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Almost geodesic mappings of affinely connected spaces that preserve the Riemannian curvature*

Vladimir Berezovski[a], **Sándor Bácsó**[b], **Josef Mikeš**[c]

[a]Uman National University of Horticulture, Ukraine
berez.volod@rambler.ru

[b]Debrecen University, Hungary
bacsos@unideb.hu

[c]Palacky University Olomouc, Czech Republic
josef.mikes@upol.cz

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Abstract

In the present paper the authors give some conditions preserved Riemannian curvature tensor with respect to almost geodesic mappings of affinely connected spaces. It is noteworthy that these conditions are valid for other types of mappings. For the almost geodesic mappings of first type, when the Riemannian curvature tensor is invariant, the authors deduce a differential equations system of Cauchy type.

In addition the authors investigate almost geodesic mappings of first type, where the Weyl tensor of projective curvature is invariant and Riemannian tensor is not invariant.

Keywords: almost geodesic mapping, Riemannian curvature tensor

MSC: 53B05

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1. Introduction

Several works [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24] have been devoted to study almost geodesic mappings. These mappings are generalization of geodesic and quasigeodesic mappings, see [11, 12, 13, 17].

The basic concepts of almost geodesic curve and almost geodesic mapping of affinely connected spaces are introduced in paper [15] and included in the monographs [17, p. 156], [12, p. 457] and surveys [18, 4, 8, 10].

Definition 1.1. A curve $x(t)$ in an affinely connected space A_n is called an *almost geodesic curve* if there exists a plane $\tau(t)$ in every tangent space of the curve $x(t)$ such that:

(1) $\tau(t)$ are parallel translated along $x(t)$, and

(2) the tangent vector $\dot{x}(t)$ of the curve lies in $\tau(t)$.

Definition 1.2. A diffeomorphism $f\colon A_n \to \bar{A}_n$ is called *almost geodesic mapping*, if under f any geodesic curve of A_n coincides with an almost geodesic curve of $\bar{A}_n$.

Theorem 1.3. *Diffeomorphism $f\colon A_n \to \bar{A}_n$ is almost geodesic mapping if and only if the deformation tensor of the affine connections $P^h_{ij}(x) \equiv \bar{\Gamma}^h_{ij}(x) - \Gamma^h_{ij}(x)$ satisfies for any vector λ^h the following conditions:*

$$A^h_{\alpha\beta\gamma}\lambda^\alpha\lambda^\beta\lambda^\gamma = a\, P^h_{\alpha\beta}\lambda^\alpha\lambda^\beta + b\,\lambda^h$$

where

$$A^h_{ijk} = P^h_{ij,k} + P^\alpha_{ij}P^h_{k\alpha}, \tag{1.1}$$

Γ^h_{ij} ($\bar{\Gamma}^h_{ij}$) are objects of affine connections of spaces A_n ($\bar{A}_n$) respectively, a and b are some functions depend on x^h and λ^h and $x = (x^1, x^2, \ldots, x^n)$ is a common system of coordinates. The symbol "," means covariant derivation with respect to A_n.

Three types of almost geodesic mapping was discovered by Sinyukov [15, 16, 17, 18], he called them π_1, π_2 and π_3. In [2] it was proved that another almost geodesic mapping, if $n > 5$ does not exist.

Almost geodesic mapping π_1 is characterized by the following conditions:

$$A^h_{(ijk)} = \delta^h_{(i}a_{jk)} + b_{(i}P^h_{jk)},$$

where a_{ij} is a symmetric tensor, b_i is a vector, and the symbol (ijk) means symmetrization without division for the indices i, j, k.

2. Mappings of affinelly connected spaces that preserve the Riemann curvature tensor

If we give a diffeomorphism $f\colon A_n \to \bar{A}_n$, then the relation between Riemann curvature tensors R^h_{ijk} and $\bar{R}^h_{ijk}$ of A_n and $\bar{A}_n$ is the following [18, p. 78], [11, p. 86], [12, p. 184]:

$$\bar{R}^h_{ijk} = R^h_{ijk} + P^h_{i[k,j]} + P^\alpha_{i[k} P^h_{j]\alpha}, \tag{2.1}$$

where the symbol $[kj]$ denotes the alternalization for the indices k and j.

Using of (1.1) and (2.1) we have

Theorem 2.1. *A mapping preserves the Riemann curvature tensor if and only if it satisfies the condition*

$$A^h_{ijk} = A^h_{ikj}, \tag{2.2}$$

that is, the tensor A^h_{ijk} is to be symmetric in the indices j and k.

If the Riemann curvature tensor is preserved by the mapping, then, of course, Ricci tensor $R_{ij} = R^\alpha_{i\alpha j}$ and Weyl tensor of projective curvature

$$W^h_{ijk} = R^h_{ikj} - \frac{1}{n+1}\,\delta^h_i R_{[jk]} + \frac{1}{n^2-1}\,[(nR_{ij} + R_{ji})\delta^h_k - (nR_{ik} + R_{ki})\delta^h_j] \tag{2.3}$$

also are invariant under this mapping.

The condition of Theorem 1.3 is sufficient condition for preserving the Ricci tensor and Weyl tensor of projective curvature, but it is not necessary. Further on we give an example.

3. Special almost geodesic mappings of first type which preserve Riemannian tensor

Let be a mapping given between affinely connected spaces A_n and $\bar{A}_n$, which satisfies the condition:

$$P^h_{ij,k} + P^h_{ik,j} = -P^\alpha_{ij} P^h_{\alpha k} - P^\alpha_{ik} P^h_{\alpha j} + \delta^h_{(i} a_{jk)} \tag{3.1}$$

This mapping is a special case of almost geodesic mapping of first type.

Alternating equation (3.1) in i and j, we get

$$P^h_{ik,j} - P^h_{jk,i} = -P^\alpha_{ik} P^h_{\alpha j} + P^\alpha_{jk} P^h_{\alpha i}. \tag{3.2}$$

At now in (3.2), we exchange the indices i and k, we obtain

$$P^h_{ik,j} - P^h_{ji,k} = -P^\alpha_{ik} P^h_{\alpha j} + P^\alpha_{ji} P^h_{\alpha k}. \tag{3.3}$$

If we subtract equation (3.3) from equation (3.1), we have

$$2P^h_{ij,k} = -2P^\alpha_{ij} P^h_{\alpha k} + \delta^h_{(i} a_{jk)},$$

that is,

$$P^h_{ij,k} + P^\alpha_{ij} P^h_{\alpha k} = \delta^h_{(i} \tilde{a}_{jk)}, \tag{3.4}$$

where $\tilde{a}_{ij} = \frac{1}{2} a_{ij}$.

In this case the tensor $A^h_{ijk} = \delta^h_{(i} \tilde{a}_{jk)}$ is symmetric in indices j and k.

Using of Theorem 1.3 we have

Theorem 3.1. *The almost geodesic mapping (determined by* (3.1)*) preserves the Riemann curvature tensor* R^h_{ijk}.

If Riemann curvature tensor vanishes in an affine space, then we have the following

Theorem 3.2. *If an affine space* A_n *admits an almost geodesic mapping (determined by* (3.1)*) into* $\bar{A}_n$*, then* $\bar{A}_n$ *is also an affine space.*

So affinely spaces are closed under almost geodesic mapping (determined by (3.1)).

From equation (3.1) we obtained the equation (3.4). Equation (3.4) is a system of Cauchy type for deformation tensor. We can find it's integrability conditions.

We differentiate covariantly equation (3.4) by x^m, further on, we change the indices k and m, using of Ricci identities we have

$$\delta^h_i \tilde{a}_{j[k,m]} + \delta^h_j \tilde{a}_{i[k,m]} + \delta^h_{[k} \tilde{a}_{ij,lm]} = P^\alpha_{ij} R^h_{\alpha km} + P^h_{\alpha(j} R^\alpha_{i)km} + a_{\tilde{j}[m} P^h_{k]i} + \tilde{a}_{i[m} P^h_{k]j}. \tag{3.5}$$

After transvecting of integrability conditions (3.5) by indices h and m we obtain

$$\begin{aligned} \tilde{a}_{jk,i} + \tilde{a}_{ik,j} - (n+1)\tilde{a}_{ij,k} = {} & -P^\alpha_{ij} R_{\alpha k} + P^\beta_{\alpha j} R^\alpha_{ik\beta} + P^\beta_{\alpha i} R^\alpha_{jk\beta} \\ & + \tilde{a}_{j\alpha} P^\alpha_{ki} - \tilde{a}_{jk} P^\alpha_{\alpha i} + \tilde{a}_{i\alpha} P^\alpha_{jk} - \tilde{a}_{ik} P^\alpha_{j\alpha}. \end{aligned} \tag{3.6}$$

Alternating equation (3.6) in k and j, we obtain

$$\begin{aligned} \tilde{a}_{ij,k} = \tilde{a}_{ik,j} + \tfrac{1}{n+2}(& -P^\alpha_{ij} R_{\alpha k} + P^\alpha_{ik} R_{\alpha j} - P^\beta_{\alpha j} R^\alpha_{ik\beta} + P^\beta_{\alpha k} R^\alpha_{ij\beta} - \\ & P^\beta_{\alpha i} R^\alpha_{jk\beta} + P^\beta_{\alpha i} R^\alpha_{kj\beta} - \tilde{a}_{j\alpha} P^\alpha_{ki} + \tilde{a}_{k\alpha} P^\alpha_{ij} + \tilde{a}_{ik} P^\alpha_{j\alpha} - \tilde{a}_{ij} P^\alpha_{k\alpha}). \end{aligned} \tag{3.7}$$

In equation we exchange the indices k and i, we get

$$\begin{aligned} \tilde{a}_{kj,i} = \tilde{a}_{ik,j} + \tfrac{1}{n+2}(& -P^\alpha_{kj} R_{\alpha i} + P^\alpha_{ki} R_{\alpha j} - P^\beta_{\alpha j} R^\alpha_{ki\beta} + P^\beta_{\alpha i} R^\alpha_{kj\beta} - \\ & P^\beta_{\alpha k} R^\alpha_{ji\beta} + P^\beta_{\alpha k} R^\alpha_{ij\beta} - \tilde{a}_{j\alpha} P^\alpha_{ik} + \tilde{a}_{i\alpha} P^\alpha_{kj} + \tilde{a}_{ki} P^\alpha_{j\alpha} - \tilde{a}_{kj} P^\alpha_{i\alpha}). \end{aligned} \tag{3.8}$$

Substituting equation (3.7) and (3.8) into (3.6), we have

$$\begin{aligned} \tilde{a}_{ik,j} = \tfrac{1}{(n-1)(n+2)}(& -n(P^\alpha_{ik} R_{\alpha j} + P^\beta_{\alpha(k} R^\alpha_{i)j\beta})) - R_{\alpha(k} P^\alpha_{i)j} - P^\beta_{\alpha j} R^\alpha_{(ik)\beta} - \\ & P^\beta_{\alpha(i} R^\alpha_{|j|k)\beta} + (n+1)(\tilde{a}_{j(i} P^\alpha_{k)\alpha} - \tilde{a}_{\alpha(i} P^\alpha_{k)j}) + 2(\tilde{a}_{ik} P^\alpha_{j\alpha} - \tilde{a}_{j\alpha} P^\alpha_{ik})). \end{aligned} \tag{3.9}$$

Equation (3.4) and (3.9) are a system of Cauchy type for the function $P^h_{ij}(x)$ and $\tilde{a}_{ij}(x)$, which satisfy the following

$$P^h_{ij}(x) = P^h_{ji}(x), \quad \tilde{a}_{ij}(x) = \tilde{a}_{ji}(x). \tag{3.10}$$

So is proved the

Theorem 3.3. *An affinely connected space A_n admits an almost geodesic mapping (determined by (3.1)) into an affinely connected space $\bar{A}_n$ if and only if in A_n exist solution functions $P^h_{ij}(x)$ and $\tilde{a}_{ij}(x)$ for equation system of Cauchy type (3.4), (3.9) and (3.10).*

4. Special almost geodesic mappings of first type which preserve Weyl tensor of projective curvature but does not preserve Riemann curvature tensor

Let be the $f\colon \bar{A}_n \to \bar{A}_n$ mapping given, which satisfies the following condition

$$P^h_{ij,k} + P^\alpha_{ij} P^h_{\alpha k} = \delta^h_k a_{ij}, \tag{4.1}$$

where a_{ij} is a symmetric tensor.

It is well known, that this above mentioned mapping is an almost geodesic mapping of first type.

The tensor A^h_{ijk} on the basis of (4.1) is equal to $\delta^h_k a_{ij}$. If the tensor $a_{ij} \not\equiv 0$, then the tensor A^h_{ijk} is not symmetric in indices j and k. So, in general the mapping (determined by (4.1)) does not preserve the Riemannian curvature tensor.

Using of (2.1) and (4.1) we get

$$\bar{R}^h_{ijk} = R^h_{ijk} - a_{i[j}\delta^h_{k]}. \tag{4.2}$$

It is easy to see, that after transvecting (4.2) in indices h and k, we have

$$a_{ij} = \frac{1}{n-1}\,(\bar{R}_{ij} - R_{ij}).$$

From the last formulae, symmetry a_{ij}, (2.3) and (4.2) we get

$$\bar{W}_{ij} = W_{ij}, \quad \bar{\tilde{W}}{}^h_{ijk} = \tilde{W}^h_{ijk}, \quad \text{and} \quad \bar{W}^h_{ijk} = W^h_{ijk}$$

where

$$W_{ij} = R_{ij} - R_{ji} \quad \text{and} \quad \tilde{W}^h_{ijk} = R^h_{ijk} + \frac{1}{n-1} R_{i[j}\delta^h_{k]}.$$

The W_{ij} and $\bar{W}_{ij}$ are tensors of type $\binom{0}{2}$ in the space A_n and $\bar{A}_n$ respectively. The $\tilde{W}^h_{ijk}$ and $\bar{\tilde{W}}{}^h_{ijk}$ are tensors of type $\binom{1}{3}$ in the space A_n and $\bar{A}_n$ respectively. The W^h_{ijk} and $\bar{W}^h_{ijk}$ are Weyl tensors of projective curvature of A_n and $\bar{A}_n$ respectively.

Finally we obtain

Theorem 4.1. *Tensors W_{ij}, $\tilde{W}^h_{ijk}$ and W^h_{ijk} are invariants under almost geodesic mapping (determined by (4.1)).*

From Theorem 3.3 follows

Theorem 4.2. *If a projective-euclidean or equiaffinely space A_n admits almost geodesic mapping (determined by (4.1)) into an affinely connected space $\bar{A}_n$, then $\bar{A}_n$ is a projective-euclidean or equiaffinely space respectively.*

The proof of Theorem 4.1 follows from facts, that the Weyl tensor vanishes in projective-euclidean space, and the tensor is equal to zero in equiaffinely space.

So, using of Theorem 4.1, we obtain, that the projective-euclidean and equiaffinely spaces are closed sets under almost geodesic mapping (determined by (4.1)).

For almost geodesic mapping of first type, which determined by (4.1), the tensor A^h_{ijk} is equal to $\delta^h_i a_{ij}$. If $a_{ij} \not\equiv 0$, then the A^h_{ijk} tensor is not symmetric in indices j and k.

So, the mapping (determined by (4.1)) does not preserve the Riemann curvature tensor, but the Weyl tensor is invariant object.

Consider the equations (4.1) as a system of Cauchy-type for unknown P^h_{ij}, find it's integrability condition. At first, we differentiate covariantly equation (4.1) in x^m, further on we alternate it in indices k and m. After transvecting integrability condition of equation (4.1) in indices h and m, we obtain

$$(n-1)a_{ij,k} = P^\alpha_{ij} R_{\alpha k} - P^\beta_{\alpha(i} R^\alpha_{j)\beta k} - (n-1)P^\alpha_{ij} a_{\alpha k} \tag{4.3}$$

So equation (4.1) and (4.3) in a space A_n give a system of Cauchy type for unknown functions $P^h_{ij}(x)$ and $a_{ij}(x)$, which satisfies algebraic conditions

$$P^h_{ij}(x) = P^h_{ji}(x), \quad a_{ij}(x) = a_{ji}(x) \tag{4.4}$$

Therefore

Theorem 4.3. *An affinely connected space admits almost geodesic mapping of first type (determined by (4.1)) into an affinely connected space $\bar{A}_n$ if and only if in A_n there exist solution $P^h_{ij}(x)$ and $a_{ij}(x)$ for system of Cauchy type equations (4.1), (4.3) and (4.4).*

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On some identities for balancing and cobalancing numbers

Paula Catarino, Helena Campos, Paulo Vasco*

Universidade de Trás-os-Montes e Alto Douro, UTAD, www.utad.pt
Quinta de Prados, 5000-801 Vila Real, Portugal
pcatarin@utad.pt, hcampos@utad.pt, pvasco@utad.pt

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Abstract

As a consequence of the Binet formula for balancing, cobalancing, square triangular, Lucas-balancing and Lucas-cobalancing numbers, we provide some formulas for these sequences explicitly, which can have certain importance or applications in most recently investigations in this area. Also we give another expression for the general term of each sequence, using the ordinary generating function.

Keywords: Balancing number, cobalancing number, Binet's formula, generating function.

MSC: 11B39, 11B83, 05A15.

1. Introduction

Some sequences of integer numbers have been studied over several years, with emphasis on studies of the well known Fibonacci sequence (and then the Lucas sequence) which is related to the golden ratio and of the Pell sequence which is related to the silver ratio. Behera and Panda [1] introduced the sequence $(B_n)_{n=0}^{\infty}$ of balancing numbers and give some interesting properties of this sequence. According Behera and Panda [1] a positive integer n is a *balancing number* with *balancer* r,

*The authors are members of the research centre CMAT – Polo da UTAD inserted in the Unidade de Investigação da Universidade do Minho, the first two authors are also members of the research centre LabDCT/CIDTFF – Centro de Investigação em Didáctica e Tecnologia na Formação de Formadores.

if it is the solution of the Diophantine equation $1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r)$.

The sequence $(B_n)_{n=0}^{\infty}$ is defined by the following recurrence relation of second order given by

$$B_{n+1}=6B_n-B_{n-1},\ n\geq 1, \tag{1.1}$$

with initial terms $B_0=0$ and $B_1=1$, where B_n denotes the nth balancing number.

On the other hand, following Panda and Ray [13] a positive integer n is a *cobalancing number* with *cobalancer* r, if it is the solution of the Diophantine equation $1+2+\cdots+n=(n+1)+(n+2)+\cdots+(n+r)$. The sequence $(b_n)_{n=1}^{\infty}$ is defined by the following recurrence relation of second order given by

$$b_{n+1}=6b_n-b_{n-1}+2,\ n\geq 2, \tag{1.2}$$

with initial terms $b_1=0$ and $b_2=2$, where b_n denotes the nth cobalancing number. Panda and Ray [13, Theorem 6.1] proved that every balancer is a cobalancing number and every cobalancer is a balancing number. Many authors have dedicated their research to the study of these sequences and also to the generalisations of the theory of the sequences of balancing, cobalancing, Lucas-balancing and Lucas-cobalancing numbers [2, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. Behera and Panda [1] observed that n is a balancing number if and only if n^2 is a triangular number, that is $8n^2+1$ is a perfect square and the square of a balancing number is a square triangular number, that is, $B_n^2=ST_n$, where ST_n denotes the nth square triangular number. Also, we know that n is a cobalancing number if and only if $8n^2+8n+1$ is a perfect square. The same way which balancing numbers are related to square triangular numbers, also, the cobalancing numbers are related to pronic triangular numbers (triangular numbers that are expressible as a product of two consecutive natural numbers) [22, 21, 23, 24]. The sequence $(ST_n)_{n=0}^{\infty}$ is defined by the following recurrence relation of second order given by

$$ST_{n+1}=34ST_n-ST_{n-1}+2,\ n\geq 1, \tag{1.3}$$

with initial terms $ST_0=0$ and $ST_1=1$, where ST_n denotes the nth square triangular number. Panda [12] gives us the identity $C_n=\sqrt{8B_n^2+1}$ that involves the nth balancing number and the nth Lucas-balancing number C_n. Also the sequence $(C_n)_{n=0}^{\infty}$ is defined by the following recurrence relation of second order given by

$$C_{n+1}=6C_n-C_{n-1},\ n\geq 1, \tag{1.4}$$

with initial terms $C_0=1$ and $C_1=3$.

In addition, Panda and Ray [14] give the identity $c_n=\sqrt{8b_n^2+8b_n+1}$ that involves the nth cobalancing number and the nth Lucas-cobalancing number c_n. The sequence $(c_n)_{n=1}^{\infty}$ is defined by the following recurrence relation of second order given by

$$c_{n+1}=6c_n-c_{n-1},\ n\geq 2, \tag{1.5}$$

with initial terms $c_1=1$ and $c_2=7$.

The Binet formula is well known for several sequences of integer numbers. The general Binet formula for a mth order linear recurrence was deduced in 1985 by Levesque in [6]. Sometimes this formula is used in the proof of basic properties of integer sequences. In the case of balancing, cobalancing, Lucas-balancing and Lucas-cobalancing sequences, their Binet formulas are respectively,

$$B_n = \frac{r_1^n - r_2^n}{r_1 - r_2}, \tag{1.6}$$

$$C_n = \frac{r_1^n + r_2^n}{2}, \tag{1.7}$$

$$b_n = \frac{\alpha_1^{2n-1} - \alpha_2^{2n-1}}{4\sqrt{2}} - \frac{1}{2}, \tag{1.8}$$

$$c_n = \frac{\alpha_1^{2n-1} + \alpha_2^{2n-1}}{2}, \tag{1.9}$$

and using the relation between balancing numbers, square triangular numbers and (1.6) we obtain

$$ST_n = \frac{r_1^{2n} + r_2^{2n}}{32} - \frac{1}{16}, \tag{1.10}$$

where $r_1 = {\alpha_1}^2 = 3+2\sqrt{2}$ and $r_2 = {\alpha_2}^2 = 3-2\sqrt{2}$ are the roots of the characteristic equation $x^2 = 6x - 1$, associated with the recurrence relations of the sequences and α_1 and α_2 are the roots of the characteristic equation, $x^2 = 2x + 1$, associated with the Pell sequence [3, 4, 14].

There is a large number of sequences indexed in *The Online Encyclopedia of Integer Sequences*, being in this case

$$\begin{aligned}
\{(B_n)_{n=0}^{\infty}\} &= \{0, 1, 6, 35, 204, 1189, 6930, \ldots\} : A001109 \\
\{(b_n)_{n=1}^{\infty}\} &= \{0, 2, 14, 84, 492, 2870, 16730, \ldots\} : A053141 \\
\{(ST_n)_{n=0}^{\infty}\} &= \{0, 1, 36, 1225, 41616, 1413721, \ldots\} : A001110 \\
\{(C_n)_{n=0}^{\infty}\} &= \{1, 3, 17, 99, 577, 3363, 19601, \ldots\} : A001541 \\
\{(c_n)_{n=1}^{\infty}\} &= \{1, 7, 41, 239, 1393, 8119, 47321, \ldots\} : A002315.
\end{aligned}$$

Many interesting properties and important identities about these sequences are available in the literature. Interested readers can follow [2, 5, 8, 10, 12, 16, 18], among many others scientific papers. The purpose of this paper is to provide some formulas of the sequences stated above to help possible applications. In the next two sections we obtain new identities and properties for these sequences, such as the famous Catalan, Cassini and d'Ocagne identities and the sums formulae for each one. The last section is devoted to explicitly give the ordinary generating function of these sequences, as well as another expression for the general term of them.

2. Balancing, Lucas-balancing and square triangular numbers: some identities

According with recurrence relations (1.1), (1.3) and (1.4) and using the well known results involving recursive sequences, consider the respective characteristic equation and note that $r_1 r_2 = 1,\ r_1 - r_2 = 4\sqrt{2},\ r_1 + r_2 = 6$.

As a consequence of the Binet formulas (1.6), (1.7) and (1.10) we get for these sequences the following interesting identities. The first one and its proof can be found in Panda [12].

Proposition 2.1 (Catalan's identities)**.** *For the sequences $(B_n)_{n=0}^{\infty}$, $(C_n)_{n=0}^{\infty}$ and $(ST_n)_{n=0}^{\infty}$ if $n \geq r$ we have*

$$B_{n-r}B_{n+r} - B_n^2 = -B_r^2, \tag{2.1}$$

$$C_{n-r}C_{n+r} - C_n^2 = C_r^2 - 1 \tag{2.2}$$

and

$$ST_{n-r}ST_{n+r} - ST_n^2 = ST_r^2 - 2ST_nST_r, \tag{2.3}$$

respectively.

Proof. For the second identity, using the Binet formula (1.7)

$$\begin{aligned} C_{n-r}C_{n+r} - C_n^2 &= \left(\frac{r_1^{n-r} + r_2^{n-r}}{2}\right)\left(\frac{r_1^{n+r} + r_2^{n+r}}{2}\right) - \left(\frac{r_1^n + r_2^n}{2}\right)^2 \\ &= \frac{(r_1r_2)^{n-r}\left(r_2^{2r} + r_1^{2r} - 2r_1^r r_2^r\right)}{2^2} \\ &= \frac{(r_1^r + r_2^r)^2 - 4r_1^r r_2^r}{2^2} \\ &= C_r^2 - 1 \end{aligned}$$

and then the result follows. To obtain the last equality we use the definition of square triangular number, that is, $B_n^2 = ST_n$ and (2.1)

$$\begin{aligned} ST_{n-r}ST_{n+r} - ST_n^2 &= B_{n-r}^2 B_{n+r}^2 - B_n^4 \\ &= (B_n^2 - B_r^2)^2 - B_n^4 \\ &= B_r^4 - 2B_n^2 B_r^2 \\ &= ST_r^2 - 2ST_nST_r. \end{aligned}$$

□

Note that for $r = 1$ in Catalan's identities obtained, we get the Cassini identities for these sequences. In fact, the equations (2.1), (2.2) and (2.3), for $r = 1$, yields, respectively

$$B_{n-1}B_{n+1} - B_n^2 = -B_1^2,$$

$$C_{n-1}C_{n+1} - C_n^2 = C_1^2 - 1$$

and

$$ST_{n-1}ST_{n+1} - ST_n^2 = ST_1^2 - 2ST_nST_1.$$

Now, using one of the initial terms of these sequences, we obtain

Proposition 2.2 (Cassini's identities)**.** *For the sequences* $(B_n)_{n=0}^{\infty}$, $(C_n)_{n=0}^{\infty}$ *and* $(ST_n)_{n=0}^{\infty}$ *we have*

$$B_{n-1}B_{n+1} - B_n^2 = -1, \tag{2.4}$$

$$C_{n-1}C_{n+1} - C_n^2 = 8 \tag{2.5}$$

and

$$ST_{n-1}ST_{n+1} - ST_n^2 = 1 - 2ST_n,$$

respectively.

Note that an equivalent identity of (2.4) can be found in Behera and Panda [1] where its proof was done using induction on n.

The d'Ocagne identity for each of these sequences can also be obtained using the Binet formula for these type of sequences. We get

Proposition 2.3 (d'Ocagne's identities)**.** *For the sequences* $(B_n)_{n=0}^{\infty}$, $(C_n)_{n=0}^{\infty}$ *and* $(ST_n)_{n=0}^{\infty}$ *if* $m > n$ *we have*

$$B_mB_{n+1} - B_{m+1}B_n = B_{m-n}, \tag{2.6}$$

$$C_mC_{n+1} - C_{m+1}C_n = -8B_{m-n}$$

and

$$ST_mST_{n+1} - ST_{m+1}ST_n = \frac{1}{8}B_{m-n}\left(C_{m+n+1} - 3C_{m-n}\right),$$

respectively.

Proof. Once more, using the Binet formula (1.6), the fact that $r_1r_2 = 1$ and $m > n$, we get that $B_mB_{n+1} - B_{m+1}B_n$ is

$$\begin{aligned}
&\left(\frac{r_1^m - r_2^m}{r_1 - r_2}\right)\left(\frac{r_1^{n+1} - r_2^{n+1}}{r_1 - r_2}\right) - \left(\frac{r_1^{m+1} - r_2^{m+1}}{r_1 - r_2}\right)\left(\frac{r_1^n - r_2^n}{r_1 - r_2}\right) = \\
&= (r_1r_2)^n\frac{(r_1 - r_2)(r_1^{m-n} - r_2^{m-n})}{(r_1 - r_2)^2} \\
&= \frac{r_1^{m-n} - r_2^{m-n}}{r_1 - r_2} \\
&= B_{m-n}.
\end{aligned}$$

For the Lucas-balancing numbers, the proof of the statement is similar to the previous one.

For the equality involving the square triangular numbers, first we apply the fact that $B_n^2 = ST_n$ and then we obtain

$$ST_mST_{n+1} - ST_{m+1}ST_n = B_m^2B_{n+1}^2 - B_{m+1}^2B_n^2$$

$$= (B_m B_{n+1} - B_{m+1} B_n)\,(B_m B_{n+1} + B_{m+1} B_n)\,.$$

Now, we write $B_m B_{n+1} + B_{m+1} B_n$ as $\frac{1}{8}(C_{m+n+1} - 3C_{m-n})$ using the Binet formulas (1.6) and (1.7), $r_1 r_2 = 1$, $r_1 + r_2 = 6$ and doing some calculations. Finally using (2.6) the result follows. □

Once more, using the Binet formulas (1.6), (1.7) and (1.10) we obtain another property of the balancing, Lucas-balancing and square triangular sequences which is stated in the following proposition.

Proposition 2.4. *If B_n and C_n are the nth terms of the balancing sequence and Lucas-balancing sequence, respectively, then*

$$\lim_{n\to\infty} \frac{B_n}{B_{n-1}} = r_1 \tag{2.7}$$

and

$$\lim_{n\to\infty} \frac{C_n}{C_{n-1}} = r_1. \tag{2.8}$$

Consequently, if ST_n is the nth term of the square triangular sequence then

$$\lim_{n\to\infty} \frac{ST_n}{ST_{n-1}} = r_1^2. \tag{2.9}$$

Proof. We have that

$$\lim_{n\to\infty} \frac{B_n}{B_{n-1}} = \lim_{n\to\infty} \left(\frac{r_1^n - r_2^n}{r_1 - r_2}\right)\left(\frac{r_1 - r_2}{r_1^{n-1} - r_2^{n-1}}\right) = \lim_{n\to\infty} \left(\frac{r_1^n - r_2^n}{r_1^{n-1} - r_2^{n-1}}\right). \tag{2.10}$$

Since $\left|\frac{r_2}{r_1}\right| < 1$, $\lim_{n\to\infty} \left(\frac{r_2}{r_1}\right)^n = 0$. Next we use this fact writing (2.10) in an equivalent form, obtaining

$$\lim_{n\to\infty} \frac{B_n}{B_{n-1}} = \lim_{n\to\infty} \frac{1 - \left(\frac{r_2}{r_1}\right)^n}{\frac{1}{r_1} - \left(\frac{r_2}{r_1}\right)^n \frac{1}{r_2}} = \frac{1}{\frac{1}{r_1}} = r_1.$$

Proceeding in a similar way with C_n we get the analogous result for the Lucas-balancing sequence. For the square triangular sequence, taking into account that $ST_n = B_n^2$ and using (2.7) the results follows. □

Note that (2.7) and (2.8) are presented in Behera and Panda [1], but the authors used different methods in their proofs.

In what follows, we can easily show the next result using basic tools of calculus of limits, (2.7), (2.8) and (2.9).

Corollary 2.5. *If B_n, C_n and ST_n are the nth terms of the balancing sequence, Lucas-balancing sequence and square triangular sequence, respectively, then*

$$\lim_{n\to\infty} \frac{B_{n-1}}{B_n} = \frac{1}{r_1} = r_2,$$

$$\lim_{n\to\infty} \frac{C_{n-1}}{C_n} = \frac{1}{r_1} = r_2$$

and

$$\lim_{n\to\infty} \frac{ST_{n-1}}{ST_n} = \frac{1}{r_1^2} = r_2^2.$$

Ray [17] establishes some new identities for the common factors of both balancing and Lucas-balancing numbers. Also we can establish more identities listed in the following proposition. Some of these identities involve both type of numbers, sums of terms, products of terms, among other relations between terms of these sequences.

Proposition 2.6. *If B_j, C_j and ST_j are the jth terms of the balancing sequence, Lucas-balancing sequence and square triangular sequence, respectively, then*

1. $B_{2n} = 2C_nB_n$;
2. $ST_n^2 = B_n^4$;
3. $C_n^2 = 8B_n^2 + 1 = 8ST_n + 1$;
4. $C_{2n} = 16B_n^2 + 1$;
5. $B_{n+2} - B_{n-2} = 12C_n$;
6. $\sum_{j=0}^{n} B_j = \frac{-1-B_n+B_{n+1}}{4}$;
7. $\sum_{j=0}^{n} C_j = \frac{2-C_n+C_{n+1}}{4}$;
8. $\sum_{j=0}^{n} ST_j = \frac{-1-ST_n+ST_{n+1}-2n}{32}$.

Proof. The first five identities are easily proved using the Binet formulas for B_n, C_n and ST_n, respectively. For the first identity, we easily have that

$$2C_nB_n = 2\left(\frac{r_1^n + r_2^n}{2}\right)\left(\frac{r_1^n - r_2^n}{r_1 - r_2}\right) = B_{2n}$$

after doing some calculations. However, we can find in Panda [11] a different proof of this identity. The second identity is easily obtained using the well know fact that $B_n^2 = ST_n$. The third equality is a well known relation between the balancing and Lucas-balancing numbers [12]. We add one more identity using the fact that $C_n = \sqrt{8ST_n + 1}$. In order to obtain the fourth identity we only need adding and subtracting appropriate terms in order to get B_n^2. About the fifth identity we use the Binet formula for the sequences involved and we immediately get the result. Next we obtain the sum of the terms of the sequences, starting with the balancing sequence. Since $B_{n+1} = 6B_n - B_{n-1}$ for every $n \geq 1$, we have

$$6B_1 - B_0 = B_2$$

$$6B_2 - B_1 = B_3$$

$$6B_3 - B_2 = B_4$$

$$\cdots$$

$$6B_n - B_{n-1} = B_{n+1}.$$

Consequently,

$$6(B_1+B_2+B_3+\cdots+B_n)-B_0-B_1-B_2-\cdots-B_{n-1} = B_2+B_3+B_4+\cdots+B_{n+1}$$

which is equivalent to

$$6\sum_{j=1}^{n} B_j - 2\sum_{j=2}^{n-1} B_j = B_0 + B_1 + B_n + B_{n+1}.$$

Therefore

$$4\sum_{j=0}^{n} B_j = 5B_0 - B_1 - B_n + B_{n+1}.$$

Now if we consider the initial terms the result follows. For the Lucas-balancing and square triangular sequences proceeding in a similar way we obtain the required results. □

Remark 2.7. One of the most usual methods for the study of the recurrence sequences is to define the so-called generating matrix. Ray [15] introduces the matrix Q_B called $Q-$matrix and defined by $Q_B = \begin{bmatrix} 6 & -1 \\ 1 & 0 \end{bmatrix}$. This matrix is a generating matrix for balancing sequence. Ray [15, Theorem 1] gives the result $Q_B^n = \begin{bmatrix} B_{n+1} & -B_n \\ B_n & -B_{n-1} \end{bmatrix}$. Also, Ray [15] defines the $R-$matrix as $R_B = \begin{bmatrix} 3 & -1 \\ 1 & -3 \end{bmatrix}$ and the author shows that $R_B Q_B^n = \begin{bmatrix} C_{n+1} & -C_n \\ C_n & -C_{n-1} \end{bmatrix}$, where C_n denotes the nth Lucas-balancing number. Note that $|Q_B| = 1$ and so $|Q_B^n| = |Q_B|^n = 1$. On the other hand $|Q_B^n| = -B_{n-1}B_{n+1} + B_n^2$ and so we obtain, for balancing sequence, the respective Cassini identity given in the equation (2.4). For the Lucas-balancing sequence, using a similar argument as we did for balancing sequence, we can obtain the respective Cassini identity given in the equation (2.5). In fact, we know that $|R_B| = -8$ and $|R_B Q_B^n| = |R_B||Q_B|^n = -8$. On the other hand, $|R_B Q_B^n| = -C_{n+1}C_{n-1} + C_n^2$ and then the Cassini identity for Lucas-balancing sequence follows.

3. Cobalancing, Lucas-cobalancing: some identities

In this section we present new identities and properties for cobalancing and Lucas-cobalancing sequences previously given by the recurrence relations (1.2) and (1.5). These identities are easily proved using the Binet formulas, (1.6), (1.8) and (1.9) for the involved sequences.

Proposition 3.1 (Catalan's identities). *For the sequences* $(b_n)_{n=1}^{\infty}$ *and* $(c_n)_{n=1}^{\infty}$ *if* $n > r$ *we have*

$$b_{n-r}b_{n+r} - b_n^2 = B_r^2 - \frac{1}{2}\left(b_{n-r} + b_{n+r} - 2b_n\right) \tag{3.1}$$

and

$$c_{n-r}c_{n+r} - c_n^2 = -8B_r^2. \tag{3.2}$$

Note that for $r = 1$ in Catalan's identities obtained, we have the Cassini identities for these sequences. In fact, using the equations (3.1) and (3.2), for $r = 1$, and the initial terms of these sequences, we proved the following result:

Proposition 3.2 (Cassini's identities). *For the sequences* $(b_n)_{n=1}^{\infty}$ *and* $(c_n)_{n=1}^{\infty}$ *if* $n \geq 2$ *we have*

$$b_{n-1}b_{n+1} - b_n^2 = -2b_n$$

and

$$c_{n-1}c_{n+1} - c_n^2 = -8.$$

Panda and Ray [13] obtained an equivalent identity using different arguments in their proof.

Proposition 3.3 (d'Ocagne's identities). *If* $m > n$ *and for the sequences* $(b_n)_{n=1}^{\infty}$ *and* $(c_n)_{n=1}^{\infty}$*, we have*

$$b_m b_{n+1} - b_{m+1} b_n = -B_{m-n} + B_m - B_n$$

and

$$c_m c_{n+1} - c_{m+1} c_n = 16B_{m-n}.$$

As we proceeded in the previous cases for balancing, Lucas-balancing and square triangular sequences we obtain analogous results for cobalancing and Lucas-cobalancing, as a consequence of the Binet formulas (1.8) and (1.9).

Proposition 3.4. *If* b_n *and* c_n *are the nth terms of the cobalancing sequence and Lucas-cobalancing sequence, respectively, then*

$$\lim_{n\to\infty} \frac{b_n}{b_{n-1}} = r_1 \tag{3.3}$$

and

$$\lim_{n\to\infty} \frac{c_n}{c_{n-1}} = r_1. \tag{3.4}$$

Corollary 3.5. *If b_n and c_n are the nth terms of the cobalancing sequence and Lucas-cobalancing sequence, respectively, then*

$$\lim_{n\to\infty} \frac{b_{n-1}}{b_n} = \frac{1}{r_1} = r_2$$

and

$$\lim_{n\to\infty} \frac{c_{n-1}}{c_n} = \frac{1}{r_1} = r_2.$$

As we did for balancing, Lucas-balancing and square triangular numbers we present in the next result new identities, where some of them involve these type of numbers, sums of terms, products of terms, among others.

Proposition 3.6. *If b_j and c_j are the jth terms of the cobalancing sequence and Lucas-cobalancing sequence, respectively, then*

1. $c_n^2 = \frac{1}{2}(C_{2n-1}-1) = 8b_n^2+8b_n+1$, *where C_j is the jth term of Lucas-balancing sequence*;
2. $b_n^2 = \frac{1}{16}C_{2n-1}-b_n-\frac{3}{16}$, *where C_j is the jth term of Lucas-balancing sequence*;
3. $b_nc_n = \frac{1}{2}(B_{2n-1} - c_n)$, *where B_j is the jth term of balancing sequence*;
4. $\sum_{j=1}^{n} b_j = \frac{b_{n+1}-b_n-2n}{4}$;
5. $\sum_{j=1}^{n} c_j = \frac{c_{n+1}-c_n-2}{4}$.

Proof. The first three identities are easily proved using the Binet formulas for the sequences involved. For the first identity, one of the equalities come from a well known relation between the cobalancing and Lucas-cobalancing sequences and the other equality is easily obtained using the Binet formula for $(c_n)_{n=1}^{\infty}$ and doing some calculations. Again using the Binet formula and doing some calculations we get the second and third identities as we refered before. For the last two identities a similar process that we applied for the sum of the first n terms of balancing sequence, can be used and the result follows. □

4. Generating functions

Next we shall give the generating functions for balancing, cobalancing, square triangular numbers, Lucas-balancing and Lucas-cobalancing sequences. These sequences can be considered as the coefficients of the power series expansion of the corresponding generating function. Recall that a sequence $(x_n)_{n=1}^{\infty}$ has a generation function given by $f(x) = \sum_{n=1}^{\infty} a_n x^n$. Behera and Panda [1] obtained

Proposition 4.1. *The ordinary generating function of the balancing sequence can be written as*

$$G(B_n; x) = \frac{x}{1-6x+x^2}. \tag{4.1}$$

Also Panda and Ray [13] established

Proposition 4.2. *The ordinary generating function of the cobalancing sequence can be written as*

$$G(b_n; x) = \frac{2x^2}{(1-x)(1-6x+x^2)}. \tag{4.2}$$

Sloane and Plouffe [20] have already obtained a generating function to the square triangular numbers sequence

Proposition 4.3. *The ordinary generating function of the square triangular numbers sequence can be written as*

$$G(ST_n; x) = \frac{x(x+1)}{(1-x)(1-34x+x^2)}. \tag{4.3}$$

Now, consider the ordinary generating function $G(C_n; x)$ associated with the sequence $(C_n)_{n=0}^{\infty}$ and defined by

$$G(C_n; x) = \sum_{n=1}^{\infty} C_n x^n$$

Using the initial terms, we get

$$\begin{aligned} G(C_n; x) &= 3x + 17x^2 + \sum_{n=3}^{\infty} C_n x^n \\ &= 3x + 17x^2 + \sum_{n=3}^{\infty} (6C_{n-1} - C_{n-2}) x^n \\ &= 3x + 17x^2 + \sum_{n=3}^{\infty} 6C_{n-1} x^n - \sum_{n=3}^{\infty} C_{n-2} x^n \\ &= 3x + 17x^2 + 6x \sum_{n=3}^{\infty} C_{n-1} x^{n-1} - x^2 \sum_{n=3}^{\infty} C_{n-2} x^{n-2} \end{aligned}$$

and considering $k = n-1$ and $j = n-2$ then the last identity can be written as

$$\begin{aligned} G(C_n; x) &= 3x + 17x^2 + 6x \left(\sum_{k=1}^{\infty} C_k x^k - 3x \right) - x^2 \sum_{j=1}^{\infty} C_j x^j \\ &= 3x - x^2 + 6x \sum_{k=1}^{\infty} C_k x^k - x^2 \sum_{j=1}^{\infty} C_j x^j. \end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty} C_n x^n (1 - 6x + x^2) = 3x - x^2$$

and so we have the following result with respect to the Lucas-balancing sequence.

Proposition 4.4. *The ordinary generating function of the Lucas-balancing sequence can be written as*

$$G(C_n; x) = \frac{3x - x^2}{1 - 6x + x^2}. \tag{4.4}$$

With a slight modification of the previous proposition we obtain a generating function for Lucas-cobalancing sequence $(c_n)_{n=1}^{\infty}$ as

Proposition 4.5. *The ordinary generating function of the Lucas-cobalancing sequence can be written as*

$$G(c_n; x) = \frac{x + x^2}{1 - 6x + x^2}. \tag{4.5}$$

Now recall that for a sequence $(a_n)_{n=0}^{\infty}$, if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$, where L is a positive real number, considering the power series $\sum_{n=1}^{\infty} a_n x^k$ its radius of convergence R is equal to $\frac{1}{L}$. Hence, for the balancing, the Lucas-balancing, the cobalancing and the Lucas-cobalancing sequences, using the results (2.7), (2.8), (3.3) and (3.4), respectively, we know that the sequences can be written as a power series with radius of convergence equal to $\frac{1}{r_1} = r_2$. In the case of the square triangular numbers sequence, according to (2.9), the radius of convergence is $\frac{1}{r_1^2} = r_2^2$. For a sequence $(a_n)_{n=0}^{\infty}$ we have $a_n = \frac{h^{(n)}(0)}{n!}$, where the derivation is meant in the convergence domain, and $h(x)$ is the corresponding generating function. Next we give another expression for the general term of all sequences using the ordinary generating function (4.1), (4.2), (4.3), (4.4) and (4.5).

Remark 4.6. Let us consider $F(x) = \sum_{n=1}^{\infty} B_n x^n$, $f(x) = \sum_{n=1}^{\infty} b_n x^n$, $G(x) = \sum_{n=1}^{\infty} C_n x^n$, $g(x) = \sum_{n=1}^{\infty} c_n x^n$ for $x \in]-r_2, r_2[$ and $t(x) = \sum_{n=1}^{\infty} ST_n x^n$ for $x \in]-r_2^2, r_2^2[$. Then we have that

$$B_n = \frac{F^{(n)}(0)}{n!},$$

$$b_n = \frac{f^{(n)}(0)}{n!},$$

$$C_n = \frac{G^{(n)}(0)}{n!},$$

$$c_n = \frac{g^{(n)}(0)}{n!}$$

and

$$ST_n = \frac{t^{(n)}(0)}{n!},$$

where $F^{(n)}(x)$, $f^{(n)}(x)$, $G^{(n)}(x)$, $g^{(n)}(x)$ and $t^{(n)}(x)$ denote the nth order derivative of the functions F, f, G, g and t, respectively.

