see [6])
- 5) σ -convexity, if $e(x, y) = \min(\sigma(x), \sigma(y))\|x - y\|$, if X is a normed space, and $\sigma : D \rightarrow \mathbb{R}$ be a nonnegative function.

In the following theorem, we would like to give a strengthening type result. This result is similar as in [3].

Theorem 1. *If the function f is e -convex on D , then the following e -convexity type inequality also holds,*

$$\begin{aligned} f(tx + (1-t)y) &\leq tf(x) + (1-t)f(y) + t(1-t)e(x, y) \\ &+ \frac{t-r}{s-r} \cdot \frac{s-t}{s-r} (e(sx + (1-s)y, rx + (1-r)y) - (s-r)^2 e(x, y)) \end{aligned} \quad (2.2)$$

for all $x, y \in D$, and $0 \leq r \leq t \leq s \leq 1$.

Proof. Write, in (2.1), x by $sx + (1-s)y$ and y by $rx + (1-r)y$, we can get that

$$f((ts + (1-t)r)x + (1 - (ts + (1-t)r))y)$$

$$\begin{aligned}
&= f(t(sx + (1-s)y) + (1-t)(rx + (1-r)y)) \\
&\leq tf(sx + (1-s)y) + (1-t)f(rx + (1-r)y) \\
&\quad + t(1-t)e(sx + (1-s)y, rx + (1-r)y) \\
&\leq t\left(sf(x) + (1-s)f(y) + s(1-s)e(x, y)\right) \\
&\quad + (1-t)\left(rf(x) + (1-r)f(y) + r(1-r)e(x, y)\right) \\
&\quad + t(1-t)e(sx + (1-s)y, rx + (1-r)y) \\
&\leq (ts + (1-t)r)f(x) + (1-(ts + (1-t)r))f(y) \\
&\quad + ts(1-s)e(x, y) + (1-t)r(1-r)e(x, y) \\
&\quad + t(1-t)e(sx + (1-s)y, rx + (1-r)y).
\end{aligned}$$

Let $u = ts + (1-t)r$, then $t = \frac{u-r}{s-r}$ and from the previous inequality we have that,

$$\begin{aligned}
f(ux + (1-u)y) &\leq uf(x) + (1-u)f(y) + u(1-u)e(x, y) \\
&\quad + (ts(1-s) + (1-t)r(1-r) - u(1-u))e(x, y) \\
&\quad + t(1-t)e(sx + (1-s)y, rx + (1-r)y).
\end{aligned}$$

Applying the previous substitution, we have (2.2), which proves the statement. $\square$

Remark 2. According to the previous theorem, we may assume that the plus error term is nonnegative, namely

$$e(sx + (1-s)y, rx + (1-r)y) - (s-r)^2 e(x, y) \geq 0 \quad \text{for all } 0 \leq r \leq s \leq 1.$$

If not, we can strengthen our error term with the e -type function in (2.2). This means that the error e has the property of superquadratic. For example, in the case of normed space, if $e(x, y) = \|x - y\|^p$, where $p > 0$. We can get $p \leq 2$.

Theorem 2. *The e -convexity of the function $f : D \rightarrow \mathbb{R}$ is equivalent with the following property: For all $x_1, \dots, x_n \in D$, $t_i \geq 0$ with $\sum_{i=1}^n t_i = 1$,*

$$\begin{aligned}
&f\left(\sum_{i=1}^n t_i x_i\right) \\
&\leq \sum_{i=1}^n t_i f(x_i) + \sum_{j=1}^n \sum_{i=1}^{j-1} \frac{t_i t_j}{\left(\sum_{k=1}^j t_k\right)^2} \cdot \\
&\quad \cdot e\left(\left(\sum_{k=1}^j t_k\right) x_i + \sum_{k=j+1}^n (t_k x_k), \left(\sum_{k=1}^j t_k\right) x_{i+1} + \sum_{k=j+1}^n (t_k x_k)\right).
\end{aligned} \tag{2.3}$$

Proof. Assume that f is e -convex on D . We will show (2.3) by induction. If $n = 2$, we have the e -convexity of f . Let us assume that (2.3) satisfies for $n \in \mathbb{N}$. Let's consider the case $n + 1$. Let $x_1, \dots, x_n, x_{n+1} \in I$ and $t_i \geq 0$ with $\sum_{i=1}^{n+1} t_i = 1$. If

$t_{n+1} = 1$, the statement is true. If it is not, then $1 - t_{n+1} = \sum_{i=1}^n t_i$. Then using the inductive assumption, and some simple computation, finally the e -convexity of f , we can get that,

$$\begin{aligned}
f\left(\sum_{j=1}^{n+1} t_j x_j\right) &= f\left(\sum_{j=1}^n \frac{t_j}{1-t_{n+1}} ((1-t_{n+1})x_j + t_{n+1}x_{n+1})\right) \\
&\leq \sum_{j=1}^n \frac{t_j}{1-t_{n+1}} f((1-t_{n+1})x_j + t_{n+1}x_{n+1}) \\
&\quad + \sum_{j=1}^n \sum_{i=1}^{j-1} \frac{\frac{t_i}{1-t_{n+1}} \frac{t_j}{1-t_{n+1}}}{\left(\sum_{k=1}^j \frac{t_k}{1-t_{n+1}}\right)^2} e\left(\left(\sum_{k=1}^j \frac{t_k}{1-t_{n+1}}\right) ((1-t_{n+1})x_i + t_{n+1}x_{n+1})\right. \\
&\quad \left.+ \sum_{k=j+1}^n \frac{t_k}{1-t_{n+1}} ((1-t_{n+1})x_k + t_{n+1}x_{n+1})\right), \\
&\quad \left(\sum_{k=1}^j \frac{t_k}{1-t_{n+1}}\right) ((1-t_{n+1})x_{i+1} + t_{n+1}x_{n+1}) + \sum_{k=j+1}^n \frac{t_k}{1-t_{n+1}} ((1-t_{n+1})x_k + t_{n+1}x_{n+1})\right) \\
&= \sum_{j=1}^n \frac{t_j}{1-t_{n+1}} f((1-t_{n+1})x_j + t_{n+1}x_{n+1}) \\
&\quad + \sum_{j=1}^n \sum_{i=1}^{j-1} \frac{t_i t_j}{\left(\sum_{k=1}^j t_k\right)^2} e\left(\left(\sum_{k=1}^j t_k\right) x_i + \sum_{k=j+1}^n (t_k x_k) + t_{n+1}x_{n+1},\right. \\
&\quad \left.\left(\sum_{k=1}^j t_k\right) x_{i+1} + \sum_{k=j+1}^n (t_k x_k) + t_{n+1}x_{n+1}\right) \\
&\leq \sum_{j=1}^n \frac{t_j}{1-t_{n+1}} ((1-t_{n+1})f(x_j) + t_{n+1}f(x_{n+1}) + t_{n+1}(1-t_{n+1})e(x_j, x_{n+1})) \\
&\quad + \sum_{j=1}^n \sum_{i=1}^{j-1} \frac{t_i t_j}{\left(\sum_{k=1}^j t_k\right)^2} e\left(\left(\sum_{k=1}^j t_k\right) x_i + \sum_{k=j+1}^{n+1} (t_k x_k), \left(\sum_{k=1}^j t_k\right) x_{i+1} + \sum_{k=j+1}^{n+1} (t_k x_k)\right) \\
&= \sum_{j=1}^{n+1} t_j f(x_j) + \sum_{j=1}^n t_j t_{n+1} e(x_j, x_{n+1}) \\
&\quad + \sum_{j=1}^n \sum_{i=1}^{j-1} \frac{t_i t_j}{\left(\sum_{k=1}^j t_k\right)^2} e\left(\left(\sum_{k=1}^j t_k\right) x_i + \sum_{k=j+1}^{n+1} (t_k x_k), \left(\sum_{k=1}^j t_k\right) x_{i+1} + \sum_{k=j+1}^{n+1} (t_k x_k)\right) \\
&= \sum_{j=1}^{n+1} t_j f(x_j)
\end{aligned}$$

$$+ \sum_{j=1}^{n+1} \sum_{i=1}^{j-1} \frac{t_i t_j}{\left(\sum_{k=1}^j t_k\right)^2} e \left(\left(\sum_{k=1}^j t_k\right) x_i + \sum_{k=j+1}^{n+1} (t_k x_k), \left(\sum_{k=1}^j t_k\right) x_{i+1} + \sum_{k=j+1}^{n+1} (t_k x_k) \right),$$

which proves the statement. The substitution $n = 2$ give that the implication (iii) $\rightarrow$ (i) also holds. $\square$

Theorem 3. Let I be an open interval in $\mathbb{R}$, then $f : I \rightarrow \mathbb{R}$ is e -convex on I , if and only if for all $x < u < y$ from I ,

$$\frac{f(u) - f(x)}{u - x} \leq \frac{f(y) - f(u)}{y - u} + \frac{e(x, y)}{y - x} \quad (2.4)$$

holds.

Proof. Assume that f is e -convex on I , then substituting $tx + (1 - t)y$ by u , $x < u < y$ in (2.1), we can get that $t = \frac{y-u}{y-x}$ and

$$f(u) \leq \frac{y-u}{y-x} f(x) + \frac{u-x}{y-x} f(y) + \frac{y-u}{y-x} \cdot \frac{u-y}{y-x} e(x, y).$$

Rearranging the above inequality we can get (2.4).

The implication (ii) $\rightarrow$ (i) is also a simple calculation. Namely with the substitution $u = tx + (1 - t)y$ we have the e -convexity of f . $\square$

Corollary 1. If $f : I \rightarrow \mathbb{R}$ is differentiable and e -convex on I , then

$$f(x) - f(y) \geq f'(y)(x - y) - e(x, y) \quad (x, y \in I). \quad (2.5)$$

Proof. Taking the limit $y \rightarrow u$ in (2.4), we have (2.5). $\square$

Corollary 2. If $f : I \rightarrow \mathbb{R}$ is differentiable and e -convex, then

$$(f'(x) - f'(y))(x - y) \geq -2e(x, y) \quad (x, y \in I). \quad (2.6)$$

Proof. Let $x, y \in I$, then using (2.5) and applying the substitution x by y and y by x , and adding the two inequalities, we have (2.6). $\square$

Proposition 1. Let $I = [a, b]$. If $e : I \times I \rightarrow [0, \infty[$ is upper semicontinuous and f is e -convex, then f is continuous.

Proof. Assume that x_0 in I and (x_n) is a sequence in $]x_0, b[$, converging to x_0 . Then,

$$x_n = \lambda_n b + (1 - \lambda_n)x_0 \quad \text{with} \quad \lambda_n \rightarrow 0.$$

On the other hand $x_n \in]a, x_0[$. Thus there exists $\lambda'_n \in [0, 1]$, such that

$$x_0 = \lambda'_n a + (1 - \lambda'_n)x_n \quad \text{with} \quad \lambda'_n \rightarrow 0.$$

Since f is e -convex, we have that

$$f(x_n) \leq \lambda_n f(b) + (1 - \lambda_n)f(x_0) + \lambda_n(1 - \lambda_n)e(b, \lambda_n b + (1 - \lambda_n)x_0).$$

Therefore, by taking the $\limsup$ in the above inequality we have that

$$\limsup_{n \rightarrow \infty} f(x_n) \leq f(x_0).$$

However,

$$f(x_0) \leq \lambda'_n f(a) + (1 - \lambda'_n) f(x_n) + \lambda'_n (1 - \lambda'_n) e(a, \lambda'_n a + (1 - \lambda_n) x_n).$$

Taking the $\liminf$, we have that

$$f(x_0) \leq \liminf f(x_n).$$

□

Remark 3. Assume that $0 \in I$ and $f : I \rightarrow \mathbb{R}$ is e -convex. If $f(0) \leq 0$ and $e(x, 0) = 0$ for all $x \in I$, then f is super-additive on $I \cap [0, \infty)$. Indeed, by the e -convexity of f , we have

$$f(tx) = f(tx + (1-t)0) \leq tf(x) + (1-t)f(0) + t(1-t)e(x, 0) \leq tf(x).$$

On the other hand, for all $x, y \in I$

$$\begin{aligned} f(x) + f(y) &= f\left((x+y)\frac{x}{x+y}\right) + f\left((x+y)\frac{y}{x+y}\right) \\ &\leq \frac{x}{x+y}f(x+y) + \frac{y}{x+y}f(x+y) = f(x+y) \end{aligned}$$

In what follows, we find connections between a lower Hermite–Hadamard type inequality and e -convexity. We will need the definition of hemi-property. The function $f : D \rightarrow \mathbb{R}$ has a *hemi-property*, if for all $x, y \in D$ the map

$$t \rightarrow f(tx + (1-t)y) \quad t \in [0, 1] \quad (2.7)$$

has got that property. For example $f : D \rightarrow \mathbb{R}$ is hemi-bounded, if for all $x, y \in D$ the function defined by (2.7) is bounded.

Now, we recall a theorem of [5].

Theorem 4. Let D be a convex set of a linear space X . Let $\mathcal{A}$ be a sigma algebra containing the Borel subsets of $[0, 1]$ and μ be a probability measure on the measure space $([0, 1], \mathcal{A})$ such that the support of μ is not a singleton. Denote

$$S(\mu) := \mu([0, \mu_1]) \int_{[\mu_1, 1]} t d\mu(t) - \mu(] \mu_1, 1]) \int_{[0, \mu_1]} t d\mu(t). \quad (2.8)$$

Assume that $f : D \rightarrow \mathbb{R}$ is an hemi- μ -integrable solution of the functional inequality

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y) + e_{x,y}(t) \quad ((x, y) \in D^2, t \in [0, 1]), \quad (2.9)$$

where, for all $(x, y) \in D^{2*}$, $e_{x,y} : [0, 1] \rightarrow \mathbb{R}$ is a function such that

$$I(x, y) := \int_{[\mu_1, 1]} \int_{[0, \mu_1]} (t'' - t') e_{(1-t')x + t'y, (1-t'')x + t''y} \left(\frac{\mu_1 - t'}{t'' - t'} \right) d\mu(t') d\mu(t'') \quad (2.10)$$

exists in $[-\infty, \infty]$ for all $(x, y) \in D^{2*}$. Then, for all $(x, y) \in D^{2*}$, the function f also satisfies the lower Hermite–Hadamard type inequality

$$f((1 - \mu_1)x + \mu_1 y) \leq \int_{[0,1]} f((1 - t)x + ty) d\mu(t) + E(x, y) \quad ((x, y) \in D^2), \quad (2.11)$$

where

$$E(x, y) := \frac{I(x, y)}{S(\mu)} \quad ((x, y) \in D^{2*}). \quad (2.12)$$

The following result gives a lower Hermite–Hadamard type inequality for e -convex functions and it is a simple connection of the previous theorem.

Corollary 3. *Let D be a convex set of a linear space X . Let $\mathcal{A}$ be a sigma algebra containing the Borel subsets of $[0, 1]$ and μ be a probability measure on the measure space $([0, 1], \mathcal{A})$ such that the support of μ is not a singleton. Let $S(\mu)$ defined by (2.8). Assume that $f : D \rightarrow \mathbb{R}$ is an hemi- μ -integrable solution of the e -convexity type inequality (2.1), moreover let $I(x, y)$ - defined by (2.10) - exist in $[-\infty, \infty]$, for all $(x, y) \in D^2$. Then f satisfies the lower Hermite–Hadamard type inequality, (2.11), where E is defined by (2.12).*

Now, we apply this corollary for Lebesgue integral.

Corollary 4. *Let D be a convex set of a linear space X . Assume that $f : D \rightarrow \mathbb{R}$ is an hemi-Lebesgue-integrable solution of the e -convexity inequality (2.1). Then f satisfies the following lower Hermite–Hadamard type inequality,*

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f((1-t)x + ty) dt + 4I(x, y) \quad ((x, y) \in D^2), \quad (2.13)$$

where

$$I(x, y) := \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} \frac{(\frac{1}{2} - t')(t'' - \frac{1}{2})}{t'' - t'} e((1-t')x + t'y, (1-t'')x + t''y) dt' dt''.$$

Proof. Denote by λ the Lebesgue measure on $[0, 1]$. Then $\lambda_1 = \int_0^1 t dt = \frac{1}{2}$. On the other hand,

$$\begin{aligned} S(\lambda) &:= \lambda([0, \lambda_1]) \int_{[\lambda_1, 1]} t dt - \lambda([\lambda_1, 1]) \int_{[0, \lambda_1]} t dt \\ &= \lambda([0, \frac{1}{2}]) \int_{\frac{1}{2}}^1 t dt - \lambda([\frac{1}{2}, 1]) \int_0^{\frac{1}{2}} t dt = \frac{1}{4}. \end{aligned}$$

□

Now, we recall a result from [4].

Theorem 5. Let μ be a Borel probability measure on $[0, 1]$, denote $\mu_1 := \int_{[0,1]} t d\mu(t)$ and assume that the support of μ is not a singleton, i.e., $\mu \neq \delta_{\mu_1}$. Assume that, for all $(x, y) \in D^2$, $f : D \rightarrow \mathbb{R}$ is an upper hemicontinuous solution of the functional inequality (2.13), where $E : D^2 \rightarrow \mathbb{R}$. Assume that, for all $(x, y) \in D^2$, $e_{x,y} : [0, 1] \rightarrow \mathbb{R}$ is a lower semicontinuous function with $e_{x,y}(0) = e_{x,y}(1) = 0$ satisfying the following system of inequalities:

$$e_{x,y}(s) \geq \begin{cases} \int_{[0,1]} e_{x,y}\left(\frac{st}{\mu_1}\right) d\mu(t) + E\left(x, \left(1 - \frac{s}{\mu_1}\right)x + \frac{s}{\mu_1}y\right) & (s \in [0, \mu_1]), \\ \int_{[0,1]} e_{x,y}\left(1 - \frac{(1-s)(1-t)}{1-\mu_1}\right) d\mu(t) + E\left(\frac{1-s}{1-\mu_1}x + \left(1 - \frac{1-s}{1-\mu_1}\right)y, y\right) & (s \in [\mu_1, 1]). \end{cases} \quad (2.14)$$

Then, for all $(x, y) \in D^2$ and $s \in [0, 1]$, the function f also satisfies the approximate convexity inequality (2.9).

The following proposition states that from Hermite–Hadamard type inequality, we can get e -convexity.

Corollary 5. Let μ be a Borel probability measure on $[0, 1]$, denote $\mu_1 := \int_{[0,1]} t d\mu(t)$, $\mu_2 = \int_{[0,1]} t^2 d\mu(t)$ and assume that the support of μ is not a singleton, i.e., $\mu \neq \delta_{\mu_1}$. Assume that, for all $(x, y) \in D^2$, $f : D \rightarrow \mathbb{R}$ is an upper hemicontinuous solution of the functional inequality (2.13), where $E : D^2 \rightarrow \mathbb{R}$. Assume that, for all $(x, y) \in D^2$, $e : D \times D \rightarrow \mathbb{R}$ is a lower semicontinuous function with satisfying the following system of inequalities:

$$\begin{cases} s^2\left(\frac{\mu_2}{\mu_1^2} - 1\right)e_{x,y}(s) \geq E\left(x, \left(1 - \frac{s}{\mu_1}\right)x + \frac{s}{\mu_1}y\right) & (s \in [0, \mu_1]), \\ (1-s)^2\left(\frac{1-2\mu_1+\mu_2}{(1-\mu_1)^2}\right)e(x, y) \geq E\left(\frac{1-s}{1-\mu_1}x + \left(1 - \frac{1-s}{1-\mu_1}\right)y, y\right) & (s \in [\mu_1, 1]). \end{cases} \quad (2.15)$$

Then, the function f is e -convex on D .

Proof. Define for $x, y \in D$ and $t \in [0, 1]$, the function $e_{x,y}$ by the following formulae:

$$e_{x,y}(t) = t(1-t)e(x, y)$$

Simple calculations shows that (2.14) reduces (2.15). Using the previous theorem, we have the e -convexity of f . $\square$

Corollary 6. Let $d : D \times D \rightarrow [0, \infty[$ be a symmetric function. Assume that $f : D \rightarrow \mathbb{R}$ is a hemi-continuous solution of the functional inequality,

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f(tx + (1-t)y)dt + d(x, y), \quad (x, y \in D)$$

with

$$\frac{1}{12}s^2e(x, y) \geq d(x, (1-s)x + sy) \quad (s \in [0, 1], x, y \in D). \quad (2.16)$$

Then f is e -convex.

Proof. In this case, μ is the Lebesgue measure, which is denoted by λ . $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = \frac{1}{3}$. Using the symmetry of the function d it is also easy to see that (2.15) reduces (2.16). Applying the previous corollary, we can get the e -convexity of f . $\square$

The following proposition states a Bernstein–Doetsch type theorem for e -convexity.

Corollary 7. *Let X is normed space and D is a nonempty, open and convex subset of X . Let $d : D \times D \rightarrow [0, \infty[$ be a symmetric function. Let $f : D \rightarrow \mathbb{R}$ be a continuous solution of the following functional inequality,*

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} + d(x,y)$$

Assume that $e : D \times D \rightarrow [0, \infty[$ is symmetric and it satisfies the following functional inequality,

$$\frac{s^2}{4}e(x,y) \geq d(x, (1-s)x + sy) \quad s \in [0, 1], s \in [0, 1].$$

Then f is e -convex.

Proof. Let μ be the Dirac-measure which concentrated to $\frac{1}{2}$. Then, from Corollary 6, we can get the statement. $\square$

ACKNOWLEDGEMENTS

The article was carried out as part of the EFOP-3.6.1-16-2016-00011 “Younger and Renewing University – Innovative Knowledge City – institutional development of the University of Miskolc aiming at intelligent specialisation” project implemented in the framework of the Szechenyi 2020 program. The realization of this project is supported by the European Union, co-financed by the European Social Fund.

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THE YANG-BAXTER MATRIX EQUATION FOR INVOLUTIONS

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Received 10 March, 2021

Abstract. In this paper we find all involutive solutions $X \in \mathbb{C}^{n \times n}$ of the Yang-Baxter matrix equation $AXA = XAX$, where $A \in \mathbb{C}^{n \times n}$ is a given involutory matrix. The construction is algorithmic. It is based on the concept of quadratic matrices. Algorithms for generating concrete involutive solutions of the Yang-Baxter matrix equation $AXA = XAX$ are also presented along with several examples.

2010 Mathematics Subject Classification: 15A06; 15A18; 15B99

Keywords: Yang-Baxter equation, involution, eigenvalues, quadratic matrices

1. INTRODUCTION

We recall that $A \in \mathbb{C}^{n \times n}$ is an involutory matrix (involution) if $A^2 = I_n$, where I_n is the $n \times n$ identity matrix. Let $\mathcal{K}_n$ denote the set of all involutory matrices of size n . The purpose of this paper then is to find all explicit involutive solutions X of the Yang-Baxter matrix equation $AXA = XAX$, where A is involution. Our research is inspired by the papers [5] and [7], where the general analysis of finding all explicit solutions of $AXA = XAX$ for given involutory A has been given. In this paper we restrict our attention to involutory matrices X . This problem is related to the quantum Yang-Baxter equation (QYBE). We say that $Z \in \mathbb{R}^{m^2 \times m^2}$ satisfies the QYBE if

$$(I_m \otimes Z)(Z \otimes I_m)(I_m \otimes Z) = (Z \otimes I_m)(I_m \otimes Z)(Z \otimes I_m) \quad (1.1)$$

where $B \otimes C$ denotes the Kronecker product (tensor product) of the matrices B and C : $B \otimes C = (b_{i,j}C)$. That is, the Kronecker product $B \otimes C$ is a block matrix whose (i, j) blocks are $b_{i,j}C$. Notice that if Z is involution and satisfies (1.1), then $A = I_m \otimes Z$ and $X = Z \otimes I_m$ are involutions as well, and we have $AXA = XAX$.

The Yang-Baxter equation has many applications of modern physics, computer science and mathematics ([6, 8, 9]), for example, in statistical mechanics, integrable

The research of Ryszard R. Andruszkiewicz is partially supported by the National Science Centre, Poland, grant no. 2019/35/B/ST1/01115.

quantum field theory, condensed matter physics, quantum integrable models, quantum computation, to name a few. Many techniques for the construction of involutive solutions of the Yang-Baxter equation have been developed, see [2–4, 8].

For given $A \in \mathcal{K}_U$ we define the set $\mathcal{S}_n(A)$ of all involutive solutions of the Yang-Baxter matrix equation thus:

$$\mathcal{S}_n(A) = \{X \in \mathcal{K}_U : AXA = XAX\}.$$

Using the concept of quadratic matrices, see Theorem 2, we are able to find the explicit expressions of such solutions for all the case, see Theorems 3–6. This enables the development of methods, easy to implement, for generating concrete involutive solutions of the Yang-Baxter matrix equation, see Section 8.

First we present easily-checkable basic properties of the set $\mathcal{S}_n(A)$.

Lemma 1. *Let $A \in \mathcal{K}_U$. Then we have*

- (a): $A \in \mathcal{S}_n(A)$,
- (b): if $X \in \mathcal{S}_n(A)$ then $-X \in \mathcal{S}_n(-A)$,
- (c): if $X \in \mathcal{S}_n(A)$ then A and X are similar; we have $X = (AX)A(AX)^{-1}$,
- (d): if $P \in \mathbb{C}^{n \times n}$ is nonsingular, then $X \in \mathcal{S}_n(A)$ if and only if $P^{-1}XP \in \mathcal{S}_n(P^{-1}AP)$,
- (e): if $A = \pm I_n$ then $\mathcal{S}_n(A) = \{A\}$.

Assume that $A \in \mathbb{C}^{n \times n}$ is a given involution. Then A is diagonalizable. According to Lemma 1 (b), there is no loss of generality in assuming that A is not equal to $\pm I_n$, and there exists a nonsingular matrix $P \in \mathbb{C}^{n \times n}$ such that

$$A = PDP^{-1}, \quad D = \text{diag}(I_p, -I_{n-p}), \quad 1 \leq p < n, \quad n \leq 2p. \quad (1.2)$$

Then we have $\mathcal{S}_n(A) = \{PYP^{-1} : Y \in \mathcal{S}_n(D)\}$. Natural questions to ask about $\mathcal{S}_n(D)$ then are: How many $Y \in \mathcal{S}_n(D)$ are there? and, how to find them? In general, the Yang-Baxter matrix equation has infinitely many solutions, see Example 1.

2. IDENTITIES FOR INVOLUTIVE SOLUTIONS OF $DYD = YDY$

We would like to find $Y \in \mathcal{S}_n(D)$, where D is defined by (1.2). Partition Y conformally with D as

$$Y = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}, \quad Y_1(p \times p). \quad (2.1)$$

From Lemma 1 it follows that D and Y are similar, so $\text{tr}Y = \text{tr}D$, where $\text{tr}Y$ denotes the trace of Y . This together with (1.2)-(2.1) gives

$$\text{tr}Y_1 + \text{tr}Y_4 = 2p - n \geq 0. \quad (2.2)$$

Lemma 2. *Let D be given by (1.2) and Y be partitioned as in (2.1). Then $DYD = YDY$ if and only if*

$$Y_1^2 - Y_1 = Y_2Y_3, \quad (2.3)$$

$$Y_4^2 + Y_4 = Y_3Y_2, \quad (2.4)$$

$$(Y_1 + I_p)Y_2 = Y_2Y_4, \quad (2.5)$$

$$Y_3(Y_1 + I_p) = Y_4Y_3. \quad (2.6)$$

Proof. Compare the blocks of the following matrices:

$$DYD = \begin{bmatrix} Y_1 & -Y_2 \\ -Y_3 & Y_4 \end{bmatrix}, \quad YDY = \begin{bmatrix} Y_1^2 - Y_2Y_3 & Y_1Y_2 - Y_2Y_4 \\ Y_3Y_1 - Y_4Y_3 & Y_3Y_2 - Y_4^2 \end{bmatrix}.$$

□

Lemma 3. *Let Y be partitioned as in (2.1). Then Y is an involution if and only if*

$$I_p - Y_1^2 = Y_2Y_3, \quad (2.7)$$

$$I_{n-p} - Y_4^2 = Y_3Y_2, \quad (2.8)$$

$$Y_1Y_2 = -Y_2Y_4, \quad (2.9)$$

$$Y_3Y_1 = -Y_4Y_3. \quad (2.10)$$

Proof. It is evident that $Y^2 = I_n$ holds if and only if

$$Y^2 = \begin{bmatrix} Y_1^2 + Y_2Y_3 & Y_1Y_2 + Y_2Y_4 \\ Y_3Y_1 + Y_4Y_3 & Y_3Y_2 + Y_4^2 \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & I_{n-p} \end{bmatrix}.$$

□

Lemma 4. *Let (1.2)-(2.2) hold. Assume that $Y \in \mathcal{S}_n(D)$. Then Y_1 and Y_4 (called the quadratic matrices) satisfy the equations*

$$(2Y_1 + I_p)(Y_1 - I_p) = 0, \quad (2Y_4 - I_{n-p})(Y_4 + I_{n-p}) = 0. \quad (2.11)$$

Moreover, we have

$$(2Y_1 + I_p)Y_2 = 0, \quad Y_3(2Y_1 + I_p) = 0, \quad (2.12)$$

$$I_p - Y_1 = 2Y_2Y_3, \quad I_{n-p} + Y_4 = 2Y_3Y_2. \quad (2.13)$$

Proof. From (2.3) and (2.7) we get $Y_1^2 - Y_1 = I_p - Y_1^2$. Similarly, from (2.4) and (2.8) we obtain $Y_4^2 + Y_4 = I_{n-p} - Y_4^2$. The above equations are equivalent to (2.11). From (2.5) and (2.9), and from (2.6) and (2.10) we get (2.12). Note that (2.3) together with (2.7), and (2.4) together with (2.8), gives (2.13). □

Theorem 1. *Under the hypotheses of Lemma 4, $Y \in \mathcal{S}_n(D)$ if and only if Y satisfies (2.11)-(2.13).*

Proof. Standard calculations show that the conditions (2.3)-(2.10) are equivalent to (2.11)-(2.13). □

To find concrete solutions of the equation $DYD = YDY$ the following Lemma 5 is useful.

Lemma 5. *Let D be defined by (1.2), and $W = \text{diag}(W_1, W_2) \in \mathbb{C}^{n \times n}$ be arbitrary nonsingular matrix, where $W_1(p \times p)$. Assume that $Y \in \mathcal{S}_n(D)$ and define $\hat{Y} = W^{-1}YW$. Then we have*

- (i): $\hat{Y} \in \mathcal{S}_n(D)$,
- (ii): if we partition $\hat{Y}$ conformally with Y as follows

$$\hat{Y} = \begin{bmatrix} \hat{Y}_1 & \hat{Y}_2 \\ \hat{Y}_3 & \hat{Y}_4 \end{bmatrix} = \begin{bmatrix} W_1^{-1}Y_1W_1 & W_1^{-1}Y_2W_2 \\ W_2^{-1}Y_3W_1 & W_2^{-1}Y_4W_2 \end{bmatrix}, \quad (2.14)$$

then we get the identities

$$(2\hat{Y}_1 + I_p)\hat{Y}_2 = 0, \quad \hat{Y}_3(2\hat{Y}_1 + I_p) = 0, \quad I_p - \hat{Y}_1 = 2\hat{Y}_2\hat{Y}_3 \quad (2.15)$$

and

$$I_{n-p} + \hat{Y}_4 = 2\hat{Y}_3\hat{Y}_2. \quad (2.16)$$

Proof. From Lemma 1 it follows that $Y \in \mathcal{S}_n(D)$ if and only if $\hat{Y} = W^{-1}YW \in \mathcal{S}_n(W^{-1}DW)$. However, since W is a block diagonal matrix, we obtain the identity $W^{-1}DW = D$, so $\hat{Y} \in \mathcal{S}_n(D)$.

To prove part (ii) of Lemma 5, we apply Lemma 4 to $\hat{Y}$ using the fact that $\hat{Y} \in \mathcal{S}_n(D)$. This completes the proof. $\square$

Remark 1. It is obvious that if W is an arbitrary nonsingular matrix then $W^{-1}DW = D$ holds only for block diagonal matrix $W = \text{diag}(W_1, W_2)$, where $W_1(p \times p)$.

Note that if there are two matrices P and Q such that $A = PDP^{-1}$ and $A = QDQ^{-1}$, where D is defined by (1.2), then $W = P^{-1}Q$ is a block diagonal matrix. This together with Lemma 5 leads to

$$\mathcal{S}_n(A) = \{PY P^{-1} : Y \in \mathcal{S}_n(D)\} = \{QY Q^{-1} : Y \in \mathcal{S}_n(D)\}.$$

Remark 2. Notice that Y_1 and Y_4 satisfying (2.11) are nonsingular because each eigenvalue of Y_1 is either 1 or $-\frac{1}{2}$, and each eigenvalue of Y_4 is either -1 or $\frac{1}{2}$. Trivial solutions of (2.11) are: $Y_1 = I_p$, $Y_1 = -\frac{1}{2}I_p$, and $Y_4 = -I_{n-p}$, $Y_4 = \frac{1}{2}I_{n-p}$. In order to characterize other matrices Y_1 and Y_4 satisfying (2.11), we need some properties of quadratic matrices.

3. QUADRATIC MATRICES

We recall that $A \in \mathbb{C}^{n \times n}$ is a quadratic matrix, if there exist $\alpha, \beta \in \mathbb{C}$ such that $(A - \alpha I_n)(A - \beta I_n) = 0$. For the convenience of the reader we repeat the relevant material from [1].

Theorem 2 ([1, Theorem 1.2]). *Let $(A - \alpha I_n)(A - \beta I_n) = 0$, where $\alpha, \beta \in \mathbb{C}$. Assume that $A \neq \alpha I_n, \beta I_n$.*

- (1) Then there exist a unitary matrix $U \in \mathbb{C}^{n \times n}$ and triangular $R \in \mathbb{C}^{n \times n}$ such that $A = URU^*$ (the Schur form), where

$$R = \begin{bmatrix} \alpha I_k & G \\ 0 & \beta I_{n-k} \end{bmatrix}, \quad 1 \leq k < n. \quad (3.1)$$

- (2) Each eigenvalue of A is either α or β .
 (3) If $\alpha \neq \beta$ then A is diagonalizable, and can be written as $A = PDP^{-1}$, where $P = UW$, and W is involution containing the eigenvectors of R , i.e. $RW = WR$, where

$$W = \begin{bmatrix} I_k & \frac{G}{\alpha - \beta} \\ 0 & -I_{n-k} \end{bmatrix}, \quad D = \begin{bmatrix} \alpha I_k & 0 \\ 0 & \beta I_{n-k} \end{bmatrix}. \quad (3.2)$$

Lemma 6. Let the quadratic matrices Y_1 and Y_4 satisfy (2.11), where $n \leq 2p$. Assume that $Y_1 \neq I_p, -\frac{1}{2}I_p$ and $Y_4 \neq -I_{n-p}, \frac{1}{2}I_{n-p}$. Then there exist nonsingular matrices P_1 and P_4 such that

$$Y_1 = P_1 D_1 P_1^{-1}, \quad D_1 = \text{diag}(-\frac{1}{2}I_r, I_{p-r}), \quad 1 \leq r < p \quad (3.3)$$

and

$$Y_4 = P_4 D_4 P_4^{-1}, \quad D_4 = \text{diag}(\frac{1}{2}I_s, -I_{n-p-s}), \quad 1 \leq s < n-p \leq p. \quad (3.4)$$

Proof. It follows from Lemma 4 and Theorem 2. $\square$

Remark 3. We see that all possible cases for Y_1 and Y_4 are: $Y_1 = I_p$ or $Y_1 = -\frac{1}{2}I_p$, or Y_1 satisfies (3.3). Similarly, Y_4 is equal to $-I_{n-p}$ or $\frac{1}{2}I_{n-p}$, or Y_4 satisfies (3.4). We consider all these cases.

4. SOLUTIONS OF $DYD = YDY$ FOR $Y_1 = I_p$

Theorem 3. Let (1.2)-(2.2) hold. If $Y_1 = I_p$ or $Y_4 = -I_{n-p}$ then $Y = D$ is the only solution of the quadratic equation $YDY = DYD$.

Proof. Let $Y_1 = I_p$. Then from (2.12) it follows that $Y_2 = 0$ and $Y_3 = 0$. This together with (2.13) leads to $Y_4 = -I_{n-p}$, hence $Y = D$.

Now assume that $Y_4 = -I_{n-p}$. Then from (2.13) we have $Y_3 Y_2 = 0$. Since $\text{tr}(Y_3 Y_2) = 0$ and $\text{tr}(2Y_3 Y_2) = \text{tr}(2Y_2 Y_3) = \text{tr}(I_p - Y_1) = p - \text{tr} Y_1$, we get $\text{tr} Y_1 = p$, so $Y_1 = I_p$. This together with (2.12) leads to $Y_2 = 0$ and $Y_3 = 0$, hence $Y = D$. $\square$

5. SOLUTIONS OF $DYD = YDY$ FOR $Y_1 = -\frac{1}{2}I_p$

Theorem 4. Let (1.2)-(2.2) hold and $Y \in \mathcal{S}_n(D)$. Assume that $Y_1 = -\frac{1}{2}I_p$. Then $n = 2p$, $Y_4 = \frac{1}{2}I_p$, and solutions of $YDY = DYD$ are

$$Y = \begin{bmatrix} -\frac{1}{2}I_p & Y_2 \\ \frac{3}{4}Y_2^{-1} & \frac{1}{2}I_p \end{bmatrix}, \quad (5.1)$$

where $Y_2(p \times p)$ is an arbitrary nonsingular matrix.

Proof. From Theorem 3 it follows that $Y_4 \neq -I_{n-p}$. Assume now that $Y_4 \neq \frac{1}{2}I_{n-p}$. By Remark 3, Y_4 should satisfy (3.4). From (3.4) it follows that $\text{tr}Y_4 = \frac{s}{2} - (n-p-s)$, where $1 \leq s < n-p$. This together with (1.2) gives $s < p$. Therefore, $\text{tr}Y_1 + \text{tr}Y_4 = \frac{s-p}{2} - (n-p-s)$, so from (2.2) it follows that $\text{tr}Y_1 + \text{tr}Y_4 = 2p-n$ if and only if $s = p$, a contradiction. We conclude that $Y_4 = \frac{1}{2}I_{n-p}$. Then $\text{tr}Y_1 + \text{tr}Y_4 = -\frac{p}{2} + \frac{n-p}{2} = \frac{n-2p}{2}$. This together with (2.2) gives $n = 2p$. From (2.13) we get $Y_2Y_3 = \frac{3}{4}I_p$. Therefore, Y_2 and Y_3 are nonsingular, so $Y_3 = \frac{3}{4}Y_2^{-1}$. Clearly, Y has a form (5.1). It is easy to check that such Y satisfies the equation $YDY = YDY$. $\square$

6. SOLUTIONS OF $DYD = YDY$ FOR Y_1 SATISFYING (3.3)

In this section we study two remaining cases: $Y_4 = \frac{1}{2}I_{n-p}$, or Y_4 satisfies (3.4). Through this section we assume that Y_1 satisfies (3.3). We apply Lemma 5 to $W = \text{diag}(P_1, W_2)$, where W_2 is a nonsingular matrix. Then from (2.14) and (3.3) we get that $\hat{Y}_1 = D_1 = \text{diag}(-\frac{1}{2}I_r, I_{p-r})$, and we have

$$2D_1 + I_p = 3\text{diag}(0, I_{p-r}), \quad I_p - D_1 = \frac{3}{2}\text{diag}(I_r, 0).$$

Now we partition $\hat{Y}_2$ and $\hat{Y}_3$ as follows

$$\hat{Y}_2 = \begin{bmatrix} B_2 \\ C_2 \end{bmatrix}, \quad B_2(r \times (n-p)), \quad \hat{Y}_3 = (B_3, C_3), \quad B_3((n-p) \times r). \quad (6.1)$$

Then from (2.15) it follows that

$$C_2 = 0, \quad C_3 = 0, \quad B_2B_3 = \frac{3}{4}I_r. \quad (6.2)$$

Theorem 5. *Let (1.2)-(2.2) hold. Let Y_1 satisfy (3.3) and $Y_4 = \frac{1}{2}I_{n-p}$. Then $r = n-p$, where $1 \leq r < p$. Moreover, $Y \in \mathcal{S}_n(D)$ if and only if $Y = W\hat{Y}W^{-1}$, where $W = \text{diag}(P_1, I_{n-p})$ and*

$$\hat{Y} = \left[\begin{array}{cc|c} -\frac{1}{2}I_{n-p} & 0 & B_2 \\ 0 & I_{2p-n} & 0 \\ \hline \frac{3}{4}B_2^{-1} & 0 & \frac{1}{2}I_{n-p} \end{array} \right], \quad (6.3)$$

where $B_2(n-p) \times (n-p)$ is an arbitrary nonsingular matrix.

Proof. Here $\hat{Y}_1 = D_1$ and $\hat{Y}_4 = Y_4$, where D_1 is defined by (3.3). Then $\text{tr}Y_1 + \text{tr}Y_4 = \text{tr}\hat{Y}_1 + \text{tr}\hat{Y}_4 = -\frac{r}{2} + (p-r) + \frac{n-p}{2}$, so by (2.2) we get $r = n-p$. From this and (6.1)-(6.2) it follows that B_2 and B_3 are nonsingular matrices. By (6.2), we get $B_3 = B_2^{-1}$. Clearly, (2.16) also holds. $\square$

Theorem 6. *Let (1.2)-(2.2) hold. Let Y_1 satisfy (3.3) and Y_4 satisfy (3.4). Then $s = r$, where $1 \leq r < n-p \leq p$. Moreover, $Y \in \mathcal{S}_n(D)$ if and only if $Y = W\hat{Y}W^{-1}$,*

where $W = \text{diag}(P_1, P_4)$ and

$$\hat{Y} = \left[\begin{array}{cc|cc} -\frac{1}{2}I_r & 0 & F_1 & 0 \\ 0 & I_{p-r} & 0 & 0 \\ \hline \frac{3}{4}F_1^{-1} & 0 & \frac{1}{2}I_r & 0 \\ 0 & 0 & 0 & -I_{n-p-r} \end{array} \right],$$

where $F_1(r \times r)$ is an arbitrary nonsingular matrix.

Proof. Here $\hat{Y}_1 = D_1$ and $\hat{Y}_4 = D_4$, where D_1 is defined by (3.3), and D_4 is given in (3.4). Then $\text{tr}Y_1 + \text{tr}Y_4 = \text{tr}\hat{Y}_1 + \text{tr}\hat{Y}_4 = -\frac{r}{2} + (p-r) + \frac{s}{2} - (n-p-s)$, so by (2.2) we get $s = r$.

We use (6.1) and partition B_2 and B_3 as follows

$$B_3 = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \quad F_1(r \times r), \quad B_2 = (G_1, G_2), \quad G_1(r \times r).$$

Then using (6.2) and (2.16) we conclude that $G_1 = \frac{3}{4}F_1^{-1}$, $G_2 = 0$ and $F_2 = 0$. $\square$

7. ALGORITHMS

For given involutory matrix $A \in \mathbb{C}^{n \times n}$ and matrices P, D defined in (1.2), we can compute the concrete involutory solutions X of the equation $AXA = XAX$ as $X = PYP^{-1}$, where Y is a solution of the equation $DYD = YDY$.

In order to help readers implement our methods, we include algorithms for finding such solutions $Y \in \mathcal{S}_n(D)$. The proposed algorithms (Algorithms 1–3) construct **all solutions** $Y \neq D$ (we omit the trivial case where $X = A$, i.e. $Y = D$).

Algorithm 1. Construction of $Y \in \mathcal{S}_{2p}(D)$, using Theorem 4.

Take any natural number p and arbitrary nonsingular matrix $Y_2 \in \mathbb{C}^{p \times p}$.

The algorithm is determined now by two steps:

- compute Y_2^{-1} ,
- form $Y \in \mathbb{C}^{2p \times 2p}$ as follows:

$$Y = \begin{bmatrix} -\frac{1}{2}I_p & Y_2 \\ \frac{3}{4}Y_2^{-1} & \frac{1}{2}I_p \end{bmatrix}.$$

Algorithm 2. Construction of $Y \in \mathcal{S}_n(D)$, using Theorem 5

Take any natural numbers n and p such that $1 \leq n-p < p$, and arbitrary nonsingular matrices $P_1 \in \mathbb{C}^{p \times p}$, $B_2 \in \mathbb{C}^{(n-p) \times (n-p)}$.

The algorithm splits into the following steps:

- compute P_1^{-1} and B_2^{-1} ,
- $Y_1 = P_1 \text{diag}(-\frac{1}{2}I_{n-p}, I_{2p-n})P_1^{-1}$,
- $Y_2 = P_1 \begin{bmatrix} B_2 \\ 0 \end{bmatrix}$,
- $Y_3 = (\frac{3}{4}B_2^{-1}, 0)P_1^{-1}$,

- form $Y \in \mathbb{C}^{n \times n}$ as follows

$$Y = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & \frac{1}{2}I_{n-p} \end{bmatrix}.$$

Algorithm 3. Construction of $Y \in \mathcal{S}_n(D)$, using Theorem 6

Take any natural numbers n, p, r such that $1 \leq r < n - p \leq p$, and arbitrary nonsingular matrices $P_1 \in \mathbb{C}^{p \times p}$, $P_4 \in \mathbb{C}^{(n-p) \times (n-p)}$, and $F_1 \in \mathbb{C}^{r \times r}$.

The algorithm consists with the following steps:

- compute P_1^{-1} , P_4^{-1} and F_1^{-1} ,
- $Y_1 = P_1 \text{diag}(-\frac{1}{2}I_r, I_{p-r})P_1^{-1}$,
- $Y_2 = P_1 \text{diag}(F_1, 0)P_4^{-1}$,
- $Y_3 = P_4 \text{diag}(\frac{3}{4}F_1^{-1}, 0)P_1^{-1}$,
- $Y_4 = P_4 \text{diag}(\frac{1}{2}I_r, -I_{n-p-r})P_4^{-1}$,
- form $Y \in \mathbb{C}^{n \times n}$ as follows:

$$Y = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}.$$

8. CONCRETE EXAMPLES

This section contains a few examples to illustrate our theoretical results. All tests are performed in MATLAB using Symbolic Math Toolbox version 8.5 (R2020a).

Example 1. Let $D = \text{diag}(1, -1)$. Then from Theorems 3 and 4 it follows that $\mathcal{S}_2(A) = \{D\} \cup \mathcal{K}$, where

$$\mathcal{K} = \left\{ \begin{bmatrix} -\frac{1}{2} & t \\ \frac{3}{4t} & \frac{1}{2} \end{bmatrix} : 0 \neq t \in \mathbb{C} \right\}.$$

Note that the set K is uniquely determined by parameter t . We have $\{D\} \cap K = \emptyset$.

Example 2. Let $D = \text{diag}(1, 1, -1)$. Here $n = 3$ and $p = 2$. We prove that $\mathcal{S}_3(D) = \mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_3$, where $\mathcal{K}_1 = \{D\}$ and $\mathcal{K}_i \cap \mathcal{K}_j = \emptyset$ for $i \neq j$.

It is obvious that $Y \in \mathcal{S}_3(D)$ iff $Y = D$ or $Y = W\hat{Y}W^{-1}$, where $W = \text{diag}(P_1, 1)$, and $P_1(2 \times 2)$ is an arbitrary nonsingular matrix. By (6.3), we get

$$\hat{Y} = \begin{bmatrix} -\frac{1}{2} & 0 & t \\ 0 & 1 & 0 \\ \frac{3}{4t} & 0 & \frac{1}{2} \end{bmatrix}, \quad 0 \neq t \in \mathbb{C}.$$

Without loss of generality we can assume that $\det P_1 = 1$, i.e. $P_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $ab - cd = 1$.

Notice that for such $W = \text{diag}(P_1, 1)$ we get

$$Y = W\hat{Y}W^{-1} = \begin{bmatrix} -\frac{(ad+2bc)}{2} & \frac{3ab}{2} & at \\ -\frac{3cd}{2} & \frac{(bc+2ad)}{2} & ct \\ \frac{3d}{4t} & -\frac{3b}{4t} & \frac{1}{2} \end{bmatrix}.$$

Case (i): $a = 0$. Then $\det P_1 = -bc = 1$, hence $b \neq 0$ and $c \neq 0$. Let $u = ct$ and $t_2 = ct$. Then $Y \in \mathcal{K}_2$ iff

$$Y = Y(u, t_2) = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3u}{2} & -\frac{1}{2} & t_2 \\ \frac{3u}{4t_2} & \frac{3}{4t_2} & \frac{1}{2} \end{bmatrix}, \quad u \neq 0, t_2 \neq 0.$$

We see that the set $\mathcal{K}_2$ is uniquely determined by parameters u and t_2 .

Case (ii): $a \neq 0$. This together with $ad - bc = 1$ leads to $ad = 1 + bc$. Define $t_3 = at, b_2 = ab, c_2 = c/a$. Then $Y \in \mathcal{K}_3$ iff

$$Y = Y(b_2, c_2, t_3) = \begin{bmatrix} -\frac{1}{2} - \frac{3b_2c_2}{2} & \frac{3b_2}{2} & t_3 \\ -\frac{3c_2(1+b_2c_2)}{2} & 1 + \frac{3b_2c_2}{2} & c_2t_3 \\ \frac{3(1+b_2c_2)}{4t_3} & -\frac{3b_2}{4t_3} & \frac{1}{2} \end{bmatrix}, \quad t_3 \neq 0.$$

Notice that the set $\mathcal{K}_3$ is uniquely determined by parameters b_2, c_2, t_3 .

Example 3. Let

$$A = \begin{bmatrix} 1 & 0 & 2 & 6 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

By Theorem 2, the matrices A and P are involutions and we have the spectral decomposition $A = PDP^{-1}$, where $D = \text{diag}(1, 1, -1, -1)$. We would like to find concrete solutions $Y \in \mathcal{S}_4(D)$ and $X = PYP \in \mathcal{S}_4(A)$. We see that here we can use only Algorithm 1 and Algorithm 3.

First, we apply Algorithm 1 for the matrix $Y_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Then we get

$$Y = \begin{bmatrix} -\frac{1}{2} & 0 & 1 & 2 \\ 0 & -\frac{1}{2} & 3 & 4 \\ -\frac{3}{2} & \frac{3}{4} & \frac{1}{2} & 0 \\ \frac{9}{8} & -\frac{3}{8} & 0 & \frac{1}{2} \end{bmatrix}, \quad X = \begin{bmatrix} \frac{11}{8} & -\frac{3}{8} & \frac{5}{8} & \frac{1}{4} \\ \frac{33}{8} & -\frac{19}{8} & \frac{55}{8} & \frac{11}{2} \\ \frac{9}{8} & -\frac{3}{8} & \frac{7}{8} & \frac{15}{4} \\ -\frac{9}{8} & \frac{3}{8} & -\frac{15}{8} & -\frac{5}{2} \end{bmatrix}.$$

Next, we apply Algorithm 3, taking $F_1 = 4$ and

$$P_1 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad P_4 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Then we obtain the following solutions $Y \in \mathcal{S}_4(D)$ and $X = PYP \in \mathcal{S}_4(A)$:

$$Y = \begin{bmatrix} -2 & \frac{3}{2} & 2 & -2 \\ -3 & \frac{5}{2} & 2 & -2 \\ \frac{3}{8} & -\frac{3}{16} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{3}{8} & \frac{3}{16} & -\frac{3}{4} & -\frac{1}{4} \end{bmatrix}, \quad X = \begin{bmatrix} -\frac{11}{4} & \frac{15}{8} & -6 & -\frac{23}{8} \\ -\frac{33}{8} & \frac{49}{16} & -12 & -\frac{137}{16} \\ -\frac{3}{8} & \frac{3}{16} & -1 & -\frac{27}{16} \\ \frac{3}{8} & -\frac{3}{16} & 0 & \frac{11}{16} \end{bmatrix}.$$

9. CONCLUSIONS

- We characterized all explicit involutive solutions of the Yang-Baxter matrix equation $AXA = XAX$.
- There are infinitely many such solutions for $A \neq \pm I_n$.
- Constructing involutive solutions of $AXA = XAX$ can be done easily by direct implementation of Algorithms 1–3.

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FIXED POINT THEOREMS UNDER LOCALLY T -TRANSITIVE BINARY RELATIONS EMPLOYING MATKOWSKI CONTRACTIONS

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Received 04 February, 2020

Abstract. In this paper, we extend the relation-theoretic contraction principle due to Alam and Imdad (J. Fixed Point Theory Appl. 17 (2015) 693-702) for Matkowski contractions employing a locally T -transitive binary relation. Our results improve and enrich several fixed point theorems of the existing literature.

2010 *Mathematics Subject Classification:* 47H10; 54H25

Keywords: locally T -transitive binary relations; comparison functions; $\mathcal{R}$ -connected sets

1. INTRODUCTION

Banach contraction principle [8] is one of the most fruitful results in nonlinear analysis. Various noted generalizations of this core result are available in the existing literature. Role of non-negative constant $\alpha < 1$ in $d(Tx, Ty) \leq \alpha d(x, y)$ for all $x, y \in X$ is that many authors generalized the Banach contraction principle by replacing the involved constant α with an appropriate mapping, say ϕ depending on the contractivity condition. A mapping $\phi : [0, \infty) \rightarrow [0, \infty)$ satisfying $\phi(t) < t$ for each $t > 0$ is said to be control function. A self-mapping T defined on a metric space (X, d) is said to be a nonlinear contraction with respect to control function ϕ (or, in short, ϕ -contraction) if

$$d(Tx, Ty) \leq \phi(d(x, y)) \quad \forall x, y \in X.$$

In fact, the idea of ϕ -contraction initiated by Browder [10] in 1968, wherein author assumed ϕ to be right continuous and increasing control function and utilized the same to generalize the Banach contraction principle. Thereafter, many authors generalized Browder fixed point theorem by varying the properties of control function ϕ (see Boyd-Wong and Matkowski contractions [9, 16]. In 2004, Ran and Reurings [22] established a variant of Banach contraction principle in the setting of ordered metric spaces, which was further refined by Nieto and Rodríguez-López [19]. In this continuation, Agarwal et al. [1] extended the results of Nieto and Rodríguez-López for Matkowski type nonlinear contractions, which was later refined by O'Regan and

Petruşel [20]. In [20], authors enlarges the class of spaces by replacing class of metric spaces with the class of L -spaces. In [20], authors also proved the order-theoretic fixed point theorems in the context of L -spaces. Petruşel and Rus [21] further proved some generalized fixed point results in L -spaces as well as metric spaces endowed with partial ordered relations.

In 2015, Alam and Imdad [4] obtained a noted generalization of Banach contraction principle employing an amorphous (arbitrary) binary relation, which was further improved by the (same) authors to Boyd-Wong type nonlinear contractions patterned after [6]. For further generalizations of these lines, we refer [2, 3, 5, 6].

In this paper, we establish a variant of the Banach contraction principle for Matkowski type nonlinear contractions under the relaxed transitivity condition by employing locally T -transitive relation. In order to establish our results, we will utilize the relation-theoretic analogues of certain involved metrical notions such as: contraction, completeness, continuity, *etc.* Indeed, under the universal relation these newly defined notions reduce to their corresponding natural analogues. Our newly established results generalize and unify a multitude of corresponding results of the existing literature. Also, we furnish some examples to demonstrate the utility of our results over corresponding existing ones.

2. PRELIMINARIES

Given a nonempty set X , a subset $\mathcal{R}$ of X^2 is called a binary relation on X . For simplicity, we sometimes write $x\mathcal{R}y$ instead of $(x, y) \in \mathcal{R}$. Given subset $E \subseteq X$ and a binary relation $\mathcal{R}$ on X , the restriction of $\mathcal{R}$ to E , denoted by $\mathcal{R}|_E$, are in fact $\mathcal{R} \cap E^2$. Indeed, $\mathcal{R}|_E$ is a relation on E induced by $\mathcal{R}$.

Out of various kind of binary relations, the following are relevant to our present discussion:

A binary relation $\mathcal{R}$ defined on a nonempty set X is called

- “amorphous” if it has no specific property at all,
- “universal” if $\mathcal{R} = X^2$,
- “empty” if $\mathcal{R} = \emptyset$,
- “reflexive” if $(x, x) \in \mathcal{R} \ \forall x \in X$,
- “symmetric” if whenever $(x, y) \in \mathcal{R}$ implies $(y, x) \in \mathcal{R}$,
- “antisymmetric” if whenever $(x, y) \in \mathcal{R}$ and $(y, x) \in \mathcal{R}$ imply $x = y$,
- “transitive” if whenever $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$ imply $(x, z) \in \mathcal{R}$,
- “complete” if $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R} \ \forall x, y \in X$,
- “partial order” if $\mathcal{R}$ is reflexive, antisymmetric and transitive.

Throughout this paper, $\mathcal{R}$ stands for a nonempty binary relation but for the sake of simplicity, we often write ‘binary relation’ instead of ‘nonempty binary relation’. Also, $\mathbb{N}$ stands for the set of natural numbers, $\mathbb{N}_0$ for the set of whole numbers (*i.e.*, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$), $\mathbb{Q}$ for the set of rational numbers and $\mathbb{Q}^+$ for set of positive rational numbers.

Definition 1. [12, 18, 25] Let X be a nonempty set equipped with partial order $\preceq$. A self-mapping T defined on X is called increasing (or isotone or order-preserving) if for any $x, y \in X$,

$$x \preceq y \Rightarrow T(x) \preceq T(y).$$

The following notion is formulated by using a suitable property with a view to relax the continuity requirement of the underlying mapping especially in the hypotheses of a fixed point theorem due to Nieto and Rodríguez-López [19].

Definition 2. [7] Let (X, d) be a metric space equipped with a partial order $\preceq$. We say that the triplet $(X, d, \preceq)$ has “*ICU* (increasing-convergence-upper bound) property” if every increasing convergent sequence in X is bounded above by its limit (as an upper bound).

Definition 3. [16] A function $\phi : [0, \infty) \rightarrow [0, \infty)$ is called “comparison function” if

- (i) ϕ is increasing,
- (ii) $\lim_{n \rightarrow \infty} \phi^n(t) = 0 \forall t > 0$.

The class of control functions of Boyd and Wong [9] is given by:

$$\Omega = \left\{ \phi : [0, \infty) \rightarrow [0, \infty) : \phi(t) < t \text{ for each } t > 0 \text{ and } \limsup_{r \rightarrow t^+} \phi(r) < t \text{ for each } t > 0 \right\}.$$

Recall that ϕ -contractions via comparison functions are known as Matkowski contractions, later it will be shown that the comparison function is control function.

Notice that, the classes of Boyd-Wong and Matkowski contractions are independent. To differentiate this fact, consider the following two functions from $[0, \infty)$ to $[0, \infty)$ defined by:

$$\phi_1(t) = \begin{cases} 0, & \text{if } t = 0, \\ \frac{1}{n+1}, & \text{if } t \in \left(\frac{1}{n+1}, \frac{1}{n}\right], n = 1, 2, 3, \dots, \\ 1, & \text{if } t > 1, \end{cases}$$

$$\phi_2(t) = \begin{cases} \frac{t}{5}, & \text{if } t < 2, \\ \frac{1}{t}, & \text{if } t \geq 2. \end{cases}$$

Then ϕ_1 is a comparison function but not lie in Ω as it is not upper semi continuous from the right (see [16]). On the other hand, the decreasing function ϕ_2 is a member of Boyd-Wong class, but it is not comparison function.

It is worth mentioning here that Ćirić [11] proved that for a member $\phi \in \Omega$,

$$\lim_{n \rightarrow \infty} \phi^n(t) = 0, \forall t > 0.$$

Unfortunately, this fact is not true in general as mentioned in Jachymski [13].

Employing comparison functions, Agarwal et al. [1] proved the following:

Theorem 1. [1] Let (X, d) be a metric space endowed with a partial order $\preceq$ and T a self-mapping on X . Suppose that the following conditions hold:

- (a) (X, d) is complete,
- (b) T is increasing,
- (c) either T is continuous or $(X, d, \preceq)$ has ICU property,
- (d) there exists $x_0 \in X$ such that $x_0 \preceq T(x_0)$,
- (e) there exists a comparison function ϕ such that

$$d(Tx, Ty) \leq \phi(d(x, y)) \quad \forall x, y \in X \text{ with } x \preceq y.$$

Then T has a fixed point.

Inspired by Roldán-López-de-Hierro et al. [23], Alam and Imdad introduced the following: (i.e., a notion originated from T -transitive subset of X^2 is essentially due to [23]).

Definition 4. [6] Let X be a nonempty set and T a self-mapping on X . A binary relation $\mathcal{R}$ defined on X is called “ T -transitive” if for any $x, y, z \in X$,

$$(Tx, Ty), (Ty, Tz) \in \mathcal{R} \Rightarrow (Tx, Tz) \in \mathcal{R}.$$

Inspired by Turinici [28, 29], Alam and Imdad [6] introduced the following notions by localizing the transitivity condition.

Definition 5. [6] Let X be a nonempty set. A binary relation $\mathcal{R}$ defined on X is called “locally transitive” if for each (effectively) $\mathcal{R}$ -preserving sequence $\{x_n\} \subset X$ (with range $E = \{x_n : n \in \mathbb{N}\}$), such that $\mathcal{R}|_E$ is transitive.

Clearly, the notions of “ T -transitivity” and “locally transitivity” both are relatively weaker than the notions of transitivity, but they are independent of each other. In order to make them compatible, Alam and Imdad [6] introduced the following notion of transitivity.

Definition 6. [6] Let X be a nonempty set and T a self-mapping on X . A binary relation $\mathcal{R}$ defined on X is called locally T -transitive if for each (effectively) $\mathcal{R}$ -preserving sequence $\{x_n\} \subset T(X)$ (with range $E = \{x_n : n \in \mathbb{N}\}$), such that $\mathcal{R}|_E$ is transitive.

The following result accomplishes the dominant idea of ‘locally T -transitivity’ over other variants of ‘transitivity’:

Proposition 1. [6] Let X be a nonempty set, $\mathcal{R}$ a binary relation on X and T a self-mapping on X . Then

- (i) $\mathcal{R}$ is T -transitive $\Leftrightarrow \mathcal{R}|_{T(X)}$ is transitive,
- (ii) $\mathcal{R}$ is locally T -transitive $\Leftrightarrow \mathcal{R}|_{T(X)}$ is locally transitive,
- (iii) $\mathcal{R}$ is transitive $\Rightarrow \mathcal{R}$ is locally transitive $\Rightarrow \mathcal{R}$ is locally T -transitive,
- (iv) $\mathcal{R}$ is transitive $\Rightarrow \mathcal{R}$ is T -transitive $\Rightarrow \mathcal{R}$ is locally T -transitive.

3. RELEVANT NOTIONS AND AUXILIARY RESULTS

In this section, for the sake of completeness, we summarize some relevant definitions and basic results for our subsequent discussions:

Definition 7. [4] Let $\mathcal{R}$ be a binary relation on a nonempty set X and $x, y \in X$. We say that x and y are $\mathcal{R}$ -comparative if either $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R}$. We denote it by $[x, y] \in \mathcal{R}$.

Definition 8. [15] Let X be a nonempty set and $\mathcal{R}$ a binary relation on X .

- (1) The inverse or transpose or dual relation of $\mathcal{R}$, denoted by $\mathcal{R}^{-1}$, is defined by $\mathcal{R}^{-1} = \{(x, y) \in X^2 : (y, x) \in \mathcal{R}\}$.
- (2) The symmetric closure of $\mathcal{R}$ (denoted by $\mathcal{R}^s$) is defined to be the set $\mathcal{R} \cup \mathcal{R}^{-1}$ (i.e., $\mathcal{R}^s := \mathcal{R} \cup \mathcal{R}^{-1}$). Indeed, $\mathcal{R}^s$ is the smallest symmetric relation on X containing $\mathcal{R}$.

Proposition 2. [4] For a binary relation $\mathcal{R}$ defined on a nonempty set X ,

$$(x, y) \in \mathcal{R}^s \iff [x, y] \in \mathcal{R}.$$

Definition 9. [4] Let $\mathcal{R}$ be a binary relation defined on a nonempty set X . A sequence $\{x_n\} \subset X$ is called “ $\mathcal{R}$ -preserving” if

$$(x_n, x_{n+1}) \in \mathcal{R} \quad \forall n \in \mathbb{N}_0.$$

Definition 10. [4] Let X be a nonempty set and T a self-mapping on X . A binary relation $\mathcal{R}$ defined on X is called T -closed if for any $x, y \in X$,

$$(x, y) \in \mathcal{R} \Rightarrow (Tx, Ty) \in \mathcal{R}.$$

Proposition 3. [5] Let X be a nonempty set endowed with a binary relation $\mathcal{R}$ and T a self-mapping on X such that $\mathcal{R}$ is T -closed, then $\mathcal{R}^s$ is also T -closed.

Proposition 4. [6] Let $\mathcal{R}$ be a binary relation defined on a nonempty set X and T a self-mapping on X . If $\mathcal{R}$ is T -closed, then for all $n \in \mathbb{N}_0$, $\mathcal{R}$ is also T^n -closed, where T^n denotes n^{th} iterate of T .

Definition 11. [5] Let $\mathcal{R}$ be a binary relation defined on a nonempty set X . We say that (X, d) is $\mathcal{R}$ -complete if every $\mathcal{R}$ -preserving Cauchy sequence in X converges.

Notice that every complete metric space is $\mathcal{R}$ -complete. Particularly, under the universal relation the notion of $\mathcal{R}$ -completeness coincides with usual completeness.

Definition 12. [5] Let $\mathcal{R}$ be a binary relation defined on a nonempty set X with $x \in X$. A mapping $T : X \rightarrow X$ is called $\mathcal{R}$ -continuous at x if for any $\mathcal{R}$ -preserving sequence $\{x_n\}$ such that $x_n \xrightarrow{d} x$, we have $T(x_n) \xrightarrow{d} T(x)$. Moreover, T is called $\mathcal{R}$ -continuous if it is $\mathcal{R}$ -continuous at each point of X .

Clearly, every continuous mapping is $\mathcal{R}$ -continuous, for any binary relation $\mathcal{R}$. Particularly, under the universal relation the notion of $\mathcal{R}$ -continuity coincides with usual continuity.

The following notion is a generalization of d -self-closedness of a partial order relation ($\preceq$) contained in Turinici [26, 27]:

Definition 13. [4] A binary relation $\mathcal{R}$ defined on a metric space (X, d) is called d -self-closed if for any $\mathcal{R}$ -preserving sequence $\{x_n\}$ such that $x_n \xrightarrow{d} x$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[x_{n_k}, x] \in \mathcal{R} \quad \forall k \in \mathbb{N}_0$.

Definition 14. [24] Let $\mathcal{R}$ be a binary relation defined on a nonempty set X . A subset E of X is called $\mathcal{R}$ -directed if for each $x, y \in E$, there exists $z \in X$ such that $(x, z) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$.

Definition 15. [14] Let $\mathcal{R}$ be a binary relation defined on a nonempty set X . For $x, y \in X$, a path of length k (where k is a natural number) in $\mathcal{R}$ from x to y is a finite sequence $\{z_0, z_1, z_2, \dots, z_k\} \subset X$ satisfying the following:

- (i) $z_0 = x$ and $z_k = y$,
- (ii) $(z_i, z_{i+1}) \in \mathcal{R}$ for each i ($0 \leq i \leq k-1$).

Notice that a path of length k involves $k+1$ elements of X , although they may or may not be distinct.

Definition 16. [5] Let $\mathcal{R}$ be a binary relation defined on a nonempty set X . A subset E of X is called $\mathcal{R}$ -connected if for each pair $x, y \in E$, there exists a path in $\mathcal{R}$ from x to y .

Given a binary relation $\mathcal{R}$ and a self-mapping T on a nonempty set X , we use the following notations:

- (i) $F(T)$:= the set of all fixed points of T ,
- (ii) $X(T, \mathcal{R}) := \{x \in X : (x, Tx) \in \mathcal{R}\}$.

The following result is a relation-theoretic extension of Banach contraction principle:

Theorem 2. [4] Let (X, d) be a metric space, $\mathcal{R}$ a binary relation on X and T a self-mapping on X . Suppose that the following conditions hold:

- (a) (X, d) is $\mathcal{R}$ -complete,
- (b) $\mathcal{R}$ is T -closed,
- (c) either T is $\mathcal{R}$ -continuous or $\mathcal{R}$ is d -self-closed,
- (d) $X(T, \mathcal{R})$ is nonempty,
- (e) there exists $\alpha \in [0, 1)$ such that

$$d(Tx, Ty) \leq \alpha d(x, y) \quad \forall x, y \in X \text{ with } (x, y) \in \mathcal{R}.$$

Then T has a fixed point. Moreover, if X is $\mathcal{R}^s$ -connected, then T has a unique fixed point.

In view of symmetry of d , we can have the following.

Proposition 5. *If (X, d) is a metric space, $\mathcal{R}$ is a binary relation on X , T is a self-mapping on X and ϕ is a comparison function, then the following contractivity conditions are equivalent:*

- (I) $d(Tx, Ty) \leq \phi(d(x, y)) \quad \forall x, y \in X \text{ with } (x, y) \in \mathcal{R}$,
- (II) $d(Tx, Ty) \leq \phi(d(x, y)) \quad \forall x, y \in X \text{ with } [x, y] \in \mathcal{R}$.

Now, we propose the two main properties of the comparison function:

Proposition 6. *[17] Let ϕ be a comparison function, then ϕ is control function.*

Proof. Let there exists $t_0 > 0$ such that $t_0 \leq \phi(t_0)$. As ϕ is increasing $\phi(t_0) \leq \phi^2(t_0)$, it follows that $t_0 \leq \phi(t_0) \leq \phi^2(t_0)$. Thus, inductively for all $n \in \mathbb{N}$, we have $t_0 \leq \phi^n(t_0)$ which on letting $n \rightarrow \infty$, gives rise, $t_0 \leq 0$, which is a contradiction. $\square$

Proposition 7. *Let ϕ be a comparison function, then $\phi(0) = 0$.*

Proof. Suppose on contrary that $\phi(0) = t$ for some $t > 0$. As $0 < t$ and ϕ is increasing, $\phi(0) \leq \phi(t)$, it follows that $t \leq \phi(t) < t$, which is contradiction, hence $\phi(0) = 0$. $\square$

4. MAIN RESULTS

Now, we state and prove the existence and uniqueness of fixed point results via a locally T -transitive binary relation using comparison functions, besides deducing some special cases.

Theorem 3. *Let (X, d) be a metric space endowed with a binary relation $\mathcal{R}$ and T a self-mapping on X . Suppose that the following conditions hold:*

- (a) (X, d) is $\mathcal{R}$ -complete,
- (b) $\mathcal{R}$ is T -closed and locally T -transitive,
- (c) either T is $\mathcal{R}$ -continuous or $\mathcal{R}$ is d -self-closed,
- (d) $X(T, \mathcal{R})$ is nonempty,
- (e) there exists a comparison function ϕ such that

$$d(Tx, Ty) \leq \phi(d(x, y)) \quad \forall x, y \in X \text{ with } (x, y) \in \mathcal{R}.$$

Then T has a fixed point.

Proof. As $X(T, \mathcal{R}) \neq \emptyset$, choose $x_0 \in X(T, \mathcal{R})$. Construct a sequence $\{x_n\}$ of iteration based at the initial point x_0 , i.e.,

$$x_n = T^n(x_0) \quad \forall n \in \mathbb{N}_0. \tag{4.1}$$

As $(x_0, Tx_0) \in \mathcal{R}$, using T -closedness of $\mathcal{R}$ and Proposition 4, we get

$$(T^n x_0, T^{n+1} x_0) \in \mathcal{R}$$

so that

$$(x_n, x_{n+1}) \in \mathcal{R} \quad \forall n \in \mathbb{N}_0. \quad (4.2)$$

Therefore the sequence $\{x_n\}$ is $\mathcal{R}$ -preserving. Now, if $d(x_{n_0+1}, x_{n_0}) = 0$ for some $n_0 \in \mathbb{N}_0$, then in view of (4.1), we have $T(x_{n_0}) = x_{n_0}$ so that x_{n_0} is a fixed point of T and hence we are done.

On the other hand, if $d(x_{n+1}, x_n) > 0 \quad \forall n \in \mathbb{N}_0$, then applying the contractivity condition (e) to (4.2), we deduce, for all $n \in \mathbb{N}_0$ that

$$d(x_{n+1}, x_n) \leq \phi(d(x_n, x_{n-1})),$$

which on using (4.2), contractive condition (e) and increasing property of ϕ , reduces to

$$d(x_{n+1}, x_n) \leq \phi^n(d(x_1, x_0)). \quad (4.3)$$

Making $n \rightarrow \infty$, in (4.3) and using the definition of comparison function, we get

$$\lim_{n \rightarrow \infty} d(x_{n+1}, x_n) = 0. \quad (4.4)$$

Fix $\varepsilon > 0$. Then in view of (4.4), we can choose $n \in \mathbb{N}_0$ such that

$$d(x_{n+1}, x_n) < \varepsilon - \phi(\varepsilon). \quad (4.5)$$

Now, we claim that $\{x_n\}$ is a Cauchy sequence. To substantiate the claim, using increasing property of ϕ , (4.2) and (4.5), we obtain

$$\begin{aligned} d(x_{n+2}, x_n) &\leq d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n) < d(Tx_{n+1}, Tx_n) + \varepsilon - \phi(\varepsilon) \\ &\leq \phi(d(x_{n+1}, x_n)) + \varepsilon - \phi(\varepsilon) \leq \phi(\varepsilon - \phi(\varepsilon)) + \varepsilon - \phi(\varepsilon) \\ &\leq \phi(\varepsilon) + \varepsilon - \phi(\varepsilon) = \varepsilon. \end{aligned}$$

Now, using the increasing property of ϕ , (4.2) and locally T -transitivity of $\mathcal{R}$, we obtain

$$\begin{aligned} d(x_{n+3}, x_n) &\leq d(x_{n+3}, x_{n+1}) + d(x_{n+1}, x_n) < d(Tx_{n+2}, Tx_n) + \varepsilon - \phi(\varepsilon) \\ &\leq \phi(d(x_{n+2}, x_n)) + \varepsilon - \phi(\varepsilon) \leq \phi(\varepsilon - \phi(\varepsilon)) + \varepsilon - \phi(\varepsilon) \\ &\leq \phi(\varepsilon) + \varepsilon - \phi(\varepsilon) = \varepsilon \end{aligned}$$

so that inductively yields,

$$d(x_{n+k}, x_n) < \varepsilon \text{ for all } k \in \mathbb{N},$$

which shows that the sequence $\{x_n\}$ is Cauchy, which is also $\mathcal{R}$ -preserving. By $\mathcal{R}$ -completeness of (X, d) , $\exists x \in X$ such that $x_n \xrightarrow{d} x$.

Now, we show that x is a fixed point of T . To do this, assume that T is $\mathcal{R}$ -continuous. As $\{x_n\}$ is $\mathcal{R}$ -preserving with $x_n \xrightarrow{d} x$, $\mathcal{R}$ -continuity of T implies that $x_{n+1} = T(x_n) \xrightarrow{d} T(x)$. Using the uniqueness of limit, we obtain $T(x) = x$, i.e., x is a fixed point of T .

Alternatively, assume that $\mathcal{R}$ is d -self-closed. As $\{x_n\}$ is $\mathcal{R}$ -preserving such that $x_n \xrightarrow{d} x$, the d -self-closedness of $\mathcal{R}$ guarantees the existence of a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[x_{n_k}, x] \in \mathcal{R} \quad (\forall k \in \mathbb{N}_0)$. On using assumption (e), Proposition 5, $[x_{n_k}, x] \in \mathcal{R}$ with $x_{n_k} \xrightarrow{d} x$ and Propositions 6 and 7 (whether $d(x_{n_k}, x)$ is zero or nonzero), we obtain

$$\begin{aligned} d(x_{n_k+1}, Tx) &= d(Tx_{n_k}, Tx) \leq \phi(d(x_{n_k}, x)) \leq d(x_{n_k}, x) \\ &\rightarrow 0 \text{ as } k \rightarrow \infty \end{aligned}$$

so that $x_{n_k+1} \xrightarrow{d} T(x)$. Owing to the uniqueness of limit, we obtain $T(x) = x$ so that x is a fixed point of T . Hence proof is completed. $\square$

Combining with Proposition 1, we deduce the following consequence of Theorem 3.

Corollary 1. *Theorem 3 remains true if locally T -transitivity of $\mathcal{R}$ (utilized in assumption (b)) is replaced by any one of the following conditions besides retaining rest of the hypotheses:*

- (i) $\mathcal{R}$ is transitive,
- (ii) $\mathcal{R}$ is T -transitive,
- (iii) $\mathcal{R}$ is locally transitive.

Now, we prove a uniqueness result corresponding to Theorem 3.

Theorem 4. *In addition to the hypotheses of Theorem 3, assume that the following condition holds:*

- (u) $T(X)$ is $\mathcal{R}^s$ -connected.

Then T has a unique fixed point.

Proof. Due to Theorem 3, $F(T) \neq \emptyset$. Choose $x, y \in F(T)$, then for all $n \in \mathbb{N}_0$, we have

$$T^n(x) = x \text{ and } T^n(y) = y. \quad (4.6)$$

By hypothesis (u), there exists a path (say $\{z_0, z_1, z_2, \dots, z_k\}$) of some finite length k in $\mathcal{R}^s$ from x to y so that

$$z_0 = x, z_k = y \text{ and } [z_i, z_{i+1}] \in \mathcal{R} \text{ for each } i \ (0 \leq i \leq k-1). \quad (4.7)$$

Since $\mathcal{R}$ is T -closed, using Propositions 3 and 4, we have

$$[T^n z_i, T^n z_{i+1}] \in \mathcal{R} \text{ for each } i \ (0 \leq i \leq k-1) \text{ and for each } n \in \mathbb{N}_0. \quad (4.8)$$

Now, for each $n \in \mathbb{N}_0$ and for each $i \ (0 \leq i \leq k-1)$, write $t_n^i := d(T^n z_i, T^n z_{i+1})$. We assert that

$$\lim_{n \rightarrow \infty} t_n^i = 0. \quad (4.9)$$

For each fixed $i \ (0 \leq i \leq k-1)$, we differentiate two cases. Firstly, assume that $t_{n_0}^i = d(T^{n_0} z_i, T^{n_0} z_{i+1}) = 0$ for some $n_0 \in \mathbb{N}_0$, i.e., $T^{n_0}(z_i) = T^{n_0}(z_{i+1})$, which yields

that $T^{n_0+1}(z_i) = T^{n_0+1}(z_{i+1})$. Consequently, we get $t_{n_0+1}^i = d(T^{n_0+1}z_i, T^{n_0+1}z_{i+1}) = 0$. Hence by induction on n , we get $t_n^i = 0 \forall n \geq n_0$, so that $\lim_{n \rightarrow \infty} t_n^i = 0$. Secondly, assume that $t_n^i > 0 \forall n \in \mathbb{N}_0$. Then on using (4.8), assumption (e), Proposition 5 and increasing property of ϕ , we have

$$t_{n+1}^i = d(T^{n+1}z_i, T^{n+1}z_{i+1}) \leq \phi(d(T^n z_i, T^n z_{i+1})) = \phi(t_n^i) \leq \phi^2(t_{n-1}^i) \leq \dots \leq \phi^n(t_1^i)$$

so that

$$t_{n+1}^i \leq \phi^n(t_1^i). \quad (4.10)$$

On making $n \rightarrow \infty$ in (4.10) and using the definition of ϕ , we have

$$\lim_{n \rightarrow \infty} t_{n+1}^i \leq \lim_{n \rightarrow \infty} \phi^n(t_1^i) = 0.$$

Thus in each case, (4.9) is proved.

In view of (4.6), (4.7), (4.9) and the triangular inequality, we have

$$d(x, y) = d(T^n z_0, T^n z_k) \leq t_n^0 + t_n^1 + \dots + t_n^{k-1} \rightarrow 0 \text{ as } n \rightarrow \infty$$

so that $x = y$. Hence T has a unique fixed point. $\square$

Corollary 2. *Theorem 4 remains true if we replace condition (u) by one of the following conditions besides retaining rest of the hypotheses:*

- (u') $\mathcal{R}|_{T(X)}$ is complete,
- (u'') $T(X)$ is $\mathcal{R}^s$ -directed.

The proof of above corollary can be completed on the lines of the proof of Corollary 3.4 contained in [6].

Now, we deduce some special cases, which are noted fixed point theorems of the existing literature.

- (1) Under the universal relation (i.e., $\mathcal{R} = X^2$), Theorem 4 deduces to Matkowski fixed point theorem.
- (2) On choosing $\mathcal{R}$ to be a partial order $\preceq$ in Theorem 3, we obtain Theorem 1. Clearly, T -closedness of $\preceq$ is equivalent to increasing property of T .
- (3) Taking $\phi(t) = \alpha t$ (where $\alpha \in [0, 1)$) in Theorem 4, we obtain Theorem 2. In this case, the requirement of locally T -transitivity on a binary relation is not necessary.

5. ILLUSTRATIVE EXAMPLES

Now, we furnish some examples to demonstrate the utility of Theorems 3 and 4 over corresponding earlier known results.

Example 1. Let $X = [0, 2]$ equipped with usual metric d . Let

$$\mathcal{R} = \{(0, 0), (1, 0), (0, 1), (1, 1), (0, 2)\}$$

be a binary relation, then $\mathcal{R}^s = \{(0,0), (0,1), (1,0), (1,1), (0,2), (2,0)\}$. On X , define a self-mapping T by

$$T(x) = \begin{cases} 0, & x \in [0,2] \cap \mathbb{Q}; \\ 1, & x \in [0,2] \cap \mathbb{Q}^c, \end{cases}$$

Next, define a comparison function ϕ by $\phi(t) = \log(1+t) \forall t \in [0, \infty)$. It can be easily seen that, for all $(x,y) \in \mathcal{R}$ hypothesis (e) satisfied. Notice that $\mathcal{R}$ is locally T -transitive but not transitive. Choose any $\mathcal{R}$ -preserving sequence $\{x_n\}$ in X , i.e.,

$$(x_n, x_{n+1}) \in \mathcal{R}, \text{ for all } n \in \mathbb{N}_0 \text{ with } x_n \xrightarrow{d} x.$$

Now, if $(x_n, x_{n+1}) \in \mathcal{R}$, for all $n \in \mathbb{N}_0$, then there exists $N \in \mathbb{N}$ such that $x_n = x \in \{0,1\}$, for all $n \geq N$. Therefore, we choose a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that $x_{n_k} = x$, for all $k \in \mathbb{N}$, which amounts to saying that $[x_{n_k}, x] \in \mathcal{R}$, for all $k \in \mathbb{N}$. Hence, $\mathcal{R}$ is d -self-closed. Further, remaining hypotheses of Theorem 4 can be easily verified. Notice that T has a unique fixed point (namely $x = 0$). As $\mathcal{R}$ is not transitive; therefore $\mathcal{R}$ is not a partial order so this example can not be covered by Theorem 1 (due to Agarwal *et al.* [1]), which substantiate the utility of our results.

Example 2. Consider $X = [0, \infty)$ endowed with usual metric d . Define a mapping $T : X \rightarrow X$ by $T(x) = \frac{x}{1+2x} \forall x \in X$. Let $\mathcal{R} := \{(x,y) \in X^2 : x - y > 0 \text{ and } x \in \mathbb{Q}^+\}$, then $\mathcal{R}$ is locally T -transitive binary relation on X . Clearly, X is $\mathcal{R}$ -complete and $\mathcal{R}$ is T -closed. Now, define a comparison function ϕ by $\phi(t) = \frac{t}{t+1} \forall t \in [0, \infty)$. Now, for all $(x,y) \in \mathcal{R}$, we have

$$\begin{aligned} d(Tx, Ty) &= \left| \frac{x}{1+2x} - \frac{y}{1+2y} \right| = \left| \frac{(x-y)}{1+2x+2y+4xy} \right| \\ &\leq \frac{(x-y)}{1+(x-y)} = \frac{d(x,y)}{1+d(x,y)} = \phi(d(x,y)). \end{aligned}$$

Hence T and ϕ satisfy the assumption (e) of Theorem 3. Further, rest of the conditions of Theorem 4 can be satisfied easily. Observe T has a unique fixed point (namely $x = 0$).

Notice that $\mathcal{R}$ is not partially ordered and T is not linear contraction, therefore Example 2 can not be covered by corresponding Theorems 1 (due to Agarwal *et al.* [1]) and 2 (due to Alam and Imdad [4]), respectively.

ACKNOWLEDGEMENT

All the authors are grateful to an anonymous learned referee for his/her critical readings and pertinent comments, besides his/her comments gave us the knowledge of compatibility among the results. The First author is thankful to University Grant Commission, New Delhi, Government of India for the financial supports in the form of MANF (Moulana Azad National Fellowship).

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EXISTENCE OF SOLUTION FOR A DIRICHLET BOUNDARY VALUE PROBLEM INVOLVING THE $p(x)$ LAPLACIAN VIA A FIXED POINT APPROACH

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Received 23 January, 2022

Abstract. In this paper, we study the existence of a non-trivial solution in $W_0^{1,p(x)}(\Omega)$ for the problem

$$\begin{cases} \Delta_{p(x)} u = f(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{in } \Omega. \end{cases}$$

The proof is based on Schaefer's fixed point theorem.

2010 *Mathematics Subject Classification:* 34B15; 34A12; 54H25

Keywords: $p(x)$ -Laplacian, generalized Sobolev space, variable exponent, fixed point.

1. INTRODUCTION

Let us consider the problem

$$\begin{cases} -\Delta_{p(x)} u = f(x, u, \nabla u), & \text{if } x \in \Omega \\ u = 0, & \text{if } x \in \partial\Omega, \end{cases} \quad (1.1)$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, $p(x) \in C(\overline{\Omega})$ is log-Hölder continuous with values in $(1, +\infty)$,

$$-\Delta_{p(x)}(u) = \operatorname{div} \left(|\nabla u(x)|^{p(x)-2} \nabla u(x) \right)$$

and $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Caratheodory function satisfying some growth conditions.

According to the behavior of f , the existence results of weak solutions for Dirichlet boundary value problems involving $p(x)$ -Laplacian have been established in [3, 4, 6, 10–12] where variational and topological methods were used in [10] while the proof relies on simple variational arguments based on Mountain-Pass Theorem in [12].

Our contribution in this area is to prove the existence of nontrivial solutions for the Problem (1.1) using variable exponent and a fixed point theorem, in particular, Schaefer's fixed point theorem. This paper is divided into three sections. In the

following section, we will introduce some tools which we need to prove our main result in the last section.

2. PRELIMINARIES

Let us summary the most important results on variable exponent Sobolev space and the basic properties of the $p(x)$ -Laplacian and the Nemytskii operators. For more details, we orient the reader to [3, 5–8, 10–14] and references therein. We also recall Schaefer's and Schauder's fixed point theorems [1, 2, 9, 15].

$\Omega \subset \mathbb{R}^N, N \geq 2$ denotes an open bounded subset with smooth boundary $\partial\Omega$ and set

$$C_+(\overline{\Omega}) := \{g : g \in C(\overline{\Omega}), g(x) > 1 \text{ for all } x \in \overline{\Omega}\},$$

$$D = \{u : \Omega \rightarrow \mathbb{R}; \text{ is a measurable real-valued function}\}.$$

For any $p \in C_+(\overline{\Omega})$, we denote $1 < p^- := \min_{x \in \Omega} p(x) \leq p^+ = \max_{x \in \Omega} p(x) < \infty$ and

$$L^{p(x)}(\Omega) = \left\{ u \in D : \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\}.$$

On this space, the so-called Luxemburg norm is defined by the following formula

$$|u|_{p(x)} = \inf \left\{ \tau > 0 : \int_{\Omega} \left| \frac{u(x)}{\tau} \right|^{p(x)} dx \leq 1 \right\}.$$

$(L^{p(x)}(\Omega), |u|_{p(x)})$ are Banach spaces. There exists a fact that they are reflexive if and only if $1 < p^- \leq p^+ < \infty$. The inclusion between Lebesgue spaces also generalizes naturality: if $0 < |\Omega| < \infty$ and $p_1(\cdot), p_2(\cdot)$ are variable exponents such that $p_1(x) \leq p_2(x)$ a.e. $x \in \Omega$. Then, there exists a continuous embedding from $L^{p_2(x)}(\Omega)$ to $L^{p_1(x)}(\Omega)$.

Moreover, if $L^{p'(x)}(\Omega)$ is the conjugate space of $L^{p(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, then we have Hölder-type inequality [3]

$$\left| \int_{\Omega} uv dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-} \right) |u|_{p(x)} |v|_{p'(x)} \leq 2 |u|_{p(x)} |v|_{p'(x)}, \quad u \in L^{p(x)}(\Omega), v \in L^{p'(x)}(\Omega). \quad (2.1)$$

Later we need to use the modular and its properties, which is a mapping

$$\rho_{p(x)} : L^{p(x)}(\Omega) \rightarrow \mathbb{R}$$

defined by

$$\rho_{p(x)}(u) := \int_{\Omega} |u|^{p(x)} dx.$$

Proposition 1. (See [3]) For all $u, v \in L^{p(x)}(\Omega)$, we have

$$|u|_{p(x)} < 1 \text{ (resp. } = 1, > 1) \Leftrightarrow \rho_{p(x)}(u) < 1 \text{ (resp. } = 1, > 1). \quad (2.2)$$

$$|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-}. \quad (2.3)$$

$$|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^+}. \quad (2.4)$$

$$\rho_{p(x)}(u - v) \rightarrow 0 \Leftrightarrow |u - v|_{p(x)} \rightarrow 0. \quad (2.5)$$

From (2.3) and (2.4), it follows that

$$|u|_{p(x)} \leq \rho_{p(x)}(u) + 1 \quad (2.6)$$

and

$$\rho_{p(x)}(u) \leq |u|_{p(x)}^{p^+} + |u|_{p(x)}^{p^-}. \quad (2.7)$$

The variable Sobolev space $W^{1,p(x)}(\Omega)$ is defined as follows:

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\}$$

and it is equipped with the norm

$$\|u\|_{1,p(x)} := |u|_{p(x)} + |\nabla u|_{p(x)}.$$

$W_0^{1,p(x)}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ and

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if } p(x) < N \\ +\infty, & \text{if } p(x) \geq N. \end{cases}$$

We have the following proposition.

Proposition 2. (See [5, 7, 12])

- (1) $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable reflexive Banach spaces.
- (2) If $\alpha \in C_+(\overline{\Omega})$ and $\alpha(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{\alpha(x)}(\Omega)$ is continuous and compact.
- (3) There exists a constant $\theta_0 > 0$ such that

$$|u|_{p(x)} \leq \theta_0 |\nabla u|_{p(x)} \quad \forall u \in W_0^{1,p(x)}(\Omega). \quad (2.8)$$

According to this last assertion, $|\nabla u|_{p(x)}$ and $\|u\|_{1,p(x)}$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$.

We close this section with the following fixed point theorem which will be used in this paper.

Theorem 1. (Classical Schaefer's fixed point theorem) [15]

Let X be a normed space, T a continuous mapping of X into X which is compact on each bounded subset of X . Then, either

- the equation $u = \lambda Tu$ has a solution for $\lambda = 1$, or
- the set of all such solutions u , for $0 < \lambda < 1$ is unbounded.

3. MAIN RESULT

In this section, we first consider the following problem

$$\begin{cases} -\Delta_{p(x)} u = f(x, u, \nabla u), & x \in \Omega \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, $p(x) \in C(\overline{\Omega})$ is log-Hölder continuous with values in $(1, +\infty)$, $1 < p^- \leq p(x) \leq p^+ < \infty$, $p'(x)$ the conjugate of $p(x)$, $-\Delta_{p(x)}(u) = \operatorname{div}(|\nabla u(x)|^{p(x)-2} \nabla u(x))$ and $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is Caratheodory function satisfying the following growth condition.

$$(A_1) \quad |f(x, s, t)| \leq \phi(x) + c|s|^{\eta(x)-1} + c|t|^{\eta(x)-1}, \quad \forall (x, s, t) \in \Omega \times \mathbb{R} \times \mathbb{R}^N,$$

where

$$\phi(x) \in L^{p'(x)}(\Omega), \quad c > 0, \quad \eta(x) \in C_+(\overline{\Omega}), \quad \eta(x) < p^*(x), \quad 1 < \eta^- \leq \eta(x) \leq \eta^+ < p^-.$$

Let θ_0 be such that $|u|_{p(x)} \leq \theta_0 |\nabla u|_{p(x)}$, for all $u \in W_0^{1,p(x)}(\Omega)$ and θ_1 the constant of the embedding of $L^{p(x)}(\Omega)$ into $L^{\eta(x)}(\Omega)$ such that

$$(A_2) \quad 0 < c < \frac{1}{4\lambda^{p^+-1} \left((\theta_0 \theta_1)^{p^-} + \theta_1^{p^-} \right)}, \quad |\phi|_{p'(x)} < \frac{1 - 4c\lambda^{p^+-1} \left((\theta_0 \theta_1)^{p^-} + \theta_1^{p^-} \right)}{2\lambda^{p^+-1} \theta_0},$$

for some $\lambda \in [0, 1]$. If u is a weak solution of the Problem (1.1), then for all $v \in W_0^{1,p(x)}(\Omega)$, we have

$$\int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla v \, dx = \int_{\Omega} f(x, u, \nabla u) v \, dx. \quad (3.1)$$

Let $K = W_0^{1,p(x)}(\Omega)$ and K^* its dual, that is $(W_0^{1,p(x)}(\Omega))^* = K^*$. We define the operators $(-\Delta_{p(x)})$ and Φ as follows:

$$\begin{aligned} -\Delta_{p(x)} : K &\rightarrow K^* \\ \langle -\Delta_{p(x)} u, v \rangle &= \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla v \, dx, \quad \forall u, v \in K \end{aligned}$$

and

$$\begin{aligned} \Phi : K^* &\rightarrow K \\ \langle \Phi u, v \rangle &= \int_{\Omega} f(x, u, \nabla u) v \, dx, \quad \forall u, v \in K. \end{aligned}$$

If (A_1) is satisfied, then the following propositions give the most properties of $-\Delta_{p(x)}$ and Φ .

Proposition 3. [3, 6]

- (1) $-\Delta_{p(x)} : W_0^{1,p(x)}(\Omega) \rightarrow \left(W_0^{1,p(x)}(\Omega)\right)'$ is a homeomorphism from $W_0^{1,p(x)}(\Omega)$ to $\left(W_0^{1,p(x)}(\Omega)\right)'$.
- (2) $-\Delta_{p(x)} : W_0^{1,p(x)}(\Omega) \rightarrow \left(W_0^{1,p(x)}(\Omega)\right)'$ is a continuous, bounded and strictly monotone operator.
- (3) $-\Delta_{p(x)} : W_0^{1,p(x)}(\Omega) \rightarrow \left(W_0^{1,p(x)}(\Omega)\right)'$ is a mapping of type S_+ .
- (4) The operator $-\Delta_{p(x)} : W_0^{1,p(x)}(\Omega) \rightarrow \left(W_0^{1,p(x)}(\Omega)\right)'$ has a continuous inverse mapping $(-\Delta_{p(x)})^{-1} : \left(W_0^{1,p(x)}(\Omega)\right)' \rightarrow W_0^{1,p(x)}(\Omega)$.
- (5) $(-\Delta_{p(x)})^{-1}$ is bounded and satisfies condition S_+ .

Proposition 4. [3] The operator Φ is compact.

One can see that if u is a weak solution for the Problem (1.1) means that

$$\langle -\Delta_{p(x)}u, v \rangle = \langle \Phi u, v \rangle, \quad \forall v \in K.$$

On the other hand, we have the following equivalence:

$$-\Delta_{p(x)}u = \Phi(u) \iff u = (-\Delta_{p(x)})^{-1}\Phi u.$$

Let T be the operator defined by

$$T : K \rightarrow K, \quad Tu = (-\Delta_{p(x)})^{-1}\Phi u.$$

Now, we state our main result which will be proved by using Schaefer's fixed point theorem.

Theorem 2. Under assumptions (A_1) and (A_2) , the Problem (1.1) admits at least a nontrivial solution.

It is easily seen that a solution of the Problem (1.1) is a fixed point of the operator T . So, we will prove that T is well defined and all the conditions of Theorem 1 are satisfied.

Proof. (1) T is well defined. Indeed, the operator T can be considered as follows:

$$T : K \xrightarrow{\Phi} K^* \xrightarrow{(-\Delta_{p(x)})^{-1}} K.$$

- (2) T is compact. In fact, let (u_n) be a bounded sequence in the reflexive space K . Then, there exist u_0 and a subsequence which we also denote (u_n) such that u_n converges weakly to u_0 in K . By the compactness of Φ (Lemma 2 in [3]), we have $\Phi(u_n) \rightarrow \Phi(u_0)$ in K^* . Using the continuity of the operator $(-\Delta_{p(x)})^{-1}$, then we get

$$(-\Delta_{p(x)})^{-1}\overline{\Phi}(u_n) \rightarrow (-\Delta_{p(x)})^{-1}\overline{\Phi}(u_0).$$

That is, $T(u_n)$ converges strongly to $T(u_0)$. We mention that T is also continuous.

(3) We claim that the set

$$B = \left\{ u \in W_0^{1,p(x)}(\Omega) : u = \lambda Tu, \lambda \in [0, 1] \right\}$$

is bounded. In fact, we will prove that for all $u \in B$, there exists $R > 0$ such that $|\nabla u|_{p(x)} \leq R$. Let $u \in B$. Then, we have to consider two cases:

- If $|\nabla u| \leq 1$, then B is bounded.
- If $|\nabla u| > 1$, by using modular's properties, we proceed as follows: If $u = \lambda Tu$ and assuming that $\lambda \neq 0$, we have $\frac{u}{\lambda} = Tu = (-\Delta_{p(x)})^{-1} \Phi u$. Then we obtain

$$-\Delta_{p(x)} \left(\frac{u}{\lambda} \right) = \Phi u$$

and consequently,

$$\int_{\Omega} |\nabla \left(\frac{u}{\lambda} \right)|^{p(x)-2} \nabla \left(\frac{u}{\lambda} \right) \nabla v \, dx = \int_{\Omega} f(x, u, \nabla u) v \, dx,$$

for all $v \in K$. Using Hölder-type inequality, Proposition 1 and Proposition 2, we have

$$\begin{aligned} \frac{1}{\lambda^{p^+}} \rho_{p(x)}(\nabla u) &\leq \rho_{p(x)} \nabla \left(\frac{u}{\lambda} \right) \leq \frac{1}{\lambda} \int_{\Omega} f(x, u, \nabla u) u \, dx, \\ \rho_{p(x)}(\nabla u) &\leq \lambda^{p^+-1} \int_{\Omega} f(x, u(x)) u(x) \, dx, \\ &\leq \lambda^{p^+-1} \left[\int_{\Omega} |\phi(x) u(x)| + c \int_{\Omega} |u(x)|^{\eta(x)} + c \int_{\Omega} |\nabla u(x)|^{\eta(x)} \, dx \right], \\ &\leq \lambda^{p^+-1} \left[2|\phi|_{p'(x)} |u|_{p(x)} + c \rho_{\eta(x)}(u) + c \rho_{\eta(x)}(\nabla u) \right], \\ &\leq \lambda^{p^+-1} \left[2|\phi|_{p'(x)} \theta_0 (\rho_{p(x)}(\nabla u) + 1) + c \left(|u|_{\eta(x)}^+ + |u|_{\eta(x)}^- \right) \right. \\ &\quad \left. + c \left(|\nabla u|_{\eta(x)}^+ + |\nabla u|_{\eta(x)}^- \right) \right], \end{aligned}$$

and so,

$$\begin{aligned} \left(1 - 2\lambda^{p^+-1} \theta_0 |\phi|_{p'(x)} \right) \rho_{p(x)}(\nabla u) &\leq 2\lambda^{p^+-1} \theta_0 |\phi|_{p'(x)} \\ &\quad + 4\lambda^{p^+-1} c \left[(\theta_0 \theta_1)^{p^-} + (\theta_1)^{p^-} \right] \left| \nabla u \right|_{p(x)}^{p^-}, \\ \left(1 - 2\lambda^{p^+-1} \theta_0 |\phi|_{p'(x)} \right) \left| \nabla u \right|_{p(x)}^{p^-} &\leq 2\lambda^{p^+-1} \theta_0 |\phi|_{p'(x)} \\ &\quad + 4\lambda^{p^+-1} c \left[(\theta_0 \theta_1)^{p^-} + (\theta_1)^{p^-} \right] \left| \nabla u \right|_{p(x)}^{p^-}, \end{aligned}$$

$$\left(1 - 2\lambda^{p^+-1}\theta_0|\phi|_{p'(x)} - 4\lambda^{p^+-1}c\left[(\theta_0\theta_1)^{p^-} + (\theta_1)^{p^-}\right]\right)\left|\nabla u\right|_{p(x)}^{p^-} \leq 2\lambda^{p^+-1}\theta_0|\phi|_{p'(x)}.$$

It follows from assumption (A_2) that

$$1 - 4\lambda^{p^+-1}c\left[(\theta_0\theta_1)^{p^-} + (\theta_1)^{p^-}\right] > 0,$$

and

$$2\lambda^{p^+-1}\theta_0|\phi|_{p'(x)} < 1 - 4\lambda^{p^+-1}c\left[(\theta_0\theta_1)^{p^-} + (\theta_1)^{p^-}\right],$$

hence we obtain

$$1 - 2\lambda^{p^+-1}\theta_0|\phi|_{p'(x)} - 4\lambda^{p^+-1}c\left[(\theta_0\theta_1)^{p^-} + (\theta_1)^{p^-}\right] > 0$$

and

$$\left|\nabla u\right|_{p(x)}^{p^-} \leq \frac{2\lambda^{p^+-1}\theta_0|\phi|_{p'(x)}}{1 - 2\lambda^{p^+-1}\theta_0|\phi|_{p'(x)} - 4\lambda^{p^+-1}c(\theta_0\theta_1)^{p^-} + (\theta_1)^{p^-}} = R > 0.$$

That is, B is bounded. By Schaefer's fixed point theorem, the operator T has a fixed point u which is the solution of the given Problem (1.1). $\square$

Remark 1. If we assume that there exists $x_0 \in \Omega$ such that $f(x_0, 0, t) \neq 0$ in Theorem 2, then the solution of the Problem (1.1) is nontrivial.

ACKNOWLEDGEMENTS

The authors express their gratitude to the anonymous referees for their helpful suggestions and corrections.

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SEMIPOSITONE FRACTIONAL BOUNDARY VALUE PROBLEMS WITH N POINT FRACTIONAL INTEGRAL BOUNDARY CONDITIONS

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Received 13 January, 2021

Abstract. In this paper, we obtain the existence of positive solutions for the semipositone fractional boundary value problem with n point fractional integral boundary conditions. The existence of positive solutions is established using the five functionals fixed point theorem. An example is given to ratify that our main result is theoretically feasible.

2010 Mathematics Subject Classification: 34B10; 34B18; 39A10

Keywords: multiple positive solution, fractional differential equation, fixed point theorem

1. INTRODUCTION

Nowadays, fractional boundary value problems sprung up dramatically due to the wide applications in various research fields such as chemistry, biology, engineering, economy and other areas. The books of Kilbas et al. [8], Podlubny [10], Samko et al. [11] are mostly cited for the theory and applications of fractional calculus. Inspired by the extensive application area of fractional boundary value problems, many works focus on the existence of positive solutions for such boundary value problems. Some kinds of procedures are applied to establish positive solutions for fractional boundary value problems such as the Krasnoselskii fixed point theorem on cones, the Leggett Williams fixed point theorem, the Avery Henderson fixed point theorem, upper and lower solutions method [2–4, 9, 12–14, 16, 18]. To apply these theories and procedures, boundary value problems admit nonnegative and continuous nonlinear term. Besides, negative term can also appear in our problems dealing with real word problems. If boundary value problems involve both negative and nonnegative nonlinearity, we say that those problems are semipositone problems, which occur in astrophysics, chemical reactions, envisagement of suspension bridges. Investigations on existence results obtained for semipositone problems are more complicated than those for positive ones. Many authors study on semipositone boundary value problems using variational methods, fixed point theory, critical point theory [5–7, 15, 17, 19]. Generally, the existence results of solutions are obtained for fractional semipositone

Riemann-Liouville differential equations for zero boundary values or Caputo fractional differential equations with boundary conditions such as three-point, m -point, integral boundary conditions. For all we know, only a few papers concerned with the existence of solutions for semipositone fractional boundary value problems subject to the Riemann-Liouville fractional integral boundary conditions.

The aim of this paper is to establish multiple positive solutions for the fractional differential equation with Caputo derivative of order $\nu \in (1, 2]$

$$D^\nu \phi(t) + f(t, \phi(t)) = 0, \quad t \in (0, 1), \quad (1.1)$$

with fractional integral boundary conditions

$$\begin{aligned} \phi(0) - \sigma \phi'(0) &= \sum_{j=1}^{n-2} a_j I^q \phi(\eta_j), \\ \phi(1) + \rho \phi'(1) &= \sum_{j=1}^{n-2} b_j I^q \phi(\xi_j), \end{aligned} \quad (1.2)$$

in which $q > 0$ and $\sigma, \rho > 0$, $a_j, b_j, \eta_j, \xi_j > 0$, $a_j \leq b_j$ and $\eta_j \leq \xi_j$ for $j \in \{1, n-2\}$, $\sum_{j=1}^{n-2} \frac{a_j \eta_j^q}{\Gamma(q+1)} < 1$, $\sum_{j=1}^{n-2} \frac{b_j \xi_j^q}{\Gamma(q+1)} < 1$, $f \in C([0, 1] \times [0, \infty), \mathbb{R})$ and $f(t, 0) \not\equiv 0$.

In order to assert our main result, we will give some background materials and the five functionals fixed point theorem.

Let Φ, κ, Θ be nonnegative continuous convex functionals on the cone P and let χ, τ be nonnegative continuous concave functionals on P . Then for nonnegative real numbers l, k, r, h and g , we define the following convex sets:

$$\begin{aligned} P(\Phi, g) &= \{\vartheta \in P : \Phi(\vartheta) < g\}, \\ P(\Phi, \chi, k, g) &= \{\vartheta \in P : k \leq \chi(\vartheta), \Phi(\vartheta) \leq g\}, \\ Q(\Phi, \kappa, h, g) &= \{\vartheta \in P : \kappa(\vartheta) \leq h, \Phi(\vartheta) \leq g\}, \\ P(\Phi, \Theta, \chi, k, r, g) &= \{\vartheta \in P : k \leq \chi(\vartheta), \Theta(\vartheta) \leq r, \Phi(\vartheta) \leq g\}, \\ Q(\Phi, \kappa, \tau, l, h, g) &= \{\vartheta \in P : l \leq \tau(\vartheta), \kappa(\vartheta) \leq h, \Phi(\vartheta) \leq g\}. \end{aligned}$$

The five functionals fixed point theorem is very significant in proving our main theorem, which is given below.

Theorem 1 ([1]). *Let P be a cone in a real Banach space E . Assume there exist $g > 0$ and $m > 0$ satisfying*

$$\chi(\vartheta) \leq \kappa(\vartheta) \quad \text{and} \quad \|\vartheta\| \leq m\Phi(\vartheta)$$

for all $\vartheta \in \overline{P(\Phi, g)}$. If

$$S : \overline{P(\Phi, g)} \rightarrow \overline{P(\Phi, g)}$$

is completely continuous and there exist nonnegative numbers k, r, h and l with $0 < h < k$ such that

- (i) $\{\vartheta \in P(\Phi, \Theta, \chi, k, r, g) : \chi(\vartheta) > k\} \neq \emptyset$ and $\chi(S\vartheta) > k$ for $\vartheta \in P(\Phi, \Theta, \chi, k, r, g)$,
- (ii) $\{\vartheta \in Q(\Phi, \kappa, \tau, l, h, g) : \kappa(\vartheta) < h\} \neq \emptyset$ and $\kappa(S\vartheta) < h$ for $\vartheta \in Q(\Phi, \kappa, \tau, l, h, g)$,
- (iii) $\chi(S\vartheta) > k$ for $\vartheta \in P(\Phi, \chi, k, g)$ with $\Theta(S\vartheta) > r$,
- (iv) $\kappa(S\vartheta) < h$ for $\vartheta \in Q(\Phi, \kappa, h, g)$ with $\tau(S\vartheta) < l$.

Then, S has at least three fixed points $\vartheta_1, \vartheta_2, \vartheta_3 \in \overline{P(\Phi, g)}$ satisfying

$$\kappa(\vartheta_1) < h, k < \chi(\vartheta_2) \text{ and } h < \kappa(\vartheta_3) \text{ with } \chi(\vartheta_3) < k.$$

2. PRELIMINARIES

In order to assert our main result, we assemble some definitions and lemmas from the fractional calculus [8, 10, 11].

Definition 1. The Riemann-Liouville fractional integral of order ν for a function y is given as

$$I^\nu y(t) = \frac{1}{\Gamma(\nu)} \int_0^1 (t-s)^{\nu-1} y(s) ds, \quad \nu > 0,$$

provided that such integral exists.

Definition 2. If $y \in C^n[0, 1]$, then the Caputo fractional derivative of order ν is defined by

$$D^\nu y(t) = \frac{1}{\Gamma(n-\nu)} \int_0^t (t-s)^{n-\nu-1} y^{(n)}(s) ds = I^{n-\nu} y^{(n)}(t),$$

where $n-1 < \nu < n$, $n = [\nu] + 1$ and $[\nu]$ denotes the integer part of the real number ν .

Lemma 1. Let $\nu > 0$ then the fractional differential equation $D^\nu u(t) = 0$ has a solution

$$u(t) = d_0 + d_1 t + d_2 t^2 + \dots + d_{n-1} t^{n-1},$$

in which $d_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n$; $n-1 < \nu < n$, $n = [\nu] + 1$.

Next, we state some auxiliary lemmas for fractional BVP (1.1)-(1.2).

Lemma 2 ([4]). If $h \in C[0, 1]$, then the fractional boundary value problem (fractional BVP for short)

$$\begin{aligned} D^\nu \vartheta(t) + h(t) &= 0, \quad t \in (0, 1), \\ \vartheta(0) - \sigma \vartheta'(0) &= 0, \\ \vartheta(1) + \rho \vartheta'(1) &= 0 \end{aligned}$$

possesses the integral expression

$$\vartheta(t) = \int_0^1 G(t,s)h(s)ds,$$

where

$$G(t,s) = \frac{1}{\Gamma(\mathfrak{v})} \begin{cases} -(t-s)^{\mathfrak{v}-1} + \frac{(t+\sigma)(1-s)^{\mathfrak{v}-1}}{1+\sigma+\rho} + \frac{(t+\sigma)(\mathfrak{v}-1)\rho(1-s)^{\mathfrak{v}-2}}{1+\sigma+\rho}, \\ 0 \leq s \leq t \leq 1, \\ \frac{(t+\sigma)(1-s)^{\mathfrak{v}-1}}{1+\sigma+\rho} + \frac{(t+\sigma)(\mathfrak{v}-1)\rho(1-s)^{\mathfrak{v}-2}}{1+\sigma+\rho}, \\ 0 \leq t \leq s \leq 1. \end{cases}$$

Lemma 3 ([9, 18]). $G(t,s)$ satisfies the following properties

- (i) $G(t,s)$ is continuous on $[0,1] \times [0,1]$,
- (ii) If $\sigma > \frac{2-\mathfrak{v}}{\mathfrak{v}-1}$, then $0 \leq G(t,s) \leq G(s,s)$ for any $t,s \in [0,1]$,
- (iii) If $\sigma > \frac{2-\mathfrak{v}}{\mathfrak{v}-1}$, then there exists $\omega > 0$ such that $\omega G(s,s) \leq G(t,s) \leq G(s,s)$ for any $t,s \in [0,1]$,

where $\omega = \min\{\omega_1, \omega_2\}$ can be given by

$$\omega_1 = \frac{4\rho[\sigma(\mathfrak{v}-1) + (\mathfrak{v}-2)]}{[\rho(\mathfrak{v}-1) + 1 - \sigma]^2 + 4\sigma[\rho(\mathfrak{v}-1) + 1]}, \quad (2.1)$$

$$\omega_2 = \frac{4\sigma\rho[\sigma(\mathfrak{v}-1) + (\mathfrak{v}-2)]}{[\rho(\mathfrak{v}-1) + 1 - \sigma]^2 + 4\sigma[\rho(\mathfrak{v}-1) + 1]}. \quad (2.2)$$

Lemma 4. For $h \in C[0,1]$, the fractional BVP

$$D^{\mathfrak{v}}\phi(t) + h(t) = 0, \quad t \in (0,1), \quad (2.3)$$

$$\begin{aligned} \phi(0) - \sigma\phi'(0) &= \sum_{j=1}^{n-2} a_j I^q \phi(\eta_j), \\ \phi(1) + \rho\phi'(1) &= \sum_{j=1}^{n-2} b_j I^q \phi(\xi_j), \end{aligned} \quad (2.4)$$

possesses the integral expression

$$\phi(t) = \int_0^1 H(t,s)h(s)ds, \quad (2.5)$$

where

$$\begin{aligned}
H(t,s) = & G(t,s) + \frac{1}{\Lambda} \left(\sum_{j=1}^{n-2} a_j I^q G(\eta_j, s) \right) \left(1 + \rho - \sum_{j=1}^{n-2} \frac{b_j \xi_j^{q+1}}{\Gamma(q+2)} \right) \\
& + \frac{1}{\Lambda} \left(\sum_{j=1}^{n-2} b_j I^q G(\xi_j, s) \right) \left(\sigma + \sum_{j=1}^{n-2} \frac{a_j \eta_j^{q+1}}{\Gamma(q+2)} \right) \\
& + \frac{t}{\Lambda} \left[\left(\sum_{j=1}^{n-2} b_j I^q G(\xi_j, s) \right) \left(1 - \sum_{j=1}^{n-2} \frac{a_j \eta_j^q}{\Gamma(q+1)} \right) \right. \\
& \left. - \left(\sum_{j=1}^{n-2} a_j I^q G(\eta_j, s) \right) \left(1 - \sum_{j=1}^{n-2} \frac{b_j \xi_j^q}{\Gamma(q+1)} \right) \right]
\end{aligned}$$

and

$$\Lambda = \left(1 - \sum_{j=1}^{n-2} \frac{a_j \eta_j^q}{\Gamma(q+1)} \right) \left(1 + \rho - \sum_{j=1}^{n-2} \frac{b_j \xi_j^{q+1}}{\Gamma(q+2)} \right) + \left(1 - \sum_{j=1}^{n-2} \frac{b_j \xi_j^q}{\Gamma(q+1)} \right) \left(\sigma + \sum_{j=1}^{n-2} \frac{a_j \eta_j^{q+1}}{\Gamma(q+2)} \right).$$

Here, $I^q G(\eta_j, s)$ and $I^q G(\xi_j, s)$ are the Riemann-Liouville fractional integrals of $G(t, s)$ with respect to $t = \eta_j$ and $t = \xi_j$ respectively.

Proof. Let

$$\vartheta(t) = \int_0^1 G(t, s) h(s) ds. \quad (2.6)$$

By employing Lemma 2, $\vartheta(t)$ holds

$$\begin{aligned}
D^v \vartheta(t) + h(t) &= 0, \quad t \in (0, 1), \\
\vartheta(0) - \sigma \vartheta'(0) &= 0, \\
\vartheta(1) + \rho \vartheta'(1) &= 0.
\end{aligned}$$

Suppose $\phi(t)$ is a solution of the BVP (2.3)-(2.4) and

$$z(t) = \phi(t) - \vartheta(t), \quad t \in [0, 1],$$

then $z(t)$ holds the fractional BVP :

$$\begin{aligned}
D^v z(t) &= 0, \quad t \in (0, 1), \\
z(0) - \sigma z'(0) &= \sum_{j=1}^{n-2} a_j I^q z(\eta_j) + \sum_{j=1}^{n-2} a_j I^q \vartheta(\eta_j), \\
z(1) + \rho z'(1) &= \sum_{j=1}^{n-2} b_j I^q z(\xi_j) + \sum_{j=1}^{n-2} b_j I^q \vartheta(\xi_j).
\end{aligned} \quad (2.7)$$

Lemma 1 implies that

$$z(t) = d_0 + d_1 t, \quad t \in [0, 1], d_0, d_1 \in \mathbb{R}, \quad (2.8)$$

and replacing $z(t)$ into (2.7) leads to

$$\begin{aligned} d_0 = \frac{1}{\Lambda} & \left[\left(\sum_{j=1}^{n-2} a_j I^q \vartheta(\eta_j) \right) \left(1 + \rho - \sum_{j=1}^{n-2} \frac{b_j \xi_j^{q+1}}{\Gamma(q+2)} \right) \right. \\ & \left. + \left(\sum_{j=1}^{n-2} b_j I^q \vartheta(\xi_j) \right) \left(\sigma + \sum_{j=1}^{n-2} \frac{a_j \eta_j^{q+1}}{\Gamma(q+2)} \right) \right] \end{aligned} \quad (2.9)$$

and

$$\begin{aligned} d_1 = \frac{1}{\Lambda} & \left[\left(1 - \sum_{j=1}^{n-2} \frac{a_j \eta_j^q}{\Gamma(q+1)} \right) \left(\sum_{j=1}^{n-2} b_j I^q \vartheta(\xi_j) \right) \right. \\ & \left. - \left(1 - \sum_{j=1}^{n-2} \frac{b_j \xi_j^q}{\Gamma(q+1)} \right) \left(\sum_{j=1}^{n-2} a_j I^q \vartheta(\eta_j) \right) \right], \end{aligned} \quad (2.10)$$

where

$$\begin{aligned} \Lambda = & \left(1 - \sum_{j=1}^{n-2} \frac{a_j \eta_j^q}{\Gamma(q+1)} \right) \left(1 + \rho - \sum_{j=1}^{n-2} \frac{b_j \xi_j^{q+1}}{\Gamma(q+2)} \right) \\ & + \left(1 - \sum_{j=1}^{n-2} \frac{b_j \xi_j^q}{\Gamma(q+1)} \right) \left(\sigma + \sum_{j=1}^{n-2} \frac{a_j \eta_j^{q+1}}{\Gamma(q+2)} \right). \end{aligned}$$

Finally, rewriting (2.9) and (2.10) into (2.8), we have

$$\begin{aligned} z(t) = \frac{1}{\Lambda} & \left(\sum_{j=1}^{n-2} a_j I^q \vartheta(\eta_j) \right) \left[\left(1 + \rho - \sum_{j=1}^{n-2} \frac{b_j \xi_j^{q+1}}{\Gamma(q+2)} \right) - t \left(1 - \sum_{j=1}^{n-2} \frac{b_j \xi_j^q}{\Gamma(q+1)} \right) \right] \\ & + \frac{1}{\Lambda} \left(\sum_{j=1}^{n-2} b_j I^q \vartheta(\xi_j) \right) \left[\left(\sigma + \sum_{j=1}^{n-2} \frac{a_j \eta_j^{q+1}}{\Gamma(q+2)} \right) + t \left(1 - \sum_{j=1}^{n-2} \frac{a_j \eta_j^q}{\Gamma(q+1)} \right) \right]. \end{aligned} \quad (2.11)$$

Thus, we can conclude from (2.6) and (2.11) that (2.5) is satisfied. Therefore, the proof of Lemma 4 is accomplished. $\square$

Lemma 5. $H(t, s)$ holds the following properties

- (i) $H(t, s) \in C([0, 1] \times [0, 1])$, $H(t, s) \geq 0$ for any $t, s \in (0, 1)$.
- (ii) There exist nonnegative numbers φ and ω such that

$$H(t, s) \leq \varphi G(s, s), \quad t, s \in [0, 1],$$

and

$$H(t, s) \geq \omega G(s, s), \quad t, s \in [0, 1], \quad (2.12)$$

in which

$$\begin{aligned} \varphi = 1 &+ \frac{1}{\Lambda} \left(\sum_{j=1}^{n-2} \frac{a_j \eta_j^q}{\Gamma(q+1)} \right) \left(1 + \rho - \sum_{j=1}^{n-2} \frac{b_j \xi_j^{q+1}}{\Gamma(q+2)} \right) \\ &+ \frac{1}{\Lambda} \left(\sum_{j=1}^{n-2} \frac{b_j \xi_j^q}{\Gamma(q+1)} \right) \left(\sigma + \sum_{j=1}^{n-2} \frac{a_j \eta_j^{q+1}}{\Gamma(q+2)} \right) \\ &+ \frac{1}{\Lambda} \left(1 - \sum_{j=1}^{n-2} \frac{a_j \eta_j^q}{\Gamma(q+1)} \right) \left(\sum_{j=1}^{n-2} \frac{b_j \xi_j^q}{\Gamma(q+1)} \right), \end{aligned} \quad (2.13)$$

and ω is defined by (2.1)-(2.2).

Proof. Apparently, (i) is satisfied using the definition of $H(t, s)$. We will show property (ii). Lemma 3 implies that

$$\begin{aligned} H(t, s) &\leq G(s, s) + \frac{1}{\Lambda} \left(\sum_{j=1}^{n-2} \frac{a_j \eta_j^q}{\Gamma(q+1)} G(s, s) \right) \left(1 + \rho - \sum_{j=1}^{n-2} \frac{b_j \xi_j^{q+1}}{\Gamma(q+2)} \right) \\ &+ \frac{1}{\Lambda} \left(\sum_{j=1}^{n-2} \frac{b_j \xi_j^q}{\Gamma(q+1)} G(s, s) \right) \left(\sigma + \sum_{j=1}^{n-2} \frac{a_j \eta_j^{q+1}}{\Gamma(q+2)} \right) \\ &+ \frac{t}{\Lambda} \left(\sum_{j=1}^{n-2} \frac{b_j \xi_j^q}{\Gamma(q+1)} G(s, s) \right) \left(1 - \sum_{j=1}^{n-2} \frac{a_j \eta_j^q}{\Gamma(q+1)} \right) \\ &\leq \left[1 + \frac{1}{\Lambda} \left(\sum_{j=1}^{n-2} \frac{a_j \eta_j^q}{\Gamma(q+1)} \right) \left(1 + \rho - \sum_{j=1}^{n-2} \frac{b_j \xi_j^{q+1}}{\Gamma(q+2)} \right) \right. \\ &\quad \left. + \frac{1}{\Lambda} \left(\sum_{j=1}^{n-2} \frac{b_j \xi_j^q}{\Gamma(q+1)} \right) \left(\sigma + \sum_{j=1}^{n-2} \frac{a_j \eta_j^{q+1}}{\Gamma(q+2)} \right) \right. \\ &\quad \left. + \frac{1}{\Lambda} \left(\sum_{j=1}^{n-2} \frac{b_j \xi_j^q}{\Gamma(q+1)} \right) \left(1 - \sum_{j=1}^{n-2} \frac{a_j \eta_j^q}{\Gamma(q+1)} \right) \right] G(s, s) \\ &= \varphi G(s, s). \end{aligned}$$

On the other hand, one can see easily that (2.12) is satisfied. Therefore, the proof of Lemma 5 is accomplished. $\square$

Theorem 1 will be utilized to prove the presence of positive solution. For this purpose, we are in position to introduce the Banach space $\mathcal{B} = C[0, 1]$ and a cone

$$P = \{ \vartheta \in \mathcal{B} : \vartheta(t) \geq \Omega \|\vartheta\|, t \in [0, 1] \},$$

where $\Omega = \frac{\omega}{\varphi}$, ω and φ are defined by (2.1)-(2.2) and (2.13).

To prove that fractional BVP (1.1)-(1.2) possesses multiple positive solutions, we choose $l = 0$, $m = 1$ and the following three functionals are defined by

$$\chi(\vartheta) = \min_{t \in [0,1]} \vartheta(t), \quad \tau(\vartheta) = 0, \quad \Phi(\vartheta) = \kappa(\vartheta) = \Theta(\vartheta) = \|\vartheta\|.$$

Furthermore

$$\chi(\vartheta) \leq \kappa(\vartheta) \quad \text{and} \quad \|\vartheta\| \leq m\Phi(\vartheta) \quad \text{for } \vartheta \in P.$$

Let us denote

$$\begin{aligned} \mu &= \omega \int_0^1 G(s, s) ds, \\ \zeta &= \varphi \int_0^1 G(s, s) ds, \\ D_1 &= \frac{\varphi^2}{\omega} \int_0^1 G(s, s) ds. \end{aligned}$$

Theorem 2. Assume that there exist constants $KD_1 < h < h + KD_1\Omega < k < \frac{k}{\Omega^2} < g$ such that $\frac{1}{\Omega} < N < \frac{g\mu}{k\zeta}$ holds.

Furthermore f verifies the following conditions:

- (C₀) There exists $K > 0$ such that $f(t, \vartheta) \geq -K$ for $(t, \vartheta) \in [0, 1] \times \mathbb{R}^+$,
- (C₁) $f(t, \vartheta) \leq \frac{g}{\zeta} - K$ for $t \in [0, 1]$, $\vartheta \in [0, g]$,
- (C₂) $f(t, \vartheta) \geq \frac{kN}{\mu} - K$ for $t \in [0, 1]$, $\vartheta \in [k - KD_1\Omega, g]$,
- (C₃) $f(t, \vartheta) < \frac{h}{\zeta} - K$ for $t \in [0, 1]$, $\vartheta \in [0, h]$.

Then fractional BVP (1.1)-(1.2) has at least two positive solutions.

Proof. Assume w is a solution of

$$\begin{aligned} D^\nu \phi(t) + 1 &= 0, \quad t \in (0, 1), \\ \phi(0) - \sigma \phi'(0) &= \sum_{j=1}^{n-2} a_j I^q \phi(\eta_j), \\ \phi(1) + \rho \phi'(1) &= \sum_{j=1}^{n-2} b_j I^q \phi(\xi_j), \end{aligned}$$

and $z(t) = Kw(t)$ for $t \in [0, 1]$. Then

$$z(t) = Kw(t) = K \int_0^1 H(t, s) ds \leq K\varphi \int_0^1 G(s, s) ds \leq KD_1\Omega.$$

We shall show that the fractional BVP

$$D^\nu \vartheta(t) + h(t, \bar{\vartheta}(t)) = 0, \quad t \in (0, 1), \quad (2.14)$$

$$\begin{aligned} \vartheta(0) - \sigma \vartheta'(0) &= \sum_{j=1}^{n-2} a_j I^q \vartheta(\eta_j), \\ \vartheta(1) + \rho \vartheta'(1) &= \sum_{j=1}^{n-2} b_j I^q \vartheta(\xi_j), \end{aligned} \quad (2.15)$$

has at least three positive solutions in which

$$h(t, \bar{\vartheta}(t)) = f(t, \bar{\vartheta}(t)) + K \text{ and } \bar{\vartheta}(t) = \max\{\vartheta(t) - z(t), 0\}.$$

For $\vartheta \in P$, denote an operator S by

$$S\vartheta(t) = \int_0^1 H(t, s) h(s, \bar{\vartheta}(s)) ds.$$

Clearly, the fractional BVP (2.14)-(2.15) has a solution provided that the operator S admits a fixed point.

Now, we check that $S(P) \subseteq P$. Indeed, for $\vartheta \in P$, Lemma 5 implies

$$\int_0^1 \omega G(s, s) h(s, \bar{\vartheta}(s)) ds \leq S\vartheta(t) \leq \int_0^1 \phi G(s, s) h(s, \bar{\vartheta}(s)) ds.$$

Hence,

$$S\vartheta(t) \geq \int_0^1 \omega G(s, s) h(s, \bar{\vartheta}(s)) ds \geq \Omega \|S\vartheta\|.$$

Furthermore by employing standard methods, the operator $S : P \rightarrow P$ is completely continuous. In what follows, we will show that all the conditions of Theorem 1 are satisfied.

We prove that $S(\overline{P(\Phi, g)}) \subseteq \overline{P(\Phi, g)}$. Let $\vartheta \in \overline{P(\Phi, g)}$ then $0 \leq \bar{\vartheta}(t) \leq \vartheta(t) \leq g$. By C_1 , we get

$$\Phi(S\vartheta) = \|S\vartheta\| = \max_{t \in [0, 1]} S\vartheta(t) \leq \phi \int_0^1 G(s, s) h(s, \bar{\vartheta}(s)) ds \leq \phi \frac{g}{\zeta} \int_0^1 G(s, s) ds \leq g.$$

So $S : \overline{P(\Phi, g)} \rightarrow \overline{P(\Phi, g)}$. In the following, we now prove that the conditions of Theorem 1 is satisfied with $r = g$.

To verify condition (i) of Theorem 1, let $\vartheta(t) = \frac{k}{\Omega^2}$, then one can see easily that

$$\{\vartheta \in P(\Phi, \Theta, \chi, k, r, g) : \chi(\vartheta) > k\} = \left\{ \vartheta \in P : \min_{t \in [0, 1]} \vartheta(t) > k, \|\vartheta\| \leq g \right\} \neq \emptyset.$$

Moreover, if $\vartheta \in P(\Phi, \Theta, \chi, k, r, g)$, then $\vartheta(t) - z(t) \leq \vartheta(t) \leq g$, that is $k - KD_1\Omega \leq \vartheta(t) - z(t) \leq g$. Applying (C₂), we get

$$\chi(S\vartheta) = \min_{t \in [0,1]} S\vartheta(t) \geq \Omega \|S\vartheta\| \geq \Omega \frac{kN}{\mu} \omega \int_0^1 G(s, s) ds = \Omega Nk > k. \quad (2.16)$$

Hence, condition (i) of Theorem 1 is satisfied.

Apparently,

$$\{\vartheta \in Q(\Phi, \kappa, \tau, l, h, g) : \kappa(\vartheta) < h\} = \{\vartheta \in P : \|\vartheta\| < h\} \neq \emptyset.$$

Using (C₃) leads that for $\vartheta \in Q(\Phi, \kappa, \tau, l, h, g)$

$$\kappa(S\vartheta) = \|S\vartheta\| = \max_{t \in [0,1]} S\vartheta(t) \leq \varphi \int_0^1 G(s, s) h(s, \bar{\vartheta}(s)) ds < \varphi \frac{h}{\zeta} \int_0^1 G(s, s) ds = h.$$

Hence (ii) of Theorem 1 is satisfied. Let $\vartheta \in P(\Phi, \chi, k, g)$. Using the same method followed in (2.16) results in $\chi(S\vartheta) > k$ for $\vartheta \in P(\Phi, \chi, k, g)$. Hence, (iii) of Theorem 1 holds.

Finally, we omit (iv) because $\tau(S\vartheta) < l = 0$ is not possible. Theorem 1 implies that fractional BVP (2.14)-(2.15) has at least three positive solutions ϑ_1^* , ϑ_2^* and ϑ_3^* such that

$$\|\vartheta_1^*\| < h, \quad k < \chi(\vartheta_2^*), \quad \|\vartheta_3^*\| > h, \quad \chi(\vartheta_3^*) < k.$$

Moreover,

$$\begin{aligned} \vartheta_2^*(t) &\geq \Omega \|\vartheta_2^*\| > \Omega \chi(\vartheta_2^*) > \Omega k > \Omega KD_1 \geq z(t), \quad t \in [0, 1], \\ \vartheta_3^*(t) &\geq \Omega \|\vartheta_3^*\| > \Omega h \geq \Omega KD_1 \geq z(t), \quad t \in [0, 1]. \end{aligned}$$

$\vartheta_2 = \vartheta_2^* - z$, $\vartheta_3 = \vartheta_3^* - z$ are two positive solutions of (1.1)-(1.2). This completes the proof. $\square$

Example 1. Consider the fractional boundary value problem

$$\begin{cases} D^{3/2}\phi(t) + f(t, \phi(t)) = 0, & t \in (0, 1), \\ \phi(0) - \frac{5}{2}\phi'(0) = \sum_{j=1}^2 a_j I^{1/2}\phi(\eta_j), \\ \phi(1) + \frac{1}{2}\phi'(1) = \sum_{j=1}^2 b_j I^{1/2}\phi(\xi_j), \end{cases} \quad (2.17)$$

in which $\nu = \frac{3}{2}$, $\sigma = \frac{5}{2}$, $\rho = \frac{1}{2}$, $q = \frac{1}{2}$, $n > 3$, $n = 4$, $a_1 = \frac{1}{4}$, $b_1 = \frac{1}{3}$, $\eta_1 = \frac{1}{4}$, $\xi_1 = \frac{1}{3}$, $a_2 = \frac{1}{8}$, $b_2 = \frac{1}{2}$, $\eta_2 = \frac{1}{8}$, $\xi_2 = \frac{1}{2}$.

$$f(t, \vartheta) = \begin{cases} -\frac{1}{2}\cos\frac{\pi}{2}t + \frac{\vartheta}{20000}, & t \in [0, 1], \vartheta \in [0, 50], \\ -\frac{1}{2}\cos\frac{\pi}{2}t + \frac{\vartheta}{20000} + \frac{100000}{29}(\vartheta - 50), & t \in [0, 1], \vartheta \in [50, 52.9], \\ -\frac{1}{2}\cos\frac{\pi}{2}t + \frac{\vartheta}{20000} + 10000, & t \in [0, 1], \vartheta \in [52.9, 23000], \\ -\frac{1}{2}\cos\frac{\pi}{2}t + \frac{23}{20} + 10000, & t \in [0, 1], \vartheta \in [23000, \infty), \end{cases}$$

Through calculation, we get $\omega_1 = 0.106$, $\omega_2 = 0.266$, $\Lambda = 2.037$, $\varphi = 2.132$, $D_1 = 42.53$, $\mu = 0.105$, $\zeta = 2.11$, $\Omega = 0.049$. Let $K = 1$, $N = 19$, $h = 50$, $k = 55$, $g = 23000$, then $f(t, \vartheta)$ satisfies

$$\begin{aligned} f(t, \vartheta) &\geq -K = -1, \text{ for } t \in [0, 1], \\ f(t, \vartheta) &\leq \frac{g}{\zeta} - K \approx 10898.47, \text{ for } t \in [0, 1], \vartheta \in [0, 23000], \\ f(t, \vartheta) &\geq \frac{kN}{\mu} - K \approx 9951.38, \text{ for } t \in [0, 1], \vartheta \in [52.9, 23000], \\ f(t, \vartheta) &\leq \frac{h}{\zeta} - K \approx 22,69, \text{ for } t \in [0, 1], \vartheta \in [0, 50]. \end{aligned}$$

We conclude that all the assumptions of Theorem 2 are verified, thus problem (2.17) has at least two positive solutions.

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SECOND-ORDER DIFFERENTIAL EQUATIONS: ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS

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Received 24 June, 2020

Abstract. In this work, we obtain necessary and sufficient conditions for the oscillation of all solutions of the second-order delay differential equation $(\pi(y')^\gamma)'(t) + p(t)f(y(\tau(t))) = 0$, under the assumption $\int^\infty (\pi(\eta))^{-1/\gamma} d\eta = \infty$, we consider two cases: when $f(v)/v^\beta$ is non-increasing, and non-decreasing. In the final section, we provide examples illustrating the results and state an open problem.

2010 *Mathematics Subject Classification:* 34K11

Keywords: Oscillation criteria, non-oscillation, delay, second-order equations, Lebesgue's dominated convergence theorem

1. INTRODUCTION

The motivation to study the oscillation of differential equations comes from several systems in the real world, like species populations and neuronal populations that exhibit oscillatory behavior. Thus, equations having delayed, advanced or both delayed and advanced arguments have been used to model lossless transmission lines in engineering, the switching of data packets in high speed networks and several other natural or artificial processes, from celestial motion and bridge design to learning and memory formation in the synaptic contacts between neurons in the brain.

In 1978, Brands [6] showed that the solutions to

$$y''(t) + p(t)y(t - \tau(t)) = 0$$

are oscillatory if and only if the solutions to $y''(t) + p(t)y(t) = 0$ are oscillatory. In 2018, Pinelas and Santra [13] have obtained necessary and sufficient conditions for the oscillations of the solutions of

$$(y(t) + b(t)y(t - \sigma))' + \sum_{i=1}^m p_i(t)f(y(t - \tau_i)) = 0,$$

for different ranges of the neutral coefficient b . Wong [18] studied necessary and sufficient conditions for the oscillation of the solutions to

$$(y(t) + by(t - \sigma))'' + p(t)f(y(t - \tau)) = 0,$$

where the constant p satisfies $-1 < b < 0$. Santra [17] has obtained several sufficient conditions for the oscillation of the solutions for the equations

$$(\pi z')'(t) + \sum_{i=1}^m p_i(t) f(y(t - \tau_i)) = 0, \quad z(t) = y(t) + b(t)y(t - \sigma).$$

Karpuz and Santra [9] have established sufficient conditions for the oscillation and asymptotic behavior of the solutions to the equation

$$(\pi z')'(t) + \sum_{i=1}^m p_i(t) f_i(y(\tau_i(t))) = 0, \quad z(t) = y(t) + b(t)y(\sigma(t)).$$

Migda et al. have studied asymptotic behaviors of solutions of second order difference equations with deviating argument. For a more detailed account of the oscillatory behavior of the solutions to this type of equations, we refer the readers to [1–5, 7–17, 19]. Note that most publications consider only sufficient conditions, and merely a few consider necessary and sufficient conditions.

In this work, we establish necessary and sufficient conditions for the oscillation of all solutions to the second-order nonlinear delay differential equation

$$(\pi(y')^\gamma)'(t) + p(t)f(y(\tau(t))) = 0 \quad (1.1)$$

by considering two cases: when $f(v)/v^\beta$ is non-increasing, and non-decreasing.

We assume that the following conditions hold:

- (A1) γ is the quotient of two odd positive integers, $\pi, p \in C(\mathbb{R}, \mathbb{R})$ with $\pi(t) > 0$ and p is not identically zero eventually, $\tau \in C([t_0, \infty), \mathbb{R})$ such that $\tau(t) \leq t$ for $t \geq t_0$, $\tau(t) \rightarrow \infty$ as $t \rightarrow \infty$.
- (A2) $f \in C(\mathbb{R}, \mathbb{R})$, f is non-decreasing and $vf(v) > 0$ for $v \neq 0$. Moreover, we assume that $f(uv) = f(u)f(v)$, $\forall u, v \in \mathbb{R}$.
- (A3) $\pi(t) > 0$ and $\int_0^\infty (\pi(\eta))^{-1/\gamma} d\eta = \infty$. Letting $\Pi(t) = \int_0^t (\pi(\eta))^{-1/\gamma} d\eta$, we have $\lim_{t \rightarrow \infty} \Pi(t) = \infty$.

Initially, we consider a single delay. In a later section, we study for the several delays. As examples of functions satisfying (A2) and (A3), we have $f(u) = u^\gamma$ with γ the quotient of two odd positive integers and $\pi(t) = e^{-t}$ or $\pi(t) = 1$ respectively.

By a solution to equation (1.1), we mean a function $y \in C([T_y, \infty), \mathbb{R})$, where $T_y \geq t_0$, such that $\pi y' \in C^1([T_y, \infty), \mathbb{R})$, and satisfying (1.1) on the interval $[T_y, \infty)$. A solution y of (1.1) is said to be proper if y is not identically zero eventually, i.e., $\sup\{|y(t)| : t \geq T\} > 0$ for all $T \geq T_y$. We assume that (1.1) possesses such solutions. A solution of (1.1) is called oscillatory if it has arbitrarily large zeros on $[T_y, \infty)$; otherwise, it is said to be non-oscillatory. (1.1) itself is said to be oscillatory if all of its solutions are oscillatory.

From [16], we know that (A2) implies f being odd. Indeed, $f(1)f(1) = f(1)$ and $f(1) > 0$ imply that $f(1) = 1$. Further, $(f(-1))^2 = f(-1)f(-1) = f(1) = 1$. Since

$f(-1) < 0$, we conclude that $f(-1) = -1$. Hence,

$$f(-u) = f(-1)f(u) = -f(u).$$

On the other hand, $f(uv) = f(u)f(v)$ for $u > 0$ and $v > 0$ and $f(-u) = -f(u)$ imply that $f(xy) = f(x)f(y)$ for every $x, y \in \mathbb{R}$.

Also from [16], we have that under assumption (A2), if $y(t)$ is a solution of (1.1), then $-y(t)$ is also a solution of (1.1).

2. PRELIMINARY RESULTS

Lemma 1. Assume that (A1)–(A3) hold and y is an eventually positive solution of (1.1). Then we have

$$y'(t) > 0 \quad \text{and} \quad (\pi(y')^\gamma)'(t) < 0,$$

for sufficiently large t .

Proof. Suppose that there exists a $t_1 \geq t_0$ such that $y(t) > 0$ and $y(\tau(t)) > 0$ for $t \geq t_1$. From (1.1) and (A2), it follows that

$$(\pi(y')^\gamma)'(t) = -p(t)f(y(\tau(t))) < 0 \quad \text{for } t \geq t_1. \quad (2.1)$$

Consequently, $(\pi(y')^\gamma)(t)$ is non-increasing on $[t_1, \infty)$. Since $\pi(t) > 0$, and thus either $y'(t) < 0$ or $y'(t) > 0$ for $t_2 \geq t_1$, where $t \geq t_2$.

We claim that $y'(t) > 0$ for $t \geq t_2$. To the contrary, assume that $y'(t) < 0$ for $t \geq t_2$, then there exists $\kappa_1 > 0$ such that $(\pi(y')^\gamma)(t) \leq -\kappa_1$ for $t \geq t_2$, which yields upon integration over $[t_2, t) \subset [t_2, \infty)$ after dividing through by π that

$$y(t) \leq y(t_2) - \kappa_1^{1/\gamma} \int_{t_2}^t (\pi(\eta))^{-1/\gamma} d\eta \quad \text{for } t \geq t_2. \quad (2.2)$$

By virtue of condition (A3), $\lim_{t \rightarrow \infty} y(t) = -\infty$. This contradicts $y(t)$ being a positive solution. So, our claim is true. This completes the proof. $\square$

3. NECESSARY AND SUFFICIENT CONDITIONS FOR OSCILLATIONS

In this section, we study necessary and sufficient conditions for oscillations of solutions of (1.1) by considering the cases when $f(v)/v^\beta$ is non-increasing and non-decreasing.

3.1. Non-increasing $f(v)/v^\beta$

We assume that there exists a constant β such that $0 < \beta < \gamma$ and

$$\frac{f(v)}{v^\beta} \geq \frac{f(u)}{u^\beta}, \quad \text{for } 0 < v \leq u. \quad (3.1)$$

A typical example of a nonlinear function satisfying (3.1) is $f(y) = |y|^\alpha \text{sgn}(y)$ with $0 < \alpha < \beta$.

Lemma 2. Assume that (A1)–(A3) hold and y is an eventually positive solution of (1.1). Then we have

$$y(t) \leq \kappa^{1/\gamma} \Pi(t) \quad (3.2)$$

$$y(t) \geq \int_{t_3}^t \left[\frac{1}{\pi(\eta)} \int_{\eta}^{\infty} p(\zeta) \frac{f(\kappa^{1/\gamma} \Pi(\tau(\zeta)))}{(\kappa^{1/\gamma} \Pi(\tau(\zeta)))^{\beta}} y^{\beta}(\tau(\zeta)) d\zeta \right]^{1/\gamma} d\eta, \quad (3.3)$$

for sufficiently large t .

Proof. Suppose that there exists $t_1 \geq t_0$ such that $y(t) > 0$ and $y(\tau(t)) > 0$ for $t \geq t_1$. Then, Lemma 1 holds true for $t \geq t_2$. Since $(\pi(y')^{\gamma})(t)$ is positive and non-increasing, there exists $\kappa > 0$ and $t_3 \geq t_2$ such that $(\pi(y')^{\gamma})(t) \leq \kappa$ for $t \geq t_3$. Integrating the inequality $y'(t) \leq (\kappa/\pi(t))^{1/\gamma}$, we have

$$y(t) \leq y(t_3) + \kappa^{1/\gamma} (\Pi(t) - \Pi(t_3)).$$

Since $\lim_{t \rightarrow \infty} \Pi(t) = \infty$, the last inequality becomes

$$y(t) \leq \kappa^{1/\gamma} \Pi(t) \quad \text{for } t \geq t_3.$$

Note that κ depends on y being evaluated at a time t_3 . Thus, (3.2) must include all possible κ 's.

By (3.1) and (3.2), we have

$$f(y(\tau(t))) = \frac{f(y(\tau(t)))}{y^{\beta}(\tau(t))} y^{\beta}(\tau(t)) \geq \frac{f(\kappa^{1/\gamma} \Pi(\tau(t)))}{(\kappa^{1/\gamma} \Pi(\tau(t)))^{\beta}} y^{\beta}(\tau(t)).$$

Integrating (1.1) from t to ∞ , we have

$$\lim_{A \rightarrow \infty} [(\pi(y')^{\gamma})(\eta)]_t^A + \int_t^{\infty} p(\eta) \frac{f(\kappa^{1/\gamma} \Pi(\tau(\eta)))}{(\kappa^{1/\gamma} \Pi(\tau(\eta)))^{\beta}} y^{\beta}(\tau(\eta)) d\eta \leq 0.$$

Using that $(\pi(y')^{\gamma})(t)$ is positive and non-increasing, we have

$$\int_t^{\infty} p(\eta) \frac{f(\kappa^{1/\gamma} \Pi(\tau(\eta)))}{(\kappa^{1/\gamma} \Pi(\tau(\eta)))^{\beta}} y^{\beta}(\tau(\eta)) d\eta \leq (\pi(y')^{\gamma})(t) \quad \text{for } t \geq t_3.$$

Therefore,

$$y'(t) \geq \left[\frac{1}{\pi(t)} \int_t^{\infty} p(\eta) \frac{f(\kappa^{1/\gamma} \Pi(\tau(\eta)))}{(\kappa^{1/\gamma} \Pi(\tau(\eta)))^{\beta}} y^{\beta}(\tau(\eta)) d\eta \right]^{1/\gamma}. \quad (3.4)$$

Integrating (3.4) from t_3 to t , we obtain

$$y(t) \geq \int_{t_3}^t \left[\frac{1}{\pi(\eta)} \int_{\eta}^{\infty} p(\zeta) \frac{f(\kappa^{1/\gamma} \Pi(\tau(\zeta)))}{(\kappa^{1/\gamma} \Pi(\tau(\zeta)))^{\beta}} y^{\beta}(\tau(\zeta)) d\zeta \right]^{1/\gamma} d\eta$$

$$\geq \int_{t_3}^t \left[\frac{1}{\pi(\eta)} \int_t^\infty p(\zeta) \frac{f(\kappa^{1/\gamma} \Pi(\tau(\zeta)))}{(\kappa^{1/\gamma} \Pi(\tau(\zeta)))^\beta} y^\beta(\tau(\zeta)) d\zeta \right]^{1/\gamma} d\eta.$$

The proof of the lemma is complete. $\square$

Theorem 1. Assume that (A1)-(A3) hold. Then every solution of (1.1) is oscillatory if and only if

$$\int_0^\infty p(\eta) f(\kappa^{1/\gamma} \Pi(\tau(\eta))) d\eta = +\infty \quad \forall \kappa > 0. \quad (3.5)$$

Proof. To prove sufficiency by contradiction, we assume that there exists a non-oscillatory solution $y(t)$ of (1.1). Since $-y(t)$ is also a solution of (1.1), we can confine our discussion only to the case where the solution $y(t)$ is eventually positive. Then there exists $t_1 \geq t_0$ such that $y(t) > 0$ and $y(\tau(t)) > 0$ for $t \geq t_1$. Since Lemmas 1 and 2 hold, (3.3) gives

$$y(t) > (\Pi(t) - \Pi(t_3)) \omega^{1/\gamma}(t) \quad \text{for } t \geq t_3,$$

where

$$\omega(t) = \int_t^\infty p(\zeta) \frac{f(\kappa^{1/\gamma} \Pi(\tau(\zeta)))}{(\kappa^{1/\gamma} \Pi(\tau(\zeta)))^\beta} y^\beta(\tau(\zeta)) d\zeta > 0.$$

Because $\lim_{t \rightarrow \infty} \Pi(t) = \infty$, there exists $t_4 \geq t_3$ such that $\Pi(t) - \Pi(t_3) \geq \frac{1}{2} \Pi(t)$ for $t \geq t_4$. Then

$$y(t) > \frac{1}{2} \Pi(t) \omega^{1/\gamma}(t) \quad \text{for } t \geq t_4,$$

and $y^\beta / (\kappa^{1/\gamma} \Pi)^\beta \geq \omega^{\beta/\gamma} / (2\kappa^{1/\gamma})^\beta$. Taking the derivative of ω we have

$$\begin{aligned} \omega'(t) &= -p(t) \frac{f(\kappa^{1/\gamma} \Pi(\tau(t)))}{(\kappa^{1/\gamma} \Pi(\tau(t)))^\beta} y^\beta(\tau(t)) \\ &\leq -p(t) f(\kappa^{1/\gamma} \Pi(\tau(t))) \omega^{\beta/\gamma}(\tau(t)) (2\kappa^{1/\gamma})^{-\beta} \leq 0. \end{aligned}$$

Therefore, $\omega(t)$ is non-increasing so $\omega^{\beta/\gamma}(\tau(t)) / \omega^{\beta/\gamma}(t) \geq 1$, and

$$(\omega^{1-\beta/\gamma}(t))' = (1 - \beta/\gamma) \omega^{-\beta/\gamma}(t) \omega'(t) \leq -\frac{(1 - \beta/\gamma)}{(2\kappa^{1/\gamma})^\beta} p(t) f(\kappa^{1/\gamma} \Pi(\tau(t))).$$

Integrating this inequality from t_4 to t , we have

$$[\omega^{1-\beta/\gamma}(\eta)]_{t_4}^t \leq -\frac{(1 - \beta/\gamma)}{(2\kappa^{1/\gamma})^\beta} \int_{t_4}^t p(\eta) f(\kappa^{1/\gamma} \Pi(\tau(\eta))) d\eta.$$

Since $\beta/\gamma < 1$ and $\omega(t)$ is positive and non-increasing, we have

$$\int_{t_4}^t p(\eta) f(\kappa^{1/\gamma} \Pi(\tau(\eta))) d\eta \leq \frac{(2\kappa^{1/\gamma})^\beta}{(1 - \beta/\gamma)} \omega^{1-\beta/\gamma}(t_4),$$

which contradicts (3.5).

Next, we show that (3.5) is necessary. Suppose that (3.5) does not hold; so for some $\kappa > 0$ the integral in (3.5) is finite. Then there exists $T \geq t_0$ such that

$$\int_T^\infty p(\eta) f(\kappa^{1/\gamma} \Pi(\tau(\eta))) d\eta \leq \frac{\kappa}{2}. \quad (3.6)$$

Let us consider the closed subset of continuous functions

$$M = \left\{ y \in C([t_0, +\infty), \mathbb{R}) : y(t) = 0 \text{ for } t_0 \leq t < T \text{ and } \left(\frac{\kappa}{2}\right)^{1/\gamma} [\Pi(t) - \Pi(T)] \leq y(t) \leq \kappa^{1/\gamma} [\Pi(t) - \Pi(T)] \text{ for } T \leq t \right\}.$$

We define the operator $\Omega : M \rightarrow C([t_0, +\infty), \mathbb{R})$ by

$$(\Omega y)(t) = \begin{cases} 0, & t_0 \leq t < T \\ \int_T^t \left[\frac{1}{\pi(\eta)} \left[\frac{\kappa}{2} + \int_\eta^\infty p(\zeta) f(y(\tau(\zeta))) d\zeta \right] \right]^{1/\gamma} d\eta, & T \leq t. \end{cases}$$

For $y \in M$ and $t \geq T$, we have

$$(\Omega x)(t) \geq \int_T^t \left[\frac{1}{\pi(\eta)} \frac{\kappa}{2} \right]^{1/\gamma} d\eta = \left(\frac{\kappa}{2}\right)^{1/\gamma} [\Pi(t) - \Pi(T)].$$

For $y \in M$ and $t \geq T$, we have $y(t) \leq \kappa^{1/\gamma} \Pi(t)$ and $f(y) \leq f(\kappa^{1/\gamma} \Pi(t))$. Using (3.6), we have

$$(\Omega x)(t) \leq \int_T^t \left[\frac{1}{\pi(\eta)} \left(\frac{\kappa}{2} + \frac{\kappa}{2} \right) \right]^{1/\gamma} d\eta = \kappa^{1/\gamma} [\Pi(t) - \Pi(T)].$$

Thus, $\Omega x \in M$. Now, we define a sequence of continuous function $v_n : [t_0, +\infty) \rightarrow \mathbb{R}$ by the recursive formula

$$v_1(t) = \begin{cases} 0, & t \in [t_0, T) \\ \left(\frac{\kappa}{2}\right)^{1/\gamma} [\Pi(t) - \Pi(T)], & t \geq T. \end{cases}$$

$$v_n(t) = (\Omega v_{n-1})(t) \quad \text{for } n > 1.$$

It is easy to verify that for $n > 1$,

$$\left(\frac{\kappa}{2}\right)^{1/\gamma} [\Pi(t) - \Pi(T)] \leq v_{n-1}(t) \leq v_n(t) \leq \kappa^{1/\gamma} [\Pi(t) - \Pi(T)].$$

Therefore, the pointwise limit of the sequence exists. Let $\lim_{n \rightarrow \infty} v_n(t) = v(t)$ for $t \geq t_0$. By Lebesgue's dominated convergence theorem $v \in M$ and $(\Omega v)(t) = v(t)$, where $v(t)$ is a solution of equation (1.1) on $[T, \infty)$. Hence, (3.5) is a necessary condition. This completes the proof. $\square$

Example 1. Consider the delay differential equation

$$(e^{-t} (y'(t))^{3/5})' + \frac{1}{t+1} (y(t-2))^{1/3} = 0, \quad t \geq 0. \quad (3.7)$$

Here $\gamma = 3/5$, $\pi(t) = e^{-t}$, $\tau(t) = t - 2$, $\Pi(t) = \int_0^t e^{5s/3} ds = \frac{3}{5}(e^{5t/3} - 1)$, $f(v) = v^{1/3}$. For $\beta = 1/2$, we have $f(v)/v^\beta = v^{-1/6}$ which is a decreasing function. To check (3.5) we have

$$\int_0^\infty p(\eta) f(\kappa^{1/\gamma} \Pi(\tau(\eta))) d\eta = \int_0^\infty \frac{1}{\eta+1} \left(\kappa^{5/3} \frac{3}{5} (e^{5(\eta-2)/3} - 1) \right)^{1/3} d\eta = \infty \quad \forall \kappa > 0,$$

because the integral approaches $+\infty$ as $\eta \rightarrow +\infty$. So that all the conditions of Theorem 1 hold. Therefore, all solutions of (3.7) are oscillatory.

3.2. Non-decreasing $f(v)/v^\beta$

We assume that there exists $\beta > \gamma > 0$ such that

$$\frac{f(v)}{v^\beta} \leq \frac{f(u)}{u^\beta}, \quad \text{for } 0 < v \leq u. \quad (3.8)$$

A typical example of a nonlinear function satisfying (3.8) is $f(y) = |y|^\alpha \text{sgn}(y)$ with $\gamma < \beta < \alpha$.

Theorem 2. *Assuming (A1)-(A3) and $\tau'(t) \geq 1$, every solution of (1.1) is oscillatory if and only if*

$$\int_0^\infty \left[\frac{1}{\pi(\eta)} \int_\eta^\infty p(\zeta) d\zeta \right]^{1/\gamma} d\eta = +\infty. \quad (3.9)$$

Proof. To prove sufficiency by contradiction, we assume that there exists a non-oscillatory solution $y(t)$ of (1.1). Since $-y(t)$ is also a solution of (1.1), we can confine our discussion only to the case where the solution $y(t)$ is eventually positive. Then there exists $t_1 \geq t_0$ such that $y(t) > 0$ and $y(\tau(t)) > 0$ for $t \geq t_1$. Then, Lemma 1 holds true for $t \geq t_3 \geq t_2$. Since $y'(t) > 0$, so y is increasing and $y(t) \geq y(t_3)$ for $t \geq t_3$. Therefore,

$$y(\tau(t)) \geq y(\tau(t_3)) := \kappa > 0 \quad \text{for } t \geq t_3.$$

From (3.8), we have

$$f(y(\tau(t))) = \frac{f(y(\tau(t)))}{y^\beta(\tau(t))} y^\beta(\tau(t)) \geq \frac{f(\kappa)}{\kappa^\beta} y^\beta(\tau(t)).$$

Integrating (1.1) from t to ∞ , we have

$$\lim_{A \rightarrow \infty} \left[(\pi(y')^\gamma)(\eta) \right]_t^A + \frac{f(\kappa)}{\kappa^\beta} \int_t^\infty p(\eta) y^\beta(\tau(\eta)) d\eta \leq 0.$$

Using that $(\pi(y')^\gamma)(t)$ is positive and non-increasing, we have

$$\frac{f(\kappa)}{\kappa^\beta} \int_t^\infty p(\eta) y^\beta(\tau(\eta)) d\eta \leq (\pi(y')^\gamma)(t) \leq (\pi(y')^\gamma)(\tau(t)) \leq \pi(t) ((y')^\gamma)(\tau(t))$$

for all $t \geq t_3$. Therefore,

$$\left(\frac{f(\kappa)}{\kappa^\beta} \right)^{1/\gamma} \left[\frac{1}{\pi(t)} \int_t^\infty p(\eta) y^\beta(\tau(\eta)) d\eta \right]^{1/\gamma} \leq y'(\tau(t))$$

implies that

$$\left(\frac{f(\kappa)}{\kappa^\beta}\right)^{1/\gamma} \left[\frac{1}{\pi(t)} \int_t^\infty p(\eta) d\eta\right]^{1/\gamma} \leq \frac{y'(\tau(t))}{y^{\beta/\gamma}(\tau(t))} \leq \frac{y'(\tau(t))\tau'(t)}{y^{\beta/\gamma}(\tau(t))} \quad (3.10)$$

Integrating (3.10) from t_3 to ∞ , we have

$$\left(\frac{f(\kappa)}{\kappa^\beta}\right)^{1/\gamma} \int_{t_3}^\infty \left[\frac{1}{\pi(\eta)} \int_\eta^\infty p(\zeta) d\zeta\right]^{1/\gamma} d\eta \leq \frac{y^{1-\beta/\gamma}(\tau(t_3))}{\beta/\gamma - 1} < \infty,$$

which contradicts (3.9).