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Some inequalities for power series with nonnegative coefficients via a reverse of Jensen inequality

Silvestru Sever Dragomir

Mathematics, School of Engineering & Science
Victoria University, Melbourne City, Australia
sever.dragomir@vu.edu.au

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Abstract

Some inequalities for power series with nonnegative coefficients via a new reverse of Jensen inequality are given. Applications for some fundamental functions defined by power series are also provided.

Keywords: Power series, Jensen's inequality, Reverse of Jensen's inequality

MSC: 26D15; 26D10

1. Introduction

On utilizing some reverses of Jensen discrete inequality for convex functions, we obtained in [5] the following result for functions defined by power series with nonnegative coefficients:

Theorem 1.1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R > 0$ or $R = \infty$. If $p \geq 1$, $0 < \alpha < R$ and $x > 0$ with $\alpha x^p, \alpha x^{p-1} < R$, then*

$$0 \leq \frac{f(\alpha x^p)}{f(\alpha)} - \left[\frac{f(\alpha x)}{f(\alpha)}\right]^p \leq p\left[\frac{f(\alpha x^p)}{f(\alpha)} - \frac{f\left(\alpha x^{p-1}\right)}{f(\alpha)}\frac{f(\alpha x)}{f(\alpha)}\right]. \tag{1.1}$$

Moreover, if $0 < x \le 1$, *then*

$$0 \le \frac{f(\alpha x^p)}{f(\alpha)} - \left[\frac{f(\alpha x)}{f(\alpha)}\right]^p \le p\left[\frac{f(\alpha x^p)}{f(\alpha)} - \frac{f\left(\alpha x^{p-1}\right)}{f(\alpha)}\frac{f(\alpha x)}{f(\alpha)}\right] \tag{1.2}$$
$$\le \frac{1}{2}p\left(\frac{f\left(\alpha x^{2(p-1)}\right)}{f(\alpha)} - \left[\frac{f\left(\alpha x^{p-1}\right)}{f(\alpha)}\right]^2\right)^{1/2} \le \frac{1}{4}p$$

and

$$0 \le \frac{f(\alpha x^p)}{f(\alpha)} - \left[\frac{f(\alpha x)}{f(\alpha)}\right]^p \le p\left[\frac{f(\alpha x^p)}{f(\alpha)} - \frac{f\left(\alpha x^{p-1}\right)}{f(\alpha)}\frac{f(\alpha x)}{f(\alpha)}\right] \tag{1.3}$$
$$\le \frac{1}{2}p\left(\frac{f\left(\alpha x^{2}\right)}{f(\alpha)} - \left[\frac{f(\alpha x)}{f(\alpha)}\right]^2\right)^{1/2} \le \frac{1}{4}p.$$

Corollary 1.2. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be a power series with nonnegative coefficients and convergent on the open disk* $D(0,R)$ *with* $R > 0$ *or* $R = \infty$. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $u, v > 0$ *with* $v^p \le u^q < R$, *then*

$$\left[\frac{f(uv)}{f(u^q)}\right]^p \le \frac{f(v^p)}{f(u^q)} \le \frac{1}{4}p + \left[\frac{f(uv)}{f(u^q)}\right]^p \tag{1.4}$$

and

$$0 \le [f(v^p)]^{1/p}[f(u^q)]^{1/q} - f(uv) \le \frac{1}{4^{1/p}}p^{1/p}f(u^q). \tag{1.5}$$

Utilising a different approach in [6] we obtained the following results as well:

Theorem 1.3. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be a power series with nonnegative coefficients and convergent on the open disk* $D(0,R)$ *with* $R > 0$ *or* $R = \infty$. *If* $p > 1$, $0 < \alpha < R$ *and* $0 < x \le 1$, *then*

$$0 \le \frac{f(\alpha x^p)}{f(\alpha)} - \left(\frac{f(\alpha x)}{f(\alpha)}\right)^p \le M_p\left(1 - \frac{f(\alpha x)}{f(\alpha)}\right)\frac{f(\alpha x)}{f(\alpha)} \le \frac{1}{4}M_p \tag{1.6}$$

and

$$0 \le \frac{f(\alpha x^p)}{f(\alpha)} - \left(\frac{f(\alpha x)}{f(\alpha)}\right)^p \le \frac{1}{4}\cdot\frac{1 - \left(\frac{f(\alpha x)}{f(\alpha)}\right)^{p-1}}{1 - \frac{f(\alpha x)}{f(\alpha)}} \le \frac{1}{4}M_p, \tag{1.7}$$

where

$$M_p := \begin{cases} 1 & \text{if } p \in (1,2], \\ p-1 & \text{if } p \in (2,\infty). \end{cases}$$

Corollary 1.4. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R>0$ or $R=\infty$. If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $u,v>0$ with $v^p \le u^q < R$, then*

$$0 \le \frac{f(v^p)}{f(u^q)} - \left(\frac{f(uv)}{f(u^q)}\right)^p \le M_p \left(1 - \frac{f(uv)}{f(u^q)}\right) \frac{f(uv)}{f(u^q)} \le \frac{1}{4} M_p \tag{1.8}$$

and

$$0 \le \frac{f(v^p)}{f(u^q)} - \left(\frac{f(uv)}{f(u^q)}\right)^p \le \frac{1}{4} \cdot \frac{1 - \left(\frac{f(uv)}{f(u^q)}\right)^{p-1}}{1 - \frac{f(uv)}{f(u^q)}} \le \frac{1}{4} M_p. \tag{1.9}$$

For some similar exponential and logarithmic inequalities see [5] and [6] where further applications for some fundamental functions were provided .

For other recent results for power series with nonnegative coefficients, see [2, 8, 12, 13]. For more results on power series inequalities, see [2] and [8]–[11].

The most important power series with nonnegative coefficients that can be used to illustrate the above results are:

$$\exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n, \ z \in \mathbb{C}, \ \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \ z \in D(0,1), \tag{1.10}$$

$$\ln \frac{1}{1-z} = \sum_{n=1}^{\infty} \frac{1}{n} z^n, \ z \in D(0,1), \ \cosh z = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n}, \ z \in \mathbb{C},$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1}, \ z \in \mathbb{C}.$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$\frac{1}{2} \ln\left(\frac{1+z}{1-z}\right) = \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1}, \ z \in D(0,1), \tag{1.11}$$

$$\sin^{-1}(z) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!} z^{2n+1}, \ z \in D(0,1),$$

$$\tanh^{-1}(z) = \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1}, \ z \in D(0,1),$$

$$ {}_2F_1(\alpha,\beta,\gamma,z) := \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)} z^n, \alpha,\beta,\gamma > 0$$
$$z \in D(0,1),$$

where Γ is *Gamma function.*

Motivated by the above results and utilizing a reverse of Jensen's inequality, we provide in this paper other inequalities for power series with nonnegative coefficients. Applications for some fundamental functions are given as well.

2. A reverse of Jensen's inequality

The following result holds:

Theorem 2.1. *Let* $f : I \to \mathbb{R}$ *be a continuous convex function on the interval of real numbers* I *and* $m, M \in \mathbb{R}$, $m < M$ *with* $[m, M] \subset \mathring{I}$, $\mathring{I}$ *is the interior of* I. *If* $x_i \in [m, M]$ *and* $w_i \geq 0$ $(i = 1, \ldots, n)$ *with* $W_n := \sum_{i=1}^{n} w_i = 1$, *then we have the inequalities*

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} w_i f(x_i) - f\left(\sum_{i=1}^{n} w_i x_i\right) \\ &\leq 2 \max\left\{\frac{M - \sum_{i=1}^{n} w_i x_i}{M - m}, \frac{\sum_{i=1}^{n} w_i x_i - m}{M - m}\right\} \\ &\times \left[\frac{f(m) + f(M)}{2} - f\left(\frac{m + M}{2}\right)\right]. \end{aligned} \tag{2.1}$$

Proof. We recall the following result obtained by the author in [4] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned} & n \min_{i \in \{1,\ldots,n\}} \{p_i\} \left[\frac{1}{n} \sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n} \sum_{i=1}^{n} x_i\right)\right] \\ &\leq \frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i) - f\left(\frac{1}{P_n} \sum_{i=1}^{n} p_i x_i\right) \\ &\leq n \max_{i \in \{1,\ldots,n\}} \{p_i\} \left[\frac{1}{n} \sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n} \sum_{i=1}^{n} x_i\right)\right], \end{aligned} \tag{2.2}$$

where $f \colon C \to \mathbb{R}$ is a convex function defined on the convex subset C of the linear space X, $\{x_i\}_{i \in \{1,\ldots,n\}} \subset C$ are vectors and $\{p_i\}_{i \in \{1,\ldots,n\}}$ are nonnegative numbers with $P_n := \sum_{i=1}^{n} p_i > 0$.

For $n = 2$ we deduce from (2.2) that

$$\begin{aligned} & 2 \min\{t, 1 - t\} \left[\frac{f(x) + f(y)}{2} - f\left(\frac{x + y}{2}\right)\right] \\ &\leq t f(x) + (1 - t) f(y) - f(tx + (1 - t) y) \\ &\leq 2 \max\{t, 1 - t\} \left[\frac{f(x) + f(y)}{2} - f\left(\frac{x + y}{2}\right)\right] \end{aligned} \tag{2.3}$$

for any $x, y \in C$ and $t \in [0, 1]$.

If we use the second inequality in (2.3) for the convex function $f \colon I \to \mathbb{R}$ where $m, M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$, we have for $t = \frac{M - \sum_{i=1}^{n} w_i x_i}{M - m}$ that

$$\frac{\left(M - \sum_{i=1}^{n} w_i x_i\right) f(m) + \left(\sum_{i=1}^{n} w_i x_i - m\right) f(M)}{M - m} \tag{2.4}$$

$$- f\left(\frac{m\left(M-\sum_{i=1}^{n} w_i x_i\right)+M\left(\sum_{i=1}^{n} w_i x_i-m\right)}{M-m}\right)$$
$$\leq 2\max\left\{\frac{M-\sum_{i=1}^{n} w_i x_i}{M-m}, \frac{\sum_{i=1}^{n} w_i x_i-m}{M-m}\right\}$$
$$\times\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right].$$

By the convexity of f we have that

$$\sum_{i=1}^{n} w_i f(x_i)-f\left(\sum_{i=1}^{n} w_i x_i\right) \tag{2.5}$$
$$=\sum_{i=1}^{n} w_i f\left[\frac{m(M-x_i)+M(x_i-m)}{M-m}\right]$$
$$- f\left(\sum_{i=1}^{n} w_i\left[\frac{m(M-x_i)+M(x_i-m)}{M-m}\right]\right)$$
$$\leq \sum_{i=1}^{n} w_i \frac{(M-x_i) f(m)+(x_i-m) f(M)}{M-m}$$
$$- f\left(\frac{m\left(M-\sum_{i=1}^{n} w_i x_i\right)+M\left(\sum_{i=1}^{n} w_i x_i-m\right)}{M-m}\right)$$
$$=\frac{\left(M-\sum_{i=1}^{n} w_i x_i\right) f(m)+\left(\sum_{i=1}^{n} w_i x_i-m\right) f(M)}{M-m}$$
$$- f\left(\frac{m\left(M-\sum_{i=1}^{n} w_i x_i\right)+M\left(\sum_{i=1}^{n} w_i x_i-m\right)}{M-m}\right).$$

Utilizing the inequality (2.5) and (2.4) we deduce the desired inequality in (2.1). □

For some related integral versions, see [4].

Remark 2.2. Since, obviously,

$$\frac{M-\sum_{i=1}^{n} w_i x_i}{M-m}, \frac{\sum_{i=1}^{n} w_i x_i-m}{M-m} \leq 1,$$

then we obtain from the first inequality in (2.1) the simpler, however coarser inequality, namely

$$0 \leq \sum_{i=1}^{n} w_i f(x_i)-f\left(\sum_{i=1}^{n} w_i x_i\right) \leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right], \tag{2.6}$$

provided that $x_i \in [m, M]$ and $w_i \geq 0$ $(i=1,\ldots,n)$ with $W_n := \sum_{i=1}^{n} w_i = 1$.

This inequality was obtained in 2008 by S. Simić in [14].

Example 2.3. a) If we write the inequality (2.1) for the convex function $f\colon [m,M]\subset[0,\infty)\to[0,\infty)$, $f(t)=t^p$, $p\geq 1$, then we have

$$
\begin{aligned}
0 &\leq \sum_{i=1}^{n} w_i x_i^p - \left(\sum_{i=1}^{n} w_i x_i\right)^p \qquad (2.7)\\
&\leq 2\max\left\{\frac{M-\sum_{i=1}^{n} w_i x_i}{M-m}, \frac{\sum_{i=1}^{n} w_i x_i - m}{M-m}\right\}\left[\frac{m^p+M^p}{2}-\left(\frac{m+M}{2}\right)^p\right]\\
&\leq 2\left[\frac{m^p+M^p}{2}-\left(\frac{m+M}{2}\right)^p\right],
\end{aligned}
$$

for any $x_i\in[m,M]$ and $w_i\geq 0$ $(i=1,\dots,n)$ with $W_n:=\sum_{i=1}^{n} w_i=1$.

b) If we apply the inequality (2.1) for the convex function $f\colon [m,M]\subset[0,\infty)\to[0,\infty)$, $f(t)=-\ln t$, then we have

$$
\begin{aligned}
0 &\leq \ln\left(\sum_{i=1}^{n} w_i x_i\right) - \sum_{i=1}^{n} w_i \ln x_i \qquad (2.8)\\
&\leq 2\max\left\{\frac{M-\sum_{i=1}^{n} w_i x_i}{M-m}, \frac{\sum_{i=1}^{n} w_i x_i - m}{M-m}\right\}\ln\left(\frac{\frac{m+M}{2}}{\sqrt{mM}}\right)\\
&\leq \ln\left(\frac{\frac{m+M}{2}}{\sqrt{mM}}\right)^2
\end{aligned}
$$

for any $x_i\in[m,M]$ and $w_i\geq 0$ $(i=1,\dots,n)$ with $W_n:=\sum_{i=1}^{n} w_i=1$.

This inequality is equivalent to

$$
\begin{aligned}
1 &\leq \frac{\sum_{i=1}^{n} w_i x_i}{\prod_{i=1}^{n} x_i^{w_i}} \leq \left(\frac{\frac{m+M}{2}}{\sqrt{mM}}\right)^{2\max\left\{\frac{M-\sum_{i=1}^{n} w_i x_i}{M-m}, \frac{\sum_{i=1}^{n} w_i x_i - m}{M-m}\right\}} \qquad (2.9)\\
&\leq \frac{(m+M)^2}{4mM}
\end{aligned}
$$

for any $x_i\in[m,M]$ and $w_i\geq 0$ $(i=1,\dots,n)$ with $W_n:=\sum_{i=1}^{n} w_i=1$.

We can state the following result connected to *Hölder's inequality*:

Proposition 2.4. *If $x_i\geq 0$, $y_i>0$ for $i\in\{1,\dots,n\}$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and such that*

$$0\leq k\leq \frac{x_i}{y_i^{q-1}}\leq K \text{ for } i\in\{1,\dots,n\}, \qquad (2.10)$$

then we have

$$0\leq \frac{\sum_{i=1}^{n} x_i^p}{\sum_{i=1}^{n} y_i^q} - \left(\frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} y_i^q}\right)^p \qquad (2.11)$$

$$\le 2\max\left\{\frac{K-\frac{\sum_{i=1}^{n}x_iy_i}{\sum_{i=1}^{n}y_i^q}}{K-k},\frac{\frac{\sum_{i=1}^{n}x_iy_i}{\sum_{i=1}^{n}y_i^q}-k}{K-k}\right\}\left[\frac{k^p+K^p}{2}-\left(\frac{k+K}{2}\right)^p\right]$$
$$\le 2\left[\frac{k^p+K^p}{2}-\left(\frac{k+K}{2}\right)^p\right].$$

Proof. The inequalities (2.11) follow from (2.7) by choosing

$$z_i=\frac{x_i}{y_i^{q-1}} \text{ and } w_i=\frac{y_i^q}{\sum_{j=1}^{n}y_j^q}, i\in\{1,\dots,n\}.$$

The details are omitted. □

Remark 2.5. Let $p>1$, $\frac{1}{p}+\frac{1}{q}=1$. Assume that

$$0\le k\le\frac{a_i}{b_i^{q-1}}\le K,\ \text{ for } i\in\{1,\dots,n\}. \tag{2.12}$$

If $p_i>0$ for $i\in\{1,\dots,n\}$, then for $x_i:=p_i^{1/p}a_i$ and $y_i:=p_i^{1/q}b_i$ we have

$$\frac{x_i}{y_i^{q-1}}=\frac{p_i^{1/p}a_i}{\left(p_i^{1/q}b_i\right)^{q-1}}=\frac{p_i^{1/p}a_i}{p_i^{(q-1)/q}b_i^{q-1}}=\frac{p_i^{1/p}a_i}{p_i^{1/p}b_i^{q-1}}=\frac{a_i}{b_i^{q-1}}\in[k,K]$$

for $i\in\{1,\dots,n\}$.

If we write the inequality (2.11) for these choices, we get the weighted inequalities

$$0\le\frac{\sum_{i=1}^{n}p_ia_i^p}{\sum_{i=1}^{n}p_ib_i^q}-\left(\frac{\sum_{i=1}^{n}p_ia_ib_i}{\sum_{i=1}^{n}p_ib_i^q}\right)^p \tag{2.13}$$
$$\le 2\max\left\{\frac{K-\frac{\sum_{i=1}^{n}p_ia_ib_i}{\sum_{i=1}^{n}p_ib_i^q}}{K-k},\frac{\frac{\sum_{i=1}^{n}p_ia_ib_i}{\sum_{i=1}^{n}p_ib_i^q}-k}{K-k}\right\}\left[\frac{k^p+K^p}{2}-\left(\frac{k+K}{2}\right)^p\right]$$
$$\le 2\left[\frac{k^p+K^p}{2}-\left(\frac{k+K}{2}\right)^p\right].$$

From this inequality we have:

$$\left(\frac{\sum_{i=1}^{n}p_ia_ib_i}{\sum_{i=1}^{n}p_ib_i^q}\right)^p\le\frac{\sum_{i=1}^{n}p_ia_i^p}{\sum_{i=1}^{n}p_ib_i^q} \tag{2.14}$$
$$\le\left(\frac{\sum_{i=1}^{n}p_ia_ib_i}{\sum_{i=1}^{n}p_ib_i^q}\right)^p+2\left[\frac{k^p+K^p}{2}-\left(\frac{k+K}{2}\right)^p\right].$$

Taking into the second inequality of (2.14) the power $1/p$ and utilizing the elementary inequality

$$(\alpha+\beta)^{1/p}\le\alpha^{1/p}+\beta^{1/p},\alpha,\beta\ge 0 \text{ and } p>1,$$

then we get the following additive reverse of Hölder inequality

$$\left(\sum_{i=1}^{n} p_i a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} p_i b_i^q\right)^{1/q} \le \sum_{i=1}^{n} p_i a_i b_i + 2^{1/p}\left[\frac{k^p+K^p}{2} - \left(\frac{k+K}{2}\right)^p\right]^{1/p} \sum_{i=1}^{n} p_i b_i^q, \tag{2.15}$$

provided

$$0 \le k \le \frac{a_i}{b_i^{q-1}} \le K, \text{ for } i \in \{1, \dots, n\}$$

and $p_i > 0$ for $i \in \{1, \dots, n\}$.

3. Power inequalities

We can state the following result for powers:

Theorem 3.1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R > 0$ or $R = \infty$. If $p > 1$, $0 < \alpha < R$ and $0 < x \le 1$, then*

$$0 \le \frac{f(\alpha x^p)}{f(\alpha)} - \left(\frac{f(\alpha x)}{f(\alpha)}\right)^p \le \frac{2^{p-1}-1}{2^{p-1}} \max\left\{1 - \frac{f(\alpha x)}{f(\alpha)}, \frac{f(\alpha x)}{f(\alpha)}\right\} \le \frac{2^{p-1}-1}{2^{p-1}}. \tag{3.1}$$

Proof. Let $m \ge 1$ and $0 < \alpha < R$, $0 < x \le 1$. If we write the inequality (2.7) for

$$w_j = \frac{a_j \alpha^j}{\sum_{k=0}^{m} a_k \alpha^k} \text{ and } z_j := x^j \in [0,1], \ j \in \{0, \dots, m\},$$

then we get

$$\begin{aligned} 0 &\le \frac{1}{\sum_{k=0}^{m} a_k \alpha^k} \sum_{j=0}^{m} a_j \alpha^j x^{pj} - \left(\frac{1}{\sum_{k=0}^{m} a_k \alpha^k} \sum_{j=0}^{m} a_j \alpha^j x^j\right)^p \\ &\le \frac{2^{p-1}-1}{2^{p-1}} \max\left\{1 - \frac{1}{\sum_{k=0}^{m} a_k \alpha^k} \sum_{j=0}^{m} a_j \alpha^j x^j, \frac{1}{\sum_{k=0}^{m} a_k \alpha^k} \sum_{j=0}^{m} a_j \alpha^j x^j\right\} \\ &\le \frac{2^{p-1}-1}{2^{p-1}}. \end{aligned} \tag{3.2}$$

Since all series whose partial sums involved in the inequality (3.2) are convergent, then by letting $m \to \infty$ in (3.2) we deduce (2.15). □

Corollary 3.2. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R>0$ or $R=\infty$. If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $u,v>0$ with $v^p\le u^q<R$, then*

$$0\le \frac{f(v^p)}{f(u^q)}-\left(\frac{f(uv)}{f(u^q)}\right)^p\le \frac{2^{p-1}-1}{2^{p-1}}\max\left\{1-\frac{f(uv)}{f(u^q)},\frac{f(uv)}{f(u^q)}\right\} \tag{3.3}$$

and

$$0\le [f(v^p)]^{1/p}[f(u^q)]^{1/q}-f(uv)\le\left(\frac{2^{p-1}-1}{2^{p-1}}\right)^{1/p}f(u^q). \tag{3.4}$$

Proof. The inequality (3.3) follows by taking into (3.1) $\alpha=u^q$ and $x=\frac{v}{u^{q/p}}$. The details are omitted.

Taking the power $1/p$ and using the inequality $(a+b)^{1/p}\le a^{1/p}+b^{1/p}$, $p\ge 1$ we get from

$$\frac{f(v^p)}{f(u^q)}\le\left(\frac{f(uv)}{f(u^q)}\right)^p+\frac{2^{p-1}-1}{2^{p-1}}$$

the desired inequality (3.4). □

Example 3.3. a) If we write the inequality (3.1) for the function $\frac{1}{1-z}=\sum_{n=0}^{\infty}z^n$, $z\in D(0,1)$, then we have

$$0\le\frac{1-\alpha}{1-\alpha x^p}-\left(\frac{1-\alpha}{1-\alpha x}\right)^p\le\frac{2^{p-1}-1}{2^{p-1}}\max\left\{\frac{\alpha(1-x)}{1-\alpha x},\frac{1-\alpha}{1-\alpha x}\right\} \tag{3.5}$$

for any $\alpha,x\in(0,1)$ and $p\ge 1$.

b) If we write the inequality (3.1)) for the function $\exp z=\sum_{n=0}^{\infty}\frac{1}{n!}z^n$, $z\in\mathbb{C}$, then we have

$$\begin{aligned}0&\le\exp[\alpha(x^p-1)]-\exp[p\alpha(x-1)] \\ &\le\frac{2^{p-1}-1}{2^{p-1}}\max\{1-\exp[\alpha(x-1)],\exp[\alpha(x-1)]\}\end{aligned} \tag{3.6}$$

for any $\alpha,p>0$ and $x\in(0,1)$.

4. Logarithmic inequalities

If we write the inequality (2.1) for the convex function $f\colon [m,M]\subset(0,\infty)\to\mathbb{R}$, $f(t)=t\ln t$, then we have

$$\begin{aligned}0&\le\sum_{i=1}^{n}w_ix_i\ln x_i-\left(\sum_{i=1}^{n}w_ix_i\right)\ln\left(\sum_{i=1}^{n}w_ix_i\right) \\ &\le 2\max\left\{\frac{M-\sum_{i=1}^{n}w_ix_i}{M-m},\frac{\sum_{i=1}^{n}w_ix_i-m}{M-m}\right\}\end{aligned} \tag{4.1}$$

$$\times\left[\frac{m\ln m+M\ln M}{2}-\left(\frac{m+M}{2}\right)\ln\left(\frac{m+M}{2}\right)\right]$$

for any $x_i\in[m,M]$ and $w_i\geq 0$ $(i=1,\dots,n)$ with $W_n:=\sum_{i=1}^n w_i=1$.

This is equivalent to

$$1\leq\frac{\prod_{i=1}^n x_i^{w_ix_i}}{\left(\sum_{i=1}^n w_ix_i\right)^{\left(\sum_{i=1}^n w_ix_i\right)}}\leq\left[\frac{m^mM^M}{\left(\frac{m+M}{2}\right)^{m+M}}\right]^{\max\left\{\frac{M-\sum_{i=1}^n w_ix_i}{M-m},\frac{\sum_{i=1}^n w_ix_i-m}{M-m}\right\}}\tag{4.2}$$

for any $x_i\in[m,M]$ and $w_i\geq 0$ $(i=1,\dots,n)$ with $W_n:=\sum_{i=1}^n w_i=1$.

If we take $M=1$ and let $m\to 0+$ in the inequality (4.1), we have

$$0\leq\sum_{i=1}^n w_ix_i\ln x_i-\left(\sum_{i=1}^n w_ix_i\right)\ln\left(\sum_{i=1}^n w_ix_i\right)\leq\max\left\{1-\sum_{i=1}^n w_ix_i,\sum_{i=1}^n w_ix_i\right\}\ln 2,\tag{4.3}$$

for any $x_i\in(0,1]$ and $w_i\geq 0$ $(i=1,\dots,n)$ with $W_n:=\sum_{i=1}^n w_i=1$.

This is equivalent to

$$1\leq\frac{\prod_{i=1}^n x_i^{w_ix_i}}{\left(\sum_{i=1}^n w_ix_i\right)^{\left(\sum_{i=1}^n w_ix_i\right)}}\leq 2^{\max\left\{1-\sum_{i=1}^n w_ix_i,\sum_{i=1}^n w_ix_i\right\}},\tag{4.4}$$

for any $x_i\in(0,1]$ and $w_i\geq 0$ $(i=1,\dots,n)$ with $W_n:=\sum_{i=1}^n w_i=1$.

Theorem 4.1. *Let $f(z)=\sum_{n=0}^\infty a_nz^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R>0$ or $R=\infty$. If $0<\alpha<R$, $p>0$ and $x\in(0,1)$, then*

$$0\leq\frac{p\alpha x^pf'(\alpha x^p)}{f(\alpha)}\ln x-\frac{f(\alpha x^p)}{f(\alpha)}\ln\left(\frac{f(\alpha x^p)}{f(\alpha)}\right)\leq\max\left\{1-\frac{f(\alpha x^p)}{f(\alpha)},\frac{f(\alpha x^p)}{f(\alpha)}\right\}\ln 2.\tag{4.5}$$

Proof. If $0<\alpha<R$ and $m\geq 1$, then by (4.3) for $x_j=(x^p)^j$, we have

$$0\leq\frac{1}{\sum_{k=0}^m a_k\alpha^k}\sum_{j=0}^m a_j\alpha^j(x^p)^j\ln(x^p)^j\tag{4.6}$$

$$
\begin{aligned}
&- \left(\frac{1}{\sum_{k=0}^{m} a_k \alpha^k} \sum_{j=0}^{m} a_j \alpha^j \left(x^p\right)^j \right) \ln \left(\frac{1}{\sum_{k=0}^{m} a_k \alpha^k} \sum_{j=0}^{m} a_j \alpha^j \left(x^p\right)^j \right) \\
&\le \max \left\{ 1 - \frac{1}{\sum_{k=0}^{m} a_k \alpha^k} \sum_{j=0}^{m} a_j \alpha^j \left(x^p\right)^j , \frac{1}{\sum_{k=0}^{m} a_k \alpha^k} \sum_{j=0}^{m} a_j \alpha^j \left(x^p\right)^j \right\} \ln 2,
\end{aligned}
$$

for $p > 0$ and $x \in (0, 1)$. This is equivalent to:

$$
\begin{aligned}
0 \le \;& \frac{p \ln (x)}{\sum_{k=0}^{m} a_k \alpha^k} \sum_{j=0}^{m} j a_j \alpha^j \left(x^p\right)^j \\
&- \left(\frac{1}{\sum_{k=0}^{m} a_k \alpha^k} \sum_{j=0}^{m} a_j \alpha^j \left(x^p\right)^j \right) \ln \left(\frac{1}{\sum_{k=0}^{m} a_k \alpha^k} \sum_{j=0}^{m} a_j \alpha^j \left(x^p\right)^j \right) \\
&\le \max \left\{ 1 - \frac{1}{\sum_{k=0}^{m} a_k \alpha^k} \sum_{j=0}^{m} a_j \alpha^j \left(x^p\right)^j , \frac{1}{\sum_{k=0}^{m} a_k \alpha^k} \sum_{j=0}^{m} a_j \alpha^j \left(x^p\right)^j \right\} \ln 2,
\end{aligned} \tag{4.7}
$$

for $p > 0$ and $x \in (0, 1)$.

Since $0 < \alpha < R$, $x \in (0, 1)$ and $p > 0$ then $0 < \alpha x^p < R$ and the series

$$
\sum_{k=0}^{\infty} a_k \alpha^k, \sum_{j=0}^{\infty} j a_j \alpha^j \left(x^p\right)^j \text{ and } \sum_{j=0}^{\infty} a_j \alpha^j \left(x^p\right)^j
$$

are convergent. Therefore by letting $m \to \infty$ in (4.7) we deduce (4.5). □

Example 4.2. a) If we write the inequality (4.5) for the function $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, $z \in D(0, 1)$, then we have for $\alpha, x \in (0, 1)$ and $p > 0$ that

$$
\begin{aligned}
0 &\le \frac{p \alpha x^p (1 - \alpha)}{(1 - \alpha x^p)^2} \ln x - \frac{1 - \alpha}{(1 - \alpha x^p)} \ln \left(\frac{1 - \alpha}{1 - \alpha x^p} \right) \\
&\le \max \left\{ \frac{\alpha (1 - x^p)}{1 - \alpha x^p}, \frac{1 - \alpha}{1 - \alpha x^p} \right\} \ln 2
\end{aligned} \tag{4.8}
$$

b) If we write the inequality (4.5) for the function $\exp z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$, $z \in \mathbb{C}$, then we have

$$
\begin{aligned}
0 &\le \left[p \alpha x^p \ln x - \alpha (x^p - 1) \right] \exp \left[\alpha (x^p - 1) \right] \\
&\le \max \left\{ 1 - \exp \left[\alpha (x^p - 1) \right], \exp \left[\alpha (x^p - 1) \right] \right\} \ln 2
\end{aligned} \tag{4.9}
$$

for $x \in (0, 1)$ and $\alpha, p > 0$.

5. Exponential inequalities

If we consider the exponential function $f\colon \mathbb{R} \to (0,\infty)$, $f(t) = \exp(t)$, then from (2.1) we have the inequalities

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} w_i \exp(x_i) - \exp\left(\sum_{i=1}^{n} w_i x_i\right) \\ &\leq 2\max\left\{\frac{M-\sum_{i=1}^{n} w_i x_i}{M-m}, \frac{\sum_{i=1}^{n} w_i x_i - m}{M-m}\right\} \\ &\times \left[\frac{\exp(m)+\exp(M)}{2} - \exp\left[\left(\frac{m+M}{2}\right)\right]\right] \end{aligned} \tag{5.1}$$

if $x_i \in [m, M]$ and $w_i \geq 0$ $(i = 1, \ldots, n)$ with $W_n := \sum_{i=1}^{n} w_i = 1$.

If we take in (5.1) $M = 0$ and let $m \to -\infty$, then we get

$$0 \leq \sum_{i=1}^{n} w_i \exp(x_i) - \exp\left(\sum_{i=1}^{n} w_i x_i\right) \leq 1 \tag{5.2}$$

for $x_i \leq 0$ and $w_i \geq 0$ $(i = 1, \ldots, n)$ with $W_n := \sum_{i=1}^{n} w_i = 1$.

Theorem 5.1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R > 0$ or $R = \infty$. If $x \leq 0$ with $\exp(x) < R$ and $0 < \alpha < R$, then*

$$0 \leq \frac{f(\alpha \exp(x))}{f(\alpha)} - \exp\left[\frac{\alpha x f'(\alpha)}{f(\alpha)}\right] \leq 1. \tag{5.3}$$

Proof. If $0 < \alpha < R$ and $m \geq 1$, then by (5.2) for $x_j = jx$, we have

$$0 \leq \frac{1}{\sum_{j=0}^{m} a_j \alpha^j} \sum_{j=0}^{m} a_j \alpha^j [\exp(x)]^j - \exp\left(\frac{x}{\sum_{j=0}^{m} a_j \alpha^j} \sum_{j=0}^{m} j a_j \alpha^j\right) \leq 1 \tag{5.4}$$

for $x \in (-\infty, 0)$.

Since all series whose partial sums involved in the inequality (5.4) are convergent, then by letting $m \to \infty$ in (5.4) we deduce (5.3). □

Example 5.2. a) If we write the inequality (5.3) for the function $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, $z \in D(0,1)$, then we have for $x \leq 0$ and $0 < \alpha < 1$, that

$$0 \leq \frac{1-\alpha}{1-\alpha\exp(x)} - \exp\left(\frac{\alpha x}{1-\alpha}\right) \leq 1. \tag{5.5}$$

b) If we write the inequality (5.3) for the function $\exp z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$, $z \in \mathbb{C}$, then we have

$$0 \leq \exp(\alpha[\exp(x) - 1]) - \exp(\alpha x) \leq 1 \tag{5.6}$$

for any $\alpha > 0$ and $x \leq 0$.

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Visualization of univariate data for comparison

Csaba Faragó

Department of Software Engineering
University of Szeged, Hungary
farago@inf.u-szeged.hu

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Abstract

"A picture is worth a thousand words." This idiom is true for research studies as well: illustrations in a paper helps the reader to better understand the findings of the authors. There are already several possibilities for visualizing data. But there always exist cases when the currently available diagram types are not useful enough. We also ran into such a situation, and created two new diagram types: Cumulative Characteristic Diagram and Quantile Difference Diagram for illustrating data sets of numeric types.

The Cumulative Characteristic Diagram is a curve, which is based on the non-ascending order of the values. It makes it easy to read many characteristics of the input data, and it is suitable to find similarities and differences between several data sets quickly.

Quantile Difference Diagram draws the differences of two ascending sets of data on the same quantiles. This diagram is suitable to illustrate in which subset the data are higher, and it also reveals some important details, which would remain hidden using statistic tests only.

We found them very useful both in explaining our actual results, and gaining ideas for further development directions. In this article we show the usefulness of these diagrams illustrating the results of Contingency Chi-Squared tests, Wilcoxon rank tests and variance tests.

Keywords: univariate data, data visualization, maintainability, Cumulative Characteristic Diagram, Quantile Difference Diagram

1. Introduction

1.1. Overview

In research data arise. Visualization of this is very important, as a diagram may reveal important characteristics. Furthermore, illustrating the statistic tests with proper diagrams might help understanding the results.

A great number of diagram types exist, but sometimes none of them are really adequate for visualization. In this study we present 2 diagram types invented and implemented by us. One of them we call Cumulative Characteristic Diagram, and abbreviate CCD. The other one we call Quantile Difference Diagram, and abbreviate QDD.

These diagrams helped us in further research, and we found them useful in illustrating the results of Contingency Chi-Squared test, the Wilcoxon test and variance test.

1.2. Motivation

The motivating examples come from our research in finding the impact of various developer interactions on software quality. In this section we explain the idea using a very high level of abstraction, concentrating on the data only.

We collected sets of numbers, and divided them according to some rules to disjoint subsets. Using some statistical tests we found that there are differences among the data in the disjoint subsets, and we revealed similar patters among several executions. However, by visualizing the data with the plain old box plots or other traditional diagram types, the statistical results could not be really supported. The box plots, which we found not to be useful in our special case are shown in Figure 1. Note that this version does not contain the outliers; the diagrams with outliers were even worse. On the diagram the leftmost box plot illustrates all the data, and the rest four represents the data falling into disjoint subsets. The data division was performed on four input data sets.

Based on this example we framed the Cumulative Characteristic Diagram, which proved to be suitable for illustrating the results. Furthermore, this diagram type helped us to identify additional connections not discovered earlier. The analysis of these earlier unrevealed findings leads us to framing the Quantile Difference Diagrams.

2. Related work

Several diagram types exists for illustrating univariate, bivariate and multivariate data. One of the fundamental works in this topic is the book *Graphical Methods for Data Analysis* (Chambers et al. [1]). For statistical package R a recommended reading is the *R Graphics* (Murrel [2]).

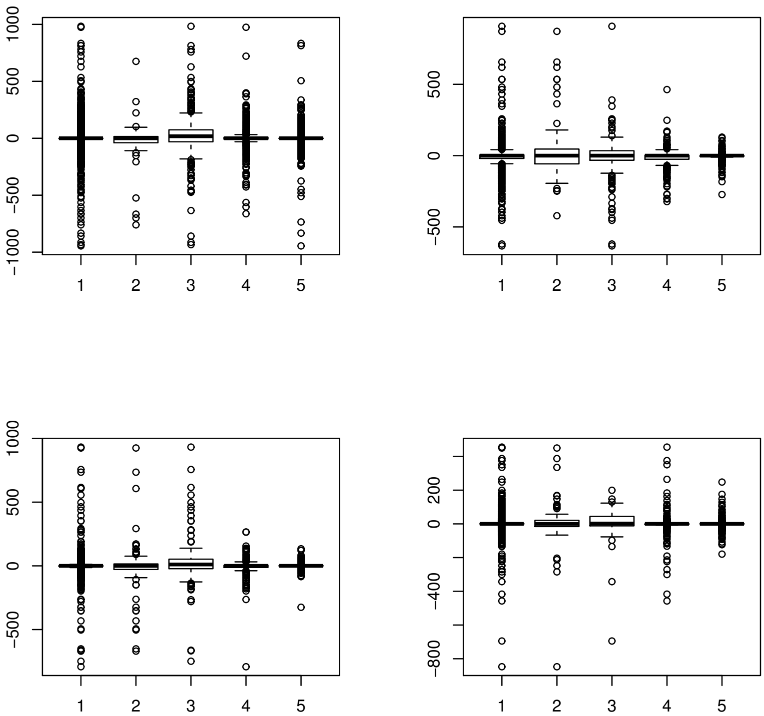

Figure 1: Illustrating case: Plots with limited usefulness

The article *Variations of Box Plots* by McGill et al. [3] suggests two extensions of the base box plots. Those times the computers were expensive and slow, and the diagrams were mostly drawn by hand. The study *Opening the Box of a Boxplot* by Benjamini [4] exploits the capability of the computer. The article *Some Implementations of the Boxplot* by Frigge et al. [5] deals mainly with the problem of outliers. The study *Methods for Presenting Statistical Information: The Box Plot* by Potter et al. [6] provides a summary of the variations of box plots.

Probably the most important problem with box plot is that it hides the local densities. The mentioned studies mostly deal with this problem. To overcome this shortcoming, in R the density plot could be a good choice in several cases (using `density()` function from `stat`, like follows: `plot(density(x))`). Other popular diagram types handling this issue are violin plots (R function `vioplot()` in package `vioplot` (see paper *Violin Plots: a Box Plot-density Trace Synergism* by Hintze and Nelson [7])) and bean plots (R function `beanplot()` in package `beanplot`, as described in paper *Beanplot: A Boxplot Alternative for Visual Comparison of Distributions* by Kampstra [8]).

The problem of illustrating bivariate data is also very common. An early proposal of a bivariate extension of boxplots is presented in article *Bivariate Extensions of the Boxplot* by Goldberg and Iglewicz [9]. An interesting two dimensional extension of the box plots is the bag plot, suggested by article *The Bagplot: a Bivariate Boxplot* by Rousseeuw et al. [10]; R function `bagplot()` in package `aplpack`.

Visualizing multivariate data is even harder. Article *The strucplot Framework: Visualizing Multi-way Contingency Tables with vcd* by Hornik et al. [11] suggests a framework for visualizing multi-way contingency tables.

The presented R functions are mainly based on base package `graphics`. Another basic visualization related package in R is `grid`. The `lattice` package is based on `grid`, see work *Lattice: Multivariate Data Visualization with R* by Sarkar [12] for details.

3. Diagrams

3.1. Cumulative Characteristic Diagram (CCD)

The input of the base diagram is a set of numbers. In the first step, these numbers are sorted non-ascending. Then cumulatives are calculated for every element: the series starts with 0, the next element will be the value of the first element of the sorted array, the second element will be the sum of the first 2 elements, and so on. In the diagrams the x coordinate represents the number of elements, and the y coordinate represents the calculated cumulatives. Instead of drawing each point one by one, these points are connected with straight lines. If the number of elements is high enough, the result will look like a continuous line without bends.

The diagram type is mostly suitable for data of normal distribution with the mean close to 0. The diagram is applicable for quick comparison of several data sets: to illustrate the similarities and differences. It can be used to illustrate quickly two or more – seemingly similar – data sets if they are really similar or not. A CCD which contains two or more characteristics on the same diagram we call *Composite Cumulative Characteristic Diagram.*

Examples are shown later in this chapter in Figure 2.

3.2. Quantile Difference Diagram (QDD)

The idea behind the Quantile Difference Diagrams is to compare the same quantiles of two sets of numbers. This means the first element of the first set should be compared to the first element of the second one, similarly the 10% to the 10%, the median to the medial, the 90% to the 90%, highest to the highest and so on.

Therefore the input of the QDD is always two sets of numbers. Every centile is determined in both subsets, i.e. the 0% (which is the lowest one), the 1% (e.g. if the set contains 1000 elements, this is the 10^{th}) etc. This results 101 values in every case, either by omitting values, or taking the same values several times. Then the differences are calculated at every centile. On the the diagram these differences are displayed as a line. Examples for QDD can be found later in this chapter in Figure 3.

3.3. The `vudc` R package

Both diagram types have been implemented as an R package [13], named `vudc`, which stand for *Visualization of Univariate Data for Comparison*. This can be installed directly from the R statistical software as any other package, either directly from the R GUI, or by downloading from CRAN (`http://cran.r-project.org/web/packages/vudc/index.html`). After installation it should be loaded as a usual R package, as follows:

```
library(vudc)
```

The package contains two functions: `ccdplot()` and `qddplot()`; furthermore, data used in our research: `projectdata`. General information can be obtained using R help command:

```
?vudc
```

`ccdplot()`

This function creates a Cumulative Characteristic Diagram. Figure 2 illustrates some examples.

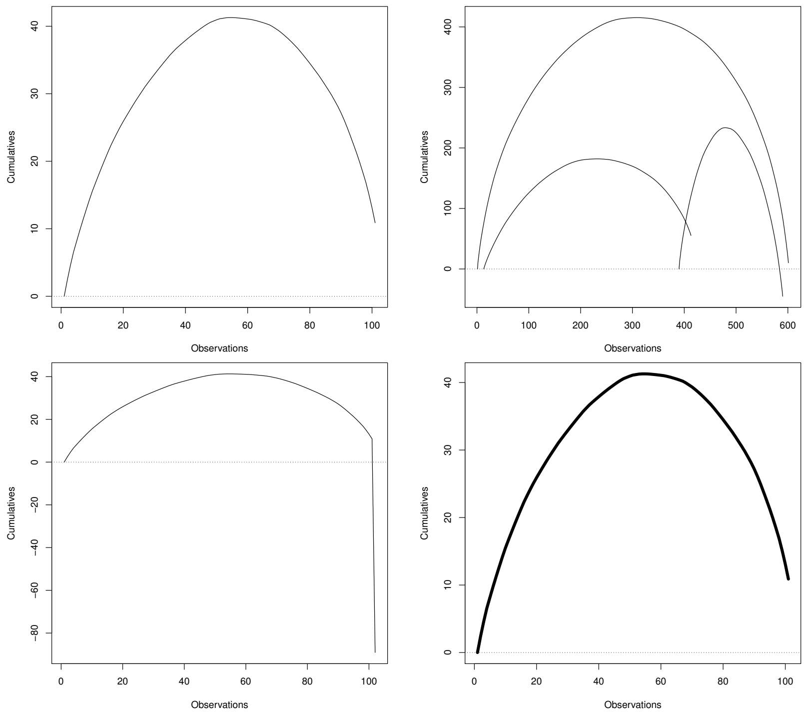

Figure 2: `ccdplot()` examples

The upper left graphics draws the *Cumulative Characteristic Diagram* of 100 random real numbers of standard normal distribution. This can be drawn with the following R function:

```
ccdplot(rnorm(100))
```

The upper right figure illustrates the *Composite Cumulative Characteristic Diagram*. This diagram contains characteristic diagram of two or more sets of numbers on the same scale, along with (the optional) the CCD of the union of the numbers. The illustration contains numbers of normal distribution, with different size, different expected values and different variances. This can be created with help of R command

```
ccdplot(list(rnorm(400, 0.1, 1), rnorm(200, -0.1, 3)))
```

The differences in width, height and the right end are spectacular.

The lower left diagram illustrates that this diagram is sensible on the outliers. A mechanism is built in to remove the outliers automatically, either by providing an absolute threshold, or a percentage; the later one is applied on both ends. To reproduce a similar the diagram, use command

```
ccdplot(c(rnorm(100), -100))
```

Finally, the lower right diagram illustrates that the function integrates into the standard R diagram functions, the standard parameters can be passed. The line is thick, which can be achieved with the following command:

```
ccdplot(rnorm(100), lwd=5)
```

Detailed information about all the possible parameters and further examples can be obtained using the R help command:

```
?ccdplot
```

`qddplot()`

This function creates Quantile Difference Diagrams. Figure 3 illustrates some examples.

The upper left diagram illustrates the comparison of two sets of random numbers of normal distribution, with different number of elements (100 vs. 200), different means (30 vs. 10) and different standard deviation. Despite the fact that the number of elements in the second subset is twice as much as in the first one, the illustration was possible. The diagram illustrates that the numbers in the first subset are higher than those in the second: the territory above the abscissa (i.e. the x-coordinate) is higher than below it. On the other hand, it also illustrates that among the lowest elements the numbers in the second subset are higher than those in the first one. The diagram can be created using the following command:

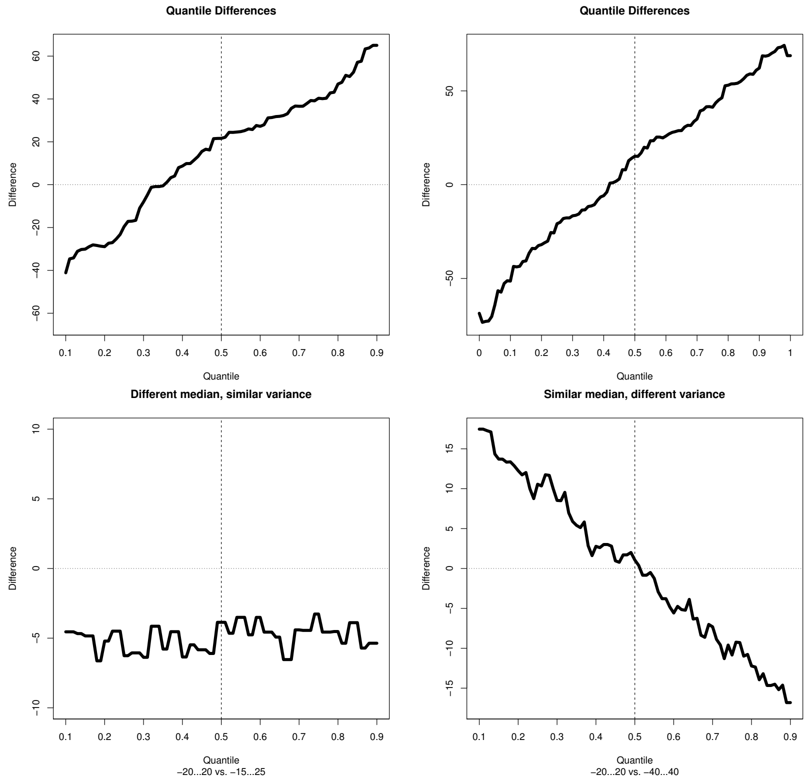

Figure 3: `qddplot()` examples

```
qddplot(rnorm(100, 30, 50), rnorm(200, 10, 10))
```

The upper right diagram illustrates that the diagram is biased at both ends. This diagram is illustrated with numbers of the same distribution as above. By default, the diagram does not display the lower and the upper 5%. This can be fine-tuned using parameters. In this example the remove ratio is set to 0:

```
qddplot(rnorm(100, 30, 50), rnorm(200, 10, 10), remove.ratio=0.0)
```

The primary usage of this diagram is intended to illustrate the comparison of two sets of numbers of the same distribution and similar variance, but different expected value. The first set contains 41 numbers, close to all the integers from -20 up to +20, and the second one similar numbers from -15 to +25. The difference is about 5, which is illustrated in the lower left diagram. The usage is intended to be converse: we have 2 sets of numbers, and the diagram reveals this property. The diagram was made using command

```
qddplot(seq(-20, 20) + rnorm(41), seq(-15, 25) + rnorm(41),
```

```
    main = "Different median, similar variance",
    sub = "-20...20 vs. -15...25")
```

The second most important usage of the diagram is intended to be the variance comparison. In this example the first set of numbers contain similar elements as above, and the second one contains 81 elements, around the integers from -40 up to +40. The comparison statistic tests, which compare the mean or median of the numbers would not show relevant deflection, however, the diagram, as shown in the lower right, indicates that there is a difference in variance. The medians are more or less the same (the difference is around 0 at the median), but in the ends the line is far from 0. Therefore such an illustration would indicate it is worth to compare the variances. The diagrams was created using the following command:

```
qddplot(seq(-20, 20) + rnorm(41), seq(-40, 40) + rnorm(81),
    main = "Similar median, different variance",
    sub = "-20...20 vs. -40...40")
```

Detailed information about all the possible parameters and further examples can be obtained using the R help command:

```
?qddplot
```

`projectdata`

The package contains information about the following software systems: the well-known open source **Ant**, **Struts 2** and **Tomcat**, and about an industrial software with name **Gremon**, which is a greenhouse monitoring system. In order to access the data first we need to issue the following command:

```
data(projectdata)
```

For each project a *data frame* is provided, containing information of every available commit. The rows of the data frame represent commits, and there are the following columns:

- `A`: number of added files
- `U`: number of updated files
- `D`: number of deleted files
- `MaintainabilityDiff`: maintainability difference caused by the commit

The number of added, updated and deleted files are non negative integers, containing information about Java files (non Java files were removed). Commits not containing Java files were removed.

The `MaintainabilityDiff` is the difference of maintainability values of 2 subsequent revisions. The maintainability of every revision was calculated with the help of Columbus Quality Model [14]. These maintainability values were normalized and the difference was calculated as described in paper by Faragó et al. [15]. The final result is a real number.

This is an example excerpt of the data (information about the first 10 commits of project Ant):

```
> projectdata$Ant[1:10,]
    A U D MaintainabilityDiff
1  44 0 0             0.00000
2   0 5 0           -14.55057
3   0 1 0             0.00000
4   0 2 1          -524.46238
5   1 1 1           -19.55645
6   0 3 0          -184.04878
7   0 3 0           -15.25897
8   0 1 0           -56.05360
9   0 2 0            16.39003
10  0 0 6           -71.82581
```

Detailed information can be obtained using the help page of the project data, using R command

```
?projectdata
```

4. Illustrating the statistic tests

In this section we provide some examples about the usage of the defined diagrams, illustrating various statistic tests.

For the illustration, first we generate sets of numbers. Both sets are of normal distribution, containing 101 elements each. The first subset's (`x` in the example) mean is 1, and the standard deviation is also 1, and the second subset's (`y` in the example) mean is -1, and the standard deviation is 3. For the data generation first we set the radom seed in order to be able to reproduce the results.

```
set.seed(1)
x <- rnorm(101, 1, 1)
y <- rnorm(101, -1, 3)
```

In this example we act as we just received these sets of numbers, and we do not know anything about them. First we generate the Cumulative Characteristic Diagram and the Quantile Difference Diagram, and then begin with the analysis. The diagram generation is performed with the following commands:

```
ccdplot(list(x, y))
qddplot(x, y)
```

The results are displayed in Figure 4.

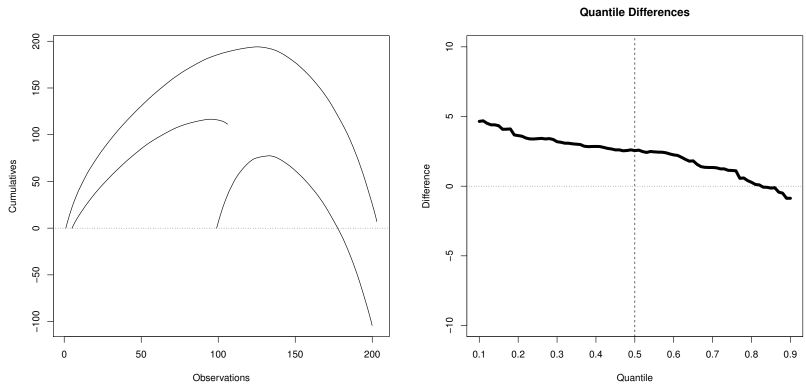

Figure 4: Examples for statistic test demonstrations

4.1. Wilcoxon rank correlation tests and CCD

First let us check the Cumulative Characteristic Diagram (the left diagram of Figure 4). Based on the difference of the altitude of the right end of the characteristic lines (the first one is far above 0, and the second one is far below 0) it indicates that it is likely that the elements in the first subset are significantly higher than those in the second. To check this, we perform a one tailed Wilcoxon rank correlation test (also known as Mann-Whitney U test). This test compares each elements of the first subset with each in the second one. The advantage of this test over the t-test (which performs the comparison on the averages) is that while t-test is very sensitive to outliers the Wilcoxon test does not.

This is the result of the Wilcoxon test:

```
> wilcox.test(x, y, alternative="greater")

        Wilcoxon rank sum test with continuity correction

data:  x and y
W = 7843, p-value = 2.046e-11
alternative hypothesis: true location shift is greater than 0
```

The preliminary assumption based on the CCD turned to be correct: the p-value is very low. Conversely, having a good result of Wilcoxon test, we can illustrate it with CCD.

4.2. Wilcoxon rank correlation tests and QDD

The results so far indicates that the numbers in the first subset are greater than those in the second one. But can we tell more about them? To answer the question, consider the QDD (the right diagram of Figure 4).

The result of the Wilcoxon test on this diagram means the following: the signed territory between the line and the x coordinate is positive. However, on the right side the line is below 0, meaning that just considering the highest values, those in the second data set are higher than in the first one. In concrete cases this worth further analysis. Without QDD, this attribute could have been bypassed.

What does it mean in practice? Let the numbers denote the knowledge of students in mathematics in different countries. It can be higher in country A compared to country B in general, but the best students in country B might be better than in country A. On the Mathematics Olympics country B is likely to gain better results over country A. On the contrary: having a better results on the Olympics does not necessarily mean that the education is on the good way.

4.3. Variance tests and CCD

Considering the CCD again (the left diagram of Figure 4) there is another spectacular difference between the left and the right curve to note: their width are the same, but the vertical lengths of the lines are different: the right hand side is much longer than the left hand side. This indicates differences in variance. Let us perform the variance test!

```
> var.test(x, y, alternative="less")

        F test to compare two variances

data:  x and y
F = 0.095434, num df = 100, denom df = 100, p-value < 2.2e-16
alternative hypothesis: true ratio of variances is less than 1
95 percent confidence interval:
 0.0000000 0.1328177
sample estimates:
ratio of variances
        0.09543427
```

It turned out that the difference (indeed: the ratio) between the variance of the two subsets are really significant with extremely low p-value (meaning: it is very unlikely that this happened by chance).

It was not a big surprise for us as we generated the values to have different variances; however if we act we do not know anything about the nature of the input data, this could be helpful. In our study such a diagram helped us to perform analysis in this direction, and we presented the result in article [16].

4.4. Variance tests and QDD

How does the difference in variance look like on the QDD?

If the line on the QDD is more or less horizontal, it indicates that there is no real difference in variance. On the other hand, if it has a slope, it is a sign of difference in variance. Considering the right diagram of Figure 4, we conclude that the line has a slope, indicating the probable significant difference of variances.

4.5. Contingency Chi-Squared tests and CCD

In the basic case of Chi-Squared tests we have a null hypothesis about the number of elements of some subsets, and real observations. For example, consider the genres of students in a university. The null hypothesis is that 50% are male and 50% are female. In the fictive example of a Technical University there are 257 students, among them 243 boys and 14 girls. With the Chi-Squared test we can check if the difference is casual, or we should reject the null hypothesis, and state an alternative one, that in the technical universities there are more males then females.

In a more general case we have a matrix of any dimension. Every observation belongs to exactly one cell in the matrix. The null hypothesis is that the observations are distributed evenly in the matrix. It does not exactly mean that the number of elements are the same in each cell, but it is calculated based on the row and the column sums.

In our example we consider a matrix of dimensions 2x2, containing the number of positive and negative elements in both subsets. The following listing contains how it was created, and then the values are displayed. Then the Chi-Squared test is performed, and the expected values, the global result of the test and the standard residuals on each cells are displayed. The meaning of the standard residual of a cell in nutshell is the following: what was the difference between the expected and the actual value if it was a number of standard normal distribution. Based on this value, p-values can be calculated for each cell.

```
> sign <- matrix(c(length(x[x>0]), length(x[x<0]),
                   length(y[y>0]), length(y[y<0])),
    2, 2, dimnames=list(c("positive", "negative"), c("x", "y")))
> sign
          x  y
positive 90 34
negative 11 67
> chisq.test.result <- chisq.test(sign)
> chisq.test.result$expected
          x  y
positive 62 62
negative 39 39
> chisq.test.result

        Pearson's Chi-squared test with Yates' continuity correction

data:  sign
X-squared = 63.177, df = 1, p-value = 1.889e-15

> chisq.test.result$stdres
                 x         y
positive  8.092926 -8.092926
negative -8.092926  8.092926
>
```

The result of the Chi Squared test indicates that the number of positive and negative elements in sets `x` and `y` is significant. This is indeed not surprising for us, as we know the nature of numbers in the sets. But in general this is not known.

The connection between the result of the Chi-Squared test and CCD is the following: if the shapes of the curves does not resemble to each other, with proper division it is likely that Chi-Squared test will show significant deflection from the null hypothesis (which in terms of CCD it means the shapes of the curves are similar).

In practice the Chi-Squared test is suggested to be executed specially in the following case: consider several observations (e.g. technical universities of different cities), and the curves on the CCD diagrams are different, but the CCD diagrams themselves are similar.

4.6. CCDs of the motivating examples

Figure 5 illustrates the CCD version of the motivating example (see 1 in Chapter 1.2). The curves within diagrams are obviously different, and there are similarities between the diagrams. Therefore we found it more useful, compared to the boxplot version.

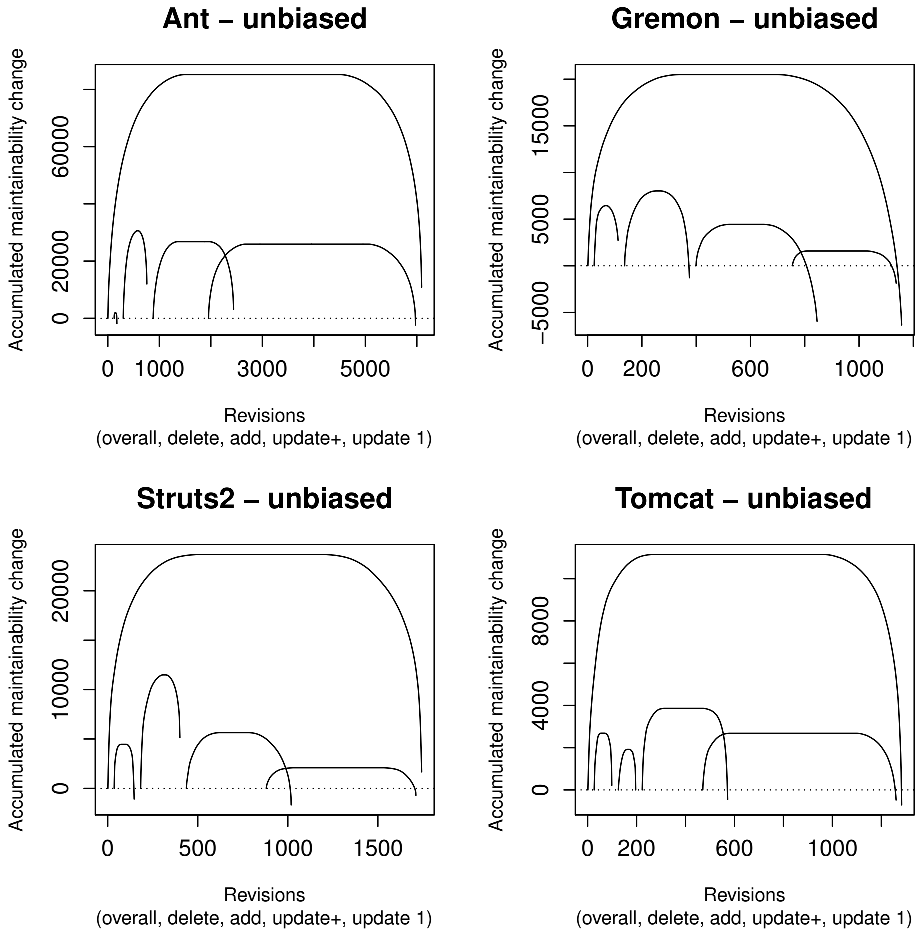

Figure 5: Composite Cumulative Characteristic Diagrams about maintainability

5. Conclusions

In our earlier studies, we faced several problems in illustrating our achieved results visually. However, visualization is very important as without it, using just text and tables, it is harder to explain and understand what we want to express. Furthermore, having just a bunch of numbers and p-values of test results it is hard to reveal patterns within the underlying data. For these reasons we introduced two new diagram types which are suitable for illustrating some of our results so far. Furthermore, we found it useful to forebode other significant connections not realized before.

In this paper we described these new diagram types, which we call *Cumulative Characteristic Diagram (CCD)*, and *Quantile Difference Diagram (QDD)*. In general, both are suitable to visualize a numeric set of data. The Cumulative Characteristic Diagram itself is a curve that is based on the non-ascending order of the values, which are accumulated and plotted. The Quantile Difference Diagram illustrates the difference of two data sets on the same quantiles.

After an introduction and giving our motivations, we defined the diagram types and illustrated some possibilities for their application. We presented how they can be used for visualizing the results of Wilcoxon rank tests, variance tests and the CCD for contingency Chi-squared tests.

We implemented the diagram in the R statistic program in package *vudc*, and we also provided technical details about the usage of the implemented `ccdplot` and `qddplot` R function, along with the data `projectdata` added to this package. We described the parameters of these function in detail and provided examples of their use in practice.

Finally, we presented how these diagrams type helped us in illustrating the results of our research for revealing the effect of developer interactions on software maintainability. With the help of these diagrams we were able to reveal some non-standard commits and outliers; finding and handling them properly helped us to achieve more meaningful results. We demonstrated how these diagrams were suitable to illustrate the result of a concrete Chi-squared contingency test, then we provided some examples for visualizing the results of a Wilcoxon test and variance test.

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Pell and Pell-Lucas numbers with only one distinct digit

Bernadette Faye[ab], Florian Luca[b]

[a]Ecole Doctorale de Mathematiques et d'Informatique
Université Cheikh Anta Diop de Dakar
BP 5005, Dakar Fann, Senegal
bernadette@aims-senegal.org

[b]School of Mathematics,
University of the Witwatersrand
Private Bag 3, Wits 2050, Johannesburg, South Africa
Florian.Luca@wits.ac.za

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Abstract

In this paper, we show that there are no Pell or Pell-Lucas numbers larger than 10 with only one distinct digit.

Keywords: Pell numbers, rep-digits.

MSC: 11B39, 11D61

1. Introduction

Let $\{P_n\}_{n\geq 0}$ be the sequence of Pell numbers given by $P_0 = 0$, $P_1 = 1$ and

$$P_{n+2} = 2P_{n+1} + P_n \quad \text{for all} \quad n \geq 0.$$

The Pell-Lucas sequence $(Q_n)_{n\geq 0}$ satisfies the same recurrence as the sequence of Pell numbers with initial condition $Q_0 = Q_1 = 2$. If $(\alpha, \beta) = (1+\sqrt{2}, 1-\sqrt{2})$ is the pair of roots of the characteristic equation $x^2 - 2x - 1 = 0$ of both the Pell and Pell-Lucas numbers, then the Binet formulas for their general terms are:

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad Q_n = \alpha^n + \beta^n \quad \text{for all} \quad n \geq 0.$$

Given an integer $g > 1$, a base g-repdigit is a number of the form

$$N = a\left(\frac{g^m - 1}{g - 1}\right) \quad \text{for some} \quad m \geq 1 \quad \text{and} \quad a \in \{1, \dots, g-1\}.$$

When $g = 10$ we refer to such numbers as *repdigits*. Here we use some elementary methods to study the presence of rep-digits in the sequences $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$. This problem leads to a Diophantine equation of the form

$$U_n = V_m \quad \text{for some} \quad m, n \geq 0, \tag{1.1}$$

where $\{U_n\}_n$ and $\{V_m\}_m$ are two non degenerate linearly recurrent sequences with dominant roots. There is a lot of literature on how to solve such equations. See, for example, [1] [4], [6], [7], [8]. The theory of linear forms in logarithms à la Baker gives that, under reasonable conditions (say, the dominant roots of $\{U_n\}_{n\geq 0}$ and $\{V_m\}_{m\geq 0}$ are multiplicatively independent), equation (1.1) has only finitely many solutions which are effectively computable. In fact, a straightforward linear form in logarithms gives some very large bounds on $\max\{m, n\}$, which then are reduced in practice either by using the LLL algorithm or by using a procedure originally discovered by Baker and Davenport [1] and perfected by Dujella and Pethő [3].

In this paper, we do not use linear forms in logarithms, but show in an elementary way that 5 and 6 are respectively the largest Pell and Pell-Lucas numbers which has only one distinct digit in their decimal expansion. The method of the proofs is similar to the method from [5], paper in which the second author determined in an elementary way the largest repdigits in the Fibonacci and the Lucas sequences. We mention that the problem of determining the repdigits in the Fibonacci and Lucas sequence was revisited in [2], where the authors determined all the repdigits in all generalized Fibonacci sequences $\{F_n^{(k)}\}_{n\geq 0}$, where this sequence starts with $k-1$ consecutive 0's followed by a 1 and follows the recurrence $F_{n+k}^{(k)} = F_{n+k-1}^{(k)} + \cdots + F_n^{(k)}$ for all $n \geq 0$. However, for this generalization, the method used in [2] involved linear forms in logarithms.

Our results are the following.

Theorem 1.1. *If*

$$P_n = a\left(\frac{10^m - 1}{9}\right) \quad \textit{for some} \quad a \in \{1, 2, \dots, 9\}, \tag{1.2}$$

then $n = 0, 1, 2, 3$.

Theorem 1.2. *If*

$$Q_n = a\left(\frac{10^m - 1}{9}\right) \quad \textit{for some} \quad a \in \{1, 2, \dots, 9\}, \tag{1.3}$$

then $n = 0, 1, 2$.

2. Proof of Theorem 1.1

We start by listing the periods of $\{P_n\}_{n\geq 0}$ modulo 16, 5, 3 and 7 since they will be useful later

$$\begin{aligned}
&0,\ 1,\ 2,\ 5,\ 12,\ 13,\ 6,\ 9,\ 8,\ 9,\ 10,\ 13,\ 4,\ 5,\ 14,\ 1,\ 0,\ 1 \pmod{16}\\
&0,\ 1,\ 2,\ 0,\ 2,\ 4,\ 0,\ 4,\ 3,\ 0,\ 3,\ 1,\ 0,\ 1 \pmod{5}\\
&0,\ 1,\ 2,\ 2,\ 0,\ 2,\ 1,\ 1,\ 0,\ 1 \pmod{3}\\
&0,\ 1,\ 2,\ 5,\ 5,\ 1,\ 0,\ 1 \pmod{7}.
\end{aligned} \tag{2.1}$$

We also compute P_n for $n \in [1, 20]$ and conclude that the only solutions in this interval correspond to $n = 1, 2, 3$. From now, we suppose that $n \geq 21$. Hence,

$$P_n \geq P_{21} = 38613965 > 10^7.$$

Thus, $m \geq 7$. Now we distinguish several cases according to the value of a. We first treat the case when $a = 5$.

Case $a = 5$.

Since $m \geq 7$, reducing equation (1.2) modulo 16 we get

$$P_n = 5\left(\frac{10^m - 1}{9}\right) \equiv 3 \pmod{16}.$$

A quick look at the first line in (2.1) shows that there is no n such that $P_n \equiv 3 \pmod{16}$.

From now on, $a \neq 5$. Before dealing with the remaining cases, let us show that m is odd. Indeed, assume that m is even. Then, $2 \mid m$, therefore

$$11 \mid \frac{10^2 - 1}{9} \mid \frac{10^m - 1}{9} \mid P_n.$$

Since, $11 \mid P_n$, it follows that $12 \mid n$. Hence,

$$2^2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 = 13860 = P_{12} \mid P_n = a \cdot \frac{10^m - 1}{9},$$

and the last divisibility is not possible since $a(10^m - 1)/9$ cannot be a multiple of 10. Thus, m is odd.

We are now ready to deal with the remaining cases.

Case $a = 1$.

Reducing equation (1.2) modulo 16, we get $P_n \equiv 7 \pmod{16}$. A quick look at the first line of (2.1) shows that there is no n such that $P_n \equiv 7 \pmod{16}$. Thus, this case is impossible.

Case $a = 2$.

Reducing equation (1.2) modulo 16, we get

$$P_n = 2\left(\frac{10^m - 1}{9}\right) \equiv 14 \pmod{16}.$$

A quick look at the first line of (2.1) gives $n \equiv 14 \pmod{16}$. Reducing also equation (1.2) modulo 5, we get $P_n \equiv 2 \pmod 5$, and now line two of (2.1) gives $n \equiv 2, 4 \pmod{12}$. Since also $n \equiv 14 \pmod{16}$, we get that $n \equiv 14 \pmod{48}$. Thus, $n \equiv 6 \pmod 8$, and now row three of (2.1) shows that $P_n \equiv 1 \pmod 3$. Thus,

$$2\left(\frac{10^m - 1}{9}\right) \equiv 1 \pmod 3.$$

The left hand side above is $2(10^{m-1} + 10^{m-2} + \cdots + 10 + 1) \equiv 2m \pmod 3$, so we get $2m \equiv 1 \pmod 3$, so $m \equiv 2 \pmod 3$, and since m is odd we get $m \equiv 5 \pmod 6$. Using also the fact that $n \equiv 2 \pmod 6$, we get from the last row of (2.1) that $P_n \equiv 2 \pmod 7$. Thus,

$$2\left(\frac{10^m - 1}{9}\right) \equiv 2 \pmod 7,$$

leading to $10^m - 1 \equiv 9 \pmod 7$, so $10^{m-1} \equiv 1 \pmod 7$. This gives $6 \mid m - 1$, or $m \equiv 1 \pmod 6$, contradicting the previous conclusion that $m \equiv 5 \pmod 6$.

Case $a = 3$.

In this case, we have that $3 \mid P_n$, therefore $4 \mid n$ by the third line of (2.1). Further,

$$P_n = 3\left(\frac{10^m - 1}{9}\right) \equiv 5 \pmod{16}.$$

The first line of (2.1) shows that $n \equiv 3, 13 \pmod{16}$, contradicting the fact that $4 \mid n$. Thus, this case in impossible.

Case $a = 4$.

We have $4 \mid P_n$, which implies that $4 \mid n$. Reducing equation (1.2) modulo 5 we get that $P_n \equiv 4 \pmod 5$. Row two of (2.1) shows that $n \equiv 5, 7 \pmod{12}$, which contradicts the fact that $4 \mid n$. Thus, this case is impossible.

Case $a = 6$.

Here, we have that $3 \mid P_n$, therefore $4 \mid n$. Hence,

$$12 \mid P_n = 6\left(\frac{10^m - 1}{9}\right),$$

which is impossible.

Case $a = 7$**.**

In this case, we have that $7 \mid P_n$, therefore $6 \mid n$ by row four of (2.1). Hence,

$$70 = P_6 \mid P_n = 7\left(\frac{10^m - 1}{9}\right),$$

which is impossible.

Case $a = 8$.

We have that $8 \mid P_n$, so $8 \mid n$. Hence,

$$8 \cdot 3 \cdot 17 = 408 = P_8 \mid P_n = 8\left(\frac{10^m - 1}{9}\right),$$

implying $17 \mid 10^m - 1$. This last divisibility condition implies that $16 \mid m$, contradicting the fact that m is odd.

Case $a = 9$**.**

We have $9 \mid P_n$, thus $12 \mid n$. Hence,

$$13860 = P_{12} \mid P_n = 10^m - 1,$$

a contradiction.

This completes the proof of Theorem 1.1.

3. The proof of Theorem 1.2

We list the periods of $\{Q_n\}_{n\geq 0}$ modulo 8, 5 and 3 getting

$$\begin{array}{l} 2,\ 2,\ 6,\ 6,\ 2,\ 2 \pmod 8 \\ 2,\ 2,\ 1,\ 4,\ 4,\ 2,\ 3,\ 3,\ 4,\ 1,\ 1,\ 3,\ 2,\ 2 \pmod 5 \\ 2,\ 2,\ 0,\ 2,\ 1,\ 1,\ 0,\ 1,\ 2,\ 2 \pmod 3 \end{array} \tag{3.1}$$

(3.2)

We next compute the first values of Q_n for $n \in [1, 20]$ and we see that there is no solution $n > 3$ in this range. Hence, from now on,

$$Q_n > Q_{21} = 109216786 > 10^8,$$

so $m \geq 9$. Further, since Q_n is always even and the quotient $(10^m - 1)/9$ is always odd, it follows that $a \in \{2, 4, 6, 8\}$. Further, from row one of (3.1) we see that Q_n is never divisible by 4. Thus, $a \in \{2, 6\}$.

Case $a = 2$**.**

Reducing equation (1.3) modulo 8, we get that

$$Q_n = 2\left(\frac{10^m - 1}{9}\right) \equiv 6 \pmod 8.$$

Row one of (3.1) shows that $n \equiv 2, 3 \pmod 4$. Reducing equation (1.3) modulo 5 we get that $Q_n \equiv 2 \pmod 5$, and now row two of (3.1) gives that $n \equiv 0, 1, 5 \pmod{12}$, so in particular $n \equiv 0, 1 \pmod 4$. Thus, we get a contradiction.

Case $a = 6$.

First $3 \mid n$, so by row three of (3.1), we have that $n \equiv 2, 6 \pmod 8$. Next reducing (1.3) modulo 8 we get

$$Q_n = 6\left(\frac{10^m - 1}{9}\right) \equiv 2 \pmod 8.$$

and by the first row of (3.1) we get $n \equiv 0, 1 \pmod 4$. Thus, this case is impossible.

This completes the proof of Theorem 1.2.

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On geometric Hermite arcs*

Imre Juhász

Department of Descriptive Geometry, University of Miskolc, Hungary
agtji@uni-miskolc.hu

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Abstract

A geometric Hermite arc is a cubic curve in the plane that is specified by its endpoints along with unit tangent vectors and signed curvatures at them. This problem has already been solved by means of numerical procedures. Based on projective geometric considerations, we deduce the problem to finding the base points of a pencil of conics, that reduces the original quartic problem to a cubic one that easier can exactly be solved. A simple solvability criterion is also provided.

Keywords: Hermite arc, geometric constraint, pencil of conincs

MSC: 65D17, 68U07

1. Introduction

In Computer Aided Geometric Design curves are often specified by means of some constraints that the required curve has to fulfill, instead of by those data that are necessary for a certain representation form (e.g. Bézier or B-spline). A classical example is the Hermite arc, where endpoints are given along with the tangent vector (i.e. the first derivative of the curve with respect to the parameter) at them from which data a cubic arc has to be determined. This task always has a unique solution.

A slight modification of the previous problem is the so called geometric Hermite arc, where endpoints of a cubic arc in plane with unit tangent vectors and signed

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curvatures at them are given. Unlike in the classical Hermite arc, a solution to this problem does not always exist.

This problem pops up in [1] in connection with equidistants of plane cubic spline curves. In [2] G^2 cubic plane interpolating splines are constructed with geometric Hermite arcs, moreover a thorough analysis and a criterion for solvability is provided. In [3] G^2 end conditions are discussed for C^2 planar cubic interpolating splines (Ferguson splines). [4] studies transition curves between circular and conic arcs meeting G^2 continuity requirements, while [5] provides G^2 transition curves between two circles. There are several generalizations of the problem. [6] generalizes the problem to rational cubics and [7] to interpolating spline surfaces, while [8] restricts the solution to Pythagorean-hodograph cubics.

No matter in what way we seek the solution to the original problem, it always ends in a system of quadratic equations in two variables. In all the cited publications this system is solved numerically. Equations of the system are quite special that gives the hope of a simple exact solution. In this contribution we provide such a solution based on projective geometric considerations.

2. Bézier points of the arc

Let us denote the endpoints by $\mathbf{p}_0$ and $\mathbf{p}_1$, the signed curvatures at them by κ_0 and κ_1, and the unit tangent vectors by $\mathbf{t}_0$ and $\mathbf{t}_1$.We want to produce the Bézier representation

$$\mathbf{b}(u) = \sum_{i=0}^{3} B_i^3(u)\,\mathbf{b}_i,\ \ u \in [0,1]$$

of the cubic Hermite arc, where $B_i^3(u)$ denotes the ith cubic Bernstein polynomial. Its control points $\{\mathbf{b}_i\}_{i=0}^3$ can be expressed in the form

$$\begin{aligned}
\mathbf{b}_0 &= \mathbf{p}_0,\\
\mathbf{b}_3 &= \mathbf{p}_1,\\
\mathbf{b}_1 &= \mathbf{p}_0 + l_0\mathbf{t}_0, && (2.1)\\
\mathbf{b}_2 &= \mathbf{p}_1 - l_1\mathbf{t}_1, && (2.2)
\end{aligned}$$

in which positive lengths l_0 and l_1 are unknown.

The signed curvature of a cubic planar Bézier curve is of the form

$$\kappa(u) = \frac{\dot{\mathbf{b}}(u) \wedge \ddot{\mathbf{b}}(u)}{\left|\dot{\mathbf{b}}(u)\right|^3},$$

where $\dot{\mathbf{b}}(u) \wedge \ddot{\mathbf{b}}(u)$ is the third component of the cross product of the vectors $\dot{\mathbf{b}}(u)$ and $\ddot{\mathbf{b}}(u)$. Its value at the first and last points are

$$\kappa_0 := \kappa(0) = \frac{2(\mathbf{b}_1 - \mathbf{b}_0) \wedge (\mathbf{b}_2 - \mathbf{b}_1)}{3l_0^3}$$

$$= \frac{2A(\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2)}{3l_0^3}$$

and

$$\kappa_1 := \kappa(1) = \frac{2(\mathbf{b}_2 - \mathbf{b}_1) \wedge (\mathbf{b}_3 - \mathbf{b}_2)}{3l_1^3}$$
$$= \frac{2A(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)}{3l_1^3}$$

respectively, where $A(\mathbf{a}, \mathbf{b}, \mathbf{c})$ stands for the signed area of the triangle determined by the sequence of vertices $\mathbf{a}, \mathbf{b}, \mathbf{c}$. Denoting the signed distance of the control point $\mathbf{b}_2$ from the directed straight line $\mathbf{b}_0, \mathbf{b}_1$ by d_0 (cf. Fig. 1) we obtain for the signed area

$$A(\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2) = \frac{l_0 d_0}{2}$$

and for the signed curvature

$$\kappa_0 = \frac{2}{3}\frac{d_0}{l_0^2}. \tag{2.3}$$

Analogously, the signed curvature κ_1 is of the form

$$\kappa_1 = \frac{2}{3}\frac{d_1}{l_1^2}, \tag{2.4}$$

where d_1 is the signed distance of the control point $\mathbf{b}_1$ from the directed line $\mathbf{b}_2, \mathbf{b}_3$.

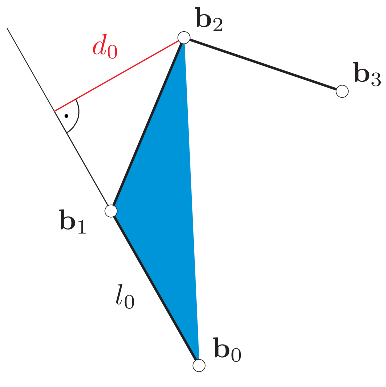

Figure 1: The signed curvature at the first point of a Bézier curve is proportional to the signed area of the triangle with vertices $\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2$

2.1. Parallel tangent vectors

If $\mathbf{t}_0 \| \mathbf{t}_1$ then Eqs. (2.1) and (2.2) imply equality $d = |d_0| = |d_1|$, where d is the distance between the parallel lines determined by point and direction vector pairs $\mathbf{p}_0, \mathbf{t}_0$ and $\mathbf{p}_1, \mathbf{t}_1$.

If $d = 0$ then on the basis of Eqs. (2.3) and (2.4) it is obvious that a solution to the problem exists if and only if $\kappa_0 = \kappa_1 = 0$, in which case the number of solutions is infinite since both l_0 and l_1 can be considered as free parameters. The resulted curve is always a straight line segment, i.e. the cubic curve degenerates.

Assumption $d \neq 0$ implies $\kappa_0 \neq 0$ and $\kappa_1 \neq 0$, and for any such a pair of signed curvatures there is the unique solution

$$l_0 = \sqrt{\frac{2}{3}\frac{d}{\kappa_0}} \text{ and } l_1 = \sqrt{\frac{2}{3}\frac{d}{\kappa_1}}$$

that can be obtained by Eqs. (2.3) and (2.4).

2.2. Generic case

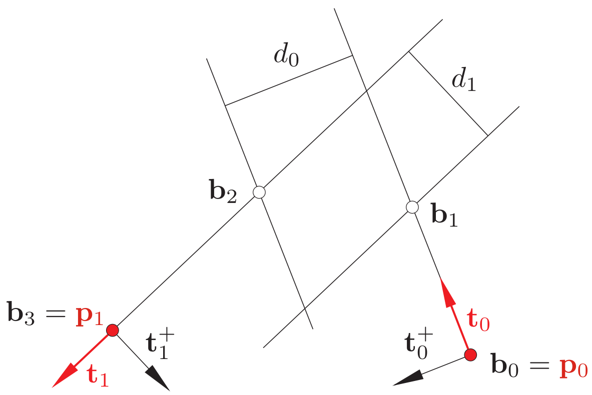

Figure 2: Determination of Bézier points

Hereafter we assume that $\mathbf{t}_0 \nparallel \mathbf{t}_1$. Using the notations of Fig. 2, distance d_1 is of the form

$$\begin{aligned} d_1 &= (\mathbf{b}_1 - \mathbf{b}_3) \cdot \mathbf{t}_1^+ = (\mathbf{p}_0 + l_0\mathbf{t}_0 - \mathbf{p}_1) \cdot \mathbf{t}_1^+ \\ &= l_0\mathbf{t}_0 \cdot \mathbf{t}_1^+ - (\mathbf{p}_1 - \mathbf{p}_0) \cdot \mathbf{t}_1^+ \end{aligned}$$

and the distance d_0 is

$$\begin{aligned} d_0 &= (\mathbf{b}_2 - \mathbf{b}_0) \cdot \mathbf{t}_0^+ = (\mathbf{p}_1 - l_1\mathbf{t}_1 - \mathbf{p}_0) \cdot \mathbf{t}_0^+ \\ &= (\mathbf{p}_1 - \mathbf{p}_0) \cdot \mathbf{t}_0^+ - l_1\mathbf{t}_1 \cdot \mathbf{t}_0^+. \end{aligned}$$

Introducing notations $a = \mathbf{t}_0 \cdot \mathbf{t}_1^+ = -\mathbf{t}_1 \cdot \mathbf{t}_0^+$, $b = (\mathbf{p}_1 - \mathbf{p}_0) \cdot \mathbf{t}_1^+$ and $c = (\mathbf{p}_1 - \mathbf{p}_0) \cdot \mathbf{t}_0^+$, where $\mathbf{t}_i^+$ denotes the positive normal vector of $\mathbf{t}_i$, i.e. $\mathbf{t}_i$ is rotated through $\pi/2$ in counterclockwise direction, we obtain equalities

$$d_0 = al_1 + c, \tag{2.5}$$

$$d_1 = al_0 - b. \tag{2.6}$$

Eqs. (2.3) and (2.5) yield equation

$$3\kappa_0 l_0^2 - 2al_1 - 2c = 0 \tag{2.7}$$

and Eqs. (2.4) and (2.6)

$$3\kappa_1 l_1^2 - 2al_0 + 2b = 0 \tag{2.8}$$

for the unknown distances l_0 and l_1. Therefore, the solution of the geometric Hermite interpolation is reduced to the solution of the system of quadratic equations (2.7), (2.8) of unknowns l_0 and l_1. This system does not always has a solution, or if it has, the the obtained values of l_0 and l_1 are not necessarily positive.

Before we solve the generic case, we have a look at those special cases when one or both of the curvatures vanish. If, e.g. $\kappa_0 = 0$ and $\kappa_1 \neq 0$ then the problem is reduced to a linear one, the solution is

$$l_0 = \frac{3}{2}\kappa_1 \frac{c^2}{a^3} + \frac{b}{a},$$
$$l_1 = -\frac{c}{a}$$

and control points $\mathbf{b}_0, \mathbf{b}_1$ and $\mathbf{b}_2$ become collinear, since $\kappa_0 = 0$ implies $d_0 = 0$. This case is illustrated in Fig. 3.

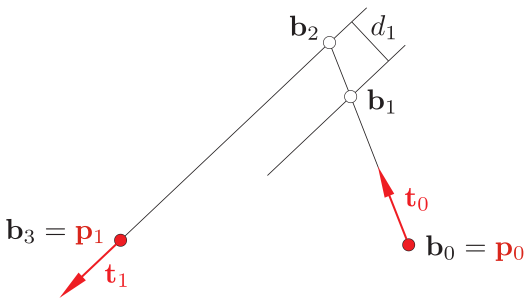

Figure 3: The case $\kappa_0 = 0$ and $\kappa_1 \neq 0$

If both curvatures vanish, then $\kappa_0 = \kappa_1 = 0$ involves $d_0 = d_1 = 0$ thus control points $\mathbf{b}_1$ and $\mathbf{b}_2$ coincide and are the point of intersection of the two tangent lines that must exist due to the assumption $\mathbf{t}_0 \nparallel \mathbf{t}_1$ (cf. Fig. 4).

3. An exact solution of the quadratic system

Hereafter we assume that $\mathbf{t}_0 \nparallel \mathbf{t}_1$ and $\kappa_0 \neq 0$, $\kappa_1 \neq 0$. In this case equalities (2.7) and (2.8) describe parabolas in the coordinate system (l_0, l_1) the axis of which are

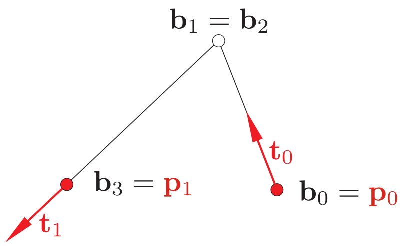

Figure 4: The case $\kappa_0 = \kappa_1 = 0$

perpendicular. Using homogeneous coordinates, the matrices of these parabolas are

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & -a \\ 0 & 3\kappa_1 & 0 \\ -a & 0 & 2b \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 3\kappa_0 & 0 & 0 \\ 0 & 0 & -a \\ 0 & -a & -2c \end{bmatrix}.$$

To solve the system means to find points of intersection of these parabolas. These two parabolas establish a pencil of conics, the elements of which are of the form

$$\mathbf{C} = \alpha\mathbf{A} + \beta\mathbf{B},\ \alpha, \beta \in \mathbb{R},\ |\alpha| + |\beta| \neq 0. \tag{3.1}$$

To find points of intersection of the parabolas is equivalent to determine the base points of the pencil. One of the parameters can be eliminated from matrix (3.1), since matrices $\mathbf{C}$ and $\gamma\mathbf{C}$, $(0 \neq \gamma \in \mathbb{R})$ determine the same conic, therefore we will study the matrix

$$\begin{aligned} \mathbf{C}(\lambda) &= \lambda\mathbf{A} + \mathbf{B} \\ &= \begin{bmatrix} 3\kappa_0 & 0 & -\lambda a \\ 0 & 3\lambda\kappa_1 & -a \\ -\lambda a & -a & 2(\lambda b - c) \end{bmatrix},\ \lambda \in \mathbb{R}. \end{aligned} \tag{3.2}$$

If we can find a degenerate element of this pencil, i.e. an element that is composed of a pair of straight lines, then we can reduce the original quartic problem to a quadratic one. The degenerate element of the pencil is provided by such a λ for which $\det(\mathbf{C}(\lambda)) = 0$, that yields the cubic polynomial

$$\kappa_1 a^2 \lambda^3 - 6b\kappa_0\kappa_1\lambda^2 + 6c\kappa_0\kappa_1\lambda + \kappa_0 a^2 = 0.$$

One real root of this cubic polynomial has to be determined, that always exists. After all, the quartic problem can only be reduced to a cubic one using this method, however this can exactly be solved in a simple way without a numerical procedure.

Parabolas (2.7) and (2.8) are of special position, their axes are perpendicular, and the pencil determined by them always contains an element which is a circle. Indeed, it is easy to check that absolute (or imaginary) circle points (points at which the line at infinity intersects any circle) $\begin{bmatrix} 1 & i & 0 \end{bmatrix}^T$ and $\begin{bmatrix} -1 & i & 0 \end{bmatrix}^T$

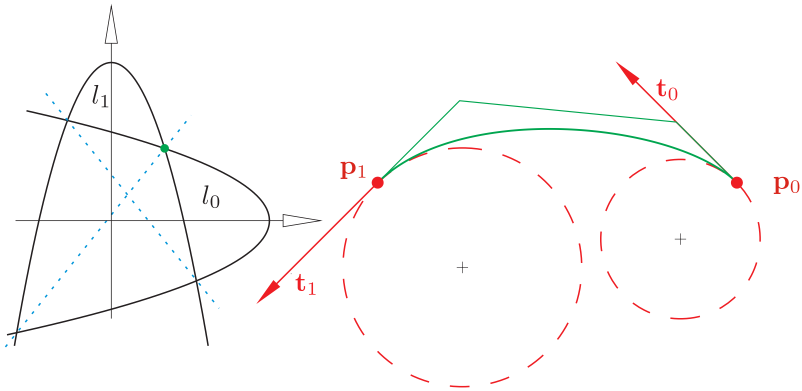

Figure 5: One positive solution (right) along with the corresponding pair of parabolas and a degenerated element (blue dotted) of the pencil determined by them (left). Settings are $\mathbf{p}_0 = [2\ 0]^T$, $\mathbf{p}_1 = [-1\ 0]^T$, $\mathbf{t}_0 = [-1\ 1]^T$, $\mathbf{t}_1 = [-1\ -1]^T$, $\kappa_0 = 1.5$, $\kappa_1 = 1$ (For interpretation of the references to color in this figure legend, the reader is referred to the pdf version of this article.)

are on the element of the pencil (3.2) that corresponds to

$$\lambda = \frac{\kappa_0}{\kappa_1},$$

provided $\kappa_1 \neq 0$, that we have already excluded. The matrix of this circle is

$$\begin{bmatrix} 3\kappa_0 & 0 & -\frac{\kappa_0}{\kappa_1}a \\ 0 & 3\kappa_0 & -a \\ -\frac{\kappa_0}{\kappa_1}a & -a & 2\left(\frac{\kappa_0}{\kappa_1}b - c\right) \end{bmatrix},$$

therefore the square of its radius is

$$r^2 = \frac{2}{3\kappa_0}\left(c - \frac{\kappa_0}{\kappa_1}b\right). \tag{3.3}$$

If (3.3) is positive (note that the circle can be imaginary as well) then parabolas (2.7) and (2.8) have real points on common, i.e. the system of quadratic equations has a solution. Note, that this is a criterion just for the solvability of the system and not for the positive solution.