Next, we show that (3.9) is necessary. Suppose that (3.9) does not hold; so for each $\kappa > 0$, there exists $T \geq t_0$ such that

$$\int_T^\infty \left[\frac{1}{\pi(\eta)} \int_\eta^\infty p(\zeta) d\zeta\right]^{1/\gamma} d\eta \leq \frac{\kappa}{2(f(\kappa))^{1/\gamma}} \quad (3.11)$$

Let us consider the closed subset of continuous functions

$$M = \left\{x \in C([t_0, +\infty), \mathbb{R}) : x(t) = \frac{\kappa}{2} \text{ for } t \in [t_0, T) \text{ and } \frac{\kappa}{2} \leq x(t) \leq \kappa \text{ for } t \geq T\right\}.$$

We define the operator $\Omega : M \rightarrow C([t_0, +\infty), \mathbb{R})$ by

$$(\Omega y)(t) = \begin{cases} \kappa/2, & t_0 \leq t < T \\ \kappa/2 + \int_T^t \left[\frac{1}{\pi(\eta)} \int_\eta^\infty p(\zeta) f(y(\tau(\zeta))) d\zeta\right]^{1/\gamma} d\eta & T \leq t. \end{cases}$$

Note that for $y \in M$, we have $(\Omega y)(t) \geq \kappa/2$. Also for $y \in M$ and $t \geq T$, we have $y(t) \leq \kappa$ and by (3.11), $(\Omega y)(t) \leq \kappa$. Therefore, $\Omega y \in M$. As in the proof of Theorem 1, the mapping Ω has a fixed point $v \in M$; that is, $(\Omega v)(t) = v(t)$ for $t \geq t_0$. It can be easily verified that $v(t)$ is a solution of (1.1), such that $\kappa/2 \leq v(t) \leq \kappa$ for $t \geq T$. Thus we have a non-oscillatory solution to (1.1). This completes the proof. $\square$

Example 2. Consider the delay differential equation

$$((y^{1/5})^{t^{5/3}})' = 0, \quad t \geq 0. \quad (3.12)$$

Here $\gamma = 1/5$, $\pi(t) = 1$, $\tau(t) = t - 1$ and $f(u) = u^{5/3}$. For $\beta = 4/3$, we have $f(v)/v^\beta = v^{1/3}$ which is an increasing function. Thus

$$\int_2^\infty \left[\int_\eta^\infty (\zeta + 1) d\zeta\right]^5 d\eta = \infty.$$

So, all the conditions of Theorem 2 are satisfied, and therefore all solution of (3.12) are oscillatory.

4. CONCLUSION

In this section, we conclude the paper by stating a Remark and presenting two examples.

Remark 1. The results of this paper also hold for equations of the form

$$(\pi(y^\gamma))'(t) + \sum_{i=1}^m p_i(t) f_i(y(\tau_i(t))) = 0,$$

where π, p_i, f_i, τ_i ($i = 1, 2, \dots, m$) satisfy the assumptions (A1)-(A3), (3.1) or (3.8). In order to extend Theorem 1 and Theorem 2, there exists an index i such that p_i, f_i, τ_i fulfill (3.5) and (3.9), respectively.

Next, we provide two examples, illustrating how Remark 3.1 can be applied.

Example 3. Consider the delay differential equation

$$(e^{-t}(y^{3/5}))' + \frac{1}{t+1}(y(t-2))^{1/3} + \frac{1}{t+2}(y(t-1))^{1/5} = 0, \quad t \geq 0. \quad (4.1)$$

Here $\gamma = 3/5$, $\pi(t) = e^{-t}$, $\tau_1(t) = t-2$, $\tau_2(t) = t-1$, $\Pi(t) = \int_0^t e^{5\eta/3} d\eta = \frac{3}{5}(e^{5t/3} - 1)$, $f_1(v) = v^{1/3}$ and $f_2(v) = v^{1/5}$. For $\beta = 1/2$, we have $f_1(v)/v^\beta = v^{-1/6}$ and $f_2(v)/v^\beta = v^{-3/10}$ which both are decreasing functions. To check (3.5) we have

$$\begin{aligned} \int_0^\infty \sum_{i=1}^m p_i(\eta) f_i(\kappa^{1/\gamma} \Pi(\tau_i(\eta))) d\eta &\geq \int_0^\infty p_1(\eta) f_1(\kappa^{1/\gamma} \Pi(\tau_1(\eta))) d\eta \\ &= \int_0^\infty \frac{1}{\eta+1} \left(\kappa^{5/3} \frac{3}{5} (e^{5(\eta-2)/3} - 1) \right)^{1/3} d\eta = \infty \quad \forall \kappa > 0, \end{aligned}$$

because the integral approaches $+\infty$ as $\eta \rightarrow +\infty$. So, all the conditions of Theorem 1 hold, and therefore all solution of (4.1) are oscillatory.

Example 4. Consider the delay differential equation

$$((y^{3/5})^{5/3} + (t+1)(y(t-1))^{7/3})' = 0, \quad t \geq 0. \quad (4.2)$$

Here $\gamma = 3/5$, $\pi(t) = 1$, $\tau_1(t) = t-2$, $\tau_2(t) = t-1$, $\Pi(t) = t$, $f_1(v) = v^{5/3}$ and $f_2(v) = v^{7/3}$. For $\beta = 4/3$, we have $f_1(v)/v^\beta = v^{1/3}$ and $f_2(v)/v^\beta = v$ which both are increasing functions. Clearly, all the conditions of Theorem 2 hold. Thus, every solution of (4.2) oscillates.

Open problem

Based on this work and [6, 9, 13, 15, 17, 18] an open problem that arises is to establish necessary and sufficient conditions for the oscillation of the solutions of the second-order nonlinear neutral differential equation.

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THE EXISTENCE OF ONE NON-TRIVIAL WEAK SOLUTION OF GENERALIZED YAMABE EQUATIONS

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Received 02 March, 2021

Abstract. In this work, we study the nonlinear problem on compact d -dimensional ($d \geq 3$) Riemannian manifolds with respect to absence of boundary. The existence of one non-trivial weak solution is established, and its application to solve Emden-Fowler equations which contain infinity nonlinear terms. We also introduce an example to illustrate the results obtained, which can be applied to many of the problems resulting in astrophysics, conformal Riemannian geometry, gas combustion, isothermal stationary gas sphere and in the theory of thermionic emission.

2010 Mathematics Subject Classification: 35J60; 35B30; 35B40

Keywords: Yamabe problem, Emden-Fowler equation, sublinear eigenvalue problem, one non-trivial solutions, Riemannian manifolds

1. INTRODUCTION

In the last decades, many researchers have focused on searching mathematical analysis of differential equations. The field of analysis of the Riemannian manifolds is fertile and under great development, and it is an important tool for solving geometrical problems. On the other hand, geometry can help solve some of the problems associated with the analysis (see [2, 12, 16]).

We consider a compact d -dimensional Riemannian manifold with absence of boundary $(\mathcal{M}, g)$, where $d \geq 3$, Δ_g is the Laplace-Beltrami operator and suppose that $\alpha, K \in C^\infty(\mathcal{M})$ are positive functions. Assume the locally Hölder continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ with *sublinear* growth and $\lambda > 0$. In this work, we investigate the existence of the (classical) solutions of the eigenvalue problem

$$-\Delta_g w + \alpha(\sigma)w = \lambda K(\sigma)f(w), \quad \sigma \in \mathcal{M}, \quad w \in H_1^2(\mathcal{M}). \quad (1.1)$$

This problem is a generalization of the Yamabe equation (see [3]):

$$4\frac{d-1}{d-2}\Delta_g \phi + R\phi = \mu\phi^{q-1} \text{ in } \mathcal{M},$$

The fifth author extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through research groups program under grant (R.G.P-2/1/42).

where $2 < q < 2d/(d-2)$ with a curvature scalar R in $\mathcal{M}$. In [20], Yamabe showed in detail that for embedded manifolds there always exists such an equivalent metric corresponding to any given scale on the manifold. By applying heterogeneous methods, we get a well determined open interval of values λ for any problem (1.1) accepts at least one nontrivial solution. Later, Aubin in [2] proved the theorem for all manifolds of dimension $d \geq 6$ that were not conformably flat.

In 1984, by applying the variational methods Schoen settled the problem completely in [18], and get a well determined open interval of values λ for any problem (1.1) accepting at least one nontrivial solution.

A case of problem (1.1) is

$$-\Delta_h w + s(1-s-d)w = \lambda K(\sigma)f(w), \quad \sigma \in S^d, \quad w \in H_1^2(S^d), \quad (1.2)$$

where S^d is the unit sphere in $\mathbb{R}^{d+1}$, the standard metric h is induced by the embedding $S^d \hookrightarrow \mathbb{R}^{d+1}$, the constant s verifies $1-d < s < 0$, and Δ_h is the Laplace-Beltrami operator on (S^d, h) .

Using a suitable switching of coordinates, the results of the existence in (1.2) give us the existence of solutions to the Emden-Fowler equation

$$-\Delta u = \lambda |x|^{s-2} K(x/|x|) f(|x|^{-s} u), \quad x \in \mathbb{R}^{d+1} \setminus \{0\}, \quad (1.3)$$

and we refer to Remark 3 and Corollary 2.

In addition, when $s = -d/2$ or $s = -d/2 + 1$, then the problem (1.2) has a smooth positive solution, and the answer $f(t) = |t|^{\frac{4}{d-2}} t$ can be considered convergent to the problem of Yamabe on S^d (for more details see [2, 9–11, 14, 15, 19, 20]).

In [19], the authors have solved the problem by applying minimization or minimum methods, provided that $f(t) = |t|^{p-1}t$, with $p > 1$. In [13] the authors were concerned with the existence of multiple solutions to problem (1.1) to obtain the solutions of Emden-Fowler equation with regard to infinity as nonlinear terms. Likewise, in [13], two non-trivial solutions to problem (1.1) were obtained, which took λ large enough.

There are also several works related to this type of problem, for example see ([1, 4–8]). Starting from these works, in this paper we will take a new path, and this is by attaching subproblems at infinity (previously studied and developed in [17]), where we get an open interval of positive parameters for which problem (1.1) admits at least one nontrivial solution.

The paper has been divided into the following parts: in the second section we lay out some preliminary and necessary facts and definitions. In the third section we examine the existence of at least one non-trivial solution to problem (1.1). In the final section we present some key results, with the solution to the Emden-Fowler equation, after which we propose an applied example for our theorems.

2. PRELIMINARIES

Starting from works [2] and [10], we present some basic concepts in Riemannian geometry.

Consider the smooth compact d -dimensional ($d \geq 3$) Riemannian manifold $(\mathcal{M}, g)$ without boundary and let g_{ij} be the components of the metric g . Denote the space of smooth functions given on $\mathcal{M}$ by $C^\infty(\mathcal{M})$. Let the positive function $\alpha \in C^\infty(\mathcal{M})$ and $\|\alpha\|_\infty := \max_{\sigma \in \mathcal{M}} \alpha(\sigma)$. For every $w \in C^\infty(\mathcal{M})$, set

$$\|w\|_{H_\alpha^2}^2 := \int_{\mathcal{M}} |\nabla w(\sigma)|^2 d\sigma_g + \int_{\mathcal{M}} \alpha(\sigma) |w(\sigma)|^2 d\sigma_g,$$

where ∇w is the covariant derivative of w , and $d\sigma_g$ is the Riemannian measure. In local coordinates $(x^1, \dots, x^d)$, the components of ∇w are given by

$$(\nabla^2 w)_{ij} = \frac{\partial^2 w}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial w}{\partial x^k},$$

where

$$\Gamma_{ij}^k := \frac{1}{2} \left(\frac{\partial g_{1j}}{\partial x^i} + \frac{\partial g_{li}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} \right) g^{lk},$$

are Christoffel symbols and g^{lk} are the elements of the inverse matrix of g .

In what follows, we adopt the Einstein summation convention.

Furthermore, the element of measure $d\sigma_g$ supposes the form $d\sigma_g = \sqrt{\det g} dx$, where dx stands for the Lebesgue volume element of $\mathbb{R}^d$. Then, let

$$\text{Vol}_g(M) := \int_{\mathcal{M}} d\sigma_g.$$

If $(\mathcal{M}, g) = (S^d, h)$, where S^d is the unit sphere in $\mathbb{R}^{d+1}$ and h is the standard metric induced by the embedding $S^d \hookrightarrow \mathbb{R}^{d+1}$, we set

$$\omega_d := \text{Vol}_h(S^d) := \int_{S^d} d\sigma_h.$$

$H_\alpha^2(\mathcal{M})$ is the Sobolev space given as the completion of $C^\infty(\mathcal{M})$ with respect to the norm $\|\cdot\|_{H_\alpha^2}$. Hence, the Hilbert space $H_\alpha^2(\mathcal{M})$ is endowed with the inner product

$$\langle w_1, w_2 \rangle_{H_\alpha^2} = \int_{\mathcal{M}} \langle \nabla w_1, \nabla w_2 \rangle_g d\sigma_g + \int_{\mathcal{M}} \alpha(\sigma) \langle w_1, w_2 \rangle_g d\sigma_g, \quad w_1, w_2 \in H_\alpha^2(\mathcal{M}),$$

where $\langle \cdot, \cdot \rangle_g$ is the inner product on covariant tensor fields associated to g .

As $\alpha > 0$, the norm $\|\cdot\|_{H_\alpha^2}$ is equivalent with the norm

$$\|w\|_{H_1^2} := \left(\int_{\mathcal{M}} |\nabla w(\sigma)|^2 d\sigma_g + \int_{\mathcal{M}} |w(\sigma)|^2 d\sigma_g \right)^{1/2}.$$

Furthermore, if $w \in H_\alpha^2(\mathcal{M})$, the following inequalities hold:

$$\min\{1, \min_{\sigma \in \mathcal{M}} \alpha(\sigma)^{1/2}\} \|w\|_{H_1^2} \leq \|w\|_{H_\alpha^2} \leq \max\{1, \|\alpha\|_\infty^{1/2}\} \|w\|_{H_1^2}. \quad (2.1)$$

From the Rellich-Kondrachov theorem for compact manifolds without boundary one has

$$H_\alpha^2(\mathcal{M}) \hookrightarrow L^q(\mathcal{M}),$$

for any $q \in [1, 2d/(d-2)]$, the embedding is compact, whenever $q \in [1, 2d/(d-2)]$.

Then, $\exists S_q > 0$ so that

$$\|w\|_q \leq S_q \|w\|_{H_\alpha^2}, \text{ for all } w \in H_\alpha^2(\mathcal{M}). \quad (2.2)$$

Now, we suppose that the nonlinear f satisfies the following structural condition: $f : \mathbb{R} \rightarrow \mathbb{R}$ is a locally Hölder continuous function sublinear at infinity, as follows

$$\lim_{|t| \rightarrow +\infty} \frac{f(t)}{t} = 0. \quad (2.3)$$

Let $K \in C^\infty(\mathcal{M})$ a positive function. Recall that $w \in H_1^2(\mathcal{M})$ is a *weak solution* of problem (1.1) if

$$\int_{\mathcal{M}} \langle \nabla w, \nabla v \rangle_g d\sigma_g + \int_{\mathcal{M}} \alpha(\sigma) \langle w, v \rangle_g d\sigma_g - \lambda \int_{\mathcal{M}} K(\sigma) f(w(\sigma)) v(\sigma) d\sigma_g = 0$$

for every $v \in H_1^2(\mathcal{M})$. Moreover, because of the regularity supposition on f , the weak solutions are classical.

Remark 1. In (1.1), Δ_g represents the Laplace-Beltrami operator applied to a function $w \in H_1^2(\mathcal{M})$, and is given (locally) by the expression

$$\Delta_g w = g^{ij} \left(\frac{\partial^2 w}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial w}{\partial x^k} \right).$$

For a fixed $\lambda > 0$, the function $w_\lambda(\sigma) = c \in \mathbb{R} \setminus \{0\}$ is a solution of (P_λ) if and only if the function $\sigma \mapsto \lambda K(\sigma)/\alpha(\sigma)$ is constant. In this case, non-trivial constant solutions of (1.1) appear as fixed points of the function $t \mapsto k_\lambda f(t)$, where k_λ denotes the constant value $\lambda K(\sigma)/\alpha(\sigma)$.

In order to obtain one non-trivial solution of (1.1) not only in the case of constant solution but also other solutions, we use the variational methods. The main tool is a critical point theorem and we recall it here in a convenient form. The result is obtained by Ricceri in [17].

Theorem 1 ([17]). *Let X be a reflexive real Banach space, and let Φ, Ψ be two Gâteaux differentiable functionals such that Φ is (strongly) continuous, sequentially weakly lower semicontinuous and coercive. Moreover, suppose that Ψ is sequentially weakly upper semicontinuous. For any $\rho > \inf_X \Phi$, put*

$$\varphi(\rho) = \inf_{u \in \Phi^{-1}([-\infty, \rho])} \frac{\left(\sup_{v \in \Phi^{-1}([-\infty, \rho])} \Psi(v) \right) - \Psi(u)}{\rho - \Phi(u)}.$$

Hence, for every $\rho > \inf_X \Phi$, and any $\lambda \in \left]0, \frac{1}{\varphi(\rho)}\right[$, the restriction of $J_\lambda := \Phi - \lambda\Psi$ to $\Phi^{-1}(\left]-\infty, \rho\right[)$ accepts a global minimum, which is a critical point (local minimum) of J_λ in X .

This result is a refinement of the variational principle of Ricceri, for more information see [17].

3. MAIN RESULTS

We give our main result in this section. Set

$$\kappa_\alpha := \left(\frac{2}{\|\alpha\|_{L^1(\mathcal{M})}}\right)^{1/2}$$

and

$$K_1 := \frac{S_1}{\sqrt{2}} \|\alpha\|_{L^1(\mathcal{M})}, \quad K_2 := \frac{S_q^q}{2^{\frac{2-q}{2}} - q} \|\alpha\|_{L^1(\mathcal{M})}.$$

Moreover, let

$$F(\xi) := \int_0^\xi f(t) dt$$

for any $\xi \in \mathbb{R}$.

The main abstract theorem in this work is the following.

Theorem 2. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function with $f(t) \neq 0$ in $\mathbb{R}$, such that (h_∞) holds and suppose that*

- (h1) $\exists a_1, a_2 > 0$ such that $|f(t)| \leq a_1 + a_2 |t|^{q-1}$, for any $t \in \mathbb{R}$, where $q \in]1, 2d/(d-2)[$.

Then $\exists \lambda^* > 0$ given by

$$\lambda^* := \frac{\|\alpha\|_{L^1(\mathcal{M})}}{2\|K\|_\infty} \sup_{\gamma > 0} \left(\frac{\bar{\gamma}}{a_1 K_1 + a_2 K_2 \bar{\gamma}^{q-1}} \right)$$

such that, for any parameter λ belonging to $\Lambda_\gamma :=]0, \lambda^*[$, problem (1.1) possesses at least one non-trivial weak solution u_λ in $H_1^2(\mathcal{M})$. Moreover,

$$\lim_{\lambda \rightarrow 0^+} \|u_\lambda\| = 0$$

and the function $\lambda \mapsto J_\lambda(u_\lambda)$ is negative and strictly decreasing in $]0, \lambda^*[$.

Proof. Our goal is to use Theorem 1. Let $X := H_1^2(\mathcal{M})$ and let the functionals $\Phi, \Psi : X \rightarrow \mathbb{R}$ defined by

$$\Phi(w) := \frac{\|w\|_{H_\alpha^2}^2}{2},$$

$$\Psi(w) := \int_{\mathcal{M}} K(\sigma) F(w(\sigma)) d\sigma_g$$

for every $w \in X$. $\Phi : X \rightarrow \mathbb{R}$ is a coercive, continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on X^* . Ψ is well defined, continuously Gâteaux differentiable and has compact derivative. Further, one has

$$\Phi'(w)(v) = \int_{\mathcal{M}} \langle \nabla w, \nabla v \rangle_g d\sigma_g + \int_{\mathcal{M}} \alpha(\sigma) \langle w(\sigma), v(\sigma) \rangle_g d\sigma_g,$$

and

$$\Psi'(w)(v) = \int_{\mathcal{M}} K(\sigma) f(w(\sigma)) v(\sigma) d\sigma_g,$$

for any $w, v \in X$. Fix $\lambda > 0$. A critical point of the functional $J_\lambda := \Phi - \lambda\Psi$ is a function $w \in X$ such that

$$\Phi'(w)(v) - \lambda\Psi'(w)(v) = 0,$$

for any $v \in X$. Then, the critical points of the functional J_λ are weak solutions (classical solutions) of problem (1.1).

Now, since (h1) and $\Phi(0) = \Psi(0) = 0$ hold, we have

$$F(\xi) \leq a_1 |\xi| + a_2 \frac{|\xi|^q}{q}, \quad (3.1)$$

for every $\xi \in \mathbb{R}$. Since $0 < \lambda < \lambda^*$, there exists $\bar{\gamma}$ such that

$$\lambda < \lambda^*(\bar{\gamma}) := \frac{\|\alpha\|_{L^1(\mathcal{M})}}{2\|K\|_\infty} \left(\frac{\bar{\gamma}}{a_1 K_1 + a_2 K_2 \bar{\gamma}^{q-1}} \right). \quad (3.2)$$

Let $\rho \in]0, +\infty[$ and let the function

$$\chi(\rho) := \frac{\sup_{w \in \Phi^{-1}(]-\infty, \rho])} \Psi(w)}{\rho}.$$

Recalling (3.1), it follows that

$$\Psi(w) = \int_{\mathcal{M}} K(\sigma) F(w(\sigma)) d\sigma_g \leq \|K\|_\infty \left(a_1 \|w\|_{L^1(\mathcal{M})} + \frac{a_2}{q} \|w\|_{L^q(\mathcal{M})}^q \right).$$

Hence, for any $w \in X$ such that $w \in \Phi^{-1}(]-\infty, \rho])$ owing to (3.2), we find

$$\Psi(w) \leq \|K\|_\infty \left((2\rho)^{1/2} S_1 a_1 + \frac{2^{q/2} S_q^q a_2}{q} \rho^{q/2} \right).$$

Then, by using the definition of Φ , we have

$$\sup_{w \in \Phi^{-1}(]-\infty, \rho])} \Psi(w) \leq \|K\|_\infty \left((2\rho)^{1/2} S_1 a_1 + \frac{2^{q/2} S_q^q a_2}{q} \rho^{q/2} \right). \quad (3.3)$$

From (3.3), the following inequality holds

$$\chi(\rho) \leq \|K\|_\infty \left(\sqrt{\frac{2}{\rho}} S_1 a_1 + \frac{2^{q/2} S_q^q a_2}{q} \rho^{q/2-1} \right)$$

for every $\rho > 0$.

Hence, in particular

$$\chi(\bar{\gamma}^2) \leq \|K\|_\infty \left(\frac{\sqrt{2}}{\bar{\gamma}} S_1 a_1 + \frac{2^{q/2} S_q^q a_2}{q} \bar{\gamma}^{q-2} \right). \quad (3.4)$$

Now, observe that

$$\varphi(\bar{\gamma}^2) := \inf_{u \in \Phi^{-1}(\bar{\gamma}^2)} \frac{\left(\sup_{v \in \Phi^{-1}(\bar{\gamma}^2)} \Psi(v) \right)}{\rho - \Phi(u)} \leq \chi(\bar{\gamma}^2),$$

because $u_0 \in \Phi^{-1}(\bar{\gamma}^2)$ and $\Phi(u_0) = \Psi(u_0) = 0$, where $u_0 \in X$ is the identically zero function. Finally, recalling to (3.3), the last inequality with (3.4) gives

$$\begin{aligned} \varphi(\bar{\gamma}^2) &\leq \chi(\bar{\gamma}^2) \leq \|K\|_\infty \left(\frac{\sqrt{2}}{\bar{\gamma}} S_1 a_1 + \frac{2^{q/2} S_q^q a_2}{q} \bar{\gamma}^{q-2} \right) \\ &= \frac{2\|K\|_\infty}{\|\alpha\|_{L^1(\mathcal{M})}} \left(a_1 \frac{K_1}{\bar{\gamma}} + a_2 K_2 \bar{\gamma}^{q-2} \right) < \frac{1}{\lambda}. \end{aligned}$$

In other words,

$$\lambda \in \left] 0, \frac{\|\alpha\|_{L^1(\mathcal{M})}}{2\|K\|_\infty} \left(\frac{\bar{\gamma}}{a_1 K_1 + a_2 K_2 \bar{\gamma}^{q-1}} \right) \right] \subseteq \left] 0, \frac{1}{\varphi(\bar{\gamma}^2)} \right].$$

By Theorem 1, $\exists u_\lambda \in \Phi^{-1}(\bar{\gamma}^2)$ such that

$$J'_\lambda(u_\lambda) = \Phi'(u_\lambda) - \lambda \Psi'(u_\lambda) = 0,$$

and, u_λ is a global minimum of the restriction of J_λ to $\Phi^{-1}(\bar{\gamma}^2)$. Further, since $f(t) \neq 0$ in $\mathbb{R}$, the function u_λ cannot be trivial, i.e. $u_\lambda \neq 0$.

Then, for any $\lambda \in]0, \lambda^*[$ problem (1.1) accepts a non-trivial solution $u_\lambda \in X$. Now, we prove that $\|u_\lambda\| \rightarrow 0$ as $\lambda \rightarrow 0^+$ and the function $\lambda \rightarrow J_\lambda(u_\lambda)$ is negative and strictly decreasing in $]0, \lambda^*[$.

For our aim, let us consider $\bar{\lambda} \in]0, \lambda^*[$. Moreover, let $\bar{\gamma} > 0$ such that $\bar{\lambda} < \lambda^*(\bar{\gamma})$. Arguing as before, for every $\lambda \in]0, \lambda^*(\bar{\gamma})[$ the functional $J_\lambda(u_\lambda)$ admits a non-trivial critical point $u_\lambda \in \Phi^{-1}(\bar{\gamma}^2)$. At this point, since Φ is coercive and $u_\lambda \in \Phi^{-1}(\bar{\gamma}^2)$ for every $\lambda \in]0, \lambda^*(\bar{\gamma})[$, $\exists \tau > 0$ such that

$$\|u_\lambda\| \leq \tau,$$

for any $\lambda \in]0, \lambda^*(\bar{\gamma})[$. So, since Ψ' is a compact operator, $\exists M > 0$ such that

$$|\Psi(u_\lambda)| \leq \|\Psi'(u_\lambda)\|_{X^*} \|u_\lambda\| < M \tau^2, \quad (3.5)$$

for every $\lambda \in]0, \lambda^*(\bar{\gamma})[$.

Now $J'_\lambda(u_\lambda) = 0$, for every $\lambda \in]0, \lambda^*(\bar{\gamma})[$ and in particular

$$J'_\lambda(u_\lambda)(u_\lambda) = 0,$$

that is,

$$\|u_\lambda\|_{H_\alpha^2}^2 = \lambda \int_{\mathcal{M}} K(\sigma) f(u_\lambda(\sigma)) u_\lambda(\sigma) dg_\sigma, \quad (3.6)$$

for every $\lambda \in]0, \lambda^*(\bar{\gamma})[$. Hence, from (3.5) and (3.6) it follows that

$$\lim_{\lambda \rightarrow 0^+} \Phi(u_\lambda) = 0,$$

implying

$$\lim_{\lambda \rightarrow 0^+} \|u_\lambda\|_{H_1^2} = 0.$$

Further, the map $\lambda \mapsto J_\lambda(u_\lambda)$ is negative in $]0, \lambda^*(\bar{\gamma})[$, since the restriction of the functional J_λ to $\Phi^{-1}(]-\infty, \bar{\gamma}^2])$ accepts a global minimum, which is a critical point (local minimum) of J_λ in X .

Finally, observe that

$$J_\lambda(u) = \lambda \left(\frac{\Phi(u)}{\lambda} - \Psi(u) \right),$$

for every $u \in X$ and fix $0 < \lambda_1 < \lambda_2 < \lambda^*(\bar{\gamma})$. Moreover, put

$$m_{\lambda_1} := \left(\frac{\Phi(u_{\lambda_1})}{\lambda_1} - \Psi(u_{\lambda_1}) \right) = \inf_{u \in \Phi^{-1}(]-\infty, \bar{\gamma}^2])} \left(\frac{\Phi(u)}{\lambda_1} - \Psi(u) \right),$$

and

$$m_{\lambda_2} := \left(\frac{\Phi(u_{\lambda_2})}{\lambda_2} - \Psi(u_{\lambda_2}) \right) = \inf_{u \in \Phi^{-1}(]-\infty, \bar{\gamma}^2])} \left(\frac{\Phi(u)}{\lambda_2} - \Psi(u) \right).$$

Since $m_{\lambda_i} < 0$, (for $i = 1, 2$), and $m_{\lambda_2} < m_{\lambda_1}$ according $\lambda_1 < \lambda_2$. Then the mapping $\lambda \mapsto J_\lambda(u_\lambda)$ is strictly decreasing in $]0, \lambda^*(\bar{\gamma})[$ owing to

$$J_{\lambda_2}(u_{\lambda_2}) = \lambda_2 m_{\lambda_2} \leq \lambda_2 m_{\lambda_1} < \lambda_1 m_{\lambda_1} = J_{\lambda_1}(u_{\lambda_1}).$$

This concludes the proof. $\square$

Remark 2. Theorem 2 ensures that f has the global growth given by (h1). Hence, for any parameter λ belonging to the real interval $\Lambda_{\bar{\gamma}} :=]0, \lambda^*[$, where

$$\lambda^* = \frac{\|\alpha\|_{L^1(\mathcal{M})}}{2\|K\|_\infty} \sup_{\gamma > 0} \left(\frac{\bar{\gamma}}{a_1 K_1 + a_2 K_2 \bar{\gamma}^{q-1}} \right),$$

problem (1.1) accepts at least one non-trivial solution $u_\lambda \in X$. Furthermore, $\|u_\lambda\|_{H_1^2} \rightarrow 0$ as $\lambda \rightarrow 0^+$. Hence, a straightforward computation shows that

$$\lambda^* := \begin{cases} +\infty, & \text{if } 1 < q < 2, \\ \frac{\|\alpha\|_{L^1(\mathcal{M})}}{2\|K\|_\infty K_1 a_1}, & \text{if } q = 2, \\ \frac{\|\alpha\|_{L^1(\mathcal{M})}}{2\|K\|_\infty} \left(\frac{\bar{\gamma}_{\max}}{a_1 K_1 + a_2 K_2 \bar{\gamma}_{\max}^{q-1}} \right), & \text{if } q \in]2, 2d/(d-2)[, \end{cases}$$

where

$$\bar{\gamma}_{\max} := \left(\frac{a_1 K_1}{(q-2)a_2 K_2} \right)^{1/(q-1)}.$$

Remark 3. The sharp Sobolev inequalities are important in the study of PDEs, especially in the study of those arising in geometry and physics. There has been much work on such inequalities and their applications. A concrete upper bound for the constants S_q in Theorem 2 is essential for a concrete evaluation of the interval Λ_γ .

In the case $(\mathcal{M}, g) = (\mathbb{S}^d, h)$, if $q \in [1, 2d/(d-2)[$, we have

$$S_q \leq \frac{\kappa_q}{\min\{1, \min_{\sigma \in \mathbb{S}^d} \alpha(\sigma)^{1/2}\}}, \quad (3.6)$$

where

$$\kappa_q := \begin{cases} \omega_d^{\frac{2-q}{2q}}, & \text{if } q \in [1, 2[, \\ \max \left\{ \left(\frac{q-2}{d \omega_d^{\frac{q-2}{q}}} \right)^{\frac{1}{2}}, \frac{1}{\omega_d^{\frac{q-2}{2q}}} \right\}, & \text{if } q \in [2, \frac{2d}{d-2}]. \end{cases}$$

Indeed, in Beckner [7], it is shown that for every $2 \leq q < 2d/(d-2)$ and every $w \in H_1^2(\mathbb{S}^d)$, we have

$$\left(\int_{\mathbb{S}^d} |w(\sigma)|^q d\sigma_h \right)^{2/q} \leq \frac{q-2}{d \omega_d^{1-2/q}} \int_{\mathbb{S}^d} |\nabla w(\sigma)|^2 d\sigma_h + \frac{1}{\omega_d^{1-2/q}} \int_{\mathbb{S}^d} |w(\sigma)|^2 d\sigma_h.$$

See also Theorem 4.28 in Hebey [10]. Then,

$$\|w\|_{L^q(\mathbb{S}^d)} \leq \max \left\{ \left(\frac{q-2}{d \omega_d^{\frac{q-2}{q}}} \right)^{\frac{1}{2}}, \frac{1}{\omega_d^{\frac{q-2}{2q}}} \right\} \left(\int_{\mathbb{S}^d} |\nabla w(\sigma)|^2 d\sigma_h + \int_{\mathbb{S}^d} |w(\sigma)|^2 d\sigma_h \right)^{1/2},$$

for every $w \in H_1^2(\mathbb{S}^d)$. And, if $q \in [1, 2[$, as a consequence of Hölder inequality, it follows that

$$\|w\|_{L^q(\mathbb{S}^d)} \leq \omega_d^{\frac{2-q}{2q}} \|w\|_{L^2(\mathbb{S}^d)}, \text{ for all } w \in L^2(\mathbb{S}^d).$$

The thesis is achieved by taking into account that

$$\|w\|_{L^2(\mathbb{S}^d)} \leq \|w\|_{H_1^2} \leq \frac{\|w\|_{H_\alpha^2}}{\min\{1, \min_{\sigma \in \mathbb{S}^d} \alpha(\sigma)^{1/2}\}}, \text{ for every } w \in H_1^2(\mathbb{S}^d).$$

4. APPLICATIONS TO NONLINEAR EIGENVALUE PROBLEMS AND EMDEN-FOWLER EQUATIONS

We set $\alpha, K \in C^\infty(S^d)$ be positive and let

$$K_1^* := \frac{\kappa_1 \|\alpha\|_{L^1(S^d)}}{\sqrt{2} \min\{1, \min_{\sigma \in S^d} \alpha(\sigma)^{1/2}\}}. \quad (4.1)$$

Further, for $q \in [1, \frac{2d}{d-2}]$ we will denote

$$K_2^* := \frac{\kappa_q^q \|\alpha\|_{L^1(S^d)}}{2^{\frac{2-q}{2}} q \min\{1, \min_{\sigma \in S^d} \alpha(\sigma)^{q/2}\}}. \quad (4.2)$$

By Theorem 2, and according to Remark 2, we find the existence of at least one non-trivial solution for nonlinear eigenvalue problems on the unit sphere S^d .

Corollary 1. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function such that (h_∞) and (h_1) hold. Moreover, there exists a positive number λ_1^* given by*

$$\lambda_1^* := \frac{\|\alpha\|_{L^1(\mathcal{M})}}{2\|K\|_\infty} \sup_{\gamma > 0} \left(\frac{\bar{\gamma}}{a_1 K_1^* + a_2 K_2^* \bar{\gamma}^{d-1}} \right)$$

such that, for every parameter λ belonging to $\Lambda_\gamma :=]0, \lambda_1^[$, the problem*

$$-\Delta_h w + \alpha(\sigma)w = \lambda K(\sigma)f(w), \quad \sigma \in S^d, w \in H_1^2(S^d) \quad (4.3)$$

possesses at least one non-trivial weak solution u_λ in $H_1^2(S^d)$. Further,

$$\lim_{\lambda \rightarrow 0^+} \|u_\lambda\| = 0.$$

Now, consider the following parametrized Emden-Fowler problem that arises in astrophysics, conformal Riemannian geometry, gas combustion, the theory of thermionic emission and isothermal stationary gas sphere :

$$-\Delta u = \lambda |x|^{s-2} K(x/|x|) f(|x|^{-s} u), x \in \mathbb{R}^{d+1} \setminus \{0\}. \quad (4.4)$$

Equation (4.4) has been studied when f has the form $f(t) = |t|^{p-1} t, p > 1$, see Cotsiolis and Iliopoulos [9], Vázquez and Véron [5].

Remark 4. The solutions of (4.4) are being expressed in the particular form

$$u(x) = r^s w(\sigma), \quad (4.5)$$

where

$$(r, \sigma) := (|x|, x/|x|) \in (0, \infty) \times S^d$$

are the spherical coordinates in $\mathbb{R}^{d+1} \setminus \{0\}$ and the smooth function w given on S^d .

Also, in [5] the authors used this type of transformation, where studied the asymptotic of a special form of (4.4). Throughout (4.5), with that in mind

$$\Delta u = r^{-d} \frac{\partial}{\partial r} (r^d \frac{\partial u}{\partial r}) + r^{-2} \Delta_h u,$$

(4.4) reduces to

$$-\Delta_h w + s(1-s-d)w = \lambda K(\sigma) f(w), \quad \sigma \in S^d, \quad w \in H_1^2(S^d),$$

(see also [13] Kristály and Rădulescu).

According to Remark 3, we give the following result.

Corollary 2. *Suppose that d and s are two constants such that $1-d < s < 0$. Moreover, let the positive function $K \in C^\infty(S)$ and $f: \mathbb{R} \rightarrow \mathbb{R}$ as in Corollary 1. Hence, for every parameter λ belonging to*

$$\Lambda_\gamma^{s,d} := \left[0, \frac{s(1-s-d)\omega_d}{2\|K\|_\infty(a_1 \frac{K_1^*}{\gamma} + a_2 K_2^* \gamma^{q-1})} \right]$$

the following problem

$$-\Delta u = \lambda |x|^{s-2} K(x/|x|) f(|x|^{-s} u), \quad x \in \mathbb{R}^{d+1} \setminus \{0\} \quad (4.6)$$

accepts at least one non-trivial weak solution.

Proof. Choose $(\mathcal{M}, g) = (S^d, h)$, and $\alpha(\sigma) := s(1-s-d)$ for any $\sigma \in S^d$ in Corollary 1. Clearly $\alpha \in C^\infty(S^d)$ and, by $1-d < s < 0$, we have $\alpha > 0$ on S^d . Hence, for any $\lambda \in \Lambda_\gamma^{s,d}$, the problem

$$-\Delta_h w + s(1-s-d)w = \lambda K(\sigma) f(w), \quad \sigma \in S^d, \quad w \in H_1^2(S^d)$$

has at least one non-trivial weak solution $w_\lambda \in H_1^2(S^d)$. According to (4.5), we have that $u_\lambda(x) = |x|^s w_\lambda(x/|x|)$ is a solution of (4.6). $\square$

Example 1. Let (S^d, h) with $d \geq 3$, and $K \in S^d$. Moreover, consider the following equation

$$-\Delta_h w + w = \lambda K(\sigma) \left(|w|^{r-2} w + |w|^{s-2} w \right) \quad (4.7)$$

for every $\sigma \in S^d$ and $w \in H_1^2(S^d)$, where $1 < r < 2$ and $2 < s < 2^*$. Hence, for any

$$\lambda \in \left[0, \frac{\omega_d^{1/2} \bar{\gamma}_{\max}}{4\|K\|_\infty (K_1 + K_2 \bar{\gamma}_{\max}^{s-1})} \right],$$

where

$$\bar{\gamma}_{\max} := \left(\frac{K_1}{K_2(s-2)} \right)^{1/(s-1)}$$

problem (4.7) accepts at least one non-trivial weak solution $w_\lambda \in H_1^2(S^d)$ such that $\|u_\lambda\|_{H_1^2} \rightarrow 0$ as $\lambda \rightarrow 0^+$. To show this, we can use Theorem 2 with

$$f(t) = \left(|t|^{r-2}t + |t|^{s-2}t \right)$$

for every $t \in \mathbb{R}$. Then $|f(t)| \leq 2 \left(1 + |t|^{s-1} \right)$ can be verified $\forall t \in \mathbb{R}$.

So, all the suppositions of Theorem 2 are verified and the conclusion follows.

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MEASURE THEORETIC GENERALIZATIONS OF JENSEN'S INEQUALITY BY FINK'S IDENTITY

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Received 17 January, 2021

Abstract. We generalize integral Jensen's inequality and its converse for real Stieltjes measure utilizing the theory of η -convex function by employing Fink's identity. We also give several versions of discrete Jensen's inequality along with its reverses and its converse for real weights. As an application we give generalized variants of Hermite-Hadamard's inequality. Also we give applications in information theory by giving new estimations of generalized divergence functional, Shannon and relative entropies. Finally we give connections to Zipf-Mandelbrot and hybrid Zipf-Mandelbrot entropies.

2010 Mathematics Subject Classification: 26D15; 26D10; 94A17

Keywords: Jensen's inequality, η -convex function, Fink's identity, divergence functional, entropy, hybrid Zipf-Mandelbrot entropy

1. INTRODUCTION

Jensen's inequality for differentiable convex functions plays a significant role in the field of inequalities as several other inequalities can be seen as special cases of it. It is used in order to make claims regarding the function while just a little is known or is needed to be known about the distribution. It is also used in defining the lower bound on the probability of a random variable. Some key applications covers derivation of AM-GM inequality, estimation of Shannon and non-Shannon entropies, convergence of maximization algorithm and non-negativity of divergence functionals. Taking into consideration the very numerous applications of Jensen's inequality in various fields of mathematics and other applied sciences, the generalizations and improvements of Jensen's inequality has been a topic of supreme interest for the researchers during the last few decades as evident from a large number of publications on the topic see [6, 14, 15, 21, 22] and the references therein.

The first author was supported by H.E.C. Pakistan under NRPU project 7906. The fourth author has been supported by the RUDN University Strategic Academic Leadership Program.

The well-known Jensen's inequality asserts that for the function Γ holds

$$\Gamma\left(\frac{1}{P_r} \sum_{j=1}^r p_j x_j\right) \leq \frac{1}{P_r} \sum_{j=1}^r p_j \Gamma(x_j), \quad (1.1)$$

if Γ is a convex function on interval $I \subset \mathbb{R}$, where p_j be positive real numbers and $x_j \in I$ ($j = 1, \dots, r$), while $P_r = \sum_{j=1}^r p_j$.

However the well known integral analogue of Jensen's inequality is:

Theorem 1. *Let $\tilde{h} : [v_1, v_2] \rightarrow [\varrho_1, \varrho_2]$ be a continuous function and $\lambda : [v_1, v_2] \rightarrow \mathbb{R}$ is an increasing and bounded function with $\lambda(v_1) \neq \lambda(v_2)$. Then for every continuous convex function $\Gamma : [\varrho_1, \varrho_2] \rightarrow \mathbb{R}$, the following inequality holds*

$$\Gamma(\tilde{h}) \leq \frac{\int_{v_1}^{v_2} \Gamma(\tilde{h}(\zeta)) d\lambda(\zeta)}{\int_{v_1}^{v_2} d\lambda(\zeta)}, \quad (1.2)$$

where

$$\tilde{h} = \frac{\int_{v_1}^{v_2} \tilde{h}(\zeta) d\lambda(\zeta)}{\int_{v_1}^{v_2} d\lambda(\zeta)} \in [\varrho_1, \varrho_2]. \quad (1.3)$$

There are several inequalities coming from Jensen's inequality both in integral and discrete cases which can be obtained by varying conditions on the function $\tilde{h}$ and measure λ defined in Theorem 1.

Jensen Steffensen's Conditions. If $\tilde{h}$ is continuous monotonic and λ is a continuous or a function of bounded variation such that

$$\lambda(v_1) \leq \lambda(x) \leq \lambda(v_2), \quad \forall x \in [v_1, v_2], \quad \lambda(v_2) > \lambda(v_1), \quad (1.4)$$

then (1.2) holds and is called Jensen Steffensen's inequality given by Boas [4]. Boas [4] also gave a generalization of the above inequality [19, p. 59].

Jensen Boas Conditions. Let λ be a continuous or a function of bounded variation such that

$$\lambda(v_1) \leq \lambda(x_1) \leq \lambda(y_1) \leq \lambda(x_2) \leq \lambda(y_2) \leq \dots \leq \lambda(y_{r-1}) \leq \lambda(x_r) \leq \lambda(v_2) \quad (1.5)$$

$\forall x_k \in (y_{k-1}, y_k)$ ($y_0 = v_1, y_r = v_2$) and $\lambda(v_2) > \lambda(v_1)$. If $\tilde{h}$ is continuous and monotonic on the r intervals (y_{k-1}, y_k) then again (1.2) holds and called Jensen-Boas inequality.

In 1992, Fink introduced a novel representation of real n -times differentiable function whose n -th derivative ($n \geq 1$) is absolutely continuous by connecting Taylor series and Peano kernel approach together in an identity given as:

Theorem 2 ([10]). *Let $\varrho_1, \varrho_2 \in \mathbb{R}$, $\Gamma : [\varrho_1, \varrho_2] \rightarrow \mathbb{R}$, $n \geq 1$ and $\Gamma^{(n-1)}$ is absolutely continuous on $[\varrho_1, \varrho_2]$. Then*

$$\begin{aligned} \Gamma(x) = & \frac{n}{\varrho_2 - \varrho_1} \int_{\varrho_1}^{\varrho_2} \Gamma(s) ds - \sum_{z=1}^{n-1} \left(\frac{n-z}{z!} \right) \left(\frac{\Gamma^{(z-1)}(\varrho_1)(x-\varrho_1)^z - \Gamma^{(z-1)}(\varrho_2)(x-\varrho_2)^z}{\varrho_2 - \varrho_1} \right) \\ & + \frac{1}{(n-1)!(\varrho_2 - \varrho_1)} \int_{\varrho_1}^{\varrho_2} (x-s)^{n-1} F_{\varrho_1}^{\varrho_2}(s, x) \Gamma^{(n)}(s) ds \end{aligned} \quad (1.6)$$

where

$$F_{\varrho_1}^{\varrho_2}(s, x) = \begin{cases} s - \varrho_1, & \varrho_1 \leq s \leq x \leq \varrho_2, \\ s - \varrho_2, & \varrho_1 \leq x < s \leq \varrho_2. \end{cases} \quad (1.7)$$

Fink's identity helps to give more insight of bounds on the deviation of a function from its averages [10]. One can get Ostrowski's inequality and formulate its new variants with improve and optimal quadrature formulas using Fink's identity [9]. It is used to generalize renowned Guessab-Schmeisser and Popoviciu's inequalities for n -times differentiable function, for instance see [1, 7].

2. GENERALIZATION OF INTEGRAL JENSEN'S INEQUALITY BY FINK'S IDENTITY

Before giving our main results, we consider the following assumptions that we use throughout our paper:

M_1 : $\hbar : [\nu_1, \nu_2] \rightarrow \mathbb{R}$ be continuous function such that $\hbar([\nu_1, \nu_2]) \subset [\varrho_1, \varrho_2]$.

M_2 : $\lambda : [\nu_1, \nu_2] \rightarrow \mathbb{R}$ be a continuous function or the functions of bounded variation such that $\lambda(\nu_1) \neq \lambda(\nu_2)$.

In our first main result we employ Fink's identity to obtain the following real Stieltjes measure theoretic representations of Jensen's inequality.

Theorem 3. *Let $\hbar, \lambda$ be as defined in M_1, M_2 and $\Gamma : [\varrho_1, \varrho_2] \rightarrow \mathbb{R}$ be such that $\Gamma^{(n-1)}$ is absolutely continuous for $n \geq 1$. Suppose, that Γ is n -convex such that*

$$\begin{aligned} (\widetilde{\hbar} - s)^{n-1} F_{\varrho_1}^{\varrho_2}(s, \widetilde{\hbar}) \leq & \frac{\int_{\nu_1}^{\nu_2} (\hbar(\zeta) - s)^{n-1} F_{\varrho_1}^{\varrho_2}(s, \hbar(\zeta)) d\lambda(\zeta)}{\int_a^b d\lambda(\zeta)}, \quad s \in [\varrho_1, \varrho_2] \end{aligned} \quad (2.1)$$

with $\widetilde{\hbar}$ and $F_{\varrho_1}^{\varrho_2}(s, \cdot)$ be as defined in (1.3) and (1.7), respectively. Then we have

$$\Gamma(\widetilde{\hbar}) - \frac{\int_{\nu_1}^{\nu_2} \Gamma(\hbar(\zeta)) d\lambda(\zeta)}{\int_{\nu_1}^{\nu_2} d\lambda(\zeta)} \leq \sum_{z=2}^{n-1} \left(\frac{n-z}{z!(\varrho_2 - \varrho_1)} \right) \quad (2.2)$$

$$\times \left\{ \Gamma^{(z-1)}(\varrho_2) \left((\tilde{h} - \varrho_2)^z - \frac{\int_{\nu_1}^{\nu_2} (\tilde{h}(\zeta) - \varrho_2)^z d\lambda(\zeta)}{\int_{\nu_1}^{\nu_2} d\lambda(\zeta)} \right) - \Gamma^{(z-1)}(\varrho_1) \left((\tilde{h} - \varrho_1)^z - \frac{\int_{\nu_1}^{\nu_2} (\tilde{h}(\zeta) - \varrho_1)^z d\lambda(\zeta)}{\int_{\nu_1}^{\nu_2} d\lambda(\zeta)} \right) \right\}.$$

Proof. As $\Gamma^{(n-1)}$ is absolutely continuous for $(n \geq 1)$, we can use the representation of Γ using Fink's identity (1.6) and can calculate

$$\begin{aligned} \Gamma(\tilde{h}) &= \frac{n}{\varrho_2 - \varrho_1} \int_{\varrho_1}^{\varrho_2} \Gamma(s) ds \\ &+ \sum_{z=1}^{n-1} \left(\frac{n-z}{z!(\varrho_2 - \varrho_1)} \right) \left\{ \Gamma^{(z-1)}(\varrho_2) \left((\tilde{h} - \varrho_2)^z \right) - \Gamma^{(z-1)}(\varrho_1) \left((\tilde{h} - \varrho_1)^z \right) \right\} \\ &+ \frac{1}{(n-1)!(\varrho_2 - \varrho_1)} \int_{\varrho_1}^{\varrho_2} \left((\tilde{h} - s)^{n-1} F_{\varrho_1}^{\varrho_2}(s, \tilde{h}) \right) \Gamma^{(n)}(s) ds. \end{aligned}$$

The integration of the composition of functions $\Gamma \circ \tilde{h}$ for the real measure λ on $[\nu_1, \nu_2]$ gives

$$\begin{aligned} \frac{\int_{\nu_1}^{\nu_2} \Gamma(\tilde{h}(\zeta)) d\lambda(\zeta)}{\int_{\nu_1}^{\nu_2} d\lambda(\zeta)} &= \frac{n}{\varrho_2 - \varrho_1} \int_{\varrho_1}^{\varrho_2} \Gamma(s) ds + \sum_{z=1}^{n-1} \left(\frac{n-z}{z!(\varrho_2 - \varrho_1)} \right) \\ &\times \left\{ \Gamma^{(z-1)}(\varrho_2) \left(\frac{\int_{\nu_1}^{\nu_2} (\tilde{h}(\zeta) - \varrho_2)^z d\lambda(\zeta)}{\int_{\nu_1}^{\nu_2} d\lambda(\zeta)} \right) - \Gamma^{(z-1)}(\varrho_1) \left(\frac{\int_{\nu_1}^{\nu_2} (\tilde{h}(\zeta) - \varrho_1)^z d\lambda(\zeta)}{\int_{\nu_1}^{\nu_2} d\lambda(\zeta)} \right) \right\} \\ &+ \frac{1}{(n-1)!(\varrho_2 - \varrho_1)} \int_{\varrho_1}^{\varrho_2} \left(\frac{\int_{\nu_1}^{\nu_2} (\tilde{h}(\zeta) - s)^{n-1} F_{\varrho_1}^{\varrho_2}(s, \tilde{h}(\zeta)) d\lambda(\zeta)}{\int_{\nu_1}^{\nu_2} d\lambda(\zeta)} \right) \Gamma^{(n)}(s) ds. \end{aligned}$$

Now computing the difference $\Gamma(\tilde{h}) - \frac{\int_{\nu_1}^{\nu_2} \Gamma(\tilde{h}(\zeta)) d\lambda(\zeta)}{\int_{\nu_1}^{\nu_2} d\lambda(\zeta)}$, we get the following generalized identity involving real Stieltjes measure:

$$\begin{aligned}
\Gamma(\tilde{h}) - \frac{\int_{v_1}^{v_2} \Gamma(\tilde{h}(\zeta)) d\lambda(\zeta)}{\int_{v_1}^{v_2} d\lambda(\zeta)} &= \sum_{z=2}^{n-1} \left(\frac{n-z}{z!(\varrho_2 - \varrho_1)} \right) \\
&\times \left\{ \Gamma^{(z-1)}(\varrho_2) \left((\tilde{h} - \varrho_2)^z - \frac{\int_{v_1}^{v_2} (\tilde{h}(\zeta) - \varrho_2)^z d\lambda(\zeta)}{\int_{v_1}^{v_2} d\lambda(\zeta)} \right) - \Gamma^{(z-1)}(\varrho_1) \left((\tilde{h} - \varrho_1)^z - \frac{\int_{v_1}^{v_2} (\tilde{h}(\zeta) - \varrho_1)^z d\lambda(\zeta)}{\int_{v_1}^{v_2} d\lambda(\zeta)} \right) \right\} \\
&+ \frac{1}{(n-1)!(\varrho_2 - \varrho_1)} \int_{\varrho_1}^{\varrho_2} \left((\tilde{h} - s)^{n-1} F_{\varrho_1}^{\varrho_2}(s, \tilde{h}) - \frac{\int_{v_1}^{v_2} (\tilde{h}(\zeta) - s)^{n-1} F_{\varrho_1}^{\varrho_2}(s, \tilde{h}(\zeta)) d\lambda(\zeta)}{\int_{v_1}^{v_2} d\lambda(\zeta)} \right) \Gamma^{(n)}(s) ds.
\end{aligned} \tag{GI.1}$$

Finally by our assumption $\Gamma^{(n-1)}$ is absolutely-continuous on $[\varrho_1, \varrho_2]$, as a result $\Gamma^{(n)}$ exists almost everywhere. Moreover, Γ is supposed to be n -convex, so we have $\Gamma^{(n)}(x) \geq 0$ almost everywhere on $[\varrho_1, \varrho_2]$. Therefore by taking into account the last term in generalized identity (GI.1) and inequality (2.1), we get (2.2). $\square$

In the later part of this section, we will employ convexity properties and Theorem 3 by alternating conditions on functions $\tilde{h}$ and Stieltjes measure $d\lambda$ to obtain generalized variants of Jensen–Steffensen’s, Jensen–Boas, Jensen–Brunk and Jensen type inequalities. We start with the following generalization of Jensen–Steffensen’s inequality for n -convex functions.

Theorem 4. *Let Γ be as defined in Theorem 3 is n -convex and $\tilde{h}$ be as defined in M_1 is monotonic. Then the following results hold.*

(i) *Let λ be as defined in M_2 satisfying*

$$\lambda(v_1) \leq \lambda(x) \leq \lambda(v_2), \quad \forall x \in [v_1, v_2], \quad \lambda(v_2) > \lambda(v_1).$$

Then for even $n \geq 3$, (2.1) is valid.

(ii) *Moreover if (2.1) is valid and the function*

$$H(x) := \sum_{z=2}^{n-1} \left(\frac{n-z}{z!\varrho_2 - \varrho_1} \right) \left(\Gamma^{(z-1)}(\varrho_1)(x - \varrho_1)^z - \Gamma^{(z-1)}(\varrho_2)(x - \varrho_2)^z \right) \tag{2.3}$$

is convex, then we get inequality (1.2) and it is called the generalized Jensen–Steffensen’s inequality for n -convex functions.

Proof. (i) We consider the following function in the remainder term of Fink's identity as:

$$\wp(x) := (x - \xi)^{n-1} F_{\varrho_1}^{\varrho_2}(\xi, x) = \begin{cases} (x - \xi)^{n-1}(\xi - \varrho_1), & \xi \leq x \leq \varrho_2, \\ (x - \xi)^{n-1}(\xi - \varrho_2), & x < \xi \leq \varrho_2. \end{cases}$$

Now taking derivative twice, we have

$$\wp''(x) := \begin{cases} (n-1)(n-2)(x - \xi)^{n-3}(\xi - \varrho_1), & \xi \leq x \leq \varrho_2, \\ (n-1)(n-2)(x - \xi)^{n-3}(\xi - \varrho_2), & x < \xi \leq \varrho_2. \end{cases}$$

By applying the second derivative test, the function $\wp$ is convex for even $n > 3$. Now using the assumed conditions one can employ Jensen–Steffensen's inequality given by Boas (see [4] or [19, p. 59]) for the convex function $\wp(x)$ to obtain (2.1).

(ii) We can rewrite the r.h.s. of (2.2) in the difference

$$H(\tilde{h}) - \frac{\int_{v_1}^{v_2} H(h(\zeta)) d\lambda(\zeta)}{\int_{v_1}^{v_2} d\lambda(\zeta)}.$$

For the convex function H and by our assumed conditions on functions h and λ , this difference is non-positive by using Jensen–Steffensen's inequality difference [4]. As a result, the r.h.s. of inequality (2.2) is non-positive and we get the generalized Jensen–Steffensen's inequality (1.2) for n -convex functions. $\square$

Now we give a similar result related to Jensen–Boas inequality [19, p. 59], that is a generalization of the Jensen–Steffensen's inequality:

Corollary 1. *Let Γ be as defined in Theorem 3 is n -convex function. Also let h be as defined in M_1 with $v_1 = y_0 < y_1 < \dots < y_k < \dots < y_{r-1} < y_r = v_2$ and h is monotonic in each of the r intervals $((y_{k-1}, y_k))$. Then the following results hold.*

(i) *Let λ be as defined in M_2 satisfying*

$$\lambda(v_1) \leq \lambda(x_1) \leq \lambda(y_1) \leq \lambda(x_2) \leq \lambda(y_2) \leq \dots \leq \lambda(y_{r-1}) \leq \lambda(x_r) \leq \lambda(v_2)$$

$\forall x_k \in (y_{k-1}, y_k)$ and $\lambda(v_2) > \lambda(v_1)$. Then for even $n \geq 3$, (2.1) is valid.

(ii) *Moreover if (2.1) is valid and the function $H(\cdot)$ defined in (2.3) is convex, then again inequality (1.2) holds and is called Jensen–Boas inequality for n -convex functions.*

Proof. We follow the similar idea as in the proof of Theorem 4 but under the conditions of this corollary, we utilize Jensen–Boas inequality (see [4] or [19, p. 59]) instead of Jensen–Steffensen's inequality. $\square$

Next we give results for Jensen–Brunk inequality.

Corollary 2. *Let Γ be as defined in Theorem 3 is $\mathfrak{n}$ -convex and $\hbar$ be as defined in M_1 is an increasing function. Then the following results hold.*

(i) *Let λ be as defined in M_2 with $\lambda(v_2) > \lambda(v_1)$,*

$$\int_{v_1}^x (\hbar(x) - \hbar(\zeta)) d\lambda(\zeta) \geq 0 \quad \text{and} \quad \int_x^{v_2} (\hbar(x) - \hbar(\zeta)) d\lambda(\zeta) \leq 0$$

$\forall x \in [v_1, v_2]$ hold. Then for even $\mathfrak{n} \geq 3$, (2.1) is valid.

(ii) *Moreover if (2.1) is valid and the function $H(\cdot)$ defined in (2.3) is convex, then again inequality (1.2) holds and is called Jensen–Brunk inequality for $\mathfrak{n}$ -convex functions.*

Proof. We follow the similar idea as in the proof of Theorem 4 but under the conditions of this corollary, we employ Jensen–Brunk inequality (see [5] or [19, p. 59]) instead of Jensen–Steffensen's inequality. $\square$

Remark 1. The similar result in Corollary 2 is also valid provided that the function $\hbar$ is decreasing. Also assuming that the function $\hbar$ is monotonic one can replace the conditions in Corollary 2(i) by

$$0 \leq \int_{v_1}^x |\hbar(x) - \hbar(\zeta)| d\lambda(\zeta) \leq \int_x^{v_2} |\hbar(x) - \hbar(\zeta)| d\lambda(\zeta).$$

Remark 2. It is interesting to see that by employing a similar method as in Theorem 4, we can also get the generalization of classical Jensen's inequality (1.2) for $\mathfrak{n}$ -convex functions by assuming the functions $\hbar$ and λ along with there respective conditions in Theorem 1.

Another important application of Theorem 3 is by setting the function $\hbar$ as $\hbar(\zeta) = \zeta$ gives the generalized version of the r. h. s. inequality of the Hermite–Hadamard inequality:

Corollary 3. *Let $\lambda : [v_1, v_2] \rightarrow \mathbb{R}$ be a function of bounded variation such that $\lambda(v_1) \neq \lambda(v_2)$ with $[v_1, v_2] \subset [\varrho_1, \varrho_2]$ and $\tilde{\zeta} = \frac{\int_{v_1}^{v_2} \zeta d\lambda(\zeta)}{\int_{v_1}^{v_2} d\lambda(\zeta)} \in [\varrho_1, \varrho_2]$. Under the assumptions of Theorem 3, if Γ is $\mathfrak{n}$ -convex such that*

$$(\tilde{\zeta} - s)^{\mathfrak{n}-1} F_{\varrho_1}^{\varrho_2}(s, \tilde{\zeta}) \leq \frac{\int_{v_1}^{v_2} (\zeta - s)^{\mathfrak{n}-1} F_{\varrho_1}^{\varrho_2}(s, \zeta) d\lambda(\zeta)}{\int_{v_1}^{v_2} d\lambda(\zeta)}, \quad s \in [\varrho_1, \varrho_2], \quad (2.4)$$

then we have

$$\begin{aligned} \Gamma(\tilde{\zeta}) &\leq \frac{\int_{\nu_1}^{\nu_2} \Gamma(\zeta) d\lambda(\zeta)}{\int_{\nu_1}^{\nu_2} d\lambda(\zeta)} + \sum_{z=2}^{n-1} \left(\frac{n-z}{z!(\varrho_2 - \varrho_1)} \right) \\ &\times \left\{ \Gamma^{(z-1)}(\varrho_2) \left((\tilde{\zeta} - \varrho_2)^z - \frac{\int_{\nu_1}^{\nu_2} (\zeta - \varrho_2)^z d\lambda(\zeta)}{\int_{\nu_1}^{\nu_2} d\lambda(\zeta)} \right) - \Gamma^{(z-1)}(\varrho_1) \left((\tilde{\zeta} - \varrho_1)^z - \frac{\int_{\nu_1}^{\nu_2} (t - \varrho_1)^z d\lambda(\zeta)}{\int_{\nu_1}^{\nu_2} d\lambda(\zeta)} \right) \right\}. \end{aligned} \quad (2.5)$$

If inequality (2.4) holds in the reverse direction, then (2.5) also holds reversely.

The special case of the above corollary can be given in the form of the following remark:

Remark 3. It is interesting to see that substituting $\lambda(\zeta) = \zeta$ gives $\int_{\nu_1}^{\nu_2} d\lambda(\zeta) = \nu_2 - \nu_1$ and $\tilde{\zeta} = \frac{\nu_1 + \nu_2}{2}$. Using these substitutions in (1.2) and by following Remark 2 we get the l.h.s. inequality of the renowned Hermite Hadamard inequality for n -convex functions.

3. GENERALIZATION OF DISCRETE JENSEN'S INEQUALITY BY FINK'S IDENTITY

In this section, we give generalizations for the discrete case by using Fink's identity. The proofs are similar to that of continuous case were given in the previous section, therefore we give results directly. In discrete case we have that $p_j > 0$ for all $j = 1, 2, \dots, r$. Here we give generalizations of results allowing p_j to be negative real numbers. Also with usual notations for $p_j x_j$ ($j = 1, 2, \dots, r$), we notate the r -tuples

$$\mathbf{x} = (x_1, x_2, \dots, x_r) \text{ and } \mathbf{p} = (p_1, p_2, \dots, p_r),$$

$$P_l = \sum_{j=1}^l p_j, \quad \bar{P}_l = P_r - P_{l-1} \quad (l = 1, 2, \dots, r)$$

and

$$\bar{x} = \frac{1}{P_r} \sum_{j=1}^r p_j x_j.$$

Using Fink's identity (1.6), we obtain the following representations of discrete Jensen's inequality.

Theorem 5. Let $\Gamma : [\varrho_1, \varrho_2] \rightarrow \mathbb{R}$ be such that for $n \geq 1$, $\Gamma^{(n-1)}$ is absolutely continuous. Also let $x_j \in [\nu_1, \nu_2] \subseteq [\varrho_1, \varrho_2]$, $p_j \in \mathbb{R}$ ($j = 1, \dots, r$), be such that $P_r \neq 0$ and $\bar{x} \in [\varrho_1, \varrho_2]$.

(i) Then the following generalized identity holds

$$\begin{aligned} \Gamma(\bar{x}) - \frac{1}{P_r} \sum_{j=1}^r p_j \Gamma(x_j) &= \sum_{z=2}^{n-1} \left(\frac{n-z}{z!(\varrho_2 - \varrho_1)} \right) \\ &\times \left\{ \Gamma^{(z-1)}(\varrho_2) \left((\bar{x} - \varrho_2)^z - \frac{1}{P_r} \sum_{j=1}^r p_j (x_j - \varrho_2)^z \right) - \Gamma^{(z-1)}(\varrho_1) \left((\bar{x} - \varrho_1)^z - \frac{1}{P_r} \sum_{j=1}^r p_j (x_j - \varrho_1)^z \right) \right\} \\ &+ \frac{1}{(n-1)!(\varrho_2 - \varrho_1)} \int_{\varrho_1}^{\varrho_2} \left[\left((\bar{x} - s)^{n-1} F_{\varrho_1}^{\varrho_2}(s, \bar{x}) - \frac{1}{P_r} \sum_{j=1}^r p_j \left((x_j - s)^{n-1} F_{\varrho_1}^{\varrho_2}(s, x_j) \right) \right) \right] \Gamma^{(n)}(s) ds, \end{aligned} \quad (\text{DGI.1})$$

where $F_{\varrho_1}^{\varrho_2}(s, \cdot)$ is defined in (1.7).

(ii) Moreover, if Γ is n -convex and the inequality

$$(\bar{x} - s)^{n-1} F_{\varrho_1}^{\varrho_2}(s, \bar{x}) \leq \frac{1}{P_r} \sum_{j=1}^r p_j \left((x_j - s)^{n-1} F_{\varrho_1}^{\varrho_2}(s, x_j) \right) \quad (3.1)$$

holds, then we have the following generalized inequality

$$\begin{aligned} \Gamma(\bar{x}) - \frac{1}{P_r} \sum_{j=1}^r p_j \Gamma(x_j) &\leq \sum_{z=2}^{n-1} \left(\frac{n-z}{z!(\varrho_2 - \varrho_1)} \right) \\ &\times \left\{ \Gamma^{(z-1)}(\varrho_2) \left((\bar{x} - \varrho_2)^z - \frac{1}{P_r} \sum_{j=1}^r p_j (x_j - \varrho_2)^z \right) - \Gamma^{(z-1)}(\varrho_1) \left((\bar{x} - \varrho_1)^z - \frac{1}{P_r} \sum_{j=1}^r p_j (x_j - \varrho_1)^z \right) \right\}. \end{aligned} \quad (3.2)$$

If inequality (3.1) holds in the reverse direction, then (3.2) also holds reversely.

Proof. Similar to that of Theorem 3. $\square$

In the later part of this section, we will vary our conditions on $p_j x_j$ ($j = 1, 2, \dots, r$) to obtain generalized discrete variants of Jensen–Steffensen's, Jensen's and Jensen–Petrović type inequalities. We start with the following generalization of discrete Jensen–Steffensen's inequality for n -convex functions:

Theorem 6. Let Γ be as defined in Theorem 5. Also let $\mathbf{x}$ be monotonic r -tuple, $x_j \in [\nu_1, \nu_2] \subseteq [\varrho_1, \varrho_2]$ and $\mathbf{p}$ be a real r -tuple such that

$$0 \leq P_l \leq P_r, \quad (l = 1, 2, \dots, r-1), \quad P_r > 0$$

is satisfied.

(i) If Γ is n -convex, then for even $n \geq 3$, (3.1) is valid.

(ii) Moreover if (3.1) is valid and the function $H(\cdot)$ defined in (2.3) is convex, then we get the following generalized discrete Jensen–Steffensen's inequality

$$\Gamma(\bar{x}) \leq \frac{1}{P_r} \sum_{j=1}^r p_j \Gamma(x_j). \quad (3.3)$$

Proof. It is interesting to see that under the assumed conditions on tuples $\mathbf{x}$ and $\mathbf{p}$, we have that $\bar{x} \in [\nu_1, \nu_2]$. Since for $x_1 \geq x_2 \geq \dots \geq x_r$,

$$P_r(x_1 - \bar{x}) = \sum_{j=2}^r p_j(x_1 - x_j) = \sum_{l=2}^r (x_{l-1} - x_l)(P_r - P_{l-1}) \geq 0.$$

This shows that $x_1 \geq \bar{x}$. Also $\bar{x} \geq x_r$, since we have

$$P_r(\bar{x} - x_r) = \sum_{j=1}^{r-1} p_j(x_j - x_r) = \sum_{l=1}^{r-1} (x_l - x_{l+1})P_l \geq 0.$$

For further details see the proof of discrete Jensen–Steffensen’s inequality [19, p. 57]. The idea of the rest of the proof is similar to that of Theorem 3, but here we employ Theorem 5 and discrete Jensen–Steffensen’s inequality. $\square$

Corollary 4. *Let Γ be as defined in Theorem 5 and let $x_j \in [\nu_1, \nu_2] \subseteq [\varrho_1, \varrho_2]$ with a positive r -tuple $\mathbf{p}$.*

- (i) *If Γ is n -convex, then for even $n \geq 3$, (3.1) is valid.*
- (ii) *Moreover if (3.1) is valid and the function $H(\cdot)$ defined in (2.3) is convex, then again we get (3.3) and is called Jensen’s inequality for n -convex functions.*

Proof. For $p_j > 0$, $x_j \in [\nu_1, \nu_2]$ ($j = 1, 2, 3, \dots, r$) ensures that $\bar{x} \in [\nu_1, \nu_2]$. So by applying classical discrete Jensen inequality (1.1) and the idea of Theorem 6 we will get the required results. $\square$

Remark 4. Under the assumptions of Corollary 4, if we choose $P_r = 1$, then Corollary 4(ii) gives the following inequality for n -convex functions:

$$\Gamma\left(\sum_{j=1}^r p_j x_j\right) \leq \sum_{j=1}^r p_j \Gamma(x_j). \quad (3.4)$$

Now we give the following reverses of Jensen–Steffensen’s and Jensen type inequalities:

Corollary 5. *Let Γ be as defined in Theorem 5. Also let $\mathbf{x}$ be a monotonic r -tuple, $x_j \in [\nu_1, \nu_2] \subseteq [\varrho_1, \varrho_2]$ and $\mathbf{p}$ be a real r -tuple such that there exist $m \in \{1, 2, \dots, r\}$ with*

$$0 \geq P_l, \text{ for } l < m \text{ and } 0 \geq \bar{P}_l, \text{ for } l > m,$$

where $P_r > 0$ and $\bar{x} \in [\varrho_1, \varrho_2]$.

- (i) *If Γ is n -convex, then for even $n \geq 3$, the reverse of inequality (3.1) holds.*
- (ii) *Moreover if (3.1) holds reversely and the function $H(\cdot)$ defined in (2.3) is convex, then we get the reverse of generalized Jensen–Steffensen inequality (3.3) for n -convex functions.*

Proof. We follow the idea of Theorem 6 but according to our assumed conditions we employ the reverse of Jensen–Steffensen inequality to obtain results. $\square$

In the next corollary we give explicit conditions on real tuple $\mathbf{p}$ such that we get the reverse of classical Jensen inequality:

Corollary 6. *Let Γ be as defined in Theorem 5 and let $x_j \in [v_1, v_2] \subseteq [\varrho_1, \varrho_2]$ such that $\bar{x} \in [\varrho_1, \varrho_2]$. Let $\mathbf{p}$ be a real r -tuple such that*

$$0 < p_1, 0 \geq p_2, p_3, \dots, p_r, 0 < P_r$$

is satisfied.

- (i) *If Γ is n -convex, then for even $n \geq 3$, the reverse of inequality (3.1) is valid.*
- (ii) *Also if reverse of (3.1) is valid and the function $H(\cdot)$ defined in (2.3) is convex, then we get the reverse of (3.3).*

Proof. We follow the idea of Theorem 6 but according to our assumed conditions we employ the reverse of Jensen inequality to obtain results. $\square$

In [2] (see also [19]) one can find the result which is equivalent to the Jensen–Steffensen and the reverse Jensen–Steffensen inequality together. It is the so-called Jensen–Petrović inequality. Here, without the proof, we give the adequate corollary which uses that result. The proof goes the same way as in the previous corollaries.

Corollary 7. *Let Γ be as defined in Theorem 5 and let $x_i \in [v_1, v_2] \subseteq [\varrho_1, \varrho_2]$ be such that $x_r \geq x_{r-1}, \dots, x_2 \geq x_1$. Let $\mathbf{p}$ be a real r -tuple with $P_r = 1$ such that*

$$0 \leq P_l, \text{ for } 1 \leq l < r-1 \text{ and } 0 \leq \bar{P}_l, \text{ for } 2 \leq l < r$$

is satisfied. Then we get the equivalent results given in Theorem 6 (i) and (ii) respectively.

Remark 5. Under the assumptions of Corollary 7, if there exist $m \in \{1, 2, \dots, r\}$ such that

$$0 \geq P_l, \text{ for } l < m \text{ and } 0 \geq \bar{P}_l, \text{ for } l > m$$

and $\bar{x} \in [\varrho_1, \varrho_2]$, then we get equivalent results for the reverse Jensen–Steffensen's inequality given in Corollary 5(i) and (ii) respectively.

Remark 6. It is interesting to see that the conditions on $p_j, j = 1, 2, \dots, r$ given in Corollary 7 and Remark 5 are coming from Jensen–Petrović inequality which becomes equivalent to conditions for $p_j, j = 1, 2, \dots, r$ for Jensen–Steffensen's results given in Theorem 6 and Corollary 5 respectively when $P_r = 1$.

Now we give results for Jensen and its reverses for r -tuples $\mathbf{x}$ and $\mathbf{p}$ when r is an odd number.

Corollary 8. *Let Γ be as defined in Theorem 5 and let $x_j \in [v_1, v_2] \subseteq [\varrho_1, \varrho_2]$ for $j = 1, 2, \dots, r$ be such that $\mathbf{x}, \mathbf{p}$ be real r -tuples, $r = 2m + 1, m \in \mathbb{N}$ and $\hat{x} = \frac{1}{\sum_{j=1}^{2k+1} p_j} \sum_{j=1}^{2k+1} p_j x_j \in [\varrho_1, \varrho_2]$ for all $k = 1, 2, \dots, m$. If for every $k = 1, 2, \dots, m$, we have*

$$(i^*) \quad p_1 > 0, \quad p_{2k} \leq 0, \quad p_{2k} + p_{2k+1} \leq 0, \quad \sum_{j=1}^{2k} p_j \geq 0, \quad \sum_{j=1}^{2k+1} p_j > 0$$

and

$$(ii^*) \quad x_{2k} \leq x_{2k+1}, \quad \sum_{j=1}^{2k} p_j (x_j - x_{2k+1}) \geq 0,$$

then we have the following statements to be valid:

(i) If Γ is n -convex, then for even $n \geq 3$,

$$(\hat{x} - s)^{n-1} F_{\varrho_1}^{\varrho_2}(s, \hat{x}) \geq \frac{1}{P_{2m+1}} \sum_{j=1}^{2m+1} p_j \left((x_j - s)^{n-1} F_{\varrho_1}^{\varrho_2}(s, x_j) \right). \quad (3.5)$$

(ii) Also if (3.5) is valid and the function $H(\cdot)$ defined in (2.3) is convex, then we get the following generalized inequality

$$\Gamma(\hat{x}) \geq \frac{1}{P_{2m+1}} \sum_{j=1}^{2m+1} p_j \Gamma(x_j). \quad (3.6)$$

Proof. We employ the idea of the proofs of Theorem 5 and Theorem 6 for $r = \text{odd}$ along with inequality of P. M. Vasić and J. R. Janić [18]. $\square$

Remark 7. We can also obtain inequalities (3.5) and (3.6) along with their reverses respectively given in Corollary 8 by employing the important cases and corresponding explicit conditions given in [18].

4. GENERALIZATION OF CONVERSE OF INTEGRAL JENSEN'S INEQUALITY BY FINK'S IDENTITY

In this section, we give results for the converse of the Jensen's inequality to hold, giving the conditions on the real Stieltjes measure $d\lambda$, such that $\lambda(v_1) \neq \lambda(v_2)$, allowing that the measure can also be negative, but employing Fink's identity.

To start we need the following assumption for the results of this section:

M_3 : Let $m, M \in [\varrho_1, \varrho_2]$ ($m \neq M$) be such that $m \leq h(\zeta) \leq M$ for all $\zeta \in [v_1, v_2]$ where h is defined in M_1 .

For a given function $\Gamma : [\varrho_1, \varrho_2] \rightarrow R$, we consider the difference

$$CJ(\Gamma, h_{\{m, M\}}; \lambda) = \frac{\int_{v_1}^{v_2} \Gamma(h(\zeta)) d\lambda(\zeta)}{\int_{v_1}^{v_2} d\lambda(\zeta)} - \frac{M - \tilde{h}}{M - m} \Gamma(m) - \frac{\tilde{h} - m}{M - m} \Gamma(M), \quad (4.1)$$

where $\tilde{h}$ is defined in (1.3).

Using Fink's identity, we obtain the following representation of the converse of Jensen's inequality.

Theorem 7. Let $\hbar, \lambda$ be as defined in M_1, M_2 and let $\Gamma : [\varrho_1, \varrho_2] \rightarrow \mathbb{R}$ be such that $\Gamma^{(n-1)}$ is absolutely continuous for $n \geq 1$. If Γ is n -convex such that

$$CJ\left((x-s)^{n-1} F_{\varrho_1}^{\varrho_2}(s, x), \hbar_{\{m, M\}}; \lambda\right) \leq 0, \quad s \in [\varrho_1, \varrho_2], \quad (4.2)$$

or

$$\begin{aligned} \frac{\int_{\nu_1}^{\nu_2} (\hbar(\zeta) - s)^{n-1} F_{\varrho_1}^{\varrho_2}(s, \hbar(\zeta)) d\lambda(\zeta)}{\int_{\nu_1}^{\nu_2} d\lambda(\zeta)} &\leq \frac{M - \tilde{\hbar}}{M - m} \left(F_{\varrho_1}^{\varrho_2}(s, m) (m - s)^{n-1} \right) \\ &\quad + \frac{\tilde{\hbar} - m}{M - m} \left(F_{\varrho_1}^{\varrho_2}(s, M) (M - s)^{n-1} \right), \quad s \in [\varrho_1, \varrho_2], \end{aligned} \quad (4.3)$$

then we get the following extension of the converse of the Jensen's difference

$$\begin{aligned} \frac{\int_{\nu_1}^{\nu_2} \Gamma(\hbar(\zeta)) d\lambda(\zeta)}{\int_{\nu_1}^{\nu_2} d\lambda(\zeta)} &\leq \frac{M - \tilde{\hbar}}{M - m} \Gamma(m) + \frac{\tilde{\hbar} - m}{M - m} \Gamma(M) + \sum_{z=2}^{n-1} \left(\frac{n-z}{z!(\varrho_2 - \varrho_1)} \right) \\ &\quad \times \left(\Gamma^{(z-1)}(\varrho_2) CJ((x - \varrho_2)^z, \hbar_{\{m, M\}}; \lambda) - \Gamma^{(z-1)}(\varrho_1) CJ((x - \varrho_1)^z, \hbar_{\{m, M\}}; \lambda) \right). \end{aligned} \quad (4.4)$$

where $F_{\varrho_1}^{\varrho_2}(s, \cdot)$ is defined in (1.7).

Proof. As $\Gamma^{(n-1)}$ is absolutely continuous for $n \geq 1$ we can use the representation of Γ using Fink's identity (1.6) in the difference $CJ(\Gamma, \hbar_{\{m, M\}}; \lambda)$:

$$\begin{aligned} CJ(\Gamma, \hbar_{\{m, M\}}; \lambda) &= CJ\left(\frac{n}{\varrho_2 - \varrho_1} \int_{\varrho_1}^{\varrho_2} \Gamma(\zeta) d\zeta, \hbar_{\{m, M\}}; \lambda\right) \\ &\quad + \sum_{z=1}^{n-1} \left(\frac{n-z}{z!(\varrho_2 - \varrho_1)} \right) \Gamma^{(z-1)}(\varrho_2) CJ((x - \varrho_2)^z, \hbar_{\{m, M\}}; \lambda) \\ &\quad - \sum_{z=1}^{n-1} \left(\frac{n-z}{z!(\varrho_2 - \varrho_1)} \right) \Gamma^{(z-1)}(\varrho_1) CJ((x - \varrho_1)^z, \hbar_{\{m, M\}}; \lambda) \\ &\quad + \frac{1}{(n-1)!(\varrho_2 - \varrho_1)} \int_{\varrho_1}^{\varrho_2} CJ\left((x-s)^{n-1} F_{\varrho_1}^{\varrho_2}(s, x), \hbar_{\{m, M\}}; \lambda\right) \Gamma^{(n)}(s) ds. \end{aligned} \quad (4.5)$$

After simplification and following the fact that $CJ(\Gamma, \hbar_{\{m, M\}}; \lambda)$ is zero for Γ to be constant or linear we get the following generalized identity

$$CJ(\Gamma, \hbar_{\{m, M\}}; \lambda) = \sum_{z=2}^{n-1} \left(\frac{n-z}{z!(\varrho_2 - \varrho_1)} \right) \quad (\text{CGI.1})$$

$$\begin{aligned} & \times \left(\Gamma^{(z-1)}(\varrho_2) C J((x - \varrho_2)^z, \hbar_{\{m, M\}}; \lambda) - \Gamma^{(z-1)}(\varrho_1) C J((x - \varrho_1)^z, \hbar_{\{m, M\}}; \lambda) \right) \\ & + \frac{1}{(n-1)!(\varrho_2 - \varrho_1)} \int_{\varrho_1}^{\varrho_2} C J((x - s)^{n-1} F_{\varrho_1}^{\varrho_2}(s, x), \hbar_{\{m, M\}}; \lambda) \Gamma^{(n)}(s) ds. \end{aligned}$$

Now using characterizations of n -convex functions like in the proof of Theorem 3, we get (4.4). $\square$

The next result gives the converse of Jensen's inequality for higher order convex functions.

Theorem 8. *Let Γ be as defined in Theorem 7 is n -convex and $\hbar$ be as defined in M_3 . Then the following results hold.*

- (i) *If λ is a non-negative measure on $[v_1, v_2]$, then for even $n \geq 3$, (4.3) is valid.*
- (ii) *Moreover if (4.3) is valid and the function $H(\cdot)$ defined in (2.3) is convex, then we get the following inequality for n -convex functions to be valid*

$$\frac{\int_{v_1}^{v_2} \Gamma(\hbar(\zeta)) d\lambda(\zeta)}{\int_{v_1}^{v_2} d\lambda(\zeta)} \leq \frac{M - \widetilde{\hbar}}{M - m} \Gamma(m) - \frac{\widetilde{\hbar} - m}{M - m} \Gamma(M). \quad (4.6)$$

Proof. The idea of the proof is similar to that of Theorem (3), but we use the converse of Jensen's inequality (see [3] or [19, p. 98]). $\square$

Another important consequence of Theorem 7 is by setting the function $\hbar$ as $\hbar(\zeta) = \zeta$ gives the generalized version of the l. h. s. inequality of the Hermite-Hadamard inequality:

Corollary 9. *Let $\lambda : [v_1, v_2] \rightarrow \mathbb{R}$ be a function of bounded variation such that $\lambda(v_1) \neq \lambda(v_2)$ with $[v_1, v_2] \subset [\varrho_1, \varrho_2]$ and $\widetilde{\zeta} = \frac{\int_{v_1}^{v_2} \zeta d\lambda(\zeta)}{\int_{v_1}^{v_2} d\lambda(\zeta)} \in [\varrho_1, \varrho_2]$. Under the assumptions of Theorem 7, if Γ is n -convex such that*

$$\begin{aligned} \frac{\int_{v_1}^{v_2} (\zeta - s)^{n-1} F_{\varrho_1}^{\varrho_2}(s, \zeta) d\lambda(\zeta)}{\int_{v_1}^{v_2} d\lambda(\zeta)} & \leq \frac{v_2 - \widetilde{\zeta}}{v_2 - v_1} \left(F_{\varrho_1}^{\varrho_2}(s, v_1) (v_1 - s)^{n-1} \right) \\ & + \frac{\widetilde{\zeta} - v_1}{v_2 - v_1} \left(F_{\varrho_1}^{\varrho_2}(s, v_2) (v_2 - s)^{n-1} \right), \quad s \in [\varrho_1, \varrho_2], \end{aligned} \quad (4.7)$$

then we have

$$\begin{aligned} \frac{\int_{\nu_1}^{\nu_2} \Gamma(\zeta) d\lambda(\zeta)}{\int_{\nu_1}^{\nu_2} d\lambda(\zeta)} &\leq \frac{\nu_2 - \tilde{\zeta}}{\nu_2 - \nu_1} \Gamma(\nu_1) + \frac{\tilde{\zeta} - \nu_1}{\nu_2 - \nu_1} \Gamma(\nu_2) + \sum_{z=2}^{n-1} \left(\frac{n-z}{z!(\varrho_2 - \varrho_1)} \right) \\ &\times \left(\Gamma^{(z-1)}(\varrho_2) C J((x - \varrho_2)^z, id_{[\nu_1, \nu_2]}; \lambda) - \Gamma^{(z-1)}(\varrho_1) C J((x - \varrho_1)^z, id_{[\nu_1, \nu_2]}; \lambda) \right). \end{aligned} \quad (4.8)$$

If the inequality (4.7) holds in the reverse direction, then (4.8) also holds reversely.

The special case of the above corollary can be given in the form of the following remark:

Remark 8. It is interesting to see that substituting $\lambda(\zeta) = \zeta$ and by following Theorem 8 we get the r.h.s. inequality of the renowned Hermite-Hadamard inequality for n -convex functions.

5. GENERALIZATION OF CONVERSE OF DISCRETE JENSEN'S INEQUALITY BY FINK'S IDENTITY

In this section, we give the results for the converse of Jensen's inequality in discrete case by using the Fink's identity.

Let $x_j \in [\nu_1, \nu_2] \subseteq [\varrho_1, \varrho_2]$, $\nu_1 \neq \nu_2$, $p_j \in \mathbb{R}$ ($j = 1, \dots, r$) be such that $P_r \neq 0$. Then we have the following difference of the converse of Jensen's inequality for $\Gamma : [\varrho_1, \varrho_2] \rightarrow \mathbb{R}$:

$$C J_{dis}(\Gamma) = \frac{1}{P_r} \sum_{j=1}^r p_j \Gamma(x_j) - \frac{\nu_2 - \bar{x}}{\nu_2 - \nu_1} \Gamma(\nu_1) - \frac{\bar{x} - \nu_1}{\nu_2 - \nu_1} \Gamma(\nu_2) \quad (5.1)$$

Similarly, we assume the Giaccardi difference given as:

$$G_{cardi}(\Gamma) = \sum_{j=1}^r p_j \Gamma(x_j) - A \Gamma \left(\sum_{j=1}^r p_j x_j \right) - B \left(\sum_{j=1}^r p_j - 1 \right) \Gamma(x_0), \quad (5.2)$$

where

$$A = \frac{\left(\sum_{j=1}^r p_j (x_j - x_0) \right)}{\left(\sum_{j=1}^r p_j x_j - x_0 \right)}, \quad B = \frac{\sum_{j=1}^r p_j x_j}{\left(\sum_{j=1}^r p_j x_j - x_0 \right)} \quad \text{and} \quad \sum_{j=1}^r p_j x_j \neq x_0.$$

Theorem 9. Let $\Gamma : [\varrho_1, \varrho_2] \rightarrow \mathbb{R}$ be such that for $n \geq 1$, $\Gamma^{(n-1)}$ is absolutely continuous. Also let $x_0, x_j \in [\nu_1, \nu_2] \subseteq [\varrho_1, \varrho_2]$, $p_j \in \mathbb{R}$ ($j = 1, \dots, r$), be such that $\sum_{j=1}^r p_j x_j \neq x_0$.

(i) Then the following generalized identity hold

$$CJ_{dis}(\Gamma) = \sum_{z=2}^{n-1} \left(\frac{n-z}{z!(\varrho_2 - \varrho_1)} \right) \left(\Gamma^{(z-1)}(\varrho_2) CJ_{dis}((x_j - \varrho_2)^z) - \Gamma^{(z-1)}(\varrho_1) CJ_{dis}((x_j - \varrho_1)^z) \right) \\ + \frac{1}{(n-1)!(\varrho_2 - \varrho_1)} \int_{\varrho_1}^{\varrho_2} CJ_{dis}((x_j - s)^{n-1} F_{\varrho_1}^{\varrho_2}(s, x_j)) \Gamma^{(n)}(s) ds, \quad (\text{DC.GI})$$

where $F_{\varrho_1}^{\varrho_2}(s, \cdot)$ is defined in (1.7).

(ii) Moreover, if Γ is n -convex and the inequality

$$CJ_{dis}((x_j - s)^{n-1} F_{\varrho_1}^{\varrho_2}(s, x_j)) \leq 0 \quad (5.3)$$

holds, then we have the following generalized inequality

$$CJ_{dis}(\Gamma) \leq \sum_{z=2}^{n-1} \left(\frac{n-z}{z!(\varrho_2 - \varrho_1)} \right) \left(\Gamma^{(z-1)}(\varrho_2) CJ_{dis}((x_j - \varrho_2)^z) - \Gamma^{(z-1)}(\varrho_1) CJ_{dis}((x_j - \varrho_1)^z) \right). \quad (5.4)$$

If inequality (5.3) holds in the reverse direction, then (5.4) also holds reversely.

Proof. Similar to that of Theorem 5. $\square$

Theorem 10. Let $\Gamma : [\varrho_1, \varrho_2] \rightarrow \mathbb{R}$ be such that for $n \geq 1$, $\Gamma^{(n-1)}$ is absolutely continuous. Also let $x_j \in [\nu_1, \nu_2] \subseteq [\varrho_1, \varrho_2]$, $p_j \in \mathbb{R}$ ($j = 1, \dots, r$), be such that $P_r \neq 0$ and $\bar{x} \in [\varrho_1, \varrho_2]$.

(i) Then the following generalized Giaccardi identity hold

$$G_{cardi}(\Gamma) = \sum_{z=2}^{n-1} \left(\frac{n-z}{z!(\varrho_2 - \varrho_1)} \right) \cdot \left(\Gamma^{(z-1)}(\varrho_2) G_{cardi}((x_j - \varrho_2)^z) - \Gamma^{(z-1)}(\varrho_1) G_{cardi}((x_j - \varrho_1)^z) \right) \\ + \frac{1}{(n-1)!(\varrho_2 - \varrho_1)} \int_{\varrho_1}^{\varrho_2} G_{cardi}((x_j - s)^{n-1} F_{\varrho_1}^{\varrho_2}(s, x_j)) \Gamma^{(n)}(s) ds \quad (\text{GIA.GI})$$

where $F_{\varrho_1}^{\varrho_2}(s, \cdot)$ is defined in (1.7).

(ii) Moreover, if Γ is n -convex and the inequality

$$G_{cardi}((x_j - s)^{n-1} F_{\varrho_1}^{\varrho_2}(s, x_j)) \leq 0 \quad (5.5)$$

holds, then we have the following generalized Giaccardi inequality

$$G_{cardi}(\Gamma) \leq \sum_{z=2}^{n-1} \left(\frac{n-z}{z!(\varrho_2 - \varrho_1)} \right) \left(\Gamma^{(z-1)}(\varrho_2) G_{cardi}((x_j - \varrho_2)^z) - \Gamma^{(z-1)}(\varrho_1) G_{cardi}((x_j - \varrho_1)^z) \right). \quad (5.6)$$

If inequality (5.5) holds in the reverse direction, then (5.6) also holds reversely.

Proof. Similar to that of Theorem 5. $\square$

In the later part of this section, we will vary our conditions on $p_j x_j$ ($j = 1, 2, \dots, r$) to obtain generalized converse discrete variants of Jensen's inequality and Giaccardi inequality for η -convex functions.

Theorem 11. *Let Γ be as defined in Theorem 9. Also let $x_j \in [v_1, v_2] \subseteq [\varrho_1, \varrho_2]$ and $\mathbf{p}$ be a positive r -tuple.*

- (i) *If Γ is η -convex, then for even $\eta \geq 3$, (5.3) is valid.*
- (ii) *Moreover if (5.3) is valid and the function $H(\cdot)$ defined in (2.3) is convex, then we get the following generalized converse of Jensen's inequality*

$$\frac{1}{P_r} \sum_{j=1}^r p_j \Gamma(x_j) \leq \frac{v_2 - \bar{x}}{v_2 - v_1} \Gamma(v_1) + \frac{\bar{x} - v_1}{v_2 - v_1} \Gamma(v_2). \quad (5.7)$$

Proof. We follow the idea of Theorem 6 but according to our assumed conditions we employ the converse of Jensen's inequality (see [3] or [19, p. 98]) to obtain results. $\square$

Finally in this section we give Giaccardi inequality for higher order convex functions.

Theorem 12. *Let Γ be as defined in Theorem 9. Also let $x_0, x_j \in [v_1, v_2] \subseteq [\varrho_1, \varrho_2]$ and $\mathbf{p}$ be a positive r -tuple such that*

$$\sum_{j=1}^r p_j x_j \neq x_0 \text{ and } (x_l - x_0) \left(\sum_{j=1}^r p_j x_j - x_l \right) \geq 0, \quad (l = 1, \dots, r).$$

- (i) *If Γ is η -convex, then for even $\eta \geq 3$, (5.5) is valid.*
- (ii) *Moreover if (5.5) is valid and the function $H(\cdot)$ defined in (2.3) is convex, then we get the following generalized Giaccardi inequality*

$$\sum_{j=1}^r p_j \Gamma(x_j) \leq A \Gamma \left(\sum_{j=1}^r p_j x_j \right) + B \left(\sum_{j=1}^r p_j - 1 \right) \Gamma(x_0), \quad (5.8)$$

where A and B are defined in (5.2).

Proof. We follow the idea of Theorem 6 but according to our assumed conditions we employ Giaccardi inequality (see [17, p. 11]) to obtain results. $\square$

6. APPLICATION IN INFORMATION THEORY FOR DISCRETE JENSEN'S INEQUALITY

Jensen's inequality plays a key role in information theory to construct lower bounds for some notable inequalities, but here we will use it to make connections between inequalities in information theory.

Consider a convex function $\Gamma : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, let $\mathbf{p} := (p_1, \dots, p_r)$ and $\mathbf{q} := (q_1, \dots, q_r)$ be positive probability distributions and Γ -divergence functional is defined in [8] as follows:

$$I_\Gamma(\mathbf{p}, \mathbf{q}) = \sum_{j=1}^r q_j \Gamma\left(\frac{p_j}{q_j}\right).$$

L. Horváth et al. in [11] defined the generalized Csiszár divergence functional in the following way:

Definition 1. Consider I is an interval in $\mathbb{R}$ and let $\Gamma : I \rightarrow \mathbb{R}$ be a function. Also let $\mathbf{p} := (p_1, \dots, p_r) \in \mathbb{R}^r$ and $\mathbf{q} := (q_1, \dots, q_r) \in (0, \infty)^r$ such that

$$\frac{p_j}{q_j} \in I, \quad j = 1, \dots, r.$$

Then let

$$\tilde{I}_\Gamma(\mathbf{p}, \mathbf{q}) = \sum_{j=1}^r q_j \Gamma\left(\frac{p_j}{q_j}\right). \quad (6.1)$$

We need the following representation for the results of this section:

$$F(\mathbf{p}, x_j, \Gamma) = \sum_{z=2}^{n-1} \left(\frac{n-z}{z!(\varrho_2 - \varrho_1)} \right) \left\{ \Gamma^{(z-1)}(\varrho_1) \left((\bar{x} - \varrho_1)^z - \frac{1}{P_r} \sum_{j=1}^r p_j (x_j - \varrho_1)^z \right) \right. \\ \left. - \Gamma^{(z-1)}(\varrho_2) \left((\bar{x} - \varrho_2)^z - \frac{1}{P_r} \sum_{j=1}^r p_j (x_j - \varrho_2)^z \right) \right\}. \quad (6.2)$$

Theorem 13. Under the assumptions of Theorem 5(ii), let (3.1) hold and Γ be n -convex. Also let $\mathbf{p} := (p_1, \dots, p_r) \in \mathbb{R}^r$ and $\mathbf{q} := (q_1, \dots, q_r) \in (0, \infty)^r$, then we have the following result:

$$\tilde{I}_\Gamma(\mathbf{p}, \mathbf{q}) \geq P_r \Gamma(1) + P_r F\left(\mathbf{q}, \frac{p_j}{q_j}, \Gamma\right). \quad (6.3)$$

Proof. From Theorem 5 we can rearrange (3.2) as

$$\frac{1}{P_r} \sum_{j=1}^r p_j \Gamma(x_j) \geq \Gamma(\bar{x}) - \sum_{z=2}^{n-1} \left(\frac{n-z}{z!(\varrho_2 - \varrho_1)} \right) \\ \times \left\{ \Gamma^{(z-1)}(\varrho_2) \left((\bar{x} - \varrho_2)^z - \frac{1}{P_r} \sum_{j=1}^r p_j (x_j - \varrho_2)^z \right) - \Gamma^{(z-1)}(\varrho_1) \left((\bar{x} - \varrho_1)^z - \frac{1}{P_r} \sum_{j=1}^r p_j (x_j - \varrho_1)^z \right) \right\}. \quad (6.4)$$

Now by replacing p_j with q_j and x_j with $\frac{p_j}{q_j}$, and following (6.2) we get (6.3). $\square$

For a positive r -tuple $\mathbf{q} = (q_1, \dots, q_r)$ such that $\sum_{j=1}^r q_j = 1$, the **Shannon entropy** [23] is defined by

$$S(\mathbf{q}) = - \sum_{j=1}^r q_j \ln q_j. \quad (6.5)$$

Corollary 10. *Under the assumptions of Theorem 13, let n be even. Then we have the following results:*

(i) *If $\mathbf{q} := (q_1, \dots, q_r) \in (0, \infty)^r$, then*

$$\sum_{j=1}^r q_j \ln q_j \geq rF\left(\mathbf{q}, \frac{1}{q_j}, -\ln(\cdot)\right). \quad (6.6)$$

(ii) *We can get bounds for the Shannon entropy of $\mathbf{q}$, if we choose $\mathbf{q} := (q_1, \dots, q_r)$ a positive probability distribution:*

$$S(\mathbf{q}) \leq -rF\left(\mathbf{q}, \frac{1}{q_j}, -\ln(\cdot)\right). \quad (6.7)$$

Proof. (i) For even n , $\Gamma(x) := -\ln x$ is n -convex and using $\mathbf{p} := (1, 1, \dots, 1)$ in Theorem 13 we get (6.6) by following (6.2).

(ii) Since we have $\sum_{j=1}^r q_j = 1$, therefore by multiplying -1 on both side to (6.6) and taking into account expression (6.5), we get (6.7). $\square$

The **Kullback-Leibler** distance [16] between the positive probability distributions $\mathbf{p} = (p_1, \dots, p_r)$ and $\mathbf{q} = (q_1, \dots, q_r)$ is defined by

$$D(\mathbf{q} \parallel \mathbf{p}) = \sum_{j=1}^r q_j \ln\left(\frac{q_j}{p_j}\right). \quad (6.8)$$

Corollary 11. *Under the assumption of Corollary 10, we have the following results:*

(i) *If $\mathbf{q} := (q_1, \dots, q_r), \mathbf{p} := (p_1, \dots, p_r) \in (0, \infty)^r$, then*

$$\sum_{j=1}^r q_j \ln\left(\frac{q_j}{p_j}\right) \geq rF\left(\mathbf{q}, \frac{p_j}{q_j}, -\ln(\cdot)\right). \quad (6.9)$$

(ii) *Now if we take positive probability distributions $\mathbf{q} := (q_1, \dots, q_r)$ and $\mathbf{p} := (p_1, \dots, p_r)$, then we have*

$$D(\mathbf{q} \parallel \mathbf{p}) \geq rF\left(\mathbf{q}, \frac{p_j}{q_j}, -\ln(\cdot)\right). \quad (6.10)$$

Proof. (i) Using $\Gamma(x) := -\ln x$ (which is n -convex for even n) in Theorem 13 and following (6.2), after simplification we get (6.9).

(ii) It is a special case of (i). $\square$

7. RESULTS FOR ZIPF AND HYBRID ZIPF-MANDELBROT ENTROPY

One of the basic laws in information science is Zipf's law [20, 24] and is highly applied in linguistics. Let $c \geq 0$, $d > 0$ and $N \in \{1, 2, 3, \dots\}$. The **Zipf-Mandelbrot**

entropy can be given as

$$Z_M(H, c, d) = \frac{d}{H_{c,d}^N} \sum_{j=1}^N \frac{\ln(j+c)}{(j+c)^d} + \ln(H_{c,d}^N), \quad (7.1)$$

where

$$H_{c,d}^N = \sum_{\sigma=1}^N \frac{1}{(\sigma+c)^d}.$$

Consider

$$q_j = \Gamma(j; N, c, d) = \frac{1}{((j+c)^d H_{c,d}^N)} \quad (7.2)$$

where $\Gamma(j; N, c, d)$ is a discrete probability distribution known as **Zipf-Mandelbrot** law. The Zipf-Mandelbrot law has many applications in linguistics and information science. Some of the recent studies about Zipf-Mandelbrot law can be seen in the listed references (see [11, 13]). Now we state our results involving entropy introduced by Mandelbrot law by establishing the relationship with Shannon and relative entropies.

Theorem 14. *Let $\mathbf{q}$ be a Zipf-Mandelbrot law as defined in (7.2) with parameters $c \geq 0$, $d > 0$ and $N \in \{1, 2, \dots\}$. Then we have*

$$Z_M(H, c, d) = S(\mathbf{q}) \leq -N \times F\left(\mathbf{q}, ((j+c)^d H_{c,d}^N), -\ln(\cdot)\right). \quad (7.3)$$

Proof. It is interesting to see that for q_j , defined as in (7.2), $\sum_{j=1}^N q_j = 1$. Therefore, using the above q_j in Shannon entropy (6.5), we get Mandelbrot entropy (7.1):

$$\begin{aligned} S(\mathbf{q}) &= -\sum_{j=1}^N q_j \ln q_j = -\sum_{j=1}^N \frac{1}{((j+c)^d H_{c,d}^N)} \ln \frac{1}{((j+c)^d H_{c,d}^N)} \\ &= \frac{d}{(H_{c,d}^N)} \sum_{j=1}^N \frac{\ln(j+c)}{(j+c)^d} + \ln(H_{c,d}^N). \end{aligned} \quad (7.4)$$

Finally, substituting $q_j = \frac{1}{((j+c)^d H_{c,d}^N)}$ in Corollary 10(ii) and following (6.2) we get the desired result. $\square$

Corollary 12. *Let $\mathbf{q}$ and $\mathbf{p}$ be Zipf-Mandelbrot laws with parameters $c_1, c_2 \in [0, \infty)$, $d_1, d_2 > 0$, let $H_{c_1, d_1}^N = \sum_{\sigma=1}^N \frac{1}{(\sigma+c_1)^{d_1}}$ and $H_{c_2, d_2}^N = \sum_{\sigma=1}^N \frac{1}{(\sigma+c_2)^{d_2}}$. Now using*

$$q_j = \frac{1}{(j+c_1)^{d_1} H_{c_1, d_1}^N} \text{ and } p_j = \frac{1}{(j+c_2)^{d_2} H_{c_2, d_2}^N}$$

in Corollary 11(ii) and following (6.2), we have

$$\begin{aligned}
 D(\mathbf{q} \parallel \mathbf{p}) &= \sum_{j=1}^N \frac{1}{(J+c_1)^{d_1} H_{c_1, d_1}^N} \ln \left(\frac{(J+c_2)^{d_2} H_{c_2, d_2}^N}{(J+c_1)^{d_1} H_{c_1, d_1}^N} \right) \\
 &= -Z_M(H, c_1, d_1) + \frac{d_2}{H_{c_1, d_1}^N} \sum_{j=1}^N \frac{\ln(J+c_2)}{(J+c_1)^{d_1}} + \ln(H_{c_2, d_2}^N) \\
 &\geq N \times F \left(\mathbf{q}, \frac{(J+c_1)^{d_1} H_{c_1, d_1}^N}{(J+c_2)^{d_2} H_{c_2, d_2}^N}, -\ln(\cdot) \right).
 \end{aligned} \tag{7.5}$$

The final results are about **hybrid Zipf-Mandelbrot entropy**, which is a further generalization of Zipf-Mandelbrot entropy. Let $N \in \{1, 2, 3, \dots\}$, $c \geq 0$, $d, \omega > 0$. Then hybrid Zipf-Mandelbrot entropy can be given as follows:

$$\widehat{Z}_M(H^*, c, d, \omega) = \frac{1}{H_{c, d, \omega}^*} \sum_{j=1}^N \frac{\omega^j}{(J+c)^d} \ln \left(\frac{(J+c)^d}{\omega^j} \right) + \ln(H_{c, d, \omega}^*), \tag{7.6}$$

where

$$H_{c, d, \omega}^* = \sum_{j=1}^N \frac{\omega^j}{(J+c)^d} \tag{7.7}$$

Consider

$$q_j = \Gamma(J; N, c, d, \omega) = \frac{\omega^j}{(J+c)^d H_{c, d, \omega}^*}, \tag{7.8}$$

which is called **hybrid Zipf-Mandelbrot law**. There is a unified approach, maximization of Shannon entropy [25], that naturally follows the path of generalization from Zipf's to hybrid Zipf's law. Extending this idea Jakšetić et al. in [12] presented a transition from Zipf-Mandelbrot to hybrid Zipf-Mandelbrot law by employing maximum entropy technique with one additional constraint. It is interesting that the examination of its densities provides some new insights of Lerch's transcendent.

Theorem 15. Let $\mathbf{q}$ be a **hybrid Zipf-Mandelbrot law** as defined in (7.8) with parameters $c \geq 0$, $d, \omega > 0$ and $N \in \{1, 2, \dots\}$, then we have

$$\widehat{Z}_M(H^*, c, d, \omega) = S(\mathbf{q}) \leq -N \times F \left(\mathbf{q}, \frac{(J+c)^d H_{c, d, \omega}^*}{\omega^j}, -\ln(\cdot) \right). \tag{7.9}$$

Proof. It is interesting to see that for q_j , defined as in (7.8), $\sum_{j=1}^N q_j = 1$. Therefore, using the above q_j in Shannon entropy (6.5), we get hybrid Zipf-Mandelbrot entropy (7.6):

$$S(\mathbf{q}) = - \sum_{j=1}^N q_j \ln q_j = - \sum_{j=1}^N \frac{\omega^j}{(J+c)^d H_{c, d, \omega}^*} \ln \frac{\omega^j}{(J+c)^d H_{c, d, \omega}^*}$$

$$\begin{aligned}
&= \frac{-1}{H_{c,d,\omega}^*} \sum_{j=1}^N \frac{\omega^j}{(j+c)^d} \left[\ln \left(\frac{\omega^j}{(j+c)^d} \right) + \ln \left(\frac{1}{H_{c,d,\omega}^*} \right) \right] \\
&= \frac{1}{H_{c,d,\omega}^*} \sum_{j=1}^N \frac{\omega^j}{(j+c)^d} \left[\ln \left(\frac{(j+c)^d}{\omega^j} \right) + \ln(H_{c,d,\omega}^*) \right] \\
&= \frac{1}{H_{c,d,\omega}^*} \sum_{j=1}^N \frac{\omega^j}{(j+c)^d} \ln \left(\frac{(j+c)^d}{\omega^j} \right) + \ln(H_{c,d,\omega}^*). \quad (7.10)
\end{aligned}$$

Finally, substituting this $q_j = \frac{\omega^j}{(j+c)^d H_{c,d,\omega}^*}$ in Corollary 10(ii) and following (6.2) we get the desired result. $\square$

Corollary 13. *Let $\mathbf{q}$ and $\mathbf{p}$ be hybrid Zipf-Mandelbrot laws with parameters $c_1, c_2 \in [0, \infty)$, $\omega_1, \omega_2, d_1, d_2 > 0$. Now using*

$$q_j = \frac{\omega_1^j}{(j+c_1)^{d_1} H_{c_1,d_1,\omega_1}^*} \text{ and } p_j = \frac{\omega_2^j}{(j+c_2)^{d_2} H_{c_2,d_2,\omega_2}^*}$$

in Corollary 11(ii) and following (6.2), then we have

$$\begin{aligned}
D(\mathbf{q} \parallel \mathbf{p}) &= \sum_{j=1}^N \frac{\omega_1^j}{(j+c_1)^{d_1} H_{c_1,d_1,\omega_1}^*} \ln \left(\frac{\omega_1^j (j+c_2)^{d_2} H_{c_2,d_2,\omega_2}^*}{\omega_2^j (j+c_1)^{d_1} H_{c_1,d_1,\omega_1}^*} \right) \\
&= -\widehat{Z}_M(H^*, c_1, d_1, \omega_1) + \frac{1}{H_{c_1,d_1,\omega_1}^*} \sum_{j=1}^N \frac{\omega_1^j}{(j+c_1)^{d_1}} \ln \left(\frac{(j+c_2)^{d_2}}{\omega_2^j} \right) + \ln(H_{c_2,d_2,\omega_2}^*). \\
&\geq N \times F \left(\mathbf{q}, \frac{\omega_2^j (j_1+c_1)^{d_1} H_{c_1,d_1,\omega_1}^*}{\omega_1^j (j_2+c_2)^{d_2} H_{c_2,d_2,\omega_2}^*}, -\ln(\cdot) \right). \quad (7.11)
\end{aligned}$$

Remark 9. Similarly, we can give results for Shannon entropy, Kullback-Leibler distance, Zipf-Mandelbrot entropy and hybrid Zipf-Mandelbrot entropy by using results for converse discrete Jensen's inequality and therefore also for generalized Giaccardi inequality defined in (5.6) on the same approach.

8. CONCLUDING REMARKS

Jensen's inequality is considered to be much fruitful for the characterization of convex functions. First we gave real Stieltjes measure theoretic representations of integral Jensen's inequality by using Fink's identity. Then we formulate results for other inequalities like Jensen Steffensen's inequality, Jensen-Boas inequality and Jensen-Brunk inequality. We can obtain Jensen Steffensen's inequality, Jensen-Boas inequality and Jensen-Brunk inequality by changing the assumptions in Jensen's inequality. We also gave generalized Jensen's inequality in discrete case having real weights. As a result we give its reverses and converses by studying conditions on

tuples. At the end we gave applications in information theory for our obtained results, specially we gave results for hybrid Zipf-Mandelbrot entropy.

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KLEENE POSETS AND PSEUDO-KLEENE POSETS

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Received 07 October, 2020

Abstract. The concept of a Kleene algebra (sometimes also called Kleene lattice) was already generalized by the first author for non-distributive lattices under the name pseudo-Kleene algebra. We extend these concepts to posets and show how (pseudo-)Kleene posets can be characterized by identities and implications of assigned commutative meet-directoids. Moreover, we prove that the Dedekind-MacNeille completion of a pseudo-Kleene poset is a pseudo-Kleene algebra and that the Dedekind-MacNeille completion of a finite Kleene poset is a Kleene algebra. Further, we introduce the concept of a strict (pseudo-)Kleene poset and show that under an additional assumption a strict Kleene poset can be organized into a residuated structure. Finally, we prove by using the so-called twist-product construction that every poset can be embedded into a pseudo-Kleene poset in some natural way.

2010 *Mathematics Subject Classification:* 06A11; 06D30; 03G25

Keywords: (pseudo-)Kleene algebra, (pseudo-)Kleene poset, strong pseudo-Kleene poset, strict (pseudo-)Kleene poset, commutative meet-directoid, Dedekind-MacNeille completion, twist-product construction

1. INTRODUCTION

Kleene algebras, or in another terminology Kleene lattices (see [7] and [11]), are special cases of De Morgan algebras, i.e. distributive lattices with an antitone involution satisfying the so-called *normality condition*, i.e. the identity $x \wedge x' \leq y \vee y'$. These algebras were also called *normal i -lattices* (see [11]) or quasi-Boolean algebras (by A. Bialynicki and H. Rasiowa). They are important models in the field of logic since they generalize Boolean algebras, Łukasiewicz algebras and Post algebras. The name “Kleene algebra” was introduced by R. Cignoli ([7]). The case when the underlying lattice need not be distributive was treated by the first author in [4] under the name pseudo-Kleene algebras.

Support of the research of the authors by the Austrian Science Fund (FWF), project I 4579-N, and the Czech Science Foundation (GAČR), project 20-09869L, entitled “The many facets of orthomodularity”, as well as by ÖAD, project CZ 02/2019, entitled “Function algebras and ordered structures related to logic and data fusion”, and, concerning the first author, by IGA, project PpF 2021 030, is gratefully acknowledged.

In some propositional logics the identification of disjunction with lattice join $\vee$ turns out to be problematic. For example, the logic of quantum mechanics was originally modeled by orthomodular lattices (which are special pseudo-Kleene algebras) and later on by orthomodular posets in which the existence of the join $x \vee y$ of two elements x and y is guaranteed only in the case when these elements are orthogonal to each other, i.e., $x \leq y'$ (or, equivalently, $y \leq x'$). Similar problems may occur also in other models of non-classical logics. Hence the question arises whether results obtained for Kleene algebras or pseudo-Kleene algebras can be generalized to posets with an antitone involution satisfying some condition analogous to normality. In this paper we solve this problem by investigating so-called Kleene posets, pseudo-Kleene posets and strong pseudo-Kleene posets. We believe that these may be successfully applied in the algebraic axiomatization of several non-classical logics. Moreover, we introduce some kind of residuation which may be applied in fuzzy logic.

For the reader's convenience, we recall several concepts concerning posets.

Let $\mathbf{P} = (P, \leq)$ be a poset, $a, b \in P$ and $A, B \subseteq P$. We write $a \parallel b$ if a and b are incomparable. We extend $\leq$ to subsets by defining

$$A \leq B \text{ if and only if } x \leq y \text{ for all } x \in A \text{ and } y \in B.$$

Instead of $\{a\} \leq B$ and $A \leq \{b\}$ we also write $a \leq B$ and $A \leq b$, respectively. Analogous notations are used for the reverse order $\geq$. Moreover, we define

$$\begin{aligned} L(A) &:= \{x \in P \mid x \leq A\}, \\ U(A) &:= \{x \in P \mid A \leq x\}. \end{aligned}$$

Instead of $L(A \cup B)$, $L(\{a\} \cup B)$, $L(A \cup \{b\})$ and $L(\{a, b\})$ we also write $L(A, B)$, $L(a, B)$, $L(A, b)$ and $L(a, b)$, respectively. Analogous notations are used for U . Instead of $L(U(A))$ we also write $LU(A)$. Analogously, we proceed in similar cases. Sometimes we identify singletons with their unique element, so we often write $L(a, b) = 0$ and $U(a, b) = 1$ instead of $L(a, b) = \{0\}$ and $U(a, b) = \{1\}$, respectively. We have

$$\begin{aligned} A \subseteq B &\text{ imply } L(B) \subseteq L(A), \\ A &\subseteq LU(A), \\ LUL(A) &= L(A), \\ LU(a) &= L(a), \\ L(A, B) &= L(A) \cap L(B), \\ L(U(a), B) &= L(a, B), \\ L(A) &= \bigcup_{x \in L(A)} L(x), \\ L(A) &= \bigcap_{x \in A} L(x). \end{aligned}$$

There also hold the corresponding statements obtained by duality. The *poset* $\mathbf{P}$ is called *downward directed* if $L(x, y) \neq \emptyset$ for all $x, y \in P$. The *poset* $\mathbf{P}$ is called *bounded* if it has a least element 0 and a greatest element 1. This fact will be expressed by notation $(P, \leq, 0, 1)$. The *poset* $\mathbf{P}$ is called *distributive* if it satisfies one of the following equivalent identities:

$$\begin{aligned} L(U(x, y), z) &\approx LU(L(x, z), L(y, z)), \\ UL(U(x, y), z) &\approx U(L(x, z), L(y, z)), \\ U(L(x, y), z) &\approx UL(U(x, z), U(y, z)), \\ LU(L(x, y), z) &\approx L(U(x, z), U(y, z)). \end{aligned}$$

In fact, the inclusions

$$\begin{aligned} LU(L(x, z), L(y, z)) &\subseteq L(U(x, y), z), \\ UL(U(x, z), U(y, z)) &\subseteq U(L(x, y), z) \end{aligned}$$

hold in every poset. Hence, to check distributivity, we need only to confirm one of the converse inclusions. A unary operation $'$ on P is called

- *antitone* if, for all $x, y \in P$, $x \leq y$ implies $y' \leq x'$,
- an *involution* if it satisfies the identity $x'' \approx x$.

For $A \subseteq P$ we define $A' := \{x' \mid x \in A\}$. If the poset is bounded and distributive, we can prove the following property of an antitone involution.

Lemma 1. *Let $\mathbf{P} = (P, \leq, ', 0, 1)$ be a bounded distributive poset with an antitone involution $'$ and $a, b \in P$ with $a \leq b$ and $L(b, a') = \{0\}$. Then the following hold:*

$$\begin{aligned} L(a, a') &= L(b, b') = \{0\}, \\ U(a, a') &= U(b, b') = \{1\}. \end{aligned}$$

Proof. Using distributivity of $\mathbf{P}$ we obtain

$$\begin{aligned} L(a, a') &= LUL(a, a') = LU(L(a, a'), 0) = LU(L(a, a'), L(b, a')) = L(U(a, b), a') \\ &= L(U(b), a') = L(b, a') = \{0\}, \\ L(b, b') &= LUL(b', b) = LU(0, L(b', b)) = LU(L(a', b), L(b', b)) = L(U(a', b'), b) \\ &= L(U(a'), b) = L(a', b) = \{0\}. \end{aligned}$$

By De Morgan's laws we finally have

$$\begin{aligned} U(a, a') &= (L(a', a))' = \{0\}' = \{1\}, \\ U(b, b') &= (L(b', b))' = \{0\}' = \{1\}. \end{aligned}$$

□

2. KLEENE POSETS AND PSEUDO-KLEENE POSETS

The concept of a Kleene algebra is taken from [7] and that of a pseudo-Kleene algebra from [4]. One can easily check that the normality condition mentioned in the introduction can be translated into the language of posets as follows.

Definition 1. A *pseudo-Kleene poset* is a poset $\mathbf{P} = (P, \leq, ')$ with an antitone involution satisfying

$$(K) \quad L(x, x') \leq U(y, y') \text{ for all } x, y \in P.$$

An element a of P is called a *fixed point* of $\mathbf{P}$ if $a' = a$. By a *Kleene poset* we mean a distributive pseudo-Kleene poset.

Lemma 2. Let $\mathbf{P} = (P, \leq, ')$ be a pseudo-Kleene poset. Then $\mathbf{P}$ has at most one fixed point.

Proof. If a and b are fixed points of $\mathbf{P}$ then

$$\begin{aligned} a \in L(a) = L(a, a) = L(a, a') \leq U(b, b') = U(b, b) = U(b), \\ b \in L(b) = L(b, b) = L(b, b') \leq U(a, a') = U(a, a) = U(a) \end{aligned}$$

and hence $b \leq a$ and $a \leq b$, i.e. $a = b$. $\square$

Example 1. The poset visualized in Figure 1 is a Kleene poset which is not a lattice:

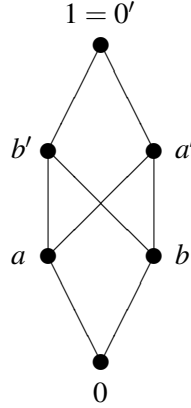


Fig. 1

Example 2. The poset $(P, \leq, ')$ visualized in Figure 2 is a pseudo-Kleene poset which is neither a lattice nor a Kleene poset since

$$\begin{aligned} L(U(a, c), b) &= L(1, b) = L(b) \neq \{0\} = L(P) = LU(0) = LU(0, 0) \\ &= LU(L(a, b), L(c, b)). \end{aligned}$$

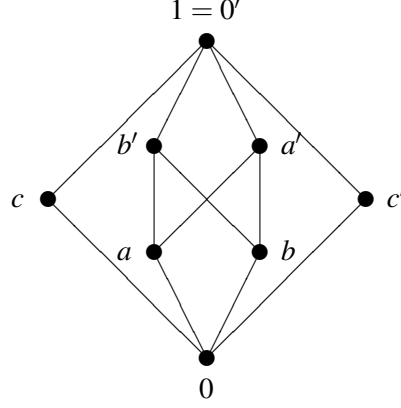


Fig. 2

Recall that a *pseudo-Kleene algebra* (see [4]) is a lattice $(L, \vee, \wedge, ')$ with an antitone involution satisfying the identity

$$x \wedge x' \leq y \vee y' \text{ for all } x, y \in L.$$

A *Kleene algebra* (see [11]) is a distributive pseudo-Kleene algebra. Observe that a lattice with an antitone involution is a (pseudo-)Kleene poset if and only if it is a (pseudo-)Kleene algebra. Hence, our concepts defined above are appropriate generalizations of pseudo-Kleene algebras and Kleene algebras as will be shown in the next section.

Example 3. The lattice visualized in Figure 3 is a pseudo-Kleene algebra which is not a Kleene algebra since the lattice is not distributive:

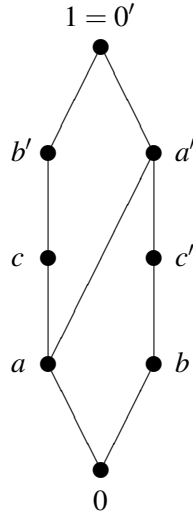


Fig. 3

Example 4. The lattice visualized in Figure 4 is a Kleene algebra:

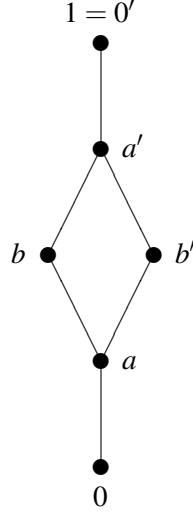


Fig. 4

3. DEDEKIND-MACNEILLE COMPLETION

It is a natural question when a Dedekind-MacNeille completion $\mathbf{DM}(\mathbf{P})$ of a poset $\mathbf{P}$ is a pseudo-Kleene or Kleene algebra. We get the answer in next two theorems. At first, we recall several necessary concepts taken e.g. from [1].

Let $\mathbf{P} = (P, \leq, ')$ be a poset with an antitone involution. Define

$$\begin{aligned} \mathbf{DM}(\mathbf{P}) &:= \{L(A) \mid A \subseteq P\}, \\ A^* &:= L(A') \text{ for all } A \in \mathbf{DM}(\mathbf{P}), \\ \mathbf{DM}(\mathbf{P}) &:= (\mathbf{DM}(\mathbf{P}), \subseteq, *). \end{aligned}$$

Then $\mathbf{DM}(\mathbf{P})$ is a complete lattice with an antitone involution, called the *Dedekind-MacNeille completion* of $\mathbf{P}$. One can show that

$$\mathbf{DM}(\mathbf{P}) = \{LU(B) \mid B \subseteq P\} = \{C \subseteq P \mid LU(C) = C\}.$$

That $*$ is an antitone involution on $(\mathbf{DM}(\mathbf{P}), \subseteq)$ can be seen as follows. Let $A, B \in \mathbf{DM}(\mathbf{P})$. If $A \subseteq B$ then $A' \subseteq B'$ and hence $B^* = L(B') \subseteq L(A') = A^*$. Moreover, $A^{**} = L((L(A'))') = LU(A) = A$ since $(L(A'))' = U(A)$. We have

$$\begin{aligned} (L(A))^* &= L((L(A))') = LU(A') \text{ (since } (L(A))' = U(A')) \text{ for all } A \subseteq P, \\ A \vee B &= LU(A, B) \text{ for all } A, B \in \mathbf{DM}(\mathbf{P}), \\ A \wedge B &= A \cap B \text{ for all } A, B \in \mathbf{DM}(\mathbf{P}). \end{aligned}$$

Theorem 1. *Let $\mathbf{P} = (P, \leq, ')$ be a poset with an antitone involution. Then $\mathbf{DM}(\mathbf{P})$ is a pseudo-Kleene algebra if and only if $\mathbf{P}$ is a pseudo-Kleene poset.*

Proof. Assume that $\mathbf{P}$ is a pseudo-Kleene poset. We use the properties of L and U mentioned in the introduction. The following are equivalent:

$$\begin{aligned} & \mathbf{DM}(\mathbf{P}) \text{ is a pseudo-Kleene algebra,} \\ & C \wedge C^* \subseteq D \vee D^* \text{ for all } C, D \in \mathbf{DM}(\mathbf{P}), \\ & L(A) \wedge (L(A))^* \subseteq L(B) \vee (L(B))^* \text{ for all } A, B \subseteq P, \\ & L(A) \cap LU(A') \subseteq LU(L(B), LU(B')) \text{ for all } A, B \subseteq P, \\ & L(A, U(A')) \subseteq L(UL(B) \cap ULU(B')) \text{ for all } A, B \subseteq P, \\ & L(A, U(A')) \subseteq L(UL(B) \cap U(B')) \text{ for all } A, B \subseteq P, \\ & L(A, U(A')) \subseteq LU(L(B), B') \text{ for all } A, B \subseteq P, \\ & L(A, U(A')) \leq U(L(B), B') \text{ for all } A, B \subseteq P. \end{aligned}$$

Now let A, B be fixed subsets of P . Then

$$\begin{aligned} L(A, U(A')) &= L(A) \cap LU(A') = \bigcup_{x \in L(A)} L(x) \cap \bigcap_{y \in U(A')} L(y) \\ &= \bigcup_{x \in L(A)} (L(x) \cap \bigcap_{y \in U(A')} L(y)) \subseteq \bigcup_{x \in L(A)} (L(x) \cap L(x')) = \bigcup_{x \in L(A)} L(x, x'), \\ U(L(B), B') &= UL(B) \cap U(B') = \bigcap_{x \in L(B)} U(x) \cap \bigcup_{y \in U(B')} U(y) \\ &= \bigcup_{y \in U(B')} (\bigcap_{x \in L(B)} U(x) \cap U(y)) \subseteq \bigcup_{y \in U(B')} (U(y') \cap U(y)) = \bigcup_{y \in U(B')} U(y, y') \end{aligned}$$

and

$$\bigcup_{x \in L(A)} L(x, x') \leq \bigcup_{y \in U(B')} U(y, y').$$

Hence $\mathbf{DM}(\mathbf{P})$ is a pseudo-Kleene algebra provided $\mathbf{P}$ is a pseudo-Kleene poset. The converse is evident since the mapping $x \mapsto L(x)$ is an embedding of $\mathbf{P}$ into $\mathbf{DM}(\mathbf{P})$. $\square$

Theorem 2. *The Dedekind-MacNeille completion of a finite Kleene poset is a Kleene algebra.*

Proof. According to a result by M. Ern  ([8]), the Dedekind-MacNeille completion of a finite distributive poset is a distributive lattice. Hence, by Theorem 1 the Dedekind-MacNeille completion of a finite Kleene poset is a Kleene algebra. $\square$

The Dedekind-MacNeille completion of the Kleene poset from Example 1 is visualized in Figure 5:

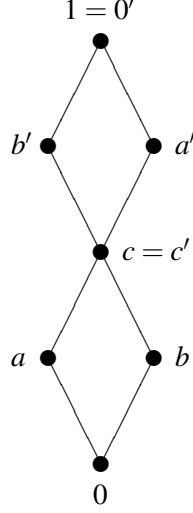


Fig. 5

4. A REPRESENTATION BY COMMUTATIVE MEET-DIRECTOIDS

Now we recall the concept of a commutative meet-directoid. This concept was introduced by J. Ježek and R. Quackenbush ([10]), an overview of results on directoids can be found in the authors' monograph [6]. We will use commutative meet-directoids for the characterization of pseudo-Kleene posets, Kleene posets, strong pseudo-Kleene posets and strict pseudo-Kleene posets. The advantage of this approach is that we characterize properties of posets by means of identities and quasi-identities of algebras. Hence, one can use algebraic tools for their investigation.

A *commutative meet-directoid* (see [6] and [10]) is a groupoid $\mathbf{D} = (D, \sqcap)$ satisfying the following identities:

$$\begin{aligned} x \sqcap x &\approx x \text{ (idempotency),} \\ x \sqcap y &\approx y \sqcap x \text{ (commutativity),} \\ (x \sqcap (y \sqcap z)) \sqcap z &\approx x \sqcap (y \sqcap z) \text{ (weak associativity).} \end{aligned}$$

If $\mathbf{P} = (P, \leq)$ is a downward directed poset, if we define $x \sqcap y := x \wedge y$ for comparable $x, y \in P$ and if we put for $x \sqcap y = y \sqcap x$ an arbitrary element of $L(x, y)$ if $x, y \in P$ are incomparable, then $\mathbb{D}(\mathbf{P}) := (P, \sqcap)$ is a commutative meet-directoid which is called a *meet-directoid assigned to $\mathbf{P}$* . Conversely, if $\mathbf{D} = (D, \sqcap)$ is a commutative meet-directoid and we define

$$x \leq y \text{ if and only if } x \sqcap y = x$$

for all $x, y \in D$ then $\mathbb{P}(\mathbf{D}) := (D, \leq)$ is a downward directed poset, the so-called *poset induced by $\mathbf{D}$* . Though the assignment $\mathbf{P} \mapsto \mathbb{D}(\mathbf{P})$ is not necessarily unique, we have

$\mathbb{P}(\mathbb{D}(\mathbf{P})) = \mathbf{P}$ for every downward directed poset $\mathbf{P}$. Sometimes we consider posets and commutative meet-directoids together with a unary operation. Let $(D, \sqcap, ')$ be a commutative meet-directoid $(D, \sqcap, ')$ with an antitone involution. We define

$$x \sqcup y := (x' \sqcap y')' \text{ for all } x, y \in D.$$

Then $\sqcup$ is also idempotent, commutative and weakly associative and we have for all $x, y \in D$

$$\begin{aligned} x \sqcup y &= x \vee y \text{ if } x, y \text{ are comparable,} \\ x \sqcup y &= y \sqcup x \in U(x, y) \text{ if } x \parallel y, \\ x \sqcap y &= x \text{ if and only if } x \sqcup y = y, \\ L(x) &= \{z \sqcap x \mid z \in P\}, \\ U(x) &= \{z \sqcup x \mid z \in P\}, \\ L(x, y) &= \{(z \sqcap x) \sqcap (z \sqcap y) \mid z \in P\}, \\ U(x, y) &= \{(z \sqcup x) \sqcup (z \sqcup y) \mid z \in P\}. \end{aligned}$$

Posets with an antitone involution can be characterized in the language of commutative meet-directoids by identities as follows.

Lemma 3. *Let $\mathbf{P} = (P, \leq, ')$ be a downward directed poset with a unary operation and $\mathbb{D}(\mathbf{P})$ an assigned meet-directoid. Then $\mathbf{P}$ is a poset with an antitone involution if and only if $\mathbb{D}(\mathbf{P})$ satisfies the identities*

$$x'' \approx x, \tag{4.1}$$

$$(x \sqcap y)' \sqcap y' \approx y'. \tag{4.2}$$

Proof. Condition (4.1) is evident by definition. Since $\{x \sqcap y \mid x \in P\} = L(y)$, the following are equivalent:

$$\begin{aligned} &(4.2), \\ &x' \sqcap y' = y' \text{ for all } x, y \in P \text{ with } x \leq y, \\ &y' \leq x' \text{ for all } x, y \in P \text{ with } x \leq y, \\ &' \text{ is antitone.} \end{aligned}$$

□

Now we characterize pseudo-Kleene posets by identities of an assigned commutative meet-directoid.

Theorem 3. *Let $\mathbf{P} = (P, \leq, ')$ be a downward directed poset with a unary operation and $\mathbb{D}(\mathbf{P})$ an assigned meet-directoid. Then $\mathbf{P}$ is a pseudo-Kleene poset if and only if $\mathbb{D}(\mathbf{P})$ satisfies identities (4.1) – (4.3):*

$$(z \sqcap x) \sqcap (z \sqcap x') \leq (w \sqcup y) \sqcup (w \sqcup y'). \tag{4.3}$$

Proof. It is easy to check that (4.3) is equivalent to $L(x, x') \leq U(y, y')$ for all $x, y \in P$. Applying Lemma 3 completes the proof. $\square$

In fact, condition (4.3) is not an identity, but it can easily be rewritten as an identity. Hence we can call it an identity. Similarly, we proceed in the following.

In order to characterize Kleene posets in a similar way we need to capture distributivity of posets in the language of commutative meet-directoids.

Theorem 4. *Let $\mathbf{P} = (P, \leq, ')$ be a downward directed poset with a unary operation and $\mathbb{D}(\mathbf{P})$ an assigned meet-directoid. Then $\mathbf{P}$ is a Kleene poset if and only if $\mathbb{D}(\mathbf{P})$ satisfies identities (4.1) – (4.3) and the following implication:*

$$\begin{aligned} w \sqcap ((t \sqcup x) \sqcup (t \sqcup y)) = w \sqcap z = w \text{ and } s \sqcup ((t \sqcap x) \sqcap (t \sqcap z)) \\ = s \sqcup ((t \sqcap y) \sqcap (t \sqcap z)) = s \text{ for all } t \in P \text{ imply } w \leq s. \end{aligned} \quad (4.4)$$

Proof. Since

$$\begin{aligned} U(x, y) &= \{(t \sqcup x) \sqcup (t \sqcup y) \mid t \in P\}, \\ w \sqcap u = w &\text{ is equivalent to } w \in L(u), \end{aligned}$$

$$w \sqcap ((t \sqcup x) \sqcup (t \sqcup y)) = w \sqcap z = w \text{ for all } t \in P \text{ is equivalent to } w \in L(U(x, y), z).$$

Further, since

$$\begin{aligned} L(x, z) &= \{(t \sqcap x) \sqcap (t \sqcap z) \mid t \in P\}, \\ L(y, z) &= \{(t \sqcap y) \sqcap (t \sqcap z) \mid t \in P\}, \\ s \sqcup u = s &\text{ is equivalent to } s \in U(u), \end{aligned}$$

$s \sqcup ((t \sqcap x) \sqcap (t \sqcap z)) = s \sqcup ((t \sqcap y) \sqcap (t \sqcap z)) = s$ for all $t \in P$ is equivalent to $s \in U(L(x, z), L(y, z))$. Hence the following are equivalent:

(4.4),

$$w \in L(U(x, y), z) \text{ and } s \in U(L(x, z), L(y, z)) \text{ imply } w \leq s,$$

$$L(U(x, y), z) \subseteq LU(L(x, z), L(y, z)),$$

$\mathbf{P}$ is distributive.

Applying Theorem 3 completes the proof. $\square$

Let us note that the class of all directoids assigned to downward directed pseudo-Kleene posets forms a variety due to Theorem 3. As shown in [6] and [5], every variety of directoids with an antitone involution is congruence distributive.

Definition 2. A *strong pseudo-Kleene poset* is a poset $(P, \leq, ')$ with an antitone involution satisfying

$$x \parallel y \text{ implies } L(x, x') = L(y, y').$$

Lemma 4. *Let $\mathbf{P} = (P, \leq, ')$ be a strong pseudo-Kleene poset. Then $\mathbf{P}$ is a pseudo-Kleene poset.*

Proof. Let $a, b \in P$.

If $a \leq b$ then $L(a, a') \leq a \leq b \leq U(b, b')$.

If $b \leq a$ then $a' \leq b'$ and hence $L(a, a') \leq a' \leq b' \leq U(b, b')$.

If $a \parallel b$ then $L(a, a') = L(b, b') \leq b \leq U(b, b')$. $\square$

Example 5. The poset visualized in Figure 6 is a strong pseudo-Kleene poset which is not a lattice and hence not a pseudo-Kleene algebra:

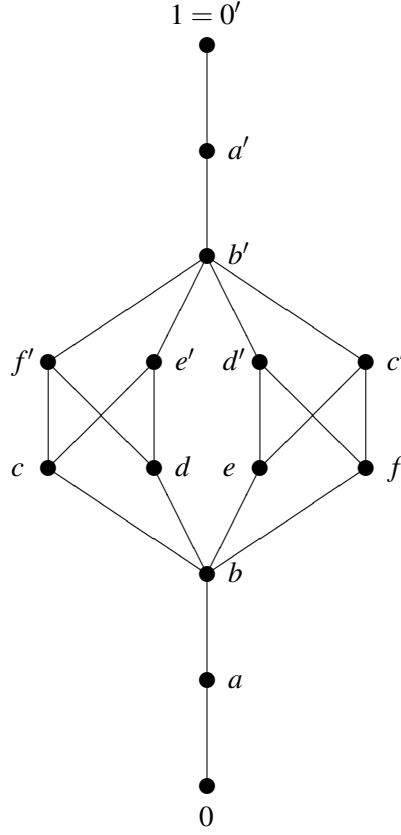


Fig. 6

We are going to determine the class of directoids assigned to strong pseudo-Kleene posets.

Theorem 5. Let $\mathbf{P} = (P, \leq, ')$ be a downward directed poset with a unary operation and $\mathbb{D}(\mathbf{P})$ an assigned meet-directoid. Then $\mathbf{P}$ is a strong pseudo-Kleene poset if and only if $\mathbb{D}(\mathbf{P})$ satisfies identities (4.1) and (4.2) and the following implication:

$$x \neq x \sqcap y \neq y \text{ and } x \sqcap z = x' \sqcap z = z \text{ imply } y \sqcap z = y' \sqcap z = z. \quad (4.5)$$

Proof. Since

$$\begin{aligned} x \sqcap y = x & \text{ is equivalent to } x \leq y, \\ x \sqcap y = y & \text{ is equivalent to } y \leq x \end{aligned}$$

we have that $x \neq x \sqcap y \neq y$ is equivalent to $x \parallel y$. Further, since

$$\begin{aligned} x \sqcap z = x' \sqcap z = z & \text{ is equivalent to } z \in L(x, x'), \\ y \sqcap z = y' \sqcap z = z & \text{ is equivalent to } z \in L(y, y'), \end{aligned}$$

the following are equivalent:

$$\begin{aligned} (4.5), \\ x \parallel y \text{ and } z \in L(x, x') & \text{ imply } z \in L(y, y'), \\ x \parallel y & \text{ implies } L(x, x') \subseteq L(y, y'), \\ x \parallel y & \text{ implies } L(x, x') = L(y, y'). \end{aligned}$$

Applying Lemma 3 completes the proof. $\square$

The following concept will be used in the sequel.

Definition 3. A *strict pseudo-Kleene poset* is a bounded poset $(P, \leq, ')$ with an antitone involution satisfying

$$x, y \neq 0, 1 \text{ implies } L(x, x') = L(y, y').$$

A *strict Kleene poset* is a distributive strict pseudo-Kleene poset.

Obviously, every strict pseudo-Kleene poset is a strong pseudo-Kleene poset and hence a pseudo-Kleene poset according to Lemma 4, but not conversely (see Example 5 where $L(a, a') = a \neq b = L(b, b')$).

Of course, every Boolean poset, i.e. every bounded distributive poset where the antitone involution is a complementation (i.e. $L(x, x') \approx \{0\}$ and $U(x, x') = \{1\}$), is a strict Kleene poset. In the next example we show a strict Kleene poset which is not Boolean.

Example 6. The poset visualized in Figure 7 is a strict Kleene poset which is not a lattice and hence not a Kleene algebra.

Analogously as above, also strict pseudo-Kleene posets and strict Kleene posets can be characterized by means of properties of assigned meet-directoids.

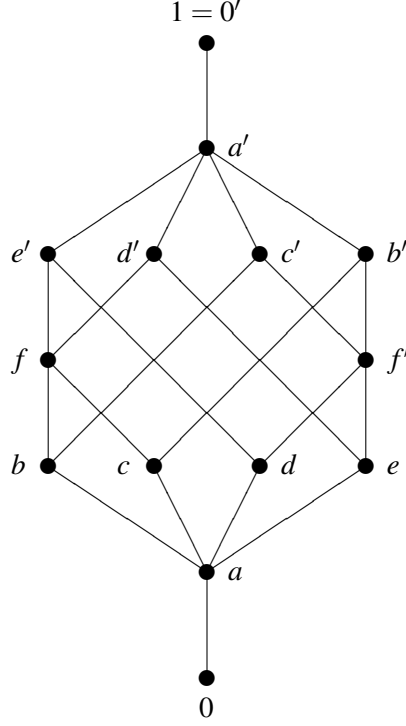


Fig. 7

Theorem 6. Let $\mathbf{P} = (P, \leq, ')$ be a downward directed poset with a unary operation and $\mathbb{D}(\mathbf{P})$ an assigned meet-directoid. Then the following hold:

- (i) $\mathbf{P}$ is a strict pseudo-Kleene poset if and only if $\mathbb{D}(\mathbf{P})$ satisfies identities (4.1) and (4.2) and the following implication:

$$x, y \notin \{0, 1\} \text{ and } x \sqcap z = x' \sqcap z = z \text{ imply } y \sqcap z = y' \sqcap z = z. \quad (4.6)$$

- (ii) $\mathbf{P}$ is a strict Kleene poset if and only if $\mathbb{D}(\mathbf{P})$ satisfies identities (4.1) and (4.2) and implications (4.4) and (4.6).

Proof. (i) The following are equivalent:

(4.6),

$$x, y \notin \{0, 1\} \text{ and } z \in L(x, x') \text{ imply } z \in L(y, y'),$$

$$x, y \notin \{0, 1\} \text{ implies } L(x, x') \subseteq L(y, y'),$$

$$x, y \notin \{0, 1\} \text{ implies } L(x, x') = L(y, y').$$

Applying Lemma 3 completes the proof.

- (ii) This follows from (i) and the proof of Theorem 4. □

5. RESIDUATION IN KLEENE POSETS

In the study of non-classical logics we prefer to have a logical connective implication which enables deduction, i.e. to derive new propositions from given ones. The question arises how to define implication in the logic based on pseudo-Kleene posets and Kleene posets. Usually implication is considered to be well-behaved if it is related with conjunction by means of adjointness. For a systematic overview on residuated structures we recommend the monographs [3] and [9]. These monographs contain both the basic concepts and results on residuated structures as well as applications to the formalization of substructural and fuzzy logics.

In the following we will study residuation in strict pseudo-Kleene posets. Contrary to the case of lattices, $\odot$ and $\rightarrow$ cannot be binary operations, they are only operators, i.e. mappings from P^2 to 2^P . We extend $\odot$ to $(2^P)^2$ by defining

$$A \odot B := \bigcap_{x \in A, y \in B} (x \odot y)$$

for all $A, B \subseteq P$.

Definition 4. A *Kleene residuated poset* is an ordered six-tuple $(P, \leq, \odot, \rightarrow, 0, 1)$, where $(P, \leq, 0, 1)$ is a bounded strict pseudo-Kleene poset and $\odot$ and $\rightarrow$ are mappings from P^2 to 2^P satisfying the following conditions for all $x, y, z \in P$:

- $x \odot y \approx y \odot x$,
- $x \odot 1 \approx 1 \odot x \approx L(x)$,
- $(x \odot y) \odot z \approx x \odot (y \odot z)$,
- $x \odot y \leq z$ if and only if $x \leq y \rightarrow z$ (adjointness).

Let $(P, \leq, ')$ be a poset with an antitone involution. Define mappings $\odot$ and $\rightarrow$ from P^2 to 2^P as follows:

$$x \odot y := \begin{cases} 0 & \text{if } x \leq y', \\ L(x, y) & \text{otherwise} \end{cases} \quad x \rightarrow y := \begin{cases} 1 & \text{if } x \leq y, \\ U(x', y) & \text{otherwise} \end{cases} \quad (5.1)$$

Theorem 7. Let $\mathbf{P} = (P, \leq, ', 0, 1)$ be a bounded strict Kleene poset satisfying

$$L(x, y) \neq 0 \text{ for all } x, y \in P \setminus \{0\} \quad (5.2)$$

and let $\odot$ and $\rightarrow$ be defined by (5.1). Then $(P, \leq, \odot, \rightarrow, 0, 1)$ is a Kleene residuated poset.

Proof. Let $a, b, c \in P$. One can easily see that $x \odot 0 \approx 0 \odot x \approx 0$. Since $0 \in L(x, y)$ for all $x, y \in P$ and therefore $0 \in x \odot y$ for all $x, y \in P$, we have $0 \in a \odot b$ and $0 \in b \odot c$ and hence $(a \odot b) \odot c = 0 = a \odot (b \odot c)$ proving associativity of $\odot$. Because $a \leq b'$ is equivalent to $b \leq a'$ and, moreover, $L(a, b) = L(b, a)$, $\odot$ is commutative. Further,

$$\begin{aligned} \text{if } a = 0 \text{ then } a \odot 1 &= 0 = L(0) = L(a), \\ \text{if } a \neq 0 \text{ then } a \odot 1 &= L(a, 1) = L(a). \end{aligned}$$

Now $x \odot 1 \approx 1 \odot x \approx x$ follows by commutativity of $\odot$. We consider the following cases:

- $a \leq b'$ and $b \leq c$.
Then $a \odot b = 0 \leq c$ and $a \leq 1 = b \rightarrow c$.
- $a \leq b'$ and $b \not\leq c$.
Then $a \odot b = 0 \leq c$ and $a \leq b' \leq U(b', c) = b \rightarrow c$.
- $a \not\leq b'$ and $b \leq c$.
Then $a \odot b = L(a, b) \leq b \leq c$ and $a \leq 1 = b \rightarrow c$.
- $a \not\leq b'$, $b \not\leq c$ and $a = 1$.
Then $a \odot b = L(a, b) = L(1, b) = L(b) \not\leq c$. Moreover, $b, c' \neq 0$ and therefore, using De Morgan's laws and (5.2), $b \rightarrow c = U(b', c) = (L(b, c'))' \neq 0' = 1$ whence $a = 1 \not\leq b \rightarrow c$.
- $a \not\leq b'$, $b \not\leq c$ and $b = 1$.
Then the following are equivalent:

$$\begin{aligned}
 & a \odot b \leq c, \\
 & L(a, b) \leq c, \\
 & L(a, 1) \leq c, \\
 & L(a) \leq c, \\
 & a \leq c, \\
 & a \leq U(c), \\
 & a \leq U(0, c), \\
 & a \leq U(b', c), \\
 & a \leq b \rightarrow c.
 \end{aligned}$$

- $a \not\leq b'$, $b \not\leq c$ and $c = 0$.
Then $a, b \neq 0$ and therefore $a \odot b = L(a, b) \neq 0$ according to (5.2) whence $a \odot b \not\leq c$. Moreover, $a \not\leq U(b') = U(b', 0) = U(b', c) = b \rightarrow c$.
- $a \not\leq b'$, $b \not\leq c$, $a, b \neq 1$ and $c \neq 0$.

$$\begin{aligned}
 & \text{Then } a, b, c \neq 0, 1. \text{ If } a \odot b \leq c \text{ then, by distributivity of } \mathbf{P} \text{ and (5.2),} \\
 & b \rightarrow c = U(b', c) \subseteq U(b', a \odot b) = U(b', L(a, b)) = UL(U(b', a), U(b', b)) \\
 & \quad = UL(U(b', a), U(a', a)) \subseteq ULU(a) = U(a)
 \end{aligned}$$

and hence $a \leq b \rightarrow c$. If, conversely, $a \leq b \rightarrow c$ then, again by distributivity of $\mathbf{P}$ and (5.2),

$$\begin{aligned}
 & a \odot b = L(a, b) \subseteq L(b \rightarrow c, b) = L(U(b', c), b) = LU(L(b', b), L(c, b)) \\
 & \quad = LU(L(c', c), L(c, b)) \subseteq LUL(c) = L(c)
 \end{aligned}$$

and hence $a \odot b \leq c$.

This shows that in any case $a \odot b \leq c$ is equivalent to $a \leq b \rightarrow c$. $\square$

Remark 1. It seems to be impossible to generalize Theorem 7 in such a way that the assumption of distributivity is replaced by the weaker assumption of modularity. If, e.g., $L(x, y)$ and $U(x', y)$ in (5.1) are replaced by $L(U(x, y'), y)$ and $U(x', L(x, y))$, respectively, then adjointness cannot be proved in the last case considered in the proof of Theorem 7.

As it is usual in logics satisfying the double negation law, the connectives conjunction (i.e. $\odot$) and implication (i.e. $\rightarrow$) can be derived one by the other by means of involution.

Theorem 8. *Let $\mathbf{P} = (P, \leq, ', 0, 1)$ be a bounded poset with an antitone involution, $\odot$ and $\rightarrow$ defined by (5.1) and $a, b \in P$. Then the following hold:*

- (i) $a \odot b = (a \rightarrow b')'$,
- (ii) $a \rightarrow b = (a \odot b')'$.

If, moreover, $\mathbf{P}$ satisfies (5.2) then

- (iii) $a \odot b = 0$ if and only if $a \leq b'$,
- (iv) $a \rightarrow b = 1$ if and only if $a \leq b$.

If, moreover, $\mathbf{P}$ is a strict Kleene poset (not necessarily satisfying (5.2)) then

- (v) *If $a \leq b$ and $L(a', b) = 0$ then $a = b$.*

Proof. (i)

If $a \leq b'$ then $a \odot b = 0 = 1' = (a \rightarrow b')'$,

if $a \not\leq b'$ then $a \odot b = L(a, b) = (U(a', b'))' = (a \rightarrow b')'$ according to De Morgan's laws.

(ii) According to (i) we have $a \rightarrow b = (a \rightarrow b'')'' = (a \odot b')'$.

(iii)

If $a \leq b'$ then $a \odot b = 0$ by definition,

if $a \not\leq b'$ then $a, b \neq 0$ and hence $a \odot b = L(a, b) \neq 0$ according to (5.2).

(iv) According to (ii) and (iii) the following are equivalent:

$$\begin{aligned} a \rightarrow b &= 1, \\ (a \odot b')' &= 1, \\ a \odot b' &= 0, \\ a &\leq b'', \\ a &\leq b. \end{aligned}$$

(v)

If $a = 0$ then $L(b) = L(1, b) = L(a', b) = 0$ and hence $a = 0 = b$,

if $a = 1$ then $b = 1$ because of $a \leq b$ and hence $a = 1 = b$,

if $b = 0$ then $a = 0$ because of $a \leq b$ and hence $a = 0 = b$,

if $b = 1$ then $L(a') = L(a', 1) = L(a', b) = 0$ and hence $a' = 0$, i.e. $a = 1 = b$,

if $a, b \neq 0, 1$ then because of the distributivity of $\mathbf{P}$ and $U(b', b) \subseteq U(b)$ we have

$$\begin{aligned} U(a) &= U(0, a) = U(L(a', b), a) = UL(U(a', a), U(b, a)) = UL(U(b', b), U(b)) \\ &= ULU(b) = U(b) \text{ which implies } a = b. \end{aligned}$$

□

6. TWIST-PRODUCT CONSTRUCTION

Now we show how to construct pseudo-Kleene posets and Kleene posets from posets and distributive posets, respectively. We embed an arbitrary given poset into a pseudo-Kleene poset by using the so-called twist-product construction known already for distributive lattices.

The concept of a twist-product was introduced recently. It is broadly used in algebra and lattice theory. As a source for this construction we mention the paper [2]. For our purposes, we modify this construction for posets as follows.

For an arbitrary poset $\mathbf{Q} = (Q, \leq)$ and an arbitrary element a of Q we define

$$P_a(\mathbf{Q}) := \{(x, y) \in Q^2 \mid L(x, y) \leq a \leq U(x, y)\}.$$

Let p_1 and p_2 denote the first and second projection from $P_a(\mathbf{Q})$ to Q , respectively. In $P_a(\mathbf{Q})$ we introduce a binary relation $\leq$ and a unary operation $'$ as follows:

$$\begin{aligned} (x, y) \leq (z, v) &: \Leftrightarrow x \leq z \text{ and } v \leq y, \\ (x, y)' &:= (y, x). \end{aligned}$$

Put $\mathbb{P}_a(\mathbf{Q}) := (P_a(\mathbf{Q}), \leq, ')$.

Theorem 9. *Let $\mathbf{Q} = (Q, \leq)$ be a poset and $a \in Q$. Then the following hold:*

- (i) $\mathbb{P}_a(\mathbf{Q})$ is a pseudo-Kleene poset with fixed point (a, a) ,
- (ii) the mapping $x \mapsto (x, a)$ is an embedding of $\mathbf{Q}$ into $\mathbb{P}_a(\mathbf{Q})$,
- (iii) $\mathbf{Q}$ is distributive if and only if $\mathbb{P}_a(\mathbf{Q})$ is a Kleene poset.

Proof. Let $(b, c), (d, e), (f, g) \in P_a(\mathbf{Q})$ and $h, i \in Q$. We have

$$L(A) = L(p_1(A)) \times U(p_2(A)) \text{ for all } A \subseteq P_a(\mathbf{Q}), \quad (6.1)$$

$$U(A) = U(p_1(A)) \times L(p_2(A)) \text{ for all } A \subseteq P_a(\mathbf{Q}). \quad (6.2)$$

(6.1) can be seen as follows: For $A \subseteq P_a(\mathbf{Q})$ the following are equivalent:

- $(h, i) \in L(A)$,
- $(h, i) \leq (x, y)$ for all $(x, y) \in A$,
- $(h \leq x \text{ and } y \leq i)$ for all $(x, y) \in A$,
- $(h \leq x \text{ for all } (x, y) \in A) \text{ and } (y \leq i \text{ for all } (x, y) \in A)$,

$$\begin{aligned}
& (h \leq x \text{ for all } x \in p_1(A)) \text{ and } (y \leq i \text{ for all } y \in p_2(A)), \\
& h \in L(p_1(A)) \text{ and } i \in U(p_2(A)), \\
& (h, i) \in L(p_1(A)) \times U(p_2(A)).
\end{aligned}$$

(6.2) follows in an analogous way.

- (i) Clearly, $\mathbb{P}_a(\mathbf{Q})$ is a poset with an antitone involution and fixed point (a, a) .
According to (6.1), (6.2) and the definition of $P_a(\mathbf{Q})$ we conclude

$$L((b, c), (c, b)) = L(b, c) \times U(b, c) \leq (a, a) \leq U(d, e) \times L(d, e) = U((d, e), (e, d)).$$

- (ii) We have $(h, a), (i, a) \in P_a(\mathbf{Q})$. Moreover, $(h, a) \leq (i, a)$ if and only if $h \leq i$.
(iii) Using (6.1) and (6.2) we obtain

$$\begin{aligned}
L(U((b, c), (d, e)), (f, g)) &= L(U(b, d) \times L(c, e), (f, g)) \\
&= L(U(b, d), f) \times U(L(c, e), g),
\end{aligned}$$

$$\begin{aligned}
LU(L((b, c), (f, g)), L((d, e), (f, g))) &= LU(L(b, f) \times U(c, g), L(d, f) \times U(e, g)) \\
&= L(U(L(b, f), L(d, f)) \times L(U(c, g), U(e, g))) \\
&= LU(L(b, f), L(d, f)) \times UL(U(c, g), U(e, g)).
\end{aligned}$$

Hence

$$L(U((b, c), (d, e)), (f, g)) = LU(L((b, c), (f, g)), L((d, e), (f, g)))$$

is equivalent to

$$L(U(b, d), f) \times U(L(c, e), g) = LU(L(b, f), L(d, f)) \times UL(U(c, g), U(e, g)),$$

i.e. to

$$L(U(b, d), f) = LU(L(b, f), L(d, f)) \text{ and } U(L(c, e), g) = UL(U(c, g), U(e, g)).$$

Here b, d, f can be arbitrary elements of $\mathbf{Q}$ since $(b, a), (d, a), (f, a) \in P_a(\mathbf{Q})$ for every $b, d, f \in \mathbf{Q}$. $\square$

It is well-known (see e.g. [11]) that for an arbitrary distributive lattice $\mathbf{Q}$ the constructed poset $\mathbb{P}_a(\mathbf{Q})$ is a Kleene algebra. We have shown that this construction can be extended to arbitrary posets.