In practice, positive solutions (both l_0 and l_1 are positive) are used in general, since only in this case will the direction of tangent vectors be kept. In the coordinate system (l_0, l_1) the axis of parabolas (2.7) and (2.8) coincide with the axes l_0 and l_1 of the coordinate system, respectively, thus the number of positive solutions can be $0, 1, 2$ or 3.

As it is specified in [2], in the generic case there can be a positive solution if products

$$\mathbf{t}_0 \wedge \mathbf{t}_1,\ (\mathbf{p}_1 - \mathbf{p}_0) \wedge \mathbf{t}_1,\ \mathbf{t}_0 \wedge (\mathbf{p}_1 - \mathbf{p}_0)$$

have the same sign that coincides with the sing of κ_0 and κ_1. In this case the existence of a positive solution can be guaranteed by adjusting the magnitude of curvatures, i.e. curvatures can be used as shape parameters. Fig. 5 illustrates a case of one positive solution.

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On the right-continuity of infimal convolutions

Gergely Pataki

Department of Mathematical Analysis
Budapest University of Technology and Economics, Budapest
pataki@math.bme.hu

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Abstract

We provide a sufficient condition for the right-continuity of the infimal convolution. First we introduce the generalized infimal convolution, and we show that this is not a proper generalization in the case of real-valued functions.

Following the main theorem, through some illustrating examples we prove that the result cannot be strengthened.

Keywords: infimal convolution, generalized infimal convolution, Darboux-property (intermediate value property), right-continuity

MSC: 26A03, 26A15, 44A35, 54C05, 54C30

1. Preliminaries

We use the following usual notions and notations in the whole paper.

Let

$$\underline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \qquad \text{and} \qquad \overline{\mathbb{R}} = \underline{\mathbb{R}} \cup \{\infty\}.$$

Note that $\overline{\mathbb{R}} = [-\infty, \infty]$ is called the extended real line.

In this paper a Darboux function is an extended real-valued function f of a real variable which has the intermediate value property, that is for any two real values a and b, and any y between $f(a)$ and $f(b)$, there is some c between a and b with $f(c) = y$.

A function $f : \mathbb{R} \to \overline{\mathbb{R}}$ is called right-continuous if for any $x_0 \in \mathbb{R}$ and for all neighborhood K of $f(x_0)$ there exists a positive real δ such that $f(x) \in K$ for all real x with $x_0 < x < x_0 + \delta$.

Now, we define $(f * g)$ and $(f \circledast g)$, the so called infimal convolution and generalized infimal convolution of f and g. (See in [5], [8] and [10])

Definition 1.1. Let $f : \mathbb{R} \to \underline{\mathbb{R}}$ be a function of $\mathbb{R}$ into $\underline{\mathbb{R}}$ and $g : V \subset \mathbb{R} \to \underline{\mathbb{R}}$ be a function of a subset V of $\mathbb{R}$ into $\underline{\mathbb{R}}$.

For any $x \in \mathbb{R}$, let

$$\Gamma(x) = \{(u, v) \in \mathbb{R} \times V : x \leq u + v\}$$

and define the functions $(f * g) : \mathbb{R} \to \overline{\mathbb{R}}$ and $(f \circledast g) : \mathbb{R} \to \overline{\mathbb{R}}$ of $\mathbb{R}$ into $\overline{\mathbb{R}}$ such that

$$(f * g)(x) = \inf_{v \in V} \big(f(x - v) + g(v)\big)$$

and

$$(f \circledast g)(x) = \inf\{f(u) + g(v) : (u, v) \in \Gamma(x)\}.$$

$(f * g)$ and $(f \circledast g)$ are called infimal convolution and generalized infimal convolution of f and g, respectively.

Remark 1.2. Note that we define the (generalized) infimal convolution only in the case, when the range of the factors f and g are subsets of $\underline{\mathbb{R}}$, and note also that if $V \neq \emptyset$, then the range of $(f \circledast g)$ and $(f * g)$ is also subset of $\underline{\mathbb{R}}$.

For all $x, y, z \in \underline{\mathbb{R}}$ we have $(-\infty) + x = x + (-\infty) = -\infty$ and $x \leq y \implies x + z \leq y + z$.

In this paper, the order of the domain and range of f and g is the usual order of (extended) real line, and the inf derives from this order also. The following theorem and corollary show that since the same ordering, the generalized infimal convolution is not a proper generalization contrary to the more abstract case, see [1] and [6].

Definition 1.3. Let $f : \mathbb{R} \to \underline{\mathbb{R}}$ and define $\hat{f} : \mathbb{R} \to \underline{\mathbb{R}}$ for all $u \in \mathbb{R}$ in the following way.

$$\hat{f}(u) = \inf_{a \geq u} f(a)$$

Proposition 1.4. *If $f : \mathbb{R} \to \underline{\mathbb{R}}$, then $\hat{f}$ is also, that is $\hat{f}(u) \neq \infty$ for all $u \in \mathbb{R}$. Moreover $\hat{f} \leq f$, $\hat{f}$ is increasing, and $\hat{f} = f$ if and only if f is increasing.*

The generalized infimal convolution could be expressed by classical one.

Theorem 1.5. *If $f : \mathbb{R} \to \underline{\mathbb{R}}$ and $g : V \subset \mathbb{R} \to \underline{\mathbb{R}}$, then for any $x \in \mathbb{R}$ we have*

$$(f \circledast g)(x) = (\hat{f} * g)(x).$$

Proof. By definition

$$(f \circledast g)(x) \leq f(u) + g(v)$$

for all $(u, v) \in \Gamma(x)$, therefore

$$(f \circledast g)(x) \leq \inf_{u \geq x-v} \big(f(u) + g(v)\big)$$

for all $v \in V$, and it follows that

$$(f \circledast g)(x) \leq \inf_{v \in V} \Big(\inf_{u \geq x-v} \big(f(u) + g(v)\big)\Big).$$

On the other hand

$$\inf_{v \in V} \Big(\inf_{u \geq x-v} \big(f(u) + g(v)\big)\Big) \leq \inf_{u \geq x-v} \big(f(u) + g(v)\big)$$

for all $v \in V$, and

$$\inf_{u \geq x-v} \big(f(u) + g(v)\big) \leq f(u) + g(v)$$

for all $(u, v) \in \Gamma(x)$, therefore

$$\inf_{v \in V} \Big(\inf_{u \geq x-v} \big(f(u) + g(v)\big)\Big) \leq (f \circledast g)(x).$$

Now we have that

$$(f \circledast g)(x) = \inf_{v \in V} \Big(\inf_{u \geq x-v} \big(f(u) + g(v)\big)\Big),$$

and by the required definitions

$$(\hat{f} * g)(x) = \inf_{v \in V} \big(\hat{f}(x-v) + g(v)\big) = \inf_{v \in V} \Big(\inf_{u \geq x-v} \big(f(u)\big) + g(v)\Big).$$

If we notice that $\inf(A + a) = \inf A + a$ for all $A \subset \underline{\mathbb{R}}$ and $a \in \underline{\mathbb{R}}$, and write $\{f(u) : u \geq x - v\}$ and $g(v)$ in place of A and a respectively, then the proof is complete. □

According to Proposition 1.4 and Theorem 1.5 we obtain:

Corollary 1.6. *If $f : \mathbb{R} \to \underline{\mathbb{R}}$ is increasing and $g : V \subset \mathbb{R} \to \underline{\mathbb{R}}$, then for any $x \in \mathbb{R}$ we have*

$$(f \circledast g)(x) = (f * g)(x).$$

The infimal convolution could not always be expressed by generalized one. Namely, there is not exist $f : \mathbb{R} \to \underline{\mathbb{R}}$ and $g : V \subset \mathbb{R} \to \underline{\mathbb{R}}$ such that $(f \circledast g) = (\sin * \sin)$, because

$$(\sin * \sin)(0) = \inf_{v \in \mathbb{R}} \big(\sin(0 - v) + \sin(v)\big) = 0,$$

and

$$(\sin * \sin)\left(\frac{\pi}{2}\right) = \inf_{v \in \mathbb{R}} \left(\sin\left(\frac{\pi}{2} - v\right) + \sin(v)\right) = \inf_{v \in \mathbb{R}} \big(\cos(v) + \sin(v)\big) = -\sqrt{2},$$

but by Proposition 2.1 $(f \circledast g)$ is increasing.

2. Main Theorem

The following proposition has been proved in [10]. Its proof is included here for the reader's convenience.

Proposition 2.1. *If* $f : \mathbb{R} \to \underline{\mathbb{R}}$ *and* $g : V \subset \mathbb{R} \to \underline{\mathbb{R}}$, *then* $(f \circledast g)$ *is increasing.*

Proof. Suppose that $x, y \in \mathbb{R}$ such that $x \leq y$. Then, by the corresponding definitions, it is clear that $\Gamma(y) \subset \Gamma(x)$ and thus $(f \circledast g)(x) \leq (f \circledast g)(y)$. □

Theorem 2.2. *If* $f : \mathbb{R} \to \underline{\mathbb{R}}$ *is a Darboux function and* $g : V \subset \mathbb{R} \to \underline{\mathbb{R}}$, *then* $(f \circledast g)$ *is right-continuous.*

Proof. Since the domain of $(f \circledast g)$ is the real line, we need only show that, for any $x \in \mathbb{R}$, we have

$$(f \circledast g)(x) = \lim_{t \to x+} (f \circledast g)(t).$$

Since Proposition 2.1 $(f \circledast g)$ is increasing, therefore by [7] Theorem 4.29. we have

$$(f \circledast g)(x) \leq \inf_{t > x} (f \circledast g)(t) = \lim_{t \to x+} (f \circledast g)(t).$$

To prove the corresponding equality, define

$$y = \inf_{t > x} (f \circledast g)(t),$$

and assume to the contrary that $(f \circledast g)(x) < y$. Then, $y \neq -\infty$.

Let $(u, v) \in \Gamma(x)$ such that

$$f(u) + g(v) < y.$$

There exists such (u, v) because y is not a lower bound of $\{f(u) + g(v) : (u, v) \in \Gamma(x)\}$.

Hence by using that $(f \circledast g)(u + v) \leq f(u) + g(v)$, we can infer that

$$(f \circledast g)(u + v) < y.$$

Thus, by the definition of y we necessarily have

$$u + v \leq x.$$

Moreover, since $(u, v) \in \Gamma(x)$, we also have $x \leq u + v$, and thus $x = u + v$.

Also by the corresponding definitions, we can note that

$$y \leq (f \circledast g)(x + 1) = (f \circledast g)(u + 1 + v) \leq f(u + 1) + g(v).$$

Hence, since $y \neq -\infty$, we can see that $g(v) \neq -\infty$.

Now, from the inequalities

$$f(u) + g(v) < y \leq f(u + 1) + g(v),$$

we can infer that

$$f(u) < y - g(v) \leq f(u+1).$$

Thus, by the Darboux property of f, there exists $s \in]u, u+1[$, such that

$$f(u) < f(s) < y - g(v),$$

and thus

$$f(s) + g(v) < y.$$

Moreover, since $x = u + v < s + v$, we can see that

$$y \leq (f \circledast g)(s+v) \leq f(s) + g(v) < y.$$

This contradiction proves the required equality. □

Corollary 2.3. *If $f : \mathbb{R} \to \underline{\mathbb{R}}$ and $g : V \subset \mathbb{R} \to \underline{\mathbb{R}}$ such that f is a continuous increasing function, then $(f * g)$ is right-continuous.*

Note that since f is increasing, hence it is a Darboux function if and only if it is continuous.

3. Some Illustrating examples

The following two examples show that $(f \circledast g)$ may be discontinuous even if $f, g : \mathbb{R} \to \underline{\mathbb{R}}$ are increasing continuous real-valued functions.

Example 3.1. Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^3$. Since f is increasing

$$(f \circledast f)(x) = (f * f)(x) = \inf_{v \in \mathbb{R}} \left(x^3 - 3x^2 v + 3xv^2\right).$$

Let $p(v) = x^3 - 3x^2 v + 3xv^2$.

If $x < 0$, then

$$\lim_{v \to \pm\infty} p(v) = -\infty$$

therefore $(f \circledast f)(x) = -\infty$.

If $x > 0$, then p is differentiable at everywhere, and $p'(v) = -3x^2 + 6xv = 3x(2v - x)$. Now we have that $p'(v) < 0$, $p'(v) = 0$ and $p'(v) > 0$ if $v < \frac{x}{2}$, $v = \frac{x}{2}$ and $v > \frac{x}{2}$, respectively. Therefore

$$(f \circledast f)(x) = \inf_{v \in \mathbb{R}} p(v) = \min_{v \in \mathbb{R}} p(v) = p\left(\frac{x}{2}\right) = \frac{x^3}{4}.$$

By Theorem 2.2, $(f \circledast f)$ is right-continuous. Hence one has

$$(f \circledast f)(0) = \lim_{x \to 0+} (f \circledast f)(x) = 0.$$

In the above example, the convolution has an infinite jump discontinuity. Since the increasingness, an arbitrary convolution has at most 2 infinite jump, and at most countable many finite jump, and has not any essential discontinuity. (See [7] Theorem 4.30.)

In the following more complicated example $(f \circledast g)$ is also real-valued, and $(f \circledast g)$ has infinitely many finite jump. For any $x \in \mathbb{R}$ let $[x]$ and $\{x\}$ denote the integer and fractional part of x.

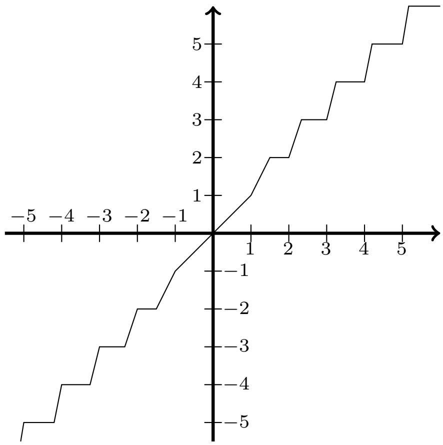
Example 3.2. Let $\alpha : \mathbb{R} \to \mathbb{R}$ such that

$$\alpha(x) = \begin{cases} 0, \text{ if} & x < 0, \\ x, \text{ if} & 0 \le x \le 1, \\ 1, \text{ if} & 1 < x. \end{cases}$$

It is easy to see that α is an increasing continuous function. Moreover let $g : \mathbb{R}_0^+ \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ such that

$$g(x) = \sum_{n=0}^{\infty} \alpha((n+1)(x-n)) \qquad \text{and} \qquad f(x) = \begin{cases} g(x), & \text{if } 0 \le x, \\ -g(-x), & \text{if } x < 0. \end{cases}$$

This figure shows the graph of f.



For all $x \ge 0$ if $n \in \mathbb{Z}$ with $n \ge [x] + 1 > x$ we have that $x - n < 0$ and $n + 1 > 0$, hence $(n+1)(x-n) < 0$, therefore $\alpha((n+1)(x-n)) = 0$. It follows that

$$g(x) = \sum_{n=0}^{[x]} \alpha((n+1)(x-n)) \tag{3.1}$$

for all $x \ge 0$, and if $a \ge 0$, then

$$g(x) = \sum_{n=0}^{[a]} \alpha((n+1)(x-n))$$

on $[0,a]$.

It means, that definition of g is correct and g is really a real-valued function. It also means that the function series is uniformly convergent on $[0,a]$ for all $a \geq 0$, hence the continuity of $\alpha((n+1)(x-n))$ follows the continuity of g on $[0,a]$ for all $a \geq 0$ and hence g is continuous. Note that by (3.1) we can also easily see the increasingness of g.

Moreover, if $n \in \mathbb{Z}$ with $0 \leq n \leq [x]-1$, then since $(n+1)(x-n) \geq 1$, hence $\alpha((n+1)(x-n)) = 1$. It follows by (3.1) that

$$g(x) = \sum_{n=0}^{[x]-1} 1 + \alpha(([x]+1)(x-[x])) = [x] + \alpha(([x]+1)\{x\}). \tag{3.2}$$

$[x]+1 \geq 1$ and $\{x\} \geq 0$ imply that $([x]+1)\{x\} \geq \{x\}$ and it follows by the increasingness of α, that $\alpha(([x]+1)\{x\}) \geq \alpha(\{x\}) = \{x\}$. Now, since (3.2) we have that

$$g(x) \geq [x] + \{x\} = x \tag{3.3}$$

for all $x \geq 0$.

If $n \in \mathbb{Z}_0^+$, then by (3.2)

$$g(n) = [n] + \alpha(([n]+1)\{n\}) = n, \tag{3.4}$$

and

$$g(x+n) - g(x) = [x+n] + \alpha(([x+n]+1)\{x+n\}) - [x] - \alpha(([x]+1)\{x\}) \geq n \tag{3.5}$$

for all $x \geq 0$.

Since the increasingness of g and (3.4) we also have that

$$g(x) \leq g([x]+1) = [x]+1 \leq x+1, \tag{3.6}$$

for all $x \geq 0$.

To prove the increasingness of f, by using the increasingness of g let $x_1 < x_2 < 0$, and see the following

$$f(x_1) = -g(-x_1) \leq -g(-x_2) = f(x_2) \leq 0 = g(0) = f(0).$$

It follows by Corollary 1.6 that

$$(f \circledast f)(x) = (f * f)(x).$$

By (3.3) and (3.6) we have that

$$x \leq f(x) \leq x+1, \qquad \text{for all } x \geq 0, \tag{3.7}$$

and

$$x-1 \leq f(x) \leq x, \qquad \text{for all } x < 0. \tag{3.8}$$

Now, we prove that $(f \circledast f)(x) = [x]$ for all $x \geq 0$.

At first, let $x \in \mathbb{Z}_0^+$. In this case one has

$$f(x-0) + f(0) = g(x) = x,$$

therefore $(f \circledast f)(x) \leq x$.

To prove the converse inequality we show that $f(x-v)+f(v) \geq x$, for all $v \in \mathbb{R}$.
If $0 \leq v \leq x$, then $f(x-v) + f(v) \geq (x-v) + v = x$ by (3.7).
If $x < v$, then $f(x-v) + f(v) = -g(v-x) + g(v) = g((v-x)+x) - g(v-x) \geq x$ by (3.5).
If $v < 0$, then $f(x-v) + f(v) = g(x-v) - g(-v) \geq x$ by (3.5).

At second, let $x \in \mathbb{R}_0^+ \setminus \mathbb{Z}$. In this case one has

$$(f \circledast f)(x) \geq (f \circledast f)([x]) = [x],$$

by Proposition 2.1.

To prove the converse inequality we show that $f(x-v)+f(v) = [x]$ with $v \in \mathbb{Z}$ such that $v \geq [x] + 1/(1-\{x\})$.

Really, for such v we have that

$$(v-[x])(1-\{x\}) \geq 1,$$

that is

$$\alpha(([v-x]+1)\{v-x\}) = 1.$$

It means by (3.2) that

$$g(v-x) = [v-x] + 1 = v - [x],$$

therefore

$$f(x-v) + f(v) = -g(v-x) + g(v) = v - (v-[x]) = [x].$$

Finally, we show that $(f \circledast f)$ is real-valued. For this, it is enough to show that if $x \in \mathbb{R}$, then by (3.7) and (3.8)

$$x - 2 = (x-v-1) + (v-1) \leq f(x-v) + f(v) \leq (x-v+1) + (v+1) = x+2,$$

for all $v \in \mathbb{R}$, therefore

$$x - 2 \leq (f \circledast f)(x) \leq x + 2.$$

Finally, the following example shows that the Darboux property in Theorem 2.2 is essential.

Example 3.3. Let $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = 1_{\mathbb{R}^+} = \begin{cases} 0, \text{ if} & x \leq 0 \\ 1, \text{ if} & 0 < x. \end{cases}$$

For all real x and v we have that $f(x-v), f(v) \in \{0,1\}$, therefore $f(x-v)+f(v) \in \{0,1,2\}$, hence $(f*f)(x) \in \{0,1,2\}$.

With $v=0$ we have that $f(x-0)+f(0)=f(x)$, hence $(f*f)(x) \leq f(x)$ for all $x \in \mathbb{R}$.

To prove the converse inequality, it is enough to show that $1 \leq (f*f)(x)$ for all $x>0$. If $0<x$, then for all $v \in \mathbb{R}$ $x-v>0$ or $v>0$, that is $f(x-v)=1$ or $f(v)=1$, hence $1 \leq f(x-v)+f(v)$. It follows that $1 \leq (f*f)(x)$ for all $x>0$.

Now, we have that

$$(f*f)=f,$$

which is not right-continuous.

Note that by Corollary 1.6, the increasingness of f follows that

$$(f \circledast f)(x)=(f*f)(x).$$

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On weak symmetries of Kenmotsu Manifolds with respect to quarter-symmetric metric connection

D. G. Prakasha, K. Vikas

Department of Mathematics, Karnatak University, Dharwad
prakashadg@gmail.com
vikasprt@gmail.com

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Abstract

The aim of this paper is to study weakly symmetries of Kenmotsu manifolds with respect to quarter-symmetric metric connection. We investigate the properties of weakly symmetric, weakly Ricci-symmetric and weakly concircular Ricci-symmetric Kenmotsu manifolds with respect to quarter symmetric metric connection and obtain interesting results.

Keywords: Kenmotsu manifold; weakly symmetric manifold; weakly Ricci-symmetric manifold; weakly concircular Ricci-symmetric manifold; quarter-symmetric metric connection.

MSC: 53C15, 53C25, 53B05;

1. Introduction

In 1924, A. Friedman and J. A. Schouten ([8, 22]) introduced the notion of a semi-symmetric metric linear connection on a differentiable manifold. H.A. Hayden [10] defined a metric connection with torsion on a Riemannian manifold. In 1970, K. Yano [29] studied some curvature and derivational conditions for semi-symmetric connections in Riemannian manifolds. In 1975, S. Golab [9] initiated the study of quarter-symmetric linear connection on a differentiable manifold. A linear connection $\widetilde{\nabla}$ in an n-dimensional differentiable manifold is said to be a quarter-symmetric connection if its torsion T is of the form

$$T(X,Y) = \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X,Y] = \eta(Y)\phi X - \eta(X)\phi Y, \tag{1.1}$$

where η is a 1-form and ϕ is a tensor of type (1, 1). In addition, a quarter-symmetric linear connection $\widetilde{\nabla}$ satisfies the condition

$$(\widetilde{\nabla}_X g)(Y,Z) = 0 \tag{1.2}$$

for all $X, Y, Z \in \chi(M)$, where $\chi(M)$ is the Lie algebra of vector fields of the manifold M, then $\widetilde{\nabla}$ is said to be a quarter-symmetric metric connection. If we replace ϕX by X and ϕY by Y in (1.1) then the connection is called a semi-symmetric metric connection [29]. In 1980, R.S. Mishra and S. N. Pandey [15] studied quarter-symmetric metric connection and in particular, Ricci quarter-symmetric metric connection on Riemannian, Sasakian and Kaehlerian manifolds. Note that a quarter-symmetric metric connection is a Hayden connection with the torsion tensor of the form (1.1). A studies on various types of quarter-symmetric metric connection and their properties included in ([1, 5, 18, 20, 21, 30]) and others.

On the other hand K. Kenmotsu [14] defined a type of contact metric manifold which is now a days called Kenmotsu manifold. It may be mentioned that a Kenmotsu manifold is not a Sasakian manifold.

The weakly symmetric and weakly Ricci-symmetric manifolds were defined by L. Tamássy and T. Q. Binh [26](1992, 1993) and studied by several authors (see [3, 4, 6, 13, 16, 19, 23, 24]. The weakly concircular Ricci symmetric manifolds were introduced by U. C. De and G. C. Ghosh (2005) [7] and these type of notion were studied with Kenmotsu structure in [11]. Many authors investigate these manifolds and their generalizations.

A non-flat Riemannian manifold $M(n > 2)$ is called a weakly symmetric if there exist 1-forms A, B, C, D and their curvature tensor R of type (0, 4) satisfies the condition

$$(\nabla_X R)(Y,Z,V) = A(X)R(Y,Z,V) + B(Y)R(X,Z,V) + C(Z)R(Y,X,V) + D(V)R(Y,Z,X) + g(R(Y,Z,V),X)P \tag{1.3}$$

for all vector fields $X, Y, Z, V \in \chi(M)$, where A, B, C, D and P are not simultaneously zero and ∇ is the operator of covariant differentiation with respect to the Riemannian metric g. The 1-forms are called the associated 1-forms of the manifold.

A non-flat Riemannian manifold $M(n > 2)$ is called weakly Ricci-symmetric if there exist 1-forms α, β and γ and their Ricci tensor S of type (0, 2) satisfies the condition

$$(\nabla_X S)(Y,Z) = \alpha(X)S(Y,Z) + \beta(Y)S(X,Z) + \gamma(Z)S(Y,X) \tag{1.4}$$

for all vector fields $X, Y, Z \in \chi(M)$, where α, β and γ are not simultaneously zero.

A non-flat Riemannian manifold $M(n > 2)$ is called weakly concircular Ricci-symmetric manifold [7] if its concircular Ricci tensor P of type (0, 2) given by

$$P(Y,Z) = \sum_{i=1}^{n} \bar{C}(Y, e_i, e_i, Z) = S(Y,Z) - \frac{r}{n} g(Y,Z) \tag{1.5}$$

is not identically zero and satisfies the condition

$$(\nabla_X P)(Y,Z) = \alpha(X)P(Y,Z) + \beta(Y)P(X,Z) + \gamma(Z)P(Y,X), \tag{1.6}$$

where α, β and γ are associated 1-forms (not simultaneously zero). In equation (5.12), $\bar{C}$ denotes the concircular curvature tensor defined by [28]

$$\bar{C}(Y,U,V,Z) = R(Y,U,V,Z) - \frac{r}{n(n-1)}[g(U,V)g(Y,Z) - g(Y,V)g(U,Z)],$$

where r is the scalar curvature of the manifold.

The paper is organized as follows: In section 2, we give a brief account of Kenmotsu manifolds. In section 3 we give the relation between Levi-Civita connection ∇ and quarter-symmetric metric connection $\widetilde{\nabla}$ on a Kenmotsu manifold. Section 4 is devoted to the study of weakly symmetries of Kenmotsu manifolds with respect to quarter-symmetric metric connection $\widetilde{\nabla}$. It is shown that, in a weakly symmetric Kenmotsu manifold M $(n > 2)$ with respect to the connection $\widetilde{\nabla}$, the sum of associated 1-forms A, C and D is zero everywhere. In the last section, we study weakly Ricci-symmetric and weakly concircular Ricci-symmetric Kenmotsu manifolds with respect to quarter-symmetric metric connection $\widetilde{\nabla}$ in that we proved the sum of associated 1-forms α, β and γ is zero everywhere. Also, it is proved that, if the weakly Ricci symmetric Kenmotsu manifold with respect to the connection $\widetilde{\nabla}$ is Ricci-recurrent with respect to the connection $\widetilde{\nabla}$ then the associated 1-forms β and γ are in opposite directions. Finally, we consider weakly concircular Ricci-symmetric Kenmotsu manifold with respect to quarter-symmetric metric connection and prove that in such a manaifold, the sum of associated 1-forms is zero if the scalar curvature of the manifold is constant.

2. Kenmotsu manifolds

An $n(= 2m+1)$-dimensional differentiable manifold M is called an almost contact Riemannian manifold if either its structural group can be reduced to $U(n) \times 1$ or equivalently, there is an almost contact structure (ϕ, ξ, η) consisting of a (1,1) tensor field ϕ, a vector field ξ and a 1-form η satisfying

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \qquad \phi\xi = 0, \qquad \eta(\phi X) = 0, \tag{2.1}$$

Let g be a compatible Riemannian metric with (ϕ, ξ, η), that is

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{2.2}$$

or equivalently,

$$g(X,\phi Y) = -g(\phi X, Y) \quad and \quad g(X,\xi) = \eta(X) \tag{2.3}$$

for any vector fields X, Y on M [2].
An almost Kenmotsu manifold become a Kenmotsu manifold if

$$g(X,\phi Y) = d\eta(X,Y) \tag{2.4}$$

for all vector fields X, Y. If moreover

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.5}$$

$$(\nabla_X \phi)(Y) = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.6}$$

for any $X, Y \in \chi(M)$ then (M, ϕ, ξ, η, g) is called an almost Kenmotsu manifold. Here ∇ denotes the Riemannian connection of g. In a Kenmotsu manifold M the following relations hold [14]:

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.7}$$

$$R(X,\xi)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.8}$$

$$(\nabla_X \eta)Y = g(X,Y) - \eta(X)\eta(Y), \tag{2.9}$$

$$S(X,\xi) = -(n-1)\eta(X), \tag{2.10}$$

$$S(\xi,\xi) = -(n-1), \tag{2.11}$$

for every vector fields X, Y on M where R and S are the Riemannian curvature tensor and the Ricci tensor with respect to LeviCivita connection, respectively.

3. Quarter symmetric metric connection on a Kenmotsu manifold

A quarter symmetric metric connection $\tilde{\nabla}$ on a Kenmotsu manifold is given by [25]

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y. \tag{3.1}$$

A relation between the curvature tensor of M with respect to the quarter symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ is given by [17, 25]

$$\begin{aligned}\tilde{R}(X,Y)Z = R(X,Y)Z - 2d\eta(X,Y)\phi Z + [\eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z)]\xi \\ + [\eta(Y)\phi X - \eta(X)\phi Y]\eta(Z),\end{aligned} \tag{3.2}$$

where $\tilde{R}$ and R are the Riemannian curvatures of the connection $\tilde{\nabla}$ and ∇, respectively. From (3.2), it follows that

$$\tilde{S}(Y,Z) = S(Y,Z) - 2d\eta(\phi Z, Y) + g(\phi Y, Z) + \psi\eta(Y)\eta(Z), \tag{3.3}$$

where $\tilde{S}$ and S are the Ricci tensors of the connection $\tilde{\nabla}$ and ∇, respectively and $\psi = \sum_{i=1}^{n} g(\phi e_i, e_i) = Trace\ \ of\ \ \phi$. Contracting (3.3), we get

$$\tilde{r} = r + 2(n-1), \tag{3.4}$$

where $\tilde{r}$ and r are the scalar curvatures of the connection $\tilde{\nabla}$ and ∇, respectively. From (3.3) it is clear that in a Kenmotsu manifold the Ricci tensor with respect to the quarter-symmetric metric connection is not symmetric.

4. Weakly symmetric Kenmotsu manifolds admitting a quarter-symmetric metric connection

Analogous to the notions of weakly symmetric, weakly Ricci-symmetric and weakly concircular Ricci-symmetric Kenmotsu manifold with respect to Levi-Civita connection, in this section we define the notions of weakly symmetric, weakly Ricci-symmetric and weakly concircular Ricci-symmetric Kenmotsu manifodls with respect to quarter-symmetric metric connection. This notions have been studied by J. P. Jaiswal [12] in the context of Sasakian manifolds.

Definition 4.1. A Kenmotsu manifold $M(n > 2)$ is called weakly symmetric with respect to quarter-symmetric metric connection $\tilde{\nabla}$ if there exist 1-forms A, B, C and D and their curvature tensor $\tilde{R}$ satisfies the condition

$$\begin{aligned}(\tilde{\nabla}_X \tilde{R})(Y,Z,V) = A(X)\tilde{R}(Y,Z,V) + B(Y)\tilde{R}(X,Z,V) + C(Z)\tilde{R}(Y,X,V)\\ + D(V)\tilde{R}(Y,Z,X) + g(\tilde{R}(Y,Z,V),X)P,\end{aligned} \tag{4.1}$$

for all vector fields $X, Y, Z, V \in \chi(M)$.

Let M be a weakly symmetric Kenmotsu manifold with respect to the connection $\tilde{\nabla}$. So equation (4.1) holds. Contracting (4.1) over Y, we have

$$\begin{aligned}(\tilde{\nabla}_X \tilde{S})(Z,V) = A(X)\tilde{S}(Z,V) + B(\tilde{R}(X,Z,V)) + C(Z)\tilde{S}(X,V)\\ + D(V)\tilde{S}(X,Z) + E(\tilde{R}(X,V,Z))\end{aligned} \tag{4.2}$$

where E is defined by $E(X) = g(X,P)$. Replacing V with ξ in the above equation and then using the relations (2.7), (2.8),(2.10) and (3.3), we get

$$\begin{aligned}&(\tilde{\nabla}_X \tilde{S})(Z,\xi)\\ &= \{\psi - (n-1)\}\{A(X)\eta(Z) + C(Z)\eta(X)\} + \eta(X)\{B(Z) - B(\phi Z)\}\\ &\quad - \eta(Z)\{B(X) - B(\phi X)\} + D(\xi)\{S(X,Z) - 2d\eta(\phi Z, X) + g(\phi X, Z)\\ &\quad + \psi\eta(X)\eta(Z)\} + E(\xi)\{g(X,Z) - g(\phi X,Z)\} - \eta(Z)\{E(X) - E(\phi X)\}.\end{aligned} \tag{4.3}$$

We know that

$$(\tilde{\nabla}_X \tilde{S})(Z,\xi) = \tilde{\nabla}_X \tilde{S}(Z,\xi) - \tilde{S}(\tilde{\nabla}_X Z,\xi) - \tilde{S}(Z,\tilde{\nabla}_X \xi). \tag{4.4}$$

By making use of (2.3), (2.5), (2.9), (3.1) and (3.3) in (4.4) we have

$$
\begin{aligned}
(\tilde{\nabla}_X \tilde{S})(Z, \xi) = & -S(X, Z) + 2d\eta(\phi Z, X) - g(\phi Z, X) \\
& + \{\psi - (n-1)\}g(X, Z) - \psi\eta(X)\eta(Z).
\end{aligned} \tag{4.5}
$$

Applying (4.5) in (4.3), we obtain

$$
\begin{aligned}
& - S(X, Z) + 2d\eta(\phi Z, X) - g(\phi Z, X) + \{\psi - (n-1)\}g(X, Z) - \psi\eta(X)\eta(Z) \\
& = \{\psi - (n-1)\}\{A(X)\eta(Z) + C(Z)\eta(X)\} + \eta(X)\{B(Z) + B(\phi Z)\} \\
& \quad - \eta(Z)\{B(X) + B(\phi X)\} + D(\xi)\{S(X, Z) - 2d\eta(\phi Z, X) + g(\phi X, Z) \\
& \quad + \psi\eta(X)\eta(Z)\} + E(\xi)\{g(X, Z) - g(\phi X, Z)\} - \eta(Z)\{E(X) - E(\phi X)\}.
\end{aligned} \tag{4.6}
$$

Setting $X = Z = \xi$ in (4.6) and using (2.1) and (2.9), we find that

$$
\{\psi - (n-1)\}\{A(\xi) + C(\xi) + D(\xi)\} = 0, \tag{4.7}
$$

which implies that (since $n > 3$)

$$
A(\xi) + C(\xi) + D(\xi) = 0 \tag{4.8}
$$

holds on M.

Next, plugging Z with ξ in (4.2) and doing the calculations it can be shown that

$$
\begin{aligned}
& - S(X, V) + 2d\eta(\phi V, X) - g(\phi V, X) + \{\psi - (n-1)\}g(X, V) - \psi\eta(X)\eta(V) \\
& = \{\psi - (n-1)\}\{A(X)\eta(V) + D(V)\eta(X)\} + B(\xi)\{g(X, V) - g(\phi X, V)\} \\
& \quad - \eta(V)\{B(X) - B(\phi X)\} + \eta(X)\{E(V) - E(\phi V)\} - \eta(V)\{E(X) - E(\phi X)\} \\
& \quad + C(\xi)\{S(X, V) - 2d\eta(\phi V, X) + g(\phi X, V) + \psi\eta(X)\eta(V)\}
\end{aligned} \tag{4.9}
$$

Setting $V = \xi$ in (4.9) and then using the relations (2.1),(2.3)and (2.10) we get

$$
\begin{aligned}
& \{\psi - (n-1)\}A(X) - \{B(X) - B(\phi X)\} + \eta(X)B(\xi) \\
& \quad + \{\psi - (n-1)\}\eta(X)C(\xi) + \{\psi - (n-1)\}\eta(X)D(\xi) \\
& \quad - \{E(X) - E(\phi X)\} + \eta(X)E(\xi) = 0.
\end{aligned} \tag{4.10}
$$

Similarly, if we set $X = \xi$ in (4.9), we obtain

$$
\begin{aligned}
& \{\psi - (n-1)\}A(\xi)\eta(V) + \{\psi - (n-1)\}C(\xi)\eta(V) \\
& \quad + \{\psi - (n-1)\}D(V) - \eta(V)E(\xi) + \{E(V) - E(\phi V)\} = 0,
\end{aligned} \tag{4.11}
$$

Replacing V with X the above equation becomes

$$
\begin{aligned}
& \{\psi - (n-1)\}A(\xi)\eta(X) + \{\psi - (n-1)\}C(\xi)\eta(X) \\
& \quad + \{\psi - (n-1)\}D(X) - \eta(X)E(\xi) + \{E(X) - E(\phi X)\} = 0,
\end{aligned} \tag{4.12}
$$

Adding (4.10) and (4.12) and using the relation (4.8) we have

$$\begin{aligned}&\{\psi-(n-1)\}\{A(X)+D(X)\}-\{B(X)-B(\phi X)\}\\&\quad+\eta(X)B(\xi)+\{\psi-(n-1)\}C(\xi)\eta(X)=0.\end{aligned}\tag{4.13}$$

Now putting $X=\xi$ in the equation (4.6) and then using (2.1), (2.3) and (2.10) it follows that

$$\begin{aligned}&\{\psi-(n-1)\}A(\xi)\eta(Z)-\eta(Z)B(\xi)+\{B(Z)-B(\phi Z)\}\\&\quad+\{\psi-(n-1)\}C(Z)+\{\psi-(n-1)\}\eta(Z)D(\xi)=0.\end{aligned}\tag{4.14}$$

Replacing Z by X the above equation becomes

$$\begin{aligned}&\{\psi-(n-1)\}A(\xi)\eta(X)-\eta(X)B(\xi)+\{B(X)-B(\phi X)\}\\&\quad+\{\psi-(n-1)\}C(X)+\{\psi-(n-1)\}\eta(X)D(\xi)=0.\end{aligned}\tag{4.15}$$

Adding the equation (4.13) and (4.15) and using the relation (4.8) we get

$$\{\psi-(n-1)\}\{A(X)+C(X)+D(X)\}=0,\tag{4.16}$$

which implies that (since $n>3$)

$$A(X)+C(X)+D(X)=0,$$

for any X on M. Hence we are able to state the following:

Theorem 4.2. *In a weakly symmetric Kenmotsu manifold $M(n>2)$ with respect to quarter-symmetric metric connection, the sum of associated 1-forms A, C and D is zero everywhere.*

5. Weakly Ricci-symmetric Kenmotsu manifolds admitting a quarter-symmetric metric connection

Definition 5.1. A Kenmotsu manifold $M(n>2)$ is called weakly Ricci-symmetric with respect to quarter-symmetric metric connection if there exist 1-forms α,β and γ and their Ricci tensor $\tilde{S}$ of type (0, 2) satisfies the condition

$$(\tilde{\nabla}_X\tilde{S})(Y,Z)=\alpha(X)\tilde{S}(Y,Z)+\beta(Y)\tilde{S}(X,Z)+\gamma(Z)\tilde{S}(Y,X)\tag{5.1}$$

for all vector fields $X,Y,Z\in\chi(M)$.

Let us consider a weakly Ricci-symmetric Kenmotsu manifold with respect to the connection $\tilde{\nabla}$. So by virtue of (5.1) yields for $Z=\xi$ that

$$(\tilde{\nabla}_X\tilde{S})(Y,\xi)=\alpha(X)\tilde{S}(Y,\xi)+\beta(Y)\tilde{S}(X,\xi)+\gamma(\xi)\tilde{S}(Y,X).\tag{5.2}$$

Equating the right hand sides of (4.5) and (5.2), it follows that

$$-S(X,Y)+2d\eta(\phi Y,X)-g(\phi Y,X)+\{\psi-(n-1)\}g(X,Y)-\psi\eta(X)\eta(Y)=\alpha(X)\tilde{S}(Y,\xi)+\beta(Y)\tilde{S}(X,\xi$$

Putting $X=Y=\xi$ in the above relation and then using the equations (2.1), (3.3) and (2.9) we get

$$\{\psi-(n-1)\}\{\alpha(\xi)+\beta(\xi)+\gamma(\xi)\}=0.$$

which implies that (since $n>3$)

$$\alpha(\xi)+\beta(\xi)+\gamma(\xi)=0. \tag{5.3}$$

Next, taking $Y=\xi$ in equation (5.3) and then using relations (2.9), (3.3) and (5.3) we get

$$\alpha(X)=\alpha(\xi)\eta(X). \tag{5.4}$$

In a similar manner we can obtain

$$\beta(X)=\beta(\xi)\eta(X). \tag{5.5}$$

and

$$\gamma(X)=\gamma(\xi)\eta(X). \tag{5.6}$$

Adding (5.4), (5.5) and (5.6) and then using (5.3) we obtain

$$\alpha(X)+\beta(X)+\gamma(X)=0, \tag{5.7}$$

for all vector field X on M. Thus, we state the following:

Theorem 5.2. *In a weakly Ricci-symmetric Kenmotsu manifold $M(n>2)$ with respect to quarter-symmetric metric connection, the sum of associated 1-forms α, β and γ is zero everywhere.*

Definition 5.3. A weakly Ricci-symmetric Kenmotsu manifold $M(n>2)$ with respect to quarter symmetric metric connection $\tilde{\nabla}$ is said to be Ricci-recurrent with respect to connection $\tilde{\nabla}$ if it satisfies the condition

$$(\tilde{\nabla}_X S)(Y,Z)=\alpha(X)S(Y,Z). \tag{5.8}$$

Suppose a weakly Ricci-symmetric Kenmotsu manifold with respect to quarter symmetric metric connection $\tilde{\nabla}$ is Ricci-recurrent with respect to the connection $\tilde{\nabla}$, then from (1.4) and definition (5.3), we have

$$\beta(Y)\tilde{S}(X,Z)+\gamma(Z)\tilde{S}(Y,X)=0. \tag{5.9}$$

Putting $X=Y=Z=\xi$ in (5.9) and then using (3.3), we obtain

$$\beta(\xi)+\gamma(\xi)=0 \tag{5.10}$$

for $\psi\neq(n-1)$. Putting $X=Y=\xi$ in (5.9), we get

$$\gamma(Z)=-\{\psi-(n-1)\}\beta(\xi)\eta(Z). \tag{5.11}$$

Similarly, we have

$$\beta(Z) = -\{\psi - (n-1)\}\gamma(\xi)\eta(Z).$$

Adding the above equation with (5.11)and using (5.10), we obtain

$$\beta(Z) + \gamma(Z) = 0.$$

for any vector field Z on M. So that β and γ are in opposite direction. Hence we state

Theorem 5.4. *If a weakly Ricci-symmetric Kenmotsu manifold $M(n > 2)$ with respect to quarter symmetric metric connection $\widetilde{\nabla}$ is Ricci-recurrent with respect to the connection $\widetilde{\nabla}$, then the 1-forms β and γ are in opposite direction.*

Definition 5.5. A Kenmotsu manifold $M(n > 2)$ is called weakly concircular Ricci-symmetric manifold with respect to quarter-symmetric metric connection $\widetilde{\nabla}$ if its concircular Ricci tensor $\widetilde{P}$ of type (0, 2) given by

$$\widetilde{P}(Y,Z) = \sum_{i=1}^{n} \widetilde{\widetilde{C}}(Y, e_i, e_i, Z) = \widetilde{S}(Y,Z) - \frac{\widetilde{r}}{n} g(Y,Z) \tag{5.12}$$

is not identically zero and satisfies the condition

$$(\nabla_X \widetilde{P})(Y,Z) = \alpha(X)\widetilde{P}(Y,Z) + \beta(Y)\widetilde{P}(X,Z) + \gamma(Z)\widetilde{P}(Y,X), \tag{5.13}$$

where α, β and γ are associated 1-forms (not simultaneously zero) and $\widetilde{\widetilde{C}}$ denotes the concircular curvature tensor with respect to the connection $\widetilde{\nabla}$.

Consider a weakly Concircular Ricci-symmetric Kenmotsu manifold $M(n > 2)$ with respect to the connection $\widetilde{\nabla}$, then the equation (5.13) holds on M.
In view of (5.12) and (5.13) yields

$$\begin{aligned}(\tilde{\nabla}_X \tilde{S})(Y,Z) - \frac{d\tilde{r}(X)}{n} g(Y,Z) = {} & \alpha(X)[\tilde{S}(Y,Z) - \frac{\tilde{r}}{n} g(Y,Z)] \\ & + \beta(Y)[\tilde{S}(X,Z) - \frac{\tilde{r}}{n} g(X,Z)] \\ & + \gamma(Z)[\tilde{S}(X,Y) - \frac{\tilde{r}}{n} g(X,Y)].\end{aligned} \tag{5.14}$$

Setting $X = Y = Z = \xi$ in (5.14), we get the relation

$$\alpha(\xi) + \beta(\xi) + \gamma(\xi) = \frac{d\tilde{r}(\xi)}{[\tilde{r} - n\{\psi - (n-1)]\}} \tag{5.15}$$

Next, substituting X and Y by ξ in (5.14) and using (2.10) and (5.15), we obtain

$$\gamma(Z) = \gamma(\xi)\eta(Z), \qquad \tilde{r} - n\{\psi - (n-1)\} \neq 0. \tag{5.16}$$

Setting $X = Z = \xi$ in (5.14) and processing in a similar manner as above we get

$$\beta(Y) = \beta(\xi)\eta(Y), \qquad \tilde{r} - n\{\psi - (n-1)\} \neq 0. \tag{5.17}$$

Again, Taking $Y = Z = \xi$ in (5.14) and using (2.11) and (5.15), we get

$$\alpha(X) = \frac{d\tilde{r}(X)}{\tilde{r} - n\{\psi - (n-1)\}} + \big[\alpha(\xi) - \frac{d\tilde{r}(\xi)}{\tilde{r} - n\{\psi - (n-1)\}}\big]\eta(X), \tag{5.18}$$

provided $\tilde{r} - n\{\psi - (n-1)\} \neq 0$. Adding (5.16), (5.17) and (5.18) and using (3.4) and (5.15), we get

$$\alpha(X) + \beta(X) + \gamma(X) = \frac{d\tilde{r}(X)}{\tilde{r} - n\{\psi - (n-1)\}} = \frac{dr(X)}{\{r - n\psi + (n-1)(n+2)\}}$$

for any vector field X on M. This leads to the following:

Theorem 5.6. *In a weakly concircular Ricci-symmetric Kenmotsu manifold $M(n > 2)$ with respect to quarter symmetric metric connection $\widetilde{\nabla}$, the sum of the associated 1-forms is zero if the scalar curvature is constant and $\{r - n\psi + (n - 1)(n+2)\} \neq 0$.*

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A note on the k-Narayana sequence

José L. Ramírez[a], Víctor F. Sirvent[b]

[a]Departamento de Matemáticas, Universidad Sergio Arboleda, Bogotá, Colombia
josel.ramirez@ima.usergioarboleda.edu.co

[b]Departamento de Matemáticas, Universidad Simón Bolívar, Caracas, Venezuela
vsirvent@usb.ve

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Abstract

In the present article, we define the k-Narayana sequence of integer numbers. We study recurrence relations and some combinatorial properties of these numbers, and of the sum of their first n terms. These properties are derived from matrix methods. We also study some relations between the k-Narayana sequence and convolved k-Narayana sequence, and permanents and determinants of one type of Hessenberg matrix. Finally, we show how these sequences arise from a family of substitutions.

Keywords: The k-Narayana Sequence, Recurrences, Generating Function, Combinatorial Identities

MSC: 11B39, 11B83, 05A15.

1. Introduction

The Narayana sequence was introduced by the Indian mathematician Narayana in the 14th century, while studying the following problem of a herd of cows and calves: *A cow produces one calf every year. Beginning in its fourth year, each calf produces one calf at the beginning of each year. How many calves are there altogether after 20 years?* (cf. [1]).

This problem can be solved in the same way that Fibonacci solved its problem about rabbits (cf. [14]). If n is the year, then the Narayana problem can be modelled by the recurrence $b_{n+1} = b_n + b_{n-2}$, with $n \geqslant 2, b_0 = 0, b_1 = 1, b_2 = 1$ (cf. [1]). The first few terms are $0, 1, 1, 1, 2, 3, 4, 6, 9, 13, \dots$, (sequence A000930[1]). This sequence is called *Narayana sequence*.

In this paper, we introduce a generalization of the Narayana numbers. Specifically, for any nonzero integer number k the *k-Narayana sequence*, say $\{b_{k,n}\}_{k=0}^{\infty}$, is defined by the recurrence relation

$$b_{k,0} = 0, \quad b_{k,1} = 1, \quad b_{k,2} = k \quad \text{and} \quad b_{k,n} = kb_{k,n-1} + b_{k,n-3}.$$

The first few terms are

$$0,\ 1,\ k,\ k^2,\ k^3+1,\ k^4+2k,\ k^5+3k^2,\ k^6+4k^3+1,\ k^7+5k^4+3k,\ \dots$$

In particular:

$$\begin{aligned}
\{b_{1,n}\}_{n=0}^{\infty} &= \{0,1,1,1,2,3,4,6,9,13,19,28,41,\dots\}, \text{A000930, Narayana Seq.}\\
\{b_{2,n}\}_{n=0}^{\infty} &= \{0,1,2,4,9,20,44,97,214,472,1041,2296,\dots\}, \quad \text{A008998.}\\
\{b_{3,n}\}_{n=0}^{\infty} &= \{0,1,3,9,28,87,270,838,2601,8073,25057,\dots\}, \quad \text{A052541.}\\
\{b_{-1,n}\}_{n=0}^{\infty} &= \{0,1,-1,1,0,-1,2,-2,1,1,-3,4,-3,0,4,-7,\dots\}, \quad \text{A050935.}
\end{aligned}$$

Let $S_{k,n} = \sum_{i=1}^{n} b_{k,i}, \quad n \geq 1$, i.e., $S_{k,n}$ is the sum of the first n terms of the k-Narayana sequence. In this article we study the sequences $\{b_{k,n}\}$ and $\{S_{k,n}\}$. In Section 2.1, we give a combinatorial representation of $\{S_{k,n}\}$. Using the methods of [11], we find Binet-type formulae for $\{b_{k,n}\}$ and $\{S_{k,n}\}$ and their generating functions. We also study some identities involving these sequences, obtained from matrix methods. Similar researches have been made for tribonacci numbers [6, 7, 11], Padovan numbers [25], and generalized Fibonacci and Pell numbers [12, 10]. In Section 3 we obtain some relation determinants and permanents of certain Hessenberg matrices. In Section 4 we define the convolved k-Narayana sequences and we show some identities. In Section 5, we show how these sequences arises from a well known family of substitutions on an alphabet of three symbols.

2. Definitions and basic constructions

In this section, we define a new generating 3×3 matrix for the k-Narayana numbers. We also show the generating function and Binet formula for the k-Narayana sequence and some identities of the sum of the first n terms of the k-Narayana sequence.

For any integer number k, $(k \neq 0)$, we define the following matrix:

$$Q_k := \begin{bmatrix} k & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{2.1}$$

[1] Many integer sequences and their properties are expounded on *The On-Line Encyclopedia of Integer Sequences*[22].

By induction on n, we show that

$$(Q_k)^n = \begin{bmatrix} k & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n = \begin{bmatrix} b_{k,n+1} & b_{k,n-1} & b_{k,n} \\ b_{k,n} & b_{k,n-2} & b_{k,n-1} \\ b_{k,n-1} & b_{k,n-3} & b_{k,n-2} \end{bmatrix}, \quad n \geq 3. \tag{2.2}$$

Then Q_k is a generating matrix of the k-Narayana sequence.

Proposition 2.1. *For all integers m, n such that $0 < m < n$, we have the following relations*

1. $b_{k,n} = b_{k,m+1}b_{k,n-m} + b_{k,m-1}b_{k,n-m-1} + b_{k,m}b_{k,n-m-2}$.
2. $b_{k,n} = b_{k,m}b_{k,n-m+1} + b_{k,m-2}b_{k,n-m} + b_{k,m-1}b_{k,n-m-1}$.

Proof. It is clear that $Q_k^n = Q_k^m Q_k^{n-m}$. Then from Equation (2.2), we have

$$\begin{bmatrix} b_{k,n+1} & b_{k,n-1} & b_{k,n} \\ b_{k,n} & b_{k,n-2} & b_{k,n-1} \\ b_{k,n-1} & b_{k,n-3} & b_{k,n-2} \end{bmatrix} = \begin{bmatrix} b_{k,m+1} & b_{k,m-1} & b_{k,m} \\ b_{k,m} & b_{k,m-2} & b_{k,m-1} \\ b_{k,m-1} & b_{k,m-3} & b_{k,m-2} \end{bmatrix} \times \begin{bmatrix} b_{k,n-m+1} & b_{k,n-m-1} & b_{k,n-m} \\ b_{k,n-m} & b_{k,n-m-2} & b_{k,n-m-1} \\ b_{k,n-m-1} & b_{k,n-m-3} & b_{k,n-m-2} \end{bmatrix}. \tag{2.3}$$

Equating the (1,3)-th and (2,1)-th elements of the equation, we obtain the relations. □

Let $B_k(z)$ be the generating function of the k-Narayana numbers $b_{k,n}$. From standard methods we can obtain that

$$B_k(z) = \frac{z}{1 - kz - z^3}. \tag{2.4}$$

Moreover, from Equation (2.4) we obtain that the k-Narayana numbers are given by the following Binet's formula:

$$b_{k,n} = \frac{\alpha_k^{n+1}}{(\alpha_k - \beta_k)(\alpha_k - \gamma_k)} + \frac{\beta_k^{n+1}}{(\beta_k - \alpha_k)(\beta_k - \gamma_k)} + \frac{\gamma_k^{n+1}}{(\gamma_k - \alpha_k)(\gamma_k - \beta_k)}, \quad n \geq 0, \tag{2.5}$$

where $\alpha_k, \beta_k, \gamma_k$ are the zeros of characteristic equation of the k-Narayana numbers, $x^3 - kx^2 - 1 = 0$. Specifically,

$$\alpha_k = \frac{1}{3}\left(k + k^2 \sqrt[3]{\frac{2}{27 + 2k^3 + 3\sqrt{81 + 12k^3}}} + \sqrt[3]{\frac{27 + 2k^3 + 3\sqrt{81 + 12k^3}}{2}} \right),$$

$$\beta_k = \frac{1}{3}\left(k - \omega k^2 \sqrt[3]{\frac{2}{27 + 2k^3 + 3\sqrt{81 + 12k^3}}} + \omega^2 \sqrt[3]{\frac{27 + 2k^3 + 3\sqrt{81 + 12k^3}}{2}} \right),$$

$$\gamma_k = \frac{1}{3}\left(k + \omega^2 k^2 \sqrt[3]{\frac{2}{27+2k^3+3\sqrt{81+12k^3}}} - \omega \sqrt[3]{\frac{27+2k^3+3\sqrt{81+12k^3}}{2}}\right),$$

where $\omega = \frac{1+i\sqrt{3}}{2}$ is the primitive cube root of unity. If $k \geq 1$, the number α_k is a *Pisot number*, i.e., α_k is algebraic integer greater than 1, whereas its Galois conjugates (β_k and γ_k) have norm smaller than 1. The number α_1 is the fourth smallest Pisot number (cf. [2]).

Proposition 2.2. *Let $n > 2$ and the integers r, s, such that $0 \leqslant s < r$. Then the following equality holds*

$$\begin{aligned} b_{k,rn+s} = \\ &(\alpha_k^r + \beta_k^r + \gamma_k^r) b_{k,r(n-1)+s} + ((\alpha_k\beta_k)^r + (\alpha_k\gamma_k)^r + (\beta_k\gamma_k)^r) b_{k,r(n-2)+s} \\ &+ (\alpha_k\beta_k\gamma_k)^r b_{k,r(n-2)+s}, \end{aligned} \quad (2.6)$$

where α_k, β_k and γ_k are the roots of the characteristic equation of the k-Narayana numbers.

Proof. By induction on n, we can show that for any positive integer r, the numbers $(\alpha_k^r + \beta_k^r + \gamma_k^r)$, $(\alpha_k\beta_k)^r + (\alpha_k\gamma_k)^r + (\beta_k\gamma_k)^r$ and $(\alpha_k\beta_k\gamma_k)^r$ are always integers. Then, from Binet formula Equation (2.6) follows. □

Let

$$S_{k,n} = \sum_{i=1}^{n} b_{k,i}, \ n \geq 1, \quad \text{and} \quad S_{k,0} = 0.$$

By induction on n, we can proof the following identities:

- If $n \geq 3$, then $S_{k,n} = kS_{k,n-1} + S_{k,n-3} + 1$.
- If $n \geq 4$, then $S_{k,n} = (k+1)S_{k,n-1} - kS_{k,n-2} + S_{k,n-3} - S_{k,n-4}$.

Since the generating function of $\{b_{k,n}\}_n$ is $B_k(z)$, given in Equation (2.4), and using the Cauchy product of series. We obtain the generating function of $\{S_{k,n}\}_n$:

$$\sum_{i=0}^{\infty} S_{k,i} z^i = \frac{z}{(1-z)(1-kz-z^3)}. \quad (2.7)$$

Moreover, using similar techniques of [11], we obtain that the sum of the k-Narayana numbers are given by the following Binet-type formula:

$$\begin{aligned} S_{k,n} = \\ &\frac{\alpha_k^{n+2}}{(\alpha_k-1)(\alpha_k-\beta_k)(\alpha_k-\gamma_k)} + \frac{\beta_k^{n+2}}{(\beta_k-1)(\beta_k-\alpha_k)(\beta_k-\gamma_k)} \\ &+ \frac{\gamma_k^{n+2}}{(\gamma_k-1)(\gamma_k-\alpha_k)(\gamma_k-\beta_k)}. \end{aligned} \quad (2.8)$$

On the other hand, using the results of [13] we obtain that

$$A_k^n = B_{k,n}, \tag{2.9}$$

where

$$A_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & k & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \text{and} \quad B_{k,n} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ S_{k,n+1} & b_{k,n+2} & b_{k,n} & b_{k,n+1} \\ S_{k,n} & b_{k,n+1} & b_{k,n-1} & b_{k,n} \\ S_{k,n-1} & b_{k,n} & b_{k,n-2} & b_{k,n-1} \end{bmatrix}, \ n \geq 2.$$

Moreover, if $n, m \geq 3$, then

$$S_{k,n+m} = S_{k,n} + b_{k,n+1}S_{k,m+1} + b_{k,n-1}S_{k,m} + b_{k,n}S_{k,m-1}. \tag{2.10}$$

Define the diagonal matrix D_k and the matrix V_k as shown, respectively:

$$D_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha_k & 0 & 0 \\ 0 & 0 & \beta_k & 0 \\ 0 & 0 & 0 & \gamma_k \end{bmatrix}, \qquad V_k = \begin{bmatrix} k & 0 & 0 & 0 \\ -1 & \alpha_k^2 & \beta_k^2 & \gamma_k^2 \\ -1 & \alpha_k & \beta_k & \gamma_k \\ -1 & 1 & 1 & 1 \end{bmatrix}.$$

Note that $A_kV_k = V_kD_k$. Moreover, since the roots $\alpha_k, \beta_k, \gamma_k$ are different, it follows that $\det V_k \neq 0$, with $k \neq 0$. Then we can write $V_k^{-1}A_kV_k = D_k$, so the matrix A_k is similar to the matrix D_k. Hence $A_k^nV_k = V_kD_k^n$. By Equation (2.9), we have $B_{k,n}V_k = V_kD_k^n$. By equating the (3,1)-th element of the last equation, the result follows.

$$kS_{k,n} = b_{k,n+1} + b_{k,n} + b_{k,n-1} - 1, \quad n \geq 1.$$

2.1. Combinatorial representation of $S_{k,n}$

Let C_m be a $m \times m$ matrix defined as follows:

$$C_m(u_1, u_2, \ldots, u_m) = \begin{bmatrix} u_1 & u_2 & \ldots & u_{m-1} & u_m \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix}.$$

This matrix, or some of its modifications, is called *companion matrix of the polynomial* $p(x) = x^n + u_1x^{n-1} + u_2x^{n-2} + \cdots + u_{m-1}x + u_m$, because its characteristic polynomial is $p(x)$.

Chen and Louck ([5]) showed the following result about the matrix power of $C_m(u_1, u_2, \ldots, u_m)$.

Theorem 2.3. *The (i,j)-th entry $c_{ij}^{(n)}(u_1,\dots,u_m)$ in the matrix $C_m^n(u_1,\dots,u_m)$ is given by the following formula:*

$$c_{ij}^{(n)}(u_1,\dots,u_m) = \sum_{(t_1,t_2,\dots,t_m)} \frac{t_j+t_{j+1}+\cdots+t_m}{t_1+t_2+\cdots+t_m} \times \binom{t_1+\cdots+t_m}{t_1,\dots,t_m} u_1^{t_1}\cdots u_m^{t_m}, \tag{2.11}$$

where the summation is over nonnegative integers satisfying $t_1+2t_2+\cdots+mt_m = n-i+j$, the coefficient in (2.11) is defined to be 1 if $n=i-j$, and

$$\binom{n}{n_1,\dots,n_m} = \frac{n!}{n_1!\ \cdots\ n_m!}$$

is the multinomial coefficient.

Let R_k and $W_{k,n}$ be the following 4×4 matrices

$$R_k = \begin{bmatrix} k+1 & -k & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad W_{k,n} = \begin{bmatrix} S_{k,n+1} & f_{k,n} & b_{k,n} & -S_{k,n} \\ S_{k,n} & f_{k,n-1} & b_{k,n-1} & -S_{k,n-1} \\ S_{k,n-1} & f_{k,n-2} & b_{k,n-2} & -S_{k,n-2} \\ S_{k,n-2} & f_{k,n-3} & b_{k,n-3} & -S_{k,n-3} \end{bmatrix},$$

where $f_{k,n} = kf_{k,n-1} + f_{k,n-3} - k$ with $f_{k,-1} = 1, f_{k,0} = 0, f_{k,1} = -k, f_{k,2} = 1-k-k^2$.

Proposition 2.4. *If $n \geq 2$, then $R_k^n = W_{k,n}$.*

Proof. We have

$$R_kW_{k,n-1} = \begin{bmatrix} k+1 & -k & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} S_{k,n} & f_{k,n-1} & b_{k,n-1} & -S_{k,n-1} \\ S_{k,n-1} & f_{k,n-2} & b_{k,n-2} & -S_{k,n-2} \\ S_{k,n-2} & f_{k,n-3} & b_{k,n-3} & -S_{k,n-3} \\ S_{k,n-3} & f_{k,n-4} & b_{k,n-4} & -S_{k,n-4} \end{bmatrix}$$

$$= \begin{bmatrix} S_{k,n+1} & f_{k,n} & b_{k,n} & -S_{k,n} \\ S_{k,n} & f_{k,n-1} & b_{k,n-1} & -S_{k,n-1} \\ S_{k,n-1} & f_{k,n-2} & b_{k,n-2} & -S_{k,n-2} \\ S_{k,n-2} & f_{k,n-3} & b_{k,n-3} & -S_{k,n-3} \end{bmatrix} = W_{k,n}.$$

Then $W_{k,n} = R_k^{n-1}W_{k,1}$. Finally, by direct computation follows $W_{k,1} = R_k$. Hence $R_k^n = W_{k,n}$. □

Note that the characteristic polynomial of the matrix R_k is $p_{R_k}(x) = p_{Q_k}(x)(x-1) = x^4-(k+1)x^3+kx^2-x+1$, where $p_{Q_k}(x)$ is the characteristic polynomial of the matrix Q_k. So the roots of R_k are $\alpha_k, \beta_k, \gamma_k, 1$.

Corollary 2.5. *Let $S_{k,n}$ be the sums of the k-Narayana numbers. Then*

$$S_{k,n} = \sum_{(t_1,t_2,t_3,t_4)} \binom{t_1+t_2+t_3+t_4}{t_1,t_2,t_3,t_4} (-1)^{t_2+t_4}(k+1)^{t_1}k^{t_2},$$

where the summation is over nonnegative integers satisfying $t_1+2t_2+3t_3+4t_4 = n-1$.

Proof. In Theorem 2.3, we consider the $(2,1)$-th entry, with $n=4, u_1=k+1, u_2=-k, u_3=1$ and $u_4=-1$. Then the proof follows from Proposition 2.4 by considering the matrices R_k and $W_{k,n}$. □

For example,

$$S_{2,5} = \sum_{t_1+2t_2+3t_3+4t_4=4} \binom{t_1+t_2+t_3+t_4}{t_1,t_2,t_3,t_4} (-1)^{t_2+t_4} 3^{t_1} 2^{t_2}$$
$$= \binom{4}{4,0,0,0}3^4 - \binom{3}{2,1,0,0}3^2\cdot 2 + \binom{2}{1,0,1,0}3 - \binom{1}{0,0,0,1} + \binom{2}{0,2,0,0}2^2 = 36.$$

3. Hessenberg matrices and the k-Narayana sequence

An *upper Hessenberg matrix*, A_n, is a $n\times n$ matrix, where $a_{i,j}=0$ whenever $i>j+1$ and $a_{j+1,j}\neq 0$ for some j. That is, all entries bellow the superdiagonal are 0 but the matrix is not upper triangular:

$$A_n = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n-1} & a_{2,n} \\ 0 & a_{3,2} & a_{3,3} & \cdots & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & 0 & \cdots & a_{n,n-1} & a_{n,n} \end{bmatrix}. \tag{3.1}$$

We consider two types of upper Hessenberg matrix whose determinants and permanents are the k-Narayana numbers. The following result about upper Hessenberg matrices, proved in [8], will be used.

Theorem 3.1. *Let* $a_1, p_{i,j}, (i\leqslant j)$ *be arbitrary elements of a commutative ring* R, *and let the sequence* $a_1, a_2, \dots$ *be defined by:*

$$a_{n+1} = \sum_{i=1}^{n} p_{i,n} a_i, \quad (n = 1, 2, \dots).$$

If

$$A_n = \begin{bmatrix} p_{1,1} & p_{1,2} & p_{1,3} & \cdots & p_{1,n-1} & p_{1,n} \\ -1 & p_{2,2} & p_{2,3} & \cdots & p_{2,n-1} & p_{2,n} \\ 0 & -1 & p_{3,3} & \cdots & p_{3,n-1} & p_{3,n} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p_{n-1,n-1} & p_{n-1,n} \\ 0 & 0 & 0 & \cdots & -1 & p_{n,n} \end{bmatrix}.$$

Then $a_{n+1} = a_1 \det A_n$, *for* $n \geq 1$.

Let $L_{k,n}$ be a n-square matrix as follow

$$L_{k,n} = \begin{bmatrix} k & 0 & 1 & & & & 0 \\ -1 & k & 0 & 1 & & & \\ & -1 & k & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & -1 & k & 0 & 1 \\ & & & & -1 & k & 0 \\ 0 & & & & & -1 & k \end{bmatrix}.$$

Then from the above Theorem, it is clear that

$$\det L_{k,n} = b_{k,n+1}, \quad \text{for} \quad n \geq 1. \tag{3.2}$$

Theorem 3.2 (Trudi's formula [17]). *Let m be a positive integer. Then*

$$\det \begin{bmatrix} a_1 & a_2 & \cdots & & a_m \\ a_0 & a_1 & \cdots & & \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & a_1 & a_2 \\ 0 & 0 & \cdots & a_0 & a_1 \end{bmatrix}$$

$$= \sum_{(t_1,t_2,\ldots,t_m)} \binom{t_1+\cdots+t_m}{t_1,\cdots,t_m} (-a_0)^{m-t_1-\cdots-t_m} a_1^{t_1} a_2^{t_2} \cdots a_m^{t_m}, \tag{3.3}$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + mt_m = m$.

From Trudi's formula and Equation (3.2), we have

$$b_{k,n+1} = \sum_{t_1+3t_3=n} \binom{t_1+t_3}{t_1,t_3} k^{t_1}.$$

For example,

$$b_{2,7} = \sum_{t_1+3t_3=6} \binom{t_1+t_3}{t_1,t_3} 2^{t_1} = \binom{2}{0,2} + \binom{4}{3,1} 2^3 + \binom{6}{6,0} 2^6 = 97.$$

The permanent of a matrix is defined in a similar manner to the determinant but all the sign used in the Laplace expansion of minors are positive. The permanent of a n-square matrix is defined by

$$\operatorname{per}A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where the summation extends over all permutations σ of the symmetric group S_n ([16]). Let $A = [a_{ij}]$ be a $m \times n$ real matrix with row vectors $r_1, r_2, \ldots, r_m$. We say A is *contractible on column* k if column k contains exactly two nonzero entries, in a similar manner we define *contractible on row.* Suppose A is contractible on column k with $a_{ik} \neq 0 \neq a_{jk}$ and $i \neq j$. Then the $(m-1) \times (n-1)$ matrix $A_{ij:k}$ obtained from A by replacing row i with $a_{jk}r_i + a_{ik}r_j$, and deleting row j and column k. The matrix $A_{ij:k}$ is called the *contraction of* A *on column* k *relative to rows* i *and* j. If A is contractible on row k with $a_{ki} \neq 0 \neq a_{kj}$ and $i \neq j$, then the matrix $A_{k:ij} = [A^T_{ij:k}]^T$ is called the *contraction of* A *on row* k *relative to columns* i *and* j.

Brualdi and Gibson [3] proved the following result about the permanent of a matrix.

Lemma 3.3. *Let* A *be a nonnegative integral matrix of order* $n > 1$ *and let* B *be a contraction of* A*. Then*

$$\operatorname{per}A = \operatorname{per}B.$$

There are a lot of relations between determinants or permanents of matrices and number sequences. For example, Yilmaz and Bozkurt [25] obtained some relations between Padovan sequence and permanents of one type of Hessenberg matrix. Kiliç [11] obtained some relations between the tribonacci sequence and permanents of one type of Hessenberg matrix. Öcal et al. [18] studied some determinantal and permanental representations of k-generalized Fibonacci and Lucas numbers. Janjić [8] considered a particular upper Hessenberg matrix and showed its relations with a generalization of the Fibonacci numbers. In [15], Li obtained three new Fibonacci-Hessenberg matrices and studied its relations with Pell and Perrin sequence. More examples can be found in [4, 9, 20, 21, 24].

Define the n-square Hessenberg matrix $J_k(n)$ as follows:

$$J_k(n) = \begin{bmatrix} k^2 & 1 & k & & & & 0 \\ 1 & k & 0 & 1 & & & \\ & 1 & k & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & k & 0 & 1 \\ & & & & 1 & k & 0 \\ 0 & & & & & 1 & k \end{bmatrix}. \tag{3.4}$$

Theorem 3.4. *Let* $J_k(n)$ *be a* n*-square matrix as in (3.4), then*

$$\operatorname{per}J_k(n) = b_{k,n+2}, \tag{3.5}$$

where $b_{k,n}$ is the n-th k-Narayana number.

Proof. Let $J_{k,r}(n)$ be the r-th contraction of $J_k(n)$, by construction is a $n-r\times n-r$ matrix. By definition of the matrix $J_k(n)$, it can be contracted on column 1, then

$$J_{k,1}(n)=\begin{bmatrix} k^3+1 & k & k^2 & & & & 0\\ 1 & k & 0 & 1 & & & \\ & 1 & k & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & k & 0 & 1\\ & & & & 1 & k & 0\\ 0 & & & & & 1 & k \end{bmatrix}.$$

After contracting $J_{k,1}(n)$ on the first column we have

$$J_{k,2}(n)=\begin{bmatrix} k^4+2k & k^2 & k^3+1 & & & & 0\\ 1 & k & 0 & 1 & & & \\ & 1 & k & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & k & 0 & 1\\ & & & & 1 & k & 0\\ 0 & & & & & 1 & k \end{bmatrix}.$$

According to this procedure, the r-th contraction is

$$J_{k,r}(n)=\begin{bmatrix} b_{k,r+3} & b_{k,r+1} & b_{k,r+2} & & & & 0\\ 1 & k & 0 & 1 & & & \\ & 1 & k & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & k & 0 & 1\\ & & & & 1 & k & 0\\ 0 & & & & & 1 & k \end{bmatrix}.$$

Hence, the $(n-3)$-th contraction is

$$J_{k,n-3}(n)=\begin{bmatrix} b_{k,n} & b_{k,n-2} & b_{k,n-1}\\ 1 & k & 0\\ 0 & 1 & k \end{bmatrix},$$

which, by contraction of $J_{k,n-3}(n)$ on column 1,

$$J_{k,n-2}(n)=\begin{bmatrix} b_{k,n+1} & b_{k,n-1}\\ 1 & k \end{bmatrix}.$$

Then from Lemma 3.3,

$$\mathrm{per}J_k(n)=\mathrm{per}J_{k,n-2}(n)=kb_{k,n+1}+b_{k,n-1}=b_{k,n+2}.$$ □

4. The convolved k-Narayana numbers

The *convolved k-Narayana numbers* $b_{k,j}^{(r)}$ are defined by

$$B_k^{(r)}(z) = (1 - kz - z^3)^{-r} = \sum_{j=0}^{\infty} b_{k,j+1}^{(r)} z^j, \ r \in \mathbb{Z}^+.$$

Note that

$$b_{k,m+1}^{(r)} = \sum_{j_1+j_2+\cdots+j_r=m} b_{k,j_1+1} b_{k,j_2+1} \cdots b_{k,j_r+1}. \tag{4.1}$$

The generating functions of the convolved k-Narayana numbers for $k = 2$ and $r = 2, 3, 4$ are

$$\begin{aligned}
B_2^{(2)}(z) &= \frac{1}{(1-2z-z^3)^2} = 1 + 4z + 12z^2 + 34z^3 + 92z^4 + 240z^5 + 611z^6 + \cdots \\
B_2^{(3)}(z) &= \frac{1}{(1-2z-z^3)^3} = 1 + 6z + 24z^2 + 83z^3 + 264z^4 + 792z^5 + 2278z^6 + \cdots \\
B_2^{(4)}(z) &= \frac{1}{(1-2z-z^3)^4} = 1 + 8z + 40z^2 + 164z^3 + 600z^4 + 2032z^5 + \cdots .
\end{aligned}$$

Let A and C be matrices of order $n \times n$ and $m \times m$, respectively, and B be an $n \times m$ matrix. Since

$$\det \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \det A \det C,$$

the principal minor $M^{(k)}(i)$ of $L_{k,n}$ is equal to $b_{k,i} b_{k,n-i+1}$. It follows that the principal minor $M^{(k)}(i_1, i_2, \ldots, i_l)$ of the matrix $L_{k,n}$ is obtained by deleting rows and columns with indices $1 \leqslant i_1 < i_2 < \cdots < i_l \leqslant n$:

$$M^{(k)}(i_1, i_2, \ldots, i_l) = b_{k,i_1} b_{k,i_2-i_1} \cdots b_{k,i_l-i_{l-1}} b_{k,n-i_l+1}. \tag{4.2}$$

Then from (4.2) we have the following theorem.

Theorem 4.1. *Let $S_{n-l}^{(k)}, (l = 0, 1, 2, \ldots, n-1)$ be the sum of all principal minors of $L_{k,n}$ of order $n - l$. Then*

$$S_{n-l}^{(k)} = \sum_{j_1+j_2+\cdots+j_{l+1}=n-l} b_{k,j_1+1} b_{k,j_2+1} \cdots b_{k,j_{l+1}+1} = b_{k,n-l+1}^{(l+1)}. \tag{4.3}$$

For example,

$$S_4^{(2)} = 2 \cdot \det \begin{bmatrix} 2 & 0 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 2 \end{bmatrix} + \det \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 2 \end{bmatrix} +$$

$$\det\begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 2 \end{bmatrix} + \det\begin{bmatrix} 2 & 0 & 1 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} = 92 = b_{2,5}^{(2)}.$$

Since the coefficients of the characteristic polynomial of a matrix are, up to the sign, sums of principal minors of the matrix, then we have the following.

Corollary 4.2. *The convolved k-Narayana number $b_{k,n-l+1}^{(l+1)}$ is equal, up to the sign, to the coefficient of x^l in the characteristic polynomial $p_n(x)$ of $L_{k,n}$.*

For example, the characteristic polynomial of the matrix $L_{2,5}$ is $x^5 - 10x^4 + 40x^3 - 83x^2 + 92x - 44$. So, it is clear that the coefficient of x is $b_{2,5}^{(2)} = 92$.

5. Sequences and substitutions

In this section, we show that the k-Narayana sequence is related to a substitution on an alphabet of 3 symbols.

A *substitution* or a *morphism* on a finite alphabet $\mathcal{A} = \{1, \ldots, r\}$ is a map ζ from $\mathcal{A}$ to the set of finite words in $\mathcal{A}$, i.e., $\mathcal{A}^* = \cup_{i\geq 0}\mathcal{A}^i$. The map ζ is extended to $\mathcal{A}^*$ by concatenation, i.e., $\zeta(\emptyset) = \emptyset$ and $\zeta(UV) = \zeta(U)\zeta(V)$, for all U, $V \in \mathcal{A}^*$. Let $A^{\mathbb{N}}$ denote the set of one-sided infinite sequences in $\mathcal{A}$. The map ζ, is extended to $\mathcal{A}^{\mathbb{N}}$ in the obvious way. We call $u \in \mathcal{A}^{\mathbb{N}}$ a *fixed point* of ζ if $\zeta(u) = u$ and *periodic* if there exists $l > 0$ so that it is fixed for ζ^l. To these fixed or periodic points we can associate dynamical systems, which have been studied extensively, see for instance [19].

We write $l_i(U)$ for the number of occurrences of the symbol i in the word U and denote the column-vector $\mathbf{l}(U) = (l_1(U), \ldots, l_r(U))^t$. The *incidence matrix* of the substitution ζ is defined as the matrix $M_\zeta = M = (m_{ij})$ whose entry $m_{ij} = l_i(\zeta(j))$, for $1 \leq i, j \leq k$. Note that $M_\zeta(\mathbf{l}(U)) = \mathbf{l}(\zeta(U))$, for all $U \in \mathcal{A}^*$.

We consider the following substitution:

$$\zeta_k = \begin{cases} 1 \to 1^k 2, \\ 2 \to 3, \\ 3 \to 1; \end{cases}$$

where 1^k is the word $\overbrace{1 \cdots 1}^{k}$, and $k \geq 1$.

The substitution ζ_k has an unique fixed point in $\{1, 2, 3\}^{\mathbb{N}}$, and the words $\zeta_k^n(1)$ are prefixes of this fixed point. The fixed point of ζ_1 (sequence A105083) starts with the symbols:

$$123112123123112311212311212312311212312311231121231231123 1 \cdots .$$

Some of the dynamical and geometrical properties associated to these sequences, have been studied in [23].

Let $a_{k,n}$ be the number of symbols of the word $\zeta_k^n(1)$. So $a_{k,0} = 1$, $a_{k,1} = k+1$ and $a_{k,2} = k^2 + k + 1$. For $n \geq 3$, we have

$$\begin{aligned}\zeta^n(1) &= \zeta^{n-1}(1^k 2) = (\zeta^{n-1}(1))^k \zeta^{n-1}(2) \\ &= (\zeta^{n-1}(1))^k \zeta^{n-2}(3) = (\zeta^{n-1}(1))^k \zeta^{n-3}(1).\end{aligned}$$

Hence

$$a_{k,n} = k a_{k,n-1} + a_{k,n-3}, \quad \text{for } n \geq 3.$$

The fist few terms of $\{a_{k,n}\}_{n=0}^{\infty}$ are

$$1,\ k+1,\ k^2+k+1,\ k^3+k^2+k+1,\ k^4+k^3+k^2+2k+1,\ k^5+k^4+k^3+3k^2+2k+1, \ldots$$

In particular:

$$\begin{aligned}&\{a_{1,n}\}_{n=0}^{\infty} = \{1,2,3,4,6,9,13,19,28,41,60,\ldots\}, \text{i.e., } a_{1,n} = b_{1,n+3} \text{ for all } n \geq 1.\\ &\{a_{2,n}\}_{n=0}^{\infty} = \{1,3,7,15,33,73,161,355,783,1727,3809,\ldots\}, \quad \text{A193641}.\\ &\{a_{3,n}\}_{n=0}^{\infty} = \{1,4,13,40,124,385,1195,3709,11512,35731,\ldots\}, \quad \text{A098183}.\end{aligned}$$

By the definition of the matrix associated to the substitution, we have M_{ζ_k} is equal to the matrix Q_k, defined in Section 2. By the recurrence of the substitution we have that the entry (i,j)-th of the matrix $M_{\zeta_k}^n$, corresponds to the number of occurrences of the symbol i in the word $\zeta_k^n(j)$, i.e., $l_i(\zeta_k^n(j))$. Since $a_{k,n}$ is the length of the word $\zeta_k^n(1)$, we have

$$\begin{aligned}a_{k,n} &= l_1(\zeta_k^n(1)) + l_2(\zeta_k^n(1)) + l_3(\zeta_k^n(1)) && (5.1)\\ &= (Q_k)_{1,1}^n + (Q_k)_{1,2}^n + (Q_k)_{1,3}^n && (5.2)\\ &= b_{k,n+1} + b_{k,n} + b_{k,n-1}. && (5.3)\end{aligned}$$

This identity shows the relation between the k-Narayana sequence and the sequence $\{a_{k,n}\}_{n=0}^{\infty}$.

If we consider the substitutions

$$\zeta_{k,i} = \begin{cases} 1 \to 1^{k-i}21^i, \\ 2 \to 3, \\ 3 \to 1; \end{cases}$$

with $0 \leq i \leq k-1$. Obviously the length of the words $\zeta_k^n(1)$ and $\zeta_{k,i}^n(1)$ coincide.

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A note on log-convexity of power means

József Sándor

Babeş-Bolyai University, Department of Mathematics, Cluj-Napoca, Romania
jsandor@math.ubbcluj.ro

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Abstract

We point out results connected with the log-convexity of power means of two arguments.

1. Introduction

Let $M_p(a,b) = \left(\frac{a^p+b^p}{2}\right)^{1/p}$ $(p \neq 0)$, $M_0(a,b) = \sqrt{ab}$ denote the power mean (or Hölder mean, see [2]) of two arguments $a, b > 0$. Recently A. Bege, J. Bukor and J. T. Tóth [1] have given a proof of the fact that for $a \neq b$, the application $p \to M_p$ is log-convex for $p \leq 0$ and log-concave for $p \geq 0$. They also proved that it is also convex for $p \leq 0$. We note that this last result follows immediately from the well-known convexity theorem, which states that all log-convex functions are convex, too (see e.g. [2]). The proof of authors is based on an earlier paper by T. J. Mildorf (see [1]).

In what follows, we will show that this result is well-known in the literature, even in a more general setting. A new proof will be offered, too.

2. Notes and results

In 1948 H. Shniad [6] studied the more general means $M_t(a,\xi) = \left(\sum_{i=1}^n \xi_i a_i^t\right)^{1/t}$ $(t \neq 0)$, $M_0(a,\xi) = \prod_{i=1}^n a_i^{\xi_i}$, $M_{-\infty}(a,\xi) = \min\{a_i : i = 1, \dots\}$, $M_{+\infty}(a,\xi) = \max\{a_i : i = 1, \dots\}$; where $0 < a_i < a_{i+1}$ $(i = 1, \dots, n-1)$ are given positive real numbers, and $\xi_i (i = \overline{1,n})$ satisfy $\xi_i > 0$ and $\sum_{i=1}^n \xi_i = 1$.

Put $\Lambda(t) = \log M_t(a,\xi)$. Among other results, in [6] the following are proved:

Theorem 2.1.
(1) *If* $\xi_1 \geq \frac{1}{2}$ *then* $\Lambda(t)$ *is convex for all* $t < 0$.

(2) *If* $\xi_n \geq \frac{1}{2}$ *then* $\Lambda(t)$ *is concave for all* $t > 0$.

Clearly, when $n = 2$, in case of M_p one has $\xi_1 = \xi_2 = \frac{1}{2}$, so the result by Bege, Bukor and Tóth [1] follows by Theorem 2.1.

Another generalization of power mean of order two is offered by the Stolarsky means (see [7]) for $a, b > 0$ and $x, y \in \mathbb{R}$ define

$$D_{x,y}(a,b) = \begin{cases} \left[\frac{y(a^x-b^x)}{x(a^y-b^y)}\right]^{1/(x-y)}, & \text{if } xy(x-y) \neq 0, \\ \exp\left(-\frac{1}{x} + \frac{a^x \ln a - b^x \ln b}{a^x - b^x}\right), & \text{if } x = y \neq 0, \\ \left[\frac{a^x - b^x}{(\ln a - \ln b)}\right]^{1/x}, & \text{if } x \neq 0, y = 0, \\ \sqrt{ab}, & \text{if } x = y = 0. \end{cases}$$

The means $D_{x,y}$ are called sometimes as the difference means, or extended means.

Let $I_x(a,b) = (I(a^x, b^x))^{1/x}$, where $I(a,b)$ denotes the identic mean (see [2, 4]) defined by

$$I(a,b) = D_{1,1}(a,b) = \frac{1}{e}(b^b/a^a)^{1/(b-a)} \quad (a \neq b),$$
$$I(a,a) = a.$$

K. Stolarsky [7] proved also the following representation formula:

$$\log D_{x,y} = \frac{1}{y-x}\int_x^y \log I_t dt \text{ for } x \neq y.$$

Now, in 2001 the author [4] proved for the first time that the application $t \to \log I_t$ is convex for $t < 0$ and concave for $t > 0$.

This in turn implies immediately (see also [3]) the following fact:

Theorem 2.2.
(1) *If* $x > 0$ *and* $y > 0$, *then* $D_{x,y}$ *is log-concave in both* x *and* y.

(2) *If* $x < 0$ *and* $y < 0$, *then* $D_{x,y}$ *is log-convex in both* x *and* y.

Now, remark that

$$M_p(a,b) = D_{2p,p}(a,b),$$

so the log-convexity properties by H. Shniad are also particular cases of Theorem 2.2.

We note that an application of log-convexity of M_p is given in [5].

3. A new elementary proof

We may assume (by homogenity properties) that $b = 1$ and $a > 1$. Let $f(p) = \ln((a^p + 1)/2)/p$, and denote $x = a^p$. Then, as $x' = \frac{dx}{dp} = a^p \ln a = x \ln a$, from the identity $pf(p) = \ln(x+1)/2$ we get by differentiation

$$f(p) + pf'(p) = \frac{x \ln a}{x+1}. \tag{3.1}$$

By differentiating once again (3.1), we get

$$2f'(p) + pf''(p) = \frac{(x \ln^2 a)(x+1) - x^2 \ln^2 a}{(x+1)^2},$$

which implies, by definition of $f(p)$ and relation (3.1):

$$\begin{aligned} p^3 f''(p) &= \frac{(x \ln^2 x)(x+1) - x^2 \ln^2 x}{(x+1)^2} - \frac{2}{x+1}\left[x \ln x - (x+1)\ln\left(\frac{x+1}{2}\right)\right] \\ &= \frac{x \ln^2 x + 2(x+1)^2 \ln(x+1) - 2x(x+1)\ln x}{(x+1)^2}, \end{aligned}$$

after some elementary computations, which we omit here. Put

$$g(x) = x \ln^2 x + 2(x+1)^2 \ln\left(\frac{x+1}{2}\right) - 2x(x+1)\ln x.$$

One has successively:

$$\begin{aligned} g'(x) &= ln^2 x + 4(x+1)\ln\left(\frac{x+1}{2}\right) - 4x \ln x, \\ g''(x) &= \frac{2\ln x}{x} + 4\ln\left(\frac{x+1}{2}\right) - 4\ln x, \\ g'''(x) &= 2\left[\frac{1-\ln x}{x^2} - \frac{2}{x(x+1)}\right] = \frac{-2}{x^2(x+1)}[x - 1 + (x+1)\ln x]. \end{aligned}$$

Now, remark that for $x > 1$, clearly $g'''(x) < 0$, so $g''(x)$ is strictly decreasing, implying $g''(x) < g''(1) = 0$. Thus $g'(x) < g'(1) = 0$, giving $g(x) < g(1) = 0$. Finally, one gets $f''(p) < 0$, which shows that for $x > 1$ the function $f(p)$ is strictly concave function of p. As $x = a^p$ with $a > 1$, this happend only when $p > 0$.

For $x < 1$, remark that $x - 1 < 0$ and $\ln x < 0$, so $g'''(x) > 0$, and all above procedure may be repeted. This shows that $f(p)$ is a strictly convex function of p for $p < 0$.

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A general strong law of large numbers and applications to associated sequences and to extreme value theory*

Harouna Sangare[a†], Gane Samb Lo[b]

[a]Lerstad, Université Gaston Berger de Saint-Louis, Senegal and Département d'Etudes et Recherches en Mathématiques et d'Informatique, Faculté des Sciences Techniques (FST), Université des Sciences Techniques et Technologies de Bamako (USTTB), Bamako, Mali-1
harounasangareusttb@gmail.com

[b]Lerstad, Université Gaston Berger de Saint-Louis, Senegal and Laboratoire de Statistique Théorique et Appliquée (LSTA), Université Pierre et Marie Curie, Paris, France-2
ganesamblo@ganesamblo.net

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Abstract

The purpose of this paper is to establish a general strong law of large numbers (SLLN) for arbitrary sequences of random variables (rv's) based on the squared indice method and to provide applications to SLLN of associated sequences. This SLLN is compared to those based on the Hájek–Rényi type inequality. Nontrivial examples are given. An interesting issue that is related to extreme value theory (EVT) is handled here.

Keywords: Positive Dependence, Association, Negatively Associated, Hájek–Rényi Inequality, Max-Variance(r) Property, Strong Law of Large Numbers, Squared Indices Method, Extreme Value Theory, Hill's Estimator.

MSC: Primary 60F15, 62G20; Secondary 62G32, 62F12

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1. Introduction

In this paper, we present a general SLLN for arbitrary rv's and particularize it for associated sequences. In the recent decades both strong law of large numbers and central limit theorem for associated sequences have received and are still receiving huge interests since Lebowitz [13] and Newman [17] results under the strict stationarity assumption. The stationarity assumption was dropped by Birkel [3], who proved a version of a SLLN that can be interpreted as a generalized Kolmogorov's one. A recent account of such researches in this topic is available in [19]. Although many results are available for such sequences, there are still many open problems, especially regarding nonstationary sequences.