Example 7. If $\mathbf{Q}$ is the poset $(Q, \leq)$ visualized in Figure 8:

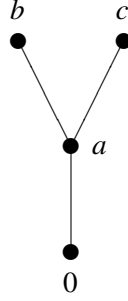


Fig. 8

then $\mathbb{P}_a(\mathbf{Q})$ is the pseudo-Kleene poset shown in Figure 9:

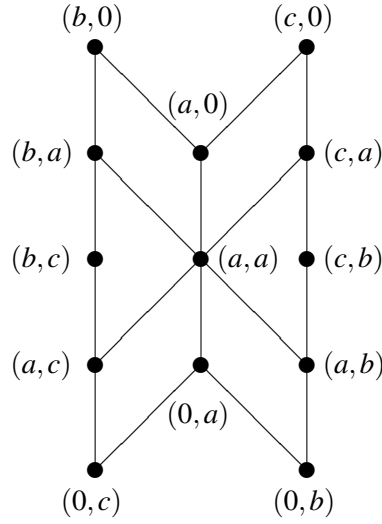


Fig. 9

which is neither a pseudo-Kleene algebra nor a Kleene poset since

$$\begin{aligned}
 L\left(U((a,c), (0,a)), (a,b)\right) &= L\left(U((a,a)), (a,b)\right) = L((a,a), (a,b)) = \{(0,b), (a,b)\} \\
 &\neq \{(0,b)\} = L(P_a(\mathbf{Q}) \setminus \{(0,c), (a,c), (b,c)\}) \\
 &= LU((0,b)) = LU\left(L((a,c), (a,b)), L((0,a), (a,b))\right).
 \end{aligned}$$

ACKNOWLEDGEMENT

The authors are grateful to the anonymous referee whose valuable remarks helped to increase the quality of the paper.

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A NOTE ON RADICALS OF ASSOCIATIVE RINGS AND ALTERNATIVE RINGS

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To the memory of professor Avirmed Nyamrinchin

Received 17 April, 2018

Abstract. Let U be a universal subclass of a universal class V of rings. We investigate connections between radicals in U and V . We define $\mathcal{T}$ and $\mathcal{T}_s$ as follows:

$$\mathcal{T} = \{A \in \mathcal{A}_{ss} \mid \text{every prime homomorphic image of } A \text{ is not a hereditary Amitsur ring}\}$$

$$\mathcal{T}_s = \{A \in \mathcal{A}_{ss} \mid \text{every prime homomorphic image of } A \text{ has no nonzero ideal which is a hereditary Amitsur ring}\}.$$

Let $\gamma \in \{\mathcal{T}, \mathcal{T}_s\}$ and let A be a commutative ring with minimum condition on ideals. We give a sufficient and necessary condition for A to be γ -semisimple.

2010 *Mathematics Subject Classification:* 16N80

Keywords: Radicals of associative or alternative rings, lower radicals

1. INTRODUCTION

In this note, all rings considered will belong to some arbitrary (but fixed) universal class V of not necessarily associative rings. A universal class of rings is a class of rings that is hereditary (that is, if $A \in V$ and I is an ideal of A , then $I \in V$) and homomorphically closed (that is, if $A \in V$ and I is an ideal of A , then $A/I \in V$). All radicals considered in this paper are in the sense of Kurosh and Amitsur. For the fundamental definitions and properties of radicals, we refer the reader to [1].

For any ring A , let A^0 denote the ring with additive group $(A, +)$ and with multiplication defined by $xy = 0$ for all $x, y \in R$. Such a ring A^0 is called a zero ring. The notation $I \trianglelefteq A$ means that I is an ideal of A . A subring B of a ring A is said to be an accessible subring of A if there exists a finite sequence $C_1, \dots, C_n$ of subrings of A such that $C_i \trianglelefteq C_{i+1}$, for $i = 1, \dots, n-1$, $C_1 = B$ and $C_n = A$. We recall that a radical γ of rings has the ADS property if $\gamma(I) \trianglelefteq A$ for every $I \trianglelefteq A$. Let

The authors was supported in part by the Science and Technology Fund of Mongolia, Grant No.Shuss2017/64.

$\mathcal{M}$ be a subclass of the universal class V . If $\mathcal{M}$ determines an upper radical, let $\mathcal{U}_V(\mathcal{M}) = \{A \in V \mid A \text{ has no nonzero homomorphic image in } \mathcal{M}\}$ that is, the upper radical generated by $\mathcal{M}$ in V . The lower radical in V , induced by the class $\mathcal{M}$, is denoted by $\mathcal{L}_V(\mathcal{M})$. For a radical γ , the semisimple class of γ in V is given by $(S_V)\gamma = \{A \in V \mid \gamma(A) = 0\}$.

2. GENERAL RINGS

In what follows, U denotes a universal subclass of the universal class V .

Proposition 1. *Let γ be a radical class in V . Then $\gamma \cap U$ is a radical class in U .*

Proof. Clear. □

Proposition 2. *Let γ be a radical class in U . Then $\gamma = \mathcal{L}_V(\gamma) \cap U$.*

Proof. It is clear that $\gamma \subseteq \mathcal{L}_V(\gamma) \cap U$.

If the semisimple class $(S_U)\gamma$ of γ is equal to 0, then $\gamma = U$ and $\gamma \supseteq \mathcal{L}_V(\gamma) \cap U$.

Suppose $(S_U)\gamma \neq 0$. Then $\mathcal{U}_V((S_U)\gamma) \supseteq \mathcal{L}_V(\gamma) \supseteq \mathcal{L}_V(\gamma) \cap U$. If there exists a nonzero ring $A \in \mathcal{L}_V(\gamma) \cap U$ with $\gamma(A) = 0$, then $A \in (S_U)\gamma$ and we have $A \in \mathcal{U}_V((S_U)\gamma) \cap (S_U)\gamma = 0$. This leads to a contradiction, so $\gamma \supseteq \mathcal{L}_V(\gamma) \cap U$ □

Now let us consider the following condition in the universal class V .

(ADS) : Every radical γ in V has the ADS property.

Theorem 1. *Let $U \subseteq V$ be universal classes of rings with property (ADS) and let γ be a radical in V . If for every ring $A \in \gamma$, there exists a nonzero accessible subring I_0 with $I_0 \in \gamma \cap U$, then $\mathcal{L}_V(\gamma \cap U) = \gamma$.*

Proof. We put $\gamma' = \mathcal{L}_V(\gamma \cap U)$.

It is clear that $\gamma' \subseteq \gamma$.

Let us suppose that $0 \neq A \in \gamma \setminus \gamma'$, then $\bar{0} \neq \bar{A} = A/\gamma'(A)$ and $\bar{A} \in \gamma$. Hence by assumption there exists a nonzero accessible subring $\bar{I}_0 \trianglelefteq \bar{I}_1 \trianglelefteq \bar{I}_2 \trianglelefteq \cdots \trianglelefteq \bar{I}_n = \bar{A}$ such that $\bar{I}_0 \in \gamma \cap U$. Therefore $\gamma'(\bar{I}_1) \neq \bar{0}$ and by ADS property we have $\gamma'(\bar{I}_1) \trianglelefteq \bar{I}_2$. Then by induction $\gamma'(\bar{I}_n) \neq \bar{0}$. Thus $\gamma'(\bar{A}) \neq \bar{0}$. But $\gamma'(A/\gamma'(A)) = \bar{0}$, which leads to a contradiction. Hence $\gamma \subseteq \gamma'$. □

3. ASSOCIATIVE RINGS

In this section, all rings are associative not necessarily with a unity element. In [4], we introduced (hereditary) Amitsur rings and also constructed radicals $\mathcal{T}$ and $\mathcal{T}_s$. A ring A is said to be a (hereditary) Amitsur ring if $\gamma(A[x]) = (\gamma(A[x]) \cap A)[x]$, for all (hereditary) radicals γ . Let us recall the definitions of $\mathcal{T}$ and $\mathcal{T}_s$.

If $\mathcal{A}_{ss}$ denotes the class of all associative rings, then

$$\mathcal{T} = \{A \in \mathcal{A}_{ss} \mid \text{every prime homomorphic image of } A \text{ is not a hereditary Amitsur ring}\}.$$

$\mathcal{T}_s = \{A \in \mathcal{A}_{ss} \mid \text{every prime homomorphic image of } A \text{ has no nonzero ideal which is a hereditary Amitsur ring}\}.$

Working in the class $\mathcal{A}$ of associative rings, we denote the semisimple class of a radical γ by $S\gamma$.

Proposition 3. *If F is a finite field, then F is not a hereditary Amitsur ring.*

Proof. Let F be a finite field and let us consider the class $\{F\}$. It is clear that $\{F\}$ is a special class of rings. Therefore the upper radical $\gamma = \mathcal{U}(\{F\})$ is a special radical class of rings. Hence γ is a hereditary radical class.

Also it is easy to check that $\gamma(F[x])$ is the ideal of $F[x]$, generated by $x^{p^n} - x$, where p^n is the number of elements in F . So $\gamma(F[x]) \subseteq xF[x]$. Therefore $\gamma(F[x]) \neq (\gamma(F[x]) \cap F)[x]$. Thus F is not a hereditary Amitsur ring. $\square$

Proposition 4. [4] *Let A be a ring without proper prime homomorphic images. If A is an infinite integral domain, then A is a hereditary Amitsur ring.*

Corollary 1. Let $J = \left\{ \frac{2x}{2y+1} \mid x, y \in \mathbb{Z}, (2x, 2y+1) = 1 \right\}$. Then J is a hereditary Amitsur ring.

Proposition 5. *Let $\gamma \in \{\mathcal{T}, \mathcal{T}_s\}$ and F be a field. Then*

- (1) $F \in S\gamma$ if and only if F is an infinite field.
- (2) $F \in \gamma$ if and only if F is a finite field.

Proof. It follows from Propositions 3 and 4. $\square$

Proposition 6. *For all simple rings A , $\mathcal{T}(A) = \mathcal{T}_s(A)$.*

Proof. If A is a simple ring, then A has no nonzero proper ideals and so it is obvious that $A \in \mathcal{T}$ if and only if $A \in \mathcal{T}_s$. $\square$

Let us denote by $\mathcal{P}$ the class of all prime rings. Let (η, ξ) be a partition of simple rings. Let us consider the following class:

$$s(\xi) = \{A \in \mathcal{P} \mid A \text{ is subdirectly irreducible with heart } H(A) \in \xi\}$$

We denote by $\mathcal{L}_{sp}\eta$ the lower special radical generated by η .

Theorem 2. [5] $\mathcal{L}_{sp}\eta \subsetneq \mathcal{U}(s(\xi))$.

Let us put:

$$\begin{aligned} \eta_{\mathcal{T}} &= \{A \in \mathcal{T} \mid A \text{ is a simple ring}\} \\ \xi_{\mathcal{T}} &= \{A \in S\mathcal{T} \mid A \text{ is a simple ring}\} \\ \eta_{\mathcal{T}_s} &= \{A \in \mathcal{T}_s \mid A \text{ is a simple ring}\} \\ \xi_{\mathcal{T}_s} &= \{A \in S\mathcal{T}_s \mid A \text{ is a simple ring}\} \end{aligned}$$

Corollary 2. $\mathcal{L}_{sp}(\eta_{\mathcal{T}}) = \mathcal{L}_{sp}(\eta_{\mathcal{T}_s}) \subseteq \mathcal{T}_s \subseteq \mathcal{T} \subsetneq \mathcal{U}(s(\xi_{\mathcal{T}})) = \mathcal{U}(s(\xi_{\mathcal{T}_s}))$.

Proof. By the notations above and Proposition 4 and 6, we have $\mathcal{L}_{sp}(\eta_{\mathcal{T}_s}) \subseteq \mathcal{T}_s \subseteq \mathcal{T} \subseteq \mathcal{U}(s(\xi_{\mathcal{T}})) = \mathcal{U}(s(\xi_{\mathcal{T}_s}))$. Let J be the ring defined on Corollary 1. Then in a similar to the proof of the Theorem 2, we can show that $J \notin \mathcal{T}$ and $J \in \mathcal{U}(s(\xi_{\mathcal{T}_s}))$. Thus $\mathcal{T} \subsetneq \mathcal{U}(s(\xi_{\mathcal{T}_s}))$. $\square$

Theorem 3. *Let A be a commutative ring, satisfying the minimum condition on ideals. Let $\gamma \in \{\mathcal{T}, \mathcal{T}_s\}$. Then $A \in S\gamma$ if and only if $A = F_1 \oplus F_2 \oplus \cdots \oplus F_n$, where F_i is an infinite field for any i , $1 \leq i \leq n$. Moreover, this statement is true for any radical γ such that $\mathcal{L}_{sp}(\eta_{\mathcal{T}_s}) \subseteq \gamma \subseteq \mathcal{U}(s(\xi_{\mathcal{T}_s}))$.*

Proof. Let $A \in S\gamma$. Since A satisfies the minimum condition on ideals, there exists a non-zero minimal ideal $I \trianglelefteq A$. $A \in S\gamma$ implies that A is a semiprime ring. Therefore I is a prime simple ring. Since A is a commutative ring, I is a commutative ring. Hence $aI = I$, for every non-zero $a \in I$. Thus I is a field. It is well known that I is a direct summand of A . So we have $A = I_1 \oplus A_1$, where $I = I_1$ and $A_1 \in S\gamma$. Also, A_1 satisfies all the conditions of the theorem. Hence if we continue this procedure, then we have $A = I_1 \oplus I_2 \oplus \cdots \oplus I_n \oplus \dots$.

Put

$$\begin{aligned} J_1 &= A, \\ J_2 &= I_2 \oplus \cdots \oplus I_n \oplus \dots \\ &\vdots \\ J_n &= I_n \oplus \cdots \oplus I_{n+s} \dots \end{aligned}$$

Then $J_1 \supsetneq J_2 \supsetneq \cdots \supsetneq J_n \supsetneq \dots$ and $J_i \trianglelefteq A$, for each $i = 1, 2, \dots$.

By assumption there exists $n \in \mathbb{N}$, such that

$$J_n = J_{n+1} = \cdots = .$$

It implies $I_{n+1} = 0$. Since I_i , $1 \leq i \leq n$, is a field, we have

$$A = F_1 \oplus \cdots \oplus F_n, \text{ where } I_i = F_i.$$

Also $A \in S\gamma$ and $F_i \in S\gamma$. Thus by Proposition 5, each F_i is an infinite field.

Let $A = F_1 \oplus \cdots \oplus F_n$, where F_i is an infinite field. Then by Proposition 4 each $F_i \in S\gamma$. Thus $A \in S\gamma$. $\square$

We denote by A_{ss} all associative rings and by C_{ss} all commutative associative rings. It is clear that C_{ss} is a universal subclass of A_{ss} .

Remark 1. It follows from Proposition 1 that $\mathcal{T} \cap C_{ss}$ and $\mathcal{T}_s \cap C_{ss}$ are radicals in C_{ss} .

Theorem 4. *Let $\gamma \in \{\mathcal{T}, \mathcal{T}_s\}$ and A be a subdirectly irreducible semiprime ring. Then A is $\sigma = \mathcal{L}_{A_{ss}}(\gamma \cap C_{ss})$ -semisimple if and only if A is not a finite field.*

Proof. It is easy to see that $\sigma \subseteq \gamma$ and A is a prime ring. Suppose that A is not a finite field and not σ -semisimple. Therefore $\sigma(A) \neq 0$ and $\sigma(A) \trianglelefteq A$. It is clear that $\sigma(A)$ is a subdirectly irreducible ring. Let $H(\sigma(A))$ denote the heart of $\sigma(A)$. Since $\sigma(A) \in \sigma$, there exists a nonzero accessible subring I_0 of $\sigma(A)$, which is in $\gamma \cap C_{ss}$. Since A is a prime ring $H(\sigma(A)) \subseteq I_0$. This implies that $H(\sigma(A))$ is a simple commutative ring. Therefore $F = H(\sigma(A))$ is a field. It is easy to see that $F = A$. Thus by Proposition 6, $F \in \mathcal{T}_s \cap C_{ss}$. Then by Proposition 5, A is a finite field, which leads to a contradiction. Hence A is a σ -semisimple ring.

Suppose A is σ -semisimple and $A = F$ is a finite field. Then again by Proposition 5, $A = F \in \gamma \cap C_{ss}$, which leads to a contradiction. $\square$

A class δ of rings is said to be a matrix-extensible class if for all natural numbers n , $A \in \delta$ if and only if $M_n(A) \in \delta$, where $M_n(A)$ is the $n \times n$ matrix ring.

Corollary 3. $\sigma = \mathcal{L}_{A_{ss}}(\gamma \cap C_{ss})$ is not a matrix-extensible class.

Proof. Let F be a finite field. Then $F \in \gamma \cap C_{ss}$. Thus, by Theorem 4, $M_n(F)$ is a σ -semisimple class. $\square$

For the next theorem we use the following notations:

β — Baer radical, $\mathcal{L}$ — Levitzki radical, $\mathcal{N}$ — Nil radical, $\mathcal{J}$ — Jacobson radical and $\mathcal{G}$ — Brown-McCoy radical. The notation $\alpha \uparrow\uparrow r$ means that α and r are not comparable radicals.

Theorem 5.

- (i) $\beta = \mathcal{L}_{A_{ss}}(\beta \cap C_{ss})$;
- (ii) $\mathcal{L} \neq \mathcal{L}_{A_{ss}}(\mathcal{L} \cap C_{ss}) = \beta$;
- (iii) $\mathcal{N} \neq \mathcal{L}_{A_{ss}}(\mathcal{N} \cap C_{ss}) = \beta$;
- (iv) $\mathcal{L}_{A_{ss}}(\mathcal{J} \cap C_{ss}) \uparrow\uparrow \mathcal{N}$;
- (v) $\mathcal{L}_{A_{ss}}(\mathcal{G} \cap C_{ss}) \uparrow\uparrow \mathcal{N}$.

Proof.

- (i) Let A be a ring in β . Then there exists a non-zero accessible subring I^0 of A , which is a zero ring. Hence $I^0 \in \beta \cap C_{ss}$. Thus by Theorem 1, $\beta = \mathcal{L}_{A_{ss}}(\beta \cap C_{ss})$. Also, we know that $\beta \cap C_{ss} = \mathcal{L} \cap C_{ss} = \mathcal{N} \cap C_{ss}$.
- (ii) and (iii) In [6], E.I. Zelmanov constructed a ring A , which is locally nilpotent as well as prime. So $\beta(A) = 0$ while $\mathcal{L}(A) = A$, which implies $\beta \neq \mathcal{L}$. Also, it is easy to see that $\beta \subseteq \mathcal{L}_{A_{ss}}(\mathcal{L} \cap C_{ss}) \subseteq \mathcal{L}_{A_{ss}}(\mathcal{N} \cap C_{ss})$ and $\mathcal{N} \cap C_{ss} \subseteq \beta$. Thus $\beta = \mathcal{L}_{A_{ss}}(\mathcal{L} \cap C_{ss}) = \mathcal{L}_{A_{ss}}(\mathcal{N} \cap C_{ss})$. Also, it is easy to see that $\mathcal{L}_{A_{ss}}(\mathcal{N} \cap C_{ss}) \neq \mathcal{N}$.
- (iv) We shall show that $\mathcal{L}_{A_{ss}}(\mathcal{J} \cap C_{ss}) \not\subseteq \mathcal{N}$. Let

$$J = \left\{ \frac{2x}{2y+1} \mid x, y \in \mathbb{Z}, (2x, 2y+1) = 1 \right\}.$$

Then we know that J is a commutative Jacobson radical ring. So $J \in \mathcal{J} \cap C_{ss}$, also $J \in \mathcal{L}_{A_{ss}}(\mathcal{J} \cap C_{ss})$. But J has no nonzero nilpotent elements. Thus $J \notin \mathcal{N}$. In [2] A.Smoktunowicz proved that there exists a simple nil prime ring A . It is clear that $A \notin \mathcal{L}_{A_{ss}}(\mathcal{J} \cap C_{ss})$. Thus $\mathcal{N} \not\subseteq \mathcal{L}_{A_{ss}}(\mathcal{J} \cap C_{ss})$.

(v) It is clear. □

Corollary 4. $\mathcal{J} \neq \mathcal{L}_{A_{ss}}(\mathcal{J} \cap C_{ss}) \neq \beta$ and $\mathcal{G} \neq \mathcal{L}_{A_{ss}}(\mathcal{G} \cap C_{ss}) \neq \beta$.

Corollary 5. Let $\gamma \in \{\mathcal{T}, \mathcal{T}_s\}$. Then $\beta \subsetneq \mathcal{L}_{A_{ss}}(\gamma \cap C_{ss})$.

Proof. All zero rings and all finite fields are in $\gamma \cap C_{ss}$. Thus $\beta \subsetneq \mathcal{L}_{A_{ss}}(\gamma \cap C_{ss})$. □

4. ALTERNATIVE RINGS

In this section all rings are alternative. An alternative rings is a ring in which multiplication need not be associative, only alternative, that is, $x^2y = x(xy)$ and $yx^2 = (yx)x$, for all $x, y \in A$. We denote the class of all alternative rings by Alt . Let $\mathcal{M}$ be a nonempty class of alternative rings and assume that $\mathcal{M}$ is homomorphically closed. Let us define $\mathcal{M}_1 = \mathcal{M}$. Assuming that $\mathcal{M}_\alpha$ has been defined for every ordinal number α such that $1 \leq \alpha < \beta$, we define $\mathcal{M}_\beta$ to be the class of all alternative rings A such that every nonzero homomorphic image of A contains a nonzero ideal I , which is in $\mathcal{M}_\alpha$ for some $\alpha < \beta$. It is clear that $\mathcal{M}_\alpha \leq \mathcal{M}_\beta$ if $\alpha \leq \beta$ and each class $\mathcal{M}_\alpha$ is homomorphically closed. Let $\mathcal{L}_{Alt}(\mathcal{M}) = \cup_\alpha \mathcal{M}_\alpha$. Then $\mathcal{L}_{Alt}(\mathcal{M})$ determines a radical property and this is the smallest radical class containing $\mathcal{M}$ (see [6]).

Lemma 1 ([3]). *If B is a nonzero accessible subring of an alternative ring A and if B is in $\mathcal{M}$, then $\bar{B}$, the ideal of A generated by B , is in $\mathcal{M}_{q \cdot w_0}$, where q is finite and w_0 is the first infinite ordinal.*

Proposition 7. *An alternative ring A is in $\gamma = \mathcal{L}_{Alt}(\mathcal{M})$ if and only if every nonzero homomorphic image of $\bar{A}$ of A contains a nonzero accessible subring B , such that B is in $\mathcal{M}$.*

Proof. Suppose that A is not in γ . Then $\bar{A} = A/\gamma(A) \neq \bar{0}$. By assumption, $\bar{A}$ contains a nonzero accessible subring B such that B is in $\mathcal{M}$. From Lemma 1, $\bar{B}$ is the ideal of $\bar{A}$ generated by B , which is in $\mathcal{M}_{q \cdot w_0}$. Hence $\bar{B}$ is a γ -radical ideal. Therefore $\bar{0} = \gamma(A/\gamma(A)) = \gamma(\bar{A}) \neq \bar{0}$, which leads to a contradiction. Let $A \in \gamma$. Then every nonzero homomorphic image $\bar{A}$ of A is in γ . By Lemma 1 of [2] and the proof of Theorem 3 of [3], $0 \neq \bar{A}$ contains a nonzero accessible subring B such that B is in $\mathcal{M}$. □

Theorem 6. *Let γ be a radical in Alt . Then $\mathcal{L}_{Alt}(\gamma \cap A_{ss}) = \gamma$ if and only if every nonzero ring $A \in \gamma$ contains a nonzero accessible subring $B \in \gamma \cap A_{ss}$.*

Proof. It is well known that every radical γ in Alt has the ADS property. Thus the result follows from Theorem 1 and Proposition 7. □

Corollary 6. *Let γ be a hereditary radical in Alt . Then $\mathcal{L}_{Alt}(\gamma \cap A_{ss}) = \gamma$ if and only if every nonzero ring $A \in \gamma$ contains a nonzero associative accessible subring B .*

Proof. Let γ be a radical in Alt such that $\mathcal{L}_{Alt}(\gamma \cap A_{ss}) = \gamma$. By Theorem 6, every nonzero ring $A \in \gamma$ contains a nonzero accessible subring B such that $B \in \gamma \cap A_{ss}$. Thus B is an associative ring.

Let A be a nonzero ring in γ . Then, by the assumption, there exists a nonzero associative accessible subring B such that $B = B_1 \trianglelefteq B_2 \trianglelefteq \cdots \trianglelefteq B_{n-1} \trianglelefteq B_n = A$. Since γ is a hereditary radical, $B_n \in \gamma$. Also $B_{n-1} \in \gamma \dots B_1 \in \gamma$. Hence $B_1 = B \in \gamma \cap A_{ss}$. Therefore, by Theorem 6, we have $\mathcal{L}_{Alt}(\gamma \cap A_{ss}) = \gamma$. $\square$

Let us denote by $\mathcal{B}$ the Baer radical in Alt . We define a chain of subsets in a ring A by setting $A^{(1)} = A^2, \dots, A^{(n)} = (A^{(n-1)})^2$. We recall that a ring A is solvable if $A^{(n)} = 0$, for some n . Also it is clear that $A^{(n)} \trianglelefteq A^{(n-1)}$.

Remark 2. From the definition of $\mathcal{B}$ it is easy to prove that $\mathcal{B}$ is generated by all solvable alternative rings. (see [6]).

Corollary 7. $\beta = \mathcal{B} \cap A_{ss}$ and $\mathcal{L}_{Alt}(\beta) = \mathcal{B}$.

Proof. By Remark 2 $\beta \subseteq \mathcal{B}$ and $\beta \subseteq A_{ss}$. Hence $\beta \subseteq \mathcal{B} \cap A_{ss}$. It is easy to see that $\mathcal{B} \cap A_{ss} \subseteq \beta$. Hence $\beta = \mathcal{B} \cap A_{ss}$. By Theorem 9 of [6] $\mathcal{B}$ is a hereditary radical. By Proposition 7 and Remark 2, for every ring $A \in \mathcal{B}$ there exists an accessible subring B of A such that B is a solvable ring, that is $B^k = 0$ and $B^{k-1} \neq 0$. Since B is an accessible subring of A , B^{k-1} is an accessible subring of A . But B^{k-1} is a zero ring. Thus $0 \neq B^{k-1} \in A_{ss}$. By Corollary 6, $\mathcal{L}_{Alt}(\beta) = \mathcal{B}$, because $\mathcal{B}$ is a hereditary radical. $\square$

Remark 3. $\mathcal{T} = \mathcal{T}_s$ if and only if $\mathcal{L}_{Alt}(\mathcal{T}) = \mathcal{L}_{Alt}(\mathcal{T}_s)$.

Indeed, by Proposition 2, $\mathcal{T} = \mathcal{L}_{Alt}(\mathcal{T}) \cap A_{ss} = \mathcal{L}_{Alt}(\mathcal{T}_s) \cap A_{ss} = \mathcal{T}_s$.

ACKNOWLEDGEMENT

The authors thanks to the referee for his/her valuable comments for improving this paper.

The second and third authors were supported in part by the Science and Technology Fund of Mongolia, Grant No.Shuss 2017/64

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SOME APPLICATIONS OF MAIA'S FIXED POINT THEOREM IN THE CASE OF OPERATORS WITH VOLTERRA PROPERTY WITH RESPECT TO A SUBINTERVAL

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Received 08 September, 2019

Abstract. The aim of this paper is to improve some fixed point results given by I. A. Rus in (*On a fixed point theorem of Maia*, Studia Univ. Babeş-Bolyai Math., 22(1977), 40-42), (*Basic problem for Maia's theorem*, Sem. on Fixed Point Theory, Preprint 3(1981), Babeş-Bolyai Univ. Cluj-Napoca, 112-115) and (*Data dependence of the fixed points in a set with two metrics*, Fixed Point Theory, 8(2007), no. 1, 115-123), by applying, instead of contraction principle, some variants of Maia's fixed point theorem.

2010 *Mathematics Subject Classification*: 47H10; 47H09; 34K05; 34K12; 45D05; 45G10; 54H25

Keywords: G -contraction, fiber contraction, Bielecki norm, Picard operator, operator with Volterra property, Maia's fixed point theorem

1. INTRODUCTION AND PRELIMINARIES

By following the paper of I. A. Rus [15], we give some variants of Maia's fixed point theorem and an application of these variants, concerning the existence and uniqueness of fixed points for some integral equations. Our results improve some fixed point results given in [6, 7] and [12].

First, let us recall gradually some notions, notations and results which will be used in the sequel of this paper. We will start with the notion of L -space, the most simple structure that allows us to present the Picard and weakly Picard operators. Notice that we do not need the context of metric space to do this, but we will use this context to recall the following notion and results: the G -contraction, the Fiber Contraction Principle and the Maia's fixed point theorem.

1.1. L -space

The notion of L -space was introduced in 1906 by M. Fréchet in [3]. It is an abstract space in which works one of the basic tools in the theory of operatorial equations, especially in the fixed point theory: the sequence of successive approximations method.

Let X be a nonempty set. Let $s(X) := \{\{x_n\}_{n \in \mathbb{N}} \mid x_n \in X, n \in \mathbb{N}\}$. Let $c(X)$ be a subset of $s(X)$ and $\text{Lim} : c(X) \rightarrow X$ be an operator. By definition the triple $(X, c(X), \text{Lim})$ is called an L -space (denoted by $(X, \rightarrow)$) if the following conditions are satisfied:

- (i) if $x_n = x$, for all $n \in \mathbb{N}$, then $\{x_n\}_{n \in \mathbb{N}} \in c(X)$ and $\text{Lim}\{x_n\}_{n \in \mathbb{N}} = x$.
- (ii) if $\{x_n\}_{n \in \mathbb{N}} \in c(X)$ and $\text{Lim}\{x_n\}_{n \in \mathbb{N}} = x$, then for all subsequences $\{x_{n_i}\}_{i \in \mathbb{N}}$ of $\{x_n\}_{n \in \mathbb{N}}$, we have that $\{x_{n_i}\}_{i \in \mathbb{N}} \in c(X)$ and $\text{Lim}\{x_{n_i}\}_{i \in \mathbb{N}} = x$.

A simple example of an L -space is the pair $(X, \xrightarrow{d})$, where X is a nonempty set and $\xrightarrow{d}$ is the convergence structure induced by a metric d on X .

In general, an L -space is any nonempty set endowed with a structure implying a notion of convergence for sequences. Other examples of L -spaces are: Hausdorff topological spaces, generalized metric spaces ($d(x, y) \in \mathbb{R}_+^m$ or $d(x, y) \in \mathbb{R}_+ \cup \{+\infty\}$), K -metric spaces ($d(x, y) \in K$, where K is a cone in an ordered Banach space), gauge spaces, 2-metric spaces, D - R -spaces, probabilistic metric spaces, syntopogenous spaces.

1.2. Picard operators and weakly Picard operators

Let $(X, \rightarrow)$ be an L -space. An operator $f : X \rightarrow X$ is called weakly Picard operator (WPO) if the sequence of successive approximations, $\{f^n(x)\}_{n \in \mathbb{N}}$, converges for all $x \in X$ and its limit (which generally depend on x) is a fixed point of f .

If an operator f is WPO and the fixed point set of f is a singleton, $F_f = \{x^*\}$, then by definition, f is called Picard operator (PO).

For a WPO , $f : X \rightarrow X$, we define the operator $f^\infty : X \rightarrow X$, by $f^\infty(x) := \lim_{n \rightarrow \infty} f^n(x)$. Notice that, $f^\infty(X) = F_f$, i.e., f^∞ is a set retraction of X on F_f .

If X is a nonempty set, then the triple $(X, \rightarrow, \leq)$ is an ordered L -space if $(X, \rightarrow)$ is an L -space and $\leq$ is a partial order relation on X which is closed with respect to the convergence structure of the L -space.

In the setting of ordered L -spaces, we have some properties concerning WPO s and PO s.

Abstract Gronwall Lemma. Let $(X, \rightarrow, \leq)$ be an ordered L -space and $f : X \rightarrow X$ be an increasing WPO . Then

- (i) $\forall x \in X, x \leq f(x) \Rightarrow x \leq f^\infty(x)$;
- (ii) $\forall x \in X, x \geq f(x) \Rightarrow x \geq f^\infty(x)$.

Abstract Comparison Lemma. Let $(X, \rightarrow, \leq)$ be an ordered L -space and $f, g, h : X \rightarrow X$ be such that

- (1) $f \leq g \leq h$;
- (2) the operators f, g, h are WPO s;
- (3) the operator g is increasing.

Then $\forall x, y, z \in X, x \leq y \leq z \Rightarrow f^\infty(x) \leq g^\infty(y) \leq h^\infty(z)$.

Regarding the theory of *WPOs* and *POs* see [8–10, 13, 14, 16, 17] and [1].

1.3. *G*-contractions

Let (X, d) be a metric space and $G \subset X \times X$ be a nonempty subset. An operator $f : X \rightarrow X$ is a *G*-contraction if there exists $l \in]0, 1[$ such that

$$d(f(x), f(y)) \leq ld(x, y), \forall (x, y) \in G.$$

Notice that if $G = X \times X$ then the notion of *G*-contraction is identical with the notion of contraction (or *l*-contraction, if we want explicitly to mention the contraction constant l), i.e., there exists $l \in]0, 1[$ such that, $d(f(x), f(y)) \leq ld(x, y)$, $\forall x, y \in X$.

Example 1. Let $a, b, c \in \mathbb{R}$, $a < c < b$. We consider the set of all continuous real-valued functions defined on the interval $[a, b]$, $X := C[a, b]$, endowed with the metric

$$\rho(x, y) := \left(\int_a^b |x(s) - y(s)|^2 ds \right)^{\frac{1}{2}}.$$

For $K, H \in C([a, b] \times [a, b] \times \mathbb{R}, \mathbb{R})$, let $f : C[a, b] \rightarrow C[a, b]$ be defined by

$$f(x)(t) := \int_a^c K(t, s, x(s)) ds + \int_a^t H(t, s, x(s)) ds, \quad t \in [a, b].$$

We suppose that there exists $L_H \in C([a, b] \times [a, b])$ such that

$$|H(t, s, u) - H(t, s, v)| \leq L_H(t, s)|u - v|, \quad \forall t, s \in [a, b], \quad \forall u, v \in \mathbb{R}.$$

If $G := \{(x, y) \in C[a, b] \times C[a, b] \mid x|_{[a, c]} = y|_{[a, c]}\}$ and $\int_c^b \int_c^b |L_H(t, s)|^2 dt ds < 1$, then f is a *G*-contraction with respect to ρ .

Indeed, let $(x, y) \in G$. Then, $\rho(f(x), f(y)) = \left(\int_c^b |f(x)(t) - f(y)(t)|^2 dt \right)^{\frac{1}{2}}$ and

$$\begin{aligned} |f(x)(t) - f(y)(t)| &\leq \int_c^t |H(t, s, x(s)) - H(t, s, y(s))| ds \\ &\leq \int_c^t L_H(t, s) |x(s) - y(s)| ds \\ &\stackrel{\text{Hölder's inequality}}{\leq} \left(\int_c^t |L_H(t, s)|^2 ds \right)^{\frac{1}{2}} \left(\int_c^t |x(s) - y(s)|^2 ds \right)^{\frac{1}{2}}. \end{aligned}$$

Hence,

$$\begin{aligned} \rho(f(x), f(y)) &\leq \left(\int_c^b \left(\int_c^t |L_H(t, s)|^2 ds \right) \rho(x, y)^2 dt \right)^{\frac{1}{2}} \\ &= \left(\int_c^b \int_c^b |L_H(t, s)|^2 dt ds \right)^{\frac{1}{2}} \rho(x, y), \quad \text{for all } (x, y) \in G. \end{aligned}$$

For other examples of *G*-contractions see [11, 15] and [16], pp. 282–284.

1.4. Fiber Contraction Principle

Another important result (see [13, 17]) concerning *WPOs* and *POs* is the following **Fiber Contraction Theorem**. Let $(X, \rightarrow)$ be an *L-space*, (Y, ρ) be a metric space, $g : X \rightarrow X$, $h : X \times Y \rightarrow Y$ and $f : X \times Y \rightarrow X \times Y$, $f(x, y) := (g(x), h(x, y))$. We suppose that

- (1) (Y, ρ) is a complete metric space;
- (2) g is *WPO*;
- (3) $h(x, \cdot) : Y \rightarrow Y$ is a contraction, $\forall x \in X$;
- (4) $h : X \times Y \rightarrow Y$ is continuous.

Then, f is *WPO*. Moreover, if g is a *PO*, then f is a *PO*.

1.5. Maia's fixed point theorem

The following result was introduced by M. G. Maia in [4].

Theorem 1. Let X be a nonempty set, d and ρ be two metrics on X and $V : X \rightarrow X$ be an operator. We suppose that

- (1) there exists $c > 0$ such that, $d(x, y) \leq c\rho(x, y)$, $\forall x, y \in X$;
- (2) (X, d) is a complete metric space;
- (3) $V : (X, d) \rightarrow (X, d)$ is continuous;
- (4) $V : (X, \rho) \rightarrow (X, \rho)$ is a contraction.

Then

- (i) $F_V = \{x^*\}$;
- (ii) $V : (X, d) \rightarrow (X, d)$ is *PO*.

Maia's Theorem 1 remains true if we replace the condition (1) with the following one:

- (1') there exists $c > 0$ such that, $d(V(x), V(y)) \leq c\rho(x, y)$, $\forall x, y \in X$.

Hence, we obtain the so called Rus' variant of Maia's fixed point theorem. More considerations can be found in [5–8, 12] and [2].

2. FIXED POINT THEOREMS FOR OPERATORS WITH VOLTERRA PROPERTY WITH RESPECT TO A SUBINTERVAL

By following [15], we present first some notions and notations.

Let $a, b, c \in \mathbb{R}$, with $a < c < b$.

Let $C[a, b] := \{f : [a, b] \rightarrow \mathbb{R} \mid f \text{ is continuous on } [a, b]\}$.

On $C[a, b]$ and $C[a, c]$ we consider norms of uniform convergence such as the max-norms, $\|\cdot\|$, or the Bielecki norm, $\|\cdot\|_\tau$.

In $C[a, b] \times C[a, b]$ we consider the subset

$$G := \{(x, y) \in C[a, b] \times C[a, b] \mid x|_{[a, c]} = y|_{[a, c]}\}$$

and for each $x \in C[a, b]$ we consider the subset

$$X_x := \{y \in C[a, b] \mid y|_{[a, c]} = x\}.$$

Definition 1. An operator, $V : C[a, b] \rightarrow C[a, b]$, has the Volterra property on the subinterval $[a, c]$, if the following implication holds

$$x, y \in C[a, b], x|_{[a, c]} = y|_{[a, c]} \Rightarrow V(x)|_{[a, c]} = V(y)|_{[a, c]}.$$

Definition 2. An operator, $V : C[a, b] \rightarrow C[a, b]$, has the Volterra property if it has the Volterra property on each subinterval $[a, t]$, for $a < t < b$.

For example, in the terms of the above considerations, let $K, H \in C([a, b] \times [a, b] \times \mathbb{R}, \mathbb{R})$ and $V : C[a, b] \rightarrow C[a, b]$ be defined by

$$V(x)(t) := \int_a^c K(t, s, x(s))ds + \int_a^t H(t, s, x(s))ds, \quad t \in [a, b].$$

This operator V has the Volterra property on the subinterval $[a, c]$, but V has not the Volterra property.

Remark 1. If $V : C[a, b] \rightarrow C[a, b]$ is an operator with Volterra property on $[a, c]$, then the operator V induces an operator V_1 on $C[a, c]$, defined by

$$V_1(x) := V(\tilde{x})|_{[a, c]}, \text{ where } \tilde{x} \in C[a, b] \text{ with } \tilde{x}|_{[a, c]} = x.$$

Remark 2. If $V : C[a, b] \rightarrow C[a, b]$ has the Volterra property on $[a, c]$ and V is a G -contraction, then the operator

$$V|_{X_x} : X_x \rightarrow X_{V_1(x)},$$

is a contraction on X_x , for all $x \in C[a, c]$.

If $x^* \in F_{V_1}$, then $V(X_{x^*}) \subset X_{x^*}$.

The first abstract main result of our paper is the following.

Theorem 2. Let $a, b, c \in \mathbb{R}$, with $a < c < b$. On $C[a, b]$ we consider the max-norm, $\|\cdot\|$ (or the Bielecki norm, $\|\cdot\|_\tau$). On $C[a, c]$ we consider the max-norm, $\|\cdot\|$ and a metric $\rho : C[a, c] \times C[a, c] \rightarrow \mathbb{R}_+$.

Let $V : C[a, b] \rightarrow C[a, b]$ be an operator. We suppose that

- (1) V has the Volterra property on $[a, c]$;
- (2) V_1 is a contraction with respect to $\|\cdot\|$;
- (3) $\exists M > 0$ such that $\|x - y\| \leq M\rho(x, y)$, $\forall x, y \in C[a, c]$;
- (4) V is a G -contraction with respect to ρ .

Then

- (i) $F_V = \{x^*\}$;
- (ii) V_1 is PO with respect to the uniform convergence and $x^*|_{[a, c]} = V_1^\infty(x)$, $\forall x \in C[a, c]$;

$$(iii) \ x^* = V^\infty(x), \forall x \in X_{x^*}|_{[a,c]}.$$

Proof. Let $x_0 \in C[a, c]$. By (1) and (4), V_1 is a contraction with respect to ρ on $C[a, c]$. By following the proof of Maia's Theorem 1, it can be shown that the sequence of successive approximations $\{V_1^n(x_0)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $(C[a, c], \rho)$. By (3), we get that $\{V_1^n(x_0)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $(C[a, c], \|\cdot\|)$, so it converges uniformly in the Banach space $(C[a, c], \|\cdot\|)$, to an element $x_1^* \in C[a, c]$, i.e., $V_1^\infty(x_0) = x_1^*$. By (2), $x_1^* \in F_{V_1}$ and by (4) we get $F_{V_1} = \{x_1^*\}$. Hence, V_1 is *PO* with respect to the uniform convergence on $(C[a, c], \|\cdot\|)$.

By (1), (4) and Remark 2, the operator $W := V|_{X_{x_1^*}} : X_{x_1^*} \rightarrow X_{x_1^*}$ is a contraction with respect to ρ , where $X_{x_1^*} = \{y \in C[a, b] \mid y|_{[a,c]} = x_1^*\}$.

Let $y_0 \in X_{x_1^*}$. Since W is a contraction with respect to ρ , it can be shown that $\{W^n(y_0)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $(X_{x_1^*}, \rho)$, and by (3), we get that $\{W^n(y_0)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in the Banach space $(X_{x_1^*}, \|\cdot\|)$, i.e., it converges uniformly to an element $x^* \in X_{x_1^*}$. Since W is a contraction, W is also continuous, so $F_W = \{x^*\}$. Hence, $x^*|_{[a,c]} = x_1^*$.

From these we have (i), (ii) and (iii). $\square$

From the above theorem, the following conjecture arises.

Conjecture 1. *In the conditions of Theorem 2, the operator V is *PO* with respect to the uniform convergence on $(C[a, b], \|\cdot\|)$, i.e., $x^* = V^\infty(x)$, $\forall x \in C[a, b]$.*

Also, we have the Rus' variant for Theorem 2.

Theorem 3. *Let $a, b, c \in \mathbb{R}$, with $a < c < b$. On $C[a, b]$ we consider the max-norm, $\|\cdot\|$ (or the Bielecki norm, $\|\cdot\|_\tau$). On $C[a, c]$ we consider the max-norm, $\|\cdot\|$ and a metric $\rho : C[a, c] \times C[a, c] \rightarrow \mathbb{R}_+$.*

Let $V : C[a, b] \rightarrow C[a, b]$ be an operator. We suppose that

- (1) *V has the Volterra property on $[a, c]$;*
- (2) *V_1 is a contraction with respect to $\|\cdot\|$;*
- (3) *$\exists M > 0$ such that $\|V_1(x) - V_1(y)\| \leq M\rho(x, y)$, $\forall x, y \in C[a, c]$;*
- (4) *V is a G -contraction with respect to ρ .*

Then

- (i) $F_V = \{x^*\}$;
- (ii) V_1 is *PO* with respect to the uniform convergence and $x^*|_{[a,c]} = V_1^\infty(x)$, $\forall x \in C[a, c]$;
- (iii) $x^* = V^\infty(x)$, $\forall x \in X_{x^*}|_{[a,c]}.$

Proof. Let $x_0 \in C[a, c]$. By (4), $\{V_1^n(x_0)\}_{n \in \mathbb{N}}$ is a Cauchy sequence with respect to ρ . On the other hand, for $n, p \in \mathbb{N}$, we have:

$$\rho(V_1^{n+p}(x_0), V_1^n(x_0)) \leq \rho(V_1^{n+p}(x_0), V_1^{n+p-1}(x_0)) + \dots + \rho(V_1^{n+1}(x_0), V_1^n(x_0))$$

$$\begin{aligned} &\leq l^{n+p-1} \rho(V_1(x_0), x_0) + \dots + l^n \rho(V_1(x_0), x_0) \\ &\leq \frac{l^n}{1-l} \rho(V_1(x_0), x_0) \rightarrow 0 \text{ as } n, p \rightarrow \infty. \end{aligned}$$

By (3), it follows that $\|V_1^{n+1}(x_0), V_1^{n+p+1}(x_0)\| \rightarrow 0$ as $n, p \rightarrow \infty$. Hence, $\{V_1^n(x_0)\}_{n \in \mathbb{N}}$ is a Cauchy sequence with respect to $\|\cdot\|$. We follow the proof of Theorem 2. $\square$

3. APPLICATIONS

In this section, we present an application for Theorem 2, Fiber Contraction Theorem and Abstract Gronwall Lemma.

Let $a, b, c \in \mathbb{R}$, with $a < c < b$.

For $K, H \in C([a, b] \times [a, b] \times \mathbb{R}, \mathbb{R})$, we consider the following functional integral equation

$$x(t) = \int_a^c K(t, s, x(s)) ds + \int_a^t H(t, s, x(s)) ds, \quad t \in [a, b]. \quad (3.1)$$

We are looking for solutions for this equation in $C[a, b]$.

We suppose that there exists $L_K \in C([a, b] \times [a, b])$ such that

$$|K(t, s, u) - K(t, s, v)| \leq L_K(t, s) |u - v|, \quad \forall t \in [a, b], \forall s \in [a, c], \forall u, v \in \mathbb{R}$$

and there exists $L_H \in C([a, b] \times [a, b])$ such that

$$|H(t, s, u) - H(t, s, v)| \leq L_H(t, s) |u - v|, \quad \forall t, s \in [a, b], \forall u, v \in \mathbb{R}.$$

In addition, we suppose that $\int_c^b \int_c^b |L_H(t, s)|^2 dt ds < 1$.

Let $V : C[a, b] \rightarrow C[a, b]$ be the operator defined by

$$V(x)(t) := \int_a^c K(t, s, x(s)) ds + \int_a^t H(t, s, x(s)) ds, \quad \forall t \in [a, b].$$

The operator V_1 , induced by V on the interval $[a, c]$ is

$$V_1(x)(t) := \int_a^c K(t, s, x(s)) ds + \int_a^t H(t, s, x(s)) ds, \quad \forall t \in [a, c].$$

On $C[a, b]$ we consider the max-norm, $\|x - y\| := \max_{t \in [a, b]} |x(t) - y(t)|$ and the metric

$$\rho(x, y) := \left(\int_a^b |x(t) - y(t)|^2 ds \right)^{\frac{1}{2}}.$$

It is clear that V has the Volterra property on $[a, c]$ and V_1 is continuous with respect to $\|\cdot\|$.

Let $x, y \in C[a, c]$. We have

$$\begin{aligned} |V_1(x)(t) - V_1(y)(t)| &\leq \int_a^c |K(t, s, x(s)) - K(t, s, y(s))| ds \\ &\quad + \int_a^t |H(t, s, x(s)) - H(t, s, y(s))| ds \end{aligned}$$

$$\begin{aligned}
&\leq \int_a^c L_K(t, s)|x(s) - y(s)|ds + \int_a^t L_H(t, s)|x(s) - y(s)|ds \\
&\stackrel{\text{Hölder's inequality}}{\leq} \left(\int_a^c |L_K(t, s)|^2 ds \right)^{\frac{1}{2}} \left(\int_a^c |x(s) - y(s)|^2 ds \right)^{\frac{1}{2}} + \\
&\quad + \left(\int_a^t |L_H(t, s)|^2 ds \right)^{\frac{1}{2}} \left(\int_a^t |x(s) - y(s)|^2 ds \right)^{\frac{1}{2}} \\
&\leq \left(\left(\int_a^c |L_K(t, s)|^2 ds \right)^{\frac{1}{2}} + \left(\int_a^t |L_H(t, s)|^2 ds \right)^{\frac{1}{2}} \right) \rho(x, y).
\end{aligned}$$

By taking the $\max_{t \in [a, c]}$ in the above inequalities, there exists

$$M := \max_{t \in [a, c]} \left(\left(\int_a^c |L_K(t, s)|^2 ds \right)^{\frac{1}{2}} + \left(\int_a^t |L_H(t, s)|^2 ds \right)^{\frac{1}{2}} \right) > 0$$

such that $\|V_1(x) - V_1(y)\| \leq M\rho(x, y)$, for all $x, y \in C[a, c]$.

On the other hand, by Example 1, the operator V is a G -contraction with respect to ρ , where $G := \{(x, y) \in C[a, b] \times C[a, b] \mid x|_{[a, c]} = y|_{[a, c]}\}$.

By applying Theorem 3, it follows that equation (3.1) has a unique solution x^* in $C[a, b]$. Moreover, for $t \in [a, c]$, $x^*(t) = \lim_{n \rightarrow \infty} x_n(t)$, for each $x_0 \in C[a, c]$, where $\{x_n\}_{n \in \mathbb{N}}$ is defined by

$$x_{n+1}(t) = \int_a^c K(t, s, x_n(s))ds + \int_a^t H(t, s, x_n(s))ds,$$

and for $t \in [a, b]$, $x^*(t) = \lim_{n \rightarrow \infty} y_n(t)$, where $\{y_n\}_{n \in \mathbb{N}}$ is defined by

$$y_0 \in C[a, b], \text{ with } y_0|_{[a, c]} = x^*|_{[a, c]},$$

and

$$y_{n+1}(t) = \int_a^c K(t, s, x^*(s))ds + \int_a^t H(t, s, y_n(s))ds.$$

Remark 3. In the case of the operator V , considered above, Conjecture 1 is a theorem. Indeed, let $X := C[a, c]$, $Y := C[c, b]$ and $C[a, b]$ be endowed with max-norm. We take, $g : C[a, c] \rightarrow C[a, c]$, defined by $g := V_1$ and $h : C[a, c] \times C[c, b] \rightarrow C[c, b]$ be defined by

$$h(x, y)(t) := \int_a^c K(t, s, x(s))ds + \int_a^c H(t, s, x(s))ds + \int_c^t H(t, s, y(s))ds.$$

We remark that g is PO when

$$\max_{t \in [a, c]} \left(\int_a^c L_K(t, s)ds + \int_a^t L_H(t, s)ds \right) < 1 \quad (3.2)$$

and the operator $h(x, \cdot) : C[c, b] \rightarrow C[c, b]$ is a contraction, when

$$l := \max_{t \in [c, b]} \int_c^t L_H(t, s) ds < 1. \quad (3.3)$$

By the Fiber Contraction Theorem, in the conditions (3.2) and (3.3), the operator $f : C[a, c] \times C[c, b] \rightarrow C[a, c] \times C[c, b]$, defined by $f(x, y) = (g(x), h(x, y))$, for all $(x, y) \in C[a, c] \times C[c, b]$ is *PO*.

Now, let

$$x_0 \in C[a, c], \quad x_{n+1} = g(x_n), \quad n \in \mathbb{N},$$

and

$$y_0 \in C[c, b], \quad y_{n+1} = h(x_n, y_n), \quad n \in \mathbb{N}.$$

Then, $x_n \rightarrow x^*|_{[a, c]}$ as $n \rightarrow \infty$ and $y_n \rightarrow x^*|_{[c, b]}$ as $n \rightarrow \infty$.

We denote

$$u_n(t) = \begin{cases} x_n(t), & t \in [a, c], \\ y_n(t), & t \in [c, b]. \end{cases}$$

Then $u_n \in C[a, b]$, for $n \in \mathbb{N}^*$, and, $u_{n+1} = V(u_n)$ with $u_n \rightarrow x^*$ as $n \rightarrow \infty$, i.e., V is a *PO*.

This result is very important because we can apply for V , the Abstract Gronwall Lemma. So we have:

Theorem 4. *Let us consider equation (3.1) in conditions (3.2), (3.3) and $K(t, s, \cdot)$, $H(t, s, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ be increasing functions, for all $t, s \in [a, b]$. Let us denote by x^* the unique solution of (3.1). Then the following implications hold:*

- (i) $x \in C[a, b]$, $x(t) \leq \int_a^c K(t, s, x(s)) ds + \int_a^t H(t, s, x(s)) ds$, $t \in [a, b]$, $\Rightarrow x \leq x^*$;
- (ii) $x \in C[a, b]$, $x(t) \geq \int_a^c K(t, s, x(s)) ds + \int_a^t H(t, s, x(s)) ds$, $t \in [a, b]$, $\Rightarrow x \geq x^*$.

Also, from the Abstract Comparison Lemma it can be established a comparison result for equation (3.1).

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GLOBAL STABILITY AND BIFURCATION ANALYSIS OF A DISCRETE TIME SIR EPIDEMIC MODEL

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Received 07 September, 2020

Abstract. In this paper, we study the complex dynamical behaviors of a discrete-time SIR epidemic model. Analysis of the model demonstrates that the Diseases Free Equilibrium (DFE) point is globally asymptotically stable if the basic reproduction number is less than one while the Endemic Equilibrium (EE) point is globally asymptotically stable if the basic reproduction number is greater than one. The results are further substantiated visually with numerical simulations. Furthermore, numerical results demonstrate that the discrete model has more complex dynamical behaviors including multiple periodic orbits, quasi-periodic orbits and chaotic behaviors. The maximum Lyapunov exponent and chaotic attractors also confirm the chaotic dynamical behaviors of the model.

2010 *Mathematics Subject Classification:* 39A30; 92D25

Keywords: discrete epidemic model, stability, equilibrium point, bifurcation, chaos

1. INTRODUCTION

For centuries, infectious diseases have ranked with wars and famine as major challenges to human progress and survival [8]. The Black Death of fourteen century and the 1918-1920, the SARS of 2003 and the global outbreak of covid-19 recently, which have a huge impact on the world economy and people's health without doubt. To prevent and control the spread of infectious disease, we must understand its pathogenesis. Epidemic dynamical model is one of the most useful tools to understand the pathogenesis of diseases. Kermack and McKendrick pioneered compartment model to predict the spread of disease. Hence then, various epidemic models are studied to control infectious disease [3],[4],[5],[6],[7],[11],[12],[15].

Generally, epidemic dynamical models have two kinds: one is continuous-time models described by differential equations, another is discrete-time models described by difference equations. When the size of population is rarely small or the population has no overlapping generation, the discrete-time models are more appropriate than the continuous ones. In reality, most fish and insect populations have no overlap between successive generations, and thus their population evolves in discrete-time steps [1],

[13],[17]. In theory, discrete-time system has much more richer dynamical behaviors than the continuous system.

Recently, many authors studied the dynamical behaviors of discrete epidemic model. Hu et al. [19] discussed the dynamical behaviors of a discrete SIR epidemic model, which derived by applying Euler scheme to corresponding continuous SIR epidemic model. Cao et al. [9] investigated a discrete SIR epidemic model with bilinear incidence rate and constant recovery. These all shows that the stability of discrete model determined by the basic reproduction number and, discrete model undergoes bifurcation and exhibit complex dynamical behaviors, such as the period-doubling bifurcation in period-2,4,8, quasi-periodic orbits and chaotic sets.

In this paper, we mainly study the stability and bifurcation of the following discrete SIR epidemic model by including vaccination to the model given by [16]

$$\begin{aligned} S_{t+1} &= (1-p)S_t - \frac{\alpha}{N}I_t S_t + \beta(R_t + I_t), \\ I_{t+1} &= \frac{\alpha}{N}I_t S_t + (1-\beta-\gamma)I_t, \\ R_{t+1} &= (1-\beta)R_t + \gamma I_t + pS_t, \end{aligned} \quad (1.1)$$

with initial conditions S_0, I_0 and R_0 , ($S_0 + I_0 + R_0 = N$), which are positive real numbers. Here $0 < p + \alpha < 1$ and $0 < \beta + \gamma < 1$. Also, β is the probability of death which is equal to the probability of birth, γ is the probability of recovery, p is the proportion vaccinated, α is the contact rate, and N is the total population size.

The organization of this paper is as follows. We first give the existence and stability of equilibria in section 2. Then, the numerical simulations are stated in section 3. In section 4, we simulate the complex dynamical behaviors including multiple periodic orbits, quasi-periodic orbits and chaotic behaviors. Finally, we give a discussion in the last section.

2. EXISTENCE AND STABILITY OF EQUILIBRIA

Since $S_n + I_n + R_n = N$ is a constant, then the dynamical behaviors of model (1.1) is equivalent to the dynamical behaviors of the following model

$$\begin{aligned} S_{t+1} &= (1-p)S_t - \frac{\alpha}{N}I_t S_t + \beta(N - S_t), \\ I_{t+1} &= \frac{\alpha}{N}I_t S_t + (1-\beta-\gamma)I_t, \end{aligned} \quad (2.1)$$

where p, α, β , and γ are positive parameters.

On examining system (2.1), the equilibrium points $P_0 = \left(\frac{\beta N}{\beta+p}, 0\right)$ and $P_1 = \left(\frac{(\beta+\gamma)N}{\alpha}, \frac{N(\alpha\beta-(\beta+\gamma)(p+\beta))}{\alpha(\beta+\gamma)}\right)$ has been obtained in [14] by using $S_t = S_{t+1} = S^*$ and $I_t = I_{t+1} = I^*$.

The condition for the equilibrium to be locally asymptotically stable is presented in the following lemma.

Lemma 1. [2] Assume that $X_{n+1} = F(X_n)$, $n = 0, 1, \dots$ be a system of difference equations and $\bar{X}$ is a steady state point of F . If all eigenvalues of the Jacobian matrix J_F about the steady state $\bar{X}$ lie inside the open unit disk $|\lambda| < 1$, then $\bar{X}$ is locally asymptotically stable. If one of them has absolute value greater than one, then $\bar{X}$ is unstable.

Definition 1. [10] Suppose x is an equilibrium of the difference equation,

$$x_{t+1} = f(x_t)$$

where $f: [0, a) \rightarrow [0, a)$, $0 < a \leq \infty$. Then $\bar{x}$ is said to be globally attractive if for all initial conditions $x_0 \in (0, a)$, $\lim_{t \rightarrow \infty} x_t = \bar{x}$ and is said to be globally asymptotically stable if $\bar{x}$ is globally attractive and if $\bar{x}$ is locally stable.

The theorem signifies the result on the roots of the polynomial based on trace and determinant of the jacobian matrix.

Theorem 1. [18] The characteristic polynomial

$$F(x) = x^2 + Bx + C,$$

has all its roots inside the unit open disk ($|x| < 1$) if and only if

- (i) $F(1) > 0$ and $F(-1) > 0$;
- (ii) $D_1^+ = 1 + C > 0$ and $D_1^- = 1 - C > 0$

where $-B$ is the trace of Jacobian matrix and C is the determinant of Jacobian matrix.

Now, we investigate the global stability of these equilibrium points. For analyzing the global stability of disease-free equilibrium $P_0 = \left(\frac{\beta N}{\beta + p}, 0\right)$, we give the following theorems.

Theorem 2. Assume that $0 < \beta + p < 1$. The disease-free equilibrium (DFE) point $P_0 = \left(\frac{\beta N}{\beta + p}, 0\right)$ of system (2.1) is globally asymptotically stable if,

$$R_0 = \frac{\alpha\beta}{(\beta + p)(\beta + \gamma)} < 1. \quad (2.2)$$

P_0 is unstable if $R_0 > 1$.

Proof. By considering (2.1), we can get the Jacobian matrix as

$$J_{P_0} = \begin{pmatrix} 1 - p - \beta & \frac{-\alpha\beta}{(\beta + p)} \\ 0 & \frac{\alpha\beta}{(\beta + p)} + (1 - \beta - \gamma) \end{pmatrix},$$

evaluated at P_0 , the eigenvalues are

$$\lambda_1 = 1 - p - \beta, \quad \lambda_2 = \frac{\alpha\beta}{(\beta + p)} + (1 - \beta - \gamma).$$

The conditions $0 < \beta + p < 1$, $0 < \beta + \gamma < 1$ and $R_0 < 1$ implies that $|\lambda_1| < 1$ and $|\lambda_2| = |1 - (\beta + \gamma)(1 - R_0)| < 1$. It follows from Lemma 1 that the DFE point P_0 is locally asymptotically stable if $R_0 < 1$. When $R_0 > 1$, we have $\lambda_2 = 1 - (\beta + \gamma)(1 - R_0) > 1$, then the DFE point P_0 is unstable. From the first equation of system (2.1), it is easy to see that S_t satisfies the following inequality:

$$S_t \leq \beta N + (1 - p - \beta)S_{t-1}.$$

The equation $u_t = \beta N + (1 - p - \beta)u_{t-1}$ has a unique equilibrium point $u^* = \frac{\beta N}{p + \beta}$, which is globally asymptotically stable. From inequality (2.2) and the comparison principle, we know that for any small ε_1 , there exists a positive integer T_1 such that $S_t \leq \frac{\beta N}{p + \beta} + \varepsilon_1$ for all $t > T_1$. From the second equation of system (2.1) and $S_t \leq \frac{\beta N}{p + \beta} + \varepsilon_1$, $t > T_1$, we have

$$I_t \leq \left(1 - (\beta + \gamma)(1 - R_0) + \varepsilon_1 \frac{\alpha}{N}\right) I_{t-1}, \text{ for } t > T_1.$$

The conditions $0 < \beta + \gamma < 1$, $R_0 < 1$ and the arbitrariness of ε_1 imply $0 < 1 - (\beta + \gamma)(1 - R_0) + \varepsilon_1 \frac{\alpha}{N} < 1$. Then we can rewrite (2.2) as

$$I_t \leq \left(1 - (\beta + \gamma)(1 - R_0) + \varepsilon_1 \frac{\alpha}{N}\right)^t I_0, \text{ for } t > T_1,$$

from which we can derive that

$$\lim_{t \rightarrow \infty} I_t = 0.$$

Therefore, for any small $\varepsilon_2 > 0$, there exists a larger positive integer $T_2 \geq T_1$ such that $I_t < \varepsilon_2$ for $t > T_2$. Then from the first equation of system (2.1), we have

$$S_t \geq \beta N + \left(1 - p - \beta - \varepsilon_2 \frac{\alpha}{N}\right) S_{t-1}, \text{ for } t > T_2.$$

Then from the comparison principle, we know that for any small $\varepsilon_3 > 0$, there exists an integer $T_3 > T_2$ such that $S_t \geq \frac{\beta N}{\beta + p + \varepsilon_2 \alpha / N} - \varepsilon_3$ for all $t > T_3$. Let $T_4 = T_1 + T_3$, then the inequalities

$$\frac{\beta N}{\beta + p + \varepsilon_2 \alpha / N} - \varepsilon_3 \leq S_t \leq \frac{\beta N}{\beta + p} + \varepsilon_1, \text{ for } t > T_4,$$

and the arbitrariness of $\varepsilon_1, \varepsilon_2$ and ε_3 imply that

$$\lim_{t \rightarrow \infty} S_t = \frac{\beta N}{\beta + p},$$

Thus the limits

$$\lim_{t \rightarrow \infty} S_t = \frac{\beta N}{\beta + p} \text{ and } \lim_{t \rightarrow \infty} I_t = 0.$$

imply that the DFE point P_0 is a global attractor if $R_0 < 1$. Consequently; since P_0 is locally asymptotically stable if $R_0 < 1$ and P_0 is global attractor, we have the DFE point P_0 of system (2.1) is globally asymptotically stable if $R_0 < 1$. $\square$

Remark 1. [3, 14] The basic reproductive ratio R_0 is referred as $\frac{\alpha\beta}{(\beta+p)(\beta+\gamma)}$. This ratio is a threshold parameter for the SIR epidemic model. If $R_0 < 1$, then there exists the DFE point which is locally asymptotically stable.

Theorem 3. The endemic equilibrium (EE) point $P_1 = \left(\frac{(\beta+\gamma)N}{\alpha}, \frac{N(\alpha\beta - (\beta+\gamma)(p+\beta))}{\alpha(\beta+\gamma)} \right)$ of the system (2.1) is locally asymptotically stable if $R_0 > 1$.

Proof. By considering (2.1), we can write the Jacobian matrix evaluated at P_1 as

$$J_{P_1} = \begin{pmatrix} 1 - p - \beta - \frac{\alpha I^*}{N} & -\beta - \gamma \\ \frac{\alpha I^*}{N} & 1 \end{pmatrix}. \quad (2.3)$$

The characteristic polynomial of the Jacobian matrix J_{P_1} is as follows:

$$G(\lambda) = \lambda^2 - \left(2 - p - \beta - \frac{\alpha I^*}{N} \right) \lambda + 1 - p - \beta - (1 - \beta - \gamma) \frac{\alpha I^*}{N}. \quad (2.4)$$

It is easy to verify that $G(1) = (\beta + \gamma) \frac{\alpha I^*}{N}$, $G(0) = 1 - p - \beta - (1 - \beta - \gamma) \frac{\alpha I^*}{N}$ and

$$\begin{aligned} G(-1) &= 2(1 - p - \beta) + 2 \left(1 - \frac{\alpha I^*}{N} \right) + (\beta + \gamma) \frac{\alpha I^*}{N} \\ &> 2(1 - p - \beta) + 2(1 - \alpha) + (\beta + \gamma) \frac{\alpha I^*}{N}. \end{aligned}$$

Since $R_0 = \frac{\alpha\beta}{(\beta+p)(\beta+\gamma)} > 1$ implies $\alpha > \beta + p$, then using the assumptions $0 < p + \alpha < 1$, $0 < \beta + \gamma < 1$, we have $G(1) > 0$, $G(-1) > 0$ and $C = G(0) < 1$ when $R_0 > 1$. It follows from Theorem 1 that the two roots, λ_1 and λ_2 , of the equation $G(\lambda) = 0$ satisfies $|\lambda_1| < 1$ and $|\lambda_2| < 1$. Therefore, from Lemma 2, we conclude that EE point P_1 is locally asymptotically stable when $R_0 > 1$. $\square$

Let $L_t = S_t + I_t$, then we can rewrite system (2.1) as follows

$$\begin{aligned} L_{t+1} &= \beta N + (1 - \beta - p)L_t + (p - \gamma)I_t, \\ I_{t+1} &= \frac{\alpha}{N}I_t(L_t - I_t) + (1 - \beta - \gamma)I_t. \end{aligned} \quad (2.5)$$

The global stability of the positive equilibrium point of system (2.5) is equivalent to that of system (2.1).

Theorem 4. The endemic equilibrium(EE) point of the system (2.5) is globally asymptotically stable if

- (i) $1 < R_0 < \frac{\beta+p+\gamma}{\beta+p} < \frac{2+\beta+\min\{p,\gamma\}}{\beta+p}$ and $p \leq \gamma$,
- (ii) $1 < \frac{\beta+p+\gamma}{\beta+p} < R_0 < \frac{2+\beta+\min\{p,\gamma\}}{\beta+p}$ and $p > \gamma$.

Proof. The global stability of the positive equilibrium point of (2.5) is quite difficult, so we consider two cases to prove it.

Case 1. $p \leq \gamma$.

From the inequality $0 < I_t < L_t$ and the first equation of system (2.5), we obtain

$$\begin{aligned} L_{t+1} &\leq \beta N + (1 - \beta - p)L_t, \\ L_{t+1} &\geq \beta N + (1 - \beta - p - \gamma)L_t. \end{aligned} \quad (2.6)$$

From these two inequalities in (2.6) and the comparison theorem [15], we know that for any small $\varepsilon > 0$, there exists a positive integer T_1 such that $L_1^l \leq L_t \leq L_1^r$ for $t > T_1$, where $L_1^l = \frac{\beta N}{\beta + p + \gamma} - \varepsilon$ and $L_1^r = \frac{\beta N}{\beta + p} + \varepsilon$.

When $t > T_1$, we substitute $L_1^l \leq L_t \leq L_1^r$ into the second equation of (2.5), then we obtain

$$\begin{aligned} I_{t+1} &\geq \frac{\alpha}{N} I_t (L_1^l - I_t) + (1 - \beta - \gamma) I_t, \\ I_{t+1} &\leq \frac{\alpha}{N} I_t (L_1^r - I_t) + (1 - \beta - \gamma) I_t. \end{aligned} \quad (2.7)$$

Let us consider the following auxiliary equations corresponding to the inequalities in (2.7)

$$\begin{aligned} I_{t+1}^l &= \frac{\alpha}{N} I_t^l (L_1^l - I_t^l) + (1 - \beta - \gamma) I_t^l, \\ I_{t+1}^r &= \frac{\alpha}{N} I_t^r (L_1^r - I_t^r) + (1 - \beta - \gamma) I_t^r. \end{aligned} \quad (2.8)$$

Define $I_t^l = \frac{(1 - \beta - \gamma)N + \alpha L_1^l}{\alpha} U_t^l$ and $I_t^r = \frac{(1 - \beta - \gamma)N + \alpha L_1^r}{\alpha} U_t^r$. Then we can rewrite (2.8) as follows

$$\begin{aligned} U_{t+1}^l &= r^l U_t^l (1 - U_t^l), \text{ with } r^l = 1 - \beta - \gamma + \frac{\alpha}{N} L_1^l, \\ U_{t+1}^r &= r^r U_t^r (1 - U_t^r), \text{ with } r^r = 1 - \beta - \gamma + \frac{\alpha}{N} L_1^r. \end{aligned} \quad (2.9)$$

It follows from [10] that the first equation of (2.8) has a positive equilibrium $I_{1*}^l = L_1^l - \frac{(\beta + \gamma)N}{\alpha}$, which is global asymptotically stable if $1 < R_0 < \frac{\beta + p + \gamma}{\beta + p} < \frac{2 + \beta + p + \gamma}{\beta + p}$. A similar argument implies that the second equation of (2.8) has a positive equilibrium $I_{1*}^r = L_1^r - \frac{(\beta + \gamma)N}{\alpha}$, which is globally asymptotically stable if $1 < R_0 < \frac{2 + \beta + p}{\beta + p}$.

The global asymptotic stability and the comparison theory imply that there exists a $T_2 \geq T_1$ such that $I_1^l < I_t < I_1^r$ for all $t > T_2$, where $I_1^l = I_{1*}^l - \varepsilon$ and $I_1^r = I_{1*}^r + \varepsilon$. When $t > T_2$, by substituting the inequality $I_1^l < I_t < I_1^r$ into the first equation of (2.5), we have

$$\begin{aligned} L_{t+1} &\geq \beta N + (1 - \beta - p)L_t + (p - \gamma)I_1^r, \\ L_{t+1} &\leq \beta N + (1 - \beta - p)L_t + (p - \gamma)I_1^l. \end{aligned} \quad (2.10)$$

From (2.10) and a similar argument, we can derive that there exists a positive integer $T_3 \geq T_2$ such that $L_2^l \leq L_t \leq L_2^r$ for all $t > T_3$, where $L_2^l = \frac{\beta N + (p - \gamma)I_1^r}{\beta + p} - \varepsilon$ and $L_2^r = \frac{\beta N + (p - \gamma)I_1^l}{\beta + p} + \varepsilon$. Obviously, $L_1^l < L_2^l < L_2^r < L_1^r$.

When $t > T_3$, the inequality $L_2^l \leq I_t \leq L_2^r$ substitute into the second equation of (2.5) implies

$$\begin{aligned} I_{t+1} &\geq \frac{\alpha}{N} I_t (L_2^l - I_t) + (1 - \beta - \gamma) I_t, \\ I_{t+1} &\leq \frac{\alpha}{N} I_t (L_2^r - I_t) + (1 - \beta - \gamma) I_t. \end{aligned} \quad (2.11)$$

A similar argument implies that there exists a $T_4 \geq T_3$ such that $I_2^l \leq I_t \leq I_2^r$ for $t > T_4$, where $I_2^l = L_2^l - \frac{(\beta+\gamma)N}{\alpha} - \varepsilon$ and $I_2^r = L_2^r - \frac{(\beta+\gamma)N}{\alpha} + \varepsilon$. By substituting L_2^l, L_2^r into I_2^l, I_2^r , we derive

$$\begin{aligned} I_2^l &= \frac{p-\gamma}{\beta+p} I_1^r + \frac{\beta+\gamma}{\alpha} N(R_0 - 1) - 2\varepsilon, \\ I_2^r &= \frac{p-\gamma}{\beta+p} I_1^l + \frac{\beta+\gamma}{\alpha} N(R_0 - 1) + 2\varepsilon. \end{aligned} \quad (2.12)$$

When $t > T_4$ and $I_2^l \leq I_t \leq I_2^r$, equations in (2.12) hold. Then from the induction, we know that there exist sequences $T_{2k-1}, T_{2k}, L_k^l, L_k^r, I_k^l, I_k^r$ such that $I_k^l \leq I_t \leq I_k^r$ for $t > T_{2k}$, and I_k^l, I_k^r satisfy that

$$\begin{aligned} I_{k+1}^l &= \frac{p-\gamma}{\beta+p} I_k^r + \frac{\beta+\gamma}{\alpha} N(R_0 - 1) - 2\varepsilon, \\ I_{k+1}^r &= \frac{p-\gamma}{\beta+p} I_k^l + \frac{\beta+\gamma}{\alpha} N(R_0 - 1) + 2\varepsilon. \end{aligned} \quad (2.13)$$

One can easily check that linear system (2.13) has a unique positive equilibrium $P_*(I_*^l(\varepsilon), I_*^r(\varepsilon))$ with

$$I_*^l(\varepsilon) = \frac{\beta+p}{\alpha} N(R_0 - 1) - \frac{2(\beta+p)}{\beta+2p-\gamma} \varepsilon, \quad I_*^r(\varepsilon) = \frac{\beta+p}{\alpha} N(R_0 - 1) + \frac{2(\beta+p)}{\beta+2p-\gamma} \varepsilon,$$

which is globally asymptotically stable. That is,

$$\lim_{k \rightarrow \infty} I_k^l = I_*^l(\varepsilon) \quad \text{and} \quad \lim_{k \rightarrow \infty} I_k^r = I_*^r(\varepsilon).$$

The arbitrary small of ε means that

$$\lim_{\varepsilon \rightarrow 0} I_*^l(\varepsilon) = \lim_{\varepsilon \rightarrow 0} I_*^r(\varepsilon) = \frac{\beta+p}{\alpha} N(R_0 - 1).$$

Then from the inequality $I_k^l < I_t < I_k^r$ and those limits, we derive that

$$\lim_{t \rightarrow \infty} I_t = \frac{\beta+p}{\alpha} N(R_0 - 1).$$

Similarly, we can prove that the sequences $\{L_k^l\}$ and $\{L_k^r\}$ is a linear system of difference equations and satisfy

$$\lim_{k \rightarrow \infty} L_k^l = \lim_{k \rightarrow \infty} L_k^r = \frac{\beta N}{\beta+p} + \frac{p-\gamma}{\alpha} N(R_0 - 1), \quad \text{as } \varepsilon \rightarrow 0.$$

The inequality $L_k^l < L_t < L_k^r$ and the limits implies that

$$\lim_{t \rightarrow \infty} L_t = \frac{\beta N}{\beta + p} + \frac{p - \gamma}{\alpha} N(R_0 - 1).$$

Finally, from $L_t = S_t + I_t$, then we have

$$\lim_{t \rightarrow \infty} S_t = \lim_{t \rightarrow \infty} L_t - \lim_{t \rightarrow \infty} I_t = \frac{(\beta + \gamma)N}{\alpha}.$$

Therefore, the endemic equilibrium P_1 of system (2.5) is globally asymptotically stable when $1 < R_0 < \frac{\beta + p + \gamma}{\beta + p} < \frac{2 + \beta + p}{\beta + p}$ and $p \leq \gamma$.

Case 2. $p > \gamma$.

For this case, from the inequality $0 < I_t < L_t$ and the first equation of system (2.5), we obtain

$$\begin{aligned} L_{t+1} &\leq \beta N + (1 - \beta - \gamma)L_t, \\ L_{t+1} &\geq \beta N + (1 - \beta - p - \gamma)L_t. \end{aligned} \quad (2.14)$$

Similarly, using the comparison theory, we have that for any small $\delta > 0$, there exists a positive integer $\tilde{T}_1$ such that $L_1^s \leq L_t \leq L_1^b$ for $t > \tilde{T}_1$, where $L_1^s = \frac{\beta N}{\beta + p + \gamma} - \delta$ and $L_1^b = \frac{\beta N}{\beta + \gamma} + \delta$.

When $t > \tilde{T}_1$, by substituting inequality $L_1^s \leq L_t \leq L_1^b$ into the second equation of (2.5), we have

$$\begin{aligned} I_{t+1} &\geq \frac{\alpha}{N} I_t (L_1^s - I_t) + (1 - \beta - \gamma)I_t, \\ I_{t+1} &\leq \frac{\alpha}{N} I_t (L_1^b - I_t) + (1 - \beta - \gamma)I_t. \end{aligned} \quad (2.15)$$

Define

$$\begin{aligned} I_{t+1} &= \frac{\alpha}{N} I_t (L_1^s - I_t) + (1 - \beta - \gamma)I_t, \\ I_{t+1} &= \frac{\alpha}{N} I_t (L_1^b - I_t) + (1 - \beta - \gamma)I_t. \end{aligned} \quad (2.16)$$

Using transform $I_t^s = \frac{(1 - \beta - \gamma)N + \alpha L_1^s}{\alpha} V_t^s$ and $I_t^b = \frac{(1 - \beta - \gamma)N + \alpha L_1^b}{\alpha} V_t^b$, then we can rewrite (2.16) as follows

$$\begin{aligned} V_{t+1}^s &= r^s V_t^s (1 - V_t^s), \text{ with } r^s = 1 - \beta - \gamma + \frac{\alpha}{N} L_1^s, \\ V_{t+1}^b &= r^b V_t^b (1 - V_t^b), \text{ with } r^b = 1 - \beta - \gamma + \frac{\alpha}{N} L_1^b. \end{aligned} \quad (2.17)$$

It follows from [10], we know that the first equation of (2.16) has a positive equilibrium $I_{1*}^s = L_1^s - \frac{(\beta + \gamma)N}{\alpha}$, which is global asymptotically stable if $\frac{\beta + p + \gamma}{\beta + p} < R_0 < \frac{2 + \beta + p + \gamma}{\beta + p}$, and that the second equation of (2.16) has a positive equilibrium $I_{1*}^b = L_1^b - \frac{(\beta + \gamma)N}{\alpha}$, which is globally asymptotically stable if $1 < R_0 < \frac{2 + \beta + \gamma}{\beta + p}$.

It follows from the asymptotic stability and the comparison theory that there exists a $\tilde{T}_2 \geq \tilde{T}_1$ such that $I_1^s < I_t < I_1^b$ for all $t > \tilde{T}_2$, where $I_1^s = I_{1*}^s - \delta$ and $I_1^b = I_{1*}^b + \delta$. When $t > \tilde{T}_2$, by substituting the inequality $I_1^s < I_t < I_1^b$ into the first equation of (2.5), we have

$$\begin{aligned} L_{t+1} &\geq \beta N + (1 - \beta - p)L_t + (p - \gamma)I_1^s, \\ L_{t+1} &\leq \beta N + (1 - \beta - p)L_t + (p - \gamma)I_1^b. \end{aligned} \quad (2.18)$$

Similarly, from (2.18), we derive that there exists a positive integer $\tilde{T}_3$ such that $L_2^s \leq L_t \leq L_2^b$ for all $t > \tilde{T}_3$, where $L_2^s = \frac{\beta N + (p - \gamma)I_1^s}{\beta + p} - \delta$ and $L_2^b = \frac{\beta N + (p - \gamma)I_1^b}{\beta + p} + \delta$. Obviously, $L_1^s < L_2^s < L_2^b < L_1^b$.

When $t > \tilde{T}_3$, the inequality $L_2^s \leq L_t \leq L_2^b$ substitute into the second equation of (2.5) implies

$$\begin{aligned} I_{t+1} &\geq \frac{\alpha}{N}I_t(L_2^s - I_t) + (1 - \beta - \gamma)I_t, \\ I_{t+1} &\leq \frac{\alpha}{N}I_t(L_2^b - I_t) + (1 - \beta - \gamma)I_t. \end{aligned} \quad (2.19)$$

Similar argument implies that there exists a $\tilde{T}_4 \geq \tilde{T}_3$ such that $I_2^s \leq I_t \leq I_2^b$ for $t > \tilde{T}_4$, where

$$\begin{aligned} I_2^s &= L_2^s - \frac{(\beta + \gamma)N}{\alpha} - \delta = \frac{p - \gamma}{\beta + p}I_1^s + \frac{\beta + \gamma}{\alpha}N(R_0 - 1) - 2\delta, \\ I_2^b &= L_2^b - \frac{(\beta + \gamma)N}{\alpha} + \delta = \frac{p - \gamma}{\beta + p}I_1^b + \frac{\beta + \gamma}{\alpha}N(R_0 - 1) + 2\delta. \end{aligned} \quad (2.20)$$

Equations in (2.20) hold if $t > \tilde{T}_4$ and $I_2^s \leq I_t \leq I_2^b$. Then by the induction, there exists sequences $\{I_k^s\}, \{I_k^b\}$ such that $I_k^s < I_t < I_k^b$ for all $t > \tilde{T}_{2k}$, where I_k^s, I_k^b satisfy that

$$\begin{aligned} I_{k+1}^s &= \frac{p - \gamma}{\beta + p}I_k^s + \frac{\beta + \gamma}{\alpha}N(R_0 - 1) - 2\delta, \\ I_{k+1}^b &= \frac{p - \gamma}{\beta + p}I_k^b + \frac{\beta + \gamma}{\alpha}N(R_0 - 1) + 2\delta. \end{aligned} \quad (2.21)$$

Note that the first and the second equation of system (2.21) are both linear difference equations and $0 < \frac{p - \gamma}{\beta + p} < 1$, then it is easy to verify that

$$\lim_{k \rightarrow \infty} I_k^s = \frac{\beta + p}{\alpha}N(R_0 - 1) - \frac{2(\beta + \gamma)}{\beta + p}\delta, \quad \lim_{k \rightarrow \infty} I_k^b = \frac{\beta + p}{\alpha}N(R_0 - 1) + \frac{2(\beta + \gamma)}{\beta + p}\delta.$$

The inequality $I_k^s < I_t < I_k^b$ for $t > \tilde{T}_{2k}$ and the arbitrary of δ suggest that

$$\lim_{t \rightarrow \infty} I_t = \frac{\beta + p}{\alpha}N(R_0 - 1).$$

Similar to the proof of case 1, we have

$$\lim_{t \rightarrow \infty} S_t = \frac{(\beta + \gamma)N}{\alpha}.$$

Therefore, the endemic equilibrium point P_1 of system (2.5) is globally asymptotically stable when $1 < \frac{\beta+p+\gamma}{\beta+p} < R_0 < \frac{2+\beta+\gamma}{\beta+p}$ and $p > \gamma$.

Therefore, we conclude that the endemic equilibrium point P_1 of system (2.5) is globally asymptotically stable when $1 < \frac{\beta+p+\gamma}{\beta+p} < R_0 < \frac{2+\beta+\min\{p,\gamma\}}{\beta+p}$. $\square$

3. NUMERICAL SIMULATIONS

In this section, our aim is to present numerical simulations to illustrate the key results of theoretical analysis and graphical representations in the form of time plots, phase portrait diagrams of system (2.5).

Example 1. For DFE point, we take the parameter values as $N = 100$, $p = 0.5$, $\beta = 0.95$, $\alpha = 0.8$, $\gamma = 0.0005$ and initial values as $(S_0, I_0) = (100, 30)$. The eigen values are $|\lambda_1| = 0.9795 < 1$, $|\lambda_2| = 0.7780 < 1$ and basic reproductive ratio $R_0 = 0.5514 < 1$ then DFE point of the system (2.5) is globally asymptotically stable (see Figure 1).

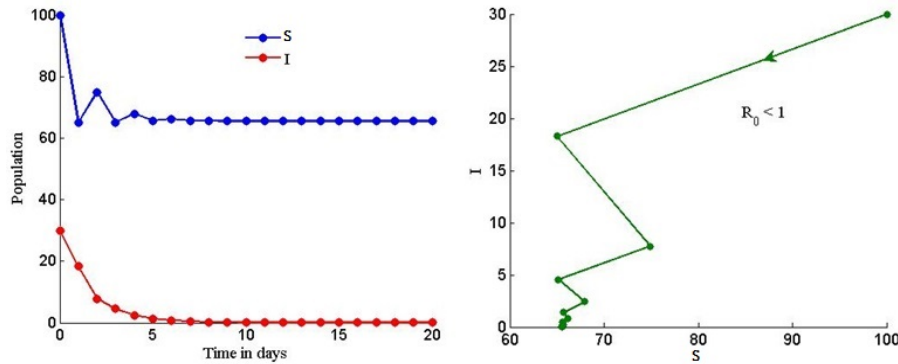


FIGURE 1. Time plots and phase portraits of DFE point of the system (2.5) with stability $R_0 < 1$.

Example 2. For EE point, we take the parameter values as $N = 100$, $p = 0.0005$, $\beta = 0.025$, $\alpha = 0.6$, $\gamma = 0.3$ and initial values as $(S_0, I_0) = (100, 30)$. We apply the conditions $1 < R_0 (= 1.81) < \frac{\beta+p+\gamma}{\beta+p} (= 12.7647) < \frac{2+\beta+p}{\beta+p} (= 79.4314)$ and $p (= 0.0005) \leq \gamma (= 0.3)$ then the endemic equilibrium point of the system (2.5) is globally asymptotically stable (see Figure 2).

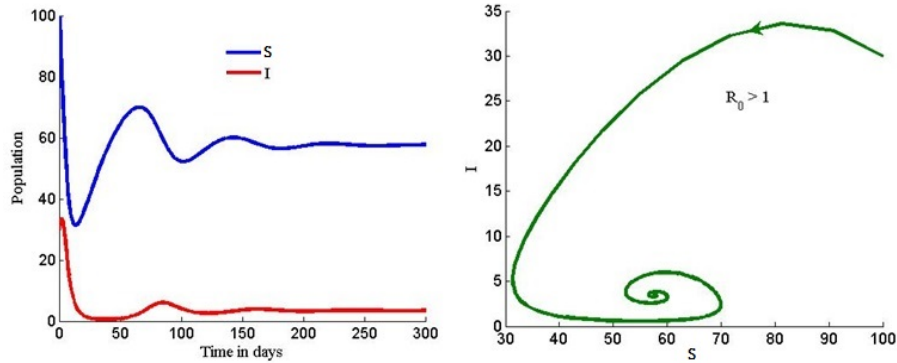


FIGURE 2. Time plots and phase portraits of EE point of the system (2.5) with stability $R_0 > 1$

Example 3. For EE point, we take the parameter values as $N = 100$, $p = 0.5$, $\beta = 0.2$, $\alpha = 0.8$, $\gamma = 0.0005$ and initial values as $(S_0, I_0) = (100, 30)$. We apply the conditions $\frac{\beta+p+\gamma}{\beta+p} (= 1.0007) < R_0 (= 1.14) < \frac{2+\beta+\gamma}{\beta+p} (= 3.1436)$ and $p > \gamma$ then the endemic equilibrium point of the system (2.5) is globally asymptotically stable (see Figure 3).

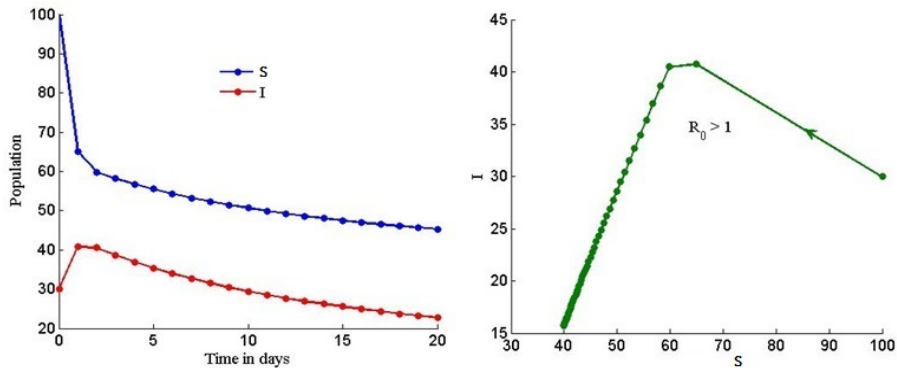
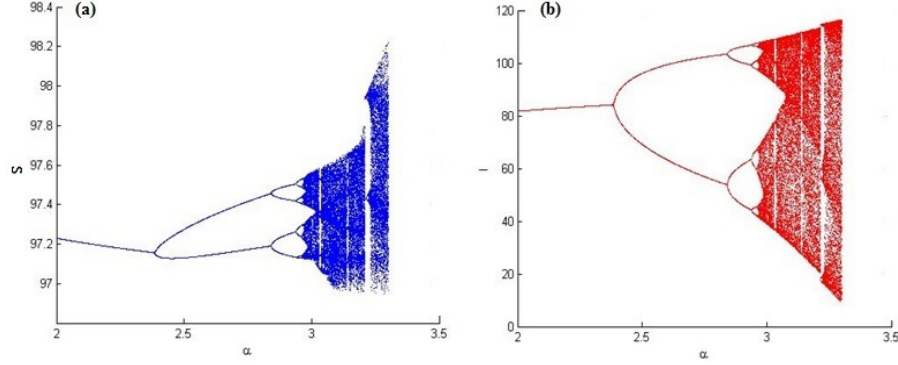
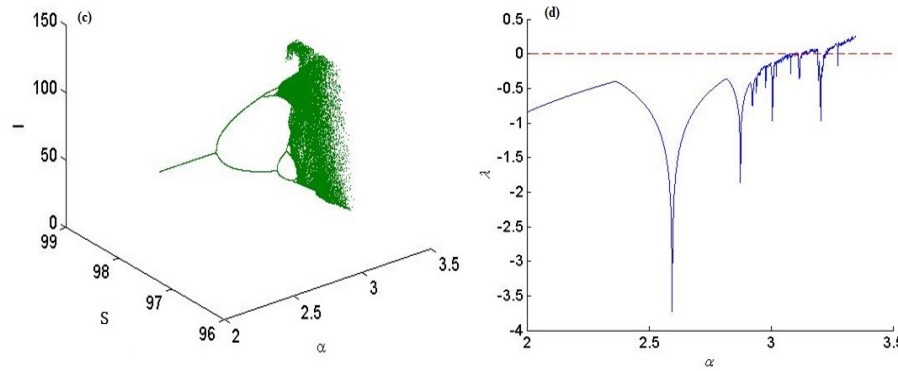


FIGURE 3. Time plots and phase portraits of EE point of the system (2.5) with stability $R_0 > 1$.

4. BIFURCATION AND CHAOTIC BEHAVIOR

In this section, we present the bifurcation diagrams and maximum Lyapunov exponent and chaotic attractors of the system (2.5). It is known that Maximum Lyapunov exponent qualifies the exponential divergence of initially close state-space trajectories and frequency employ to identify a chaotic behavior.

FIGURE 4. The bifurcation diagrams of the system (2.5) for $\alpha \in (2, 3.4)$.FIGURE 5. The bifurcation diagram in $(\alpha - S - I)$ space and the Maximum Lyapunov exponent corresponding to Figure 4(a-b).

The bifurcation diagrams are considered for three cases:

Case 1. Fixing parameters $N = 100$, $\beta = 0.3$, $p = 0.0009$, $\gamma = 0.01$ and varying $\alpha \in (2, 3.4)$.

The bifurcation diagrams of the system (2.5) plotted in particular range of $\alpha \in (2, 3.4)$ with the contact rate α as the bifurcation parameter are given in Figures 4(a-b). The bifurcation diagram of the system (2.5) in $(\alpha - S - I)$ space is given in Figure 5(c). The Maximum Lyapunov exponent corresponding to Figures 4(a-b) are computed and plotted in Figure 5(d) confirming the existence of the chaotic regions and period orbits in the parametric space. Figures 6(a-b) is the local amplification corresponding to Figures 4(a-b) for $\alpha \in [3, 3.3]$. The phase portraits for various α -values corresponding to Figure 4(a-b) are plotted in Figures 7(a)-(f) to illustrate the observations.

Furthermore, the stable equilibria at $\alpha = 2.2$, the emergence of periodic-2,4,8 orbits are observed when $\alpha = 2.6, 2.9, 2.95$ in Figures 7(b)-(d). For instance, when $\alpha = 2.99, 3.0$ chaotic attractors appears in Figures 7(e)-(f). Some interesting phenomena are also seen in different chaotic attractors in the range $\alpha = 3.1$ to $\alpha = 3.3$. The occurrence of chaotic regions is observed in Figures 6(a)-(b), these phenomena are illustrated by the phase portraits in Figures 8(g)-(l).

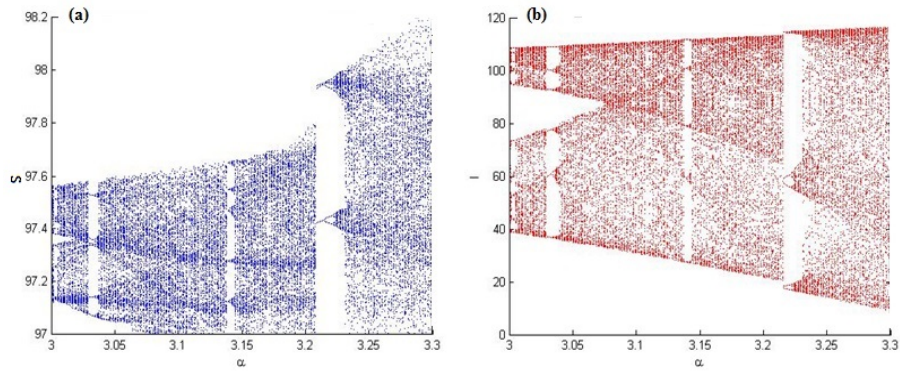


FIGURE 6. Local amplification corresponding to Figure 4(a-b).

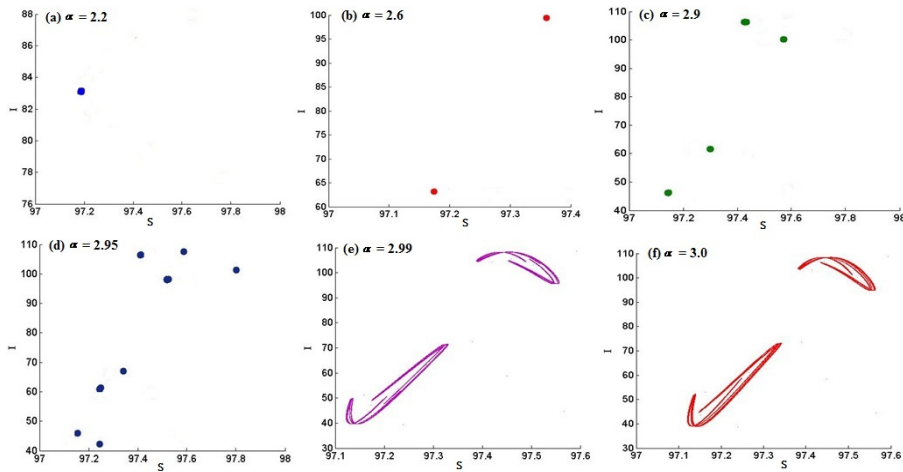


FIGURE 7. Phase portrait diagrams of the system (2.5) for various α corresponding to Figure 4.

Case 2. Fixing parameters $N = 100$, $p = 0.0005$, $\gamma = 0.1$, $\alpha = 4.16$ and varying $\beta \in (1, 2.5)$. The bifurcation diagrams of the system (2.5) plotted in particular range

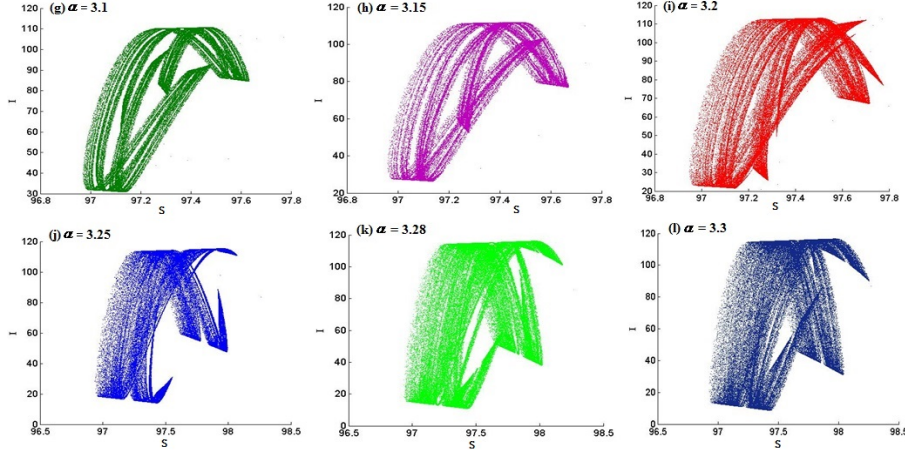


FIGURE 8. Phase portrait diagrams of the system (2.5) for various α corresponding to Figure 4.

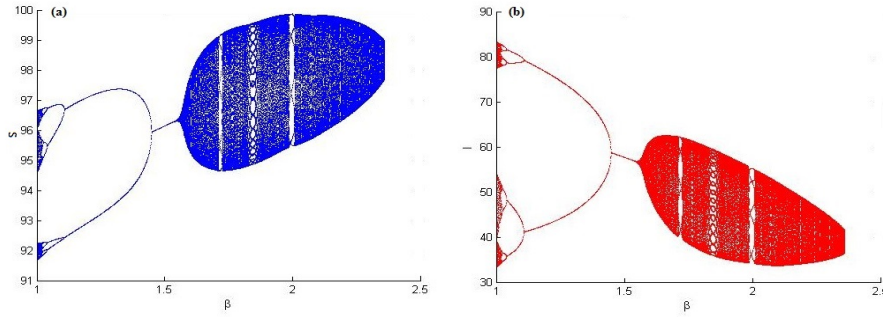


FIGURE 9. Bifurcation diagrams of the system (2.5) with varying $\beta \in (1, 2.5)$.

of $\beta \in (1, 2.5)$ with birth rate as the bifurcation parameter are given in Figures 9(a-b). The bifurcation diagram of the system (2.5) in $(\beta - S - I)$ space are given in Figure 10(c). The Maximum Lyapunov exponent corresponding to Figures 9(a-b) are computed and plotted in Figure 10(d) confirming the existence of the chaotic regions and periodic orbits in the parametric space. The chaotic regions are also observed in Figures 9(a-b): these phenomena are illustrated by the phase portrait in Figure 11(a). The phase portraits for various β -values corresponding to Figures 9(a-b) are plotted in Figures 11(b)-(f) to illustrate the observations. Furthermore, the emergence of periodic-8,4,2 orbits are observed when $\beta = 1.03, 1.05, 1.25$ in Figure 11(b)-(d), the stable equilibria at $\beta = 1.56$ in Figure 11(e). Attracting invariant circle appears

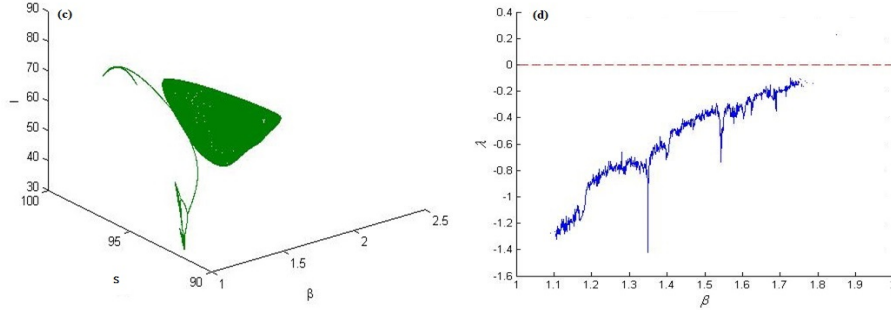


FIGURE 10. Bifurcation diagram in $(\beta - S - I)$ space and Maximum Lyapunov exponents of the system (2.5) with varying $\beta \in (1, 2.5)$.

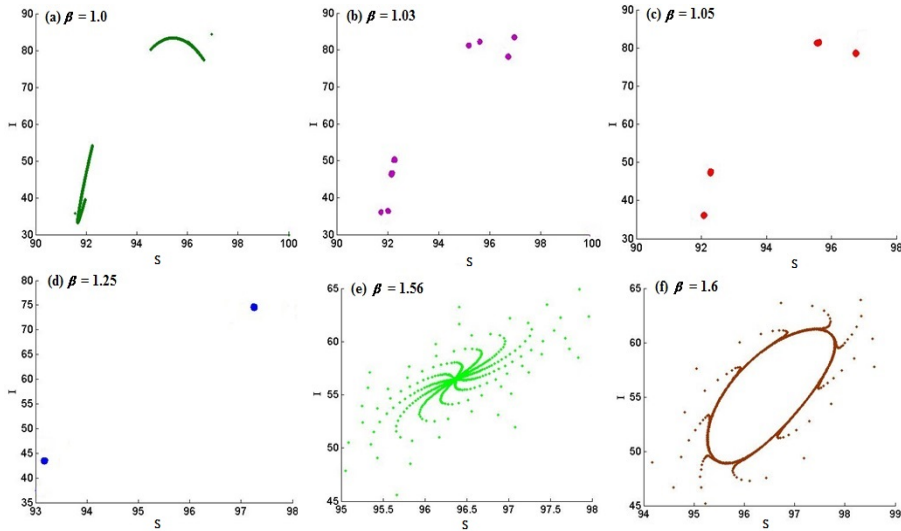


FIGURE 11. Phase portrait diagrams of the system (2.5) for various β corresponding to figure 9.

when $\beta = 1.6$ in Figure 11(f) and when $\beta = 1.7, 1.71, 1.85, 1.95, 2.3$ in Figures 12(g)-(j) and Figure 12(l). Furthermore, quasi-periodic orbits appears when $\beta = 2.0$ in Figure 12(k).

Case 3. Fixing parameters $N = 100$, $p = 0.0009$, $\beta = 0.3$, $\alpha = 3.3$ and varying $\gamma \in (0, 1)$. The bifurcation diagrams of the system (2.5) plotted in particular range of $\gamma \in (0, 1)$ recovery rate as the bifurcation parameter are given in Figure 13(a). The Maximum Lyapunov exponent corresponding to Figure 13(a) are computed and plotted in Figure 13(b) confirming the existence of the chaotic regions and periodic orbits in the parametric space.

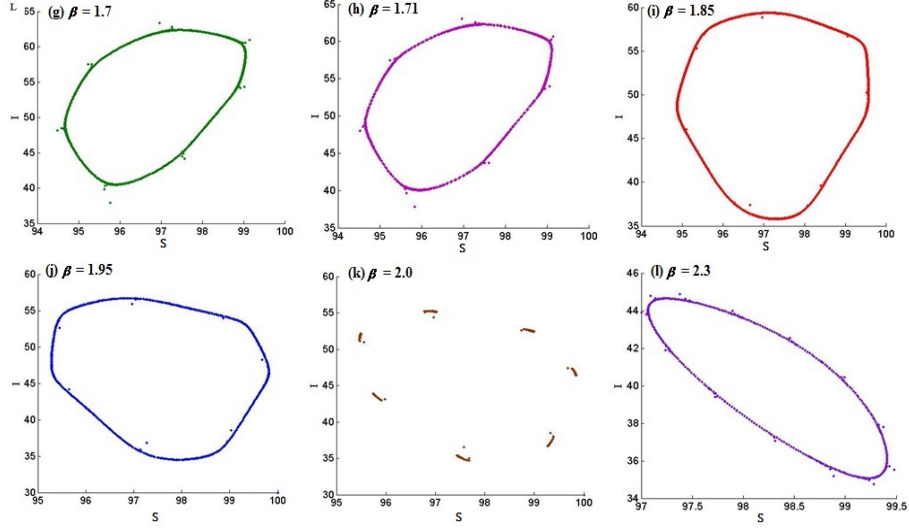


FIGURE 12. Phase portrait diagrams of the system (2.5) for various β corresponding to figure 9.

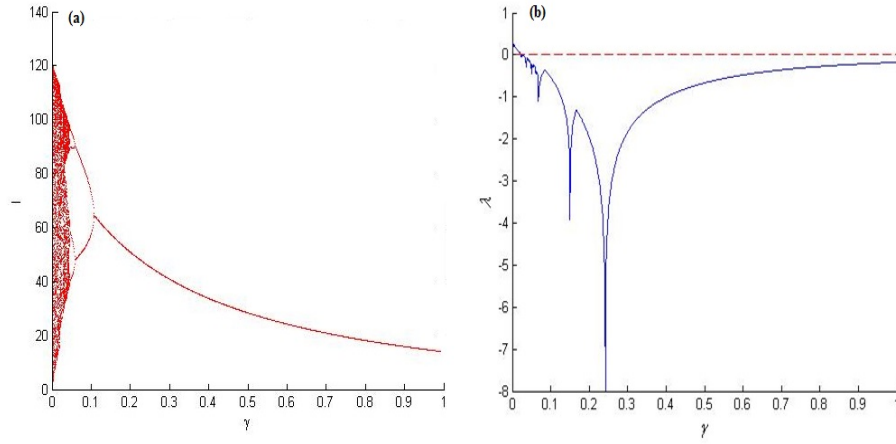


FIGURE 13. Bifurcation diagram and Maximum Lyapunov exponent of the system (2.5) with varying $\gamma \in (0, 1)$.

5. CONCLUSION

In this paper, we considered the dynamical behaviors of a discrete-time SIR epidemic model (1.1). The basic reproduction number completely determine the stability of discrete model (1.1). If the basic reproduction number is less than one, model

(1.1) only has a DFE and it is globally asymptotically stable. If the basic reproduction number is bigger than one, model has a EE except DFE, and EE is globally asymptotically stable. Numerical simulations are provided to justify our analytical findings. Furthermore, let α, β, γ be the bifurcation parameters, numerical simulations show that model (1.1) has periodic orbits, period-2,4,8 orbits, and chaotic sets, which implies that the susceptible and infective can coexists in the stable period-n orbits and cycle. We also present the maximum Lyapunov exponent, when the maximum Lyapunov exponent is positive which is an evidence for chaos. These results reveal far richer dynamics of the discrete model compared to the continuous model. The theoretical analysis of bifurcation and chaos will be studied in the further.

ACKNOWLEDGEMENT

This work was supported by the Natural Science Project of Ningxia Higher Institutions (No. NGY2020005), the National Natural Science of China (No. 12001305) and the Natural Science Foundation of Ningxia (No. 2020AAC03060).

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L^p –REGULARITY RESULTS FOR $2m$ -TH ORDER PARABOLIC EQUATIONS IN TIME-VARYING DOMAINS

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Received 13 January, 2020

Abstract. This paper is devoted to the analysis of the following linear $2m$ -th order parabolic equation $\partial_t u + (-1)^m \sum_{k=1}^N \partial_{x_k}^{2m} u = f$, subject to Dirichlet type condition $\partial_\nu^l u = 0$, $l = 0, 1, \dots, m-1$, on the lateral boundary, where m is a positive integer. The right-hand side f of the equation is taken in the Lebesgue space L^p , $1 < p < +\infty$. The problem is set in a domain of the form $Q = \left\{ (t, x_1, \dots, x_N) \in \mathbb{R}^{N+1} : 0 \leq \sqrt{x_1^2 + \dots + x_N^2} < t^\alpha \right\}$ with $\alpha > 1/2m$.

2010 *Mathematics Subject Classification:* 35K05, 35K55, 35K65, 35K90.

Keywords: High order parabolic equation, non-cylindrical domains, anisotropic Sobolev spaces, sum of linear operators, interpolation spaces

1. INTRODUCTION

Let Q be an open set of $\mathbb{R}^{N+1}$ defined by

$$Q = \left\{ (t, x_1, x_2, \dots, x_N) \in \mathbb{R}^{N+1} : (x_1, x_2, \dots, x_N) \in \Omega_t, 0 < t < 1 \right\}$$

where for a fixed t in the interval $]0, 1[$, Ω_t is a bounded domain of $\mathbb{R}^N$, $N > 1$, defined by

$$\Omega_t = \left\{ (x_1, x_2, \dots, x_N) \in \mathbb{R}^N : 0 \leq \sqrt{x_1^2 + x_2^2 + \dots + x_N^2} < t^\alpha \right\}$$

with $\alpha > 1/2m$ and m belongs to the set of all nonzero natural numbers $\mathbb{N}^*$. In Q , consider the boundary value problem

$$\begin{cases} \partial_t u + Mu &= f \in L^p(Q), \\ \partial_\nu^l u|_{\partial Q \setminus \Gamma_1} &= 0, l = 0, 1, \dots, m-1, \end{cases} \quad (1.1)$$

where $M = (-1)^m \sum_{k=1}^N \partial_{x_k}^{2m}$, ∂Q is the boundary of Q , Γ_1 is the part of the boundary of Q where $t = 1$ and ∂_ν^l stands for the derivative of order l throughout the normal

This work is supported in part by the General Direction of Scientific Research and Technological Development of Algeria.

vector v on $(\partial Q \setminus \Gamma_1)$. Here, $L^p(Q)$, $1 < p < +\infty$, denotes the space of p -integrable functions on Q with the measure $dt dx_1 \dots dx_N$.

If the domain under investigation is a cylinder, the solvability of the corresponding problem is known over the scales of anisotropic Sobolev-Slobodetskii or Hölder spaces since the mid of the last century. Indeed, classical results on the resolution of Problem (1.1) can be found in [20] and [21] and in the references therein. Some recent regularity results are given in [15], [10], [8], [11], [26], [27], [25] and [2].

Besides being interesting in itself, the study of Problem (1.1) is motivated by the interest of researchers for many mathematical questions related to non-regular domains. During the last decades and since many applied problems lead directly to boundary-value problems in "bad" domains, numerous authors studied partial differential equations in non-smooth domains. Among these which are related to higher order parabolic equations we can cite Baderko [1], Cherepova [3], Sadallah [29], Galaktionov [12], Mikhailov [23], [24], Cherfaoui *et al.* [5], Grimaldi [13] and the references therein.

The L^2 -solvability of Problem (1.1) has been investigated in [6] by the domain decomposition method, see also [16] and [4]. The difficulty with the space L^p , $p \neq 2$, is that this space is not a Hilbert space. So, the domain decomposition method used in [6] does not seem to be appropriate for our study and cannot be generalized in this sense. An idea for this extension (to the case L^p , $p \in (1, \infty)$) can be found in [8] and [17], in which the operators sum method was used. This method is interesting because it may be generalized to Banach spaces instead of Hilbert spaces. For more details and recent results concerning this method, see [7], [28] and the references therein.

In this work, we are especially interested in the question of what sufficient conditions, as weak as possible, the dimension N , the exponent p and the type of the domain Q must verify in order that Problem (1.1) has a solution with optimal regularity, that is a solution u belonging to the anisotropic Sobolev space

$$H_{0,p}^{1,2m}(Q) := \left\{ u \in H_p^{1,2m}(Q) : \partial_\nu^l u|_{\partial Q \setminus \Gamma_1} = 0, l = 0, 1, \dots, m-1 \right\},$$

with

$$H_p^{1,2m}(Q) = \{ u : \partial_t u, \partial^\alpha u \in L^p(Q), |\alpha| \leq 2m \}$$

where $\alpha = (i_1, i_2, \dots, i_N) \in \mathbb{N}^N$, $|\alpha| = i_1 + i_2 + \dots + i_N$ and $\partial^\alpha u = \partial_{x_1}^{i_1} \partial_{x_2}^{i_2} \dots \partial_{x_N}^{i_N} u$. The space $H_p^{1,2m}(Q)$ is equipped with the natural norm, that is

$$\|u\|_{H_p^{1,2m}(Q)} = \left(\|\partial_t u\|_{L^p(Q)}^p + \sum_{|\alpha| \leq 2m} \|\partial^\alpha u\|_{L^p(Q)}^p \right)^{1/p}.$$

The main assumption is

$$\frac{1}{2m} < \alpha < \frac{p-1}{N}. \quad (1.2)$$

The outline of this paper is as follows. In Section 2 we recall the essential of the sum theory we will have to apply. In Section 3 we perform a change of variables conserving (modulo a weight) the spaces L^p and $H_p^{1,2m}$, and transforming Problem (1.1) into a degenerate parabolic problems in a cylindrical domain. Section 4 is concerned with the application of the sum operators method to the transformed problem. We can find in the Favini-Yagi book [9] an important study of abstract problems of parabolic type with degenerated terms in the time derivative. They used the notion of multi-valued linear operators and constructed fundamental solutions when the right-hand side has a Hölder regularity with respect to the time. Our approach is based on the direct use of operators sums in a weighted L^p -Sobolev space. Finally, in Section 5 we give results concerning the transformed problem and we return to our initial problem by using the inverse change of variables.

Note that this approach may be extended at least in the following directions:

- (1) The high order operator M may be replaced by the following constant coefficient operator:

$$L = \sum_{|\delta|=|\beta|=m} (-1)^m a_{\delta\beta} \partial^\delta \partial^\beta$$

with $a_{\delta\beta} = a_{\beta\delta}$ and there exists a constant $C > 0$ such that

$$a_{\delta\beta} \xi^\delta \xi^\beta > C |\xi|^{2m}, \quad \xi \in \mathbf{R}^N.$$

- (2) The function f on the right-hand side of the equation of Problem (1.1), may be taken in Hölder or little Hölder spaces.

These questions will be developed in forthcoming works.

2. ON THE NON-COMMUTATIVE SUM OF LINEAR OPERATORS

Let Λ be a closed linear operator in a complex Banach space E . Then, Λ is said to be sectorial if

- (i) $D(\Lambda)$ and $Im(\Lambda)$ are dense in E ,
- (ii) $\ker(\Lambda) = \{0\}$,
- (iii) $]-\infty, 0[\subset \rho(\Lambda)$ ($\rho(\Lambda)$ is the resolvent set of Λ) and there exists a constant $K \geq 1$ such that $\forall t > 0$, $\left\| t(\Lambda + tI)^{-1} \right\|_{L(E)} \leq K$. If Λ is sectorial it follows easily that $\rho(-\Lambda)$ contains an open sector $\Sigma_\varphi := \{z \in \mathbb{C} : z \neq 0, |\arg z| < \varphi\}$, with $\varphi \in]0, \pi[$.

Consider two closed linear operators A and B with dense domains $D(A)$ and $D(B)$ respectively in E . Assume that both operators satisfy the following assumptions of Da Prato-Grisvard's type [7].

There exist positive numbers $r, C_A, C_B, \theta_A, \theta_B$ such that

$$\theta_A + \theta_B < \pi, \tag{2.1}$$

$$\rho(-A) \supset \Sigma_{\pi-\theta_A} := \{z \in \mathbb{C} : |z| \geq r, |\arg z| < \pi - \theta_A\} \tag{2.2}$$

$$\begin{aligned}
& \text{and } \forall \lambda \in \Sigma_{\pi-\theta_A}, \left\| (A + \lambda I)^{-1} \right\|_{L(E)} \leq \frac{C_A}{|\lambda|}, \\
& \rho(-B) \supset \Sigma_{\pi-\theta_B} := \{z \in \mathbb{C} : |z| \geq r, |\arg z| < \pi - \theta_B\} \quad (2.3) \\
& \text{and } \forall \mu \in \Sigma_{\pi-\theta_B}, \left\| (B + \mu I)^{-1} \right\|_{L(E)} \leq \frac{C_B}{|\mu|}.
\end{aligned}$$

We also assume that there are constants $C > 0$, $\lambda_0 > 0$, (with $\lambda_0 \in \rho(-A)$), τ and ρ such that

$$\begin{cases} (i) \left\| (A + \lambda_0 I) (A + \lambda I)^{-1} \left[(A + \lambda_0 I)^{-1}; (B + \mu I)^{-1} \right] \right\|_{L(E)} \\ \leq \frac{C}{|\lambda|^{1-\tau} \cdot |\mu|^{1+\rho}} \quad \forall \lambda \in \rho(-A), \forall \mu \in \rho(-B), \\ (ii) 0 \leq \tau < \rho \leq 1. \end{cases} \quad (2.4)$$

For more details concerning this last Labbas-Terreni commutator assumption, see [18], [19].

For any $\sigma \in]0, 1[$ and $1 \leq p \leq +\infty$, let us introduce the real Banach interpolation spaces $D_A(\sigma, p)$ between $D(A)$ and E (or $D_B(\sigma, p)$ between $D(B)$ and E) which are characterized (for $1 \leq p < +\infty$) by

$$D_A(\sigma, p) = \left\{ \xi \in E : t \mapsto \left\| t^\sigma A (A + tI)^{-1} \xi \right\|_E \in L^p_* \right\},$$

where L^p_* denotes the space of p -integrable functions on $(0, +\infty)$ with the measure dt/t . For $p = +\infty$,

$$D_A(\sigma, +\infty) = \left\{ \xi \in E : \sup_{t>0} \left\| t^\sigma A (A + tI)^{-1} \xi \right\|_E < \infty \right\}.$$

For these spaces, see [14]. Then the main result proved in Labbas-Terreni [18] is

Theorem 1. *Under assumptions (2.1), (2.2), (2.3) and (2.4), there exists $\lambda^* > 0$ such that for any $\lambda \geq \lambda^*$ and for any $h \in D_A(\sigma, p)$, equation*

$$Aw + Bw + \lambda w = h,$$

has a unique solution $w \in D(A) \cap D(B)$ with the regularities $Aw, Bw \in D_A(\theta, p)$ and $Aw \in D_B(\theta, p)$ for any θ verifying $\theta \leq \min(\sigma, \rho - \tau)$.

3. CHANGE OF VARIABLES AND OPERATIONAL SETTING OF THE PROBLEM

3.1. Change of variables

We make the following change of variables and functions

$$\begin{aligned}
\Pi : Q & \longrightarrow G \\
(t, x_1, x_2, \dots, x_N) & \longmapsto (t, y_1, y_2, \dots, y_N) = \left(t, \frac{x_1}{t^\alpha}, \frac{x_2}{t^\alpha}, \dots, \frac{x_N}{t^\alpha} \right)
\end{aligned}$$

where $G =]0, 1[\times B(0, 1)$, with $B(0, 1)$ is the unit ball of $\mathbb{R}^N$ centered at the origin. Set $u(t, x_1, x_2, \dots, x_N) = v(t, y_1, y_2, \dots, y_N)$ and $f(t, x_1, x_2, \dots, x_N) = g(t, y_1, y_2, \dots, y_N)$, then Problem (1.1) is transformed, in G , into the following degenerate evolution problem

$$\begin{cases} t^{2m\alpha} \partial_t v + Mv - \alpha t^{2m\alpha-1} \sum_{k=1}^N y_k \partial_{y_k} v &= t^{2m\alpha} g = h, \\ v|_{\Sigma_0} &= 0, \\ \partial_v^l v|_{\partial G \setminus (\Sigma_0 \cup \Sigma_1)} &= 0, l = 0, 1, \dots, m-1, \end{cases} \quad (3.1)$$

where $M = \sum_{k=1}^N \partial_{y_k}^{2m}$, Σ_j , $j = 0, 1$ is the part of the boundary of G where $t = j$. It is easy to see that $f \in L^p(Q)$ if and only if $t^{N\alpha/p} g \in L^p(G)$. Indeed,

$$\begin{aligned} f \in L^p(Q) &\Leftrightarrow \int_0^1 \int_{\Omega_t} |f(t, x_1, \dots, x_N)|^p dt dx_1 \dots dx_N < +\infty \\ &\Leftrightarrow \int_0^1 \int_{B(0,1)} \left| t^{N\alpha/p} g(t, y_1, \dots, y_N) \right|^p dt dy_1 \dots dy_N < +\infty \\ &\Leftrightarrow t^{N\alpha/p} g \in L^p(G). \end{aligned}$$

Consequently, $f \in L^p(Q)$ if and only if $t^{-2m\alpha+(N\alpha/p)} h \in L^p(G)$ which implies that $h \in L^p(G)$, since $h = (t^{-2m\alpha+(N\alpha/p)} h) t^{2m\alpha-(N\alpha/p)}$ and $2m\alpha - (N\alpha/p) > 0$. Then, the function $h = t^{2m\alpha} g$ lies in the closed subspace of $L^p(G)$ defined by

$$E = \left\{ h \in L^p(0, 1; L^p(B(0, 1))) : t^{-2m\alpha+(N\alpha/p)} h \in L^p(0, 1; L^p(B(0, 1))) \right\}.$$

This space is equipped with the norm $\|h\|_E = \|t^{-2m\alpha+(N\alpha/p)} h\|_{L^p(0,1;L^p(B(0,1)))}$.

3.2. Operational formulation of Problem (3.1)

Recall that $\alpha > 1/2m$ and assume

$$p > 1 + N\alpha. \quad (3.2)$$

Set $X = L^p(B(0, 1))$ and define the functions

$$\begin{aligned} v &: [0, 1] \longrightarrow X; \quad t \longmapsto v(t); \quad v(t)(y_1, y_2, \dots, y_N) = v(t, y_1, y_2, \dots, y_N), \\ h &: [0, 1] \longrightarrow X; \quad t \longmapsto h(t); \quad h(t)(y_1, y_2, \dots, y_N) = h(t, y_1, y_2, \dots, y_N). \end{aligned}$$

Consider the family of operators $(L(t))_{t \in [0,1]}$ defined by

$$\begin{cases} D(L(t)) &= \left\{ \psi \in W^{2m,p}(B(0, 1)) : \partial_v^l \psi|_{\partial B(0,1)} = 0, l = 1, \dots, m-1 \right\}, \\ (L(t)\psi) &= M\psi - \alpha t^{2m\alpha-1} \sum_{k=1}^N y_k \partial_{y_k} \psi \text{ for a.e. } t \in (0, 1), \end{cases}$$

then Problem (3.1) is equivalent to the following operational degenerate Cauchy problem in X

$$\begin{cases} t^{2m\alpha} v'(t) + L(t) v(t) &= h(t), & t \in (0, 1), \\ v(0) &= 0. \end{cases} \quad (3.3)$$

Observe that $\overline{D(L(t))} = X$. Set

$$\begin{cases} w(t) &= e^{\lambda \frac{t^{1-2m\alpha}}{1-2m\alpha}} v(t) \\ k(t) &= e^{\lambda \frac{t^{1-2m\alpha}}{1-2m\alpha}} h(t), \end{cases}$$

where λ is some positive number. Then, w verifies

$$\begin{cases} t^{2m\alpha} w'(t) + L(t) w(t) + \lambda w(t) &= k(t), & t \in (0, 1), \\ w(0) &= 0, \end{cases}$$

where k belongs to the space

$$E = \left\{ h \in L^p(0, 1; X) : t^{-2m\alpha + (N\alpha/p)} h \in L^p(0, 1; X) \right\}.$$

We obtain then the new operational form of the previous problem, mainly

$$Aw + Bw + \lambda w = k,$$

where

$$(Aw)(t) = L(t) w(t), \quad t \in [0, 1],$$

with domain

$$D(A) = \left\{ w \in E : t^{-2m\alpha + (N\alpha/p)} w \in L^p(0, 1; W^{2m,p}(B(0, 1)) \cap W_0^{m,p}(B(0, 1))) \right\}$$

and

$$(Bw)(t) = t^{2m\alpha}(t) w'(t), \quad t \in [0, 1],$$

with domain

$$D(B) = \left\{ w \in E : t^{(N\alpha/p)} w' \in L^p(0, 1; X) \text{ and } w(0) = 0 \right\}.$$

Note that the trace $w(0)$ is well defined in $D(B)$. In fact, we have

$$t^{N\alpha/p} w \in L^p(0, 1; X), \quad t^{N\alpha/p} w' \in L^p(0, 1; X),$$

and in virtue of (3.2) $(N\alpha/p) + (1/p) < 1$. Then, w is continuous on $[0, 1]$, (see [31, Lemma, p. 42]).

4. APPLICATION OF THE SUMS

Now we are in position to apply the result of the sums of operators. For this purpose, we must verify the assumptions of Theorem 1. The spectral properties of A and B are as follows.

Proposition 1. *A and B are linear closed operators and their domains are dense in E . Moreover, they satisfy assumptions (2.1), (2.2) and (2.3).*

Proof. 1. Let us study the spectral equation related to the operator B

$$Bw + zw = k.$$

Fix some positive μ_0 and let z such that $\operatorname{Re} z \geq \mu_0$. Then the general solution of the problem

$$\begin{cases} t^{2m\alpha} w'(t) + zw(t) = k(t), & t \in [0, 1], \\ w(0) = 0, \end{cases}$$

is given by

$$w(t) = d \exp\left(z \int_t^1 \frac{ds}{s^{2m\alpha}}\right) + \int_0^t \left(\frac{k(\sigma)}{\sigma^{2m\alpha}} \exp\left(-z \int_\sigma^t \frac{ds}{s^{2m\alpha}}\right)\right) d\sigma,$$

where d is an arbitrary constant. The hypothesis $p > 1 + N\alpha$ implies that the function

$$t \mapsto t^{-2m\alpha + (N\alpha/p)} \exp\left(z \int_t^1 \frac{ds}{s^{2m\alpha}}\right)$$

does not belong to $L^p(B(0, 1))$. So, we will take $d = 0$ to obtain $w \in E$. Consequently

$$\begin{aligned} w(t) &= \left((B + zI)^{-1} k\right)(t) \\ &= \int_0^t \left(\frac{k(\sigma)}{\sigma^{2m\alpha}} \exp\left(-z \int_\sigma^t \frac{ds}{s^{2m\alpha}}\right)\right) d\sigma \\ &= \exp\left(\frac{z}{(2m\alpha-1)t^{2m\alpha-1}}\right) \int_0^t \frac{k(\sigma)}{\sigma^{2m\alpha}} \exp\left(\frac{-z}{(2m\alpha-1)\sigma^{2m\alpha-1}}\right) d\sigma \end{aligned}$$

Let us check that this formula is well defined on $[0, 1]$ and gives $w(0) = 0$. Set $\mu = \frac{z}{(2m\alpha-1)}$, then

$$\begin{aligned} \|w(t)\| &\leq \exp\left(\frac{\operatorname{Re} \mu}{t^{2m\alpha-1}}\right) \int_0^t \left\| \sigma^{-2m\alpha + (N\alpha/p)} k(\sigma) \right\| \sigma^{-N\alpha/p} \exp\left(\frac{-\operatorname{Re} \mu}{\sigma^{2m\alpha-1}}\right) d\sigma \\ &\leq \left(\int_0^t \left\| \sigma^{-2m\alpha + (N\alpha/p)} k(\sigma) \right\|^p d\sigma \right)^{1/p} \left(\int_0^t \sigma^{-qN\alpha/p} d\sigma \right)^{1/q} \\ &\leq \left(\frac{1}{1 - (qN\alpha)/p} \right)^{1/q} t^{(1/q) - (N\alpha/p)} \|k\|_E \end{aligned}$$

where $(1/p) + (1/q) = 1$. Hence $w(t)$ is defined and $w(0) = 0$ since

$$\frac{1}{q} - \frac{N\alpha}{p} = 1 - \frac{1}{p} - \frac{N\alpha}{p}$$

means $p > 1 + N\alpha$. On the other hand we can write

$$\begin{aligned} t^{-2m\alpha+(N\alpha/p)} w(t) &= \int_0^t \left(\frac{k(\sigma)}{t^{2m\alpha-(N\alpha/p)} \sigma^{2m\alpha}} \exp \mu(t^{-2m\alpha+1} - \sigma^{-2m\alpha+1}) \right) d\sigma \\ &= \int_0^t \left(\frac{k(\sigma)}{\sigma^{2m\alpha-(N\alpha/p)}} \left(\frac{1}{t^{2m\alpha-(N\alpha/p)} \sigma^{N\alpha/p}} \exp \mu(t^{-2m\alpha+1} - \sigma^{-2m\alpha+1}) \right) \right) d\sigma. \end{aligned}$$

Putting

$$K_\mu(t, \sigma) = \begin{cases} \frac{1}{t^{2m\alpha-(N\alpha/p)} \sigma^{N\alpha/p}} \exp \mu(t^{-2m\alpha+1} - \sigma^{-2m\alpha+1}) & \text{if } t > \sigma, \\ 0 & \text{if } t < \sigma, \end{cases}$$

we deduce that

$$t^{-2m\alpha+(N\alpha/p)} w(t) = \int_0^1 \frac{k(\sigma)}{\sigma^{2m\alpha-(N\alpha/p)}} K_\mu(t, \sigma) d\sigma.$$

We need the following classical interpolation result, the so-called Schur's Lemma.

Lemma 1. *If there exists a constant C such that*

- a) $\left| \int_0^1 K_\mu(t, \sigma) d\sigma \right| \leq C$ for every $t \in]0, 1[$,
- b) $\left| \int_0^1 K_\mu(t, \sigma) dt \right| \leq C$ for every $\sigma \in]0, 1[$,

then

$$\left\| t^{-2m\alpha+(N\alpha/p)} w \right\|_{L^p(0,1;X)} \leq C \left\| t^{-2m\alpha+(N\alpha/p)} k \right\|_{L^p(0,1;X)}.$$

Now, we have to check that the conditions a) and b) are satisfied.

Condition a)

We have

$$\begin{aligned} \int_0^1 |K_\mu(t, \sigma)| d\sigma &= \frac{1}{t^{2m\alpha-(N\alpha/p)}} \exp(t^{-2m\alpha+1} \operatorname{Re} \mu) \int_0^t \frac{\exp(-\sigma^{-2m\alpha+1} \operatorname{Re} \mu)}{\sigma^{N\alpha/p}} d\sigma \\ &\leq \frac{1}{2m\alpha-1} \exp(t^{-2m\alpha+1} \operatorname{Re} \mu) \int_{t^{-2m\alpha+1}}^{+\infty} \exp(-s \operatorname{Re} \mu) ds \\ &\leq \frac{1}{\operatorname{Re} z}. \end{aligned}$$

Consequently

$$\max_{t \in [0,1]} \int_0^1 |K_\mu(t, \sigma)| d\sigma \leq \frac{1}{\operatorname{Re} z}. \quad (4.1)$$

This shows that the condition a) of Lemma 1 holds true.

Condition b)

We have

$$\begin{aligned} \int_0^1 |K_\mu(t, \sigma)| dt &= \frac{1}{\sigma^{\frac{N\alpha}{p}}} \exp(-\sigma^{-2m\alpha+1} \operatorname{Re} \mu) \int_\sigma^1 \frac{\exp(t^{-2m\alpha+1} \operatorname{Re} \mu)}{t^{2m\alpha-(N\alpha/p)}} dt \\ &= \frac{1}{2m\alpha-1} \frac{\exp(-\sigma^{-2m\alpha+1} \operatorname{Re} \mu)}{\sigma^{\frac{N\alpha}{p}}} \int_1^{\sigma^{-2m\alpha+1}} \frac{1}{s^{\frac{N\alpha}{p(2m\alpha-1)}}} \exp(s \operatorname{Re} \mu) ds, \end{aligned}$$

and

$$\begin{aligned} \int_1^{\sigma^{-2m\alpha+1}} \frac{1}{s^{\frac{N\alpha}{p(2m\alpha-1)}}} \exp(s \operatorname{Re} \mu) ds &= \int_1^{\frac{1+\sigma^{-2m\alpha+1}}{2}} \frac{1}{s^{\frac{N\alpha}{p(2m\alpha-1)}}} \exp(s \operatorname{Re} \mu) ds \\ &\quad + \int_{\frac{1+\sigma^{-2m\alpha+1}}{2}}^{\sigma^{-2m\alpha+1}} \frac{1}{s^{\frac{N\alpha}{p(2m\alpha-1)}}} \exp(s \operatorname{Re} \mu) ds \\ &\leq \int_1^{\frac{1+\sigma^{-2m\alpha+1}}{2}} \exp(s \operatorname{Re} \mu) ds \\ &\quad + \frac{1}{\left(\frac{1+\sigma^{-2m\alpha+1}}{2}\right)^{\frac{N\alpha}{p(2m\alpha-1)}}} \int_{\frac{1+\sigma^{-2m\alpha+1}}{2}}^{\sigma^{-2m\alpha+1}} \exp(s \operatorname{Re} \mu) ds, \\ &= I_1 + I_2. \end{aligned}$$

Then

$$\begin{aligned} I_1 &\leq \frac{1}{\operatorname{Re} \mu} [\exp(\operatorname{Re} \mu^{\frac{(1+\sigma^{-2m\alpha+1})}{2}}) - \exp(\operatorname{Re} \mu)] \\ &\leq \frac{1}{\operatorname{Re} \mu} \exp(\operatorname{Re} \mu^{\frac{(1+\sigma^{-2m\alpha+1})}{2}}), \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2m\alpha-1} \frac{\exp(-\sigma^{-2m\alpha+1} \operatorname{Re} \mu)}{\sigma^{\frac{N\alpha}{p}}} I_1 &\leq \frac{1}{\operatorname{Re} z} \frac{\exp\left(\frac{-(\sigma^{-2m\alpha+1}-1)}{2} \operatorname{Re} \mu\right)}{\sigma^{\frac{N\alpha}{p}}} \\ &\leq \frac{1}{\operatorname{Re} z} \frac{\exp\left(\frac{-(\sigma^{-2m\alpha+1}-1)}{2} \mu_0\right)}{\sigma^{\frac{N\alpha}{p}}} \\ &\leq \frac{C_1(\alpha, p)}{\operatorname{Re} z}, \end{aligned}$$

since the function

$$\sigma \mapsto \frac{\exp\left(\frac{-(\sigma^{-2m\alpha+1}-1)}{2} \mu_0\right)}{\sigma^{\frac{N\alpha}{p}}}$$

is continuous on $[0, 1]$. Moreover

$$\begin{aligned} \frac{1}{2m\alpha-1} \frac{\exp(-\sigma^{-2m\alpha+1} \operatorname{Re} \mu)}{\sigma^{\frac{N\alpha}{p}}} I_2 &\leq \frac{1}{2m\alpha-1} \frac{\exp(-\sigma^{-2m\alpha+1} \operatorname{Re} \mu)}{\sigma^{\frac{N\alpha}{p} \left(\frac{1+\sigma^{-2m\alpha+1}}{2} \right)^{\frac{N\alpha}{p(2m\alpha-1)}}}} \int_{\frac{1+\sigma^{-2m\alpha+1}}{2}}^{\sigma^{-2m\alpha+1}} \exp(s \operatorname{Re} \mu) ds \\ &\leq \frac{1}{\operatorname{Re} z} \frac{C_2(\alpha, p)}{\sigma^{\frac{N\alpha}{p} \left(\frac{1+\sigma^{-2m\alpha+1}}{2} \right)^{\frac{N\alpha}{p(2m\alpha-1)}}}} \\ &\leq \frac{C_3(\alpha, p)}{\operatorname{Re} z} \end{aligned}$$

in virtue of the fact that

$$\lim_{\sigma \rightarrow 0} \frac{1}{\sigma^{\frac{N\alpha}{p} (1 + \sigma^{-2m\alpha+1})^{\frac{N\alpha}{p(2m\alpha-1)}}}} = 1.$$

Consequently, there exists some constant $C(\alpha, p) > 0$ such that

$$\max_{\sigma \in [0,1]} \int_0^1 |K_z(t, \sigma)| dt \leq \frac{C(\alpha, p)}{\operatorname{Re} z}. \quad (4.2)$$

This shows that the condition b) of Lemma 1 holds also true. Now, using Lemma 1 together with (4.1) and (4.2), we obtain

$$\left\| t^{-2m\alpha+(N\alpha/p)} w \right\|_{L^p(0,1;X)} \leq \frac{C(\alpha, p)}{\operatorname{Re} z} \left\| t^{-2m\alpha+(N\alpha/p)} k \right\|_{L^p(0,1;X)},$$

from which it follows

$$\left\| (B + zI)^{-1} \right\|_{L(E)} \leq \frac{C(\alpha, p)}{\operatorname{Re} z}.$$

Thus, we can take $\theta_B = \frac{\pi}{2} - \theta_0$, (for each $\theta_0 \in]0, \frac{\pi}{2}[$).

2. Now, we are concerned with the operator A which has the same properties as its realization $L(t)$. The study uses the following perturbation result due to Lunardi ([22, Proposition 2.4.3, p. 65]).

Proposition 2. *Let L_0 be a linear operator of domain $D(L_0)$ dense in E . Assume that L_0 is sectorial and P a linear continuous operator on $D(L_0)$ which is compact. Then operator $L_0 + P : D(L_0) \rightarrow X$ is sectorial.*

For each $t \in [0, 1]$, we write

$$L(t)\psi = L_0\psi + P(t)\psi$$

with

$$\begin{cases} D(L_0) &= \left\{ \psi \in W^{2m,p}(B(0,1)) : \partial_\nu^l \psi|_{\partial B(0,1)} = 0, l = 0, 1, \dots, m-1 \right\} \\ L_0\psi &= M\psi = \sum_{k=1}^N \partial_{y_k}^{2m} \psi \end{cases}$$

and

$$\begin{cases} D(P(t)) &= W^{1,p}(B(0,1)) \\ P(t)\psi &= -\alpha t^{2m\alpha-1} \sum_{k=1}^N y_k \partial_{y_k} \psi, \end{cases}$$

It is well known that $\overline{D(L_0)} = L^p(B(0,1))$. The fact that L_0 is sectorial can be proved as in [8, Lemma 5.2 and Lemma 5.3, pp. 18-19]. Observe that

$$\psi^{(l)}(y) = \int_0^y (-s\psi^{(l+1)}(s))ds + \int_y^1 (1-s)\psi^{(l+1)}(s)ds - \int_0^1 \psi^{(l)}(s)ds,$$

$l = 1, 2, \dots, 2m-1$, where $\psi^{(l)}$ denotes the derivative of order l of ψ . Thanks to Hölder inequality, for $\psi \in D(L_0) \subset D(P(t))$ and by using the previous equality we have

$$\begin{aligned} \|P(t)\psi\|_{L^p(B(0,1))} &= \left(\int_{B(0,1)} |-\alpha t^{2m\alpha-1} \sum_{k=1}^N y_k \partial_{y_k} \psi(y_1, y_2, \dots, y_N)|^p dy_1 dy_2 \dots dy_N \right)^{\frac{1}{p}} \\ &\leq \sum_{k=1}^N \left(\int_{B(0,1)} |-\alpha t^{2m\alpha-1} y_k (\int_0^{y_k} s_k \partial_{s_k}^2 \psi ds_k + \int_{y_k}^1 (1-s_k) \partial_{s_k}^2 \psi ds_k)|^p dy_1 \dots dy_N \right)^{\frac{1}{p}} \\ &\leq \alpha t^{2m\alpha-1} \left[C_1(p) \|M\psi\|_{L^p(B(0,1))} + C_2(p) \|M\psi\|_{L^p(B(0,1))} \right] \\ &\leq C_3(\alpha, p) \|\psi\|_{D(L_0)}. \end{aligned}$$

On the other hand, let us set

$$\begin{aligned} m_k(t) : L^p(B(0,1)) &\rightarrow L^p(B(0,1)) \\ \psi &\mapsto (m_k(t)\psi) = -\alpha t^{2m\alpha-1} y_k \psi, \quad k = 1, \dots, N, \\ i : W^{1,p}(B(0,1)) &\rightarrow L^p(B(0,1)) \\ \psi &\mapsto \psi, \\ d_k : W^{2m,p}(B(0,1)) &\rightarrow W^{1,p}(B(0,1)) \\ \psi &\mapsto d_k(\psi) = \partial_{y_k} \psi, \quad k = 1, \dots, N. \end{aligned}$$

Then one has

$$P(t) = \sum_{k=1}^N P_k(t) = \sum_{k=1}^N m_k(t) \circ i \circ d_k.$$

Thus, $P(t)$ is compact from $D(L_0)$ into E since i is compact and $d_k, m_k(t), k = 1, \dots, N$ are continuous. So for any $t \in [0, 1]$, the operator $L(t)$ is sectorial and consequently there exist some $r_0 > 0$ and $\theta_1 \in]0, \frac{\pi}{2}[$ such that

$$\rho(-L(t)) \supset \Sigma_{\pi-\theta_1} = \{z : |z| \geq r_0, |\arg z| < \pi - \theta_1\}.$$

Now, for $k \in E$ and $z \in \Sigma_{\pi-\theta_1}$ the spectral equation

$$Aw + zw = k,$$

is equivalent to

$$L(t)w(t) + zw(t) = k(t), \quad t \in [0, 1],$$

which admits a unique solution

$$w(t) = (L(t) + zI)^{-1} k(t).$$

Hence

$$\|w(t)\|_{L^p(B(0,1))} \leq \frac{K}{|z|} \|k(t)\|_{L^p(B(0,1))},$$

which implies

$$\|w\|_E = \left(\int_0^1 \left\| t^{-2m\alpha + (N\alpha/p)} w(t) \right\|_X^p d\tau \right)^{1/p} \leq \frac{K}{|z|} \|k\|_E.$$

This ends the proof of Proposition 1. $\square$

Proposition 3. *Operators A and B satisfy the Labbas-Terreni condition (2.4).*

Proof. In our case, since the domains $D(L(t))$ are constant, the condition (2.4) holds whenever the following so-called estimate of Sobolevskii [30] is fulfilled: There exists $K > 0$ such that for all $t, \sigma \in [0, 1]$,

$$\|(L(t)L(\sigma)^{-1} - I)\|_{L(X)} \leq K |t - \sigma|^\rho \quad (4.3)$$

For $g \in X = L^p(B(0, 1))$, the equation $\psi = L(\sigma)^{-1}g$ is equivalent to

$$\begin{cases} (L(t)\psi)(y) = M\psi - \alpha t^{2m\alpha-1} \sum_{k=1}^N y_k \partial_{y_k} \psi(y) = g(y), \\ \partial_v^l \psi|_{\partial B(0,1)} = 0, \quad l = 0, 1, \dots, m-1, \end{cases}$$

and

$$[(L(t) - L(\sigma))L(\sigma)^{-1}g](y) = \alpha(\sigma^{2m\alpha-1} - t^{2m\alpha-1}) \sum_{k=1}^N y_k \partial_{y_k} \psi(y).$$

where $y = (y_1, y_2, \dots, y_N)$. Then, we get

$$\begin{aligned} \|(L(t) - L(\sigma))L(\sigma)^{-1}g\|_X &\leq \alpha |t^{2m\alpha-1} - \sigma^{2m\alpha-1}| \left\| \sum_{k=1}^N y_k \partial_{y_k} \psi \right\|_X \\ &\leq M_1 |t - \sigma|^{\min(1, 2m\alpha-1)} \|M\psi\|_{L^p(B(0,1))} \\ &\leq M_2 |t - \sigma|^{\min(1, 2m\alpha-1)} \|\psi\|_{W^{2m,p}(B(0,1))} \\ &\leq K |t - \sigma|^{\min(1, 2m\alpha-1)} \|g\|_{L^p(B(0,1))}. \end{aligned}$$

So, the condition (4.3) is satisfied with $\rho = \min(1, 2m\alpha - 1)$. To prove (2.4), it is sufficient to estimate

$$\left\| A(A + \lambda)^{-1} \left[A^{-1}; (B + z)^{-1} \right] \right\|_{L(E)}$$

where $\lambda \in \rho(-A)$ and $z \in \rho(-B)$. Let $k \in E$, then

$$\begin{aligned}
 D &= \left(t^{-2m\alpha+(N\alpha/p)} A(A+\lambda)^{-1} \left[A^{-1}; (B+z)^{-1} \right] k \right)(t) \\
 &= t^{-2m\alpha+(N\alpha/p)} A(A+\lambda)^{-1} \left(A^{-1} (B+z)^{-1} - (B+z)^{-1} A^{-1} \right) k(t) \\
 &= t^{-2m\alpha+(N\alpha/p)} L(t) (L(t)+\lambda)^{-1} \\
 &\quad \times \left[L(t)^{-1} \left((B+z)^{-1} k \right)(t) - \left((B+z)^{-1} L(t)^{-1} k \right)(t) \right] \\
 &= L(t) (L(t)+\lambda)^{-1} \int_0^1 \sigma^{-2m\alpha+(N\alpha/p)} K_\mu(t, \sigma) \left(L(t)^{-1} - L(\sigma)^{-1} \right) k(\sigma) d\sigma \\
 &= \int_0^1 \sigma^{-2m\alpha+(N\alpha/p)} K_\mu(t, \sigma) L(t) (L(t)+\lambda)^{-1} \left(L(t)^{-1} - L(\sigma)^{-1} \right) k(\sigma) d\sigma \\
 &= \int_0^1 \sigma^{-2m\alpha+(N\alpha/p)} K_\mu(t, \sigma) (L(t)+\lambda)^{-1} \left(I - L(t) L(\sigma)^{-1} \right) k(\sigma) d\sigma
 \end{aligned}$$

since the domains $D(L(t))$ are constant, where (we recall)

$$K_\mu(t, \sigma) = \begin{cases} \frac{1}{t^{2m\alpha-(N\alpha/p)} \sigma^{N\alpha/p}} \exp \mu(t^{-2m\alpha+1} - \sigma^{-2m\alpha+1}) & \text{if } t > \sigma, \\ 0 & \text{if } t < \sigma, \end{cases}$$

with $\mu = \frac{z}{(2m\alpha-1)}$. Then

$$\|D\|_X \leq \frac{K}{|\lambda|} \int_0^1 |K_\mu(t, \sigma)| |t - \sigma|^\rho \sigma^{-2m\alpha+(N\alpha/p)} \|k(\sigma)\|_X d\sigma,$$

with $\rho = \min(1, 2m\alpha - 1)$. We have

$$\begin{aligned}
 \int_0^1 |K_\mu(t, \sigma)| |t - \sigma|^\rho d\sigma &= \frac{1}{t^{2m\alpha-(N\alpha/p)}} \exp(t^{-2m\alpha+1} \operatorname{Re} \mu) \\
 &\quad \times \int_0^t \sigma^{-N\alpha/p} (t - \sigma)^\rho \exp(-\sigma^{-2m\alpha+1} \operatorname{Re} \mu) d\sigma.
 \end{aligned}$$

Then by Hölder inequality, one has

$$\begin{aligned}
 &\int_0^t \sigma^{-N\alpha/p} (t - \sigma)^\rho \exp(-\sigma^{-2m\alpha+1} \operatorname{Re} \mu) d\sigma \\
 &\leq \left(\int_0^t \sigma^{-N\alpha/p} \exp(-\sigma^{-2m\alpha+1} \operatorname{Re} \mu) d\sigma \right)^{1-\rho} \\
 &\quad \times \left(\int_0^t \sigma^{-N\alpha/p} (t - \sigma) \exp(-\sigma^{-2m\alpha+1} \operatorname{Re} \mu) d\sigma \right)^\rho
 \end{aligned}$$

and

$$\begin{aligned}
 J_1 &= \left(\int_0^t \sigma^{2m\alpha-(N\alpha/p)} \sigma^{-2m\alpha} \exp(-\sigma^{-2m\alpha+1} \operatorname{Re} \mu) d\sigma \right)^{1-\rho} \\
 &\leq \frac{(t^{2m\alpha-(N\alpha/p)})^{1-\rho}}{(2m\alpha-1)^{1-\rho}} \frac{1}{(\operatorname{Re} \mu)^{1-\rho}} \left(\exp(-t^{-2m\alpha+1} \operatorname{Re} \mu) \right)^{1-\rho} \\
 J_2 &= \left(\int_0^t \sigma^{2m\alpha-(N\alpha/p)} \sigma^{-2m\alpha} (t - \sigma) \exp(-\sigma^{-2m\alpha+1} \operatorname{Re} \mu) d\sigma \right)^\rho \\
 &\leq \frac{(t^{2m\alpha-(N\alpha/p)})^\rho}{(2m\alpha-1)^\rho} \frac{1}{(\operatorname{Re} \mu)^\rho} \left(\int_0^t (t - \sigma) \chi'(\sigma) d\sigma \right)^\rho
 \end{aligned}$$

where $\chi(\sigma) = \exp(-\sigma^{-2m\alpha+1} \text{Re } \mu)$. Using an integration by parts, we obtain

$$\begin{aligned} \int_0^t (t-\sigma) \chi'(\sigma) d\sigma &= \int_0^t \exp(-\sigma^{-2m\alpha+1} \text{Re } \mu) d\sigma \\ &= \int_0^t \sigma^{2m\alpha} \sigma^{-2m\alpha} \exp(-\sigma^{-2m\alpha+1} \text{Re } \mu) d\sigma \\ &\leq \frac{t^{2m\alpha}}{2m\alpha-1} \frac{1}{\text{Re } \mu} \exp(-t^{-2m\alpha+1} \text{Re } \mu) \end{aligned}$$

from which we deduce that

$$J_2 \leq \frac{(t^{2m\alpha-(N\alpha/p)})^\rho}{(2m\alpha-1)^\rho} \frac{1}{(\text{Re } \mu)^\rho} \frac{(t^{2m\alpha})^\rho}{(2m\alpha-1)^\rho} \frac{1}{(\text{Re } \mu)^\rho} (\exp(-t^{-2m\alpha+1} \text{Re } \mu))^\rho.$$

Finally we have

$$\begin{aligned} \int_0^1 |K_\mu(t, \sigma)| |t-\sigma|^\rho d\sigma &\leq \frac{\exp(t^{-2m\alpha+1} \text{Re } \mu)}{t^{2m\alpha-(N\alpha/p)}} \frac{(t^{2m\alpha-(N\alpha/p)})^{1-\rho}}{(2m\alpha-1)^{1-\rho}} \frac{(\exp(-t^{-2m\alpha+1} \text{Re } \mu))^{1-\rho}}{(\text{Re } \mu)^{1-\rho}} \\ &\quad \times \frac{(t^{2m\alpha-(N\alpha/p)})^\rho}{(2m\alpha-1)^\rho} \frac{1}{(\text{Re } \mu)^\rho} \frac{\rho}{(2m\alpha-1)^\rho} \frac{1}{(\text{Re } \mu)^\rho} [\exp(-t^{-2m\alpha+1} \text{Re } \mu)]^\rho \\ &\leq \frac{(t^{2m\alpha})^\rho}{(2m\alpha-1)^{1+\rho}} \frac{1}{(\text{Re } \mu)^{1+\rho}}, \end{aligned}$$

and

$$\max_{t \in [0,1]} \int_0^1 |K_\mu(t, \sigma)| |t-\sigma|^\rho d\sigma \leq \frac{C}{(\text{Re } \mu)^{1+\rho}}. \quad (4.4)$$

In a similar manner we obtain

$$\max_{\sigma \in [0,1]} \int_0^1 |K_\mu(t, \sigma)| |t-\sigma|^\rho dt \leq \frac{C}{(\text{Re } \mu)^{1+\rho}}. \quad (4.5)$$

Now, using Schur interpolation Lemma together with (4.4) and (4.5), we obtain

$$\left\| A(A+\lambda)^{-1} \left[A^{-1}; (B+z)^{-1} \right] \right\|_{L(E)} \leq \frac{C}{|\lambda| (\text{Re } \mu)^{1+\rho}} = \frac{C}{|\lambda| (\text{Re } z)^{1+\rho}}$$

which implies

$$\left\| A(A+\lambda)^{-1} \left[A^{-1}; (B+z)^{-1} \right] \right\|_{L(E)} \leq \frac{C}{|\lambda| |z|^{1+\rho}}$$

for any $\lambda \in \rho(-A)$ and any z belonging to a simple path γ joining $\infty e^{-i\theta_2}$ to $\infty e^{i\theta_2}$ for some $\theta_2 \in]\pi - \theta_B, \theta_1[$, γ lies to $\Sigma_{\pi-\theta_1} \cap \Sigma_{\pi-\theta_B}$. Then (2.4) is verified with $(\tau, \rho) = (0, \min(1, 2m\alpha - 1))$. $\square$

5. REGULARITY RESULTS FOR THE ORIGINAL PROBLEM

5.1. Regularity results for the transformed problem (3.1)

Using Theorem 1, we deduce the following result

Proposition 4. *There exists λ^* such that for all $\lambda \geq \lambda^*$ and for all $k \in D_A(\sigma, p)$ (respectively, $k \in D_B(\sigma, p)$), Problem (3.2) admits a unique solution $w \in D(A) \cap D(B)$ such that for all $\theta \leq \min(\sigma, 2m\alpha - 1)$*

- i) $L(\cdot)w \in D_A(\theta, p)$,
- ii) $t^\alpha w' \in D_A(\theta, p)$,
- iii) $L(\cdot)w \in D_B(\theta, p)$

(respectively,

- i) $L(\cdot)w \in D_B(\theta, p)$,
- ii) $t^\alpha w' \in D_B(\theta, p)$,
- iii) $L(\cdot)w \in D_A(\theta, p)$).

Now, let us specify the space $D_A(\sigma, p)$. One has

$$D_A(\sigma, p) =$$

$$\left\{ \begin{array}{l} \left\{ w \in E : t^{-2m\alpha + (N\alpha/p)} w \in L^p(0, 1; W^{2m\sigma, p}(B(0, 1))) , \partial_\nu^l w|_{\partial B(0, 1)} = 0 \right\} \\ \text{where } l = 0, 1, \dots, m-1; \text{ if } 2m\sigma > 1/p, \\ \left\{ w \in E : t^{-2m\alpha + (N\alpha/p)} w \in L^p(0, 1; W^{2m\sigma, p}(B(0, 1))) \right\} \text{ if } 2m\sigma < 1/p. \end{array} \right\}$$

Indeed, we know that

$$D_A(\sigma, p) = \left\{ w \in E : \left\| \zeta^{1-\sigma} A e^{-\zeta A} w \right\|_E \in L_*^p \right\},$$

because $(-A)$ is a generator of the analytic semigroup $\left\{ e^{-\zeta A} \right\}_{\zeta \geq 0}$. Now, $w \in D_A(\sigma, p)$ implies

$$\left\| \zeta^{1-\sigma} A e^{-\zeta A} w \right\|_E \in L_*^p.$$

Or $\left\| \zeta^{1-\sigma} A e^{-\zeta A} w \right\|_E \in L_*^p$ is equivalent to

$$\begin{aligned} & \int_0^\infty \left\| \zeta^{1-\sigma} A e^{-\zeta A} w \right\|_E^p \frac{d\zeta}{\zeta} \\ &= \int_0^\infty \left\| t^{-2m\alpha + (N\alpha/p)} \zeta^{1-\sigma} A e^{-\zeta A} w \right\|_{L^p(0, 1; L^p(B(0, 1)))}^p \frac{d\zeta}{\zeta} \\ &= \int_0^\infty \left(\int_0^1 \left\| t^{-2m\alpha + (N\alpha/p)} \zeta^{1-\sigma} (A e^{-\zeta A} w)(t) \right\|_{L^p(B(0, 1))}^p dt \right) \frac{d\zeta}{\zeta} < +\infty. \end{aligned}$$

On the other hand, thanks to the Dunford representation of the semigroup $\{e^{-\zeta A}\}_{\zeta \geq 0}$, we have

$$e^{-\zeta A} = \frac{1}{2i\pi} \int_{\gamma} e^{\zeta \lambda} (A + \lambda I)^{-1} d\lambda,$$

where γ is a sectorial curve lying in $\rho(-A)$ such that $\operatorname{Re}(-\lambda) < 0$ for a larger $\lambda \in \gamma$. Moreover

$$(Ae^{-\zeta A} w)(t) = L(t) e^{\zeta L(t)} (w(t)).$$

Then, by Fubini's Theorem, we obtain

$$\begin{aligned} & \int_0^\infty \left\| \zeta^{1-\sigma} A e^{-\zeta A} w \right\|_E^p \frac{d\zeta}{\zeta} \\ &= \int_0^\infty \left[\int_0^1 \left\| t^{-2m\alpha + (N\alpha/p)} \zeta^{1-\sigma} L(t) e^{\zeta L(t)} (w(t)) \right\|_{L^p(B(0,1))}^p dt \right] \frac{d\zeta}{\zeta} \\ &= \int_0^1 \left\| t^{-2m\alpha + (N\alpha/p)} \right\|^p \left[\int_0^\infty \left\| \zeta^{1-\sigma} L(t) e^{\zeta L(t)} w(t) \right\|_{L^p(B(0,1))}^p \frac{d\zeta}{\zeta} \right] dt < +\infty, \end{aligned}$$

which means that, for almost every t , the function

$$(y_1, y_2, \dots, y_N) \mapsto t^{-2m\alpha + (N\alpha/p)}(t) w(t) (y_1, y_2, \dots, y_N)$$

is in $D_{L(t)}(\sigma, p)$. It is well known that this last space is the following:

$$D_{L(t)}(\sigma, p) = (W^{2m,p}(B(0,1)) \cap W_0^{m,p}(B(0,1)); L^p(B(0,1)))_{1-\sigma,p}$$

and

$$\begin{aligned} & (W^{2m,p}(B(0,1)) \cap W_0^{2,p}(B(0,1)); L^p(B(0,1)))_{1-\sigma,p} \\ &= \begin{cases} \left\{ w \in W^{2m\sigma,p}(B(0,1)) : \partial_\nu^l w|_{\partial B(0,1)} = 0, l = 0, 1, \dots, m-1 \right\} \\ \text{if } 2m\sigma > 1/p, \\ W^{2m\sigma,p}(B(0,1)) \text{ if } 2m\sigma < 1/p. \end{cases} \end{aligned}$$

Let σ be a fixed positive number satisfying $\sigma < 1/2mp$ and $\sigma \leq 2m\alpha - 1$. From the above proposition, we deduce the following result.