We intend to provide a more general SLLN for associated sequences as applications of a new general SLLN for arbitrary rv's. This new general SLLN is used to solve a remarkable issue of extreme value theory by using a pure probabilistic method.

Here is how this paper is organized. Since association is the central notion used here, we first make a quick reminder of it in Section 2. In Section 3, we make a round up of SLLN's available in the literature with the aim of comparing them to our findings. In Section 4, we state our general SLLN for arbitrary rv's and derive some classical cases. In Section 5, we give an application to EVT where the continuous Hill's estimator is studied by our method. The Section 6 concerns the conclusion and some perspectives are given. The paper is ended by the Appendix, where are postponed the proofs of Propositions 2 and 3 stated in Section 5.

To begin with, we give a short reminder of the concept of association.

2. A brief reminder of the concept of association

The notion of positive dependence for random variables was introduced by Lehmann (1966) (see [14]) in the bivariate case. Later this idea was extended to multivariate distributions by Esary, Proschan and Walkup (1967) (see [7]) under the name of association. The concept of association for rv's generalizes that of independence and seems to model a great variety of stochastic models. This property also arises in Physics, and is quoted under the name of FKG property (Fortuin, Kastelyn and Ginibre (1971), see [9]), in percolation theory and even in Finance (see [11]). The definite definition is given by Esary, Proschan and Walkup (1967) (see [7]) as follows.

Definition 2.1. A finite sequence of random variables $(X_1, \ldots, X_n)$ is associated if for any couple of real and coordinate-wise non-decreasing functions f and g defined on $\mathbb{R}^n$, we have

$$\mathrm{Cov}(f(X_1, \ldots, X_n),\ g(X_1, \ldots, X_n)) \geq 0$$

whenever the covariance exists. An infinite sequence of random variables is associated whenever all its finite subsequences are associated.

We have a few number of interesting properties to be found in ([19]): **(P1)** A sequence of independent rv's is associated. **(P2)** Partial sums of associated rv's are associated. **(P3)** Order statistics of independent rv's are associated. **(P4)** Non-decreasing functions and non-increasing functions of associated variables are associated. **(P5)** Let the sequence $Z_1, Z_2, \ldots, Z_n$ be associated and let $(a_i)_{1\leq i\leq n}$ be positive numbers and $(b_i)_{1\leq i\leq n}$ real numbers. Then the rv's $a_i(Z_i - b_i)$ are associated.

As immediate other examples of associated sequences, we may cite Gaussian random vectors with nonnegatively correlated components (see [18]) and homogeneous Markov chains (see [4]).

The negative association was introduced by Joag-Dev and Proschan (1983) (see [12]) as follows

Definition 2.2. The variables $X_1, \ldots, X_n$ are negatively associated if, for every pair of disjoint subsets nonempty A, B of $\{1, \ldots, n\}$, $A = \{i_1, \ldots, i_m\}$, $B = \{i_{m+1}, \ldots, i_n\}$ and for every pair of coordinatewise nondecreasing functions $f : \mathbb{R}^m \to \mathbb{R}$ and $g \colon \mathbb{R}^{n-m} \to \mathbb{R}$,

$$\operatorname{Cov}(f(X_i,\ i \in A),\ g(X_i,\ i \in B)) \leq 0 \tag{2.1}$$

whenever the covariance exists. An infinite collection is said to be negatively associated if every finite sub-collection is negatively associated.

Remark 2.3. For negatively associated sequences, we have (2.1), so the covariances are non-positive. This remark will be used in Subsubsection 4.1.2.

A usefull result of Newman (see [15]) on assocation, that is used in this paper, is the following

Lemma 2.4 (Newman [15])**.** *Suppose that X and Y are two random variables with finite variance and, f and g are $\mathbb{C}^1$ complex valued functions on $\mathbb{R}^1$ with bounded derivatives f' and g'. Then*

$$|\operatorname{Cov}(f(X), g(Y))| \leq ||f'||_\infty ||g'||_\infty \operatorname{Cov}(X, Y).$$

Here, we point out that strong laws of large numbers and, central limit theorem and invariance principle for associated rv's are available. Many of these results in that field are reviewed in [19]. Such studies go back to Lebowitz (1972) (see [13]) and Newman (1984) (see [17]). As Glivenko-classes for the empirical process for associated data, we may cite Yu (1993) (see [22]). We remind the results of such authors in this:

Theorem 2.5 (Lebowitz [13] and Newman [17])**.** *Let $X_1, X_2, \ldots$ be a strictly stationary sequence which is either associated or negatively associated, and let T denote the usual shift transformation, defined so that*

$$T(f(X_{j_1}, \ldots, X_{j_m})) = f(X_{j_1+1}, \ldots, X_{j_m+1}).$$

Then T is ergodic (i.e., every T-invariant event in the σ-field generated by the X_j's has probability 0 or 1) if and only if

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{j=1}^{n}\mathrm{Cov}(X_1,X_j)=0. \tag{2.2}$$

In particular, if (2.2) is valid, then for any f such that $f(X_1)$ is L_1,

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{i=1}^{n}f(X_i)=\mathbb{E}\left(f(X_1)\right)\quad \textit{almost surely (a.s).}$$

Now we are going to state some classical SLLN's for arbitrary rv's in relation with Hájek–Rényi's scheme.

3. Strong laws of large numbers

For independent rv's, two approaches are mainly used to get SLLN's. A direct method using squared indice method seems to be the oldest one. Another one concerns the Kolmogorov's law based on the maximal inequality of the same name. Many SLLN's for dependent data are kinds of generalization of these two methods. Particularly, the second approach that has been developed to become the Hájek–Rényi's method (see [10]), seems to give the most general SLLN to handle dependent data. Since we will use such results to compare our findings to, we recall one of the most sophisticated forms of the Hájek–Rényi setting given by Tómács and Líbor (see [21]) denoted by (GCHR). These authors introduced a Hájek–Rényi's inequality for probabilities and, subsequently, got from it SLLN's for random sequences. They obtained first:

Theorem 3.1. *Let r be a positive real number, a_n be a sequence of nonnegative real numbers. Then the following two statements are equivalent.*
(i) *There exists $C>0$ such that for any $n\in\mathbb{N}$ and any $\varepsilon>0$*

$$\mathbb{P}\left(\max_{\ell\leq n}|S_\ell|\geq\varepsilon\right)\leq C\varepsilon^{-r}\sum_{\ell\leq n}a_\ell.$$

(ii) *There exists $C>0$ such that for any nondecreasing sequence $(b_n)_{n\in\mathbf{N}}$ of positive real numbers, for any $n\in\mathbb{N}$ and any $\varepsilon>0$*

$$\mathbb{P}\left(\max_{\ell\leq n}|S_\ell|b_\ell^{-1}\geq\varepsilon\right)\leq C\varepsilon^{-r}\sum_{\ell\leq n}a_\ell b_\ell^{-r}$$

where $S_n=\sum_{i=1}^{n}X_i$ for all $n\in\mathbb{N}$.

And next, they derived this SLLN from it.

Theorem 3.2. *Let a_n and b_n be non-negative sequences of real numbers and let $r > 0$. Suppose that b_n is a positive non-decreasing, unbounded sequence of positive real numbers. Let us assume that*

$$\sum_n \frac{a_n}{b_n^r} < +\infty$$

and there exists $C > 0$ such that for any $n \in \mathbb{N}$ and any $\varepsilon > 0$

$$\mathbb{P}\left(\max_{m \leq n} |S_m| \geq \varepsilon\right) \leq C\, \varepsilon^{-r} \sum_{m \leq n} a_m.$$

Then

$$\lim_{n \to +\infty} \frac{S_n}{b_n} = 0 \quad a.s.$$

For convenience, introduce these three notations. We say that a sequence of random variables $X_1, X_2, \ldots$ has the $\mathbb{P}$-*max-variance*(r) property, with $r > 0$, if and only if there exists a constant $C > 0$ such that for any fixed $n \geq 1$, for any $\lambda > 0$,

$$\mathbb{P}\left(\max\left(|S_1|, \ldots, |S_n|\right) \geq \lambda\right) \leq C \lambda^{-r}\, \mathbb{V}\mathrm{ar}(S_n).$$

It has the $\mathbb{V}$ar-*max-variance*(r) property, with $r > 0$, if and only if there exists a constant $C > 0$ such that for any fixed $n \geq 1$,

$$\mathbb{V}\mathrm{ar}(\max(|S_1|, \ldots, |S_n|))^{2/r} \leq C\, \mathbb{V}\mathrm{ar}(S_n)$$

and it has the $\mathbb{E}$-*max-variance*(r) property, with $r > 0$, if and only if there exists a constant $C > 0$ such that for any fixed $n \geq 1$,

$$\left(\mathbb{E}\left(\max(|S_1|, \ldots, |S_n|)\right)^2\right)^{2/r} \leq C\, \mathbb{V}\mathrm{ar}(S_n).$$

In the sequel we will say that *max-variance* property is satisfied if one of the three above max-variance properties holds.

Theorem 3.1 leads to these general laws.

Proposition 1. *Let $X_1, X_2, \ldots$ be a sequence of centered random variables. Let $(b_k)_{k \geq 1}$ be an increasing and nonbounded sequence of positive real numbers. Assume that*

$$\lim \sup_{n \to +\infty} \sum_{1 \leq i \leq n} b_i^{-r}\, \mathrm{Cov}(X_i, S_n) < +\infty \tag{3.1}$$

and the sequence has the $\mathbb{P}$-max-variance(r) *property, $r > 0$. Then $S_n / b_n \to 0$ a.s. as $n \to +\infty$.*

If the sequence has the $\mathbb{V}$ar-*max-variance*(2) *property or the* $\mathbb{E}$-*max-variance*(2) *property and if $\sum_{i \geq 1} b_i^{-2} \sum_{j \geq 1} \mathrm{Cov}(X_i, X_j) < +\infty$, then $S_n / b_n \to 0$ a.s. as $n \to +\infty$.*

Remark 3.3. Here, (3.1) is called the general condition of Hájek–Rényi ($GCHR$).

Proof. If the sequence has the $\mathbb{E}$-max-variance(r) property, then there exists a constant $C > 0$ such that for any fixed $n \geq 1$, for any $\lambda > 0$, and for $r = 2$,

$$\begin{aligned}\mathbb{P}(\max(|S_1|, \dots, |S_n|) \geq \lambda) &\leq \lambda^{-r}\,\mathbb{V}\mathrm{ar}(\max(|S_1|, \dots, |S_n|)) \\ &\leq \lambda^{-r}\mathbb{E}\left(\max(|S_1|, \dots, |S_n|)\right)^2 \\ &\leq C\lambda^{-r}\,\mathbb{V}\mathrm{ar}(S_n) = C\lambda^{-r}\sum_{i=1}^{n}\left[\sum_{j=1}^{n}\mathrm{Cov}(X_i, X_j)\right].\end{aligned}$$

The conclusion comes out by taking $a_i = \left[\sum_{j=1}^{n}\mathrm{Cov}(X_i, X_j)\right] = \mathrm{Cov}(X_i,\, S_n)$ in the Hájek–Rényi's Theorem 3.1 and applying Theorem 3.2. □

It is worth mentioning that the Hájek–Rényi's inequality is indeed very powerfull but, unfortunately, it works only if we have the max-variance property. For example, the $\mathbb{E}$-max property holds for strictly stationary and associated sequences (see [16]).

As to the squared indice method, it seems that it has not been sufficiently standed to provide general strong laws for dependent data. We aim at filling such a gap.

Indeed, in the next section, we provide a new general SLLN that inspired by the squared indice method. This SLLN will be showed to have interesting applications when comparing to the results of the present section.

4. Our results

In this section, we present a general SLLN based on the squared indice method and give different forms in specific types of dependent data including association with comparison with available results. The result will be used in Section 5 to establish the strong convergence for the continuous Hill's estimator with in the frame of EVT.

Theorem 4.1. *Let $X_1, X_2, \dots$ be an arbitrary sequence of rv's, and let $(f_{i,n})_{i\geq 1, n\geq 1}$ be a sequence of measurable functions such that $\mathbb{V}\mathrm{ar}[f_{i,n}(X_i)] < +\infty$, for $i \geq 1$ and $n \geq 1$. Let us suppose that for some $\delta, 0 < \delta < 3$,*

$$C_1 = \sup_{n\geq 1}\sup_{q\geq 1}\mathbb{V}\mathrm{ar}\left(\frac{1}{q^{(3-\delta)/4}}\sum_{i=1}^{q} f_{i,n}(X_i)\right) < +\infty \tag{4.1}$$

and that for some δ, $0 < \delta < 3$,

$$C_2 < +\infty, \tag{4.2}$$

where C_2 is defined by

$$\sup_{n>0}\sup_{k\geq 1}\sup_{q\,:\,q^2+1\leq k\leq (q+1)^2}\ \sup_{k\leq j\leq (q+1)^2}\mathbb{V}\mathrm{ar}\left(\frac{1}{q^{(3-\delta)/2}}\sum_{i=1}^{j-q^2} f_{q^2+i,n}\left(X_{q^2+i}\right)\right).$$

Then

$$\frac{1}{n}\sum_{i=1}^{n}\left(f_{i,n}(X_i) - \mathbb{E}\left(f_{i,n}(X_i)\right)\right) \to 0 \quad a.s.\ as \quad n \to +\infty.$$

Remark 4.2. We say that the sequence $X_1, X_2, \ldots, X_n$ satisfies the (GCIP) whenever (4.1) and (4.2) hold.

Proof. It suffices to prove the announced results for $Y_i = f_{i,n}(X_i)$ and $\mathbb{E}(Y_i) = 0$, $i \geq 1$. Observe that omitting the subscript n does not cause any ambiguity in the proof below. We have for any positive real number β,

$$\mathbb{P}\left(\left|\frac{1}{k}\sum_{i=1}^{k} Y_i\right| \geq k^{-\beta}\right) \leq \mathbb{P}\left(\left|\sum_{i=1}^{k} Y_i\right| \geq k^{1-\beta}\right) \leq \frac{1}{k^{2(1-\beta)}}\,\mathbb{V}\mathrm{ar}\left(\sum_{i=1}^{k} Y_i\right).$$

We apply this formula for $k = q^2$ and get for $0 < \delta < 3$,

$$\mathbb{P}\left(\left|\frac{1}{q^2}\sum_{i=1}^{q^2} Y_i\right| \geq q^{-2\beta}\right) \leq \frac{1}{q^{4(1-\beta)}}\,\mathbb{V}\mathrm{ar}\left(\sum_{i=1}^{q^2} Y_i\right)$$
$$\leq \frac{1}{q^{1+\delta-4\beta}}\,\mathbb{V}\mathrm{ar}\left(\frac{1}{q^{(3-\delta)/2}}\sum_{i=1}^{q^2} Y_i\right) \leq \frac{C_1}{q^{1+\delta-4\beta}}.$$

Then we have for $0 < \beta < \delta/4$, $\sum_{q=1}^{+\infty}\mathbb{P}\left(\left|\frac{1}{q^2}\sum_{i=1}^{q^2} Y_i\right| > q^{-2\beta}\right) < +\infty$. We conclude that

$$\frac{1}{q^2}\sum_{i=1}^{q^2} Y_i \to 0 \quad \text{a.s. as} \quad q \to +\infty. \tag{4.3}$$

Now set $q^2 \leq k \leq (q+1)^2$ and $\epsilon_{k,q} = 0$ if $k = q^2$ and 1 otherwise. We have

$$\begin{aligned}\frac{1}{k}\sum_{i=1}^{k} Y_i - \frac{1}{q^2}\sum_{i=1}^{q^2} Y_i &= \frac{1}{k}\sum_{i=1}^{k} Y_i - \frac{1}{k}\sum_{i=1}^{q^2} Y_i + \frac{1}{k}\sum_{i=1}^{q^2} Y_i - \frac{1}{q^2}\sum_{i=1}^{q^2} Y_i \\ &= \frac{\epsilon_{k,q}}{k}\left(\sum_{i=1}^{k} Y_i - \sum_{i=1}^{q^2} Y_i\right) + \frac{1}{q^2}\sum_{i=1}^{q^2} Y_i\left(\frac{q^2-k}{k}\right) \\ &= \frac{\epsilon_{k,q}}{k}\left(\sum_{i=q^2+1}^{k} Y_i\right) + \frac{1}{q^2}\sum_{i=1}^{q^2} Y_i\left(\frac{q^2-k}{k}\right). \end{aligned} \tag{4.4}$$

But $(q^2 - k)/k \to 0$ as $q \to +\infty$. This combined with (4.3) proves that the second term of (4.4) converges to zero a.s. It remains to handle the first term. For $0 < \delta < 3$,

$$\mathbb{P}\left(\frac{1}{k}\left|\epsilon_{k,q}\sum_{i=q^2+1}^{k} Y_i\right| \geq k^{-\beta}\right) \leq \mathbb{P}\left(\left|\epsilon_{k,q}\sum_{i=q^2+1}^{k} Y_i\right| \geq k^{1-\beta}\right)$$

$$\leq \mathbb{P}\left(\left|\epsilon_{k,q}\sum_{i=q^2+1}^{k} Y_i\right| \geq q^{2(1-\beta)}\right) \leq \frac{\epsilon_{k,q}}{q^{4-4\beta}}\,\mathbb{V}\mathrm{ar}\left(\sum_{i=q^2+1}^{k} Y_i\right)$$

$$\leq \frac{\epsilon_{k,q}}{q^{1+\delta-4\beta}}\,\mathbb{V}\mathrm{ar}\left(\frac{1}{q^{(3-\delta)/2}}\sum_{i=q^2+1}^{k} Y_i\right) \leq \frac{\epsilon_{k,\,q}C_2}{q^{1+\delta-4\beta}}.$$

Now for $0 < \beta < \delta/4$, $\sum_{k=1}^{+\infty}\mathbb{P}\left(\epsilon_{k,q}\left|\sum_{i=q^2+1}^{k} Y_i\right| \geq k^{1-\beta}\right) < +\infty$. Then

$$\frac{\epsilon_{k,q}}{k}\left[\sum_{i=1}^{k} Y_i - \sum_{i=1}^{q^2} Y_i\right] \to 0 \quad \text{a.s. as} \quad q \to +\infty. \tag{4.5}$$

Now in view of (4.3), (4.4) and (4.5) and since $(q^2-k)/k \to 0$, we may conclude the proof. □

Remark 4.3. In most cases, conditions (4.1) and (4.2) are used for $\delta = 1$, as it is the case for the independent and indentically distributed random variables. We will exhibit a situation in Proposition 2 that cannot be handled without using (4.1) and (4.2) for $\delta < 1$.

4.1. Comparison and particular cases

Let us see how (GCIP), that is fulfilment of conditions (4.1) and (4.2), works in special cases. We have to compare our (GCIP) to (GCHR). But (GCHR) is used only when max-variance property is satisfied. We only consider the case where $X_1, X_2, \ldots$ are real and the $f_{i,n}$'s are identity functions.

4.1.1. Independence case.

By using Theorem 3.1, we observe that we have the $\mathbb{P}$-max-variance(2) property, that is the Kolmogorov's maximal inequality. By using the Hájek–Rényi's general condition, we have the strong law of large numbers of Kolmogorov: $S_n/n \to 0$ a.s. whenever

$$\sum_{n\geq 1}\mathbb{V}\mathrm{ar}(X_n)/n^2 < +\infty.$$

To apply Theorem 4.1 here, we notice that the sequence of variances $\mathbb{V}\mathrm{ar}(S_n)$ is non-decreasing in n. Then (4.1) and (4.2) are implied by, for some $0 < \nu_1$ and

$0 < \nu_2$,

$$\sup_{k \geq 1} \frac{1}{k^{1+\nu_1}} \sum_{i=1}^{k} \mathbb{V}\mathrm{ar}(X_i) < +\infty \text{ and } \sup_{k \geq 1} \frac{1}{k^{2+\nu_2}} \sum_{i=k^2+1}^{(k+1)^2} \mathbb{V}\mathrm{ar}(X_i) < +\infty.$$

But, by observing that the latter is

$$k^{-(2+\nu_2)} \sum_{i=k^2}^{(k+1)^2} \mathbb{V}\mathrm{ar}(X_i) = k^{-(2+\nu_2)} \left[\sum_{i=1}^{(k+1)^2} \mathbb{V}\mathrm{ar}(X_i) - \sum_{i=1}^{k^2} \mathbb{V}\mathrm{ar}(X_i) \right],$$

we conclude that the SLLN is implied by

$$\sup_{k \geq 1} \frac{1}{k^{1+\nu}} \sum_{i=1}^{k} \mathbb{V}\mathrm{ar}(X_i) < +\infty, \tag{4.6}$$

some $\nu > 0$. In the independent case, one has the SLLN for $k^{-1} \sum_{i=1}^{k} \mathbb{V}\mathrm{ar}(X_i) \to \sigma^2$. And the parameter ν in (4.6) is useless in that case. But the availability of the parameter ν is important for situations beyond the classical cases. As a first example, let us use the Kolmogorov's Theorem and construct a probability space holding a sequence of independent centered rv's $X_1, X_2, \ldots$ with $\mathbb{E}X_n^2 = n^{1/3}$. But (4.6) does not hold for $\nu = 0$ since

$$\frac{1}{n} \sum_{i=1}^{n} i^{1/3} \geq \frac{1}{n} \int_{1}^{n} x^{1/3} dx \geq \frac{3}{4} \left(n^{1/3} - 1 \right) \to +\infty, \quad \text{as} \quad n \to +\infty$$

while (GCHR) entails the SLLN.

We will consider in proposition 2 below an important other example which cannot be concluded unless we use a positive value of ν. Now, if we may take $\nu = 1/3$, we have that $n^{-(1+\nu)} \sum_{i=1}^{n} i^{1/3}$ is bounded and our Theorem also ensures the SLLN.

Now if the sequence is second order stationary, then (4.1) and (4.2) are both valid. Also, if the variances are bounded by a common constant C_0, both (4.1) and (4.2) are valid.

4.1.2. Pairwise negatively dependent variables.

In that case, we may drop the covariances in (GCIP) and then (4.1) and (4.2) lead to (4.6) as a general condition for the validity of the SLLN in the independent case. As to (GCHR), we don't have any information whether or not the max-variance property holds.

4.1.3. Associated sequences

Here $\mathbb{V}\mathrm{ar}(S_n)$ is non-decreasing in n and (GCIP) becomes for $\nu = (1-\delta)/2 \geq 0$ with $0 < \delta < 1$

$$\sup_{q\geq 1} \frac{1}{q^{1+\nu}} \mathbb{V}\mathrm{ar}\left(\sum_{i=1}^{q} X_i\right) < +\infty \tag{4.7}$$

and

$$\sup_{q\geq 1} \frac{1}{q^{2(1+\nu)}} \mathbb{V}\mathrm{ar}\left(\sum_{i=q^2+1}^{(q+1)^2} X_i\right) < +\infty. \tag{4.8}$$

If the sequence is second order stationary, then (4.7) implies (4.8), since

$$\begin{aligned}\frac{1}{q^{2(1+\nu)}} \mathbb{V}\mathrm{ar}\left(\sum_{i=q^2+1}^{(q+1)^2} X_i\right) &= \frac{(2q+1)^{1+\nu}}{q^{2(1+\nu)}}\left[\frac{1}{(2q+1)^{1+\nu}} \mathbb{V}\mathrm{ar}\left(\sum_{i=1}^{2q+1} X_i\right)\right] \\ &\sim \frac{2}{q^{(1+\nu)}} \mathbb{V}\mathrm{ar}\left(\frac{1}{k^{(1+\nu)/2}}\sum_{i=1}^{k} X_i\right),\end{aligned}$$

for $k = 2q+1$. And (4.7) may be witten as

$$\sup_{q\geq 1} \frac{1}{q^{\nu}}\left[\mathbb{V}\mathrm{ar}(X_1) + \frac{2}{q}\sum_{i=2}^{q}(q-i+1)\,\mathrm{Cov}(X_1, X_i)\right] < +\infty. \tag{4.9}$$

This is our general condition under which SLLN holds for second order stationary associated sequence. Then, by the Kronecker lemma, we have the SLLN if

$$\sigma^2 = \mathbb{V}\mathrm{ar}(X_1) + 2\sum_{i=2}^{+\infty}\mathrm{Cov}(X_1, X_i) < +\infty. \tag{4.10}$$

Condition (4.10) is obtained by Newman [16]. Clearly, by the Cesàro lemma, (4.10) implies

$$\lim_{q\to+\infty} \frac{1}{q}\sum_{i=1}^{q}\mathrm{Cov}(X_1, X_i) \to 0.$$

And, in fact, the latter is a necessary condition of strong law of large numbers as proved in Theorem 7 in [17], from the original result of Lebowitz (see [13]).

The reader may find a larger review on this subject in [19]. Our result seems more powerful since we may still have the strong law of large numbers even if $\sigma^2 = +\infty$.

We only need to check condition (4.9). We will comment this again after Proposition 2.

For strictly stationary associated sequences with finite variance, we have the $\mathbb{E}$-max-variance(2) property (see [16]). Then (GCHR) may be used. It becomes

$$\limsup_{n} \sum_{i=1}^{n} \frac{1}{i^2}\,\mathrm{Cov}(X_i, S_n) < +\infty, \tag{4.11}$$

which is equivalent to

$$\limsup_n \left[\sum_{i=1}^{n} \frac{\mathbb{V}\mathrm{ar}(X_i)}{i^2} + \sum_{j=2}^{n} \left(\sum_{i=1}^{n-j+1} \frac{1}{i^2} + \sum_{i=j}^{n} \frac{1}{i^2} \right) \mathrm{Cov}(X_1, X_j) \right] < +\infty$$

and reduces to

$$\sum_{j=2}^{+\infty} \mathrm{Cov}(X_1, X_j) < +\infty.$$

We then see that (GCHR) gives weaker results than ours. Indeed, in our formula (4.9), we did not require that $\frac{2}{q}\sum_{i=2}^{q}(q-i+1)\,\mathrm{Cov}(X_1, X_i)$ is bounded. It may be allowed to go to infinity at a slower convergence rate than $q^{-\nu}$. Then our condition (4.9) besides being more general, applies to any associated sequences and is significantly better than the (GCHR) for strictly stationary sequences.

Nevertheless, for (4.11), it is itself more powerfull than Theorem 6.3.6 and Corollary 6.3.7 in [19], due to the use of Theorem 3.1 and Proposition 1, of Tómács and Líbor (see [21]). Such a result is also obtained by Yu (1993) (see [22]) for the strong convergence of empirical distribution function for associated sequence with identical and continuous distribution.

Birkel (see [3]) used direct computations on the convariance structure for associated variables and got the following condition

$$\limsup_n \sum_{i=1}^{n} \frac{1}{i^2}\,\mathrm{Cov}(X_i, S_i) < +\infty$$

for SLLN for associated variables.

Now, to sum up, the comparison between (GCIP) and (GCHR) is as follows:

1. For independent case the two conditions are equivalent.

2. In negatively associated case, the form of (GCIP) for independent case remains valid. And we have no information whether the max-variance property holds to be able to apply (GCHR).

3. For association with strictly stationary of sequences, (GCIP) gives a better condition than (GCHR).

4. For association with no information on stationarity, so (GCHR) cannot be applied unless a max-variance property is proved. Our condition still works and is the same as for the stationary associated sequences in point 3.

5. For arbitrary sequences with finite variances, point 4 may be recontacted.

In conclusion, our method effectively brings a significant contribution to SLLN for associated random variables. And we are going to apply it to an associated sequence in the extreme value theory fields.

5. Applications

5.1. Application to extreme value theory

The EVT offers us the opportunuity to directly apply our general conditions (4.1) and (4.2) to a sum of dependent and non-stationary random variables and to show how to proceed in such a case.

We already emphasized the importance of the parameter $\nu = (1-\delta)/2$ in (GCIP). In the example we are going to treat, we will see that a conclusion cannot be achieved with $\nu = 0$.

Let $E_1, E_2, \dots$ be an infinite sequence of independent standard exponential random variables, $f(j)$ is an increasing function of the integer $j \geq 0$ with $f(0) = 0$ and $\gamma > 0$ a real parameter. Define the following sequences of random variables

$$W_k = \sum_{j=1}^{k-1} f(j) \left[\exp\left(-\gamma \sum_{h=j+1}^{k-1} E_h/h \right) - \exp\left(-\gamma \sum_{h=j}^{k-1} E_h/h \right) \right], \ k \geq 1. \quad (5.1)$$

The characterization of the asymptotic behavior of (5.1) has important applications and consequences in two important fields: the extreme value theory in statistics and the central limit theorem issue for sum of non stationary associated random variables. Let us highlight each of these points.

On one side, let $X, X_1, X_2, \dots$ be independent and identically random variables in Weibull extremal domain of parameter $\gamma > 0$ such that $X > 0$ and let $X_{1,n} \leq X_{2,n} \leq \cdots \leq X_{n,n}$ denote the order statistics based on the $n \geq 1$ observations. The distribution function G of $Y = \log X$ has a finite upper endpoint y_0 and admits the following representation:

$$y_0 - G^{-1}(1-u) = c u^{1/\gamma} (1 + p(u)) \exp\left(\int_u^1 t^{-1} b(t) dt \right), \ u \in (0,1)$$

where c is a constant and, $p(u)$ and $b(u)$ are functions of $u \in (0,1)$ such that $(p(u), b(u)) \to 0$ as $u \to 0$. This is called a representation of a sequence of random variables in the Weibull domain of attraction.

To stay simple, suppose that $p(u) = b(u) = 0$ for all $u \in (0,1)$ consider the simplest case

$$y_0 - G^{-1}(1-u) = u^{\gamma}, \ u \in (0,1). \quad (5.2)$$

The so-called Hill's statistic, based on the identity function $id(x) = x$ and the k largest values with $1 \leq k \leq n$,

$$T_n(id) = \frac{1}{id(k)} \sum_{j=1}^{k} id(j) \left(\log X_{n-j+1,n} - \log X_{n-j,n} \right)$$

is an estimator of γ in the sense that

$$\frac{T_n(id)}{(y_0 - \log X_{n-k,n})} \to^{\mathbb{P}} (\gamma+1)^{-1},$$

as $n \to +\infty$. When we replace the identity function with an increasing function $f(j)$ of the integer $j \geq 0$ with $f(0) = 0$, we get the functional Hill's estimator defined as

$$T_n(f) = \frac{1}{f(k)} \sum_{j=1}^{k} f(j) \left(\log X_{n-j+1,n} - \log X_{n-j,n} \right)$$

introduced by Dème E., Lo G.S. and Diop, A. (2012) (see [5]). From this processus is derived the Diop and Lo (2006) (see [6]) generalization of Hill's statistic. We are going to highlight that $f(k)T_n(f)/(y_0 - \log X_{n-k,n})$ is of the form of (5.1) when (5.2) holds. We have to use two representations. The Rényi's representation allows to find independent standard uniform random variables $U_1, U_2, \ldots$ such that the following equalities in distribution hold

$$\{\log Y_j,\ j \geq 1\} =_d \{G^{-1}(1 - U_j),\ j \geq 1\}$$

and

$$\{\{\log X_{n-j+1,n},\ 1 \leq j \leq n\},\ n \geq 1\} =_d \left\{\{G^{-1}(1 - U_{j,n}),\ 1 \leq j \leq n\},\ n \geq 1\right\}.$$

Next, by the Malmquist representation (see ([20]), p. 336), we have for each $n \geq 1$, the following equality in distribution holds

$$\{j^{-1} \log(U_{j+1,n}/U_{j,n}),\ 1 \leq j \leq n\} =_d \{E_j^{(n)},\ 1 \leq j \leq n\},$$

where $E_j^{(n)}$, $1 \leq j \leq n$, are independent exponential random variables. We apply these two tools to get that for each fixed n and $k = k(n)$

$$\frac{T_n(f)}{(y_0 - \log X_{n-k,n})} =_d W_{k(n)}. \tag{5.3}$$

For an arbitrary element of the Weibull extremal domain of attraction, it may be easily showed that $f(k)T_n(f)/(y_0 - \log X_{n-k,n})$ also behaves as (5.1) if some extra conditions are imposed of the auxiliary functions p and b. Hence a complete characterization of the asymptotic behavior of (5.1) provides asymptotic laws in extreme value theory.

On another side, easy algebra leads to

$$W_k = f(k-1) + \sum_{j=1}^{k-1} \Delta f(j) \exp\left(-\gamma \sum_{h=j}^{k-1} E_h/h \right),$$

where $\Delta f(j) = f(j) - f(j-1)$, $j \geq 1$. We consider

$$W_k^* = W_k - \mathbb{E}(W_k) = \sum_{j=1}^{k-1} \Delta f(j) \left[\exp\left(-\gamma \sum_{h=j}^{k-1} E_h/h \right) - \mathbb{E} \exp\left(-\gamma \sum_{h=j}^{k-1} E_h/h \right) \right].$$

This is a sum of non stationary dependent random variables. In fact the rv's

$$\Delta f(j)\left[\exp\left(-\gamma\sum_{h=j}^{k-1}E_h/h\right)-\mathbb{E}\exp\left(-\gamma\sum_{h=j}^{k-1}E_h/h\right)\right]$$

are associated.

Now, we are going to apply our general conditions to (5.4), defined below

$$S_k^*=\sum_{j=1}^{k-1}\Delta f(j)\left[\exp\left(-\gamma\sum_{h=j}^{k-1}E_h/h\right)-\mathbb{E}\exp\left(-\gamma\sum_{h=j}^{k-1}E_h/h\right)\right]\alpha(k), \tag{5.4}$$

where $\alpha(k)$ is a sequence of positive real numbers. Next, we will particularize the result for $f(j)=j^{\tau}$, $\tau>0$. Our results depend on computation techniques developed in [8]. Here are our results:

Proposition 2. *Suppose that, for L and q large enough such that $L\leq q^2$, the following conditions hold for some δ, $0<\delta<3$.*

$$\sup_{k\geq L}\frac{\alpha^2(k)}{k^{2\gamma+1+\nu}}\sum_{j=L}^{k-1}\Delta^2 f(j)j^{2\gamma}<+\infty, \tag{5.5}$$

$$\sup_{k\geq L}\frac{\alpha^2(k)}{k^{1+\nu}}\sum_{j=L+1}^{k-1}\left[\sum_{i=L}^{j-1}\Delta f(i)\right]\Delta f(j)\frac{1}{j}<+\infty, \tag{5.6}$$

$$\sup_{k\geq L}\frac{\alpha^2(k)}{k^{1+\nu}}\sum_{L\leq j\leq k-1}\Delta f(j)/j<+\infty, \tag{5.7}$$

$$\sup_{k\geq 1}\frac{1}{q^{(3-\delta)}}\sum_{i=1}^{2q+1}\alpha^2(k)\Delta^2 f(q^2+i)\left(\frac{q^2+i}{k}\right)^{2\gamma}<+\infty \tag{5.8}$$

and

$$\sup_{k\geq 1}\sup_{(q^2+1)\leq k\leq (q+1)^2}\frac{\alpha^2(k)}{q^{(3-\delta)}}\sum_{j=2}^{2q+1}\left[\sum_{i=1}^{j-1}\Delta f(q^2+i)\right]\Delta f(q^2+j)\frac{1}{q^2+j}<+\infty. \tag{5.9}$$

Then

$$\frac{S_k^*}{k}\to 0 \quad a.s.$$

Further, if

$$\mu_k=\sum_{j=1}^{k-1}\alpha(k)\Delta f\,(j)\,\mathbb{E}\exp\left(-\gamma\sum_{h=j}^{k-1}E_h/h\right)\to\mu,$$

where μ is a finite, then

$$k^{-1}\sum_{j=1}^{k-1}\alpha(k)\Delta f\,(j)\exp\left(-\gamma\sum_{h=j}^{k-1}E_h/h\right)\to\mu \quad a.s.$$

Proposition 3. *For $f(j) = j^\tau$, if* (5.5), (5.6), (5.7), (5.8) *and* (5.9) *hold,* $\alpha(k) = 1/k^{\tau-1}$ *and if* $\mu = \tau/(\tau+\gamma)$. *Then*

$$\frac{1}{k^\tau}\sum_{j=1}^{k-1}(j^\tau - (j-1)^\tau)\exp\left(-\gamma\sum_{h=j}^{k-1}E_h/h\right) \to \frac{\tau}{\gamma+1} \quad \textit{a.s. as} \quad k \to +\infty.$$

Remark 5.1. Since these results are only based on moments, the a.s. convergence remains true for $T_n(f)/(y_0 - \log X_{n-k,n})$ in vertue of (5.3). We get under the model that

$$\frac{T_n(f)}{(y_0 - \log X_{n-k,n})} \to \frac{\tau}{\gamma+1} \text{ a.s. as } n \to +\infty \text{ and } k = k(n) \to +\infty \text{ and } k/n \to 0$$

under the assumptions (5.5), (5.6), (5.7), (5.8) and (5.9), in the general case.

Remark 5.2. This strong law may be easily checked by Monte Carlo simulations. For example, consider $\gamma = 2$ and $\tau = 1$. We observe the following errors corresponding to the values of 50, 75 and 100 of k: 0.358, 0.321 and 0.3332. This shows the good performance of this strong law for the particular values $\gamma = 2$ and $\tau = 1$.

5.1.1. Proofs

Both proofs of the two propositions are postponed in the Appendix.

6. Conclusion and perspectives

We have established a general SLLN and applied it to associated variables. Comparison with SLLN's derived from the Hájek-Rényi inequality proved that this SLLN is not trivial. We have also used it to find the strong convergence of statistical estimators under non-stationary associated samples in EVT.

It seems that it has promising applications in non-parametric statistic, when dealing with the strong convergence of the empirical process and the non-parametric density estimator for a stationary sequence with an arbitrary parent distribution function.

7. Appendix

7.1. Proofs of Proposition 2 and Proposition 3

7.1.1. Assumptions

We have to show that the assumptions of Proposition 2 entail the general condition (GCIP). We first remind that

$$S_k^* = \sum_{j=1}^{k-1}\Delta f(j)\left[\exp\left(-\gamma\sum_{h=j}^{k-1}E_h/h\right) - \mathbb{E}\exp\left(-\gamma\sum_{h=j}^{k-1}E_h/h\right)\right]\alpha(k)$$

that we write as

$$S_k^* = \sum_{j=1}^{k-1} \alpha(k)\Delta f(j)\left(S_{j,k} - s_{j,k}\right), \tag{7.1}$$

where $S_{j,k} = \exp\left(-\gamma \sum_{h=j}^{k-1} E_h/h\right)$ and $s_{j,k} = \mathbb{E}\exp\left(-\gamma \sum_{h=j}^{k-1} E_h/h\right)$. Next, we are going to check (4.1) and (4.2) for this sum of random variables. Fix δ, $0 < \delta < 3$. Let us split (7.1) into

$$S_k^* = \sum_{j=1}^{L-1} \alpha(k)\Delta f(j)\left(S_{j,k} - s_{j,k}\right) + \sum_{j=L}^{k-1} \alpha(k)\Delta f(j)\left(S_{j,k} - s_{j,k}\right) =: S_L^1 + S_L^2.$$

Then for $\nu = (1-\delta)/2$ with $0 < \delta < 1$,

$$\begin{aligned} \frac{1}{k^{1+\nu}}\,\mathbb{V}\mathrm{ar}(S_k^*) &= \frac{1}{k^{1+\nu}}\mathbb{V}ar\left(S_L^1\right) + \frac{1}{k^{1+\nu}}\,\mathbb{V}\mathrm{ar}\left(S_L^2\right) + \frac{2}{k^{1+\nu}}\,\mathrm{Cov}\left(S_L^1, S_L^2\right) \\ &=: A_k + B_k + 2C_k. \end{aligned}$$

Let us treat each term in the above equality. Here, we use Formulas 18 and 21 in [8] and take L large enough to ensure

$$\mathbb{V}\mathrm{ar}\left(S_{j,k}\right) = \left(\frac{j}{k-1}\right)^{2\gamma} V(1,j)V(2,j), \tag{7.2}$$

with

$$|V(1,j)| = 1 + O(j^{-1}) \text{ and } 0 \leq V(2,j) \leq \frac{2\gamma^2\,|a_1(\in)|}{j}$$

and

$$\mathrm{Cov}(S_{j,k}, S_{j+\ell,k}) = \mathbb{V}\mathrm{ar}\left(S_{j+\ell,k}\right)\left(\frac{j}{j+\ell-1}\right)^{\gamma}(1 + O(j^{-1})).$$

We suppose that L is large enough so that $|V(1,j)| \leq 1/2$, for $j \geq L$.

First we see that

$$A_k \to 0, \quad \text{as} \quad k \to +\infty, \tag{7.3}$$

since $\mathbb{V}\mathrm{ar}(S_L^1)$ is let constant with L. Next, split B_k into

$$\begin{aligned} B_k &= \frac{1}{k^{1+\nu}} \sum_{j=L}^{k-1} \alpha^2(k)\Delta^2 f(j)\,\mathbb{V}\mathrm{ar}\left(S_{j,k} - s_{j,k}\right) \\ &\quad + \frac{1}{k^{1+\nu}} \sum_{L \leq i \neq j \leq k-1} \alpha^2(k)\Delta f(j)\Delta f(i)\,\mathrm{Cov}\left(S_{i,k}, S_{j,k}\right) \\ &=: B_{k,1} + B_{k,2}. \end{aligned}$$

By (7.2) we get

$$B_{k,1} = \frac{1}{k^{1+\nu}} \sum_{j=L}^{k-1} \alpha^2(k)\Delta^2 f(j)\, \mathbb{V}\mathrm{ar}\,(S_{j,k}) \leq (1/2)\frac{\alpha^2(k)}{k^{2\gamma+1+\nu}} \sum_{j=L}^{k-1} \Delta^2 f(j) j^{2\gamma}. \quad (7.4)$$

Now let us turn to the term $B_{k,2}$. Let us remark that the rv′s $S_{j,k}$ are non increasing functions of independent rv's E_j. So they are associated. We then use the Lemma 3 of Newman [15] stated in Lemma 2.4 to get

$$\left|\mathrm{Cov}\left(\exp\left(-\gamma\sum_{h=i}^{k-1} E_h/h\right),\ \exp\left(-\gamma\sum_{h=j}^{k-1} E_h/h\right)\right)\right|$$
$$\leq \mathrm{Cov}\left(\gamma\sum_{h=i}^{k-1} E_h/h,\ \gamma\sum_{h=j}^{k-1} E_h/h\right),$$

where we use the one-value bound of $\exp(-x)$. For $i \leq j$,

$$\mathrm{Cov}\left(\gamma\sum_{h=i}^{k-1} E_h/h,\ \gamma\sum_{h=j}^{k-1} E_h/h\right) = \mathbb{V}\mathrm{ar}\left(\gamma\sum_{h=j}^{k-1} E_h/h\right) = \gamma^2\sum_{h=j}^{k-1} h^{-2} \leq \frac{\gamma^2}{j}, \quad (7.5)$$

the latter inequality is directly obtained by comparing $\sum_{h=j}^{k-1} h^{-2}$ and $\int_j^k x^{-2}dx$. We get

$$\begin{aligned}|B_{k,2}| &\leq \frac{1}{k^{1+\nu}} \sum_{L\leq i\neq j\leq k} \alpha^2(k)\Delta f(j)\Delta f(i)\,\mathrm{Cov}\left(\gamma\sum_{h=i}^{k-1} E_h/h,\ \gamma\sum_{h=j}^{k-1} E_h/h\right)\\ &\leq \frac{2\gamma^2}{k^{1+\nu}} \sum_{L\leq i<j\leq k} \alpha^2(k)\Delta f(j)\Delta f(i)/j\\ &= \frac{2\gamma^2}{k^{1+\nu}}\alpha^2(k) \sum_{j=L+1}^{k-1}\left[\sum_{i=L}^{j-1}\Delta f(i)\right]\Delta f(j)\frac{1}{j}. \quad (7.6)\end{aligned}$$

Finally, by using the techniques of (7.5) and (7.6), we get

$$\begin{aligned}C_k &= \sum_{1\leq i\leq L-1}\ \sum_{L\leq j\leq k-1} \alpha^2(k)\Delta f(i)\Delta f(j)\,\mathrm{Cov}(S_{i,k}, S_{j,k})\\ &\leq \frac{\alpha^2(k)\gamma^2}{k^{1+\nu}} \sum_{L\leq j\leq k-1}\left[\sum_{1\leq i\leq L-1}\Delta f(i)\right]\Delta f(j)/j, \quad (7.7)\end{aligned}$$

where $\left[\sum_{1\leq i\leq L-1}\Delta f(i)\right]$ is a constant. By putting together (7.3), (7.4), (7.6) and (7.7), we get that assumptions (5.5), (5.6) and (5.7) entail (4.1) in Theorem 4.1.

We are going to check for (4.2) now. We already noticed that the rv's $\alpha(k)\Delta f(q^2+i)(S_{k,q^2+i}-s_{k,q^2+i})$ are associated and partial sums of associated rv's have non decreasing variances. Then for $j \leq 2q+1$, we have

$$\begin{aligned}&\mathbb{V}\mathrm{ar}\left(\sum_{i=1}^{j}\alpha(k)\Delta f(q^2+i)\left(S_{k,q^2+i}-s_{k,q^2+i}\right)\right)\\&\leq \mathbb{V}\mathrm{ar}\left(\sum_{i=1}^{2q+1}\alpha(k)\Delta f(q^2+i)\left(S_{k,q^2+i}-s_{k,q^2+i}\right)\right).\end{aligned}$$

And (4.2) becomes

$$\sup_{k\geq 1}\sup_{(q^2+1)\leq k\leq (q+1)^2}\frac{1}{q^{(3-\delta)}}\mathbb{V}\mathrm{ar}\left(\sum_{i=1}^{2q+1}\alpha(k)\Delta f(q^2+i)\left(S_{k,q^2+i}-s_{k,q^2+i}\right)\right). \tag{7.8}$$

We fix q but large enough to ensure $q^2 \geq L$, where L is introduced in (7.2). So (7.8) is bounded by

$$\sup_{(q^2+1)\leq k\leq (q+1)^2}\frac{1}{q^{(3-\delta)}}\mathbb{V}\mathrm{ar}\left(\sum_{i=1}^{2q+1}\alpha(k)\Delta f(q^2+i)\left(S_{k,q^2+i}-s_{k,q^2+i}\right)\right).$$

Now, we only have to show that

$$D=\sup_{(q^2+1)\leq k\leq (q+1)^2}\frac{1}{q^{(3-\delta)}}\mathbb{V}\mathrm{ar}\left(\sum_{i=1}^{2q+1}\alpha(k)\Delta f(q^2+i)\left(S_{k,q^2+i}-s_{k,q^2+i}\right)\right)$$

is bounded for $q^2 \geq L$. Let us split term in the brackets into

$$\begin{aligned}D&=\frac{1}{q^{(3-\delta)}}\sum_{i=1}^{2q+1}\alpha^2(k)\Delta^2 f(q^2+i)\,\mathbb{V}\mathrm{ar}\left(S_{k,q^2+i}\right)\\&+\frac{1}{q^{(3-\delta)}}\sum_{1\leq i\neq j\leq 2q+1}\alpha^2(k)\Delta f(q^2+i)\Delta f(q^2+j)\,\mathrm{Cov}\left(S_{k,q^2+i},S_{k,q^2+j}\right)\\&=: D_1+D_2.\end{aligned}$$

We have, by (7.2),

$$\begin{aligned}D_1&=\frac{1}{q^{(3-\delta)}}\sum_{i=1}^{2q+1}\alpha^2(k)\Delta^2 f(q^2+i)\,\mathbb{V}\mathrm{ar}\left(S_{k,q^2+i}\right)\\&\leq (1/2)\frac{1}{q^{(3-\delta)}}\sum_{i=1}^{2q+1}\alpha^2(k)\Delta^2 f(q^2+i)\left(\frac{q^2+i}{k}\right)^{2\gamma}.\end{aligned}\tag{7.9}$$

Now, we handle D_2. We use again the techniques that lead to (7.6) based on the Newman's Lemma to get, for $i \leq j$,

$$|D_2| \leq \frac{1}{q^{(3-\delta)}} \sum_{1 \leq i \neq j \leq 2q+1} \alpha^2(k) \Delta f(q^2+i) \Delta f(q^2+j) \, \mathbb{V}\mathrm{ar}\left(\gamma \sum_{h=q^2+j}^{2q+1} E_h/h\right).$$

We remind, as in (7.5), that

$$\mathbb{V}\mathrm{ar}\left(\gamma \sum_{h=q^2+j}^{2q+1} E_h/h\right) \leq \gamma^2/(q^2+j)$$

and then

$$\begin{aligned} |D_2| &\leq \frac{2\gamma^2}{q^{(3-\delta)}} \alpha^2(k) \sum_{1 \leq i < j \leq 2q+1} \Delta f(q^2+i) \Delta f(q^2+j) \frac{1}{q^2+j} \\ &= \frac{2\gamma^2}{q^{(3-\delta)}} \alpha^2(k) \sum_{j=2}^{2q+1} \left[\sum_{i=1}^{j-1} \Delta f(q^2+i)\right] \Delta f(q^2+j) \frac{1}{q^2+j}. \end{aligned} \tag{7.10}$$

By putting together (7.9) and (7.10), we get that assumptions (5.8) and (5.9) entail (4.2) in Theorem 4.1. We may conclude that the strong law of large numbers holds for S_k^*.

7.1.2. Special case for $f(j) = j^\tau$

We are going to check the conditions (5.5), (5.6), (5.7), (5.8) and (5.9) for the special function $f(j) = j^\tau$, $\tau > 0$. We fix L as indicated, consider $q \geq L$ and work with $k \geq q^2 + 1$. We notice that $\Delta f(j)$ is equivalent to $\tau j^{\tau-1}$ and $\Delta f(q^2+j)$ is uniformly equivalent to $\tau j^{\tau-1}$ uniformly in $j \geq L$. Here $\alpha(k) = k^{-(\tau-1)}$. Then (5.5) holds when

$$\sup_{k \geq L} \frac{\tau^2}{k^{2\gamma+2\tau-1+\nu}} \sum_{j=L}^{k-1} j^{2\gamma+2\tau-2}$$

is bounded. But if $2\gamma + 2\tau - 1 = 0$, we get

$$\frac{1}{k^\nu} \sum_{j=L}^{k-1} j^{-1} \sim k^{-\nu} \log k \to 0$$

and for $2\gamma + 2\tau - 1 \neq 0$, we get

$$\frac{1}{k^{2\gamma+2\tau-1+\nu}} \sum_{j=L}^{k-1} j^{2\gamma+2\tau-2} \sim k^{-\nu}(2\gamma+2\tau-1)^{-1}$$

and (5.5) holds. (5.6) holds with boundedness of

$$\sup_{k\geq L} \frac{1}{k^{2\tau-1+\nu}} \sum_{j=L+1}^{k-1} j^{2\tau-2}$$

which is, for $2\tau - 1 \neq 0$

$$\frac{1}{2\tau-1} k^{-\nu} \to 0,$$

and for $2\tau = 1$

$$k^{-\nu} \ln k \to 0.$$

Next (5.7) is equivalent to the boundedness of

$$\frac{1}{k^{2\tau-1+\nu}} \sum_{j=L}^{k-1} j^{\tau-2},$$

which is equivalent to the boundedness of $k^{-(\tau+\nu)} \log k$, for $\tau - 1 = 0$ and to that of $k^{-(\tau+\nu)}$ for $\tau - 1 \neq 0$. Let us now handle (5.8) which is equivalent to the boundedness of

$$\begin{aligned} &\frac{1}{q^{(3-\delta)}} \alpha^2(k) \sum_{j=2}^{2q+1} \left[\sum_{i=1}^{j-1} \Delta f(q^2+i) \right] \Delta f(q^2+j) \frac{1}{q^2+j} \\ &\leq \frac{1}{q^{(3-\delta)}} \alpha^2(k) \sum_{j=1}^{2q+1} \left[\sum_{i=1}^{j-1} \Delta f(q^2+i) \right] \Delta f(q^2+j) \frac{1}{q^2+j}, \end{aligned}$$

for enough large q. We have to establish that

$$\sup_{k\geq 1} \sup_{(q^2+1)\leq k\leq (q+1)^2} \frac{1}{q^{(3-\delta)}} \frac{1}{k^{2\gamma+2\tau-2}} \sum_{j=1}^{2q+1} (q^2+j)^{2\gamma+2\tau-2} < +\infty.$$

If $2\gamma + 2\tau - 1 \neq 0$, then $\frac{1}{q^{(3-\delta)}} \frac{1}{k^{2\gamma+2\tau-2}} \sum_{j=1}^{k-(2k+1-q^2)} (q^2+j)^{2\gamma+2\tau-2}$ is bounded whenever

$$\frac{1}{q^{(3-\delta)}} \frac{1}{k^{2\gamma+2\tau-2}} \frac{k^{2\gamma+2\tau-1}}{2\gamma+2\tau-1} = \frac{1}{2\gamma+2\tau-1} (k/q^2) q^{-(1-\delta)}$$

is bounded. And if $2\gamma+2\tau-1 = 0$, $\sum_{j=1}^{k-(2k+1-q^2)} (q^2+j)^{2\gamma+2\tau-2}$ is bounded along with

$$\frac{k}{q^{(3-\delta)}} \log k \leq \left(k/q^2\right) q^{-(1-\delta)} \log k.$$

In both cases, $\left(k/q^2\right) q^{-(1-\delta)} \sim q^{-(1-\delta)} \to 0$ as k (and q) goes to infinity. The proof is now complete.

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The experience of applying a method to develop the use of color theory

Erika Perge

University of Debrecen, Faculty of Engineering
perge@eng.unideb.hu

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Abstract

As is suggested by experience, young people beginning their technical studies in higher education know little about color theory, their knowledge is not systematized and the majority are incapable of applying what they learnt from a number of sources. In this article a method will be introduced which aims at systematizing what secondary-school schoolchildren learn about color theory and allows them to apply the methods of IT consciously. Part of the methodological research is to elaborate an interactive, multimedia color theory educational software and apply it in Short Color Courses. It is recommended that applying our elaborated chromatics teaching software should be integrated into teaching IT in the secondary schools, so as to teach schoolchildren how to handle colours, as well as lay down the foundations of IT sciences such as computer graphics and digital image processing. We are summarizing the experience and results gained in the application of the method in schools.

Keywords: color theory, educational software, skills development

MSC: 97D40, 97U50

1. Introduction

"Colour is life; for a world without colours appears to us as dead." (Johannes Itten)

In education (in all types of school), from the aspect of the of color theory, our main task is to prepare young people for a world dominated by colors, colorful, natural and built surroundings, colorful objects, pictures, posters and printed documents as well as color films, etc. Color theory is taught and learnt and applied creatively in teaching several subjects (such as drawing, physics, chemistry, biology, art history, information technology) in primary and secondary schools based on the National Curriculum (see [15]), on which professional training is built at a later stage.

We have experienced that young people can hardly use their knowledge of color theory that might involve several problems in learning and mastering certain professions (see [16]). This deficiency can especially be observed in technical higher education, more specifically in the field of architectural planning and product design, computer graphics and digital picture processing, where creating a harmony of colors is part of daily work. One reason for the deficiencies is that the knowledge gained from various fields is not arranged into a unified whole and does not turn into a consciously used means. The deficiencies can be experienced at several levels: lack of basic knowledge, lack of knowledge arranged in a unified way, lack of color defining, color-distinguishing skills, sensing colors, aesthetic sense as well as a low level of adapting the knowledge available.

A developing method needs to be elaborated and used to motivate the schoolchildren attending secondary-schools so as to be able to prepare young people for the requirements of higher education.In our well-elaborated method, we suggest that teaching colour theory should be organically and well-concertedly linked to informatics, applying means of IT and teaching IT. By applying our method, IT sciences such as computer graphics and picture processing can be made more popular, and their foundations can be laid down. In addition, the programme mentioned above must exploit inherent potentials of computers to represent and manage colors.The schoolchildren should be enabled to get to know concepts, abbreviations such as RGB and master their colour theory backgrounds.

2. Potentials and means of developing the knowledge and skills of color theory

A research project aimed at developing the knowledge and skills of color theory has been going on at the Faculty of Engineering of Debrecen University for years.

The research is aimed at developing applicable means as well as applying them under control, where the efficiency of methods can be measured. Part of the research is to survey students' problem-solving skills in terms of color theory by measuring the conscious application of the color theory, and color perception regarding the ([16]). The test revealed that the overwhelming majority of students did not actually have the competencies prescribed by the National Curriculum ([15]), so students do need developing courses. They preferably have to attend the courses in the secondary-school because developing color theory is rather difficult with university students due to their age.

To achieve that aim a method has been developed for secondary schools that focuses on the importance of color theory and applicability within the Short Color Course of a few hours, which offers rich experience while pointing out how they are related to each other. It also highlights the tools needed to develop the skills needed for their creative work such as color sensitivity, and their ability to define and distinguish colors.

The unique nature of the training method is especially obvious in defining the themes of processing, assembling the means facilitating mastering knowledge as well as in the way it is applied.

3. Historical review

The fact that colors are important in our life is also borne out by the fact that a great number of scientists, physicists, doctors, painters, poets and philosophers have done research into colors and the secrets of colors.

The origin of the complex science dealing with colors goes back to ancient times. Chinese, Egyptians, Greek canons and scientific views raised the question: what is color? How can or does it have to be used and interpreted? In the Middle-Ages, branches of science were separate so optics had a chance to become partially independent.

It was the Renaissance Period when color emerged as an independent issue. Questions of science and art related to colors soon merged, but it required a person who could examine the origin of colors as a scientist, and one who was an artist at the same time was curious about the context that was so definitive in practice (see [8]).

Personalities from different scientific fields doing research into color included Pithagorasz, Platon, Arisztotelész, Grosseteste, Alberti, Leonardo da Vinci, Forsius, Kirschner, Newton, Mayer, Lambert, Goethe, Scopenhauer, Hegel, Runge, Young, Chevreul, Grassmann, Maxwell, Delacroix, Helmholtz, Bezold, Seurat, Ostwald, Rood, Hofler, Munsell, Itten, Nemcsics, etc.

Leonardo summarized his research related to this field in Trattato della pittura.

Newton began to research colors during his research into optics and light optics. He summarizes the results of his research in Principia (1685) and Opticae (1704).

One of the least-known features of the life of Goethe, the prince of poets, is his research into colors and his comprehensive conclusions that he drew from art practice, chemistry, physiology and psychology (see [7]). Although the poet is mostly celebrated for his poems all over the world, it was not his literary work that he considered lasting. He said towards the end of his life: *"I do not think much of what I created as a poet. Many a great poet lived in my era, even greater before and many more will live after me. But the fact that I am the only one to know the truth in the complex field of color theory in this century is something that I am proud of and even makes me feel I am above others."*

The main work of Ostwald, the chemist, is Die Farbenlehre. The subjects of the five volumes published are: the color theory of mathematics, physics, chemistry, physiology and psychology. Hemholtz, a doctor and mathematician elaborated the theory of additive and subtractive mixing of colors. In the second half of the XIX. century, Bezold studied chromatopsy, color appearance, Rood and Hofler focused on sensed scene. However, the ultimate and most important features of color theory emerged in XX century physics, physiology and psychology.

It would be time-consuming to list the research and practice fields but it includes all visual arts, the world of symbols and language, ergonomics, sciences and several industries. The results of tests related to the system of colors and color collections used today were summarized by Billmeyer, and it is reported by the annotation bibliography AIC (Associatione Internationale de la Couleur) containing 435 publications published in 1986 ([14]). The Color-teaching special group called Associatione Internationale de la Couleur (SGCE) teaches color theory and develops methods in education. Szín-Vonal a professional working party, focuses on color theory in primary, secondary and higher education.

4. Short Color Course

The method of Short Color Courseaimed at developing knowledge and skills of color theory was applied in seven secondary-schools in Debrecen between 2013-2015 involving 240 schoolchildren. The lessons were held 17 times in normal and irregular lessons, as well as study circles and special school activities. An occasion meant 4×45 minutes.

It was already suggested by Comenius in his ”Didactica Magna” that things should be presented by sensing and experiencing, reasoning, through our own considerations and by accounting for others' opinions. We suggest that processing the material be based on experience, whereby practice precedes summarizing and classifying theory.

5. The color theory themes of Short Color Courses

In this section we present a proposed order of processing the knowledge belonging to different disciplines and considered relevant and useful by ourselves.

Studying the main characteristics of color (hue, saturation, brightness) and classifying colors should be started with color-comparison, color-matching tasks, as our experience suggests that students are rather uncertain in distinguishing hues, dark-light, and saturated-unsaturated color values. This should be followed by the practice of color blending tasks, blending paint colors and light colors. Based on the experiential results of these tasks, the teaching material of optics and light physics is summarized (electromagnetic radiation and light, refraction of light, reflection of light, blending light colors), as well as the explanation of the physiology of visual sensing, the process of colored vision, light source, light reflection, the operation of the eyes and brain. This is to be followed by defining the color-concepts of various sciences (see [2, 3, 5, 6, 10, 13, 19]).

As colors are influenced by their environment, that is the colors surrounding them, we study their interactions and misleading behaviour. For instance, one color should seem two, two colors should seem identical, the color-changing effect of background is to be studied in terms of hue, saturation and brightness, different backgrounds - identical color pattern identical backgrounds-different color patterns (see [1, 14]). Colors look different from what they are really like, so color contrasts, after-images and the subtraction of color and optical illusions can be explained by summing up the causes of color illusion.

Color systems (e.g. Colouroid, HSL, RGB, CMY, CIE) and color collections (e.g. Panton, Munsell) can be reached by classifying the explanation for color relations into a system. Color systems suggest that certain color assemblies within the system harmonize. The main goal and sense of color combinations is to create harmony. In fact, this allows students to get to know monochrome, dichrome, trichrome and tetrachrome harmony types and the way they are created. Then they study the effects of the color relations created, the way they are applied, options of usage, as knowing how to apply color harmonies is indispensible for students in the course of planning.

6. Means supporting learning the teaching material

Apart from traditional means and materials used forprocessing color theory material and its creative application it is also supported with an interactive computerized color theory educational software of our own development. By applying them synchronically, a means system has been developed for simultaneous obtainment of information, while being complementary to each other and making the discovery of reality fuller.

6.1. Traditional means

Traditional means applied in Short Color Courses are paints (such as waterpaint, tempera, print paint), bright light lamps, colored paper, printed color cards, colorful banner, and a rotation disc used for blending paint colors, which is brought about by converting an old tumble-dryer.

Colored paper (which may be pre-produced colored sheets, or pieces of paper clippedout of magazines, journals, advertisements, posters, illustrations and various catalogues) or is preferred for preparing color comparing tasks or creating color collections in the lessons. Paper allows for using a wide-range of hue, lightness andsaturation scale to be used directly, if a wide range of paper is available, and the same color can be used several times without changing its color properties, or its surface propeties changing at all.

6.2. Interactive color theory teaching program

"Applying computer technique makes the learning process unique." (Karvalics)

The development of the Color theory educational software was launched at the Faculty of Technology of Debrecen University in 2011. As a result, it allowed for mastering color theory, a deeper understanding and classification of the material.

Elaborating the software is part of the methodological research whose aim is to reveal efficient application of computer use in teaching mathematics, phisics, geometry and color theory. Computer applications have to be involved into education so that it would help us to recognize the link between theory and practice while maintaining interest in the subject in question by fostering the experimental spirit of students (see [11]).

In Short Color Courses mediated knowledge appears verbally and through picture coding while texts, pictures and interactive animation are used. The educational software is suitable for developing students' color sensitivity, their color-defining, color-distinguishing skills and for measuring the knowledge and skills mastered.

Sovány says in ([20]): *"Interactive multimedia systems are especially suitable for creating an effect mechanism perfectly fitting the information recording and registering mechanism of the human brain, while mediating knowledge."*

The efficiency of multimedia programs is owing to the fact that desired content can be reached very quickly and conveniently (see [12]). The advantage of the educational software developed by us compared to traditional means is as follows:

- it allows students to prepare on their own from the teaching material available;
- students can carry out basic experiments and tasks by themselves as well as obtain further information depending on their interests;
- the material presented can be used by several age groups of various standards;
- it contains a variety of tasks;
- it contains tasks of various complexity, that is its approach is;
- there is a chance for reproduction;
- it can test students (pre and post tests);
- both the instructor and the students use it in the teaching process.

Furthermore the software is simple, easy to use and platform-free (can be used on any operational-system).

7. Introducing tasks of the lessons

In this section, the main tasks are introduced, which were carried out with a multimedia teaching program in the lessons. Each task is analysed to reveal how the software has contributed to solving the problem, how the deficiencies of traditional technique and tools could be improved and how the difficulties cropping up in their use could be handled.

7.1. Color-blending tasks in the HSB system

As the three main properties of colors are their hue, saturation and brightness, it is recommended to analyse, compare and define color-patterns based on these properties. This is vital, as these experience and impressions define our choice of colors, our relationship to colors, in everyday life, during our planning, creative and implementing work.

The students get a sample of color, which they have to blend by using paints. Obviously, this is a rather difficult task as it is not easy to tell which of the three properties diverts from the given one, and to what extent the size and direction differ. In fact, this is where the ”Color-blending in HSB system” of our software can help.

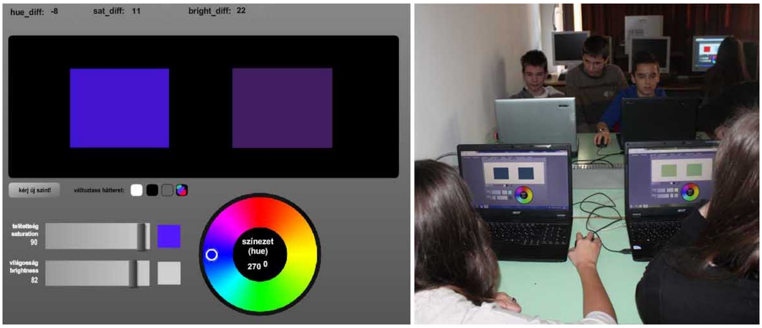

Figure 1: The HSB surface of educational software and its application

The color generated by the computer can be seen in the rectangle on the left of the panel of the module, and the students have to match the color-stock of the rectangle on the right with the color on the left. To achieve that, one can choose the right color from the color-circle below the colors (0–360°), fine-tuning its saturation and brightness using the sliders (0–100%). The black background currently running ban behind the rectangles can be modified to white, grey, mutlicolor or colored background. The values of difference between the sample defined in the program and the values of the blended paint (hue, saturation and brightness) can be seen above. The + and – signs indicate the direction of digression. The values of the digression in the initial stage of the blending practice offer many a + help, but this function can be switched off. While blending the colors, we are continuously comparing the two color samples by their three properties (see Figure 1).

7.2. Additive and subtractive color-blending tasks

“It is revealed by our experiences that newer and newer colors can be blended from any two or more colors. It is also a fact that any color can be created by blending the right number of base-paints. Any color can be created from the base colors in

two different ways, by additive or subtractive color-blending" (see [14]).This is the two-color-blending theory of crometics, which is applied in our daily lives.

In the blending of paint-colors (subtractive color blending), our students have had earlier practice. As a matter of fact this task can be made more spectacular by means of using a color-blending machine converted from a tumble-drier. By using only the three base-colors (Cyan, Magenta, Yellow) we can experience and study the formation of the blended colors (see Figure 2). Three kinds of print paint are the most suitable for his task. Students have no experience in blending light colors

Figure 2: Subtractive color blending in practice

(additive color blending). Three lamps are used for the experiment and the three base paints are Red, Green and Blue. Two-two colors covering each other create a lighter one, so the three colors blended produce white (see Figure 3). We try

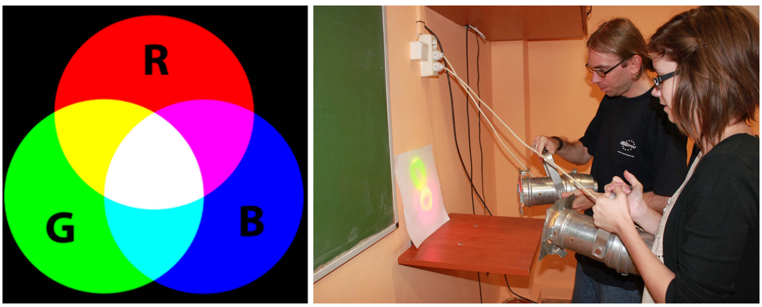

Figure 3: Additive color-blending in practice

to find out what the regularities of color blending are through experiments. On the other hand, if the color qualityis not the right one with the previous tasks, paint blending will not be suitable, i.e. the problem of material quality might prove misleading. When using colored lights the intensity of colors cannot be modified, our tool limits options of variability, so we cannot blend colors of all hues. These problems are eliminated by the task “Color blending in different color systems” of our educational software (see Figure 4).

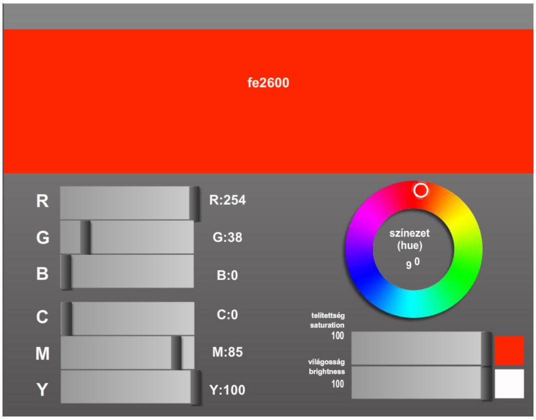

Figure 4: Colour blending in different color systems (RGB, CMY, HSB)

Additive and subtractive color blending is simulated in the educational software. The user can adjust the ”light intensity” of certain light displayswith the slider, and can observe the blending colors formed as well as their color codes. Now, we have a chance to introduce the regularities of creating colors (see Figure 5).

Figure 5: Testing the regularities of color blending

7.3. Color comparing tasks

The task is to compare identical hue – different brightness, or identical hue – different saturation colorpatterns, and also different hue – different brightness, different hue — identical brightness color patterns. The students' color distinguishing skills and their color sensitivity can be developed by doing color comparison tasks.

With traditional means we have as many experiments available as the number of color cards we have made by painting or printing (see Figure 6), so the number of experiments is limited. There are several experiments available for colour

Figure 6: Color comparison tasks with colored cards

stimulation coordination in our educational software, which differ either in terms of brightness or saturation values.

Figure 7: Color comparison tasks in our educational software

The task is to decide which one is a brighter with an identical color and identical saturation value, and which one is more saturated with identical color and identical brightness (see Figure 7). The educational software generates both color patterns at random, so that the difference in value between the colors to be compared is more and more narrowed down to a random number.

7.4. Creating saturation and light scales and their combination

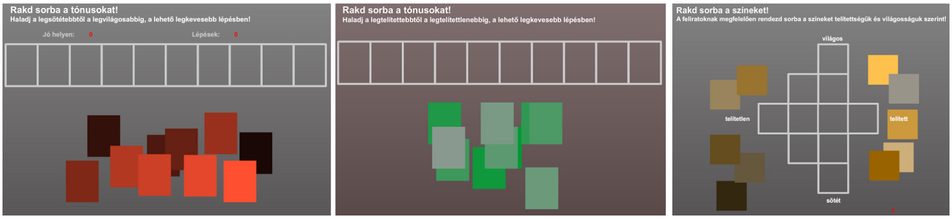

Figure 8: Saturation and light scales in the educational software

The task is to sequence the color patterns by brightness or saturation in rising or falling order, and then place the color patterns in a dark-bright-saturated-unsaturated system in the given formation. The latter task is rather difficult as the color patterns need to be compared from various aspects (see Figure 8). This task actually develops students combining skills (see [15]).

7.5. Color harmonies

Applying color harmony is of primary importance in ornamental art and in planning and implementing work (in the case of an object, a document or even a colorful building). In terms of the number of colors in the color assembly, there are monochrome, dichrome, trichrome and tetrachrome color harmonies. The ed-

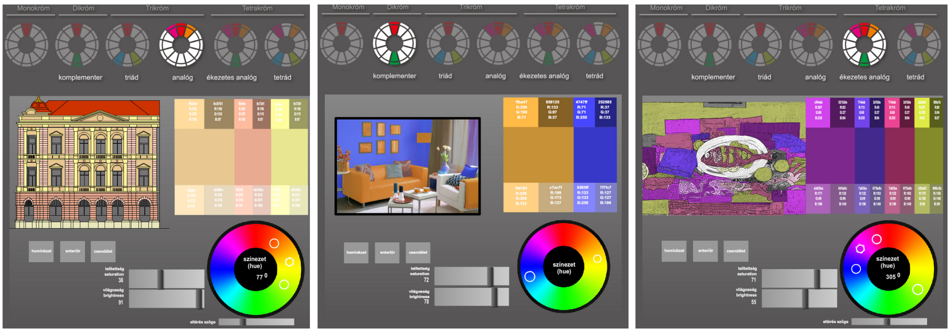

Figure 9: Color harmony types

ucational software allows one to get to know different types of harmonies and to test their applicability in different themes (see Figure 9). A possible means of raising enthusiasm about the subject and theme is experiments. By selecting the color from the set of colors, the software generates five color patterns of different saturation and brightness. The saturation and brightness values can be found by sliders. Our software is especially unique because it can make the color assembly we have created appear in different themes immediately. Thereby the color relations can appear on facades, inside buildings, in still life and grafics right away and can be studied (see Figure 10). György Pólya said that *"We should not only pass on*

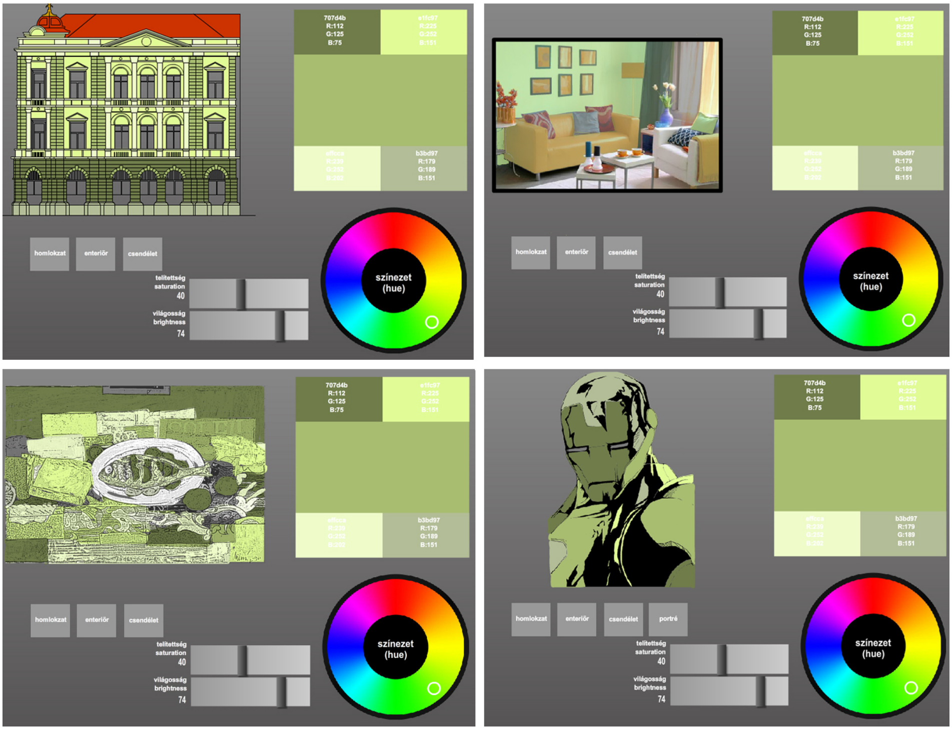

Figure 10: Monochrome harmony in different themes

knowledge to our students but we should also develop their capacity to think". Following his line of thoughts ([18]), we assume that it is not enough to know harmony types and we cannot fully be satisfied with harmony color assemblies either. The program actually offers an opportunity to examine what saturity and brightness of the chosen colors is reasonable to use in the chosen theme (see Figure 11). In fact, this allows our students to gain a lot of experience from observations and testing, which can be applied later in the planning process. An idea of the "Ten commandments for teachers ", written down by Mr Pólya in the 1950s should be mentioned here: *"Start looking for what can prove useful to solve problems to come in the current problems, and try to unfold the general solution behind the specific situations."*

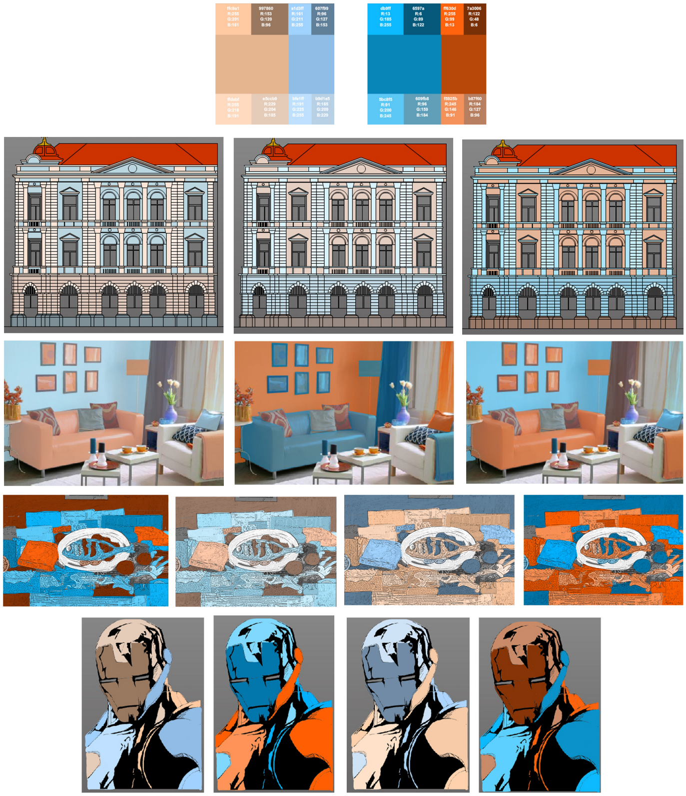

Figure 11: Dichrome harmony in different themes

8. Results and conclusions

It is important from the aspect of effects influencing the success of studying to encourage students to be actively involved in the classroom, since it is learning from your own experience in the classes that is the most lasting. Based on the experience gained from the experiments attendants of the Short Color Courses brushed up their knowledge of color theory in the different subjects, made up for their deficiencies and classified their knowledge. By developing their abilities (color sensitivity), they became capable of creating sensual color complexes, color harmonies from fine hues. It can be pointed out that the attendees of the Short Color Courses could apply colors better at the end of the course owing to the work carried out. At the end of the Short Color Courses we discussed with students what they had learnt from the classes and how useful they considered their experiences. Their positive feedback suggested that they had gained very rich experiences and found it very interesting that the very same theme had been approached from a completely different aspect. In fact, some of them actually realized that they had learnt about the same issue before. Also, they found the tasks pretty varied, the students liked the tools applied found the educational software user-friendly and varied as well as a great experience. They also added they would like to use the software later on in teaching and would have liked to spend more time learning about it.

We also asked teachers offering special classes to test how efficiently the students had learnt to apply their knowledge in practice. The feedback pointed out that secondary-school students managed to incorporate the knowledge mastered in Short Color Courses, and they integrate their knowledge into their new tasks of design and implementation in the lessons creatively. When it comes to solving specific tasks the color assemblies of color harmonies are employed in the process of planning. Further, they enjoy using the software even after the lessons to learn new things, revise their knowledge, practise and experiment.

Physics, biology, drawing and informatics teachers of several schools have let us know that we should help their work by offering lessons. That is why we would like to contribute to students' mastering color theory in the different years.

The question of which subject students like or not is an important signal evaluating the standards of pedagogical-methodological culture that can be experienced in the teaching of the subject (see [4]).

While Short Color Courses for secondary-school students were in progress, the method was also tested at the University of Debrecen, Faculty of Engineering. Statistical methods were applied to verify whether our novel approach resulted in improvement of students' color aptitude and their ability to match and distinguish colors. A paper reporting the results of the experiment is currently under review at Teaching Mathematics and Computer Science (see [17]).

Our method proved successful after several years of testing. During the methodological test the functions of the software were also tested. The training material is being further developed and new modules are being integrated.

While agreeing with views emphasizing the importance of the teaching-aid function of computers, we suggest that the method should be applied widely. To facilitate achieving that we wish to make the software available when all its functions are activated and it has proved useful both technically and methodologically.

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Exploring Hermite interpolation polynomials using recursion*

Róbert Vajda

Bolyai Institute, University of Szeged
vajdar@math.u-szeged.hu

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Abstract

In this paper we consider the teaching of Hermite interpolation. We propose here two nonstandard approaches for exploring Hermite interpolation polynomials in a computer supported environment.

As an extension to the standard construction of the interpolation polynomials based on either on the fundamental polynomials or the triangular shaped divided difference table, we first investigate the generalization of the Neville type recursive scheme which may be familiar to the reader or to the student from the chapter about Lagrangian interpolation.

Second, we propose an interactive demo tool where by the step-by-step construction of the interpolation polynomial, the interpolation constraints can be considered in an almost arbitrary order. Thus the same interpolating polynomial can be constructed in several different ways. As a by-product, one can also ask an interesting combinatorial problem from the students about the number of compatible orders of the constraints depending on the cardinality of node system.

Keywords: Hermite interpolation, Neville's recursion, divided differences, linear algebra, interpolation sequence, enumeration, sequence A000680, computer supported learning environment, Mathematica.

MSC: 65D05, 97N50, 97N80

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1. Introduction

Numerical methods are taught to a wide variety of university students. The author itself teaches numerical mathematics at the University of Szeged for several years. Approximation of functions and thus polynomial interpolation is a standard part of numerical mathematics. However, the instructor can choose between different approaches when introducing the simplest polynomial interpolation problems such as the univariate *Lagrange* and *Hermite* interpolation. The classic approach usually considers the Lagrangian and Newtonian form of the interpolation polynomial [3, 5, 7, 9, 10] and mostly because of time constraints, some very powerful ideas and interesting relations remain hidden related to the topic. Some argued for a linear algebra approach [8] and it is worth to mention that the topic can be also seen as a special case of Chinese remaindering from the more abstract point of view [11]. As an extension to the classic approach, in this paper we propose to show the students Neville's simple but powerful idea to solve the Hermite interpolation problem recursively.

Linking a Neville type algorithm with Hermite interpolation is not new in computational mathematics and is often emphasized by computer aided geometric design (CAGD) researchers, see e.g. [2, Chap. 3]. However, to our best knowledge, in teaching Hermite interpolation Neville is not the most frequent approach. Therefore, based on the Neville recursive algorithm in the literature, in this article we propose to explore the construction of Hermite interpolation polynomials by combining small degree Lagrange interpolation and Taylor polynomials as basic building blocks into an interpolating polynomial. The exploration path is aided by interactive graphical tools which were developed in *Wolfram Mathematica* [12].

We also reconsider the recursive algorithm which leads to the Newtonian form of the Hermite interpolation polynomial. We investigate all the bases in a vector space which arise when by the step-by-step construction of the interpolation polynomial the interpolation constraints are reordered [6]. For investigating the possible orders and finite polynomial sequences, another interactive graphical tool is offered for the students. As a by-product, we also come across an interesting elementary combinatorial problem.

The paper is structured as follows. Section 2.1 introduces the problem, Section 2.2 reviews Neville's solution for the Lagrange problem. Section 2.3 contains the generalization of Neville's scheme to the Hermite problem. Section 2.4 considers the Hermite interpolation polynomials in different bases. The last section concludes.

2. The Hermite problem

2.1. Problem specification

First we give some basic definitions and specify the problem exactly, that we want to solve together with the students. We want to keep the technicalities in a minimum level and therefore we do not consider here problems with higher order derivatives and we assume that in all nodes information about both the zero-order and first-order derivatives are given. We work over the reals, that is, we assume that both the nodes and the given function values are real numbers. This approach is taken e.g. in [3, 9].

Given three finite lists X, F, DF with cardinality $|X| = |F| = |DF| = n \in \mathbb{N}$.

Definition 2.1. Assume that the list X consists of n distinct real numbers, we may also assume that elements of X are ordered:

$$X = \{x_1, x_2, \ldots, x_{n-1}, x_n\},\ x_1 < x_2 < \cdots < x_{n-1} < x_n. \tag{2.1}$$

We call the x_i's in (2.1) the *interpolation nodes* and the grid X the *node system.*

No further assumptions over the values in $F = \{f_1, \ldots, f_n\}$ (function values, zeroth derivatives) and $DF = \{df_1, \ldots, df_n\}$ (first derivatives) are made. We wish to construct an interpolating polynomial H with the following constraints:

$$H(x_j) = f_j \wedge H'(x_j) = df_j \quad (1 \leq j \leq n). \tag{2.2}$$

Definition 2.2. (A) polynomial H which satisfies all constraints is called (an) *Hermite interpolation polynomial* and we also denote it by $H^{(1,1,\ldots,n,n)}$ to emphasize that it satisfies two interpolation constraints at all nodes.

In the sequel we explore the existence and uniqueness of the interpolation polynomial in a computer supported environment. The learning unit about Hermite interpolation usually follows the Lagrange interpolation unit, which is the simplest interpolation problem type, since only function values f_j's (zeroth derivatives) are known in the nodes. We assume that the students are already familiar with Lagrange interpolation. For the exploration we use the general purpose CAS Mathematica, but the exploration can be adapted to any other computer algebra system. We wish to challenge the students to generalize the Neville recursive scheme to the solve the Hermite problem. Therefore as a first step, we quickly review the Neville's main idea for solving Lagrange interpolation in the next subsection. All steps are elementary.

2.2. Neville solution for the Lagrange problem

Assume that we wish to solve the Lagrangian problem with three nodes, i.e., the Lagrange interpolation problem with initial data $(x_1, f_1), (x_2, f_2), (x_3, f_3)$ is given. We wish to find a polynomial L that satisfies all the three constraints:

$$L(x_1) = f_1,\ L(x_2) = f_2,\ L(x_3) = f_3.$$

Assume moreover, that $L^{(1,2)}$ is the linear polynomial which fits to (x_1, f_1) and (x_2, f_2) and $L^{(2,3)}$ is the linear polynomial which fits to (x_2, f_2) and (x_3, f_3).

Neville's beautiful and powerful idea is to combine $L^{(1,2)}$ and $L^{(2,3)}$ into a polynomial $L = L^{(1,2,3)}$ which satisfies all the three initial interpolation constraints: As a first step we ask the student to construct the quadratics

$$q_1 = \frac{x - x_3}{x_1 - x_3} L^{(1,2)}, \quad q_2 = \frac{x - x_1}{x_3 - x_1} L^{(2,3)}. \tag{2.3}$$

It is easy to see (as Figure 1 indicates) that

$$q_1(x_1) = f_1,\ q_1(x_3) = 0; \quad q_2(x_1) = 0,\ q_2(x_3) = f_3,$$

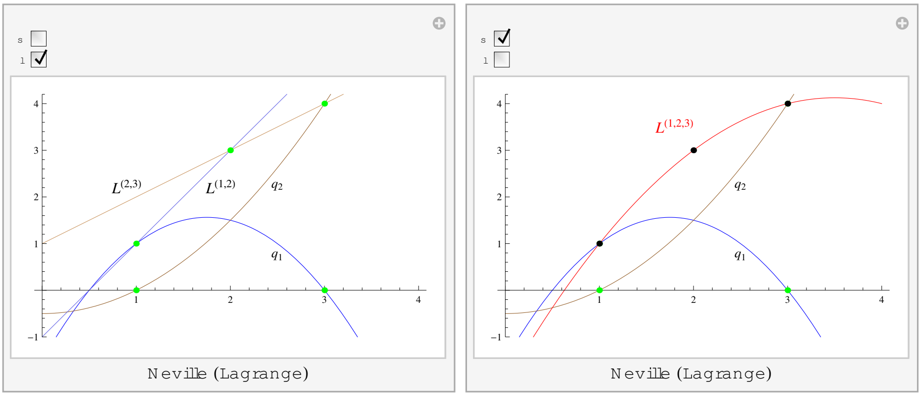

Figure 1: Combining two quadratics into one Lagrange interpolation polynomial

and therefore their sum $q_1 + q_2$ satisfies the first and the last initial interpolation constraints, that is,

$$(q_1 + q_2)(x_1) = f_1 + 0 = f_1, \quad (q_1 + q_2)(x_3) = 0 + f_3 = f_3.$$

But what about the middle point x_2? Is it true that $(q_1 + q_2)(x_2) = f_2$? A short calculation shows that

$$\begin{aligned} L(x_2) &= L^{(1,2,3)}(x_2) = (q_1 + q_2)(x_2) \\ &= \frac{x_2 - x_3}{x_1 - x_3} f_2 + \frac{x_2 - x_1}{x_3 - x_1} f_2 = f_2 \frac{(x_3 - x_2) + (x_2 - x_1)}{x_3 - x_1} = f_2. \end{aligned}$$

In a similar way, we can combine the two polynomials $L^{(1,\ldots,n-1)}$ and $L^{(2,\ldots,n)}$ satisfying all except for the last, and all except for the first interpolating constraints into

$$L = \frac{x - x_n}{x_1 - x_n} L^{(1,\ldots,n-1)} + \frac{x - x_1}{x_n - x_1} L^{(2,\ldots,n)},$$

which satisfies all the n constraints,

$$L(x_j) = f_j \quad (j = 1, \ldots, n).$$

In the next subsection we try to challenge the students to apply the same recursive scheme in order to solve the Hermite interpolation problem given above. We start again with building up an interpolation polynomial satisfying three interpolation constraints by combining two smaller interpolation polynomials. Both smaller polynomials satisfy two interpolation constraints from the initial Hermite constraint-set. However, this time some of the constraint corresponds to the slope of the tangent.

2.3. Neville solution for the Hermite problem

The problem specification of the Hermite interpolation as given in Section 2.1 may make it difficult to come up with the right recursive analog of the idea discussed in the previous Section 2.2. It is clear that uniqueness can be guaranteed by a suitable degree bound. Let us denote the (formally) cubic polynomial which satisfies the constraints of (2.2) on two nodes by $H^{(1,1,2,2)}$, i.e., $H^{(1,1,2,2)}(x_1) = f_1, (H^{(1,1,2,2)})'(x_1) = df_1$, $H^{(1,1,2,2)}(x_2) = f_2, (H^{(1,1,2,2)})'(x_2) = df_2$. It can be shown that we cannot combine the two Hermite (Taylor) polynomials $H^{(1,1)} = T^{(1,1)}$ and $H^{(2,2)} = T^{(2,2)}$ into $H^{(1,1,2,2)}$, but we should rather consider linear Taylor and Lagrange polynomials as atoms for the construction (cf. [7, p. 53]). Therefore we will work with intermediate polynomials which were not present in the initial problem specification.

Definition 2.3. We extend the notation introduced in Definition 2.2 to cover the intermediate polynomials. $H^{(1,1,2)}$ denotes the polynomial which satisfies only the three constraints $H^{(1,1,2)}(x_1) = f_1$, $(H^{(1,1,2)})'(x_1) = df_1$, $H^{(1,1,2)}(x_2) = f_2$ and $H^{(1,2)}$ is simply $L^{(1,2)}$.