Proposition 5. *For all h such that $t^{-2m\alpha + (N\alpha/p)} h \in L^p(0, 1; W^{2m\sigma,p}(B(0,1)))$, Problem (3.3) admits a unique solution fulfilling the following regularity properties:*

- (i) $w \in L^p([0, 1] \times B(0, 1))$, $t^{-2m\alpha + (N\alpha/p)} w \in L^p([0, 1] \times B(0, 1))$, $w(0) = 0$,
- (ii) $t^{-2m\alpha + (N\alpha/p)} M_w \in L^p([0, 1] \times B(0, 1))$,
- (iii) $t^{N\alpha/p} \partial_t w \in L^p([0, 1] \times B(0, 1))$,
- (iv) $t^{-2m\alpha + (N\alpha/p)} M_w \in L^p(0, 1; W^{2m\sigma,p}(B(0,1)))$,
- (v) $t^{N\alpha/p} \partial_t w \in L^p(0, 1; W^{2m\sigma,p}(B(0,1)))$.

5.2. Going back to the original problem (1.1)

We go back to our original domain Q by using the inverse change of variables

$$\begin{aligned} \Pi^{-1} : G =]0, 1[\times B(0, 1) &\longrightarrow Q \\ (t, y_1, y_2, \dots, y_N) &\longmapsto (t, x_1, x_2, \dots, x_N) = (t, t^\alpha y_1, t^\alpha y_2, \dots, t^\alpha y_N). \end{aligned}$$

Let us recall that

$$\begin{cases} h(t, y_1, y_2, \dots, y_N) = t^{2m\alpha} g(t, y_1, y_2, \dots, y_N) \\ g(t, y_1, y_2, \dots, y_N) = f(t, x_1, x_2, \dots, x_N), \\ w(t, y_1, y_2, \dots, y_N) = u(t, x_1, x_2, \dots, x_N). \end{cases}$$

First, we see that

$$\begin{cases} \partial_{y_k} w = t^\alpha \partial_{x_k} u, \quad k = 1, 2, \dots, N, \\ \partial_{y_k}^{2m} w = t^{2m\alpha} \partial_{x_k}^{2m} u, \quad k = 1, 2, \dots, N, \\ \partial_t w = \partial_t u + (\alpha/t) \sum_{k=1}^N x_k \partial_{x_k} u. \end{cases}$$

The assumption $t^{-2m\alpha + (N\alpha/p)} h \in L^p(0, 1; W^{2m\sigma, p}(B(0, 1)))$ means that

$$\int_0^1 \left\| t^{-2m\alpha + (N\alpha/p)}(t) h(t, \cdot) \right\|_{W^{2m\sigma}(B(0, 1))}^p dt < \infty.$$

So, by setting

$$y = (y_1, y_2, \dots, y_N), \quad y' = (y'_1, y'_2, \dots, y'_N), \quad dy = dy_1 \dots dy_N, \quad dy' = dy'_1 \dots dy'_N,$$

we have

$$\begin{aligned} &\int_0^1 \left\| t^{-2m\alpha + (N\alpha/p)}(t) h(t, \cdot) \right\|_{W^{2m\sigma}(B(0, 1))}^p dt \\ &= \int_0^1 t^{N\alpha - 2m\alpha p} \int_{B(0, 1)} \int_{B(0, 1)} \frac{|h(t, y) - h(t, y')|^p}{\|y - y'\|^{2m\sigma p + N}} dy dy' dt \\ &= \int_0^1 t^{2m\sigma\alpha p} \int_{\Omega_t} \int_{\Omega_t} \frac{|f(t, x) - f(t, x')|^p}{\|x - x'\|^{2m\sigma p + N}} dx dx' dt, \end{aligned}$$

where

$$x = (t^\alpha y_1, t^\alpha y_2, \dots, t^\alpha y_N), \quad x' = (t^\alpha y'_1, t^\alpha y'_2, \dots, t^\alpha y'_N), \quad dx = dx_1 \dots dx_N, \quad dx' = dx'_1 \dots dx'_N$$

and

$$\Omega_t = \left\{ (x_1, x_2, \dots, x_N) \in \mathbb{R}^N : 0 \leq \sqrt{x_1^2 + x_2^2 + \dots + x_N^2} < t^\alpha \right\}.$$

Let us introduce the following subspace of $L^p(Q)$:

$$\begin{aligned} &L_{t^{2m\sigma\alpha}}^p(0, 1; W_{t^\alpha}^{2m\sigma, p}) = \\ &\left\{ f \in L^p(Q) : \int_0^1 t^{2m\sigma\alpha p} \int_{\Omega_t} \int_{\Omega_t} \frac{|f(t, x) - f(t, x')|^p}{\|x - x'\|^{2m\sigma p + N}} dx dx' dt < \infty \right\}. \end{aligned}$$

Then, we are in position to prove the main result of this work.

Theorem 2. For given $\sigma \in]0, 1[$ such that $0 < \sigma < \frac{1}{2mp}$ (such that p verifies (1.2)), and for any $f \in L_{t^{2m\sigma}}^p(0, 1; W_{t^\alpha}^{2m\sigma, p})$, Problem (1.1) has a unique solution $u \in H_p^{1, 2m}(Q)$ with the regularities: $u, \partial_t u, \partial_{x_k} u, k = 1, \dots, N$ and Mu belong to $L_{t^{2m\sigma}}^p(0, 1; W_{t^\alpha}^{2m\sigma, p})$.

The proof of Theorem 2 can be easily deduced from the following equivalences.

Proposition 6.

- (i) $t^{-2m\alpha + (N\alpha/p)} h \in L^p(0, 1; W^{2m\sigma, p}(B(0, 1)))$ if and only if
$$f \in L_{t^{2m\sigma}}^p(0, 1; W_{t^\alpha}^{2m\sigma, p}),$$
- (ii) $t^{-2m\alpha + (N\alpha/p)} w \in L^p(0, 1; L^p(B(0, 1)))$ if and only if $u \in L^p(Q)$,
- (iii) $t^{-2m\alpha + (N\alpha/p)} Mw \in L^p(0, 1; W^{2m\sigma, p}(B(0, 1)))$ if and only if
$$Mu \in L_{t^{2m\sigma}}^p(0, 1; W_{t^\alpha}^{2m\sigma, p}),$$
- (iv) $t^{N\alpha/p} \partial_t w \in L^p(0, 1; W^{2m\sigma, p}(B(0, 1)))$ if and only if
$$\partial_t u \in L_{t^{2m\sigma}}^p(0, 1; W_{t^\alpha}^{2\sigma, p}).$$

ACKNOWLEDGEMENT

The authors are grateful to anonymous reviewers for carefully reading this paper and for their valuable comments and suggestions, which have improved the paper.

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LOCAL EXISTENCE AND BLOW UP FOR P-LAPLACIAN EQUATION WITH LOGARITHMIC NONLINEARITY

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Received 22 September, 2020

Abstract. This paper deals with a problem of a wave equation with p-Laplacian and logarithmic nonlinearity term. Firstly, local existence of weak solutions have been obtained by applying Banach fixed theorem. Later, the finite-time blow up of the solutions have been obtained for negative initial energy.

2010 Mathematics Subject Classification: 35A01; 35B44; 35L05

Keywords: existence, blow up, p-Laplacian equation, logarithmic nonlinearity

1. INTRODUCTION

In this work, we consider the following p-Laplacian hyperbolic type equation with logarithmic nonlinearity

$$\begin{cases} u_{tt} - \Delta u - \operatorname{div} \left(|\nabla u|^{p-2} \nabla u \right) - \Delta u_t + |u_t|^{k-2} u_t = |u|^{p-2} u \ln |u|, & x \in \Omega, t > 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in \Omega, \\ u(x, t) = 0, & x \in \partial\Omega, t \geq 0, \end{cases} \quad (1.1)$$

where $p > k > 2$ are real numbers and $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, the functions u_0, u_1 are given initial data and exponent p satisfies

$$\begin{cases} 2 < p < \infty, & \text{if } n = 1, 2, \\ 2 < p < \frac{2(n-1)}{n-2} & \text{if } n \geq 3. \end{cases} \quad (1.2)$$

In absence of p-Laplacian operator $\operatorname{div} \left(|\nabla u|^{p-2} \nabla u \right)$, the problem (1.2) can be reduced to the following wave equation with damping and logarithmic source terms:

$$u_{tt} - \Delta u + h(u_t) = |u|^{p-2} u \ln |u|. \quad (1.3)$$

Problems like equation (1.3) is encountered naturally in quantum mechanics, inflation cosmology, supersymmetric field theories, and a lot of different areas of physics such as, optics, geophysics and nuclear physics [3, 10]. Let us review some works with related to the problem (1.3). Many authors have investigated the local existence,

blow up, and asymptotic behaviour of solutions for wave equation with logarithmic nonlinearity, see in this regard [1, 2, 5, 7, 10, 11, 14, 16–18, 21–23, 26] and a long list of references therein.

The p-Laplacian parabolic problems with logarithmic nonlinearity was investigated by [4, 6, 8, 9, 12, 13, 15, 24].

In [24], Le et al. carried out the research on pseudo p-Laplacian evolution equations with logarithmic nonlinearity

$$u_t - \operatorname{div}(|\nabla u|^{p-2} \nabla u) - \Delta u_t = |u|^{p-2} u \ln |u|, \quad (1.4)$$

and established results of existence or nonexistence of global weak solutions when $p > 2$. They proved the large time decay of the global weak solutions. Finally, authors showed that the solution $u(t)$ is not global, that means, it blows up at finite time.

In [4], the authors studied the following equation

$$u_t - \operatorname{div}(|\nabla u|^{p-2} \nabla u) - k \Delta u_t = |u|^{p-2} u \ln |u|, \quad (1.5)$$

where $1 < p < 2$, $\Omega \subset \mathbb{R}^n (n \geq 1)$, $k \geq 0$. They showed that the weak solutions of (1.5) are global and can not blow up in finite time when different from the case $p > 2$. They found the sufficient conditions to divide the global boundedness and blowing-up at infinity of the weak solutions.

To best of our knowledge, there are not enough works related to the hyperbolic type equation with p-Laplacian operator and logarithmic nonlinearity. Recently, this type of problems was studied by [20, 25].

In [20] the same author of this paper investigated the following hyperbolic type equation with logarithmic nonlinearity

$$u_{tt} - \operatorname{div}(|\nabla u|^{p-2} \nabla u) - \Delta u + u_t = k u \ln |u|. \quad (1.6)$$

and proved the local existence of solutions for problem (1.6) by using potential well theory combined with Galerkin method.

By the motivation of the above works, we decided to study the local existence and blow up of solution for problem (1.1). The remaining part of this paper is organized as follows: In Section 2, we state some notations and lemmas which will be useful for our main results. In Section 3, the local existence of the solution is given. In Section 4, we established the finite time blow up when the initial energy is negative.

2. PRELIMINARIES

In this section we give some notations and lemmas which will be used throughout this paper. For simplify notations, we adopt the following abbreviations:

$$\|u\|_p = \|u\|_{L^p(\Omega)}, \quad \|u\|_{1,s} = \|u\|_{W_0^{1,p}(\Omega)} = \left(\|u\|_p + \|\nabla u\|_p \right)^{\frac{1}{p}},$$

for $2 < p < \infty$. We use C and C_i ($i = 1, 2, \dots$) to denote various positive constants and they can have different values in different places. In order to state our main results, we define the corresponding energy to problem (1.1) as follows

$$E(t) = \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|\nabla u\|^2 + \frac{1}{p} \|\nabla u\|_p^p - \frac{1}{p} \int_{\Omega} \ln |u| u^p dx + \frac{1}{p^2} \|u\|_p^p. \quad (2.1)$$

For proof of Lemma 1, we refer the reader to Pişkin [19]. The proof of Lemma 2 is clear and straightforward, and will be omitted.

Lemma 1. For any $u \in H_0^1(\Omega)$, we get

$$\|u\|_q \leq C_q \|\nabla u\|_2,$$

for all $1 \leq q \leq \frac{2n}{n-2}$ if $n \geq 3$; $1 \leq q < \infty$ if $n \leq 2$, where C_p is the best embedding constant.

Lemma 2. For each $q > 0$

$$|r^q \ln r| \leq \frac{1}{eq} \text{ for } 0 < r < 1 \text{ and } 0 \leq r^{-q} \ln r \leq \frac{1}{eq} \text{ for } r \geq 1.$$

Lemma 3. $E(t)$ is a nonincreasing function, for $t \geq 0$

$$E'(t) = -\|\nabla u_t\|^2 - \|u_t\|_k^k \leq 0. \quad (2.2)$$

Proof. Multiplying equation (1.1) by u_t and integrating on Ω by using Green formula, we have

$$\begin{aligned} \int_{\Omega} u_{tt} u_t dx + \int_{\Omega} \nabla u \nabla u_t dx - \int_{\Omega} \operatorname{div}(|\nabla u|^{p-2} \nabla u) u_t dx + \int_{\Omega} \nabla u_t \nabla u_t dx \\ + \int_{\Omega} |u_t|^{k-1} u_t dx = \int_{\Omega} u^{p-2} u \ln |u| u_t dx, \end{aligned}$$

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|\nabla u\|^2 + \frac{1}{p} \|\nabla u\|_p^p - \frac{1}{p} \int_{\Omega} \ln |u| u^p dx + \frac{1}{p^2} \|u\|_p^p \right) \\ = -\|\nabla u_t\|^2 - \|u_t\|_k^k, \end{aligned}$$

$$E'(t) = -\|\nabla u_t\|^2 - \|u_t\|_k^k.$$

□

Lemma 4 ([14]). Let ϑ be a positive number. Then the following inequality holds

$$|s|^{p-2} \log |s| \leq A + |s|^{p-2+\vartheta}, \quad p > 2 \quad (2.3)$$

for $A > 0$.

For reader's straightforwardness, we notice the definition of weak solutions of problem (1.1).

Definition 1. A function $u(t)$ is called a weak solution of problem (1.1) on $\Omega \times [0, T)$, if

$$u \in C\left((0, T); W_0^{1,p}(\Omega)\right) \cap C^1\left((0, T); L^2(\Omega)\right)$$

and

$$u_t \in L^\infty\left((0, T); H_0^1(\Omega)\right)$$

which satisfies

$$\begin{cases} \int_{\Omega} u_{tt}(x, t) w(x) dx + \int_{\Omega} |\nabla u(x, t)|^{p-2} \nabla u(x, t) \nabla w(x) dx \\ \quad + \int_{\Omega} \nabla u(x, t) \nabla w(x) dx + \int_{\Omega} \nabla u_t(x, t) \nabla w(x) dx \\ \quad + \int_{\Omega} |u_t|^{k-2}(x, t) |u_t|(x, t) w(x) dx \\ = \int_{\Omega} \ln |u(x, t)| u^{p-2}(x, t) w(x) dx, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x). \end{cases}$$

for $\forall w \in H_0^1(\Omega)$.

3. EXISTENCE OF LOCAL SOLUTION

In this part, we prove the local existence results by combining contraction mapping principle and Faedo-Galerkin method. Firstly, we state linear problem

$$\begin{cases} v_{tt} - \Delta v - \Delta v_t + |v_t|^{k-2} v_t - \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) = |u|^{p-2} u \ln |u|, & (x, t) \in \Omega \times (0, T), \\ v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x), & x \in \Omega, \\ v = \frac{\partial v}{\partial n} = 0, & x \in \partial\Omega \times \mathbb{R}^+, \end{cases} \quad (3.1)$$

in which $T > 0$. We consider the space

$$\mathcal{L} := C\left([0, T]; W_0^{1,p}(\Omega)\right) \cap C^1\left([0, T]; H_0^1(\Omega)\right)$$

endowed with the norm

$$\|u\|_{\mathcal{L}}^2 = \max_{t \in [0, T]} \left(\|\nabla u(t)\|_{\mathcal{L}}^2 + \|u_t(t)\|_2^2 \right).$$

To prove the existence and uniqueness of local solution of problem (1.1), we obtain the following result.

Lemma 5. Let $(u_0, u_1) \in W_0^{1,p}(\Omega) \times H_0^1(\Omega)$ and $u \in \mathcal{L}$ for every $T > 0$. Then problem (3.1) has a unique weak solution

$$v \in C\left([0, T]; W_0^{1,p}(\Omega)\right), \quad v_t \in C\left([0, T]; H_0^1(\Omega)\right).$$

Proof. Let $\{w_j\}_{j=1}^{\infty}$ be the orthogonal complete system of eigenfunctions of the Laplace operator in $H_0^1(\Omega)$ with $\|w_j\| = 1$ for all j . According to their multiplicity of

$$-\Delta w_i = \lambda_i w_i, \quad w_i|_{\partial\Omega} = 0. \quad (3.2)$$

We denote by λ_j the related eigenvalues repeated. Then, we choose an orthogonal basis $\{w_j\}_{j=1}^{\infty}$ in $H_0^1(\Omega)$ which is orthonormal in $L^2(\Omega)$. Let

$$u_0^m(x, t) = \sum_{j=1}^m \left(\int_{\Omega} \nabla u_0 \nabla w_j \right) w_j,$$

and

$$u_1^m(x, t) = \sum_{j=1}^m \left(\int_{\Omega} u_1 w_j \right) w_j.$$

So that there exist subsequences $u_0^m \in W_m$, $u_1^m \in W_m$, $u_0^m \rightarrow u_0$ in $W_0^{1,p}(\Omega)$ and $u_1^m \rightarrow u_1$ in $H_0^1(\Omega)$ as $m \rightarrow \infty$. For all $m \geq 1$ we will seek an approximate solution such that

$$v_m(t) = \sum_{i=1}^m \gamma_i^m(t) w_i \quad (3.3)$$

which satisfies the following Cauchy problem

$$\begin{cases} \int_{\Omega} \left(v_m''(t) - \Delta v_m - \Delta v_m' + |v_m'|^{k-2} v_m' - \operatorname{div} \left(|\nabla u|^{p-2} \nabla u \right) - |u|^{p-2} u \ln |u| \right) \eta dx = 0, \\ v_m(0) = v_0^m, \quad v_m' = v_1^m, \end{cases} \quad (3.4)$$

where $t \geq 0$ and $\eta \in W_m$. Let $\eta = w_i$ for $i = 1, 2, \dots, m$ in (3.4), for the first term, we obtain

$$\int_{\Omega} v_m''(t) w_i dx = \int_{\Omega} \left(\sum_{j=1}^m \ddot{\gamma}_j^m(t) w_j \right) w_i dx = \ddot{\gamma}_i^m(t) \int_{\Omega} |w_i|^2 dx = \ddot{\gamma}_i^m(t). \quad (3.5)$$

Similarly we obtain the second term as follows

$$\begin{aligned} \int_{\Omega} -\Delta v_m w_i dx &= - \int_{\Omega} \sum_{j=1}^m \gamma_j^m(t) \Delta w_j w_i dx \\ &= \int_{\Omega} \sum_{j=1}^m \gamma_j^m(t) \lambda_j w_j w_i dx \\ &= \gamma_i^m(t) \lambda_i \int_{\Omega} |w_i|^2 dx \\ &= \gamma_i^m(t) \lambda_i. \end{aligned} \quad (3.6)$$

For the third term we have

$$\begin{aligned} \int_{\Omega} -\Delta v'_m(t) w_i dx &= \int_{\Omega} \left(\sum_{j=1}^m -\dot{\Delta} \gamma_j^m(t) w_j \right) w_i dx \\ &= \nabla \dot{\gamma}_j^m(t). \end{aligned} \quad (3.7)$$

For the fourth term, we get

$$\begin{aligned} \int_{\Omega} |v'_m(t)|^{k-2} v'_m(t) w_i dx &= \int_{\Omega} \left(\sum_{j=1}^m |\dot{\gamma}_j^m(t)|^{k-2} \dot{\gamma}_j^m(t) w_j \right) w_i dx \\ &= |\dot{\gamma}_j^m(t)|^{k-2} \dot{\gamma}_j^m(t) \int_{\Omega} |w_i|^2 dx \\ &= |\dot{\gamma}_j^m(t)|^{k-2} \dot{\gamma}_j^m(t). \end{aligned} \quad (3.8)$$

Then, we insert (3.5)-(3.8) in (3.4). So that (3.4) yields the following Cauchy problem for a linear ordinary differential equation for unknown functions $\gamma_i^m(t)$;

$$\begin{cases} \ddot{\gamma}_i^m(t) + \gamma_i^m(t) \lambda_i + \nabla \dot{\gamma}_j^m(t) + |\dot{\gamma}_j^m(t)|^{k-2} \dot{\gamma}_j^m(t) = G_i(t), \quad i = 1, 2, \dots, m, \\ \gamma_i^m(0) = \int_{\Omega} u_0 w_i dx, \quad \dot{\gamma}_i^m(0) = \int_{\Omega} u_1 w_i dx, \quad i = 1, 2, \dots, m, \end{cases} \quad (3.9)$$

where

$$G_i(t) = \int_{\Omega} \operatorname{div} \left(|\nabla u|^{p-2} \nabla u \right) w_i dx + \int_{\Omega} |u|^{p-2} u \ln |u| w_i, \quad i = 1, 2, \dots, m, t \in [0, T]. \quad (3.10)$$

Then the above problem possesses a unique local solution $\gamma_i^m \in C^2[0, T]$ for all i , which implies a unique v_m defined by (3.3) satisfies (3.4). Replacing η by v'_m in (3.4) and then integrating this over $[0, t] \subset [0, T]$, we obtain

$$\begin{aligned} &\|v'_m(t)\|^2 + \|\nabla v_m(t)\|^2 + 2 \int_0^t \|\nabla v'_m(\tau)\|^2 d\tau + 2 \int_0^t \|v'_m(\tau)\|_k^k d\tau \\ &= \|v_{1m}(t)\|^2 + \|\nabla v_{0m}\|^2 + 2 \int_0^t \int_{\Omega} |u(s)|^{p-2} u(s) \ln |u(s)| v'_m(s) dx ds \\ &\quad + 2 \int_0^t \int_{\Omega} \operatorname{div} \left(|\nabla u|^{p-2} \nabla u \right) (s) v'_m(s) dx ds, \end{aligned} \quad (3.11)$$

for each $m \geq 1$.

Our aim is to estimate the last two terms in the right-hand side of (3.11). First, we estimate the third term by applying Sobolev's and Young's inequalities, we infer that

$$\begin{aligned} 2 \int_0^t \int_{\Omega} |u(s)|^{p-2} u(s) \ln |u(s)| v'_m(s) dx ds &\leq 2 \int_0^t \int_{\Omega} \left| |u(s)|^{p-1} \ln |u(s)| \right| |v'_m(s)| dx ds \\ &\leq \int_0^t \int_{\Omega} \left| |u(s)|^{p-1} \ln |u(s)| \right|^{\frac{m}{m-1}} dx ds + \int_0^t \|v'_m(s)\|_k^k ds. \end{aligned} \quad (3.12)$$

In order to estimate (3.12), we deal with logarithmic term. We define

$$\Omega_1 = \{x \in \Omega; |u(x)| < 1\} \text{ and } \Omega_2 = \{x \in \Omega; |u(x)| \geq 1\},$$

where $\Omega = \Omega_1 \cup \Omega_2$. Then we obtain

$$\begin{aligned} \int_{\Omega} \left| |u(s)|^{p-1} \ln |u(s)| \right|^{\frac{k}{k-1}} dx \\ = \int_{\Omega_1} \left| |u(s)|^{p-1} \ln |u(s)| \right|^{\frac{k}{k-1}} dx + \int_{\Omega_2} \left| |u(s)|^{p-1} \ln |u(s)| \right|^{\frac{k}{k-1}} dx. \end{aligned} \quad (3.13)$$

By make use of Lemma 2, we have

$$\int_{\Omega_1} \left| |u(s)|^{p-1} \ln |u(s)| \right|^2 dx \leq [e(p-1)]^{-\frac{k}{k-1}} |\Omega| = C, \quad (3.14)$$

where

$$\inf_{r \in (0,1)} r^{p-1} \ln r = [e(p-1)]^{-1}.$$

Let

$$\theta = \frac{2n}{n-2} \cdot \frac{k}{k-1} - p + 1 > 0 \text{ for } n \geq 3; \text{ each positive } \theta \text{ for } n = 1, 2.$$

By the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$ if $n \geq 3$ and to $L^q(\Omega)$ for any $q \geq 1$ if $n = 1, 2$, recalling $u \in \mathcal{L} := C([0, T]; H_0^1(\Omega))$, we obtain

$$\begin{aligned} \int_{\Omega_2} \left| |u(s)|^{p-1} \ln |u(s)| \right|^{\frac{k}{k-1}} dx &\leq \int_{\Omega_2} \theta^{-\frac{k}{k-1}} \theta^{\frac{k}{k-1}} \left(|u(s)|^{p-1} \ln |u(s)| \right)^{\frac{k}{k-1}} dx \\ &\leq \theta^{-\frac{k}{k-1}} \int_{\Omega_2} \left(|u(s)|^{p-1} \ln |u(s)| \right)^{\theta} dx \\ &\leq \theta^{-\frac{k}{k-1}} \int_{\Omega_2} \left(|u(s)|^{p-1+\theta} \right)^{\frac{k}{k-1}} dx \end{aligned}$$

$$\begin{aligned}
&\leq \theta^{-\frac{k}{k-1}} \int_{\Omega_2} |u(s)|^{\frac{2n}{n-2}} dx \\
&\leq \theta^{-\frac{k}{k-1}} \int_{\Omega=\Omega_1 \cup \Omega_2} |u(s)|^{\frac{2n}{n-2}} dx \\
&= \theta^{-\frac{k}{k-1}} \|u\|_{\frac{2n}{n-2}}^{\frac{2n}{n-2}} \\
&\leq C \|u\|_{\mathcal{L}}^{\frac{2n}{n-2}} \leq C.
\end{aligned} \tag{3.15}$$

The proof of the case $n = 1, 2$ is similar.

By using of (3.14), (3.15) and embedding theorem, (3.12) yields

$$2 \int_0^t \int_{\Omega} |u(s)|^{p-2} u(s) \ln |u(s)| v'_m(s) dx ds \leq CT + \int_0^t \|v'_m(s)\|_k^k ds. \tag{3.16}$$

By similar calculations we have the second term of (3.11) in the right-hand side as follows

$$\begin{aligned}
2 \int_0^t \int_{\Omega} \operatorname{div} \left(|\nabla u|^{p-2} \nabla u \right) (s) v'_m(s) dx ds &\leq 2 \int_0^t \left(\frac{1}{p} \|\nabla u(s)\|_p^p \|v'_m(s)\| \right) ds \\
&\leq \int_0^t \left(\left(\frac{1}{p} \|\nabla u(s)\|_p^p \right)^2 + \|v'_m(s)\|_2^2 \right) ds \\
&\leq CT + C \int_0^t \|\nabla v'_m(s)\|_2^2 ds.
\end{aligned} \tag{3.17}$$

Adapting (3.17) and (3.16) into (3.11), we have

$$\begin{aligned}
&\|v'_m(t)\|^2 + \|\nabla v_m(t)\|^2 + \int_0^t \|\nabla v'_m(\tau)\|^2 d\tau + \int_0^t \|v'_m(\tau)\|_k^k d\tau, \\
&\leq \|v_{1m}(t)\|^2 + \|\nabla v_{0m}\|^2 + CT \leq C,
\end{aligned} \tag{3.18}$$

where C is a positive constant and independent of m . We have from (3.18) that

$$\begin{cases} v_m, \text{ is bounded in } L^\infty([0, T]; H_0^1(\Omega)), \\ v'_m, \text{ is bounded in } L^\infty([0, T]; H_0^1(\Omega)). \end{cases} \tag{3.19}$$

Thus, up to subsequence, we can pass to the limit in (3.4) satisfies the above regularity. We obtained a weak local solution of the problem (3.4).

To prove the uniqueness, arguing by contradiction, we assume that there are two solutions such that w and v of problem (3.4) which have the same initial data. Subtracting these two equations and testing result by $w_t - v_t$, we can obtain

$$\begin{aligned} \|w_t - v_t\|^2 + \|\nabla w - \nabla v\|^2 + 2 \int_0^T \int_{\Omega} (\nabla w_{\tau} - \nabla v_{\tau}) (\nabla w_{\tau} - \nabla v_{\tau}) d\tau \\ + 2 \int_0^T \int_{\Omega} \left(|w_{\tau}|^{k-2} w_{\tau} - |v_{\tau}|^{k-2} v_{\tau} \right) (w_{\tau} - v_{\tau}) d\tau = 0. \end{aligned} \quad (3.20)$$

We define $f : R \rightarrow R$, $f(x) = x$ and $g : R \rightarrow R$, $g(y) = y$. It follows from the following element inequality

$$\left(|f(x)|^{k-2} f(x) - |g(y)|^{k-2} g(y) \right) (f(x) - g(y)) \geq C |f(x) - g(y)|^k \text{ for } k \geq 2,$$

that (3.20) yields

$$\|w_t - v_t\|^2 + \|\nabla w - \nabla v\|^2 + c \left(\int_0^T \|\nabla w_{\tau} - \nabla v_{\tau}\|^2 + \int_0^T \|\nabla w_{\tau} - \nabla v_{\tau}\|_k^k \right) \leq 0.$$

□

Theorem 1. *There exist $T > 0$, such that the problem (1.1) has a unique local weak solution on $[0, T]$.*

Proof. Let $(u_0, u_1) \in W_0^{1,p}(\Omega) \times L^2(\Omega)$ and

$$r^2 = \|u_1\|^2 + \|\nabla u_0\|^2.$$

For $T > 0$, we denote

$$B_r = \{u \in \mathcal{L} : u(0, x) = u_0(x), u_t(0, x) = u_1(x), \|u(t)\|_{\mathcal{L}} \leq r\}.$$

By Lemma 5, we define a nonlinear mapping $\Psi : u \rightarrow v = \Psi(u)$ where v is the unique solution of problem (3.1). We claim that Ψ is a contraction map satisfying $\Psi(B_r) \subseteq B$ for small $T > 0$.

Indeed, suppose that $u \in B_r$, the corresponding solution $v = \Psi(u)$ satisfies the energy identity for all $t \in [0, T]$ as follows

$$\begin{aligned} \frac{1}{2} \left(\|v_t\|^2 + \|\nabla v\|^2 \right) &\leq \frac{1}{2} \left(\|u_1\|^2 + \|\nabla u_0\|^2 \right) + \int_0^t \int_{\Omega} |u(s)|^{p-2} u(s) \ln |u(s)| v_t(s) dx ds \\ &+ \int_0^t \int_{\Omega} \operatorname{div} \left(|\nabla u|^{p-2} \nabla u \right) v_t(s) dx ds. \end{aligned} \quad (3.21)$$

Next we estimate the last terms in (3.21) as

$$\begin{aligned}
\frac{1}{2} \left(\|v_t\|^2 + \|\nabla v\|^2 \right) &\leq \frac{r^2}{2} + \int_0^t \left(r^{\frac{2n}{n-2}} + \frac{1}{p} r^p \right) \|v_t(s)\| ds \\
&= \frac{r^2}{2} + \sqrt{2} \left(r^{\frac{2n}{n-2}} + \frac{1}{p} r^p \right) \int_0^t \left(\frac{1}{2} \|v_t(s)\|^2 \right)^{\frac{1}{2}} ds \\
&\leq \frac{r^2}{2} + \sqrt{2} \left(r^{\frac{2n}{n-2}} + \frac{1}{p} r^p \right) \int_0^t \left(\frac{1}{2} \|v_t(s)\|^2 + \frac{1}{2} \|\nabla v(s)\|^2 \right)^{\frac{1}{2}} ds.
\end{aligned} \tag{3.22}$$

By Gronwall inequality, we have

$$\left(\frac{1}{2} \|v_t(s)\|^2 + \frac{1}{2} \|\nabla v(s)\|^2 \right)^{\frac{1}{2}} \leq \frac{r}{2} + \sqrt{2} \left(r^{\frac{2n}{n-2}} + \frac{1}{p} r^p \right) T. \tag{3.23}$$

If we choose $T > 0$ small enough, $\|u(t)\|_{\mathcal{L}} \leq r$, which implies that $\Psi(B_r) \subseteq B$.

Next, we will verify that Ψ is a contraction mapping. Let $v_1, v_2 \in X_{r_0, T}$ and $u_1 = \Psi v_1$, $u_2 = \Psi v_2$, be the corresponding solution for problem (3.1). Setting $v_1 = \Psi w_1$, $v_2 = \Psi w_2$, $w_1, w_2 \in B_r$, and $v = v_1 - v_2$, then, v satisfies

$$\left\{ \begin{aligned} &(v_{1t}, \eta) + (\nabla v, \nabla \eta) + \int_{\Omega} (\nabla v_{1t} - \nabla v_{2t}) \nabla \eta dx \\ &\quad + \int_{\Omega} (\nabla v_{1t} - \nabla v_{2t}) \nabla \eta dx + \int_{\Omega} (|v_{1t}|^{k-2} v_{1t} - |v_{2t}|^{k-2} v_{2t}) \eta dx \\ &= \int_{\Omega} \left[\operatorname{div} (|\nabla w_1|^{p-2} \nabla w_1) - \operatorname{div} (|\nabla w_2|^{p-2} \nabla w_2) \right] \eta dx \\ &\quad + \int_{\Omega} (|w_1|^{p-2} w_1 \ln |w_1| - |w_2|^{p-2} w_2 \ln |w_2|) \eta dx, \end{aligned} \right. \tag{3.24}$$

for any $\eta \in H_0^1(\Omega)$ and a.e. $t \in [0, T]$.

Taking $\eta = v_t = v_{1t} - v_{2t}$ and using Green formula, noticing

$$\begin{aligned}
\int_{\Omega} (|v_{1t}|^{k-2} v_{1t} - |v_{2t}|^{k-2} v_{2t}) (v_{1t} - v_{2t}) dx &\geq 0, \\
\int_{\Omega} (\nabla v_{1t} - \nabla v_{2t}) (\nabla v_{1t} - \nabla v_{2t}) dx &\geq 0,
\end{aligned} \tag{3.25}$$

and integrating both sides of (3.24) over $(0, t)$, we have

$$\left(\|v_t\|^2 + \|\nabla v\|^2 \right) = 2 \int_0^t \int_{\Omega} (|w_1|^{p-2} w_1 \ln |w_1| - |w_2|^{p-2} w_2 \ln |w_2|) v_t dx ds$$

$$+ 2 \int_0^t \int_{\Omega} \left[\operatorname{div} \left(|\nabla w_1|^{p-2} \nabla w_1 \right) - \operatorname{div} \left(|\nabla w_2|^{p-2} \nabla w_2 \right) \right] v_t dx ds. \quad (3.26)$$

Firstly we note that from the Mean Value Theorem there exists $C_p > 0$ such that

$$||x|^{p-2}x - |y|^{p-2}y| \leq C_p(|x|^{p-2} + |y|^{p-2})|x - y|, \forall x, y \in \mathbb{R}^2. \quad (3.27)$$

Then using Hölder inequality with $\frac{p-2}{2(p-1)} + \frac{1}{2(p-1)} + \frac{1}{2} = 1$ and embedding theorems, we have

$$\begin{aligned} & \left| \int_0^t \int_{\Omega} \left[\operatorname{div} \left(|\nabla w_1|^{p-2} \nabla w_1 \right) - \operatorname{div} \left(|\nabla w_2|^{p-2} \nabla w_2 \right) \right] v_t dx ds \right| \\ & \leq C \int_0^t \int_{\Omega} \left(|\nabla w_1|^{p-2} + |\nabla w_2|^{p-2} \right) |\nabla w_1 - \nabla w_2| |v_t| dx ds \\ & \leq C \int_0^t \left(\|\nabla w_1\|_{2(p-1)}^{p-2} + \|\nabla w_2\|_{2(p-1)}^{p-2} \right) \|v_t\|_{2(p-1)} \|\nabla w_1 - \nabla w_2\|_2^2 dx ds \\ & \leq C \int_0^t r^{2(p-2)} \|\nabla w_1 - \nabla w_2\|_2^2 \|v_t\|_2 \\ & \leq C \int_0^t r^{2(p-2)} \|\nabla w_1 - \nabla w_2\|_2^2 (\|v_t\|_2 + \|\nabla v\|_2), \end{aligned} \quad (3.28)$$

where $C > 0$.

Now, we need estimate the logarithmic term in (3.26). If we set

$$G(s) = |s|^{p-2} s \ln |s|,$$

then

$$\begin{aligned} G'(s) &= (p-1)|s|^{p-2} \ln |s| + |s|^{p-2} \\ &= (1 + (p-1) \ln |s|) |s|^{p-2}. \end{aligned}$$

Making use of mean value theorem, we get

$$\begin{aligned} |G(w_1) - G(w_2)| &= |G'(\vartheta w_1 + (1-\vartheta)w_2)(w_1 - w_2)| \\ &\leq [1 + (p-1) \ln |(\vartheta w_1 + (1-\vartheta)w_2)|] |(\vartheta w_1 + (1-\vartheta)w_2)|^{p-2} |w_1 - w_2| \end{aligned} \quad (3.29)$$

where $0 < \vartheta < 1$.

Then, by recalling Lemma 4 we conclude that

$$|G(w_1) - G(w_2)| \leq |(\vartheta w_1 + (1-\vartheta)w_2)|^{p-2} |w_1 - w_2| + (p-1)A |w_1 - w_2|$$

$$\begin{aligned}
& + (p-1) |w_1 - w_2| |(\vartheta w_1 + (1-\vartheta) w_2)|^{p-2+\varepsilon} \\
& \leq |(w_1 + w_2)|^{p-2} |w_1 - w_2| + (p-1) A |w_1 - w_2| \\
& + (p-1) |w_1 - w_2| |w_1 + w_2|^{p-2+\varepsilon}
\end{aligned} \tag{3.30}$$

As $w_1, w_2 \in B_r$, then it follows from Hölder inequality (for $\frac{1}{p-1} + \frac{p-2}{p-1} = 1$) and Sobolev embedding that

$$\begin{aligned}
\int_{\Omega} \left[|(w_1 + w_2)|^{p-2} |w_1 - w_2| \right]^2 dx & \leq C \left[\|w_1\|_{2(p-1)}^{2(p-1)} + \|w_2\|_{2(p-1)}^{2(p-1)} \right]^{\frac{p-2}{p-1}} \|w_1 - w_2\|_{2(p-1)}^2 \\
& \leq C \left[\|\nabla w_1\|_2^{2(p-1)} + \|\nabla w_2\|_2^{2(p-1)} \right]^{\frac{p-2}{p-1}} \|\nabla w_1 - \nabla w_2\|_2^2 \\
& \leq C r^{2(p-2)} \|\nabla w_1 - \nabla w_2\|_2^2.
\end{aligned} \tag{3.31}$$

By similar calculation, we obtain

$$\begin{aligned}
& \int_{\Omega} \left[|w_1 - w_2| |w_1 + w_2|^{p-2+\varepsilon} \right]^2 dx \\
& \leq \left(\int_{\Omega} |w_1 + w_2|^{\frac{2(p-2+\varepsilon)(p-1)}{p-2}} dx \right)^{\frac{p-2}{p-1}} \left(\int_{\Omega} |w_1 - w_2|^{2(p-1)} dx \right)^{\frac{1}{p-1}} \\
& \leq \left(\int_{\Omega} |w_1 + w_2|^{2(p-1) + \frac{2\varepsilon(p-1)}{p-2}} dx \right)^{\frac{p-2}{p-1}} \|w_1 - w_2\|_{2(p-1)}^2.
\end{aligned} \tag{3.32}$$

Because of (1.2) ($2p-2 < \frac{2n}{n-2}$), we choose $\varepsilon > 0$ small enough such that

$$p_* = 2(p-1) + \frac{2\varepsilon(p-1)}{p-2} < \frac{2n}{n-2}.$$

Therefore, by using of Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{2(p-1)}(\Omega)$ and $H_0^1(\Omega) \hookrightarrow L^{p_*}(\Omega)$ (3.32) yields that

$$\begin{aligned}
\int_{\Omega} \left[|w_1 - w_2| |w_1 + w_2|^{p-2+\varepsilon} \right]^2 dx & \leq C \left[\|\nabla w_1\|_2^{p_*} + \|\nabla w_2\|_2^{p_*} \right]^{\frac{p-2}{p-1}} \|\nabla w_1 - \nabla w_2\|_2^2 \\
& \leq C r^{\frac{p_*(p-2)}{p-1}} \|\nabla w_1 - \nabla w_2\|_2^2.
\end{aligned} \tag{3.33}$$

Noticing that $\|w_1 - w_2\|_2 \leq C \|\nabla w_1 - \nabla w_2\|_2$, by using of (3.31) and (3.33), we can infer that

$$\int_0^t \int_{\Omega} \left(|w_1|^{p-2} w_1 \ln |w_1| - |w_2|^{p-2} w_2 \ln |w_2| \right) v_t dx ds$$

$$\leq C \int_0^t \left(1 + r^{\frac{p^*(p-2)}{p-1}} + r^{2(p-2)}\right) \|\nabla w_1 - \nabla w_2\|_2^2 (\|v_t\|_2 + \|\nabla v\|_2) ds. \quad (3.34)$$

Consequently, by inserting (3.34), (3.28) into (3.26), we arrive at

$$\begin{aligned} & \left(\|v_t\|^2 + \|\nabla v\|^2 \right) \\ & \leq C \int_0^t \left(1 + r^{\frac{p^*(p-2)}{p-1}} + 2r^{2(p-2)}\right) \|\nabla w_1 - \nabla w_2\|_2^2 (\|v_t\|_2 + \|\nabla v\|_2) ds \\ & \leq C \int_0^t \left(1 + r^{\frac{p^*(p-2)}{p-1}} + 2r^{2(p-2)}\right) \|\nabla w_1 - \nabla w_2\|_2^2 \left(\|v_t\|^2 + \|\nabla v\|^2 \right)^{\frac{1}{2}} ds. \end{aligned}$$

By Gronwall inequality, we obtain

$$\left(\|v_t\|^2 + \|\nabla v\|^2 \right)^{\frac{1}{2}} \leq CT \|\nabla w_1 - \nabla w_2\|_2^2.$$

We choose T small enough such that $C \left(1 + r^{\frac{p^*(p-2)}{p-1}} + 2r^{2(p-2)}\right) T < 1$. Thus, we have Ψ is a contraction mapping in B_r . The proof is completed. $\square$

4. BLOW UP

In this part, we prove the blow up result of solution for the problem (1.1) with negative initial energy. We give some lemmas which be used in our proof. For proof of Lemmas 6-8, we refer the readers to Kafini and Messaoudi [14].

Lemma 6. Suppose that (1.2) holds. There exists a positive constant depending on Ω only such that

$$\left(\int_{\Omega} u^p \ln |u| dx \right)^{\frac{s}{p}} \leq C \left[\int_{\Omega} u^p \ln |u| dx + \|\nabla u\|_2^2 \right], \quad (4.1)$$

for any $u \in L^{p+1}(\Omega)$ and $2 \leq s \leq p$, provided that $\int_{\Omega} u^p \ln |u| dx \geq 0$.

Lemma 7. Suppose that (1.2) holds. There exists a positive constant depending on Ω only such that

$$\|u\|_p^p \leq C \left[\int_{\Omega} u^p \ln |u| dx + \|\nabla u\|_2^2 \right], \quad (4.2)$$

for any $u \in L^p(\Omega)$, provided that $\int_{\Omega} u^p \ln |u| dx \geq 0$.

Thus, the result was obtained.

Corollary 1. *Let the assumptions of the Lemma 7 and $k < p$ hold. Using the fact that $\|u\|_k^k \leq C \|u\|_p^k \leq C (\|u\|_k^p)^{\frac{k}{p}}$. Then we obtain the following*

$$\|u\|_k^k \leq C \left[\left(\int_{\Omega} u^p \ln |u| dx \right)^{\frac{k}{p}} + \|\nabla u\|^{\frac{2k}{p}} \right]. \quad (4.3)$$

Lemma 8. *Suppose that (1.2) holds. There exists a positive constant depending on Ω only such that*

$$\|u\|_p^s \leq C \left[\|u\|_p^p + \|\nabla u\|_2^2 \right], \quad (4.4)$$

for any $u \in L^p(\Omega)$ and $2 \leq s \leq p$.

Theorem 2. *Assume that $E(0) < 0$. Let the conditions in Lemma 8 hold. Then the solutions of (1.1) blow up in finite time*

$$T^* \leq \frac{1 - \alpha}{\xi^{\frac{\alpha}{1-\alpha}} L^{\frac{\alpha}{1-\alpha}}(0)}, \quad (4.5)$$

where ξ and α positive constant.

Proof. For this purpose, we denote

$$H(t) = -E(t). \quad (4.6)$$

By using the definition of $H(t)$ and (2.2) we obtain

$$H'(t) = -E'(t) = \|\nabla u_t\|^2 + \|u_t\|_k^k \geq 0. \quad (4.7)$$

Consequently by virtue of (2.1), (4.6) and (4.7) we get

$$0 < H(0) \leq H(t) \leq \frac{1}{p} \int_{\Omega} u^p \ln |u| dx. \quad (4.8)$$

We set

$$L(t) = H^{1-\alpha}(t) + \varepsilon \int_{\Omega} uu_t dx + \varepsilon \frac{1}{2} \|\nabla u\|^2, \quad (4.9)$$

for ε small to be chosen later and

$$\frac{2(p^2 - 2k)}{(k-1)p^3} < \alpha < \frac{p-k}{(k-1)p}. \quad (4.10)$$

Now, differentiating $L(t)$ with respect to t , we obtain

$$\begin{aligned} L'(t) &= (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon \int_{\Omega} |u_t|^2 dx + \varepsilon \int_{\Omega} uu_{tt} dx + \varepsilon \int_{\Omega} \nabla u \nabla u_t \\ &= (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon \|u_t\|^2 + \varepsilon \int_{\Omega} \nabla u \nabla u_t \end{aligned}$$

$$\begin{aligned}
& + \varepsilon \int_{\Omega} u \left(\Delta u + \operatorname{div} \left(|\nabla u|^{p-2} \nabla u \right) - |u_t|^{k-2} u_t + \Delta u_t + |u|^{p-2} u \ln |u| \right) dx \\
& = (1 - \alpha) H^{-\alpha}(t) H'(t) + \varepsilon \|u_t\|^2 - \varepsilon \|\nabla u\|^2 - \varepsilon \|\nabla u\|_p^p \\
& \quad - \varepsilon \int_{\Omega} |u_t|^{k-2} u_t u dx + \varepsilon \int_{\Omega} u^p \ln |u| dx. \tag{4.11}
\end{aligned}$$

Adding and subtracting $\varepsilon p(1 - \alpha)H(t)$ for some $0 < \alpha < 1$ in (4.11), we obtain

$$\begin{aligned}
L'(t) & = (1 - \alpha) H^{-\alpha}(t) H'(t) + \varepsilon \left(\frac{p(1 - \alpha) + 2}{2} \right) \|u_t\|^2 \\
& \quad - \varepsilon \left(\frac{2 - p(1 - \alpha)}{2} \right) \|\nabla u\|^2 - \varepsilon \alpha \|\nabla u\|_p^p + \varepsilon \frac{(1 - \alpha)}{p} \|u\|_p^p \\
& \quad + \varepsilon \alpha \int_{\Omega} u^p \ln |u| dx - \varepsilon \int_{\Omega} |u_t|^{k-2} u_t u dx + \varepsilon p(1 - \alpha) H(t).
\end{aligned}$$

Exploiting Young's inequality, we get

$$\begin{aligned}
\int_{\Omega} |u_t|^{k-2} u_t u dx & \leq \frac{\delta^k}{k} \|u\|_k^k + \frac{k-1}{k} \delta^{-\frac{k}{k-1}} \|u_t\|_k^k \\
& \leq \frac{\delta^k}{k} \|u\|_k^k + \frac{k-1}{k} \delta^{-\frac{k}{k-1}} \left(\|u_t\|_k^k + \|\nabla u_t\|^2 \right),
\end{aligned}$$

for any $\delta > 0$, (4.11) takes form

$$\begin{aligned}
L'(t) & = \left((1 - \alpha) H^{-\alpha}(t) - \varepsilon \frac{k-1}{k} \delta^{-\frac{k}{k-1}} \right) H'(t) - \varepsilon \frac{\delta^k}{k} \|u\|_k^k \\
& \quad + \varepsilon \left(\frac{p(1 - \alpha) + 2}{2} \right) \|u_t\|^2 - \varepsilon \left(\frac{2 - p(1 - \alpha)}{2} \right) \|\nabla u\|^2 \\
& \quad - \varepsilon \alpha \|\nabla u\|_p^p + \varepsilon p(1 - \alpha) H(t) + \varepsilon \frac{(1 - \alpha)}{p} \|u\|_p^p + \varepsilon \alpha \int_{\Omega} u^p \ln |u| dx. \tag{4.12}
\end{aligned}$$

Of course (4.12) holds even if δ is time dependent since the integral is taken over the x -variable. Therefore by choosing δ so that $\delta^{-\frac{k}{k-1}} = M_1 H^{-\alpha}(t)$, for M_1 to be specified later, and substituting in (4.12), we have

$$\begin{aligned}
L'(t) & \geq \left(1 - \alpha - \varepsilon \frac{k-1}{k} M_1 \right) H^{-\alpha}(t) H'(t) + \varepsilon \frac{(1 - \alpha)}{p} \|u\|_p^p \\
& \quad + \varepsilon \left(\frac{p(1 - \alpha) + 2}{2} \right) \|u_t\|^2 - \varepsilon \alpha \|\nabla u\|_p^p - \varepsilon \left(\frac{2 - p(1 - \alpha)}{2} \right) \|\nabla u\|^2 \\
& \quad - \varepsilon \frac{(M_1)^{1-k}}{k} H^{\alpha(k-1)}(t) \|u\|_k^k + \varepsilon \alpha \int_{\Omega} u^p \ln |u| dx + \varepsilon p(1 - \alpha) H(t). \tag{4.13}
\end{aligned}$$

By using of Corollary 1, embedding theorem and Young's inequality, we obtain

$$\begin{aligned}
H^{\alpha(k-1)} \|u\|_k^k &\leq \left(\frac{1}{p} \int_{\Omega} u^p \ln |u| dx \right)^{\alpha(k-1)} \|u\|_k^k \\
&\leq C \left(\int_{\Omega} u^p \ln |u| dx \right)^{\alpha(k-1)} \left[\left(\int_{\Omega} u^p \ln |u| dx \right)^{\frac{k}{p}} + \|\nabla u\|^{\frac{2k}{p}} \right] \\
&\leq C \left[\left(\int_{\Omega} u^p \ln |u| dx \right)^{\alpha(k-1) + \frac{k}{p}} + \left(\int_{\Omega} u^p \ln |u| dx \right)^{\alpha(k-1)} \|\nabla u\|^{\frac{2k}{p}} \right] \\
&\leq C \left[\left(\int_{\Omega} u^p \ln |u| dx \right)^{\alpha(k-1) + \frac{k}{p}} + \left(\int_{\Omega} u^p \ln |u| dx \right)^{\alpha(k-1)} \|\nabla u\|^{\frac{2k}{p}} \right] \\
&\leq C \left[\left(\int_{\Omega} u^p \ln |u| dx \right)^{\alpha(k-1) + \frac{k}{p}} + \left(\int_{\Omega} u^p \ln |u| dx \right)^{\alpha(k-1) \frac{p^2}{p^2-2k}} + \|\nabla u\|_p^p \right]
\end{aligned} \tag{4.14}$$

Make use of (4.10), we find

$$2 < \alpha(k-1)p + k \leq p \text{ and } 2 < \frac{\alpha(k-1)p^3}{p^2-2k} \leq p.$$

Therefore, Lemma 6 yields that

$$H^{\alpha(k-1)} \|u\|_k^k \leq C \left[\int_{\Omega} u^p \ln |u| dx + \|\nabla u\|_p^p \right]. \tag{4.15}$$

Inserting (4.15) into (4.13), we arrive at

$$\begin{aligned}
L'(t) &\geq \left(1 - \alpha - \varepsilon \frac{k-1}{k} M_1 \right) H^{-\alpha}(t) H'(t) + \varepsilon \left(\frac{p(1-\alpha)+2}{2} \right) \|u_t\|^2 \\
&\quad - \varepsilon \left[\alpha + \frac{(M_1)^{1-k}}{k} C \right] \|\nabla u\|_p^p - \varepsilon \left(\frac{2-p(1-\alpha)}{2} \right) \|\nabla u\|^2 + \varepsilon \frac{(1-\alpha)}{p} \|u\|_p^p \\
&\quad + \varepsilon \left[\alpha - \frac{(M_1)^{1-k}}{k} C \right] \int_{\Omega} u^p \ln |u| dx + \varepsilon p(1-\alpha) H(t).
\end{aligned} \tag{4.16}$$

Since $0 < \frac{2}{p} < 1$, now using the following inequality

$$x^v \leq x + 1 \leq \left(1 + \frac{1}{\beta}\right)(x + \beta), \quad \forall x \geq 0, 0 < v < 1, \beta \geq 0 \quad (4.17)$$

especially taking $x = \|\nabla u\|_p^p$, $d = 1 + \frac{1}{H(0)}$, $\beta = H(0)$ and by using of the (4.8), we have

$$\|\nabla u\|_2^2 \leq \left(\|\nabla u\|_p^p\right)^{\frac{2}{p}} \leq \left(1 + \frac{1}{H(0)}\right) \left(\|\nabla u\|_p^p + H(0)\right) \leq d \left(\|\nabla u\|_p^p + H(t)\right). \quad (4.18)$$

Inserting (4.15)-(4.18) into (4.16) we deduce

$$\begin{aligned} L'(t) &\geq \left(1 - \alpha - \varepsilon \frac{k-1}{k} M_1\right) H^{-\alpha}(t) H'(t) + \varepsilon \left(\frac{p(1-\alpha)+2}{2}\right) \|u_t\|^2 \\ &\quad + \varepsilon \left[d \left(\frac{p(1-\alpha)-2}{2}\right) - \alpha - \frac{(M_1)^{1-k}}{k} C\right] \|\nabla u\|_p^p \\ &\quad + \varepsilon \frac{(1-\alpha)}{p} \|u\|_p^p + \varepsilon \left[\alpha - \frac{(M_1)^{1-k}}{k} C\right] \int_{\Omega} u^p \ln |u| dx \\ &\quad + \varepsilon \left[p(1-\alpha) + d \left(\frac{p(1-\alpha)-2}{2}\right)\right] H(t), \end{aligned} \quad (4.19)$$

where we used $L^p(\Omega) \hookrightarrow L^2(\Omega)$, $2 < p$.

At this point, we choose $0 < \alpha < \frac{p-2}{2p}$ enough small that

$$d \left(\frac{p(1-\alpha)-2}{2}\right) > 0$$

and M_1 sufficiently large that

$$d \left(\frac{p(1-\alpha)-2}{2}\right) - \alpha - \frac{(M_1)^{1-k}}{k} C > 0 \quad \text{and} \quad \alpha - \frac{(M_1)^{1-k}}{k} C > 0.$$

Once M_1 and α are fixed, we pick $0 < \varepsilon < \frac{1-\alpha}{M_1}$ so that

$$L(0) = H^{1-\alpha}(0) + \varepsilon \int_{\Omega} u_0 u_1 dx > 0. \quad (4.20)$$

Therefore, (4.19) has the form

$$L'(t) \geq \lambda \left[H(t) + \|u_t\|^2 + \|\nabla u\|_p^p + \|u\|_p^p + \int_{\Omega} u^p \ln |u| dx \right], \quad (4.21)$$

where $\lambda > 0$ is the minimum of the coefficients of $H(t), \|u_t\|^2, \|\nabla u\|_p^p, \|u\|_p^p, \int_{\Omega} u^p \ln|u| dx$.

Consequently we obtain $L(t) > L(0), t \geq 0$. On the other hand, by $(a+b)^p \leq 2^{p-1}(a^p + b^p)$, we have

$$\begin{aligned} L(t)^{\frac{1}{1-\alpha}} &= \left[H^{1-\alpha}(t) + \varepsilon \int_{\Omega} uu_t dx + \varepsilon \frac{1}{2} \|\nabla u\|^2 \right]^{\frac{1}{1-\alpha}} \\ &\leq C \left[H(t) + \left(\int_{\Omega} |uu_t| dx \right)^{\frac{1}{1-\alpha}} + \|\nabla u\|^{\frac{2}{1-\alpha}} \right]. \end{aligned} \quad (4.22)$$

Now make use of the following inequality,

$$x^v \leq x + 1 \leq \left(1 + \frac{1}{\beta} \right) (x + \beta), \quad \forall x \geq 0, \quad 0 < v < 1, \quad \beta \geq 0$$

for $x = \|\nabla u\|_p^p, v = \frac{2}{p(1-\alpha)} < 1$, since $\alpha < \frac{p-2}{2p}, d = 1 + \frac{1}{H(0)}, \beta = H(0)$, we get

$$\|\nabla u\|^{\frac{2}{1-\alpha}} \leq \left(\|\nabla u\|_p^p \right)^{\frac{2}{p(1-\alpha)}} \leq \left(1 + \frac{1}{H(0)} \right) \left(\|\nabla u\|_p^p + H(0) \right) \leq d \left(\|\nabla u\|_p^p + H(t) \right), \quad (4.23)$$

where we used $L^p(\Omega) \hookrightarrow L^2(\Omega)$.

Hölder's inequality gives us

$$\left| \int_{\Omega} uu_t dx \right| \leq \left(\int_{\Omega} |u_t|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |u|^2 dx \right)^{\frac{1}{2}} \leq c \left(\int_{\Omega} |u_t|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |u|^p dx \right)^{\frac{1}{p}},$$

where c is the positive constant which comes from the embedding $L^p(\Omega) \hookrightarrow L^2(\Omega)$. This inequality implies that there exists a positive constant $C > 0$ such that

$$\left| \int_{\Omega} uu_t dx \right|^{1/(1-\alpha)} \leq \left(\int_{\Omega} |u_t|^2 dx \right)^{\frac{1}{2(1-\alpha)}} \left(\int_{\Omega} |u|^p dx \right)^{\frac{1}{p(1-\alpha)}}. \quad (4.24)$$

Applying Young's inequality to the right-hand side of the (4.24), we get

$$\left| \int_{\Omega} uu_t dx \right|^{1/(1-\alpha)} \leq C \left[\left(\int_{\Omega} |u_t|^2 dx \right)^{\frac{\kappa}{2(1-\alpha)}} + \left(\int_{\Omega} |u|^p dx \right)^{\frac{\mu}{p(1-\alpha)}} \right], \quad (4.25)$$

for $\frac{1}{\mu} + \frac{1}{\kappa} = 1$.

To be able to use Lemma 8, we take $\kappa = 2/(1-\alpha)$, which gives $\mu = 2(1-\alpha)/(1-2\alpha)$, (4.25) has the form

$$\left| \int_{\Omega} uu_t dx \right|^{1/(1-\alpha)} \leq C \left[\int_{\Omega} |u_t|^2 dx + \left(\int_{\Omega} |u|^p dx \right)^{\frac{2}{p(1-2\alpha)}} \right].$$

By using of Poincare's inequality we get

$$\left| \int_{\Omega} uu_t dx \right|^{1/(1-\alpha)} \leq C \left[\int_{\Omega} |u_t|^2 dx + \left(\int_{\Omega} |\nabla u|^p dx \right)^{\frac{2}{p(1-2\alpha)}} \right].$$

With the re-use of the inequality

$$x^v \leq x + 1 \leq \left(1 + \frac{1}{\beta}\right) (x + \beta), \quad \forall x \geq 0, \quad 0 < v < 1, \quad \beta \geq 0,$$

where $x = \|\nabla u\|_p^p$, $d = 1 + \frac{1}{H(0)}$, $\beta = H(0)$, $v = \frac{2}{p(1-2\alpha)} < 1$, since $\alpha < \frac{p-2}{2p}$, we obtain

$$\left(\int_{\Omega} |\nabla u|^p dx \right)^{\frac{2}{p(1-2\alpha)}} \leq \left(1 + \frac{1}{H(0)}\right) (\|\nabla u\|_p^p + H(0)) \leq d [\|\nabla u\|_p^p + H(t)]. \quad (4.26)$$

Thus, (4.24) becomes

$$\left| \int_{\Omega} uu_t dx \right|^{1/(1-\alpha)} \leq C (H(t) + \|u_t\|^2 + \|\nabla u\|_p^p). \quad (4.27)$$

Inserting (4.23) and (4.27) into (4.22), it follows that

$$\begin{aligned} L(t)^{\frac{1}{1-\alpha}} &\leq C (H(t) + \|u_t\|^2 + \|\nabla u\|_p^p), \\ &\leq \left(H(t) + \|u_t\|^2 + \|\nabla u\|_p^p + \|u\|_p^p + \int_{\Omega} u^p \ln |u| dx \right). \end{aligned} \quad (4.28)$$

By associating (4.28) and (4.21) we arrive at

$$L'(t) \geq \xi L^{\frac{1}{1-\alpha}}(t), \quad (4.29)$$

where ξ is a positive constant which depends only on λ , $C > 0$.

Integration of (4.29) over $(0, t)$ we reach

$$\frac{dL}{dt} \geq \xi L^{\frac{1}{1-\alpha}}(t),$$

$$\begin{aligned}\int_0^t \frac{dL}{L^{\frac{1}{1-\alpha}}(t)} &\geq \int_0^t \xi dt, \\ L^{-\frac{\alpha}{1-\alpha}}(t) - L^{-\frac{\alpha}{1-\alpha}}(0) &\geq \xi t, \\ L^{-\frac{\alpha}{1-\alpha}}(t) &\geq L^{-\frac{\alpha}{1-\alpha}}(0) + \xi t, \\ L^{\frac{\alpha}{1-\alpha}}(t) &\geq \frac{1}{L^{-\frac{\alpha}{1-\alpha}}(0) - \frac{\xi \alpha}{1-\alpha}}\end{aligned}$$

Therefore the solutions blow up within a time given by the estimate (4.5) above. Consequently we completed our proof. $\square$

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POSITIVE SOLUTIONS FOR SECOND ORDER IMPULSIVE DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS ON AN INFINITE INTERVAL

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Received 07 September, 2020

Abstract. This paper is concerned with the existence of positive solutions of second-order impulsive differential equations with integral boundary conditions on an infinite interval. As an application, an example is given to demonstrate our main results.

2010 Mathematics Subject Classification: 34B37; 34B15; 34B18; 34B40

Keywords: impulsive differential equations, positive solutions, fixed point theorems in cones, impulsive boundary value problems, integral boundary conditions

1. INTRODUCTION

Consider the following second-order impulsive integral boundary value problem (IBVP) with integral boundary conditions,

$$\begin{cases} \frac{1}{p(t)}(p(t)x'(t))' + f(t, x(t), x'(t)) = 0, & \forall t \in J'_+ \\ \Delta x|_{t=t_k} = I_k(x(t_k)), & k = 1, 2, \dots, n \\ \Delta x'|_{t=t_k} = -\bar{I}_k(x(t_k)), & k = 1, 2, \dots, n \\ a_1 x(0) - b_1 \lim_{t \rightarrow 0^+} p(t)x'(t) = \int_0^\infty g_1(x(s))\psi(s)ds, \\ a_2 \lim_{t \rightarrow \infty} x(t) + b_2 \lim_{t \rightarrow \infty} p(t)x'(t) = \int_0^\infty g_2(x(s))\psi(s)ds, \end{cases} \quad (1.1)$$

where $J = [0, \infty)$, $J_+ = (0, \infty)$, $J'_+ = J_+ \setminus \{t_1, \dots, t_n\}$, $0 < t_1 < t_2 < \dots < t_n$, and note $J_0 = [0, t_1)$, $J_i = (t_i, t_{i+1}]$, $(i = 1, 2, \dots, n)$, $\Delta x|_{t=t_k}$ and $\Delta x'|_{t=t_k}$ denote the jump of $x(t)$ and $x'(t)$ at $t = t_k$, i.e.,

$$\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-), \quad \Delta x'|_{t=t_k} = p(t_k)[x'(t_k^+) - x'(t_k^-)],$$

where $x(t_k^+)$, $x'(t_k^+)$ and $x(t_k^-)$, $x'(t_k^-)$ denote the right-hand limit and left-hand limit of $x(t)$ and $x'(t)$ at $t = t_k$, $k = 1, 2, \dots, n$, respectively.

Throughout this paper, we assume that the following conditions hold;

(H1) $a_1, a_2, b_1, b_2 \in J$ with $D = a_2b_1 + a_1b_2 + a_1a_2B(0, \infty) > 0$ in which

$$B(t, s) = \int_t^s \frac{d\sigma}{p(\sigma)}.$$

(H2) $f \in C(J_+ \times J_+ \times \mathbb{R}, J_+)$ and also, $f(t, x, y) \leq k(t)h(x, y)$, $(t \in J_+)$, where $h \in C(J_+ \times \mathbb{R}, J_+)$ and for $t \in J_+$, x, y in a bounded set, $h(x, y)$ is bounded and $k \in C(J_+, J_+)$.

(H3) $g_1, g_2 : J_+ \rightarrow J_+$ are continuous, nondecreasing functions, and for $t \in J_+$, x in a bounded set, $g_1(x), g_2(x)$ are bounded.

(H4) $I_k, \bar{I}_k \in C(J_+, J_+)$ are bounded functions where

$$[b_2 + a_2B(t_k, \infty)]\bar{I}_k(x(t_k)) - \frac{a_2}{p(t_k)}I_k(x(t_k)) > 0, \quad (k = 1, 2, \dots, n).$$

(H5) $\psi : J \rightarrow J$ is a continuous function with $\int_0^\infty \psi(s)ds < \infty$.

(H6) $p \in C(J, J_+) \cap C^1(J_+, J_+)$ with $p > 0$ on J_+ , and $\int_0^\infty \frac{ds}{p(s)} < \infty$.

Impulsive differential equations encountered in physics, chemical technology, population dynamics, biotechnology, economics etc. (see [2] and the references there in) have become more important in recent years due to the appearance of some important in recent years due to the appearance of some mathematical models of the actual processes. A significant development was observed in theory of impulsive differential equations with fixed time of pulses; see the monographs by Bainov and Simeonov [5], Lakshmikantham, et al. [1], Samoilenko and Prestyuk [18], Benchohra, et al. [4] and the papers [3, 6, 7, 9–11, 13–17, 19].

The existence and multiplicity of positive solutions for linear and nonlinear second-order impulsive dynamic equations have been extensively studied, see [11, 12, 21–23]. Due to the fact that an infinite interval is noncompact, the discussion about boundary value problems on the half-line more complicated, in particular, for impulsive IBVP on an infinite interval, few works were done, see [8, 25]. There is not work on positive solutions for double impulsive IBVP on an infinite interval expect that in [24, 27].

In [27], Zhang, Yang and Feng studied the following double impulsive IBVP:

$$\begin{cases} -x''(t) = f(t, x(t), x'(t)), & t \in J, \quad t \neq t_k, \\ \Delta x|_{t=t_k} = I_k(x(t_k)), & k = 1, 2, \dots \\ \Delta x'|_{t=t_k} = \bar{I}_k(x(t_k)), & k = 1, 2, \dots \\ x(0) = \int_0^\infty g(t)x(t)dt, & x'(\infty) = 0. \end{cases}$$

Using the fixed point theorem in cones, they obtained criteria for existence of the multiple positive solutions.

In [24], Yu, Wang and Guo discussed the existence and multiple positive solutions for the following nonlinear second-order double impulsive integral boundary value

problems:

$$\begin{cases} (\phi_p(x'(t)))' + a(t)f(t, x(t), x'(t)) = 0, & t \in J, \quad t \neq t_k, \\ \Delta x|_{t=t_k} = I_k(x(t_k)), & k = 1, 2, \dots \\ \Delta \phi_p(x')|_{t=t_k} = \bar{I}_k(x(t_k)), & k = 1, 2, \dots \\ x(0) = \int_0^\infty g(t)x(t)dt, & x'(\infty) = 0. \end{cases}$$

Motivated by the above works, in this study, we consider the existence of two positive solutions for the second-order double impulsive integral boundary value problem (1.1). Our boundary conditions are more general. Hence, these results can be considered as a contribution to this field.

The present paper is organized as follows. In Section 2, we present some preliminaries and lemmas which are key tools for our main results. We give and prove our main results in Section 3. Finally, in Section 4, we give an example to demonstrate our results.

2. PRELIMINARIES AND LEMMAS

In this section, we will employ several lemmas to prove the main results in this paper. Set

$$PC(J) = \{x : J \rightarrow \mathbb{R} : x \in C(J'), x(t_k^+) \text{ and } x(t_k^-) \text{ exist and } x(t_k^-) = x(t_k), \quad 1 \leq k \leq n\}.$$

$$PC^1(J) = \{x \in PC(J) : x' \in C(J'), x'(t_k^+) \text{ and } x'(t_k^-) \text{ exist and } x'(t_k^-) = x'(t_k)\}.$$

$$BPC^1(J) = \{x \in PC^1(J) : \lim_{t \rightarrow \infty} x(t) \text{ exists, and } \sup_{t \in J} |x'(t)| < \infty\}.$$

It is easy to see that $BPC^1(J)$ is a Banach space with the norm

$$\|x\| = \sup_{t \in J} \{|x(t)| + |x'(t)|\}.$$

A function $x \in PC^1(J) \cap C^2(J'_+)$ is called a positive solution of the impulsive IBVP (1.1) if $x(t) > 0$ for all $t \in J$ and $x(t)$ satisfies (1.1).

We define a cone $K \subset BPC^1(J)$ as follows:

$$K = \{x \in BPC^1(J) : x(t) > 0, \quad t \in J_+\}.$$

K is a positive cone in $BPC^1(J)$.

By θ and φ we denote the solutions of the corresponding homogeneous equation

$$\frac{1}{p(t)}(p(t)x'(t))' = 0, \quad t \in (0, \infty), \quad (2.1)$$

under the initial conditions,

$$\begin{aligned} \theta(0) &= b_1, \quad \lim_{t \rightarrow 0^+} p(t)\theta'(t) = a_1, \\ \lim_{t \rightarrow \infty} \varphi(t) &= b_2, \quad \lim_{t \rightarrow \infty} p(t)\varphi'(t) = -a_2. \end{aligned} \quad (2.2)$$

Using the initial conditions (2.2), we can deduce from equation (2.1) for $\theta(t)$ and $\varphi(t)$, the following equations:

$$\theta(t) = b_1 + a_1 \int_0^t \frac{ds}{p(s)}, \quad (2.3)$$

$$\varphi(t) = b_2 + a_2 \int_t^\infty \frac{ds}{p(s)}. \quad (2.4)$$

Let $G(t, s)$ be the Green Function for (1.1) is given by

$$G(t, s) = \frac{1}{D} \begin{cases} \theta(t)\varphi(s), & 0 \leq t \leq s < \infty, \\ \theta(s)\varphi(t), & 0 \leq s \leq t < \infty. \end{cases} \quad (2.5)$$

where $\theta(t)$ and $\varphi(t)$ are given in (2.3) and (2.4) respectively.

Lemma 1. Suppose that (H1) – (H6) are satisfied. Then $x \in PC^1(J) \cap C^2(J'_+)$ is a solution of the impulsive IBVP (1.1) if and only if $x(t)$ is a solution of the following integral equation

$$\begin{aligned} x(t) = & \int_0^\infty G(t, s)p(s)f(s, x(s), x'(s))ds + \frac{\varphi(t)}{D} \int_0^\infty g_1(x(s))\psi(s)ds \\ & + \frac{\theta(t)}{D} \int_0^\infty g_2(x(s))\psi(s)ds + \sum_{k=1}^n G(t, t_k)\bar{I}_k(x(t_k)) + \sum_{k=1}^n p(t_k)G_s(t, s)|_{s=t_k}I_k(x(t_k)), \end{aligned}$$

where $G(t, s)$ is given by (2.5).

Remark 1. Under the conditions (H1) and (H6), the Green function $G(t, s)$ in equation (2.5) possesses the following properties:

- (1) $G(t, s)$ is continuous on $J_+ \times J_+$,
- (2) for each $s \in J_+$, $G(t, s)$ is continuously differentiable on J_+ except $t = s$,
- (3) $\frac{\partial G(t, s)}{\partial t}|_{t=s^+} - \frac{\partial G(t, s)}{\partial t}|_{t=s^-} = \frac{1}{p(s)}$,
- (4) $G(t, s) \leq G(s, s) < \infty$, and $G_s(t, s) \leq G_s(t, s)|_{t=s} < \infty$,
- (5) $|G_t(t, s)| \leq \frac{c}{p(t)}G(s, s)$, and $|G_{st}(t, s)| \leq \frac{c}{p(t)}G_s(t, s)|_{t=s}$, where

$$c = \frac{\max\{a_1, a_2\}}{\min\{b_1, b_2\}}, \quad (2.6)$$

$$(6) \quad \bar{G}(s) = \lim_{t \rightarrow \infty} G(t, s) = \frac{b_2}{D} \theta(s) \leq G(s, s) < \infty,$$

$$(7) \quad \bar{G}'(s) = \lim_{t \rightarrow \infty} G_s(t, s) = \frac{b_2}{D} \theta'(s) \leq G_s(t, s)|_{t=s} < \infty,$$

$$(8) \quad \text{for any } t \in [a, b] \subset (0, \infty) \text{ and } s \in [0, \infty), \text{ we have}$$

$$G(t, s) \geq wG(s, s),$$

where

$$w = \min \left\{ \frac{b_1 + a_1 B(0, a)}{b_1 + a_1 B(0, \infty)}, \frac{b_2 + a_2 B(b, \infty)}{b_2 + a_2 B(0, \infty)} \right\}. \quad (2.7)$$

Obviously, $0 < w < 1$.

Define

$$\begin{aligned} (Tx)(t) = & \int_0^\infty G(t, s) p(s) f(s, x(s), x'(s)) ds + \frac{\varphi(t)}{D} \int_0^\infty g_1(x(s)) \psi(s) ds \\ & + \frac{\theta(t)}{D} \int_0^\infty g_2(x(s)) \psi(s) ds + \sum_{k=1}^n G(t, t_k) \bar{I}_k(x(t_k)) \\ & + \sum_{k=1}^n p(t_k) G_s(t, s)|_{s=t_k} I_k(x(t_k)), \end{aligned} \quad (2.8)$$

where G is defined by as in (2.5).

Obviously, the impulsive IBVP (1.1) has a solution x if and only if $x \in K$ is a fixed point of the operator T defined by (2.8).

It is convenient to list the following condition which is to be used in our theorems:

$$(H7) \quad 0 < \int_0^\infty G(s, s) p(s) k(s) ds < \infty.$$

As we know that the Ascoli-Arzelà Theorem does not hold in infinite interval J , we need the following compactness criterion:

Lemma 2 ([20]). *Let $M \subset BPC^1(J)$. Then M is relatively compact in $BPC^1(J)$ if the following conditions hold.*

- (i) *M is uniformly bounded in $BPC^1(J)$.*
- (ii) *The function belonging to M are equicontinuous on any compact interval of $[0, \infty)$.*
- (iii) *The functions from M are equiconvergent, that is, for any given $\varepsilon > 0$, there exist a $T = T(\varepsilon) > 0$ such that $|f(t) - f(\infty)| < \varepsilon$ for any $t > T$, $f \in M$.*

The main tool of this work is a fixed point theorem in cones.

Lemma 3 ([26]). *Let X be an Banach space and K is a positive cone in X . Assume that Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$. Let $T : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow K$ be a completely continuous operator such that*

- (i) *$\|Tx\| \leq \|x\|$ for all $x \in K \cap \partial\Omega_1$.*
- (ii) *There exists a $\Phi \in K$ such that $x \neq Tx + \lambda\Phi$, for all $x \in K \cap \partial\Omega_2$ and $\lambda > 0$.*

Then T has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

Lemma 4. *If (H1)-(H7) are satisfied, then for any bounded open set $\Omega \subset BPC^1(J)$, $T : K \cap \bar{\Omega} \rightarrow K$ is a completely continuous operator.*

Proof. For any bounded open set $\Omega \subset BPC^1(J)$, there exists a constant $M > 0$ such that $\|x\| \leq M$ for any $x \in \overline{\Omega}$.

First, we show $T : K \cap \overline{\Omega} \rightarrow K$ is well defined. Let $x \in K \cap \overline{\Omega}$. From (H2), (H3) and (H4), we have

$$S_M = \sup\{S_1, S_2, S_3, S_4, S_5\}, \quad (2.9)$$

where

$$\begin{aligned} S_1 &= \sup\{h(x, y) : |x| + |y| \leq M\} < \infty, & S_2 &= \sup\{I_k(x) : 0 \leq x \leq M\}, \\ S_3 &= \sup\{\bar{I}_k(x) : 0 \leq x \leq M\}, & S_4 &= \sup\{g_1(x) : 0 \leq x \leq M\}, \\ S_5 &= \sup\{g_2(x) : 0 \leq x \leq M\}. \end{aligned}$$

Let $t_1, t_2 \in J$, $t_1 < t_2$, then

$$\int_0^\infty |G(t_1, s) - G(t_2, s)| p(s) k(s) ds \leq 2 \int_0^\infty G(s, s) p(s) k(s) ds < \infty. \quad (2.10)$$

Hence, by the Lebesgue dominated convergence theorem, we have for any $t_1, t_2 \in J$, $x \in K \cap \overline{\Omega}$, we have

$$\begin{aligned} & |(Tx)(t_1) - (Tx)(t_2)| \\ & \leq \int_0^\infty |G(t_1, s) - G(t_2, s)| p(s) f(s, x(s), x'(s)) ds \\ & \quad + \frac{|\varphi(t_1) - \varphi(t_2)|}{D} \int_0^\infty g_1(x(s)) \psi(s) ds + \frac{|\theta(t_1) - \theta(t_2)|}{D} \int_0^\infty g_2(x(s)) \psi(s) ds \\ & \quad + \sum_{k=1}^n |G(t_1, t_k) - G(t_2, t_k)| \bar{I}_k(x(t_k)) \\ & \quad + \sum_{k=1}^n |p(t_k) G_s(t, s)|_{s=t_1} - p(t_k) G_s(t, s)|_{s=t_2} | I_k(x(t_k)) \\ & \leq S_M \left\{ \int_0^\infty |G(t_1, s) - G(t_2, s)| p(s) k(s) ds \right. \\ & \quad + \left[\frac{|\varphi(t_1) - \varphi(t_2)|}{D} + \frac{|\theta(t_1) - \theta(t_2)|}{D} \right] \int_0^\infty \psi(s) ds + \sum_{k=1}^n |G(t_1, t_k) - G(t_2, t_k)| \\ & \quad + \frac{1}{D} \sum_{t_k \leq t_1} p(t_k) \theta'(t_k) |\varphi(t_1) - \varphi(t_2)| \Big\} + \frac{1}{D} \sum_{t_2 \leq t_k} p(t_k) |\varphi'(t_k)| |\theta(t_1) - \theta(t_2)| \\ & \quad + \frac{1}{D} \sum_{t_1 \leq t_k \leq t_2} p(t_k) |\theta'(t_1) \varphi'(t_k) - \theta'(t_k) \varphi'(t_2)| \Big\} \\ & \rightarrow 0 \text{ as } t_1 \rightarrow t_2, \end{aligned} \quad (2.11)$$

$$|(Tx)'(t_1) - (Tx)'(t_2)|$$

$$\begin{aligned}
 &\leq S_M \left\{ \frac{a_2}{D} \left| \frac{1}{p(t_1)} - \frac{1}{p(t_2)} \right| \int_0^{t_1} \theta(s)p(s)k(s)ds + \frac{a_1}{Dp(t_1)} \int_{t_1}^{t_2} \varphi(s)p(s)k(s)ds \right. \\
 &\quad + \frac{a_1}{D} \left| \frac{1}{p(t_1)} - \frac{1}{p(t_2)} \right| \int_{t_2}^{\infty} \varphi(s)p(s)k(s)ds + \frac{a_2}{Dp(t_2)} \int_{t_1}^{t_2} \theta(s)p(s)k(s)ds \\
 &\quad + \frac{a_2}{D} \left| \frac{1}{p(t_2)} - \frac{1}{p(t_1)} \right| \int_0^{\infty} \psi(s)ds + \frac{a_1}{D} \left| \frac{1}{p(t_1)} - \frac{1}{p(t_2)} \right| \int_0^{\infty} \psi(s)ds \\
 &\quad + \frac{a_1}{D} \left| \frac{1}{p(t_1)} - \frac{1}{p(t_2)} \right| \sum_{t_2 \leq t_k} [\varphi(t_k) + p(t_k)|\varphi'(t_k)|] \\
 &\quad + \frac{a_2}{D} \left| \frac{1}{p(t_1)} - \frac{1}{p(t_2)} \right| \sum_{t_k \leq t_1} [\theta(t_k) + p(t_k)\theta'(t_k)] \\
 &\quad \left. + \frac{a_1}{Dp(t_1)} \sum_{t_1 \leq t_k \leq t_2} [\varphi(t_k) + p(t_k)|\varphi'(t_k)|] + \frac{a_2}{Dp(t_2)} \sum_{t_1 \leq t_k \leq t_2} [\theta(t_k) + p(t_k)\theta'(t_k)] \right\} \\
 &\rightarrow 0 \text{ as } t_1 \rightarrow t_2.
 \end{aligned} \tag{2.12}$$

Thus, $Tx \in PC^1(J)$. We can show that $Tx \in BPC^1(J)$.

Then by (H5), (H7), the properties (5), (6), (7) of Remark 1 and the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}
 \lim_{t \rightarrow \infty} (Tx)(t) &= \int_0^{\infty} \bar{G}(s)p(s)f(s, x(s), x'(s))ds + \frac{\varphi(\infty)}{D} \int_0^{\infty} g_1(x(s))\psi(s)ds \\
 &\quad + \frac{\theta(\infty)}{D} \int_0^{\infty} g_2(x(s))\psi(s)ds + \sum_{k=1}^n \bar{G}(t_k)\bar{I}_k(x(t_k)) \\
 &\quad + \sum_{k=1}^n p(t_k)\bar{G}'(t_k)I_k(x(t_k)) < \infty
 \end{aligned} \tag{2.13}$$

and

$$\begin{aligned}
 |(Tx)'(t)| &\leq S_M \left\{ \frac{c}{p(t)} \int_0^{\infty} G(s, s)p(s)k(s)ds + \frac{\max\{a_1, a_2\}}{Dp(t)} \int_0^{\infty} \psi(s)ds \right. \\
 &\quad \left. + \frac{c}{p(t)} \sum_{k=1}^n G(t_k, t_k) + \frac{c}{p(t)} \sum_{k=1}^n p(t_k)G_s(t, s)|_{s=t_k} \right\} < \infty.
 \end{aligned} \tag{2.14}$$

Therefore, $\sup_{t \in J} |(Tx)'(t)| < \infty$. Hence $T : K \cap \bar{\Omega} \rightarrow K$ is well defined.

Next, we prove that T is continuous. Let $x_n \rightarrow x$ in $K \cap \bar{\Omega}$, then $\|x_n\| \leq M$, ($n = 1, 2, \dots$). We will show that $Tx_n \rightarrow Tx$. For any $\varepsilon > 0$, by (H7) there exists a constant $A_0 > 0$ such that

$$S_M \int_{A_0}^{\infty} G(s, s)p(s)k(s)ds \leq \frac{\varepsilon}{12}. \tag{2.15}$$

On the other hand, by the continuity of $f(t, u, v)$ on $(0, A_0] \times J_+ \times \mathbb{R}$, the continuities of g_1, g_2 on J_+ and the continuities of $I_k, \bar{I}_k$ on J_+ , for the above $\varepsilon > 0$, there exists a $\delta > 0$ such that, for any u, v, u_1, v_1 , satisfying $|u| + |v| < M$, and $|u_1| + |v_1| < M$, $|u - u_1| + |v - v_1| < \delta$,

$$\begin{aligned}
|f(s, u, v) - f(s, u_1, v_1)| &< \frac{\varepsilon}{6} \left(\int_0^{A_0} G(s, s) p(s) ds \right)^{-1}, \\
|g_1(u) - g_1(u_1)| &< \frac{\varepsilon}{6} \left(\frac{\Phi(0)}{D} \int_0^\infty \Psi(s) ds \right)^{-1}, \\
|g_2(u) - g_2(u_1)| &< \frac{\varepsilon}{6} \left(\frac{\Theta(\infty)}{D} \int_0^\infty \Psi(s) ds \right)^{-1}, \\
|I_k(u(t_k)) - I_k(u_1(t_k))| &< \frac{\varepsilon}{6} \left(\sum_{k=1}^n p(t_k) G_s(t, s)|_{t=t_k}^{s=t_k} \right)^{-1}, \\
|\bar{I}_k(u(t_k)) - \bar{I}_k(u_1(t_k))| &< \frac{\varepsilon}{6} \left(\sum_{k=1}^n G(t_k, t_k) \right)^{-1}.
\end{aligned} \tag{2.16}$$

From the fact that $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$, for above δ , there exists a sufficiently large number N such that, when $n > N$, we have, for $t \in (0, A_0]$,

$$|x_n(t) - x(t)| + |x'_n(t) - x'(t)| \leq \|x_n - x\| < \delta. \tag{2.17}$$

By (2.15)-(2.16), we have, for $n > N$,

$$\begin{aligned}
|(Tx_n)(t) - (Tx)(t)| &\leq \left| \int_0^\infty G(s, s) p(s) [f(s, x_n(s), x'_n(s)) - f(s, x(s), x'(s))] ds \right. \\
&\quad + \frac{\Phi(0)}{D} \int_0^\infty [g_1(x_n(s)) - g_1(x(s))] \Psi(s) ds \\
&\quad + \frac{\Theta(\infty)}{D} \int_0^\infty [g_2(x_n(s)) - g_2(x(s))] \Psi(s) ds \\
&\quad + \sum_{k=1}^n G(t_k, t_k) [\bar{I}_k(x_n(t_k)) - \bar{I}_k(x(t_k))] \\
&\quad \left. + \sum_{k=1}^n p(t_k) G_s(t, s)|_{t=t_k}^{s=t_k} [I_k(x_n(t_k)) - I_k(x(t_k))] \right| \\
&\leq \int_0^{A_0} G(s, s) p(s) |f(s, x_n(s), y_n(s)) - f(s, x(s), y(s))| ds \\
&\quad + 2S_M \int_{A_0}^\infty G(s, s) p(s) k(s) ds \\
&\quad + \frac{\Phi(0)}{D} \int_0^\infty |g_1(x_n(s)) - g_1(x(s))| \Psi(s) ds
\end{aligned}$$

$$\begin{aligned}
 & + \frac{\theta(\infty)}{D} \int_0^\infty |g_2(x_n(s)) - g_2(x(s))| \psi(s) ds \\
 & + \sum_{k=1}^n G(t_k, t_k) |\bar{I}_k(x_n(t_k)) - \bar{I}_k(x(t_k))| \\
 & + \sum_{k=1}^n p(t_k) G_s(t, s) \Big|_{s=t_k}^{t=t_k} |I_k(x_n(t_k)) - I_k(x(t_k))| \\
 & < \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \varepsilon.
 \end{aligned}$$

Similarly, we can see that when $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$, $|(Tx_n)'(t) - (Tx)'(t)| \rightarrow 0$ as $n \rightarrow \infty$. This implies that T is a continuous operator.

Finally we show that $T : K \cap \bar{\Omega} \rightarrow K$ is a compact operator. In fact for any bounded set $D \subset \bar{\Omega}$, there exists a constant $R > 0$ such that $\|x\| \leq R$ for any $x \in K \cap D$. Hence, we have

$$\begin{aligned}
 |(Tx)(t)| & \leq \left| \int_0^\infty G(s, s) p(s) f(s, x(s), x'(s)) ds + \frac{\varphi(0)}{D} \int_0^\infty g_1(x(s)) \psi(s) ds \right. \\
 & \quad + \frac{\theta(\infty)}{D} \int_0^\infty g_2(x(s)) \psi(s) ds \\
 & \quad \left. + \sum_{k=1}^n G(t_k, t_k) \bar{I}_k(x(t_k)) + \sum_{k=1}^n p(t_k) G_s(t, s) \Big|_{s=t_k}^{t=t_k} I_k(x(t_k)) \right| \\
 & \leq S_R \left(\int_0^\infty G(s, s) p(s) k(s) ds + \frac{\varphi(0)}{D} \int_0^\infty \psi(s) ds \right. \\
 & \quad \left. + \frac{\theta(\infty)}{D} \int_0^\infty \psi(s) ds + \sum_{k=1}^n G(t_k, t_k) + \sum_{k=1}^n p(t_k) G_s(t, s) \Big|_{s=t_k}^{t=t_k} \right) < \infty.
 \end{aligned}$$

From (2.14), we get $|(Tx)'(t)| < \infty$, for $t \in J$. Therefore, $T(K \cap D)$ is uniformly bounded in $BPC^1(J)$.

Given $r > 0$, for any $t_1, t_2 \in J$, $x \in K \cap D$, as the proof of (2.11), (2.12), we can get $|(Tx)(t_1) - (Tx)(t_2)| \rightarrow 0$ and $|(Tx)'(t_1) - (Tx)'(t_2)| \rightarrow 0$ as $t_1 \rightarrow t_2$, i.e., $\|(Tx)(t_1) - (Tx)(t_2)\| \rightarrow 0$ as $t_1 \rightarrow t_2$. Thus $F = \{Tx : x \in K \cap D\}$ is equicontinuous on $[0, r]$. Since $r > 0$ arbitrary, F is locally equicontinuous on J_+ . By (H5), (H7) the properties (5), (6), (7) and the Lebesgue dominated converges theorem, we get

$$\begin{aligned}
 |(Tx)(t) - (Tx)(\infty)| & \leq S_R \left(\int_0^\infty |G(t, s) - \bar{G}(s)| p(s) k(s) ds \right. \\
 & \quad + \frac{|\varphi(t) - \varphi(\infty)|}{D} \int_0^\infty \psi(s) ds + \frac{|\theta(t) - \theta(\infty)|}{D} \int_0^\infty \psi(s) ds \\
 & \quad \left. + \sum_{k=1}^n |G(t, t_k) - \bar{G}(t_k)| + \frac{1}{D} \sum_{t_k \leq t} p(t_k) \theta'(t_k) |\varphi(t) - \varphi(\infty)| \right)
 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{D} \sum_{t \leq t_k} p(t_k) |\theta(t) \phi'(t_k) - \theta(t_k) \phi'(\infty)| \\
& \rightarrow 0 \text{ as } t \rightarrow \infty
\end{aligned}$$

and

$$\begin{aligned}
|(Tx)'(t) - (Tx)'(\infty)| & \leq S_R \left\{ \frac{a_2}{D} \left| \frac{1}{p(t)} - \frac{1}{p(\infty)} \right| \int_0^t \theta(s) p(s) k(s) ds \right. \\
& + \frac{a_1}{D} \left| \frac{1}{p(t)} - \frac{1}{p(\infty)} \right| \int_t^\infty \phi(s) p(s) k(s) ds \\
& + \frac{a_2}{D} \left| \frac{1}{p(t)} - \frac{1}{p(\infty)} \right| \int_0^\infty \psi(s) ds \\
& + \frac{a_1}{D} \left| \frac{1}{p(t)} - \frac{1}{p(\infty)} \right| \int_0^\infty \psi(s) ds \\
& + \frac{a_1}{D p(t)} \sum_{t \leq t_k} [\phi(t_k) + p(t_k) |\phi'(t_k)|] \\
& + \frac{a_2}{D} \left| \frac{1}{p(t)} - \frac{1}{p(\infty)} \right| \sum_{t_k \leq t} [\theta(t_k) + p(t_k) \theta'(t_k)] \\
& \left. + \frac{a_2}{D p(\infty)} \sum_{t \leq t_k} p(t_k) [\theta(t_k) + p(t_k) \theta'(t_k)] \right\} \\
& \rightarrow 0 \text{ as } t \rightarrow \infty.
\end{aligned} \tag{2.18}$$

Hence $T(K \cap D)$ is equiconvergent in $BPC^1(J)$. By Lemma 2, we have that F is relatively compact in $BPC^1(J)$. Therefore, $T : K \cap \bar{\Omega} \rightarrow K$ is completely continuous. $\square$

3. MAIN RESULTS

For convenience and simplicity in the following discussion, we use following notations:

$$\begin{aligned}
f_0 &= \lim_{|x|+|y| \rightarrow 0} \inf \min_{t \in [a,b]} \frac{f(t,x,y)}{|x|+|y|}, & f_\infty &= \lim_{|x|+|y| \rightarrow \infty} \inf \min_{t \in [a,b]} \frac{f(t,x,y)}{|x|+|y|}, \\
g_{i_0} &= \lim_{x \rightarrow 0} \inf \frac{g_i(x)}{x} \quad (1 \leq i \leq 2), & g_{i_\infty} &= \lim_{x \rightarrow \infty} \inf \frac{g_i(x)}{x} \quad (1 \leq i \leq 2), \\
h^q &= \lim_{|x|+|y| \rightarrow q} \sup \frac{h(x,y)}{|x|+|y|}, & g_i^q &= \lim_{x \rightarrow q} \sup \frac{g_i(x)}{x} \quad (1 \leq i \leq 2), \\
I_0(k) &= \lim_{x \rightarrow 0} \inf \frac{I_k(x)}{x}, & I_\infty(k) &= \lim_{x \rightarrow \infty} \inf \frac{I_k(x)}{x}, \\
\bar{I}_0(k) &= \lim_{x \rightarrow 0} \inf \frac{\bar{I}_k(x)}{x}, & \bar{I}_\infty(k) &= \lim_{x \rightarrow \infty} \inf \frac{\bar{I}_k(x)}{x},
\end{aligned}$$

$$I^q(k) = \limsup_{x \rightarrow q} \frac{I_k(x)}{x}, \quad \bar{I}^q(k) = \limsup_{x \rightarrow q} \frac{\bar{I}_k(x)}{x}.$$

Theorem 1. Assume that the conditions (H1)-(H7) are satisfied. Then the impulsive IBVP (1.1) has at least two positive solutions satisfying $0 < \|x_1\| < q < \|x_2\|$ if for $[a, b] \subset (0, \infty)$, the following conditions hold:

$$\begin{aligned} (A1) \quad & w \left(f_0 \int_a^b G(s, s) p(s) ds + \frac{\min\{\varphi(\infty), \theta(0)\}}{D} (g_{1_0} + g_{2_0}) \int_a^b \psi(s) ds \right. \\ & \left. + \sum_{k=1}^n G(t_k, t_k) \bar{I}_0(k) + \sum_{k=1}^n p(t_k) G_s(t, s) \Big|_{s=t_k} I_0(k) \right) > 1, \\ & w \left(f_\infty \int_a^b G(s, s) p(s) ds + \frac{\min\{\varphi(\infty), \theta(0)\}}{D} (g_{1_\infty} + g_{2_\infty}) \int_a^b \psi(s) ds \right. \\ & \left. + \sum_{k=1}^n G(t_k, t_k) \bar{I}_\infty(k) + \sum_{k=1}^n p(t_k) G_s(t, s) \Big|_{s=t_k} I_\infty(k) \right) > 1, \\ (A2) \quad & \text{There exists a } q > 0 \text{ such that} \\ & \left[1 + c \sup_{t \in J} \frac{1}{p(t)} \right] \left\{ h^q \int_0^\infty G(s, s) p(s) k(s) ds + \frac{\max\{\varphi(0), \theta(\infty)\}}{D} (g_1^q + g_2^q) \right. \\ & \left. \times \int_0^\infty \psi(s) ds + \sum_{k=1}^n G(t_k, t_k) \bar{I}^q(k) + \sum_{k=1}^n p(t_k) G_s(t, s) \Big|_{s=t_k} I^q(k) \right\} < 1, \\ & \text{for all } 0 < |x| + |x'| \leq q, \text{ a.e. } t \in [0, \infty). \end{aligned}$$

Proof. By the definition of $f_0, I_0, \bar{I}_0, g_{1_0}$ and g_{2_0} for any $\varepsilon > 0$, there exist $r \in (0, q)$ such that,

$$\begin{aligned} f(t, x, y) &\geq (1 - \varepsilon) f_0(|x| + |y|), & (|x| + |y| \leq r, \quad t \in [a, b]) \\ g_1(x) &\geq (1 - \varepsilon) g_{1_0} x, & g_2(x) \geq (1 - \varepsilon) g_{2_0} x, \\ I_k(x) &\geq (1 - \varepsilon) I_0(k) x, & \bar{I}_k(x) \geq (1 - \varepsilon) \bar{I}_0(k) x, \end{aligned}$$

$$\begin{aligned} (1 - \varepsilon) w \left(f_0 \int_a^b G(s, s) p(s) ds + \frac{\min\{\varphi(\infty), \theta(0)\}}{D} (g_{1_0} + g_{2_0}) \int_a^b \psi(s) ds \right. \\ \left. + \sum_{k=1}^n G(t_k, t_k) \bar{I}_0(k) + \sum_{k=1}^n p(t_k) G_s(t, s) \Big|_{s=t_k} I_0(k) \right) \geq 1. \end{aligned}$$

Define the open sets

$$\Omega_r = \{x \in BPC^1(J) : \|x\| < r\}.$$

Let $\Phi = 1$ then, $\Phi \in K$. Now we prove that

$$x \neq Tx + \lambda \Phi, \quad \forall x \in K \cap \partial \Omega_r, \quad \lambda > 0. \quad (3.1)$$

We assume that $x_0 = Tx_0 + \lambda_0 \Phi$ where $x_0 \in K \cap \partial\Omega_r$ and $\lambda_0 > 0$. Let $\mu = \min_{t \in [a, b]} x_0(t)$, then for any $t \in [a, b]$, we have

$$\begin{aligned} x_0(t) &= Tx_0(t) + \lambda_0 \Phi \\ &= \int_0^\infty G(t, s) p(s) f(s, x_0(s), x_0'(s)) ds \\ &\quad + \frac{\varphi(t)}{D} \int_0^\infty g_1(x_0(s)) \psi(s) ds + \frac{\theta(t)}{D} \int_0^\infty g_2(x_0(s)) \psi(s) ds \\ &\quad + \sum_{k=1}^n G(t, t_k) \bar{I}_k(x_0(t_k)) + \sum_{k=1}^n p(t_k) G_s(t, s)|_{s=t_k} I_k(x_0(t_k)) + \lambda_0 \\ &> w\mu(1 - \varepsilon) \left\{ f_0 \int_a^b G(s, s) p(s) ds + \frac{\min\{\varphi(\infty), \theta(0)\}}{D} (g_{1_0} + g_{2_0}) \right. \\ &\quad \left. \times \int_a^b \psi(s) ds + \sum_{k=1}^n G(t_k, t_k) \bar{I}_0(k) + \sum_{k=1}^n p(t_k) G_s(t, s)|_{s=t_k} I_0(k) \right\} + \lambda_0 \\ &\geq \mu + \lambda_0. \end{aligned}$$

This implies $\mu > \mu + \lambda_0$ a contradiction. Therefore (3.1) holds.

By the definition of $f_\infty, I_\infty, \bar{I}_\infty, g_{1_\infty}$ and g_{2_∞} for any $\varepsilon > 0$, there exist $R > q$ such that,

$$\begin{aligned} f(t, x, y) &\geq (1 - \varepsilon) f_\infty(|x| + |y|), & (|x| + |y| \geq R, \quad t \in [a, b]), \\ g_1(x) &\geq (1 - \varepsilon) g_{1_\infty} x, & (\forall |x| \geq R), & g_2(x) \geq (1 - \varepsilon) g_{2_\infty} x, \\ I_k(x) &\geq (1 - \varepsilon) I_\infty(k) x, & \bar{I}_k(x) \geq (1 - \varepsilon) \bar{I}_\infty(k) x, \end{aligned}$$

$$\begin{aligned} (1 - \varepsilon) w \left(f_\infty \int_0^\infty G(s, s) p(s) ds + \frac{\min\{\varphi(\infty), \theta(0)\}}{D} (g_{1_\infty} + g_{2_\infty}) \int_0^\infty \psi(s) ds \right. \\ \left. + \sum_{k=1}^n G(t_k, t_k) \bar{I}_\infty(k) + \sum_{k=1}^n p(t_k) G_s(t, s)|_{s=t_k} I_\infty(k) \right) \geq 1. \end{aligned}$$

Define the open sets

$$\Omega_R = \{x \in BPC^1(J) : \|x\| < R\}. \quad (3.2)$$

As the proof of (3.1), we can get that

$$x \neq Tx + \lambda \Phi, \quad \forall x \in K \cap \partial\Omega_R, \quad \lambda > 0. \quad (3.3)$$

On the other hand, for any $\varepsilon > 0$, choose q in (A2) such that

$$(1 + \varepsilon) \left[1 + c \sup_{t \in J} \frac{1}{p(t)} \right] \left\{ h^q \int_0^\infty G(s, s) p(s) k(s) ds + \frac{\max\{\varphi(0), \theta(\infty)\}}{D} (g_1^q + g_2^q) \right\}$$

$$\times \int_0^\infty \psi(s) ds + \sum_{k=1}^n G(t_k, t_k) \bar{I}^q(k) + \sum_{k=1}^n p(t_k) G_s(t, s) \Big|_{s=t_k} I^q(k) \Big\} \leq 1. \quad (3.4)$$

By the definition of h^q , I^q , $\bar{I}^q$, g_1^q and g_2^q , for the above $\varepsilon > 0$, there exists $\delta > 0$, when $|x|, |x| + |y| \in (q - \delta, q + \delta)$; thus, we get

$$\begin{aligned} h(x, y) &\leq (1 + \varepsilon) h^q(|x| + |y|), & g_i(x) &\leq (1 + \varepsilon) g_i^q x, \quad 1 \leq i \leq 2 \\ I_k(x) &\leq (1 + \varepsilon) I^q(k) x, & \bar{I}_k(x) &\leq (1 + \varepsilon) \bar{I}^q(k) x. \end{aligned}$$

Define the open sets

$$\Omega_q = \{x \in BPC^1(J) : \|x\| < q\}. \quad (3.5)$$

Then for any $x \in K \cap \partial\Omega_q$ and $t \in J$ we obtain that

$$\begin{aligned} |(Tx)(t)| + |(Tx)'(t)| &\leq \int_0^\infty G(s, s) p(s) f(s, x(s), x'(s)) ds \\ &\quad + \frac{\max\{\varphi(0), \theta(\infty)\}}{D} \int_0^\infty [g_1(x(s)) + g_2(x(s))] \psi(s) ds \\ &\quad + \sum_{k=1}^n G(t_k, t_k) \bar{I}_k(x(t_k)) + \sum_{k=1}^n p(t_k) G_s(t, s) \Big|_{s=t_k} I_k(x(t_k)) \\ &\quad + c \sup_{t \in J} \frac{1}{p(t)} \int_0^\infty G(s, s) p(s) f(s, x(s), y(s)) ds \\ &\quad + \frac{\max\{a_1, a_2\}}{D} \sup_{t \in J} \frac{1}{p(t)} \int_0^\infty [g_1(x(s)) + g_2(x(s))] \psi(s) ds \\ &\quad + c \sup_{t \in J} \frac{1}{p(t)} \left[\sum_{k=1}^n G(t_k, t_k) \bar{I}_k(x(t_k)) \right. \\ &\quad \left. + \sum_{k=1}^n p(t_k) G_s(t, s) \Big|_{s=t_k} I_k(x(t_k)) \right] \\ &\leq \left[1 + c \sup_{t \in J} \frac{1}{p(t)} \right] \int_0^\infty G(s, s) p(s) k(s) h(x(s), x'(s)) ds \\ &\quad + \left[\frac{\max\{\varphi(0), \theta(\infty)\}}{D} + \frac{\max\{a_1, a_2\}}{D} \sup_{t \in J} \frac{1}{p(t)} \right] \\ &\quad \times \int_0^\infty [g_1(x(s)) + g_2(x(s))] \psi(s) ds \\ &\quad + \left[1 + c \sup_{t \in J} \frac{1}{p(t)} \right] \left[\sum_{k=1}^n G(t_k, t_k) \bar{I}_k(x(t_k)) \right. \\ &\quad \left. + \sum_{k=1}^n p(t_k) G_s(t, s) \Big|_{s=t_k} I_k(x(t_k)) \right] \end{aligned}$$

$$\begin{aligned}
&\leq (1 + \varepsilon) \left[1 + c \sup_{t \in J} \frac{1}{p(t)} \right] \left[h^q \int_0^\infty G(s, s) p(s) k(s) ds \right. \\
&\quad + \frac{\max\{\varphi(0), \theta(\infty)\}}{D} \times (g_1^q + g_2^q) \int_0^\infty \Psi(s) ds \\
&\quad \left. + \sum_{k=1}^n G(t_k, t_k) \bar{I}^q(k) + \sum_{k=1}^n p(t_k) G_s(t, s) \Big|_{t=t_k} I^q(k) \right] \|x\| \leq \|x\|.
\end{aligned}$$

Therefore $\|Tx\| \leq \|x\|$.

So, we can get the existence of two positive solutions x_1 and x_2 satisfying $0 < \|x_1\| < q < \|x_2\|$ by using Lemma 3. $\square$

Using a similar proof of Theorem 1, we can get the following theorem.

Theorem 2. Assume that the conditions (H1)-(H7) are satisfied. Then the impulsive IBVP (1.1) has at least two positive solutions satisfying $0 < \|x_1\| < q < \|x_2\|$ if for $[a, b] \subset (0, \infty)$, the following conditions hold;

$$\begin{aligned}
(A3) \quad &\left[1 + c \sup_{t \in J} \frac{1}{p(t)} \right] \left\{ h^0 \int_0^\infty G(s, s) p(s) k(s) ds + \frac{\max\{\varphi(0), \theta(\infty)\}}{D} (g_1^0 + g_2^0) \right. \\
&\quad \times \int_0^\infty \Psi(s) ds + \sum_{k=1}^n G(t_k, t_k) \bar{I}^0(k) + \sum_{k=1}^n p(t_k) G_s(t, s) \Big|_{t=t_k} I^0(k) \left. \right\} < 1, \\
&\left[1 + c \sup_{t \in J} \frac{1}{p(t)} \right] \left\{ h^\infty \int_0^\infty G(s, s) p(s) k(s) ds + \frac{\max\{\varphi(0), \theta(\infty)\}}{D} (g_1^\infty + g_2^\infty) \right. \\
&\quad \times \int_0^\infty \Psi(s) ds + \sum_{k=1}^n G(t_k, t_k) \bar{I}^\infty(k) + \sum_{k=1}^n p(t_k) G_s(t, s) \Big|_{t=t_k} I^\infty(k) \left. \right\} < 1,
\end{aligned}$$

(A4) There exists a $q > 0$ such that

$$\begin{aligned}
&w \left\{ f_q \int_a^b G(s, s) p(s) ds + \frac{\min\{\varphi(\infty), \theta(0)\}}{D} (g_{1_q} + g_{2_q}) \int_a^b \Psi(s) ds \right. \\
&\quad \left. + \sum_{k=1}^n G(t_k, t_k) \bar{I}_q(k) + \sum_{k=1}^n p(t_k) G_s(t, s) \Big|_{t=t_k} I_q(k) \right\} > 1,
\end{aligned}$$

for all $0 < |x| + |x'| \leq q$, a.e. $t \in [0, \infty)$.

4. EXAMPLE

To illustrate how our main results can be used in practise we present the following example.

Consider the following boundary value problem:

$$\left\{ \begin{array}{l} e^{-t}(e^t x'(t))' + f(t, x(t), x'(t)) = 0, \quad t \in J_+, t \neq \frac{1}{2}, \\ \Delta x|_{t=\frac{1}{2}} = \frac{1}{100}x(\frac{1}{2}), \\ \Delta x'|_{t=\frac{1}{2}} = \frac{1}{2-e^{-\frac{1}{2}}}x^{-1}(\frac{1}{2}), \\ x(0) = \frac{1}{40\pi} \int_0^\infty \frac{x^2(s)}{1+s^2} ds, \\ \lim_{t \rightarrow \infty} e^t x'(t) = \frac{1}{8000\pi} \int_0^\infty \frac{x^2(s)}{1+s^2} ds, \end{array} \right. \quad (4.1)$$

where

$$f(t, x(t), x'(t)) = \frac{e^{-t}}{e^t - 2}, \quad a_1 = 1, a_2 = 0, \quad b_1 = 1, \quad b_2 = 1, \quad p(t) = e^t, \quad \psi(t) = \frac{1}{1+t^2},$$

$$I_k(x(t)) = \frac{x(t)}{100}, \quad \bar{I}_k(x(t)) = \frac{x^{-1}(t)}{2-e^{-\frac{1}{2}}}, \quad g_1(x(s)) = \frac{x^2(s)}{40\pi}, \quad g_2(x(s)) = \frac{x^4(s)}{8000\pi}.$$

Set $k(t) = \frac{e^{-t}}{e^t - 2}$, $h(x(t), y(t)) = 1$ and $q = 10$. It follows from a direct calculation that

$$g_1^q = \frac{1}{4\pi}, \quad g_2^q = \frac{1}{8\pi}, \quad g_{1_\infty} = \infty, \quad g_{2_\infty} = \infty, \quad g_{1_0} = 0, \quad g_{2_0} = 0, \quad I^q(k) = \frac{1}{100},$$

$$\bar{I}^q(k) = \frac{1}{100(1-e^{-\frac{1}{2}})}, \quad I_0(k) = \frac{1}{100}, \quad \bar{I}_0(k) = \infty, \quad I_\infty(k) = \frac{1}{100}, \quad \bar{I}_\infty(k) = 0.$$

Furthermore, $f_0 = \infty$, $f_\infty = 1$, $\int_0^\infty G(s, s)p(s)k(s)ds = 1$ and $\int_0^\infty \psi(s)ds = \frac{\pi}{2}$. Thus (A1) and (A2) are satisfied. Therefore, by Theorem 1, the impulsive IBVP (4.1) has at least two positive solutions x_1, x_2 satisfying $0 < \|x_1\| < 10 < \|x_2\|$.

ACKNOWLEDGEMENT

The second author was supported in part by Ege University Scientific Research Project, Grant No.18-FEN-033.

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ABOUT ONE CHARACTERISTIC INITIAL VALUE PROBLEM WITH PREHISTORY AND ITS INVESTIGATION

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Received 12 October, 2020

Abstract. We have built a constructive method for investigation approximate solution of nonlinear characteristic initial value problem with prehistory and suggested the one algorithm acceleration convergence of consequent approximation, established sufficient condition of subsistence the unique regular solution of the investigated problem and its equality.

2010 Mathematics Subject Classification: 34B10; 34B15

Keywords: two-sided method, boundary value problem, "free" curves, initial value problem

1. INTRODUCTION

In examining the process of different nature (gas sorption, drying by the air flow, pipes heating by a stream of hot water, etc.) there are cases, when in relevant mathematical models (problems) not all output data (boundary or initial condition) are known, thus they need to be found through the solution some auxiliary problems, which are mathematical models of processes that proceed to research.

These problems should be named as problems with prehistory. This work is dedicated to investigation of the such problem.

2. PROBLEM SETTINGS AND REASONING

Let in phases space xOt there is specified domain $D = D_1 \cup D_2$, where

$$D_1 = \{(x, t) | x \in (x_0, x_1], t \in (g(x), t_1]\},$$
$$D_2 = \{(x, t) | x \in (x_0, x_1], t \in (t_1, t_2]\},$$

and $t = g(x) \iff x = \kappa(t)$ — "free" curve and $g'(x) > 0$, $t_i = g(x_i)$, $i = 0, 1$, $0 \leq x_0 < x_1$, $0 \leq t_0 < t_1 < t_2$.

Problem settings [1]: in the function space $C^*(\overline{D}_2) = C^{(1,1)}(D_2) \cap C(\overline{D}_2)$ to find the solution of non-linear differential equation,

$$D^{(1,1)}U(x, t) = f\left(x, t, U(x, t), D^{(0,1)}U(x, t)\right) := f[U(x, t)], \quad (2.1)$$

which satisfies the conditions

$$\begin{aligned} U(x_0, t) &= \varphi(t), \quad t \in [t_1, t_2], \\ U(x, t_1) &= V(x, t_1), \quad x \in [x_0, x_1], \end{aligned} \quad (2.2)$$

where $V(x, t) \in C^*(\bar{D}_1)$ is the solution the Darboux problem [8]

$$\begin{aligned} D^{(1.1)}V(x, t) + a_1(x, t)D^{(1.0)}V(x, t) + a_2(x, t)D^{(0.1)}V(x, t) \\ = \omega(x, t, V(x, t)) := \omega[V(x, t)], \end{aligned} \quad (2.3)$$

$$\begin{aligned} V(x_0, t) &= \mu(t), \quad t \in (g(x), t_1], \\ V(x, g(x)) &= \psi(x), \quad x \in (x_0, x_1], \end{aligned} \quad (2.4)$$

and the appropriate condition of consistency is true

$$\varphi(t_1) = \mu(t_1), \quad \psi(x_0) = \mu(t_0). \quad (2.5)$$

Under the following conditions, we consider, that $f[U(x, t)] \in C(\bar{B}_2)$, $f: \bar{B}_2 \rightarrow \mathbb{R}$, $\bar{B}_2 \subset \mathbb{R}^4$, $\Pi p_{xOt} \bar{B}_2 = \bar{D}_2$, $\omega[V(x, t)] \in C(\bar{B}_1)$, $\omega: \bar{B}_1 \rightarrow \mathbb{R}$, $\bar{B}_1 \subset \mathbb{R}^3$, $\Pi p_{xOt} \bar{B}_1 = \bar{D}_1$, $a_1(x, t) \in C^{(1.0)}(D_1)$, $a_2(x, t) \in C^{(0.1)}(D_1)$, and the given functions are $\varphi(t) \in C^1[t_1, t_2]$, $\mu(t) \in C^1[t_0, t_1]$, $\psi(x) \in C^1[x_0, x_1]$.

It's easy to make sure in the equity of the following.

Lemma 1. *Let $f[U(x, t)] \in C(\bar{B}_2)$, $\omega[V(x, t)] \in C(\bar{B}_1)$, $a_1(x, t) \in C^{(1.0)}(D_1)$, $a_2(x, t) \in C^{(0.1)}(D_1)$ and the condition has been fulfilled.*

$$D^{(1.0)}a_1(x, t) = D^{(0.1)}a_2(x, t), \quad (x, t) \in D_1. \quad (2.6)$$

Thus, boundary value problem (2.1)–(2.5) can be represented in equivalent integral form [7]

$$\begin{aligned} U_s(x, t) &= \Phi_s(x, t) + \varepsilon_s T_{1,s} f_1[U_1(\xi, \eta)] \\ &\quad + T_s f_s[U_s(\xi, \eta)], \quad (x, t) \in \bar{D}_s, \quad s = 1, 2, \end{aligned} \quad (2.7)$$

where

$$\varepsilon_s = \begin{cases} 0, & \text{for } s = 1, \\ 1, & \text{for } s = 2, \end{cases}$$

and

$$\begin{aligned} \Phi_1(x, t) &:= \mu(t) \exp \left(\int_x^{x_0} a_2(\xi, t) d\xi \right) + [\psi(x) \\ &\quad - \mu(g(x)) \exp \left(\int_x^{x_0} a_2(\xi, g(x)) d\xi \right)] \exp \left(\int_t^{g(x)} a_1(x, \eta) d\eta \right), \end{aligned}$$

$$\begin{aligned}
\Phi_2(x, t) &:= \varphi(t) + \mu(t_1) \left[\exp \left(\int_x^{x_0} a_2(\xi, t_1) d\xi \right) - 1 \right] + [\psi(x) \\
&\quad - \mu(g(x)) \exp \left(\int_x^{x_0} a_2(\xi, g(x)) d\xi \right)] \exp \left(\int_{t_1}^{g(x)} a_1(x, \eta) d\eta \right), \\
T_1 f_1 [U_1(\xi, \eta)] &:= \int_{x_0}^x \int_{g(x)}^t K(x, t; \xi, \eta) f_1 [U_1(\xi, \eta)] d\eta d\xi \\
U_1(t, x) &:= V(t, x), K(x, t; \xi, \eta) := \exp \left(\int_t^\eta a_1(\xi, \tau) d\tau + \int_x^\xi a_2(\zeta, t) d\zeta \right), \\
f_1 [U_1(x, t)] &:= \omega [U_1(x, t)] + \left[D^{(0.1)} a_2(x, t) + a_1(x, t) a_2(x, t) \right] U_1(x, t), \\
U_2(x, t) &:= U(x, t), f_2 [U_2(x, t)] := f [U(x, t)], \\
T_2 f_2 [U_2(\xi, \eta)] &:= \int_{t_1}^t \int_{x_0}^x f_2 [U_2(\xi, \eta)] d\xi d\eta, \\
T_{1,2} f_1 [U_1(\xi, \eta)] &:= \int_{x_0}^x \int_{g(x)}^{t_1} K(x, t_1; \xi, \eta) f_1 [U_1(\xi, \eta)] d\eta d\xi
\end{aligned}$$

3. CONSTRUCTION INVESTIGATION METHOD AND APPROXIMATE SOLUTION FOR SYSTEM (2.7)

Let's set the sufficient condition of existence and one way solution for integral equation systems (2.7) and make up the method of their approximate solution.

For this effect let's introduce the space of function $C_1(\overline{B}_s)$.

Definition 1. Supposing that, functions $f_s [U_s(x, t)] \in C_1(\overline{B}_s)$, if they satisfy following condition [3, 5]:

- (1) $f_s [U_s(x, t)] \in C(\overline{B}_s)$, $s = 1, 2$;
- (2) in the space of function $C(\overline{B}_{1,s})$, $\overline{B}_{1,s} \subset \mathbb{R}^{2+2s}$, $\Pi_{p_{xOt}} \overline{B}_{1,s} = \overline{D}_s$, $s = 1, 2$, there exist functions

$$\begin{aligned}
H_1(x, t, V_1(x, t); Z_1(x, t)) &:= H_1[V_1(x, t); Z_1(x, t)], \\
H_2(x, t, V_2(x, t), D^{(0.1)} V_2(x, t); Z_2(x, t), D^{(0.1)} Z_2(x, t)) &:= H_2[V_2(x, t); Z_2(x, t)],
\end{aligned}$$

such as

- (a) $H_s[U_s(x, t); U_s(x, t)] \equiv f_s [U_s(x, t)]$, $(x, t) \in \overline{D}_s$, $s = 1, 2$,

- (b) for any continuously differential pair of functions $V_s(x, t)$, $Z_s(x, t) \in \bar{B}_{1,s}$, which satisfy conditions

$$\begin{aligned} V_s(x, t) &\geq Z_s(x, t), D^{(0,1)}V_2(x, t) \geq D^{(0,1)}Z_2(x, t), \\ (x, t) &\in \bar{D}_s, \text{ in } \bar{B}_{1,s} \text{ the inequality} \\ H_s[V_s(x, t); Z_s(x, t)] &\geq H_s[Z_s(x, t); V_s(x, t)], (x, t) \in \bar{D}_s, \end{aligned} \quad (3.1)$$

is proven;

- (3) the functions $H_s[V_s(x, t); Z_s(x, t)]$ in $\bar{B}_{1,s}$ satisfy the Lipschitz condition, i.e. for any functions $V_{s,\kappa}(x, t), Z_{s,\kappa}(x, t) \in \bar{B}_{1,s}$, $\kappa = 1, 2$, from $C^{(0,1)}(\bar{D}_s)$ space the conditions

$$\begin{aligned} |H_1[V_{1,1}(x, t); V_{1,2}(x, t)] - H_1[Z_{1,1}(x, t); Z_{1,2}(x, t)]| \\ \leq L_1 (|W_{1,1}(x, t)| + |W_{1,2}(x, t)|), \\ |H_2[V_{2,1}(x, t); V_{2,2}(x, t)] - H_2[Z_{2,1}(x, t); Z_{2,2}(x, t)]| \\ \leq L_2 \left(|W_{2,1}(x, t)| + |W_{2,2}(x, t)| + \left| D^{(0,1)}W_{2,1}(x, t) \right| \right. \\ \left. + \left| D^{(0,1)}W_{2,2}(x, t) \right| \right), \end{aligned}$$

are preserved, where L_s is the Lipschitz constant and

$$W_{s,\kappa}(x, t) = V_{s,\kappa}(x, t) - Z_{s,\kappa}(x, t), \quad \kappa, s = 1, 2.$$

Note, that if functions $f_s[U_s(x, t)] \in C(\bar{B}_s)$ and have in the domain $\bar{B}_s$ first order partial derivatives, then they always belong to space $C_1(\bar{B}_s)$. The opposite statement is not true.

Let functions $Z_{1,p}(x, t), V_{1,p}(x, t) \in C(\bar{D}_1), Z_{2,p}(x, t), V_{2,p}(x, t) \in C^*(\bar{D}_2)$, belong to the domain $\bar{B}_{1,s}, s = 1, 2, p \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

We will introduce the following:

$$\begin{aligned} f_s^p(x, t) &:= H_s[Z_{s,p}(x, t); V_{s,p}(x, t)], \\ f_{s,p}(x, t) &:= H_s[V_{s,p}(x, t); Z_{s,p}(x, t)], \\ \alpha_{s,p}(x, t) &:= Z_{s,p}(x, t) - \Phi_{s,p}(x, t) - \varepsilon_s T_{1,s} f_1^p(\xi, \eta) - T_s f_s^p(\xi, \eta), \\ \beta_{s,p}(x, t) &:= V_{s,p}(x, t) - \Phi_{s,p}(x, t) - \varepsilon_s T_{1,s} f_{1,p}(\xi, \eta) - T_s f_{s,p}(\xi, \eta), \\ W_{s,p}(x, t) &= Z_{s,p}(x, t) - V_{s,p}(x, t), (x, t) \in \bar{D}_s, s = 1, 2, p \in \mathbb{N}_0. \end{aligned} \quad (3.2)$$

Let's build the sequences of functions $\{Z_{s,p}(x, t)\}, \{V_{s,p}(x, t)\}$ according to [4, 6]

$$\begin{aligned} Z_{s,p+1}(x, t) &= \Phi_s(x, t) + \varepsilon_s T_{1,s} f_1^p(\xi, \eta) + T_s f_s^p(\xi, \eta), \\ V_{s,p+1}(x, t) &= \Phi_s(x, t) + \varepsilon_s T_{1,s} f_{1,p}(\xi, \eta) + T_s f_{s,p}(\xi, \eta), \\ (x, t) &\in \bar{D}_s, s = 1, 2, p \in \mathbb{N}_0, \end{aligned} \quad (3.3)$$

where zero approximation $Z_{s,0}(x,t), V_{s,0}(x,t) \in \bar{B}_{1,s}$ could be considered as arbitrary functions from $C(\bar{D}_1), C^*(\bar{D}_2)$, which satisfies conditions (2.4), (2.5), (2.2) and the inequalities

$$\begin{aligned} W_{s,0}(x,t) &\geq 0, D^{0,1}W_{2,0}(x,t) \geq 0, \alpha_{s,0}(x,t) \geq 0, \beta_{s,0}(x,t) \leq 0, \\ D^{(0,1)}\alpha_{2,0}(x,t) &\geq 0, D^{(0,1)}\beta_{2,0}(x,t) \leq 0, \\ (x,t) &\in \bar{D}_s, s = 1, 2. \end{aligned} \quad (3.4)$$

Definition 2. The functions $Z_{1,0}(x,t), V_{1,0}(x,t) \in C(\bar{D}_1), Z_{2,0}(x,t), V_{2,0}(x,t) \in C^*(\bar{D}_2)$, of the domain $\bar{B}_{1,s}, s = 1, 2$, and satisfy the conditions (2.4), (2.5), (2.2) and inequalities (3.4), could be called the comparison function of the problem (2.1)–(2.5) [7].

From (3.2), (3.3) the validity of the following equations follows:

$$\begin{aligned} Z_{s,p}(x,t) - Z_{s,p+1}(x,t) &= \alpha_{s,p}(x,t), \\ V_{s,p}(x,t) - V_{s,p+1}(x,t) &= \beta_{s,p}(x,t), \\ W_{s,p+1}(x,t) &= \varepsilon_s T_{1,s}(f_1^p(\xi, \eta) - f_{1,p}(\xi, \eta)) + T_s(f_s^p(\xi, \eta) - f_{s,p}(\xi, \eta)), \\ \alpha_{s,p+1}(x,t) &= \varepsilon_s T_{1,s}(f_1^p(\xi, \eta) - f_1^{p+1}(\xi, \eta)) + T_s(f_s^p(\xi, \eta) - f_s^{p+1}(\xi, \eta)), \\ \beta_{s,p+1}(x,t) &= \varepsilon_s T_{1,s}(f_{1,p}(\xi, \eta) - f_{1,p+1}(\xi, \eta)) + T_s(f_{s,p}(\xi, \eta) - f_{s,p+1}(\xi, \eta)), \\ (x,t) &\in \bar{D}_s, s = 1, 2, p \in \mathbb{N}_0 \end{aligned} \quad (3.5)$$

From the last equations considering the conditions (3.1), (3.4), by the method of mathematical induction, we have confirmed that following inequalities are true

$$\begin{aligned} \alpha_{s,p}(x,t) &\geq 0, \beta_{s,p}(x,t) \leq 0, \\ D^{(0,1)}\alpha_{2,p}(x,t) &\geq 0, D^{(0,1)}\beta_{2,p}(x,t) \leq 0, \\ D^{(0,\kappa-1)}V_{s,p}(x,t) &\leq D^{(0,\kappa-1)}V_{s,p+1}(x,t) \leq D^{(0,\kappa-1)}Z_{s,p+1}(x,t) \\ &\leq D^{(0,\kappa-1)}Z_{s,p}(x,t), \end{aligned} \quad (3.6)$$

for all $p \in \mathbb{N}_0$ and $(x,t) \in \bar{D}_s, s = 1, 2$ ($\kappa = 1$ in $s = 1$ and $\kappa = 1, 2$ in $s = 2$)

The following lemma takes place

Theorem 1. Let the functions $f_s[U_s(x,t)] \in C_1(\bar{B}_s), a_1(x,t) \in C^{(1,0)}(D_1), a_2(x,t) \in C^{(0,1)}(D_1)$ and the condition (2.6) holds, and for the domain $\bar{B}_{1,s}, s = 1, 2$ there exist comparison functions $Z_{s,0}(x,t)$ and $V_{s,0}(x,t)$ of the problem (2.1)–(2.5).

Then for any $p \in \mathbb{N}$ and $s = 1, 2$ functions $Z_{s,p}(x,t)$ and $V_{s,p}(x,t)$, which had been built according to (3.3), the inequalities (3.6) are true in the domain $\bar{D}_s$.

Let show that the built this way sequence of functions $\{Z_{s,p}(x,t)\}$ and $\{V_{s,p}(x,t)\}$ converge uniformly at $(x,t) \in \bar{D}_s$ to the unique regular solution of problem (2.1)–(2.5).

Because of inequalities (3.6) at $(x, t) \in \bar{D}_s$ in order to

$$\lim_{p \rightarrow \infty} Z_{s,p}(x, t) = \lim_{p \rightarrow \infty} V_{s,p}(x, t)$$

it is enough to show, that $\lim_{p \rightarrow \infty} W_{s,p}(x, t) = 0$.

Let

$$\begin{aligned} \sup_{\bar{D}_1 \times \bar{D}_1} K(x, t; \xi, \eta) &\leq K, \max_s \left[\sup_{\bar{D}_s} |W_{s,0}(x, t)|, \sup_{\bar{D}_2} |D^{(0.1)} W_{2,0}(x, t)| \right] \leq d, \\ \sup_{\bar{D}} [1, x - x_0 + t - t_0] &= l, \max(2KL_1, 4L_2) = L. \end{aligned}$$

Indeed: from (3.5), by the method of mathematical induction, we insure that the estimates are true

$$\max_s \left\{ \sup_{\bar{D}_s} W_{s,p}(x, t), \sup_{\bar{D}_2} |W_{2,p}(x, t)| \right\} \leq \frac{[Ll(t - t_0 + x - x_0)]^p}{p!} d, \quad p \in \mathbb{N}. \quad (3.7)$$

From the estimates (3.7) follows, that

$$\lim_{p \rightarrow \infty} W_{s,p}(x, t) = 0, \quad \lim_{p \rightarrow \infty} D^{(0.1)} W_{2,p}(x, t) = 0,$$

that

$$\begin{aligned} U_s(x, t) &:= \lim_{p \rightarrow \infty} Z_{s,p}(x, t) = \lim_{p \rightarrow \infty} V_{s,p}(x, t), \quad (x, t) \in \bar{D}_s, s = 1, 2, \\ \gamma(x, t) &:= \lim_{p \rightarrow \infty} D^{(0.1)} Z_{2,p}(x, t) = \lim_{p \rightarrow \infty} D^{(0.1)} V_{2,p}(x, t), \quad (x, t) \in \bar{D}_2. \end{aligned}$$

Is not difficult to make sure that $\gamma(x, t) = D^{(0.1)} U_2(x, t)$, for it in (3.3) it would be enough to go the limit for $p \rightarrow \infty$ and differentiate the obtained result in t under $s = 2$. After that the limit functions $U_s(x, t)$ are the solutions of the system integral equations (2.7) and under $(x, t) \in \bar{D}_s, s = 1, 2$, and so the boundary-value problem (2.1)–(2.5).

Theorem 2. *Let the condition of Theorem 1 to be true. Then, the sequences of functions $\{Z_{s,p}(x, t)\}$ and $\{V_{s,p}(x, t)\}$, have been built according to (3.3), (3.4):*

- (1) *uniformly convergence to the unique solution of the corresponding integral equation in system (2.7) for $(x, t) \in \bar{D}_s, s = 1, 2$;*
- (2) *the true estimates (3.7);*
- (3) *in $\bar{B}_{s,1}$ the inequalities hold*

$$\begin{aligned} D^{(0, \kappa-1)} V_{s,p}(x, t) &\leq D^{(0, \kappa-1)} V_{s,p+1}(x, t) \leq D^{(0, \kappa-1)} U_s(x, t) \\ &\leq D^{(0, \kappa-1)} Z_{s,p+1}(x, t) \leq D^{(0, \kappa-1)} Z_{s,p}(x, t), \\ (x, t) &\in \bar{D}_s, s = 1, 2, \end{aligned} \quad (3.8)$$

$$\kappa = \begin{cases} 1, s = 1, \\ 1, 2, s = 2, \end{cases} \quad (3.9)$$

for all $p \in \mathbb{N}_0$, where $U_s(x, t)$ – is the unique solution of the corresponding integral equation in (2.7) for $(x, t) \in \overline{D}_s, s = 1, 2$.

The uniqueness of the integral equations system solution (2.7) for $(x, t) \in \overline{D}_s, s = 1, 2$, and appropriateness of inequalities (3.8) have been proved from the opposite.

Corollary 1. *Let the conditions of Theorem 1 to be hold. Then in the domain $\overline{D}$ there exist a unique regular solution of the boundary-value problem (2.1)–(2.5) and for it the inequalities (3.8) are true.*

Corollary 2. *Let the conditions of Theorem 1 to be hold, $\mu(t) = \psi(x) = 0, t \in [g(x), t_1], x \in [x_0, x_1], \varphi(t) = 0, t \in [t_1, t_2]$, in particular $f_s[U_s(x, t)] = H_s[U_s(x, t); 0], (x, t) \in \overline{D}_s, s = 1, 2$.*

Then, if $f_s[0] \leq (\geq) 0$ in the domain $\overline{B}_s, s = 1, 2$, the solution of the boundary-value problem (2.1)–(2.5) for $(x, t) \in \overline{D}_s, s = 1, 2$ satisfies the inequalities [7]

$$U_s(x, t) \leq (\geq) 0, (x, t) \in \overline{D}_s, s = 1, 2.$$

4. ACCELERATION CONVERGENCE OF THE METHOD (3.3), (3.4)

Let's show one approach of acceleration convergence iteration method (3.3), (3.4).

With this aim let's build sequence of functions $Z_{s,p}(x, t), V_{s,p}(x, t)$ according to [2, 7]

$$\begin{aligned} Z_{s,p+1}(x, t) &= \Phi_s(x, t) + \varepsilon_s T_{1,s} \bar{f}_1^p(\xi, \eta) + T_s \bar{f}_s^p(\xi, \eta), \\ V_{s,p+1}(x, t) &= \Phi_s(x, t) + \varepsilon_s T_{1,s} \bar{f}_{1,p}(\xi, \eta) + T_s \bar{f}_{s,p}(\xi, \eta), (x, t) \in \overline{D}_s, s = 1, 2, p \in \mathbb{N}_0, \end{aligned} \quad (4.1)$$

where

$$\begin{aligned} \bar{f}_s^p(x, t) &:= H_s [\bar{Z}_{s,p}(x, t); \bar{V}_{s,p}(x, t)], \\ \bar{f}_{s,p}(x, t) &:= H_s [\bar{V}_{s,p}(x, t); \bar{Z}_{s,p}(x, t)], s = 1, 2, (x, t) \in \overline{D}_s. \\ D^{(0, \kappa-1)} \bar{Z}_{s,p}(x, t) &:= D^{(0, \kappa-1)} Z_{s,p}(x, t) - d_{s,p}^{(\kappa-1)}(x, t) D^{(0, \kappa-1)} W_{s,p}(x, t), \\ D^{(0, \kappa-1)} \bar{V}_{s,p}(x, t) &:= D^{(0, \kappa-1)} V_{s,p}(x, t) + q_{s,p}^{(\kappa-1)}(x, t) D^{(0, \kappa-1)} W_{s,p}(x, t), \\ s &= 1, 2, (x, t) \in \overline{D}_s, \\ \kappa &= \begin{cases} 1, s = 1, \\ 1, 2, s = 2, \end{cases} \end{aligned}$$

and consistent functions $d_{s,p}^{(\kappa-1)}(x,t)$, $q_{s,p}^{(\kappa-1)}(x,t)$, $(x,t) \in \overline{D}_s$, satisfy conditions

$$0 \leq d_{s,p}^{(\kappa-1)}(x,t) \leq 0,5; 0 \leq q_{s,p}^{(\kappa-1)}(x,t) \leq 0,5, \quad (4.2)$$

$(x,t) \in \overline{D}_s$, $s = 1, 2, p \in \mathbb{N}_0$.

As the zero approximation $Z_{s,0}(x,t)$, $V_{s,0}(x,t)$ we take comparison function of the problem (2.1)–(2.5).

We have noted, that if $D^{(0,\kappa-1)}Z_{s,p}(x,t)$ and $D^{(0,\kappa-1)}V_{s,p}(x,t)$ belong to the domain $\overline{B}_{1,s}$, then

$$D^{(0,\kappa-1)}\overline{Z}_{s,p}(x,t), D^{(0,\kappa-1)}\overline{V}_{s,p}(x,t) \in \overline{B}_{1,s}.$$

If at every iteration step (4.1), (3.4), (4.2) functions $d_{s,p}^{(\kappa-1)}(x,t)$, $q_{s,p}^{(\kappa-1)}(x,t)$ satisfy the conditions

$$\begin{aligned} D^{(0,\kappa-1)}[Z_{s,p}(x,t) - Z_{s,p+1}(x,t)] - d_{s,p}^{(\kappa-1)}(x,t)D^{(0,\kappa-1)}W_{s,p}(x,t) &\geq 0, \\ D^{(0,\kappa-1)}[V_{s,p}(x,t) - V_{s,p+1}(x,t)] + q_{s,p}^{(\kappa-1)}(x,t)D^{(0,\kappa-1)}W_{s,p}(x,t) &\leq 0, \\ (x,t) &\in \overline{D}_s, s = 1, 2, \end{aligned} \quad (4.3)$$

then it's easy to see, that functions $Z_{s,p}(x,t)$ and $V_{s,p}(x,t)$ built according to (4.1), (3.4), (4.2), (4.4), satisfy the inequalities (3.6)

Lemma 2. *Let the conditions of Theorem 1 to be hold. The set of functions $d_{s,p}^{(\kappa-1)}(x,t)$ and $q_{s,p}^{(\kappa-1)}(x,t)$, $(x,t) \in \overline{D}_s$, $s = 1, 2, p \in \mathbb{N}_0$, which belong to space $C(\overline{D}_s)$ and satisfy conditions (4.2), (4.4), is non-empty.*

Proof. Indeed, let us put on each step of iteration in (4.1), (3.4)

$$\begin{aligned} d_{s,p}^{(\kappa-1)}(x,t) &= \begin{cases} D^{(0,\kappa-1)}\alpha_{s,p}(x,t) [\rho_{s,p}^{(\kappa-1)}(x,t)]^{-1}, & \text{for } D^{(0,\kappa-1)}W_{s,p}(x,t) \neq 0, \\ 0, & \text{for } D^{(0,\kappa-1)}W_{s,p}(x,t) = 0, \end{cases} \\ q_{s,p}^{(\kappa-1)}(x,t) &= \begin{cases} -D^{(0,\kappa-1)}\beta_{s,p}(x,t) [\rho_{s,p}^{(\kappa-1)}(x,t)]^{-1}, & \text{for } D^{(0,\kappa-1)}W_{s,p}(x,t) \neq 0, \\ 0, & \text{for } D^{(0,\kappa-1)}W_{s,p}(x,t) = 0, \end{cases} \end{aligned} \quad (4.4)$$

where $\rho_{s,p}^{(\kappa-1)}(x,t) := D^{(0,\kappa-1)}[\alpha_{s,p}(x,t) - \beta_{s,p}(x,t) + W_{s,p}(x,t)]$.

It is obvious, that chosen functions $d_{s,p}^{(\kappa-1)}(x,t)$ and $q_{s,p}^{(\kappa-1)}(x,t)$ according to (4.4) satisfy the conditions (4.2) and under (3.1), (4.2) we have received

$$\begin{aligned} D^{(0,\kappa-1)}[Z_{s,p}(x,t) - Z_{s,p-1}(x,t)] - \frac{D^{(0,\kappa-1)}\alpha_{s,p}(x,t)}{\rho_{s,p}^{(\kappa-1)}(x,t)}D^{(0,\kappa-1)}W_{s,p}(x,t) \\ = D^{(0,\kappa-1)}\alpha_{s,p}(x,t) + D^{(0,\kappa-1)}[\epsilon_s T_{1,s}(f_1^p(\xi, \eta) - \overline{f}_1^p(\xi, \eta))] \end{aligned}$$

$$\begin{aligned}
 & +T_s \left(f_s^p(\xi, \eta) - \bar{f}_s^p(\xi, \eta) \right) \Big] - \frac{D^{(0, \kappa-1)} \alpha_{s,p}(x, t)}{\rho_{s,p}^{(\kappa-1)}(x, t)} D^{(0, \kappa-1)} W_{s,p}(x, t) \\
 & \geq D^{(0, \kappa-1)} \alpha_{s,p}(x, t) \left[1 - \frac{D^{(0, \kappa-1)} W_{s,p}(x, t)}{\rho_{s,p}^{(\kappa-1)}(x, t)} \right] \geq 0, \quad (x, t) \in \bar{D}_s.
 \end{aligned}$$

All other inequalities (4.4) are proofed analogically for $(x, t) \in \bar{D}_s, s = 1, 2$, and Lemma 2 could have been proven. $\square$

Let $Z_{s,p}(x, t)$ and $V_{s,p}(x, t)$ are constructed by some method of the problem comparison function (2.1)–(2.5). We could denote the following approximations to the solution of the system (2.7), constructed according to the method (4.1)–(4.4) via $Z_{s,p+1}^*(x, t)$ and $V_{s,p+1}^*(x, t)$, then from (3.3), (4.1) we have obtained

$$\begin{aligned}
 Z_{s,p+1}(x, t) - Z_{s,p+1}^*(x, t) &= \varepsilon_s T_{1,s} \left[f_1^p(\xi, \eta) - \bar{f}_1^p(\xi, \eta) \right] \\
 &+ T_s \left[f_s^p(\xi, \eta) - \bar{f}_s^p(\xi, \eta) \right] \geq 0,
 \end{aligned}$$

similarly, that $V_{s,p+1}(x, t) \leq V_{s,p+1}^*(x, t)$, for $(x, t) \in \bar{D}_s$ the following inequalities take place

$$V_{s,p+1}(x, t) \leq V_{s,p+1}^*(x, t) \leq Z_{s,p+1}^*(x, t) \leq Z_{s,p+1}(x, t),$$

thus the convergence of the method (4.1)–(4.4) is not slower than the convergence of the iterative process (3.3), (3.4).

Theorem 3. *Let the conditions of Theorem 1 to be true.*

Then the sequences of function $\{Z_{s,p}(x, t)\}$ and $\{V_{s,p}(x, t)\}$, built according to (3.4), (4.1)–(4.4); have the following characteristics

- (1) *are uniform convergent to the unique regular solution of the problem (2.1)–(2.5) for $(x, t) \in \bar{D}$;*
- (2) *the inequalities (3.6) are true in the domain $\bar{B}_{s,1}$;*
- (3) *the convergence of the method (3.4), (4.1)–(4.4) is not slower than the convergence of the iterative process (3.3), (3.4).*

Note, that depending on the choice of functions $d_{s,p}^{(\kappa-1)}(x, t)$ and $q_{s,p}^{(\kappa-1)}(x, t)$, which satisfy the condition (4.2), (4.4), we could obtain different approximate research methods and approximate solution for the differential equations theory problems.

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SOME COMBINATORIAL IDENTITIES VIA k -ORDER FIBONACCI MATRICES

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Received 28 January, 2021

Abstract. Matrix factorizations have been widely used in recent years, especially in engineering problems, to facilitate performance-requiring computations. In this paper, we investigate some interesting relationships between some combinatorial matrices such as Pascal matrix, Stirling matrices and k -order Fibonacci matrices. We give factorizations and inverse factorizations of the Pascal and Stirling matrices via k -order Fibonacci matrices. Moreover, we derive various combinatorial properties by using relationships between these matrices. Finally, compared to previous studies, we present more general results for specific values of k .

2010 *Mathematics Subject Classification:* 15A23; 11B39; 05A10

Keywords: Pascal matrix, Stirling matrix, k -order Fibonacci matrix, factorization

1. INTRODUCTION

The Pascal matrix arises in linear algebra, probability and combinatorial analysis. Until now, several researchers have studied the factorizations, generalizations and eigenvalues of some combinatorial matrices [1, 2, 4, 6, 11, 13–15]. In [2], the authors produced some combinatorial identities and an existence theorem for diophantine equation systems by applying linear algebra. Lee et al. gave a factorization of the Pascal matrix via Fibonacci matrix and they obtained some combinatorial identities by using Pascal matrix, the Stirling matrices of the first kind and the second kind and the Fibonacci matrix [11]. In another way, Zhang and Wang presented a factorization of the Pascal matrix and they found various combinatorial properties with the help of the Fibonacci matrix [15]. For more information about Pascal matrices, Stirling matrices and properties of the binomial coefficients, the reader is referred to book such as [5].

For any positive integer n , the $n \times n$ lower triangular Pascal matrix $\mathcal{P}_n = [p_{ij}]$ and its inverse $\mathcal{P}_n^{-1} = [p'_{ij}]$ are defined by

$$p_{ij} = \begin{cases} \binom{i-1}{j-1} & \text{if } i \geq j, \\ 0 & \text{otherwise,} \end{cases} \quad p'_{ij} = \begin{cases} (-1)^{i-j} \binom{i-1}{j-1} & \text{if } i \geq j, \\ 0 & \text{otherwise,} \end{cases} \quad (1.1)$$

where the binomial coefficient $\binom{i}{j}$ counts the number of lattice paths from $(0,0)$ to $(i-j, j)$ with steps $(1,0)$ and $(0,1)$.

For example

$$\mathcal{P}_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} \quad \text{and} \quad \mathcal{P}_5^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ -1 & 3 & -3 & 1 & 0 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix}.$$

For nonnegative integers n, k and $n \geq k \geq 0$, the Stirling numbers of the first kind $s(n, k)$ and of the second kind $S(n, k)$, which are the coefficients of polynomials that commonly arise in combinatorial problems, are defined as the coefficients in the following expansions (see [3]):

$$[x]_n = \sum_{k=0}^n s(n, k) x^k \quad \text{and} \quad x^n = \sum_{k=0}^n S(n, k) [x]_k$$

where

$$[x]_n = \begin{cases} x(x-1) \cdots (x-n+1) & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases}$$

Moreover, the $S(n, k)$ satisfy the following formula

$$S(n, k) = \sum_{l=k-1}^{n-1} \binom{n-1}{l} S(l, k-1). \quad (1.2)$$

For the Stirling numbers $s(i, j)$ and $S(i, j)$ of the first kind and of the second kind, respectively, define $\mathcal{S}_n^{(1)} = [s_{ij}]$ and $\mathcal{S}_n^{(2)} = [S_{ij}]$ to be the $n \times n$ matrices by

$$s_{ij} = \begin{cases} s(i, j) & \text{if } i \geq j, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad S_{ij} = \begin{cases} S(i, j) & \text{if } i \geq j, \\ 0 & \text{otherwise.} \end{cases}$$

We call the matrices $\mathcal{S}_n^{(1)}$ and $\mathcal{S}_n^{(2)}$ Stirling matrix of the first kind and of the second kind, respectively (see [5]). For example,

$$\mathcal{S}_5^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 2 & -3 & 1 & 0 & 0 \\ -6 & 11 & -6 & 1 & 0 \\ 24 & -50 & 35 & -10 & 1 \end{bmatrix} \quad \text{and} \quad \mathcal{S}_5^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 \\ 1 & 7 & 6 & 1 & 0 \\ 1 & 15 & 25 & 10 & 1 \end{bmatrix}.$$

The Fibonacci numbers are one of the most well-known numbers, and have many important applications to a wide variety of research areas such as mathematics, computer science, physics, biology, and statistics [8]. The Fibonacci numbers, $\{F_n\}_{n=0}^{\infty}$, are the terms of the sequence $0, 1, 1, 2, 3, 5, \dots$ where each term is the sum of the two previous terms, and starting with the initial values $F_0 = 0$ and $F_1 = 1$. Up to

this time, some researchers have studied the generalizations and applications of the k -generalized Fibonacci numbers [7, 9, 12].

For $n > k \geq 2$, an interesting generalization of the Fibonacci sequence, k -order Fibonacci sequence, k -order Fibonacci sequence $\{g(k)_n\}_{n=0}^\infty$, were defined in [9] by

$$g(k)_n = g(k)_{n-1} + g(k)_{n-2} + \cdots + g(k)_{n-k},$$

with the initial conditions $g(k)_0 = \cdots = g(k)_{k-2} = 0, g(k)_{k-1} = g(k)_k = 1$. For example, if $k = 2$, then $\{g(2)_n\}_{n=0}^\infty$ is the Fibonacci sequence, $\{F_n\}_{n=0}^\infty$ and if $k = 3$, then $\{g(3)_n\}_{n=0}^\infty$ is the 3-Fibonacci sequence, namely the Tribonacci sequence $\{T_n\}_{n=0}^\infty$.

For any positive integer n , the $n \times n$ k -order Fibonacci matrix, $\mathcal{F}_n^{(k)} = [f_{ij}]$, is defined by [10]

$$f_{ij} = \begin{cases} g_{i-j+1} & \text{if } i-j \geq 0, \\ 0 & \text{otherwise.} \end{cases} \quad (1.3)$$

where $g_t = g(k)_{t+k-2}$, $1 \leq t \leq n$. Here we note that $f_{n1} = g_n$ and each column of $\mathcal{F}_n^{(k)}$ is the vector of k -order Fibonacci numbers. For example, the Fibonacci and 4-Fibonacci matrices are as follows:

$$\mathcal{F}_6^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 \\ 3 & 2 & 1 & 1 & 0 & 0 \\ 5 & 3 & 2 & 1 & 1 & 0 \\ 8 & 5 & 3 & 2 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathcal{F}_6^{(4)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 \\ 4 & 2 & 1 & 1 & 0 & 0 \\ 8 & 4 & 2 & 1 & 1 & 0 \\ 15 & 8 & 4 & 2 & 1 & 1 \end{bmatrix}.$$

From (1.3), the matrix $\mathcal{F}_n^{(k)}$ is lower triangular and $\det(\mathcal{F}_n^{(k)}) = 1 \neq 0$. Thus, the matrix $\mathcal{F}_n^{(k)}$ has an inverse. Therefore, for any positive integer n , the $n \times n$ inverse k -order Fibonacci matrix, $(\mathcal{F}_n^{(k)})^{-1} = [f'_{ij}]$, is defined by [10]

$$f'_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -1 & \text{if } i-k \leq j \leq i-1, \\ 0 & \text{otherwise.} \end{cases} \quad (1.4)$$

The relations between the Pascal matrix, Stirling matrices and Fibonacci matrix motivate us to study a more generalized situation. Thus, motivated by the above cited works, we give the factorizations of the Pascal and Stirling matrices via k -order Fibonacci matrices. We derive various generalized combinatorial identities by using these matrices. As compared to earlier works [11], [15] and [14], we give k -extended results.

2. FIRST FACTORIZATION OF THE PASCAL MATRIX

In this section, we consider the first factorization of the Pascal matrix by means of the k -order Fibonacci matrices.

Definition 1. For $n \in \mathbb{N}$, the $n \times n$ matrix $\mathcal{L}_n^{(k)} = [l_{ij}]$ is defined by

$$l_{ij} = \binom{i-1}{j-1} - \sum_{t=1}^k \binom{i-t-1}{j-1}. \quad (2.1)$$

By Definition 1, we observe that $l_{11} = 1$, $l_{1j} = 0$, for $j \geq 2$, $l_{21} = 0$, $l_{22} = 1$, $l_{2j} = 0$, for $j \geq 3$, $l_{31} = -1$, $l_{32} = 1$, $l_{33} = 1$, $l_{3j} = 0$, for $j \geq 4$ and

$$l_{i1} = \begin{cases} 2-i, & 1 \leq i \leq k, \\ 1-k, & i > k. \end{cases} \quad (2.2)$$

Moreover, for $i, j \geq 2$ we have the recursion $l_{ij} = l_{i-1,j-1} + l_{i-1,j}$. In particular, if we take $n = 8$ and $k = 5$, we get

$$\mathcal{L}_8^{(5)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ -3 & -2 & 2 & 3 & 1 & 0 & 0 & 0 \\ -4 & -5 & 0 & 5 & 4 & 1 & 0 & 0 \\ -4 & -9 & -5 & 5 & 9 & 5 & 1 & 0 \\ -4 & -13 & -14 & 0 & 14 & 14 & 6 & 1 \end{bmatrix}.$$

By virtue of the equations (1.1), (1.3) and (2.1), we can derive the following theorem.

Theorem 1. Let $\mathcal{L}_n^{(k)}$ be the matrix as in (2.1). For the Pascal matrix $\mathcal{P}_n$ and the k -order Fibonacci matrix $\mathcal{F}_n^{(k)}$, we have

$$\mathcal{P}_n = \mathcal{F}_n^{(k)} \mathcal{L}_n^{(k)}.$$

Proof. Since the matrix $\mathcal{F}_n^{(k)}$ is invertible, we will prove that $\left(\mathcal{F}_n^{(k)}\right)^{-1} \mathcal{P}_n = \mathcal{L}_n^{(k)}$.

Let $\left(\mathcal{F}_n^{(k)}\right)^{-1} = [f'_{ij}]$ be the inverse of $\mathcal{F}_n^{(k)}$. Since $f'_{1j} = 0$ for $j \geq 2$, $f'_{11}p_{11} = 1$ and $l_{11} = 1 = \sum_{t=1}^n f'_{1t}p_{t1}$. Since $p_{1j} = 0$ and $f'_{1j} = 0$ for $j \geq 2$, $\sum_{t=1}^n f'_{1t}p_{tj} = 0 = l_{1j}$ for $j \geq 2$. Since $f'_{2j} = 0$ for $j \geq 3$, $f'_{21} = -1$ and $f'_{22} = 1$, we have $\sum_{t=1}^n f'_{2t}p_{t1} = l_{21} = 0$. By virtue of (1.4), we have, for $i = 3, 4, \dots, n$, $\sum_{t=1}^n f'_{it}p_{t1} = l_{i1}$.

Now, we consider $i \geq 3$ and $j \geq 2$. From (1.4) and the recurrence relation of l_{ij} , we have $\sum_{t=1}^n f'_{it}p_{tj} = l_{ij}$. Therefore, we have $\left(\mathcal{F}_n^{(k)}\right)^{-1} \mathcal{P}_n = \mathcal{L}_n^{(k)}$, and the proof is completed. $\square$

By virtue of Theorem 1, we have the following Corollary.

Corollary 1. For $1 \leq r \leq n$, we have

$$\binom{n-1}{r-1} = \sum_{t=r}^n g_{n-t+1} \left[\binom{t-1}{r-1} - \sum_{s=1}^k \binom{t-s-1}{r-1} \right]. \quad (2.3)$$

In particular, if $r = 1$ then

$$g_n = 1 + \sum_{i=2}^k (i-2)g_{n-i+1} + (k-1) \sum_{i=k+1}^n g_{n-i+1}.$$

Proof. Since $\mathcal{P}_n = \mathcal{F}_n^{(k)} \mathcal{L}_n^{(k)}$, $g_1 = 1, g_2 = 1$ and $l_{ij} = 0$ for $i \leq j-1$, we have

$$\binom{n-1}{r-1} = p_{nr} = g_n l_{1r} + g_{n-1} l_{2r} + g_{n-2} l_{3r} + \cdots + g_3 l_{n-2,r} + g_2 l_{n-1,r} + g_1 l_{nr}.$$

From (2.2) and Theorem 1, we know $l_{11}, l_{21}, l_{22}, l_{i1}$ and the recurrence $l_{ij} = l_{i-1,j-1} + l_{i-1,j}$. Thus we obtain

$$p_{nr} = \sum_{t=r}^n g_{n-t+1} l_{t,r},$$

which is equal to (2.3). In particular, if we take $r = 1$, we have

$$p_{n1} = g_n l_{11} + g_{n-1} l_{21} + g_{n-2} l_{31} + \cdots + g_{n-k} l_{k+1,1} + \cdots + g_1 l_{n1}.$$

By virtue of (2.2), we get

$$1 = g_n + (0)g_{n-1} + (-1)g_{n-2} + (-2)g_{n-3} + (-3)g_{n-4} + \cdots + (2-k)g_{n-k+1} \\ + (1-k)(g_{n-k} + g_{n-k+1} + \cdots + g_2 + g_1).$$

By arranging the above expressions, we find that

$$g_n = 1 + \sum_{i=2}^k (i-2)g_{n-i+1} + (k-1) \sum_{i=k+1}^n g_{n-i+1}.$$

Hence the proof is completed. $\square$

From (2.1), we can find the inverse of the matrix $(\mathcal{L}_n^{(k)})^{-1} = [l'_{ij}]$ as

$$l'_{ij} = \sum_{t=j}^i (-1)^{i+t} \binom{i-1}{t-1} g_{t-j+1}. \quad (2.4)$$

We observe that $l'_{11} = 1, l'_{1j} = 0$, for $j \geq 2$, $l'_{21} = 0, l'_{22} = 1, l'_{2j} = 0$, for $j \geq 3$ and the recursion $l'_{ij} = l'_{i-1,j-1} - l'_{i-1,j}$ holds for $i, j \geq 2$. Since $\mathcal{F}_n^{(k)} = \mathcal{P}_n (\mathcal{L}_n^{(k)})^{-1}$, we have

$$f_{nr} = p_{n1} l'_{1r} + p_{n2} l'_{2r} + p_{n3} l'_{3r} + \cdots + p_{n,n-1} l'_{n-1,r} + p_{n,n} l'_{nr}.$$

From (1.1) and (2.4), we get

$$g_{n-r+1} = \sum_{t=r}^n \left[\binom{n-1}{t-1} \left[\sum_{s=r}^t (-1)^{t+s} \binom{t-1}{s-1} g_{s-r+1} \right] \right].$$

In particular, if $r = 1$, we obtain

$$g_n = \sum_{t=1}^n \left[\binom{n-1}{t-1} \left[\sum_{s=1}^t (-1)^{t+s} \binom{t-1}{s-1} g_s \right] \right].$$

Let I_n be the identity matrix of order n , and let L_k be the $k \times k$ lower triangular matrix as follows:

$$L_k = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

Set $S_l = L_{k+1} \oplus I_l$, $l = 1, 2, \dots$. Further we define n by n matrices $\overline{\mathcal{F}}_n^{(k)} = [1] \oplus \mathcal{F}_{n-1}^{(k)}$, $G_1 = I_n$, $G_2 = I_{n-2} \oplus L_2$, $G_3 = I_{n-3} \oplus L_3, \dots, G_k = I_{n-k} \oplus L_k$, $G_{k+1} = I_{n-k-1} \oplus L_{k+1}$, and, for $k+2 \leq l \leq n$, $G_l = I_{n-l} \oplus S_{l-k-1}$. In particular $S_0 = L_{k+1}$ and $G_n = S_{n-k-1}$. In [10], the authors gave the factorization of the k -order Fibonacci matrix as: $\mathcal{F}_n^{(k)} = G_1 G_2 G_3 \cdots G_n$.

For $k \geq 2$, we define the $n \times n$ matrices $H_n^{(k)}$ and $\overline{H}_i$ by

$$H_n^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 1 & 0 & \cdots & 0 \\ -1 & 1 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ -1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}, \quad H_n^{(k)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ -2 & 1 & 1 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 1-k & 1 & 1 & 1 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1-k & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \end{bmatrix},$$

$\overline{H}_i = I_{n-i} \oplus H_i^{(k)}$. From the definition of $\overline{H}_i$, we have $\overline{H}_1 = \overline{H}_2 = I_n$. Thus, we have the following lemma by the definition of the matrix product.

Lemma 1. The matrix $\mathcal{L}_n^{(k)}$ can be factored by the $\overline{H}_i$'s as follows:

$$\mathcal{L}_n^{(k)} = \overline{H}_n \overline{H}_{n-1} \overline{H}_{n-2} \cdots \overline{H}_2 \overline{H}_1.$$

By virtue of Lemma 1, we have the following theorem.

Theorem 2. For the $n \times n$ Pascal matrix, $\mathcal{P}_n$, we have

$$\mathcal{P}_n = \prod_{i=1}^n G_i \prod_{i=1}^n \bar{H}_{n-i+1}.$$

3. SECOND FACTORIZATION OF THE PASCAL MATRIX

In this section, we consider the second factorization of the Pascal matrix by means of the k -order Fibonacci matrix.

Definition 2. For $n \in \mathbb{N}$, the $n \times n$ matrix $\mathcal{R}_n^{(k)} = [r_{ij}]$ is defined by

$$r_{ij} = \binom{i-1}{j-1} - \sum_{t=1}^k \binom{i-1}{j+t-1}. \quad (3.1)$$

From Definition 2, we observe that $r_{11} = 1$, $r_{1j} = 0$, for $j \geq 2$, $r_{21} = 0$, $r_{22} = 1$, $r_{2j} = 0$, for $j \geq 3$, $r_{31} = -2$, $r_{32} = 1$, $r_{33} = 1$, $r_{3j} = 0$, for $j \geq 4$ and

$$r_{i1} = \begin{cases} 2 - 2^{i-1}, & 2 \leq i \leq k+1, \\ 1 - \sum_{t=1}^k \binom{i-1}{t}, & i > k+1. \end{cases}$$

Moreover, for $i, j \geq 2$ we have the recursion $r_{ij} = r_{i-1,j-1} + r_{i-1,j}$. For example, if we take $n = 8$ and $k = 5$, we get

$$\mathcal{R}_8^{(5)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -6 & -1 & 2 & 1 & 0 & 0 & 0 & 0 \\ -14 & -7 & 1 & 3 & 1 & 0 & 0 & 0 \\ -30 & -21 & -6 & 4 & 4 & 1 & 0 & 0 \\ -61 & -51 & -27 & -2 & 8 & 5 & 1 & 0 \\ -118 & -112 & -78 & -29 & 6 & 13 & 6 & 1 \end{bmatrix}.$$

By virtue of the equations (1.1), (1.3) and (3.1), we can derive the following theorem.

Theorem 3. Let $\mathcal{R}_n^{(k)}$ be the matrix as in (3.1). For the Pascal matrix $\mathcal{P}_n$ and the k -order Fibonacci matrix $\mathcal{F}_n^{(k)}$, we have

$$\mathcal{P}_n = \mathcal{R}_n^{(k)} \mathcal{F}_n^{(k)}.$$

Proof. Since the matrix $\mathcal{F}_n^{(k)}$ is invertible, we will prove that $\mathcal{P}_n \left(\mathcal{F}_n^{(k)} \right)^{-1} = \mathcal{R}_n^{(k)}$.

Let $\left(\mathcal{F}_n^{(k)} \right)^{-1} = [f'_{ij}]$ be the inverse of $\mathcal{F}_n^{(k)}$. Since $f'_{1j} = 0$ for $j \geq 2$, $p_{11}f'_{11} = 1$ and $r_{11} = 1 = \sum_{t=1}^n p_{1t}f'_{t1}$. Since $p_{1j} = 0$ and $f'_{1j} = 0$ for $j \geq 2$, $\sum_{t=1}^n p_{1t}f'_{tj} = 0 = r_{1j}$ for $j \geq 2$. Since $f'_{2j} = 0$ for $j \geq 3$, $f'_{21} = -1$ and $f'_{22} = 1$, we have $\sum_{t=1}^n p_{2t}f'_{t1} = r_{21} = 0$. By virtue of (1.4), we have, for $i = 3, 4, \dots, n$, $\sum_{t=1}^n p_{it}f'_{t1} = r_{i1}$.

Now, we consider $i \geq 3$ and $j \geq 2$. From (1.4) and the recurrence relation of r_{ij} , we have $\sum_{t=1}^n p_{it} f'_{tj} = r_{ij}$. Therefore, we have $\mathcal{P}_n \left(\mathcal{F}_n^{(k)} \right)^{-1} = \mathcal{R}_n^{(k)}$, and the proof is completed. $\square$

By virtue of Theorem 3, we have the following Corollary.

Corollary 2. For $1 \leq s \leq n$, we have

$$\binom{n-1}{s-1} = \sum_{t=s}^n \left[\binom{n-1}{t-1} - \sum_{m=1}^k \binom{n-1}{t+m-1} \right] g_{t-s+1}. \quad (3.2)$$

Proof. Since $\mathcal{P}_n = \mathcal{R}_n^{(k)} \mathcal{F}_n^{(k)}$, we have $p_{ns} = \sum_{t=s}^n r_{n,t} g_{t-s+1}$ which is equal to (3.2). In particular, if we take $s = 1$, we have

$$p_{n1} = r_{n1} g_1 + r_{n2} g_2 + r_{n3} g_3 + \cdots + r_{n,n-1} g_{n-1} + r_{n,n} g_n$$

By virtue of (3.1), we get

$$\begin{aligned} 1 &= g_1 \left[\binom{n-1}{0} - \binom{n-1}{1} - \cdots - \binom{n-1}{k} \right] \\ &\quad + g_2 \left[\binom{n-1}{1} - \binom{n-1}{2} - \cdots - \binom{n-1}{k+1} \right] + \cdots \\ &\quad + g_{n-1} \left[\binom{n-1}{n-2} - \binom{n-1}{n-1} \right] + g_n \left[\binom{n-1}{n-1} \right]. \end{aligned}$$

By simplifying the above expressions, we find that

$$g_n = 1 - (n-2)g_{n-1} - \left(\frac{n^2 - 5n + 2}{2} \right) g_{n-2} - \sum_{t=1}^{n-3} \left[\binom{n-1}{t-1} - \sum_{m=1}^k \binom{n-1}{t+m-1} \right] g_t.$$

Hence the proof is completed. $\square$

From (3.1), we can find the inverse of the matrix $\left(\mathcal{R}_n^{(k)} \right)^{-1} = [r'_{ij}]$ as

$$r'_{ij} = \sum_{t=j}^i (-1)^{j+t} \binom{t-1}{j-1} g_{i-t+1}. \quad (3.3)$$

Since $\mathcal{F}_n^{(k)} = \left(\mathcal{R}_n^{(k)} \right)^{-1} \mathcal{P}_n$, we have

$$f_{ns} = r'_{n1} p_{1s} + r'_{n2} p_{2s} + r'_{n3} p_{3s} + \cdots + r'_{n,n-1} p_{n-1,s} + r'_{nn} p_{ns}.$$

By virtue of (1.1) and (3.3), we get

$$g_{n-s+1} = \sum_{t=s}^n \left[\sum_{m=t}^n (-1)^{t+m} \binom{m-1}{t-1} g_{n-m+1} \right] \binom{t-1}{s-1}.$$

In particular, if $s = 1$ then,

$$g_n = \sum_{t=1}^n \left[\sum_{m=t}^n (-1)^{t+m} \binom{m-1}{t-1} g_{n-m+1} \right].$$

For $k \geq 2$, we define the $n \times n$ matrices $W_n^{(k)}$ and $\overline{W}_i$ by

$$W_n^{(k)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ -2 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ -6 & 1 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 2-2^k & 1 & 1 & 1 & 1 & 1 & 0 & \cdots & 0 \\ 1-\sum_{t=1}^k \binom{k+1}{t} & 1 & 1 & 1 & 1 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1-\sum_{t=1}^k \binom{n-1}{t} & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

$\overline{W}_i = I_{n-i} \oplus W_i^{(k)}$. From the definition of $\overline{W}_i$, we have $\overline{W}_1 = \overline{W}_2 = I_n$. Thus, we have the following lemma by the definition of the matrix product.

Lemma 2. *The matrix $\mathcal{R}_n^{(k)}$ can be factored by the $\overline{W}_i$'s as follows:*

$$\mathcal{R}_n^{(k)} = \overline{W}_n \overline{W}_{n-1} \overline{W}_{n-2} \cdots \overline{W}_2 \overline{W}_1.$$

By virtue of Lemma 2, we have the following theorem.

Theorem 4. *For the $n \times n$ Pascal matrix, $\mathcal{P}_n$, we have*

$$\mathcal{P}_n = \prod_{i=1}^n \overline{W}_{n-i+1} \prod_{i=1}^n G_i.$$

4. STIRLING NUMBER OF THE SECOND KIND

In this section, we consider relationships between Stirling matrices of the second kind and k -order Fibonacci matrices.

We define an $n \times n$ matrix $\mathcal{M}_n^{(k)} = [m_{ij}]$ by using the Stirling numbers of the second kind as follows:

$$m_{ij} = S(i, j) - \sum_{t=1}^k S(i-t, j). \quad (4.1)$$

From (4.1), we observe that $m_{11} = 1, m_{1j} = 0$ for $j \geq 2, m_{21} = 0, m_{22} = 1, m_{2j} = 0$ for $j \geq 3$,

$$m_{i1} = \begin{cases} 2-i, & 1 \leq i \leq k, \\ 1-k, & i > k, \end{cases} \quad (4.2)$$

and, for $i, j \geq 2, m_{ij} = m_{i-1, j-1} + jm_{i-1, j}$. From the definition of $\mathcal{S}_n^{(2)}, \mathcal{F}_n^{(k)}$ and $\mathcal{M}_n^{(k)}$, we have the following theorem.

Theorem 5. Let $\mathcal{M}_n$ be the $n \times n$ matrix as in (4.1). For the Stirling matrix of the second kind $\mathcal{S}_n^{(2)}$ and the Fibonacci matrix $\mathcal{F}_n^{(k)}$, we have $\mathcal{S}_n^{(2)} = \mathcal{F}_n^{(k)} \mathcal{M}_n^{(k)}$.

Proof. Since the matrix $\mathcal{F}_n^{(k)}$ is invertible, we will prove that $\left(\mathcal{F}_n^{(k)}\right)^{-1} \mathcal{S}_n^{(2)} = \mathcal{M}_n^{(k)}$.

Let $\left(\mathcal{F}_n^{(k)}\right)^{-1} = [f'_{ij}]$ be the inverse of $\mathcal{F}_n^{(k)}$. Since $f'_{1j} = 0$ for $j \geq 2, f'_{11}S_{11} = 1 = m_{11}$. As $S_{1j} = 0$ and $f'_{1j} = 0$ for $j \geq 2, \sum_{t=1}^n f'_{1t}S_{tj} = 0 = m_{1j}$ for $j \geq 2$. As $f'_{2j} = 0$ for $j \geq 3, f'_{21} = -1$ and $f'_{22} = 1$, we have $\sum_{t=1}^n f'_{2t}S_{t1} = m_{21}$. By virtue of (1.4), we have, for $i = 3, 4, \dots, n, \sum_{t=1}^n f'_{it}S_{t1} = m_{i1}$ which is given in (4.2).

Now we consider $i \geq 3$ and $j \geq 2$. By (1.4) and (4.1), we have $\sum_{t=1}^n f'_{it}S_{tj} = m_{ij}$. Thus, we have $\mathcal{M}_n^{(k)} = \left(\mathcal{F}_n^{(k)}\right)^{-1} \mathcal{S}_n^{(2)}$, i.e. $\mathcal{S}_n^{(2)} = \mathcal{F}_n^{(k)} \mathcal{M}_n^{(k)}$ which completes the proof. $\square$

For example, for $k = 5$ and $n = 8$, we have

$$\begin{aligned} \mathcal{S}_8^{(2)} &= \mathcal{F}_8^{(5)} \mathcal{M}_8^{(5)} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 4 & 2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 8 & 4 & 2 & 1 & 1 & 0 & 0 & 0 \\ 16 & 8 & 4 & 2 & 1 & 1 & 0 & 0 \\ 31 & 16 & 8 & 4 & 2 & 1 & 1 & 0 \\ 61 & 31 & 16 & 8 & 4 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 3 & 5 & 1 & 0 & 0 & 0 & 0 \\ -3 & 4 & 18 & 9 & 1 & 0 & 0 & 0 \\ -4 & 5 & 58 & 54 & 14 & 1 & 0 & 0 \\ -4 & 6 & 179 & 274 & 124 & 20 & 1 & 0 \\ -4 & 8 & 543 & 1275 & 894 & 244 & 27 & 1 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 7 & 6 & 1 & 0 & 0 & 0 & 0 \\ 1 & 15 & 25 & 10 & 1 & 0 & 0 & 0 \\ 1 & 31 & 90 & 65 & 15 & 1 & 0 & 0 \\ 1 & 63 & 301 & 350 & 140 & 21 & 1 & 0 \\ 1 & 127 & 966 & 1701 & 1050 & 266 & 28 & 1 \end{bmatrix}.$$

Since $S_{nt} = S(n, t) = \sum_{r=1}^n f_{nr} m_{rt}$ and for $i > k$

$$m_{ij} = \frac{1}{j!} \sum_{l=0}^j (-1)^l \binom{j}{l} \left((j-l)^i - \sum_{t=1}^k (j-l)^{i-t} \right),$$

we have the following corollary.

Corollary 3. For $k < j \leq n$, we have

$$S(n, j) = \sum_{i=j}^n g_{n-i+1} \left[\frac{1}{j!} \sum_{l=0}^j (-1)^l \binom{j}{l} \left((j-l)^i - \sum_{t=1}^k (j-l)^{i-t} \right) \right].$$

Lemma 3. For the $(n-1) \times (n-1)$ Stirling matrix of the second kind, $\mathcal{S}_{n-1}^{(2)}$, we have

$$\mathcal{M}_n^{(k)} = \mathcal{L}_n^{(k)} \left([1] \oplus \mathcal{S}_{n-1}^{(2)} \right).$$

Proof. Let $\Phi_n = [\varphi_{ij}] = \mathcal{L}_n^{(k)} \left([1] \oplus \mathcal{S}_{n-1}^{(2)} \right)$. By Theorem 1, we know that $l_{11} = 1 = m_{11}$, $l_{21} = 0 = m_{21}$ and $l_{22} = S(1, 1) = 1 = m_{22}$. So, we have $\varphi_{ij} = m_{ij}$ for $i = 1, 2$.

Now we consider the case $i \geq 3$. Since

$$\varphi_{ij} = \sum_{t=j-1}^{i-1} \left[\binom{i-1}{t} S(t, j-1) - \sum_{s=1}^k \binom{i-s-1}{t} S(t, j-1) \right]$$

and from (1.2), we have $\varphi_{ij} = S(i, j) - \sum_{t=1}^k S(i-t, j) = m_{ij}$.

Therefore, $\mathcal{M}_n^{(k)} = \mathcal{L}_n^{(k)} \left([1] \oplus \mathcal{S}_{n-1}^{(2)} \right)$. □

The following corollary is an immediate consequence of Lemma 3.

Corollary 4. For $n \geq 2$, $\mathcal{S}_n^{(2)} = \mathcal{F}_n^{(k)} \mathcal{L}_n^{(k)} \left([1] \oplus \mathcal{S}_{n-1}^{(2)} \right)$.

For $(s \times s)$ matrices $\mathcal{F}_s^{(k)}$ and $\mathcal{L}_s^{(k)}$, we define the $n \times n$ matrix $\overline{\mathcal{F}_n^{(k)} \mathcal{L}_n^{(k)}}$ by $\overline{\mathcal{F}_s^{(k)} \mathcal{L}_s^{(k)}} = I_{n-s} \oplus \mathcal{F}_s^{(k)} \mathcal{L}_s^{(k)}$. So, $\overline{\mathcal{F}_n^{(k)} \mathcal{L}_n^{(k)}} = \mathcal{F}_n^{(k)} \mathcal{L}_n^{(k)}$ and $\overline{\mathcal{F}_1^{(k)} \mathcal{L}_1^{(k)}} = I_n$. Therefore, we can give the following corollary.

Corollary 5. *The Stirling matrix of the second kind, $\mathcal{S}_n^{(2)}$, can be factorized by the matrices $\overline{\mathcal{F}_n^{(k)} \mathcal{L}_n^{(k)}}$'s as :*

$$\mathcal{S}_n^{(2)} = \overline{\left(\mathcal{F}_n^{(k)} \mathcal{L}_n^{(k)}\right)} \overline{\left(\mathcal{F}_{n-1}^{(k)} \mathcal{L}_{n-1}^{(k)}\right)} \cdots \overline{\left(\mathcal{F}_2^{(k)} \mathcal{L}_2^{(k)}\right)} \overline{\left(\mathcal{F}_1^{(k)} \mathcal{L}_1^{(k)}\right)}.$$

Since $P_n = \mathcal{F}_n^{(k)} \mathcal{L}_n^{(k)}$ and $\bar{P}_s = \overline{\mathcal{F}_s^{(k)} \mathcal{L}_s^{(k)}}$, we can give the following corollary as an immediate consequence of Corollary 5.

Corollary 6. *The Stirling matrix of the second kind, $\mathcal{S}_n^{(2)}$, can be factorized by the Pascal matrices $\bar{P}_n$'s as :*

$$\mathcal{S}_n^{(2)} = \bar{P}_n \bar{P}_{n-1} \cdots \bar{P}_2 \bar{P}_1.$$

5. STIRLING NUMBER OF THE FIRST KIND

In this section, we consider the relationships between the Stirling matrix of the first type $\mathcal{S}_n^{(1)}$ and k -order Fibonacci matrix $\mathcal{F}_n^{(k)}$.

We define an $n \times n$ matrix $\mathcal{C}_n^{(k)} = [c_{ij}]$ by using the Stirling number of the first kind as follows:

$$c_{ij} = s(i, j) - \sum_{t=1}^k s(i, j+t). \quad (5.1)$$

From (5.1), we see that $c_{11} = 1$, $c_{1j} = 0$, for $j \geq 2$, $c_{21} = -2$, $c_{22} = 1$, $c_{2j} = 0$, for $j \geq 3$, and for $i, j \geq 2$, $c_{ij} = c_{i-1, j-1} - (i-1)c_{i-1, j}$. By virtue of the definition of $\mathcal{S}_n^{(1)}$, $\mathcal{C}_n^{(k)}$ and $\mathcal{F}_n^{(k)}$, we can derive the following Theorem.

Theorem 6. *Let $\mathcal{C}_n^{(k)}$ be the $n \times n$ matrix as in (5.1). For the Stirling matrix of the first kind $\mathcal{S}_n^{(1)}$ and the Fibonacci matrix $\mathcal{F}_n^{(k)}$, we have $\mathcal{S}_n^{(1)} = \mathcal{C}_n^{(k)} \mathcal{F}_n^{(k)}$.*

Proof. As $\mathcal{F}_n^{(k)}$ is an invertible matrix, it suffices to prove that $\mathcal{C}_n^{(k)} = \mathcal{S}_n^{(1)} \cdot \left(\mathcal{F}_n^{(k)}\right)^{-1}$.

Let $\left(\mathcal{F}_n^{(k)}\right)^{-1} = [f'_{ij}]$ be the inverse of $\mathcal{F}_n^{(k)}$. Since $f'_{1j} = 0$ for $j \geq 2$, $s_{11}f'_{11} = 1 = c_{11}$. Since $s_{1j} = 0$ and $f'_{1j} = 0$ for $j \geq 2$, $\sum_{t=1}^n s_{1t}f'_{tj} = 0 = c_{1j}$ for $j \geq 2$. Since $f'_{2j} = 0$ for $j \geq 3$, $f'_{21} = -1$ and $f'_{22} = 1$, we have $\sum_{t=1}^n s_{2t}f'_{t1} = c_{21} = -2$ and $c_{22} = 1$. By virtue of (1.4), we have, for $i = 3, 4, \dots, n$, $\sum_{t=1}^n s_{it}f'_{t1} = c_{i1}$.

Now, we consider $i \geq 3$ and $j \geq 2$. From (1.4) and the recurrence relation $c_{ij} = c_{i-1, j-1} - (i-1)c_{i-1, j}$, we have $\sum_{t=1}^n s_{it}f'_{tj} = c_{ij}$. Therefore, we have $\mathcal{C}_n^{(k)} = \mathcal{S}_n^{(1)} \cdot \left(\mathcal{F}_n^{(k)}\right)^{-1}$, and the proof is completed. $\square$

For example, for $k = 4$ and $n = 6$, we have

$$\begin{aligned}
\mathcal{S}_6^{(1)} &= \mathcal{C}_6^{(4)} \mathcal{F}_6^{(4)} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 \\ 4 & -4 & 1 & 0 & 0 & 0 \\ -12 & 16 & -7 & 1 & 0 & 0 \\ 48 & -76 & 44 & -11 & 1 & 0 \\ -239 & 428 & -296 & 99 & -16 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 \\ 4 & 2 & 1 & 1 & 0 & 0 \\ 8 & 4 & 2 & 1 & 1 & 0 \\ 15 & 8 & 4 & 2 & 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 2 & -3 & 1 & 0 & 0 & 0 \\ -6 & 11 & -6 & 1 & 0 & 0 \\ 24 & -50 & 35 & -10 & 1 & 0 \\ -120 & 274 & -225 & 85 & -15 & 1 \end{bmatrix}.
\end{aligned}$$

By virtue of Theorem 6, we can arrive at the following interesting identity:

$$(n-1)! = \sum_{j=1}^n (-1)^{n+1} \left(s(n, j) - \sum_{t=1}^k s(n, j+t) \right) g_j.$$

Furthermore, by analogy to the Corollaries 4, 5 and 6, we can derive the following theorem.

Theorem 7. For the Stirling matrix $\mathcal{S}_n^{(1)}$ of the first kind,

$$\mathcal{S}_n^{(1)} = ([1] \oplus \mathcal{S}_{n-1}^{(1)}) \mathcal{F}_n^{(k)} \mathcal{L}_n^{(k)} = \bar{P}_1 \bar{P}_2 \cdots \bar{P}_n.$$

6. ACKNOWLEDGMENT

We thank the anonymous reviewers for their careful reading of our manuscript and their many insightful comments and suggestions.

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ON THE EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR NON-AUTONOMOUS SEMI-LINEAR SYSTEMS WITH NON-INSTANTANEOUS IMPULSES, DELAY, AND NON-LOCAL CONDITIONS

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Received 28 March, 2021

Abstract. A non-autonomous evolution semi-linear differential system under non-instantaneous impulses, delays, and perturbed by non-local conditions is studied. Its piece-wise continuous solutions belong to a finite dimensional Banach space. The existence and uniqueness of solutions on the interval $[-r, \tau]$ are obtained by applying Karakostas' fixed-point theorem. Further results concerning solution prolongation are developed. An example is presented, and several remarks on the infinite-dimensional case are included.

2010 Mathematics Subject Classification: 93C10; 93C23

Keywords: non-instantaneous impulses, non-autonomous systems, Karakostas' fixed-point theorem, evolution equations, delay, non-local conditions

1. INTRODUCTION

Impulsive systems are of vital importance on most scientific fields. They can be found in applications ranging from biology and population dynamics to economics and engineering. Usually, many situations are modeled by differential equations. Controls, delays, impulses, and non-linear perturbations are added to capture either feedback or activity characterization.

The interest of this article is the non-autonomous non-instantaneous impulsive semi-linear system involving state-delay and non-local conditions, which is motivated by applications, such as species population, nanoscale electronic circuits consisting of single-electron tunneling junctions, and mechanical systems with impacts [13, 24, 26]. In particular, impulses represent sudden deviations of the states at specific times, by either instantaneous jumps or continuous intervals.

Our mathematical motivation is to extend the existence and uniqueness of solutions on a finite-dimensional Banach space [2, 5, 20] for the aforementioned semi-linear system. It is worth to highlight that some existence and controllability results on the impulsive autonomous case have been done by [4, 21, 22, 25], and [14]. The latter

authors are the pioneer in implementing Karakostas' fixed-point theorem to prove the existence of solutions on semi-linear equations with instantaneous impulses.

Furthermore, techniques from [6] and Rothe's fixed-point theorem have been used in [8, 9, 19] for studying the approximate and exact controllability of this family of systems with delays, instantaneous impulses, and memory considerations. Important results on autonomous impulsive systems involving delay were developed by [7, 10, 18]. In addition, problems with non-local conditions including impulses can be found in [16].

Finally, [11] introduced the class of non-instantaneous impulsive systems, and [23] showed the existence of solutions for these systems. Later, [1, 3, 17] showcased relevant studies for models on non-instantaneous impulsive differential equations. However, it is not of our knowledge that there are results on the existence of solutions for semi-linear non-autonomous systems including all conditions simultaneously. This is the center of our research work.

This article is structured as follows. Section 2 describes the analyzed system and notation. Section 3 deals with preliminary concepts, definitions, and hypotheses used throughout this work. Section 4 is devoted to the existence and uniqueness of solutions for the system in the light of Karakostas' fixed-point theorem, which is an extension of the fixed-point theorem due to M. A. Krasnosel'skiĭ developed in [12]. Finally, sections 5 and 6 illustrate these results with an example of the considered system and present conclusions and guidelines for open problems.

2. SYSTEM DESCRIPTION

Let $N \in \mathbb{N}$, and denote I_N as the set $\{1, 2, \dots, N\}$. In this article, the existence and uniqueness of solutions for the following semi-linear non-autonomous system are proved:

$$\begin{cases} z'(t) = A(t)z(t) + f(t, z_t), & t \in \bigcup_{i=0}^N (s_i, t_{i+1}], \\ z(t) = G_i(t, z(t)), & t \in (t_i, s_i], i \in I_N, \\ z(t) = \phi(t) - g(z_{\theta_1}, z_{\theta_2}, \dots, z_{\theta_q})(t), & t \in [-r, 0], \end{cases} \quad (2.1)$$

where $s_i, t_i, \theta_j, r \in (0, \tau)$, with $t_i \leq s_i < t_{i+1}$, $\theta_j < \theta_{j+1}$, for $i \in I_N$ and $j \in I_q$, $s_0 = t_0 = 0$, and $t_{N+1} = \theta_{q+1} = \tau$, all fixed real numbers. The system solutions are denoted by $z: \mathcal{J} = [-r, \tau] \rightarrow \mathbb{R}^n$ and the non-instantaneous impulses are represented by $G_i: (t_i, s_i] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $i \in I_N$. A is a continuous matrix such that $A(t) \in \mathbb{R}^{n \times n}$, $t \in \mathbb{R}$. z_t stands for the translated function of z defined by $z_t(s) = z(t+s)$, with $s \in [-r, 0]$. The function $f: \overline{\mathbb{R}_+} \times PC_r([-r, 0]; \mathbb{R}^n) \rightarrow \mathbb{R}^n$ represents the non-linear perturbation of the differential equation in the system, where $\overline{\mathbb{R}_+} = [0, +\infty)$, and $g: PC_r^q([-r, 0]; (\mathbb{R}^n)^q) \rightarrow PC_r([-r, 0]; \mathbb{R}^n)$ indicates the behavior in the non-local conditions. The function

$$\phi: [-r, 0] \rightarrow \mathbb{R}^n \quad (2.2)$$

represents the historical pass on the time interval $[-r, 0]$.

In order to properly set system (2.1), the following Banach spaces are considered. Denote $C(\mathcal{U}; \mathbb{R}^n)$ as the space of continuous functions on a set $\mathcal{U} \subset \mathbb{R}$. $PC_r = PC_r([-r, 0]; \mathbb{R}^n)$ is the space of continuous functions of the form (2.2) except on a finite number of points r_i , $i \in I_l$, with $l \leq N$, where the side limits $\phi(r_i^+)$, $\phi(r_i^-)$ exist, and $\phi(r_i) = \phi(r_i^-)$, for all $i \in I_N$, endowed with the supremum norm.

The natural Banach space for the solutions of system (2.1) is defined as:

$$PC_{r\tau} = PC_{r\tau}(\mathcal{J}; \mathbb{R}^n) = \left\{ z: \mathcal{J} \longrightarrow \mathbb{R}^n \mid z|_{[-r, 0]} \in PC_r, z|_{[0, \tau]} \in C(\mathcal{J}'; \mathbb{R}^n), \right. \\ \left. \text{there exist } z(t_k^+), z(t_k^-), \text{ and } z(t_k) = z(t_k^-), k \in I_N \right\},$$

where $\mathcal{J}' = [0, \tau] \setminus \{t_1, t_2, \dots, t_N\}$, and endowed with the norm

$$\|z\| = \|z\|_0 = \sup_{t \in \mathcal{J}} \|z(t)\|_{\mathbb{R}^n}, \quad z \in PC_{r\tau}.$$

The cartesian product space given by $(\mathbb{R}^n)^q = \mathbb{R}^n \times \mathbb{R}^n \times \dots \times \mathbb{R}^n = \prod_{i=1}^q \mathbb{R}^n$ is

equipped with the norm $\|z\|_{(\mathbb{R}^n)^q} = \sum_{i=1}^q \|z_i\|_{\mathbb{R}^n}$, for $z \in (\mathbb{R}^n)^q$.

The space $PC_r^q = PC_r^q([-r, 0]; (\mathbb{R}^n)^q)$ is defined analogously and endowed with the norm

$$\|z\|_{PC_r^q} = \sup_{t \in [-r, 0]} \|z(t)\|_{(\mathbb{R}^n)^q}, \quad z \in PC_r^q.$$

3. PRELIMINARY THEORY AND HYPOTHESES

In this section, the evolution operator based on the corresponding linear system is defined. This work can be extended to infinite-dimensional Banach spaces. Thus, the properties of the evolution operator are included, aiming to establish their similarities with the latter case, where uniform continuity is lost unless the evolution operator is assumed to be compact. Finally, the system solutions are characterized, and hypotheses for applying Karakostas' fixed-point theorem are presented.

Let U be the evolution operator corresponding to system (2.1)

$$U(t, s) = \Phi(t)\Phi^{-1}(s), \quad \text{for all } t, s \in \mathbb{R}, \quad (3.1)$$

where Φ is the fundamental matrix solution of the associated linear system

$$z'(t) = A(t)z(t). \quad (3.2)$$

Therefore, there exist constants $\hat{M}$, $\omega > 0$ and $M \geq 1$ such that:

$$\|U(t, s)\| \leq \hat{M}e^{\omega(t-s)} \leq M, \quad 0 \leq s \leq t \leq \tau,$$

The following proposition exhibits a characterization of solutions of the system (2.1), and is based on the works done in [16] and [23].

Proposition 1. *The semi-linear system (2.1) has a solution $z \in PC_{r\tau}(\mathcal{J}; \mathbb{R}^n)$ if, and only if,*

$$z(t) = \begin{cases} U(t, 0)[\phi(0) - g(z_{\theta_1}, z_{\theta_2}, \dots, z_{\theta_q})(0)] \\ \quad + \int_0^t U(t, s)f(s, z_s)ds, & t \in (0, t_1], \\ U(t, s_i)G_i(s_i, z(s_i)) + \int_{s_i}^t U(t, s)f(s, z_s)ds, & t \in (s_i, t_{i+1}], i \in I_N, \\ G_i(t, z(t)), & t \in (t_i, s_i], i \in I_N, \\ \phi(t) - g(z_{\theta_1}, z_{\theta_2}, \dots, z_{\theta_q})(t), & t \in [-r, 0]. \end{cases} \quad (3.3)$$

Observe that on some interval $[-r, p_1)$, if a solution z of the form (3.3) is defined, and there is no $p_2 > p_1$ such that a solution can be defined on $[-r, p_2)$, then, $[-r, p_1)$ is a *maximal interval* of existence.

In this work, the following hypotheses are assumed:

H1 The next conditions hold:

- (i) The function g fulfills that $g(0) = 0$, and there exists $N_q > 0$ such that, for all $y, z \in PC_r^q$ and $t \in [-r, 0]$,

$$\|g(y)(t) - g(z)(t)\|_{\mathbb{R}^n} \leq N_q \|y(t) - z(t)\|_{(\mathbb{R}^n)^q}.$$

- (ii) There exists a constant $L > 0$ such that, for all $i \in I_N$, the functions G_i satisfy $G_i(\cdot, 0) = 0$, and, if $\varphi_1, \varphi_2 \in PC_{r\tau}$, for $t \in (t_i, s_i]$, then,

$$\|G_i(t, \varphi_1(t)) - G_i(t, \varphi_2(t))\|_{\mathbb{R}^n} \leq L \|\varphi_1 - \varphi_2\|, \quad \text{where } L + N_q q < \frac{1}{2}.$$

H2 The function f satisfies the following conditions:

$$\begin{aligned} \|f(t, \varphi_1) - f(t, \varphi_2)\|_{\mathbb{R}^n} &\leq K(\|\varphi_1\|, \|\varphi_2\|) \|\varphi_1 - \varphi_2\|, \\ \|f(t, \varphi)\|_{\mathbb{R}^n} &\leq \Psi(\|\varphi\|), \end{aligned}$$

where $K: \overline{\mathbb{R}_+} \times \overline{\mathbb{R}_+} \rightarrow \overline{\mathbb{R}_+}$ and $\Psi: \overline{\mathbb{R}_+} \rightarrow \overline{\mathbb{R}_+}$ are continuous and non-decreasing functions in their arguments, and $\varphi, \varphi_1, \varphi_2 \in PC_r([-r, 0]; \mathbb{R}^n)$.

H3 The following relations hold for τ and $\rho > 0$:

- (i) $MN_q q (\|\tilde{\phi}\| + \rho) + M\tau\Psi(\|\tilde{\phi}\| + \rho) \leq \rho$,
(ii) $ML(\|\tilde{\phi}\| + \rho) + \|\alpha\|_{\mathbb{R}^n} + M\tau\Psi(\|\tilde{\phi}\| + \rho) \leq \rho$,
(iii) $L(\|\tilde{\phi}\| + \rho) + \|\beta\|_{\mathbb{R}^n} \leq \rho$,

where $\alpha, \beta \in \mathbb{R}^n$ are arbitrarily fixed, and the function $\tilde{\phi}$ is defined as:

$$\tilde{\phi}(t) = \begin{cases} U(t, 0)\phi(0), & t \in (0, t_1], \\ \alpha, & t \in \bigcup_{i=1}^N (s_i, t_{i+1}], \\ \beta, & t \in \bigcup_{i=1}^N (t_i, s_i], \\ \phi(t), & t \in [-r, 0]. \end{cases} \quad (3.4)$$

H4 The following relations hold for τ and $\rho > 0$:

- (i) $MN_q q + M\tau K (\|\tilde{\phi}\| + \rho, \|\tilde{\phi}\| + \rho) < 1$,
- (ii) $ML + M\tau K (\|\tilde{\phi}\| + \rho, \|\tilde{\phi}\| + \rho) < 1$.

Before the statement of Karakostas' fixed-point theorem (see [12, 14]), we first recall the meaning of equicontractivity

Definition 1. The family $\{T(\cdot, x) : x\}$ is called equicontractive if there is a $l \in [0, 1)$ such that

$$\|T(y_1, x) - T(y_2, x)\| \leq l \|y_1 - y_2\|,$$

for all $(y_1, x), (y_2, x)$ in the domain of T .

Theorem 1 (Karakostas). *Let $\mathcal{P}$ and $\mathcal{Q}$ be Banach spaces, $D \subset \mathcal{P}$ a closed and convex subset, and $J : D \rightarrow \mathcal{Q}$ a continuous compact operator. Let $F : D \times \overline{J(D)} \rightarrow D$ be a continuous operator such that the family given by $\{F(\cdot, y) : y \in \overline{J(D)}\}$ is equicontractive. Then, $F(z, J(z)) = z$ admits a solution in D .*

4. EXISTENCE AND UNIQUENESS OF SOLUTIONS

In this section, the proofs of the existence and uniqueness of the solution for system (2.1) are presented. To apply Karakostas' fixed-point theorem, the operators J and F are defined. Then, a fixed point on a subset of $PC_{r\tau}$ for equation (4.3) is proved. Therefore, the problem of finding a solution of the form (3.3) becomes a fixed-point problem.

Consider the following continuous operators

$$J : PC_{r\tau}(\mathcal{J}; \mathbb{R}^n) \rightarrow PC_{r\tau}(\mathcal{J}; \mathbb{R}^n),$$

$$F : PC_{r\tau}(\mathcal{J}; \mathbb{R}^n) \times PC_{r\tau}(\mathcal{J}; \mathbb{R}^n) \rightarrow PC_{r\tau}(\mathcal{J}; \mathbb{R}^n),$$

and a fixed $\eta \in \mathbb{R}^n$. For $y, z \in PC_{r\tau}$,

$$J(y)(t) = \begin{cases} U(t, 0) [\phi(0) - g(y_{\theta_1}, y_{\theta_2}, \dots, y_{\theta_q})(0)] \\ \quad + \int_0^t U(t, s) f(s, y_s) ds, & t \in (0, t_1], \\ U(t, s_i) G_i(s_i, y(s_i)) \\ \quad + \int_{s_i}^t U(t, s) f(s, y_s) ds, & t \in (s_i, t_{i+1}], i \in I_N, \\ \eta, & t \in \bigcup_{i=1}^N (t_i, s_i], \\ \phi(t), & t \in [-r, 0], \end{cases} \quad (4.1)$$

$$F(z, y)(t) = \begin{cases} y(t), & t \in \bigcup_{i=0}^N (s_i, t_{i+1}], \\ G_i(t, z(t)), & t \in (t_i, s_i], i \in I_N, \\ \phi(t) - g(z_{\theta_1}, z_{\theta_2}, \dots, z_{\theta_q})(t), & t \in [-r, 0]. \end{cases} \quad (4.2)$$

From the J and F definition, the following fixed-point equation is equivalent to solve system (2.1):

$$F(z, J(z)) = z, \quad z \in PC_{r\tau}. \quad (4.3)$$

First, it is observed that J is compact, and the set $\{F(\cdot, y) : y \in \overline{J(D_\rho)}\}$ is equicontractive, where

$$D_\rho = D_\rho(\tau, \phi) := \{\varphi \in PC_{r\tau}(\mathcal{I}; \mathbb{R}^n) : \|\varphi - \tilde{\phi}\| \leq \rho\}, \quad \text{for } \rho > 0. \quad (4.4)$$

This set is closed and convex, and $\tilde{\phi}$ is given by (3.4). So, the hypotheses of Theorem 1 are satisfied. Lemma 1 highlights the relevance of hypotheses **H1**, **H2**, and how they fit into the main results. Theorems 2, 3 and 4 follow on this foundation.

Lemma 1. *Let hypotheses **H1** and **H2** be satisfied. Then, the operators J and F satisfy the following assertions:*

- (i) J is continuous.
- (ii) J maps bounded sets onto bounded sets.
- (iii) J maps bounded sets onto equicontinuous sets.
- (iv) J is a compact operator.
- (v) The set $\{F(\cdot, y) : y \in \overline{J(D_\rho)}\}$ is comprised of equicontractive operators, with D_ρ as in (4.4).

Proof. (i) J is continuous.

Taking $y, z \in PC_{r\tau}$, trivially, for $t \in [-r, 0]$,

$$\|J(z)(t) - J(y)(t)\|_{\mathbb{R}^n} = \|\phi(t) - \phi(t)\|_{\mathbb{R}^n} = 0.$$

Thus,

$$\sup_{t \in [-r, 0]} \|J(z)(t) - J(y)(t)\|_{\mathbb{R}^n} = 0. \quad (4.5)$$

By **H1**, **H2**, and $t \in (0, t_1]$, the following estimate holds:

$$\begin{aligned} \|J(z)(t) - J(y)(t)\|_{\mathbb{R}^n} &\leq M \|g(y_{\theta_1}, \dots, y_{\theta_q})(0) - g(z_{\theta_1}, \dots, z_{\theta_q})(0)\|_{\mathbb{R}^n} \\ &\quad + M \int_0^t \|(f(s, z_s) - f(s, y_s))\|_{\mathbb{R}^n} ds \\ &\leq MN_q \sum_{i=1}^q \|y - z\| \\ &\quad + M \int_0^t K(\|z_s\|, \|y_s\|) \|z_s - y_s\| ds \\ &\leq MN_q q \|z - y\| + Mt_1 K(\|z\|, \|y\|) \|z - y\|. \end{aligned}$$

Taking the sup,

$$\sup_{t \in (0, t_1]} \|J(z)(t) - J(y)(t)\| \leq M [N_q q + t_1 K(\|z\|, \|y\|)] \|z - y\|. \quad (4.6)$$

Now, for $i \in I_N$ and $t \in (s_i, t_{i+1}]$,

$$\begin{aligned} \|J(z)(t) - J(y)(t)\|_{\mathbb{R}^n} &\leq M \|G_i(s_i, z(s_i)) - G_i(s_i, y(s_i))\|_{\mathbb{R}^n} \\ &\quad + M \int_{s_i}^t \|(f(s, z_s) - f(s, y_s))\|_{\mathbb{R}^n} ds \\ &\leq ML \|z(s_i) - y(s_i)\|_{\mathbb{R}^n} \\ &\quad + M \int_{s_i}^t K(\|z_s\|, \|y_s\|) \|z_s - y_s\| ds \\ &\leq ML \|z - y\| + M\tau K(\|z\|, \|y\|) \|z - y\|. \end{aligned}$$

Thus,

$$\sup_{t \in (s_i, t_{i+1}]} \|J(z)(t) - J(y)(t)\| \leq M [L + \tau K(\|z\|, \|y\|)] \|z - y\|.$$

Together with (4.5), (4.6), and since J is constant on $\bigcup_{i=1}^N (t_i, s_i]$, it yields that there exists $N_{y,z} > 0$ such that:

$$\|J(z) - J(y)\| \leq N_{y,z} \|z - y\|.$$

Hence, J is continuous. And, in fact, it is Lipschitz continuous.

(ii) J maps bounded sets onto bounded sets.

Without loss of generality, set $R > 0$ arbitrarily and prove that there exists $d > 0$ such that, for every $y \in B_R = \overline{B_R(0)} = \{z \in PC_{r\tau} : \|z\| \leq R\}$, it follows that $\|J(y)\| \leq d$.

For $t \in [-r, 0]$, it gives that:

$$\|J(y)(t)\|_{\mathbb{R}^n} = \|\phi(t)\|_{\mathbb{R}^n} \leq \|\phi\| =: d_0.$$

Let $y \in B_R$ and $t \in (0, t_1]$. **H2** yields

$$\begin{aligned} \|J(y)(t)\|_{\mathbb{R}^n} &\leq \|U(t, 0) [\phi(0) - g(y_{\theta_1}, \dots, y_{\theta_q})(0)]\|_{\mathbb{R}^n} \\ &\quad + \int_0^t \|U(t, s) f(s, y_s)\|_{\mathbb{R}^n} ds \\ &\leq M (\|\phi(0)\|_{\mathbb{R}^n} + N_q q \|y\|) + Mt_1 \Psi(\|y\|) \\ &\leq M (\|\phi(0)\|_{\mathbb{R}^n} + N_q q R) + Mt_1 \Psi(R) =: d_1. \end{aligned}$$

Similarly, for each $i \in I_N$, if $t \in (s_i, t_{i+1}]$, then,

$$\begin{aligned} \|J(y)(t)\|_{\mathbb{R}^n} &\leq \|U(t, s_i) G_i(s_i, y(s_i))\|_{\mathbb{R}^n} + \int_{s_i}^t \|U(t, s) f(s, y_s)\|_{\mathbb{R}^n} ds \\ &\leq ML \|y\| + M(t_{i+1} - s_i) \Psi(\|y\|) \\ &\leq MLR + M\tau \Psi(R) =: d_2. \end{aligned}$$

Finally, whenever $t \in (t_i, s_i]$, for $i \in I_N$, it follows that:

$$\|J(t)\|_{\mathbb{R}^n} = \|\eta\| =: d_3.$$

Taking $d = \max_{0 \leq i \leq 3} \{d_i\}$, boundedness is proved.

(iii) J maps bounded sets onto equicontinuous sets.

Let B_R as in (ii), and $y \in B_R$, arbitrary. For some $0 < v_1 < v_2 \leq t_1$, the following estimate holds:

$$\begin{aligned} & \|J(y)(v_2) - J(y)(v_1)\|_{\mathbb{R}^n} \\ & \leq \left\| [U(v_2, 0) - U(v_1, 0)] [\phi(0) - g(y_{\theta_1}, \dots, y_{\theta_q})(0)] \right\|_{\mathbb{R}^n} \\ & \quad + \left\| \int_0^{v_2} U(v_2, s) f(s, y_s) ds - \int_0^{v_1} U(v_1, s) f(s, y_s) ds \right\|_{\mathbb{R}^n} \\ & \leq \|U(v_2, 0) - U(v_1, 0)\| \left(\|\phi(0)\|_{\mathbb{R}^n} + N_q \sum_{i=1}^q \|y_{\theta_i}(0)\|_{\mathbb{R}^n} \right) \\ & \quad + \int_0^{v_1} \| (U(v_2, s) - U(v_1, s)) f(s, y_s) \|_{\mathbb{R}^n} ds \\ & \quad + \int_{v_1}^{v_2} \| U(v_2, s) f(s, y_s) \|_{\mathbb{R}^n} ds \\ & \leq \|U(v_2, 0) - U(v_1, 0)\| (\|\phi(0)\|_{\mathbb{R}^n} + N_q q R) \\ & \quad + \Psi(R) \int_0^{v_1} \|U(v_2, s) - U(v_1, s)\| ds + M \Psi(R) (v_2 - v_1). \end{aligned} \tag{4.7}$$

Similarly, for each $i \in I_N$ and every v_1, v_2 , with $s_i < v_1 < v_2 \leq t_{i+1}$, it follows that:

$$\begin{aligned} & \|J(y)(v_2) - J(y)(v_1)\|_{\mathbb{R}^n} \\ & \leq \|U(v_2, s_i) - U(v_1, s_i)\| \|G_i(s_i, y(s_i))\|_{\mathbb{R}^n} \\ & \quad + \left\| \int_{s_i}^{v_2} U(v_2, s) f(s, y_s) ds - \int_{s_i}^{v_1} U(v_1, s) f(s, y_s) ds \right\|_{\mathbb{R}^n} \\ & \leq \|U(v_2, s_i) - U(v_1, s_i)\| L \|y\| \\ & \quad + \int_{s_i}^{v_1} \|U(v_2, s) - U(v_1, s)\| \Psi(\|y_s\|) ds + M \int_{v_1}^{v_2} \Psi(\|y_s\|) ds \\ & \leq \|U(v_2, s_i) - U(v_1, s_i)\| LR \\ & \quad + \Psi(R) \int_{s_i}^{v_1} \|U(v_2, s) - U(v_1, s)\| ds + M \Psi(R) (v_2 - v_1). \end{aligned} \tag{4.8}$$

By (4.7) and (4.8), the continuity and boundedness of $U(t, s)$ yield that, as v_2 approaches to v_1 , $\|J(y)(v_2) - J(y)(v_1)\|_{\mathbb{R}^n}$ goes to zero, independently

of y . Therefore, $J(B_R)$ is equicontinuous on the set $\bigcup_{i=0}^N (s_i, t_{i+1}]$. In the same fashion, equicontinuity on $[-r, 0]$ and $\bigcup_{i=1}^N (t_i, s_i]$ is obtained. And, the family of functions $J(B_R)$ is equicontinuous on the interval $J \setminus \{t_1, \dots, t_N\}$.

(iv) J is a compact operator.

Let $B \subset PC_{r\tau}$ be a bounded subset, and $\{\omega_n\}_{n \in \mathbb{N}}$, a sequence on $J(B)$. Then, (ii) and (iii) imply that it is uniformly bounded and equicontinuous on $[-r, t_1]$. Note that $\{\omega_n|_{[-r, 0]}\}_{n \in \mathbb{N}} = \{\phi\}$. Arzelà-Ascoli theorem on $\{\omega_n|_{[0, t_1]}\}_{n \in \mathbb{N}} \subset C([0, t_1]; \mathbb{R}^n)$ implies there is a uniformly convergent subsequence $\{\omega_n^1\}_{n \in \mathbb{N}}$ on $[-r, t_1]$.

Consider the sequence $\{\omega_n^1\}_{n \in \mathbb{N}}$ on the interval $[s_1, t_2]$. It is uniformly bounded and equicontinuous, and as before, it has a convergent subsequence $\{\omega_n^2\}_{n \in \mathbb{N}}$ on $[s_1, t_2]$. Therefore, a uniformly convergent subsequence $\{\omega_n^2\}_{n \in \mathbb{N}}$ of $\{\omega_n\}_{n \in \mathbb{N}}$ on the interval $[-r, t_2]$ is obtained, since each ω_n^2 has the same definition on $[t_1, s_1]$.

Continuing this process on the intervals $[t_2, s_2]$, $[s_2, t_3]$, $[t_3, s_3]$, $\dots$, $[s_N, \tau]$, it is concluded that there is a subsequence $\{\omega_n^{N+1}\}_{n \in \mathbb{N}}$ of $\{\omega_n\}_{n \in \mathbb{N}}$, uniformly convergent on $[-r, \tau]$. Thus, the set $\overline{J(B)}$ is compact, and by the characterization of sequentially compact spaces, J is compact.

(v) The set $\{F(\cdot, y) : y \in \overline{J(D_\rho)}\}$ is comprised of equicontractive operators.

Let $\rho > 0$, $y \in \overline{J(D_\rho)}$, $x, z \in PC_{r\tau}$, and $t \in [-r, 0]$. Thus, **H1** yields

$$\begin{aligned} \|F(z, y)(t) - F(x, y)(t)\|_{\mathbb{R}^n} &\leq \|g(x_{\theta_1}, \dots, x_{\theta_q})(t) - g(z_{\theta_1}, \dots, z_{\theta_q})(t)\|_{\mathbb{R}^n} \\ &\leq N_q q \|z - x\|. \end{aligned} \quad (4.9)$$

For each $i \in I_N$ and $t \in (t_i, s_i]$, it follows that:

$$\begin{aligned} \|F(z, y)(t) - F(x, y)(t)\|_{\mathbb{R}^n} &\leq \|G_i(t, z(t)) - G_i(t, x(t))\|_{\mathbb{R}^n} \\ &\leq L \|z - x\|. \end{aligned} \quad (4.10)$$

Moreover, on the intervals $(s_i, t_{i+1}]$, $i \in \{0\} \cup I_N$, it follows that:

$$\|F(z, y)(t) - F(x, y)(t)\|_{\mathbb{R}^n} = \|y(t) - y(t)\|_{\mathbb{R}^n} = 0. \quad (4.11)$$

Combining (4.9)-(4.11), the next estimate holds:

$$\|F(z, y) - F(x, y)\| \leq \frac{1}{2} \|z - x\|.$$

Hence, F is a contraction on the first variable, independently of $y \in \overline{J(D_\rho)}$. $\square$

Theorem 2. Assume **H1** - **H3**. Then, problem (2.1) has, at least, one solution on the interval $\mathcal{J} = [-r, \tau]$.

Proof. For $\rho > 0$, let D_ρ as in (4.4), and define the operators $\tilde{J}$ and $\tilde{F}$ as:

$$\tilde{J} = J|_{D_\rho} : D_\rho \longrightarrow PC_{r\tau}(\mathcal{J}; \mathbb{R}^n) \quad \text{and} \quad \tilde{F} = F|_{D_\rho \times \overline{\tilde{J}(D_\rho)}} : D_\rho \times \overline{\tilde{J}(D_\rho)} \longrightarrow D_\rho.$$

Because of Lemma 1, $\tilde{J}$ is continuous and compact, and the family $\{F(\cdot, y) : y \in \overline{\tilde{J}(D_\rho)}\}$ is equicontractive. Continuity of $\tilde{F}$ follows analogously. The goal is to prove that, indeed, $\tilde{F}(D_\rho, \overline{\tilde{J}(D_\rho)}) \subset D_\rho$. Thus, Theorem 1 assumptions will be satisfied, and an equivalent solution will be obtained.

Take an arbitrary $z \in D_\rho$, for $t \in [-r, 0]$, it yields

$$\begin{aligned} \left\| \tilde{F}(z, \tilde{J}(z))(t) - \tilde{\Phi}(t) \right\|_{\mathbb{R}^n} &= \|g(z_{\theta_1}, \dots, z_{\theta_q})(t)\|_{\mathbb{R}^n} \\ &\leq N_q \sum_{j=1}^q \|z_{\theta_j}(t)\|_{\mathbb{R}^n} \\ &\leq MN_q q \|z\| \\ &\leq MN_q q (\|\tilde{\Phi}\| + \rho) \leq \rho. \end{aligned} \tag{4.12}$$

Similarly, $t \in (0, t_1]$ imply

$$\begin{aligned} \left\| \tilde{F}(z, \tilde{J}(z))(t) - \tilde{\Phi}(t) \right\|_{\mathbb{R}^n} &\leq M \|g(z_{\theta_1}, \dots, z_{\theta_q})(0)\|_{\mathbb{R}^n} \\ &\quad + \int_0^t \|U(t, s)f(s, z_s)\|_{\mathbb{R}^n} ds \\ &\leq MN_q \sum_{i=1}^q \|z\| + M \int_0^t \|f(s, z_s)\| ds \\ &\leq MN_q q \|z\| + Mt_1 \Psi(\|z\|) \\ &\leq MN_q q (\|\tilde{\Phi}\| + \rho) + M\tau \Psi(\|\tilde{\Phi}\| + \rho) \leq \rho. \end{aligned} \tag{4.13}$$

Likewise, $t \in (s_i, t_{i+1}]$, $i \in I_N$, gives

$$\begin{aligned} \left\| \tilde{F}(z, \tilde{J}(z))(t) - \tilde{\Phi}(t) \right\|_{\mathbb{R}^n} &\leq \|U(t, s_i)G_i(s_i, z(s_i)) - \alpha\|_{\mathbb{R}^n} \\ &\quad + \int_{s_i}^t \|U(t, s)f(s, z_s)\|_{\mathbb{R}^n} ds \\ &\leq ML\|z\| + \|\alpha\|_{\mathbb{R}^n} + M(t_{i+1} - s_i)\Psi(\|z\|) \\ &\leq ML(\|\tilde{\Phi}\| + \rho) + \|\alpha\|_{\mathbb{R}^n} + M\tau \Psi(\|\tilde{\Phi}\| + \rho) \\ &\leq \rho. \end{aligned} \tag{4.14}$$

Additionally, for $i \in I_N$, if $t \in (t_i, s_i]$, then,

$$\begin{aligned} \left\| \tilde{F} \left(z, \tilde{J}(z) \right) (t) - \tilde{\Phi}(t) \right\|_{\mathbb{R}^n} &= \|G_i(t, z(t)) - \beta\|_{\mathbb{R}^n} \\ &\leq L\|z\| + \|\beta\|_{\mathbb{R}^n} \\ &\leq L(\|\tilde{\Phi}\| + \rho) + \|\beta\|_{\mathbb{R}^n} \leq \rho. \end{aligned} \quad (4.15)$$

Thus, equations (4.12) through (4.15) give

$$\left\| \tilde{F} \left(z, \tilde{J}(z) \right) - \tilde{\Phi} \right\| = \sup_{t \in \mathcal{J}} \left\| \tilde{F} \left(z, \tilde{J}(z) \right) (t) - \tilde{\Phi}(t) \right\|_{\mathbb{R}^n} \leq \rho.$$

Applying Theorem 1 to $\tilde{J}$ and $\tilde{F}$, it follows $\tilde{F} \left(z, \tilde{J}(z) \right) = z$, i.e., there exists a fixed-point solution $z \in D_\rho \subset PC_{r\tau}$, equivalent to the system (2.1) solution given by Proposition 1. $\square$

The following theorem proves the uniqueness of the solution to system (2.1).

Theorem 3. *Asuming **H1** - **H4**, system (2.1) has a unique solution on $\mathcal{J} = [-r, \tau]$.*

Proof. Consider two solutions z_1 and z_2 to (2.1), which satisfy (3.3). Let $\rho > 0$ such that $z_1, z_2 \in D_\rho$. Then, for $t \in [-r, 0]$, the following estimate holds:

$$\begin{aligned} \|z_1(t) - z_2(t)\|_{\mathbb{R}^n} &\leq \left\| g \left(z_{2\theta_1}, \dots, z_{2\theta_q} \right) (t) - g \left(z_{1\theta_1}, \dots, z_{1\theta_q} \right) (t) \right\|_{\mathbb{R}^n} \\ &\leq N_q q \|z_1 - z_2\| \\ &\leq \frac{1}{2} \|z_1 - z_2\|. \end{aligned} \quad (4.16)$$

If $t \in (0, t_1]$, **H2** implies

$$\begin{aligned} \|z_1(t) - z_2(t)\|_{\mathbb{R}^n} &\leq \|U(t, 0)\| \left\| g \left(z_{2\theta_1}, \dots, z_{2\theta_q} \right) (0) - g \left(z_{1\theta_1}, \dots, z_{1\theta_q} \right) (0) \right\|_{\mathbb{R}^n} \\ &\quad + \int_0^t \|U(t, s)\| (f(s, z_{1s}) - f(s, z_{2s})) ds \\ &\leq [MN_q q + Mt_1 K(\|z_1\|, \|z_2\|)] \|z_1 - z_2\| \\ &\leq [MN_q q + M\tau K(\|\tilde{\Phi}\| + \rho, \|\tilde{\Phi}\| + \rho)] \|z_1 - z_2\|, \end{aligned} \quad (4.17)$$

and, $t \in (s_i, t_{i+1}]$, $i \in I_N$, yields

$$\begin{aligned} \|z_1(t) - z_2(t)\|_{\mathbb{R}^n} &\leq \|U(t, s_i)\| \|G_i(s_i, z_1(s_i)) - G_i(s_i, z_2(s_i))\|_{\mathbb{R}^n} \\ &\quad + \int_{s_i}^t \|U(t, s)\| (f(s, z_{1s}) - f(s, z_{2s})) ds \\ &\leq [ML + M(t_{i+1} - s_i)K(\|z_1\|, \|z_2\|)] \|z_1 - z_2\| \\ &\leq [ML + M\tau K(\|\tilde{\Phi}\| + \rho, \|\tilde{\Phi}\| + \rho)] \|z_1 - z_2\|. \end{aligned} \quad (4.18)$$

Lastly, if $t \in (t_i, s_i]$, $i \in I_N$, then,

$$\begin{aligned} \|z_1(t) - z_2(t)\|_{\mathbb{R}^n} &\leq \|G_i(t, z_1(t)) - G_i(t, z_2(t))\|_{\mathbb{R}^n} \\ &\leq L\|z_1 - z_2\| \\ &\leq \frac{1}{2}\|z_1 - z_2\|. \end{aligned} \quad (4.19)$$

Therefore, taking the sup limit of equations (4.16)-(4.19) and **H4** imply that there exists a constant m , with $0 < m < 1$, such that:

$$\|z_1 - z_2\| = \sup_{t \in \mathcal{J}} \|z_1(t) - z_2(t)\|_{\mathbb{R}^n} \leq m\|z_1 - z_2\|.$$

Hence, $z_1 = z_2$. □

Finally, the next theorem and corollary extend the system solution towards $[-r, +\infty)$.

Theorem 4. Assume **H1** - **H4** are satisfied, and consider the solution z over a maximal interval $[-r, p_1)$. Then, $p_1 = +\infty$, or there exists a convergent sequence $\{\tau_n\}_{n \in \mathbb{N}}$ to p_1 , such that:

$$\lim_{n \rightarrow \infty} z(\tau_n) = \tilde{z} \in \partial B_{\|\tilde{\phi}\| + \rho} \subset \mathbb{R}^n. \quad (4.20)$$

Proof. Assume $p_1 < +\infty$, and suppose that there exists a neighborhood V of the boundary of $B_{\|\tilde{\phi}\| + \rho}$ such that if $t \in [p_2, p_1)$, with $s_N < p_2 < p_1$, then, $z(t) \notin V$.

Without loss of generality, assume that $V = B_{\|\tilde{\phi}\| + \rho} \setminus E$, with $E \subset B_{\|\tilde{\phi}\| + \rho}$ is a closed set and $z(t) \in E$, for $t \in [p_2, p_1)$.

Consider $p_2 \leq s < t < p_1$. It follows that:

$$\begin{aligned} \|z(t) - z(s)\|_{\mathbb{R}^n} &\leq \|U(t, s_N) - U(s, s_N)\| \|G_N(s_N, z(s_N))\|_{\mathbb{R}^n} \\ &\quad + \int_s^t \|U(t, \xi)\| \|f(\xi, z_\xi)\|_{\mathbb{R}^n} d\xi \\ &\quad + \int_{s_N}^s \|U(t, \xi) - U(s, \xi)\| \|f(\xi, z_\xi)\|_{\mathbb{R}^n} d\xi \\ &\leq \|U(t, s_N) - U(s, s_N)\| L\|z\| + M(t-s)\Psi(\|\tilde{\phi}\| + \rho) \\ &\quad + \Psi(\|\tilde{\phi}\| + \rho) \int_{s_N}^s \|U(t, \xi) - U(s, \xi)\| d\xi. \end{aligned}$$

Then, uniform continuity of the evolution operator yields

$$\lim_{s \rightarrow p_1^-} \|z(t) - z(s)\|_{\mathbb{R}^n} = 0.$$

Thus, there exists $\tilde{z} \in \mathbb{R}^n$ such that $z(p_1^-) = \tilde{z} \in E$, and a solution can be defined at p_1 through extending z by continuity, which contradicts the maximality of $[-r, p_1)$. Thus, either $p_1 = +\infty$, or a sequence $\{\tau_n\}_n$ exists and fulfills (4.20). □

Corollary 1. *Under Theorem 4 assumptions, suppose that:*

$$\|f(t, \varphi)\|_{\mathbb{R}^n} \leq h(t) (1 + \|\varphi(0)\|_{\mathbb{R}^n}),$$

for $\varphi \in PC_r$ and $h: \overline{\mathbb{R}_+} \rightarrow \overline{\mathbb{R}_+}$ continuous. Then, there exists a unique solution to problem (2.1) on $[-r, +\infty)$.

Proof. Consider $t \in [s_N, p_1)$. It follows that:

$$\begin{aligned} \|z(t)\|_{\mathbb{R}^n} &\leq \|U(t, s_N)\| \|G_N(s_N, z(s_N))\|_{\mathbb{R}^n} + \int_{s_N}^t \|U(t, s)\| \|f(s, z_s)\| ds \\ &\leq ML \|z(s_N)\|_{\mathbb{R}^n} + \int_{s_N}^{p_1} Mh(s) ds + \int_{s_N}^t Mh(s) \|z(s)\|_{\mathbb{R}^n} ds. \end{aligned}$$

Hence, Grönwall's inequality yields

$$\|z(t)\|_{\mathbb{R}^n} \leq M \left(L \|z(s_N)\|_{\mathbb{R}^n} + \int_{s_N}^{p_1} h(s) ds \right) \exp \left(\int_{s_N}^{p_1} Mh(s) ds \right).$$

By Theorem 4, the solution stays bounded, as desired. $\square$

5. EXAMPLE

In this section, particular definitions for functions G_i , g and f , $i \in I_N$, exemplify the results of this work. To this end, consider an arbitrary finite-dimensional continuous operator A , such that $A(t)$ is a $n \times n$ matrix.

Given $N, R \in \mathbb{N}$, the non-linear term, $f: \overline{\mathbb{R}_+} \times PC_r([-r, 0]; \mathbb{R}^n) \rightarrow \mathbb{R}^n$, the functions describing non-instantaneous impulses, $G_i: (t_i, s_i] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, and non-local conditions, $g: PC_r^q([-r, 0]; (\mathbb{R}^n)^q) \rightarrow PC_r([-r, 0]; \mathbb{R}^n)$, are given as follows, for $z \in PC_{r\tau}$ and $i \in I_N$,

$$f(t, \varphi) = \frac{1}{R} \begin{pmatrix} (\varphi_1(-r))^2 \\ (\varphi_2(-r))^2 \\ \vdots \\ (\varphi_n(-r))^2 \end{pmatrix}, \quad G_i(t, z(t)) = \frac{\cos(s_i)}{R} \begin{pmatrix} \sin(z_1(t)) \\ \sin(z_2(t)) \\ \vdots \\ \sin(z_n(t)) \end{pmatrix},$$

$$g(\varphi) = \sum_{i=1}^q \frac{1}{R} \varphi_i.$$

Clearly, g verifies that $g(0) = 0$, and if $t \in [-r, 0]$, then,

$$\|g(y)(t) - g(z)(t)\|_{\mathbb{R}^n} \leq \frac{1}{R} \|y(t) - z(t)\|_{(\mathbb{R}^n)^q}, \quad \text{for all } y, z \in PCp^q.$$

The functions G_i , $i \in I_N$, satisfy $G_i(\cdot, 0) = 0$, and, for any $y, z \in PC_{r\tau}$, given $t \in (t_i, s_i]$,

$$\begin{aligned} \|G_i(t, y(t)) - G_i(t, z(t))\|_{\mathbb{R}^n} &\leq \frac{|\cos(s_i)|}{R} \left(\sum_{k=1}^N |\sin(y_k(t)) - \sin(z_k(t))|^2 \right)^{1/2} \\ &\leq \frac{|\cos(s_i)|}{R} \|y - z\|. \end{aligned}$$

For R sufficiently large, it yields

$$\frac{|\cos(s_i)|}{R} + \frac{1}{R}q < \frac{1}{2}.$$

Finally, given $t \geq 0$, $y, z \in PC_{r\tau}$, and $\phi \in PC_r$, the function f satisfies

$$\begin{aligned} \|f(t, y_t) - f(t, z_t)\|_{\mathbb{R}^n} &\leq \frac{1}{R} \left(\sum_{k=1}^n (|y_k(t-r)| + |z_k(t-r)|)^2 \right)^{1/2} \|y - z\| \\ &\leq K(\|y\|, \|z\|) \|y - z\|, \end{aligned}$$

and

$$\|f(t, \phi)\|_{\mathbb{R}^n} = \frac{1}{R} \left\| \begin{pmatrix} (\phi_1(-r))^2 \\ (\phi_2(-r))^2 \\ \vdots \\ (\phi_n(-r))^2 \end{pmatrix} \right\|_{\mathbb{R}^n} \leq \Psi(\|\phi\|),$$

where K and Ψ are continuous non-decreasing functions. Hence, hypotheses **H1** and **H2** are satisfied. For R sufficiently large, conditions **H3** and **H4** are similarly verified. Then, by Theorem 3 and Corollary 1, system (2.1), with the foregoing definitions, admits a unique solution on $[-r, +\infty)$.

6. FINAL REMARKS

In this work, existence and uniqueness of solutions for semi-linear systems of non-autonomous differential equations considering non-instantaneous impulses, delay, and non-local conditions simultaneously were proved. The technique used was based on Karakostas' fixed-point theorem, by transforming the existence of solutions problem into a fixed-point existence problem of a certain operator equation satisfying the specific conditions. This led to choose the adequate hypotheses to meet the requirements of that theorem. Observe that this work can be generalized to infinite-dimensional Banach spaces. However, proving equicontinuity of specific operator families and the main operator compactness must be carefully treated before applying a fixed-point theorem. The strongly continuous semigroup in the non-autonomous system requires compactness to ensure the uniform continuity away from zero. A different version of Arzelà-Ascoli theorem must be considered on the corresponding functional spaces. To end, the controllability of these systems is part of our outgoing

research. In particular, the exact and approximate controllability of this system can be proven using Rothe's fixed-point theorem [15] and the techniques developed in [6].

ACKNOWLEDGMENTS

The authors would like to express their thanks to the editor and anonymous referees for constructive comments and suggestions that improved the quality of this manuscript.

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WEIGHTED SIMPSON'S TYPE INTEGRAL INEQUALITIES FOR HARMONICALLY-PREINVEX FUNCTIONS

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Received 02 November, 2019

Abstract. In this paper, some new weighted Simpson's type integral inequalities are presented for the class of harmonically-preinvex functions by using power-mean integral inequality and Hölder's integral inequality.

2010 Mathematics Subject Classification: 26A33; 41A55; 26D15; 26E60

Keywords: Simpson's inequality; Convex functions; harmonically-preinvex function; Power-mean inequality; Hölder's inequality.

1. INTRODUCTION

It is known that a function $\vartheta : I \subset \mathbb{R} \rightarrow \mathbb{R}$ is convex if the inequality

$$\vartheta(vx + (1-v)y) \leq v\vartheta(x) + (1-v)\vartheta(y)$$

holds for all $x, y \in I$ and $v \in [0, 1]$.

Convex functions can also be characterized by an important inequality known as Hermite-Hadamard inequality. This inequality is stated as follows.

Let $\vartheta : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a convex function defined on a convex interval I of real numbers and $\lambda, \mu \in I$ with $\lambda < \mu$. Then

$$\vartheta\left(\frac{\lambda + \mu}{2}\right) \leq \frac{1}{\mu - \lambda} \int_{\lambda}^{\mu} \vartheta(x) dx \leq \frac{\vartheta(\lambda) + \vartheta(\mu)}{2}.$$

Convex functions play a key role in the development of the field of mathematical inequalities. The field of mathematical inequalities using generalizations of the convex functions is growing rapidly.

Harmonically-preinvex functions is one of the generalizations of the convex functions stated as follows.

Definition 1. [19] A function $\vartheta : I_{h\varphi} = [\lambda, \lambda + \varphi(\mu, \lambda)] \subset \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ is said to be harmonic preinvex function with respect to the bifunction $\varphi(.,.)$ if

$$\vartheta\left(\frac{x(x + \varphi(y, x))}{x + (1-v)\varphi(y, x)}\right) \leq v\vartheta(y) + (1-v)\vartheta(x)$$

for all $x, y \in I_{h\varphi}$ and $v \in [0, 1]$. The function ϑ is said to be harmonic preconcave function with respect to $\varphi(.,.)$, if and only if, $-\vartheta$ is harmonic preinvex function.

Let us recall the notion of harmonically symmetric functions that will also be useful to establish our results.

Definition 2. [19] A function $\vartheta : I_{h\varphi} = [\lambda, \lambda + \varphi(\mu, \lambda)] \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ is said to be harmonically symmetric with respect to $\frac{2\lambda(\lambda + \varphi(\mu, \lambda))}{2\lambda + \varphi(\mu, \lambda)}$ if

$$\vartheta(x) = \vartheta\left(\frac{\lambda(\lambda + \varphi(\mu, \lambda))x}{(2\lambda + \varphi(\mu, \lambda))x - \lambda(\lambda + \varphi(\mu, \lambda))}\right)$$

holds for all $x \in [\lambda, \lambda + \varphi(\mu, \lambda)]$.

One of the important quadrature formulae for functions of one real variable is the Simpson's rule.

The Simpson's inequality states that if $\vartheta : [\lambda, \mu] \rightarrow \mathbb{R}$ is a four times continuously differentiable mapping on (λ, μ) and $\|\vartheta^{(4)}\|_{\infty} = \sup_{v \in (\lambda, \mu)} |\vartheta^{(4)}(v)| < \infty$, then

$$\left| \int_{\lambda}^{\mu} \vartheta(x) dx - \frac{\mu - \lambda}{3} \left[\frac{\vartheta(\lambda) + \vartheta(\mu)}{2} + 2\vartheta\left(\frac{\lambda + \mu}{2}\right) \right] \right| \leq \frac{1}{2880} \|\vartheta^{(4)}\|_{\infty} \cdot (\mu - \lambda)^5. \quad (1.1)$$

The inequality (1.1) has attracted a considerable attention of a number of researchers since it is very important and remarkable in the field of mathematical inequalities and numerical analysis. Plenty of new Simpson's type inequalities using different types of convex functions have been established. There is a substantial literature on the extensions, generalizations and refinements of Simpson's inequality and on Simpson's type integral inequalities using a variety of convexity conditions, see for example [1–11], [12–18], [20–31] and the references cited in these papers.

In Section 2, we prove new Simpson's type integral inequalities using harmonically-preinvex and harmonically symmetric weight functions. The results of our manuscript provide extensions of previously established Simpson's type integral inequalities.

2. WEIGHTED SIMPSON'S TYPE INEQUALITIES FOR HARMONICALLY-PREINVEX FUNCTIONS

Throughout in this section, we will use the notations $U_1(v)$ and $U_2(v)$ respectively for $\frac{2\lambda(\lambda + \varphi(\mu, \lambda))}{2\lambda + (1+v)\varphi(\mu, \lambda)}$ and $\frac{2\lambda(\lambda + \varphi(\mu, \lambda))}{2\lambda + (1-v)\varphi(\mu, \lambda)}$.

In order to prove the results of this paper, we need the following lemma.

Lemma 1. Let $\vartheta : I_{h\varphi} \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ be a differentiable function on $I_{h\varphi}^\circ$ and $\lambda, \lambda + \varphi(\mu, \lambda) \in I_{h\varphi}^\circ$ with $\varphi(\mu, \lambda) > 0$ and let $w : [\lambda, \lambda + \varphi(\mu, \lambda)] \rightarrow [0, \infty)$ be a continuous positive mapping and harmonically symmetric to $\frac{2\lambda(\lambda + \varphi(\mu, \lambda))}{2\lambda + \varphi(\mu, \lambda)}$.

If $\vartheta' \in L([\lambda, \lambda + \varphi(\mu, \lambda)])$, the following equality holds

$$\begin{aligned} & \frac{1}{8} \left(\frac{\lambda(\lambda + \varphi(\mu, \lambda))}{\varphi(\mu, \lambda)} \right) \left[\vartheta(\lambda) + 6\vartheta \left(\frac{2\lambda(\lambda + \varphi(\mu, \lambda))}{2\lambda + \varphi(\mu, \lambda)} \right) + \vartheta(\lambda + \varphi(\mu, \lambda)) \right] \\ & \times \int_{\lambda}^{\lambda + \varphi(\mu, \lambda)} \frac{w(z)}{z^2} dz - \frac{\lambda(\lambda + \varphi(\mu, \lambda))}{\varphi(\mu, \lambda)} \int_{\lambda}^{\lambda + \varphi(\mu, \lambda)} \frac{w(z)\vartheta(z)}{z^2} dz = \frac{\varphi(\mu, \lambda)}{4(\lambda(\lambda + \varphi(\mu, \lambda)))} \\ & \times \left\{ \int_0^1 p_1(v)(U_1(v))^2 \vartheta'(U_1(v)) dv + \int_0^1 p_2(v)(U_2(v))^2 \vartheta'(U_2(v)) dv \right\}, \end{aligned} \quad (2.1)$$

where

$$p_1(v) = \frac{3}{4} \int_0^1 w(U_1(s)) ds - \int_0^v w(U_1(s)) ds$$

and

$$p_2(v) = -\frac{3}{4} \int_0^1 w(U_2(s)) ds + \int_0^v w(U_2(s)) ds.$$

Proof. By integration by parts, we have

$$\begin{aligned} I_1 &= \int_0^1 p_1(v)(U_1(v))^2 \vartheta'(U_1(v)) dv \\ &= - \left(\frac{2\lambda(\lambda + \varphi(\mu, \lambda))}{\varphi(\mu, \lambda)} \right) \int_0^1 p_1(v) \left[- \left(\frac{\varphi(\mu, \lambda)}{2\lambda(\lambda + \varphi(\mu, \lambda))} \right) \right] (U_1(v))^2 \vartheta'(U_1(v)) dv \\ &= - \left(\frac{2\lambda(\lambda + \varphi(\mu, \lambda))}{\varphi(\mu, \lambda)} \right) \int_0^1 \left[\frac{3}{4} \int_0^1 w(U_1(s)) ds - \int_0^v w(U_1(s)) ds \right] d[\vartheta(U_1(v))] dv \\ &= - \left(\frac{2\lambda(\lambda + \varphi(\mu, \lambda))}{\varphi(\mu, \lambda)} \right) \left[\frac{3}{4} \int_0^1 w(U_1(s)) ds - \int_0^v w(U_1(s)) ds \right] \vartheta(U_1(v)) \Big|_0^1 \\ &\quad - \left(\frac{2\lambda(\lambda + \varphi(\mu, \lambda))}{\varphi(\mu, \lambda)} \right) \int_0^1 w(U_1(v)) \vartheta(U_1(v)) dv = \left(\frac{2\lambda(\lambda + \varphi(\mu, \lambda))}{\varphi(\mu, \lambda)} \right) \end{aligned}$$

$$\begin{aligned} & \times \left[\frac{1}{4} \vartheta(\lambda) \int_0^1 w(U_1(v)) dv + \frac{3}{4} \vartheta \left(\frac{2\lambda(\lambda + \varphi(\mu, \lambda))}{2\lambda + \varphi(\mu, \lambda)} \right) \int_0^1 w(U_1(v)) dv \right] \\ & - \left(\frac{2\lambda(\lambda + \varphi(\mu, \lambda))}{\varphi(\mu, \lambda)} \right) \int_0^1 w(U_1(v)) \vartheta(U_1(v)) dv. \end{aligned}$$

By making the substitution $z = U_1(v)$, we get

$$\begin{aligned} I_1 &= \left(\frac{\lambda(\lambda + \varphi(\mu, \lambda))}{\varphi(\mu, \lambda)} \right)^2 \left[\vartheta(\lambda) + 3\vartheta \left(\frac{2\lambda(\lambda + \varphi(\mu, \lambda))}{2\lambda + \varphi(\mu, \lambda)} \right) \right] \int_{\lambda}^{\frac{2\lambda(\lambda + \varphi(\mu, \lambda))}{2\lambda + \varphi(\mu, \lambda)}} \frac{w(z)}{z^2} dz \\ & - \left(\frac{2\lambda(\lambda + \varphi(\mu, \lambda))}{\varphi(\mu, \lambda)} \right)^2 \int_{\lambda}^{\frac{2\lambda(\lambda + \varphi(\mu, \lambda))}{2\lambda + \varphi(\mu, \lambda)}} \frac{w(z) \vartheta(z)}{z^2} dz. \end{aligned}$$

Similarly, we can have

$$\begin{aligned} I_2 &= \int_0^1 p_2(v) (U_2(v))^2 \vartheta'(U_2(v)) dv \\ &= \left(\frac{\lambda(\lambda + \varphi(\mu, \lambda))}{\varphi(\mu, \lambda)} \right)^2 \left[\vartheta(\lambda + \varphi(\mu, \lambda)) + 3\vartheta \left(\frac{2\lambda(\lambda + \varphi(\mu, \lambda))}{2\lambda + \varphi(\mu, \lambda)} \right) \right] \\ & \times \int_{\frac{2\lambda(\lambda + \varphi(\mu, \lambda))}{2\lambda + \varphi(\mu, \lambda)}}^{\lambda + \varphi(\mu, \lambda)} \frac{w(z)}{z^2} dz - \left(\frac{2\lambda(\lambda + \varphi(\mu, \lambda))}{\varphi(\mu, \lambda)} \right)^2 \int_{\frac{2\lambda(\lambda + \varphi(\mu, \lambda))}{2\lambda + \varphi(\mu, \lambda)}}^{\lambda + \varphi(\mu, \lambda)} \frac{w(z) \vartheta(z)}{z^2} dz. \end{aligned}$$

Since $w(z)$ is harmonically symmetric with respect to $\frac{2\lambda(\lambda + \varphi(\mu, \lambda))}{2\lambda + \varphi(\mu, \lambda)}$, we have

$$\int_{\lambda}^{\frac{2\lambda(\lambda + \varphi(\mu, \lambda))}{2\lambda + \varphi(\mu, \lambda)}} \frac{w(z)}{z^2} dz = \int_{\frac{2\lambda(\lambda + \varphi(\mu, \lambda))}{2\lambda + \varphi(\mu, \lambda)}}^{\lambda + \varphi(\mu, \lambda)} \frac{w(z)}{z^2} dz = \frac{1}{2} \int_{\lambda}^{\lambda + \varphi(\mu, \lambda)} \frac{w(z)}{z^2} dz.$$

Thus, we have

$$\frac{\varphi(\mu, \lambda)}{4\lambda(\lambda + \varphi(\mu, \lambda))} (I_1 + I_2)$$

$$\begin{aligned}
&= \frac{1}{8} \left(\frac{\lambda(\lambda + \varphi(\mu, \lambda))}{\varphi(\mu, \lambda)} \right) \left[\vartheta(\lambda) + 6\vartheta \left(\frac{2\lambda(\lambda + \varphi(\mu, \lambda))}{2\lambda + \varphi(\mu, \lambda)} \right) + \vartheta(\lambda + \varphi(\mu, \lambda)) \right] \\
&\times \int_{\lambda}^{\lambda + \varphi(\mu, \lambda)} \frac{w(z)}{z^2} dz - \frac{\lambda(\lambda + \varphi(\mu, \lambda))}{\varphi(\mu, \lambda)} \int_{\lambda}^{\lambda + \varphi(\mu, \lambda)} \frac{w(z) \vartheta(z)}{z^2} dz.
\end{aligned}$$

Hence the proof of the theorem is done. $\square$

Remark 1. Throughout this manuscript we will use the following notation for the sake of convenience

$$\begin{aligned}
\vartheta(\lambda, \lambda + \varphi(\mu, \lambda); \vartheta, w) &= \frac{1}{8} \left(\frac{\lambda(\lambda + \varphi(\mu, \lambda))}{\varphi(\mu, \lambda)} \right) \\
&\times \left[\vartheta(\lambda) + 6\vartheta \left(\frac{2\lambda(\lambda + \varphi(\mu, \lambda))}{2\lambda + \varphi(\mu, \lambda)} \right) + \vartheta(\lambda + \varphi(\mu, \lambda)) \right] \\
&\times \int_{\lambda}^{\lambda + \varphi(\mu, \lambda)} \frac{w(z)}{z^2} dz - \frac{\lambda(\lambda + \varphi(\mu, \lambda))}{\varphi(\mu, \lambda)} \int_{\lambda}^{\lambda + \varphi(\mu, \lambda)} \frac{w(z) \vartheta(z)}{z^2} dz.
\end{aligned}$$

Corollary 1. Under the assumptions of Lemma 1, the following inequality holds

$$\begin{aligned}
\vartheta(\lambda, \lambda + \varphi(\mu, \lambda); \vartheta, w) &\leq \frac{\varphi(\mu, \lambda)}{4\lambda(\lambda + \varphi(\mu, \lambda))} \|w\|_{[\lambda, \lambda + \varphi(\mu, \lambda)], \infty} \\
&\times \left\{ \int_0^1 \left(\frac{3}{4} - v \right) (U_1(v))^2 \vartheta'(U_1(v)) dv + \int_0^1 \left(v - \frac{3}{4} \right) (U_2(v))^2 \vartheta'(U_2(v)) dv \right\}.
\end{aligned} \tag{2.2}$$

Proof. Proof follows from the fact that

$$\|w\|_{\left[\lambda, \frac{2\lambda(\lambda + \varphi(\mu, \lambda))}{2\lambda + \varphi(\mu, \lambda)}\right], \infty} \leq \|w\|_{[\lambda, \lambda + \varphi(\mu, \lambda)], \infty}$$

and

$$\|w\|_{\left[\frac{2\lambda(\lambda + \varphi(\mu, \lambda))}{2\lambda + \varphi(\mu, \lambda)}, \lambda + \varphi(\mu, \lambda)\right], \infty} \leq \|w\|_{[\lambda, \lambda + \varphi(\mu, \lambda)], \infty}.$$

$\square$

Theorem 1. Let $\vartheta : I_{h\varphi} \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a differentiable mapping on $I_{h\varphi}^\circ$, where $\lambda, \lambda + \varphi(\mu, \lambda) \in I_{h\varphi}^\circ$ with $\varphi(\mu, \lambda) > 0$ and let $w : [\lambda, \lambda + \varphi(\mu, \lambda)] \rightarrow [0, \infty)$ be a continuous positive mapping and harmonically symmetric to $\frac{2\lambda(\lambda + \varphi(\mu, \lambda))}{2\lambda + \varphi(\mu, \lambda)}$.

If $\vartheta', w \in L_1([\lambda, \lambda + \varphi(\mu, \lambda)])$ and $|\vartheta'|^q$ is harmonically-preinvex on $[\lambda, \lambda + \varphi(\mu, \lambda)]$ for $q \geq 1$, then

$$\begin{aligned} |\vartheta(\lambda, \lambda + \varphi(\mu, \lambda); \vartheta, w)| &\leq \frac{\varphi(\mu, \lambda)}{4\lambda(\lambda + \varphi(\mu, \lambda))} \|w\|_{[\lambda, \lambda + \varphi(\mu, \lambda)], \infty} \\ &\times \left(\frac{5}{16}\right)^{1-\frac{1}{q}} m_{\varphi}^{\frac{2}{q}}(\lambda, \mu) \left\{ \left[o_1(n_{\varphi}(\lambda, \mu)) |\vartheta'(\lambda)|^q + o_2(n_{\varphi}(\lambda, \mu)) |\vartheta'(\mu)|^q \right]^{\frac{1}{q}} \right. \\ &\left. + \left[o_2(-n_{\varphi}(\lambda, \mu)) |\vartheta'(\lambda)|^q + o_1(-n_{\varphi}(\lambda, \mu)) |\vartheta'(\mu)|^q \right]^{\frac{1}{q}} \right\}, \end{aligned} \quad (2.3)$$

where

$$\begin{aligned} o_1(n_{\varphi}(\lambda, \mu)) &= \frac{3n_{\varphi}^2(\lambda, \mu) - 3n_{\varphi}(\lambda, \mu) - 4}{8n_{\varphi}^2(\lambda, \mu)(1 + n_{\varphi}(\lambda, \mu))} + \frac{(n_{\varphi}(\lambda, \mu) - 8)}{8n_{\varphi}^3(\lambda, \mu)} \ln \left(\frac{16(1 + n_{\varphi}(\lambda, \mu))}{(4 + 3n_{\varphi}(\lambda, \mu))^2} \right), \\ o_2(n_{\varphi}(\lambda, \mu)) &= \frac{4 + 3n_{\varphi}(\lambda, \mu)}{8n_{\varphi}^2(\lambda, \mu)} + \frac{(8 + 7n_{\varphi}(\lambda, \mu))}{8n_{\varphi}^3(\lambda, \mu)} \ln \left(\frac{16(1 + n_{\varphi}(\lambda, \mu))}{(4 + 3n_{\varphi}(\lambda, \mu))^2} \right), \\ n_{\varphi}(\lambda, \mu) &= \frac{\varphi(\mu, \lambda)}{2\lambda + \varphi(\mu, \lambda)} \text{ and } m_{\varphi}(\lambda, \mu) = \frac{2\lambda(\lambda + \varphi(\mu, \lambda))}{2\lambda + \varphi(\mu, \lambda)}. \end{aligned}$$

Proof. Taking the absolute value on both sides of (2.2) and using the power-mean inequality, we have

$$\begin{aligned} |\vartheta(\lambda, \lambda + \varphi(\mu, \lambda); \vartheta, w)| &\leq \frac{\varphi(\mu, \lambda)}{4\lambda(\lambda + \varphi(\mu, \lambda))} \|w\|_{[\lambda, \lambda + \varphi(\mu, \lambda)], \infty} \\ &\times \left\{ \left(\int_0^1 \left| \frac{3}{4} - v \right| dv \right)^{1-\frac{1}{q}} \left(\int_0^1 \left| \frac{3}{4} - v \right| (U_1(v))^2 |\vartheta'(U_1(v))|^q dv \right)^{\frac{1}{q}} \right. \\ &\left. + \left(\int_0^1 \left| v - \frac{3}{4} \right| dv \right)^{1-\frac{1}{q}} \left(\int_0^1 \left| v - \frac{3}{4} \right| (U_2(v))^2 |\vartheta'(U_2(v))|^q dv \right)^{\frac{1}{q}} \right\}. \end{aligned} \quad (2.4)$$

By using the harmonically-preinvexity of $\left|\vartheta'\right|^q$ on $[\lambda, \lambda + \varphi(\mu, \lambda)]$ for $q \geq 1$, we get

$$\begin{aligned} & \int_0^1 \left| \frac{3}{4} - v \right| (U_1(v))^2 \left| \vartheta'(U_1(v)) \right|^q dv \\ & \leq \left| \vartheta'(\lambda) \right|^q \left[\int_0^{\frac{3}{4}} \left(\frac{3}{4} - v \right) \left(\frac{1+v}{2} \right) (U_1(v))^2 dv + \int_{\frac{3}{4}}^1 \left(v - \frac{3}{4} \right) \left(\frac{1+v}{2} \right) (U_1(v))^2 dv \right] \\ & + \left| \vartheta'(\mu) \right|^q \left[\int_0^{\frac{3}{4}} \left(\frac{3}{4} - v \right) \left(\frac{1-v}{2} \right) (U_1(v))^2 dv + \int_{\frac{3}{4}}^1 \left(v - \frac{3}{4} \right) \left(\frac{1-v}{2} \right) (U_1(v))^2 dv \right] \\ & = m_\varphi^2(\lambda, \mu) \left[\frac{3n_\varphi(\lambda, \mu)^2 - 3n_\varphi(\lambda, \mu) - 4}{8n_\varphi^2(\lambda, \mu)(1+n_\varphi(\lambda, \mu))} + \frac{(n_\varphi(\lambda, \mu) - 8)}{8n_\varphi^3(\lambda, \mu)} \ln \left(\frac{16(1+n_\varphi(\lambda, \mu))}{(4+3n_\varphi(\lambda, \mu))^2} \right) \right] \\ & \times \left| \vartheta'(\lambda) \right|^q + m_\varphi^2(\lambda, \mu) \left[\frac{4+3n_\varphi(\lambda, \mu)}{8n_\varphi^2(\lambda, \mu)} + \frac{(8+7n_\varphi(\lambda, \mu))}{8n_\varphi^3(\lambda, \mu)} \ln \left(\frac{16(1+n_\varphi(\lambda, \mu))}{(4+3n_\varphi(\lambda, \mu))^2} \right) \right] \left| \vartheta'(\mu) \right|^q \end{aligned} \quad (2.5)$$

and

$$\begin{aligned} & \int_0^1 \left| v - \frac{3}{4} \right| (U_2(v))^2 \left| \vartheta'(U_2(v)) \right|^q dv \\ & \leq \left| \vartheta'(\lambda) \right|^q \left[\int_0^{\frac{3}{4}} \left(\frac{3}{4} - v \right) \left(\frac{1-v}{2} \right) (U_2(v))^2 dv + \int_{\frac{3}{4}}^1 \left(v - \frac{3}{4} \right) \left(\frac{1-v}{2} \right) (U_2(v))^2 dv \right] \\ & + \left| \vartheta'(\mu) \right|^q \left[\int_0^{\frac{3}{4}} \left(\frac{3}{4} - v \right) \left(\frac{1+v}{2} \right) (U_2(v))^2 dv + \int_{\frac{3}{4}}^1 \left(v - \frac{3}{4} \right) \left(\frac{1+v}{2} \right) (U_2(v))^2 dv \right] \\ & = m_\varphi^2(\lambda, \mu) \left[\frac{4-3n_\varphi(\lambda, \mu)}{8n_\varphi^2(\lambda, \mu)} - \frac{(8-7n_\varphi(\lambda, \mu))}{8n_\varphi^3(\lambda, \mu)} \ln \left(\frac{16(1-n_\varphi(\lambda, \mu))}{(4-3n_\varphi(\lambda, \mu))^2} \right) \right] \left| \vartheta'(\lambda) \right|^q \\ & + m_\varphi^2(\lambda, \mu) \left[\frac{3n_\varphi^2(\lambda, \mu) + 3n_\varphi(\lambda, \mu) - 4}{8n_\varphi^2(\lambda, \mu)(1-n_\varphi(\lambda, \mu))} + \frac{(n_\varphi(\lambda, \mu) + 8)}{8n_\varphi^3(\lambda, \mu)} \ln \left(\frac{16(1-n_\varphi(\lambda, \mu))}{(4-3n_\varphi(\lambda, \mu))^2} \right) \right] \left| \vartheta'(\mu) \right|^q, \end{aligned} \quad (2.6)$$

where

$$n_\varphi(\lambda, \mu) = \frac{\varphi(\mu, \lambda)}{2\lambda + \varphi(\mu, \lambda)}$$

and

$$m_\varphi(\lambda, \mu) = \frac{2\lambda(\lambda + \varphi(\mu, \lambda))}{2\lambda + \varphi(\mu, \lambda)}.$$

Using (2.5) and (2.6) in (2.4) we get (2.3). $\square$

Theorem 2. Let $\vartheta : I_{h\varphi} \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a differentiable mapping on $I_{h\varphi}^\circ$, where $\lambda, \lambda + \varphi(\mu, \lambda) \in I_{h\varphi}^\circ$ with $\varphi(\mu, \lambda) > 0$ and let $w : [\lambda, \lambda + \varphi(\mu, \lambda)] \rightarrow [0, \infty)$ be a continuous positive mapping and harmonically symmetric to $\frac{2\lambda(\lambda + \varphi(\mu, \lambda))}{2\lambda + \varphi(\mu, \lambda)}$. If $\vartheta', w \in L_1([\lambda, \lambda + \varphi(\mu, \lambda)])$ and $|\vartheta'|^q$ is harmonically-preinvex on $[\lambda, \lambda + \varphi(\mu, \lambda)]$ for $q > 1$, then

$$\begin{aligned} |\vartheta(\lambda, \lambda + \varphi(\mu, \lambda); \vartheta, w)| &\leq \frac{m_\varphi^2(\lambda, \mu) \varphi(\mu, \lambda)}{4\lambda(\lambda + \varphi(\mu, \lambda))} \|w\|_{[\lambda, \lambda + \varphi(\mu, \lambda)], \infty} \left(\frac{1-q}{1+q} \right)^{1-\frac{1}{q}} \\ &\times \left\{ [\rho_1(n_\varphi(\lambda, \mu))]^{1-\frac{1}{q}} [\rho_2(q) |\vartheta'(\lambda)|^q + \rho_3(q) |\vartheta'(\mu)|^q]^{\frac{1}{q}} \right. \\ &\left. + [\rho_1(-n_\varphi(\lambda, \mu))]^{1-\frac{1}{q}} [\rho_3(q) |\vartheta'(\lambda)|^q + \rho_2(q) |\vartheta'(\mu)|^q]^{\frac{1}{q}} \right\}, \end{aligned} \quad (2.7)$$

where

$$\begin{aligned} \rho_1(n_\varphi(\lambda, \mu)) &= n_\varphi^{-1}(\lambda, \mu) \left((1 + n_\varphi(\lambda, \mu))^{\frac{1+q}{1-q}} - 1 \right), \\ \rho_2(q) &= \frac{2^{-2q-5}(8q+15) + 2^{-2q-5} \times 3^{q+1}(4q+11)}{(q+1)(q+2)}, \\ \rho_3(q) &= \frac{2^{-2q-5} + 2^{-2q-5} \times 3^{q+1}(4q+5)}{(q+1)(q+2)}, \\ n_\varphi(\lambda, \mu) &= \frac{\varphi(\mu, \lambda)}{2\lambda + \varphi(\mu, \lambda)} \text{ and } m_\varphi(\lambda, \mu) = \frac{2\lambda(\lambda + \varphi(\mu, \lambda))}{2\lambda + \varphi(\mu, \lambda)}. \end{aligned}$$

Proof. Taking the absolute value on both sides of (2.2) and using the Hölder integral inequality, we have

$$\begin{aligned} |\vartheta(\lambda, \lambda + \varphi(\mu, \lambda); \vartheta, w)| &\leq \frac{\varphi(\mu, \lambda)}{4\lambda(\lambda + \varphi(\mu, \lambda))} \|w\|_{[\lambda, \lambda + \varphi(\mu, \lambda)], \infty} \\ &\times \left\{ \left(\int_0^1 (U_1(v))^{\frac{2q}{q-1}} dv \right)^{1-\frac{1}{q}} \left(\int_0^1 \left| \frac{3}{4} - v \right|^q |\vartheta'(U_1(v))|^q dv \right)^{\frac{1}{q}} \right. \\ &\left. + \left(\int_0^1 (U_2(v))^{\frac{2q}{q-1}} dv \right)^{1-\frac{1}{q}} \left(\int_0^1 \left| v - \frac{3}{4} \right|^q |\vartheta'(U_2(v))|^q dv \right)^{\frac{1}{q}} \right\}. \end{aligned} \quad (2.8)$$

By using the harmonically-preinvexity of $\left|\vartheta'\right|^q$ on $[\lambda, \lambda + \varphi(\mu, \lambda)]$ for $q \geq 1$, we get

$$\begin{aligned}
 & \int_0^1 \left| \frac{3}{4} - v \right|^q \left| \vartheta'(U_1(v)) \right|^q dv \\
 & \leq \left| \vartheta'(\lambda) \right|^q \left[\int_0^{\frac{3}{4}} \left(\frac{3}{4} - v \right)^q \left(\frac{1+v}{2} \right) dv + \int_{\frac{3}{4}}^1 \left(v - \frac{3}{4} \right)^q \left(\frac{1+v}{2} \right) dv \right] \\
 & + \left| \vartheta'(\mu) \right|^q \left[\int_0^{\frac{3}{4}} \left(\frac{3}{4} - v \right)^q \left(\frac{1-v}{2} \right) dv + \int_{\frac{3}{4}}^1 \left(v - \frac{3}{4} \right)^q \left(\frac{1-v}{2} \right) dv \right] \\
 & = \left(\frac{2^{-2q-5} \times 3^{q+1} (4q+11) + 2^{-2q-5} (8q+15)}{(q+1)(q+2)} \right) \left| \vartheta'(\lambda) \right|^q \\
 & + \left(\frac{2^{-2q-5} + 2^{-2q-5} \times 3^{q+1} (4q+5)}{(q+1)(q+2)} \right) \left| \vartheta'(\mu) \right|^q
 \end{aligned} \tag{2.9}$$

and

$$\begin{aligned}
 & \int_0^1 \left| v - \frac{3}{4} \right|^q \left| \vartheta'(U_2(v)) \right|^q dv \\
 & \leq \left| \vartheta'(\lambda) \right|^q \left[\int_0^{\frac{3}{4}} \left(\frac{3}{4} - v \right)^q \left(\frac{1-v}{2} \right) dv + \int_{\frac{3}{4}}^1 \left(v - \frac{3}{4} \right)^q \left(\frac{1-v}{2} \right) dv \right] \\
 & + \left| \vartheta'(\mu) \right|^q \left[\int_0^{\frac{3}{4}} \left(\frac{3}{4} - v \right)^q \left(\frac{1+v}{2} \right) dv + \int_{\frac{3}{4}}^1 \left(v - \frac{3}{4} \right)^q \left(\frac{1+v}{2} \right) dv \right] \\
 & = \left(\frac{2^{-2q-5} \times 3^{q+1} (4q+5) + 2^{-2q-5}}{(q+1)(q+2)} \right) \left| \vartheta'(\lambda) \right|^q \\
 & + \left(\frac{2^{-2q-5} (8q+15) + 2^{-2q-5} \times 3^{q+1} (4q+11)}{(q+1)(q+2)} \right) \left| \vartheta'(\mu) \right|^q.
 \end{aligned} \tag{2.11}$$

We also observe that

$$\int_0^1 (U_1(v))^{\frac{2q}{q-1}} dv = \left(\frac{1-q}{1+q} \right) m_{\varphi}^{\frac{2q}{q-1}}(\lambda, \mu) n_{\varphi}^{-1}(\lambda, \mu) \left((1+n_{\varphi}(\lambda, \mu))^{\frac{1+q}{1-q}} - 1 \right) \quad (2.12)$$

and

$$\int_0^1 (U_2(v))^{\frac{2q}{q-1}} dv = \left(\frac{1-q}{1+q} \right) m_{\varphi}^{\frac{2q}{q-1}}(\lambda, \mu) n_{\varphi}^{-1}(\lambda, \mu) \left(1 - (1-n_{\varphi}(\lambda, \mu))^{\frac{1+q}{1-q}} \right), \quad (2.13)$$

where

$$n_{\varphi}(\lambda, \mu) = \frac{\varphi(\mu, \lambda)}{2\lambda + \varphi(\mu, \lambda)} \text{ and } m_{\varphi}(\lambda, \mu) = \frac{2\lambda(\lambda + \varphi(\mu, \lambda))}{2\lambda + \varphi(\mu, \lambda)}.$$

Applying (2.9)-(2.13) in (2.8), we get (2.7). $\square$

Theorem 3. Let $\vartheta : I_{h\varphi} \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a differentiable mapping on $I_{h\varphi}^{\circ}$, where $\lambda, \lambda + \varphi(\mu, \lambda) \in I_{h\varphi}^{\circ}$ with $\varphi(\mu, \lambda) > 0$ and let $w : [\lambda, \lambda + \varphi(\mu, \lambda)] \rightarrow [0, \infty)$ be a continuous positive mapping and harmonically symmetric to $\frac{2\lambda(\lambda + \varphi(\mu, \lambda))}{2\lambda + \varphi(\mu, \lambda)}$. If $\vartheta', w \in L_1([\lambda, \lambda + \varphi(\mu, \lambda)])$ and $|\vartheta'|^q$ is harmonically-preinvex on $[\lambda, \lambda + \varphi(\mu, \lambda)]$ for $q > 1$, then

$$\begin{aligned} |\vartheta(\lambda, \lambda + \varphi(\mu, \lambda); \vartheta, w)| &\leq \frac{m_{\varphi}^2(\lambda, \mu) \varphi(\mu, \lambda)}{4\lambda(\lambda + \varphi(\mu, \lambda))} \|w\|_{[\lambda, \lambda + \varphi(\mu, \lambda)], \infty} \left(\frac{q-1}{2q-1} \right)^{1-\frac{1}{q}} \\ &\times \left[4^{-\frac{2q-1}{q-1}} \left(3^{\frac{2q-1}{q-1}} + 1 \right) \right]^{1-\frac{1}{q}} \left(\frac{1}{4(2q-1)(q-1)\mu^2} \right)^{\frac{1}{q}} \\ &\times \left\{ \left[\rho_1(n_{\varphi}(\lambda, \mu)) |\vartheta'(\lambda)|^q + \rho_2(n_{\varphi}(\lambda, \mu)) |\vartheta'(\mu)|^q \right]^{\frac{1}{q}} \right. \\ &\left. + \left[\rho_2(-n_{\varphi}(\lambda, \mu)) |\vartheta'(\lambda)|^q + \rho_1(-n_{\varphi}(\lambda, \mu)) |\vartheta'(\mu)|^q \right]^{\frac{1}{q}} \right\}, \end{aligned} \quad (2.14)$$

where

$$\begin{aligned} \rho_1(n_{\varphi}(\lambda, \mu)) &= 1 + 2(q-1)n_{\varphi}(\lambda, \mu) - (1+n_{\varphi}(\lambda, \mu))^{1-2q} (1 + (4q-3)n_{\varphi}(\lambda, \mu)), \\ \rho_2(n_{\varphi}(\lambda, \mu)) &= (1+n_{\varphi}(\lambda, \mu))^{2-2q} + 2(q-1)n_{\varphi}(\lambda, \mu) - 1 \\ n_{\varphi}(\lambda, \mu) &= \frac{\varphi(\mu, \lambda)}{2\lambda + \varphi(\mu, \lambda)} \text{ and } m_{\varphi}(\lambda, \mu) = \frac{2\lambda(\lambda + \varphi(\mu, \lambda))}{2\lambda + \varphi(\mu, \lambda)}. \end{aligned}$$

Proof. Taking the absolute value on both sides of (2.2) and using the Hölder integral inequality, we have

$$\begin{aligned} |\vartheta(\lambda, \lambda + \varphi(\mu, \lambda); \vartheta, w)| &\leq \frac{\varphi(\mu, \lambda)}{4\lambda(\lambda + \varphi(\mu, \lambda))} \|w\|_{[\lambda, \lambda + \varphi(\mu, \lambda)], \infty} \\ &\times \left\{ \left(\int_0^1 \left| \frac{3}{4} - v \right|^{\frac{q}{q-1}} dv \right)^{1-\frac{1}{q}} \left(\int_0^1 (U_1(v))^{2q} \left| \vartheta'(U_1(v)) \right|^q dv \right)^{\frac{1}{q}} \right. \\ &\quad \left. + \left(\int_0^1 \left| v - \frac{3}{4} \right|^{\frac{q}{q-1}} dv \right)^{1-\frac{1}{q}} \left(\int_0^1 (U_2(v))^{2q} \left| \vartheta'(U_2(v)) \right|^q dv \right)^{\frac{1}{q}} \right\}. \end{aligned} \quad (2.15)$$

By using the harmonically-preinvexity of $|\vartheta'|^q$ on $[\lambda, \lambda + \varphi(\mu, \lambda)]$ for $q \geq 1$, we get

$$\int_0^1 (U_1(v))^{2q} \left| \vartheta'(U_1(v)) \right|^q dv \quad (2.16)$$

$$\begin{aligned} &\leq \left| \vartheta'(\lambda) \right|^q \int_0^1 \left(\frac{1+v}{2} \right) (U_1(v))^{2q} dv + \left| \vartheta'(\mu) \right|^q \int_0^1 \left(\frac{1-v}{2} \right) (U_1(v))^{2q} dv \\ &= \frac{m_{\varphi}^{2q}(\lambda, \mu) \left[1 + 2(q-1)n_{\varphi}(\lambda, \mu) - (1+n_{\varphi}(\lambda, \mu))^{1-2q} (1 + (4q-3)n_{\varphi}(\lambda, \mu)) \right]}{4(2q-1)(q-1)n_{\varphi}^2(\lambda, \mu)} \\ &\quad \times \left| \vartheta'(\lambda) \right|^q + \frac{m_{\varphi}^{2q}(\lambda, \mu) \left[(1+n_{\varphi}(\lambda, \mu))^{2-2q} + 2(q-1)n_{\varphi}(\lambda, \mu) - 1 \right]}{4(2q-1)(q-1)n_{\varphi}^2(\lambda, \mu)} \left| \vartheta'(\mu) \right|^q \end{aligned}$$

and

$$\int_0^1 (U_2(v))^{2q} \left| \vartheta'(U_2(v)) \right|^q dv \quad (2.17)$$

$$\begin{aligned} &\leq \left| \vartheta'(\lambda) \right|^q \int_0^1 \left(\frac{1-v}{2} \right) (U_2(v))^{2q} dv + \left| \vartheta'(\mu) \right|^q \int_0^1 \left(\frac{1+v}{2} \right) (U_2(v))^{2q} dv \\ &= \frac{m_{\varphi}^{2q}(\lambda, \mu) \left[(1-n_{\varphi}(\lambda, \mu))^{2-2q} + 2(1-q)n_{\varphi}(\lambda, \mu) - 1 \right]}{4(2q-1)(q-1)n_{\varphi}^2(\lambda, \mu)} \left| \vartheta'(\lambda) \right|^q + m_{\varphi}^{2q}(\lambda, \mu) \\ &\quad \times \frac{\left[1 + 2(1-q)n_{\varphi}(\lambda, \mu) - (1-n_{\varphi}(\lambda, \mu))^{1-2q} (1 + (3-4q)n_{\varphi}(\lambda, \mu)) \right]}{4(2q-1)(q-1)n_{\varphi}^2(\lambda, \mu)} \left| \vartheta'(\mu) \right|^q. \end{aligned}$$

We notice that

$$\int_0^1 \left| \frac{3}{4} - v \right|^{\frac{q}{q-1}} dv = \int_0^1 \left| v - \frac{3}{4} \right|^{\frac{q}{q-1}} dv = 4^{-\frac{2q-1}{q-1}} \left(\frac{q-1}{2q-1} \right) \left(3^{\frac{2q-1}{q-1}} + 1 \right). \quad (2.18)$$

Applying (2.16)-(2.18) in (2.15), we get (2.14). $\square$

Theorem 4. Let $\vartheta : I_{h\varphi} \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a differentiable mapping on $I_{h\varphi}^\circ$, where $\lambda, \lambda + \varphi(\mu, \lambda) \in I_{h\varphi}^\circ$ with $\varphi(\mu, \lambda) > 0$ and let $w : [\lambda, \lambda + \varphi(\mu, \lambda)] \rightarrow [0, \infty)$ be a continuous positive mapping and harmonically symmetric to $\frac{2\lambda(\lambda + \varphi(\mu, \lambda))}{2\lambda + \varphi(\mu, \lambda)}$. If $\vartheta', w \in L_1([\lambda, \lambda + \varphi(\mu, \lambda)])$ and $|\vartheta'|^q$ is harmonically-preinvex on $[\lambda, \lambda + \varphi(\mu, \lambda)]$ for $q \geq 1$, then

$$\begin{aligned} & |\vartheta(\lambda, \lambda + \varphi(\mu, \lambda); \vartheta, w)| \\ & \leq \frac{m_\varphi^2(\lambda, \mu) \varphi(\mu, \lambda)}{4\lambda(\lambda + \varphi(\mu, \lambda))} \|w\|_{[\lambda, \lambda + \varphi(\mu, \lambda)], \infty} \left\{ [\sigma_1(n_\varphi(\lambda, \mu))]^{1-\frac{1}{q}} \right. \\ & \quad \times \left[o_1(n_\varphi(\lambda, \mu)) |\vartheta'(\lambda)|^q + o_2(n_\varphi(\lambda, \mu)) |\vartheta'(\mu)|^q \right]^{\frac{1}{q}} + [\sigma_1(-n_\varphi(\lambda, \mu))]^{1-\frac{1}{q}} \\ & \quad \times \left[o_2(-n_\varphi(\lambda, \mu)) |\vartheta'(\lambda)|^q + o_1(-n_\varphi(\lambda, \mu)) |\vartheta'(\mu)|^q \right]^{\frac{1}{q}} \Big\}, \end{aligned} \quad (2.19)$$

where

$$\begin{aligned} \sigma_1(n_\varphi(\lambda, \mu)) &= \frac{3n_\varphi(\lambda, \mu) + 2}{4n_\varphi(\lambda, \mu)(1 + n_\varphi(\lambda, \mu))} + \frac{1}{n_\varphi^2(\lambda, \mu)} \ln \left(\frac{16(1 + n_\varphi(\lambda, \mu))}{(4 + 3n_\varphi(\lambda, \mu))^2} \right), \\ n_\varphi(\lambda, \mu) &= \frac{\varphi(\mu, \lambda)}{2\lambda + \varphi(\mu, \lambda)} \text{ and } m_\varphi(\lambda, \mu) = \frac{2\lambda(\lambda + \varphi(\mu, \lambda))}{2\lambda + \varphi(\mu, \lambda)}. \end{aligned}$$

Proof. Taking the absolute value on both sides of (2.2) and using the power-mean inequality, we have

$$\begin{aligned} & |\vartheta(\lambda, \lambda + \varphi(\mu, \lambda); \vartheta, w)| \leq \frac{\varphi(\mu, \lambda)}{4\lambda(\lambda + \varphi(\mu, \lambda))} \|w\|_{[\lambda, \lambda + \varphi(\mu, \lambda)], \infty} \\ & \quad \times \left\{ \left(\int_0^1 \left| \frac{3}{4} - v \right| (U_1(v))^2 dv \right)^{1-\frac{1}{q}} \left(\int_0^1 \left| \frac{3}{4} - v \right| (U_1(v))^2 |\vartheta'(U_1(v))|^q dv \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\int_0^1 \left| v - \frac{3}{4} \right| (U_2(v))^2 dv \right)^{1-\frac{1}{q}} \left(\int_0^1 \left| \frac{3}{4} - v \right| (U_2(v))^2 |\vartheta'(U_2(v))|^q dv \right)^{\frac{1}{q}} \right\}. \end{aligned} \quad (2.20)$$

By using the harmonically-preinvexity of $\left|\vartheta'\right|^q$ on $[\lambda, \lambda + \varphi(\mu, \lambda)]$ for $q \geq 1$, we get

$$\begin{aligned}
 & \int_0^1 \left| \frac{3}{4} - v \right| (U_1(v))^2 \left| \vartheta'(U_1(v)) \right|^q dv \\
 & \leq \left| \vartheta'(\lambda) \right|^q \left[\int_0^{\frac{3}{4}} \left(\frac{3}{4} - v \right) \left(\frac{1+v}{2} \right) (U_1(v))^2 dv + \int_{\frac{3}{4}}^1 \left(v - \frac{3}{4} \right) \left(\frac{1+v}{2} \right) (U_1(v))^2 dv \right] \\
 & + \left| \vartheta'(\mu) \right|^q \left[\int_0^{\frac{3}{4}} \left(\frac{3}{4} - v \right) \left(\frac{1-v}{2} \right) (U_1(v))^2 dv + \int_{\frac{3}{4}}^1 \left(v - \frac{3}{4} \right) \left(\frac{1-v}{2} \right) (U_1(v))^2 dv \right] \\
 & = \left[\frac{3n_\varphi^2(\lambda, \mu) - 3n_\varphi(\lambda, \mu) - 4}{8n_\varphi(\lambda, \mu)^2 (1 + n_\varphi(\lambda, \mu))} + \frac{(n_\varphi(\lambda, \mu) - 8)}{8n_\varphi^3(\lambda, \mu)} \ln \left(\frac{16(1 + n_\varphi(\lambda, \mu))}{(4 + 3n_\varphi(\lambda, \mu))^2} \right) \right] \\
 & \times m_\varphi^2(\lambda, \mu) \left| \vartheta'(\lambda) \right|^q + m_\varphi^2(\lambda, \mu) \left[\frac{4 + 3n_\varphi(\lambda, \mu)}{8n_\varphi^2(\lambda, \mu)} \right. \\
 & \left. + \frac{(8 + 7n_\varphi(\lambda, \mu))}{8n_\varphi^3(\lambda, \mu)} \ln \left(\frac{16(1 + n_\varphi(\lambda, \mu))}{(4 + 3n_\varphi(\lambda, \mu))^2} \right) \right] \left| \vartheta'(\mu) \right|^q
 \end{aligned} \tag{2.21}$$

and

$$\begin{aligned}
 & \int_0^1 \left| v - \frac{3}{4} \right| (U_2(v))^2 \left| \vartheta'(U_2(v)) \right|^q dv \\
 & \leq \left| \vartheta'(\lambda) \right|^q \left[\int_0^{\frac{3}{4}} \left(\frac{3}{4} - v \right) \left(\frac{1-v}{2} \right) (U_2(v))^2 dv + \int_{\frac{3}{4}}^1 \left(v - \frac{3}{4} \right) \left(\frac{1-v}{2} \right) (U_2(v))^2 dv \right] \\
 & + \left| \vartheta'(\mu) \right|^q \left[\int_0^{\frac{3}{4}} \left(\frac{3}{4} - v \right) \left(\frac{1+v}{2} \right) (U_2(v))^2 dv + \int_{\frac{3}{4}}^1 \left(v - \frac{3}{4} \right) \left(\frac{1+v}{2} \right) (U_2(v))^2 dv \right] \\
 & = \left[\frac{4 - 3n_\varphi(\lambda, \mu)}{8n_\varphi^2(\lambda, \mu)} - \frac{(8 - 7n_\varphi(\lambda, \mu))}{8n_\varphi^3(\lambda, \mu)} \ln \left(\frac{16(1 - n_\varphi(\lambda, \mu))}{(4 - 3n_\varphi(\lambda, \mu))^2} \right) \right]
 \end{aligned}$$

$$\begin{aligned} & \times m_{\varphi}^2(\lambda, \mu) \left| \vartheta'(\lambda) \right|^q + m_{\varphi}^2(\lambda, \mu) \left[\frac{3n_{\varphi}(\lambda, \mu)^2 + 3n_{\varphi}(\lambda, \mu) - 4}{8n_{\varphi}^2(\lambda, \mu)(1 - n_{\varphi}(\lambda, \mu))} \right. \\ & \left. + \frac{(n_{\varphi}(\lambda, \mu) + 8)}{8n_{\varphi}^3(\lambda, \mu)} \ln \left(\frac{16(1 - n_{\varphi}(\lambda, \mu))}{(4 - 3n_{\varphi}(\lambda, \mu))^2} \right) \right] \left| \vartheta'(\mu) \right|^q. \end{aligned} \quad (2.22)$$

We also have

$$\begin{aligned} \int_0^1 \left| \frac{3}{4} - v \right| (U_1(v))^2 dv &= \int_0^{\frac{3}{4}} \left(\frac{3}{4} - v \right) (U_1(v))^2 dv + \int_{\frac{3}{4}}^1 \left(v - \frac{3}{4} \right) (U_1(v))^2 dv \quad (2.23) \\ &= m_{\varphi}^2(\lambda, \mu) \left[\frac{3n_{\varphi}(\lambda, \mu) + 2}{4n_{\varphi}(\lambda, \mu)(1 + n_{\varphi}(\lambda, \mu))} + \frac{1}{n_{\varphi}^2(\lambda, \mu)} \ln \left(\frac{16(1 + n_{\varphi}(\lambda, \mu))}{(4 + 3n_{\varphi}(\lambda, \mu))^2} \right) \right] \end{aligned}$$

and

$$\begin{aligned} \int_0^1 \left| \frac{3}{4} - v \right| (U_2(v))^2 dv &= \int_0^{\frac{3}{4}} \left(\frac{3}{4} - v \right) (U_2(v))^2 dv + \int_{\frac{3}{4}}^1 \left(v - \frac{3}{4} \right) (U_2(v))^2 dv \quad (2.24) \\ &= m_{\varphi}^2(\lambda, \mu) \left[\frac{3n_{\varphi}(\lambda, \mu) - 2}{4n_{\varphi}(\lambda, \mu)(1 - n_{\varphi}(\lambda, \mu))} + \frac{1}{n_{\varphi}^2(\lambda, \mu)} \ln \left(\frac{16(1 - n_{\varphi}(\lambda, \mu))}{(4 - 3n_{\varphi}(\lambda, \mu))^2} \right) \right], \end{aligned}$$

where

$$n_{\varphi}(\lambda, \mu) = \frac{\varphi(\mu, \lambda)}{2\lambda + \varphi(\mu, \lambda)} \text{ and } m_{\varphi}(\lambda, \mu) = \frac{2\lambda(\lambda + \varphi(\mu, \lambda))}{2\lambda + \varphi(\mu, \lambda)}.$$

Using (2.21)-(2.24) in (2.20) we get (2.19). $\square$

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GENERAL HISTORY-DEPENDENT OPERATORS WITH APPLICATIONS TO DIFFERENTIAL EQUATIONS

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Received 27 March, 2019

Abstract. In this paper, we introduce a class of nonlinear operators—the class of general history-dependent operators. These are the operators defined on spaces of functions endowed with a structure of Banach space (the case of bounded interval of time) or Fréchet space (the case of unbounded interval of time). We state and prove various properties of such operators, including fixed point properties. Moreover, we also study several classes of differential equations in Banach spaces, for which our previous results can be applied.

2010 Mathematics Subject Classification: 34A08, 34A40, 37C25, 47G10, 35R11

Keywords: general history-dependent operator, fixed point theorem, differential equation, fractional derivative.

1. INTRODUCTION

As we know, the term “history-dependent operator” was firstly introduced by Sofonea–Matei [15] on spaces of functions endowed with a structure of Banach space (the case of bounded interval of time) or Fréchet space (the case of unbounded interval of time), because history-dependent operator is useful to represent the models in Contact Mechanics involving both quasistatic frictional and frictionless contact conditions with elastic or viscoelastic materials, for example, mechanical impact problems, electrical circuits with ideal diodes, the Coulomb friction problems for contacting bodies and so on. After the work [15], more and more scholars are attracted to boost the development of theory and applications for history-dependent operator. For instance, see [2, 9, 13, 14, 16, 18]. Very recently, Sofonea and Migórski [17] considered a generalized “history-dependent operator”, which is called “almost history-dependent operator”.

Our work is motivated by the development of the mathematical theory in Contact Mechanics, which requires new mathematical tools needed for the study of the differential equations. In particular, the theory requires results for new classes of general history-dependent operators under specific assumptions on functionals and operators. Several differential equations, for instance, in [1, 3–6, 8, 12], lead to general history-dependent operators of the form studied in this paper, in which is either the

integer-order differential equations or the fractional-order differential equations. The abstract results presented in the paper can be applied to the study of these problems and, therefore, they can be used to prove the unique solvability of the corresponding differential equations. Any progress in the study of above mentioned problems will open avenues for new advances and applications of the history-dependent operators.

The aim of this manuscript is two folds. The first one is to provide the definition for general history-dependent operators, and then we prove an existence and uniqueness result for a new class of general history-dependent operators on spaces of functions endowed with a structure of Banach space (the case of bounded interval of time) or Fréchet space (the case of unbounded interval of time). The second aim is to illustrate our main results to study several classes of fractional differential equations. This gives rise to a new approach on the analysis of the corresponding mathematics models.

The rest of the manuscript is structured as follows. In Section 2, we present a definition for general history-dependent operator and then prove a general fixed point principle in the study of differential equations which gives rise to generic existence and uniqueness results in Theorems 1 and 2. In Section 3, we particularize Theorems 1 and 2 in the study of a specific class of differential equations for which we obtain new results of unique solvability for fractional-order differential equations in Theorem 3 and 4.

2. GENERAL HISTORY-DEPENDENT OPERATORS

In this section we review some prerequisites that are necessary in the next sections. Throughout this paper, the norm of a Banach space X will be denoted by $\|\cdot\|_X$. Let $\mathbb{R}_+ = [0, +\infty)$. Below, I denotes either a bounded interval of the form $[0, T]$ with $T > 0$, or the unbounded interval $\mathbb{R}_+$. For, $L^p(I; X)$ we denote the Banach space of all Bochner integrable functions from I into X with the norm $\|f\|_{L^p(I, X)} = (\int_I \|f(t)\|_X^p ds)^{\frac{1}{p}}$. Next, in the case $I = [0, T]$ the space $C(I; X)$ will be equipped with the norm $\|x\|_{C(I; X)} = \max_{t \in I=[0, T]} \|x(t)\|_X$. It is well known that if X is a Banach space, then $C(I; X)$ is also a Banach space. Assume now that $I = \mathbb{R}_+$. It is well known that, if X is a Banach space, then $C(I; X)$ can be organized in a canonical way as a Fréchet space, i.e., a complete metric space in which the corresponding topology is induced by a countable family of seminorms. Recall that the convergence of a sequence $\{x_k\}_{k \geq 1}$ to the element x , in the space $C(\mathbb{R}_+; X)$, can be described as follows

$$\begin{cases} x_k \rightarrow x \text{ in } C(\mathbb{R}_+; X), \text{ as } n \rightarrow \infty \text{ if and only if} \\ \max_{t \in [0, n]} \|x_k(t) - x(t)\|_X \rightarrow 0, \text{ as } k \rightarrow \infty, \quad \forall n \in \mathbb{N}. \end{cases} \quad (2.1)$$

Moreover, we also note that

$$\left\{ \begin{array}{l} \{x_k\} \subset C(\mathbb{R}_+; X) \text{ is a Cauchy sequence if and only if} \\ \forall \varepsilon > 0, \forall n \in \mathbb{N}, \exists N = N(\varepsilon, n) \text{ such that} \\ \|x_{k_1} - x_{k_2}\|_{C([0, n]; X)} < \varepsilon \quad \forall k_1, k_2 > N. \end{array} \right.$$

As we know, the notion of history-dependent operator was introduced in [15] and the definition of history-dependent operator was considered in [17], which have found numerous applications. Here, we define the class of general history-dependent operator which is valid in the case of both bounded and unbounded intervals I as follows.

Definition 1. An operator $F : C(I; X) \rightarrow C(I; X)$ is called a general history-dependent operator if for any compact set $K \subset I$, there exist constants $\ell_K \in [0, 1)$, $L_K \geq 0$ and $q > -1$ such that

$$\|(Fx)(t) - (Fy)(t)\|_X \leq \ell_K \|x(t) - y(t)\|_X + L_K \int_0^t (t-s)^q \|x(s) - y(s)\|_X ds \quad (2.2)$$

for all $x, y \in C(I; X)$, $t \in K$.

Remark 1. We specialize the previous definition of general history-dependent operators in the cases $I = [0, T]$ and $I = \mathbb{R}_+$, respectively.

- (i) An operator $F : C([0, T]; X) \rightarrow C([0, T]; X)$ is called a general history-dependent operator if there exist constants $\ell \in [0, 1)$, $L \geq 0$ and $q > -1$ such that

$$\|(Fx)(t) - (Fy)(t)\|_X \leq \ell \|x(t) - y(t)\|_X + L \int_0^t (t-s)^q \|x(s) - y(s)\|_X ds \quad (2.3)$$

for all $x, y \in C([0, T]; X)$, $t \in [0, T]$.

- (ii) An operator $F : C(\mathbb{R}_+; X) \rightarrow C(\mathbb{R}_+; X)$ is called a general history-dependent operator if for any $n \in \mathbb{N}$, there exist constants $\ell_n \in [0, 1)$, $L_n \geq 0$ and $q > -1$ such that

$$\|(Fx)(t) - (Fy)(t)\|_X \leq \ell_n \|x(t) - y(t)\|_X + L_n \int_0^t (t-s)^q \|x(s) - y(s)\|_X ds \quad (2.4)$$

for all $x, y \in C(\mathbb{R}_+; X)$, $t \in [0, n]$.

Next, let us now consider three special cases of general history-dependent operators.

- (i) If $L_K = 0$, then (2.2) is reduced to

$$\|(Fx)(t) - (Fy)(t)\|_X \leq \ell_K \|x(t) - y(t)\|_X,$$

which implies that

$$\|Fx - Fy\|_{C(I; X)} \leq \ell_K \|x - y\|_{C(I; X)},$$

i.e., F is a Lipschitz continuous operator on $C(I; X)$ with constant $\ell_K \in [0, 1)$.

(ii) If $\ell_K = 0$, $q = 0$, then (2.2) is reduced to

$$\|(Fx)(t) - (Fy)(t)\|_X \leq L_K \int_0^t \|x(s) - y(s)\|_X ds,$$

i.e., F is a history-dependent operator on $C(I; X)$ with constant $L_K \geq 0$ (cf. [15]).

(iii) If $q = 0$, then (2.2) is reduced to

$$\|(Fx)(t) - (Fy)(t)\|_X \leq \ell_K \|x(t) - y(t)\|_X + L_K \int_0^t \|x(s) - y(s)\|_X ds,$$

i.e., F is an almost history-dependent operator on $C(I; X)$ with constant $\ell_K \in [0, 1)$, $L_K \geq 0$ (cf. [17]).

Similarly, general history-dependent operators also have important fixed point properties which are very useful to prove the solvability of various classes of non-linear equations and variational inequalities. Our aim in what follows is to present the fixed point properties for general history-dependent operators, in the cases of both bounded and unbounded intervals of time. We shall consider separately the case of the bounded interval $I = [0, T]$ and the case of the unbounded interval $I = \mathbb{R}_+$, since the arguments in proof are different. We start with the case of the bounded interval $I = [0, T]$ as follows.

Theorem 1. *If $F : C([0, T]; X) \rightarrow C([0, T]; X)$ is a general history-dependent operator, then F has a fixed point in $C([0, T]; X)$.*

Proof. For $\lambda > 0$, we introduce the Bielecki norm

$$\|x\|_\lambda = \max_{t \in [0, T]} e^{-\lambda t} \|x(t)\|_X \quad \text{for } x \in C([0, T]; X). \quad (2.5)$$

Clearly, $\|\cdot\|_\lambda$ defines a norm on the space $C([0, T]; X)$ which is equivalent to the usual norm $\|\cdot\|_{C([0, T]; X)}$. As a consequence, it results that $C([0, T]; X)$ is a Banach space with the norm $\|\cdot\|_\lambda$, too. Let $t \in [0, T]$. From the definition of the general history-dependent operator, it follows that

$$\|(Fx)(t) - (Fy)(t)\|_X \leq \ell \|x(t) - y(t)\|_X + L \int_0^t (t-s)^q \|x(s) - y(s)\|_X ds$$

for all $x, y \in C([0, T]; X)$ with $\ell \in [0, 1)$, $L \geq 0$ and $q > -1$. Then for $\lambda > 0$,

$$\begin{aligned} & e^{-\lambda t} \|(Fx)(t) - (Fy)(t)\|_X \\ & \leq \ell e^{-\lambda t} \|x(t) - y(t)\|_X + L e^{-\lambda t} \int_0^t (t-s)^q \|x(s) - y(s)\|_X ds \\ & = \ell e^{-\lambda t} \|x(t) - y(t)\|_X + L e^{-\lambda t} \int_0^t (t-s)^q e^{\lambda s} (e^{-\lambda s} \|x(s) - y(s)\|_X) ds \\ & \leq \ell \|x - y\|_\lambda + L e^{-\lambda t} \|x - y\|_\lambda \int_0^t (t-s)^q e^{\lambda s} ds \end{aligned}$$

$$\begin{aligned}
&= \ell \|x - y\|_\lambda + L e^{-\lambda t} \|x - y\|_\lambda \int_0^t r^q e^{\lambda(t-r)} dr \quad (r = t - s) \\
&= \ell \|x - y\|_\lambda + L \|x - y\|_\lambda \int_0^t r^q e^{-\lambda r} dr \\
&= \ell \|x - y\|_\lambda + L \|x - y\|_\lambda \lambda^{-q-1} \int_0^{\lambda t} \theta^q e^{-\theta} d\theta \quad (\theta = \lambda r) \\
&\leq \ell \|x - y\|_\lambda + L \|x - y\|_\lambda \lambda^{-q-1} \int_0^\infty \theta^q e^{-\theta} d\theta \\
&= \ell \|x - y\|_\lambda + L \|x - y\|_\lambda \lambda^{-q-1} \Gamma(q+1),
\end{aligned}$$

and hence

$$\|Fx - Fy\|_\lambda \leq (\ell + L\lambda^{-q-1}\Gamma(q+1))\|x - y\|_\lambda.$$

Next, since $\ell \in [0, 1)$, we choose λ such that $\lambda > \sqrt[q+1]{\frac{L\Gamma(q+1)}{1-\ell}}$. Then $\ell + L\lambda^{-q-1}\Gamma(q+1) < 1$, which shows that the operator F is a contraction on the space $C([0, T]; X)$ endowed with the norm $\|\cdot\|_\lambda$. By applying the Banach fixed point theorem we obtain that F has a unique fixed point $x^* \in C([0, T]; X)$, which concludes the proof. $\square$

We now move to the case of the unbounded interval $I = \mathbb{R}_+$.

Theorem 2. *If $F : C(\mathbb{R}_+; X) \rightarrow C(\mathbb{R}_+; X)$ is a general history-dependent operator, then F has a fixed point in $C(\mathbb{R}_+; X)$.*

Proof. The proof is based on the following two claims.

Claim (1) For every $n \in \mathbb{N}$ there exist two constants $\ell_n \in [0, 1)$, $L_n > 0$, which depend on n , such that

$$\|Fx - Fy\|_{\lambda, n} \leq (\ell_n + L_n \lambda^{-q-1} \Gamma(q+1)) \|x - y\|_{\lambda, n}, \quad (2.6)$$

where $x, y \in C(\mathbb{R}_+; X)$ and

$$\|x\|_{\lambda, n} = \max_{t \in [0, n]} e^{-\lambda t} \|x(t)\|_X, \quad \text{for } x \in C([0, n]; X).$$

The proof of this claim follows from the proof of Theorem 1.

Claim (2) For every $n \in \mathbb{N}$, there exists a unique function $x_n \in C([0, n]; X)$ such that

$$(Fx_n)(t) = x_n(t), \quad \forall t \in [0, n]. \quad (2.7)$$

Moreover, if $m, n \in \mathbb{N}$ are such that $m \geq n$, then

$$x_m(t) = x_n(t) \quad \forall t \in [0, n]. \quad (2.8)$$

The proof of this claim is based on recurrence and is a consequence of **Claim (1)**. First, we choose λ_1 such that $\lambda_1 > \sqrt[q+1]{\frac{L_1\Gamma(q+1)}{1-\ell_1}}$ and denote $\alpha_1 = \ell_1 + L_1\lambda_1^{-q-1}\Gamma(q+1)$. Then it follows that $\alpha_1 \in (0, 1)$ and, using (2.6) for $n = 1$, we have

$$\|Fx - Fy\|_{\lambda_1, 1} \leq \alpha_1 \|x - y\|_{\lambda_1, 1}, \quad \forall x, y \in C([0, 1]; X).$$

Next, we choose $\lambda_2 > \max\{\lambda_1, \sqrt[q+1]{\frac{L_2\Gamma(q+1)}{1-\ell_2}}\}$ and define $\alpha_2 = \ell_2 + L_2\lambda_2^{-q-1}\Gamma(q+1)$, and we continue by recurrence. As a result, we obtain a sequence $\{\lambda_n\}$ which satisfies $0 < \lambda_1 < \lambda_2 < \dots < \lambda_n < \dots$ and a sequence $\{\alpha_n\}$ which satisfies $\alpha_n \in (0, 1)$ for all $n \in \mathbb{N}$. Moreover,

$$\|Fx - Fy\|_{\lambda_n, n} \leq \alpha_n \|x - y\|_{\lambda_n, n}, \quad \forall x, y \in C([0, n]; X), n \in \mathbb{N}. \quad (2.9)$$

The first part of the claim is now a consequence of the Banach fixed point theorem. Assume now that $m, n \in \mathbb{N}$ are such that $m \geq n$. Then, equality (2.8) is a consequence of the uniqueness of the fixed point of the operator F on the space $C([0, n]; X)$ for all $n \in \mathbb{N}$.

We now have all the ingredients to proceed with the proof of this Theorem.

Existence. **Claim (2)** allows us to consider the sequence $\{\tilde{x}_n\} \subset C(\mathbb{R}_+; X)$ given by

$$\tilde{x}_n(t) = \begin{cases} x_n(t) & \text{if } t \in [0, n], \\ x_n(n) & \text{if } t \geq n. \end{cases} \quad (2.10)$$

Let $n \in \mathbb{N}$. It follows from (2.8) and (2.10) that, if $m \in \mathbb{N}$ satisfies $m \geq n$, then

$$\tilde{x}_m(t) = x_n(t), \quad \forall t \in [0, n], \quad (2.11)$$

which implies that if $m_1, m_2 \in \mathbb{N}$ are such that $m_1, m_2 \geq n$, and hence

$$\|\tilde{x}_{m_1} - \tilde{x}_{m_2}\|_{\lambda_n, n} = 0, \quad \forall t \in [0, n]. \quad (2.12)$$

Since n is arbitrary, we deduce from (2.12) that $\{\tilde{x}_m\}$ is a Cauchy sequence in $C(\mathbb{R}_+; X)$. By the completeness of the space $C(\mathbb{R}_+; X)$, there exists $x \in C(\mathbb{R}_+; X)$ such that $\tilde{x}_m \rightarrow x$ as $m \rightarrow \infty$. Then we have

$$\lim_{m \rightarrow \infty} \|\tilde{x}_m - x\|_{\lambda_n, n} = 0, \quad (2.13)$$

i.e., the sequence $\{\tilde{x}_m\}$ converges uniformly on $[0, n]$, for all $n \in \mathbb{N}$. Since the uniform convergence implies the pointwise convergence, we have

$$\lim_{m \rightarrow \infty} \tilde{x}_m(t) = x(t), \quad \forall t \in [0, n], n \in \mathbb{N}. \quad (2.14)$$

We now combine (2.11) and (2.14) to obtain that

$$x_n(t) = x(t), \quad \forall t \in [0, n], n \in \mathbb{N}. \quad (2.15)$$

On the other hand, taking into account (2.9) and (2.13) we deduce that

$$\lim_{m \rightarrow \infty} \|F\tilde{x}_m - Fx\|_{\lambda_n, n} = 0 \quad n \in \mathbb{N}, \quad (2.16)$$

which shows that

$$\lim_{m \rightarrow \infty} F\tilde{x}_m = Fx \quad \text{in } C(\mathbb{R}_+; X). \quad (2.17)$$

Let $t \in \mathbb{R}_+$ be fixed. Obviously, there exists $n \geq 1$ such that $t \in [0, n]$ and, again, since the uniform convergence on $C([0, n]; X)$ implies the pointwise convergence, by (2.17) we have

$$\lim_{m \rightarrow \infty} (F\tilde{x}_m)(t) = (Fx)(t) \quad \text{in } X. \quad (2.18)$$

Now, using (2.7), (2.11) and (2.15) for $m \geq n$, we obtain

$$(F\tilde{x}_m)(t) = (Fx_m)(t) = x_n(t) = x(t). \quad (2.19)$$

We pass to the limit, as $m \rightarrow \infty$ in (2.19) and use (2.18) to find that $(Fx)(t) = x(t)$. Since t is an arbitrary real positive number, we conclude that $Fx = x$, i.e., $x \in C(\mathbb{R}_+; X)$ is a fixed point of the operator F .

Uniqueness. Assume that there exist $x, x' \in C(\mathbb{R}_+; X)$ such that $x \neq x'$, and

$$Fx = x \quad \text{and} \quad Fx' = x'. \quad (2.20)$$

Then, there exists $t_0 \in \mathbb{R}_+$ such that

$$x(t_0) \neq x'(t_0). \quad (2.21)$$

We choose $n \in \mathbb{N}$ such that $t_0 \in [0, n]$. Equations (2.20) imply that the functions $x, x' : [0, n] \rightarrow X$ are two fixed points of the operator F on the space $C([0, n]; X)$, and therefore, by the uniqueness of the function $x_n \in C([0, n]; X)$ introduced in **Claim 2**, we have $x(t) = x'(t) = x_n(t)$ for all $t \in [0, n]$, which contradicts (2.21). We conclude that the fixed point of the operator F is unique. $\square$

3. FRACTIONAL-ORDER DIFFERENTIAL EQUATIONS

In this section, we give several fractional-order differential equations to illustrate the applications of the generalized history-dependent operators. We will apply Theorem 1 and Theorem 2 to show the existence and uniqueness results for the equations.

Let us recall some basic statements of fractional calculus from [7, 11].

Definition 2. The fractional integral of order q with the lower limit zero for a function f is defined as

$$I_t^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds, \quad t > 0, q > 0,$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where Γ is the gamma function.

Definition 3. The Caputo fractional derivative of order q for a function $f : [0, \infty) \rightarrow R$ is defined as

$${}^c D_t^q f(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} f(s) ds, \quad t > 0, 0 < q < 1.$$

Remark 2. (i) If $f \in AC([0, \infty))$, then

$${}^c D_t^q f(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} f'(s) ds = I_t^{1-q} f'(t), \quad t > 0, 0 < q < 1.$$

- (ii) The Caputo derivative of a constant is equal to zero.
- (iii) If f is an abstract function with values in X , then integrals which appear in Definitions 2 and 3 are taken in Bochner's sense.

Consider the fractional neutral differential equations having the following form:

$$\begin{cases} {}^c D_t^q (x(t) - g(t, x(t))) = f(t, x(t)), & t \in I, 0 < q < 1, \\ x(0) = x_0. \end{cases} \quad (3.1)$$

Definition 4. A function $x \in C(I; X)$ is said to be a solution of problem (3.1) on I if

$$x(t) = x_0 - g(0, x_0) + g(t, x(t)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s)) ds, \quad t \in I.$$

The hypotheses on the functions f and g are the following:

$$\left\{ \begin{array}{l} g : I \times X \rightarrow X \text{ is such that} \\ \quad \text{(a) } g(\cdot, x) \text{ is measurable for every } x \in X; \\ \quad \text{(b) there exists a constant } L_g \in (0, 1) \text{ such that} \\ \qquad \|g(t, x) - g(t, y)\|_X \leq L_g \|x - y\|_X \\ \qquad \text{for all } x, y \in X, \text{ a.e. } t \in I. \end{array} \right. \quad (3.2)$$

$$\left\{ \begin{array}{l} f : I \times X \rightarrow X \text{ is such that} \\ \quad \text{(a) } f(\cdot, x) \text{ is measurable for every } x \in X; \\ \quad \text{(b) there exists a constant } L_f > 0 \text{ such that} \\ \qquad \|f(t, x) - f(t, y)\|_X \leq L_f \|x - y\|_X \\ \qquad \text{for all } x, y \in X, \text{ a.e. } t \in I. \end{array} \right. \quad (3.3)$$

Define an operator $F_1 : C(I; X) \rightarrow C(I; X)$ as follows:

$$(F_1 x)(t) = x_0 - g(0, x_0) + g(t, x(t)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s)) ds, \quad t \in I. \quad (3.4)$$

Then, we have the following existence and uniqueness result.

Theorem 3. If (3.2), (3.3) hold and $x_0 \in X$, then problem (3.1) has a unique mild solution on $C(I; X)$.

Proof. It is clear that the operator F_1 defined by (3.4) is a generalized history-dependent operator. $\square$

In what follows, we will study the fractional evolution equations having the following form:

$$\begin{cases} {}^c D_t^q x(t) = Ax(t) + f(t, x(t)), & t \in I, 0 < q < 1, \\ x(0) = x_0. \end{cases} \quad (3.5)$$

Based on [10, 19], we shall define the following concept.

Definition 5. A function $x \in C(I; X)$ is said to be a mild solution of problem (3.5) on I if

$$x(t) = S_q(t)x_0 + \int_0^t (t-s)^{q-1} T_q(t-s) f(s, x(s)) ds, \quad t \in I,$$

where

$$S_q(t) = \int_0^\infty \xi_q(\theta) T(t^q \theta) d\theta,$$

$$T_q(t) = q \int_0^\infty \theta \xi_q(\theta) T(t^q \theta) d\theta,$$

$$\xi_q(\theta) = \frac{1}{q} \theta^{-1-\frac{1}{q}} \varpi_q(\theta^{-\frac{1}{q}}),$$

$$\varpi_q(\theta) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-nq-1} \frac{\Gamma(nq+1)}{n!} \sin(n\pi q), \quad \theta \in (0, \infty),$$

ξ_q is a probability density function defined on $(0, \infty)$ (see [19]), that is

$$\xi_q(\theta) \geq 0, \quad \theta \in (0, \infty), \quad \text{and} \quad \int_0^\infty \xi_q(\theta) d\theta = 1.$$

Due to the papers [10, 19] again, we can obtain the following results.

Lemma 1. $S_q(t)$, $T_q(t)$ have the following properties.

- (i) For any fixed $t \geq 0$, $S_q(t)$, $T_q(t)$ are linear and bounded operators, such that, for any $x \in X$,

$$\|S_q(t)x\| \leq M\|x\|,$$

$$\|T_q(t)x\| \leq \frac{M}{\Gamma(q)}\|x\|.$$

- (ii) $S_q(t)$, $T_q(t)$ ($t \geq 0$) are strongly continuous.

Define an operator $F_2 : C(I; X) \rightarrow C(I; X)$ as follows:

$$(F_2x)(t) = S_q(t)x_0 + \int_0^t (t-s)^{q-1} T_q(t-s) f(s, x(s)) ds, \quad t \in I. \quad (3.6)$$

Then, we have the following existence and uniqueness result.

Theorem 4. If (3.3) holds and $x_0 \in X$, then problem (3.5) has a unique mild solution on $C(I; X)$.

Proof. For all $x, y \in C(I; X)$ and $t \in I$, we have

$$\begin{aligned} \|(F_2x)(t) - (F_2y)(t)\| &= \left\| \int_0^t (t-s)^{q-1} T_q(t-s) (f(s, x(s)) - f(s, y(s))) ds \right\| \\ &\leq \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|f(s, x(s)) - f(s, y(s))\| ds \end{aligned}$$

$$\leq \frac{ML_f}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|x(s) - y(s)\| ds,$$

which shows that the operator F_2 is a general history-dependent operator. $\square$

ACKNOWLEDGEMENT

The first author is supported by NNSF of China (No. 11661001), the Project of Guangxi Education Department (No. KY2016YB417). The second author is supported by the Natural Science Foundation of Guangxi Province (No. 2018JJB110060), the Start-up Project of Scientific Research on Introducing talents at school level in Guangxi University for Nationalities (No. 2019KJQD04) and the Xiangsihu Young Scholars Innovative Research Team of Guangxi University for Nationalities (No. 2019RSCXSHQN02).

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NEW GEOMETRIC VIEWPOINTS TO CHEN CHAOTIC SYSTEM

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Received 29 March, 2021

Abstract. This paper presents new geometric viewpoints to Chen chaotic system. Firstly, the existence of two nontransverse homoclinic orbits in Chen system is rigorously proved beyond the classical parameters. Secondly, combined with the theory of tangent bundle, a new geometric viewpoint is given to explore chaos mechanism of Chen system.

The fundamental geometric definition of tangent bundle and the essential role of nonlinear connection between the tangent space and the base space are described. By introducing the geometrical viewpoints of second order system governed by Lie-Poisson equation, some geometric invariants of Chen system can be obtained. Furthermore, the torsion tensor as one of the geometric invariants is obtained, and it gives the geometrical interpretation to the chaotic behaviour of Chen system.

Finally, the torsion tensor of Chen system and Lorenz system are also compared. The obtaining results show that torsion tensor change will lead the Chen system from periodic to chaotic, which is not found in Lorenz system.

2010 Mathematics Subject Classification: 34A26; 34A34; 34C37; 53B40; 53C60

Keywords: Chen system, homoclinic orbit, torsion tensor, tangent bundle, chaos

1. INTRODUCTION

In the second half of the 20th century, nonlinear science has achieved unprecedented vigorous development, which is regarded as the "third revolution" in the history of natural science. Chaos as one of the subjects of nonlinear science has been always widely concerned, which has been widely applied in secure communication, biological science, engineering technology and other fields. The Lorenz system is the first classical chaos system that was proposed by the famous meteorologist E. N. Lorenz in 1963 [19], which has a fundamental significance in the development of chaos. Since then, a increase number of literature are devoted to the study of Lorenz system and Lorenz-like system (e.g. Lorenz system [19], Chen system [4], Lorenz system family [3]). Among them, the most peculiar is the Chen system with a similar

This work is supported by the National Natural Science Foundation of China (Grant No. 11961074), Natural Science Foundation of Guangxi Province (Grant No. 2018GXNSFDA281028), the High Level Innovation Team Program from Guangxi Higher Education Institutions of China (Document No. [2018] 35) and the Senior Talent Research Foundation of Yulin Normal University (Grant No. G2019ZK51).

but topologically nonequivalent to Lorenz system [4, 5, 23], which is the dual system of Lorenz system [3]. It was discovered by G.R. Chen in 1999 and has more complex topological structure and dynamic behaviour than Lorenz system. Chen system is given by

$$\begin{cases} \dot{x}_1 = a(x_2 - x_1) \\ \dot{x}_2 = (c - a)x_1 + cx_2 - x_1x_3 \\ \dot{x}_3 = x_1x_2 - bx_3. \end{cases} \quad (1.1)$$

where a, b, c are positive real numbers. When $a = 35, b = 3$ and $c = 28$, the corresponding typical numerical chaotic attractor is shown in Fig. 1.

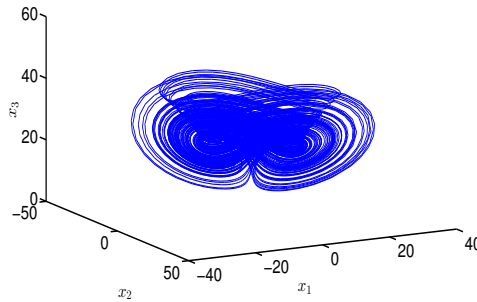


FIGURE 1. The chaotic attractor of Chen system (1.1) with $(a, b, c) = (35, 3, 28)$ and initial conditions $(x_1(0), x_2(0), x_3(0)) = (-10, 0, 37)$.

There have been already a sea of investigations and studies of Chen system (1.1), for example, bifurcation analyses [14, 20, 23], global analyses [17, 21, 30], the compound topological structure [16, 30], chaotic attractors [23, 31]. In particular, the literature [7], based on KCC-theory, revealed that the underlying chaotic evolution of Chen system by quantitatively described the behavior of the second KCC-invariant.

On the one hand, homoclinic orbits and heteroclinic orbits are important concepts in the study of the bifurcation of vector fields and chaos. Many chaotic behaviours of a complex system are related to the existence or nonexistence of these kinds of orbits in the system. In the literatures [10, 11, 13, 18], the existence of nontransverse homoclinic orbits of Chen system has been well proved. Using Fishing principle, which ideas are from [17], the obtained results showed that when $a > c$, Chen system either has homoclinic orbits with $2a > b$ or has no homoclinic orbits with $b > 2a$. But what happens when $c > a$, which is rarely reported. On the other hand, nowadays, differential geometry has become a practical tool in the study of the complexity of typical chaotic systems. Some good results have been reported in analysing the trajectory behaviour of chaotic systems from the viewpoint of tangent bundle. The literature

[22] has successfully described the chaotic behaviour of Lorenz system on the tangent space. In [27], the aperiodic behaviour of Rikitake system can be expressed by torsion tensor based on the geometrical unified theory. Moreover, the literature [28] shown that a torsion tensor as one of the geometrical invariants, relates to the chaotic behaviour characterized by the Rayleigh number, and results from the decomposition from the second order system to the tangent space (state space) and base space (configuration space), respectively. The above studies indicate that it is feasible and beneficial to discuss the dynamic behaviour of chaotic systems on the tangent bundle. To our knowledge, the dynamic analysis of the Chen system on the tangent bundle has rarely been studied in the literature.

In this paper, on the one hand, the existence of nontransverse homoclinic orbits in Chen system under specific parameter conditions with $c > a$ has been rigorously proven.

On the other hand, we mainly focus on analyzing the relationship between the characteristics of orbit of Chen system and the torsion tensor from the viewpoint of tangent bundle. According to geometric theory, the solution of Chen system can be regarded as a trajectory on the tangent bundle, from which we can calculate the geometric invariants of the Chen system. The advantage of this method is that it uses geometric invariants to qualitatively explore the chaos mechanism of the Chen system in 6+1 dimensional tangent space, and the results obtained are more detailed and intuitive than in original 3+1 dimensional Euclidean space. Moreover, the changing trend and degree of the system trajectory can be observed more clearly in the tangent space. Different from the way of parameter selection of Lorenz system [28], one has been considered selecting three regions near the classical value of Chen system to discuss the relationship between the change of torsion tensor and the dynamic behaviour. The differences of dynamics and torsion tensor between Lorenz system and Chen system has been explored, the obtained results has shown that Chen system has more complex dynamics than Lorenz system, which can be explained geometrically by the change of torsion tensor.

These results may give a contribution in an understanding of the geometric mechanism for chaos of the system.

The present paper is organized as follows. In Section 2, geometric definitions for tangent bundle and geometric quantities in the non-holonomic system are described. In Section 3, the existence of two nontransverse homoclinic orbits in Chen system is rigorously proved. In Section 4, the geometric quantities of Chen system are given and a relation between the chaotic behaviour of Chen system and the torsion tensor is discussed. Moreover, the torsion tensors of Lorenz and Chen systems are compared. In Section 5, the conclusion of this paper is given.

2. TANGENT BUNDLE VIEWPOINT OF THE CHEN SYSTEM

In this section, we introduce the related definitions, notions and formulas of the tangent bundle [28], including nonlinear connection, connection coefficient, torsion tensors, curvature tensors, et.al. One will need them in Section 4. In this article, Einstein's summation convention is used.

In the following, a non-holonomic system and a holonomic system are given by $(\mathbf{x})$ -field and $(\dot{\mathbf{p}})$ -field, respectively. In addition, the coordinate indices in the non-holonomic system is represented by $\alpha, \beta, \gamma, \delta, \varepsilon, \dots$, while the coordinate indexes in the holonomic system is represented by $i, j, k, \dots$.

2.1. Geometric definitions for tangent bundle

Let M be the m -dimensional C^k manifold, $T_p(M)$ be the tangent space of M at point p and mark $T(M) = \bigcup_{p \in M} T_p(M) = \{X_p \in T_p(M) \mid p \in M\}$. According to the definition in [1], $(T(M), \pi, M)$, or simply $T(M)$, is called the tangent bundle on M . π is called the natural mapping. $T_p(M)$ is the fiber of $T(M)$ at point p .

Firstly, the coordinate systems of holonomic system are defined on a tangent bundle, and the geometric descriptions are obtained. Then, the coordinates on a base manifold M and the fiber coordinates are denoted by (p^i) and $(\dot{p}^i) = (dp^i/dt)$, respectively. We also call that the manifold are spanned by the coordinates (p^i) and $(\dot{p}^i)$, which are regarded as $(\mathbf{p})$ -field and $(\dot{\mathbf{p}})$ -field, respectively. Then, $(p^i, \dot{p}^i)$ represents the coordinates that they are defined on the total space or the tangent bundle TM .

In order to illustrate the essence of the nonlinear connection, a definition is introduced as follows.

Definition 1 ([2]). A non-linear connection on the tangent bundle TM is a sub-bundle (HTM, τ_H, TM) of the tangent bundle (TTM, τ, TM) such that on fibers, we have

$$T_u T(M) = H_u TM \oplus V_u TM, \forall u \in TM, \quad (2.1)$$

where $H_u TM$ and $V_u TM$ are the horizontal distribution and the vertical distribution, respectively. $\oplus$ is the Whitney sum.

Thus, a non-linear connection N on TM induces the tangent space $T_u T(M)$ to produce as the direct sum decomposition of relation (2.1). For a given non-linear connection (HTM) or N , we always have a natural basis and its dual basis to adapt the relation (2.1). They are given as following

$$(dp^i, \delta \dot{p}^i) = (dp^i, dp^i + N_j^i dp^j), \quad (2.2)$$

$$\left(\frac{\delta}{\delta p^i}, \frac{\partial}{\partial \dot{p}^i} \right) = \left(\frac{\partial}{\partial p^i} - N_i^j \frac{\partial}{\partial p^j}, \frac{\partial}{\partial \dot{p}^i} \right), \quad (2.3)$$

where N_j^i is the component of the nonlinear connection.

According to the result in the literature [2], there is canonical isomorphism between two tangent spaces $T_p M$ and $T_q M$ at $p, q \in M$, which will be called a parallel transport it is equivalent to existing a linear connection on the manifold M . This means that we can define a parallel transport between $T_u TM$ and $T_v TM$ for two points $u, v \in TM$ that preserves the above decomposition of (2.1). The linear connection, corresponding to such a parallel transport, is called a Finsler connection (or a N-linear connection) on TM . These functions $N_j^i, F_{jk}^i, C_{jk}^i$ are called the local coefficients of a Finsler connection $D\Gamma = (F_{jk}^i, N_j^i, C_{jk}^i)$.

With respect to the adapted frames (2.2) and (2.3), the connection coefficients F_{jk}^i and C_{jk}^i are called horizontal and vertical connection coefficients, respectively [2], and are denoted by (F_{jk}^i, C_{jk}^i) on TM . These connections preserve the horizontal and the vertical distributions, and are given by the covariant derivative for an arbitrary vector field V^i [2]:

$$\begin{aligned} DV^i &= dV^i + V^j (F_{jk}^i dp^k + C_{jk}^i D\dot{p}^k). \\ D\dot{p}^i &= d\dot{p}^i + N_j^i dp^j. \end{aligned} \quad (2.4)$$

In this study, Chen system is regarded as a system of second order differential equations. Then, by using equations (2.7) and (2.8) in section 2.2, the second-order differential equations can be related to the Berwald connection[1]. Thus, in this article, one considers that the connection coefficients F_{jk}^i and C_{jk}^i of the Chen system are the Berwald connection. A Finsler connection $D\Gamma = (F_{jk}^i, N_j^i, C_{jk}^i)$ is reduced to a Berwald connection when $C_{jk}^i = 0$, and sometimes is represented by $(F_{jk}^i, N_j^i, 0)$. Moreover, in this case, F_{jk}^i is a function of p^i or constant.

2.2. Geometric quantities in the non-holonomic system

The equation of motion of the differential equation is given by

$$\frac{dp^i}{dt} = k_{(i)} \dot{p}^i, \quad (2.5)$$

$$\frac{d\dot{p}^i}{dt} + 2G^i(p^j, \dot{p}^j) = 0, \quad (2.6)$$

where p^i and $\dot{p}^i$ can be interpreted geometrically as the coordinates of the base space and tangent space, respectively. G^i is a smooth function and $k_{(i)}$ is a constant, where $k_{(i)} \dot{p}^i$ does not sum.