As a useful hint, we suggest the students to build up the polynomial $H^{(1,1,2)}$ from $T^{(1,1)}$ and $L^{(1,2)}$. Forming again the two quadratics $Q_1 = \frac{x-x_2}{x_1-x_2}T^{(1,1)}$ and $Q_2 = \frac{x-x_1}{x_2-x_1}L^{(1,2)}$ by just multiplying $T^{(1,1)}$ and $L^{(1,2)}$ with a suitable linear factor as in (2.3), we easily verify that (see also Figure 2).

$$(Q_1 + Q_2)(x_1) = Q_1(x_1) = f_1, (Q_1 + Q_2)(x_2) = Q_2(x_2) = f_2.$$

But what about the derivative at x_1? Let us compute it!

$$(Q_1 + Q_2)'(x_1) = \frac{f_1}{x_1 - x_2} + df_1 + Q_2'(x_1) = \frac{f_1}{x_1 - x_2} + df_1 + \frac{f_1}{x_2 - x_1} = df_1$$

Thus

$$H^{(1,1,2)} = Q_1 + Q_2 = \frac{x - x_2}{x_1 - x_2}H^{(1,1)} + \frac{x - x_1}{x_2 - x_1}H^{(1,2)}.$$

At this point we may already see the generalization of the recursive scheme (Figure 3). In a similar way we construct $H^{(1,2,2)}$ and finally combine $H^{(1,1,2)}$ and $H^{(1,2,2)}$ to get the polynomial $H^{(1,1,2,2)}$.

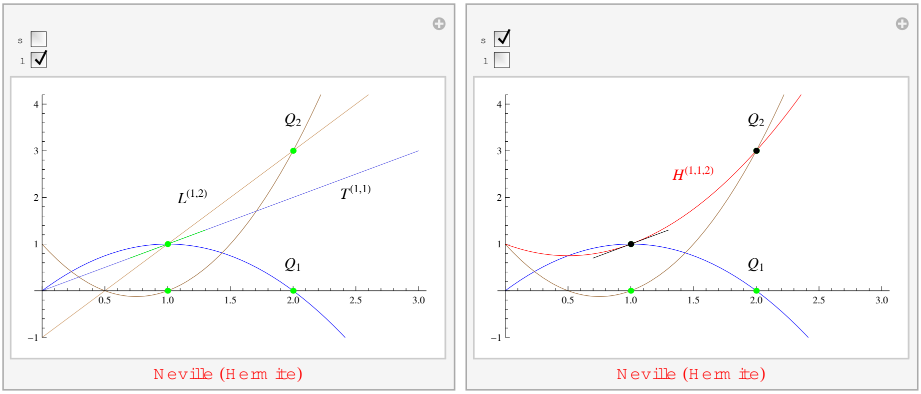

Figure 2: Combining two quadratics into one Hermite interpolation polynomial

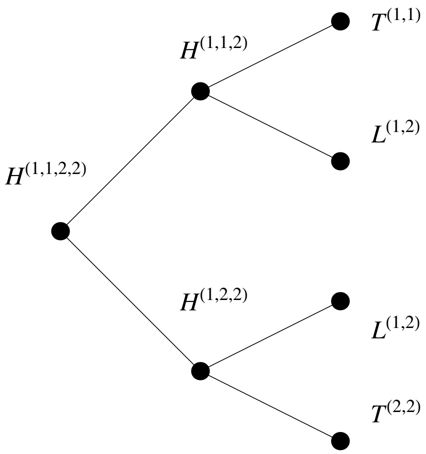

Figure 3: Recursive scheme for the Hermite polynomial $H^{(1,1,2,2)}$

Especially with computer science students, we develop a prototype code in *Mathematica* for the construction of the Hermite polynomial following the Neville's scheme (H0 is the main function, the base cases and the general recursive case are separated using *Mathematica*'s pattern matching capabilities):

```
H0[X_,F_,DF_,var_]:=H[Flatten[Transpose[{X,X}]],Flatten[Transpose[{F,DF}]],var]
H[{x_},{ff_},var_]=ff;
H[{x_,x_},{ff_,df_},var_]=ff+df(var-x);

H[X_,FF_,var_]:=1/(Last(X)-First[X])
((var-First[X]) H[Drop[X,1],Drop[FF,If[X[[1]] === X[[2]], {2}, {1}]],var] -
(var-Last[X]) H[Drop[X,-1],Drop[FF,-1],var])
```

2.4. Order of the interpolation constraints

The classic Newtonian form of the interpolation polynomial using a non-monomial basis

$$B : b_0 = 1, b_1 = (x - x_1)^1, b_2 = (x - x_1)^2, b_3 = (x - x_1)^2(x - x_2)^1, \ldots$$

is widely known and taught, see [7, 9, 10]. The construction of the triangular shaped difference table and the Newton polynomial corresponds to a specific order of the $2n$ interpolation constraints. We build up the polynomial $H = H^{(1,1,\ldots,n,n)}$ starting from a constant polynomial by successively considering one constraint at a time and forming the finite polynomial sequence

$$h : h_0 = H^{(1)} = f_1, h_1 = H^{(1,1)} = f_1 + df_1(x - x_1), \ldots, h_{2n-1} = H.$$

Recall that the first raw of the triangular table containing the generalized divided differences is the coefficient-sequence of the Hermite interpolation polynomial in the basis B. Computation of the coefficients means solving linear equations successively.

In this subsection we reconsider the Hermite interpolation sequence leading to H: We challenge the students to build up H in many different ways, e.g., by extending the Lagrange interpolation polynomial $L^{(1,2,\ldots,n)}$ to $H^{(1,1,\ldots,n,n)}$. To consider the options we have, we introduce the following definition:

Definition 2.4. We call the order of the interpolation constraints *admissible* or *compatible*, if at each node x_j, we use the constraint on the first derivative df_j only after the constraint on f_j.

To be able to play interactively with the possible interpolation sequences, we developed a Mathematica graphical tool. The user can select successively constraints via a button controller on the right. At the beginning of the experiment, no constraint is selected. Once a constraint is selected and added to the list of used constraints, the user gets a real-time visual feedback: The application shows the graphs of the previous and current interpolation polynomial in the sequence h. Interpolation constraints are also visualized. Moreover, a graph in the middle of the main panel indicates the already used constraints, the last used constraint and the still available constraints. The formula of the last element of the polynomial sequence h is also shown. The following screenshots (Figure 4 and Figure 5) show two possible starting sequences for the same three-node Hermite problem ($X = \{1, 2, 4\}$, $F = \{3, 5, 2\}$, $DF = \{1, 0, -1\}$).

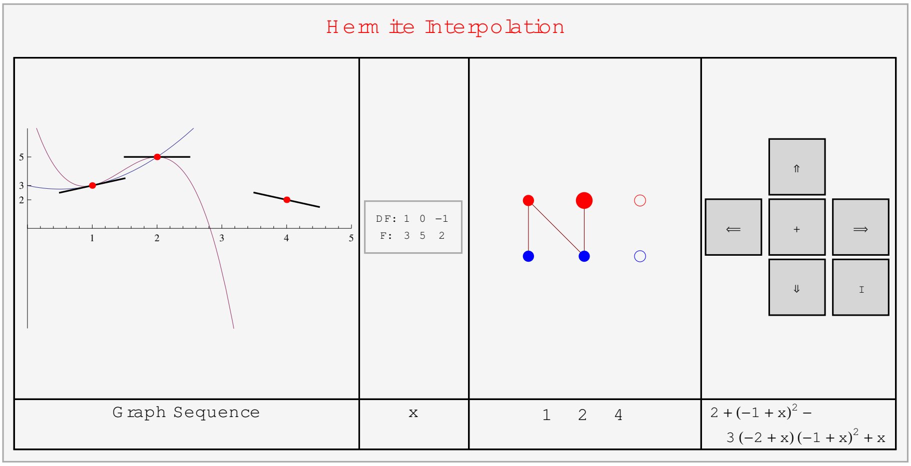

Figure 4: Starting sequence $H^{(1)}, H^{(1,1)}, H^{(1,1,2)}, H^{(1,1,2,2)}$ corresponding to the classic difference scheme ("zigzag")

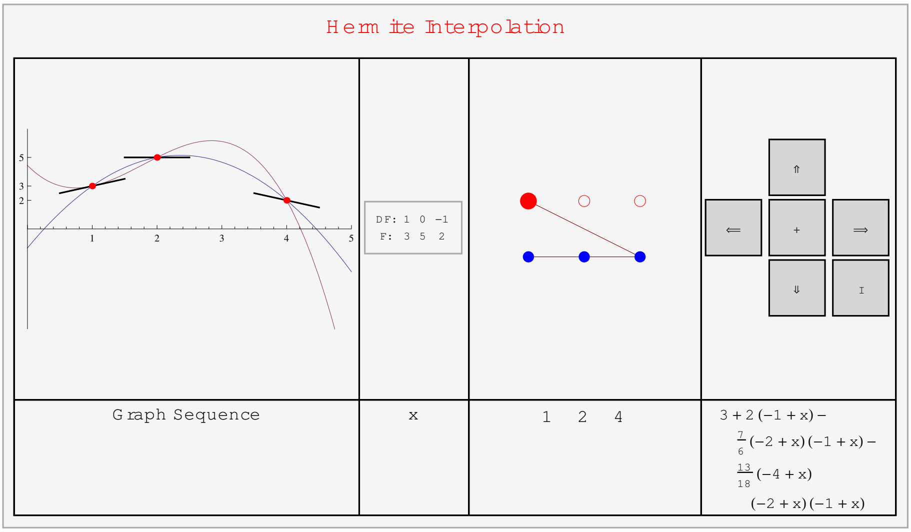

Figure 5: Starting sequence $H^{(1)}, H^{(1,2)}, H^{(1,2,3)}, H^{(1,1,2,3)}$ corresponding to the extension of the Lagrangian polynomial

After the experiment, an interesting question about the number of compatible constraint orderings may pop up among the students. This leads to an elementary combinatorial problem. We identify this scenario as an important by-product of the experiments above: The (oriented) exploration *may lead unexpectedly to new* *nice mathematical questions and conjectures* and *can connect seemingly distinct areas.*

We close this subsection with resolving this question by a different type of computer support. Assuming that the node system X consists of $n = 1, 2, 3, 4, \dots$ distinct elements, we have $1, 6, 90, 2520, \dots$ many compatible orders. Doesn't that integer sequence look familiar? Typing the first few elements into On-Line Encyclopedia of Integer Sequences (OEIS) [4], we learn that the sequence can be found as A600680 and has a closed form

$$a_n = \frac{(2n)!}{2^n}.$$

It is also very insightful that nth element of the number sequence can be interpreted as the *number of ways that $2n$ people of different heights can be arranged (for a photograph) in two rows of equal length so that every person in the front row is shorter than the person immediately behind them in the back row.*

3. Conclusion and future work

We reviewed the possible approaches to the teaching of Hermite interpolation and as an extension to the classic Lagrangian and Newtonian solutions, we proposed to show the students the generalization of Neville's simple and nice recursive idea. We developed several tools which facilitate the exploration of Hermite interpolation in a computer supported environment. One of the tools enabled the interactive exploration of the different forms of the Hermite interpolation polynomial depending on the order in which the interpolation constraints are considered. In the future, in the frame of an Austro-Hungarian project we are also interested in the computer supported verification of the correctness of the generalized Neville algorithm.

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Introduction of differential calculus in the class 10 with graphical calculator

Ferenc Várady

Budapest Business School, CCCT, Budapest, Hungary
varady.ferenc@kvifk.bgf.hu

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Abstract

In the German mathematical educational system the graphical calculator is not only allowed but also recommended in secondary grammar school. Therefore in the Deutsche Schule Budapest, where the graphical calculator based on German curriculum is also allowed and required, an experiment were conducted with two groups in grade 10. This experiment was performed in these groups for 46 lectures. The parallel teaching with and without graphical calculator was the main emphasis in the lectures. We consistently used the "old" with the "new" method simultaneously. The aim of the experiment was to determine how the graphical calculator can be helpful in the mathematics courses when it is used for introduction of differential calculus. To apply calculators for mathematical problems secondary-school students have to understand the meaning and the concepts of mathematical formulas confidently, they have to convert and interpret the parameters and substitute them into the right formulas. Besides, they have to know correctly the functions of the calculator in practice. Even better students can have difficulty in using the calculator because they need different notations on paper and booklet operation and on the software or graphical calculator. While it has been suggested that appropriate use of the graphical calculator can support students in differential calculus, the following issues were considered: how and how deep the differential calculus should be instructed with and without graphical calculator so that the students can build the right concept image.

Keywords: difference quotient, differential quotient, graphing calculator, representations, secondary grammar school

MSC: 97D43, 97D53, 97I43, 97R23

1. Introduction

Up until now the graphical calculator has been introduced in a lot of countries, including Germany, but unfortunately not in Hungary. Many studies show (see [2, 5, 11]) that the graphical calculator can have considerably important role in learning mathematics and can improve the pupils' knowledge and skills e.g. in the following areas: concept development, problem solving and computation skills. "*Using graphing calculators in mathematics education brings also new methods of work – especially the possibility of exploration and modelling of mathematical problems, multiple representation of mathematical problems (numerical, algebraic, graphic, algorithmic representation) and graphic support of the results obtained by algebraic procedures.*" (See [13].)

Earlier, before using GC in the lessons the student had no opportunist to see the dynamic of e.g. difference quotient. The teacher drew an function on the blackboard with more secant lines. The students had to be able to imagine the moving these in tangent line and to understand the mathematical meaning of their concept in order to apply it by calculating. With GC it is no problem to realise the connection between e.g. difference quotient and derivative. However with this technology pupils have not to calculate so much, as well. The pupils calculating with GC can use modern IT technique precisely but it often realised they have no idea what kind of mathematics exist in the background. The conducted experiment was *innovative* because we consistently used the "old" with the "new" method simultaneously. That means, pupils had to know the mathematical meaning of the new concepts, to work and to calculate with them without GC. At the same time they had to use GC and to know their necessary functions in this topic. During the lessons and in the post-tests pupils solved the tasks with and without GC.

This experiment evaluated two mathematical domains: the concept of difference and differential quotient and the increase and decrease of function in a given interval, concerning the fact that representation has very important role. The appropriate concept image of the difference quotient can be developed with dual representation (see [16]). Pupils have to learn changing between the enactive, iconic, and later the symbolic meaning of representation, and applying them to problem solving. In this process the graphical calculator can be helpful, according to Kamarulhaili and Sim (see [8]): "*The cognitive gain in number sense, conceptual development, and visualization can empower and motivate students to engage the true mathematical problem solving at a level preciously denied to all but the most talented. The calculator is an essential tool to all students in mathematics.*"

Nowadays, practical strategies in the mathematical teaching system are widely encouraged, and sometimes, overemphasised, thus during the experiment the following important questions were considered: Is it possible that these technical applications overshadow the theoretical knowledge? Is it possible that students cannot apply mathematical relations to different contexts, if the theory was not evidently recognizable and can forget the basic mathematical operations using the opportunities provided by the graphical calculator? How could they be trained to solve basic exercises properly also without a calculator? Considering all angles of the questions, the answers to the first two questions pointed in the same direction. It is often observed that if pupils do not get sufficient theoretical background, most of them do not recognise the basic relations in practical exercises, and they cannot generalise it. If the same question that they have already known were asked in different context they cannot recognise the relation between them, and generally, they cannot use the rules when the question is asked in different words. Meir Ben-Hur basically defined the five phases of learning Practice, De-contextualisation, Encapsulating a generalization in words, Re-contextualisation, Realisation (see [3]).The third phase, encapsulating process, where generalization has to be reinforced both in words and mathematically at the same time was highlighted during the experiment.

2. Theoretical background

2.1. The role of representation

According to the developmental theory by Bruner, the formal operational stage begins at about age 11. As adolescents enter this stage, they gain the ability to think in an abstract manner, the ability to combine and classify items in a more sophisticated way, and the capacity for higher-order reasoning (see [4]).

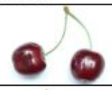
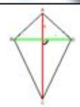
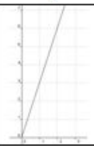
Modes:	Enactive	Iconic	Symbolic
Natural numbers	One pair of cherry		2, two, II
Geometry	Kite		Deltoid
Functions	A walker goes 3 km an hour. What is the contact between time and trip?		f: $\mathbf{R}^{+,0} \to \mathbf{R}$, f(x) = 3x

Figure 1: Modes of representation

Based on this idea, it should be expectable that tenth grade (16 years old) pupils understand and apply discretely phrased definitions and theorems. Generally many of the most talented students can reach this stage after a while, however even they need a lot of support during the process. This can be a well-chosen introductory exercise and an adequate representation. It is obvious in the case of functions where these three models of representation, enactive representation (action-based), iconic representation (image-based), and symbolic representation (language-based) can be efficiently connected (see [12]). Although the sequence of representations is very important, it is only worth going on the next level when the previous one is already well-understood.

2.2. Using information technology in teaching

The technological advancement makes the usage of the tools of information technology and the media possible. Today the range of these tools is rather diverse. We may talk about different computer applications, software, educational videos, interactive boards. In recent times the easily portable tools have come to the foreground. These might be "simple" or graphic calculators, voting-machines, and we can download several useful mathematical and scientific applications even on smart phones.

Our graphical calculator, TI-Nspire CAS, provides several functions. The main functions are:

- Calculator: Perform computations and enter expressions, equations and formulas in proper math notation.
- Graphs: Plot and explore functions, equations and inequalities, animate points on objects and graphs, use sliders to explain their behaviour and more.
- Geometry: Construct and explore geometric figures and create animations.
- Lists & Spreadsheet: Perform mathematical operations on data and visualise the connections between the data and their plots.
- Notes: Enter notes, steps, instructions and other comments on the screen alongside the math.
- Data & Statistics: Summarise and analyse data using different graphical methods such as histograms, box plots, bar and pie charts and more.
- Vernier Data Quest Application: Create a hypothesis graphically and replay data collection experiments all in a single application.

The TI-Nspire CAS handheld's innovative capabilities support teaching strategies that research has found accelerate understanding of complex mathematic and scientific concepts. Multiple representations of expressions in problems are presented simultaneously, enabling students to visualise how algebraic, graphical, geometric, numeric and written forms of those expressions relate to one another.[1] During the lessons we used the advantages of simultaneously presentations entirely.

Tablets or smart phones can have the similar function like TI-Nspire CAS depending on the used applications. There are many applications, some of them are gratis -like GeoGebra, some of them not -like e.g. WolframAlpha. This application gives functions in mathematics tool like: elementary mathematics, numbers,

[1] `https://education.ti.com/en/us/products/calculators/graphing-calculators/ti-nspire-cas-with-touchpad/tabs/overview`

plotting, algebra, matrices, calculus, geometry, trigonometry, discrete mathematics, number theory, applied mathematics, logic functions, definitions. This software is very useful and user friendly. However, similar experiment can be run on computers or other mobile devices theoretically, but using mobile devices during the lessons have disadvantages. Students have opportunities to use other applications – e.g. games and to make on-line communication which is not allowed not even during the tests.

Timo Leuders claims that applying these new technologies may be especially important in mathematics: "*Neue Technologien und neue Medien (gemeint ist meist: Computer) bieten für den Mathematikunterricht – mehr noch als die meisten anderen Schulfächer – die Chance zu einer grundlegenden inhaltlichen und methodischen Reform. Sie ermöglichen eine Entlastung von Routinearbeiten und bahnen daher exploratives und kreatives Arbeiten, ebenso die Behandlung realistischer Anwendungssituationen und das Vernetzen von Inhalten.*"[2] (See [10].) The question is how much, or to what extent is it necessary and possible to use these new opportunities in teaching. According to Tulodziecky, the different tool of information technology have to be used as support and encouragement in school education, if the teaching process is problem, decision and organisation oriented (see [19]). Tulodziecky considers the application of the media especially important in five cases:

1. Difficult exercises with decomplex initial conditions;
2. If we want to exemplify the goal or the route to the solution;
3. If the individual or cooperative work form comes to the foreground while solving a difficult task;
4. When comparing different modes of solution;
5. During the application of the theory learned and reflections to them.

In our experience, pupils use the graphic calculator with pleasure, and they even use such applications that they do not need. Finally, we would like to mention the viewpoints of Erwin Abfalterer, which have to be considered by all means when planning a lesson with the computer (graphic calculator), so that the lesson flows with the gratest efficiency in the time available:

1. The software has to be prepared and tested;
2. The flow of the lesson is planned and the goald are set;
3. The exercises are given, the role of the teacher is clear;
4. A short feedback always needs to be possible (see [1]).

[2] *"New technologies and new media (is usually meant: computer) provide for teaching mathematics - more so than most other school subjects – the opportunity for a fundamental substantive and methodological reform. They provide relief from routine tasks and thus pave explorative and creative work, as well as the treatment of realistic situations and use the cross-linking of content."*

2.3. About the use of graphical calculator

According to Horton graphical calculators can be helpful to make connection among representations in the mathematics education and therefore it can "*permit realism through the use of authentic data*" (see [6]). In 2000 the National Council of Teachers of Mathematics summarised the results from many resources (see [14, 15, 18]). Data have shown that using graphical calculators had potential benefits:

- *Speed*: after the students could handle the tool appropriately, they had opportunity to compute, graph, or create a table of values quickly.
- *Leaping Hurdles*: without technology, it was nearly impossible for students who had few skills and little understanding of fractions and integers to study algebra in a meaningful way. Consequently, lower level high school courses often became arithmetic remediation courses. With technology, all students have opportunity to study rich mathematics. They can use their calculators to perform the skills that they are unable to do themselves.
- *Connections*: sophisticated use of graphing calculator promotes students to make connections among different representations of mathematical models. Users can quickly manoeuvre among tabular, graphical, and algebraic forms.
- *Realism*: No longer are teachers restricted to using contrived data that lead to only integer or other simplistic solutions. Graphical calculators permit the creation of several types of best-fitting regression models. This capability allows data analysis to become integrated within the traditional curriculum; the tedium and difficulty of calculating a best-fit model are no longer factors in introducing data analysis into the curriculum. (pp. 24-32)

Tiwari also proved that the connection between algebraic and geometric representations with graphical calculators can be deeper in calculus; the graphical calculator can support the understanding "when it is used as a supplementary instructional tool in achieving conceptual understanding and enhancing problem solving abilities of students in learning differential calculus" (see [17]). Van Streun, Harskamp and Suhre showed (see [21]) that the use of graphical calculators could lead to changes in students' approaches in problem solving. These positive changes have affected students' successfulness. Jones got similar results (see [7]). He thought that using the graphical calculator pupils can approach problems graphically, numerically and algebraically. Ng Wee Leng found that the use of various problem-solving approaches can support students' visualization in order to find the solution and allow them to explore problem situations that they probably could not handle otherwise (see [9]).

3. Research methodology

In the Deutsche Schule Budapest, the curriculum of the German Baden- Württenberg province is applied. There, smart boards can be found in every classroom, therefore the students have the opportunity to use the software GeoGebra during the lessons. They also have to download it at home, and to do homework with the help of the software in lower classes as well. With that method, they can practice the sketching functions. Later on, in class 10, it can support them using the graphical calculator.

TI-nspire CAS type graphical calculators were used by the pupils and they can use it constantly in class and at home as well. The tenth grade mathematics curriculum starts with the topic functions. As a first step the most important definitions were reviewed followed by five more lessons soon afterwards. On these lessons the notions of the difference quotient, the derivative, derivative function, and the derivative rules were introduced together with some simple related definitions. In this chapter the graphical calculator and handmade calculation were used in parallel.

During the lessons we always had the opportunity to communicate with pupils and to help them. Furthermore, if some of them had problems, they could help each other as well. We had altogether 46 lessons for the examined topic – 11 weeks, four lessons a week. During this period the following chapters were covered:

- Lessons 1–4: Reviewing functional elements. Domain, range, zero point, function value. Practice.
- Lesson 5–8: Introducing the difference quotient. Practice.
- Lessons 9–12: The differential quotient. Practice.
- Lessons 13–16: Calculation of the differential quotient. Practice.
- Lessons 17–20: Derivative function. Practice.
- Lessons 21–24: Derivative rules. Practice.
- Lessons 25–26: Summary, practice.
- Lessons 27–28: First test.
- Lessons 29–32: Intervals of increase and decrease. Practice.
- Lessons 33–36: Maxima and minima of function (local, absolute). Practice.
- Lessons 37–38: Sketch polynomial functions. Practice.
- Lessons 39–42: All day problems with (polynomial) functions. Practice.
- Lessons 43–44: Summary, practice. Practice.
- Lessons 45–46: Second test.

4. Discussion of Experiment

The 10a class consisted of 18, the 10b consisted of 19 pupils in the observation period. The mathematical knowledge of the pupils was considered to be common. The averages of the mathematics grades was 2.73 (standard deviation: 1.46) and 2.89 (1.24) in the two classes (according to the German system, where 1 is the best and 6 is the worst.) The first test was written by 12 and 13 pupils,respectively, the rest of the class was sick on the day of the test. The averages were 2.50 (standard deviation: 1.26) and 2.57 (1.17). After the 12 students became healthy again, they wrote a similar test. The results of this post-test (2.71, standard deviation: 1.22) are not included in the average of the experiment. The second test was written by every pupil. The averages of the second test were 2.37 (standard deviation: 1.21) and 2,58 (standard deviation: 1.62). The grading scheme was following:

To ... %	95	90	85	80	75	70	65	60	55
Grade	1+	1	1-	2+	2	2-	3+	3	3-
Value	0.66	1	1.33	1.66	2	2.33	2.66	3	3.33

To ... %	50	45	40	33.3	26.7	20	0
Grade	4+	4	4-	5+	5	5-	6
Value	3.66	4	4.33	4.66	5	5.33	6

Table 1: Grading scheme

After the first 26 lessons the students wrote the first 90-minute test. It was not allowed to use a graphical calculator in part A. In this part, the tasks could be solved with simple calculation, however, the students had to know the exact definition. Since part B could not be solved without graphical calculator, the pupils had to handle it confidently.

4.1. First test

Group A

Part A
It is NOT ALLOWED to use a GC!

1. *Given the function* $f\colon f(x) = \sqrt{x}$ $(\in R^+)$. *Give the derivative function and prove it. (1+3)*
2. *Given the function* $f\colon f(x) = \frac{1}{2}x^2 + 1$.
   a) *Give the difference quotient in interval* $[0; 2]$. *(3)*
   b) *Give the geometric meaning of the result. (1)*
   c) *Give the differential of* f *at* $x_0 = 2$ *with* $h \to 0$. *(3)*
   d) *Give the derivative of function. (2)*

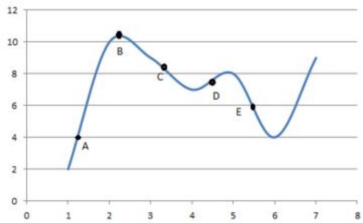
e) Give the geometric meaning of the result in c). (1)

3. See the function graph in interval ($D = [1; 7]$) and answer the questions.

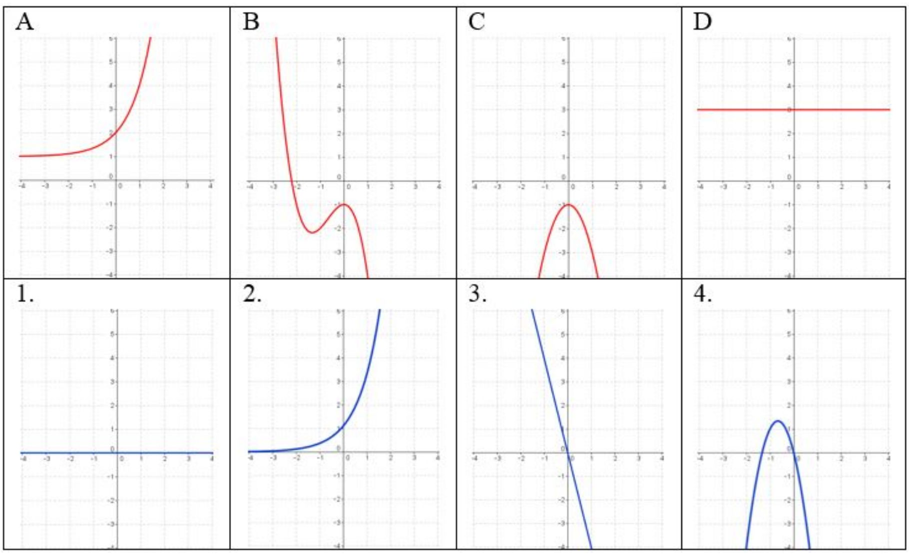
a) Give the intervals when the slope of the function (of the tangent line) is positive. (3)

b) Arrange the slopes of the tangents at the points marked in a growing order. (4)

c) Draw the approximate tangent lines at the points B, C and D, and evaluate their slopes. (1.5+1.5)



4. Couple the graphs of the functions (red) with the derivative functions (blue) and explain shortly why only the specified pairs were chosen. (4+4)



Group A

Part B
It is ALLOWED to use a GC!

5. *Give the derivative functions of the following functions, and calculate the slope at the point $x_0 = 3$. Suggestion: calculate the derivative function and the value function without a GC, and then you can check them with a GC.*

   a) $f(x) = 3x^2 - 2x + 3$ *(3+2)*

   b) $g(x) = \frac{2}{x}x^{-3}$ *(4+2)*

6. *Determine the tangent term of the function using GC:*

$$f\colon f(x) = 2x^4 - 3x^2 + x - 5$$

   at $x_0 = 2$. Describe briefly how you proceeded! Sketch the function and the tangent line using the GC on the sheet. (3+2+3)

7. *Given the function $f\colon f(x) = -2x^3 + 5x + 4$.*

   a) *Plot the function AND its derivative function with a GC, and sketch them appropriately on the sheet. (3+3)*

   b) *Explain briefly why the derivative function is just seen at the zero points, or why the derivative function is only positive / negative where you see it. (3)*

Just slight differences were measured between the two groups (group A and group B) and between the two classes, therefore we assessed the results of the two classes altogether – about the detailed results see Várady [20] and appendix. Pupils wrote the test with good outcomes which was surprisingly unexpected: the result in the theoretical part (Part A) was 71.1%. In this part lots of questions were about the meaning of tangent line and about the slope of functions without using the graphical calculator. In the graphical calculator part (Part B) the result was 78%, where pupils got questions inter alia about the meaning of tangent line and about the slope of functions. Both of them were considered a good rate in an average class and they were also balanced.

1.		2.					3.			4.	
Satz	Bew.	a)	b)	c)	d)	e)	a)	b)	c)	a)	b)
84.0	46.7	69.3	68.0	53.3	72.0	68.0	73.3	69.0	71.3	95.0	78.0

5.				6.			7.			**Average**
a1)	a2)	b1)	b2)	a)	b)	c)	a1)	a2)	b)	
82.7	80.0	68.0	60.0	74.7	76.0	85.3	92.0	84.0	74.7	**75.2 %**

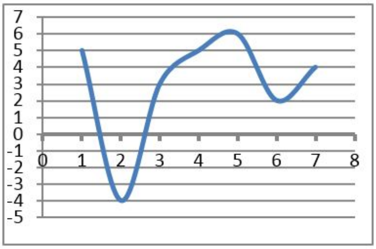
Table 2: Results of first test

4.2. Second test

The second chapter included the most important characteristics of functions and their ways of calculation - domain, range, interval of increase, interval of decrease, constant, maximum, minimum, zero of a function, end behavior. During the lessons and in the homework pupils were encouraged to do exercises with and also without the graphical calculator. There were some situations where they were not allowed to use it, and there were some, where the function describing the process was such that they were not able to do the task without a calculator. Of course, pupils wrote a test at the end of the chapter, where they were not allowed to use the calculator for some exercises, one task had to be done with and also without the calculator, and the last exercise could only be done with the help of the calculator. There were two groups in which the tasks were similar to each other. One of the task sheets looked like the following:

Part A
It is NOT ALLOWED to use a GC!

1. *Give the intervals of increase and decrease and the maxima and minima of this function. The function is defined in the interval* $[1; 7]$.



2. *Given is the function:* $f(x) = -x^3 + 6x^2$.
   a) *Give the maximal domain!*
   b) *Calculate the zero points!*
   c) *Calculate the intervals of increase and decrease and the maxima and minima of this function with the derivative function.*
   d) *Determine the y intercepts of the curve.*
   e) *Sketch the function!*

Part B
It is ALLOWED to use a GC!

3. *Determine the zero points of function! First you have to calculate without the calculator, after that you have to check them with GC. Describe your steps with GC!*

$$f(x) = (x^3 - 20)(x^4 - 28x^2 + 75)$$

4. *The next function gives the profile of a landscape in interval [1; 7]. The value x is km, the value of $f(x)$ is 10 meters.*

$$h(x) = 0.3x^4 - 5.1x_3 + 29.5x^2 - 65.1x + 45.3$$

   a) *Sketch the function with GC and draw the curve on the sheet.*

   b) *Calculate the maxima and minima of the landscape and their values with GC.*

   c) *Give the intervals of increase and decrease.*

1st task:

Perfectly done by 8 pupils. Nine pupils did not consider the closed inteval, so two end points were missing from the maxima, but the other solutions were good. The remaining 3 pupils had other mistakes as well, they defined the intervals with wrong starting and end points and/or the maxima and minima were not good either.

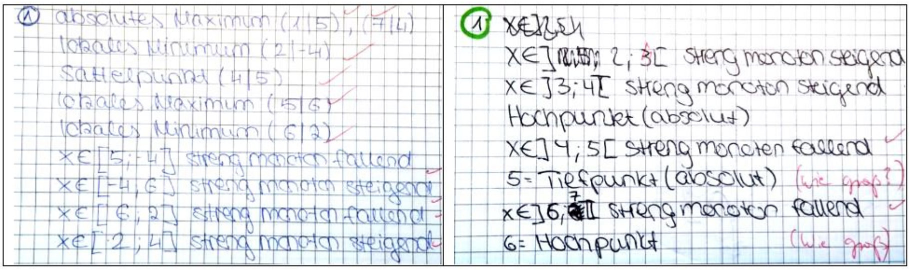


Figure 2: Pupils' solutions

2nd task:

a) Everybody answered well.

b) 18 pupils gave the derivative function well, 15 among them counted the zero points of derivative function well. The 3 pupils missed 0 as zero point (they divided by x without checking the oppurtinity $x = 0$). Two pupils derivated in the wrong way.

c) Thirteen among fifteen pupils drew the table well, two of them wrongly substituted back into the $f(x)$ function, so they got partly wrong results. 13 pupils did this part well (Figure 3).

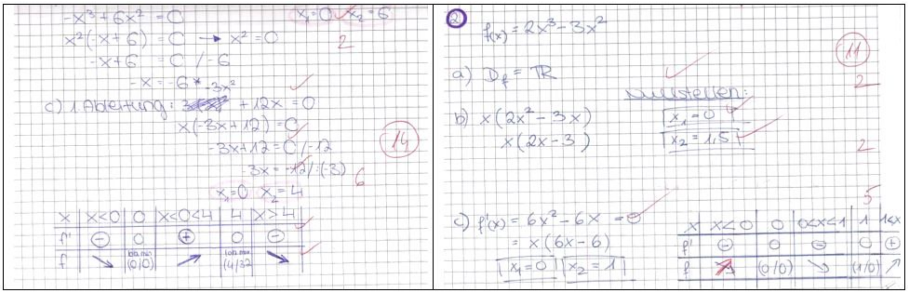


Figure 3: Pupils' solutions, table

d) Everybody determined the y intercepts of the curve well.

e) Those who did not drew the table well were naturally not able to do the sketching well. Among the 13 pupils with good tables 12 did the sketching well, one was still wrong (Figure 4).

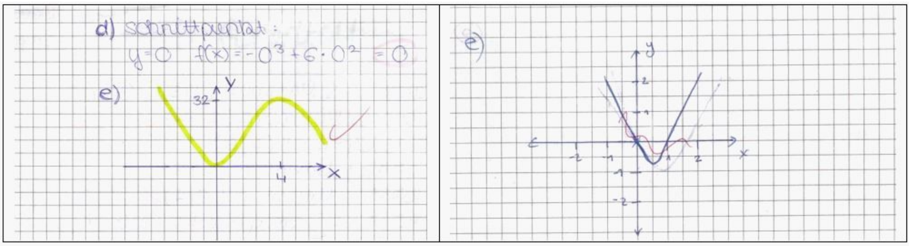


Figure 4: Pupils' results, sketching

3rd task:

Everybody was able to use the GC, they gave the command well, they got good zero points. However, five pupils got partly different results through calculations, still they did not examine the reasons for the differences (Figure 5).

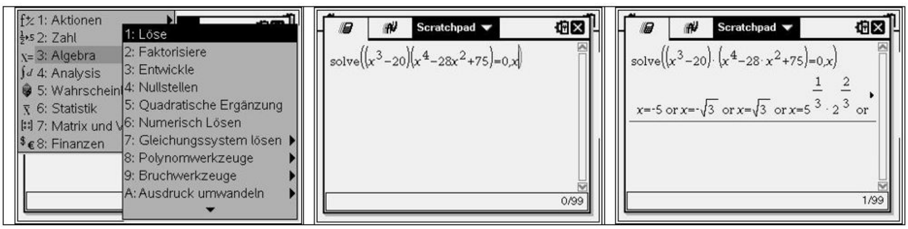


Figure 5: Calculation of zero points with GC

The following solution (Figure 6) is one of the above mentioned ones done in the other group, where the function given was the following:

$$f(x) = (x^4 - 12x^2 + 27)(x^2 - 4)$$

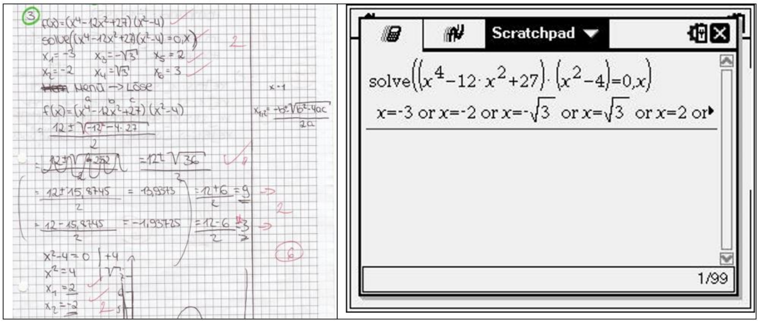


Figure 6: Partly good pupil's solution, done with GC

4th task:

In task 4 the difficulties for some pupils were caused by the fact that the graphical calculator did not sketch the function in such interval that would have been appropriate for them. First the division of the axes should have been changed and then they would have been able to sketch it correctly. Those who did not do this, got partly wrong solutions. Still, 14 pupils did the sketching well which, in our opinion, was a very good result. The maxima, minima were correctly determined within the interval by all the 14 pupils who could use the calculator, whereas only six of them paid attention to examining the end points of the closed interval. Consequently, they could determine the intervals of increase and decrease perfectly.

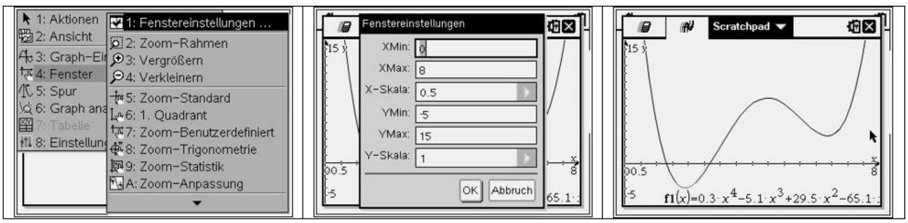


Figure 7: Calculation of zero points with graphical calculator

The results of tasks in the second test given in percentages were the following:

1.	2.					3.	4.			**Average**
	a)	b)	c)	d)	e)		a)	b)	c)	
60.3	100	92.8	61.2	100	57.1	77.4	66.7	47.6	47.6	71.07%

Table 3: Results of second test

The averages of the second test were 2.37 (standard deviation: 1.21) and 2,58 (standard deviation: 1.62). The results of the parts done with and without GC were similar. In comparison with the previous results, we can say that pupils applied theory well in the exercises and they also used the calculator well mostly. All these results were better than the expected outcome. After both tests we discussed the tasks in details. In the parts where the pupils had questions the mathematical concept and the correct use of GC was explained again. At the end the pupils could separate the real properties of the functions from their graphical, visual properties. Because GC was used before and after this experiment, the pupils' skills and their confidence were growing during the school year.

5. Conclusions

The graphical calculator during teaching mathematics is still ambiguous. In German education system mathematics teachers allow and even encourage their students to use graphical calculators in education. On the one hand, they can provide support to do exercises that need a lot of calculations. It offers a comprehensive range of opportunities that need to be managed well. It has been supposed that it could improve student success in area such as introduction of differential calculus and at the same time could provide opportunities for visualization. The results from our experiment strongly indicated that graphical calculator can be an effective tool in this examined topic of mathematical education. The technology had a positive impact on students' successfulness and triggered their ability to improve their conceptual understanding. The students manipulated it with pleasure, though the proper use of each application required a lot of attention. They gave very positive feedback about parallel using of the "old" with the "new" method. Pupils appreciated that they could understand the mathematical meaning of concepts and at the same time they could use IT technology by calculations and representations.

On the other hand, teachers have to make sure that pupils are fully aware of the mathematical background as well. It was obvious that the students could understand the mathematical concepts with this "dual" method of teaching and learning. In spite of the fact that the technology can provide multiple representation of mathematical problems, such as in our experiment, the students have to be confident about the learned processes without the using graphical calculator where it is possible. While it is necessary to develop various ways to improve the students' skills in teaching and learning mathematics, taking into consideration our results, applying graphical calculators for mathematical problems can be a successful strategy in schools.

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Appendix

			10.a												10.b													
			♥					♣							♥							♣						%
1.	Satz	*1*	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	0	1	1	1	1	0	1	1	0	84,0
	Bew.	*3*	1	0	3	3	3	3	0	0	3	0	3	0	3	3	0	0	2	0	0	0	3	0	2	3	0	46,7
2.	a)	*3*	3	3	3	3	3	3	3	1	3	0	3	3	3	2	0	3	1	1	0	1	3	2	2	3	0	69,3
	b)	*1*	1	1	1	0	1	1	1	1	1	0	0	0	1	1	0	1	1	0	1	0	1	1	1	1	0	68,0
	c)	*3*	2	3	1	3	0	2	2	1	2	0	3	0	1	3	1	2	2	1	2	2	3	0	1	3	0	53,3
	d)	*2*	2	2	2	2	1	2	2	2	2	0	2	0	1	1	1	2	2	0	2	2	2	1	1	2	0	72,0
	e)	*1*	0	1	1	0	1	1	1	1	1	0	0	0	1	1	0	1	1	0	1	1	1	1	1	1	0	68,0
3.	a)	*3*	2	2	2	3	2	3	3	3	3	0	3	1	3	3	3	2	0	2	2	3	3	1	3	3	0	73,3
	b)	*4*	4	3	0	3	4	0	4	0	4	0	4	4	4	4	1	4	0	4	4	4	4	3	3	4	0	69,0
	c)	*3*	1,5	1,5	3	2,5	3	3	1	2	3	0	3	2	2	2,5	2	2,5	0,5	2,5	2	2,5	3	2	3	3	0,5	71,3
4.	a)	*4*	4	4	4	4	4	4	2	4	4	1	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	95,0
	b)	*4*	4	4	4	4	4	4	2	4	4	0	4	2	4	4	2	4	0	2	4	2	4	3	4	4	1	78,0
5.	a1)	*3*	3	3	3	3	3	3	3	3	3	0	3	3	3	3	0	3	3	0	3	3	3	0	3	3	2	82,7
	a2)	*2*	2	2	2	2	2	2	2	2	2	0	2	0	2	2	0	2	2	0	2	2	2	0	2	2	2	80,0
	b1)	*4*	4	4	4	4	4	4	1	3	4	0	4	0	4	2	0	2	4	0	4	4	4	0	4	4	0	68,0
	b2)	*2*	2	2	2	2	2	2	0	2	2	0	2	0	2	0	0	0	2	0	2	0	2	0	2	2	0	60,0
6.	a) Gl.	*3*	3	3	3	3	0	3	3	3	3	0	3	3	3	2	0	3	3	3	3	0	3	0	3	3	0	74,7
	b) Beschr	*2*	2	2	2	2	2	2	2	2	2	0	2	2	2	0	0	2	2	2	2	0	2	0	2	2	0	76,0
	c) Dar	*3*	3	2	3	3	3	3	3	3	3	0	3	3	3	3	0	3	3	1	3	3	3	3	3	3	1	85,3
7.	a1) f	*3*	3	3	3	3	3	3	3	3	3	0	3	3	3	3	3	3	3	3	3	3	3	3	3	3	0	92,0
	a2) f'	*3*	3	3	3	3	3	3	3	0	3	0	3	0	3	3	3	3	3	3	3	3	3	3	3	3	0	84,0
	b) Beschr	*3*	3	3	3	3	3	3	3	0	3	0	3	2	3	3	0	1	3	0	3	2	3	3	3	3	0	74,7
Total		60	53,5	53	53	56,5	52	55	45	41	59	1	58	33	56	50,5	21	48,5	41,5	29,5	51	42,5	60	30	54	60	10,5	44,6
Part A %		32	79,7	79,7	78,1	89,1	84,4	84,4	68,8	62,5	96,9	3,1	93,8	53,1	87,5	92,2	46,9	82,8	42,2	54,7	71,9	70,3	100	56,3	81,3	100	17,2	71,1
Part B %		28	100,0	96,4	100,0	100,0	89,3	100,0	82,1	75,0	100,0	0,0	100,0	57,1	100	75,0	21,4	78,6	100	42,9	100	71,4	100	42,9	100	100	17,9	78,0
Total %			89,2	87,5	88,3	94,2	86,7	91,7	75,0	68,3	98,3	1,7	96,7	55,0	93,3	84,2	35,0	80,8	69,2	49,2	85,0	70,8	100	50,0	90,0	100	17,5	74,3
Grade			1	1-	1-	1	1-	1	2	3+	1+	6	1+	4+	1	2+	5+	2+	2-	4	1-	2-	1+	4	1	1+	6	