Introducing the non-holonomic transformation defined on TM as follows

$$p^\alpha = p^\alpha(p^i), \quad (2.7)$$

$$x^\alpha = X_i^\alpha \dot{p}^i, \quad \dot{p}^i = X_\alpha^i x^\alpha, \quad (2.8)$$

where $X_j^\alpha X_\beta^j = \delta_\beta^\alpha$, $X_\alpha^i = (X_1^i, X_2^i, X_3^i)$, and $X_i^\alpha = (X_i^1, X_i^2, X_i^3)$. Using equations (2.7) and (2.8), equations (2.5) and (2.6) are converted to

$$\frac{dp^i}{dt} = X_\alpha^i x^\alpha, \quad (2.9)$$

$$\frac{dx^\alpha}{dt} + 2\bar{G}^\alpha(p^\beta, x^\gamma) = 0. \quad (2.10)$$

Then, taking the derivation of both sides of Eq. (2.8) with respect to t , and comparing the obtained result with Eq. (2.10) to get

$$\bar{G}^\alpha = X_\alpha^i G^i - \frac{1}{2} \frac{\partial X_\alpha^i}{\partial p^j} p^i p^j. \quad (2.11)$$

The nonlinear connection and the connection coefficient are defined by the function $\bar{G}^\alpha$ as

$$\bar{N}_\beta^\alpha = \frac{\partial \bar{G}^\alpha}{\partial x^\beta}, \bar{F}_{\beta\gamma}^\alpha = \frac{\partial^2 \bar{G}^\alpha}{\partial x^\beta \partial x^\gamma}. \quad (2.12)$$

The paths of equations(2.9) and (2.10) are defined by

$$\frac{dx^\alpha}{dt} + \bar{N}_\beta^\alpha x^\beta + \bar{N}_0^\alpha = 0, \quad (2.13)$$

here, $\bar{N}_0^\alpha = 2\bar{G}^\alpha - \bar{N}_\beta^\alpha x^\beta$ is called the first invariant which indicates external force. If the nonlinear connection is first-order homogeneous in x^α , then the nonlinear connection becomes $\bar{N}_\beta^\alpha = \bar{F}_{\beta\gamma}^\alpha x^\gamma$. The torsion tensor as one of the geometric invariants is defined by

$$\bar{R}_{\beta\gamma}^\alpha = \frac{\delta \bar{N}_\beta^\alpha}{\delta p^\gamma} - \frac{\delta \bar{N}_\gamma^\alpha}{\delta p^\beta}. \quad (2.14)$$

According to the literature [1], similarly to the holonomic system, a Finsler connection $D\Gamma = (\bar{F}_{\beta\gamma}^\alpha, \bar{N}_\beta^\alpha, \bar{C}_{\beta\gamma}^\alpha)$ has only three local components $\bar{R}_{\delta\beta\gamma}^\alpha$, $\bar{P}_{\delta\beta\gamma}^\alpha$ and $\bar{S}_{\delta\beta\gamma}^\alpha$, which are given by

$$\begin{aligned} \bar{R}_{\delta\beta\gamma}^\alpha &= \frac{\delta \bar{F}_{\delta\beta}^\alpha}{\delta p^\gamma} - \frac{\delta \bar{F}_{\delta\gamma}^\alpha}{\delta p^\beta} + \bar{F}_{\delta\beta}^\epsilon \bar{F}_{\epsilon\gamma}^\alpha - \bar{F}_{\delta\gamma}^\epsilon \bar{F}_{\epsilon\beta}^\alpha + \bar{C}_{\delta\epsilon}^\alpha \bar{R}_{\beta\gamma}^\epsilon, \\ \bar{P}_{\delta\beta\gamma}^\alpha &= \frac{\partial \bar{F}_{\delta\beta}^\alpha}{\partial x^\gamma} - \frac{\delta \bar{C}_{\delta\gamma}^\alpha}{\delta p^\beta} + \bar{F}_{\delta\beta}^\epsilon \bar{C}_{\epsilon\gamma}^\alpha + \bar{F}_{\gamma\beta}^\epsilon \bar{C}_{\delta\epsilon}^\alpha + \bar{C}_{\delta\epsilon}^\alpha \bar{P}_{\alpha\gamma}^\epsilon, \\ \bar{S}_{\delta\beta\gamma}^\alpha &= \frac{\partial \bar{C}_{\delta\beta}^\alpha}{\partial x^\gamma} - \frac{\partial \bar{C}_{\delta\gamma}^\alpha}{\partial x^\beta} + \bar{C}_{\delta\beta}^\epsilon \bar{C}_{\epsilon\gamma}^\alpha - \bar{C}_{\delta\gamma}^\epsilon \bar{C}_{\epsilon\beta}^\alpha. \end{aligned}$$

When $\bar{C}_{\beta\gamma}^\alpha = 0$ in a Finsler connection $D\Gamma = (\bar{F}_{\beta\gamma}^\alpha, \bar{N}_\beta^\alpha, \bar{C}_{\beta\gamma}^\alpha)$, this Finsler connection is reduced to a Berwald connection. Therefore, the Berwald connection $D\Gamma =$

$(\bar{F}_{\beta\gamma}^\alpha, \bar{N}_\beta^\alpha, 0)$ has only two nonzero components of curvature $\bar{P}_{\delta\beta\gamma}^\alpha$ and $\bar{R}_{\delta\beta\gamma}^\alpha$, and these are given by

$$\bar{P}_{\delta\beta\gamma}^\alpha = \frac{\partial \bar{F}_{\delta\beta}^\alpha}{\partial x^\gamma}, \quad (2.15)$$

$$\bar{R}_{\delta\beta\gamma}^\alpha = \frac{\delta \bar{F}_{\delta\beta}^\alpha}{\delta p^\gamma} - \frac{\delta \bar{F}_{\delta\gamma}^\alpha}{\delta p^\beta} + \bar{F}_{\delta\beta}^\epsilon \bar{F}_{\epsilon\gamma}^\alpha - \bar{F}_{\delta\gamma}^\epsilon \bar{F}_{\epsilon\beta}^\alpha. \quad (2.16)$$

3. NONTRANSVERSE HOMOCLINIC ORBITS

In this section, one investigates homoclinic orbits in Chen system (1.1) within a specific parameter range by the generalized Melnikov method, which was developed by Wiggins and Holmes [25]. It's important to point out that a rigorous proof that the existence of periodic orbits and homoclinic chaos in the periodic forced Chen system under certain parameter conditions has been studied in literature [6]. There are similar results in Lorenz system [15], the diffusionless Lorenz system [24] and a special system with two stable focal points [29], etc. It should be pointed out that these focus on the case that $c > a$, which is different from the previous rich results [10, 11, 13, 17, 18] that $c < a$.

Under the transformation $z \rightarrow z - a + c$, system (1.1) becomes

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = -xz + cy \\ \dot{z} = xy - bz - b(c - a). \end{cases} \quad (3.1)$$

Introducing the rescaling $x \rightarrow \frac{ax}{\epsilon}$, $y \rightarrow \frac{ay}{-\epsilon^2}$, $z \rightarrow \frac{az}{-\epsilon^2}$, $t \rightarrow \epsilon t$, $\epsilon = \frac{a}{\sqrt{b(c-a)}}$, system (3.1) becomes

$$\begin{cases} \dot{x} = -y - \epsilon x \\ \dot{y} = -xz + \epsilon \frac{c}{a} y \\ \dot{z} = xy + \epsilon \left(-\frac{b}{a} z + 1\right). \end{cases} \quad (3.2)$$

To ensure the sufficient smallness of ϵ , one studies this dependency here. When the parameters $c \gg a$, one obtains that $0 < \epsilon \ll 1$. Thus, the approximate homoclinic orbits are obtained as analytical solutions of the Chen system at the value $\epsilon = 0$. Moreover, unlike the Lorenz system, which is known to be Levinson dissipative (i.e there exists a global bounded absorbing set containing global Lorenz attractor) for all positive values of parameters [9], the Chen system is Levinson dissipative only when $a + b > c > 0$ [12] and in general could have unbounded trajectories tending to infinity. The dissipative condition of the Chen system should be taken into account when studying possible existence of homoclinic orbits, which is a crucial difference between Lorenz system and Chen system. When the parameters $c \gg a$, Chen system is not dissipative in the sense of Levinson dissipative, In other words, Chen system has unbounded trajectories tending to infinity. That is, the above definition of ϵ is reasonable.

When $\varepsilon = 0$, system (3.2) can be seen as a three dimensional generalized Hamiltonian system

$$\frac{d}{dt} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & -y \\ -z & 0 & 0 \\ y & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ 0 \\ 1 \end{pmatrix} = J \begin{pmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial y} \\ \frac{\partial H}{\partial z} \end{pmatrix}$$

with Hamiltonian function

$$H(x, y, z) = z + \frac{x^2}{2} = A$$

and Casimir function

$$C(x, y, z) = y^2 + z^2 = r^2.$$

Let $r > 0$ and make a polar-coordinate transformation

$$\begin{cases} x = x \\ y = r \cos(\theta - \frac{\pi}{2}) \\ z = r \sin(\theta - \frac{\pi}{2}). \end{cases} \quad (3.3)$$

Then, system (3.2) is transformed into

$$\begin{cases} \dot{\theta} = x + \varepsilon(\frac{c}{a} \sin \theta \cos \theta + \frac{b}{a} \sin \theta \cos \theta + \frac{\sin \theta}{r}) \\ \dot{x} = -r \sin \theta - \varepsilon x \\ \dot{r} = \varepsilon(\frac{rc}{a} \sin^2 \theta - \frac{br}{a} \cos^2 \theta - \cos \theta). \end{cases} \quad (3.4)$$

Then, the following conclusion holds.

Theorem 1. *Let a, b, c are positive real numbers and $3a + c > b/2$. If $c > a$, for ε sufficiently small near the two homoclinic orbits $\Gamma_{h\pm}$ of system $(3.4)_{\varepsilon=0}$, system (3.4) possesses two nontransverse homoclinic orbits near $y^2 + z^2 = r_*^2$, where*

$$r_* = \frac{3a}{2(3a + c) - b}.$$

Proof. System $(3.4)_{\varepsilon=0}$ is a Hamiltonian system, and its Hamiltonian function is $H(x, r, \theta)$, where $H(x, r, \theta) = \frac{x^2}{2} - r \cos \theta = A$. Therefore, when $A = r$, the two homoclinic orbits $\Gamma_{h\pm}^1$ of the system $(3.4)_{\varepsilon=0}$ are connected to the saddle point $(r, \pi, 0)$, and the parameter expression is

$$\begin{cases} \theta_h(t) = \pm 2 \arctan(\sinh \sqrt{r}t) \\ x_h(t) = \pm 2\sqrt{r}(\operatorname{sech} \sqrt{r}t). \end{cases} \quad (3.5)$$

From (3.4) and (3.5), Melnikov function of (3.4)

$$\begin{aligned} M(r) = & \int_{-\infty}^{+\infty} \left[-x_h^2(t) + \frac{br}{a} \sin^2 \theta_h(t) \cos \theta_h(t) + \sin^2 \theta_h(t) + \cos^2 \theta_h(t) \right] dt \\ & + \int_{-\infty}^{+\infty} \left[\frac{br}{a} \cos^3 \theta_h(t) - \frac{cr}{a} \sin^2 \theta_h(t) + \cos \theta_h(t) + \frac{br}{a} \cos^2 \theta_h(t) \right] dt \end{aligned}$$

$$= -8\sqrt{r} + \frac{4b\sqrt{r}}{3a} - \frac{8c\sqrt{r}}{3a} + \frac{4}{\sqrt{r}}.$$

Thus, $M(r_*) = 0$, $\frac{\partial M}{\partial r} \big|_{r=r_*} \neq 0$, where $r = r_* = \frac{3a}{2(3a+c)-b}$. For $3a + c > b/2$, the results can be obtained from Lemma 4.2 in the literature [25]. Namely, for ε sufficiently small near the two homoclinic orbits $\pm \Gamma_{h_{\pm}}^1$ of system $(3.4)_{\varepsilon=0}$, system (3.4) possesses two non-transverse homoclinic orbits. $\square$

Remark 1. From [15], it is known that if $r \gg \sigma$, near the two homoclinic orbits $\Gamma_{h_{\pm}}^1$ of the perturbed system $(2.5)_{\varepsilon=0}$ of the Lorenz system on the phase cylinder $w^2 + z^2 = 1$, there exist two homoclinic orbits $\Gamma_{h_{\pm}}^1$ connecting the saddle point $(1, 0, 0)$ of $(2.5)_{\varepsilon=0}$ when $\sigma = \frac{1}{3}$. In this paper, the condition for the existence of non-transverse homoclinic orbit in the Chen system is stronger than that in the Lorenz system, which can be revealed from the fact that the dissipation condition of the Chen system is stronger than that of the Lorenz system.

4. CHAOTIC BEHAVIOR AND GEOMETRIC QUANTITIES

In this section, from the geometric viewpoint of tangent bundle, we analyze the chaotic complexity of Chen system. The relevant preliminary is briefly introduced in Section 2. These may give a contribution in an understanding of the geometric mechanism for chaotic systems.

4.1. Convert to a second order differential equation

Rewrite Chen system into a set of second order differential equations. Firstly, system (2.8) is equivalent to

$$\begin{cases} \dot{x}^1 = X_1^1 \dot{p}^1 + X_2^1 \dot{p}^2 + X_3^1 \dot{p}^3 = \dot{p}^1 \\ \dot{x}^2 = X_1^2 \dot{p}^1 + X_2^2 \dot{p}^2 + X_3^2 \dot{p}^3 = \dot{p}^2 \\ \dot{x}^3 = X_1^3 \dot{p}^1 + X_2^3 \dot{p}^2 + X_3^3 \dot{p}^3 = \dot{p}^3. \end{cases} \quad (4.1)$$

where $X_j^\alpha X_\beta^j = \delta_\beta^\alpha$. Secondly, making the following transformation to system (1.1)

$$x_1 \rightarrow x^1, x_2 \rightarrow x^2, x_3 \rightarrow x^3. \quad (4.2)$$

Then system (1.1) can be rewritten as

$$\begin{cases} \ddot{p}^1 = a(\dot{p}^2 - \dot{p}^1) \\ \ddot{p}^2 = (c - a)\dot{p}^1 + c\dot{p}^2 - \dot{p}^1\dot{p}^3 \\ \ddot{p}^3 = \dot{p}^1\dot{p}^2 - b\dot{p}^3. \end{cases} \quad (4.3)$$

From (4.1), one sees that system (4.3) is equivalent to

$$\begin{cases} \dot{x}^1 = a(x^2 - x^1) \\ \dot{x}^2 = (c - a)x^1 + cx^2 - x^1x^3 \\ \dot{x}^3 = x^1x^2 - bx^3. \end{cases} \quad (4.4)$$

Introduce a linear substitution of variables to system (4.4)

$$x^1 \rightarrow x^1, x^2 \rightarrow x^2, x^3 \rightarrow x^3 + c. \quad (4.5)$$

Then system (4.4) can be rewritten as follows

$$\begin{cases} \frac{dx^1}{dt} = ax^2 - ax^1 \\ \frac{dx^2}{dt} = -ax^1 + cx^2 - x^1x^3 \\ \frac{dx^3}{dt} = x^1x^2 - bx^3 - bc. \end{cases} \quad (4.6)$$

Note that x^1, x^2, x^3 are defined in the tangent space, and $x^1 = \dot{p}^1, x^2 = \dot{p}^2, x^3 = \dot{p}^3$.

According to literature [8], one sets principle moments of inertia for system (4.6) as follows

$$I = \text{diag}(I_1, I_2, I_3) = \text{diag}(1, (1 + e_3)^{-1}, (1 + e_1 + e_3)^{-1}) = \text{diag}\left(1, \frac{1}{2}, \frac{1}{2}\right).$$

Hamiltonian function are defined as $H = K + U$, $K = \frac{1}{2}((x^1)^2 + 2(x^2)^2 + 2(x^3)^2)$, $U = ax^3$. The Lie-Poisson brackets are given by

$$\{x, H\} = x \times \nabla H = \begin{pmatrix} ax^2 \\ -x^1x^3 - ax^1 \\ x^1x^2 \end{pmatrix}.$$

Therefore, system (4.6) can be described as the Kolmogorov system. The specific form of the Kolmogorov system is rewritten from Chen system as follows

$$\dot{x} = \begin{pmatrix} \dot{x}^1 \\ \dot{x}^2 \\ \dot{x}^3 \end{pmatrix} = \begin{pmatrix} ax^2 \\ -x^1x^3 - ax^1 \\ x^1x^2 \end{pmatrix} - \begin{pmatrix} ax^1 \\ -cx^2 \\ bx^3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -bc \end{pmatrix} = \{x, H\} - \Lambda x + f,$$

where $\Lambda = \text{diag}\{a, -c, b\}$ and $f = (0, 0, -bc)^T$.

4.2. Geometric quantities

Chen system has been rewritten as the Lie-Poisson equation in Section 4.1, which means that the Chen system is a non-holonomic system with torsion tensor. Similar to Lorenz system [28], Chen system can be regarded as a unified system of the tangent space and the base space.

By comparing equation (2.10) with equations (4.6), Chen system can be given by three equations of the form

$$\frac{dx^\alpha}{dt} + 2\bar{G}^\alpha(p^\beta, x^\gamma) = 0, \quad (\alpha, \beta, \gamma = 1, 2, 3),$$

where

$$\bar{G}^1 = \frac{1}{2}(ax^1 - ax^2), \bar{G}^2 = \frac{1}{2}(x^1x^3 + ax^1 - cx^2)$$

and

$$\bar{G}^3 = \frac{1}{2}(-x^1x^2 + bx^3 + bc).$$

Therefore, the nonlinear connection and the connection coefficient are given by

$$\begin{aligned} \bar{N}_1^1 &= \frac{1}{2}a, \bar{N}_2^1 = -\frac{1}{2}a, \bar{N}_3^1 = 0, \bar{N}_1^2 = \frac{1}{2}(x^3 + a), \bar{N}_2^2 = -\frac{1}{2}c, \\ \bar{N}_3^2 &= \frac{1}{2}x^1, \bar{N}_1^3 = -\frac{1}{2}x^2, \bar{N}_2^3 = -\frac{1}{2}x^1, \bar{N}_3^3 = \frac{1}{2}b \end{aligned} \quad (4.7)$$

and

$$\begin{aligned} \bar{F}_{11}^2 &= \bar{F}_{12}^2 = \bar{F}_{32}^2 = \bar{F}_{33}^2 = 0, \bar{F}_{11}^3 = \bar{F}_{11}^3 = \bar{F}_{13}^3 = \bar{F}_{23}^3 = \bar{F}_{22}^3 = 0, \bar{F}_{\beta\gamma}^1 = 0, \\ \bar{F}_{2\gamma}^2 &= 0, \bar{F}_{3\gamma}^3 = 0, (\beta, \gamma = 1, 2, 3), \bar{F}_{13}^2 = \bar{F}_{31}^2 = \frac{1}{2}, \bar{F}_{12}^3 = \bar{F}_{21}^3 = -\frac{1}{2}, \end{aligned} \quad (4.8)$$

respectively. The first invariants of Chen system are obtained as follows

$$\bar{N}_0^1 = \frac{1}{2}(ax^1 - ax^2), \quad \bar{N}_0^2 = \frac{1}{2}(ax^1 - cx^2), \quad \bar{N}_0^3 = bc - \frac{1}{2}bx^3.$$

Just to make it easier to compute the torsion tensor, we firstly give the following result.

Theorem 2. *The form of $\bar{G}^\alpha$ in Chen system is given by*

$$\bar{G}^\alpha = \bar{F}_{\beta\gamma}^\alpha x^\beta x^\gamma + \Lambda_\beta^\alpha x^\beta + f^\alpha, \quad (4.9)$$

where Λ_β^α and f^α represent the dissipation term and external force, respectively.

Proof. Noting that $\alpha = 1$, $\bar{N}_1^1 = \frac{1}{2}a$, $\bar{N}_2^1 = -\frac{1}{2}a$, $\bar{N}_3^1 = 0$, $\bar{F}_{\beta\gamma}^1 = 0$, $(\beta, \gamma = 1, 2, 3)$. The first equation of the system (4.6) can be seen as

$$\frac{dx^1}{dt} + 2\bar{F}_{\beta\gamma}^1 x^\beta x^\gamma + 2\bar{N}_1^1 x^1 - 2\bar{N}_2^1 x^2 + 2f^1 = 0.$$

Further,

$$\frac{dx^1}{dt} + 2\bar{F}_{\beta\gamma}^1 x^\beta x^\gamma + 2\Lambda_1^1 x^1 + 2\Lambda_2^1 x^2 + 2\Lambda_3^1 x^3 + 2f^1 = 0,$$

where $\Lambda_1^1 = \bar{N}_1^1$, $\Lambda_2^1 = -\bar{N}_2^1$, $\Lambda_3^1 = 0$, $f^1 = 0$. Consequently,

$$\frac{dx^1}{dt} + 2(\bar{F}_{\beta\gamma}^1 x^\beta x^\gamma + \Lambda_\beta^1 x^\beta + f^1) = 0.$$

From equation (2.10), one obtains $\bar{G}^1 = \bar{F}_{\beta\gamma}^1 + \Lambda_\beta^1 x^\beta + f^1$.

When $\alpha = 2$ or 3 , similarly results are obtained as follows

$$\frac{dx^2}{dt} + 2(\bar{F}_{\beta\gamma}^2 x^\beta x^\gamma + \Lambda_\beta^2 x^\beta + f^2) = 0$$

and

$$\frac{dx^3}{dt} + 2(\bar{F}_{\beta\gamma}^3 x^\beta x^\gamma + \Lambda_\beta^3 x^\beta + f^3) = 0,$$

respectively. Therefore, we obtain $\bar{G}^\alpha$ in Chen system as the following form

$$\bar{G}^\alpha = \bar{F}_{\beta\gamma}^\alpha x^\beta x^\gamma + \Lambda_\beta^\alpha x^\beta + f^\alpha.$$

□

From the above Theorem 2, formula (2.14) can be rewritten as follows

$$\bar{R}_{\beta\gamma}^\alpha = \bar{N}_\beta^\delta \bar{F}_{\delta\gamma}^\alpha - \bar{N}_\gamma^\delta \bar{F}_{\delta\beta}^\alpha. \quad (4.10)$$

The curvature tensors of Chen system are given by

$$\bar{P}_{\delta\beta\gamma}^\alpha = \frac{\partial \bar{F}_{\delta\beta}^\alpha}{\partial x^\gamma} = 0, \quad (\alpha, \beta, \delta, \gamma = 1, 2, 3).$$

From equation (2.16), another the curvature tensors of the Chen system are obtained:

$$\bar{R}_{\delta\beta\gamma}^\alpha = \begin{cases} 0, & \text{for } \delta, \beta, \gamma = 1, 2, 3, \text{ and } \alpha = 1; \\ -\frac{1}{4}, & \text{for } \delta = 1, \beta = 2, \gamma = 1, \text{ and } \alpha = 2; \\ 0, & \text{for } \delta \neq 1, \beta \neq 2, \gamma \neq 1, \text{ and } \alpha = 2; \\ -\frac{1}{4}, & \text{for } \delta = 1, \beta = 3, \gamma = 1, \text{ and } \alpha = 3; \\ 0, & \text{for } \delta \neq 1, \beta \neq 3, \gamma \neq 1, \text{ and } \alpha = 3. \end{cases}$$

4.3. Discuss the chaotic behavior and torsion

In this subsection, the relations between the chaotic behaviour and the torsion for the Chen system are discussed on the base space and the tangent space, respectively. Some good results have been reported in discussing the relationship between the torsion and the trajectory behaviour of dynamical systems in the base space [26–28].

Firstly, one analyses Chen system on the base space. From equation (2.13), the torsion and nonlinear connection with the first invariant can describe the discrepancy of Chen system as follows

$$\Delta x^\alpha = \oint_c dx^\alpha = - \oint_c (\bar{N}_\beta^\alpha X_i^\beta dp^i + \bar{N}_0^\alpha dt) = - \oint_c \bar{N}_\beta^\alpha X_i^\beta dp^i \neq 0, \quad (4.11)$$

where Δx^α expresses the discrepancy along the trajectory $c = (x^\alpha(t))$ in the $(\mathbf{x})$ -field. Here, we put $(p^i) = (p^i, p^0) = (p^i, t)$, $X_i^\alpha = (X_i^\alpha, X_0^\alpha) = (X_i^\alpha, 1)$ and $X_0^\alpha = X_i^0 = 0$. Therefore, using the Stokes' theorem, relation (4.11) can be rewritten as

$$\Delta x^\alpha = - \oint_c \bar{N}_\beta^\alpha X_i^\beta dp^i = \frac{1}{2} \iint_S \bar{R}_{\beta\gamma}^\alpha dp^\beta \wedge dp^\gamma \neq 0, \quad (4.12)$$

where S is a region inside the trajectory c . Here, we put $(p^{\hat{\beta}}) = (p^{\beta}, t)$,

$$\bar{R}_{0\beta}^{\alpha} = \frac{\delta \bar{N}_0^{\alpha}}{\delta p^{\beta}} - \frac{\delta \bar{N}_{\beta}^{\alpha}}{\delta p^0}, \text{ and } \frac{\delta}{\delta p^0} = \frac{\partial}{\partial t} - \bar{N}_0^{\gamma} \frac{\partial}{\partial x^{\gamma}}. \quad (4.13)$$

Thus, the torsion tensor $\bar{R}_{\beta\gamma}^{\alpha}$ on the base spaces expresses the discrepancy which implies the trajectory of Chen system is not periodic.

In order to do better next analysis, the components of the torsion tensor are calculated as follows.

Theorem 3. *If the torsion tensor $\bar{R}_{32}^2$ satisfy $a \neq 0$, then the torsion tensor $\bar{R}_{\beta\gamma}^{\alpha}$ of Chen system does not vanish.*

Proof. Combine equations (4.7), (4.8) and (4.10), one can calculate that $\bar{R}_{\beta\gamma}^1 = \bar{N}_{\beta}^{\delta} \bar{F}_{\delta\gamma}^1 - \bar{N}_{\gamma}^{\delta} \bar{F}_{\delta\beta}^1 = 0$ holds for all $\delta, \beta, \gamma \in \{1, 2, 3\}$.

Similarly, one can obtain that $\bar{R}_{\beta\gamma}^2 = \bar{N}_{\beta}^{\delta} \bar{F}_{\delta\gamma}^2 - \bar{N}_{\gamma}^{\delta} \bar{F}_{\delta\beta}^2$. From $\bar{N}_1^3 \bar{F}_{32}^2 - \bar{N}_2^3 \bar{F}_{31}^2 = 0$, $(-\frac{1}{2}x^2) - (-\frac{1}{2}x^1) \cdot \frac{1}{2} = \frac{1}{4}x^1$ and $\bar{N}_1^{\delta} \bar{F}_{\delta 2}^2 - \bar{N}_2^{\delta} \bar{F}_{\delta 1}^2 = 0$ for $\delta \in \{1, 2\}$, one obtains that $\bar{R}_{12}^2 = \frac{1}{4}x^1$. The others can be given by $\bar{R}_{13}^2 = \frac{1}{4}(a - b)$ and $\bar{R}_{32}^2 = \frac{1}{4}a$.

From $\bar{R}_{\beta\gamma}^3 = \bar{N}_{\beta}^{\delta} \bar{F}_{\delta\gamma}^3 - \bar{N}_{\gamma}^{\delta} \bar{F}_{\delta\beta}^3$, one gains that $\bar{N}_1^2 \bar{F}_{23}^3 - \bar{N}_3^2 \bar{F}_{21}^3 = \frac{1}{4}x^1$ and $\bar{R}_{\beta\gamma}^3 = \bar{N}_{\beta}^{\delta} \bar{F}_{\delta\gamma}^3 - \bar{N}_{\gamma}^{\delta} \bar{F}_{\delta\beta}^3 = 0$ for $\delta \in \{1, 3\}$, then, $\bar{R}_{13}^3 = \frac{1}{4}x^1$. One also acquires that $\bar{N}_2^1 \bar{F}_{11}^3 - \bar{N}_1^1 \bar{F}_{12}^3 = \frac{1}{4}a$, $\bar{N}_1^2 \bar{F}_{23}^3 - \bar{N}_3^2 \bar{F}_{21}^3 = \frac{1}{4}x^1$ and $\bar{N}_2^3 \bar{F}_{31}^3 - \bar{N}_1^3 \bar{F}_{32}^3 = 0$, thus, $\bar{R}_{21}^3 = \frac{1}{4}(a + c)$. From the above analysis, the torsion tensor $\bar{R}_{\beta\gamma}^{\alpha}$ does not disappear because of $\bar{R}_{32}^2 = \frac{1}{4}a \neq 0$ ($a \neq 0$). $\square$

Secondly, one will discuss a relation between the dynamics behavior of Chen system and the torsion under three sets parameter conditions by means of the numerical study. Based on the values of the components of the torsion tensor of under each parameter condition, the corresponding Lyapunov exponent spectrum and bifurcation diagram are drawn to describe complex dynamics characteristics of Chen system. After transformation (4.5), the first two components of one of the non-zero equilibrium points of the original Chen system (1.1) and Chen system (4.6) are the same, i.e., $(x^1, x^2) = (\bar{x}^1, \bar{x}^1) = (\sqrt{b(2c-a)}, \sqrt{b(2c-a)})$. So we know from our previous calculations that the torsion tensor doesn't change under coordinate transformation (4.5). In other words, studying the relationship between the dynamics of Chen system (1.1) and the torsion tensor is equivalent to studying the Chen system (4.6). Next, we only consider the torsion around at $(x^1, x^2) = (\bar{x}^1, \bar{x}^1) = (\sqrt{b(2c-a)}, \sqrt{b(2c-a)})$ of Chen system (1.1) in the tangent space. From Theorem 3, one knows that $\bar{R}_{12}^2 = \bar{R}_{13}^3 = \frac{1}{4}x^1$. For brevity, we remark that $4\bar{R}_{\beta\gamma}^{\alpha} = \hat{R}_{\beta\gamma}^{\alpha}$.

It should be noted that the three set of parameter values considered below are all Levinson dissipative, i.e., all satisfying the parameters $a + b > c > 0$ in the Chen system.

First case, fixing the parameters $a = 35$, $b = 3$ and letting $c \in [20, 28]$ in Chen system, the values of the components of the torsion tensor with the change of the parameter c are obtained, as shown in Fig. 2. It reveals that as $\hat{R}_{12}^2$, $\hat{R}_{13}^3$ and $\hat{R}_{21}^3$ go up, $\hat{R}_{13}^2$ and $\hat{R}_{32}^2$ remain unchanged with increasing the value of parameter c . And it can also be seen intuitively that the torsion $\hat{R}_{12}^2$, $\hat{R}_{13}^3$ and $\hat{R}_{21}^3$ are proportional to the value of parameter c .

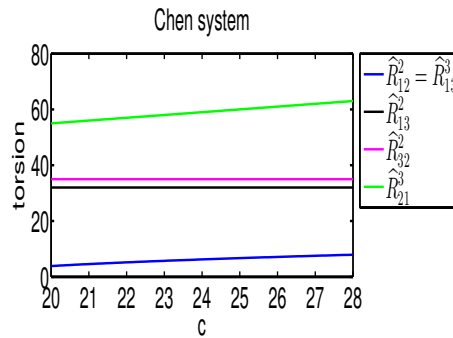


FIGURE 2. The values of the components of the torsion tensor of Chen system (1.1) with $a = 35$, $b = 3$ and $c \in [20, 28]$.

In Figs. 4, 6 and 8, computation of the Lyapunov exponent spectrums use wolf algorithm, the period of time is 0.1, start and finish values of time are 0 and 4000, respectively. When $c \in [20, 28]$, the corresponding Lyapunov exponent spectrum and the bifurcation diagram of Chen system are shown in Figs. 3(a) and 3(b).

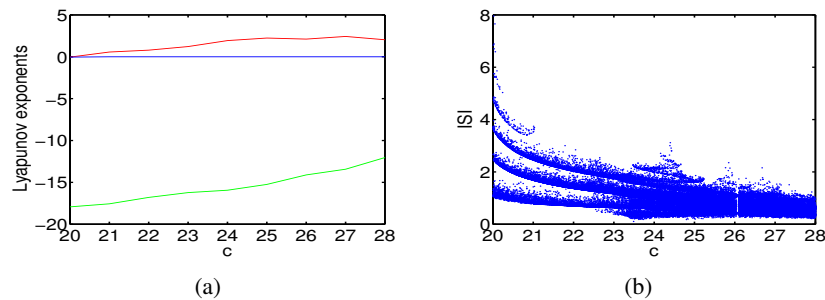


FIGURE 3. (a) Lyapunov exponent spectrum with the initial conditions $(x^1(0), x^2(0), x^3(0)) = (1.3, 1.3, 0.038)$ and (b) bifurcation diagram with the initial conditions $(x^1(0), x^2(0), x^3(0)) = (1.15, 3.5, 3.3)$ for Chen system with $a = 35$, $b = 3$ and $c \in [20, 28]$.

It can be observed that the bifurcation diagram coincides with the spectrum of Lyapunov exponents. Fig. 3(a) shows that Chen system evolves from the periodic to

chaotic for a very wide range of c . Combining Figs. 3(a) and 3(b), the dynamical behaviors of Chen system can be clearly observed. When $c = 20$, the largest Lyapunov exponents is almost equal zero, which implies that Chen system is periodic or quasi-periodic. When $c \in (20, 28]$, the largest Lyapunov exponents is positive, which means that Chen system is chaotic.

Second case, fixing the parameters $a = 35$, $c = 28$, and letting $b \in [3, 15]$, the values of the components of the torsion tensor with the change of the parameter b are given, as shown in Fig. 4. It displays that as $\hat{R}_{12}^2$ and $\hat{R}_{13}^3$ go up, $\hat{R}_{13}^2$ goes down, and $\hat{R}_{13}^2, \hat{R}_{32}^2$ stay the same with increasing the value of b . And one can see that $\hat{R}_{12}^2$ and $\hat{R}_{13}^3$ are proportional to parameter b but $\hat{R}_{13}^2$ is inversely proportional to parameter b .

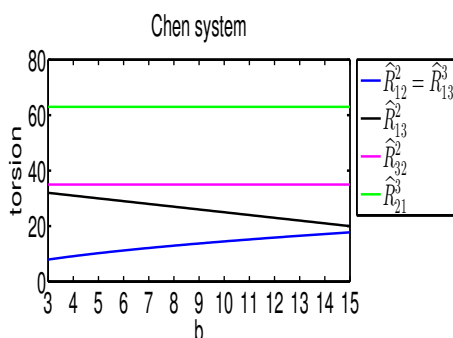


FIGURE 4. The values of the components of the torsion tensor of Chen system with $a = 35$, $c = 28$ and $b \in [3, 15]$.

When $b \in [3, 15]$, the corresponding Lyapunov exponent spectrum and the bifurcation diagram of Chen system are shown in Figs. 5(a) and 5(b). It can be observed that the bifurcation diagram coincides with the spectrum of Lyapunov exponents. Fig. 5(a) shows that Chen system evolves from the periodic to chaotic for a very wide range of b . Combining Figs. 5(a) and 5(b), the dynamical behaviors of Chen system can be clearly observed. When $b \in [4, 15]$, the largest Lyapunov exponents is almost equal zero, which implies that Chen system is periodic or quasi-periodic. The Chen system is chaotic in the sense that the largest Lyapunov exponents is positive when $b \in [3, 4)$.

Third case, fixing the parameters $b = 3$, $c = 28$ in Chen system and letting $a \in [30, 40]$ varies, the values of the components of the torsion tensor with the change of the parameter a are obtained, as shown in Fig. 6. It shows that $\hat{R}_{12}^2$ and $\hat{R}_{13}^3$ go down slowly and others go up quickly with increasing the value of a . And it can also be seen intuitively that the torsion $\bar{R}_{\beta\gamma}^\alpha$ has a linear relationship with the parameter a .

When $a \in [30, 40]$, the corresponding Lyapunov exponent spectrum and the bifurcation diagram of Chen system are shown in Figs. 7(a) and 7(b). It can be observed

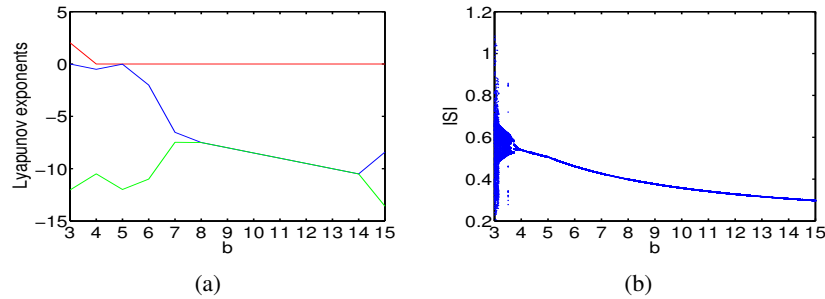


FIGURE 5. (a) Lyapunov exponent spectrum with the initial conditions $(x^1(0), x^2(0), x^3(0)) = (1.3, 1.3, 0.038)$ and (b) bifurcation diagram with the initial conditions $(x^1(0), x^2(0), x^3(0)) = (1.15, 3.5, 3.3)$ for Chen system with $a = 35$, $c = 28$ and $b \in [3, 15]$.

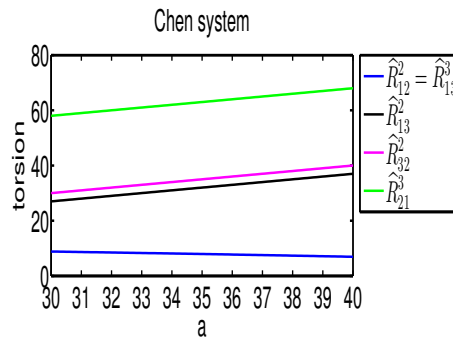


FIGURE 6. The values of the components of the torsion tensor of Chen system with $b = 3$, $c = 28$ and $a \in [30, 40]$.

that the bifurcation diagram coincides with the spectrum of Lyapunov exponents. Fig. 7(a) shows that Chen system evolves from the periodic to chaotic for a very wide range of b . Combining Figs. 7(a) and 7(b), the dynamical behaviours of Chen system can be clearly observed. When $a \in [30, 34.3]$ and $a = 37.4$, the largest Lyapunov exponents is almost equal zero, which implies that Chen system is periodic or quasi-periodic. When $a \in (34.3, 37.4) \cup (37.4, 40]$, the largest Lyapunov exponents is positive, which means that Chen system is chaotic.

Therefore, basing on Figs. 3, 5, and 7 and then fixing

$$(a, b, c) = (35, 3, 20), (33, 3, 28), (35, 10, 28), (35, 3, 28),$$

respectively, the phase diagrams of Chen system are drawn as in Figs. 8, 9 and 10. These figures display that the behaviour of Chen system evolves from the periodic to chaotic coincide with the change of torsion.

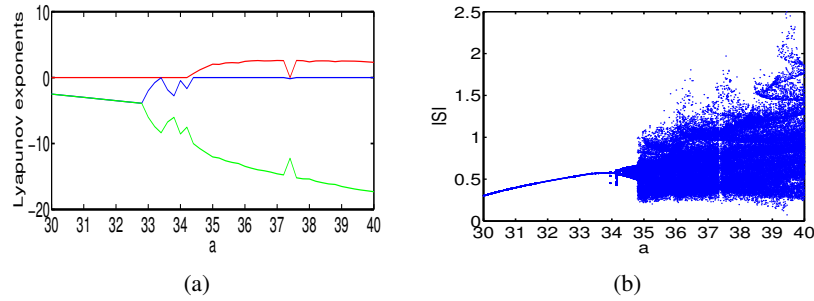


FIGURE 7. (a) Lyapunov exponent spectrum with the initial conditions $(x^1(0), x^2(0), x^3(0)) = (1.3, 1.3, 0.038)$ and (b) bifurcation diagram with the initial conditions $(x^1(0), x^2(0), x^3(0)) = (1.15, 3.5, 3.3)$ for Chen system with $b = 3$, $c = 28$ and $a \in [30, 40]$.

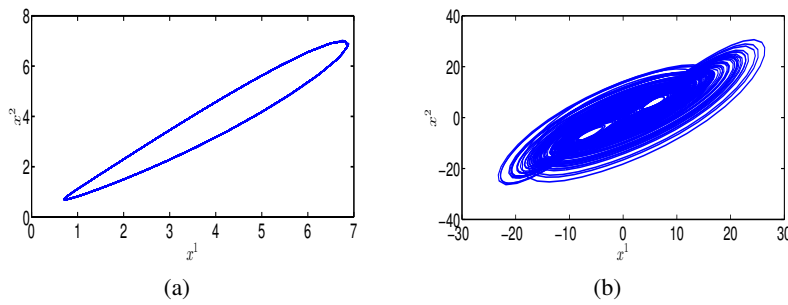


FIGURE 8. Trajectory of Chen system with $(a, b) = (35, 3)$ and the initial values $(x^1(0), x^2(0), x^3(0)) = (-2, 3, 8)$: (a) periodic for $c = 20$ and the corresponding the components of the $\hat{R}_{12}^2 = \hat{R}_{13}^3 \approx 3.9$; (b) chaos for $c = 28$ and the corresponding the components of the $\hat{R}_{12}^2 = \hat{R}_{13}^3 \approx 7.9$.

In conclusion, these figures numerically show that the complex dynamic behavior of the Chen system is expressed by the change of torsion tensor which change with the parameters, and among them $\hat{R}_{12}^2, \hat{R}_{13}^3$ are obtained in the $(\mathbf{x})$ -field. That is to say, the torsion tensor $\hat{R}_{\beta\gamma}^\alpha$ gives the geometrical interpretation of the complex dynamic behavior of Chen system.

In order to better explore the complexity of dynamics and torsion tensor between Lorenz system and Chen system, one concretely compares that the values of the components of the torsion tensor of two systems in the form of a table as shown in Table 1.

The results show that the four components of the torsion tensor of the two systems are consistent, but the component of the torsion tensor $\hat{R}_{21}^3$ is particularly different,

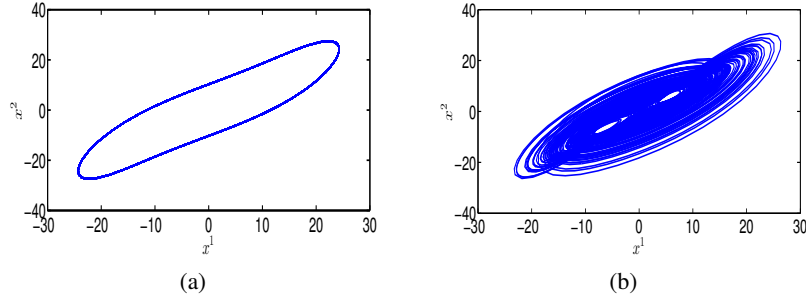


FIGURE 9. Trajectory of the Chen system with $(a, c) = (35, 28)$ and initial values $(x^1(0), x^2(0), x^3(0)) = (-2, 3, 8)$: (a) periodic for $b = 10$ and the corresponding the components of the $\hat{R}_{12}^2 = \hat{R}_{13}^3 \approx 14.5$; (b) chaos for $b = 3$ and the corresponding the components of the $\hat{R}_{12}^2 = \hat{R}_{13}^3 \approx 7.9$.

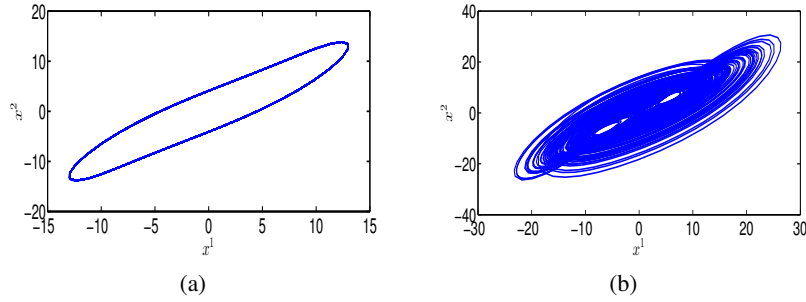


FIGURE 10. Trajectory of the Chen system with $(b, c) = (3, 28)$ and initial values $(x^1(0), x^2(0), x^3(0)) = (-2, 3, 8)$: (a) periodic for $a = 33$ and the corresponding the components of the $\hat{R}_{12}^2 = \hat{R}_{13}^3 \approx 8.3$; (b) chaos for $a = 35$ and the corresponding the components of the $\hat{R}_{12}^2 = \hat{R}_{13}^3 \approx 7.9$.

which is equal to $a - 1$ in Lorenz system, and is equal to $a + c$ in Chen system. Therefore, we want to know what happens to the torsion tensor and dynamics of the two systems when a and b are fixed and c takes different values?

Based on the torsion tensor is equal to the first component of the equilibrium point of the two systems, let's compare two systems from two perspectives.

In order to compare with Lorenz system, we fix the parameters $a = 35$ and $b = 3$. With reference to Lorenz system (4.14) in tangent space [28] rewrite the system (4.4)

TABLE 1. The values of the torsion tensor of Lorenz system and Chen system

the components of the torsion tensor	Lorenz system	Chen system
$\widehat{R}_{12}^2$	x_0^1	x^1
$\widehat{R}_{13}^3$	x_0^2	x^2
$\widehat{R}_{13}^2$	$a - b$	$a - b$
$\widehat{R}_{32}^2$	a	a
$\widehat{R}_{21}^3$	$a - 1$	$a + c$

into (4.15) as follows

$$\begin{cases} \dot{x}^1 = 35(x^2 - x^1) \\ \dot{x}^2 = c_0 x^1 - x^1 x^3 - x^2 \\ \dot{x}^3 = x^1 x^2 - 3x^3, \end{cases} \quad (4.14)$$

$$\begin{cases} \dot{x}^1 = 35(x^2 - x^1) \\ \dot{x}^2 = c_0 x^1 - x^1 x^3 - x^2 + (c + 1)x^2 + (c - 10 - c_0)x^1 \\ \dot{x}^3 = x^1 x^2 - 3x^3. \end{cases} \quad (4.15)$$

Then the component of the torsion tensor corresponding to Table 1 is changed as follows

TABLE 2. The values of the torsion tensor of Lorenz system and Chen system with $a = 35$, $b = 3$ at the equilibrium point

the components of the torsion tensor	Lorenz system	Chen system
$\widehat{R}_{12}^2$	$x_0^1 = \sqrt{3(c_0 - 1)}$	$x^1 = \sqrt{3((c_0 - 1) + (2c - 34 - c_0))}$
$\widehat{R}_{13}^3$	$x_0^2 = \sqrt{3(c_0 - 1)}$	$x^2 = \sqrt{3((c_0 - 1) + (2c - 34 - c_0))}$
$\widehat{R}_{13}^2$	32	32
$\widehat{R}_{32}^2$	35	35
$\widehat{R}_{21}^3$	34	$35 + c$

As we all know, the Chen system is constructed by adding a simple state feedback to the second equation of the Lorenz system [4]. Although the Lorenz system and the Chen system have similar properties, there is difference between the two systems. From Table 2, one obtains that $(x^1)^2 = (x_0^1)^2 + 3(2c - 34 - c_0)$. The values of the component $\widehat{R}_{12}^2 = \widehat{R}_{13}^3$ of the original Lorenz system only have a simple linear relationship with c_0 , but in the Chen system their square have one more term, i.e., $3(2c - 34 - c_0)$ than the Lorenz system. Moreover, in this case, we find that when

$c_0 = c \in [24, 32]$, the two systems show different dynamic characteristics under the same parameters and initial conditions as shown in Fig. 11. The Lorenz system is always stable, while the Chen system shows more complex dynamic behavior, that is, from chaos to periodic. These indicate that the Chen system exhibits more complex torsion tensor and dynamics than the Lorenz system.

Comparing Chen system with $a = 35$, $b = 3$ and Lorenz system with $a = 10$, $b = 8/3$, the values of the components of the torsion tensor with the change of the parameter c are given as shown in Figs. 2 and 12. And it's clearly observed that the torsion tensor $\hat{R}_{21}^3$ increases obviously from periodic to chaos of the Chen system in Fig. 2, whereas it is constant of the Lorenz system in Fig. 12. In other words, the Chen system evolves from the periodic to chaos is expressed by the change of torsion $\hat{R}_{21}^3$, which is not found in Lorenz system. The above shows that the chaotic mechanism of Chen system may be more complex than Lorenz system.

5. CONCLUSION

This paper presents a new perspective of dynamics analysis of Chen system. On the one hand, by using the generalized Meinikov method, the existence of two non-transverse homoclinic orbits is proved of the Chen system (1.1) for $3a + c > b/2$ and $c > a$.

Based on the theory of tangent bundle, a new geometric method is proposed, and the relationship between the complex dynamic behaviour in Chen system and the torsion tensor has been discussed in this paper. Firstly, Chen system is transformed into Kolmogorov system, which can be regarded as a set of second-order differential equations controlled by Lie-Poisson equations. Then, some geometric invariants of Chen system are obtained by associating non-linear connection and connection coefficients. Furthermore, through numerical study, the results show that the complex dynamic behaviour of Chen system is expressed by the change of torsion tensor which are proportional or inversely proportional to the parameters. And it should be noted that $\hat{R}_{12}^2 = \hat{R}_{13}^3 = x^1 = \sqrt{b(2c - a)}$ are obtained in the $(\mathbf{x})$ -field (i.e., tangent space), which indicate that it is meaningful and effective to explore the dynamic characteristic of Chen system on the tangent bundle. These mean that the torsion tensor $\hat{R}_{\beta\gamma}^\alpha$ gives the geometrical interpretation of the complex dynamic behaviour of the Chen system. Finally, one concretely compares that the values of the components of the torsion tensor of two systems as shown in Table 1 for exploring the complexity of dynamics and torsion tensor between Lorenz system and Chen system. The results show that torsion tensor change will lead the Chen system from periodic to chaotic, which is not found in Lorenz system. These indicate that the chaotic mechanism of Chen system may be more complex than Lorenz system. It is also hoped that this paper will help to reveal the most essential geometrical structure of the chaotic attractors.

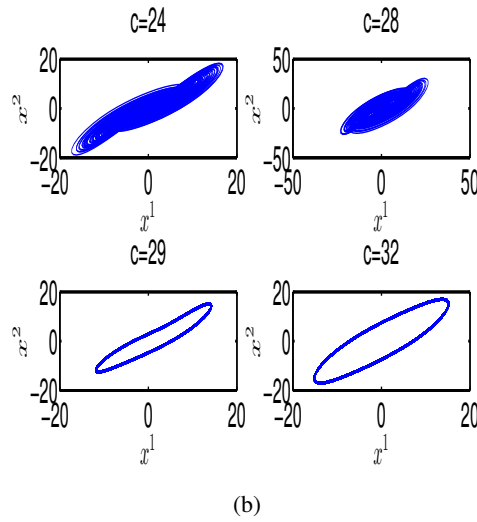
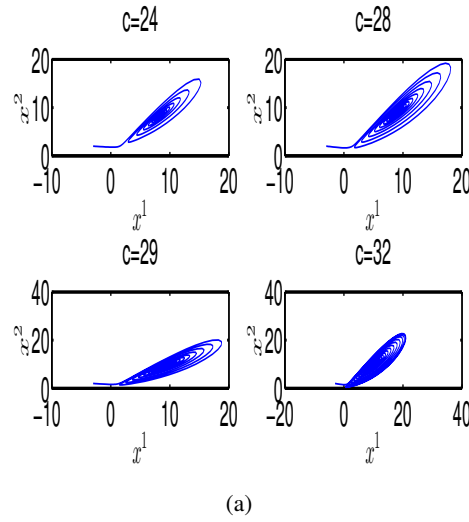


FIGURE 11. Trajectory of the system for $(a, b) = (35, 3)$ and $c_0 = c \in [24, 32]$ with initial values $(x^1(0), x^2(0), x^3(0)) = (-3, 2, 20)$: (a) stable for Lorenz system; (b) $c \in [24, 28]$ chaos and $c \in [29, 32]$ periodic for Chen system.

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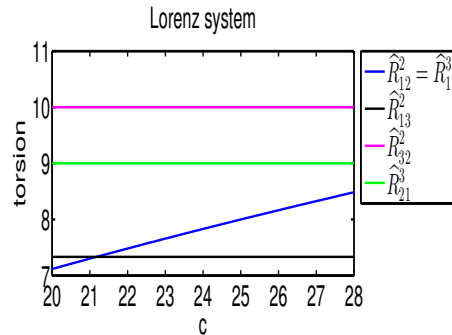


FIGURE 12. The values of the components of the torsion tensor of Lorenz system with $a = 3$, $b = 8/3$ and $c \in [20, 28]$.

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GALERKIN METHODS FOR THE NUMERICAL SOLUTION OF THE SCHRÖDINGER EQUATION BY USING TRIGONOMETRIC B-SPLINES

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Received 16 September, 2020

Abstract. This paper includes four finite element methods which are based on quadratic, cubic, quartic and quintic trigonometric B-spline functions for space discretization and Crank-Nicolson method for time discretization, to be achieved the numerical solution of the Schrödinger equation (SE). The algorithms obtained by different degrees trigonometric B-spline Galerkin methods are new for getting numerical solution of the SE. To see the accuracy of the proposed methods, two numerical experiments are investigated and the comparison of the methods are given in the test problem section.

2010 Mathematics Subject Classification: 35C08; 41A15; 65M60

Keywords: Schrödinger equation, Galerkin finite element method, quadratic trigonometric B-spline function, cubic trigonometric B-spline function, quartic trigonometric B-spline function, quintic trigonometric B-spline function

1. INTRODUCTION

The SE with the conditions is

$$iu_t + u_{xx} + q|u|^2 u = 0, \quad i = \sqrt{-1}, \quad (1.1)$$

$$u(a, t) = 0, \quad u(b, t) = 0, \quad t \geq 0, \quad (1.2)$$

$$u_x(a, t) = 0, \quad u_x(b, t) = 0, \quad (1.3)$$

$$u(x, 0) = f(x), \quad (1.4)$$

where u is a complex function, q is a real number corresponds to a focusing ($q > 0$) or defocusing ($q < 0$) effect of the cubic nonlinearity and $f(x)$ will be defined later. The SE has application to problems in nonlinear physical phenomena such as the propagation of optical pulses, waves in water and plasmas, superconductivity, self-focussing effects in laser pulses, quantum hydro dynamics and Bose-Einstein condensates. The lack of an analytical solution for the general initial conditions of

This work was supported by the Scientific Research Council of Eskisehir Osmangazi University under Project No. 2017-1529.

the SE has led to the search for numerical solutions. Although the studies [3, 5, 6, 8, 9, 12, 16, 18, 19], which obtain the numerical solutions of the SE by using Galerkin method, have been included in the literature before, the use of trigonometric B-splines base for the Galerkin method to solve the SE is new.

The trigonometric B-spline function was worked out by Schoenberg in 1964 [17]. Lyche and Winther showed that a recurrence relation similar to the one for polynomial splines was satisfied by trigonometric B-splines [13]. The some properties of the trigonometric B-spline were studied by Walz in 1997 [20]. Cubic trigonometric B-spline was used for numerical solution of an important equation system in celestial mechanics [15]. Abbas et al. obtained the approximate solution of nonclassical diffusion problem using the two-time level implicit technique based on cubic trigonometric B-spline function [2]. The wave equation was solved using the cubic trigonometric B-spline collocation method in study [1]. Dag et al. solved the regularized long wave equation by using the trigonometric B-spline collocation method [7]. The studies [4] included the different finite element method based on trigonometric B-spline function to solve numerically the Burgers' equation. The cubic and quadratic trigonometric B-spline Galerkin finite element methods were introduced by Irk and Keskin to achieve the numerical solution of regularized long wave equation [10, 11]. Cubic trigonometric B-spline collocation method was applied to have the numerical solution of the fractional sub-diffusion equation and second-order hyperbolic telegraph equation [14, 21]. Zhu et al. suggested the cubic trigonometric B-spline based differential quadrature method for the numerical solution of fractional advection-diffusion equation [23]. Zahra interested the numerical solutions of the PHI-Four and Allen-Cahn equations by applying the trigonometric B-spline collocation method in study [22]. In this paper, we develop four finite element methods which are based on quadratic, cubic, quartic and quintic trigonometric B-spline functions. In the first section, the trigonometric B-spline functions are introduced. In the next section, the space and time discretization of the equation are described to obtain a fully discretized form of the SE for both of the methods. In the test problems section, the accuracy of the proposed methods are discussed with the help of the results obtained from the two test problems, and the obtained results are compared with each others.

Consider $\Omega = [a, b] \times [0, T]$ be smooth region with the grid points (x_m, t_n) , where

$$x_m = a + mh, \quad m = 0, 1, 2, \dots, N, \quad t_n = n\Delta t, \quad n = 0, 1, 2, \dots,$$

h and Δt are mesh size in the space and time direction respectively.

2. TRIGONOMETRIC B-SPLINES

The trigonometric B-spline which is of order 0 is indicated as:

$$T_i^0(x) = \begin{cases} 1, & x \in [x_i, x_{i+1}) \\ 0 & \text{otherwise.} \end{cases}$$

The others which are of orders $k = 1, 2, 3, \dots$ can be obtained by using the following recursive equation

$$T_i^k(x) = \sin\left(\frac{x - x_i}{2}\right) B_i^{k-1}(x) + \sin\left(\frac{x_{i+k} - x}{2}\right) B_{i+1}^{k-1}(x),$$

where

$$B_j^{k-1}(x) = \begin{cases} \frac{T_j^{k-1}(x)}{\sin\left(\frac{x_{j+k-1} - x_j}{2}\right)}, & x_j < x_{j+k-1}, \\ 0, & x_j = x_{j+k-1}, \end{cases}$$

for $j = i$ or $j = i + 1$. Accordingly, for $k = 2$ the quadratic trigonometric B-spline is obtained as:

$$T_i^2(x) = \frac{1}{\psi} \begin{cases} \varphi^2(x_{i-1}), & x \in [x_{i-1}, x_i), \\ -\varphi(x_{i-1})\varphi(x_{i+1}) - \varphi(x_{i+2})\varphi(x_i), & x \in [x_i, x_{i+1}), \\ \varphi^2(x_{i+2}), & x \in [x_{i+1}, x_{i+2}), \\ 0 & \text{otherwise,} \end{cases} \quad (2.1)$$

and for $k = 3$ the cubic trigonometric B-spline is obtained as:

$$T_i^3(x) = \frac{1}{\phi} \begin{cases} \varphi^3(x_{i-2}), & x \in [x_{i-2}, x_{i-1}), \\ -\varphi^2(x_{i-2})\varphi(x_i) - \varphi(x_{i-2})\varphi(x_{i-1})\varphi(x_{i+1}) \\ -\varphi(x_{i+2})\varphi^2(x_{i-1}), & x \in [x_{i-1}, x_i), \\ \varphi(x_{i-2})\varphi^2(x_{i+1}) + \varphi(x_{i+2})\varphi(x_{i+1})\varphi(x_{i-1}) \\ +\varphi^2(x_{i+2})\varphi(x_i), & x \in [x_i, x_{i+1}), \\ -\varphi^3(x_{i+2}), & x \in [x_{i+1}, x_{i+2}), \\ 0 & \text{otherwise,} \end{cases} \quad (2.2)$$

and for $k = 4$ the quartic trigonometric B-spline is obtained as:

$$T_i^4(x) = \frac{1}{\vartheta} \begin{cases} \varphi^4(x_{i-2}), & x \in [x_{i-2}, x_{i-1}), \\ -\varphi^3(x_{i-2})\varphi(x_i) - \varphi^2(x_{i-2})\varphi(x_{i+1})\varphi(x_{i-1}) & \\ -\varphi(x_{i+2})\varphi(x_{i-2})\varphi^2(x_{i-1}) - \varphi(x_{i+3})\varphi^3(x_{i-1}), & x \in [x_{i-1}, x_i), \\ \varphi^2(x_{i-2})\varphi^2(x_{i+1}) + \varphi(x_{i+2})\varphi(x_{i+1})\varphi(x_{i-1})\varphi(x_{i-2}) & \\ +\varphi^2(x_{i+2})\varphi(x_i)\varphi(x_{i-2}) + \varphi(x_{i+3})\varphi(x_{i+1})\varphi^2(x_{i-1}) & x \in [x_i, x_{i+1}), \\ +\varphi(x_{i+3})\varphi(x_{i+2})\varphi(x_i)\varphi(x_{i-1}) + \varphi^2(x_{i+3})\varphi^2(x_i), & \\ -\varphi(x_{i-2})\varphi^3(x_{i+2}) - \varphi(x_{i+3})\varphi^2(x_{i+2})\varphi(x_{i-1}) & x \in [x_{i+1}, x_{i+2}), \\ -\varphi^2(x_{i+3})\varphi(x_{i+2})\varphi(x_i) - \varphi^3(x_{i+3})\varphi(x_{i+1}), & \\ \varphi^4(x_{i+3}), & x \in [x_{i+2}, x_{i+3}), \\ 0 & \text{otherwise,} \end{cases} \quad (2.3)$$

and for $k = 5$ the quintic trigonometric B-spline is obtained as:

$$T_m^5(x) = \frac{1}{\tau} \begin{cases} \varphi^5(x_{m-3}), & x_{m-3} \leq x < x_{m-2} \\ -\varphi^4(x_{m-3})\varphi(x_{m-1}) - \varphi^3(x_{m-3})\varphi(x_m)\varphi(x_{m-2}) & \\ -\varphi^2(x_{m-3})\varphi(x_{m+1})\varphi^2(x_{m-2}) & x_{m-2} \leq x < x_{m-1} \\ -\varphi(x_{m-3})\varphi(x_{m+2})\varphi^3(x_{m-2}) - \varphi(x_{m+3})\varphi^4(x_{m-2}), & \\ \varphi^3(x_{m-3})\varphi^2(x_m) + \varphi^2(x_{m-3})\varphi(x_{m+1})\varphi(x_{m-2})\varphi(x_m) & \\ +\varphi^2(x_{m-3})\varphi^2(x_{m+1})\varphi(x_{m-1}) & \\ +\varphi(x_{m-3})\varphi(x_{m+2})\varphi^2(x_{m-2})\varphi(x_m) & \\ +\varphi(x_{m-3})\varphi(x_{m+2})\varphi(x_{m-2})\varphi(x_{m+1})\varphi(x_{m-1}) & \\ +\varphi(x_{m-3})\varphi^2(x_{m+2})\varphi^2(x_{m-1}) + \varphi(x_{m+3})\varphi^3(x_{m-2})\varphi(x_m) & x_{m-1} \leq x < x_m \\ +\varphi(x_{m+3})\varphi^2(x_{m-2})\varphi(x_{m+1})\varphi(x_{m-1}) & \\ +\varphi(x_{m+3})\varphi(x_{m-2})\varphi(x_{m+2})\varphi^2(x_{m-1}) & \\ +\varphi^2(x_{m+3})\varphi^3(x_{m-1}), & \\ -\varphi^2(x_{m-3})\varphi^3(x_{m+1}) - \varphi(x_{m-3})\varphi(x_{m+2})\varphi(x_{m-2})\varphi^2(x_{m+1}) & \\ -\varphi(x_{m-3})\varphi^2(x_{m+2})\varphi(x_{m-1})\varphi(x_{m+1}) & \\ -\varphi(x_{m-3})\varphi^3(x_{m+2})\varphi(x_m) - \varphi(x_{m+3})\varphi^2(x_{m-2})\varphi^2(x_{m+1}) & \\ -\varphi(x_{m+3})\varphi(x_{m-2})\varphi(x_{m+2})\varphi(x_{m-1})\varphi(x_{m+1}) & x_m \leq x < x_{m+1} \\ -\varphi(x_{m+3})\varphi(x_{m-2})\varphi^2(x_{m+2})\varphi(x_m) & \\ -\varphi^2(x_{m+3})\varphi^2(x_{m-1})\varphi(x_{m+1}) & \\ -\varphi^2(x_{m+3})\varphi(x_{m-1})\varphi(x_{m+2})\varphi(x_m) - \varphi^3(x_{m+3})\varphi^2(x_m), & \\ \varphi(x_{m-3})\varphi^4(x_{m+2}) + \varphi(x_{m+3})\varphi(x_{m-2})\varphi^3(x_{m+2}) & \\ +\varphi^2(x_{m+3})\varphi(x_{m-1})\varphi^2(x_{m+2}) & x_{m+1} \leq x < x_{m+2} \\ +\varphi^3(x_{m+3})\varphi(x_m)\varphi(x_{m+2}) + \varphi^4(x_{m+3})\varphi(x_{m+1}), & \\ -\varphi^5(x_{m+3}), & x_{m+2} \leq x < x_{m+3} \\ 0 & \text{otherwise,} \end{cases} \quad (2.4)$$

where

$$\begin{aligned} h &= x_i - x_{i-1}, \varphi(x_i) = \sin\left(\frac{x - x_i}{2}\right), \Psi = \sin\left(\frac{h}{2}\right) \sin(h), \phi = \sin\left(\frac{h}{2}\right) \sin(h) \sin\left(\frac{3h}{2}\right), \\ \vartheta &= \sin\left(\frac{h}{2}\right) \sin(h) \sin\left(\frac{3h}{2}\right) \sin(2h), \tau = \sin\left(\frac{h}{2}\right) \sin(h) \sin\left(\frac{3h}{2}\right) \sin(2h) \sin\left(\frac{5h}{2}\right). \end{aligned}$$

3. FINITE ELEMENT SOLUTIONS OF SE

Let take the complex function u as

$$u(x, t) = r(x, t) + is(x, t), \quad (3.1)$$

where $r(x, t)$ and $s(x, t)$ are real functions. With substituting (3.1) into (1.1), the real and imaginary parts of the obtained function can be rewritten as two real differential equations:

$$r_t + s_{xx} + q(r^2 + s^2)s = 0, \quad (3.2)$$

$$s_t - r_{xx} - q(r^2 + s^2)r = 0. \quad (3.3)$$

Using the Crank-Nicolson method the time discretized forms of (3.2) and (3.3) are obtained as:

$$r^{n+1} + \frac{\Delta t}{2}(s_{xx})^{n+1} + q\frac{\Delta t}{2}((r^2 + s^2)s)^{n+1} = r^n - \frac{\Delta t}{2}(s_{xx})^n - q\frac{\Delta t}{2}((r^2 + s^2)s)^n, \quad (3.4)$$

$$s^{n+1} - \frac{\Delta t}{2}(r_{xx})^{n+1} - q\frac{\Delta t}{2}((r^2 + s^2)r)^{n+1} = s^n + \frac{\Delta t}{2}(r_{xx})^n + q\frac{\Delta t}{2}((r^2 + s^2)r)^n. \quad (3.5)$$

3.1. Quadratic trigonometric B-spline Galerkin method (Method 1)

By applying Galerkin finite element method to (3.4) and (3.5) and using integration by parts, we get

$$\begin{aligned} & \int_a^b \left[W(x) \left(r^{n+1} + q\frac{\Delta t}{2}((r^2 + s^2)s)^{n+1} \right) - \frac{\Delta t}{2} W_x(x)(s_x)^{n+1} \right] dx \\ &= \int_a^b \left[W(x) \left(r^n - q\frac{\Delta t}{2}((r^2 + s^2)s)^n \right) + \frac{\Delta t}{2} W_x(x)(s_x)^n \right] dx, \end{aligned} \quad (3.6)$$

$$\begin{aligned} & \int_a^b \left[W(x) \left(s^{n+1} - q\frac{\Delta t}{2}((r^2 + s^2)r)^{n+1} \right) + \frac{\Delta t}{2} W_x(x)(r_x)^{n+1} \right] dx \\ &= \int_a^b \left[W(x) \left(s^n + q\frac{\Delta t}{2}((r^2 + s^2)r)^n \right) - \frac{\Delta t}{2} W_x(x)(r_x)^n \right] dx, \end{aligned} \quad (3.7)$$

where $W(x)$ is a weight function. Let divide $[a, b]$ by N equally subintervals with the knots $x_j, j = 1, \dots, N$. The approximate function U over all elements $[x_m, x_{m+1}]$ is

$$u(x, t) \simeq U(x, t) = \sum_{j=m-1}^{m+1} \delta_j(t) T_j^2(x), \quad (3.8)$$

where $\delta_j = \alpha_j + i\beta_j$. The approximate function and its first derivative can be written by using the quadratic trigonometric B-spline as

$$U(x_m, t) = \frac{\sin^2(\frac{h}{2})}{\Psi} (\delta_{m-1} + \delta_m), \quad U'(x_m, t) = \frac{\sin(h)}{\Psi} (-\delta_{m-1} + \delta_m).$$

Let choose the weight function $W(x)$ as the quadratic trigonometric B-spline shape function (2.1) and by substituting approximate function (3.8) into (3.6) and (3.7), we have the following approximation over the subinterval $[x_m, x_{m+1}]$:

$$\begin{aligned} & \sum_{j=m-1}^{m+1} \left\{ \left(\int_{x_m}^{x_{m+1}} T_i T_j dx \right) \alpha_j^{n+1} - \frac{1}{2} \Delta t \left(\int_{x_m}^{x_{m+1}} T_i' T_j' dx \right) \beta_j^{n+1} \right. \\ & + \frac{1}{2} q \Delta t \sum_{j=m-1}^{m+1} \sum_{l=m-1}^{m+1} \left(\int_{x_m}^{x_{m+1}} T_i [T_k \beta_k^{n+1} T_l \beta_l^{n+1} + T_k \alpha_k^{n+1} T_l \alpha_l^{n+1}] T_j dx \right) \beta_j^{n+1} \Big\} \\ & - \sum_{j=m-1}^{m+1} \left\{ \left(\int_{x_m}^{x_{m+1}} T_i T_j dx \right) \alpha_j^n + \frac{1}{2} \Delta t \left(\int_{x_m}^{x_{m+1}} T_i' T_j' dx \right) \beta_j^n \right. \\ & - \frac{1}{2} q \Delta t \sum_{k=m-1}^{m+1} \sum_{l=m-1}^{m+1} \left(\int_{x_m}^{x_{m+1}} T_i [T_k \beta_k^n T_l \beta_l^n + T_k \alpha_k^n T_l \alpha_l^n] T_j dx \right) \beta_j^n \Big\} \end{aligned} \quad (3.9)$$

and

$$\begin{aligned} & \sum_{j=m-1}^{m+1} \left\{ \left(\int_{x_m}^{x_{m+1}} T_i T_j dx \right) \beta_j^{n+1} + \frac{1}{2} \Delta t \left(\int_{x_m}^{x_{m+1}} T_i' T_j' dx \right) \alpha_j^{n+1} \right. \\ & - \frac{1}{2} q \Delta t \sum_{k=m-1}^{m+1} \sum_{l=m-1}^{m+1} \left(\int_{x_m}^{x_{m+1}} T_i [T_k \beta_k^{n+1} T_l \beta_l^{n+1} + T_k \alpha_k^{n+1} T_l \alpha_l^{n+1}] T_j dx \right) \alpha_j^{n+1} \Big\} \\ & - \sum_{j=m-1}^{m+1} \left\{ \left(\int_{x_m}^{x_{m+1}} T_i T_j dx \right) \beta_j^n - \frac{1}{2} \Delta t \left(\int_{x_m}^{x_{m+1}} T_i' T_j' dx \right) \alpha_j^n \right. \\ & + \frac{1}{2} q \Delta t \sum_{k=m-1}^{m+1} \sum_{l=m-1}^{m+1} \left(\int_{x_m}^{x_{m+1}} T_i [T_k \beta_k^n T_l \beta_l^n + T_k \alpha_k^n T_l \alpha_l^n] T_j dx \right) \alpha_j^n \Big\}. \end{aligned} \quad (3.10)$$

Denoting the integrals in the systems (3.9) and (3.10) by

$$\begin{aligned} A_{ij}^e &= \int_{x_m}^{x_{m+1}} T_i T_j dx, & B_{ij}^e &= \int_{x_m}^{x_{m+1}} T_i' T_j' dx, \\ C_{ij}^e &= \int_{x_m}^{x_{m+1}} T_i [T_k \beta_k^{n+1} T_l \beta_l^{n+1} + T_k \alpha_k^{n+1} T_l \alpha_l^{n+1}] T_j dx, \\ D_{ij}^e &= \int_{x_m}^{x_{m+1}} T_i [T_k \beta_k^n T_l \beta_l^n + T_k \alpha_k^n T_l \alpha_l^n] T_j dx \end{aligned}$$

and the systems (3.9), (3.10) are gathered over all elements, the matrix form of (3.9) and (3.10) can be written as

$$\mathbf{A}\alpha^{n+1} - \frac{\Delta t}{2}\mathbf{B}\beta^{n+1} + \frac{\Delta t q}{2}\mathbf{C}\beta^{n+1} = \mathbf{A}\alpha^n + \frac{\Delta t}{2}\mathbf{B}\beta^n - \frac{\Delta t q}{2}\mathbf{D}\beta^n, \quad (3.11)$$

$$\mathbf{A}\beta^{n+1} + \frac{\Delta t}{2}\mathbf{B}\alpha^{n+1} - \frac{\Delta t q}{2}\mathbf{C}\alpha^{n+1} = \mathbf{A}\beta^n - \frac{\Delta t}{2}\mathbf{B}\alpha^n + \frac{\Delta t q}{2}\mathbf{D}\alpha^n, \quad (3.12)$$

where $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ are the element matrices, $\alpha = (\alpha_{-1}, \alpha_0, \dots, \alpha_{N-1}, \alpha_N)^T$ and $\beta = (\beta_{-1}, \beta_0, \dots, \beta_{N-1}, \beta_N)^T$. The set of equations consists of $2N + 4$ equations with $2N + 4$ unknown parameters. For the nonlinear terms in these systems, we apply an inner iteration procedure. Before starting the procedure, boundary conditions must be adapted into the system. For this purpose, we delete first and last equations from the systems (3.11) and (3.12), and eliminate the terms α_{-1}, β_{-1} and α_N, β_N from the remaining systems (3.11) and (3.12) by using boundary conditions (1.2). So, we obtain a new matrix system with the dimension $(2N) \times (2N)$.

Using the boundary conditions (1.3) and initial condition (1.4), the initial parameters $(\alpha_{-1}^0, \beta_{-1}^0, \alpha_0^0, \beta_0^0, \dots, \alpha_{N-1}^0, \beta_{N-1}^0, \alpha_N^0, \beta_N^0)$ can be obtained. Once the initial parameters α^0 and β^0 are determined, the unknown parameters α^n and β^n at time $t^n = n\Delta t, n = 1, 2, 3, \dots$ are obtained. Thus the approximate solution (3.8) can be determined by using these values.

3.2. Cubic trigonometric B-spline Galerkin method (Method 2)

Let take the approximate function U over all elements $[x_m, x_{m+1}]$ as

$$u(x, t) \simeq U(x, t) = \sum_{j=m-1}^{m+2} \delta_j(t) T_j^3(x) \quad (3.13)$$

where $\delta_j = \alpha_j + i\beta_j$. The approximate function U and its first two derivatives can be obtained by using the cubic trigonometric B-spline functions (2.2). When we repeat the same procedure with selecting the cubic trigonometric B-spline shape function (2.2) as weight function $W(x)$ in the Galerkin discretized form of SE (3.6) and (3.7),

and using the approximate function (3.13) and its derivatives in (3.6) and (3.7), we have the following approximation over the subinterval $[x_m, x_{m+1}]$:

$$\begin{aligned}
& \sum_{j=m-1}^{m+2} \left\{ \left(\int_{x_m}^{x_{m+1}} T_i T_j dx \right) \alpha_j^{n+1} + \frac{1}{2} \Delta t \left(\int_{x_m}^{x_{m+1}} T_i T_j'' dx \right) \beta_j^{n+1} \right. \\
& + \frac{1}{2} q \Delta t \sum_{k=m-1}^{m+2} \sum_{l=m-1}^{m+2} \left(\int_{x_m}^{x_{m+1}} T_i [T_k (\beta_k^{n+1}) T_l (\beta_l^{n+1}) + T_k (\alpha_k^{n+1}) T_l (\alpha_l^{n+1})] T_j dx \right) \beta_j^{n+1} \Bigg\} \\
& - \sum_{j=m-1}^{m+2} \left\{ \left(\int_{x_m}^{x_{m+1}} T_i T_j dx \right) \alpha_j^n - \frac{1}{2} \Delta t \left(\int_{x_m}^{x_{m+1}} T_i T_j'' dx \right) \beta_j^n \right. \\
& - \frac{1}{2} q \Delta t \sum_{k=m-1}^{m+2} \sum_{l=m-1}^{m+2} \left(\int_{x_m}^{x_{m+1}} T_i [T_k (\beta_k^n) T_l (\beta_l^n) + T_k (\alpha_k^n) T_l (\alpha_l^n)] T_j dx \right) \beta_j^n \Bigg\}. \quad (3.14)
\end{aligned}$$

$$\begin{aligned}
& \sum_{j=m-1}^{m+2} \left\{ \left(\int_{x_m}^{x_{m+1}} T_i T_j dx \right) \beta_j^{n+1} - \frac{1}{2} \Delta t \left(\int_{x_m}^{x_{m+1}} T_i T_j'' dx \right) \alpha_j^{n+1} \right. \\
& - \frac{1}{2} q \Delta t \sum_{k=m-1}^{m+2} \sum_{l=m-1}^{m+2} \left(\int_{x_m}^{x_{m+1}} T_i [T_k (\beta_k^{n+1}) T_l (\beta_l^{n+1}) + T_k (\alpha_k^{n+1}) T_l (\alpha_l^{n+1})] T_j dx \right) \alpha_j^{n+1} \Bigg\} \\
& - \sum_{j=m-1}^{m+2} \left\{ \left(\int_{x_m}^{x_{m+1}} T_i T_j dx \right) \beta_j^n + \frac{1}{2} \Delta t \left(\int_{x_m}^{x_{m+1}} T_i T_j'' dx \right) \alpha_j^n \right. \\
& + \frac{1}{2} q \Delta t \sum_{k=m-1}^{m+2} \sum_{l=m-1}^{m+2} \left(\int_{x_m}^{x_{m+1}} T_i [T_k (\beta_k^n) T_l (\beta_l^n) + T_k (\alpha_k^n) T_l (\alpha_l^n)] T_j dx \right) \alpha_j^n \Bigg\}. \quad (3.15)
\end{aligned}$$

Denoting the integrals in (3.14) and (3.15) by

$$\begin{aligned}
A_{ij}^e &= \int_{x_m}^{x_{m+1}} T_i T_j dx, & B_{ij}^e &= \int_{x_m}^{x_{m+1}} T_i T_j'' dx, \\
C_{ij}^e &= \int_{x_m}^{x_{m+1}} T_i [T_k (\beta_k^{n+1}) T_l (\beta_l^{n+1}) + T_k (\alpha_k^{n+1}) T_l (\alpha_l^{n+1})] T_j dx, \\
D_{ij}^e &= \int_{x_m}^{x_{m+1}} T_i [T_k (\beta_k^n) T_l (\beta_l^n) + T_k (\alpha_k^n) T_l (\alpha_l^n)] T_j dx
\end{aligned}$$

and collecting the element matrices over all elements $[x_m, x_{m+1}]$, we get the following matrix form of (3.14) and (3.15):

$$\mathbf{A}\alpha^{n+1} + \frac{\Delta t}{2}\mathbf{B}\beta^{n+1} + \frac{\Delta t q}{2}\mathbf{C}\beta^{n+1} = \mathbf{A}\alpha^n - \frac{\Delta t}{2}\mathbf{B}\beta^n - \frac{\Delta t q}{2}\mathbf{D}\beta^n, \quad (3.16)$$

$$\mathbf{A}\beta^{n+1} - \frac{\Delta t}{2}\mathbf{B}\alpha^{n+1} - \frac{\Delta t q}{2}\mathbf{C}\alpha^{n+1} = \mathbf{A}\beta^n + \frac{\Delta t}{2}\mathbf{B}\alpha^n + \frac{\Delta t q}{2}\mathbf{D}\alpha^n, \quad (3.17)$$

where $\alpha = (\alpha_{-1}, \alpha_0, \dots, \alpha_N, \alpha_{N+1})^T$ and $\beta = (\beta_{-1}, \beta_0, \dots, \beta_N, \beta_{N+1})^T$. The linearization of the obtained systems are done in the same way as in method 1. Similarly, by eliminating the terms α_{-1}, β_{-1} and $\alpha_{N+1}, \beta_{N+1}$ from the systems (3.16) and (3.17), the obtained matrix system with dimension $(2N+2) \times (2N+2)$ is solved by Gauss elimination method. Again, firstly we attain the initial parameters α^0 and β^0 , then the value of the approximate function U becomes computable.

3.3. Quartic trigonometric B-spline Galerkin method (Method 3)

In this method, the approximate function U over all elements $[x_m, x_{m+1}]$ with the quartic trigonometric B-spline functions is taken as

$$u(x, t) \simeq U(x, t) = \sum_{j=m-2}^{m+2} \delta_j(t) T_j^4(x) \quad (3.18)$$

where $\delta_j = \alpha_j + i\beta_j$. This approximate function and first and second derivatives of this can be obtained by the quartic trigonometric B-spline functions (2.3). With a similar procedure applied in the previous methods, by choosing the weight function as the quartic trigonometric B-spline functions in (3.6) and (3.7), the following expressions are obtained over the subinterval $[x_m, x_{m+1}]$:

$$\begin{aligned} & \sum_{j=m-2}^{m+2} \left\{ \left(\int_{x_m}^{x_{m+1}} T_i T_j dx \right) \alpha_j^{n+1} + \frac{1}{2} \Delta t \left(\int_{x_m}^{x_{m+1}} T_i T_j'' dx \right) \beta_j^{n+1} \right. \\ & + \frac{1}{2} q \Delta t \sum_{k=m-2}^{m+2} \sum_{l=m-2}^{m+2} \left(\int_{x_m}^{x_{m+1}} T_i [T_k (\beta_k^{n+1}) T_l (\beta_l^{n+1}) + T_k (\alpha_k^{n+1}) T_l (\alpha_l^{n+1})] T_j dx \right) \beta_j^{n+1} \Big\} \\ & - \sum_{j=m-2}^{m+2} \left\{ \left(\int_{x_m}^{x_{m+1}} T_i T_j dx \right) \alpha_j^n - \frac{1}{2} \Delta t \left(\int_{x_m}^{x_{m+1}} T_i T_j'' dx \right) \beta_j^n \right. \\ & \left. - \frac{1}{2} q \Delta t \sum_{k=m-2}^{m+2} \sum_{l=m-2}^{m+2} \left(\int_{x_m}^{x_{m+1}} T_i [T_k (\beta_k^n) T_l (\beta_l^n) + T_k (\alpha_k^n) T_l (\alpha_l^n)] T_j dx \right) \beta_j^n \right\}, \quad (3.19) \end{aligned}$$

$$\begin{aligned}
& \sum_{j=m-2}^{m+2} \left\{ \left(\int_{x_m}^{x_{m+1}} T_i T_j dx \right) \beta_j^{n+1} - \frac{1}{2} \Delta t \left(\int_{x_m}^{x_{m+1}} T_i T_j'' dx \right) \alpha_j^{n+1} \right. \\
& - \frac{1}{2} q \Delta t \sum_{k=m-2}^{m+2} \sum_{l=m-2}^{m+2} \left(\int_{x_m}^{x_{m+1}} T_i [T_k (\beta_k^{n+1}) T_l (\beta_l^{n+1}) + T_k (\alpha_k^{n+1}) T_l (\alpha_l^{n+1})] T_j dx \right) \alpha_j^{n+1} \Big\} \\
& - \sum_{j=m-2}^{m+2} \left\{ \left(\int_{x_m}^{x_{m+1}} T_i T_j dx \right) \beta_j^n + \frac{1}{2} \Delta t \left(\int_{x_m}^{x_{m+1}} T_i T_j'' dx \right) \alpha_j^n \right. \\
& + \frac{1}{2} q \Delta t \sum_{k=m-2}^{m+2} \sum_{l=m-2}^{m+2} \left(\int_{x_m}^{x_{m+1}} T_i [T_k (\beta_k^n) T_l (\beta_l^n) + T_k (\alpha_k^n) T_l (\alpha_l^n)] T_j dx \right) \alpha_j^n \Big\}. \quad (3.20)
\end{aligned}$$

Denoting the above integrals by

$$\begin{aligned}
A_{ij}^e &= \int_{x_m}^{x_{m+1}} T_i T_j dx, & B_{ij}^e &= \int_{x_m}^{x_{m+1}} T_i T_j'' dx, \\
C_{ij}^e &= \int_{x_m}^{x_{m+1}} T_i [T_k (\beta_k^{n+1}) T_l (\beta_l^{n+1}) + T_k (\alpha_k^{n+1}) T_l (\alpha_l^{n+1})] T_j dx, \\
D_{ij}^e &= \int_{x_m}^{x_{m+1}} T_i [T_k (\beta_k^n) T_l (\beta_l^n) + T_k (\alpha_k^n) T_l (\alpha_l^n)] T_j dx
\end{aligned}$$

and collecting the element matrices over all elements $[x_m, x_{m+1}]$, we get the following matrix forms of (3.19) and (3.20):

$$\mathbf{A} \alpha^{n+1} + \frac{\Delta t}{2} \mathbf{B} \beta^{n+1} + \frac{\Delta t q}{2} \mathbf{C} \beta^{n+1} = \mathbf{A} \alpha^n - \frac{\Delta t}{2} \mathbf{B} \beta^n - \frac{\Delta t q}{2} \mathbf{D} \beta^n, \quad (3.21)$$

$$\mathbf{A} \beta^{n+1} - \frac{\Delta t}{2} \mathbf{B} \alpha^{n+1} - \frac{\Delta t q}{2} \mathbf{C} \alpha^{n+1} = \mathbf{A} \beta^n + \frac{\Delta t}{2} \mathbf{B} \alpha^n + \frac{\Delta t q}{2} \mathbf{D} \alpha^n, \quad (3.22)$$

where $\alpha = (\alpha_{-2}, \alpha_{-1}, \alpha_0, \dots, \alpha_N, \alpha_{N+1})^T$ and $\beta = (\beta_{-2}, \beta_{-1}, \beta_0, \dots, \beta_N, \beta_{N+1})^T$. By the same linearization procedure as shown in Methods 1 and 2, and the elimination of the terms α_{-2}, β_{-2} and $\alpha_{N+1}, \beta_{N+1}$ from the systems (3.21) and (3.22), a matrix system with dimension $(2N+4) \times (2N+4)$ is obtained. Then the value of the approximate function (3.18) is found similarly to the Methods 1 and 2.

3.4. Quintic trigonometric B-spline Galerkin method (Method 4)

By the fourth method, the approximate function U over all elements $[x_m, x_{m+1}]$ is chosen as

$$u(x, t) \simeq U(x, t) = \sum_{j=m-2}^{m+3} \delta_j(t) T_j^5(x) \quad (3.23)$$

where $\delta_j = \alpha_j + i\beta_j$. The approximate function, first and second derivatives of this can be obtained by the quintic trigonometric B-spline function (2.4). By choosing the weight function as the quintic trigonometric B-spline functions in (3.6) and (3.7), over the subinterval $[x_m, x_{m+1}]$ the following expressions are obtained:

$$\begin{aligned} & \sum_{j=m-2}^{m+3} \left\{ \left(\int_{x_m}^{x_{m+1}} T_i T_j dx \right) \alpha_j^{n+1} + \frac{1}{2} \Delta t \left(\int_{x_m}^{x_{m+1}} T_i T_j'' dx \right) \beta_j^{n+1} \right. \\ & + \frac{1}{2} q \Delta t \sum_{k=m-2}^{m+3} \sum_{l=m-2}^{m+3} \left(\int_{x_m}^{x_{m+1}} T_i [T_k (\beta_k^{n+1}) T_l (\beta_l^{n+1}) + T_k (\alpha_k^{n+1}) T_l (\alpha_l^{n+1})] T_j dx \right) \beta_j^{n+1} \Big\} \\ & - \sum_{j=m-2}^{m+3} \left\{ \left(\int_{x_m}^{x_{m+1}} T_i T_j dx \right) \alpha_j^n - \frac{1}{2} \Delta t \left(\int_{x_m}^{x_{m+1}} T_i T_j'' dx \right) \beta_j^n \right. \\ & \left. - \frac{1}{2} q \Delta t \sum_{k=m-2}^{m+3} \sum_{l=m-2}^{m+3} \left(\int_{x_m}^{x_{m+1}} T_i [T_k (\beta_k^n) T_l (\beta_l^n) + T_k (\alpha_k^n) T_l (\alpha_l^n)] T_j dx \right) \beta_j^n \right\}, \quad (3.24) \end{aligned}$$

$$\begin{aligned} & \sum_{j=m-2}^{m+3} \left\{ \left(\int_{x_m}^{x_{m+1}} T_i T_j dx \right) \beta_j^{n+1} - \frac{1}{2} \Delta t \left(\int_{x_m}^{x_{m+1}} T_i T_j'' dx \right) \alpha_j^{n+1} \right. \\ & - \frac{1}{2} q \Delta t \sum_{k=m-2}^{m+3} \sum_{l=m-2}^{m+3} \left(\int_{x_m}^{x_{m+1}} T_i [T_k (\beta_k^{n+1}) T_l (\beta_l^{n+1}) + T_k (\alpha_k^{n+1}) T_l (\alpha_l^{n+1})] T_j dx \right) \alpha_j^{n+1} \Big\} \\ & - \sum_{j=m-2}^{m+3} \left\{ \left(\int_{x_m}^{x_{m+1}} T_i T_j dx \right) \beta_j^n + \frac{1}{2} \Delta t \left(\int_{x_m}^{x_{m+1}} T_i T_j'' dx \right) \alpha_j^n \right. \\ & \left. + \frac{1}{2} q \Delta t \sum_{k=m-2}^{m+3} \sum_{l=m-2}^{m+3} \left(\int_{x_m}^{x_{m+1}} T_i [T_k (\beta_k^n) T_l (\beta_l^n) + T_k (\alpha_k^n) T_l (\alpha_l^n)] T_j dx \right) \alpha_j^n \right\}. \quad (3.25) \end{aligned}$$

Denoting the above integrals with

$$\begin{aligned} A_{ij}^e &= \int_{x_m}^{x_{m+1}} T_i T_j dx, & B_{ij}^e &= \int_{x_m}^{x_{m+1}} T_i T_j'' dx, \\ C_{ij}^e &= \int_{x_m}^{x_{m+1}} T_i [T_k(\beta_k^{n+1}) T_l(\beta_l^{n+1}) + T_k(\alpha_k^{n+1}) T_l(\alpha_l^{n+1})] T_j dx, \\ D_{ij}^e &= \int_{x_m}^{x_{m+1}} T_i [T_k(\beta_k^n) T_l(\beta_l^n) + T_k(\alpha_k^n) T_l(\alpha_l^n)] T_j dx \end{aligned}$$

and collecting the element matrices over all elements $[x_m, x_{m+1}]$, we get the following matrix forms of (3.24) and (3.25):

$$\mathbf{A}\alpha^{n+1} + \frac{\Delta t}{2}\mathbf{B}\beta^{n+1} + \frac{\Delta t q}{2}\mathbf{C}\beta^{n+1} = \mathbf{A}\alpha^n - \frac{\Delta t}{2}\mathbf{B}\beta^n - \frac{\Delta t q}{2}\mathbf{D}\beta^n, \quad (3.26)$$

$$\mathbf{A}\beta^{n+1} - \frac{\Delta t}{2}\mathbf{B}\alpha^{n+1} - \frac{\Delta t q}{2}\mathbf{C}\alpha^{n+1} = \mathbf{A}\beta^n + \frac{\Delta t}{2}\mathbf{B}\alpha^n + \frac{\Delta t q}{2}\mathbf{D}\alpha^n, \quad (3.27)$$

where $\alpha = (\alpha_{-2}, \alpha_{-1}, \alpha_0, \dots, \alpha_{N+1}, \alpha_{N+2})^T$ and $\beta = (\beta_{-2}, \beta_{-1}, \beta_0, \dots, \beta_{N+1}, \beta_{N+2})^T$. Then, by eliminating the terms α_{-2}, β_{-2} and $\alpha_{N+2}, \beta_{N+2}$ from the matrix systems (3.26) and (3.27), the obtained matrix system with dimension $(2N+6) \times (2N+6)$ is solved similarly to the previous methods and the approximate function (3.23) can be computed.

4. TEST PROBLEMS

In this section, we investigate the propagation of single soliton and the interaction of two solitons to demonstrate the efficiency of the given algorithms. Accuracy of the methods are measured by the error norm

$$L_\infty = \|u - U_M\|_\infty = \max_{0 \leq j \leq N} \left| \| (u)_j \| - \| (U_M)_j \| \right|.$$

The conservation laws which are verified by the SE are

$$C_1 = \int_a^b |U|^2 dx, \quad C_2 = \int_a^b \left[|U_x|^2 - \frac{1}{2}q|U|^4 \right] dx,$$

which are calculated by using the trapezoidal rule. The order of convergence for space and time steps are computed by the formulas

$$\text{order} = \frac{\log |(L_\infty)_{h_i} / (L_\infty)_{h_{i+1}}|}{\log |h_i / h_{i+1}|}, \quad \text{order} = \frac{\log |(L_\infty)_{\Delta t_i} / (L_\infty)_{\Delta t_{i+1}}|}{\log |\Delta t_i / \Delta t_{i+1}|},$$

where $(L_\infty)_{h_i}$ is the error norm for space step and $(L_\infty)_{\Delta t_i}$ is the error norm for time step.

4.1. The propagation of single soliton

The single soliton solution of the SE, which is formed by the equilibrium between nonlinearity and dispersion is

$$U(x, t) = \lambda \sqrt{\frac{2}{q}} e^{i\eta} \operatorname{sech} \lambda (x - \xi t),$$

where $\eta = \frac{\xi}{2}x - \frac{1}{4}(\xi^2 - \lambda^2)t$. This soliton, whose magnitude depends on the parameters ξ , λ and q , travels with the velocity ξ . In this experiment, we choose $\xi = 4$, $\lambda = 1$, $q = 2$, $\Delta t = 0.00001$ and $h = 0.05$ with range $[-30, 30]$, the simulation is done up to $t = 1$, the initial condition is obtained from the solution for $t = 0$ and the conditions $U(-30, t)$, $U(30, t)$, $U_x(-30, t)$ and $U_x(30, t)$ are selected as zero. The values of the error norm L_∞ and the conservation invariants are listed in Table 1. Method 3 has the smallest value of the error norm. The simulation of the travelling wave for Method 3 is plotted in Figure 1 and it is observed that there is no change in its form throughout the simulation period. The absolute errors calculated for the wave up to $t = 1$ can be seen in Figure 2 for all of the proposed methods.

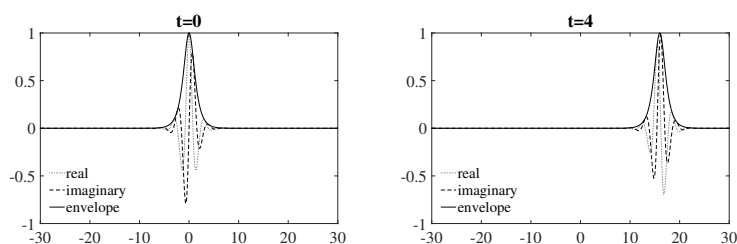


Fig. 1 The simulation of the travelling wave for Method 3

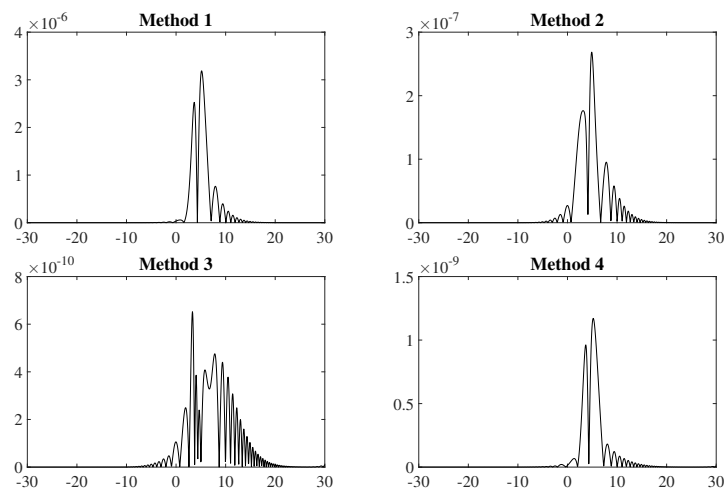


Fig. 2 The absolute error profiles

The rate of convergence of the proposed methods for space and time are shown in Tables 2 and 3. For all of the methods the rate of convergence for time is almost same and 2, and the rate of convergence for space is over 4 for Methods 1 and 2, over 6 for Methods 3 and 4. In addition, it can be said that all of the proposed methods give more accurate results as the space and time steps become smaller in Tables 2 and 3.

TABLE 1. The error norm L_∞ and the conservation invariants with range $[-30, 30]$, $h = 0.05$ and $\Delta t = 0.00001$ at time $t = 1$.

Method	C_1	C_2	L_∞
Method 1	2.0000000000	7.3501541167	3.19e-06
Method 2	2.0000000000	7.3333173459	2.69e-07
Method 3	2.0000000000	7.3333357360	6.52e-10
Method 4	2.0000000000	7.3333333281	1.17e-09
Exact	2.0000000000	7.3333333333	0

TABLE 2. The error norm L_∞ and the order of convergence with range $[-30, 30]$, $h = 0.05$ at time $t = 1$.

Δt	Method 1		Method 2		Method 3		Method 4	
	L_∞	order	L_∞	order	L_∞	order	L_∞	order
1e-1	1.12e-1		1.12e-1		1.12e-1		1.12e-1	
5e-2	2.72e-2	2.04	2.72e-2	2.04	2.72e-2	2.04	2.72e-2	2.04
2e-2	4.22e-3	2.03	4.22e-3	2.04	4.22e-3	2.03	4.22e-3	2.03
1e-1	1.05e-3	2.01	1.05e-3	2.01	1.05e-3	2.01	1.05e-3	2.01
5e-3	2.62e-4	2.00	2.62e-4	2.00	2.62e-4	2.00	2.62e-4	2.00
2e-3	4.20e-5	2.00	4.20e-5	2.00	4.19e-5	2.00	4.19e-5	2.00
1e-3	1.05e-5	2.00	1.05e-5	2.00	1.05e-5	2.00	1.05e-5	2.00

TABLE 3. The error norm L_∞ and the order of convergence with range $[-30, 30]$, $\Delta t = 0.00001$ at time $t = 1$.

h	Method 1		Method 2		Method 3		Method 4	
	L_∞	order	L_∞	order	L_∞	order	L_∞	order
1	5.93e-1		1.94e-1		3.80e-1		1.54e-1	
0.5	4.90e-2	3.60	4.95e-3	5.29	5.07e-3	6.23	8.98e-4	7.42
0.2	8.96e-4	4.37	7.16e-5	4.62	5.06e-6	7.54	1.27e-6	7.16
0.1	5.19e-5	4.11	4.33e-6	4.05	6.71e-8	6.24	1.30e-8	6.61

4.2. The interaction of two solitons

As a second test problem, let take the initial condition

$$U(x, 0) = U_1(x, 0) + U_2(x, 0), \quad U_j(x, 0) = \lambda_j \sqrt{\frac{2}{q}} e^{i\eta_j} \operatorname{sech} \lambda_j (x - x_j),$$

$$\eta_j = \frac{\xi_j}{2} (x - x_j).$$

By choosing the parameters $q = 2, h = 0.1, \Delta t = 0.01, \lambda_1 = \lambda_2 = 1, \xi_1 = 4, x_1 = -10, \xi_2 = -4, x_2 = 10$ over the interval $[-30, 30]$, two solitons appear and they move in opposite directions, so a collision is occurred. These interacted profiles obtained by Method 3 are shown in Figure 3 at various times. It can be seen in Figure 3 that two waves moving in the opposite direction collide, then separate and they appear to maintain their first forms.

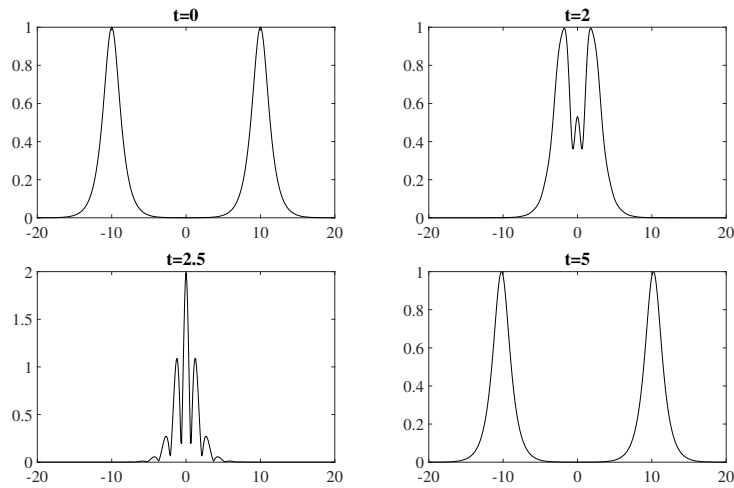


Fig. 3 The interaction of two solitons for Method 3

TABLE 4. The conservation invariants with $\Delta t = 0.01$ at time $t = 5$.

Method	$h = 0.1$		$h = 0.01$	
	C_1	C_2	C_1	C_2
Method 1	3.9999994111	14.8023658179	3.9999994314	14.6679992477
Method 2	3.9999994469	14.6661385046	3.9999994314	14.6666558174
Method 3	3.9999994311	14.6667354303	3.9999994314	14.6666558762
Method 4	3.9999994314	14.6666551815	3.9999994314	14.6666558685
Exact	4	14.6666666667	4	14.6666666667

The calculated values of the conservation constants are tabulated in Table 4. The best approximation value to the conservation constants has been obtained by Method 3 when $h = 0.01$ is chosen and Method 4 when $h = 0.1$ is chosen.

5. CONCLUSION

In this study, the fully discretized form of the SE has been obtained by the help of the four various trigonometric B-spline Galerkin methods and the accuracy of the proposed methods are investigated by comparing with each others. According to the results obtained in the first test problem in which the motion of soliton wave is examined, it is seen that by the too small selection for the time step Δt , the errors due to time discretization are minimized. Also it is seen that the errors decreases when the degree of spline function used in the calculations increases. By the use of high degree spline functions the better results have been obtained, although the cost of the process increases. From the second test problem in which the interaction of two solitary wave is examined, it can be said that the conservation conditions are nearly the same by selecting the various values of the h for all of the methods. In conclusion, the methods give the considerable results for both of the test problems.

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COFINITELY ESSENTIAL G-SUPPLEMENTED MODULES

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Received 17 January, 2021

Abstract. Let M be an R -module. If every cofinite essential submodule of M has a g -supplement in M , then M is called a cofinitely essential g -supplemented (or briefly cofinitely eg -supplemented) module. In this work, some properties of these modules are investigated. It is proved that every factor module and every homomorphic image of a cofinitely eg -supplemented module are cofinitely eg -supplemented. Let M be a cofinitely eg -supplemented module. Then every M -generated R -module is cofinitely eg -supplemented.

2010 Mathematics Subject Classification: 16D10; 16D70

Keywords: cofinite submodules, essential submodules, g -small submodules, g -supplemented modules

1. INTRODUCTION

Throughout this paper all rings are associative with identity and all modules are unital left modules.

Let R be a ring and M be an R -module. We denote a submodule N of M by $N \leq M$. A submodule U of an R -module M is called a *cofinite submodule* of M if M/U is finitely generated. Let M be an R -module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a *small* (or *superfluous*) submodule of M and denoted by $N \ll M$. A submodule N of an R -module M is called an *essential* submodule, denoted by $N \trianglelefteq M$, in case $K \cap N \neq 0$ for every submodule $K \neq 0$, or equivalently, $N \cap L = 0$ for $L \leq M$ implies that $L = 0$. Let M be an R -module and K be a submodule of M . K is called a *generalized small* (briefly, *g -small*) submodule of M if for every essential submodule T of M with the property $M = K + T$ implies that $T = M$, we denote this by $K \ll_g M$ (in [15], it is called an *e -small submodule* of M and denoted by $K \ll_e M$). Let M be an R -module and $U, V \leq M$. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M . M is said to be *supplemented* if every submodule of M has a supplement in M . M is said to be *cofinitely supplemented* if every cofinite submodule of M has a supplement in M . M is said to be *essential supplemented* (briefly, *e -supplemented*) if every essential submodule of M has a supplement in M . M is said to be *cofinitely*

essential supplemented (briefly, *cofinitely e-supplemented*) if every cofinite essential submodule of M has a supplement in M . Let M be an R -module and $U, V \leq M$. If $M = U + V$ and $M = U + T$ with $T \leq V$ implies that $T = V$, or equivalently, $M = U + V$ and $U \cap V \ll_g V$, then V is called a *g-supplement* of U in M . M is said to be *g-supplemented* if every submodule of M has a g-supplement in M . M is said to be *essential g-supplemented* if every essential submodule of M has a g-supplement in M . M is said to be *cofinitely g-supplemented* if every cofinite submodule of M has a g-supplement in M . The intersection of all maximal submodules of an R -module M is called the *radical* of M and denoted by $RadM$. If M have no maximal submodules, then we denote $RadM = M$. The intersection of all essential maximal submodules of an R -module M is called the *generalized radical* (briefly, *g-radical*) of M and denoted by $Rad_g M$ (in [15], it is denoted by $Rad_e M$). If M have no essential maximal submodules, then we denote $Rad_g M = M$. Let M be an R -module and $K \leq V \leq M$. We say V lies above K in M if $V/K \ll M/K$.

More details about supplemented modules are in [3, 14]. More informations about cofinitely supplemented modules are in [1]. More details about essential supplemented modules are in [11, 12]. More details about cofinitely essential supplemented modules are in [7, 8]. More informations about g-small submodules and g-supplemented modules are in [5, 6]. The definition of cofinitely g-supplemented modules and more informations about these modules are in [4]. The definition of essential g-supplemented modules and some properties of them are in [9].

Lemma 1. *Let M be an R -module and $K, N \leq M$. Consider the following conditions.*

- (1) *If $K \leq N$ and N is a generalized small submodule of M , then K is a generalized small submodule of M .*
- (2) *If K is contained in N and a generalized small submodule of N , then K is a generalized small submodule in submodules of M which contain N .*
- (3) *If $K \ll_g L$ and $N \ll_g T$ with $L, T \leq M$, then $K + N \ll_g L + T$.*
- (4) $Rad_g M = \sum_{L \ll_g M} L$.
- (5) *Let T be an R -module and $f : M \rightarrow T$ be an R -module homomorphism. If $K \ll_g M$, then $f(K) \ll_g T$. Here $f(Rad_g M) \leq Rad_g T$.*

Proof. See [6, Lemma 1 and Lemma 3]. □

2. COFINITELY ESSENTIAL G-SUPPLEMENTED MODULES

Definition 1. Let M be an R -module. If every cofinite essential submodule of M has a g-supplement in M , then M is called a *cofinitely essential g-supplemented* (or briefly *cofinitely eg-supplemented*) module. (See also [10]).

Clearly we can see that every essential g-supplemented module is cofinitely eg-supplemented. But the converse is not true in general (see Example 1 and Example 2). Every cofinitely essential supplemented module is cofinitely eg-supplemented.

Proposition 1. *Let M be a cofinitely eg-supplemented R -module. If every nonzero submodule of M is essential in M , then M is cofinitely supplemented.*

Proof. Clear from definitions. $\square$

Lemma 2. *Let M be a finitely generated R -module. Then M is essential g-supplemented if and only if M is cofinitely eg-supplemented.*

Proof. Clear, since every submodule of M is cofinite. $\square$

Lemma 3. *Let M be a cofinitely eg-supplemented module. Then $M/\text{Rad}_g M$ have no proper cofinite essential submodules.*

Proof. Let $U/\text{Rad}_g M$ be a cofinite essential submodule $M/\text{Rad}_g M$. Then $U \trianglelefteq M$ and since $\frac{M}{U} \cong \frac{M/\text{Rad}_g M}{U/\text{Rad}_g M}$, U is a cofinite essential submodule of M . Since M is cofinitely eg-supplemented, U has a g-supplement V in M . Here $M = U + V$ and $U \cap V \ll_g V$. Since $U \cap V \ll_g V$, by Lemma 1, $U \cap V \leq \text{Rad}_g M$. Then $M/\text{Rad}_g M = (U + V)/\text{Rad}_g M = U/\text{Rad}_g M + (V + \text{Rad}_g M)/\text{Rad}_g M$ and $U/\text{Rad}_g M \cap (V + \text{Rad}_g M)/\text{Rad}_g M = (U \cap V + \text{Rad}_g M)/\text{Rad}_g M = \text{Rad}_g M/\text{Rad}_g M = 0$. Hence $M/\text{Rad}_g M = U/\text{Rad}_g M \oplus (V + \text{Rad}_g M)/\text{Rad}_g M$ and since $U/\text{Rad}_g M \trianglelefteq M/\text{Rad}_g M$, $U/\text{Rad}_g M = M/\text{Rad}_g M$. Thus $M/\text{Rad}_g M$ have no proper cofinite essential submodules. $\square$

Lemma 4. *Let M be an R -module, U be a cofinite essential submodule of M and $N \leq M$. If $U + N$ has a g-supplement in M and N is cofinitely eg-supplemented, then U has a g-supplement in M .*

Proof. Let X be a g-supplement of $U + N$ in M . Then $M = U + N + X$ and $(U + N) \cap X \ll_g X$. Since $U \trianglelefteq M$, $U + X \trianglelefteq M$ and $N \cap (U + X) \trianglelefteq N$. Since U is a cofinite submodule of M , $U + X$ is a cofinite submodule of M . Hence by $\frac{M}{U+X} = \frac{U+N+X}{U+X} \cong \frac{N}{N \cap (U+X)}$, $N \cap (U + X)$ is a cofinite essential submodule of M . Since N is cofinitely eg-supplemented, $N \cap (U + X)$ has a g-supplement Y in N . Since Y is a g-supplement of $N \cap (U + X)$ in N , $N = N \cap (U + X) + Y$ and $(U + X) \cap Y = N \cap (U + X) \cap Y \ll_g Y$. Then $M = U + N + X = U + N \cap (U + X) + Y + X = U + X + Y$ and, by Lemma 1, $U \cap (X + Y) \leq (U + Y) \cap X + (U + X) \cap Y \leq (U + N) \cap X + (U + X) \cap Y \ll_g X + Y$. Hence $X + Y$ is a g-supplement of U in M . $\square$

Corollary 1. *Let M be an R -module, U be a cofinite essential submodule of M and $N_1, N_2, \dots, N_k \leq M$. If $U + N_1 + N_2 + \dots + N_k$ has a g-supplement in M and N_i is cofinitely eg-supplemented for $i = 1, 2, \dots, k$, then U has a g-supplement in M .*

Proof. Clear from Lemma 4. $\square$

Lemma 5. Let $M = \sum_{\lambda \in \Lambda} M_\lambda$. If M_λ is cofinitely eg-supplemented for every $\lambda \in \Lambda$, then M is also cofinitely eg-supplemented.

Proof. Let U be a cofinite essential submodule of M . Since U is a cofinite submodule of M , it is easy to see that there exist $\lambda_1, \lambda_2, \dots, \lambda_n \in \Lambda$ such that $M = U + M_{\lambda_1} + M_{\lambda_2} + \dots + M_{\lambda_n}$. Then $U + M_{\lambda_1} + M_{\lambda_2} + \dots + M_{\lambda_n}$ has a trivial g-supplement 0 in M and since M_{λ_i} is cofinitely eg-supplemented for every $i = 1, 2, \dots, n$, by Corollary 1, U has a g-supplement in M . Hence M is cofinitely eg-supplemented. $\square$

Lemma 6. Let $f : M \longrightarrow N$ be an R -module epimorphism, $U, V \leq M$ and $Ke f \leq U$. If V is a g-supplement of U in M , then $f(V)$ is a g-supplement of $f(U)$ in N .

Proof. Since V is a g-supplement of U in M , $M = U + V$ and $U \cap V \ll_g V$. Then $N = f(M) = f(U + V) = f(U) + f(V)$. Let $x \in f(U) \cap f(V)$. Then there exist $u \in U$ and $v \in V$ with $x = f(u) = f(v)$. Here $f(v - u) = f(v) - f(u) = 0$ and $v - u \in Ke f \leq U$. Then $v = v - u + u \in U$ and since $v \in V$, $v \in U \cap V$. Hence $x = f(v) \in f(U \cap V)$ and $f(U) \cap f(V) \leq f(U \cap V)$. Here clearly we can see that $f(U \cap V) \leq f(U) \cap f(V)$ and $f(U) \cap f(V) = f(U \cap V)$. Since $U \cap V \ll_g V$, by Lemma 1, $f(U) \cap f(V) = f(U \cap V) \ll_g f(V)$. Hence $f(V)$ is a g-supplement of $f(U)$ in N , as desired. $\square$

Lemma 7. Every homomorphic image of a cofinitely eg-supplemented module is cofinitely eg-supplemented.

Proof. Let M be a cofinitely eg-supplemented R -module and $f : M \longrightarrow N$ be an R -module epimorphism. Let U be a cofinite essential submodule of N . Since $U \trianglelefteq N$, by [14, 17.3 (3)], $f^{-1}(U) \trianglelefteq M$. Let $p : N \longrightarrow N/U$ be a canonical epimorphism. Since $(pf)(x) = p(f(x)) = f(x) + U = U$ for every $x \in f^{-1}(U)$, $x \in Ke(pf)$ and $f^{-1}(U) \leq Ke(pf)$. Let $y \in Ke(pf)$. Then $U = (pf)(y) = p(f(y)) = f(y) + U$ and $f(y) \in U$. Hence $y \in f^{-1}(U)$ and $Ke(pf) \leq f^{-1}(U)$. Since $f^{-1}(U) \leq Ke(pf)$, $Ke(pf) = f^{-1}(U)$. Hence $M/f^{-1}(U) \cong N/U$ and $f^{-1}(U)$ is a cofinite submodule of M . Moreover, $f^{-1}(U) \trianglelefteq M$. Since M is cofinitely eg-supplemented, $f^{-1}(U)$ has a g-supplement V in M . Since $Ke f \leq f^{-1}(U)$, by Lemma 6, $f(V)$ is a g-supplement of $f(f^{-1}(U)) = U$ in N . Hence N is cofinitely eg-supplemented, as desired. $\square$

Corollary 2. Every factor module of a cofinitely eg-supplemented module is cofinitely eg-supplemented.

Proof. Clear from Lemma 7. $\square$

Lemma 8. Let M be a cofinitely eg-supplemented R -module. Then every M -generated R -module is cofinitely eg-supplemented.

Proof. Let N be a M -generated R -module. Then there exist an index set Λ and an R -module epimorphism $f : M^{(\Lambda)} \longrightarrow N$. Since M is cofinitely eg-supplemented,

by Lemma 5, $M^{(\Lambda)}$ is cofinitely eg-supplemented. Then by Lemma 7, N is cofinitely eg-supplemented, as desired. $\square$

Proposition 2. *Let R be a ring. Then the R -module ${}_R R$ is essential g-supplemented if and only if every R -module is cofinitely eg-supplemented.*

Proof. $(\implies)$ Clear from Lemma 8.

$(\impliedby)$ Clear from Lemma 2, since ${}_R R$ is finitely generated. $\square$

Definition 2. Let M be an R -module and $X \leq M$. If X is a g-supplement of a cofinite essential submodule of M , then X is called a ceg-supplement submodule in M .

Let M be an R -module. It is defined the relation ' β^* ' on the set of submodules of an R -module M by $X\beta^*Y$ if and only if $Y + K = M$ for every $K \leq M$ such that $X + K = M$ and $X + T = M$ for every $T \leq M$ such that $Y + T = M$ (See [2]). It is defined the relation ' β_g^* ' on the set of submodules of an R -module M by $X\beta_g^*Y$ if and only if $Y + K = M$ for every $K \trianglelefteq M$ such that $X + K = M$ and $X + T = M$ for every $T \trianglelefteq M$ such that $Y + T = M$ (See [13]).

Lemma 9. *Let M be an R -module. If every cofinite essential submodule of M is β_g^* equivalent to a ceg-supplement submodule in M , then M is cofinitely eg-supplemented.*

Proof. Let X be a cofinite essential submodule of M . By hypothesis, there exists a ceg-supplement submodule V in M with $X\beta_g^*V$. Let V be a g-supplement of a cofinite essential submodule U in M . Then $M = U + V$ and $U \cap V \ll_g V$. Since U is a cofinite essential submodule of M , by hypothesis, there exists a ceg-supplement submodule Y in M with $U\beta_g^*Y$. Let S be a cofinite essential submodule of M and Y be a g-supplement of S in M . Then $M = S + Y$ and $S \cap Y \ll_g Y$. Since $X\beta_g^*V$ and $M = U + V$, $M = X + U$ and since $U\beta_g^*Y$ and $X \trianglelefteq M$, $M = X + Y$. Assume $X + T = M$ with $T \trianglelefteq Y$. Then $Y = Y \cap M = Y \cap (X + T) = X \cap Y + T$. By using [3, Lemma 1.24], we can see that $M = S + Y = X \cap Y + S + T = Y + X \cap (S + T) = U + X \cap (S + T) = X + U \cap (S + T) = V + U \cap (S + T) = U \cap V + S + T$. Since $U \cap V \ll_g M$ and $S + T \trianglelefteq M$, $M = S + T$ and since Y is a g-supplement of S in M and $T \trianglelefteq Y$, $T = Y$. Hence Y is a g-supplement of X in M . Thus M is cofinitely eg-supplemented. $\square$

Corollary 3. *Let M be an R -module. If every cofinite essential submodule of M is β^* equivalent to an ceg-supplement submodule in M , then M is cofinitely eg-supplemented.*

Proof. Clear from Lemma 9. $\square$

Corollary 4. *Let M be an R -module. If every cofinite essential submodule of M lies above an ceg-supplement submodule in M , then M is cofinitely eg-supplemented.*

Proof. Clear from Corollary 3. $\square$

Corollary 5. *Let M be an R -module. If every cofinite essential submodule of M is a ceg-supplement submodule in M , then M is cofinitely eg-supplemented.*

Proof. Clear from Corollary 4. □

Lemma 10. *Let M be an R -module. If every cofinite submodule of M is β^* equivalent to a ceg-supplement submodule in M , then M is cofinitely g-supplemented.*

Proof. Let X be a cofinite submodule of M . By hypothesis, there exists a ceg-supplement submodule V in M with $X\beta^* V$. Let V be a g-supplement of a cofinite essential submodule U in M . Then $M = U + V$ and $U \cap V \ll_g V$. Since U is a cofinite submodule of M , by hypothesis, there exists a ceg-supplement submodule Y in M with $U\beta^* Y$. Let S be a cofinite essential submodule of M and Y be a g-supplement of S in M . Then $M = S + Y$ and $S \cap Y \ll_g Y$. Since $X\beta^* V$ and $M = U + V$, $M = X + U$ and since $U\beta^* Y$, $M = X + Y$. Assume $X + T = M$ with $T \trianglelefteq Y$. Then $Y = Y \cap M = Y \cap (X + T) = X \cap Y + T$. By using [3, Lemma 1.24], we can see that $M = S + Y = X \cap Y + S + T = Y + X \cap (S + T) = U + X \cap (S + T) = X + U \cap (S + T) = V + U \cap (S + T) = U \cap V + S + T$. Since $U \cap V \ll_g M$ and $S + T \trianglelefteq M$, $M = S + T$ and since Y is a g-supplement of S in M and $T \trianglelefteq Y$, $T = Y$. Hence Y is a g-supplement of X in M . Thus M is cofinitely g-supplemented. □

Corollary 6. *Let M be an R -module. If every cofinite submodule of M lies above a ceg-supplement submodule in M , then M is cofinitely g-supplemented.*

Proof. Clear from Lemma 10. □

Corollary 7. *Let M be an R -module. If every cofinite submodule of M is a ceg-supplement submodule in M , then M is cofinitely g-supplemented.*

Proof. Clear from Lemma 10. □

Example 1. Consider the $\mathbb{Z}$ -module $\mathbb{Q}$. Since ${}_{\mathbb{Z}}\mathbb{Q}$ have no proper cofinite essential submodules, ${}_{\mathbb{Z}}\mathbb{Q}$ is cofinitely eg-supplemented. But, since ${}_{\mathbb{Z}}\mathbb{Q}$ is not supplemented and every nonzero submodule of ${}_{\mathbb{Z}}\mathbb{Q}$ is essential in ${}_{\mathbb{Z}}\mathbb{Q}$, ${}_{\mathbb{Z}}\mathbb{Q}$ is not essential g-supplemented.

Example 2. Consider the $\mathbb{Z}$ -module $\mathbb{Q} \oplus \mathbb{Z}_{p^2}$ for a prime p . It is easy to check that $\text{Rad}_g \mathbb{Z}_{p^2} \neq \mathbb{Z}_{p^2}$. By [6, Lemma 4], $\text{Rad}_g (\mathbb{Q} \oplus \mathbb{Z}_{p^2}) = \text{Rad}_g \mathbb{Q} \oplus \text{Rad}_g \mathbb{Z}_{p^2} \neq \mathbb{Q} \oplus \mathbb{Z}_{p^2}$. Since $\mathbb{Q}$ and $\mathbb{Z}_{p^2}$ are cofinitely eg-supplemented, by Lemma 5, $\mathbb{Q} \oplus \mathbb{Z}_{p^2}$ is cofinitely eg-supplemented. But $\mathbb{Q} \oplus \mathbb{Z}_{p^2}$ is not essential g-supplemented.

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LIPSCHITZ STABILITY OF n -CUBIC FUNCTIONAL EQUATIONS AND APPLICATIONS

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Received 23 January, 2021

Abstract. In this paper, we establish the stability of n -cubic functional equations in Lipschitz spaces and as a consequence we give the stability of cubic functional equations.

2010 Mathematics Subject Classification: 39B82; 39B52

Keywords: n -cubic functional equation, Lipschitz space, stability

1. INTRODUCTION

Lipschitz spaces have a rich and beautiful algebra structure and these spaces possess various universal properties. Some of open problems in this area are given in Chapter 7 of [25]. Much work has been done on the Banach space of Lipschitz functions (cf. [6]). The Lipschitz condition is one of the central concepts of some important subjects. The Lipschitz condition was used in the elementary theory of ordinary differential equations, real analysis, and metric geometry. There are also striking applications to topology. Every topological manifold outside dimension four admits a unique Lipschitz structure, while such a manifold may have no smooth or piecewise linear structures or it may have many. On a practical side, questions about Lipschitz functions arise in image processing and in the study of internet search engines. (cf. [10, 21]).

Let G be an Abelian group and V a vector space. We say that $S(V)$, a family of subsets of V , is linearly invariant if it is closed under the addition and scalar multiplication defined as usual sense and translation invariant, in the sense that $x + A \in S(V)$, for every $A \in S(V)$ and every $x \in V$ (see [3]). Note that $S(V)$ contains all singleton subsets of V . For instance, $CB(V)$ the family of all closed balls is a linearly invariant family in a normed vector space V . By $B(G^n, S(V))$ we denote the family of all functions $f : G^n \rightarrow V$ such that $\mathbb{I}_m f \subset A$ for some $A \in S(V)$, where G^n is the Cartesian product of G . This family is a vector space and contains all constant functions.

The problem of the stability of functional equations has been posed by Ulam in [24]. Hyers in [5] gave an affirmative partial answer for the stability of the linear

functional equation for Banach spaces. For more detailed definitions of such terminology one can refer to [8, 9, 15, 18] and [16, 17, 19, 20] and references therein.

In Lipschitz spaces, Czerwik et al. [3] and Tabor [22, 23] studied the stability type problems for some functional equations. The author of the present paper proved the stability of quadratic, cubic and quartic functional equations in Lipschitz spaces (cf. [4, 11–14]). Czerwik et al. [3] considered the stability of the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \quad (1.1)$$

Jun and Kim [7] introduced the following cubic functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x), \quad (1.2)$$

which are somewhat different from (1.1). They established the general solution and the Hyers-Ulam-Rassias stability problem of (1.2) for mapping from a real vector space to a Banach space. Bae and Park [1] proved the general solution and the stability of the following 2-variable quadratic functional equation

$$f(x+z, y+w) + f(x-z, y-w) = 2f(x, y) + 2f(z, w)$$

in complete normed spaces.

In this paper, we introduce the n -variable cubic functional equations as follows

$$\begin{aligned} 2f(x_1+y_1, \dots, x_n+y_n) + 2f(x_1-y_1, \dots, x_n-y_n) + 12f(x_1, \dots, x_n) \\ = f(2x_1+y_1, \dots, 2x_n+y_n) + f(2x_1-y_1, \dots, 2x_n-y_n). \end{aligned} \quad (1.3)$$

We say that a function $f : G^n \rightarrow V$ is n -cubic if f satisfies (1.3). We verify the stability of the n -cubic functional equation in Lipschitz spaces and as a consequence we give the stability of the cubic functional equation.

2. APPROXIMATION WITH $\mathbf{D}$ -LIPSCHITZ FUNCTIONS

Suppose that $f : G^n \rightarrow V$ is a function. We say that f is an odd function if $f(-x_1, \dots, -x_n) = -f(x_1, \dots, x_n)$ for all $(x_1, \dots, x_n) \in G^n$. We consider the n -variable cubic difference as follows:

$$\begin{aligned} Sf(x_1, \dots, x_n; y_1, \dots, y_n) := 2f(x_1+y_1, \dots, x_n+y_n) + 2f(x_1-y_1, \dots, x_n-y_n) \\ + 12f(x_1, \dots, x_n) - f(2x_1+y_1, \dots, 2x_n+y_n) - f(2x_1-y_1, \dots, 2x_n-y_n) \end{aligned}$$

for all $(x_1, \dots, x_n), (y_1, \dots, y_n) \in G^n$. A function f is n -cubic if

$$Sf(x_1, \dots, x_n; y_1, \dots, y_n) = 0.$$

It is clear that if a function f is n -cubic, then $f(0, \dots, 0) = 0$. Let $S(V)$ be a family of subsets of V . It is easy to verify that $S(V)$ contains all singleton subsets of V . Following [2, 22] let $\mathbf{d} : G^n \times G^n \rightarrow S(V)$ be a set-valued function such that

$$\begin{aligned} \mathbf{d}((x_1+a_1, \dots, x_n+a_n), (y_1+a_1, \dots, y_n+a_n)) &= \mathbf{d}((a_1+x_1, \dots, a_n+x_n), (a_1+y_1, \dots, a_n+y_n)) \\ &= \mathbf{d}((x_1, \dots, x_n), (y_1, \dots, y_n)) \end{aligned}$$

for all $(x_1, \dots, x_n), (y_1, \dots, y_n), (a_1, \dots, a_n) \in G^n$. A function $f : G^n \rightarrow V$ is called **d**-Lipschitz if

$$f(x_1, \dots, x_n) - f(y_1, \dots, y_n) \in \mathbf{d}((x_1, \dots, x_n), (y_1, \dots, y_n))$$

for all $(x_1, \dots, x_n), (y_1, \dots, y_n) \in G^n$. Let (G^n, d) be a metric group and V a normed space. A function $\mathbf{mc}_f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a module of continuity of $f : G^n \rightarrow V$ if for all $\varepsilon > 0$ and all $(x_1, \dots, x_n), (y_1, \dots, y_n) \in G^n$ the condition $d((x_1, \dots, x_n), (y_1, \dots, y_n)) \leq \varepsilon$ implies $\|f(x_1, \dots, x_n) - f(y_1, \dots, y_n)\| \leq \mathbf{mc}_f(\varepsilon)$. A function $f : G^n \rightarrow V$ is called Lipschitz function if it satisfies the condition

$$\|f(x_1, \dots, x_n) - f(y_1, \dots, y_n)\| \leq Ld((x_1, \dots, x_n), (y_1, \dots, y_n)), \quad (2.1)$$

where $L > 0$ is a constant and $(x_1, \dots, x_n), (y_1, \dots, y_n) \in G^n$. Let $Lip(G^n, V)$ be the Lipschitz space consisting of all bounded Lipschitz functions with the norm

$$\|f\|_{Lip} := \|f\|_\infty + P(f),$$

where $\|\cdot\|_\infty$ is the supremum norm and

$$P(f) = \sup \left\{ \frac{\|f(x_1, \dots, x_n) - f(y_1, \dots, y_n)\|}{d((x_1, \dots, x_n), (y_1, \dots, y_n))} : (x_1, \dots, x_n), (y_1, \dots, y_n) \in G^n, \right. \\ \left. (x_1, \dots, x_n) \neq (y_1, \dots, y_n) \right\}.$$

Definition 1. We say that $B(G^n, S(V))$ admits a left invariant mean (briefly LIM), if the family $S(V)$ is linearly invariant and there exists a linear operator η from $B(G^n, S(V))$ to V such that

- (i) if $\mathbb{I}mf \subset A$ for some $A \in S(V)$, then $\eta[f] \in A$,
- (ii) if $f \in B(G^n, S(V))$ and $(a_1, \dots, a_n) \in G^n$, then $\eta[f^{a_1, \dots, a_n}] = \eta[f]$, where we define $f^{a_1, \dots, a_n}(x_1, \dots, x_n) = f(x_1 + a_1, \dots, x_n + a_n)$.

Definition 2. Consider an Abelian group $(G^n, +)$ with a metric d invariant under translation, i.e., satisfying the condition

$$\begin{aligned} d((x_1 + a_1, \dots, x_n + a_n), (y_1 + a_1, y_n + a_n)) \\ = d((a_1 + x_1, \dots, a_n + x_n), (a_1 + y_1, \dots, a_n + y_n)) \\ = d((x_1, \dots, x_n), (y_1, \dots, y_n)) \end{aligned}$$

for all $(x_1, \dots, x_n), (y_1, \dots, y_n), (a_1, \dots, a_n) \in G^n$. We say that a metric ρ is a product metric in G^{2n} if it is an invariant metric and the following condition holds

$$\begin{aligned} \rho((x_1, \dots, x_n; a_1, \dots, a_n), (y_1, \dots, y_n; a_1, \dots, a_n)) \\ = \rho((a_1, \dots, a_n; x_1, \dots, x_n), (a_1, \dots, a_n; y_1, \dots, y_n)) \\ = d((x_1, \dots, x_n), (y_1, \dots, y_n)) \end{aligned}$$

for all $(x_1, \dots, x_n; a_1, \dots, a_n), (y_1, \dots, y_n; a_1, \dots, a_n) \in G^{2n}$.

Theorem 1. *Let G be an Abelian group, V a vector space, and $f : G^n \longrightarrow V$ an odd function. If the family $B(G^n, S(V))$ admits LIM and $Sf(t_1, \dots, t_n; \cdot, \dots, \cdot) : G^n \longrightarrow V$ is $\mathbf{d}$ -Lipschitz for all $(t_1, \dots, t_n) \in G^n$, then there exists an n -cubic function Γ such that $f - \Gamma$ is $\frac{1}{12}\mathbf{d}$ -Lipschitz.*

Proof. For every $(a_1, \dots, a_n) \in G^n$ we define $F_{a_1, \dots, a_n} : G^n \longrightarrow V$ by

$$F_{a_1, \dots, a_n}(x_1, \dots, x_n) := \frac{1}{12}f(2a_1 + x_1, \dots, 2a_n + x_n) + \frac{1}{12}f(2a_1 - x_1, \dots, 2a_n - x_n) \\ - \frac{1}{6}f(a_1 + x_1, \dots, a_n + x_n) - \frac{1}{6}f(a_1 - x_1, \dots, a_n - x_n).$$

Fix $(a_1, \dots, a_n) \in G^n$. We see that

$$\begin{aligned} F_{a_1, \dots, a_n}(x_1, \dots, x_n) &= \frac{1}{12}f(2a_1 + x_1, \dots, 2a_n + x_n) + \frac{1}{12}f(2a_1 - x_1, \dots, 2a_n - x_n) \\ &\quad - \frac{1}{6}f(a_1 + x_1, \dots, a_n + x_n) - \frac{1}{6}f(a_1 - x_1, \dots, a_n - x_n) \\ &= f(0, \dots, 0) + \frac{1}{12}f(2a_1 + x_1, \dots, 2a_n + x_n) \\ &\quad + \frac{1}{12}f(2a_1 - x_1, \dots, 2a_n - x_n) \\ &\quad - \frac{1}{6}f(a_1 + x_1, \dots, a_n + x_n) - \frac{1}{6}f(a_1 - x_1, \dots, a_n - x_n) \\ &\quad - f(a_1, \dots, a_n) + f(a_1, \dots, a_n) - f(0, \dots, 0) \\ &= \frac{1}{12}Sf(0, \dots, 0; x_1, \dots, x_n) - \frac{1}{12}Sf(a_1, \dots, a_n; x_1, \dots, x_n) \\ &\quad + f(a_1, \dots, a_n) - f(0, \dots, 0). \end{aligned}$$

From the fact that $Sf(t_1, \dots, t_n; \cdot, \dots, \cdot)$ is $\mathbf{d}$ -Lipschitz and $S(V)$ is translation invariant, we detect that $\text{Im}F_{a_1, \dots, a_n} \subset A$ for some $A \in S(V)$ and hence $F_{a_1, \dots, a_n} \in B(G^n, S(V))$. By assumption the family $B(G^n, S(V))$ admits LIM and so there exists a linear operator $\eta : B(G^n, S(V)) \longrightarrow V$ such that

- (i) $\eta[F_{a_1, \dots, a_n}] \in A$ for some $A \in S(V)$,
- (ii) if for $(y_1, \dots, y_n) \in G^n$, $F_{a_1, \dots, a_n}^{y_1, \dots, y_n} : G^n \longrightarrow V$ is defined by

$$F_{a_1, \dots, a_n}^{y_1, \dots, y_n}(t_1, \dots, t_n) := F_{a_1, \dots, a_n}(t_1 + y_1, \dots, t_n + y_n)$$

for every $(t_1, \dots, t_n) \in G^n$, then $F_{a_1, \dots, a_n}^{y_1, \dots, y_n} \in B(G^n, S(V))$ and

$$\eta[F_{a_1, \dots, a_n}] = \eta[F_{a_1, \dots, a_n}^{y_1, \dots, y_n}].$$

Since $Sf(t_1, \dots, t_n; \cdot, \dots, \cdot)$ is a $\mathbf{d}$ -Lipschitz function,

$$\frac{1}{12}Sf(t_1, \dots, t_n; x_1, \dots, x_n) - \frac{1}{12}Sf(t_1, \dots, t_n; y_1, \dots, y_n) \in \frac{1}{12}\mathbf{d}((x_1, \dots, x_n), (y_1, \dots, y_n))$$

for all $(x_1, \dots, x_n), (y_1, \dots, y_n) \in G^n$ and so we observe that

$$\text{Im}\left(\frac{1}{12}Sf(\cdot, \dots, \cdot; x_1, \dots, x_n) - \frac{1}{12}Sf(\cdot, \dots, \cdot; y_1, \dots, y_n)\right) \subseteq \frac{1}{12}\mathbf{d}((x_1, \dots, x_n), (y_1, \dots, y_n)).$$

From property (i) of η it becomes that

$$\eta\left[\frac{1}{12}Sf(\cdot, \dots, \cdot; x_1, \dots, x_n) - \frac{1}{12}Sf(\cdot, \dots, \cdot; y_1, \dots, y_n)\right] \in \frac{1}{12}\mathbf{d}((x_1, \dots, x_n), (y_1, \dots, y_n)) \quad (2.2)$$

for all $(x_1, \dots, x_n), (y_1, \dots, y_n) \in G^n$. Define the function $\Gamma: G^n \longrightarrow V$ by $\Gamma(x_1, \dots, x_n) := \eta[F_{x_1, \dots, x_n}]$ for $(x_1, \dots, x_n) \in G^n$. Note that $B(G^n, S(V))$ contains constant functions and hence using property (i) of η we see that if $f: G^n \longrightarrow V$ is constant, i.e., $f(x_1, \dots, x_n) = c$ for $(x_1, \dots, x_n) \in G^n$, where $c \in V$, then $\eta[f] = c$. Fix $(x_1, \dots, x_n) \in G^n$ and define the constant function $C_{x_1, \dots, x_n}: G^n \longrightarrow V$ by $C_{x_1, \dots, x_n}(\cdot, \dots, \cdot) := f(x_1, \dots, x_n)$. One has

$$\begin{aligned} (f(x_1, \dots, x_n) - \Gamma(x_1, \dots, x_n)) - (f(y_1, \dots, y_n) - \Gamma(y_1, \dots, y_n)) \\ = (\eta[C_{x_1, \dots, x_n}] - \eta[F_{x_1, \dots, x_n}]) - (\eta[C_{y_1, \dots, y_n}] - \eta[F_{y_1, \dots, y_n}]) \\ = \eta[C_{x_1, \dots, x_n} - F_{x_1, \dots, x_n}] - \eta[C_{y_1, \dots, y_n} - F_{y_1, \dots, y_n}] \\ = \eta\left[\frac{1}{12}Sf(\cdot, \dots, \cdot; x_1, \dots, x_n) - \frac{1}{12}Sf(\cdot, \dots, \cdot; y_1, \dots, y_n)\right] \end{aligned} \quad (2.3)$$

for all $(x_1, \dots, x_n), (y_1, \dots, y_n) \in G^n$. From (2.2) and (2.3) it follows that

$$\begin{aligned} (f(x_1, \dots, x_n) - \Gamma(x_1, \dots, x_n)) - (f(y_1, \dots, y_n) - \Gamma(y_1, \dots, y_n)) \\ \in \frac{1}{12}\mathbf{d}((x_1, \dots, x_n), (y_1, \dots, y_n)) \end{aligned}$$

for all $(x_1, \dots, x_n), (y_1, \dots, y_n) \in G^n$. This means that $f - \Gamma$ is a $\frac{1}{12}\mathbf{d}$ -Lipschitz function. By applying property (ii) of η , we see that

$$\begin{aligned} & 2\Gamma(y_1 + z_1, \dots, y_n + z_n) + 2\Gamma(y_1 - z_1, \dots, y_n - z_n) + 12\Gamma(y_1, \dots, y_n) \\ &= 2\eta[F_{y_1 + z_1, \dots, y_n + z_n}(x_1, \dots, x_n)] + 2\eta[F_{y_1 - z_1, \dots, y_n - z_n}(x_1, \dots, x_n)] \\ & \quad + 12\eta[F_{y_1, \dots, y_n}(x_1, \dots, x_n)] \\ &= \eta[F_{y_1 + z_1, \dots, y_n + z_n}^{2y_1, \dots, 2y_n}(x_1, \dots, x_n)] + \eta[F_{y_1 - z_1, \dots, y_n - z_n}^{-2y_1, \dots, -2y_n}(x_1, \dots, x_n)] \\ & \quad + \eta[F_{y_1 - z_1, \dots, y_n - z_n}^{2y_1, \dots, 2y_n}(x_1, \dots, x_n)] + \eta[F_{y_1 + z_1, \dots, y_n + z_n}^{-2y_1, \dots, -2y_n}(x_1, \dots, x_n)] \\ & \quad + 2\eta[F_{y_1, \dots, y_n}^{y_1 + z_1, \dots, y_n + z_n}(x_1, \dots, x_n)] + 2\eta[F_{y_1, \dots, y_n}^{y_1 - z_1, \dots, y_n - z_n}(x_1, \dots, x_n)] \\ & \quad + 2\eta[F_{y_1, \dots, y_n}^{-y_1 - z_1, \dots, -y_n - z_n}(x_1, \dots, x_n)] \\ & \quad + 2\eta[F_{y_1, \dots, y_n}^{-y_1 + z_1, \dots, -y_n + z_1}(x_1, \dots, x_n)] + 2\eta[F_{y_1, \dots, y_n}^{z_1, \dots, z_n}(x_1, \dots, x_n)] \\ & \quad + 2\eta[F_{y_1, \dots, y_n}^{-z_1, \dots, -z_n}(x_1, \dots, x_n)]. \end{aligned} \quad (2.4)$$

On the other hand,

$$\begin{aligned} & \eta[F_{y_1+z_1, \dots, y_n+z_n}^{2y_1, \dots, 2y_n}(x_1, \dots, x_n)] \\ &= \eta[\frac{1}{12}f(4y_1+2z_1+x_1, \dots, 4y_n+2z_n+x_n) + \frac{1}{12}f(2z_1-x_1, \dots, 2z_n-x_n) \\ & \quad - \frac{1}{6}f(3y_1+x_1+z_1, \dots, 3y_n+x_n+z_n) - \frac{1}{6}f(z_1-x_1-y_1, \dots, z_n-x_n-y_n)], \quad (2.5) \end{aligned}$$

$$\begin{aligned} & \eta[F_{y_1+z_1, \dots, y_n+z_n}^{-2y_1, \dots, -2y_n}(x_1, \dots, x_n)] \\ &= \eta[\frac{1}{12}f(2z_1+x_1, \dots, 2z_n+x_n) + \frac{1}{12}f(4y_1+2z_1-x_1, \dots, 4y_n+2z_n-x_n) \\ & \quad - \frac{1}{6}f(x_1-y_1+z_1, \dots, x_n-y_n+z_n) - \frac{1}{6}f(3y_1+z_1-x_1, \dots, 3y_n+z_n-x_n)], \quad (2.6) \end{aligned}$$

$$\begin{aligned} & \eta[F_{y_1-z_1, \dots, y_n-z_n}^{2y_1, \dots, 2y_n}(x_1, \dots, x_n)] \\ &= \eta[\frac{1}{12}f(4y_1-2z_1+x_1, \dots, 4y_n-2z_n+x_n) + \frac{1}{12}f(-2z_1-x_1, \dots, -2z_n-x_n) \\ & \quad - \frac{1}{6}f(3y_1+x_1-z_1, \dots, 3y_n+x_n-z_n) - \frac{1}{6}f(-z_1-x_1-y_1, \dots, -z_n-x_n-y_n)], \quad (2.7) \end{aligned}$$

$$\begin{aligned} & \eta[F_{y_1-z_1, \dots, y_n-z_n}^{-2y_1, \dots, -2y_n}(x_1, \dots, x_n)] \\ &= \eta[\frac{1}{12}f(-2z_1+x_1, \dots, -2z_n+x_n) + \frac{1}{12}f(4y_1-2z_1-x_1, \dots, 4y_n-2z_n-x_n) \\ & \quad - \frac{1}{6}f(-z_1+x_1-y_1, \dots, -z_n+x_n-y_n) - \frac{1}{6}f(3y_1-z_1-x_1, \dots, 3y_n-z_n-x_n)]. \quad (2.8) \end{aligned}$$

We can also find that

$$\begin{aligned} & 2\eta[F_{y_1, \dots, y_n}^{y_1+z_1, \dots, y_n+z_n}(x_1, \dots, x_n)] \\ &= \eta[\frac{1}{6}f(3y_1+x_1+z_1, \dots, 3y_n+x_n+z_n) + \frac{1}{6}f(y_1-x_1-z_1, \dots, y_n-x_n-z_n) \\ & \quad - \frac{1}{3}f(2y_1+x_1+z_1, \dots, 2y_n+x_n+z_n) - \frac{1}{3}f(-x_1-z_1, \dots, -x_n-z_n)], \quad (2.9) \end{aligned}$$

$$\begin{aligned} & 2\eta[F_{y_1, \dots, y_n}^{y_1-z_1, \dots, y_n-z_n}(x_1, \dots, x_n)] \\ &= \eta[\frac{1}{6}f(3y_1+x_1-z_1, \dots, 3y_n+x_n-z_n) + \frac{1}{6}f(y_1-x_1+z_1, \dots, y_n-x_n+z_n) \\ & \quad - \frac{1}{3}f(2y_1+x_1-z_1, \dots, 2y_n+x_n-z_n) - \frac{1}{3}f(-x_1+z_1, \dots, -x_n+z_n)], \quad (2.10) \end{aligned}$$

$$2\eta[F_{y_1, \dots, y_n}^{-y_1-z_1, \dots, -y_n-z_n}(x_1, \dots, x_n)]$$

$$= \eta \left[\frac{1}{6} f(x_1 + y_1 - z_1, \dots, x_n + y_n - z_n) + \frac{1}{6} f(3y_1 - x_1 + z_1, \dots, 3y_n - x_n + z_n) \right. \\ \left. - \frac{1}{3} f(x_1 - z_1, \dots, x_n - z_n) - \frac{1}{3} f(2y_1 - x_1 + z_1, \dots, 2y_n - x_n + z_n) \right], \quad (2.11)$$

$$2\eta[F_{y_1, \dots, y_n}^{-y_1+z_1, \dots, -y_n+z_n}(x_1, \dots, x_n)] \\ = \eta \left[\frac{1}{6} f(x_1 + y_1 + z_1, \dots, x_n + y_n + z_n) + \frac{1}{6} f(3y_1 - x_1 - z_1, \dots, 3y_n - x_n - z_n) \right. \\ \left. - \frac{1}{3} f(x_1 + z_1, \dots, x_n + z_n) - \frac{1}{3} f(2y_1 - x_1 - z_1, \dots, 2y_n - x_n - z_n) \right], \quad (2.12)$$

$$2\eta[F_{y_1, \dots, y_n}^{z_1, \dots, z_n}(x_1, \dots, x_n)] \\ = \eta \left[\frac{1}{6} f(2y_1 + x_1 + z_1, \dots, 2y_n + x_n + z_n) + \frac{1}{6} f(2y_1 - x_1 - z_1, \dots, 2y_n - x_n - z_n) \right. \\ \left. - \frac{1}{3} f(x_1 + y_1 + z_1, \dots, x_n + y_n + z_n) - \frac{1}{3} f(y_1 - x_1 - z_1, \dots, y_n - x_n - z_n) \right], \quad (2.13)$$

$$2\eta[F_{y_1, \dots, y_n}^{-z_1, \dots, -z_n}(x_1, \dots, x_n)] \\ = \eta \left[\frac{1}{6} f(2y_1 + x_1 - z_1, \dots, 2y_n + x_n - z_n) + \frac{1}{6} f(2y_1 - x_1 + z_1, \dots, 2y_n - x_n + z_n) \right. \\ \left. - \frac{1}{3} f(x_1 + y_1 - z_1, \dots, x_n + y_n - z_n) - \frac{1}{3} f(y_1 - x_1 + z_1, \dots, y_n - x_n + z_n) \right]. \quad (2.14)$$

By summing Eqs. (2.5)–(2.14) and using (2.4) we conclude that

$$2\Gamma(y_1 + z_1, \dots, y_n + z_n) + 2\Gamma(y_1 - z_1, \dots, y_n - z_n) + 12\Gamma(y_1, \dots, y_n) \\ = \eta \left[\frac{1}{12} f(4y_1 + 2z_1 + x_1, \dots, 4y_n + 2z_n + x_n) \right. \\ \left. + \frac{1}{12} f(4y_1 + 2z_1 - x_1, \dots, 4y_n + 2z_n - x_n) \right. \\ \left. - \frac{1}{6} f(2y_1 + x_1 + z_1, \dots, 2y_n + x_n + z_n) - \frac{1}{6} f(2y_1 + z_1 - x_1, \dots, 2y_n + z_n - x_n) \right] \\ + \eta \left[\frac{1}{12} f(4y_1 - 2z_1 + x_1, \dots, 4y_n - 2z_n + x_n) \right. \\ \left. + \frac{1}{12} f(4y_1 - 2z_1 - x_1, \dots, 4y_n - 2z_n - x_n) \right. \\ \left. - \frac{1}{6} f(2y_1 + x_1 - z_1, \dots, 2y_n + x_n - z_n) - \frac{1}{6} f(2y_1 - z_1 - x_1, \dots, 2y_n - z_n - x_n) \right] \\ = \eta[F_{2y_1+z_1, \dots, 2y_n+z_n}(x_1, \dots, x_n)] + \eta[F_{2y_1-z_1, \dots, 2y_n-z_n}(x_1, \dots, x_n)] \\ = \Gamma(2y_1 + z_1, \dots, 2y_1 + z_1) + \Gamma(2y_1 - z_1, \dots, 2y_n - z_n).$$

This entails that $S\Gamma(y_1, \dots, y_n; z_1, \dots, z_n) = 0$ and so Γ is n -cubic. $\square$

By $\Delta(G)$ we denote the diagonal set on G , i.e.,

$$\Delta(G) := \{(x, \dots, x) \in G^n : x \in G\}$$

and throughout the paper we define $F := f|_{\Delta(G)}$.

Corollary 1. *Under the hypotheses of Theorem 1, there exists a cubic function $C : G \rightarrow V$ such that $F - C$ is $\frac{1}{12}\mathbf{d}$ -Lipschitz.*

Proof. Define the function $\gamma : G^n \rightarrow V$ by $\gamma(x_1, \dots, x_n) := \eta[F_{x_1, \dots, x_n}]$ for $(x_1, \dots, x_n) \in G^n$. Then, γ is n -cubic as in the proof of Theorem 1. Let $C : G \rightarrow V$ be a function defined by $C(x) := \gamma(x, \dots, x)$. We have

$$F - C = f|_{\Delta(G)} - \gamma|_{\Delta(G)} = (f - \gamma)|_{\Delta(G)}.$$

The function $f - \gamma$ is $\frac{1}{12}\mathbf{d}$ -Lipschitz and so is $F - C$. The following equality now entails that C is cubic:

$$\begin{aligned} 2C(x+y) + 2C(x-y) + 12C(x) &= 2\gamma(x+y, \dots, x+y) \\ &\quad + 2\gamma(x-y, \dots, x-y) + 12\gamma(x, \dots, x) \\ &= \gamma(2x+y, \dots, 2x+y) + \gamma(2x-y, \dots, 2x-y) \\ &= C(2x+y) + C(2x-y). \end{aligned}$$

□

Lemma 1. *Under the hypotheses of Theorem 1, if $\text{Im} Sf \subset A$ for some $A \in S(V)$, then $\text{Im}(f - \Gamma) \subset \frac{1}{12}A$.*

Proof. It is clear that

$$\text{Im}\left(\frac{1}{12}Sf(x_1, \dots, x_n; \cdot, \dots, \cdot)\right) \subset \text{Im}\left(\frac{1}{12}Sf\right) \subset \frac{1}{12}A$$

and so $\frac{1}{12}Sf(x_1, \dots, x_n; \cdot, \dots, \cdot) \in B(G^n, S(V))$ for all $(x_1, \dots, x_n) \in G^n$. Thus, property (i) of η implies

$$f(x_1, \dots, x_n) - \Gamma(x_1, \dots, x_n) = \eta\left[\frac{1}{12}Sf(x_1, \dots, x_n; \cdot, \dots, \cdot)\right] \in \frac{1}{12}A$$

for all $(x_1, \dots, x_n) \in G^n$. Therefore, $\text{Im}(f - \Gamma) \subset \frac{1}{12}A$. □

Let $CB(V)$ be a family of closed balls of V .

Theorem 2. *Let $(G^n, +, d, \rho)$ be a product metric, V a normed space, and $f : G^n \rightarrow E$ an odd function. If $B(G^n, CB(V))$ admits LIM and $Sf \in \text{Lip}(G^n \times G^n, E)$, then there exists an n -cubic function Γ such that*

$$\|f - \Gamma\|_{\text{Lip}} \leq \frac{1}{12}\|Sf\|_{\text{Lip}}.$$

Proof. Define the set-valued function $\mathbf{d} : G \times G \longrightarrow CB(V)$ by

$$\mathbf{d}((x_1, \dots, x_n), (y_1, \dots, y_n)) := \inf_{d((x_1, \dots, x_n), (y_1, \dots, y_n)) \leq \delta} \text{mc}_{Sf}(\delta)B(0, 1),$$

where $B(0, 1)$ is the closed unit ball with center at zero. Let $\text{mc}_{Qf} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be the module of continuity of $Sf : G^n \times G^n \rightarrow E$ with the product metric ρ on $G^n \times G^n$. Then,

$$\begin{aligned} & \|Sf(t_1, \dots, t_n; x_1, \dots, x_n) - Sf(t_1, \dots, t_n; y_1, \dots, y_n)\| \\ & \leq \inf_{\rho((t_1, \dots, t_n; x_1, \dots, x_n), (t_1, \dots, t_n; y_1, \dots, y_n)) \leq \delta} \text{mc}_{Sf}(\delta) = \inf_{d((x_1, \dots, x_n), (y_1, \dots, y_n)) \leq \delta} \text{mc}_{Sf}(\delta) \end{aligned}$$

for all $(t_1, \dots, t_n), (x_1, \dots, x_n), (y_1, \dots, y_n) \in G^n$. This inequality entails that $Sf(t_1, \dots, t_n; \cdot, \dots, \cdot)$ is $\mathbf{d}$ -Lipschitz and so Theorem 1 implies that there exists an n -cubic function Γ such that $f - \Gamma$ is $\frac{1}{12}\mathbf{d}$ -Lipschitz. Hence,

$$\begin{aligned} & \|(f(x_1, \dots, x_n) - \Gamma(x_1, \dots, x_n)) - (f(y_1, \dots, y_n) - \Gamma(y_1, \dots, y_n))\| \\ & \leq \inf_{d((x_1, \dots, x_n), (y_1, \dots, y_n)) \leq \delta} \frac{1}{12} \text{mc}_{Sf}(\delta), \end{aligned}$$

which shows that $\text{mc}_{f-\Gamma} = \frac{1}{12} \text{mc}_{Sf}$. Since $\text{Im} Sf \subseteq \|Sf\|_\infty B(0, 1)$, Lemma 1 implies that

$$\text{Im}(f - \Gamma) \subset \frac{1}{12} \|Sf\|_\infty B(0, 1),$$

which is equivalent to

$$\|f - \Gamma\|_\infty \leq \frac{1}{12} \|Sf\|_\infty. \quad (2.15)$$

We may also prove that $\text{mc}_{Sf} = \mathbb{P}(Sf)$ and so $\mathbb{P}(f - \Gamma) \leq \frac{1}{12} \mathbb{P}(Sf)$. The inequality (2.15) now ensures

$$\|f - \Gamma\|_{Lip} = \|f - \Gamma\|_\infty + \mathbb{P}(f - \Gamma) \leq \frac{1}{12} \|Sf\|_\infty + \frac{1}{12} \mathbb{P}(Sf) = \frac{1}{12} \|Sf\|_{Lip}.$$

□

Corollary 2. *Under the hypotheses of Theorem 2, there exists a cubic function $C : G \longrightarrow V$ such that*

$$\|F - C\|_{Lip} \leq \frac{1}{12} \|Sf\|_{Lip}.$$

Proof. It follows from Theorem 2 that there exists an n -cubic function $\gamma : G^n \longrightarrow V$ such that

$$\|f - \gamma\|_{Lip} \leq \frac{1}{12} \|Sf\|_{Lip}.$$

Define $C : G \longrightarrow V$ by $C(x) := \gamma(x, \dots, x)$. Corollary 1 entails that C is cubic and

$$F - C = (f - \gamma)_{|\Delta(G)}.$$

On the other hand, the function $f - \gamma$ is $\frac{1}{12}\mathbf{d}$ -Lipschitz and so is $F - C$. The following inequality now ensures the result:

$$\|F - C\|_{Lip} \leq \|f - \gamma\|_{Lip} \leq \frac{1}{12} \|Sf\|_{Lip}.$$

□

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FEJÉR AND HERMITE-HADAMARD TYPE INEQUALITIES FOR DIFFERENTIABLE h -CONVEX AND QUASI CONVEX FUNCTIONS WITH APPLICATIONS

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Received 08 October, 2019

Abstract. In this paper, we establish new weighted integral inequalities for differentiable h -convex and quasi-convex functions. Applications of our findings to continuous random variables and approximation of integrals are given.

2010 Mathematics Subject Classification: 26D15; 26A51; 26A33

Keywords: convex functions, Hölder inequality, quasi-convex function, Hermite-Hadamard inequality

1. INTRODUCTION

The following two types of convexity of functions are well known in the literature.

Definition 1 ([7]). A function $\alpha : J \subset \mathbb{R} \rightarrow \mathbb{R}$ is said to be h -convex, where J is an interval, if the inequality

$$\alpha(tx + (1-t)y) \leq h(t)\alpha(x) + h(1-t)\alpha(y),$$

holds for all $x, y \in J$ and $t \in [0, 1]$.

Definition 2 ([4]). A function $\alpha : J \subset \mathbb{R} \rightarrow \mathbb{R}$ is said to be quasi convex, where J is an interval, if the inequality

$$\alpha(tx + (1-t)y) \leq \max\{\alpha(x), \alpha(y)\}$$

holds for all $x, y \in J$ and $t \in [0, 1]$.

Varšanec, in [9], introduced the concept of h -convex functions, which generalizes the concept of a nonnegative convex function, and other types of convexity of functions such as Godunova–Levin, and s -convex functions. We refer the reader to [7] and [8] for further properties of h -convex functions.

The Hermite-Hadamard inequality is one of the most important mathematical inequalities due to its applications in different contexts. It is stated as follows: For an

interval $[c, d]$ with $c < d$,

$$\alpha\left(\frac{c+d}{2}\right) \leq \frac{1}{d-c} \int_c^d \alpha(x) dx \leq \frac{\alpha(c) + \alpha(d)}{2},$$

where $\alpha : [c, d] \rightarrow \mathbb{R}$ is a convex function. For extensions, refinements and applications of this inequality, see [3] and [6].

Recently, several authors established a number of inequalities to estimate the difference between $\alpha\left(\frac{c+d}{2}\right)$ and $\frac{1}{d-c} \int_c^d \alpha(x) dx$ for differentiable convex mappings, see for example [1, 4] and [5]. In [2], Hwang obtained the following two theorems.

Theorem 1 ([2]). *Let $\alpha : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on J° , where J is an interval, and $w : [u, v] \rightarrow [0, \infty)$ be a continuous and symmetric mapping with respect to $\frac{u+v}{2}$, where $u, v \in J^\circ$ with $u < v$. If $\alpha' \in L_1([u, v])$ and $|\alpha'|$ is convex on $[u, v]$, then*

$$\begin{aligned} \left| \alpha\left(\frac{u+v}{2}\right) \int_u^v w(x) dx - \int_u^v w(x) \alpha(x) dx \right| \\ \leq \frac{(v-u)}{2} [|\alpha'(u)| + |\alpha'(v)|] \int_0^1 M(w, u, v, t) dt, \quad (1.1) \end{aligned}$$

where

$$M(w, u, v, t) = \int_u^{l(u,v,t)} w(x) dx \text{ and } l(u, v, t) = \frac{1+t}{2}u + \frac{1-t}{2}v.$$

Theorem 2 ([2]). *Let $\alpha : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on J° , where J is an interval, and $w : [u, v] \rightarrow [0, \infty)$ be a continuous and symmetric mapping with respect to $\frac{u+v}{2}$, where $u, v \in J^\circ$ with $u < v$. If $\alpha' \in L_1([u, v])$ and $|\alpha'|^q$ is convex on $[u, v]$ for $q \geq 1$, then*

$$\begin{aligned} \left| \alpha\left(\frac{u+v}{2}\right) \int_u^v w(x) dx - \int_u^v w(x) \alpha(x) dx \right| \\ \leq (v-u) \left[\frac{|\alpha'(u)|^q + |\alpha'(v)|^q}{2} \right]^{\frac{1}{q}} \int_0^1 M(w, u, v, t) dt, \quad (1.2) \end{aligned}$$

where

$$M(w, u, v, t) = \int_u^{l(u,v,t)} w(x) dx \text{ and } l(u, v, t) = \frac{1+t}{2}u + \frac{1-t}{2}v.$$

Clearly, inequalities (1.1) and (1.2) give an upper bound for the difference between $\int_u^v w(x) \alpha(x) dx$ and $\alpha\left(\frac{u+v}{2}\right) \int_u^v w(x) dx$. In this paper, we generalize the inequalities obtained by Hwang in [2] for h -convex and quasi convex functions. Then we introduce an application of these results to continuous random variables whose probability density functions are continuous, and give another application to approximation of integrals.

Throughout this paper, $\mathbb{R}$ denotes the set of all real numbers, $J \subset \mathbb{R}$ denotes an interval, and $h : (0, 1) \rightarrow \mathbb{R}$ denotes a non-negative function and non-zero function. In addition, for any positive integer k , and $u, v \in J^\circ$ with $u < v$, the functions $\theta_{k,u,v}, \theta_{k,u,v}^* : [0, k] \rightarrow \mathbb{R}$ are defined as:

$$\theta_{k,u,v}(t) = \left(\frac{k+t}{2k}\right)u + \frac{k-t}{2k}v, \theta_{k,u,v}^*(t) = u + v - \theta_{k,u,v}(t).$$

The discrete power mean inequality will be used in the next section, and it is stated as follows:

$$x_1^\gamma + x_2^\gamma \leq 2^{1-\gamma}(x_1 + x_2)^\gamma, \quad (1.3)$$

where $x_1 > 0, x_2 > 0, \gamma < 1$.

2. MAIN RESULTS

We start this section with the following lemma which will be used repeatedly in the sequel.

Lemma 1. Let $\alpha : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on J° and $u, v \in J^\circ$ with $u < v$. Suppose that $g : [u, v] \rightarrow [0, \infty)$ is a differentiable mapping. If $\alpha' \in L_1([u, v])$, then for any positive integer k ,

$$\begin{aligned} & g(u) \left(\frac{\alpha(u) + \alpha(v)}{2} \right) - g(v) \alpha \left(\frac{u+v}{2} \right) \\ & + \left(\frac{v-u}{4k} \right) \int_0^k [g'(\theta_{k,u,v}(t)) + g'(\theta_{k,u,v}^*(t))] [\alpha(\theta_{k,u,v}(t)) + \alpha(\theta_{k,u,v}^*(t))] dt \\ & = \left(\frac{v-u}{4k} \right) \int_0^k [g(\theta_{k,u,v}(t)) - g(\theta_{k,u,v}^*(t)) + g(v)] \\ & \quad \times [\alpha'(\theta_{k,u,v}^*(t)) - \alpha'(\theta_{k,u,v}(t))] dt. \end{aligned} \quad (2.1)$$

Proof. Let

$$S_1 = - \int_0^k [g(\theta_{k,u,v}(t)) - g(\theta_{k,u,v}^*(t)) + g(v)] \alpha'(\theta_{k,u,v}(t)) dt$$

and

$$S_2 = \int_0^k [g(\theta_{k,u,v}(t)) - g(\theta_{k,u,v}^*(t)) + g(v)] \alpha'(\theta_{k,u,v}^*(t)) dt.$$

Using integration by parts,

$$\begin{aligned}
 S_1 &= \frac{-2k}{u-v} \left[g(\theta_{k,u,v}(t)) - g(\theta_{k,u,v}^*(t)) + g(v) \right] \alpha(\theta_{k,u,v}(t)) \Big|_0^k \\
 &\quad + \frac{2k}{u-v} \int_0^k \left[g'(\theta_{k,u,v}(t)) \frac{u-v}{2k} - g'(\theta_{k,u,v}^*(t)) \frac{v-u}{2k} \right] \alpha(\theta_{k,u,v}(t)) dt \\
 &= \frac{2k}{v-u} \left(g(u) \alpha(u) - g(v) \alpha\left(\frac{u+v}{2}\right) \right) \\
 &\quad + \int_0^k [g'(\theta_{k,u,v}(t)) + g'(\theta_{k,u,v}^*(t))] \alpha(\theta_{k,u,v}(t)) dt,
 \end{aligned}$$

and

$$\begin{aligned}
 S_2 &= \frac{2k}{v-u} \left(g(u) \alpha(v) - g(v) \alpha\left(\frac{u+v}{2}\right) \right) \\
 &\quad + \int_0^k [g'(\theta_{k,u,v}(t)) + g'(\theta_{k,u,v}^*(t))] \alpha(\theta_{k,u,v}^*(t)) dt.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 \left(\frac{v-u}{4k} \right) [S_1 + S_2] &= g(u) \left(\frac{\alpha(u) + \alpha(v)}{2} \right) - g(v) \alpha\left(\frac{u+v}{2}\right) \\
 &\quad + \left(\frac{v-u}{4k} \right) \int_0^k [g'(\theta_{k,u,v}(t)) + g'(\theta_{k,u,v}^*(t))] [\alpha(\theta_{k,u,v}(t)) + \alpha(\theta_{k,u,v}^*(t))] dt,
 \end{aligned}$$

and hence the result follows. $\square$

Theorem 3. Let $\alpha: J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on J° and $u, v \in J^\circ$ with $u < v$. Suppose that $p: [u, v] \rightarrow [0, \infty)$ is a continuous mapping which is symmetric about $\frac{u+v}{2}$. If $\alpha' \in L_1([u, v])$ and $|\alpha'|$ is h -convex on $[u, v]$, then for any positive integer k ,

$$\begin{aligned}
 &\left| \int_u^v p(x) \alpha(x) dx - \alpha\left(\frac{u+v}{2}\right) \int_u^v p(x) dx \right| \\
 &\leq \left(\frac{v-u}{k} \right) \left(\frac{|\alpha'(u)| + |\alpha'(v)|}{2} \right) \int_0^k W_{k,u,v}(t) \left(h\left(\frac{k+t}{2k}\right) + h\left(\frac{k-t}{2k}\right) \right) dt \quad (2.2)
 \end{aligned}$$

where $W_{k,u,v}(t) = \int_u^{\theta_{k,u,v}(t)} p(x) dx$.

Proof. Define $g : [u, v] \rightarrow [0, \infty)$ as follows:

$$g(t) = \int_u^t p(x) dx, t \in [u, v].$$

Clearly, w is differentiable with $w' = p$. Using Lemma 1, we have

$$\begin{aligned} & g(u) \left(\frac{\alpha(u) + \alpha(v)}{2} \right) - g(v) \alpha \left(\frac{u+v}{2} \right) \\ & + \left(\frac{v-u}{4k} \right) \int_0^k [g'(\theta_{k,u,v}(t)) + g'(\theta_{k,u,v}^*(t))] [\alpha(\theta_{k,u,v}(t)) + \alpha(\theta_{k,u,v}^*(t))] dt \\ & = \left(\frac{v-u}{4k} \right) \int_0^k [g(\theta_{k,u,v}(t)) - g(\theta_{k,u,v}^*(t)) + g(v)] [\alpha'(\theta_{k,u,v}^*(t)) - \alpha'(\theta_{k,u,v}(t))] dt. \end{aligned}$$

But

$$g(u) = 0, g(v) = \int_u^v p(x) dx,$$

and

$$g'(\theta_{k,u,v}(t)) = p(\theta_{k,u,v}(t)), g'(\theta_{k,u,v}^*(t)) = p(\theta_{k,u,v}^*(t)),$$

which imply that

$$\begin{aligned} & \left(\frac{v-u}{4k} \right) \int_0^k [p(\theta_{k,u,v}(t)) + p(\theta_{k,u,v}^*(t))] [\alpha(\theta_{k,u,v}(t)) + \alpha(\theta_{k,u,v}^*(t))] dt \\ & - \alpha \left(\frac{u+v}{2} \right) \int_u^v p(x) dx \\ & = \left(\frac{v-u}{4k} \right) \times \int_0^k [g(\theta_{k,u,v}(t)) - g(\theta_{k,u,v}^*(t)) + g(v)] [\alpha'(\theta_{k,u,v}^*(t)) - \alpha'(\theta_{k,u,v}(t))] dt. \end{aligned} \quad (2.3)$$

Note that,

$$\begin{aligned} g(\theta_{k,u,v}(t)) - g(\theta_{k,u,v}^*(t)) + g(v) &= \int_u^{\theta_{k,u,v}(t)} p(x) dx - \int_u^{\theta_{k,u,v}^*(t)} p(x) dx + \int_u^v p(x) dx \\ &= \int_u^{\theta_{k,u,v}(t)} p(x) dx + \int_{\theta_{k,u,v}^*(t)}^v p(x) dx. \end{aligned} \quad (2.4)$$

Since p is symmetric about $\frac{u+v}{2}$, we have

$$\begin{aligned} & \left(\frac{v-u}{4k} \right) \int_0^k [p(\theta_{k,u,v}(t)) + p(\theta_{k,u,v}^*(t))] [\alpha(\theta_{k,u,v}(t)) + \alpha(\theta_{k,u,v}^*(t))] dt \\ & = \left(\frac{v-u}{2k} \right) \int_0^k p(\theta_{k,u,v}(t)) \alpha(\theta_{k,u,v}(t)) dt + \left(\frac{v-u}{2k} \right) \int_0^k p(\theta_{k,u,v}^*(t)) \alpha(\theta_{k,u,v}^*(t)) dt \end{aligned}$$

$$= \int_u^c p(x) \alpha(x) dx + \int_c^v p(x) \alpha(x) dx = \int_u^v p(x) \alpha(x) dx, \quad (2.5)$$

and

$$\int_u^{\theta_{k,u,v}(t)} p(x) dx = \int_{\theta_{k,u,v}^*(t)}^v p(x) dx, \quad (2.6)$$

for each $t \in [0, k]$. This implies that

$$\begin{aligned} & \left(\frac{v-u}{4k} \right) \int_0^k [g(\theta_{k,u,v}(t)) - g(\theta_{k,u,v}^*(t)) + g(v)] [\alpha'(\theta_{k,u,v}^*(t)) - \alpha'(\theta_{k,u,v}(t))] dt \\ &= \left(\frac{v-u}{2k} \right) \int_0^k \left[\int_u^{\theta_{k,u,v}(t)} p(x) dx \right] \times [\alpha'(\theta_{k,u,v}^*(t)) - \alpha'(\theta_{k,u,v}(t))] dt. \end{aligned} \quad (2.7)$$

Combining (2.3), (2.4), (2.5), (2.6) and (2.7), we get that

$$\begin{aligned} & \int_u^v p(x) \alpha(x) dx - \alpha\left(\frac{u+v}{2}\right) \int_u^v p(x) dx \\ &= \left(\frac{v-u}{2k} \right) \int_0^k W_{k,u,v}(t) [\alpha'(\theta_{k,u,v}^*(t)) - \alpha'(\theta_{k,u,v}(t))] dt. \end{aligned}$$

Using the triangle inequality and the h -convexity of $|\alpha'|$, we have

$$\begin{aligned} & \left| \int_u^v p(x) \alpha(x) dx - \alpha\left(\frac{u+v}{2}\right) \int_u^v p(x) dx \right| \\ & \leq \left(\frac{v-u}{2k} \right) \int_0^k W_{k,u,v}(t) (|\alpha'(\theta_{k,u,v}^*(t))| + |\alpha'(\theta_{k,u,v}(t))|) dt \\ & \leq \left(\frac{v-u}{2k} \right) (|\alpha'(u)| + |\alpha'(v)|) \int_0^k W_{k,u,v}(t) \left(h\left(\frac{k+t}{2k}\right) + h\left(\frac{k-t}{2k}\right) \right) dt. \end{aligned}$$

□

Corollary 1. Under the same conditions as in Theorem 3, if h is super-additive, i.e., $h(x) + h(y) \leq h(x+y)$ for each $x, y \in [0, 1]$, then the inequality

$$\begin{aligned} & \left| \int_u^v p(x) \alpha(x) dx - \alpha\left(\frac{u+v}{2}\right) \int_u^v p(x) dx \right| \\ & \leq h(1) \left(\frac{v-u}{k} \right) \left(\frac{|\alpha'(u)| + |\alpha'(v)|}{2} \right) \int_0^k W_{k,u,v}(t) dt, \end{aligned} \quad (2.8)$$

holds for each positive integer k , where $W_{k,u,v}(t) = \int_u^{\theta_{k,u,v}(t)} p(x) dx$.

Corollary 2. Under the same conditions as in Theorem 3, if h is symmetric about $\frac{1}{2}$ then

$$\left| \int_u^v p(x) \alpha(x) dx - \alpha\left(\frac{u+v}{2}\right) \int_u^v p(x) dx \right|$$

$$\begin{aligned} &\leq \left(\frac{v-u}{k}\right) (|\alpha'(u)| + |\alpha'(v)|) \int_0^k W_{k,u,v}(t) h\left(\frac{k+t}{2k}\right) dt \\ &= \left(\frac{v-u}{k}\right) (|\alpha'(u)| + |\alpha'(v)|) \int_0^k W_{k,u,v}(t) h\left(\frac{k-t}{2k}\right) dt, \end{aligned}$$

for each positive integer k , where $W_{k,u,v}(t) = \int_u^{\theta_{k,u,v}(t)} p(x) dx$.

Remark 1. In Lemma 1, if $h(t) = t$ for all $t \in [0, 1]$, then Inequality (2.2) reduces to

$$\begin{aligned} &\left| \int_u^v p(x) \alpha(x) dx - \alpha\left(\frac{u+v}{2}\right) \int_u^v p(x) dx \right| \\ &\leq \left(\frac{v-u}{2k}\right) (|\alpha'(u)| + |\alpha'(v)|) \int_0^k W_{k,u,v}(t) dt. \quad (2.9) \end{aligned}$$

Theorem 4. Let $\alpha: J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on J° and $u, v \in J^\circ$ with $u < v$. Suppose that $p: [u, v] \rightarrow [0, \infty)$ is a continuous mapping which is symmetric about $\frac{u+v}{2}$. If $\alpha' \in L_1([u, v])$ and $|\alpha'|^q$ is h -convex on $[u, v]$ for $q > 1$, then for any positive integer k ,

$$\begin{aligned} &\left| \int_u^v p(x) \alpha(x) dx - \alpha\left(\frac{u+v}{2}\right) \int_u^v p(x) dx \right| \\ &\leq \left(\frac{v-u}{k}\right) \left(\int_0^k W_{k,u,v}(t) dt\right)^{\frac{1}{p}} \left(\frac{|\alpha'(u)|^q + |\alpha'(v)|^q}{2}\right)^{\frac{1}{q}} \\ &\quad \times \left(\int_0^k W_{k,u,v}(t) \left(h\left(\frac{k+t}{2k}\right) + h\left(\frac{k-t}{2k}\right)\right) dt\right)^{\frac{1}{q}}, \quad (2.10) \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $W_{k,u,v}(t) = \int_u^{\theta_{k,u,v}(t)} p(x) dx$.

Proof. As in the proof of Theorem 3, we have

$$\begin{aligned} &\left| \int_u^v p(x) \alpha(x) dx - \alpha\left(\frac{u+v}{2}\right) \int_u^v p(x) dx \right| \\ &\leq \left(\frac{v-u}{2k}\right) \int_0^k W_{k,u,v}(t) (|\alpha'(\theta_{k,u,v}^*(t))| + |\alpha'(\theta_{k,u,v}(t))|) dt. \end{aligned}$$

Applying Holder's inequality, we get that

$$\begin{aligned} &\left| \int_u^v p(x) \alpha(x) dx - \alpha\left(\frac{u+v}{2}\right) \int_u^v p(x) dx \right| \\ &\leq \left(\frac{v-u}{2k}\right) \left(\int_0^k W_{k,u,v}(t) dt\right)^{\frac{1}{p}} \left[\left(\int_0^k W_{k,u,v}(t) |\alpha'(\theta_{k,u,v}^*(t))|^q dt\right)^{\frac{1}{q}}\right] \end{aligned}$$

$$+ \left(\int_0^k W_{k,u,v}(t) |\alpha'(\theta_{k,u,v}(t))|^q dt \right)^{\frac{1}{q}} \Bigg].$$

Using inequality (1.3) and the h -convexity of $|\alpha'|^q$ on $[u, v]$, we have

$$\begin{aligned} & \left| \int_u^v p(x) \alpha(x) dx - \alpha\left(\frac{u+v}{2}\right) \int_u^v p(x) dx \right| \\ & \leq \left(\frac{v-u}{2^{\frac{1}{q}} k} \right) \left(\int_0^k W_{k,u,v}(t) dt \right)^{\frac{1}{p}} \\ & \quad \times \left(\int_0^k W_{k,u,v}(t) (|\alpha'(\theta_{k,u,v}^*(t))|^q + |\alpha'(\theta_{k,u,v}^*(t))|^q) dt \right)^{\frac{1}{q}} \\ & \leq \left(\frac{v-u}{k} \right) \left(\int_0^k W_{k,u,v}(t) dt \right)^{\frac{1}{p}} \left(\frac{|\alpha'(u)|^q + |\alpha'(v)|^q}{2} \right)^{\frac{1}{q}} \\ & \quad \times \left(\int_0^k W_{k,u,v}(t) \left(h\left(\frac{k+t}{2k}\right) + h\left(\frac{k-t}{2k}\right) \right) dt \right)^{\frac{1}{q}}. \end{aligned}$$

□

Corollary 3. Under the same conditions as in Theorem 4, if $h(t) = t$ for each $t \in [0, 1]$, then inequality (2.10) reduces to

$$\begin{aligned} & \left| \int_u^v p(x) \alpha(x) dx - \alpha\left(\frac{u+v}{2}\right) \int_u^v p(x) dx \right| \\ & \leq \left(\frac{v-u}{k} \right) \left(\frac{|\alpha'(u)|^q + |\alpha'(v)|^q}{2} \right)^{\frac{1}{q}} \left(\int_0^k W_{k,u,v}(t) dt \right). \end{aligned} \quad (2.11)$$

Corollary 4. Under the same conditions as in Theorem 4, if $p(x) = \frac{4}{k(v-u)}$ for each $x \in [u, v]$, then inequality (2.10) reduces to

$$\begin{aligned} & \left| \alpha\left(\frac{u+v}{2}\right) - \frac{1}{v-u} \int_u^v p(x) \alpha(x) dx \right| \\ & \leq \frac{(v-u)}{2^{2-\frac{1}{q}} k^{\frac{2}{q}}} \left[\frac{|\alpha'(u)|^q + |\alpha'(v)|^q}{2} \right]^{\frac{1}{q}} \\ & \quad \times \left(\int_0^k (k-t) \left(h\left(\frac{k+t}{2k}\right) + h\left(\frac{k-t}{2k}\right) \right) dt \right)^{\frac{1}{q}}. \end{aligned} \quad (2.12)$$

Corollary 5. Under the same conditions as in Theorem 4, if h is super-additive, then the inequality

$$\left| \int_u^v p(x) \alpha(x) dx - \alpha\left(\frac{u+v}{2}\right) \int_u^v p(x) dx \right|$$

$$\leq (h(1))^{\frac{1}{q}} \left(\frac{v-u}{k} \right) \left(\frac{|\alpha'(u)|^q + |\alpha'(v)|^q}{2} \right)^{\frac{1}{q}} \left(\int_0^k W_{k,u,v}(t) dt \right), \quad (2.13)$$

holds for each positive integer k , where $W_{k,u,v}(t) = \int_u^{\theta_{k,u,v}(t)} p(x) dx$.

Corollary 6. Suppose that the assumptions of Theorem 4 are satisfied. If h is symmetric about $\frac{1}{2}$ then for any positive integer k ,

$$\begin{aligned} & \left| \int_u^v p(x) \alpha(x) dx - \alpha\left(\frac{u+v}{2}\right) \int_u^v p(x) dx \right| \\ & \leq \left(\frac{v-u}{k} \right) \left(\int_0^k W_{k,u,v}(t) dt \right)^{\frac{1}{p}} (|\alpha'(u)|^q + |\alpha'(v)|^q)^{\frac{1}{q}} \\ & \quad \times \left(\int_0^k W_{k,u,v}(t) h\left(\frac{k+t}{2k}\right) dt \right)^{\frac{1}{q}} \\ & = \left(\frac{v-u}{k} \right) \left(\int_0^k W_{k,u,v}(t) dt \right)^{\frac{1}{p}} (|\alpha'(u)|^q + |\alpha'(v)|^q)^{\frac{1}{q}} \\ & \quad \times \left(\int_0^k W_{k,u,v}(t) h\left(\frac{k-t}{2k}\right) dt \right)^{\frac{1}{q}}, \end{aligned} \quad (2.14)$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $W_{k,u,v}(t) = \int_u^{\theta_{k,u,v}(t)} p(x) dx$.

Theorem 5. Let $\alpha: J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on J° and $u, v \in J^\circ$ with $u < v$. Suppose that $p: [u, v] \rightarrow [0, \infty)$ is a continuous mapping which is symmetric about $\frac{u+v}{2}$. If $\alpha' \in L_1([u, v])$ and $|\alpha'|$ is quasi-convex on $[u, v]$, then for any positive integer k ,

$$\begin{aligned} & \left| \int_u^v p(x) \alpha(x) dx - \alpha\left(\frac{u+v}{2}\right) \int_u^v p(x) dx \right| \\ & \leq \left(\frac{v-u}{2k} \right) \left[\max \left\{ |\alpha'(v)|, \left| \alpha'\left(\frac{u+v}{2}\right) \right| \right\} \right. \\ & \quad \left. + \max \left\{ |\alpha'(u)|, \left| \alpha'\left(\frac{u+v}{2}\right) \right| \right\} \right] \int_0^k W_{k,u,v}(t) dt. \end{aligned} \quad (2.15)$$

where $W_{k,u,v}(t) = \int_u^{\theta_{k,u,v}(t)} p(x) dx$.

Proof. As in the proof of Theorem 3, we have

$$\begin{aligned} & \left| \int_u^v p(x) \alpha(x) dx - \alpha\left(\frac{u+v}{2}\right) \int_u^v p(x) dx \right| \\ & \leq \left(\frac{v-u}{2k} \right) \int_0^k W_{k,u,v}(t) (|\alpha'(\theta_{k,u,v}^*(t))| + |\alpha'(\theta_{k,u,v}(t))|) dt. \end{aligned}$$

Using the quasi-convexity of $|\alpha'|$ on $[u, v]$, we get

$$\begin{aligned} & \left| \int_u^v p(x) \alpha(x) dx - \alpha\left(\frac{u+v}{2}\right) \int_u^v p(x) dx \right| \\ & \leq \left(\frac{v-u}{2k}\right) \left[\max \left\{ |\alpha'(v)|, \left| \alpha'\left(\frac{u+v}{2}\right) \right| \right\} \right. \\ & \quad \left. + \max \left\{ |\alpha'(u)|, \left| \alpha'\left(\frac{u+v}{2}\right) \right| \right\} \right] \int_0^k W_{k,u,v}(t) dt. \end{aligned}$$

□

Remark 2. In Theorem 5,

(1) If $|\alpha'|$ is non-decreasing, then inequality (2.15) reduces to

$$\begin{aligned} & \left| \int_u^v p(x) \alpha(x) dx - \alpha\left(\frac{u+v}{2}\right) \int_u^v p(x) dx \right| \\ & \leq \left(\frac{v-u}{2k}\right) \left(|\alpha'(v)| + \left| \alpha'\left(\frac{u+v}{2}\right) \right| \right) \int_0^k W_{k,u,v}(t) dt. \quad (2.16) \end{aligned}$$

(2) If $|\alpha'|$ is non-increasing, then inequality (2.15) reduces to

$$\begin{aligned} & \left| \int_u^v p(x) \alpha(x) dx - \alpha\left(\frac{u+v}{2}\right) \int_u^v p(x) dx \right| \\ & \leq \left(\frac{v-u}{2k}\right) \left(|\alpha'(u)| + \left| \alpha'\left(\frac{u+v}{2}\right) \right| \right) \int_0^k W_{k,u,v}(t) dt. \quad (2.17) \end{aligned}$$

Theorem 6. Let $\alpha: J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on J° and $u, v \in J^\circ$ with $u < v$. Suppose that $p: [u, v] \rightarrow [0, \infty)$ is a continuous mapping which is symmetric about $\frac{u+v}{2}$. If $\alpha' \in L_1([u, v])$ and $|\alpha'|^q$ is quasi-convex on $[u, v]$ for $q > 1$, then for any positive integer k ,

$$\begin{aligned} & \left| \int_u^v p(x) \alpha(x) dx - \alpha\left(\frac{u+v}{2}\right) \int_u^v p(x) dx \right| \\ & \leq \left(\frac{v-u}{4k^{1-\frac{1}{q}}}\right) \left[\left(\max \left\{ |\alpha'(v)|^q, \left| \alpha'\left(\frac{u+v}{2}\right) \right|^q \right\} \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\max \left\{ |\alpha'(u)|^q, \left| \alpha'\left(\frac{u+v}{2}\right) \right|^q \right\} \right)^{\frac{1}{q}} \right] \left(\int_0^k W_{k,u,v}(t) dt \right), \quad (2.18) \end{aligned}$$

where $W_{k,u,v}(t) = \int_u^{\Theta_{k,u,v}(t)} p(x) dx$.

Proof. As in the proof of Theorem 4, we have

$$\left| \int_u^v p(x) \alpha(x) dx - \alpha\left(\frac{u+v}{2}\right) \int_u^v p(x) dx \right|$$

$$\begin{aligned} &\leq \left(\frac{v-u}{2^{\frac{1}{q}}k} \right) \left(\int_0^k W_{k,u,v}(t) dt \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_0^k W_{k,u,v}(t) (|\alpha'(\theta_{k,u,v}^*(t))|^q + |\alpha'(\theta_{k,u,v}^*(t))|^q) dt \right)^{\frac{1}{q}}, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Using the quasi-convexity of $|\alpha'|^q$ on $[u, v]$, we get that

$$\begin{aligned} &\left| \int_u^v p(x) \alpha(x) dx - \alpha \left(\frac{u+v}{2} \right) \int_u^v p(x) dx \right| \\ &\leq \left(\frac{v-u}{4k^{1-\frac{1}{q}}} \right) \left[\left(\max \left\{ |\alpha'(v)|^q, \left| \alpha' \left(\frac{u+v}{2} \right) \right|^q \right\} \right)^{\frac{1}{q}} \right. \\ &\quad \left. + \left(\max \left\{ |\alpha'(u)|^q, \left| \alpha' \left(\frac{u+v}{2} \right) \right|^q \right\} \right)^{\frac{1}{q}} \right] \left(\int_0^k W_{k,u,v}(t) dt \right). \end{aligned}$$

□

Remark 3. In Theorem 6,

(1) If $|\alpha'|$ is non-decreasing, then inequality (2.18) reduces to

$$\begin{aligned} &\left| \int_u^v p(x) \alpha(x) dx - \alpha \left(\frac{u+v}{2} \right) \int_u^v p(x) dx \right| \\ &\leq \left(\frac{v-u}{4k^{1-\frac{1}{q}}} \right) \left(|\alpha'(v)| + \left| \alpha' \left(\frac{u+v}{2} \right) \right| \right) \int_0^k W_{k,u,v}(t) dt. \quad (2.19) \end{aligned}$$

(2) If $|\alpha'|$ is non-increasing, then inequality (2.18) reduces to

$$\begin{aligned} &\left| \int_u^v p(x) \alpha(x) dx - \alpha \left(\frac{u+v}{2} \right) \int_u^v p(x) dx \right| \\ &\leq \left(\frac{v-u}{4k^{1-\frac{1}{q}}} \right) \left(|\alpha'(u)| + \left| \alpha' \left(\frac{u+v}{2} \right) \right| \right) \int_0^k W_{k,u,v}(t) dt. \quad (2.20) \end{aligned}$$

3. SOME APPLICATIONS

We start this section with an application to continuous random variables whose probability density functions are continuous.

Proposition 1. Let X be a continuous random variable taking its values in the finite interval $[c, d]$, where $0 < c < d$, with a continuous probability density function $w : [c, d] \rightarrow [0, 1]$ which is symmetric about $\frac{c+d}{2}$. Let $q > 1$, and $r \geq 1 + \frac{2}{q}$. If the

r -moment of X is finite, i.e., $E_r(X) = \int_c^d t^r p(t) dt < \infty$, and $\tau \in (0, 1]$ then

$$\left| E_r(X) - \left(\frac{c+d}{2} \right)^r \right| \leq \frac{r(d-c)}{2^{1-\frac{1}{q}}} \left(\frac{c^{q(r-1)} + d^{q(r-1)}}{2(\tau+1)} \right)^{\frac{1}{q}}. \quad (3.1)$$

Proof. Let $\alpha(x) = \frac{1}{r}x^r$ for $x \in [c, d]$. Since $|\alpha'(x)|^q = x^{q(r-1)}$ is h -convex for $h(t) = t^\tau, t \in (0, 1]$, applying Theorem 4, we have

$$\begin{aligned} & \left| \int_c^d p(x) \alpha(x) dx - \alpha\left(\frac{c+d}{2}\right) \int_c^d p(x) dx \right| \\ & \leq \left(\frac{d-c}{k} \right) \left(\int_0^k W_{k,c,d}(t) dt \right)^{\frac{1}{p}} \left(\frac{|\alpha'(c)|^q + |\alpha'(d)|^q}{2} \right)^{\frac{1}{q}} \\ & \quad \times \left(\int_0^k W_{k,c,d}(t) \left(h\left(\frac{k+t}{2k}\right) + h\left(\frac{k-t}{2k}\right) \right) dt \right)^{\frac{1}{q}}, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $W_{k,c,d}(t) = \int_c^{\theta_{k,c,d}(t)} p(x) dx$. The result follows using the facts that

$$\int_c^d p(x) dx = 1,$$

$$\int_c^d p(x) \alpha(x) dx = \frac{1}{r} E_r(X),$$

$$\int_0^k W_{k,c,d}(t) dt = \int_0^k \int_c^{\theta_{k,c,d}(t)} p(x) dx dt \leq \frac{k}{2},$$

and

$$\begin{aligned} & \int_0^k W_{k,c,d}(t) \left(h\left(\frac{k+t}{2k}\right) + h\left(\frac{k-t}{2k}\right) \right) dt \\ & \leq \frac{1}{2} \int_0^k \left(h\left(\frac{k+t}{2k}\right) + h\left(\frac{k-t}{2k}\right) \right) dt = \frac{k}{\tau+1}. \end{aligned}$$

□

The second application will be devoted to approximating an integral. Recall that a partition D of a finite interval $[c, d]$, $c < d$, is a finite sequence of numbers $c = c_0 < c_1 < \dots < c_n = d$.

Proposition 2. Let $\alpha : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on J° and $c, d \in J^\circ$ with $c < d$. Suppose that $p : [c, d] \rightarrow [0, \infty)$ is a continuous mapping which is

symmetric about $\frac{c+d}{2}$. Let $D : c = c_0 < c_1 < \cdots < c_n = d$ be a partition of $[c, d]$. If $\alpha' \in L_1([c, d])$ and $\left| \alpha' \right|$ is h -convex on $[c, d]$, then for any positive integer k

$$\int_c^d p(x) \alpha(x) dx = A(\alpha, p, D) + E(\alpha, p, D),$$

where

$$A(\alpha, p, D) = \sum_{j=0}^{n-1} \alpha\left(\frac{c_j + c_{j+1}}{2}\right) P_{c_j, c_{j+1}},$$

$$P_{c_j, c_{j+1}} = \int_{c_j}^{c_{j+1}} p(x) dx,$$

and

$$|E(\alpha, p, D)| \leq \sum_{j=0}^{n-1} \left(\frac{c_{j+1} - c_j}{k} \right) \left[\frac{|\alpha'(c_j)| + |\alpha'(c_{j+1})|}{2} \right]$$

$$\times \int_0^k W_{k, c_j, c_{j+1}}(t) \left(h\left(\frac{k+t}{2k}\right) + h\left(\frac{k-t}{2k}\right) \right) dt. \quad (3.2)$$

Proof. For each $j = 0, 1, \dots, n-1$, applying Theorem 3 over the interval $[c_j, c_{j+1}]$, we have

$$\left| \int_{c_j}^{c_{j+1}} p(x) \alpha(x) dx - \alpha\left(\frac{c_j + c_{j+1}}{2}\right) \int_{c_j}^{c_{j+1}} p(x) dx \right|$$

$$\leq \left(\frac{c_{j+1} - c_j}{k} \right) \left[\frac{|\alpha'(c_j)| + |\alpha'(c_{j+1})|}{2} \right]$$

$$\times \int_0^k W_{k, c_j, c_{j+1}}(t) \left(h\left(\frac{k+t}{2k}\right) + h\left(\frac{k-t}{2k}\right) \right) dt. \quad (3.3)$$

where $W_{k, c_j, c_{j+1}} = \int_{c_j}^{\theta_{k, c_j, c_{j+1}}(t)} p(x) dx$. Note that

$$E(\alpha, p, D) = \int_c^d p(x) \alpha(x) dx - A(\alpha, p, D)$$

$$= \sum_{j=0}^{n-1} \int_{c_j}^{c_{j+1}} p(x) \alpha(x) dx - \sum_{j=0}^{n-1} \alpha\left(\frac{c_j + c_{j+1}}{2}\right) P_{c_j, c_{j+1}}$$

$$= \sum_{j=0}^{n-1} \left[\int_{c_j}^{c_{j+1}} p(x) \alpha(x) dx - \left(\frac{\alpha(c_j) + \alpha(c_{j+1})}{2} \right) \int_{c_j}^{c_{j+1}} p(x) dx \right].$$

Using the triangle inequality and inequality (3.3), we get that

$$|E(\alpha, p, D)| \leq \sum_{j=0}^{n-1} \left(\frac{c_{j+1} - c_j}{k} \right) \left[\frac{|\alpha'(c_j)| + |\alpha'(c_{j+1})|}{2} \right] \\ \times \int_0^k W_{k, c_j, c_{j+1}}(t) \left(h\left(\frac{k+t}{2k}\right) + h\left(\frac{k-t}{2k}\right) \right) dt.$$

□

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ON (p, φ_h) -CONVEX FUNCTION AND ITS APPLICATIONS

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Received 1 November, 2019

Abstract. In this paper, we introduce the notion of (p, φ_h) -convex functions and present some properties and representation of such functions. Finally, a version of Hermite Hadamard-type inequalities for (p, φ_h) -convex functions are established.

2010 Mathematics Subject Classification: 26A51; 26D15; 26D10

Keywords: Convex function, (p, φ_h) -convex function, Hermite–Hadamard inequality

1. INTRODUCTION

Recently theory of convexity has received much attentions by many researchers. Consequently the classical concepts of convex sets and convex functions have been extended and generalized in several directions using novel and innovative ideas, see [16]. In [23], Varoşneac introduced the notion of h -convex functions. It is worth to mention here that besides the classical convex functions, the class of h -convex functions also includes the class of some convex functions which are s -convex functions, Godunova-Levin functions and p -functions [7]. For some recent investigations on h -convex functions, see [19, 20]. The interrelationship between theory of convex functions and theory of inequalities has attracted many researchers. One of the most extensively studied inequality for convex functions is the Hermite–Hadamard inequality. This inequality provides the necessary and sufficient condition for a function to be convex. The inequalities discovered by C. Hermite and J. Hadamard for convex functions are very important in literature (see, e.g. [7]). These inequalities state that if $f: I \rightarrow \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2}. \quad (1.1)$$

The inequality (1.1) has evoked the interest of many mathematicians who used many different convex functions. Especially in the last three decades numerous generalizations, variants and extensions of this inequality have been obtained by using h -convex, φ_h -convex, p -convex etc., to mention a few (see [1–7, 13, 17, 21, 22]) and the references cited therein.

In this paper, we define new general version of p -convex functions name as the (p, φ_h) -convex function and we give some important properties of the (p, φ_h) -convex function. Then by using convexity we establish the Hermite–Hadamard type inequalities.

2. PRELIMINARIES AND LEMMAS

In this section, we recall some previously known concepts and derive some new results which play an important role in the development of our main results.

In [25], Zhang and Wan define a new convexity class called p -convex function and in [12], İşcan give a different version of the definition of p -convex function as follows:

Definition 1. [12, Theorem 3] Let $I \subset (0, \infty)$ be a real interval and $p \in \mathbb{R} \setminus \{0\}$. A function $f: I \rightarrow \mathbb{R}$ is said to be a p -convex function or belong the class $PC(I)$, if

$$f\left([tx^p + (1-t)y^p]^{\frac{1}{p}}\right) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $t \in [0, 1]$.

It is clear that p -convex functions includes convex functions and Harmonically convex functions as special cases. According to Definition 1, it can be easily seen that for $p = 1$ and $p = -1$, p -convexity reduces to ordinary convexity and harmonically convexity of functions defined on $I \subset (0, \infty)$, respectively. For some results related to p -convex functions and its generalizations, we refer the reader to see [9, 12–15, 25].

Let us consider a function $\varphi: [a, b] \rightarrow [a, b]$ where $[a, b] \subset \mathbb{R}$. Youness have defined the φ -convex functions in [24], but we work here with the improved definition, according to [4]:

Definition 2. [4, Definition 1] A function $f: [a, b] \rightarrow \mathbb{R}$ is said to be φ -convex on $[a, b]$ if for every two point $x \in [a, b]$, $y \in [a, b]$ and $t \in [0, 1]$ the following inequality holds:

$$f(t\varphi(x) + (1-t)\varphi(y)) \leq tf(\varphi(x)) + (1-t)f(\varphi(y)). \quad (2.1)$$

In [23] Varošanec introduced the notion of h -convex functions as follows:

Definition 3. [23, Definition 4] Let I be an interval in $\mathbb{R}$ and $h: (0, 1) \rightarrow (0, \infty)$ be a given function. We say that a function $f: I \rightarrow \mathbb{R}$ is h -convex if

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y) \quad (2.2)$$

for all $x, y \in I$ and $t \in (0, 1)$.

This notion unifies and generalizes the known classes of convex functions, s -convex functions, Godunova-Levin functions and p -functions, which are obtained putting in (2.2), $h(t) = t$, $h(t) = t^s$, $h(t) = \frac{1}{t}$ and $h(t) = 1$ respectively. Many properties of them can be found for instance, in [7, 8, 10, 11, 16, 19, 20].

In [17], Sarikaya has defined the following φ_h -convex functions which is a generalization of the above convex functions:

Definition 4. Let I be an interval in R and $h: (0, 1) \rightarrow (0, \infty)$ be a given function. We say that a function $f: I \rightarrow R$ is φ_h -convex if

$$f(t\varphi(x) + (1-t)\varphi(y)) \leq h(t)f(\varphi(x)) + h(1-t)f(\varphi(y)) \quad (2.3)$$

for all $x, y \in I$ and $t \in (0, 1)$. If the inequality (2.3) is reversed, then f is said to be φ_h -concave. In particular if (2.3) with $h(t) = t$, $h(t) = t^s$, $h(t) = \frac{1}{t}$ and $h(t) = 1$, then f is said to be φ -convex, φ_s -convex, φ -Godunova-Levin function and φ - P -function, respectively (see e.g. [17, 18]).

Definition 5 ([7]). A function $h: I \rightarrow R$ is called a super-multiplicative function if

$$h(xy) \geq h(x)h(y) \quad (2.4)$$

for all $x, y \in I$. If the inequality sign in (2.4) is reversed, then h is multiplicative function. If $c > 1$, then h is a sub-multiplicative function and if the equality holds in (2.4), then h is called a multiplicative function.

3. MAIN RESULTS

In this section, we give new definitions and properties of the (p, φ_h) -convex function. Throughout this paper we assume that $(0, 1) \subseteq J$, $I \subset (0, \infty)$, f and h are real non-negative functions defined on I and J , respectively, $p \in R \setminus \{0\}$ and $\varphi: [a, b] \rightarrow [a, b]$ is continuous where $[a, b] \subset \mathbb{R}$. We first give a definition of the new class of convex functions.

Definition 6. Let $h: J \rightarrow R$ be a non-negative and non-zero function. We say that $f: I \rightarrow R$ is a (p, φ_h) -convex or that f belongs to the class $fsx(p, \varphi_h, I)$, if f is non-negative and

$$f([t\varphi^p(x) + (1-t)\varphi^p(y)]^{\frac{1}{p}}) \leq h(t)f(\varphi(x)) + h(1-t)f(\varphi(y)) \quad (3.1)$$

for all $x, y \in I$ and $t \in (0, 1)$. Similarly, if the inequality sign in (3.1) is reversed, then f is said to be a (p, φ_h) -concave function or to the class $fsv(p, \varphi_h, I)$.

Remark 1. It can be obviously seen that if f is satisfies (2.1) with $p = 1$, f is said to be φ_h -convex, $h(t) = t^s$ and $p = 1$, f is said to be φ_s -convex, $h(t) = \frac{1}{t}$ and $p = 1$ f is said to be φ -Godunova-Levin function, $h(t) = 1$ and $p = 1$, f is said to be φ - P -function.

Lemma 1. Let $h: (0, 1) \rightarrow (0, \infty)$. If $f, g \in fsx(p, \varphi_h, I)$ and $\lambda > 0$, then $f + g, \lambda f \in fsx(p, \varphi_h, I)$. Similarly, if $f, g \in fsv(p, \varphi_h, I)$ and $\lambda > 0$, then $f + g, \lambda f \in fsv(p, \varphi_h, I)$.

Proof. Let identify $m(x) = f(x) + g(x)$, then we get

$$\begin{aligned} m\left([t\varphi^p(x) + (1-t)\varphi^p(y)]^{\frac{1}{p}}\right) &= f\left([t\varphi^p(x) + (1-t)\varphi^p(y)]^{\frac{1}{p}}\right) + g\left([t\varphi^p(x) + (1-t)\varphi^p(y)]^{\frac{1}{p}}\right) \\ &\leq h(t)f(\varphi(x)) + h(1-t)f(\varphi(y)) + h(t)g(\varphi(x)) + h(1-t)g(\varphi(y)) \\ &= h(t)[f(\varphi(x)) + g(\varphi(x))] + h(1-t)[f(\varphi(y)) + g(\varphi(y))] \\ &= h(t)m(\varphi(x)) + h(1-t)m(\varphi(y)) \end{aligned}$$

and hence, $f + g \in fsx(p, \varphi_h, I)$. Similarly, it is proved that $\lambda f \in fsx(p, \varphi_h, I)$. $\square$

Lemma 2. Let $h_1, h_2: (0, 1) \rightarrow (0, \infty)$ be such that $h_2(t) \leq h_1(t)$ for all $t \in (0, 1)$. If $f \in fsx(p, \varphi_{h_2}, I)$, then $f \in fsx(p, \varphi_{h_1}, I)$.

Proof. If $f \in fsx(p, \varphi_{h_2}, I)$, then for any $x, y \in I$ and $t \in (0, 1)$ we have

$$\begin{aligned} f\left([t\varphi^p(x) + (1-t)\varphi^p(y)]^{\frac{1}{p}}\right) &\leq h_2(t)f(\varphi(x)) + h_2(1-t)f(\varphi(y)) \\ &\leq h_1(t)f(\varphi(x)) + h_1(1-t)f(\varphi(y)) \end{aligned}$$

and hence $f \in fsx(p, \varphi_{h_1}, I)$. $\square$

Lemma 3. Let I be an interval such that $0 \in I$ and let $p > 0$. Then, we have the following

- (1) If $f \in fsx(p, \varphi_h, I)$, $f(0) = 0$, $\varphi(0) = 0$ and h is super-multiplicative, then the inequality

$$f\left([\alpha\varphi^p(a) + \beta\varphi^p(b)]^{\frac{1}{p}}\right) \leq h(\alpha)f(\varphi(x)) + h(\beta)f(\varphi(y))$$

holds for all $x, y \in I$ and $\alpha, \beta > 0$ such that $\alpha + \beta \leq 1$.

- (2) If $f \in fsv(p, \varphi_h, I)$, $f(0) = 0$, $\varphi(0) = 0$ and h is sub-multiplicative, then the inequality

$$f\left([\alpha\varphi^p(a) + \beta\varphi^p(b)]^{\frac{1}{p}}\right) \geq h(\alpha)f(\varphi(x)) + h(\beta)f(\varphi(y))$$

holds for all $x, y \in I$ and $\alpha, \beta > 0$ such that $\alpha + \beta \leq 1$.

Proof.

- (1) Let $\alpha, \beta > 0$, $\alpha + \beta = \gamma < 1$ and let a and b be numbers such that $a = \frac{\alpha}{\gamma}$ and $b = \frac{\beta}{\gamma}$. Then, we have $a + b = 1$. Since $f \in fsx(p, \varphi_h, I)$, $f(0) = 0$, $\varphi(0) = 0$, and h is super-multiplicative, we obtain

$$\begin{aligned} f\left([\alpha\varphi^p(x) + \beta\varphi^p(y)]^{\frac{1}{p}}\right) &= f\left([a\gamma\varphi^p(x) + b\gamma\varphi^p(y)]^{\frac{1}{p}}\right) \\ &\leq h(a)f\left(\gamma^{\frac{1}{p}}\varphi(x)\right) + h(b)f\left(\gamma^{\frac{1}{p}}\varphi(y)\right) \\ &= h(a)f\left([\gamma\varphi^p(x) + (1-\gamma)\varphi^p(0)]^{\frac{1}{p}}\right) \end{aligned}$$

$$\begin{aligned}
& + h(b) f \left([\gamma \varphi^p(y) + (1 - \gamma) \varphi^p(0)]^{\frac{1}{p}} \right) \\
& \leq h(a) h(\gamma) f(\varphi(x)) + h(b) h(\gamma) f(\varphi(y)) \\
& \leq h(\alpha \gamma) f(\varphi(x)) + h(b \gamma) f(\varphi(y)) \\
& = h(\alpha) f(\varphi(x)) + h(\beta) f(\varphi(y))
\end{aligned}$$

for all $x, y \in I$.

- (2) Similarly, let $\alpha, \beta > 0$, $\alpha + \beta = \gamma < 1$, and let a and b be numbers such that $a = \frac{\alpha}{\gamma}$ and $b = \frac{\beta}{\gamma}$. Then, we have $a + b = 1$. Since $f \in f_{sv}(p, \varphi_h, I)$, $f(0) = 0$, $\varphi(0) = 0$ and h is sub-multiplicative, we have

$$\begin{aligned}
f \left([\alpha \varphi^p(x) + \beta \varphi^p(y)]^{\frac{1}{p}} \right) & = f \left([a \gamma \varphi^p(x) + b \gamma \varphi^p(y)]^{\frac{1}{p}} \right) \\
& \geq h(a) f \left(\gamma^{\frac{1}{p}} \varphi(x) \right) + h(b) f \left(\gamma^{\frac{1}{p}} \varphi(y) \right) \\
& = h(a) f \left([\gamma \varphi^p(x) + (1 - \gamma) \varphi^p(0)]^{\frac{1}{p}} \right) \\
& \quad + h(b) f \left([\gamma \varphi^p(y) + (1 - \gamma) \varphi^p(0)]^{\frac{1}{p}} \right) \\
& \geq h(a) h(\gamma) f(\varphi(x)) + h(b) h(\gamma) f(\varphi(y)) \\
& \geq h(\alpha \gamma) f(\varphi(x)) + h(b \gamma) f(\varphi(y)) \\
& = h(\alpha) f(\varphi(x)) + h(\beta) f(\varphi(y))
\end{aligned}$$

for all $x, y \in I$.

□

Lemma 4. Let $f: [a, b] \rightarrow \mathbb{R}$, $h: (0, 1) \rightarrow (0, \infty)$ and $\varphi: [a, b] \rightarrow [a, b]$. Then the following statements are equivalent:

- (i) The function f is a (p, φ_h) -convex on $[a, b]$.
- (ii) For every $x, y \in [a, b]$, the mapping $g: [0, 1] \rightarrow \mathbb{R}$,
 $g(t) = f \left([t \varphi^p(x) + (1 - t) \varphi^p(y)]^{\frac{1}{p}} \right)$ is h -convex on $[a, b]$.

Proof. Assume that f is (p, φ_h) -convex function and let us consider two points $x, y \in [a, b]$, $\lambda \in (0, 1)$ and $t_1, t_2 \in [0, 1]$, then we obtain

$$\begin{aligned}
g(\lambda t_1 + (1 - \lambda) t_2) & = f \left([(\lambda t_1 + (1 - \lambda) t_2) \varphi^p(x) + (1 - \lambda t_1 - (1 - \lambda) t_2) \varphi^p(y)]^{\frac{1}{p}} \right) \\
& = f \left([\lambda (t_1 \varphi^p(x) + (1 - t_1) \varphi^p(y)) + (1 - \lambda) (t_2 \varphi^p(x) + (1 - t_2) \varphi^p(y))]^{\frac{1}{p}} \right) \\
& \leq h(\lambda) f \left([t_1 \varphi^p(x) + (1 - t_1) \varphi^p(y)]^{\frac{1}{p}} \right) \\
& \quad + h(1 - \lambda) f \left([t_2 \varphi^p(x) + (1 - t_2) \varphi^p(y)]^{\frac{1}{p}} \right) \\
& = h(\lambda) g(t_1) + h(1 - \lambda) g(t_2)
\end{aligned}$$

which gives that g is a h -convex function.

Conversely, if g is a h -convex function, then for $x, y \in [a, b]$, $\lambda \in (0, 1)$ and t_1 and $t_2 = 0$, we get

$$\begin{aligned} f\left([\lambda\varphi^p(x) + (1-\lambda)\varphi^p(y)]^{\frac{1}{p}}\right) &= g(\lambda 1 + (1-\lambda)0) \\ &\leq h(\lambda)g(1) + h(1-\lambda)g(0) \\ &= h(\lambda)f\left([\varphi^p(x)]^{\frac{1}{p}}\right) + h(1-\lambda)f\left([\varphi^p(y)]^{\frac{1}{p}}\right) \\ &= h(\lambda)f(\varphi(x)) + h(1-\lambda)f(\varphi(y)) \end{aligned}$$

which shows that f is (p, φ_h) -convex function. This completes the proof. $\square$

Theorem 1. Let $h: (0, 1) \rightarrow (0, \infty)$ and f belongs to the class $fsx(p, \varphi_h, I)$. Let $\varphi(a), \varphi(b) \in I$ with $\varphi(a) < \varphi(b)$. Then the following inequalities hold:

$$\begin{aligned} \frac{1}{2h\left(\frac{1}{2}\right)} f\left(\left[\frac{\varphi^p(a) + \varphi^p(b)}{2}\right]^{\frac{1}{p}}\right) &\leq \frac{p}{\varphi^p(b) - \varphi^p(a)} \int_{\varphi(a)}^{\varphi(b)} x^{p-1} f(x) dx \\ &\leq [f(\varphi(a)) + f(\varphi(b))] \int_0^1 h(t) dt. \end{aligned} \quad (3.2)$$

Proof. By using the (p, φ_h) -convexity of f , we have

$$\begin{aligned} &f\left(\left[\frac{\varphi^p(a) + \varphi^p(b)}{2}\right]^{\frac{1}{p}}\right) \\ &= f\left(\left[\frac{t\varphi^p(a) + (1-t)\varphi^p(b)}{2} + \frac{(1-t)\varphi^p(a) + t\varphi^p(b)}{2}\right]^{\frac{1}{p}}\right) \\ &\leq h\left(\frac{1}{2}\right) \left[f\left([t\varphi^p(a) + (1-t)\varphi^p(b)]^{\frac{1}{p}}\right) + f\left([(1-t)\varphi^p(a) + t\varphi^p(b)]^{\frac{1}{p}}\right)\right]. \end{aligned} \quad (3.3)$$

Integrating both side of (3.3) over the interval $(0, 1)$, it follows that

$$\begin{aligned} f\left(\left[\frac{\varphi^p(a) + \varphi^p(b)}{2}\right]^{\frac{1}{p}}\right) &\leq h\left(\frac{1}{2}\right) \cdot \left[\int_0^1 f\left([t\varphi^p(a) + (1-t)\varphi^p(b)]^{\frac{1}{p}}\right) dt \right. \\ &\quad \left. + \int_0^1 f\left([(1-t)\varphi^p(a) + t\varphi^p(b)]^{\frac{1}{p}}\right) dt\right]. \end{aligned}$$

In the first integral, we substitute $x = [t\varphi^p(a) + (1-t)\varphi^p(b)]^{\frac{1}{p}}$. Meanwhile, in the second integral, we also use the substitution $x = [(1-t)\varphi^p(a) + t\varphi^p(b)]^{\frac{1}{p}}$, we have

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\left[\frac{\varphi^p(a) + \varphi^p(b)}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{\varphi^p(b) - \varphi^p(a)} \int_{\varphi(a)}^{\varphi(b)} x^{p-1} f(x) dx.$$

In order to prove second inequality of (3.2), we start from the (p, φ_h) -convexity of f , meaning that for every $t \in (0, 1)$, one has

$$f\left([t\varphi^p(a) + (1-t)\varphi^p(b)]^{\frac{1}{p}}\right) \leq h(t)f(\varphi(a)) + h(1-t)f(\varphi(b))$$

integrating both sides of the above inequality over $(0, 1)$, we obtain

$$\int_0^1 f\left([t\varphi^p(a) + (1-t)\varphi^p(b)]^{\frac{1}{p}}\right) dt \leq f(\varphi(a)) \int_0^1 h(t) dt + f(\varphi(b)) \int_0^1 h(1-t) dt.$$

The previous substitution in the first side of this inequality leads to

$$\frac{p}{\varphi^p(b) - \varphi^p(a)} \int_{\varphi(a)}^{\varphi(b)} x^{p-1} f(x) dx \leq [f(\varphi(a)) + f(\varphi(b))] \int_0^1 h(t) dt$$

which gives the second inequality of (3.2). This completes the proof. $\square$

Remark 2. If $p = 1$, then inequality (3.2) gives the Hermite–Hadamard inequality for φ_h -convex proved by Sarikaya in [17, Theorem 1].

Remark 3. If $p = 1$, $h(t) = t$, $t \in (0, 1)$, then in (3.2) coincide with the Hermite–Hadamard type inequalities for φ -convex functions proved by Cristescu in [4, Theorem 3].

Remark 4. If $p = 1$, $h(t) = t^s$ ($s \in (0, 1)$), $t \in (0, 1)$, then in (3.2) coincide with the Hermite–Hadamard type inequalities for φ_s -convex functions proved by Sarikaya in [17, Corollary 1].

Theorem 2. Suppose that f and g are functions such that $f \in fsx(p, \varphi_{h_1}, I)$, $g \in fsx(p, \varphi_{h_2}, I)$, $f, g \in L_1([a, b])$ and $\varphi(a), \varphi(b) \in I$ with $\varphi(a) < \varphi(b)$ and $h_1, h_2 \in L_1([0, 1])$ with $a, b \in I$ and $a < b$. We then have

$$\begin{aligned} & \frac{p}{\varphi^p(b) - \varphi^p(a)} \int_{\varphi(a)}^{\varphi(b)} x^{p-1} f(x) g(x) dx \\ & \leq M_\varphi(a, b) \int_0^1 h_1(t) h_2(t) dt + N_\varphi(a, b) \int_0^1 h_1(t) h_2(1-t) dt \end{aligned}$$

where

$$\begin{aligned} M_\varphi(a, b) &= f(\varphi(a))g(\varphi(a)) + f(\varphi(b))g(\varphi(b)) \\ N_\varphi(a, b) &= f(\varphi(b))g(\varphi(a)) + f(\varphi(a))g(\varphi(b)). \end{aligned}$$

Proof. Since $f \in fsx(p, \varphi_{h_1}, I)$ and $g \in fsx(p, \varphi_{h_2}, I)$, we have

$$\begin{aligned} f\left([t\varphi^p(a) + (1-t)\varphi^p(b)]^{\frac{1}{p}}\right) &\leq h_1(t)f(\varphi(a)) + h_1(1-t)f(\varphi(b)) \\ g\left([t\varphi^p(a) + (1-t)\varphi^p(b)]^{\frac{1}{p}}\right) &\leq h_2(t)g(\varphi(a)) + h_2(1-t)g(\varphi(b)) \end{aligned}$$

for all $t \in [0, 1]$. Because f and g are non-negative, we get the inequality

$$\begin{aligned} & f\left([t\varphi^p(a) + (1-t)\varphi^p(b)]^{\frac{1}{p}}\right) g\left([t\varphi^p(a) + (1-t)\varphi^p(b)]^{\frac{1}{p}}\right) \\ & \leq h_1(t)h_2(t)f(\varphi(a))g(\varphi(a)) + h_1(1-t)h_2(t)f(\varphi(b))g(\varphi(a)) \\ & \quad + h_1(t)h_2(1-t)f(\varphi(a))g(\varphi(b)) + h_1(1-t)h_2(1-t)f(\varphi(b))g(\varphi(b)). \end{aligned}$$

Integrating both sides of the above inequality over $(0, 1)$ we obtain the inequality

$$\begin{aligned} & \int_0^1 f\left([t\varphi^p(a) + (1-t)\varphi^p(b)]^{\frac{1}{p}}\right) g\left([t\varphi^p(a) + (1-t)\varphi^p(b)]^{\frac{1}{p}}\right) dt \\ & \leq f(\varphi(a))g(\varphi(a)) \int_0^1 h_1(t)h_2(t)dt + f(\varphi(b))g(\varphi(a)) \int_0^1 h_1(1-t)h_2(t)dt \\ & \quad + f(\varphi(a))g(\varphi(b)) \int_0^1 h_1(t)h_2(1-t)dt \\ & \quad + f(\varphi(b))g(\varphi(b)) \int_0^1 h_1(1-t)h_2(1-t)dt. \end{aligned}$$

Setting $x^p = t\varphi^p(a) + (1-t)\varphi^p(b)$, we get

$$\begin{aligned} & \frac{p}{\varphi^p(b) - \varphi^p(a)} \int_{\varphi(a)}^{\varphi(b)} x^{p-1} f(x) g(x) dx \\ & \leq M_\varphi(a, b) \int_0^1 h_1(t)h_2(t)dt + N_\varphi(a, b) \int_0^1 h_1(t)h_2(1-t)dt. \end{aligned}$$

□

Theorem 3. Let $f \in fsx(p, \varphi_h, I)$, $g \in fsx(p, \varphi_h, I)$ be functions such that $fg \in L_1([a, b])$ and $h \in L_1([a, b])$ and let $\varphi(a), \varphi(b) \in I$ with $\varphi(a) < \varphi(b)$. Then we have

$$\begin{aligned} & \frac{1}{2h^2(\frac{1}{2})} f\left(\left[\frac{\varphi^p(a) + \varphi^p(b)}{2}\right]^{\frac{1}{p}}\right) g\left(\left[\frac{\varphi^p(a) + \varphi^p(b)}{2}\right]^{\frac{1}{p}}\right) \\ & \quad - \frac{p}{\varphi^p(b) - \varphi^p(a)} \int_{\varphi(a)}^{\varphi(b)} x^{p-1} f(x) g(x) dx \\ & \leq N_\varphi(a, b) \int_0^1 h^2(t)dt + M_\varphi(a, b) \int_0^1 h(t)h(1-t)dt. \end{aligned}$$

Proof. Since

$$\frac{\varphi^p(a) + \varphi^p(b)}{2} = \frac{t\varphi^p(a) + (1-t)\varphi^p(b)}{2} + \frac{(1-t)\varphi^p(a) + t\varphi^p(b)}{2},$$

we have

$$f\left(\left[\frac{\varphi^p(a) + \varphi^p(b)}{2}\right]^{\frac{1}{p}}\right) g\left(\left[\frac{\varphi^p(a) + \varphi^p(b)}{2}\right]^{\frac{1}{p}}\right)$$

$$\begin{aligned}
&= f\left(\left[\frac{t\varphi^p(a) + (1-t)\varphi^p(b)}{2} + \frac{(1-t)\varphi^p(a) + t\varphi^p(b)}{2}\right]^{\frac{1}{p}}\right) \\
&\quad \cdot g\left(\left[\frac{t\varphi^p(a) + (1-t)\varphi^p(b)}{2} + \frac{(1-t)\varphi^p(a) + t\varphi^p(b)}{2}\right]^{\frac{1}{p}}\right) \\
&\leq h\left(\frac{1}{2}\right) \left[f\left([t\varphi^p(a) + (1-t)\varphi^p(b)]^{\frac{1}{p}}\right) + f\left([(1-t)\varphi^p(a) + t\varphi^p(b)]^{\frac{1}{p}}\right)\right] \\
&\quad \cdot h\left(\frac{1}{2}\right) \left[g\left([t\varphi^p(a) + (1-t)\varphi^p(b)]^{\frac{1}{p}}\right) + g\left([(1-t)\varphi^p(a) + t\varphi^p(b)]^{\frac{1}{p}}\right)\right] \\
&\leq h^2\left(\frac{1}{2}\right) f\left([t\varphi^p(a) + (1-t)\varphi^p(b)]^{\frac{1}{p}}\right) g\left([t\varphi^p(a) + (1-t)\varphi^p(b)]^{\frac{1}{p}}\right) \\
&\quad + h^2\left(\frac{1}{2}\right) f\left([(1-t)\varphi^p(a) + t\varphi^p(b)]^{\frac{1}{p}}\right) g\left([(1-t)\varphi^p(a) + t\varphi^p(b)]^{\frac{1}{p}}\right) \\
&\quad + h^2\left(\frac{1}{2}\right) [h(t)f(\varphi(a)) + h(1-t)f(\varphi(b))] \\
&\quad \cdot [h(1-t)g(\varphi(a)) + h(t)g(\varphi(b))] \\
&\quad + h^2\left(\frac{1}{2}\right) [h(1-t)f(\varphi(a)) + h(t)f(\varphi(b))] \\
&\quad \cdot [h(t)g(\varphi(a)) + h(1-t)g(\varphi(b))] \\
&= h^2\left(\frac{1}{2}\right) f\left([t\varphi^p(a) + (1-t)\varphi^p(b)]^{\frac{1}{p}}\right) g\left([t\varphi^p(a) + (1-t)\varphi^p(b)]^{\frac{1}{p}}\right) \\
&\quad + h^2\left(\frac{1}{2}\right) f\left([(1-t)\varphi^p(a) + t\varphi^p(b)]^{\frac{1}{p}}\right) g\left([(1-t)\varphi^p(a) + t\varphi^p(b)]^{\frac{1}{p}}\right) \\
&\quad + h^2\left(\frac{1}{2}\right) [(h(t)h(1-t) + h(1-t)h(t))] M_\varphi(a, b) \\
&\quad + h^2\left(\frac{1}{2}\right) [(h^2(t) + h^2(1-t))] N_\varphi(a, b).
\end{aligned}$$

Integrating above inequality over $[0, 1]$, we obtain

$$\begin{aligned}
&\frac{1}{2h^2\left(\frac{1}{2}\right)} f\left(\left[\frac{\varphi^p(a) + \varphi^p(b)}{2}\right]^{\frac{1}{p}}\right) g\left(\left[\frac{\varphi^p(a) + \varphi^p(b)}{2}\right]^{\frac{1}{p}}\right) \\
&\quad - \frac{p}{\varphi^p(b) - \varphi^p(a)} \int_{\varphi(a)}^{\varphi(b)} x^{p-1} f(x) g(x) dx \\
&\quad \leq N_\varphi(a, b) \int_0^1 h^2(t) dt + M_\varphi(a, b) \int_0^1 h(t) h(1-t) dt.
\end{aligned}$$

This completes the proof. □

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GEODESIC BALL PACKINGS GENERATED BY REGULAR PRISM TILINGS IN NIL GEOMETRY

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Received 07 May, 2019

Abstract. In this paper we study the regular prism tilings and construct ball packings by geodesic balls related to the above tilings in the projective model of **Nil** geometry. Packings are generated by action of the discrete prism groups $\mathbf{pq2}_1$. We prove that these groups are realized by prism tilings in **Nil** space if $(p, q) = (3, 6), (4, 4), (6, 3)$ and determine packing density formulae for geodesic ball packings generated by the above prism groups. Moreover, studying these formulae we determine the conjectured maximal dense packing arrangements and their densities and visualize them in the projective model of **Nil** geometry. We get a dense (conjectured locally densest) geodesic ball arrangement related to the parameters $(p, q) = (6, 3)$ where the kissing number of the packing is 14, similarly to the densest lattice-like **Nil** geodesic ball arrangement investigated by the second author in [11].

2000 *Mathematics Subject Classification:* 52C17, 52C22, 53A35, 51M20

1. INTRODUCTION

In mathematics sphere packing problems concern the arrangements of non-overlapping equal spheres which fill a space. Usually the space involved is the three-dimensional Euclidean space where the famous *Kepler conjecture* was proved by T. C. Hales and S. P. Ferguson in [5].

However, ball (sphere) packing problems can be generalized to the other 3-dimensional Thurston geometries.

In an n -dimensional space of constant curvature $\mathbf{E}^n, \mathbf{H}^n, \mathbf{S}^n$ ($n \geq 2$) let $d_n(r)$ be the density of $n + 1$ spheres of radius r mutually touching one another with respect to the simplex spanned by the centres of the spheres. L. Fejes Tóth and H. S. M. Coxeter conjectured that in an n -dimensional space of constant curvature the density of packing spheres of radius r can not exceed $d_n(r)$. This conjecture has been proved by C. Roger in the Euclidean space. The 2-dimensional case has been solved by L. Fejes Tóth. In an 3-dimensional space of constant curvature the problem has been investigated by Böröczky and Florian in [2] and it has been studied by K. Böröczky in [1] for n -dimensional space of constant curvature ($n \geq 4$).

In [14] we generalized the above problem of finding the densest geodesic and translation ball (or sphere) packing to the other 3-dimensional homogeneous geometries (Thurston geometries) $\widehat{\mathrm{SL}_2\mathbf{R}}, \mathbf{Nil}, \mathbf{S}^2 \times \mathbf{R}, \mathbf{H}^2 \times \mathbf{R}, \mathbf{Sol}$, and in the papers [11], [13], [12], [14] we investigated several interesting ball packing and covering problems in the above geometries. We described in $\mathbf{S}^2 \times \mathbf{R}$ geometry (see [14]) a candidate of the densest geodesic and translation ball arrangement whose density is ≈ 0.8750 .

In this paper we study the regular prism tilings and construct ball packings by geodesic balls related to the prism tilings in the projective model of $\mathbf{Nil}$ geometry where the packings are generated by action of the discrete prism groups $\mathbf{pq2}_1$. We obtain density formulae for calculations for geodesic ball packings. Analyzing these density functions we obtain a conjecture for optimal geodesic ball packing configurations and determine their densities related to the above prismatic tessellations.

The paper is organized as follows: in Section 2 we summarize the notions of $\mathbf{Nil}$ geometry using the projective model. We introduce the translation and rotation formulas in this model, then define and review the basic fact on the geodesic curves and spheres of $\mathbf{Nil}$.

In Section 3 we consider the problem of prism-like tilings in a $\mathbf{Nil}$ geometry context. We define the infinite and bounded prisms of $\mathbf{Nil}$, then consider the existence of regular prism tilings. Theorem 4 gives proof of the regular prism tilings corresponding to space group $\mathbf{pq2}_1$.

Finally in Section 4 we take a look at the geodesic ball packings corresponding to the prism-like tilings of $\mathbf{Nil}$. After the necessary definitions we investigate the optimal ball packings generated by the prism tilings of $\mathbf{pq2}_1$. The main results are summarized in Theorem 5 and Conjecture 1.

2. BASIC NOTIONS OF THE $\mathbf{Nil}$ GEOMETRY

Nilmanifolds are extremely important geometric objects and consequently there is a great literature of nilpotent Lie groups and their geometry. The study of two-step nilpotent metric Lie algebras with left-invariant metrics has a special importance (for example see [9] and [3]), especially those that are created from Heisenberg groups. In our case we investigate the geometry of the homogeneous 3-space derived from the 3-dimensional Heisenberg real matrix group (for example [4]). To this group we can attach multiple Riemann metrics, which define different geometries. The choice of this metric can change the geometry to a degree, as seen in [6]. In our previous work we have also used other model and metric of the 3-dimensional $\mathbf{Nil}$ geometry, as seen in [8].

In our paper we define the $\mathbf{Nil}$ space using one of these left invariant metrics of the Heisenberg group, but other metrics are also investigated in the literature. According to [6], the space of left invariant Riemannian metrics on the Heisenberg group is 3-dimensional. Here, as in our previous works we shall use the standard Riemannian

metric of **Nil**, obtained by pull back transform to the infinitesimal arc-length-square at the origin. We will now introduce the projective model of **Nil** geometry.

The left (row-column) multiplication of Heisenberg matrices

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+x & c+xb+z \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{pmatrix} \quad (2.1)$$

defines "translations" $\mathbf{L}(\mathbf{R}) = \{(x, y, z) : x, y, z \in \mathbf{R}\}$ on the points of the space $\mathbf{Nil} = \{(a, b, c) : a, b, c \in \mathbf{R}\}$. These translations are not commutative in general. The matrices $\mathbf{K}(z) \triangleleft \mathbf{L}$ of the form

$$\mathbf{K}(z) \ni \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto (0, 0, z) \quad (2.2)$$

constitute the one parametric centre, i.e. each of its elements commutes with all elements of $\mathbf{L}$. The elements of $\mathbf{K}$ are called *fibre translations*. **Nil** geometry of the Heisenberg group can be projectively (affinely) interpreted by the "right translations" on points as the matrix formula

$$(1; a, b, c) \rightarrow (1; a, b, c) \begin{pmatrix} 1 & x & y & z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix} = (1; x+a, y+b, z+bx+c) \quad (2.3)$$

shows, according to (1.1). Here we consider $\mathbf{L}$ as projective collineation group with right actions in homogeneous coordinates. We will use the Cartesian homogeneous coordinate simplex $E_0(\mathbf{e}_0), E_1^\infty(\mathbf{e}_1), E_2^\infty(\mathbf{e}_2), E_3^\infty(\mathbf{e}_3)$, $(\{\mathbf{e}_i\} \subset \mathbf{V}^4$ with the unit point $E(\mathbf{e} = \mathbf{e}_0 + \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3))$ which is distinguished by an origin E_0 and by the ideal points of coordinate axes, respectively. Moreover, $\mathbf{y} = c\mathbf{x}$ with $0 < c \in \mathbb{R}$ (or $c \in \mathbb{R} \setminus \{0\}$) defines a point $(\mathbf{x}) = (\mathbf{y})$ of the projective 3-sphere $\mathcal{PS}^3$ (or that of the projective space $\mathcal{P}^3$ where opposite rays $(\mathbf{x})$ and $(-\mathbf{x})$ are identified). The dual system $\{(e^i)\}$, $(\{\mathbf{e}^i\} \subset V_4)$ describes the simplex planes, especially the plane at infinity $(e^0) = E_1^\infty E_2^\infty E_3^\infty$, and generally, $v = u_c^1$ defines a plane $(u) = (v)$ of $\mathcal{PS}^3$ (or that of $\mathcal{P}^3$). Thus $0 = \mathbf{x}u = \mathbf{y}v$ defines the incidence of point $(\mathbf{x}) = (\mathbf{y})$ and plane $(u) = (v)$, as $(\mathbf{x})I(u)$ also denotes it. Thus **Nil** can be visualized in the affine 3-space $\mathbf{A}^3$ (so in $\mathbf{E}^3$) as well.

The translation group $\mathbf{L}$ defined by formula (2.3) can be extended to a larger group $\mathbf{G}$ of collineations, preserving the fibering, that will be equivalent to the (orientation preserving) isometry group of **Nil**. In [7] E. Molnár has shown that a rotation trough angle ω about the z -axis at the origin, as isometry of **Nil**, keeping invariant the Riemann metric everywhere, will be a quadratic mapping in x, y to z -image $\bar{z}$ as

follows:

$$\begin{aligned} \mathbf{r}(O, \omega) : (1; x, y, z) &\rightarrow (1; \bar{x}, \bar{y}, \bar{z}); \\ \bar{x} &= x \cos \omega - y \sin \omega, \quad \bar{y} = x \sin \omega + y \cos \omega, \\ \bar{z} &= z - \frac{1}{2}xy + \frac{1}{4}(x^2 - y^2) \sin 2\omega + \frac{1}{2}xy \cos 2\omega. \end{aligned} \quad (2.4)$$

This rotation formula, however, is conjugate by the quadratic mapping $\mathcal{M}$

$$\begin{aligned} x \rightarrow x' = x, \quad y \rightarrow y' = y, \quad z \rightarrow z' = z - \frac{1}{2}xy \quad \text{to} \\ (1; x', y', z') \rightarrow (1; x', y', z') \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \omega & \sin \omega & 0 \\ 0 & -\sin \omega & \cos \omega & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (1; x'', y'', z''), \end{aligned} \quad (2.5)$$

$$\text{with } x'' \rightarrow \bar{x} = x'', \quad y'' \rightarrow \bar{y} = y'', \quad z'' \rightarrow \bar{z} = z'' + \frac{1}{2}x''y'',$$

i.e. to the linear rotation formula. This quadratic conjugacy modifies the **Nil** translations in (2.3), as well. We shall use the following important classification theorem.

- Theorem 1** (E. Molnár [7]). (1) *Any group of **Nil** isometries, containing a 3-dimensional translation lattice, is conjugate by the quadratic mapping in (2.5) to an affine group of the affine (or Euclidean) space $\mathbf{A}^3 = \mathbf{E}^3$ whose projection onto the (x, y) plane is an isometry group of $\mathbf{E}^2$. Such an affine group preserves a plane $\rightarrow$ point polarity of signature $(0, 0, \pm 0, +)$.*
- (2) *Of course, the involutive line reflection about the y axis*

$$(1; x, y, z) \rightarrow (1; -x, y, -z),$$

*preserving the Riemann metric, and its conjugates by the above isometries in 1 (those of the identity component) are also **Nil**-isometries. There does not exist orientation reversing **Nil**-isometry.*

Remark 1. We obtain from the above described projective model a new model of **Nil** geometry derived by the quadratic mapping $\mathcal{M}$. This is the *linearized model of **Nil** space* (see [2]).

2.1. Geodesic curves and spheres

The geodesic curves of the **Nil** geometry are generally defined as having locally minimal arc length between their any two (near enough) points. The equation systems of the parametrized geodesic curves $g(x(t), y(t), z(t))$ in our model can be determined by the general theory of Riemann geometry. We can assume, that the starting point of a geodesic curve is the origin because we can transform a curve into an arbitrary starting point by translation (2.1);

$$\begin{aligned} x(0) = y(0) = z(0) = 0; \quad \dot{x}(0) = c \cos \alpha, \quad \dot{y}(0) = c \sin \alpha, \\ \dot{z}(0) = w; \quad -\pi \leq \alpha \leq \pi. \end{aligned}$$

The arc length parameter s is introduced by

$$s = \sqrt{c^2 + w^2} \cdot t, \text{ where } w = \sin \theta, c = \cos \theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},$$

i.e. unit velocity can be assumed.

Remark 2. Thus we have harmonized the scales along the coordinate axes.

The equation systems of a helix-like geodesic curves $g(x(t), y(t), z(t))$ if $0 < |w| < 1$:

$$\begin{aligned} x(t) &= \frac{2c}{w} \sin \frac{wt}{2} \cos \left(\frac{wt}{2} + \alpha \right), \quad y(t) = \frac{2c}{w} \sin \frac{wt}{2} \sin \left(\frac{wt}{2} + \alpha \right), \\ z(t) &= wt \cdot \left\{ 1 + \frac{c^2}{2w^2} \left[\left(1 - \frac{\sin(2wt + 2\alpha) - \sin 2\alpha}{2wt} \right) + \right. \right. \\ &\quad \left. \left. + \left(1 - \frac{\sin(2wt)}{wt} \right) - \left(1 - \frac{\sin(wt + 2\alpha) - \sin 2\alpha}{2wt} \right) \right] \right\} = \\ &= wt \cdot \left\{ 1 + \frac{c^2}{2w^2} \left[\left(1 - \frac{\sin(wt)}{wt} \right) + \left(\frac{1 - \cos(2wt)}{wt} \right) \sin(wt + 2\alpha) \right] \right\}. \end{aligned} \quad (2.6)$$

In the cases $w = 0$ the geodesic curve is the following:

$$x(t) = c \cdot t \cos \alpha, \quad y(t) = c \cdot t \sin \alpha, \quad z(t) = \frac{1}{2} c^2 \cdot t^2 \cos \alpha \sin \alpha. \quad (2.7)$$

The cases $|w| = 1$ are trivial: $(x, y) = (0, 0)$, $z = w \cdot t$.

Definition 1. The distance $d(P_1, P_2)$ between the points P_1 and P_2 is defined by the arc length of geodesic curve from P_1 to P_2 .

In our work [11] we introduced the following definitions:

Definition 2. The geodesic sphere of radius R with centre at the point P_1 is defined as the set of all points P_2 in the space with the condition $d(P_1, P_2) = R$. Moreover, we require that the geodesic sphere is a simply connected surface without self-intersection in the **Nil** space.

Remark 3. We shall see that this last condition depends on radius R .

Definition 3. The body of the geodesic sphere of centre P_1 and of radius R in the **Nil** space is called geodesic ball, denoted by $B_{P_1}(R)$, i.e. $Q \in B_{P_1}(R)$ iff $0 \leq d(P_1, Q) \leq R$.

Remark 4. Henceforth, typically we choose the origin as centre of the sphere and its ball, by the homogeneity of **Nil**.

We have denoted by $B(S)$ the body of the **Nil** sphere S , furthermore we have denoted their volumes by $Vol(B(S))$.

In [11] we have proved the the following theorem:

Theorem 2. *The geodesic sphere and ball of radius R exists in the **Nil** space if and only if $R \in [0, 2\pi]$.*

We obtain the volume of the geodesic ball of radius R by the following integral (see 2.8):

$$\begin{aligned} Vol(B(S)) &= 2\pi \int_0^{\frac{\pi}{2}} X^2 \frac{dZ}{d\theta} d\theta \\ &= 2\pi \int_0^{\frac{\pi}{2}} \left(\frac{2\cos\theta}{\sin\theta} \sin \frac{(R\sin\theta)}{2} \right)^2 \cdot \left(-\frac{1}{2} \frac{R\cos^3\theta}{\sin^2\theta} + \frac{\cos\theta \sin(R\sin\theta)}{\sin\theta} \right. \\ &\quad \left. + \frac{\cos^3\theta \sin(R\sin\theta)}{\sin^3\theta} - \frac{1}{2} \frac{R\cos^3\theta \cos(R\sin\theta)}{\sin^2\theta} \right) d\theta. \end{aligned} \quad (2.8)$$

The parametric equation system of the geodesic sphere $S(R)$ in our model (see [11]):

$$\begin{aligned} x(R, \theta, \phi) &= \frac{2c}{w} \sin \frac{wR}{2} \cdot \cos \phi = \frac{2\cos\theta}{\sin\theta} \sin \frac{R\sin\theta}{2} \cdot \cos \phi, \\ y(R, \theta, \phi) &= \frac{2c}{w} \sin \frac{wR}{2} \cdot \sin \phi = \frac{2\cos\theta}{\sin\theta} \sin \frac{R\sin\theta}{2} \cdot \sin \phi, \\ z(R, \theta, \phi) &= wR + \frac{c^2 R}{2w} - \frac{c^2}{2w^2} \sin wR + \frac{1}{4} \left(\frac{2c}{w} \sin \frac{wR}{2} \right)^2 \sin 2\phi \\ &= R \sin \theta + \frac{R \cos^2 \theta}{2 \sin \theta} - \frac{\cos^2 \theta}{2 \sin^2 \theta} \sin(R \sin \theta) + \frac{1}{4} \left(\frac{2 \cos \theta}{\sin \theta} \sin R \frac{\sin \theta}{2} \right)^2 \sin 2\phi \\ &\quad - \pi < \phi \leq \pi, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \text{ and } \theta \neq 0. \end{aligned}$$

if $\theta = 0$ then

$$x(R, 0, \phi) = R \cos \phi, \quad y(R, 0, \phi) = R \sin \phi, \quad z(R, 0, \phi) = \frac{1}{2} R^2 \cos \phi \sin \phi. \quad (2.9)$$

We have obtained by the derivatives of these parametrically represented functions (by intensive and careful computations with *Maple* through the second fundamental form) the following theorem (see [11]):

Theorem 3. *The geodesic **Nil** ball $B(S(R))$ is convex in affine-Euclidean sense in our model if and only if $R \in [0, \frac{\pi}{2}]$.*

We also use the following basic definition:

Definition 4. For a ball packing associated with a prism tiling the kissing number of the ball packing is defined as the number of non-overlapping spheres in the packing, that each touch a common sphere.

3. **Nil** PRISMS AND PRISM TILINGS

The prisms and prism-like tilings have been thoroughly investigated in $\mathbf{S}^2 \times \mathbf{R}$, $\mathbf{H}^2 \times \mathbf{R}$ and $\mathbf{SL}_2\mathbf{R}$ spaces in papers [10], [16]. Here we consider the analogous problem in **Nil** space. We will use the in 2. section described projective model of **Nil** geometry. In the following the plane of x, y axis are called *base plane* of the model and if we say *plane* then it is a plane in Euclidean sense.

Definition 5. Let $\mathcal{P}^i$ be an infinite solid bounded by planes, that are determined by fibre-lines passing through the points of a p -gon ($p \geq 3$, integer parameter) $\mathcal{P}^b$ lying in the base-plane. The images of $\mathcal{P}^i$ by **Nil** isometries are called infinite p -sided prisms.

The common part of $\mathcal{P}^i$ with the base plane is defined as the *base figure* $\mathcal{P}^b$ of the prism.

Let $\mathcal{F}$ be the $\mathcal{M}^{-1}$ image of the base plane in the **Nil**-space (see Remark 2.2) and let τ be a fibre translation (2.2).

Definition 6. Let $\mathcal{P}^i$ be an infinite p -sided prism, that is trimmed by the surface $\mathcal{F}$ and its translated copy $\mathcal{F}^\tau$. The parts of $\mathcal{F}$ and $\mathcal{F}^\tau$ inside the infinite prism are called cover faces and are denoted by $C_{\mathcal{F}}$ and $C_{\mathcal{F}^\tau}$.

The p -sided bounded prism is the part of $\mathcal{P}^i$ between the cover faces $C_{\mathcal{F}}$ and $C_{\mathcal{F}^\tau}$.

Definition 7. A bounded or infinite p -sided prism is said to be *regular* if its side surfaces are congruent to each other under **Nil** rotations with angle $\frac{2\pi}{p}$ (see (2.4) and (2.5)) about the central fibre line of the prism.

3.1. Regular bounded prism tilings

In this section we will investigate the existence of regular bounded prism tilings $\mathcal{T}_p(q)$ of **Nil** space. In this case the prism tiles are regular bounded prisms having p -gonal base figures ($p \geq 3$). The prism itself is a *topological polyhedron* with $2p$ vertices, and having at every vertex one p -gonal cover face and two quadrangle side faces (traced by fibre lines). We are looking such prism tilings of **Nil** space where at each side edge of the prism (which are fibre lines going through vertices of the base figure) meet q prisms regularly, by **Nil** rotations with angle $\frac{2\pi}{q}$ ($q \geq 3$, integer parameter).

We shall see in Theorem 3.4 that the regular prism tiling $\mathcal{T}_p(q)$ exists for some parameters (p, q) . Let $\mathcal{P}_p(q)$ one of its tiles with vertices $A_1A_2 \dots A_p B_1B_2 \dots B_p$. We may assume that A_1 lies on the x -axis. It is clear that the side curves $c_{A_iA_{i+1}}$ ($i = 1 \dots p$, $A_{p+1} \equiv A_1$) are derived from each other by $\frac{2\pi}{p}$ rotation about the x axis. The corresponding vertices $B_1B_2 \dots B_p$ are generated by a fibre translation τ with a positive real parameter. The cover faces $A_1, \dots, A_p, B_1, \dots, B_p$ and the side surfaces form a p -sided regular prism $\mathcal{P}_p(q)$ in **Nil**. $\mathcal{T}_p(q)$ will be generated by its rotational isometry group $\Gamma_p(q) = \mathbf{pq2}_1 \subset \text{Isom}(\mathbf{Nil})$ (if these tiling there exist see Theorem

3.4) which is given by its fundamental domain $\mathcal{F}_p(q) = A_1 A_2 O A_1^s A_2^s O^s$, $A_1^s = B_p$. Here, $A_2^s = B_1$, $O^s = O^r$, and $\mathcal{F}_p(q)$ is a piece-wise linear topological polyhedron. The group presentation can be determined by a standard procedure, called Poincaré algorithm. The generators will pair the bent (piecewise linear) faces of $\mathcal{F}$:

$$\begin{aligned} \mathbf{a} &: OA_1 B_p O^s(O) \rightarrow OA_2 B_1 O^s(O), \\ \mathbf{b} &: A_1 A_2 B_1(A_1) \rightarrow A_1 B_p B_1(A_1), \mathbf{s} : OA_1 A_2(O) \rightarrow O^s B_p B_1(O^s) \end{aligned}$$

mapping $\mathcal{F}_p(q)$ onto its neighbours $\mathcal{F}_p(q)^{\mathbf{a}}$, $\mathcal{F}_p(q)^{\mathbf{b}}$, $\mathcal{F}_p(q)^{\mathbf{s}}$, respectively. E.g. for the face a^{-1} a point A (relative freely, e.g. in the segment OB_p) is taken. Then the union of triangles AOO^s , $AO^s B_p$, $AB_p A_1$, $OA_1 O$ will be the face a^{-1} .

Then the $\mathbf{a}$ -image $A^{\mathbf{a}}$ is taken in OB_1 for the face $a = A^{\mathbf{a}} O O^s \cup A^{\mathbf{a}} O^s B_1 \cup A^{\mathbf{a}} B_1 A_2 \cup A^{\mathbf{a}} A_2 O$, as usual. The relations are induced by the edge equivalence classes $\{OO'\}$; $\{A_1 B_1\}$; $\{OA_1, OA_2, O'B_1, O'B_p\}$; $\{A_1 A_2, A_1 B_p, A_2 B_1, B_p B_1\}$. So we get the group

$$\mathbf{pq2_1} = \{\mathbf{a}, \mathbf{b} : \mathbf{a}^p = \mathbf{b}^q = \mathbf{ababa}^{-1} \mathbf{b}^{-1} \mathbf{a}^{-1} \mathbf{b}^{-1} = 1\}, \quad (3.1)$$

where $\mathbf{a}$ is a p rotation about the fibre line through the origin (the z -axis), $\mathbf{b}$ is a q rotation about a side fibre line of $\mathcal{P}_p(q)$ (through a vertex of its base figure). Notice, that $\mathbf{bab}$ is a screw motion, and thus $\tau := \mathbf{abab} = \mathbf{baba}$ is the fibre translation connecting the cover faces.

Our first question is the following: For which $3 \leq p, q \in \mathbb{N}$ is $\Gamma_p(q) = \mathbf{pq2_1} \subset \text{Isom}(\mathbf{Nil})$?

The following Theorem answers it:

Theorem 4. *In $\mathbf{Nil}$ there exist 3 regular p -gonal non-face-to-face prism tilings $\mathcal{T}_p(q)$ with $\mathbf{Nil}$ isometry group $\Gamma_p(q) = \mathbf{pq2_1}$ for integer parameters $p, q \geq 3$:*

the regular triangular prism tiling with $(p, q) = (3, 6)$,

the regular square prism tiling with $(p, q) = (4, 4)$,

the regular hexagonal prism tiling with $(p, q) = (6, 3)$,

and each group $\Gamma_p(q)$ has a free parameter $x_p(q) \in \mathbb{R}^+$.

Proof. Let $A_1 = (1; x_p(q), 0, 0)$ be a "bottom" vertex of the regular bounded prism. Then the other vertices of the bottom cover face can be generated by the $\mathbf{Nil}$ rotation formula (see (2.4), (2.5)):

$$\begin{aligned} A_2 &= A_1^{\mathbf{a}} = \left(1; x_p(q) \cos\left(\frac{2\pi}{p}\right), x_p(q) \sin\left(\frac{2\pi}{p}\right), \frac{1}{4}x_p(q)^2 \sin\left(\frac{4\pi}{p}\right)\right), \\ A_3 &= A_2^{\mathbf{a}} = A_1^{\mathbf{a}^2} = \left(1; x_p(q) \cos\left(\frac{4\pi}{p}\right), x_p(q) \sin\left(\frac{4\pi}{p}\right), \frac{1}{4}x_p(q)^2 \sin\left(\frac{8\pi}{p}\right)\right), \\ &\dots \\ A_p &= A_{p-1}^{\mathbf{a}} = A_{p-1}^{\mathbf{a}^{p-1}}. \end{aligned}$$

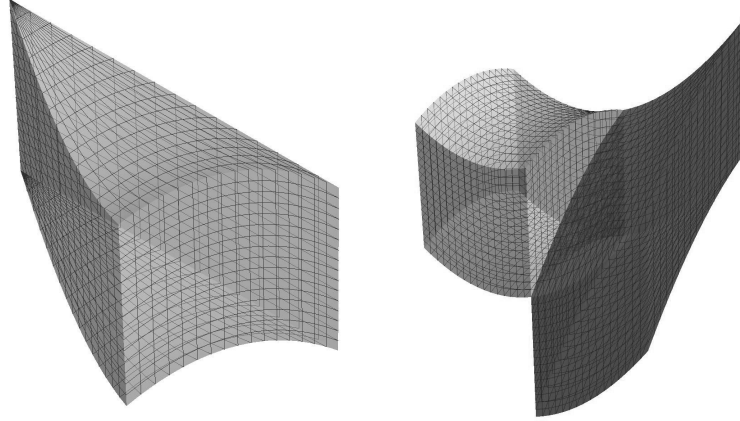


FIGURE 1. The regular triangular and rectangular prisms in Nil-space, with parameters $x_3(6) = \frac{2}{\sqrt{3}}$ and $x_4(4) = \sqrt{2}$ respectively

Then the condition for the existence of the tiling is the following:

$$A_3 = A_2^{\mathbf{a}} \equiv A_1^{\mathbf{b}^{-1}}, \quad (3.2)$$

where $\mathbf{a}$ is a p rotation about the fibre line through the origin and $\mathbf{b}$ is a q rotation about the side fibre line of $\mathcal{P}_p(q)$ through the vertex A_2 . $\equiv$ means that the corresponding points lie on the same fibre lines.

$$1 = \cos^2\left(\frac{\pi}{p}\right) + \cos^2\left(\frac{\pi}{q}\right),$$

where p and q are positive integers. This equation only has the following integer solutions:

$$(p, q) = (4, 4), (3, 6) \text{ or } (6, 3).$$

We obtain from the above computations, that the existence of the above regular prism tilings is independent from the parameter $x_p(q) \in \mathbb{R}^+$, so we have proven the Theorem. $\square$

Remembering that $\tau = \mathbf{abab}$ is the "vertical" translation of the group, we can also compute the height of the regular bounded prism corresponding to the group tiling, since: $O^{\mathbf{abab}} = O^\tau$, where O is the origin. Using this, we can also give a metric representation of the group, allowing the visualization of the corresponding prism and prism tiling (see Fig. 1. and Fig. 2.).

4. THE OPTIMAL GEODESIC BALL PACKINGS UNDER GROUP $\mathbf{pq2_1}$

The sphere packing problem deals with the arrangements of non-overlapping equal spheres, or balls, which fill the space. While the usual problem is in the n -dimensional

Euclidean-space ($n \geq 2$), it can be generalized to the other 3-dimensional Thurston spaces (see [15]). In this paper we investigate the optimal ball packings of **Nil** generated by the above described **pq2₁** group.

Let $\mathcal{T}_p(x_p(q), q)$ (where $(p, q) = (3, 6), (4, 4)$ or $(6, 3)$ as stated above and $x_p(q) \in \mathbb{R}^+$) be a regular prism tiling, and let $\mathcal{P}_p(x_p(q), q)$ be one of its tiles that is centered at the origin, with a base face given by the vertices $A_1, A_2, \dots, A_p$. The corresponding vertices $B_1, B_2, \dots, B_p$ of the prism are generated by fibre translations $\tau = \mathbf{abab}$.

We can assume by symmetry, that the optimal geodesic ball is centered at the origin. The volume of a geodesic ball with radius R can be determined by the formula (2.8).

We study only one case of the multiply transitive geodesic ball packings where the fundamental domains of the **Nil** space groups **pq2₁** are not prisms. Let the fundamental domains be derived by the Dirichlet — Voronoi cells (D-V cells) where their centers are images of the origin. The volume of the p -times fundamental domain and of the D-V cell is the same, respectively, as in the prism case (for any above $(p, q, x_p(q))$ fixed). It is easy to see by the formulas (2.5), using the quadratic mapping $\mathcal{M}$, that the volume of the Dirichlet — Voronoi cell (or the corresponding prism) is

$$\text{Vol}(\mathcal{P}_p(x_p(q), q)) = \frac{p}{2} x_p^2(q) \sin\left(\frac{2\pi}{p}\right) d(OO^\tau). \quad (4.1)$$

These locally densest geodesic ball packings can be determined for all possible fixed integer parameters $p, q, x_p(q)$. The optimal radius $R_{opt}(x_p(q))$ is

$$R_{opt}(x_p(q), p, q) = \min \left\{ d(OA_1), \frac{d(OO^\tau)}{2}, \frac{d(O, O^{\mathbf{ab}})}{2} \right\}, \quad (4.2)$$

where d is the geodesic distance function of **Nil** geometry (see Definition 2.4).

Since the congruent images of $\mathcal{P}_p(x_p(q), q)$ under the discrete group **pq2₁** cover the **Nil** space, therefore for the density of the ball packing it is sufficient to relate the volume of the ball to the volume of the prism:

Definition 8. The maximal density $\delta_p(x_p(q), q)$ of the above multiply transitive ball packing for given parameters $(p, q, x_p(q))$ ($(p, q) = (3, 6), (4, 4), (6, 3)$ and $x_p(q) \in \mathbb{R}^+$):

$$\delta_p(x_p(q), q) = \frac{\text{Vol}(B(R_{opt}))}{\text{Vol}(\mathcal{P}_p(x_p(q), q))} = \frac{\text{Vol}(B(R_{opt}))}{\frac{p}{2} x_p^2(q) \sin\left(\frac{2\pi}{p}\right) d(OO^\tau)}. \quad (4.3)$$

For every $p, q, x_p(q)$ parameters the locally densest geodesic ball packing can be determined.

If we fixed the parameters p and q then the distance function $d(x_p(q))$ is a continuous function. Therefore it is easy to prove the following Lemma:

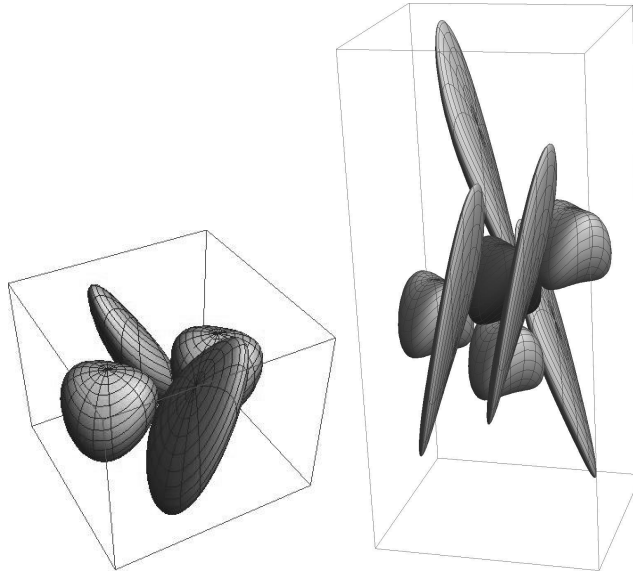


FIGURE 2. Some balls of the optimal ball arrangements for the square and hexagonal tilings with parameters $(p, q) = (4, 4)$ and $(p, q) = (3, 6)$.

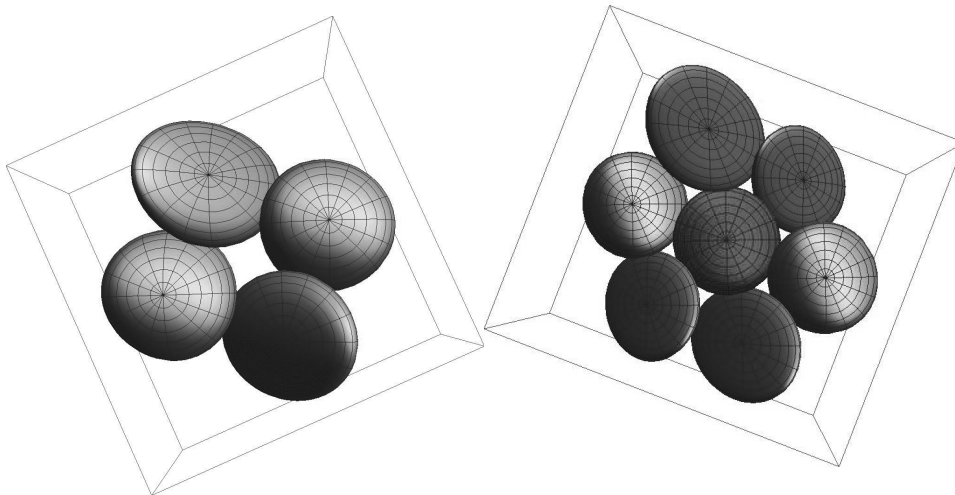


FIGURE 3. Some balls of the optimal ball arrangements for the square and hexagonal tilings shown from the direction of the z -axis.

Lemma 1. *In Nil-space for the rotation group $\Gamma_p(q) = \mathbf{pq2_1}$ there always exist $x_p(q) \in \mathbb{R}^+$ for given parameters $(p, q) = (4, 4), (3, 6), (6, 3)$ where*

$$d(O, O^{\mathbf{ab}}) = d(O, O^{\mathbf{abab}}) = d(O, O^{\mathbf{r}}). \quad (4.4)$$

The system of equations (4.4) in Lemma 4.2 can be solved by numerical methods and the corresponding ball arrangements are denoted by $\mathcal{B}_p(q)$. We obtain - using the formulas (4.1-3) - that in Nil-space for the rotation group $\Gamma_p(q) = \mathbf{pq2_1}$ the metric data of the godesic ball arrangements $\mathcal{B}_p(q)$ are the following:

Theorem 5. *If the system of equation (4.4) holds then the maximal radii and densities of the optimal ball packings are the following:*

- If $(p, q) = (3, 6)$, then $\delta_p(q) \approx 0.2593$, with $R_{opt}(p, q) \approx 0.7389$,
- If $(p, q) = (4, 4)$, then $\delta_p(q) \approx 0.6512$, with $R_{opt}(p, q) \approx 1.2154$,
- If $(p, q) = (6, 3)$, then $\delta_p(q) \approx 0.7272$, with $R_{opt}(p, q) \approx 1.9601$.

If we vary the parameter $x_p(q)$ in the above cases then the corresponding radius $R_{opt}(p, q)$ and the density $\delta_p(q)$ also change. The following table shows that probably the $\mathcal{B}_p(q)$ ball packings with maximal kissing numbers provide the optimal ball packing densities.

(p, q)	Radius	Prism volume	Density	Kissing number
(3,6)	0.5876	4.1446	0.2063	2
	0.6392	4.9032	0.2246	2
	0.6929	5.7616	0.2438	2
	0.7389	6.5517	0.2593	8
	0.7787	7.8111	0.2558	6
	0.8132	9.0201	0.2525	6
	0.8481	10.3641	0.2495	6
(4,4)	0.9927	7.8849	0.5283	2
	1.0644	9.0650	0.5678	2
	1.1386	10.3729	0.6090	2
	1.2154	11.8175	0.6512	10
	1.2594	13.4079	0.6404	8
	1.3036	15.1538	0.6295	8
	1.3480	17.0647	0.6194	8
(6,3)	1.6934	34.4141	0.6190	2
	1.7801	38.0287	0.6537	2
	1.8690	41.9209	0.6897	2
	1.9601	46.1044	0.7272	14
	2.0087	50.5935	0.7153	12
	2.0573	55.4028	0.7038	12
	2.1059	60.5470	0.6929	12

Therefore, we can formulate by the above results the following conjecture:

Conjecture 1. *The ball arrangements $\mathcal{B}_p(q)$ provide the densest ball packing arrangements related to $\Gamma_p(q) = \mathbf{pq2}_1$ Nil isometry group with parameters $(p, q) = (4, 4), (3, 6), (6, 3)$.*

Remark 5. The optimal ball packing in the case of $(p, q) = (6, 3)$ has a kissing number of 14, which is greater than the maximal kissing number 12 in the Euclidean 3-dimensional space. In fact, this is the second ball packing arrangement in Nil that has this high of a kissing number (see [11]).

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DEFERRED CESÀRO STATISTICAL CONVERGENCE OF MARTINGALE SEQUENCE AND KOROVKIN-TYPE APPROXIMATION THEOREMS

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Received 29 December, 2020

Abstract. In the present paper, we introduce and study the concepts of statistical convergence and statistical summability for martingale sequences of random variables via deferred Cesàro mean. We then establish an inclusion theorem concerning the relation between these two beautiful and potentially useful concepts. Also, based upon our proposed notions, we state and prove new Korovkin-type approximation theorems with algebraic test functions for a martingale sequence over a Banach space. Moreover, we demonstrate that our theorems effectively extend and improve most (if not all) of the previously existing results (in statistical and classical versions). Finally, by using the generalized Bernstein polynomials, we present an illustrative example of a martingale sequence in order to demonstrate that our established theorems are stronger than their traditional and statistical versions.

2010 Mathematics Subject Classification: 40A05; 40G15; 41A36; 33C45

Keywords: stochastic sequences, martingale sequences, statistical convergence of martingale sequences, deferred Cesàro mean, Banach space, Korovkin-type theorem, Bernstein polynomials and positive linear operators

1. INTRODUCTION AND MOTIVATION

Let $(\Omega, \mathcal{F}, P)$ be a probability measurable space and suppose that (X_n) is a random variable defined over this space. Also let $\mathcal{F}_n \subseteq \mathcal{F}$ ($n \in \mathbb{N}$) be a monotonically increasing sequence of σ -fields of measurable sets. Now, considering the random variable (X_n) and the measurable functions $(\mathcal{F}_n)$, we adopt a stochastic sequence $(X_n, \mathcal{F}_n; n \in \mathbb{N})$.

A given stochastic sequence $(X_n, \mathcal{F}_n; n \in \mathbb{N})$ is said to be a martingale sequence if

(i) $\mathbb{E}|X_n| < \infty$,

(ii) $\mathbb{E}(X_{n+1} | \mathcal{F}_n) = X_n$ almost surely (a.s.) and

(iii) $(\mathcal{F}_n)$ is a measurable sequence of functions,

where $\mathbb{E}$ is the mathematical expectation.

We now recall the definition of convergence of martingale sequences of random variables.

Definition 1. A martingale sequence $(X_n, \mathcal{F}_n; n \in \mathbb{N})$ with $\mathbb{E}|X_n|$ is bounded and $\text{Prob}(X_n) = 1$ (that is, with probability 1) is said to be convergent to a martingale $(X_0, \mathcal{F}_0)$, if

$$\lim_{n \rightarrow \infty} (X_n, \mathcal{F}_n) \longrightarrow (X_0, \mathcal{F}_0) \quad (\mathbb{E}|X_0| < \infty).$$

Recently, the study of statistical convergence has been one of the beautiful aspects of the theory of sequence spaces. The investigation and study of statistical convergence are potentially useful in sequence space because it is more general than the usual convergence. Such a concept was first introduced independently by two eminent mathematicians, Fast [6] and Steinhaus [24]. Subsequently, by using this nice concept with different settings, various researchers developed many interesting and useful results in several fields of mathematics such as summability theory, Fourier series, approximation theory, probability theory, measure theory, and so on. Moreover, the introduction of statistical probability convergence has enhanced the glory of this development. For some recent research works in this direction, see [2–4, 7–9, 11, 17, 21] and [22].

Let $\mathcal{U} \subseteq \mathbb{N}$. Also let

$$\mathcal{U}_n = \{i : i \leq n \text{ and } i \in \mathcal{U}\}.$$

Then the natural density $d(\mathcal{U})$ of $\mathcal{U}$ is defined by

$$d(\mathcal{U}) = \lim_{n \rightarrow \infty} \frac{|\mathcal{U}_n|}{n} = b,$$

where b is a real and finite number and $|\mathcal{U}_n|$ is the cardinality of $\mathcal{U}_n$.

A given sequence (a_n) is statistically convergent to λ if, for each $\varepsilon > 0$,

$$\mathcal{U}_\varepsilon = \{i : i \in \mathbb{N} \text{ and } |a_i - \lambda| \geq \varepsilon\}$$

has zero natural density (see [6] and [24]). Thus, for each $\varepsilon > 0$, we have

$$d(\mathcal{U}_\varepsilon) = \lim_{n \rightarrow \infty} \frac{|\mathcal{U}_\varepsilon|}{n} = 0.$$

We write

$$\text{stat} \lim_{n \rightarrow \infty} a_n = \lambda.$$

We now introduce the definition of statistical convergence of martingale sequence.

Definition 2. A bounded martingale sequence $(X_n, \mathcal{F}_n, n \in \mathbb{N})$ having its probability 1 is said to be statistically convergent to a martingale $(X_0, \mathcal{F}_0)$ with $\mathbb{E}|X_0| < \infty$ if, for all $\varepsilon > 0$,

$$\mathcal{U}_\varepsilon = \{i : i \leq n \text{ and } |(X_i, \mathcal{F}_i) - (X_0, \mathcal{F}_0)| \geq \varepsilon\}$$

has zero natural density. This means that, for every $\varepsilon > 0$, we have

$$d(\mathfrak{U}_\varepsilon) = \lim_{n \rightarrow \infty} \frac{|\mathfrak{U}_\varepsilon|}{n} = 0.$$

We write

$$\text{stat}_{\text{mart}} \lim_{n \rightarrow \infty} (X_n, \mathcal{F}_n) = (X_0, \mathcal{F}_0).$$

Example 1. Let $(\mathcal{F}_n, n \in \mathbb{N})$ be a monotonically increasing sequence of 0-mean independent random variables over σ -fields. Also let $(X_n) \in \mathcal{F}_n$ be such that

$$X_n = \begin{cases} 1 & (n = m^2; m \in \mathbb{N}) \\ 0 & (\text{otherwise}). \end{cases}$$

It is easy to see that the martingale sequence $(X_n, \mathcal{F}_n; n \in \mathbb{N})$ is statistically convergent to zero, but not simply martingale convergent.

Based on our proposed definition, we now establish a theorem concerning a relation between the ordinary and statistical versions of convergence of martingale sequences.

Theorem 1. *If a martingale sequence $(X_n, \mathcal{F}_n, n \in \mathbb{N})$ is convergent to a martingale $(X_0, \mathcal{F}_0)$ with $\mathbb{E}|X_0| < \infty$, then it is statistically convergent to the same martingale.*

Proof. Let the martingale sequence $(X_n, \mathcal{F}_n; n \in \mathbb{N})$ be bounded and convergent with probability 1, then there exists a martingale $(X_0, \mathcal{F}_0)$ with $\mathbb{E}|X_0| < \infty$, that is,

$$\lim_{n \rightarrow \infty} (X_n, \mathcal{F}_n) \longrightarrow (X_0, \mathcal{F}_0).$$

As the given martingale sequence $(X_n, \mathcal{F}_n; n \in \mathbb{N})$ is bounded with probability 1, then, for every $\varepsilon > 0$, we have

$$\frac{1}{n} \{i : i \leq n \text{ and } |(X_i, \mathcal{F}_i) - (X_0, \mathcal{F}_0)| \geq \varepsilon\} \subseteq \lim_{n \rightarrow \infty} |(X_n, \mathcal{F}_n) - (X_0, \mathcal{F}_0)| < \varepsilon.$$

Consequently, by Definition 2, we obtain

$$d(\mathfrak{U}_\varepsilon) = \lim_{n \rightarrow \infty} \frac{|\mathfrak{U}_\varepsilon|}{n} = 0,$$

where, just as in Definition 2, we have

$$\mathfrak{U}_\varepsilon := \{i : i \leq n \text{ and } |(X_i, \mathcal{F}_i) - (X_0, \mathcal{F}_0)| \geq \varepsilon\}.$$

This evidently completes the demonstration of Theorem 1. $\square$

Motivated essentially by the above-mentioned investigations, here we introduce and study the concepts of statistical convergence and statistical summability for martingale sequences of random variables via deferred Cesàro mean. We then establish an inclusion theorem with associated example concerning a relation between these new concepts. Also, based upon our proposed methods, we state and prove the new Korovkin-type approximation theorems, with algebraic test functions, involving a

martingale sequence over a Banach space. Moreover, we demonstrate that our theorems effectively extend and improve most (if not all) of the previously existing results (in both the statistical and classical versions). Finally, by considering the generalized Bernstein polynomials, we present an illustrative example of a martingale sequence in order to demonstrate that our established theorems are stronger than their traditional and statistical versions.

2. DEFERRED CESÀRO MARTINGALE SEQUENCE

Let (u_n) and (v_n) be sequences of non-negative integers such that $u_n < v_n$ and

$$\lim_{n \rightarrow \infty} v_n = +\infty.$$

Then the deferred Cesàro mean for the martingale sequence $(X_n, \mathcal{F}_n; n \in \mathbb{N})$ is defined by

$$\begin{aligned} \mathcal{D}(X_n, \mathcal{F}_n) &= \frac{(X_{u_n+1}, \mathcal{F}_{u_n+1}) + (X_{u_n+2}, \mathcal{F}_{u_n+2}) + \cdots + (X_{v_n}, \mathcal{F}_{v_n})}{v_n - u_n} \\ &= \frac{1}{v_n - u_n} \sum_{k=u_n+1}^{v_n} (X_k, \mathcal{F}_k). \end{aligned}$$

We now present the definitions of deferred Cesàro statistical convergence and statistically deferred Cesàro summability of martingale sequences.

Definition 3. Let (u_n) and (v_n) be sequences of non-negative integers. A bounded martingale sequence $(X_n, \mathcal{F}_n; n \in \mathbb{N})$ having probability 1 is deferred Cesàro statistically convergent to a martingale $(X_0, \mathcal{F}_0)$ with $\mathbb{E}|X_0| < \infty$ if, for all $\varepsilon > 0$,

$$\mathfrak{U}_\varepsilon = \{i : u_n < i \leq v_n \text{ and } |(X_i, \mathcal{F}_i) - (X_0, \mathcal{F}_0)| \geq \varepsilon\}$$

has zero natural density. This means that, for every $\varepsilon > 0$, we have

$$\lim_{n \rightarrow \infty} \frac{|\{i : u_n < i \leq v_n \text{ and } |(X_i, \mathcal{F}_i) - (X_0, \mathcal{F}_0)| \geq \varepsilon\}|}{u_n - v_n} = 0.$$

We write

$$\text{DM}_{\text{stat}} \lim_{n \rightarrow \infty} (X_n, \mathcal{F}_n) = (X_0, \mathcal{F}_0).$$

Definition 4. Let (u_n) and (v_n) be sequences of non-negative integers. A bounded martingale sequence $(X_n, \mathcal{F}_n; n \in \mathbb{N})$ having probability 1 is statistically deferred Cesàro summable to a martingale $(X_0, \mathcal{F}_0)$ with $\mathbb{E}|X_0| < \infty$ if, for all $\varepsilon > 0$,

$$\mathfrak{U}_\varepsilon = \{i : u_n < i \leq v_n \text{ and } |\mathcal{D}(X_i, \mathcal{F}_i) - (X_0, \mathcal{F}_0)| \geq \varepsilon\}$$

has zero natural density. This means that, for every $\varepsilon > 0$, we have

$$\lim_{n \rightarrow \infty} \frac{|\{i : u_n < i \leq v_n \text{ and } |\mathcal{D}(X_i, \mathcal{F}_i) - (X_0, \mathcal{F}_0)| \geq \varepsilon\}|}{u_n - v_n} = 0.$$

We write

$$\text{stat}_{\text{DM}} \lim_{n \rightarrow \infty} \mathcal{D}(X_i, \mathcal{F}_i) = (X_0, \mathcal{F}_0).$$

We now establish an inclusion theorem concerning the two new and interesting notions that every deferred Cesàro statistical convergent martingale sequence is statistically deferred Cesàro summable, but the converse is not generally true.

Theorem 2. *If a given martingale sequence $(X_n, \mathcal{F}_n; n \in \mathbb{N})$ is deferred Cesàro statistically convergent to a martingale $(X_0, \mathcal{F}_0)$ with $\mathbb{E}|X_0| < \infty$, then it is statistically deferred Cesàro summable to the same martingale, but not conversely.*

Proof. Suppose the given martingale sequence $(X_n, \mathcal{F}_n; n \in \mathbb{N})$ is deferred Cesàro statistically convergent to a martingale $(X_0, \mathcal{F}_0)$ with $\mathbb{E}|X_0| < \infty$. Then, by Definition 3, we have

$$\lim_{n \rightarrow \infty} \frac{|\{i : u_n < i \leq v_n \text{ and } |(X_i, \mathcal{F}_i) - (X_0, \mathcal{F}_0)| \geq \varepsilon\}|}{u_n - v_n} = 0.$$

Now, for the following two sets:

$$\mathcal{W}_\varepsilon = \{i : u_n < i \leq v_n \text{ and } |(X_i, \mathcal{F}_i) - (X_0, \mathcal{F}_0)| \geq \varepsilon\}$$

and

$$\mathcal{W}_\varepsilon^c = \{i : u_n < i \leq v_n \text{ and } |(X_i, \mathcal{F}_i) - (X_0, \mathcal{F}_0)| < \varepsilon\},$$

we find that

$$\begin{aligned} |\mathcal{D}(X_n, \mathcal{F}_n) - (X_0, \mathcal{F}_0)| &= \left| \frac{1}{v_n - u_n} \sum_{k=u_n+1}^{v_n} (X_k, \mathcal{F}_k) - (X_0, \mathcal{F}_0) \right| \\ &\leq \left| \frac{1}{v_n - u_n} \sum_{k=u_n+1}^{v_n} [(X_k, \mathcal{F}_k) - (X_0, \mathcal{F}_0)] \right| \\ &\quad + \left| \frac{1}{v_n - u_n} \sum_{k=u_n+1}^{v_n} (X_0, \mathcal{F}_0) - (X_0, \mathcal{F}_0) \right| \\ &\leq \frac{1}{v_n - u_n} \sum_{\substack{k=u_n+1 \\ (i \in \mathcal{W}_\varepsilon)}}^{v_n} |\mathcal{D}(X_n, \mathcal{F}_n) - (X_0, \mathcal{F}_0)| \\ &\quad + \frac{1}{v_n - u_n} \sum_{\substack{k=u_n+1 \\ (i \in \mathcal{W}_\varepsilon^c)}}^{v_n} |\mathcal{D}(X_n, \mathcal{F}_n) - (X_0, \mathcal{F}_0)| \\ &\quad + |(X_0, \mathcal{F}_0)| \left| \frac{1}{v_n - u_n} \sum_{\lambda=u_n+1}^{v_n} -1 \right| \\ &\leq \frac{1}{v_n - u_n} |\mathcal{W}_\varepsilon| + \frac{1}{v_n - u_n} |\mathcal{W}_\varepsilon^c| = 0. \end{aligned}$$

Thus, clearly, we obtain

$$|\mathcal{D}(X_n, \mathcal{F}_n) - (X_0, \mathcal{F}_0)| < \varepsilon.$$

Therefore, the martingale sequence $(X_n, \mathcal{F}_n; n \in \mathbb{N})$ is statistically deferred Cesàro summable to the martingale $(X_0, \mathcal{F}_0)$ with $\mathbb{E}|X_0| < \infty$.

Next, in support of the non-validity of the converse statement, we present here an example demonstrating that a statistically deferred Cesàro summable martingale sequence is not necessarily deferred Cesàro statistically convergent.

Example 2. Let us set

$$u_n = 2n \quad \text{and} \quad v_n = 4n \quad (n \in \mathbb{N}).$$

Also let $(\mathcal{F}_n, n \in \mathbb{N})$ be a monotonically increasing sequence of 0-mean independent random variables of σ -fields with $(X_n) \in \mathcal{F}_n$ such that

$$X_n = \begin{cases} 1 & (n = 2m; m \in \mathbb{N}) \\ -1 & (n = 2m + 1; m \in \mathbb{N}). \end{cases}$$

It is easy to see that, the martingale sequence $(X_n, \mathcal{F}_n, n \in \mathbb{N})$ is neither convergent nor deferred Cesàro statistically convergent; however, it is deferred Cesàro summable to $\frac{1}{2}$. Therefore, it is statistically deferred Cesàro summable to $\frac{1}{2}$.

□

3. KOROVKIN-TYPE THEOREMS FOR MARTINGALE SEQUENCE

Quite recently, a number of researchers worked toward extending (or generalizing) the approximation aspect of the Korovkin-type theorems in different fields of mathematics such as (for example) sequence spaces, Banach space, probability space, measurable space, and so on. This concept is extremely valuable in real analysis, functional analysis, harmonic analysis, and other related areas. Here, in this connection, we choose to refer the interested readers to the recent works [5, 12–15, 17–19] and [20].

We establish here the statistical versions of new Korovkin-type approximation theorems for martingale sequences of positive linear operators via deferred Cesàro mean.

Let $\mathcal{C}([0, 1])$ be the space of all real-valued continuous functions defined on $[0, 1]$ under the norm $\|\cdot\|_\infty$. Also let $C[0, 1]$ be a Banach space. Then, for $f \in C[0, 1]$, the norm of f denoted by $\|f\|$ is given by

$$\|f\|_\infty = \sup_{x \in [0, 1]} \{|f(x)|\}.$$

We say that the operator $\mathfrak{L}$ is a martingale sequence of positive linear operators, provided that

$$\mathfrak{L}(f; x) \geq 0 \quad \text{whenever} \quad f \geq 0 \quad \text{with} \quad \mathfrak{L}(f; x) < \infty \quad \text{and} \quad \text{Prob}(\mathfrak{L}(f; x)) = 1.$$

Theorem 3. *Let*

$$\mathfrak{L}_m : C[0, 1] \rightarrow C[0, 1]$$

be a martingale sequence of positive linear operators. Then, for all $f \in C[0, 1]$,

$$\text{DM}_{\text{stat}} \lim_{m \rightarrow \infty} \|\mathfrak{L}_m(f; x) - f(x)\|_{\infty} = 0 \quad (3.1)$$

if and only if

$$\text{DM}_{\text{stat}} \lim_{m \rightarrow \infty} \|\mathfrak{L}_m(1; x) - 1\|_{\infty} = 0, \quad (3.2)$$

$$\text{DM}_{\text{stat}} \lim_{m \rightarrow \infty} \|\mathfrak{L}_m(2x; x) - 2x\|_{\infty} = 0 \quad (3.3)$$

and

$$\text{DM}_{\text{stat}} \lim_{m \rightarrow \infty} \|\mathfrak{L}_m(3x^2; x) - 3x^2\|_{\infty} = 0. \quad (3.4)$$

Proof. Since each of the following functions:

$$f_0(x) = 1, \quad f_1(x) = 2x \quad \text{and} \quad f_2(x) = 3x^2$$

belong to $C[0, 1]$ and are continuous, the implication given by (3.1) implies that conditions (3.2) to (3.4) are obvious.

In order to complete the proof of Theorem 3, we first assume that conditions (3.2) to (3.4) hold true. If $f \in C[0, 1]$, then there exists a constant $\mathcal{A} > 0$ such that

$$|f(x)| \leq \mathcal{A} \quad (\forall x \in [0, 1]).$$

We thus find that

$$|f(t) - f(x)| \leq 2\mathcal{A} \quad (t, x \in [0, 1]). \quad (3.5)$$

Clearly, for a given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(t) - f(x)| < \varepsilon \quad (3.6)$$

whenever

$$|t - x| < \delta \quad \text{for all } t, x \in [0, 1].$$

Let us choose

$$\varphi_1 = \varphi_1(t, x) = 4(t - x)^2.$$

If $|t - x| \geq \delta$, then we find that

$$|f(t) - f(x)| < \frac{2\mathcal{A}}{\delta^2} \varphi_1(t, x). \quad (3.7)$$

Thus, from the equations (3.6) and (3.7), we get

$$|f(t) - f(x)| < \varepsilon + \frac{2\mathcal{A}}{\delta^2} \varphi_1(t, x),$$

which implies that

$$-\varepsilon - \frac{2\mathcal{A}}{\delta^2} \varphi_1(t, x) \leq f(t) - f(x) \leq \varepsilon + \frac{2\mathcal{A}}{\delta^2} \varphi_1(t, x). \quad (3.8)$$

Now, since $\mathfrak{L}_m(1; x)$ is monotone and linear, by applying the operator $\mathfrak{L}_m(1; x)$ to this inequality, we have

$$\begin{aligned}\mathfrak{L}_m(1; x) \left(-\varepsilon - \frac{2\mathcal{A}}{\delta^2} \varphi_1(t, x) \right) &\leq \mathfrak{L}_m(1; x)(f(t) - f(x)) \\ &\leq \mathfrak{L}_m(1; x) \left(\varepsilon + \frac{2\mathcal{A}}{\delta^2} \varphi_1(t, x) \right).\end{aligned}$$

We note that x is fixed and so $f(x)$ is a constant number. Therefore, we have

$$\begin{aligned}-\varepsilon \mathfrak{L}_m(1; x) - \frac{2\mathcal{A}}{\delta^2} \mathfrak{L}_m(\varphi_1; x) &\leq \mathfrak{L}_m(f; x) - f(x) \mathfrak{L}_m(1; x) \\ &\leq \varepsilon \mathfrak{L}_m(1; x) + \frac{2\mathcal{A}}{\delta^2} \mathfrak{L}_m(\varphi_1; x).\end{aligned}\quad (3.9)$$

We know also that that

$$\mathfrak{L}_m(f; x) - f(x) = [\mathfrak{L}_m(f; x) - f(x) \mathfrak{L}_m(1; x)] + f(x) [\mathfrak{L}_m(1; x) - 1]. \quad (3.10)$$

So, by using (3.9) and (3.10), we have

$$\mathfrak{L}_m(f; x) - f(x) < \varepsilon \mathfrak{L}_m(1; x) + \frac{2\mathcal{A}}{\delta^2} \mathfrak{L}_m(\varphi_1; x) + f(x) [\mathfrak{L}_m(1; x) - 1]. \quad (3.11)$$

We now estimate $\mathfrak{L}_m(\varphi_1; x)$ as follows:

$$\begin{aligned}\mathfrak{L}_m(\varphi_1; x) &= \mathfrak{L}_m((2t - 2x)^2; x) = \mathfrak{L}_m(2t^2 - 8xt + 4x^2; x) \\ &= \mathfrak{L}_m(4t^2; x) - 8x \mathfrak{L}_m(t; x) + 4x^2 \mathfrak{L}_m(1; x) \\ &= 4[\mathfrak{L}_m(t^2; x) - x^2] - 8x[\mathfrak{L}_m(t; x) - x] + 4x^2[\mathfrak{L}_m(1; x) - 1].\end{aligned}$$

Thus, by using (3.11), we obtain

$$\begin{aligned}\mathfrak{L}_m(f; x) - f(x) &< \varepsilon \mathfrak{L}_m(1; x) + \frac{2\mathcal{A}}{\delta^2} \{4[\mathfrak{L}_m(t^2; x) - x^2] \\ &\quad - 8x[\mathfrak{L}_m(t; x) - x] + 4x^2[\mathfrak{L}_m(1; x) - 1]\} + f(x) [\mathfrak{L}_m(1; x) - 1]. \\ &= \varepsilon [\mathfrak{L}_m(1; x) - 1] + \varepsilon + \frac{2\mathcal{A}}{\delta^2} \{4[\mathfrak{L}_m(t^2; x) - x^2] \\ &\quad - 8x[\mathfrak{L}_m(t; x) - x] + 4x^2[\mathfrak{L}_m(1; x) - 1]\} + f(x) [\mathfrak{L}_m(1; x) - 1].\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we can write

$$\begin{aligned}|\mathfrak{L}_m(f; x) - f(x)| &\leq \varepsilon + \left(\varepsilon + \frac{8\mathcal{A}}{\delta^2} + \mathcal{A} \right) |\mathfrak{L}_m(1; x) - 1| \\ &\quad + \frac{16\mathcal{A}}{\delta^2} |\mathfrak{L}_m(t; x) - x| + \frac{8\mathcal{A}}{\delta^2} |\mathfrak{L}_m(t^2; x) - x^2| \\ &\leq \mathcal{E}(|\mathfrak{L}_m(1; x) - 1| + |\mathfrak{L}_m(t; x) - x| + |\mathfrak{L}_m(t^2; x) - x^2|),\end{aligned}\quad (3.12)$$

where

$$E = \max \left(\varepsilon + \frac{8\mathcal{A}}{\delta^2} + \mathcal{A}, \frac{16\mathcal{A}}{\delta^2}, \frac{8\mathcal{A}}{\delta^2} \right).$$

Now, for a given $r > 0$, there exists $\varepsilon > 0$ ($\varepsilon < r$), we get

$$\mathfrak{G}_m(x; r) = \{m : u_n < m \leq v_n \text{ and } |\mathfrak{L}_m(f; x) - f(x)| \geq r\}.$$

Furthermore, for $j = 0, 1, 2$, we have

$$\mathfrak{G}_{j,m}(x; r) = \left\{ m : u_n < m \leq v_n \text{ and } |\mathfrak{L}_m(f; x) - f_j(x)| \geq \frac{r - \varepsilon}{3\mathcal{K}} \right\},$$

so that

$$\mathfrak{G}_m(x; r) \leq \sum_{j=0}^2 \mathfrak{G}_{j,m}(x; r).$$

Clearly, we obtain

$$\frac{\|\mathfrak{G}_m(x; r)\|_{C[0,1]}}{v_n - u_n} \leq \sum_{j=0}^2 \frac{\|\mathfrak{G}_{j,m}(x; r)\|_{C[0,1]}}{v_n - u_n}. \quad (3.13)$$

Now, using the above assumption about the implications in (3.2) to (3.4) and, by Definition 3, the right-hand side of (3.13) is seen to tend to 0 as $n \rightarrow \infty$. Consequently, we get

$$\lim_{n \rightarrow \infty} \frac{\|\mathfrak{G}_m(x; r)\|_{C[0,1]}}{v_n - u_n} = 0 \quad (\delta, r > 0).$$

Therefore, implication (3.1) holds true. This completes the proof of Theorem 3. $\square$

Next, by using Definition 4, we present the following theorem.

Theorem 4. *Let $\mathfrak{L}_m : C[0, 1] \rightarrow C[0, 1]$ be a martingale sequence of positive linear operators. Also let $f \in C[0, 1]$. Then*

$$\text{stat}_{\text{DM}} \lim_{m \rightarrow \infty} \|\mathfrak{L}_m(f; x) - f(x)\|_{\infty} = 0 \quad (3.14)$$

if and only if

$$\text{stat}_{\text{DM}} \lim_{m \rightarrow \infty} \|\mathfrak{L}_m(1; x) - 1\|_{\infty} = 0, \quad (3.15)$$

$$\text{stat}_{\text{DM}} \lim_{m \rightarrow \infty} \|\mathfrak{L}_m(2x; x) - 2x\|_{\infty} = 0 \quad (3.16)$$

and

$$\text{stat}_{\text{DM}} \lim_{m \rightarrow \infty} \|\mathfrak{L}_m(3x^2; x) - 3x^2\|_{\infty} = 0. \quad (3.17)$$

Proof. The proof of Theorem 4 is similar to the proof of Theorem 3. We, therefore, choose to skip the details involved. $\square$

We present below an illustrative example for the martingale sequence of positive linear operators that does not satisfy the conditions of the statistical convergence versions of Korovkin-type approximation Theorem 3, and also the result of Srivastava *et al.* [21], but it satisfies the conditions of statistical summability versions of our Korovkin-type approximation Theorem 4. Thus, clearly, our Theorem 4 is stronger than the results asserted by Theorem 3 and also the result of Srivastava *et al.* [21].

We now recall the operator

$$\mathbf{v}(1 + \mathbf{v}D) \quad \left(D = \frac{d}{d\mathbf{v}} \right),$$

which was used by Al-Salam [1] and, more recently, by Viskov and Srivastava [25] (see [16] and the monograph by Srivastava and Manocha [23] for various general families of operators and polynomials of this kind). Here, in our Example 3 below, we use this operator in conjunction with the Bernstein polynomials.

Example 3. Let us consider the Bernstein polynomials $\mathcal{B}_m(f; \mathbf{v})$ on $C[0, 1]$ given by

$$\mathcal{B}_m(f; \mathbf{v}) = \sum_{m=0}^n f\left(\frac{m}{n}\right) \binom{n}{m} \mathbf{v}^m (1 - \mathbf{v})^{n-m} \quad (\mathbf{v} \in [0, 1]). \quad (3.18)$$

Next, we present the martingale sequences positive linear operators on $C[0, 1]$ defined as follows:

$$\mathfrak{L}_m(f; \mathbf{v}) = [1 + (X_n, \mathcal{F}_n)] \mathbf{v}(1 + \mathbf{v}D) \mathcal{B}_m(f; \mathbf{v}) \quad (\forall f \in C[0, 1]) \quad (3.19)$$

with $(X_n, \mathcal{F}_n)$ mentioned already in Example 2 above.

Now, by using our proposed operators (3.19), calculate the values of the functions 1, $2\mathbf{v}$ and $3\mathbf{v}^2$ as follows:

$$\mathfrak{L}_m(1; \mathbf{v}) = [1 + (X_m, \mathcal{F}_m)] \mathbf{v}(1 + \mathbf{v}D) 1 = [1 + (X_m, \mathcal{F}_m)] \mathbf{v},$$

$$\mathfrak{L}_m(2\mathbf{v}; \mathbf{v}) = [1 + (X_m, \mathcal{F}_m)] \mathbf{v}(1 + \mathbf{v}D) 2\mathbf{v} = [1 + (X_m, \mathcal{F}_m)] \mathbf{v}(1 + 2\mathbf{v}),$$

and

$$\begin{aligned} \mathfrak{L}_m(3\mathbf{v}^2; \mathbf{v}) &= [1 + (X_m, \mathcal{F}_m)] \mathbf{v}(1 + \mathbf{v}D) 3 \left\{ \mathbf{v}^2 + \frac{\mathbf{v}(1 - \mathbf{v})}{m} \right\} \\ &= [1 + (X_m, \mathcal{F}_m)] \left\{ \mathbf{v}^2 \left(6 - \frac{9\mathbf{v}}{m} \right) \right\}, \end{aligned}$$

so that we have

$$\begin{aligned} \text{stat}_{\text{DM}} \lim_{m \rightarrow \infty} \|\mathfrak{L}_m(1; \mathbf{v}) - 1\|_{\infty} &= 0, \\ \text{stat}_{\text{DM}} \lim_{m \rightarrow \infty} \|\mathfrak{L}_m(2\mathbf{v}; \mathbf{v}) - 2\mathbf{v}\|_{\infty} &= 0 \end{aligned}$$

and

$$\text{stat}_{\text{DM}} \lim_{m \rightarrow \infty} \|\mathfrak{L}_m(3v^2; v) - 3v^2\|_\infty = 0.$$

Consequently, the sequence $\mathfrak{L}_m(f; v)$ satisfies the conditions (3.15) to (3.17). Therefore, by Theorem 4, we have

$$\text{stat}_{\text{DM}} \lim_{m \rightarrow \infty} \|\mathfrak{L}_m(f; v) - f\|_\infty = 0.$$

Here, clearly, the given martingale sequence $(X_m, \mathcal{F}_m)$ of functions in Example 2 is statistically deferred Cesàro summable but not deferred Cesàro statistically convergent. Thus, the martingale operators defined by (3.19) satisfy Theorem 4. However, they do not satisfy Theorem 3.

4. CONCLUDING REMARKS AND OBSERVATIONS

In this concluding section of our investigation, we present several further remarks and observations concerning the various results which we have proved in this article.

Remark 1. Let $(X_n, \mathcal{F}_n; n \in \mathbb{N})$ be a martingale sequence given in Example 2. Then, since

$$\text{stat}_{\text{DM}} \lim_{m \rightarrow \infty} X_m = \frac{1}{2} \text{ on } [0, 1],$$

we have

$$\text{stat}_{\text{DM}} \lim_{m \rightarrow \infty} \|\mathfrak{L}_m(f_j; x) - f_j(x)\|_\infty = 0 \quad (j = 0, 1, 2). \quad (4.1)$$

Thus, by Theorem 3, we can write

$$\text{stat}_{\text{DM}} \lim_{m \rightarrow \infty} \|\mathfrak{L}_m(f; x) - f(x)\|_\infty = 0, \quad (4.2)$$

where

$$f_0(x) = 1, \quad f_1(x) = 2x \quad \text{and} \quad f_2(x) = 3x^2.$$

Here the martingale sequence $(X_n, \mathcal{F}_n; n \in \mathbb{N})$ is neither statistically convergent nor it converges uniformly in the ordinary sense; thus, clearly, the classical and statistical versions of Korovkin-type theorems do not work here for the operators defined by (3.19). Hence, this application indicates that our Theorem 4 is a non-trivial generalization of the classical as well as statistical versions of Korovkin-type theorems (see [6] and [10]).

Remark 2. Let $(X_n, \mathcal{F}_n; n \in \mathbb{N})$ be a martingale sequence given in Example 2. Then, since

$$\text{stat}_{\text{DM}} \lim_{m \rightarrow \infty} X_m = \frac{1}{2} \text{ on } [0, 1],$$

so (4.1) holds true. Now, by applying (4.1) and Theorem 4, condition (4.2) also holds true. However, since the martingale sequence $(X_n, \mathcal{F}_n; n \in \mathbb{N})$ is not deferred Cesàro statistically convergent, but it is statistically deferred Cesàro summable. Thus,

Theorem 4 is certainly a non-trivial extension of Theorem 3. Therefore, Theorem 4 is stronger than Theorem 3.

Remark 3. It is sincerely hoped that the various developments, which we have presented in this article as well as the associated bibliographical items, will motivate the interested readers in further related researches involving other approximation operators, other applied aspects of the Laguerre and associated special functions and polynomials which we have used here, the usages of wavelet frames, and so on.

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MINIMAL DOUBLY RESOLVING SETS OF ANTIPRISM GRAPHS AND MÖBIUS LADDERS

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Received 06 April, 2016

Abstract. Consider a simple connected graph $G = (V(G), E(G))$, where $V(G)$ represents the vertex set and $E(G)$ represents the edge set respectively. A subset W of $V(G)$ is called a resolving set for a graph G if for every two distinct vertices $x, y \in V(G)$, there exist some vertex $w \in W$ such that $d(x, w) \neq d(y, w)$, where $d(u, v)$ denotes the distance between vertices u and v . A resolving set of minimal cardinality is called a metric basis for G and its cardinality is called the metric dimension of G , which is denoted by $\beta(G)$. A subset D of $V(G)$ is called a doubly resolving set of G if for every two distinct vertices x, y of G , there are two vertices $u, v \in D$ such that $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$. A doubly resolving set with minimum cardinality is called minimal doubly resolving set. This minimum cardinality is denoted by $\psi(G)$.

In this paper, we determine the minimal doubly resolving sets for antiprism graphs denoted by A_n with $n \geq 3$ and for Möbius ladders denoted by M_n , for every even positive integer $n \geq 8$. It has been proved that $\psi(A_n) = 3$ for $n \geq 3$ and

$$\psi(M_n) = \begin{cases} 3, & \text{if } n \equiv 0 \text{ or } 4 \pmod{8} \\ 4, & \text{if } n \equiv 2 \text{ or } 6 \pmod{8} \end{cases}$$

for every even positive integer $n \geq 8$.

2010 Mathematics Subject Classification: 05C12, 05C15.

Keywords: resolving set, metric dimension, minimal doubly resolving set, antiprism graph, Möbius ladder.

1. INTRODUCTION AND PRELIMINARY RESULTS

The concept of metric dimension was introduced by Slater in [14] and also independently by Harary and Melter in [6]. This concept has different applications in the diverse areas of network discovery and verification [2], robot navigation [13], and chemistry.

Consider a simple connected undirected graph $G = (V(G), E(G))$, where $|V(G)| = n$ and $|E(G)| = m$. Let $d(x, y)$ denote the distance between vertices x and y . A vertex v of graph G is said to resolve two vertices x and y of G if $d(v, x) \neq d(v, y)$. A vertex set $W = \{w_1, w_2, \dots, w_k\}$ of G is a *resolving set* or a *locating set* of G if every two distinct vertices of G are resolved by some vertex of W . The k -tuple

$r(v, W) = (d(v, w_1), d(v, w_2), \dots, d(v, w_k))$ is called the vector of metric coordinates of v with respect to W . A resolving set of minimum cardinality is called a *metric basis* of G . The cardinality of metric basis, denoted by $\beta(G)$, is called the *metric dimension* of G .

Let $\mathcal{F}$ be a family of connected graphs $G_n : \mathcal{F} = (G_n)_{n \geq 1}$ depending on n as follows: the order $|V(G)| = \varphi(n)$ and $\lim_{n \rightarrow \infty} \varphi(n) = \infty$. If there exists a constant $C > 0$ such that $\beta(G_n) \leq C$ for every $n \geq 1$ then we shall say that $\mathcal{F}$ has bounded metric dimension; otherwise $\mathcal{F}$ has an unbounded metric dimension. If all graphs in $\mathcal{F}$ have the same metric dimension (which does not depend on n), $\mathcal{F}$ is called a family with constant metric dimension. For recent results on metric dimension of certain classes of graphs, please consult [7–11].

The doubly resolving set of a graph G is a recent concept introduced by Cáceres et al.[3]. They proved that the metric dimension of the Cartesian product $G \square G$ is related in a strong way to doubly resolving sets of G with the minimum cardinality. The vertices x, y of the graph G , $n \geq 2$, are said to doubly resolve vertices u, v of G if $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$. A vertex set D of G is a doubly resolving set of G if every two distinct vertices of G are doubly resolved by some two vertices of D , i.e., if there are no two distinct vertices of G with the same difference between their corresponding metric coordinates with respect to D . The minimal doubly resolving set is a doubly resolving set with minimum cardinality. The cardinality of minimum doubly resolving set is denoted by $\psi(G)$. The problem of minimal doubly resolving set is NP-hard [12].

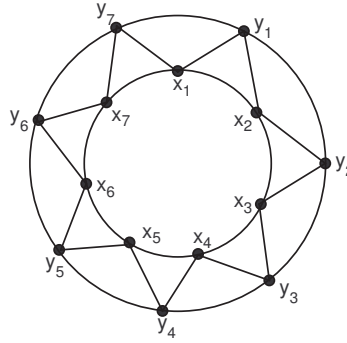
Note that we have $\beta(G) \leq \psi(G)$ always by the definition. Since if x, y doubly resolve u, v , then $d(u, x) - d(v, x) \neq 0$ or $d(u, y) - d(v, y) \neq 0$, and hence x or y resolve u, v , which follows that a doubly resolving set is also a resolving set. In [11], it has been proved that metric dimension of antiprism denoted by A_n , is 3. The metric dimension of Möbius ladder M_n has been discussed in [1] and it was proved that Möbius ladder M_n has bounded metric dimension.

The problem of determining the minimal cardinality of double resolving sets for prism graphs denoted by $C_n \square P_2$ was studied in [4]. It was proved that the minimal cardinality is equal to four if n is even and equal to three if n is odd. For more study on computing the doubly resolving sets of graphs, see [12].

In this paper, we extend this study to antiprism graphs denoted by A_n and to Möbius ladders denoted by M_n . We determine the cardinality of minimal doubly resolving set $\psi(A_n)$ and $\psi(M_n)$.

2. THE MINIMAL DOUBLY RESOLVING SETS FOR ANTIPRISM GRAPHS A_n

In this section, we determine the minimal doubly resolving set for antiprism graphs denoted by A_n . The antiprism A_n [5] is a 4-regular graph and for $n = 3$, it is the octahedron. The graph of the antiprism A_n for $n \geq 3$ consists of $V(A_n) = \{x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n\}$ and $E(A_n) = \{x_i x_{i+1} : 1 \leq i \leq n\} \cup \{y_i y_{i+1} : 1 \leq i \leq n\} \cup \{x_i y_i : 1 \leq i \leq n\}$.


 FIGURE 1. The antiprism graph A_7 .

$n\} \cup \{x_{i+1}y_i : 1 \leq i \leq n\}$ with indices taken modulo n . As a convention, $\{y_1, y_2, \dots, y_n\}$ forms outer cycle and $\{x_1, x_2, \dots, x_n\}$ forms inner cycle and we have $|V(A_n)| = 2n$ and $|E(A_n)| = 4n$ for $n \geq 3$. Figure 1 displays A_7 . It has been proved that metric dimension $\beta(A_n)$ is equal to 3, i.e., $\beta(A_n) = 3$ for $n \geq 3$. So we have $\psi(A_n) \geq 3$, for every $n \geq 3$, since $\psi(A_n) \geq \beta(A_n)$.

Lemma 1. Let $S_i(y_1) = \{w \in V(A_n) : d(y_1, w) = i\}$ be the set of vertices in V at distance i from y_1 . For the graph of antiprism A_n with $n \geq 8$, the sets $S_i(y_1)$ are given in Table 1 where x_{-k}, y_{-k} denote x_{n-k}, y_{n-k} , respectively.

Proof. By the definition of $S_i(y_1)$, it is easy to see that $S_i(y_1)$ is the set of all $v \in V(A_n)$ such that $va \in E(A_n)$ for some $a \in S_{i-1}(y_1)$ and $v \notin S_l(y_1), l < i - 1$. Now using the definition of the antiprism A_n , it immediately follows that

$$S_1(y_1) = \{x_1, x_2, y_2, y_n\}.$$

Let us prove by induction that, for $2 \leq i < \lfloor \frac{n}{2} \rfloor = t$

$$S_i(y_1) = \{x_{i+1}, x_{n-(i-2)}, y_{i+1}, y_{n-(i-1)}\}. \quad (2.1)$$

For $i = 2$, starting from $S_1(y_1)$, we obtain

$$S_2(y_1) = \{x_3, x_n, y_3, y_{n-1}\}.$$

Suppose that (2.1) holds for $2 \leq i \leq l < \lfloor \frac{n}{2} \rfloor - 1$. Starting from $S_l(y_1)$, we obtain the following candidates for the members of $S_{l+1}(y_1)$:

$$x_l, x_{n-l+1}, y_l, y_{n-l}, x_{l+2}, x_{n-l+3}, y_{l+2}, y_{n-l+2}.$$

The first four vertices belong to $S_{l-1}(y_1)$ and consequently cannot belong to $S_{l+1}(y_1)$. For example $x_l = x_{(l-1)+1} \in S_{l-1}(y_1)$. The set of remaining four vertices can be expressed in the form (2.1) for $i = l + 1$. For $i \geq \lfloor \frac{n}{2} \rfloor$, $S_i(y_1)$ depends on n .

Case 1: If $n = 2t, t \geq 4$.

Starting from $S_{t-1}(y_1)$, we obtain the following candidates for $S_t(y_1)$:

$$x_{t-1}, x_{n-t+4}, y_{t-1}, y_{n-t+3}, x_{t+1}, x_{n-t+2}, y_{t+1}.$$

The first four vertices belongs to $S_{t-2}(y_1)$ and can be neglected. The set of remaining three vertices represents set $S_t(y_1)$ for $n = 2t$. Starting from $S_t(y_1)$, $S_{t+1}(y_1) = \emptyset$.

Case 2: If $n = 2t + 1, t \geq 4$.

Starting from $S_{t-1}(y_1)$, we obtain the following candidates for $S_t(y_1)$:

$$x_{t-1}, x_{n-t+4}, y_{t-1}, y_{n-t+3}, x_{t+1}, x_{n-t+2}, y_{t+1}, y_{n-t+1}.$$

The first four vertices belongs to $S_{t-2}(y_1)$ and can be neglected. The set of remaining four vertices represents set $S_t(y_1)$ for $n = 2t + 1$. Starting from $S_t(y_1)$, the candidates for $S_{t+1}(y_1)$ are:

$$x_t, x_{n-t+3}, y_t, y_{n-t+2}, x_{t+2}.$$

As $x_t, x_{n-(t-3)}, y_t, y_{n-(t-2)} \in S_{t-1}(y_1)$. So $S_{t+1}(y_1) = \{x_{t+2}\}$. It is easy to see that similar construction applied to $S_{t+1}(y_1)$ lead to $S_{t+2}(y_1) = \emptyset$. The following table displays $S_i(y_1)$. $\square$

n	i	$S_i(y_1)$
$2t$	1	$\{x_1, x_2, y_2, y_n\}$
	$2 \leq i < \lfloor \frac{n}{2} \rfloor$	$\{x_{i+1}, x_{-(i-2)}, y_{i+1}, y_{-(i-1)}\}$
	t	$\{x_{t+1}, x_{-(t-2)}, y_{t+1}\}$
	$t+1$	$\{x_{t+2}\}$

TABLE 1. $S_i(y_1)$ for A_n

Theorem 1. For $n \geq 3$, $\psi(A_n) = 3$.

Proof. There are two cases to be discussed.

Case 1: If $n = 2t, t \geq 4$.

Consider Table 2 which shows the vectors of metric coordinates of vertices of A_n with respect to set $D^* = \{y_1, y_3, y_{t+2}\}$ in a special way.

Starting from Table 2, note that $y_1 \in D^*$ therefore the first metric coordinate of a vector from $S_i(y_1)$ with respect to D^* is equal to i . It is easy to check that for each $i \in \{1, 2, \dots, t\}$, there do not exist two vertices $x, y \in S_i(y_1)$ such that all coordinates of the vector $r(x, D^*) - r(y, D^*)$ are equal to 0. Also, it can be easily seen that for each $i, j \in \{1, 2, \dots, t\}, i \neq j$, there do not exist two vertices $x \in S_i(y_1)$ and $y \in S_j(y_1)$ such that all coordinates of the vector $r(x, D^*) - r(y, D^*)$ are equal to $i - j$.

Case 2: If $n = 2t + 1, t \geq 4$.

Consider Table 3 which gives the vectors of metric coordinates of vertices of A_n with respect to set $D^* = \{y_1, y_3, y_{t+2}\}$ in a special way.

i	$S_i(y_1)$	$D^* = \{y_1, y_3, y_{t+2}\}$
0	y_1	$(0, 2, t-1)$
1	x_1	$(1, 3, t-1)$
	x_2	$(1, 2, t)$
	y_2	$(1, 1, t)$
	y_n	$(1, 3, t-2)$
2	x_3	$(2, 1, t)$
	x_n	$(2, 4, t-2)$
	y_3	$(2, 0, t-1)$
	y_{n-1}	$(2, 4, t-3)$
$3 \leq i \leq t-3$ ($t \geq 6$)	x_{i+1}	$(i, i-2, t-i+2)$
	x_{n-i+2}	$(i, i+2, t-i)$
	y_{i+1}	$(i, i-2, t-i+1)$
	y_{n-i+1}	$(i, i+2, t-i-1)$
$t-2$	x_{t-1}	$(t-2, t-4, 4)$
	x_{n-t+4}	$(t-2, t, 2)$
	y_{t-1}	$(t-2, t-4, 3)$
	y_{n-t+3}	$(t-2, t, 1)$
$t-1$	x_t	$(t-1, t-3, 3)$
	x_{n-t+3}	$(t-1, t, 1)$
	y_t	$(t-1, t-3, 2)$
	y_{n-t+2}	$(t-1, t-1, 0)$
t	x_{n-t+1}	$(t, t-2, 2)$
	x_{n-t+2}	$(t, t-1, 1)$
	y_{n-t+1}	$(t, t-2, 1)$

TABLE 2. Vectors of metric coordinates for A_n , $n = 2t$, $t \geq 4$

Starting from Table 3, it is easy to check that for each $i \in \{1, 2, \dots, t+1\}$, there do not exist two vertices $x, y \in S_i(y_1)$ such that all coordinates of the vector $r(x, D^*) - r(y, D^*)$ are equal to 0. Also, it can be easily seen that for each $i, j \in \{1, 2, \dots, t+1\}$, $i \neq j$, there do not exist two vertices $x \in S_i(y_1)$ and $y \in S_j(y_1)$ such that all coordinates of the vector $r(x, D^*) - r(y, D^*)$ are equal to $i - j$.

Enumeration technique shows that minimal doubly resolving set for A_3 , A_4 is $D^* = \{y_1, y_2, y_3\}$, for A_5 , it is $D^* = \{y_1, y_3, y_4\}$, and for A_6, A_7 , it is $D^* = \{y_1, y_3, y_5\}$ respectively.

In this way, D^* is a minimal doubly resolving set for A_n , $n \geq 3$ and Theorem 1 holds. $\square$

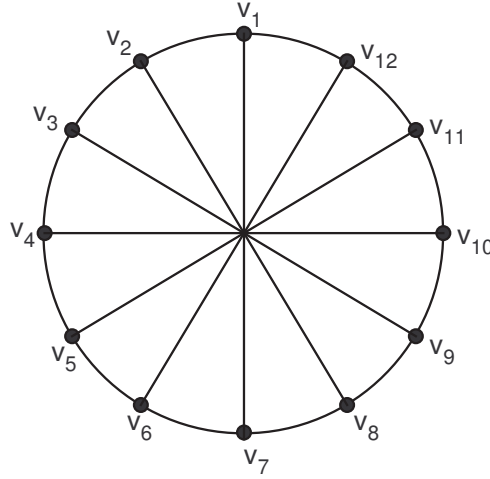
i	$S_i(y_1)$	$D^* = \{y_1, y_3, y_{t+2}\}$
0	y_1	$(0, 2, t)$
1	x_1	$(1, 3, t)$
	x_2	$(1, 2, t+1)$
	y_2	$(1, 1, t)$
	y_n	$(1, 3, t-1)$
2	x_3	$(2, 1, t)$
	x_n	$(2, 4, t-1)$
	y_3	$(2, 0, t-1)$
	y_{n-1}	$(2, 4, t-2)$
$3 \leq i \leq t-2$ ($t \geq 5$)	x_{i+1}	$(i, i-2, t-i+2)$
	x_{n-i+2}	$(i, i+2, t-i+1)$
	y_{i+1}	$(i, i-2, t-i+1)$
	y_{n-i+1}	$(i, i+2, t-i)$
$t-1$	x_t	$(t-1, t-3, 3)$
	x_{n-t+3}	$(t-1, t+1, 2)$
	y_t	$(t-1, t-3, 2)$
	y_{n-t+2}	$(t-1, t, 1)$
t	x_{n-t}	$(t, t-2, 2)$
	x_{n-t+2}	$(t, t, 1)$
	y_{n-t}	$(t, t-2, 1)$
	y_{n-t+1}	$(t, t-1, 0)$
$t+1$	x_{n-t+1}	$(t+1, t-1, 1)$

TABLE 3. Vectors of metric coordinates for A_n , $n = 2t + 1$, $t \geq 4$ 3. THE MINIMAL DOUBLY RESOLVING SETS FOR MÖBIUS LADDERS M_n

The Möbius ladder M_n is a cubic circulant graph which consist of an even number of vertices. It can be constructed from an n -cycle by adding new edges (called "rungs") which connects the opposite pair of vertices in the cycle. The applications of Möbius ladders can be found in electronics, computer science, chemistry and chemical stereography. We assume that the vertices of Möbius ladder M_n are numbered $\{v_1, v_2, \dots, v_n\}$ counter clockwise in cycle C_n and opposite pairs of vertices are connected by adding edges between them, where n is an even number. In [1], it has been proved that the metric dimension $\beta(M_n) \geq 3$ for $n \geq 8$. Thus, we have $\psi(M_n) \geq \beta(M_n) \geq 3$, i.e., $\psi(M_n) \geq 3$ for every $n \geq 8$.

Lemma 2. Let $S_i(v_1) = \{w \in V(M_n) : d(v_1, w) = i\}$ be the set of vertices in V at distance i from v_1 . For M_n with $n \geq 8$, sets $S_i(v_1)$ are given in Table 4.

Proof. There are four cases to be discussed $n \equiv 0, 2, 4$ or $6 \pmod{8}$ i.e. $n = 8k, 8k+2, 8k+4$ or $8k+6$ for $k \geq 1$. Let us consider the case when $n \equiv 0 \pmod{8}$,

FIGURE 2. The Möbius ladder M_{12} .

so $n = 8k$ for $k \geq 1$. By definition of $S_i(v_1)$, it is easy to see that $S_i(v_1)$ is the set of all $v \in V(M_n)$ such that $va \in E(M_n)$ for some $a \in S_{i-1}(v_1)$ and $v \notin S_l(v_1)$, $l < i - 1$. Now using the definition of M_n , it immediately follows that

$$S_1(v_1) = \{v_2, v_{4k+1}, v_n\}.$$

Let us prove by induction that, for $2 \leq i \leq 2k$

$$S_i(v_1) = \{v_{i+1}, v_{n-i+1}, v_{4k+2-i}, v_{4k+i}\}. \quad (3.1)$$

For $i = 2$, starting from $S_1(v_1)$, we obtain

$$S_2(v_1) = \{v_3, v_{n-1}, v_{4k}, v_{4k+2}\}.$$

Suppose that (3.1) holds for $2 \leq l < 2k$. Starting from $S_l(v_1)$, we obtain the following candidates for members of $S_{l+1}(v_1)$:

$$v_l, v_{n-l}, v_{4k+1-l}, v_{4k-l+1}, v_{l+2}, v_{n-l+2}, v_{4k+3-l}, v_{4k+1+l}.$$

The first four vertices belong to $S_{l-1}(v_1)$ and consequently cannot belong to $S_{l+1}(v_1)$. For example $v_l = v_{(l-1)+1} \in S_{l-1}(v_1)$. The set of remaining four vertices can be expressed in form (3.1) for $i = l + 1$. The sets $S_i(v_1)$ for other three cases can be obtained in the same way. The following table displays $S_i(v_1)$. $\square$

Note that the sets $S_i(v_1)$, defined in Lemma 2 can be used to determine the distance between two arbitrary vertices in $V(M_n)$ in the following way.

n	i	$S_i(v_1)$
$8k$	1	$\{v_2, v_{4k+1}, v_n\}$
	$2 \leq i \leq 2k$	$\{v_{i+1}, v_{n-i+1}, v_{4k+2-i}, v_{4k+i}\}$
$8k+2$	1	$\{v_2, v_{4k+2}, v_n\}$
	$2 \leq i \leq 2k$	$\{v_{i+1}, v_{n-i+1}, v_{4k+3-i}, v_{4k+1+i}\}$
$8k+4$	$2k+1$	$\{v_{2k+2}, v_{6k+2}\}$
	1	$\{v_2, v_{4k+3}, v_n\}$
$8k+6$	$2 \leq i \leq 2k+1$	$\{v_{i+1}, v_{n-i+1}, v_{4k+4-i}, v_{4k+2+i}\}$
	1	$\{v_2, v_{4k+4}, v_n\}$
	$2 \leq i \leq 2k+1$	$\{v_{i+1}, v_{n-i+1}, v_{4k+5-i}, v_{4k+3+i}\}$
	$2k+2$	$\{v_{2k+3}, v_{6k+5}\}$

TABLE 4. Vectors of metric coordinates for M_n , $n = 8k$, $k \geq 1$

As symmetry of M_n displays the fact that $d(v_i, v_j) = d(v_1, v_{j-i+1})$ for $j > i$. Consequently if we know the distance $d(v_1, w)$ for every $w \in V(M_n)$ then we can reconstruct the distance between every two vertices from $V(M_n)$.

Lemma 3. For $n \geq 8$, $\psi(M_n) = 3$ when $n \equiv 0$ or $4 \pmod{8}$.

Proof. **Case 1:** If $n \equiv 0 \pmod{8}$.

We have $n = 8k$ for $k \geq 1$. Consider the following table which shows the vectors of metric coordinates of vertices of M_n with respect to set $D^* = \{v_1, v_{2k+1}, v_{4k+1}\}$ in the following way.

i	$S_i(v_1)$	$D^* = \{v_1, v_{2k+1}, v_{4k+1}\}$
0	v_1	$(0, 2k, 1)$
1	v_2	$(1, 2k-1, 2)$
	v_{4k+1}	$(1, 2k, 0)$
	v_n	$(1, 2k, 2)$
	v_{i+1}	$(i, 2k-i, i+1)$
$2 \leq i \leq 2k-1$	v_{n-i+1}	$(i, 2k-i+1, i+1)$
	v_{4k+2-i}	$(i, 2k-i+1, i-1)$
	v_{4k+i}	$(i, 2k-i+2, i-1)$
	v_{2k+1}	$(2k, 0, 2k)$
$2k$	v_{6k+1}	$(2k, 1, 2k)$
	v_{2k+2}	$(2k, 1, 2k-1)$
	v_{6k}	$(2k, 2, 2k-1)$

TABLE 5. Vectors of metric coordinates for M_n , $n = 8k$, $k \geq 1$

Case 2: If $n \equiv 4 \pmod{8}$.

We have $n = 8k + 4$ for $k \geq 1$. Consider the following table which shows the vectors of metric coordinates of vertices of M_n with respect to set $D^* = \{v_1, v_{2k+2}, v_{4k+3}\}$ in the following way.

i	$S_i(v_1)$	$D^* = \{v_1, v_{2k+2}, v_{4k+3}\}$
0	v_1	$(0, 2k+1, 1)$
1	v_2	$(1, 2k, 2)$
	v_{4k+3}	$(1, 2k+1, 0)$
	v_n	$(1, 2k+1, 2)$
$2 \leq i \leq 2k$	v_{i+1}	$(i, 2k-i+1, i+1)$
	v_{n-i+1}	$(i, 2k-i+2, i+1)$
	v_{4k+4-i}	$(i, 2k-i+2, i-1)$
	v_{4k+2+i}	$(i, 2k-i+3, i-1)$
$2k+1$	v_{2k+2}	$(2k+1, 0, 2k+1)$
	v_{6k+4}	$(2k+1, 1, 2k+1)$
	v_{2k+3}	$(2k+1, 1, 2k)$
	v_{6k+3}	$(2k+1, 2, 2k)$

TABLE 6. Vectors of metric coordinates for M_n , $n = 8k + 4$, $k \geq 1$

Since $v_1 \in D^*$, Tables 5 and 6 show us that the first metric coordinate of a vector from $S_i(v_1)$ with respect to D^* is equal to i . It is easy to check that, there do not exist two vertices $x, y \in S_i(v_1)$ such that all coordinates of the vector $r(x, D^*) - r(y, D^*)$ are equal to 0. Also, it can be verified that there do not exist two vertices x, y such that $x \in S_i(v_1)$ and $y \in S_j(v_1)$ with $i \neq j$, so that all coordinates of the vector $r(x, D^*) - r(y, D^*)$ are equal to $i - j$.

In this way, we have found minimal doubly resolving set $D^* = \{v_1, v_{2k+1}, v_{4k+1}\}$ for M_n when $n \equiv 0 \pmod{8}$ and $D^* = \{v_1, v_{2k+2}, v_{4k+3}\}$ when $n \equiv 4 \pmod{8}$. This proves the statement of Lemma 3. $\square$

Lemma 4. $\psi(M_n) > 3$ for $n \equiv 2 \pmod{8}$.

Proof. We know that $\psi(M_n) \geq 3$ and thus we should prove that every subset D of vertex set $V(M_n)$ with $|D| = 3$ is not a doubly resolving set for M_n . We may assume that $v_1 \in D$. In Table 7, one can find all possible types of such set D and the corresponding non-doubly resolving pair of vertices from $V(M_n)$.

Let us prove that the vertices v_{4k+1}, v_{4k+2} are not doubly resolved by any two vertices from set $\{v_1, v_i, v_j\}$, $2 \leq i \leq 2k$, $4k+3 \leq j \leq 6k+2$. Using distances from Table 4, it follows that

- (i) $d(v_1, v_{4k+1}) = 2$ and $d(v_1, v_{4k+2}) = 1$.
- (ii) $d(v_i, v_{4k+1}) = d(v_1, v_{4k+2-i}) = i+1$ and $d(v_i, v_{4k+2}) = d(v_1, v_{4k+3-i}) = i$ for $2 \leq i \leq 2k-1$.
- (iii) $d(v_{2k}, v_{4k+1}) = d(v_1, v_{2k+2}) = 2k+1$ and $d(v_{2k}, v_{4k+2}) = d(v_1, v_{2k+3}) = 2k$.

D	Non-doubly resolved pair
$\{v_1, v_i, v_j\}, 2 \leq i, j \leq 4k+1, i \neq j$	$\{v_{4k+2}, v_n\}$
$\{v_1, v_i, v_j\}, 4k+3 \leq i, j \leq n, i \neq j$	$\{v_2, v_{4k+2}\}$
$\{v_1, v_{4k+2}, v_i\}, 2 \leq i \leq 2k+1 \text{ or } 6k+3 \leq i \leq n$	$\{v_{2k+2}, v_{6k+2}\}$
$\{v_1, v_{4k+2}, v_i\}, 2k+2 \leq i \leq 4k+1 \text{ or } 4k+3 \leq i \leq 6k+2$	$\{v_{2k+1}, v_{6k+3}\}$
$\{v_1, v_i, v_j\}, 2 \leq i \leq 2k, 4k+3 \leq j \leq 6k+2$	$\{v_{4k+1}, v_{4k+2}\}$
$\{v_1, v_i, v_j\}, 2 \leq i \leq 2k, 6k+3 \leq j \leq n$	$\{v_1, v_{4k+2}\}$
$\{v_1, v_{2k+1}, v_i\}, 4k+3 \leq i \leq 6k+1$	$\{v_1, v_n\}$
$\{v_1, v_{2k+1}, v_{6k+2}\}$	$\{v_{4k+2}, v_n\}$
$\{v_1, v_{2k+1}, v_i\}, 6k+3 \leq i \leq n$	$\{v_{2k+2}, v_{6k+2}\}$
$\{v_1, v_{2k+2}, v_i\}, 4k+3 \leq i \leq 6k+2$	$\{v_{2k+1}, v_{6k+3}\}$
$\{v_1, v_{2k+2}, v_{6k+3}\}$	$\{v_2, v_{4k+2}\}$
$\{v_1, v_{2k+2}, v_i\}, 6k+4 \leq i \leq n$	$\{v_{4k+2}, v_{4k+3}\}$
$\{v_1, v_i, v_j\}, 2k+3 \leq i \leq 4k+1, 4k+3 \leq j \leq 6k+2$	$\{v_{2k+1}, v_{6k+3}\}$
$\{v_1, v_i, v_j\}, 2k+3 \leq i \leq 4k+1, 6k+3 \leq j \leq n$	$\{v_1, v_2\}$

TABLE 7. Non-doubly resolved pairs of M_n for $n = 8k+2, k \geq 1$

- (iv) $d(v_j, v_{4k+1}) = d(v_1, v_{j-4k}) = j - 4k - 1$ and $d(v_j, v_{4k+2}) = d(v_1, v_{j-4k-1}) = j - 4k - 2$ for $4k+3 \leq j \leq 6k+2$.

From (i), (ii), (iii) and (iv), we have

$$\begin{aligned} d(v_1, v_{4k+1}) - d(v_1, v_{4k+2}) &= d(v_i, v_{4k+1}) - d(v_i, v_{4k+2}) \\ &= d(v_j, v_{4k+1}) - d(v_j, v_{4k+2}) = 1, \end{aligned}$$

i.e., $\{v_1, v_i, v_j\}, 2 \leq i \leq 2k, 4k+3 \leq j \leq 6k+2$ is not a doubly resolving set of M_n . Similarly, we can consider all other types of D from Table 7 and verify their corresponding non-doubly resolved pairs of vertices. $\square$

Lemma 5. $\psi(M_n) > 3$ for $n \equiv 6 \pmod{8}$.

Proof. The proof is similar to the proof of Lemma 4. The following table displays all possible types of set D with $|D| = 3$ and the corresponding non-doubly resolving pair of vertices. $\square$

Lemma 6. For every $n \geq 8$, $\psi(M_n) = 4$ when $n \equiv 2$ or $6 \pmod{8}$.

Proof. From Lemmas 4 and 5, it is clear that $\psi(M_n) > 3$, therefore it is enough to show that there exist a subset of $V(M_n)$ with cardinality 4 such that it doubly resolves each pair of vertices from $V(M_n)$. Consider the following two tables:

(i) $n \equiv 2 \pmod{8}$.

We have $n = 8k+2$ for $k \geq 1$. Consider the following table which shows the vectors of metric coordinates of vertices of M_n with respect to set $D^* = \{v_1, v_{2k+1}, v_{4k+2}, v_{6k+2}\}$ in the following way.

D	Non-doubly resolved pair
$\{v_1, v_i, v_j\}, 2 \leq i, j \leq 4k+3, i \neq j$	$\{v_{4k+4}, v_n\}$
$\{v_1, v_i, v_j\}, 4k+5 \leq i, j \leq n, i \neq j$	$\{v_2, v_{4k+4}\}$
$\{v_1, v_{4k+4}, v_i\}, 2 \leq i \leq 2k+2 \text{ or } 6k+6 \leq i \leq n$	$\{v_{2k+3}, v_{6k+5}\}$
$\{v_1, v_{4k+4}, v_i\}, 2k+3 \leq i \leq 4k+3 \text{ or } 4k+5 \leq i \leq 6k+5$	$\{v_{2k+2}, v_{6k+6}\}$
$\{v_1, v_i, v_j\}, 2 \leq i \leq 2k+2, 4k+5 \leq j \leq 6k+5$	$\{v_{2k+2}, v_{2k+3}\}$
$\{v_1, v_i, v_j\}, 2 \leq i \leq 2k+2, 6k+6 \leq j \leq n$	$\{v_1, v_{4k+4}\}$
$\{v_1, v_i, v_j\}, 2k+3 \leq i \leq 4k+3, 4k+5 \leq j \leq 6k+5$	$\{v_{2k+2}, v_{6k+6}\}$
$\{v_1, v_i, v_j\}, 2k+3 \leq i \leq 4k+3, 6k+6 \leq j \leq n$	$\{v_{6k+5}, v_{6k+6}\}$

TABLE 8. Non-doubly resolved pairs of M_n for $n = 8k+6, k \geq 1$

i	$S_i(v_1)$	$D^* = \{v_1, v_{2k+1}, v_{4k+2}, v_{6k+2}\}$
0	v_1	$(0, 2k, 1, 2k+1)$
1	v_2	$(1, 2k-1, 2, 2k)$
	v_{4k+2}	$(1, 2k+1, 0, 2k)$
	v_n	$(1, 2k+1, 2, 2k)$
$2 \leq i \leq 2k$	v_{i+1}	$(i, 2k-i, i+1, 2k+1-i)$
	v_{n-i+1}	$(i, 2k+2-i, i+1, 2k+1-i)$
	v_{4k+3-i}	$(i, 2k+2-i, i-1, 2k+3-i)$
	v_{4k+1+i}	$(i, 2k+2-i, i-1, 2k+1-i)$
$2k+1$	v_{2k+2}	$(2k+1, 1, 2k, 2)$
	v_{6k+2}	$(2k+1, 1, 2k, 0)$

TABLE 9. Vectors of metric coordinates for $M_n, n = 8k+2, k \geq 1$

(ii) $n \equiv 6 \pmod{8}$.

We have $n = 8k+6$ for $k \geq 1$. Consider the following table which shows the vectors of metric coordinates of vertices of M_n with respect to set $D^* = \{v_1, v_{2k+2}, v_{4k+4}, v_{6k+5}\}$ in the following way.

Since $v_1 \in D^*$, Tables 9 and 10 show us that the first metric coordinate of a vector from $S_i(v_1)$ with respect to D^* is equal to i . It is easy to check that, there do not exist two vertices $x, y \in S_i(v_1)$ such that all coordinates of the vector $r(x, D^*) - r(y, D^*)$ are equal to 0. Also, it can be verified that there do not exist two vertices x, y such that $x \in S_i(v_1)$ and $y \in S_j(v_1)$ with $i \neq j$, so that all coordinates of the vector $r(x, D^*) - r(y, D^*)$ are equal to $i - j$.

In this way, we have found minimal doubly resolving set $D^* = \{v_1, v_{2k+1}, v_{4k+2}, v_{6k+2}\}$ for M_n when $n \equiv 2 \pmod{8}$ and $D^* = \{v_1, v_{2k+2}, v_{4k+4}, v_{6k+5}\}$ when $n \equiv 6 \pmod{8}$. This proves the statement of Lemma 6. $\square$

i	$S_i(v_1)$	$D^* = \{v_1, v_{2k+2}, v_{4k+4}, v_{6k+5}\}$
0	v_1	$(0, 2k+1, 1, 2k+2)$
1	v_2	$(1, 2k, 2, 2k+1)$
$2 \leq i \leq 2k+1$	v_{4k+4}	$(1, 2k+2, 0, 2k+1)$
	v_n	$(1, 2k+2, 2, 2k+1)$
	v_{i+1}	$(i, 2k+1-i, i+1, 2k+2-i)$
	v_{n-i+1}	$(i, 2k+3-i, i+1, 2k+2-i)$
	v_{4k+5-i}	$(i, 2k+3-i, i-1, 2k+4-i)$
$2k+2$	v_{4k+3+i}	$(i, 2k+3-i, i-1, 2k+2-i)$
	v_{2k+3}	$(2k+2, 1, 2k+1, 2)$
	v_{6k+5}	$(2k+2, 1, 2k+1, 0)$

TABLE 10. Vectors of metric coordinates for M_n , $n = 8k+6$, $k \geq 1$

Theorem 2. For every even positive integer $n \geq 8$, we have

$$\psi(M_n) = \begin{cases} 3, & \text{if } n \equiv 0 \text{ or } 4 \pmod{8} \\ 4, & \text{if } n \equiv 2 \text{ or } 6 \pmod{8} \end{cases}.$$

Proof. The proof follows by combining the results of Lemmas 3 and 6. $\square$

4. CONCLUSION

In this paper, we theoretically determine the minimal cardinality $\psi(A_n)$ and $\psi(M_n)$ of doubly resolving sets for the antiprism graphs and Möbius ladders. The future work will be focused on determining some other invariants of generalized Petersen graphs related to the metric dimension and doubly resolving sets.

ACKNOWLEDGEMENT

The research for this article was supported by VEGA 1/0233/18.

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GLOBAL DYNAMICAL BEHAVIOURS AND PERIODICITY OF A CERTAIN QUADRATIC-RATIONAL DIFFERENCE EQUATION WITH DELAY

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Received 17 November, 2021

Abstract. Our aim in this paper is to deal with the dynamics of following higher order difference equation

$$x_{n+1} = A + B \frac{x_{n-m}}{x_n^2}$$

where $A, B > 0$, and initial values are positive, and $m = \{1, 2, \dots\}$. Furthermore, we discuss the periodicity, boundedness, semi-cycles, global asymptotic stability of solutions of these equations. We also handle the rate of convergence of solutions of these difference equations.

2010 *Mathematics Subject Classification:* 39A10; 39A23; 39A30

Keywords: Difference equations, periodicity, boundedness, semi-cycle, global asymptotic stability, rate of convergence, delay.

1. INTRODUCTION

Last few decades, rational difference equations and their systems have attracted the interest of many researchers for varied reasons. One of the reason of this rapid growth of interest is, these equations provided a natural description of many discrete mathematical models. Such discrete mathematical models are often scrutinized in various fields of science and technology for instance, biology, ecology, physiology, physics, engineering, economics, probability theory, genetics, psychology, resource management and population dynamics. We believe that the interest of studying rational difference equations will increase in future years as more fascinating and interesting results are obtaining. Although difference equations are very simple in form, it is extremely difficult to understand thoroughly the behaviors of their solutions. Furthermore, higher-order rational difference equations and systems of rational equations have also been widely studied but still have many aspects to be investigated. There are many papers related to the rational difference equations and higher-order rational difference equations (see, for example, [2–4, 11, 13] and references therein).

In [5], Amleh et al. discussed the stability, boundedness and periodic character of solutions of difference equation

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n},$$

where the initial values are positive numbers, and $\alpha \geq 0$.

In [9], Devault et al. studied periodicity, global stability and the boundedness of solutions of the following higher order difference equation

$$x_{n+1} = p + \frac{x_{n-k}}{x_n},$$

where the initial conditions are positive numbers, and $p > 0$.

In [16], Saleh et al. handled the dynamical behaviours of following higher order difference equation

$$y_{n+1} = A + \frac{y_{n-k}}{y_n}, \quad (1.1)$$

with $k \in \{2, 3, \dots\}$ and A is positive. The authors especially discussed the global asymptotic stability, semi-cycle analysis and periodicity of the unique positive equilibrium of Eq.(1.1).

In [1], Abu-Saris et al. dealt with the global asymptotic stability of positive equilibrium point of difference equations

$$y_{n+1} = A + \frac{y_n}{y_{n-k}}, \quad (1.2)$$

where $k \in \{2, 3, \dots\}$ and A is positive. Moreover, in [17], Saleh et al. studied the global stability of the negative equilibrium of the difference equation (1.2) where $k \in \{1, 2, \dots\}$ and $A < 0$.

In [12], Hamza et al. discussed the dynamics of following difference equation

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n^k}, \quad (1.3)$$

where α, k and the initial values are positive real numbers. The authors dealt with the boundedness, oscillation behaviours and stability analysis of unique equilibrium point of Eq.(1.3).

In [20], Yalçinkaya handled the oscillation behaviours, bounded solutions, periodic solutions and global stability of solutions of difference equation

$$x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k},$$

where the initial values are positive real numbers and $\alpha, k > 0$.

In [18], Stevic investigated the dynamical properties of difference equation

$$x_{n+1} = \frac{x_{n-1}}{g(x_n)},$$

where $x_{-1}, x_0 > 0$.

In [6], Bešo et al. showed the Neimark–Sacker bifurcation, boundedness and global attractivity of following difference equation

$$x_{n+1} = \gamma + \delta \frac{x_n}{x_{n-1}^2},$$

with the initial conditions and γ, δ are positive real numbers.

In [19], Taşdemir investigated the boundedness, rate of convergence, global asymptotic stability and periodicity of the following higher order difference equations

$$x_{n+1} = A + B \frac{x_n}{x_{n-m}^2}, \quad (1.4)$$

where the initial conditions and A, B are positive real numbers and $m \in \{2, 3, \dots\}$.

Our aim in this work is to deal with the dynamics of following higher order difference equation

$$x_{n+1} = A + B \frac{x_{n-m}}{x_n^2}, \quad (1.5)$$

with $m = \{1, 2, \dots\}$, and the initial conditions are positive numbers, and $A, B > 0$. We first handle the periodicity, boundedness and oscillation behaviors of solutions of Eq.(1.5). Moreover, we analyze the local and global asymptotic stability of the solutions of Eq.(1.5). Finally, we study the rate of convergence of Eq.(1.5) and we present some numerical examples to verify our theoretical results.

Here, we summarize the significant results and definitions on the theory of difference equations. For more information, see [8, 10, 14] and references therein.

Let I be some interval of real numbers and let $f : I^{k+1} \rightarrow I$ be a continuously differentiable function. A difference equation of order $(k+1)$ is an equation of the form

$$x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-k}), \quad n = 0, 1, \dots. \quad (1.6)$$

A solution of Eq.(1.6) is a sequence $\{x_n\}_{n=-k}^{\infty}$ that satisfies Eq.(1.6) for all $n \geq -k$.

Suppose that the function f is continuously differentiable in some open neighborhood of an equilibrium point $\bar{x}$. Let

$$q_i = \frac{\partial f}{\partial u_i}(\bar{x}, \bar{x}, \dots, \bar{x}), \quad \text{for } i = 0, 1, \dots, k$$

denote the partial derivative of $f(u_0, u_1, \dots, u_k)$ with respect to u_i evaluated at the equilibrium point $\bar{x}$ of Eq.(1.6).

Definition 1. The equation

$$z_{n+1} = q_0 z_n + q_1 z_{n-1} + \dots + q_k z_{n-k}, \quad n = 0, 1, \dots, \quad (1.7)$$

is called the linearized equation of Eq.(1.6) about the equilibrium point $\bar{x}$.

Theorem 1 (Clark's Theorem). Consider Eq.(1.7). Then,

$$\sum_{i=0}^k |q_i| < 1.$$

is a sufficient condition for the locally asymptotically stability of Eq.(1.6).

Consider the scalar k th-order linear difference equation

$$x_{n+k} + p_1(n)x_{n+k-1} + \cdots + p_k(n)x_n = 0, \quad (1.8)$$

where k is a positive integer and $p_i : \mathbb{Z}^+ \rightarrow \mathbb{C}$ for $i = 1, \dots, k$. Assume that

$$q_i = \lim_{k \rightarrow \infty} p_i(n), i = 1, \dots, k, \quad (1.9)$$

exist in $\mathbb{C}$. Consider the limiting equation of (1.8):

$$x_{n+k} + q_1x_{n+k-1} + \cdots + q_kx_n = 0. \quad (1.10)$$

Theorem 2 (Poincaré's Theorem). *Consider (1.8) subject to condition (1.9). Let $\lambda_1, \dots, \lambda_k$ be the roots of the characteristic equation*

$$\lambda^k + q_1\lambda^{k-1} + \cdots + q_k = 0 \quad (1.11)$$

of the limiting equation (1.10) and suppose that $|\lambda_i| \neq |\lambda_j|$ for $i \neq j$. If x_n is a solution of (1.8), then either $x_n = 0$ for all large n or there exists an index $j \in \{1, \dots, k\}$ such that

$$\lim_{n \rightarrow \infty} \frac{x_{n+1}}{x_n} = \lambda_j.$$

The following results were obtained by Perron, and one of Perron's results was improved by Pituk, see [15].

Theorem 3. *Suppose that (1.9) holds. If x_n is a solution of (1.8), then either $x_n = 0$ eventually or*

$$\limsup_{n \rightarrow \infty} (|x_j(n)|)^{1/n} = \lambda_j.$$

where $\lambda_1, \dots, \lambda_k$ are the (not necessarily distinct) roots of the characteristic equation (1.11).

Theorem 4 (See [7]). *Let $n \in N_{n_0}^+$ and $g(n, u, v)$ be a nondecreasing function in u and v for any fixed n . Suppose that, for $n \geq n_0$, the inequalities*

$$\begin{aligned} y_{n+1} &\leq g(n, y_n, y_{n-1}), \\ u_{n+1} &\geq g(n, u_n, u_{n-1}) \end{aligned}$$

hold. Then

$$\begin{aligned} y_{n_0-1} &\leq u_{n_0-1}, \\ y_{n_0} &\leq u_{n_0} \end{aligned}$$

implies that

$$y_n \leq u_n, n \geq n_0.$$

Firstly, we are in a position to study the dynamics of the higher order difference equation (1.5). The Eq.(1.5) which by the change of variables

$$y_n = \frac{x_n}{A},$$

reduces to the following difference equation

$$y_{n+1} = 1 + p \frac{y_{n-m}}{y_n^2}, \quad (1.12)$$

where $p = \frac{B}{A^2}$. Henceforth, we consider the difference equation (1.12). Note that Eq.(1.12) has an unique positive equilibrium point such that

$$\bar{y} = \frac{1 + \sqrt{1+4p}}{2}.$$

2. BOUNDEDNESS OF SOLUTIONS OF EQ.(1.12)

Now, we handle the bounded solutions of Eq.(1.12). We also find out that Eq.(1.12) has bounded solutions.

Theorem 5. *If $p > 0$, then $y_n > 1$ for $n \geq 1$. Moreover, if $0 < p < 1$, then*

$$1 < y_n \leq \frac{1}{1-p} + {}^{m+1}\sqrt{p^n} \sum_{j=1}^{m+1} c_j \left(e^{\frac{2\pi i}{m+1}(j-1)} \right)^n,$$

where $c_j, j = 1, 2, \dots, m+1$, are arbitrary constants, and $n \in \{0, 1, 2, \dots\}$, and ${}^{m+1}\sqrt{p}$ is one of the $(m+1)$ th roots of p .

Proof. Let $p > 0$, and $\{y_n\}_{n=-m}^\infty$ be a positive solution of Eq.(1.12). Then, we obtain from Eq.(1.12)

$$\begin{aligned} y_1 &= 1 + p \frac{y_{-m}}{y_0^2} > 1, \\ y_2 &= 1 + p \frac{y_{1-m}}{y_1^2} > 1. \end{aligned}$$

Hence, by induction, we get $y_n > 1$ for $n \geq 1$.

Now, we handle the other case. We have from Eq.(1.12)

$$y_{n+1} = 1 + p \frac{y_{n-m}}{y_n^2} \leq 1 + p y_{n-m}.$$

According to Theorem 4, we consider a sequence $\{u_n\}_{n=0}^\infty$, and $y_n \leq u_n, n = 0, 1, \dots$, and

$$u_{n+1} = 1 + p u_{n-m}, n \geq 1, \quad (2.1)$$

such that

$$u_{s+i} = y_{s+i}, s \in \{-m, -m+1, \dots\}, i = \{0, 1, 2, \dots\}, n \geq s. \quad (2.2)$$

The characteristic polynomial to Eq.(2.1) is

$$P_{m+1}(\lambda) = \lambda^{m+1} - p.$$

Thus, we have the roots of characteristic polynomial as follows:

$$\lambda_j = \sqrt[m+1]{p} e^{\frac{2\pi i}{m+1}(j-1)},$$

where $j = 1, 2, \dots, m+1$. The homogeneous solution of Eq.(2.1) is

$$u_h = \sum_{j=1}^{m+1} c_j \sqrt[m+1]{p^n} \left(e^{\frac{2\pi i}{m+1}(j-1)} \right)^n,$$

where c_j are arbitrary constants for $j = 1, 2, \dots, m+1$. Now, we handle the equilibrium solution of Eq.(2.1). From Eq.(2.1), we get that

$$\bar{u} = \frac{1}{1-p}.$$

Therefore, the solution of Eq.(2.1) is

$$u_n = \frac{1}{1-p} + \sum_{j=1}^{m+1} c_j \sqrt[m+1]{p^n} \left(e^{\frac{2\pi i}{m+1}(j-1)} \right)^n, \quad (2.3)$$

where c_j are arbitrary constants for $j = 1, 2, \dots, m+1$. Furthermore, we have from (2.2) and (2.3)

$$y_{n+1} - u_{n+1} \leq p(y_n - u_n),$$

where $n > s$, and $p \in (0, 1)$. Therefore, we get $y_n \leq u_n, n > s$. So, the proof completed. $\square$

3. OSCILLATION BEHAVIORS OF EQ.(1.12)

In this section, we discuss the semi-cycles of Eq.(1.12). We also reveal the oscillation behaviours of solutions of Eq.(1.12) in detail.

Theorem 6. *Let $\{y_n\}_{n=-m}^{\infty}$ be a positive solution of Eq.(1.12). Then, the following statements are true:*

- (i) *The every semi-cycle at most m terms.*
- (ii) *Every solution of Eq.(1.12) oscillates about the positive equilibrium $\bar{y}$.*

Proof. We firstly handle the positive semi-cycle of solution of Eq.(1.12). The negative semi-cycle is similar and can be omitted. Assume that Eq.(1.12) has a positive semi-cycle with m terms. Suppose that y_N is the first term in this positive semi-cycle. Therefore, we get

$$y_N, y_{N+1}, \dots, y_{N+m-1} > \bar{y}.$$

Hence, we obtain from Eq.(1.12)

$$y_{N+m} = 1 + p \frac{y_{N-1}}{y_{N+m-1}^2} < 1 + p \frac{y_{N-1}}{\bar{y}^2} < \bar{y}.$$

So, we have that a semi-cycle consists at most m terms. We also get that every solution of Eq.(1.12) oscillates about $\bar{y}$. The proof completed as desired. $\square$

Theorem 7. *Let m be an odd number and let*

$$y_{-m}, y_{-m+2}, \dots, y_{-1} \leq \bar{y} \text{ and } y_{-m+1}, y_{-m+3}, \dots, y_0 > \bar{y}. \quad (3.1)$$

Then, every semi-cycle of Eq.(1.12) has lenght one. Additionally, the solution $\{y_n\}_{n=-m}^{\infty}$ of Eq.(1.12) is oscillatory about unique positive equilibrium point $\bar{y}$.

Proof. We will prove this theorem by induction method. Let $\{y_n\}_{n=-m}^{\infty}$ be a positive solution of Eq.(1.12). Assume that (3.1) holds. Hence, we obtain from Eq.(1.12),

$$\begin{aligned} y_1 &= 1 + p \frac{y_{-m}}{y_0^2} < \bar{y}, \\ y_2 &= 1 + p \frac{y_{-m+1}}{y_1^2} > \bar{y}, \\ y_3 &= 1 + p \frac{y_{-m+2}}{y_2^2} < \bar{y}. \end{aligned}$$

Therefore, we have by induction

$$y_{2n+1} = 1 + p \frac{y_{2n-m}}{y_{2n}^2} < \bar{y},$$

and

$$y_{2n} = 1 + p \frac{y_{2n-(m+1)}}{y_{2n-1}^2} > \bar{y}.$$

$\square$

4. PERIODICITY OF EQ.(1.12)

Now, we study the existence of periodic solutions of Eq.(1.12).

Theorem 8. *Assume that m is an even number. Then, Eq.(1.12) has no two periodic solution.*

Proof. Let m is an even number. We also suppose that Eq.(1.12) has two periodic solution such that

$$\dots, \alpha, \beta, \alpha, \beta, \dots$$

where α and β are positive numbers and $\alpha \neq \beta$. Hence, we get the followings

$$\begin{aligned} y_{2n+1} &= 1 + p \frac{y_{2n-m}}{y_{2n}^2} \Rightarrow \alpha = 1 + \frac{p}{\beta}, \\ y_{2n+2} &= 1 + p \frac{y_{2n+1-m}}{y_{2n+1}^2} \Rightarrow \beta = 1 + \frac{p}{\alpha}. \end{aligned} \quad (4.1)$$

where $n \geq 1$. Therefore, we obtain that

$$(\alpha - \beta) \left(1 - \frac{p}{\alpha\beta} \right) = 0.$$

Now, from our supposition, we have $\alpha \neq \beta$. Thus, we get

$$1 - \frac{p}{\alpha\beta} = 0 \Rightarrow p = \alpha\beta.$$

From (4.1), we have

$$\alpha\beta = \beta + p \Rightarrow \beta = 0.$$

So, we have a contradiction. The proof completed as desired. $\square$

5. STABILITY OF SOLUTIONS OF EQ.(1.12)

In this section, we deal with the asymptotic stability of the solutions of Eq.(1.5). Firstly, we find the linearized equation associated with Eq.(1.12) about its positive equilibrium point.

The function f is continuously differentiable in some open neighborhood of unique positive equilibrium point $\bar{y}$ as follow:

$$f(y_n, y_{n-1}, \dots, y_{n-m}) = 1 + p \frac{y_{n-m}}{y_n^2}$$

Thus, we obtain that

$$q_0 = \frac{\partial f}{\partial y_n}(\bar{y}, \bar{y}, \dots, \bar{y}) = -\frac{2p}{\bar{y}^2},$$

$$q_1 = q_2 = \dots = q_{m-1} = 0,$$

$$q_m = \frac{\partial f}{\partial y_{n-m}}(\bar{y}, \bar{y}, \dots, \bar{y}) = \frac{p}{\bar{y}^2}.$$

Then the linearized equation of Eq.(1.12) about its unique positive equilibrium point $\bar{y}$ is:

$$z_{n+1} = q_0 z_n + q_1 z_{n-1} + \dots + q_m z_{n-m},$$

for $n = 0, 1, \dots$. Therefore, we get

$$z_{n+1} + \frac{2p}{\bar{y}^2} z_n - \frac{p}{\bar{y}^2} z_{n-m} = 0. \quad (5.1)$$

Hence, the characteristic equation of Eq.(1.12) is as follows,

$$\lambda^{m+1} + \frac{2p}{\bar{y}^2} \lambda^m - \frac{p}{\bar{y}^2} = 0. \quad (5.2)$$

Theorem 9. *The positive equilibrium $\bar{y}$ of Eq.(1.12) is locally asymptotically stable when $p \in (0, \frac{3}{4})$.*

Proof. From Theorem 1, all roots of the characteristic equation of Eq.(5.1) lie in an open disc $|\lambda| < 1$, if

$$|q_0| + |q_1| + |q_2| + \cdots |q_m| < 1.$$

It follows from (5.2) that

$$|q_0| + |q_1| + |q_2| + \cdots |q_m| = \frac{3p}{\bar{y}^2}.$$

Note that

$$\frac{p}{\bar{y}^2} = \frac{2p+1-\sqrt{4p+1}}{2p}.$$

With many numerical calculations, we get that

$$\begin{aligned} |q_0| + |q_1| + |q_2| + \cdots |q_m| &= \frac{3p}{\bar{y}^2} < 1, \\ \frac{3(2p+1-\sqrt{4p+1})}{2p} &< 1, \\ \frac{4p+3-3\sqrt{4p+1}}{2p} &< 0. \end{aligned}$$

Hence, we obtain from $p > 0$,

$$\left(\sqrt{4p+1}-1\right)\left(\sqrt{4p+1}-2\right) < 0.$$

Therefore, we get that $0 < p < \frac{3}{4}$. And the proof is complete. $\square$

Theorem 10. The equilibrium point $\bar{y}$ of Eq.(1.12) is globally asymptotically stable if $0 < p < \frac{1}{2}$.

Proof. From Theorem 5, we know that there exist I and S such that

$$1 < I = \liminf_{n \rightarrow \infty} y_n \leq S = \limsup_{n \rightarrow \infty} y_n.$$

Hence, we get from Eq.(1.12)

$$I \geq 1 + p \frac{I}{S^2} \text{ and } S \leq 1 + p \frac{S}{I^2}.$$

Therefore, we obtain that

$$S + p \frac{I}{S} \leq IS \leq I + p \frac{S}{I}.$$

Thus, we have

$$(S-I) \left(1 - p \left(\frac{1}{S} + \frac{1}{I}\right)\right) \leq 0.$$

From $S \geq I > 1$ and $0 < p < \frac{1}{2}$, we also get

$$1 - p \left(\frac{1}{S} + \frac{1}{I}\right) > 0.$$

So, we have $S \leq I$ which the result follows. Hence, the equilibrium point $\bar{y}$ of Eq.(1.12) is globally asymptotically stable if $0 < p < \frac{1}{2}$. $\square$

Conjecture 1. *Many numerical simulations show that If $\frac{1}{2} \leq p < \frac{3}{4}$, then the equilibrium point $\bar{y}$ of Eq.(1.12) is globally asymptotically stable.*

6. RATE OF CONVERGENCE OF EQ.(1.12)

Here, we investigate the rate of convergence of solutions of Eq.(1.12).

Theorem 11. *Let λ_j be roots of characteristic equation (5.2) where $j \in \{1, \dots, k\}$. Then, every solution of Eq.(1.12) ensures the following relations:*

$$\lim_{n \rightarrow \infty} \left| \frac{y_{n+1} - \bar{y}}{y_n - \bar{y}} \right| = |\lambda_j|,$$

and

$$\limsup_{n \rightarrow \infty} (|y_n - \bar{y}|)^{1/n} = |\lambda_j|.$$

Proof. According to Eq.(1.12), we have that

$$y_{n+1} - \bar{y} = -\frac{p(y_n + \bar{y})}{\bar{y}y_n^2} (y_n - \bar{y}) + \frac{p}{y_n^2} (y_{n-m} - \bar{y}).$$

Now, we consider $e_n = y_n - \bar{y}$. Then, we obtain

$$e_{n+1} + p_n e_n + q_n e_{n-m} = 0,$$

such that

$$p_n = -\frac{p(y_n + \bar{y})}{\bar{y}y_n^2},$$

and

$$q_n = \frac{p}{y_n^2}.$$

Therefore, we get from globally asymptotic stability

$$\lim_{n \rightarrow \infty} p_n = -\frac{2p}{\bar{y}^2},$$

and

$$\lim_{n \rightarrow \infty} q_n = \frac{p}{\bar{y}^2}.$$

So, the proof is completed. $\square$

7. NUMERICAL SIMULATIONS

In an attempt to support our theoretical results, we handle three numerical examples. These examples include three figures which drawn by Mathematica.

Example 1. With $A = \sqrt{5}$, $B = 4$ and $m = 3$, we handle Eq.(1.5). Thus, we get following fourth order difference equation

$$x_{n+1} = \sqrt{5} + 4 \frac{x_{n-3}}{x_n^2}. \quad (7.1)$$

Here, we apply the following change of the variables for Eq.(7.1) $y_n = \frac{x_n}{\sqrt{5}}$. Therefore, we have that $p = \frac{B}{A^2} = 0.8$, and we obtain the following difference equation

$$y_{n+1} = 1 + 0.8 \frac{y_{n-3}}{y_n^2}. \quad (7.2)$$

Let the initial conditions are $y_{-3} = 8$, $y_{-2} = 5$, $y_{-1} = 4$ and $y_0 = 2$. Then, every solution of Eq.(7.2) oscillates about the equilibrium point $\bar{y}$. Additionally, every solution of Eq.(7.2) has bounded from below and above. Figure 1 shows the first 300 terms of Eq.(7.2).

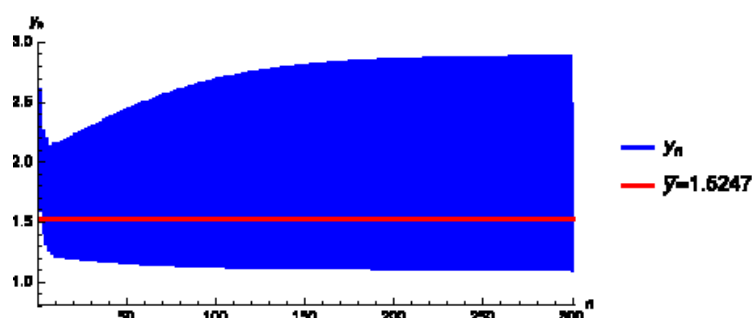


FIGURE 1. Plot of Eq.(1.5) with $A = \sqrt{5}$, $B = 4$ and $m = 3$.

Example 2. Let the Eq.(1.5) with $A = \sqrt{20}$, $B = 9$ and $m = 2$. Then we obtain the following third order quadratic-rational difference equation

$$x_{n+1} = \sqrt{20} + 9 \frac{x_{n-2}}{x_n^2}. \quad (7.3)$$

Hence, we get $p = \frac{B}{A^2} = 0.45$ and we obtain the difference equation

$$y_{n+1} = 1 + 0.45 \frac{y_{n-2}}{y_n^2}. \quad (7.4)$$

Assume that the initial conditions are $y_{-2} = 200$, $y_{-1} = 0.1$ and $y_0 = 0.07$. Hence, the positive equilibrium point $\bar{y} = 1.3367$ of Eq.(7.4) is globally asymptotically stable.

Furthermore, the every solution of Eq.(7.4) bounded, and also oscillate about positive equilibrium point. Figure 2 presents the first 100 terms of Eq.(7.4).

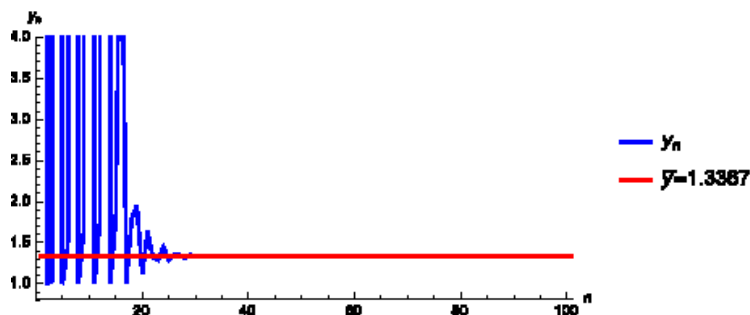


FIGURE 2. Plot of Eq.(1.5) with $A = \sqrt{20}$, $B = 9$ and $m = 2$.

Example 3. With $A = 10$, $B = 39$ and $m = 5$, we consider Eq.(1.5). So, we obtain the sixth order difference equation

$$x_{n+1} = 10 + 39 \frac{x_{n-5}}{x_n^2}. \quad (7.5)$$

Hence, we get that $p = \frac{B}{A^2} = 0.39$ and we obtain the following difference equation

$$y_{n+1} = 1 + 0.39 \frac{y_{n-5}}{y_n^2}. \quad (7.6)$$

Let the initial values are $y_{-5} = 2$, $y_{-4} = 6$, $y_{-3} = 1$, $y_{-2} = 4$, $y_{-1} = 2$ and $y_0 = 3$. So, the positive equilibrium point $\bar{y} = 1.3$ of Eq.(7.6) is globally asymptotically stable. Furthermore, the every solution of Eq.(7.6) has bounded, and also oscillate about positive equilibrium point. Figure 3 demonstrates the first 100 terms of Eq.(7.6).

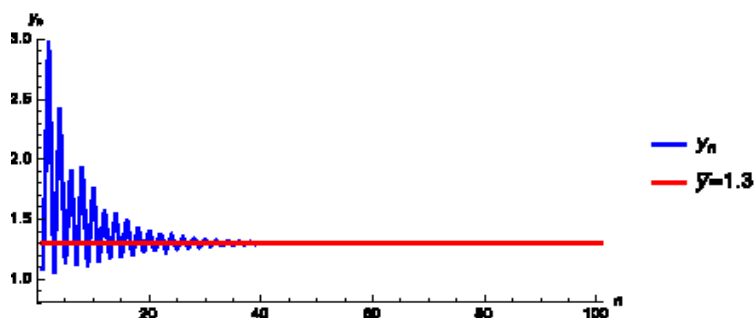


FIGURE 3. Plot of Eq.(1.5) with $A = 10$, $B = 39$ and $m = 5$.

8. CONCLUSION

In this paper, we firstly study the boundedness of solutions of Eq.(1.5). For the choice of p , if $p \in (0, 1)$, then every solution of Eq.(1.5) has bounded from below and above. Moreover, we examine the oscillation behaviors of solutions of Eq.(1.5). Next, we handle the existence of periodic solutions of Eq.(1.5). According to this result, there are no two periodic solutions of Eq.(1.5) if m is an even number. We also investigate the global asymptotic stability and rate of convergence of Eq.(1.5). Lastly, we demonstrate three numerical simulations and graphed by using Mathematica.

ACKNOWLEDGEMENTS

The authors would like to express their thanks to the referees for their constructive advices and comments that helped to improve this paper.

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FRACTIONAL DIFFERENTIAL EQUATIONS OF VARIABLE ORDER: EXISTENCE RESULTS, NUMERICAL METHOD AND ASYMPTOTIC STABILITY CONDITIONS

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Received 01 November, 2018

1. INTRODUCTION

The concept of variable and distributed order fractional derivative firstly appeared in [9] since many physical processes exhibited memory effects that may vary with time or space variables. Some new variable order fractional derivatives and applications were suggested, for example, Hamilton's principle [2], variable-order mechanics [3], constitutive relation for vis-coelasticity [12], fractional diffusion equations [8, 13, 14]. Although the variable-order fractional derivative provides more freedom degrees and new ways to understand the complicated dynamics, the main difficulty is to consider the qualitative theories. Hence, it is a challenging work to define a variable-order function not only can be efficient in explanation of physical phenomena but also for convenience of mathematical analysis.

In this paper, we propose a kind of short memory fractional differential equations and try to address this problem which is our main purpose. We investigate the following fractional differential equation and give existence results

$$\begin{cases} {}^C_{t_k} D_t^{\alpha_{k+1}} x = f(x, t), t \in [t_k, t_{k+1}] \\ x(t_{l_0}) = \eta, \quad 0 < \alpha_{k+1} \leq 1, k = 0, 1, \dots, m-1, l_0 = 0, t_m = T, 1 \leq m, \end{cases} \quad (1.1)$$

where t_{l_k} is the initial point, ${}^C_{t_k} D_t^{\alpha_{k+1}} x$ is the Caputo derivative of the function $x(t)$, $f: \mathbb{R} \times [t_0, T] \rightarrow \mathbb{R}$ and the fractional order α_{k+1} is a piecewise constant defined over each $[t_k, t_{k+1}]$.

The paper is organized in following sections. Section 2 compares the classical fractional differential equations with Eq. (1.1). Then it gives existence results. Section 3 applies predictor-corrector method to obtain numerical solutions. Section 4 derives the exact solution of linear equations. Section 5 investigates the linear fractional variable-order system's asymptotic stability. Finally, conclusion is made in Section 6 and some possible applications are discussed.

2. EXISTENCE RESULTS

We need to point out Eq. (1.1) is totally different from classical fractional differential equations with initial conditions

$$\begin{cases} {}^C_{t_0}D_t^\alpha x(t) = f(x, t) \\ x(t_0) = \eta, \quad 0 < \alpha \leq 1. \end{cases} \quad (2.1)$$

Eq. (1.1) has “moving” initial points. We call it as a short memory fractional differential equation since the solution $x(t)$ only depends on the information from $x(t_{l_k})$ for $t \in [t_{l_k}, t_{l_{k+1}}]$. There is no need to start from t_0 in fractional modelling and this provides more freedom degrees in real-world applications. Besides, this feature is much easier for mathematical analysis of variable-order problems. In the rest of the paper, we give existence results and numerical solutions of Eq. (1.1).

Now, let's revisit some results in the fractional calculus and introduce the following definitions in [11].

Definition 1. For $\alpha > 0$, the Riemann-Liouville integral of α order for function y on $[t_0, +\infty)$ is defined as

$${}_t I_t^\alpha y = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} y(s) ds, \quad t > t_0. \quad (2.2)$$

Definition 2. For $0 < \alpha < 1$ and $y(t) \in AC^1[t_0, +\infty)$, the Caputo derivative of α order is defined by

$${}^C_{t_0}D_t^\alpha y := \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^t (t-s)^{-\alpha} y'(s) ds, \quad t > t_0. \quad (2.3)$$

For $\alpha = 1$, then ${}^C_{t_0}D_t^\alpha y(t) = y'(t)$.

Assume that $B(b, a) = \{(x, t) : |x - x^*| \leq b, |t - t^*| \leq a\}$. Let the function $f : B(b, a) \rightarrow \mathbb{R}$ be bounded by M^* , and f is Lipschitz continuous with respect to x with the constant L^* .

Lemma 1 ([1, 7]). $x(t)$ is a solution of the fractional differential equation

$$\begin{cases} {}^C_{t^*}D_t^\alpha x = f(x, t), \quad (x, t) \in B(b, a), \quad 0 < \alpha \leq 1, \\ x(t^*) = x^*. \end{cases} \quad (2.4)$$

if and only if $x(t)$ is a solution of the following equivalent integral equation

$$x(t) = x(t^*) + {}_{t^*}I_t^\alpha f(x, t). \quad (2.5)$$

Lemma 2 ([1, 7]). The system (2.4) has a unique solution over the interval $[t^*, t^* + h^*]$ if f satisfies the Lipschitz condition

$$|f(x, t) - f(y, t)| \leq L^* |x - y|, \quad (x, t), (y, t) \in B(b, a). \quad (2.6)$$

where

$$h^{*\alpha} = \min \left\{ a^\alpha, \frac{\Gamma(1+\alpha)}{L^*}, \frac{\Gamma(1+\alpha)b}{M^*} \right\}. \quad (2.7)$$

and $M^* = \max_{(x,t) \in B(b,a)} (|f(x,t)|)$.

Considering the fractional variable order system (1.1), let $t^* = t_{l_k}$, $k = 0, \dots, m-1$, $l_0 = 0$ and the initial condition becomes (t_{l_k}, x_{l_k}) . For example $t^* = t_0$, we can determine h_0 and get the interval $[t_0, t_{l_1}]$ where $t_{l_1} = t_0 + h_0$. With the new initial condition (t_{l_1}, x_{l_1}) and by use of the existence condition (2.7), we can determine h_1 , $[t_{l_1}, t_{l_1} + h_1]$ and (t_{l_2}, x_{l_2}) . More generally, we can obtain each h_k and $[t_{l_k}, t_{l_k} + h_k]$ successively in this way. Hence, we now arrive at existence results of Eq. (1.1).

Theorem 1. $f(x,t)$ is globally Lipschitz continuous with respect to x

$$|f(x_2^*, t) - f(x_1^*, t)| \leq L^* |x_2^* - x_1^*|, (x_1^*, t), (x_2^*, t) \in B(b, a), k = 0, 1, \dots, m-1. \quad (2.8)$$

Eq. (1.1) has a unique solution on $[t_0, t_0 + \sum_{k=0}^{m-1} h_k]$, where h_k is defined

$$h_k^{\alpha_{k+1}} = \min \left\{ a^{\alpha_{k+1}}, \frac{\Gamma(1 + \alpha_{k+1})}{L^*}, \frac{\Gamma(1 + \alpha_{k+1})b}{M^*} \right\}. \quad (2.9)$$

Theorem 2. Eq. (1.1) has a unique solution for $t \in [t_0, t_0 + ml]$ where

$$l = \min \{h_0, \dots, h_{m-1}\}. \quad (2.10)$$

3. NUMERICAL METHOD

Although we can use Picard's method to obtain series solutions, the accuracy is not high enough to get the update initial conditions (t_{kl}, x_{kl}) . Hence, in this section, we consider the numerical solutions. Let us first illustrate general steps for exact solutions of the linear equations. Then we consider the predictor-corrector method for the nonlinear case.

The predictor-corrector method developed in [6] is the most popular numerical method for chaotic analysis of fractional differential equations. Recently, several improved versions and other applications are considered [4, 5]. Eq. (2.1) is equal to

$$x(t) = x(t_0) + t_0 I_t^\alpha f(x, t). \quad (3.1)$$

Diethelm proposed the rectangle and trapezoid formulae for the fractional integral [6] where the coefficients were derived as

$$b_{j,n+1} = \frac{1}{\alpha} ((n+1-j)^\alpha - (n-j)^\alpha) \quad (3.2)$$

and

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^\alpha & \text{if } j = 0; \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1} & \text{if } 1 \leq j \leq n; \\ 1 & \text{if } j = n+1. \end{cases} \quad (3.3)$$

For the variable-order fractional differential equation,

$$\begin{cases} {}^C D_t^{\alpha_{k+1}} x(t) = f(x, t), t \in [t_{kl}, t_{(k+1)l}] , \\ x(t_0) = \eta, \quad 0 < \alpha_{k+1} \leq 1, k = 0, 1, \dots, m-1, \end{cases} \quad (3.4)$$

it has the same numerical formulae on the first interval $[t_0, t_l]$. From $t \in [t_{kl}, t_{(k+1)l}]$, $1 \leq k \leq m-1$, $m = 2, 3, \dots$, and $\Delta t = \frac{l}{s}$ where s is a positive integer, we obtain the numerical formula

$$\begin{cases} x_{ks+i+1}^p = x_{ks} + \frac{\Delta t^{\alpha_{k+1}}}{\Gamma(\alpha_{k+1})} \sum_{j=0}^i b_{j,i+1} f(x_{j+ks}, t_{j+ks}), i = 0, \dots, s-1, \\ x_{ks+i+1} = x_{ks} + \frac{\Delta t^{\alpha_{k+1}}}{\Gamma(\alpha_{k+1}+2)} \sum_{j=0}^i a_{j,i+1} f(x_{j+ks}, t_{j+ks}) + \frac{\Delta t^{\alpha_{k+1}}}{\Gamma(\alpha_{k+1}+2)} f(x_{ks+i+1}^p, t_{ks+i+1}). \end{cases} \quad (3.5)$$

Here x_n is the numerical solution, $x_n := x(t_n)$ and Δt is the step-length of the numerical formulae. The error estimation is $O(\Delta t^p)$ and $p = 1 + \alpha_{k+1}$.

Example 1. Consider the following fractional differential equation

$$\begin{cases} {}^C D_t^{\alpha_{k+1}} x = \sin(x), t \in [t_{kl}, t_{(k+1)l}], \\ x(t_0) = 0.1, \quad 0 < \alpha_{k+1} \leq 1, k = 0, 1, \dots, m-1. \end{cases} \quad (3.6)$$

We adopt the following parameters: $m = 3$, $L = 1$, $\alpha_1 = 0.7$, $\alpha_2 = 0.8$ and $\alpha_3 = 0.9$. According to Theorem 2, we can use solutions' interval as $[0, 3l]$ and $l = 0.8$.

By use of the numerical method, the numerical solutions are given in Figs. 1 and 2. With different time domains, the fractional order is varied in Fig. 1. And the constant order case is compared in Fig. 2 where we set the order to 0.8. From the solution's behavior, we can see that although the fractional order is the same on different domains, the solution is not differentiable at the ends t_{kl} due to the short memory effects.

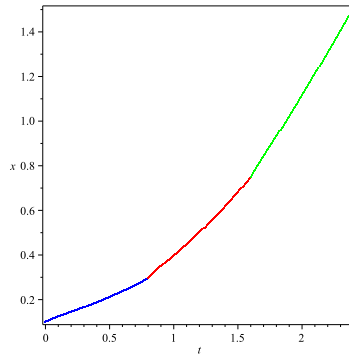


FIGURE 1. Numerical solutions of variable order system (3.6) (the blue: $\alpha_1 = 0.7$ and $t \in [0, 0.8]$; the red: $\alpha_2 = 0.8$ and $t \in [0.8, 1.6]$ the green: $\alpha_3 = 0.9$ and $t \in [1.6, 2.4]$).

4. EXACT SOLUTIONS OF LINEAR EQUATIONS

In this subsection, we discuss two linear equations.

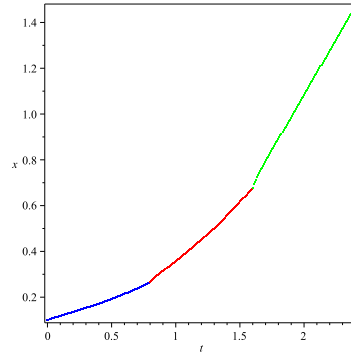


FIGURE 2. Numerical solutions of constant order system (3.6) on different time domains (the blue: $\alpha_1 = 0.8$ and $t \in [0, 0.8]$; the red: $\alpha_2 = 0.8$ and $t \in [0.8, 1.6]$; the green: $\alpha_3 = 0.8$ and $t \in [1.6, 2.4]$).

Theorem 3. *The fractional differential equation*

$$\begin{cases} {}^C D_t^{\alpha_{k+1}} x(t) &= \lambda x(t), t \in [t_{kl}, t_{(k+1)l}] , \\ x(t_0) &= \eta, \quad 0 < \alpha_{k+1} \leq 1, k = 0, 1, \dots \end{cases} \quad (4.1)$$

has a unique solution as

$$x(t) = \eta \left[\prod_{i=1}^k E_{\alpha_i} \left(\lambda, (t_{il} - t_{(i-1)l})^{\alpha_i} \right) \right] E_{\alpha_{k+1}} \left(\lambda, (t - t_{kl})^{\alpha_{k+1}} \right), t \in [t_{kl}, t_{(k+1)l}]. \quad (4.2)$$

Proof. For $t \in [t_0, t_l]$, we derive that

$$\begin{aligned} x(t) &= x(t_0) + \lambda {}_t I_t^{\alpha_1} x(t), \\ x(t) &= \eta E_{\alpha_1} \left(\lambda, (t - t_0)^{\alpha_1} \right) \end{aligned}$$

and

$$x(t_l) = \eta E_{\alpha_1} \left(\lambda, (t_l - t_0)^{\alpha_1} \right)$$

where $E_\alpha(\lambda, t)$ is the Mittag-Leffler function defined by

$$E_\alpha(\lambda, t) = \sum_{k=0}^{+\infty} \frac{\lambda^k t^{k\alpha}}{\Gamma(1 + k\alpha)}.$$

For $t \in [t_l, t_{2l}]$, we have

$$\begin{aligned} x(t) &= x(t_l) + \lambda {}_t I_t^{\alpha_2} x(t), \\ x(t) &= \eta E_{\alpha_1} \left(\lambda, (t_l - t_0)^{\alpha_1} \right) E_{\alpha_2} \left(\lambda, (t - t_l)^{\alpha_2} \right). \end{aligned}$$

Finally, we get

$$x(t) = x(t_{kl}) + \lambda {}_t I_t^{\alpha_{k+1}} x(t),$$

$$x(t) = \eta \left[\prod_{i=1}^k E_{\alpha_i} \left(\lambda, (t_{il} - t_{(i-1)l})^{\alpha_i} \right) \right] E_{\alpha_{k+1}} \left(\lambda, (t - t_{kl})^{\alpha_{k+1}} \right), t \in [t_{kl}, t_{(k+1)l}].$$

which completes the proof. $\square$

We can use the predictor corrector method to derive the numerical solutions in Figs. 3 and 4 where $\lambda = 0.8$ and $\lambda = -0.8$, respectively. Other parameters are set to $\eta = 1$, $l = 3$, $m = 3$, $\alpha_1 = 0.7$, $\alpha_2 = 0.8$ and $\alpha_3 = 0.9$.

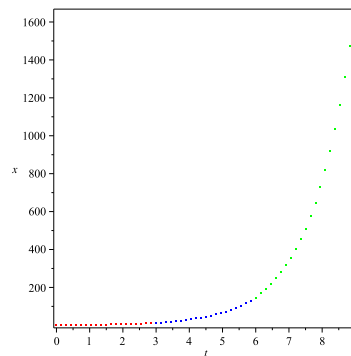


FIGURE 3. Mittag-Leffler function of variable order (5.1) (the red: $\alpha_1 = 0.7$ and $t \in [0, 3]$; the blue: $\alpha_2 = 0.8$ and $t \in [3, 6]$; the green: $\alpha_3 = 0.9$ and $t \in [6, 9]$).

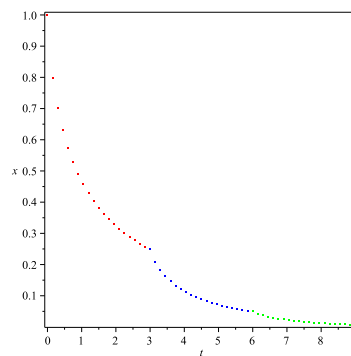


FIGURE 4. Mittag-Leffler function of variable order (5.1) on different time domains (the blue: $\alpha_1 = 0.7$ and $t \in [0, 3]$; the red $\alpha_2 = 0.8$ and $t \in [3, 6]$; the green: $\alpha_3 = 0.9$ and $t \in [6, 9]$).

5. ASYMPTOTIC STABILITY

We can define a Mittag–Leffler function of variable order as

$$\varepsilon_{\alpha_{k+1}}(\lambda, t) := \left[\prod_{i=1}^k E_{\alpha_i} \left(\lambda, (t_{il} - t_{(i-1)l})^{\alpha_i} \right) \right] E_{\alpha_{k+1}} \left(\lambda, (t - t_{kl})^{\alpha_{k+1}} \right), t \in [t_{kl}, t_{(k+1)l}] \quad (5.1)$$

where $0 < \alpha_{k+1} \leq 1$, for $t \in [t_{kl}, t_{(k+1)l}]$ and $k = 0, 1, \dots, m-1$.

If m is a positive integer number, for $\lambda < 0$, $t \in [t_{ml}, \infty)$ and $t \rightarrow +\infty$, we can obtain

$$x(t) = \eta \varepsilon_{\alpha_{k+1}}(\lambda, t) \rightarrow 0. \quad (5.2)$$

Much more generally, according to Matignon's stability conditions [10], we know the following stability result of the standard fractional linear systems.

Lemma 3. [10] *Suppose λ is an eigenvalue of the coefficient matrix A . The fractional linear autonomous system*

$$\begin{cases} {}^C_{t_0} D_t^\alpha x = Ax, & 0 < \alpha \leq 1 \\ x(t_0) = \eta, \end{cases}$$

where $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$ is asymptotically stable if and on if $|\arg(\lambda)| > \frac{\alpha\pi}{2}$ is satisfied for all eigenvalues of matrix A .

We can extend Theorem 3 and the exact solution can be presented in form of a matrix Mittag–Leffler function of variable order $\varepsilon_{\alpha_{k+1}}(A, t)$. We now give the stability theorem.

Theorem 4. *The fractional linear system*

$$\begin{cases} {}^C_{t_{kl}} D_t^{\alpha_{k+1}} x(t) = Ax(t), & t \in [t_{kl}, t_{(k+1)l}], \\ x(t_0) = \eta, & 0 < \alpha_{k+1} \leq 1, k = 0, 1, \dots \end{cases}$$

is asymptotically stable if there exists a positive integer N such that $|\arg(\lambda)| > \frac{\alpha_{k+1}\pi}{2}$ for $k > N$.

6. CONCLUSIONS

Fractional derivative has non-locality or memory effects. This feature has made it be a powerful tool in various applied sciences and the fractional differential equation has become one of the popular directions. The concept of variable-order fractional derivative was proposed about fifteen years ago and it was efficient to reveal complicated fractional dynamics. However, less theories contributed except some numerical methods for numerical solutions. This paper provides a new concept of short memory which is very convenient to define a variable-order function. We then give existence conditions of such equations with variable orders. The predictor-corrector method

is used to show the new concept both suitable for theoretical analysis and numerical calculation. We only give the existence results in this paper and we believe the following topics are important in future:

1) Numerical methods of high accuracy. We only illustrate the application of the predictor-corrector method. There are many numerical methods developed and available. They also can be used in this study. Besides, we notice that the computational time is saved a lot, particularly when the m becomes very large. The fractional differential equation itself is a short memory one and it saves much storage space in numerical calculations.

2) New applications of the short memory. Many applications now all considered the memory or non-locality of the whole interval. However, we may only need some of the information or data. That means we need a fractional approach between non-locality and locality. This study gives some a possible way for fractional modeling. We now can consider some other applications such as short memory Euler-Lagrange equations, fractional diffusion equations and signal processing.

7. ACKNOWLEDGEMENTS

This study was supported in part by the National Natural Science Foundation of China (Grant No. 62076141), Sichuan Youth Science and Technology Foundation (Grant No. 22JCQN0197) and Open Research Fund Program of Data Recovery Key Laboratory of Sichuan Province (Grant No. DRN2101).

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BOUNDS FOR THE GENERALIZED ELLIPTIC INTEGRAL OF THE SECOND KIND

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Received 04 May, 2021

Abstract. For $a \in (0, 1)$ and $r \in (0, 1)$, let $\mathcal{E}_a(r)$ be the generalized elliptic integral of the second kind and $R(a)$ be Ramanujan's constant. In this paper, we prove the following inequalities

$$\begin{aligned} \frac{\sin(\pi a)}{2(1-a)} + r'^2 \left((1-a) \sin(\pi a) \log \left(\frac{e^{R(a)/2}}{r'} \right) - \sigma \right) &< \mathcal{E}_a(r) \\ &< \frac{\sin(\pi a)}{2(1-a)} + r'^2 \left((1-a) \sin(\pi a) \log \left(\frac{e^{R(a)/2}}{r'} \right) - \delta \right) \end{aligned}$$

with the best possible constants $\sigma = \frac{1}{4} \sin(\pi a)$ and $\delta = \frac{\sin(\pi a)}{2(1-a)} + (1-a) \sin(\pi a) \frac{R(a)}{2} - \frac{\pi}{2}$.

2010 *Mathematics Subject Classification:* 33C05; 33E05

Keywords: generalized elliptic integrals, Ramanujan's constant, Gaussian hypergeometric function, Hersch-Pfluger distortion function

1. INTRODUCTION

For real numbers a, b and c with $c \neq 0, -1, -2, \dots$, the *Gaussian hypergeometric function* is defined by

$${}_2F_1(a, b; c; x) \equiv \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{x^n}{n!}, \quad |x| < 1.$$

Here $(a, 0) = 1$ for $a \neq 0$, and (a, n) is the *shifted factorial function*

$$(a, n) \equiv a(a+1)(a+2) \cdots (a+n-1)$$

for $n \in \mathbb{N} \equiv \{k : k \text{ is a positive integer}\}$.

This research was supported by the National Natural Science Foundation of China (11771400) and the Natural Science Foundation of Zhejiang Province (LY22A010004).

The generalized elliptic integrals can be represented by the Gaussian hypergeometric function as follows [2]: for $a \in (0, 1)$,

$$\begin{cases} \mathcal{K}_a = \mathcal{K}_a(r) = \frac{\pi}{2} {}_2F_1(a, 1-a; 1; r^2), \\ \mathcal{K}'_a = \mathcal{K}'_a(r) = \mathcal{K}_a(r'), \\ \mathcal{K}_a(0) = \frac{\pi}{2}, \quad \mathcal{K}_a(1) = \infty, \end{cases}$$

and

$$\begin{cases} \mathcal{E}_a = \mathcal{E}_a(r) = \frac{\pi}{2} {}_2F_1(1-a, a-1; 1; r^2), \\ \mathcal{E}'_a = \mathcal{E}'_a(r) = \mathcal{E}_a(r'), \\ \mathcal{E}_a(0) = \frac{\pi}{2}, \quad \mathcal{E}_a(1) = 1. \end{cases}$$

Here and hereafter, we always let $r' = (1 - r^2)^{1/2}$. It is easy to see that $\mathcal{K}_a$ is strictly increasing and $\mathcal{E}_a$ is strictly decreasing on $(0, 1)$. The generalized elliptic integrals satisfy the following Legendre relation:

$$\mathcal{K}_a(r)\mathcal{E}'_a(r) + \mathcal{K}'_a(r)\mathcal{E}_a(r) - \mathcal{K}_a(r)\mathcal{K}'_a(r) = \frac{\pi \sin(\pi a)}{4(1-a)}.$$

For $a = 1/2$, the generalized elliptic integrals reduce to the classical complete elliptic integrals $\mathcal{K}$ and $\mathcal{E}$, respectively. It is well known that the complete elliptic integrals $\mathcal{K}$ and $\mathcal{E}$ have many applications in several fields of mathematics as well as in physics and engineering. Numerous properties have been obtained for $\mathcal{K}$ and $\mathcal{E}$ (for instance, see [1, 4–7, 9, 18]).

In 2000, Anderson, et al [2] investigated certain combinations of the generalized elliptic integrals which occur in Ramanujan's modular equations and approximations to π . They showed the monotonicity and convexity properties of these quantities and obtained various sharp inequalities for them. Recently, the generalized elliptic integrals have attracted the attention of many mathematicians. In particular, many remarkable properties and inequalities for the generalized elliptic integrals can be found in the literature [2, 6, 8, 10–13, 15, 17, 19].

During the past few decades, many authors obtained various sharp elementary estimates for $\mathcal{E}$ and $\mathcal{E}_a$. For instance, in 2004, Alzer and Qiu [1] proved the inequality

$$\frac{\pi}{2} \left(\frac{1+r'^{\eta}}{2} \right)^{\eta} < \mathcal{E}(r) < \frac{\pi}{2} \left(\frac{1+r'^{\theta}}{2} \right)^{\theta},$$

where $\eta = 3/2$ and $\theta = \log 2 / \log(\pi/2)$ are best possible. This result improves previous known bounds and gives symmetric upper and lower bounds.

In 2011, Wang et al. [13, Theorem 1.2] obtained the following sharp bounds for $\mathcal{E}(r)$:

$$\frac{\pi}{2} - \frac{3\pi}{16} \frac{r - r'^2 \operatorname{arth} r}{r} < \mathcal{E}(r) < \frac{\pi}{2} - \left(\frac{\pi}{2} - 1\right) \frac{r - r'^2 \operatorname{arth} r}{r} \quad (1.1)$$

and

$$\frac{\pi}{2} - \log r' - r \operatorname{arth} r < \mathcal{E}(r) < \frac{\pi}{2} - \left(2 - \frac{\pi}{4}\right) \log r' - r \operatorname{arth} r. \quad (1.2)$$

The purpose of this paper is to study some monotonicity properties for the generalized elliptic integrals of the second kind and obtain the following theorem.

Theorem 1. *Given $a \in (0, 1)$, let $R(a)$ be Ramanujan's constant defined as (2.1). Then, for all $r \in (0, 1)$,*

$$\begin{aligned} \frac{\sin(\pi a)}{2(1-a)} + r'^2 \left((1-a) \sin(\pi a) \log \left(\frac{e^{R(a)/2}}{r'} \right) - \sigma \right) &< \mathcal{E}_a(r) \\ &< \frac{\sin(\pi a)}{2(1-a)} + r'^2 \left((1-a) \sin(\pi a) \log \left(\frac{e^{R(a)/2}}{r'} \right) - \delta \right), \end{aligned}$$

where the constants $\sigma = \frac{1}{4} \sin(\pi a)$ and $\delta = \frac{\sin(\pi a)}{2(1-a)} + (1-a) \sin(\pi a) \frac{R(a)}{2} - \frac{\pi}{2}$ are best possible.

As an application, we obtain a sharp functional inequality for the generalized Hersch-Pfluger distortion function

$$\Phi_k^a(r) < r^{1/K} \exp \left\{ \left(1 - \frac{1}{K}\right) \frac{R(a)}{\pi - 2} r'^2 \left(\frac{1}{2} \log \frac{2}{r'} + \frac{\pi}{2} - 1 \right) \right\}.$$

The definition of the generalized Hersch-Pfluger distortion function is in Section 3.

2. PROOF OF MAIN RESULT

Throughout this paper, let $\gamma = 0.577215 \dots$ be the Euler constant and let

$$R(a, b) = -2\gamma - \psi(a) - \psi(b), \quad R(a) = R(a, 1-a), \quad R(1/2) = \log 16 \quad (2.1)$$

be Ramanujan's constant, where ψ is the classical psi function.

The functions $\mathcal{K}_a$ and $\mathcal{E}_a$ satisfy the following derivative formulas [2]:

$$\begin{aligned} \frac{d\mathcal{K}_a}{dr} &= \frac{2(1-a)}{rr'^2} (\mathcal{E}_a - r'^2 \mathcal{K}_a), \\ \frac{d\mathcal{E}_a}{dr} &= \frac{-2(1-a)}{r} (\mathcal{K}_a - \mathcal{E}_a), \\ \frac{d(\mathcal{E}_a - r'^2 \mathcal{K}_a)}{dr} &= 2ar \mathcal{K}_a, \\ \frac{d(\mathcal{K}_a - \mathcal{E}_a)}{dr} &= \frac{2(1-a)r \mathcal{E}_a}{r'^2}. \end{aligned}$$

We shall prove monotonicity properties of some functions defined by the generalized elliptic integrals.

Lemma 1. *The function*

$$v(a) = (1-a)^2(4a^2-8a)\pi + 8(1-a)\sin(\pi a)$$

is positive on $(0,1)$.

Proof. Let $w(a) = v(a)/(1-a) = (1-a)(4a^2-8a)\pi + 8\sin(\pi a)$. We need to prove $w(a)$ is positive on $(0,1)$. Simple computations lead to

$$w(0) = 0 = w(1). \quad (2.2)$$

By differentiation, we have

$$w'(a) = (1-a)(8a-8)\pi - (4a^2-8a)\pi + 8\pi\cos(\pi a)$$

with

$$w'(0+) = 0, \quad w'(1-) = -4\pi, \quad (2.3)$$

$$w''(a) = (24-24a)\pi - 8\pi^2\sin(\pi a)$$

with

$$w''(0+) = 24\pi, \quad w''(1-) = 0, \quad (2.4)$$

and

$$w'''(a) = -24\pi - 8\pi^3\cos(\pi a)$$

with

$$w'''(0+) = -24\pi - 8\pi^3 < 0, \quad w'''(1-) = -24\pi + 8\pi^3 > 0. \quad (2.5)$$

Since $w'''(a)$ is strictly increasing in a , we see from (2.5) that there exists $a_0 \in (0,1)$ such that $w'''(a) < 0$ for $a \in (0, a_0)$ and $w'''(a) > 0$ for $a \in (a_0, 1)$. Thus $w''(a)$ is strictly decreasing on $(0, a_0)$ and strictly increasing on $(a_0, 1)$. The limiting values (2.4) and the piecewise monotonicity of $w''(a)$ imply that there exists $a_1 \in (0, a_0)$ such that $w''(a) > 0$ for $a \in (0, a_1)$ and $w''(a) < 0$ for $a \in (a_1, 1)$. Then $w'(a)$ is strictly increasing on $a \in (0, a_1)$ and strictly decreasing on $(a_1, 1)$. It follows from (2.3) together with the piecewise monotonicity of $w'(a)$ that there exists $a_2 \in (a_1, 1)$ such that $w'(a) > 0$ for $a \in (0, a_2)$ and $w'(a) < 0$ for $a \in (a_2, 1)$. Thus $w(a)$ is strictly increasing on $(0, a_2)$ and strictly decreasing on $(a_2, 1)$. The limiting values (2.2) and the piecewise monotonicity of $w(a)$ imply that $w(a) > 0$ for $a \in (0, 1)$ and hence $v(a) > 0$. $\square$

Lemma 2. *The function*

$$p(a) = (1-(1-a)^2)\pi - \frac{\sin(\pi a)}{1-a} - (1-a)\sin(\pi a)$$

is negative on $(0,1)$.

Proof.

$$\begin{aligned} p(a) &= \pi \left(1 - (1-a)^2 - \frac{1 + (1-a)^2}{\pi(1-a)} \sin(\pi a) \right) \\ &= \pi \left(1 - (1-a)^2 - \frac{1 + (1-a)^2}{\pi(1-a)} \sin(\pi(1-a)) \right) \\ &< \pi \left(1 - (1-a)^2 - (1 + (1-a)^2) \frac{1 - (1-a)^2}{1 + (1-a)^2} \right) = 0, \end{aligned}$$

where the inequality follows from the well-known Redheffer inequality [16]:

$$\frac{\sin(\pi x)}{\pi x} > \frac{1-x^2}{1+x^2} \quad \text{for } x \in (0, 1).$$

□

Lemma 3. For given $a \in (0, 1)$, the function

$$u(r) = -2(1-a)(\mathcal{K}_a(r) - \mathcal{E}_a(r))r'^2 + 2r^2\mathcal{E}_a(r) - \frac{1 + (1-a)^2r'^2}{1-a}r^2 \sin(\pi a)$$

is negative on $(0, 1)$.

Proof. We denote f and g respectively the functions

$$f(r) = 4(1 - (1-a)^2)\mathcal{E}_a(r) - \frac{2\sin(\pi a)}{1-a} - 2(1-a)\sin(\pi a)(r'^2 - r^2)$$

and

$$g(r) = -8(1 - (1-a)^2)(1-a) \frac{\mathcal{K}_a(r) - \mathcal{E}_a(r)}{r^2} + 8(1-a)\sin(\pi a).$$

Applying the derivative formulas, we obtain

$$u'(r) = rf(r) \quad \text{and} \quad f'(r) = rg(r).$$

Since the function $r \rightarrow (\mathcal{K}_a(r) - \mathcal{E}_a(r))/r^2$ is strictly increasing from $(0, 1)$ onto $(\pi(1-a)/2, \infty)$ (see [2]), the function g is strictly decreasing from $(0, 1)$ onto $(-\infty, (a-1)^2(4a^2 - 8a)\pi + 8(1-a)\sin(\pi a))$. It follows from Lemma 1 that $(a-1)^2(4a^2 - 8a)\pi + 8(1-a)\sin(\pi a) > 0$. We see that there exists a number $r_0 \in (0, 1)$ such that $rg(r)$ is positive on $(0, r_0)$ and negative on $(r_0, 1)$. Hence, the function f is strictly increasing on $(0, r_0)$ and decreasing on $(r_0, 1)$. Since $f(0^+) = 2(1 - (1-a)^2)\pi - 2\sin(\pi a)/(1-a) - 2(1-a)\sin(\pi a) < 0$ by Lemma 2 and $f(1^-) = 0$, we conclude that there exists a number $r_1 \in (0, 1)$ such that $rf(r)$ is negative on $(0, r_1)$ and positive on $(r_1, 1)$. Therefore the function u is strictly decreasing on $(0, r_1)$ and increasing on $(r_1, 1)$. Considering the fact $u(0^+) = 0 = u(1^-)$, we conclude that $u(r) < 0$ for $r \in (0, 1)$. □

Proof of Theorem 1. Given $a \in (0, 1)$. For $r \in (0, 1)$, we construct a function

$$H(r) = \frac{1}{r'^2} \left(\mathcal{E}_a(r) - \frac{\sin(\pi a)}{2(1-a)} \right) - (1-a) \sin(\pi a) \log \frac{e^{R(a)/2}}{r'}.$$

We aim to prove that the function H is strictly decreasing from $(0, 1)$ onto $\left(-\frac{1}{4} \sin(\pi a), -\delta\right)$, where $\delta = \frac{\sin(\pi a)}{2(1-a)} + (1-a) \sin(\pi a) \frac{R(a)}{2} - \frac{\pi}{2}$. In fact, differentiation yields that

$$\begin{aligned} rr'^4 H'(r) &= -2(1-a)(\mathcal{K}_a(r) - \mathcal{E}_a(r))r'^2 + 2r^2 \mathcal{E}_a(r) \\ &\quad - \frac{r^2 \sin(\pi a)}{1-a} - (1-a) \sin(\pi a) r^2 r'^2 < 0 \end{aligned}$$

by Lemma 3. Hence, the function H is strictly decreasing on $(0, 1)$. For the limiting values, it is clear that

$$H(0^+) = -\frac{\sin(\pi a)}{2(1-a)} - (1-a) \sin(\pi a) \frac{R(a)}{2} + \frac{\pi}{2} = -\delta.$$

Write

$$\begin{aligned} h_1(r) &= \mathcal{E}_a(r) - \frac{\sin(\pi a)}{2(1-a)} - (1-a)r'^2 \sin(\pi a) \log \left(\frac{e^{R(a)/2}}{r'} \right), \\ h_2(r) &= r'^2. \end{aligned}$$

Then $h_1(1^-) = 0 = h_2(1^-)$, and by l'Hôpital's rule

$$\begin{aligned} H(1^-) &= \lim_{r \rightarrow 1^-} \frac{h_1'(r)}{h_2'(r)} \\ &= \lim_{r \rightarrow 1^-} \frac{2(1-a)}{2r^2} (\mathcal{K}_a(r) - \mathcal{E}_a(r)) - (1-a) \sin(\pi a) \log \left(\frac{e^{R(a)/2}}{r'} \right) + \frac{1}{4} \sin(\pi a) \\ &= \lim_{r \rightarrow 1^-} -\frac{1}{2} \sin(\pi a) + (1-a)(\mathcal{K}_a(r) - \sin(\pi a) \log \left(\frac{e^{R(a)/2}}{r'} \right)) + \frac{1}{4} \sin(\pi a) \\ &= -\frac{1}{2} \sin(\pi a) + \frac{1}{4} \sin(\pi a) \\ &= -\frac{1}{4} \sin(\pi a), \end{aligned}$$

where we used the fact

$$\lim_{r \rightarrow 1^-} \mathcal{K}_a(r) - \sin(\pi a) \log \left(\frac{e^{R(a)/2}}{r'} \right) = 0.$$

The desired inequalities follow from the monotonicity of the function $H(r)$ and its limiting values. $\square$

Setting $a = 1/2$ in Theorem 1, we have the following inequality for the complete elliptic integral of the second kind.

Corollary 1. *For $r \in (0, 1)$, then*

$$1 + r'^2 \left(\frac{1}{2} \log \frac{4}{r'} - \gamma \right) < \mathcal{E}(r) < 1 + r'^2 \left(\frac{1}{2} \log \frac{4}{r'} - \delta \right),$$

where the constants $\gamma = \frac{1}{4}$ and $\delta = 1 + \log 2 - \frac{\pi}{2}$ are best possible.

Remark 1. Computational and numerical experiments show that, on the whole interval $(0, 1)$, neither the upper bound nor lower bound in Corollary 1 is comparable with the bounds in (1.1) and (1.2).

3. APPLICATION TO THE GENERALIZED HERSCH-PFLUGER FUNCTION

Three related functions μ_a , m_a , and φ_K^a are defined as follows [2]: for $a \in (0, 1/2)$, $r \in (0, 1)$, and $K \in (0, \infty)$,

$$\begin{aligned} \mu_a(r) &= \frac{\pi}{2 \sin \pi a} \frac{\mathcal{K}_a'(r)}{\mathcal{K}_a(r)}, \\ m_a(r) &= \frac{2}{\pi \sin \pi a} r'^2 \mathcal{K}_a(r) \mathcal{K}_a'(r), \\ \varphi_K^a(r) &= \mu_a^{-1}(\mu_a(r)/K). \end{aligned}$$

The function φ_K^a is called the generalized Hersch-Pfluger function. When $a = 1/2$, the function reduces to the Hersch-Pfluger distortion function. These functions have many important applications in the distortion theory of quasiconformal mappings and Ramanujan's modular equations [2–6, 12, 14].

We have the following sharp functional inequalities (see [13] and [10], respectively): for all $r \in (0, 1)$,

$$2(\mathcal{E}(r) - 1) < m_a(r) + \log r < \frac{R(a)}{\pi - 2} (\mathcal{E}(r) - 1), \quad (3.1)$$

and

$$\varphi_K^a(r) < r^{1/K} \exp \left\{ \left(1 - \frac{1}{K} \right) (m_a(r) + \log r) \right\} \quad (3.2)$$

for $K \in [1, \infty)$.

Combining the above inequalities (3.1), (3.2) and Corollary 1, we obtain a sharp functional inequalities for the generalized Hersch-Pfluger function.

Corollary 2. *For $a \in (0, 1/2]$ and $K \in [1, +\infty)$, the inequality*

$$\varphi_K^a(r) < r^{1/K} \exp \left\{ \left(1 - \frac{1}{K} \right) \frac{R(a)}{\pi - 2} r'^2 \left(\frac{1}{2} \log \frac{2}{r'} + \frac{\pi}{2} - 1 \right) \right\}$$

holds for all $r \in (0, 1)$.

ACKNOWLEDGMENTS

The authors are indebted to the anonymous referees for the valuable suggestions.

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OPTIMAL CONTROL PROBLEM GOVERNED BY WEAKLY COUPLED SYSTEM ON INDUCTION HEATING

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Received 14 September, 2020

Abstract. In induction heating process, there is a problem of uniform heating. In this paper, an optimal applied electric field control problem for induction heating process of conductive materials is considered. The cost function is defined such that the temperature profile at the final stage has a relative uniform distribution in the field. The controlled system is a coupled by Maxwell's equations with nonlinear heat equation. The existence of solutions for weakly coupled system is proved. We show that there exists an optimal applied electric field which minimizes the cost functional. Moreover, a first-order necessary condition is derived.

2010 Mathematics Subject Classification: 35Q60, 35K55, 49J20

Keywords: Maxwell's equations, heating equation, coupled system, optimal control, induction heating.

1. INTRODUCTION

Induction heating uses induction eddy current for heating materials. As the characteristics of high heating speed, high heat efficiency and no pollution, induction heating is widely used in metal smelting, casting, welding, heat treatment, hot forging and other modern industrial manufacturing. Meanwhile, the output power of the power source has good controllable conditions. Since 1990s, induction heating and its corresponding control problems have attracted extensive attention (see [1, 17, 18]).

In the process of induction heating, the temperature distribution of the heated object is not uniform. In recent years, many researchers have studied the uniform distribution of heating workpiece temperature and the shortening of heating time through simulation control, which has improved the quality and efficiency of heating (see [14, 21]). However, the mathematical model of induction heating involves the coupling of three-dimensional electromagnetic field and nonlinear temperature field, which is difficult to simulate. Therefore, it is inevitable for the control mode to develop from traditional analog control to digital control. In view of this, we can select suitable applied electric field to improve the uniformity of temperature distribution. In this paper, we formulate this model as an optimal control problem in which the underlying dynamics is governed by Maxwell's equations coupled with nonlinear

heat conduction associated with a heat source generated by induction waves. The goal of the control is to reach a relatively uniform heat profile at a specific time by controlling the applied field on the boundary of the domain.

Optimal control associated with partial differential equations or systems as underlying states have been studied by many researchers. There are several classical books, see e.g., [13, 20], and more recent ones, see e.g., [4, 11, 12, 23]. Especially for optimal control problems of microwave heating, Li, Tang and Yin [9] considered a similar problem where the control is chosen from the heat source. Wei and Yin [10, 22] considered a boundary control problem where the external electric field is chosen as the control function. But induction heating is different from microwave heating, induction heating is a parabolic system with the unknown quantity of magnetic field intensity. Therefore, we study induction heating in different ways.

Since the local induction heating model of conductive materials is expressed by Maxwell's equations with the mixed boundary value coupled with nonlinear heat conduction associated with a heat source in this paper, this poses numerous mathematical challenges. Maxwell's equations in a quasi-stationary electromagnetic field with only one boundary value have been studied by Yin in [6]. But we study the corresponding mixed boundary value problem, which needs to deal with some related problems by introducing two trace operators. In addition, the heat equation is strongly nonlinear. Invoking similar ideas in reference [24], we prove the existence of a solution for the coupled system. One of the difficulties in the present paper is the nonlinear heat source often belongs to $L^1(Q_T)$ for this type of coupled systems. This causes a serious problem when one needs to prove the existence of an optimal control. By applying properties of space $H(\text{curl}, \Omega)$ and Sobolev's imbedding theorems, we can overcome the difficulties to establish the existence result.

This paper is organized as follows. In Section 2, the mathematical model for induction heating is derived. Moreover, we present the optimal control problem, in which the control function is the applied electric field on a partial boundary. In Section 3, we prove that the controlled system has a unique weak solution for any applied electric field. In Section 4, we prove an existence theorem for the optimal control problem. Finally in Section 5 the state variables are differentiable with respect to the control variable. Based on the previous results the necessary condition is derived.

2. THE FORMULATION OF AN OPTIMAL CONTROL FOR UNIFORMLY TEMPERATURE

Recalling Maxwell's equations, a simplified version has already appeared in [5]. Suppose that a targeted substance is placed in an inductive processor cavity, denoted by $\Omega \subset \mathbf{R}^3$ with C^1 -boundary $S = \partial\Omega$. Let $\mathbf{H}(x, t)$ denote the magnetic fields at $x \in \Omega$ and time t . Hereafter, a bold letter represents a vector function in $\mathbf{R}^3$. Noting that the induction material is highly conductive, therefore from electromagnetic theory [2, 16],

Maxwell's equations in Ω can be expressed by

$$\mu \mathbf{H}_t + \nabla \times [\rho(x, t) \nabla \times \mathbf{H}] = 0,$$

where the constant $\mu = \mu_1 - i\mu_2$ ($\mu_1 > 0, \mu_2 > 0$) and $\rho(x, t) = \frac{1}{\sigma(x, t)}$ (σ is the electric conductivity) are a complex value coefficient of permeability and the electric resistivity, respectively.

The heat source produced by Joule's heat is $Q(x, t) = \rho(x, t) |\nabla \times \mathbf{H}|^2$. With this, by using Fourier's law and the conservation of energy, one can easily see that the temperature $u(x, t)$ satisfies a nonlinear heat equation with an internal source generated by induction:

$$\rho_0 c_0 u_t - \nabla \cdot (k(x, u) \nabla u) = \rho(x, t) |\nabla \times \mathbf{H}|^2, (x, t) \in Q_T,$$

where $Q_T = \Omega \times (0, T)$, ρ_0 is the density, c_0 the specific heat, and $k(x, u)$ the heat conductivity.

We sum up the above derivation and normalize certain physical constants to obtain the following mathematical model:

$$\mathbf{H}_t + \nabla \times [\rho(x, t) \nabla \times \mathbf{H}] = 0, \quad (x, t) \in Q_T, \quad (2.1)$$

$$u_t - \nabla \cdot (k(x, u) \nabla u) = \rho(x, t) |\nabla \times \mathbf{H}|^2, \quad (x, t) \in Q_T, \quad (2.2)$$

$$\mathbf{n} \times \mathbf{H} = 0, \quad (x, t) \in S_{\Gamma_1} = \Gamma_1 \times (0, T), \quad (2.3)$$

$$\mathbf{n} \times [\rho(x, t) \nabla \times \mathbf{H}(x, t)] = \mathbf{n} \times \mathbf{G}(x, t), (x, t) \in S_{\Gamma_2} = \Gamma_2 \times (0, T), \quad (2.4)$$

$$\mathbf{H}(x, 0) = \mathbf{H}_0(x), \quad x \in \Omega, \quad (2.5)$$

$$u_{\mathbf{n}}(x, t) = 0, \quad (x, t) \in S_{\Gamma} = \partial\Omega \times (0, T), \quad (2.6)$$

$$u(x, 0) = u_0(x), \quad x \in \Omega, \quad (2.7)$$

where the boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$ is split into two disjoint measurable subsets Γ_1 and Γ_2 , both of which are nonempty, $\mathbf{n}$ is the outward unit normal on $S = \partial\Omega$, $u_{\mathbf{n}} = \nabla u \cdot \mathbf{n}$ is the normal derivative on S , and $\mathbf{G}(x, t)$ is the electric field generated by external optoelectronic devices which is considered as a control variable.

Remark 1. The boundary condition (2.4) on Γ_2 is the external electric field applied to the system [8].

The admissible control set is

$$U_{ad} = \{\mathbf{G} \in L^2(0, T; L^2(\Gamma_2)) : \|\mathbf{G}\|_{L^2(0, T; L^2(\Gamma_2))} \leq A_0 < \infty\},$$

where A_0 is a constant.

Optimal control problem (P): Given $T > 0$ and a desired temperature $u_T(\cdot) \in L^2(\Omega)$ at time T , find an optimal control $\mathbf{G}^* \in U_{ad}$ such that the cost functional

$$J(\mathbf{G}; \mathbf{H}, u) = \frac{1}{2} \int_{\Omega} |u(x, T) - u_T(x)|^2 dx + \frac{\lambda}{2} \int_0^T \int_{\Gamma_2} |\mathbf{G}(x, t)|^2 ds dt \quad (2.8)$$

reaches its minimum at $(\mathbf{H}^*, u^*)$ for all $\mathbf{G} \in U_{ad}$, where $(\mathbf{H}, u)$ and $(\mathbf{H}^*, u^*)$ are weak solutions of the coupled system (2.1)-(2.7) corresponding to $\mathbf{G}$ and $\mathbf{G}^*$, respectively. Where the number $\lambda > 0$ is a typical regularization parameter.

3. EXISTENCE AND UNIQUENESS OF SOLUTION FOR THE UNDERLYING SYSTEM

We recall some standard Banach spaces which we will use in this paper. For convenience, a product space D^n is often simply denoted by D . Let

$$\begin{aligned} H(curl, \Omega) &= \{\mathbf{M} \in L^2(\Omega) : \nabla \times \mathbf{M} \in L^2(\Omega)\}, \\ X &= \{\mathbf{M} \in L^2(\Omega) : \nabla \times \mathbf{M} \in L^2(\Omega), \mathbf{n} \times \mathbf{M} = 0 \text{ on } \Gamma_1\}. \end{aligned}$$

Obviously, X is a linear subspace of $H(curl, \Omega)$. $H(curl, \Omega)$ is a Hilbert space equipped with inner product

$$(\mathbf{M}, \mathbf{N}) = \int_{\Omega} [(\nabla \times \mathbf{M}) \times (\nabla \times \mathbf{N}^*) + \mathbf{M} \cdot \mathbf{N}^*] dx$$

where $\mathbf{N}^*$ represents the complex conjugate of $\mathbf{N}$. A norm on $H(curl, \Omega)$ is given by $\|\cdot\|_{H(curl, \Omega)} = \sqrt{(\cdot, \cdot)}$ (see[19]).

Since the boundary consists of two parts, we need to introduce the two trace mapping $\Upsilon_t : H(curl, \Omega) \rightarrow Y(\partial\Omega)$ and $\Upsilon_T : H(curl, \Omega) \rightarrow Y'(\partial\Omega)$ (the dual space of $Y(\partial\Omega)$) defined by $\Upsilon_t(\mathbf{M}) = \mathbf{n} \times \mathbf{M}|_{\partial\Omega}$ and $\Upsilon_T(\mathbf{M}) = \mathbf{n} \times (\mathbf{n} \times \mathbf{M})|_{\partial\Omega}$ for every $\mathbf{M} \in H(curl, \Omega)$, respectively. Where $\mathbf{n}$ is the above description and $Y(\partial\Omega)$ is a Hilbert space (see[19]) as follow

$$Y(\partial\Omega) = \{\mathbf{f} \in H^{-\frac{1}{2}}(\Omega) : \text{there exists } \mathbf{M} \in H(curl, \Omega) \text{ with } \Upsilon_t(\mathbf{M}) = \mathbf{f}\}.$$

with norm

$$\|\mathbf{f}\|_{Y(\partial\Omega)} = \inf_{\mathbf{M} \in H(curl, \Omega), \Upsilon_t(\mathbf{M}) = \mathbf{f}} \|\mathbf{M}\|_{H(curl, \Omega)}.$$

We impose some basic assumptions which ensure the well-posedness of the underlying system.

- H(1)** Functions $u_0(\cdot)$ and $u_T(\cdot)$ are nonnegative with $u_0(\cdot), u_T(\cdot) \in L^2(\Omega)$.
- H(2)** The function $k(x, u)$ is measurable in x , uniformly Lipschitz continuous with respect to u , and $0 < k_1 \leq k(x, u) \leq k_2$, for positive constants k_1 and k_2 .
- H(3)** (a) The function $\rho(x, t)$ is real, measurable and bounded. Moreover,

$$0 < \rho_1 \leq \rho(x, t) \leq \rho_2$$

for some constants $\rho_1 > 0, \rho_2 > 0$.

- (b) The vector function $\mathbf{H}_0(\cdot) \in L^2(\Omega)$.

In the following, we will show that the system (2.1) and (2.3)-(2.5) have a solution under the condition **H(3)** for any given $\mathbf{G} \in U_{ad}$.

Definition 1. A vector function $\mathbf{H} \in L^2(0, T; X)$ is said to be a weak solution of problem (2.1) and (2.3)-(2.5) if

$$\begin{aligned} & - \int_0^T \int_{\Omega} \mathbf{H} \cdot \Phi_t dx dt + \int_0^T \int_{\Omega} \rho(x, t) (\nabla \times \mathbf{H}) \cdot (\nabla \times \Phi) dx dt \\ & = \int_{\Omega} \mathbf{H}_0 \cdot \Phi(x, 0) dx - \int_0^T \langle \mathbf{n} \times \mathbf{G}, \Upsilon_T(\Phi) \rangle_{\Gamma_2} dt, \end{aligned}$$

for any vector function $\Phi \in H^1(0, T; X)$ with $\Phi(x, T) = 0$ a.e. $x \in \Omega$.

Remark 2. Note that $\partial\Omega = \Gamma_1 \cup \Gamma_2$, we need to consider the boundary conditions in Definition 1. On Γ_1 the boundary condition (2.3) gives no information about $\mathbf{n} \times [\rho(x, t) \nabla \times \mathbf{H}]$, so we choose Φ such that $\Upsilon_T(\Phi) = 0$ on Γ_1 . Using this fact, the integral of boundary contains only the portion on Γ_2 .

We will use the Galerkin's method to derive the existence of solutions for the system (2.1).

Note that we have a compactness embedding $H(\text{curl}, \Omega) \hookrightarrow L^2(\Omega)$ and X is a linear subspace of $H(\text{curl}, \Omega)$, we may choose a countable set of linearly independent elements $\{\mathbf{w}_k(x)\}_{k=1}^{\infty}$ in X . After possibly performing an orthogonalization process with respect to the scalar product of $L^2(\Omega)$, we may assume that $\{\mathbf{w}_k(x)\}_{k=1}^{\infty}$ forms an orthonormal system in $L^2(\Omega)$ which is also complete in $L^2(\Omega)$.

Now fix a positive integer m , we will look for a vector-valued function $\mathbf{H}_m : [0, T] \rightarrow X$ of the form

$$\mathbf{H}_m(t) = \sum_{k=1}^m d_m^k(t) \mathbf{w}_k. \quad (3.1)$$

We hope to select the coefficients $d_m^k(t) (0 \leq t \leq T, k = 1, 2, \dots, m)$ so that

$$\left(\frac{d\mathbf{H}_m(t)}{dt}, \mathbf{w}_k \right) + a(\mathbf{H}_m, \mathbf{w}_k; t) = -\langle \mathbf{n} \times \mathbf{G}, \Upsilon_T(\mathbf{w}_k) \rangle_{\Gamma_2}, \quad (3.2)$$

$$d_m^k(0) = (\mathbf{H}_0, \mathbf{w}_k), \quad (3.3)$$

where we write $(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle_{\Gamma_2}$ for two scalar products, respectively, in $L^2(\Omega)$ and $L^2(\Gamma_2)$, $a(\mathbf{H}_m, \mathbf{w}_k; t) = \int_{\Omega} \rho(x, t) (\nabla \times \mathbf{H}) \cdot (\nabla \times \mathbf{w}_k) dx dt$.

Theorem 1. (Galerkin approximate). For each integer $m = 1, 2, \dots$, there exists a unique vector function $\mathbf{H}_m$ of the form (3.1) satisfying (3.2) and (3.3).

Proof. Assuming $\mathbf{H}_m$ has the structure (3.1), we first note that

$$\left(\frac{d\mathbf{H}_m(t)}{dt}, \mathbf{w}_k \right) = \left(d_m^k(t), \mathbf{w}_k \right)'$$

Furthermore

$$a(\mathbf{H}_m, \mathbf{w}_k; t) = \sum_{l=1}^m e^{kl} d_m^l(t)$$

for $e^{kl} = a(\mathbf{w}_l, \mathbf{w}_k; t)$, $(k, l = 1, 2, \dots, m)$. Let us further write

$$b^k(t) := -\langle \mathbf{n} \times \mathbf{G}, \Upsilon_T(\mathbf{w}_k) \rangle_{\Gamma_2} \quad (k = 1, 2, \dots, m).$$

Then (3.2) becomes the linear system of ODE

$$\left(d_m^k(t) \right)' + \sum_{l=1}^m e^{kl} d_m^l(t) = b^k(t) \quad (k = 1, 2, \dots, m),$$

subject to the initial condition (3.3). Owing to Carathéodory's theorem, this initial value problem for a system of n linear ordinary differential equations on $[0, T]$ for the unknown vector function $d^m = (d_m^1(t), d_m^2(t), \dots, d_m^m(t))$ has a unique absolutely continuous solution $d^m \in X^m$. Then $\mathbf{H}_m$ defined by (3.1) satisfying (3.2) and (3.3). $\square$

Theorem 2. (Estimates for $\{\mathbf{H}_m\}$.) *There exists constant C_3 , depending only on Ω, T and the known data $\mathbf{p}_1$, such that*

$$\|\mathbf{H}_m\|_{C([0,T];L^2(\Omega))} + \|\mathbf{H}_m\|_{L^2(0,T;H(\text{curl},\Omega))} \leq C_3(\|\mathbf{H}_0\|_{L^2(\Omega)} + \|\mathbf{G}\|_{L^2(0,T;L^2(\Gamma_2))}). \quad (3.4)$$

Proof. Multiplying (3.2) by $d_m^k(t)$ and adding the resulting equations from $k = 1$ to $k = m$, we obtain, for almost every $t \in (0, T)$,

$$\left(\frac{d\mathbf{H}_m}{dt}, \mathbf{H}_m \right) + a(\mathbf{H}_m, \mathbf{H}_m; t) = -\langle \mathbf{n} \times \mathbf{G}, \Upsilon_T(\mathbf{H}_m) \rangle_{\Gamma_2}. \quad (3.5)$$

For any arbitrary but fixed $\tau \in (0, T]$, we have the identity

$$\int_0^\tau \left(\frac{d\mathbf{H}_m(t)}{dt}, \mathbf{H}_m(t) \right) dt = \frac{1}{2} \|\mathbf{H}_m(\tau)\|_{L^2(\Omega)}^2 - \frac{1}{2} \|\mathbf{H}_m(0)\|_{L^2(\Omega)}^2.$$

Integration of (3.5) over $[0, \tau]$ therefore yields that

$$\begin{aligned} \frac{1}{2} \|\mathbf{H}_m(\tau)\|_{L^2(\Omega)}^2 + \int_0^\tau a[\mathbf{H}_m(t), \mathbf{H}_m(t); t] dt \\ = \frac{1}{2} \|\mathbf{H}_m(0)\|_{L^2(\Omega)}^2 - \int_0^\tau \langle \mathbf{n} \times \mathbf{G}, \Upsilon_T(\mathbf{H}_m) \rangle_{\Gamma_2} dt. \end{aligned} \quad (3.6)$$

By virtue of Bessel's inequality, we have

$$\|\mathbf{H}_m(0)\|_{L^2(\Omega)}^2 = \sum_{k=1}^m |d_m^k(0)|^2 = \sum_{k=1}^m |(\mathbf{H}_0, \mathbf{w}_k)|^2 \leq \|\mathbf{H}_0\|_{L^2(\Omega)}^2. \quad (3.7)$$

Choosing δ sufficiently small and using the assumption **H(3)**, we have

$$\begin{aligned} |\langle \mathbf{n} \times \mathbf{G}, \Upsilon_T(\mathbf{H}_m) \rangle| &\leq \delta \|\mathbf{G}\|_{L^2(\Gamma_2)}^2 + C(\delta) \|\mathbf{H}_m\|_{L^2(\Gamma_2)}^2 \\ &\leq \delta \|\mathbf{G}\|_{L^2(\Gamma_2)}^2 + C_0 C(\delta) \|\mathbf{H}_m\|_{H(\text{curl},\Omega)}^2. \end{aligned} \quad (3.8)$$

for almost every $t \in (0, T)$. Invoking Gronwall's inequality, we can infer from (3.6)-(3.8) that

$$\max_{t \in [0, T]} \|\mathbf{H}_m\|_{L^2(\Omega)} \leq K \|\mathbf{H}_0\|_{L^2(\Omega)} + \|\mathbf{G}\|_{L^2(0, T; L^2(\Gamma_2))}.$$

It's easy to get inequalities $C \|\mathbf{H}\|_{H(\text{curl}, \Omega)}^2 \leq a(\mathbf{H}, \mathbf{H}; t) + \|\mathbf{H}\|_{L^2(\Omega)}^2$, where C depends on known data ρ_1 . Inserting this estimate for $\mathbf{H}_m$ in $C([0, T], L^2(\Omega))$ into (3.6) with $\tau = T$, we see that there is a constant $C_3 > 0$ such that

$$\|\mathbf{H}_m\|_{C([0, T]; L^2(\Omega))} + \|\mathbf{H}_m\|_{L^2(0, T; H(\text{curl}, \Omega))} \leq C_3 (\|\mathbf{H}_0\|_{L^2(\Omega)} + \|\mathbf{G}\|_{L^2(0, T; L^2(\Gamma_2))}).$$

□

Theorem 3. (*Existence and uniqueness of weak solution*). Assume that condition **H(3)** holds, the problem (2.1) and (2.3)-(2.5) has a unique weak solution $\mathbf{H} \in L^2(0, T; X)$.

Proof. First, we shall show that the existence of weak solution. From (3.4) and in view of the orthonormality, we have

$$\sum_{k=1}^m |d_m^k(t)|^2 = \|\mathbf{H}_m(t)\|_{L^2(\Omega)}^2 \leq C^2, \forall t \in [0, T], m \in \mathbf{N}, \quad (3.9)$$

where C is a constant. Hence, we have $|d_m^k(t)| \leq C$ for all t, k , and m . Moreover, it follows from (3.2) by integration over time that for any fixed $k \in \mathbf{N}$ the sequence $\{d_m^k(t)\}_{m=1}^\infty$ forms an equicontinuous set in $C[0, T]$. Recalling the Arzelà-Ascoli theorem, there exists a subsequence of $\{d_m^k(t)\}_{m=1}^\infty$, denoted by $\{d_{m_l}^k(t)\}_{m=1}^\infty$, such that

$$\lim_{l \rightarrow \infty} d_{m_l}^k(t) = d^k(t) \text{ strongly in } C[0, T] \quad \forall k \in \mathbf{N}.$$

With the limit functions $d^k(t)$, we define the function

$$\mathbf{H}(x, t) := \sum_{k=1}^\infty d^k(t) \mathbf{w}_k(x), (x, t) \in Q_T.$$

It can be shown that $\mathbf{H}_{m_l}(\cdot, t) \rightarrow \mathbf{H}(\cdot, t)$ weakly in $L^2(\Omega)$, uniformly with respect to $t \in [0, T]$. Owing to the weak lower sequential semicontinuity of the norm, we infer from the estimate (3.4) that

$$\|\mathbf{H}\|_{L^2(\Omega)} \leq C(\|\mathbf{G}\|_{L^2(0, T; L^2(\Gamma_2))}^2 + \|\mathbf{H}_0\|_{L^2(\Omega)}^2)$$

for almost all t , which means that $\mathbf{H} \in L^\infty(0, T; L^2(\Omega))$.

Since $\sum_{k=1}^\infty |d_m^k(0)|^2 < \infty$ by (3.7), we have

$$\|\mathbf{H}_{m_l}(0) - \mathbf{H}_0\|_{L^2(\Omega)} = \left\| \sum_{k=1}^{m_l} d_{m_l}^k(0) \mathbf{w}_k - \sum_{k=1}^\infty (\mathbf{H}_0, \mathbf{w}_k) \mathbf{w}_k \right\|_{L^2(\Omega)} = \sum_{k=m_l+1}^\infty |d_{m_l}^k(0)|^2 \rightarrow 0$$

as $l \rightarrow \infty$. This means that $\mathbf{H}_{m_l}(0)$ converges strongly in $L^2(\Omega)$ to $\mathbf{H}_0(0)$.

From (3.4), we can assume that $\mathbf{H}_{m_l}(t) \rightarrow \mathbf{H}$ weakly in $L^2(0, T; H(\text{curl}, \Omega))$. We take as test function in (3.2) any function $\Phi_n(x, t)$ of the form

$$\Phi_n(x, t) = \sum_{j=1}^n \alpha_j(t) \mathbf{w}_j(x), n \leq m$$

where $\alpha_j \in C^1[0, T]$ satisfies $\alpha_j(T) = 0$ for $1 \leq j \leq n$. It then follows from (3.5) that

$$\left(\frac{d\mathbf{H}_{m_l}}{dt}, \Phi_n(x, t) \right) + a(\mathbf{H}_{m_l}, \Phi_n; t) = -\langle \mathbf{n} \times \mathbf{G}, \Upsilon_T(\Phi_n) \rangle_{\Gamma_2}.$$

Furthermore, we integrate over $[0, T]$ by parts,

$$\begin{aligned} - \int_0^T \left(\mathbf{H}_{m_l}, \frac{d}{dt} \Phi_n(x, t) \right) dt + \int_0^T a(\mathbf{H}_{m_l}, \Phi_n; t) dt \\ = - \int_0^T \langle \mathbf{n} \times \mathbf{G}, \Upsilon_T(\Phi_n) \rangle_{\Gamma_2} dt + (\mathbf{H}_{m_l}(0), \Phi_n(0)). \end{aligned}$$

Note that $\mathbf{H}_{m_l} \rightarrow \mathbf{H}$ weakly in $L^2(0, T; H(\text{curl}, \Omega))$ and $\mathbf{H}_{m_l}(0) \rightarrow \mathbf{H}_0$ strongly in $L^2(\Omega)$. Passage to the limit as $l \rightarrow \infty$ in the above equation therefore yields

$$\begin{aligned} - \int_0^T \left(\mathbf{H}, \frac{d}{dt} \Phi_n(x, t) \right) dt + \int_0^T a(\mathbf{H}, \Phi_n; t) dt \\ = - \int_0^T \langle \mathbf{n} \times \mathbf{G}, \Upsilon_T(\Phi_n) \rangle_{\Gamma_2} dt + (\mathbf{H}_0, \Phi_n(0)). \end{aligned}$$

Since the set of all functions Φ_n is dense in the class of all functions from $H^1(0, T; H(\text{curl}, \Omega))$, $\mathbf{H}$ satisfies the variational formulation of Definition 1 and is thus a weak solution.

Finally, we shall show that the uniqueness of weak solution. Note again that $\{\mathbf{H}_{m_l}(t)\}_{l=1}^\infty$ converges weakly in $L^2(0, T; H(\text{curl}, \Omega))$ to $\mathbf{H}$ and the norm has the weak lower sequential semicontinuity, we infer from the estimate (3.4) that

$$\|\mathbf{H}\|_{L^2(0, T; H(\text{curl}, \Omega))} \leq C_4(\|\mathbf{G}\|_{L^2(0, T; L^2(\Gamma_2))}^2 + \|\mathbf{H}_0\|_{L^2(\Omega)}^2). \quad (3.10)$$

To prove the uniqueness, it suffices to check that the only weak solution of (2.1) and (2.3)-(2.5) with $\mathbf{H}_0 \equiv \mathbf{G} \equiv \mathbf{0}$ is $\mathbf{H} \equiv \mathbf{0}$. However, this is obvious from (3.10). $\square$

Invoking (3.4) and (3.10), we immediately obtain the following theorem.

Theorem 4. *Under the assumption H(3), the problem (2.1) and (2.3)-(2.5) has a unique weak solution $\mathbf{H} \in L^2(0, T; X)$ and the following estimates is hold.*

$$\sup_{t \in [0, T]} \|\mathbf{H}(\cdot, t)\|_{L^2(\Omega)} + \|\mathbf{H}\|_{L^2(0, T; H(\text{curl}, \Omega))} \leq C_5, \quad (3.11)$$

where C_5 depend only on known data.

Next fix the weak $\mathbf{H} \in L^2(0, T; X)$ of problem (2.1) and (2.3)-(2.5), we now turn to the problem (2.2), (2.6) and (2.7). Let $V = H^1(\Omega)$, $H = L^2(\Omega)$, then $V^* = H^{-1}(\Omega)$ and $V \hookrightarrow H \hookrightarrow V^*$ is an evolution triple.

Define space $W(0, T)$ as

$$W(0, T) = \{u \in L^2(0, T; V) : u_t \in L^2(0, T; V^*)\},$$

with the norm

$$\|u\|_{W(0, T)}^2 = \|u\|_{L^2(0, T; V)}^2 + \|u_t\|_{L^2(0, T; V^*)}^2.$$

Then $W(0, T)$ is a Banach space, $W(0, T) \hookrightarrow C([0, T]; H)$ is continuous and $W(0, T) \hookrightarrow L^2(0, T; H)$ is compact, where $\frac{d}{dt}$ means the generalized derivative of real functions on $[0, T]$.

For almost every $t \in [0, T]$, we can see that $\nabla \times \mathbf{H}(\cdot, t) \in H(\text{curl}, \Omega)$ when $\mathbf{H}(\cdot, t)$ lies in $X \subset H(\text{curl}, \Omega)$ (see [19]). By using Sobolev's embedding with the dimension $N = 3$, we know that the $\nabla \times \mathbf{H}(\cdot, t) \in L^6(\Omega)$. Finally, recalling assumption **H(3)**, we have that $F(t) = \rho(\cdot, t)|\nabla \times \mathbf{H}(\cdot, t)|^2 \in L^2(\Omega)$. With this result, the weak solution of the problem (2.2), (2.6) and (2.7) will be well defined.

Theorem 5. (see [3, 24]) Assume that conditions **H(1)–H(3)** hold, the system (2.2), (2.6) and (2.7) has unique weak solution $u \in W(0, T)$ for any given $\mathbf{H} \in L^2(0, T; X)$, which satisfies the following integral identity

$$\frac{d}{dt} \int_{\Omega} u(x, t)v(x)dx + \int_{\Omega} k(x, u)\nabla u(x, t) \cdot \nabla v(x)dx = \int_{\Omega} \rho(x, t)|\nabla \times \mathbf{H}(x, t)|^2 v(x)dx$$

with the initial condition $u(0) = u_0$, for all $v \in V$ and for almost all $t \in [0, T]$, where $\frac{d}{dt}$ means the generalized derivative.

Moreover, we have estimate

$$\|u\|_{W(0, T)} \leq C_6, \quad (3.12)$$

where positive constant C_6 depends on known data.

Summarizing the above considerations, we obtain the existence of solutions of the coupled system (2.1)-(2.7).

Theorem 6. If the assumptions **H(1)–H(3)** hold, then the coupled system (2.1)-(2.7) has a unique weak solution $(\mathbf{H}, u) \in L^2(0, T; X) \times W(0, T)$.

4. EXISTENCE OF AN OPTIMAL CONTROL

In this section, we will prove the existence of an optimal control for problem **(P)**. Invoking Theorem 6, for any fixed $\mathbf{G} \in L^2(0, T; L^2(\Gamma_2))$, there is unique weak solution $(\mathbf{H}, u) \in L^2(0, T; X) \times W(0, T)$. Hence we can define an control-to-state mapping $P : \mathbf{G} \rightarrow (\mathbf{H}, u)$ from $L^2(0, T; L^2(\Gamma_2))$ into $L^2(0, T; X) \times W(0, T)$ for the system (2.1)-(2.7).

Theorem 7. Under the assumptions **H(1)**-**H(3)**, and for any fixed $\mathbf{G} \in U_{ad}$, the control-to-state mapping P is weakly sequentially continuous.

Proof. Let a sequences $\{\mathbf{G}_m\}_{m=1}^\infty \subset L^2(0, T; L^2(\Gamma_2))$ such that

$$\mathbf{G}_m \rightarrow \mathbf{G}^* \text{ weakly in } L^2(0, T; L^2(\Gamma_2)), \text{ as } m \rightarrow +\infty. \quad (4.1)$$

We shall show that

$$(\mathbf{H}_m, u_m) = P(\mathbf{G}_m) \rightarrow P(\mathbf{G}^*) = (\mathbf{H}^*, u^*) \text{ weakly in } L^2(0, T; X) \times W(0, T), \text{ as } m \rightarrow \infty.$$

According to the definition of mapping P , $(\mathbf{H}_m, u_m)$ is the weak solution of system (2.1)-(2.7) corresponding to the control $\mathbf{G}_m$ for $m = 1, 2, \dots$. That is,

$$\begin{cases} -\int_0^T \int_\Omega \mathbf{H}_m \cdot \Phi_t dx dt & + \int_0^T \int_\Omega \rho(x, t) \nabla \times \mathbf{H}_m \cdot \nabla \times \Phi dx dt \\ & = \int_\Omega \mathbf{H}_m(x, 0) \cdot \Phi(x, 0) dx - \int_0^T \langle \mathbf{n} \times \mathbf{G}_m, \Upsilon_T(\Phi) \rangle_{\Gamma_2} dt, \\ \mathbf{n} \times \mathbf{H}_m(x, t) & = 0, \quad (x, t) \in S_{\Gamma_1}, \\ \mathbf{n} \times [\rho(x, t) \nabla \times \mathbf{H}_m] & = \mathbf{n} \times \mathbf{G}_m(x, t), \quad (x, t) \in S_{\Gamma_2}, \\ \mathbf{H}_m(x, 0) & = \mathbf{H}_0(x), \quad x \in \Omega, \\ \frac{d}{dt} \int_\Omega u_m v dx + \int_\Omega k(x, u_m) \nabla u_m \cdot \nabla v dx & = \int_\Omega \rho(x, t) |\nabla \times \mathbf{H}_m|^2 v dx, \quad \text{a.e. } t \in [0, T], \\ (u_m)_n(x, t) & = 0, \quad (x, t) \in S_\Gamma, \\ u_m(x, 0) & = u_0(x), \quad x \in \Omega, \end{cases}$$

where $\Phi \in H^1(0, T; X)$, $v \in V = H^1(\Omega)$.

From Theorems 4 and 5, $(\mathbf{H}_m, u_m)$ is bounded in reflexive space $L^2(0, T; H(\text{curl}, \Omega)) \times W(0, T)$. Hence, there exists a subsequence of $(\mathbf{H}_m, u_m)$, again denoted by $(\mathbf{H}_m, u_m)$, such that

$$\mathbf{H}_m \rightarrow \mathbf{H}^* \text{ weakly in } L^2(0, T; H(\text{curl}, \Omega)), \quad (4.2)$$

$$u_m \rightarrow u^* \text{ weakly in } L^2(Q_T), \quad (4.3)$$

$$\nabla u_m \rightarrow \nabla u^* \text{ weakly in } L^2(0, T; L^2(\Omega)), \quad (4.4)$$

$$\nabla \times \mathbf{H}_m \rightarrow \nabla \times \mathbf{H}^* \text{ weakly in } L^2(0, T; L^2(\Omega)). \quad (4.5)$$

Moreover, the compactness embedding $H(\text{curl}, \Omega) \hookrightarrow L^2(\Omega)$ and $W(0, T) \hookrightarrow L^2(0, T; L^2(\Omega))$ imply that

$$\mathbf{H}_m \rightarrow \mathbf{H}^* \text{ strongly in } L^2(0, T; L^2(\Omega)), \quad (4.6)$$

$$u_m \rightarrow u^* \text{ strongly in } L^2(Q_T). \quad (4.7)$$

The definition of the spaces $L^6(\Omega)$ and $\nabla \times \mathbf{H}_m(\cdot, t) \in L^6(\Omega)$ imply that

$$\nabla \times \mathbf{H}_m(\cdot, t) \rightarrow \nabla \times \mathbf{H}^*(\cdot, t) \text{ weakly in } L^6(\Omega). \quad (4.8)$$

From (4.6), we know

$$-\lim_{m \rightarrow \infty} \int_0^T \int_\Omega \mathbf{H}_m \cdot \Phi_t dx dt = - \int_0^T \int_\Omega \mathbf{H}^* \cdot \Phi_t dx dt,$$

$$\lim_{m \rightarrow \infty} \int_{\Omega} \mathbf{H}_m(x, 0) \cdot \Phi(x, 0) dx = \int_{\Omega} \mathbf{H}_0^*(x) \cdot \Phi(x, 0) dx.$$

Computing

$$\begin{aligned} & \left| \int_0^T \int_{\Omega} \rho(x, t) (\nabla \times \mathbf{H}_m) \cdot (\nabla \times \Phi) dx dt - \int_0^T \int_{\Omega} \rho(x, t) (\nabla \times \mathbf{H}^*) \cdot (\nabla \times \Phi) dx dt \right| \\ & \leq \rho_2 \int_0^T \int_{\Omega} |(\nabla \times \mathbf{H}_m - \nabla \times \mathbf{H}^*) \cdot (\nabla \times \Phi)| dx dt. \end{aligned}$$

and using (4.5), we have

$$\int_0^T \int_{\Omega} (\nabla \times \mathbf{H}_m) \cdot (\nabla \times \Phi) dx dt \rightarrow \int_0^T \int_{\Omega} (\nabla \times \mathbf{H}^*) \cdot (\nabla \times \Phi) dx dt$$

as $m \rightarrow \infty$.

We can infer from (4.1) that

$$\int_0^T \langle \mathbf{n} \times \mathbf{G}_m, \Upsilon_T(\Phi)_T \rangle_{\Gamma_2} dt \rightarrow \int_0^T \langle \mathbf{n} \times \mathbf{G}^*, \Upsilon_T(\Phi)_T \rangle_{\Gamma_2} dt$$

as $m \rightarrow \infty$.

By performing integration by parts, for all $\eta \in C^\infty(0, T)$ and $\Psi \in H^1(\Omega)$, we have

$$\begin{aligned} & \int_0^T \int_{\Omega} [(\mathbf{H}_m)_t \cdot \eta_t \Psi(x) + \mathbf{H}_m \cdot \eta_t(t) \Psi(x)] dx dt \\ & = \int_{\Omega} \mathbf{H}_m(x, T) \cdot \eta(T) \Psi(x) dx - \int_{\Omega} \mathbf{H}_m(x, 0) \cdot \eta(0) \Psi(x) dx. \end{aligned} \quad (4.9)$$

Passage to the limit as $m \rightarrow \infty$ in (4.9) therefore yields

$$\begin{aligned} & \int_{\Omega} \mathbf{H}^*(x, T) \cdot \eta_t(T) \Psi(x) dx - \int_{\Omega} \mathbf{H}_0(x) \cdot \eta_t(0) \Psi(x) dx \\ & = \int_{\Omega} \mathbf{H}^*(x, T) \cdot \eta(T) \Psi(x) dx - \int_{\Omega} \mathbf{H}^*(x, 0) \cdot \eta(0) \Psi(x) dx. \end{aligned} \quad (4.10)$$

Taking $\eta(T) = 0$ and $\eta(0) = 1$ in (4.10), we have

$$\int_{\Omega} [\mathbf{H}_0(x) - \mathbf{H}^*(x, 0)] \cdot \eta(0) \Psi(x) dx = 0.$$

Since $H^1(\Omega)$ is dense in $L^2(\Omega)$, this implies

$$\mathbf{H}^*(\cdot, 0) = \mathbf{H}_0(\cdot) \text{ in } L^2(\Omega).$$

Owing to the continuity of embedding $L^6(\Omega) \hookrightarrow L^4(\Omega)$ and (4.8), for any $t \in [0, T]$, we have

$$\int_{\Omega} \left| |\nabla \times \mathbf{H}_m(x, t)|^2 - |\nabla \times \mathbf{H}^*(x, t)|^2 \right| v(x) dx \rightarrow 0 \text{ for all } v \in H^1(\Omega).$$

Now recall that $0 < \rho_1 \leq \rho(x, t) \leq \rho_2$. Then, for any $t \in [0, T]$, we obtain

$$\left| \int_{\Omega} \rho(x, t) |\nabla \times \mathbf{H}_m(x, t)|^2 v(x) dx - \int_{\Omega} \rho(x, t) |\nabla \times \mathbf{H}^*(x, t)|^2 v(x) dx \right|$$

$$\leq \rho_2 \int_{\Omega} \left| \left(|\nabla \times \mathbf{H}_m(x, t)|^2 - |\nabla \times \mathbf{H}^*(x, t)|^2 \right) v(x) \right| dx \rightarrow 0, \quad (4.11)$$

as $m \rightarrow \infty$. From Majorized convergence theorem and (4.11), for all $\varphi \in C_0^\infty(0, T)$, we have

$$\begin{aligned} & \lim_{m \rightarrow \infty} \int_0^T \int_{\Omega} \rho(x, t) |\nabla \times \mathbf{H}_m(x, t)|^2 v(x) \varphi(t) dx dt \\ &= \int_0^T \int_{\Omega} \rho(x, t) |\nabla \times \mathbf{H}^*(x, t)|^2 v(x) \varphi(t) dx dt. \end{aligned}$$

We can obtain from (4.7) that

$$-\lim_{m \rightarrow \infty} \int_0^T \int_{\Omega} u_m(x, t) v(x) \varphi_t(t) dx dt = - \int_0^T \int_{\Omega} u^*(x, t) v(x) \varphi_t(t) dx dt$$

for all $\varphi \in C_0^\infty(0, T)$ and $v \in H^1(\Omega)$ (see [24]).

By virtue of $\mathbf{H}(2)$, (4.4) and (4.7), one can easy to see that

$$\begin{aligned} & \lim_{m \rightarrow \infty} \int_0^T \int_{\Omega} k(x, u_m) \nabla u_m(x, t) \cdot \nabla v(x) \varphi(t) dx dt \\ &= \int_0^T \int_{\Omega} k(x, u^*) \nabla u^*(x, t) \cdot \nabla v(x) \varphi(t) dx dt \end{aligned}$$

for all $\varphi \in C_0^\infty(0, T)$ and $v \in H^1(\Omega)$ (see [24]).

Since the condition $u^*(\cdot, 0) = u_0(\cdot) \in L^2(\Omega)$ has the same structure as condition $\mathbf{H}^*(\cdot, 0) = \mathbf{H}_0(\cdot) \in L^2(\Omega)$, it may be shown by the same way.

Therefore, $(\mathbf{H}^*, u^*)$ is a weak solution of (2.1)-(2.7) corresponding to $\mathbf{G}^*$. That is, $P(\mathbf{G}^*) = (\mathbf{H}^*, u^*)$. In the step above we have shown that $(\mathbf{H}_m, u_m) \rightarrow (\mathbf{H}^*, u^*)$ for a subsequence. With the arguments above, we can prove that every subsequence has a subsequence converging to the same $(\mathbf{H}^*, u^*)$. Therefore, the entire sequence $(\mathbf{H}_m, u_m)$ converges to $(\mathbf{H}^*, u^*)$ weakly. $\square$

Theorem 8. (Existence of an Optimal Control). Under the assumptions $\mathbf{H}(1)$ - $\mathbf{H}(3)$, there exists at least one global minimizer $(\mathbf{G}^*; \mathbf{H}^*, u^*)$ of the problem (P) such that

$$\mathbf{G}^* \in U_{ad}, \mathbf{H}^* \in L^2(0, T; X), u^* \in W(0, T).$$

Proof. Using the control-to-state map $P : \mathbf{G} \rightarrow (\mathbf{H}, u)$, we reduce the functional $J(\mathbf{G})$ in (2.8) such that $J(\mathbf{G}) := J(\mathbf{G}; \mathbf{H}, u)$. Obviously, $J(\mathbf{G})$ is bounded from (2.8). Hence, we get the existence of an infimum J , that is,

$$J^* := \inf_{\mathbf{G} \in U_{ad}} J(\mathbf{G}) \in \mathbf{R}.$$

Put $\{\mathbf{G}_m\}_{m=1}^\infty \subset U_{ad}$ be a minimizing sequence such that $\lim_{m \rightarrow \infty} J(\mathbf{G}_m) = J^*$.

Since $\{\mathbf{G}_m\}_{m=1}^\infty \subset U_{ad}$ and $\|\mathbf{G}_m\|_{L^2(0, T; L^2(\Gamma_2))} < \infty$, there exists a subsequence of $\{\mathbf{G}_m\}_{m=1}^\infty \subset U_{ad}$, again denoted by $\{\mathbf{G}_m\}_{m=1}^\infty \subset U_{ad}$, such that

$$\mathbf{G}_m \rightarrow \mathbf{G}^* \text{ weakly in } L^2(0, T; L^2(\Gamma_2)).$$

where $\mathbf{G}^* \in U_{ad}$ from the Mazur's lemma.

Observe that J is weakly sequentially lower semi-continuous in (2.8) and P is weakly sequentially continuous from Theorem 7. Therefore, we find

$$J^* = \lim_{m \rightarrow \infty} J(\mathbf{G}_m) \geq J(\mathbf{G}^*) \geq J^*.$$

This means that $\mathbf{G}^*$, also $(\mathbf{G}^*, \mathbf{H}^*, u^*)$ is a global minimizer. $\square$

5. NECESSARY OPTIMALITY CONDITION

Next, we will show the necessary optimality condition of (2.1)–(2.7). First, a variational inequality will be derived that still involves the states $\mathbf{H}$ and u ; then, $\mathbf{H}$ and u will be eliminated by means of the adjoint states to deduce a variational inequality for the control.

Theorem 9. *Suppose that the assumption $\mathbf{H}(3)$ holds. Then the mapping $\Lambda_1 : U_{ad} \rightarrow L^2(0, T; X)$ is differentiable and the following limit*

$$\hat{\mathbf{H}}(\mathbf{G}; \mathbf{K}) = \lim_{\varepsilon \rightarrow 0} \frac{\mathbf{H}(\mathbf{G} + \varepsilon \mathbf{K}) - \mathbf{H}(\mathbf{G})}{\varepsilon} \text{ in } L^2(0, T; X),$$

exists for any $\mathbf{G}, \mathbf{K} \in U_{ad}$ such that $\mathbf{G} + \varepsilon \mathbf{K} \in U_{ad}$ for small ε . Moreover, in the weak sense, $\hat{\mathbf{H}}$ satisfies

$$\begin{cases} \hat{\mathbf{H}}_t + \nabla \times [\rho(x, t) \nabla \times \hat{\mathbf{H}}] = 0, (x, t) \in Q_T, \\ \mathbf{n} \times \hat{\mathbf{H}} = 0, (x, t) \in S_{\Gamma_1}, \\ \mathbf{n} \times [\rho(x, t) \nabla \times \hat{\mathbf{H}}] = \mathbf{n} \times \mathbf{K}, (x, t) \in S_{\Gamma_2}, \\ \hat{\mathbf{H}}(x, 0) = 0, x \in \Omega, \end{cases} \quad (5.1)$$

Proof. Let $\mathbf{H} := \mathbf{H}(\mathbf{G})$; $\mathbf{H}_\varepsilon := \mathbf{H}(\mathbf{G} + \varepsilon \mathbf{K})$ and put $\hat{\mathbf{H}}_\varepsilon = (\mathbf{H}_\varepsilon - \mathbf{H})/\varepsilon$, $(x, t) \in Q_T$, where $\mathbf{H}_\varepsilon$ and $\mathbf{H}$ is a solutions of (2.1) and (2.3)–(2.7) corresponding to $\mathbf{G} + \varepsilon \mathbf{K}$ and $\mathbf{G}$, respectively. By computing, in the weak sense, we easily obtain that $\hat{\mathbf{H}}_\varepsilon$ satisfies

$$\begin{cases} (\hat{\mathbf{H}}_\varepsilon)_t + \nabla \times [\rho(x, t) \nabla \times \hat{\mathbf{H}}_\varepsilon] = 0, (x, t) \in Q_T, \\ \mathbf{n} \times \hat{\mathbf{H}}_\varepsilon = 0, (x, t) \in S_{\Gamma_1}, \\ \mathbf{n} \times [\rho(x, t) \nabla \times \hat{\mathbf{H}}_\varepsilon] = \mathbf{n} \times \mathbf{K}, (x, t) \in S_{\Gamma_2}, \\ \hat{\mathbf{H}}_\varepsilon(x, 0) = 0, x \in \Omega, \end{cases} \quad (5.2)$$

As several estimates were shown in Theorem 3.7, therefore under the condition $\mathbf{H}(3)$, $\hat{\mathbf{H}}_\varepsilon$ can be estimated similarly:

$$\sup_{t \in [0, T]} \|\hat{\mathbf{H}}_\varepsilon\|_{L^2(\Omega)} + \|\hat{\mathbf{H}}_\varepsilon\|_{L^2(0, T; H(\text{curl}, \Omega))} \leq C_7, \quad (5.3)$$

where C_7 depend only on known data. From estimate (5.3), there exists a subsequence of $\hat{\mathbf{H}}_\varepsilon$, again denoted by $\hat{\mathbf{H}}_\varepsilon$, and there exist $\hat{\mathbf{H}} \in L^2(0, T; H(\text{curl}, \Omega))$ such that

$$\hat{\mathbf{H}}_\varepsilon \rightarrow \hat{\mathbf{H}} \text{ weakly in } L^2(0, T; H(\text{curl}, \Omega)), \quad (5.4)$$

$$\nabla \times \hat{\mathbf{H}}_\varepsilon \rightarrow \nabla \times \hat{\mathbf{H}} \text{ weakly in } L^2(0, T; L^2(\Omega)) \quad (5.5)$$

as $\varepsilon \rightarrow 0$. Again remember that two embeddings $H(\text{curl}, \Omega) \hookrightarrow L^2(\Omega)$ is compact. Thus we have

$$\hat{\mathbf{H}}_\varepsilon \rightarrow \hat{\mathbf{H}} \text{ strongly in } L^2(0, T; L^2(\Omega)), \quad (5.6)$$

In view of the system (5.2) and limits (5.4)–(5.6), we can obtain the system (5.1) by taking limit as $\varepsilon \rightarrow 0$ and using a similar technique as in Theorem 7.

Note that the system (5.1) and the boundary with respect to $\mathbf{K}$ are linear. Hence, the solution $\hat{\mathbf{H}}$ is unique and the mapping

$$\wedge_1 : U_{ad} \rightarrow L^2(0, T; H(\text{curl}, \Omega))$$

is Fréchet differentiable, which Fréchet derivative is $\hat{\mathbf{H}}$. $\square$

Secondly, in analogy to Theorem 9, we have the following theorem:

Theorem 10. *Suppose that the assumptions $\mathbf{H}(1)$ – $\mathbf{H}(3)$ hold. Then the mapping $\wedge_2 : U_{ad} \rightarrow W(0, T)$ is differentiable and the following limit*

$$\hat{u}(\mathbf{G}; \mathbf{K}) = \lim_{\varepsilon \rightarrow 0} \frac{u(\mathbf{G} + \varepsilon \mathbf{K}) - u(\mathbf{G})}{\varepsilon} \text{ in } W(0, T),$$

exists for any $\mathbf{G}, \mathbf{K} \in U_{ad}$ such that $\mathbf{G} + \varepsilon \mathbf{K} \in U_{ad}$ for small ε . Moreover, in the weak sense, $\hat{u}$ satisfies

$$\left\{ \begin{array}{l} \hat{u}_t - \nabla \cdot [k(x, u) \nabla \hat{u}] - \nabla \cdot [k_u(x, u) \hat{u} \nabla u] \\ \quad = 2\rho(x, t) \nabla \times \hat{\mathbf{H}} \cdot \nabla \times \mathbf{H}, (x, t) \in Q_T, \\ \hat{u}_{\mathbf{n}}(x, t) = 0, (x, t) \in S_\Gamma, \\ \hat{u}(x, 0) = 0, x \in \Omega, \end{array} \right. \quad (5.7)$$

where $(\mathbf{H}, u)$ is a weak solution of (2.1)–(2.7) corresponding to $\mathbf{G}$.

Theorem 11. *Suppose that the assumptions $\mathbf{H}(1)$ – $\mathbf{H}(3)$ hold and $(\mathbf{H}^0, u^0)$ is the optimal solution of the system (2.1)–(2.7) corresponding to the optimal control $\mathbf{G}^0 \in U_{ad}$. Then there exists a pair of functions $(\mathbf{N}, p) \in L^2(0, T; X) \times W(0, T)$, which satisfy the adjoint system*

$$\left\{ \begin{array}{ll} \mathbf{N}_t - \nabla \times [\rho(x, t) \nabla \times \mathbf{N}] = -\nabla \times [2\rho(x, t) p \nabla \times \mathbf{H}^0], & (x, t) \in Q_T, \\ p_t + \nabla \cdot [k(x, u^0) \nabla p] - k_u(x, u^0) \nabla u^0 \cdot \nabla p = 0, & (x, t) \in Q_T, \\ \mathbf{n} \times \mathbf{N} = 0, & (x, t) \in S_{\Gamma_1}, \\ \mathbf{n} \times [\rho(x, t) \nabla \times \mathbf{N}] = \mathbf{n} \times [2\rho(x, t) p \nabla \times \mathbf{H}^0], & (x, t) \in S_{\Gamma_2}, \\ \mathbf{N}(x, T) = 0, & x \in \Omega, \\ p_{\mathbf{n}}(x, t) = 0, & (x, t) \in S_\Gamma, \\ p(x, T) = u^0(T) - u_T, & x \in \Omega. \end{array} \right. \quad (5.8)$$

Moreover, the following inequality is satisfied:

$$\int_0^T \int_{\Gamma_2} [-\mathbf{n} \times (\mathbf{G} - \mathbf{G}^0) \cdot \Upsilon_T(\mathbf{N}) + \lambda \mathbf{G}^0 \cdot (\mathbf{G} - \mathbf{G}^0)] ds dt \geq 0, \forall \mathbf{G} \in U_{ad}. \quad (5.9)$$

Proof. Using the Lagrange technique, the adjoint system (5.8) is derived. According to similar ideas in some literatures [4, 7], one can show that there exists a unique solution for the adjoint system. It remains to derive that the inequality (5.9) holds. Since a similar derivation process of the inequality can be seen [15], we skip the details here. $\square$

ACKNOWLEDGEMENTS

This work was supported by the National Natural Science Foundation of China Nos.11761021.

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SINGULAR DEGENERATE NORMAL OPERATORS FOR FIRST-ORDER

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Received 29 April, 2020

Abstract. In this paper, all normal extensions of the minimal operator associated with the singular degenerate linear differential expressions for first order in the Hilbert space of vector-functions are investigated in terms of boundary values. As well, it analyses the spectrum parts of any normal extension.

2010 Mathematics Subject Classification: 47A20; 47A10

Keywords: degenerate differential operator, formally normal and normal operator, minimal and maximal operators, spectrum of an operator

1. INTRODUCTION

A densely defined closed operator $T : D(T) \subset H \rightarrow H$ in a Hilbert space H is said formally normal if $D(T) \subset D(T^*)$ and $\|Tx\| = \|T^*x\|$ for all $x \in D(T)$. This definition is equal to $(Tx, Ty) = (T^*x, T^*y)$ for all $x, y \in D(T) \subset D(T^*)$. A formally normal operator T in H is a normal operator such that $D(T) = D(T^*)$. An unbounded normal operator was given the first satisfactory description by Neuman [25]. The fundamental results in the normal extension of unbounded formally normal operators in a Hilbert space are attributable to Kilpi [14–16] and Davis [7]. Besides, Biriuk and Coddington [4], Coddington [6], Stochel and Szafraniec [20–22] developed it as a general theory, and advanced it. Some results of this idea can also be found in [10–13, 27].

Differential operator analysis is partly inspired by problems in physics, geometry, applied mathematics, quantum mechanics, quantum field theory, and so on [17, 24, 26]. In addition, several authors [1, 2, 9, 18, 19, 23] have studied the theory of the degenerate Cauchy problem in Banach spaces. Carrol and Showalter [5] performed the first research in this area, which has many important theoretical results and applications.

In this study, we give the boundary conditions for any normal extension of first order singular operator with a degeneration. We consider singularities to occur only at the end points of the domain of the Hilbert valued functions in $L^2(H_1, (-\infty, a)) \oplus$

$L^2(H_2, (b, +\infty))$. The difference of this study from [27] is that the differential expression is degenerated.

Throughout the article, we use the notations $\text{Range}(T)$, $\text{Ker}(T)$, $\sigma_c(T)$, $\sigma_p(T)$ for the range of T , the kernel of T , the continuous spectrum of T and the point spectrum of T , respectively.

2. NORMAL EXTENSIONS OF THE MINIMAL OPERATOR

Let H_1 and H_2 be separable complex Hilbert spaces, $A_k : H_k \rightarrow H_k$ be linear bounded positive selfadjoint operators with closed range and $\text{Ker} A_k \neq \{0\}$ and $B_k : H_k \rightarrow H_k$ be linear bounded selfadjoint operators for $k = 1, 2$. Consider the Hilbert space of vector-functions

$$\mathcal{H} := L^2(H_1, (-\infty, a)) \oplus L^2(H_2, (b, +\infty)), a < b.$$

with the inner product given by for all $u(t) = (u_1(t), u_2(t))$, $v(t) = (v_1(t), v_2(t)) \in \mathcal{H}$

$$(u(t), v(t))_{\mathcal{H}} := \int_{-\infty}^a (u_1(t), v_1(t))_{H_1} dt + \int_b^{+\infty} (u_2(t), v_2(t))_{H_2} dt$$

where $u_1(t), v_1(t) \in L^2(H_1, (-\infty, a))$ and $u_2(t), v_2(t) \in L^2(H_2, (b, +\infty))$. We deal with the linear degenerate differential expression for $u(t) = (u_1(t), u_2(t)) \in \mathcal{H}$

$$l(u(t)) = \begin{cases} (A_1 u_1(t))' + B_1 u_1(t), & t < a \\ (A_2 u_2(t))' + B_2 u_2(t), & t > b \end{cases} \quad (2.1)$$

where $\dim \text{Range}(A_1) = \dim \text{Range}(A_2) > 0$ and $(-1)^k B_k \geq 0$, for $k = 1, 2$. Formally adjoint of this differential expression in the space $\mathcal{H}$ is in the form

$$l^+(v(t)) = \begin{cases} -(A_1 v_1(t))' + B_1 v_1(t), & t < a \\ -(A_2 v_2(t))' + B_2 v_2(t), & t > b \end{cases} \quad (2.2)$$

where $v(t) = (v_1(t), v_2(t)) \in \mathcal{H}$.

Now, consider the operator L'_0 on the dense linear manifold in $\mathcal{H}$

$$D'_0 := \{u(t) \in \mathcal{H} : u(t) = \sum_{i=1}^n \varphi_i(t) f_i, \varphi_i(t) f_i = (\varphi_{i1}(t) f_{i1}, \varphi_{i2}(t) f_{i2}),$$

$$\varphi_{i1}(t) \in C_0^\infty(-\infty, a), \varphi_{i2}(t) \in C_0^\infty(b, \infty), f_{ik} \in H_k, k = 1, 2, n \in \mathbb{N}\}$$

and $L'_0 u(t) := l(u(t))$. L'_0 has closure because the domain of L_0^{l*} contains D'_0 . We say that the closure operator of L'_0 in $\mathcal{H}$ is called the minimal operator associated with the linear differential expression (2.1) and denote by L_0 .

Additionally, the minimal operator L_0^+ in $\mathcal{H}$ generated by the linear differential expression (2.2) can be given. The adjoint operator of L_0^+ (L_0) in $\mathcal{H}$ is called the maximal operator generated by (2.1) ((2.2)) and is denoted by L (L^+) [3, 8]. We note that if T is a closable operator on a Hilbert space H and $S : H \rightarrow H$ is a bounded

operator, then $\overline{T+S} = \overline{T} + \overline{S}$ and $D(\overline{T+S}) = D(\overline{T})$ where $\overline{T}$ is the smallest closed extension of T . In this fact and [27] it can be easily seen that $L_0 \subset L$, $L_0^+ \subset L^+$,

$$D(L_0) = \{u = (u_1, u_2) \in \mathcal{H} : ((A_1 u_1)', (A_2 u_2)') \in \mathcal{H}, u_1(a) \in \text{Ker} A_1, u_2(b) \in \text{Ker} A_2\}$$

and $D(L) = \{u = (u_1, u_2) \in \mathcal{H} : ((A_1 u_1)', (A_2 u_2)') \in \mathcal{H}\}$.

Also, we denote $H_{k0} := \text{Ker} A_k$, $H_{k1} := \text{Range} A_k$, $A_{k1} := A|_{H_{k1}} : H_{k1} \rightarrow H_{k1}$ and $B_{kj} := B|_{H_{kj}} : H_{kj} \rightarrow H_{kj}$, for $k = 1, 2$, and $j = 0, 1$.

Theorem 1. *The following statements are equivalent:*

- (i) L_0 is a formally normal operator on $\mathcal{H}$.
- (ii) $A_k B_k = B_k A_k$, $k = 1, 2$.

Proof. (i) $\rightarrow$ (ii) Let L_0 be a formally normal operator on $\mathcal{H}$, then for each $u(t) = (u_1(t), u_2(t))$, $v(t) = (v_1(t), v_2(t)) \in D(L_0) \subset D(L_0^*)$ the following equality holds

$$\begin{aligned} & (L_0 u(t), L_0 v(t))_{\mathcal{H}} - (L_0^* u(t), L_0^* v(t))_{\mathcal{H}} \\ &= 2 \left[((A_1 u_1(t))', B_1 v_1(t))_{L^2(H_1, (-\infty, a))} + (B_1 u_1(t), (A_1 v_1(t))')_{L^2(H_1, (-\infty, a))} \right. \\ & \quad \left. + ((A_2 u_2(t))', B_2 v_2(t))_{L^2(H_2, (b, +\infty))} + (B_2 u_2(t), (A_2 v_2(t))')_{L^2(H_2, (b, +\infty))} \right] = 0. \end{aligned}$$

Also, if it is used before the inequation for $e^{it} u(t)$, $e^{it} v(t) \in D(L_0)$, then

$$((B_1 A_1 - A_1 B_1) u_1(t), v_1(t))_{L^2(H_1, (-\infty, a))} + ((B_2 A_2 - A_2 B_2) u_2(t), v_2(t))_{L^2(H_2, (b, +\infty))} = 0$$

Since $D(L_0)$ is dense in $\mathcal{H}$, we have

$$A_k B_k = B_k A_k \text{ for } k = 1, 2.$$

(ii) $\rightarrow$ (i) In this case, since B_k are bounded linear operator

$$B_k (A_k u_k(t))' = (B_k A_k u_k(t))'$$

is true for $k = 1, 2$. Therefore, from the first equation and this result it is can be obtained that L_0 is a formally normal operator. $\square$

Theorem 2. *Let $A_k B_k = B_k A_k$, $k = 1, 2$ be hold. L_n is any normal extension of the minimal operator L_0 iff the domain of L_n is equal to a restriction of $D(L)$ satisfied the following boundary condition*

$$\begin{cases} A_2^{1/2} u(b) = W A_1^{1/2} u(a), \\ A_1^{1/2} u(a) \in \text{Ker}(-B_1)^{1/2}, A_2^{1/2} u(b) \in \text{Ker}(B_2)^{1/2} \end{cases}$$

where $W : H_{11} \rightarrow H_{21}$ is a unitary operator. Moreover, the unitary operator W is defined uniquely by the extension L_n , i.e. $L_n = L_W$.

Proof. Let L_n be a normal extension of the minimal operator L_0 . In this case, we have

$$(L_n u(t), u(t))_{\mathcal{H}} - (u(t), L_n^* u(t))_{\mathcal{H}} = \|A_1^{1/2} u(a)\|_{H_1}^2 - \|A_2^{1/2} u(b)\|_{H_2}^2 = 0.$$

for all $u \in D(L_n) = D(L_n^*)$. Therefore, there is an isometric operator $WA_1^{1/2} u(a) = A_2^{1/2} u(b)$, $u \in D(L_n)$. Moreover,

$$H_a = \{u(a) \in H_1 : u(t) \in D(L_n)\},$$

$$H_b = \{u(b) \in H_2 : u(t) \in D(L_n)\}$$

are subspaces of H_1 and H_2 , respectively. We claim that $H_a = H_1$ and $H_b = H_2$. Otherwise, there exist a nonzero element $f_1 \in H_a^\perp$ or $f_2 \in H_b^\perp$ and it can be defined two functions such that $u_1(t) = (e^t A_{11}^{-1} f_1, 0)$ and $u_2(t) = (0, e^{-t} A_{21}^{-1} f_2)$ in $\mathcal{H}$. Moreover, for all $u \in D(L_n)$

$$(L_n u(t), u_k(t))_{\mathcal{H}} - (u(t), L_n^* u_k(t))_{\mathcal{H}} = (A_1 u(a), u_k(a))_{H_1} - (A_2 u(b), u_k(b))_{H_2} = 0$$

is true. This result implies $u_k \in D(L_n^*)$, but $u_k \notin D(L_n)$, $k = 1, 2$. Because L_n is a normal operator, it is a contradiction. Therefore, $W : H_{12} \rightarrow H_{21}$ is a unitary operator and determined uniquely by the extension of L_n .

On the other hand, L_n is a normal extension operator of L_0 , for every $u(t) \in D(L_n)$ the following inequalities hold

$$\begin{aligned} & (L_n u(t), L_n u(t))_{\mathcal{H}} - (L_n^* u(t), L_n^* u(t))_{\mathcal{H}} \\ &= 2 \left[\left(B_1 A_1^{1/2} u(a), A_1^{1/2} u(a) \right)_{H_1} - \left(B_2 A_2^{1/2} u(b), A_2^{1/2} u(b) \right)_{H_2} \right] \\ &= -2 \|(-B_1 A_1)^{1/2} u(a)\|_{H_1}^2 - 2 \|(B_2 A_2)^{1/2} u(b)\|_{H_2}^2 = 0. \end{aligned}$$

Hence, $u(a) \in \text{Ker}(-B_1)^{1/2}$ and $u(b) \in \text{Ker}(B_2)^{1/2}$.

Conversely, suppose that $W : H_{11} \rightarrow H_{21}$ is a unitary operator and the boundary conditions

$$A_2^{1/2} u(b) = WA_1^{1/2} u(a), u(a) \in \text{Ker}(-B_1)^{1/2}, u(b) \in \text{Ker}(B_2)^{1/2}$$

are satisfied. In this case, the adjoint operator L_W^* is generated by the differential-operator expression (2.2) and $D(L_W) = D(L_W^*)$. Also, the other condition of normal extensions in $\mathcal{H}$ can be easily obtained. $\square$

Corollary 1. *If any of B_1 or B_2 is a one to one operator, then there isn't any normal extension of L_0 in $\mathcal{H}$.*

Corollary 2. *If there exists a normal extension of L_0 in $\mathcal{H}$, then*

$$\dim \text{Ker}(-B_{11})^{1/2} = \dim \text{Ker}(B_{21})^{1/2} > 0.$$

3. THE SPECTRUM PARTS OF NORMAL EXTENSIONS

In this section, the spectrum parts of any normal extension L_W of minimal operator L_0 generated by (2.1) in $\mathcal{H}$ and boundary conditions with the unitary operator $W : H_{11} \rightarrow H_{21}$ and

$$\begin{cases} A_2^{1/2}u(b) = WA_1^{1/2}u(a), \\ A_1^{1/2}u(a) \in \text{Ker}(-B_1)^{1/2}, A_2^{1/2}u(b) \in \text{Ker}(B_2)^{1/2} \end{cases}$$

will be investigated. Also, it will be considered that $0 \in \sigma_p((-B_{11})^{1/2}) \cap \sigma_p((B_{21})^{1/2})$.

Theorem 3. *Letting L_W be a normal extension of L_0 , then the point spectrum $\sigma_p(L_W)$ of L_W is the following form*

$$\sigma_p(L_W) = \sigma_p(B_{10}) \cup \sigma_p(B_{20}).$$

Proof. Consider a problem for the point spectrum for the normal extension L_W , i.e.

$$L_W u(t) = \lambda u(t), u(t) \in D(L_W), \lambda \in \mathbb{C}.$$

It can be written $u(t) = (u_1(t), u_2(t)) = (u_{10}(t) + u_{11}(t), u_{20}(t) + u_{21}(t))$, $u_{ki}(t) \in H_{ki}$, $k = 1, 2$, $i = 0, 1$. Also, since the restriction operator A_{k1} on H_{k1} , $k = 1, 2$ have a bounded inverse, $u'_{k1}(t)$ exists. The solution of this problem in $\mathcal{H}$

$$\begin{aligned} u_{11}(t) &= e^{-A_{11}^{-1}(B_1 - \lambda)(t-a)} f_1, t < a, f_1 \in H_{11}, \\ u_{21}(t) &= e^{-A_{21}^{-1}(B_2 - \lambda)(t-b)} f_2, t > b, f_2 \in H_{21}, \\ A_2^{1/2} f_2 &= WA_1^{1/2} f_1, \\ B_1 u_{10}(t) &= \lambda u_{10}(t), B_2 u_{20}(t) = \lambda u_{20}(t). \end{aligned}$$

Also, $A_1^{1/2} f_1 \in \text{Ker}(B_1)$ and $A_2^{1/2} f_2 \in \text{Ker}(B_2)$, from Theorem 1 it must be

$$\begin{aligned} u_{11}(t) &= e^{\lambda A_{11}^{-1}(t-a)} f_1, t < a, f_1 \in H_{11}, \\ u_{21}(t) &= e^{\lambda A_{21}^{-1}(t-b)} f_2, t > b, f_2 \in H_{21}, \\ A_2^{1/2} f_2 &= WA_1^{1/2} f_1, \\ B_1 u_{10}(t) &= \lambda u_{10}(t), B_2 u_{20}(t) = \lambda u_{20}(t). \end{aligned}$$

But $(u_{11}(t), u_{21}(t)) \in \mathcal{H}$ iff $f_k = 0$, $k = 1, 2$, as a result of this

$$\sigma_p(L_W) = \sigma_p(B_{10}) \cup \sigma_p(B_{20}).$$

□

Theorem 4. *Letting L_W be any normal extension of L_0 , then the continuous spectrum $\sigma_c(L_W)$ of L_W is in form*

$$\sigma_c(L_W) = i\mathbb{R} \cup \sigma_c(B_{10}) \cup \sigma_c(B_{20}).$$

Proof. It is known that the spectrum of normal operators [8],

$$\sigma(L_W) \subset \sigma(\operatorname{Re} L_W) + i\sigma(\operatorname{Im} L_W),$$

where $\operatorname{Re} L_W$ and $\operatorname{Im} L_W$ are real part and imaginary part of L_W . If $\lambda = \lambda_r + i\lambda_i \in \sigma_c(L_W)$, then $\lambda_r \in \sigma(\operatorname{Re} L_W)$ and $\lambda_i \in \sigma(\operatorname{Im} L_W)$.

Let us deal with the solution of the following problem of the normal extension L_W ,

$$L_W u(t) = \lambda u(t) + f(t), u(t) \in D(L_W), f(t) \in \mathcal{H}, \lambda \in \mathbb{C}.$$

For $u(t) = (u_1(t), u_2(t)) = (u_{11}(t) + u_{10}(t), u_{21}(t) + u_{20}(t))$, $u_{ki}(t) \in H_{ki}$, $k = 1, 2, i = 0, 1$

$$A_1 u'_{11}(t) + B_1 u_{11}(t) = \lambda u_{11}(t) + f_{11}(t), u_{11}(t), f_{11}(t) \in L^2(H_{11}, (-\infty, a)),$$

$$A_2 u'_{21}(t) + B_2 u_{21}(t) = \lambda u_{21}(t) + f_{21}(t), u_{21}(t), f_{21}(t) \in L^2(H_{21}, (b, \infty)),$$

$$A_2^{1/2} u_{21}(b) = W A_1^{1/2} u_{11}(a), u_{11}(a) \in \operatorname{Ker}(-B_1)^{1/2}, u_{21}(b) \in \operatorname{Ker}(B_2)^{1/2},$$

$$B_{10} u_{10}(t) = \lambda u_{10}(t) + f_{10}(t), u_{10}(t), f_{10}(t) \in L^2(H_{10}, (-\infty, a)),$$

$$B_{20} u_{20}(t) = \lambda u_{20}(t) + f_{20}(t), u_{20}(t), f_{20}(t) \in L^2(H_{20}, (b, \infty)).$$

In this case, when $\lambda_r \neq 0$ or $\lambda_r \in \sigma_c(B_{10}) \cup \sigma_c(B_{20})$ there isn't any solution of this problem. Also, because of the boundary conditions and Theorem 1 the general solution is in the following form

$$u_{11}(t) = e^{i\lambda_i A_{11}^{-1}(t-a)} f_1 - \int_t^a e^{-A_{11}^{-1}(B_1 - i\lambda_i)(t-s)} A_{11}^{-1} f_{11}(s) ds, t < a, f_1 \in H_{11},$$

$$u_{21}(t) = e^{i\lambda_i A_{21}^{-1}(t-b)} f_2 + \int_b^t e^{-A_{21}^{-1}(B_2 - i\lambda_i)(t-s)} A_{21}^{-1} f_{21}(s) ds, t > b, f_2 \in H_{21}.$$

On the other hand, there exists a function

$$f(t) = (e^{(1+i\lambda_i)A_{11}^{-1}(t-a)} f_1^*, e^{(-1+i\lambda_i)A_{21}^{-1}(t-b)} f_2^*),$$

$A_{k1}^{-1} f_k^* \in \operatorname{Ker}(B_{k1})$, $k = 1, 2$ in $\mathcal{H}$, then

$$u_{11}(t) = e^{i\lambda_i A_{11}^{-1}(t-a)} f_1 - \int_t^a e^{i\lambda_i A_{11}^{-1}(t-a)} e^{A_{11}^{-1}(s-a)} A_{11}^{-1} f_1^* ds$$

$$= e^{i\lambda_i A_{11}^{-1}(t-a)} f_1 + e^{i\lambda_i A_{11}^{-1}(t-a)} (e^{A_{11}^{-1}(t-a)} - E) f_1^*, t < a, f_1 \in H_{11},$$

$$u_{21}(t) = e^{i\lambda_i A_{21}^{-1}(t-b)} f_2 + \int_b^t e^{i\lambda_i A_{21}^{-1}(t-b)} e^{-A_{21}^{-1}(s-b)} A_{21}^{-1} f_2^* ds$$

$$= e^{i\lambda_i A_{21}^{-1}(t-b)} f_2 - e^{i\lambda_i A_{21}^{-1}(t-b)} (e^{-A_{21}^{-1}(t-b)} - E) f_2^*, t > b, f_2 \in H_{21}.$$

Also, $u(t) = (u_{11}(t), u_{21}(t)) \in D(L_W)$ iff $f_1^* = f_1$, $f_2^* = -f_2$ and $A_2^{1/2} f_2^* = W A_1^{1/2} f_1^*$. Because of this result for $\alpha \in \mathbb{C} \setminus \{1\}$, we have

$$f_\alpha(t) = (e^{(1+i\lambda_i)A_{11}^{-1}(t-a)} f_1^*, \alpha e^{(-1+i\lambda_i)A_{21}^{-1}(t-b)} f_2^*),$$

where $A_{k1}^{-1} f_k^* \in \text{Ker}(B_{k1})$, for $k = 1, 2$. There is no solution $L_W u(t) = i\lambda_i u(t) + f_\alpha(t)$ in $D(L_W)$. This result completes of the proof. $\square$

COMPETING INTERESTS

The authors declare that they have no competing interests.

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Miskolc Mathematical Notes

Abstracted in *Zentralblatt für Mathematik*, *Mathematical Reviews*,
Реферативный журнал „Математика”

Indexed in *Dimensions*, *EBSCO Academic Search Ultimate*, *EBSCO Academic Search Complete*, *EBSCO Academic Search Premier*, *EBSCO Academic Search Elite*, *EBSCO STM Source*, *Google Scholar*, *Naver Academic*, *ProQuest Central*, *ProQuest SciTech Premium Collection*, *ProQuest Technology Collection*, *ROAD*, *SCImago*, *SCOPUS*, *Web of Science Core Collection: Science Citation Index Expanded Additional*, *Web of Science Indexes: Essential Science Indicators*, *UGC-CARE List*

Secretariat of the Vice-Rector for Research and International Relations, University of Miskolc

Responsible for publication: Prof. Dr. Zita Horváth, rector

Published by the Miskolc University Press under the leadership of Attila Szendi

Responsible for duplication: Works manager Erzsébet Pásztor

Number of copies printed: 70

Put to the Press on 28 March, 2022

Number of permission: MERT 2022-60 ME

HU e-ISSN 1787-2413